# Quantization of Algebras Defined by Ultradifferentiable Group Actions

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# Abstract

In this thesis we extend the approach of Cordes to characterize the symbols $\mathscr{S}^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$ via their Kohn-Nirenberg operators $T$ and the smoothness of the map $\rho_\lambda(-)T\rho_\lambda(-)^*$ for the Schrödinger representations $\rho_\lambda$. For this purpose we introduce generalizations $\mathscr{F}(\pi)$ of the spaces of smooth vectors $\mathscr{E}(\pi)$ and analytic vectors $\mathscr{A}(\pi)$ of representations $\pi$ and discuss properties of associated algebras $\mathscr{F}(\mathrm{Ad}_\pi)$ for the representation

$$\mathrm{Ad}_\pi T = \pi(-) \circ T \circ \pi(-)^{-1}$$

on the continuous operators. In order to apply these concepts to the ultradifferentiable case, we built on top of the existing theory of ultradifferentiable functions and create a framework for vector valued ultradifferentiable functions defined by the action of analytic frames.

We apply our results to the ultradifferentiable operators $\mathscr{E}^{[M]}_D(\mathrm{Ad}_\pi)$ and identify the corresponding spaces of symbols for the Schrödinger representations $\pi = \rho_\lambda$, for the left-regular representation $\pi = \boldsymbol{L}_2$ on compact Lie groups and for Schrödinger-type representations $\pi = \Theta_\lambda$ on the Dynin-Folland group $\mathbb{H}_2$.

We create new Gelfand triples that work well with the Fourier transform and the Kohn-Nirenberg quantizations on general homogeneous Lie groups. The new framework enables us to rely more heavily on topological tensor products. We hope this will be useful for the task of integrating the Kohn-Nirenberg quantization on homogenous Lie groups into the approach of Cordes in future research.

# Zusammenfassung


In dieser Arbeit erweitern wir den Ansatz von Cordes, in dem die Symbole $\mathscr{S}^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$ mit Hilfe ihrer Kohn-Nirenberg-Operatoren $T$ und der Differenzierbarkeit der Abbildung $\rho_\lambda(-)T\rho_\lambda(-)^{-1}$ charakterisiert werden. Hierbei bezeichnet $\rho_\lambda$ die Schrödinger-Darstellung. Dafür führen wir Verallgemeinerungen $\mathscr{F}(\pi)$ der Räume der glatten Vektoren $\mathscr{E}(\pi)$ und analytischen Vektoren $\mathscr{A}(\pi)$ ein. Außerdem diskutieren wir Eigenschaften der zugehörigen Algebren $\mathscr{F}(\mathrm{Ad}_\pi)$ zur Darstellung

$$\mathrm{Ad}_\pi T = \pi(-) \circ T \circ \pi(-)^{-1}$$

auf den stetigen Operatoren. Um diese Konzepte im ultradifferenzierbaren Fall anwenden zu können, knüpfen wir an die existierende Theorie der ultradifferenzierbaren Funktionen an und konstruieren und diskutieren Räume von vektorwertigen ultradifferenzierbaren Funktionen mit Hilfe von analytischen Rahmen.

Wir wenden diese Resultate auf die ultradifferenzierbaren Operatoren $\mathscr{E}^{[M]}_D(\mathrm{Ad}_\pi)$ an und identifizieren die zugehörigen Symbolräume. Dabei betrachten wir die Schödinger-Darstellungen $\pi = \rho_\lambda$, die linksreguläre Darstellung $\pi = \boldsymbol{L}_2$ einer kompakten Lie Gruppe und Darstellungen $\pi = \Theta_\lambda$ der Dynin-Folland-Gruppe $\mathbb{H}_2$.

Wir haben neue Gelfand-Tripel konstruiert, die sich gut in die Arbeit mit der Gruppen-Fouriertransformation und Kohn-Nirenberg-Quantisierung einfügen. Diese neuen Gelfand-Tripel ermöglichen es uns stärker von der Theorie der topologischen Tensorprodukte zu zehren. Wir hoffen, dass dies für zukünftige Forschung bei der Integration der Kohn-Nirenberg-Quantisierung auf homogenen Lie-Gruppen in den Ansatz von Cordes hilfreich sein wird.

# Contents

# Introduction

In the theory of pseudodifferential operators, one often uses spaces of operators, which are defined via the Kohn-Nirenberg quantization $\mathrm{Op}_{\mathbb{R}^n}$ of spaces of symbols $\mathcal{S}^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$. But the usual symbol classes $\mathcal{S}^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ can also be characterized on the operator side via the behaviour of their commutators with derivatives and multiplication operators. In [67] an overview of these criteria can be found. There, these characterizations are used to show that the algebras of operators on $L^p(\mathbb{R}^n)$, which are induced by the symbol spaces $\mathcal{S}^0_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$, $0 \leq \delta < 1$, via the Kohn-Nirenberg quantization, are spectrally invariant in $\mathcal{L}(L^p(\mathbb{R}^n))$. As written in [67], in the $L^2$-case there is more leeway and for $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$ the algebra $\mathrm{Op}_{\mathbb{R}^n}\left(\mathcal{S}^0_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)\right)$ is spectrally invariant in $\mathcal{L}(L^2(\mathbb{R}^n))$, see [5]. Cordes used a slightly different point of view in [15], in which $\mathrm{Op}_{\mathbb{R}^n}\left(\mathcal{S}^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)\right)$ is characterized as the smooth vectors to time-frequency shifts on the operator space $\mathcal{L}(L^2(\mathbb{R}^n))$. This can be written in terms of the Heisenberg group $\mathbb{H}$, the Schrödinger representation $\rho_1$ of $\mathbb{H}$ on $L^2(\mathbb{R}^n)$ and the corresponding adjoint representation $\mathrm{Ad}_{\rho_1} T := \rho_1(-)T\rho_1(-)^*$ on $\mathcal{L}(L^2(\mathbb{R}^n)$. Namely, the space of smooth vectors $\mathscr{E}(\mathrm{Ad}_{\rho_1})$ to $\mathrm{Ad}_{\rho_1}$ and the operator algebra $\mathrm{Op}_{\mathbb{R}^n}\left(\mathcal{S}^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)\right)$ coincide as locally convex spaces.

Using this approach, one automatically gets a simple argument for the spectral invariance of $\mathrm{Op}_{\mathbb{R}^n}\left(\mathcal{S}^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)\right)$, since the space of smooth functions with values in a Banach algebra is closed under inversion. Moreover, various continuity properties of the operators in $\mathrm{Op}_{\mathbb{R}^n}\left(\mathcal{S}^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)\right)$ can be seen as a natural consequence of the continuous multiplication between vector valued differentiable functions. Here one may use the fact that various forms of Sobolev spaces can be seen as differentiable vectors to translations or time-frequency shifts on $L^2(\mathbb{R}^n)$.

This approach can be extended to different Lie groups, as the description of the space of operators $\mathrm{Op}_{\mathbb{G}}\left(\mathcal{S}^m_{\delta,\rho}(\mathbb{G}\times\widehat{\mathbb{G}})\right)$ in [23] shows, where compact Lie groups $\mathbb{G}$ and the Kohn-Nirenberg calculus developed in [59] were considered. Here, $\mathrm{Op}_{\mathbb{G}}\left(\mathcal{S}^0_{0,0}(\mathbb{G}\times\widehat{\mathbb{G}})\right)$ and $\mathscr{E}(\mathrm{Ad}_{\boldsymbol{L}_2})$ coincide for the left regular representation $\boldsymbol{L}_2$ on $L^2(\mathbb{G},\mu)$. In [9], the approach of Cordes is modified to encompass *analytic* vectors $\mathscr{A}(\mathrm{Ad}_{\boldsymbol{L}_2})$ to $\mathrm{Ad}_{\boldsymbol{L}_2}$ on $\mathcal{L}(L^2(\mathbb{T}^n))$ for the $n$-dimensional torus $\mathbb{T}^n$. Analogously to the other cases, this algebra of operators corresponds to a space of analytic symbols via the Kohn-Nirenberg quantization.

These results are the motivations for our attempt to generalize Cordes' approach. Our goal is a generalization that encompasses the Kohn-Nirenberg quantization $\mathrm{Op}_{\mathbb{G}}$ on a larger class of Lie groups $\mathbb{G}$ and more general constructions $\mathscr{F}(\mathrm{Ad}_\pi)$ than smooth vectors $\mathscr{E}(\mathrm{Ad}_\pi)$ or analytic vectors $\mathscr{A}(\mathrm{Ad}_\pi)$, but keeps the aforementioned benefits. For this purpose, we use the theorems of Schwartz on bilinear maps on topological tensor products of locally convex spaces [64]. We are especially interested in cases, where we can identify spaces of symbols, which are homeomorphic to the operator algebra $\mathscr{F}(\mathrm{Ad}_\pi)$ via the Kohn-Nirenberg quantization.

We were able to construct a general enough approach to use ultradifferentiable vectors $\mathscr{E}^{[M]}(\mathrm{Ad}_\pi)$ together with the Kohn-Nirenberg quantization on $\mathbb{R}^n$ or arbitrary compact Lie groups $\mathbb{G}$. We generalize the concept of differentiable, smooth and analytic vectors to general representations. This way, a definition of ultradifferentiable vectors appears as a special case. The concept of ultradifferentiable vectors is not new [13], though our approach encompasses a wider variety of ultradifferentiable vectors. We built on top of the theory of vector valued ultradifferentiable functions from [39, 40, 41] and prove a description in terms of left invariant vector fields for general Lie groups as introduced in [17, 18, 19] for compact Lie groups. We identify the preimage of the constructed operators algebras under the Kohn-Nirenberg quantization, which are spaces of ultradifferentiable symbols. As mentioned, the spectral invariance and various continuity properties of these operator algebras follow immediately from our approach. Also, the statements from [9] and the cited statements concerning $\mathcal{S}^0_{0,0}(\mathbb{R}^n\times\mathbb{R}^n)$ resp. $\mathcal{S}^0_{0,0}(\mathbb{G}\times\widehat{\mathbb{G}})$ appear as special cases in this regime.

We develop new spaces of test functions, which work well with the group Fourier transform and the Kohn-Nirenberg quantization. Even for the Heisenberg group $\mathbb{G} = \mathbb{H}$ there seems to be no simple characterization for the Fourier image of the Schwartz space of rapidly decreasing functions, $\mathscr{S}(\mathbb{G})$, see [2, 31]. By using a subspace $\mathscr{S}_*(\mathbb{G})$ of $\mathscr{S}(\mathbb{G})$ instead of the whole space, we get a Fourier image that is easy to characterize and even splits into a tensor product of a space of smooth functions and a space of operators. This enables us to use the theory of bilinear maps on tensor products of locally convex spaces due to Schwartz for the multiplication operators on the Fourier side. We restrict our considerations to the case, where $\mathbb{G}$ is homogeneous and admits irreducible unitary representations that are square integrable modulo the center $Z(\mathbb{G})$ of $\mathbb{G}$, and where $\dim Z(\mathbb{G}) = 1$. This enables us to use an easy to handle characterization of the irreducible unitary representations that are square integrable modulo $Z(\mathbb{G})$ [53, 54, 33]. Also, this setting combines very well with Pedersen's machinery [57, 54]. Furthermore, using these new spaces we are able to construct Gelfand triples around $L^2(\mathbb{G}, \mu)$ and its Fourier image. Associated to this we have fitting Gelfand triples of operators such that the Kohn-Nirenberg quantization is a Gelfand triple isomorphism. We hope that these results will prove helpful in integrating the Kohn-Nirenberg quantization on homogeneous Lie groups into the approach of Cordes in future research. In this regime we are also able to recover the formula $a(-, \xi) = \xi(-)^* \cdot (A \otimes \mathrm{I})(\xi)$ for the Kohn-Nirenberg symbol $a = \mathrm{Op}_{\mathbb{G}}^{-1}(A)$, which is well known for compact Lie groups $\mathbb{G}$ or the case $\mathbb{G} = \mathbb{R}^n$ [5, 59].

This thesis is structured as follows.

In **Chapter 1**, we will introduce and revisit basic notations and concepts from functional analysis. Our main focus lies on the topic of tensor products of bilinear maps, which we will later need for the multiplication between vector valued functions. In preparation for our later analysis we start by proving the first general statements and collecting resp. adjusting theorems from the literature about topological tensor products. Another focus of this chapter is the topic of Gelfand triples and real structures. Here we build a general foundation for using Gelfand triples in context with real structures, which we will use in Chapter 3.

**Chapter 2** is dedicated to the theory of vector valued function spaces. We focus on spaces of ultradifferentiable vector valued functions and their application for the generalization of differentiable, smooth or analytic vectors of representations. Inspired by [17, 18, 19], we show that we may use limits of Banach spaces of ultradifferentiable functions defined by analytic frames in order to construct the usual Denjoy-Carleman classes of Roumieu or Beurling type. We also discuss further properties of spaces of vector valued ultradifferentiable functions. Afterwards, we show in what way the same holds for the vector valued analogues of said spaces. Equipped with this structure, we introduce and discuss ultradifferentiable vectors of representations. Here the definition via left invariant frames on Lie groups comes into play.

Furthermore, we extend the topic of rapidly decreasing and slowly increasing functions, polynomials and tempered distributions on polynomial manifolds [56]. This topic is of importance for our later introduction of new Gelfand triples for the Kohn-Nirenberg quantization on homogeneous Lie groups in Chapter 3. For this purpose, we pay special attention to the polynomial manifold $\mathbb{R}^{\times}$.

In **Chapter 3** we give an introduction of the Kohn-Nirenberg quantization on Lie groups of the Pedersen quantization in the context of Gelfand triples. Next, we use our preliminary work on functions and distributions on polynomial manifolds to first define and discuss new Gelfand triples for the group Fourier transform and afterwards define and discuss new Gelfand triples for the Kohn-Nirenberg quantization. We close the chapter by showing that the formula $\mathrm{Op}_{\mathbb{G}}^{-1}(A)(-,\pi) = \pi(-)^* \cdot (A \otimes \mathrm{I})(\pi)$ for Kohn-Nirenberg symbols holds in our regime. At the same time we discuss the usual integral formula

$$\mathrm{Op}_{\mathbb{G}}(\sigma)f = \int_{\widehat{\mathbb{G}}} \mathrm{Tr}[\pi\, \sigma(-,\pi)\, \pi(f)]\, \mathrm{d}\widehat{\mu}([\pi])$$

for the action of Kohn-Nirenberg operators in this setting.

Finally, in **Chapter 4**, we tie together Chapter 2 and Chapter 3. First we prove general properties of algebras of operators $T$ defined by regularity requirements on the maps $\mathrm{Ad}_\pi T = \pi(-)T\pi(-)^{-1}$ for representations $\pi$. Our focus lies on operators $T$, which induce ultradifferentiable maps $\mathrm{Ad}_\pi T$. Here our discussion of general ultradifferentiable functions and vectors from Chapter 2 comes into play. As an application, we discuss the operator algebras defined as smooth or ultradifferentiable vectors to the representation $\mathrm{Ad}_\pi$, in which $\pi$ is a Schrödinger representation or the left regular representation on a compact Lie group. Finally, we find corresponding spaces of symbols such that the Kohn-Nirenberg quantization is a homeomorphism onto said algebras of operators.

# Chapter 1

# Notation and basic concepts

For four sets $A_1$, $B_1$, $A_2$ and $B_2$ and two maps $f_j \colon A_j \to B_j$ we define

$$f_1 \times f_2 \colon A_1 \times A_2 \to B_1 \times B_2 \colon (x, y) \mapsto (f_1(x), f_2(y)) .$$

Furthermore, if $A, B, C$ and $D$ are sets, $f \colon A \to B$ is a function and $C \subset A$, then the restriction of $f$ to $C$ will be denoted by $f \restriction_C$. If $f(C) \subset D$ we will also write $f \restriction_C^D$,

$$f \colon C \to D \quad \text{or} \quad C \xrightarrow{f} D$$

for the restricted function $x \mapsto f(x)$ with domain $C$ and codomain $D$. If $A, B, C$ and $D$ are topological spaces, we will also write that $f \restriction_C$ resp. $f \restriction_C^D$,

$$f \colon C \to D \quad \text{resp.} \quad C \xrightarrow{f} D$$

is continuous (open, a homeomorphism) if it is so with respect to the topologies on $C$ and $B$ resp. the topologies on $C$ and $D$. The interior resp. closure of a set $M$ in a topological space $A$ is denoted by $\operatorname{Int} M$ resp. $\overline{M}$. We will reserve the notation $M^\circ$ for the **polar** of a subset $M$ of a locally convex space.

In general, we will denote by $\mathrm{I}_A$ the identity on a set $A$. If there is no risk of confusion, we will also just write $\mathrm{I}$ instead of $\mathrm{I}_A$.

We will call a linear map

$$T \colon E \to F$$

between locally convex spaces $E$ and $F$ a **linear homeomorphism** if $T$ is bijective and $T$ and $T^{-1}$ are continuous. Even though such maps are often called isomorphisms (resp. isomorphisms onto $F$), we chose the above terminology in order to avoid confusions with isomorphisms in the sense of groups, vector spaces or algebras. If $T$ is injective and the map with restricted codomain

$$T\colon E \to T(E)$$

is a linear homeomorphism with respect to the subspace topology of $T(E)$ in $F$, then we will call $T$ a linear homeomorphism onto its image.

## 1.1 Some concepts from functional analysis

For any two locally convex spaces $E$, $F$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ we denote by $\mathcal{L}(E;F)$ the space of continuous linear maps from $E$ to $F$. As usual we write $\mathcal{L}(E) := \mathcal{L}(E;E)$ and $E' := \mathcal{L}(E;\mathbb{K})$ for the dual of $E$. We equip $\mathcal{L}(E;F)$ (and $\mathcal{L}(E)$, $E'$) with the topology of uniform convergence on bounded sets of $E$.

In general, we will reserve the symbol $\|-\|$ for norms and by $\|-\|_E$ we always mean the chosen norm in a normed space $E$. The symbol $(-,-)$ is reserved for inner products and by $(-,-)_E$ we always mean the chosen inner product in a Hilbert space $E$. The dual pairing between a locally convex space $E$ and its dual $E'$ will always be denoted by

$$\langle -,- \rangle : E \times E' \colon (e, e') \mapsto \langle e, e' \rangle \ .$$

If $E$ and $F$ are normed spaces, then the topology in $\mathcal{L}(E;F)$ is normable. In this case we will equip $\mathcal{L}(E;F)$ with the operator norm $\|-\|_{\mathcal{L}(E;F)}$. If $E$, $F$ are Hilbert spaces, then we also consider the space of trace class operators $\mathcal{N}(E;F)$ and Hilbert-Schmidt operators $\mathcal{HS}(E;F)$ with norms

$$\|T\|_{\mathcal{N}(E;F)} := \mathrm{Tr}[(T^*T)^{\frac{1}{2}}] \quad \text{and} \quad \|T\|_{\mathcal{HS}(E;F)} := \mathrm{Tr}[T^*T]^{\frac{1}{2}} \ ,$$

in which Tr is the trace functional on $\mathcal{N}(E)$ and $T^*$ is the adjoint of $T$.

We will also use the following topologies and notations on the above spaces. For $\mathcal{L}(E;F)$ equipped with the topology of pointwise convergence (resp. uniform convergence on absolutely convex compact sets, resp. uniform convergence on bounded sets), we will write $\mathcal{L}_s(E;F)$ (resp. $\mathcal{L}_c(E;F)$, resp. $\mathcal{L}_b(E;F)$), with short-hands $\mathcal{L}_\alpha(E)$ and $E'_\alpha$ for $\alpha \in \{s,c,b\}$ as before.

The canonical evaluation map

$$E \to (E'_\alpha)' : e \mapsto [e' \mapsto \langle e, e'\rangle] \tag{1.1.1}$$

is well-defined and injective for each $\alpha \in \{s,c,b\}$. For $\alpha \in \{s,c\}$ it is also onto [66, Theorem 36.1].

We denote by $\mathcal{L}_\varepsilon(E'_\alpha;F)$, for $\alpha \in \{s,c,b\}$, the space $\mathcal{L}(E'_\alpha;F)$ equipped with the topology of uniform convergence on equicontinuous subsets of $E'$. Naturally, we will use the short-hand $(E'_\alpha)'_\varepsilon$ as before. Note that with respect to the evaluation map (1.1.1), we have $E \simeq (E'_\alpha)'_\varepsilon$ for any locally convex space $E$ and $\alpha \in \{s,c\}$ [66, Proposition 36.1]. If we have $E \simeq (E'_b)'_b = E''$ by the same map, $E$ is called **reflexive**.

If $E$ and $F$ are locally convex spaces and $T \in \mathcal{L}(E;F)$, then the transpose of $T$ will be denoted by $T' \in \cap_{\alpha\in\{s,c,b\}}\mathcal{L}(F'_\alpha;E'_\alpha)$, where

$$\langle e, T'f'\rangle := \langle Te, f'\rangle \qquad \text{for all } e \in E\,,\ f' \in F'\,.$$

If $T\colon E \to F$ is a continuous **antilinear** operator, then we will also define a continuous antilinear operator $T'\colon F' \to E'$, by

$$\langle e, T'f'\rangle := \overline{\langle Te, f'\rangle} \qquad \text{for all } e \in E\,,\ f' \in F'\,.$$

**Lemma 1.1.1.** *Suppose $E$ and $F$ are reflexive locally convex spaces. Then*

$$(-)'\colon \mathcal{L}(E;F) \to \mathcal{L}(F';E')\colon T \mapsto T'$$

*is a linear homeomorphism.*

*Proof.* The topology in $\mathcal{L}(F';E')$ is defined by seminorms

$$S \mapsto \sup_{f'\in B', e\in B} |\,\langle e, S\,f'\rangle\,|,$$

for bounded sets $B' \subset F'$ and $B \subset E$. Since $F$ is reflexive

$$p(f) := \sup_{f'\in B'} |\,\langle f, f'\rangle\,| \qquad \text{for} \qquad f \in F$$

defines a continuous seminorm on $F$. For $T \in \mathcal{L}(E, F)$ we get

$$\sup_{f' \in B', e \in B} |\langle e, T' f' \rangle| = \sup_{e \in B} p(T\, e).$$

Thus, $T \mapsto T'$ is continuous. Of course, $E'$ and $F'$ are reflexive as well. By identifying $E \simeq (E')'$ and $F \simeq (F')'$ the inverse of $T \mapsto T'$ is simply

$$\mathcal{L}(F'; E') \to \mathcal{L}(E; F) \colon S \mapsto S'$$

and hence continuous by the above. □

Now we will introduce a bit of notation for the topic of integration. If $X$ is a measurable space equipped with a measure $\mu$ we will write $L^p(X, \mu)$ (resp. $L^p(A, \mu)$ for a measurable subset $A \subset X$) for usual $L^p$-space with respect to the base space $X$ (resp. to the base space $A$) with respect to the measure $\mu$ (resp. to the restriction of $\mu$ to $A$). For a $f$ measurable, nonnegative function we will invoke the notation $f(x)\ \mathrm{d}\mu(x)$[1] if we mean the measure defined by

$$A \mapsto \int_A f(x)\ \mathrm{d}\mu(x)\,.$$

For $\mathbb{R}^n$ and its dual space $\mathbb{R}_n$ we will shorten the notation. If $e_1, \ldots, e_n$ is the standard basis in $\mathbb{R}^n$ and $e^1, \ldots, e^n$ is the corresponding dual basis, we will write $\mathrm{d}x$ for the unique Lebesgue measure on $\mathbb{R}^n$ resp. on $\mathbb{R}_n$ that prescribes to $[0,1]^n$ resp. $\{\sum_j t_j e^j \mid t \in [0,1]^n\}$ the volume 1. More generally, if $A$ is a measurable subset of any Cartesian product of spaces $\mathbb{R}^k$, $\mathbb{R}_k$, for $k \in \mathbb{N}$, we will use $\mathrm{d}x$ for the corresponding product measure on $A$. We will also write $L^p(A) := L^p(A,\ \mathrm{d}x)$.

For a Banach space $(E, \|-\|_E)$ and a $\sigma$-finite measure space $(X, \nu)$ we will also use the Lebesgue-Bochner spaces. We equip $E$ with its Borel algebra. A function $f \colon X \to E$ is called $\mu$-measurable if there is a sequence of measurable functions $s_n \colon X \to E$ such that $s_n(X)$ is finite and $\lim_{n\to\infty} s_n(x) = f(x)$ for $\nu$-almost all $x \in X$. For any $\nu$-measurable function we denote

$$\|f\|_{L^p(X,\nu;E)} := \left(\int_X \|f(x)\|_E^p\ \mathrm{d}\nu(x)\right)^{1/p} \quad \text{and} \quad \|f\|_{L^\infty(X,\nu;E)} := \operatorname*{ess\,sup}_{x\in X} \|f(x)\|_E$$

[1]We change the variable name $x$ depending on the context.

for $p \in [1, \infty)$. If $\mathcal{N}$ is the vector space of $\nu$-measurable functions that vanish $\nu$-almost everywhere, then the Lebesgue-Bochner spaces are defined as

$$L^p(X, \nu; E) := \left\{ f\colon X \to E \;\middle|\; f \text{ is } \nu\text{-measurable and } \|f\|_{L^p(X,\nu;E)} < \infty \right\} \Big/ \mathcal{N}$$

equipped with the norms defined by $\|f + \mathcal{N}\|_{L^p(X,\nu;E)} := \|f\|_{L^p(X,\nu;E)}$ for any $p \in [1, \infty]$.

We often need to integrate vector valued functions. For this purpose, we will use the concept of weak integrals.

**Definition 1.1.2.** *Suppose $(X, \nu)$ is a measure space and $E$ is a locally convex vector space. We will call a function $f\colon X \to E$ integrable iff there is some $e \in E$ such that for each $e' \in E'$ we have $e' \circ f \in L^1(X, \nu)$ and*

$$\langle e, e' \rangle = \int_X e' \circ f \; \mathrm{d}\nu.$$

*The element $\int_X f \; \mathrm{d}\mu := e$ is called integral over $f$. If $f$ is integrable, we will also just say that $\int_X f \; \mathrm{d}\mu$ exists (or converges) in $E$.*

From this definition, it automatically follows that

$$T \int_X f \; \mathrm{d}\nu = \int_X T \circ f \; \mathrm{d}\nu.$$

for any continuous linear or antilinear operator $T\colon E \to F$ into another locally convex space $F$. Here, the integrability of $f$ implies the integrability of $T \circ f$.

**Definition 1.1.3.** *Let $(A, \leq)$ be a directed set and let $(E_\alpha)_{\alpha \in A}$ be a family of locally convex spaces.*

*For a collection of continuous maps $j_{\alpha,\beta} \in \mathcal{L}(E_\beta; E_\alpha)$ for $\alpha, \beta \in A$ with $\alpha \leq \beta$ which also fulfil $j_{\alpha,\beta}\, j_{\beta,\gamma} = j_{\alpha,\gamma}$ and $j_{\alpha,\alpha} = \mathrm{I}_{E_\alpha}$ for $\alpha \leq \beta \leq \gamma$, the **projective limit** with respect to the $E_\alpha$, $j_{\alpha,\beta}$ is defined by*

$$\varprojlim_{\alpha \in A} (E_\alpha, j_{\alpha,\beta}) := \left\{ (e_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} E_\alpha \;\middle|\; \forall_{\alpha \leq \beta \in A}\; e_\alpha = j_{\alpha,\beta} e_\beta \right\}$$

*and equipped with the subspace topology in $\prod_{\alpha \in A} E_\alpha$.*

*Let $j_{\alpha,\beta} \in \mathcal{L}(E_\beta; E_\alpha)$, in which $\alpha, \beta \in A$ with $\alpha \geq \beta$, be a collection of continuous maps that fulfil $j_{\alpha,\beta}\, j_{\beta,\gamma} = j_{\alpha,\gamma}$ and $j_{\alpha,\alpha} = \mathrm{I}_{E_\alpha}$ for $\alpha \geq \beta \geq \gamma$. Let $j_\beta$ be the inclusion map of $E_\beta$ into $\bigoplus_{\alpha\in A} E_\alpha$ and let $H$ be the linear hull of*

$$\{j_\beta e - j_\alpha j_{\alpha,\beta} e \mid \alpha, \beta \in A\,,\ \alpha \geq \beta\,,\ e \in E_\beta\} \subset \bigoplus_{\alpha\in A} E_\alpha\,.$$

*Then the* ***inductive limit*** *with respect to the $E_\alpha$, $j_{\alpha,\beta}$ is defined as the locally convex space*

$$\varinjlim_{\alpha\in A}(E_\alpha, j_{\alpha,\beta}) := \left(\bigoplus_{\alpha\in A} E_\alpha\right)\Big/ H\,.$$

For applications, it is helpful to know that $\varinjlim_{\alpha\in A}(E_\alpha, j_{\alpha,\beta})$ carries the finest locally convex topology such that for each $\beta$ the canonical map

$$\tilde{j}_\beta \colon E_\beta \to \varinjlim_{\alpha\in A}(E_\alpha, j_{\alpha,\beta}) \colon e \mapsto j_\beta(e) + H$$

is continuous. If $F$ is a locally convex space and $T\colon \varinjlim_{\alpha\in A}(E_\alpha, j_{\alpha,\beta}) \to F$ a linear map, then $T$ is continuous iff $T \circ \tilde{j}_\beta$ is continuous for each $\beta \in A$.

## 1.2 Topological tensor products

For two locally convex spaces $E$ and $F$ over $\mathbb{C}$, we denote by $E\,\varepsilon\,F$ the $\varepsilon$-product of L. Schwartz [63]. It is defined to be the set of bilinear maps $u\colon E' \times F' \to \mathbb{C}$ such that the collections of linear maps $\big(u(-,b)\big)_{b\in B}$ and $\big(u(a,-)\big)_{a\in A}$ are equicontinuous for all equicontinuous subsets $A \subset E'$ and $B \subset F'$. We equip it with the topology of uniform convergence on products of equicontinuous sets, i.e. with the topology induced by seminorms

$$u \quad\longmapsto \sup_{e'\in A, f'\in B} |u(e', f')|$$

for equicontinuous subsets $A \subset E'$ and $B \subset F'$. If $E$ and $F$ are complete, then so is $E\,\varepsilon\,F$. By $E \otimes F$ we denote the algebraic tensor product between $E$ and $F$. By

$$(e \otimes f)(e', f') := \langle e, e'\rangle\ \langle f, f'\rangle \tag{1.2.2}$$

we identify $E \otimes F$ with a subspace $E\,\varepsilon\,F$. The **injective topology** on $E \otimes F$ is the subspace topology derived from $E\,\varepsilon\,F$. The completion with respect said topology is the

**complete injective tensor product** $E \hat{\otimes}_\varepsilon F$ of $E$ and $F$. We may always consider $E \hat{\otimes}_\varepsilon F$ and $E \varepsilon F$ to be a subspaces of the separately continuous bilinear forms on $E'_s \times F'_s$ equipped with the topology of uniform convergence on products of equicontinuous sets. The space $E$ is said to have **the approximation property** if $E \otimes F \subset E \varepsilon F$ is dense for all locally convex spaces $F$ [63, Proposition 11]. With (1.2.2) we get

$$E \hat{\otimes}_\varepsilon F = E \varepsilon F$$

for complete locally convex spaces $E$, $F$, where $E$ or $F$ has the approximation property [63, Proposition 3]. Any Hilbert space has the approximation property [36, Satz 10.16]. We will also equip $E \otimes F$ with another topology. The projective topology is the finest locally convex topology such that the canonical bilinear map

$$E \times F \to E \otimes F : (e, f) \mapsto e \otimes f$$

is continuous. Now the completion with respect to this topology is the **complete projective tensor product** $E \hat{\otimes}_\pi F$ of $E$ and $F$. The projective topology on $E \otimes F$ is finer then the injective topology, i.e. we have a continuous map

$$E \hat{\otimes}_\pi F \to E \hat{\otimes}_\varepsilon F,$$

that is just the identity on $E \otimes F$. If $E$ and $F$ are Hilbert spaces, then this map is injective, i.e. it is a continuous dense embedding [44, §43.2 (8)]. Note, in [44] Grothendieck's notion of the approximation property [34, I §5] is used, whereas we use Schwartz' notion of the approximation property [63]. However, whenever $E$ is quasi-complete, those two variants coincide on $E$.

The above discussed products between locally convex spaces $E$, $F$ and $G$ all have the following associativity property. We have the linear homeomorphism

$$(E \boxtimes F) \boxtimes G \simeq E \boxtimes (F \boxtimes G) \qquad \text{for } \boxtimes \in \{\varepsilon, \hat{\otimes}_\varepsilon, \hat{\otimes}_\pi\}\,, \tag{1.2.3}$$

in the sense that both sides are linearly homeomorphic to (a completion of) a corresponding space of trilinear maps [34, I p. 51], [63, Proposition 7]. Hence, in the future we will not distinguish between both sides. Instead we will just write $E \boxtimes F \boxtimes G$.

A locally convex space $E$ is called **nuclear** if $E \hat{\otimes}_\pi F = E \hat{\otimes}_\varepsilon F$ for each locally convex space $F$. Hence we will merely write $E \hat{\otimes} F$ for the complete projective/injective tensor product if $E$ or $F$ is nuclear. If *both* $E$ and $F$ are nuclear, then so is $E \hat{\otimes} F$ [66, Proposition 50.1]. A convenient property of a nuclear space $E$ is that any bounded set $B \subset E$ is relatively compact in the completion of $E$ [66, Proposition 50.2]. Also, all nuclear spaces have the approximation property [36, Satz 11.18]. For Fréchet spaces, the situation is especially tame. The following Proposition lists several properties of nuclear Fréchet spaces we will use throughout this thesis.

**Proposition 1.2.1.** *Suppose $E$ and $F$ are Fréchet spaces. Then*

**(i)** *$E \,\varepsilon\, F$, $E \hat{\otimes}_\varepsilon F$ and $E \hat{\otimes}_\pi F$ are Fréchet spaces.*

*If $E$ is nuclear, then the following holds.*

**(ii)** *$E$ is reflexive.*

**(iii)** *$E'$ is nuclear, barrelled and complete.*

**(iv)** *$(E \hat{\otimes} F)' \simeq E' \hat{\otimes} F'$.*

*Proof.* **(i)**: This is [44, §41.2. (7) on p. 178], [44, §44.2. (5) on p. 267] and [44, §44.2. (7) on p. 269].

**(ii)** and **(iii)**: By [66, Corollary 3 to Proposition 50.2] $E$ is Montel, i.e. $E$ is barrelled and any closed bounded set in $E$ is compact. Hence $E$ is reflexive and $E'$ is Montel due to [61, IV 5.9 and the preceding paragraph] thus especially barrelled. The strong dual $E'$ is complete by [61, IV 6.1].

**(iv)**: This can be found in [61, IV 9.9] □

If $E$ and $F$ are Hilbert spaces, we will also consider the inner product on $E \otimes F$ defined by

$$(e \otimes f, e' \otimes f') := (e, e')_E \, (f, f')_F, \qquad \text{for} \qquad e, e' \in E\,,\; f, f' \in F,$$

where $(-,-)_E$ resp. $(-,-)_F$ is the inner product in $E$ resp. $F$. The Hilbert space we get by completing $E \otimes F$ with respect to this inner product will be denoted by $E \hat{\otimes}_{\mathrm{H}} F$. Note that the analogue of (1.2.3) holds for the Hilbert space product as well.

If $E_j$, $F_j$, $j = 1, 2$ are four locally convex spaces and $S \in \mathcal{L}(E_1; E_2)$, $T \in \mathcal{L}(F_1; F_2)$, then the $\varepsilon$-product of $S$ and $T$ is defined by

$$S \,\varepsilon\, T \in \mathcal{L}(E_1 \,\varepsilon\, F_1; E_2 \,\varepsilon\, F_2) \qquad \text{with} \qquad S \,\varepsilon\, T\,(u) := u \circ (S' \times T')$$

for all $u \in E_1 \,\varepsilon\, F_1$. The following property of the $\varepsilon$-product of continuous linear maps can be found in [44, §44.4. (5) and (6) on p. 277-278].

**Lemma 1.2.2.** *Suppose $E_j$, $F_j$, $j = 1, 2$ are four locally convex spaces and $S \in \mathcal{L}(E_1; E_2)$, $T \in \mathcal{L}(F_1; F_2)$ are injective (resp. homeomorphisms onto their images), then $S \,\varepsilon\, T$ is injective (resp. a homeomorphism onto its image).*

This implies especially that for (topological) subspaces $E_0 \subset E$ and $F_0 \subset F$ we can identify $E_0 \,\varepsilon\, F_0$ with a (topological) subspace of $E \,\varepsilon\, F$ via the $\varepsilon$-product of the inclusion mappings.

The tensor product of $S$ and $T$ is the linear operator on $E_1 \otimes F_1$ defined by

$$S \otimes T(e \otimes f) = (S\, e) \otimes (T\, f) \qquad \text{for all} \qquad e \in E_1\,,\ f \in F_1\,,$$

i.e. $T \otimes S = (S \,\varepsilon\, T) \restriction_{E_1 \otimes F_1}$. We will denote the extension of $S \otimes T$ to the different types of completions of $E_1 \otimes F_1$ by the same symbol. These extensions are continuous operators

$$S \otimes T \in \mathcal{L}(E_1 \boxtimes F_1; E_2 \boxtimes F_2) \qquad \text{for} \qquad \boxtimes \in \{\hat{\otimes}_\pi, \hat{\otimes}_\varepsilon, \hat{\otimes}_{\mathrm{H}}\},$$

in which we only consider Hilbert spaces $E_j$, $F_j$ for the case $\boxtimes = \hat{\otimes}_{\mathrm{H}}$. We will also use the following notation for the Lebesgue-Bochner spaces. If $(X_j, \nu_j)$ are $\sigma$-finite measure spaces and $E_j$ Hilbert spaces for $j = 1, 2$, then it is easy to see that

$$L^2(X_j, \nu_j) \otimes E_j \to L^2(X_j, \nu_j; E_j) \colon f \otimes e \mapsto [x \mapsto f(x)\, e]$$

extends to a unitary operator

$$\psi \colon L^2(X_j, \nu_j) \,\hat{\otimes}_{\mathrm{H}}\, E_j \to L^2(X_j, \nu_j; E_j)\,. \tag{1.2.4}$$

Also, the same approach leads to the identity

$$L^2(X_1 \times X_2, \nu_1 \otimes \nu_2) = L^2(X_1, \nu_1; L^2(X_2, \nu_2))\,.$$

**Lemma 1.2.3.** *If $S$ and $T$ are linear homeomorphisms and $E_j$, $F_j$, $j = 1, 2$ are locally convex spaces, then $S \,\varepsilon\, T$ and*

$$S \otimes T \colon E_1 \boxtimes F_1 \to E_2 \boxtimes F_2 \quad \textit{for} \quad \boxtimes \in \{\hat{\otimes}_\pi, \hat{\otimes}_\varepsilon\}$$

*are linear homeomorphisms. If the $E_j$, $F_j$ are Hilbert spaces and $S$, $T$ are linear homeomorphisms (resp. unitary operators) then $S \otimes T$ is a homeomorphism (resp. a unitary operator) between $E_1 \hat{\otimes}_{\mathrm{H}} F_1$ and $E_2 \hat{\otimes}_{\mathrm{H}} F_2$.*

*Proof.* For any $S_1, S_2 \in \mathcal{L}(E_1; E_2)$ and $T_1, T_2 \in \mathcal{L}(F_1; F_2)$ we have

$$(S_1 \,\varepsilon\, T_1)(S_2 \,\varepsilon\, T_2) = (S_1 S_2) \,\varepsilon\, (T_1 T_2) \quad \text{and} \quad (S_1 \otimes T_1)(S_2 \otimes T_2) = (S_1 S_2) \otimes (T_1 T_2)\,.$$

Furthermore, $(S_1 \otimes T_1)^* = (S_1^* \otimes T_1^*)$ in the Hilbert space case. Also, the tensor or $\varepsilon$ product of identities is undoubtedly the identity. Now we can simply choose in the above $S_j$, $T_j$ as $S$, $T$, $S^{-1}$ or $T^{-1}$ in the necessary combinations and prove the statements. □

Note that we may define products of continuous *antilinear* maps $S$ and $T$. In this case, we put

$$S \,\varepsilon\, T(u)(e', f') := \overline{u(S' e', T' f')} \qquad \text{for all} \qquad u \in E_1 \,\varepsilon\, F_1\,,\ e' \in E_2'\,,\ f' \in F_2'\,. \tag{1.2.5}$$

The result is a continuous antilinear map $E_1 \varepsilon F_1 \to E_2 \varepsilon F_2$ with $S \varepsilon T(e \otimes f) = (Se) \otimes (Tf)$ for all $e \in E_1$, $f \in F_1$. As before, this also leads to continuous antilinear operators $S \otimes T \colon E_1 \boxtimes F_1 \to E_2 \boxtimes F_2$ for $\boxtimes \in \{\hat{\otimes}_\pi, \hat{\otimes}_\varepsilon, \hat{\otimes}_{\mathrm{H}}\}$, where $S \otimes T \colon E_1 \boxtimes F_1 \to T \colon E_1 \boxtimes F_1$ is an antilinear homeomorphism for $\boxtimes = \hat{\otimes}_\varepsilon$ (resp. antiunitary for $\boxtimes = \hat{\otimes}_{\mathrm{H}}$) if $S$ and $T$ are antilinear homeomorphisms (resp. antiunitary operators).

**Lemma 1.2.4.** *Let $E, F$ and $G$ be locally convex spaces, then*

$$\mathcal{L}(E; G) \to \mathcal{L}(E \,\hat{\otimes}_\varepsilon\, F; G \,\hat{\otimes}_\varepsilon\, F) \colon T \mapsto T \otimes 1$$

*is continuous.*

*Proof.* The topology on $\mathcal{L}(E \,\hat{\otimes}_\varepsilon\, F; G \,\hat{\otimes}_\varepsilon\, F)$ is induced by seminorms of the form

$$T \mapsto \sup_{u \in B} p(T\, u)$$

where $B$ is a bounded set in $E \hat{\otimes}_\varepsilon F$ and $p$ is a continuous seminorm on $G \hat{\otimes}_\varepsilon F$. These seminorms $p$ have the form

$$p(v) := \sup_{g' \in A_1} \sup_{f' \in C_1} |v(g', f')|$$

where $A_1 \subset G'$ and $C_1 \subset F'$ are equicontinuous sets. Note that the set $B \subset E \hat{\otimes}_\varepsilon F$ is bounded iff

$$\sup_{u \in B} \sup_{e' \in A_2} \sup_{f' \in C_2} |u(e', f')| < \infty$$

for all equicontinuous sets $A_2 \subset E'$ and $C_2 \subset F'$. In general, a subset $\tilde{B} \subset E$ is bounded iff $\sup_{e' \in A_2} \sup_{e' \in \tilde{B}} | \langle e, e' \rangle | < \infty$ for all equicontinuous sets $A_2 \subset E'$. Hence the set $B_{C_1} := \cup_{f' \in C_1} (1 \otimes f')(B)$ is a bounded subset of $E$. We arrive at

$$\sup_{u \in B} p((A \otimes 1)u) = \sup_{u \in B} \sup_{g' \in A_1} \sup_{f' \in C_1} |(g' A \otimes f')(u)| = \sup_{g' \in A_1} \sup_{e \in B_{C_1}} | \langle Ae, g' \rangle | ,$$

where the right hand side defines a continuous seminorm on $\mathcal{L}(E; G)$. □

The following lemma lists further properties of $E \,\varepsilon\, F$, which we need later.

**Lemma 1.2.5.** *Let $E$ and $F$ be locally convex spaces, then*

**(i)** *The maps*

$$\mathcal{L}_\varepsilon(E'_c; F) \to E \,\varepsilon\, F : T \mapsto [(e', f') \mapsto \langle Te', f' \rangle]$$
$$\mathcal{L}_\varepsilon(F'_c; E) \to E \,\varepsilon\, F : T \mapsto [(e', f') \mapsto \langle Tf', e' \rangle]$$

*are linear homeomorphisms.*

**(ii)** *If $E = \varprojlim_{\alpha \in A} (E_\alpha, j_{\alpha,\beta})$ is a projective limit, then $E \,\varepsilon\, F \simeq \varprojlim_{\alpha \in A} (E_\alpha \,\varepsilon\, F, j_{\alpha,\beta} \,\varepsilon\, \mathrm{I}_F)$.*

**(iii)** *If $E$ carries the initial topology with respect to linear maps $T_j : E \to E_j$, $j \in J$, into locally convex spaces $E_j$, then $E \,\varepsilon\, F$ carries the initial topology with respect to $T_j \,\varepsilon\, \mathrm{I}_F : E \,\varepsilon\, F \to E_j \,\varepsilon\, F$, $j \in J$.*

*Proof.* **(i):** This is proven in [63, p. 35, Corollaire 2]

**(ii):** This is shortly explained in [42, Proposition 1.5]. We will take a closer look at the involved arguments. By [44, §44.5. (4)] (or again a very short exposition in [42]), we can construct the linear homeomorphism

$$(\prod_{\alpha\in A} E_\alpha)\,\varepsilon\, F \to \prod_{\alpha\in A}(E_\alpha\,\varepsilon\, F)\colon v \mapsto \big((p_\alpha\,\varepsilon\,\mathrm{I}_F)v\big)_{\alpha\in A}\,, \tag{1.2.6}$$

in which $p_\beta\colon \prod_\alpha E_\alpha \ni (e_\alpha)_\alpha \mapsto e_\beta \in E_\beta$ for $\beta\in A$. For $\alpha,\beta\in A$, $\alpha\le\beta$ denote $J_{\alpha,\beta} := j_{\alpha,\beta}\,\varepsilon\,\mathrm{I}_F$ and $P_\alpha := p_\alpha\,\varepsilon\,\mathrm{I}_F$. Then $J_{\alpha,\alpha} = \mathrm{I}_{E\varepsilon F}$ and $J_{\alpha,\beta}J_{\beta,\gamma} = J_{\alpha,\gamma}$ for $\alpha\le\beta\le\gamma$. Thus we may construct the projective limit $\varprojlim_{\alpha\in A}(E\,\varepsilon\,F, J_{\alpha,\beta})$. Since $E$ is a subspace of $\prod_\alpha E_\alpha$ we may identify $E\varepsilon F$ with a subspace of $(\prod_\alpha E_\alpha)\,\varepsilon\,F$. Hence, it is enough to show that the above homeomorphism (1.2.6) maps $E\varepsilon F$ onto $\varprojlim_\alpha(E_\alpha\,\varepsilon\,F, J_{\alpha,\beta})$.

Using **(i)**, we may identify each $v\in(\prod_\alpha E_\alpha)\,\varepsilon\,F$ with an operator $T\in\mathcal{L}(F'_c;\prod_\alpha E_\alpha)$. For these $T$, we have $T\in\mathcal{L}(F'_c;E)$ iff $p_\alpha\circ T = j_{\alpha,\beta}\circ p_\beta\circ T$ for all $\alpha\le\beta$. Hence

$$v\in E\,\varepsilon\,F \qquad\Leftrightarrow\qquad \forall_{\alpha\le\beta}\colon J_{\alpha,\beta}P_\beta v = P_\alpha v\,.$$

For $(v_\alpha)_\alpha\in\prod_\alpha E_\alpha$ we have

$$(v_\alpha)_\alpha \in \varprojlim_\alpha(E_\alpha\,\varepsilon\,F, J_{\alpha,\beta}) \qquad\Leftrightarrow\qquad \forall_{\alpha\le\beta}\colon J_{\alpha,\beta}v_\beta = v_\alpha$$

by definition of the projective limit. Thus we get **(ii)**.

**(iii):** This is stated and proven in [43, §44.5. (4)][2]. □

A major reason to use tensor products are kernel theorems, i.e. the description of certain spaces of linear operators by tensor products.

**Definition 1.2.6.** *Let $E$ and $F$ be locally convex spaces. We define the linear map*

$$\mathcal{J}\colon F\otimes E' \to \mathcal{L}(E;F) \qquad \textit{where} \qquad \langle \mathcal{J}\,(f\otimes e')\,e, f'\rangle := \langle e, e'\rangle\,\langle f, f'\rangle\ ,$$

*for all $e\in E$, $e'\in E$, $f\in F$ and $f'\in F'$. Via $\mathcal{J}$, the tensor product $F\otimes E'$ is mapped bijectively onto the continuous finite rank operators $\mathcal{F}(E;F)$. The* ***kernel map*** *will be denoted by $\mathcal{K}$ and is defined as the inverse $\mathcal{K} := \mathcal{J}^{-1}$ defined on $\mathcal{F}(E;F)$.*

[2] Alternatively, one could express $E$ as a space linearly homeomorphic to a projective limit indexed by the finite subsets of $J$ and use **(ii)**.

*Suppose $\boxtimes \in \{\hat{\otimes}_\varepsilon, \hat{\otimes}_\pi, \hat{\otimes}_{\mathrm{H}}\}$, in which $E$ and $F$ are supposed to be Hilbert spaces for $\boxtimes = \hat{\otimes}_{\mathrm{H}}$. By equipping $\mathcal{F}(E;F)$ with a suitable topology, we may extend the domain of $\mathcal{J}$ to $F \boxtimes E'$ such that $\mathcal{M} := \mathcal{J}(F \boxtimes E')$ is in the completion of $\mathcal{F}(E;F)$.*

*If the extended map $\mathcal{J}$ is injective and its image $\mathcal{M}$ is a subspace of $\mathcal{L}(E;F)$, we will still use the symbol $\mathcal{K}$ for the inverse*

$$\mathcal{K} = \mathcal{J}^{-1} \colon \mathcal{M} \to F \boxtimes E'$$

*and call it* ***kernel map****.*

Of course, for us the cases where $\mathcal{K}$ and $\mathcal{J}$ are linear homeomorphisms are of interest. Nuclear spaces are tailor-made for the usage of kernel maps. The following proposition can be found in [66, Propositions 50.5].

**Proposition 1.2.7.** *If $E$ and $F$ are complete locally convex spaces such that $E$ is barrelled and $E'$ is nuclear and complete, then the kernel map*

$$\mathcal{K} \colon \mathcal{L}(E;F) \to F \hat{\otimes} E'$$

*is a linear homeomorphism.*

If $E$ is a nuclear Fréchet space, then the above theorem is applicable for $E$ and its dual. In this case the kernel maps

$$\mathcal{K} \colon \mathcal{L}(E;F) \to F \hat{\otimes} E' \qquad \text{and} \qquad \mathcal{K} \colon \mathcal{L}(E';F) \to F \hat{\otimes} E \tag{1.2.7}$$

are linear homeomorphisms where we used $E \simeq E''$. If $F$ is a Fréchet space, then we use the isomorphism of nuclear Fréchet spaces

$$\mathcal{L}(E;F')' \simeq \mathcal{L}(E';F) \tag{1.2.8}$$

via Proposition 1.2.1 **(iv)** and (1.2.7).

The following well known statements can be found in [44, §43.2. (7)] and [37, 2.6.9 Proposition].

**Proposition 1.2.8.** *If $H$ and $K$ are Hilbert spaces, then the kernel map*

$$\mathcal{K}\colon \mathcal{N}(H;K) \to K \mathbin{\hat{\otimes}_\pi} H' \qquad \textit{resp.} \qquad \mathcal{K}\colon \mathcal{HS}(H;K) \to K \mathbin{\hat{\otimes}_{\mathrm{H}}} H'$$

*is a linear homeomorphism resp. a unitary operator.*

Often bilinear maps occurring in the theory of locally convex spaces fail to be continuous. One example of this would be the composition map between spaces of continuous operators

$$\mathcal{L}_\alpha(E) \times \mathcal{L}_\alpha(E) \to \mathcal{L}_\alpha(E),$$

for a nonnormable locally convex space $E$, which is not continuous for $\alpha \in \{s, c, b\}$ or any other sensible topology due to an old result of B. Maissen [51]. Similarly the multiplication between spaces of smooth functions and spaces of distributions in the regime of L. Schwartz [65] is discontinuous more often than not. For example, the multiplications

$$\mathscr{S}(\mathbb{R}^n) \times \mathscr{O}_{\mathrm{M}}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n) \qquad \text{and} \qquad \mathscr{S}'(\mathbb{R}^n) \times \mathscr{O}_{\mathrm{M}}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n) \tag{1.2.9}$$

are discontinuous [49].

However, in many cases we may use hypocontinuity instead of continuity.

**Definition 1.2.9.** *Let $E, F$ and $G$ be locally convex spaces. A bilinear map $u\colon E \times F \to G$ is defined to be* ***hypocontinuous*** *if for all bounded sets $B_E \subset E$ and $B_F \subset F$ the two sets of linear maps*

$$\{u(e, -) \mid e \in B_E\} \quad \textit{and} \quad \{u(-, f) \mid f \in B_F\}$$

*are equicontinuous. We denote by $E \mathbin{\hat{\otimes}_\beta} F$ the completion of $E \otimes F$ with respect to the finest locally convex topology that makes $\otimes\colon E \times F \to E \otimes F$ hypocontinuous.*

Note that a bilinear map $u\colon E \times F \to G$ is hypocontinuous if and only if $u \restriction_{E \times B_F}$ and $u \restriction_{B_E \times F}$ is continuous for all bounded sets $B_E \subset E$ and $B_F \subset F$.

Furthermore, in the following theorem we will use the fact that for any continuous bilinear map $u$ from $E \times F$ into another locally convex space there is a unique continuous linear map $\overline{u}\colon E \mathbin{\hat{\otimes}_\pi} F \to G$ such that $u = \overline{u} \circ \otimes$. Analogously, if $u$ is hypocontinuous, there is a unique continuous map $\overline{u}\colon E \mathbin{\hat{\otimes}_\beta} F \to G$ such that $u = \overline{u} \circ \otimes$ [64, p. 10-11].

Now our first example of a bilinear map

$$\mathcal{L}(E) \times \mathcal{L}(E) \to \mathcal{L}(E) \colon (S, T) \mapsto S\,T\,,$$

which is separately continuous but not continuous, is indeed hypocontinuous for barrelled spaces $E$. This is a standard implication of the Banach-Steinhaus Theorem [66, Theorem 33.1].

**Lemma 1.2.10.** *Suppose $E, F, G$ are locally convex space and suppose $F$ is barrelled then the bilinear maps*

$$F \times \mathcal{L}(F; G) \to G \colon (f, T) \mapsto Tf \qquad \textit{and}$$
$$\mathcal{L}(E; F) \times \mathcal{L}(F; G) \to \mathcal{L}(E; G) \colon (S, T) \mapsto T \circ S$$

*are hypocontinuous.*

*Proof.* Since $\mathcal{L}(E; F)$ is equipped with the topology of uniform convergence on bounded sets, $B_{\mathcal{L}} \subset \mathcal{L}(E; F)$ is bounded iff $B_{\mathcal{L}}(B_E)$ is bounded in $F$ for every bounded $B_E \subset E$. Hence

$$T \mapsto \sup_{f \in B_F} p(Tf) \quad \text{and} \quad T \mapsto \sup_{S \in B_{\mathcal{L}}} \sup_{e \in B_E} p(TSe)$$

are continuous seminorms on $\mathcal{L}(F; G)$ for any continuous seminorm $p$ on $G$ and any bounded sets $B_F \subset F$, $B_E \subset E$ and $B_{\mathcal{L}} \subset \mathcal{L}(E; F)$.

We complete the proof by using the Banach-Steinhaus Theorem [66, Theorem 33.1], which states that any bounded set in $\mathcal{L}(F; G)$ is equicontinuous. $\square$

Linear maps on tensor factors can easily be combined to construct a linear map on the complete tensor product. The situation for bilinear maps is not as simple. However, in the context of nuclear spaces, we may use the following theorems. The first one is very similar to a corollary due to L. Schwartz [64, Corollaire on p. 38]. It was adjusted in [3, Proposition 1'] to almost the exact form we are going to use. In that case, though, the authors used quasi-complete spaces. We prefer to formulate the statement with complete spaces, because almost all spaces we will encounter are complete and by using this stricter requirement we do not need to require the *strict approximation property* [63,

p. 5]. L. Schwartz formulated his version of the statement for quasi-complete spaces, but explained in [64, Remarques on p. 38] how it may easily be adjusted to the complete setting. This argumentation can be applied to [3, Proposition 1'] as well.

**Theorem 1.2.11.** *Let $\mathscr{H}$, $\mathscr{K}$, $\mathscr{L}$, $E$, $F$ and $G$ be complete locally convex spaces and let $\mathscr{H}$ be nuclear. Suppose that*

$$u\colon \mathscr{H}\times\mathscr{K}\to\mathscr{L} \quad \text{and} \quad b\colon E\times F\to G$$

*are bilinear maps with $u$ continuous and $b$ hypocontinuous. Then there is a hypocontinuous bilinear map*

$$\overset{b}{\underset{u}{}}\colon (\mathscr{H}\,\varepsilon\, E)\times(\mathscr{K}\,\varepsilon\, F)\to\mathscr{L}\,\varepsilon\, G,$$

*that fulfils the consistency property*

$$\overset{b}{\underset{u}{}}(S\otimes e, T\otimes f) = u(S,T)\otimes b(e,f).$$

*If $\mathscr{K}$ or $F$ has the approximation property, then $\overset{b}{\underset{u}{}}$ is the unique separately continuous bilinear map fulfilling the above consistency property.*

*Proof.* Essentially, we merely need to exchange all *quasi-completions* with *completions* in the proof to [3, Proposition 1]. Since $\mathscr{H}$ is nuclear it has the approximation property, which is sufficient in this case. For the convenience of the reader we will elaborate.

By the [64, Proposition 2] there exists a hypocontinuous[3] bilinear map

$$\Gamma\colon (E\mathbin{\hat{\otimes}_\varepsilon}\mathscr{H})\times(F\,\varepsilon\,\mathscr{K})\to(E\mathbin{\hat{\otimes}_\beta}F)\,\varepsilon\,(\mathscr{H}\mathbin{\hat{\otimes}_\varepsilon}\mathscr{K}),$$

such that $\Gamma(e\otimes T, f\otimes S) = (e\otimes f)\otimes(T\otimes S)$ for all $e\in E$, $f\in F$, $T\in\mathscr{H}$ and $S\in\mathscr{K}$. Since $\mathscr{H}$ is nuclear, we have $E\mathbin{\hat{\otimes}_\varepsilon}\mathscr{H} = E\,\varepsilon\,\mathscr{H}$ and $\mathscr{H}\mathbin{\hat{\otimes}_\varepsilon}\mathscr{K} = \mathscr{H}\mathbin{\hat{\otimes}_\pi}\mathscr{K}$. Since the $\varepsilon$-product is symmetric, we can consider $\Gamma$ as a bilinear hypocontinuous map

$$\Gamma\colon (\mathscr{H}\,\varepsilon\,E)\times(\mathscr{K}\,\varepsilon\,F)\to(\mathscr{H}\mathbin{\hat{\otimes}_\pi}\mathscr{K})\,\varepsilon\,(E\mathbin{\hat{\otimes}_\beta}F).$$

Now let

$$\overline{u}\colon \mathscr{H}\mathbin{\hat{\otimes}_\pi}\mathscr{K}\to\mathscr{L} \qquad \text{and} \qquad \overline{b}\colon E\mathbin{\hat{\otimes}_\beta}F\to G$$

[3] Note that in [64] hypocontinuous maps are also called $\beta$-continuous.

be the unique continuous linear maps fulfilling $\overline{u} \circ \otimes = u$ and $\overline{b} \circ \otimes = b$. Then

$$\,^{b}_{u} := (\overline{u} \,\varepsilon\, \overline{b}) \circ \Gamma \colon (\mathscr{H} \,\varepsilon\, E) \times (\mathscr{K} \,\varepsilon\, F) \to \mathscr{L} \,\varepsilon\, G$$

is a well-defined hypocontinuous map with $\,^{b}_{u}(T \otimes e, S \otimes f) = u(S,T) \otimes b(e,f)$ for all $e \in E$, $f \in F$, $T \in \mathscr{H}$ and $S \in \mathscr{K}$.

If either $\mathscr{K}$ or $F$ has the approximation property, then $\mathscr{K} \otimes F$ is dense in $\mathscr{K} \,\varepsilon\, F$. Since $\mathscr{H}$ is nuclear, $\mathscr{H} \otimes E$ is dense $\mathscr{H} \,\varepsilon\, E$. Now the uniqueness of $\,^{b}_{u}$ follows, since any separately continuous bilinear map $v \colon H \times \tilde{H} \to K$ is already completely defined by its action on $H_0 \times \tilde{H}_0$ for dense subset $H_0 \subset H$, $\tilde{H}_0 \subset \tilde{H}$. □

The second theorem we will use is due to C. Bargetz and N. Ortner [4, Proposition 1].

**Theorem 1.2.12.** *Let $\mathscr{H}$, $\mathscr{K}$, $\mathscr{L}$, $E$, $F$ and $G$ be complete locally convex spaces and let $\mathscr{H}$ be nuclear. Suppose that*

$$u \colon \mathscr{H} \times \mathscr{K} \to \mathscr{L} \quad \text{and} \quad b \colon E \times F \to G$$

*are two hypocontinuous bilinear maps. Suppose furthermore that either one of the two properties*

- *$\mathscr{H}$ and $E$ are Fréchet spaces*
- *$\mathscr{H}$ and $E$ are duals to Fréchet spaces*

*is fulfilled. Then there is a unique hypocontinuous bilinear map*

$$\,^{b}_{u} \colon (\mathscr{H} \,\varepsilon\, E) \times (\mathscr{K} \,\varepsilon\, F) \to \mathscr{L} \,\varepsilon\, G,$$

*that fulfils the consistency property*

$$\,^{b}_{u}(S \otimes e, T \otimes f) = u(S,T) \otimes b(e,f).$$

*If $\mathscr{K}$ has the approximation property, then $\,^{b}_{u}$ is the unique separately continuous bilinear map fulfilling the above property.*

If both bilinear maps $u$ and $b$ are continuous, then one can also construct the tensor product ${}^{b}_{u}$ of these bilinear maps on the projective tensor product. This construction is much simpler than the approach due to Schwartz. Yet, since we mostly use $\varepsilon$-products of spaces, Theorem 1.2.11 and Theorem 1.2.12 are more suitable even if both $u$ and $b$ are continuous. Only if we deal with tensor products of locally m-convex algebras do we need a corresponding theorem for the projective case.

**Definition 1.2.13.** *An algebra $A$ equipped with a locally convex topology is called **locally m-convex** iff there is a set of continuous seminorms $\mathcal{P}$ on $A$ defining the topology such that*

$$p(ab) \leq p(a)\, p(b) \quad \textit{for all} \quad a, b \in A\,,\ p \in \mathcal{P}\,.$$

*In this case the multiplication is a continuous bilinear map.*

Equivalently, the algebra $A$ is locally m-convex iff there is a basis of absolutely convex and closed neighbourhoods of zero $\mathcal{U}$ such that

$$U \cdot U \subset U \quad \text{for all} \quad U \in \mathcal{U}\,.$$

The corresponding proof can be found in [52, Chapter I, Theorem 3.1].

Candidates for $\mathcal{U}$ and $\mathcal{P}$ can be constructed from each other. If $\mathcal{P}$ is given, then $\mathcal{U}$ can be constructed by using the subbasis $\{p^{-1}([0,\varepsilon]) \mid p \in \mathcal{P}\,,\ 0 < \varepsilon \leq 1\}$. Similarly, if $\mathcal{U}$ is given, then $\mathcal{P}$ can be defined as the set of gauge functions for the neighbourhoods in $\mathcal{U}$.

**Proposition 1.2.14.** *Suppose $A$ and $B$ are locally m-convex algebras. Then there is a unique multiplication on $A \hat{\otimes}_\pi B$ such that*

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)\,, \qquad \textit{for all } a_1, a_2 \in A\,,\ b_1, b_2 \in B\,.$$

*Equipped with this topology $A \hat{\otimes}_\pi B$ is a locally m-convex algebra.*

*Proof.* $A \otimes B$ equipped with the subspaces topology in $A \hat{\otimes}_\pi B$ is a locally m-convex algebra [52, Chapter X, Proposition 3.1]. Of course, its completion, $A \hat{\otimes}_\pi B$, is locally m-convex as well [52, Chapter I, Lemma 4.1]. $\square$

## 1.3 Real structures

**Definition 1.3.1.** *Suppose $E$ is a locally convex space over $\mathbb{C}$. A* ***real Structure*** *on $E$ is an antilinear homeomorphism $\mathcal{C}\colon E \to E$ with $\mathcal{C}^2 = \mathrm{I}_E$. We will call such a pair $(E, \mathcal{C})$ a* ***locally convex space with a real structure****.*

Let us denote by $E_{\mathbb{R}}$ the locally convex space we get by restricting the scalar multiplication from $\mathbb{C} \times E$ to $\mathbb{R} \times E$. Connected to a real structure $\mathcal{C}$ on $E$ is always a splitting

$$E_{\mathbb{R}} = \mathrm{Re}(E) \oplus \mathrm{i}\,\mathrm{Im}(E), \quad \text{where} \quad \mathrm{Re}(E) = \mathrm{Im}(E), \tag{1.3.10}$$

into a real and an imaginary subspace with respect to projections

$$\mathrm{Re} = \frac{1}{2}(\mathrm{I}_E + \mathcal{C}) \quad \text{and} \quad \mathrm{i}\,\mathrm{Im} = \frac{1}{2}(\mathrm{I}_E - \mathcal{C}).$$

In the other direction, each splitting (1.3.10) defines a real structure $\mathcal{C}(v + \mathrm{i}w) := v - \mathrm{i}w$, for $v, w \in \mathrm{Re}(E)$, on $E$. We may also define a canonical real structure on the dual $E'_\alpha$ for $\alpha \in \{s, c, b\}$ by

$$\mathcal{C}'\colon E' \to E', \quad \text{where} \quad \langle e, \mathcal{C}'\, e' \rangle = \overline{\langle \mathcal{C}\, e, e' \rangle}, \quad \text{for } e \in E,\ e' \in E'. \tag{1.3.11}$$

This real structure induces an isomorphism

$$\mathrm{Re}(E'_\alpha) \cong \mathrm{Re}(E)'_\alpha \quad \text{where} \quad \mathrm{Re}(e') \mapsto \mathrm{Re}(e') \restriction_{\mathrm{Re}(E)} . \tag{1.3.12}$$

If we take a Hilbert space $H$, then a real structure on $H$ is connected with a unitary map between $H$ and its dual $H'$. Namely, each unitary isomorphism $H \simeq H'$ corresponds to an antiunitary map on $H$. Note that for continuous antilinear maps $T\colon H_1 \to H_2$ the adjoint is defined via $(Th_1, h_2)_{H_2} = (T^* h_2, h_1)_{H_1}$ and $T$ is called antiunitary if $T$ is bijective with $T^* = T^{-1}$. The Fréchet-Riesz map

$$\mathcal{R}\colon H \to H'\colon h \mapsto (-, h)_H$$

is an example of an antiunitary map.

We use unitary maps $\mathcal{I}\colon H \to H'$ between Hilbert spaces and their duals, because on the one hand they are convenient in concert with *Gelfand triples* as described in the following section and on the other hand they simplify the kernel map $\mathcal{K}$ in Proposition 1.2.8. We will use unitary maps $\mathcal{HS}(H;K) \stackrel{\mathcal{K}}{\to} K \,\hat{\otimes}_{\mathrm{H}}\, H$ resp. linear homeomorphisms $\mathcal{N}(H;K) \stackrel{\mathcal{K}}{\to} K \,\hat{\otimes}_{\pi}\, H$ defined via

$$\mathcal{J}(k \otimes h)\,\tilde{h} := \mathcal{K}^{-1}(k \otimes h)\,\tilde{h} := \langle \tilde{h}, \mathcal{I}\, h\rangle \; k \tag{1.3.13}$$

for $h, \tilde{h} \in H$ and $k \in K$. The following lemma describes why it is natural to consider real structure in order to get an identification $H \simeq H'$.

**Lemma 1.3.2.** *Let $H$ be a Hilbert space. Each unitary map $\mathcal{I}\colon H \to H'$ with $\langle h_2, \mathcal{I}\, h_1\rangle = \langle h_1, \mathcal{I}\, h_2\rangle$, for $h_1, h_2 \in H$, defines an antiunitary real structure $\mathcal{C}$ on $H$ and vice versa. Both maps are related by $\mathcal{I} = \mathcal{R}\,\mathcal{C}$.*

*Proof.* For $h_1, h_2 \in H$ and an antiunitary real structure $\mathcal{C}$ we have

$$\langle h_1, \mathcal{RC}\, h_2\rangle = (h_1, \mathcal{C}\, h_2) = (\mathcal{C}^2\, h_2, \mathcal{C}\, h_1) = \langle h_2, \mathcal{RC}\, h_1\rangle\,.$$

Hence $\mathcal{I} = \mathcal{RC}\colon H \to H'$ is unitary with the appropriate symmetry properties.

Suppose $\mathcal{I}$ is some unitary map as described in the lemma. Then $\mathcal{C} = \mathcal{R}^{-1}\mathcal{I}$ is antiunitary and

$$(h_1, \mathcal{C}h_2) = \langle h_1, \mathcal{I}\, h_2\rangle = (h_2, \mathcal{C}h_1).$$

Hence $\mathcal{C}^* = \mathcal{C}$ and $\mathcal{C}$ is an antiunitary real structure. □

## 1.4 Gelfand triples

Gelfand triples are a convenient setting for both distributions and the Fourier transform. We start by defining the class of Gelfand triples we are going to use.

**Definition 1.4.1.** *A **Gelfand triple** is a tuple of spaces $\mathcal{G} = (E, H, E')$ fulfilling the following properties:*

**(i)** *$E$ is a nuclear Fréchet space and $E'$ is its strong dual.*

**(ii)** *$H$ is a Hilbert space, with dense and continuous embedding $E \hookrightarrow H$.*

Because the embedding $E \hookrightarrow H$ is continuous and dense, the dual map $H' \hookrightarrow E'$ is a continuous dense embedding as well. Classically in Gelfand triples an antilinear embedding of $E$ and $H$ into $E'$ [30] is used. Here the embedding is defined via the Fréchet-Riesz isomorphism $\mathcal{R}\colon H \xrightarrow{\cong} H'$ and the dual embedding $H' \hookrightarrow E'$. However, for us this approach would be unwieldy, because we will use Gelfand triples in concert with tensor products. Since there is no canonical unitary map between $H$ and $H'$, we are going to use real structures to fix one.

**Definition 1.4.2.** *A **real structure** on a Gelfand triple $\mathcal{G} = (E, H, E')$ over $\mathbb{C}$ is a triple of real structures $\mathcal{C}_E$ on $E$, $\mathcal{C}_H$ on $H$ and $\mathcal{C}_{E'}$ on $E'$ such that*

**(i)** $\langle e, \mathcal{C}_{E'}\, e'\rangle = \overline{\langle \mathcal{C}_E\, e, e'\rangle}$ *for all $e \in E$, $e' \in E'$.*

**(ii)** *$\mathcal{C}_H$ is antiunitary,*

**(iii)** *if $\iota\colon E \hookrightarrow H$ is the Gelfand triple embedding, then $\mathcal{C}_H\, \iota = \iota\, \mathcal{C}_E$.*

*The map $\mathcal{C} = \mathcal{C}_{E'}$ will be called the real structure of $\mathcal{G}$.*

Each real structure on the Gelfand triple $\mathcal{G} = (E, H, E')$ defines a unitary map $\mathcal{I}$ in the sense of Lemma 1.3.2, i.e.

$$\mathcal{I} = \mathcal{R} \circ \mathcal{C}_H\colon H \to H', \tag{1.4.14}$$

where $\mathcal{R}$ is the Fréchet-Riesz map $H \to H'$. If $\iota\colon E \hookrightarrow H$ is the embedding defined by the Gelfand triple, this results in a continuous, dense embedding

$$\mathcal{I}\colon H \hookrightarrow E' \quad \text{in which} \quad \mathcal{I} = \iota'\, \mathcal{R}\, \mathcal{C}_H \tag{1.4.15}$$

Using a real structure on the Gelfand triple in order to define such an isomorphism is quite natural. Indeed any unitary isomorphism $H \cong H'$ that fulfils the symmetry condition from Lemma 1.3.2 induces a real structure on $H$. If this real structure is supposed to pull back to a homeomorphism on $E$, then this already fixes a unique real structure $\mathcal{G}$.

Also, if $\|\cdot\|_H$ is the norm on $H$, it is clear that any real structure $\mathcal{C}_E$ on $E$ such that $\|\iota\, \mathcal{C}_E\, e\|_H = \|\iota\, e\|_H$ for all $e \in E$ already fixes a unique real structure on the whole Gelfand triple $\mathcal{G} = (E, H, E')$.

**Convention 1.4.3.** *From now on, we will only consider Gelfand triples* $\mathcal{G} = (E, H, E')$ *over* $\mathbb{C}$ *if we do not say otherwise. We will think of any such Gelfand triple to be equipped with a real structure. We consider* $E$ *to be a subspace of* $H$ *and* $H$ *to be a subspace of* $E'$ *via the embeddings*

$$E \overset{\iota}{\hookrightarrow} H \overset{\mathcal{I}}{\hookrightarrow} E'$$

*in the sense of* (1.4.15)*. Similarly, we will regard* $H$ *and* $H'$ *to be the same topological vector space via* (1.4.14)*. Furthermore, we will not distinguish between the real structures on* $E$*,* $H$ *and* $E'$ *and denote them by the same letter, say* $\mathcal{C}$*. In this sense we will just write*

$$\langle e, e' \rangle \quad \text{and} \quad (e, e') := \langle e, \mathcal{C}e' \rangle$$

*for* $(e, e')$ *in* $E \times E'$*,* $E' \times E$ *or* $H \times H$*, where* $\langle \cdot, \cdot \rangle$ *denotes (depending on the situation) the dual pairing on* $E \times E'$*,* $E' \times E$ *or* $H \times H$ *induced by* (1.4.14)*. Note that for* $e, e' \in H$ *the pairing* $(e, e')$ *is just the inner product on* $H$*.*

The commutative diagrams below elucidate this notion. Using the real structure $\mathcal{C}$ and the corresponding projection Re we may identify real subspaces which form a Gelfand triple over $\mathbb{R}$ with the isomorphism $\mathrm{Re}(E') \cong \mathrm{Re}(E)'$ from (1.3.12).

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, "\mathcal{C}"'] & H \arrow[r, hook] \arrow[d, "\mathcal{C}"'] & E' \arrow[d, "\mathcal{C}"'] \\ E \arrow[r, hook] & H \arrow[r, hook] & E' \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, hook] \arrow[d, "\mathrm{Re}"'] & H \arrow[r, hook] \arrow[d, "\mathrm{Re}"'] & E' \arrow[d, "\mathrm{Re}"'] \\ \mathrm{Re}(E) \arrow[r, hook] & \mathrm{Re}(H) \arrow[r, hook] & \mathrm{Re}(E') \end{tikzcd}$$

The diagram on the right hand side justifies the choice of the term *real structure* on a Gelfand triple. Also, using the dual pairings $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$ not only with arguments $(e, e') \in E \times E'$, but also with arguments $(e, e') \in E' \times E$ seems to clash with our previous convention, where we used functionals in the second argument. In order to remedy this, we use the canonical isomorphism $E \simeq E''$.

**Definition 1.4.4.** *Let* $\mathcal{G} = (E, H, E')$ *be a Gelfand triple with real structure* $\mathcal{C}$*. Suppose* $F$ *is a locally convex space equipped with continuous dense embeddings,*

$$\begin{tikzcd} E \arrow[r, hook] \arrow[rr, hook, bend right, "\subset"'] & F \arrow[r, hook] & E' \end{tikzcd}$$

*that commute with the inclusion map as described in the commutative diagram above. If in addition the restriction of $\mathcal{C}$ from $E'$ to $F$ is a homeomorphism from $F$ to $F$, then we will call $F$ a $\mathcal{G}$-**regular space**.*

If $F$ is $\mathcal{G}$-regular and reflexive, then $F'$ is $\mathcal{G}$-regular as well. In order to prove this, we may consider the dense dual embeddings

$$\begin{tikzcd} E \arrow[r, hook] \arrow[rr, hook, bend right] & F' \arrow[r, hook] & E', \end{tikzcd}$$

where we used the canonical isomorphism $E \simeq E''$. The real structure acts on $E'$ by

$$\langle e, \mathcal{C}\, e' \rangle = \overline{\langle \mathcal{C}\, e, e' \rangle} \qquad \text{for} \qquad e, \in E, e' \in E'.$$

Thus $\mathcal{C}$ also restricts to a homeomorphism from $F'_\alpha$ onto $F'_\alpha$ for each $\alpha \in \{s, c, b\}$, because $\mathcal{C}$ restricts to a homeomorphism from $F$ onto $F$. Hence $F'$ is $\mathcal{G}$-regular.

If $\mathcal{G}_j = (E_j, H_j, E'_j)$ $j = 1, 2$, are Gelfand triples with real structure $\mathcal{C}_1$, $\mathcal{C}_2$, $F$ is $\mathcal{G}_1$-regular and $G$ is $\mathcal{G}_2$-regular spaces, then $F'$ resp. $G'$ are $\mathcal{G}_1$- resp. $\mathcal{G}_2$-regular. By the corresponding embeddings due to the $\mathcal{G}_j$-regularity, we may embed $\mathcal{L}(F;G)$ continuously into $\mathcal{L}(E_1;E'_2)$ and $\mathcal{L}(G';F')$ continuously into $\mathcal{L}(E_2;E'_1)$. Using these identifications we formulate the following definition.

**Definition 1.4.5.** *Let $\mathcal{G}_j$, $\mathcal{C}_j$, $F$ and $G$ be be above. We define the **adjoint** $T^* \in \mathcal{L}(G';F')$ of $T \in \mathcal{L}(F;G)$ by*

$$T^* := \mathcal{C}_1\, T'\, \mathcal{C}_2 \qquad i.e. \qquad (T^*\, e, e') = (e, T e') \qquad \textit{for all} \qquad e \in E_2\,,\ e' \in E_1.$$

On continuous operators between $H_1$ and $H_2$ this definition coincides with the usual adjoint, via the identification of $H_j$ with $H'_j$.

**Lemma 1.4.6.** *Suppose $\mathcal{G}_j$ is a Gelfand triple with real structure $\mathcal{C}_j$ for $j = 1, 2$. If $F$ is $\mathcal{G}_1$-regular and $G$ is $\mathcal{G}_2 -$ regular and both are reflexive, then the adjoint defines an antilinear homeomorphism*

$$\mathcal{L}(F;G) \to \mathcal{L}(G';F') \colon T \mapsto T^*.$$

*Proof.* Since $G$ is reflexive, the map

$$\mathcal{L}(F;G) \to \mathcal{L}(G';F') \colon T \to T'$$

is a homeomorphism by Lemma 1.1.1. The rest follows with the fact that $\mathcal{C}_1$ restricts to a homeomorphism from $F'$ to $F'$ and $\mathcal{C}_2$ restricts to a homeomorphism from $G'$ to $G'$. $\square$

It is helpful to have a term for maps that behave well with the Gelfand triple structure.

**Definition 1.4.7.** *Let $\mathcal{G}_j = (E_j, H_j, E_j')$, $j = 1, 2$, be Gelfand triples and let $T \colon E_1' \to E_2'$ be linear. We write*

$$T \colon \mathcal{G}_1 \to \mathcal{G}_2,$$

*if $T(E_1) \subset E_2$ and $T(H_1) \subset H_2$ with respect to the above described embeddings. We will call $T$ a* ***Gelfand triple isomorphism*** *if $T \restriction_{E_1}^{E_2} \colon E_1 \to E_2$, $T \colon E_1' \to E_2'$ are homeomorphisms and $T \restriction_{H_1}^{H_2} \colon H_1 \to H_2$ is unitary.*

The above definition implies that writing $T \colon \mathcal{G}_1 \to \mathcal{G}_2$ is equivalent to saying that the diagram

$$\begin{array}{ccccc} E_1 & \hookrightarrow & H_1 & \hookrightarrow & E_1' \\ {\scriptstyle T}\downarrow & & {\scriptstyle T}\downarrow & & {\scriptstyle T}\downarrow \\ E_2 & \hookrightarrow & H_2 & \hookrightarrow & E_2' \end{array}$$

is commutative. In order to identify a Gelfand triple isomorphism $T \colon \mathcal{G}_1 \to \mathcal{G}_2$, it is enough to examine its action on $E_1$. To be more precise, if $S \colon E_1 \to E_2$ is a linear homeomorphism such that $\|S\,e\|_{H_2} = \|e\|_{H_1}$ for all $e \in E_1$, then there is exactly one Gelfand triple isomorphism $T$ with $T \restriction_{E_1}^{E_2} = S$. This Gelfand triple is defined by

$$T = (S^{-1})^* = \mathcal{C}_2\,(S^{-1})'\,\mathcal{C}_1\,,$$

where $\mathcal{C}_j$ is the real structure to $\mathcal{G}_j$ for $j = 1, 2$.

In the following, we denote by $H_1 \oplus_{\mathrm{H}} H_2$ the Hilbert space sum of two Hilbert spaces $H_1$ and $H_2$. In other words $H_1 \oplus_{\mathrm{H}} H_2$ is the direct sum $H_1 \oplus H_2$ equipped with the norm defined by $\|h_1 \oplus h_2\|^2_{H_1 \oplus_{\mathrm{H}} H_2} := \|h_2\|^2_{H_1} + \|h_2\|^2_{H_2}$ for $h_1 \in H_1$ and $h_2 \in H_2$.

Now we will describe how we may construct new Gelfand triples by using tensor products and direct sums.

**Definition 1.4.8.** *Let $\mathcal{G}_j = (E_j, H_j, E_j')$ be Gelfand triples with structure maps $\mathcal{C}_j$ for $j = 1, 2$. Using the identifications $E_1' \oplus E_2' \simeq (E_1 \oplus E_2)'$ and $E_1' \hat{\otimes} E_2' \simeq (E_1 \hat{\otimes} E_2)'$ due to Proposition 1.2.1 resp. $\mathcal{L}(E_1, E_2') \simeq \mathcal{L}(E_1', E_2)'$ due to* (1.2.8) *we may define the following Gelfand triples.*

*The sum resp. tensor product of $\mathcal{G}_1$ and $\mathcal{G}_2$ is defined by*

$$\mathcal{G}_1 \oplus \mathcal{G}_2 := \begin{pmatrix} E_1 \oplus E_2 \\ H_1 \oplus_{\mathrm{H}} H_2 \\ E_1' \oplus E_2' \end{pmatrix} \quad \textit{resp.} \quad \mathcal{G}_1 \otimes \mathcal{G}_2 := \begin{pmatrix} E_1 \hat{\otimes} E_2 \\ H_1 \hat{\otimes}_{\mathrm{H}} H_2 \\ E_1' \hat{\otimes} E_2' \end{pmatrix}$$

*with structure maps $\mathcal{C}_1 \oplus \mathcal{C}_2$ resp. $\mathcal{C}_1 \otimes \mathcal{C}_2$.*

*The operator Gelfand triple from $\mathcal{G}_1$ to $\mathcal{G}_2$ is defined as*

$$\mathcal{L}(\mathcal{G}_1; \mathcal{G}_2) := \begin{pmatrix} \mathcal{L}(E_1', E_2) \\ \mathcal{HS}(H_1, H_2') \\ \mathcal{L}(E_1, E_2') \end{pmatrix}$$

*with structure map $T \mapsto \mathcal{C}_2 \, T \, \mathcal{C}_1$.*

Let us now discuss why $\mathcal{G}_1 \otimes \mathcal{G}_2$ and $\mathcal{L}(\mathcal{G}_1; \mathcal{G}_2)$ are indeed Gelfand triples. Integral to our argumentation are the real structures and kernel maps $\mathcal{K} = \mathcal{J}^{-1}$. Here we use the adjustment (1.3.13) for the kernel maps

$$\mathcal{N}(H_1, H_2) \simeq H_2 \hat{\otimes}_\pi H_1 \quad \text{resp.} \quad \mathcal{HS}(H_1, H_2) \simeq H_2 \hat{\otimes}_{\mathrm{H}} H_1,$$

from Proposition 1.2.8. To be precise, these unitary isomorphisms are defined by extending the linear map

$$\mathcal{J} \colon H_2 \otimes H_1 \to \mathcal{F}(H_1; H_2), \quad \text{where} \quad \mathcal{J}(h_2 \otimes h_1)(\tilde{h}_1) := \langle h_1, \tilde{h}_2 \rangle \; h_2 \,.$$

These isomorphisms, together with (1.2.7), can be used to construct the following chain of injective continuous maps with dense ranges

$$\begin{array}{ccccc} E_2 \hat{\otimes} E_1 & \hookrightarrow & H_2 \hat{\otimes}_\pi H_1 & & H_2 \hat{\otimes}_{\mathrm{H}} H_1 \\ \mathcal{K} \big\uparrow \simeq & & \mathcal{K}^{-1} \big\downarrow \simeq & & \mathcal{K} \big\uparrow \simeq \\ \mathcal{L}(E_1'; E_2) & & \mathcal{N}(H_1; H_2) & \hookrightarrow & \mathcal{HS}(H_1; H_2) \end{array}$$

We get additional continuous embeddings with dense ranges

$$\mathcal{L}(E_1'; E_2) \hookrightarrow \mathcal{N}(H_1; H_2) \qquad \text{and} \qquad H_2 \mathbin{\hat{\otimes}_\pi} H_1 \hookrightarrow H_2 \mathbin{\hat{\otimes}_{\mathrm{H}}} H_1$$

by completing the commutative diagram horizontally. This both shows that $\mathcal{L}(\mathcal{G}_1; \mathcal{G}_2)$ and $\mathcal{G}_2 \otimes \mathcal{G}_1$ are Gelfand triples, and proves the following Proposition.

**Proposition 1.4.9.** *Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are the Gelfand triples from above, then the canonical kernel map*

$$\mathcal{K}\colon \mathcal{L}(E_1; E_2') \to E_2' \mathbin{\hat{\otimes}} E_1'$$

*is a Gelfand triple isomorphism $\mathcal{K}\colon \mathcal{L}(\mathcal{G}_1, \mathcal{G}_2) \to \mathcal{G}_2 \otimes \mathcal{G}_1$.*

# Chapter 2

# Spaces of vector valued smooth and ultradifferentiable functions

Before we approach the topic of differentiable vector valued functions, we first need to fix the notation for the standard spaces of scalar valued functions.

Suppose $\mathbb{M}$ is a locally compact second countable topological space, e.g. a smooth manifold, then $\mathscr{C}(\mathbb{M})$ will denote the space of continuous functions on $\mathbb{M}$ with values in $\mathbb{C}$, equipped with the topology of uniform convergence on compact sets. I.e. the topology of $\mathscr{C}(\mathbb{M})$ is defined by seminorms of the form

$$f \mapsto \|f \restriction_K \|_\infty = \sup_{x\in K} |f(x)| \qquad \text{for compact} \qquad K \subset M\,.$$

## 2.1 Spaces of continuous vector valued functions

For any quasi-complete locally convex space $E$ we define the space of $E$ valued continuous functions

$$\mathscr{C}(\mathbb{M}; E) := \{f\colon \mathbb{M} \to E \mid f \text{ is continuous}\}$$

equipped with the topology of uniform convergence on compact sets, i.e. the topology induced by the set of seminorms

$$f \mapsto \sup_{x\in K} p(f(x)) \qquad \text{for compact } K \subset \mathbb{M} \text{ and continuous seminorms } p\colon E \to \mathbb{R}\,.$$

The space $\mathscr{C}(\mathbb{M};E)$ can be identified with $\mathscr{C}(\mathbb{M})\,\varepsilon\, E$ via the linear homeomorphism

$$f \mapsto [\mathscr{C}'(\mathbb{M}) \times E' \ni (\mu, e') \mapsto \langle e' \circ f, \mu\rangle \in \mathbb{C}]\,.$$

A proof of this fact can be found in [39, Theorem 1.10]. This homeomorphism also motivates the following definition of more general spaces of vector valued continuous functions [39, p. 235].

**Definition 2.1.1.** *We call a locally convex space $\mathscr{F}(\mathbb{M})$ a $\mathscr{C}(\mathbb{M})$-function space if it is a linear subspace of $\mathscr{C}(\mathbb{M})$ equipped with a topology which is finer than or equal to the subspace topology from $\mathscr{C}(\mathbb{M})$.*

*If $E$ is a locally convex space over $\mathbb{C}$ and $\mathscr{G}(\mathbb{M})$ a $\mathscr{C}(\mathbb{M})$-function space, we will denote by $\mathscr{G}(\mathbb{M};E)$ the space of functions $f\colon \mathbb{M} \to E$ such that*

$$e' \circ f \in \mathscr{G}(\mathbb{M}) \text{ for all } e' \in E' \quad \text{and} \quad [e' \mapsto e' \circ f] \in \mathcal{L}(E'_c; \mathscr{G}(\mathbb{M}))\,,$$

*and equip it with the topology induced by the seminorms*

$$f \mapsto \sup_{e' \in W} p(e' \circ f)$$

*for continuous seminorms $p$ on $\mathscr{G}(\mathbb{M})$ and equicontinuous and $W \subset E'$.*

From this definition follows with Lemma 1.2.5 that

$$\mathscr{G}(\mathbb{M};E) \to \mathscr{G}(\mathbb{M})\,\varepsilon\, E\colon f \mapsto [(\phi, e') \mapsto \langle e' \circ f, \phi\rangle] \tag{2.1.1}$$

is a linear homeomorphism for any locally convex space $E$. This homeomorphism also motivates the following convention. If

$$v \in \mathscr{G}_1(\mathbb{M}_1)\,\varepsilon \cdots \varepsilon\, \mathscr{G}_n(\mathbb{M}_n)\,\varepsilon\, E\,,$$

in which the $\mathscr{G}_j(\mathbb{M}_j)$ are $\mathscr{C}(\mathbb{M}_j)$-function spaces, and if $\varepsilon_{x_j} \in \mathscr{G}_j(\mathbb{M}_j)'$ with $\langle f, \varepsilon_{x_j}\rangle = f(x_j)$, then we will write

$$v(x_1, \ldots, x_n, e') := v(\varepsilon_{x_1}, \ldots, \varepsilon_{x_n}, e')$$

and also define

$$v(x_1, \ldots, x_n) \in E \quad \text{by} \quad \langle v(x_1, \ldots, x_n), e'\rangle := v(x_1, \ldots, x_n, e')\,.$$

Note that any $v \in \mathscr{G}_1(\mathbb{M}_1)\,\varepsilon \cdots \varepsilon\, \mathscr{G}_n(\mathbb{M}_n)\,\varepsilon\, E$ is completely defined by its values $v(x)$ for $x \in \prod_{j=1}^{n} \mathbb{M}_j$, since the sets of functionals $\operatorname{span}_{\mathbb{C}}\{\varepsilon_{x_j} \mid x_j \in \mathbb{M}_j\}$ are dense in $\mathscr{G}_j(\mathbb{M}_j)'_c$.

The above approach to vector valued functions is inverse to the one usually used for spaces of vector valued functions. More commonly, one defines a space of vector valued functions $f\colon \mathbb{M} \to E$, say $F(\mathbb{M}; E)$, in which the corresponding space scalar valued functions $F(\mathbb{M})$ are just the special case for $E = \mathbb{C}$. Afterwards, one checks whether $F(\mathbb{M})\,\varepsilon\, E \simeq F(\mathbb{M}; E)$ holds. However, the homeomorphism $F(\mathbb{M})\,\varepsilon\, E \simeq F(\mathbb{M}; E)$ is integral to our approach, so using it as a definition is more convenient. An immediate benefit of this homeomorphism is that continuous operators on a $\mathscr{C}(\mathbb{M})$-function space $\mathscr{F}(\mathbb{M})$ and on a locally convex space $E$ automatically correspond to continuous operators on $\mathscr{G}(\mathbb{M}; E)$.

Let the homeomorphism (2.1.1) be denoted by $\psi$ and let $T \in \mathcal{L}(\mathscr{F}_1(\mathbb{M}_1); \mathscr{F}_2(\mathbb{M}_2))$ and $S \in \mathcal{L}(E_1, E_2)$, in which the $\mathscr{F}_j(\mathbb{M}_j)$ are $\mathscr{C}(\mathbb{M}_j)$-function spaces and the $E_j$ are locally convex. Then $\psi^{-1} \circ (T \,\varepsilon\, S) \circ \psi$ defines an operator in $\mathcal{L}(\mathscr{F}_1(\mathbb{M}_1; E_1); \mathscr{F}_2(\mathbb{M}_2; E_2))$. If there is no risk of confusion, we will denote this operator by $T \,\varepsilon\, S$ as well. We will use such operators especially often in the case where $T = P \in \operatorname{Diff}(\mathbb{M})$ is a differential operator and $S = \mathrm{I}$ is an identity. Here, we will merely write $Pf(x) := P_x f(x) := (P \,\varepsilon\, \mathrm{I}) f(x)$.

### 2.1.1 Limits of $\mathscr{C}(\mathbb{M})$-function spaces

Often, we will encounter certain spaces of continuous functions that are isomorphic to inductive or projective limits. We will make a few general observations and introduce simplified notation.

If $(A, \leq)$ is a directed set and $(\mathscr{F}_\alpha(\mathbb{M}))_{\alpha \in A}$ is a family of $\mathscr{C}(\mathbb{M})$-function spaces, such $\mathscr{F}_\alpha(\mathbb{M}) \supset \mathscr{F}_\beta(\mathbb{M})$ and the inclusion maps $\mathrm{i}_{\alpha,\beta}\colon \mathscr{F}_\beta(\mathbb{M}) \to \mathscr{F}_\alpha(\mathbb{M})$ are continuous for $\alpha \leq \beta$. We may then always identify the projective limit $\varprojlim_{\alpha \in A}(\mathscr{F}_\alpha(\mathbb{M}), \mathrm{I}_{\alpha,\beta})$ with the linear subspace $\bigcap_{\alpha \in A} \mathscr{F}_\alpha(\mathbb{M}) \subset \mathscr{C}(\mathbb{M})$ via $(f_\alpha)_\alpha \mapsto f_\beta$ (this definition does not depend on the choice $\beta \in A$). Equipped with the topology transported from $\varprojlim_{\alpha \in A}(\mathscr{F}_\alpha(\mathbb{M}), \mathrm{I}_{\alpha,\beta})$, this is a $\mathscr{C}(\mathbb{M})$-function space, which we will just denote by $\varprojlim_{\alpha \in A} \mathscr{F}_\alpha(\mathbb{M})$. For a complete locally convex space $E$ we define $\varprojlim_{\alpha \in A} \mathscr{F}_\alpha(\mathbb{M}; E)$ analogously as a space of $E$-valued continuous functions linearly homeomorphic to $\varprojlim_{\alpha\in A}(\mathscr{F}_\alpha(\mathbb{M};E), \mathrm{I}_{\alpha,\beta}\,\varepsilon\,\mathrm{I}_E)$.

Similarly, if $(\mathscr{F}_\alpha(\mathbb{M}))_{\alpha\in A}$ is a family of $\mathscr{C}(\mathbb{M})$-function spaces such that $\mathscr{F}_\beta(\mathbb{M})$ is contianed in $\mathscr{F}_\alpha(\mathbb{M})$ and the inclusion maps $\mathrm{i}_{\alpha,\beta}\colon \mathscr{F}_\beta(\mathbb{M}) \to \mathscr{F}_\alpha(\mathbb{M})$ are continuous for $\alpha \geq \beta$, then we can identify the inductive limit $\varinjlim_{\alpha\in A}(\mathscr{F}_\alpha(\mathbb{M}), \mathrm{I}_{\alpha,\beta}) = (\bigoplus_{\alpha\in A}\mathscr{F}_\alpha(\mathbb{M}))/H$ with the linear subspace $\bigcup_{\alpha\in A}\mathscr{F}_\alpha(\mathbb{M}) \subset \mathscr{C}(\mathbb{M})$ via

$$\psi\colon \bigcup_{\alpha\in A}\mathscr{F}_\alpha(\mathbb{M}) \to \left(\bigoplus_{\alpha\in A}\mathscr{F}_\alpha(\mathbb{M})\right)\Big/H\,, \quad \psi(f) = j_\alpha(f) + H \quad \text{for } f \in \mathscr{F}_\alpha(\mathbb{M})\,,$$

in which we used the $H$ and $j_\alpha$ from Definition 1.1.3. We denote by $\varinjlim_{\alpha\in A}\mathscr{F}_\alpha(\mathbb{M})$ the linear space $\bigcup_{\alpha\in A}\mathscr{F}_\alpha(\mathbb{M})$ equipped with the topology transported from $\varinjlim_{\alpha\in A}(\mathscr{F}_\alpha(\mathbb{M}), \mathrm{I}_{\alpha,\beta})$. Analogously, $\varinjlim_{\alpha\in A}\mathscr{F}_\alpha(\mathbb{M};E)$ is defined by exchanging $\mathscr{F}_\alpha(\mathbb{M})$ with $\mathscr{F}_\alpha(\mathbb{M};E)$ in the above.

Suppose $(\mathcal{K}, \subset)$ is a directed set of compact subsets of $\mathbb{M}$, ordered by the inclusion relation such that $\mathbb{M} = \bigcup_{K\in\mathcal{K}} K$ [1]. Suppose $\mathscr{F}(K)$ is a $\mathscr{C}(K)$-function space for each $K \in \mathcal{K}$ such that

$$\mathrm{I}_{K',K}\colon \mathscr{F}(K) \to \mathscr{F}(K')\colon f \mapsto f\restriction_{K'}\,, \qquad \text{for } K' \subset K\,,$$

is well-defined and continuous. Then $\varprojlim_{K\in\mathcal{K}}(\mathscr{F}(K), \mathrm{I}_{K,K'})$ can be identified with a $\mathscr{C}(\mathbb{M})$-function space, denoted by $\varprojlim_{K\in\mathcal{K}}\mathscr{F}(K)$, via

$$(f_K)_{K\in\mathcal{K}} \mapsto f \quad \text{where} \quad f(x) = f_K(x) \qquad \text{for } x \in K\,.$$

Analogously, we define $\varprojlim_{K\in\mathcal{K}}\mathscr{F}(K;E)$ as a space of $E$ valued continuous functions, which is linearly homeomorphic to $\varprojlim_{K\in\mathcal{K}}(\mathscr{F}(K;E), \mathrm{I}_{K,K'}\,\varepsilon E)$.

### 2.1.2 Differentiable functions and differential operators on manifolds and regular compact sets

Suppose $\mathbb{M}$ is an $N$-dimensional smooth manifold. We will always identify the tangent space at $x \in \mathbb{M}$, denoted by $T_x\mathbb{M}$, with the space of derivations at $x \in \mathbb{M}$. By $\mathscr{E}(\mathbb{M})$ resp. $\mathscr{C}^k(\mathbb{M})$, $k \in \mathbb{N}_0$, we will denote the space of smooth function resp. $k$-times continuously

[1]Later, we will just use the family of regular compact sets from Definition 2.1.4

differentiable functions on $\mathbb{M}$. The topology of these spaces is the initial topology with respect to the maps

$$\mathscr{E}(\mathbb{M}) \to \mathscr{E}(\phi(U)) \qquad \text{resp.} \qquad \mathscr{C}^k(\mathbb{M}) \to \mathscr{C}^k(\phi(U)) \quad : \quad f \mapsto f \circ \phi^{-1}$$

for smooth charts $(\phi, U)$, in which $\phi \colon \mathbb{M} \supset U \to \phi(U) \subset \mathbb{R}^N$. As usual $\mathscr{D}(\mathbb{M})$ denotes the space of smooth functions with compact support on $\mathbb{M}$. If $\mathbb{M}$ is an analytic manifold, then we denote the vector space of analytic functions by $\mathscr{A}(\mathbb{M})$. A differential operator $P$ on an open set $V \subset \mathbb{R}^N$ is a linear operator on $\mathscr{E}(V)$ defined by

$$P\,f := \sum_{\alpha \in \mathbb{N}_0^N} a_\alpha \cdot \partial^\alpha f$$

for some smooth functions $a_\alpha \in \mathscr{E}(V)$ such that on any compact $K \subset V$ only finitely many $a_\alpha \restriction_K$ are non-zero. The ring of differential operators on $V$ will be denoted by $\mathrm{Diff}(V)$. The number

$$\deg P := \sup\{|\alpha| \colon \exists_{x \in V}\ a_\alpha(x) \neq 0\} \in \mathbb{N}_0 \cup \{\infty\}$$

is the degree of the differential operator $P$. The set of differential operators $P$ with $\deg P \leq k$ will be denoted by $\mathrm{Diff}^k(V)$. Now the same concepts are defined on smooth manifolds $\mathbb{M}$ by

$$\mathrm{Diff}(\mathbb{M}) := \left\{ P \in \mathcal{L}(\mathscr{E}(\mathbb{M})) \;\middle|\; \begin{array}{c} \text{for each smooth chart } (\phi, U) \text{ there exists} \\ P_\phi \in \mathrm{Diff}(\phi(U)) \text{ with } P\,(-) \circ \phi^{-1} = P_\phi\,(- \circ \phi^{-1}) \end{array} \right\}.$$

The sets of differentiable operators $\mathrm{Diff}^k(\mathbb{M})$ are defined analogously, i.e. we just exchange $\mathrm{Diff}(\phi(U))$ with $\mathrm{Diff}^k(\phi(U))$. Note that $\mathrm{Diff}^k(\mathbb{M})$ is a module over $\mathscr{E}(\mathbb{M}) \simeq \mathrm{Diff}^0(\mathbb{M})$. If $\mathbb{M}$ is an analytic manifold, we will also need the space of real analytic sections $\mathbb{M} \to T\mathbb{M}$, i.e. analytic vector fields, denoted by $\mathcal{V}_{\mathrm{a}}(\mathbb{M})$. The action of some vector field $X$ on $f \in \mathscr{E}(\mathbb{M})$ is denoted by $(Xf)(x) := X(x)(f)$.

Naturally, each differential operator $P \in \mathrm{Diff}(\mathbb{M})$ can not only be seen as a linear operator on $\mathscr{E}(\mathbb{M})$, but also as a continuous linear operator on $\mathscr{E}(U)$ and between $\mathscr{C}^n(U)$ and $\mathscr{C}^m(U)$ for open $U \subset \mathbb{M}$ and $n - \deg P \geq m$. In general, we will also use the notation $Pf(x) := P_x f(x)$ for any differential operator or vector field $P$ and a function $f$.

**Differentiable functions and differential operators on regular compact sets**

Let $|c|$ be the length of a rectifiable curve $c$. We use the following property for subsets $A \subset \mathbb{R}^N$ from [69]:

$$\begin{array}{l}\text{There is } C > 0 \text{ such that any } x, y \in A \text{ are connected by a} \\ \text{rectifiable curve } c \subset A \text{ with } |c| \leq C|x-y|.\end{array} \tag{P}$$

This property plays a role in the extension of differentiable functions (resp. jets) [6, 69]. Obviously, convex sets have property **(P)** and if $A$ has property **(P)** then so does the closure $\overline{A}$ by [69, Lemma 2]. In [40], Komatsu defines his spaces of ultradifferentiable functions on finite disjoint unions of compact sets with property **(P)**. Since we need to use spaces of functions on manifolds, we need to be able to define our compact sets via charts.

Indeed we also have a certain invariance with respect to diffeomorphisms. Suppose $U, V$ are open in $\mathbb{R}^N$ and suppose $\psi\colon U \to V$ is a diffeomorphism. If $c$ is a rectifiable curve in $A$ and $\overline{A} \subset U$, then

$$\inf_{x\in A} \|\mathrm{d}_x\psi^{-1}\|^{-1}_{\mathcal{L}(\mathbb{R}^N)} \cdot |c| \leq |\psi(c)| \leq \sup_{x\in A} \|\mathrm{d}_x\psi\|_{\mathcal{L}(\mathbb{R}^N)} \cdot |c| \,.$$

So if we assume $\overline{A}$ is a compact subset of $U$, then $A$ has property **(P)** iff $\psi(A)$ has property **(P)**. More generally, the same holds for any bijective Lipschitz function $\psi\colon A \to \psi(A)$ with Lipschitz inverse. Hence the following definition will be helpful.

**Definition 2.1.2.** *A subset $A \subset \mathbb{R}^n$ will be called **bounded Lipschitz domain**, iff $A$ is open, bounded, connected and for any $x \in \partial A$ there is an open set $U \ni x$, and a bijective Lipschitz map $\psi\colon U \to (-1,1)^n$ with Lipschitz inverse, such that*

$$\psi(U \cap \partial A) = (-1,1)^{n-1} \times \{0\} \quad \textit{and} \quad \psi(U \cap A) = (-1,1)^{n-1} \times (0,1)\,.$$

**Lemma 2.1.3.**

**(i)** *Suppose $A \subset \mathbb{R}^N$ is relatively compact and for each $x \in \overline{A}$ there is an open neighbourhood $U \ni x$ in $\mathbb{R}^N$ such that $U \cap A$ has property **(P)**. Then all connected components of $A$ fulfil **(P)**. Also, $A$ has finitely many connected components.*

**(ii)** *Suppose $A \subset \mathbb{R}^N$ is a bounded Lipschitz domain. If $A \subset B \subset \overline{A}$, then $B$ fulfils **(P)**.*

*Proof.* ***(i)*:** Since $\overline{A}$ is compact, we find a finite open cover $U_1, \ldots, U_n$ such that $A \cap U_j$ has property **(P)** for each $j$. Since each $U_j \cap A$ is connected, $A$ has only finitely many connected components.

We show that the connected components of $A$ have property **(P)**. Without loss of generality we may assume that $A$ is connected and that $\overline{A} \setminus \bigcup_{i \neq j} U_i \neq \emptyset$ for each $j$. We define $d := \sup_{x,y \in A} |x - y|$ and

$$\mu := \inf \left\{ |x - y| \;\middle|\; j \in \{1, \ldots, n\}\,,\ x \in \overline{A} \setminus U_j\,,\ y \in \overline{A} \setminus \bigcup_{i \neq j} U_i \right\} \in (0, \infty)\,.$$

Let $C_1, \ldots, C_n > 0$ be constants such that for each pair $x, y \in A \cap U_j$ there is a rectifiable curve $c \subset U_j \cap A$ with endpoints $x, y$ and $|c| \leq C_j |x - y|$. We put $C := \frac{d\,n}{\mu} \max_j C_j$.

Suppose $x, y \in A \cap U_j$, then $x, y$ are of course linked by a rectifiable path $c$ with

$$|c| \leq C_j |x - y| \leq C |x - y|\,.$$

If $x, y \in A$ and there is no $j$ such that $x, y \in U_j$, then there is $j$ with $x \in A \setminus U_j$ and $y \in A \setminus \bigcup_{i \neq j} U_i$. We find a selection of at most $n$ rectifiable paths $c_k \subset U_{i_k} \cap A$, $i_k \in \{1, \ldots, n\}$, $k = 1, 2, \ldots K \leq n$, such that

- $c_k$ links $z_{k-1}$ and $z_k$ with $z_k \in U_{i_k} \cap U_{i_{k-1}} \cap A$,
- $|c_k| \leq C_{i_k} |z_{k-1} - z_k|$ and
- $z_0 = x$ and $z_K = y$.

The $c_k$ combine to a rectifiable curve $c$ linking $x$ with $y$ such that

$$|c| \leq \sum_{k=1}^{K} C_{i_k} |z_{k-1} - z_k| \leq C\mu \leq C|x - y|\,.$$

In conclusion, for any pair $x, y \in A$ there is a rectifiable curve $c \subset A$ connecting $x$ and $y$ with $|c| \leq C|x - y|$.

***(ii)*:** For any $x \in \partial A = \partial B$ we choose $U$ and $\psi$ as in the last definition. Then

$$(-1, 1)^{n-1} \times (0, 1) \subset \psi(U \cap B) \subset (-1, 1)^{n-1} \times [0, 1)$$

and thus $\psi(U \cap B)$ has property **(P)**. This also implies that $U \cap B$ has property **(P)**.

If $x \in \operatorname{Int} B$, then there is some open ball $U$ with $x \in U \subset B$. Obviously $U = B \cap U$ has property **(P)**.

Hence $B$ has property **(P)** by **(i)**.

$\square$

A closed set $K \subset \mathbb{M}$ of a topological space $\mathbb{M}$ such that $K = \overline{\operatorname{Int} K}$ is called *regular closed.*

**Definition 2.1.4.** *Let $\mathbb{M}$ be a smooth manifold. A subset $K \subset \mathbb{M}$ will be called **regular compact**, in symbols $K \overset{\text{rc}}{\subset} \mathbb{M}$, iff $K$ is compact and regular closed in $\mathbb{M}$ and for each $x \in K$ there is a chart $(\psi, U)$ and a bounded Lipschitz domain $A$ such that $x \in U$ and $A \subset \psi(K \cap U) \subset \overline{A}$.*

By Lemma 2.1.3, our notion of regular compact subsets of $\mathbb{R}^n$ is stronger than the one from [40]. We require regular closedness in order to use functions instead of jets on regular compact sets. Using bounded Lipschitz domains ensures that we have a nice local description of regular compact sets.

It is clear that any regular compact set has finitely many connected components. Moreover, for open $U \subset \mathbb{M}$ and $K \subset U$ we have $K \overset{\text{rc}}{\subset} U$ iff $K \overset{\text{rc}}{\subset} \mathbb{M}$.

**Lemma 2.1.5.** *Suppose $\mathbb{M}$ is a smooth manifold. Then the following holds.*

**(i)** *For any open $U \subset \mathbb{M}$ and compact $\tilde{K} \subset U$ there is some $K \overset{\text{rc}}{\subset} \mathbb{M}$ with*

$$\tilde{K} \subset \operatorname{Int} K \subset K \subset U\,.$$

**(ii)** *There is a sequence $K_1 \subset K_2 \subset \dots$ of subsets $K_j \overset{\text{rc}}{\subset} \mathbb{M}$ with $\mathbb{M} = \bigcup_j \operatorname{Int} K_j$.*

**(iii)** *If $K \overset{\text{rc}}{\subset} \mathbb{M}$ then there is a finite collection of charts $(\phi_j, U_j)_j$ and corresponding regular closed and compact $V_j \subset U_j$ such that $K \cap V_j \overset{\text{rc}}{\subset} \mathbb{M}$ and $K \subset \bigcup_j \operatorname{Int} V_j$. If $\mathbb{M}$ is an analytic manifold, the charts $(U_j, \phi_j)_{j \in J}$ can be chosen to be analytic.*

*Proof.* ***(i):*** Without loss of generality we may assume $U = \mathbb{M}$. There exists some real valued function $f \in \mathscr{D}(\mathbb{M})$ with $f(x) = 1$ for $x \in \tilde{K}$. By Sard's Theorem [50, Theorem

6.10] there exists a regular value $w$ to $f$ with $0 < w < 1$. Since $w$ is a regular value, $f$ does not take on any local maxima on $f^{-1}(\{w\})$. So any open set that intersects $f^{-1}(\{w\})$ must also intersect $f^{-1}((w, \infty))$. This means that $f^{-1}(\{w\})$ is the topological boundary of the open set $f^{-1}((w, \infty))$. Thus $K := f^{-1}([w, \infty)) \supset \tilde{K}$ is a compact and regular closed subset of $U$. Furthermore, $f^{-1}(\{w\})$ is a smooth submanifold of $\mathbb{M}$ by the regular value theorem. Thus for any $x \in K$ there is some chart $(\varphi, V)$, with $x \in V$, such that $\varphi(V \cap K) = \varphi(V) \cap \mathbb{R}^{n-1} \times [0, \infty)$. We can just choose $V$ such that $\varphi(V)$ is an open ball, in which case $A \subset \varphi(V \cap K) \subset \overline{A}$ for the Lipschitz domain $A := \{(y, t) \in \varphi(V) \mid t > 0\}$. Hence $K$ is regular compact.

***(ii):*** Since $\mathbb{M}$ is a manifold, it is especially second countable and locally compact. Thus there exists a sequence of open sets $(V_j)_j$ with

$$\overline{V}_1 \subset V_2 \subset \overline{V}_2 \subset V_3 \subset \dots$$

such that each $\overline{V}_j$ is compact and $\mathbb{M} = \bigcup_j V_j$. Now we just need to use ***(i)*** in order to get $K_j \overset{\text{rc}}{\subset} \mathbb{M}$ with $\overline{V}_j \subset K_j \subset V_{j+1}$.

***(iii):*** Suppose $x \in K$ and $(\phi, U)$ is any chart (smooth or analytic) around $x$. Since $K \overset{\text{rc}}{\subset} \mathbb{M}$, there is some chart $(f, B)$ around $x \in K$ and a bounded Lipschitz domain $A$ such that $A \subset f(B \cap K) \subset \overline{A}$.

If $f(x) \in \operatorname{Int}(A)$, then $x \in \operatorname{Int} K$. Hence we may choose a closed Ball $V \subset \operatorname{Int}(K) \cap U$ with $x \in \operatorname{Int} V$. In this case $V = V \cap K \overset{\text{rc}}{\subset} U$.

If $f(x) \in \partial A$, we proceed similarly as in the proof of Lemma 2.1.3 ***(ii)***. We find an open set $W \subset f(B \cap U)$ with $f(x) \in W$ and a bijective Lipschitz function $\psi \colon W \to (-2, 2)^n$ with Lipschitz inverse such that $\psi(W \cap A) = (-2, 2)^{n-1} \times (0, 2)$ and $\psi(W \cap \partial A) = (-2, 2)^{n-1} \times \{0\}$. Now define the regular closed and compact set $V := f^{-1} \circ \psi^{-1}\big([-1, 1]^n\big)$. Then $V \subset U \cap B$ and $\Omega := \psi^{-1}\big((-1, 1)^{n-1} \times (0, 1)\big)$ is a bounded Lipschitz domain with

$$\Omega \subset \psi^{-1}\big([-1, 1]^{n-1} \times (0, 1]\big) \subset f(V \cap K) \subset \psi^{-1}\big([-1, 1]^{n-1} \times [0, 1]\big) = \overline{\Omega}\,.$$

Thus $V \cap K \overset{\text{rc}}{\subset} U$.

Finally we can use any covering by sets of the form $\operatorname{Int} V$ for corresponding $(\phi, U)$ and $V$ and choose a finite subcover. $\square$

**Definition 2.1.6.** *Let* $\mathbb{M}$ *be a smooth manifold,* $k \in \mathbb{N}_0$ *and* $K \overset{\text{rc}}{\subset} \mathbb{M}$*. We define the spaces*

$$\mathscr{C}^k(K) := \{f \restriction_K \mid f \in \mathscr{C}^k(\mathbb{M})\} \quad \textit{resp.} \quad \mathscr{E}(K) := \{f \restriction_K \mid f \in \mathscr{E}(\mathbb{M})\}$$

*equipped with the topology defined by the seminorms*

$$f \restriction_K \mapsto \sup_{x \in K} |Pf(x)| \,, \qquad P \in \mathrm{Diff}^k(\mathbb{M}) \quad \textit{resp.} \quad P \in \mathrm{Diff}(\mathbb{M})\,.$$

*Similarly, we define*

$$\mathcal{V}_{\mathrm{a}}(K) := \{X \restriction_K \mid X \in \mathcal{V}_{\mathrm{a}}(U) \textit{ for some open } U \textit{ with } K \subset U \subset \mathbb{M}\}$$

*for an analytic manifold* $\mathbb{M}$*.*

Suppose $U \subset \mathbb{M}$ is open with $K \overset{\text{rc}}{\subset} U$. Then, the existence of a bump function $f \in \mathscr{D}(U)$ with $f(x) = 1$ for $x \in K$ ensures that we get the same notion of $\mathscr{C}^k(K)$ or $\mathscr{E}(K)$ whether we use $U$ or $\mathbb{M}$ for its definition. Of course, each $P \in \mathrm{Diff}(\mathbb{M})$ can be seen as a linear operator on $\mathscr{E}(K)$ or between $\mathscr{C}^l(K)$ and $\mathscr{C}^k(K)$ for appropriate integers $l, k$. With the next lemma we ensure that $\mathscr{C}^k(K)$ and $\mathscr{E}(K)$ behave analogously to their counterparts on manifolds.

**Lemma 2.1.7.** *Let* $\mathbb{M}$ *be a smooth manifold, let* $k \in \mathbb{N}_0$ *and let* $K \overset{\text{rc}}{\subset} \mathbb{M}$*. Then the following holds*

**(i)** *For a function* $f \colon K \to \mathbb{C}$ *we have* $f \in \mathscr{C}^k(K)$ *iff*

$$f \restriction_{\mathrm{Int}\,K} \in \mathscr{C}^k(\mathrm{Int}\,K) \textit{ and for all } P \in \mathrm{Diff}^k(\mathbb{M}) \textit{ the function } Pf \restriction_{\mathrm{Int}\,K} \textit{ extends continuously to } K. \tag{$*$}$$

**(ii)** *For a function* $f \colon K \to \mathbb{C}$ *we have* $f \in \mathscr{E}(K)$ *iff*

$$f \restriction_{\mathrm{Int}\,K} \in \mathscr{E}(\mathrm{Int}\,K) \textit{ and for all } P \in \mathrm{Diff}(\mathbb{M}) \textit{ the function } Pf \restriction_{\mathrm{Int}\,K} \textit{ extends continuously to } K. \tag{$**$}$$

**(iii)** $\mathscr{C}^k(K)$ *and* $\mathscr{E}(K)$ *are Fréchet.*

**(iv)** $\mathscr{E}(K)$ *is nuclear.*

*Proof.* **(i)** and **(ii):** For the case where $K$ is the closure of a region in $\mathbb{R}^n$ **(i)** is proven in [69]. Using [68, Theorem I] we also get **(ii)** in this case. As a direct conclusion we get both **(i)** and **(ii)** if $K$ is mapped onto a regular compact subset of $\mathbb{R}^n$ by a single chart. Now for general $K \overset{\text{rc}}{\subset} \mathbb{M}$ for some smooth manifold $\mathbb{M}$ we can just use a partition of unity.

We choose any finite family of charts $(U_j, \phi_j)_j$ and regular closed compact subset $(V_j)_j$ as in Lemma 2.1.5 ***(iii)***. Then we choose a partition of unity $(\chi_j)_j \subset \mathscr{D}(\mathbb{M})$ with $\operatorname{supp} \chi_j \subset V_j$ and $\sum_j \chi_j(x) = 1$ for each $x \in K$.

Let $f\colon K \to \mathbb{C}$ fulfil $(*)$ resp. $(**)$. For each $j$ the function $f \restriction_{V_j \cap K}$ extends to a function $F_j$ in $\mathscr{C}^k(\mathbb{M})$ resp. $\mathscr{E}(\mathbb{M})$. Then $F := \sum_j \chi_j F_j$ is an extension of $f$ and $F$ is in $\mathscr{C}^k(\mathbb{M})$ resp. $\mathscr{E}(\mathbb{M})$.

**(iii):** By definition $\mathscr{E}(K)$ is metrizable and $\mathscr{C}^k(K)$ is normable. With the help of **(i)** and **(ii)**, it is easy to see that they are also complete and thus Fréchet.

**(iv):** The linear space $\mathscr{N}_K(\mathbb{M}) := \{f \in \mathscr{E}(\mathbb{M}) \mid f(x) = 0 \text{ for } x \in K\}$ is closed in $\mathscr{E}(\mathbb{M})$. Hence the quotient space $\mathscr{E}(\mathbb{M})/\mathscr{N}_K(\mathbb{M})$ is a nuclear Fréchet space [66, Proposition 50.1]. Since $f + \mathscr{N}_K(\mathbb{M}) \mapsto f \restriction_K$ defines a linear continuous bijection from the Fréchet space $\mathscr{E}(\mathbb{M})/\mathscr{N}_K(\mathbb{M})$ onto the Fréchet space $\mathscr{E}(K)$, the space $\mathscr{E}(K)$ is nuclear by the open mapping theorem. □

## Frames and compositions of vector fields

An analytic frame $D = (D_1, \dots, D_N)$ is an ordered family of analytic vector fields defined on some open set $U \subset \mathbb{M}$ such that $(D_1(x), \dots, D_N(x))$ is a basis in $T_x\mathbb{M}$ for each $x \in U$. For regular closed compact $K \subset \mathbb{M}$ we will also call $D = (D_1, \dots, D_N) \subset \mathcal{V}_{\mathrm{a}}(K)$ a frame if $(D_1(x), \dots, D_N(x))$ is a basis of $T_x\mathbb{M}$ for each $x \in K$. The frame corresponding to the standard derivative on $\mathbb{R}^N$ (on any regular compact or any open set) will always be denoted by $\partial = (\partial_1, \dots, \partial_N)$. If $\phi\colon \mathbb{M} \supset U \to \mathbb{R}^N$ is any chart we will denote by

$$\partial_\phi = (\partial_{\phi_1}, \dots, \partial_{\phi_N}) \tag{2.1.2}$$

the family of vector fields defined by $\partial_{\phi_j} f := \big(\partial_j (f \circ \phi^{-1})\big) \circ \phi$. If $\phi$ is an analytic chart, then $\partial_\phi$ is an analytic frame.

We will use two distinct notations for compositions of analytic vector fields. Let $D = (D_1, \ldots, D_N)$ be an ordered family of vector fields defined on some open subset of $\mathbb{M}$. Then we will define the $D^\alpha := D_1^{\alpha_1} \circ \cdots \circ D_N^{\alpha_N}$ for any multi-index $\alpha \in \mathbb{N}_0^N$.

In many cases it will be convenient to use a notation that can represent any possible composition of the invariant differential operators $D$. For this purpose we define

$$\mathcal{S}_N := \{a \colon \mathbb{N} \to \{0, 1, \ldots, N\} \mid \operatorname{supp} a := \mathbb{N} \setminus a^{-1}(\{0\}) \text{ is finite}\}$$

and together with the convention $D_0 := \mathrm{I}_{\mathscr{E}(\mathbb{G})}$ we denote

$$D^a := \cdots D_{a_3} \circ D_{a_2} \circ D_{a_1} \qquad \text{for} \qquad a \in \mathcal{S}_N.$$

On the new type of indices $a \in \mathcal{S}_N$ we define the degree $|a| = \# \operatorname{supp} a$. Also, the following sets of tuples of indices will be convenient when using Leibniz rule. For $k \in \mathbb{N}$ we define

$$\mathcal{S}_{N,k}(a) = \left\{ (a^j)_{j=1}^k \in (\mathcal{S}_N)^k \;\middle|\; a = \sum_{j=1}^k a^j \text{ and } \operatorname{supp} a^j \cap \operatorname{supp} a^i = \emptyset \text{ for } i \neq j \right\},$$

where the sum of the function $a^j$ are taken to be the pointwise sum in $\mathbb{N}_0$. Now, if $f_1, \ldots f_k \colon U \to \mathbb{C}$ are smooth enough functions, we have

$$D^a(f_1 \cdot f_2 \cdots f_k) = \sum_{(a^j)_j \in \mathcal{S}_{N,k}(a)} (D^{a^1} f_1) \cdot (D^{a^2} f_2) \cdots (D^{a^k} f_k).$$

Note that we have a lot of redundancies that do not appear in the formulation of the Leibniz rule for multi indices $\alpha \in \mathbb{N}_0^N$

$$D^\alpha(f_1 \cdot f_2) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (D^\alpha f_1) \cdot (D^\beta f_2)\,.$$

For any analytic chart $\phi$, the corresponding frame $\partial_\phi$ is composed of commuting vector fields. Hence, it is sufficient to use multi-indices $\alpha \in \mathbb{N}_0^N$ instead of $a \in \mathcal{S}_N$. To be precise, for each $a \in \mathcal{S}_N$ there is exactly one $\alpha \in \mathbb{N}_0^N$ such that $\partial_\phi^a = \partial_\phi^\alpha$ and $|a| = |\alpha|$.

In general we will also use the notation $Pf(x) := P_x f(x)$ for any differential operator or vector field $P$ and a function $f$.

### Lie groups and invariant differential operators

We denote by $\mathbb{G}$ always a Lie group. Its unit will be denoted by $1_{\mathbb{G}}$ and its center by $Z(\mathbb{G})$. We write $\mathfrak{g}$ for the (abstract) Lie algebra of $\mathbb{G}$. The corresponding exponential map will be denoted by $\exp_{\mathbb{G}}\colon \mathfrak{g} \to \mathbb{G}$ and its center will be denoted by $Z(\mathfrak{g})$.

Let $\ell_x(y) := x\,y$ and $r_x(y) := y\,x$ for $x, y \in \mathbb{G}$. A differential operator $P \in \mathrm{Diff}(\mathbb{G})$ is called left resp. right invariant iff

$$P\,(f \circ \ell_x) = (P\,f) \circ \ell_x \quad \text{resp.} \quad P\,(f \circ r_x) = (P\,f) \circ r_x$$

for all $x \in \mathbb{G}$ and $f \in \mathscr{E}(\mathbb{G})$. The left resp. right invariant subset of $\mathrm{Diff}(\mathbb{G})$ and $\mathrm{Diff}^k(\mathbb{G})$ will be denoted by $\mathrm{Diff}_{\mathrm{L}}(\mathbb{G})$ and $\mathrm{Diff}^k_{\mathrm{L}}(\mathbb{G})$ resp. $\mathrm{Diff}_{\mathrm{R}}(\mathbb{G})$ and $\mathrm{Diff}^k_{\mathrm{R}}(\mathbb{G})$.

We will denote the usual realizations of $\mathfrak{g}$ as $\mathbb{R}$-linear subspaces of $\mathrm{Diff}_{\mathrm{L}}(\mathbb{G})$ and $\mathrm{Diff}_{\mathrm{R}}(\mathbb{G})$ by

$$\mathfrak{g}_{\mathrm{L}} := \{X_{\mathrm{L}} \mid X \in \mathfrak{g}\} \quad \text{and} \quad \mathfrak{g}_{\mathrm{R}} := \{X_{\mathrm{R}} \mid X \in \mathfrak{g}\}, \quad \text{in which}$$
$$X_{\mathrm{L}} f(x) = \partial_t f(x \exp_{\mathbb{G}}(tX))\big|_{t=0} \quad \text{and} \quad X_{\mathrm{R}} f(x) = \partial_t f(\exp_{\mathbb{G}}(tX)x)\big|_{t=0}$$

for $f \in \mathscr{E}(\mathbb{G})$, $x \in \mathbb{G}$ and $X \in \mathfrak{g}$. Now suppose $D = (D_1, \ldots, D_N)$ is a basis of $\mathfrak{g}_{\mathrm{L}}$ resp. in $\mathfrak{g}_{\mathrm{R}}$. For $\alpha \in \mathbb{N}_0^N$ the differential operator $D^\alpha := D_1^{\alpha_1} \circ \cdots \circ D_N^{\alpha_N}$ is a left resp. right invariant differential operator. Depending on whether the basis $D$ is left or right invariant, $(D^\alpha)_{\alpha \in \mathbb{N}_0^N}$ is a basis of the $\mathbb{R}$-vector space $\mathrm{Diff}_{\mathrm{L}}(\mathbb{G})$ resp. $\mathrm{Diff}_{\mathrm{R}}(\mathbb{G})$ and $(D^\alpha)_{|\alpha| \le k}$ is a basis of the $\mathbb{R}$-vector space $\mathrm{Diff}^k_{\mathrm{L}}(\mathbb{G})$ resp. $\mathrm{Diff}^k_{\mathrm{R}}(\mathbb{G})$. Furthermore by using the appropriate charts, we may see that both the left and the right invariant differential operators span the modules of differential operators of the corresponding degree, i.e.

$$\operatorname{span}_{\mathscr{E}(\mathbb{G})} \mathrm{Diff}_{\mathrm{L/R}}(\mathbb{G}) = \mathrm{Diff}(\mathbb{G}) \quad \text{and} \quad \operatorname{span}_{\mathscr{E}(\mathbb{G})} \mathrm{Diff}^k_{\mathrm{L/R}}(\mathbb{G}) = \mathrm{Diff}^k(\mathbb{G})\,.$$

## 2.1.3 Vector valued differentiable functions

Let $U \subset \mathbb{R}^N$ be open and $E$ be an arbitrary locally convex space. A function $f\colon U \to E$ is said to be continuously differentiable if for each $j = 1, \ldots, n$ and each $x \in \mathbb{R}^n$ the limit

$$\partial_j f(x) = \lim_{t \to 0} \frac{1}{t}(f(x + te_j) - f(x))$$

exists in $E$, where $(e_j)_j$ is the standard basis in $\mathbb{R}^N$, and each partial derivative, $\partial_j f$, is a continuous function. The function $f$ is called $k$-times continuously differentiable if $f$ and $\partial_j f$, for $j = 1, \dots, N$, are $k-1$ times continuously differentiable. Also, $f$ is called smooth iff it is $k$ times continuously differentiable for all $k \in \mathbb{N}$. Of course for a smooth manifold $\mathbb{M}$ a function $f\colon \mathbb{M} \to E$ is $k$ times continuously differentiable resp. smooth iff $f \circ \phi^{-1}$ is $k$ times continuously differentiable resp. smooth for all smooth charts $\phi$. Suppose $K \overset{\mathrm{rc}}{\subset} \mathbb{M}$. As in the scalar case, we call a function $f\colon K \to E$ smooth resp. $k$ times continuously differentiable if $f \restriction_{\operatorname{Int} K}$ is smooth resp. $k$ times continuously differentiable such that $Pf$ can be continuously extended to $K$ for all $P \in \operatorname{Diff}(\mathbb{M})$ resp. $P \in \operatorname{Diff}^k(\mathbb{M})$.

Suppose $U \subset \mathbb{R}^n$ is open, $E$ is a locally convex space and $f\colon U \to E$ is smooth. Then $f$ is called analytic iff for each $x \in U$ there exists some $\varepsilon > 0$ such that for each $|y - x| < \varepsilon$

$$f(y) = \sum_{\alpha \in \mathbb{N}_0^n} \frac{\partial^\alpha f(x)}{\alpha!}(y - x)^\alpha$$

converges in $E$. If $\mathbb{M}$ is an analytic manifold, then $f\colon \mathbb{M} \to E$ is called analytic iff $f \circ \phi^{-1}$ is analytic for each analytic chart $\phi$.

Suppose $E$ is a complete locally convex space and $\mathscr{G}(\mathbb{M})$ is a $\mathscr{C}(\mathbb{M})$-function space. Then, by [36, Satz 10.5][2], $\mathscr{G}(\mathbb{M}; E)$ is precisely the set of functions $f\colon \mathbb{M} \to E$ such that $e' \circ f \in \mathscr{G}(\mathbb{M})$ for all $e' \in E'$ and such that $\{e' \circ F \mid e' \in W\}$ is relatively compact in $\mathscr{G}(\mathbb{M})$ for any equicontinuous $W \subset E'$. With this criterion one can easily recover the usual characterization of the spaces $\mathscr{C}^k(\mathbb{M}; E)$ and $\mathscr{E}(\mathbb{M}; E)$.

**Lemma 2.1.8.** *Let $E$ be a quasi-complete locally convex space, $\mathbb{M}$ a smooth manifold, $K \overset{\mathrm{rc}}{\subset} \mathbb{M}$ and $\mathbb{X} \in \{K, \mathbb{M}\}$. Then the spaces $\mathscr{C}^k(\mathbb{X}; E)$ resp. $\mathscr{E}(\mathbb{X}; E)$ are precisely the spaces of $k$ times continuously differentiable functions resp. smooth functions from $\mathbb{X}$ to $E$, equipped with the topologies defined by the seminorms*

$$f \mapsto \sup_{x \in K'} p(Pf(x)) \tag{2.1.3}$$

*for compact $K' \subset \mathbb{X}$, continuous seminorms $p$ on $E$ and $P \in \operatorname{Diff}^k(\mathbb{M})$ resp. $P \in \operatorname{Diff}(\mathbb{M})$.*

[2] See also [42, Theorem 1.12] for a more general statement involving quasi-complete spaces $E$.

*Proof.* By [36, p. 236] the statement is true for an open subset $\mathbb{X}$ of $\mathbb{R}^n$. For the other cases, it seems to be quicker to show the statements directly instead of adjusting the cited proof.

Naturally, the topologies of $\mathscr{C}^k(\mathbb{X}; E)$ and $\mathscr{E}(\mathbb{X}; E)$ are induced by (2.1.3) as a consequence of Lemma 1.2.5 **(iii)**.

By using charts, it is obvious that $\mathscr{C}^k(\mathbb{M}; E)$ resp. $\mathscr{E}(\mathbb{M}; E)$ is a space of $k$ times continuously differentiable functions resp. smooth functions. Since, any differential operator $P \in \mathrm{Diff}^k(\mathbb{M}; E)$ maps $\mathscr{C}^k(K; E)$ continuously to $\mathscr{C}(K; E)$, this also implies that $\mathscr{C}^k(K; E)$ resp. $\mathscr{E}(K; E)$ is a space of $k$ times continuously differentiable functions resp. smooth functions.

Now let $f\colon \mathbb{X} \to E$ be $k$-times continuously differentiable and $V \subset E'$ be equicontinuous. We define $V_f := \{e' \circ f \mid e' \in V\} \subset \mathscr{C}^k(\mathbb{X})$ and

$$T\colon \mathscr{C}^k(\mathbb{X}) \to \prod_{P \in \mathrm{Diff}^k(\mathbb{M})} \mathscr{C}(\mathbb{X})\colon f \mapsto (Pf)_P \,.$$

The operator $T$ is a homeomorphism onto its range. Now, $V_f$ is relatively compact in $\mathscr{C}^k(\mathbb{X})$ iff $TV_f$ is relatively compact in $T\mathscr{C}^k(\mathbb{X})$. Since $T\mathscr{C}^k(\mathbb{X})$ is closed in $\prod_P \mathscr{C}(\mathbb{X})$, this holds iff $TV_f$ is relatively compact in $\prod_P \mathscr{C}(\mathbb{X})$. By the Arzelà-Ascoli theorem, $V_f$ is relatively compact in $\mathscr{C}(\mathbb{X})$. By $PV_f = V_{Pf}$ we know that $PV_f$ is relatively compact in $\mathscr{C}(\mathbb{X})$ for each $P$, hence $\prod_P PV_f$ and thus also $TV_f \subset \prod_P PV_f$ are relatively compact in $\prod_P \mathscr{C}(\mathbb{X})$ by Tychonoff's theorem. In conclusion $V_f$ is relatively compact in $\mathscr{C}^k(\mathbb{X})$ for any equicontinuous $V$, which implies $f \in \mathscr{C}^k(\mathbb{X}; E)$.

The analogous argumentation ensures that any smooth function $f\colon \mathbb{X} \to E$ is in $\mathscr{E}(\mathbb{X}; E)$. □

Later we will also need the following Lemma.

**Lemma 2.1.9.** *For any smooth manifolds* $\mathbb{M}$, $\mathbb{M}'$ *and regular compact* $K \subset \mathbb{M}$, $K' \subset \mathbb{M}'$, *we have*

$$\mathscr{E}(\mathbb{M} \times \mathbb{M}') = \mathscr{E}(\mathbb{M}; \mathscr{E}(\mathbb{M}')) \quad \textit{and} \quad \mathscr{E}(K \times K') = \mathscr{E}(K; \mathscr{E}(K')) \,.$$

*Proof.* By [66, p.530 Corollary] and [66, Theorem 51.6] we have $\mathscr{E}(U \times V) = \mathscr{E}(U; \mathscr{E}(V))$ for open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$. Using charts, we get the corresponding homeomorphism for general manifolds. Let $(\varphi_i, U_i)$ resp. $(\psi_j, V_j)$ be an atlas for $\mathbb{M}$ resp. $\mathbb{M}'$. Now $f \mapsto (f \restriction_{U_i \times V_j})_{i,j}$ defines a homeomorphism of $\mathscr{E}(\mathbb{M} \times \mathbb{M}')$ onto a subspace of $\prod_{i,j} \mathscr{E}(U_i \times V_j)$ resp. of $\mathscr{E}(\mathbb{M} : \mathscr{E}(\mathbb{M}'))$ onto a subspace of $\prod_{i,j} \mathscr{E}(U_i; \mathscr{E}(V_j))$. We have $\mathscr{E}(U_i \times V_j) = \mathscr{E}(U_i; (\mathscr{E}(V_j))$ and both homeomorphisms have the same range. Hence $\mathscr{E}(\mathbb{M} \times \mathbb{M}') = \mathscr{E}(\mathbb{M}; \mathscr{E}(\mathbb{M}'))$ as topological vector spaces.

For the compact case we use $\mathscr{C}(K; \mathscr{C}(K')) = \mathscr{C}(K \times K')$ [44, §44.7. (3) and (4)].

By Lemma 2.1.8, it is clear that $\mathscr{E}(K; \mathscr{E}(K'))$ is the set of functions $f$ such that $f \restriction_{\mathrm{Int}(K) \times K'} \in \mathscr{E}(\mathrm{Int}\, K; \mathscr{E}(K'))$ and $P \,\varepsilon\, \mathrm{I}\, f \in \mathscr{C}(K; \mathscr{E}(K'))$ for all $P \in \mathrm{Diff}(\mathbb{M}))$. Inserting the description of $\mathscr{E}(K')$ gives us $f \in \mathscr{E}(K; \mathscr{E}(K'))$ iff $f \restriction_{\mathrm{Int}(K \times K')} \in \mathscr{E}(\mathrm{Int}\, K; \mathscr{E}(\mathrm{Int}\, K')) = \mathscr{E}(\mathrm{Int}(K \times K'))$ and $P \,\varepsilon\, P' f \in \mathscr{C}(K; \mathscr{C}(K')) = \mathscr{C}(K \times K')$. And the other way around, we can describe the action of $\mathrm{Diff}(\mathbb{M} \times \mathbb{M}')$ on $\mathscr{E}(\mathrm{Int}(K \times K'))$ by

$$\begin{aligned}
&\{Q \restriction_{\mathscr{E}(\mathrm{Int}(K \times K'))} \mid Q \in \mathrm{Diff}(\mathbb{M} \times \mathbb{M}')\} \\
&\quad = \mathrm{span}_{\mathbb{R}} \{\varphi(\boldsymbol{m})\, (P \,\varepsilon\, P') \restriction_{\mathscr{E}(\mathrm{Int}(K \times K'))} \mid P \in \mathrm{Diff}(\mathbb{M})\,,\ P' \in \mathrm{Diff}(\mathbb{M}')\,,\ \varphi \in \mathscr{E}(K \times K')\}\,,
\end{aligned}$$

in which $\varphi(\boldsymbol{m})$ denotes the multiplication operator $f \mapsto \varphi f$. Hence

$$\mathscr{E}(K; \mathscr{E}(K')) = \mathscr{E}(K \times K')$$

as topological vector spaces. $\square$

### 2.1.4 Multiplication of vector valued functions

Now we will discuss the possible multiplication maps between spaces of vector valued functions. The foundation to this will be Theorem 1.2.11.

**Proposition 2.1.10.** *Let $\mathbb{M}$ be a locally compact topological space and let $\mathscr{F}(\mathbb{M})$, $\mathscr{G}(\mathbb{M})$ and $\mathscr{H}(\mathbb{M})$ be complete $\mathscr{C}(\mathbb{M})$-function spaces such that the pointwise multiplication is a hypocontinuous map $m\colon \mathscr{F}(\mathbb{M}) \times \mathscr{G}(\mathbb{M}) \to \mathscr{H}(\mathbb{M})$. Furthermore let $F$, $G$ and $H$ be complete locally convex spaces and let $u\colon F \times G \to H$ be hypocontinuous bilinear map. Suppose $\mathscr{G}(\mathbb{M})$ or $G$ has the approximation property. If either*

**(i)** $\mathscr{C}(\mathbb{M}) = \mathscr{F}(\mathbb{M}) = \mathscr{G}(\mathbb{M}) = \mathscr{H}(\mathbb{M})$,

**(ii)** *$m$ is continuous and $\mathscr{F}(\mathbb{M})$ is nuclear,*

**(iii)** *$\mathscr{G}(\mathbb{M})$ and $G$ are Fréchet spaces and $\mathscr{F}(\mathbb{M})$ is nuclear or*

**(iv)** *$\mathscr{G}(\mathbb{M})$ and $G$ are strong duals of Fréchet spaces and $\mathscr{F}(\mathbb{M})$ is nuclear,*

*then the bilinear map*

$$\dot{u}\colon \mathscr{F}(\mathbb{M}; F) \times \mathscr{G}(\mathbb{M}; G) \to \mathscr{H}(\mathbb{M}; H) \quad \textit{where} \quad \dot{u}(f, g) := [x \mapsto u(f(x), g(x))],$$

*is well-defined and hypocontinuous.*

*Proof.* For $f \in \mathscr{C}(\mathbb{M}; F)$ and $g \in \mathscr{C}(\mathbb{M}; G)$ we define the function

$$u_{\mathscr{C}}(f, g)\colon \mathbb{M} \to H\colon x \mapsto u(f(x), g(x)) .$$

The subsets $B_f := \{f(x) \mid x \in K\} \subset E$ and $B_g := \{g(x) \mid x \in K\} \subset G$ are compact and thus bounded for each compact $K \subset \mathbb{M}$. Since $u$ is hypocontinuous, this implies that $u_{\mathscr{C}}(f, g) \restriction_K$ is continuous for each compact $K \subset \mathbb{M}$. But $\mathbb{M}$ is locally compact, so $u_{\mathscr{C}}(f, g)$ is continuous. Hence

$$u_{\mathscr{C}}\colon \mathscr{C}(\mathbb{M}; F) \times \mathscr{C}(\mathbb{M}; G) \to \mathscr{C}(\mathbb{M}; H)\colon (f, g) \mapsto u_{\mathscr{C}}(f, g)$$

is well-defined. Also, let $B_F \subset \mathscr{C}(\mathbb{M} : F)$ and $B_G \subset \mathscr{C}(\mathbb{M}; G)$ be bounded, i.e.

$$B_{F,K} := \bigcup_{f \in B_F} f(K) \qquad \text{and} \qquad B_{G,K} := \bigcup_{g \in B_G} g(K)$$

are bounded in $F$ and $G$ for each compact $K \subset \mathbb{M}$. Suppose $p$ is any continuous seminorm on $H$. Then there are continuous seminorms $p_{F,K}$ on $F$ and $p_{G,K}$ on $G$ such that pointwise

$$\sup_{\xi \in B_{F,K}} p(\xi, -) \leq p_{F,K}(-) \qquad \text{and} \qquad \sup_{\eta \in B_{G,K}} p(-, \eta) \leq p_{G,K}(-) .$$

Hence for all $f \in \mathscr{C}(\mathbb{M}; F)$ and all $g \in \mathscr{C}(\mathbb{M}; G)$

$$\sup_{x \in K} \sup_{\varphi \in B_F} p\big(u(\varphi(x), g(x))\big) \leq \sup_{x \in K} p_{F,K}(g(x)) \quad \text{and}$$
$$\sup_{x \in K} \sup_{\varphi \in B_G} p\big(u(f(x), \varphi(x))\big) \leq \sup_{x \in K} p_{G,K}(g(x)) .$$

In conclusion $u_{\mathscr{C}}$ is hypocontinuous. Note that the above also works for *quasi-complete* spaces $F$, $G$ and $H$.

Since $\mathscr{F}(\mathbb{M})$, $\mathscr{G}(\mathbb{M})$ and $\mathscr{H}(\mathbb{M})$ are $\mathscr{C}(\mathbb{M})$-function spaces, we have the continuous embeddings

$$\mathscr{F}(\mathbb{M};F) \hookrightarrow \mathscr{C}(\mathbb{M};F)\,, \quad \mathscr{F}(\mathbb{M};F) \hookrightarrow \mathscr{C}(\mathbb{M};F) \quad \text{and} \quad \mathscr{F}(\mathbb{M};F) \hookrightarrow \mathscr{C}(\mathbb{M};F)\,.$$

Thus, we may restrict $u_{\mathscr{C}}$ to a hypocontinuous bilinear map

$$\tilde{u}_{\mathscr{C}} := u_{\mathscr{C}} \restriction_{\mathscr{F}(\mathbb{M};F)\times\mathscr{G}(\mathbb{M};G)} \colon \mathscr{F}(\mathbb{M};F) \times \mathscr{G}(\mathbb{M};G) \to \mathscr{C}(\mathbb{M};H)\,.$$

Of course the multiplication can also be seen as a continuous (resp. hypocontinuous) map

$$\mathscr{F}(\mathbb{M}) \times \mathscr{G}(\mathbb{M}) \to \mathscr{C}(\mathbb{M})\,.$$

Now by Theorem 1.2.11 (resp. Theorem 1.2.12) $\tilde{u}_{\mathscr{C}}$ is the unique hypocontinuous bilinear map between $\mathscr{F}(\mathbb{M};F) \times \mathscr{G}(\mathbb{M};G)$ and $\mathscr{C}(\mathbb{M};H)$ that fulfils the consistency property

$$\tilde{u}_{\mathscr{C}}(f\,\xi, g\,\eta) = f\,g\,u(\xi,\eta) \quad \text{for all} \quad f \in \mathscr{F}(\mathbb{M})\,,\ g \in \mathscr{G}(\mathbb{M})\,,\ \xi \in F \text{ and } \eta \in G\,. \tag{2.1.4}$$

But by Theorem 1.2.11 (resp. Theorem 1.2.12) there is also a unique hypocontinuous bilinear map

$$\dot{u} \colon \mathscr{F}(\mathbb{M};F) \times \mathscr{G}(\mathbb{M};G) \to \mathscr{H}(\mathbb{M};H)$$

fulfilling the consistency property (2.1.4). If we extend the codomain of $\dot{u}$ from $\mathscr{H}(\mathbb{M};H)$ to $\mathscr{C}(\mathbb{M};H)$ the map stays hypocontinuous. Thus, since $\tilde{u}_{\mathscr{C}}$ is unique, we have

$$\dot{u}(f,g)(x) = u(f(x),g(x)) \quad \text{for all} \quad x \in \mathbb{M}\,.$$

$\square$

Of course these bilinear maps can also be defined between spaces of continuous differentiable functions. In the above proposition we used complete spaces $F$, $G$ and $H$. Below we extend this to quasi-complete spaces $F$, $G$ and $H$ and also prove the product rule for differentiation.

**Lemma 2.1.11.** *Suppose $k \in \mathbb{N}$, $\mathbb{M}$ is a smooth manifold, $K \overset{\text{rc}}{\subset} \mathbb{M}$ and $u\colon F \times G \to E$ is a hypcontinuous bilinear map between quasi-complete spaces $E, F, G$. The bilinear map*

$$\dot{u}\colon \mathscr{C}^k(\mathbb{X}; F) \times \mathscr{C}^k(\mathbb{X}; G) \to \mathscr{C}^k(\mathbb{X}; E) \quad \textit{where} \quad \dot{u}(f,g)(x) := u(f(x), g(x))$$

*is well-defined and hypocontinuous for $\mathbb{X} \in \{K, \mathbb{M}\}$. Furthermore, if $f \in \mathscr{C}^k(\mathbb{X}; F)$, $g \in \mathscr{C}^k(\mathbb{X}; G)$, then the product rule*

$$X\dot{u}(f,g) = \dot{u}(Xf, g) + \dot{u}(f, Xg)$$

*holds for any vector field $X$.*

*Proof.* First suppose $\mathbb{X}$ is an open subset of $\mathbb{R}^N$. Let us denote by $o(h)$, $h \in \mathbb{R}^N$, any terms that fulfil $\frac{1}{|h|}o(h) \xrightarrow{h\to 0} 0$ in $E$, $F$ or $G$. Let $x \in \mathbb{X}$ and $U \subset \mathbb{R}^N$ be open and bounded such that $x + U \subset \mathbb{X}$. If $h \in U$, then

$$\dot{u}(f,g)(x+h) - \dot{u}(f,g)(x) = u(f(x+h) - f(x), g(x+h)) + u(f(x), g(x+h) - g(x))$$

At once, we get

$$u(f(x), g(x+h) - g(x)) = u(f(x), \mathrm{d}_x g(h) + o(h)) = u(f(x), \mathrm{d}_x g(h)) + o(h)\,.$$

Since $u$ is hypocontinuous, the sets of linear maps

$$\Big\{u(-, g(x+h)) \mid h \in U\Big\} \quad \text{and} \quad \left\{\frac{1}{|h|}u(\mathrm{d}_x f(h), -) \;\middle|\; h \in U \setminus \{0\}\right\}$$

are equicontinuous. Furthermore, $g$ is continuous, i.e. $g(x+h) - g(x) \xrightarrow{h\to 0} 0$, thus

$$u(\mathrm{d}_x f(h) + o(h), g(x+h)) = u(\mathrm{d}_x f(h), g(x)) + o(h)\,.$$

Any smooth vectorfield $X$ on $\mathbb{X}$ is of the form $X = \sum_{j=1}^N a_j \partial_j$, where $a_j \in \mathscr{E}(L)$, hence

$$X\dot{u}(f,g) = \dot{u}(Xf, g) + \dot{u}(f, Xg)\,. \tag{2.1.5}$$

The same formula follows for $\mathbb{X} = \mathbb{M}$ from the above by using charts. For $\mathbb{X} = K$ we first write down (2.1.5) for functions restricted to $\operatorname{Int} K$. But clearly, for $f \in \mathscr{C}^1(K; F)$, $g \in \mathscr{C}^1(K; G)$ formula (2.1.5) can also be applied for arguments in $K$.

Formula (2.1.5) also implies that $X\dot{u}(f,g)$ is continuous by Proposition 2.1.10 and that

$$\mathscr{C}^1(\mathbb{X};F)\times\mathscr{C}^1(\mathbb{X};G)\to\mathscr{C}(\mathbb{X};E)\colon (f,g)\mapsto X\dot{u}(f,g)$$

is hypocontinuous[3]. By induction we can show that

$$\dot{u}\colon \mathscr{C}^k(\mathbb{X};F)\times\mathscr{C}^k(\mathbb{X};G)\to\mathscr{C}^k(\mathbb{X};E)$$

is well-defined and hypocontinuous. □

Suppose $A$ is a locally convex space that is also an algebra with hypocontinuous multiplication and suppose $\mathscr{G}(\mathbb{M})$ is a $\mathscr{C}(\mathbb{M})$-function space that is also an algebra with continuous multiplication. Then Theorem 1.2.11, Proposition 2.1.10 and Lemma 2.1.11 give us cases in which $\mathscr{G}(\mathbb{M};A)$ is also an algebra. In these cases we can ask the question, for which $A$ and $\mathscr{G}(\mathbb{M})$ a pointwise invertible function $f$ is invertible in the algebra $\mathscr{G}(\mathbb{M};A)$. This topic is connected with the invertibility criteria we will discuss for algebras of operators in Chapter 4.1. For this we will use the following definitions.

**Definition 2.1.12.** *For any Algebra $A$ with unit element $1_A\in A$ we denote by $A^\times$ the set of invertible elements. The spectrum $\sigma_A(a)$ of an element $a\in A$ is defined by*

$$\sigma_A(a) := \{z\in\mathbb{C}\mid z\,1_A - a\notin A^\times\}\,.$$

**Definition 2.1.13.** *Let $\mathscr{F}(\mathbb{M})$ be a $\mathscr{C}(\mathbb{M})$-function space that is also a subalgebra of $\mathscr{C}(\mathbb{M})$ and contains all constant functions. If for each complete locally m-convex algebra $A$ the space $\mathscr{F}(\mathbb{M};A)$ is an algebra with respect to the pointwise multiplication and if for each function $f\colon\mathbb{M}\to A$*

$$\textbf{\textit{(IC)}}\quad f\in\mathscr{F}(\mathbb{M};A)\wedge\forall_{x\in\mathbb{M}}\ f(x)\in A^\times\quad\Leftrightarrow\quad f\in\mathscr{F}(\mathbb{M};A)^\times,$$

*then we will say $\mathscr{F}(\mathbb{M})$ has the property **(IC)**.*

Even though, the above definition of **(IC)** uses locally m-convex algebras, it is actually enough to test on Banach algebras for this property.

[3]We use the remark made in the proof that we may also choose quasi-complete spaces.

**Lemma 2.1.14.** *Suppose $\mathscr{F}(\mathbb{M})$ a $\mathscr{C}(\mathbb{M})$-function space that is also a subalgebra of $\mathscr{C}(\mathbb{M})$ and contains all constant functions. If for each Banach algebra $A$ the space $\mathscr{F}(\mathbb{M};A)$ is an algebra with respect to the pointwise multiplication and if for each function $f \in \mathscr{C}(\mathbb{M};A)$*

$$\forall_{x\in\mathbb{M}} f(x) \in A^{\times} \quad \Leftrightarrow \quad f \in \mathscr{F}(\mathbb{M};A)^{\times}$$

*holds, then $\mathscr{F}(\mathbb{M})$ has property **(IC)**.*

*Proof.* Suppose $A$ is a complete locally m-convex algebra. As in the proof to Lemma 2.2.17 we define for a continuous seminorm $p$ the Banach space $A_p$ as the completion of $A/p^{-1}(0)$ with respect the norm $v + p^{-1}(0) \mapsto p(v)$. If $p$ is a submultiplicative seminorm, then $p^{-1}(0)$ is a closed ideal and $A_p$ is a Banach algebra. Since $A$ is locally m-convex, there is a basis of absolutely convex neighbourhoods of zero $U$ with $U \cdot U \subset U$. If $p$ is the gauge to such a neighbourhood $U$, then it is a submultiplicative continuous seminorm. Now, combining the above with [61, Chapter II 5.4] we get the representation $A \simeq \varprojlim_p (A_p, \iota_{p,q})$, with respect to

$$A \xrightarrow{\iota_p} A_p \xrightarrow{\iota_{q,p}} A_q\,, \quad \text{in which} \quad \iota_p(v) = v + p^{-1}(0)\,,\ \iota_{q,p}(v + p^{-1}(0)) = v + q^{-1}(0)$$

and $A_p$ are indexed by the submultiplicative continuous seminorms. Also, the above linear homeomorphism between $A$ and $\varprojlim_p (A_p, \iota_{p,q})$ is given by

$$A \to \varprojlim_p (A_p, \iota_{p,q})\colon a \mapsto (\iota_p a)_p\,.$$

Furthermore, all the maps $\iota_p$ and $\iota_{q,p}$ are multiplicative and for any $v \in A$ we have $v \in A^{\times}$ iff $v + p^{-1}(0) \in A_p^{\times}$ for all continuous submultiplicative seminorms $p$ on $A$.

Now let $f, g\colon \mathbb{X} \to A^{\times}$ such that $g(x) = f(x)^{-1}$ for all $x \in \mathbb{X}$. Due to the linear homeomorphism $\mathscr{F}(\mathbb{M};A) \simeq \varprojlim_p (\mathscr{F}(\mathbb{X};A_p), \mathrm{I}\,\varepsilon\,\iota_{q,p})$ from Lemma 1.2.5 we have

$$f \in \mathscr{F}(\mathbb{M};A) \ \big(\text{resp. } \ g \in \mathscr{F}(\mathbb{M};A)\big)$$
$$\Leftrightarrow \quad \forall_p\colon \iota_p \circ f \in \mathscr{F}(\mathbb{M};A_p) \ \big(\text{resp. } \iota_p \circ g \in \mathscr{F}(\mathbb{M};A_p)\big)\,.$$

So in order to prove that $\mathscr{F}(\mathbb{X})$ has the property **(IC)**, it is enough to show that

$$f \in \mathscr{F}(\mathbb{M};A) \quad \text{implies} \quad g \in \mathscr{F}(\mathbb{M};A)$$

for any Banach algebra $A$. $\square$

Now let

$$\widetilde{\mathcal{S}_{N,k}(a)} := \{(a^i) \in \mathcal{S}_{N,k}(a) \mid a^i \neq 0 \text{ for } i = 1, \dots, k\}$$

We will give the first example of a space having property **(IC)** and introduce a version of the iterated quotient rule from [38].

**Lemma 2.1.15.** *Let $\mathbb{M}$ be a smooth manifold of dimension $N$, let $K \overset{\mathrm{rc}}{\subset} \mathbb{M}$ and let $\mathbb{X} \in \{K, \mathbb{M}\}$. Then $\mathscr{E}(\mathbb{X})$ and $\mathscr{C}^k(\mathbb{X})$ have the property **(IC)** for any $k \in \mathbb{N}_0$.*

*If $A$ is a locally m-convex algebra and if $f \in \mathscr{C}^k(\mathbb{X}; A)$ such that $f(x) \in A^\times$, then the iterated quotient rule*

$$D^a g = \sum_{k=1}^{|a|} (-1)^k \sum_{(a^i) \in \widetilde{\mathcal{S}_{N,k}(a)}} g\,(D^{a^1} f)\, g\,(D^{a^2} f)\, g \cdots (D^{a^k} f)\, g.$$

*holds for $g(x) := f(x)^{-1}$ and any smooth frame $D$.*

*Proof.* Suppose $f \colon \mathbb{X} \to A$ such that $f(x) \in A^\times$ for all $x \in \mathbb{X}$. Denote $g(x) := f(x)^{-1}$ for $x \in \mathbb{X}$. First of all, if $A$ is locally m-convex, then $\mathrm{inv} \colon A^\times \to A^\times \colon a \mapsto a^{-1}$ is continuous with respect to the subspace topology [35, Proposition V.1.6]. Hence $g \in \mathscr{C}(\mathbb{X}; A)$ iff $f \in \mathscr{C}(\mathbb{X}; A)$. Suppose $f \in \mathscr{C}^1(\mathbb{X}; A)$. Let $(\varphi, U)$ be a smooth chart with $U \subset \operatorname{Int} \mathbb{X}$ and let $\tilde{g} := g \circ \varphi^{-1}$, $\tilde{f} := f \circ \varphi^{-1}$. If we denote by $o(h)$ terms, for which $o(h)/h \xrightarrow{h \to 0} 0$ in $A$, then

$$\begin{aligned}
\tilde{g}(x+h) &= (\tilde{f}(x) + \mathrm{d}_x \tilde{f}.h + o(h))^{-1} \\
&= \tilde{g}(x) - \tilde{g}(x)\,(1 + (\mathrm{d}_x \tilde{f}.h)\tilde{g}(x) + o(h))^{-1}((\mathrm{d}_x \tilde{f}.h)\,\tilde{g}(x) + o(h)) \\
&= \tilde{g}(x) - \tilde{g}(x)\,(\mathrm{d}_x \tilde{f}.h)\,\tilde{g}(x) + o(h)
\end{aligned}$$

for all $x \in \varphi(U)$. The above shows that $g \in \mathscr{C}^1(\operatorname{Int} \mathbb{X}; A)$ and that $Xg \restriction_{\operatorname{Int} \mathbb{X}} = -g\, X\, g \restriction_{\operatorname{Int} \mathbb{X}}$ extends uniquely to a function in $\mathscr{C}(\mathbb{X}; A)$ for any vector field $X$ defined on some neighbourhood of $\mathbb{X}$. By induction we get $g \in \mathscr{C}^k(\mathbb{X}; A)$ (resp. $g \in \mathscr{E}(\mathbb{X}; A)$), for $f \in \mathscr{C}^k(\mathbb{X}; A)$ (resp. $f \in \mathscr{E}(\mathbb{X}; A)$), because the point-wise multiplication is a well-defined bilinear map

$$\mathscr{C}^k(\mathbb{X}; A) \times \mathscr{C}^k(\mathbb{X}; A) \to \mathscr{C}^k(\mathbb{X}; A)$$

by Lemma 2.1.11.

The iterated quotient rule is proven in [38, Lemma 17]. Although this lemma is written for the situation of commutative derivations acting on a Banach algebra, the proof works just as well in this context. Since this can be proven by a simple inductive argument, we will quickly do this in our notation. Suppose

$$D^a g = \sum_{k=1}^{|a|} (-1)^k \sum_{(a^i)\in\mathcal{S}_{N,k}(a)} g\,(D^{a^1} f)\, g\,(D^{a^2} f)\, g \cdots (D^{a^k} f)\, g$$

holds for $|a| \leq n$. For $b \in \mathcal{S}_N$ with $|b| = n$ the product rule implies

$$0 = D^b(f\,g) = \sum_{(b^i)\in\mathcal{S}_{N,2}(b)} (D^{b^1} f)(D^{b^2} g)$$

And thus

$$\begin{aligned}
D^b g &= -g\,(D^b f)\, g \;-\; \sum_{(b^i)\in\widetilde{\mathcal{S}_{N,2}(b)}} (D^{b^1} f)(D^{b^2} g) \\
&= -g\,(D^b f)\, g \;-\; \sum_{(b^i)\in\widetilde{\mathcal{S}_{N,2}(b)}} (D^{b^1} f) \sum_{k=1}^{|b^2|} (-1)^k \sum_{(a^i)\in\widetilde{\mathcal{S}_{N,k}(b^2)}} g\,(D^{a^1} f)\, g\,(D^{a^2} f)\, g \cdots (D^{a^k} f)\, g \\
&= -g\,(D^b f)\, g \;+ \sum_{k=2}^{|b|} (-1)^k \sum_{(a^i)\in\widetilde{\mathcal{S}_{N,k}(b)}} g\,(D^{a^1} f)\, g\,(D^{a^2} f)\, g \cdots (D^{a^k} f)\, g\,.
\end{aligned}$$

□

## 2.2 Ultradifferentiable functions

**Definition 2.2.1.** *Suppose $\mathbb{M}$ is an analytic manifold of dimension $N$ and $M \in \mathbb{R}_+^{\mathbb{N}_0} = (0,\infty)^{\mathbb{N}_0}$ is a sequence. A function $f \in \mathscr{E}(\mathbb{M})$ is called **ultradifferentiable of class** $M$ iff for each analytic chart $(\phi, U)$ and any compact set $K \subset U$ there is $h > 0$ such that*

$$\lim_{\substack{|\alpha|\to\infty \\ \alpha\in\mathbb{N}_0^N}} \frac{\|(h\partial_\phi)^\alpha f\restriction_K\|_\infty}{|\alpha|!\, M_{|\alpha|}} = 0\,. \tag{2.2.6}$$

The definition implies especially that any $f \in \mathscr{E}(\mathbb{M})$ is ultradifferentiable of class $M$ iff for any analytic chart $(\phi, U)$ the function $f \circ \phi^{-1}$ is ultradifferentiable of class $M$. The space of analytic functions can be defined as the space of ultradifferentiable functions of class $\mathbb{1} := (1, 1, 1, \dots)$. Indeed, a smooth function $f\colon \mathbb{M} \to \mathbb{C}$ is analytic iff for any analytic chart $(\phi, U)$ and regular closed compact $K \subset U$

$$\exists_{h>0}\colon \lim_{|\alpha|\to\infty} \frac{\|(h\partial_\phi)^\alpha f \restriction_K \|_\infty}{|\alpha|!} = 0\,. \tag{2.2.7}$$

In reality we need to test $f$ for each $x$ only with *one* analytic chart $(\phi, U)$, where $x \in U$, in order to know if $f$ is analytic. This is also true for general ultradifferentiable functions with minor assumptions on the sequence $M$. Although this is well known, it can be seen as special case of Lemma 2.2.4. Also, this lemma will show that we may use general analytic frames to define ultradifferentiability and test for ultradifferentiability.

For open subsets $\mathbb{M} \subset \mathbb{R}^N$ these definitions coincide with the ones used in the classical sequence of papers by Komatsu [40, 41, 42] with the one slight difference. We follow the convention of [45, 46, 47] and incorporate faculties in (2.2.6).

Later we will use spaces of ultradifferentiable vectors to Lie group representation. But instead of using local charts as in [13], it is more convenient for use to use bases of invariant vector fields to define and build our spaces of ultradifferentiable vectors and functions. In [17, 18, 19] this global approach was developed for compact manifolds. In particular it was shown that this leads to the same concept of ultradifferentiabilty as a local approach like in Definition 2.2.1. We will prove the same with Proposition 2.2.10.

### 2.2.1 Ultradifferentiable function spaces defined by frames

Now we begin by defining the core Banach spaces of ultradifferentiable functions from which we build all other space of ultradifferentiable functions.

**Definition 2.2.2.** *For an analytic manifold $\mathbb{M}$, a regular compact subset $K \subset \mathbb{M}$, a finite family $D = (D_1, \dots, D_N) \subset \mathcal{V}_{\mathrm{a}}(K)$ and sequence $M \in \mathbb{R}_{>0}^{\mathbb{N}_0}$ we define the Banach space*

$$\mathscr{E}_D^M(K) := \left\{ f \in \mathscr{E}(K) \,\middle|\, \lim_{\substack{|a|\to\infty \\ a\in\mathcal{S}_N}} \sup_{x\in K} \frac{\|D^a f\|_\infty}{|a|!\, M_{|a|}} = 0 \right) \qquad \text{with norm}$$

$$\|f\|_{D,M} := \sup_{a\in\mathcal{S}_N} \frac{\|D^a f\|_\infty}{|a|!\, M_{|a|}}\,.$$

Usually such Banach spaces of ultradifferentiable functions are defined by requiring boundedness of the sequences $(\|(h\partial)^\alpha f\|_\infty)/(M_{|\alpha|}\,|\alpha|!)$. We demand convergence to zero instead of just boundedness in (2.2.6), since this ensures an easier description of the vector valued spaces $\mathscr{E}^M_D(K;E)$. Also, this convention ensures that the left resp. right translation acts continuously on $\mathscr{E}^M_D(\mathbb{G}) = \varprojlim_{K \overset{\mathrm{rc}}{\subset} \mathbb{G}} \mathscr{E}^M_D(K)$ for a basis of left resp. right invariant vector fields on a Lie group $\mathbb{G}$. Of course using convergence to zero instead of boundedness is just a minor difference, as a slight perturbation of $D$ to $hD$ for some $0 < h < 1$ is enough to move from boundedness to convergence. Hence, this change does not affect the definition of the Carleman classes in Definition 2.2.7.

We list a few basic relations between the defined spaces. For any regular closed, compact set $K \subset \mathbb{M}$, any frame $D \subset \mathcal{V}_{\mathrm{a}}(K)$ and any $h > 1$ the identity on $\mathscr{E}(K)$ restricts to continuous embeddings

$$\mathscr{E}^M_{hD}(K) \hookrightarrow \mathscr{E}^M_D(K) \hookrightarrow \mathscr{E}^M_{h^{-1}D}(K)\,.$$

For two sequences $M, N \in \mathbb{R}_+^{\mathbb{N}_0}$ we will write $N \subset M$ if

$$\sup_{k \in \mathbb{N}_0} \left(\frac{N_k}{M_k}\right)^{\frac{1}{k}} =: h < \infty \tag{2.2.8}$$

holds. In this case the identity induces the continuous embedding

$$\mathscr{E}^N_{hD}(K) \hookrightarrow \mathscr{E}^M_D(K)\,.$$

This is especially important for the sequence $\mathbb{1} := (1,1,1,\dots)$. Here we have

$$\mathscr{E}^{\mathbb{1}}_D(K) \hookrightarrow \mathscr{E}^M_D(K)$$

for any sequence $M$ with $\liminf_{k\to\infty} M_k^{\frac{1}{k}} > 1$. We will write $N \prec M$ if

$$\lim_{k \to \infty} \left(\frac{N_k}{M_k}\right)^{\frac{1}{k}} = 0\,. \tag{2.2.9}$$

In this case we even have

$$\mathscr{E}^N_{hD}(K) \hookrightarrow \mathscr{E}^M_{h'D}(K)\,.$$

for *any* $h, h' > 0$.

If $\mathbb{M}$, $\mathbb{M}'$ are analytic manifolds and $\phi\colon \mathbb{M} \to \mathbb{M}'$ is an analytic diffeomorphism, then we can define the pullback frame $\phi^* D = (\phi^* D_1, \ldots, \phi^* D_N) \subset \mathcal{V}_{\mathrm{a}}(K)$ for any $K \overset{\mathrm{rc}}{\subset} \mathbb{M}$ and any frame $D \subset \mathcal{V}_{\mathrm{a}}(\phi(K))$ by

$$(\phi^* D_j f) \circ \phi^{-1} := D_j(f \circ \phi^{-1}) , \qquad \text{for } f \in \mathscr{E}(K) .$$

Trivially, this leads to the linear homeomorphism

$$\mathscr{E}^M_{\phi^* D}(K) \to \mathscr{E}^M_D(\phi(K))\colon f \mapsto f \circ \phi^{-1} . \tag{2.2.10}$$

Together with Lemma 2.2.4, we will be able to see that the spaces $\mathscr{E}^M_D(K)$ describe the same notion of ultradifferentiability as Definition 2.2.1.

In the following, we list the properties that we will consider for $M$.

**Definition 2.2.3.** *A **weight sequence** is a sequence $M = (M_k)_k \in \mathbb{R}_+^{\mathbb{N}_0}$ which has the following four properties.*

***(N)*** *$M$ is **normalized** $:\Leftrightarrow M_0 = 1$.*

***(I)*** *$M$ is **increasing** $:\Leftrightarrow M_k \geq M_j$ for $k \geq j$.*

***(D)*** *$M$ is **stable under differential operators** $:\Leftrightarrow k \mapsto \left(\frac{M_{k+1}}{M_k}\right)^{\frac{1}{k}}$ is bounded.*

***(LC)*** *$M$ is **log-convex** $:\Leftrightarrow k \mapsto \log(M_k)$ is convex.*

*in which we call a sequence $a\colon \mathbb{N}_0 \to \mathbb{R}$ **convex** iff it can be extended to a convex map $\hat{a}\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$. We will consider the following possible other properties of a sequence $M$.*

***(LC')*** *$M$ is **weakly log-convex** $:\Leftrightarrow k \mapsto \log(M_k\, k!)$ is convex,*

***(AI)*** *the sequence $k \mapsto M_k^{\frac{1}{k}}$ is **almost increasing** $:\Leftrightarrow \sup\limits_{j,k\colon k \geq j} \frac{M_j^{\frac{1}{j}}}{M_k^{\frac{1}{k}}} < \infty$.*

***(MG)*** *$M$ **has moderate growth** $:\Leftrightarrow (k,j) \mapsto \left(\frac{M_{j+k}}{M_j\, M_k}\right)^{\frac{1}{j+k}}$ is bounded.*

***(nQA)*** *$M$ is **non quasi-analytic** $:\Leftrightarrow \sum_{n=1}^\infty \frac{M_{n-1}}{n\, M_n} < \infty$.*

***(AF)*** *$M$ **allows analytic functions** $:\Leftrightarrow \lim_{k\to\infty} M_k^{\frac{1}{k}} = \infty \Leftrightarrow \mathbb{1} \prec M$.*

Let us give an overview about these properties. Although [45, 46, 47] deal with the much more complicated topic of ultradifferentiable maps between infinite dimensional spaces, the author found the summary given in these publications very helpful. So we mainly reference [47, pp. 553–554] and also [45, 46] for the following. If **(LC)** holds, then **(N)** and **(I)** are mostly a question of convenience and do not impact the properties of the spaces of ultradifferentiable functions we create from the basic building blocks in Definition 2.2.2. Namely, if $M$ is not increasing but log-convex, then $L := [k \mapsto Ch^k M_k]$ fulfils **(N)**, **(I)** and **(LC)** for appropriate $C, h > 0$ and we have $L \subset M$ and $M \subset L$. Also, $L$ fulfils either of the properties **(D)**, **(AI)**, **(MG)**, **(nQA)** or **(AF)** if $M$ does so.

**(D)** ensure that for any frame $D \subset \mathcal{V}_{\mathrm{a}}(\mathbb{M})$ there exist $h > 0$ such that $f \mapsto D_j f$ is a continuous operator from $\mathscr{E}^M_D(K)$ to $\mathscr{E}^M_{hD}(K)$. Namely, we have

$$\sup_{\substack{|a|=n \\ a\in\mathcal{S}_{\dim M}}} \frac{\|D_j D^a f\|_\infty}{|a|! M_{|a|}} \leq \sup_{\substack{|a|=n+1 \\ a\in\mathcal{S}_{\dim M}}} \frac{\|D^a f\|_\infty}{(|a|-1)! M_{|a|-1}} \leq C \sup_{\substack{|a|=n+1 \\ a\in\mathcal{S}_{\dim M}}} \frac{\|(hD)^a f\|_\infty}{|a|! M_{|a|}},$$

where $h, C > 0$ are chosen such that $(k+1)M_{k+1} \leq Ch^{k+1} k M_k$. These constants exist, since $\sup_{k\in\mathbb{N}} (k+1)^{1/k} (M_{k+1}/M_k)^{1/k} < \infty$.

**(LC)** implies $M_k M_l \leq M_0 M_{k+l}$ for any $k, l \in \mathbb{N}_0$. **(LC')** ensures that the pointwise multiplication is a continuous map $\mathscr{E}^M_D(K) \times \mathscr{E}^M_D(K) \to \mathscr{E}^M_{hD}(K)$ for an appropriate $h > 0$. Note that in [40] the factorials in (2.2.6) are not used for the definition of ultradifferentiable functions. This means that our notion of smoothness defined by some chosen $M$ with **(LC')** corresponds to the notion of smoothness from [40] defined by some appropriate $\tilde{M}$ with **(LC)**. But we prefer to require the slightly stronger property **(LC)**, since not only **(LC)** $\Rightarrow$ **(LC')** but also **(LC)** $\Rightarrow$ **(AI)**. Indeed, if $L \in \mathbb{R}_+^{\mathbb{N}_0}$ fulfils **(LC)**, **(N)** and if $L_{n-1}^{\frac{1}{n-1}} \leq L_n^{\frac{1}{n}}$, then $L_n^2 \leq L_n^{\frac{n-1}{n}} L_{n+1}$ and thus $L_n^{\frac{1}{n}} \leq L_{n+1}^{\frac{1}{n+1}}$. Now for $M$ with **(LC)** put $L_n := M_n/M_0$, then $n \mapsto L_n^{\frac{1}{n}}$ is increasing and thus $n \mapsto M_n^{\frac{1}{n}}$ is almost increasing.

**(AI)** is connected to the ultradifferentiability of the pointwise multiplicative inverse of an ultradifferentiable function. In the context of Banach algebras, an analogous fact was proven in [38]. Of course, this property will be important for the introduction of spectrally invariant operator algebras defined by ultradifferentiable group actions.

**(MG)** ensures that in the context of ultradifferentiable spaces we have the analogous fact to $\mathscr{E}(\mathbb{M}; \mathscr{E}(\mathbb{M}')) = \mathscr{E}(\mathbb{M} \times \mathbb{M}')$. In [40, 41, 42] the property **(MG)** is called *stability* *under ultradifferential operators.*

**(nQA)** ensures that there are non-trivial compactly supported functions and especially a partition of unity of this class. For this see [40].

Finally, **(AF)** ensures that real analytic functions are contained in $\mathscr{E}^M_D(K)$.

A common example is the weight sequence defined by $M_k := (k!)^{s-1}$ for $s > 1$, which corresponds the Gevrey class of ultradifferentiable functions. This sequence fulfils all the above properties.

**Lemma 2.2.4.** *Suppose $\mathbb{M}$ is an analytic manifold, $M \in \mathbb{R}_+^{\mathbb{N}_0}$ is monotonously increasing, i.e. fulfils **(I)**, and $K \overset{\text{rc}}{\subset} \mathbb{M}$ is regular compact. Let $D, E \in \mathcal{V}_{\text{a}}(K)^N$ be tuples of analytic vector fields. If there is an analytic function $A = (A_{i,j})_{i,j} \colon \mathbb{M} \to \mathbb{R}^{N\times N}$ with*

$$E = AD := (\sum_j A_{i,j} D_j)_i \,,$$

*then there exists some $\mu > 0$ such that the identity induces a continuous embedding*

$$\mathscr{E}^M_D(K) \hookrightarrow \mathscr{E}^M_{\mu E}(K) \,.$$

*Proof.* **Step 1:** We prove the lemma for the case $A \in \mathscr{E}^{\mathbb{1}}_{h'D}(K; \mathbb{R}^{N\times N})$ for some $h' > 0$.

Suppose $f \in \mathscr{E}^M_D(K)$, i.e. there exists $C > 0$ such that for all $a \in \mathcal{S}_N$ and for $h = \min\{h', 1\}$

$$\max_{i,j} \|D^a A_{i,j}\|_\infty \leq C\, h^{-|a|}\, |a|! \qquad \text{and} \qquad \|D^a f\|_\infty \leq \|f\|_{D,M}\, h^{-|a|}\, M_{|a|}\, |a|! \,. \tag{2.2.11}$$

This implies that for any $a \in \mathcal{S}_N$ where $n := |a|$ and for any $(a^j)_j \in \mathcal{S}_{N,n}(a)$ and any $\alpha, \beta \in \{1, \dots, N\}^{n-1}$

$$\|(D^{a^1} A_{\alpha_1,\beta_1}) \cdots (D^{a^{n-1}} A_{\alpha_{n-1},\beta_{n-1}}) \cdot (D^{a^n} f)\|_\infty \leq \|f\|_{D,M}\, C^{n-1}\, |a^1|! \cdots |a^n|!\, M_n\, h^{-n} \,, \tag{2.2.12}$$

where we also used that $M$ is monotonously increasing. If $g \in \mathscr{E}(K)$ we denote by $g(\boldsymbol{m})$ the corresponding multiplication operator from $\mathscr{E}(K)$ to itself. For $a \in \mathcal{S}_N$ and $n := |a|$

we rewrite $E^a f$ in terms of derivatives with respect to $D$

$$E^a f = (AD)^a f = \sum_{b_1,\dots,b_n=1}^{N} A_{a_n,b_n}(\boldsymbol{m}) D_{b_n} \cdots A_{a_1,b_1}(\boldsymbol{m}) D_{b_1} f \tag{2.2.13}$$

$$= \sum_{b_1,\dots,b_n=1}^{N} \sum_{(b^i)_i \in \mathcal{S}_{N,n}(b)} p_{b,(b^i)_i}\, A_{a_n,b_n}\, (D^{b^n} f) \cdot (D^{b^{n-1}} A_{a_{n-1},b_{n-1}}) \cdots (D^{b^1} A_{a_1,b_1})\,, \tag{2.2.14}$$

where $b = (b_1, \dots, b_{n-1}, b_n, 0, 0, \dots)$ and $p_{b,(b^i)_i}$ is a non-negative integer. The number $p_{b,(b^i)_i}$ describes how often the corresponding summands occur when using the product rule successively in (2.2.13). For $n \in \mathbb{N}$ we define

$$\mathcal{B}_n := \left\{ \alpha \in \mathbb{N}_0^n \;\middle|\; \forall_{j=1,\dots,n} \sum_{i=1}^{j} \alpha_i \le j \text{ and } |\alpha| = n \right\} \quad \text{and} \quad B_k := \sum_{\substack{(b^i)_i \in \mathcal{S}_{N,n}(b) \\ \forall_i\ |b^i| = k_i}} p_{b,(b^i)_i}\,,$$

for $k \in \mathbb{N}_0^n$. Here $B_k$ is the same for any possible index sequence $b$ such that $n = |b|$. With the $\mathcal{B}_n$, $B_k$ and (2.2.14) we may estimate $E^a f$ by

$$\begin{aligned}
\|E^a f\|_\infty &\le \sum_{b_1,\dots,b_n=1}^{N} \sum_{(b^i)_i \in \mathcal{S}_{N,n}(b)} p_{b,(b^i)_i}\, \|f\|_{D,M}\, |b^n|! \cdots |b^1|!\, M_n\, (C\, h^{-1})^n \\
&\le \|f\|_{D,M} \sum_{b_1,\dots,b_n=1}^{N} \sum_{k \in \mathcal{B}_n} k!\, B_k\, M_n (C\, h^{-1})^n \\
&= \|f\|_{D,M} \sum_{k \in \mathcal{B}_n} k!\, B_k\, M_n\, (C\, h^{-1}\, N)^n\,.
\end{aligned} \tag{2.2.15}$$

We can calculate $B_k$ by counting how many summands of derivatives of order $k$ occur in (2.2.13) when applying the product rule. Its value is

$$B_k = \prod_{j=2}^{n} \binom{j - \sum_{i=1}^{j-1} k_i}{k_j} \text{ for } k \in \mathcal{B}_n \quad \text{and} \quad B_k = 0 \text{ for } k \in \mathbb{N}_0^n \setminus \mathcal{B}_n\,.$$

Thus we have

$$k!\, B_k = \prod_{j=2}^{n} \frac{\left(j - \sum_{i=1}^{j-1} k_i\right)!}{\left(j - \sum_{i=1}^{j} k_i\right)!} = \prod_{j=1}^{n-1} \left( j + 1 - \sum_{i=1}^{j} k_i \right) \qquad \text{for any } k \in \mathcal{B}_n\,.$$

Again, we introduce new sets of indices and connected integers by

$$\mathcal{C}_m := \left\{ k \in \mathbb{N}_0^m \;\middle|\; \forall_j \sum_{i=1}^{j} k_i \le j \right\} \quad \text{and} \quad C_k := \prod_{j=1}^{m} \left( j + 1 - \sum_{i=1}^{j} k_i \right) \quad \text{for } k \in \mathcal{C}_m\,,$$

in which $m \in \mathbb{N}$. For any $m \in \mathbb{N}_{\geq 2}$, $k \in \mathcal{C}_{m-1}$ and $l \in \{0, \dots, m - |k|\}$ we have $(k, l) \in \mathcal{C}_m$ and we may use

$$C_{(k,l)} = C_k \, (m + 1 - |k| - l) \, .$$

Thus, by the hockey stick identity, we have for any $p \in \mathbb{N}_0$

$$\begin{aligned} \sum_{l=0}^{m-|k|} C_{(k,l)} \binom{m+1-|k|-l+p}{m+1-|k|-l} &= C_k \sum_{l=0}^{m-|k|} (p+1) \binom{l+1+p}{l} \\ &= C_k \, (p+1) \binom{m-|k|+p+2}{m-|k|} \, . \end{aligned} \tag{2.2.16}$$

By iterating (2.2.16) we get

$$\begin{aligned} \sum_{k \in \mathcal{B}_n} k! \, B_k &= \sum_{k_1=0}^{1} \cdots \sum_{k_j=0}^{j-\sum_{i=1}^{j-1} k_i} \cdots \sum_{k_{n-1}=0}^{n-1-\sum_{i=1}^{n-2} k_i} C_{(k_1\dots,k_{n-1})} \binom{n-\sum_{i=1}^{n-1} k_i + 0}{n - \sum_{i=1}^{n-1} k_i} \\ &= \sum_{k_1=0}^{1} \cdots \sum_{k_j=0}^{j-\sum_{i=1}^{j-1} k_i} \cdots \sum_{k_{n-2}=0}^{n-2-\sum_{i=1}^{n-3} k_i} C_{(k_1\dots,k_{n-2})} \, 1! \binom{n-1-\sum_{i=1}^{n-2} k_i + 2}{n-1-\sum_{i=1}^{n-2} k_i} \\ &= \sum_{k_1=0}^{1} \cdots \sum_{k_j=0}^{j-\sum_{i=1}^{j-1} k_i} C_{(k_1,\dots,k_j)} \, \frac{(2n-2-2j)!}{2^{n-1-j}(n-1-j)!} \binom{j+1-\sum_{i=1}^{j} k_i + 2(n-1-j)}{j+1-\sum_{i=1}^{j} k_i} \\ &= \sum_{k_1=0}^{1} C_{k_1} \, \frac{(2n-4)!}{2^{n-2}(n-2)!} \binom{2-k_1+2(n-2)}{2-k_1} \\ &\leq \frac{(n-2)!}{2^{n-2}} \sum_{l=0}^{1} (2-l) \binom{2-l+(n+2)}{2-l} \\ &\leq r 2^n n! \, , \end{aligned} \tag{2.2.17}$$

for some $r > 0$ and all $n \geq 2$. Now, by using (2.2.15) and (2.2.17), we may bound $|E^a f|$ by

$$\|E^a f\|_\infty \leq r \, \|f\|_{D,M} \, n! \, M_n \, (2 \, C \, h^{-1} \, N)^n$$

and thus for $0 < \mu < h \, (2 \, N \, C)^{-1} =: \hat{\mu}$ and $f \in \mathscr{E}_D^M(K)$ we have

$$\|f\|_{\mu E,M} := \sup_{a \in \mathcal{S}_N} \frac{\|(\mu E)^a f\|_\infty}{|a|! \, M_{|a|}} \leq r \, \|f\|_{D,M} \quad \text{and} \quad \lim_{\substack{|a| \to \infty \\ a \in \mathcal{S}_N}} \frac{\|(\mu E)^a f\|_\infty}{|a|! \, M_{|a|}} = 0 \, .$$

Thus we have proven the continuous embedding

$$\mathscr{E}_D^M(K) \hookrightarrow \mathscr{E}_{\mu E}^M(K) \, .$$

**Step 2:** We prove that for any tuple $D \in \mathcal{V}_{\mathrm{a}}(K)^N$ and any $\mathbb{R}^{N\times N}$-valued, analytic map $A$ on $\mathbb{M}$ there is $h' > 0$ such that $A \restriction_K \in \mathscr{E}^{\mathbb{1}}_{h'D}(K; \mathbb{R}^{N\times N})$.

Suppose $(\phi_1, U_1), \ldots, (\phi_n, U_n)$ are analytic charts and $K_1, \ldots, K_n$ regular compact such that $K = \bigcup_{i=1}^n K_i$ and $K_j \subset U_j$ for $j = 1, \ldots, n$. Let us denote by $D^{(k)} \in \mathcal{V}_{\mathrm{a}}(K_k)$ the frame induced by $\phi_k$ on $K_k$ as described in (2.1.2). As in (2.2.7) there exists $\tau > 0$ such that

$$A_{i,j} \restriction_{K_k} \in \mathscr{E}^{\mathbb{1}}_{\tau D^{(k)}}(K_k) \qquad \text{for all} \qquad i, j = 1, \ldots N \text{ and } k = 1, \ldots, n\,.$$

Hence, by Step 1, we also have

$$A_{i,j} \restriction_{K_k} \in \mathscr{E}^{\mathbb{1}}_{h'D}(K_k) \qquad \text{for all} \qquad i, j = 1, \ldots N$$

for some $h' > 0$, in which we denoted the restriction of $D$ to the sets $K_k$ by $D$ as well. This implies $A \restriction_K \in \mathscr{E}^{\mathbb{1}}_{h'D}(K; \mathbb{R}^{N\times N})$. □

The above lemma can be applied to any pair of analytic frames $D, E \in \mathcal{V}_{\mathrm{a}}(K)$, because we always find an analytic matrix valued map $A$ with $E = AD$. Naturally, the situation is better if the frames $D$ and $E$ are connected by a *constant* linear transformation.

**Lemma 2.2.5.** *Suppose* $\mathbb{M}$ *is an analytic manifold,* $K \overset{\mathrm{rc}}{\subset} \mathbb{M}$*,* $D \subset \mathcal{V}_{\mathrm{a}}(K)$ *is a frame. Let* $AD := (\sum_j A_{i,j} D_j)_i$ *of an invertible matrix* $A = (A_{i,j})_{i,j} \in \mathbb{R}^{N\times N}$*. Then there exists some* $h > 1$ *such that*

$$\mathscr{E}^M_{hD}(K) \xrightarrow{\mathrm{I}} \mathscr{E}^M_{AD}(K) \quad \textit{and} \quad \mathscr{E}^M_{D}(K) \xrightarrow{\mathrm{I}} \mathscr{E}^M_{h^{-1}AD}(K)$$

*are well-defined and continuous.*

*Proof.* Let $f \in \mathscr{E}(K)$ and put $\mu = \max_{i,j} |A_{i,j}|$. Then

$$\max_{|a|=k} \|(AD)^a f\|_\infty \leq (N\mu)^k \max_{|a|=k} \|D^a f\|_\infty\,.$$

Hence $\mathscr{E}^M_{N\mu D}(K) \xrightarrow{\mathrm{I}} \mathscr{E}^M_{AD}(K)$ is continuous. Now consider $\tilde{D} := (N\mu)^{-1} D$ Then

$$\mathscr{E}^M_{D}(K) = \mathscr{E}^M_{N\mu\tilde{D}}(K) \xrightarrow{\mathrm{I}} \mathscr{E}^M_{A\tilde{D}}(K) = \mathscr{E}^M_{(N\mu)^{-1}AD}(K)$$

is well-defined and continuous. □

We already mentioned that property **(D)** ensures the continuity of differential operators between spaces of ultradifferentiable functions. But it also ensures that the following lemma holds, which is equally important for our later discussion.

**Lemma 2.2.6.** *Let $M$ be a weight sequence, let $\mathbb{M}$ be an analytic manifold, let $X \overset{\mathrm{rc}}{\subset} \mathbb{M}$ and let $D \subset \mathcal{V}_{\mathrm{a}}(K)$ be a frame. Then there exist $h > 1$ such that*

$$\mathscr{E}^M_{hD}(K) \xrightarrow{\mathrm{I}} \mathscr{E}^M_D(K) \quad \textit{and} \quad \mathscr{E}^M_D(K) \xrightarrow{\mathrm{I}} \mathscr{E}^M_{h^{-1}D}(K)$$

*are nuclear embeddings.*

*Proof.* We may use the corresponding proof from Komatsu [40, Proposition 2.4] if we make minor changes. We exchange the partial derivatives $\partial$ with the frame $D$ and make an adjustment for the noncommutativity of $D$.

We use the concept of quasi-nuclear maps. A linear operator $T\colon E \to F$ between Banach spaces $E$, $F$ is called quasi-nuclear if there is a sequence $(e'_j)_j \subset E'$ with

$$\sum_j \|e'_j\|_{E'} < \infty \quad \text{and} \quad \|Te\|_F \leq \sum_j |\langle e, e'_j\rangle|, \quad \text{for all } e \in E.$$

By Komatsu [40, Lemma 2.3] the identity $\mathscr{C}^{N+1}(L) \xrightarrow{\mathrm{I}} \mathscr{C}(L)$ is nuclear for $L \overset{\mathrm{rc}}{\subset} \mathbb{R}^N$. By Lemma 2.1.5 we may use charts to get the nuclear embedding $\mathscr{C}^{N+1}(K) \xrightarrow{\mathrm{I}} \mathscr{C}(K)$ for $K \overset{\mathrm{rc}}{\subset} \mathbb{M}$ and $N := \dim \mathbb{M}$. So there exist $(v_j)_j \subset \mathscr{C}^{N+1}(K)'$ with

$$C_0 := \sum_j \|v_j\|_{\mathscr{C}^{N+1}(K)'} < \infty \quad \text{and} \quad \|f\|_\infty \leq \sum_j |\langle f, v_j\rangle|, \quad \text{for all } f \in \mathscr{C}^{N+1}(K).$$

We define the *finite* sets of differential operators

$$\mathcal{P}_k := \{D^a \mid a \in \mathcal{S}_{\dim \mathbb{M}},\ |a| = k\}, \qquad \text{for } k \in \mathbb{N}_0.$$

For some chosen $h, \lambda > 0$ we define the linear functionals $u_{P,j} \in \mathscr{E}^M_{\lambda D}(K)'$ by

$$\langle f, u_{P,j}\rangle := \frac{\mu^k \langle Pf, v_j\rangle}{M_k\, k!}, \qquad \text{for } P \in \mathcal{P}_k,\ f \in \mathscr{E}^M_{\lambda D}(K),$$

This sequence of functionals fulfils

$$\|f\|_{\mu D, M} \leq \sum_{k=0}^{\infty} \sum_{P \in \mathcal{P}_k} \frac{\mu^k \|Pf\|_\infty}{M_k\, k!} \leq \sum_{k=0}^{\infty} \sum_{P \in \mathcal{P}_k} \sum_j |\langle f, u_{P,j}\rangle|, \qquad \text{for } f \in \mathscr{E}^M_{\lambda D}(K).$$

Since $M$ fulfils **(D)** there exist $C, H \geq 1$ with $(k+1)\, M_{k+1}/M_k \leq C H^k$ for each $k \in \mathbb{N}_0$. Thus we get

$$
\begin{aligned}
|\,\langle f, u_{P,j}\rangle\,| \leq \|v_j\|_{\mathscr{C}^{N+1}(K)'} \max_{\substack{|a|\leq N+1\\ a\in\mathcal{S}_N}} \frac{\mu^k \|D^a P f\|_\infty}{M_k\, k!} \\
&\leq \max_{0\leq l\leq N+1} \frac{\mu^k}{\lambda^{k+l}} \cdot \frac{(l+k)!\, M_{l+k}}{k!\, M_k} \|v_j\|_{\mathscr{C}^{N+1}(K)'} \|f\|_{\lambda D, M} \\
&\leq (\lambda^{-N-1}+1)\, \frac{\mu^k}{\lambda^k}\, C^{N+1} H^{(N+1)\,(k+N)}\, \|v_j\|_{\mathscr{C}^{N+1}(K)'} \|f\|_{\lambda D, M} \\
&=: C_1 \cdot \frac{\mu^k}{\lambda^k} H^{(N+1)k}\, \|v_j\|_{\mathscr{C}^{N+1}(K)'} \|f\|_{\lambda D, M}\,.
\end{aligned}
$$

So for $\lambda, \mu > 0$ such that $\kappa := \frac{\mu}{\lambda} \cdot H^{N+1} < 1$ we have

$$
\begin{aligned}
\sum_{k=0}^\infty \sum_{P\in\mathcal{P}_k} \sum_j \|u_{P,j}\|_{\mathscr{E}^M_{\lambda D}(K)'} &\leq \sum_{k=0}^\infty \sum_{P\in\mathcal{P}_k} \sum_j C_1 \cdot \frac{\mu^k}{\lambda^k} H^{(N+1)k}\, \|v_j\|_{\mathscr{C}^{N+1}(K)'} \\
&\leq \sum_{k=0}^\infty k^N C_0\, C_1\, \kappa^k < \infty\,.
\end{aligned}
$$

So in this case $\mathscr{E}^M_{\lambda D}(K) \xrightarrow{\mathrm{I}} \mathscr{E}^M_{\mu D}(K)$ is quasi-nuclear. As cited in [40], the composition of two quasi-nuclear maps is nuclear. Hence, we always find $h > 1$ such that

$$
\mathscr{E}^M_{hD}(K) \xrightarrow{\mathrm{I}} \mathscr{E}^M_{D}(K) \quad \text{and} \quad \mathscr{E}^M_{D}(K) \xrightarrow{\mathrm{I}} \mathscr{E}^M_{h^{-1}D}(K)
$$

are nuclear.

$\square$

### 2.2.2 Ultradifferentiable functions of Roumieu and Beurling type

Finally, we will define the main spaces of ultradifferentiable functions.

**Definition 2.2.7.** *Suppose $\mathbb{M}$ is an analytic manifold and $M \in \mathbb{R}_+^{\mathbb{N}_0}$ a sequence. If $K \subset \mathbb{M}$ is regular closed compact and $D \subset \mathcal{V}_{\mathrm{a}}(K)$ a frame, then we define $\mathscr{C}(K)$-function spaces*

$$
\mathscr{E}^{\{M\}}_D(K) := \varinjlim_{h>0} \mathscr{E}^M_{hD}(K) \qquad \textit{and} \qquad \mathscr{E}^{(M)}_D := \varprojlim_{h>0} \mathscr{E}^M_{hD}(K)\,,
$$

*in which we use* $\mathbb{R}_+ = (0,\infty)$ *with the standard ordering resp. in the second case the* inverted *ordering.*

*Furthermore, if* $D \subset \mathcal{V}_{\mathrm{a}}(\mathbb{M})$ *is a global frame, we will also define the* $\mathscr{C}(\mathbb{M})$*-function spaces* $\mathscr{E}_D^M(\mathbb{M})$, $\mathscr{E}_D^{(M)}(\mathbb{M})$ *and* $\mathscr{E}_D^{\{M\}}(\mathbb{M})$ *by*

$$\mathscr{F}(\mathbb{M}) = \varprojlim_{K \overset{\mathrm{rc}}{\subset} \mathbb{M}} \mathscr{F}(K)\,, \qquad \textit{for } \mathscr{F} \in \{\mathscr{E}_D^M, \mathscr{E}_D^{(M)}, \mathscr{E}_D^{\{M\}}\}\,.$$

*For* $\mathbb{X} \in \{K, \mathbb{M}\}$ *and for the frame* $D \in \mathcal{V}_{\mathrm{a}}(\mathbb{X})$ *the space* $\mathscr{E}^{\{M\}}(\mathbb{X})$ *resp.* $\mathscr{E}^{(M)}(\mathbb{X})$ *is* ***Carleman class of Roumieu type*** *resp.* ***of Beurling type*** *on* $\mathbb{X}$ *associated to* $M$ *and* $D$.

In cases where we can treat the Beurling and Roumieu classes at the same time, the following convention has merit.

**Convention 2.2.8.** *For a weight sequence* $M$ *we will use the symbol* $[M]$ *for an undetermined variable* $[M] \in \{(M), \{M\}\}$*. Any statement involving* $[M]$ *is meant to be true for both the case* $[M] = (M)$ *and the case* $[M] = \{M\}$*.*

Note that by definition we have the continuous embeddings

$$\mathscr{E}_D^{(M)}(\mathbb{X}) \xrightarrow{\mathrm{I}} \mathscr{E}_D^{\{M\}}(\mathbb{X})\,, \quad \mathscr{E}_D^{(M)}(\mathbb{X}) \xrightarrow{\mathrm{I}} \mathscr{E}_D^{M}(\mathbb{X}) \quad \text{and} \quad \mathscr{E}_D^{M}(\mathbb{X}) \xrightarrow{\mathrm{I}} \mathscr{E}_D^{\{M\}}(\mathbb{X})$$

for any monotonously increasing sequence $M \in \mathbb{R}_+^{\mathbb{N}_0}$, any $\mathbb{X} \in \{K, \mathbb{M}\}$ for an analytic manifold $\mathbb{M}$ and $K \overset{\mathrm{rc}}{\subset} \mathbb{M}$ and for any frame $D \subset \mathcal{V}_{\mathrm{a}}(\mathbb{X})$. If $N \subset M$ then we have the continuous embeddings

$$\mathscr{E}_D^{(N)}(\mathbb{X}) \xrightarrow{\mathrm{I}} \mathscr{E}_D^{(M)}(\mathbb{X}) \quad \text{and} \quad \mathscr{E}_D^{\{N\}}(\mathbb{X}) \xrightarrow{\mathrm{I}} \mathscr{E}_D^{\{M\}}(\mathbb{X})\,.$$

If $N \prec M$ then we have the continuous embedding

$$\mathscr{E}_D^{\{N\}}(\mathbb{X}) \xrightarrow{\mathrm{I}} \mathscr{E}_D^{(M)}(\mathbb{X})\,.$$

This motivates the following definition, in order to effectively use the variable $[M]$ for embeddings.

**Definition 2.2.9.** *Let $M$ and $N$ be weight sequences. We will write $(M) \subset (N)$, $\{M\} \subset \{N\}$ or $(M) \subset \{N\}$ iff $M \subset N$ i.e. $\sup_j (M_j/N_j)^{\frac{1}{j}} < \infty$. We will write $\{M\} \subset (N)$ iff $M \prec N$ i.e. $\lim_{j\to\infty}(M_j/N_j)^{\frac{1}{j}} = 0$.*

This way, the above can be shortened to

$$[M] \subset [N] \qquad \Rightarrow \qquad \mathscr{E}_D^{[M]}(\mathbb{X}) \xrightarrow{\mathrm{I}} \mathscr{E}_D^{[N]}(\mathbb{X}) \qquad \text{is continuous}$$

Later it will be convenient to use this notation for the inclusion of analytic functions, i.e. $\{\mathbb{1}\} \subset [N]$ implies the continuous inclusion $\mathscr{E}_D^{\{\mathbb{1}\}}(\mathbb{X}) \xrightarrow{\mathrm{I}} \mathscr{E}_D^{[N]}(\mathbb{X})$. Naturally, $M$ fulfils **(AF)** iff $\{\mathbb{1}\} \subset (M)$.

Now, we will discuss some general properties of the spaces of ultradifferentiable functions. We will especially prove that we may describe the Denjoy-Carleman classes of Roumieu type with the help of analytic charts. This also ensures that we could easily use a lot of the statements proved for ultradifferentiable functions on open subsets $\mathbb{M} \subset \mathbb{R}^n$ by H. Komatsu. But since this approach does not work for the Beurling case, we will use the description via limits of the spaces $\mathscr{E}_D^M(K)$ to prove e.g. nuclearity.

**Proposition 2.2.10.** *Let $\mathbb{M}$ be an analytic manifold, let $K \overset{\mathrm{rc}}{\subset} \mathbb{M}$ and let $M \in \mathbb{R}_+^{\mathbb{N}_0}$ be a weight sequence. Suppose $\mathbb{X} \in \{K, \mathbb{M}\}$ and suppose $D \subset \mathcal{V}_{\mathrm{a}}(\mathbb{X})$ is a frame. Then the following holds.*

**(i)** *The space $\mathscr{E}_D^{\{M\}}(\mathbb{X})$ does not depend on the choice of frame $D$. If there is some invertible matrix $A = (A_{i,j})_{i,j}$ with $E = (\sum_j A_{i,j} D_j)_i$, then $\mathscr{E}_E^{(M)}(\mathbb{X}) = \mathscr{E}_D^{(M)}(\mathbb{X})$ as well.*

**(ii)** *Suppose $(\phi_j, U_j)$, $j \in J$, is a family of analytic charts, with regular compact subsets $V_j \subset U_j$ such that $\mathbb{X} \subset \cup_{j\in J} \operatorname{Int} V_j$ and $\mathbb{X} \cap V_j \overset{\mathrm{rc}}{\subset} \mathbb{M}$. Then*

$$f \in \mathscr{E}_D^{\{M\}}(\mathbb{X}) \quad \Leftrightarrow \quad \forall_{j\in J} f \circ \phi_j^{-1} \in \mathscr{E}_\partial^{\{M\}}(\phi_j(\mathbb{X} \cap V_j))$$

*and the topology of $\mathscr{E}_D^{\{M\}}(\mathbb{X})$ carries the initial topology with respect to the above maps.*[4]

[4]We can always find such a family $(\phi_j, U_j)$ of charts and subsets $V_j$, as Lemma 2.1.5 shows.

**(iii)** $\mathscr{E}_D^{\{M\}}(K)$ *is the strong dual of a nuclear Fréchet space.* $\mathscr{E}_D^{(M)}(K)$ *is a nuclear Fréchet space.*

**(iv)** $\mathscr{E}_D^{[M]}(\mathbb{X})$ *is nuclear and complete and* $\mathscr{E}_D^{(M)}(\mathbb{X})$ *is Fréchet.*

*Proof.* **(i):** This is a direct result of Lemma 2.2.4 and Lemma 2.2.5.

**(ii):** Let $K_j := \mathbb{X} \cap V_j$, then $K_j$ is regular compact. Note that $D$ and $\partial_{\phi_j}$ always restrict to frames in $\mathcal{V}_{\mathrm{a}}(K_j)$. Now, by (2.2.10) and Lemma 2.2.4

$$\mathscr{E}_D^{\{M\}}(K_j) \to \mathscr{E}_\partial^{\{M\}}(\phi_j(K_j)) \colon f \mapsto f \circ \phi_j^{-1}$$

is a linear homeomorphism for each $j \in J$. Moreover, we have $f \in \mathscr{E}(\mathbb{X})$ iff $f \restriction_{K_j} \in \mathscr{E}(K_j)$ for all $j \in J$ and

$$\mathscr{E}_D^{\{M\}}(\mathbb{X}) \to \prod_{j \in J} \mathscr{E}_D^{\{M\}}(K_j) \colon f \mapsto (f \restriction_{K_j})_{j \in J}$$

is a homeomorphism onto its image. Together, the two arguments prove the statement.

**(iii):** By Lemma 2.2.6 there are a sequences $h_n \searrow 0$, $n \to \infty$, and $k_n \nearrow \infty$, $n \to \infty$ such that $\mathscr{E}_{h_n D}^M(K) \xrightarrow{\mathrm{I}} \mathscr{E}_{h_{n+1} D}^M(K)$ and $\mathscr{E}_{k_{n+1} D}^M(K) \xrightarrow{\mathrm{I}} \mathscr{E}_{k_n D}^M(K)$ are nuclear for each $n \in \mathbb{N}$. Hence, $\mathscr{E}_D^{(M)}(K) = \varprojlim_{n \to \infty} \mathscr{E}_{k_n D}^{(M)}(K)$ and $\mathscr{E}^{\{M\}}(K) = \varinjlim_{n \to \infty} \mathscr{E}_{h_n}^M(K)$ with nuclear and injective linking maps. This means $\mathscr{E}_D^{(M)}(K)$ is a nuclear Fréchet space and $\mathscr{E}_D^{\{M\}}(K)$ is a complete nuclear (DF) space as described in [40, p. 34]. Thus, by [61, Exercise 32(b) on p. 199], $\mathscr{E}_D^{\{M\}}(K)$ is Montel and hence reflexive by [61, p. 147]. The strong dual of a complete (DF) space is a Fréchet space by [36, Satz 8.19], hence $\mathscr{E}_D^{\{M\}}(K)$ is the dual of a nuclear Fréchet space by [66, Proposition 50.6].

**(iv):** For $\mathbb{X} = K$ only the Roumieu-case is left. Since $\mathscr{E}_D^{\{M\}}(K)$ is the dual of a nuclear Fréchet space, it is complete and nuclear as discussed in Proposition 1.2.1.

The space $\mathscr{E}_D^{[M]}(\mathbb{M})$ is linearly homeomorphic to the projective limit of the complete nuclear spaces $\mathscr{E}_D^{[M]}(K')$, $K' \overset{\mathrm{rc}}{\subset} \mathbb{M}$. Hence $\mathscr{E}^{[M]}(\mathbb{M})$ is complete and nuclear by [61, Chapter II, 5.3] and [66, Proposition 50.1]. Of course, it is enough to consider a countable family of regular compact subset of $\mathbb{M}$ for this limit. Thus $\mathscr{E}_D^{(M)}(\mathbb{M})$ is Fréchet, since any countable projective limit of Fréchet spaces is Fréchet. $\square$

Note that **(ii)** especially works with $\mathbb{X} = \mathbb{M}$ and the family of *all* pairs $((\varphi, U), V)$ of analytic charts $(\varphi, U)$ of $\mathbb{M}$ and $V \overset{\mathrm{rc}}{\subset} U$. I.e. we have

$$f \in \mathscr{E}_D^{\{M\}}(\mathbb{M}) \quad \Leftrightarrow \quad f \circ \varphi^{-1} \in \mathscr{E}_\partial^{\{M\}}(\varphi(U)) \text{ for all analytic charts } (\varphi, U)$$

and $\mathscr{E}_D^{\{M\}}(\mathbb{M})$ carries the initial topology with respect to the maps

$$\mathscr{E}_D^{\{M\}}(\mathbb{M}) \to \mathscr{E}_\partial^{\{M\}}(\varphi(U)) \colon f \mapsto f \circ \varphi^{-1}\,.$$

This also elucidates that $\mathscr{A}(\mathbb{M}) = \mathscr{E}_D^{\{\mathbb{1}\}}(\mathbb{M})$ as linear spaces.

Often it is more convenient to represent $\mathscr{E}_D^{\{M\}}(\mathbb{M})$ purely by a *projective* limit of Banach spaces. One reason is that $\varepsilon$-products play much more nicely with projective limits than with inductive limits. Moreover in Proposition 2.2.15, we will later get a better result concerning the continuity of the multiplication in $\mathscr{E}_D^{\{M\}}(\mathbb{M})$ using this approach.

**Definition 2.2.11.** *We define the set*

$$\Lambda := \{(c_0\, c_1 \cdots c_n)_{n\in\mathbb{N}_0} \mid \textit{ for some monotone } (c_n)_n \in \mathbb{R}_{>0}^{\mathbb{N}_0} \textit{ with } \lim_{n\to\infty} c_n = +\infty\}\,.$$

*and equip it with the preorder*

$$h' \gtrsim h \qquad :\Leftrightarrow \qquad \sup_{n\in\mathbb{N}_0} h_n/h'_n < \infty \qquad \textit{for } h, h' \in \Lambda\,.$$

*Furthermore, for an analytic manifold* $\mathbb{M}$, $K \overset{\mathrm{rc}}{\subset} \mathbb{M}$, $M \in \mathbb{R}_{>0}^{\mathbb{N}_0}$ *and a frame* $D \subset \mathcal{V}_a(\mathbb{X})$ *we define the* $\mathscr{C}(\mathbb{X})$*-function space*

$$\mathscr{E}_{D,\mathrm{proj}}^{\{M\}}(\mathbb{X}) := \varprojlim_{h\in\Lambda} \mathscr{E}_D^{hM}(\mathbb{X})\,.$$

*In some cases it will also be convenient to use the notation* $\mathscr{E}_{D,\mathrm{proj}}^{(M)}(\mathbb{X}) := \mathscr{E}_D^{(M)}(\mathbb{X})$.

Let us quickly make sure that the above is well-defined. For $h, k \in \Lambda$ there are positive, monotone and diverging sequences $b, c$ with $h_n = c_0 \cdots c_n$ and $k_n = b_0 \cdots b_n$ for all $n \in \mathbb{N}_0$. Now put $a = (a_n)_{n\in\mathbb{N}_0}$ with $a_n := \min\{b_n, c_n\}$. Then $a$ is monotone, positive and diverging and hence $l := (a_0 \cdots a_n)_{n\in\mathbb{N}_0} \in \Lambda$ and $h, k \gtrsim l$. Thus $(\Lambda, \gtrsim)$ is a directed set. Also, the preorder on $\Lambda$ is defined in such a way that we have the continuous embedding $\mathscr{E}_D^{\{hM\}}(\mathbb{X}) \xrightarrow{\mathrm{I}} \mathscr{E}_D^{\{kM\}}(\mathbb{X})$ for $k \gtrsim h$. Thus the projective limit $\varprojlim_{h\in\Lambda} \mathscr{E}_D^{\{hM\}}(\mathbb{X})$ is well-defined.

Note that for any weight sequence $M$ and any $h \in \Lambda$ we have $M \prec hM$. If $h_0 = 1$ then the sequence $hM$ is a weight sequence as well. Also $\mathscr{E}_{\mu D,\mathrm{proj}}^{\{M\}}(\mathbb{X}) = \mathscr{E}_{D,\mathrm{proj}}^{\{M\}}(\mathbb{X})$ for any $\mu > 0$. Thus we especially have

$$\mathscr{E}_{D,\mathrm{proj}}^{\{M\}}(\mathbb{X}) = \varprojlim_{\mu>0} \varprojlim_{h\in\Lambda} \mathscr{E}_{\mu D}^{hM}(\mathbb{X}) = \varprojlim_{h\in\Lambda} \mathscr{E}_D^{(hM)}(\mathbb{X})\,. \tag{2.2.18}$$

So $\mathscr{E}_{D,\mathrm{proj}}^{\{M\}}(\mathbb{X})$ is nuclear as a projective limit of nuclear spaces [66, Proposition 50.1]. Of course this space is complete as well. Though in general we do not know if the space $\mathscr{E}_{D,\mathrm{proj}}^{\{M\}}(\mathbb{X})$ is barrelled. For application that need barrelled or bornological spaces we instead need to use $\mathscr{E}_D^{\{M\}}(K)$. If one can show $\mathscr{E}_D^{\{M\}}(\mathbb{M}) = \mathscr{E}_{D,\mathrm{proj}}^{\{M\}}(\mathbb{M})$ the situation is especially convenient. For this purpose Komatsu uses the following [42, Lemma 3.4].

**Lemma 2.2.12.** *Let $c = (c_n)_n$ be a positive sequence. Then the following two statements are equivalent.*

- **(i)** *There is some $h > 0$ such that $\sup_n \dfrac{c_n}{h^n} < \infty$.*
- **(ii)** *For all $h \in \Lambda$ we have $\sup_n \dfrac{c_n}{h_n} < \infty$.*

*The following two complementary statements are equivalent as well.*

- **(iii)** *For all $h > 0$ we have $\sup_n c_n\, h^n < \infty$.*
- **(iv)** *There exists some $h \in \Lambda$ such that $\sup_n c_n\, h_n < \infty$.*

The above implies that for two weight sequences $M$, $L$ we have $M \prec L$ iff there is some $h \in \Lambda$, $C > 0$ such that for all $n \in \mathbb{N}_0$

$$\frac{1}{L_n} \le \frac{C}{h_n\, M_n}\,.$$

**Lemma 2.2.13.** *Suppose $M$ is a weight sequence, $\mathbb{M}$ is an analytic manifold and $K \overset{\mathrm{rc}}{\subset} \mathbb{M}$. Suppose furthermore that $\mathbb{X} \in \{K, \mathbb{M}\}$ and $D \subset \mathcal{V}_{\mathrm{a}}(\mathbb{X})$ is a frame. Then*

$$\mathscr{E}_D^{\{M\}}(\mathbb{X}) = \bigcap_{h\in\Lambda} \mathscr{E}_D^{hM}(\mathbb{X})$$

*as vector spaces and* $\mathscr{E}^{\{M\}}(\mathbb{X}) \xrightarrow{\mathrm{I}} \mathscr{E}^{\{M\}}_{D,\mathrm{proj}}(\mathbb{X})$ *resp.* $\mathscr{E}^{\{M\}}(\mathbb{X}) \xrightarrow{\mathrm{I}} \mathscr{E}^{hM}_{D}(\mathbb{X})$ *is continuous for any* $h \in \Lambda$*. Moreover, if* $M$ *fulfils* ***(nQA)****, then*

$$\mathscr{E}^{\{M\}}_{D}(\mathbb{M}) = \mathscr{E}^{\{M\}}_{D,\mathrm{proj}}(\mathbb{M}) \tag{2.2.19}$$

*as locally convex vector spaces.*

*Proof.* If $\mathbb{X}$ is an open subset of $\mathbb{R}^n$, then the statement is proven in [42, Proposition 3.5]. Note that in the corresponding chapter in [42] the property **(nQA)** is a global assumption. However, in the proof of the continuity (from left to right) and bijectivity of the identity (2.2.19) this property is not used.

For arbitrary analytic manifolds $\mathbb{M}$ we get $\mathscr{E}^{\{M\}}_{D}(\mathbb{M}) = \mathscr{E}^{\{M\}}_{D,\mathrm{proj}}(\mathbb{M})$ in the sense of locally convex spaces by using analytic charts and Lemma 2.2.10.

For general $\mathbb{X}$ we can use the same approach. First of all, $\mathscr{E}^{\{M\}}_{D}(\mathbb{X})$ and $\mathscr{E}^{\{M\}}_{D,\mathrm{proj}}(\mathbb{X})$ coincide in the sense of vector spaces due to Lemma 2.2.12. Since $M \prec hM$ for any $h \in \Lambda$, we have continuous embeddings $\mathscr{E}^{\{M\}}_{D}(\mathbb{X}) \xrightarrow{\mathrm{I}} \mathscr{E}^{hM}_{D}(\mathbb{X})$ for any $h \in \Lambda$.

□

Note that in [47, Theorem 8.2] it is shown that for any weight sequence $M$, any open subset $\mathbb{M}$ of $\mathbb{R}^n$ and a set $B \subset \mathscr{E}(\mathbb{M})$ we have

$$B \subset \mathscr{E}^{\{M\}}_{\partial}(\mathbb{M}) \text{ is bounded} \quad \Leftrightarrow \quad \underset{\substack{M \prec L \\ L \text{ fulfils } \textbf{(LC')}}}{\forall} \quad B \subset \mathscr{E}^{(L)}_{\partial}(\mathbb{M}) \text{ is bounded.}$$

This fits (2.2.19), since $M \prec hM$ and $hM$ fulfils **(LC')** (and even **(LC)**[5]) for any weight sequence $M$ and any $h \in \Lambda$. Naturally, this can also be generalized to arbitrary analytic manifolds $\mathbb{M}$ in the same manner as above.

The following definition will be useful whenever we need to use projective limits for our spaces of ultradifferentiable functions.

**Definition 2.2.14.** *Suppose* $M$ *is a weight sequence. We will introduce the property* ***(PL)*** *for* $[M]$*.*

***(PL)*** *Either* $[M] = \{M\}$ *and* $M$ *has* ***(nQA)*** *or* $[M] = (M)$*.*

[5] $h = (h_n) \in \Lambda$ itself is log-convex since $h_{n+1} = c_{n+1} h_n$ with a *monotonously increasing* sequence $(c_n)$.

Now let us discuss the multiplication on $\mathscr{E}_D^{[M]}(\mathbb{M})$.

**Proposition 2.2.15.** *Let $\mathbb{M}$ be an analytic manifold, let $K \overset{\mathrm{rc}}{\subset} \mathbb{M}$, let $\mathbb{X} \in \{K, \mathbb{M}\}$ and let $M \in \mathbb{R}_+^{\mathbb{N}_0}$ be a weight sequence. Suppose $D \subset \mathcal{V}_{\mathrm{a}}(\mathbb{X})$ is a frame. Then the following holds.*

**(i)** *Let $h > 0$ and let $L$ be a weight sequence such that $[M] \subset (L)$. Then the multiplication*

$$\mathscr{E}_D^{[M]}(\mathbb{X}) \times \mathscr{E}_{hD}^{L}(\mathbb{X}) \to \mathscr{E}_{hD}^{L}(\mathbb{X})$$

*is well-defined and continuous.*

**(ii)** *$\mathscr{E}_D^{[M]}(\mathbb{X})$ and $\mathscr{E}_{D,\mathrm{proj}}^{\{M\}}(\mathbb{X})$ are algebras with continuous multiplication.*

**(iii)** *$\mathscr{E}_{D,\mathrm{proj}}^{[M]}(\mathbb{X})$ is a locally m-convex algebra. If $[M]$ has **(PL)** and $\mathbb{X} = \mathbb{M}$, this specifically means that the algebra $\mathscr{E}_D^{[M]}(\mathbb{M})$ is a locally m-convex.*

*Proof.* **(i):** For any $L$ and any $h > 0$ let us define $L_h := (h^{-n} L_n)_{n \in \mathbb{N}_0}$. We have $\mathscr{E}_{hD}^{L}(\mathbb{X}) = \mathscr{E}_D^{L_h}(\mathbb{X})$. Furthermore, we have $L \subset L_h$, thus $[M] \subset (L)$ implies $[M] \subset (L_h)$. So it is enough to prove the statement for $h = 1$.

First, we will discuss the case $\mathbb{X} = K$. Suppose $f, g \in \mathscr{E}(K)$. We define

$$\|f\|_{k,D,L} := \max_{\substack{a \in \mathcal{S}_N \\ |a| = k}} \frac{\|D^a f\|_\infty}{|a|!\, L_{|a|}} .$$

Then

$$\begin{aligned}
\|D^a (f\, g)\|_\infty &\leq \sum_{(a^1, a^2) \in \mathcal{S}_{N,2}(a)} \|D^{a^1} f\|_\infty \|D^{a^2} g\|_\infty \\
&\leq \sum_{(a^1, a^2) \in \mathcal{S}_{N,2}(a)} \|f\|_{|a^1|,\mu D, L} \|g\|_{|a^2|, D, L}\, \mu^{-|a^1|} |a^1|! L_{|a^1|} |a^2|! L_{|a^2|} \\
&= \sum_{k=0}^{|a|} \|f\|_{k,\mu D, L} \|g\|_{|a|-k, D, L} |a|! \mu^{-k} L_k L_{|a|-k} .
\end{aligned}$$

Due to the log-convexity and $L_0 = 1$, we have $L_k L_{|a|-k} \leq L_{|a|}$, thus

$$\|f\, g\|_{n,D,L} \leq \|f\|_{\mu D, L} \sum_{k=0}^{n} \mu^{-k}\, \|g\|_{n-k,D,L} \xrightarrow{n \to \infty} 0$$

for any $\mu > 1$ and any $f \in \mathscr{E}^L_{\mu D}(K)$ and $g \in \mathscr{E}^L_D(K)$. It follows that the multiplication

$$\mathscr{E}^L_{\mu D}(K) \times \mathscr{E}^L_D(K) \to \mathscr{E}^L_D(K) \tag{2.2.20}$$

is well-defined and continuous for any $\mu > 1$. Next we use $[M] \subset (L)$. Since the identity induces a continuous embedding $\mathscr{E}^{[M]}_D(K) \hookrightarrow \mathscr{E}^L_{\mu D}(K)$, the multiplication

$$\mathscr{E}^{[M]}_D(K) \times \mathscr{E}^L_D(K) \to \mathscr{E}^L_D(K)$$

is well-defined and continuous.

For the case $\mathbb{X} = \mathbb{M}$ we can use the fact that the above is true for *any* $K \subset \mathbb{X}$. Since multiplication commutes with the restriction to regular compact subsets of $\mathbb{X}$, we get a continuous multiplication

$$\mathscr{E}^{[M]}_D(\mathbb{X}) \times \mathscr{E}^L_D(\mathbb{X}) \to \mathscr{E}^L_D(\mathbb{X})\,. \tag{2.2.21}$$

**(ii):** First let us consider $[M] = (M)$. We use (2.2.20). For all $h > 0$, we have the following chain of continuous maps

$$\mathscr{E}^{(M)}_D(\mathbb{X}) \times \mathscr{E}^{(M)}_D(\mathbb{X}) \xrightarrow{\mathrm{I}} \mathscr{E}^M_{2hD}(\mathbb{X}) \times \mathscr{E}^M_{hD}(\mathbb{X}) \xrightarrow{\cdot} \mathscr{E}^M_{hD}(\mathbb{X})\,,$$

which already implies the continuity of the multiplication on $\mathscr{E}^{(M)}(\mathbb{X})$.

Now we consider $[M] = \{M\}$. We start with $\mathscr{E}^{\{M\}}_{D,\mathrm{proj}}(\mathbb{X})$. Here we can argue in the same manner as above. For each $h > 1$ and $\lambda \in \Lambda$ we have $M \prec (h^{-n}\lambda_n M_n)_n$. Thus

$$\mathscr{E}^{\{M\}}_{D,\mathrm{proj}}(\mathbb{X}) \times \mathscr{E}^{\{M\}}_{D,\mathrm{proj}}(\mathbb{X}) \xrightarrow{\mathrm{I}} \mathscr{E}^{\lambda M}_{hD}(\mathbb{X}) \times \mathscr{E}^{\lambda M}_{D}(\mathbb{X}) \xrightarrow{\cdot} \mathscr{E}^{\lambda M}_{D}(\mathbb{X})$$

is continuous for all $\lambda \in \Lambda$ which already implies the continuity of the multiplication on $\mathscr{E}^{\{M\}}_{D,\mathrm{proj}}(\mathbb{X})$.

Finally, we discuss $\mathscr{E}^{\{M\}}_D(\mathbb{X})$. For open $\mathbb{M} \subset \mathbb{R}^n$ the continuity of the multiplication is proven [40, Theorem 2.8], but the same proof also holds for general analytic manifolds $\mathbb{M}$.

For any $h, \mu > 0$ there is a $\lambda > 0$ such that the multiplication

$$\mathscr{E}^L_{\mu D}(K) \times \mathscr{E}^L_{hD}(K) \to \mathscr{E}^L_{\lambda D}(K)$$

is well-defined and continuous. Hence each fixed $f \in \mathscr{E}^{\{M\}}(K)$ induces a continuous operator

$$\mathscr{E}^{M}_{hD}(K) \to \mathscr{E}^{\{M\}}_{D}(K)\colon g \mapsto f\, g$$

for any $h > 0$. By a standard property of inductive limits [61, II 6.1], the extended map

$$\mathscr{E}^{\{M\}}_{D}(K) \to \mathscr{E}^{\{M\}}_{D}(K)\colon g \mapsto f\, g$$

is continuous. Hence the multiplication on $\mathscr{E}^{\{M\}}_{D}(K)$ is separately continuous. We complete the proof by using [66, Theorem 41.1], which states that separately continuous bilinear maps between strong duals of reflexive Fréchet spaces are continuous.

**(iii):** In this instance we can use a method from [52, Chapter I, Theorem 5.2]. It is enough to consider $\mathbb{X} = K$, since the rest follows from the description via the projective limit over the regular compact subsets. Suppose $L$ is any weight sequence. We define

$$U^L := \{f \in \mathscr{E}^{(M)}_{D}(K) \mid \|f\|_{D,L} \leq 1\}\,. \tag{2.2.22}$$

By **(i)** there is some $c > 0$ with

$$f \cdot U^L \subset c \cdot U^L$$

for any $f \in \mathscr{E}^{(M)}_{D}(K)$ and any $L$ with $(M) \subset (L)$. Next we define

$$V^L := \{f \in U^L \mid f \cdot U^L \subset U^L\}\,.$$

Then we have $V^L \cdot V^L \subset V^L$ and $V^L$ is a barrel, i.e. it is absolutely convex, absorbing and closed. Since the space $\mathscr{E}^{(M)}_{D}(K) = \mathscr{E}^{(M)}_{D,\mathrm{proj}}(K)$ is barrelled, $V^L$ is a neighbourhood of zero. The set

$$\mathcal{U} := \{\varepsilon \cdot U^L \mid M \subset L\,,\ \varepsilon \in [0,1]\}$$

is a basis of neighbourhoods of zero in $\mathscr{E}^{(M)}_{D}(K)$, so

$$\mathcal{V} := \{\varepsilon \cdot V^L \mid M \subset L\,,\ \varepsilon \in [0,1]\}$$

is also a basis of neighbourhoods of zero, because $V^L \subset U^L$ for any $L$. This implies that $\mathscr{E}^{(M)}_{D}(K)$ is locally m-convex.

Next we use the description

$$\mathscr{E}^{\{M\}}_{D,\mathrm{proj}}(K) = \varprojlim_{h\in\Lambda} \mathscr{E}^{(hM)}_D(K)$$

discussed in (2.2.18). Since any $\mathscr{E}^{(hM)}_D(K)$ is locally m-convex, the algebra $\mathscr{E}^{\{M\}}_{D,\mathrm{proj}}(K)$ is locally m-convex as well.

$\square$

By Lemma 2.2.4 we have $\mathscr{E}^{\{M\}}_D(\mathbb{X}) = \mathscr{E}^{\{M\}}_E(\mathbb{X})$ for arbitrary frames $D, E \subset \mathcal{V}_{\mathrm{a}}(\mathbb{X})$. Thus, we can adjust the definition of $\mathscr{E}^{\{M\}}_D(\mathbb{X})$ such that we are not dependent on the existence of a global frame $D \subset \mathcal{V}_{\mathrm{a}}(\mathbb{X})$.

For any $x \in \mathbb{X}$ we can find some $K' \overset{\mathrm{rc}}{\subset} \mathbb{M}$ with $x \in K' \subset \mathbb{X}$ with a frame $D' \subset \mathcal{V}_{\mathrm{a}}(K')$, by using analytic charts. Moreover, by Lemma 2.2.4, for a weight sequence $M$ and any two $K', K'' \overset{\mathrm{rc}}{\subset} \mathbb{M}$ with frames $D' \subset \mathcal{V}_{\mathrm{a}}(K')$, $D'' \subset \mathcal{V}_{\mathrm{a}}(K'')$ and with $K'' \subset K'$, the restriction defines a continuous map

$$\mathscr{E}^{\{M\}}_{D'}(K') \to \mathscr{E}^{\{M\}}_{D''}(K'') \colon f \mapsto f \restriction_{K''} .$$

This leads us to the following definition, which is consistent with the prior definition of $\mathscr{E}^{\{M\}}_D(\mathbb{X})$.

**Definition 2.2.16.** *Suppose* $\mathbb{M}$ *is an analytic manifold,* $K \overset{\mathrm{rc}}{\subset} \mathbb{M}$, $\mathbb{X} \in \{K, \mathbb{M}\}$ *and* $M$ *is a weight sequence. Let*

$$F_{\mathbb{M}}(\mathbb{X}) := \{(K', D) \mid K' \overset{\mathrm{rc}}{\subset} \mathbb{M}\,,\ K' \subset \mathbb{X} \textit{ and } D \subset \mathcal{V}_{\mathrm{a}}(K') \textit{ is a frame}\}\,.$$

*We define* $\mathscr{E}^{\{M\}}(\mathbb{X})$ *as the space*

$$\mathscr{E}^{\{M\}}(\mathbb{X}) := \{f \in \mathscr{E}(\mathbb{X}) \mid \forall_{(K',D)\in F_{\mathbb{M}}(\mathbb{X})} \colon f \restriction_{K'} \in \mathscr{E}^{\{M\}}_D(K')\}$$

*equipped with the initial topology with respect to the maps*

$$\mathscr{E}^{\{M\}}(\mathbb{X}) \to \mathscr{E}^{\{M\}}_D(K') \colon f \mapsto f \restriction_{K'} \qquad \textit{for } (K', D) \in F_{\mathbb{M}}(\mathbb{X})\,.$$

*The spaces* $\mathscr{E}^{\{M\}}(\mathbb{X})$ *are also called Denjoy-Carleman class of Roumieu type.*

According to the above definition we especially have

$$\mathscr{E}^{\{M\}}(\mathbb{M}) = \varprojlim_{K \overset{\mathrm{rc}}{\subset} \mathbb{M}} \mathscr{E}^{\{M\}}(K) \quad \text{and} \quad \mathscr{E}^{\{M\}}(K) = \mathscr{E}_D^{\{M\}}(K)\,,$$

for a frame $D \subset \mathcal{V}_{\mathrm{a}}(K)$.

By (2.2.7) and Lemma 2.2.4, $\mathscr{E}^{\{\mathbb{1}\}}(\mathbb{M})$ is exactly the space of analytic functions $\mathscr{A}(\mathbb{M})$ on an analytic manifold $\mathbb{M}$ [6] (see also Proposition 2.2.10 **(iii)** for more context).

We cannot do the same for the Beurling case. Here the defined spaces depend on the choice of frame. This is not surprising, since the spaces $\mathscr{E}_\partial^{(\mathbb{1})}(U)$ for open $U \subset \mathbb{R}^n$ are the functions that extend to entire functions on $\mathbb{C}^n$. Certainly, the composition of entire functions with arbitrary analytic functions might not be entire. Consider the analytic manifold $\mathbb{R}^+$ and the vector field $D$, defined by $Df(x) = x^2 \partial f(x)$. If we take $g(x) := x$ for $x \in \mathbb{R}^+$, then

$$\sup_{x \in K} |D^k g(x)| = k! \cdot t^k \qquad \text{for } K \overset{\mathrm{rc}}{\subset} \mathbb{R}^+ \text{ and } t := \max K$$

and thus $g \in \mathscr{E}_\partial^{(\mathbb{1})}(\mathbb{R}^+) \setminus \mathscr{E}_D^{(\mathbb{1})}(\mathbb{R}^+)$. Even a very well behaved change of frame might not be sufficient to guarantee that the corresponding Carleman classes of Beurling type coincide. Indeed, in the example above we have $A \in \mathscr{E}_\partial^{(\mathbb{1})}(\mathbb{R}^+)$ for $D = A\partial$.

### 2.2.3 Vector valued ultradifferentiable functions

Let us now turn our attention to the spaces of vector valued ultradifferentiable functions. In order to characterize the spaces of ultradifferentiable vector valued functions of Roumieu and Beurling type, we will first represent the spaces $\mathscr{E}_D^M(\mathbb{M}; E)$ resp. $\mathscr{E}_D^M(K; E)$ in a more convenient way.

**Proposition 2.2.17.** *Let $E$ be a complete locally convex space, let $\mathbb{M}$ be an analytic manifold of dimension $N$ with $K \overset{\mathrm{rc}}{\subset} \mathbb{M}$, $\mathbb{X} \in \{K, \mathbb{M}\}$ and let $D \subset \mathcal{V}_{\mathrm{a}}(\mathbb{X})$ be a frame. Let $F(E)$ be the linear space of all $f \in \mathscr{E}(\mathbb{X}; E)$ such that*

$$\lim_{\substack{|a| \to \infty \\ a \in \mathcal{S}_N}} \sup_{x \in K'} \frac{p(D^a f(x))}{M_{|a|}\,|a|!} = 0$$

[6] Naturally we still use the definition $\mathbb{1} = (1, 1, 1, \dots)$.

*for all compact $K' \subset \mathbb{X}$ and all continuous seminorms $p$ on $E$, equipped with the topology defined by the seminorms*

$$f \mapsto \sup_{x\in K'} \sup_{a\in\mathcal{S}_N} \frac{p(D^a f(x))}{M_{|a|}\,|a|!}\,,$$

*in which $K'$ runs through the compact subsets of $\mathbb{X}$ and $p$ runs through continuous seminorms on $E$. Then $\mathscr{E}_D^M(\mathbb{X};E) = F(E)$ as topological vector spaces.*

*Proof.* By definition we have $F(E) \subset \mathscr{E}(\mathbb{X};E)$ and we have $\mathscr{E}_D^M(\mathbb{X};E) \subset \mathscr{E}(\mathbb{X};E)$ since $\mathscr{E}_D^M(\mathbb{X}) \subset \mathscr{E}(\mathbb{X})$.

First, suppose $E$ is a Banach space and let $N := \dim \mathbb{M}$.

Let $f \in \mathscr{E}(\mathbb{X};E)$ and put

$$T_f \colon E' \to \mathscr{E}(\mathbb{X})\colon e' \mapsto e' \circ f\,.$$

Lemma 1.2.5 **(i)** implies that $f \in \mathscr{E}_D^M(\mathbb{X};E)$ iff

(1) $T_f e' \in \mathscr{E}_D^M(\mathbb{X})$ for all $e' \in E'$ and

(2) for all compact $K' \subset \mathbb{X}$ there is some compact and absolutely convex $C \subset E$ with $\forall_{e'\in E'}\ \sup_{x\in K'} \sup_{a\in\mathcal{S}_N} \frac{|\langle D^a f(x), e'\rangle|}{M_{|a|}\,|a|!} \le \sup_{e\in C} |\langle e, e'\rangle|$.

Let us define $c_a(x) := \frac{1}{M_{|a|}\,|a|!} D^a f(x)$, for $x \in \mathbb{X}$ and $a \in \mathcal{S}_N$, and let

$$M(K') := \{c_a(x) \mid x \in K'\,,\ a \in \mathcal{S}_N\}$$

for each compact $K' \subset \mathbb{X}$. By identifying $(E'_c)'_\varepsilon \simeq E$ via the canonical map, we get

(1) $\Leftrightarrow c_a(x) \xrightarrow{|a|\to\infty} 0$ in $(E'_c)'_s$ uniformly in $x \in K'$ for each compact $K' \subset \mathbb{X}$ and

(2) $\Leftrightarrow M(K')$ is a set of equicontinuous functionals on $E'_c$ for each compact $K' \subset \mathbb{X}$.

If $M \subset (E'_c)'$ is a set of equicontinuous functionals on $E'_c$, then the topologies inherited from $(E'_c)'_s$ and $(E'_c)'_c$ coincide on $M$ by [36, Satz 1.4]. However, for an equicontinuous $M$, the set $M - M = \{e_1 - e_2 \mid e_1, e_2 \in M\}$ is equicontinuous as well. Consequently, even the uniform structures inherited from $(E'_c)'_s$ and $(E'_c)'_c$ coincide on $M$. This enables us to conclude that together (1) and (2) are equivalent to

(3) $c_a(x) \xrightarrow{|a|\to\infty} 0$ in $(E'_c)'_c$ uniformly in $x \in K'$ for each compact $K' \subset \mathbb{X}$.

Since $E$ is a Banach space, it carries the Mackey topology by [36, p. 176] and thus $(E'_c)'_c \simeq E$ via the canonical map $E \to (E'_c)'_c$ by [63, p. 17]. This means (3) is equivalent to $f \in F(E)$ and thus $F(E) = \mathscr{E}^M_D(\mathbb{X}; E)$ as vector spaces.

The topology on $\mathscr{E}^M(\mathbb{X}; E)$ is defined by the seminorms

$$f \mapsto \sup_{e' \in V} p(e' \circ f)$$

as $p$ runs through the continuous seminorms in $\mathscr{E}^M_D(\mathbb{X})$ and $V$ runs though the equicontinuous subsets of $E'$. A seminorm $p$ on $\mathscr{E}^M_D(\mathbb{X})$ is continuous iff compact $K' \subset \mathbb{X}$ and $C > 0$ exist such that $p(f) \leq C \, \|f \restriction_{K'}\|_{D,M}$ for all $f \in \mathscr{E}^M_D(\mathbb{X})$. Hence,

$$\sup_{e' \in V} p(e' \circ f) \leq C \sup_{x \in K'} \sup_{a \in \mathcal{S}_N} \sup_{e' \in V} \frac{|\langle D^a f(x), e' \rangle|}{M_{|a|} \, |a|!} .$$

Since the topology on $E$ can be defined by the seminorms $e \mapsto \sup_{e' \in V} |\langle e, e' \rangle|$ as $V$ runs through the equicontinuous subsets of $E'$, we get $F(E) = \mathscr{E}^M_D(\mathbb{X}; E)$ as topological vector spaces.

Now let $E$ be any complete locally convex space.

For any continuous seminorm $p$ on $E$ let $E_p$ be the Banach space defined as the completion of $E/p^{-1}(0)$ equipped with the norm $e + p^{-1}(0) \mapsto p(e)$. For any continuous seminorms $p, q$, such that pointwise $p \geq q$, we may extend the maps $e \mapsto e + p^{-1}(0) \mapsto e + q^{-1}(0)$ to continuous surjective maps

$$E \xrightarrow{j_p} E_p \xrightarrow{j_{q,p}} E_q.$$

We use [61, Ch. II, 5.4], which states that the map

$$E \to \prod_p E_p \colon e \mapsto (e + p^{-1}(0))_p$$

is a linear homeomorphism onto the projective limit $\varprojlim_p (E_p, j_{p,q})$ in which $p$ and $q$ run through the continuous seminorms on $E$. With Lemma 1.2.5 **(ii)** we get

$$\mathscr{E}^M_D(\mathbb{X}; E) \simeq \varprojlim_p \left(E^M_D(\mathbb{X}; E_p), \mathrm{I}\, \varepsilon j_{p,q}\right) = \varprojlim_p (F(E_p), \mathrm{I}\, \varepsilon j_{p,q}) \, ,$$

in which the linear homeomorphism is given by $f \mapsto (j_p \circ f)_p$. We complete the proof by pointing out that by the definition of $F(E)$ we also have $F(E) \simeq \varprojlim_p (F(E_p), \mathrm{I}\,\varepsilon j_{p,q})$ via $f \mapsto (j_p \circ f)_p$. $\square$

Now that we have a representation for $\mathscr{E}^M_D(\mathbb{X}; E)$, we will use various limits to represent the vector valued Denjoy-Carleman classes of Beurling and Roumieu type.

**Proposition 2.2.18.** *Let $E$ be a complete locally convex space, let $M$ be a weight sequence, let $\mathbb{M}$ be an analytic manifold with $K \overset{\mathrm{rc}}{\subset} \mathbb{M}$, let $\mathbb{X} \in \{K, \mathbb{M}\}$ and let $D \subset \mathcal{V}_{\mathrm{a}}(\mathbb{X})$ be a frame.*

**(i)** *A function $f \colon \mathbb{X} \to E$ is in $\mathscr{E}^{[M]}_D(\mathbb{X}; E)$ iff $e' \circ f \in \mathscr{E}^{[M]}_D(\mathbb{X})$ for all $e' \in E'$.*

**(ii)** *Let $\mathcal{W}_{[M]} := \{L \mid L \text{ is a weight sequence, } [M] \subset (L)\}$ equipped with the partial order*

$$L \gtrsim K \quad :\Leftrightarrow \quad \sup_{k\in\mathbb{N}} \frac{K_k}{L_k} < \infty\,.$$

*The identities*

$$\mathscr{E}^{(M)}_D(\mathbb{X}; E) = \varprojlim_{h>0} \mathscr{E}^M_{hD}(\mathbb{X}; E) = \varprojlim_{h>0} \varprojlim_{K' \overset{\mathrm{rc}}{\subset} \mathbb{M},\, K' \subset \mathbb{X}} \mathscr{E}^M_{hD}(K'; E) \tag{2.2.23}$$

$$\mathscr{E}^{\{M\}}_{D,\mathrm{proj}}(\mathbb{X}; E) = \varprojlim_{h\in\Lambda} \mathscr{E}^{hM}_D(\mathbb{X}; E) = \varprojlim_{h\in\Lambda} \varprojlim_{K' \overset{\mathrm{rc}}{\subset} \mathbb{M},\, K' \subset \mathbb{X}} \mathscr{E}^{hM}_D(K'; E) \tag{2.2.24}$$

$$\mathscr{E}^{[M]}_{D,\mathrm{proj}}(\mathbb{X}; E) = \varprojlim_{L\in\mathcal{W}_{[M]}} \mathscr{E}^L_D(\mathbb{X}; E) = \varprojlim_{L\in\mathcal{W}_{[M]}} \varprojlim_{K' \overset{\mathrm{rc}}{\subset} \mathbb{M},\, K' \subset \mathbb{X}} \mathscr{E}^L_D(K'; E) \tag{2.2.25}$$

*hold in the sense of topological vector spaces.*

**(iii)** *The identity*

$$\mathscr{E}^{\{M\}}_D(\mathbb{X}; E) = \mathscr{E}^{\{M\}}_{D,\mathrm{proj}}(\mathbb{X}; E)\,, \tag{2.2.26}$$

*is valid in the sense of vector spaces. In (2.2.26) the topology on the left-hand side is finer than the topology on the right-hand side.*

*If $M$ fulfils **(nQA)**, then the identity is valid in the sense of topological vector spaces.*

**(iv)** *If $E$ is a Banach space, then the equalities*

$$\mathscr{E}_D^{\{M\}}(\mathbb{X};E) = \varprojlim_{K' \overset{\mathrm{rc}}{\subset} \mathbb{M}\,,\, K' \subset \mathbb{X}} \mathscr{E}_D^{\{M\}}(K';E) = \varprojlim_{K' \overset{\mathrm{rc}}{\subset} \mathbb{M}\,,\, K' \subset \mathbb{X}} \varinjlim_{h>0} \mathscr{E}_{hD}^{M}(K';E) \tag{2.2.27}$$

*hold in the sense of vector spaces.*

*Proof.* **(i)**: Suppose $K' \subset \mathbb{X}$ is compact. $\mathscr{E}_D^{[M]}(K')$ is Montel as a nuclear Fréchet space (resp. dual of a nuclear Fréchet space). Furthermore, $\mathscr{E}_D^{[M]}(K')$ is a *webbed* space by [36, pp.162-163], since it is a countable projective limit (resp. countable inductive limit) of Banach spaces. Hence we may apply [36, Satz 10.5], which states that $f \in \mathscr{E}_D^{[M]}(K';E)$ iff $e' \circ f \in \mathscr{E}_D^{[M]}(K')$ for all $e' \in E'$. Thus we also have $f \in \mathscr{E}_D^{[M]}(\mathbb{X};E)$ iff $e' \circ f \in \mathscr{E}_D^{[M]}(\mathbb{X})$ for all $e' \in E'$ by Lemma 1.2.5 **(ii)** and by the isomorphism $\mathscr{E}_D^{[M]}(\mathbb{X};E) \simeq \mathscr{E}_D^{[M]}(\mathbb{X}) \,\varepsilon\, E$.

**(ii)**: For each $h > 0$ we define a weight sequence $M(h)$ by $M(h)_k := M_k h^{-k}$. With this definition we have $\mathscr{E}_{hD}^{M}(\mathbb{X}) = \mathscr{E}_D^{M(h)}(\mathbb{X})$ and

$$\mathscr{E}_D^{(M)}(\mathbb{X}) = \varprojlim_{h>0} \mathscr{E}_D^{M(h)}(\mathbb{X})$$
$$\mathscr{E}_{D,\mathrm{proj}}^{\{M\}}(\mathbb{X}) = \varprojlim_{h \in \Lambda\,,\, h_0 = 1} \mathscr{E}_D^{hM}(\mathbb{X})\,.$$

Moreover, $M(h) \gtrsim M(h') \in \mathcal{W}_{(M)}$ for $h, h' > 0$ with $h' \leq h$. For $K \gtrsim L$ the embedding $\mathscr{E}_D^{L}(\mathbb{X}) \xrightarrow{\mathrm{I}} \mathscr{E}_D^{K}(\mathbb{X})$ is well-defined and continuous. This results in

$$\mathscr{E}_{D,\mathrm{proj}}^{(M)}(\mathbb{X}) = \varprojlim_{L \in \mathcal{W}_{(M)}} \mathscr{E}_D^{L}(\mathbb{X})\,.$$

Similarly, we have $hM \gtrsim h'M \in \mathcal{W}_{\{M\}}$ for $h, h' \in \Lambda$ with $h \gtrsim h'$ and $h_0 = h'_0 = 1$. If $M \prec L$, then $L \gtrsim hM$ for some $h \in \Lambda$ by Lemma 2.2.12. Thus the embedding $\mathscr{E}_{D,\mathrm{proj}}^{[M]}(\mathbb{X}) \xrightarrow{\mathrm{I}} \mathscr{E}_D^{L}(\mathbb{X})$ is well-defined and continuous for each $L \in \mathcal{W}_{\{M\}}$. This gives us finally

$$\mathscr{E}_{D,\mathrm{proj}}^{\{M\}}(\mathbb{X}) = \varprojlim_{L \in \mathcal{W}_{\{M\}}} \mathscr{E}_D^{L}(\mathbb{X})\,.$$

The rest follows directly from the definition of $\mathscr{E}_D^{(M)}(\mathbb{X})$ resp. $\mathscr{E}_{D,\mathrm{proj}}^{\{M\}}(\mathbb{X})$, Lemma 1.2.5 **(ii)** and the isomorphism $\mathscr{F}(\mathbb{X};E) \simeq \mathscr{F}(\mathbb{X}) \,\varepsilon\, E$ for $\mathscr{F} \in \{\mathscr{E}_D^{(M)}, \mathscr{E}_{D,\mathrm{proj}}^{\{M\}}\}$.

**(iii)**: By definition we have $\mathscr{E}_D^{\{M\}}(\mathbb{X};E) \simeq \mathscr{E}_D^{\{M\}}(\mathbb{X})\,\varepsilon\, E$. From Lemma 2.2.13 we know that

$$\widetilde{\mathrm{I}} := \left[\mathscr{E}_D^{\{M\}}(\mathbb{X}) \xrightarrow{\mathrm{I}} \varprojlim_{h\in\Lambda} \mathscr{E}_D^{hM}(\mathbb{X})\right] \tag{2.2.28}$$

is continuous and bijective. Hence

$$\widetilde{\mathrm{I}}\,\varepsilon\,\mathrm{I}_E \colon \mathscr{E}_D^{\{M\}}(\mathbb{X};E) \to \varprojlim_{h\in\Lambda} \mathscr{E}_D^{hM}(\mathbb{X};E)$$

is continuous and injective. For any $f \in \bigcap_{h\in\Lambda} \mathscr{E}_D^{hM}(\mathbb{X};E)$ and for any $e' \in E'$ the scalar valued function $e' \circ f$ is in $\bigcap_{h\in\Lambda} \mathscr{E}_D^{hM}(\mathbb{X})$ and thus in $\mathscr{E}_D^{\{M\}}(\mathbb{X})$. As a consequence, $f \in \mathscr{E}_D^{\{M\}}(\mathbb{X};E)$ according to **(i)** and hence $\widetilde{\mathrm{I}}\,\varepsilon\,\mathrm{I}_E$ is bijective.

If $M$ fulfils **(nQA)** and $\mathbb{X} = \mathbb{M}$, then $\widetilde{\mathrm{I}}$ is a homeomorphism by Lemma 2.2.13, hence $\widetilde{\mathrm{I}}\,\varepsilon\,\mathrm{I}_E$ is a homeomorphism.

**(iv)**: Since $\mathscr{E}_D^{\{M\}}(\mathbb{X};E) = \varprojlim_{K' \overset{\mathrm{rc}}{\subset} \mathbb{M}, K' \subset \mathbb{X}} \mathscr{E}_D^{\{M\}}(K';E)$ via Lemma 1.2.5, it is enough to consider solely a single $K' \overset{\mathrm{rc}}{\subset} \mathbb{M}$. We proved in **(iii)** that $\mathscr{E}_D^{\{M\}}(K';E)$ coincides with $\bigcap_{h\in\Lambda} \mathscr{E}_D^{hM}(K';E)$. For a smooth function $f\colon K' \to E$ we define the sequence

$$c^f := (c_k^f)_k := \left(\sup_{x\in K'} \sup_{\substack{|a|=k \\ a\in\mathcal{S}_{\dim\mathbb{M}}}} \frac{\|D^a f(x)\|_E}{M_k\, k!}\right)_{k\in\mathbb{N}_0}.$$

A function $f\colon K' \to E$ is in $\bigcap_{h\in\Lambda} \mathscr{E}_D^{hM}(K';E)$ iff both $f \in \mathscr{E}(K';E)$ and for all $h \in \Lambda$ the (pointwise) quotient $c^f/h$ converges to zero. Similarly, a function $f\colon K' \to E$ is in $\bigcup_{h>0} \mathscr{E}_D^{M}(K';E)$ iff both $f \in \mathscr{E}(K';E)$ and there is some $h > 0$ such that $c_k^f h^k \xrightarrow{k\to\infty} 0$. By Lemma 2.2.12 the two statements are equivalent. Hence $\mathscr{E}_D^{\{M\}}(K';E) = \varinjlim_{h>0} \mathscr{E}_{hD}^{M}(K';E)$ in the sense of vector spaces. $\square$

In special cases, we can regard vector valued analytic functions as ultradifferentiable functions. This way, the above proposition gives us tools to deal with vector valued analytic functions as well.

**Corollary 2.2.19.** *Suppose $\mathbb{M}$ is an analytic manifold and $E$ is a Banach space. Then $\mathscr{E}^{\{\mathbb{1}\}}(\mathbb{M};E)$ is exactly the space of analytic functions from $\mathbb{M}$ to $E$.*

*Proof.* If $f\colon \mathbb{M} \to E$ is analytic, then $e' \circ f$ is analytic for each $e' \in E'$. Thus $f$ is an element of $\mathscr{E}^{\{\mathbb{1}\}}(\mathbb{M};E)$ by **(i)** of Proposition 2.2.18 [7].

[7] We apply the result for $\mathscr{E}_{\partial_\phi}^{\{\mathbb{1}\}}(U;E)$ for each analytic chart $(\phi, U)$.

Conversely, if $f \in \mathscr{E}^{\{\mathbb{1}\}}(\mathbb{M}; E)$, then for each analytic chart $(\phi, U)$ and for each compact $K \subset U$ there is some $h > 0$ such that

$$\sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^{\dim \mathbb{M}}} \frac{\| h^\alpha \partial_\phi^\alpha f(x) \|_E}{|\alpha|!} < \infty \, .$$

By using the inequality $\binom{n}{k} \leq (ne/k)^k$ we can see that there are constants $C, \mu > 0$ such that $|\alpha|!/\alpha! \leq C\mu^{|\alpha|}$ for all $\alpha \in \mathbb{N}_0^{\dim \mathbb{M}}$. Hence, the Taylor expansion

$$\sum_\alpha \frac{\partial^\alpha (f \circ \phi^{-1})(x)}{\alpha!} (y - x)^\alpha$$

converges for all analytic charts $(\phi, U)$ and all $x, y \in \phi(U)$ with $|x - y|$ small enough. In conclusion, $f$ is analytic. □

Next, we will discuss in what way the identity $\mathscr{E}(\mathbb{M} \times \mathbb{M}') = \mathscr{E}(\mathbb{M}; \mathscr{E}(\mathbb{M}'))$ can be applied to the Carleman classes of Beurling or Roumieu type.

**Proposition 2.2.20.** *Let $\mathbb{M}$ and $\mathbb{M}'$ be analytic manifolds and let $M$ be a weight sequence with **(MG)**. Suppose $D \subset \mathcal{V}_{\mathrm{a}}(\mathbb{X})$ and $D' \subset \mathcal{V}_{\mathrm{a}}(\mathbb{X}')$ are frames. We denote by $E$ the frame in $\mathcal{V}_{\mathrm{a}}(\mathbb{X} \times \mathbb{X}')$ defined by*

$$E = (D_1 \, \varepsilon \, \mathrm{I}, D_2 \, \varepsilon \, \mathrm{I} \, \ldots, \mathrm{I} \, \varepsilon D_1', \mathrm{I} \, \varepsilon D_2', \ldots) \, .$$

*If $[M]$ has **(PL)**, then*

$$\mathscr{E}_E^{[M]}(\mathbb{M} \times \mathbb{M}') = \mathscr{E}_D^{[M]}(\mathbb{M}; \mathscr{E}_{D'}^{[M]}(\mathbb{M}'))$$

*in the sense of topological vector spaces. If $[M]$ does not have **(PL)**, then this identity still holds in the sense of vector spaces.*

*Proof.* Let $N = \dim \mathbb{M}$ and $N' = \dim \mathbb{M}'$. By Lemma 2.1.9 all involved function spaces are continuously embedded into $\mathscr{E}(\mathbb{M} \times \mathbb{M}') = \mathscr{E}(\mathbb{M}; \mathscr{E}(\mathbb{M}'))$. Furthermore, it is clear that $E$ indeed defines an analytic frame on $\mathbb{M} \times \mathbb{M}'$.

Due to **(LC)** and **(MG)** there are $C, \lambda > 0$ such that

$$M_k \, k! \, M_l \, l! \leq M_{k+l} \, (k + l)! \leq C\lambda^{k+l} M_k \, k! \, M_l \, l!$$

for all $k, l \in \mathbb{N}_0$.

First, we take a look at the Beurling case. Following standard procedure, we start by considering spaces of functions on $K \overset{\mathrm{rc}}{\subset} \mathbb{M}$ and $K' \overset{\mathrm{rc}}{\subset} \mathbb{M}'$ and then use projective limits. Due to Proposition 2.2.18 **(ii)** and the closed graph theorem it is sufficient to show that for any $\mu > 0$ there are $h, h' > 0$ such that

$$\mathscr{E}^M_{hD}(K; \mathscr{E}^M_{h'D'}(K')) \subset \mathscr{E}^M_{\mu E}(K \times K') \tag{2.2.29}$$

and for any $h, h' > 0$ there is a $\mu > 0$ with

$$\mathscr{E}^M_{\mu E}(K \times K') \subset \mathscr{E}^M_{hD}(K; \mathscr{E}^M_{h'D'}(K'))\,. \tag{2.2.30}$$

For a chosen $\mu > 0$ let $h = h' = \mu$. Due to **(LC)**, we have

$$\sup_{c \in \mathcal{S}_{N+N'}} \sup_{(x,y) \in K \times K'} \frac{|(\mu E)^c f(x,y)|}{M_{|c|}\,|c|!} \leq \sup_{a \in \mathcal{S}_N} \sup_{b \in \mathcal{S}_{N'}} \sup_{(x,y) \in K \times K'} \frac{|(hD)^a_x (h'D)'^{\,b}_y f(x,y)|}{M_{|a|}\, M_{|b|}\, |a|!\, |b|!}$$

for any $f \in \mathscr{E}(K \times K')$. By Proposition 2.2.17 this implies (2.2.29). For a choice $h, h' > 0$ we may use **(MG)** and put $\mu = \max\{h, h'\} \cdot \lambda$. Then

$$\sup_{a \in \mathcal{S}_N} \sup_{b \in \mathcal{S}_{N'}} \sup_{(x,y) \in K \times K'} \frac{|(hD)^a_x (h'D)'^{\,b}_y f(x,y)|}{M_{|a|}\, M_{|b|}\, |a|!\, |b|!} \leq C \sup_{c \in \mathcal{S}_{N+N'}} \sup_{(x,y) \in K \times K'} \frac{|(\mu E)^c f(x,y)|}{M_{|c|}\,|c|!}$$

and consequently (2.2.30) holds true.

Secondly, we move onto the Roumieu case. We can follow a similar procedure as above. For any choice $h = (h_n)_n = (c_0 \cdots c_n)_n, h' = (h'_n)_n = (c'_0 \cdots c'_n)_n \in \Lambda$ we may put $\mu := (\tilde{c}_0 \cdots \tilde{c}_n)_n$ with $\tilde{c}_k = \min\{c_k, c'_k\}/\lambda$. Then $\mu \in \Lambda$ and

$$\sup_{a \in \mathcal{S}_N} \sup_{b \in \mathcal{S}_{N'}} \sup_{(x,y) \in K \times K'} \frac{|D^a_x D'^{\,b}_y f(x,y)|}{h_{|a|}\, h'_{|b|}\, M_{|a|}\, M_{|b|}\, |a|!\, |b|!} \leq C \sup_{c \in \mathcal{S}_{N+N'}} \sup_{(x,y) \in K \times K'} \frac{|E^c f(x,y)|}{\mu_{|c|}\, M_{|c|}\, |c|!}\,.$$

For a chosen $\mu = (\mu_n)_n \in \Lambda$ we can put $h = h' = \mu$. Then $h_n h'_n \leq \mu_n$ for all $n \in N_0$ and thus

$$\sup_{c \in \mathcal{S}_{N+N'}} \sup_{(x,y) \in K \times K'} \frac{|E^c f(x,y)|}{\mu_{|c|}\, M_{|c|}\, |c|!} \leq \sup_{a \in \mathcal{S}_N} \sup_{b \in \mathcal{S}_{N'}} \sup_{(x,y) \in K \times K'} \frac{|D^a_x D'^{\,b}_y f(x,y)|}{h_{|a|}\, h'_{|b|}\, M_{|a|}\, M_{|b|}\, |a|!\, |b|!}\,.$$

As before this implies

$$\begin{aligned} &\mathscr{E}^{\{M\}}_{E,\mathrm{proj}}(K \times K') = \mathscr{E}^{\{M\}}_{D,\mathrm{proj}}(K; \mathscr{E}^{\{M\}}_{D',\mathrm{proj}}(K')) \quad \text{and thus} \\ &\mathscr{E}^{\{M\}}_{E,\mathrm{proj}}(\mathbb{M} \times \mathbb{M}') = \mathscr{E}^{\{M\}}_{D,\mathrm{proj}}(\mathbb{M}; \mathscr{E}^{\{M\}}_{D',\mathrm{proj}}(\mathbb{M}')) \end{aligned}$$

as topological vector spaces. By Lemma 2.2.13 and Proposition 2.2.18 **(iii)** we have

$$\mathscr{E}_E^{\{M\}}(\mathbb{M}\times\mathbb{M}') = \mathscr{E}_{E,\mathrm{proj}}^{\{M\}}(\mathbb{M}\times\mathbb{M}') = \mathscr{E}_{D,\mathrm{proj}}^{\{M\}}(\mathbb{M};\mathscr{E}_{D',\mathrm{proj}}^{\{M\}}(\mathbb{M}')) = \mathscr{E}_D^{\{M\}}(\mathbb{M};\mathscr{E}_{D'}^{\{M\}}(\mathbb{M}'))$$

as vector spaces and, if $M$ fulfils **(nQA)**, also as topological vector spaces. $\square$

We will also prove that $\mathscr{E}^{\{M\}}(\mathbb{X})$ has the property **(IC)** for a weight sequence $M$. In [38] inverse closed subalgebras of Banach algebras of a ultradifferentiable type were discussed. These algebras were constructed with the help of a family of commutating derivations. However, the proof for [38, Theorem 16] does not rely on this commutativity, which suits us very well. Nevertheless, we will reiterate and slightly adjust the proof of [38, Theorem 16], since our situation is slightly different.

**Proposition 2.2.21.** *Suppose $M$ is a weight sequence, $\mathbb{M}$ is an analytic manifold, $K$ is regular compact in $\mathbb{M}$, $\mathbb{X}\in\{K,\mathbb{M}\}$ and $D\subset\mathcal{V}_{\mathrm{a}}(\mathbb{X})$ is a frame. Then $\mathscr{E}_D^{\{M\}}(\mathbb{X})$ and $\mathscr{E}_{D,\mathrm{proj}}^{\{M\}}(\mathbb{X})$ have the property **(IC)**.*

*Proof.* It is enough to test for **(IC)** by using a Banach algebra $A$ with norm $p$. Both $\mathscr{E}_D^{\{M\}}(\mathbb{X};A)$ and $\mathscr{E}_{D,\mathrm{proj}}^{\{M\}}(\mathbb{X};A)$ are algebras with respect to the pointwise multiplication due to Proposition 2.1.10 and Proposition 2.2.15. Due to Proposition 2.2.18 we have

$$\mathscr{E}_D^{\{M\}}(\mathbb{X};A) = \mathscr{E}_{D,\mathrm{proj}}^{\{M\}}(\mathbb{X};A) = \varprojlim_{\substack{K\subset\mathbb{X}\\ K'\overset{\mathrm{rc}}{\subset}\mathbb{M}}} \varinjlim_{h>0} \mathscr{E}_{hD}^{M}(K';A) \tag{2.2.31}$$

in the sense of vector spaces.

Let us put

$$p_h(f)(x) := \sup_{a\in\mathcal{S}_N} \frac{p\big((hD)^a f(x)\big)}{M_{|a|}\,|a|!}\,, \qquad \text{for } f\in\mathscr{E}(\mathbb{X};A)\,.$$

Let now $f\in\mathscr{E}(\mathbb{X};A)$ such that $f(x)\in A^\times$ and $g(x) := f(x)^{-1}$ for all $x\in\mathbb{X}$. Then by

Lemma 2.1.15, we have $g \in \mathscr{E}(\mathbb{X}; A)$ and

$$\begin{aligned} p(D^a g(x)) &\leq \sum_{m=1}^{|a|} \sum_{(a^{(j)})_j \in \widetilde{\mathcal{S}_{N,m}(a)}} p(g(x))^{m+1} p(D^{a^1} f(x)) \cdots p(D^{a^m} f(x)) \\ &\leq \sum_{m=1}^{|a|} p(g(x))^{m+1} p_h(f)(x)^m \sum_{(a^{(j)})_j \in \widetilde{\mathcal{S}_{N,m}(a)}} |a^1|! \cdots |a^n|!\, h^{|a|}\, M_{|a^1|} \cdots M_{|a^1|} \\ &= \sum_{m=1}^{|a|} p(g(x))^{m+1} p_h(f)(x)^m \sum_{\substack{(k_j)_j \in \mathbb{N}^m, \\ k_1+\ldots+k_m=|a|}} |a|!\, h^{|a|} M_{k_1} \cdots M_{k_m}\,. \end{aligned}$$

for any $x \in \mathbb{X}$, in which we used

$$\#\left\{ (a^j)_j \in \widetilde{\mathcal{S}_{N,m}(a)} \;\middle|\; k_j = |a^j| \text{ for } j = 1, \ldots, m \right\} = \frac{|a|!}{k_1! \cdots k_m!}$$

for $k \in \mathbb{N}^m$ such that $|k| = |a|$. Since $M$ fulfils **(LC)**, it also fulfils **(AI)**. Thus

$$M_{k_1} \cdots M_{k_m} \leq C^{|a|}\, M_{|a|}, \quad \text{where} \quad C := \sup_{j \leq k,\ j,k \in \mathbb{N}} \frac{M_j^{1/j}}{M_k^{1/k}} < \infty.$$

for $k_1 + \cdots + k_m = |a|$. Hence

$$\begin{aligned} p(D^a g(x)) &\leq \sum_{m=1}^{|a|} p(g(x))^{m+1} p_h(f)(x)^m \sum_{\substack{(k_j)_j \in \mathbb{N}^m, \\ k_1+\ldots+k_m=|a|}} h^{|a|}\, C^{|a|}\, M_{|a|}\, |a|! \\ &= h^{|a|}\, C^{|a|}\, M_{|a|}\, |a|! \sum_{m=1}^{|a|} p(g(x))^{m+1}\, p_h(f)(x)^m \binom{|a|-1}{m-1} \\ &= p(g(x))^2\, p_h(f)(x) \left[h\, C\, (p(g(x)) + p_h(f)(x))\right]^{|a|} M_{|a|}\, |a|!\,. \end{aligned}$$

Now suppose $f \restriction_{K'} \in \mathscr{E}^M_{hD}(K'; A)$, then $p(g(-))$ and $p_h(f)(-)$ are bounded on $K'$. With (2.2.31) we get $g \restriction_{K'} \in \mathscr{E}^{\{M\}}_D(K'; A)$ in this case. Hence $\mathscr{E}^{\{M\}}_D(\mathbb{X})$ and $\mathscr{E}^{\{M\}}_{D,\mathrm{proj}}(\mathbb{X})$ have the property **(IC)**. □

## 2.3 Function spaces on polynomial manifolds

We will need polynomial manifolds for the Pedersen quantization, a generalization of the Weyl quantization, and for the generalizations of the spaces $\mathscr{S}(\mathbb{R}^n; E)$ and $\mathscr{O}_{\mathrm{M}}(\mathbb{R}^n; E)$, see [65, 34]. It is convenient to have one notion of rapidly decreasing and slowly increasing functions that can be applied to simply connected nilpotent Lie groups, Lie algebras and coadjoint orbits in a consistent way. Furthermore, this will lead to a notion of rapidly decreasing and slowly increasing function on $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$, which we will use frequently in Chapter 3.

The definition given below corresponds to the polynomial manifolds used by Pedersen in [56], the only difference being that we also admit non connected manifolds.

**Definition 2.3.1.** *Suppose $\mathbb{M}$ is an $n$-dimensional smooth manifold with finitely many connected components. An atlas $\mathcal{A}$ of $M$ will be called a polynomial atlas iff each two charts $(\phi, U), (\psi, V) \in \mathcal{A}$ fulfil*

**(i)** *$U$, $V$ are connected components of $\mathbb{M}$ and $\phi(U) = \psi(V) = \mathbb{R}^n$,*

**(ii)** *and if $U = V$, then $\phi \circ \psi^{-1}$ is a polynomial function on $\mathbb{R}^n$.*

*Two polynomial atlases $\mathcal{A}, \mathcal{A}'$ are said to be equivalent iff $\mathcal{A} \cup \mathcal{A}'$ is a polynomial atlas. A polynomial structure is an equivalence class of polynomial atlases.*

*Together with a polynomial structure $\mathbb{M}$ will be called a polynomial manifold. A chart of a polynomial structure of $\mathbb{M}$ will be called a polynomial chart on $\mathbb{M}$.*

### 2.3.1 Polynomials, rapidly decreasing and slowly increasing functions

By using polynomial charts, we may generalize polynomials and definitions that depend on the set of polynomials.

Suppose $E$ is a vector space. We denote by $\mathscr{P}(\mathbb{R}^n)$ (resp. $\mathscr{P}(\mathbb{R}^n; E)$) the vector space

$$\left\{ x \mapsto \sum_{|\alpha| \leq k} c_\alpha x^\alpha \;\middle|\; k \in \mathbb{N}_0\,,\ c_\alpha \in \mathbb{C} \text{ (resp. } c_\alpha \in E) \text{ for } \alpha \in \mathbb{N}_0^n \right\}$$

of polynomial functions from $\mathbb{R}^n$ to $\mathbb{C}$ (resp. to $E$) and by $\mathrm{Diff}_{\mathscr{P}}(\mathbb{R}^n)$ the set of differential operators with polynomial coefficients on $\mathbb{R}^n$.

Now suppose $E$ is a locally convex space. The space of $E$-valued rapidly decreasing functions, $\mathscr{S}(\mathbb{R}^n;E)$, is the space of functions $\varphi \in \mathscr{E}(\mathbb{R}^n;E)$ such that $P\varphi$ is bounded in $E$ for each $P \in \mathrm{Diff}_{\mathscr{P}}(\mathbb{R}^n)$. Its topology is defined by the seminorms

$$\varphi \mapsto \sup_{x\in\mathbb{R}^n} p(P\varphi(x)) \qquad \text{for } P \in \mathrm{Diff}_{\mathscr{P}}(\mathbb{R}^n) \text{ and continuous seminorms } p \text{ on } E.$$

Note that $\mathscr{S}(\mathbb{R}^n) := \mathscr{S}(\mathbb{R}^n;\mathbb{C})$ is a $\mathscr{C}(\mathbb{R}^n)$-function space and the above is consistent with Definition 2.1.1. $\mathscr{S}(\mathbb{R}^n)$ is a nuclear Fréchet space [66, Corollary on p.530], thus $\mathscr{S}(\mathbb{R}^n)$ is reflexive and its dual $\mathscr{S}'(\mathbb{R}^n) := \mathscr{S}(\mathbb{R}^n)' = \mathscr{S}(\mathbb{R}^n)'_c$ is nuclear as well. Furthermore, the reflexivity of $\mathscr{S}(\mathbb{R}^n)$ implies that the equicontinuous subsets of $\mathscr{S}'(\mathbb{R}^n)'_c \simeq \mathscr{S}(\mathbb{R}^n)$ correspond to the bounded subsets of $\mathscr{S}(\mathbb{R}^n)$.

The space of slowly increasing functions is the $\mathscr{C}(\mathbb{R}^n)$-function space $\mathscr{O}_{\mathrm{M}}(\mathbb{R}^n)$, defined as the set of smooth functions $f$ such that

$$[\varphi \mapsto f\cdot\varphi] \in \mathcal{L}(\mathscr{S}(\mathbb{R}^n))$$

equipped with the subspace topology in $\mathcal{L}(\mathscr{S}(\mathbb{R}^n))$. Since $\mathcal{L}(\mathscr{S}(\mathbb{R}^n))$ is nuclear, $\mathscr{O}_{\mathrm{M}}(\mathbb{R}^n)$ is nuclear as well by [66, Proposition 50.1]. We have the canonical linear homeomorphisms

$$\begin{aligned}\mathcal{L}(\mathscr{S}(\mathbb{R}^n))\,\varepsilon\, E \simeq \mathscr{S}'(\mathbb{R}^n)\,\varepsilon\,\mathscr{S}(\mathbb{R}^n)\,\varepsilon\, E \simeq \mathscr{S}'(\mathbb{R}^n)\,\varepsilon\,\mathscr{S}(\mathbb{R}^n;E) &\simeq \mathcal{L}_\varepsilon(\mathscr{S}'(\mathbb{R}^n)'_c;\mathscr{S}(\mathbb{R}^n;E)) \\ &\simeq \mathcal{L}(\mathscr{S}(\mathbb{R}^n);\mathscr{S}(\mathbb{R}^n;E))\,,\end{aligned}$$

for any locally convex space $E$. Hence, we may identify $\mathscr{O}_{\mathrm{M}}(\mathbb{R}^n;E)$ with a subspace of $\mathcal{L}(\mathscr{S}(\mathbb{R}^n);\mathscr{S}(\mathbb{R}^n;E))$ equipped with the corresponding subspace topology. Evaluating the above homeomorphisms shows that $\mathscr{O}_{\mathrm{M}}(\mathbb{R}^n;E)$ is precisely the space of all $f \in \mathscr{E}(\mathbb{R}^n;E)$ such that

$$[\varphi \mapsto f\cdot\varphi] \in \mathcal{L}(\mathscr{S}(\mathbb{R}^n);\mathscr{S}(\mathbb{R}^n;E))\,.$$

**Definition 2.3.2.** *Suppose* $\mathbb{M},\mathbb{M}'$ *are polynomial manifolds and* $E$ *is a complete locally convex space.*

*A function* $f\colon M\to N$ *will be called polynomial resp. slowly increasing, iff* $f$ *is continuous and* $\psi\circ f\circ\phi^{-1}$ *is a polynomial resp. slowly increasing for any pair of polynomial* *charts $(\phi, U)$ on $\mathbb{M}$ and $(\psi, V)$ on $\mathbb{M}'$ with $f(U) \subset V$. The function $f$ will be called polynomial resp. tempered diffeomorphism, iff $f$ is bijective and both $f$ and $f^{-1}$ are polynomial resp. slowly increasing.*

*For $\mathscr{F} \in \{\mathscr{P}, \mathscr{S}, \mathscr{O}_{\mathrm{M}}\}$ we define*

$$\mathscr{F}(\mathbb{M}) := \{f\colon M \to E \mid f \circ \phi^{-1} \in \mathscr{F}(\mathbb{R}^n) \text{ for all polynomial charts } \phi\}\,.$$

*We equip $\mathscr{S}(\mathbb{M}; E)$ resp. $\mathscr{O}_{\mathrm{M}}(\mathbb{M}; E)$ with the initial topology with respect to the maps $f \mapsto f \circ \phi^{-1}$ into $\mathscr{S}(\mathbb{R}^n; E)$ resp. $\mathscr{O}_{\mathrm{M}}(\mathbb{R}^n; E)$ for polynomial charts $\phi$ on $\mathbb{M}$.*

*The set of polynomial differential operators on $\mathbb{M}$ is defined to be*

$$\mathrm{Diff}_{\mathscr{P}}(\mathbb{M}) := \left\{ P \in \mathcal{L}(\mathscr{E}(\mathbb{M})) : \begin{matrix} [\varphi \mapsto P(\varphi \circ \phi) \circ \phi^{-1}] \in \mathrm{Diff}_{\mathscr{P}}(\mathbb{R}^n) \\ \text{for all polynomial charts } \phi \end{matrix} \right\}.$$

Identical to the euclidean case, $\mathscr{S}(\mathbb{M}; E)$ is the space of all $f \in \mathscr{E}(\mathbb{M}; E)$ such that the function $Pf$ has bounded image in $E$ for any $P \in \mathrm{Diff}_{\mathscr{P}}(\mathbb{M})$ equipped with the seminorms

$$f \mapsto p(Pf) \quad \text{for continuous seminorms} \quad p \text{ on } E \text{ and } P \in \mathrm{Diff}_{\mathscr{P}}(\mathbb{M})\,.$$

Similarly, $\mathscr{O}_{\mathrm{M}}(\mathbb{M}; E)$ is the space of all $f \in \mathscr{E}(\mathbb{M}; E)$ such that

$$[\varphi \mapsto f \cdot \varphi] \in \mathcal{L}(\mathscr{S}(\mathbb{M}); \mathscr{S}(\mathbb{M}; E))$$

equipped with the subspace topology in $\mathcal{L}(\mathscr{S}(\mathbb{M}); \mathscr{S}(\mathbb{M}; E))$.

We may construct new polynomial manifolds by disjoint unions $\mathbb{M}\dot{\cup}\mathbb{M}'$ of polynomial manifolds $\mathbb{M}$, $\mathbb{M}'$ with the same dimension and products $\mathbb{M} \times \mathbb{M}'$ of arbitrary polynomial manifolds $\mathbb{M}$, $\mathbb{M}'$. The corresponding polynomial structure on $\mathbb{M}\dot{\cup}\mathbb{M}'$ is induced by the polynomial charts on $\mathbb{M}$ and $\mathbb{M}'$. On $\mathbb{M}\times\mathbb{M}'$ we choose the canonical polynomial structure defined by combining charts $\phi$ on $M$ and $\psi$ on $N$ to polynomial charts $\phi \times \psi$ on $M \times N$. Directly from our definition and well known facts from the euclidean case [66, Theorem 51.6] follows that

$$\mathscr{S}(\mathbb{M}) \oplus \mathscr{S}(\mathbb{M}') \simeq \mathscr{S}(\mathbb{M}\dot{\cup}\mathbb{M}') \quad \text{and} \quad \mathscr{S}(\mathbb{M}) \hat{\otimes} \mathscr{S}(\mathbb{M}') \simeq \mathscr{S}(\mathbb{M} \times \mathbb{M}') \tag{2.3.32}$$

via the linear homeomorphisms

$$f \oplus g \mapsto h\colon h(x) := \begin{cases} f(x), & x \in \mathbb{M} \\ g(x), & x \in \mathbb{M}' \end{cases} \quad \text{and} \quad v \mapsto [\mathbb{M} \times \mathbb{M}' \ni (x, x') \mapsto v(x, x') \in \mathbb{C}]\,.$$

The identities also hold if we exchange $\mathscr{S}$ with $\mathscr{O}_{\mathrm{M}}$ [62, p. 115][8].

We will call a Radon measure $\nu$ on $\mathbb{R}^n$ tempered, iff it is mutually absolutely continuous to the Lebesgue measure $\mathrm{d}x$ and the Radon-Nikodym derivatives $\frac{\mathrm{d}x}{\mathrm{d}\nu}$ and $\frac{\mathrm{d}\nu}{\mathrm{d}x}$ are slowly increasing almost everywhere. A Radon measure on a polynomial manifold will be called tempered if each pushforward by a polynomial chart is tempered.

**Definition 2.3.3.** *Suppose $\mathbb{M}$ is a polynomial manifold and $\nu$ a tempered measure on $\mathbb{M}$. Then $\mathcal{G}(\mathbb{M}, \nu)$ is defined to be the Gelfand triple*

$$\mathscr{S}(\mathbb{M}) \hookrightarrow L^2(\mathbb{M}, \nu) \hookrightarrow \mathscr{S}'(\mathbb{M}),$$

*equipped with the real structure defined by the usual complex conjugation $\varphi \mapsto \overline{\varphi}$.*

If $f\colon \mathbb{M}_1 \to \mathbb{M}$ is a tempered diffeomorphism, then for each $\phi \in \mathscr{S}'(\mathbb{M})$ the pull back $\wp_f \phi(\varphi) := \phi(\varphi \circ f^{-1})$ is well-defined and induces a Gelfand triple isomorphism

$$\mathcal{G}(\mathbb{M}, \nu) \to \mathcal{G}(\mathbb{M}_1, \nu \circ f^{-1}).$$

Indeed, we defined tempered measures and polynomial manifolds in such a way that we have a very simple Gelfand-Triple isomorphism

$$\mathcal{G}(\mathbb{M}, \nu) \simeq \bigoplus_{j=1}^{k} \mathcal{G}(\mathbb{R}^n, \mathrm{d}x),$$

given by (2.3.32), pullbacks and multiplications with slowly increasing functions, provided that $\mathbb{M}$ is an $n$-dimensional polynomial manifold with $k$ connected components. Moreover, for any two polynomial manifolds $\mathbb{M}$ and $\mathbb{M}'$ with tempered measures $\nu$ and $\nu'$ we have a canonical Gelfand triple isomorphism

$$\mathcal{G}(\mathbb{M}, \nu) \otimes \mathcal{G}(\mathbb{M}', \nu') \simeq \mathcal{G}(\mathbb{M} \times \mathbb{M}', \nu \otimes \nu')$$

that is an extension of the linear homeomorphism in (2.3.32).

[8] In this reference the topology of $\mathscr{O}_{\mathrm{M}}(\mathbb{R}^n)$ is described by the seminorms $f \mapsto |g\,\partial^\alpha f|$ for $\alpha \in \mathbb{N}_0^n$ and $g \in \mathscr{S}(\mathbb{R}^n)$. Both approaches result in the same topology, because any bounded set $B \subset \mathscr{S}(\mathbb{R}^n)$ is pointwise bounded by some $g \in \mathscr{S}(\mathbb{R}^n)$.

The most basic examples are given by vector resp. affine spaces, whose polynomial charts are given by the linear resp. affine charts. This means that we may and will consider Lie algebras as polynomial manifolds with respect to the linear charts. The following two examples are the reason for the introduction of polynomial manifolds in [56].

**Definition 2.3.4.** *Suppose $\mathbb{G}$ is a Lie group and $\mathfrak{g}$ is its Lie algebra. The adjoint representation of $\mathbb{G}$ on $\mathfrak{g}$ will be denoted by $\mathrm{Ad}_{\mathbb{G}}$ and the adjoint representation of $\mathfrak{g}$ on itself by $\mathrm{ad}_x y = [x, y]$, in which $[-, -]$ is the Lie bracket on $\mathfrak{g}$.*

*$\mathbb{G}$ and $\mathfrak{g}$ are called **nilpotent** iff there is some $n \in \mathbb{N}$ such that*

$$\{\mathrm{ad}_{x_1} \cdots \mathrm{ad}_{x_{n-1}} x_n \mid x_1, \dots, x_n \in \mathfrak{g}\} = \{0\} .$$

*The coadjoint representation of $\mathbb{G}$ on the dual $\mathfrak{g}'$ will be denoted by $\mathrm{Ca}_{\mathbb{G}}(x) := \mathrm{Ad}(x^{-1})'$. A coadjoint orbit of $\mathbb{G}$ is a set of the form $\Omega = \mathrm{Ca}_{\mathbb{G}}(\mathbb{G})\xi$ for some $\xi \in \mathfrak{g}$.*

For a simply connected, connected Lie group $\mathbb{G}$ the exponential map $\exp_{\mathbb{G}}$ is a diffeomorphism $\exp_{\mathbb{G}} \colon \mathfrak{g} \to \mathbb{G}$ [16, Theorem 1.2.1]. Thus

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \colon (x, y) \mapsto \exp_{\mathbb{G}}^{-1}(\exp_{\mathbb{G}} x)(\exp_{\mathbb{G}} y)$$

is polynomial.

In this case we will also consider $\mathbb{G}$ as a polynomial manifold with respect to the chart $\exp_{\mathbb{G}}$. This automatically implies the following corollary.

**Corollary 2.3.5.** *For any simply connected, connected nilpotent Lie group $\mathbb{G}$ the maps*

$$\mathbb{G} \times \mathbb{G} \to \mathbb{G} \colon (x, y) \mapsto xy \quad \textit{and} \quad \mathbb{G} \to \mathbb{G} \colon x \mapsto x^{-1}$$

*are polynomial.*

Furthermore, by [56] each coadjoint orbit $\Omega = \mathrm{Ca}_{\mathbb{G}}(\mathbb{G})\xi$ of a simply connected, connected nilpotent Lie group $\mathbb{G}$ can be equipped with a canonical polynomial structure. We will introduce and use this structure in Chapter 3.

### 2.3.2 Functions and distributions on the polynomial manifold $\mathbb{R}^\times$

Next to nilpotent Lie groups and coadjoint orbits to nilpotent Lie groups, the two most important examples of polynomial manifolds will be the half lines $\mathbb{R}_+ = (0,\infty)$ and $\mathbb{R}_- = (-\infty, 0)$. Here the polynomial structure is induced by the chart $\sigma\colon \lambda \mapsto |\lambda| - 1/|\lambda|$. On $\mathbb{R}_+$ the inverse reads $\sigma^{-1}(y) = (y + \sqrt{y^2+4})/2$.

**Lemma 2.3.6.** *If we extend each function in $\mathscr{S}(\mathbb{R}_\pm)$ by zero to the whole real line, then*

$$\mathscr{S}(\mathbb{R}_\pm) = \{\varphi \in \mathscr{S}(\mathbb{R}) \mid \varphi \equiv 0 \ \textit{on}\ \mathbb{R}^\mp\}$$

*and $\mathscr{S}(\mathbb{R}_\pm)$ carries the subspace topology with respect to $\mathscr{S}(\mathbb{R})$.*

*Proof.* We will prove the statement for the $\mathbb{R}_+$ case, for $\mathbb{R}_-$ the proof is analogous. Since $\sigma$ is a polynomial diffeomorphism from $\mathbb{R}_+$ to $\mathbb{R}$, the map

$$\varphi \mapsto \varphi \circ \sigma$$

is a linear homeomorphism between $\mathscr{D}(\mathbb{R})$ and $\mathscr{D}(\mathbb{R}_+)$ resp. between $\mathscr{S}(\mathbb{R})$ and $\mathscr{S}(\mathbb{R}_+)$. Hence $\mathscr{D}(\mathbb{R}_+)$ is dense in $\mathscr{S}(\mathbb{R}_+)$. Let us define

$$\mathscr{S}_+(\mathbb{R}) := \{\varphi \restriction_{\mathbb{R}_+} \mid \ \varphi \in \mathscr{S}(\mathbb{R}) \text{ with } \varphi \equiv 0 \text{ on } \mathbb{R}^-\},$$

equipped with the subspace topology with respect to $\mathscr{S}(\mathbb{R})$. Let $f\colon \mathbb{R} \to [0,1]$ be smooth such that $\operatorname{supp} f \subset \mathbb{R}_+$ and $f \equiv 1$ on $[1,\infty)$. For each $\varphi \in \mathscr{S}_+(\mathbb{R})$ and $\alpha \in \mathbb{N}_0$ we have $\partial^\alpha \varphi(x) = o(x^N)$ for $x \to 0$ of arbitrary high order $N \in \mathbb{N}$. Hence for each $\alpha, \beta \in \mathbb{N}_0$ there are some $C_1, C_2 > 0$ and $N > \alpha$ with

$$\begin{aligned}\sup_{x\in\mathbb{R}_+} |x^\beta \partial_x^\alpha (f(n\,x)\,\varphi(x) - \varphi(x))| &\leq C_1 \sum_{0<\gamma\leq\alpha} n^\gamma \sup_{y\in\mathbb{R}} |\partial^\gamma f(y)| \sup_{0<x\leq 1/n} x^\beta |\partial^{\alpha-\gamma}\varphi(x)| \\ &\leq C_2 \sum_{0<\gamma\leq\alpha} n^{\gamma-\beta-N} \xrightarrow{n\to\infty} 0.\end{aligned}$$

By employing the usual cut-off functions, we realize that $\mathscr{D}(\mathbb{R}_+)$ is dense in $\mathscr{S}_+(\mathbb{R})$ as well. Therefore, it is sufficient to show that the topologies of $\mathscr{S}(\mathbb{R}_+)$ and $\mathscr{S}_+(\mathbb{R})$ coincide on $\mathscr{D}(\mathbb{R}_+)$. The $\mathscr{S}(\mathbb{R}_+)$-topology is induced by seminorms of the form

$$\mathscr{D}(\mathbb{R}_+) \to \mathbb{R}\colon \varphi \mapsto \sup_{x>0} |A^k B^j \varphi(x)|, \quad k, j \in \mathbb{N}_0,$$

where $A\varphi := (\partial(\varphi \circ \sigma^{-1})) \circ \sigma$ and $B\varphi := (\boldsymbol{m}(\varphi \circ \sigma^{-1})) \circ \sigma = \sigma \cdot \varphi$. First of all, we have

$$A\varphi(x) = \frac{x^2}{x^2+1}\partial_x \varphi(x) =: \eta(x) \cdot \partial_x \varphi(x), \quad \varphi \in \mathscr{D}(\mathbb{R}_+), x \in \mathbb{R}_+.$$

The rational function $\eta$ and all of its derivatives are bounded. Hence $A$ can be extended to an operator in $\mathcal{L}(\mathscr{S}_+(\mathbb{R}))$.

We show that $B$ has an extension in $\mathcal{L}(\mathscr{S}_+(\mathbb{R}))$. For this purpose it is enough to prove that $\frac{1}{\boldsymbol{m}} \in \mathcal{L}(\mathscr{S}_+(\mathbb{R}))$, where $\frac{1}{\boldsymbol{m}}\varphi(x) = \varphi(x)/x$. First of all, for each $\varphi \in \mathscr{D}(\mathbb{R}_+)$ and each $x > 1$

$$\begin{aligned} |x^k \partial_x^n (\frac{1}{x}\varphi(x))| &\leq \sum_{j=0}^n \frac{n!}{(n-j)!} x^{k-j-1} |\varphi^{(n-j)}(x)| \\ &\leq \sum_{j=0}^n \frac{n!}{(n-j)!} \sup_y |y^k \varphi^{(n-j)}(y)|, \end{aligned}$$

for arbitrary $k, n \in \mathbb{N}_0$. Now we only need to bound the left-hand side for $0 < x < 1$. For $k > n$ we can use roughly the same inequality as above. We assume now $n \geq k$. For $0 < x < 1$ and each $m \in \mathbb{N}$

$$|\varphi(x)/x^m| = |\frac{1}{x^m}\int_0^x \frac{(x-t)^{m-1}}{(m-1)!}\varphi^{(m)}(t)\ \mathrm{d}t| \leq \frac{1}{m!}\sup_y |\varphi^{(m)}(y)|.$$

Hence

$$\begin{aligned} |x^k \partial_x^n (\frac{1}{x}\varphi(x))| &\leq \sum_{j=0}^n \frac{n!}{(n-j)!} x^{k-j-1} |\varphi^{(n-j)}(x)| \\ &\leq \sum_{j=0}^n \frac{n!}{(n-j)!} x^{-n-1} |\varphi^{(n-j)}(x)| \\ &\leq \sum_{j=0}^n \frac{1}{(n-j)!(n+1)} \sup_y |\varphi^{(2n+1-j)}(y)|, \end{aligned} \tag{2.3.33}$$

for all $0 < x < 1$, $n \leq k$ and $\varphi \in \mathscr{D}(\mathbb{R}_+)$. In conclusion, $\frac{1}{\boldsymbol{m}} \in \mathcal{L}(\mathscr{S}_+(\mathbb{R}))$ and subsequently also $B \in \mathcal{L}(\mathscr{S}_+(\mathbb{R}))$. Due to the continuity of $A$ and $B$ we arrive at

$$\mathscr{S}_+(\mathbb{R}) \hookrightarrow \mathscr{S}(\mathbb{R}_+),$$

i.e. the $\mathscr{S}_+(\mathbb{R})$-topology is finer than the $\mathscr{S}(\mathbb{R}_+)$-topology.

For the reverse embedding we will transport our situation to the whole real line by

$$\varphi \mapsto \varphi \circ \sigma^{-1},$$

which is an isomorphism $\mathscr{D}(\mathbb{R}_+) \to \mathscr{D}(\mathbb{R})$ and $\mathscr{S}(\mathbb{R}_+) \to \mathscr{S}(\mathbb{R})$. We denote the image of $\mathscr{S}_+(\mathbb{R})$ by this map by $\mathscr{S}_\oplus(\mathbb{R})$ and equip it with the transported $\mathscr{S}_+(\mathbb{R})$-topology. Then $\mathscr{S}_\oplus(\mathbb{R})$ is a space of smooth functions on $\mathbb{R}$ with

$$\mathscr{D}(\mathbb{R}) \hookrightarrow \mathscr{S}_\oplus(\mathbb{R}) \hookrightarrow \mathscr{S}(\mathbb{R}),$$

where both embeddings are dense. The topology in $\mathscr{S}_\oplus(\mathbb{R})$ is induced by seminorms of the form

$$\mathscr{S}_\oplus(\mathbb{R}) \to \mathbb{R} \colon \varphi \mapsto \sup_{y\in\mathbb{R}} |C^k E^j \varphi(y)|, \quad k, j \in \mathbb{N}_0,$$

where $C\varphi := (\partial(\varphi \circ \sigma)) \circ \sigma^{-1}$ and $E\varphi := (\boldsymbol{m}(\varphi \circ \sigma)) \circ \sigma^{-1} = \sigma^{-1} \cdot \varphi$. The operator $C$ can be rewritten as

$$C\varphi(y) = \left(1 + \frac{2}{(y + \sqrt{y^2+4})^2}\right)\varphi'(y) =: \psi(y) \cdot \varphi'(y), \quad \varphi \in \mathscr{S}_\oplus(\mathbb{R}), y \in \mathbb{R}.$$

Because $\sigma^{-1}, \psi \in \mathscr{O}_{\mathrm{M}}(\mathbb{R})$, both $C$ and $E$ have extensions in $\mathcal{L}(\mathscr{S}(\mathbb{R}))$. Thus $\mathscr{S}_\oplus(\mathbb{R}) = \mathscr{S}(\mathbb{R})$ and finally $\mathscr{S}_+(\mathbb{R}) = \mathscr{S}(\mathbb{R}_+)$. □

The most important property of $\mathscr{S}(\mathbb{R}_\pm)$ (apart from being a closed subspace of $\mathscr{S}(\mathbb{R})$) is the continuity of the multiplication operator $f \mapsto |{-}|^\nu f$.

**Corollary 2.3.7.** *The map $x \mapsto |x|^v$ is in $\mathscr{O}_{\mathrm{M}}(\mathbb{R}_\pm)$ for each $v \in \mathbb{R}$.*

*Proof.* The continuity of $\frac{1}{\boldsymbol{m}}$ was already shown in the proof of the last lemma with inequalities (2.3.33). Of course $\boldsymbol{m}\varphi(x) := x\varphi(x)$ defines a continuous operator on $\mathscr{S}(\mathbb{R}_\pm)$ as well. The derivatives of $x \mapsto |x|^v$ are bounded by terms of the form $x \mapsto x^k$ for $k \in \mathbb{Z}$, which concludes the proof. □

We now find a characterisation for the functions in $\mathscr{O}_{\mathrm{M}}(\mathbb{R}_\pm \times \mathbb{M}; E)$. This space will be of importance later on, when we examine the Fourier image of $\mathscr{S}(\mathbb{G})$ in further detail, as well as when we want to discuss the integral formula for the Kohn-Nirenberg quantization.

**Corollary 2.3.8.** *Let $\mathbb{M}$ be a polynomial manifold. A smooth function $f\colon \mathbb{R}_\pm \times \mathbb{M} \to E$ is in $\mathscr{O}_{\mathrm{M}}(\mathbb{R}_\pm \times \mathbb{M}; E)$, iff for each $k \in \mathbb{N}_0$, each $P \in \mathrm{Diff}_{\mathscr{P}}(\mathbb{M})$ and each continuous seminorm $p$ on $E$ there exist an $l \in \mathbb{N}$ and a $q \in \mathscr{P}(\mathbb{M})$ such that*

$$p(\partial_\lambda^k P_x f(\lambda, x)) \leq (1 + |\lambda|^l + |\lambda|^{-l}) q(x)\,.$$

*Proof.* We know that $\mathscr{O}_{\mathrm{M}}(\mathbb{R}_+ \times \mathbb{M}; E)$ is the space of all smooth functions $f$ on $\mathbb{R}_+$ such that

$$[\varphi \mapsto f \cdot \varphi] \in \mathcal{L}(\mathscr{S}(\mathbb{R}_\pm \times \mathbb{M}), \mathscr{S}(\mathbb{R}_\pm \times \mathbb{M}; E)).$$

Once we prove the statement for $\mathbb{R}_+$, the other statement follows at once, since $\mathbb{R}_-$ is isomorphic to $\mathbb{R}_+$ by $x \mapsto -x$. Furthermore, it is enough to consider $\mathbb{M} = \mathbb{R}^n$, as the more general case follows by simply using polynomial coordinate charts.

Suppose $f \in \mathscr{O}_{\mathrm{M}}(\mathbb{R}_+ \times \mathbb{R}^n; E)$. Since $f$ induces a continuous multiplication operator and since $\mathscr{S}(\mathbb{R}_+)$ is a subspace of $\mathscr{S}(\mathbb{R})$, for each $k \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^n$ and each continuous seminorm $p$ on $E$, there are some $m \in \mathbb{N}$ and $C > 0$ with

$$\begin{aligned}\sup_{\lambda \in \mathbb{R}_+, x \in M} & p(\partial_\lambda^k \partial_x^\alpha (f(\lambda, x)\varphi(\lambda, x))) \\ & \leq C \max_{|\beta|, l \leq m} \sup_{\lambda \in \mathbb{R}_+, x \in M} (1 + |\lambda|^m)(1 + |x|^2)^m |\partial_\lambda^l \partial_x^\beta \varphi(\lambda, x)|,\end{aligned}$$

for all $\varphi \in \mathscr{S}(\mathbb{R}_+ \times \mathbb{R}^n)$. We choose $\varphi \in \mathscr{S}(\mathbb{R}_+ \times \mathbb{R}^n)$, such that $\varphi \equiv 1$ on some neighbourhood around $(\lambda, x) = (1, 0)$, and define $\varphi_{a,y}(x) := \varphi(xa^{-1}, x - y)$ for $a > 0$, $y \in \mathbb{R}^n$. Then

$$\begin{aligned} p(\partial^{(k,\alpha)} f(a, y)) &= p(\partial_\lambda^k \partial_x^\alpha (f(\lambda, x)\varphi_{a,y}(\lambda, x)))\big|_{(\lambda,x)=(a,y)} \\ &\leq C \max_{|\beta|, l \leq m} \sup_{\lambda \in \mathbb{R}_+, x \in \mathbb{R}^n} (1 + |\lambda|^m)(1 + |x|^2)^m |\partial_\lambda^l \partial_x^\beta \varphi_{a,y}(\lambda, x)| \\ &= C \max_{|\beta|, l \leq m} \sup_{\lambda \in \mathbb{R}_+, x \in \mathbb{R}^n} a^{-l}(1 + |a\lambda|^m)(1 + |x + y|^2)^m |\partial_\lambda^l \partial_x^\beta \varphi(\lambda, x)| \\ &\leq C'(1 + a^m + a^{-m})(1 + |y|^2)^m, \end{aligned}$$

where $k$, $\alpha$, $m$ and $C$ are as above. Naturally this implies that for each $k \in \mathbb{N}_0$, each differential operator $P \in \mathrm{Diff}_{\mathscr{P}}(\mathbb{R}^n)$ and each continuous seminorm $p$ on $E$, there exists an $l \in \mathbb{N}$ and a $q \in \mathscr{P}(\mathbb{R}^n)$ such that

$$p(\partial_\lambda^k P_x f(\lambda, x)) \leq (1 + |\lambda|^l + |\lambda|^{-l}) q(x). \tag{2.3.34}$$

For the converse implication let $f\colon \mathbb{R}_+\times\mathbb{R}^n \to \mathbb{C}$ be any smooth function such that for $p$, $k$ and $P$ we find $m$ and $q$ for the inequality (2.3.34). Then for arbitrary $\varphi \in \mathscr{S}(\mathbb{R}_+\times M)$,

$$\begin{aligned}
&\sup_{\lambda\in\mathbb{R}_+, x\in\mathbb{R}^n} (1+|\lambda|^k)(1+|x|^2)^k p(\partial^\alpha(f\,\varphi)(\lambda,x)) \\
&\qquad\le C \sup_{\lambda\in\mathbb{R}_+, x\in\mathbb{R}^n} (1+\lambda^k)(1+|x|^2)^k \sum_{\beta\le\alpha} |\partial^{\alpha-\beta} f(\lambda,x)\,\partial^\beta\varphi(\lambda,x)| \\
&\qquad\le C' \sup_{x\in\mathbb{R}_+} (1+|x|^2)^{k+m}(1+\lambda^{k+m}+\lambda^{k-m}) \sum_{\beta\le\alpha} |\partial^\beta\varphi^{(j)}(x)|.
\end{aligned}$$

Since $\frac{1}{\boldsymbol{m}}$ is a continuous operator on $\mathscr{S}(\mathbb{R}_+)$, the last line defines a continuous seminorm on $\mathscr{S}(\mathbb{R}_+\times\mathbb{R}^n)$. Thus the operator $\varphi\mapsto f\cdot\varphi$ is continuous. □

From the polynomial structures on $\mathbb{R}_+$ and $\mathbb{R}_-$ we construct the polynomial manifold $\mathbb{R}^\times = \mathbb{R}_+ \dot\cup \mathbb{R}_-$. Its Schwartz space $\mathscr{S}(\mathbb{R}^\times) = \mathscr{S}(\mathbb{R}_+)\oplus\mathscr{S}(\mathbb{R}_-)$ can be seen as the closed subspace of $\mathscr{S}(\mathbb{R})$ of functions $f$, which vanish to infinite order in 0, i.e. $\partial^k f(0) = 0$ for all $k\in\mathbb{N}_0$. The dual space and the Fourier image of $\mathscr{S}(\mathbb{R}^\times)$ will play a significant role in the coming discussion. The first statement requires no further proof.

**Lemma 2.3.9.** *The image of $\mathscr{S}(\mathbb{R}^\times)$ under the Fourier transform on $\mathbb{R}$, $\mathcal{F}_\mathbb{R}$, is $\mathscr{S}_*(\mathbb{R})$, which is defined to be the subspace of Schwartz functions $f$ with vanishing moments of infinite order, i.e.*

$$\int_\mathbb{R} f(x)\,p(x)\,\mathrm{d}x = 0, \quad \textit{for all} \quad p\in\mathscr{P}(\mathbb{R}).$$

The next lemma is less obvious. It is an extension of the well-known fact that $\mathscr{S}'_*(\mathbb{R})$, as a vector space, can be identified with the quotient $\mathscr{S}'(\mathbb{R})/\mathscr{P}(\mathbb{R})$ e.g. [32, Proposition 1.1.3].

**Lemma 2.3.10.** *Let $E$ be a nuclear Fréchet space and $\mathscr{E}'_0(\mathbb{R})$ the space of distributions on $\mathbb{R}$ with support in $\{0\}$. Then $\mathscr{E}'_0(\mathbb{R})\otimes E'$ is a closed subspace of $\mathscr{S}'(\mathbb{R})\,\hat\otimes\,E'$ and*

$$(\mathscr{S}(\mathbb{R}^\times)\,\hat\otimes\,E)' \simeq (\mathscr{S}'(\mathbb{R})\,\hat\otimes\,E')/(\mathscr{E}'_0(\mathbb{R})\otimes E')\,.$$

*Proof.* First, we will prove that $Z := \mathscr{E}'_0(\mathbb{R})\otimes E'$ is a closed subspace of $X'\simeq\mathscr{S}'(\mathbb{R})\,\hat\otimes\,E'$, where $X := \mathscr{S}(\mathbb{R})\,\hat\otimes\,E$. The family $(\partial^k\delta_0)_{k\in\mathbb{N}_0}$ is a basis for $\mathscr{E}'_0(\mathbb{R})$ where $\delta_0$ is the delta distribution [66, Theorem 24.6]. We use Lemma 1.2.4 on the sequence $P_N$ of projections onto the subspaces spanned by $\{\delta_0, \dots, \partial^N \delta_0\}$ and conclude that $Z$ is sequentially dense in its closure $\overline{Z}$. Furthermore, we realize that for any $\phi \in \overline{Z}$ there is a sequence $(e'_k) \subset E'$ such that

$$\phi = \lim_{N\to\infty} \phi_N := \lim_{N\to\infty} \sum_{k=0}^{N} (\partial^k \delta_0) \otimes e'_k.$$

Because $X$ is a Fréchet space and $Z \subset X'$, we can apply the Banach-Steinhaus Theorem. Hence, there exist a continuous seminorm $q$ on $E$ and $M \in \mathbb{N}$ such that

$$|\phi_N(f)| \leq \max_{k\leq M} \sup_{x\in\mathbb{R}} \langle x\rangle^M q(\partial_x^k f(x))$$

for all functions $f \in X = \mathscr{S}(\mathbb{R}) \hat{\otimes} E$ and all $N \in \mathbb{N}$.

Next let us assume that there is one $l > M$ such that $e'_l \neq 0$. Let us define the sequence of Schwartz functions $f_m(x) := \mathrm{e}^{\mathrm{i}mx}\psi(x)e/m^{l-1}$, where $\psi$ is a rapidly decreasing function equal to one near zero and $e \in E$ with $e'_l(e) = 1$. We arrive at

$$|\phi_l(f_m)| = \left| \sum_{k=0}^{l} \frac{(\mathrm{i}m)^k}{m^{(l-1)}} e'_k(e) \right| \xrightarrow{m\to\infty} \infty.$$

But also

$$\sup_{m\in\mathbb{N}} \max_{k\leq M} \sup_{x\in\mathbb{R}} \langle x\rangle^M q(\partial_x^k f_m(x)) < \infty,$$

which is a contradiction. Hence $\phi \in Z$, i.e. $\phi$ is in the finite span of the $\partial^k\delta$ and $e'_k$.

Now let $Y := Z^\circ$ be the polar of $Z$. Because $X$ is reflexive, we may identify $Y \subset X$. Since $Z$ is a closed subspace, we also have $Y^\circ = Z^{\circ\circ} = Z$. Since $\partial^k\delta_0 \otimes e' \in Z$ for all $k \in \mathbb{N}_0$, $e' \in E'$ and

$$(\partial^k\delta_0 \otimes e')(\varphi) = \langle (-1)^k \partial^k \varphi(0), e' \rangle\,, \quad \text{for} \quad \varphi \in X = \mathscr{S}(\mathbb{R}; E),$$

it is apparent that $Y = \mathscr{S}(\mathbb{R}^\times) \hat{\otimes} E$.

Since $E$ is a nuclear Fréchet space, $X$ is a nuclear Fréchet space. That also means that $X$ is an (FS) space, i.e. it is linearly homeomorphic to a projective limit $\varprojlim_{k\to\infty}(X_k, u_{k,j})$ of a sequence of Banach spaces $(X_k)_k$ with compact maps $u_{k,j} \colon X_j \to X_k$ [61, Chapter 3, Corollary 3 to Theorem 7.3]. Note that the maps $u_{k,k+1} \colon X_{k+1} \to X_k$ are weakly compact as well. This enables us to use Theorem 13 of [39]. The theorem states that in our situation – $Y$ is closed and $X$ is an (FS) space – we have $Y' \simeq X'/Y^\circ$. $\square$

By using the euclidean Fourier transform in combination with the last lemma, we get the following corollary.

**Corollary 2.3.11.** *Let $E$ be a nuclear Fréchet space, then*

$$(\mathscr{S}_*(\mathbb{R}) \hat{\otimes} E)' \simeq (\mathscr{S}'(\mathbb{R}) \hat{\otimes} E')/(\mathscr{P}(\mathbb{R}) \otimes E')$$

*and $\mathscr{P}(\mathbb{R}) \otimes E'$ is closed in $\mathscr{S}'(\mathbb{R}) \hat{\otimes} E'$.*

Furthermore, this characterization of the dual spaces of $\mathscr{S}(\mathbb{R}^\times) \hat{\otimes} E$ and $\mathscr{S}_*(\mathbb{R}) \hat{\otimes} E$ by quotient spaces enables us to find subspaces of $\mathscr{S}'(\mathbb{R}) \hat{\otimes} E'$ which are embedded into these dual spaces. Suppose $F$ is a Banach space such that there is a continuous embedding $E \hookrightarrow F$ with dense range. Then we may see that the Lebesgue-Bochner spaces $L^p(\mathbb{R}; F')$ are embedded both into $\mathscr{S}_*'(\mathbb{R}) \hat{\otimes} E'$ and into $\mathscr{S}(\mathbb{R}^\times) \hat{\otimes} E'$ for $p \in (1, \infty)$. Here we define the distribution corresponding to $f \in L^p(\mathbb{R}; F')$ by

$$T_f(\varphi) := \int_{\mathbb{R}} \langle f(x), \varphi(x) \rangle \, \mathrm{d}x, \quad \varphi \in \mathscr{S}(\mathbb{R}; E),$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing on $F' \times F$. Note that $f \mapsto T_f$ is indeed an injective map into $\mathscr{S}'(\mathbb{R}; E')$, since $f = 0$ almost everywhere iff $T_f(\varphi \otimes e) = 0$ for all $\varphi \in \mathscr{S}(\mathbb{R})$ and all $e \in E$.

Nevertheless, we can make a much more general claim. For this purpose we define the following subspaces of $\mathscr{S}'(\mathbb{R}) \hat{\otimes} E'$.

$$\dot{\mathscr{B}}'(\mathbb{R}; E') := \{\phi \in \mathscr{S}'(\mathbb{R}) \hat{\otimes} E' \mid \forall_{\varphi \in \mathscr{S}(\mathbb{R}) \hat{\otimes} E} \ \phi\big(\varphi(- + x)\big) \xrightarrow{|x| \to \infty} 0\}$$
$$\widetilde{\mathscr{B}}'(\mathbb{R}; E') := \{\phi \in \mathscr{S}'(\mathbb{R}) \hat{\otimes} E' \mid \forall_{\varphi \in \mathscr{S}(\mathbb{R}) \hat{\otimes} E} \ \phi(\varphi(\lambda^{-1} \cdot (-))) \xrightarrow{\lambda \to 0} 0\}$$

**Lemma 2.3.12.** *Let $F$ be a Banach space as described above. The Lebesgue-Bochner space $L^p(\mathbb{R}; F')$ is a subspace of $\dot{\mathscr{B}}'(\mathbb{R}; E')$ for $p \in [1, \infty)$ and a subspace of $\widetilde{\mathscr{B}}'(\mathbb{R}; E')$ for $p \in [1, \infty]$ with respect to the embedding $f \mapsto T_f$.*

*Proof.* Let $f \in L^p(\mathbb{R}; F')$ and let $\varphi \in \mathscr{S}(\mathbb{R}) \hat{\otimes} E$ then

$$[x \mapsto (1 + x^2)\, \varphi(x)] \in L^q(\mathbb{R}; F)$$

also holds true for each $q \in [1,\infty]$ with $1 = 1/p + 1/q$. First, let us suppose $p \in [1,\infty)$, then for some $C > 0$ independent of $x \in \mathbb{R}$

$$|T_f(\varphi(\cdot - x))| \leq \int_{\mathbb{R}} |\langle f(y), \varphi(y-x)\rangle| \,\mathrm{d}y \leq C \left( \int_{\mathbb{R}} \frac{\|f(y)\|_F^p}{(1+(x-y)^2)^p} \,\mathrm{d}y \right)^{\frac{1}{p}} .$$

Now let $\varepsilon > 0$ be arbitrary and let $R > 0$ be big enough such that

$$\int\limits_{\{y\in\mathbb{R}\colon |y|\geq R\}} \|f(y)\|_{F'}^p \,\mathrm{d}y \leq \varepsilon,$$

With this inequality we get

$$\left( \int_{\mathbb{R}} \frac{\|f(y)\|_{F'}^p}{(1+(x-y)^2)^p} \,\mathrm{d}y \right)^{\frac{1}{p}} \leq \left( \varepsilon + \int\limits_{\{y\in\mathbb{R}\colon |y|\leq R\}} \frac{\|f(y)\|_{F'}^p}{(1+(x-y)^2)^p} \,\mathrm{d}y \right)^{\frac{1}{p}} \xrightarrow{x\to\pm\infty} \varepsilon^{\frac{1}{p}} .$$

Hence $T_f \in \dot{\mathscr{B}}'(\mathbb{R}; E')$, because $\varepsilon > 0$ can be arbitrarily small.

Suppose $p = 1$. Applying the same calculation as before, we get

$$|T_f(\varphi(\cdot/\lambda))| \leq C \left( \varepsilon + \int\limits_{\{y\in\mathbb{R}\colon |y|\leq R\}} \frac{\|f(y)\|_{F'}}{1+(y/\lambda)^2} \,\mathrm{d}y \right) \xrightarrow{\lambda\to 0} C\varepsilon\,.$$

Thus $T_f \in \widetilde{\mathscr{B}}'(\mathbb{R}; E')$. Next, suppose $p = (1,\infty]$ and $1/p + 1/q = 1$. In that case we have

$$|T_f(\varphi(\cdot/\lambda))| \leq \lambda^{\frac{1}{q}} \, \|f(x)\|_{L^p(R;F')} \, \|\varphi(y)\|_{L^q(\mathbb{R};F)} \xrightarrow{\lambda\to 0} 0.$$

Hence also $T_f \in \widetilde{\mathscr{B}}'(\mathbb{R}; E')$ for this case. ☐

Note that the distributions in $\dot{\mathscr{B}}'(\mathbb{R}; E')$ can have any form in a bounded region, whereas distributions in $\widetilde{\mathscr{B}}'(\mathbb{R}; E')$ can have any form away from zero, as long as they are tempered.

**Proposition 2.3.13.** *The quotient maps*

$$\mathscr{S}'(\mathbb{R}) \,\hat{\otimes}\, E' \to \mathscr{S}'(\mathbb{R}^\times) \,\hat{\otimes}\, E',$$
$$\mathscr{S}'(\mathbb{R}) \,\hat{\otimes}\, E' \to \mathscr{S}'_*(\mathbb{R}) \,\hat{\otimes}\, E',$$

*restrict to embeddings*

$$\widetilde{\mathscr{B}}'(\mathbb{R}; E') \hookrightarrow \mathscr{S}'(\mathbb{R}^\times) \,\hat{\otimes}\, E',$$
$$\dot{\mathscr{B}}'(\mathbb{R}; E') \hookrightarrow \mathscr{S}'_*(\mathbb{R}) \,\hat{\otimes}\, E'.$$

*Proof.* A short calculation yields

$$\widetilde{\mathscr{B}}'(\mathbb{R}; E') \cap \mathscr{E}_0'(\mathbb{R}) \otimes E' = \{0\} = \dot{\mathscr{B}}'(\mathbb{R}; E') \cap \mathscr{P}(\mathbb{R}) \otimes E'.$$

Combined with the above lemma and corollary, this already concludes the proof. □

## 2.4 Smooth and ultradifferentiable vectors of representations

In the following section we always assume that $\mathbb{G}$ is a Lie group. Thus $\mathbb{G}$ is also an analytic manifold.

**Definition 2.4.1.** *A tuple $(\pi, E)$ of a locally convex space $E$ and a group homomorphism*

$$\pi \colon \mathbb{G} \to \mathcal{L}(E)^{\times} : x \mapsto \pi(x)$$

*will be called a locally convex representation.*

*A locally convex representation $(\pi, E)$ is called*

**(i)** *strongly continuous iff $\pi \colon \mathbb{G} \to \mathcal{L}_s(E)$ is continuous,*

**(ii)** *locally equicontinuous iff for each compact $K \subset \mathbb{G}$ the set of operators $\pi(K)$ is equicontinuous and*

**(iii)** *admissible iff it is locally equicontinuous and strongly continuous.*

*Furthermore, if $\pi$ is a unitary representation, we will always denote the corresponding representation Hilbert space by $H_\pi$.*

Automatically, for an admissible representation $(\pi, E)$, the map

$$\pi \colon \mathbb{G} \to \mathcal{L}_c(E)$$

is continuous [36, Satz 1.4]. Moreover, if $(\pi, E)$ is a locally convex representation, $E$ is barrelled and $\pi \colon \mathbb{G} \to \mathcal{L}_s(E)$ is continuous, then $(\pi, E)$ is locally equicontinuous by the Banach-Steinhaus Theorem [66, Theorem 33.1].

### 2.4.1 Vectors associated to $\mathscr{C}(\mathbb{G})$-function spaces

If $(\pi, E)$ is a locally convex representation on a Lie group $\mathbb{G}$, a vector $e \in E$ is called differentiable (resp. smooth, resp. analytic) if the map

$$\mathbb{G} \to E\colon x \mapsto \pi(x)e$$

is differentiable (resp. smooth, resp. analytic). Analogously, a vector $e \in E$ is called ultradifferentiable if the above map is ultradifferentiable. The following definition introduces a generalization of this concept and equips the vectors of a given type with a topology.

**Definition 2.4.2.** *Suppose $(\pi, E)$ is a locally convex representation and $\mathscr{F}(\mathbb{G})$ a $\mathscr{C}(\mathbb{G})$-function space. We will define the subspace*

$$\mathscr{F}(\pi) := \{e \in E \mid \pi(-)e \in \mathscr{F}(\mathbb{G}; E) \simeq \mathscr{F}(\mathbb{G})\, \varepsilon\, E\}$$

*and equip it with the initial topology with respect to the map*

$$\Phi_\pi^{\mathscr{F}} : \mathscr{F}(\pi) \to \mathscr{F}(\mathbb{G})\, \varepsilon\, E \quad \textit{where} \quad (\Phi_\pi^{\mathscr{F}} e)(x, e') = \langle \pi(x)e, e' \rangle \ .$$

*If $F$ is a linear subspace of $E$ equipped with any locally convex topology such that $\pi(\mathbb{G})F = \{\pi(x)f \mid x \in \mathbb{G}\,,\ f \in F\} \subset F$, i.e. $F$ is $\pi$-invariant, then we will denote by*

$$\pi{\downarrow_F}\colon \mathbb{G} \to \mathcal{L}(F)\colon x \mapsto \pi(x) \restriction_F^F$$

*the subrepresentation of $\pi$ on $F$.*

By definition $\pi{\downarrow_F}$ is a locally convex representation for a $\pi$-invariant subspace $F \subset E$ equipped with some locally convex topology.

The left translation resp. right translation on $\mathscr{C}(\mathbb{G})$ defined by

$$\boldsymbol{L}(x)f(y) := f(x^{-1}y) \quad \text{resp.} \quad \boldsymbol{R}(x)f(y) := f(yx) \tag{2.4.35}$$

for all $f \in \mathscr{C}(\mathbb{G})$ and $x, y \in \mathbb{G}$ will be particularly important. If $\mu$ is some biinvariant Haar measure on $\mathbb{G}$, we will also use the left resp. right regular representation on $L^2(\mathbb{G}, \mu)$ defined by

$$\boldsymbol{L}_2(x)f(y) := f(x^{-1}y) \quad \text{resp.} \quad \boldsymbol{R}_2(x)f(y) := f(yx)$$

for all $f \in L^2(\mathbb{G}, \mu)$, $x \in \mathbb{G}$ and for $\mu$-almost all $y \in \mathbb{G}$.

We start by introducing a general theory for locally convex representations, we will later need for the discussion of spaces of ultradifferentiable vectors $\mathscr{E}_D^{[M]}(\pi)$.

**Lemma 2.4.3.** *Suppose $(\pi, E)$ and $(\sigma, F)$ are locally convex representations and suppose $\mathscr{F}(\mathbb{G})$, $\mathscr{G}(\mathbb{G})$ and $\mathscr{H}(\mathbb{G})$ are $\mathscr{C}(\mathbb{G})$-function spaces. Then the following holds true.*

- **(i)** *If $(\pi, E)$ is locally equicontinuous, then $\mathscr{C}(\pi)$ carries the subspace topology in $E$.*

  *$(\pi, E)$ is strongly continuous iff $\mathscr{C}(\pi) = E$ in the sense of vector spaces.*

  *$(\pi, E)$ is admissible iff $\mathscr{C}(\pi) = E$ in the sense of topological vector spaces.*

- **(ii)** *$\pi \,\varepsilon\, \mathrm{I}_F \colon x \mapsto \pi(x) \,\varepsilon\, \mathrm{I}_F$ defines a locally convex representation on $E \,\varepsilon\, F$. If $\pi$ is locally equicontinuous (resp. admissible), then so is $\pi \,\varepsilon\, \mathrm{I}_F$.*

- **(iii)** *If $\mathscr{F}(\mathbb{G})$ is $\boldsymbol{R}$-invariant, then $\pi{\downarrow}_{\mathscr{F}(\pi)}$ is a well-defined locally convex representation. If $\boldsymbol{R}{\downarrow}_{\mathscr{F}(\mathbb{G})}$ is locally equicontinuous (resp. admissible), then so is $\pi{\downarrow}_{\mathscr{F}(\pi)}$.*

- **(iv)** *If $T \in \mathcal{L}(E; F)$ such that $T\pi(x) = \sigma(x)T$ for all $x \in \mathbb{G}$, then $T \restriction_{\mathscr{F}(\pi)}^{\mathscr{F}(\sigma)} \in \mathcal{L}(\mathscr{F}(\pi); \mathscr{F}(\sigma))$ is well-defined.*

- **(v)** *Suppose $\mathscr{F}(\mathbb{G})$, $\mathscr{G}(\mathbb{G})$ and $\mathscr{H}(\mathbb{G})$ are $\boldsymbol{L}$-invariant and $T \in \mathcal{L}(\mathscr{F}(\mathbb{G}); \mathscr{G}(\mathbb{G}))$ such that $T\,\boldsymbol{L}(x) = \boldsymbol{L}(x)\,T$ for all $x \in \mathbb{G}$. Then there is a unique operator $\pi(T) \in \mathcal{L}(\mathscr{F}(\pi); \mathscr{G}(\pi))$ with $\Phi_\pi^{\mathscr{G}} \pi(T) = (T \,\varepsilon\, \mathrm{I}_E)\Phi_\pi^{\mathscr{F}}$. If $S \in \mathcal{L}(\mathscr{G}(\mathbb{G}); \mathscr{H}(\mathbb{G})$ is another $\boldsymbol{L}$-invariant operator, then $\pi(S)\pi(T) = \pi(ST)$.*

*Proof.* **(i)**: For arbitrary locally convex representations $(\pi, E)$ we have $\mathscr{C}(\pi) \subset E$ and the topology on $\mathscr{C}(\pi)$ is defined by the seminorms

$$p_K \colon e \mapsto \sup_{x \in K} p(\sigma(x)e) \qquad \text{for compact } K \subset \mathbb{G} \text{ and continuous seminorms } p \text{ on } E.$$

Hence $\mathscr{C}(\pi)$ is always equipped with a topology stricter than the subspace topology with respect to $E$. If $\pi$ is locally equicontinuous, then for each compact $K$ the seminorm $p_K$ is clearly continuous on $E$. So in this case $\mathscr{C}(\pi)$ carries the subspace topology.

We have $\mathscr{C}(\pi) = E$ as linear spaces if and only if the map $\pi(-)e \colon \mathbb{G} \to E$ is continuous for each $e \in E$.

Suppose $\mathscr{C}(\pi)$ and $E$ coincide as linear spaces. Then $\mathscr{C}(\pi) = E$ as topological vector spaces iff $p_K$ is a continuous seminorm on $E$ for each continuous seminorm $p$ on $E$ and each compact $K \subset \mathbb{G}$. This is true by definition iff $\{\pi(x) \mid x \in K\}$ is equicontinuous on $E$ for each compact $K \subset \mathbb{G}$.

**(ii)**: Since $(\pi(x) \,\varepsilon\, \mathrm{I}_F)(\pi(y) \,\varepsilon\, \mathrm{I}_F) = \pi(xy) \,\varepsilon\, \mathrm{I}_F$, $\pi \,\varepsilon\, \mathrm{I}_F$ is a locally convex representation. Let $K \subset \mathbb{G}$ be compact. We use the identification $E \,\varepsilon\, F \simeq \mathcal{L}_\varepsilon(F'_c; E)$. This way $\pi(x) \,\varepsilon\, \mathrm{I}_F$ acts on $\mathcal{L}_\varepsilon(F'_c; E)$, by $T \mapsto \pi(x) \circ T$. Hence $(\pi \,\varepsilon\, \mathrm{I}_F)(K)$ is equicontinuous if $\pi(K)$ is equicontinuous. By using a suitable continuous embedding $E \hookrightarrow \mathcal{L}_\varepsilon(F'_c; E)$, we can see that the converse is true as well. So $\pi \,\varepsilon\, \mathrm{I}_F$ is locally equicontinuous iff $\pi$ is locally equicontinuous.

Next we will move on to strong continuity. First of all, the map

$$\mathcal{L}_c(E) \to \mathcal{L}_\varepsilon(E'_c) \colon T \mapsto T'$$

is well-defined and injective. Since $T'$ maps equicontinuous sets to equicontinuous sets we have $T'' \in \mathcal{L}((E'_c)'_\varepsilon)$ for any $T \in \mathcal{L}(E)$. For any equicontinuous $V \subset E'$ there is some continuous seminorm $p$ on $E$ such that for any absolutely convex compact set $C \subset E$

$$\sup_{e' \in V} \sup_{e \in C} |\, \langle e, Te' \rangle \,| \leq \sup_{e \in C} p(Te) = \sup_{e' \in W} \sup_{e \in C} |\, \langle e, Te' \rangle \,| \,,$$

in which $W = \big(p^{-1}([0,1])\big)^\circ$ is equicontinuous. Thus the map $\mathcal{L}_c(E) \ni T \mapsto T' \in \mathcal{L}_\varepsilon(E'_c)$ is a linear homeomorphism onto its image. It is in fact a linear homeomorphism, since for each $S \in \mathcal{L}(E'_c)$ the dual operator fulfils $S' \in \mathcal{L}_c((E'_c)'_\varepsilon) \simeq \mathcal{L}_c(E)$ and consequently $T''$ is identified with $T$ for $T \in \mathcal{L}(E)$.

If $\pi$ is locally equicontinuous and strongly continuous, then $\pi(y) \xrightarrow{y \to x} \pi(x)$ in $\mathcal{L}_c(E)$, hence $\pi(y)' \xrightarrow{y \to x} \pi(x)'$ in $\mathcal{L}_\varepsilon(E'_c)$. It follows that for any equicontinuous sets $A \subset E'$, $B \subset F'$ and any $u \in E \,\varepsilon\, F$, we get

$$\sup_{(e',f') \in A \times B} |(\pi(x) - \pi(y)) \,\varepsilon\, \mathrm{I}_F \, u(e', f')| = \sup_{(e',f') \in A \times B} |u((\pi(x) - \pi(y))e', f')| \xrightarrow{x \to y} 0 \,,$$

since $u(-, B)$ is equicontinuous on $E'_c$. In other words $\pi \varepsilon \mathrm{I}_F \colon \mathbb{G} \to \mathcal{L}_s(E \varepsilon F)$ is continuous.

**(iii)**: We have $\Phi^{\mathscr{C}}_\pi \pi(x)e = (\boldsymbol{R}(x) \,\varepsilon\, \mathrm{I}_E)\Phi^{\mathscr{C}}_\pi e$ for all $e \in E$. If $\mathscr{F}(\mathbb{G})$ is $\boldsymbol{R}$-invariant, then $\Phi^{\mathscr{C}}_\pi(E) \cap \mathscr{F}(\mathbb{G}) \,\varepsilon\, E$ (seen as a subspace of $\mathscr{C}(\mathbb{G}) \,\varepsilon\, E$) is $\boldsymbol{R} \,\varepsilon\, \mathrm{I}_E$-invariant. Since $\Phi^{\mathscr{F}}_\pi$ is a linear homeomorphism onto its image in $\mathscr{F}(\mathbb{G})\,\varepsilon\, E$ and $\Phi_\pi^{\mathscr{F}}(\mathscr{F}(\pi)) = \Phi_\pi^{\mathscr{C}}(E) \cap \mathscr{F}(\mathbb{G})\,\varepsilon\, E$, this also implies that $\pi{\downarrow}_{\mathscr{F}(\pi)}$ is a locally convex representation.

If additionally the restriction $\boldsymbol{R}{\downarrow}_{\mathscr{F}(\mathbb{G})}$ is a locally equicontinuous (resp. admissible) representation, then $(\boldsymbol{R}{\downarrow}_{\mathscr{F}(\mathbb{G})})\,\varepsilon\,\mathrm{I}_E$ is a locally equicontinuous (resp. admissible) representation on $\mathscr{F}(\mathbb{G})\,\varepsilon\, E$ by **(ii)**. Analogous to the argumentation above, $\pi{\downarrow}_{\mathscr{F}(\pi)}$ is locally equicontinuous (resp. admissible), because $\Phi_\pi^{\mathscr{F}}\pi(x)e = (\boldsymbol{R}(x)\,\varepsilon\,\mathrm{I}_E)\Phi_\pi^{\mathscr{F}} e$ for all $e \in \mathscr{F}(\pi)$ and $\Phi_\pi^{\mathscr{F}}$ is a homeomorphism onto its image.

**(iv)**: This follows directly with $\mathrm{I}_{\mathscr{F}(\mathbb{G})}\,\varepsilon T \in \mathcal{L}(\mathscr{F}(\mathbb{G})\,\varepsilon\, E; \mathscr{F}(\mathbb{G})\,\varepsilon\, F)$ and $\Phi_\sigma^{\mathscr{F}} \circ T = (\mathrm{I}_{\mathscr{F}(\mathbb{G})}\,\varepsilon T) \circ \Phi_\pi^{\mathscr{F}}$.

**(v)**: For all $e \in \mathscr{F}(\pi)$, $e' \in E'$ and $x \in \mathbb{G}$ we have

$$\begin{aligned}
(T\,\varepsilon\,\mathrm{I}_E)\Phi_\pi^{\mathscr{F}} e(x,e') &= (\boldsymbol{L}(x^{-1})T\,\varepsilon\,\mathrm{I}_E)\Phi_\pi^{\mathscr{F}} e(1_{\mathbb{G}},e') \\
&= (T\,\boldsymbol{L}(x^{-1})\,\varepsilon\,\mathrm{I}_E)\Phi_\pi^{\mathscr{F}} e(1_{\mathbb{G}},e') \\
&= (T\,\varepsilon\,\mathrm{I}_E)(\mathrm{I}_{\mathscr{F}(\mathbb{G})}\,\varepsilon\,\pi(x))\Phi_\pi^{\mathscr{F}} e(1_{\mathbb{G}},e') \\
&= (\mathrm{I}_{\mathscr{F}(\mathbb{G})}\,\varepsilon\,\pi(x))(T\,\varepsilon\,\mathrm{I}_E)\Phi_\pi^{\mathscr{F}} e(1_{\mathbb{G}},e')\,.
\end{aligned}$$

Hence $T\,\varepsilon\,\mathrm{I}_E\,\Phi_\pi^{\mathscr{F}}\mathscr{F}(\pi) \subset \Phi_\pi^{\mathscr{G}}\mathscr{G}(\pi)$. Now $\pi(T) := (\Phi_\pi^{\mathscr{G}})^{-1}(T\,\varepsilon\,\mathrm{I}_E)\Phi_\pi^{\mathscr{F}}$ defines the unique operator $\pi(T) \in \mathcal{L}(\mathscr{F}(\mathbb{G});\mathscr{G}(\mathbb{G}))$ with $\Phi_\pi^{\mathscr{G}}\pi(T) = (T\,\varepsilon\,\mathrm{I}_E)\Phi_\pi^{\mathscr{F}}$, because $\Phi_\pi^{\mathscr{G}}$ and $\Phi_\pi^{\mathscr{F}}$ are linear homeomorphisms onto their respective images. In addition to that, $\pi(S)\pi(T) = \pi(ST)$ follows from $\Phi^{\mathscr{H}}\pi(S)\pi(T) = (ST)\,\varepsilon\,\mathrm{I}_E\,\Phi_\pi^{\mathscr{F}}$. □

The following statement is an extension of Lemma 2.4.3 **(iii)**.

**Lemma 2.4.4.** *Suppose $\mathscr{F}(\mathbb{G})$ and $\mathscr{G}(\mathbb{G})$ are $\mathscr{C}(\mathbb{G})$-function spaces such that $\mathscr{F}(\mathbb{G}) = \mathscr{F}(\boldsymbol{R}{\downarrow}_{\mathscr{G}(\mathbb{G})})$. Then $\mathscr{F}(\pi) = \mathscr{F}(\pi{\downarrow}_{\mathscr{G}(\pi)})$ for any locally convex representation $\pi$.*

*Proof.* Let us denote the representation space to $\pi$ by $E$. Furthermore, let $G := \mathscr{G}(\pi)$ and $\sigma := \pi{\downarrow}_G$. $\mathscr{F}(\sigma)$ is a continuously embedded subspace of $\mathscr{F}(\pi)$, since $G \subset E$ is continuously embedded. We will show that this embedding is in fact a linear homeomorphism. We define

$$\mathscr{H} := \Phi_\pi^{\mathscr{F}}\mathscr{F}(\pi)\,, \qquad \mathscr{K} := \Phi_\sigma^{\mathscr{F}}\mathscr{F}(\sigma) \qquad \text{and}$$
$$\mathscr{L} := \{v \in \mathscr{F}(\mathbb{G})\,\varepsilon\,\mathscr{G}(\mathbb{G})\,\varepsilon\, E \mid \exists_{e\in E}\forall_{x,y\in\mathbb{G}}\ v(x,y) = \pi(yx)e\}\,.$$

We equip $\mathscr{H}$, $\mathscr{K}$ and $\mathscr{L}$ with with the subspace topologies in $\mathscr{F}(\mathbb{G})\,\varepsilon\, E$, $\mathscr{F}(\mathbb{G})\,\varepsilon\, G$ and $\mathscr{F}(\mathbb{G})\,\varepsilon\,\mathscr{G}(\mathbb{G})\,\varepsilon\, E$, respectively. The space $\mathscr{L}$ is the image of $\mathscr{H}$ under $\Phi^{\mathscr{F}}_{\boldsymbol{R}\downarrow_{\mathscr{G}(\mathbb{G})}}\,\varepsilon\,\mathrm{I}_E$, yet $\mathscr{L}$ is also the image of $\mathscr{K}$ under $\mathrm{I}_{\mathscr{F}(\mathbb{G})}\,\varepsilon\Phi^{\mathscr{G}}_{\pi}$. Since $\Phi^{\mathscr{F}}_{\pi}$, $\Phi^{\mathscr{F}}_{\sigma}$, $\mathrm{I}_{\mathscr{F}(\mathbb{G})}\,\varepsilon\Phi^{\mathscr{G}}_{\pi}$ and $\Phi^{\mathscr{F}}_{\boldsymbol{R}\downarrow_{\mathscr{G}(\mathbb{G})}}\,\varepsilon\,\mathrm{I}_E$ are all homeomorphisms onto their respective images, we get the following scheme of linear homeomorphism

$$\begin{tikzcd}
\mathscr{F}(\pi) \arrow[r, "\Phi^{\mathscr{F}}_{\pi}"] & \mathscr{H} \arrow[dr, "\Phi^{\mathscr{F}}_{\boldsymbol{R}\downarrow_{\mathscr{G}(\mathbb{G})}}\varepsilon\mathrm{I}_E"] & \\
& & \mathscr{L} \\
\mathscr{F}(\sigma) \arrow[r, "\Phi^{\mathscr{F}}_{\sigma}"'] & \mathscr{K} \arrow[ur, "\mathrm{I}_{\mathscr{F}(\mathbb{G})}\,\varepsilon\Phi^{\mathscr{G}}_{\pi}"'] &
\end{tikzcd}$$

and thus $\mathscr{F}(\pi) = \mathscr{F}(\sigma)$ as topological vector spaces. $\square$

In order to use Lemma 2.4.4 and **(ii)** of Lemma 2.4.3 for our spaces of ultradifferentiable functions, we need the following statement.

**Lemma 2.4.5.** *Let $M$ be a weight sequence and $k \in N_0$.*

- **(i)** $\boldsymbol{R}\downarrow_{\mathscr{F}(\mathbb{G})}$ *resp.* $\boldsymbol{L}\downarrow_{\mathscr{F}(\mathbb{G})}$ *is an admissible representation for any basis $D \subset \mathfrak{g}_{\mathrm{R}}$ resp. $D \subset \mathfrak{g}_{\mathrm{L}}$ and any choice*

$$\mathscr{F}(\mathbb{G}) \in \{\mathscr{C}^k(\mathbb{G}), \mathscr{E}(\mathbb{G}), \mathscr{E}^M_D(\mathbb{G}), \mathscr{E}^{(M)}_D, \mathscr{E}^{\{M\}}_{D,\mathrm{proj}}(\mathbb{G})\}\,.$$

- **(ii)** $\mathscr{F}(\mathbb{G}) = \mathscr{F}(\boldsymbol{R}\downarrow_{\mathscr{E}(\mathbb{G})})$ *for* $\mathscr{F}(\mathbb{G}) \in \{\mathscr{E}(\mathbb{G}), \mathscr{E}^{(M)}_D(\mathbb{G}), \mathscr{E}^{\{M\}}_{D,\mathrm{proj}}(\mathbb{G})\}$*, in which we can choose any basis $D \subset \mathfrak{g}_{\mathrm{L}}$.*

*Proof.* **(i)**: The proofs for the right invariant case and the left invariant case work exactly the same. Hence, it is enough to only prove the right invariant case. Suppose $\mathscr{F}(\mathbb{G})$ is $\boldsymbol{R}$-invariant and there exists a set of $\boldsymbol{R}$-invariant maps $\mathcal{P} \ni P\colon \mathscr{F}(\mathbb{G}) \to \mathscr{C}(\mathbb{G})$ and a grouping $\mathcal{Q} \subset 2^{\mathcal{P}}$ such that for each $f \in \mathscr{F}(\mathbb{G})$, $Q \in \mathcal{Q}$ the set $\{Pf \mid P \in Q\}$ is relatively compact in $\mathscr{C}(\mathbb{G})$ (i.e. bounded and equicontinuous) and the topology of $\mathscr{F}(\mathbb{G})$ is induced by the seminorms

$$f \mapsto \sup_{P\in Q}\sup_{x\in K}|Pf(x)| \qquad \text{for compact } K \subset \mathbb{G} \text{ and } Q \in \mathcal{Q}\,.$$

This structure implies at once that $\boldsymbol{R}{\downarrow_{\mathscr{F}(\mathbb{G})}}$ is admissible.

Indeed most of the considered spaces are of this type. For $\mathscr{C}^k(\mathbb{G})$ we use $\mathcal{P} = \mathrm{Diff}^k_{\mathrm{R}}(\mathbb{G})$, for $\mathscr{E}(\mathbb{G})$ we use $\mathcal{P} = \mathrm{Diff}_{\mathrm{R}}(\mathbb{G})$. In both cases we may take $\mathcal{Q} = \{\{P\} \mid P \in \mathcal{P}\}$.

For $\mathscr{E}^{(M)}_D(\mathbb{G})$ we use

$$\mathcal{P} = \cup_{h>0}\mathcal{P}_h := \cup_{h>0}\{(hD)^a/(M_{|a|}|a|!) \mid a \in \mathcal{S}_{\dim \mathbb{G}}\}$$

with $\mathcal{Q} := \{\mathcal{P}_h \mid h > 0\}$ and for $\mathscr{E}^{\{M\}}_{D,\mathrm{proj}}(\mathbb{G})$ we use

$$\mathcal{P} = \cup_{h\in\Lambda}\mathcal{P}_h := \cup_{h\in\Lambda}\{D^a/(h_{|a|}M_{|a|}|a|!) \mid a \in \mathcal{S}_{\dim \mathbb{G}}\}$$

with $\mathcal{Q} := \{\mathcal{P}_h \mid h \in \Lambda\}$.

In the case of $\mathscr{E}^M_D(\mathbb{G})$ we may use $\mathcal{P} = \{\frac{D^a}{M_{|a|}|a|!} \mid a \in \mathcal{S}_{\dim \mathbb{G}}\}$ and $\mathcal{Q} = \{\mathcal{P}\}$.

**(ii)**: The case $\mathscr{F}(\mathbb{G}) = \mathscr{E}(\mathbb{G})$ is simpler.

Naturally for $f \in \mathscr{E}(\mathbb{G})$ the map $[(x, y) \mapsto \boldsymbol{R}(x)f(y) = f(yx)]$ is in $\mathscr{E}(\mathbb{G} \times \mathbb{G})$. By Lemma 2.1.9 we have $\mathscr{E}(\mathbb{G}\times\mathbb{G}) = \mathscr{E}(\mathbb{G}; \mathscr{E}(\mathbb{G}))$ as topological vector spaces. This already implies $\mathscr{E}(\mathbb{G}) = \mathscr{E}(\boldsymbol{R}{\downarrow_{\mathscr{E}(\mathbb{G})}})$ as vector spaces. We only need to show that the topologies derived from $\mathscr{E}(\mathbb{G})$ and $\mathscr{E}(\boldsymbol{R}{\downarrow_{\mathscr{E}(\mathbb{G})}})$ coincide. The topology on $\mathscr{E}(\boldsymbol{R}{\downarrow_{\mathscr{E}(\mathbb{G})}})$ is the initial topology with respect to $\mathscr{E}(\mathbb{G}) \ni f \mapsto P\,\boldsymbol{R}(P')f \in \mathscr{C}(\mathbb{G})$ for $P, P' \in \mathrm{Diff}_{\mathrm{L}}(\mathbb{G})$. But $\boldsymbol{R}(P') = P'$, so $\mathscr{E}(\mathbb{G}) = \mathscr{E}(\boldsymbol{R}{\downarrow_{\mathscr{E}(\mathbb{G})}})$ as topological vector spaces.

Now we we check the case $\mathscr{F}(\mathbb{G}) \in \{\mathscr{E}^{(M)}_D(\mathbb{G}), \mathscr{E}^{\{M\}}_{D,\mathrm{proj}}(\mathbb{G})\}$. For any $P \in \mathrm{Diff}_{\mathrm{R}}(\mathbb{G})$ there is a unique $\widetilde{P} \in \mathrm{Diff}_{\mathrm{L}}(\mathbb{G})$ such that

$$P_x f(yx) = \widetilde{P}_y f(yx) \qquad \text{for all } f \in \mathscr{E}(\mathbb{G})\,.$$

Since $\widetilde{P}$ maps $\mathscr{F}(\mathbb{G})$ continuously into itself and since $\mathscr{E}(\mathbb{G}) = \mathscr{E}(\boldsymbol{R}{\downarrow_{\mathscr{E}(\mathbb{G})}})$, we get $\mathscr{F}(\mathbb{G}) = \mathscr{F}(\boldsymbol{R}{\downarrow_{\mathscr{E}(\mathbb{G})}})$ due to the representation of $\mathscr{F}(\mathbb{G}; \mathscr{E}(\mathbb{G}))$ via the projective limits from Proposition 2.2.18 **(ii)**.

□

We have the following general denseness properties for spaces of ultradifferentiable vectors.

**Lemma 2.4.6.** *Suppose $M$ is a weight sequence, $D \subset \mathcal{V}_{\mathrm{a}}(\mathbb{G})$ is a frame and $(\pi, E)$ is an admissible representation of a Lie group $\mathbb{G}$ in a Banach space $E$. Then the following holds.*

**(i)** *$\mathscr{E}^{\{M\}}(\pi)$ is dense in $E$.*

**(ii)** *If $M$ fulfils **(AF)**, then $\mathscr{E}_D^{(M)}(\pi)$ is dense in $E$.*

**(iii)** *Suppose $\mathscr{F}(\mathbb{G})$ is* any *$\boldsymbol{R}$-invariant $\mathscr{C}(\mathbb{G})$-function space. If $\pi$ is a unitary irreducible representation, then $\mathscr{F}(\pi)$ is either dense in $H_\pi$ or $\mathscr{F}(\pi) = \{0\}$.*

*Proof.* **(i):** Since $M$ is a weight sequence, we have $\mathscr{E}^{\{\mathbb{1}\}}(\mathbb{G}) \subset \mathscr{E}^{\{M\}}(\mathbb{G})$, thus also $\mathscr{E}^{\{\mathbb{1}\}}(\pi) \subset \mathscr{E}^{\{M\}}(\pi)$. By Corollary 2.2.19, $\mathscr{E}^{\{\mathbb{1}\}}(\mathbb{G})$ is exactly the space of analytic vectors to $\pi$. Due to [55, Theorem 4], the space of analytic vectors $\mathscr{E}^{\{\mathbb{1}\}}(\pi)$ is dense in $E$.

**(ii):** The property **(AF)** ensures that $\mathscr{E}^{\{\mathbb{1}\}}(\mathbb{G}) \subset \mathscr{E}_D^{(M)}(\mathbb{G})$, thus $\mathscr{E}^{\{\mathbb{1}\}}(\pi) \subset \mathscr{E}_D^{(M)}(\pi)$. Hence $\mathscr{E}_D^{(M)}(\pi)$ is dense in $E$.

**(iii):** $\mathscr{F}(\pi)$ is a $\pi$-invariant subspace of $E$ by Lemma 2.4.3. Now the rest follows directly from the irreducibility of $\pi$. □

In the following we will also discuss the denseness of ultradifferentiable vectors in different spaces of differentiable vectors.

**Lemma 2.4.7.** *Suppose $\mathbb{G}$ is a Lie group, $(\pi, E)$ is a locally convex representation of $\mathbb{G}$ on a Banach space $E$ and $M$ is a weight sequence. Furthermore, suppose $\{\mathbb{1}\} \subset [M]$ and let $D \subset \mathfrak{g}_{\mathrm{L}}$ be a frame. Then*

$$\mathscr{E}_D^{[M]}(\pi) \quad \text{is dense in} \quad \mathscr{E}(\pi)\,,\ \mathscr{C}^k(\pi)\,,\ k \in \mathbb{N}\,.$$

*Proof.* First of all, with the requirements of the lemma, $\mathscr{E}_D^{[M]}(\sigma)$ is dense in $F$ by Lemma 2.4.6 for any admissible representation $(\sigma, F)$ on a Banach space $F$. The space $\mathscr{C}^k(\pi)$ is Fréchet, since it can be identified with a closed subspace of $\mathscr{C}^k(\mathbb{G}; E)$. Take any compact neighbourhood $U$ of the unit $1_{\mathbb{G}}$ in $\mathbb{G}$. Then for any compact $K \subset \mathbb{G}$ there is some $n \in \mathbb{G}$ and $x_1, \ldots, x_n \in \mathbb{G}$ with $K \subset \bigcup_j x_j U$. If we put $C_K := \max_j \|\pi(x_j)\|_{\mathcal{L}(E)}$, then

$$\max_{|\alpha| \le k} \sup_{x \in K} \|\pi(x)\pi(D^\alpha)e\|_E \le C_k \max_{|\alpha| \le k} \sup_{x \in U} \|\pi(x)\pi(D^\alpha)e\|_E$$

for any $e \in \mathscr{C}^k(\pi)$ any any frame $D \subset \mathfrak{g}_{\mathrm{L}}$. Hence $\mathscr{C}^k(\pi)$ is a Banach space. Thus $\mathscr{E}_D^{[M]}(\pi{\downarrow_{\mathscr{C}^k(\pi)}})$ is dense in $\mathscr{C}^k(\pi)$ for any $k \in \mathbb{N}_0$, since $\pi{\downarrow_{\mathscr{C}^k(\pi)}}$ is admissible by Lemma 2.4.3.

Now we will show that $\mathscr{E}_D^{[M]}(\pi) = \mathscr{E}_D^{[M]}(\pi{\downarrow_{\mathscr{C}^k(\pi)}})$ in the sense of linear spaces.

By Lemma 2.4.5 we have $\mathscr{E}_{D,\mathrm{proj}}^{[M]}(\boldsymbol{R}) = \mathscr{E}_{D,\mathrm{proj}}^{[M]}(\boldsymbol{R}{\downarrow_{\mathscr{E}(\mathbb{G})}})$, thus $\mathscr{E}_{D,\mathrm{proj}}^{[M]}(\pi) = \mathscr{E}_{D,\mathrm{proj}}^{[M]}(\pi{\downarrow_{\mathscr{E}(\pi)}})$ for *any* weight sequence $M$. Due to the continuous embeddings $\mathscr{E}(\pi) \hookrightarrow \mathscr{C}^k(\pi) \hookrightarrow E$ we also have $\mathscr{E}_{D,\mathrm{proj}}^{[M]}(\pi) = \mathscr{E}_{D,\mathrm{proj}}^{[M]}(\pi{\downarrow_{\mathscr{C}^k(\pi)}})$.

Hence, $\mathscr{E}_{D,\mathrm{proj}}^{[M]}(\pi) = \mathscr{E}_{D,\mathrm{proj}}^{[M]}(\pi{\downarrow_{\mathscr{C}^k(\pi)}})$ is dense in $\mathscr{C}^k(\pi)$ for any $k \in \mathbb{N}_0$. Thus $\mathscr{E}_{D,\mathrm{proj}}^{[M]}(\pi)$ is dense in $\mathscr{E}(\pi) = \varprojlim_{k\to\infty} \mathscr{C}^k(\pi)$. We complete the proof by using $\mathscr{E}_{D,\mathrm{proj}}^{[M]}(\pi) = \mathscr{E}_D^{[M]}(\pi)$ in the sense of vector spaces.

□

### 2.4.2 Examples of spaces of smooth and ultradifferentiable vectors

We will now discuss a few examples of spaces of smooth resp. differentiable vectors. To be precise, we will take a look at a few common function spaces and show that they may be seen as certain spaces of differentiable vectors. This perspective will simplify proofs in Chapter 4. If $(\pi, E)$ is a locally equicontinuous and locally convex representation, then the description of the topologies associated to differentiable and smooth vectors is especially simple. By using $D_x^a \pi(x)e = \pi(x)\pi(D^a)e$ for $e \in \mathscr{C}^{|a|}(\pi)$ and $a \in \mathcal{S}_{\dim \mathbb{G}}$, we get that the topology on $\mathscr{C}^k(\pi)$ resp. $\mathscr{E}(\pi)$ is induced by the seminorms

$$e \mapsto \pi(P)e\,, \quad \text{for continouous seminorms } p \text{ on } E \text{ and } P \in \mathrm{Diff}_{\mathrm{L}}^k(\mathbb{G}) \text{ resp. } P \in \mathrm{Diff}_{\mathrm{L}}\,.$$

Also, we have a rather easy characterization of ultradifferentiable vectors to admissible representations.

**Lemma 2.4.8.** *Suppose $(E, \pi)$ is a complete locally equicontinuous representation on a Lie group $\mathbb{G}$ with Lie algebra $\mathfrak{g}$. If $D \subset \mathfrak{g}_{\mathrm{L}}$ is a frame, $M$ a weight sequence and $\mathcal{P}$ the* *set of continuous seminorms on $E$, then*

$$\mathscr{E}_D^M(\pi) = \left\{ e \in \mathscr{E}(\pi) \;\middle|\; \forall_{p\in\mathcal{P}}\colon \lim_{\substack{|a|\to\infty\\ a\in\mathcal{S}_N}} \frac{p(\pi(D^a)e)}{M_{|a|}\,|a|!} = 0 \right\}$$
$$\textit{with topology defined by} \quad f \mapsto \sup_{a\in\mathcal{S}_N} \frac{p(\pi(D^a)e)}{M_{|a|}\,|a|!}\,,$$

*in which $p$ runs through all continuous seminorms in $\mathcal{P}$.*

*Proof.* By Proposition 2.2.17, we have

$$\mathscr{E}_D^M(\mathbb{G};E) = \left\{ f \in \mathscr{E}(\mathbb{G};E) \;\middle|\; \forall_{K \overset{\mathrm{rc}}{\subset} \mathbb{G}} \forall_{p\in\mathcal{P}}\colon \lim_{\substack{|a|\to\infty\\ a\in\mathcal{S}_N}} \sup_{x\in K} \frac{p(D^a f(x))}{M_{|a|}\,|a|!} = 0 \right\}$$
$$\text{with topology defined by} \quad f \mapsto \sup_{a\in\mathcal{S}_N} \sup_{x\in K} \frac{p(D^a f(x))}{M_{|a|}\,|a|!}\,,$$

in which $K$ runs through the compact subsets of $\mathbb{G}$ and $p$ runs through all continuous seminorms in $\mathcal{P}$. Since $(\pi, E)$ is locally equicontinuous, $\{\pi(x) \mid x \in K\}$ is equicontinuous. Since $e \in \mathscr{E}(\pi)$ and $D^a f(x) = \pi(x)\pi(D^a)e$ for $f = \pi(-)e \in \mathscr{E}(\mathbb{G};E)$, we get

$$\mathscr{E}_D^M(\pi) := \left\{ e \in \mathscr{E}(\pi) \;\middle|\; \forall_{p\in\mathcal{P}}\colon \lim_{\substack{|a|\to\infty\\ a\in\mathcal{S}_N}} \frac{p(\pi(D^a)e)}{M_{|a|}\,|a|!} = 0 \right\}$$
$$\text{with topology defined by} \quad e \mapsto \sup_{a\in\mathcal{S}_N} \frac{p(\pi(D^a)e)}{M_{|a|}\,|a|!}\,,$$

in which $p$ runs through all $p \in \mathcal{P}$.

□

Before considering spaces of ultradifferentiable vectors of Roumieu or Beurling type, we will show that projective limits of $\mathscr{C}(\mathbb{G})$-function spaces play nicely with our construction of generalized differentiable vectors.

**Lemma 2.4.9.** *Let $(E, \pi)$ be a locally convex representation of a Lie group $\mathbb{G}$ and let $(A, \leq)$ be a directed set. If for each $\alpha \in A$ the spaces $\mathscr{F}(\mathbb{G})$, $\mathscr{F}_\alpha(\mathbb{G})$ are $\mathscr{C}(\mathbb{G})$-function spaces with $\mathscr{F}(\mathbb{G}) = \varprojlim_{\alpha\in A} \mathscr{F}_\alpha(\mathbb{G})$, then*

$$\mathscr{F}(\pi) = \bigcap_{\alpha\in A} \mathscr{F}_\alpha(\pi)$$

*and $\mathscr{F}(\pi)$ is equipped with the initial topology with respect to the maps $\mathscr{F}(\pi) \xrightarrow{\mathrm{I}} \mathscr{F}_\alpha(\pi)$. Since this implies that $\mathscr{F}(\pi)$ is linearly homeomorphic to $\varprojlim_{\alpha\in A}(\mathscr{F}_\alpha(\pi), \mathrm{I})$, we will also write*

$$\varprojlim_{\alpha\in A} \mathscr{F}_\alpha(\pi) := \mathscr{F}(\pi)\,.$$

*Proof.* Let us define the locally convex space $\varprojlim_{\alpha\in A} \mathscr{F}_\alpha(\pi) := \bigcap_{\alpha\in A} \mathscr{F}_\alpha(\pi)$ equipped with the initial topology with respect to the maps $\varprojlim_{\alpha\in A} \mathscr{F}_\alpha(\pi) \xrightarrow{\mathrm{I}} \mathscr{F}_\beta(\pi)$.

By Lemma 1.2.5 we have

$$e \in \mathscr{F}(\pi) \;\Leftrightarrow\; \pi(-)e \in \bigcap_{\alpha\in A} \mathscr{F}_\alpha(\mathbb{G}; E) \;\Leftrightarrow\; e \in \bigcap_{\alpha\in A} \mathscr{F}_\alpha(\pi) = \varprojlim_{\alpha\in A} \mathscr{F}_\alpha(\pi)\,.$$

Furthermore a net $(e_j)_j$ converges in $\mathscr{F}(\pi)$, iff there is some $e \in E$ such that the net $(\pi(-)e_j)_j$ converges to $\pi(-)e$ in every $\mathscr{F}_\alpha(\mathbb{G}; E)$. By definition, this is equivalent to the existence of $e \in E$ with $\lim_j e_j = e$ in every $\mathscr{F}_\alpha(\pi)$. Hence $\varprojlim_{\alpha\in A} \mathscr{F}_\alpha(\pi) = \mathscr{F}(\pi)$ as topological vector spaces. ☐

The above Lemma also gives a concrete description of the spaces $\mathscr{E}_D^{[M]}(\pi)$ via the following limit description.

**Lemma 2.4.10.** *Suppose $M$ is a weight sequence, $D \subset \mathfrak{g}_{\mathrm{L}}$ a basis and $(\pi, E)$ a locally equicontinuous and locally convex representation on $\mathbb{G}$. Then the following holds.*

- **(i)** $\mathscr{E}^{(M)}(\pi) = \varprojlim_{h>0} \mathscr{E}_{hD}^{M}(\pi) = \varprojlim_{L\in\mathcal{W}_{(M)}} \mathscr{E}_D^{L}(\pi)$
- **(ii)** $\mathscr{E}_{D,\mathrm{proj}}^{\{M\}}(\pi) = \varprojlim_{h\in\Lambda} \mathscr{E}_D^{hM}(\pi) = \varprojlim_{L\in\mathcal{W}_{\{M\}}} \mathscr{E}_D^{L}(\pi)$.
- **(iii)** $\mathscr{E}_D^{\{M\}}(\pi) = \mathscr{E}_{D,\mathrm{proj}}^{\{M\}}(\pi)$ *in the sense of vector spaces. If $M$ fulfils **(nQA)**, then this identity holds in the sense of topological vector spaces.*

*Proof.* This is just a consequence of Proposition 2.2.18. ☐

Our first examples are the $L^2$-Sobolev spaces.

**Definition 2.4.11.** *Let $\mathbb{G}$ be an unimodular Lie group with biinvariant Haar measure $\mu$. We define $H_{\mathrm{L}}^k(\mathbb{G})$, for $k \in \mathbb{N}_0 \cup \{\infty\}$, to be the locally convex space of $k$-times weakly* *differentiable* $f \in L^2(\mathbb{G},\mu)$ *such that* $Pf \in L^2(\mathbb{G},\mu)$ *for all* $P \in \mathrm{Diff}^k_{\mathrm{L}}(\mathbb{G})$. *The topology on* $H^k_{\mathrm{L}}(\mathbb{G})$ *is defined by the seminorms*

$$f \mapsto \|Pf\|_{L^2(\mathbb{G},\mu)} \quad \textit{for} \quad P \in \mathrm{Diff}^k_{\mathrm{L}}(\mathbb{G})\,.$$

*The spaces* $H^k_{\mathrm{R}}(\mathbb{G})$, *for* $k \in \mathbb{N}_0 \cup \{\infty\}$, *are defined analogously to the above. We just exchange* $\mathrm{Diff}^k_{\mathrm{L}}(\mathbb{G})$ *with* $\mathrm{Diff}^k_{\mathrm{R}}(\mathbb{G})$.

It is clear that $H^\infty_{\mathrm{L/R}}(\mathbb{G})$ is a space of smooth functions due to the Sobolev embeddings.

For compact Lie groups $\mathbb{G}$ it is not necessary to distinguish between left- or right-invariant Sobolev spaces. In fact, in this case any differential operator $P$ on $\mathbb{G}$ can be written as a linear combination

$$P = \sum_\alpha c_\alpha D^\alpha = \sum_\beta d_\beta L^\beta$$

for frames $D \subset \mathfrak{g}_{\mathrm{L}}$, $L \subset \mathfrak{g}_{\mathrm{R}}$ and bounded smooth functions $c_\alpha$, $d_\beta$. Hence

$$H^k_{\mathrm{L}}(\mathbb{G}) = H^k_{\mathrm{R}}(\mathbb{G}) =: H^k(\mathbb{G})\,.$$

Furthermore, by using Sobolev embeddings, we get $H^\infty(\mathbb{G}) = \mathscr{E}(\mathbb{G})$ as topological vector spaces for all compact Lie groups $\mathbb{G}$.

Of course, the identity $H^k_{\mathrm{L}}(\mathbb{G}) = H^k_{\mathrm{R}}(\mathbb{G}) =: H^k(\mathbb{G})$ holds for abelian Lie groups as well.

**Lemma 2.4.12.** *Let* $\mathbb{G}$ *be an unimodular Lie group with biinvariant Haar measure* $\mu$ *and let* $k \in \mathbb{N}_0$. *Then*

$$\mathscr{C}^k(\boldsymbol{L}_2) = H^k_{\mathrm{R}}(\mathbb{G}) \quad \textit{and} \quad \mathscr{C}^k(\boldsymbol{R}_2) = H^k_{\mathrm{L}}(\mathbb{G})\,,$$

*in which* $\boldsymbol{R}_2$ *resp.* $\boldsymbol{L}_2$ *denotes the right resp. left regular representation of* $\mathbb{G}$ *on* $L^2(\mathbb{G},\mu)$. *We also have*

$$\mathscr{E}(\boldsymbol{L}_2) = H^\infty_{\mathrm{R}}(\mathbb{G}) \quad \textit{and} \quad \mathscr{E}(\boldsymbol{R}_2) = H^\infty_{\mathrm{L}}(\mathbb{G})\,.$$

*Proof.* It is clear that $P \mapsto \boldsymbol{L}_2(P)$ defines a bijection between $\mathrm{Diff}^k_{\mathrm{L}}(\mathbb{G})$ and $\mathrm{Diff}^k_{\mathrm{R}}(\mathbb{G})$. Thus $\mathscr{C}^k(\boldsymbol{L}_2)$ is a closed subspace in $H^k_{\mathrm{R}}(\mathbb{G})$. Since $\mathscr{D}(\mathbb{G}) \subset \mathscr{E}(\boldsymbol{L}_2)$ is dense in $H^k(\mathbb{G})$, we get $\mathscr{C}^k(\boldsymbol{L}_2) = H^k_{\mathrm{R}}(\mathbb{G})$. Since this holds for all $k \in \mathbb{N}_0$, we also have $\mathscr{E}(\boldsymbol{L}_2) = H^\infty_{\mathrm{R}}(\mathbb{G})$.

The identities $\mathscr{C}^k(\boldsymbol{R}_2) = H^k_{\mathrm{L}}(\mathbb{G})$ and $\mathscr{E}(\boldsymbol{R}_2) = H^\infty_{\mathrm{L}}(\mathbb{G})$ can be proven analogously. $\square$

Now we will work on a similar statement for ultradifferentiable vectors of $\boldsymbol{L}_2$ and $\boldsymbol{R}_2$.

**Definition 2.4.13.** *Suppose $\mathbb{G}$ is a Lie group, $(\pi, E)$ is a locally convex representation, $(A, \alpha)$ is a directed set and $\mathscr{F}_\alpha(\mathbb{G})$ a collection of $\mathscr{C}(\mathbb{G})$-function spaces with continuous embeddings $\mathscr{F}_\alpha(\mathbb{G}) \xrightarrow{\mathrm{I}} \mathscr{F}_\beta(\mathbb{G})$ for $\alpha \leq \beta$. Then we define the linear space*

$$\varinjlim_{\alpha \in A} \mathscr{F}_\alpha(\pi) = \bigcup_{\alpha \in A} \mathscr{F}_\alpha(\pi)$$

*and equip it with the finest locally convex topology such that all the embeddings*

$$\mathscr{F}_\beta(\pi) \xrightarrow{\mathrm{I}} \varinjlim_{\alpha \in A} \mathscr{F}_\alpha(\pi)$$

*are continuous.*

Note that $\varinjlim_{\alpha \in A} \mathscr{F}_\alpha(\pi)$ is continuously embedded into $\mathscr{C}(\pi)$ and is linearly homeomorphic to the inductive limit $\varinjlim_{\alpha \in A} (\mathscr{F}_\alpha(\pi), \mathrm{I})$. In general, $\varinjlim_{\alpha \in A} \mathscr{F}_\alpha(\pi)$ may be a different space than $\mathscr{F}(\pi)$ for $\mathscr{F}(\mathbb{G}) = \varinjlim_{\alpha \in A} \mathscr{F}_\alpha(\mathbb{G})$.

**Definition 2.4.14.** *Suppose $\mathbb{G}$ is a Lie group with Haar measure $\mu$. For any frame $D \subset \mathcal{V}_{\mathrm{a}}(\mathbb{G})$ and any weight sequence $M$, we define the spaces*

$$H_D^M(\mathbb{G}) := \left\{ f \in \mathscr{E}(\mathbb{G}) \;\middle|\; \lim_{\substack{|a| \to \infty \\ a \in \mathcal{S}_N}} \frac{\|D^a f\|_{L^2(\mathbb{G}, \mu)}}{M_{|a|}\, |a|!} = 0 \right\} \quad \textit{with topology defined by}$$

$$f \mapsto \|f\|_{2,D,M} := \sup_{a \in \mathcal{S}_N} \frac{\|D^a f\|_{L^2(\mathbb{G}, \mu)}}{M_{|a|}\, |a|!}\,.$$

*Similarly, we will define the Sobolev spaces*

$$H_D^{(M)}(\mathbb{G}) = \varprojlim_{h>0} H_{hD}^M(\mathbb{G})\,, \qquad H_D^{\{M\}}(\mathbb{G}) := \varinjlim_{h>0} H_{hD}^M(\mathbb{G})\,,$$

$$H_{D,\mathrm{proj}}^{[M]}(\mathbb{G}) = \varprojlim_{L \in \mathcal{W}_{[M]}} H_D^L(\mathbb{G})\,.$$

As in the differentiable resp. smooth case, we get the following characterization of these Sobolev spaces.

**Lemma 2.4.15.** *Let $\mathbb{G}$ be an unimodular Lie group with biinvariant Haar measure $\mu$ and Lie algebra $\mathfrak{g}$. Let $M$ be a weight sequence and $D \subset \mathfrak{g}_{\mathrm{L}}$ be a frame. We define the frame of right invariant vector fields $\tilde{D} = \boldsymbol{L}(D)$ and denote by $\boldsymbol{R}_2$ resp. $\boldsymbol{L}_2$ the right resp. left regular representation of $\mathbb{G}$ on $L^2(\mathbb{G}, \mu)$. Then the following holds.*

**(i)** $\mathscr{E}^M_D(\boldsymbol{R}_2) = H^M_D(\mathbb{G})$ *and* $\mathscr{E}^M_D(\boldsymbol{L}_2) = H^M_{\tilde{D}}(\mathbb{G})$.

**(ii)** $\mathscr{E}^{[M]}_{D,\mathrm{proj}}(\boldsymbol{R}_2) = H^{[M]}_{D,\mathrm{proj}}(\mathbb{G})$ *and* $\mathscr{E}^{[M]}_{D,\mathrm{proj}}(\boldsymbol{L}_2) = H^{[M]}_{\tilde{D},\mathrm{proj}}(\mathbb{G})$.

**(iii)** *For any frame* $E \subset \mathcal{V}_\mathrm{a}(\mathbb{G})$ *we have* $H^{\{M\}}_{E,\mathrm{proj}}(\mathbb{G}) = H^{\{M\}}_E(\mathbb{G})$ *in the sense of vector spaces. Moreover, the bounded sets on both sides coincide.*

*Proof.* **(i):** It is evident that $H^M_D(\mathbb{G}) \subset H^\infty_\mathrm{L}(\mathbb{G}) = \mathscr{E}(\boldsymbol{R}_2)$ and $H^M_{\tilde{D}}(\mathbb{G}) \subset H^\infty_\mathrm{R}(\mathbb{G}) = \mathscr{E}(\boldsymbol{L}_2)$. We also have $D = \boldsymbol{R}_2(D)$ and $\tilde{D} = \boldsymbol{L}_2(D)$. Combining the above with Lemma 2.4.8 results in

$$\mathscr{E}^M_D(\boldsymbol{R}_2) = H^M_D(\mathbb{G}) \quad \text{and} \quad \mathscr{E}^M_D(\boldsymbol{L}_2) = H^M_{\tilde{D}}(\mathbb{G})\,.$$

**(ii):** Here we just need to use Lemma 2.4.10.

**(iii):** The space $H^{\{M\}}_E(\mathbb{G})$ is continuously embedded into $H^{\{M\}}_{E,\mathrm{proj}}(\mathbb{G})$. So each set that is bounded in $H^{\{M\}}_E(\mathbb{G})$ is bounded in $H^{\{M\}}_{E,\mathrm{proj}}(\mathbb{G})$. Conversely, each bounded set $B \subset H^{\{M\}}_{E,\mathrm{proj}}(\mathbb{G})$ is bounded in $H^M_{hE}(\mathbb{G})$ for some $h > 0$ by Lemma 2.2.12. Thus $B$ is bounded in $H^{\{M\}}_E(\mathbb{G})$ as well. $\square$

Note that either of the spaces $\mathscr{E}^{[M]}_D(\pi)$ or $\mathscr{E}^{[M]}_{D,\mathrm{proj}}(\pi)$ are the same space for *all* frames $D \subset \mathfrak{g}_\mathrm{L}$ (resp. for *all* frames $D \subset \mathfrak{g}_\mathrm{R}$). This is due to Lemma 2.2.5 and the fact that any pair of left resp. right invariant frames $E, D$ are connected by a constant matrix $A$ such that $E = AD$. Thus for each $[M]$ there is exactly one right invariant and one left invariant version of each of the Sobolev spaces $H^{[M]}_{D,\mathrm{proj}}(\mathbb{G})$ and $H^{[M]}_D(\mathbb{G})$.

We already saw that for compact Lie groups the Sobolev spaces $H^\infty_{\mathrm{R/L}}(\mathbb{G})$ are just the space of smooth functions on $\mathbb{G}$. We also have the analogous relation for the ultradifferentiable Sobolev spaces.

**Lemma 2.4.16.** *Suppose* $\mathbb{G}$ *is a compact Lie group,* $M$ *is a weight sequence and* $D$ *is a frame in* $\mathcal{V}_\mathrm{a}(\mathbb{G})$. *Then*

$$\mathscr{E}^{[M]}_D(\mathbb{G}) = H^{[M]}_D(\mathbb{G}) \quad \textit{and} \quad \mathscr{E}^{[M]}_{D,\mathrm{proj}}(\mathbb{G}) = H^{[M]}_{D,\mathrm{proj}}(\mathbb{G})\,.$$

*Proof.* It is clear that

$$\mathscr{E}^{(M)}_D(\mathbb{G}) \xrightarrow{\mathrm{I}} H^{(M)}_D(\mathbb{G}) \quad \text{and} \quad \mathscr{E}^{\{M\}}_D(\mathbb{G}) \xrightarrow{\mathrm{I}} H^{\{M\}}_D(\mathbb{G})$$

are injective and continuous, since $\|f\|_{L^2(\mathbb{G},\mu)} \leq \mu(\mathbb{G})^{1/2}\|f\|_\infty$ for all $f \in \mathscr{E}(\mathbb{G})$. Now by the Sobolev embedding, there is some $k \in \mathbb{N}$ such that $\|f\|_\infty \leq C \max_{\substack{|a|\leq k \\ a\in\mathcal{S}_{\dim\mathbb{G}}}} \|D^a f\|_{L^2(\mathbb{G},\mu)}$ for all $f \in \mathscr{E}(\mathbb{G})$. Since all of the $D^a$ are continuous operators from $H_D^{[M]}(\mathbb{G})$ into itself, we get continuous injective maps

$$H_D^{(M)}(\mathbb{G}) \xrightarrow{\mathrm{I}} \mathscr{E}_D^{(M)}(\mathbb{G}) \quad \text{and} \quad H_D^{\{M\}}(\mathbb{G}) \xrightarrow{\mathrm{I}} \mathscr{E}_{D,}^{\{M\}}(\mathbb{G})\,.$$

The proof for $\mathscr{E}_{D,\mathrm{proj}}^{\{M\}}(\mathbb{G})$ and $H_{D,\mathrm{proj}}^{\{M\}}(\mathbb{G})$ works analogously. $\square$

Later, in Chapter 4, we will also see that these spaces are always dense in $L^2(\mathbb{G},\mu)$.

Irreducible representations are an integral ingredient for the quantization of both compact Lie groups and connected, simply connected nilpotent Lie groups. In the latter case the smooth vectors are of great importance as well.

**Definition 2.4.17.** *For a Lie group $\mathbb{G}$, we denote by $\mathrm{Irr}(\mathbb{G})$ the set of admissible, unitary and irreducible representations of the group $\mathbb{G}$. By $\mathrm{Irr}^{\mathbb{R}}(\mathbb{G})$ we denote the set of all pairs $(\pi, \mathcal{C}_\pi)$ of representations $\pi \in \mathrm{Irr}(\mathbb{G})$ and antiunitary maps $\mathcal{C}_\pi \colon H_\pi \to H_\pi$ such that $\mathcal{C}_\pi \mathscr{E}(\pi) = \mathscr{E}(\pi)$. Usually we will just write $\pi \in \mathrm{Irr}^{\mathbb{R}}(G)$ and mean that we chose some $\mathcal{C}_\pi$ for $\pi$.*

If $\mathbb{G}$ is a compact or a connected, simply connected nilpotent Lie group, we may find a Gelfand triple for each $\pi \in \mathrm{Irr}(\mathbb{G})$. For this purpose, we first formulate the following lemma.

**Lemma 2.4.18.** *If $\mathbb{G}$ is a compact Lie group or a connected, simply connected nilpotent Lie group and $\pi \in \mathrm{Irr}(\mathbb{G})$, then $\mathscr{E}(\pi)$ is a nuclear Fréchet space and we find some antiunitary operator $\mathcal{C}_\pi$ on $H_\pi$ such that $(\pi, \mathcal{C}_\pi) \in \mathrm{Irr}^{\mathbb{R}}(\mathbb{G})$.*

*Proof.* First suppose $\mathbb{G}$ is connected, simply connected and nilpotent. By the discussion on pages 124 and 125 in [16] and also [16, Corollary 4.1.2] each $\pi \in \mathrm{Irr}(\mathbb{G})$ is unitary equivalent to some representation $\pi'$ such that either $H_{\pi'} = L^2(\mathbb{R}^k)$ and $\mathscr{E}(\pi') = \mathscr{S}(\mathbb{R}^k)$ for some $k \in \mathbb{N}$ or $H_{\pi'} = \mathbb{C}$. In this case we can just use the pointwise complex conjugation on $L^2(\mathbb{R}^k)$ or $\mathbb{C}$ to define $\mathcal{C}_\pi$. Also, the unitary operators taking $\pi$ to $\pi'$ restricts to a linear homeomorphism from $\mathscr{E}(\pi)$ onto $\mathscr{S}(\mathbb{R}^k)$. Since $\mathscr{S}(\mathbb{R}^k)$ is a nuclear Fréchet space, so is $\mathscr{E}(\pi)$.

Now suppose $\mathbb{G}$ is compact. Then for each $\pi \in \mathrm{Irr}(\mathbb{G})$, $H_\pi$ is finite dimensional and $\mathscr{E}(\pi) = H_\pi$. Hence we can choose any antiunitary operator on $H_\pi$ as $\mathcal{C}_\pi$. Obviously $H_\pi$ is also a nuclear Fréchet space. $\square$

Now we are finally able to define the Gelfand triple corresponding to a representation $\pi \in \mathrm{Irr}^{\mathbb{R}}(\mathbb{G})$.

**Definition 2.4.19.** *Let $\mathbb{G}$ be a connected, simply connected nilpotent Lie group. For each $(\pi, \mathcal{C}_\pi) \in \mathrm{Irr}^{\mathbb{R}}(\mathbb{G})$ we define the Gelfand triples*

$$\mathcal{G}(\pi) := (\mathscr{E}(\pi), H_\pi, \mathscr{E}(\pi)') \quad \textit{and} \quad \mathcal{G}_{\mathrm{op}}(\pi) := \mathcal{L}(\mathcal{G}(\pi); \mathcal{G}(\pi))$$

*with respect to the real structure defined by $\mathcal{C}_\pi$ on $\mathcal{G}(\pi)$.*

For us, the Schrödinger representations of the Heisenberg group $\mathbb{H}$ on $L^2(\mathbb{R}^n)$ is the default environment for these constructions.

**Definition 2.4.20.** *For any finite dimensional vector space $V$ with Haar measure $\mu$, we will denote by $\boldsymbol{M}_2$ the representation*

$$\boldsymbol{M}_2(x')f(y) := \mathrm{e}^{2\pi\mathrm{i}\langle y, x'\rangle} f(y)\,, \qquad \textit{for } f \in L^2(V, \mu)\,,$$

*of $V'$ on $L^2(V, \mu)$.*

*For $n \in \mathbb{N}$ the Heisenberg group $\mathbb{H}$ is defined as the smooth manifold*

$$\begin{aligned} &\mathbb{R} \times \mathbb{R}_n \times \mathbb{R}^n \quad \textit{equipped with the multiplication} \\ &(t, x', x)(s, y', y) := \left( t + s + \frac{1}{2}(\langle x, y'\rangle - \langle y, x'\rangle), x' + y', x + y \right) . \end{aligned}$$

*For $\lambda \in \mathbb{R}^\times$, the Schrödinger representation $\rho_\lambda$ of $\mathbb{H}$ on $L^2(\mathbb{R}^n)$ are defined by*

$$\rho_1(t, x', x) := \mathrm{e}^{2\pi\mathrm{i}t}\, \boldsymbol{R}_2(x/2) \boldsymbol{M}_2(x')\, \boldsymbol{R}_2(x/2) \quad \textit{and} \quad \rho_\lambda(t, x', x) := \rho_1\big(\delta^{\mathbb{H}}_\lambda(t, x', x)\big)$$

*in which $\boldsymbol{R}_2$ is the right regular representation of $\mathbb{R}^n$ and*

$$\delta^{\mathbb{H}}_\lambda(t, x', x) := (\lambda t, \mathrm{sgn}(\lambda)|\lambda|^{\frac{1}{2}} x', |\lambda|^{\frac{1}{2}} x)$$

*and* $\operatorname{sgn}(\lambda)$ *is the sign of* $\lambda$.

*We always equip* $\mathbb{H}$ *and* $L^2(\mathbb{H})$ *with the standard Haar measure* $\mathrm{d}(t,x',x)$. *Furthermore, we always use the linear space* $\mathbb{R}\times\mathbb{R}_n\times\mathbb{R}^n$ *itself as the Lie algebra* $\mathfrak{h}$ *of* $\mathbb{H}$.

The Heisenberg group is the unique connected, simply connected Lie group that arises from the Lie algebra of operators spanned by all partial derivatives and multiplications by coordinate functions on $L^2(\mathbb{R}^n)$. The same construction can by done for the left invariant differential operators and multiplications by coordinate functions on $L^2(\mathbb{H})$. As remarked in [58], the corresponding Lie group $\mathbb{H}_2$ was first investigated by A. S. Dynin [21] and later by G. B. Folland [26]. As in the case of the Heisenberg group, the action of the Lie algebra of $\mathbb{H}_2$ on $L^2(\mathbb{H})$ can be integrated to a Schrödinger-type representation of the Dynin-Folland group $\mathbb{H}_2$ on $L^2(\mathbb{H})$. Before we define these terms, we need to remark that $\mathbb{H}$ in itself is also a vector space. Hence, we may define $\boldsymbol{M}_2(x')$ on $L^2(\mathbb{H})$ for $x'\in\mathbb{H}'$. For any connected, simply connected nilpotent Lie group the exponential map is a polynomial diffeomorphism. In this case, we even have $\exp_{\mathbb{H}}=\mathrm{I}$. Using $\exp_{\mathbb{H}}$ we define the adjoint representation on $\mathbb{H}=\mathfrak{h}$ and put

$$\operatorname{ad}_x y := [x,y] = \partial_t\partial_s (tx)(sy)(tx)^{-1}\big|_{s=t=0}\,,$$

in which $t,s\in\mathbb{R}$, $x,y\in\mathbb{H}$ and both scalar multiplication and group multiplication on $\mathbb{H}$ are used. This induces the corresponding dual map

$$\operatorname{ca}_x := (\operatorname{ad}_{-x})' \in \mathcal{L}(\mathbb{H}') \qquad \text{for } x\in\mathbb{H}\,.$$

Note that $\operatorname{ca}_x$ can equivalently be defined by

$$\operatorname{ca}_x y' := \partial_t \operatorname{Ca}_{\mathbb{H}}(tx)y'\big|_{t=0}\,, \qquad \text{for } y'\in\mathbb{H}'\,,\ x\in\mathbb{H}\,,$$

in which we use $t\in\mathbb{R}$, the dual map $\exp'_{\mathbb{H}}=\mathrm{I}$ and the coadjoint representation Ca of $\mathbb{H}$ on the dual $\mathbb{H}'=\mathfrak{h}'$. For the next definition we follow mostly [58].

**Definition 2.4.21.** *The Dynin-Folland group is defined as the smooth manifold*

$$\mathbb{R} \times \mathbb{H}' \times \mathbb{H} \qquad \textit{equipped with the multiplication}$$

$$(t, x', x)(s, y', y) := \bigg( t + s + \frac{1}{2}(\langle x, y' \rangle - \langle y, x' \rangle) - \frac{1}{8} \langle \mathrm{ad}_x\, y, x' - y' \rangle\,, \\ x' + y' + \frac{1}{4}(\mathrm{ca}_x\, y' - \mathrm{ca}_y\, x'), xy \bigg)\,.$$

*For $\lambda \in \mathbb{R}^\times$, we define the representation $\Theta_\lambda$ of $\mathbb{H}_2$ on $L^2(\mathbb{H})$ by*

$$\Theta_1(t, x', x) := \mathrm{e}^{2\pi \mathrm{i} t}\, \boldsymbol{R}_2(x/2) \boldsymbol{M}_2(x')\, \boldsymbol{R}_2(x/2) \quad \textit{and} \quad \Theta_\lambda(t, x', x) := \Theta_1\big(\delta_\lambda^{\mathbb{H}_2}(t, x', x)\big)$$

*in which $\boldsymbol{R}_2$ is the right regular representation of $\mathbb{H}$ and*

$$\delta_\lambda^{\mathbb{H}_2}(t, x', x) := \bigg( \lambda t, \lambda \big(\delta^{\mathbb{H}}_{|\lambda|^{-\frac{1}{2}}}\big)' x', \delta^{\mathbb{H}}_{|\lambda|^{\frac{1}{2}}} x \bigg)\,.$$

*We always equip $\mathbb{H}_2$ and $L^2(\mathbb{H}_2)$ with the standard Haar measure $\mathrm{d}(t, x', x)$. Furthermore, we always use the linear space $\mathbb{R} \times \mathbb{H}' \times \mathbb{H}$ itself as the Lie algebra $\mathfrak{h}_2$ of $\mathbb{H}_2$.*

We defined the Schrödinger-type representations $\Theta_\lambda$ slightly differently than the representations $T_\lambda$ used in [25, 58]. The reason for this is that we want to use $\Theta_\lambda$ as an example in Sections 3.3 and 3.4. Though, we can quickly find the unitary operator $D_\lambda$ on $L^2(\mathbb{H})$, defined by $D_\lambda f(x) := |\lambda|^{-\frac{n+1}{4}} f(\delta^{\mathbb{H}}_{|\lambda|^{\frac{1}{2}}} x)$ for $x \in \mathbb{H}$, $f \in L^2(\mathbb{H})$, which fulfils

$$D_\lambda^{-1} T_\lambda(t, x', x) D_\lambda = \Theta_\lambda(t, x', x) \qquad \text{for } (t, x', x) \in \mathbb{H}_2\,.$$

With an analogous calculation, we can show that our version of the Schrödinger representation $\rho_\lambda$ is equivalent to the one defined in [25, p. 22]. Because $\mathrm{Diff}_{\mathscr{P}}(\mathbb{R}^n) = \rho_\lambda(\mathrm{Diff}_{\mathrm{L}}(\mathbb{R}^n))$ and due to [58, 26] and [25, Proposition 1.43], we may use the following statement.

**Lemma 2.4.22.** *Both $\rho_\lambda$ and $\Theta_\lambda$ are irreducible and we have*

$$\mathscr{E}(\rho_\lambda) = \mathscr{S}(\mathbb{R}^n) \quad \textit{and} \quad \mathscr{E}(\Theta_\lambda) = \mathscr{S}(\mathbb{H})$$

*for all $\lambda \in \mathbb{R}^\times$.*

We will remark now why the spaces $\mathscr{C}^k(\rho_\lambda)$ and $\mathscr{C}^k(\Theta_\lambda)$ are polynomially weighted Sobolev spaces.

**Definition 2.4.23.** *Suppose $\mathbb{G} \in \{\mathbb{R}^n, \mathbb{H}\}$ with corresponding Lie algebra $\mathfrak{g}$. For $k \in \mathbb{N}_0$ and any bases $q \subset \mathbb{G}'$ and $D \subset \mathfrak{g}_{\mathrm{L}}$ we define*

$$H^k_{\mathscr{P}}(\mathbb{G}) := \{f \in H^k_{\mathrm{L}}(\mathbb{G}) \mid \forall_{a,b \in \mathcal{S}_{\dim \mathbb{G}}} \colon |a|, |b| \leq k \;\Rightarrow\; q^a D^b f \in L^2(\mathbb{G})\}$$

$$\textit{equipped with the topology induced by} \quad f \mapsto \|q^a D^b f\|_{L^2(\mathbb{G})},$$

*in which $a, b$ run through the elements of $\mathcal{S}_{\dim \mathbb{G}}$ with $|a|, |b| \leq k$. In the above, we use the notation $q^a := \cdots q_{a_3}\, q_{a_2}\, q_{a_1}$ for $q = (q_1, \ldots, q_N)$, $q_0 := 1$ and $a \in \mathcal{S}_N$.*

Note that this definition does not depend on the chosen bases $q \subset \mathbb{G}'$ and $D \subset \mathfrak{g}_{\mathrm{L}}$.

**Lemma 2.4.24.** *Suppose $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}^\times$. Then*

$$H^k_{\mathscr{P}}(\mathbb{R}^n) = \mathscr{C}^k(\rho_\lambda) \quad \textit{and} \quad H^k_{\mathscr{P}}(\mathbb{H}) = \mathscr{C}^k(\Theta_\lambda).$$

*Proof.* Suppose $(\mathbb{G}, \mathbb{G}_2, \pi) \in \{(\mathbb{R}^n, \mathbb{H}, \rho_\lambda), (\mathbb{H}, \mathbb{H}_2, \Theta_\lambda)\}$ and let $m = \dim \mathbb{G}_2$.

There exists a frame $(\tilde{\partial}, \tilde{D}, \tilde{D}') \subset \mathrm{Diff}_{\mathrm{L}}(\mathbb{G}_2)$ such that $\pi(\tilde{\partial}^k) f = \mathrm{i}^k f$, $\pi(\tilde{D}^a) f = D^a f$ and $\pi(\tilde{D}'^b) f = \mathrm{i}^{|b|} q^b f$. But this also means that for any $P \in \mathrm{Diff}^k_{\mathrm{L}}(\mathbb{G}_2)$ there are coefficients $c_a \in \mathbb{C}$, for $a \in \mathcal{S}_m$, such that

$$\pi(P) f = \sum_{a \in \mathcal{S}_m} c_a\, q^a\, D^a f \qquad \text{for } f \in \mathscr{C}^k(\pi),$$

as a finite linear combination. Hence $\mathscr{C}^k(\pi)$ is a closed subspace of $H^k_{\mathscr{P}}(\mathbb{G})$. By the usual convolution and cut-off arguments, $\mathscr{S}(\mathbb{G})$ is dense in $H^k_{\mathscr{P}}(\mathbb{G})$. But also $\mathscr{S}(\mathbb{G}) \subset \mathscr{E}(\pi) \subset \mathscr{C}^k(\mathbb{G})$, hence $H^k_{\mathscr{P}}(\mathbb{G}) = \mathscr{C}^k(\pi)$. □

The next group of spaces we will consider, are the Gelfand-Shilov spaces [29]. These spaces are continuously embedded subspaces of the Schwartz space $\mathscr{S}(\mathbb{R}^n)$. We will use an analogous concept to also construct Gelfand-Shilov spaces of functions on $\mathbb{H}$.

**Definition 2.4.25.** *Suppose $\mathbb{G} \in \{\mathbb{R}^n, \mathbb{H}\}$ with Lie algebra $\mathfrak{g}$ and let $N := \dim \mathbb{G}$. For weight sequences $M, L \in \mathbb{R}^{\mathbb{N}_0}_{>0}$, a frame $D \subset \mathfrak{g}_{\mathrm{L}}$ and a basis $q \subset \mathbb{G}'$, we define the Banach spaces*

$$H^{M,D}_{L,q}(\mathbb{G}) := \left\{ f \in \mathscr{E}(\mathbb{G}) \;\middle|\; \lim_{\substack{|a|+|b| \to \infty \\ a,b \in \mathcal{S}_N}} \frac{\|q^b D^a f\|_{L^2(\mathbb{G})}}{M_{|a|}\, |a|!\, L_{|b|}\, |b|!} = 0 \right\} \quad \textit{with norm}$$

$$\|f\|^{M,D}_{L,q} := \sup_{a,b \in \mathcal{S}_N} \frac{\|q^b D^a f\|_{L^2(\mathbb{G})}}{M_{|a|}\, |a|!\, L_{|b|}\, |b|!},$$

*in which of course $q^a$ is defined as in the last definition.*

*The Gelfand-Shilov spaces are defined by*

$$\mathscr{S}^{(M)}_{(L)}(\mathbb{G}) := \varprojlim_{h,k>0} H^{M,kD}_{L,hq}(\mathbb{G}) \quad \textit{and} \quad \mathscr{S}^{\{M\}}_{\{L\}}(\mathbb{G}) := \varinjlim_{h,k>0} H^{M,kD}_{L,hq}(\mathbb{G})\,.$$

*For convenience, we also define the mixed Gelfand-Shilov spaces*

$$\mathscr{S}^{[M]}_{[L],\mathrm{proj}}(\mathbb{G}) := \varprojlim_{M'\in\mathcal{W}_{[M]}\,,\,L'\in\mathcal{W}_{[L]}} H^{M',D}_{L',q}(\mathbb{G})\,.$$

With an analogous argument as in Proposition 2.2.18, we get

$$\mathscr{S}^{(M)}_{(L)}(\mathbb{G}) = \mathscr{S}^{(M)}_{(L),\mathrm{proj}}(\mathbb{G})\,.$$

Also, the Gelfand-Shilov space of Roumieu type, $\mathscr{S}^{\{M\}}_{\{L\}}(\mathbb{R}^n)$, coincides with $\mathscr{S}^{\{M\}}_{\{L\},\mathrm{proj}}(\mathbb{R}^n)$ as a vector space. This can be seen with Lemma 2.2.12, which also implies that the bounded sets in both topological vector spaces coincide. Moreover, the topology in $\mathscr{S}^{\{M\}}_{\{L\}}(\mathbb{R}^n)$ is finer than in the space $\mathscr{S}^{\{M\}}_{\{L\},\mathrm{proj}}(\mathbb{R}^n)$.

The above definition is a generalization for the definition given in [29]. See for example [14] for a definition more in line with the one given here. Still, our version of Gelfand-Shilov spaces differs in some aspects from the Gelfand-Shilov spaces defined in [14]. First of all, we used factorials in addition to weight sequences for the definition. Of course, this is motivated by the convention we follow for the spaces of ultradifferentiable functions $\mathscr{E}^{[M]}_{\partial}(\mathbb{R}^n)$. Secondly, we used $L^2$-Sobolev spaces for the limit description of the Gelfand-Shilov spaces. But since the weight sequences $M$ and $L$ fulfil **(D)**, we may exchange the space $H^{L,q}_{M,D}(\mathbb{G})$ by say

$$\mathscr{S}^{M,D}_{L,q}(\mathbb{G}) := \left\{ f \in \mathscr{E}(\mathbb{G}) \;\middle|\; \|f\|^{M,D}_{L,q,\infty} := \sup_{a,b\in\mathcal{S}_N} \frac{\|q^b D^a f\|_\infty}{M_{|a|}\,|a|!\,L_{|b|}\,|b|!} < \infty \right\}$$
$$\text{equipped with the norm } \|-\|^{L,q}_{M,D,\infty}$$

without changing the Gelfand-Shilov spaces $\mathscr{S}^{[M]}_{[L]}(\mathbb{G})$ due to the usual Sobolev embeddings.

Similar to the case of $L^2$-Sobolev spaces, the Gelfand-Shilov spaces can be equivalently defined as subspaces of the smooth vectors with respect to the representations $\rho_\lambda$ resp. $\Theta_\lambda$.

Also, the definition of $\mathscr{S}_{[L]}^{[M]}(\mathbb{G})$ does not depend on the involved left invariant frame $D$ or basis $q$. This can be seen as a direct implication of Lemma 2.2.10 and the following Lemma 2.4.28.

**Definition 2.4.26.** *Let $\mathbb{M}$ be an analytic manifold, let $M$, $L$, $K$ be weight sequences and let $T, E, D \subset \mathcal{V}_{\mathrm{a}}(\mathbb{M})$ be families of vector fields composed of $\#T$ resp. $\#E$ resp. $\#D$ vector fields such that $(T, E, D)$ is a frame. Then we denote by $\mathscr{E}_{T,E,D}^{K,L,M}(\mathbb{M})$ the $\mathscr{C}(\mathbb{M})$-function space defined by*

$$\mathscr{E}_{T,E,D}^{K,L,M}(\mathbb{M}) := \left\{ f \in \mathscr{E}(\mathbb{M}) \;\middle|\; \lim_{|a|,|b|,|c| \to \infty} \frac{\|T^a E^b D^c f\|_\infty}{K_{|a|}\,|a|!\,L_{|b|}\,|b|!\,M_{|c|}\,|c|!} = 0 \right\}$$

*equipped with the norm*

$$f \mapsto \sup_{a,b,c} \frac{\|T^a E^b D^c f\|_\infty}{K_{|a|}\,|a|!\,L_{|b|}\,|b|!\,M_{|c|}\,|c|!}\,,$$

*in which $a \in \mathcal{S}_{\#T}$, $b \in \mathcal{S}_{\#E}$ and $c \in \mathcal{S}_{\#D}$. As usual we define limit spaces*

$$\mathscr{E}_{T,E,D}^{[K],[L],[M]}(\mathbb{M}) := \varprojlim_{K' \in \mathcal{W}_{[K]}} \varprojlim_{L' \in \mathcal{W}_{[L]}} \varprojlim_{M' \in \mathcal{W}_{[M]}} \mathscr{E}_{T,E,D,\mathrm{proj}}^{K,L,M}(\mathbb{M})\,.$$

These spaces can not be dealt with Proposition 2.2.20, since the families $T$, $E$ and $D$ might not commute with each other. But the situation is at least similar. If $N$ is a weight sequence that fulfils **(MG)**, then there are constants $C, H > 0$ such that

$$\frac{N_{k+l+m}}{N_k\,N_l\,N_m} \le C\,H^{k+l+m}\,.$$

This implies the inequality

$$\sup_{a,b,c} \frac{\|T^a E^b D^c f\|_\infty}{N_{|a|}\,|a|!\,N_{|b|}\,|b|!\,N_{|c|}\,|c|!} \le C \sup_d H^{|d|} \frac{\|(T,E,D)^d f\|_\infty}{N_{|d|}\,|d|!}\,.$$

Now if $[N] \subset [K], [L], [M]$, then the continuous embedding

$$\mathscr{E}_{T,E,D}^{[N],[N],[N]}(\mathbb{M}) \xrightarrow{\mathrm{I}} \mathscr{E}_{T,E,D,\mathrm{proj}}^{[K],[L],[M]}(\mathbb{M})$$

and the above inequality give us the continuous embedding

$$\mathscr{E}_{(T,E,D)}^{[N]}(\mathbb{M}) \xrightarrow{\mathrm{I}} \mathscr{E}_{T,E,D,\mathrm{proj}}^{[K],[L],[M]}(\mathbb{M})\,.$$

**Lemma 2.4.27.** *Let $\mathbb{G} \in \{\mathbb{R}^n, \mathbb{H}\}$, let $M$ be a weight sequence, and let $D$ be a basis of left invariant vector fields on $G'$. Then $\mathscr{E}_D^{[M]}(\boldsymbol{M}_2)$ is $\boldsymbol{R}_2$-invariant.*

*Proof.* Denote $\chi_x(x') := \mathrm{e}^{2\pi\mathrm{i}\langle x,x'\rangle}$ for $x \in \mathbb{G}$ and $x' \in \mathbb{G}'$. Note that $\chi_x \in \mathscr{E}_D^{(\mathbb{1})}(\mathbb{G}')$. If $f \in L^2(\mathbb{G})$, then $f \in \mathscr{E}_D^M(\boldsymbol{M}_2)$ iff $x' \mapsto \boldsymbol{M}_2(x')f$ is in $\mathscr{E}_D^M(\mathbb{G}'; L^2(\mathbb{G}))$.

For $\mathbb{G} = \mathbb{R}^n$, we have $F_x(x') := \boldsymbol{M}_2(x')\,\boldsymbol{R}_2(x)f = \chi_{-x}(x')\,\boldsymbol{R}_2(x)\boldsymbol{M}_2(x')f$. Since $\boldsymbol{R}_2(x)$ is unitary on $L^2(\mathbb{G})$ and by the continuity of multiplication in Proposition 2.2.15, $F_x$ is in $\mathscr{E}_D^M(\mathbb{G}'; L^2(\mathbb{G}))$ iff $x' \mapsto \boldsymbol{M}_2(x')f$ is in $\mathscr{E}_D^M(\mathbb{G}'; L^2(\mathbb{G}))$. Thus $\mathscr{E}_D^{[M]}(\boldsymbol{M}_2)$ is $\boldsymbol{R}_2$-invariant.

Now let $\mathbb{G} = \mathbb{H}$ and let $f \in \mathscr{E}_D^{[M]}(\boldsymbol{M}_2)$. We have

$$F_x(x') = \chi_{-x}(x')\,\boldsymbol{R}_2(x)\boldsymbol{M}_2(x' + \mathrm{ca}_x\, x'/2)f\,.$$

Note that $A\colon x' \mapsto x' + \mathrm{ca}_x\, x'/2$ is a linear map, so $x' \mapsto \boldsymbol{M}_2(Ax')f$ is in $\mathscr{E}_D^{[M]}(\mathbb{G}'; L^2(\mathbb{G}))$ by Proposition 2.2.10 $\square$

**Lemma 2.4.28.** *Let $\lambda \in \mathbb{R}^\times$ and let either $(\mathbb{G}, \pi, \mathbb{M}) = (\mathbb{R}^n, \rho_\lambda, \mathbb{H})$ or $(\mathbb{G}, \pi, \mathbb{M}) = (\mathbb{H}, \Theta_\lambda, \mathbb{H}_2)$. Furthermore, let $M$, $L$, $K$ be weight sequences. If $q \subset \mathbb{G}'$ is a basis and $D \subset \mathrm{Diff}_{\mathrm{L}}(\mathbb{G})$ a frame, then the following holds.*

**(i)** *There exist families of vector fields $T, E, F \subset \mathrm{Diff}_{\mathrm{L}}(\mathbb{M})$ such that $(T, E, F)$ is a frame on $\mathbb{M}$ with*

$$H_{L,q}^{M,D}(\mathbb{G}) = \mathscr{E}_{T,E,F}^{K,L,M}(\pi) \quad \textit{and} \quad \mathscr{S}_{[L],\mathrm{proj}}^{[M]}(\mathbb{G}) = \mathscr{E}_{T,E,F,\mathrm{proj}}^{[K],[L],[M]}(\pi)\,.$$

**(ii)** *There is a frame $E \subset \mathrm{Diff}_{\mathrm{L}}(\mathbb{G}')$ such that*

$$\mathscr{S}_{[L],\mathrm{proj}}^{[M]}(\mathbb{G}) = \mathscr{E}_D^{[M]}(\boldsymbol{R}_2{\downarrow_{\mathscr{E}_E^{[L]}(\boldsymbol{M}_2)}})$$

*in the sense of vector spaces. If $[M]$ and $[N]$ have **(PL)**, then this equality holds in the sense of locally convex spaces.*

*Proof.* **(i):** We may define $T$ such that it is just composed of one vector field which acts along the *center* of $\mathbb{M}$. Furthermore, we may choose $T$ such that $\pi(T) = \mathrm{i}\,\mathrm{I}$. Similarly, we find vector fields $E = (E_1, \dots, E_N)$ such that $\pi(E_j)f = \mathrm{i}\,q_j\,f$ and $F = (F_1, \dots, F_N)$ such that $\pi(F_j)f = D_j f$ for $f \in H_{\mathrm{L}}^\infty(\mathbb{G})$ (see e.g. [26] for the Dynin-Folland group). The argumentation from Lemma 2.4.8 can be slightly adjusted to show $H_{L,q}^{M,D}(\mathbb{G}) = \mathscr{E}_{T,E,F}^{K,L,M}(\pi)$, which already implies the rest.

**(ii):** Obviously we have $\boldsymbol{R}_2(D^a)f = D^a f$ for all $a \in \mathcal{S}_n$ and $f \in \mathscr{E}(\boldsymbol{R}_2)$. Also, we may choose $E$ such that $\boldsymbol{M}_2(E^a)f = \mathrm{i}^a\, q^a f$ for all $a \in \mathcal{S}_n$ and $f \in \mathscr{E}(\boldsymbol{M}_2)$.

Let $H := L^2(\mathbb{G})$. By Proposition 2.2.18 we may employ the following chain of identities and bijections (resp. homeomorphisms)

$$\begin{aligned}
\mathscr{E}^{[M]}_{D,\mathrm{proj}}(\mathbb{G};\mathscr{E}^{[L]}_{E,\mathrm{proj}}(\mathbb{G}';H)) &= \mathscr{E}^{[M]}_{D}(\mathbb{G};\mathscr{E}^{[L]}_{E,\mathrm{proj}}(\mathbb{G}';H)) \\
&\simeq \mathscr{E}^{[L]}_{E,\mathrm{proj}}(\mathbb{G}';\mathscr{E}^{[M]}_{D}(\mathbb{G};H)) \\
&= \mathscr{E}^{[L]}_{E}(\mathbb{G}';\mathscr{E}^{[M]}_{D}(\mathbb{G};H)) \simeq \mathscr{E}^{[M]}_{D}(\mathbb{G};\mathscr{E}^{[L]}_{E}(\mathbb{G}';H))
\end{aligned}$$

to get $\mathscr{E}^{[M]}_{D,\mathrm{proj}}(\mathbb{G};\mathscr{E}^{[L]}_{E,\mathrm{proj}}(\mathbb{G}';H)) = \mathscr{E}^{[M]}_{D}(\mathbb{G};\mathscr{E}^{[L]}_{E}(\mathbb{G}';H))$ in the sense of vector spaces. If $[M]$ and $[L]$ have **(PL)**, then this holds in the sense of topological vector spaces. This gives us

$$\mathscr{E}^{[M]}_{D}(\boldsymbol{R}_2{\downarrow}_{\mathscr{E}^{[L]}_{E}(\boldsymbol{M}_2)}) = \mathscr{E}^{[M]}_{D,\mathrm{proj}}(\boldsymbol{R}_2{\downarrow}_{\mathscr{E}^{[L]}_{E,\mathrm{proj}}(\boldsymbol{M}_2)})$$

in the sense of vector spaces (resp. in the sense of topological vector space if $[M]$ and $[L]$ have **(PL)**). Now we only need to use

$$\begin{aligned}
\mathscr{S}^{[M]}_{[L],\mathrm{proj}}(\mathbb{G}) = \varprojlim_{M'\in\mathcal{W}_{[M]}} \varprojlim_{L'\in\mathcal{W}_{[L]}} H^{M,D}_{L,q}(\mathbb{G}) &= \varprojlim_{M'\in\mathcal{W}_{[M]}} \mathscr{E}^{M'}_{D}(\boldsymbol{R}_2{\downarrow}_{\mathscr{E}^{[L]}_{E,\mathrm{proj}}(\boldsymbol{M}_2)}) \\
&= \mathscr{E}^{[M]}_{D,\mathrm{proj}}(\boldsymbol{R}_2{\downarrow}_{\mathscr{E}^{[L]}_{E,\mathrm{proj}}(\boldsymbol{M}_2)})\,.
\end{aligned}$$

$\square$

From the above especially follows that for $\{\mathbb{1}\} \subset [M], [L]$ we have

$$\mathscr{E}^{\{\mathbb{1}\}}(\pi) \subset \mathscr{S}^{[M]}_{[L],\mathrm{proj}}(\mathbb{G})\,.$$

So in this case we know that $\mathscr{S}^{[M]}_{[L],\mathrm{proj}}(\mathbb{G})$ is dense in $L^2(\mathbb{G})$, because the space of analytic vectors is dense (see Lemma 2.4.6).

# Chapter 3

# Quantization on Gelfand triples

The following chapter is largely based on a prior publication of the author [7]. Quantization is a procedure, by which functions and distributions are mapped to operators. Among the many types of quantizations, we will only use the Kohn-Nirenberg quantization and the Pedersen quantization (resp. the Weyl quantization). Both are closely connected to types of Fourier transformations. Historically, the Fourier transformation on $\mathbb{R}^n$ was used for this purpose. Namely, for a function $a \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}_n)$ the corresponding Kohn-Nirenberg operator is defined by

$$\mathrm{Op}_{\mathbb{R}^n}(a)\varphi(x) = \mathcal{F}_{\mathbb{R}^n}^{-1}(a(x,-)\,\mathcal{F}_{\mathbb{R}^n}\varphi)(x) \qquad \text{for } \varphi \in \mathscr{S}(\mathbb{R}^n)\,,\ x \in \mathbb{R}^n$$

and the Weyl operator is defined as the integral

$$\int_{\mathbb{R}_n}\int_{\mathbb{R}^n} (\mathcal{F}_{\mathbb{R}^n\times\mathbb{R}_n} a)(x',x)\,\rho_1(0,x',x)\,\mathrm{d}x\,\mathrm{d}x'\,,$$

which exists in $\mathcal{L}_s(L^2(\mathbb{R}^n))$. Both quantization procedures induce a Gelfand triple isomorphism

$$\mathcal{G}(\mathbb{R}^n \times \mathbb{R}_n,\, \mathrm{d}x\,\mathrm{d}x') \to \mathcal{L}(\mathcal{G}(\mathbb{R}^n,\, \mathrm{d}x); \mathcal{G}(\mathbb{R}^n,\, \mathrm{d}x))\,.$$

The Pedersen quantization [57] is a generalization of the Weyl quantization. By swapping $\mathcal{F}_{\mathbb{R}^n}$ with the group Fourier transform on some Lie group $\mathbb{G}$ with biinvariant Haar measure $\mu$, one can generalize the Kohn-Nirenberg quantization to map certain operator valued distributions on $\mathbb{G} \times \widehat{\mathbb{G}}$ to operators defined on a subspace of $L^2(\mathbb{G},\mu)$. A calculus for compact and nilpotent Lie groups was developed in [59, 24]. We start with an exposition on the Kohn-Nirenberg quantization, after which we will define and discuss the Pedersen quantization. In the third section, we will use the Pedersen quantization in order to define new Gelfand triples for homogeneous Lie group such that the Kohn-Nirenberg quantization operates as a Gelfand triple isomorphism between them.

## 3.1 The Kohn-Nirenberg quantization

Let $\mathbb{G}$ be either a compact or a connected, simply connected nilpotent Lie group with Haar measure $\mu$. For each separable Hilbert space $H$ let $\mathrm{Irr}_H(\mathbb{G})$ be the set of irreducible admissible unitary representations. We define on $\mathrm{Irr}_H(\mathbb{G})$ the smallest $\sigma$-algebra such that

$$\mathrm{Irr}_H(\mathbb{G}) \to \mathbb{C} \colon \pi \mapsto \big(\pi(x)h, h'\big)_H$$

is measurable for all $x \in \mathbb{G}$ and $h, h' \in H$. We equip $\mathrm{Irr}(\mathbb{G})$ with the biggest $\sigma$-algebra such that all the inclusions $\mathrm{Irr}_H(\mathbb{G}) \hookrightarrow \mathrm{Irr}(\mathbb{G})$ are measurable.

The dual of $\mathbb{G}$, denoted by $\widehat{\mathbb{G}}$, is the quotient of $\mathrm{Irr}(\mathbb{G})$ under the equivalence relation of unitary equivalence. We equip $\widehat{\mathbb{G}}$ with the hull kernel topology[1]. The general definition of the topology is not important to us, as for our special cases we have a better, more concrete description of the hull-kernel topology. Here we use that for any *compact* Lie group $\mathbb{G}$ the dual $\widehat{\mathbb{G}}$ is carrying the discrete topology [20, Corollary 18.4.3]. For *nilpotent* Lie groups we will use Theorem 3.2.1 and Proposition 3.3.7. The Mackey-Borel structure on $\widehat{\mathbb{G}}$ is the largest $\sigma$-algebra on $\widehat{\mathbb{G}}$ with respect to which the quotient map

$$\mathrm{Irr}(\mathbb{G}) \to \widehat{\mathbb{G}} \colon \pi \mapsto [\pi]$$

is measurable. Here $[\pi]$ denotes the class of unitary equivalent representations to $\pi$. This $\sigma$-algebra coincides with the Borel $\sigma$-algebra defined by the hull-kernel topology [20, 18.5.3]. A prerequisite for this statement is that the group $\mathbb{G}$ is *postliminal.* Though in our case, this is always fulfilled, since compact resp. connected, simply connected nilpotent

[1]In [20, 3.1.2] this topology is introduced as the Jacobson topology. See for example [27, p. 225] or [22] for the fact that it is just another name for the hull-kernel topology.

Lie groups are type I as described in [28, p. 72]. For Lie groups, type I implies postliminal [20, 13.9.4]. Note that the definition of a type I group is approached differently in [28] and [20], but both definitions are equivalent due to [28, Theorem 3.23].

Let $\widehat{\mathbb{G}}_n := \{[\pi] \mid \dim H_\pi = n\,,\ \pi \in \mathrm{Irr}(\mathbb{G})\}$. There exists a measurable map

$$\eta \colon \widehat{\mathbb{G}} \to \mathrm{Irr}(\mathbb{G})\,, \quad \eta([\pi]) \in [\pi] \quad \text{and} \quad H_{\eta([\pi])} = H_{\eta([\tilde{\pi}])} =: H_n \tag{3.1.1}$$

for all $n \in \mathbb{N} \cup \{\aleph_0\}$ and all $[\pi], [\tilde{\pi}] \in \widehat{\mathbb{G}}_n$ [20, Proposition 8.6.2].

Let $\nu$ be a measure on $\widehat{\mathbb{G}}$. Now we define $\Sigma(\widehat{\mathbb{G}})$ as the set of operator valued maps $\sigma \colon \mathrm{Irr}(\mathbb{G}) \to \cup_{\pi \in \mathrm{Irr}(\mathbb{G})} \mathcal{L}(H_\pi)$ fulfilling

**(i)** $\sigma(\pi) \in \mathcal{L}(\mathcal{H}_\pi)$ for all $\pi \in \mathrm{Irr}(\mathbb{G})$,

**(ii)** $\sigma(U\pi U^{-1}) = U\sigma(\pi)U^{-1}$ for all $(\pi, H_\pi) \in \mathrm{Irr}(\mathbb{G})$ and unitary operators $U$ with domain $H_\pi$ and

**(iii)** the map $\widehat{\mathbb{G}}_n \ni [\pi] \mapsto \sigma\big(\eta([\pi])\big)h$ is $\nu$-measurable for all $n \in \mathbb{N} \cup \{\aleph_0\}$ and $h \in H_n$. This automatically ensures that this holds for *all* $\eta$ which fulfil (3.1.1).

If $H$ is a Hilbert space and $T \in \mathcal{L}(T)$, then we use the convention $\|T\|_{\mathcal{HS}(H)} = \infty$ (resp. $\|T\|_{\mathcal{N}(H)} = \infty$) iff $T \notin \mathcal{HS}(H)$ (resp. $T \notin \mathcal{N}(H)$). If $\nu$ is a measure on $\widehat{\mathbb{G}}$, then we define $B^\infty(\widehat{\mathbb{G}}, \nu)$, $B^2(\widehat{\mathbb{G}}, \nu)$ and $B^1(\widehat{\mathbb{G}}, \nu)$ as the quotient spaces

$$\begin{aligned}
B^1(\widehat{\mathbb{G}}, \nu) :=& \left\{ \sigma \in \Sigma(\widehat{\mathbb{G}}) \;\middle|\; \|\sigma\|_{B^1(\widehat{\mathbb{G}},\nu)} := \int_{\widehat{\mathbb{G}}} \|\sigma(\pi)\|_{\mathcal{N}(H_\pi)} \,\mathrm{d}\nu([\pi]) < \infty \right\} \Big/ N(\widehat{\mathbb{G}}, \nu)\\
B^2(\widehat{\mathbb{G}}, \nu) :=& \left\{ \sigma \in \Sigma(\widehat{\mathbb{G}}) \;\middle|\; \|\sigma\|_{B^2(\widehat{\mathbb{G}},\nu)} := \left( \int_{\widehat{\mathbb{G}}} \|\sigma(\pi)\|^2_{\mathcal{HS}(H_\pi)} \,\mathrm{d}\nu([\pi]) \right)^{\frac{1}{2}} < \infty \right\} \Big/ N(\widehat{\mathbb{G}}, \nu)\\
B^\infty(\widehat{\mathbb{G}}, \nu) :=& \left\{ \sigma \in \Sigma(\widehat{\mathbb{G}}) \;\middle|\; \|\sigma\|_{B^\infty(\widehat{\mathbb{G}},\nu)} := \nu\text{-}\operatorname*{ess\,sup}_{[\pi]\in\widehat{\mathbb{G}}} \|\sigma(\pi)\|_{\mathcal{L}(H_\pi)} < \infty \right\} \Big/ N(\widehat{\mathbb{G}}, \nu)
\end{aligned}$$

with respect to the subspace $N(\widehat{\mathbb{G}}, \nu) := \{\sigma \in \Sigma(\widehat{\mathbb{G}}) \mid \|\sigma\|_{B^2(\widehat{\mathbb{G}},\nu)} = 0\}$. We equip the space $B^p(\widehat{\mathbb{G}}, \nu)$ with the norm $\|\sigma + N(\widehat{\mathbb{G}}, \nu)\|_{B^p(\widehat{\mathbb{G}},\mu)} := \|\sigma\|_{B^p(\widehat{\mathbb{G}},\mu)}$ for $p \in \{1, 2, \infty\}$. As with the Lebesgue-Bochner spaces we will write $\sigma \in B^p(\widehat{\mathbb{G}}, \nu)$ for a function $\sigma$ iff $\sigma \in \Sigma(\widehat{\mathbb{G}})$ and its equivalence class is in $B^p(\widehat{\mathbb{G}}, \nu)$. From our definitions follows that $B^2(\widehat{\mathbb{G}}, \eta)$ is equivalent to the more commonly used approach using direct integrals[2]. To be precise, for any measurable map $\eta\colon \widehat{\mathbb{G}} \to \mathrm{Irr}(\mathbb{G})$ such that $\eta([\pi]) \in [\pi]$ for all $\pi \in \mathrm{Irr}(\mathbb{G})$ the map

$$B^2(\widehat{\mathbb{G}}, \nu) \to \int_{\widehat{\mathbb{G}}}^{\oplus} \mathcal{HS}(H_{\eta([\pi])}) \,\mathrm{d}\nu([\pi])\,, \quad \text{defined by} \quad \sigma \mapsto \sigma \circ \eta$$

for functions $\sigma \in B^2(\widehat{\mathbb{G}}, \eta)$, is unitary.

The **group Fourier transform** of $f \in L^1(\mathbb{G}, \mu)$ at $\pi \in \mathrm{Irr}(\mathbb{G})$ with representation space $H$ is defined by

$$\pi(f) := \mathcal{F}_{\mathbb{G}} f(\pi) := \int_{\mathbb{G}} f(x)\,\pi(x)^*\,\mathrm{d}\mu(x)\,,$$

where the integral exists in $\mathcal{L}_s(H)$. The Fourier transform of $f$ fulfils $\mathcal{F}_{\mathbb{G}} f \in \Sigma(\widehat{\mathbb{G}})$. As for $\mathbb{R}$, there exists a Plancherel theorem for more general groups $\mathbb{G}$. Since compact and connected, simply connected nilpotent Lie groups are unimodular and of type I they fulfil the requirements for [28, Theorem 3.31]. Thus we get the following.

**Theorem 3.1.1.** *If $\mathbb{G}$ is compact or nilpotent with Haar measure $\mu$, there exists a unique Borel measure $\widehat{\mu}$, the* ***Plancherel measure****, such that*

$$\int_{\mathbb{G}} |f(x)|^2\,\mathrm{d}\mu(x) = \int_{\widehat{\mathbb{G}}} \|\pi(f)\|^2_{\mathcal{HS}(H_\pi)}\,\mathrm{d}\widehat{\mu}([\pi])$$

*for all $f \in L^1(\mathbb{G}, \mu) \cap L^2(\mathbb{G}, \mu)$. The extension of*

$$L^1(\mathbb{G}, \mu) \cap L^2(\mathbb{G}, \mu) \to B^2(\widehat{\mathbb{G}}, \widehat{\mu})\colon f \mapsto \mathcal{F}_{\mathbb{G}} f$$

*to $L^2(\mathbb{G}, \mu)$ is a unitary operator onto $B^2(\widehat{\mathbb{G}}, \widehat{\mu})$.*

There is also an analogue for the Fourier inversion formula. We use the formulation given in [28, Theorem 4.4].

**Theorem 3.1.2.** *Let $\mathbb{G}$ and $\mu$ be as above. If $\mathcal{F}_{\mathbb{G}} f \in B^1(\widehat{\mathbb{G}}, \widehat{\mu}) \cap B^2(\widehat{\mathbb{G}}, \widehat{\mu})$, then*

$$f(x) = \int_{\widehat{\mathbb{G}}} \mathrm{Tr}[\pi(x)\,\pi(f)]\,\mathrm{d}\widehat{\mu}([\pi])$$

*for $\mu$-almost all $x \in \mathbb{G}$ and the right-hand side defines a continuous function.*

[2] See for example [28] for a definition of the direct integral of Hilbert spaces and a formulation of the Plancherel theorem involving direct integrals.

As in the Euclidean case, the group Fourier transform becomes a Gelfand triple isomorphism. For any **compact** Lie group $\mathbb{G}$ with Haar measure $\mu$, we will write

$$\mathscr{S}(\mathbb{G}) := \mathscr{E}(\mathbb{G}) = \mathscr{D}(\mathbb{G})$$

and define the Gelfand triple

$$\mathcal{G}(\mathbb{G},\mu) := \left(\mathscr{S}(\mathbb{G}), L^2(\mathbb{G},\mu), \mathscr{S}'(\mathbb{G})\right),$$

equipped with the real structure defined by pointwise complex conjugation on $\mathscr{S}(\mathbb{G})$ resp. $L^2(\mathbb{G},\mu)$.

**Definition 3.1.3.** *For any compact or simply connected, connected nilpotent Lie group $\mathbb{G}$ with Haar measure $\mu$ we define*

$$\mathscr{S}(\widehat{\mathbb{G}}) := \mathcal{F}_{\mathbb{G}}\mathscr{S}(\mathbb{G})$$

*and equip it the topology transported from $\mathscr{S}(\mathbb{G})$ via $\mathcal{F}_{\mathbb{G}}$. Furthermore, its dual will be denoted by $\mathscr{S}'(\widehat{\mathbb{G}}) := \mathscr{S}(\widehat{\mathbb{G}})'$ and we define the Gelfand triple*

$$\mathcal{G}(\widehat{\mathbb{G}},\widehat{\mu}) := (\mathscr{S}(\widehat{\mathbb{G}}), B^2(\widehat{\mathbb{G}},\widehat{\mu}), \mathscr{S}'(\widehat{G}))$$

*equipped with the real structure defined by the pointwise conjugation*

$$\sigma \mapsto \sigma^* := [\pi \mapsto \sigma(\pi)^*]$$

*on $\mathscr{S}(\widehat{\mathbb{G}})$ resp. $B^2(\widehat{\mathbb{G}},\widehat{\mu})$.*

If $f^*(x) := \overline{f(x^{-1})}$, then $(\mathcal{F}_{\mathbb{G}}f)^* = \mathcal{F}_{\mathbb{G}}(f^*)$ for all $f \in L^1(\mathbb{G},\mu) \cap L^2(\mathbb{G},\mu)$. This implies that $\sigma \mapsto \sigma^*$ defines an antilinear homeomorphism from $\mathscr{S}(\widehat{\mathbb{G}})$ to itself. Thus the Gelfand triple $\mathscr{G}(\widehat{\mathbb{G}},\widehat{\mu})$ is well-defined.

The unitary operator from Theorem 3.1.1 restricts to a linear homeomorphism from $\mathscr{S}(\mathbb{G})$ to $\mathscr{S}(\widehat{\mathbb{G}})$ by definition of $\mathscr{S}(\widehat{\mathbb{G}})$. This enables us to define the group Fourier transform as the following Gelfand triple isomorphism.

**Definition 3.1.4.** *The group Fourier transform $\mathcal{F}_{\mathbb{G}}$ is the unique Gelfand triple isomorphism*

$$\mathcal{F}_{\mathbb{G}}\colon \mathcal{G}(\mathbb{G},\mu) \to \mathcal{G}(\widehat{\mathbb{G}},\widehat{\mu}) \tag{3.1.2}$$

*that extends the map $L^1(\mathbb{G},\mu) \cap L^2(\mathbb{G},\mu) \to B^2(\widehat{\mathbb{G}},\widehat{\mu})\colon f \mapsto \mathcal{F}_{\mathbb{G}}f$.*

With the help of the group Fourier transform we will define the Kohn-Nirenberg quantization for a compact or connected, simply connected nilpotent group $\mathbb{G}$. We will use the symbol $\mathcal{K}_{\mathbb{G}}$ specifically for the kernel map

$$\mathcal{K}_{\mathbb{G}} \colon \mathcal{L}(\mathcal{G}(\mathbb{G},\mu);\mathcal{G}(\mathbb{G},\mu)) \to \mathcal{G}(\mathbb{G},\mu) \otimes \mathcal{G}(\mathbb{G},\mu)$$

from Proposition 1.4.9. Now consider the map

$$\mathscr{S}(\mathbb{G}) \mathbin{\hat{\otimes}} \mathscr{S}(\mathbb{G}) \to \mathscr{S}(\mathbb{G}) \mathbin{\hat{\otimes}} \mathscr{S}(\mathbb{G}) \colon f \mapsto \mathcal{T}_{\mathbb{G}} f\,, \quad \text{in which} \quad \mathcal{T}_{\mathbb{G}} f(x,y) := f(x, xy^{-1})$$

for all $x, y \in \mathbb{G}$. This map is a linear homeomorphism and extends to a unitary map from $L^2(\mathbb{G},\mu) \mathbin{\hat{\otimes}_{\mathrm{H}}} L^2(\mathbb{G},\mu)$ onto itself. Thus we may extend the above map $f \mapsto \mathcal{T}_{\mathbb{G}} f$ to a Gelfand triple isomorphism

$$\mathcal{T}_{\mathbb{G}} \colon \mathcal{G}(\mathbb{G},\mu) \otimes \mathcal{G}(\mathbb{G},\mu) \to \mathcal{G}(\mathbb{G},\mu) \otimes \mathcal{G}(\mathbb{G},\mu)\,.$$

Now we are able to define the Kohn-Nirenberg quantization as a Gelfand triple isomorphism.

**Definition 3.1.5.** *The Kohn-Nirenberg quantization is the Gelfand triple isomorphism*

$$\mathrm{Op}_{\mathbb{G}} \colon \mathcal{G}(\mathbb{G},\mu) \otimes \mathcal{G}(\widehat{\mathbb{G}},\widehat{\mu}) \to \mathcal{L}(\mathcal{G}(\mathbb{G},\mu);\mathcal{G}(\mathbb{G},\mu))\,,$$

*defined by* $\mathrm{Op}_{\mathbb{G}} := \mathcal{K}_{\mathbb{G}}^{-1} \mathcal{T}_{\mathbb{G}}^{-1} (\mathrm{I} \otimes \mathcal{F}_{\mathbb{G}})^{-1}$.

The object $\sigma \in \mathscr{S}'(\mathbb{G}) \mathbin{\hat{\otimes}} \mathscr{S}'(\widehat{\mathbb{G}})$ is called Kohn-Nirenberg symbol to the operator $\mathrm{Op}_{\mathbb{G}}(\sigma)$ and $\mathrm{Op}_{\mathbb{G}}(\sigma)$ is called the Kohn-Nirenberg operator to $\sigma$.

## 3.2 The Pedersen quantization

Let $\mathbb{G}$ be a connected, simply connected Lie group. We already introduced the notation $\mathrm{Ad}_{\mathbb{G}}$ for the adjoint action of $\mathbb{G}$ on $\mathfrak{g}$ and $\mathrm{Ca}_{\mathbb{G}}(x)\xi := \xi \circ \mathrm{Ad}_{\mathbb{G}}(x^{-1})$ for the coadjoint action of $x \in \mathbb{G}$ on linear functionals $\xi \in \mathfrak{g}'$. We will now discuss how coadjoint orbits relate to the Pedersen quantization and in which way the Pedersen quantization can be understood as a Gelfand triple isomorphism.

We start this endeavour, by revisiting the correspondence between coadjoint orbits and unitary irreducible admissible representations. A subalgebra $\mathfrak{m} \subset \mathfrak{g}$ is called polarizing to $\xi \in \mathfrak{g}'$, iff $\xi([\mathfrak{m}, \mathfrak{m}]) = \{0\}$ and $\mathfrak{m}$ is a maximal algebra fulfilling this condition. For any $\xi \in \mathfrak{g}'$ we can find at least one polarizing algebra. There is a bijection between the coadjoint Orbits and the irreducible unitary representations of $\mathbb{G}$. It can be described by $[\pi] \leftrightarrow \Omega = \mathrm{Ca}_{\mathbb{G}}\, \xi$, where $\pi$ is unitarily equivalent to the induced representation of $\chi(m) = \mathrm{e}^{2\pi\mathrm{i}\xi(m)}$ for $m \in \mathfrak{m} \subset \mathbb{G}$ for some maximal subordinate algebra $\mathfrak{m}$ of $\xi$ [16, Theorems 2.2.1 - 2.2.4]. This correspondence only depends on the orbit $\Omega$ and not on the choice of element $\xi$ spanning $\Omega$ or the choice of polarizing algebra $\mathfrak{m}$. We will write $\pi \sim \xi$ or $\pi \sim \Omega$, if the equivalence class of $\pi$ corresponds to the orbit $\Omega = \mathrm{Ca}_{\mathbb{G}}(\mathbb{G})\xi$. For any $\xi$ the orbit $\Omega = \mathrm{Ca}_{\mathbb{G}}(\mathbb{G})\xi$ is an even dimensional polynomial manifold [56, page 521] and [16, Lemma 1.3.2]. The following theorem from [8] shows that we may use the correspondence between orbits and irreducible representations in order to describe the topology on $\widehat{\mathbb{G}}$.

**Theorem 3.2.1.** *Let $\sim$ be the equivalence relation on $\mathfrak{g}'$ defined by*

$$\xi \sim \eta \quad :\Leftrightarrow \quad \exists_{x \in \mathbb{G}} \colon \mathrm{Ca}_{\mathbb{G}}(x)\xi = \eta\,.$$

*Then the bijection from $\widehat{\mathbb{G}}$ to the quotient space $\mathfrak{g}'/\sim$ defined by*

$$[\pi] \mapsto \Omega \quad \textit{for} \quad \pi \sim \Omega$$

*is a homeomorphism.*

A Jordan-Hölder basis of $\mathfrak{g}$ is a basis $(e_j)_j$ such that the linear hull $\mathfrak{g}_k = \mathrm{span}\{e_1, \dots e_k\}$ is an ideal in $\mathfrak{g}$ for each $k \leq \dim \mathbb{G}$. Let $q_k$ be the quotient map $\mathfrak{g}' \to \mathfrak{g}'/\mathfrak{g}_k^\circ$. The set of jump indices $J$ is the set of $j > 1$ such that

$$\dim q_j(\Omega) - \dim q_{j-1}(\Omega) = 1 \tag{3.2.3}$$

Let us denote $\mathfrak{g}_J := \mathrm{span}\{e_j \mid j \in J\}$. From Corollary 3.1.5 of [16] follows that a polynomial chart of $\Omega$ is given by

$$\sigma_\Omega \colon \Omega \to \mathfrak{g}_J' \colon \xi \mapsto \xi \restriction_{\mathfrak{g}_J}\,.$$

This correspondence between orbits and the spaces of functionals $\mathfrak{g}'_J$ leads to the definition of the orbital Fourier transform

$$\mathcal{F}_\Omega \varphi(x) := \int_\Omega \mathrm{e}^{-2\pi \mathrm{i}\xi(x)} \varphi(\xi)\, \mathrm{d}\theta_\Omega(\xi), \quad x \in \mathfrak{g}_J,\ \varphi \in \mathscr{S}(\Omega),$$

where $\theta_\Omega \circ \sigma_\Omega^{-1}$ is a Haar measure on $\mathfrak{g}'_J$. The Pedersen quantization (see [57]) is the equivalent of the Weyl quantization for general connected, simply connected nilpotent Lie groups. It is defined by the integral

$$\mathrm{op}_\pi(\varphi) := \int_{\mathfrak{g}_J} \pi(\exp_{\mathbb{G}} x) \int_\Omega \mathrm{e}^{-2\pi \mathrm{i}\xi(x)} \varphi(\xi)\, \mathrm{d}\theta_\Omega(\xi)\, \mathrm{d}\nu_\Omega(x)\,, \qquad \text{for } \varphi \in \mathscr{S}(\Omega)$$

for some representation $\pi \sim \Omega$ and a fitting Haar measure $\nu_\Omega$ on $\mathfrak{g}_J$. We can easily see that the outermost integral converges in $\mathcal{L}_s(H_\pi)$. The following theorem fixes the choice of $\nu_\Omega$.

**Theorem 3.2.2.** *For each $\theta_\Omega$ as above, there is a unique $\nu_\Omega$ such that the Pedersen quantization to $\pi \sim \Omega$ extends to a Gelfand triple isomorphism*

$$\mathrm{op}_\pi \colon \mathcal{G}(\Omega, \theta_\Omega) \to \mathcal{G}_{\mathrm{op}}(\pi).$$

*Proof.* This is essentially stated in [57, Theorem 4.1.4]. Here Pedersen proves that

$$\mathscr{S}(\Omega) \to \mathcal{B}(H_\pi)_\infty \colon a \mapsto \mathrm{op}_\pi(a)$$

is a homeomorphism, where $\mathcal{B}(H_\pi)_\infty$ is the space of smooth operators with respect to $\pi$. The spaces of smooth operators is introduced as $\mathcal{B}(H_\pi)_\infty = \mathscr{E}(\Pi)$, where $\Pi$ is the unitary representation of $\mathbb{G} \times \mathbb{G}$ on $\mathcal{HS}(H_\pi)$ defined by $\Pi(x, y)T = \pi(x) \circ T \circ \pi(y)^{-1}$. Furthermore Pedersen shows that

$$\int_\Omega a\, \overline{b}\, \mathrm{d}\theta_\Omega = \mathrm{Tr}[\mathrm{op}_\pi(a)\ \mathrm{op}_\pi(b)^*] \quad \text{for } a, b \in \mathcal{S}(\Omega)$$

and for a suitable $\nu_\Omega$. Note that Pedersen uses the convention $\xi \leftrightarrow \chi(\cdot) = \mathrm{e}^{\mathrm{i}\xi \circ \log_{\mathbb{G}}(\cdot)}$ for the bijection between functionals and characters. Though adjusting the formulas just results in additional constants, which may be hidden away inside the measures $\nu_\Omega$ and $\theta_\Omega$.

In order to fit this result into our scheme, we will make sure that $\mathcal{L}(\mathscr{E}(\pi)', \mathscr{E}(\pi))$ and $\mathcal{B}(H_\pi)_\infty$ coincide as topological vector spaces. Using the embeddings $\mathscr{E}(\pi) \hookrightarrow H_\pi$

and $H_\pi \hookrightarrow \mathscr{E}(\pi)'$ defined by some real structure $\mathcal{C}_\pi$, we will consider $\mathcal{B}(H_\pi)_\infty$, $\mathcal{L}(H_\pi)$, $\mathcal{HS}(H_\pi)$ and $\mathcal{L}(\mathscr{E}(\pi)', \mathscr{E}(\pi))$ as linear subspaces of $\mathcal{L}(\mathscr{E}(\pi); \mathscr{E}(\pi)')$. With respect to these embeddings $\mathcal{L}(\mathscr{E}(\pi)'; \mathscr{E}(\pi))$ is exactly the subspace of operators $T$ such that

$$\pi(P')T\pi(P) \in \mathcal{L}(H_\pi) \quad \text{for all} \quad P, P' \in \mathrm{Diff_L}(\mathbb{G})$$

equipped with the corresponding seminorms

$$T \mapsto \|\pi(P')T\pi(P)\|_{\mathcal{L}(H_\pi)} .$$

Similarly, $\mathcal{B}(H_\pi)_\infty$ is identified with all operators $T$ such that $\pi(P')T\pi(P) \in \mathcal{HS}(H_\pi)$ for all $P, P' \in \mathrm{Diff_L}(\mathbb{G})$ equipped with the corresponding seminorms $T \mapsto \|\pi(P')T\pi(T)\|_{\mathcal{HS}(H_\pi)}$. Thus we have the continuous embedding $\mathcal{B}(H_\pi)_\infty \hookrightarrow \mathcal{L}(\mathscr{E}(\pi)'; \mathscr{E}(\pi))$.

Due to [12, Théorème 2.6] we know there is $P \in \mathrm{Diff_L}(\mathbb{G})$ such that $\pi(P)$ is invertible on $\mathscr{E}(\pi)$ and $\pi(P)^{-1}$ can be extended to a nuclear operator on $H_\pi$. Thus

$$\|\pi(P'')T\pi(P')\|_{\mathcal{HS}(H_\pi)} \leq \|\pi(P)^{-1}\|_{\mathcal{HS}(H_\pi)} \|\pi(P'')T\pi(P'P)\|_{\mathcal{L}(H_\pi)}$$

for each $T \in \mathcal{L}(\mathscr{E}(\pi)'; \mathscr{E}(\pi))$ and as a result $\mathcal{B}(H_\pi)_\infty = \mathcal{L}(\mathscr{E}(\pi)'; \mathscr{E}(\pi))$ as topological vector spaces.

□

## 3.3 Alternative Gelfand triples for the Fourier transform on homogeneous Lie groups

In the following section $\mathbb{G}$ is always a connected, simply connected nilpotent Lie group with Haar measure $\mu$ and Lie algebra $\mathfrak{g}$ and corresponding center $\mathfrak{z} := Z(\mathfrak{g})$. Note that always $\exp_{\mathbb{G}} \mathfrak{z} = Z(\mathbb{G})$.

If $\mathbb{X}$ is a compact group with Haar measure $\nu$, then $\widehat{\mathbb{X}}$ is discrete and for each $\pi \in \mathrm{Irr}(\mathbb{X})$ the space $H_\pi$ is finitely dimensional. This ensures that one can work with the Fourier image $\mathcal{G}(\widehat{\mathbb{X}}, \widehat{\nu})$ of the Gelfand triple $\mathcal{G}(\mathbb{X}, \nu)$ relatively easy (see [59]). Here the simple characterization of $\mathscr{S}(\widehat{\mathbb{X}})$ and the simple identification of multiplication operators on

$\mathscr{S}'(\widehat{\mathbb{X}})$ are very convenient. This also results in the equation

$$(\mathrm{I}_{\mathscr{S}'(\widehat{\mathbb{X}})}\,\varepsilon\zeta_\pi)(a) := a(-,\pi) = \pi(-)^* \cdot A\,\varepsilon\,\mathrm{I}_{\mathcal{L}(H_\pi)}(\pi)\,, \qquad \text{for } \pi \in \mathrm{Irr}(\mathbb{X})\,, \tag{3.3.4}$$

in which $A = \mathrm{Op}_{\mathbb{X}}(a) \in \mathcal{L}(\mathscr{S}(\mathbb{X}); \mathscr{S}'(\mathbb{X}))$ and we use the notation $\zeta_\pi \in \mathcal{L}(\mathscr{S}'(\widehat{\mathbb{X}}); \mathcal{L}(H_\pi))$ for $\zeta_\pi \colon \sigma \mapsto \sigma(\pi)$. Here $H_\pi$ is finite dimensional, so $\pi \in \mathscr{S}(\mathbb{X}; \mathcal{L}(H_\pi))$ and the multiplication of $\pi(-)^*$ with the $\mathcal{L}(H_\pi)$-valued distribution $A\,\varepsilon\,\mathrm{I}_{\mathcal{L}(H_\pi)}(\pi)$ can be understood componentwise with respect to some basis[3].

For the nilpotent Lie group $\mathbb{G}$ the situation is much more complicated. Even for the Heisenberg group $\mathbb{G} = \mathbb{H}$ there seems to be no simple intrinsic characterization for the Fourier image of the Schwartz space of rapidly decreasing smooth functions $\mathscr{S}(\mathbb{G})$, see [31, 2]. But we may derive a simple characterization of the Fourier image for a certain subspace $\mathscr{S}_*(\mathbb{G})$ of $\mathscr{S}(\mathbb{G})$. This characterization not only induces a Gelfand triple $(\mathscr{S}_*(\mathbb{G}), L^2(\mathbb{G},\mu), \mathscr{S}_*'(\mathbb{G}))$ but will also enable us to identify a large class of well behaved multiplication operators on the Fourier image of $\mathscr{S}_*(\mathbb{G})$. Using these multiplication operators, we can prove an analogue of (3.3.4) in Section 3.4.

### 3.3.1 Generic and flat orbits of homogeneous Lie groups

In order to get a better description of the group Fourier transform on homogeneous Lie groups, we will use the Pedersen quantization. Though in our case, we can first simplify the Pedersen quantization, since we are only interested in representations derived from a special class of orbits, the *generic orbits*. We start by introducing and discussing a certain subset of $\mathrm{Irr}(\mathbb{G})$.

**Definition 3.3.1.** *A representation $\pi \in \mathrm{Irr}(\mathbb{G})$ is* square integrable modulo the center*, if $x \mapsto |(\pi(x)v, w)_{H_\pi}|$ is square integrable on $\mathbb{G}/Z(\mathbb{G})$ with respect to the Haar measure for all $v, w \in H_\pi$. Let us denote the set of irreducible representations, that are square integrable modulo the center, by $\mathrm{SI/Z}(\mathbb{G}) \subset \mathrm{Irr}(\mathbb{G})$ and pairs of such representations together with some matching real structure by $\mathrm{SI/Z}_{\mathbb{R}}(\mathbb{G})$.*

[3]This is also consistent with our prior discussions of bilinear maps between tensor products in Theorem 1.2.11 or Theorem 1.2.12.

Suppose $\pi \sim \Omega = \mathrm{Ca}_{\mathbb{G}}(\mathbb{G})\xi$, then $\pi \in \mathrm{SI/Z}(\mathbb{G})$ if and only if $\Omega = \xi + \mathfrak{z}^\circ$ [53, Theorem 1]. Orbits of this type are called *flat*. Furthermore, if $\mathrm{SI/Z}(\mathbb{G}) \neq \emptyset$, then the orbits to representations in $\mathrm{SI/Z}(\mathbb{G})$ are exactly those having the maximal possible dimension [16, Corollary 4.5.6]. Also, for $\pi \in \mathrm{SI/Z}(\mathbb{G})$ the equivalence class $[\pi] \in \widehat{\mathbb{G}}$ is uniquely determined by the central character $\pi \restriction_{Z(\mathbb{G})} = e^{2\pi \mathrm{i}\xi \circ \exp_{\mathbb{G}}^{-1}(-)} \mathrm{id}_{H_\pi}$, where $\xi \in Z(\mathfrak{g})'$ [16, Proposition 4.5.7]. Now, if $(e_j)_j$ is a Jordan Hölder-Basis with $Z(\mathfrak{g}) = \mathrm{span}_{\mathbb{R}}\{e_1, e_2, \dots, e_k\}$ and $\pi \in \mathrm{SI/Z}(\mathbb{G})$, then the corresponding jump indices are given by $J = \{k+1, k+2, \dots, \dim \mathbb{G}\}$.

Now we will describe, why the representations in $\mathrm{SI/Z}(\mathbb{G})$ are very convenient when working with the Pedersen quantization. We use the notation from Theorem 3.2.2. For all $\pi \in \mathrm{SI/Z}(\mathbb{G})$ the Pedersen quantization is simpler, because we can just take *one* Haar measure $\theta$ on $\mathfrak{z}^\circ$ and translate it to a measure $\theta_\Omega$ on $\Omega \sim \pi$ for each $\pi \in \mathrm{SI/Z}(\mathbb{G})$. The subspace $\omega := \mathfrak{g}_J$ complements $\mathfrak{z}$ in $\mathfrak{g}$ and is the same for each representation in $\mathrm{SI/Z}(\mathbb{G})$. We get a Gelfand triple isomorphism

$$T_\Omega \colon \mathcal{G}(\mathfrak{z}^\circ, \theta) \to \mathcal{G}(\Omega, \theta_\Omega) \quad \text{defined by} \quad T_\Omega \varphi := \varphi \circ P_{\mathfrak{z}^\circ} \restriction_\Omega \quad \text{for } \varphi \in \mathscr{S}(\mathfrak{z}^\circ),$$

where $P_{\mathfrak{z}^\circ}$ is the projection onto $\mathfrak{z}^\circ$ along $\omega^\circ$. Using this isomorphism, we adjust the Pedersen quantization.

**Definition 3.3.2.** *We will use the Pedersen quantization $\mathfrak{op}_\pi$ on $\mathcal{G}(\mathfrak{z}^\circ, \theta)$ with respect to $\pi \in \mathrm{SI/Z}(\mathbb{G})$ defined by*

$$\mathfrak{op}_\pi \colon \mathcal{G}(\mathfrak{z}^\circ, \theta) \to \mathcal{G}_{\mathrm{op}}(\pi), \ \phi \mapsto \mathrm{op}_\pi(T_\Omega \phi).$$

This version of the Pedersen quantization takes on the form

$$\mathfrak{op}_\pi(\varphi) = \int_\omega \pi(\exp_{\mathbb{G}} x) \int_{\mathfrak{z}^\circ} \mathrm{e}^{-2\pi \mathrm{i}\xi(x)} \varphi(\xi) \, \mathrm{d}\theta(\xi) \, \mathrm{d}\nu(x) \qquad \text{for } \varphi \in \mathscr{S}(\mathfrak{z}^\circ),$$

where $\nu = \nu_\Omega$ depends on $\theta$. Naturally, $\mathfrak{op}_\pi$ is a Gelfand triple isomorphism as well.

Now we will discuss the concept of generic orbits and square integrable (modulo the center) representation in context with homogeneous groups.

**Definition 3.3.3.** *A connected, simply connected Lie group $\mathbb{G}$ is called a homogeneous Lie group if its Lie algebra $\mathfrak{g}$ is equipped with a group of dilations*

$$(0, \infty) \to \mathrm{Hom}(\mathfrak{g}) \colon \lambda \mapsto \delta_\lambda,$$

*where* $\delta_\lambda x = \mathrm{e}^{\log(\lambda)A}x$ *is also a Lie algebra isomorphism and* $A$ *is a diagonalizable map with positive eigenvalues. The number* $Q := \mathrm{Tr}[A]$ *is the* homogeneous dimension *of* $\mathbb{G}$*. We will equip* $\mathbb{G}$ *with a family of group automorphisms, also denoted by* $(\delta_\lambda)_{\lambda>0}$*, defined by* $\delta_\lambda \circ \exp_{\mathbb{G}} := \exp_{\mathbb{G}} \circ \delta_\lambda$*.*

We may always decompose $\mathfrak{g}$ into eigenspaces $\mathcal{E}_\kappa$ of $A$ to eigenvalues $\kappa > 0$, i.e.

$$\mathfrak{g} = \bigoplus_{\kappa>0} \mathcal{E}_\kappa, \quad \text{where} \quad [\mathcal{E}_\kappa, \mathcal{E}_{\kappa'}] \subset \mathcal{E}_{\kappa+\kappa'}.$$

Thus, a homogeneous Lie group is always *nilpotent.* Note that the center $\mathfrak{z}$ of $\mathfrak{g}$ is always invariant to both $\delta_\lambda$ and $A$, since

$$[\delta_\lambda z, x] = \delta_\lambda[z, \delta_{\lambda^{-1}}x] = 0 \quad \text{for all} \quad \lambda > 0, z \in \mathfrak{z} \text{ and } x \in \mathfrak{g}.$$

For every $\mu > 0$ the space $\bigoplus_{\kappa\geq\mu} \mathcal{E}_\kappa$ is an ideal in $\mathfrak{g}$. We may always choose a Jordan-Hölder basis $(e_j)_j$ through these ideals [16, Theorem 1.1.13], i.e. we can choose a Jordan-Hölder basis $(e_j)_j$ such that $e_j$ is an eigenvector to $\delta_\lambda$ for any $j$. If $\mathfrak{z}$ is an eigenspace, e.g. if $\dim \mathfrak{z} = 1$, we also have the unique decomposition

$$\mathfrak{g} = \mathfrak{z} \oplus \omega, \quad \omega \text{ is } A\text{-invariant.}$$

**Definition 3.3.4.** *A coadjoint orbit* $\Omega$ *of a connected, simply connected nilpotent Lie group is called* generic *with respect to a given Jordan-Hölder basis* $(e_j)_j$ *if for each* $k$ *the dimension of the manifold* $q_k(\Omega)$ *is maximal compared to all other orbits, in which* $q_k$ *is defined as in* (3.2.3)*.*

*Let* $\mathbb{G}$ *be a connected, simply connected homogeneous Lie group. If* $(e_j)_j$ *is a Jordan-Hölder basis of eigenvectors to* $A$ *and the* $\delta_\lambda$*, then we will denote the set of equivalence classes derived from generic orbits by* $\widehat{\mathbb{G}}_{\mathrm{gen}} \subset \widehat{\mathbb{G}}$*.*

The first convenient property of the generic orbits is that the Plancherel measure $\widehat{\mu}$ is concentrated on $\widehat{\mathbb{G}}_{\mathrm{gen}}$ by [16, Theorem 4.3.16].

Next, we will discuss the interaction between the concept of generic orbits and the concept of square integrable representations (modulo the center). If $[\pi] \in \widehat{\mathbb{G}}_{\mathrm{gen}}$ and $\mathrm{SI/Z}(\mathbb{G}) \neq \emptyset$, then $\pi \in \mathrm{SI/Z}(\mathbb{G})$, since the representations in $\mathrm{SI/Z}(\mathbb{G})$ correspond to the orbits of maximal dimension. Also, if $\mathrm{SI/Z}(\mathbb{G}) \neq \emptyset$ and $\dim \mathfrak{z} = 1$, then situation is especially easy. Here $\mathfrak{z} = \mathcal{E}_\mu$ for $\mu = \max\{\kappa > 0 \mid \mathcal{E}_\kappa \neq \{0\}\}$ and for a Jordan-Hölder basis $(e_j)_j$ of eigenvectors, we always have $\mathfrak{z} = \mathfrak{g}_1 = \mathbb{R} \cdot e_1$. Thus the set $\widehat{\mathbb{G}}_{\mathrm{gen}}$ does not depend on the concrete choice of Jordan-Hölder basis of eigenvectors to $A$. If $J$ is the set of jump indices to any Jordan-Hölder basis of eigenvectors, then we also have $\mathfrak{g}_J = \omega$ by the above discussion, which is important for handling the Pedersen quantization.

Now for $\lambda < 0$ denote

$$\delta_\lambda x := -\delta_{|\lambda|} x \text{ for } x \in \mathfrak{z}, \quad \text{and} \quad \delta_\lambda x := \delta_{|\lambda|} x \text{ for } x \in \omega.$$

Furthermore, let $\delta_\lambda \xi := \xi \circ \delta_\lambda$ for $\lambda \in \mathbb{R}^\times$ and $\xi \in \mathfrak{g}'$.

The question arises whether generic orbits are mapped to generic orbits by $\delta_\lambda$. The dilation $\delta_\lambda$ on $\mathfrak{g}'/\mathfrak{g}_k^\circ$ is a well-defined vector space isomorphism by $\delta_\lambda \circ q_j := q_j \circ \delta_\lambda$, since $\mathfrak{g}_k$ and thus also $\mathfrak{g}_k^\circ$ are $\delta_\lambda$-invariant. Furthermore,

$$\dim q_j(\delta_\lambda \Omega) = \dim \delta_\lambda \circ q_j(\Omega) = \dim q_j(\Omega). \tag{3.3.5}$$

Thus $\delta_\lambda \Omega$ is generic for each $\lambda \in \mathbb{R}^\times$.

**Definition 3.3.5.** *For any connected, simply connected homogeneous Lie group $\mathbb{G}$ and any $\pi \in \mathrm{Irr}_{\mathbb{R}}(\mathbb{G})$ with real structure $\mathcal{C}_\pi$ we put $\overline{\pi} := \mathcal{C}_\pi \pi \mathcal{C}_\pi \in \mathrm{Irr}_{\mathbb{R}}(\mathbb{G})$ equipped with the same real structure $\mathcal{C}_\pi$. We define the representations $\pi_\lambda$ for $\lambda \in \mathbb{R}^\times$ by*

$$\pi_\lambda(x) := \pi(\delta_\lambda x) \textit{ for } \lambda > 0 \quad \textit{and} \quad \pi_\lambda(x) := \overline{\pi}_{|\lambda|}(x) := \overline{\pi}(\delta_{|\lambda|} g) \textit{ for } \lambda < 0$$

*for all $x \in \mathbb{G}$.*

All the representations $\pi_\lambda$ are admissible irreducible unitary representations acting on $H_\pi$ with $\mathscr{E}(\pi_\lambda) = \mathscr{E}(\pi)$.

We already used two sets of examples in Definition 2.4.20 and Definition 2.4.21. Indeed $\delta_\lambda^{\mathbb{H}}$ resp. $\delta_\lambda^{\mathbb{H}_2}$ makes the Heisenberg group $\mathbb{H}$ resp. the Dynin-Folland group $\mathbb{H}_2$ into a homogeneous Lie group. Moreover, the representations $\rho_\lambda$ and $\Theta_\lambda$ fulfil

$$\rho_\lambda = (\rho_1)_\lambda \quad \text{and} \quad \Theta_\lambda = (\Theta_1)_\lambda \quad \text{for all } \lambda \in \mathbb{R}^\times .$$

With these definitions and the discussion above we get the three equivalences

- $\pi \in \mathrm{SI/Z}(\mathbb{G})$ if and only if $\pi_\lambda \in \mathrm{SI/Z}(\mathbb{G})$,
- $[\pi] \in \widehat{\mathbb{G}}_{\mathrm{gen}}$ if and only if $[\pi_\lambda] \in \widehat{\mathbb{G}}_{\mathrm{gen}}$,
- $\pi \sim \xi$ if and only if $\pi_\lambda \sim \delta_\lambda \xi$.

Let $\mathrm{SI/Z}(\mathbb{G}) \neq \emptyset$ and $\dim \mathfrak{z} = 1$. As every equivalence class of representations in $\mathrm{SI/Z}(\mathbb{G})$ only depends on its central character on $\mathfrak{z} \simeq \mathbb{R}$, we get a bijection between $\mathbb{R}^\times$ and $\widehat{\mathbb{G}}_{\mathrm{gen}}$ resp. $\{[\pi] \mid \pi \in \mathrm{SI/Z}(\mathbb{G})\}$. Thus $\pi \in \mathrm{SI/Z}(\mathbb{G})$ if and only if $[\pi] \in \widehat{\mathbb{G}}_{\mathrm{gen}}$.

We can even go one step further. The dilations $\delta_\lambda$ help us to understand $\widehat{\mathbb{G}}$ as a measure space. For this purpose we need the Pfaffian $\mathit{Pf}(\xi)$ to a coadjoint orbit $\Omega = \mathrm{Ca}_{\mathbb{G}}(\mathbb{G})\xi$, which is defined by $\mathit{Pf}(\xi)^2 = \det B_\xi$ up to a sign. Here we use the matrix $B_\xi := (\xi([e_j, e_i]))_{i,j=2}^{\dim \mathbb{G}}$, in which $(e_j)_{j=1}^{\dim \mathbb{G}}$ is a Jordan-Hölder basis of eigenvectors to $A$ and the $(e_j)_{j=2}^{\dim \mathbb{G}}$ span the complementary space $\omega$ to $\mathfrak{z}$.

**Definition 3.3.6.** *Let $\kappa > 0$ be the real number such that $\delta_\lambda \eta := \mathrm{sgn}(\lambda)|\lambda|^\kappa \eta$ for $\eta \in \omega^\circ$, let $\omega^\times \ni \ell \sim \pi \in \mathrm{SI/Z}(\mathbb{G})$ and let $(e_j)_{j=1}^{\dim \mathbb{G}}$ be a Jordan-Hölder basis of eigenvectors to the dilations $\delta_\lambda$ resp. $A$. Suppose furthermore that $\langle e_j, \ell \rangle = \delta_{j,1}$ and $\mu(E) = 1$ for $E = \{\sum_j t_j e_j \mid t \in [0,1]^{\dim \mathbb{G}}\}$. Then we define the measure $\widehat{\mu}_\pi$ on $\mathbb{R}^\times$ by*

$$\mathrm{d}\widehat{\mu}_\pi(\lambda) := \kappa |\lambda|^{Q-1} |\mathit{Pf}(\ell)| \, \mathrm{d}\lambda \,,$$

*in which the Pfaffian $|\mathit{Pf}(\ell)|$ is calculated with respect to $(e_j)_j$.*

The measure $\widehat{\mu}_\pi$ depends on $\pi$ and $\mu$, but does not depend on the concrete choice of Jordan-Hölder basis $(e_j)_j$ as long as it fulfils the criteria for the definition above. This statement is a direct conclusion of the following proposition.

**Proposition 3.3.7.** *Suppose $\mathbb{G}$ is a homogeneous Lie group, $\pi \in \mathrm{SI/Z}_{\mathbb{R}}(\mathbb{G})$ and $\dim \mathfrak{z} = 1$ and let $\widehat{\mu}_\pi$ be defined as above. Then*

$$(\mathbb{R}^\times, \widehat{\mu}_\pi) \to (\widehat{\mathbb{G}}_{\mathrm{gen}}, \widehat{\mu}) \colon \lambda \mapsto [\pi_\lambda],$$

*where $\pi \sim \ell \in \omega^\circ$, is a homeomorphism and an isomorphism between the Borel measure spaces. Furthermore, if $\Omega$ is a fixed generic orbit, then $\lambda \mapsto \delta_\lambda \Omega$ defines a bijection between $\mathbb{R}^\times$ and the generic orbits.*

*Proof.* Let $U$ be the set of functionals $\xi \in \mathfrak{g}'$ such that $\mathrm{Ca}_{\mathbb{G}}\,\xi$ is a generic orbit with respect to our basis. For $\xi \in U$ we have $\delta_\lambda \xi \in U$ for each $\lambda \in \mathbb{R}^\times$ by equation (3.3.5). Each orbit meets $U \cap \omega^\circ$ in exactly one point [16, Theorem 3.1.9 and Theorem 4.5.5]. Furthermore, for any $\xi \in \omega^\times := \omega^\circ \setminus \{0\}$, we have that

$$\mathbb{R}^\times \to \omega^\times \colon \lambda \mapsto \delta_\lambda \xi$$

is a homeomorphism. Thus also $U \cap \omega^\circ = \omega^\times = \{\delta_\lambda \ell \mid \lambda \in \mathbb{R}^\times\}$. But $\omega^\times$ induces all maximal flat orbits, so they coincide with the generic orbits. Since the correspondence of $\mathfrak{g}'/_\sim$ with $\widehat{\mathbb{G}}$ is a homeomorphism by Theorem 3.2.1, we also have $U/_\sim \simeq \widehat{\mathbb{G}}_{\mathrm{gen}}$ with respect to the subspace topologies. Let $q \colon U \to U/_\sim$ be the quotient map. Now $q \restriction_{\omega^\times}$ is a continuous bijection. We show that it is also open. By [16, Theorem 3.1.9], there is a well-defined map $\psi \colon \omega^\times \times \mathfrak{z}^\circ \to U$ such that

$$\psi(u, v) = w \quad \Leftrightarrow \quad w \in \mathrm{Ca}_{\mathbb{G}}(\mathbb{G})u \text{ and } P_{\mathfrak{z}^\circ} w = v,$$

where $P_{\mathfrak{z}^\circ}$ is the projection onto $\mathfrak{z}^\circ$ along $\omega^\circ$. The map $\psi$ is a rational, non singular bijection with rational non singular inverse. Hence $\psi$ is a homeomorphism. If $V \subset \omega^\times$ is open in $\omega^\times$, then $\mathrm{Ca}_{\mathbb{G}}(\mathbb{G})V$ is open in $U$, since

$$\psi(V \times \mathfrak{z}^\circ) = \mathrm{Ca}_{\mathbb{G}}\, V.$$

Now, since $q$ is open and $q(\mathrm{Ca}_{\mathbb{G}}(\mathbb{G})V) = q(V)$, the restriction $q \restriction_{\omega^\times}$ is an open map and thus a homeomorphism. If we now denote

$$\sigma \colon \mathbb{R}^\times \to \widehat{\mathbb{G}}_{\mathrm{gen}} \colon \lambda \mapsto [\delta_\lambda \pi],$$

then $\sigma$ is a homeomorphism by the discussion above. Let $\varphi \colon \widehat{\mathbb{G}} \to [0, \infty)$ be Borel measurable. Then, by [16, Theorem 4.3.10] and the subsequent discussion,

$$\int_{\widehat{\mathbb{G}}} \varphi([\pi]) \,\mathrm{d}\widehat{\mu}([\pi]) = \int_{U \cap \omega^\circ} \varphi([\pi_\xi]) |\mathit{Pf}(\xi)| \,\mathrm{d}\widetilde{\mu}(\xi),$$

where $\widetilde{\mu}$ is the Haar measure on $\omega^\circ$such that $\{t\ell \mid t \in [0,1]\}$ has measure equal to one and $\pi_\xi \sim \mathrm{Ca}_{\mathbb{G}}(\mathbb{G})\xi$. Let $B := A \restriction_{\omega}^{\omega}$. Since our chosen Jordan-Hölder basis is an eigenbasis to $A$ resp. $\delta_\lambda$, we have

$$|\mathit{Pf}(\delta_\lambda \ell)| = |\det(\delta_\lambda \ell([e_j, e_i]))_{j,i}|^{\frac{1}{2}} = |\det(|\lambda|^{n_i + n_j} \ell([e_j, e_i]))_{j,i}|^{\frac{1}{2}} = |\lambda|^{\mathrm{Tr}\, B} |\mathit{Pf}(\ell)|,$$

where $|\lambda|^{n_j}$ is the eigenvalue of $e_j$ to $\delta_\lambda$ for $j \in J$. Both $\sigma$ and $\sigma^{-1}$ are measurable and we have $\mathrm{d}(\widetilde{\mu} \circ \sigma)(\lambda) = \kappa |\lambda|^{\kappa - 1}\, \mathrm{d}\lambda$. Hence

$$\begin{aligned} \int_{U \cap \omega^\circ} \varphi([\pi_\xi]) |Pf(\xi)|\, \mathrm{d}\widetilde{\mu}(\xi) &= \int_{\mathbb{R}^\times} \varphi([\pi_\lambda]) |Pf(\delta_\lambda \ell)|\, \mathrm{d}(\widetilde{\mu} \circ \sigma)(\lambda) \\ &= \int_{\mathbb{R}^\times} \varphi([\pi_\lambda]) \kappa |\lambda|^{-1 + \operatorname{Tr} A} |Pf(\ell)|\, \mathrm{d}\lambda \end{aligned}$$

and $\sigma$ is a strict isomorphism of measure spaces. □

Since the homeomorphism between $\mathbb{R}^\times$ and $\widehat{\mathbb{G}}_{\mathrm{gen}}$ does not depend on the concrete Jordan-Hölder basis used in the construction in Definition 3.3.6, the measure $\widehat{\mu}_\pi$ is also invariant with respect to this choice.

Again we have suitable examples in the Heisenberg group $\mathbb{H}$ and the Dynin-Folland group $\mathbb{H}_2$. Here a Jordan-Hölder basis of eigenvectors to the dilations is given by the standard basis

$$e_k := (0, \dots, 0, 1, 0, \dots, 0)\,, \quad \text{in which the 1 is at the } k\text{th position.}$$

Let $(e^k)_k$ be the dual basis to this Jordan-Hölder basis. Then

$$(\langle [e_j, e_k], e^1 \rangle)_{j,k \geq 2} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}\,,$$

in which $E$ is the identity matrix. Because $e^1 \sim \rho_1$ resp. $e^1 \sim \Theta_1$, we have

$$\mathrm{d}\widehat{\mu}_{\rho_1}(\lambda) = |\lambda|^{\frac{\dim \mathbb{H} - 1}{2}}\, \mathrm{d}\lambda \quad \text{resp.} \quad \mathrm{d}\widehat{\mu}_{\Theta_1}(\lambda) = |\lambda|^{\frac{\dim \mathbb{H}_2 - 1}{2}}\, \mathrm{d}\lambda\,,$$

in which of course $\mu$ is the standard Haar measure with $\mathrm{d}\mu(t, x', x) = \mathrm{d}(t, x', x)$.

### 3.3.2 The Fourier transform on $\mathscr{S}_*(\mathbb{G})$

The discussion in the prior subsection and especially the last proposition motivate us to make the following convention.

**Convention 3.3.8.** *If not otherwise stated, we will assume for the rest of this chapter that the following holds.*

**(i)** $\mathbb{G}$ *is a connected, simply connected homogeneous Lie group* $\mathbb{G}$ *with Haar measure* $\mu$*, Lie algebra* $\mathfrak{g}$ *and corresponding center* $\mathfrak{z} = Z(\mathfrak{g})$ *such that* $\dim \mathfrak{z} = 1$.

**(ii)** *We will restrict ourselves to the case* $\mathbb{G} = \mathfrak{g}$ *as sets* [4]*. Depending on which property we want to emphasize, we will switch between the symbols* $\mathbb{G}$ *and* $\mathfrak{g}$.

**(iii)** $\omega$ *is the A resp.* $\delta_\lambda$ *invariant subspace of* $\mathfrak{g}$ *such that* $\mathfrak{g} = \mathfrak{z} \oplus \omega$*. We put* $\omega^\times = \omega^\circ \setminus \{0\}$.

**(iv)** *We assume* $\mathrm{SI/Z}(\mathbb{G}) \neq \emptyset$.

We will denote the euclidean Fourier transform on $\mathfrak{g}$ by

$$\mathcal{F}_\mathfrak{g}\varphi(\xi) = \int_\mathfrak{g} \mathrm{e}^{2\pi\mathrm{i}\xi(x)} \varphi(x)\,\mathrm{d}\mu(x), \quad \varphi \in \mathscr{S}(\mathfrak{g}),\ \xi \in \mathfrak{g}'.$$

Naturally, there is exactly one Haar measure $\mu'$ on $\mathfrak{g}'$ such that the Fourier transform is a Gelfand triple isomorphism $\mathcal{G}(\mathfrak{g},\mu) \to \mathcal{G}(\mathfrak{g}',\mu')$. Suppose $\ell \in \omega^\times$. Together with the euclidean Fourier transform and the Pedersen quantization, the map

$$\wp_\ell f(\lambda,\xi) := f(\delta_\lambda(\ell+\xi)) \quad \text{for} \quad \xi \in \mathfrak{z}^\circ, \lambda \in \mathbb{R}^\times \text{ and } f\colon \mathfrak{g}' \to \mathbb{C}$$

will enable us to describe the group Fourier transform on $\mathbb{G}$ (see also [54] for a similar statement).

Now we will use the isomorphism from Proposition 3.3.7 in order to find a new representation of the group Fourier transform on $L^2(\mathbb{G},\mu)$. This will be the basis for the definition of our new Gelfand triples and a Gelfand triple isomorphism in the form of an equivalent Fourier transform.

**Proposition 3.3.9.** *Suppose* $\varphi \in \mathscr{S}(\mathbb{G})$ *and* $\pi \in \mathrm{SI/Z}_\mathbb{R}(\mathbb{G})$ *with* $\pi \sim \ell \in \omega^\times$*, then*

$$\mathcal{F}_\mathbb{G}\varphi(\pi_\lambda) = \begin{cases} \mathfrak{op}_\pi\big(\wp_\ell \mathcal{F}_\mathfrak{g}\varphi(\lambda,-)\big), & \lambda > 0,\\ \mathfrak{op}_{\overline{\pi}}\big(\wp_\ell \mathcal{F}_\mathfrak{g}\varphi(\lambda,-)\big), & \lambda < 0.\end{cases}$$

[4]For connected, simply connected Lie groups $\exp_\mathbb{G}$ is always a polynomial diffeomorphism, so $\mathfrak{g}$, equipped with the group multiplication transported via $\exp_\mathbb{G}$, is isomorphic to $\mathbb{G}$. Hence $\mathbb{G} = \mathfrak{g}$ is not a real restriction. Note that $\mu$ is a biinvariant Haar measure for both the group multiplication *and* the vector addition in $\mathbb{G} = \mathfrak{g}$.

*Proof.* First of all, for any $\varphi \in \mathscr{S}(\mathbb{G})$ we have

$$\mathcal{F}_{\mathbb{G}}\varphi(\pi_\lambda) = \int_{\mathbb{G}} \lambda^{-\operatorname{Tr} A}\varphi(\delta_\lambda^{-1}x)\pi(x)^* \,\mathrm{d}\mu(x) = \lambda^{-\operatorname{Tr} A}\mathcal{F}_{\mathbb{G}}(\varphi \circ \delta_\lambda^{-1})(\pi),$$

for $\lambda > 0$. Also

$$\mathcal{F}_{\mathfrak{g}}(\varphi \circ \delta_\lambda^{-1}) = \lambda^{\operatorname{Tr} A}(\mathcal{F}_{\mathfrak{g}}\varphi) \circ \delta_\lambda$$

for $\lambda > 0$. Note that for $x \in \mathfrak{g}$ and $z \in \mathfrak{z}$ we have $x \cdot z = x + z$ and thus

$$\mathrm{e}^{2\pi\mathrm{i}\ell(z)}\pi(x) = \pi(z)\pi(x) = \pi(z \cdot x) = \pi(z + x).$$

Let $\mu_{\mathfrak{z}}$ resp. $\nu$ be Haar measures on $\mathfrak{z}$ resp $\omega$ such that $\mu = \mu_{\mathfrak{z}} \otimes \nu$, then by the above calculation

$$\begin{aligned}\mathcal{F}_{\mathbb{G}}\varphi(\pi) &= \int_\omega \pi(x) \int_{\mathfrak{z}} \mathrm{e}^{-2\pi\mathrm{i}\ell(z)}\varphi(z - x) \,\mathrm{d}\mu_{\mathfrak{z}}(z)\,\mathrm{d}\nu(x)\\ &= \int_\omega \pi(x) \int_{\mathfrak{z}^\circ} \mathrm{e}^{-2\pi\mathrm{i}\xi(X)}\mathcal{F}_{\mathfrak{g}}\varphi(\xi) \,\mathrm{d}\theta(\xi)\,\mathrm{d}\nu(x).\end{aligned}$$

Here $\theta$ is the measure associated to $\nu$ as described in Definition 3.3.2. This formula indeed holds pointwise. Hence

$$\begin{aligned}\mathcal{F}_{\mathbb{G}}\varphi(\pi_\lambda) &= \lambda^{-\operatorname{Tr} A}\mathfrak{op}_\pi(\mathcal{F}_{\mathfrak{g}}(\varphi \circ \delta_\lambda^{-1})) = \mathfrak{op}_\pi((\mathcal{F}_{\mathfrak{g}}\varphi) \circ \delta_\lambda)\\ &= \mathfrak{op}_\pi\big(\wp_\ell\mathcal{F}_{\mathfrak{g}}\varphi(\lambda, -)\big)\end{aligned}$$

for all $\lambda > 0$. For $\lambda < 0$ we get

$$\mathcal{F}_{\mathbb{G}}\varphi(\pi_\lambda) = \mathcal{F}_{\mathbb{G}}(\overline{\pi}_{-\lambda}) = \mathfrak{op}_{\overline{\pi}}\big(\wp_{-\ell}\mathcal{F}_{\mathfrak{g}}\varphi(-\lambda, -)\big),$$

since $\overline{\pi} \sim -\ell$. Now we can complete the proof by using $\delta_{-\lambda}(-\ell + \xi) = \delta_\lambda(\ell + \xi)$ for any $\xi \in \mathfrak{z}^\circ$. $\square$

The above proposition (c.f. [54, Theorem 3.3]) shows that the group Fourier transform splits into operators which are easy to handle in the $L^2$-setting. If we use the isomorphism[5] $(\widehat{\mathbb{G}}, \widehat{\mu}) \simeq (\mathbb{R}^\times, \widehat{\mu}_\pi)$, then we can see $\mathcal{F}_{\mathbb{G}}$ as the composition of unitary operators

$$\begin{aligned}\mathcal{F}_{\mathfrak{g}}\colon L^2(\mathbb{G}, \mu) &\to L^2(\mathfrak{g}', \mu'),\\ \wp_\ell\colon L^2(\mathfrak{g}', \mu') &\to L^2(\mathbb{R}^\times, \widehat{\mu}_\pi) \mathbin{\hat{\otimes}_{\mathrm{H}}} L^2(\mathfrak{z}^\circ, \theta),\\ \mathfrak{Op}_\pi\colon L^2(\mathbb{R}^\times, \widehat{\mu}_\pi) \mathbin{\hat{\otimes}_{\mathrm{H}}} L^2(\mathfrak{z}^\circ, \theta) &\to L^2\big(\mathbb{R}^\times, \widehat{\mu}_\pi\big) \mathbin{\hat{\otimes}_{\mathrm{H}}} \mathcal{HS}(H_\pi),\end{aligned}$$

[5]Here we mean a measurable map with measurable inverse that is defined between $\widehat{\mathbb{G}} \setminus N_1$ and $\mathbb{R}^\times \setminus N_2$ for two null sets $N_1$ and $N_2$.

in which $\theta$ is an appropriate Haar measure on $\mathfrak{z}^\circ$ and $\mathfrak{Op}_\pi$ corresponds to the operator $P_+ \otimes \mathfrak{op}_\pi + P_- \otimes \mathfrak{op}_{\overline{\pi}}$ for the projection $P_\pm$ of $L^2(\mathbb{R}^\times)$ onto $L^2(\mathbb{R}_\pm)$. Moreover, we used the canonical unitary map (1.2.4) in order to reinterpret $\wp_\ell$ as a map from $L^2(\mathfrak{g}', \mu')$ into $L^2(\mathbb{R}^\times, \widehat{\mu}_\pi) \hat{\otimes}_{\mathrm{H}} L^2(\mathfrak{z}^\circ, \theta)$.

It is very convenient that the operator component emerges as a tensor product factor, which in turn enables us to understand multiplication operators on the Fourier side more easily. Though we run into problems if we try the same for the Fourier transform on $\mathscr{S}(\mathbb{G})$. Here we are not able to describe $\mathscr{S}(\widehat{\mathbb{G}})$ as the tensor product of a space of functions with a space of operators. This motivates us to define alternative spaces of test functions.

In order to know which function space is a good choice, we will first take a look at the pull back $\wp_\ell$. Here our earlier discussion of polynomial manifolds comes into play again. Remember that $\mathbb{R}^\times$ is equipped with a polynomial structure defined by $\mathbb{R}^\times = \mathbb{R}_+ \dot\cup \mathbb{R}_-$, i.e. defined by the polynomial structures on $\mathbb{R}_\pm$. Similarly, for $\ell \in \omega^\times$ we define $\mathfrak{g}_\ell^+$, $\mathfrak{g}_\ell^-$, $\mathfrak{g}^\times$ by

$$\mathfrak{g}_\ell^+ := \{t\ell + \eta \mid t > 0, \eta \in \mathfrak{z}^\circ\}\,,\ \mathfrak{g}_\ell^- = -\mathfrak{g}_\ell^+\,, \quad \mathfrak{g}^\times = \mathfrak{g}_\ell^+ \dot\cup \mathfrak{g}_\ell^-$$

and equip $\mathfrak{g}_\ell^\pm$, $\mathfrak{g}^\times$ with the polynomial structure analogously to the one on $\mathbb{R}_\pm$, $\mathbb{R}^\times$, i.e. the polynomial structure induced by the map

$$\mathfrak{g}_\ell^\pm \to \mathbb{R} \times \mathfrak{z}^\circ \colon (t\ell + \eta) \mapsto (t - 1/t, \eta).$$

Then $\delta_\lambda$ induces a tempered diffeomorphism as written in the following lemma. Note that we just have $\mathfrak{g}^\times = \mathfrak{g}' \setminus \mathfrak{z}^\circ$ as a set.

**Lemma 3.3.10.** *Let $\ell \in \omega^\times$. The Map $w_\ell \colon \mathbb{R}_\pm \times \mathfrak{z}^\circ \to \mathfrak{g}_\ell^\pm \colon (\lambda, \xi) \mapsto \delta_\lambda(\ell + \xi)$ is a tempered diffeomorphism.*

*Proof.* We prove that $\mathbb{R}_{>0} \times \mathfrak{z}^\circ \simeq \mathfrak{g}_\ell^+$ via $w_\ell$. The proof to the second statement is analogous. Suppose $(\xi^j)_j$ is the dual basis to our Jordan–Hölder basis $(e_j)_{j=0}^{2n}$ of eigenvectors, in which $e_0 \in \mathfrak{z}$. Here $(\xi_j)_{j=1}^{2n}$ is a basis of $\mathfrak{z}^\circ$. Let $\kappa_j$ be the positive number such that $\delta_\lambda \xi^j = \lambda^{\kappa_j} \xi^j$ for $\lambda > 0$.

We use the charts $\sigma$ resp. $\sigma_1$, defined by

$$(\lambda, \sum_{j=1}^{2n} c_j \xi^j) \text{ resp. } (\lambda\ell + \sum_{j=1}^{2n} c_j \xi^j) \mapsto (\lambda - 1/\lambda, c_1, \ldots, c_{2n}).$$

Then

$$\sigma_1 \circ w_\ell \circ \sigma^{-1}(t, c_1, \ldots, c_{2n}) = \left( \frac{(t + \sqrt{t^2+4})^{\kappa_0}}{2^{\kappa_0}} - \frac{2^{\kappa_0}}{(t+\sqrt{t^2+4})^{\kappa_0}}, \right. \\ \left. \frac{(t+\sqrt{t^2+4})^{\kappa_1}}{2^{\kappa_1}} c_1, \ldots, \frac{(t+\sqrt{t^2+4})^{\kappa_{2n}}}{2^{\kappa_{2n}}} c_{2n} \right),$$

which is a slowly increasing function. Similarly

$$\sigma \circ w_\ell^{-1} \circ \sigma_1^{-1}(t, c_1, \ldots, c_{2n}) = \left( \frac{(t + \sqrt{t^2+4})^{\frac{1}{\kappa_0}}}{2^{\frac{1}{\kappa_0}}} - \frac{2^{\frac{1}{\kappa_0}}}{(t+\sqrt{t^2+4})^{\frac{1}{\kappa_0}}}, \right. \\ \left. \frac{(t+\sqrt{t^2+4})^{\frac{-\kappa_1}{\kappa_0}}}{2^{\frac{-\kappa_1}{\kappa_0}}} c_1, \ldots, \frac{(t+\sqrt{t^2+4})^{\frac{-\kappa_{2n}}{\kappa_0}}}{2^{\frac{-\kappa_{2n}}{\kappa_0}}} c_{2n} \right)$$

is slowly increasing. □

By Lemma 2.3.6, we can see $\mathscr{S}(\mathfrak{g}_\ell^\pm)$ as the space

$$\mathscr{S}(\mathfrak{g}_\ell^\pm) = \{\varphi \in \mathscr{S}(\mathfrak{g}') \mid \varphi \equiv 0 \text{ on } \mathfrak{g}_\ell^\mp\}$$

equipped with the subspace topology in $\mathscr{S}(\mathfrak{g}')$.

The tempered diffeomorphism from the last lemma induces a Gelfand triple isomorphism.

**Lemma 3.3.11.** *Suppose $\omega^\times \ni \ell \sim \pi \in \mathrm{SI}/\mathrm{Z}_{\mathbb{R}}(\mathbb{G})$ and $\widehat{\mu}_\pi$ is defined as in Definition 3.3.6. The pullback $\wp_\ell\varphi(\lambda, \xi) := \varphi \circ w_\ell(\lambda, \xi)$ for $\varphi \in \mathscr{S}(\mathfrak{g}^\times)$, $\xi \in \mathfrak{z}^\circ$ and $\lambda \in \mathbb{R}^\times$ defines a Gelfand triple isomorphism*

$$\wp_\ell \colon \mathcal{G}(\mathfrak{g}^\times, \mu') \to \mathcal{G}(\mathbb{R}^\times, \widehat{\mu}_\pi) \otimes \mathcal{G}(\mathfrak{z}^\circ, \theta)\,.$$

*Furthermore, $\wp_\ell$ restricts to Gelfand triple isomorphisms*

$$\wp_\ell \colon \mathcal{G}(\mathfrak{g}_\ell^\pm, \mu') \to \mathcal{G}(\mathbb{R}^\pm, \widehat{\mu}_\pi) \otimes \mathcal{G}(\mathfrak{z}^\circ, \theta)$$

*if we use the canonical Gelfand triple isomorphism $\mathcal{G}(\mathfrak{g}^\times, \mu') \simeq \mathcal{G}(\mathfrak{g}_\ell^+, \mu') \oplus \mathcal{G}(\mathfrak{g}_\ell^-, \mu')$.*

*Proof.* We take an arbitrary continuous function $f\colon \mathfrak{g}_\ell^\pm \to \mathbb{C}$ with compact support. We define $\mathit{Pf}(\ell)$ as in Proposition 3.3.7 resp. Definition 3.3.6 and let $\omega^\pm := \mathbb{R}_\pm \cdot \ell$. Then

$$\begin{aligned}
&\int_{\mathbb{R}_\pm}\int_{\mathfrak{z}^\circ} f(\delta_\lambda(\ell+\xi))\kappa|\mathit{Pf}(\ell)|\,|\lambda|^{Q-1}\,\mathrm{d}\lambda\,\mathrm{d}\theta(\xi)\\
&\qquad\qquad = \int_{\mathbb{R}_\pm}\int_{\mathfrak{z}^\circ} f((\delta_\lambda\ell)+\xi)\kappa|\mathit{Pf}(\ell)|\,|\lambda|^{\kappa_0-1}\,d\lambda\,\mathrm{d}\theta(\xi)\\
&\qquad\qquad = \int_{\omega^\pm}\int_{\mathfrak{z}^\circ} f(\eta+\xi))\,|\mathit{Pf}(\ell)|\,d\mu_{\omega^\circ}(\eta)\,\mathrm{d}\theta(\xi)\\
&\qquad\qquad = \int_{\mathfrak{g}_\ell^\pm} f(\xi)\,\mathrm{d}\mu'(\xi).
\end{aligned}$$

For the last two equalities we used that the measure $\mu_{\omega^\circ}$ on $\omega^\circ$ is defined by the Lebesgue measure and $\ell$ and that $\theta$ is defined by $\mu' = |\mathit{Pf}(\ell)|\,\mu_{\omega^\circ}\otimes\theta$. The rest follows with the fact that $\wp_\ell f(\lambda,\xi) = f\circ w_\ell(\lambda,\xi)$, where $w_\ell$ is the tempered diffeomorphism from Lemma 3.3.10 and the canonical Gelfand triple isomorphism

$$\mathcal{G}(\mathbb{R}_\pm,\widehat{\mu}_\pi)\otimes\mathcal{G}(\mathfrak{z}^\circ,\theta)\simeq\mathcal{G}(\mathbb{R}_\pm\times\mathfrak{z}^\circ,\widehat{\mu}_\pi\otimes\theta)\,.$$

□

We also proved that the restriction of the Haar measure $\mu'$ to $\mathfrak{g}_\ell^\pm$ is actually a tempered measure with respect to our chosen polynomial structure.

Now we are ready to define Gelfand triples, with respect to which we get a convenient theory for the group Fourier transform.

**Definition 3.3.12.** *We define the following reduced Schwartz space*

$$\mathscr{S}_*(\mathbb{G}) := \{\varphi\in\mathscr{S}(\mathbb{G}) \mid [\mathbb{R}\times\omega\ni(\lambda,x)\mapsto\varphi(\lambda z+x)]\in\mathscr{S}_*(\mathbb{R};\mathscr{S}(\omega))\}$$

*for any choice $z\in\mathfrak{z}\setminus\{0\}$, equipped with the subspace topology in $\mathscr{S}(\mathbb{G})$, and the corresponding Gelfand triple*

$$\mathcal{G}_*(\mathbb{G},\mu) := (\mathscr{S}_*(\mathbb{G}),L^2(\mathbb{G},\mu),\mathscr{S}_*'(\mathbb{G})),$$

*equipped with the real structure given by the pointwise complex conjugation. Furthermore,* *we define the Gelfand triple*

$$\mathcal{G}(\mathbb{R}^\times;\pi) := \begin{pmatrix} \mathscr{S}(\mathbb{R}^\times;\pi) \\ L^2(\mathbb{R}^\times;\pi) \\ \mathscr{S}'(\mathbb{R}^\times;\pi) \end{pmatrix} := \mathcal{G}(\mathbb{R}^\times, \widehat{\mu}_\pi) \otimes \mathcal{G}_{\mathrm{op}}(\pi).$$

*for each* $\pi \in \mathrm{SI}/\mathrm{Z}_{\mathbb{R}}(\mathbb{G})$*.*

That $\mathcal{G}_*(\mathbb{G},\mu)$ is indeed a Gelfand triple can be seen by using Proposition 2.3.13. We use any linear isomorphism $\mathbb{R} \simeq \mathfrak{z}$ to define $\dot{\mathcal{B}}'_*(\mathfrak{z};\mathscr{S}'(\omega))$. Then we may see, since $L^2(\mathbb{G},\mu) \subset \dot{\mathcal{B}}'_*(\mathfrak{z};\mathscr{S}'(\omega))$ by Lemma 2.3.12 that the space $L^2(\mathbb{G},\mu)$ is embedded into $\mathscr{S}'_*(\mathbb{G}) = \mathscr{S}'_*(\mathfrak{z};\mathcal{S}'(\omega))$. This embedding is continuous, since $L^2(\mathbb{G},\mu) \hookrightarrow \mathscr{S}'(\mathbb{G})$ is continuous. Of course, the canonical map of $\mathscr{S}_*(\mathbb{G})$ into $L^2(\mathbb{G},\mu)$ is a continuous embedding as well. Now the Hahn–Banach theorem implies that both embeddings are also dense, for they are dual to each other.

To be more precise, if $\mathscr{S}_*(\mathbb{G})^\circ$ is the polar of $\mathscr{S}_*(\mathbb{G})$ in $L^2(\mathbb{G},\mu) \simeq L^2(\mathbb{G},\mu)'$, then it is also the kernel of the map

$$j' \colon L^2(\mathbb{G},\mu) \to \mathscr{S}'_*(\mathbb{G}) \colon f \mapsto \left[\mathscr{S}_*(\mathbb{G}) \ni \varphi \mapsto \int_{\mathbb{G}} f\,\varphi \,\mathrm{d}\mu(x)\right].$$

But this map has a trivial kernel by Lemma 2.3.12. Hence $\mathscr{S}_*(\mathbb{G})^\circ = \{0\}$ and $\mathscr{S}_*(\mathbb{G})$ is dense in $L^2(\mathbb{G},\mu)$. Now denote by $Y$ the image of $L^2(\mathbb{G},\mu)$ in $\mathscr{S}'_*(\mathbb{G})$. Since $\mathscr{S}_*(\mathbb{G})$ is reflexive, $Y^\circ$ can be identified with the kernel of the embedding $j \colon \mathscr{S}_*(\mathbb{G}) \hookrightarrow L^2(\mathbb{G},\mu)$, which is trivial. Hence $Y \subset \mathscr{S}'_*(\mathbb{G})$ is dense as well.

Note that $\mathcal{G}_*(\mathbb{G},\mu)$ does not depend on the choice of $\pi \in \mathrm{SI}/\mathrm{Z}_{\mathbb{R}}(\mathbb{G})$ or $z \in \mathfrak{z}$. The Gelfand triple $\mathcal{G}(\mathbb{R}^\times;\pi)$ does depend on $\pi \in \mathrm{SI}/\mathrm{Z}_{\mathbb{R}}(\mathbb{G})$ but each different choice of $\pi$ leads to an isomorphic Gelfand triple as the theorem below shows.

**Theorem 3.3.13.** *Let* $\mathrm{SI}/\mathrm{Z}_{\mathbb{R}}(\mathbb{G}) \ni \pi \sim \ell \in \omega^\times$ *and let* $P_+ = \mathrm{I} - P_- \in \mathcal{L}(\mathscr{S}(\mathbb{R}^\times))$ *be the projection of* $\mathscr{S}(\mathbb{R}^\times)$ *onto* $\mathscr{S}(\mathbb{R}_+)$ *along* $\mathscr{S}(\mathbb{R}_-)$*. Let the* ***Fourier transform in*** $\pi$***-picture****,* $\mathcal{F}_\pi$*, be defined by*

$$\mathcal{F}_\pi := \mathfrak{Op}_\pi \circ \wp_\ell \circ \mathcal{F}_{\mathfrak{g}}\,,$$

*where* $\mathfrak{Op}_\pi$ *is the Gelfand triple isomorphism onto* $\mathcal{G}(\mathbb{R}^\times;\pi)$ *defined by*

$$\mathfrak{Op}_\pi\varphi = P_+ \otimes \mathfrak{op}_\pi\varphi + P_- \otimes \mathfrak{op}_{\overline{\pi}}\varphi \quad \textit{for} \quad \varphi \in \mathscr{S}(\mathbb{R}^\times)\,\hat{\otimes}\,\mathcal{L}(\mathscr{E}(\pi)';\mathscr{E}(\pi))\,.$$

*Then $\mathcal{F}_\pi$ is a Gelfand triple isomorphism*

$$\mathcal{F}_\pi \colon \mathcal{G}_*(\mathbb{G}, \mu) \to \mathcal{G}(\mathbb{R}^\times; \pi).$$

*Proof.* The proof essentially writes itself by now and is a summary of previous statements. The euclidean Fourier transform $\mathcal{F}_\mathfrak{g}$ is a Gelfand triple isomorphism between $\mathcal{G}_*(\mathbb{G}, \mu)$ and $\mathcal{G}(\mathfrak{g}^\times, \mu') \simeq \mathcal{G}(\omega^\times, |Pf(\ell)|\, \mu_{\omega^\circ}) \otimes \mathcal{G}(\mathfrak{z}^\circ, \theta)$ by Lemma 2.3.9, where we choose the Haar measures $\mu_{\omega^\circ}$ and $\theta$ such that $\mu' = |Pf(\ell)|\, \mu_{\omega^\circ} \otimes \theta$ and $\mu_{\omega^\circ}$ is induced by the Lebesgue measure $\mathrm{d}\lambda$ via the map $\mathbb{R} \ni \lambda \mapsto \lambda\ell \in \omega^\circ$.

By Lemma 3.3.11, the pull back $\wp_\ell$ is a Gelfand triple isomorphism between $\mathcal{G}(\mathfrak{g}^\times, \mu_{\mathfrak{g}'})$ and $\mathcal{G}(\mathbb{R}^\times, \widehat{\mu}_\pi) \otimes \mathcal{G}(\mathfrak{z}^\circ, \theta)$.

For the last step we just need to use the canonical Gelfand triple isomorphism

$$\mathcal{G}(\mathbb{R}^\times, \widehat{\mu}_\pi) \simeq \mathcal{G}(\mathbb{R}_+, \widehat{\mu}_\pi) \oplus \mathcal{G}(\mathbb{R}_-, \widehat{\mu}_\pi)$$

and the fact that $\mathfrak{op}_\pi$ and $\mathfrak{op}_{\overline{\pi}}$ are Gelfand triple isomorphisms by Theorem 3.2.2 and Definition 3.3.2. Thus $\mathfrak{Op}_\pi$ is a Gelfand triple isomorphism between $\mathcal{G}(\mathbb{R}^\times, \widehat{\mu}_\pi) \otimes \mathcal{G}(\mathfrak{z}^\circ, \mu_{\mathfrak{z}^\circ})$ and $\mathcal{G}(\mathbb{R}^\times; \pi)$. □

Let us now discuss a few properties of $\mathscr{S}_*(\mathbb{G})$ and $\mathscr{S}(\mathbb{R}^\times; \pi)$. Their duals can be identified with quotient spaces, in particular

$$\mathscr{S}'_*(\mathbb{G}) \simeq \mathscr{S}'(\mathbb{G})/(\mathscr{P}(\mathfrak{z}) \otimes \mathscr{S}'(\omega)) \quad \text{and}$$
$$\mathscr{S}'(\mathbb{R}^\times; \pi) \simeq \mathscr{S}(\mathbb{R}^\times) \mathbin{\hat{\otimes}} \mathcal{L}(\mathscr{E}(\pi); \mathscr{E}(\pi)')/(\mathscr{E}'_0(\mathbb{R}) \otimes \mathcal{L}(\mathscr{E}(\pi); \mathscr{E}(\pi)'),$$

by Lemma 2.3.10 and Corollary 2.3.11. By employing Proposition 2.3.13, we can identify a large space of distributions on $\mathbb{G}$ resp. $\mathbb{R}$ that are embedded into $\mathscr{S}'_*(\mathbb{G})$ resp. $\mathscr{S}'(\mathbb{R}^\times; \pi)$. I.e. if we define $\dot{\mathscr{B}}'(\mathfrak{z}; \mathscr{S}'(\omega))$ by using any isomorphism $\mathbb{R} \simeq \mathfrak{z}$, then

$$\dot{\mathscr{B}}'(\mathfrak{z}; \mathscr{S}'(\omega)) \hookrightarrow \mathscr{S}'_*(\mathbb{G}) \quad \text{and} \quad \widetilde{\mathscr{B}}'(\mathbb{R}; \mathcal{L}(\mathscr{E}(\pi), \mathscr{E}(\pi)')) \hookrightarrow \mathscr{S}'(\mathbb{R}^\times; \pi).$$

We may, for example, identify $L^p(\mathbb{G}, \mu)$, for $p \in [1, \infty)$, and also $\mathscr{S}(\mathbb{G})$ as a subspaces of $\dot{\mathscr{B}}'(\mathfrak{z}; \mathscr{S}'(\omega))$ and the Bochner-Lebesgue spaces $L^p(\mathbb{R}; \mathcal{L}(H_\pi))$, for $p \in (1, \infty]$, and also $\mathscr{S}(\mathbb{R}; \mathcal{L}(\mathscr{E}(\pi)'; \mathscr{E}(\pi)))$ as subspaces of $\widetilde{\mathscr{B}}'(\mathbb{R}; \mathcal{L}(\mathscr{E}(\pi), \mathscr{E}(\pi)'))$.

The definition of $\mathscr{S}(\mathbb{R}^\times; \pi)$ and $\mathscr{S}'(\mathbb{R}^\times; \pi)$ enables us to define a multiplication with a large class of smooth functions via Theorem 1.2.11.

**Proposition 3.3.14.** *For any $\pi \in \mathrm{SI/Z}(\mathbb{G})$, the multiplications*

$$\mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times; \mathcal{L}(\mathscr{E}(\pi))) \times \mathscr{S}(\mathbb{R}^\times; \pi) \to \mathscr{S}(\mathbb{R}^\times; \pi) \colon (f, \varphi) \mapsto f\,\varphi$$
$$\mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times; \mathcal{L}(\mathscr{E}(\pi)')) \times \mathscr{S}(\mathbb{R}^\times; \pi) \to \mathscr{S}(\mathbb{R}^\times; \pi) \colon (f, \varphi) \mapsto \varphi\, f,$$

*defined via*

$$(f\,\varphi)(\lambda) := f(\lambda) \circ \varphi(\lambda) \quad \textit{and} \quad (\varphi\, f)(\lambda) := \varphi(\lambda) \circ f(\lambda)$$

*for any $\lambda \in \mathbb{R}^\times$, are hypocontinuous bilinear maps.*

*Proof.* We just need show that we may apply Proposition 2.1.10. The spaces $\mathscr{E}(\pi)$ and $\mathscr{E}(\pi)'$ are barrelled, bornological and complete since $\mathscr{E}(\pi) \simeq \mathscr{S}(\mathbb{R}^n)$. Thus the compositions of operators

$$\mathcal{L}(\mathscr{E}(\pi)) \times \mathcal{L}(\mathscr{E}(\pi)'; \mathscr{E}(\pi)) \to \mathcal{L}(\mathscr{E}(\pi)'; \mathscr{E}(\pi)) \colon (A, B) \mapsto A\,B$$
$$\mathcal{L}(\mathscr{E}(\pi)') \times \mathcal{L}(\mathscr{E}(\pi)'; \mathscr{E}(\pi)) \to \mathcal{L}(\mathscr{E}(\pi)'; \mathscr{E}(\pi)) \colon (A, B) \mapsto B\,A$$

are hypocontinuous by Lemma 1.2.10 and all involved spaces are complete. Also, the multiplication of slowly increasing functions and rapidly decreasing functions is hypocontinuous. This follows directly from the identification of $\mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times)$ with a closed (topological) subspace of $\mathcal{L}(\mathscr{S}(\mathbb{R}^\times))$. Now we just need to remind ourselves that $\mathscr{S}(\mathbb{R}^\times; \pi)$ is a tensor product of nuclear Fréchet spaces. Thus we may apply Proposition 2.1.10. □

Now, we will prove the analogous result for the multiplication with the operator valued tempered distributions $\mathscr{S}'(\mathbb{R}^\times; \pi)$. As we used in the proof above, $\mathscr{E}(\pi)$ is reflexive for any $\pi \in \mathrm{SI/Z}(\mathbb{G})$. Thus, by using the adjoint in the sense of Definition 1.4.5 with respect to the Gelfand triple $\mathcal{G}_{\mathrm{op}}(\pi)$, we get the two antilinear homeomorphisms

$$\mathcal{L}(\mathscr{E}(\pi)) \ni A \mapsto A^* \in \mathcal{L}(\mathscr{E}(\pi)') \quad \text{and} \quad \mathcal{L}(\mathscr{E}(\pi)') \ni B \mapsto B^* \in \mathcal{L}(\mathscr{E}(\pi)).$$

Denote for $f$ in $\mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times; \mathcal{L}(\mathscr{E}(\pi)))$ or in $\mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times; \mathcal{L}(\mathscr{E}(\pi)'))$ the operator valued function $f^*(\lambda) := f(\lambda)^*$. Then we may define multiplications on $\mathscr{S}'(\mathbb{R}^\times; \pi)$ by

$$(\varphi, f\,\phi) := (f^*\,\varphi, \phi) \quad \text{and} \quad (\varphi, \phi\, g) := (\varphi\, g^*, \phi)$$

for all $\phi \in \mathscr{S}'(\mathbb{R}^\times; \pi)$ and $\varphi \in \mathscr{S}(\mathbb{R}^\times; \pi)$, if we choose $g \in \mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times; \mathcal{L}(\mathscr{E}(\pi)))$ and if we choose $f \in \mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times; \mathcal{L}(\mathscr{E}(\pi)'))$. We get the following corollary.

**Corollary 3.3.15.** *For any $\pi \in \mathrm{SI/Z}(\mathbb{G})$, the multiplications*

$$\mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times; \mathcal{L}(\mathscr{E}(\pi))) \times \mathscr{S}'(\mathbb{R}^\times; \pi) \to \mathscr{S}'(\mathbb{R}^\times; \pi) \colon (f, \phi) \mapsto f\,\phi,$$
$$\mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times; \mathcal{L}(\mathscr{E}(\pi)')) \times \mathscr{S}'(\mathbb{R}^\times; \pi) \to \mathscr{S}'(\mathbb{R}^\times; \pi) \colon (f, \phi) \mapsto \phi\, f$$

*are hypocontinuous.*

*Proof.* This follows directly from the definition of the multiplication and the fact that the dual pairing is hypocontinuous. Equivalently, we could also directly employ Theorem 1.2.11. □

Let us now relate the Fourier transform in $\pi$ picture with the group Fourier transform.

**Lemma 3.3.16.** *Suppose $\omega^\circ \ni \ell \sim \pi \in \mathrm{SI/Z}_{\mathbb{R}}(\mathbb{G})$, then*

$$j_\pi \colon B^2(\widehat{\mathbb{G}}, \widehat{\mu}) \to L^2(\mathbb{R}^\times, \widehat{\mu}_\pi; \mathcal{HS}(H_\pi)) \quad \textit{defined by} \quad \sigma \mapsto [\lambda \mapsto \sigma(\pi_\lambda)]\,,$$

*is unitary with $j_\pi \mathcal{F}_{\mathbb{G}} = \mathcal{F}_\pi$.*

*Proof.* As noted before $\mathfrak{Op}_\pi$, $\wp_\ell$ and $\mathcal{F}_{\mathfrak{g}}$ are unitary, so $\mathcal{F}_\pi$ is unitary from $L^2(\mathbb{G}, \mu)$ onto $L^2\big(\mathbb{R}^\times, \widehat{\mu}_\pi\big) \mathbin{\hat{\otimes}_{\mathrm{H}}} \mathcal{HS}(H_\pi)\big)$.

The map $\mathbb{R}^\times \ni \lambda \mapsto \pi_\lambda \in \mathrm{Irr}_{H_\pi}(\mathbb{G})$ is measurable, since $\lambda \mapsto \big(\pi_\lambda(x)h, h'\big)_{H_\pi}$ is continuous from $\mathbb{R}^\times$ to $\mathbb{C}$ for all $h, h' \in H_\pi$ and $x \in \mathbb{G}$. For any $\sigma \in B^2(\widehat{\mathbb{G}}, \widehat{\mu})$ define

$$j_\pi\sigma \colon \mathbb{R}^\times \to \mathcal{HS}(H_\pi) \colon \lambda \mapsto \sigma(\pi_\lambda)\,.$$

Then $\lambda \mapsto \big(j_\pi\sigma(\lambda)h, h'\big)_{H_\pi}$ is measurable for all $h, h' \in H_\pi$. Operators of the form

$$h \mapsto \sum_j h_j \big(h, h'_j\big)_{H_\pi} \qquad \text{for } h_j, h'_j \in H_\pi$$

are dense in the Hilbert space $\mathcal{HS}(H_\pi)$, thus $j_\pi\sigma$ is weakly measurable, i.e. the map $\psi \circ j_\pi\sigma \colon \mathbb{R}^\times \to \mathbb{C}$ is measurable for each $\psi \in \mathcal{HS}(H_\pi)'$. Since $\mathcal{HS}(H_\pi)$ is separable, we may use the Pettis Measurability Theorem [60, Proposition 2.15] on $j_\pi\sigma$, which ensures that $j_\pi\sigma$ is $\widehat{\mu}_\pi$-measurable.

By Proposition 3.3.7 and $\widehat{\mu}(\widehat{\mathbb{G}} \setminus \widehat{\mathbb{G}}_{\mathrm{gen}}) = 0$, we have

$$\|\sigma\|_{B^2(\widehat{\mathbb{G}}, \widehat{\mu})} = \|j_\pi\sigma\|_{L^2(\mathbb{R}^\times, \widehat{\mu}_\pi; \mathcal{HS}(H_\pi))}\,,$$

and thus

$$j_\pi \colon B^2(\widehat{\mathbb{G}}, \widehat{\mu}) \to L^2(\mathbb{R}^\times, \widehat{\mu}_\pi; \mathcal{HS}(H_\pi))$$

is a well-defined isometric operator. Now Proposition 3.3.9 ensures that $j_\pi \mathcal{F}_{\mathbb{G}} = \mathcal{F}_\pi$ and thus $j_\pi$ is surjective. $\square$

Let us define $\mathscr{S}(\widehat{\mathbb{G}}_{\mathrm{gen}}) := j_\pi^{-1}\mathscr{S}(\mathbb{R}^\times;\pi)$ with the corresponding Fréchet topology transported via $j_\pi^{-1}$. The space $\mathscr{S}(\widehat{\mathbb{G}}_{\mathrm{gen}})$ is invariant under under taking pointwise adjoints, because $\mathscr{S}(\mathbb{R}^\times;\pi)$ is invariant under this operation. This way we get a Gelfand triple

$$\mathcal{G}(\widehat{\mathbb{G}}_{\mathrm{gen}}, \widehat{\mu}) = (\mathscr{S}(\widehat{\mathbb{G}}_{\mathrm{gen}}), B^2(\widehat{\mathbb{G}}, \widehat{\mu}), \mathscr{S}'(\widehat{\mathbb{G}}_{\mathrm{gen}}))$$

equipped with the real structure defined by $\mathcal{C}\sigma(\pi) := \sigma(\pi)^*$ for $\sigma \in \mathscr{S}(\widehat{\mathbb{G}}_{\mathrm{gen}})$. We denote the inclusion of $\mathscr{S}(\widehat{\mathbb{G}}_{\mathrm{gen}})$ into $\mathscr{S}(\widehat{\mathbb{G}})$ by $j_0$ and the inclusion of $\mathscr{S}_*(\mathbb{G})$ into $\mathscr{S}(\mathbb{G})$ by $j_*$.

**Proposition 3.3.17.** *The Fourier transform $\mathcal{F}_{\mathbb{G}} \restriction_{\mathscr{S}_*(\mathbb{G})}^{\mathscr{S}(\widehat{\mathbb{G}}_{\mathrm{gen}})}$ extends into a Gelfand triple isomorphism*

$$\mathcal{F}_{\mathbb{G},*} \colon \mathcal{G}_*(\mathbb{G}, \mu) \to \mathcal{G}(\widehat{\mathbb{G}}_{\mathrm{gen}}, \widehat{\mu})\,.$$

*We have the commutative diagrams*

$$\begin{tikzcd}
L^2(\mathbb{G},\mu) \arrow[r, "\mathcal{F}_{\mathbb{G}}", "\simeq"'] \arrow[dr, "\simeq", "\mathcal{F}_\pi"'] & B^2(\widehat{\mathbb{G}},\widehat{\mu}) \arrow[d, "j_\pi", "\simeq"'] \\
 & L^2(\mathbb{R}^\times;\pi)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{S}(\mathbb{G}) \arrow[r, "\mathcal{F}_{\mathbb{G}}", "\simeq"'] & \mathscr{S}(\widehat{\mathbb{G}}) \\
\mathscr{S}_*(\mathbb{G}) \arrow[u, "j_*", "\subset"'] \arrow[r, "\mathcal{F}_{\mathbb{G},*}", "\simeq"'] & \mathscr{S}(\widehat{\mathbb{G}}_{\mathrm{gen}}) \arrow[u, "\subset", "j_0"'] \\
 & \mathscr{S}(\mathbb{R}^\times;\pi) \arrow[ul, "\simeq"', "\mathcal{F}_\pi^{-1}"] \arrow[u, "\simeq", "j_\pi^{-1}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{S}'(\mathbb{G}) \arrow[r, "\mathcal{F}_{\mathbb{G}}", "\simeq"'] \arrow[d, "j'_*"'] & \mathscr{S}'(\widehat{\mathbb{G}}) \arrow[d, "j'_0"] \\
\mathscr{S}'_*(\mathbb{G}) \arrow[r, "\mathcal{F}_{\mathbb{G},*}", "\simeq"'] \arrow[dr, "\simeq", "\mathcal{F}_\pi"'] & \mathscr{S}'(\widehat{\mathbb{G}}_{\mathrm{gen}}) \arrow[d, "(j'_\pi)^{-1}", "\simeq"'] \\
 & \mathscr{S}'(\mathbb{R}^\times;\pi)
\end{tikzcd}$$

*in which $j'_*$ and $j'_0$ are surjective and open.*

*Proof.* Lemma 3.3.16 implies the $L^2$-diagram. The commutative diagram for the $L^2$-spaces implies the commutative diagram for the spaces of rapidly decreasing functions. Together, the two diagrams imply that $\mathcal{F}_{\mathbb{G},*}$ is a well-defined Gelfand triples isomorphism. Also, by duality, we get the commutative diagram for the tempered distributions.

By Corollary 2.3.11 the map $j'_*$ can be seen as the quotient map

$$\mathscr{S}'(\mathbb{G}) \to \mathscr{S}'_*(\mathbb{G}) \simeq \mathscr{S}'(\mathbb{G})/(\mathscr{P}(\mathfrak{z}) \otimes \mathscr{S}'(\omega)),$$

which is an open map. This also implies that $j_0'$ is surjective and open. $\square$

Also of interest is the Fourier transform in $\pi$-picture defined on complex measures. If we denote by $M(\mathbb{G})$ the space of complex Borel measures on $\mathbb{G}$, equipped with the norm given by the total variation $\|\mu\|_{M(\mathbb{G})} := |\mu|(\mathbb{G})$ for $\mu \in M(\mathbb{G})$, then

$$M(\mathbb{G}) \to \mathscr{S}'(\mathbb{G}) \colon \nu \mapsto \left[\varphi \mapsto \int_{\mathbb{G}} \varphi \,\mathrm{d}\nu\right]$$

defines a continuous embedding. For any $\nu \in M(\mathbb{G})$, $\varphi \in \mathscr{S}(\mathbb{G})$ and $\varepsilon > 0$ there is a compact $K \subset \mathbb{G}$ with $|\nu|(\mathbb{G} \setminus K) < \varepsilon$ and $\sup_{x \in \mathbb{G} \setminus K} |\varphi(x)| < \varepsilon$. Now for any $\xi \in \mathfrak{z}$ and for $h > 0$ big enough, we have $K + h\xi \cap K = \emptyset$ and hence

$$\left| \int_{\mathbb{G}} \varphi(x + h\xi)| \,\mathrm{d}\nu(x) \right| \le \varepsilon \|\varphi\|_\infty + \varepsilon \|\nu\|_{M(\mathbb{G})} .$$

Thus $M(\mathbb{G}) \subset \dot{\mathscr{B}}'(\mathfrak{z}; \mathscr{S}'(\omega))$ via the standard embedding

$$M(\mathbb{G}) \subset \mathscr{S}'(\mathbb{G}) \simeq \mathscr{S}'(\mathfrak{z}) \,\hat{\otimes}\, \mathscr{S}'(\omega) .$$

In this sense, we can see $M(\mathbb{G})$ as a subspace of $\mathscr{S}_*'(\mathbb{G})$.

As common, we may calculate the group Fourier transform for each element $\nu \in M(\mathbb{G})$ by

$$\mathcal{F}_{\mathbb{G}}\nu(\pi) = \pi(\nu) := \int_{\mathbb{G}} \pi(x) \,\mathrm{d}\nu(x) \quad \text{for } \pi \in \mathrm{Irr}(\mathbb{G}) ,$$

in which the integral exists in $\mathcal{L}_s(H_\pi)$. Furthermore, we can see that $\mathcal{F}_{\mathbb{G}}\nu \in B^\infty(\widehat{\mathbb{G}}, \widehat{\mu})$.

**Corollary 3.3.18.** *The Fourier transform in $\pi$-picture restricts to a continuous map*

$$\mathcal{F}_\pi \colon M(\mathbb{G}) \to \mathscr{C}(\mathbb{R}^\times; \mathcal{L}_s(H_\pi)) \quad \textit{with inequality} \quad \sup_{\lambda \in \mathbb{R}^\times} \|\mathcal{F}_\pi \nu(\lambda)\|_{\mathcal{L}(H_\pi)} \le \|\nu\|_{M(\mathbb{G})}$$

*for all $\nu \in M(\mathbb{G})$.*

*Proof.* The inequality follows at once by the integral defining $\mathcal{F}_{\mathbb{G}}\nu(\pi_\lambda)$ for $\lambda \in \mathbb{R}^\times$. Applying the dominated convergence theorem to

$$\|\mathcal{F}_\pi\nu(\lambda_n)h - \mathcal{F}_\pi\nu(\lambda)h\|_{H_\pi} \le \int_{\mathbb{G}} \|\pi_{\lambda_n}(x)h - \pi_\lambda(x)h\|_{H_\pi} \,\mathrm{d}|\nu|(x) , \quad h, h \in H_\pi ,$$

for a convergent sequence $(\lambda_n) \subset \mathbb{R}^\times$, where $\lim_{n\to\infty} \lambda_n = \lambda$, results in the continuity property. $\square$

Since $w_\ell\colon \mathbb{R}^\times \times \mathfrak{z}^\circ \to \mathfrak{g}^\times$, $w_\ell(\lambda,\xi) = \delta_\lambda(\ell+\xi)$ is a tempered diffeomorphism, we can also see $\wp_\ell$ as an isomorphism between $\mathscr{O}_\mathrm{M}(\mathfrak{g}^\times)$ and $\mathscr{O}_\mathrm{M}(\mathbb{R}^\times \times \mathfrak{z}^\circ)$ resp. between $\mathscr{S}(\mathfrak{g}^\times)$ and $\mathscr{S}(\mathbb{R}^\times \times \mathfrak{z}^\circ)$. However, in order to examine the Fourier image on $\mathscr{S}(\mathbb{G})$, it is even better to consider mixed spaces. We equip $\omega^\times = \mathbb{R}^\times \cdot \ell$ with the polynomial structure transported from $\mathbb{R}^\times$. The space $\mathscr{O}_\mathrm{M}(\omega^\times)\,\hat{\otimes}\,\mathscr{S}(\mathfrak{z}^\circ)$ can be seen as a subspace of $\mathscr{O}_\mathrm{M}(\mathfrak{g}^\times)$. Since for any polynomial manifold $\mathbb{M}$ with tempered measure $\nu$ we have the continuous inclusion

$$\mathscr{O}_\mathrm{M}(\mathbb{M}) \hookrightarrow \mathscr{S}'(\mathbb{M})\colon f \mapsto [\varphi \mapsto \int_\mathbb{M} f\varphi \, \mathrm{d}\nu]\,,$$

we can consider $\mathscr{O}_\mathrm{M}(\omega^\times;\mathscr{S}(\mathfrak{z}^\circ))$ as a subspace of $\mathscr{S}'(\mathfrak{g}^\times) \simeq \mathscr{S}'(\omega^\times)\,\hat{\otimes}\,\mathscr{S}'(\mathfrak{z}^\circ)$ and we cam consider $\mathscr{O}_\mathrm{M}(\mathbb{R}^\times)\,\hat{\otimes}\,\mathscr{S}(\mathfrak{z}^\circ)$ as a subspace of $\mathscr{S}'(\mathbb{R}^\times)\,\hat{\otimes}\,\mathscr{S}'(\mathfrak{z}^\circ)$.

**Lemma 3.3.19.** *If we use the identifications above, the Gelfand-Triple isomorphism $\wp_\ell$ restricts to a linear homeomorphism*

$$\wp_\ell\colon \mathscr{O}_\mathrm{M}(\omega^\times;\mathscr{S}(\mathfrak{z}^\circ)) \to \mathscr{O}_\mathrm{M}(\mathbb{R}^\times)\,\hat{\otimes}\,\mathscr{S}(\mathfrak{z}^\circ).$$

*Proof.* We identify $\omega^\times \simeq \mathbb{R}^\times$ and $\mathfrak{z}^\circ \simeq \mathbb{R}^{2n}$ and $\mathfrak{z}^\circ \simeq \mathbb{R}^{2n}$ via our basis of eigenvectors to the dilations. It is enough to consider the $\mathbb{R}_+$-part, since $\mathscr{O}_\mathrm{M}(\mathbb{R}^\times) = \mathscr{O}_\mathrm{M}(\mathbb{R}_+) \oplus \mathscr{O}_\mathrm{M}(\mathbb{R}_-)$. With these adjustments, we need to exchange $\wp_\ell$ by the map $\wp$, where

$$\wp g(\lambda, x) = g(\lambda^{\kappa_0}, (\lambda^{\kappa_j} x_j)_{j=1}^{2n}).$$

First of all, we realize that $\lambda \mapsto \lambda^{\kappa_0}$ is a tempered diffeomorphism. Hence $T \in \mathcal{L}(\mathscr{O}_\mathrm{M}(\mathbb{R}_+))$ defined by $T\psi(\lambda) := \psi(\lambda^{\kappa_0})$ is a linear homeomorphism.

Now let us define linear isomorphisms $f_\lambda(x) = (\lambda^{\kappa_j/\kappa_0} x_j)_{j=1}^{2n}$ on $\mathbb{R}^{2n}$. Then it is easy to see that both $\lambda \mapsto f_\lambda$ and $\lambda \mapsto f_\lambda^{-1}$ define functions in $\mathcal{O}_\mathrm{M}(\mathbb{R}_+;\mathcal{L}(\mathbb{R}^{2n}))$ with values in the invertible matrices. We denote by $F_\lambda$ the corresponding operator $F_\lambda\varphi := \varphi \circ f_\lambda$ and set $F\colon \lambda \mapsto F_\lambda$ resp. $F^{-1}\colon \lambda \mapsto F_\lambda^{-1}$. A standard calculation shows that for any continuous seminorm $p$ on $\mathcal{L}(\mathscr{S}(\mathbb{R}^{2n}))$ and any $k \in \mathbb{N}_0$ there is a polynomial $q$ on $\mathcal{L}(\mathbb{R}^{2n})^{k+2}$ such that

$$p(\partial_\lambda^k F_\lambda) \le q(f_\lambda^{-1}, f_\lambda, \partial_\lambda f_\lambda, \ldots, \partial_\lambda^k f_\lambda).$$

Of course, an analogous inequality is valid for $F^{-1}$. Hence, we may conclude

$$F, F^{-1} \in \mathcal{O}_{\mathrm{M}}(\mathbb{R}_+; \mathcal{L}(\mathscr{S}(\mathbb{R}^{2n}))).$$

Here $F^{-1}$ is indeed the inverse of $F$ in the algebra $\mathcal{O}_{\mathrm{M}}(\mathbb{R}_+; \mathcal{L}(\mathscr{S}(\mathbb{R}^{2n})))$. Due to Theorem 1.2.11, we know that the multiplication

$$\mathscr{O}_{\mathrm{M}}(\mathbb{R}_+; \mathscr{S}(\mathbb{R}^{2n})) \ni g \mapsto F\, g \in \mathscr{O}_{\mathrm{M}}(\mathbb{R}_+) \mathbin{\hat{\otimes}} \mathscr{S}(\mathbb{R}^{2n}), \quad (F\, g)(\lambda, x) = H_\lambda(g(\lambda, -))(x),$$

is continuous and in fact a linear homeomorphism.

Because $\wp g = (T \otimes 1)(F\, g)$, we can conclude that $\wp$ is an isomorphism. $\square$

Using the above lemma, we may now prove the following continuity property for the Fourier transform in $\pi$-picture on $\mathscr{S}(\mathbb{G})$.

**Proposition 3.3.20.** *The Fourier transform in $\pi$-picture restricts to a continuous map*

$$\mathcal{F}_\pi \colon \mathscr{S}(\mathbb{G}) \to \mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times) \mathbin{\hat{\otimes}} \mathcal{L}(\mathscr{E}(\pi)', \mathscr{E}(\pi)).$$

*Proof.* This statement follows from the continuity of the maps

$$\mathscr{S}(\mathbb{G}) \xrightarrow{\mathcal{F}_{\mathfrak{g}}} \mathscr{S}(\mathfrak{g}') \hookrightarrow \mathscr{O}_{\mathrm{M}}(\omega^\times; \mathscr{S}(\mathfrak{z}^\circ)) \xrightarrow{\wp_\ell} \mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times) \mathbin{\hat{\otimes}} \mathscr{S}(\mathfrak{z}^\circ),$$

in which we use the continuous inclusion $\mathscr{S}(\omega^\circ) \subset \mathscr{O}_{\mathrm{M}}(\omega^\times)$, and also from the continuity of

$$\mathfrak{Op}_\pi = P_+ \otimes \mathfrak{op}_\pi + P_- \otimes \mathfrak{op}_{\overline{\pi}} \colon \mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times) \mathbin{\hat{\otimes}} \mathscr{S}(\mathfrak{z}^\circ) \to \mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times) \mathbin{\hat{\otimes}} \mathcal{L}(\mathscr{E}(\pi)', \mathscr{E}(\pi)),$$

in which $P_\pm f(x) := 0$ for $\pm x < 0$ and $P_\pm f(x) = f(x)$ for $\pm x > 0$. $\square$

## 3.4 Alternative Gelfand triples for the Kohn-Nirenberg quantization on homogeneous Lie groups

We will keep using Convention 3.3.8 for this section.

We already introduced the Kohn-Nirenberg quantization as a Gelfand triple isomorphism

$$\mathrm{Op}_{\mathbb{G}} \colon \mathcal{G}(\mathbb{G},\mu) \to \mathcal{G}(\widehat{\mathbb{G}},\widehat{\mu}) .$$

We will now introduce the Gelfand triples $\mathcal{G}_*(\mathbb{G},\mu)$ and $\mathcal{G}(\mathbb{R}^\times;\pi)$ into this context. This approach will lend itself to prove the already mentioned formula for the symbol motivated by the compact case.

In [24] the term symbol[6] is used for a map

$$\sigma \colon \mathbb{G} \times \mathrm{Irr}(\mathbb{G}) \to \bigcup_{\pi \in \mathrm{Irr}(\mathbb{G})} \{T \mid T \colon \mathscr{E}(\pi) \to H_\pi\}$$

that fulfils the following two properties.

**(i)** There exist $n, m \in \mathbb{Z}$ such that

$$\mathbb{G} \to B^\infty(\widehat{\mathbb{G}},\widehat{\mu}) \colon x \mapsto [\pi \mapsto \pi(\mathrm{I}+R)^m \sigma(x,\sigma)\pi(\mathrm{I}+R)^n]$$

is well-defined[7] and continuous in $x$ for a Rockland operator[8] $R \in \mathrm{Diff}_{\mathrm{L}}(\mathbb{G})$.

**(ii)** For each $\pi \in \mathrm{Irr}(\mathbb{G})$ and each $v \in \mathscr{E}(\pi)$ the map

$$\mathbb{G} \to H_\pi \colon x \mapsto \sigma(x,\pi)v$$

is smooth and for any $P \in \mathrm{Diff}(\mathbb{G})$ the corresponding derivatives $(x,\pi) \mapsto P_x\sigma(x,\pi)$ fulfil **(i)** for some $m, n \in \mathbb{Z}$.

Ruzhansky and Fischer use such symbols $\sigma$ in [24] in order to define the Kohn-Nirenberg quantization by the convergent integral

$$\mathrm{Op}_{\mathbb{G}}(\sigma)\varphi(x) := \int_{\widehat{\mathbb{G}}} \mathrm{Tr}[\pi(x)\, a(x,\pi)\, \pi(\varphi)]\, \mathrm{d}\widehat{\mu}([\pi]) , \quad \text{for } \varphi \in \mathscr{S}(\mathbb{G}), x \in \mathbb{G} . \tag{3.4.6}$$

Note that an equation analogous to (3.4.6) can be recovered rather quickly from $\mathrm{Op}_{\mathbb{G}} = \mathcal{K}_{\mathbb{G}}^{-1}\mathcal{T}_{\mathbb{G}}^{-1}(1 \otimes \mathcal{F}_{\mathbb{G}}^{-1})$ for operators in $\mathcal{HS}(L^2(\mathbb{G},\mu))$. For $\mathrm{Op}_{\mathbb{G}}(a) \in \mathcal{HS}(L^2(\mathbb{G},\mu))$, we have $a \in L^2(\mathbb{G},\mu) \hat{\otimes}_{\mathrm{H}} B^2(\widehat{\mathbb{G}},\widehat{\mu})$ and

$$\big( \mathrm{Op}(a)f, g\big)_{L^2(\mathbb{G},\mu)} = \int_{\widehat{\mathbb{G}}} \int_{\mathbb{G}} \mathrm{Tr}[a(x,\pi)\, ((1 \otimes \mathcal{F}_{\mathbb{G}}\, \mathrm{inv})\mathcal{T}_{\mathbb{G}} g \otimes \overline{f})(x,\pi)^*]\, \mathrm{d}\mu(x)\, \mathrm{d}\widehat{\mu}([\pi])$$

[6]See Definition 1.8.13, Definition 5.1.21 and Definition 5.1.34 in [24]

[7]In the sense that each operator $\pi(\mathrm{I}+R)^m\sigma(x,\sigma)\pi(\mathrm{I}+R)^n$ extends to an operator in $\mathcal{L}(H_\pi)$.

[8]See Definition 4.1.2, Definition 4.4.2 and Definition 5.1.12 in [24]

for all $f, g \in L^2(\mathbb{G}, \mu)$, where $\operatorname{inv} f(x) := f(-x)$ and $(\cdot,\cdot)_{L^2(\mathbb{G},\mu)}$ is the inner product in $L^2(\mathbb{G}, \mu)$. Because

$$\begin{aligned}
\operatorname{Tr}[a(x,\pi)\,((1\otimes \mathcal{F}_{\mathbb{G}}\operatorname{inv})\mathcal{T}_{\mathbb{G}} g\otimes \overline{f})(x,\pi)^*] &= \operatorname{Tr}\left[a(x,\pi)\left(g(x)\,(\mathcal{F}_{\mathbb{G}}\,\boldsymbol{R}_2(x)^{-1}\operatorname{inv}\overline{f})(\pi)\right)^*\right] \\
&= \overline{g(x)}\operatorname{Tr}\left[a(x,\pi)\,(\mathcal{F}_{\mathbb{G}}\operatorname{inv}\overline{f})(\pi)^*\,\pi(x)\right] \\
&= \overline{g(x)}\ \operatorname{Tr}[a(x,\pi)\,\mathcal{F}_{\mathbb{G}} f(\pi)\,\pi(x)]
\end{aligned}$$

for almost all $(x, [\pi]) \in \mathbb{G} \times \widehat{\mathbb{G}}$, we may write the operator $\operatorname{Op}_{\mathbb{G}}(a)$ as

$$\operatorname{Op}_{\mathbb{G}}(a)\varphi = \int_{\widehat{\mathbb{G}}} \operatorname{Tr}[\pi(-)\,a(-,\pi)\,\mathcal{F}_{\mathbb{G}} f(\pi)]\,\mathrm{d}\widehat{\mu}([\pi]), \quad \text{for } f \in L^2(\mathbb{G}, \mu),$$

where the integral converges in $L^2(\mathbb{G}, \mu)$.

For other spaces of operators we will approach this from a different direction. We will first reinterpret the Kohn-Nirenberg quantization as a Gelfand-triple isomorphism $\operatorname{Op}_\pi$ involving the Gelfand triples $\mathcal{G}_*(\mathbb{G}, \mu)$ and $\mathcal{G}(\mathbb{R}^\times; \pi)$ for $\pi \in \mathrm{SI}/\mathrm{Z}_{\mathbb{R}}(\mathbb{G})$. Afterwards we prove the formula $\operatorname{Op}_\pi^{-1}(A)(x, \pi) = \pi(x)^*(A \otimes \mathrm{I})(\pi(-))(x)$ and show we also get a representation corresponding to (3.4.6).

### 3.4.1 The Kohn-Nirenberg quantization for operators defined on $\mathscr{S}_*(\mathbb{G})$

We start by determining in what way $\mathcal{T}_{\mathbb{G}}$ is a Gelfand triple isomorphism in this context. Although $\mathcal{T}_{\mathbb{G}}$ cannot be seen as a map from $\mathcal{G}_*(\mathbb{G}, \mu) \otimes \mathcal{G}_*(\mathbb{G}, \mu)$ onto itself, it defined a map from $\mathcal{G}(\mathbb{G}, \mu) \otimes \mathcal{G}_*(\mathbb{G}, \mu)$ onto itself.

**Lemma 3.4.1.** *The map $\mathcal{T}_{\mathbb{G}} \restriction^{\mathscr{S}(\mathbb{G})\hat{\otimes}\mathscr{S}(\mathbb{G})}_{\mathscr{S}(\mathbb{G})\hat{\otimes}\mathscr{S}(\mathbb{G})}$ extends to a Gelfand triple isomorphism*

$$\mathcal{T}_{\mathbb{G},*} \colon \mathcal{G}(\mathbb{G}, \mu) \otimes \mathcal{G}_*(\mathbb{G}, \mu) \to \mathcal{G}(\mathbb{G}, \mu) \otimes \mathcal{G}_*(\mathbb{G}, \mu)\,.$$

*Proof.* Suppose $\varphi \in \mathscr{S}(\mathbb{G}) \mathbin{\hat{\otimes}} \mathscr{S}_*(\mathbb{G})$ and $q \in \mathscr{P}(\mathbb{G})$, then for all $x \in \mathbb{G}$ and $y \in \omega$

$$\int_{\mathfrak{z}} q(z)\varphi(x, x(-z-y))\,\mathrm{d}\mu_{\mathfrak{z}}(Z) = \int_{\mathfrak{z}} q((-x)(-z-y))\varphi(x,z)\,\mathrm{d}\mu_{\mathfrak{z}}(z) = 0$$

Because $[z \mapsto q((-x)(-z-y)] \in \mathscr{P}(\mathfrak{z})$. Hence $\mathcal{T}_{\mathbb{G}}\varphi \in \mathscr{S}(\mathbb{G}) \mathbin{\hat{\otimes}} \mathscr{S}_*(\mathbb{G})$. Analogously we may prove that $\mathcal{T}_{\mathbb{G}}^{-1}$ maps $\mathscr{S}(\mathbb{G}) \mathbin{\hat{\otimes}} \mathscr{S}_*(\mathbb{G})$ onto itself. Because $\mathscr{S}(\mathbb{G}) \mathbin{\hat{\otimes}} \mathscr{S}_*(\mathbb{G})$ carries the subspace topology in $\mathscr{S}(\mathbb{G})\,\hat{\otimes}\,\mathscr{S}(\mathbb{G})$, the continuity of $\mathcal{T}_{\mathbb{G}}$ and $\mathcal{T}_{\mathbb{G}}^{-1}$ on $\mathscr{S}(\mathbb{G})\,\hat{\otimes}\,\mathscr{S}_*(\mathbb{G})$ is evident. Since also

$$\int_{\mathbb{G}\times\mathbb{G}} \psi\, \mathcal{T}_{\mathbb{G}}\varphi \,\mathrm{d}(\mu\otimes\mu) = \int_{\mathbb{G}\times\mathbb{G}} \varphi\, \mathcal{T}_{\mathbb{G}}^{-1}\psi \,\mathrm{d}(\mu\otimes\mu),$$

for all $\varphi, \psi \in \mathscr{S}(\mathbb{G}\times\mathbb{G})$, we may extend $\mathcal{T}_{\mathbb{G}} \restriction^{\mathscr{S}(\mathbb{G})\hat{\otimes}\mathscr{S}(\mathbb{G})}_{\mathscr{S}(\mathbb{G})\hat{\otimes}\mathscr{S}(\mathbb{G})}$ to a Gelfand triple isomorphism. $\square$

Now a direct conclusion is the formulation of the Kohn-Nirenberg quantization as a Gelfand triple isomorphism that incorporates the new Gelfand triples $\mathcal{G}_*(\mathbb{G},\mu)$ and $\mathcal{G}(\mathbb{R}^\times;\pi)$.

**Proposition 3.4.2.** *Let $\mathcal{K}_{\mathbb{G},*}$ be the kernel map*

$$\mathcal{K}_{\mathbb{G},*} \colon \mathcal{L}(\mathcal{G}_*(\mathbb{G},\mu), \mathcal{G}(\mathbb{G},\mu)) \to \mathcal{G}(\mathbb{G},\mu)\otimes\mathcal{G}_*(\mathbb{G},\mu)\,.$$

*The* ***Kohn-Nirenberg quantization in $\pi$-picture***

$$\mathrm{Op}_\pi := \mathcal{K}_{\mathbb{G},*}^{-1}\mathcal{T}_{\mathbb{G},*}^{-1}(1\otimes\mathcal{F}_\pi^{-1}) \colon \mathcal{G}(\mathbb{G},\mu)\otimes\mathcal{G}(\mathbb{R}^\times;\pi) \to \mathcal{L}(\mathcal{G}_*(\mathbb{G},\mu),\mathcal{G}(\mathbb{G},\mu)),$$

*is a Gelfand triple isomorphism.*

As for the Fourier transformation in $\pi$-picture, we may relate $\mathrm{Op}_\pi$ to the original Kohn-Nirenberg quantization $\mathrm{Op}_{\mathbb{G}}$ via the diagrams on page 158.

### 3.4.2 The integral formula

Representations in $\mathrm{SI}/\mathrm{Z}_{\mathbb{R}}(\pi)$ can also be seen as slowly increasing functions. This is integral to our approach and will be proven in the proposition following the next lemma.

**Lemma 3.4.3.** *Suppose $E$ is a complete locally convex space and $f\in\mathscr{O}_{\mathrm{M}}(\mathbb{G};E)$ and let $F(\lambda,x) := f(\delta_\lambda x)$. Then $F\in\mathscr{O}_{\mathrm{M}}(\mathbb{R}_\pm\times\mathbb{G};E)$.*

*Proof.* We only consider the case $\mathbb{R}_+$. It is enough to show that for each continuous seminorm $p$ on $E$, each $k\in\mathbb{N}_0$ each $P\in\mathrm{Diff}_{\mathscr{P}}(\mathbb{G})$ there is a polynomial $q\in\mathscr{P}(\mathbb{G})$ and $l>0$, for which

$$p(\partial_\lambda^k P_x F(\lambda,x)) \le (1+|\lambda|^l+|\lambda|^{-l})q(x).$$

We realize that there are polynomial differential operators $P_v$ such that

$$\partial_\lambda^k P_x F(\lambda, x) = \sum_{v\in\mathbb{R}} \lambda^v (P_v f)(\delta_\lambda x),$$

as a finite linear combination. Since each $p(P_v f)$ is bounded by a polynomial $\widetilde{q}_v$, we may find polynomials $q_v$ such that

$$p(\partial_\lambda^k P_x F(\lambda, x)) \leq \sum_{v\in\mathbb{R}} |\lambda|^v \widetilde{q}_v(\delta_\lambda x) = \sum_{v\in\mathbb{R}} |\lambda|^v q_v(x).$$

This concludes the proof. $\square$

**Proposition 3.4.4.** *If $\pi \in \mathrm{SI}/\mathrm{Z}_{\mathbb{R}}(\mathbb{G})$, then the operator valued function $(x, \lambda) \mapsto \pi_\lambda(x)$ is both in $\mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times \times \mathbb{G}; \mathcal{L}(\mathscr{E}(\pi)))$ and in $\mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times \times \mathbb{G}; \mathcal{L}(\mathscr{E}(\pi)'))$*

*Proof.* By Lemma 3.4.3 it is enough to show that $x \mapsto \pi(x)$ is slowly increasing. For this purpose we choose an equivalent representation that is more easily understood. There is a representation $\sigma \sim \pi$ on $H_\sigma = L^2(\mathbb{R}^n)$ such that $\mathscr{E}(\sigma) = \mathscr{S}(\mathbb{R}^n)$ and

$$\sigma(x)f(t) = \mathrm{e}^{2\pi \mathrm{i}\xi(a(x,t))} f(x^{-1} \cdot t)$$

where $\xi$ is a linear functional on a subalgebra $\mathfrak{m}$ of $\mathfrak{g}$, $a\colon \mathbb{G} \times \mathbb{R}^n \to \mathfrak{m}$ is polynomial and $\mathbb{G} \times \mathbb{R}^n \ni (x, t) \mapsto x \cdot t \in \mathbb{R}^n$ is a polynomial action of $\mathbb{G}$ on $\mathbb{R}^n$ by [56] and [16, Corollary 4.1.2]. Because $(x, t) \mapsto x^{-1} \cdot t$ is polynomial, we may represent the action of $\mathbb{G}$ on $\mathbb{R}^n$ by a linear combination

$$x \cdot t = \sum_{j,k} s_{k,j}(x)\, u_{k,j}(t)\, e_j, \qquad \text{for } x \in \mathbb{G}\,,\ t \in \mathbb{R}^n\,,$$

where $(e_j)_j$ is the standard basis on $\mathbb{R}^n$ and $s_{k,j}$, $u_{k,j}$ are polynomials. Thus, we also have

$$t_j\, \sigma(x)f(t) = \sum_k s_{k,j}(x)\, \sigma(x)(u_{k,j}\, f)(t).$$

For the same reason, there are polynomials $q_{j,k}$, $\widetilde{q}_{j,k}$ on $\mathbb{G}$, $r_{j,k}$, $\widetilde{r}_{j,k}$ on $\mathbb{R}^n$ such that

$$\begin{aligned}\partial_{t_j} f(x^{-1} \cdot t) &= \sum_k \widetilde{q}_{j,k}(x)\, \widetilde{r}_{j,k}(t)\, (\partial_k f)(x^{-1} \cdot t) \\ &= \sum_k q_{j,k}(x)\, r_{j,k}(x^{-1} \cdot t)\, (\partial_k f)(x^{-1} \cdot t).\end{aligned}$$

Hence, for all $\alpha, \beta \in \mathbb{N}_0$ we find operators $A_k \in \mathcal{L}(\mathscr{S}(\mathbb{R}^n))$ and polynomials $v_k \in \mathscr{P}(\mathbb{G})$ such that

$$t^\beta \partial_t^\alpha \sigma(x) f(t) = \sum_k v_k(x)\, \sigma(x)(A_k f)(t)$$

as a finite linear combination.

The topology on $\mathcal{L}(\mathscr{S}(\mathbb{R}^n))$ is induced by the seminorms

$$p\colon A \mapsto \sup_{f\in B} \sup_{t\in\mathbb{R}^n} |t^\beta \partial_t^\alpha A f(t)|, \quad B \subset \mathscr{S}(\mathbb{R}^n) \text{ bounded}, \alpha, \beta \in \mathbb{N}_0^n.$$

Now if $L \in \mathrm{Diff}_{\mathrm{L}}(\mathbb{G})$ is any left invariant differential operator on $\mathbb{G}$ and $p$ is a seminorm as above, we get

$$\begin{aligned} p(L_x \sigma(x)) &\leq \sum_k v_k(x) \sup_{f\in B} \sup_{t\in\mathbb{R}^n} |\sigma(x)(A_k \sigma(L) f)(t)| \\ &= \sum_k v_k(x) \sup_{f\in B} \sup_{t\in\mathbb{R}^n} |(A_k \sigma(L) f)(t)|. \end{aligned}$$

The right-hand side of the above inequality is a sum of continuous seminorms times polynomials, since $\sigma(L) \in \mathcal{L}(\mathscr{S}(\mathbb{R}^n))$. Thus $x \mapsto \sigma(x)$ is slowly increasing, because $\mathrm{Diff}_{\mathscr{P}}(\mathbb{G}) = \mathrm{span}_{\mathscr{P}(\mathbb{G})} \mathrm{Diff}_{\mathrm{L}}(\mathbb{G})$ by [16, Lemma A.2.2]. Due to $\pi \sim \sigma$ the map $x \mapsto \pi(x)$ is slowly increasing, too. Now $(x, \lambda) \mapsto \pi_\lambda(x)$ is slowly increasing with values in $\mathcal{L}(\mathscr{E}(\pi))$ due to Lemma 3.4.3. We finish the proof by remarking that $\mathcal{L}(\mathscr{E}(\pi))$ and $\mathcal{L}(\mathscr{E}(\pi)')$ are antilinearly homeomorphic by the adjoint map and $\pi_\lambda(x)^* = \pi_\lambda(x^{-1})$. This implies that $\pi$ is also slowly increasing with values in $\mathcal{L}(\mathscr{E}(\pi)')$, since $x \mapsto x^{-1}$ is polynomial. $\square$

With the help of the above proposition, we want to write the inverse Fourier transform as an integral, which converges in $\mathscr{O}_{\mathrm{M}}(\mathbb{G})$. For this purpose, we need to explain a small fact about the dual space $\mathscr{O}'_{\mathrm{M}}(\mathbb{G})$. Denote by $D_1, D_2, \ldots$ the directional derivative to any basis $v_1, v_2, \ldots$ of $\mathfrak{g}$. Each continuous linear functional on $\mathscr{O}_{\mathrm{M}}(\mathfrak{g})$ can be represented by some element in

$$\mathscr{O}'_{\mathrm{M}}(\mathbb{G}) = \mathrm{span}_{\mathbb{C}}\{D^\alpha f \mid \alpha \in \mathbb{N}_0^{\dim(\mathbb{G})} \text{ and } f \in \mathscr{C}(\mathbb{G}) \text{ where } f(x) = \mathcal{O}(|x|^{-\infty}),\ |x| \to \infty\}, \tag{3.4.7}$$

see [34, p. 130 of chapter 2], if we use the dual pairing

$$\langle g, D^\alpha f \rangle := \int_{\mathbb{G}} f\, (-D)^\alpha g \,\mathrm{d}\mu.$$

Here we say $f(x) = \mathcal{O}(|x|^{-\infty})\,,\ |x| \to \infty$, iff $qf$ is a bounded function for any $q \in \mathscr{P}(\mathbb{G})$. The differential operators $D^\alpha$, $\alpha \in \mathbb{N}_0^{\dim(\mathbb{G})}$ span the $\mathscr{P}(\mathbb{G})$-module $\mathrm{Diff}_{\mathcal{P}}(\mathbb{G})$. Since the multiplication of rapidly decreasing functions with polynomials is continuous, we may exchange $D^\alpha$ with arbitrary $P \in \mathrm{Diff}_{\mathscr{P}}(\mathbb{G})$ in the pairing above. I.e. each continuous linear functional on $\mathscr{O}_{\mathrm{M}}(\mathbb{G})$ has a representation in

$$\mathrm{span}_{\mathbb{C}}\{Pf \mid P \in \mathrm{Diff}_{\mathscr{P}}(\mathbb{G}) \text{ and } f \in \mathscr{C}(\mathbb{G}) \text{ where } f(x) = \mathcal{O}(|x|^{-\infty})\,,\ |x| \to \infty\},$$

with respect to the pairing

$$\langle g, Pf \rangle := \int_{\mathbb{G}} f\, P^t g \,\mathrm{d}\mu\,,$$

where $P^t$ is the formal transpose of $P$ defined by

$$\int_{\mathbb{G}} (P^t\varphi)\,\psi \,\mathrm{d}\mu := \int_{\mathbb{G}} \varphi\, P\psi \,\mathrm{d}\mu \quad \text{for all } \varphi, \psi \in \mathscr{D}(\mathbb{G})\,.$$

Note that $P \mapsto P^t$ is a bijection from $\mathrm{Diff}_{\mathscr{P}}(\mathbb{G})$ resp. $\mathrm{Diff}_{\mathrm{L}}(\mathbb{G})$ onto itself.

By [16, Lemma A.2.2] the $\mathscr{P}(\mathbb{G})$-span of the left invariant differential operators $\mathrm{Diff}_{\mathrm{L}}(\mathbb{G})$ is equal to $\mathrm{Diff}_{\mathscr{P}}(\mathbb{G})$. Now let $w^1, w^2, \ldots$ be the dual basis to $v_1, v_2, \ldots$ and let $X_1, X_2, \ldots$ be the left invariant vector fields associated to $v_1, v_2, \ldots$. A quick calculation shows that for all $\phi \in \mathscr{S}'(\mathbb{G})$ and all $j, k$ there exists a polynomial $q \in \mathscr{P}(\mathbb{G})$ with

$$w^j\, X_k\phi = q\,\phi + X_k(w^j\,\phi).$$

Of course, the set of functions $f$ with $f(x) = \mathcal{O}(|x|^{-\infty})\,,\ |x| \to \infty$, is invariant under the multiplication with polynomials. In conclusion, we may represent the dual to $\mathscr{O}_{\mathrm{M}}(\mathbb{G})$ by

$$\mathscr{O}'_{\mathrm{M}}(\mathbb{G}) = \mathrm{span}_{\mathbb{C}}\{Pf \mid P \in \mathrm{Diff}_{\mathrm{L}}(\mathbb{G}) \text{ and } f \in \mathscr{C}(\mathbb{G}) \text{ with } f(x) = \mathcal{O}(|x|^{-\infty})\,,\ |x| \to \infty\}.$$

**Lemma 3.4.5.** *If $\varphi \in \mathscr{S}(\mathfrak{g})$ and $\omega^\times \ni \ell \sim \pi \in \mathrm{SI/Z}(\mathbb{G})$, then the integral*

$$\varphi = \int_{\mathbb{R}^\times} \mathrm{Tr}[\pi_\lambda(-)\,\mathcal{F}_\pi\varphi(\lambda)]\;\mathrm{d}\widehat{\mu}_\pi(\lambda)$$

*exists in $\mathscr{O}_{\mathrm{M}}(\mathbb{G})$.*

*Proof.* Let $f \in \mathscr{C}(\mathbb{G})$ with $f(x) = \mathcal{O}(|x|^{-\infty})\,,\ |x| \to \infty$, let $P \in \mathrm{Diff}_{\mathrm{L}}(\mathbb{G})$ and let $\varphi \in \mathscr{S}(\mathbb{G})$. Then, $f$ and $P^{\mathrm{t}}\varphi$ are $L^2$ functions and we may apply Plancherel for $\mathcal{F}_\pi$.

Hence

$$\langle \varphi, P^{\mathrm{t}}\overline{f}\rangle = \int_{\mathbb{G}} \overline{f}\, P\varphi \,\mathrm{d}\mu = \int_{\mathbb{R}^\times} \mathrm{Tr}[\pi_\lambda(f)^* \,\mathcal{F}_\pi(P\varphi)(\lambda)]\;\mathrm{d}\widehat{\mu}_\pi(\lambda)\,.$$

Since $f \in L^1(\mathbb{G},\mu)$, we know that the integral that evaluates $\pi_\lambda(f)$ converges in $\mathcal{L}_s(H_\pi)$. That means for each pair $u, v \in H_\pi$ we have

$$\big(\pi_\lambda(f)^* u, v\big)_{H_\pi} = \int_{\mathbb{G}} \overline{f(x)}\,\big(\pi_\lambda(x)u, v\big)_{H_\pi}\,\mathrm{d}\mu(x).$$

Because $P\varphi \in \mathscr{S}(\mathbb{G})$, we have $\mathcal{F}_\pi(P\varphi)(\lambda) = \pi_\lambda(P)\,\pi_\lambda(\varphi) \in \mathcal{L}(\mathscr{E}(\pi)', \mathscr{E}(\pi))$, which is a nuclear operator on $H_\pi$ for each $\lambda \in \mathbb{R}^\times$. Hence for each orthonormal basis $(e_k)_{k\in\mathbb{N}} \subset H_\pi$

$$\begin{aligned}\int_{\mathbb{G}} \sum_{k\in\mathbb{N}} \big|\overline{f(x)}\,\big(\pi_\lambda(x)\,\pi_\lambda(P)\,\pi_\lambda(\varphi)e_k, e_k\big)_{H_\pi}\big|\,\mathrm{d}\mu(x)& \\ \leq \|f\|_{L^1(\mathbb{G},\mu)}\,\|\pi_\lambda(P)\,\pi_\lambda(\varphi)\|_{\mathcal{N}(H_\pi)} &< \infty,\end{aligned}$$

where $\|\cdot\|_{\mathcal{N}(H_\pi)}$ is the trace-norm on the space of nuclear operators on $H_\pi$. Using Fubini with respect to the counting measure and $\mu$ results in

$$\mathrm{Tr}[\pi_\lambda(f)^* \,\mathcal{F}_\pi(P\varphi)(\lambda)] = \int_{\mathbb{G}} \overline{f(x)}\,\mathrm{Tr}[\pi_\lambda(x)\,\pi_\lambda(P)\,\pi_\lambda(\varphi)]\,\mathrm{d}\mu(x),$$

since $f \in L^1(\mathbb{G},\mu)$. Naturally, we have $\pi_\lambda(x)\,\pi_\lambda(P) = P_x\pi_\lambda(x)$. By the embedding of $\mathcal{L}(\mathscr{E}(\pi)', \mathscr{E}(\pi))$ into the nuclear operators $\mathcal{N}(H_\pi)$, we may see Tr as a continuous functional on $\mathcal{L}(\mathscr{E}(\pi)', \mathscr{E}(\pi))$. Because the operator valued function $\pi_\lambda(-)\,\pi_\lambda(\varphi)$ is a slowly increasing map from $\mathbb{G}$ to $\mathcal{L}(\mathscr{E}(\pi)', \mathscr{E}(\pi))$, we get

$$\begin{aligned}\mathrm{Tr}[\pi_\lambda(x)\,\pi_\lambda(P)\,\pi_\lambda(\varphi)] &= P_x\,\mathrm{Tr}[\pi_\lambda(x)\,\pi_\lambda(\varphi)],\\ \mathrm{Tr}[\pi_\lambda(-)\,\pi_\lambda(\varphi)] &\in \mathscr{O}_{\mathrm{M}}(\mathbb{G}).\end{aligned}$$

Finally we get

$$\begin{aligned}\langle \varphi, P^{\mathrm{t}}\overline{f}\rangle &= \int_{\mathbb{R}^\times}\int_{\mathbb{G}} \overline{f(x)}\,P_x\,\mathrm{Tr}[\pi_\lambda(x)\,\pi_\lambda(\varphi)]\,\mathrm{d}\mu(x)\mathrm{d}\lambda_\pi\\ &= \int_{\mathbb{R}^\times} \langle \mathrm{Tr}[\pi_\lambda(-)\,\pi_\lambda(\varphi)], P^{\mathrm{t}}\overline{f}\rangle\,\mathrm{d}\lambda_\pi,\end{aligned}$$

which completes the proof. □

Let us now define

$$\rho, \rho^* \in \mathscr{O}_{\mathrm{M}}(\mathbb{G}) \hat{\otimes} \mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times; \mathscr{E}(\pi))$$

by $\rho(x,\lambda) := \pi_\lambda(x)$ and $\rho^*(\lambda, x) := \pi_\lambda(x^{-1})$ for some fixed $\pi \in \mathrm{SI}/\mathrm{Z}_{\mathbb{R}}(\mathbb{G})$. With Lemma 1.2.4, we already proved the continuity of the map

$$\mathcal{L}(\mathscr{O}_{\mathrm{M}}(\mathbb{G})) \to \mathcal{L}(\mathscr{O}_{\mathrm{M}}(\mathbb{G}) \hat{\otimes} \mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times; \mathscr{E}(\pi))),\ A \mapsto A \otimes 1.$$

Of course, the evaluation map

$$\mathcal{L}(\mathscr{O}_{\mathrm{M}}(\mathbb{G}) \hat{\otimes} \mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times; \mathscr{E}(\pi))) \to \mathscr{O}_{\mathrm{M}}(\mathbb{G}) \hat{\otimes} \mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times; \mathscr{E}(\pi)),\ T \mapsto T(\rho)$$

is continuous as well. Finally, since the multiplication in $\mathscr{O}_{\mathrm{M}}(\mathbb{G} \times \mathbb{R}^\times)$ is continuous [65, p. 248] and because of Proposition 2.1.10, we can construct a hypocontinuous multiplication on $\mathscr{O}_{\mathrm{M}}(\mathbb{G} \times \mathbb{R}^\times; \mathcal{L}(\mathscr{E}(\pi)))$ given by the pointwise operator composition. Using the canonical linear homeomorphism

$$\mathscr{O}_{\mathrm{M}}(\mathbb{G} \times \mathbb{R}^\times; \mathcal{L}(\mathscr{E}(\pi))) \simeq \mathscr{O}_{\mathrm{M}}(\mathbb{G}) \hat{\otimes} \mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times; \mathscr{E}(\pi))$$

we get a continuous multiplication map

$$\begin{aligned} \mathscr{O}_{\mathrm{M}}(\mathbb{G}) \hat{\otimes} \mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times; \mathscr{E}(\pi)) \to \mathscr{O}_{\mathrm{M}}(\mathbb{G}) \hat{\otimes} \mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times; \mathscr{E}(\pi))\colon F \mapsto \rho^* \cdot F\,, \\ \text{where} \quad (\rho^* \cdot F)(x,\lambda) = \rho^*(x,\lambda) \circ F(x,\lambda)\,. \end{aligned}$$

Finally, we define the continuous map $S$ by

$$S\colon \mathcal{L}(\mathscr{O}_{\mathrm{M}}(\mathbb{G})) \to \mathscr{O}_{\mathrm{M}}(\mathbb{G} \times \mathbb{R}^\times; \mathcal{L}(\mathscr{E}(\pi))),\ A \mapsto \rho^* \cdot (A \otimes 1)(\rho)\,. \tag{3.4.8}$$

Now this map looks exactly like the inverse Kohn-Nirenberg quantization on compact Lie groups $\mathbb{X}$ from [59]. Namely, for any $B \in \mathcal{L}(\mathscr{S}(\mathbb{X}))$ the unique Kohn-Nirenberg symbol $b$ with $B = \mathrm{Op}_{\mathbb{X}}(b)$, evaluated at the irreducible unitary representation $\xi \in \mathrm{Irr}(\mathbb{X})$, is given by $b(-,\xi) = \xi(-)^* \cdot (B \otimes \mathrm{I})(\xi) \in \mathscr{S}(\mathbb{X}; \mathcal{L}(H_\xi))$.

Before proving that $S$ coincides with $\mathrm{Op}_\pi^{-1}$, we need a bit of preparation.

**Lemma 3.4.6.** *Suppose $a \in \mathscr{S}'(\mathbb{G}) \hat{\otimes} \mathscr{S}'(\mathbb{R}^\times; \pi)$, then*

$$\rho \cdot a = (1 \otimes \mathcal{F}_\pi \operatorname{inv}) \mathcal{T}_{\mathbb{G},*}^{-1} (1 \otimes \mathcal{F}_\pi^{-1}) a,$$

*where* inv *is the Gelfand triple isomorphism from* $\mathcal{G}_*(\mathbb{G},\mu)$ *onto itself, defined by* $\operatorname{inv} f(x) = f(x^{-1})$ *for* $f \in \mathscr{S}(\mathbb{G})$*,* $x \in \mathbb{G}$*.*

*Proof.* First, we take $a \in \mathscr{S}(\mathbb{G}) \hat{\otimes} \mathscr{S}(\mathbb{R}^\times;\pi)$. Then we just have

$$(1 \otimes \mathcal{F}_\pi^{-1})(\rho \cdot a)(x,y) = (1 \otimes \mathcal{F}_\pi^{-1})a(x,yx) = (1 \otimes \operatorname{inv})\mathcal{T}_{\mathbb{G},*}^{-1}(1 \otimes \mathcal{F}_{\mathbb{G}}^{-1})a(x,y),$$

by the integral formula for the inverse Fourier transform from Lemma 3.4.5. Now the rest simply follows due to the continuity of the involved maps. □

**Lemma 3.4.7.** *Define* $\chi_x(\xi) := \mathrm{e}^{2\pi \mathrm{i}\xi(x)}$ *for* $x \in \mathfrak{g}$ *and* $\xi \in \mathfrak{g}'$*. Then*

$$\big(\mathfrak{Op}_\pi \wp_\ell \, \chi_x\big)(\lambda) = \pi_\lambda(x)$$

*for any* $\lambda \in \mathbb{R}^\times$ *and* $\mathrm{SI}/\mathrm{Z}_{\mathbb{R}}(\mathbb{G}) \ni \pi \sim \ell \in \omega^\times$*.*

*Proof.* Let $x \in \mathbb{G}$ and let $\varepsilon_x$ be the functional on $\mathscr{S}_*(\mathbb{G})$ defined by $\varepsilon_x \colon \varphi \mapsto \varphi(x)$. By Corollary 3.3.18 we have

$$\mathcal{F}_\pi \varepsilon_x \in \mathscr{C}(\mathbb{R}^\times; \mathcal{L}_s(H_\pi)) \quad \text{with} \quad \mathcal{F}_\pi(\varepsilon_x)(\lambda) = \pi_\lambda(x)$$

for any $\lambda \in \mathbb{R}^\times$. Also we have $\mathcal{F}_{\mathfrak{g}}^{-1}\varepsilon_x = \chi_x$. Thus

$$\big(\mathfrak{Op}_\pi \wp_\ell \, \chi_x\big)(\lambda) = \big(\mathcal{F}_\pi \mathcal{F}_{\mathfrak{g}}^{-1} \chi_x\big)(\lambda) = \pi_\lambda(x)\,.$$

□

**Lemma 3.4.8.** *The embedding* $\mathscr{S}_*(\mathbb{G}) \hookrightarrow \mathscr{O}_{\mathrm{M}}(\mathbb{G})$ *is continuous and has dense range.*

*Proof.* The multiplication on $\mathscr{S}(\mathbb{G})$ is a continuous bilinear map. This implies the continuity of the canonical embedding $\tau \colon \mathscr{S}_*(\mathbb{G}) \hookrightarrow \mathscr{O}_{\mathrm{M}}(\mathbb{G})$, since $\mathscr{S}_*(\mathbb{G})$ carries the subspace topology in $\mathscr{S}(\mathbb{G})$. Now consider the dual map

$$\tau' \colon \mathscr{O}_{\mathrm{M}}'(\mathbb{G}) \to \mathscr{S}_*'(\mathbb{G}), \quad \text{where} \quad \langle \varphi, \tau'\phi \rangle = \langle \varphi, \phi \rangle, \quad \text{for all } \varphi \in \mathscr{S}_*(\mathbb{G}).$$

That this is indeed an embedding, can be seen from Proposition 2.3.13 and the representation (3.4.7) of the dual space $\mathscr{O}_{\mathrm{M}}'(\mathbb{G})$. By the Hahn-Banach theorem the operator $\tau$ has dense image. □

In the Lemma above we saw that $\mathscr{S}_*(\mathbb{G}) \hookrightarrow \mathscr{O}_\mathrm{M}(\mathbb{G})$ has dense range. Naturally, we also have $\mathscr{O}_\mathrm{M}(\mathbb{G}) \hookrightarrow \mathscr{S}'(\mathbb{G})$, $\mathscr{S}(\mathbb{G}) \hookrightarrow \mathscr{S}'(\mathbb{G})$ and $\mathscr{O}_\mathrm{M}(\mathbb{R}^\times) \hookrightarrow \mathscr{S}'(\mathbb{R}^\times)$, thus we get embeddings

$$\begin{aligned}
\mathcal{L}(\mathscr{O}_\mathrm{M}(\mathbb{G}); \mathscr{F}(\mathbb{G})) &\hookrightarrow \mathcal{L}(\mathscr{S}_*(\mathbb{G}), \mathscr{S}'(\mathbb{G})), \\
\mathscr{F}(\mathbb{G}) \mathbin{\hat{\otimes}} \mathscr{O}_\mathrm{M}(\mathbb{R}^\times; \mathcal{L}(\mathscr{E}(\pi))) &\hookrightarrow \mathscr{S}'(\mathbb{G}) \mathbin{\hat{\otimes}} \mathscr{S}'(\mathbb{R}^\times; \pi))
\end{aligned}$$

for $\mathscr{F}(\mathbb{G}) \in \{\mathscr{S}(\mathbb{G}), \mathscr{O}_\mathrm{M}(\mathbb{G})\}$. Note that we can exchange $\mathcal{L}(\mathscr{E}(\pi))$ with $\mathcal{L}(\mathscr{E}(\pi)')$ in the paragraph above. We can even go one step further. For $A \in \mathcal{L}(\mathscr{O}_\mathrm{M}(\mathbb{G}); \mathscr{S}(\mathbb{G}))$ we can still define the map $S$, since $\mathscr{S}(\mathbb{G}) \hookrightarrow \mathscr{O}_\mathrm{M}(\mathbb{G})$. However, we are lacking the tools to check whether $S(A) \in \mathscr{S}(\mathbb{G}) \mathbin{\hat{\otimes}} \mathscr{O}_\mathrm{M}(\mathbb{R}^\times; \mathcal{L}(\mathscr{E}(\pi)))$ or not, since we cannot apply Theorem 1.2.11 or Theorem 1.2.12. We run into the same problem if we define $S$ for operators $A \in \mathcal{L}(\mathscr{O}_\mathrm{M}(\mathbb{G}); \mathscr{S}'(\mathbb{G}))$.

With the above embeddings, we finally prove that the map $S$ does indeed reproduce the Kohn-Nirenberg symbol.

**Theorem 3.4.9.** *Let $\mathscr{F}(\mathbb{G}) \in \{\mathscr{S}(\mathbb{G}), \mathscr{O}_\mathrm{M}(\mathbb{G}))$. The inverse of the Kohn-Nirenberg quantization in $\pi$-picture defines a continuous map*

$$\mathrm{Op}_\pi^{-1} \colon \mathcal{L}(\mathscr{O}_\mathrm{M}(\mathbb{G}); \mathscr{F}(\mathbb{G})) \to \mathscr{F}(\mathbb{G}) \mathbin{\hat{\otimes}} \mathscr{O}_\mathrm{M}(\mathbb{R}^\times; \mathcal{L}(\mathscr{E}(\pi)).$$

*For any $A \in \mathcal{L}(\mathscr{O}_\mathrm{M}(\mathbb{G}), \mathscr{F}(\mathbb{G}))$ the equality $a := \mathrm{Op}_\pi^{-1}(A) = \rho^* \cdot (A \otimes \mathrm{I})(\rho)$ is valid. Furthermore,*

$$A\,\varphi = \int_{\mathbb{R}^\times} \mathrm{Tr}[\pi_\lambda(-)\, a(-,\lambda)\, \mathcal{F}_\pi\varphi(\lambda)]\, \mathrm{d}\widehat{\mu}_\pi(\lambda) \quad \textit{for} \quad \varphi \in \mathscr{S}(\mathbb{G}),$$

*in which the integral exists in $\mathscr{F}(\mathbb{G})$.*

*Proof.* As in (3.4.8), we define $S(A) := \rho^* \cdot (A \otimes \mathrm{I})(\rho)$, where the multiplication is defined via the hypocontinuous multiplication

$$\mathscr{F}(\mathbb{G}) \mathbin{\hat{\otimes}} \mathscr{O}_\mathrm{M}(\mathbb{R}^\times; \mathcal{L}(\mathscr{E}(\pi)) \times \mathscr{O}_\mathrm{M}(\mathbb{G}) \mathbin{\hat{\otimes}} \mathscr{O}_\mathrm{M}(\mathbb{R}^\times; \mathcal{L}(\mathscr{E}(\pi)) \to \mathscr{O}_\mathrm{M}(\mathbb{G}) \mathbin{\hat{\otimes}} \mathscr{O}_\mathrm{M}(\mathbb{R}^\times; \mathcal{L}(\mathscr{E}(\pi))\,.$$

Via the argumentation leading up to (3.4.8), we get the continuity of $S$.

Now we will prove the integral formula for $A \in \mathcal{L}(\mathscr{O}_{\mathrm{M}}(\mathbb{G}); \mathscr{F}(\mathbb{G}))$. From Lemma 3.4.5 we know that for $\varphi \in \mathscr{S}(\mathbb{G})$

$$A\,\varphi = A \int_{\mathbb{G}} \mathrm{Tr}[\pi_\lambda(-)\,\pi_\lambda(\varphi)]\,\mathrm{d}\widehat{\mu}_\pi(\lambda) = \int_{\mathbb{G}} A\big(\mathrm{Tr}[\pi_\lambda(-)\,\pi_\lambda(\varphi)]\big)\,\mathrm{d}\widehat{\mu}_\pi(\lambda)\,,$$

where the integral converges in $\mathscr{F}(\mathbb{G})$. Due to Proposition 3.3.20 and Proposition 3.4.4, we know that

$$\pi_\lambda(-)\,\pi_\lambda(\varphi) \in \mathscr{O}_{\mathrm{M}}(\mathbb{G}) \,\hat{\otimes}\, \mathcal{L}(\mathscr{E}(\pi)', \mathscr{E}(\pi)).$$

The trace operator Tr, restricted from the nuclear operators on $H_\pi$, is a continuous functional on $\mathcal{L}(\mathscr{E}(\pi)', \mathscr{E}(\pi))$, so we may use the tensor product structure of the above expression to get

$$A\big(\mathrm{Tr}[\pi_\lambda(-)\,\pi_\lambda(\varphi)]\big) = (A \otimes \mathrm{Tr})\big(\pi_\lambda(-)\,\pi_\lambda(\varphi)\big) = (1 \otimes \mathrm{Tr})(A \otimes 1)\big(\pi_\lambda(-)\,\pi_\lambda(\varphi)\big)$$

for each $\lambda \in \mathbb{R}^\times$. Furthermore,

$$(A \otimes 1)(\pi_\lambda(-)\,\pi_\lambda(\varphi)) = \pi_\lambda \cdot \pi_\lambda^* \cdot (A \otimes 1)(\pi_\lambda) \cdot \pi_\lambda(\varphi),$$

in which the multiplication is defined pointwise by the multiplication in $\mathcal{L}(\mathscr{E}(\pi))$. Hence, we can represent $A\,\varphi$ by the integral

$$A\,\varphi = \int_{\mathbb{R}^\times} \mathrm{Tr}[\pi_\lambda(-)\,a(-,\lambda)\,\pi_\lambda(\varphi)]\,\mathrm{d}\widehat{\mu}_\pi(\lambda)$$

with $a := S(A)$.

Now it is left to check that indeed $A = \mathrm{Op}_\pi(a)$. First of all, due to Lemma 3.4.6

$$\mathcal{T}_{\mathbb{G},*}^{-1}(1 \otimes \mathcal{F}_\pi^{-1})a = (1 \otimes \mathrm{inv}\,\mathcal{F}_\pi^{-1})(\rho \cdot a).$$

We define the function $\chi(x,\xi) := \mathrm{e}^{2\pi\mathrm{i}\xi(x)}$ for $\xi \in \mathfrak{g}'$, $x \in \mathfrak{g}$, then $\chi \in \mathscr{O}_{\mathrm{M}}(\mathfrak{g} \times \mathfrak{g}^\times)$. Because $(1 \otimes \mathfrak{O}\mathfrak{p}_\pi \wp_\ell)\chi(x,\lambda) = \pi_\lambda(x) = \rho(x,\lambda)$, due to Lemma 3.4.7, and $\mathcal{F}_\pi = \mathfrak{O}\mathfrak{p}_\pi \wp_\ell \mathcal{F}_{\mathfrak{g}}$, we know that

$$(1 \otimes \mathrm{inv}\,\mathcal{F}_\pi^{-1})(A \otimes 1)(\rho) = (A \otimes \mathrm{inv}\,\mathcal{F}_{\mathfrak{g}}^{-1})(\chi) = (A \otimes \mathcal{F}_{\mathfrak{g}'})(\chi).$$

We choose arbitrary $\varphi \in \mathscr{S}(\mathfrak{g})$ and $\psi \in \mathscr{S}_*(\mathbb{G})$. The integral

$$\varphi = \int_{\mathfrak{g}'} \chi(-,\xi)\,\mathcal{F}_{\mathfrak{g}}\varphi(\xi)\,\mathrm{d}\mu'(\xi)$$

converges in $\mathscr{O}_{\mathrm{M}}(\mathfrak{g})$. Hence,

$$\begin{aligned}\langle (A\otimes\mathcal{F}_{\mathfrak{g}'})\chi,\psi\otimes\varphi\rangle &= \langle (A\otimes 1)\chi,\psi\otimes\mathcal{F}_{\mathfrak{g}}\varphi\rangle\\ &= \int_{\mathfrak{g}'}\langle A(\chi(-,\xi)),\psi\rangle\,\mathcal{F}_{\mathfrak{g}}\varphi(\xi)\,\mathrm{d}\mu'(\xi)\\ &= \langle A\varphi,\psi\rangle.\end{aligned}$$

Combining the calculations above implies

$$\mathcal{K}_{\mathbb{G},*}A=\mathcal{T}_{\mathbb{G},*}^{-1}(1\otimes\mathcal{F}_{\pi}^{-1})a,$$

for the kernel map

$$\mathcal{K}_{\mathbb{G},*}\colon\mathcal{L}(\mathcal{G}_*(\mathbb{G},\mu);\mathcal{G}(\mathbb{G},\mu))\to\mathcal{G}(\mathbb{G},\mu)\otimes\mathcal{G}_*(\mathbb{G},\mu)\,,$$

i.e. $\mathrm{Op}_{\pi}^{-1}(A)=S(A)$. $\square$

We can even go one step further in the description of the Kohn-Nirenberg symbol. Namely, we can describe the symbol of operators in $\mathcal{L}(\mathscr{O}_{\mathrm{M}}(\mathbb{G});\mathscr{S}'(\mathbb{G}))$ in a similar manner as above.

**Corollary 3.4.10.** *Suppose $A\in\mathcal{L}(\mathscr{O}_{\mathrm{M}}(\mathbb{G});\mathscr{S}'(\mathbb{G}))$, then the inverse to the Kohn-Nirenberg in $\pi$-picture can be expressed by*

$$\mathrm{Op}_{\pi}^{-1}(A)=\rho^*\cdot(A\otimes\mathrm{I})(\rho)\,,$$

*in which the multiplication is defined via Theorem 1.2.12 as the vector valued bilinear multiplication*

$$\mathscr{O}_{\mathrm{M}}(\mathbb{G})\,\hat{\otimes}\,\mathscr{O}_{\mathrm{M}}(\mathbb{R}^{\times};\mathcal{L}(\mathscr{E}(\pi)')\times\mathscr{S}'(\mathbb{G})\,\hat{\otimes}\,\mathscr{S}'(\mathbb{R}^{\times};\pi)\to\mathscr{S}'(\mathbb{G})\,\hat{\otimes}\,\mathscr{S}'(\mathbb{R}^{\times};\pi)\,.$$

*Proof.* The operator valued functions $\rho$, $\rho^*$ are in $\mathscr{O}_{\mathrm{M}}(\mathbb{G})\,\hat{\otimes}\,\mathscr{O}_{\mathrm{M}}(\mathbb{R}^{\times};\mathcal{L}(\mathscr{E}(\pi)))$. The operator $A\otimes\mathrm{I}$ maps $\rho$ into $\mathscr{S}'(\mathbb{G})\,\hat{\otimes}\,\mathscr{O}_{\mathrm{M}}(\mathbb{R}^{\times};\mathcal{L}(\mathscr{E}(\pi)')$, which in turn is embedded into $\mathscr{S}'(\mathbb{G})\,\hat{\otimes}\,\mathscr{S}'(\mathbb{R}^{\times};\pi)$. This induces a continuous map

$$\mathcal{L}(\mathscr{O}_{\mathrm{M}}(\mathbb{G});\mathscr{S}'(\mathbb{G}))\to\mathscr{S}'(\mathbb{G})\,\hat{\otimes}\,\mathscr{S}'(\mathbb{R}^{\times};\pi)\colon A\mapsto(A\otimes\mathrm{I})(\rho)\,.$$

Moreover, the composition with the multiplication $A \mapsto \rho^* \cdot (A \otimes \mathrm{I})(\rho)$ is continuous and well-defined by Theorem 1.2.12.

Now we use the continuity of $\mathrm{Op}_\pi^{-1}$ from $\mathcal{L}(\mathscr{O}_{\mathrm{M}}(\mathbb{G}); \mathscr{S}'(\mathbb{G})) \hookrightarrow \mathcal{L}(\mathscr{S}_*(\mathbb{G}); \mathscr{S}'(\mathbb{G}))$ into $\mathscr{S}'(\mathbb{G}) \mathbin{\hat{\otimes}} \mathscr{S}'(\mathbb{R}^\times; \pi)$. The space $\mathscr{L}(\mathscr{O}_{\mathrm{M}}(\mathbb{G}))$ is dense in $\mathcal{L}(\mathscr{O}_{\mathrm{M}}(\mathbb{G}); \mathscr{S}'(\mathbb{G}))$ and for all operators $A \in \mathscr{L}(\mathscr{O}_{\mathrm{M}}(\mathbb{G}))$ the expression $\mathrm{Op}_\pi^{-1}(A) = \rho^* \cdot (A \otimes \mathrm{I})(\rho)$ holds with the multiplication defined in $\mathscr{O}_{\mathrm{M}}(\mathbb{G}) \mathbin{\hat{\otimes}} \mathscr{O}_{\mathrm{M}}(\mathbb{R}^\times; \mathcal{L}(\mathscr{E}(\pi)'))$. Since the multiplication commutes with the embedding of the left-hand side into $\mathscr{S}'(\mathbb{G}) \mathbin{\hat{\otimes}} \mathscr{S}'(\mathbb{R}^\times; \pi)$, the formula for the symbol $\mathrm{Op}_\pi^{-1}(A) = \rho^* \cdot (A \otimes \mathrm{I})(\rho)$ holds for all $A \in \mathcal{L}(\mathscr{O}_{\mathrm{M}}(\mathbb{G}); \mathscr{S}'(\mathbb{G}))$. $\square$

# Chapter 4

# Operator spaces characterized by ultradifferentiable group actions

For any locally convex representation $(\pi, E)$ on a Lie group $\mathbb{G}$ we will define a corresponding locally convex representation $(\mathrm{Ad}_\pi, \mathcal{L}(E))$ by

$$\mathrm{Ad}_\pi(x)T := \pi(x) \circ T \circ \pi(x)^{-1} \qquad \text{for } T \in \mathcal{L}(E), x \in \mathbb{G}\,.$$

Note that even for admissible $\pi$ the representation $\mathrm{Ad}_\pi$ might not be admissible. But of course $\mathrm{Ad}_\pi \downarrow_{\mathscr{C}(\mathrm{Ad}_\pi)}$ is admissible in any case. But if $\pi$ is admissible, then $\mathrm{Ad}_\pi$ is still locally equicontinuous, i.e. $\mathscr{C}(\mathrm{Ad}_\pi)$ carries the subspace topology with respect to $\mathcal{L}(E)$.

The spaces of ultradifferentiable vectors $\mathscr{E}_D^{[M]}(\pi)$, for frames $D \subset \mathrm{Diff}_{\mathrm{L}}(\mathbb{G})$, are characterized via the decay of the family of vectors

$$\pi(D^a)e = \cdots \pi(D_{a_3})\pi(D_{a_2})\pi(D_{a_1})e\,, \quad a \in \mathcal{S}_{\dim \mathbb{G}}$$

in $E$. This implies that the ultradifferentiable vectors the representation $\mathrm{Ad}_\pi$, i.e. the ultradifferentiable operators, are characterized by the decay and existence of the family of higher order commutators

$$\begin{aligned}\mathrm{Ad}_\pi(D^a)T &= \cdots \mathrm{Ad}_\pi(D_{a_3})\,\mathrm{Ad}_\pi(D_{a_2})\,\mathrm{Ad}_\pi(D_{a_1})T \\ &= \ldots [\pi(D_{a_3}), [\pi(D_{a_2}), [\pi(D_{a_1}), T]]] \ldots\,, \quad a \in \mathcal{S}_{\dim \mathbb{G}}\,,\end{aligned}$$

in $\mathcal{L}(E)$.

## 4.1 Operator spaces defined by adjoint representations

The operator spaces $\mathscr{F}(\mathrm{Ad}_\pi)$ often have very convenient properties. Most notably, continuous multiplications between $\mathscr{F}(\mathbb{G})$ and other $\mathscr{C}(\mathbb{G})$-function spaces imply mapping properties of the operators in $\mathscr{F}(\mathrm{Ad}_\pi)$.

**Lemma 4.1.1.** *Suppose $\mathscr{F}(\mathbb{G})$, $\mathscr{G}(\mathbb{G})$ and $\mathscr{H}(\mathbb{G})$ are complete $\mathscr{C}(\mathbb{G})$-function spaces with continuous multiplication $\mathscr{F}(\mathbb{G}) \times \mathscr{G}(\mathbb{G}) \to \mathscr{H}(\mathbb{G})$ such that $\mathscr{F}(\mathbb{G})$ is nuclear. Suppose furthermore $(E, \pi)$ is a bornologic, complete locally convex representation and that either $E$ or $\mathscr{G}(\mathbb{G})$ has the approximation property. Then the restriction map*

$$\mathscr{F}(\mathrm{Ad}_\pi) \to \mathcal{L}(\mathscr{G}(\pi), \mathscr{H}(\pi)) \colon T \mapsto T \restriction_{\mathscr{G}(\pi)}^{\mathscr{H}(\pi)}$$

*is well-defined and continuous. Moreover, if the right translation $\boldsymbol{R}$ acts as an admissible representation on $\mathscr{G}(\mathbb{G})$ and $\mathscr{H}(\mathbb{G})$, then for each $T \in \mathscr{F}(\mathrm{Ad}_\pi)$ the vector valued map*

$$\mathbb{G} \to \mathcal{L}_c(\mathscr{G}(\pi); \mathscr{H}(\pi)) \colon x \mapsto \sigma(x)\, T\, \omega(x)^{-1}\,,$$

*in which $\sigma(x) = \pi(x) \restriction_{\mathscr{H}(\pi)}^{\mathscr{H}(\pi)}$ and $\omega(x) = \pi(x) \restriction_{\mathscr{G}(\pi)}^{\mathscr{G}(\pi)}$, is continuous*

*Proof.* Let us denote by $\mathrm{ev} \colon \mathcal{L}(E) \times E \to E$ the evaluation map $\mathrm{ev}(T, e) = Te$. Since $E$ is both bornologic and complete, it is barrelled by [36, Satz 7.14]. Hence ev is a hypocontinuous bilinear map as described in Lemma 1.2.10. Furthermore, $\mathcal{L}(E)$ is complete by [44, §39.6 (4)]. Hence the bilinear map

$$\dot{\mathrm{ev}} \colon \mathscr{F}(\mathbb{M}; \mathcal{L}_b(E)) \times \mathscr{G}(\mathbb{M}; E) \to \mathscr{H}(\mathbb{M}; E) \colon \dot{\mathrm{ev}}(T, e)(x) := \mathrm{ev}(T(x), e(x)) \text{ for } x \in \mathbb{G}$$

is well-defined and hypocontinuous by Proposition 2.1.10. Of course this map pulls back to a hypcontinuous bilinear map

$$\tilde{\mathrm{ev}} \colon \mathscr{F}(\mathrm{Ad}_\pi) \times \mathscr{G}(\pi) \to \mathscr{H}(\pi)\,,$$

which is also a restriction of ev, i.e.

$$\dot{\mathrm{ev}}(\mathrm{Ad}_\pi\, T, \pi\, v) = \pi\, T v = \pi\, \tilde{\mathrm{ev}}(T, v) = \pi\ \mathrm{ev}(T, v) \qquad \text{for } T \in \mathscr{F}(\mathrm{Ad}_\pi)\,,\ v \in \mathscr{G}(\pi)\,.$$

Now the continuity of $T \mapsto T \restriction^{\mathscr{H}(\mathbb{G})}_{\mathscr{G}(\pi)}$ follows immediately. Moreover, if $\boldsymbol{R}{\downarrow}_{\mathscr{G}(\mathbb{G})}$ and $\boldsymbol{R}{\downarrow}_{\mathscr{F}(\mathbb{G})}$ are admissible representations, then $\sigma$ and $\omega$ are admissible representations by Lemma 2.4.3 **(iii)** and the families of operators $(\omega(y))_{y\in K}$ and $(\sigma(y))_{y\in K}$ are equicontinuous for each compact neighbourhood $K$ of $x$. Thus we have $\omega(y) \xrightarrow{y\to x} \omega(x)$ in $\mathcal{L}_c(\mathscr{G}(\pi))$, $\sigma(y) \xrightarrow{y\to x} \sigma(x)$ in $\mathcal{L}_c(\mathscr{H}(\pi))$ and

$$\sigma(x)S\omega(x) - \sigma(y)S\omega(y) = \sigma(x)S\big(\omega(x)-\omega(y)\big) + \big(\sigma(x)-\sigma(y)\big)S\omega(y) \quad \xrightarrow{y\to x} \quad 0$$

in $\mathcal{L}_c(\mathscr{G}(\pi);\mathscr{H}(\pi))$. □

Note that without additional requirements, the map $T \mapsto T \restriction^{\mathscr{H}(\pi)}_{\mathscr{G}(\pi)}$ might not be injective or it could even be trivial. Of course, whenever $\mathscr{G}(\pi)$ is dense in $E$, we get an embedding.

If we want to prove continuity on Sobolev spaces defined by inductive sequences, the above lemma is not directly applicable. In these cases the following lemma will bridge that gap.

**Lemma 4.1.2.** *Let $\mathbb{G}$ be a Lie group, let $(\pi, E)$ be a locally convex representation on Fréchet space $E$. Furthermore, let $(A,\leq)$ and $(B,\leq)$ be directed sets and let $\mathscr{F}_\alpha(\mathbb{G})$, $\mathscr{G}_\beta(\mathbb{G})$ be $\mathscr{C}(\mathbb{G})$-function spaces and Fréchet spaces for $\alpha\in A$ and $\beta\in B$. Suppose there are continuous embeddings $\mathscr{F}_\alpha(\mathbb{G}) \xrightarrow{\mathrm{I}} \mathscr{G}_\beta(\mathbb{G})$, $\mathscr{F}_\alpha(\mathbb{G}) \xrightarrow{\mathrm{I}} \mathscr{F}_{\alpha'}(\mathbb{G})$ and $\mathscr{G}_\beta(\mathbb{G}) \xrightarrow{\mathrm{I}} \mathscr{F}_{\beta'}(\mathbb{G})$ for all $\alpha\leq\alpha'$ and $\beta'\leq\beta$ and*

$$\exists_{\alpha\in A}\colon B\subset \mathscr{F}_\alpha(\pi) \text{ is bounded} \quad\Leftrightarrow\quad \forall_{\beta\in B}\colon B\subset\mathscr{G}_\beta(\pi) \text{ is bounded}. \tag{4.1.1}$$

*Then*

$$\mathcal{L}\Big(\varprojlim_{\beta\in B}\mathscr{G}_\beta(\pi)\Big) \xrightarrow{\mathrm{I}} \mathcal{L}\Big(\varinjlim_{\alpha\in A}\mathscr{F}_\alpha(\pi)\Big)$$

*is a well-defined embedding. Furthermore, if*

$$\tilde{\mathcal{L}} := \Big\{T\in\mathcal{L}\Big(\varinjlim_{\alpha\in A}\mathscr{F}_\alpha(\pi)\Big) \;\Big|\; \forall_{\alpha\in A}\ T\restriction^{\mathscr{F}_\alpha(\pi)}_{\mathscr{F}_\alpha(\pi)} \text{ is continuous}\Big\}$$

*equipped with the initial topology with respect to the restriction maps $\tilde{\mathcal{L}}\to\mathcal{L}(\mathscr{F}_\alpha(\pi))$ then*

$$\tilde{\mathcal{L}} \xrightarrow{\mathrm{I}} \mathcal{L}\Big(\varinjlim_{\alpha\in A}\mathscr{F}_\alpha(\pi)\Big)$$

*is a continuous embedding.*

*Proof.* We define

$$F := \varinjlim_{\alpha \in A} \mathscr{F}_\alpha(\pi) \quad \text{and} \quad G := \varprojlim_{\beta \in B} \mathscr{G}_\beta(\pi)\,.$$

First of all, the assumptions of the lemma imply that $F = G$ in the sense of vector spaces. Moreover, since $F \xrightarrow{\mathrm{I}} G$ is continuous, all subsets which are bounded on the left hand side are also bounded on the right hand side. From (4.1.1) follows that the bounded sets on both sides coincide.

For each $\alpha \in A$ the space $\mathscr{F}_\alpha(\pi)$ can be identified with a closed subspace of the Fréchet space $\mathscr{F}_\alpha(\mathbb{G})\,\varepsilon\, E$, so $\mathscr{F}_\alpha(\pi)$ is Fréchet. Since $F$ is the inductive limit of Fréchet spaces, it is bornological [61, II 8.2 Corollary 1]. A linear operator $T\colon F \to F$ is called bounded iff it maps bounded sets to bounded sets. Let us denote by $\mathcal{B}(F)$ the space of bounded operators on $F$ equipped with the topology of uniform convergence on bounded sets. Of course any continuous operator is bounded. Since $F$ is bornological, the converse is true [61, II 8.3] and we have $\mathcal{L}(F) = \mathcal{B}(F)$ as locally convex spaces.

Finally, since the bounded sets in $F$ and $G$ coincide, we have an embedding $\mathcal{L}(G) \xrightarrow{\mathrm{I}} \mathcal{B}(F) = \mathcal{L}(F)$.

Now we prove the continuity of the second embedding. If $(T_j)_j$ is a convergent net in $\tilde{\mathcal{L}}$, then $(T_j x)_j$ converges in $F$ uniformly in $x \in B$ for any bounded $B \subset \mathscr{F}_\alpha(\pi)$. Since the bounded sets in $F$ and $G$ coincide and due to (4.1.1), a subset $B \subset F$ is bounded, iff $B \subset \mathscr{F}_\alpha(\pi)$ is bounded for some $\alpha \in A$. Hence $(T_j)_j$ converges in $\mathcal{L}(F)$. $\square$

Now, we will make a few general observations regarding the algebra structures of the operator spaces defined by adjoint representations.

**Definition 4.1.3.** **(i)** *Suppose $A$ is an algebra with unit $1_A$ and $B \subset A$ is a subalgebra. Then $B$ is called **spectrally invariant** in $A$ iff $1_A \in B$ and*

$$\forall_{T \in B}\colon \sigma_A(T) = \sigma_B(T)\,.$$

**(ii)** *If $A$ is a Banach algebra with involution $T \mapsto T^*$, then $B$ is called $*$**-subalgebra** of $A$ if it is a subalgebra of $A$ invariant under the involution, i.e. $B^* = B$.*

Note that $B$ is spectrally invariant in $A$ iff

$$B^\times = A^\times \cap B\,.$$

The following Lemma gives a criterion for the spectral invariance of $\mathscr{G}(\mathrm{Ad}_\pi)$ in $\mathcal{L}(E)$ for a $\mathscr{C}(\mathbb{G})$-function space $\mathscr{G}(\mathbb{G})$.

**Lemma 4.1.4.** *Let $\mathscr{G}(\mathbb{G})$ be a $\mathscr{C}(\mathbb{G})$-function space, let $E$ be a Banach space and let $(E,\pi)$ be a locally convex representation.*

**(i)** *If $\mathscr{G}(\mathbb{G})$ has property **(IC)**, then*

$$\sigma_{\mathcal{L}(E)}(T) = \sigma_{\mathscr{G}(\mathrm{Ad}_\pi)}(T)$$

*for all $T \in \mathscr{G}(\mathrm{Ad}_\pi)$.*

**(ii)** *If the pointwise complex conjugation defines a continuous map of $\mathscr{G}(\mathbb{G})$ into itself, if $E$ is a Hilbert space and if $\pi(x)$ is unitary for all $x \in \mathbb{G}$, then $\mathscr{G}(\mathrm{Ad}_\pi)$ is a $*$-subalgebra of $\mathcal{L}(E)$.*

*Proof.* **(i):** Let $T \in \mathscr{G}(\mathrm{Ad}_\pi)$. Since $E$ is a Banach space, $\mathcal{L}(E)$ is locally m-convex.

First of all, we have $\sigma_{\mathcal{L}(E)}(T) \subset \sigma_{\mathscr{G}(\mathrm{Ad}_\pi)}(T)$, since $\mathscr{G}(\mathrm{Ad}_\pi)$ is a subalgebra of $\mathcal{L}(E)$.

Now let $\lambda \in \mathbb{C}$ such that $\lambda\,\mathrm{I} - T$ is invertible in $\mathcal{L}(E)$. That means $\mathrm{Ad}_\pi(x)(\lambda\,\mathrm{I} - T) = \mathrm{Ad}_\pi(x)(\lambda\,\mathrm{I} - T)$ is invertible for each $x \in G$ and hence, due to **(IC)**, $\lambda\,\mathrm{I} - \Phi_{\mathrm{Ad}_\pi}(T)$ is invertible in $\mathscr{G}(\mathbb{M}; \mathcal{L}(E))$. The inverse $(\lambda\,\mathrm{I} - \Phi_{\mathrm{Ad}_\pi}(T))^{-1} = \Phi_{\mathrm{Ad}_\pi}((\lambda\,\mathrm{I} - T)^{-1})$ is in $\Phi_{\mathrm{Ad}_\pi}\mathcal{L}(E)$. Hence $(\lambda\,\mathrm{I} - T)^{-1} \in \mathscr{G}(\mathrm{Ad}_\pi)$. In conclusion $\sigma_{\mathcal{L}(E)}(T) = \sigma_{\mathscr{G}(\mathrm{Ad}_\pi)}(T)$.

**(ii):** Let $A\colon \mathcal{L}(E) \to \mathcal{L}(E)$ with $AT = T^*$ for $T \in \mathcal{L}(E)$ and put $\mathcal{C} \in \mathcal{L}(\mathscr{G}(\mathbb{G}))$ with $\mathcal{C}f(x) := \overline{f(x)}$ for all $x \in \mathbb{G}$ and $f \in \mathscr{G}(\mathbb{G})$. Then $\mathcal{C}\,\varepsilon\, A$ defines a continuous antilinear operator from $\mathscr{G}(\mathbb{G}; \mathcal{L}(E))$ into itself as described in Section 1.1. Since

$$\mathcal{C}\,\varepsilon\, A(\mathrm{Ad}_\pi(-)T)(x) = (\mathrm{Ad}_\pi(x)T)^* = \mathrm{Ad}_\pi(x)(T^*)$$

for all $T \in \mathscr{G}(\mathrm{Ad}_\pi)$ and $x \in \mathbb{G}$, the algebra $\mathscr{G}(\mathrm{Ad}_\pi)$ is a $*$-subalgebra of $\mathcal{L}(E)$. $\square$

Closely related to spectral invariance of subalgebras $B$ in a Banach algebra $A$ is the invariance under the holomorphic functional calculus [36, p. 330]. Let $1_A$ be the unit element in $A$. Suppose $T \in A$ and $U \subset \mathbb{C}$ is open with $\sigma_A(T) \subset U$. Since $\sigma_A(T)$ is compact, there is a bounded open set $D$ with $\sigma_A(T) \subset D \subset \overline{D} \subset U$ and with smooth boundary. If $F\colon U \to \mathbb{C}$ is a holomorphic function, then the element $F(T) \in A$ is defined via the complex integral

$$F(T) = \frac{1}{2\pi \mathrm{i}} \int_{\partial D} F(z)\, (z\, 1_A - T)^{-1}\, \mathrm{d}z\,,$$

This integral does not depend on the choice of $D$.

**Definition 4.1.5.** *Suppose $A$ is a unital Banach algebra with unit $1_A$ and $B \subset A$ is a subalgebra. We call $B$ **invariant under the holomorphic functional calculus** iff for all $T \in B$, all open $U \supset \sigma_A(T)$ and all holomorphic $F\colon U \to \mathbb{C}$ we also have $F(T) \in B$.*

Clearly, if $B$ is invariant under the holomorphic functional calculus, then

$$1_A = \frac{1}{2\pi \mathrm{i}} \int_{\partial D} (z\, 1_A - T)^{-1}\, \mathrm{d}z \in B \quad \text{resp.} \quad T^{-1} = \frac{1}{2\pi \mathrm{i}} \int_{\partial D} \frac{1}{z} (z\, 1_A - T)^{-1}\, \mathrm{d}z \in B\,,$$

in which $D$ is as above and $T \in B$ resp. $T \in A^\times \cap B$. So this implies that subalgebras $B$ which are invariant under the holomorphic functional calculus are also spectrally invariant. For complete locally m-convex subalgebras $B$ for which the embedding $B \xrightarrow{\mathrm{I}} A$ is continuous, the converse is true as well.

**Lemma 4.1.6.** *Suppose $\mathscr{F}(\mathbb{G})$ is a complete nuclear $\mathscr{C}(\mathbb{G})$-function space with **(IC)** and $(E, \pi)$ is a representation on a Banach space $E$. If $\mathscr{F}(\mathbb{G})$ is a locally m-convex algebra for the pointwise multiplication, then $\mathscr{F}(\mathrm{Ad}_\pi)$ is invariant under the holomorphic functional calculus in $\mathcal{L}(E)$.*

*Proof.* **Step 1:** First we will reduce this situation to a more general context. Note that

$$\mathscr{F}(\mathbb{G}; \mathcal{L}(E)) \simeq \mathscr{F}(\mathbb{G})\, \hat{\otimes}_\pi\, \mathcal{L}(E)\,,$$

since $\mathscr{F}(\mathbb{G})$ is complete and nuclear. Hence $\mathscr{F}(\mathbb{G}; \mathcal{L}(E))$ is a complete locally m-convex algebra by Proposition 1.2.14. The algebra $\mathscr{F}(\mathrm{Ad}_\pi)$ can be seen as a closed subalgebra of $\mathscr{F}(\mathbb{G}; \mathcal{L}(E))$, so $\mathscr{F}(\mathrm{Ad}_\pi)$ is a complete locally m-convex algebra. Furthermore, $\mathscr{F}(\mathrm{Ad}_\pi)$ is spectrally invariant in $\mathcal{L}(E)$ and the embedding $\mathscr{F}(\mathrm{Ad}_\pi) \to \mathcal{L}(E)$ is continuous.

**Step 2:** We will prove the following general statement: Suppose $A$ is a unital Banach algebra and $B$ is a spectrally invariant, complete and locally m-convex subalgebra with continuous embedding $B \xrightarrow{\mathrm{I}} A$. Then $B$ is invariant under the holomorphic functional calculus.

Since $B$ is locally m-convex, there is a set of submultiplicative seminorms $\mathcal{Q}$ defines the topology on $B$ and which is directed via the preorder

$$q_1 \leq q_1 \quad :\Leftrightarrow \quad \forall_{T \in B}\, q_1(T) \leq q_2(T)\,.$$

Suppose $\|-\|$ is a submultiplicative continuous norm that defines the topology on $A$. Then

$$\mathcal{P} := \{q + \|-\| \mid q \in \mathcal{Q}\}$$

defines a set of submultiplicative norms that defines the topology on $B$. Moreover, $(\mathcal{P}, \leq)$ is a directed set. For each $p \in \mathcal{P}$ denote by $B_p$ the completion of $B$ in $A$ with respect to the norm $p$. Then for each $p$ the embeddings $B \xrightarrow{\mathrm{I}} B_p$, $B_p \xrightarrow{\mathrm{I}} A$ are continuous, $B_p$ is a Banach algebra with unit $1_A$ and $B \simeq \varprojlim_{p \in \mathcal{P}} B_p$ by [61, Chapter II, 5.4]. This also implies $B = \bigcap_{p \in \mathcal{P}} B_p$ and we have $\sigma_A(T) = \sigma_{B_p}(T) = \sigma_B(T)$ for each $T \in B$. Now we choose some $T \in B$. Let $U \supset \sigma_A(T)$ be open, let $F \colon U \to \mathbb{C}$ be holomorphic and let $D$ be bounded and open with $\sigma_A(T) = \sigma_{B_p}(T) \subset D \subset \overline{D} \subset U$ and with smooth boundary $\partial D$. Then the integral

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial D} F(z)\,(z\,1_A - T)^{-1}\,\mathrm{d}z$$

exists in *all* the Banach algebras $B_p$, $p \in \mathcal{P}$, and $A$. Furthermore, in all cases this gives us the same element $F(T) \in A$, due to the continuous embeddings. Now we just use $F(T) \in \bigcap_{p \in \mathcal{P}} B_p = B$. □

Note that even if $\mathscr{F}(\mathbb{G})$ does not have **(IC)**, it is helpful to know that $\mathscr{F}(\mathbb{G})$ is locally m-convex. As noted in the proof above, if $\mathscr{F}(\mathbb{G})$ is nuclear and locally m-convex and if $(\pi, E)$ is a representation on a Banach pace $E$, then $\mathscr{F}(\mathrm{Ad}_\pi)$ is locally m-convex. This implies that for any $T \in \mathscr{F}(\mathrm{Ad}_\pi)$ and any *entire* function $F \colon \mathbb{C} \to \mathbb{C}$ we have $F(T) \in \mathscr{F}(\mathrm{Ad}_\pi)$. This follows from the convergence of the series

$$F(T) = \sum_{k=0}^{\infty} \frac{(\partial^k F)(0)}{k!} T^k \quad \text{in} \quad \mathscr{F}(\mathrm{Ad}_\pi)\,.$$

Now we will finally start to discuss operator spaces defined by ultradifferentiable adjoint group actions. Note the proof below can be easily adapted to other $\mathscr{C}(\mathbb{G})$-function spaces, due to the general nature of Lemma 4.1.1, Lemma 4.1.4 and Lemma 4.1.6.

**Theorem 4.1.7.** *Suppose $M$, $L$ and $N$ are weight sequences, $\mathbb{G}$ is a Lie group and $(\pi, E)$ is a bornologic and complete locally convex representation with the approximation property. Furthermore, let $D \subset \mathcal{V}_{\mathrm{a}}(\mathbb{G})$ be a frame, let $[M] \subset [L]$ and let $[M] \subset (N)$. Then the map*

$$R^F_{\mathscr{F}(\mathbb{G})} \colon \mathscr{F}(\mathrm{Ad}_\pi) \to \mathcal{L}(F) \colon T \mapsto T \upharpoonright^F_F \tag{4.1.2}$$

*is well-defined and continuous for any choice from Table 4.1.*

| $\mathscr{F}(\mathbb{G})$ | $F$ |
|---|---|
| $\mathscr{E}(\mathbb{G})$ | $\mathscr{C}^k(\pi)$, $\mathscr{E}(\pi)$ |
| $\mathscr{E}^{[M]}_D(\mathbb{G})$ | $\mathscr{C}^k(\pi)$, $\mathscr{E}(\pi)$, $\mathscr{E}^N_D(\pi)$, $\mathscr{E}^{[L]}_D(\pi)$, $\mathscr{E}^{[L]}_{D,\mathrm{proj}}(\pi)$ |

Table 4.1: Possible choices of locally convex spaces $F$ and $\mathscr{F}(\mathbb{G})$ for Theorem 4.1.7

*Proof.* **(i):** Obviously, the multiplication $\mathscr{E}(\mathbb{G}) \times \mathscr{C}^k(\mathbb{G}) \to \mathscr{C}^k(\mathbb{G})$ and the multiplication $\mathscr{E}(\mathbb{G}) \times \mathscr{E}(\mathbb{G}) \to \mathscr{E}(\mathbb{G})$ are continuous. Due to Proposition 2.2.15 and the continuous embeddings

$$\mathscr{E}^{[M]}_D(\mathbb{G}) \hookrightarrow \mathscr{E}(\mathbb{G})\,, \quad \mathscr{E}^{[M]}_D(\mathbb{G}) \hookrightarrow \mathscr{E}^{[L]}_{D,}(\mathbb{G}) \quad \text{and} \quad \mathscr{E}^{[M]}_D(\mathbb{G}) \hookrightarrow \mathscr{E}^{[L]}_{D,\mathrm{proj}}(\mathbb{G})\,,$$

the multiplication $\mathscr{E}^{[M]}_D(\mathbb{G}) \times \mathscr{G}(\mathbb{G}) \to \mathscr{G}(\mathbb{G})$ is continuous for any choice of

$$\mathscr{G}(\mathbb{G}) \in \{\mathscr{C}^k(\mathbb{G}), \mathscr{E}(\mathbb{G}), \mathscr{E}^N_D(\mathbb{G}), \mathscr{E}^{[L]}_D(\mathbb{G}), \mathscr{E}^{[L]}_{D,\mathrm{proj}}(\mathbb{G})\}\,.$$

Moreover, all involved $\mathscr{C}(\mathbb{G})$-function spaces are complete and $\mathscr{E}(\mathbb{G})$ and $\mathscr{E}^{[M]}_D(\mathbb{G})$ are nuclear. Thus we may apply Lemma 4.1.1 and

$$\mathscr{F}(\mathrm{Ad}_\pi) \ni T \mapsto T \upharpoonright^F_F \in \mathcal{L}(F)$$

is well-defined and continuous for the choices described in the theorem.

□

The above theorem holds especially if $(\pi, E)$ is a representation on a Fréchet or Banach space $E$ with the approximation property. Since any Hilbert space has the approximation property, this includes the canonical case, where $(\pi, E)$ is a unitary representation.

## 4.2 Operator algebras defined by the left regular representation on compact Lie groups

In this section we will apply our approach to compact Lie groups $\mathbb{G}$ with Haar measure $\mu$ and relate the outcome to known results. The following discussion will contain the results from [9] and the characterization of $\operatorname{Op}_{\mathbb{G}} \mathcal{S}^0_{0,0}(\mathbb{G} \times \widehat{\mathbb{G}})$ in terms of commutators from [23] as special cases. These results will also imply $\operatorname{Op}_{\mathbb{G}} \mathcal{S}^0_{0,0}(\mathbb{G} \times \widehat{\mathbb{G}})$ is a spectrally invariant $*$-subalgebra of $\mathcal{L}(L^2(\mathbb{G}, \mu))$.

First, we will prove a statement about the denseness of ultradifferentiable functions on compact groups in the usual Hilbert space of $L^2$-functions. For this purpose we will use the following representation. On $B^2(\widehat{\mathbb{G}}, \widehat{\mu})$ let $\widehat{\boldsymbol{L}}_2$ resp. $\widehat{\boldsymbol{R}}_2$ be the unitary representation such that $\widehat{\boldsymbol{L}}_2(x)\sigma(\pi) := \sigma(\pi)\pi(x^{-1})$ resp. $\widehat{\boldsymbol{R}}_2(x)\sigma(\pi) := \pi(x)\sigma(\pi)$ for $\pi \in \operatorname{Irr}(\mathbb{G})$, $x \in \mathbb{G}$ and functions $\sigma \in B^2(\widehat{\mathbb{G}}, \widehat{\mu})$.

**Lemma 4.2.1.** *Let $M$, $L$ be weight sequences and let $\mathbb{G}$ be a compact Lie group with Haar measure $\mu$ and Lie algebra $\mathfrak{g}$. If $[M] \subset [L]$, then the space $\mathscr{E}^{[M]}_D(\mathbb{G})$ is dense in $L^2(\mathbb{G}, \mu)$ and in $\mathscr{E}^{[L]}_D(\mathbb{G})$ for any basis $D \subset \mathfrak{g}_{\mathrm{L}}$ or $D \subset \mathfrak{g}_{\mathrm{R}}$.*

*Proof.* Let $D \subset \mathfrak{g}_{\mathrm{L}}$ be a basis. We may describe the space $\mathscr{E}^M_D(\widehat{\boldsymbol{L}}_2)$ by

$$\mathscr{E}^M_{hD}(\widehat{\boldsymbol{L}}_2) = \left\{ \sigma \in \Sigma(\widehat{\mathbb{G}}) \;\middle|\; \lim_{\substack{|a| \to \infty \\ a \in \mathcal{S}_{\dim \mathbb{G}}}} \frac{h^a}{M_{|a|}\, |a|!} \|\pi \mapsto \sigma(\pi)\pi(D^a)\|_{B^2(\widehat{\mathbb{G}}, \widehat{\mu})} = 0 \right\}$$

equipped with the norm defined by

$$\sigma \mapsto \sup_{a \in \mathcal{S}_{\dim \mathbb{G}}} \frac{h^a}{M_{|a|}\, |a|!} \|\pi \mapsto \sigma(\pi)\pi(D^a)\|_{B^2(\widehat{\mathbb{G}}, \widehat{\mu})}\,.$$

Note that $\pi(D^a) \in \mathcal{L}(H_\pi)$, since $\dim H_\pi < \infty$. If we put

$$\Sigma_f(\widehat{\mathbb{G}}) := \{\sigma \in \Sigma(\widehat{\mathbb{G}}) \mid \text{there is some finite } U \subset \widehat{\mathbb{G}} \text{ with } [\pi] \in \widehat{\mathbb{G}} \setminus U \;\Rightarrow\; \sigma(\pi) = 0\}\,,$$

then $\Sigma_f(\widehat{\mathbb{G}})$ is dense in $\mathscr{E}^M_{hD}(\widehat{\boldsymbol{L}}_2)$ and in $B^2(\widehat{\mathbb{G}},\widehat{\mu})$. Let us denote $\mathscr{T} := \mathcal{F}_G^{-1}\Sigma_f(\widehat{\mathbb{G}})$.

$\mathcal{F}_{\mathbb{G}}$ is a linear homeomorphism from $L^2(\mathbb{G},\mu)$ onto $L^2(\widehat{\mathbb{G}},\widehat{\mu})$, hence $1 \,\varepsilon\, \mathcal{F}_{\mathbb{G}}$ is a linear homeomorphism from $\mathscr{E}^M_D(\mathbb{G}; L^2(\mathbb{G},\mu))$ onto $\mathscr{E}^M_D(\mathbb{G}; B^2(\widehat{\mathbb{G}},\widehat{\mu}))$ by Lemma 1.2.3. Finally, due to $\mathcal{F}_{\mathbb{G}}\, \boldsymbol{L}_2 = \widehat{\boldsymbol{L}}_2 \mathcal{F}_{\mathbb{G}}$, we see that

$$\mathcal{F}_{\mathbb{G}} \colon \mathscr{E}^M_D(\boldsymbol{L}_2) \to \mathscr{E}^M_D(\widehat{\boldsymbol{L}}_2)$$

defines a linear homeomorphism. Note that $D \mapsto \tilde{D} := \boldsymbol{L}_2(D)$ defines a bijection between $\mathfrak{g}_{\mathrm{L}}$ and $\mathfrak{g}_{\mathrm{R}}$ and that $\mathscr{E}^M_D(\boldsymbol{L}_2) = H^M_{\tilde{D}}(\mathbb{G})$. Hence, $\mathscr{T}$ is dense in $H^M_{\tilde{D}}(\mathbb{G})$ and thus also in $\mathscr{E}^{[M]}_{\tilde{D},\mathrm{proj}}(\mathbb{G}) = H^{[M]}_{\tilde{D},\mathrm{proj}}(\mathbb{G})$. For the other Roumieu case, $\mathscr{E}^{\{M\}}_{\tilde{D}}(\mathbb{G})$, note that the embedding $H^M_{\tilde{D}}(\mathbb{G}) \xrightarrow{\mathrm{I}} \mathscr{E}^{\{M\}}_{\tilde{D}}(\mathbb{G})$ is continuous for each $h > 0$ and that $\bigcup_{h>0} H^M_{\tilde{D}}(\mathbb{G}) = \mathscr{E}^{\{M\}}_{\tilde{D}}(\mathbb{G})$. This implies that $\mathscr{T}$ is dense in $\mathscr{E}^{\{M\}}_{\tilde{D}}(\mathbb{G})$ as well.

For the left invariant spaces we exchange $\widehat{\boldsymbol{L}}_2$ and $\boldsymbol{L}_2$ with $\widehat{\boldsymbol{R}}_2$ and $\boldsymbol{R}_2$. The analogous argumentation as above proves the rest of the lemma.

$\square$

As before, a lot of the following statements can be adapted to other $\mathscr{C}(\mathbb{G})$-function spaces, due to the general formulation of Lemma 4.1.1, Lemma 4.1.4 and Lemma 4.1.6. The next theorem and Theorem 4.2.6 can be seen as a generalization of [9]. It is mainly a summary of the preceding general lemmata applied to the left regular representation on compact Lie groups.

**Theorem 4.2.2.** *Suppose $M$, $N$ and $L$ are weight sequences, $\mathbb{G}$ is a compact Lie group with Haar measure $\mu$ and Lie algebra $\mathfrak{g}$ and $\boldsymbol{L}_2$ is the left-regular representation on $L^2(\mathbb{G},\mu)$. Let $D \in \mathfrak{g}_{\mathrm{L}}$ and $\tilde{D} \subset \mathfrak{g}_{\mathrm{R}}$ be a frames. Then the following holds.*

- **(i)** *For $\mathscr{F}(\mathbb{G}) \in \{\mathscr{E}^{[M]}_D(\mathbb{G}), \mathscr{E}(\mathbb{G})\}$ the algebra $\mathscr{F}(\mathrm{Ad}_{\boldsymbol{L}_2})$ is a $*$-subalgebra of $\mathcal{L}(L^2(\mathbb{G},\mu))$, i.e.*

  $$\mathscr{F}(\mathrm{Ad}_{\boldsymbol{L}_2})^* = \mathscr{F}(\mathrm{Ad}_{\boldsymbol{L}_2})\,.$$

  *Moreover, if $[M]$ has **(PL)**, then $\mathscr{F}(\mathrm{Ad}_{\boldsymbol{L}_2})$ is locally m-convex.*

- **(ii)** *For $\mathscr{F}(\mathbb{G}) \in \{\mathscr{E}^{\{M\}}_D(\mathbb{G}), \mathscr{E}(\mathbb{G})\}$ the algebra $\mathscr{F}(\mathrm{Ad}_{\boldsymbol{L}_2})$ is even invariant under the holomorphic functional calculus in $\mathcal{L}(L^2(\mathbb{G}))$.*

**(iii)** *Let $[M] \subset [L]$ and $[M] \subset (N)$. Then we have embeddings*

$$\mathscr{F}(\mathrm{Ad}_{\boldsymbol{L}_2}) \hookrightarrow \mathcal{L}(E) \colon T \mapsto T \restriction_E^E$$

*if we choose $\mathscr{F}(\mathbb{G})$ and $E$ from Table 4.2. These embeddings are continuous for $E \neq \mathscr{E}_{\tilde{D}}^{\{L\}}(\mathbb{G})$ or $[M] \subset (L)$.*

| $\mathscr{F}(\mathbb{G})$ | $E$ |
|---|---|
| $\mathscr{E}(\mathbb{G})$, $\mathscr{E}_D^{[M]}(\mathbb{G})$ | $H^k(\mathbb{G})$, $\mathscr{E}(\mathbb{G})$ |
| $\mathscr{E}_D^{[M]}(\mathbb{G})$ | $H_{\tilde{D}}^N(\mathbb{G})$, $\mathscr{E}_{\tilde{D}}^{[L]}(\mathbb{G})$, $\mathscr{E}_{\tilde{D},\mathrm{proj}}^{[L]}(\mathbb{G})$ |

Table 4.2: Possible choices of locally convex spaces $E$ and $\mathscr{F}(\mathbb{G})$ for Theorem 4.2.2 **(iii)**

*Proof.* **(i):** Since $(\boldsymbol{L}_2, L^2(\mathbb{G},\mu))$ is a complete admissible unitary representation, we may apply $(E,\pi) = (\boldsymbol{L}_2, L^2(\mathbb{G},\mu))$ to Lemma 4.1.4.

If $[M]$ has **(PL)**, then $\mathscr{F}(\mathrm{Ad}_{\boldsymbol{L}_2})$ is locally m-convex, since $\mathscr{F}(\mathrm{Ad}_{\boldsymbol{L}_2})$ can be identified with a subalgebra of the locally m-convex algebra $\mathscr{F}(\mathbb{G}) \hat{\otimes} \mathcal{L}(L^2(\mathbb{G},\mu))$.

**(ii):** $\mathscr{E}_{D,\mathrm{proj}}^{\{M\}}(\mathbb{G})$ and $\mathscr{E}(\mathbb{G})$ are nuclear, locally m-convex and have **(IC)**. Hence, we may apply Lemma 4.1.6 and use that $\mathscr{E}_{D,\mathrm{proj}}^{[M]}(\mathrm{Ad}_\pi) = \mathscr{E}_D^{[M]}(\mathrm{Ad}_\pi)$ as vector spaces.

**(iii):** We may choose $D$ such that $\tilde{D} = \boldsymbol{L}(D)$ without changing $\mathscr{E}_D^{[M]}(\mathrm{Ad}_{\boldsymbol{L}_2})$ by Proposition 2.2.10. We apply Theorem 4.1.7 to $(E,\pi) = (\boldsymbol{L}_2, L^2(\mathbb{G},\mu))$. By Lemma 2.4.12, Lemma 2.4.15 and Lemma 2.4.16 we have

$$\mathscr{C}^k(\boldsymbol{L}_2) = H^k(\mathbb{G})\,,\ \mathscr{E}(\boldsymbol{L}_2) = \mathscr{E}(\mathbb{G})\,,\ \mathscr{E}_{\tilde{D}}^L(\boldsymbol{L}_2) = H_D^L(\mathbb{G})\,,\ \mathscr{E}_{D,\mathrm{proj}}^{[L]}(\boldsymbol{L}_2) = \mathscr{E}_{\tilde{D},\mathrm{proj}}^{[L]}(\mathbb{G})\,.$$

Furthermore, for the case $E = \mathscr{E}_{\tilde{D}}^{\{L\}}(\mathbb{G})$ we can use Lemma 4.1.2.

Due to Lemma 4.2.1, we may use that $E$ is dense in $L^2(\mathbb{G},\mu)$. Thus $T \mapsto T \restriction_E^E$ is an embedding. $\square$

The above theorem especially shows that the algebra of analytic vectors $\mathscr{E}^{\{\mathbb{1}\}}(\mathrm{Ad}_{\boldsymbol{L}_2})$ to the representation $\mathrm{Ad}_{\boldsymbol{L}_2}$ is a spectrally invariant $*$-subalgebra of $\mathcal{L}(L^2(\mathbb{T}^n,\mu))$ for the $n$-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z}^n)$, which was proven in [9, Corollary 1].

Using Gelfand triples, we can also prove that many of the considered operators extend to continuous operators between spaces of ultradistributions.

**Corollary 4.2.3.** *Let $\mathbb{G}$ be a compact Lie group with Haar measure $\mu$, $D \subset \mathfrak{g}_{\mathrm{L}}$ a frame, $\tilde{D} := \boldsymbol{L}(D)$ and let $M$ be a weight sequence. Then*

$$\mathcal{G}^{(M)}_{\tilde{D}}(\mathbb{G},\mu) := (\mathscr{E}^{(M)}_{\tilde{D}}(\mathbb{G}), L^2(\mathbb{G},\mu), \mathscr{E}^{(M)}_{\tilde{D}}(\mathbb{G})')$$

*is a Gelfand triple with real structure defined by the pointwise complex conjugation. Suppose $E = \mathscr{E}^{[L]}_{\tilde{D}}(\mathbb{G})$ for a weight sequence $L$ with $M \subset L$, then $E$ and $E'$ are $\mathcal{G}^{(M)}_{\tilde{D}}(\mathbb{G},\mu)$-regular and each operator $T \in \mathscr{E}^{[M]}_{D}(\mathrm{Ad}_{\boldsymbol{L}_2})$ extends uniquely to an operator in $\mathcal{L}(E')$.*

*Proof.* $\mathcal{G}^{(M)}_{\tilde{D}}(\mathbb{G},\mu)$ is a Gelfand triple, since $\mathscr{E}^{(M)}_{\tilde{D}}(\mathbb{G})$ is a nuclear Fréchet space and dense in $L^2(\mathbb{G},\mu)$ and the pointwise complex conjugation maps $\mathscr{E}^{(M)}_{\tilde{D}}(\mathbb{G})$ continuously to itself. Naturally, we have continuous dense embeddings

$$\mathscr{E}^{(M)}_{\tilde{D}}(\mathbb{G}) \hookrightarrow E \hookrightarrow L^2(\mathbb{G},\mu)\,,$$

and the pointwise complex conjugation maps $E$ continuously to itself. Furthermore, $E$ is reflexive as a nuclear Fréchet space (resp. dual to a nuclear Fréchet space) as written in Proposition 2.2.10. Hence $E$ is $\mathcal{G}^{(M)}_{\tilde{D}}(\mathbb{G},\mu)$-regular. This automatically implies that $E'$ is $\mathcal{G}^{(M)}_{\tilde{D}}(\mathbb{G},\mu)$-regular.

Now let $T \in \mathscr{E}^{[M]}_{D}(\mathrm{Ad}_{\boldsymbol{L}_2}) \subset \mathcal{L}(L^2(\mathbb{G},\mu))$, then $S := T^* \restriction^{E}_{E}$ is well-defined. We may use the adjoint of the $\mathcal{G}^{(M)}_{D}(\mathbb{G},\mu)$ Gelfand triple and define $R := S^* \in \mathcal{L}(E')$. It is easy to see that $R$ is an extension of $T$, i.e. $R \restriction^{L^2(\mathbb{G},\mu)}_{L^2(\mathbb{G},\mu)} = T$. This extension is unique, since $L^2(\mathbb{G},\mu) \subset E'$ is dense. $\square$

For any compact Lie group $\mathbb{G}$ with Haar measure $\mu$, we define the representations $\boldsymbol{L}_{\mathscr{S}} := \boldsymbol{L}{\downarrow_{\mathscr{S}(\mathbb{G})}}$ on $\mathscr{S}(\mathbb{G})$, $\boldsymbol{L}_{\mathscr{S}'}$ on $\mathscr{S}'(\mathbb{G})$ where $\boldsymbol{L}_{\mathscr{S}'}(x) := \boldsymbol{L}_{\mathscr{S}}(x^{-1})'$ for all $x \in \mathbb{G}$ and $\mathrm{Ad}_{\boldsymbol{L}_{\mathscr{S},\mathscr{S}'}}$ on $\mathcal{L}(\mathscr{S}(\mathbb{G});\mathscr{S}'(\mathbb{G}))$ where $\mathrm{Ad}_{\mathscr{S},\mathscr{S}'}(x)T := \boldsymbol{L}_{\mathscr{S}'}(x)T\,\boldsymbol{L}_{\mathscr{S}}(x)^{-1}$ for $x \in \mathbb{G}$, $T \in \mathcal{L}(\mathscr{S}(\mathbb{G});\mathscr{S}'(\mathbb{G}))$. Then, using the density of

$$\mathcal{L}(\mathscr{S}'(\mathbb{G});\mathscr{S}(\mathbb{G})) \subset \mathcal{L}(\mathscr{S}(\mathbb{G});\mathscr{S}'(\mathbb{G})) \quad \text{and} \quad \mathscr{S}(\mathbb{G}\times\mathbb{G}) \subset \mathscr{S}'(\mathbb{G}\times\mathbb{G})\,,$$

a quick calculation yields

$$\mathcal{K}_{\mathbb{G}}\,\mathrm{Ad}_{\boldsymbol{L}_{\mathscr{S},\mathscr{S}'}} = (\boldsymbol{L}_{\mathscr{S}'}\,\varepsilon\,\boldsymbol{L}_{\mathscr{S}'})\mathcal{K}_{\mathbb{G}} \quad \text{and} \quad \mathcal{T}_{\mathbb{G}}(\boldsymbol{L}_{\mathscr{S}'}\,\varepsilon\,\boldsymbol{L}_{\mathscr{S}'}) = (\boldsymbol{L}_{\mathscr{S}'}\,\varepsilon\,\mathrm{I}_{\mathscr{S}'(\mathbb{G})})\mathcal{T}_{\mathbb{G}}\,.$$

Hence $\mathrm{Op}_{\mathbb{G}}$ intertwines the left translation on symbols with the representation $\mathrm{Ad}_{\boldsymbol{L}_{\mathscr{S},\mathscr{S}'}}$, i.e.

$$\mathrm{Ad}_{\boldsymbol{L}_{\mathscr{S},\mathscr{S}'}}(x) \circ \mathrm{Op}_{\mathbb{G}} = \mathrm{Op}_{\mathbb{G}} \circ (\boldsymbol{L}_{\mathscr{S}'}(x) \,\varepsilon\, \mathrm{I}_{\mathscr{S}'(\widehat{G})}) \,, \qquad \text{for all } x \in \mathbb{G} \,. \tag{4.2.3}$$

Of course this equation fits to the perspective of Gelfand triples, since $\mathrm{Ad}_{\boldsymbol{L}_{\mathscr{S},\mathscr{S}'}}(x)$ is a Gelfand triple isomorphism from $\mathcal{L}(\mathcal{G}(\mathbb{G},\mu);\mathcal{G}(\mathbb{G},\mu))$ onto itself and $\boldsymbol{L}_{\mathscr{S}'}(x) \otimes \mathrm{I}_{\mathscr{S}'(\widehat{G})}$ is a Gelfand triple isomorphism from $\mathcal{G}(\mathbb{G},\mu) \otimes \mathcal{G}(\mathbb{G},\mu)$ onto itself for each $x \in \mathbb{G}$.

Identity (4.2.3) and a criterium for the boundedness of $\mathrm{Op}_{\mathbb{G}}(\sigma)$ on $L^2(\mathbb{G},\mu)$ will help us to characterize operators in $\mathscr{G}(\mathrm{Ad}_{\boldsymbol{L}_2})$ by corresponding spaces of Kohn-Nirenberg symbols.

Obviously, $\mathrm{Op}_{\mathbb{G}}(\sigma)$ is bounded on $L^2(\mathbb{G},\mu)$ for $\sigma \in L^2(\mathbb{G},\mu) \mathbin{\hat{\otimes}_{\mathrm{H}}} L^2(\widehat{\mathbb{G}},\widehat{\mu})$ but also boundedness of sufficiently many derivatives of $\sigma$ is a viable criterium for this. Before citing the corresponding facts, we need to relate bounded $B^\infty(\widehat{\mathbb{G}},\widehat{\mu})$-valued functions with elements in the Gelfand triple $\mathcal{G}(\mathbb{G},\mu) \otimes \mathcal{G}(\widehat{\mathbb{G}},\widehat{\mu})$.

For any compact Lie group $\mathbb{G}$ with Haar measure $\mu$ the measure space $(\widehat{\mathbb{G}},\widehat{\mu})$ is discrete, i.e. each equivalence class $[\pi]$ is an atom and any function $f\colon \widehat{\mathbb{G}} \mapsto \mathbb{C}$ is measurable. $B^\infty(\widehat{\mathbb{G}},\widehat{\mu})$ can be embedded into $\mathscr{S}'(\widehat{\mathbb{G}})$ as described in [24, 2.1.3 and 2.1.4]. Thus any bounded function

$$\tilde{\sigma}\colon \mathbb{G} \to B^\infty(\widehat{\mathbb{G}},\widehat{\mu})\colon x \mapsto [\tilde{\sigma}(x)\colon \pi \mapsto \tilde{\sigma}(x,\pi)] \,,$$

such that for any $\pi \in \mathrm{Irr}(\mathbb{G})$ the function $\tilde{\sigma}(-,\pi)$ is $\mu$-measurable, can be identified with a unique element $\sigma \in \mathscr{S}'(\mathbb{G}) \mathbin{\hat{\otimes}} \mathscr{S}'(\widehat{\mathbb{G}})$ via

$$\sigma\colon \mathscr{S}(\mathbb{G};\mathscr{S}(\widehat{\mathbb{G}})) \to \mathbb{C}\colon \omega \mapsto \int_{\mathbb{G}} \langle \omega(x), \tilde{\sigma}(x) \rangle \, \mathrm{d}\mu(x) \,, \tag{4.2.4}$$

in which we used the standard homeomorphism $\mathscr{S}'(\mathbb{G}) \mathbin{\hat{\otimes}} \mathscr{S}'(\widehat{\mathbb{G}}) \simeq \mathscr{S}(\mathbb{G};\mathscr{S}(\widehat{\mathbb{G}}))'$ from Proposition 1.2.1 **(iv)**. We may use the following boundedness result for the Kohn-Nirenberg quantization found in [59, Theorem 10.5.5].

**Proposition 4.2.4.** *Suppose $\mathbb{G}$ is a compact Lie group, $k > \dim \mathbb{G}/2$ is an integer and $D \subset \mathfrak{g}_{\mathrm{L}}$ is a basis. There is a constant $C > 0$ such that for any $\sigma \in \mathscr{C}^k(\mathbb{G}; B^\infty(\widehat{\mathbb{G}};\widehat{\mu}))$ the Kohn-Nirenberg operator $\mathrm{Op}(a)$ defines a bounded operator on $L^2(\mathbb{G},\mu)$ and*

$$\| \mathrm{Op}(\sigma) \|_{\mathcal{L}(L^2(\mathbb{G},\mu))} \leq C \max_{|a| \leq k} \sup_{x \in \mathbb{G}} \| D_x^a \sigma(x) \|_{\mathcal{B}^\infty(\widehat{\mathbb{G}},\widehat{\mu})} \,.$$

Complementing the above proposition, we can also bound the symbol $\sigma$ by continuous seminorms in $\mathscr{E}(\mathrm{Ad}_{\boldsymbol{L}_2})$ evaluated on $\mathrm{Op}_{\mathbb{G}}(\sigma)$. A proof for almost the exact statement below can be found in [23, Proposition 8.11]. We use the cited proof with just minor adjustments.

**Proposition 4.2.5.** *Let $\mathbb{G}$ be a compact Lie group with Haar measure $\mu$, let $k > \dim \mathbb{G}/2$ be an integer and let $D \subset \mathfrak{g}_{\mathrm{L}}$ be a basis. Then there is some constant $C > 0$ such that for any $T \in \mathscr{C}^{k+1}(\mathrm{Ad}_{\boldsymbol{L}_2})$ the symbol $\sigma = \mathrm{Op}_{\mathbb{G}}^{-1}(T)$ can be identified with a continuous function $\sigma \colon \mathbb{G} \mapsto B^\infty(\widehat{\mathbb{G}}, \widehat{\mu})$ via (4.2.4) with*

$$\sup_{x\in\mathbb{G}} \|\sigma(x)\|_{B^\infty(\widehat{\mathbb{G}},\widehat{\mu})} \leq C \max_{|a|\leq k} \| \mathrm{Ad}_{\boldsymbol{L}_2}(D^a)T\|_{\mathcal{L}(L^2(\mathbb{G},\mu)} . \tag{4.2.5}$$

*Proof.* Note that the statement as we use it is not formulated in [23, Proposition 8.11]. However, in the proof to [23, Proposition 8.11] on page 3459 it is shown that the inequality (4.2.5) holds for $T \in \mathscr{C}^k(\mathrm{Ad}_{\boldsymbol{L}_2})$ and $\sigma := \mathrm{Op}_{\mathbb{G}}^{-1}(T)$, where $\sigma$ can be identified with a bounded map $\mathbb{G} \mapsto B^\infty(\widehat{\mathbb{G}}, \widehat{\mu})$. For the map $x \mapsto \sigma(x) \in B^\infty(\widehat{\mathbb{G}}, \widehat{\mu})$ we denote by $\sigma(x, \pi)$ the evaluation of $\sigma(x)$ at $\pi \in \mathrm{Irr}(\mathbb{G})$. Now, if we choose $T \in \mathscr{C}^{k+1}(\mathrm{Ad}_{\boldsymbol{L}_2})$ and if we take any $X \in \mathfrak{g}_{\mathrm{L}}$, then $\mathrm{Ad}_{\boldsymbol{L}_2}(X)A \in \mathscr{C}^k(\mathrm{Ad}_{\boldsymbol{L}_2})$. Due to (4.2.3), we know that $\mathrm{Op}^{-1}(\mathrm{Ad}_{\boldsymbol{L}_2}(X)T) = \boldsymbol{L}(X)\sigma$ is a bounded map and that $x \mapsto \big(\sigma(x, \pi)v, w\big)_{H_\pi}$ is differentiable[1] with derivative $\big(\, \boldsymbol{L}(X)\sigma(-, \pi)v, w\big)_{H_\pi}$ for each fixed $\pi \in \mathrm{Irr}(\mathbb{G})$ and all pairs $v, w \in H_\pi$. Thus

$$\begin{aligned} |\big(\sigma(\exp_{\mathbb{G}}(-tX)x, \pi) - \sigma(x, \pi)\big)_{H_\pi}| &\leq \sup_{x\in\mathbb{G}} |t\ \boldsymbol{L}(X)_x\big(\sigma(x, \pi)v, w\big)_{H_\pi}| \\ &\leq \sup_{x\in\mathbb{G}} |t\,\big(\, \boldsymbol{L}(X)_x\sigma(x, \pi)v, w\big)_{H_\pi}| \\ &\leq |t|\, C \max_{|a|\leq k+1} \| \mathrm{Ad}_{\boldsymbol{L}_2}(D^a)T\|_{\mathcal{L}(L^2(\mathbb{G},\mu)} \end{aligned}$$

for all $\pi \in \mathrm{Irr}(\mathbb{G})$, all $t \in \mathbb{R}$ with $|t|$ small enough and all $X \in \mathfrak{g}_{\mathrm{L}}$ with $\|X\|$ small enough with respect to some norm $\|-\|$ on $\mathfrak{g}_{\mathrm{L}}$. Thus $\sigma \in \mathscr{C}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}}, \widehat{\mu}))$. □

Combining the last two propositions, we may realize that the Kohn-Nirenberg quantization can be restricted to a linear homeomorphism between the smooth resp. ultradifferentiable vectors to $\boldsymbol{L}\,\varepsilon\,\mathrm{I}_{B^\infty(\widehat{\mathbb{G}},\widehat{\mu})}$ and smooth resp. ultradifferentiable vectors to $\mathrm{Ad}_{\boldsymbol{L}_2}$. This results in the following theorem.

[1] $a \mapsto (a(\pi)v, w)_{H_\pi}$ is a continuous functional on $\mathscr{S}'(\widehat{\mathbb{G}})$.

**Theorem 4.2.6.** *Let $\mathbb{G}$ be a compact Lie group with Haar measure $\mu$ and let $\mathscr{F}(\mathbb{G})$ be a $\mathscr{C}(\mathbb{G})$-function space such that $\mathscr{F}(\mathbb{G}) = \mathscr{F}(\boldsymbol{R}{\downarrow_{\mathscr{E}(\mathbb{G})}})$. If we define the $\mathscr{C}(\mathbb{G})$-function space $\breve{\mathscr{F}}(\mathbb{G}) := \mathscr{F}(\boldsymbol{L})$, then*

$$\mathrm{Op}\colon \breve{\mathscr{F}}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}}, \widehat{\mu})) \to \mathscr{F}(\mathrm{Ad}_{\boldsymbol{L}_2}) \tag{4.2.6}$$

*is a linear homeomorphism.*

*Let $M$ be a weight sequence, let $[M]$ have **(PL)**, let $D \subset \mathfrak{g}_{\mathrm{L}}$ and $\tilde{D} \subset \mathfrak{g}_{\mathrm{R}}$ be bases. Then the above holds for $\mathscr{F}(\mathbb{G}) = \mathscr{E}_D^{[M]}(\mathbb{G})$ and $\breve{\mathscr{F}}(\mathbb{G}) = \mathscr{E}_{\tilde{D}}^{[M]}(\mathbb{G})$.*

*Proof.* Due to Proposition 4.2.4 and Proposition 4.2.5, $\mathrm{Op}_{\mathbb{G}}$ and $\mathrm{Op}_{\mathbb{G}}^{-1}$ restrict to continuous maps

$$\mathrm{Op}_{\mathbb{G}}\colon \mathscr{E}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}}, \widehat{\mu})) \to \mathcal{L}(L^2(\mathbb{G}, \mu)) \quad \text{and} \quad \mathrm{Op}_{\mathbb{G}}^{-1}\colon \mathscr{E}(\mathrm{Ad}_{\boldsymbol{L}_2}) \to \mathscr{C}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}}, \widehat{\mu}))\,.$$

By using identity (4.2.3) we see that $\mathrm{Op}_{\mathbb{G}} \restriction_{\mathscr{E}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}}, \widehat{\mu}))}$ intertwines the representation $(\boldsymbol{L}\,\varepsilon\,\mathrm{I}_{B^\infty(\widehat{\mathbb{G}}, \widehat{\mu})}){\downarrow_{\mathscr{E}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}}, \widehat{\mu}))}}$ with $\mathrm{Ad}_{\boldsymbol{L}_2}$ and $\mathrm{Op}_{\mathbb{G}}^{-1} \restriction \mathscr{E}(\mathrm{Ad}_{\boldsymbol{L}_2})$ intertwines $\mathrm{Ad}_{\boldsymbol{L}_2}{\downarrow_{\mathscr{E}(\mathrm{Ad}_{\boldsymbol{L}_2})}}$ with $\boldsymbol{L}\,\varepsilon\,\mathrm{I}_{B^\infty(\widehat{\mathbb{G}}, \widehat{\mu})}$. Lemma 2.4.3 **(iv)** implies that $\mathrm{Op}_{\mathbb{G}}$ and $\mathrm{Op}_{\mathbb{G}}^{-1}$ restrict to continuous maps

$$\begin{aligned} \mathrm{Op}_{\mathbb{G}}\colon \mathscr{F}(\boldsymbol{L}\,\varepsilon\,\mathrm{I}_{B^\infty(\widehat{\mathbb{G}}, \widehat{\mu})}{\downarrow_{\mathscr{E}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}}, \widehat{\mu}))}}) &\to \mathscr{F}(\mathrm{Ad}_{\boldsymbol{L}_2})\,, \\ \mathrm{Op}_{\mathbb{G}}^{-1}\colon \mathscr{F}(\mathrm{Ad}_{\boldsymbol{L}_2}{\downarrow_{\mathscr{E}(\mathrm{Ad}_{\boldsymbol{L}_2})}}) &\to \mathscr{F}(\boldsymbol{L}\,\varepsilon\,\mathrm{I}_{B^\infty(\widehat{\mathbb{G}}, \widehat{\mu})})\,. \end{aligned}$$

Moreover, we have

$$\mathscr{E}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}}, \widehat{\mu})) = \mathscr{E}(\boldsymbol{L}\,\varepsilon\,\mathrm{I}_{B^\infty(\widehat{\mathbb{G}}, \widehat{\mu})})\,.$$

Then, due to Lemma 2.4.4 and $\mathscr{F}(\mathbb{G}) = \mathscr{F}(\boldsymbol{R}{\downarrow_{\mathscr{E}(\mathbb{G})}})$,

$$\mathrm{Op}_{\mathbb{G}}\colon \mathscr{F}(\boldsymbol{L}\,\varepsilon\,\mathrm{I}_{B^\infty(\widehat{\mathbb{G}}, \widehat{\mu})}) \to \mathscr{F}(\mathrm{Ad}_{\boldsymbol{L}_2})$$

is a linear homeomorphism. Now we use the linear homeomorphism

$$\breve{\mathscr{F}}(\mathbb{G})\,\varepsilon\,B^\infty(\widehat{\mathbb{G}}, \widehat{\mu}) = \mathscr{F}(\boldsymbol{L})\,\varepsilon\,B^\infty(\widehat{\mathbb{G}}, \widehat{\mu}) \simeq \mathscr{F}(\boldsymbol{L}\,\varepsilon\,\mathrm{I}_{B^\infty(\widehat{\mathbb{G}}, \widehat{\mu})})$$

and the canonical isomorphism $\breve{\mathscr{F}}(\mathbb{G})\,\varepsilon\,B^\infty(\widehat{\mathbb{G}}, \widehat{\mu}) \simeq \breve{\mathscr{F}}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}}, \widehat{\mu}))$. Together they imply $\breve{\mathscr{F}}(\boldsymbol{L}\,\varepsilon\,\mathrm{I}_{B^\infty(\widehat{\mathbb{G}}, \widehat{\mu})}) = \breve{\mathscr{F}}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}}, \widehat{\mu}))$.

We finish the proof by using $\mathscr{E}_{\tilde{D}}^{[M]}(\mathbb{G}) = \mathscr{E}_D^{[M]}(\boldsymbol{L})$ and Lemma 2.4.5, which gives us $\mathscr{E}_D^{[M]}(\mathbb{G}) = \mathscr{E}_D^{[M]}(\boldsymbol{R}{\downarrow_{\mathscr{E}(\mathbb{G})}})$ . □

We are left with one blemish in the theorem above. Since we used the description of $\mathscr{E}^{\{M\}}(\mathrm{Ad}_{\boldsymbol{L}_2})$ and $\mathscr{E}^{\{M\}}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}}, \widehat{\mu}))$ as projective limits to prove the statements above, we needed **(nQA)** in this case. This especially excludes the spaces of analytic functions resp. analytic vectors corresponding to $\{M\} = \{\mathbb{1}\}$. But if we discard this assumption, we still get the following.

**Corollary 4.2.7.** *Suppose $M$ is a weight sequence. Then*

$$\mathrm{Op}_{\mathbb{G}} \colon \mathscr{E}^{\{M\}}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}}, \widehat{\mu})) \to \mathscr{E}^{\{M\}}(\mathrm{Ad}_{\boldsymbol{L}_2})$$

*is a linear bijection.*

*Proof.* We just use Theorem 4.2.6 with $\mathscr{F}(\mathbb{G}) := \mathscr{E}^{\{M\}}_{D,\mathrm{proj}}(\mathbb{G})$ for some frame $D \subset \mathrm{Diff}_{\mathrm{L}}(\mathbb{G})$. With Proposition 2.2.18 and Lemma 2.4.10 we get $\breve{\mathscr{F}}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}}, \widehat{\mu})) = \mathscr{E}^{\{M\}}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}}, \widehat{\mu}))$ and $\mathscr{F}(\mathrm{Ad}_{\boldsymbol{L}_2}) = \mathscr{E}^{\{M\}}(\mathrm{Ad}_{\boldsymbol{L}_2})$ in the sense of vector spaces. $\square$

By setting $M = \mathbb{1}$ and $\mathbb{G} = \mathbb{T}^n$ the above corollary can be used to recover [9, Theorem 3], which states that the algebra of analytic vectors $\mathscr{E}^{\{\mathbb{1}\}}(\mathrm{Ad}_{\boldsymbol{L}_2})$ are in one-to-one correspondence with the analytic maps $\mathbb{G} \to B^\infty(\widehat{\mathbb{T}^n}, \widehat{\mu})$.

Let us now relate the statements of this section to some results concerning the space of symbols $\mathcal{S}^0_{0,0}(\mathbb{G} \times \widehat{\mathbb{G}})$. First, we need the definition of $\mathcal{S}^0_{0,0}(\mathbb{G} \times \widehat{\mathbb{G}})$.

**Definition 4.2.8.** *Let $\mathbb{G}$ be a compact Lie group and $V$ some finite dimensional vector space. For any $\varphi \in \mathscr{E}(\mathbb{G}; \mathcal{L}(V))$ the difference operator $\Delta_\varphi$ is defined by*

$$\Delta_\varphi \colon \mathscr{S}'(\widehat{\mathbb{G}}) \to \mathscr{S}'(\widehat{\mathbb{G}}) \otimes \mathcal{L}(V) \colon \sigma \mapsto (\mathcal{F}_{\mathbb{G}} \otimes \mathrm{I}_{\mathcal{L}(V)})(\varphi \cdot \mathcal{F}_{\mathbb{G}}^{-1}\sigma)\,.$$

*For a finite family $\Pi = (\pi^1, \ldots, \pi^n) \subset \mathrm{Irr}(\mathbb{G})$ put $H_\Pi := H_{\pi^1} \otimes \cdots \otimes H_{\pi^n}$ and define $\varphi_\Pi \in \mathscr{E}(\mathbb{G}; \mathcal{L}(H_\Pi))$ by*

$$\varphi_\Pi(x) := (\mathrm{I}_{\mathcal{L}(H_{\pi^1})} - \pi^1(x)) \otimes \cdots \otimes (\mathrm{I}_{\mathcal{L}(H_{\pi^n})} - \pi^n(x))\,.$$

*The space $\mathcal{S}^0_{0,0}(\mathbb{G} \times \widehat{\mathbb{G}})$ is defined to be the the set of symbols $\sigma \in \mathscr{E}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}}, \widehat{\mu})) \subset \mathscr{S}'(\mathbb{G}) \hat{\otimes} \mathscr{S}'(\widehat{\mathbb{G}})$ such that for any $P \in \mathrm{Diff}(\mathbb{G})$ and any finite collection $\Pi \subset \mathrm{Irr}(\mathbb{G})$*

$$P \otimes \Delta_{\varphi_\Pi}\sigma \in \mathscr{E}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}}, \widehat{\mu})) \otimes \mathcal{L}(H_\Pi)\,.$$

$\mathcal{S}^0_{0,0}(\mathbb{G}\times\widehat{\mathbb{G}})$ *is equipped with the topology defined by the seminorms*

$$\sigma \mapsto \sup_{\pi\in\mathrm{Irr}(\mathbb{G})}\sup_{x\in\mathbb{G}} \|(P_x\otimes\Delta_{\varphi_\Pi})\sigma(x,\pi)\|_{\mathcal{L}(H_\pi\otimes H_\Pi)}$$

*for* $P\in\mathrm{Diff}(\mathbb{G})$ *and finite families* $\Pi\subset\mathrm{Irr}(\mathbb{G})$.

Note that for finite families $\Pi\subset\mathrm{Irr}(\mathbb{G})$ the difference operator $\Delta_{\varphi_\Pi}$ is continuous on $\mathscr{S}'(\widehat{\mathbb{G}})$ and restricts to a continuous operator in $\mathcal{L}(\mathscr{S}(\widehat{\mathbb{G}});\mathscr{S}(\widehat{\mathbb{G}})\otimes\mathcal{L}(H_\Pi))$.

In order to use our previous work, we will show that $\mathcal{S}^0_{0,0}(\mathbb{G}\times\widehat{\mathbb{G}})$ is just the space of smooth vectors $\mathscr{E}(\mathbb{G};B^\infty(\mathbb{G},\widehat{\mu}))$ to the representation $\boldsymbol{L}\,\varepsilon\,\mathrm{I}_{B^\infty(\mathbb{G},\widehat{\mu})}$. For the torus $\mathbb{G}=\mathbb{T}^n=\mathbb{R}^n/(2\pi\mathbb{Z}^n)$ this is especially easy.

Suppose $\#$ is the counting measure on $\mathbb{Z}^n$ and $\mu$ is the Haar measure on $\mathbb{T}^n$ such that

$$\mu(E+2\pi\mathbb{Z}^n)=\frac{1}{2\pi}\int_E \mathrm{d}x \qquad \text{for Borel sets } E\subset[0,2\pi]^n\,.$$

Then $(\mathbb{Z}^n,\#)\simeq(\widehat{\mathbb{T}}^n,\widehat{\mu})$ with respect to the bijection $k\mapsto[e_k]$, in which $e_k(x+2\pi\mathbb{Z}^n):=\mathrm{e}^{\mathrm{i}(x,k)}$ for $x\in\mathbb{R}^n$ and the inner product $(-,-)$ in $\mathbb{R}^n$. Using this identification, we get $B^\infty(\widehat{\mathbb{G}},\widehat{\mu})\simeq\ell^\infty(\mathbb{Z}^n)$ and the difference operators take on the form

$$\Delta_{e_k}\sigma(\ell)=\sigma(\ell-k) \quad \text{for} \quad \sigma\in\ell^\infty(\mathbb{Z}^n)\,,\ \ell,k\in\mathbb{Z}^n\,.$$

Thus the difference operators $\Delta_{\varphi_\Pi}$ act as continuous operators on $\ell^\infty(\mathbb{Z}^n)$ and we get $\mathscr{E}(\mathbb{T}^n;B^\infty(\widehat{\mathbb{T}}^n,\widehat{\mu}))=\mathcal{S}^0_{0,0}(\mathbb{T}^n\times\widehat{\mathbb{T}}^n)$.

For general compact $\mathbb{G}$, we can do something similar by using the homeomorphism from Theorem 4.2.6. This yields the following proposition.

**Proposition 4.2.9.** *For any compact Lie group* $\mathbb{G}$ *with Haar measure* $\mu$ *the identity*

$$\mathscr{E}(\mathbb{G};B^\infty(\widehat{\mathbb{G}},\widehat{\mu}))=\mathcal{S}^0_{0,0}(\mathbb{G}\times\widehat{\mathbb{G}})$$

*holds in the sense of topological vector spaces.*

Although the same can also be shown with [23, Lemma 8.7], the following proof needs less additional resources.

*Proof.* It is obvious that $\mathcal{S}^0_{0,0}(\mathbb{G}\times\widehat{\mathbb{G}}) \subset \mathscr{E}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}},\widehat{\mu}))$ is equipped with a finer topology. For any $\pi \in \mathrm{Irr}(\mathbb{G})$ we define the continuous maps

$$T_\pi\colon \mathscr{S}'(\mathbb{G}) \to \mathscr{S}'(\mathbb{G}) \otimes \mathcal{L}(H_\pi)\colon f \mapsto T_\pi f := \pi(-)^{-1} \cdot f$$
$$R_\pi\colon \mathscr{S}'(\mathbb{G}) \otimes \mathcal{L}(H_\pi) \to \mathscr{S}'(\mathbb{G}) \otimes \mathcal{L}(H_\pi)\colon f \mapsto R_\pi f := f \cdot \pi(-)\,.$$

Both $T_\pi$ and $R_\pi$ stay well-defined and continuous if we exchange $\mathscr{S}'(\mathbb{G})$ with $L^2(\mathbb{G},\mu)$ or with $\mathscr{S}(\mathbb{G}) = \mathscr{E}(\mathbb{G})$. Now let $a \in \mathscr{S}(\mathbb{G}) \,\hat{\otimes}\, \mathscr{S}(\widehat{\mathbb{G}})$, $A := \mathrm{Op}_{\mathbb{G}}(a)$ and $K := \mathcal{K}_{\mathbb{G}}(A)$. First of all

$$\mathcal{K}_{\mathbb{G}} \otimes \mathrm{I}\left[R_\pi \circ (A \otimes \mathrm{I}_{\mathcal{L}(H_\pi)}) \circ T_\pi\right](x,y) = K(x,y)\cdot \pi(y^{-1}x)\,.$$

We use $K(x,y) = \mathrm{I}\otimes\mathcal{F}_{\mathbb{G}}^{-1} a(x, y^{-1}x)$ in order to get

$$\begin{aligned}
K(x,y)\cdot\pi(y^{-1}x) &= \big(\mathrm{I}\otimes\mathcal{F}_{\mathbb{G}}^{-1} a\big)(x,y^{-1}x)\cdot\pi(y^{-1}x)\\
&= \big(\mathrm{I}\otimes\mathcal{F}_{\mathbb{G}}^{-1}\otimes\mathrm{I}\big)\big(\mathrm{I}\otimes\Delta_\pi a\big)(x,y^{-1}x)\\
&= (\mathcal{K}_{\mathbb{G}}\,\mathrm{Op}_{\mathbb{G}}\otimes\mathrm{I})(\mathrm{I}\otimes\Delta_\pi a)\,.
\end{aligned}$$

In summary, we have

$$R_\pi(A\otimes\mathrm{I}_{\mathcal{L}(H_\pi)})T_\pi = (\mathrm{Op}_{\mathbb{G}}\otimes\mathrm{I}_{\mathcal{L}(H_\pi)})(\mathrm{I}\otimes\Delta_\pi a)\,.$$

By the continuity of $R_\pi$, $T_\pi$, $\mathrm{Op}_{\mathbb{G}}$ and $\Delta_\pi$, this can be extended to arbitrary $a \in \mathscr{S}'(\mathbb{G}) \,\hat{\otimes}\, \mathscr{S}'(\widehat{\mathbb{G}})$. Since $\pi$ and $\pi(-)^{-1}$ are smooth maps,

$$\mathscr{E}(\mathrm{Ad}_{\boldsymbol{L_2}}) \ni A \mapsto R_\pi(A\otimes\mathrm{I}_{\mathcal{L}(H_\pi)})T_\pi \in \mathscr{E}(\mathrm{Ad}_{\boldsymbol{L_2}})\otimes\mathcal{L}(H_\pi)$$

is a linear homeomorphism onto its range. By Theorem 4.2.6, this implies that $\mathrm{I}_{\mathscr{S}'(\mathbb{G})}\otimes\Delta_\pi$ restricts to a continuous map from $\mathscr{E}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}},\widehat{\mu}))$ to $\mathscr{E}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}},\widehat{\mu}))\otimes\mathcal{L}(H_\pi)$. Thus

$$\mathrm{I}\,\varepsilon\Delta_{\varphi_\Pi}\colon \mathscr{E}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}},\widehat{\mu})) \to \mathscr{E}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}},\widehat{\mu}))\otimes\mathcal{L}(H_\Pi)$$

is well-defined and continuous for any finite family $\Pi \subset \mathrm{Irr}(\mathbb{G})$. This results in the identity $\mathcal{S}^0_{0,0}(\mathbb{G}\times\widehat{\mathbb{G}}) = \mathscr{E}(\mathbb{G}; B^\infty(\widehat{\mathbb{G}},\widehat{\mu}))$ as locally convex spaces. $\square$

The above lemma shows that Theorem 4.2.6 contains the statement from [23, Proposition 8.11] resp. [23, Corollary 8.6] concerning $\mathcal{S}^0_{0,0}(\mathbb{G}\times\widehat{\mathbb{G}})$. Together with Theorem 4.2.2 and Theorem 4.2.6 we get the following corollary.

**Corollary 4.2.10.** $\operatorname{Op}_{\mathbb{G}} \mathcal{S}^0_{0,0}(\mathbb{G} \times \widehat{\mathbb{G}})$ *is a $*$-subalgebra of $\mathcal{L}(L^2(\mathbb{G},\mu))$ for any compact Lie group $\mathbb{G}$ with Haar measure $\mu$. Furthermore, each operator in $\operatorname{Op}_{\mathbb{G}} \mathcal{S}^0_{0,0}(\mathbb{G} \times \widehat{\mathbb{G}})$ can be identified with an operator in $\mathcal{L}(H^k(\mathbb{G}))$ for $k \in \mathbb{N}_0 \cup \{\infty\}$.*

By [9] the set of operators $\mathscr{E}^{\{\mathbb{1}\}}(\operatorname{Ad}_{\boldsymbol{L}_2})$ is dense in $\operatorname{Op}_{\mathbb{T}^n} \mathcal{S}^0_{0,0}(\mathbb{T}^n \times \widehat{\mathbb{T}}^n)$ with respect to the subspace topology in $\mathcal{L}(L^2(\mathbb{T}^n,\mu))$. We can even go one step further and prove the denseness of $\mathscr{E}^{\{\mathbb{1}\}}(\operatorname{Ad}_{\boldsymbol{L}_2})$ in $\operatorname{Op}_{\mathbb{T}^n} \mathcal{S}^0_{0,0}(\mathbb{T}^n \times \widehat{\mathbb{T}}^n)$ with respect to its own Fréchet topology. Here we can use that $\mathscr{E}^{[M]}_D(\operatorname{Ad}_{\boldsymbol{L}_2})$ is dense in $\mathscr{E}(\operatorname{Ad}_{\boldsymbol{L}_2})$ by Lemma 2.4.7.

**Corollary 4.2.11.** *Suppose $\mathbb{G}$ is a compact Lie group with Haar measure $\mu$. Then $\mathscr{E}^{\{\mathbb{1}\}}(\operatorname{Ad}_{\boldsymbol{L}_2})$ is dense in $\mathscr{E}(\operatorname{Ad}_{\boldsymbol{L}_2}) = \operatorname{Op}_{\mathbb{G}} \mathcal{S}^0_{0,0}(\mathbb{G} \times \widehat{\mathbb{G}})$.*

## 4.3 Operators defined by Schrödinger type representations

In this subsection we will consider the representations $\rho_\lambda$ and $\Theta_\lambda$. For the most part, we can prove statements for the operators in $\mathscr{E}^{[M]}_D(\operatorname{Ad}_{\rho_\lambda})$ and $\mathscr{E}^{[M]}_D(\operatorname{Ad}_{\Theta_\lambda})$ that are analogous to the operators in $\mathscr{F}(\operatorname{Ad}_{\boldsymbol{L}_2})$, where $\boldsymbol{L}_2$ is the left regular representation for some compact Lie group.

**Lemma 4.3.1.** *If $M$ and $L$ are weight sequences then $\mathscr{S}^{[M]}_{[L],\mathrm{proj}}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$.*

*Proof.* Since $\mathscr{S}^{(\mathbb{1})}_{(\mathbb{1})}(\mathbb{R}^n) \subset \mathscr{S}^{[M]}_{[L],\mathrm{proj}}(\mathbb{R}^n)$, it is enough for $\mathscr{S}^{(\mathbb{1})}_{(\mathbb{1})}(\mathbb{R}^n)$ to be dense in $L^2(\mathbb{R}^n)$. In [48] it was proven that all Hermite functions are contained in $\mathscr{S}^{(\mathbb{1})}_{(\mathbb{1})}(\mathbb{R}^n)$. This implies the denseness because they form an orthonormal basis of $L^2(\mathbb{R}^n)$. $\square$

As for the case of the left regular representation, the following theorem is a summary of preceding lemmata applied the Schrödinger representation on $\mathbb{H}$.

**Theorem 4.3.2.** *Let $M$, $L$ and $K$ be weight sequences, let $\lambda \in \mathbb{R}^\times$ and let $D \subset \operatorname{Diff}_{\mathrm{L}}(\mathbb{H})$ be a frame. Then the following holds.*

**(i)** *If $\mathscr{F}(\mathbb{H}) \in \{\mathscr{E}(\mathbb{H}), \mathscr{E}_D^{[M]}(\mathbb{H})\}$, then the algebra $\mathscr{F}(\mathrm{Ad}_{\rho_\lambda})$ is a $*$-subalgebra of $\mathcal{L}(L^2(\mathbb{R}^n))$, i.e.*

$$\mathscr{F}(\mathrm{Ad}_{\rho_\lambda})^* = \mathscr{F}(\mathrm{Ad}_{\rho_\lambda}).$$

*Moreover, if $[M]$ has **(PL)**, then $\mathscr{F}(\mathrm{Ad}_{\rho_\lambda})$ is locally m-convex.*

**(ii)** *If $\mathscr{F}(\mathbb{H}) \in \{\mathscr{E}(\mathbb{H}), \mathscr{E}_D^{\{M\}}(\mathbb{H})\}$, then the algebra $\mathscr{F}(\mathrm{Ad}_{\rho_\lambda})$ is invariant under the holomorphic functional calculus.*

**(iii)** *If $F \subset \mathrm{Diff_L}(\mathbb{R}^n)$ is a frame and $q \subset \mathbb{R}_n$ a basis, then we have embeddings*

$$\mathscr{F}(\mathrm{Ad}_{\rho_\lambda}) \hookrightarrow \mathcal{L}(E) \colon T \mapsto T\restriction_E^E$$

*if we choose $\mathscr{F}(\mathbb{H})$ and $E$ from Table 4.3. These embeddings are continuous for $E \neq \mathscr{S}_{\{L\}}^{\{K\}}(\mathbb{R}^n)$ or $[M] \subset (K),(L)$.*

| $\mathscr{F}(\mathbb{H})$ | $E$ |
|---|---|
| $\mathscr{E}(\mathbb{H})$, $\mathscr{E}_D^{[M]}(\mathbb{H})$ | $H^k_{\mathscr{P}}(\mathbb{R}^n)$ for $k \in \mathbb{N}_0$, $\mathscr{S}(\mathbb{R}^n)$ |
| $\mathscr{E}_D^{[M]}(\mathbb{H})$ and $M$ fulfils **(MG)** | $H^{K,F}_{L,q}(\mathbb{R}^n)$, $\mathscr{S}^{(K)}_{(L)}(\mathbb{R}^n)$ for $[M] \subset (L)$ and $[M] \subset (K)$ |
| | $\mathscr{S}^{\{K\}}_{\{L\}}(\mathbb{R}^n)$ for $[M] \subset \{L\}$ and $[M] \subset \{K\}$ |
| | $\mathscr{S}^{[K]}_{[L],\mathrm{proj}}(\mathbb{R}^n)$ for $[M] \subset [L]$ and $[M] \subset [K]$ |

Table 4.3: Possible choices of locally convex spaces $E$ and $\mathscr{F}(\mathbb{H})$ for Theorem 4.2.2 **(iii)**

*Proof.* We omit the proof for **(i)** and **(ii)**, since these statements can be proven exactly as in Theorem 4.2.2.

**(iii):** By Lemma 2.4.22 and Lemma 2.4.24 we have

$$\mathscr{C}^k(\rho_\lambda) = H^k_{\mathscr{P}}(\mathbb{R}^n) \quad \text{and} \quad \mathscr{E}(\rho_\lambda) = \mathscr{S}(\mathbb{R}^n).$$

Thus, for the first row of Table 4.3, we may apply Theorem 4.1.7 for $(\pi, E) = (\rho_\lambda, L^2(\mathbb{R}^n))$.

Now we construct the proof for the second row of Table 4.3. The space $\mathscr{E}_D^{[M]}(\mathrm{Ad}_{\rho_\lambda})$ does not depend on the choice $D$ by Proposition 2.2.10. So we may assume that $D = (D^{(0)}, D^{(1)}, D^{(2)})$ with $\rho_\lambda(D^{(0)})f = \mathrm{i}f$, $\rho_\lambda(D^{(1)}_j)f = \mathrm{i}\,q_j\,f$ and $\rho_\lambda(D^{(2)}_j)f = F_j f$ for all

$f \in \mathscr{S}(\mathbb{R}^n)$. Let $N$ be *any* weight sequence with $[M] \subset (N)$. By a slight adjustment to the proof of Proposition 2.2.15 **(i)**[2], we get the continuous multiplication

$$\mathscr{E}^{[M],[M],[M]}_{D^{(0)},D^{(1)},D^{(2)}\mathrm{proj}}(\mathbb{H}) \times \mathscr{E}^{N,L,K}_{D^{(0)},D^{(1)},D^{(2)}}(\mathbb{H}) \to \mathscr{E}^{N,L,K}_{D^{(0)},D^{(1)},D^{(2)}}(\mathbb{H})$$

for all $L, K$ with $[M] \subset (L), (K)$ and thus also a continuous multiplication

$$\mathscr{E}^{[M],[M],[M]}_{D^{(0)},D^{(1)},D^{(2)}\mathrm{proj}}(\mathbb{H}) \times \mathscr{E}^{[N],[L],[K]}_{D^{(0)},D^{(1)},D^{(2)},\mathrm{proj}}(\mathbb{H}) \to \mathscr{E}^{[N],[L],[K]}_{D^{(0)},D^{(1)},D^{(2)},\mathrm{proj}}(\mathbb{H})$$

for all $[L]$, $[K]$ with $[M] \subset [L], [K]$. Since $M$ fulfils **(MG)**, we have the continuous embeddings

$$\mathscr{E}^{[M]}_{D}(\mathbb{H}) \xrightarrow{\mathrm{I}} \mathscr{E}^{[M]}_{D,\mathrm{proj}}(\mathbb{H}) \xrightarrow{\mathrm{I}} \mathscr{E}^{[M],[M],[M]}_{D^{(0)},D^{(1)},D^{(2)}\mathrm{proj}}(\mathbb{H}) .$$

Now, combining the embedding with the continuous multiplications, Lemma 2.4.28 and Lemma 4.1.1 gives us the continuous map

$$\mathscr{E}^{[M]}_{D}(\mathrm{Ad}_{\rho_\lambda}) \to \mathcal{L}(E)$$

for $E = H^{N,F}_{K,q}(\mathbb{R}^n)$ or $E = \mathscr{S}^{[K]}_{[L],\mathrm{proj}}(\mathbb{R}^n)$. By using $\mathscr{S}^{(K)}_{(L),\mathrm{proj}}(\mathbb{R}^n) = \mathscr{S}^{(K)}_{(L)}(\mathbb{R}^n)$ and Lemma 4.1.2 we get the continuity for all other choices for $E$.

Finally $\mathscr{E}^{[M]}_{D}(\mathrm{Ad}_{\rho_\lambda}) \to \mathcal{L}(E)\colon T \mapsto T \upharpoonright^{E}_{E}$ is an embedding for all considered choices, since $\mathscr{S}^{(\mathbb{1})}_{(\mathbb{1})}(\mathbb{R}^n) \subset E$ is dense in $L^2(\mathbb{R}^n)$ by Lemma 4.3.1 □

Let $\mathscr{C}_b(\mathbb{R}^n \times \mathbb{R}_n)$ be the space of all continuous bounded functions $f\colon \mathbb{R}^n \times \mathbb{R}_n \to \mathbb{C}$ equipped with the topology defined by the supremum norm

$$f \mapsto \|f\|_\infty = \sup_{x \in \mathbb{R}^n \times \mathbb{R}_n} |f(x)| .$$

If we define the representation $\boldsymbol{R}_{b,\lambda}$ of $\mathbb{H}$ on $\mathscr{C}_b(\mathbb{R}^n \times \mathbb{R}_n)$ by

$$\boldsymbol{R}_{b,\lambda}(t, x', x) f(y, y') := f\big(y + \sqrt{|\lambda|}x, y' - \mathrm{sgn}(\lambda)\sqrt{|\lambda|}x'\big)$$

for $t \in \mathbb{R}$, $x, y \in \mathbb{R}^n$, $x', y' \in \mathbb{R}_n$ and $f \in \mathscr{C}_b(\mathbb{R}^n \times \mathbb{R}_n)$, then we have

$$\mathscr{E}(\boldsymbol{R}_{b,\lambda}) = \{f \in \mathscr{E}(\mathbb{R}^n \times \mathbb{R}_n) \mid Pf \text{ is bounded for all } P \in \mathrm{Diff}_{\mathrm{L}}(\mathbb{R}^n \times \mathbb{R}_n)\} .$$

[2] Essentially we just need to exchange the seminorms $\|-\|_{k,D,L}$ with seminorms that are more suited to the spaces $\mathscr{E}^{N,L,K}_{D^{(0)},D^{(1)},D^{(2)}}(\mathbb{H})$.

Since this locally convex space does not depend on the choice $\lambda \in \mathbb{R}^\times$ we will also just write $\mathscr{E}_b(\mathbb{R}^n \times \mathbb{R}_n) := \mathscr{E}(\boldsymbol{R}_{b,\lambda})$. By identifying $\mathbb{R}_n$ with $\mathbb{R}^n$, the space $\mathscr{E}_b(\mathbb{R}^n \times \mathbb{R}_n)$ can be identified with the usual space of symbols $\mathcal{S}^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$ [15, 67]. Note $(\boldsymbol{R}_{b,\lambda}, \mathscr{C}_b(\mathbb{R}^n \times \mathbb{R}_n))$ is not an admissible representation, since $\mathscr{C}(\boldsymbol{R}_{b,\lambda})$ are the bounded *uniformly* continuous functions on $\mathbb{R}^n \times \mathbb{R}_n$. Now for any $f \in \mathscr{E}(\boldsymbol{R}_{b,\lambda})$ we have

$$\mathrm{Op}_{\mathbb{R}^n}(\boldsymbol{R}_{b,\lambda}(x)f) = \mathrm{Ad}_{\rho_\lambda}(x)\, \mathrm{Op}_{\mathbb{R}^n}(f) \qquad \text{for all } x \in \mathbb{H}\,.$$

Here we identify $\mathbb{R}_n$ with $\widehat{\mathbb{R}^n}$ via $x' \mapsto [x \mapsto \mathrm{e}^{2\pi \mathrm{i}\langle x, x'\rangle}]$. This way we identify any function $f \in \mathscr{E}(\boldsymbol{R}_{b,\lambda})$ with a distribution in $\mathscr{S}'(\mathbb{R}^n) \mathbin{\hat{\otimes}} \mathscr{S}'(\widehat{\mathbb{R}^n}) \simeq \mathscr{S}'(\mathbb{R}^n) \mathbin{\hat{\otimes}} \mathscr{S}'(\mathbb{R}_n)$ via

$$\mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}_n) \ni (\varphi, \psi) \mapsto \int_{\mathbb{R}^n} \int_{\mathbb{R}_n} f(x, x')\varphi(x)\psi(x')\, \mathrm{d}x'\, \mathrm{d}x\,.$$

For any weight sequence $M$ and any frame $D \subset \mathcal{V}_{\mathrm{a}}(\mathbb{R}^n \times \mathbb{R}_n)$ we define

$$\mathscr{E}^M_{b,D}(\mathbb{R}^n \times \mathbb{R}_n) := \left\{ f \in \mathscr{E}(\boldsymbol{R}_{b,\lambda}) \;\middle|\; \sup_{a \in \mathcal{S}_{2n}} \frac{\|D^a f\|_\infty}{M_{|a|}\, |a|!} < \infty \right\}$$

equipped with the norm

$$f \mapsto \sup_{a \in \mathcal{S}_{2n}} \frac{\|D^a f\|_\infty}{M_{|a|}\, |a|!}$$

and also

$$\mathscr{E}^{(M)}_b(\mathbb{R}^n \times \mathbb{R}_n) := \varprojlim_{h>0} \mathscr{E}^M_{b,h\partial}(\mathbb{R}^n \times \mathbb{R}_n) \quad \text{and} \quad \mathscr{E}^{\{M\}}_b(\mathbb{R}^n \times \mathbb{R}_n) := \varprojlim_{h\in\Lambda} \mathscr{E}^{hM}_{b,\partial}(\mathbb{R}^n \times \mathbb{R}_n)\,.$$

Note that similar as in Proposition 2.2.18 we have

$$\mathscr{E}^{\{M\}}_b(\mathbb{R}^n \times \mathbb{R}_n) = \varinjlim_{h>0} \mathscr{E}^M_{b,h\partial}(\mathbb{R}^n \times \mathbb{R}_n) = \mathscr{E}^{\{M\}}_\partial(\boldsymbol{R}_{b,\lambda})$$

as vector spaces. Due to Lemma 2.4.8, we have $\mathscr{E}^M_{b,|\lambda|^{\frac{1}{2}}h\partial}(\mathbb{R}^n \times \mathbb{R}_n) = \mathscr{E}^M_{h\partial}(\boldsymbol{R}_{b,\lambda})$ and thus also $\mathscr{E}^{[M]}_b(\mathbb{R}^n \times \mathbb{R}_n) = \mathscr{E}^{[M]}_{\partial,\mathrm{proj}}(\boldsymbol{R}_{b,\lambda})$. If $M$ fulfils **(nQA)**, then $\mathscr{E}^{\{M\}}_{b,\partial}(\mathbb{R}^n \times \mathbb{R}_n) = \mathscr{E}^{\{M\}}_\partial(\boldsymbol{R}_{b,\lambda})$ even holds in the sense of topological vector spaces.

In the following theorem, we built on top and expand the results of Cordes concerning the description of smooth operators via their symbols.

**Theorem 4.3.3.** *Suppose $M$ is a weight sequence and $\lambda \in \mathbb{R}^\times$. Then*

$$\mathrm{Op}_{\mathbb{R}^n} \colon \mathscr{F}(\boldsymbol{R}_{b,\lambda}) \to \mathscr{F}(\mathrm{Ad}_{\rho_\lambda}) \tag{4.3.7}$$

*is a linear homeomorphism for any $\mathscr{C}(\mathbb{H})$-function space $\mathscr{F}(\mathbb{H})$ with $\mathscr{F}(\mathbb{H}) = \mathscr{F}(\boldsymbol{R}{\downarrow_{\mathscr{E}(\mathbb{H})}})$.*

*Let $D \subset \mathrm{Diff}_{\mathrm{L}}(\mathbb{H})$ be a frame. For $\mathscr{F}(\mathbb{H}) \in \{\mathscr{E}(\mathbb{H}), \mathscr{E}_D^{[M]}(\mathbb{H})\}$ this is especially true if $[M]$ has **(PL)**, in which case also*

$$\mathscr{E}_D^{[M]}(\boldsymbol{R}_{b,\lambda}) = \mathscr{E}_b^{[M]}(\mathbb{R}^n \times \mathbb{R}_n) \quad \textit{and} \quad \mathscr{E}(\boldsymbol{R}_{b,\lambda}) = \mathscr{E}_b(\mathbb{R}^n \times \mathbb{R}_n)\,. \tag{4.3.8}$$

*If $\mathscr{F} = \mathscr{E}^{[M]}$ with $[M] = \{M\}$ and $M$ does not fulfil **(nQA)**, then (4.3.7) is still a bijection and (4.3.8) holds in the sense of vector spaces.*

*Proof.* The proof is closely related to the proofs of 4.2.6 and Corollary 4.2.7.

First of all, by the discussion immediately before this theorem we have

$$\mathscr{E}_{D,\mathrm{proj}}^{[M]}(\boldsymbol{R}_{b,\lambda}) = \mathscr{E}_b^{[M]}(\mathbb{R}^n \times \mathbb{R}_n) \quad \text{and also} \quad \mathscr{E}(\boldsymbol{R}_{b,\lambda}) = \mathscr{E}_b(\mathbb{R}^n \times \mathbb{R}_n)\,.$$

And also $\mathscr{E}_D^{[M]}(\boldsymbol{R}_{b,\lambda}) = \mathscr{E}_{D,\mathrm{proj}}^{[M]}(\boldsymbol{R}_{b,\lambda})$ in the sense of vector spaces (resp. in the sense of topological vector spaces if $[M]$ has **(PL)**). Now let $\mathscr{F}(\mathbb{H}) \in \{\mathscr{E}(\mathbb{H}), \mathscr{E}_{D,\mathrm{proj}}^{[M]}(\mathbb{H})\}$. By Lemma 2.4.5, we have

$$\mathscr{F}(\boldsymbol{R}_{b,\lambda}{\downarrow_{\mathscr{E}(\boldsymbol{R}_{b,\lambda})}}) = \mathscr{F}(\boldsymbol{R}_{b,\lambda}) \quad \text{and} \quad \mathscr{F}(\mathrm{Ad}_{\rho_\lambda}{\downarrow_{\mathscr{E}(\mathrm{Ad}_{\rho_\lambda})}}) = \mathscr{F}(\mathrm{Ad}_{\rho_\lambda})\,.$$

Now we discuss the general case in which (4.3.7) is a homeomorphism. The map

$$\mathrm{Op}_{\mathbb{R}^n} \colon \mathscr{E}(\boldsymbol{R}_{b,\lambda}) = \mathscr{E}_b(\mathbb{R}^n \times \mathbb{R}_n) \to \mathscr{E}(\mathrm{Ad}_{\rho_\lambda}) \tag{4.3.9}$$

is a well-defined linear homeomorphism by Theorem 4.2 and Theorem 4.3 of [15, Chapter 8]. Since $\mathrm{Op}_{\mathbb{R}^n} \boldsymbol{R}_{b,\lambda}{\downarrow_{\mathscr{E}(\boldsymbol{R}_{b,\lambda})}} = \mathrm{Ad}_{\rho_\lambda}{\downarrow_{\mathscr{E}(\mathrm{Ad}_{\rho_\lambda})}} \mathrm{Op}_{\mathbb{R}^n}$ the restriction

$$\mathrm{Op}_{\mathbb{R}^n} \colon \mathscr{F}(\boldsymbol{R}_{b,\lambda}{\downarrow_{\mathscr{E}(\boldsymbol{R}_{b,\lambda})}}) \to \mathscr{F}(\mathrm{Ad}_{\rho_\lambda}{\downarrow_{\mathscr{E}(\mathrm{Ad}_{\rho_\lambda})}})$$

is a well-defined homeomorphism due to Lemma 2.4.3. Finally, due to Lemma 2.4.4, we know that

$$\mathscr{F}(\boldsymbol{R}_{b,\lambda}) = \mathscr{F}(\boldsymbol{R}_{b,\lambda}{\downarrow_{\mathscr{E}(\boldsymbol{R}_{b,\lambda})}}) \quad \text{and} \quad \mathscr{F}(\mathrm{Ad}_{\rho_\lambda}) = \mathscr{F}(\mathrm{Ad}_{\rho_\lambda}{\downarrow_{\mathscr{E}(\mathrm{Ad}_{\rho_\lambda})}})\,.$$

□

The above theorem especially implies that the operator algebras $\mathscr{E}_D^{[M]}(\mathrm{Ad}_{\rho_\lambda})$ considered in Theorem 4.3.2 are dense in the usual algebra of pseudodifferential operators with symbols in $\mathscr{E}_b(\mathbb{R}^n \times \mathbb{R}_n) \simeq \mathcal{S}^0_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$ for the case $\{\mathbb{1}\} \subset [M]$. For this statement we only need to apply Lemma 2.4.7.

We will now relate the above to other results. In [10] pseudodifferential operators with symbols $a \in \Gamma^m_{\mu,\nu}$ fulfilling the boundedness requirements

$$\sup_{\alpha,\beta\in\mathbb{N}_0^n}\ \sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}_n} C^{|\alpha|+|\beta|}\frac{|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|}{(\alpha!)^\mu\,\langle x\rangle^{m_1-|\alpha|}\,(\beta!)^\nu\,\langle \xi\rangle^{m_2-|\beta|}} < \infty \quad \text{for some} \quad C>0$$

are considered, where $\langle x\rangle = (1+|x|)^{\frac{1}{2}}$ and $m=(m_1,m_2)$. In the case $m=0$, we have $\mathrm{Op}_{\mathbb{R}^n}\,\Gamma^0_{\mu,\mu} \subset \mathscr{E}^{\{G^{(\mu)}\}}(\mathrm{Ad}_{\rho_\lambda})$ for the Gevrey sequence $G_k^{(\mu)} = (k!)^{1-\mu}$ due to Theorem 4.3.3. This way the continuity property $\mathrm{Op}_{\mathbb{R}^n}\,\Gamma^0_{\mu,\mu} \subset \mathcal{L}(\mathscr{S}^{\{M\}}_{\{M\}}(\mathbb{R}^n))$ from [10, Theorem 2.2] is included in Theorem 4.3.2. But in later papers [1, 11], more general symbols have been considered. The corresponding statements about the continuity on Gelfand-Shilov spaces are no longer contained Theorem 4.3.2. However, there do not seem to be any results concerning the spectral invariance of these algebras of pseudodifferential operators. Here the characterization via regularity conditions on the adjoint representations has merit.

Next to the cases discussed before, we could also use the flexibility of our approach on other representations $(\pi, E)$. For example, we could formulate an equivalent of Theorem 4.3.3 for $(\pi, E) = (\rho_\lambda, L^p(\mathbb{R}^n))$. Though, in this case we can not use (4.3.9) for a characterization via symbols. Another field of application would be to use other Lie groups that are neither compact nor abelian. An example is given in the following theorem, which uses the Schrödinger-type representations $\Theta_\lambda$ on $L^2(\mathbb{H})$. Though, here we do not have a homeomorphism of the type (4.3.9) that intertwines $\mathrm{Ad}_{\Theta_\lambda}\downarrow_{\mathscr{E}(\mathrm{Ad}_{\Theta_\lambda})}$ with a corresponding group action on bounded smooth symbols as well.

**Theorem 4.3.4.** *Let $M$ be a weight sequence, let $\lambda \in \mathbb{R}^\times$ and let $D \subset \mathrm{Diff}_{\mathrm{L}}(\mathbb{H}_2)$ be a frame. Then the following holds.*

- **(i)** *If $\mathscr{F}(\mathbb{H}_2) \in \{\mathscr{E}(\mathbb{H}_2), \mathscr{E}_D^{[M]}(\mathbb{H}_2)\}$, then the algebra $\mathscr{F}(\mathrm{Ad}_{\Theta_\lambda})$ is a $*$-subalgebra of $\mathcal{L}(L^2(\mathbb{H}))$, i.e.*

$$\mathscr{F}(\mathrm{Ad}_{\Theta_\lambda})^* = \mathscr{F}(\mathrm{Ad}_{\Theta_\lambda})\,.$$

*Moreover, if $[M]$ has **(PL)**, then $\mathscr{F}(\mathrm{Ad}_{\Theta_\lambda})$ is locally m-convex.*

**(ii)** *If $\mathscr{F}(\mathbb{H}_2) \in \{\mathscr{E}(\mathbb{H}_2), \mathscr{E}_D^{\{M\}}(\mathbb{H}_2)\}$, then the algebra $\mathscr{F}(\mathrm{Ad}_{\Theta_\lambda})$ is a spectrally invariant in $\mathcal{L}(L^2(\mathbb{H}))$, i.e.*

$$\mathcal{L}(L^2(\mathbb{H}))^\times \cap \mathscr{F}(\mathrm{Ad}_{\Theta_\lambda}) = \mathscr{F}(\mathrm{Ad}_{\Theta_\lambda})^\times .$$

*Moreover, if $M$ fulfils **(nQA)**, then $\mathscr{F}(\mathrm{Ad}_{\rho_\lambda})$ is invariant under the holomorphic functional calculus.*

**(iii)** *If $F \subset \mathrm{Diff}_{\mathrm{L}}(\mathbb{H})$ is a frame and $q \subset \mathbb{H}'$ a basis and if $\{\mathbb{1}\} \subset [M]$, then we have embeddings*

$$\mathscr{F}(\mathrm{Ad}_{\Theta_\lambda}) \hookrightarrow \mathcal{L}(E) \colon T \mapsto T \restriction_E^E$$

*if we choose $\mathscr{F}(\mathbb{H}_2)$ and $E$ from Table 4.4. These embeddings are continuous for $E \neq \mathscr{S}_{\{L\}}^{\{K\}}(\mathbb{H})$ or $[M] \subset (K), (L)$.*

| $\mathscr{F}(\mathbb{H}_2)$ | $E$ |
|---|---|
| $\mathscr{E}(\mathbb{H}_2)$, $\mathscr{E}_D^{[M]}(\mathbb{H}_2)$ | $H^k_{\mathscr{P}}(\mathbb{H})$ for $k \in \mathbb{N}_0$, $\mathscr{S}(\mathbb{H})$ |
| $\mathscr{E}_D^{[M]}(\mathbb{H}_2)$ and $M$ fulfils **(MG)** | $H^{K,F}_{L,q}(\mathbb{H})$, $\mathscr{S}^{(K)}_{(L)}(\mathbb{H})$ for $[M] \subset (L)$ and $[M] \subset (K)$ |
| | $\mathscr{S}^{\{K\}}_{\{L\}}(\mathbb{H})$ for $[M] \subset \{L\}$ and $[M] \subset \{K\}$ |
| | $\mathscr{S}^{[K]}_{[L],\mathrm{proj}}(\mathbb{H})$ for $[M] \subset [L]$ and $[M] \subset [K]$ |

Table 4.4: Possible choices of locally convex spaces $E$ and $\mathscr{F}(\mathbb{H}_2)$ for Theorem 4.2.2 **(iii)**

*Proof.* The proofs for **(i)**, **(ii)** and **(iii)** work exactly as in Theorem 4.3.2. The only difference for **(iii)** is that we can not cite sources for the denseness of $E$ in $L^2(\mathbb{H})$. Here we use $\{\mathbb{1}\} \subset [M]$, which ensures that $\mathscr{E}^{\{\mathbb{1}\}}(\Theta_\lambda) \subset E$. Thus

$$\mathscr{F}(\mathrm{Ad}_{\Theta_\lambda}) \hookrightarrow \mathcal{L}(E) \colon T \mapsto T \restriction_E^E ,$$

is an embedding, since $\mathscr{E}^{\{\mathbb{1}\}}(\Theta_\lambda)$ is dense in $L^2(\mathbb{H})$. □

# List of symbols

## Notations and basic concepts



## Spaces of vector valued smooth and ultradifferentiable functions

# Quantization on Gelfand triples

# Operator spaces characterized by ultradifferentiable group actions

# Bibliography


[1] Ahmed Abdeljawad, Marco Cappiello, and Joachim Toft. Pseudo-differential calculus in anisotropic Gelfand-Shilov setting. *Integral Equations Operator Theory*, 91(3):Paper No. 26, 33, 2019.

[2] Francesca Astengo, Bianca Di Blasio, and Fulvio Ricci. Fourier transform of Schwartz functions on the Heisenberg group. *Studia Math.*, 214(3):201–222, 2013.

[3] C. Bargetz and N. Ortner. Convolution of vector-valued distributions: A survey and comparison. *Dissertationes Math.*, 495:51, 2013.

[4] Christian Bargetz and Norbert Ortner. Characterization of L. Schwartz' convolutor and multiplier spaces $\mathscr{O}'_C$ and $\mathscr{O}_M$ by the short-time Fourier transform. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, 108(2):833–847, 2014.

[5] Richard Beals. Characterization of pseudodifferential operators and applications. *Duke Math. J.*, 44(1):45–57, 1977.

[6] Edward Bierstone. Differentiable functions. *Bol. Soc. Brasil. Mat.*, 11(2):139–189, 1980.

[7] Jonas Brinker and Jens Wirth. Gelfand triples for the Kohn-Nirenberg quantization on homogeneous lie groups. In *Advances in harmonic analysis and partial differential equations*, Trends Math., pages 51–97. Birkhäuser/Springer, Cham, 2020.

[8] Ian D. Brown. Dual topology of a nilpotent Lie group. *Ann. Sci. École Norm. Sup. (4)*, 6:407–411, 1973.

[9] Rodrigo A. H. M. Cabral and Severino T. Melo. Operators with analytic orbit for the torus action. *Studia Math.*, 243(3):243–250, 2018.

[10] Marco Cappiello and Luigi Rodino. SG-pseudodifferential operators and Gelfand-Shilov spaces. *Rocky Mountain J. Math.*, 36(4):1117–1148, 2006.

[11] Marco Cappiello and Joachim Toft. Pseudo-differential operators in a Gelfand-Shilov setting. *Math. Nachr.*, 290(5-6):738–755, 2017.

[12] Pierre Cartier. Vecteurs différentiables dans les représentations unitaires des groupes de Lie. In *Séminaire Bourbaki (1974/1975), Exp. No. 454*, pages 20–34. Lecture Notes in Math., Vol. 514. 1976.

[13] Ion Colojoară. Ultradifferentiable representations. *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie*, 41 (89)(1):1–14, 1998.

[14] Elena Cordero, Stevan Pilipović, Luigi G. Rodino, and Nenad Teofanov. Quasianalytic Gelfand-Shilov spaces with application to localization operators. *The Rocky Mountain Journal of Mathematics*, 40(4):1123–1147, 2010.

[15] H. O. Cordes. *The technique of pseudodifferential operators*, volume 202 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1995.

[16] Lawrence J. Corwin and Frederick P. Greenleaf. *Representations of nilpotent Lie groups and their applications. Part I*, volume 18 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990. Basic theory and examples.

[17] Aparajita Dasgupta and Michael Ruzhansky. Gevrey functions and ultradistributions on compact Lie groups and homogeneous spaces. *Bull. Sci. Math.*, 138(6):756–782, 2014.

[18] Aparajita Dasgupta and Michael Ruzhansky. Eigenfunction expansions of ultradifferentiable functions and ultradistributions. *Trans. Amer. Math. Soc.*, 368(12):8481–8498, 2016.

[19] Aparajita Dasgupta and Michael Ruzhansky. Eigenfunction expansions of ultradifferentiable functions and ultradistributions. II. Tensor representations. *Trans. Amer. Math. Soc. Ser. B*, 5:81–101, 2018.

[20] Jacques Dixmier. *$C^*$-algebras*. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. Translated from the French by Francis Jellett, North-Holland Mathematical Library, Vol. 15.

[21] A. Dynin. Pseudodifferential operators on Heisenberg groups. In *Pseudodifferential operator with applications (Bressanone, 1977)*, pages 5–18. Liguori, Naples, 1978.

[22] J. M. G. Fell. The dual spaces of $C^*$-algebras. *Trans. Amer. Math. Soc.*, 94:365–403, 1960.

[23] Véronique Fischer. Intrinsic pseudo-differential calculi on any compact Lie group. *J. Funct. Anal.*, 268(11):3404–3477, 2015.

[24] Veronique Fischer and Michael Ruzhansky. *Quantization on nilpotent Lie groups*, volume 314 of *Progress in Mathematics*. Birkhäuser/Springer, [Cham], 2016.

[25] Gerald B. Folland. *Harmonic analysis in phase space*, volume 122 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989.

[26] Gerald B. Folland. Meta-Heisenberg groups. In *Fourier analysis (Orono, ME, 1992)*, volume 157 of *Lecture Notes in Pure and Appl. Math.*, pages 121–147. Dekker, New York, 1994.

[27] Gerald B. Folland. *A course in abstract harmonic analysis*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, second edition, 2016.

[28] Hartmut Führ. *Abstract harmonic analysis of continuous wavelet transforms*, volume 1863 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2005.

[29] I. M. Ge'fand and G. E. Shilov. *Generalized functions. Vol. 2. Spaces of fundamental and generalized functions.* Translated from the Russian by Morris D. Friedman, Amiel Feinstein and Christian P. Peltzer. Academic Press, New York-London, 1968.

[30] I. M. Gel'fand and N. Ya. Vilenkin. *Generalized functions. Vol. 4: Applications of harmonic analysis.* Translated by Amiel Feinstein. Academic Press, New York - London, 1964.

[31] Daryl Geller. Fourier analysis on the Heisenberg group. I. Schwartz space. *J. Functional Analysis*, 36(2):205–254, 1980.

[32] Loukas Grafakos. *Modern Fourier analysis*, volume 250 of *Graduate Texts in Mathematics.* Springer, New York, third edition, 2014.

[33] Karlheinz Gröchenig and David Rottensteiner. Orthonormal bases in the orbit of square-integrable representations of nilpotent Lie groups. *J. Funct. Anal.*, 275(12):3338–3379, 2018.

[34] Alexandre Grothendieck. Produits tensoriels topologiques et espaces nucléaires. *Mem. Amer. Math. Soc.*, 16:Chapter 1: 196 pp.; Chapter 2: 140, 1955.

[35] A. Ya. Helemskii. *Banach and locally convex algebras.* Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993. Translated from the Russian by A. West.

[36] Winfried Kaballo. *Aufbaukurs Funktionalanalysis und Operatortheorie.* Springer Spektrum, Berlin, 2014. Distributionen—lokalkonvexe Methoden—Spektraltheorie [Distributions, locally convex methods, spectral theory].

[37] Richard V. Kadison and John R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. I*, volume 15 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 1997. Elementary theory, Reprint of the 1983 original.

[38] Andreas Klotz. Inverse closed ultradifferential subalgebras. *J. Math. Anal. Appl.*, 409(2):615–629, 2014.

[39] Hikosaburo Komatsu. Projective and injective limits of weakly compact sequences of locally convex spaces. *J. Math. Soc. Japan*, 19:366–383, 1967.

[40] Hikosaburo Komatsu. Ultradistributions. I. Structure theorems and a characterization. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 20:25–105, 1973.

[41] Hikosaburo Komatsu. Ultradistributions. II. The kernel theorem and ultradistributions with support in a submanifold. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 24(3):607–628, 1977.

[42] Hikosaburo Komatsu. Ultradistributions. III. Vector-valued ultradistributions and the theory of kernels. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 29(3):653–717, 1982.

[43] Gottfried Köthe. *Topological vector spaces. I.* Translated from the German by D. J. H. Garling. Die Grundlehren der mathematischen Wissenschaften, Band 159. Springer-Verlag New York Inc., New York, 1969.

[44] Gottfried Köthe. *Topological vector spaces. II*, volume 237 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science]*. Springer-Verlag, New York-Berlin, 1979.

[45] Andreas Kriegl, Peter W. Michor, and Armin Rainer. The convenient setting for non-quasianalytic Denjoy-Carleman differentiable mappings. *J. Funct. Anal.*, 256(11):3510–3544, 2009.

[46] Andreas Kriegl, Peter W. Michor, and Armin Rainer. The convenient setting for quasianalytic Denjoy-Carleman differentiable mappings. *J. Funct. Anal.*, 261(7):1799–1834, 2011.

[47] Andreas Kriegl, Peter W. Michor, and Armin Rainer. The convenient setting for Denjoy-Carleman differentiable mappings of Beurling and Roumieu type. *Rev. Mat. Complut.*, 28(3):549–597, 2015.

[48] Michael Langenbruch. Hermite functions and weighted spaces of generalized functions. *Manuscripta Math.*, 119(3):269–285, 2006.

[49] Julian Larcher. Multiplications and convolutions in L. Schwartz' spaces of test functions and distributions and their continuity. *Analysis (Berlin)*, 33(4):319–332, 2013.

[50] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[51] B. Maissen. Über Topologien im Endomorphismenraum eines topologischen Vektorraumes. *Math. Ann.*, 151:283–285, 1963.

[52] Anastasios Mallios. *Topological algebras. Selected topics*, volume 124 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1986. Notas de Matemática [Mathematical Notes], 109.

[53] Calvin C. Moore and Joseph A. Wolf. Square integrable representations of nilpotent groups. *Trans. Amer. Math. Soc.*, 185:445–462 (1974), 1973.

[54] M. Măntoiu and M. Ruzhansky. Quantizations on nilpotent Lie groups and algebras having flat coadjoint orbits. *J. Geom. Anal.*, 29(3):2823–2861, 2019.

[55] Edward Nelson. Analytic vectors. *Ann. of Math. (2)*, 70:572–615, 1959.

[56] Niels Vigand Pedersen. Geometric quantization and the universal enveloping algebra of a nilpotent Lie group. *Trans. Amer. Math. Soc.*, 315(2):511–563, 1989.

[57] Niels Vigand Pedersen. Matrix coefficients and a Weyl correspondence for nilpotent Lie groups. *Invent. Math.*, 118(1):1–36, 1994.

[58] David Rottensteiner. *Time-frequency analysis on the Heisenberg groups.* PhD thesis, Imperial College London, 2014.

[59] Michael Ruzhansky and Ville Turunen. *Pseudo-differential operators and symmetries*, volume 2 of *Pseudo-Differential Operators. Theory and Applications.* Birkhäuser Verlag, Basel, 2010. Background analysis and advanced topics.

[60] Raymond A. Ryan. *Introduction to tensor products of Banach spaces.* Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2002.

[61] H. H. Schaefer and M. P. Wolff. *Topological vector spaces*, volume 3 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1999.

[62] Laurent Schwartz. Espaces de fonctions différentiables à valeurs vectorielles. *J. Analyse Math.*, 4:88–148, 1954/55.

[63] Laurent Schwartz. Théorie des distributions à valeurs vectorielles. I. *Ann. Inst. Fourier (Grenoble)*, 7:1–141, 1957.

[64] Laurent Schwartz. Théorie des distributions à valeurs vectorielles. II. *Ann. Inst. Fourier (Grenoble)*, 8:1–209, 1958.

[65] Laurent Schwartz. *Théorie des distributions.* Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X. Nouvelle édition, entiérement corrigée, refondue et augmentée. Hermann, Paris, 1966.

[66] François Trèves. *Topological vector spaces, distributions and kernels.* Academic Press, New York-London, 1967.

[67] Johannes Ueberberg. Zur Spektralinvarianz von Algebren von Pseudodifferentialoperatoren in der $L^p$-Theorie. *Manuscripta Math.*, 61(4):459–475, 1988.

[68] Hassler Whitney. Analytic extensions of differentiable functions defined in closed sets. *Trans. Amer. Math. Soc.*, 36(1):63–89, 1934.

[69] Hassler Whitney. Functions differentiable on the boundaries of regions. *Ann. of Math. (2)*, 35(3):482–485, 1934.