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RESEARCH ARTICLE

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On the limit of a two-phase flow problem in thin porous media domains of Brinkman type

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We study the process of two-phase flow in thin porous media domains of Brinkman type. This is generally described by a model of coupled, mixed-type differential equations of fluids' saturation and pressure. To reduce the model complexity, different approaches that utilize the thin geometry of the domain have been suggested. We focus on a reduced model that is formulated as a single nonlocal evolution equation of saturation. It is derived by applying standard asymptotic analysis to the dimensionless coupled model; however, a rigid mathematical derivation is still lacking. In this paper, we prove that the reduced model is the analytical limit of the coupled two-phase flow model as the geometrical parameter of domain's width–length ratio tends to zero. Precisely, we prove the convergence of weak solutions for the coupled model to a weak solution for the reduced model as the geometrical parameter approaches zero.

KEYWORDS

Brinkman regimes, mathematical convergence, model reduction in thin domains, two-phase flow, weak solution

MSC CLASSIFICATION

00**A**69

1 | INTRODUCTION

We study the process of fluid displacement by another fluid in nondeformable saturated porous media domains of thin structure. This is crucial for many environmental and industrial applications. Examples are enhanced oil recovery in oil reservoirs and carbon dioxide sequestration in saline aquifers. Such processes are typically described by the two-phase flow model, which is a coupled system of mixed-type differential equations.[1] The complexity of the model and the large volume of such domains in the subsurface lead to high computational complexity. However, different approaches that utilize the thin geometry of these domains have been suggested to reduce the model's complexity. An example is the dimensional reduction approach by vertical integration in the field of petroleum studies,[2] hydrogeology,[1,3] and carbon dioxide sequestration.[4,5] Another example is the asymptotic approach for two-phase flows in Darcy[6] and Brinkman regimes[7] and for single-phase flow in deformable porous materials.[8] We also refer to the multiscale model approach in Guo et al.[9] A comparative study on the accuracy and efficiency of the asymptotic approach over the others is preformed in Armiti-Juber and Rohde,[7] where also an equivalence result with the multiscale approach is proved. A recent approach suggests an adaptive algorithm that couples the dimensional reduction approach with the full model.[10] It is based on a local criterion that determines the applicability of the reduced model.

In this paper, we provide a mathematically rigorous derivation of the reduced model from the asymptotic approach in Armiti-Juber and Rohde[7] by exploring the limit of the full two-phase flow model in porous media domains of Brinkman type as the width–length ratio of the domain tends to zero. Precisely, we prove the analytical convergence of weak solutions for the full model to weak solutions for the reduced model. Such mathematically rigorous derivations of reduced models have been applied for other different applications. As an example, we refer to the derivation of a Richards-like equation by investigating the analytical limit of the two-phase flow model when the viscosity of the displaced fluid is considered as a vanishing parameter.[11] Another example is the derivation of a reduced model as the limit of a classical model that describes crystal dissolution in thin strips by letting the thickness tend to zero.[12] We refer also to the rigorously derived reduced models in fractured porous media as the thickness of the fractures approaches zero for both single-phase[13–15] and unsaturated flows.[16] Other reduced models for fractured porous media have been derived in previous works[17–21] using formal dimensionality reduction techniques, such averaging across the fracture thickness or asymptotic analysis.

We consider the homogenized flow of two incompressible immiscible fluids in the rectangular domain $\Omega_\gamma = (0, L) \times (0, H)$ such that $H \ll L$ (Figure 1), where $\gamma := H/L$ is the geometrical parameter. Using dimensionless variables, governing equations for such flows are given by the so-called Brinkman two-phase flow model (BTP model),

$$
\begin{aligned}
\partial_t S + \partial_x \left(f(S)U \right) + \partial_z \left(f(S)Q \right) - \beta_1 \partial_{txx} S - \beta_2 \partial_{tzz} S = 0,& \\
U = -\lambda_{tot}(S) \partial_x p,& \\
\gamma^2 Q = -\lambda_{tot}(S) \partial_z p,& \\
\partial_x U + \partial_z Q = 0&
\end{aligned}
\tag{1}
$$

in the dimensionless domain $\Omega \times (0, T)$, where $\Omega = (0, 1) \times (0, 1)$ and $T > 0$.[7] We refer to Appendix A1 for details on the derivation of this model. The unknowns here are the saturation $S = S(x, z, t) \in [0, 1]$ of the wetting (or invading) phase and the global pressure $p = p(x, z, t) \in \mathbb{R}$. The component $U = U(x, z, t) \in \mathbb{R}$, for any $(x, z, t) \in \Omega \times (0, T)$ is the horizontal velocity and $Q = Q(x, z, t) \in \mathbb{R}$ is the vertical one. The total mobility function $\lambda_{tot} = \lambda_{tot}(S) \in (0, \infty)$ is the mobility sum of both phases. We refer to[1] for possible choices for the mobilities. The function $f = f(S) \in [0, 1]$ is the given fractional flow function, which is determined using the fluids' mobilities and viscosities. The parameters β_1 and β_2 determine the flow regime. The case $\beta_1 = \beta_2 = 0$ results in the so-called Darcy regime. Otherwise, it is called Brinkman regime.[1]

The reduced model resulting from the asymptotic approach in Brinkman regimes is derived in Armiti-Juber and Rohde.[7] It is a nonlocal nonlinear evolution equation of saturation. For details on the derivation of this model we refer to Appendix B1. The model is given as

$$
\partial_t S + \partial_x \left(f(S)U[S] \right) + \partial_z \left(f(S)Q[S] \right) - \beta_1 \partial_{txx} S - \beta_2 \partial_{tzz} S = 0 \tag{2}
$$

in $\Omega \times (0, T)$, with the velocity components

$$
U[S] = \frac{\hat{U}_{\text{inflow}} \lambda_{tot}(S)}{\int_0^1 \lambda_{tot}(S) dz}, \quad Q[S] = -\partial_x \int_0^z U[S(\cdot, r, \cdot)] dr. \tag{3}
$$

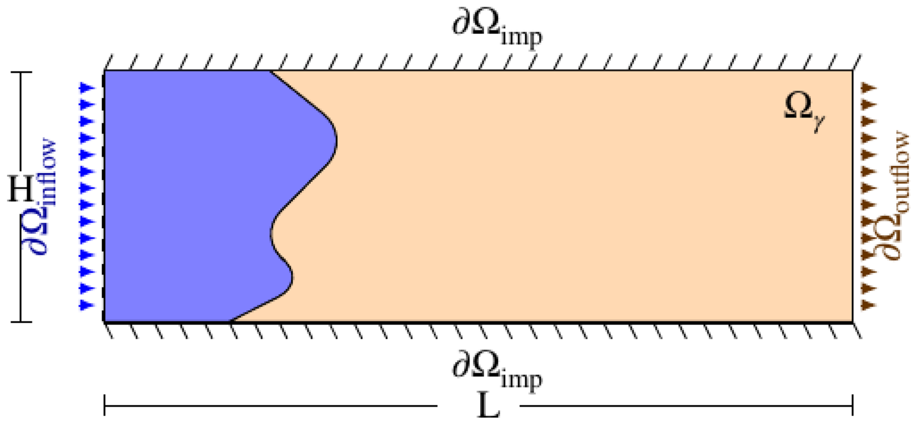


FIGURE 1 An illustration of the displacement process of a wetting phase into a thin domain Ω_γ[7] [Colour figure can be viewed at wileyonlinelibrary.com]

Here, $\hat{U}_{\text{inflow}} = \hat{U}_{\text{inflow}}(t)$ is the vertically averaged horizontal velocity at the left boundary of the domain. This inflow velocity can be evaluated using a one-dimensional elliptic equation of the vertically averaged pressure $\hat{p} = \int_0^1 p(.,z,.)\,dz$. However, it is set in Yortsos[6] to be a constant $\hat{U}_{\text{inflow}} = 1$. In Armiti-Juber and Rohde[7] it is eliminated from the model by rescaling the time t variable using $t \mapsto \bar{t} = \int_0^t \hat{U}_{\text{inflow}}(r)dr + \hat{U}_{\text{inflow}}(0)t$. In addition, the definition of the velocity components U and Q in (3) still fulfills the incompressibility constraint

$$\partial_x U + \partial_z Q = 0. \tag{4}$$

Equations 2 and (3) are called here as in Armiti-Juber and Rohde,[7] the Brinkman Vertical Equilibrium model (BVE model). This model is a proper reduction of the full BTP model (1) in thin domains as it describes the vertical dynamics in the domain. Moreover, it is computationally more efficient than the full mixed BTP model for saturation and global pressure (see Armiti-Juber & Rohde[7]). This is a consequence of the velocity equations in (3) computed from saturation directly, without solving an elliptic equation for the global pressure as in full BTP model (1).

The main goal of this paper is a rigid mathematical derivation of the reduced BVE model (2) and (3) from the full BTP model (1). We do this by proving that the reduced model is the analytical limit of the full BTP model in domains with vanishing width–length ratio γ. For the well-posedness of the models we refer to Coclite et al[22] and Armiti-Juber and Rohde,[23] where the existence of weak solutions for the BTP model and the BVE model, respectively, is proved. Other models with a similar structure to the BTP and BVE models have been developed to describe the two-phase flow including dynamic effects and/or hysteresis in the phase-pressure difference. The well-posedness of such models has been proved in previous works[24,25] for saturated porous media and in previous works[26,27] for unsaturated ones.

This paper is structured as follows. In Section 2 we choose the initial and boundary conditions that fit to the two-phase displacement process. Then, we give the definitions of weak solutions for the full BTP model (1) and the reduced BVE model (2) and (3). After that, Section 3 proves a set a priori estimates on a sequence of weak solutions for the BTP model. These are essential to prove the convergence of the sequence in Section 4 as the ratio γ approaches zero. Section 5 presents an example that shows the numerical convergence of full BTP model to the reduced BVE model as γ vanishes. Section 6 concludes the paper. Finally, the derivation of the dimensionless BTP model (1) is summarized in Appendix A1, while the derivation of the BVE model (2), (3) using the asymptotic approach as in Armiti-Juber and Rohde[7] is summarized in Appendix B1.

2 | PRELIMINARIES

In this section, we give the initial and boundary conditions associated with the displacement process in the dimensionless domain Ω. Then we provide the definition of weak solution for the BTP model and the BVE model.

The BTP model is closed with the initial and boundary conditions

$$\begin{aligned}
S(\cdot,\cdot,0) &= S^0 && \text{in } \Omega,\\
S &= S_{\text{inflow}} && \text{on } \partial\Omega_{\text{inflow}} \times [0,T],\\
S &= 0 && \text{on } \partial\Omega_{\text{imp}} \cup \partial\Omega_{\text{outflow}} \times [0,T],\\
\nabla p \cdot \mathbf{n} &= q && \text{on } \partial\Omega \times [0,T],\\
\int_\Omega p(x,z,t)\,dx\,dz &= 0 && \text{on } t \in (0,T),\\
p &= p_D && \text{on } \partial\Omega_{\text{imp}} \times [0,T],\\
Q &= 0 && \text{on } \partial\Omega_{\text{imp}} \times [0,T],
\end{aligned} \tag{5}$$

where $S_{\text{inflow}} = S_{\text{inflow}}(z)$ and $q = q(x,z,t)$ are given functions and p_D is a constant. Note that $\partial\Omega_{\text{inflow}} = \{0\} \times (0,1)$ is the inflow boundary, $\Omega_{\text{outflow}} = \{1\} \times (0,1)$ is the outflow boundary, and $\Omega_{\text{imp}} = (0,1) \times \{0,1\}$ corresponds to the impermeable lower and upper boundaries (Figure 1). We also use the notations $\Omega_T = \Omega \times (0,T)$ and impose the following assumptions.

Assumption 1.

1. The bounded domain $\Omega \subset \mathbb{R}^2$ has a Lipschitz continuous boundary $\partial\Omega$ and $0 < T < \infty$.
2. The inflow saturation satisfies $S_{\text{inflow}}(., t) \in H^{1/2}(\Omega_{\text{inflow}})$ for any $t \in [0, T]$ with $S_{\text{inflow}}(0, z, t) \in [0, 1]$ for almost every $z \in [0, 1]$ and every $t \in [0, T]$.
3. We require $S^0 \in H^1(\Omega)$ and $S^0(x, z) \in [0, 1]$ for almost all $(x, z) \in \Omega$. In addition, it satisfies $S^0 = S_{\text{inflow}}$ at $\partial\Omega_{\text{inflow}}$.
4. The function q satisfies $q \in L^2(\partial\Omega \times (0, T))$.
5. The fractional flow function $f \in C^1((0, 1))$ is Lipschitz continuous, bounded, nonnegative, monotone increasing and $f(0) = 0$, such that there exist numbers $M, L > 0$ with $f \le M, f' \le L$.
6. The total mobility function $\lambda_{tot} \in C^1((0, 1))$ is Lipschitz continuous, bounded and strictly positive, such that there exist numbers $a, M, L > 0$ with $0 < a < \lambda_{tot} \le M$ and $|\lambda'_{tot}| \le L$.

Remark 1.

1. Assumption 1(5) and 1(6) on the fractional flow function f and the total mobility λ_{tot}, respectively, are naturally satisfied by their definitions, see, for example, Helmig.[1]
2. The boundary condition $Q = 0$ at $\partial\Omega_{\text{imp}}$ in Equation (5) and the strict positivity of the total mobility λ_{tot} imply that the function q has to satisfy the compatibility condition

$$q = 0 \quad \text{at} \quad \partial\Omega_{\text{imp}}. \tag{6}$$

3. Note that the velocity components at the boundaries of the domain can be evaluated using the velocity equations in (1) and the boundary conditions on saturation and pressure in (5) in the sense of traces. For example, we define the velocity $U_{\text{inflow}} = U_{\text{inflow}}(z, t)$ at the inflow boundary as

$$U_{\text{inflow}} = \lambda_{tot}(S_{\text{inflow}}) q|_{\partial\Omega_{\text{inflow}}}, \tag{7}$$

and the velocity $U_{\text{outflow}} = U_{\text{outflow}}(z, t)$ at the outflow boundary as

$$U_{\text{outflow}} = \lambda_{tot}(0) q|_{\partial\Omega_{\text{outflow}}}. \tag{8}$$

Using Assumption 1(4) and 1(6), we have $U_{\text{inflow}}, U_{\text{outflow}} \in L^2((0, 1) \times (0, T))$. In addition, the constant pressure p_D at the boundary $\partial\Omega_{\text{imp}}$ leads to

$$U = 0 \text{ on } \partial\Omega_{\text{imp}}. \tag{9}$$

4. The incompressibility Equation (1)(d) and the zero outflow $Q = 0$ at the upper and lower boundaries $\partial\Omega_{\text{imp}}$ in Equation (5)(g) imply that the horizontal velocities U_{inflow} and U_{outflow} at the left and right boundaries, respectively, satisfy the compatibility condition

$$\int_0^1 (U_{\text{outflow}} - U_{\text{inflow}})\, dz = 0.$$

Consequently, the flux function q at the boundaries $\partial\Omega_{\text{inflow}}$ and $\partial\Omega_{\text{outflow}}$ has also to satisfy the compatibility condition

$$\int_0^1 \lambda_{tot}(S_{\text{inflow}}) q|_{\partial\Omega_{\text{inflow}}}\, dz = \int_0^1 \lambda_{tot}(0) q|_{\partial\Omega_{\text{outflow}}}\, dz.$$

Definition 1. For any $\gamma > 0$, we call $(S^\gamma, p^\gamma, U^\gamma, Q^\gamma)$ a weak solution of the BTP model (1) with the initial and boundary conditions (5) if

1. $S^\gamma \in H^1(0,T;H^1(\Omega))$, $p^\gamma \in L^2(0,T;H^1(\Omega))$, and $U^\gamma, Q^\gamma \in L^2(\Omega\times(0,T))$ with

$$\begin{aligned}\int_0^T\int_\Omega (\partial_t S^\gamma \phi - f(S^\gamma)U^\gamma\partial_x\phi - f(S^\gamma)Q^\gamma\partial_z\phi)\,dx\,dz\,dt + \beta_1\int_0^T\int_\Omega \partial_{tx}S^\gamma\partial_x\phi\,dx\,dz\,dt + \beta_2\int_0^T\int_\Omega \partial_{tz}S^\gamma\partial_z\phi\,dx\,dz\,dt \\ = \int_0^T\int_{\partial\Omega_{\mathrm{inflow}}} f(S^\gamma_{\mathrm{inflow}})U^\gamma_{\mathrm{inflow}}\phi|_{\partial\Omega_{\mathrm{inflow}}}\,dz\,dt,\end{aligned} \tag{10}$$

for any test function $\phi \in L^2(0,T;H^1(\Omega))$.

2. The velocity components satisfy

$$\int_\Omega U^\gamma\psi\,dx\,dz = -\int_\Omega \lambda_{tot}(S^\gamma)\partial_x p^\gamma\psi\,dx\,dz, \tag{11}$$

and

$$\gamma^2\int_\Omega Q^\gamma\psi\,dx\,dz = -\int_\Omega \lambda_{tot}(S^\gamma)\partial_z p^\gamma\psi\,dx\,dz, \tag{12}$$

for any test function $\psi \in L^2(\Omega)$ and almost everywhere in $(0,T)$.

3. The following two weak incompressibility relations

$$\int_\Omega \lambda_{tot}(S^\gamma)\partial_x p^\gamma\partial_x\theta + \frac{1}{\gamma^2}\lambda_{tot}(S^\gamma)\partial_z p^\gamma\partial_z\theta\,dx\,dz = \int_{\partial\Omega}\lambda_{tot}(S^\gamma)q\theta\,d\sigma, \tag{13}$$

and

$$\int_\Omega (U^\gamma\partial_x\theta + Q^\gamma\partial_z\theta)\,dx\,dz = \int_0^1\left(U_{\mathrm{outflow}}\theta|_{\partial\Omega_{\mathrm{outflow}}} - U_{\mathrm{inflow}}\theta|_{\partial\Omega_{\mathrm{inflow}}}\right)dz, \tag{14}$$

hold for any test function $\theta \in H^1(\Omega)$ almost everywhere in $(0,T)$.

4. $S^\gamma(.,.,0) = S^0$ almost everywhere in Ω.

Remark 2. Definition 1 implies that weak solutions for the BTP model (1) satisfy

$$S^\gamma \in C([0,T];H^1(\Omega)). \tag{15}$$

Definition 2. A function $S \in H^1(0,T;H^1(\Omega))$ is called a weak solution of the BVE model (2), (3) and (4) with the initial and boundary conditions (5) whenever the following conditions are fulfilled,

1. $U[S], Q[S] \in L^2(\Omega_T)$ and

$$\begin{aligned}\int_0^T\int_\Omega (\partial_t S\phi - f(S)U[S]\partial_x\phi - f(S)Q[S]\partial_z\phi)\,dx\,dz\,dt + \beta_1\int_0^T\int_\Omega \partial_{tx}S\partial_x\phi\,dx\,dz\,dt + \beta_2\int_0^T\int_\Omega \partial_{tz}S\partial_z\phi\,dx\,dz\,dt \\ = \int_0^T\int_{\partial\Omega_{\mathrm{inflow}}} f(S_{\mathrm{inflow}})U_{\mathrm{inflow}}\phi|_{\partial\Omega_{\mathrm{inflow}}}\,dz\,dt,\end{aligned} \tag{16}$$

holds for all test functions $\phi \in L^2(0,T;H^1(\Omega))$.

2. The velocity components satisfy

$$\int_{\Omega} U\psi \, dx\, dz = \int_{\Omega} \frac{\hat{U}_{\text{inflow}} \lambda_{tot}(S)}{\int_0^1 \lambda_{tot}(S) dz} \psi \, dx\, dz, \tag{17}$$

$$\int_{\Omega} Q\psi \, dx\, dz = -\int_{\Omega} \partial_x \int_0^z U[S(\cdot, r, \cdot)] dr \psi \, dx\, dz, \tag{18}$$

for any $\psi \in L^2(\Omega)$ and almost everywhere in $(0, T)$.

3. The weak incompressibility property

$$\int_{\Omega} (U[S]\partial_x \theta + Q[S]\partial_z \theta)\, dx\, dz = \int_0^1 \left(U_{\text{outflow}} \theta|_{\partial\Omega_{\text{outflow}}} - U_{\text{inflow}} \theta|_{\partial\Omega_{\text{inflow}}} \right) dz, \tag{19}$$

holds for all test functions $\theta \in H^1(\Omega)$ almost everywhere in time.

4. $S(.,.,0) = S^0$ almost everywhere in Ω.

Remark 3. Note that most of the boundary terms in Equation (10) and (16) cancel out due to the choice of boundary conditions (5)(b), (c), and (g). To demonstrate this, we refer to the time-independent boundary function S_{inflow} at $\partial\Omega_{\text{inflow}}$ and $S = 0$ (with $f(S) = f(0) = 0$) at $\partial\Omega_{\text{imp}}$ and $\partial\Omega_{\text{outflow}}$. Moreover, we have $Q = 0$ at $\partial\Omega_{\text{imp}}$.

3 | A PRIORI ESTIMATES

In the following, we prove a set of a priori estimates on the components of the sequence of weak solutions $\{(S^\gamma, p^\gamma, U^\gamma, Q^\gamma)\}_{\gamma>0}$ for the BTP model (1). These are essential for the convergence analysis as γ tends to zero in the next section.

Lemma 1. *Let $\{(S^\gamma, p^\gamma, U^\gamma, Q^\gamma)\}_{\gamma>0}$ be a sequence of weak solutions for the BTP model (1). If Assumption 1 holds, then the sequence $\{S^\gamma\}_{\gamma>0}$ satisfies the estimate*

$$\sup_{t\in[0,T]} \left(\|S^\gamma(t)\|_{L^2(\Omega)} + \beta_1 \|\partial_x S^\gamma(t)\|_{L^2(\Omega)} + \beta_2 \|\partial_z S^\gamma(t)\|_{L^2(\Omega)} \right) \leq \|S^0\|^2_{L^2(\Omega)} + \beta_1 \|\partial_x S^0\|^2_{L^2(\Omega)} + \beta_2 \|\partial_z S^0\|^2_{L^2(\Omega)} + C_{inflow},$$

where C_{inflow} is a constant depending on the data at the inflow boundary only.

Proof. We choose the test function $\phi = S^\gamma \chi_{[0,t)}$ in Equation (10), where $\chi_{[0,t)}$ is the characteristic function and $t \in (0, T]$ is arbitrary. Then, we obtain

$$\begin{aligned} &\int_0^t \int_{\Omega} (\partial_t S^\gamma S^\gamma - f(S^\gamma) U^\gamma \partial_x S^\gamma - f(S^\gamma) Q^\gamma \partial_z S^\gamma)\, dx\, dz\, dt + \beta_1 \int_0^t \int_{\Omega} \partial_{tx} S^\gamma \partial_x S^\gamma \, dx\, dz\, dt + \beta_2 \int_0^t \int_{\Omega} \partial_{tz} S^\gamma \partial_z S^\gamma \, dx\, dz\, dt \\ &\quad = \int_0^t \int_{\partial\Omega_{\text{inflow}}} f(S^\gamma_{\text{inflow}}) U^\gamma_{\text{inflow}} S^\gamma_{\text{inflow}} \, dz\, dt. \end{aligned} \tag{20}$$

Using the incompressibility relation (14) and the boundary condition on the outflow boundary, the second and third terms on the left side of the above equation satisfy,

$$\int_0^t \int_{\Omega} f(S^\gamma) \mathbf{V}^\gamma \cdot \nabla S^\gamma \, dx\, dz\, dt = \int_0^t \int_{\Omega} \mathbf{V}^\gamma \cdot \nabla F(S^\gamma)\, dx\, dz\, dt, = -\int_0^t \int_{\partial\Omega_{\text{inflow}}} U_{\text{inflow}} F(S_{\text{inflow}})\, dz\, dt, \tag{21}$$

where $\mathbf{V}^\gamma = (U^\gamma, Q^\gamma)^T$ and $F(S) = \int_0^S f(q)dq$. Integrating the first term on the left side of (20) and using the time-continuity of S^γ in Remark 2, we obtain

$$\int_0^t \int_\Omega (\partial_t S^\gamma S^\gamma \, dx dz dt = \frac{1}{2}\int_0^t \int_\Omega \partial_t (S^\gamma)^2 \, dx dz dt = \frac{1}{2}\left(\|S^\gamma(t)\|^2_{L^2(\Omega)} - \|S^0\|^2_{L^2(\Omega)}\right). \tag{22}$$

In the same way, we have

$$\beta_1 \int_0^t \int_\Omega \partial_{tx} S^\gamma \partial_x S^\gamma \, dx dz dt = \frac{\beta_1}{2}\left(\|\partial_x S^\gamma(t)\|^2_{L^2(\Omega)} - \|\partial_x S^0\|^2_{L^2(\Omega)}\right), \tag{23}$$

and

$$\beta_2 \int_0^t \int_\Omega \partial_{tz} S^\gamma \partial_z S^\gamma \, dx dz dt = \frac{\beta_2}{2}\left(\|\partial_z S^\gamma(t)\|^2_{L^2(\Omega)} - \|\partial_z S^0\|^2_{L^2(\Omega)}\right). \tag{24}$$

Substituting Equations (21)–(24) into (20) yields

$$\begin{aligned}\sup_{t\in[0,T]} \left(\frac{1}{2}\|S^\gamma(t)\|^2_{L^2(\Omega)} + \frac{\beta_1}{2}\|\partial_x S^\gamma(t)\|^2_{L^2(\Omega)} + \frac{\beta_2}{2}\|\partial_z S^\gamma(t)\|^2_{L^2(\Omega)}\right) + \int_0^t \int_{\partial\Omega_{\text{inflow}}} U_{\text{inflow}} F(S_{\text{inflow}})\, dz dt \\ = \frac{1}{2}\|S^0\|^2_{L^2(\Omega)} + \frac{\beta_1}{2}\|\partial_x S^0\|^2_{L^2(\Omega)} + \frac{\beta_2}{2}\|\partial_z S^0\|^2_{L^2(\Omega)} + \int_0^t \int_{\partial\Omega_{\text{inflow}}} U_{\text{inflow}} f(S_{\text{inflow}}) S_{\text{inflow}}\, dz dt.\end{aligned} \tag{25}$$

The boundedness of S_{inflow}, U_{inflow}, and f by Assumption 1(2), 1(5), and 1(6), respectively, implies

$$\begin{aligned}\sup_{t\in[0,T]} \left(\|S^\gamma(t)\|_{L^2(\Omega)} + \beta_1\|\partial_x S^\gamma(t)\|_{L^2(\Omega)} + \beta_2\|\partial_z S^\gamma(t)\|_{L^2(\Omega)}\right) \\ \le \|S^0\|^2_{L^2(\Omega)} + \beta_1\|\partial_x S^0\|^2_{L^2(\Omega)} + \beta_2\|\partial_z S^0\|^2_{L^2(\Omega)} + C_{\text{inflow}},\end{aligned}$$

where $C_{\text{inflow}} = 2M\|U_{\text{inflow}}\|_{L^\infty(\partial\Omega_{\text{inflow}}\times(0,T))}\|S_{\text{inflow}}\|_{L^\infty(\partial\Omega_{\text{inflow}}\times(0,T))}$. □

The following lemma proves an estimate on the sequence of pressure's gradient. In the limit $\gamma \to 0$, the estimate is equivalent to the vertical equilibrium assumption (see, e.g., Guo et al.[9]). It is also essential to formulate the limit pressure as an operator of saturation.

Lemma 2. *Let $\{(S^\gamma, p^\gamma, U^\gamma, Q^\gamma)\}_{\gamma>0}$ be a sequence of weak solutions for the BTP model (1). If Assumption 1 holds, then there exists a constant $c > 0$, independent of the parameter γ, such that the sequence $\{p^\gamma\}_{\gamma>0}$ satisfies the estimate*

$$(1-\gamma^2)\|\partial_z p^\gamma\|^2_{L^2(\Omega)} + \gamma^2\|\partial_x p^\gamma\|^2_{L^2(\Omega)} \le \frac{2cM^2\gamma^2}{a^2}\|q\|^2_{L^2(\partial\Omega)}.$$

Proof. We choose the test function $\theta = p^\gamma$ in Equation (13), then we have

$$\int_\Omega \lambda_{tot}(S^\gamma)\left((\partial_x p^\gamma)^2 + \frac{1}{\gamma^2}(\partial_z p^\gamma)^2\right) dx dz = \int_{\partial\Omega} \lambda_{tot}(S^\gamma) q p^\gamma \, d\sigma.$$

Using Assumption 1(6) on the total mobility then applying Cauchy's inequality to the right side yields

$$a\|\partial_x p^\gamma\|^2_{L^2(\Omega)} + \frac{a}{\gamma^2}\|\partial_z p^\gamma\|^2_{L^2(\Omega)} \le \frac{M^2}{2\epsilon}\int_{\partial\Omega} q^2 \, d\sigma + \frac{\epsilon}{2}\int_{\partial\Omega} (p^\gamma)^2 \, d\sigma,$$

for any constant $\epsilon > 0$. Applying the Trace theorem to the second term on the right side, then using Poincaré's inequality with the zero mean condition on the pressure (see the boundary conditions in (5)) produces

$$a\|\partial_x p^\gamma\|^2_{L^2(\Omega)} + \frac{a}{\gamma^2}\|\partial_z p^\gamma\|^2_{L^2(\Omega)} \le \frac{M^2}{2\epsilon}\int_{\partial\Omega} q^2\, d\sigma + \frac{c\epsilon}{2}\|\nabla p^\gamma\|_{L^2(\Omega)},$$

where $c > 0$ is a constant resulting from the above two Sobolev embedding theorems. Choosing $\epsilon = \frac{a}{c}$ and noting that $\gamma < 1$, yields

$$\frac{a}{2}\gamma^2\|\partial_x p^\gamma\|^2_{L^2(\Omega)} + \frac{a}{2}(1-\gamma^2)\|\partial_z p^\gamma\|^2_{L^2(\Omega)} \le \frac{cM^2}{2a}\gamma^2\int_{\partial\Omega} q^2\, d\sigma.$$

This simplifies to

$$\gamma^2\|\partial_x p^\gamma\|^2_{L^2(\Omega)} + (1-\gamma^2)\|\partial_z p^\gamma\|^2_{L^2(\Omega)} \le \frac{cM^2\gamma^2}{a^2}\int_{\partial\Omega} q^2\, d\sigma,$$

which is the required estimate. □

Corollary 1. *If Assumption 1 holds, then there exists a constant $C > 0$, independent of the parameter γ, such that the velocity components U^γ and W^γ satisfy*

$$\|U^\gamma\|_{L^2(\Omega_T)} \le C\|q\|^2_{L^2(\partial\Omega_T)},$$
$$\|Q^\gamma\|_{L^2(\Omega_T)} \le \frac{C}{1-\gamma^2}\|q\|^2_{L^2(\partial\Omega_T)}.$$

Proof. The definition of U^γ and Lemma 2 implies that

$$\|U^\gamma\|_{L^2(\Omega_T)} = \int_0^T\int_\Omega |\lambda_{tot}(S^\gamma)\partial_x p^\gamma| \le M^2\|\partial_x p^\gamma\|_{L^2(\Omega_T)} \le C\|q\|^2_{L^2(\partial\Omega_T)},$$

where $C = \frac{cM^4}{a^2}$. Similarly, the component Q^γ satisfies

$$\begin{aligned}\gamma^2\|Q^\gamma\|_{L^2(\Omega_T)} = \int_0^T\int_\Omega |\lambda_{tot}(S^\gamma)\partial_z p^\gamma| &\le M^2\|\partial_z p^\gamma\|_{L^2(\Omega_T)}\\ &\le \frac{cM^4\gamma^2}{a^2(1-\gamma^2)}\|q\|^2_{L^2(\partial\Omega_T)}.\end{aligned}$$

Hence, we have

$$\|Q^\gamma\|_{L^2(\Omega_T)} \le \frac{C}{1-\gamma^2}\|q\|^2_{L^2(\partial\Omega_T)}.$$

□

In the following lemma we prove an estimate on the time-partial derivative of the weak solution S^γ and its derivative $\partial_x S^\gamma$.

Lemma 3. *Let $\{(S^\gamma, p^\gamma, U^\gamma, Q^\gamma)\}_{\gamma>0}$ be a sequence of weak solutions for the BTP model (1). If Assumption 1 holds, then there exists a constant $C > 0$, independent of the parameters γ and μ_e, such that the sequence $\{S^\gamma\}_{\gamma>0}$ satisfies the estimate*

$$\|\partial_t S^\gamma\|_{L^2(\Omega_T)} + \frac{3\beta_1}{4}\|\partial_{tx} S^\gamma\|_{L^2(\Omega_T)} + \frac{3\beta_2}{4}\|\partial_{tz} S^\gamma\|_{L^2(\Omega_T)} \le \frac{M^2}{\mu_e}\left(C + \frac{C}{1-\gamma^2}\right)\|q\|^2_{L^2(\partial\Omega_T)}.$$

Proof. We consider the weak formulation (10) in Definition 1 with the test function $\phi = \partial_t S^\gamma$. Then, using Cauchy's inequality we obtain

$$\int_0^T \int_\Omega (\partial_t S^\gamma)^2 + \beta_1 (\partial_{tx} S^\gamma)^2 + \beta_2 (\partial_{tz} S^\gamma)^2 \, dx\, dz\, dt = \int_0^T \int_\Omega f(S^\gamma)\left(U^\gamma \partial_{xt} S^\gamma + Q^\gamma \partial_{zt} S^\gamma\right) dx\, dz\, dt,$$
$$\leq \frac{1}{\beta_1} \| f(S^\gamma) U^\gamma \|^2_{L^2(\Omega)} + \frac{\beta_1}{4} \| \partial_{tx} S^\gamma \|^2_{L^2(\Omega)} + \frac{1}{\beta_2} \| f(S^\gamma) Q^\gamma \|^2_{L^2(\Omega)} + \frac{\beta_2}{4} \| \partial_{tz} S^\gamma \|^2_{L^2(\Omega)}.$$

Note that the term on the inflow boundary vanished as a result of the time-independent choice for the inflow saturation S_{inflow}. This reduces to

$$\| \partial_t S^\gamma \|_{L^2(\Omega_T)} + \frac{3\beta_1}{4} \| \partial_{tx} S^\gamma \|_{L^2(\Omega_T)} + \frac{3\beta_2}{4} \| \partial_{tz} S^\gamma \|_{L^2(\Omega_T)} \leq \frac{1}{\beta_1} \| f(S^\gamma) U^\gamma \|^2_{L^2(\Omega)} + \frac{1}{\beta_2} \| f(S^\gamma) Q^\gamma \|^2_{L^2(\Omega)}. \tag{26}$$

Now, using Corollary 1, we obtain

$$\| \partial_t S^\gamma \|_{L^2(\Omega_T)} + \frac{3\beta_1}{4} \| \partial_{tx} S^\gamma \|_{L^2(\Omega_T)} + \frac{3\beta_2}{4} \| \partial_{tz} S^\gamma \|_{L^2(\Omega_T)} \leq \frac{M^2}{\beta} \left(C + \frac{C}{1-\gamma^2} \right) \| q \|^2_{L^2(\partial \Omega_T)},$$

where $\beta = \min\{\beta_1, \beta_2\}$ and $C > 0$ is a constant defined as in Corollary 1. □

4 | CONVERGENCE ANALYSIS

In this section we prove the analytical convergence of the sequence of weak solutions $\{(S^\gamma, p^\gamma, U^\gamma, Q^\gamma)\}_{\gamma>0}$ for the BTP model (1) to a weak solution of the BVE model (2), (3) as the geometrical parameter γ tends to 0. The main result of paper is summarized in this theorem.

Theorem 1. *Let $\{(S^\gamma, p^\gamma, U^\gamma, Q^\gamma)\}_{\gamma>0}$ be a sequence of weak solutions for the BTP model (1) with the initial and boundary conditions (5). If Assumption 1 holds, then there exists a subsequence of the weak solutions $\{S^\gamma, p^\gamma, U^\gamma, Q^\gamma\}_{\gamma>0}$, denoted in the same way, and functions $S \in H^1(0, T; H^1(\Omega))$, $p \in L^2((0, T); H^1(0, 1))$, $U \in L^2(\Omega_T)$ and $Q \in L^2(\Omega_T)$ such that*

$$\begin{aligned} S^\gamma &\to S && \text{in } L^2(\Omega_T), \\ \nabla S^\gamma &\rightharpoonup \nabla S && \text{in } H^1(0, T; L^2(\Omega)), \\ p^\gamma &\rightharpoonup p && \text{in } L^2(0, T; H^1(\Omega)), \\ U^\gamma &\rightharpoonup U && \text{in } L^2(\Omega_T), \\ Q^\gamma &\rightharpoonup Q && \text{in } L^2(\Omega_T) \end{aligned}$$

as γ tends to zero. Further, the limit pressure p is independent of the z coordinate and satisfies $\partial_x p = -\frac{\hat{U}_{inflow}}{\int_0^1 \lambda_{tot}(S)\, dz}$. The functions S, U, Q satisfy the Equations 16, (17) and (18) in Definition 2, respectively.

Proof. The estimate in Lemma 1 implies the existence of a weakly convergent subsequence of $\{S^\gamma\}_{\gamma>0}$, denoted in the same way, and a function $S \in L^2(0, T; H^1(\Omega))$ with

$$S^\gamma \rightharpoonup S \ \text{ in } \ L^2(0, T; H^1(\Omega)), \tag{27}$$

as $\gamma \to 0$. In addition, the estimate in Lemma 3 implies

$$\nabla S^\gamma \rightharpoonup \nabla S \ \text{ in } \ H^1(0, T; L^2(\Omega)), \tag{28}$$

as $\gamma \to 0$. The Rellich-Kondrachov compactness theorem and the boundedness of the domain imply the embedding $H^1(0,T;H^1(\Omega)) \Subset L^2(\Omega_T)$. Thus, the weak convergence results (27) and (28) lead to the strong convergence

$$S^\gamma \to S \in L^2(\Omega_T). \tag{29}$$

This strong convergence and the a priori estimate from Lemma 1 imply that the limit S also satisfies

$$S,\ \nabla S \in L^\infty(0,T;H^1(\Omega)). \tag{30}$$

Moreover, we have

$$S \in C([0,T];H^1(\Omega)). \tag{31}$$

The strong convergence result in (29) and the Lipschitz continuity of f and λ_{tot} imply

$$\begin{aligned} f(S^\gamma) &\to f(S) \quad \text{in} \quad L^2(\Omega_T), \\ \lambda_{tot}(S^\gamma) &\to \lambda_{tot}(S) \quad \text{in} \quad L^2(\Omega_T). \end{aligned} \tag{32}$$

Now, we consider the estimate in Lemma 2 and let $\gamma \to 0$. Then, we have

$$\|\partial_z p^\gamma\|^2_{L^2(\Omega)} \to 0, \tag{33}$$

as $\gamma \to 0$. This, consequently, leads to the uniform estimate

$$\|\partial_x p^\gamma\|^2_{L^2(\Omega)} \le \frac{cM^2}{a^2}\|q\|^2_{L^2(\partial\Omega)}.$$

Hence, there exists a weakly convergent subsequence of $\{p^\gamma\}_{\gamma>0}$, denoted in the same way, and a z-independent function $p = p(x)$ with $p \in L^2(0,T;H^1((0,1)))$ such that

$$p^\gamma \rightharpoonup p \ \text{in} \ L^2(0,T;H^1(\Omega)). \tag{34}$$

This convergence result corresponds to the vertical equilibrium assumption for almost horizontal flows in thin domains.

The strong convergence of λ_{tot} in (32) and the weak convergence of p in (34) imply the weak convergence of $U^\gamma = \lambda_{tot}(S^\gamma)\partial_x p^\gamma$ to the limit $U = \lambda_{tot}(S)\partial_x p$ such that

$$U^\gamma \rightharpoonup U = \lambda_{tot}(S)\partial_x p \ \text{in} \ L^2(\Omega_T). \tag{35}$$

Corollary 1 implies the boundedness of Q^γ in $L^2(\Omega_T)$. Hence, up to a subsequence, there exists a function $Q \in L^2(\Omega_T)$ with

$$\int_0^T\int_\Omega Q^\gamma \phi \, dx\, dz\, dt \to \int_0^T\int_\Omega Q\phi \, dx\, dz\, dt, \tag{36}$$

for any test function $\phi \in L^2(\Omega_T)$. We also have the weak convergence of the products

$$\begin{aligned} f(S^\gamma)U^\gamma &\rightharpoonup f(S)U \quad \text{in} \quad L^2(\Omega_T), \\ f(S^\gamma)Q^\gamma &\rightharpoonup f(S)Q \quad \text{in} \quad L^2(\Omega_T). \end{aligned} \tag{37}$$

All above convergence results imply that Equation (10) in Definition 1 converge to

$$\int_0^T\int_\Omega \partial_t S\phi - f(S)U\partial_x\phi - f(S)Q\partial_z\phi + \beta_1\partial_{tx}S\partial_x\phi + \beta_2\partial_{tz}S\partial_z\phi \, dx\, dz\, dt = \int_0^T\int_{\partial\Omega_{\text{inflow}}} f(S_{\text{inflow}})U_{\text{inflow}}\phi|_{\partial\Omega_{\text{inflow}}} dz\, dt, \tag{38}$$

for any $\phi \in L^2(0,T;H^1(\Omega))$. Further, the velocity component U satisfies

$$\int_{\Omega} U\psi \, dx\, dz = -\int_{\Omega} \lambda_{tot}(S)\partial_x p \psi \, dx\, dz, \tag{39}$$

for any $\psi \in L^2(\Omega)$ almost everywhere in $(0,T)$. Also the limit Q satisfies

$$\int_{\Omega} U\partial_x\phi \, dx\, dz + \int_{\Omega} Q\partial_z\phi \, dx\, dz = \int_0^1 \left(U_{\text{outflow}}\phi|_{\partial\Omega_{\text{outflow}}} - U_{\text{inflow}}\phi|_{\partial\Omega_{\text{inflow}}} \right) dz, \tag{40}$$

for any test function $\phi \in H^1(\Omega)$ almost everywhere in $(0,T)$.

In the following, we evaluate the limit pressure p using the limit saturation S and the velocity at the inflow boundary U_{inflow}. This consequently leads to limit velocities U and Q depending on S and U_{inflow} only. So, we consider Equation (40) with a test function $\phi = \phi(x)$ that satisfies $\phi \in H^1((0,1))$ (implying also that $\phi \in C^0((0,1))$), with $\phi(1) = 0$, $\phi(0) = 1$ and $\int_0^1 \phi' \, dx = -1$. Then, Equation (40) reduces to

$$\int_{\Omega} U\phi' \, dx\, dz = -\int_{\partial\Omega_{\text{inflow}}} U_{\text{inflow}} \, dz. \tag{41}$$

We also define the vertically-averaged operator

$$\hat{U}(x,t) = \int_0^1 U(x,z,t)\, dz, \tag{42}$$

for almost all $x \in (0,1)$ and $t \in (0,T)$ and choose the test function $\psi = \phi'$. Then, Equation (41) is reformulated as

$$\int_0^1 \hat{U}\phi' \, dx = -\hat{U}_{\text{inflow}}. \tag{43}$$

Similarly, Equation (39) with the z-independent test function $\psi = \phi' \in L^2((0,1))$ reduces to

$$\int_0^1 \hat{U}\phi' \, dx = -\int_0^1 \partial_x p \hat{\lambda}_{tot}(S)\phi' \, dx, \tag{44}$$

where $\hat{\lambda}_{tot} := \int_0^1 \lambda_{tot}(S)\, dz$ is the vertically-averaged total mobility. Substituting Equation (43) into (44) yields

$$\hat{U}_{\text{inflow}} = \int_0^1 \partial_x p \hat{\lambda}_{tot}(S)\phi' \, dx, \tag{45}$$

As the limit pressure $p = p(x,t)$ and the vertically averaged mobility $\hat{\lambda}_{tot}(S)$ are independent of the z-coordinate, then using $\int_0^1 \phi' \, dx = -1$ we obtain

$$\partial_x p = -\frac{\hat{U}_{\text{inflow}}}{\hat{\lambda}_{tot}(S)}. \tag{46}$$

Substituting this formula into (39) allows reformulating the horizontal velocity U component as

$$\int_{\Omega} U\psi \, dx\, dz = \int_{\Omega} \frac{\hat{U}_{\text{inflow}}\lambda_{tot}(S)}{\hat{\lambda}_{tot}(S)} \psi \, dx\, dz, \tag{47}$$

for any $\psi \in L^2(\Omega)$ and almost everywhere in $(0, T)$.

The last step in the proof is to evaluate the limit velocity Q. For this, it is necessary first to prove the claim

$$\int_\Omega U[S]\partial_x\phi\, dx\, dz = -\int_\Omega \int_0^z U[., r; S]\, dr\partial_{xz}\phi\, dx\, dz, \tag{48}$$

for any test function $\phi \in H^1(\Omega)$. The proof starts with applying Gauss' theorem to the right side of the equation above together with Equation (9) in Remark 1. Then, we have

$$\int_\Omega \int_0^z U[., r; S]\, dr\partial_{xz}\phi\, dx\, dz = -\int_\Omega \partial_z \int_0^z U[., r; S]\, dr\partial_x\phi\, dx\, dz. \tag{49}$$

Using summation by parts, it holds that

$$\int_\Omega \frac{\int_0^z U[S(x,r,t)]\, dr - \int_{-\Delta z}^{z-\Delta z} U[S(x,r,t)]\, dr}{\Delta z} \partial_x\phi\, dx\, dz\, dt$$
$$= \int_\Omega \frac{1}{\Delta z}\int_{z-\Delta z}^{z} U[S(x,r,t)]\, dr\, \partial_x\phi\, dx\, dz - \int_\Omega \frac{1}{\Delta z}\int_{-\Delta z}^{0} U[S(x,r,t)]\, dr\, \partial_x\phi\, dx\, dz.$$

Letting $\Delta z \to 0$ and using Lebesgue's Differentiation theorem[28] together with Equation (9), we obtain

$$\int_\Omega \partial_z \int_0^z U[S(x,r,t)]\, dr\, \partial_x\phi\, dx\, dz\, dt = \int_\Omega U[S(x,z,t)]\, \partial_x\phi\, dx\, dz, \tag{50}$$

for almost all $z \in (0, 1)$, which proves the claim. Thus, substituting (48) into the weak incompressibility relation (40) yields

$$\int_\Omega Q\partial_z\phi\, dx\, dz = \int_\Omega \int_0^z U[S(., r, .)]\, dr\, \partial_{xz}\phi\, dx\, dz + \int_0^1 \left(U_{\mathrm{outflow}}\, \phi|_{\partial\Omega_{\mathrm{outflow}}} - U_{\mathrm{inflow}}\, \phi|_{\partial\Omega_{\mathrm{inflow}}}\right) dz,$$

for any test function $\phi \in H^1(\Omega)$. We apply again Gauss' theorem to the first term on the right side the equation above. Then, we have

$$\begin{aligned}\int_\Omega Q\,\partial_z\phi\, dx\, dz = &-\int_\Omega \partial_x \int_0^z U[S(., r, .)]\, dr\, \partial_z\phi\, dx\, dz + \int_0^1\int_0^z \left(U_{\mathrm{outflow}}(r,\cdot)\, dr\, \partial_z\phi|_{\partial\Omega_{\mathrm{outflow}}} - U_{\mathrm{inflow}}(r,\cdot)\, dr\, \partial_z\phi|_{\partial\Omega_{\mathrm{inflow}}}\right) dz \\ &+ \int_0^1 \left(U_{\mathrm{outflow}}\, \phi|_{\partial\Omega_{\mathrm{outflow}}} - U_{\mathrm{inflow}}\, \phi|_{\partial\Omega_{\mathrm{inflow}}}\right) dz.\end{aligned} \tag{51}$$

Similar to the proof of claim (48), we can show

$$\int_0^1\int_0^z U_{\mathrm{outflow}}(r,\cdot)\, dr\partial_z\phi|_{\partial\Omega_{\mathrm{outflow}}}\, dz = -\int_0^1 U_{\mathrm{outflow}}\, \phi|_{\partial\Omega_{\mathrm{outflow}}}\, dz,$$

and

$$\int_0^1\int_0^z U_{\mathrm{inflow}}(r,\cdot)\, dr\partial_z\phi|_{\partial\Omega_{\mathrm{inflow}}}\, dz = -\int_0^1 U_{\mathrm{inflow}}\, \phi|_{\partial\Omega_{\mathrm{inflow}}}\, dz.$$

Thus, Equation (51) reduces to

$$\int_{\Omega} Q\partial_z\phi\, dx\, dz = -\int_{\Omega} \partial_x \int_0^z U[S(.,r,.)]\, dr\partial_z\phi\, dx\, dz, \tag{52}$$

for any test function $\phi \in H^1(\Omega)$. Hence, the vertical velocity Q satisfies

$$Q = -\partial_x \int_0^z U[S(.,r,.)]\, dr.$$

Equations (47) and (52) show that the limit velocity components U and Q are nonlinear nonlocal operators of the limit saturation S together with the horizontal velocity at the inflow boundary. Consequently, Equations (38), (47) and (52) imply that the limit (S, U, Q) of the sequence of weak solutions $(S^\gamma, p^\gamma, U^\gamma, Q^\gamma,)_{\gamma>0}$ for the BTP model (1) satisfies Definition 2 and is, c for BVE model (2), (3). □

Remark 4. For the convergence analysis in this paper, the parameters β_1 and β_2 in (1) are assumed to be independednt of the geometrical paramter γ to preserve the regularizing effect of the higher-order terms $\partial_{txx}S^\gamma$ and $\partial_{tzz}S^\gamma$ in the limit as $\gamma \to 0$. However, this assumption is of purely mathematical nature as these coefficients depend on the geometrical dimensions H and L of the domain, see Appendix A1. In this remark, we highlight the consequences of considering these dependencies. For this, we distinguish between two cases for thin domains with geometrical ratio $\gamma = H/L \ll 1$:

- *Case* 1: Porous media domains with width $H = \mathcal{O}(1)$ and length $L \gg 1$ have the parameters $\beta_1 = \mathcal{O}(\gamma^2)$ and $\beta_2 = \mathcal{O}(1)$. In the limit, this will lead to a solution S with reduced regularity in the x-direction, as the estimates on $\partial_x S^\gamma$ in Lemma 1 and $\partial_{tx}S^\gamma$ in Lemma 3 will vanish and the well-posedness proof of the BVE model in Armiti-Juber and Rohde[23] will not be valid.
- *Case* 2: Porous media domains with width $H = \mathcal{O}(\gamma^2)$ and length $L = \mathcal{O}(1)$. Thus, the parameter $\beta_2 = \mathcal{O}(\gamma^{-2})$ and the higher-order term $\partial_{tzz}S^\gamma$ will dominate. In such a case, the estimates in Lemma 1 and 3 will blow up unless we consider a z-independent initial condition $S^0 = S^0(x)$. Consequently, the limit solution S will be independent of the z direction and the velocity components will reduce to $U = 1$ and $Q = 0$. This implies that the BVE model will reduce further into a higher-order extension of the one-dimensional Buckley–Leverett equation.

5 | NUMERICAL EXAMPLE

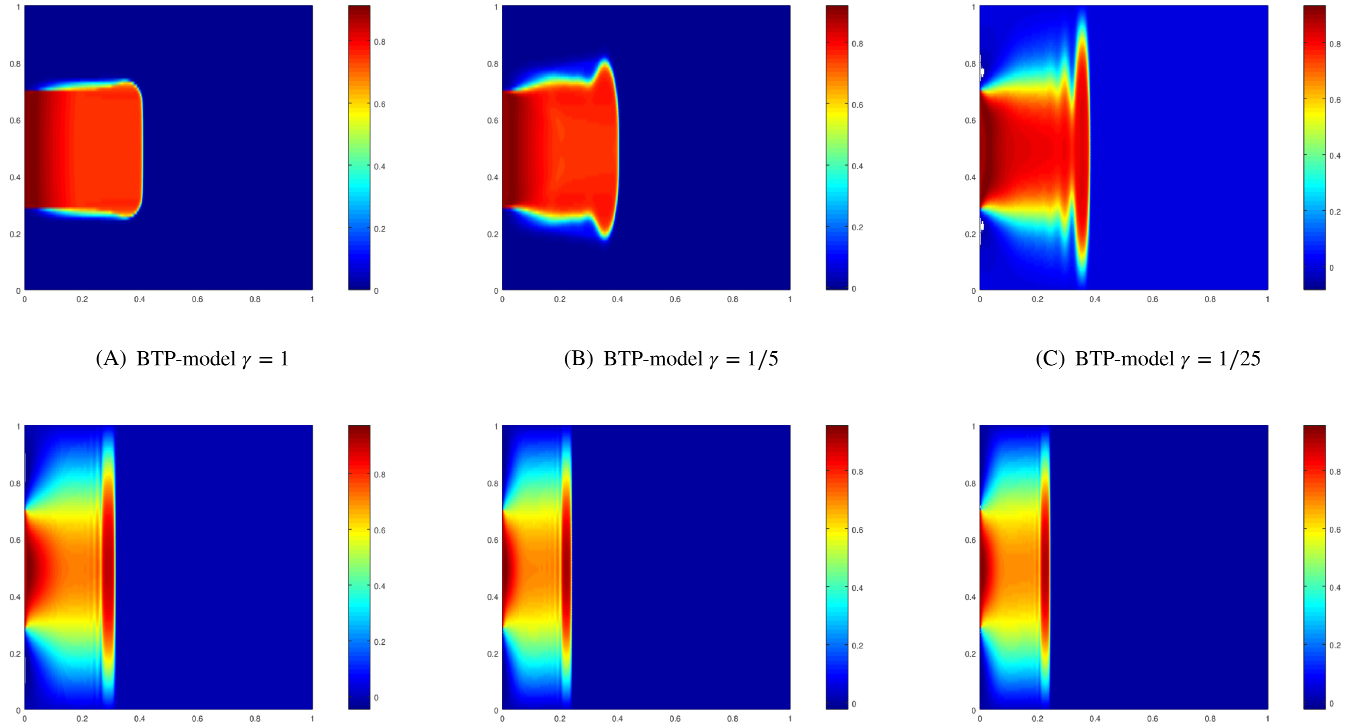
In this section, we present a numerical example that shows the convergence of numerical solutions for the dimensionless BTP model (1) to numerical solutions for the reduced BVE model (2), (3) as the geometrical parameter γ reduces. We consider the dimensionless BTP model (1) with the fractional flow function

$$f(S) = \frac{MS^2}{MS^2 + (1-S)^2}, \tag{53}$$

where M is the viscosity ratio of the defending phase and the invading phase. The model is also assumed to be satisfied in the domains $\Omega_\gamma = (0, L) \times (0, H)$ with decreasing geometrical parameter $\gamma \in \{1, 1/5, 1/25, 1/125\}$, such that the domains' length is fixed $L = 5$ and the widths are decreasing $H \in \{5, 1, 1/5, 1/15, 1/25\}$.

The initial and boundary conditions are given as

$$\begin{aligned} S^\gamma(\cdot,\cdot,0) &= S_0 && \text{in } \Omega, \\ S^\gamma &= S_{\text{inflow}} && \text{on } \{0\}\times(0,1)\times[0,T], \\ p^\gamma &= 1 && \text{on } \{0\}\times(0,1)\times[0,T], \\ p^\gamma &= 0 && \text{on } \{1\}\times(0,1)\times[0,T], \\ W^\gamma &= 0 && \text{on } (0,1)\times\{0,1\}\times[0,T]. \end{aligned} \tag{54}$$



(A) BTP-model $\gamma = 1$ (B) BTP-model $\gamma = 1/5$ (C) BTP-model $\gamma = 1/25$

(D) BTP-model $\gamma = 1/75$ (E) BTP-model $\gamma = 1/125$ (F) BVE-model

FIGURE 2 Numerical solutions for the BTP model (1) in (A)–(E), with decreasing parameter $\gamma \in \{1, 1/5, 1/25, 1/75, 1/125\}$, converge to numerical solution for the BVE model (2), (3) in figure (F), using a 1000×100 grid, $M = 2$, $\mu_e = 10^{-2}$ and $T = 0.3$ [Colour figure can be viewed at wileyonlinelibrary.com]

In the following examples we choose the initial condition

$$S_0(x, z) = g(x) S_{\text{inflow}}(z),$$

where

$$g(x) = \frac{(1-x)^2}{10^5 x^2 + (1-x)^2} \quad \text{and} \quad S_{\text{inflow}}(z) = \begin{cases} 0 & : z \leq \frac{3}{10} \text{ and } z > \frac{7}{10}, \\ 0.9 & : \quad \frac{3}{10} < z \leq \frac{7}{10}. \end{cases} \tag{55}$$

We discretize the dimensionless BTP model and the nonlocal BVE model (2), (3) by applying mass-conservative finite-volume schemes as described in Armiti-Juber and Rohde.[7] The schemes are based on Cartesian grids with number of vertical cells N_z significantly less than that in the horizontal direction N_x that fits to the case of thin domains. In the following example, we use a grid of 1000×100 elements, viscosity ratio $M = 2$, end time $T = 0.3$ and we set $\hat{U}_{\text{inflow}} = 1$ in Equation (3).

In Figure 2A–2E, we present the numerical solutions of the BTP model (1) using the parameters $\gamma \in \{1, 1/5, 1/25, 1/75, 1/125\}$, respectively, such that $L = 5$ and $H \in \{5, 1, 1/5, 1/15, 1/25\}$. Figure 2F presents the numerical solution of the BVE model in the limit case with $L = 5$ and $H = 1/25$. The results in Figure 2 suggest that numerical solutions for the BTP model (1) converge to the corresponding numerical solutions for the reduced BVE model (2), (3) as the geometrical parameter γ tends to zero. This numerical convergence supports the theoretical results in Theorem 1.

6 | CONCLUSION

We studied the limit of the two-phase flow model in porous media domains of Brinkman type as the domain's width–length ratio vanishes. We proved that weak solutions for this model converge to a weak limit. Further, we showed that the limit satisfies the definition of weak solutions for a model, in which pressure gradient is formulated as a nonlocal operator of saturation.

The nonlocal model was first suggested in Armiti-Juber and Rohde[7] as a proper reduction of the full two-phase flow model in thin domains. It was derived using standard asymptotic analysis. However, the convergence analysis in this paper contributes to this model with a first rigid mathematical derivation.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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APPENDIX A: THE DIMENSIONLESS BRINKMAN TWO-PHASE FLOW MODEL

We consider the displacement process of two incompressible immiscible fluids in a saturated nondeformable porous medium of Brinkman type. The invading phase $\alpha = i$ is displacing the defending phase $\alpha = d$ under the assumption of negligible gravity and capillary forces. Then, the two-phase flow model consists of the continuity equation, the Brinkman equations and the incompressibility equation

$$\begin{aligned} \partial_t S_\alpha + \nabla \cdot \mathbf{v}_\alpha &= 0, \\ -\mu_e \Delta \mathbf{v}_\alpha + \mathbf{v}_\alpha &= -\lambda_\alpha(S_\alpha)\mathbf{K}\nabla p_\alpha, \\ \nabla \cdot \mathbf{v} &= 0 \end{aligned} \tag{A1}$$

in $\Omega_\gamma \times (0,T)$, where $\Omega_\gamma = (0,L)\times(0,H)$ is a rectangular domain with the parameter $\gamma = H/L$. Note that the incompressibility equation in (A1) is a consequence of the saturation condition $S_i + S_d = 1$ and the continuity equation. The intrinsic permeability tensor $\mathbf{K} = \mathbf{K}(x,z)$ is defined as $\mathbf{K}(x,z) = \begin{pmatrix} K_x(x,z) & 0 \\ 0 & K_z(x,z) \end{pmatrix}$, where the components K_x and K_z correspond to the premeability in the x and z directions, respectively. We define the vector of generalized velocities

$$\mathbf{V}_\alpha = -\mu_e \Delta \mathbf{v}_\alpha + \mathbf{v}_\alpha, \tag{A2}$$

such that $\mathbf{V}_\alpha = (U_\alpha, W_\alpha)^T$. For this vector, we also define $\mathbf{V} = \mathbf{V}_i + \mathbf{V}_d$, which satisfies the incompressibility-like equation

$$\nabla \cdot \mathbf{V} = -\mu_e \,\Delta(\nabla \cdot \mathbf{v}) + \nabla \cdot \mathbf{v} = 0. \tag{A3}$$

Remark 5. It is well-known in the literature that the principal components of the permeability tensor, the velocity and pressure gradient are in the same direction[29,30] with $K_x > K_z$. This fits the displacement problem considered in the paper, see Figure 1, where fluids flow almost in the horizontal direction x. Hence, and to keep the presentation simple, we assume here that the geometrical dimensions of the domain match the principal components of permeability.

To derive the dimensionless BVE model (1) we rescale Equation (A1) using the dimensionless variables

$$\begin{aligned} &\bar{x} = \frac{x}{L}, \quad \bar{z} = \frac{z}{H}, \quad \bar{t} = \frac{t}{L/q}, \quad \kappa_j = \frac{K_j}{k_j} \\ &\bar{u}_\alpha = \frac{u_\alpha}{q}, \quad \bar{w}_\alpha = \frac{w_\alpha}{q}, \quad \bar{p} = \frac{p}{Lq\mu_d/k_x}, \end{aligned} \tag{A4}$$

for $j \in \{x,z\}$ and $\alpha \in \{i,d\}$. Here, $q > 0$ is the inflow speed at the inflow boundary $\partial\Omega_{\text{inflow}}$, μ_d is the viscosity of the defending phase, k_j is the mean value of the corresponding permeability function K_j and u_α and w_α are the horizontal and vertical components of the velocity field $\mathbf{v}_\alpha$, respectively. Applying the chain rule to (A2), then defining the dimensionless components

$$\bar{U}_\alpha := \frac{U_\alpha}{q}, \quad \bar{W}_\alpha := \frac{W_\alpha}{q}, \tag{A5}$$

where U_α and W_α are the horizontal and vertical components of $\mathbf{V}_\alpha$, respectively, yield

$$\begin{aligned} \bar{U}_\alpha &= \bar{u}_\alpha - \frac{\mu_e}{L^2}\partial_{\bar{x}\bar{x}}\bar{u}_\alpha - \frac{\mu_e}{H^2}\partial_{\bar{z}\bar{z}}\bar{u}_\alpha, \\ \bar{W}_\alpha &= \bar{w}_\alpha - \frac{\mu_e}{L^2}\partial_{\bar{x}\bar{x}}\bar{w}_\alpha - \frac{\mu_e}{H^2}\partial_{\bar{z}\bar{z}}\bar{w}_\alpha. \end{aligned} \tag{A6}$$

Applying the chain rule to Equations (A1) and (A3), using Equations A4 and (A5), then omitting the bar-signs leads to

$$\begin{aligned} \partial_t S_\alpha + \partial_x u_\alpha + (1/\gamma)\partial_z w_\alpha &= 0, \\ U_\alpha &= -\lambda_\alpha(S_\alpha)\,\kappa_x\,\partial_x p_\alpha, \\ (\gamma/\sigma)W_\alpha &= -\lambda_\alpha(S_\alpha)\,\kappa_z\,\partial_z p_\alpha, \\ \partial_x u + (1/\gamma)\partial_z w &= 0, \\ \partial_x U + (1/\gamma)\partial_z W &= 0 \end{aligned} \tag{A7}$$

in $\Omega \times (0, T)$, for both invading and defending phases $\alpha \in \{i, d\}$ and $\sigma = k_z/k_x$. Here U and W are the horizontal and vertical components of $\mathbf{V}$, respectively. Now, applying the operator $1 - \beta_1\partial_{xx} - \beta_2\partial_{zz}$ to the continuity equation, where $\beta_1 = \frac{\mu_e}{L^2}$ and $\beta_2 = \frac{\mu_e}{H^2}$, transforms model (A7) to

$$\begin{aligned} \partial_t S_\alpha - \beta_1\partial_{xxt} - \beta_2\partial_{zzt}S_\alpha + \partial_x U_\alpha + (1/\gamma)\partial_z W_\alpha &= 0, \\ U_\alpha &= -\lambda_\alpha(S_\alpha)\,\kappa_x\,\partial_x p_\alpha, \\ (\gamma/\sigma)W_\alpha &= -\lambda_\alpha(S_\alpha)\,\kappa_z\,\partial_z p_\alpha, \\ \partial_x U + (1/\gamma)\partial_z W &= 0. \end{aligned} \tag{A8}$$

The assumption of negligible capillary pressure implies $p_i = p_d =: p$ and the phases' velocities satisfy

$$U_\alpha = f(S_\alpha)U, \quad W_\alpha = f(S_\alpha)W. \tag{A9}$$

We set $\kappa_x = \kappa_z = 1$ and $\sigma = 1$ to simplify the analysis in this paper, and define the variable $Q = W/\gamma$. Then, the dimensionless model (A7) is summarized such that the unknown variables S, p, U, and Q are associated with the parameter γ,

$$\begin{aligned} \partial_t S^\gamma - \beta_1\partial_{xxt}S - \beta_2\partial_{zzt}S + \partial_x\left(f(S^\gamma)U^\gamma\right) + \partial_z\left(f(S^\gamma)Q^\gamma\right) &= 0, \\ U^\gamma &= -\lambda_{tot}(S^\gamma)\partial_x p^\gamma, \\ \gamma^2 Q^\gamma &= -\lambda_{tot}(S^\gamma)\partial_z p^\gamma, \\ \partial_x U^\gamma + \partial_z Q^\gamma &= 0, \end{aligned} \tag{A10}$$

where $S = S_i$ is the saturation of the invading fluid.

APPENDIX B: ASYMPTOTIC ANALYSIS

The BVE model is derived in Armiti-Juber and Rohde[7] by applying formal asymptotic analysis, with respect to γ, to the dimensionless BTP model (A10). We assume that each component in $(S^\gamma, p^\gamma, U^\gamma, Q^\gamma)$ is smooth and can be written in terms of the asymptotic expansions

$$Z^\gamma = Z_0 + \gamma Z_1 + \mathcal{O}(\gamma^2),\ Z^\gamma \in \{S^\gamma, p^\gamma, U^\gamma, Q^\gamma\}. \tag{B11}$$

Using the asymptotic expansion of S^γ in (B11) and Assumption 1, we have the Taylor expansions

$$G(S^\gamma) = G(S_0) + G'(S_0)(\gamma S_1) + \mathcal{O}(\gamma^2), \tag{B12}$$

for $G \in \{\lambda_{tot}, f\}$. The incompressibility relation in (A10) allows writing the continuity equation in nonconservative form. Substituting Equation (B11) and (B12) into (A10), the terms of order $\mathcal{O}(1)$ satisfy

$$\begin{aligned} \partial_t S_0 - \beta_1 \partial_{xxt} S_0 - \beta_2 \partial_{zzt} S_0 + \partial_x \left(f(S_0) U_0 \right) + \partial_z \left(f(S_0) Q_0 \right) &= \mathcal{O}(\gamma), \\ U_0 &= -\lambda_{tot}(S_0) \partial_x p_0, \\ \lambda_{tot}(S_0) \partial_z p_0 &= \mathcal{O}(\gamma^2), \\ \partial_x U_0 + \partial_z Q_0 &= \mathcal{O}(\gamma). \end{aligned} \tag{B13}$$

Using the positivity of the total mobility λ_{tot} (see Assumptions 1(6)), the third equation of (B13) implies that p_0 is independent of the z-coordinate,

$$p_0 = p_0(x, t). \tag{B14}$$

Integrating the last equation in (B13) over the vertical direction from 0 to 1 and using the assumption of impermeable upper and lower boundaries of the domain $\partial_{\text{imp}}\Omega$ in (5), we obtain

$$\partial_x \int_0^1 U_0 \, dz = -\int_0^1 \partial_z Q_0 \, dz = 0.$$

Integrating this equation from 0 to x yields

$$\int_0^1 U_0(x, z, t) \, dz - h(t) = 0, \tag{B15}$$

for any $x \in (0, 1)$ and $t \in [0, T]$, where $h(t) = \int_0^1 U_0(0, z, t) \, dz$ is the averaged horizontal velocity at the inflow boundary. Substituting the second equation in (B13) into Equation (B15) yields

$$-\int_0^1 \lambda_{tot}(S_0) \partial_x p_0 \, dz = h(t).$$

Then, using Equation (B14), we have

$$\partial_x p_0(x, t) = -\frac{h(t)}{\int_0^1 \lambda_{tot}(S_0(x, z, t)) \, dz}, \tag{B16}$$

for all $x \in (0, 1)$ and $t \in (0, T)$. Substituting (B16) into the second equation in (B13), we obtain a nonlocal saturation-dependent formula for U_0,

$$U_0[S_0] = \frac{h(t) \lambda_{tot}(S_0)}{\int_0^1 \lambda_{tot}(S_0) \, dz}, \tag{B17}$$

for all $(x, z) \in \Omega$ and $t \in (0, T)$. Consequently, the incompressibility relation in (B13) yields also a nonlocal saturation-dependent formula for Q_0,

$$Q_0[S_0] = -\partial_x \int_0^z U_0[S_0(\cdot, r, \cdot)] \, dr, \tag{B18}$$

for all $(x, z) \in \Omega$ and $t \in (0, T)$. Using Equation (B17) and (B18), omitting the subscript {0}, system (B13) reduces to a third-order nonlocal nonlinear equation of saturation

$$\partial_t S + \partial_x \left(f(S) U \right) + \partial_z \left(f(S) Q \right) - \beta_1 \partial_{xxt} S - \beta_2 \partial_{zzt} S_0 = 0, \tag{B19}$$

in $\Omega \times (0, T)$ where we have for all $z \in (0, 1)$

$$\begin{aligned} U[S] &= \frac{\lambda_{tot}(S)}{\int_0^1 \lambda_{tot}(S)\,dz}, \\ Q[S] &= -\partial_x \int_0^z U[S(\cdot, r, \cdot)]\,dr. \end{aligned} \tag{B20}$$