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Towards Understanding the Dynamics of a Braess Paradox Model

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Abstract

This work studies a recently proposed model of the Braess paradox, which has four parameters and is defined piecewise continuous on three regions. The Braess paradox describes the fact that the introduction of a new choice into a system of rational agents may not improve the situation but can make it even worse. To investigate the system and its undergoing border collision bifurcations, we are going to fix some parameters and thereby get new results on the overall dynamics. Also part of the coexistence and the period adding scenario between the cycles are analyzed.

Keywords: Braess paradox, discontinuous 2-dim map, border-collision bifurcations

Contents

1 Introduction **5**

2 Model **7**

2.1 Definition 7
2.2 Properties 7

3 Binary choice **9**

3.1 Numerics on two partitions 9
3.2 Our model as a 1D map 9
3.3 Period Adding 11

4 Ternary choice **13**

4.1 Numeric 13
4.2 Cycles on two partitions 13
4.3 possible bifurcations 15
4.4 Period Adding 17
4.5 Points in a row in one region 18

5 The case $\delta_{\mathcal{L}} = 1$ **21**

5.1 Properties 21
5.2 Partitioning of the $\delta_{\mathcal{R}} \times k$ plane 22
5.3 One R 22
5.4 Two R's 24
5.5 Three R's 33
5.6 The $O_{\mathcal{LR}^n}$ family 34

6 The $\mathcal{LRM}^n$ islands **37**

6.1 How are the islands created? 37
6.2 Coexistence 39
6.3 Period Adding between the islands 39

7 Summary and outlook **43**

List of Figures **44**

Bibliography **45**

1 Introduction

The Braess Paradox

In 1968, Braess proposed a paradox [Bra68] where another choice in a system of rational agents does not improve the overall situation but makes it even worse. As an example, let us regard a simple road network with four towns, named A, B, C and D. All roads are one-way roads. There are two roads to get from A to D. Either one agent could choose the street via B or via C. Thus, our agents which are forced to go from A to D have two choices, but will always choose the fastest way. The cost (time) of one street segment is fixed for the road from B to D and from A to C. However, the time needed to get from A to B and from C to D depends on the number of travelers on this segments. Hence, we get time c_1 for streets $A \to B$ and $C \to D$ and time $c_2 \cdot x$, where $x \in [0, 1]$ is the ratio of travelers choosing that road for $A \to C$ and $B \to D$. We could insert numbers as c_1 and c_2 and compute a Nash equilibrium.

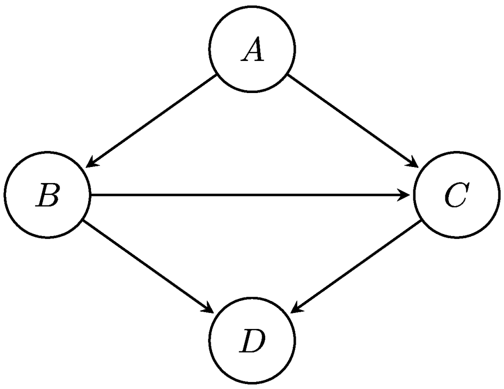


Figure 1.1: The road network

Now we build a street which allows to travel very fast from B to C with travel time 0, such that the two ratio-dependent streets are connected. The Nash Equilibrium will be lower as agents that choose the $A \to B \to C \to D$ path increase both travel times for $A \to B$ and $C \to D$. This means we have worsen the situation by introducing another choice, which is counter-intuitive.

One may think that this example may seem a bit constructed and does not occur in reality. However, Knödel gave some example in [Knö69, p. 59] where the opening of a new road network in 1968 at the Schloßplatz in Stuttgart caused a traffic snarl during the rush hour. The officials then closed one street (Untere Königsstraße) to remedy the situation.

Dal Forno and Merlone proposed a model of the Braess Paradox in [DMar]. They introduced a four parameter 2D system. The 2D state space is divided into three regions, where each region represents that the majority of the agents uses one of the three possible paths on the graph. The parameters are the switching rate to any path and the cost of the added resulting link. The goal of this study thesis is to get a step further in understanding the undergoing border-collision bifurcations of the cycles in this system.

Outline

We will first have a look at the model and its basic properties. Then we will regard a special case where our normally three partitioned system, has only two partitions. After that, we will have a quick look at the general case. Regardless if we are in the case of two or three partitions, we will see that cycles on two partitions can be investigated using a 1D model.

Since our system has four parameters, our next step will be to set another one to some limit case. As we will see, the bifurcation structure there is significantly easier to study. Hence we do so. Since the model function is linear on any partition, we expect that it preserves some of the results we get from this case. Indeed, after having studied this case we know which cycles we will see in the bifurcation diagram for given parameters.

Last, we want to have a look at the "islands", which may occur in one of the bifurcation planes. We will see how they are structured and which cycles coexist on them. We will also have a look at the period adding scenario with the coexisting cycles on the islands.

2 Model

2.1 Definition

The model proposed for the Braess Paradox in [DMar] is described as follows. We have four parameters. Three parameters of the switching rate, $\delta_{\mathcal{L}}, \delta_{\mathcal{R}}$ and $\delta_{\mathcal{M}}$ for travelers switching to the $\mathcal{L}$, $\mathcal{R}$, $\mathcal{M}$ route respectively. Thereby, $\mathcal{L}$ stands for the route $A \to B \to D$, $\mathcal{R}$ for $A \to C \to D$ and $\mathcal{M}$ for the path $A \to B \to C \to D$. The parameters $\delta_{\mathcal{L}}, \delta_{\mathcal{R}}$ and $\delta_{\mathcal{M}}$ are chosen from the interval $[0,1]$. The other parameter is $k \in [0, 0.5]$ for the cost of the new resulting link. With these parameters, the system is defined as follows:

$$x_{n+1} = \begin{cases} (1-\delta_{\mathcal{L}})x_n + \delta_{\mathcal{L}} & \text{if } (x_n, y_n)^T \in \mathcal{D}_{\mathcal{L}} \\ (1-\delta_{\mathcal{R}})x_n & \text{if } (x_n, y_n)^T \in \mathcal{D}_{\mathcal{R}} \\ (1-\delta_{\mathcal{M}})x_n & \text{if } (x_n, y_n)^T \in \mathcal{D}_{\mathcal{M}} \end{cases}$$

$$y_{n+1} = \begin{cases} (1-\delta_{\mathcal{L}})x_n & \text{if } (x_n, y_n)^T \in \mathcal{D}_{\mathcal{L}} \\ (1-\delta_{\mathcal{R}})x_n + \delta_{\mathcal{R}} & \text{if } (x_n, y_n)^T \in \mathcal{D}_{\mathcal{R}} \\ (1-\delta_{\mathcal{M}})x_n & \text{if } (x_n, y_n)^T \in \mathcal{D}_{\mathcal{M}} \end{cases}$$

Whereby the regions are given by

$$\mathcal{D}_{\mathcal{L}} = \left\{(x,y)^T \mid (x < k) \text{ and } (y > x)\right\}$$
$$\mathcal{D}_{\mathcal{R}} = \left\{(x,y)^T \mid (x > y) \text{ and } (y < k)\right\}$$
$$\mathcal{D}_{\mathcal{M}} = \left\{(x,y)^T \mid (x > k) \text{ and } (y > k)\right\}$$

2.2 Properties

Limit cases for k The $\mathcal{D}_{\mathcal{M}}$ region vanishes in the case $k = 0.5$. Thus only two regions remain. For $k = 0$, the $\mathcal{D}_{\mathcal{L}}$ and $\mathcal{D}_{\mathcal{R}}$ region vanish and we only have $\mathcal{D}_{\mathcal{M}}$ left. In this case we have just one stable fixpoint, namely $(0,0)^T$.

Linearity We immediately see that our model is linear in any partition. The points in $\mathcal{D}_{\mathcal{M}}$ will be mapped linear to the point $(0,0)^T$, the points from $\mathcal{D}_{\mathcal{R}}$ to $(0,1)^T$ and the points from $\mathcal{D}_{\mathcal{L}}$ to $(1,0)^T$. This represents the situations where the whole population of agents chooses either the $\mathcal{M}, \mathcal{R}$ or the $\mathcal{L}$ route.

Every cycle has at least to go through $\mathcal{D}_{\mathcal{L}}$ and $\mathcal{D}_{\mathcal{R}}$ Because of the linearity, we can only have a fixed point in the limit case $k = 0$. In any other case, we cannot have a fixpoint. Can we have a cycle of period two? This is possible, but because of the accumulation points mentioned above only between the $\mathcal{D}_{\mathcal{L}}$

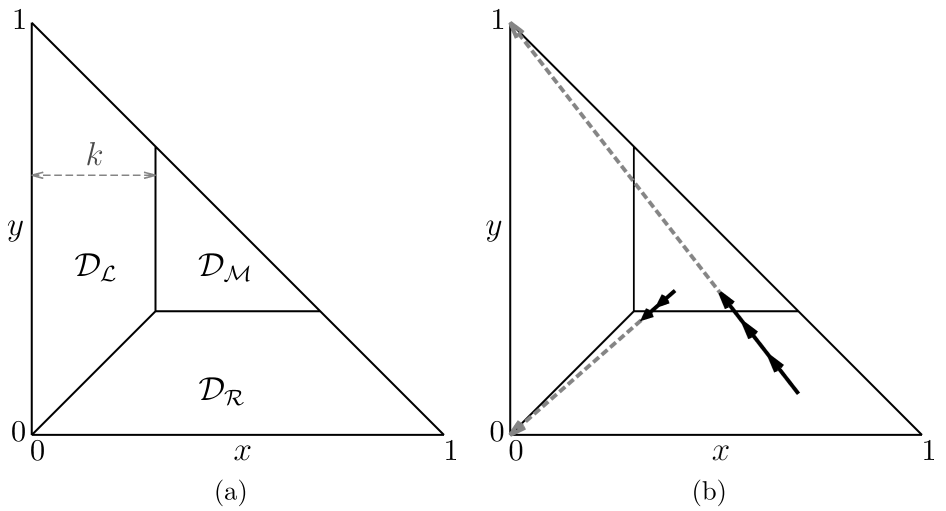


Figure 2.1: In 2.1a we see the state space with the partitions $\mathcal{D}_\mathcal{L}, \mathcal{D}_\mathcal{M}$ and $\mathcal{D}_\mathcal{R}$. Figure 2.1b shows that points in $\mathcal{D}_\mathcal{R}$ converge against $(1, 0)^T$ and points in $\mathcal{D}_\mathcal{M}$ towards $(0, 0)^T$. In this example we have $\delta_\mathcal{R} = \delta_\mathcal{M} = 0.1$ and we can have three points in the same region from the example point in $\mathcal{D}_\mathcal{R}$ and two points from the point in $\mathcal{D}_\mathcal{M}$.

and $\mathcal{D}_\mathcal{R}$ cycle. As we will see in the following chapters, all cycles on the $\mathcal{D}_\mathcal{L}$ and $\mathcal{D}_\mathcal{R}$ partition exist on the straight $y = 1 - x$. This could also be seen in Figure 2.1b.

Virtual regions We can see in the model definition and in Figure 2.1b, the points in $\mathcal{D}_\mathcal{M}$ converge on a straight to the origin as long as they are in $\mathcal{D}_\mathcal{M}$. This means, that they preserve their predicate of being above or below the main diagonal line. If the first point of a series of points in $\mathcal{D}_\mathcal{M}$ lays above the diagonal line, we will reach $\mathcal{D}_\mathcal{L}$. If it lays below the diagonal, we will reach $\mathcal{D}_\mathcal{R}$ after some steps respectively.

In the following chapters, we are going to analyze the border collision bifurcations of this system. The discussion above shows us that we have to regard only one possible border collision of a point in $\mathcal{D}_\mathcal{M}$ for a specific cycle. Either the next point should be in $\mathcal{D}_\mathcal{R}$, then we can have a collision of $y = k$, or it should be in $\mathcal{D}_\mathcal{L}$, then the $x = k$ border can cause a collision. However, if the next point should be in $\mathcal{D}_\mathcal{M}$, we will see that it cannot collide with anything as the next point is then closer to the border.

3 Binary choice

As already mentioned, for $k = 0.5$, only two regions remain. Because the $\mathcal{D}_{\mathcal{M}}$ region disappeared, our system is independent of $\delta_{\mathcal{M}}$. Thus, only the $\delta_{\mathcal{R}}$ and $\delta_{\mathcal{M}}$ parameter remains of our four parameter system. We will see, that every cycle exists on the line $y = 1 - x$ and that we can therefore regard it as a 1D piecewise linear discontinuous map on two partitions. A system with nearly the same 1D model as we use has also been studied in [BGM09]. To have a rough overview, we will first have a look at the numerics of the $\delta_{\mathcal{R}} \times \delta_{\mathcal{L}}$ plane of the respective bifurcation diagram.

3.1 Numerics on two partitions

In Figure 3.1 we see the numerics of the $\delta_{\mathcal{R}} \times \delta_{\mathcal{L}}$ plane. In this case, the $O_{\mathcal{LR}}$ cycle fills the whole region around the main diagonal line. This will change immediately as we decrease k. With decreasing $\delta_{\mathcal{L}}$ we can first observe the $O_{\mathcal{LR}}$ cycle, then the $O_{\mathcal{LR}^2}$ and so on. For decreasing $\delta_{\mathcal{L}}$, we get also the $O_{\mathcal{LR}}$ cycle first and then the $O_{\mathcal{RL}^2}$ cycle. Later in this chapter, we will see that we will have a period adding scenario between the $O_{\mathcal{LR}^n}$ as well as between the $O_{\mathcal{RL}^n}$ family.

3.2 How our model can be viewed as a 1D map

To get to the 1D system we will first show that every cycle exists on a straight line. We will then choose one component of our 2D state space and therefore get a 1D system.

All cycles exist on a straight line

Proposition 1. *All cycles involving the $\mathcal{D}_{\mathcal{L}}$ and $\mathcal{D}_{\mathcal{R}}$ regions only exist on the straight line $y = 1 - x$.*

Proof. Let (x, y) be a point in $\mathcal{D}_{\mathcal{R}}$ and already on the $y = 1 - x$ line. Then we get the next point (x', y') with

$$x' = (1 - \delta_{\mathcal{R}})x$$
$$y' = (1 - \delta_{\mathcal{R}})y + \delta_{\mathcal{R}}$$

We show that this point lies on the line, too.

$$(1 - \delta_{\mathcal{R}})(1 - x) + \delta_{\mathcal{R}} = 1 - (1 - \delta_{\mathcal{R}})x$$
$$\Rightarrow \quad (1 - \delta_{\mathcal{R}}) - (1 - \delta_{\mathcal{R}})x + (1 - \delta_{\mathcal{R}})x = 1 - \delta_{\mathcal{R}}$$
$$\Rightarrow \quad 1 - \delta_{\mathcal{R}} = 1 - \delta_{\mathcal{R}}$$

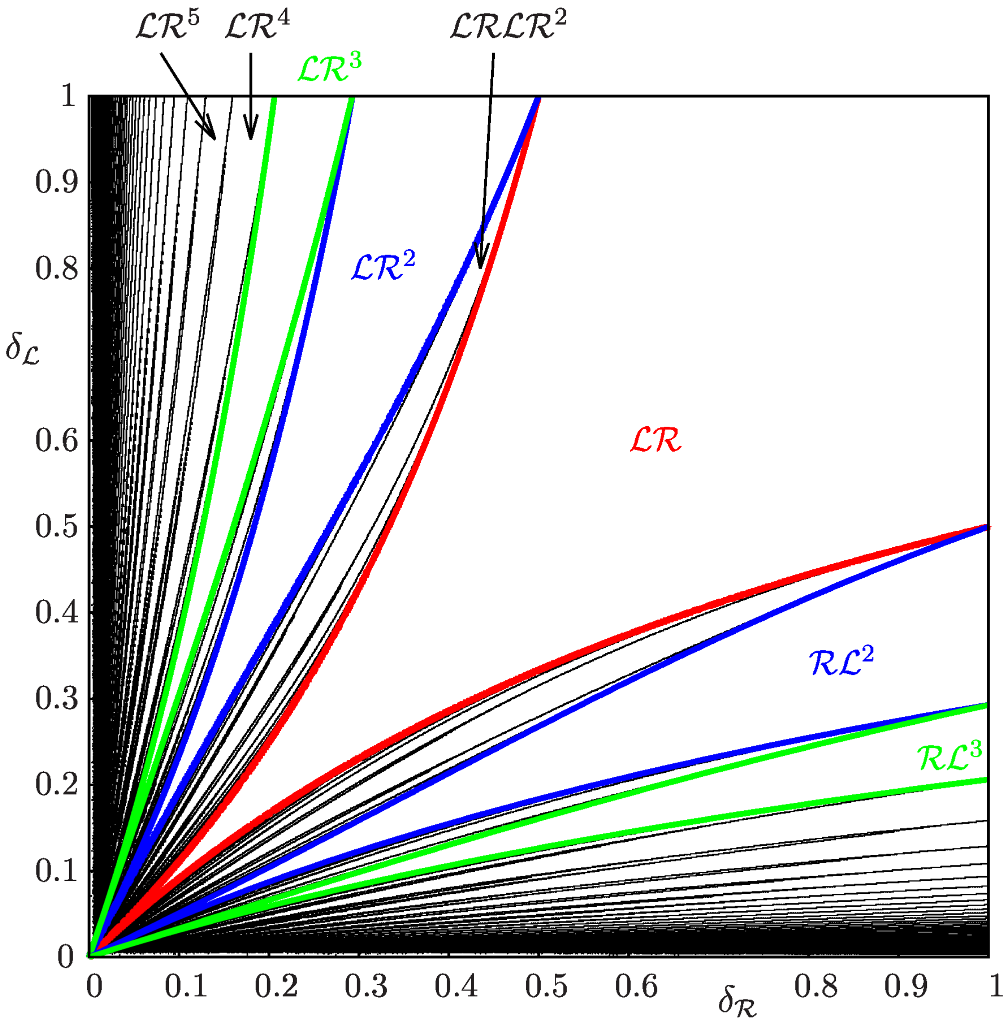


Figure 3.1: Numerics of the binary choice model with some analytical border collision curves. Between the $O_{\mathcal{LR}}$ and the $O_{\mathcal{LR}^2}$ we can see the $O_{\mathcal{LRLR}^2}$ cycle as part of the undergoing adding scenario.

We have already seen, that the mapping on $\mathcal{D}_\mathcal{R}$ (and $\mathcal{D}_\mathcal{L}$) converge against $(0,1)^T$ $((0,1)^T$ respectively). Thus every point in the state space converges against this line. □

Transforming the system into a 1D piecewise linear discontinuous map on two partitions

As we know that all cycles involving $\mathcal{D}_\mathcal{L}$ and $\mathcal{D}_\mathcal{R}$ only exist on a non-axis-aligned straight line, we can now get a bijective mapping into 1D by using only one coordinate of every point. This gives us the model of the piecewise linear discontinuous map on two partitions with one discontinuity, this has already been studied in [PSASar]. The model used there is

$$x_{n+1} = \begin{cases} a_\mathcal{L} x_n + \mu_\mathcal{R}, & x_n \leq d \\ a_\mathcal{R} x_n + \mu_\mathcal{R}, & x_n > d \end{cases} \tag{3.1}$$

We want the partitions to be preserved. Thus, we will set $d = k = 0.5$ and get for a point $x < d$ that the point lies in $\mathcal{D}_\mathcal{L}$ and for $x > d$ that x lies in $\mathcal{D}_\mathcal{R}$. Hence, we have to set the parameters of the model in [PSASar] as follows:

$$\begin{array}{ll} a_\mathcal{L} = 1 - \delta_\mathcal{L} & a_\mathcal{R} = 1 - \delta_\mathcal{R} \\ \mu_\mathcal{L} = \delta_\mathcal{L} & \mu_\mathcal{R} = 0 \\ d = k = 0.5 & \end{array} \tag{3.2}$$

3.3 Period Adding

Let us have another look at Figure 3.1. We can see the $O_{\mathcal{L}\mathcal{R}\mathcal{L}\mathcal{R}^2}$ cycle between the $O_{\mathcal{L}\mathcal{R}}$ and the $O_{\mathcal{L}\mathcal{R}^2}$ cycle. This also holds in general: Between the $O_{\mathcal{L}\mathcal{R}^n}$ and the $O_{\mathcal{L}\mathcal{R}^{n+1}}$ cycle, we can observe the $O_{\mathcal{L}\mathcal{R}^n\mathcal{L}\mathcal{R}^{n+1}}$ cycle. If we go on, we can see the $O_{(\mathcal{L}\mathcal{R})^2\mathcal{L}\mathcal{R}^2}$ cycle between the $O_{\mathcal{L}\mathcal{R}\mathcal{L}\mathcal{R}^2}$ and the $O_{\mathcal{L}\mathcal{R}}$ cycle. This indeed is the well known period adding scenario.

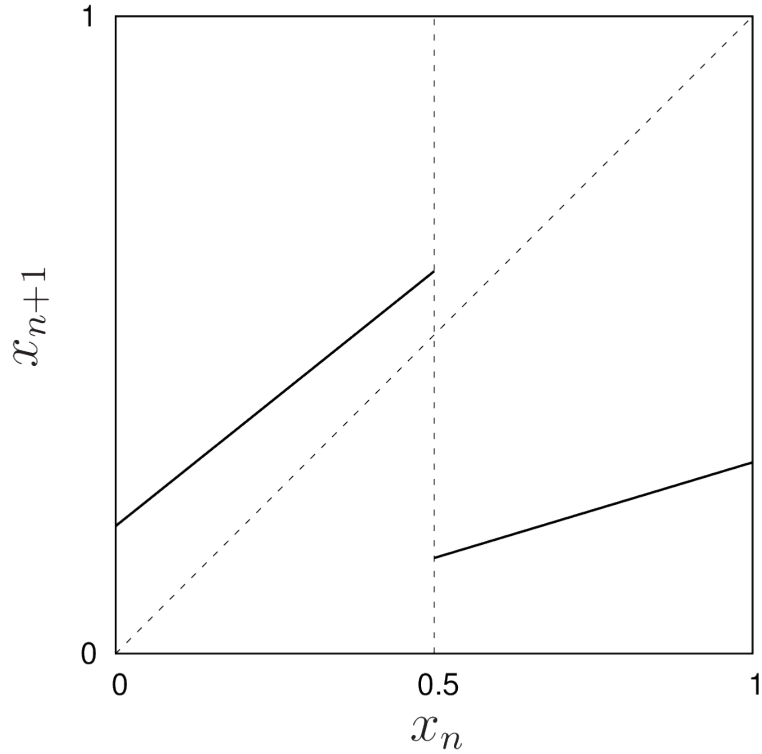


Figure 3.2: Example of our model with $\delta_{\mathcal{R}} = 0.3$ and $\delta_{\mathcal{L}} = 0.2$ transformed into the 1D model.

4 Ternary choice

Regarding the case where there are three partitions, the situation gets a lot more complicated. In this chapter, we will first have a look at the numerics of the bifurcation diagrams in the $\delta_{\mathcal{R}} \times \delta_{\mathcal{L}}$ plane to get a bit of an overview. After that, we are going to investigate the cycles on two partitions – the cycles which involve $\mathcal{D}_{\mathcal{L}}$ and $\mathcal{D}_{\mathcal{R}}$ only. Next, we want to regard the $O_{\mathcal{LRM}}$ cycle and its possible border collision bifurcations. Last, we will see how the maximum points in one region in a row are bounded by some function.

4.1 Numeric

Let us start with the numerics. Figure 4.1 and 4.2 give a rough overview of the $\delta_{\mathcal{R}} \times \delta_{\mathcal{L}}$ plane. Especially the later one seems to be much clearer.

What can we mainly see? The $O_{\mathcal{LR}}$ family has been split up into the $O_{\mathcal{LRM}^n}$ family. In both examples, the rest of the $O_{\mathcal{LR}^n}$ family does not even exist anymore. It seems like we get higher periods for lower $\delta_{\mathcal{M}}$.

4.2 Cycles on two partitions

We already discussed the special case with only two partitions and saw in Proposition 1 that a cycle on two partitions has to be on a straight line. If we decrease k, we will get our three partitions. Luckily, we can again use results from [PSASar], because the original model studied there, the continuous piecewise linear map, has also three partitions and fits perfectly for our three partitions on a straight. For simplicity, we will write just $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{M}$ for the three partitions of the 1D system in this section.

$$x_{n+1} = \begin{cases} a_{\mathcal{L}} x_n + \mu_{\mathcal{L}}, & x_n \le d_{\mathcal{L}} \\ a_{\mathcal{M}} x_n + \mu_{\mathcal{M}}, & d_{\mathcal{L}} < x_n \le d_{\mathcal{R}} \\ a_{\mathcal{R}} x_n + \mu_{\mathcal{R}}, & x_n > d_{\mathcal{R}} \end{cases} \tag{4.1}$$

We can set $d_{\mathcal{L}} = 0.5 - k$ and $d_{\mathcal{R}} = 0.5 + k$, choose $a_{\mathcal{L}}, \mu_{\mathcal{L}}, a_{\mathcal{R}}$ and $\mu_{\mathcal{R}}$ like in the previous chapter (equations 3.2) and use $\mu_{\mathcal{M}}$ and $a_{\mathcal{M}}$ to make sure that the map is continuous.

With this 1D model, we can now study all cycles on two partitions. If a cycle on the $\mathcal{L}$ and $\mathcal{R}$ region of the 1D model exists, it has to exist on $\mathcal{D}_{\mathcal{L}}$ and $\mathcal{D}_{\mathcal{R}}$ on our 2D model, also. Vice versa, if a cycle exists on two partitions of our 2D model, it exists on the 1D system, as well. However, we cannot use our model to study the 1D map, because we have $\delta_{\mathcal{L}}, \delta_{\mathcal{R}} \in [0, 1]$ and $k \in [0, 0.5]$ and the 1D model has $a_{\mathcal{L}}, a_{\mathcal{R}}, \mu_{\mathcal{L}}$ and μ_r in $\mathbb{R}$. But it neither would make things easier to study 2D systems to get results for 1D systems.

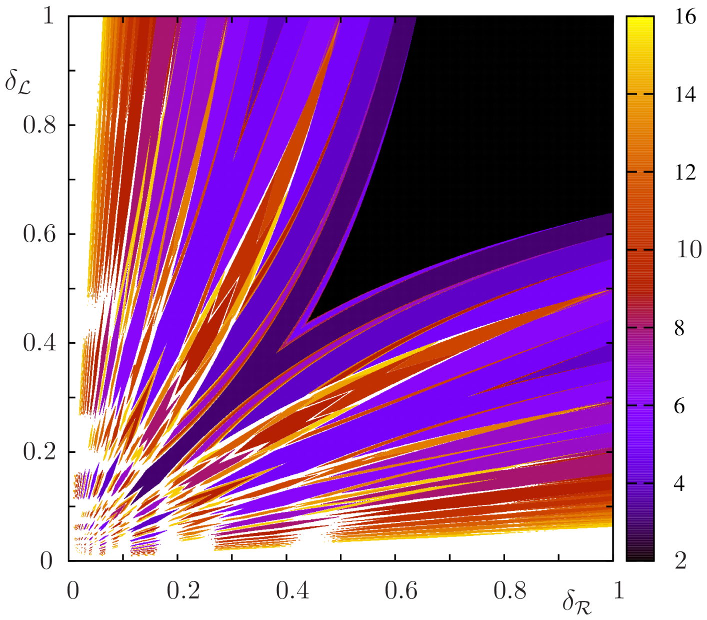


Figure 4.1: Bifurcation diagram in the $\delta_{\mathcal{R}} \times \delta_{\mathcal{L}}$ plane. On the top right we can see the period two $O_{\mathcal{LR}}$ cycle. Next to it, after some period adding structure we have to the left of it the $O_{\mathcal{LRM}}$ cycle and below the $O_{\mathcal{RLM}}$ cycle.
Parameters: $k = 0.36$, $\delta_{\mathcal{M}} = 0.1$

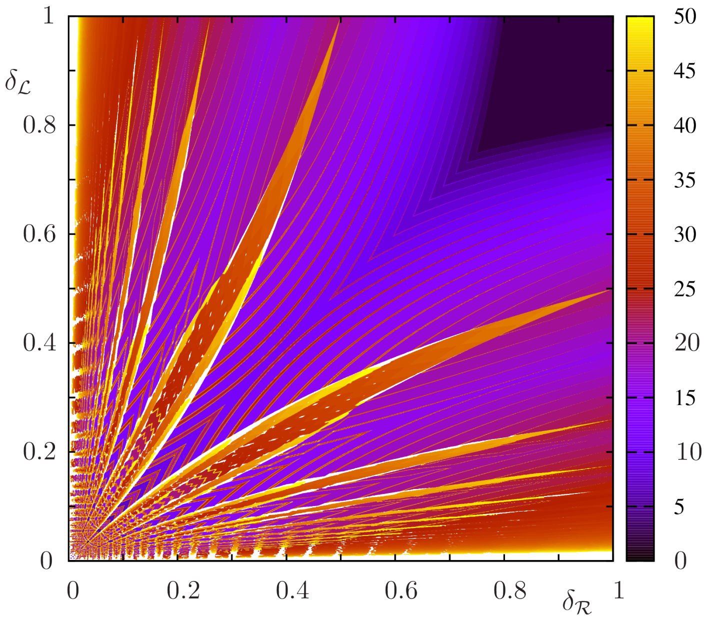


Figure 4.2: Bifurcation diagram in the $\delta_{\mathcal{R}} \times \delta_{\mathcal{L}}$ plane with $\delta_{\mathcal{M}} = 0.05$ and $k = 0.2$. In contrast to figure 4.1, we have much higher periods. Also the structure seems to be a lot more clear for smaller $\delta_{\mathcal{M}}$.

As a side note I want to mention that we could use this 1D map to find cycles on only two partitions. Because the map has to be continuous and the point in $\mathcal{D}_{\mathcal{R}}$ (or $\mathcal{D}_{\mathcal{L}}$) has to map to something below 0.5 (something above 0.5 respectively), we have to have a fixpoint in the $\mathcal{M}$ region of the 1D system (see also Figure 3.2). Therefore, all cycles of period one (fixpoints) have points on all three partitions. A cycle of every other period has to be on the two partitions $\mathcal{D}_{\mathcal{L}}$ and $\mathcal{D}_{\mathcal{R}}$ and has also to exist in our 2D model.

To sum things up, we can say that the cycles on two partitions have the same border collision bifurcations as in the case of $k = 0.5$. Only k has changed. But there are still the same collisions, namely $x = k$ for $(x, y)^T \in \mathcal{D}_{\mathcal{L}}$ and $y = k$ for $(x, y)^T \in \mathcal{D}_{\mathcal{R}}$. We have seen how we can transform our system into a 1D system. Thus, everything on two partitions can be explained with a continuous piecewise linear map on three partitions.

4.3 Cycles on three partitions: possible bifurcations

Let us now study the general case: a cycle on all three partitions. We will first have a look at the possible border collision bifurcations.

The $O_{\mathcal{LRM}}$ cycle as an example

The cycle with lowest period on all three partitions is clearly $O_{\mathcal{LRM}}$. So it should be a good example to show which collisions can happen and which cannot.

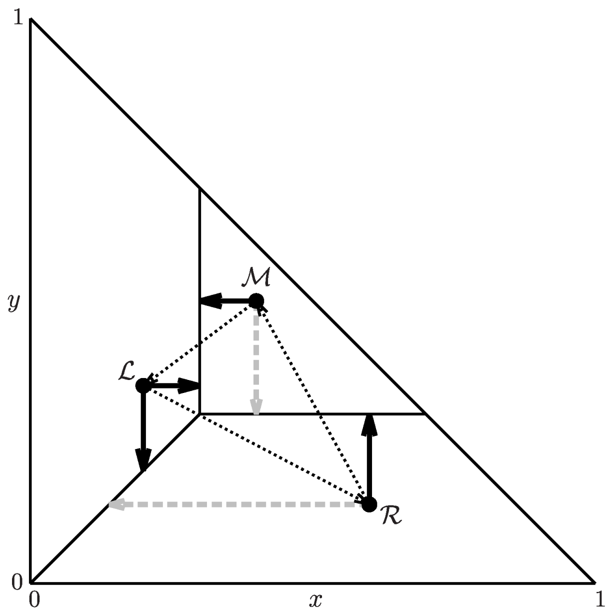


Figure 4.3: The $O_{\mathcal{LRM}}$ cycle in state space with its possible border collisions. We see that the point in $\mathcal{D}_{\mathcal{M}}$ is above the main diagonal line and the point in $\mathcal{D}_{\mathcal{R}}$ has to have $k < x$.

Possible border collision bifurcations for the $O_{\mathcal{LRM}}$ cycle Let $(x_0, y_0)^T$ be in $\mathcal{D}_{\mathcal{L}}$, (x_1, y_1) in $\mathcal{D}_{\mathcal{R}}$ and (x_2, y_2) in $\mathcal{D}_{\mathcal{M}}$. As we can see in figure 4.3, we have the following four border collision bifurcations $\xi_{\mathcal{LRM}}^{(x_0=y_0)}$, $\xi_{\mathcal{LRM}}^{(x_0=k)}$, $\xi_{\mathcal{LRM}}^{(y_1=k)}$ and $\xi_{\mathcal{LRM}}^{(x_2=k)}$.
Why can there be no more?

- $x_1 = y_1$ cannot occur, because we map from $\mathcal{D}_{\mathcal{R}}$ into direction of $(0,1)^T$ and therefore the next point could not be in $\mathcal{D}_{\mathcal{M}}$.
- $y_2 = k$ cannot happen, because we have to be in $\mathcal{D}_{\mathcal{L}}$ after this point. We map into direction of the origin. If we would be below the main diagonal line, we could only map into $\mathcal{D}_{\mathcal{R}}$ and not into $\mathcal{D}_{\mathcal{L}}$.

Computation of the bifurcation curves To compute the actual bifurcation lines, we simple compute the point $(x_0, y_0)^T$ with

$$\begin{aligned} & x_0 = (1-\delta_{\mathcal{M}})(1-\delta_{\mathcal{R}})\left((1-\delta_{\mathcal{L}})x_0 + \delta_{\mathcal{L}}\right) \\ \Rightarrow \quad & x_0 = \frac{(1-\delta_{\mathcal{M}})(1-\delta_{\mathcal{R}})\delta_{\mathcal{L}}}{1-(1-\delta_{\mathcal{M}})(1-\delta_{\mathcal{R}})(1-\delta_{\mathcal{L}})} \end{aligned}$$

$$\begin{aligned} & y_0 = (1-\delta_{\mathcal{M}})\left((1-\delta_{\mathcal{R}})(1-\delta_{\mathcal{L}})y_0 + \delta_{\mathcal{R}}\right) \\ \Rightarrow \quad & y_0 = \frac{(1-\delta_{\mathcal{M}})\delta_{\mathcal{R}}}{1-(1-\delta_{\mathcal{M}})(1-\delta_{\mathcal{R}})(1-\delta_{\mathcal{L}})} \end{aligned}$$

With this points, we can simply compute $x_1 = (1-\delta_{\mathcal{L}})x_0 + \delta_{\mathcal{L}}$ and so on. We can then insert this points into the border collision equations and get the following bifurcation lines of the $O_{\mathcal{LRM}}$ cycle.

Proposition 2. *The bifurcation lines of the $O_{\mathcal{LRM}}$ cycle are given by*

$$\begin{aligned} \xi_{\mathcal{LRM}}^{(x_0=y_0)} &= \left\{\delta_{\mathcal{L}} = \frac{\delta_{\mathcal{R}}}{1-\delta_{\mathcal{R}}}\right\} \\ \xi_{\mathcal{LRM}}^{(x_0=k)} &= \left\{k = \frac{(1-\delta_{\mathcal{M}})(1-\delta_{\mathcal{R}})\delta_{\mathcal{L}}}{1-(1-\delta_{\mathcal{M}})(1-\delta_{\mathcal{R}})(1-\delta_{\mathcal{L}})}\right\} \\ \xi_{\mathcal{LRM}}^{(y_1=k)} &= \left\{k = \frac{(1-\delta_{\mathcal{L}})(1-\delta_{\mathcal{M}})\delta_{\mathcal{R}}}{1-(1-\delta_{\mathcal{M}})(1-\delta_{\mathcal{R}})(1-\delta_{\mathcal{L}})}\right\} \\ \xi_{\mathcal{LRM}}^{(x_2=k)} &= \left\{k = \frac{(1-\delta_{\mathcal{L}})(1-\delta_{\mathcal{M}})(1-\delta_{\mathcal{R}})^2\delta_{\mathcal{L}}}{1-(1-\delta_{\mathcal{M}})(1-\delta_{\mathcal{R}})(1-\delta_{\mathcal{L}})} + (1-\delta_{\mathcal{R}})\delta_{\mathcal{L}}\right\} \end{aligned}$$

A more general example: The $O_{\mathcal{LRM}^n}$ cycle

The computation of the $O_{\mathcal{LRM}}$ cycle by hand was not that difficult. Also we could have replaced every $(1-\delta_{\mathcal{M}})$ simply by $(1-\delta_{\mathcal{M}})^n$. It would have been exactly the same computation. Thus, we already have the point $(x_0, y_0)^T$ of the $O_{\mathcal{LRM}^n}$ cycle given by

$$\begin{aligned} x_0 &= \frac{(1-\delta_{\mathcal{M}})^n(1-\delta_{\mathcal{R}})\delta_{\mathcal{L}}}{1-(1-\delta_{\mathcal{M}})^n(1-\delta_{\mathcal{R}})(1-\delta_{\mathcal{L}})} \\ y_0 &= \frac{(1-\delta_{\mathcal{M}})^n\delta_{\mathcal{R}}}{1-(1-\delta_{\mathcal{M}})^n(1-\delta_{\mathcal{R}})(1-\delta_{\mathcal{L}})} \end{aligned}$$

Possible bifurcations of the $O_{\mathcal{LRM}^n}$ cycle In the previous example we had exactly four border collisions. The arguments we used there why the three points could not collide with the other two borders still work for the $O_{\mathcal{LRM}^n}$ cycle:

- The points in $\mathcal{D}_{\mathcal{M}}$ still have to be above the main diagonal line, thus the $y = k$ collision cannot happen for all of these points.
- For the second point, $(x_1, y_1)^T$ in $\mathcal{D}_{\mathcal{R}}$, the $x = y$ collision cannot happen because then the next point would not be in $\mathcal{D}_{\mathcal{M}}$.

However, regarding the n points in $\mathcal{D}_{\mathcal{M}}$, only the very last one can actually collide with $x = k$: Consider x_i of some i-th point in $\mathcal{D}_{\mathcal{M}}$ which is not the last one in this region. We have the x-coordinate of the following point, which is $x_{i+1} = (1 - \delta_{\mathcal{M}})x_i$. As $(1 - \delta_{\mathcal{M}}) < 1$, we have $x_{i+1} < x_i$. Hence the next point is closer to the border and will collide first.

The bifurcation lines We have seen that there are also just four border collisions for the $O_{\mathcal{LRM}^n}$, just like for the $O_{\mathcal{LRM}}$ cycle. We can now simply compute the bifurcation lines for this cycle.

Proposition 3. *The bifurcation lines of the $O_{\mathcal{LRM}^n}$ cycle are given by*

$$\begin{aligned}
\xi^{(x_0=y_0)}_{\mathcal{LRM}^n} &= \left\{ \delta_{\mathcal{L}} = \frac{\delta_{\mathcal{R}}}{1 - \delta_{\mathcal{R}}} \right\} \\
\xi^{(x_0=k)}_{\mathcal{LRM}^n} &= \left\{ k = \frac{(1 - \delta_{\mathcal{M}})(1 - \delta_{\mathcal{R}})\delta_{\mathcal{L}}}{1 - (1 - \delta_{\mathcal{M}})^n(1 - \delta_{\mathcal{R}})(1 - \delta_{\mathcal{L}})} \right\} \\
\xi^{(y_1=k)}_{\mathcal{LRM}^n} &= \left\{ k = \frac{(1 - \delta_{\mathcal{L}})(1 - \delta_{\mathcal{M}})^n\delta_{\mathcal{R}}}{1 - (1 - \delta_{\mathcal{M}})^n(1 - \delta_{\mathcal{R}})(1 - \delta_{\mathcal{L}})} \right\} \\
\xi^{(x_{n+1}=k)}_{\mathcal{LRM}^n} &= \left\{ k = \frac{(1 - \delta_{\mathcal{L}})(1 - \delta_{\mathcal{M}})^{2n-1}(1 - \delta_{\mathcal{R}})^2\delta_{\mathcal{L}}}{1 - (1 - \delta_{\mathcal{M}})^n(1 - \delta_{\mathcal{R}})(1 - \delta_{\mathcal{L}})} + (1 - \delta_{\mathcal{M}})^{n-1}(1 - \delta_{\mathcal{R}})\delta_{\mathcal{L}} \right\}
\end{aligned}$$

4.4 Period Adding

In the previous chapter we already observed the period adding scenario between the $O_{\mathcal{LR}^n}$ and the $O_{\mathcal{RL}^n}$ families. Regarding the general case with three partitions, the respective families are separated by cycles involving the $\mathcal{D}_{\mathcal{M}}$ region.

Let us take the $O_{\mathcal{LR}}$ and the $O_{\mathcal{LR}^2}$ cycle as an example. In the binary choice model on two partitions, we saw that the bifurcation lines of this cycles collide in the $\mathcal{D}_{\mathcal{R}} \times \mathcal{D}_{\mathcal{L}}$ plane in the limit case of $\mathcal{D}_{\mathcal{L}} \to 1$. With all three partitions this might be no longer the case as there are various $O_{\mathcal{LRM}^n}$ and $O_{\mathcal{LRM}^n\mathcal{R}}$ cycles in between. What we could observe instead the regular period adding between these families is the period adding scenario between the $O_{\mathcal{LRM}^n}$ cycles and some totally strange scenario between the $O_{\mathcal{LRM}^n}$ cycle with highest n and the $O_{\mathcal{LRM}^m\mathcal{R}}$ cycle with highest m.[1]

[1] As we will see later, n and m are always the same for fixed k. Hence, we will just write n.

Regarding the $O_{\mathcal{LRM}^n}$ family, we can observe between the $O_{\mathcal{LRM}^n}$ and the $O_{\mathcal{LRM}^{n+1}}$ cycle the $O_{\mathcal{LRM}^n\mathcal{LRM}^{n+1}}$ cycle and so on. This is a regular period adding scenario. However, the scenario between the "neighboring" cycles of the $O_{\mathcal{LRM}^n}$ and the $O_{\mathcal{LRM}^n\mathcal{R}}$ families is much more complicated. We expect some blanket of different adding scenarios as we observe the $O_{\mathcal{LRM}^n\mathcal{LRM}^n\mathcal{R}}$ for the largest n for which the $O_{\mathcal{LRM}^n}$ and $O_{\mathcal{LRM}^n\mathcal{R}}$ cycles exist.

4.5 About the maximum number of points in a row in one region

Last in this chapter, we want to regard the length of a series of points in one region. This is useful if we want to know if we can have e.g. a $O_{\mathcal{LRM}^n}$ cycle where n is given and somehow $\delta_{\mathcal{M}}$ or k is fixed. However, this is just the upper bound for n. It does not say anything about the existence of such a cycle.

Proposition 4. *The maximum possible number of points in a row in $\mathcal{D}_{\mathcal{M}}$ is*

$$n = \left\lfloor \frac{1 + \log_2 k}{\log_2(1 - \delta_{\mathcal{M}})} \right\rfloor + 1$$

Proof. We get the maximum points in a row in $\mathcal{D}_{\mathcal{M}}$, when we have the most "space" to fill. This happens at $(0.5, 0.5)^T$. As we regard a limit case, there are two possibilities. Either

1. we will get into $\mathcal{D}_{\mathcal{R}}$ after n points and collide with $x = k$ or
2. we will get into $\mathcal{D}_{\mathcal{L}}$ after n points and collide with $y = k$.

As the computation steps are exactly the same, let us assume the first case: We will collide with $y = k$ after n points in $\mathcal{D}_{\mathcal{M}}$. This gives us the equation

$$\begin{aligned} & k = (1 - \delta_{\mathcal{M}})^n \frac{1}{2} \\ \Rightarrow \quad & 2k = (1 - \delta_{\mathcal{M}})^n \\ \Rightarrow \quad & n = \log_{1-\delta_{\mathcal{M}}}(2k) \\ & \quad = \frac{\log_2(2k)}{\log_2(1 - \delta_{\mathcal{M}})} \\ & \quad = \frac{1 + \log_2(k)}{\log_2(1 - \delta_{\mathcal{M}})}. \end{aligned}$$

We should not forget the first point, which also has been in $\mathcal{D}_{\mathcal{M}}$. Thus we need to add 1. Note also that we have to round down, as we can only do integer steps. □

Next we will regard the points in a row in $\mathcal{D}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{L}}$.

Proposition 5. *The maximum possible number of points in a row in $\mathcal{D}_{\mathcal{R}}$ is*

$$n = \left\lfloor \frac{\log(1 - k)}{\log(1 - \delta_{\mathcal{R}})} \right\rfloor + 1$$

and the maximum possible number of points in a row in $\mathcal{D}_\mathcal{L}$ is

$$n = \left\lfloor \frac{\log(1-k)}{\log(1-\delta_\mathcal{L})} \right\rfloor + 1$$

Proof. The maximum number of points in $\mathcal{D}_\mathcal{R}$ can be reached from the point $(1,0)^T$. We will jump along the $y = 1 - x$ line until we hit k. Therefore we get the following equation

$$\begin{aligned} 1 - k &= (1-\delta_\mathcal{R})^n \cdot 1 \\ \Rightarrow \quad n &= \log_{1-\delta_\mathcal{R}}(1-k) \\ \Rightarrow \quad n &= \frac{\log(1-k)}{\log(1-\delta_\mathcal{R})} \end{aligned}$$

The equation for points in $\mathcal{D}_\mathcal{L}$ is exactly the same. Just replace $\delta_\mathcal{R}$ by $\delta_\mathcal{L}$ and the point $(1,0)^T$ by $(0,1)^T$. □

In this section, we proofed the statements by always choosing the best possible points. However, in the next chapter we will have such a case for the point $(1,0)^T$. The number of $\mathcal{R}$'s after this point is exactly the number of points the above formula gives us.

5 The case $\delta_{\mathcal{L}} = 1$

In this chapter we are going to study one limit case of our model: We will set $\delta_{\mathcal{L}} = 1$. This describes the upper line as we have seen on the $\delta_{\mathcal{L}} \times \delta_{\mathcal{R}}$ bifurcation diagrams.

Because our map on $\mathcal{D}_{\mathcal{L}}$ is linear (as every map on every partition in the model), we hope that the bifurcation structure will be preserved to some extend. On the other side, there are only three parameters left, and our system is now a lot easier, as we will see in the following section.

5.1 Properties

Let (x_0, y_0) be a point in $\mathcal{D}_{\mathcal{L}}$. We can easily compute the point (x_1, y_1), where it will be mapped to:

$$x_1 = (1 - \delta_{\mathcal{L}})x + \delta_{\mathcal{L}} = (1-1)x_n + 1 = 1$$
$$y_1 = (1 - \delta_{\mathcal{L}})y = (1-1)y_n = 0$$

As we have seen, every point in $\mathcal{D}_{\mathcal{L}}$ will be immediately mapped to $(1,0)^T = (x_1, y_1)^T =: p_1$. This gives us some nice properties which we want to phrase in the following proposition.

Proposition 6. *If $\delta_{\mathcal{L}} = 1$, the following statements hold:*

(i) Every cycle goes through the points $p_1 := (1,0)^T$ and $p_2 := (1 - \delta_{\mathcal{R}}, \delta_{\mathcal{R}})^T$.

(ii) Every cycle has one unique point in $\mathcal{D}_{\mathcal{L}}$.

(iii) Every cycle has a $\mathcal{LR}$ prefix in the symbolic sequence.

Proof. We already know that every cycle has to go through $\mathcal{D}_{\mathcal{L}}$. We also know that for every point in $\mathcal{D}_{\mathcal{L}}$, the point p_1 has to follow. So every cycle has to go through $p_1 = (1,0)^T$. Since p_1 has to be in $\mathcal{D}_{\mathcal{R}}$, we can easily compute the point that will follow, which has to be $(1 - \delta_{\mathcal{R}}, \delta_{\mathcal{R}}) = p_2$.

Our model is deterministic. A cycle cannot go through one point several times. So the point p_1 and its previous point has to be unique in the cycle. In other words: Every cycle has a unique point in $\mathcal{D}_{\mathcal{L}}$.

Last, we already discussed that p_1 is in $\mathcal{D}_{\mathcal{R}}$ and the point before is in $\mathcal{D}_{\mathcal{L}}$. So every cycle has a $\mathcal{LR}$ prefix. □

One thing that follows immediately is that we cannot have any coexistence in this case. If every cycle has to go through two specific points, there cannot be any coexisting cycles. Another thing is that the computation of the cycles can be done very easily. We can just start with the point p_1 and compute the rest of the points according to our model until we reach a point in $\mathcal{D}_{\mathcal{L}}$.

Because we know that the point in $\mathcal{D}_{\mathcal{L}}$ is unique, we will write symbolic sequences starting with the $\mathcal{LR}$ prefix.

5.2 Partitioning according to the number of $\mathcal{R}$'s

As we can see in Figure 5.1, it seems to be natural to study the bifurcation structure on the $\delta_{\mathcal{R}} \times k$ plane according to the number of $\mathcal{R}$'s which are included in the cycles. This number increases for lower $\delta_{\mathcal{R}}$. For smaller k, we will get more points in $\mathcal{D}_{\mathcal{M}}$.

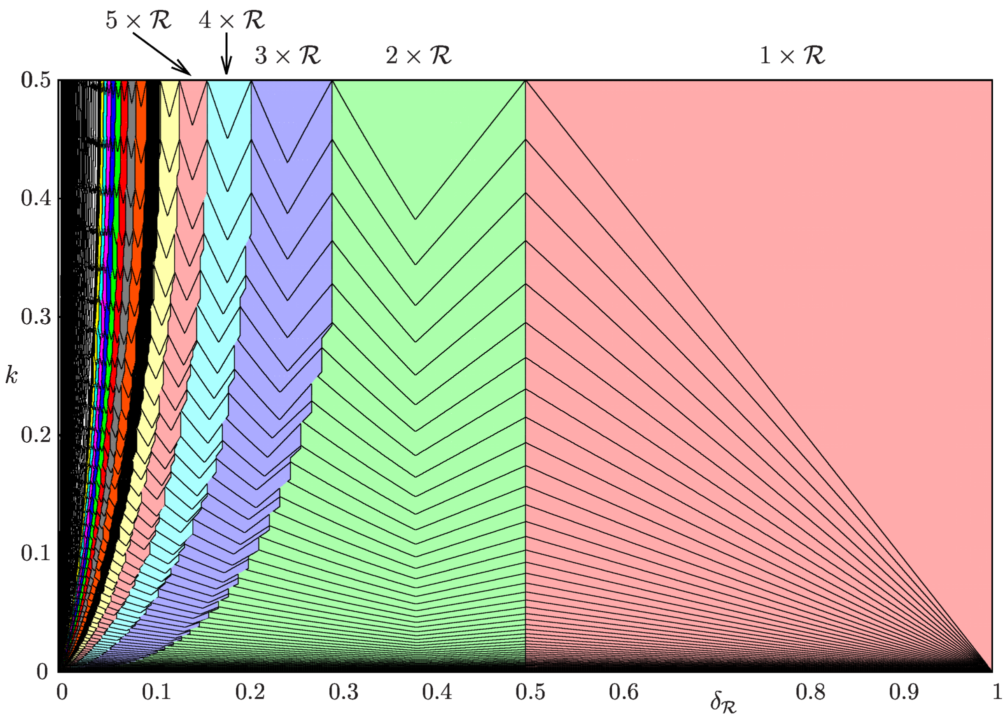


Figure 5.1: Bifurcation diagram of the $\delta_{\mathcal{R}} \times k$ plane in the case $\delta_{\mathcal{L}} = 1$ with $\delta_{\mathcal{M}} = 0.05$. We see that the number of $\mathcal{R}$'s in the cycle increases for smaller $\delta_{\mathcal{R}}$.

In the following sections, we will have a closer look at the families of the cycles with one and two $\mathcal{R}$'s. For three $\mathcal{R}$'s we will just give a rough overview, what kind of cycles can happen.

5.3 One R

By Proposition 6 we already know that we have a $\mathcal{LR}$ prefix on the symbolic sequence of any cycle. With only one $\mathcal{R}$, we can just have the $O_{\mathcal{LRM}^n}$ family with $n \geq 0$. However, the case $n = 0$ should be treated separately as there will be only one border collision bifurcation that could happen, while in the $\{O_{\mathcal{LRM}^n} : n > 0\}$ family we will see that we have three border collisions per cycle.

5.3.1 The $O_{\mathcal{LR}}$ cycle

We already know the point $p_2 = (1 - \delta_{\mathcal{R}}, \delta_{\mathcal{R}})^T$ which follows the point p_1 in $\mathcal{D}_{\mathcal{R}}$. If we are on a $O_{\mathcal{LR}}$ cycle, this point has to be in $\mathcal{D}_{\mathcal{L}}$. The point p_2 may be a bit misnamed, so we will call it $p_0 = (x_0, y_0)^T$ here.

Our points are on two partitions only. So they exist on a straight and we have only one constraint to check if we want to know whether we are in $\mathcal{D}_{\mathcal{L}}$ or in $\mathcal{D}_{\mathcal{M}}$. This one constraint is $x_0 < k$. Since we already know what x_0 is, we can simply write down the border collision bifurcation

$$\xi_{\mathcal{LR}}^{(x_0=k)} = \{x_0 = k\} = \{1 - \delta_{\mathcal{R}} = k\}.$$

This is the only border collision that could happen for $O_{\mathcal{LR}}$, because p_1 is fixated on $(1, 0)^T$ and has to be in $\mathcal{D}_{\mathcal{R}}$ for any k.

5.3.2 The $O_{\mathcal{LRM}^n}$ family

For the $O_{\mathcal{LR}}$ cycle, the fixated point $p_2 = (1 - \delta_{\mathcal{R}}, \delta_{\mathcal{R}})^T$ (which is not a fixpoint) had to be in the $\mathcal{D}_{\mathcal{L}}$ region. What happens if it crosses the border $x = k$ for some $\delta_{\mathcal{R}}$ and k? It will be in $\mathcal{D}_{\mathcal{M}}$ and we will get a cycle which involves the $\mathcal{D}_{\mathcal{M}}$ region. If we move the point p_2 deeper and deeper into the $\mathcal{D}_{\mathcal{M}}$ region, we will get more and more points in $\mathcal{D}_{\mathcal{M}}$ for our cycle. However, something totally different will happen if we cross the "virtual" border at the diagonal line. We can no longer return into $\mathcal{D}_{\mathcal{L}}$ then, but will get into $\mathcal{D}_{\mathcal{R}}$ and thus have a second $\mathcal{R}$ in our cycle. We will cover this in the next section.

Collisions of the point in $\mathcal{D}_{\mathcal{L}}$

We first want to investigate the collision that will add another point in $\mathcal{D}_{\mathcal{R}}$. If the point p_2 is below the main diagonal line, we will get – after a series of points in $\mathcal{D}_{\mathcal{M}}$ – another point in $\mathcal{D}_{\mathcal{R}}$. This happens at $x_2 = 0.5$. Note that x_2 does not collide with any real border, but it causes the border collision bifurcation of $x_0 = y_0$. We will thus write this border collision as

$$\xi_{\mathcal{LRM}^n}^{(x_0=y_0)} = \{1 - \delta_{\mathcal{R}} = 0.5\} = \{\delta_{\mathcal{R}} = 0.5\}.$$

The other collision that could happen is the $x_0 = k$ collision. If this happens, we will get another point in $\mathcal{D}_{\mathcal{M}}$. When does this happen? We already computed p_2, which has to be in $\mathcal{D}_{\mathcal{M}}$, since $n > 0$. We can easily compute the following n points which also have to be in $\mathcal{D}_{\mathcal{M}}$.

$$x_0 = x_{n+2} = (1 - \delta_{\mathcal{M}})^n x_2 = (1 - \delta_{\mathcal{M}})^n (1 - \delta_{\mathcal{R}})$$

Therefore, we will write down this border collision bifurcation as

$$\xi_{\mathcal{LRM}^n}^{(x_0=k)} = \{(1 - \delta_{\mathcal{M}})^n (1 - \delta_{\mathcal{R}}) = k\}.$$

Collision of the last point in $\mathcal{D}_{\mathcal{M}}$

The third border collision happens with the last point in $\mathcal{D}_{\mathcal{M}}$. If this point will collide with k, we will get directly into $\mathcal{D}_{\mathcal{L}}$ and have a cycle with one $\mathcal{M}$ less. Again we can simply compute this point:

$$x_{n+1} = (1 - \delta_{\mathcal{M}})^{n-1} x_2 = (1 - \delta_{\mathcal{M}})^{n-1}(1 - \delta_{\mathcal{R}}).$$

Note that this also just holds because $n > 0$. So we have the third and last border collision curve

$$\xi_{\mathcal{LRM}^n}^{x_{n+1}=k} = \{(1 - \delta_{\mathcal{M}})^{n-1}(1 - \delta_{\mathcal{R}}) = k\}.$$

5.3.3 Some special properties of the $O_{\mathcal{LRM}^n}$ family

The $O_{\mathcal{LRM}^n}$ family is a well behaved one. We have some special properties that will not necessary hold for the other families we study in this chapter.

- If a cycle of the $O_{\mathcal{LRM}^n}$ family exists for any k_0, it will exist for every $k < k_0$, too.
- The members are adjacent.
- There are infinite many members of the $O_{\mathcal{LRM}^n}$ family.

5.4 Two R's

If $\delta_{\mathcal{R}} < 0.5$, the point $p_2 = (1 - \delta_{\mathcal{R}}, \delta_{\mathcal{R}})^T$ will be below the main diagonal line. Therefore, after a few points in $\mathcal{D}_{\mathcal{M}}$, we have to return back into the $\mathcal{D}_{\mathcal{R}}$ region. This adds another $\mathcal{R}$ in the symbolics. Let us begin with the question, which cycles with two points in $\mathcal{D}_{\mathcal{R}}$ can exist in the $\delta_{\mathcal{L}} = 1$ case. From Proposition 6 we already know that we have a $\mathcal{LR}$ prefix and that there has to be one $\mathcal{L}$ in the sequence. Thus, we have only one $\mathcal{R}$ left which could have some $\mathcal{M}$'s before or after it. We get

- the $O_{\mathcal{LR}^2}$ cycle
- the $\{O_{\mathcal{LR}^2\mathcal{M}^n} : n > 0\}$ family
- the $\{O_{\mathcal{LRM}^n\mathcal{R}} : n > 0\}$ family
- and last the $\{O_{\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}} : n_1, n_2 > 0\}$ family

5.4.1 The location of the families

In Figure 5.3 we see the location of the families on the $\delta_{\mathcal{R}} \times k$ bifurcation diagram. Notice that the members of the families are adjacent to each other. Also notice that we already can see in the figure that the $O_{\mathcal{LR}^2\mathcal{M}^n}$ family is finite, which we will proof later. The other families are infinite.

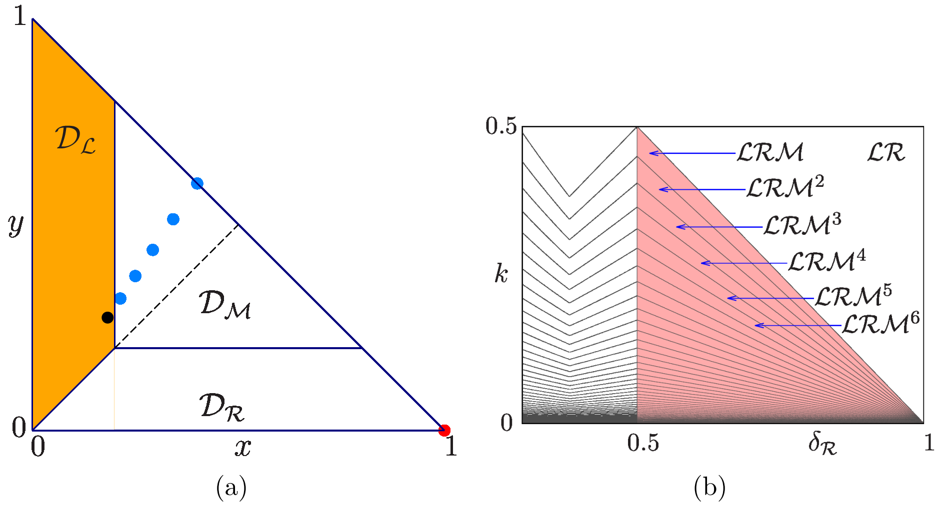


Figure 5.2: In 5.2a we see an example of the $O_{\mathcal{LRM}}$ cycle. The point p_2 has to be above the main diagonal line in this case. 5.2b shows the location of the members of the $O_{\mathcal{LRM}^n}$ family.

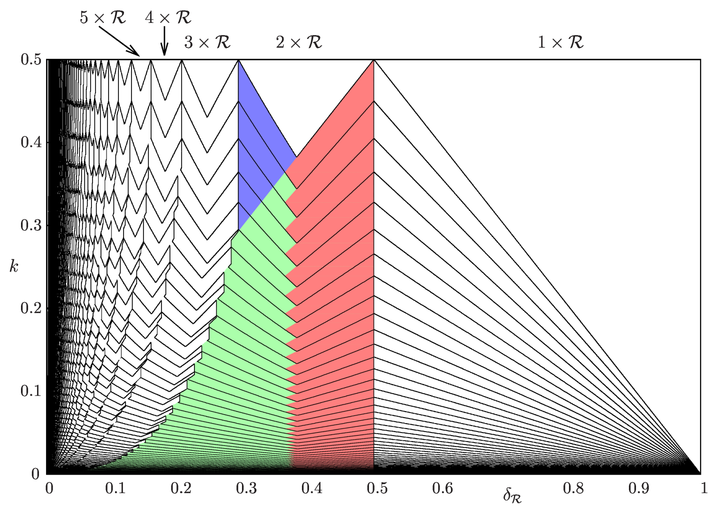


Figure 5.3: The location of the families with two $\mathcal{R}$'s. In red the $O_{\mathcal{LRM}^n\mathcal{R}}$ family, blue is the $O_{\mathcal{LR}^2\mathcal{M}^n}$ family and green is the general $O_{\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}}$ family. The $O_{\mathcal{LR}^2}$ cycle exists in the white top triangle.

5.4.2 The $O_{\mathcal{LR}^2}$ cycle

Regarding the $O_{\mathcal{LR}^2}$ cycle, we already know that $\delta_{\mathcal{R}}$ has to be smaller than 0.5. On the other hand, $p_2 = (1 - \delta_{\mathcal{R}}, \delta_{\mathcal{R}})$ still has to be in $\mathcal{D}_{\mathcal{R}}$. Another thing we already know is that this cycles lies on the $y = 1 - x$ line. This means that we can regard the system as a 1D system by using only the x-components of any point.

Collisions of p_2 in $\mathcal{D}_{\mathcal{R}}$

The point $p_1 = (1, 0)^T$ is the point where every cycle has to go through. As this point is fixated and for every k in $\mathcal{D}_{\mathcal{R}}$, it cannot collide with anything. So we will have a look at the following point, p_2 which is given by $(1 - \delta_{\mathcal{R}}, \delta_{\mathcal{R}})$. The border collision bifurcation line is given by

$$\xi_{\mathcal{LR}^2}^{(y_2=k)} = \{\delta_{\mathcal{R}} = k\}.$$

Collisions of p_0 in $\mathcal{D}_{\mathcal{L}}$

The other point of this cycle that can collide is the first one, p_0, which has to be in $\mathcal{D}_{\mathcal{L}}$. It is simply the following point of $p_2 \in \mathcal{D}_{\mathcal{R}}$. Note that we just need the x-value for checking if this point is in $\mathcal{D}_{\mathcal{L}}$, because it has to be on the $y = 1 - x$ line and cannot collide with $x_0 = y_0$:

$$x_0 = (1 - \delta_{\mathcal{R}})x_2 = (1 - \delta_{\mathcal{R}})^2.$$

This gives us the second border bifurcation line of the $O_{\mathcal{LR}^2}$ cycle:

$$\xi_{\mathcal{LR}^2}^{(x_0=k)} = \{(1 - \delta_{\mathcal{R}})^2 = k\}.$$

We have seen that there are only two bifurcation curves. As we already pointed out, there cannot be a collision with the point p_1. There also cannot be a collision with $x = y$ in $\mathcal{D}_{\mathcal{L}}$ or $\mathcal{D}_{\mathcal{R}}$, because we are on the straight $y = 1 - x$. We will cover the whole $O_{\mathcal{LR}^n}$ family later in this chapter.

5.4.3 The $O_{\mathcal{LR}^2\mathcal{M}^n}$ family

Now we want to analyze the $O_{\mathcal{LR}^2\mathcal{M}^n}$ family. Since we already did this for the $O_{\mathcal{LRM}^n}$ family we expect something similar. However, the point p_2 has now to be in $\mathcal{D}_{\mathcal{R}}$ and therefore can collide with the $y = k$ line. So we should expect at least one more border collision as for the $O_{\mathcal{LRM}^n}$ family. Indeed, we get four border collisions.

Collision of p_2 in $\mathcal{D}_{\mathcal{R}}$

As we pointed out earlier, $p_1 = (1, 0)^T$ cannot collide with anything. So let us consider $p_2 = (1 - \delta_{\mathcal{R}}, \delta_{\mathcal{R}})^T$. This point has to be in $\mathcal{D}_{\mathcal{R}}$. But it is on the straight $y = 1 - x$. Thus it can only collide with $y = k$. This gives us

$$\xi^{(y_2=k)}_{\mathcal{L}\mathcal{R}^2\mathcal{M}^n} = \{\delta_{\mathcal{R}} = k\}$$

Note that this equation is independent on $\delta_{\mathcal{M}}$ and n. Hence it somehow restricts the whole family.

Collision of p_{n+2} in $\mathcal{D}_{\mathcal{M}}$

The point p_3 has to be in $\mathcal{D}_{\mathcal{M}}$. Notice that it has to lay above the main diagonal line, because we have to end up in $\mathcal{D}_{\mathcal{L}}$ after a series of points in $\mathcal{D}_{\mathcal{M}}$. Therefore, we can have the $x = k$ collision only for any point in this series of points in $\mathcal{D}_{\mathcal{M}}$. If $n > 1$, the point that follows p_3, p_4 has to be in $\mathcal{D}_{\mathcal{M}}$, too. Because $x_4 = (1 - \delta_{\mathcal{R}})x_3 < x_3$, it will collide with the k-line first. Hence, we only have to consider the last point of the series of points in $\mathcal{D}_{\mathcal{M}}$. This point is named p_{n+2}. We get the bifurcation line with

$$\xi^{(x_{n+1}=k)}_{\mathcal{L}\mathcal{R}^2\mathcal{M}^n} = \{(1 - \delta_{\mathcal{M}})^{n-1}(1 - \delta_{\mathcal{R}})^2 = k\}.$$

Collisions of p_0 in $\mathcal{D}_{\mathcal{L}}$

After the series of points in $\mathcal{D}_{\mathcal{M}}$, we want to return to the unique point p_0 in $\mathcal{D}_{\mathcal{L}}$. This point is now given by

$$\begin{aligned} x_0 &= (1 - \delta_{\mathcal{M}})^n (1 - \delta_{\mathcal{R}})^2 \\ y_0 &= (1 - \delta_{\mathcal{M}})^n \delta_{\mathcal{R}} (2 - \delta_{\mathcal{R}}) \end{aligned}$$

It can collide with both borders of the $\mathcal{D}_{\mathcal{L}}$ region. However, we can express the $y = x$ collision also with the point p_3 being above the main diagonal line. This gives us

$$\begin{aligned} \xi^{(x_0=y_0)}_{\mathcal{L}\mathcal{R}^2\mathcal{M}^n} &= \{(1 - \delta_{\mathcal{R}})^2 = \delta_{\mathcal{R}}(2 - \delta_{\mathcal{R}})\} \\ &= \left\{\delta_{\mathcal{R}} = 1 - \frac{1}{\sqrt{2}}\right\}. \end{aligned}$$

The other possible border collision of this point is

$$\xi^{(x_0=k)}_{\mathcal{L}\mathcal{R}^2\mathcal{M}^n} = \{(1 - \delta_{\mathcal{M}})^n (1 - \delta_{\mathcal{R}})^2 = k\}.$$

The $O_{\mathcal{L}\mathcal{R}^2\mathcal{M}^n}$ family is finite

Something that is totally different from the $O_{\mathcal{L}\mathcal{R}\mathcal{M}^n}$ family is that the $O_{\mathcal{L}\mathcal{R}^2\mathcal{M}^n}$ family does not have infinite members. It also stops to exist for some k. We will formulate this as a proposition.

Proposition 7. *The maximum n such that $O_{\mathcal{LR}^2\mathcal{M}^n}$ exists in the case $\delta_{\mathcal{L}} = 1$ is given by*

$$n = \left\lceil \frac{\log_2(1 - \frac{1}{\sqrt{2}}) + 1}{\log_2(1 - \delta_{\mathcal{M}})} \right\rceil .$$

The cycle exists with $k = \delta_{\mathcal{R}} = 1 - \frac{1}{\sqrt{2}}$.

Proof. Let $p_1 = (1, 0)^T$ be the unique point in $\mathcal{D}_{\mathcal{R}}$, p_2 be in $\mathcal{D}_{\mathcal{R}}$ and p_3 be in $\mathcal{D}_{\mathcal{M}}$. To get the maximum steps in $\mathcal{D}_{\mathcal{M}}$, we need to start from the point $(0.5, 0.5)^T$. Recall that p_3 has to be on the $y = 1 - x$ line and can be at this position. It is there exactly for $k = \delta_{\mathcal{R}} = 1 - \frac{1}{\sqrt{2}}$.

As we want to end up in $\mathcal{D}_{\mathcal{M}}$, the last point in $\mathcal{D}_{\mathcal{M}}$ can only collide with the $y = k$ border. We will use this equation to compute n.

$$\begin{aligned} k &= (1 - \delta_{\mathcal{M}})^n 0.5 \\ \Rightarrow \quad 2k &= (1 - \delta_{\mathcal{M}})^n \\ \Rightarrow \quad n &= \log_{1-\delta_{\mathcal{M}}}(2k) \\ &= \frac{\log_2(2k)}{\log_2(1 - \delta_{\mathcal{M}})} \\ &= \frac{\log_2(k) + 1}{\log_2(1 - \delta_{\mathcal{M}})} \end{aligned}$$

□

5.4.4 The $O_{\mathcal{LRM}^n\mathcal{R}}$ family

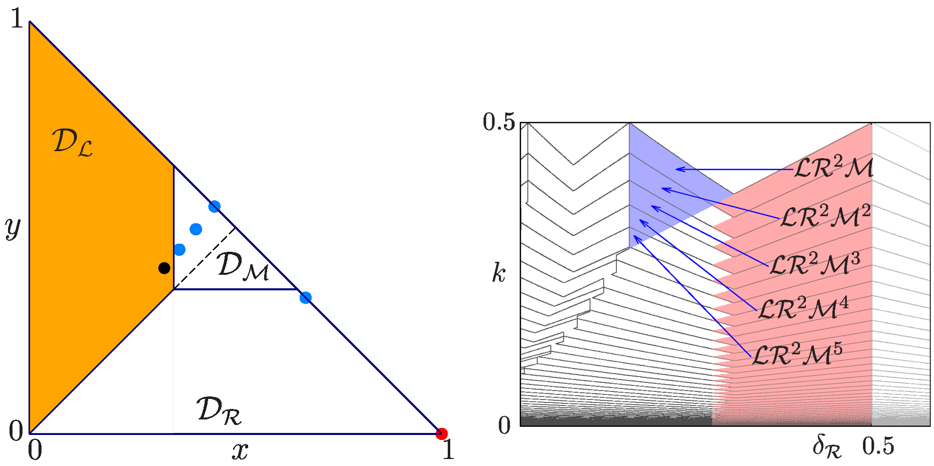
If the point $p_1 = (1 - \delta_{\mathcal{R}}, \delta_{\mathcal{R}})$ lays below the main diagonal line, but still in $\mathcal{D}_{\mathcal{M}}$, we will reach (maybe after some points in $\mathcal{D}_{\mathcal{M}}$) the $\mathcal{D}_{\mathcal{R}}$ region again. Here we want to cover the case that we map from this second point in $\mathcal{D}_{\mathcal{R}}$ directly into $\mathcal{D}_{\mathcal{L}}$.

Collision of p_{n+1} with $y = k$

Let us start again from $p_1 = (1, 0)^T$. This point still cannot collide as it is fixed for any cycle in this case. The next point $p_2 = (1 - \delta_{\mathcal{R}}, \delta_{\mathcal{R}})^T$ hast to be in $\mathcal{D}_{\mathcal{M}}$. However, we have more than just one point in $\mathcal{D}_{\mathcal{M}}$. Because the following point in $\mathcal{D}_{\mathcal{M}}$ is always closer to the border, it will collide first. Thus we just need to regard the last point of a series of points in $\mathcal{D}_{\mathcal{M}}$. In this case this is p_{n+1}.

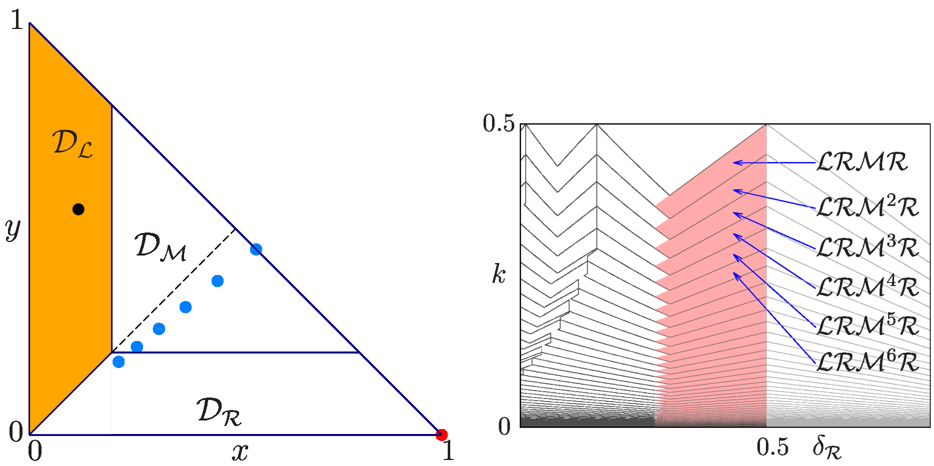
Note that this point cannot collide with $x = k$ as it has to be below the main diagonal line as the next point has to be in $\mathcal{D}_{\mathcal{R}}$ (we called this a "virtual" border collision before). But it can collide with $y = k$. This gives us

$$\xi^{(y_{n+1}=k)}_{\mathcal{LRM}^n\mathcal{R}} = \{(1 - \delta_{\mathcal{M}})^{n-1}\delta_{\mathcal{R}} = k\}$$



(a) (b)

Figure 5.4: In 5.4a we see an example of the $O_{\mathcal{L}\mathcal{R}^2\mathcal{M}}$ cycle. In this cycle, the point p_1 has to be below the k-line, but the following point has to be already above the main diagonal. 5.4b shows the location of the members of the $O_{\mathcal{L}\mathcal{R}^2\mathcal{M}^n}$ family.



(a) (b)

Figure 5.5: Figure 5.5a shows an example of the $O_{\mathcal{L}\mathcal{R}\mathcal{M}^n\mathcal{R}}$ cycle. Here, the point p_1 has to be below the main diagonal line. In 5.5b, we see the location of the members of the $O_{\mathcal{L}\mathcal{R}\mathcal{M}^n\mathcal{R}}$ family.

Collisions of p_{n+2} with $x = y$ and $y = k$

The following point, p_{n+2} in $\mathcal{D}_{\mathcal{R}}$ can collide with both, the $y = k$ and $x = y$ border:

$$\xi^{(x_{n+2}=y_{n+2})}_{\mathcal{LRM}^n\mathcal{R}} = \{\delta_{\mathcal{R}} = 0.5\}$$
$$\xi^{(y_{n+2}=k)}_{\mathcal{LRM}^n\mathcal{R}} = \{(1-\delta_{\mathcal{M}})^n\delta_{\mathcal{R}} = k\}$$

Collision of p_0 with $x = k$

The next point of this cycle, p_0 in $\mathcal{D}_{\mathcal{L}}$ could only collide with $x = k$. Just assume the “worst case”, when p_{n+2} lays on the x-axis. Since it would get above the k line from p_1 it will do the same here. So we not only know that we are above the diagonal line but also above the $y = k$ line.

For the $x = k$ border collision we get

$$\xi^{(x_0=k)}_{\mathcal{LRM}^n\mathcal{R}} = \{(1-\delta_{\mathcal{R}})^2(1-\delta_{\mathcal{M}})^n = k\}$$

Properties of the $O_{\mathcal{LRM}^n\mathcal{R}}$ cycle

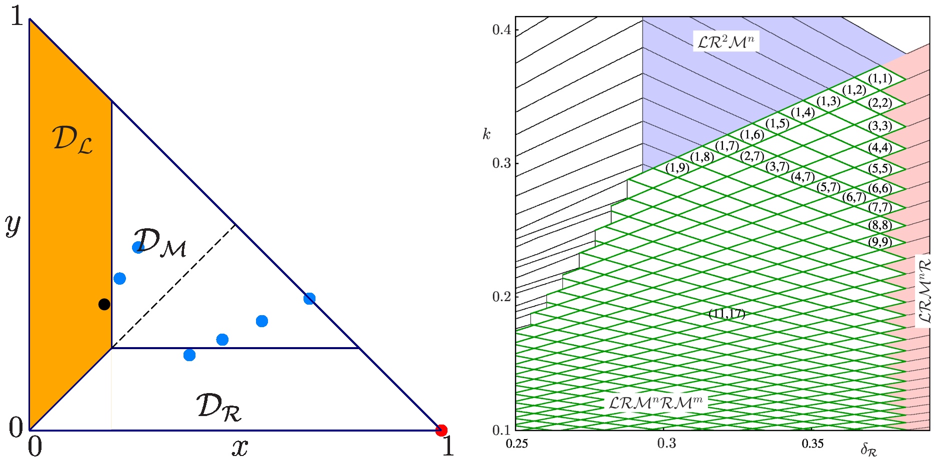
The $O_{\mathcal{LRM}^n\mathcal{R}}$ family has infinite members. The region it exists in gets smaller for higher periods and lower k. If the $O_{\mathcal{LRM}^n\mathcal{R}}$ cycle exists for any n, the $O_{\mathcal{LRM}^n}$ cycle will exist, too, for fix k. This statement does not hold the other way around. We have already seen that if a cycle of the $\{O_{\mathcal{LRM}^n}\}$ family will exist for given k_0, it will also exist for $k < k_0$. But this is not true for the $\{O_{\mathcal{LRM}^n\mathcal{R}}\}$ family.

5.4.5 The $O_{\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}}$ family

Now let us consider the most general case, the $O_{\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}}$ family with $n_1, n_2 > 0$. We will see that the bifurcation structure of this cycles forms sort of a fishnet. Every cycle of this family has in the general case four bifurcation lines. However, on the border of the adjacent family, we can see up to five lines.

The first possible border collision: $y_{n+1} = k$

The point p_1 cannot collide with anything in the case $\delta_{\mathcal{L}} = 1$. So let us start with $p_2 = (1-\delta_{\mathcal{R}}, \delta_{\mathcal{R}})^T$. This point has to be in $\mathcal{D}_{\mathcal{M}}$. Since we map towards the origin in $\mathcal{D}_{\mathcal{M}}$, we need to be below the main diagonal line to reach the $\mathcal{D}_{\mathcal{R}}$ region after n_1 steps. Therefore, the $x = k$ border collision cannot happen for the first n_1+1 points. If $n_1 = 1$, the point p_2 can collide with $y_2 = k$ to the lower border of the $\mathcal{D}_{\mathcal{M}}$ region. However, if $n_1 > 1$, a point closer to the origin will follow and it will collide with $y_3 = k$ before, since $y_3 = (1-\delta_{\mathcal{M}})y_2 < y_2$. But if $n_1 > 2$, we will have another point which will be even closer to the k-line. Hence, the first border collision that could happen has to be the last point in this sequence of points in $\mathcal{D}_{\mathcal{M}}$. Namely it is p_{n_1+1} with the collision $y_{n_1+1} = k$. Since we already



(a) (b)

Figure 5.6: An example of the $O_{\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}}$ cycle is shown in 5.6a. In this case, the point p_1 has to be in $\mathcal{D}_{\mathcal{M}}$ and below the main diagonal line. After the second point in $\mathcal{D}_{\mathcal{R}}$, we need to map above the main diagonal line.
In 5.6b we see the location of the members of the $O_{\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}}$ family. We will call this structure the fishnet. Cycles in the fishnet have four bifurcation lines. However, cycles on the margin of the fishnet can have up to five.

know $y_2 = \delta_{\mathcal{R}}$, we get $y_{n_1+1} = (1-\delta_{\mathcal{M}})^{n_1-1} y_2 = (1-\delta_{\mathcal{M}})^{n_1-1}\delta_{\mathcal{R}}$. Thus, this bifurcation is given by

$$\xi^{(y_{n_1+1}=k)}_{\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}} = \{(1-\delta_{\mathcal{M}})^{n_1-1}\delta_{\mathcal{R}} = k\}.$$

The second collision with the second point in $\mathcal{D}_{\mathcal{R}}$

We had the first border collision with p_{n_1+1}, which is the last point of the first series of points in $\mathcal{D}_{\mathcal{M}}$. Now we step over the k-line into $\mathcal{D}_{\mathcal{R}}$. This is also the next possible bifurcation. We have

$$\xi^{(y_{n_1+2}=k)}_{\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}} = \{(1-\delta_{\mathcal{M}})^{n_1}\delta_{\mathcal{R}} = k\}$$

What is with the x-component of the point p_{n_1+2}? Can it also collide with $x = y$? Since we map towards $(0,1)^T$ from $\mathcal{D}_{\mathcal{R}}$, the following point would then be in $\mathcal{D}_{\mathcal{L}}$. But it has to be in $\mathcal{D}_{\mathcal{M}}$. Thus this collision cannot happen.

The third possible border collision

After p_{n_1+2} we will get another series of points in $\mathcal{D}_{\mathcal{M}}$. They have to lay above the main diagonal line, to get into the $\mathcal{D}_{\mathcal{L}}$ region after n_2 steps. So we cannot have the $y = k$ border collision. What happens with p_{n_1+3}? This is the first point in $\mathcal{D}_{\mathcal{M}}$ after the second point in $\mathcal{D}_{\mathcal{R}}$. Can it collide with anything? It cannot if $n_2 > 1$. We can reuse the argument why only the last of the n_1 points in $\mathcal{D}_{\mathcal{M}}$ can collide with $y = k$ in the first series of points in $\mathcal{D}_{\mathcal{M}}$. Now it is the $x = k$ bifurcation, but as our system does the same for the x and y coordinate in $\mathcal{D}_{\mathcal{M}}$, we also have the same result for $y = k$. Hence, only the last point in $\mathcal{D}_{\mathcal{M}}$ of this series of points can collide with $y = k$. We get

$$\xi^{(x_{n_1+n_2+2}=k)}_{\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}} = \{(1-\delta_{\mathcal{M}})^{n_1+n_2-1}(1-\delta_{\mathcal{R}})^2 = k\}.$$

The forth and fifth possible border collision

The last two possible border collision bifurcations happen with the point in $\mathcal{D}_{\mathcal{L}}$. It can collide with both of $\mathcal{D}_{\mathcal{L}}$'s borders, $x_0 = y_0$ and $x_0 = k$. To write them down, we first have to compute the point p_0:

$$\begin{aligned} x_0 &= (1-\delta_{\mathcal{M}})^{n_1+n_2}(1-\delta_{\mathcal{R}})^2 \\ y_0 &= (1-\delta_{\mathcal{M}})^{n_2}\left((1-\delta_{\mathcal{R}})y_{n_1+1} + \delta_{\mathcal{R}}\right) \\ &= (1-\delta_{\mathcal{M}})^{n_2}\left((1-\delta_{\mathcal{R}})(1-\delta_{\mathcal{M}})^{n_1}\delta_{\mathcal{R}} + \delta_{\mathcal{R}}\right) \\ &= (1-\delta_{\mathcal{M}})^{n_2}\delta_{\mathcal{R}}\left((1-\delta_{\mathcal{R}})(1-\delta_{\mathcal{M}})^{n_1} + 1\right) \end{aligned}$$

This gives us the last two bifurcation curves

$$\begin{aligned} \xi^{(x_0=y_0)}_{\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}} &= \{(1-\delta_{\mathcal{M}})^{n_1}(1-\delta_{\mathcal{R}})^2 = \delta_{\mathcal{R}}\left((1-\delta_{\mathcal{R}})(1-\delta_{\mathcal{M}}) + 1\right)\} \\ \xi^{(x_0=k)}_{\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}} &= \{(1-\delta_{\mathcal{M}})^{n_1+n_2}(1-\delta_{\mathcal{R}})^2 = k\} \end{aligned}$$

Note that the next to last collision does not contain any k. Hence, it will look – for a specific member of the family – like a vertical line in the $\delta_{\mathcal{R}} \times k$ bifurcation diagram.

The fishnet

The bifurcation lines of the $O_{\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}}$ family form sort of a fishnet (see Figure 5.6b). In between this fishnet, every cycle has four border collision bifurcations. The cycles on the left margin of the fishnet can be described with all five border collision bifurcations.

To write the cycles in a more compact form, we introduce the notation (n_1, n_2) for the $O_{\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}}$ cycle. Regarding a cycle (n_1, n_2) of this family, which lays not on the margin, we see that it has exactly four adjacent neighbors of the same family:

- To the top right, we get the (n_1, n_2-1) cycle after the $\xi^{(x_{n_1+n_2+2}=k)}_{\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}}$ border collision.
- To the bottom right, after the $\xi^{(y_{n_1+2}=k)}_{\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}}$ line, we get the (n_1+1, n_2) cycle.
- To the top left, the $\xi^{(y_{n_1+1}=k)}_{\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}}$ border collision separates the cycle from the (n_1-1, n_2) cycle.
- To the bottom left, we get the (n_1, n_2+1) cycle after the $\xi^{(x_0=k)}_{\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}}$ collision line.

The only bifurcation we did not observe until now is the $\xi^{(x_0=y_0)}_{\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}}$ collision. Indeed, it does only occur on the left margin of the fishnet, for smaller $\delta_{\mathcal{R}}$. The cells of the fishnet then get truncated by this line. Depending on where they get truncated we can have up to five border collisions of one cycle.

When does a (n_1, n_2) cycle exist

A cycle does only exist when $n_1 \leq n_2$. But also n_2 is bound. It is bound by the bifurcation line, that defines the left margin of the fishnet. Recall this bifurcation line:

$$\xi^{(x_0=y_0)}_{\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}} = \{(1-\delta_{\mathcal{M}})^{n_1}(1-\delta_{\mathcal{R}})^2 = \delta_{\mathcal{R}}\left((1-\delta_{\mathcal{R}})(1-\delta_{\mathcal{M}})+1\right)\}$$

It only depends on n_1 and not on n_2, nor on k. As already mentioned, it is therefore just a vertical line in the $\delta_{\mathcal{R}} \times k$ bifurcation diagram. This line defines the maximum n_2 per specific n_1.

The $O_{\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}}$ family is infinite

Last for this family, we want to mention that there are infinite many members, as for smaller k, we can get more points in $\mathcal{D}_{\mathcal{M}}$ and k could be arbitrary small.

Another argument is the one given right before: Since (n_1, n_2) exists for $n_1 \leq n_2$, we will have a (n, n) cycle for any n.

5.5 Three R's

After having seen the cycles with two $\mathcal{R}$'s, we will shortly have a look at the cycles with three $\mathcal{R}$'s. Notice that the procedure to get all cycles that can happen is always the same. However, with three $\mathcal{R}$'s we will see for the first time that some combinations of cycles do not exist.

To get a list of all possible sequences with three $\mathcal{R}$'s and one $\mathcal{L}$, we can simply write down the most general one, here $\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}\mathcal{RM}^{n_3}$ and than simply use binary counting and set n_i to zero, if the i-th bit is 0.

$(0,0,0) \rightarrow \mathcal{LR}^3$ (just a cycle, not a family)

$(0,0,1) \rightarrow \mathcal{LR}^3\mathcal{M}^n$

$(0,1,0) \rightarrow \mathcal{LR}^2\mathcal{M}^n\mathcal{R}$

$(0,1,1) \rightarrow \mathcal{LR}^2\mathcal{M}^{n_1}\mathcal{RM}^{n_3}$

$(1,0,0) \rightarrow \mathcal{LRM}^n\mathcal{R}^2$

$(1,0,1) \rightarrow \mathcal{LRM}^{n_1}\mathcal{R}^2M^{n_2}$

$(1,1,0) \rightarrow \mathcal{LRM}^{n_2}\mathcal{RM}^{n_2}\mathcal{R}$

$(1,0,1) \rightarrow \mathcal{LRM}^{n_1}\mathcal{R}^2\mathcal{M}^{n_2}$

$(1,1,1) \rightarrow \mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}\mathcal{RM}^{n_2}$

Notice that $(0, 0, 0) \to \mathcal{LR}^3$ is not a family, but a cycle. As we now have all possible sequences, we can strike out those that cannot happen. We already saw all those families in the $\delta_R \times k$ plane, except the ones with a $\mathcal{LRM}^n\mathcal{R}^2$ prefix for $n > 0$. In fact, those cannot happen, as we will see in the following proposition.

Proposition 8. *$\mathcal{LRM}^n\mathcal{R}^2$ cannot be a prefix of any cycle in the case $\delta_{\mathcal{L}} = 1$.*

Proof. First we will remark, that because the point $(x_1, y_1)^T = (1, 0)^T$ always lies in $\mathcal{D}_{\mathcal{R}}$, we know that the point after it has to be $(x_2, y_2)^T = (1 - \delta_{\mathcal{R}}, \delta_{\mathcal{R}})^T$. Because $(x_1, y_1)^T$ has to be in $\mathcal{D}_{\mathcal{M}}$, we have $y_2 > k$ and so $\delta_{\mathcal{R}} > k$.

We will now compute only the y-coordinates of our $\mathcal{LRM}^n\mathcal{R}^2$ prefix.

$$\begin{aligned}
\mathcal{D}_{\mathcal{M}} &\ni y_2 = \delta_{\mathcal{R}} \\
\mathcal{D}_{\mathcal{M}} &\ni y_3 = (1 - \delta_{\mathcal{M}})y_2 = (1 - \delta_{\mathcal{M}})\delta_{\mathcal{R}} \\
\mathcal{D}_{\mathcal{M}} &\ni y_4 = (1 - \delta_{\mathcal{M}})^2\delta_{\mathcal{R}} \\
&\vdots \\
\mathcal{D}_{\mathcal{M}} &\ni y_{n+1} = (1 - \delta_{\mathcal{M}})^{n-1}\delta_{\mathcal{R}} \\
\mathcal{D}_{\mathcal{R}} &\ni y_{n+2} = (1 - \delta_{\mathcal{M}})^{n}\delta_{\mathcal{R}} \\
\mathcal{D}_{\mathcal{R}} &\ni y_{n+3} = (1 - \delta_{\mathcal{M}})^{n}\delta_{\mathcal{R}}^2 + \delta_{\mathcal{R}}
\end{aligned}$$

Let us now have a look on y_{n+3}. We have

$$y_{n+3} = \underbrace{(1 - \delta_{\mathcal{M}})^{n}\delta_{\mathcal{R}}^2}_{>0} + \underbrace{\delta_{\mathcal{R}}}_{>k} > k.$$

Thus, the point $(x_{n+3}, y_{n+3})^T$ is still in $\mathcal{D}_{\mathcal{M}}$ and not in $\mathcal{D}_{\mathcal{R}}$ and the $\mathcal{LRM}^n\mathcal{R}^2$ prefix cannot exist in the case $\delta_{\mathcal{L}} = 1$. □

Note that this proposition also strikes out a lot of possibilities of cycles with 4 or even more $\mathcal{R}$'s. However, for the proof we needed that the cycle goes through the point $p_1 = (1, 0)^T$. This will not hold for $\mathcal{D}_{\mathcal{L}} < 1$.

5.6 The $O_{\mathcal{LR}^n}$ family

Let us have a look at a family of cycles on two partitions: The $O_{\mathcal{LR}^n}$ family. Recall, that we can convert our model into a 1D model in this case (see 4.2). Thereby, the slope on the left side will be zero, and every point in $\mathcal{D}_{\mathcal{L}}$ will be mapped to 1.

In this section, we will first study the bifurcation lines analytically, and then show the main result: If the $O_{\mathcal{LR}^n}$ does not exist on $\mathcal{D}_{\mathcal{L}} = 1$ it will not exist on any $\mathcal{D}_{\mathcal{L}} < 1$. Thus we have some statement about this cycle that holds for any $\mathcal{D}_{\mathcal{L}}$.

5.6.1 Possible border collision bifurcations

Again, we use the fact that cycles on two partitions exist only on the $y = 1 - x$ straight. We will also use Proposition 6 with the statement that every cycle has to go through $p_1 = (1, 0)^T$. So what can happen? There are two points that can collide. Either p_0 in $\mathcal{D}_\mathcal{L}$ can collide with k (as in $x_0 = k$) or the last point in $\mathcal{D}_\mathcal{R}$, p_n can collide with k (as in $y_n = k$). The other points are on this straight. So they cannot collide with anything as long as p_n has not had its collision.

Collision of the point p_n in $\mathcal{D}_\mathcal{R}$

From the well known point $p_1 = (1, 0)^T \in \mathcal{D}_\mathcal{R}$ we can compute the point $p_n = (x_n, y_n)^T$ with

$$x_n = (1 - \delta_\mathcal{R})^{n-1} x_0 = (1 - \delta_\mathcal{R})^{n-1}$$
$$y_n = 1 - x_n = 1 - (1 - \delta_\mathcal{R})^{n-1}$$

The equation of y_n follows because we have to be on the $y = 1 - x$ line. So we can simply write down the border collision bifurcation line as

$$\xi_{\mathcal{LR}^n}^{(y_n = k)} = \{1 - (1 - \delta_\mathcal{R})^{n-1} = k\}.$$

Collision of the point p_0 in $\mathcal{D}_\mathcal{L}$

If k gets too small, we can no longer reach the $\mathcal{D}_\mathcal{L}$ region from the last point of the series of points in $\mathcal{D}_\mathcal{R}$. Since we already know x_n, we can easily compute the x-coordinate of the following point, $x_0 = (1 - \delta_\mathcal{R}) x_n = (1 - \delta_\mathcal{R})^n$. This leads us to

$$\xi_{\mathcal{LR}^n}^{(x_0 = k)} = \{(1 - \delta_\mathcal{R})^n = k\}.$$

5.6.2 If $O_{\mathcal{LR}^n}$ does not exist for $\delta_\mathcal{L} = 1$ it will not exist for any $\delta_\mathcal{L} < 1$

In the $\delta_\mathcal{L} = 1$ situation, we had the point $p_1 = (1, 0)^T$. For any $\delta_\mathcal{L} < 1$ we get a point close to it, but still slightly a bit up the $y = 1 - x$ line, where any cycle on two regions has to exist. Let this point be $\hat{p}_1 = (1 - \varepsilon, \varepsilon)^T$ with $\varepsilon > 0$. Thus, if a cycle $O_{\mathcal{LR}^n}$ exists for any n in the case $\delta_\mathcal{L} = 1$, it does not necessary exist for $\delta_\mathcal{L} < 1$. The other way around: A cycle $O_{\mathcal{LR}^n}$ which does not exist for $\delta_\mathcal{L} = 1$ cannot exist for any $\delta_\mathcal{L} < 1$.

6 The $\mathcal{LRM}^n$ islands

We have seen in the previous chapter that once a $O_{\mathcal{LRM}^n}$ cycle has been created for any k_0, it will exist for every $k < k_0$. For some k, the areas in the $\delta_{\mathcal{R}} \times \delta_{\mathcal{L}}$ plane where the cycle exists is split up into two different regions. We will call the lower one "island" and the upper one "mainland". To make things a bit more clear, we will first follow the "life" of a member of the $O_{\mathcal{LRM}^n}$ family. After that we are going to have a deeper look at the coexistence. Last we want to sketch the period adding scenario with the coexistence.

6.1 How are the islands created?

Let us follow the $O_{\mathcal{LRM}^2}$ as an example of a family member of the $O_{\mathcal{LRM}^n}$ family. We want to see how it develops for decreasing k starting with $k = 0.5$. Note that we have already computed the bifurcation lines of the whole $O_{\mathcal{LRM}^n}$ family in chapter 4. Also see figures 6.1 and 6.2.

1. In the first stage, our cycle does not exist (see also figure 5.1). We know from the last chapter that it has to exist with $\delta_{\mathcal{L}} = 1$ before it can exist for any other $\delta_{\mathcal{L}}$.

2. For $k = k_0$, the cycle is "born". It exists and has immediately two possible border collision bifurcations, namely $\xi^{(x_3=k)}_{\mathcal{LRM}^n}$ and $\xi^{(x_0=y_0)}_{\mathcal{LRM}^n}$.

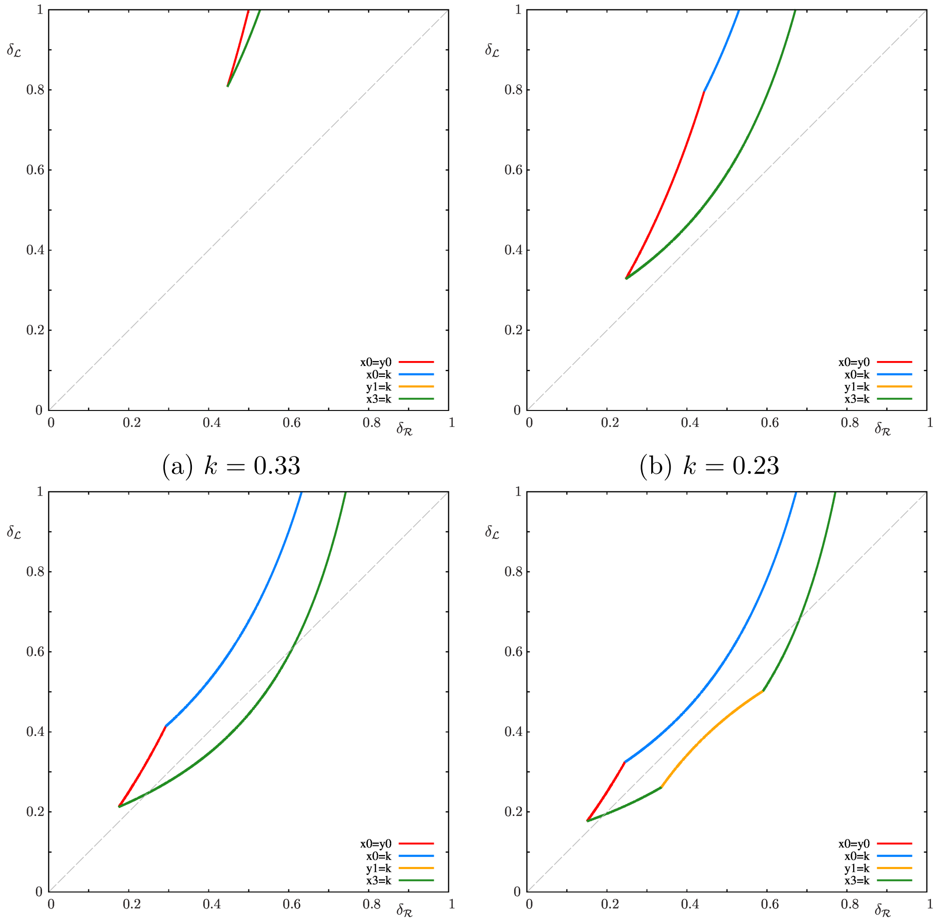
3. While k decreases slightly, we will get a third border collision bifurcation with $\xi^{(x_0=k)}_{\mathcal{LRM}^2}$.

4. Next, our cycle will coexist with its "sister" cycle, $O_{\mathcal{RLM}^2}$, as it exists below the main diagonal line.

5. We can have another border collision bifurcation, $\xi^{(y_1=k)}_{\mathcal{LRM}^2}$.

6. Our cycle is split into two separated areas. We call the one with lower $\delta_{\mathcal{L}}$ and $\delta_{\mathcal{R}}$ the "island" and the one with higher $\delta_{\mathcal{L}}$ and $\delta_{\mathcal{R}}$ the "mainland". The cycle still coexists with its sister cycle on both, the mainland and the island. However, we will see later that it coexists with a lot more cycles, but still has some area where it exists on its own.

7. Last, both areas of the cycle get smaller and smaller. But as we have pointed out in the last chapter, the cycle will exist until $k = 0$.



(a) $k = 0.33$ (b) $k = 0.23$

(c) $k = 0.18$ (d) $k = 0.16$

Figure 6.1: Development of the $O_{\mathcal{LRM}^2}$ cycle with $\delta_{\mathcal{M}} = 0.3$ from stage 2 to stage 5. In 6.1a we first see only two border collisions, namely $\xi^{(x_0=y_0)}_{\mathcal{LRM}^2}$ and $\xi^{(x_3=k)}_{\mathcal{LRM}^2}$. When we decrease k, we can see the third border collision, $\xi^{(x_0=k)}_{\mathcal{LRM}^2}$ coming in (6.1b). For even lower k, we see in 6.1c that the cycle steps over the main diagonal line. This causes coexistence with its sister cycle, which is simply mirrored on the diagonal. Figure 6.1d shows that the remaining fourth possible border collision, $\xi^{(y_1=k)}_{\mathcal{LRM}^2}$ comes in as last for low k values.

6.2 Coexistence

If the $O_{\mathcal{LRM}^n}$ cycle and the $O_{\mathcal{RLM}^n}$ cycle overlap, we get coexistence of all possible cycles with one $\mathcal{L}$, one $\mathcal{R}$ and n points in $\mathcal{M}$ at the same time.

Therefore, every island has three zones: One where only the $O_{\mathcal{LRM}^n}$ exists, another one where only the $O_{\mathcal{RLM}^n}$ exists and one where every possible combination exists.

To make things more concrete, we will use the $\mathcal{LRM}^5$ island as an example.

The $\mathcal{LRM}^5$ island as an example

Figure 6.3 gives a site plan of the $\mathcal{LRM}^5$ island. We can clearly see the three regions: In the borders of the red line we have the $O_{\mathcal{LRM}^5}$, on the borders of the blue line the $O_{\mathcal{RLM}^5}$ cycle and in both borders all possible cycles with five $\mathcal{M}$'s, one $\mathcal{R}$ and one $\mathcal{L}$.

The point in $\mathcal{D}_{\mathcal{L}}$ Let $(x_0, y_0)^T$ be the point in $\mathcal{D}_{\mathcal{L}}$ of these cycles. All coexisting cycles have the same x_0 coordinate. It is given by

$$x_0 = \frac{(1-\delta_{\mathcal{R}})(1-\delta_{\mathcal{M}})^5\delta_{\mathcal{L}}}{1-(1-\delta_{\mathcal{R}})(1-\delta_{\mathcal{M}})^5(1-\delta_{\mathcal{L}})}.$$

However, the y_0 coordinate is depending on n. We could easily compute it for the $O_{\mathcal{LM}^{5-n}\mathcal{RM}^n}$ cycle and get

$$y_0 = \frac{\delta_{\mathcal{R}}(1-\delta_{\mathcal{M}})^n}{1-(1-\delta_{\mathcal{R}})(1-\delta_{\mathcal{M}})^5(1-\delta_{\mathcal{L}})}.$$

As we could see in figure 6.4, some more points share the same x or y coordinate. That might cause some headache if one tries to study the system numerically and just uses one coordinate of the state space.

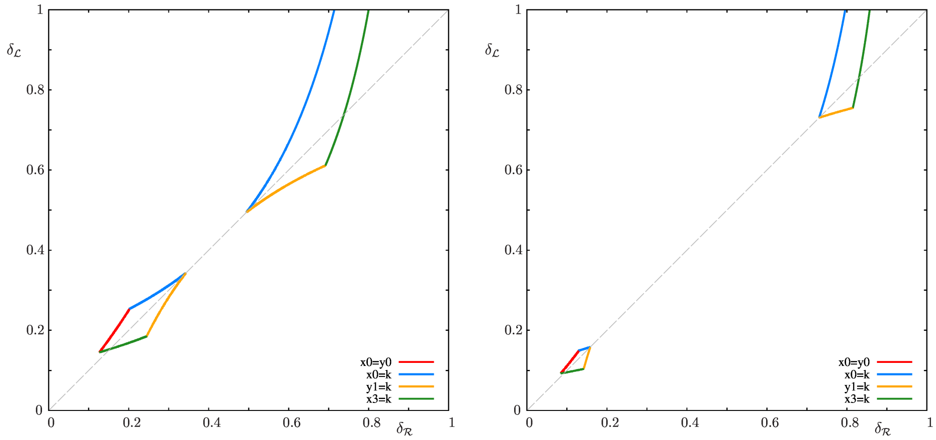
6.3 Period Adding between the islands

Between the islands we have a period adding scheme. In the part where the $O_{\mathcal{LRM}^n}$ cycle exists on its own, we get the $O_{\mathcal{LRM}^n\mathcal{LRM}^{n+1}}$ cycle between the islands.

The situation between the coexisting parts of the islands is much more interesting. We get the cycle with the symbolic sequence

$$\mathcal{LM}^{n_1}\mathcal{RM}^{n_2}\mathcal{LM}^{n_3}\mathcal{RM}^{n_4}$$

if and only if $n_1 = n_3$ or $n_2 = n_4$.



(a) $k = 0.14$ (b) $k = 0.1$

Figure 6.2: The last two stages of the $O_{\mathcal{LRM}^2}$ cycle with $\delta_{\mathcal{M}} = 0.3$. In 6.2a we see the areas in which the cycle exists is split up. We call the lower left the "island" and the upper right the "mainland". Figure 6.2b shows the last stage. The cycle gets smaller and smaller but will exist till $k = 0$.

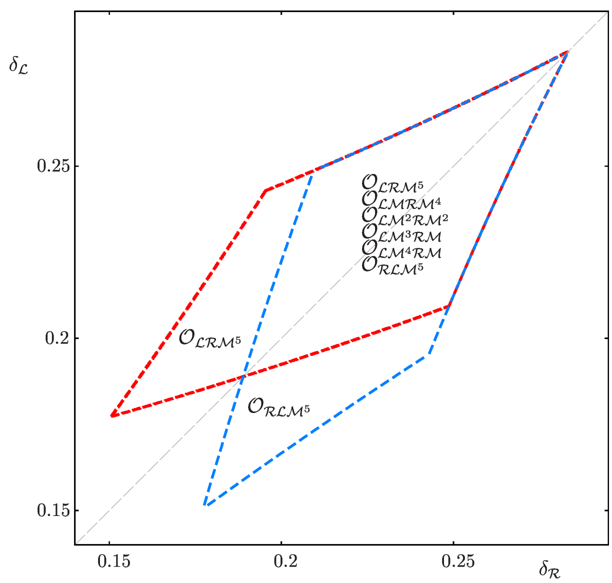


Figure 6.3: Site plan of the $\mathcal{LRM}^5$ island. In red we see the bifurcation lines of the $O_{\mathcal{LRM}^5}$ cycle and in blue the ones of the $O_{\mathcal{RLM}^5}$ cycle. Both have a region where each cycle exists on its own. However, when they overlap we get every possible cycle with five $\mathcal{M}$'s, one $\mathcal{R}$ and one $\mathcal{L}$. Parameter: $\delta_{\mathcal{M}} = 0.2$, $k = 0.08$

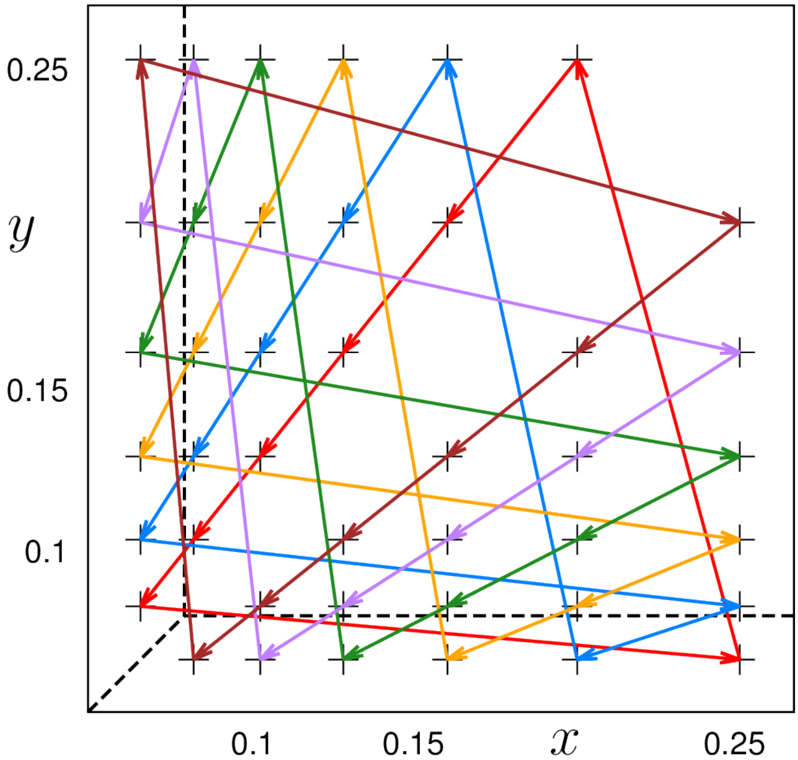


Figure 6.4: All coexisting cycles on the $\mathcal{LRM}^5$ island in the state space. The dashed line represents the border of the regions.

7 Summary and outlook

At first, we had some basic statements and proposition. We had formulas for the maximum number of points in a row in one region as well as the proposition that cycles on two partition exist on the straight $y = 1 - x$. After that, we investigated the system using limit cases. We thereby explained most of the bifurcation diagram of the $\delta_{\mathcal{R}} \times \delta_{\mathcal{L}}$ plane.

Regardless if we are in the special case $k = 0.5$ or not, we have seen that we can turn our model into a 1D model, if we want to investigate cycles on two partitions. Thankfully, this model, the continuous piecewise linear map on three partitions, has already been studied to some extend.

From the case $\delta_{\mathcal{L}} = 1$ we know which cycles can exist for which values of k. We saw that we can partition the $\delta_{\mathcal{R}} \times k$ bifurcation plane in the case $\delta_{\mathcal{L}} = 1$ using the number of $\mathcal{R}$'s in the symbolic sequence of the points and studied the families of one and two $\mathcal{R}$'s completely. For three $\mathcal{R}$'s we just gave the possible families. We also excluded a whole symbolic prefix, which also excludes some possible cycles for higher number of $\mathcal{R}$'s.

After studying the limit case for $\delta_{\mathcal{L}}$, we now know which cycles will come into the $\delta_{\mathcal{R}} \times \delta_{\mathcal{L}}$ plane and for which k. We know that in the area around the diagonal line of the $\delta_{\mathcal{R}} \times \delta_{\mathcal{L}}$ plane, we can have only cycles with one point in $\mathcal{D}_{\mathcal{L}}$ and one point in $\mathcal{D}_{\mathcal{R}}$, as well as some adding structure of them. We also saw how this cycles may create the introduced islands and that we have a lot of coexistence of the same period on them as well as on the mainland.

Outlook

Although we have explained most of the $\delta_{\mathcal{R}} \times \delta_{\mathcal{L}}$ plane, there are a few things left for this model that could be studied further.

Until now we have just observed coexistence between cycles with one point in $\mathcal{D}_{\mathcal{R}}$ and one point in $\mathcal{D}_{\mathcal{L}}$ as well as on the adding scenario between them. It is still left to show for which other cycles coexistence occurs.

Another point is the adding structure between the highest period $O_{\mathcal{LRM}^n}$ cycle and the $O_{\mathcal{LRM}^n\mathcal{R}}$ family. In the case $k = 0.5$ we already saw that this is just the period adding scenario between the $O_{\mathcal{LR}}$ and the $O_{\mathcal{LR}^2}$. However, for $k < 0.5$ this structure seems to be blanked by some other bifurcation scenario caused by the cycles with points in $\mathcal{D}_{\mathcal{M}}$.

List of Figures

1.1 The road network 5

2.1 The state space and where points in one region converges against 8

3.1 Numerics on two partitions 10
3.2 Example of the 1D model 12

4.1 Bifurcation diagram of the $\delta_{\mathcal{R}} \times \delta_{\mathcal{L}}$ plane with $k = 0.36$ and $\delta_{\mathcal{M}} = 0.1$ 14
4.2 Bifurcation diagram of the $\delta_{\mathcal{R}} \times \delta_{\mathcal{L}}$ plane with $k = 0.2$ and $\delta_{\mathcal{M}} = 0.05$ 14
4.3 The $O_{\mathcal{LRM}}$ cycle in state space with its possible border collisions 15

5.1 Bifurcation diagram of the $\delta_{\mathcal{R}} \times k$ plane with $\delta_{\mathcal{L}} = 1$ and $\delta_{\mathcal{M}} = 0.05$ 22
5.2 Example of the $O_{\mathcal{LRM}}$ cycle and where it occurs in the $\delta_{\mathcal{R}} \times k$ plane in the case $\delta_{\mathcal{L}} = 1$ 25
5.3 Location of the families with two $\mathcal{R}$'s 25
5.4 Example of the $O_{\mathcal{LR}^2\mathcal{M}}$ cycle with bifurcation diagram in $\delta_{\mathcal{R}} \times k$ plane in the case $\delta_{\mathcal{L}} = 1$ 29
5.5 Example of the $O_{\mathcal{LRM}^{n_\mathcal{R}}}$ cycle with bifurcation diagram in the $\delta_{\mathcal{R}} \times k$ plane with $\delta_{\mathcal{L}} = 1$ 29
5.6 Example of the $O_{\mathcal{LRM}^{n_1}\mathcal{RM}^{n_2}}$ cycle with bifurcation diagram in the $\delta_{\mathcal{R}} \times k$ plane with $\delta_{\mathcal{L}} = 1$ 31

6.1 Development of the $O_{\mathcal{LRM}^2}$ cycle (1) 38
6.2 Development of the $O_{\mathcal{LRM}^2}$ cycle (2) 40
6.3 Site plan of the $\mathcal{LRM}^5$ island 40
6.4 All coexisting cycles on the $\mathcal{LRM}^5$ island in the state space. The dashed line represents the border of the regions. 41

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Declaration

I hereby declare that the work presented in this thesis is entirely my own and that I did not use any other sources and references than the listed ones. I have marked all direct or indirect statements from other sources contained therein as quotations. Neither this work nor significant parts of it were part of another examination procedure. I have not published this work in whole or in part before. The electronic copy is consistent with all submitted copies.

(Christoph Dibak)