Chapter 7

Sensitivity Analysis

7.1 Introduction

Once a model has been selected, calibrated and validated, it is of crucial importance to study how changes in variables, parameters, and model structure would affect the behaviour of the model output. Such a study is generally known as sensitivity analysis (Gilchrist, 1984). As to the model user, the sensitivity analysis will provide him/her all required information and insight about the model performance and its limitations, which, in turn, will contribute to reduce the risk of an inappropriate application of the model.

It should be noted that the sensitivity with regard to model structure was already considered during the model selection (see Chapter 4 and Chapter 5). Therefore, the present chapter will go through the remaining issues, namely: 1) sensitivity of model parameters to a given variable, 2) model sensitivity to a given parameter, and, 3) sensitivity of the significance probability ($p-\text{values}$) as to the number of replicate simulations R.

7.2 Sensitivity of Parameters to Catchment Size

One of the major concerns in the present study is to investigate the effects of the spatial scale at which the model is optimised with regard to the model parameters and its overall performance. In other words, it would be necessary to answer the question: are the model- parameters invariant with regard to the spatial scale? In this case, the spatial scale is represented by the catchment size x_1, which ranges from 4.5 to 4002.0 km^2.

In order to illustrate the procedure presented below, the model No. 3733 fitted for the annual specific discharge in winter (see Section 4.1.4) will be used as an example. In this case, the model estimates can be written as

$$\hat{Q}_{i2}^{t} = f\left(x_{i7}^{t}, x_{i8}^{t}, x_{i11}^{t}, x_{i15}^{t}, x_{i17}^{t}, x_{i19}^{t}, x_{i21}^{t}, \hat{\boldsymbol{\beta}}\right) \quad i = 1,\ldots,46 \quad t = 1961,\ldots,1993\,, \tag{7.1}$$

using the vector $\hat{\boldsymbol{\beta}}$ as in Section 4.1.4, Table 4.5.

Algorithm 7

1. For all $a = 10, 25, 50, 100, 200, 250, 500, 1000, 2000, 3500, 4500$, where a is a threshold for the variable x_1 given in [km²].
   a. Build a sample $\mathcal{D}_a$ of size n_{0a} so that $x_{i1} < a \quad \forall i = 1, \ldots, 46$.
   b. Use $\mathcal{D}_a$ to estimate $\hat{\boldsymbol{\beta}}_a$ for the model $Q_{i2}^t = f\left(x_{i7}^t, x_{i8}^t, x_{i11}^t, x_{i15}^t, x_{i17}^t, x_{i19}^t, x_{i21}^t, \hat{\boldsymbol{\beta}}_a\right) + \varepsilon_i^t$ so that $\Phi_a \rightarrow \min!$
   c. Estimate the Akaike's Information Criterion AIC_a for the previous model.
2. Repeat step 1. if needed.
3. Plot n_{0a}, AIC_a, and $\hat{\boldsymbol{\beta}}_a$ versus a.

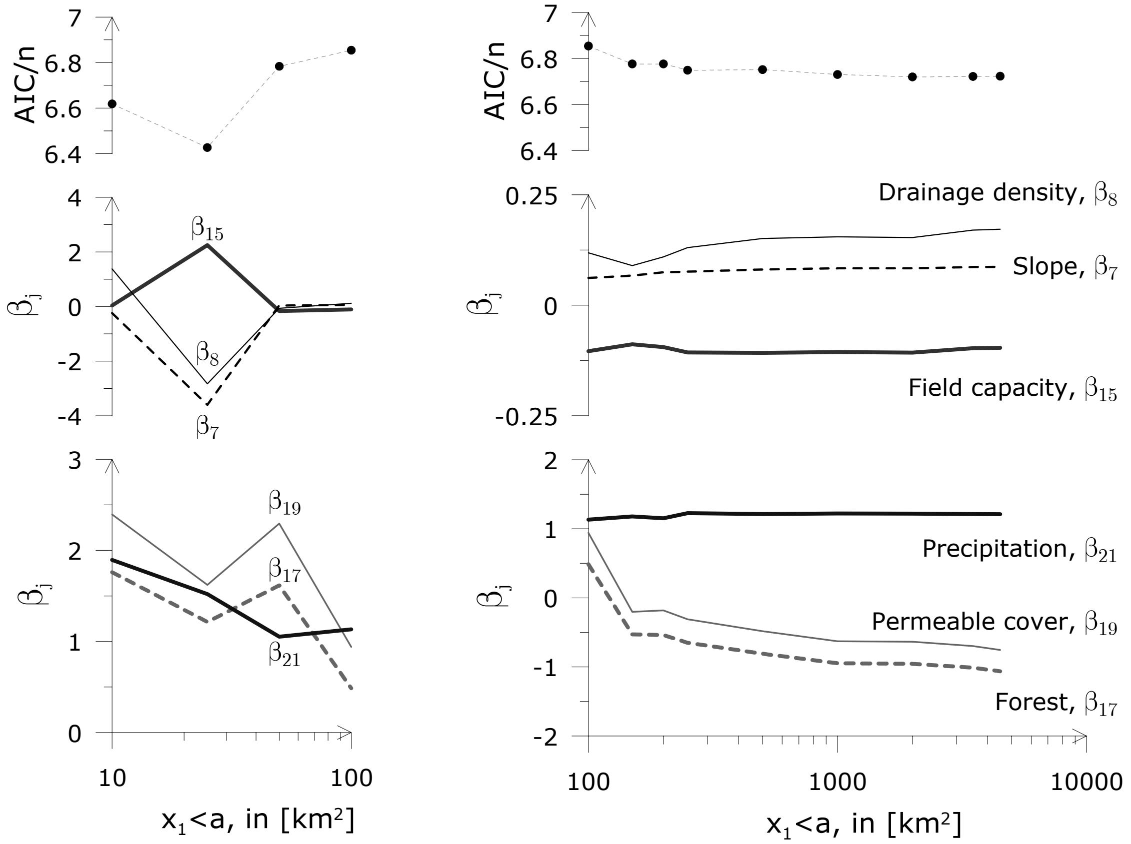

Figure 7.1 Parameter sensitivity to catchment size for the multi-linear potential model (No. 3733) selected for the annual specific discharge in winter Q_2. Samples are from the period from 1961 to 1993.

Based on the results of the previous Algorithm, which are depicted in Figure 7.1, the following remarks can be formulated.

1. Since AIC is proportional to the sample size, the ratio AIC_a / n_{0a} can be used to compare the results obtained by the previous Algorithm with regard to the quality of the model with respect to the amount of information provided. As can be seen in the Figure above, this indicator reaches a peak at around 100 km² and then decreases slowly. Based on this finding, it can be inferred that the amount of information for those samples with spatial units whose area is less than $a\,(\sim 100\,\mathrm{km}^2)$ is not as complete as for those derived with thresholds greater than or equal to $a\,(\sim 100\,\mathrm{km}^2)$.

2. Parameters for the morphological variables (i.e. x_7, x_8, x_{11}, and x_{15}) exhibit an irregular behaviour (as to their sign and magnitude) when the samples used for the model-calibration have spatial units with an area less than $a\,(\sim 100\,\text{km}^2)$. This can be regarded as a direct consequence of what has been mentioned above since the data here perhaps reflects a case specific situation unwanted for a model supposed to describe the phenomenon at a much broader scale. However, values for the analyzed parameters tend to stabilize for threshold values a greater than 100 km² (see right panel of Figure 7.1). As this example has shown, parameters of morphological variables in a model conceived to explain the specific discharge for a given catchment can be considered as scale invariant if $a > 100\,\text{km}^2$.
3. The parameter β_{21}, which is linked with the climatological variable total precipitation, has a downward trend within the interval $a \in [10, 50][\text{km}^2]$ and becomes asymptotic when it reaches a magnitude of about 1.2 (see left panel of Figure 7.1). Thus, it can be stated that this parameter is scale invariant for values of $a > 50\,\text{km}^2$. Additionally, it should be noticed that its order of magnitude is several times greater than that of the morphological variables. Such a fact just points out how important this variable is with regard to discharge predictions at a mesoscale level.
4. Finally, those parameters associated with land cover variables such as forest and permeable cover (β_{17} and β_{19}) exhibit in general a downward tendency, keeping an almost constant relationship between them. Because of this fact, it can be inferred that these variables have a complex relationship to the water system, which depends greatly on the scale at which the analysis is carried out. Consequently, their corresponding parameters appear to be scale dependent as shown in the Figure 7.1.

7.3 Model Sensitivity to a Given Parameter

In many cases, it would be desirable to know how changes of a parameter (e.g. due to errors of estimation caused by data quality) would influence the behaviour of the model output. In other words, to assess how the uncertainty of one parameter can influence the model results (Mein and Brown, 1978).

A simple procedure to assess the percentage rate of change in the expected output $\hat{Q}_{il}^t$ per unit of percentage change in the parameter β_j, frequently referred as relative sensitivity, is presented below.

Let

$$\hat{Q}_{il}^t = f\left(x_{i1}^t, \ldots, x_{iJ}^t, \hat{\boldsymbol{\beta}}\right) \qquad i = 1, \ldots, 46 \quad j = 1, \ldots, J \quad t = 1961, \ldots, 1993, \tag{7.2}$$

be a general model for a given runoff characteristic l, which depends on J predictors x_j, and where $f(\bullet)$ and $\hat{\boldsymbol{\beta}}$ are a known functional form and a vector of estimated parameters respectively. Based on these definitions, the rate of change of $\hat{Q}_{il}^t$ with β_j or simply the <u>absolute</u> sensitivity coefficient c_{ij}^t, can be computed as (McCuen, 1973, Leavesley et al. 1983, Gilchrist, 1984)

$$c_{ij}^t = \frac{\partial \hat{Q}_{il}^t}{\partial \beta_j}\Big|_{\hat{\boldsymbol{\beta}}}, \tag{7.3}$$

where the partial derivative is evaluated at $\hat{\boldsymbol{\beta}}$. Absolute sensitivities, however, have the serious disadvantage that the values estimated for two different parameters cannot be directly compared because their values largely depend on the magnitudes of each parameter respectively. Therefore, dimensionless-relative sensitivities are preferred in practice. The relative sensitivity of the model output, e_{ij}^{t}, with respect to the parameter $\hat{\beta}_j$ can be written as

$$e_{ij}^{t} = c_{ij}^{t} \frac{\hat{\beta}_j}{E\left[\hat{Q}_{il}^{t}\right]} = \left(\frac{\partial \hat{Q}_{il}^{t}}{E\left[\hat{Q}_{il}^{t}\right]} \right) \left(\frac{\partial \beta_j}{\hat{\beta}_j} \right)^{-1}, \tag{7.4}$$

where $E\left[\hat{Q}_{il}^{t}\right]$ is the expectation of the output given by $E\left[\hat{Q}_{il}^{t}\right] = f\left(x_{i1}^{t}, \ldots, x_{iJ}^{t}, \hat{\boldsymbol{\beta}}\right)$.

Figure 7.2 illustrates for a specific case how the two factors shown in parenthesis in (7.4) are related to each other, considering three different parameters. Based on this Figure, it can be concluded that the most sensitive parameter in this case is β_{22}, which is associated with the variable cumulative precipitation in summer, and the least β_{17}, which is associated with the share of forest of a given basin. These results show that the system is highly sensitive to precipitation and much less sensitive to land cover or slopes. These results are not surprising because the system is mainly governed by climatic variables, and only modulated by the morphology and the land cover of a given basin. However, it should be noted that the magnitude of the relative sensitivity of the parameter β_7 associated with the mean slope in the buffer zones of the stream network is quite similar to that of β_{17}. This result suggests that the sensitivity of the model to a change of the parameter value for land cover is as important as that corresponding to mean slopes. Nevertheless, the sign of the changes of the output will be the opposite because these variables (x_{17} and x_7) associated with these parameters have an inverse and a direct relationship with the model output respectively as can be seen in the Figure below.

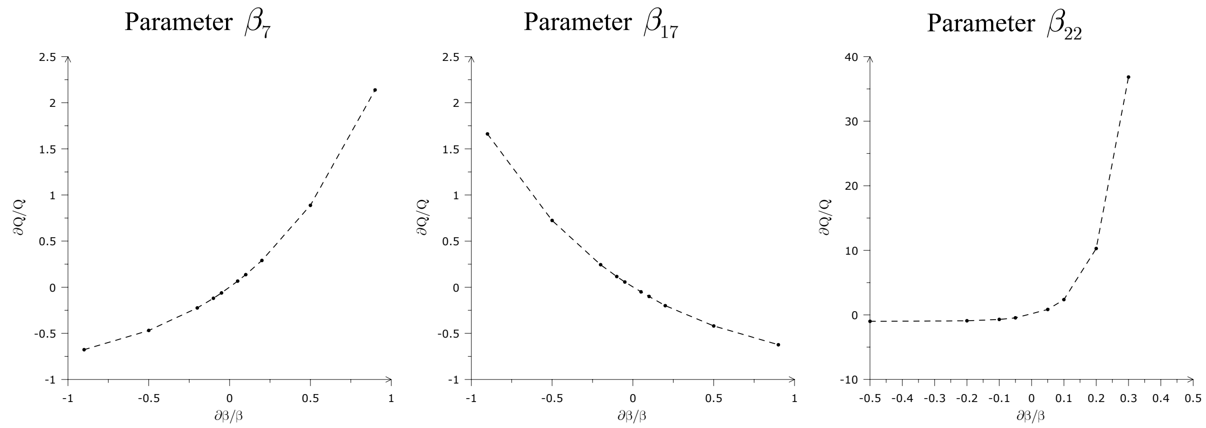

Figure 7.2 Relationship between $\partial \hat{Q}_{il}^{t} / E\left[\hat{Q}_{il}^{t}\right]$ and $\partial \beta_j / \hat{\beta}_j$ for model No. 3965 obtained for the specific discharge in summer (Q_3). The relative sensitivities for each parameter at a given level can be obtained as the quotient between an ordinate and its corresponding abscise. The dots represent the geometric mean of the relative changes taking into account all observations is the sample.

7.4 Convergence of the Monte Carlo Simulations

A randomization test is to be performed in order to assess whether an explained variable Q is either statistically independent (H_0) or dependent on a given variable x_j under a joint distribution of J predictors. As mentioned in Section 3.3.7, the estimator Φ measures the level of interdependence between Q and x_j (ceteris paribus), and the $p-\text{value}$ indicates whether to accept or reject the null hypothesis in favour of the alternative one at a given label of significance α, say 5%. However, this procedure can be executed only if one knows in advance how many replicates of the statistical test have to be carried out in order to have a conclusive result, which, in turn, leads to take the right decision.

Of course, the more replicates the better, but a large value (say $R > 10\,000$) still constitutes a great hindrance at the actual state of development of desktop computers, namely a dramatic increase of computing time. This side effect would then make this procedure too time consuming to be applied for practical purposes. Therefore, it would be advantageous to establish a certain minimum number of simulations required to guarantee a stable result.

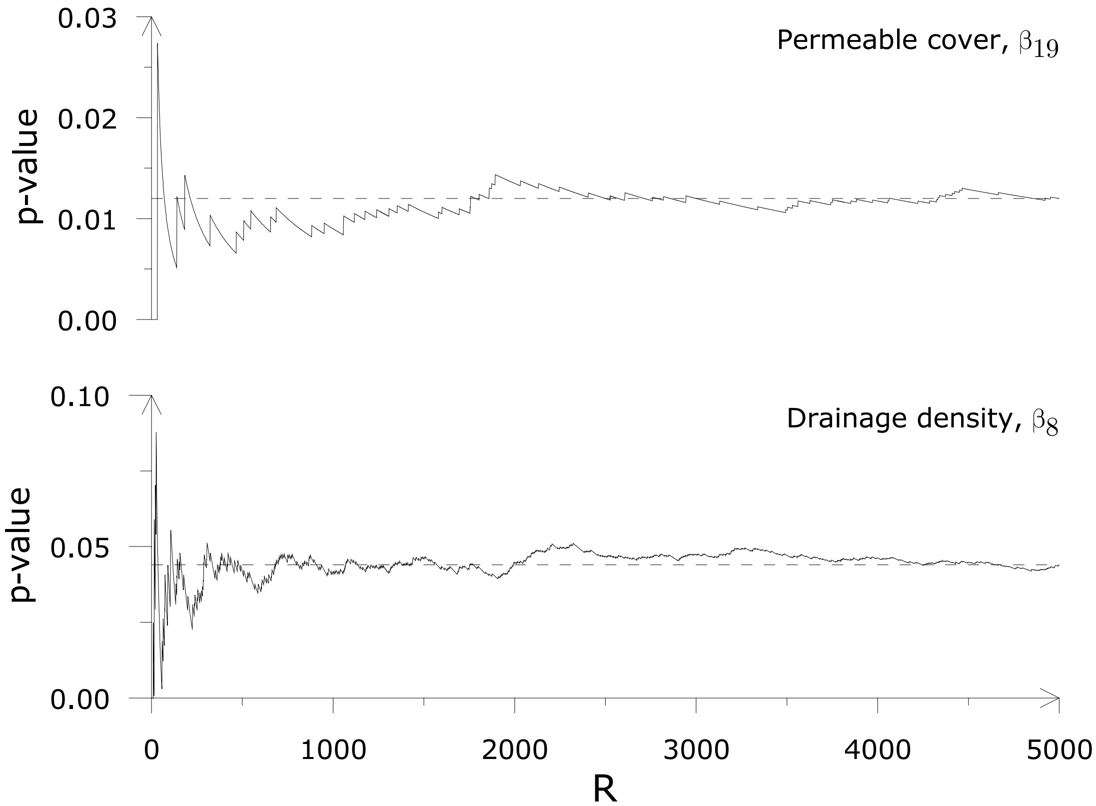

Figure 7.3 Sensitivity of the $p-\text{value}$ with respect to the number of replicate simulations. Panel on top shows the results for variable x_{19}; and panel down depicts the results for variable x_8. The simulations are carried out for model No. 3733 fitted for the annual specific discharge in winter.

As shown in Figure 7.3, the $p-\text{value}$ obtained for two variables x_{19} and x_8 that are part of the model No. 3733 calibrated for the annual specific discharge in winter [see (7.1)] tend to converge to a certain limit, which is the value that would be obtained if R would tend to infinite. As a rule of thumb, it is suggested that reasonable estimates can be obtained when R varies between 100 and 1000

(Davison and Hinkley 1997). The convergence of the $p-\text{value}$ is commonly achieved when $R \simeq 500$ as actually it happened with the variable x_8 shown in the Figure below. However, there are cases in which convergence only occurs for values of $R > 1000$ as it is the case of the variable x_{19}. Since this is not known in advance, a good recommendation would be to continue with the simulations if the obtained $p-\text{value}$ is too close (about 10%) to the level of significance decided in advance for the test of independence.

In the present case, since the $p-\text{value}$ obtained for the variable x_{19} is much less than 5%, the simulation could have been stopped at $R = 500$.

A great advantage of this test with relation to the parametric tests is that in this case the PDF of the test statistic (e.g. the estimator) is not assumed but rather built up from the simulation results as shown in Figure 7.4. The $p-\text{value}$ estimated by the Algorithm 4 (see Section 3.3.7) is the area on the left tail of the empirical distribution function of Φ that is less than or equal to the value of the estimator given the original sample, for the model 3733, this value is $\Phi = 0.9342$. As seen in the Figure below, the EDF is not symmetrical and skewed to the left.

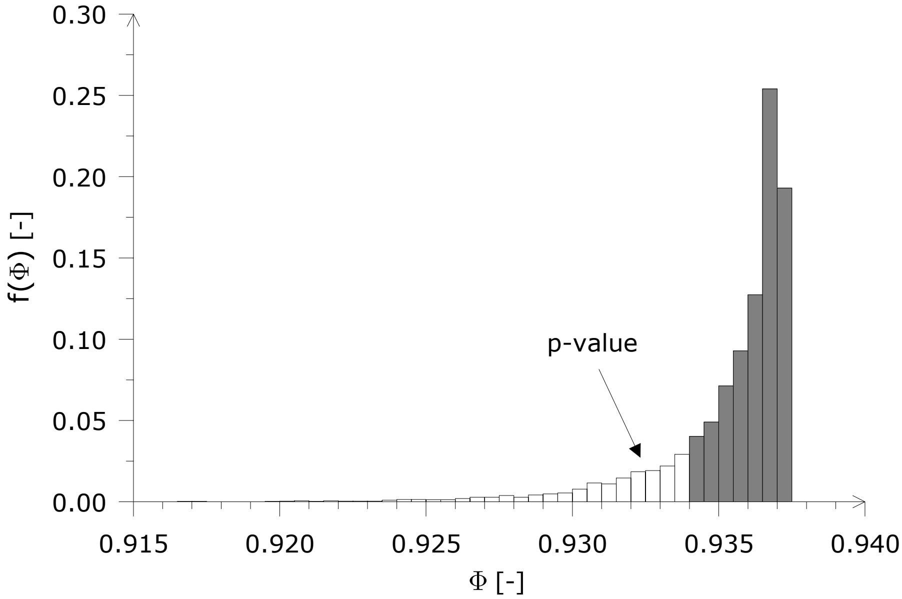

Figure 7.4 Histogram of $R = 5000$ Monte Carlo replicates of the estimator Φ for the model No. 3733 when the variable x_8 has been tested for independence. The unshaded area in the left tail correspond to the $p-\text{value}$.