Gröbner bases, multipolynomial resultants and the Gauss-Jacobi combinatorial algorithms -adjustment of nonlinear GPS/LPS observations-

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Abstract

The algebraic techniques of *Gröbner bases* and *Multipolynomial resultants* are presented as efficient algebraic tools for solving explicitly the nonlinear geodetic problems. In particular, these algebraic tools are used to provide symbolic solutions to the problems of *GPS pseudo-ranging four-point P4P*, *Minimum Distance Mapping* and the *threedimensional resection.* The various forward and backward substitution steps inherent in the classical closed form solutions for these selected geodetic problems are avoided. Similar to the Gauss elimination technique in linear systems of equations, the *Gröbner bases* and *Multipolynomial resultants* eliminate several variables in a multivariate system of nonlinear equations in such a manner that the end product normally consists of *univariate polynomial* equations whose roots can be determined by existing programs such as the *roots* command in MATLAB.

The capability of *Gröbner bases* and *Multipolynomial resultants* to solve explicitly nonlinear geodetic problems enables us to use them as the computational engine in the *Gauss-Jacobi combinatorial algorithm* proposed by *C. F. Gauss* (published posthumously e.g *Appendix A.4*) and *C. G. I. Jacobi* (1841) to solve the *nonlinear Gauss-Markov model*. With the nonlinear geodetic observation equations converted into algebraic (polynomial) form, the *Gauss-Jacobi combinatorial algorithm* is carried out in two steps.

In the first step all m combinations of minimal subsets of the observation equations are formed, and the n unknowns of each subset are rigorously solved by means of the Gröbner bases and the Multipolynomial resultants. The m solution sets can be represented as points in an n-dimensional space $\mathbb{R}^n$. In a second step the solution of the overdetermined Gauß-Markov-model is constructed as weighted arithmetic mean of those m solution points. Hereby the weights are obtained from the Error propagation law/variance-covariance propagation. Using the developed *Gauss-Jacobi combinatorial algorithm*, the *overdetermined* three-dimensional resection problem based on the test network *"Stuttgart Central"* is solved.

The algorithms are finally applied successfully to two case studies; transforming in a closed form geocentric coordinates to Gauss ellipsoidal coordinates (geodetic) and to obtain the seven datum transformation parameters from two sets of coordinates. By means of *Gröbner basis*, the *scale parameter* (in the seven datum transformation parameters problem) is shown to fulfill be a *univariate algebraic equation of fourth order (quartic polynomial)* while the rotation parameters are functions in scale and the coordinates differences.

Zusammenfassung

Die Methode der *Gröbner-Basen* (**GB**) und der *Multipolynomialen Resultante* (**MR**) wird als wirksames algebraisches Hilfsmittel zur expliziten Lösung nichtlinearer geodätischer Probleme vorgestellt. Insbesondere wird diese Methode dazu benutzt, um das *Vierpunkt-pseudo-ranging-Problem beim GPS*, das Problem des *minimalen Abstands eines vorgegebenen Punktes zu einer gekrümmten Fläche* sowie das Problem des *dreidimensionalen Rückwärtsschnittes* analytisch zu lösen. Die verschiedenen Schritte der Vorwärts- und Rückwärts-Einsetzung, die bei den klassischen geschlossenen Lösungen dieser ausgewählten Probleme unumgänglich sind, werden dabei vermieden. In ähnlicher Weise wie bei der Gauß'schen Elimination bei linearen Gleichungssystemen werden durch die Methode der **GB** und der **MR** die Variablen eines multivariaten nichtlinearen Gleichungssystems so eliminiert, dass eine univariate Gleichung höherer Ordnung entsteht, deren Lösungsmenge mit existierenden Formelmanipulationsprogrammen wie MATLAB bestimmt werden kann.

Wir nutzen die **GB** und **MR** als Rechenhilfsmittel bei der Lösung des nichtlinearen Gauß-Markov-Modells mit Hilfe des kombinatorischen *Gauß-Jacobi-Algorithmus*, der von C.F. Gauß (posthum veröffentlicht, *Anhang A.4*) und von C.G.I. Jacobi (1841) vorgeschlagen wurde. Sind die nichtlinearen geodätischen Beobachtungsgleichungen in algebraische (polynomiale) Form gebracht, wird der Gauß-Jacobi-Algorithmus in zwei Schritten durchgeführt. Im ersten Schritt werden alle m Kombinationen von minimalen Untermengen der Beobachtungsgleichungen gebildet, daraus werden mit Hilfe von **GB**- und **MR**-Algorithmen jeweils streng die n Unbekannten des Problems bestimmt. Die m Lösungsmengen können als Punkte in einem n-dimensionalen Raum $\mathbb{R}^n$ dargestellt werden. Im zweiten Schritt wird die Lösung des überbestimmten Gauß-Markov-Modells als gewichtetes arithmetisches Mittel dieser m Lösungspunkte konstruiert. Dabei ergeben sich die Gewichte aus dem Fortpflanzungsgesetz der Varianzen/Kovarianzen. Mit Hilfe des so erweiterten kombinatorischen *Gauß-Jacobi-Algorithmus* wird das Testnetz *"Stuttgart Stadtmitte"* als überbestimmter Rückwärtsschnitt ausgeglichen.

Die Algorithmen werden schließlich auf zwei Fallstudien erfolgreich angewandt: es werden zum einen geozentrische Koordinaten in geschlossener Form in Gauß'sche ellipsoidische (geodätische) Koordinaten transformiert, zum anderen werden aus zwei entsprechenden Koordinaten-Datensätzen, die sich auf Systeme unterschiedlicher Datumsfestlegung

beziehen, die sieben Transformationsparameter bestimmt. Mit Hilfe der **GB** wird gezeigt, dass bei dem letztgenannten Problem der Maßstabsfaktor als eine *univariates algebraische Gleichung* vierter Ordnung erfüllt, während die Rotationsparameter Funktionen des Maßstabsfaktors und der Koordinatendifferenzen sind.

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First and foremost, let me begin by quoting from the book of Psalms 121:1-2 (NIV) *"I lift up my eyes to the hills, where does my help come from? My help comes from the Lord, the Maker of heaven and earth"*. To you my God shall I offer my praise, for it was your hand that led me through. The precious gift of life that you have granted unto us manifests your true love, grace and mercy. We are truly a living proof of you grace and mercy. May your name be lifted high both in heaven and on earth. Glory and honour be to you, my most holy God, for through Jesus Christ, we have been made sons and part takers of the kingdom.

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Chapter 1

Introduction

1-1 Statement of the problem

In Geodesy, Photogrammetry and Computer Vision, *nonlinear equations* are often encountered in several applications as they often relate the observations (measurements) to the unknown parameters to be determined. In case the number of observations n and the number of unknowns m are equal ($n = m$), the unknown parameters may be obtained by solving explicitly or in a closed form the system of equations relating observations to the unknown parameters. For example, *D. Cox et al.* (1998, pp.28-32) has illustrated that for systems of equations with exact solution, the system become vulnerable to small errors introduced during root findings and in case of extending the partial solution to the complete solution of the system, the errors may accumulate and thus become so large. If the partial solution was derived by iterative procedures, then the errors incurred during the root-finding may blow up during the extension of the partial solution to the complete solution (back substitution).

In some applications, symbolic rather than numerical solution are desired. In such cases, explicit procedures are usually employed. The resulting symbolic expressions often consists of univariate polynomials relating the unknown parameters (unknown variables) to the known variables (observations). By inserting known values into these univariate polynomials, numerical solutions are readily computed for the unknown variables. Advantages of explicit solutions have been listed by *E. L. Merritt* (1949) as; *provision of satisfaction to the users (Photogrammetrist and Mathematicians) of the methods, provision of data tools for checking the iterative methods, desired by Geodesist whose task of control densification does not favor iterative procedures, provision of solace* and *the requirement of explicit solutions rather than iterative by some applications*. In Geodesy for example, the *Minimum Distance Mapping* problem considered by *E. Grafarend* and *P. Lohse* (1991) relates a point on the Earth's topographical surface uniquely (one-to-one) to a point on the *International Reference Ellipsoid*. The solution of such an optimization problem requires that the equations be solved explicitly.

The draw back that was experienced with explicit solutions was that they were like rare jewel. The reason for this was partly because the methods required extensive computations for the results to be obtained and partly because the resulting symbolic expressions were too large and required computers with large storage capacity. Until recently, the computers that were available could hardly handle large computations due to lack of faster Central Processing Unit (CPU), shortage of Random Access Memory (RAM) and limited hard disk space for storage. The other set back experienced by the explicit procedures was that some of the methods, especially those from algebraic fields, were formulated based on theoretical concepts that were hard to realize or comprehend without the help of computers. For a long time, these setbacks hampered progress of the explicit procedures. The advances made in computer technology in recent years however has changed the tides and led to improvements in explicit computational procedures which hitherto were difficult to achieve. Apart from the improvements in existing computational procedures, new computational techniques are continously being added to the increasing list of computational methods with an aim of optimizing computational speed and efficiency. In this category are the algebraic methods of *Gröbner bases* and *Multipolynomial resultants*.

The present study examines the suitability of algebraic computational techniques of *Gröbner bases* and *Multipolynomial resultants* in solving explicitly nonlinear systems of observation equations that have been converted into algebraic (polynomial) form in Geodesy. The algebraic techniques of *Gröbner bases* and *Sylvester resultant* (resultant of two polynomials) for solving polynomial equations in Geodesy have been mentioned and examples of their applications to the two-dimensional case given in the work of *P. Lohse* (1994, pp.36-39, 71-76). The present study considers the *Gröbner bases* and *Multipolynomial resultants (*resultant of more than two polynomials*)* in the solution of three-dimensional problems. *E. Grafarend* (1989) has already suggested the use of *Gröbner bases* approach in solving the perspective center problem in photogrammetry.

Other than revolutionizing computation procedures, the advances made in computer technology have also led to improvement in instrumentation for data acquisition as exemplified in the case of GPS positioning satellites. Since its inception as a positioning tool, the *Global Positioning System (GPS)* -referred to by *E. Grafarend* and *J. Shan* (1996) as the *Global Problem Solver*- has revolutionized geodetic positioning techniques and maintained its supremacy as a positioning tool. These improvements on instrumentation for data acquisition have led to improved data collection procedures and increase in accuracy. Lengthy geodetic procedures such as triangulation that required a lot of time, intervisibility between stations and a large manpower are rapidly being replaced by satellite positioning techniques which require shorter observation periods, no intervisibility requirement, weather independent and less manpower leading to optimization of time and money. Whereas improved instrumentation is applauded, it comes a long with its own difficulties. One of the difficulties is that a lot of data is collected than required to determine unknown parameters leading to redundancies. In positioning with GPS for example, due to its constellation that offer a wider coverage, more than four satellites can be observed at any point of the earth. In the minimal case, only four satellites are required to fix the receiver position and the receiver clock bias assuming that the transmitter clock bias and the transmitter-receiver bias have been neglected for the *pseudo-range* type of observations. More than four satellites therefore lead to superfluous observations. In such cases, where $n > m$, the explicit solutions give way to optimization procedures such as l*east squares solution* which work very well for *linear models* under specified conditions.

In Geodesy however, the observation equations are normally *nonlinear* thus requiring the use of *nonlinear Gauss-Markov model* which is normally solved either by first *linearizing* the observation equations using Taylor series expansion to the second order terms about approximate values of the unknowns then applying linear models estimation procedures or by using *iterative procedures* such as the *Gauss-Newton* approach. The *linearization approach* has the disadvantage that the linearized approximation of the nonlinear models may still suffer from nonlinearity and thus resulting in the estimates of such models being far from the real estimates of the *nonlinear models*. This can easily be checked by re-substituting the estimates from the *linearized model* into the original nonlinear model.

For the *iterative procedures*, the greatest undoing may be the requirement of the initial approximate values to start off the iterations which may not be available for some applications. For simper models, the approximate initial values may be computed, for others however, the approximate values may be impossible to compute. Apart from the problem of getting the initial approximate values, there also exists the problem that poor choice of approximate values may lead to lack of convergence or if the approximate value be far from the real solution, then a large number of iterations may be required to get close to the real solution thus rendering the whole procedure to be quite slow, especially where multiple roots are available. For other procedures such as the *7-parameter datum transformation* that requires *linearization* and *iterative* methods, it is not feasible to take into account the stochasticity of both systems involved. Clearly, a procedure for solving *nonlinear Gauss-Markov model* that can avoid the requirement of initial approximate starting values for *iteration* and *linearization* approaches and also take into consideration the *stochasticity* of the systems involved is the desire of modern day Geodesist and Photogrammetrist.

With this background, the present study aims at answering the following questions:

- For geodetic problems requiring explicit solutions, can the algebraic tools of *Gröbner bases* and *Multipolynomial resultants* that have found applications in other fields such as Robotics (for kinematic modelling of robots), Visions, Computer Aided Design (CAD), Engineering (offset surface construction in solid modelling), Computer Science (automated theorem proving) e.t.c. be used to solve *systems of nonlinear observation equations* of algebraic (polynomial) type?
- Is there any alternative for solving the *nonlinear Gauss-Markov model* without resorting to *linearization* or *iterative procedures* that require approximate starting values?

To answer the first question, the present study uses the *Gröbner bases* and *Multipolynomial resultants* to solve explicitly the problems of *GPS four-point pseudo-ranging*, *Minimum Distance Mapping* and the *threedimensional resection.* The answer to the second problem becomes clear once the first question has been answered. Should the algebraic techniques of *Gröbner bases* and *Multipolynomial resultants* be successful in solving explicitly the selected geodetic problems, then they are used as the computational engine of the *combinatorial algorithm* that was first suggested by *C. F. Gauss* (Published posthumously e.g. in *Appendix A.4*) and later on by *C. G. I. Jacobi* (1841) and extended by *P. Werkmeister* (1920). We refer to this algorithm as the *Gauss-Jacobi combinatorial algorithm*. In attempting to answer the questions above, the objectives of the present study are formulate as:

(1) Analyze the algebraic computational procedures of type *Gröbner bases* and *Multipolynomial resultants* with the aim of establishing their *suitability* in solving explicitly (in closed form) *geodetic nonlinear problems*. In this respect, the *Gröbner bases* and *Multipolynomial resultants* techniques are used to solve explicitly (symbolically) geodetic problems of *GPS pseudo-ranging four-point P4P*, *Minimum Distance Mapping* and the *threedimensional resection*. By converting the *nonlinear observation equations* of these selected geodetic problems into *algebraic* (*polynomial*), the study aims at using the *Gröbner bases* and *Multipolynomial resultants* techniques to eliminate several variables in a multivariate system of nonlinear polynomial equations in such a manner that the end product from the initial system of nonlinear observation equations normally consist of a *univariate polynomial*. The elimination procedure is similar to the Gauss elimination approach in linear systems.

(2) From the principle of weighted arithmetic mean and using the *Gauss-Jacobi combinatorial lemma* (*C. G. I. Jacobi* 1841), develop an adjustment procedure that neither *linearizes* the *nonlinear observation equations* nor uses *iterative procedures* to solve the *nonlinear Gauss-Markov* model. Linearization is permitted only for *nonlinear error propagation/variance-covariance propagation*. Such procedure is to use the *univariate polynomial* generated by the algebraic computational procedures of type *Gröbner bases* or *Multipolynomial resultants* as the computing engine for its *minimal combinatorial set*.

(3) Test the procedures in solving real geodetic problems of determining the 7 transformation parameters and transforming in a closed form the “geocentric Cartesian coordinates” to “Gauss ellipsoidal coordinates (geodetic)”.

1-2 Solution of the problem

The current known techniques for solving *nonlinear polynomial equations* can be classified into *symbolic*, *numeric* and *geometric* methods (*D. Manocha* 1994c). Symbolic methods, which we consider in this study for solving closed form geodetic problems, apply the *Gröbner bases* and the *Multipolynomial resultants techniques* to eliminate several variables in a multivariate system of equations in such a manner that the end product often consist of *univariate polynomials* whose roots can be determined by existing programs such as the roots command in MATLAB. The current available programs however are efficient only for sets of low degree polynomial systems consisting of upto three to four polynomials due to the fact that computing the roots of the *univariate polynomials* can be ill conditioned for polynomials of degree greater than 14 or 15 (*D. Manocha* 1994c).

Elaborate literature on *Gröbner bases* can be found in the works of *B. Buchberger* (1965, 1970), *J. H. Davenport et al.* (1988, pp.95-103), *F. Winkler* (1996), *D. Cox et al.* (1997, pp.47-99), *H. M. Möller* (1998), *W. V. Vasconcelos* (1998), *T. Becker* and *V. Weispfenning* (1993,1998), *B. Sturmfels* (1996), *G. Pistone* and *H. P. Wynn* (1996), *D. A. Cox* (1998) and *D. Cox et al.*(1998, pp.1-50), while literature on *Multipolynomial resultants procedure* include the works of *G. Salmon* (1876), *F. Macaulay* (1902, 1921), *A. L. Dixon* (1908), *B. L. van Waerden* (1950), *C. Bajaj et al.* (1988), *J. F. Canny* (1988), *J. F. Canny et al*. (1989), *I. M. Gelfand et al.* (1990), *J. Weiss* (1993), *D. Manocha* (1992, 1993, 1994a,b,c, 1998), *D. Manocha* and *J. F. Canny* (1991, 1992, 1993), *I. M. Gelfand et al*. (1994), *G. Lyubeznik* (1995), *S. Krishna and D. Manocha* (1995), *J. Guckenheimer et al.*(1997), *B. Sturmfels* (1994, 1998), *E. Cattani et al.* (1998) and *D. Cox et al.* (1998, pp.71-122). Besides the *Gröbner bases* and *resultant techniques*, there exists another approach for variable elimination developed by WU Wen Tsün (*W. T. Wu* 1984) using the ideas proposed by *J. F. Ritt* (1950). This approach is based on Ritts characteristic set construction and was successfully applied to automated geometric theorem by Wu. This algorithm is referred by *X. S. Gao* and *S. C. Chou* (1990) as the *Ritt-Wu's algorithm* (*D. Manocha* and *F. Canny* 1993). *C. L. Cheng* and *J. W. Van Ness* (1999) have presented polynomial measurement error models.

Numeric methods for solving polynomials can be grouped into *iterative* and *homotopy* methods. For *homotopy* we refer to *A. P. Morgan* (1992). Also in this category are geometric methods which have found application in curve and surface intersection whose convergence are however said to be slow (*D. Manocha* 1994c). In general, for low degree curve intersection, the algebraic methods have been found to be the fastest in practice. In Sections (2-321) and (2-322) of Chapter 2, we present in a nut shell the theories of *Gröbner bases* and *Multipolynomial resultants.*

The problem of *nonlinear adjustment* in Geodesy as in other fields continues to attract more attention from the modern day researchers as evidenced in the works of *R. Mautz* (2001) and *L. Guolin* (2000) who presents a procedure that tests using the F-distribution whether a *nonlinear model* can be linearized or not. The solution to the minimization problem of the *nonlinear Gauss-Markov model* unlike its linear counter part does not have a direct method for solving it and as such, always relies on the iterative procedures such as the Steepest-descent method, Newton's method and the

Gauss-Newton's method discussed by *P. Teunissen* (1990). In particular, *P. Teunissen* (1990) recommends the Gauss-Newton's method as it exploits the structure of the objective function (sum of squares) that is to be minimized. *P. Teunissen* and *E. H. Knickmeyer* (1988) considers in a statistical way how the nonlinearity of a function manifests itself during the various stages of adjustment. *E. Grafarend* and *B. Schaffrin* (1989, 1991) while extending the work of *T. Krarup* (1982) on *nonlinear adjustment* with respect to geometric interpretation have presented the *necessary* and *sufficient* conditions for least squares adjustment of *nonlinear Gauss-Markov model* and provided the geometrical interpretation of these conditions.

Other geometrical approaches include the works of *G. Blaha* and *R. P. Besette* (1989) and *K. Borre* and *S. Lauritzen* (1983) while non geometrically treatment of nonlinear problems have been presented by *K. K. Kubik* (1967), *T. Saito* (1973), *H. J. Schek* and *P. Maier* (1976), *A. Pope* (1982) and *H. G. Bähr* (1988). A comprehensive review to the iterative procedures for solving the *nonlinear equations* is presented in the work of *P. Lohse* (1994). *M. Gullikson* and *I. Söderkvist* (1995) have developed algorithms for fitting surfaces which have been explicitly or implicitly defined to some measured points with negative weights being acceptable by the algorithm.

Our approach in the present study goes back to the work of *C. F. Gauss* (Published posthumously e.g. *Appendix A.4*) and *C. G. I. Jacobi* (1841). Within the framework of arithmetic mean, *C. F. Gauss* (Published posthumously e.g. *Appendix A.4*) and *C. G. I. Jacobi* (1841) suggest that given n linear(ized) observation equations with m unknowns $(n > m)$, σ combinations, each consisting of m equations be solved for the unknown elements and the weighted arithmetic mean be applied to get the final solution. Whereas *C. F. Gauss* (Published posthumously e.g. *Appendix A.4*) proposes weighting by using the products of the square of the measured distances from the unknown point to known points and the distances of the side of the error triangles, *C. G. I. Jacobi* (1841) proposed the use of the square of the determinants as weights. In tracing the method of least squares to the arithmetic mean, *A. T. Hornoch* (1950) shows that the weighted arithmetic mean proposed by *C. G. I. Jacobi* (1841) leads to the least squares solution only if the weights used are the actual weights and the pseudo-observations formed by the combinatorial pairs are uncorrelated. Using a numerical example, *S. Wellisch* (1910, pp. 41-49) has shown the results of least squares solution to be identical to those of the *Gauss-Jacobi combinatorial algorithm* once proper weighting is applied.

P. Werkmeister (1920) illustrated that for planar resection case with three directional observations from the unknown point to three known stations, the area of the triangle (error figure) formed by the resulting three combinatorial coordinates of the new point is proportional to the determinant of the dispersion matrix of the coordinates of the new station. In the present study, these concepts of the combinatorial *linear adjustment* are extended to *nonlinear adjustment*. In chapter 2, the *linear* and *nonlinear Gauss-Markov models* are introduced in Section (2-1). Section (2-2) presents and proves the *Gauss-Jacobi combinatorial lemma* which is required for the solution of the *nonlinear Gauss-Markov model*. We illustrate using a leveling network and planar ranging problem that the results of *Gauss-Jacobi combinatorial algorithm* are identical to those of *linear Gauss-Markov model* if the actual variance-covariance matrices are used.

To test the algebraic computational tools of *Gröbner bases* and *Multipolynomial resultants* presented in Chapter 2, geodetic problems of *threedimensional resection*, *Minimum Distance Mapping* and *GPS pseudo-ranging four-point P4P* have been used in Chapter 3. In general, the search towards the solution of the three-dimensional resection problem traces its origin to the work of a German mathematician *J. A. Grunert* (1841) whose publication appeared in the year 1841. *J. A. Grunert* (1841) solved the threedimensional resection problem -what was then known as the *"Pothenot's"* problem- in a closed form by solving an algebraic equation of degree four. The problem had hitherto been solved by iterative means mainly in Photogrammetry and Computer Vision. Procedures developed later for solving the three-dimensional resection problem revolved around the improvements of the approach of *J. A. Grunert* (1841) with the aim of searching for the optimal means of distances determination. Whereas *J. A. Grunert* (1841) solves the problem by substitution approach in *three steps*, the more recent desire has been to solve the distance equations in lesser steps as exemplified in the works of *S. Finsterwalder* and *W. Scheufele* (1937), *E. L. Merritt* (1949), *M. A. Fischler* and *R. C. Bolles* (1981), *S. Linnainmaa et al*. (1988) and *E. Grafarend, P. Lohse* and *B. Schaffrin* (1989). Other research done on the subject of resection include the works of *F. J. Müller* (1925), *E. Grafarend* and *J. Kunz* (1965), *R. Horaud et al*. (1989), *P. Lohse* (1990), and *E. Grafarend* and *J. Shan* (1997a, 1997b). An extensive review of some of the procedures above are presented by *F. J. Müller* (1925) and *R. M. Haralick et al*. (1991, 1994).

R. M. Haralick et al. (1994) reviewed the performance of six direct solutions (*J. A. Grunert* 1841, *S. Finsterwalder* and *W. Scheufele* 1937, *E. L. Merritt* 1949, *M. A. Fischler* and *R. C. Bolles* 1981, *Linnainmaa et al.* 1988, and *E. Grafarend, P. Lohse* and *B. Schaffrin* 1989) with the aim of evaluating the numerical stability of the solutions and the effect of permutation within the various solutions. All the six approaches follow the outline adopted by *J. A. Grunert* (1841) with the major differences being the change in variables, the reduction of these variables, and the combination of different pairs of equations. The study revealed that the higher the order of the polynomials, the more complex the computations became and thus the less accurate the solutions were due numerical instabilities. Consequently, S.

Finsterwalder's (SF) procedure which solves a third order polynomial is ranked first, J. A Grunert (JG), Fischler and Bolles (FB), and Grafarend et al. (GLS) solutions are ranked second, Linnainmaa et al. solution which generates an eighth order polynomial is ranked third. Though it does not solve the eight order polynomial, the complexity of the polynomial is still found to be higher than those of the other procedures. An amazing result is that of Merritt's procedure which is ranked last despite the fact that it is a fourth order polynomial and is similar to Grunert's approach except for the pairs of equations used. *R. M. Haralick et al.* (1994) attributes the poor performance of Merritt's procedure to the conversion procedure adopted by *E. L Merritt* (1949) in reducing the equations from forth to third order. For planar resection problem, solutions have been proposed e.g. by *D. Werner* (1913), *G. Brandstätter* (1974) and *J. van Mierlo* (1988).

In Sections (3-21), we present a solution of the *three-dimensional resection problem* by solving the *Grunert distances equations* using *Gröbner bases* and *Multipolynomial resultants* techniques. The resulting *fourth order univariate polynomial* is solved for the unknown distance and the admissible solution substituted in other elements of *Gröbner basis* to determine the remaining two distances. Once we have obtained the spatial distances, the position is computed using any of the four approaches of the three-dimensional ranging problem ("Bogenschnitt"); *Gröbner basis*, *Multipolynomial resultants*, *elimination by substitution* and *elimination using matrix* approach. For the *orientation step* which concludes the solution of the three-dimensional resection problem, we refer to the works of *J. L. Awange* (1999) and *E. Grafarend* and *J. L. Awange* (2000) who solved the threedimensional orientation problem by using the *simple Procrustes algorithm*. Here the simple Procrustes problem or partial Procrustes problem, a special case of the general Procrustes problem, is understood to mean the fit of the rotation matrix which transforms a set of Cartesian coordinates into another set of Cartesian coordinates. The general Procrustes problem includes besides the rotational elements also translation, dilatation and reflection. A list of references is *I. Borg* and *P. Groenen* (1997), *F. B. Brokken* (1983), *T. F. Cox* and *M. A. Cox* (1994), *F. Crosilla* (1983a,1983b), *B. Green* (1952), *M. Gullikson* (1995a, 1995b), *S. Kurz* (1996), *R. Mathar* (1997), *P. H. Schönemann* (1996), *L. N. Trefethen* and *D. Bau* (1997), and *I. L. Dryden* (1998).

In order to relate a point on the Earth's topographical surface uniquely (one-to-one) to a point on the *International Reference Ellipsoid*, *E. Grafarend* and *P. Lohse* (1991) have proposed the use of the *Minimum Distance Mapping*. Other procedures that have been proposed are either iterative, approximate "closed", closed or higher order equations. Iterative procedures include the works of *N. Bartelme* and *P. Meissl.* (1975), *W. Benning* (1987), *K. M. Borkowski* (1987, 1989), *N. Croceto* (1993), *A. Fitzgibbon et al.* (1999), *T. Fukushima* (1999), *W. Gander et al.* (1994), *B. Heck* (1987), *W. Heiskannen* and *H. Moritz* (1976), *R. Hirvonen* and *H. Moritz* (1963), *P. Loskowski* (1991), *K. C. Lin* and *J. Wang* (1995), *M. K. Paul* (1973), *L. E. Sjoeberg* (1999), *T. Soler* and *L. D. Hothem* (1989), *W. Torge* (1976), *T. Vincenty* (1978) and *R. J. You* (2000).

Approximate "closed" procedures include *B. R. Bowring* (1976, 1985), *A. Fotiou* (1998), *B. Hofman-Wellenhof et al.* (1992), *M. Pick* (1985) and *T. Vincenty* (1980), while closed procedures include *W. Benning* (1987), *H. Fröhlich* and *H. H. Hansen* (1976), *E. Grafarend et al.* (1995) and *M. Heikkinen* (1982). Procedures that required the solution of higher order equations include *M. Lapaine* (1990), *M. J. Ozone* (1985), *P. Penev* (1978), *H. Sünkel* (1976), and *P. Vaniceck and E. Krakiwski* (1982*)*. In Section (3-22), the *Minimum Distance Mapping problem* is solved using the *Gröbner bases* approach. A *univariate polynomial* of fourth order is obtained together with 13 other elements of the *Gröbner basis*. The obtained *univariate polynomial* and the linear terms are compared to those of *E. Grafarend* and *P. Lohse* (1991). Other reference include *E. Grafarend* and *W. Keller* (1995) and *Mukherjee, K.* (1996)

The *GPS four-point pseudo-ranging problem* is concerned with the determination of the coordinates of a stationary receiver together with its range bias. Several closed form procedures have been put forward for obtaining closed form solution of the problem. Amongst the procedures include the vectorial approach as evidenced in the works of *L. O. Krause* (1987), *J. S. Abel* and *J. W. Chaffee* (1991), *P. Singer et al.* (1993), *J. W. Chaffee* and *J. S. Abel* (1994), *H. Lichtenegger* (1995) and *A. Kleusberg* (1994, 1999). *E. Grafarend* and *J. Shan* (1996) propose two approaches; one approach is based on a closed form solution of the nonlinear pseudo-ranging equations in geocentric coordinates while the other approach solves the same equations in barycentric coordinates.

S. C. Han et al. (2001) have developed an algorithm for very accurate absolute positioning through Global Positioning System (GPS) satellite clock estimation while *S. Bancroft* (1985) provides an algebraic closed form solution of the overdetermined GPS pseudo-ranging observations. In Section (3-231) we solve using *Gröbner basis* and *Multipolynomial resultants GPS four-point pseudo-ranging problem*. Chapter 3 ends by using the *Gauss-Jacobi combinatorial algorithm* in Section (3-232) to solve the overdetermined *GPS point pseudo-ranging problem*. The LPS and GPS *Reference Frames* relevant for these selected geodetic problems have been considered in Section (3-1). For literature on three-dimensional positioning models, we refer to *E. W. Grafarend* (1975), *V. Ashkenazi* and *S, Grist* (1982), *G. W. Hein* (1982a, 1982b), *J. Zaiser* (1986), *F. W. O. Aduol* (1989), *U. Klein* (1997) and *S. M. Musyoka* 1999).

Chapter 4 extends Chapter 3 by solving the three-dimensional resection problem based on the test network *"Stuttgart Central"*. In Section (4-2), using the computed symbolic solutions obtained in Section (3-21), the coordinates of

Dach K1 are determined explicitly. Three stations *Liederhalle*, *Eduardpfeiffer* and *Haussmanstr.* are used. In Section (4-3) all the 7 network points are used to determine the coordinates of Dach K1, thus leading to the solution of *overdetermined three-dimensional resection problem*.

Overdetermined planar resection problem has been treated graphically by *E. Hammer* (1896), *C. Runge* (1900), *P. Werkmeister* (1916) and *P. Werkmeister* (1920). *E. Gotthardt* (1940,1974) dealt with the overdetermined two dimensional resection where more than four points were considered with the aim of studying the critical configuration that would yield a solution. The work was later to be extended by *K. Killian* (1990). A special case of an overdetermined two-dimensional resection has also been considered by *H. G. Bähr* (1991) who uses six known stations and proposes the measuring of three horizontal angles which are related to the two unknown coordinates by nonlinear equations. By adopting approximate coordinates of the unknown point, an iterative adjustment procedure is performed to get the improved two-dimensional coordinates of the unknown point. It should be noted that the procedure is based on the coordinate system of the six known stations. *K. Rinner* (1962) has also contributed to the problem of overdetermined two-dimensional resection.

Once we have developed and tested the algorithms in Chapters 2 to 4, Chapter 5 considers two case studies; the conversion of the geocentric GPS points for the Baltic Sea level Project into the Gauss-Jacobi ellipsoidal coordinates and the determination of the seven datum transformation parameters. We have used the skew symmetric matrix to construct the orthogonal matrix $\boldsymbol{X}_3$ in Chapter 5. Other approaches have been presented in *G. H. Shut* (1958/59) and *E. H. Thompson* (1959 a, b). Datum transformation models have been dealt with e.g. in *E. Grafarend*, *F. Okeke* (1998), *F. Krumm* and *F. Okeke* (1995), *E. Grafarend* and *F. Okeke* (1998) and *E. Grafarend* and *R. Syffus* (1998).

Chapter 6 summarizes and concludes the study.

Chapter 2

Nonlinear Adjustment

In the present Chapter, we depart from the traditional iterative procedures for estimating the unknown fixed parameters of the *nonlinear Gauss-Markov model* and present a *combinatorial approach* that traces its roots back to the work of *C. F. Gauss* that was published posthumously (*Appendix A.4*). *C. F. Gauss* first proposed the combinatorial approach using the products of squared distances (from unknown point to known points) and the square of the perpendicular distances from the sides of the error triangle to the unknown point as the weights. According to *W. K. Nicholson* (1999, pp. 272-273), the motto in Gauss seal read *"pauca des matura"* meaning *few but ripe*. This belief led *C. F. Gauss* not to publish most of his important contributions. For instance, *W. K. Nicholson* (1999, pp. 272-273) writes "Although not all his results were recorded in the diary (many were set down only in letters to friends), several entries would have each given fame to their author if published. Gauss new about the quartenions before Hamilton...". The combinatorial method, like many of his works, was later to be published after his death. Several years later, the method was to be developed further by *C. G. I. Jacobi* (1841) who used the square of the determinants as the weights in estimating the unknown parameters from the arithmetic mean. *P. Werkmeister* (1920) later established the relationship between the area of the error figure formed by the combinatorial approach and the standard error of the determined point. We will refer to this *combinatorial approach* as the *Gauss-Jacobi combinatorial algorithm*.

First we define both the *linear* (both fixed and the random effect model) and *nonlinear Gauss-Markov models* in section (2-1). While the solution of the *linear Gauss-Markov model* by **B***est* **L***inear* **U***niformly* **U***nbiased* **E***stimator* (BLUUE) is straight forward, the solution of the *nonlinear Gauss-Markov model* has no straight forward procedure owing to the *nonlinearity* of the *injective function* (or map function) that maps $\mathbb{R}^m$ to $\mathbb{R}^n$. We therefore introduce in Section (2-2) the *Gauss-Jacobi combinatorial algorithm* which we propose to use in solving *nonlinear Gauss-Markov model.* In Section (2-3), we demonstrate how the procedure can be used to solve the *nonlinear Gauss-Markov model*.

2-1 Linear and nonlinear Gauss-Markov models

Presented in this Section are both the *linear* and *nonlinear Gauss-Markov models*. We start by the definition of the *linear Gauss-Markov model* as follows

Definition 2-0a (*Special linear Gauss-Markov model*):

Given a real $n \times 1$ random vector $\boldsymbol{y} \in \mathbb{R}^n$ of observations, a real $m \times 1$ vector $\boldsymbol{\xi} \in \mathbb{R}^m$ of unknown fixed parameters over a real $n \times m$ coefficient matrix $\boldsymbol{A} \in \mathbb{R}^{n\times m}$, a real $n \times n$ positive-definite dispersion matrix $\boldsymbol{\Sigma}$, the functional model

$$\boldsymbol{A\xi} = E\{\boldsymbol{y}\}, E\{\boldsymbol{y}\} \in R(\boldsymbol{A}), rk\boldsymbol{A} = m, \boldsymbol{\Sigma} = D\{\boldsymbol{y}\}, rk\boldsymbol{\Sigma} = n \tag{2-1}$$

is called the a *special linear Gauss-Markov* model with full rank.

The unknown vector $\boldsymbol{\xi}$ of fixed parameters in the *special linear Gauss-Markov model* (2-1) is normally estimated by **B***est* **L***inear* **U***niformly* **U***nbiased* **E***stimation* BLUUE, defined in *E. Grafarend* and *B. Schaffrin* (1993, p. 93) as

Definition 2-0b (**B***est* **L***inear* **U***niformly* **U***nbiased* **E**stimation BLUUE):

An $m \times 1$ vector $\hat{\boldsymbol{\xi}} = \boldsymbol{Ly} + \boldsymbol{\kappa}$ is $V - BLUUE$ for $\boldsymbol{\xi}$ (**B***est* **L***inear* **U***niformly* **U***nbiased* ***E****stimation* respectively the $(V - Norm)$ in (2-1) when on one hand it is uniformly unbiased in the sense of

$$E\{\hat{\boldsymbol{\xi}}\} = E\{\boldsymbol{Ly} + \boldsymbol{\kappa}\} = \boldsymbol{\xi}\, for\, all\, \boldsymbol{\xi} \in \mathbb{R}^m \tag{2-2}$$

and on the other hand in comparison to all other linear uniformly unbiased estimators give the minimum variance and therefore the minimum mean estimation error in the sense of

$$\begin{aligned} trD\{\hat{\boldsymbol{\xi}}\} = E\{(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi})'(E\{\hat{\boldsymbol{\xi}} - \boldsymbol{\xi})\} = \\ = \sigma^2 \boldsymbol{L\Sigma L} = \|\boldsymbol{L}\|_V^2 = \min_L, \end{aligned} \tag{2-3}$$

where $\boldsymbol{L}$ is a real $m \times n$ matrix and $\boldsymbol{\kappa}$ an $m \times 1$ vector.

Using (2-3) to estimate the unknown fixed parameter vector $\boldsymbol{\xi}$ in (2-1) leads to

$$\hat{\boldsymbol{\xi}} = (\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{y} \tag{2-4}$$

with its regular dispersion matrix

$$D\{\hat{\boldsymbol{\xi}}\} = (\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A})^{-1}. \tag{2-5}$$

The dispersion matrix (variance-covariance matrix) $\boldsymbol{\Sigma}$ is *unknown* and is obtained by means of estimators of type MINQUE, BIQUUE or BIQE as in *C. R. Rao* (1967, 1971, 1973 and 1978), *C. R. Rao* and *J. Kleffe* (1979), *B. Schaffrin* (1983), and *E. Grafarend* (1985). In the event that $\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A}$ is not regular (i.e.$\boldsymbol{A}$ has a rank deficiency), the rank deficiency can be overcome by procedures such as those presented by *E. Mittermayer* (1972), *E. Grafarend* and *B. Schaffrin* (1974), *E. Grafarend* and *B. Schaffrin* (1993, pp. 107-165), *F. K. Brunner* (1979), *A. Peremulter* (1979), *P. Meissl* (1982), *E. Grafarend* and *F. Sanso* (1985) and *K. R. Koch* (1999, pp.181-197) among others.

Definition 2-1a (*Gauss-Markov model* with random effects):
The model

$$\begin{gathered} \boldsymbol{y} = \boldsymbol{Cz} + \boldsymbol{e}_y - \boldsymbol{Ce}_z \\ E\{\boldsymbol{y}\} = \boldsymbol{C}E\{\boldsymbol{z}\} \in \mathbb{R}^n \\ D\{\boldsymbol{y}\} = D\{\boldsymbol{y} - \boldsymbol{Cz}\} + \boldsymbol{C}D\{\boldsymbol{z}\}\boldsymbol{C}' \in \mathbb{R}^{n\times n} \\ C\{\boldsymbol{y}, \boldsymbol{z}\} = \boldsymbol{0} \\ \boldsymbol{z}, E\{\boldsymbol{z}\}, E\{\boldsymbol{y}\}, \boldsymbol{\Sigma}_{\boldsymbol{y}-\boldsymbol{Cz}}, \boldsymbol{\Sigma}_{\boldsymbol{z}}\ unknown \\ dim\, \Re(\boldsymbol{C}') = rk\boldsymbol{C} = l \end{gathered} \tag{2-6}$$

with a real $n \times 1$ random vector $\boldsymbol{y} \in \mathbb{R}^n$ of observations, a real $l \times 1$ vector $\boldsymbol{z}$ of unknown random effects (*"zufallseffekte"*), a non stochastic real valued matrix $\boldsymbol{C} \in \mathbb{R}^{n\times l}$ of rank $rk\boldsymbol{C} = l$ is called the *Gauss-Markov model with random effects*.

The $l \times 1$ random effect vector $\boldsymbol{z}$ of the model (2-6) can be predicted by the **B***est* **L***inear* **U***niformly* **U***nbiased* **P***rediction* BLUUP defined by (*E. Grafarend* and *B. Schaffrin* 1993, p.304) as

Definition 2-1b (**B***est* **L***inear* **U***niformly* **U***nbiased* **P***rediction* BLUUP):
An $l \times 1$ vector $\tilde{\boldsymbol{z}}$ is called h omogenious **BLUUP** (**B***est* **L***inear* **U***niformly* **U***nbiased* **P***rediction)* of $\boldsymbol{z}$ in the special linear *Gauss-Markov model* with random effects (2-6) if
(1st) $\tilde{\boldsymbol{z}}$ is a homogenious linear form

$$\tilde{\boldsymbol{z}} = \boldsymbol{Ly} \tag{2-7}$$

and uniformly unbiased in the sense

$$E\{\tilde{z}\} = E\{\boldsymbol{Ly}\} = E\{z\} \tag{2-8}$$

(2nd) in comparison to all other linear uniformly unbiased predictors give the minimum mean square predictor error in the sense of

$$\left[\begin{aligned} \|\boldsymbol{MSPE}\|^2 := \\ := E\{(\tilde{\boldsymbol{z}} - \boldsymbol{z})(\tilde{\boldsymbol{z}} - \boldsymbol{z})^{'}\} = trD\{\tilde{\boldsymbol{z}} - \boldsymbol{z}\} = \\ = tr\boldsymbol{L}D\{\boldsymbol{y}\}\boldsymbol{L}^{'} + tr\, D\{\boldsymbol{z}\}\Big|_{C\{\boldsymbol{y},\boldsymbol{z}\}=\boldsymbol{0}} = \\ = \|\boldsymbol{L}^{'}\|^2_{\boldsymbol{\Sigma}_{\boldsymbol{y}}} + E\{(\tilde{\boldsymbol{z}} - \boldsymbol{z})(\tilde{\boldsymbol{z}} - \boldsymbol{z})^{'}\} = \min_L \end{aligned}\right. \tag{2-9}$$

with MSPE symbolizing "**M**ean **S**quare **P**redictor **E**rror" and $\boldsymbol{L}$ is a real $l \times n$ matrix .

Using (2-9) to predict the unknown random parameter vector $\boldsymbol{z}$ in (2-6) leads to

$$\tilde{\boldsymbol{z}} = (\boldsymbol{C}'\boldsymbol{\Sigma}_{\boldsymbol{y}}^{-1}\boldsymbol{C})^{-1}\boldsymbol{C}'\boldsymbol{\Sigma}_{\boldsymbol{y}}^{-1}\boldsymbol{y} \tag{2-10}$$

with its regular dispersion matrix

$$D\{\tilde{\boldsymbol{z}}\} = (\boldsymbol{C}'\boldsymbol{\Sigma}_{\boldsymbol{y}}^{-1}\boldsymbol{C})^{-1}, \tag{2-11}$$

if $D\{\boldsymbol{y}\} = D\{\boldsymbol{y} - \boldsymbol{C}\boldsymbol{z}\} + \boldsymbol{C}D\{\boldsymbol{z}\}\boldsymbol{C}' \in \mathbb{R}^{n\times n}$ exist.

Definition 2-2 (*Nonlinear Gauss-Markov model*):
The model

$$E\{\boldsymbol{y}\} = \boldsymbol{y} - \boldsymbol{e} = \boldsymbol{A}(\boldsymbol{\xi}), D\{\boldsymbol{y}\} = \boldsymbol{\Sigma} \tag{2-12}$$

with a real $n \times 1$ random vector $\boldsymbol{y} \in \mathbb{R}^n$ of observations, a real $m \times 1$ vector $\boldsymbol{\xi} \in \mathbb{R}^m$ of unknown fixed parameters, $n \times 1$ vector $\boldsymbol{e}$ of random errors (with zero mean and dispersion matrix $\boldsymbol{\Sigma}$), $\boldsymbol{A}$ being an *injective function* from an open domain into $n-$dimensional space $\mathbb{R}^n(m < n)$ and E the "expectation" operator is said to be a *nonlinear Gauss-Markov* model (*E. Grafarend* and *B. Schaffrin* 1989).

The difference between the *linear* and *nonlinear Gauss-Markov models* defined above lies on the *injective function* $\boldsymbol{A}$. In the *linear Gauss-Markov model*, $\boldsymbol{A}$ is linear and thus satisfies the algebraic axiom

$$\boldsymbol{A}(\alpha\xi_1 + \beta\xi_2) = \alpha\boldsymbol{A}(\xi_1) + \beta\boldsymbol{A}(\xi_2), \alpha, \beta \in \mathbb{R}, \xi_1, \xi_2 \in \mathbb{R}^m, \tag{2-13}$$

with the m -dimensional manifold traced by $\boldsymbol{A}(.)$ for varying values of $\boldsymbol{\xi}$ being flat. For the *nonlinear Gauss-Markov model* on the other hand, $\boldsymbol{A}(.)$ is a *nonlinear* vector function that maps $\mathbb{R}^m$ to $\mathbb{R}^n$ tracing an m -dimensional manifold that is curved.

In Geodesy, many nonlinear functions are normally assumed to be moderately nonlinear thus permitting linearization by Taylor series expansion and then applying the linear models to estimate the unknown fixed parameters (e.g. *K. R. Koch* 1999, pp.155-156). Whereas this may often hold, the effect of the nonlinearity of the models may still be significant on the outcome and as such, we revert in the next Section to the *Gauss-Jacobi combinatorial algorithm*.

2-2 The Gauss-Jacobi combinatorial algorithm

In this Section we present the *Gauss-Jacobi combinatorial algorithm* which is neither iterative nor requires linearization of the nonlinear observation equations for the solution of *nonlinear Gauss-Markov model.* Linearization is permitted only for the *nonlinear error propagation/variance-covariance propagation* in order to generate the dispersion matrixn (i.e. the second moments). We start by stating the *Gauss-Jacobi combinatorial lemma* in *Box* (2-1) and refer to *S. Wellisch* (1910, pp. 46-47) and *T. Hornoch* (1950) for the proof that the results are equivalent to those of least squares solution. The levelling network is presented to illustrate the *Gauss-Jacobi combinatorial* approach. We conclude the Section by *Theorem* (2-1) which allows the applicability of the *Gauss-Jacobi combinatorial algorithm* in solving *nonlinear* Geodetic observation equations once they have been converted into *algebraic (polynomial) equations*.

We state the *C. F. Gauss* and *C. G. I Jacobi* (1841) combinatorial *lemma* as follows:

<u>Box</u> <u>2-1</u> (*Lemma 2-1: Gauss-Jacobi combinatorial*):
For n algebraic observation equations in m unknowns

$$\begin{array}{l} a_1x + b_1y - y_1 = 0 \\ a_2x + b_2y - y_2 = 0 \\ a_3x + b_3y - y_3 = 0 \\ \quad\quad .... \end{array} \tag{2-14}$$

for the determination of the coordinates x and y of the unknown point $P, y_i|i \in \{1, 2, \ldots, n\}$ being the observable and $a_i, b_i|i \in \{1, 2, \ldots, n\}$ being the elements of the design matrix $\boldsymbol{A} \in R,^{n\times m}$ there exist no set of solution $\{x, y\}$ from any combinatorial pair in the equations above that satisfy the entire system of equations. This is because the solution obtained from each *combinatorial pair* of equations differs from the others due to the unavoidable random measuring errors. If the solutions from the pair of the combinatorial equations are designated $x_{1,2}, x_{2,3}, \ldots$ and $y_{1,2}, y_{2,3}, \ldots$ with the subscript indicating the combinatorials, then the combined solution is the *sum of the weighted arithmetic* mean

$$\boxed{x = \frac{p_{1,2}x_{1,2} + p_{2,3}x_{2,3} + \ldots.}{p_{1,2} + p_{2,3} + \ldots.}, \; y = \frac{p_{1,2}y_{1,2} + p_{2,3}y_{2,3} + \ldots.}{p_{1,2} + p_{2,3} + \ldots.}} \tag{2-15}$$

with $p_{1,2}, p_{2,3}, \ldots$ being the weights of the combinatorial solutions given by the square of the determinants as

$$\begin{array}{l} p_{1,2} = (a_1b_2 - a_2b_1)^2 \\ p_{2,3} = (a_2b_3 - a_3b_2)^2 \\ \quad\quad .... \end{array} \tag{2-16}$$

The results (2-15) coincides with the center of the error figure (*see Figure in Appendix A.4* for $n = 3$) formed by the coordinates of the combinatorial solutions and are identical to those of *least squares solution for linear observation equations.*

We illustrate the *Gauss-Jacobi combinatorial approach* using the levelling network below.

<u>Levelling network:</u> Consider a levelling network with four-points in *Figure* (2.1) below.

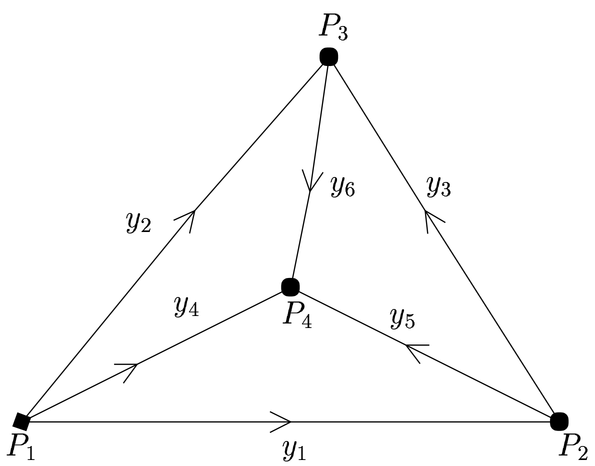


Figure 2.1: Levelling Network

Let the height of point P_1 be given as $h_1$and those of P_2 and P_3 be sought. We have

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \frac{3!}{2!(3-2)!} = 3$$

number of combinatorial routes that can be used to obtain the heights of points P_2 and P_3. These are levelling routes $P_1 - P_2 - P_4 - P_1$, $P_2 - P_3 - P_4 - P_2$, and $P_3 - P_1 - P_4 - P_3$. These combinatorials sum up to the outside loop of the network $P_1 - P_2 - P_3 - P_1$. The observation equations are written as

$$\begin{aligned} x_2 - h_1 &= y_1 \\ x_3 - h_1 &= y_2 \\ x_3 - x_2 &= y_3 \\ x_4 - h_1 &= y_4 \\ x_4 - x_2 &= y_5 \\ x_4 - x_3 &= y_6 \end{aligned} \tag{2-17}$$

which can be expressed in the form of the *special linear Gauss-Markov model* (2-1) in page (7) as

$$E\left\{\begin{bmatrix} y_1 + h_1 \\ y_2 + h_1 \\ y_3 \\ y_4 + h_1 \\ y_5 \\ y_6 \end{bmatrix}\right\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix}, \tag{2-18}$$

where $y_1, y_2, \dots, y_6$ are the observed height differences, x_2, x_3, x_4 are the unknown heights of points P_2, P_3, P_4 respectively. Let the dispersion matrix $D\{\boldsymbol{y}\} = \boldsymbol{\Sigma}$ be chosen such that the correlation matrix is unit (i.e.$\boldsymbol{\Sigma} = \boldsymbol{I}_3 = \boldsymbol{\Sigma}^{-1} Positive-definite, rk\boldsymbol{\Sigma}^{-1} = 3 = n$), the decomposition matrix $\boldsymbol{Y}$ and the normal equation matrix $\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A}$ be given respectively by

$$\boldsymbol{Y} = \begin{bmatrix} y_1 + h_1 & 0 & 0 \\ 0 & 0 & y_2 + h_1 \\ 0 & y_3 & 0 \\ -(y_4 + h_1) & 0 & y_2 + h_1 \\ y_5 & -y_5 & 0 \\ 0 & y_6 & -y_6 \end{bmatrix}, \boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}. \tag{2-19}$$

We compute the heights of points P_2 and P_3 using the combinatorial procedure as follows:

1st, 2nd and 3nd Combinatorials (levelling along routes: $route(1) := P_1 - P_2 - P_4 - P_1$, $route(2) := P_2 - P_3 - P_4 - P_2$ and $route(3) := P_3 - P_1 - P_4 - P_3$):
(2-19) and (2-4) leads to the partial solutions

$$\left.\begin{aligned} &\hat{\boldsymbol{\xi}}_{route(1)} = \tfrac{1}{2}\begin{bmatrix} y_1 + \frac{h_1}{2} - \frac{y_5}{2} - \frac{y_4}{2} \\ \frac{y_1}{2} - \frac{y_4}{2} \\ \frac{y_1}{2} - \frac{h_1}{2} + \frac{y_5}{2} - y_4 \end{bmatrix}, \hat{\boldsymbol{\xi}}_{route(2)} = \tfrac{1}{2}\begin{bmatrix} \frac{y_5}{2} - \frac{y_3}{2} \\ \frac{y_3}{2} - \frac{y_6}{2} \\ \frac{y_6}{2} - \frac{y_5}{2} \end{bmatrix} \\ &\hat{\boldsymbol{\xi}}_{route(3)} = \tfrac{1}{2}\begin{bmatrix} \frac{y_4}{2} - \frac{y_2}{2} \\ \frac{y_4}{2} + \frac{y_6}{2} - \frac{h_1}{2} - y_2 \\ \frac{h_1}{2} - \frac{y_2}{2} - \frac{y_6}{2} + y_4 \end{bmatrix} \end{aligned}\right\} \tag{2-20}$$

The heights of the stations x_2, x_3, x_4 would then be given by the summation of the combinatorial solutions

$$\hat{\boldsymbol{\xi}}_l = \hat{\boldsymbol{\xi}}_{P_1-P_2-P_4-P_1} + \hat{\boldsymbol{\xi}}_{P_2-P_3-P_4-P_2} + \hat{\boldsymbol{\xi}}_{P_3-P_1-P_4-P_3} = \tfrac{1}{2}\begin{bmatrix} y_1 + \frac{h_1}{2} - \frac{y_3}{2} - \frac{y_2}{2} \\ \frac{y_1}{2} + \frac{y_3}{2} - \frac{h_1}{2} - y_2 \\ \frac{y_1}{2} - \frac{y_2}{2} \end{bmatrix}. \tag{2-21}$$

Levelling along route $P_1 - P_2 - P_3 - P_1$) and with (2-4) we have

$$\hat{\boldsymbol{\xi}}_{P_1-P_2-P_3-P_1} = (\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\begin{bmatrix} y_1+h_1 \\ -(y_2+h_1) \\ y_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} y_1+\dfrac{h_1}{2}-\dfrac{y_3}{2}-\dfrac{y_2}{2} \\ \dfrac{y_1}{2}+\dfrac{y_3}{2}-\dfrac{h_1}{2}-y_2 \\ \dfrac{y_1}{2}-\dfrac{y_2}{2} \end{bmatrix}, \tag{2-22}$$

which are identical to the *Gauss-Jacobi combinatorial solution* in (2-21) ♣.

Having stated and illustrated using a levelling example the *Gauss-Jacobi combinatorial lemma* for linear equations, we state below the *theorem* that allows the solution of *nonlinear equations* in Geodesy.

Box 2-2 (*Theorem* 2-1):
Given algebraic (polynomial) observational equations (n observations, where n is the dimension of the observation space $\mathbb{Y}$) of order l in m variables (unknowns) (m is the dimension of the parameter space $\mathbb{X}$), the application of least squares solution (LESS) to the algebraic observation equations gives $(2l-1)$ as the order of the set of *nonlinear algebraic normal equations*. There exists m normal equations of the polynomial order $(2l-1)$ to be solved.
Proof: Given nonlinear algebraic equations $f_i \in k\{\xi_1, \ldots, \xi_m\}$ expressed as

$$\begin{bmatrix} f_1 \in k\{\xi_1, \ldots, \xi_m\} \\ f_2 \in k\{\xi_1, \ldots, \xi_m\} \\ . \\ . \\ . \\ f_n \in k\{\xi_1, \ldots, \xi_m\}. \end{bmatrix} \tag{2-23}$$

and the order considered as l, we write the objective function to be minimized as

$$\|f\|^2 = f_1^2 + \ldots. + f_n^2 \mid \forall f_i \in k\{\xi_1, \ldots, \xi_m\} \tag{2-24}$$

and obtain the partial derivatives (first derivatives) of (2-24) with respect to the unknown variables $\{\xi_1, \ldots, \xi_m\}$. The order of (2-24) which is l^2 then reduces to $(2l-1)$ upon differentiating the objective function with respect to the variables $\xi_1, \ldots, \xi_m$. Thus resulting in m normal equations of the polynomial order $(2l-1)$.

Example (pseudo-ranging):

For pseudo-ranging or distance equations, the order of the polynomials in the algebraic observational equations is $l = 2$. If we take the "pseudo-ranges squared" or "distances squared", a necessary procedure in order to make the observational equations "algebraic" or "polynomial", and implement LESS, the objective function which is of order $l = 4$ reduces by one to order $l = 3$ upon differentiating once. The normal equations are of order $l = 3$ as expected.
♣

The significance of the theorem above is that by using the *Gauss-Jacobi combinatorial approach* to solve the *nonlinear Gauss-Markov model*, all observation equations of geodetic interest are successfully converted to *"algebraic"* or *"polynomial"* equations.

2-3 Solution of nonlinear Gauss-Markov model

Having presented and proved the *Gauss-Jacobi combinatorial lemma* using simple linear levelling and linearized ranging examples, we proceed to solve the *nonlinear Gauss-Markov model* in two steps:

- Step 1: Combinatorial minimal subsets of observations are constructed and rigorously solved by means of the *Multipolynomial resultant* or *Groebner basis* (*J. L. Awange* and *E. Grafarend* 2001).

- Step 2: The combinatorial solution points of step 1 are reduced to their final adjusted values by means of an adjustment procedure where the **B**est **L**inear **U**niformly **U**nbiased **E**stimator (BLUUE) is used to estimate the vector of fixed parameters within the linear Gauss-Markov model with the dispersion matrix of the real valued random vector of pseudo-observations from *Step 1* generated via the *nonlinear error propagation law* also known in this case as the *nonlinear variance-covariance propagation*.

2-31 Construction of the minimal combinatorial subsets

Since $n > m$ we construct the minimal combinatorial subsets comprising m equations solvable in closed form using either *Gröbner bases* or *Multipolynomial resultants* which we present in Section (2-32). We begin by the following elementary definitions:

Definition 2-3 (Permutation):
Given a set S with elements $\{i, j, k\} \in S$, the arrangement obtained by placing $\{i, j, k\} \in S$ in some sequence is called *permutation*. If we choose any of the elements say i first, then each of the remaining elements j, k can be put in the second position, while the third position is occupied by the unused letter either j or k. For the set S, the following permutations can be made:

$$\begin{bmatrix} ijk & ikj & jik \\ jki & kij & kji \end{bmatrix} \tag{2-25}$$

From (2-25) there exist three ways of filling the first position, two ways of filling the second position and one way of filling the third position. Thus the number of *permutations* is given by $3 \times 2 \times 1 = 6$. In general, for n different elements, the number of permutation is equal to $n \times \ldots \times 3 \times 2 \times 1 = n!$

Definition 2-4 (Combination):
If for n elements only m elements are used for the permutation, then we have a *combination* of the mth order. If we follow the definition above, then the first position can be filled in n ways, the second in $n - 1$ ways and the mth in $n - (m - 1)$ ways. In (2-25) above the combinations are identical and contain the same elements in different sequences. If the arrangement is to be neglected, then we have for n elements, a combination of mth order being given by

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{n(n-1)\ldots(n-m+1)}{m \times \ldots \times 3 \times 2 \times 1}. \tag{2-26}$$

Given n nonlinear equations to be solved, we first form $\binom{n}{m}$ minimal combinatorial subsets each consisting of m elements (where m is the number of the unknown elements). Each minimal combinatorial subset is solved using the algebraic procedures discussed in (2-32). In geodesy, the number of elements n normally consist of the observations in the vector $\boldsymbol{y}$, while the number of elements m normally consist of the unknown fixed parameters in the vector $\boldsymbol{\xi}$.

2-32 Closed form solution of the minimal combinatorial subsets

In this Section, we present two algebraic algorithms that are used to solve in closed form the minimal combinatorial subsets constructed in Section (2-31). We first present the approach based on the *Gröbner bases* and thereafter consider the *Multipolynomial resultants* approach.

2-321 Closed form solution of nonlinear equations by Gröbner bases

As a recipe to what *Gröbner bases* can do, consider that most problems in nature, here in Geodesy, Photogrammetry, Machine Vision, Robotics, Surveying etc. can be modelled by *sets of nonlinear equations* forming polynomials. These *nonlinear systems of equations* that have to be solved can be used to form linear combinations of other polynomials called *Ideals* by being multiplied by *arbitrary polynomials* and summed up. In this case, a collection of these *nonlinear algebraic equations* forming *Ideals* are referred to us the set of polynomials generating the *Ideal* and forms the elements of this *Ideal*. The *B. Buchberger algorithm* then takes this set of generating polynomials and derive, using a procedure that will be explained shortly, another set of polynomials called the *Gröbner basis* which has some special properties . One of the special properties of the *Gröbner bases* is that its elements can divide the elements of the generating set giving zero remainder. This property is achieved by the *B. Buchberger algorithm* by canceling the *leading terms* of the polynomials in the generating set and in so doing deriving the *Gröbner basis* of the *Ideal* (whose elements are the generating *nonlinear algebraic equations*). With the *lexicographic* type of ordering chosen, one of the elements of the *Gröbner basis* is often a *univariate polynomial* whose roots can be obtained by the *roots* command in *MATLAB*. The other special property is that two sets of polynomial equations will generate the same *Ideal* if and only if their *Gröbner bases* are equal with respect to any term ordering. This property is important in that the solution of the *Gröbner basis* will satisfy the original solution required by the generating set of nonlinear equations.

With this brief outline, we now define the term *Gröbner basis*. In the outline above and definitions below, we have used several terms that require definitions. In *Appendix A.1* (p.97), we present the definitions of the terms as follows; *monomial* (*Definition A-1*), *polynomial* (*Definition A-2*), *linear algebra* (*Definition A-3*), *ring* (*Definition A-4*), *polynomial ring* (*Definition A-5*), *Ideal* (*Definition A-6*), *monomial ordering* (*Definition A-7*) and *leading coefficients* (LC), *leading terms* (LT), *leading monomials* (LM) and *multideg* (*Definition A-8*). Indeed what we present here is just a definition of the term *Gröbner basis* and as such, any reader interested on the subject may consult text books such as *J. H. Davenport et al.* (1988), *T. Becker* and *V. Weispfenning* (1993, 1998), *B. Sturmfels* (1996), *F. Winkler* (1996), *D. Cox et al.* (1997), *B. Buchberger* and *F. Winkler* (1998), and *W. V. Vasconcelos* (1998). We start by first defining an Ideal and the *Hilbert Basis Theorem* that guarantees that every *Ideal* has a finite set of *Gröbner basis* before defining the term *Gröbner basis* and briefly looking at the *B. Buchberger algorithm* which is the engine behind the computation of *Gröbner bases*.

Definition 2-5 (*J. H. Davenport et al*. 1988, p.96, *D. Cox et al*. 1997, p.29):
An *Ideal* is generated by a family of generators as consisting of the *set of linear combinations* of these *generators* with *polynomial coefficients. L*et $f_1, \ldots, f_s$ be polynomials in $k[x_1, \ldots, x_n]$ then

$$< f_1, \ldots, f_s > \; = \left\{ \sum_{i=1}^{s} h_i f_i, \quad h_1, \ldots, h_s \in k[x_1, \ldots, x_n]. \right\} \tag{2-27}$$

$< f_1, \ldots, f_s >$ is an Ideal and if a subset $I \subset k[x_1, \ldots, x_n]$ is an Ideal, it must satisfy the following conditions;
(a) $0 \in I$.
(b) *If $f, g \in I$, then $f + g \in I$ (i.e. I* is an *additive subgroup* of the *additive group* of the *field k).*
(c) *If $f \in I$* and $h \in k[x_1, \ldots, x_n]$, then $hf \in I$ *(i.e.I* is closed under multiplication ring element*).*

The definition of an *Ideal c*an be presented in terms of polynomial equations $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$. *O*ne begins by expressing the system of polynomial equations

$$\left[\begin{array}{l} f_1 = 0 \\ f_2 = 0 \\ \quad . \\ \quad . \\ f_s = 0, \end{array} \right. \tag{2-28}$$

which can be used to derive others by multiplication of the individual equation f_i by another polynomial $h_i \in k[x_1, \ldots, x_n]$ and adding to get $h_1 f_1 + h_2 f_2 + \ldots + h_s f_s = 0$ (cf. 2-27). The *Ideal* $< f_1, \ldots, f_s >$ thus consists of a system of equations $f_1 = f_2 = \ldots = f_s = 0$ thus indicating that if $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$, then $< f_1, \ldots, f_s >$ is an *Ideal* generated by $f_1, \ldots, f_s$ *(*thus forming the *basis* of the *Ideal I*), being the *basis* of the *Ideal* I.

Perhaps a curious reader may begin to wonder why the term *Ideal* was selected. To quench this curiosity we refer to *W. K. Nicholson* (1999, p.220) and quote from *T. Becker* and *V. Weispfenning* (1993, p.59) on how the term *Ideal* and *ring* came to be. They write:

> "On the origin of the term *Ideal*, the concept is attributed to *Dedekind* who introduced it as a set theoretical version of *Kummer's "Ideal number"* to circumvent the failure of unique factorization in certain natural extension of the domain $\mathbb{Z}$. The relevance of *Ideal* in the theory of *polynomial rings* was highlighted by the *Hilbert Basis Theorem*. The systematic development of *Ideal* theory in more general rings is largely due to *E. Noether*. In the older literature the term "module" is sometimes used for *"Ideal"* (cf. *F. Macaulay* 1916). The term "ring" seems to be due to *D. Hilbert*; *Kronecker* used the term "order" for ring".

Example 2-1: Given the polynomials in $\mathbb{R}[x_1, x_2, x_3]$ as

$$\left[\begin{array}{l} x_1^2 + 2a_{12}x_1x_2 + x_2^2 + a_{oo} = 0 \\ x_2^2 + 2b_{23}x_2x_3 + x_3^2 + b_{oo} = 0 \\ x_3^2 + 2c_{31}x_3x_1 + x_1^2 + c_{oo} = 0 \end{array}\right. \quad (2\text{-}29)$$

then
Ideal $I =< x_1^2 + 2a_{12}x_1x_2 + x_2^2 + a_{oo}, x_2^2 + 2b_{23}x_2x_3 + x_3^2 + b_{oo}, x_3^2 + 2c_{31}x_3x_1 + x_1^2 + c_{oo} >$.

Example 2-2: Given the polynomials in $\mathbb{R}[x_1, x_2, x_3, x_4]$ as

$$\left[\begin{array}{l} x_1^2 - 2a_0x_1 + x_2^2 - 2b_0x_2 + x_3^2 - 2c_0x_3 - x_4^2 + 2d_0x_4 + a_0^2 + b_0^2 + c_0^2 + d_0^2 = 0 \\ x_1^2 - 2a_1x_1 + x_2^2 - 2b_1x_2 + x_3^2 - 2c_1x_3 - x_4^2 + 2d_1x_4 + a_1^2 + b_1^2 + c_1^2 + d_1^2 = 0 \\ x_1^2 - 2a_2x_1 + x_2^2 - 2b_2x_2 + x_3^2 - 2c_2x_3 - x_4^2 + 2d_2x_4 + a_2^2 + b_2^2 + c_2^2 + d_2^2 = 0 \\ x_1^2 - 2a_3x_1 + x_2^2 - 2b_3x_2 + x_3^2 - 2c_3x_3 - x_4^2 + 2d_3x_4 + a_3^2 + b_3^2 + c_3^2 + d_3^2 = 0 \end{array}\right. \quad (2\text{-}30)$$

then *Ideal* I is given by

$$< \begin{array}{l} x_1^2 - 2a_0x_1 + x_2^2 - 2b_0x_2 + x_3^2 - 2c_0x_3 - x_4^2 + 2d_0x_4 + a_0^2 + b_0^2 + c_0^2 + d_0^2, \\ x_1^2 - 2a_1x_1 + x_2^2 - 2b_1x_2 + x_3^2 - 2c_1x_3 - x_4^2 + 2d_1x_4 + a_1^2 + b_1^2 + c_1^2 + d_1^2, \\ x_1^2 - 2a_2x_1 + x_2^2 - 2b_2x_2 + x_3^2 - 2c_2x_3 - x_4^2 + 2d_2x_4 + a_2^2 + b_2^2 + c_2^2 + d_2^2, \\ x_1^2 - 2a_3x_1 + x_2^2 - 2b_3x_2 + x_3^2 - 2c_3x_3 - x_4^2 + 2d_3x_4 + a_3^2 + b_3^2 + c_3^2 + d_3^2. \end{array} > \quad (2\text{-}31)$$

Example 2-3: Given the polynomials in $\mathbb{R}[x_1, x_2, x_3, x_4]$ as

$$\left[\begin{array}{l} -(X - x_1) + b^2x_1x_4 = 0 \\ -(Y - x_2) + b^2x_2x_4 = 0 \\ -(Z - x_3) + b^2x_3x_4 = 0 \\ b^2x_1^2 + b^2x_2^2 + a^2x_3^2 - a^2b^2 = 0 \end{array}\right. \quad (2\text{-}32)$$

The
Ideal $I =< -(X - x_1) + b^2x_1x_4, -(Y - x_2) + b^2x_2x_4, -(Z - x_3) + b^2x_3x_4, b^2x_1^2 + b^2x_2^2 + a^2x_3^2 - a^2b^2 >$

Having defined and given examples of an *Ideal*, we present the definitions of the *division algorithm* exploited by *B. Buchberger algorithm* before defining the *Gröbner basis (Standard Basis)* of an *Ideal*. We begin by stating the *Hilbert Basis Theorem*, which gives the assurance of the existence of the *Gröbner basis.* In general, we will denote by $k[x_1, \ldots, x_n]$ a *collection of polynomials* $f_1, \ldots, f_s$ with variables in $f_1, \ldots, f_n$ and coefficients in any field k. $k[x_1, \ldots, x_n]$ forms a *polynomial ring* (*Definition A-5, Appendix A.1*).

Lemma 2-2 (*Division Algorithm, D. Cox et al.* 1997, *theorem* 3, p.61, *theorem* 4, p.47):
Fix a monomial order > on $\mathbb{Z}^n_{\geq 0}$, and let $F = (f_1, \ldots, f_s)$ be an ordered S-tuple polynomial in $k[x_1, \ldots, x_n]$. Then every $f \in k[x_1, \ldots, x_n]$ can be written as $f = a_1f_1 + a_2f_2 + \ldots + a_sf_s + r$, where $a_i, r \in k[x_1, \ldots, x_n]$ and either $r = 0$ or a linear combination with coefficients in k of monomials, none of which is divisible by any of $LT(f_1), \ldots, LT(f_s)$. The *Hilbert Basis Theorem* assures that every *Ideal* $I \subset k[x_1, \ldots, x_n]$ has a finite generating set, that is $I =< g_1, \ldots, g_s >$ for some $g_1, \ldots, g_s \in I$.

The finite generating set G in *Hilbert Basis Theorem* is what is known as a *basis*. Suppose every non-zero polynomial is written in decreasing order of its monomials:

$$\sum_{i=1}^{n} a_i x_i, \;\; a_i \neq 0, \; x_i > x_{i+1}, \quad (2\text{-}33)$$

if we let the system of generators of the *Ideal* be in a set G , a polynomial f is reduced with respect to G if no leading monomial of an element of G (LM (G)) divides the leading monomial of f (LM(f)). The polynomial f is said to be *completely reduced* with respect to G if no monomials of f is divisible by the leading monomial of an element of G (*J. H. Davenport et al.* 1988, pp. 96-97).

The introduction of the *division algorithm* given in *Lemma 2-6* above fits well to the case of *univariate polynomials* as the remainder r can uniquely be determined. In the case of *multivariate polynomials*, the remainder may not be uniquely determined as this depends on the order of the divisors. The division of the polynomial F by $\{f_1, f_2\}$ may not necessarily give the same remainder as the division of F by $\{f_2, f_1\}$ in whose case the order has been changed. This problem is overcome if we pass over to the *Standard basis* having been assured of its existence for every *Ideal* by the *Hilbert Basis Theorem* in *Lemma 2*-6.

The *Standard Basis* G which completely reduces the polynomial f and uniquely determines the remainder r is also known as the *Gröbner basis* and is defined as follows:

Definition 2-6 (*J. H. Davenport et al.* 1988 p.97, *D. Cox et al.* 1997, Definition 1, p.100):
A system of generators (or basis) G of an *Ideal* I is called a *Standard Basis* or *Gröbner basis* (with respect to the order <) if every reduction of $f \in I$ to a reduced polynomial (with respect to G) always gives zero as a remainder. The above definition is a special case of a more general definition given as: Fix a monomial order and let $G = \{g_1, \ldots, g_s\} \subset k[x_1, \ldots, x_n]$. Given $f \in k[x_1, \ldots, x_n]$, then f reduces to zero Modulo G, written as

$$f \to_G 0, \tag{2-34}$$

if f can be written in the form

$$f = a_1 g_1 + \ldots + a_t g_t \tag{2-35}$$

such that whenever $a_i g_i \neq 0$, we have multideg(f)$\geq$ multideg($a_i g_i$).

Suggested by *W. Gröbner* in 1949 and developed by his student at the time *B. Buchberger* in 1965, B. Buchberger decided to honour his thesis supervisor *W. Gröbner*(1899-1980) by naming the *Standard basis* for *Ideals* in polynomial ring $k[x_1, \ldots, x_n]$ as *Gröbner basis* (*B. Buchberger* 1965). *Gröbner bases* has become a household name in algebraic manipulations and finds application in fields such as Statistics and Engineering. It has found use as a tool for discovering and proving theorems to solving systems of polynomial equations as elaborated in publications by *B. Buchberger* and *F. Winkler* (1998). *Gröbner bases* also give a solution to the *Ideal* membership problem. By reducing a given polynomial f with respect to the *Gröbner basis* G, f is said to be a member of the *Ideal* if the zero remainder is obtained. Thus let $G = \{g_1, \ldots, g_s\}$ be a *Gröbner basis* of an *Ideal* $I \subset k[x_1, \ldots, x_n]$ and let $f \in k[x_1, \ldots, x_n]$ be a polynomial, then $f \in I$ if and only if the remainder on division of f by G is zero. *Gröbner bases* can also be used to show the equivalence of polynomial equations. Two sets of polynomial equations will generate the same *Ideal* if and only if their *Gröbner bases* are equal with respect to any term ordering. This implies that a system of polynomial equations $f_1(x_1, \ldots, x_n) = 0, \ldots, f_s(x_1, \ldots, x_n) = 0$ will have the same solution with a system arising from any *Gröbner basis* of $f_1, \ldots, f_s$ with respect to any term ordering. This is the main property of the *Gröbner bases* that is used to solve a system of polynomial equations as will be explained below.

B. Buchberger algorithm

Given polynomials $g_1, \ldots, g_s \in I$, the *B. Buchberger algorithm* seeks to derive the standard generators or the Groebner basis of this *Ideal*. Systems of equations $g_1 = 0, \ldots, g_s = 0$ to be solved in practise are normally formed by these same polynomials which here generating the *Ideal*. The *B. Buchberger algorithm* computes the *Groebner basis* by making use of pairs of polynomials from the original polynomials $g_1, \ldots, g_s \in I$and computes the subtraction polynomial known as the $S-polynomial$ explained in *D. Cox et al.* (1997; p. 81**)** as follows:

Definition 2-7 ($S-polynomial$):
Let $f, g \in k[x_1, \ldots x_n]$ be two non-zero polynomials. If multideg $(f) = \alpha$ and multideg $(g) = \beta$, then let $\gamma = \gamma_1, \ldots, \gamma_n$, where $\gamma_i = \max\{\alpha_i, \beta_i\}$ for each i . x^γ is called the *Least Common Multiple* (LCM) of LM(f) and LM(g) expressed as $x^\gamma = LCM\{LM(f), LM(g)\}$. The $S-polynomial$ of f and g is given as

$$S(f,g) = \frac{x^\gamma}{LT(f)} f - \frac{x^\gamma}{LT(g)} g. \tag{2-36}$$

The expression above gives S as a linear combination of the monomials $\frac{x^\gamma}{LT(f)}, \frac{x^\gamma}{LT(g)}$ with polynomial coefficients f and g and thus belongs to the *Ideal* generated by f and g.

Definition 2-8 (*Gröbner basis* in terms of $S-polynomial$):
A basis G is *Gröbner basis* if and only if for every pair of polynomials f and g of G, $S(f,g)$ reduces to zero with respect to G. More generally a basis $G = \{g_1, \ldots, g_s\}$ for an *Ideal* I is a *Gröbner basis* if and only if $S(f,g) \to_G 0, \quad i \neq j$.

The implication of *Definition 2-8* is the following: Given two polynomials $f, g \in G$ such that $LCM(LM(f), LM(g)) = LM(f).LM(g)$ then the leading monomials of f and g are relatively prime leading to $S(f, g) \rightarrow_G 0$. The concept of prime integer is clearly documented in *K. Ireland* and *M. Rosen* (1990 pp. 1-17).

Example 2-4 ($S - Polynomial$):
Consider the two polynomials

$$\left[\begin{array}{l} g_1 = x_1^2 + 2a_{12}x_1x_2 + x_2^2 + a_{oo} \\ g_2 = x_2^2 + 2b_{23}x_2x_3 + x_3^2 + b_{oo} \end{array}\right. \tag{2-37}$$

of *Example A-1* in *Appendix A.1*, the $S-polynomial$ can then be computed as follows: First we choose a *lexicographic ordering* $\{x_1 > x_2 > x_3\}$ then

$$\left[\begin{array}{c} LM(g_1) = x_1^2,\ LM(g_2) = x_2^2, LT(g_1) = x_1^2,\ LT(g_2) = x_2^2 \\ LCM(LM(g_1), LM(g_2)) = x_1^2 x_2^2 \\ S(g_1, g_2) = \dfrac{x_1^2 x_2^2}{x_1^2}(x_1^2 + 2a_{12}x_1x_2 + x_2^2 + a_{oo}) - \dfrac{x_1^2 x_2^2}{x_2^2}(x_2^2 + 2b_{23}x_2x_3 + x_3^2 + b_{oo}) \\ = x_2^2x_1^2 + 2a_{12}x_1x_2^3 + x_2^4 + a_{oo}x_2^2 - x_1^2x_2^2 - 2b_{23}x_1^2x_2x_3 - x_1^2x_3^2 - b_{oo}x_1^2) \\ = -b_{oo}x_1^2 - 2b_{23}x_1^2x_2x_3 - x_1^2x_3^2 + 2a_{12}x_1x_2^3 + x_2^4 + a_{oo}x_2^2 \end{array}\right. \tag{2-38}$$

Example 2-5 ($S - Polynomial$):
Consider *Example A-2* in *Appendix A.1* for pseudo-ranging problem. The first two polynomials equations are given as

$$\left[\begin{array}{l} g_3 = x_1^2 - 2a_0x_1 + x_2^2 - 2b_0x_2 + x_3^2 - 2c_0x_3 - x_4^2 + 2d_0x_4 + a_0^2 + b_0^2 + c_0^2 + d_0^2 \\ g_4 = x_1^2 - 2a_1x_1 + x_2^2 - 2b_1x_2 + x_3^2 - 2c_1x_3 - x_4^2 + 2d_1x_4 + a_1^2 + b_1^2 + c_1^2 + d_1^2. \end{array}\right. \tag{2-39}$$

By choosing the *lexicographic ordering* $\{x_1 > x_2 > x_3 > x_4\}$, the $S-$ polynomial is computed as follows

$$\left[\begin{array}{l} LM(g_3) = x_1^2, LM(g_4) = x_1^2, LT(g_3) = x_1^2, LT(g_4) = x_1^2 \\ LCM(LM(g_3), LM(g_4)) = x_1^2 \\ S(g_3, g_4) = \dfrac{x_1^2}{x_1^2}(g_3) - \dfrac{x_1^2}{x_1^2}(g_4) = g_3 - g_4 \\ S(g_3, g_4) = 2(a_1 - a_o)x_1 + 2(b_1 - b_o)x_2 + 2(c_1 - c_o)x_3 + \\ +2(d_o - d_1)x_4 + a_o - a_1 + b_o - b_1 + c_o - c_1 + d_0 - d_1 \end{array}\right. \tag{2-40}$$

Example 2-6 ($S - Polynomial$):
As an additional example on the computation of $S-$ polynomial, let us consider the minimum distance mapping problem of *Example A-3* in *Appendix A.1*. The last two polynomials equations are given as

$$\left[\begin{array}{l} g_5 = x_3 + b^2x_3x_4 - Z \\ g_6 = b^2x_1^2 + b^2x_2^2 + a^2x_3^2 - a^2b^2 \end{array}\right. \tag{2-41}$$

By choosing the *lexicographic ordering* $\{x_1 > x_2 > x_3 > x_4\}$, the $S-$ polynomial is computed as follows

$$\left[\begin{array}{l} LM(g_5) = x_3, LM(g_6) = x_1^2, LT(g_5) = x_3, LT(g_6) = b^2x_1^2 \\ LCM(LM(g_5), LM(g_6)) = x_1^2x_3 \\ S(g_5, g_6) = \dfrac{x_1^2x_3}{x_3}(x_3 + b^2x_3x_4 - Z) - \dfrac{x_1^2x_3}{b^2x_1^2}(b^2x_1^2 + b^2x_2^2 + a^2x_3^2 - a^2b^2) \\ S(g_5, g_6) = -Zx_1^2 + b^2x_1^2x_3x_4 - x_2^2x_3 - \frac{a^2}{b^2}x_3^3 + a^2x_3 \end{array}\right. \tag{2-42}$$

Example 2-7 (computation of *Gröbner basis* from the $S - polynomials$):
By means of an Example given by *J. H. Davenport et al.* (1988, pp. 101-102) , we illustrate how the *B. Buchberger algorithm* works. Let us consider the *Ideal* generated by the polynomial equations

$$\left[\begin{array}{l} g_1 = x^3yz - xz^2 \\ g_2 = xy^2z - xyz \\ g_3 = x^2y^2 - z \end{array}\right. \tag{2-43}$$

with the *lexicographic ordering* $x > y > z$ adopted. The S -polynomials to be considered are $S(g_1, g_2)$, $S(g_2, g_3)$ and $S(g_1, g_3)$. We consider first $S(g_2, g_3)$ and show that the result is used to supress g_1 upon which any pair $S(g_1, g_i)$ (e.g. $S(g_1, g_2)$ and $S(g_1, g_3)$) containing g_1 will not be considered. $LT(g_2) = xy^2z,\ LT(g_3) = x^2y^2,$ then $LCM(g_2, g_3) = x^2y^2z$ respectively

$$\left[\begin{array}{l} S(g_2, g_3) = \frac{x^2y^2z}{xy^2z} g_2 - \frac{x^2y^2z}{x^2y^2} g_3 \\ = (x^2y^2z - x^2yz) - (x^2y^2z - z^2) \\ = -x^2yz + z^2 \end{array}\right. \tag{2-44}$$

We immediately note that the leading term of the resulting polynomial LT($S(g_2, g_3)$) is not divisible by any of the leading terms of the elements of G, and thus the remainder upon the division of $S(g_2, g_3)$ by the polynomials in G is not zero (i.e. when reduced with respect to G) thus G is *not* a *Gröbner basis*. This resulting polynomial after the formation of the $S-$ *polynomial* is denoted g_4 and its negative (to make calculations more reliable) added to the initial set of G leading to

$$\left[\begin{array}{l} g_1 = x^3yz - xz^2 \\ g_2 = xy^2z - xyz \\ g_3 = x^2y^2 - z \\ g_4 = x^2yz - z^2. \end{array}\right. \tag{2-45}$$

The $S-$ *polynomials* to be considered are now $S(g_1, g_2)$, $S(g_1, g_3)$, $S(g_1, g_4)$, $S(g_2, g_4)$ and $S(g_3, g_4)$. In the set of G, one can write $g_1 = xg_4$ leading without any change in G to the suppression of g_1 leaving only $S(g_2, g_4)$ and $S(g_3, g_4)$ to be considered. Then

$$\left[\begin{array}{l} S(g_2, g_4) = xg_2 - yg_4 \\ = -x^2yz + yz^2 \end{array}\right. \tag{2-46}$$

which can be reduced by adding g_4 to give $g_5 = yz^2 - z^2$, a non zero value thus the set G is *not* a *Gröbner basis*. This value is added to the set of G to give

$$\left[\begin{array}{l} g_2 = xy^2z - xyz, \\ g_3 = x^2y^2 - z, \\ g_4 = x^2yz - z^2, \\ g_5 = yz^2 - z^2, \end{array}\right. \tag{2-47}$$

the $S-$ *polynomials* to be considered are now $S(g_3, g_4)$, $S(g_2, g_5)$, $S(g_3, g_5)$ and $S(g_4, g_5)$. We then compute

$$\left[\begin{array}{l} S(g_3, g_4) = zg_3 - yg_4 \\ = yz^2 - z^2 \end{array}\right. \tag{2-48}$$

which upon subtraction from g_5 reduces to zero. Further,

$$\left[\begin{array}{l} S(g_2, g_5) = zg_2 - xyg_5 \\ = -xyz^2 + xyz^2 \\ = 0 \end{array}\right. \tag{2-49}$$

and

$$\left[\begin{array}{l} S(g_4, g_5) = zg_4 - x^2yg_5 \\ = x^2z^2 - z^3, \end{array}\right. \tag{2-50}$$

which is added to G as g_6 giving

$$\left[\begin{array}{l} g_2 = xy^2z - xyz, \\ g_3 = x^2y^2 - z, \\ g_4 = x^2yz - z^2, \\ g_5 = yz^2 - z^2, \\ g_6 = x^2y^2 - z^3, \end{array}\right. \tag{2-51}$$

The S polynomials to be considered are now $S(g_3, g_5)$, $S(g_2, g_6)$, $S(g_3, g_6)$, $S(g_4, g_6)$ and $S(g_5, g_6)$. We now illustrate that all this $S - polynomials$ reduces to zero as follows

$$\left[\begin{array}{l} S(g_3, g_5) = z^2g_3 - x^2yg_5 = x^2yz^2 - z^3 - zg_4 = 0 \\ S(g_2, g_6) = xzg_2 - y^2g_6 = -x^2y^2z^2 + y^2z^3 + y^2g_4 = 0 \\ S(g_3, g_6) = z^2g_3 - y^2g_6 = y^2z^3 - z^3 - (yz - z)g_5 = 0 \\ S(g_4, g_6) = zg_4 - yg_6 = yz^3 - z^3 - zg_5 = 0 \\ S(g_5, g_6) = x^2g_5 - yg_6 = -x^2z^2 + yz^3 + g_6 - zg_5 = 0. \end{array}\right. \tag{2-52}$$

Thus equation (2-52) comprise the *Gröbner basis* of the original set in (2-43).

The importance of the *S-polynomials* is that they lead to the cancellation of the leading terms of the polynomial pairs involved. In so doing the polynomial variables are systematically eliminated according to the polynomial ordering chosen. For example if the *lexicographic ordering* $x > y > z$ is chosen, x will be eliminated first, followed by y and the final expression may consist only of the variable z. *D. Cox et al.* (1998, p.15) has indicated the advantage of *lexicographic ordering* as being the ability to produce *Gröbner basis* with systematic elimination of variables. *Graded lexicographic ordering* on the other hand has the advantage of minimizing the amount of computational space needed to produce the *Gröbner basis*. The procedure is thus a *generalisation* of the *Gauss elimination procedure* for linear systems of equations. If we now put our system of polynomial equations to be solved in a set G, $S-pair$ combinations can be formed from the set of G as explained in the *definitions* above. The *theorem*, known as the *Buchberger's* $S-pair$ *polynomial criterion*, gives the criterion for deciding whether a given basis is a *Gröbner basis* or not. It suffices to compute all the *S-polynomials* and check whether they reduce to zero. Should one of the polynomials not reduce to zero, then the basis fails to be a *Gröbner basis*. Since the reduction is a linear combination of the elements of G, it can be added to the set G without changing the *Ideal* generated. *B. Buchberger* (1979) gives an *optimisation criterion* that reduces the number of the *S-polynomials* already considered in the algorithm. The criterion states that if there is an element p of G such that the leading monomial of p (LM(p)) divides the LCM($f, g \in G$), and if $S(f,p)$, $S(p,g)$ have already been considered, then there is no need of considering $S(f,g)$ as this reduces to zero.

The essential observation in using the *Gröbner bases* to solve a system of polynomial equations is that the variety (simultaneous solution of system of polynomial equations) does not depend on the original system of the polynomials $F := \{f_1, \ldots, f_s\}$ but instead on the *Ideal* I generated by F. This therefore means that the variety $V = V(I)$ and one makes use of the special generating set (*Gröbner basis*) instead of the actual system F. Since the *Ideal* is generated by F, the solution obtained by solving for the affine variety of this *Ideal* satisfies the original system F of equations. *B. Buchberger* (1970) proved that $V(I)$ is void, and thus giving a test as to whether the system of polynomial F can be solved , if and only if the computed *Gröbner basis* of polynomial Ideal I has $\{1\}$ as its element. *B. Buchberger* (1970) further gives the criterion for deciding if $V(I)$ is finite. If the system has been proved to be solvable and finite then *F. Winkler* (1996, *theorem 8.4.4*, p.192) gives a theorem for deciding whether the system has finitely or infinitely many solutions. The *theorem* states that if G is a *Gröbner basis*, then a solvable system of polynomial equations has finitely many solutions if and only if for every x_i, $1 \leq i \leq n$, there is a polynomial $g_i \in G$ such that $LM(g_i)$ is a pure power of x_i. The process of addition of the remainder after the reduction by the $S-polynomials$ and thus expanding the generating set is shown by *B. Buchberger* (1970), *D. Cox et al.* (1997 p.88) and *J. H. Davenport et al.* (1988 p.101) to terminate.

The *B. Buchberger algorithm*, more or less a generalization of the *Gauss elimination* procedure, makes use of the subtraction polynomials known as the $S-polynomials$ in *Definition 2-8* to eliminate the leading terms of a pair of polynomials. In so doing and if *lexicographic ordering* is chosen, the process end up with one of the computed $S-polynomials$ being a univariate polynomial which can be solved and substituted back in the other $S-polynomials$ using the *Extension Theorem* (*D. Cox et al.* 1998, pp.25-26) to obtain the other variables. The *Gröbner bases* approach adds to the treasures of methods that are used to solve *nonlinear algebraic systems of equations* in Geodesy, Photogrammetry, Machine Vision, Robotics and Surveying.

Having defined the term *Gröbner basis* and illustrated how the *B. Buchberger algorithm* computes the *Gröbner bases*, we briefly mention here how the *Gröbner bases* can be computed using algebraic softwares of Mathematica and Maple. In Mathematica Versions 2 or 3, the *Gröbner basis* command is executed by writing *In[1]:=GroebnerBasis[{polynomials}, {variables}, {options}]* (where In[1]:= is the mathematica prompt) which computes the *Gröbner basis* for the *ideal* generated by the *polynomials* with respect to the *monomial order s*pecified by *monomial order options* with the *variables* specified as in the executable command giving the reduced *Groebner basis*. Without specifying the options part, one gets too many elements of the *Gröbner basis* which may not be relevant. In Maple Version 5 the command is accessed by typing > *with (grobner);* (where > is the Maple prompt and the semicolon ends the Maple command). Once the *Gröbner basis* package has been loaded, the execution command then becomes > *gbasis (polynomials, variables, termorder)* which computes the *Gröbner basis* for the *ideal* generated by the *polynomials* with respect to the *monomial ordering s*pecified by *termorder* and *variables* in the executable command. Following suggestions from *B. Buchberger* (1999), we Mathematica software is adopted in the present study.

2-322 Closed form solution of nonlinear equations by Multipolynomial Resultants

Whereas the resultant of two polynomials is well known and algorithms for computing it are well incorporated into computer algebra packages such as *Maple*, the *Multipolynomial resultant*, i.e. the resultant of more than two polynomials still remain an active area of research. In Geodesy, the use of the two polynomial resultant also known as the *Sylvester resultant* is exemplified in the work of *P. Lohse* (1994, pp.72-76). The present study therefore extends on the use of *Sylvester resultants* to resultants of more than two polynomials of multiple variables *(Multipolynomial resultant)* and illustrates in Chapter 3 how the tool can be exploited in Geodesy to solve *nonlinear system of equations*.

The necessity of *Multipolynomial resultant* method in Geodesy is due to the fact that many geodetic problems involve the solution of more than two polynomials of multiple variables. This is true since we are living in a threedimensional world. We shall therefore understand the term *multipolynomial resultants* to mean resultants of more than two polynomials. We treat it as a tool besides the *Gröbner bases* and perhaps more powerful to eliminate variables in solution of polynomial systems. Publications on the subject can be found in the works of *G. Salmon* (1876), *F. Macaulay* (1902, 1916), *A. L. Dixon* (1908), *C. Bajaj et al.* (1988), *J. F. Canny* (1988), *J. F. Canny et al.* (1989), *I. M. Gelfand et al.* (1990, 1994), *D. Manocha* (1992, 1993, 1994a,b,c), *D. Manocha* and *J.F. Canny* (1991, 1992, 1993), *G. Lyubeznik* (1995), *S. Krishna and D. Manocha* (1995), *J. Guckenheimer et al.*(1997), *B. Sturmfels* (1994, 1998) and *E. Cattani et al.* (1998). Text books on the subject have been written by *I. Gelfand et al*. (1994) and, more recently, by *D. Cox et al.* (1998, pp.71-122) who provides interesting material.

In order to understand the *Multipolynomial resultants technique*, we first present the simple case; the resultant of two polynomial also known as the *Sylvester resultant*.

Resultant of two polynomials

Definition 2-10 (Homogeneous polynomial): If monomials of a polynomial p with non zero coefficients have the same *total degree*, the polynomial p is said to be *homogeneous*.

Example 2-8 (Homogeneous polynomial equation):
A homogeneous polynomial equation of total degree 2 is $p = x^2 + y^2 + z^2 + xy + xz + yz$ since the monomials $\{x, y, z, xy, xz, yz\}$ all have the sum of their powers (total degree) being 2.

To set the ball rolling, we examine next the resultant of two univariate polynomials $p,\ q \in k[x]$ of *positive degree* as

$$\left.\begin{array}{c} p = k_0x^i + \ldots . + k_i,\ k_0 \neq 0,\ i > 0 \\ q = l_0x^j + \ldots . + l_j,\ l_0 \neq 0,\ j > 0 \end{array}\right\} \tag{2-53}$$

the resultant of p and q, denoted Res$(p,\ q)$, is the $(i+j) \times (i+j)$ determinant

$$Res\,(p,\ q) = det \begin{bmatrix} k_0 & k_1 & k_2 & . & . & . & k_i & 0 & 0 & 0 & 0 & 0 \\ 0 & k_0 & k_1 & k_2 & . & . & . & k_i & 0 & 0 & 0 & 0 \\ 0 & 0 & k_0 & k_1 & k_2 & . & . & . & k_i & 0 & 0 & 0 \\ 0 & 0 & 0 & k_0 & k_1 & k_2 & . & . & . & k_i & 0 & 0 \\ 0 & 0 & 0 & 0 & k_0 & k_1 & k_2 & . & . & . & k_i & 0 \\ 0 & 0 & 0 & 0 & 0 & k_0 & k_1 & k_2 & . & . & . & k_i \\ l_0 & l_1 & l_2 & . & . & . & l_j & 0 & 0 & 0 & 0 & 0 \\ 0 & l_0 & l_1 & l_2 & . & . & . & l_j & 0 & 0 & 0 & 0 \\ 0 & 0 & l_0 & l_1 & l_2 & . & . & . & l_j & 0 & 0 & 0 \\ 0 & 0 & 0 & l_0 & l_1 & l_2 & . & . & . & l_j & 0 & 0 \\ 0 & 0 & 0 & 0 & l_0 & l_1 & l_2 & . & . & . & l_j & 0 \\ 0 & 0 & 0 & 0 & 0 & l_0 & l_1 & l_2 & . & . & . & l_j \end{bmatrix} \tag{2-54}$$

where the coefficients of the first polynomial p of (2-53) occupies j rows while those of the second polynomial q occupies i rows. The empty spaces are occupied by zeros as shown above such that a square matrix is obtained. This resultant is also known as the *Sylvester resultant* and has the following properties (*B. Sturmfels* 1998, *D. Cox et al.* 1998, §3.5)

1. Res$(p,\ q)$ is a polynomial in $k_0,\ \ldots, k_i,\ l_0,\ \ldots, l_j$ with integer coefficients
2. Res$(p,\ q) = 0$ if and only if $p(x)$ and $q(x)$ have a common factor in $Q[x]$
3. There exist a polynomial $r,\ s \in Q[x]$ such that $rp + sq = Res(p,\ q)$

Sylvester resultants can be used to solve two polynomial equations as shown in the example below

Example 2-9 (D. Cox et al.1998, p.72):
Consider the two equations

$$\left.\begin{array}{r} p := xy - 1 = 0 \\ q := x^2 + y^2 - 4 = 0. \end{array}\right\} \tag{2-55}$$

In order to eliminate one variable, we use the *hide variable technique* i.e. we consider one variable say y as a constant (of degree zero). We then have the *Sylvester resultant* from (2-54) as

$$Res\,(p,\, q, x) = det \begin{bmatrix} y & -1 & 0 \\ 0 & y & -1 \\ 1 & 0 & y^2 - 4 \end{bmatrix} = y^4 - 4y^2 + 1 \tag{2-56}$$

which can be readily solved for the variable y and substituted back in any of the equations in (2-55) to get the values of the other variable x. For two polynomials, the construction of resultant is relatively simpler and algorithms for the execution are incorporated in computer algebra algorithms. For the resultant of more than 2 polynomials of multiple variables, we turn to the *Multipolynomial resultants*.

Multipolynomial Resultants

In defining the term *multipolynomial resultant*, *D. Manocha* (1994c) writes:

"Elimination theory, a branch of classical algebraic geometry, deals with conditions for common solutions of a system of polynomial equations. Its main result is the construction of a single resultant polynomial of n homogeneous polynomial equations in n unknowns, such that the vanishing of the resultant is a *necessary* and *sufficient* condition for the given system to have a non-trivial solution. We refer to this resultant as the *multipolynomial resultant* and use it in the algorithm presented in the paper".

We present here two approaches for the formation of the design matrix whose determinant we need; first the approach based on *F. Macaulay* (1902) formulation and then a more modern approach based on *B. Sturmfels* (1998) formulation.

Approach based on *F. Macaulay* (1902) formulation: With n polynomials, the construction of the matrix whose entries are the coefficients of the polynomials $f_1, \ldots, f_n$ can be done in five steps as illustrated in the following approach of *F. Macaulay* (1902) :

Step 1: The given polynomials $f_1 = 0, \ldots, f_n = 0$ are considered to be homogeneous equations in the variables $x_1, \ldots, x_n$ and if not, they are homogenized. Let the degree of the polynomial f_i be d_i. The first step involves the determination of the *critical degree* given by *C. Bajaj et al.* (1988) as

$$d = 1 + \sum (d_i - 1). \tag{2-57}$$

Step 2: Once the *critical degree* has been established, the given monomials of the polynomial equations are multiplied with each other to generate a set X whose elements consists of monomials whose total degree equals the *critical degree*. Thus if we are given the polynomial equations $f_1 = 0, \ldots, f_n = 0$, then each monomial of f_1 is multiplied by those of $f_2, \ldots, f_n$, those of f_2 are multiplied by those of $f_3, \ldots, f_n$ until those of f_{n-1} are multiplied by those of f_n. The set X of monomials generated in this form is

$$X^d = \{x^d \mid d = \alpha_1 + \alpha_2 + \ \ldots \ + \alpha_n\} \tag{2-58}$$

and the variable $x^d = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$.

Step 3: The set X containing the monomials each of total degree d is now partitioned according to the following criteria (*J. F. Canny* 1988, p.54)

$$\begin{cases} X_1^d &= \{x^\alpha \in X^d \mid \alpha_1 \geq d_1\} \\ X_2^d &= \{x^\alpha \in X^d \mid \alpha_2 \geq d_2 \, and \, \alpha_1 < d_1\} \\ . & . \quad . \\ . & . \quad . \\ . & . \quad . \\ X_n^d &= \{x^\alpha \in X^d \mid \alpha_n \geq d_n \, and \, \alpha_i < d_i, \, for \, i = 1, \ldots, n-1\}. \end{cases} \tag{2-59}$$

The resulting sets of X_i^d are disjoint and every element of X^d is contained in exactly one of them.

Step 4: From the resulting subsets $X_i^d \subset X^d$, a set of polynomials F_i which are homogeneous in n variables are defined as follows

$$F_i = \frac{X_i^d}{x_i^{d_i}} f_i \tag{2-60}$$

from which a *square matrix* $\boldsymbol{A}$ is now formed with the row elements being the *coefficients* of the monomials of the polynomials $F_i \mid_{i=1,\ldots,n}$ and the columns correspond to the N monomials of the set X^d. The formed square matrix is of the order $\begin{pmatrix} d+n-1 \\ d \end{pmatrix} \times \begin{pmatrix} d+n-1 \\ d \end{pmatrix}$ and is such that for a given polynomial F_i in (2-60), the row is made up of the symbolic coefficients of each polynomial. The square matrix $\boldsymbol{A}$ has a special property that the non trivial solution of the homogeneous equations F_i which also form the solution of the original equations f_i are in its null space. This implies that the matrix must be singular or its determinant, $det(\boldsymbol{A})$, must be zero. For the determinant to vanish therefore, the original equations f_i and their homogenized counterparts F_i must have the same non trivial solutions.

Step 5: After computing the determinant of the square matrix $\boldsymbol{A}$ above, *F. Macaulay* (1902) suggests the computation of *extraneous factor* in order to obtain the resultant. *D. Cox et al.* (1998, *Proposition* 4.6, p.99) explains the *extraneous factors* to be integer polynomials in the coefficients of $\bar{F}_0, \ldots, \bar{F}_{n-1}$, where $\bar{F}_i = F_i(x_0, \ldots, x_{n-1}, 0)$ and is related to the determinant via

$$determinant = Res(F_1, \ldots, F_n).Ext \tag{2-61}$$

with the determinant computed as in step 4, $Res(F_1, \ldots, F_n)$ being the *Multipolynomial resultant* and Ext the extraneous factor. This expression was established as early as 1902 by *F. Macaulay* (1902) and this procedure of *resultant* formulation named after him. *F. Macaulay* (1902) determines the extraneous factor from the sub-matrix of the $N \times N$ square matrix $\boldsymbol{A}$ and calls it a factor of minor obtained by deleting rows and columns of the $N \times N$ matrix $\boldsymbol{A}$. A monomial x^α of total degree d is said to be reduced if $x_i^{d_i}$ divides x^α for exactly one i. The *extraneous factor* is obtained by computing the determinant of the sub-matrix of the coefficient matrix $\boldsymbol{A}$ obtained by deleting rows and columns corresponding to reduced monomials x^α.

From the relationship of step 5, it suffices for our purpose to solve for the unknown variable hidden in the coefficients of the polynomials f_i by obtaining the determinant of the $N \times N$ square matrix $\boldsymbol{A}$ and equating it to zero neglecting the extraneous factor. This is because the extraneous factor is an integer polynomial and as such not related to the variable in the determinant of $\boldsymbol{A}$. The existence of the non-trivial solution provides the *necessary* and *sufficient* conditions for the vanishing of the determinant. The procedure becomes clear when we consider the example of the *threedimensional resection problem* in Chapter 3. It should be pointed out that there exists several procedures for computing the resultants as exemplified in the works of *D. Manocha* (1992, 1993, 1994a,b,c) who solves the *Multipolynomial resultants* using the eigenvalue-eigenvector approach, *J. F. Canny* (1988) who solves the resultant using the characteristic polynomial approach and *B. Sturmfels* (1994, 1998) who proposes a more compact approach for solving the resultants of a ternary quadric using the Jacobian determinant approach which we present below.

Approach based on *B. Sturmfels* (1998, p.26) formulation: Given three homogeneous equations of degree two as follows

$$\begin{aligned} F_1 &:= a_{11}x^2 + a_{12}y^2 + a_{13}z^2 + a_{14}xy + a_{15}xz + a_{16}yz &= 0 \\ F_2 &:= a_{21}x^2 + a_{22}y^2 + a_{23}z^2 + a_{24}xy + a_{25}xz + a_{26}yz &= 0 \\ F_3 &:= a_{31}x^2 + a_{32}y^2 + a_{33}z^2 + a_{34}xy + a_{35}xz + a_{36}yz &= 0 \end{aligned} \tag{2-62}$$

we compute the Jacobian determinants of (2-62) by

$$J = det \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{bmatrix} \tag{2-63}$$

which is a cubic polynomial in the coefficients $\{x, y, z\}$. Since the determinant polynomial J in (2-63) is a cubic polynomial, its partial derivatives will be quadratic polynomials in variables $\{x, y, z\}$ and can be written in the form

$$\begin{aligned} \frac{\partial J}{\partial x} := b_{11}x^2 + b_{12}y^2 + b_{13}z^2 + b_{14}xy + b_{15}xz + b_{16}yz &= 0 \\ \frac{\partial J}{\partial y} := b_{21}x^2 + b_{22}y^2 + b_{23}z^2 + b_{24}xy + b_{25}xz + b_{26}yz &= 0 \\ \frac{\partial J}{\partial z} := b_{31}x^2 + b_{32}y^2 + b_{33}z^2 + b_{34}xy + b_{35}xz + b_{36}yz &= 0. \end{aligned} \tag{2-64}$$

The coefficients b_{ij} in (2-64) are quadratic polynomials in a_{ij} of equation (2-62). The final step in computing the resultant of the initial system (2-62) involves the computation of the determinant of a 6×6 matrix given by

$$Res_{222}(F_1, F_2, F_3) = det \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{12} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & a_{36} \end{bmatrix}. \tag{2-65}$$

The resultant (2-65) vanishes if and only if (2-62) have a common solution $\{x, y, z\}$, where $\{x, y, z\}$ are complex numbers or real numbers not all equal zero. In Chapter 3, we use the procedure to solve the Grunert equation and the GPS pseudo-range equations.

2-33 Adjustment of the combinatorial minimal subsets solutions

Once the *combinatorial minimal subsets* have been solved using either the *Gröbner bases* or the *Multipolynomial resultant* approach, the resulting sets of solution are considered as pseudo-observations. For each combinatorial, the obtained minimal subset solutions considered as pseudo-observations are used as the approximate values to generate the dispersion matrix via the nonlinear error propagation law/variance-covariance propagation (*e.g. E. Grafarend* and *B. Schaffrin*, 1993, pp.469-471) as follows:

From the nonlinear geodetic observation equations that have been converted into its algebraic (polynomial) form, the *combinatorial minimal subsets* will consist of polynomials $f_1, \ldots, f_m \in k[x_1, \ldots, x_m]$ with $\{x_1, \ldots, x_m\}$ being the unknown variables (fixed parameters) to be determined and $\{y_1, \ldots, y_n\}$ the known variables comprising the observations or pseudo-observations. We write the polynomials as

$$\left.\begin{array}{c} f_1 := g(x_1, \ldots, x_m, y_1, \ldots, y_n) = 0 \\ f_2 := g(x_1, \ldots, x_m, y_1, \ldots, y_n) = 0 \\ . \\ . \\ . \\ f_m := g(x_1, \ldots, x_m, y_1, \ldots, y_n) = 0 \end{array}\right\} \tag{2-66}$$

which is expressed in matrix form as

$$\boldsymbol{f} := \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0} \tag{2-67}$$

where the unknown variables $\{x_1, \ldots, x_m\}$ are placed in a vector $\boldsymbol{x}$ and the known variables $\{y_1, \ldots, y_n\}$ are placed in the vector $\boldsymbol{y}$. Next, we implement the error propagation from the observations (pseudo-observations) $\{y_1, \ldots, y_n\}$ to the parameters $\{x_1, \ldots, x_m\}$ that are to be explicitly determined namely characterized by the *first moments*, the expectation $E\{\boldsymbol{x}\} = \boldsymbol{\mu}_x$ and $E\{\boldsymbol{y}\} = \boldsymbol{\mu}_y$, as well as the *second moments*, the variance-covariance matrices/dispersion matrices $D\{\boldsymbol{x}\} = \boldsymbol{\Sigma}_x$ and $D\{\boldsymbol{y}\} = \boldsymbol{\Sigma}_y$. From *E. Grafarend* and *B. Schaffrin* (1993, pp.470-471), we have upto nonlinear terms

$$D\{\boldsymbol{x}\} = \boldsymbol{J}_x^{-1}\boldsymbol{J}_y\boldsymbol{\Sigma}_y\boldsymbol{J}_y'(\boldsymbol{J}_x^{-1})' \tag{2-68}$$

with $\boldsymbol{J}_x, \boldsymbol{J}_y$ being the partial derivatives of (2-67) with respect to $\boldsymbol{x}$,$\boldsymbol{y}$ respectively at the Taylor points $(\boldsymbol{\mu}_x, \boldsymbol{\mu}_y)$. The approximate values of unknown parameters $\{x_1, \ldots, x_m\} \in \boldsymbol{x}$ appearing in the Jacobi matrices $\boldsymbol{J}_x, \boldsymbol{J}_y$ are obtained from *Gröbner bases* or *Multipolynomial resultants* solution of the *nonlinear system of equations* (2-66).

Given $\boldsymbol{J}_i = \boldsymbol{J}_{x_i}^{-1}\boldsymbol{J}_{y_i}$ from the ith combination and $\boldsymbol{J}_j = \boldsymbol{J}_{x_j}^{-1}\boldsymbol{J}_{y_j}$ from the jth combinatorials, the correlation between the ith and jth combination is given by

$$\boldsymbol{\Sigma}_{ij} = \boldsymbol{J}_j\boldsymbol{\Sigma}_{y_j y_i}\boldsymbol{J}_i^{'} \tag{2-69}$$

The sub-matrices variance-covariance matrix for the individual combinatorials $\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2, \boldsymbol{\Sigma}_3, \ldots, \boldsymbol{\Sigma}_k$ (where k is the number of combinations) obtained via (2-68) and the correlations between combinatorials obtained from (2-69) form the variance-covariance/dispersion matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_1 & \boldsymbol{\Sigma}_{12} & . & . & . & \boldsymbol{\Sigma}_{1k} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_2 & . & . & . & \boldsymbol{\Sigma}_{2k} \\ . & & \boldsymbol{\Sigma}_3 & & & \\ . & & & . & & \\ . & & & & . & \\ \boldsymbol{\Sigma}_{k1} & . & . & . & & \boldsymbol{\Sigma}_k \end{bmatrix} \tag{2-70}$$

for the entire k combinations. The obtained dispersion matrix Σ is then used in the *linear Gauss-Markov model* (2-4) to obtain the estimates $\hat{\boldsymbol{\xi}}$ of the unknown parameters $\boldsymbol{\xi}$ with the combinatorial solution points in a polyhedron considered as pseudo-observations in the vector $\boldsymbol{y}$ of observations while the design matrix $\boldsymbol{A}$ comprises of integer values 1 being the coefficients of the unknowns as in (2-73). In order to understand the adjustment process, we consider the following example.

Example 2-10**:** Consider that one has four observations in three unknowns in a nonlinear problem. Let the observations be given by $\{y_1, y_2, y_3, y_4\}$ leading to four combinations giving the solutions and the $\boldsymbol{z}_I(y_1, y_2, y_3)$,$\boldsymbol{z}_{II}(y_2, y_3, y_4)$, $\boldsymbol{z}_{III}(y_1, y_3, y_4)$ and $\boldsymbol{z}_{IV}(y_1, y_2, y_4)$. If the solutions are placed in a vector $\boldsymbol{z}_J = [\ \boldsymbol{z}_I \quad \boldsymbol{z}_{II} \quad \boldsymbol{z}_{III} \quad \boldsymbol{z}_{IV}\ ]^{'}$, the adjustment model is then defined as

$$E\{\boldsymbol{z}_J\} = \boldsymbol{I}_{12\times3}\boldsymbol{\xi}_{3\times1}, D\{\boldsymbol{z}_J\}\ from\,Variance/Covariance\,propagation. \tag{2-71}$$

Let

$$\boldsymbol{\xi}^n = \boldsymbol{L}\boldsymbol{z}_J\ subject\,to\,\boldsymbol{z}_J := \begin{bmatrix} \boldsymbol{z}_I \\ \boldsymbol{z}_{II} \\ \boldsymbol{z}_{III} \\ \boldsymbol{z}_{IV} \end{bmatrix} \in \mathbb{R}^{12\times1} \tag{2-72}$$

such that the postulations $trD\{\boldsymbol{\xi}^n\} = min$ i.e. "*best*" and $E\{\boldsymbol{\xi}^n\} = \boldsymbol{\xi}\,for\,all\,\boldsymbol{\xi}^n \in \mathbb{R}^m$ i.e. "*uniformly unbiased*" holds. We then have from (2-70), (2-71) and (2-72) the result

$$\hat{\boldsymbol{\xi}} = (\boldsymbol{I}_{3\times12}^{'}\boldsymbol{\Sigma}_{\boldsymbol{z}_J}\boldsymbol{I}_{12\times3})\boldsymbol{I}_{3\times12}^{'}\boldsymbol{\Sigma}_{\boldsymbol{z}_J}^{-1}\boldsymbol{z}_J \tag{2-73}$$

$$\hat{\boldsymbol{L}} = arg\{trD\{\boldsymbol{\xi}^n\} = tr\,\boldsymbol{L}\boldsymbol{\Sigma}_y\boldsymbol{L}^{'} = min \mid UUE\}$$

The dispersion matrix $D\{\hat{\boldsymbol{\xi}}\}$ of the estimates $\hat{\boldsymbol{\xi}}$ is obtained via (2-5). The shift from arithmetic weighted mean to the use of *linear Gauss Markov model* is necessitated as we do not readily have the weights of the minimal combinatorial subsets but their dispersion which we obtain via *error propagation/Variance-Covariance propagation*. If we employ the equivalence *theorem* of *E. Grafarend* and *B. Schaffrin* (1993, pp. 339-341), an adjustment using linear Gauss markov model instead of weighted arithmetic mean in *Box* (2-1) is permissible. In *Appendix A.3* in page (108), we present the error propagation based on the nonlinear random effect model (multivariate) where we illustrate the effect of the biased term.

Example 2-10 (*error propagation for planar ranging problem*):
Let us consider a simple case of the planar ranging problem. From an unknown point $P(X,Y) \in \mathbb{E}^2$, let distances S_1 and S_2 be measured to two known points $P_1(X_1,Y_1) \in \mathbb{E}^2$ and $P_2(X_2,Y_2) \in \mathbb{E}^2$ respectively. We have the distance equations expressed as

$$\left[\begin{array}{l} S_1^2 = (X_1 - X)^2 + (Y_1 - Y)^2 \\ S_2^2 = (X_2 - X)^2 + (Y_2 - Y)^2 \end{array}\right. \tag{2-74}$$

which we express in algebraic form (2-66) as

$$\left[\begin{array}{l} f_1 := (X_1 - X)^2 + (Y_1 - Y)^2 - S_1^2 = 0 \\ f_2 := (X_2 - X)^2 + (Y_2 - Y)^2 - S_2^2 = 0 \end{array}\right. . \tag{2-75}$$

On taking total differential of (2-75) we have

$$\left[\begin{array}{l} df_1 := 2(X_1 - X)dX_1 - 2(X_1 - X)dX + 2(Y_1 - Y)dY_1 - \\ \qquad -2(Y_1 - Y)dY - 2S_1 dS_1 = 0 \\ df_2 := 2(X_2 - X)dX_2 - 2(X_2 - X)dX + 2(Y_2 - Y)dY_2 - \\ \qquad -2(Y_2 - Y)dY - 2S_2 dS_2 = 0 \end{array}\right. \tag{2-76}$$

which on arranging the differential vector of the unknown terms $\{X,Y\} = \{x_1,x_2\} \in \boldsymbol{x}$ on the left hand side and that of the known terms $\{X_1,Y_1,X_2,Y_2,S_1,S_2\} = \{y_1,y_2,y_3,y_4,y_5,y_6\} \in \boldsymbol{y}$ on the right hand side leads to

$$\boldsymbol{J}_x \left[\begin{array}{c} dX \\ dY \end{array}\right] = \boldsymbol{J}_y \left[\begin{array}{c} dS_1 \\ dX_1 \\ dY_1 \\ dS_2 \\ dX_2 \\ dY_2 \end{array}\right] \tag{2-77}$$

with

$$\boldsymbol{J}_x = \left[\begin{array}{cc} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} \end{array}\right] = \left[\begin{array}{cc} -2(X_1 - X) & -2(Y_1 - Y) \\ -2(X_2 - X) & -2(Y_2 - Y) \end{array}\right] \tag{2-78}$$

and

$$\begin{array}{l} \boldsymbol{J}_y = \left[\begin{array}{cccccc} \frac{\partial f_1}{\partial S_1} & \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial Y_1} & 0 & 0 & 0 \\ 0 & 0 & & \frac{\partial f_2}{\partial S_2} & \frac{\partial f_2}{\partial X_2} & \frac{\partial f_2}{\partial Y_2} \end{array}\right] = \\ = \left[\begin{array}{cccccc} 2S_1 & -2(X_1 - X) & -2(Y_1 - Y) & 0 & 0 & 0 \\ 0 & 0 & 0 & 2S_2 & -2(X_2 - X) & -2(Y_2 - Y) \end{array}\right] \end{array} \tag{2-79}$$

If we consider that

$$D\{\boldsymbol{x}\} = \boldsymbol{\Sigma}_x = \left[\begin{array}{cc} \sigma_X^2 & \sigma_{XY} \\ \sigma_{YX} & \sigma_Y^2 \end{array}\right] \tag{2-80}$$

and

$$D\{\boldsymbol{y}\} = \boldsymbol{\Sigma}_y = \left[\begin{array}{cccccc} \sigma_{S_1}^2 & \sigma_{S_1X_1} & \sigma_{S_1Y_1} & \sigma_{S_1X_2} & \sigma_{S_1S_2} & \sigma_{S_1Y_2} \\ \sigma_{X_1S_1} & \sigma_{X_1}^2 & \sigma_{X_1Y_1} & \sigma_{X_1S_2} & \sigma_{X_1X_2} & \sigma_{X_1Y_2} \\ \sigma_{Y_1S_1} & \sigma_{Y_1X_1} & \sigma_{Y_1}^2 & \sigma_{Y_1S_2} & \sigma_{Y_1X_2} & \sigma_{Y_1Y_2} \\ \sigma_{S_2S_1} & \sigma_{S_2X_1} & \sigma_{S_2Y_1} & \sigma_{S_2}^2 & \sigma_{S_2X_2} & \sigma_{S_2Y_2} \\ \sigma_{X_2S_1} & \sigma_{X_2X_1} & \sigma_{X_2Y_1} & \sigma_{X_2S_2} & \sigma_{X_2}^2 & \sigma_{X_2Y_2} \\ \sigma_{Y_2S_1} & \sigma_{Y_2X_1} & \sigma_{Y_2Y_1} & \sigma_{Y_2S_2} & \sigma_{Y_2X_2} & \sigma_{Y_2}^2 \end{array}\right] \tag{2-81}$$

we obtain with (2-77), (2-78) and (2-79) the dispersion (2-68) of the unknown variables $\{X,Y\} = \{x_1,x_2\} \in \boldsymbol{x}$. The unknown values of $\{X,Y\} = \{x_1,x_2\} \in \boldsymbol{x}$ appearing in the Jacobi matrices (2-78) and (2-79) are obtained from *Gröbner bases* or *Multipolynomial resultants* solution of the *nonlinear system of equations* (2-74).

2-4 Concluding remarks

The Chapter has presented the closed form algebraic tools of *Gröbner Bases* and the *Multipolynomial resultants* that are used in the next Chapters to solve closed form geodetic problems. Using these algebraic closed form tools, the *Gauss-Jacobi combinatorial algorithm* is presented as an alternative to estimate the fixed parameters in the nonlinear Gauss-Markov model. In Chapter 3, we consider the solution of selected geodetic problems using the algebraic tools of *Gröbner Bases* and the *Multipolynomial resultants* while in Chapter 4, we use the *Gauss-Jacobi combinatorial algorithm* to estimate the fixed parameters in the nonlinear Gauss-Markov model in the overdetermined three-dimensional resection case.

Chapter 3

Solution of selected Geodetic problems

In this Chapter, we solve in a closed form the geodetic problems of; *three-dimensional resection, minimum distance mapping* and *GPS point positioning with observations of type pseudo-range*. The *Gröbner bases* and the *multi-polynomial resultants* techniques presented in Chapter 2 are used as the computational tools with the *Gauss-Jacobi combinatorial algorithm* employed in the solution of the *overdetermine GPS code pseudo-ranging problem*. We start in Section (3-1) by giving an overview of the GPS and LPS systems before considering the solution of the selected geodetic problems in Section (3-2).

3-1 LPS and GPS positioning systems

In this section we highlight the main positioning systems; the *Global Positioning System* GPS and the *Local Positioning System* LPS and in particular, the reference frames upon which they operate. An exposition on their relationship is given and the term *"three-dimensional orientation problem"* defined.

3-11 GPS Positioning

The results of the *three-dimensional positioning* by GPS are the three-dimensional geodetic coordinates $\{\lambda, \phi, h\}$ of a point. When positioning with GPS, the outcome is the geocentric position for an individual receiver or the relative positions between co-observing receivers. The mode of operation and techniques are well-documented in GPS related publications and books (e.g. *B. Hofman-Wellenhof et al.* (1994), *A. Leick* (1995), *V. S. Schwarze (1995), G. Strang* and *K. Borre* (1997), *A. Mathes* (1998), *R. Dach* (2000) and *M. S. Grewal et al.* (2001) among others).

The *Global Reference Frame* $\mathbb{F}^\bullet$ upon which the GPS observations are based is defined by the base vectors $\mathbb{F}_{1^\bullet}, \mathbb{F}_{2^\bullet}, \mathbb{F}_{3^\bullet}$ with the origin being the center of mass. The fundamental vector is defined by the base vector $\mathbb{F}_{3^\bullet}$ and coincides with the mean axis of rotation of the Earth and points to the direction of the CIO. $\mathbb{F}_{1^\bullet}$ is oriented such that the plane formed by $\mathbb{F}_{1^\bullet}$ and $\mathbb{F}_{3^\bullet}$ points to the direction of the Greenwich in England. $\mathbb{F}_{2^\bullet}$ completes the right handed system by being perpendicular to $\mathbb{F}_{1^\bullet}$ and $\mathbb{F}_{3^\bullet}$. The Geocentric Cartesian coordinates of a positional vector $\boldsymbol{X}$ is given by

$$\boldsymbol{X} = \mathbb{F}_{1^\bullet} X + \mathbb{F}_{2^\bullet} Y + \mathbb{F}_{3^\bullet} Z \tag{3-1}$$

where $\{X, Y, Z\}$ are the components of the vector $\boldsymbol{X}$ in the system $\mathbb{F}_{1^\bullet}, \mathbb{F}_{2^\bullet}, \mathbb{F}_{3^\bullet}\,|o$.

3-12 LPS positioning

3-121 Local Positioning Systems (LPS)

E. W. Grafarend (1991) defines the *Local Level System* as a three-dimensional reference frame *at the hand* of an experimenter in an engineering network. When one is positioning using an electronic theodolite or a total station, one first centers the instrument. When the instrument is properly centered and ready for operation, the vertical axis of the instrument at this moment coincides with the direction of the local gravity vector at that particular point, hence the term *direction of the local gravity vector*, and points in the direction opposite to that of the gravity vector at the theodolite station (i.e. to the zenith). The system can now be used to measure observations of the type horizontal directions T_i , vertical directions B_i and the spatial distances S_i with the triplet $\{S_i, T_i, B_i\}$ being measured in the *Local Level Reference Frame*. These systems as opposed to GPS are only used within the local networks and are referred to as the *Local Positioning Systems* (LPS). In Section (3-122) below, we consider the choice of the local datum for such systems in a three-dimensional network.

3-122 Local datum choice in an LPS 3-D network

In positioning with the *Local Positioning System* (LPS), one has two options, namely; using a known orientation or defining the orientation arbitrarily. When the first option is adopted, in which case the azimuths are known, one operates in the *Local Level Reference Frame* of type $\mathbb{E}^*$ discussed in (a) below. Should the second approach be opted for, then one operates in the *Local Level Reference Frame* of type $\mathbb{F}^*$ discussed in (b).

(a) Local Level Reference Frame of type $\mathbb{E}^*$
The origin of the $\mathbb{E}^*$ system is a point P whose coordinates are

$$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_P = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3-2}$$

defined by base vectors $\mathbb{E}_{1^*}$,$\mathbb{E}_{2^*}$,$\mathbb{E}_{3^*}$ of type south, east, vertical. $\mathbb{E}_{3^*}$ points to the direction opposite to that of the local gravity vector $\mathbf{\Gamma}$ at point P. $\mathbb{E}_{1^*}$ points south, while $\mathbb{E}_{2^*}$ completes the system by pointing east. The datum spherical coordinates of the direction point P_i in the *Local Level Reference Frame* $\mathbb{E}^*$ are

$$\begin{array}{lll} P \longrightarrow & X^* = Y^* = Z^* = & 0 \\ PP_i \longrightarrow & \begin{bmatrix} X^* \\ Y^* \\ Z^* \end{bmatrix}_{\mathbb{E}^*} = & S_i \begin{bmatrix} cos\, A_i\, cos B_i \\ sin\, A_i \,cos\, B_i \\ sin\, B_i \end{bmatrix} \end{array} \tag{3-3}$$

with azimuths A_i, vertical directions B_i, and spatial distances S_i.

(b) Local Level Reference Frame of type $\mathbb{F}^*$
Defined by the base vectors $\mathbb{F}_{1^*}$, $\mathbb{F}_{2^*}$, $\mathbb{F}_{3^*}$, with $\mathbb{F}_{1^*}$ within the local horizontal plane spanned by the base vectors $\mathbb{E}_{1^*}$ and $\mathbb{E}_{2^*}$ directed from P to P_i in vacuo. The angle between the base vectors $\mathbb{E}_{1^*}$ and $\mathbb{F}_{1^*}$ is the *"unknown orientation parameter"* Σ in the horizontal plane. $\mathbb{E}_{1^*}$, $\mathbb{E}_{2^*}$, $\mathbb{E}_{3^*}$ is related to $\mathbb{F}_{1^*}$,$\mathbb{F}_{2^*}$,$\mathbb{F}_{3^*}$ by a *"Karussel-Transformation"* as follows

$$\left[\begin{array}{l} \mathbb{F}_{1^*} = \mathbb{E}_{1^*} \cos\Sigma + \mathbb{E}_{2^*} \sin\Sigma \\ \mathbb{F}_{2^*} = -\mathbb{E}_{1^*} \sin\Sigma + \mathbb{E}_{2^*} \cos\Sigma \\ \mathbb{F}_{3^*} = \mathbb{E}_{3^*} \end{array} \right. \tag{3-4}$$

or

$$[\mathbb{F}_{1^*}, \mathbb{F}_{2^*}, \mathbb{F}_{3^*}] = [\mathbb{E}_{1^*}, \mathbb{E}_{2^*}, \mathbb{E}_{3^*}] \begin{bmatrix} \cos\Sigma & -\sin\Sigma & 0 \\ \sin\Sigma & \cos\Sigma & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3-5}$$

thus the *Local Level Reference Frame* of type $\mathbb{F}^*$ is related to the *Local Level Reference Frame* of type $\mathbb{E}^*$ by

$$[\mathbb{E}_{1^*}, \mathbb{E}_{2^*}, \mathbb{E}_{3^*}] = [\mathbb{F}_{1^*}, \mathbb{F}_{2^*}, \mathbb{F}_{3^*}]\ \mathbf{R}_3^{\mathrm{T}}(\Sigma). \tag{3-6}$$

The datum spherical coordinates of point P_i in the *Local Level Reference Frame* $\mathbb{F}^*$ are given as

$$\begin{array}{l} P \rightarrow X^* = Y^* = Z^* = 0 \\ PP_i \rightarrow \begin{bmatrix} X^* \\ Y^* \\ Z^* \end{bmatrix}_{\mathbb{F}^*} = S_i \begin{bmatrix} \cos T_i \cos B_i \\ \sin T_i \cos B_i \\ \sin B_i \end{bmatrix} \end{array} \tag{3-7}$$

where T_i and B_i are the horizontal and vertical directions respectively, while S_i are the spatial distances.

The local cartesian coordinates of a point whose positional vector is $\mathbf{x}$ in the $\mathbb{F}^*$ system is given by

$$\mathbf{x} = \mathbb{F}_{1^*}x + \mathbb{F}_{2^*}y + \mathbb{F}_{3^*}z \tag{3-8}$$

x, y, z) are the components of the vector $\mathbf{x}$ in the system $\{\mathbb{F}_{1^*}, \mathbb{F}_{2^*}, \mathbb{F}_{3^*} \, |P\}$. More insight on the topic can be found in *B. Richter* (1986, p.28) and *E. Grafarend* and *B. Richter* (1977).

3-13 Relationship between Global and Local Level Reference Frames

In positioning within the LPS *threedimensional positioning* framework , one is interested not only on the geometrical quantities $\{X, Y, Z\}_{GPS}$ for position of the new point but also on the physical quantities $\{\Lambda_\Gamma, \Phi_\Gamma\}$ being the direction of the local gravity vector $\mathbf{\Gamma}$ at the positioned point. When positioning with LPS within the *Global Reference Frame*, the direction $\{\Lambda_\Gamma, \Phi_\Gamma\}$ of the local gravity vector $\mathbf{\Gamma}$ is obtained by solving the *threedimensional orientation problem*. This is achieved by the transformation of coordinates from the *Local Level Reference Frame* to the *Global Terrestrial Reference Frame* (e.g. ITRF97). It is conventionally solved by a means of a 3 x 3 rotation matrix, which is represented by a triplet $\{\Lambda_\Gamma, \Phi_\Gamma, \Sigma_\Gamma\}$ of orientation parameters called the *astronomical longitude* Λ_Γ , *astronomical latitude* Φ_Γ , and the *"orientation unknown"* Σ_Γ in the horizontal plane. With respect to the local gravity vector $\mathbf{\Gamma}$ the triplets $\{\Lambda_\Gamma, \Phi_\Gamma, \Gamma = \|\mathbf{\Gamma}\|\}$ are its spherical coordinates, in particular $\{\Lambda_\Gamma, \Phi_\Gamma\}$ its direction parameters. The *three-dimensional orientation problem* therefore solves the problem of determining (i) the 3 x 3 rotation matrix and (ii) the triplet $\{\Lambda_\Gamma, \Phi_\Gamma, \Sigma_\Gamma\}$ of orientation parameters from GPS/LPS measurements. As soon as the *astronomical longitude* Λ_Γ and *astronomical latitude* Φ_Γ are determined - no astronomical observations are needed anymore - the vertical deflections with respect to a well-chosen reference frame, e.g. the ellipsoidal normal vector field can be obtained (e.g. *E. Grafarend* and *J. L. Awange* (2000)). The *three-dimensional orientation problem* is formulated by relating the $\mathbb{F}^*$ frame to the $\mathbb{F}^\bullet$ frame as follows

$$[\mathbb{F}_{1^*}, \mathbb{F}_{2^*}, \mathbb{F}_{3^*}] = [\mathbb{F}_{1^\bullet}, \mathbb{F}_{2^\bullet}, \mathbb{F}_{3^\bullet}]\, \mathbf{R}_E\, (\Lambda_\Gamma, \Phi_\Gamma, \Sigma_\Gamma) \tag{3-9}$$

where the Euler rotation matrix $\mathbf{R}_E$ is parameterised by

$$\boxed{\mathbf{R}_E\,(\Lambda_\Gamma, \Phi_\Gamma, \Sigma_\Gamma) := \mathbf{R}_3\,(\Sigma_\Gamma)\,\mathbf{R}_2\left(\tfrac{\pi}{2} - \Phi_\Gamma\right)\mathbf{R}_3(\Lambda_\Gamma)} \tag{3-10}$$

i.e. the *threedimensional orientation parameters*, the *astronomical longitude* Λ, *astronomical latitude* Φ, and the *"orientation unknown"* Σ in the horizontal plane. In-terms of Cartesian co-ordinates x, y, z of the station point, the target points x_i, y_i, z_i in the *Local Level Reference Frame* $\mathbb{F},^*$ and Cartesian co-ordinates X, Y, Z of the station point, the target points X_i, Y_i, Z_i in the *Local Level Reference Frame* $\mathbb{F}^\bullet$ one writes

$$\begin{bmatrix} x_i - x \\ y_i - y \\ z_i - z \end{bmatrix}_{\mathbb{F}^*} = \mathbf{R}_E(\Lambda_\Gamma, \Phi_\Gamma, \Sigma_\Gamma) \begin{bmatrix} X_i - X \\ Y_i - Y \\ Z_i - Z \end{bmatrix}_{\mathbb{F}^\bullet} \tag{3-11}$$

with

$$\begin{bmatrix} x_i - x \\ y_i - y \\ z_i - z \end{bmatrix}_{\mathbb{F}^*} = S_i \begin{bmatrix} cos\, T_i\, cos B_i \\ sin\, T_i\, cos\, B_i \\ sin\, B_i \end{bmatrix}, \forall_i \in \{1, 2, \ldots, n\} \tag{3-12}$$

The question then arises: How can the parameters relating the *Local Level Reference Frame* $\mathbb{F}^*$ to the *Global Reference Frame* $\mathbb{F}^\bullet$ *be obtained?*

1. The conventional approach has been to determine the direction (Φ, Λ) of the local gravity vector $\mathbf{\Gamma}$ at the origin of the network and the *"orientation unknown"* Σ in the horizontal plane by the use of steller astronomical observations.

2. *E. W. Grafarend, P. Lohse* and *B. Schaffrin* (1989) have proposed an approach based on the solution of *three-dimensional resection* problem. In the approach, directional measurements are performed to the neighbouring 3 points in the *Global Reference Frame* and used to derive the distances by solving the *Grunert equations*. From these derived distances, a closed form solution of the six unknowns $\{X, Y, Z, \Lambda_\Gamma, \Phi_\Gamma, \Sigma_\Gamma\}$ by means of the *Hamilton-Quaternion procedure* is achieved.

3. *J. L. Awange* (1999) and *E. Grafarend* and *J. L. Awange* (2000) solved the overdetermined form of the problem by using the *simple Procrustes algorithm* to determine the *three-dimensional orientation parameters* $\{\Lambda_\Gamma, \Phi_\Gamma, \Sigma_\Gamma\}$ and the *deflection of the vertical* for a point whose geometrical positional quantities $\{X, Y, Z\}_{GPS}$ are known.

4. The fourth approach would be first to determine the geometrical values $\{X, Y, Z\}_{GPS}$ of the unknown point through the *three-dimensional resection* approach presented in the present study and then back substitute the obtained geometrical values $\{X, Y, Z\}$ in (3-11) to obtain the Euler rotation matrix. The Euler rotation angles can then be deduced via an inverse map presented by *E. Grafarend* and *J. L. Awange* (2000, lemma 2.3, p.286).

Here the *three-dimensional orientation problem* is understood as the fundamental problem to determine the rotation matrix as well as its three parameters called {*astronomical longitude, astronomical latitude, horizontal orientation unknown* in the horizontal plane} which relates the *Local Level Reference Frame* to the *Global Terrestrial Reference Frame* (e.g. ITRF 97) from GPS position measurements both at target points as well as the station point and LPS direction measurements (horizontal directions, vertical directions by a theodolite) from the station point to the target points (at least three). If a reference direction parameterized in terms of “surface normal” {ellipsoidal longitude Λ, ellipsoidal latitude Φ} is subtracted from the local vertical parameterized in terms of {astronomical longitude Λ_Γ, astronomical latitude Φ_Γ}, namely $\{\Lambda_\Gamma - \Lambda,\ \Phi_\Gamma - \Phi\}$ we have access to the vertical deflections.

3-14 Observation Equations

In this section, we consider the equations with respect to a stationary theodolite for the purpose of point positioning. For the case where the theodolite moves from point to point, i.e. moving horizontal triad, we refer to *E. Grafarend* (1975, 1988 and 1991). In general a more elaborate literature on observations in three-dimensional positioning is given by *E. Grafarend* (1981). Stationed at the point $P \in \mathbb{E}^3$ and with the theodolite properly centred, we sight the target points, $P_i \in \mathbb{E}^3$ where i = 1,2,3,.....,n. There exist three types of measurements that will be taken from $P \in \mathbb{E}^3$ to $P_i \in \mathbb{E}^3$ in the LPS system (*Local Level Reference Frame* $\mathbb{F}^*$). These are; the horizontal directions T_i , vertical directions B_i and spatial distances S_i whose equations are given as

$$T_i = \arctan\left(\frac{\Delta y_i}{\Delta x_i}\right)_{\mathbb{F}^*} - \Sigma_\Gamma(P) \tag{3-13}$$

$$B_i = \arctan\left(\frac{\Delta z_i}{\sqrt{\Delta x_i^2 + \Delta y_i^2}}\right)_{\mathbb{F}^*} \tag{3-14}$$

$$S_i = \sqrt{\Delta x_i^2 + \Delta y_i^2 + \Delta z_i^2}\,|_{\mathbb{F}^*} \tag{3-15}$$

where $\Delta x_i = (x_i - x),\ \ \Delta y_i = (y_i - y),\ \ \Delta z_i = (z_i - z)$ denote the coordinate difference in the *Local Level Reference Frame* $\mathbb{F}^*$ and $\Sigma_\Gamma(P)$ is the *"unknown orientation"* in the horizontal plane after setting the zero reading of the theodolite in the direction $P \to P_i$. The relationship between the *Local Level Reference Frame* $\mathbb{F}^*$ and the *Global Reference Frame* $\mathbb{F}^\bullet$ is then given by

$$\begin{bmatrix} \Delta x_i \\ \Delta y_i \\ \Delta z_i \end{bmatrix}_{\mathbb{F}^*} = \mathbf{R}_E(\Lambda_\Gamma, \Phi_\Gamma, 0) \begin{bmatrix} \Delta X_i \\ \Delta Y_i \\ \Delta Z_i \end{bmatrix}_{\mathbb{F}^\bullet} \tag{3-16}$$

with

$$\mathbf{R}_E(\Lambda_\Gamma, \Phi_\Gamma, 0) = \begin{bmatrix} \sin\Phi_\Gamma\cos\Lambda_\Gamma & \sin\Phi_\Gamma\sin\Lambda_\Gamma & -\cos\Phi_\Gamma \\ -\sin\Lambda_\Gamma & \cos\Lambda_\Gamma & 0 \\ \cos\Phi_\Gamma\cos\Lambda_\Gamma & \cos\Phi_\Gamma\sin\Lambda_\Gamma & \sin\Phi_\Gamma \end{bmatrix} \tag{3-17}$$

we now express the observations (3-13), (3-14) and (3-15) in the *Global Reference Frame* as

$$T_i = \arctan\left\{\frac{-\sin\Lambda_\Gamma\Delta X_i + \cos\Lambda_\Gamma\Delta Y_i}{\sin\Phi_\Gamma\cos\Lambda_\Gamma\Delta X_i + \sin\Phi_\Gamma\sin\Lambda_\Gamma\Delta Y_i - \cos\Phi_\Gamma\Delta Z_i}\right\} - \Sigma_\Gamma(P) \tag{3-18}$$

$$B_i = \arctan\left\{\frac{\cos\Phi_\Gamma\cos\Lambda_\Gamma\Delta X_i + \cos\Phi_\Gamma\sin\Lambda_\Gamma\Delta Y_i + \sin\Phi_\Gamma\Delta Z_i}{\sqrt{(\sin\Phi_\Gamma\cos\Lambda_\Gamma\Delta X_i + \sin\Phi_\Gamma\sin\Lambda_\Gamma\Delta Y_i - \cos\Phi_\Gamma\Delta Z_i)^2 + D_2}}\right\} \tag{3-19}$$

with $D_2 = (\cos\Lambda_\Gamma\Delta Y_i - \sin\Lambda_\Gamma\Delta X_i)^2$, $\Delta X_i = (X_i - X)$, $\Delta Y_i = (Y_i - Y)$, $\Delta Z_i = (Z_i - Z)$ in the *Global Reference Frame* $\mathbb{F}^\bullet$ and $\{\Sigma_\Gamma(P), \Lambda_\Gamma(P), \Phi_\Gamma(P)\}$ are the three orientation unknowns at the unknown point P

3-2 Selected geodetic problems

In this section, we consider the application of the techniques discussed in Chapter 2 to solve;

(1) the three-dimension resection problem
(2) the minimum distance mapping problem
(3) the four-point GPS pseudo-ranging equations.

3-21 Threedimensional resection problem

Here we consider the closed form solution of the threedimensional resection problem using *Gröbner basis* and *Multipolynomial resultant* techniques. The closed form threedimensional resection procedure is carried out in three steps namely: distance-derivation step (solution of the *Grunert equations*), the position-derivation step and the orientation-derivation step. One may argue that the distance derivation step is irrelevant in light of the modern distance measuring equipments such as EDM. We should however not forget that in forest areas, one may be lucky to measure angles but the blockage of EDM signals by tree leaves and branches may hamper accurate distance measurements. In photogrammetry, the distances have to be derived from the image coordinates in order to obtain the perspective center coordinates and the orientation parameters, i.e. the elements of exterior orientation. These two examples validate the necessity of still having procedures for deriving the spatial distances despite the existence of Electromagnetic Distance Measuring (EDM) equipments.

In the closed form threedimensional resection problem, we are interested in determining the position and orientation of the point P connected by angular observations of type horizontal directions T_i and vertical directions B_i to three other known points P_1, P_2, P_3 as in Fig (3.1) below. From the angular measurements, the distances are derived in the distance derivation step by solving the *Grunert equations*. Once the distances have been established, the unknown position P is determined in the position derivation step. The closed form threedimensional resection problem is completed in the orientation derivation step by solving the orientation parameters that relate the *Global Reference System* $\mathbb{F}^\bullet$ to the *Local Level Reference System* of type $\mathbb{F}^*$.

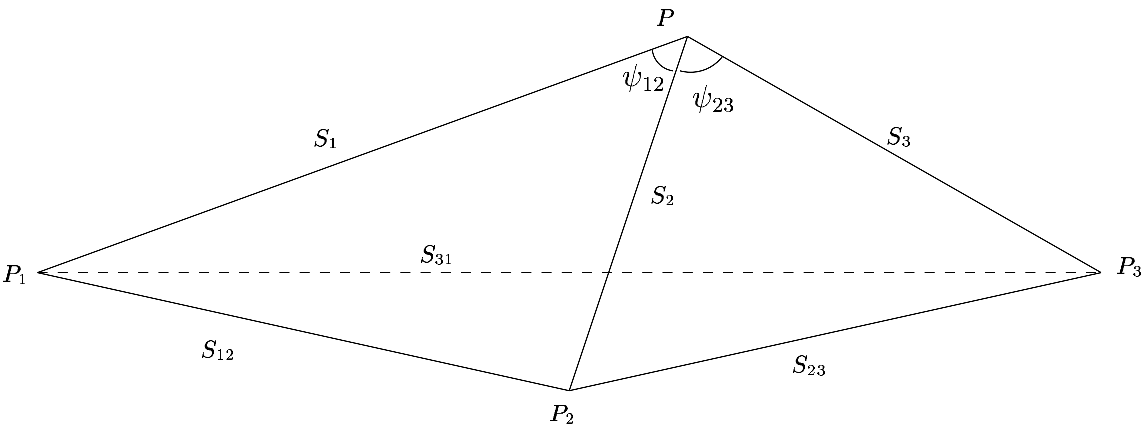


Figure 3.1: Tetrahedron for closed form 3d-resection

(a) The distance-derivation step (solution of the *Grunert equations*):
We begin in *Box* (3-1) by presenting the derivation of the distance equations (also known the *Grunert equations*) relating the known distances $S_{ij},\ i,j = 1,2,3\,|i \neq j$ between the known GPS stations in the *Global Reference Frame* $\mathbb{F},^\bullet$ the unknown distances $S_i,\ i = 1,2,3$ between the unknown station $P \in \mathbb{E}^3$ and the known stations $P_i \in \mathbb{E}^3$ and the spatial angles $\psi_{ij},\ i,j = 1,2,3\,|i \neq j$. The spatial angles $\psi_{ij},\ i,j = 1,2,3\,|i \neq j$ are obtained from observations of type horizontal directions T_i and vertical directions B_i as shown in *Box* (3-1) in the *Local Level Reference Frame* $\mathbb{F}.^*$ The relationship between observations in the *Local Level Reference Frame* and the *Global Reference Frame* $\mathbb{F}^\bullet$ in *Box* (3-1) is presented in (3-22) where $X_i, Y_i, Z_i \mid i \in \{1,2,3\}$ are GPS Cartesian coordinates of known points $P_i \in \mathbb{E}^3 \mid i \in \{1,2,3\}$, $S_i, T_i, B_i \mid i \in \{1,2,3\}$ are spherical coordinates of type: spatial distances,

horizontal and vertical directions respectively linking the old and new GPS points, while X, Y, Z are the required GPS coordinates of the unknown point $P \in \mathbb{E}^3$ (*Figure 3.1*). $\mathbf{R}$ is the rotation matrix containing the three-dimensional orientation parameters (see (3-11) in Section (3-13)). Multiplying (3-22) by (3-24) leading to (3-25) derive the relationship between the spherical coordinates of type horizontal directions T_i, vertical directions B_i and the space angles ψ_{ij}. After manipulations of (3-26),(3-27) and (3-28), the spatial angle ψ_{ij} can be written in terms of the spherical coordinates $\{T_i, B_i\}$, $\{T_j, B_j\}$ of points P_i and P_j with respect to a theodolite orthogonal Euclidean frame $\mathbb{F}^*$ as in equation (3-29). The *Grunert equations* for the three unknown distances S_1, S_2, S_3 can now be written in terms of the known distances S_{12}, S_{23}, S_{31} and the space angles $\psi_{12}, \psi_{23}, \psi_{31}$ illustrated in (*Figure 3.1*) as in (3-31). In photogrammetry, the relationship between the space angles and the measured image coordinates with respect to orthogonal Euclidean frame centred at the perspective centre is given by *E. Grafarend* and *J. Shan* (1997a, equation (1.1), p.218) as

$$\cos\psi_{ij} = \frac{x_i x_j + y_i y_j + f^2}{\sqrt{x_i^2 + y_i^2 + f^2}\sqrt{x_j^2 + y_j^2 + f^2}} \tag{3-20}$$

Box 3-1 (*derivation of the Grunert equations*):

"GPS and LPS coordinate systems"

$$\begin{bmatrix} x_i - x \\ y_i - z \\ z_i - z \end{bmatrix}_{\mathbb{F}^*} = \boldsymbol{R} \begin{bmatrix} X_i - X \\ Y_i - Y \\ Z_i - Z \end{bmatrix}_{\mathbb{F}^\bullet} \tag{3-21}$$

$$S_i \begin{bmatrix} \cos T_i \cos B_i \\ \sin T_i \cos B_i \\ \sin B_i \end{bmatrix} = \boldsymbol{R} \begin{bmatrix} X_i - X \\ Y_i - Y \\ Z_i - Z \end{bmatrix} \tag{3-22}$$

$$\boldsymbol{R} \in SO(3) := \{\boldsymbol{X} \in \mathbb{R}^{3\times 3} \mid \boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{I}_3, |\boldsymbol{X}| = +1\} \tag{3-23}$$

$$(-2)\,[\cos T_j \cos B_j, \sin T_j \cos B_j, \sin B_j]\, S_j \tag{3-24}$$

$$\left[\begin{array}{c} (-2)[cosT_j cosB_j, sinT_j cosB_j, sinB_j] S_i S_j \begin{bmatrix} cosT_i cosB_i \\ sinT_i cosB_i \\ sinB_i \end{bmatrix} = \\ \\ (-2)[(X_j - X), (Y_j - Y), (Z_j - Z)] \begin{bmatrix} X_i - X \\ Y_i - Y \\ Z_i - Z \end{bmatrix} \end{array} \right. \tag{3-25}$$

Box 3-1 (*derivation of the Grunert equations* continued):

$$\left.\begin{array}{r}(X_j-X)(X_i-X)=X_jX_i-X_jX-X_iX+X^2\\(X_i-X_j)(X_i-X_j)=X_i^2-2X_iX_j+X_j^2\\(X_i-X)(X_i-X)=X_i^2-2X_iX+X^2\\(X_j-X)(X_j-X)=X_j^2-2X_jX+X^2\end{array}\right]\Rightarrow \quad (3\text{-}26)$$
$$\Rightarrow (X_i-X_j)^2-(X_i-X)^2-(X_j-X)^2=-2(X_j-X)(X_i-X)$$

$$\left[\begin{array}{c}(-2)[cosT_jcosB_j, sinT_jcosB_j, sinB_j]S_iS_j\begin{bmatrix}cosT_icosB_i\\sinT_icosB_i\\sinB_i\end{bmatrix}=\\ \left\{\begin{array}{c}(X_i-X_j)^2+(Y_i-Y_j)^2+(Z_i-Z_j)^2-\\-(X_i-X)^2-(Y_i-Y)^2-(Z_i-Z)^2-\\-(X_j-X)^2-(Y_j-Y)^2-(Z_j-Z)^2\end{array}\right\}\end{array}\right. \quad (3\text{-}27)$$

$$-2\left\{\sin B_j\sin B_i+\cos B_j\cos B_i\cos(T_j-T_i)\right\}S_iS_j=$$
$$=\left\{\begin{array}{c}(X_i-X_j)^2+(Y_i-Y_j)^2+(Z_i-Z_j)^2-\\-(X_i-X)^2-(Y_i-Y)^2-(Z_i-Z)^2-\\-(X_j-X)^2-(Y_j-Y)^2-(Z_j-Z)^2\end{array}\right\} \quad (3\text{-}28)$$

$$\cos\psi_{ij}=\cos B_i\cos B_j\cos(T_j-T_i)+\sin B_i\sin B_j \quad (3\text{-}29)$$

$$\left[\begin{array}{l}-2\cos\psi_{ij}S_iS_j=S_{ij}^2-S_i^2-S_j^2\\S_{ij}^2=S_i^2+S_j^2-2S_iS_j\cos\psi_{ij}\end{array}\right. \quad (3\text{-}30)$$

$$\left[\begin{array}{l}S_{12}^2=S_1^2+S_2^2-2S_1S_2\cos\psi_{12}\\S_{23}^2=S_2^2+S_3^2-2S_2S_3\cos\psi_{23}\\S_{31}^2=S_3^2+S_1^2-2S_3S_1\cos\psi_{31}\end{array}\right. \quad (3\text{-}31)$$

Several procedures have been put forward to solve the *Grunert equations* for the unknown distances S_1, S_2, S_3 in (3-31) above. These include *J. A. Grunert* (1841), *S. Finsterwalder* and *W. Scheufele* (1937), *E. L. Merritt* (1949), *M. A. Fischler* and *R. C. Bolles* (1981), *S. Linnainmaa et al.* (1988), *E. Grafarend et al.* (1989), *P. Lohse* (1990) and *E. Grafarend* and *J. Shan* (1997a, 1997b). *F. J. Müller* (1925) and *R. M. Haralick et al.* (1991, 1994) present an extensive review of these procedures. The present study considers the use of algebraic computational tools (*Gröbner bases* and *Multipolynomial resultants*) discussed in Chapter 2 to solve the *Grunert equations* for the unknown distances S_1, S_2, S_3 in (3-31).

In order to understand the usefullness of *Gröbner basis* in the solution of the three-dimensional resection problem, we present first the hand computation of the *Grunert distances* for a *regular tetrahedron* using *Gröbner basis* before considering the general case of the *Grunert equations* for distances in the general threedimensional resection problem whose *Gröbner basis* is computed using *Mathematica software*.

(i) Gröbner basis solution of the Grunert equations for a regular tetrahedron

We begin by expressing the *Grunert distances equations* (3-31) -whose geometrical behaviour has been studied by *E. Grafarend, P. Lohse* and *B. Schaffrin* (1989)- in the form

$$\left[\begin{array}{l}a_0=x_1^2+x_2^2-2a_{12}x_1x_2\\b_0=x_2^2+x_3^2-2b_{23}x_2x_3\\c_0=x_3^2+x_1^2-2c_{31}x_3x_1\end{array}\right. \quad (3\text{-}32)$$

where

$$\left[\begin{array}{l}x_1^2-2a_{12}x_1x_2+x_2^2-a_0=0\\x_2^2-2b_{23}x_2x_3+x_3^2-b_0=0\\x_1^2-2c_{31}x_1x_3+x_3^2-c_0=0\end{array}\right. \quad (3\text{-}33)$$

E. Grafarend, P. Lohse and *B. Schaffrin* (1989) demonstrate that for each of the quadratic equation in (3-32), there exists an elliptical cylinder in the planes $\{x_1, x_2\}$, $\{x_2, x_3\}$ and $\{x_3, x_1\}$ for the first, second and third equations respectively . These cylinders are constrained to their first quadrant since the distances are positive thus $\{x_1 \in \mathbb{R}^+\}$, $\{x_2 \in \mathbb{R}^+\}$ and $\{x_3 \in \mathbb{R}^+\}$. In *Box* (3-2) below, we apply the *Gröbner basis* technique to solve for the distances $x_i,\ i = 1,2,3$ between the unknown station $P \in \mathbb{E}^3$ and the known stations $P_i \in \mathbb{E}^3$. For a *regular tetrahedron*, the distances $x_1 = x_2 = x_3$ joining the unknown point $P \in \mathbb{E}^3$ to the known points $P_i \in \mathbb{E}^3$ are equal to the distances $S_{12} = S_{23} = S_{31}$ between the known stations. Let us consider these distances to be equal to $+\sqrt{d}$. The spatial angles are also equal (i.e. $\psi_{12} = \psi_{23} = \psi_{31} = 60°$). The task now is to compute by hand the *Gröbner basis* of (3-32) and use them to find the *Grunert distances* for the regular tetrahedron (i.e. show that the desired solutions for $\{x_1, x_2, x_3\} \in \mathbb{R}^+$ are $x_1 = x_2 = x_3 = +\sqrt{d}$.

Box 3-2 (*Hand computation of Gröbner basis of the Grunert equations for a regular tetrahedron*):

Upon lexicographic ordering $x_1 > x_2 > x_3$ and subtracting the left hand side of (3-32) from the right hand side we have

$$\begin{bmatrix} f_1 := x_1^2 - 2a_{12}x_1x_2 + x_2^2 - a_0 = 0 \\ f_2 := x_2^2 - 2b_{23}x_2x_3 + x_3^2 - b_0 = 0 \\ f_3 := x_1^2 - 2c_{31}x_1x_3 + x_3^2 - c_0 = 0 \end{bmatrix} \tag{3-34}$$

as polynomials in $\mathbb{R}[x_1, x_2, x_3]$. These equations form (following *Definition A-6* in *Appendix A.1*) the *Ideal* I as

$$I = \langle f_1, f_2, f_3 \rangle \subset [x_1, x_2, x_3] \tag{3-35}$$

For a regular tetrahedron, where $\psi_{ij} = 60°$, $2\cos(60°) = 1$ and $a_0 = b_0 = c_0 = d$. (3-34) is then written as

$$\begin{bmatrix} x_1^2 - x_1x_2 + x_2^2 - d = 0 \\ x_2^2 - x_2x_3 + x_3^2 - d = 0 \\ x_1^2 - x_1x_3 + x_3^2 - d = 0 \end{bmatrix} \tag{3-36}$$

giving rise from (3-35) to the Ideal I as

$$\begin{matrix} I = \langle x_1^2 - x_1x_2 + x_2^2 - d, x_2^2 - x_2x_3 + x_3^2 - d, x_1^2 - x_1x_3 + x_3^2 - d \rangle \\ \subset \mathbb{R}[x_1, x_2, x_3] \end{matrix} \tag{3-37}$$

whose generators G are written as

$$\begin{bmatrix} g_1 = x_1^2 - x_1x_2 + x_2^2 - d \\ g_2 = x_1^2 - x_1x_3 + x_3^2 - d \\ g_3 = x_2^2 - x_2x_3 + x_3^2 - d. \end{bmatrix} \tag{3-38}$$

Desired now are the *Gröbner basis* for the for the generators (3-38) of the *Ideal* I in (3-35). We then proceed to compute the S pair polynomials $(g_1, g_2), (g_1, g_3), (g_2, g_3)$ from the generators (3-38) of (3-35). From *B. Buchberger's* third criterion explained in Chapter 2, we notice that $\mathrm{LM}(g_2) = x_1^2$ divides the $\mathrm{LCM}(g_1, g_3) = x_1^2x_2^2$. It suffices therefore to suppress the consideration of (g_1, g_3) and instead consider only $(g_1, g_2), (g_2, g_3)$. $S(g_1, g_2)$ gives

$$S(g_1, g_2) = -x_1x_2 + x_1x_3 + x_2^2 - x_3^2 \tag{3-39}$$

which is reduced with respect to G by subtracting g_3 to obtain

$$-x_1x_2 + x_1x_3 - 2x_3^2 + x_2x_3 + d, \tag{3-40}$$

which does not reduce to zero and is added in the original list G of the generating set of the *Ideal* I as g_4 . The $S - polynomial$ pairs to be considered next are $S(g_2, g_3), S(g_2, g_4), S(g_3, g_4)$ from the new generating set $G = \{g_2, g_3, g_4\}$. Since $LM(g_2)$ and $LM(g_3)$ are relatively prime, $S(g_2, g_3)$ reduces to zero modulo G ($S(g_2, g_3) \rightarrow_G 0$). The S pair polynomials remaining for consideration are (g_2, g_4) and (g_3, g_4). $S(g_2, g_4)$ gives

$$S(g_2, g_4) = x_1^2x_3 + x_1d - 2x_1x_3^2 + x_2x_3^2 - x_2d \tag{3-41}$$

which is reduced with respect to G by subtracting x_3g_2 to give

$$x_1d - x_1x_3^2 + x_2x_3^2 - x_2d - x_3^3 + x_3d \tag{3-42}$$

Box 3-2 (*Hand computation of Gröbner basis of the Grunert equations continued*):

which does not reduce to zero and is added to the list G of the generating set of the *Ideal* I as g_5. The $S-polynomial$ sets to be considered next is $S(g_3, g_4)$ from the new generating set $G = \{g_2, g_3, g_4, g_5\}$. $S(g_3, g_4)$ gives

$$S(g_3,g_4) = -x_1x_3^2 + x_1d + 2x_2x_3^2 - x_2^2x_3 - x_2d \tag{3-43}$$

which is reduced with respect to G by subtracting g_5 and adding x_3g_3 to give

$$2x_3^3 - 2x_3d \tag{3-44}$$

which is a *univariate polynomial* and completes the set of the *reduced Gröbner basis* of the set G summarised as follows

$$G := \begin{bmatrix} g_2 = x_1^2 - x_1x_3 + x_3^2 - d \\ g_3 = x_2^2 - x_2x_3 + x_3^2 - d \\ g_4 = -x_1x_2 + x_1x_3 - 2x_3^2 + x_2x_3 + d \\ g_5 = x_1d - x_1x_3^2 + x_2x_3^2 - x_2d - x_3^3 + x_3d \\ g_6 = 2x_3^3 - 2x_3d \end{bmatrix} \tag{3-45}$$

From the computed *reduced Gröbner basis* in (3-45) we note that the element $g_6 = 2x_3^3 - 2x_3d$ is a *univariate polynomial* in x_3 and readily gives the values of $x_3 = \left\{0, \pm\sqrt{d}\right\}$. We then proceed to derive the solution to the *Grunert distance equations* (3-34) as follows: Since $S_3 = x_3 \in \mathbb{R}^+$, the value of $S_3 = +\sqrt{d}$. This is substituted back in $g_3 = x_2^2 - x_2x_3 + x_3^2 - d$ and $g_2 = x_1^2 - x_1x_3 + x_3^2 - d$ to give $x_2 = \left\{0, +\sqrt{d}\right\}$ and $x_1 = \left\{0, +\sqrt{d}\right\}$ respectively. This complets the solution of *Grunert equations* (3-34) for the unknown distances $x_1 = x_2 = x_3 = +\sqrt{d}$ as we had initially assumed.

(ii) Gröbner basis solution of the Grunert equations for the general three-dimension resection problem

We next present the application of *Gröbner basis* technique to the solution of the *Grunert equations* (3-31) expressed as follows:

$$\begin{bmatrix} g_1 := x_1^2 + x_2^2 + 2a_{12}x_1x_2 + a_0 = 0 \\ g_2 := x_2^2 + x_3^2 + 2b_{23}x_2x_3 + b_0 = 0 \\ g_3 := x_3^2 + x_1^2 + 2c_{31}x_3x_1 + c_0 = 0, \end{bmatrix} \tag{3-46}$$

where

$$\begin{cases} S_1 = x_1 \in \mathbb{R}^+, S_2 = x_2 \in \mathbb{R}^+, S_3 = x_3 \in \mathbb{R}^+, \\ -2\cos\psi_{12} = a_{12}, -2\cos\psi_{23} = b_{23}, -2\cos\psi_{31} = c_{31}, \\ -S_{12}^2 = a_0, -S_{23}^2 = b_0, -S_{31}^2 = c_0. \end{cases} \tag{3-47}$$

We then have the *Ideal* formed from (3-46) as

$$Ideal\ I =< x_1^2 + x_2^2 + 2a_{12}x_1x_2 + a_0, x_2^2 + x_3^2 + 2b_{23}x_2x_3 + b_0, x_3^2 + x_1^2 + 2c_{31}x_3x_1 + c_0 > \tag{3-48}$$

whose *Gröbner basis* are computed in *Mathematica 3.0* after *Lexicographic ordering* $(x_1 > x_2 > x_3)$ by the command *GroebnerBasis* as

$$GroebnerBasis[\{g_1, g_2, g_3\}, \{x_1, x_2, x_3\}]. \tag{3-49}$$

The execution of the *Mathematica 3.0* command above gives the computed *Gröbner basis* of the *Ideal* (3-48) as expressed in *Boxes* (3-3a) and (3-3b) below.

<u>Box</u> 3-3a (*Computed Gröbner basis for the Grunert distance equations-univariate term*):

$g_1 = (16 - 8a_{12}^2 + a_{12}^4 - 8b_{23}^2 + 2a_{12}^2 b_{23}^2 + b_{23}^4 - 8a_{12}b_{23}c_{31} + 2a_{12}^3 b_{23}c_{31} + 2a_{12}b_{23}^3 c_{31} - 8c_{31}^2 + 2a_{12}^2 c_{31}^2 + 2b_{23}^2 c_{31}^2 + a_{12}^2 b_{23}^2 c_{31}^2 + 2a_{12}b_{23}c_{31}^3 + c_{31}^4)x_3^8 +$

$+(- 32a_0 + 8a_0 a_{12}^2 + 32b_0 - 16a_{12}^2 b_0 + 2a_{12}^4 b_0 + 16a_0 b_{23}^2 - 2a_0 a_{12}^2 b_{23}^2 - 8b_0 b_{23}^2 + 2a_{12}^2 b_0 b_{23}^2 - 2a_0 b_{23}^4 + 32c_0 - 16a_{12}^2 c_0 + 2a_{12}^4 c_0 - 16b_{23}^2 c_0 + 4a_{12}^2 b_{23}^2 c_0 + 2b_{23}^4 c_0 + 4a_0 a_{12} b_{23} c31 + a_0 a_{12}^3 b_{23} c_{31} - 12a_{12} b_0 b_{23} c_{31} + 3a_{12}^3 b_0 b_{23} c_{31} - a_0 a_{12} b_{23}^3 c_{31} + a_{12} b_0 b_{23}^3 c_{31} - 12a_{12} b_{23} c_0 c_{31} + 3a_{12}^3 b_{23} c_0 c_{31} + 3a_{12} b_{23}^3 c_0 c_{31} + 16a_0 c_{31}^2 - 2a_0 a_{12}^2 c_{31}^2 - 16b_0 c_{31}^2 + 4a_{12}^2 b_0 c_{31}^2 - 8a_0 b_{23}^2 c_{31}^2 + 2a_0 a_{12}^2 b_{23}^2 c_{31}^2 + 2b_0 b_{23}^2 c_{31}^2 + a_{12}^2 b_0 b_{23}^2 c_{31}^2 + a_0 b_{23}^4 c_{31}^2 - 8c_0 c_{31}^2 + 2a_{12}^2 c_0 c_{31}^2 + 2b_{23}^2 c_0 c_{31}^2 + a_{12}^2 b_{23}^2 c_0 c_{31}^2 - a_0 a_{12} b_{23} c_{31}^3 + 3a_{12} b_0 b_{23} c_{31}^3 + a_0 a_{12} b_{23}^3 c_{31}^3 + a_{12} b_{23} c_0 c_{31}^3 - 2a_0 c_{31}^4 + 2b_0 c_{31}^4 + a_0 b_{23}^2 c_{31}^4)x_3^6 +$

$+(24a_0^2 - 2a_0^2 a_{12}^2 - 48a_0 b_0 + 12a_0 a_{12}^2 b_0 + 24b_0^2 - 10a_{12}^2 b_0^2 + a_{12}^4 b_0^2 - 10a_0^2 b_{23}^2 + a_0^2 a_{12}^2 b_{23}^2 + 12a_0 b_0 b_{23}^2 - 2b_0^2 b_{23}^2 + a_{12}^2 b_0^2 b_{23}^2 + a_0^2 b_{23}^4 - 48a_0 c_0 + 12a_0 a_{12}^2 c_0 + 48b_0 c_0 - 28a_{12}^2 b_0 c_0 + 4a_{12}^4 b_0 c_0 + 20a_0 b_{23}^2 c_0 - 4a_0 a_{12}^2 b_{23}^2 c_0 - 12b_0 b_{23}^2 c_0 + 2a_{12}^2 b_0 b_{23}^2 c_0 - 2a_0 b_{23}^4 c_0 + 24c_0^2 - 10a_{12}^2 c_0^2 + a_{12}^4 c_0^2 - 10b_{23}^2 c_0^2 + 3a_{12}^2 b_{23}^2 c_0^2 + b_{23}^4 c_0^2 - 2a_0^2 a_{12} b_{23} c_{31} + 4a_0 a_{12} b_0 b_{23} c_{31} + a_0 a_{12}^3 b_0 b_{23} c_{31} - 2a_{12} b_0^2 b_{23} c_{31} + a_{12}^3 b_0^2 b_{23} c_{31} + a_0^2 a_{12} b_{23}^3 c_{31} + a_0 a_{12} b_0 b_{23}^3 c_{31} + 4a_0 a_{12} b_{23} c_0 c_{31} + a_0 a_{12}^3 b_{23} c_0 c_{31} - 20a_{12} b_0 b_{23} c_0 c_{31} + 4a_{12}^3 b_0 b_{23} c_0 c_{31} - 2a_0 a_{12} b_{23}^3 c_0 c_{31} + a_{12} b_0 b_{23}^3 c_0 c_{31} - 2a_{12} b_{23} c_0^2 c_{31} + a_{12}^3 b_{23} c_0^2 c_{31} + a_{12} b_{23}^3 c_0^2 c_{31} - 10a_0^2 c_{31}^2 + a_0^2 a_{12}^2 c_{31}^2 + 20a_0 b_0 c_{31}^2 - 4a_0 a_{12}^2 b_0 c_{31}^2 - 10b_0^2 c_{31}^2 + 3a_{12}^2 b_0^2 c_{31}^2 + 3a_0^2 b_{23}^2 c_{31}^2 - 4a_0 b_0 b_{23}^2 c_{31}^2 + a_0 a_{12}^2 b_0 b_{23}^2 c_{31}^2 + b_0^2 b_{23}^2 c_{31}^2 + 12a_0 c_0 c_{31}^2 - 12b_0 c_0 c_{31}^2 + 2a_{12}^2 b_0 c_0 c_{31}^2 - 4a_0 b_{23}^2 c_0 c_{31}^2 + a_0 a_{12}^2 b_{23}^2 c_0 c_{31}^2 + a_{12}^2 b_0 b_{23}^2 c_0 c_{31}^2 - 2c_0^2 c_{31}^2 + a_{12}^2 c_0^2 c_{31}^2 + b_{23}^2 c_0^2 c_{31}^2 + a_0^2 a_{12} b_{23} c_{31}^3 - 2a_0 a_{12} b_0 b_{23} c_{31}^3 + a_{12} b_0^2 b_{23} c_{31}^3 + a_0 a_{12} b_{23} c_0 c_{31}^3 + a_{12} b_0 b_{23} c_0 c_{31}^3 + a_0^2 c_{31}^4 - 2a_0 b_0 c_{31}^4 + b_0^2 c_{31}^4)x_3^4 +$

$+(-8a_0^3 + 24a_0^2 b_0 - 2a_0^2 a_{12}^2 b_0 - 24a_0 b_0^2 + 4a_0 a_{12}^2 b_0^2 + 8b_0^3 - 2a_{12}^2 b_0^3 + 2a_0^3 b_{23}^2 - 4a_0^2 b_0 b_{23}^2 + 2a_0 b_0^2 b_{23}^2 + 24a_0^2 c_0 - 2a_0^2 a_{12}^2 c_0 - 48a_0 b_0 c_0 + 16a_0 a_{12}^2 b_0 c_0 + 24b_0^2 c_0 - 14a_{12}^2 b_0^2 c_0 + 2a_{12}^4 b_0^2 c_0 - 6a_0^2 b_{23}^2 c_0 + a_0^2 a_{12}^2 b_{23}^2 c_0 + 8a_0 b_0 b_{23}^2 c_0 - 2b_0^2 b_{23}^2 c_0 + a_{12}^2 b_0^2 b_{23}^2 c_0 - 24a_0 c_0^2 + 4a_0 a_{12}^2 c_0^2 + 24b_0 c_0^2 - 14a_{12}^2 b_0 c_0^2 + 2a_{12}^4 b_0 c_0^2 + 6a_0 b_{23}^2 c_0^2 - 2a_0 a_{12}^2 b_{23}^2 c_0^2 - 4b_0 b_{23}^2 c_0^2 + 8c_0^3 - 2a_{12}^2 c_0^3 - 2b_{23}^2 c_0^3 + a_{12}^2 b_{23}^2 c_0^3 + a_0^3 a_{12} b_{23} c_{31} - a_0^2 a_{12} b_0 b_{23} c_{31} - a_0 a_{12} b_0^2 b_{23} c_{31} + a_{12} b_0^3 b_{23} c_{31} - a_0^2 a_{12} b_{23} c_0 c_{31} + 6a_0 a_{12} b_0 b_{23} c_0 c_{31} + a_0 a_{12}^3 b_0 b_{23} c_0 c_{31} - 5a_{12} b_0^2 b_{23} c_0 c_{31} + a_{12}^3 b_0^2 b_{23} c_0 c_{31} - a_0 a_{12} b_{23} c_0^2 c_{31} - 5a_{12} b_0 b_{23} c_0^2 c_{31} + a_{12}^3 b_0 b_{23} c_0^2 c_{31} + a_{12} b_{23} c_0^3 c_{31} + 2a_0^3 c_{31}^2 - 6a_0^2 b_0 c_{31}^2 + a_0^2 a_{12}^2 b_0 c_{31}^2 + 6a_0 b_0^2 c_{31}^2 - 2a_0 a_{12}^2 b_0^2 c_{31}^2 - 2b_0^3 c_{31}^2 + a_{12}^2 b_0^3 c_{31}^2 - 4a_0^2 c_0 c_{31}^2 + 8a_0 b_0 c_0 c_{31}^2 - 4b_0^2 c_0 c_{31}^2 + 2a_0 c_0^2 c_{31}^2 - 2b_0 c_0^2 c_{31}^2 + a_{12}^2 b_0 c_0^2 c_{31}^2)x_3^2$

$+ (a_0^4 - 4a_0^3 b_0 + 6a_0^2 b_0^2 - 4a_0 b_0^3 + b_0^4 - 4a_0^3 c_0 + 12a_0^2 b_0 c_0 - 2a_0^2 a_{12}^2 b_0 c_0 - 12a_0 b_0^2 c_0 + 4a_0 a_{12}^2 b_0^2 c_0 + 4b_0^3 c_0 - 2a_{12}^2 b_0^3 c_0 + 6a_0^2 c_0^2 - 12a_0 b_0 c_0^2 + 4a_0 a_{12}^2 b_0 c_0^2 + 6b_0^2 c_0^2 - 4a_{12}^2 b_0^2 c_0^2 + a_{12}^4 b_0^2 c_0^2 - 4a_0 c_0^3 + 4b_0 c_0^3 - 2a_{12}^2 b_0 c_0^3 + c_0^4)$

<u>Box</u> 3-3b (*Computed Gröbner basis for the Grunert distances equations-multivariate terms*):

$$g_2 = \begin{cases} (a_0 a_{12} b_{23} + a_{12} b_0 b_{23} - a_{12} b_{23} c_0 + 2a_0 c_{31} - 2b_0 c_{31} - 2c_0 c_{31})x_1 + \\ (2b_0 b_{23} - 2a_0 b_{23} + 2b_{23} c_0 - a_{12}^2 b_{23} c_0 - 2a_{12} c_0 c_{31})x_2 + \\ (b_{23}^2 c_0 - a_0 b_{23}^2 - b_0 b_{23}^2 + a_0 a_{12} b_{23} c_{31} - a_{12} b_0 b_{23} c_{31} + 2a_0 c_{31}^2 \\ -2b_0 c_{31}^2)x_3 + (b_{23}^2 c_{31} - 4c_{31})x_1 x_3^2 + (4b_{23} - a_{12}^2 b_{23} - b_{23}^3 \\ -2a_{12} c_{31} - a_{12} b_{23}^2 c_{31} - 2b_{23} c_{31}^2)x_2 x_3^2 - (a_{12} b_{23} c_{31} + 2c_{31}^2)x_3^3 \end{cases}$$

$$g_3 = \begin{cases} (a_{12} b_0 - a_0 a_{12} + a_{12} c_0)x_1 + a_{12}^2 c_0 x_2 + (b_0 b_{23} - a_0 b_{23} + b_{23} c_0 - \\ a_0 a_{12} c_{31} + a_{12} b_0 c_{31})x_3 + (2a_{12} + b_{23} c_{31})x_1 x_3^2 + (a_{12}^2 + \\ b_{23}^2 + a_{12} b_{23} c_{31})x_2 x_3^2 + (2b_{23} + a_{12} c_{31})x_3^3 \end{cases}$$

$$g_4 = \begin{cases} (2a_0 b_0 - a_0^2 + b_0^2 + 2a_0 c_0 - 2b_0 c_0 - c_0^2)x_1 + (a_0 a_{12} c_0 - \\ a_{12} b_0 c_0 - a_{12} c_0^2)x_2 + (2a_0 b_0 c_{31} - a_{12} b_0 b_{23} c_0 - a_0^2 c_{31} - b_0^2 c_{31} + \\ a_0 c_0 c_{31} - b_0 c_0 c_{31})x_3 + (4a_0 - 4b_0 - a_0 b_{23}^2 - 4c_0 + b_{23}^2 c_0)x_1 x_3^2 + \\ (a_0 a_{12} - a_{12} b_0 - 3a_{12} c_0 - b_{23} c_0 c_{31})x_2 x_3^2 + (-a_{12} b_0 b_{23} - \\ a_{12} b_{23} c_0 + 3a_0 c_{31} - 3b_0 c_{31} - a_0 b_{23}^2 c_{31} - c_0 c_{31})x_3^3 + \\ (-4 + b_{23}^2)x_1 x_3^4 + (-2a_{12} - b_{23} c_{31})x_2 x_3^4 + (-a_{12} b_{23} - 2c_{31})x_3^5 \end{cases}$$

$$g_5 = a_0 - b_0 - c_0 + a_{12} x_1 x_2 - c_{31} x_1 x_3 - b_{23} x_2 x_3 - 2x_3^2$$

$$g_6 = \begin{cases} a_{12} b_0 c_0 c_{31} + (a_0 c_{31} - b_0 c_{31} - c_0 c_{31})x_1 x_2 + (a_0 a_{12} + a_{12} b_0 - a_{12} c_0 \\ +a_0 b_{23} c_{31} - b_{23} c_0 c_{31})x_1 x_3 + (-2a_0 + 2b_0 + 2c_0 - a_{12}^2 c_0 + a_0 c_{31}^2 \\ -b_0 c_{31}^2)x_2 x_3 + (-a_0 b_{23} - b_0 b_{23} + b_{23} c_0 + a_0 a_{12} c_{31} + a_{12} c_0 c_{31} \\ +a_0 b_{23} c_{31}^2)x_3^2 + (4 - a_{12}^2 - b_{23}^2 - a_{12} b_{23} c_{31} - c_{31}^2)x_2 x_3^3 \end{cases}$$

<u>Box</u> <u>3-3b</u> (*Computed Gröbner basis for the Grunert distances equations-multivariate terms* continued):

$$g_7 = \begin{cases} a_{12}b_0b_{23}c_0 + (a_0b_{23} - b_0b_{23} - b_{23}c_0)x_1x_2 + (-2a_0 + 2b_0 + a_0b_{23}^2 \\ +2c_0 - b_{23}^2c_0)x_1x_3 + (2a_{12}c_0 + a_0b_{23}c_{31} - b_0b_{23}c_{31})x_2x_3 + \\ (a_{12}b_0b_{23} + a_{12}b_{23}c_0 - 2a_0c_{31} + 2b_0c_{31} + a_0b_{23}^2c_{31})x_3^2 + \\ (4 - b_{23}^2)x_1x_3^3 + (2a_{12} + b_{23}c_{31})x_2x_3^3 + (a_{12}b_{23} + 2c_{31})x_3^4 \end{cases}$$

$$g_8 = \begin{cases} (a_0a_{12} + a_{12}b_0 - a_{12}c_0)x_1 + (-2a_0 + 2b_0 + 2c_0 - a_{12}^2c_0)x_2 + \\ (-a_0b_{23} - b_0b_{23} + b_{23}c_0 + a_0a_{12}c_{31} - a_{12}b_0c_{31})x_3 + 2c_{31}x_1x_2x_3 \\ +b_{23}c_{31}x_1x_3^2 + (4 - a_{12}^2 - b_{23}^2 - a_{12}b_{23}c_{31})x_2x_3^2 - a_{12}c_{31}x_3^3 \end{cases}$$

$$g_9 = (-a_0+b_0+c_0)x_1+a_{12}c_0x_2+(-a_0c_{31}+b_0c_{31})x_3+b_{23}x_1x_2x_3+2x_1x_3^2+(a_{12}+b_{23}c_{31})x_2x_3^2+ c_{31}x_3^3$$

$$g_{10} = \begin{cases} -a_{12}b_0c_0+(-a_0+b_0+c_0)x_1x_2+(-a_0b_{23}+b_{23}c_0)x_1x_3+ \\ (-a_0c_{31}+b_0c_{31})x_2x_3+(-a_{12}b_0-a_{12}c_0-a_0b_{23}c_{31})x_3^2+2x_1x_2x_3^2+ \\ (b_{23}+c_{31})x_2x_3^3-a_{12}x_3^4 \end{cases}$$

$$g_{11} = c_0 + x_1^2 + c_{31}x_1x_3 + x_3^2$$

From the computed *Gröbner basis* of the *Ideal* $I \subset \mathbb{R}[x_1, x_2, x_3]$ above, We note that the element g_1 in *Box* (3-3a) is a *univariate polynomial* in x_3. With the coefficients of g_1 known, the *univariate polynomial* is then solved for $x_3 \in \mathbb{R}^+$ and the admissible values inserted in g_{11} in *Box* (3-3b) to obtain $x_1 \in \mathbb{R}.^+$ The obtained values of $x_3 \in \mathbb{R}^+$ and $x_1 \in \mathbb{R}^+$ are now inserted in any of the remaining elements of the *Gröbner basis* $g_2, \ldots, g_{10}$ in *Box* (3-3b) to obtain the remaining variable $x_2 \in \mathbb{R}.^+$ The correct distances are finally deduced with the help of prior information (e.g. from an existing map).

(iii) Multipolynomial resultants solution of the Grunert distance equations for the general three-dimension resection problem

Besides the use of *Gröbner bases* approach as demonstrated above, the *Multipolynomial resultants* technique can also be used to solve the *Grunert equations* for distances. We illustrate the solution of the problem first using the *F. Macaulay* (1902) formulation of the coefficient matrix and then the *B. Sturmfels* (1998) formulation of the coefficient matrix. We start with the *Grunert equations* expressed in the form (3-46) as

$$\begin{aligned} R_1 &:= x_1^2 + x_2^2 + a_{12}x_1x_2 + a_0 &= 0 \\ R_2 &:= x_2^2 + x_3^2 + b_{23}x_2x_3 + b_0 &= 0 \\ R_3 &:= x_1^2 + x_3^2 + c_{31}x_1x_3 + c_0 &= 0. \end{aligned} \quad (3\text{-}50)$$

Clearly equation (3-50) is not homogeneous. It is therefore homogenized by introducing the fourth variable x_4 and treating the variable which is to be solved first, say x_1, as a constant (i.e. hiding it by giving it degree zero). The resulting homogenized polynomial is

$$\begin{aligned} R_{11} &:= x_2^2 + a_{12}x_1x_2x_4 + (a_0 + x_1^2)x_4^2 &= 0 \\ R_{21} &:= x_2^2 + x_3^2 + b_{23}x_2x_3 + b_0x_4^2 &= 0 \\ R_{31} &:= x_3^2 + c_{31}x_1x_3x_4 + (x_1^2 + c_0)x_4^2 &= 0 \end{aligned} \quad (3\text{-}51)$$

which we simplify as

$$\begin{aligned} R_{11} &:= x_2^2 + a_1x_2x_4 + a_2x_4^2 &= 0 \\ R_{21} &:= x_2^2 + x_3^2 + b_1x_2x_3 + b_2x_4^2 &= 0 \\ R_{31} &:= x_3^2 + c_1x_3x_4 + c_2x_4^2 &= 0 \end{aligned} \quad (3\text{-}52)$$

with the coefficients given as $a_1 = a_{12}x_1,\ a_2 = (a_0 + x_1^2),\ b_1 = b_{23},\ b_2 = b_0,\ c_1 = c_{31}x_1,\ c_2 = (c_0 + x_1^2)$. We now formulate the coefficient matrix of (3-52) by first using the *F. Macaulay* (1902) approach followed by the *B. Sturmfels* (1998) approach.

Approach 1 (*F. Macaulay* 1902):

The *first step* involves the determination of the total degree of (3-52) by (2-57) in page (21) of Chapter 2 giving the value of $d = 4$. In the *second step*, one formulates the general set comprising the monomials of degree 4 in three variables by multiplying the monomials of (3-52) by each other. These monomials form the elements of the set X^d (2-58) in page (21) as

$$X^d = \left\{ \begin{array}{c} x_2^4,\, x_2^3x_4,\, x_2^2x_3^2,\, x_2^3x_3,\, x_2^2x_4^2,\, x_2^2x_3x_4,\, x_2x_3^3 \\ \\ x_2x_4^3,\, x_2x_3^2x_4,\, x_2x_3x_4^2,\, x_3^2x_4^2,\, x_3x_4^3,\, x_4^4,\, x_3^4,\, x_3^3x_4 \end{array} \right\} \tag{3-53}$$

which is now partitioned in *step 3* according to (2-59) in page (22) as

$$\begin{array}{rcc} X_i^d & = & \{x^\alpha \mid \alpha_i \geq d_i \, and \, \alpha_j < d_j, \forall_j < i\} \\ X_2^4 & = & \{x_2^4,\, x_2^3x_4,\, x_2^2x_3^2,\, x_2^3x_3,\, x_2^2x_4^2,\, x_2^2x_3x_4\} \\ X_3^4 & = & \{x_2x_3^2x_4,\, x_3^2x_4^2,\, x_2x_3^3,\, x_3^4,\, x_3^3x_4\} \\ X_4^4 & = & \{x_2x_4^3,\, x_2x_3x_4^2,\, x_3x_4^3,\, x_4^4\}. \end{array} \tag{3-54}$$

In the *fourth step*, we form the polynomials F_i using the sets above according to equation (2-60) in page (22) giving rise to

$$\begin{array}{rcl} F_1 := \dfrac{X_2^4}{x_2^2} f_1 & = & \{x_2^2f_1,\, x_2x_4f_1,\, x_3^2f_1,\, x_2x_3f_1,\, x_4^2f_1,\, x_3x_4f_1\} \\ \\ F_2 := \dfrac{X_3^4}{x_3^2} f_2 & = & \{x_2x_4f_2,\, x_4^2f_2,\, x_2x_3f_2,\, x_3^2f_2,\, x_3x_4f_2\} \\ \\ F_3 := \dfrac{X_4^4}{x_4^2} f_3 & = & \{x_2x_4f_3,\, x_2x_3f_3,\, x_3x_4f_3,\, x_4^2f_3\}. \end{array} \tag{3-55}$$

Finally we can now form a matrix

$$\boldsymbol{A} \text{ of dimension } \left(\begin{array}{c} d+n-1 \\ d \end{array} \right) \times \left(\begin{array}{c} d+n-1 \\ d \end{array} \right)$$

(in our case 15×15) whose rows are the coefficients of the f_i in (3-55) above and the columns are the monomials $\{c_1 = x_2^4, c_2 = x_2^3x_3, c_3 = x_2^3x_4, c_4 = x_2^2x_3^2, c_5 = x_2^2x_3x_4, c_6 = x_2^2x_4^2, c_7 = x_2x_3^3$, $c_8 = x_2x_3^2x_4, c_9 = x_2x_3x_4^2,$ $c_{10} = x_2x_4^3, c_{11} = x_3^4, c_{12} = x_3^3x_4, c_{13} = x_3^2x_4^2, c_{14} = x_3x_4^3$ and $c_{15} = x_4^4\}$ elements of the sets formed in (3-54) as

$$\boldsymbol{A} = \left[\begin{array}{cccccccccccccccc} & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ x_2^2f_1 & 1 & 0 & a_1 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_3^2f_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 \\ x_2x_3f_1 & 0 & 1 & 0 & 0 & a_1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_4^2f_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & a_2 \\ x_3x_4f_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & a_2 & 0 \\ x_2x_4f_1 & 0 & 0 & 1 & 0 & 0 & a_1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ x_2x_4f_2 & 0 & 0 & 1 & 0 & b_1 & 0 & 0 & 1 & 0 & b_2 & 0 & 0 & 0 & 0 & 0 \\ x_4^2f_2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & b_1 & 0 & 0 & 0 & 1 & 0 & b_2 \\ x_3^2f_2 & 0 & 0 & 0 & 1 & 0 & 0 & b_1 & 0 & 0 & 0 & 1 & 0 & b_2 & 0 & 0 \\ x_3x_4f_2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & b_1 & 0 & 0 & 0 & 1 & 0 & b_2 & 0 \\ x_2x_3f_2 & 0 & 1 & 0 & b_1 & 0 & 0 & 1 & 0 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_2x_3f_3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & c_1 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_3x_4f_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & c_1 & c_2 & 0 \\ x_4^2f_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & c_1 & c_2 \\ x_2x_4f_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & c_1 & c_2 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

The determinant of the matrix $\boldsymbol{A}$ above is a univariate polynomial of degree 8 in the variable x_1 given in *Box* (3-3c) below:

Box 3-3c (*F. Macaulay multipolynomial resultants solution of the Grunert distance equations-univariate polynomial*):

$$(8b_{23}^2 + 8b_{23}a_{12}c_{31} - 16 - 2b_{23}^2c_{31}^2 - 2b_{23}^2a_{12}^2 - b_{23}^2a_{12}^2c_{31}^2 - a_{12}^4 - 2b_{23}a_{12}c_{31}^3 - b_{23}^4 - 2b_{23}a_{12}^3c_{31} - c_{31}^4 + 8a_{12}^2 + 8c_{31}^2 - 2a_{12}^2c_{31}^2 - 2a_{12}b_{23}^3c_{31})x_1^8 +$$

$$+(-a_{12}^4c_{31}^2b_0 - 2a_{12}^2c_{31}^2a_0 - 32c_0 + 32b_0 - 32a_0 + 8b_0a_{12}^2c_{31}^2 - a_{12}^2c_{31}^4b_0 - 2b_{23}^2a_{12}^2a_0 - 2a_{12}^2c_{31}^2c_0 - 4b_{23}^2a_{12}^2c_0 + 2b_{23}^2b_0a_{12}^2 - 2b_{23}^2c_{31}^2c_0 + 2b_{23}^2b_0c_{31}^2 - 2b_{23}^4c_0 - 2a_{12}^4c_0 + 2a_{12}^4b_0 + 2b_0c_{31}^4 - 2c_{31}^4a_0 - 16b_0a_{12}^2 - 16b_0c_{31}^2 + 16a_{12}^2c_0 + 16c_{31}^2a_0 + 8c_{31}^2c_0 + 8a_{12}^2a_0 - 2b_{23}^4a_0 + 16b_{23}^2c_0 + 16b_{23}^2a_0 - 8b_{23}^2b_0 - 3a_{12}b_{23}^3c_{31}c_0 - 3a_{12}b_{23}^3a_0c_{31} - a_{12}b_{23}^3b_0c_{31} - a_{12}^3c_{31}^3b_{23}b_0 - a_{12}c_{31}^3b_{23}c_0 - a_{12}^2c_{31}^2b_{23}^2c_0 - 2b_{23}^2b_0a_{12}^2c_{31}^2 - 3b_{23}a_{12}^3c_{31}c_0 - 4b_{23}b_0a_{12}c_{31} + 12b_{23}a_{12}c_{31}a_0 + 12b_{23}a_{12}c_{31}c_0 + b_{23}b_0a_{12}^3c_{31} - b_{23}a_{12}^3c_{31}a_0 - 3b_{23}a_{12}c_{31}^3a_0 - b_{23}^2a_{12}^2c_{31}^2a_0 + b_{23}b_0a_{12}c_{31}^3 - 4b_{23}^2c_{31}^2a_0)x_1^6 +$$

$$+(-3b_0^2a_{12}^2c_{31}^2 - 12b_{23}^2b_0a_0 - 3a_{12}^2b_{23}^2c_0^2 - b_{23}^2b_0^2c_{31}^2 + 2a_{12}^4c_0b_0 + 2b_0c_{31}^4a_0 + 48b_0c_0 - 24b_0^2 + 48b_0a_0 - 48a_0c_0 - 24a_0^2 - 12b_{23}^2b_0c_0 + 28b_{23}^2a_0c_0 - 4b_{23}^4a_0c_0 - b_{23}^2c_0^2c_{31}^2 - a_{12}^2c_{31}^2c_0^2 - 20a_{12}^2b_0c_0 - 24c_0^2 - a_0^2a_{12}^2c_{31}^2 - b_{23}^2a_{12}^2a_0^2 - 12b_0a_{12}^2a_0 - 20b_0c_{31}^2a_0 + 12c_{31}^2a_0c_0 + 12a_{12}^2c_0a_0 - b_{23}^4a_0^2 - b_{23}^4c_0^2 - b_0^2c_{31}^4 + 10b_0^2c_{31}^2 - a_{12}^4b_0^2 - a_0^2c_{31}^4 + 2c_{31}^2c_0^2 - a_{12}^4c_0^2 + 10a_{12}^2c_0^2 + 10a_0^2c_{31}^2 + 2a_{12}^2a_0^2 + 2b_{23}^2b_0^2 + 10b_{23}^2a_0^2 + 10a_{12}^2b_0^2 + 10b_{23}^2c_0^2 - 12b_0c_{31}^2c_0 - a_{12}c_{31}b_{23}^3c_0^2 - a_{12}c_{31}b_{23}^3b_0c_0 - a_{12}b_{23}^3a_0b_0c_{31} - 2b_{23}^2a_{12}^2c_0a_0 + 2b_{23}c_{31}a_{12}c_0^2 - 4a_{12}b_{23}^3a_0c_{31}c_0 - b_{23}b_0^2c_{31}^3a_{12} + 4a_{12}^2c_0c_{31}^2b_0 + 4b_0a_{12}^2b_{23}^2c_0 - b_0^2a_{12}^2b_{23}^2 - a_{12}c_{31}^3b_{23}a_0c_0 - a_{12}^2c_{31}^2b_{23}^2a_0c_0 - b_{23}c_0c_{31}^3b_0a_{12} - b_{23}^2c_0c_{31}^2b_0a_{12}^2 + 2a_{12}^3c_{31}b_0b_{23}c_0 - a_{12}^3c_{31}b_{23}c_0^2 - a_{12}^3c_{31}b_{23}b_0^2 - b_{23}a_0^2a_{12}c_{31}^3 - b_{23}^2b_0a_{12}^2c_{31}^2a_0 - b_{23}a_{12}^3c_{31}c_0a_0 - 4b_{23}b_0a_{12}c_{31}a_0 - 4b_{23}b_0a_{12}c_{31}c_0 + 4b_0a_{12}^2c_{31}^2a_0 + 2b_{23}a_0^2a_{12}c_{31} + 20b_{23}a_{12}c_{31}c_0a_0 - a_{12}b_{23}^3a_0^2c_{31} + 4b_{23}^2b_0c_{31}^2a_0 + 2b_{23}b_0^2c_{31}a_{12} + 2b_{23}b_0a_{12}c_{31}^3a_0 - 2b_{23}^2c_{31}^2a_0c_0 - b_{23}b_0a_{12}^3c_{31}a_0 - 3b_{23}^2a_0^2c_{31}^2)x_1^4 +$$

$$+(6a_{12}^2c_0b_0^2 + 4a_{12}^2c_0^2a_0 + 6b_0^2c_{31}^2a_0 - 2b_0c_{31}^2c_0^2 + 4b_0^2c_{31}^2c_0 - b_{23}^2a_0^3c_{31}^2 - 6a_{12}^2c_0^2b_0 - 6b_0c_{31}^2a_0^2 - 2b_0a_{12}^2a_0^2 + 2a_0c_{31}^2c_0^2 - 16b_0c_0b_{23}^2a_0 + 2b_0^2b_{23}^2a_0 - 4b_0a_0^2b_{23}^2 + 14a_0^2c_0b_{23}^2 + 2a_0^3b_{23}^2 + 8b_0^3 - 8a_0^3 - 8c_0^3 + 2b_0^2b_{23}^2c_0 - 2b_{23}^4a_0c_0^2 - 4b_0b_{23}^2c_0^2 + 14a_0b_{23}^2c_0^2 - 2b_{23}^4a_0^2c_0 + 4a_0^2c_{31}^2c_0 + 48b_0c_0a_0 + 2a_{12}^2a_0^2c_0 - 2b_0^3c_{31}^2 + 2a_{12}^2c_0^3 + 2a_0^3c_{31}^2 + 24b_0a_0^2 + 24b_0c_0^2 - 24b_0^2a_0 - 2b_0^3a_{12}^2 - 24b_0^2c_0 - 24c_0^2a_0 - 24c_0a_0^2 + 2b_{23}^2c_0^3 - a_{12}c_0^2b_{23}^3a_0c_{31} - b_{23}^2b_0^2c_{31}^2a_0 + 2b_{23}^2b_0c_{31}^2a_0^2 + b_{23}b_0^2c_{31}c_0a_{12} + b_{23}b_0c_{31}a_{12}c_0^2 - b_{23}b_0^3c_{31}a_{12} - b_0^2a_{12}^2b_{23}^2c_0 + 2b_0a_{12}^2b_{23}^2c_0^2 + 5b_{23}a_0c_{31}a_{12}c_0^2 - 8a_{12}^2b_0c_0a_0 - 8b_0c_{31}^2a_0c_0 + b_{23}b_0a_0^2a_{12}c_{31} - b_{23}^2c_0^3a_{12}^2 - b_{23}^2c_0^2c_{31}^2a_0 - b_{23}c_0^3a_{12}c_{31} - b_{23}^2a_{12}^2a_0^2c_0 - b_{23}a_0^3a_{12}c_{31} + 5b_{23}a_0^2a_{12}c_{31}c_0 - 6b_{23}b_0a_{12}c_{31}c_0a_0 - a_{12}c_0b_{23}^3a_0b_0c_{31} - a_{12}c_0b_{23}^3a_0^2c_{31} + b_{23}b_0^2a_0c_{31}a_{12} + 4a_{12}^2b_0^2a_0)x_1^2$$

$$-a_0^4 - c_0^4 - b_0^4 + 12b_0c_0^2a_0 + 12b_0c_0a_0^2 - 4b_{23}^2b_0a_0^2c_0 + 2b_{23}^2b_0^2c_0a_0 - 12b_0^2c_0a_0 - b_{23}^4c_0^2a_0^2 + 4b_0^3a_0 - 6b_0^2a_0^2 + 4b_0a_0^3 - 6a_0^2c_0^2 - 4a_0^3c_0 + 2b_{23}^2a_0^3c_0 + 4b_{23}^2a_0^2c_0^2 - 4c_0^3a_0 + 2b_{23}^2c_0^3a_0 + 4c_0^3b_0 + 4c_0b_0^3 - 6c_0^2b_0^2 - 4b_{23}^2b_0c_0^2a_0.$$

The *univariate polynomial* in *Box* (3-3c) is then solved by the algebraic algorithm such as *roots* command in MATLAB to obtain the roots. Once these roots have been obtained, the admissible solution is substituted in the third equation of (3-50) in page (37) to obtain h value of $x_3 \in \mathbb{R}.^+$ The obtained value of $x_3 \in \mathbb{R}^+$ is in turn substituted in the second equation of (3-50) to obtain the last variable $x_2 \in \mathbb{R}.^+$The correct values of distances are deduced with the help of prior information.

Approach 2 (*B. Sturmfels* 1998):

From (3-52) in page (37), we compute the determinant of the Jacobi matrix as

$$J = det \begin{bmatrix} \frac{\partial R_{11}}{\partial x_2} & \frac{\partial R_{11}}{\partial x_3} & \frac{\partial R_{11}}{\partial x_4} \\ \frac{\partial R_{21}}{\partial x_2} & \frac{\partial R_{21}}{\partial x_3} & \frac{\partial R_{21}}{\partial x_4} \\ \frac{\partial R_{31}}{\partial x_2} & \frac{\partial R_{31}}{\partial x_3} & \frac{\partial R_{31}}{\partial x_4} \end{bmatrix} \tag{3-56}$$

respectively

$$J = det \begin{bmatrix} 2x_2 + a_1x_4 & 0 & 2a_2x_4 + a_1x_2 \\ 2x_2 + b_1x_3 & 2x_3 + b_1x_2 & 2b_2x_4 \\ 0 & 2x_3 + c_1x_4 & 2c_2x_4 + c_1x_3 \end{bmatrix} \tag{3-57}$$

which gives a cubic polynomial in $x_2,\ x_3,\ x_4$ as

$J = 8x_2x_3c_2x_4 + 4x_2c_1x_3^2 + 4b_1x_2^2c_2x_4 + 2b_1x_2^2c_1x_3 - 8x_2b2x_4x_3 - 4x_2b2x_4^2c_1 + 4a_1x_4^2x_3c_2 + 2a_1x_4c_1x_3^2 + 2a_1x_4^2b_1x_2c_2 + 2a_1x_4b_1x_2c_1x_3 - 4a_1x_4^2b2x_3 - 2a_1x_4^3b2c_1 + 8x_2a_2x_4x_3 + 4x_2a_2x_4^2c_1 + 4a_1x_2^2x_3 + 2a_1x_2^2c_1x_4 + 4b_1x_3^2a_2x_4 + 2b_1x_3a_2x_4^2c_1 + 2b_1x_3^2a_1x_2$

whose partial derivatives with respect to $x_2,\ x_3,\ x_4$ can be written in the form (2-64) in page (23) with the coefficients of b_{ij} as

Coefficients:

$b_{11} = 0$, $b_{12} = (4c_1 + 2b_1a_1)$, $b_{13} = (-4b_2c_1 + 4a_2c_1 + 2a_1b_1c_2)$, $b_{14} = (4b_1c_1 + 8a_1)$,

$b_{15} = (8b_1c_2 + 4a_1c_1)$, $b_{16} = (8c_2 - 8b_2 + 8a_2 + 2a_1b_1c_1)$, $b_{21} = (4a_1 + 2b_1c_1)$, $b_{22} = 0$,

$b_{23} = (-4a_1b_2 + 2b_1a_2c_1 + 4a_1c_2)$, $b_{24} = (8c_1 + 4b_1a_1)$, $b_{25} = (8c_2 + 2a_1b_1c_1 + 8a_2 - 8b_2)$,

$b_{26} = (4a_1c_1 + 8b_1a_2)$, $b_{31} = (2a_1c_1 + 4b_1c_2)$, $b_{32} = (2a_1c_1 + 4b_1a_2)$, $b_{33} = -6a_1b_2c_1$

$b_{34} = (-8b_2 + 2a_1b_1c_1 + 8a_2 + 8c_2)$, $b_{35} = (-8b_2c_1 + 4a_1b_1c_2 + 8a_2c_1)$, $b_{36} = (8a_1c_2 - 8a_1b_2 + 4b_1a_2c_1)$

and those of the original equations a_{ij} as

$a_{11} = 1, a_{12} = 0, a_{13} = a_2, a_{14} = 0, a_{15=a_1, a_{16}=0}$

$a_{21} = 1, a_{22} = 1, a_{23} = b_2, a_{24} = b_1, a_{25} = 0, a_{26} = 0$

$a_{31} = 0, a_{32} = 1, a_{33} = c_2, a_{34} = 0, a_{35} = 0, a_{36} = c_1$

The computation of the resultant of the matrix (2-65) in page (23) with the coefficients as given above leads to a univariate polynomial in x_1 of degree eight as

Box 3-3d (*B. Sturmfels multipolynomial resultants solution of the Grunert distances equations-univariate polynomial*):

$16b_{23}^2c_0^2a_0^2 - 16b_{23}^2b_0a_0c_0^2 - 24b_0^2c_0^2 + 16b_0a_0^3 - 24b_0^2a_0^2 - 4b_{23}^4c_0^2a_0^2 - 48b_0^2c_0a_0 + 48b_0a_0c_0^2 + 8b_{23}^2c_0^3a_0 + 8b_{23}^2c_0a_0^3 + 48b_0c_0a_0^2 - 4b_0^4 - 16b_{23}^2b_0c_0a_0^2 - 24c_0^2a_0^2 + 16b_0^3a_0 - 16c_0a_0^3 + 16b_0^3c_0 + 8b_{23}^2b_0^2c_0a_0 - 16c_0^3a_0 + 16b_0c_0^3 - 4c_0^4 - 4a_0^4 +$

$+(-8b_{23}^4c_0a_0^2 + 96b_0c_0^2 + 32b_0^3 + 8a_{12}^2c_0^3 + 8b_{23}^2b_0^2c_0 + 56b_{23}^2c_0^2a_0 + 8b_{23}^2b_0^2a_0 - 24a_{12}^2b_0c_0^2 + 56b_{23}^2c_0a_0^2 - 16b_{23}^2b_0c_0^2 + 8c_{31}^2a_0^3 + 20c_{31}b_{23}a_{12}c_0a_0^2 + 4b_{23}a_{12}b_0^2c_{31}a_0 - 4c_{31}b_{23}a_{12}a_0^3 - 32a_{12}^2b_0c_0a_0 + 192b_0c_0a_0 + 24a_{12}^2b_0^2c_0 - 24b_0c_{31}^2a_0^2 + 16a_{12}^2b_0^2a_0 - 8b_{23}^4c_0^2a_0 - 96b_0^2a_0 - 4b_0^2c_{31}^2b_{23}^2a_0 - 4b_{23}^2a_{12}^2b_0^2c_0 - 64b_{23}^2b_0c_0a_0 - 4b_{23}^2a_{12}^2c_0a_0^2 - 32b_0c_{31}^2c_0a_0 - 4b_{23}a_{12}b_0^3c_{31} + 20c_{31}b_{23}a_{12}a_0c_0^2 + 8b_{23}^2a_{12}^2b_0c_0^2 - 4c_{31}^2b_{23}^2a_0c_0^2 - 24c_{31}b_{23}a_{12}b_0c_0a_0 + 16a_{12}^2a_0c_0^2 - 16b_{23}^2b_0a_0^2 + 8c_{31}^2a_0c_0^2 + 96b_0a_0^2 + 8a_{12}^2c_0a_0^2 - 4b_{23}^2a_{12}^2c_0^3 - 4c_{31}^2b_{23}^2a_0^3 + 24b_0^2c_{31}^2a_0 - 8b_0c_{31}^2c_0^2 + 16b_0^2c_{31}^2c_0 + 8b_{23}^2a_0^3 - 8b_0^3c_{31}^2 - 8a_{12}^2b_0^3 + 8b_{23}^2c_0^3 - 96b_0^2c_0 - 8a_{12}^2b_0a_0^2 + 16c_{31}^2c_0a_0^2 - 32c_0^3 - 32a_0^3 - 96c_0a_0^2 - 96a_0c_0^2 + 4c_{31}b_{23}a_{12}b_0a_0^2 - 4b_{23}^3a_{12}c_{31}a_0c_0^2 + 4b_{23}a_{12}b_0^2c_{31}c_0 - 4b_{23}^3a_{12}c_{31}c_0a_0^2 - 4b_{23}^3a_{12}b_0c_{31}c_0a_0 - 4c_{31}b_{23}a_{12}c_0^3 + 4c_{31}b_{23}a_{12}b_0c_0^2 + 8b_0c_{31}^2b_{23}^2a_0^2)x_1^2 +$

Box 3-3d (*B. Sturmfels multipolynomial resultants solution of the Grunert distances equations-univariate polynomial* continued):

$$+(-96b_0^2 - 80b_0c_{31}^2a_0 - 4c_{31}^2a_{12}^2c_0^2 - 4b_0^2c_{31}^2b_{23}^2 + 192b_0a_0 - 12a_{12}^2c_{31}^2b_0^2 + 40c_{31}^2a_0^2 - 4a_{12}^4c_0^2 + 112b_{23}^2c_0a_0 - 80a_{12}^2b_0c_0 + 40a_{12}^2b_0^2 - 4b_0^2c_{31}^4 - 48b_{23}^2b_0c_0 - 48b_0c_{31}^2c_0 + 8c_{31}^2c_0^2 + 8c_{31}b_{23}a_{12}a_0^2 - 4a_{12}c_{31}^3b_{23}c_0a_0 + 16a_{12}^2c_{31}^2b_0a_0 - 16b_{23}^4c_0a_0 + 8a_{12}c_{31}^3b_{23}b_0a_0 + 16b_{23}^2a_{12}^2b_0c_0 - 4b_{23}^2a_{12}^2c_{31}^2b_0c_0 + 8a_{12}^2a_0^2 - 4c_{31}^2a_{12}^2a_0^2 - 48b_{23}^2b_0a_0 + 40a_{12}^2c_0^2 + 192b_0c_0 - 4a_{12}^4b_0^2 - 4b_{23}^2a_{12}^2b_0^2 - 48a_{12}^2b_0a_0 + 48a_{12}^2c_0a_0 - 12c_{31}^2b_{23}^2a_0^2 + 40b_{23}^2a_0^2 - 4c_{31}b_{23}a_{12}^3c_0^2 + 80c_{31}b_{23}a_{12}c_0a_0 + 16a_{12}^2c_{31}^2b_0c_0 - 8b_{23}^2a_{12}^2c_0a_0 + 8c_{31}b_{23}a_{12}^3b_0c_0 - 4a_{12}b_{23}c_{31}^3b_0^2 - 4c_{31}b_{23}a_{12}^3b_0^2 - 4c_{31}b_{23}a_{12}^3c_0a_0 - 8c_{31}^2b_{23}^2c_0a_0 + 8b_0c_{31}^4a_0 - 4c_{31}^4a_0^2 - 4b_{23}^4a_0^2 - 4b_{23}^4c_0^2 + 40b_0^2c_{31}^2 - 4c_{31}^2b_{23}^2c_0^2 - 4b_{23}^2a_{12}^2a_0^2 + 40b_{23}^2c_0^2 + 48c_{31}^2c_0a_0 + 8a_{12}^4b_0c_0 - 12b_{23}^2a_{12}^2c_0^2 - 192c_0a_0 - 96a_0^2 - 96c_0^2 + 8b_{23}^2b_0^2 - 4b_{23}^2a_{12}^2c_{31}^2b_0a_0 - 16c_{31}b_{23}a_{12}b_0a_0 - 4b_{23}^3a_{12}c_{31}c_0^2 - 4c_{31}b_{23}a_{12}^3b_0a_0 - 4a_{12}c_{31}^3b_{23}a_0^2 + 8b_{23}a_{12}b_0^2c_{31} - 4b_{23}^2a_{12}^2c_{31}^2c_0a_0 - 4b_{23}^3a_{12}c_{31}a_0^2 - 16b_{23}^3a_{12}c_{31}c_0a_0 - 4a_{12}c_{31}^3b_{23}b_0c_0 - 4b_{23}^3a_{12}b_0c_{31}a_0 - 4b_{23}^3a_{12}b_0c_{31}c_0 + 8c_{31}b_{23}a_{12}c_0^2 - 16c_{31}b_{23}a_{12}b_0c_0 + 16b_0c_{31}^2b_{23}^2a_0)x_1^4 +$$

$$+(128b_0 - 8b_{23}^4a_0 - 128a_0 - 4a_{12}^2b_0c_{31}^4 + 64c_{31}^2a_0 - 8c_{31}^2a_{12}^2c_0 - 128c_0 + 8b_0c_{31}^4 + 8b_{23}^2a_{12}^2b_0 - 64a_{12}^2b_0 - 16c_{31}^2b_{23}^2a_0 - 8c_{31}^2a_{12}^2a_0 - 4a_{12}^3b_0c_{31}^3b_{23} + 8a_{12}^4b_0 - 4a_{12}c_{31}^3b_{23}c_0 - 8b_{23}^2a_{12}^2c_{31}^2b_0 + 64b_{23}^2c_0 + 32a_{12}^2c_{31}^2b_0 + 64b_{23}^2a_0 + 48c_{31}b_{23}a_{12}a_0 + 4c_{31}b_{23}a_{12}^3b_0 - 4c_{31}b_{23}a_{12}^3a_0 - 16b_{23}^2a_{12}^2c_0 - 64b_0c_{31}^2 - 8c_{31}^4a_0 - 4a_{12}^4b_0c_{31}^2 + 32a_{12}^2a_0 - 8a_{12}^4c_0 + 32c_{31}^2c_0 + 64a_{12}^2c_0 - 8b_{23}^4c_0 - 8c_{31}^2b_{23}^2c_0 - 8b_{23}^2a_{12}^2a_0 + 8b_0c_{31}^2b_{23}^2 - 32b_{23}^2b_0 - 12a_{12}c_{31}^3b_{23}a_0 - 12c_{31}b_{23}a_{12}^3c_0 - 4b_{23}^2a_{12}^2c_{31}^2a_0 - 4b_{23}^2a_{12}^2c_{31}^2c_0 - 12b_{23}^3a_{12}c_{31}a_0 - 12b_{23}^3a_{12}c_{31}c_0 + 4a_{12}c_{31}^3b_{23}b_0 - 4b_{23}^3a_{12}b_0c_{31} + 48c_{31}b_{23}a_{12}c_0 - 16c_{31}b_{23}a_{12}b_0)x_1^6 +$$

$$+(-4c_{31}^4 - 4b_{23}^2a_{12}^2c_{31}^2 - 8a_{12}c_{31}^3b_{23} - 8c_{31}b_{23}a_{12}^3 + 32a_{12}^2 - 4a_{12}^4 - 64 + 32b_{23}^2 - 8c_{31}^2a_{12}^2 + 32c_{31}b_{23}a_{12} - 4b_{23}^4 - 8b_{23}^3a_{12}c_{31} + 32c_{31}^2 - 8b_{23}^2a_{12}^2 - 8c_{31}^2b_{23}^2)x_1^8$$

M. A. Fischler and *R. Bolles* (1981, pp. 386-387, Figure 5) have demonstrated that because every term in (3-31) is either a constant or of degree 2, for every real positive solution, there exist a geometrically isomorphic negative solution. Thus there are at most four positive solutions to (3-31). This is because (3-31) has eight solutions according to *G. Chrystal* (1964, p.415) who states that for n independent polynomial equations in n unknowns, there can be no more solution than the product of their respective degree. Since each equation of (3-31) is of degree 2 there can only be upto eight solutions.

Finally, in comparing the two methods, the *Gröbner bases* approach in most cases is slow and there is always a risk of the computer breaking down during computations. Besides, the *Gröbner bases* approach computes unwanted intermediary elements which occupy more space and thus leads to storage problem. The overall speed of computation is said to be proportional to twice exponential the number of variables (*D. Manocha* 1994 a, b, c, *D. Manocha* and *F. Canny* 1991). This has led to various studies advocating for the use of the alternate method, the *resultant* and specifically *multipolynomial resultant* approach. The *Gröbner bases* can be made a bit faster by computing the reduced *Gröbner bases* as explained in Chapter 2. The *Multipolynomial resultants* on the other hand involve computing with larger matrices which may require a lot of work. For linear systems and ternary quadrics, *B. Sturmfels* (1998) offers a remedy through the application of the Jacobi determinants.

(b) The position-derivation step (3d-ranging or “3d-Bogenschnitt”):

This step is commonly referred to in German literature as *"Bogenschnitt"* problem and in english literature as the "ranging problem" or "Arc section" *(H. Kahmen* and *W. Faig* 1988, p.215) and is the problem of establishing the position of a point given the distances from the unknown point $P \in \mathbb{E}^3$ to three other known stations $P_i \in \mathbb{E}^3 \mid i = 1, 2, 3$. In general the threedimensional *"Bogenschnitt"* problem can be formulated as follows: Given distances as observations or pseudo-observations from an unknown point $P \in \mathbb{E}^3$ to a minimum of three known points $P_i \in \mathbb{E}^3 \mid i = 1, 2, 3$, determine the position $\{X, Y, Z\}$ of the unknown point $P \in \mathbb{E}.^3$ When only three known stations are used to determine the position of the unknown station in threedimension, the problem reduces to that of 3d closed form solution. We present below four approaches that can be used to solve the *"3d-Bogenschnitt"* problem in a closed form. This problem is a traditional problem both in Geodesy, Photogrammetry and Robotics. In all the three areas, the determination of the coordinates of the unknown point given the distances from this point to three other known points is the key issue.

Starting from three nonlinear 3d Pythagorus distance observation equations (3-58) in *Box* (3-4) relating to the three unknowns $\{X, Y, Z\}$, two equations with three unknowns are derived. Equation (3-58) is expanded in the form given by (3-59) and differenced in (3-60) to eliminate the quadratic terms $\{X^2, Y^2, Z^2\}$. Collecting all the known terms of equation (3-60) to the right hand side and those relating to the unknowns on the left hand side leads to equation (3-61) with the terms $\{a, b\}$ given by (3-62). The solution of the unknown terms $\{X, Y, Z\}$ now involves solving equation (3-61), which has two equations with three unknowns. To circumvent the problem of having more unknowns than the equations, two of the unknowns are sought in terms of the third unknown (e.g. $X = g(Z), Y = g(Z)$).

(i) *Gröbner bases* approach:

We express equation (3-61) in the algebraic form (3-63) in *Box* (3-5)with the coefficients given as in (3-64). The *Gröbner basis* is then obtained using the *GroebnerBasis* command in Mathematica 3.0 as illustrated by (3-65) giving the computed *Gröbner basis* as in (3-66). The first equation of (3-66) is solved for $Y = g(Z)$ and is as presented in (3-67). This value is substituted in the second equation of (3-66) to give $X = g(Z)$ presented in the first equation of (3-68). The obtained values of Y and X are substituted in the first equation of (3-58) to give a quadratic equation in Z. Once this quadratic has been solved for Z, The values of Y and X can be obtained from (3-67) and (3-68) respectively. We mention here that the direct solution of $X = g(Z)$ as presented in the second equation of (3-68) could be obtained by computing the *reduced Gröbner basis* as explained in Chapter 2 rather than solving for $Y = g(Z)$ and substituting in the second equation of (3-66) to give $X = g(Z)$ presented in first equation of (3-68). Similarly we could obtain $Y = g(Z)$ alone by replacing Y with X in the option section of the *reduced Gröbner basis* discussed in Chapter 2.

(ii) Multipolynomial resultant (the Sylvester resultant) approach:

The problem is solved in *four steps* as illustrated in *Box* (3-6). In the *first step*, we solve for the first variable X in (3-63) by hiding it as a constant and homogenizing the equation using a variable W as in (3-69). In the *second step*, the *Sylvester resultant* or the *Jacobian determinant* is obtained as in (3-70). The resulting determinant (3-71) is solved for $X = g(Z)$ and presented in (3-72). The procedure is repeated for *steps three* and *four* as in equations (3-73) to (3-76) to solve for $Y = g(Z)$. The obtained values of $X = g(Z)$ and $Y = (Z)$ are substituted in the first equation of (3-58) to give a quadratic equation in Z. Once this quadratic has been solved for Z, The values of X and Y can be obtained from (3-72) and (3-76) respectively.

(iii) Solution by elimination approach-1:

In the elimination approach presented in *Box* (3-7), equations (3-77) is a simultaneous equation version of equations (3-61) with two equations and two unknowns $\{X, Y\}$ written in terms of the unknown Z . By first eliminating Y the value of X is obtained in terms of the unknown value Z and substituted in either of the two equations of (3-77) to give the value of Y . The values of $\{X, Y\}$ are as depicted by (3-78) which are expressed in a simplified form (3-79) with the coefficients $\{c, d, e, f\}$ given by (3-80). The values of $\{X, Y\}$ in (3-79) are substituted in the first equation of (3-58) in *Box* (3-4) to get the quadratic equation (3-81) respectively (3-82) in terms of Z as the unknown. The two solutions of Z are now given by the second equation of (3-82) with the coefficients $\{g, h, i\}$ given in (3-84). Once we solve (3-82) for Z, we substitute in (3-79) to obtain the corresponding pair of solutions for $\{X, Y\}$.

(iv) Solution by elimination approach-2:

The second approach presented in *Box* (3-8) involves first writing equation (3-61) in the simultaneous form (3-85), which is expressed in matrix form as (3-86). We now seek the matrix solution of $\{Y, Z\}$ in terms of the unknown element X as expressed by equation (3-87), which is written in a simpler form (3-89) whose the elements are given by (3-88). The solution of equation (3-87) for $\{Y, Z\}$ in terms of X is given by (3-90) respectively (3-91) and (3-92) with the coefficients of (3-92) given by (3-93). Substituting the obtained values of $\{Y, Z\}$ in terms of X in the first equation of (3-58) we obtain a quadratic equation (3-94) in terms of X as the unknown. The two solutions for X are given by the second equation in (3-94) and substituted back in (3-92) to obtain the values of $\{Y, Z\}$. The coefficients $\{l, m, n\}$ in equation (3-94) are given by (3-96).

A pair of solution (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) are obtained. The correct solution from this pair is obtained with the help of prior information e.g. from an existing map. Of importance is the problem of *bifurcation*, that is, to identify the point where the quadratic equation has only one solution, i.e. *bifurcates*. *S. Bancroft* (1985), *J. S. Abel* and *J. W.*

Chaffee (1991), *J. W. Chaffee* and *J. S. Abel* (1994), and *E. Grafarend* and *J. Shan* (1996) have already treated this problem. In the present study, the *bifurcation* point for (3-82) and (3-94) will be $h^2 = 4gi$ and $m^2 = 4in$ respectively.

In *Boxes* (3-9) and (3-10) the critical configuration of the two-dimensional and three-dimensional ranging problems are presented. First, the derivatives of the ranging equations (3-97) and (3-104) respectively are computed as in (3-98) and (3-105). The determinants of the matrices formed by the derivatives are obtained as in (3-100) for the twodimensional case, while a triple scalar product is obtained for the three-dimensional case to give (3-107). indicate the critical configuration to be the case when the points $P(X,Y),\ P_1(X_1,Y_1)$ and $P_2(X_2,Y_2)$ lie in a line with gradient $-\frac{a}{b}$ and intercept $-\frac{c}{b}$ for the two-dimensional case and when the points $P(X,Y,Z),\ P_1(X_1,Y_1,Z_1),\ P_2(X_2,Y_2,Z_2)$ and $P_3(X_3,Y_3,Z_3)$ lie on a plane for the three-dimensional case.

Box 3-4 (*differencing of the nonlinear distance equations*):

$$\left[\begin{array}{l} S_1^2 = (X_1-X)^2+(Y_1-Y)^2+(Z_1-Z)^2 \\ S_2^2 = (X_2-X)^2+(Y_2-Y)^2+(Z_2-Z)^2 \\ S_3^2 = (X_3-X)^2+(Y_3-Y)^2+(Z_3-Z)^2 \end{array}\right. \tag{3-58}$$

$$\left[\begin{array}{l} S_1^2 = X_1^2+Y_1^2+Z_1^2+X^2+Y^2+Z^2-2X_1X-2Y_1Y-2Z_1Z \\ S_2^2 = X_2^2+Y_2^2+Z_2^2+X^2+Y^2+Z^2-2X_2X-2Y_2Y-2Z_2Z \\ S_3^2 = X_3^2+Y_3^2+Z_3^2+X^2+Y^2+Z^2-2X_3X-2Y_3Y-2Z_3Z \end{array}\right. \tag{3-59}$$

differencing above

$$\left[\begin{array}{l} S_1^2-S_2^2 = X_1^2-X_2^2+Y_1^2-Y_2^2+Z_1^2-Z_2^2+a \\ S_2^2-S_3^2 = X_2^2-X_3^2+Y_2^2-Y_3^2+Z_2^2-Z_3^2+b \end{array}\right. \tag{3-60}$$

$$\left[\begin{array}{l} 2X(X_2-X_1)+2Y(Y_2-Y_1)+2Z(Z_2-Z_1) = a \\ 2X(X_3-X_2)+2Y(Y_3-Y_2)+2Z(Z_3-Z_2) = b \end{array}\right. \tag{3-61}$$

$$\left[\begin{array}{l} a = S_1^2-S_2^2-X_1^2+X_2^2-Y_1^2+Y_2^2-Z_1^2+Z_2^2 \\ b = S_2^2-S_3^2-X_2^2+X_3^2-Y_2^2+Y_3^2-Z_2^2+Z_3^2 \end{array}\right. \tag{3-62}$$

Box 3-5 (*Gröbner basis approach*):

$$\begin{array}{c} a_{02}X+b_{02}Y+c_{02}Z+f_{02}=0 \\ a_{12}X+b_{12}Y+c_{12}Z+f_{12}=0 \end{array} \tag{3-63}$$

$$\begin{array}{c} a_{02}=2(X_1-X_2), b_{02}=2(Y_1-Y_2), c_{02}=2(Z_1-Z_2) \\ a_{12}=2(X_2-X_3), b_{12}=2(Y_2-Y_3), c_{12}=2(Z_2-Z_3) \\ f_{02}=(S_1^2-X_1^2-Y_1^2-Z_1^2)-(S_2^2-X_2^2-Y_2^2-Z_2^2) \\ f_{12}=(S_2^2-X_2^2-Y_2^2-Z_2^2)-(S_3^2-X_3^2-Y_3^2-Z_3^2). \end{array} \tag{3-64}$$

$$GroebnerBasis[\{a_{02}X+b_{02}Y+c_{02}Z+f_{02}, a_{12}X+b_{12}Y+c_{12}Z+f_{12}\},\{X,Y\}] \tag{3-65}$$

$$\begin{array}{c} g_1=a_{02}b_{12}Y-a_{12}b_{02}Y-a_{12}c_{02}Z+a_{02}c_{12}Z+a_{02}f_{12}-a_{12}f_{02} \\ g_2=a_{12}X+b_{12}Y+c_{12}Z+f_{12} \\ g_3=a_{02}X+b_{02}Y+c_{02}Z+f_{02}. \end{array} \tag{3-66}$$

$$Y=\frac{\{(a_{12}c_{02}-a_{02}c_{12})Z+a_{12}f_{02}-a_{02}f_{12}\}}{(a_{02}b_{12}-a_{12}b_{02})} \tag{3-67}$$

$$X=\frac{-(b_{12}Y+c_{12}Z+f_{12})}{a_{12}}\ or\ X=\frac{\{(b_{02}c_{12}-b_{12}c_{02})Z+b_{02}f_{12}-b_{12}f_{02}\}}{(a_{02}b_{12}-a_{12}b_{02})} \tag{3-68}$$

Box 3-6 (*Multipolynomial resultants approach*):

Step 1: Solve for X in terms of Z

$$\begin{array}{l} f_1 := (a_{02}X + c_{02}Z + f_{02})W + b_{02}Y \\ f_2 := (a_{12}X + c_{12}Z + f_{12})W + b_{12}Y \end{array} \quad (3\text{-}69)$$

Step 2: Obtain the sylvester resultant

$$J_X = det \begin{bmatrix} \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial W} \\ \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial W} \end{bmatrix} = det \begin{bmatrix} b_{02} & (a_{02}X + c_{02}Z + f_{02}) \\ b_{12} & (a_{12}X + c_{12}Z + f_{12}) \end{bmatrix} \quad (3\text{-}70)$$

$$J_X = b_{02}a_{12}X + b_{02}c_{12}Z + b_{02}f_{12} - b_{12}a_{02}X - b_{12}c_{02}Z - b_{12}f_{02} \quad (3\text{-}71)$$

from (3-71)

$$X = \frac{\{(b_{12}c_{02} - b_{02}c_{12})Z + b_{12}f_{02} - b_{02}f_{12}\}}{(b_{02}a_{12} - b_{12}a_{02})} \quad (3\text{-}72)$$

Step 3: Solve for Y in terms of Z

$$\begin{array}{l} f_3 := (b_{02}Y + c_{02}Z + f_{02})W + b_{02}X \\ f_4 := (b_{12}Y + c_{12}Z + f_{12})W + a_{12}X \end{array} \quad (3\text{-}73)$$

Step 4: Obtain the sylvester resultant

$$J_Y = det \begin{bmatrix} \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial W} \\ \frac{\partial f_4}{\partial X} & \frac{\partial f_4}{\partial W} \end{bmatrix} = det \begin{bmatrix} a_{02} & (b_{02}Y + c_{02}Z + f_{02}) \\ a_{12} & (b_{12}Y + c_{12}Z + f_{12}) \end{bmatrix} \quad (3\text{-}74)$$

$$J_Y = a_{02}b_{12}Y + a_{02}c_{12}Z + a_{02}f_{12} - a_{12}b_{02}Y - a_{12}c_{02}Z - a_{12}f_{02} \quad (3\text{-}75)$$

from (3-75)

$$Y = \frac{\{(a_{12}c_{02} - a_{02}c_{12})Z + a_{12}f_{02} - a_{02}f_{12}\}}{(a_{02}b_{12} - a_{12}b_{02})} \quad (3\text{-}76)$$

Box 3-7 (*solution of the simultaneous equation by elimination*):

$$\begin{bmatrix} 2X(X_2-X_1)+2Y(Y_2-Y_1)=a-2Z(Z_2-Z_1) \\ 2X(X_3-X_2)+2Y(Y_3-Y_2)=b-2Z(Z_3-Z_2) \end{bmatrix} \quad (3\text{-}77)$$

$$\begin{bmatrix} X=\dfrac{a(Y_3-Y_2)-b(Y_2-Y_1)}{2\{(X_2-X_1)(Y_3-Y_2)-(X_3-X_2)(Y_2-Y_1)\}}-h_1 \\ Y=\dfrac{a(X_3-X_2)-b(X_2-X_1)}{2\{(Y_2-Y_1)(X_3-X_2)-(Y_3-Y_2)(X_2-X_1)\}}-h_2 \end{bmatrix} \quad (3\text{-}78)$$

$$\begin{bmatrix} X=c-dZ \\ Y=e-fZ \end{bmatrix} \quad (3\text{-}79)$$

$$\begin{bmatrix} h_1=\dfrac{\{(Z_2-Z_1)(Y_3-Y_2)-(Z_3-Z_2)(Y_2-Y_1)\}Z}{\{(X_2-X_1)(Y_3-Y_2)-(X_3-X_2)(Y_2-Y_1)\}} \\ h_2=\dfrac{\{(Z_2-Z_1)(X_3-X_2)-(Z_3-Z_2)(X_2-X_1)\}Z}{\{(Y_2-Y_1)(X_3-X_2)-(Y_3-Y_2)(X_2-X_1)\}} \\ c=\dfrac{a(Y_3-Y_2)-b(Y_2-Y_1)}{2\{(X_2-X_1)(Y_3-Y_2)-(X_3-X_2)(Y_2-Y_1)\}} \\ d=\dfrac{\{(Z_2-Z_1)(Y_3-Y_2)-(Z_3-Z_2)(Y_2-Y_1)\}Z}{\{(X_2-X_1)(Y_3-Y_2)-(X_3-X_2)(Y_2-Y_1)\}} \\ e=\dfrac{a(X_3-X_2)-b(X_2-X_1)}{2\{(Y_2-Y_1)(X_3-X_2)-(Y_3-Y_2)(X_2-X_1)\}} \\ f=\dfrac{\{(Z_2-Z_1)(X_3-X_2)-(Z_3-Z_2)(X_2-X_1)\}Z}{\{(Y_2-Y_1)(X_3-X_2)-(Y_3-Y_2)(X_2-X_1)\}} \end{bmatrix} \quad (3\text{-}80)$$

considering $q=(dX_1+fY_1-Z_1-cd-ef)$ and substituting (3-79) in (3-58)i, one gets a quadratic equation in Z as

$$(d^2+f^2+1)Z^2+2qZ+X_1^2+Y_1^2+Z_1^2-2X_1c-2Y_1e-S_1^2+c^2+e^2=0 \quad (3\text{-}81)$$

$$\left\{ gZ^2+hZ+i=0,\ Z_{1,2}=\frac{-h\pm\sqrt{h^2-4gi}}{2g} \right. \quad (3\text{-}82)$$

$$q^2=(d^2+f^2+1)(X_1^2+Y_1^2+Z_1^2-2X_1c-2Y_1e-S_1^2+c^2+e^2) \quad (3\text{-}83)$$

where

$$\begin{bmatrix} g=d^2+f^2+1 \\ h=2q \\ i=X_1^2+Y_1^2+Z_1^2-2X_1c-2Y_1e-S_1^2+c^2+e^2 \end{bmatrix} \quad (3\text{-}84)$$

Box 3-8 (*solution of the simultaneous equation by the matrix approach*):

$$\left[\begin{array}{l} 2Y(Y_2-Y_1)+2Z(Z_2-Z_1)=a-2X(X_2-X_1) \\ 2Y(Y_3-Y_2)+2Z(Z_3-Z_2)=b-2X(X_3-X_2) \end{array}\right. \qquad (3\text{-}85)$$

$$\begin{bmatrix} Y_2-Y_1 & Z_2-Z_1 \\ Y_3-Y_2 & Z_3-Z_2 \end{bmatrix}\begin{bmatrix} Y \\ Z \end{bmatrix}=\frac{1}{2}\left\{\begin{bmatrix} a \\ b \end{bmatrix}-2\begin{bmatrix} X_2-X_1 \\ X_3-X_2 \end{bmatrix}X\right\} \qquad (3\text{-}86)$$

$$\begin{bmatrix} Y \\ Z \end{bmatrix}=\frac{1}{2}d\begin{bmatrix} Z_3-Z_2 & -(Z_2-Z_1) \\ -(Y_3-Y_2) & (Y_2-Y_1) \end{bmatrix}\left\{\begin{bmatrix} a \\ b \end{bmatrix}-2\begin{bmatrix} X_2-X_1 \\ X_3-X_2 \end{bmatrix}X\right\} \qquad (3\text{-}87)$$

with $d=\{(Y_2-Y_1)(Z_3-Z_2)-(Y_3-Y_2)(Z_2-Z_1)\}^{-1}$

$$\left[\begin{array}{l} a_{11}=Y_2-Y_1,\ a_{12}=Z_2-Z_1,\ a_{21}=Y_3-Y_2,\ a_{22}=Z_3-Z_2 \\ c_1=-(X_2-X_1),\ c_2=-(X_3-X_2),\ \ b_1=\tfrac{1}{2}a,\ \ b_2=\tfrac{1}{2}b \end{array}\right. \qquad (3\text{-}88)$$

$$\begin{bmatrix} Y \\ Z \end{bmatrix}=\{a_{11}a_{22}-a_{12}a_{21}\}^{-1}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}\left\{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}+\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}X\right\} \qquad (3\text{-}89)$$

$$\left[\begin{array}{l} Y=\{a_{11}a_{22}-a_{12}a_{21}\}^{-1}\{a_{22}(b_1+c_1X)-a_{12}(b_2+c_2X)\} \\ Z=\{a_{11}a_{22}-a_{12}a_{21}\}^{-1}\{a_{11}(b_2+c_2X)-a_{21}(b_1+c_1X)\} \end{array}\right. \qquad (3\text{-}90)$$

$$\left[\begin{array}{l} Y=e\left[\{a_{22}b_1-a_{12}b_2\}+\{a_{22}c_1-a_{12}c_2\}X\right] \\ Z=e\left[\{a_{11}b_2-a_{21}b_1\}+\{a_{11}c_2-a_{21}c_1\}X\right] \end{array}\right. \qquad (3\text{-}91)$$

$$\left[\begin{array}{l} Y=e(f+gX) \\ Z=e(h+iX) \end{array}\right. \qquad (3\text{-}92)$$

$$\left[\begin{array}{l} e=(a_{11}a_{22}-a_{12}a_{21})^{-1},\ \ f=a_{22}b_1-a_{12}b_2,\ g=a_{22}c_1-a_{12}c_2 \\ h=a_{11}b_2-a_{21}b_1,\ \ i=a_{11}c_2-a_{21}c_1,\ k=X_1^2+Y_1^2+Z_1^2 \end{array}\right. \qquad (3\text{-}93)$$

with $s=e^2fg+e^2hi-X_1-egY_1-eiZ_1$ and substituting (3-92) in (3-58) i

$$\left\{ lX^2+mX+n=0, X=\frac{-m\pm\sqrt{m^2-4ln}}{2l}\right. \qquad (3\text{-}94)$$

$$(s)^2=(e^2i^2+e^2g^2+1)(k-S_1^2-2Y_1ef+e^2f^2-2Z_1eh+e^2h^2) \qquad (3\text{-}95)$$

where

$$\left[\begin{array}{l} l=e^2i^2+e^2g^2+1 \\ m=2(s) \\ n=k-S_1^2-2Y_1ef+e^2f^2-2Z_1eh+e^2h^2 \end{array}\right. \qquad (3\text{-}96)$$

Box 3-9 (*critical configuration of the twodimensional ranging problem*):

$$\left[\begin{array}{l} f_1(X,Y;X_1,Y_1,S_1) = (X_1-X)^2 + (Y_1-Y)^2 - S_1^2 \\ f_2(X,Y;X_2,Y_2,S_2) = (X_2-X)^2 + (Y_2-Y)^2 - S_2^2 \end{array}\right. \quad (3\text{-}97)$$

$$\left[\begin{array}{l} \dfrac{\partial f_1}{\partial X} = -2(X_1-X), \quad \dfrac{\partial f_2}{\partial X} = -2(X_2-X) \\ \dfrac{\partial f_1}{\partial Y} = -2(Y_1-Y), \quad \dfrac{\partial f_2}{\partial Y} = -2(Y_2-Y) \end{array}\right. \quad (3\text{-}98)$$

$$\left[\begin{array}{l} D = \left|\dfrac{\partial f_i}{\partial X_j}\right| = 4\left|\begin{array}{cc} X_1-X & X_2-X \\ Y_1-Y & Y_2-Y \end{array}\right| \\ D \Leftrightarrow \left|\begin{array}{cc} X_1-X & X_2-X \\ Y_1-Y & Y_2-Y \end{array}\right| = \left|\begin{array}{ccc} X & Y & 1 \\ X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \end{array}\right| = 0 \end{array}\right. \quad (3\text{-}99)$$

$$\left[\begin{array}{c} \frac{1}{4}D = (X_1-X)(Y_2-Y) - (X_2-X)(Y_1-Y) \\ = X_1Y_2 - X_1Y - XY_2 + XY - X_2Y_1 + X_2Y + XY_1 - XY \\ = X(Y_1-Y_2) + Y(X_2-X_1) + X_1Y_2 - X_2Y_1 \end{array}\right. \quad (3\text{-}100)$$

thus

$$\left|\begin{array}{ccc} X & Y & 1 \\ X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \end{array}\right| = 2 \times \; Area\ of\ triangle\ P(X,Y), P_1(X_1,Y_1) and P_2(X_2,Y_2) \quad (3\text{-}101)$$

$$D = \left|\begin{array}{ccc} X & Y & 1 \\ X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \end{array}\right| \left[\begin{array}{c} a \\ b \\ c \end{array}\right] = 0 \quad (3\text{-}102)$$

results in a system of homogeneous equations

$$\left[\begin{array}{l} aX + bY + c = 0 \\ aX_1 + bY_1 + c = 0 \\ aX_2 + bY_2 + c = 0 \end{array}\right. \quad (3\text{-}103)$$

Box 3-10 (*critical configuration of the three-dimensional ranging*):

$$\begin{bmatrix} f_1(X,Y,Z;X_1,Y_1,Z_1,S_1) = (X_1-X)^2+(Y_1-Y)^2+(Z_1-Z)^2-S_1^2 \\ f_2(X,Y,Z;X_2,Y_2,Z_2,S_2) = (X_2-X)^2+(Y_2-Y)^2+(Z_2-Z)^2-S_2^2 \\ f_3(X,Y,Z;X_3,Y_3,Z_3,S_3) = (X_3-X)^2+(Y_3-Y)^2+(Z_3-Z)^2-S_3^2 \end{bmatrix} \tag{3-104}$$

$$\begin{bmatrix} \frac{\partial f_1}{\partial X} = -2(X_1-X), & \frac{\partial f_2}{\partial X} = -2(X_2-X), & \frac{\partial f_3}{\partial X} = -2(X_3-X) \\ \frac{\partial f_1}{\partial Y} = -2(Y_1-Y), & \frac{\partial f_2}{\partial Y} = -2(Y_2-Y), & \frac{\partial f_3}{\partial Y} = -2(Y_3-Y) \\ \frac{\partial f_1}{\partial Z} = -2(Z_1-Z), & \frac{\partial f_2}{\partial Z} = -2(Z_2-Z), & \frac{\partial f_3}{\partial Zv} = -2(Z_3-Z) \end{bmatrix} \tag{3-105}$$

$$\begin{bmatrix} D = \left|\frac{\partial f_i}{\partial X_j}\right| = -8 \begin{vmatrix} X_1-X & Y_1-Y & Z_1-Z \\ X_2-X & Y_2-Y & Z_1-Z \\ X_3-X & Y_3-Y & Z_1-Z \end{vmatrix} \\ D \Leftrightarrow \begin{vmatrix} X_1-X & Y_1-Y & Z_1-Z \\ X_2-X & Y_2-Y & Z_1-Z \\ X_3-X & Y_3-Y & Z_1-Z \end{vmatrix} = \begin{vmatrix} X & Y & Z & 1 \\ X_1 & Y_1 & Z_1 & 1 \\ X_2 & Y_2 & Z_2 & 1 \\ X_3 & Y_3 & Z_3 & 1 \end{vmatrix} = 0 \end{bmatrix} \tag{3-106}$$

$$\begin{bmatrix} -\frac{1}{8}D = \{-Z_1Y_3+Y_1Z_3-Y_2Z_3+Y_3Z_2-Y_1Z_2+Y_2Z_1\}X+ \\ +\{-Z_1X_2-X_1Z_3+Z_1X_3+X_1Z_2-X_3Z_2+X_2Z_3\}Y+ \\ +\{Y_1X_2-Y_1X_3+Y_3X_1-X_2Y_3-X_1Y_2+Y_2X_3\}Z+ \\ +X_1Y_2Z_3-X_1Y_3Z_2-X_3Y_2Z_1+X_2Y_3Z_1-X_2Y_1Z_3+X_3Y_1Z_2 \end{bmatrix} \tag{3-107}$$

$$\begin{vmatrix} X & Y & Z & 1 \\ X_1 & Y_1 & Z_1 & 1 \\ X_2 & Y_2 & Z_2 & 1 \\ X_3 & Y_3 & Z_3 & 1 \end{vmatrix} \tag{3-108}$$

describes six times volume of the tetrahedron formed by the points $P(X,Y,Z), P_1(X_1,Y_1,Z_1)$, $P_2(X_2,Y_2,Z_2)$, and $P_3(X_3,Y_3,Z_3)$. Therefore

$$D = \begin{vmatrix} X & Y & Z & 1 \\ X_1 & Y_1 & Z_1 & 1 \\ X_2 & Y_2 & Z_2 & 1 \\ X_3 & Y_3 & Z_3 & 1 \end{vmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0, \tag{3-109}$$

results in a system of homogeneous equations

$$\begin{bmatrix} aX+bY+cZ+d=0 \\ aX_1+bY_1+cY_1+d=0 \\ aX_2+bY_2+cZ_2+d=0 \\ aX_3+bY_3+cZ_3+d=0 \end{bmatrix} \tag{3-110}$$

(c) The orientation step
The fourth procedure presented in Section (3-13) in page (29) of Chapter 2 can then be applied following the determination of position. This step concludes the solution of three-dimensional resection problem in a closed form.

3-22 The minimum distance mapping problem

In the minimum distance mapping problem introduced in *Example* A-3 in *Appendix A.1* in page (97), the Cartesian coordinates $\{X,Y,Z\}$ of a point on the topographical surface are given. The task now is to project this point to a (reference) ellipsoid of revolution under the constrain that the projection distance be minimum. Desired are the Cartesian ellipsoidal coordinates of this projected point so as to be able to derive the Jacobi ellipsoidal coordinates (geodetic coordinates) (i.e. ellipsoidal longitude L, ellipsoidal latitute B and height H) as will be seen in Chapter 5. The solution to the problem involves solving a system of nonlinear equations of *Example* A-3 in *Appendix A.1* in page (97) either in closed form or iteratively (see *E. Grafarend* and *P. Lohse* 1991). In this section, we present the application of *B. Buchberger algorithm* to solving the same problem by computing the *Gröbner basis* of the *Ideal* formed by these equations (see Chapter 2). We begin by the equations as presented in *E. Grafarend* and *P. Lohse* (1991) *(Example* A-3 in *Appendix A.1)* as follows

$$\left[\begin{array}{l} -(X-x_1)+b^2x_1x_4=0 \\ -(Y-x_2)+b^2x_2x_4=0 \\ -(Z-x_3)+a^2x_3x_4=0 \\ b^2x_1^2+b^2x_2^2+a^2x_3^2-a^2b^2=0 \end{array}\right. \tag{3-111}$$

being the partial derivative of the constrained minimization problem

$$\begin{array}{l} 2L(x_1,x_2,x_3,x_4)=\|\mathbf{X}-\mathbf{x}\|^2+x_4\left[b^2(x_1^2+x_2^2)+a^2x_3^2-a^2b^2\right]= \\ =(X-x_1)^2+(Y-x_2)^2+(Z-x_3)^2+x_4\left[b^2(x_1^2+x_2^2)+a^2x_3^2-a^2b^2\right] \end{array} \tag{3-112}$$

e $\mathbf{X}$ is the displacement vector from the origin $\mathbf{O}$ of $\mathbb{R}^3$ to a point $\{X,Y,Z\}$ on the topographical surface along the orthonomal triad of base vectors $\{\mathbb{E}_1,\mathbb{E}_2,\mathbb{E}_3\}$ while $\mathbf{x}$ denotes the displacement vector from the origin $\mathbf{o}$ of $\mathbb{R}^3$ to a point $\{x_1,x_2,x_3\}$ on the ellipsoid of revolution along the orthonomal triad of base vectors $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ generated by

$$bx \in \mathbb{X}^3 := \left\{x \in \mathbb{R}^3 \,\middle|\, \frac{x_1^2+x_2^2}{a^2}+\frac{x_3^2}{b^2}=1\right\}. \tag{3-113}$$

The problem can now be formulated as follows: We are given the Cartesian coordinates $\{X,Y,Z\}$ of a point on the topographic surface, semimajor axis $\{a\}$ and semiminor axis $\{b\}$. If for simplicity the origins $\mathbf{O}$ and $\mathbf{o}$ are assumed to coincide i.e. $=\mathbf{o}$, $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}=\{\mathbb{E}_1,\mathbb{E}_2,\mathbb{E}_3\}$ and that the minimum of the distance $\|\mathbf{X}-\mathbf{x}\|$ is constrained by the condition of equation (3-113). The task at hand is to find the ellipsoidal Cartesian coordinates $\{x_1,x_2,x_3\}$ of the topographic point and the Lagrange factor $\{x_4\}$. We have obtained (3-111) by taking the partial derivatives of (3-112) with respect to the unknowns $\{x_1,x_2,x_3,x_4\}$. In order to solve the nonlinear system of equation for the unknowns $\{x_1,x_2,x_3,x_4\}$ using *B. Buchberger algorithm*, we write down the generators of the *Ideal* I formed by the polynomials (3-111) following the lexicographic order $\{x_1>x_2>x_3>x_4\}$ as

$$Ideal\ I = \begin{array}{l} <x_1+b^2x_1x_4-X,x_2+b^2x_2x_4-Y,x_3+a^2x_3x_4-Z, \\ b^2x_1^2+b^2x_2^2+a^2x_3^2-a^2b^2>. \end{array} \tag{3-114}$$

The *Gröbner basis* can now be computed by using either *Mathematica software* or *Maple software*. For this problem, Mathematica 2.2 for DOS 387 was used. The executable command is *GroebnerBasis[Polynomials,Variables in a specified ordering]*. In this case, the *Gröbner basis* of the *Ideal* (3-114) is computed following lexicographic ordering $\{x_1>x_2>x_3>x_4\}$ as

$$GroebnerBasis\left[\begin{array}{l} \{x_1+b^2x_1x_4-X,x_2+b^2x_2x_4-Y,x_3+a^2x_3x_4-Z, \\ b^2x_1^2+b^2x_2^2+a^2x_3^2-a^2b^2\},\{x_1,x_2,x_3,x_4\} \end{array}\right]. \tag{3-115}$$

The results of the executable command above are the computed *Gröbner basis* in *Box* (3-11) .

Box 3-11 (*Computed Gröbner basis for the Minimum Distance Mapping problem*):

1. $\left[\begin{array}{l} a^6b^6x_4^4+(2a^6b^4+2a^4b^6)x_4^3+(a^6b^2+4a^4b^4+a^2b^6-a^4b^2X^2- \\ a^4b^2Y^2-a^2b^4Z^2)x_4^2+(2a^4b^2+2a^2b^4-2a^2b^2X^2-2a^2b^2Y^2- \\ 2a^2b^2Z^2)x_4+(a^2b^2-b^2X^2-b^2Y^2-a^2Z^2). \end{array}\right.$

2. $\left[\begin{array}{l} (a^4Z-2a^2b^2Z+b^4Z)x_3-a^6b^6x_4^3-(2a^6b^4+a^4b^6)x_4^2- \\ (a^6b^2+2a^4b^4-a^4b^2X^2-a^4b^2Y^2-a^2b^4Z^2)x_4-a^4b^2+ \\ a^2b^2X^2+a^2b^2Y^2+2a^2b^2Z^2-b^4Z^2. \end{array}\right.$

3. $\left[\begin{array}{l} (2b^2Z+b^4x_4Z-a^2Z)x_3+a^4b^6x_4^3+(2a^4b^4+a^2b^6)x_4^2+ \\ (a^4b^2+2a^2b^4-a^2b^2X^2-a^2b^2Y^2-b^4Z^2)x_4+ \\ a^2b^2-b^2X^2-b^2Y^2-2b^2Z^2. \end{array}\right.$

4. $(1+a^2x_4)x_3-Z$

5. $\left[\begin{array}{l} (a^4-2a^2b^2+b^4)x_3^2+(2a^2b^2Z-2b^4Z)x_3-a^4b^6x_4^2-2a^4b^4x_4- \\ a^4b^2+a^2b^2X^2+a^2b^2Y^2+b^4Z^2). \end{array}\right.$

6. $\left[\begin{array}{l} (2b^2-a^2+b^4x_4)x_3^2-a^2Zx_3+a^4b^6x_4^3+(2a^4b^4+2a^2b^6)x_4^2+ \\ +(a^4b^2+4a^2b^4-a^2b^2X^2-a^2b^2Y^2-b^4Z^2)x_4+2a^2b^2-2b^2X^2 \\ -2b^2Y^2-2b^2Z^2. \end{array}\right.$

7. $\left[\begin{array}{l} (X^2+Y^2)x_2+a^2b^4Yx_4^2+Y(a^2b^2+b^2x_3^2-b^2Zx_3)x_4+Yx_3^2 \\ -Y^3-YZx_3-YX^2. \end{array}\right.$

8. $(1+b^2x_4)x_2-Y$

9. $(a^2x_3-b^2x_3+b^2Z)x_2-a^2x_3Y$

10. Yx_1-Xx_2

11. $Xx_1+a^2b^4x_4^2+(a^2b^2+b^2x_3^2-b^2Zx_3)x_4+x_3^2-Zx_3+Yx_2-X^2-Y^2.$

12. $(1+b^2x_4)x_1-X$

13. $(a^2x_3-b^2x_3+b^2Z)x_1-a^2Xx_3$

14. $x_1^2+a^2b^4x_4^2+(2a^2b^2+b^2x_3^2-b^2Zx_3)x_4+2x_3^2-2Zx_3+x_2^2-X^2-Y^2.$

From the computed *Gröbner basis* above, it is clearly seen that the first equation is a *univariate polynomial* of order four in x_4 which is identical to that obtained by *E. Grafarend* and *P. Lohse* (1991, p.94) and can easily be solved for the four roots using the available software such as Matlab or Maple. Once the four roots of x_4 have been obtained, they are substituted in the polynomial equations (4, 8 and 12) to obtain the unknown variables $\{x_1, x_2, x_3\}$ respectively thus concluding the solution of the *Minimum Distance Mapping* problem. In Chapter 5, we apply these computed *Gröbner basis* to solve the *Minimum Distance Mapping* problem for a real case study in order to obtain the Jacobi ellipsoidal coordinates given geocentric cartesian coordinates. In this section, we test these computed *Gröbner basis* to solve the *Minimum Distance Mapping* problem presented by *E. Grafarend* and *P. Lohse* (1991) below:

Example 3-1 (*E. Grafarend* and *P. Lohse* 1991, p.108):
Given are the geometric parameters of the ellipsoid of revolution; *semi-major axis* $\{a\}$ = 6378137.000m and *first numerical eccentricity* $e^2 = 0.00669437999013$ from which the *semi-minor axis* $\{b\}$ can be computed. The Input are Cartesian coordinates of 8 points on the surface of the earth presented in *Table* (3.1). From the computed *Gröbner*

Table 3.1: Cartesian coordinates of topographic points

Point	$X(m)$	$Y(m)$	$Z(m)$
1	3980192.960	0	4967325.285
2	0	0	6356852.314
3	0	0	-6357252.314
4	4423689.486	529842.355	4555616.169
5	4157619.145	664852.698	4775310.888
6	-2125699.324	6012793.226	-91773.648
7	5069470.828	3878707.846	-55331.828
8	213750.930	5641092.098	2977743.624

basis in *Box* (3-11), we have the first polynomial equation as a *univariate polynomial* equation given as

$$\left[\begin{array}{l} c_4x_4^4+c_3x_4^3+c_2x_4^2+c_1x_4+c_o=0 \\ c_4=a^6b^6 \\ c_3=(2a^6b^4+2a^4b^6) \\ c_2=(a^6b^2+4a^4b^4+a^2b^6-a^4b^2X^2-a^4b^2Y^2-a^2b^4Z^2) \\ c_1=(2a^4b^2+2a^2b^4-2a^2b^2X^2-2a^2b^2Y^2-2a^2b^2Z^2) \\ c_o=(a^2b^2-b^2X^2-b^2Y^2-a^2Z^2). \end{array}\right. \tag{3-116}$$

We then proceed to compute the coefficients of the given *univariate polynomial* (3-116) using the input data of *Table* (3.1). The computed coefficients are as given in *Table* (3.2) . With the computed coefficients, the poly-

Table 3.2: Computed polynomial coefficients

Point	c_0	c_1	c_2	c_3	c_4
1	-7.7479e+22	1.3339e+41	1.3515e+55	4.3824e+68	4.4420e+81
2	-5.1720e+22	1.3374e+41	1.3529e+55	4.3824e+68	4.4420e+81
3	-2.5861e+23	1.3372e+41	1.3529e+55	4.3824e+68	4.4420e+81
4	-2.5311e+24	1.3310e+41	1.3507e+55	4.3824e+68	4.4420e+81
5	-1.8076e+23	1.3334e+41	1.3513e+55	4.3824e+68	4.4420e+81
6	-5.1549e+21	1.3285e+41	1.3493e+55	4.3824e+68	4.4420e+81
7	-2.6815e+24	1.3263e+41	1.3488e+55	4.3824e+68	4.4420e+81
8	-4.5942e+24	1.3267e+41	1.3493e+55	4.3824e+68	4.4420e+81

nomial roots can be computed in Matlab by the *roots* command *(D. Hanselman* and *B. Littlefield* 1997, p.146) as $x_4 = roots\left[\begin{array}{ccccc} c_4 & c_3 & c_2 & c_1 & c_0 \end{array}\right]$. The obtained roots are then substituted in the polynomials (4, 8, and 12) of the computed *Gröbner basis*

$$\left[\begin{array}{l} (1+a^2x_4)x_3-Z \\ (1+b^2x_4)x_2-Y \\ (1+b^2x_4)x_1-X \end{array}\right. \tag{3-117}$$

to give the values of $\{x_3, x_2, x_1\}$ respectively. The computed results presented in Table (3.3) are identical to those obtained by *E. Grafarend* and *P. Lohse* (1991, Table 4, p.108). Once the ellipsoidal Cartesian coordinates have been derived, the Jacobi ellipsoidal coordinates (ellipsoidal longitude L, ellipsoidal latitute B and height H) can be computed as in *Box* (5-1) of Chapter 5.

3-23 GPS positioning with observations of type pseudo-ranges

3-231 The pseudo-ranging four-point problem in GPS positioning

E. Grafarend and *J. Shan* (1996) have defined the *GPS pseudo-ranging four-point problem* ("pseudo 4P") as the problem of determining the four unknowns comprising the *three components of the receiver position* and the stationary receiver *range bias* from four observed pseudo-ranges to four satellite transmitter of given geocentric position. Geometrically, the four unknowns are obtained from the intersection of four spherical cones given by the pseudo-ranging

Table 3.3: Computed ellipsoidal cartesian coordinates and the Lagrange factor

Point	$x_1(m)$	$x_2(m)$	$x_3(m)$	$x_4(m^{-2})$
1	3980099.549	0.000	4967207.921	5.808116e-019
2	0.000	0.000	6356752.314	3.867016e-019
3	0.000	0.000	-6356752.314	1.933512e-018
4	4420299.446	529436.317	4552101.519	1.897940e-017
5	4157391.441	664816.285	4775047.592	1.355437e-018
6	-2125695.991	6012783.798	-91773.503	3.880221e-020
7	5065341.132	3875548.170	-55286.450	2.017617e-017
8	213453.298	5633237.315	2973569.442	3.450687e-017

equations. Several procedures have been put forward for obtaining closed form solution of the problem. Amongst the procedures include the vectorial approach evidenced in the works of *S. Bancroft* (1985), *P. Singer et al.* (1993), *H. Lichtenegger* (1995) and *A. Kleusberg* (1994,1999). *E. Grafarend* and *J. Shan* (1996) propose two approaches.

One approach is based on the inversion of a 3×3 coefficient matrix of a linear system formed by differencing of the *nonlinear pseudo-ranging equations* in geocentric coordinates, while the other approach uses the coefficient matrix from the linear system to solve the same equations in barycentric coordinates. In this section we present both the approaches of *Gröbner bases* and *Multipolynomial resultants* (*B. Sturmfel* 1998 approach) to solve the same problem. We demonstrate our algorithms by solving the *Pseudo-ranging four-point problem* already solved by *A. Kleusberg* (1994) and *E. Grafarend* and *J. Shan* (1996). It will be illustrated that both the *Gröbner bases* and the *Multipolynomial resultant* solve the same linear equations as those of *E. Grafarend* and *J. Shan* (1996) and lead to identical results (see also *J. L. Awange* and *E. Grafarend* 2002). We start with the pseudo-ranging equations written algebraically as

$$\left[\begin{array}{l} (x_1-a_0)^2+(x_2-b_0)^2+(x_3-c_0)^2-(x_4-d_0)^2=0 \\ (x_1-a_1)^2+(x_2-b_1)^2+(x_3-c_1)^2-(x_4-d_1)^2=0 \\ (x_1-a_2)^2+(x_2-b_2)^2+(x_3-c_2)^2-(x_4-d_2)^2=0 \\ (x_1-a_3)^2+(x_2-b_3)^2+(x_3-c_3)^2-(x_4-d_3)^2=0 \\ \qquad where \;\; x_1,x_2,x_3,x_4 \in \\ \qquad (a_0,b_0,c_0)=(x^0,y^0,z^0)\sim P^0 \\ \qquad (a_1,b_1,c_1)=(x^1,y^1,z^1)\sim P^1 \\ \qquad (a_2,b_2,c_2)=(x^2,y^2,z^2)\sim P^2 \\ \qquad (a_3,b_3,c_3)=(x^3,y^3,z^3)\sim P^3 \end{array}\right. \tag{3-118}$$

with $\{P^0, P^1, P^2, P^3\}$ being the position of the four GPS satellites, their ranges to the stationary receiver at P given by $\{d_0, d_1, d_2, d_3\}$. The parameters $(\{a_0,b_0,c_0\},\{a_1,b_1,c_1\},\{a_2,b_2,c_2\},\{a_3,b_3,c_3\},\{d_0,d_1,d_2,d_3\})$ are elements of the spherical cone that intersect at P to give the coordinates $\{x_1, x_2, x_3\}$ of the receiver and the stationary receiver range bias x_4 . The equations above can be expanded and arranged in the *lexicographic order* $\{x_1 > x_2 > x_3 > x_4\}$ as follows

$$\left[\begin{array}{l} x_1^2-2a_0x_1+x_2^2-2b_0x_2+x_3^2-2c_0x_3-x_4^2+2d_0x_4+a_0^2+b_0^2+c_0^2-d_0^2=0 \\ x_1^2-2a_1x_1+x_2^2-2b_1x_2+x_3^2-2c_1x_3-x_4^2+2d_1x_4+a_1^2+b_1^2+c_1^2-d_1^2=0 \\ x_1^2-2a_2x_1+x_2^2-2b_2x_2+x_3^2-2c_2x_3-x_4^2+2d_2x_4+a_2^2+b_2^2+c_2^2-d_2^2=0 \\ x_1^2-2a_3x_1+x_2^2-2b_3x_2+x_3^2-2c_3x_3-x_4^2+2d_3x_4+a_3^2+b_3^2+c_3^2-d_3^2=0 \end{array}\right. \tag{3-119}$$

with the variables $\{x_1, x_2, x_3, x_4\}$, the other terms being known constants. We re-write (3-119) above with the linear terms on one side and the nonlinear terms on the other as

$$\left[\begin{array}{l} x_1^2+x_2^2+x_3^2-x_4^2=2a_0x_1+2b_0x_2+2c_0x_3-2d_0x_4+d_0^2-a_0^2-b_0^2-c_0^2 \\ x_1^2+x_2^2+x_3^2-x_4^2=2a_1x_1+2b_1x_2+2c_1x_3-2d_1x_4+d_1^2-a_1^2-b_1^2-c_1^2 \\ x_1^2+x_2^2+x_3^2-x_4^2=2a_2x_1+2b_2x_2+2c_2x_3-2d_2x_4+d_2^2-a_2^2-b_2^2-c_2^2 \\ x_1^2+x_2^2+x_3^2-x_4^2=2a_3x_1+2b_3x_2+2c_3x_3-2d_3x_4+d_3^2-a_3^2-b_3^2-c_3^2. \end{array}\right. \tag{3-120}$$

On subtracting (3-120iv) from (3-120i), (3-120ii), and (3-120iii) we obtain

$$\begin{array}{l} f_{03} := a_{03}x_1+b_{03}x_2+c_{03}x_3+d_{30}x_4+e_{03}=0 \\ f_{13} := a_{13}x_1+b_{13}x_2+c_{13}x_3+d_{31}x_4+e_{13}=0 \\ f_{23} := a_{23}x_1+b_{23}x_2+c_3x_3+d_{32}x_4+e_{23}=0 \end{array} \tag{3-121}$$

with:

$a_{03}=2(a_0-a_3),\ b_{03}=2(b_0-b_3),\ c_{03}=2(c_0-c_3),\ d_{30}=2(d_3-d_0),$
$a_{13}=2(a_1-a_3),\ b_{13}=2(b_1-b_3),\ c_{13}=2(c_1-c_3),\ d_{31}=2(d_3-d_1),$
$a_{23}=2(a_2-a_3),\ b_{23}=2(b_2-b_3),\ c_{23}=2(c_2-c_3),\ d_{32}=2(d_3-d_2),$
$e_{03}=(d_0^2-a_0^2-b_0^2-c_0^2)-(d_3^2-a_3^2-b_3^2-c_3^2),$
$e_{13}=(d_1^2-a_1^2-b_1^2-c_1^2)-(d_3^2-a_3^2-b_3^2-c_3^2),$
$e_{23}=(d_2^2-a_2^2-b_2^2-c_2^2)-(d_3^2-a_3^2-b_3^2-c_3^2).$

We note immediately that (3-121) comprises three equations which are linear with four unknowns. Treating the unknown variable x_4 as a constant, we apply the *Gröbner bases* and the *Multipolynomial resultant* techniques to solve the linear system of equation for $x_1=g(x_4), x_2=g(x_4), x_3=g(x_4)$, where $g(x_4)$ is a linear function.

Approach 1 (*B. Sturmfels* 1998 *Multipolynomial resultants approach*):
Depending on which variable we want to solve, we can re-write equation (3-121) such that this particular variable is hidden (i.e. is treated as a constant). If we are interested in solving $x_1=g(x_4)$ for instant, we write (3-121) by hiding x_1 as

$$\begin{aligned} f_1 &:= (a_{03}x_1+d_{30}x_4+e_{03})x_5+b_{03}x_2+c_{03}x_3\\ f_2 &:= (a_{13}x_1+d_{31}x_4+e_{13})x_5+b_{13}x_2+c_{13}x_3\\ f_3 &:= (a_{23}x_1+d_{32}x_4+e_{13})x_5+b_{23}x_2+c_{23}x_3 \end{aligned} \tag{3-122}$$

with x_5 being a homogenizing factor. The Jacobian determinant of (3-122) then becomes

$$J_{x_1}=det\begin{bmatrix} \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_5}\\ \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_5}\\ \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_5} \end{bmatrix} = det\begin{bmatrix} b_{03} & c_{03} & a_{03}x_1+d_{30}x_4+e_{03}\\ b_{13} & c_{13} & a_{13}x_1+d_{31}x_4+e_{13}\\ b_{23} & c_{23} & a_{23}x_1+d_{32}x_4+e_{23} \end{bmatrix}. \tag{3-123}$$

The determinant obtained in (3-123) gives the variable $x_1=g(x_4)$ as

$$x_1=-(e_{03}b_{13}c_{23}+d_{32}x_4b_{03}c_{13}+d_{30}x_4b_{13}c_{23}-d_{30}x_4c_{13}b_{23}-d_{31}x_4b_{03}c_{23}-e_{03}c_{13}b_{23}- e_{13}b_{03}c_{23}+e_{13}c_{03}b_{23}+e_{23}b_{03}c_{13}+d_{31}x_4c_{03}b_{23}-d_{32}x_4c_{03}b_{13}-e_{23}c_{03}b_{13})/(a_{23}c_{13}b_{03}+ a_{13}b_{23}c_{03}-a_{13}c_{23}b_{03}-a_{23}b_{13}c_{03}-a_{03}c_{13}b_{23}+a_{03}c_{23}b_{13}).$$

For $x_2=g(x_4)$ we have

$$\begin{aligned} f_4 &:= (b_{03}x_2+d_{30}x_4+e_{03})x_5+a_{03}x_1+c_{03}x_3\\ f_5 &:= (b_{13}x_2+d_{31}x_4+e_{13})x_5+a_{13}x_1+c_{13}x_3\\ f_6 &:= (b_{23}x_2+d_{32}x_4+e_{23})x_5+a_{23}x_1+c_{23}x_3 \end{aligned} \tag{3-124}$$

whose Jacobian determinant is given by

$$J_{x_2}=det\begin{bmatrix} \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_5}\\ \frac{\partial f_5}{\partial x_1} & \frac{\partial f_5}{\partial x_3} & \frac{\partial f_5}{\partial x_5}\\ \frac{\partial f_6}{\partial x_1} & \frac{\partial f_6}{\partial x_3} & \frac{\partial f_6}{\partial x_5} \end{bmatrix} = det\begin{bmatrix} a_{03} & c_{03} & b_{03}x_2+d_{30}x_4+e_{03}\\ a_{13} & c_{13} & b_{13}x_2+d_{31}x_4+e_{13}\\ a_{23} & c_{23} & b_{23}x_2+d_{32}x_4+e_{23} \end{bmatrix}. \tag{3-125}$$

The determinant obtained in (3-125) gives the variable $x_2=g(x_4)$ as

$$x_2=-(a_{23}c_{13}d_{30}x_4+a_{03}c_{23}d_{31}x_4+a_{03}c_{23}e_{13}-a_{23}c_{03}d_{31}x_4-a_{03}c_{13}d_{32}x_4-a_{03}c_{13}e_{23}+ a_{13}c_{03}d_{32}x_4-a_{13}c_{23}d_{30}x_4-a_{13}c_{23}e_{03}-a_{23}c_{03}e_{13}+a_{23}c_{13}e_{03}+a_{13}c_{03}e_{23})/(a_{23}c_{13}b_{03}+ a_{13}b_{23}c_{03}-a_{13}c_{23}b_{03}-a_{23}b_{13}c_{03}-a_{03}c_{13}b_{23}+a_{03}c_{23}b_{13}).$$

For $x_3 = g(x_4)$ we have

$$\begin{array}{l} f_7 := (c_{03}x_3 + d_{30}x_4 + e_{03})x_5 + a_{03}x_1 + b_{03}x_2 \\ f_8 := (c_{13}x_3 + d_{31}x_4 + e_{13})x_5 + a_{13}x_1 + b_{13}x_2 \\ f_9 := (c_{23}x_3 + d_{32}x_4 + e_{23})x_5 + a_{23}x_1 + b_{23}x_2 \end{array} \quad (3\text{-}126)$$

whose Jacobian determinant is given by

$$J_{x_3} = det \begin{bmatrix} \frac{\partial f_7}{\partial x_1} & \frac{\partial f_7}{\partial x_2} & \frac{\partial f_7}{\partial x_5} \\ \frac{\partial f_8}{\partial x_1} & \frac{\partial f_8}{\partial x_2} & \frac{\partial f_8}{\partial x_5} \\ \frac{\partial f_9}{\partial x_1} & \frac{\partial f_9}{\partial x_2} & \frac{\partial f_9}{\partial x_5} \end{bmatrix} = det \begin{bmatrix} a_{03} & b_{03} & c_{03}x_3 + d_{30}x_4 + e_{03} \\ a_{13} & b_{13} & c_{13}x_3 + d_{31}x_4 + e_{13} \\ a_{23} & b_{23} & c_{23}x_3 + d_{32}x_4 + e_{23} \end{bmatrix}. \quad (3\text{-}127)$$

The determinant obtained in (3-123) gives the variable $x_3 = g(x_4)$ as

$$x_3 = -(a_{23}b_{03}d_{31}x_4 + a_{03}b_{13}d_{32}x_4 + a_{03}b_{13}e_{23} - a_{23}b_{13}d_{30}x_4 - a_{03}b_{23}d_{31}x_4 - a_{03}b_{23}e_{13} + a_{13}b_{23}d_{30}x_4 - a_{13}b_{03}d_{32}x_4 - a_{13}b_{03}e_{23} - a_{23}b_{13}e_{03} + a_{23}b_{03}e_{13} + a_{13}b_{23}e_{03})/(a_{23}b_{03}c_{13} + a_{13}b_{23}c_{03} - a_{13}b_{03}c_{23} - a_{23}b_{13}c_{03} - a_{03}b_{23}c_{13} + a_{03}b_{13}c_{23}).$$

On substituting the obtained values of $x_1 = g(x_4), x_2 = g(x_4)$ and $x_3 = g(x_4)$ in (3-119i) we a get a quadratic function in x_4 given in *Box* (3-12).

Box 3-12 (*Univariate (quadratic) polynomial obtained from Multipolymomial resultants solution of GPS pseudo-range equations*):

$$((-a_{23}b_{03}d_{31} + a_{23}b_{13}d_{30} + a_{03}b_{23}d_{31} - a_{03}b_{13}d_{32} - a_{13}b_{23}d_{30} + a_{13}b_{03}d_{32})^2/(a_{23}c_{13}b_{03} + a_{13}b_{23}c_{03} - a_{13}c_{23}b_{03} - a_{23}b_{13}c_{03} - a_{03}c_{13}b_{23} + a_{03}c_{23}b_{13})^2 + (-a_{23}c_{13}d_{30} + a_{23}c_{03}d_{31} + a_{03}c_{13}d_{32} - a_{03}c_{23}d_{31} - a_{13}c_{03}d_{32} + a_{13}c_{23}d_{30})^2/(a_{23}c_{13}b_{03} + a_{13}b_{23}c_{03} - a_{13}c_{23}b_{03} - a_{23}b_{13}c_{03} - a_{03}c_{13}b_{23} + a_{03}c_{23}b_{13})^2 + (-d_{32}b_{03}c_{13} - d_{30}b_{13}c_{23} + d_{31}b_{03}c_{23} - d_{31}c_{03}b_{23} + d_{32}c_{03}b_{13} + d_{30}c_{13}b_{23})^2/(a_{23}c_{13}b_{03} + a_{13}b_{23}c_{03} - a_{13}c_{23}b_{03} - a_{23}b_{13}c_{03} - a_{03}c_{13}b_{23} + a_{03}c_{23}b_{13})^2 - 1)x_4^2 +$$

$$+ (2((a_{13}c_{23}e_{03} + a_{03}c_{13}e_{23} - a_{03}c_{23}e_{13} - a_{13}c_{03}e_{23} + a_{23}c_{03}e_{13} - a_{23}c_{13}e_{03})/(a_{23}c_{13}b_{03} + a_{13}b_{23}c_{03} - a_{13}c_{23}b_{03} - a_{23}b_{13}c_{03} - a_{03}c_{13}b_{23} + a_{03}c_{23}b_{13}) - b_0)(-a_{23}c_{13}d_{30} + a_{23}c_{03}d_{31} + a_{03}c_{13}d_{32} - a_{03}c_{23}d_{31} - a_{13}c_{03}d_{32} + a_{13}c_{23}d_{30})/(a_{23}c_{13}b_{03} + a_{13}b_{23}c_{03} - a_{13}c_{23}b_{03} - a_{23}b_{13}c_{03} - a_{03}c_{13}b_{23} + a_{03}c_{23}b_{13}) + 2((-e_{03}b_{13}c_{23} + e_{03}c_{13}b_{23} + e_{13}b_{03}c_{23} - e_{13}c_{03}b_{23} - e_{23}b_{03}c_{13} + e_{23}c_{03}b_{13})/(a_{23}c_{13}b_{03} + a_{13}b_{23}c_{03} - a_{13}c_{23}b_{03} - a_{23}b_{13}c_{03} - a_{03}c_{13}b_{23} + a_{03}c_{23}b_{13}) - a_0)(-d_{32}b_{03}c_{13} - d_{30}b_{13}c_{23} + d_{31}b_{03}c_{23} - d_{31}c_{03}b_{23} + d_{32}c_{03}b_{13} + d_{30}c_{13}b_{23})/(a_{23}c_{13}b_{03} + a_{13}b_{23}c_{03} - a_{13}c_{23}b_{03} - a_{23}b_{13}c_{03} - a_{03}c_{13}b_{23} + a_{03}c_{23}b_{13}) + 2d_0 + 2((a_{13}b_{03}e_{23} + a_{03}b_{23}e_{13} - a_{03}b_{13}e_{23} - a_{13}b_{23}e_{03} + a_{23}b_{13}e_{03} - a_{23}b_{03}e_{13})/(a_{23}c_{13}b_{03} + a_{13}b_{23}c_{03} - a_{13}c_{23}b_{03} - a_{23}b_{13}c_{03} - a_{03}c_{13}b_{23} + a_{03}c_{23}b_{13}) - c_0)(-a_{23}b_{03}d_{31} + a_{23}b_{13}d_{30} + a_{03}b_{23}d_{31} - a_{03}b_{13}d_{32} - a_{13}b_{23}d_{30} + a_{13}b_{03}d_{32})/(a_{23}c_{13}b_{03} + a_{13}b_{23}c_{03} - a_{13}c_{23}b_{03} - a_{23}b_{13}c_{03} - a_{03}c_{13}b_{23} + a_{03}c_{23}b_{13}))x_4 +$$

$$+ ((-e_{03}b_{13}c_{23} + e_{03}c_{13}b_{23} + e_{13}b_{03}c_{23} - e_{13}c_{03}b_{23} - e_{23}b_{03}c_{13} + e_{23}c_{03}b_{13})/(a_{23}c_{13}b_{03} + a_{13}b_{23}c_{03} - a_{13}c_{23}b_{03} - a_{23}b_{13}c_{03} - a_{03}c_{13}b_{23} + a_{03}c_{23}b_{13}) - a_0)^2 - d_0^2 + ((a_{13}b_{03}e_{23} + a_{03}b_{23}e_{13} - a_{03}b_{13}e_{23} - a_{13}b_{23}e_{03} + a_{23}b_{13}e_{03} - a_{23}b_{03}e_{13})/(a_{23}c_{13}b_{03} + a_{13}b_{23}c_{03} - a_{13}c_{23}b_{03} - a_{23}b_{13}c_{03} - a_{03}c_{13}b_{23} + a_{03}c_{23}b_{13}) - c_0)^2 + ((a_{13}c_{23}e_{03} + a_{03}c_{13}e_{23} - a_{03}c_{23}e_{13} - a_{13}c_{03}e_{23} + a_{23}c_{03}e_{13} - a_{23}c_{13}e_{03})/(a_{23}c_{13}b_{03} + a_{13}b_{23}c_{03} - a_{13}c_{23}b_{03} - a_{23}b_{13}c_{03} - a_{03}c_{13}b_{23} + a_{03}c_{23}b_{13}) - b_0)^2$$

Approach 2 (*Gröbner bases* technique):

Using the *Gröbner bases* approach, we compute the *Gröbner basis* of the linear system of equation (3-121) as follows:

$$GroebnerBasis \begin{bmatrix} \{a_{03}x_1 + b_{03}x_2 + c_{03}x_3 + d_{30}x_4 + e_{03}, \\ a_{13}x_1 + b_{13}x_2 + c_{13}x_3 + d_{31}x_4 + e_{13}, \\ a_{23}x_1 + b_{23}x_2 + c_3x_3 + d_{32}x_4 + e_{23}\}, \{x_1, x_2, x_3, x_4\} \end{bmatrix} \quad (3\text{-}128)$$

giving the computed *Groebner basis* as in Box (3-13).

Box 3-13 (*computed Gröbner basis*):

$g_1 := (-a_{23})b_{13}e_{03} + a_{13}b_{23}e_{03} + a_{23}b_{03}e_{13} - a_{03}b_{23}e_{13} - a_{13}b_{03}e_{23} + a_{03}b_{13}e_{23} - a_{23}b_{13}c_{03}x_3 + a_{13}b_{23}c_{03}x_3 + a_{23}b_{03}c_{13}x_3 - a_{03}b_{23}c_{13}x_3 - a_{13}b_{03}c_{23}x_3 + a_{03}b_{13}c_{23}x_3 - a_{23}b_{13}d_{30}x_4 + a_{13}b_{23}d_{30}x_4 + a_{23}b_{03}d_{31}x_4 - a_{03}b_{23}d_{31}x_4 - a_{13}b_{03}d_{32}x_4 + a_{03}b_{13}d_{32}x_4$

$g_2 := (-a_{23})e_{13} + a_{13}e_{23} - a_{23}b_{13}x_2 + a_{13}b_{23}x_2 - a_{23}c_{13}x_3 + a_{13}c_{23}x_3 - a_{23}d_{31}x_4 + a_{13}d_{32}x_4$

$g_3 := (-a_{23})e_{03} + a_{03}e_{23} - a_{23}b_{03}x_2 + a_{03}b_{23}x_2 - a_{23}c_{03}x_3 + a_{03}c_{23}x_3 - a_{23}d_{30}x_4 + a_{03}d_{32}x_4$

$g_4 := (-a_{13})e_{03} + a_{03}e_{13} - a_{13}b_{03}x_2 + a_{03}b_{13}x_2 - a_{13}c_{03}x_3 + a_{03}c_{13}x_3 - a_{13}d_{30}x_4 + a_{03}d_{31}x_4$

$g_5 := e_{23} + a_{23}x_1 + b_{23}x_2 + c_{23}x_3 + d_{32}x_4$

$g_6 := e_{13} + a_{13}x_1 + b_{13}x_2 + c_{13}x_3 + d_{31}x_4$

$g_7 := e_{03} + a_{03}x_1 + b_{03}x_2 + c_{03}x_3 + d_{30}x_4.$

We notice from the computed *Groebner basis* in *Box* (3-2) that g_1 is a polynomial with only x_3 and x_4 as variables. With g_1 expressed as $x_3 = g(x_4)$, it is substituted in g_2 to obtain $x_2 = g(x_4)$, which together with $x_3 = g(x_4)$ are substituted in g_5 to give $x_1 = g(x_4)$. On substituting the obtained values of $x_1 = g(x_4), x_2 = g(x_4)$ and $x_3 = g(x_4)$ in (3-119i) we get a quadratic equation in x_4 given in *Box* (3-14) as

Box 3-14 (*Univariate (quadratic) polynomial obtained from Gröbner basis solution of GPS pseudo-range equations*):

$(-a_{23}b_{13}d_{30} + a_{13}b_{23}d_{30} + a_{23}b_{03}d_{31} - a_{03}b_{23}d_{31} - a_{13}b_{03}d_{32} + a_{03}b_{13}d_{32})^2/(a_{23}b_{13}c_{03} -$
$a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} + a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23})^2 + (-a_{23}c_{13}(-a_{23}b_{13}d_{30} + a_{13}b_{23}d_{30} +$
$a_{23}b_{03}d_{31} - a_{03}b_{23}d_{31} - a_{13}b_{03}d_{32} + a_{03}b_{13}d_{32})/(a_{23}b_{13}c_{03} - a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} +$
$a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}) + a_{13}c_{23}(-a_{23}b_{13}d_{30} + a_{13}b_{23}d_{30} + a_{23}b_{03}d_{31} - a_{03}b_{23}d_{31} - a_{13}b_{03}d_{32} +$
$a_{03}b_{13}d_{32})/(a_{23}b_{13}c_{03} - a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} + a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}) - a_{23}d_{31} +$
$a_{13}d_{32})^2/(a_{23}b_{13} - a_{13}b_{23})^2 + (b_{23}(-a_{23}c_{13}(-a_{23}b_{13}d_{30} + a_{13}b_{23}d_{30} + a_{23}b_{03}d_{31} - a_{03}b_{23}d_{31} -$
$a_{13}b_{03}d_{32} + a_{03}b_{13}d_{32})/(a_{23}b_{13}c_{03} - a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} + a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}) +$
$a_{13}c_{23}(-a_{23}b_{13}d_{30} + a_{13}b_{23}d_{30} + a_{23}b_{03}d_{31} - a_{03}b_{23}d_{31} - a_{13}b_{03}d_{32} + a_{03}b_{13}d_{32})/(a_{23}b_{13}c_{03} -$
$a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} + a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}) - a_{23}d_{31} + a_{13}d_{32})/(a_{23}b_{13} - a_{13}b_{23}) +$
$c_{23}(-a_{23}b_{13}d_{30} + a_{13}b_{23}d_{30} + a_{23}b_{03}d_{31} - a_{03}b_{23}d_{31} - a_{13}b_{03}d_{32} + a_{03}b_{13}d_{32})/(a_{23}b_{13}c_{03} - a_{13}b_{23}c_{03} -$
$a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} + a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}) + d_{32})^2/a_{23}^2 - 1)x_4^2+$

$+(2((-a_{23}e_{13} + a_{13}e_{23} - a_{23}c_{13}(-a_{23}b_{13}e_{03} + a_{13}b_{23}e_{03} + a_{23}b_{03}e_{13} - a_{03}b_{23}e_{13} - a_{13}b_{03}e_{23} +$
$a_{03}b_{13}e_{23})/(a_{23}b_{13}c_{03} - a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} + a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}) +$
$a_{13}c_{23}(-a_{23}b_{13}e_{03} + a_{13}b_{23}e_{03} + a_{23}b_{03}e_{13} - a_{03}b_{23}e_{13} - a_{13}b_{03}e_{23} + a_{03}b_{13}e_{23})/(a_{23}b_{13}c_{03} -$
$a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} + a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}))/(a_{23}b_{13} - a_{13}b_{23}) -$
$b_0)(-a_{23}c_{13}(-a_{23}b_{13}d_{30} + a_{13}b_{23}d_{30} + a_{23}b_{03}d_{31} - a_{03}b_{23}d_{31} - a_{13}b_{03}d_{32} + a_{03}b_{13}d_{32})/(a_{23}b_{13}c_{03} -$
$a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} + a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}) + a_{13}c_{23}(-a_{23}b_{13}d_{30} + a_{13}b_{23}d_{30} +$
$a_{23}b_{03}d_{31} - a_{03}b_{23}d_{31} - a_{13}b_{03}d_{32} + a_{03}b_{13}d_{32})/(a_{23}b_{13}c_{03} - a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} +$
$a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}) - a_{23}d_{31} + a_{13}d_{32})/(a_{23}b_{13} - a_{13}b_{23}) - 2(-(e_{23} + b_{23}(-a_{23}e_{13} + a_{13}e_{23} -$
$a_{23}c_{13}(-a_{23}b_{13}e_{03} + a_{13}b_{23}e_{03} + a_{23}b_{03}e_{13} - a_{03}b_{23}e_{13} - a_{13}b_{03}e_{23} + a_{03}b_{13}e_{23})/(a_{23}b_{13}c_{03} -$
$a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} + a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}) + a_{13}c_{23}(-a_{23}b_{13}e_{03} + a_{13}b_{23}e_{03} +$
$a_{23}b_{03}e_{13} - a_{03}b_{23}e_{13} - a_{13}b_{03}e_{23} + a_{03}b_{13}e_{23})/(a_{23}b_{13}c_{03} - a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} +$
$a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}))/(a_{23}b_{13} - a_{13}b_{23}) + c_{23}(-a_{23}b_{13}e_{03} + a_{13}b_{23}e_{03} + a_{23}b_{03}e_{13} -$
$a_{03}b_{23}e_{13} - a_{13}b_{03}e_{23} + a_{03}b_{13}e_{23})/(a_{23}b_{13}c_{03} - a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} + a_{13}b_{03}c_{23} -$
$a_{03}b_{13}c_{23}))/a_{23} - a_0)(b_{23}(-a_{23}c_{13}(-a_{23}b_{13}d_{30} + a_{13}b_{23}d_{30} + a_{23}b_{03}d_{31} - a_{03}b_{23}d_{31} - a_{13}b_{03}d_{32} +$
$a_{03}b_{13}d_{32})/(a_{23}b_{13}c_{03} - a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} + a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}) +$
$a_{13}c_{23}(-a_{23}b_{13}d_{30} + a_{13}b_{23}d_{30} + a_{23}b_{03}d_{31} - a_{03}b_{23}d_{31} - a_{13}b_{03}d_{32} + a_{03}b_{13}d_{32})/(a_{23}b_{13}c_{03} -$
$a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} + a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}) - a_{23}d_{31} + a_{13}d_{32})/(a_{23}b_{13} - a_{13}b_{23}) +$
$c_{23}(-a_{23}b_{13}d_{30} + a_{13}b_{23}d_{30} + a_{23}b_{03}d_{31} - a_{03}b_{23}d_{31} - a_{13}b_{03}d_{32} + a_{03}b_{13}d_{32})/(a_{23}b_{13}c_{03} -$
$a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} + a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}) + d_{32})/a_{23} + 2d_0 + 2((-a_{23}b_{13}e_{03} +$
$a_{13}b_{23}e_{03} + a_{23}b_{03}e_{13} - a_{03}b_{23}e_{13} - a_{13}b_{03}e_{23} + a_{03}b_{13}e_{23})/(a_{23}b_{13}c_{03} - a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} +$
$a_{03}b_{23}c_{13} + a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}) - c_0)(-a_{23}b_{13}d_{30} + a_{13}b_{23}d_{30} + a_{23}b_{03}d_{31} - a_{03}b_{23}d_{31} -$
$a_{13}b_{03}d_{32} + a_{03}b_{13}d_{32})/(a_{23}b_{13}c_{03} - a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} + a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}))x_4+$

Box 3-14 (*Univariate (quadratic) polynomial obtained from Gröbner basis solution of GPS pseudo-range equations* continued):

$+(-(e_{23} + b_{23}(-a_{23}e_{13} + a_{13}e_{23} - a_{23}c_{13}(-a_{23}b_{13}e_{03} + a_{13}b_{23}e_{03} + a_{23}b_{03}e_{13} - a_{03}b_{23}e_{13} - a_{13}b_{03}e_{23} + a_{03}b_{13}e_{23})/(a_{23}b_{13}c_{03} - a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} + a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}) + a_{13}c_{23}(-a_{23}b_{13}e_{03} + a_{13}b_{23}e_{03} + a_{23}b_{03}e_{13} - a_{03}b_{23}e_{13} - a_{13}b_{03}e_{23} + a_{03}b_{13}e_{23})/(a_{23}b_{13}c_{03} - a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} + a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}))/(a_{23}b_{13} - a_{13}b_{23}) + c_{23}(-a_{23}b_{13}e_{03} + a_{13}b_{23}e_{03} + a_{23}b_{03}e_{13} - a_{03}b_{23}e_{13} - a_{13}b_{03}e_{23} + a_{03}b_{13}e_{23})/(a_{23}b_{13}c_{03} - a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} + a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}))/a_{23} - a_0)^2 - d_0^2 + ((-a_{23}b_{13}e_{03} + a_{13}b_{23}e_{03} + a_{23}b_{03}e_{13} - a_{03}b_{23}e_{13} - a_{13}b_{03}e_{23} + a_{03}b_{13}e_{23})/(a_{23}b_{13}c_{03} - a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} + a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}) - c_0)^2 + ((-a_{23}e_{13} + a_{13}e_{23} - a_{23}c_{13}(-a_{23}b_{13}e_{03} + a_{13}b_{23}e_{03} + a_{23}b_{03}e_{13} - a_{03}b_{23}e_{13} - a_{13}b_{03}e_{23} + a_{03}b_{13}e_{23})/(a_{23}b_{13}c_{03} - a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} + a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}) + a_{13}c_{23}(-a_{23}b_{13}e_{03} + a_{13}b_{23}e_{03} + a_{23}b_{03}e_{13} - a_{03}b_{23}e_{13} - a_{13}b_{03}e_{23} + a_{03}b_{13}e_{23})/(a_{23}b_{13}c_{03} - a_{13}b_{23}c_{03} - a_{23}b_{03}c_{13} + a_{03}b_{23}c_{13} + a_{13}b_{03}c_{23} - a_{03}b_{13}c_{23}))/(a_{23}b_{13} - a_{13}b_{23}) - b_0)^2$

The algorithms for solving the unknown value x_4 (i.e. the range bias) from *Boxes* (3-12) or (3-14) and the respective stationary receiver coordinates are **Awange-Grafarend Groebner basis algorithm** and **Awange-Grafarend Multipolynomial Resultant algorithm** and can be accessed in the GPS toolbox (*http://www.ngs.noaa.gov/gps-toolbox*). The distinction between the *Multipolynomial resultant approach* and the approach of *E. Grafarend* and *J. Shan* (1996) is that; for *Multipolynomial resultant approach*, we do not have to invert the coefficient matrix but instead use the *necessary* and *sufficient* conditions that the determinant has to vanish if the three equations have a nontrivial solution. We next consider an example that has been considered already by *A. Kleusberg* (1994) and *E. Grafarend* and *J. Shan* (1996).

Example 3-2**:** From the coordinates of four GPS satellites given in *Table* (3.4) below, we apply the **Awange-Grafarend Groebner basis algorithm** and **Awange-Grafarend Multipolynomial Resultant algorithm** to compute the coordinates of the stationary GPS receiver and the receiver range bias term. The **Awange-Grafarend Groebner basis algorithm** and **Awange-Grafarend Multipolynomial Resultant algorithm** compute the coefficients of the quadratic equations in *Boxes* (3-12) and (3-14) respectively as $c_2 = d_2$ =-9.104704113943708e-1, $c_1 = d_1$ =5.233385578536521e+7 and $c_0 = d_0$ =-5.233405293375e+9.

Table 3.4: Geocentric coordinates of four GPS satellites and the pseudo-ranging observations

i	$x^i = a_i$	$y^i = b_i$	$z^i = c_i$	d_i
0	1.483230866e+7	-2.046671589e+7	-7.42863475e+6	2.4310764064e+7
1	-1.579985405e+7	-1.330112917e+7	1.713383824e+7	2.2914600784e+7
2	1.98481891e+6	-1.186767296e+7	2.371692013e+7	2.0628809405e+7
3	-1.248027319e+7	-2.338256053e+7	3.27847268e+6	2.3422377972e+7

Once these coefficients have been computed, the algorithms proceed to solve (using these coefficients) the roots $\{x_4\}$ of the quadratic equations in *Boxes* (3-12) and (3-14) respectively giving the stationary receiver range bias term. The admissible value of the stationary receiver range bias term is then substituted in the expressions $x_1 = g(x_4), x_2 = g(x_4), x_3 = g(x_4)$ to give the values of stationary receiver coordinates $\{x_1, x_2, x_3\}$ respectively. The complete pair of solutions are

$$\begin{array}{l} x_1 = -2892123.412m, x_2 = 7568784.349m, x_3 = -7209505.102m \\ x_4 = -57479918.164m \,|x_4^- \\ \textit{or} \quad x_1 = 1111590.460m, x_2 = -4348258.631m, x_3 = 4527351.820m \\ x_4 = -100.0006m \,|x_4^+ \end{array}$$

From the results above, we note that the solution space is non unique. In order to decide between the correct solution from the pair above, we compute the *norm* (length) of the positional vector $\{x_3, x_2, x_1\} \,|x_4^-$ and $\{x_3, x_2, x_1\} \,|x_4^+$. If the receiver coordinates are in the *Global Reference Frame*, the norm of the positional vector of the receiver station will approximate the value of the Earth's radius while the norm for the other pair of solution will be in space. The computed norms are

$$\left[\begin{array}{l} \{x_3, x_2, x_1\} \,|x_4^- = 10845636.826m \\ \{x_3, x_2, x_1\} \,|x_4^+ = 6374943.214m \end{array} \right.$$

thus clearly giving the second solution $\{x_3, x_2, x_1\} \,|x_4^+$ as the admissible solution of the receiver position.

3-232 Overdetermined GPS positioning with observations of type pseudo-ranges

In this section, we consider the case where more than four satellites have been observed. We will apply the *Gauss-Jacobi combinatorial* algorithm to the example provided by *G. Strang* and *K. Borre* (1997) and compare the results obtained to those obtained by the *linear Gauss-Markov Model* after linearization of the observations by Taylor series expansion. Pseudo-ranges are measured to six satellites whose coordinates are also given as in *Table* (3.5).

Table 3.5: Geocentric coordinates of six GPS satellites and the pseudo-range observations

PRN	$x^i = a_i$	$y^i = b_i$	$z^i = c_i$	d_i
23	14177553.47	-18814768.09	12243866.38	21119278.32
9	15097199.81	-4636088.67	21326706.55	22527064.18
5	23460342.33	-9433518.58	8174941.25	23674159.88
1	-8206488.95	-18217989.14	17605231.99	20951647.38
21	1399988.07	-17563734.90	19705591.18	20155401.42
17	6995655.48	-23537808.26	-9927906.48	24222110.91

From the data above and using (2-26) in page (13), we obtain 15 possible combinations listed in *Table* (3.6) whose PDOP are computed as in *B. Hofmann et al* (1994, pp. 249-253) below.

Table 3.6: Possible combinations and the computed PDOP

Combination Number	Combination	Computed PDOP
1	23-9-5-1	4.8
2	23-9-5-21	8.6
3	23-9-5-17	4.0
4	23-9-1-21	6.5
5	23-9-1-17	3.3
6	23-9-21-17	3.6
7	23-5-1-21	6.6
8	23-5-1-17	6.6
9	23-5-21-17	4.8
10	23-1-21-17	137.8
11	9-5-1-21	5.6
12	9-5-1-17	14.0
13	9-5-21-17	6.6
14	9-1-21-17	5.2
15	5-1-21-17	6.6

It is clearly seen from the computed PDOP that the 10th combination had a poor geometry. We plot the PDOP versus the combination number to have a clear picture in *Figure* (3.2).

Figure (3.2) indicates clearly that the 10th combination had a weaker geometry. The *Gauss-Jacobi combinatorial algorithm* takes care of this weaker geometry during the adjustment process by the use of the variance-covariance matrix computed through *nonlinear error propagation* for that respective set. We next use the derived quadratic formulae by *Gröbner basis* in *Box* (3-12) or by *Multipolynomial resultants* in *Box* (3-14) for the minimal case to compute the coefficients presented in *Table* (3.7).

From the computed coefficients in *Table* (3.7), the 10th combination is once again identified as having significantly different values from the rest. Using the coefficients of *Table* (3.7), we compute the solution of the minimal combinatorial sets (each combination) being the receiver position $\{X, Y, Z\}$ and the range bias *cdt* and present them in *Table* (3.8).

The final adjustment is performed using the *linear Gauss-Markov model* with the random values of *Table* (3.8) as pseudo-observations and the dispersion matrix obtained from the *nonlinear error propagation* (Chapter 2).

Figure (3.3) gives the plot of the scatter of the *15 Gauss-Jacobi combinatorial* solutions (shown by points) around the adjusted value (indicated by a star), while *Figure* (3.4) gives the magnification of the scatter of *14 Gauss-Jacobi combinatorial* solutions (shown by points) that are very close to the adjusted value (indicated by a star) ignoring the outlying point in *Figure* (3.3).

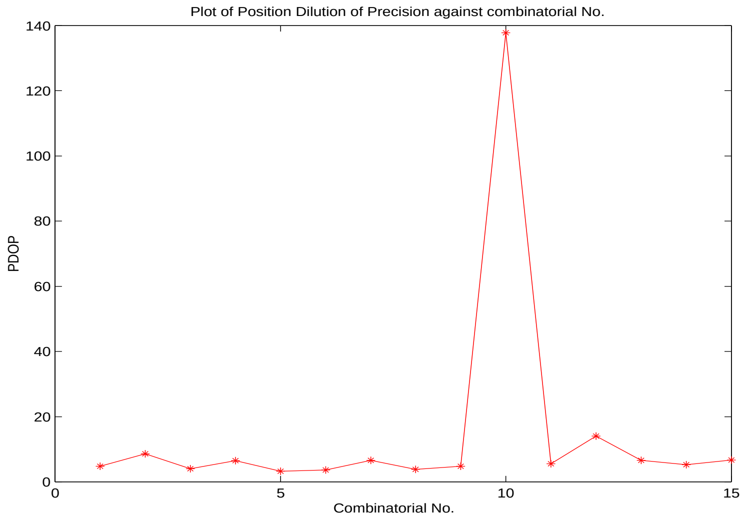


Figure 3.2: Computed PDOP for respective combinations

Table 3.7: Computed coefficients of the combinations

C/No.	c_2	c_1	c_0
1	-0.914220949236445	52374122.9848733	49022682.3125
2	-0.934176403102736	50396827.4998945	7915541824.84375
3	-0.921130625833683	51741826.0147786	343282824.25
4	-0.865060899130107	54950460.2842167	-10201105114.5
5	-0.922335616484969	51877166.0451888	280298481.625
6	-0.919296962706157	51562232.9601199	1354267366.4375
7	-0.894980063579044	53302005.6927825	-3642644147.5625
8	-0.917233949644576	52194946.1124139	132408747.46875
9	-0.925853049262193	51140847.6331213	3726719112.1875
10	3369.83293928593	-1792713339.80277	6251615074927.06
11	-0.877892756651551	54023883.5656926	-6514735288.13762
12	-0.942581538318523	50793361.5303674	784684294.241371
13	-0.908215141659006	52246642.0794924	-2499054749.05572
14	-0.883364070549387	53566554.3869961	-5481411035.37882
15	-0.866750765656126	54380648.2092251	-7320871488.80859

Table 3.8: Computed combinatorial solution points in a polyhedron

C/No.	$X(m)$	$Y(m)$	$Z(m)$	$cdt(m)$
1	596925.3485	-4847817.3618	4088206.7822	-0.9360
2	596790.3124	-4847765.7637	4088115.7092	-157.0638
3	596920.4198	-4847815.4785	4088203.4581	-6.6345
4	596972.8261	-4847933.4365	4088412.0909	185.6424
5	596924.2118	-4847814.5827	4088201.8667	-5.4031
6	596859.9715	-4847829.7585	4088228.8277	-26.2647
7	596973.5779	-4847762.4719	4088399.8670	68.3398
8	596924.2341	-4847818.6302	4088202.3205	-2.5368
9	596858.7650	-4847764.5341	4088221.8468	-72.8716
10	596951.5275	-4852779.5675	4088758.6420	3510.4002
11	597004.7562	-4847965.2225	4088300.6135	120.5901
12	596915.8657	-4847799.7045	4088195.5770	-15.4486
13	596948.5619	-4847912.9549	4088252.1599	47.8319
14	597013.7194	-4847974.1452	4088269.3206	102.3292
15	597013.1300	-4848019.6766	4088273.9565	134.6230

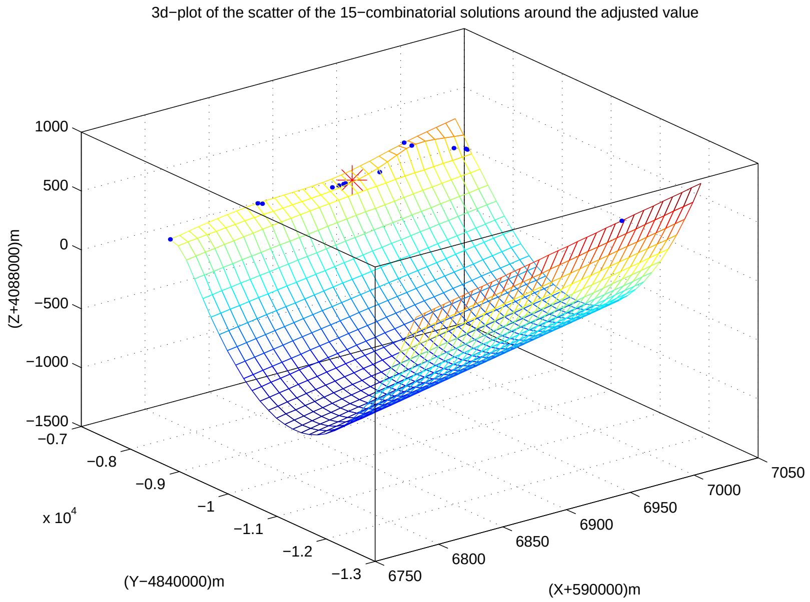


Figure 3.3: Scatter of the *15* Gauss-Jacobi combinatorial solutions (•) around the adjusted value (⋆).

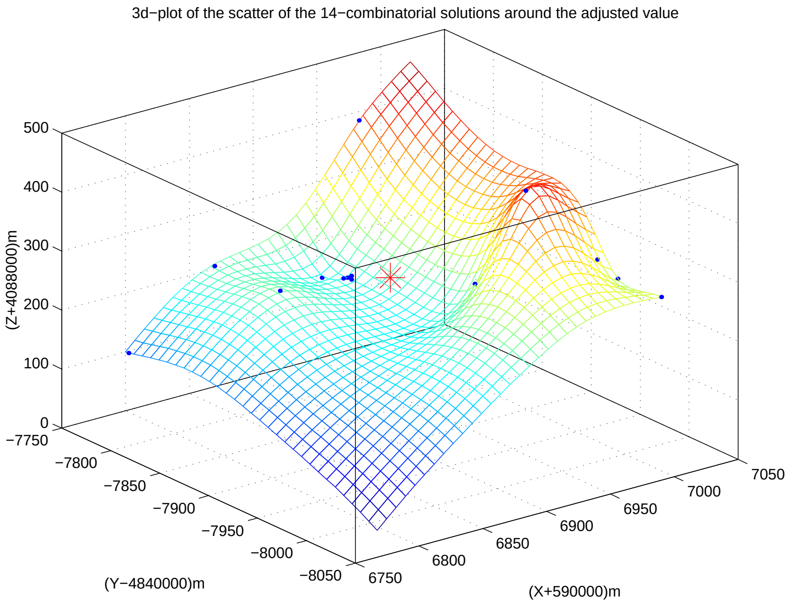


Figure 3.4: Magnification of the scatter of *14* Gauss-Jacobi combinatorial solutions (•) around the adjusted value (⋆).

3-233 Linearized least squares versus Gauss-Jacobi combinatorial algorithm

We conclude in this section by comparing between the *linearized least squares approach* and the *Gauss-Jacobi combinatorial approach*. Using the Gauss-Jacobi combinatorial approach, the stationary receiver position and the stationary receiver range bias are computed as discussed in Section (3-232). For the *linearized least squares approach*, the non-linear observation equations (3-118) are linearized using Taylor series expansion for the 6 satellites in *Table* (3.5) to generate the Jacobi matrix required for the *linearized least squares approach*. As approximate starting values, we first initiated the stationary receiver position and the stationary receiver range bias as zero and set the convergence limit to 1×10^{-8} as the difference of values between two successive iterations. With approximate values taken as zero, 6 iterations were required for convergence limit $1 \times 10.^{-8}$ In the second case, we considered the values of the

Gauss-Jacobi combinatorial algorithm as the starting (approximate) values for the linearized least squares solution. This time round, only two iterations were required to achieve convergence.

From the nonlinear equations (3-118) and the results of both *linearized least squares approach* and *Gauss-Jacobi combinatorial algorithm,* we computed the residuals and obtained the squares of these residuals and finally the error norm from

$$norm = \sqrt{\left\{\sum_{i=1}^{6}\left(d_i - [\sqrt{(\hat{x}_1 - a_i)^2 + (\hat{x}_2 - b_i)^2 + (\hat{x}_3 - c_i)^2} - \hat{x}_4]\right)^2\right\}}, \quad (3\text{-}129)$$

where $\{\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4\}$ are the computed values of the stationary receiver position and the stationary receiver range bias, $\{a_i, b_i, c_i\} \mid \forall i = \{1, \ldots, 6\}$ are the coordinates of the six satellites in *Table* (3.5) and $\{d_i\} \mid \forall i = \{1, \ldots, 6\}$ are the measured pseudo-ranges.

Table (3.9) gives the obtained results from the *Gauss-Jacobi combinatorial* algorithm and those obtained from *linearized Least squares approach* by first linearizing using Taylor series expansion. *Table* (3.10) presents the *root-mean-square-errors* and the differences between the solutions of the two procedures. In *Table* (3.11), we present the computed residuals, sum of squares of these residuals and the computed error norm from (3-129). The computed error norm are identical for both procedures.

Further comparison of the two procedures is to be found in Chapter (5) where the procedures are used to compute the 7-datum parameter transformation. Once the parameters have been computed, they are used to transform the Cartesian coordinates from the *Local Reference System (Table* 5.5) to the *Global Reference System* (WGS 84, *Table* 5.6) as shown in *Tables* (5.10) and *Table* (5.11). The residuals from both *Gauss-Jacobi combinatorial algorithm* and *Linearized Least Squares Solution* are in the same range in magnitude. We also compute the residual norm (square root of the sum of squares of residuals) and present them in *Table* (5.12) in page (93). In this case, the error norm from the *Gauss-Jacobi combinatorial* algorithm is somewhat better than those of the *linearized least squares solution.*

Table 3.9: Computed receiver position and stationary receiver range bias

	$X(m)$	$Y(m)$	$Z(m)$	$cdt(m)$
GJ- Approach	596929.6542	-4847851.5021	4088226.7858	-15.5098
(Linearized)LS	596929.6535	-4847851.5526	4088226.7957	-15.5181
Difference	0.0007	0.0505	-0.0098	0.0083

Table 3.10: Computed root-mean-square errors and the difference in solution between the two approaches

	$\sigma_X(m)$	$\sigma_Y(m)$	$\sigma_Z(m)$	$\sigma_{cdt}(m)$
GJ-Approach	6.4968	11.0141	5.4789	8.8071
GM solution	34.3769	58.2787	28.9909	46.6018

Table 3.11: Computed residuals, squares of residuals and error norm

PRN	Gauss-Jacobi (m)	Linearized Least Squares (m)
23	-16.6260	-16.6545
9	-1.3122	-1.3106
5	2.2215	2.2189
1	-16.4369	-16.4675
21	26.8623	26.8311
17	5.4074	5.3825
Sum of Squares	1304.0713	1304.0680
Error norm	36.1119	36.1119

3-234 Outlier diagnosis

In this section, we demonstrate the capability of the Gauss-Jacobi combinatorial algorithm to diagnose outliers. Consider that a satellite signal meant to travel straight from the satellite to the receiver is reflected by the reflecting surfaces (multipath effect) in built up areas for instance. The measured pseudorange reaching the receiver ends up being longer than the actual would be pseudorange. The scanned Figure from B. Hoffman et al. (1994, p.124, Figure 6.5) illustrates the case of multipath.

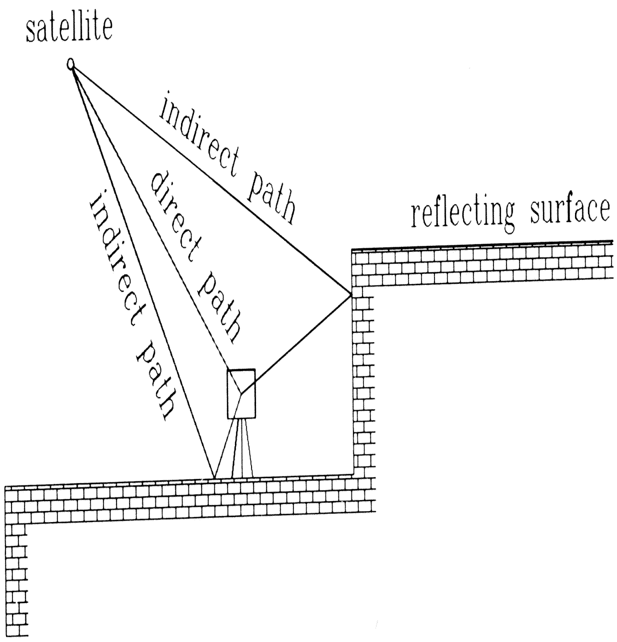


Fig. 6.5. Multipath effect

Let us now suppose that satellite number 23 had its pseudorange measurement longer by 500m owing to multipath effect. This particular satellite is chosen as it is the first in the combinatorial formation and as such the left most element of the combination and thus easy to identify in the combinatorial list of Table (3.12). For each combination, we compute the position by using either the *Gröbner basis* or *Multipolynomial resultant* algebraic tools. Once the positions have been obtained, the positional norm are then computed for each combination and written besides the combinations as in Table (3.12).

Table 3.12: Computed positional norm

Combination Number	Combination	Positional norm (km)
1	23-9-5-1	6368.126
2	23-9-5-21	6367.147
3	23-9-5-17	6368.387
4	23-9-1-21	6370.117
5	23-9-1-17	6368.474
6	23-9-21-17	6368.638
7	23-5-1-21	6368.894
8	23-5-1-17	6368.256
9	23-5-21-17	6368.005
10	23-1-21-17	6398.053
11	9-5-1-21	6369.723
12	9-5-1-17	6369.522
13	9-5-21-17	6369.647
14	9-1-21-17	6369.711
15	5-1-21-17	6369.749

It is clearly seen that the computed positional norms of the first 10 combinatorials were varying while the variation of the computed positional norm of the last 5 combinatorial was to a lesser degree. The possible explanation could be that satellite number 23 whose pseudorange is contaminated appeared in the first 10 combinations. The last 5 combinations are not affected with satellite number 23. In order to view the variation of the computed positional norm, we have computed the mean of these positional norms and subtracted it from the other norms. In *Figure* (3.5), we have plotted these deviations of the computed positional norm from the mean positional norm. Whereas the deviations of the first 10 combinations are seen to fluctuate, those of the last 5 combinations are seen to be almost constant and with minimum deviation from the mean thus clearly indicating the presence of outlier in the first 10 combinations. Since the satellite 23 is the only common satellite in the first 10 combinations, this outlier could be attributed to it.

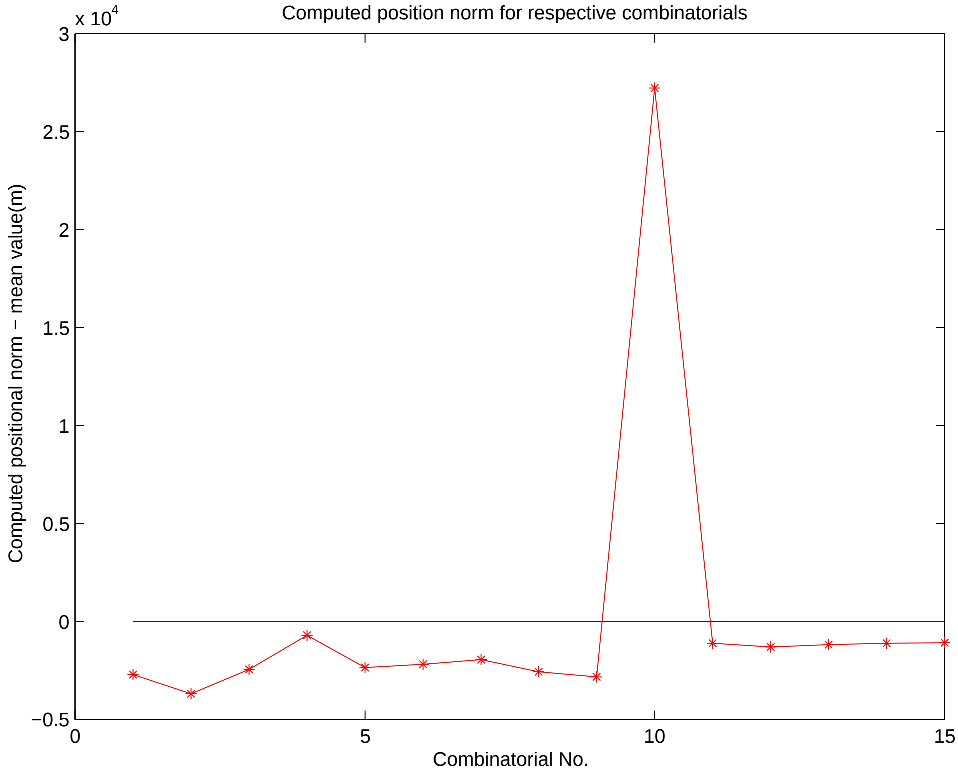


Figure 3.5: Deviation of positional norm from the mean value

Chapter 4

Test network Stuttgart Central

As an extension of Chapter 3 we consider in this Chapter the *closed form solution* and the *overdetermined solution* of the *three-dimensional resection problem* using the *Gröbner bases*, *Multipolynomial resultants* and the *Gauss-Jacobi combinatorial algorithms* discussed in Chapter 2. The test network *"Stuttgart Central"* in *Figure* (4.1) below is selected for study. In Section (4-1) we consider the observations of the test network *"Stuttgart Central"* of types *GPS coordinates*, *horizontal directions* T_i and *vertical directions* B_i that will be used in subsequent sections. Section (4-2) considers the closed form *three-dimensional resection* solution of the test network *"Stuttgart Central"* while Section (4-3) considers the *overdetermined three-dimensional resection* solution of the test network *"Stuttgart Central."*

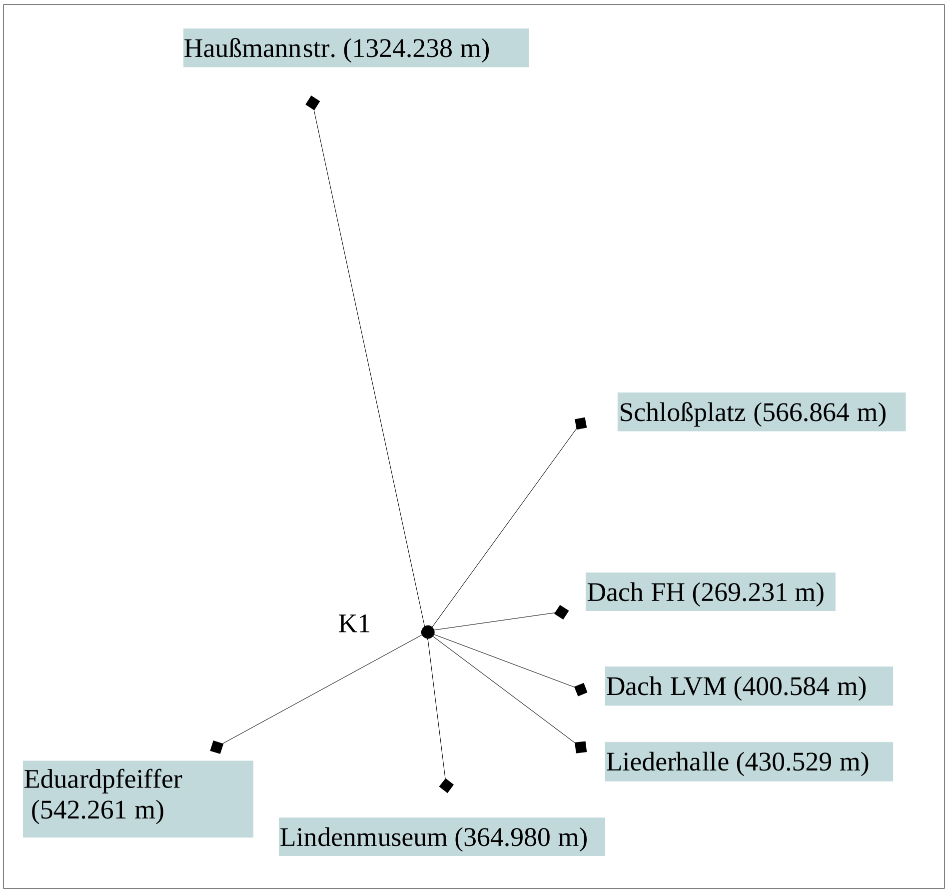


Figure 4.1: Graph of the Test network *"Stuttgart Central"*

4-1 Observations

The following experiment was performed at the centre of Stuttgart on one of the pillars of the University buildings along Kepler Strasse 11 as depicted by *Figure* (4.1). The test-network *"Stuttgart Central"* consisted of 8 GPS points listed in *Table* (4.1). A theodolite has been stationed at pillar K1 whose astronomical longitude Λ_Γ as well as astronomic latitude Φ_Γ were known from previous astrogeodetic observations made by the Department of Geodesy and GeoInformatics, Stuttgart University. Since theodolite observations of type *horizontal directions* T_i as well as *vertical directions* B_i from the pillar K1 to the target points $i,\ i = 1, 2, \ldots, 6, 7$, were only partially available we decided to simulate the horizontal and vertical directions from the given values of $\{\Lambda_\Gamma, \Phi_\Gamma\}$ as well as the Cartesian coordinates of the station point $(\mathrm{X}, \mathrm{Y}, \mathrm{Z})$ and target points $(\mathrm{X_i}, \mathrm{Y_i}, \mathrm{Z_i})$ using (3-18) and (3-19). The relationship between the observations of type *horizontal directions* T_i, *vertical directions* B_i, values of $\{\Lambda_\Gamma, \Phi_\Gamma\}$ and the Cartesian coordinates of the station point $(\mathrm{X}, \mathrm{Y}, \mathrm{Z})$ and target points $(\mathrm{X_i}, \mathrm{Y_i}, \mathrm{Z_i})$ that enabled generation of the observation data sets 1 to 11 is presented in Chapter 3, Section (3-14). Such a procedure had also the advantage that we had full control of the algorithms that we discussed in Chapter 2. In detail, the directional parameters $\{\Lambda_\Gamma, \Phi_\Gamma\}$ of the local gravity vector were adopted from the astrogeodetic observations reported by *S. Kurz* (1996 p.46) with a root-mean-square error $\sigma_\Lambda = \sigma_\Phi = 10"$. *Table* (4.1) contains the $(\mathrm{X}, \mathrm{Y}, \mathrm{Z})$ coordinates obtained from a GPS survey of the test-network *"Stuttgart Central"*, in particular with root-mean-square errors $(\sigma_\mathrm{X}, \sigma_\mathrm{Y}, \sigma_\mathrm{Z})$ neglecting the covariances $(\sigma_\mathrm{XY}, \sigma_\mathrm{YZ}, \sigma_\mathrm{ZX})$. The spherical coordinates of the relative position vector, namely of the coordinate differences $(\mathrm{x_i} - \mathrm{x}, \mathrm{y_i} - \mathrm{y}, \mathrm{z_i} - \mathrm{z})$, are called *horizontal directions* T_i, *vertical directions* B_i and *distances* S_i and are given by *Table* (4.2). The standard deviations/root-mean-square errors were fixed to $\sigma_T = 6"$, $\sigma_B = 6"$. Such root mean square errors can be obtained on the basis of a proper refraction model. Since the horizontal and vertical directions of *Table* (4.2) were simulated data, with zero noise level, we used a random generator *randn* in MATLAB version 5.3 (*e.g. D. Hanselman* and *B. Littlefield* 1997, pp. 84, 144) to produce additional observational data sets within the framework of the given root-mean-square errors. For each observable of type T_i and B_i, 30 randomly simulated data were obtained and the mean taken. Let us refer to the observational data sets $\{T_i, B_i\}, i = 1, 2, \ldots, 6, 7$, of *Table* (4.3) to *Table* (4.13) which were enriched by the root-mean-square errors of the individual randomly generated observations as well as by the differences $\Delta T_i := T_i - T_i(\text{generated})$, $\Delta B_i := B_i - B_i(\text{generated})$. Such differences $(\Delta T_i, \Delta B_i)$ indicate the difference between the ideal values of *Table* (4.2) and those randomly generated.

ervations are thus designed such that by observing the other seven GPS stations, the orientation of the *Local Level Reference Frame* $\mathbb{F}^*$ whose origin is station K1, to the *Global Reference Frame* $\mathbb{F}^\bullet$ is obtained. The relationship between the $\mathbb{F}^*$ *Reference Frame* and the $\mathbb{F}^\bullet$ *Reference Frame* is presented in Chapter 3, Section (3-13). The direction of Schlossplatz is chosen as the zero direction of the theodolite and this leads to the determination of the third component Σ_Γ of the *three-dimensional orientation parameters*. To each of the GPS target points i, the observations of the type *horizontal directions* T_i and the *vertical directions* B_i are measured. The spatial distances $S_i^2(\mathbf{X}, \mathbf{X}_i) = \|\mathbf{X}_i - \mathbf{X}\|$ are readily obtained from the observation of type *horizontal directions* T_i and *vertical directions* B_i using the algebraic computational techniques discussed in Chapter 2. Once we have *Euclidean distances* S_i computed from the observations of type *horizontal directions* T_i and *vertical directions* B_i, a foward computation using any of the three-dimensional ranging (*"Bogenschnitt")* procedures discussed in Section (3-21) is used to compute the coordinates $\{X, Y, Z\}_{GPS}$ from which the direction parameters $(\Lambda_\Gamma, \Phi_\Gamma)$ of the local gravity vector at K1 and the *"orientation unknown"* element Σ_Γ are finally computed as discussed in *J. L. Awange* (1999) and *E. Grafarend* and *J. L. Awange* (2000). The obtained values are then compared to the starting values. The following symbols have been used: $\sigma_\mathrm{X}, \sigma_\mathrm{Y}, \sigma_\mathrm{Z}$ are the standard errors of the GPS Cartesian coordinates. Covariances $\sigma_\mathrm{XY}, \sigma_\mathrm{YZ}, \sigma_\mathrm{ZX}$ were neglected. σ_T, σ_B are the standard deviation of horizontal and vertical directions respectively after an adjustment, Δ_T, Δ_B the magnitude of the noise on the horizontal and vertical directions, respectively.

Table 4.1: GPS Coordinates in the *Global Reference Frame* $\mathbb{F}^\bullet(X, Y, Z)$, (X_i, Y_i, Z_i), $i = 1, 2, \ldots, 7$

Station	$X(m)$	$Y(m)$	$Z(m)$	σ_X mm	σ_X mm	σ_X mm
Dach K1	4157066.1116	671429.6655	4774879.3704	1.07	1.06	1.09
1	4157246.5346	671877.0281	4774581.6314	0.76	0.76	0.76
2	4156749.5977	672711.4554	4774981.5459	1.77	1.59	1.61
3	4156748.6829	671171.9385	4775235.5483	1.93	1.84	1.87
4	4157066.8851	671064.9381	4774865.8238	1.38	1.29	1.38
5	4157266.6181	671099.1577	4774689.8536	1.29	1.28	1.34
6	4157307.5147	671171.7006	4774690.5691	0.20	0.10	0.30
7	4157244.9515	671338.5915	4774699.9070	2.80	1.50	3.10

Table 4.2: Ideal spherical coordinates of the relative position vector in the *Local Horizontal Reference Frame* $\mathbb{F}^*$: Spatial distances, horizontal directions, vertical directions

Station Observed from K1	Distances (m)	Horizontal directions(gon)	Vertical directions(gon)
Schlossplatz (1)	566.8635	52.320062	-6.705164
Haussmanstr. (2)	1324.2380	107.160333	0.271038
Eduardpfeiffer (3)	542.2609	224.582723	4.036011
Lindenmuseum (4)	364.9797	293.965493	-8.398004
Liederhalle (5)	430.5286	336.851237	-6.941728
Dach LVM (6)	400.5837	347.702846	-1.921509
Dach FH (7)	269.2309	370.832476	-6.686951

Table 4.3: Randomly generated spherical coordinates of the relative position vector: horizontal directions T_i and vertical directions $B_i, i = 1, 2, \ldots, 6, 7$, root-mean-square errors of individual observations, differences $\Delta T_i := T_i - T_i(\text{generated})$, $\Delta B_i := B_i - B_i(\text{generated})$ with respect to (T_i, B_i) ideal data of Table 4.2, first data set: set 1

St.	H/Dir.(gon)	V/Dir.(gon)	$\sigma_T(gon)$	$\sigma_B(gon)$	$\Delta_T(gon)$	$\Delta_B(gon)$
1	0.000000	-6.705138	0.0025794	0.0024898	-0.000228	-0.000039
2	54.840342	0.271005	0.0028756	0.0027171	-0.000298	0.000033
3	172.262141	4.035491	0.0023303	0.0022050	0.000293	0.000520
4	241.644854	-8.398175	0.0025255	0.0024874	0.000350	0.000171
5	284.531189	-6.942558	0.0020781	0.0022399	-0.000024	0.000830
6	295.382909	-1.921008	0.0029555	0.0024234	0.000278	-0.000275
7	318.512158	-6.687226	0.0026747	0.0024193	-0.000352	0.000500

Table 4.4: Second data set: set 2

St.	H/Dir.(gon)	V/Dir.(gon)	$\sigma_T(gon)$	$\sigma_B(gon)$	$\Delta_T(gon)$	$\Delta_B(gon)$
1	0.0000000	-6.705636	0.0029467	0.0022479	0.000655	0.000459
2	54.841828	0.270494	0.0023740	0.0018085	-0.000902	0.000544
3	172.262016	4.035712	0.0025738	0.0025891	0.001300	0.000300
4	241.645929	-8.397520	0.0025012	0.0027585	0.000156	-0.000484
5	284.531106	-6.940833	0.0025388	0.0021120	0.000723	-0.000895
6	295.382535	-1.920744	0.0024122	0.0022379	0.000904	-0.000765
7	318.512615	-6.686485	0.0024235	0.0027708	0.000453	-0.000466

Table 4.5: Third data set: set 3

St.	H/Dir.(gon)	V/Dir.(gon)	$\sigma_T(gon)$	$\sigma_B(gon)$	$\Delta_T(gon)$	$\Delta_B(gon)$
1	0.000000	-6.704849	0.0025794	0.0023074	-0.001470	-0.000329
2	54.839743	0.272062	0.0027316	0.0036383	0.000524	-0.001024
3	172.261715	4.036063	0.0022680	0.0025318	0.000942	-0.000052
4	241.645032	-8.398128	0.0031452	0.0030835	0.000395	0.000124
5	284.530697	-6.941783	0.0025214	0.0024290	0.000473	0.000055
6	295.382921	-1.922053	0.0024296	0.0028454	-0.000141	0.000544
7	318.511249	-6.686536	0.0024345	0.00227063	0.001174	-0.000415

Table 4.6: Fourth data set: set 4

St.	H/Dir.(gon)	V/Dir.(gon)	$\sigma_T(gon)$	$\sigma_B(gon)$	$\Delta_T(gon)$	$\Delta_B(gon)$
1	0.000000	-6.704682	0.0023308	0.0027599	0.000862	-0.000496
2	54.841145	0.271960	0.0026907	0.0021463	-0.000011	-0.000922
3	172.264284	4.036170	0.0028699	0.0024486	-0.000760	-0.000159
4	241.645972	-8.397035	0.0035089	0.0024921	0.000322	-0.000969
5	284.532505	-6.941248	0.0026110	0.0033665	-0.000468	-0.000480
6	295.384465	-1.921296	0.0027294	0.0026283	-0.000818	-0.000213
7	318.513839	-6.686125	0.0020477	0.0030185	-0.000562	-0.000826

Table 4.7: Fifth data set: set 5

St.	H/Dir.(gon)	V/Dir.(gon)	$\sigma_T(gon)$	$\sigma_B(gon)$	$\Delta_T(gon)$	$\Delta_B(gon)$
1	0.000000	-6.705407	0.0023550	0.0026607	0.000275	0.000229
2	54.839952	0.271814	0.0027139	0.0024570	0.000594	-0.000775
3	172.262789	4.036099	0.0028628	0.0020811	0.000148	-0.000088
4	241.645827	-8.398001	0.0027261	0.0027714	-0.000121	-0.000003
5	284.530609	-6.940954	0.0029166	0.0024115	0.000840	-0.000774
6	295.383197	-1.921506	0.0032741	0.0025684	-0.000138	-0.000003
7	318.513393	-6.686562	0.0031545	0.0028330	-0.000705	-0.000993

Table 4.8: Sixth data set: set 6

St.	H/Dir.(gon)	V/Dir.(gon)	$\sigma_T(gon)$	$\sigma_B(gon)$	$\Delta_T(gon)$	$\Delta_B(gon)$
1	0.000000	-6.705699	0.0032227	0.0026362	-0.000230	0.000522
2	54.841100	0.272198	0.0028716	0.0032300	-0.001059	-0.001160
3	172.262254	4.035556	0.0027485	0.0022965	0.000177	0.000455
4	241.645033	-8.398092	0.0028093	0.0030335	0.000167	0.000088
5	284.531250	-6.941579	0.0022418	0.0023971	-0.000306	-0.000149
6	295.383176	-1.921632	0.0028193	0.0031391	-0.000622	0.000123
7	318.512147	-6.687006	0.0026446	0.0018992	0.000037	0.000055

Table 4.9: seventh data set: set 7

St.	H/Dir.(gon)	V/Dir.(gon)	$\sigma_T(gon)$	$\sigma_B(gon)$	$\Delta_T(gon)$	$\Delta_B(gon)$
1	0.000000	-6.704710	0.0032501	0.0021664	0.000796	-0.000467
2	54.840622	0.271205	0.0025500	0.0026468	0.000446	-0.000167
3	172.262586	4.035479	0.0028646	0.0030243	0.000872	0.000532
4	241.645766	-8.397192	0.0020303	0.0026158	0.000461	-0.000811
5	284.533069	-6.940859	0.0026240	0.0022506	-0.001098	-0.000869
6	295.383199	-1.920591	0.0029904	0.0026217	0.000381	-0.000918
7	318.512078	-6.686979	0.0024550	0.0023116	0.001132	0.000028

Table 4.10: Eigth data set: set 8

St.	H/Dir.(gon)	V/Dir.(gon)	$\sigma_T(gon)$	$\sigma_B(gon)$	$\Delta_T(gon)$	$\Delta_B(gon)$
1	0.000000	-6.705117	0.0019401	0.0025817	0.001199	-0.000060
2	54.841233	0.271160	0.0020984	0.0028927	0.000238	-0.000122
3	172.263880	4.036182	0.0026151	0.0022965	-0.000019	-0.000170
4	241.645783	-8.397963	0.0029220	0.0022676	0.000847	-0.000041
5	284.532564	-6.941989	0.0024886	0.0021962	-0.000188	0.000261
6	295.383289	-1.920585	0.0025328	0.0025163	0.000694	-0.000924
7	318.514380	-6.686908	0.0028717	0.0028983	-0.000767	-0.000043

Table 4.11: Ninth data set: set 9

St.	H/Dir.(gon)	V/Dir.(gon)	$\sigma_T(gon)$	$\sigma_B(gon)$	$\Delta_T(gon)$	$\Delta_B(gon)$
1	0.000000	-6.705021	0.0023099	0.0029693	-0.000872	-0.000156
2	54.838686	0.271061	0.0025460	0.0024923	0.000714	-0.000023
3	172.261443	4.035575	0.0027183	0.0026865	0.000347	0.000436
4	241.645374	-8.397707	0.0024564	0.0024467	-0.000815	-0.000296
5	284.530552	-6.941628	0.0034024	0.0027446	-0.000249	-0.000100
6	295.381778	-1.921467	0.0022630	0.0027665	0.000134	-0.000042
7	318.511475	-6.686698	0.0022266	0.0025736	0.000068	-0.000253

Table 4.12: Tenth data set: set 10

St.	H/Dir.(gon)	V/Dir.(gon)	$\sigma_T(gon)$	$\sigma_B(gon)$	$\Delta_T(gon)$	$\Delta_B(gon)$
1	0.000000	-6.705515	0.0024938	0.0032987	0.000299	0.000338
2	54.841489	0.270932	0.0029717	0.0022950	-0.000918	0.000106
3	172.263665	4.036147	0.0032672	0.0024499	-0.000704	-0.000136
4	241.645336	-8.397823	0.0028515	0.0025473	0.000395	-0.000181
5	284.531567	-6.941534	0.0022931	0.0021688	-0.000093	-0.000194
6	295.383055	-1.922041	0.0033986	0.0028467	0.000028	0.000532
7	318.512017	-6.686773	0.0024359	0.0021356	0.000696	-0.000178

Table 4.13: Eleventh data set: set 11

St.	H/Dir.(gon)	V/Dir.(gon)	$\sigma_T(gon)$	$\sigma_B(gon)$	$\Delta_T(gon)$	$\Delta_B(gon)$
1	0.000000	-6.704889	0.0024962	0.0031604	0.000459	-0.000288
2	54.840818	0.271340	0.0027559	0.0028895	-0.000088	0.001659
3	172.263416	4.035779	0.0023929	0.0032068	-0.000296	0.000232
4	241.645322	-8.398136	0.0031984	0.0019623	0.000568	0.000132
5	284.532013	-6.942079	0.0027289	0.0032386	-0.000379	0.000351
6	295.383571	-1.921888	0.0026898	0.0023682	-0.000328	0.000379
7	318.513029	-6.686424	0.0027481	0.0026191	-0.000157	-0.000527

4-2 Closed form solution

In order to position point K1 in the GPS network of “Stuttgart Central” using LPS observable of types horizontal directions T_i and vertical directions B_i, three known stations (Haussmanstr., Eduardpfeiffer and Liederhalle) of the test network *“Stuttgart Central”* in *Figure* (4.1) are used. We proceed in three steps: The first step considers the computation of the *spatial distances*, the second step is the computation of the *coordinates* of the unknown station, and the final step is the computation of the *threedimensional orientation parameters*. In this section, the threedimensional resection method is considered with the aim of providing the *threedimensional geocentric GPS coordinates* in the *Global Reference Frame.* The *threedimensional orientation parameters* of type *astronomical longitude* Λ_Γ, *astronomical latitude* Φ_Γ, and the *“orientation unknown”* Σ_Γ in the horizontal plane and the deflection of the vertical can be obtained as in *J. L. Awange* (1999) and *E. Grafarend* and *J. L. Awange* (2000).

The solution of *Grunert equations* is achieved using the algebraic computational techniques i.e. *Gröbner bases* or *Multipolynomial resultant*, the *position-derivation step* involves computing the desired threedimensional GPS Cartesian coordinates $\{X, Y, Z\}_{GPS}$ of the unknown point $P \in \mathbb{E}^3$ in the *Global Reference Frame*. This is achieved by analytically solving the three-dimensional ranging problem (also known in German literature as *"dreidimensionales Bogenschnitt")* as discussed in Section (3-21).

4-21 Experiment

Known GPS stations (*Haussmanstr.*, *Eduardpfeiffer* and *Liederhalle*) of the test network *“Stuttgart Central”* in *Figure* (4.1) together with K1 form the tetrahedron $\{PP_1P_2P_4\}$ in the *Appendix A.2* in page (104). Algebraic computational tools *Gröbner bases* or *Multipolynomial resultants* discussed in Chapter 2 are used to determine the distances of the tetrahedron.

Using the computed *univariate polynomial (element of Gröbner basis* of the Ideal subset R[x_1, x_2, x_3]) in *Box* (3-3a) in Section (3-21, page 36), we determine the distances $S_i = x_i \in \mathbb{R},^+ \; i = \{1,2,3\} \in \mathbb{Z}^3_+$ etwen the unknown station $P \in \mathbb{E}^3$ and the known stations $P_i \in \mathbb{E}^3$ expressed in (3-31) for the test network *"Stuttgart Central"* in *Figure* (4.1). The unknown point P in this case is the pillar K1 on top of the University building at Kepler Strasse 11. Points $,_o f the tetrahedron \{PP_1P_2P_3\}$ in *Figure* (3.1) correspond to the chosen known GPS stations *Haussmannstr.*, *Eduardpfeiffer*, and *Liederhalle*. The distance from *K1 to Haussmannstr*. is designated $S_1 = x_1 \in \mathbb{R},^+$ *K1 to Eduardpfeiffer* $S_2 = x_2 \in \mathbb{R},^+$ while that of *K1 to Liederhalle* is designated $S_3 = x_3 \in \mathbb{R}.^+$ The distances between the known stations $\{S_{12},\, S_{23},\, S_{31}\} \in \mathbb{R}^+$ are computed from their respective GPS coordinates as indicated in *Box* (4-1) below. Their corresponding space angles $\psi_{12,}\, \psi_{23,}\, \psi_{31}$ are computed from (3-29). In order to control the computations, the Cartesian GPS coordinates of point K1 are also known. *Box* (4-1) below gives the complete solution of the unknowns $\{x_1, x_2, x_3\} \in \mathbb{R}^+$ from the computed *Gröbner basis* of *Boxes* (3-3a) and (3-3b) in Section (3-21, pages 36 and 37 respectively). The *univariate polynomial* in x_3 has eight roots, four of which are complex and four real. Of the four real roots two are positives and two are negative. The desired distance $x_3 \in \mathbb{R}^+$ is thus chosen from the two positive roots with the help of prior information and substituted in g_{11} in *Box* (3-3b) in page (37) to give two solutions of x_1, one of which is positive. Finally the obtained values of $\{x_1, x_3\} \in \mathbb{R}^+$ are substituted in g_5 *Box* (3-3b) in page (37) to obtain the remaining indeterminate x_2. Using this procedure, we have in *Box* (4-1) below that $S_3 = \{430.5286, 153.7112\}$. Since $S_3 = x_3 \in \mathbb{R},^+$ from apriori information we choose $S_3 = 430.5286$, leading to $S_1 = 1324.2381$, and $S_2 = 542.2608$. These values compare well with their real values depicted in *Figure* (4.1).

Box 4-1 (*computation of distances for test network "Stuttgart Central"*):

Using the entries of *Table* (4.1) in page (64), we computed inter-station distances by pythagorus $S_{ij} = \sqrt{(X_j - X_i)^2 + (Y_j - Y_i)^2 + (Z_j - Z_i)^2}$ and spatial angles from (3-29) are given as

$$\left[\begin{array}{l} S_{12} = 1560.3302m \\ S_{23} = 755.8681m \\ S_{31} = 1718.1090m \end{array}\right. \quad and \quad \left[\begin{array}{l} \psi_{12} = 1.843620 \\ \psi_{23} = 1.768989 \\ \psi_{31} = 2.664537 \end{array}\right.$$

and substituted in (3-47) to compute the terms $\{a_{12}, b_{23}, c_{31}, a_0, b_0, c_0\}$ which are needed to compute the coefficients of the *Gröbner basis* element g_1 in *Box* (3-3a) in Section (3-21, page 36). Expressing the *univariate polynomial* g_1 in *Box* (3-3a) in Section (3-21, page 36) as $A_8x_3^8 + A_6x_3^6 + A_4x_3^4 + A_2x_3^2 + A_0 = 0$, the computed coefficients are

$$\left[\begin{array}{l} \mathrm{A}_0 = 4.833922266706213\mathrm{e}+023 \\ \mathrm{A2} = -2.306847176510587\mathrm{e}+019 \\ \mathrm{A4} = 1.104429253262719\mathrm{e}+014 \\ \mathrm{A6} = -3.083017244255380\mathrm{e}+005 \\ \mathrm{A8} = 4.323368172460818\mathrm{e}-004. \end{array}\right.$$

The solution to the *univariate polynomial* equation is then obtained from the Matlab command *"roots"* (*e.g. D. Hanselman* and *B. Littlefield* 1997, p.146) as

$$\left[\begin{array}{l} c = [A_8\; A_7\; A_6\; A_5\; A_4\; A_3\; A_2\; A_1\; A_0] \\ x_3 = roots(c) \end{array}\right.$$

the other coefficients being zero. The obtained values of x_3 are

$$x_3 = \left\{\begin{array}{l} -20757.2530734872 + 8626.43262759353\mathrm{i} \\ -20757.2530734872 - 8626.43262759353\mathrm{i} \\ 20757.2530734872 + 8626.4326275935\mathrm{i} \\ 20757.2530734872 - 8626.4326275935\mathrm{i} \\ 430.528578109464 \\ -430.528578109464 \\ 153.711222705295 \\ -153.711222705295 \end{array}\right\}$$

where the chosen value 430.5286 of $x_3 \in \mathbb{R}^+$ using prior information is substituted in g_{11} in *Box* (3-3b) in page (37) to give

$$x_1 = \{-2089.15882397074, 1324.23808451951\}$$

and finally the values of $\{x_1, x_3\} \in \mathbb{R}^+$ are substituted in g_5 in *Box* (3-3b) in page (37) to give x_2= 542.260767703842

The computation procedure using the *B. Buchberger algorithm (Gröbner bases)* is summarized as follows:

- Arrange the given polynomial equations using a chosen monomial order as in (3-46)
- Determine the polynomial *Ideal* as in (3-48)
- Compute the *Gröbner basis* of this *Ideal* using either *Mathematica* or *Maple* softwares.
- From the computed *Gröbner basis* of the *Ideal,* solve the *univariate polynomial* for the desired roots using the roots command of MATLAB.
- Substitute the admissible value of the *univariate polynomial* solution in the other *Gröbner basis* elements to obtain the remaining variables.

Alternatively, the approach below based on *Multipolynomial resultants* technique can be used to solve the *Grunert equations*.

(a) The *F. Macaulay* (1902) approach discussed in Section (2-322) solves for the determinant of the matrix $\boldsymbol{A}$ leading to a *univariate polynomial* in x_1. The solution of the obtained *univariate polynomial* equation expressed in *Box* (3-3c) in page (39) leads to similar results as those of *Gröbner basis* i.e.

$$
\left\{\begin{array}{rcl}
det(A) & = & A_8x_1^8 + A_6x_1^6 + A_4x_1^4 + A_2x_1^2 + A_0 \\
A_0 & = & -4.87154987980622^{26},\ A_2 = 4.74815547158708^{20} \\
 & & A_4 = -113109755605017 \\
A_8 & = & -0.000432336817247789,\ A_6 = 435283.472057364 \\
x_1 & = & -22456.4891074245 + 1735.29702574406i \\
 & & -22456.4891074245 - 1735.29702574406i \\
 & & 22456.4891074245 + 1735.29702574406i \\
 & & 22456.4891074245 - 1735.29702574406i \\
 & & 1580.10924379877 \\
 & & -1580.10924379877 \\
 & & 1324.23808451944 \\
 & & -1324.23808451944 \\
x_3 & = & 430.528578109536, -2783.30427366986 \\
x_2 & = & 542.260767703823,\ -711.800947103387
\end{array}\right.
$$

(b) The *B. Sturmfels* (1998) approach discussed in Section (2-322) using the *Jacobian determinant* solves the determinant of the 6×6 matrix and leads to a *univariate polynomial* in x_1. The solution of the obtained *univariate polynomial* equation expressed in *Box* (3-3d) in page (41) leads to similar results as those of *Gröbner basis* i.e.

$$
\left\{\begin{array}{rcl}
det(A) & = & A_8x_1^8 + A_6x_1^6 + A_4x_1^4 + A_2x_1^2 + A_0 \\
A_0 & = & -1.94861995192249^{27},\ A_2 = 1.89926218863483^{21} \\
 & & A_4 = -452439022420067 \\
A_8 & = & -0.00172934726897456,\ A_6 = 1741133.88822977 \\
x_1 & = & -22456.4891075064 + 1735.29702538544i \\
 & & -22456.4891075064 - 1735.29702538544i \\
 & & 22456.4891075064 + 1735.29702538544i \\
 & & 22456.4891075064 - 1735.29702538544i \\
 & & 1580.10924379877 \\
 & & -1580.10924379877 \\
 & & 1324.23808451944 \\
 & & -1324.23808451944 \\
x_3 & = & 430.528578109535, -2783.30427366986 \\
x_2 & = & 542.260767703824,\ -711.800947103388
\end{array}\right.
$$

The computed distances from *F. Macaulay* (1902) and *B. Sturmfels* (1998) above tallies. The required solutions $\{x_1, x_2, x_3\}$ obtained from *Gröbner basis* computation and those of *Multipolynomial resultants* are the same {i.e $1324.2381, 542.2608, 430.5286$} respectively.

For each observational data sets 0-11 in *Tables* (4.2)-(4.13), distances are obtained using either *Gröbner basis* or *Multipolynomial resultant techniques* as illustrated above. The results of the computed distances for the closed form solutions are presented in *Table* (4.14) of Section (4-41). The computed distances are used with the help of *Gröbner basis* approach to determine the position of K1 for each observational data set 0-11in *Tables* (4.2)-(4.13) as discussed in Section (3-21, page 43). The results for the computed position of K1 for the closed form approach are presented in Section (4-42). For the orientation elements we refer to *J. L. Awange* (1999) and *E. Grafarend* and *J. L. Awange* (2000).

4-3 Overdetermined solution

In the preceding sections, only three known points were required to solve the closed form three-dimension resection problem for position and orientation of the unknown point K1. If superfluous observations are available made possible by the availability of several known points as in the case of the test network *"Stuttgart Central"*, the *closed form three-dimensional resection procedure* gives way to the *overdetermined three-dimension resection* case. In this case therefore, all the known GPS network stations (Haussmanstr., Eduardpfeiffer, Lindenmuseum, Liederhalle, Dach LVM, Dach FH, and Schlossplatz) of the test network *"Stuttgart Central"* in *Figure* (4.1) are used.

4-31 Experiment

Using the observation data in *Tables* (4.2)-(4.13), we proceed in six steps as follows:

Step 1 (construction of minimal combinatorial subsets for determination of distances):
From (2-26) in page (13), 35 minimal combinatorials are formed for the test network *"Stuttgart Central"* and are as presented by the 35 combinatorial simplexes in *Appendix A.2* (page 104). For each minimal combinatorial simplex, the distances are computed from the *univariate polynomials* obtained using either *Gröbner basis* or *Multipolynomial resultants algorithms* and presented in *Boxes* (3-3a, page 36), (3-3c, page 39) or (3-3d, page 41) in Chapter 3. The admissible solution from the *univariate polynomials* are substituted into other polynomials (as explained in Section 3-21 in page 31 and Section 4-21 in page 67) to get the remaining two distances. Each combinatorial minimal subset results in 3 distances thus giving rise to a total of (3×35) 105 distances which we consider in the subsequent steps as pseudo-observations. The computed distances S_i link the known points $P_i|i = 1, \ldots, 7$ to the unknown point P (K1) in *Figure* (4.1).

Step 2 (nonlinear error propagation to determine the dispersion matrix $\mathbf{\Sigma}$):
In this step, the dispersion matrix $\mathbf{\Sigma}$ is sought. This is achieved via the Error propagation law/variance-covariance propagation for each of the combinatorial set $j = 1, \ldots, 35$ above. The closed form observational equations for the first combinatorial subset $j = 1$ (tetrahedron $PP_1P_2P_3$) in *Appendix A.2* are written algebraically with (3-30) as

$$\left\{\begin{array}{l} f_1 := S_1^2 + S_2^2 - 2S_1S_2(cosB_1cosB_2cos(T_2 - T_1) + sinB_1sinB_2) - S_{12}^2 \\ f_2 := S_2^2 + S_3^2 - 2S_2S_3(cosB_2cosB_3cos(T_3 - T_2) + sinB_2sinB_3) - S_{23}^2 \\ f_3 := S_1^2 + S_3^2 - 2S_1S_3(cosB_1cosB_3cos(T_1 - T_3) + sinB_1sinB_3) - S_{31}^2 \end{array}\right. \tag{4-1}$$

where $S_{ij}\,|i,j \in \{1,2,3\}, i \neq j$ are the distances between known GPS stations of the test network *"Stuttgart Central"*, $S_k\,|k \in \{1,2,3\}$ are the unknown distances measured from the unknown GPS point $P \in \mathbb{E}^3$ to the known GPS stations $P_i \in \mathbb{E}^3\,|i \in \{1,2,3\}$ and $T_i, B_i\,|i \in \{1,2,3\}$ are the LPS observable of types horizontal and vertical directions from the unknown point $P \in \mathbb{E}^3$ to the known GPS stations $P_i \in \mathbb{E}^3\,|i \in \{1,2,3\}$ respectively. With (2-77) and (2-68) in pages (25 and 24 respectively) we have the Jacobi matrices as

$$\boldsymbol{J}_x = \begin{bmatrix} \frac{\partial f_1}{\partial S_3} & \frac{\partial f_1}{\partial S_1} & \frac{\partial f_1}{\partial S_2} \\ \frac{\partial f_2}{\partial S_3} & \frac{\partial f_2}{\partial S_1} & \frac{\partial f_2}{\partial S_2} \\ \frac{\partial f_3}{\partial S_3} & \frac{\partial f_3}{\partial S_1} & \frac{\partial f_3}{\partial S_2} \end{bmatrix} \tag{4-2}$$

and

$$
\boldsymbol{J}_y = \begin{bmatrix} \frac{\partial f_1}{\partial S_{12}} & \frac{\partial f_1}{\partial S_{23}} & \frac{\partial f_1}{\partial S_{31}} & \frac{\partial f_1}{\partial B_1} & \frac{\partial f_1}{\partial B_2} & \frac{\partial f_1}{\partial B_3} & \frac{\partial f_1}{\partial T_1} & \frac{\partial f_1}{\partial T_2} & \frac{\partial f_1}{\partial T_3} \\ \frac{\partial f_2}{\partial S_{12}} & \frac{\partial f_2}{\partial S_{23}} & \frac{\partial f_2}{\partial S_{31}} & \frac{\partial f_2}{\partial B_1} & \frac{\partial f_2}{\partial B_2} & \frac{\partial f_2}{\partial B_3} & \frac{\partial f_2}{\partial T_1} & \frac{\partial f_2}{\partial T_2} & \frac{\partial f_2}{\partial T_3} \\ \frac{\partial f_3}{\partial S_{12}} & \frac{\partial f_3}{\partial S_{23}} & \frac{\partial f_3}{\partial S_{31}} & \frac{\partial f_3}{\partial B_1} & \frac{\partial f_3}{\partial B_2} & \frac{\partial f_3}{\partial B_3} & \frac{\partial f_3}{\partial T_1} & \frac{\partial f_3}{\partial T_2} & \frac{\partial f_3}{\partial T_3} \end{bmatrix} \tag{4-3}
$$

The values $\{S_1, S_2, S_3\}$ appearing in the Jacobi matrices $\boldsymbol{J}_x, \boldsymbol{J}_y$ are obtained from the closed form solution of the first combinatorial set in step 1. From the dispersion $\boldsymbol{\Sigma}_y$ of the vector of observations $\boldsymbol{y}$ and with (4-2) and (4-3) forming $\boldsymbol{J} = \boldsymbol{J}_x^{-1}\boldsymbol{J}_y$, the variance-covariance matrix $\boldsymbol{\Sigma}_x$

$$
\boldsymbol{\Sigma}_x = \begin{bmatrix} \sigma^2_{S_1} & \sigma_{S_1 S_2} & \sigma_{S_1 S_3} \\ \sigma_{S_2 S_1} & \sigma^2_{S_2} & \sigma_{S_2 S_3} \\ \sigma_{S_3 S_1} & \sigma_{S_3 S_2} & \sigma^2_{S_3} \end{bmatrix}
$$

is finally obtained from (2-68) as

$$
\boldsymbol{\Sigma}_x = \boldsymbol{J} \begin{bmatrix} \sigma^2_{S_{12}} & \sigma_{S_{12}S_{23}} & \sigma_{S_{12}S_{31}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma_{S_{23}S_{12}} & \sigma^2_{S_{23}} & \sigma_{S_{23}S_{31}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma_{S_{31}S_{12}} & \sigma_{S_{31}S_{23}} & \sigma^2_{S_{31}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2_{B_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma^2_{B_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma^2_{B_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2_{T_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2_{T_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2_{T_3} \end{bmatrix} \boldsymbol{J}' \tag{4-4}
$$

with the 3×3 elements of $\boldsymbol{\Sigma}_y$ on the right hand side of (4-4) given by

$$
\begin{bmatrix} \sigma^2_{S_{12}} & \sigma_{S_{12}S_{23}} & \sigma_{S_{12}S_{31}} \\ \sigma_{S_{23}S_{12}} & \sigma^2_{S_{23}} & \sigma_{S_{23}S_{31}} \\ \sigma_{S_{31}S_{12}} & \sigma_{S_{31}S_{23}} & \sigma^2_{S_{31}} \end{bmatrix} = \tag{4-5}
$$

$$
\boldsymbol{J}_d\{diagonal\{[\ \sigma^2_{X_1}\ \ \sigma^2_{X_2}\ \ \sigma^2_{X_3}\ \ \sigma^2_{Y_1}\ \ \sigma^2_{Y_2}\ \ \sigma^2_{Y_3}\ \ \sigma^2_{Z_1}\ \ \sigma^2_{Z_2}\ \ \sigma^2_{Z_3}\ ]\}\boldsymbol{J}'_d
$$

and $\boldsymbol{J}_d$ in (4-5) being the Jacobi matrix of the partial derivatives of the distance equations

$$
\begin{bmatrix} S^2_{12} = (X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2 \\ S^2_{23} = (X_3 - X_2)^2 + (Y_3 - Y_2)^2 + (Z_3 - Z_2)^2 \\ S^2_{31} = (X_1 - X_3)^2 + (Y_1 - Y_3)^2 + (Z_1 - Z_3)^2 \end{bmatrix} \tag{4-6}
$$

connecting the known GPS points $P_i \in \mathbb{E}^3 \,|i \in \{1,2,3\}$ with respect to the known coordinates $P_1\{X_1, Y_1, Z_1\}$, $P_2\{X_2, Y_2, Z_2\}$ and $P_3\{X_3, Y_3, Z_3\}$ of the GPS stations involved in the minimal combinatorial set. The variance-covariance matrix computed above is obtained for every combinatorial subset. Finally we obtained the variance-covariance matrix $\boldsymbol{\Sigma}$ from (2-70) in Chapter 2 page (24).

Step 3 (rigorous adjustment of the combinatorial solution points in a polyhedron):
Once the 105 combinatorial solution points in a polyhedron have been obtained in step 1, they are finally adjusted using the *linear Gauss-Markov model* (2-1), page 7) with the dispersion matrix $\boldsymbol{\Sigma}$ obtained via the error propagation law or variance-covariance propagation in step 2. Expressing each of the 105 pseudo-observation distances as

$$
S_i^j = S_i + \varepsilon_i^j \,|i \in \{1,2,3,4,5,6,7\}, j \in \{1,2,3,4,5,6,7,\ldots,35\},
$$

and placing the pseudo-observation distances S_i^j in the vector of observation $\boldsymbol{y}$, the coefficients of the unknown seven distances S_i of the test network *"Stuttgart Central"* forming the coefficient matrix $\boldsymbol{A}$ and $\boldsymbol{x}$ comprising the vector of unknowns S_i, the adjusted solution is obtained via (2-4) and the dispersion of the estimated parameters through (2-5) in Chapter 2 page (8).

Step 4 (construction of minimal combinatorial subsets for position determination):
Once the adjusted distances and their dispersion matrix have been estimated using (2-4) and (2-5) respectively in step 3, the position of the unknown point is then determined using either *Gröbner basis* approach (*Box* 3-5 in page 43) or *Multipolynomial resultant* approach (*Box* 3-6 in page 44). Similar to the distances, we have 35 combinatorial subsets giving 35 different positions $X, Y, Z|_P$ of the same point P (for each simplex of *Appendix A.2,* we get the position value $\{X, Y, Z\}$). In total we have 105 (35×3) values of $X, Y,$ and Z which will be treated as pseudo-observations.

Step 5 (nonlinear error propagation to determine the dispersion matrix $\mathbf{\Sigma}$):
The variance-covariance matrix are computed for each of the combinatorial set $j = 1, \ldots, 35$ using error propagation. The closed form observational equations for the first combinatorial subset $j = 1$ (i.e. tetrahedron $PP_1P_2P_3$) *Appendix A.2* are written algebraically as

$$\begin{cases} f_1 := (X_1 - X)^2 + (Y_1 - Y)^2 + (Z_1 - Z)^2 - S_1^2 \\ f_2 := (X_2 - X)^2 + (Y_2 - Y)^2 + (Z_2 - Z)^2 - S_2^2 \\ f_3 := (X_3 - X)^2 + (Y_3 - Y)^2 + (Z_3 - Z)^2 - S_3^2 \end{cases} \tag{4-7}$$

where $S_i^j \,|i \in \{1, 2, 3\} \mid j = 1$ are the distances between known GPS stations $P_i \in \mathbb{E}^3 \,|i \in \{1, 2, 3\}$ of the test network *"Stuttgart Central"* and the unknown GPS point $P \in \mathbb{E}^3$ for first combination set $j = 1$. With (2-77) and (2-68) in pages (25 and 24 respectively) we have the Jacobi matrices as

$$\boldsymbol{J}_x = \begin{bmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial Z} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial Z} \\ \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial Z} \end{bmatrix} \tag{4-8}$$

and

$$\boldsymbol{J}_y = \begin{bmatrix} \frac{\partial f_1}{\partial S_1} & \frac{\partial f_1}{\partial S_2} & \frac{\partial f_1}{\partial S_3} & \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial Y_1} & \frac{\partial f_1}{\partial Z_1} & \frac{\partial f_1}{\partial X_2} & \frac{\partial f_1}{\partial Y_2} & \frac{\partial f_1}{\partial Z_2} & \frac{\partial f_1}{\partial X_3} & \frac{\partial f_1}{\partial Y_3} & \frac{\partial f_1}{\partial Z_3} \\ \frac{\partial f_2}{\partial S_1} & \frac{\partial f_2}{\partial S_2} & \frac{\partial f_2}{\partial S_3} & \frac{\partial f_2}{\partial X_1} & \frac{\partial f_2}{\partial Y_1} & \frac{\partial f_2}{\partial Z_1} & \frac{\partial f_2}{\partial X_2} & \frac{\partial f_2}{\partial Y_2} & \frac{\partial f_2}{\partial Z_2} & \frac{\partial f_2}{\partial X_3} & \frac{\partial f_2}{\partial Y_3} & \frac{\partial f_2}{\partial Z_3} \\ \frac{\partial f_3}{\partial S_1} & \frac{\partial f_3}{\partial S_2} & \frac{\partial f_3}{\partial S_3} & \frac{\partial f_3}{\partial X_1} & \frac{\partial f_3}{\partial Y_1} & \frac{\partial f_3}{\partial Z_1} & \frac{\partial f_3}{\partial X_2} & \frac{\partial f_3}{\partial Y_2} & \frac{\partial f_3}{\partial Z_2} & \frac{\partial f_3}{\partial X_3} & \frac{\partial f_3}{\partial Y_3} & \frac{\partial f_3}{\partial Z_3} \end{bmatrix} \tag{4-9}$$

The values $\{X, Y, Z\}$ appearing in the Jacobi matrices $\boldsymbol{J}_x, \boldsymbol{J}_y$ are obtained from the closed form solution of the first combinatorial set in *step 4.* From the dispersion matrix $\mathbf{\Sigma}_y$ of the vector of observations $\boldsymbol{y}$ and with (4-8) and (4-9) forming $\boldsymbol{J} = \boldsymbol{J}_x^{-1}\boldsymbol{J}_y$, the variance-covariance matrix $\mathbf{\Sigma}_x$

$$\mathbf{\Sigma}_x = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} & \sigma_{XZ} \\ \sigma_{YX} & \sigma_Y^2 & \sigma_{YZ} \\ \sigma_{ZX} & \sigma_{ZY} & \sigma_Z^2 \end{bmatrix}$$

is finally obtained from (2-68) as

$$\Sigma_x = J \begin{pmatrix} \begin{bmatrix} \sigma^2_{S_1} & \sigma_{S_1S_2} & \sigma_{S_1S_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma_{S_2S_1} & \sigma^2_{S_2} & \sigma_{S_2S_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma_{S_3S_1} & \sigma_{S_3S_2} & \sigma^2_{S_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2_{X_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma^2_{Y_1} & 0 & 0 & 00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma^2_{Z_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2_{X_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2_{Y_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2_{Z_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2_{X_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2_{Y_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2_{Z_3} \end{bmatrix} \end{pmatrix} J' \tag{4-10}$$

with the 3×3 elements of Σ_y on the right hand side of (4-10) given by (4-4) in page (71). The variance-covariance matrix computed as explained above is obtained for every combinatorial set $j = 1, \ldots, 35$. Finally we obtained the dispersion matrix Σ from (2-70) in Chapter 2 page (24).

Step 6 (rigorous adjustment of the combinatorial solution points in a polyhedron):
For each of the 35 computed coordinates of point K1 in *Figure* (4.1) in step 4, we write the observation equations as

$$\begin{bmatrix} X^j = X + \varepsilon^j_X \,|, j \in \{1,2,3,4,5,6,7,\ldots,35\} \\ Y^j = Y + \varepsilon^j_Y \,| j \in \{1,2,3,4,5,6,7,\ldots,35\} \\ Z^j = Z + \varepsilon^j_Z \,|, j \in \{1,2,3,4,5,6,7,\ldots,35\}. \end{bmatrix} \tag{4-11}$$

With the values $\{X^j, Y^j, Z^j\}$ treated as pseudo-observation and placed in the vector of observation $\boldsymbol{y}$, the coefficients of the unknown position $\{X, Y, Z\}$ being placed in the coefficient matrix $\boldsymbol{A}$ and $\boldsymbol{x}$ comprising the vector of unknowns $\{X, Y, Z\}$, The solution is obtained via (2-4) and the dispersion of the estimated parameters through (2-5) in Chapter 2 page (8).

4-4 Results

4-41 Distances

Presented in *Table* (4.14) are the results of the computed closed form three-dimensional resection distances from *K1 to Haussmanstr.*$(PP_1 := S_1)$, *K1-Eduardpfeiffer* $(PP_2 := S_2)$ and *K1-Liederhalle* $(PP_4 := S_3)$ (e.g. tetrahedron $\{PP_1P_2P_4\}$ in *Appendix A.2* page 104) and their deviation ΔS obtained by subtracting the computed distance S_i from its ideal value S in *Table* (4.2). Observational set number 0* comprised the ideal values of *Table* (4.2) which are used to control the Experiment.

In *Tables* (4.15), (4.16) and (4.17) are presented the results of the adjusted distances, root-mean-square-errors and the deviations in distances computed using the *Gauss-Jacobi combinatorial* algorithm. The root-mean-square error are computed from (2-5) in Chapter 2 page (8). The deviation in distances are obtained by subtracting the computed distance S_i from its ideal value S (in *Table* 4.2, page 65). The adjusted distances in *Tables* (4.15) were: *K1-Haussmanstr.*(S_1) , *K1-Eduardpfeiffer* (S_2), *K1-Lindenmuseum* (S_3) , *K1-Liederhalle* (S_4), *K1-Dach LVM* (S_5) , *K1-Dach FH* (S_6) *and K1-Haussmanstr.*(S_7).

4-42 Position

The obtained position of station K1 from the 11 sets under study are presented in *Tables* (4.18) and (4.19). Set 0* indicate the results of the theoretical set. Since the value of K1 is known (e.g. *Table* (4.1), the deviations $\{\Delta X, \Delta Y, \Delta Z\}$ of the computed positions from their real values are computed for both the *closed form three-dimensional resection* and the *overdetermined three-dimensional resection* for each observational data set and are plotted as in *Figure* (4.2). *Figure* (4.2) indicates the results of the *overdetermined three-dimension resection* for the test network *"Stuttgart Central"* computed from the *Gauss-Jacobi algorithm* to be better than those computed from closed form procedures. The root-mean-square errors are computed from (2-5) in Chapter 2 page (8). *Figures* (4.3)-(4.8) illustrates the plotted three-dimensional positional scatter of the 35 minimal combinatorial subsets (indicated by doted points (•)) around for the adjusted value of position indicated by a star (⋆). The plot is done for each observational data set in *Tables* (4.3) to (4.13).

Table 4.14: Distances computed by *Gröbner basis* or *Multipolynomial resultants*

set no.	$S_1(m)$	$S_2(m)$	$S_3(m)$	$\Delta S_1(m)$	$\Delta S_2(m)$	$\Delta S_3(m)$
0*	1324.2380	542.2609	430.5286	0.0000	0.0000	0.0000
1	1324.2420	542.2613	430.5247	-0.0040	-0.0004	0.0039
2	1324.2479	542.2707	430.5138	-0.0099	-0.0098	0.0148
3	1324.2412	542.2615	430.5247	-0.0032	-0.0006	0.0039
4	1324.2342	542.2588	430.5330	0.0038	0.0021	-0.0043
5	1324.2340	542.2626	430.5313	0.0040	-0.0017	-0.0027
6	1324.2442	542.2664	430.5193	-0.0063	-0.0055	0.0093
7	1324.2453	542.2540	430.5241	-0.0073	0.0069	0.0045
8	1324.2376	542.2585	430.5301	0.0003	0.0024	-0.0015
9	1324.2382	542.2560	430.5307	-0.0002	0.0049	-0.0021
10	1324.2369	542.2652	430.5276	0.0011	-0.0043	0.0010
11	1324.2375	542.2594	430.5299	0.0005	0.0015	-0.0013

Table 4.15: Distances computed by *Gauss-Jacobi combinatorial algorithm*

No.	$S_1(m)$	$S_2(m)$	$S_3(m)$	$S_4(m)$	$S_5(m)$	$S_6(m)$	$S_7(m)$
1	1324.2394	542.2598	364.9782	430.5281	400.5834	269.2303	566.8641
2	1324.2387	542.2606	364.9801	430.5274	400.5818	269.2292	566.8635
3	1324.2381	542.2604	364.9791	430.5267	400.5847	269.2296	566.8632
4	1324.2363	542.2545	364.9782	430.5355	400.5931	269.2385	566.8664
5	1324.2396	542.2611	364.9779	430.5259	400.5834	269.2306	566.8658
6	1324.2378	542.2584	364.9791	430.5300	400.5868	269.2320	566.8637
7	1324.2368	542.2558	364.9790	430.5328	400.5857	269.2345	566.8644
8	1324.2388	542.2575	364.9779	430.5324	400.5845	269.2342	566.8664
9	1324.2393	542.2646	364.9794	430.5232	400.5770	269.2265	566.8623
10	1324.2337	542.2598	364.9832	430.5350	400.5904	269.2346	566.8608
11	1324.2375	542.2573	364.9787	430.5326	400.5884	269.2344	566.8650

Table 4.16: Root-mean-square-errors of distances in *Table* (4.15)

set no.	$S_1(m)$	$S_2(m)$	$S_3(m)$	$S_4(m)$	$S_5(m)$	$S_6(m)$	$S_7(m)$
1	0.0004	0.0004	0.0004	0.0005	0.0004	0.0006	0.0003
2	0.0011	0.0012	0.0013	0.0015	0.0014	0.0019	0.0009
3	0.0008	0.0008	0.0009	0.0010	0.0009	0.0013	0.0006
4	0.0007	0.0007	0.0008	0.0010	0.0009	0.0013	0.0006
5	0.0008	0.0008	0.0009	0.0010	0.0010	0.0013	0.0006
6	0.0008	0.0008	0.0009	0.0010	0.0010	0.0013	0.0006
7	0.0012	0.0012	0.0013	0.0016	0.0014	0.0020	0.0010
8	0.0010	0.0010	0.0011	0.0013	0.0012	0.0016	0.0008
9	0.0009	0.0009	0.0010	0.0012	0.0011	0.0015	0.0007
10	0.0006	0.0006	0.0006	0.0008	0.0007	0.0010	0.0005
11	0.0005	0.0005	0.0005	0.0006	0.0006	0.0008	0.0004

Table 4.17: Deviations of distances in *Table* (4.15) from values in Table (4.2)

set no.	$\Delta S_1(m)$	$\Delta S_2(m)$	$\Delta S_3(m)$	$\Delta S_4(m)$	$\Delta S_5(m)$	$\Delta S_6(m)$	$\Delta S_7(m)$
1	-0.0014	0.0011	0.0015	0.0005	0.0002	0.0006	-0.0006
2	-0.0007	0.0003	-0.0004	0.0012	0.0019	0.0017	0.0000
3	-0.0002	0.0005	0.0006	0.0019	- 0.0010	0.0013	0.0004
4	0.0017	0.0064	0.0015	-0.0069	-0.0094	-0.0075	-0.0028
5	-0.0016	-0.0002	0.0018	0.0027	0.0003	0.0003	-0.0023
6	0.0002	0.0025	0.0006	-0.0014	-0.0031	-0.0011	-0.0002
7	0.0012	0.0051	0.0007	-0.0042	-0.0020	-0.0035	-0.0009
8	-0.0009	0.0034	0.0018	-0.0037	-0.0008	-0.0033	-0.0028
9	-0.0013	-0.0037	0.0003	0.0054	0.0067	0.0044	0.0013
10	0.0042	0.0011	-0.0035	-0.0063	-0.0067	-0.0037	0.0027
11	0.0004	0.0036	0.0010	-0.0040	-0.0047	-0.0034	-0.0015

Table 4.18: K1 computed by *Gröbner basis* or *Multipolynomial resultants*

Set No.	$X(m)$	$Y(m)$	$Z(m)$	$\Delta X(m)$	$\Delta Y(m)$	$\Delta Z(m)$
0	4157066.1116	671429.6655	4774879.3704	0	0	0
1	4157066.1166	671429.6625	4774879.3720	0.0050	0.0030	-0.0016
2	4157066.1220	671429.6586	4774879.3599	-0.0104	0.0069	0.0105
3	4157066.1105	671429.6622	4774879.3661	0.0011	0.0033	0.0043
4	4157066.1045	671429.6678	4774879.3688	0.0071	-0.0023	0.0016
5	4157066.1068	671429.6688	4774879.3658	0.0048	-0.0033	0.0046
6	4157066.1149	671429.6606	4774879.3614	-0.0033	0.0049	0.0090
7	4157066.1074	671429.6569	4774879.3708	0.0042	0.0086	-0.0004
8	4157066.1099	671429.6653	4774879.3724	0.0017	0.0002	-0.0020
9	4157066.1084	671429.6642	4774879.3740	0.0032	0.0013	-0.0036
10	4157066.1138	671429.6674	4774879.3673	-0.0022	-0.0019	0.0031
11	4157066.1107	671429.6657	4774879.3721	0.0009	-0.0002	-0.0017

Table 4.19: Position of K1 computed by *Gauss-Jacobi combinatorial algorithm*

Set No.	$X(m)$	$Y(m)$	$Z(m)$	$\sigma_X(m)$	$\sigma_Y(m)$	$\sigma_Z(m)$
1	4157066.1142	671429.6642	4774879.3705	0.00007	0.00002	0.00007
2	4157066.1150	671429.6656	4774879.3695	0.00009	0.00001	0.00008
3	4157066.1100	671429.6650	4774879.3676	0.00010	0.00002	0.00010
4	4157066.1040	671429.6648	4774879.3688	0.00008	0.00002	0.00008
5	4157066.1089	671429.6635	4774879.3699	0.00016	0.00003	0.00015
6	4157066.1127	671429.6651	4774879.3684	0.00017	0.00003	0.00016
7	4157066.1089	671429.6655	4774879.3699	0.00009	0.00002	0.00009
8	4157066.1102	671429.6643	4774879.3720	0.00009	0.00002	0.00008
9	4157066.1106	671429.6649	4774879.3699	0.00004	0.00001	0.00003
10	4157066.1121	671429.6694	4774879.3697	0.00005	0.00001	0.00005
11	4157066.1100	671429.6654	4774879.3705	0.00005	0.00001	0.00005

Table 4.20: Deviation of K1 in Table (4.19) from the real value in Table (4.1)

Set No.	$\Delta X(m)$	$\Delta Y(m)$	$\Delta Z(m)$
1	-0.0026	0.0013	-0.0001
2	-0.0034	-0.0001	0.0009
3	0.0016	0.0005	0.0028
4	0.0076	0.0007	0.0016
5	0.0027	0.0020	0.0005
6	-0.0011	0.0004	0.0020
7	0.0027	-0.0000	0.0005
8	0.0014	0.0012	-0.0016
9	0.0010	0.0006	0.0005
10	-0.0005	-0.0039	0.0007
11	0.0016	0.0001	-0.0001

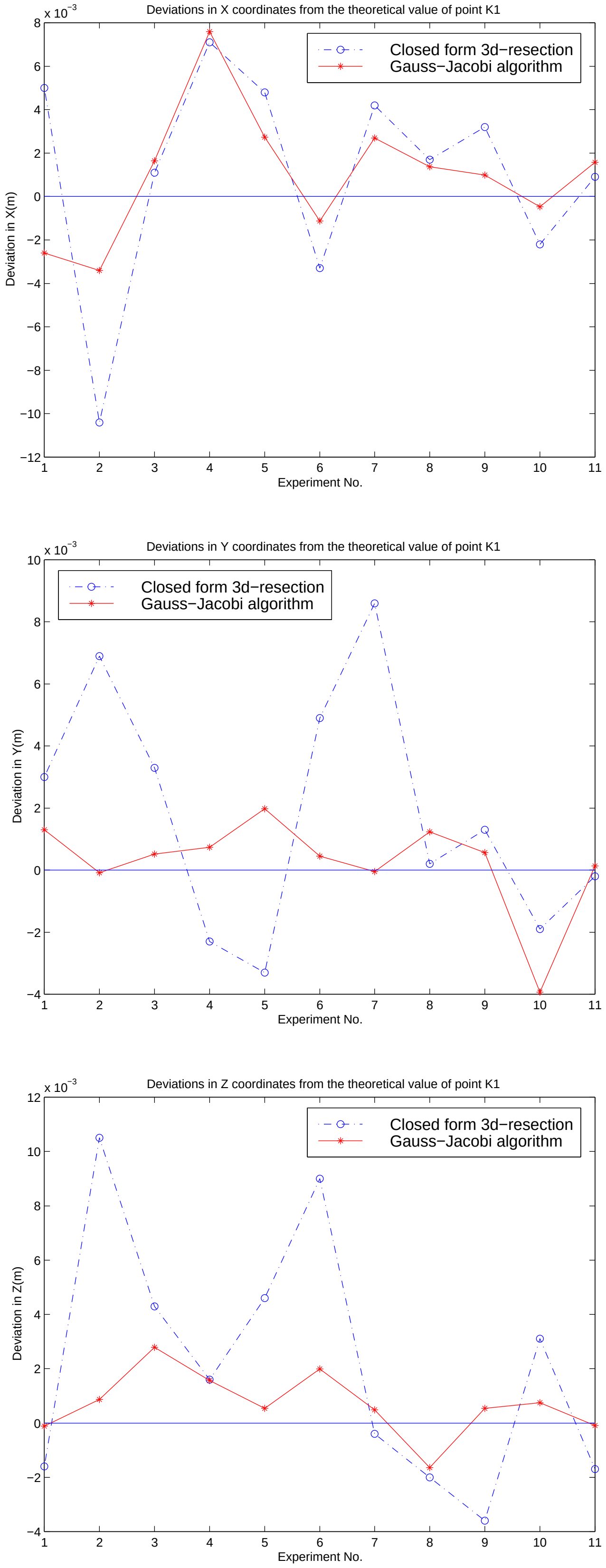
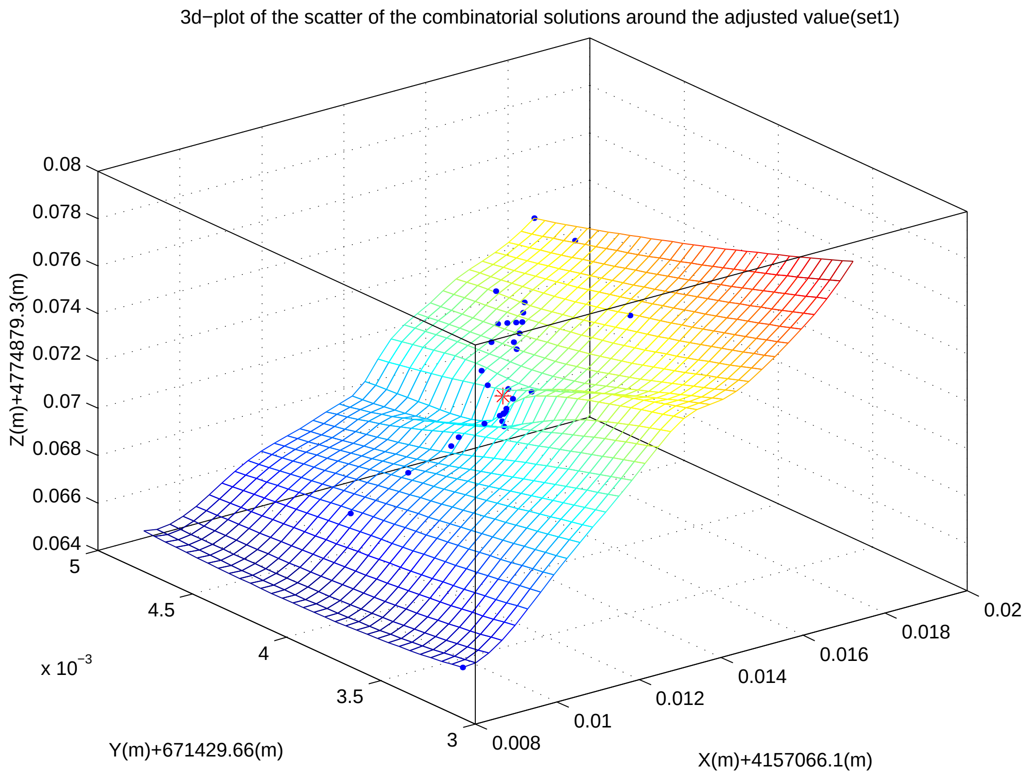


Figure 4.2: Deviation of computed position of station K1 in Tables (4.18) and (4.20) from the real value in Table (4.1)



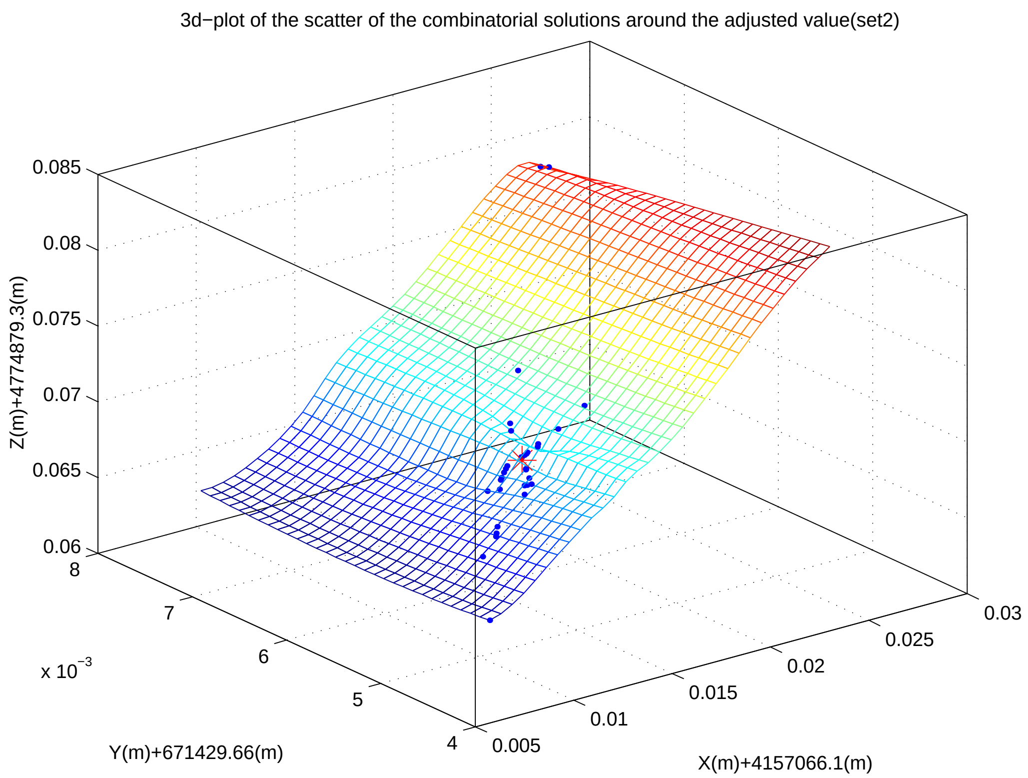
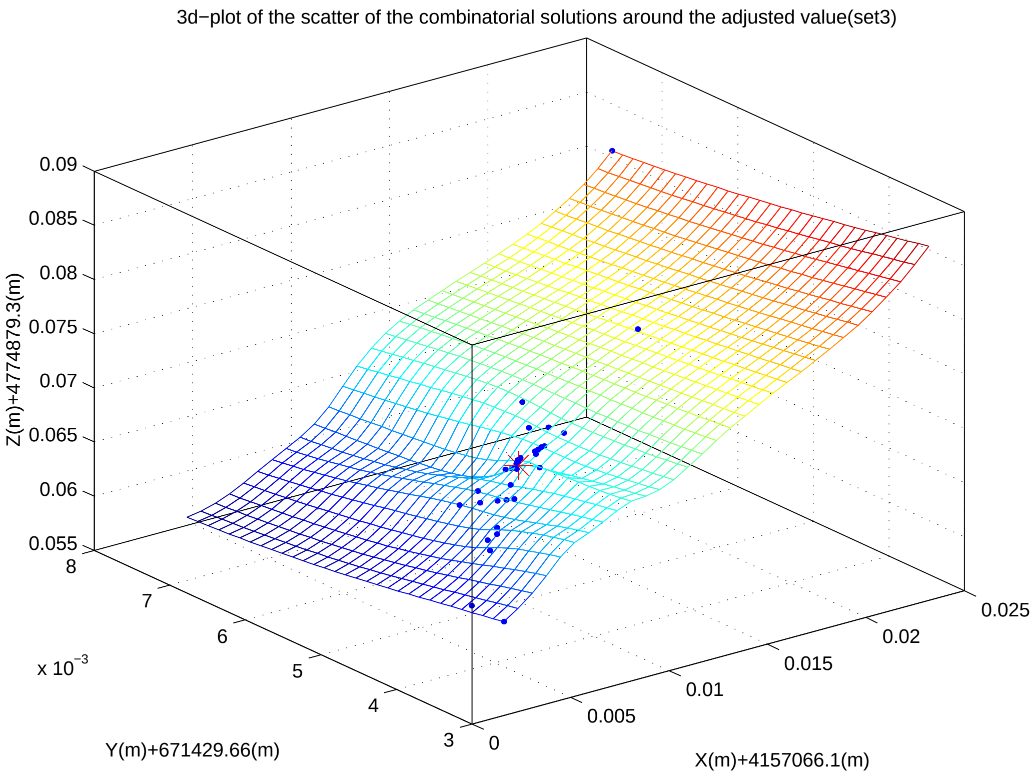


Figure 4.3: Scatter of combinatorial solutions for data sets 1 and 2 in *Tables* (4.3) and (4.4)



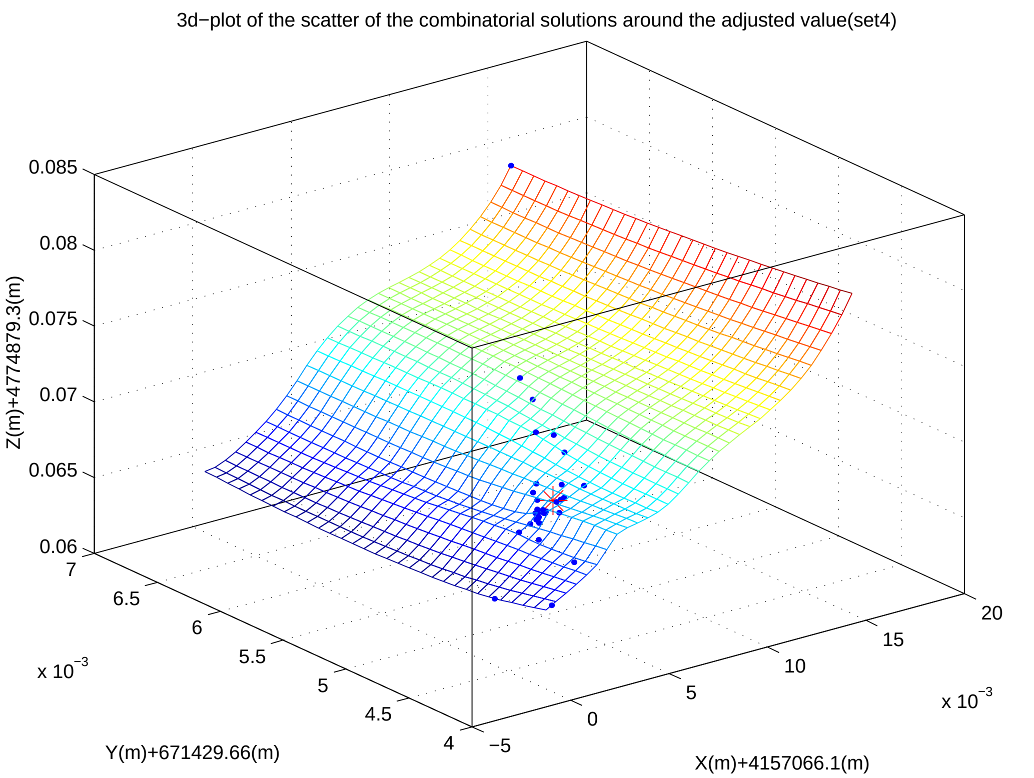
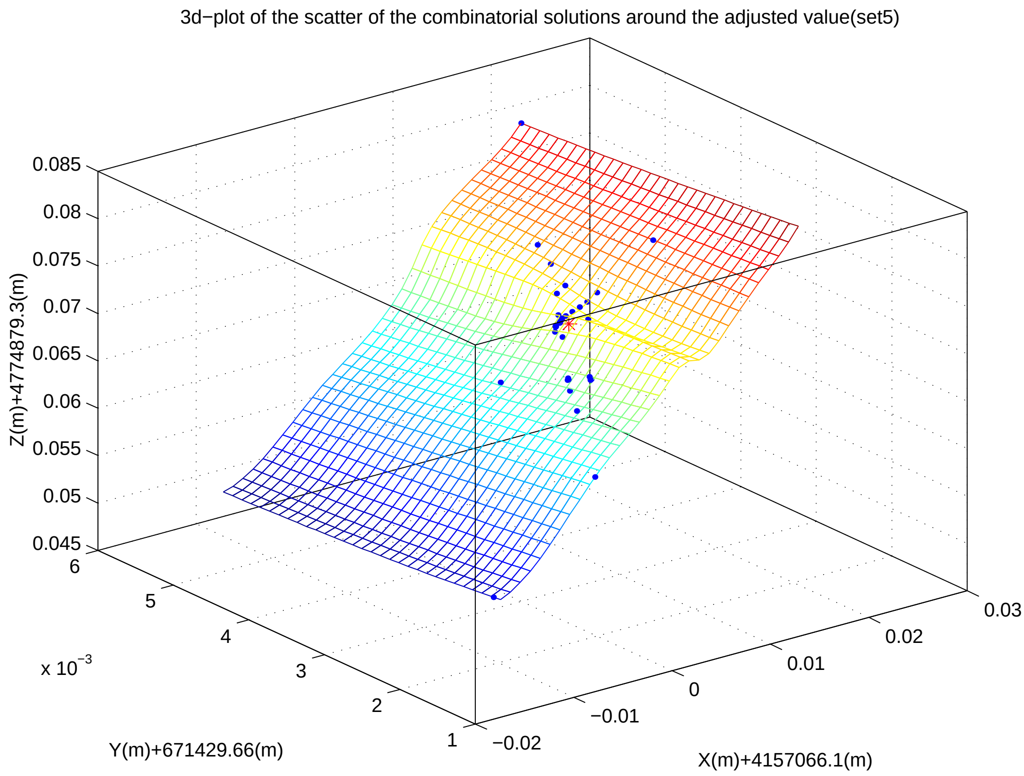


Figure 4.4: Scatter of combinatorial solutions for data sets 3 and 4 in *Tables* (4.5) and (4.6)



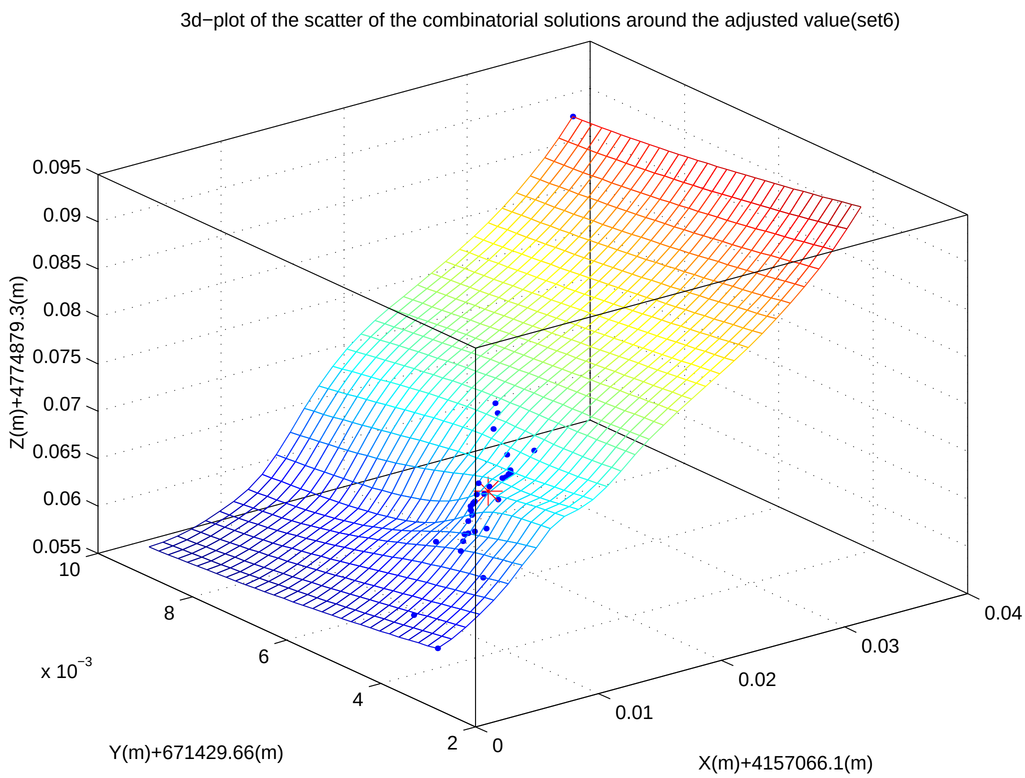
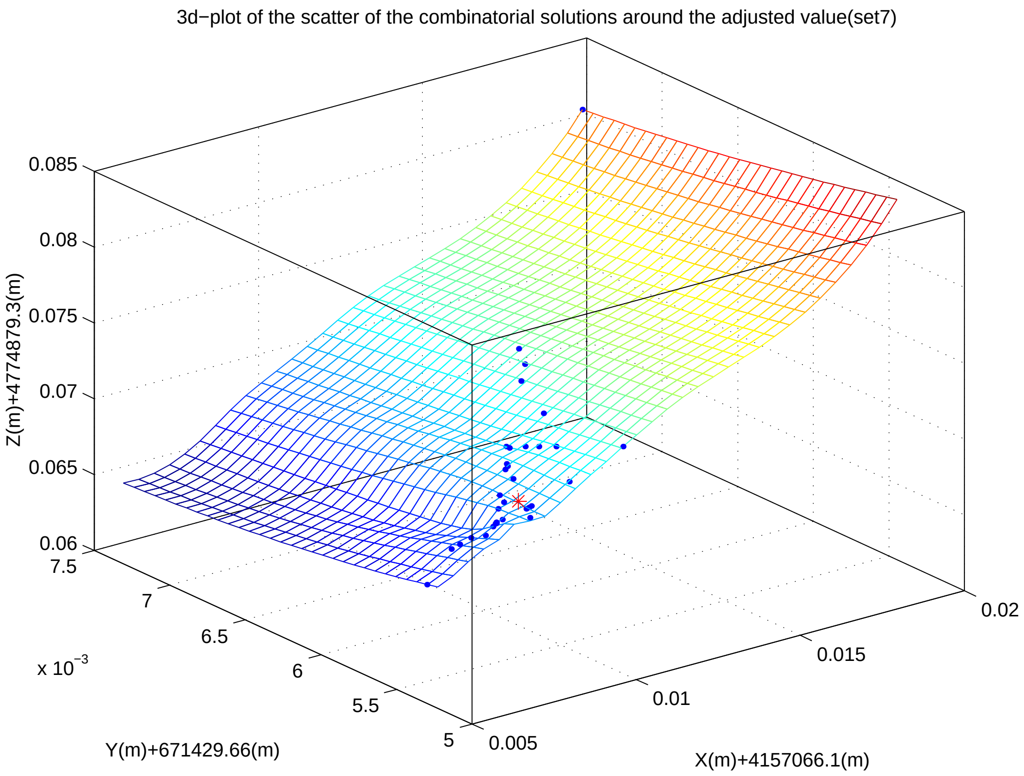


Figure 4.5: Scatter of combinatorial solutions for data sets 5 and 6 in *Tables* (4.7) and (4.8)



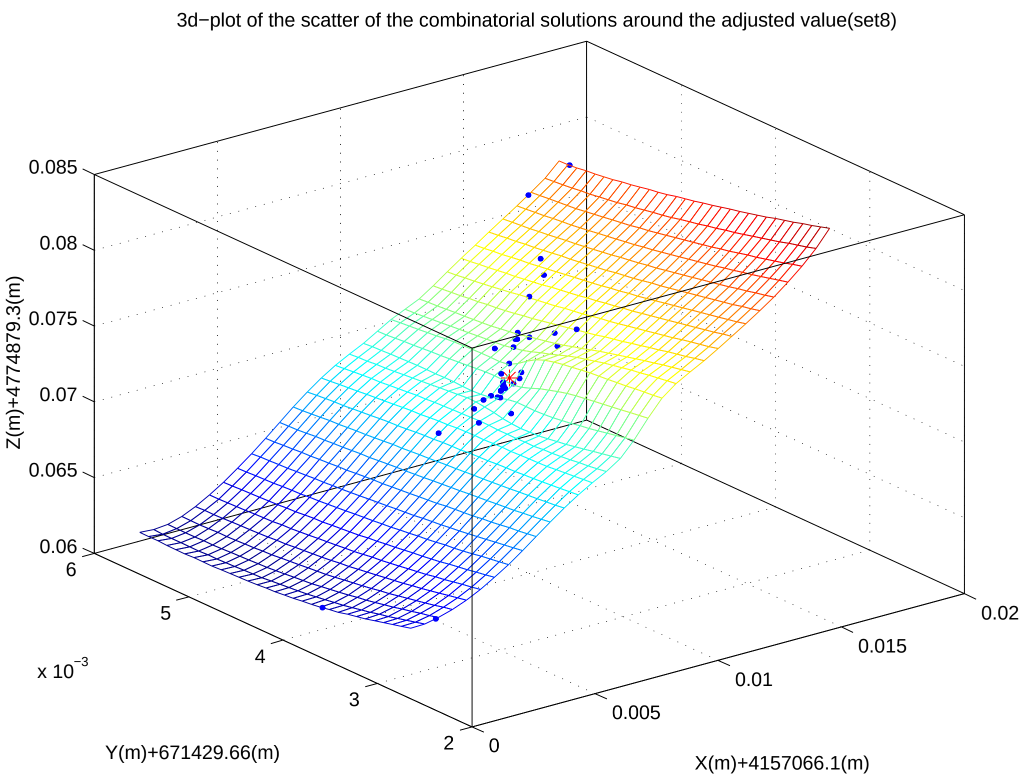
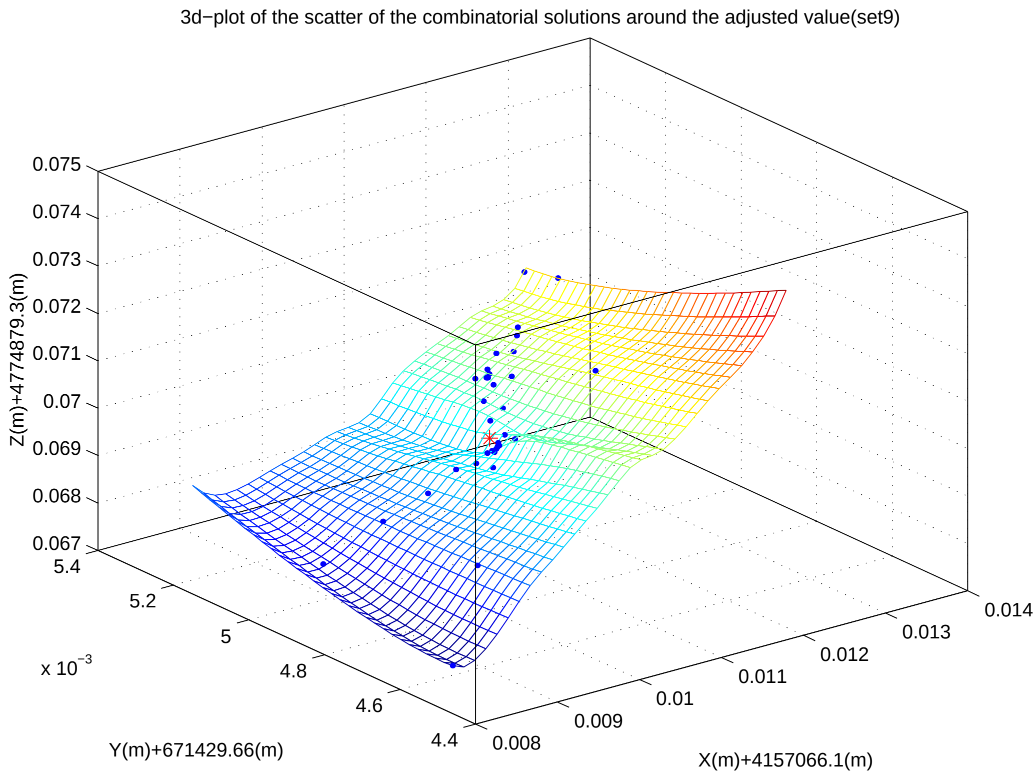


Figure 4.6: Scatter of combinatorial solutions for data sets 7 and 8 in *Tables* (4.9) and (4.10)



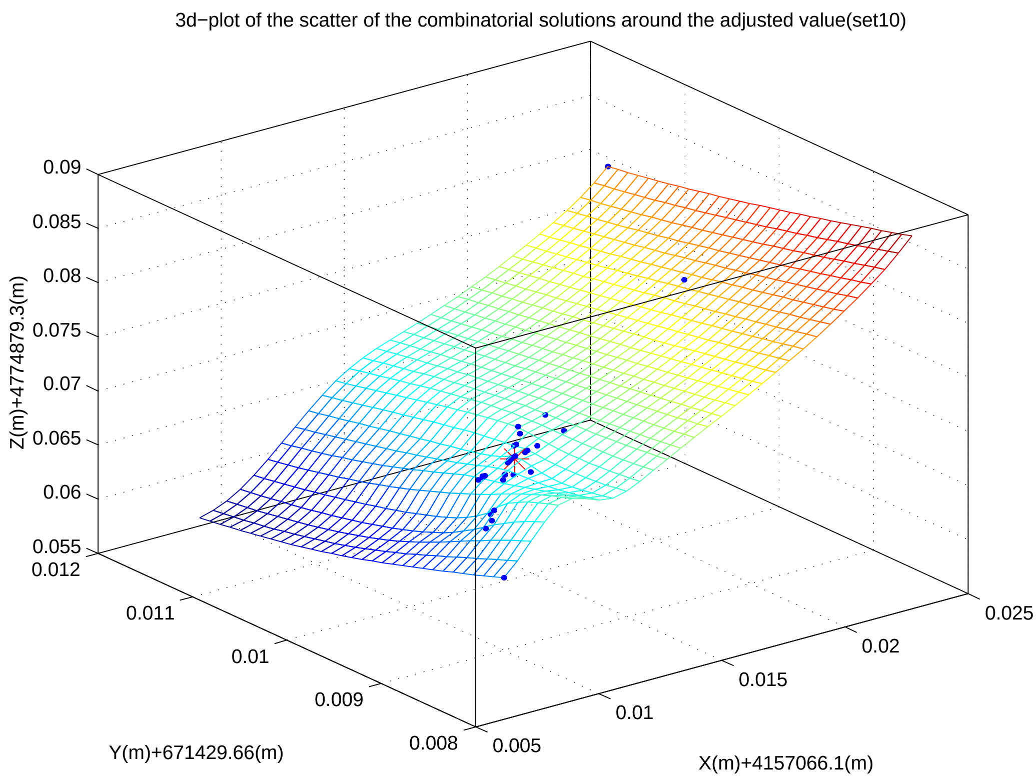


Figure 4.7: Scatter of combinatorial solutions for data sets 9 and 10 in *Tables* (4.11) and (4.12)

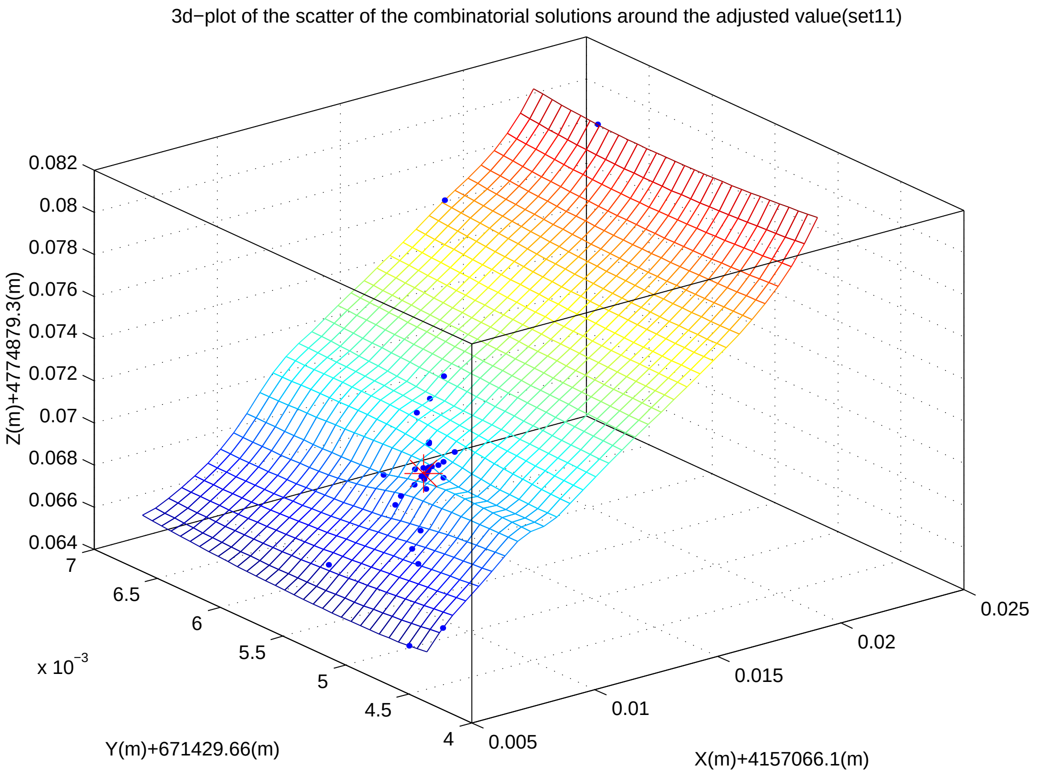


Figure 4.8: Scatter of combinatorial solutions for data set 11 *Table* (4.13)

Chapter 5

Case studies

5-1 Geocentric Cartesian to Gauss ellipsoidal coordinates

In order to relate a point P on the Earth's topographic surface to a point p on the *International Reference Ellipsoid* $\mathbb{E}^2_{a,a,b}$ we work with a bundle of half straight lines so called *projection lines* which depart from P and intersect $\mathbb{E}^2_{a,a,b}$ either not at all or in two points. There is *one projection line* which is at minimum distance relating P to p. *Figure* (5.1) is an illustration of such a *Minimum Distance Mapping*. Such an *optimization problem* is formulated by means of the Lagrangean $\pounds(x_1, x_2, x_3, x_4)$ as in (3-112) respectively (3-113).

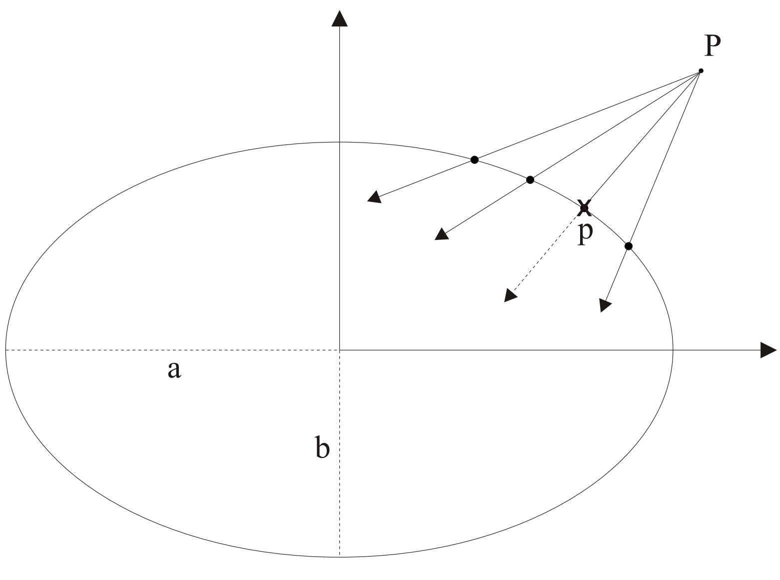


Figure 5.1: Minimum distance mapping of a point P on the Earth's topographic surface to a point p on the *International Reference Ellipsoid* $\mathbb{E}^2_{a,a,b}$

In the first case we represent the *Euclidean distance* between the points P and p in terms of *Cartesian coordinates* of $P(X, Y, Z)$ and of $p(x_1, x_2, x_3)$. The *Cartesian coordinates* (x_1, x_2, x_3) of the projection point P are *unknown*. The constraint that the n is an element of the *ellipsoid-of-revolution* $\mathbb{E}^2_{a,a,b} := \{\mathbf{x} \in \mathbb{R}^3 | b^2(x_1^2 + x_2^2) + a^2 x_2^3 - a^2 b^2 = 0, \mathbb{R}^+ \ni a > b \in \mathbb{R}^+\}$ is substituted into the *Lagrangean* by means of the *Lagrange multiplier* x_4 which is unknown, too. $(x_1^\wedge, x_2^\wedge, x_3^\wedge, x_4^\wedge) = arg\{\pounds(x_1, x_2, x_3, x_4) = min\}$ is the argument of the minimum of the *constrained Lagrangean* $\pounds(x_1, x_2, x_3, x_4)$. The result of the minimization procedure is presented by *Lemma (5-1)*. (LC1) provides the necessary conditions to constitute an extremum: The normal equations are of bilinear type. *Products* of the unknowns for instance $x_1 x_4, x_2 x_4, x_3 x_4$ and *squares* of the unknowns, for instance x_1^2, x_2^2, x_3^2 appear. Finally the matrix of second derivatives H_3 in (LC2) to be *positive definite* constitutes the *necessary condition* to obtain a minimum. Fortunately the matrix of second derivatives H_3 is *diagonal*. The eigenvalues of H_3 are $\Lambda_1 = \Lambda_2 = X \backslash x_1^\wedge, \Lambda_3 = Z \backslash x_3^\wedge$ and *must be positive*.

Lemma 5-1 (*constrained minimum distance mapping*):
The functional $\pounds(x_1, x_2, x_3, x_4)$ ist minimal, if the conditions (LC1) and (LC2) hold.

$$\text{(LC1)} \quad \frac{\partial \pounds}{\partial x_i}(x_1^\wedge, x_2^\wedge, x_3^\wedge, x_4^\wedge) = 0 \quad \forall \quad \text{i=1,2,3,4}$$

$$\begin{aligned} (i) \quad & \frac{\partial \pounds}{\partial (\mathbf{x}^\wedge)} = -(X - x_1^\wedge) + b^2 x_1^\wedge x_4^\wedge = 0 \\ (ii) \quad & \frac{\partial \pounds}{\partial (\mathbf{x}^\wedge)} = -(Y - x_2^\wedge) + b^2 x_2^\wedge x_4^\wedge = 0 \\ (iii) \quad & \frac{\partial \pounds}{\partial (\mathbf{x}^\wedge)} = -(Z - x_3^\wedge) + a^2 x_3^\wedge x_4^\wedge = 0 \\ (iv) \quad & \frac{\partial \pounds}{\partial (\mathbf{x}^\wedge)} = \frac{1}{2}[b^2(x_1^{\wedge 2} + x_4^{\wedge 2})] + a^2 x_3^{\wedge 2} - a^2 b^2 = 0 \end{aligned} \tag{5-1}$$

$$\text{(LC2)} \quad \frac{\partial^2 \pounds}{\partial x_i \partial x_j}(x_1^\wedge, x_2^\wedge, x_3^\wedge, x_4^\wedge) = 0 \quad \forall \quad \text{i,j} \in \{1,2,3\} \tag{5-2}$$

$$\begin{aligned} \boldsymbol{H}_3 \quad := \quad & \left[\frac{\partial^2 \pounds}{\partial x_i \partial x_j}(\mathbf{x}^\wedge)\right] \\ = \quad & \begin{bmatrix} 1 + b_2 x_4^\wedge & 0 & \\ 0 & 1 + b_2 x_4^\wedge & 0 \\ 0 & 0 & 1 + a_2 x_4^\wedge \end{bmatrix} \quad \in \quad \mathbb{R}^{3 \times 3} \end{aligned} \tag{5-3}$$

"*eigenvalues*"

$$|\boldsymbol{H}_3 - \Lambda \boldsymbol{I}_3| = 0 \quad \Longleftrightarrow \tag{5-4}$$

$$\Lambda_1 = \Lambda_2 := 1 + b^2 x_4^\wedge = \frac{X}{x_1^\wedge} = \frac{Y}{x_2^\wedge} \tag{5-5}$$

$$\Lambda_3 := 1 + a^2 x_4^\wedge = \frac{Z}{x_3^\wedge} \tag{5-6}$$

Box 5-1 (*Conversion from Cartesian coordinates to Gauss ellipsoidal coordinates*):

Closed form solution:
$\{X, Y, Z\} \in \mathbb{T}^2 \quad \{x_1, x_2, x_3\} \in E^2_{a,a,b}$
to $\{L, B, H\}$

"Pythagoras in three dimension"

$$H := \sqrt{(X - x_1)^2 + (Y - x_2)^2 + (Z - x_3)^2} \tag{5-7}$$

"convert $\{x_1, x_2, x_3\}$ and $\{X, Y, Z\}$
to $\{L, B\}$"

$$tanL = \frac{Y - x_2}{X - x_1} = \frac{Y - y}{X - x} \tag{5-8}$$

$$tanB = \frac{Z - x_3}{\sqrt{(X - x_1)^2 + (Y - x_2)^2}} = \frac{Z - x_3}{\sqrt{(X - x)^2 + (Y - y)^2}} \tag{5-9}$$

The nonlinear (algebraic: bilinear) *normal equations* (LC1) of *Minimum distance Mapping* $\mathbb{T}^2 \longrightarrow \mathbb{E}^2_{a,a,b}$ was solved in Section (3-22) in a closed form by means of *Gröbner basis*. The computed elements of the *Gröbner basis* are presented in *Box* (3-11) in page (50). Using (3-116) and (3-117) in page (51), the Lagrangean multiplier x_4 and ellipsoidal Cartesian coordinates $\{x_1, x_2, x_3\}$ are computed. Finally, by means of *Box* (5-1) we convert the Cartesian coordinates $(X, Y, Z) \in \mathbb{T}^2$ and $(x_1, x_2, x_3) \in \mathbb{E}^2_{a,a,b}$ to *Gauss ellipsoidal coordinates* L, B, H.

Case study:

Let us adopt the *World Geodetic Datum 2000* with the data $a = 6378136.602m$ and $b = 6356751.860m$ of type *semi-major axis* and *semi-minor axis respectively*, of the *International Reference Ellipsoid (E. Grafarend and A. Ardalan 1999).* Here we take advantage of given *Cartesian coordinates of 21 points* of the topographic surface of the Earth presented in *Table* (5.1). From the algorithm of *Box* (3-11) in page (50), the first polynomial equation of fourth order of the *Gröbner basis* (3-116) in page (51) is solved. We compute the coefficients from the input data *Table* (5.1) and solve for x_4 according to *Table* (5.2). With the admissible values x_4 substituted in the computed *Gröbner basis* (3-117) in page (51) we finally produce the values $(x_1, x_2, x_3) = (x, y, z)$ in *Table* (5.3). At this end *Table* (5.4) converts by means of (5-1) the Cartesian coordinates (X, Y, Z) and (x, y, z) to (L, B, H).

Table 5.1: Cartesian coordinates of topographic point (Baltic Sea Level Project)

Station	$X(m)$	$Y(m)$	$Z(m)$
Borkum (Ger)	3770667.9989	446076.4896	5107686.2085
Degerby (Fin)	2994064.9360	1112559.0570	5502241.3760
Furuögrund (Swe)	2527022.8721	981957.2890	5753940.9920
Hamina (Fin)	2795471.2067	1435427.7930	5531682.2031
Hanko (Fin)	2959210.9709	1254679.1202	5490594.4410
Helgoland (Ger)	3706044.9443	513713.2151	5148193.4472
Helsinki (Fin)	2885137.3909	1342710.2301	5509039.1190
Kemi (Fin)	2397071.5771	1093330.3129	5789108.4470
Klagshamn (Swe)	3527585.7675	807513.8946	5234549.7020
Klaipeda (Lit)	3353590.2428	1302063.0141	5249159.4123
List/Sylt (Ger)	3625339.9221	537853.8704	5202539.0255
Molas (Lit)	3358793.3811	1294907.4149	5247584.4010
Mäntyluoto (Fin)	2831096.7193	1113102.7637	5587165.0458
Raahe (Fin)	2494035.0244	1131370.9936	5740955.4096
Ratan (Swe)	2620087.6160	1000008.2649	5709322.5771
Spikarna (Swe)	2828573.4638	893623.7288	5627447.0693
Stockholm (Swe)	3101008.8620	1013021.0372	5462373.3830
Ustka (Pol)	3545014.3300	1073939.7720	5174949.9470
Vaasa (Fin)	2691307.2541	1063691.5238	5664806.3799
Visby (Swe)	3249304.4375	1073624.8912	5364363.0732
Ölands N. U. (Swe)	3295551.5710	1012564.9063	5348113.6687

Table 5.2: Polynomial coefficients $c_0,\ c_1,\ c_2,\ c_3,\ c_4$ of the univariate polynomial of order four in x_4

Point	c_0	c_1	c_2	c_3	c_4
1	-2.3309099e+22	1.334253e+41	1.351627e+55	4.38e+68	4.44e+81
2	-1.142213e+22	1.3351890e+41	1.352005e+55	4.38e+68	4.44e+81
3	-1.720998e+22	1.335813e+41	1.352259e+55	4.38e+68	4.44e+81
4	-8.871288e+21	1.335264e+41	1.352035e+55	4.38e+68	4.44e+81
5	-1.308070e+22	1.335160e+41	1.351993e+55	4.38e+68	4.44e+81
6	-2.275210e+22	1.334345e+41	1.351665e+55	4.38e+68	4.44e+81
7	-1.272935e+22	1.335205e+41	1.352012e+55	4.38e+68	4.44e+81
8	-1.373946e+22	1.335906e+41	1.352296e+55	4.38e+68	4.441e+81
9	-1.981047e+22	1.334546e+41	1.351746e+55	4.38e+68	4.44e+81
10	-2.755981e+22	1.334574e+41	1.351758e+55	4.38e+68	4.44e+81
11	-2.330047e+22	1.334469e+41	1.351715e+55	4.38e+68	4.44e+81
12	-1.538357e+22	1.334580e+41	1.351759e+55	4.38e+68	4.44e+81
13	-1.117760e+22	1.335399e+41	1.352090e+55	4.38e+68	4.44e+81
14	-1.124559e+22	1.335785e+41	1.352246e+55	4.38e+68	4.44e+81
15	-1.200556e+22	1.335704e+41	1.352214e+55	4.38e+68	4.44e+81
16	-1.427443e+22	1.335496e+41	1.352130e+55	4.38e+68	4.44e+81
17	-1.836471e+22	1.335087e+41	1.351965e+55	4.38e+68	4.44e+81
18	-1.772332e+22	1.334410e+41	1.351690e+55	4.38e+68	4.44e+81
19	-1.012020e+22	1.335593e+41	1.352168e+55	4.38e+68	4.44e+81
20	-1.427711e+22	1.334856e+41	1.351870e+55	4.38e+68	4.44e+81
21	-1.644250e+22	1.334815e+41	1.351854e+55	4.38e+68	4.44e+81

Table 5.3: Computed Cartesian coordinates $(x_1, x_2, x_3) = (x, y, z)$ and Lagrange multiplier x_4

Station	$x_1(m)$	$x_2(m)$	$x_3(m)$	$x_4(m^{-2})$
Borkum	3770641.3815	446073.3407	5107649.9100	1.746947e-019
Degerby	2994054.5862	1112555.2111	5502222.2279	8.554612e-020
Furuögrund	2527009.7166	981952.1770	5753910.8356	1.288336e-019
Hamina	2795463.7019	1435423.9394	5531667.2524	6.643801e-020
Hanko	2959199.2560	1254674.1532	5490572.5584	9.797001e-020
Helgoland	3706019.4100	513709.6757	5148157.7376	1.705084e-019
Helsinki	2885126.2764	1342705.0575	5509017.7534	9.533532e-020
Kemi	2397061.6153	1093325.7692	5789084.2263	1.028464e-019
Klagshamn	3527564.6083	807509.0510	5234518.0924	1.484413e-019
Klaipeda	3353562.2593	1302052.1493	5249115.3164	2.065021e-019
List/Sylt	3625314.3442	537850.0757	5202502.0726	1.746017e-019
Molas	3358777.7367	1294901.3835	5247559.7944	1.152676e-019
Mäntyluoto	2831087.1439	1113098.9988	5587146.0214	8.370165e-020
Raahe	2494026.5401	1131367.1449	5740935.7483	8.418639e-020
Ratan	2620078.1000	1000004.6329	5709301.7015	8.988111e-020
Spikarna	2828561.2473	893619.8693	5627422.6007	1.068837e-019
Stockholm	3100991.6259	1013015.4066	5462342.8173	1.375524e-019
Ustka	3544995.3045	1073934.0083	5174921.9867	1.328158e-019
Vaasa	2691299.0138	1063688.2670	5664788.9183	7.577249e-020
Visby	3249290.3945	1073620.2512	5364339.7330	1.069551e-019
Ölands N. U.	3295535.1675	1012559.8663	5348086.8692	1.231803e-019

Table 5.4: Geodetic Coordinates computed from ellipsoidal Cartesian coordinates in closed form (Baltic Sea Level Project)

Station	Longitude L $\circ$ $'$ $''$	Latitude B $\circ$ $'$ $''$	height H m
Borkum (Ger)	6 44 48.5914	53 33 27.4808	45.122
Degerby (Fin)	20 23 4.0906	60 1 52.8558	22.103
Furuögrund (Swe)	21 14 6.9490	64 55 10.2131	33.296
Hamina (Fin)	27 10 47.0690	60 33 52.9819	17.167
Hanko (Fin)	22 58 35.4445	59 49 21.6459	25.313
Helgoland (Ger)	7 53 30.3480	54 10 29.3979	44.042
Helsinki (Fin)	24 57 24.2446	60 9 13.2416	24.633
Kemi (Fin)	24 31 5.6737	65 40 27.7029	26.581
Klagshamn (Swe)	12 53 37.1597	55 31 20.3311	38.345
Klaipeda (Lit)	21 13 9.0156	55 45 16.5952	53.344
List/Sylt (Ger)	8 26 19.7594	55 1 3.0992	45.101
Molas (Lit)	21 4 58.8931	55 43 47.2453	29.776
Mäntyluoto (Fin)	21 27 47.7777	61 35 39.3552	21.628
Raahe (Fin)	24 24 1.8197	64 38 46.8352	21.757
Ratan (Swe)	20 53 25.2392	63 59 29.5936	23.228
Spikarna (Swe)	17 31 57.9060	62 21 48.7645	27.620
Stockholm (Swe)	18 5 27.2528	59 19 20.4054	35.539
Ustka (Pol)	16 51 13.8751	54 35 15.6866	34.307
Vaasa (Fin)	21 33 55.9146	63 5 42.8394	19.581
Visby (Swe)	18 17 3.9292	57 38 21.3487	27.632
Ölands N. U. (Swe)	17 4 46.8542	57 22 3.4508	31.823

5-2 7-parameter datum transformation

The seven parameters datum transformation $\mathbb{C}_7(3)$:

$$\begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = x_1 \boldsymbol{X}_3 \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix} + \boldsymbol{x}_2 \mid i = 1, 2, 3, \ldots, n \tag{5-10}$$

subject to

$$\boxed{\boldsymbol{X}_3^T \boldsymbol{X}_3 = \boldsymbol{I}_3} \tag{5-11}$$

with $\{a_i, b_i, c_i\}\, and \,\{X_i, Y_i, Z_i\}$ being coordinates of the same points in both systems and $x_1 \in \mathbb{R}$, $\boldsymbol{x}_2 \in \mathbb{R}^3$, $\boldsymbol{X}_3 \in \mathbb{R}^{3\times3}$ has always been solved using least squares solution. We illustrate in this section how the seven parameters datum transformation $\mathbb{C}_7(3)$ of a non-overdetermined and overdetermined cases can be solved by the use of *Gröbner bases* and *Gauss-Jacobi combinatorial algorithms* respectively. First, we solve analytically for $x_1 \in \mathbb{R}$, $\boldsymbol{X}_3 \in \mathbb{R}^{3\times3}$ ad then use them to transform the coordinates of one system in order to obtain the corresponding transformed values in the other system. By making use of the *skew-symmetric* matrix $\boldsymbol{S}$, the rotation matrix is expressed as

$$\boldsymbol{X}_3 = (\boldsymbol{I}_3 - \boldsymbol{S})^{-1}(\boldsymbol{I}_3 + \boldsymbol{S}) \tag{5-12}$$

where $\boldsymbol{I}_3$ is the identity matrix and the *skew-symmetric* matrix $\boldsymbol{S}$ given by

$$\boldsymbol{S} = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}. \tag{5-13}$$

The rotation matrix $\boldsymbol{X}_3 \in \mathbb{R}^{3\times3}$ is parameterized using *Euler* or *Cardan* angles. With Cardan angles, we have:

Box 5-2 (*Parametrization of the rotation matrix by Cardan angles*):

$$\boldsymbol{X}_3 = \boldsymbol{R}_1(\alpha)\boldsymbol{R}_2(\beta)\boldsymbol{R}_3(\gamma) \tag{5-14}$$

with

$$\boldsymbol{R}_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\alpha & sin\alpha \\ 0 & -sin\alpha & cos\alpha \end{bmatrix}, \boldsymbol{R}_2(\beta) = \begin{bmatrix} cos\beta & 0 & -sin\beta \\ 0 & 1 & 0 \\ sin\beta & 0 & cos\beta \end{bmatrix}, \boldsymbol{R}_3(\gamma) = \begin{bmatrix} cos\gamma & sin\gamma & 0 \\ -sin\gamma & cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

leading to

$$\boldsymbol{R}_1(\alpha)\boldsymbol{R}_2(\beta)\boldsymbol{R}_3(\gamma) = \begin{bmatrix} cos\beta cos\gamma & cos\beta sin\gamma & -sin\beta \\ sin\alpha sin\beta cos\gamma - cos\alpha sin\gamma & sin\alpha sin\beta sin\gamma + cos\alpha cos\gamma & sin\alpha cos\beta \\ cos\alpha sin\beta cos\gamma + sin\alpha sin\gamma & cos\alpha sin\beta sin\gamma - sin\alpha cos\gamma & cos\alpha cos\beta \end{bmatrix}. \tag{5-15}$$

The Cardan angles can be obtained from the rotation matrix $\boldsymbol{X}_3 \in \mathbb{R}^{3\times 3}$ through:

$$\left\{ \begin{array}{c} \alpha = tan(\frac{r_{23}}{r_{33}}), \;\; \gamma = tan(\frac{r_{12}}{r_{11}}) \\ \beta = tan(\frac{-r_{31}}{\sqrt{r_{11}^2 + r_{12}^2}}) \; or \; tan(\frac{-r_{31}}{\sqrt{r_{23}^2 + r_{33}^2}}) \end{array} \right. . \tag{5-16}$$

For parameterization using Euler angles we refer to *E. Grafarend* and *J. L. Awange* (2000). The properties of the rotation matrix $\boldsymbol{X}_3 \in \mathbb{R}^{3\times 3}$ expressed as in (5-12) have been examined by *S. Zhang* (1994) and shown to fulfill (5-11). Equation (5-10) is now written for $i = 1, 2, 3$ using (5-12) as

$$\begin{bmatrix} 1 & c & -b \\ -c & 1 & a \\ b & -a & 1 \end{bmatrix} \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = x_1 \begin{bmatrix} 1 & -c & b \\ c & 1 & -a \\ -b & a & 1 \end{bmatrix} \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix} + \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} \tag{5-17}$$

with $\{X_0, Y_0, Z_0\} \in \boldsymbol{x}_2$. For three corresponding points in both systems, the minimum observation equations required to solve the seven datum transformation parameter are expressed from (5-17) as:

$$\left[\begin{array}{r} f_1 := x_1X_1 - x_1cY_1 + x_1bZ_1 + X_0 - a_1 - cb_1 + bc_1 = 0 \\ f_2 := x_1cX_1 + x_1Y_1 - x_1aZ_1 + Y_0 + ca_1 - b_1 - ac_1 = 0 \\ f_3 := -x_1bX_1 + x_1aY_1 + x_1Z_1 + Z_0 - ba_1 + ab_1 - c_1 = 0 \\ f_4 := x_1X_2 - x_1cY_2 + x_1bZ_2 + X_0 - a_2 - cb_2 + bc_2 = 0 \\ f_5 := x_1cX_2 + x_1Y_2 - x_1aZ_2 + Y_0 + ca_2 - b_2 - ac_2 = 0 \\ f_6 := -x_1bX_2 + x_1aY_2 + x_1Z_2 + Z_0 - ba_2 + ab_2 - c_2 = 0 \\ f_7 := -x_1bX_3 + x_1aY_3 + x_1Z_3 + Z_0 - ba_3 + ab_3 - c_3 = 0 \end{array} \right. , \tag{5-18}$$

where $\{x_i, y_i, z_i\} := \{a_i, b_i, c_i\} \mid i \in \{1, 2, 3\}$, equations $\{f_1, f_2, f_3\}$ being equations formed from the *first point* with coordinates in both systems, $\{f_4, f_5, f_6\}$ being equations formed from the *second point* with coordinates in both systems and $\{f_7\}$ being the third equation extracted from the three equations formed from the *third point* with coordinates in both systems. In order to eliminate the translation parameters in (5-18), the following differencing is performed:

$$\left[\begin{array}{l} f_{14} := f_1 - f_4 = x_1X_{12} - x_1cY_{12} + x_1bZ_{12} - a_{12} - cb_{12} + bc_{12} \\ f_{25} := f_2 - f_5 = x_1cX_{12} + x_1Y_{12} - x_1aZ_{12} + ca_{12} - b_{12} - ac_{12} \\ f_{37} := f_3 - f_7 = -x_1bX_{13} + x_1aY_{13} + x_1Z_{13} - ba_{13} + ab_{13} - c_{13} \\ f_{67} := f_6 - f_7 = -x_1bX_{23} + x_1aY_{23} + x_1Z_{23} - ba_{23} + ab_{23} - c_{23} \end{array} \right. \tag{5-19}$$

with

$$\left. \begin{array}{lll} X_{ij} = X_i - X_j & Y_{ij} = Y_i - Y_j & Z_{ij} = Z_i - Z_j \\ a_{ij} = a_i - a_j & b_{ij} = b_i - b_j & c_{ij} = c_i - c_j \end{array} \right\} \mid i, j \in \{1, 2, 3\}, i \neq j.$$

The *reduced Gröbner basis* of (5-19) is then obtained using *Mathematica 3.0* Software by using the command *GroebnerBasis*[$\{f_{14}, f_{25}, f_{37}, f_{67}\}$,$\{x_1, a, b, c\}$, $\{a, b, c\}$] as explained in Chapter 2, Section (2-321). This gives only the element of *Gröbner basis* in which the variables a, b, c have been eliminated and only the scale factor x_1 left. The scale parameter is then given by the following *univariate polynomial* of order four:

Box 5-3 (*quartic polynomial for computing scale parameter*):

$$\boxed{a_4x_1^4 + a_3x_1^3 + a_2x_1^2 + a_1x_1 + a_0 = 0}$$

$a_4 = (X_{13}Y_{12}^2Y_{23} + X_{12}^2X_{13}Y_{23} - X_{12}^2X_{23}Y_{13} - X_{12}Y_{13}Z_{12}Z_{23} - X_{23}Y_{12}^2Y_{13} + X_{13}Y_{12}Z_{12}Z_{23} - X_{23}Y_{12}Z_{12}Z_{13} + X_{12}Y_{23}Z_{12}Z_{13})$

$a_3 = (c_{12}X_{13}Y_{12}Z_{23} - b_{13}X_{12}Z_{12}Z_{23} - c_{13}X_{12}Y_{23}Z_{12} + b_{23}X_{13}Y_{12}^2 - a_{23}X_{12}^2Y_{13} + c_{23}X_{12}Y_{13}Z_{12} + c_{12}X_{12}Y_{23}Z_{13} + a_{13}Y_{12}^2Y_{23} - b_{12}X_{23}Z_{12}Z_{13} - a_{23}Y_{12}^2Y_{13} + b_{12}X_{13}Z_{12}Z_{23} - c_{12}X_{12}Y_{13}Z_{23} + a_{13}X_{12}^2Y_{23} - c_{23}X_{13}Y_{12}Z_{12} - a_{23}Y_{12}Z_{12}Z_{13} + c_{13}X_{23}Y_{12}Z_{12} - b_{13}X_{23}Y_{12}^2 - b_{13}X_{12}^2X_{23} + b_{23}X_{12}Z_{12}Z_{13} + a_{13}Y_{12}Z_{12}Z_{23} + b_{23}X_{12}^2X_{13} + a_{12}Y_{23}Z_{12}Z_{13} - a_{12}Y_{13}Z_{12}Z_{23} - c_{12}X_{23}Y_{12}Z_{13})$

$a_2 = (a_{13}b_{23}X_{12}^2 + b_{12}^2X_{23}Y_{13} + b_{12}c_{13}X_{23}Z_{12} - c_{12}c_{23}X_{13}Y_{12} + b_{13}c_{23}X_{12}Z_{12} - a_{23}b_{12}Z_{12}Z_{13} + a_{12}^2X_{23}Y_{13} - b_{12}^2X_{13}Y_{23} - a_{12}^2X_{13}Y_{23} - a_{23}b_{13}X_{12}^2 + a_{13}b_{23}Y_{12}^2 - a_{23}b_{13}Y_{12}^2 + a_{12}b_{23}Z_{12}Z_{13} + a_{23}c_{13}Y_{12}Z_{12} + a_{12}c_{12}Y_{23}Z_{13} - b_{12}c_{12}X_{23}Z_{13} - b_{23}c_{13}X_{12}Z_{12} - a_{12}b_{13}Z_{12}Z_{23} - a_{23}c_{12}Y_{12}Z_{13} + a_{12}c_{23}Y_{13}Z_{12} + b_{12}c_{12}X_{13}Z_{23} - a_{12}c_{12}Y_{13}Z_{23} - a_{13}c_{23}Y_{12}Z_{12} - c_{12}c_{13}X_{12}Y_{23} + c_{12}c_{13}X_{23}Y_{12} + c_{12}c_{23}X_{12}Y_{13} - b_{13}c_{12}X_{12}Z_{23} + b_{23}c_{12}X_{12}Z_{13} + a_{13}b_{12}Z_{12}Z_{23} - a_{12}c_{13}Y_{23}Z_{12} + a_{13}c_{12}Y_{12}Z_{23} - b_{12}c_{23}X_{13}Z_{12})$

$a_1 = (-a_{12}b_{13}c_{12}Z_{23} + b_{12}c_{12}c_{13}X_{23} + b_{12}^2b_{13}X_{23} - a_{13}b_{12}^2Y_{23} - a_{12}^2b_{23}X_{13} + a_{23}c_{12}c_{13}Y_{12} - a_{13}c_{12}c_{23}Y_{12} + a_{12}b_{13}c_{23}Z_{12} - a_{12}^2a_{13}Y_{23} - b_{23}c_{12}c_{13}X_{12} + a_{12}b_{23}c_{12}Z_{13} - a_{23}b_{12}c_{12}Z_{13} - b_{12}^2b_{23}X_{13} + a_{23}b_{12}c_{13}Z_{12} - a_{12}c_{12}c_{13}Y_{23} + a_{12}^2b_{13}X_{23} + a_{12}^2a_{23}Y_{13} - a_{12}b_{23}c_{13}Z_{12} - a_{13}b_{12}c_{23}Z_{12} - b_{12}c_{12}c_{23}X_{13} + b_{13}c_{12}c_{23}X_{12} + a_{12}c_{12}c_{23}Y_{13} + a_{23}b_{12}^2Y_{13} + a_{13}b_{12}c_{12}Z_{23})$

$a_0 = a_{12}b_{13}c_{12}c_{23} - a_{13}b_{12}^2b_{23} + a_{12}^2a_{23}b_{13} - a_{12}^2a_{13}b_{23} - a_{12}b_{23}c_{12}c_{13} + a_{23}b_{12}^2b_{13} + a_{23}b_{12}c_{12}c_{13} - a_{13}b_{12}c_{12}c_{23}$

Once the admissible value of scale parameter has been chosen amongst the four roots above as $x_1 \in \mathbb{R}^+$ in *Box* (5-3), the elements of the *skew symmetric* matrix $\boldsymbol{S}$ can then be obtained via the linear functions in *Box* (5-4).

Box 5-4 (*Linear functions for computing the parameters of the skew-symmetric matrix* $\boldsymbol{S}$):

$f(a) = (-x_1^3X_{13}Y_{12}Z_{12} - b_{12}x_1^2X_{13}Z_{12} - a_{13}b_{12}c_{12} - a_{13}x_1^2Y_{12}Z_{12} + b_{13}x_1^2X_{12}Z_{12} + c_{12}x_1^2X_{12}Y_{13} - a_{13}c_{12}x_1Y_{12} + a_{12}b_{13}c_{12} + a_{12}b_{13}x_1Z_{12} - a_{13}b_{12}x_1Z_{12} - b_{12}c_{12}x_1X_{13} + a_{12}x_1^2Y_{13}Z_{12} + x_1^3X_{12}Y_{13}Z_{12} + a_{12}c_{12}x_1Y_{13} - c_{12}x_1^2X_{13}Y_{12} + b_{13}c_{12}x_1X_{12})a + (-a_{12}^2a_{13} - a_{12}c_{13}x_1Z_{12} + a_{12}x_1^2Z_{12}Z_{13} + a_{12}c_{12}x_1Z_{13} - c_{12}c_{13}x_1X_{12} - a_{12}c_{12}c_{13} + x_1^3X_{13}Y_{12}^2 - c_{13}x_1^2X_{12}Z_{12} + x_1^3X_{12}^2X_{13} + x_1^3X_{12}Z_{12}Z_{13} - a_{12}^2x_1X_{13} - b_{12}^2x_1X_{13} - a_{13}b_{12}^2 + a_{13}x_1^2X_{12}^2 + a_{13}x_1^2Y_{12}^2 + c_{12}x_1^2X_{12}Z_{13})$

$f(b) = (-b_{12}c_{12}x_1X_{13} + b_{13}c_{12}x_1X_{12} - a_{13}b_{12}x_1Z_{12} - x_1^3X_{13}Y_{12}Z_{12} - a_{13}c_{12}x_1Y_{12} + a_{12}b_{13}c_{12} + a_{12}x_1^2Y_{13}Z_{12} - a_{13}x_1^2Y_{12}Z_{12} + a_{12}b_{13}x_1Z_{12} + x_1^3X_{12}Y_{13}Z_{12} - a_{13}b_{12}c_{12} + c_{12}x_1^2X_{12}Y_{13} + a_{12}c_{12}x_1Y_{13} - b_{12}x_1^2X_{13}Z_{12} - c_{12}x_1^2X_{13}Y_{12} + b_{13}x_1^2X_{12}Z_{12})b + (-a_{12}^2b_{13} - c_{12}c_{13}x_1Y_{12} + b_{13}x_1^2Y_{12}^2 + c_{12}x_1^2Y_{12}Z_{13} + x_1^3Y_{12}Z_{12}Z_{13} + x_1^3X_{12}^2Y_{13} - b_{12}c_{13}x_1Z_{12} + b_{12}x_1^2Z_{12}Z_{13} - b_{12}^2b_{13} - a_{12}^2x_1Y_{13} + b_{13}x_1^2X_{12}^2 - c_{13}x_1^2Y_{12}Z_{12} + b_{12}c_{12}x_1Z_{13} + x_1^3Y_{12}^2Y_{13} - b_{12}^2x_1Y_{13} - b_{12}c_{12}c_{13})$

$f(c) = (-b_{12}x_1X_{13} - a_{13}x_1Y_{12} - x_1^2X_{13}Y_{12} - a_{13}b_{12} + x_1^2X_{12}Y_{13} + a_{12}b_{13} + a_{12}x_1Y_{13} + b_{13}x_1X_{12})c + (-a_{12}a_{13} - b_{12}b_{13} + x_1^2Z_{12}Z_{13} + x_1^2Y_{12}Y_{13} - c_{12}c_{13} + a_{13}x_1X_{12} + b_{13}x_1Y_{12} - a_{12}x_1X_{13} + x_1^2X_{12}X_{13} + c_{12}x_1Z_{13} - b_{12}x_1Y_{13} - c_{13}x_1Z_{12})$.

Substituting the *skew symmetric* matrix $\boldsymbol{S}$ in (5-10) gives the rotation matrix $\boldsymbol{X}_3$ from which the Cardan rotation angles are deduced from (5-16) in *Box* (5-2). The translation elements $\boldsymbol{x}_2$ can then be computed by substituting the scale parameter x_1 and the rotation matrix $\boldsymbol{X}_3$ in (5-10). Three sets of translation parameters are then obtained for the three points under consideration and the mean taken.

Gauss-Jacobi combinatorial algorithm

When more than three points are given and the transformation parameters are to be solved, the *Gauss-Jacobi combinatorial algorithm* is applied in which case the dispersion matrix has to be obtained via the nonlinear error propagation law/variance-covariance propagation law. From the algebraic system of equations (5-18), we have with (2-77) and (2-68) in pages (25 and 24 respectively) the Jacobi matrices as

$$
\boldsymbol{J}_x = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial a} & \frac{\partial f_1}{\partial b} & \frac{\partial f_1}{\partial c} & \frac{\partial f_1}{\partial X_0} & \frac{\partial f_1}{\partial Y_0} & \frac{\partial f_1}{\partial Z_0} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial a} & \frac{\partial f_2}{\partial b} & \frac{\partial f_2}{\partial c} & \frac{\partial f_2}{\partial X_0} & \frac{\partial f_2}{\partial Y_0} & \frac{\partial f_2}{\partial Z_0} \\
\frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial a} & \frac{\partial f_3}{\partial b} & \frac{\partial f_3}{\partial c} & \frac{\partial f_3}{\partial X_0} & \frac{\partial f_3}{\partial Y_0} & \frac{\partial f_3}{\partial Z_0} \\
\frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial a} & \frac{\partial f_4}{\partial b} & \frac{\partial f_4}{\partial c} & \frac{\partial f_4}{\partial X_0} & \frac{\partial f_4}{\partial Y_0} & \frac{\partial f_4}{\partial Z_0} \\
\frac{\partial f_5}{\partial x_1} & \frac{\partial f_5}{\partial a} & \frac{\partial f_5}{\partial b} & \frac{\partial f_5}{\partial c} & \frac{\partial f_5}{\partial X_0} & \frac{\partial f_5}{\partial Y_0} & \frac{\partial f_5}{\partial Z_0} \\
\frac{\partial f_6}{\partial x_1} & \frac{\partial f_6}{\partial a} & \frac{\partial f_6}{\partial b} & \frac{\partial f_6}{\partial c} & \frac{\partial f_6}{\partial X_0} & \frac{\partial f_6}{\partial Y_0} & \frac{\partial f_6}{\partial Z_0} \\
\frac{\partial f_7}{\partial x_1} & \frac{\partial f_7}{\partial a} & \frac{\partial f_7}{\partial b} & \frac{\partial f_7}{\partial c} & \frac{\partial f_7}{\partial X_0} & \frac{\partial f_7}{\partial Y_0} & \frac{\partial f_7}{\partial Z_0}
\end{bmatrix} \tag{5-20}
$$

and

$$
\boldsymbol{J}_y = \begin{bmatrix}
\frac{\partial f_1}{\partial a_1} & \frac{\partial f_1}{\partial b_1} & \frac{\partial f_1}{\partial c_1} & \frac{\partial f_1}{\partial a_2} & \frac{\partial f_1}{\partial b_2} & \frac{\partial f_1}{\partial c_2} & \frac{\partial f_1}{\partial a_3} & \cdots\cdots & \frac{\partial f_1}{\partial Z_3} \\
\frac{\partial f_2}{\partial a_1} & \frac{\partial f_2}{\partial b_1} & \frac{\partial f_2}{\partial c_1} & \frac{\partial f_2}{\partial a_2} & \frac{\partial f_2}{\partial b_2} & \frac{\partial f_2}{\partial c_2} & \frac{\partial f_2}{\partial a_3} & \cdots\cdots & \frac{\partial f_2}{\partial Z_3} \\
\frac{\partial f_3}{\partial a_1} & \frac{\partial f_3}{\partial b_1} & \frac{\partial f_3}{\partial c_1} & \frac{\partial f_3}{\partial a_2} & \frac{\partial f_3}{\partial b_2} & \frac{\partial f_3}{\partial c_2} & \frac{\partial f_3}{\partial a_3} & \cdots\cdots & \frac{\partial f_3}{\partial Z_3} \\
\frac{\partial f_4}{\partial a_1} & \frac{\partial f_4}{\partial b_1} & \frac{\partial f_4}{\partial c_1} & \frac{\partial f_4}{\partial a_2} & \frac{\partial f_4}{\partial b_2} & \frac{\partial f_4}{\partial c_2} & \frac{\partial f_4}{\partial a_3} & \cdots\cdots & \frac{\partial f_4}{\partial Z_3} \\
\frac{\partial f_5}{\partial a_1} & \frac{\partial f_5}{\partial b_1} & \frac{\partial f_5}{\partial c_1} & \frac{\partial f_5}{\partial a_2} & \frac{\partial f_5}{\partial b_2} & \frac{\partial f_5}{\partial c_2} & \frac{\partial f_5}{\partial a_3} & \cdots\cdots & \frac{\partial f_5}{\partial Z_3} \\
\frac{\partial f_6}{\partial a_1} & \frac{\partial f_6}{\partial b_1} & \frac{\partial f_6}{\partial c_1} & \frac{\partial f_6}{\partial a_2} & \frac{\partial f_6}{\partial b_2} & \frac{\partial f_6}{\partial c_2} & \frac{\partial f_6}{\partial a_3} & \cdots\cdots & \frac{\partial f_6}{\partial Z_3} \\
\frac{\partial f_7}{\partial a_1} & \frac{\partial f_7}{\partial b_1} & \frac{\partial f_7}{\partial c_1} & \frac{\partial f_7}{\partial a_2} & \frac{\partial f_7}{\partial b_2} & \frac{\partial f_7}{\partial c_2} & \frac{\partial f_7}{\partial a_3} & \cdots\cdots & \frac{\partial f_7}{\partial Z_3}
\end{bmatrix} \tag{5-21}
$$

where the doted points in $\boldsymbol{J}_y$ represents the partial derivative of (5-18) with respect $\{b_3, c_3, X_1, Y_1, Z_1, X_2, Y_2, Z_2, X_3, Y_3\}$. From the dispersion $\boldsymbol{\Sigma}_y$ of the vector of observations $\boldsymbol{y}$ and with (5-20) and (5-21) forming $\boldsymbol{J} = \boldsymbol{J}_x^{-1}\boldsymbol{J}_y$, the dispersion matrix $\boldsymbol{\Sigma}_x$ is then obtained from (2-68). Finally we obtained the dispersion matrix $\boldsymbol{\Sigma}$ from (2-70) in Chapter 2 page (24).

Case Study:

We consider Cartesian coordinates of seven stations given in the Local and Global Reference Systems (WGS 84) as in *Tables* (5.5) and (5.6). Desired are the seven parameters of datum transformation. Using the *Gröbner bases algorithm*, we obtain the 7-transformation parameters given in *Table* (5.7) which are used to transform the three points involved in the computations from the *Local Reference System* (*Table* 5.5) to the *Global Reference System* (WGS 84, *Table* 5.6) in *Table* (5.8).

Table 5.5: Coordinates for system A (Local system)

Station Name	$X(m)$	$Y(m)$	$Z(m)$
Solitude	4157222.543	664789.307	4774952.099
Buoch Zeil	4149043.336	688836.443	4778632.188
Hohenneuffen	4172803.511	690340.078	4758129.701
Kuehlenberg	4177148.376	642997.635	4760764.800
Ex Mergelaec	4137012.190	671808.029	4791128.215
Ex Hof Asperg	4146292.729	666952.887	4783859.856
Ex Kaisersbach	4138759.902	702670.738	4785552.196

Table 5.6: Coordinates for system B (WGS 84)

Station Name	$X(m)$	$Y(m)$	$Z(m)$
Solitude	4157870.237	664818.678	4775416.524
Buoch Zeil	4149691.049	688865.785	4779096.588
Hohenneuffen	4173451.354	690369.375	4758594.075
Kuehlenberg	4177796.064	643026.700	4761228.899
Ex Mergelaec	4137659.549	671837.337	4791592.531
Ex Hof Asperg	4146940.228	666982.151	4784324.099
Ex Kaisersbach	4139407.506	702700.227	4786016.645

Table 5.7: Computed 7-parameter datum transformation by Gröbner basis

Transformation parameter	Value	unit
Scale $k-1$	-1.4343	[ppm]
Rotation $\mathbf{X}_1(a)$	0.32575149	[“]
Rotation $\mathbf{X}_2(b)$	-0.46037399	[“]
Rotation $\mathbf{X}_3(c)$	-0.00810606	[“]
Translation ΔX	643.0953	[m]
Translation ΔY	22.6163	[m]
Translation ΔZ	481.6023	[m]

Table 5.8: Transformed Cartesian coordinates of System A (*Table* 5.5) into System B (*Table* 5.6) using the 7-datum transformation parameters of *Table* (5.7) computed by Gröbner basis

Site	$X(m)$	$Y(m)$	$Z(m)$
System A: Solitude	4157222.5430	664789.3070	4774952.0990
System B	4157870.2370	664818.6780	4775416.5240
Transformed value	4157870.3070	664818.6742	4775416.5240
Residual	- 0.0700	0.0038	0.0000
System A: Buoch Zeil	4149043.3360	688836.4430	4778632.1880
System B	4149691.0490	688865.7850	4779096.5880
Transformed value	4149691.1190	688865.7812	4779096.5880
Residual	- 0.0700	0.0038	0.0000
System A: Hohenneuffen	4172803.5110	690340.0780	4758129.7010
System B	4173451.3540	690369.3750	4758594.0750
Transformed value	4173451.2141	690369.3826	4758594.0750
Residual	0.1399	-0.0076	0.0000

Transformation parameters obtained in the overdetermined case by the *Gauss-Jacobi combinatorial algorithm* are presented in *Table* (5.9) and used to transform the Cartesian coordinates from the *Local Reference System (Table* 5.5) to the *Global Reference System* (WGS 84, *Table* 5.6) as shown in *Table* (5.10). *Table* (5.11) gives for comparison purposes the transformed values from the 7-datum transformation parameters obtained via *least squares solution*. The residuals from both *Gauss-Jacobi combinatorial algorithm* and *least squares solution* are of the same magnitude. We also compute the residual norm (square root of the sum of squares of residuals) and present them in *Table* (5.12). The computed norm from the *Gauss-Jacobi combinatorial* solution is somewhat better than those of the *linearized least squares solution*. *Figures* (5.2) and (5.3) indicate the scatter of the computed 36 minimal combinatorial solutions of scale (indicated by doted points (•)) around for the adjusted value indicated by a line (−). *Figures* (5.3) indicate the scatter of the computed 36 minimal combinatorial solutions of translation and rotation parameters (indicated by doted points (•)) around the adjusted values indicated by a star (⋆). The Figures clearly identifies the outlying combinations from which the respective (suspected outlying points) points can be deduced.

Table 5.9: Computed 7-parameter datum transformation using Gauss-Jacobi combinatorial algorithm

Transformation parameter	Value	Root-mean-square	unit
Scale $k-1$	4.92377597	0.350619414	[ppm]
Rotation $\mathbf{X}_1(a)$	-0.98105498"	0.040968549	["]
Rotation $\mathbf{X}_2(b)$	0.68869774"	0.047458707	["]
Rotation $\mathbf{X}_3(c)$	0.96671738"	0.044697434	["]
Translation ΔX	639.9785	2.4280	[m]
Translation ΔY	68.1548	3.0123	[m]
Translation ΔZ	423.7320	2.7923	[m]

Table 5.10: Transformed Cartesian coordinates of System A (*Table* 5.5) into System B (*Table* 5.6) using the 7-datum transformation parameters of *Table* (5.9) computed by the Gauss-Jacobi combinatorial algorithm

Site	$X(m)$	$Y(m)$	$Z(m)$
System A: Solitude	4157222.5430	664789.3070	4774952.0990
System B	4157870.2370	664818.6780	4775416.5240
Transformed value	4157870.1631	664818.5399	4775416.3843
Residual	0.0739	0.1381	0.1397
System A: Buoch Zeil	4149043.3360	688836.4430	4778632.1880
System B	4149691.0490	688865.7850	4779096.5880
Transformed value	4149691.0162	688865.8151	4779096.5785
Residual	0.0328	-0.0301	0.0095
System A: Hohenneuffen	4172803.5110	690340.0780	4758129.7010
System B	4173451.3540	690369.3750	4758594.0750
Transformed value	4173451.3837	690369.4437	4758594.0770
Residual	-0.0297	-0.0687	-0.0020
System A: Kuelenberg	4177148.3760	642997.6350	4760764.8000
System B	4177796.0640	643026.7000	4761228.8990
Transformed value	4177796.0394	643026.7347	4761228.9783
Residual	0.0246	-0.0347	-0.0793
System A: Ex Mergelaec	4137012.1900	671808.0290	4791128.2150
System B	4137659.5490	671837.3370	4791592.5310
Transformed value	4137659.6895	671837.3142	4791592.5458
Residual	-0.1405	0.0228	-0.0148
System A: Ex Hof Asperg	4146292.7290	666952.8870	4783859.8560
System B	4146940.2280	666982.1510	4784324.0990
Transformed value	4146940.2757	666982.1394	4784324.1589
Residual	-0.0477	0.0116	-0.0599
System A: Ex Keisersbach	4138759.9020	702670.7380	4785552.1960
System B	4139407.5060	702700.2270	4786016.6450
Transformed value	4139407.5733	702700.1935	4786016.6520
Residual	-0.0673	0.0335	-0.0070

Table 5.11: Transformed Cartesian coordinates of System A (*Table* 5.5) into System B (*Table* 5.6) using the 7-datum transformation parameters computed by Least Squares Solution

Site	$X(m)$	$Y(m)$	$Z(m)$
System A: Solitude	4157222.5430	664789.3070	4774952.0990
System B	4157870.2370	664818.6780	4775416.5240
Transformed value	4157870.1430	664818.5429	4775416.3838
Residual	0.0940	0.1351	0.1402
System A: Buoch Zeil	4149043.3360	688836.4430	4778632.1880
System B	4149691.0490	688865.7850	4779096.5880
Transformed value	4149690.9902	688865.8347	4779096.5743
Residual	0.0588	-0.0497	0.0137
System A: Hohenneuffen	4172803.5110	690340.0780	4758129.7010
System B	4173451.3540	690369.3750	4758594.0750
Transformed value	4173451.3939	690369.4629	4758594.0831
Residual	-0.0399	-0.0879	-0.0081
System A: Kuelenberg	4177148.3760	642997.6350	4760764.8000
System B	4177796.0640	643026.7000	4761228.8990
Transformed value	4177796.0438	643026.7220	4761228.9864
Residual	0.0202	-0.0220	-0.0874
System A: Ex Mergelaec	4137012.1900	671808.0290	4791128.2150
System B	4137659.5490	671837.3370	4791592.5310
Transformed value	4137659.6409	671837.3231	4791592.5365
Residual	-0.0919	0.0139	-0.0055
System A: Ex Hof Asperg	4146292.7290	666952.8870	4783859.8560
System B	4146940.2280	666982.1510	4784324.0990
Transformed value	4146940.2398	666982.1445	4784324.1536
Residual	-0.0118	0.0065	-0.0546
System A: Ex Keisersbach	4138759.9020	702670.7380	4785552.1960
System B	4139407.5060	702700.2270	4786016.6450
Transformed value	4139407.5354	702700.2229	4786016.6433
Residual	-0.0294	0.0041	-0.0017

Table 5.12: Computed residual norms

Method	$X(m)$	$Y(m)$	$Z(m)$
Linearized Least Squares Solution	0.1541	0.1708	0.1748
Nonlinear *Gauss-Jacobi Combinatorial*	0.1859	0.1664	0.1725

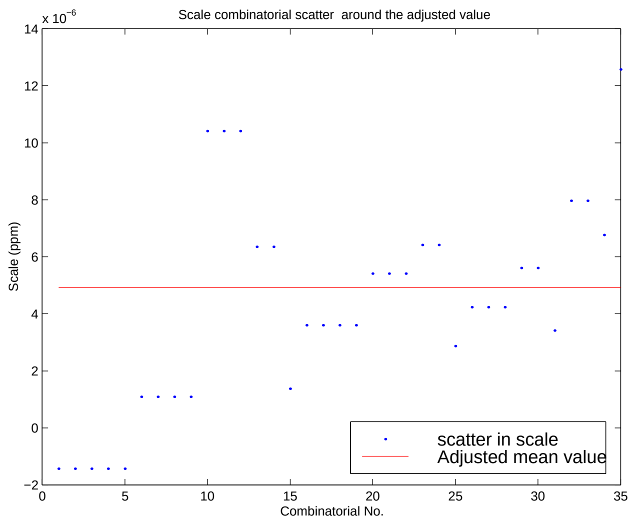


Figure 5.2: Scatter of the computed 36 minimal combinatorial values of scale around the adjusted value

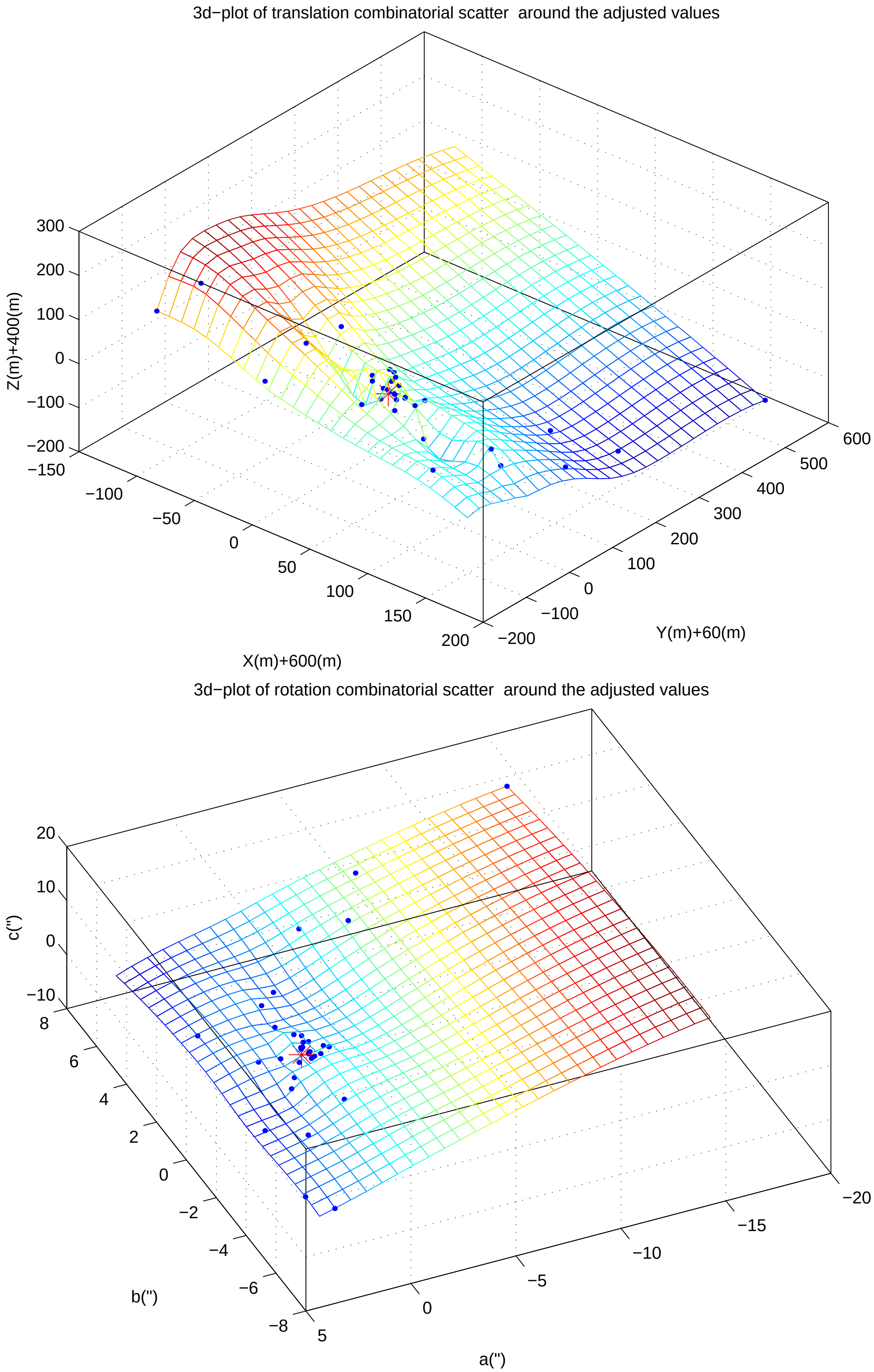


Figure 5.3: Scatter of the 36 computed translations and rotations around the adjusted values

Chapter 6

Summary and Conclusions

In summary, Chapter 3 has demonstrated the power of the algebraic computational tools of *Gröbner bases* and the *Multipolynomial resultants* in solving selected geodetic problems. In the case of the *minimal combinatorial set*, we have demonstrated how the problems of *three-dimensional resection*, *minimum distance mapping* and the *pseudo-ranging four-point problem* could be solved in a closed form using either *Gröbner bases* approach or the *Multipolynomial resultants* approach. We have succeeded in demonstrating that by converting the *nonlinear observation equations* of the selected geodetic problems above into algebraic (polynomials), the *multivariate* system of polynomial equations relating the unknown variables (indeterminate) to the known variables can be reduced into polynomial equations consisting of *a univariate polynomial* once *lexicographic monomial ordering* is specified. We have therefore managed to provide *symbolic solutions* to the problems of *three-dimensional resection*, *minimum distance mapping* and the *GPS pseudo-ranging four-point P4P* by obtaining in each case *a univariate polynomial* that can readily be solved numerically once the observations are available. Although the algebraic techniques of *Gröbner bases* and *Multipolynomial resultants* were tested for these selected geodetic problems, the tools can be used to solve explicitly any closed form problem in Geodesy. The only limitation may be the storage and computational speed of the computers when compounded with problems involving many variables. The ability of the algebraic tools (*Gröbner bases* and the *Multipolynomial resultants*) to solve closed form solutions gave the *Gauss-Jacobi combinatorial algorithm* the required operational engine as evidenced in the solution of the *overdetermined GPS pseudo-ranging problem* which was achieved without reverting to iterative or linearization procedures. The results compared well with the solutions obtained using the *linearized Gauss-Markov model* giving legitimacy to the *Gauss-Jacobi combinatorial* procedure.

By solving the test network *"Stuttgart Central"* both explicitly *(*using the *Gröbner bases* and the *Multipolynomial resultants* algorithms) and in overdetermined form (using the *Gauss-Jacobi combinatorial algorithm*) in Chapter 4, the study has highlighted the capability of the *Gröbner basis* and the *Multipolynomial resultant* algorithms to solve explicitly both the *Grunert distance equations* and the *three-dimension ranging ("Bogenschnitt")* problem as well as the capability of the *Gauss-Jacobi combinatorial algorithm* to solve the overdetermined three-dimensional resection. The deviation of the computed distances $\Delta S_i \mid i = 1, \ldots, 7$ and position $\{\Delta X, \Delta Y, \Delta Z\}$ from the real values (*Tables* 4.1 and 4.2 respectively) obtained using closed form procedures (*Gröbner basis* and the *Multipolynomial resultant*) were in millimeter range for distances and position as depicted in *Tables* (4.14) and (4.18) respectively.

The application of the *Gauss-Jacobi combinatorial algorithm* to solve the overdetermined three-dimensional resection problem based on the test network "Stuttgart Central" improves the results as seen from the deviation of the computed distances $\Delta S_i \mid i = 1, \ldots, 7$ and position $\{\Delta X, \Delta Y, \Delta Z\}$ from the real values (*Tables* 4.1 and 4.2 respectively) as depicted in *Tables* (4.17) and (4.20). In comparing the deviations of the computed values of position of station K1 using both closed form and *Gauss-Jacobi combinatorial algorithms* in *Tables* (4.18) and (4.20), It is evident that the deviations from the *Gauss-Jacobi combinatorial algorithm* were smaller. This becomes clearer once we plot these deviations as in *Figure* (4.2). One therefore concludes that better accuracies would be achieved if the *three-dimensional resection problem* were to be solved in an overdetermined form as opposed to closed form procedures which are traditionally used. The *Gauss-Jacobi combinatorial algorithm* offers the solution to the problem of lacking the approximate values normally required by the *linearized Gauss-Markov model* and *iterative procedures*. Besides, the *Gauss-Jacobi combinatorial algorithm* makes use of the full information of the observational data set through the error propagation/variance-covariance propagation.

For the case studies, the *Gröbner bases algorithm* successfully determines explicitly the 7-parameters of the datum transformation problem and shows the scale parameter to be represented by a *univariate polynomial* of *fourth order* while the rotation parameters are represented by linear functions comprising the coordinates of the two systems and the scale parameter. The admissible value of the scale is readily obtained by solving the *univariate polynomial* and restricting the scale to a positive real value $x_1 \in \mathbb{R}.^+$ This eliminates the negative components of the roots and the complex values. The admissible value $x_1 \in \mathbb{R}^+$ of the scale parameter is chosen and substituted in the linear functions that characterize the three elements of the *skew-symmetric matrix* S leading to the solution of the elements of the rotation matrix $\boldsymbol{X}_3$. The translation elements are then deduced from the transformation equation. The advantage of using the *Gröbner bases algorithm* is the fact that there exists no requirement for prior knowledge of the approximate values of the 7-transformation parameters as is usually the case.

The *Gröbner bases algorithm* managed to solve the *Minimum Distance Mapping problem* and in so doing, enabling the mapping of points from the *topographical surface* to the *International Reference Ellipsoid of Revolution*. The *univariate polynomial* obtained was identical to that obtained by *E. Grafarend* and *P. Lohse* (1991). This implies that the algebraic tools of *Gröbner bases* and the *Multipolynomial resultants* can also be used to check the validity of existing closed form procedures in Geodesy .

The *Gauss-Jacobi combinatorial algorithm highlighted* one important fact while solving the overdetermined 7 parameter transformation problem; that the stochasticity of both systems involved can be taken into account. This has been the bottleneck of the problem of *7-datum transformation parameters*.

In conclusion, the present study has contributed towards the solution of nonlinear GPS/LPS observations in Geodesy. By testing the *Gröbner bases* and the *Multipolynomial resultants* techniques on the selected geodetic problems and case studies, the study has *established that the methods are suitable tools to be applied in solving closed form problems in Geodesy*. The only requirements is that the nonlinear geodetic observation equations have to be converted into algebraic (polynomial) form. Besides the solution of nonlinear equations in closed form, the *Gröbner bases* and the *Multipolynomial resultants* approaches *can be used to check the validity of the existing closed form procedures in Geodesy.* The study has further established that the *Gauss-Jacobi combinatorial algorithm* offers an alternative procedure to *iterative* and *linearization* approaches that normally require approximate starting values. With these advantages, the overdetermined problems of GPS pseudo-ranging, three-dimensional resection and 7 parameter datum transformation were solved. Besides, the *Gauss-Jacobi combinatorial algorithm* takes into account the full information of the observations and parameter space via the nonlinear error propagation/variance-covariance propagation.

Appendices

Appendix A-1: Definitions

To enhance the understanding of the theory of *Gröbner bases* presented in Chapter 2, the following definitions supplemented with examples will help in developing ideas that lead to the definition of the *basis* of an *Ideal*, in particular the *Standard Basis* known as the *Gröbner basis* of an *Ideal* in a polynomial ring.

Definition A-1 (monomial):
A monomial is a multivariate product of the form $x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}$, $(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ in the variables $x_1, \ldots, x_n$.

Example A-1 (*E. Grafarend, P. Lohse* and *B. Schaffrin* 1989, p.128):
Consider equation (1.30) used for the solution of distances in the three-dimensional resection problem given below as

$$\left[\begin{array}{l} x_1^2 + 2a_{12}x_1x_2 + x_2^2 + a_{oo} = 0 \\ x_2^2 + 2b_{23}x_2x_3 + x_3^2 + b_{oo} = 0 \\ x_3^2 + 2c_{31}x_3x_1 + x_1^2 + c_{oo} = 0 \\ where \;\; x_1 \in \mathbb{R}^+,\, x_2 \in \mathbb{R}^+, x_3 \in \mathbb{R}^+ \end{array}\right.$$

with the variables $\{x_1, x_2, x_3,\}$ the other terms being known constants, then $\{x_1^2, x_1x_2, x_2^2, x_2x_3, x_3^2, x_3x_1\}$ is said to be a set of monomials in the variables $\{x_1, x_2, x_3.\}$

Example A-2 (*E. Grafarend* and *J. Shan* 1996, p.138):
Consider the pseudo-ranging four-point problem for determining the unknown coordinates of the stationary receiver and the stationary receiver range bias given by equation (1.6) as follows

$$\left[\begin{array}{c} (x_1 - a_0)^2 + (x_2 - b_0)^2 + (x_3 - c_0)^2 - (x_4 - d_0)^2 = 0 \\ (x_1 - a_1)^2 + (x_2 - b_1)^2 + (x_3 - c_1)^2 - (x_4 - d_1)^2 = 0 \\ (x_1 - a_2)^2 + (x_2 - b_2)^2 + (x_3 - c_2)^2 - (x_4 - d_2)^2 = 0 \\ (x_1 - a_3)^2 + (x_2 - b_3)^2 + (x_3 - c_3)^2 - (x_4 - d_3)^2 = 0 \\ where \;\; x_1, x_2, x_3, x_4 \in \mathbb{R} \\ (a_0, b_0, c_0) = (x^0, y^0, z^0) \sim P^0 \\ (a_1, b_1, c_1) = (x^1, y^1, z^1) \sim P^1 \\ (a_2, b_2, c_2) = (x^2, y^2, z^2) \sim P^2 \\ (a_3, b_3, c_3) = (x^3, y^3, z^3) \sim P^3 \end{array}\right.$$

with $\{P^0, P^1, P^2, P^3\}$ being the position of the four GPS satellites, their ranges to the stationary receiver at P given by $\{d_0, d_1, d_2, d_3\}$. The parameters $(\{a_0, b_0, c_0\}, \{a_1, b_1, c_1\}, \{a_2, b_2, c_2\}, \{a_3, b_3, c_3\}, \{d_0, d_1, d_2, d_3\})$ are elements of the spherical cone that intersect at P to give the coordinates $\{x_1, x_2, x_3\}$ of the receiver and the stationary receiver range bias x_4. The equations above can be expanded as follows

$$\left[\begin{array}{l} x_1^2 - 2a_0x_1 + x_2^2 - 2b_0x_2 + x_3^2 - 2c_0x_3 - x_4^2 + 2d_0x_4 + a_0^2 + b_0^2 + c_0^2 + d_0^2 = 0 \\ x_1^2 - 2a_1x_1 + x_2^2 - 2b_1x_2 + x_3^2 - 2c_1x_3 - x_4^2 + 2d_1x_4 + a_1^2 + b_1^2 + c_1^2 + d_1^2 = 0 \\ x_1^2 - 2a_2x_1 + x_2^2 - 2b_2x_2 + x_3^2 - 2c_2x_3 - x_4^2 + 2d_2x_4 + a_2^2 + b_2^2 + c_2^2 + d_2^2 = 0 \\ x_1^2 - 2a_3x_1 + x_2^2 - 2b_3x_2 + x_3^2 - 2c_3x_3 - x_4^2 + 2d_3x_4 + a_3^2 + b_3^2 + c_3^2 + d_3^2 = 0 \end{array}\right.$$

with the variables $\{x_1, x_2, x_3, x_4\}$, the other terms being known constants, then $\{x_1^2, x_1, x_2^2, x_2, x_3^2, x_3, x_4^2, x_4\}$ is said to be a set of monomials in the variables $\{x_1, x_2, x_3, x_4.\}$

Example A-3 (*E. Grafarend* and *P. Lohse* 1991, equation 1(3), p.93):

Consider equation 1(3) used to map the topographical surface point embedded into a three-dimensional Euclidean space $\mathbb{R}^3$ onto a (reference) ellipsoid of revolution subject to the constrain that the projection point is a point on the (reference) ellipsoid of revolution. Equation 1(3) given by

$$\left[\begin{array}{l} -(X - x_1) + b^2 x_1 x_4 = 0 \\ -(Y - x_2) + b^2 x_2 x_4 = 0 \\ -(Z - x_3) + b^2 x_3 x_4 = 0 \\ b^2 x_1^2 + b^2 x_2^2 + a^2 x_3^2 - a^2 b^2 = 0 \end{array}\right.$$

where (X, Y, Z) are known topographical coordinates from e.g. GPS, three-dimensional resection e.t.c., $\{a, b\}$ the *semi-major* and *semi-minor* axis respectively. Desired are the ellipsoidal Cartesian coordinates $\{x_1 = x, x_2 = y, x_3 = z\}$ of the projected topographical point. $\{x_1, x_2, x_3, x_4\}$ in this case are the variables whose monomials are given as $\{x_1^2, x_1, x_1x_4, x_2^2, x_2, x_2x_4, x_3^2, x_3, x_3x_4\}$.

Definition A-2 (polynomial):

A polynomial $f \in k[x_1, \ldots, x_n]$ in $x_1, \ldots, x_n$ with coefficients in the field k is a finite linear combination of monomials with pairwise different terms expressed as

$$f = \sum_{\alpha} a_\alpha x^\alpha, \quad a_\alpha \in k, \quad x^\alpha = (x^{\alpha_1}, \ldots, x^{\alpha_n}), \, \alpha = (\alpha_1, \ldots, \alpha_n)$$

Example A-4: The equations

$$\left[\begin{array}{l} x_1^2 + 2a_{12}x_1x_2 + x_2^2 + a_{oo} = 0 \\ x_2^2 + 2b_{23}x_2x_3 + x_3^2 + b_{oo} = 0 \\ x_3^2 + 2c_{31}x_3x_1 + x_1^2 + c_{oo} = 0 \end{array}\right.$$

in *Example* A-1 in page 97 are *multivariate polynomials* with the first equation being a multivariate polynomial in two variables $\{x_1, x_2\}$ and a linear combination of the monomials $\{x_1^2, x_1x_2, x_2^2.\}$ The second equation is a *multivariate polynomial* in two variables $\{x_2, x_3\}$ and a linear combination of the monomials $\{x_2^2, x_2x_3, x_3^2, \}$ while the third equation is a *multivariate polynomial* in two variables $\{x_3, x_1\}$ and a linear combination of the monomials $\{x_3^2, x_3x_1, x_1^2.\}$

Example A-5: The four Equations

$$\left[\begin{array}{l} x_1^2 - 2a_0x_1 + x_2^2 - 2b_0x_2 + x_3^2 - 2c_0x_3 - x_4^2 + 2d_0x_4 + a_0^2 + b_0^2 + c_0^2 + d_0^2 = 0 \\ x_1^2 - 2a_1x_1 + x_2^2 - 2b_1x_2 + x_3^2 - 2c_1x_3 - x_4^2 + 2d_1x_4 + a_1^2 + b_1^2 + c_1^2 + d_1^2 = 0 \\ x_1^2 - 2a_2x_1 + x_2^2 - 2b_2x_2 + x_3^2 - 2c_2x_3 - x_4^2 + 2d_2x_4 + a_2^2 + b_2^2 + c_2^2 + d_2^2 = 0 \\ x_1^2 - 2a_3x_1 + x_2^2 - 2b_3x_2 + x_3^2 - 2c_3x_3 - x_4^2 + 2d_3x_4 + a_3^2 + b_3^2 + c_3^2 + d_3^2 = 0 \end{array}\right.$$

presented in Example A-2 are multivariate polynomials with each equation being a *multivariate polynomial* in four variables $\{x_1, x_2, x_3, x_4\}$ and a linear combination of the monomials $\{x_1^2, x_1, x_2^2, x_2, x_3^2, x_3, x_4^2, x_4\}$.

Example A-6: The four Equations

$$\left[\begin{array}{l} -(X - x_1) + b^2 x_1 x_4 = 0 \\ -(Y - x_2) + b^2 x_2 x_4 = 0 \\ -(Z - x_3) + b^2 x_3 x_4 = 0 \\ b^2 x_1^2 + b^2 x_2^2 + a^2 x_3^2 - a^2 b^2 = 0 \end{array}\right.$$

presented in Example A-3 of page 98 are *multivariate polynomials* with the first equation being a *multivariate polynomial* in two variables $\{x_1, x_4\}$ and a linear combination of the monomials $\{x_1, x_1x_4\}$. The second equation is a *multivariate polynomial* in two variables $\{x_2, x_4\}$ and a linear combination of the monomials $\{x_2, x_2x_4\}$. The third equation is a *multivariate polynomial* in two variables $\{x_3, x_4\}$ and a linear combination of the monomials $\{x_3, x_3x_4\}$, while the fourth equation is a *multivariate polynomial* in three variables $\{x_1, x_2, x_3\}$ and a linear combination of the monomials $\{x_1^2, x_2^2, x_3^2\}$.

Having defined the terms *"monomial"* and *"polynomial"* above, we next define the term *"polynomial ring"* upon which the *Gröbner basis* is computed using the *B. Buchberger algorithm*. In computing the *Gröbner bases*, one computes the *Gröbner basis* of an *Ideal* ($I \in k[x_1, \dots x_n]$) which belongs to a *Polynomial ring* of the field k . First the term *"ring"* is defined in order to understand what the term *"Polynomial ring"* means. We begin by considering the definition of *linear Algebra.* For literature in *linear Algebra* we refer to *T. Becker* and *V. Weispfenning* (1993 Chapter 1, pp.1-60, 1998).

Definition A-3 (linear algebra):

Algebra can be defined as a set S of elements and a finite set M of operations. In *linear algebra* the elements of the set S are *vectors* over the field $\mathbb{R}$ of real numbers, while the set M is basically made up of two elements of *internal relation* namely "*additive*" and "*multiplicative*". An additional definition of the *external relation* expounds on the term *linear algebra* as follows: A *linear algebra* over the field of real numbers $\mathbb{R}$ consists of a set R of objects, two internal relation elements (either *"additive"* or *"multiplicative"*) and one external relation as follows

$$(opera)_1 =: \alpha : R \times R \to R$$
$$(opera)_2 =: \beta : \mathbb{R} \times R \to R \, or \, R \times \mathbb{R} \to R$$
$$(opera)_3 =: \gamma : R \times R \to R$$

The three cases are outlined as follows:

* With respect to the *internal relation* α ("*join*"), R as a linear space in a vector space over $\mathbb{R}$, an *Abelian group* written *"additively"* or *"multiplicatively"*:

$$\mathbf{a}, \mathbf{b}, \mathbf{c} \in R$$

	Axiom	"Additively" written Abelian group	"Multiplicatively" written Abelian group
		$\alpha(\mathbf{a}, \mathbf{b}) =: \mathbf{a} + \mathbf{b}$	$\alpha(\mathbf{a}, \mathbf{b}) =: \mathbf{a} \circ \mathbf{b}$
1	Associativity	$G1+ : (\mathbf{a} + \mathbf{b}) + \mathbf{c} =$ $= \mathbf{a} + (\mathbf{b} + \mathbf{c})$ (additive associativity)	$G1\circ : (\mathbf{a} \circ \mathbf{b}) \circ \mathbf{c} =$ $= \mathbf{a} \circ (\mathbf{b} \circ \mathbf{c})$ (multiplicative associativity)
2	Identity	$G2+ : \mathbf{a} + \mathbf{0} = \mathbf{a}$ (additive identity, neutral element)	$G2\circ : \mathbf{a} \circ 1 = \mathbf{a}$ (multiplicative identity, neutral element)
3	Inverse	$G3+ : \mathbf{a} + (-\mathbf{a}) = \mathbf{0}$ (additive inverse)	$G3\circ : \mathbf{a} \circ \mathbf{a}^{-1} = \mathbf{1}$ (multiplicative inverse)
4	Commutativity	$G4+ : \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (additive commutativity, Abelian axiom)	$G4\circ : \mathbf{a} \circ \mathbf{b} = \mathbf{b} \circ \mathbf{a}$ (multiplicative commutativity, Abelian axiom)

with the triplet of axioms $\{G1+, G2+, G3+\}$ or $\{G1\circ, G2\circ, G3\circ\}$ constituting the set of *group axioms* and $\{G4+, G4\circ\}$ the *Abelian axioms*. Examples of groups include
(a) The group of integer $\mathbb{Z}$ under addition.
(b) The group of non-zero rational number $\mathbb{Q}$ under multiplication.
(c) The set of rotation about the origin in the Euclidean plane under the operation of composite function.

* With respect to the external relation β the following compatibility conditions are satisfied

$$\mathbf{a}, \mathbf{b} \in R, \, t, u \in \mathbb{R}$$
$$\beta(t, \mathbf{a}) =: t \times \mathbf{a}$$

1	distr.	$D1+ : t \times (\mathbf{a} + \mathbf{b}) = (\mathbf{a} + \mathbf{b}) \times t =$ $= t \times \mathbf{a} + t \times \mathbf{b} = \mathbf{a} \times t + \mathbf{b} \times t$ 1st additive distributivity	$D1\circ : t \times (\mathbf{a} \circ \mathbf{b}) = (\mathbf{a} \circ \mathbf{b}) \times t$ $= (t \times \mathbf{a}) \circ \mathbf{b} = \mathbf{a} \circ (\mathbf{b} \times t)$ 1st multiplicative distributivity
2	distr.	$D2+ : (t + u) \times \mathbf{a} = \mathbf{a} \times (t + u) =$ $= t \times \mathbf{a} + u \times \mathbf{a} = \mathbf{a} \times t + \mathbf{a} \times u$ 2nd additive distributivity	$D2\circ : (t \circ u) \times \mathbf{a} = \mathbf{a} \times (t \circ u)$ $= t \circ (u \times \mathbf{a}) = (\mathbf{a} \times t) \circ u$ 2nd multiplicative distributivity

D3: $1 \times \mathbf{a} = \mathbf{a} \times 1 = \mathbf{a} (Left \; and \; Right \; identity)$

* With respect to the *internal relation* γ (*"meet"*) the following conditions are satisfied

$$\mathbf{a}, \mathbf{b}, \mathbf{c} \in R,\ t \in \mathbb{R}$$
$$\gamma(\mathbf{a}, \mathbf{b}) =: \ \mathbf{a} * \mathbf{b}$$

Axiom			Comments
1	Ass.	$G1* :\ (\mathbf{a} * \mathbf{b}) * \mathbf{c} = \mathbf{a} * (\mathbf{b} * \mathbf{c})$	Associativity w.r.t internal multiplication
1	dist.	$D1 * +;\ \mathbf{a} * (\mathbf{b} + \mathbf{c}) = \mathbf{a} * \mathbf{b} + \mathbf{a} * \mathbf{c}$ $(\mathbf{a} + \mathbf{b}) * \mathbf{c} = \mathbf{a} * \mathbf{c} + \mathbf{b} * \mathbf{c}$	Left and Right additive distributivity w.r.t internal multiplication
1	dist.	$D1 * \circ;\ \mathbf{a} * (\mathbf{b} \circ \mathbf{c}) = (\mathbf{a} * \mathbf{b}) \circ \mathbf{c}$ $(\mathbf{a} \circ \mathbf{b}) * \mathbf{c} = \mathbf{a} \circ (\mathbf{b} * \mathbf{c})$	Left and Right multiplicative distributivity w.r.t internal multiplication
2	dist.	$D2 * \times;\ t \times (\mathbf{a} * \mathbf{b}) = (t \times \mathbf{a}) * \mathbf{b}$ $(\mathbf{a} * \mathbf{b}) \times t = \mathbf{a} * (\mathbf{b} \times t)$	Left and Right distributivity of internal and external multiplication

<u>**Definition A-4**</u> (ring):

A *sub-algebra* is called *a ring with identity* if the following two conditions encompassing (seven conditions) hold:
(a) The set R is an *Abelian group* with respect to addition, i.e.four conditions $\{G1+, G2+, G3+, G4+\}$ of *Abelian group* hold.
(b) The set R is a *semi-group* with respect to multiplication; that is, $\{G1*, G2*\}$ holds. In other words, the set R comprises a monoid (i.e. a set with two operations, associativity and identity with respect to multiplication). The last condition is the left and right additive distributivity with respect to internal multiplication $\{D1 * +\}$ which connects the *Abelian Group* and the *monoid*. b In total the four conditions forming the *Abelian group* (a) and the three forming the *semi-group* in (b) add up to form seven conditions enclosed in a ring in Figure 6.

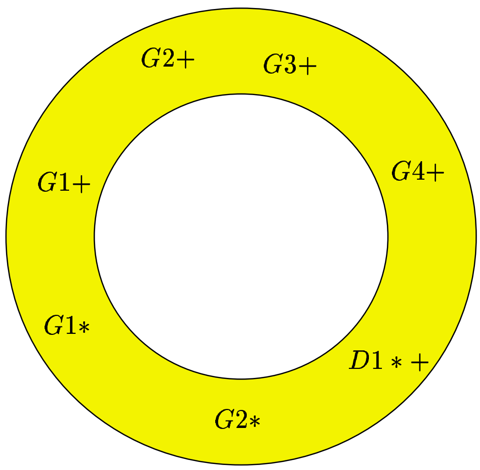


Figure A1: Ring

Condition $G2*$ makes R a *"ring with identity"* while the inclusion of $G3*$ makes the ring be known as the *"division ring"* if every non-zero element of the ring has a multiplicative inverse. The ring becomes a *"commutative ring"* if it has the commutative multiplicative $G4*$. Examples of *rings* include:
* Field k of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ and rational numbers $\mathbb{Q}$. In particular, a *ring* becomes a field if every non zero element of the *ring* has an inverse.
* Integers $\mathbb{Z}$

* Polynomial function P in n variables over a ring R expressed as $P = R[x_1, \ldots, x_n]$.
In the present study, we will consider a ring to be commutative and to include identity element. The field k will be used in subsequent definition to refer he field of an arbitrary ring $\{\mathbb{R}, \mathbb{C}, \mathbb{Q} \in k\}$. Having defined a ring, we next expound on the term *"Polynomial ring"*

Definition A-5 (polynomial ring):

Let us consider a ring R. If we consider an indeterminate $x \notin R$, a *univariate polynomial* is formed by assigning coefficients $a_i \in R$ to the indeterminate and obtaining the summation over finite number of distinct integers. Thus

$f(x) = \sum_{\alpha} a_\alpha x^\alpha, a_\alpha \in R, \alpha \geq 0$

is said to be a *univariate polynomial* over R. If two polynomials be given such that $f_1(x) = \sum_i a_i x^i$ and $f_2(x) = \sum_j b_j x^j$, then two binary operation addition and multiplication can be defined on these polynomials such that:

(a) Addition: $f_1(x) + f_2(x) = \sum_k c_k x^k, c_k = a_k + b_k, c_k \in R$

(b) Multiplicative: $f_1(x).f_2(x) = \sum_k c_k x^k, c_k = \sum_{i+j=k} a_i b_j, c_k \in R$.

A collection of polynomials with these *additive* and *multiplicative* rules forms a commutative ring with zero element and identity 1. A *univariate polynomial* $f(x)$ obtained by assigning elements a_i belonging to the ring R to the variable $\{x\}$ is called a *polynomial ring* and is expressed as $f(x) = R[x]$. In general the entire collection of all polynomials in $x_1, \ldots, x_n$ with coefficients in the field k that satisfy the definition of a *ring* above such that the operations *addition* and *multiplication* can be carried out is called a *polynomial ring*. Designated P, *polynomial ring* is represented by n unknown variables x_i over k and is expressed as $P := k[x_1, \ldots, x_n]$. Its elements are polynomials known as *univariate* when $n = 1$ and *multivariate* otherwise. To complete the definition of a *polynomial ring*, we distinguish it from a polynomial. A polynomial, belonging to a polynomial ring, is the sum of a finite set of monomials (see Definition A-1, page 97 for the definition of a monomial). A complete definition of a polynomial is given in Definition A-2, page 98.

Example A-7: Consider the equations of *Example* A-1 in page 97. The three equations

$$\begin{bmatrix} x_1^2 + 2a_{12}x_1x_2 + x_2^2 + a_{oo} = 0 \\ x_2^2 + 2b_{23}x_2x_3 + x_3^2 + b_{oo} = 0 \\ x_3^2 + 2c_{31}x_3x_1 + x_1^2 + c_{oo} = 0 \end{bmatrix}$$

are said to be polynomial elements of the *polynomial ring* in three variables $[x_1, x_2, x_3]$ over the field of real numbers $\mathbb{R}$ expressed as $P := \mathbb{R}[x_1, x_2, x_3]$.

Example A-8: Consider equations of *Example* A-2 expressed below

$$\begin{bmatrix} x_1^2 - 2a_0x_1 + x_2^2 - 2b_0x_2 + x_3^2 - 2c_0x_3 - x_4^2 + 2d_0x_4 + a_0^2 + b_0^2 + c_0^2 + d_0^2 = 0 \\ x_1^2 - 2a_1x_1 + x_2^2 - 2b_1x_2 + x_3^2 - 2c_1x_3 - x_4^2 + 2d_1x_4 + a_1^2 + b_1^2 + c_1^2 + d_1^2 = 0 \\ x_1^2 - 2a_2x_1 + x_2^2 - 2b_2x_2 + x_3^2 - 2c_2x_3 - x_4^2 + 2d_2x_4 + a_2^2 + b_2^2 + c_2^2 + d_2^2 = 0 \\ x_1^2 - 2a_3x_1 + x_2^2 - 2b_3x_2 + x_3^2 - 2c_3x_3 - x_4^2 + 2d_3x_4 + a_3^2 + b_3^2 + c_3^2 + d_3^2 = 0. \end{bmatrix}$$

The four equations are said to be polynomial elements of the *polynomial ring* in four variables $[x_1, x_2, x_3, x_4]$ over the field of real numbers $\mathbb{R}$ expressed as $P := \mathbb{R}[x_1, x_2, x_3, x_4]$.

Example A-9: Consider equation in *Example* A-3, page 98 expressed below

$$\begin{bmatrix} -(X - x_1) + b^2x_1x_4 = 0 \\ -(Y - x_2) + b^2x_2x_4 = 0 \\ -(Z - x_3) + b^2x_3x_4 = 0 \\ b^2x_1^2 + b^2x_2^2 + a^2x_3^2 - a^2b^2 = 0. \end{bmatrix}$$

The four equations are said to be polynomial elements of the *polynomial ring* in four variables $[x_1, x_2, x_3, x_4]$ over the field of real numbers $\mathbb{R}$ expressed as $P := \mathbb{R}[x_1, x_2, x_3, x_4]$.

In the case of *univariate polynomials*, the format of presentation is usually that of either ascending or descending orders with respect to the power of the variable involved. The situation is however complicated in the case of *multivariate polynomials* where an *ordering* system has to be defined. We mention here the three commonly used monomial ordering systems namely; *lexicographic ordering*, *graded lexicographic ordering* and *graded reverse lexicographic ordering*. First we define the monomial ordering before considering the three types of monomial ordering.

Definition A-6 (monomial ordering):

A monomial ordering on $k[x_1, \ldots, x_n]$ is any relation > on $\mathbb{Z}^n_{\geq 0}$ or equivalently any relation on the set x^α, $\alpha \in \mathbb{Z}^n_{\geq 0}$ satisfying the following conditions:
(a) is total (or linear) ordering on $\mathbb{Z}^n_{\geq 0}$
(b) If $\alpha > \beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$, then $\alpha + \gamma > \beta + \gamma$
(c) > is a well ordering on $\mathbb{Z}^n_{\geq 0}$.
This condition is satisfied if and only if every strictly decreasing sequence in $\mathbb{Z}^n_{\geq 0}$ eventually terminates.

Definition A-6a (*Lexicographic ordering*):

This is akin to the ordering of words used in dictionaries. If we define a polynomial in three variables as $P = k[x, y, z]$ and specify an ordering $x > y > z$, then any term with x will supersede that of y which in tern supersedes that of z. If the powers of the variables for respective monomials are given as $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n), \alpha, \beta \in \mathbb{Z}^n_{\geq 0}$, then $\alpha >_{lex} \beta$ if in the vector difference $\alpha - \beta \in \mathbb{Z}^n$, the most left non-zero entry is positive. For the same variable (e.g. x) this subsequently means $x^\alpha >_{lex} x^\beta$.

Example A-10: $x > y^5 z^9$ is an example of lexicographic ordering.

As a second example, consider the polynomial $f = 2x^2y^8 - 3x^5yz^4 + xyz^3 - xy^4$, we have the *lexicographic order*; $f = -3x^5yz^4 + 2x^2y^8 - xy^4 + xyz^3 \,|x > y > z$.

Definition A-6b (*Graded lexicographic ordering*):

In this case, the *total degree* of the monomials is taken into account. First, one considers which monomial has the highest *total degree* before looking at the *lexicographic ordering*. This ordering looks at the left most (or largest) variable of a monomial and favours the largest power. Let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$, then $\alpha >_{grlex} \beta$ if $|\alpha| = \sum_{i=1}^{n} \alpha_i > |\beta| = \sum_{i=1}^{n} \beta_i$ or $|\alpha| = |\beta|$, and $\alpha >_{lex} \beta$, in $\alpha - \beta \in \mathbb{Z}^n$, the most left non zero entry is positive.

Example A-11: $x^8y^3z^2 >_{grlex} x^6y^2z^3 \,|(8, 3, 2) >_{grlex} (6, 2, 3)$, since $|(8, 3, 2)| = 13 > |(6, 2, 3)| = 11$ and $\alpha - \beta = (2, 1, -1)$. Since the left most term of the difference (2) is positive, the ordering is *graded lexicographic*. As a second example, consider the polynomial $f = 2x^2y^8 - 3x^5yz^4 + xyz^3 - xy^4$, we have the *graded lexicographic order*; $f = -3x^5yz^4 + 2x^2y^8 - xy^4 + xyz^3 \,|x > y > z$.

Definition A-6c (*Graded reverse lexicographic ordering*):

In this case, the *total degree* of the monomials is taken into account as in the case of *graded lexicographic ordering*. First, one considers which monomial has the highest *total degree* before looking at the lexicographic ordering. In contrast to the *graded lexicographic ordering*, one looks at the right most (or largest) variable of a monomial and favours the smallest power. Let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$, then $\alpha >_{grevlex} \beta$ if $|\alpha| = \sum_{i=1}^{n} \alpha_i > |\beta| = \sum_{i=1}^{n} \beta_i$ or $|\alpha| = |\beta$, and $\alpha >_{grevlex} \beta$, and in $\alpha - \beta \in \mathbb{Z}^n$ the right most non zero entry is negative.

Example A-12: $x^8y^3z^2 >_{grevlex} x^6y^2z^3 \,|(8, 3, 2) >_{grevlex} (6, 2, 3)$, since $|(8, 3, 2)| = 13 > |(6, 2, 3)| = 11$ and $\alpha - \beta = (2, 1, -1)$. Since the right most term of the difference (-1) is negative, the ordering is *graded reverse lexicographic*. As a second example, consider the polynomial $f = 2x^2y^8 - 3x^5yz^4 + xyz^3 - xy^4$, we have the *graded reverse lexicographic order*: $f = 2x^2y^8 - 3x^5yz^4 - xy^4 + xyz^3 \,|x > y > z$.

If we consider a non-zero polynomial $f = \sum_\alpha a_\alpha x^\alpha$ in $k[x_1, \ldots, x_n]$ and fix the monomial order, the following additional terms can be defined:

Definition A-7

Multidegree of f : Multideg (f)=max($\alpha \in \mathbb{Z}^n_{\geq 0} \,|a_\alpha \neq 0$)

Leading coefficient of f : LC (f)=$a_{\mathrm{multideg}(f)} \in k$

Leading monomial of f : LM (f)=$x^{\mathrm{multideg}(f)}$ (with coefficient 1)

Leading term of f : LT (f)=LC (f) LM (f)

<u>Example A-13:</u> Consider the polynomial $f = 2x^2y^8 - 3x^5yz^4 + xyz^3 - xy^4$ with respect to *lexicographic order* $\{x > y > z\}$, we have
Multideg (f)=(5,1,4)
LC (f)= -3
LM (f)= x^5yz^4
LT (f)= $-3x^5yz^4$

The definitions of polynomial ordering above have been adopted from *D. Cox et. al.* (1997 pp.52-58). Most computer algebra makes use of the *lexicographic ordering* (*D. Cox et al.* 1997 pp. 56-57, *J. H. Davenport et al.* 1988 p. 72). The present study restricts itself to *lexicographic ordering.*

Appendix A-2: Simplexes for "Stuttgart Central"

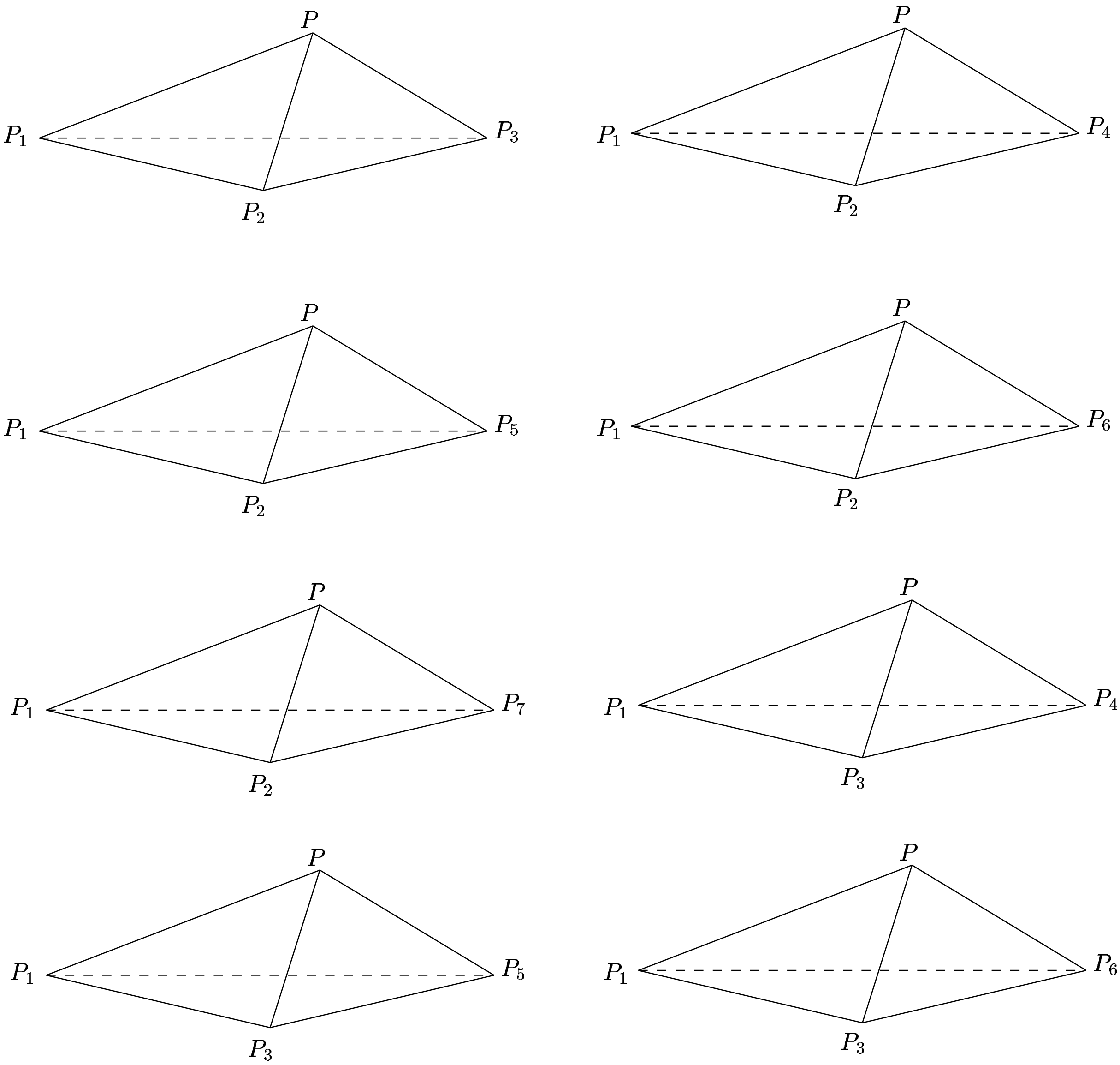


Figure A2: Combinatorial simplexes for the for test network *"Stuttgart Central"*

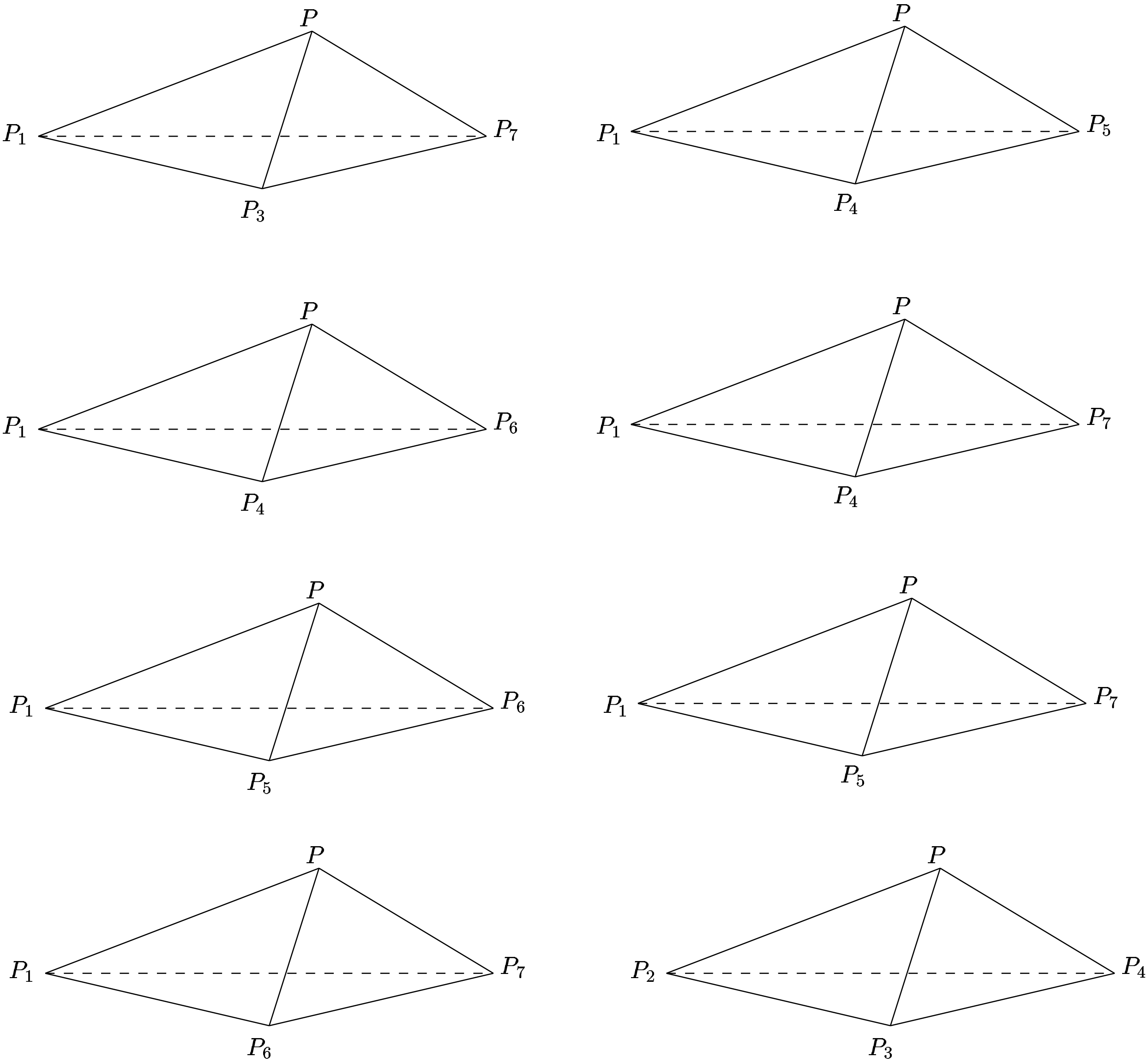


Figure A3: Combinatorial simplexes for the for test network *"Stuttgart Central"* continues

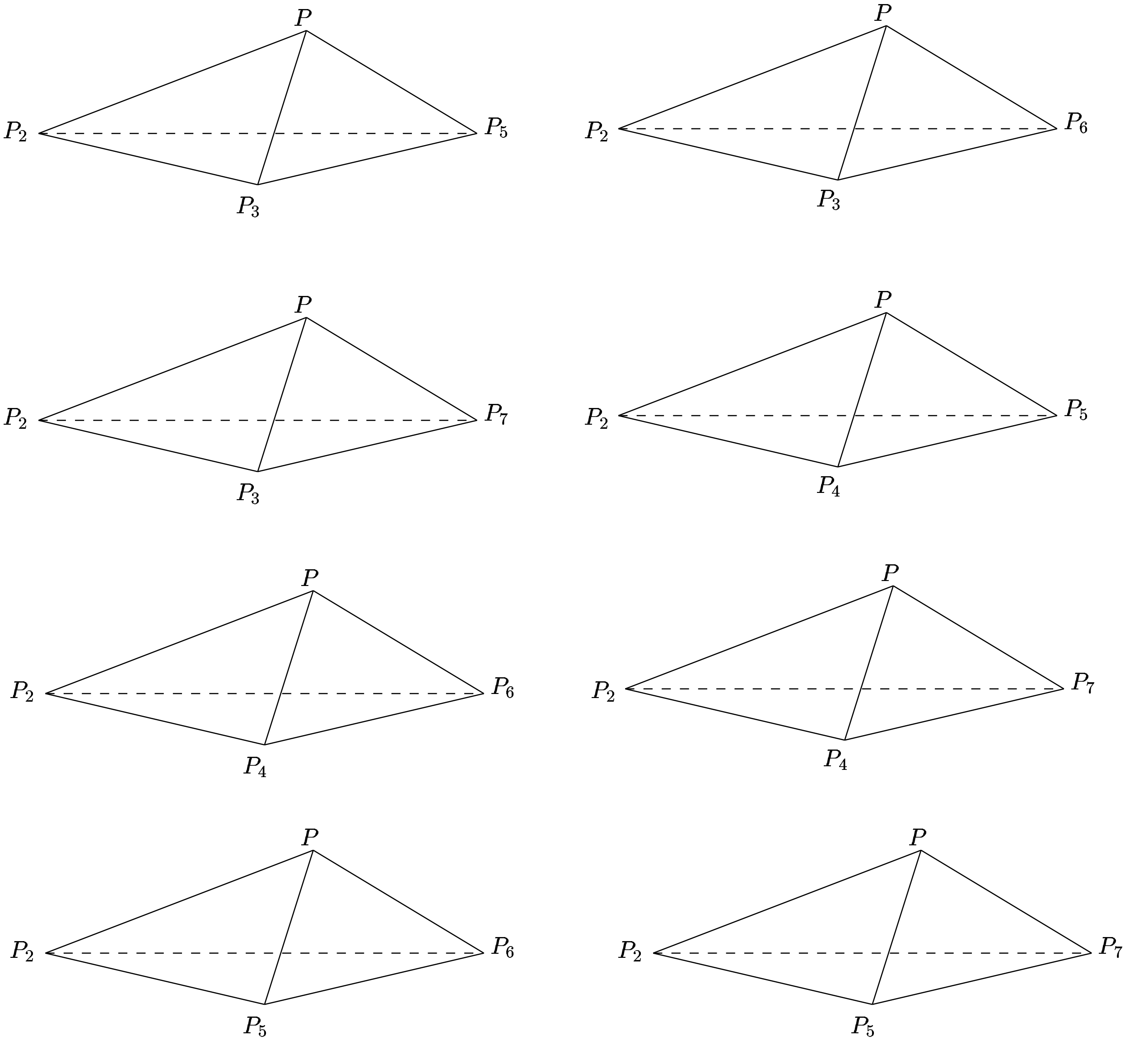


Figure A4: Combinatorial simplexes for the for test network *"Stuttgart Central"* continues

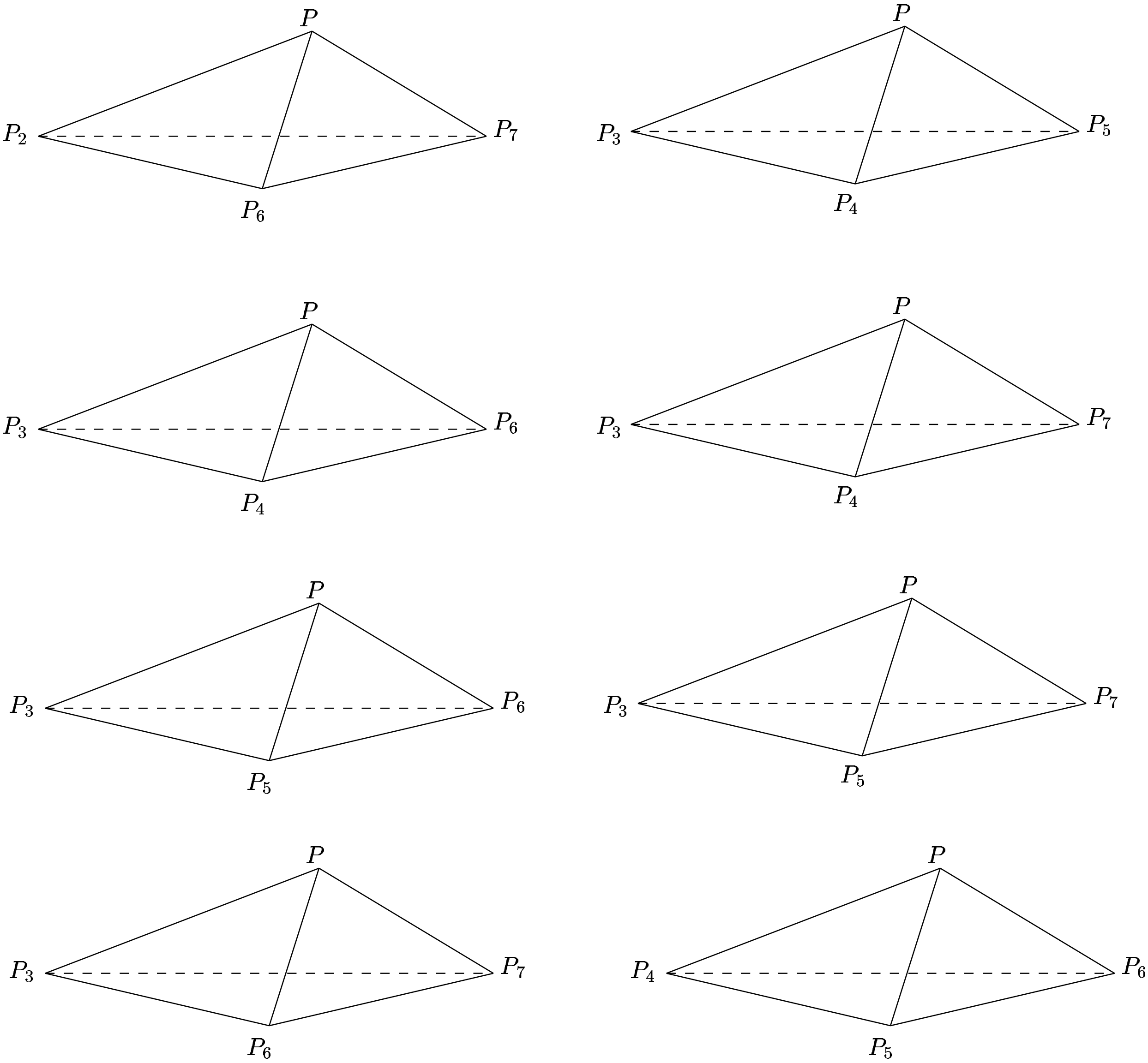


Figure A5: Combinatorial simplexes for the for test network *"Stuttgart Central"* continues

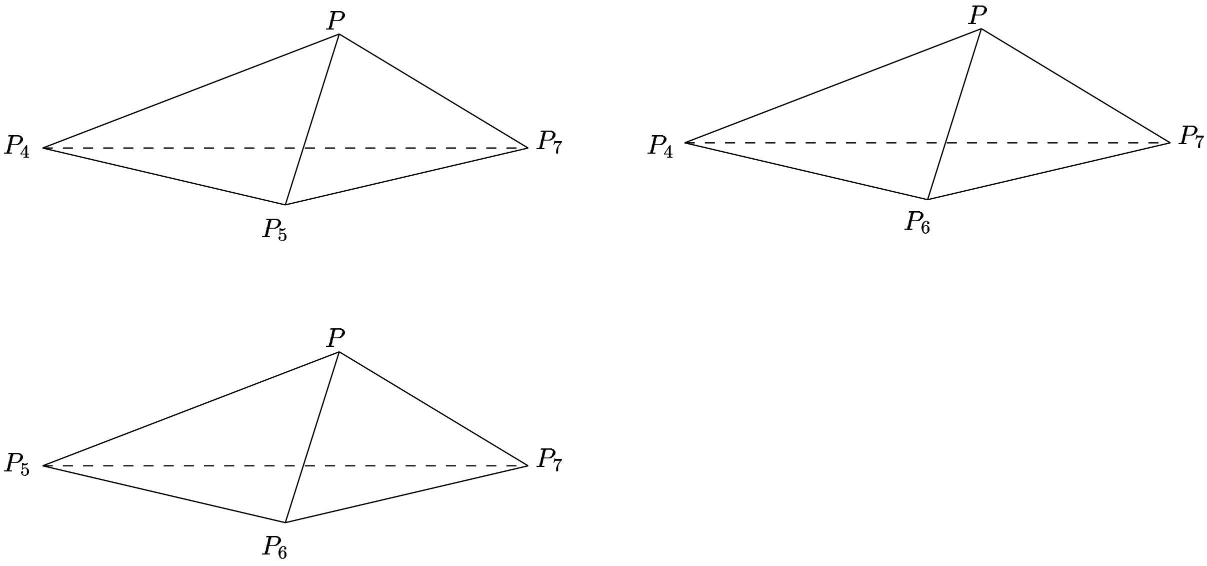


Figure A6: Combinatorial simplexes for the test network *"Stuttgart Central"* continues

Appendix A-3: Error propagation

Nonlinear random effect model (univariate)

Consider a function $y = f(z)$ where y is a scalar valued observation and z a random effect. Three cases can be specified as follows:

<u>Case 1</u> (μ_z assumed to be known):
By Taylor series expansion we have

$$f(z) = f(\mu_z) + \frac{1}{1!}f'(\mu_z)(z-\mu_z) + \frac{1}{2!}f''(\mu_z)(z-\mu_z)^2 + O(3)$$

$$E\{y\} = E\{f(z)\} = f(\mu_z) + \frac{1}{2!}f''(\mu_z)E\{(z-\mu_z)^2\} + O(3)$$

leading to (cf. *E. Grafarend* and *B. Schaffrin* 1983, p.470)

$$\boxed{E\{y\} = f(\mu_z) + \tfrac{1}{2!}f''(\mu_z)\sigma_z^2 + O(3)}.$$

$$E\{(y-E\{y\})^2\} = E\{[f'(\mu_z)(z-\mu_z) + \frac{1}{2!}f''(\mu_z)(z-\mu_z)^2 + O(3) - \\ -\frac{1}{2!}f''(\mu_z)\sigma_z^2 - O(3)]^2\},$$

hence $E\{[y-E\{y\}][y-E\{y\}]\}$ is given by

$$\sigma_y^2 = f'^2(\mu_z)\sigma_z^2 - \frac{1}{4}f''^2(\mu_z)\sigma_z^4 + f'f''(\mu_z)E\{(z-\mu_z)^3\} + \\ +\frac{1}{4}f''^2E\{(z-\mu_z)^4\} + O(3).$$

Finally if z is *quasi-normally distributed*, we have from *E. Grafarend* and *B. Schaffrin* (1983, p.468) that $\pi_3 = E\{(z-\mu_z)^3\} = 0$ and $\pi_4 = E\{(z-\mu_z)^4\} = 3\pi_2^2 = 3\sigma_z^4$ leading to

$$\boxed{\sigma_y^2 = f'^2(\mu_z)\sigma_z^2 + \tfrac{1}{2}f''^2(\mu_z)\sigma_z^4 + O(3)}$$

Case 2 (μ_z unknown, but ξ_0 known as a fixed effect approximation (this implies in *E. Grafarend* and *B. Schaffrin* 1983, p.470 that $\xi_0 \neq \mu_z$)):
By Taylor series expansion we have

$$f(z) = f(\xi_0) + \frac{1}{1!}f'(\xi_0)(z-\xi_0) + \frac{1}{2!}f''(\xi_0)(z-\xi_0)^2 + O(3)$$

using

$$\xi_0 = \mu_z + (\xi_0 - \mu_z) \Rightarrow z - \xi_0 = z - \mu_z + (\mu_z - \xi_0)$$

we have

$$\begin{aligned} f(z) = f(\xi_0) + \tfrac{1}{1!}f'(\xi_0)(z-\mu_z) + \tfrac{1}{1!}f'(\xi_0)(\mu_z-\xi_0) + \\ + \tfrac{1}{2!}f''(\xi_0)(z-\mu_z)^2 + \tfrac{1}{2!}f''(\xi_0)(\mu_z-\xi_0)^2 + \\ + f''(\xi_0)(z-\mu_z)(\mu_z-\xi_0) + O(3) \end{aligned}$$

and

$$\begin{aligned} E\{y\} = E\{f(z)\} = f(\xi_0) + f'(\xi_0)(\mu_z-\xi_0) + \frac{1}{2}f''(\xi_0)\sigma_z^2 + \\ + \frac{1}{2}f''(\xi_0)(\mu_z-\xi_0)^2 + O(3) \end{aligned}$$

leading to $E\{[y-E\{y\}][y-E\{y\}]\}$ as

$$\begin{aligned} \sigma_y^2 = f'^2(\xi_0)\sigma_z^2 + f'f''(\xi_0)E\{(z-\mu_z)^3\} + 2f'f''(\xi_0)\sigma_z^2(\mu_z-\xi_0) + \\ + \tfrac{1}{4}f''^2(\xi_0)E\{(z-\mu_z)^4\} + f''^2(\xi_0)E\{(z-\mu_z)^3\}(\mu_z-\xi_0) - \\ - \tfrac{1}{4}f''^2(\xi_0)\sigma_z^4 + f''^2(\xi_0)\sigma_z^2(\mu_z-\xi_0)^2 + O(3) \end{aligned}$$

and with z being *quasi-normally distributed*, thus $\pi_3 = E\{(z-\mu_z)^3\} = 0$ and $\pi_4 = E\{(z-\mu_z)^4\} = 3\pi_2^2 = 3\sigma_z^4$, we have

$$\boxed{\sigma_y^2 = f'^2(\xi_0)\sigma_z^2 + 2f'f''(\xi_0)\sigma_z^2(\mu_z-\xi_0) + \tfrac{1}{2}f''^2(\xi_0)\sigma_z^4 + f''^2(\xi_0)\sigma_z^2(\mu_z-\xi_0)^2 + O(3)}$$

with the *first* and *third* terms (on the right hand side) being the right hand side terms of case 1 (cf. *E. Grafarend* and *B. Schaffrin* 1983, p.470).

Case 3 (μ_z unknown, but z_0 known as a random effect approximation):
By Taylor series expansion we have

$$\begin{aligned} f(z) = f(\mu_z) + \frac{1}{1!}f'(\mu_z)(z-\mu_z) + \frac{1}{2!}f''(\mu_z)(z-\mu_z)^2 + \\ + \frac{1}{3!}f'''(\mu_z)(z-\mu_z)^3 + O(4) \end{aligned}$$

changing

$$z - \mu_z = z_0 - \mu_z = z_0 - E\{z_0\} - (\mu_z - E\{z_0\})$$

and the *initial bias*

$$-(\mu_z - E\{z_0\}) = E\{z_0\} - \mu_z =: \beta_0$$

leads to

$$\boxed{z - \mu_z = z_0 - E\{z_0\} + \beta_0}.$$

If we also change $f(\mu_z) = f(z_0) + f'(z_0)(\mu_z - z_0) + O(2)$, $f'(\mu_z) = f'(z_0) + f''(z_0)(\mu_z - z_0) + O(3)$ etc, the *derivatives became random effects* and cannot any longer be separated in $E\{f'(\mu_z)(z-\mu_z)\} = f'(\mu_z)E\{(z-\mu_z)\}$ etc. Consider

$$(z-\mu_z)^2 = (z_0 - E\{z_0\})^2 + \beta_0^2 + 2(z_0 - E\{z_0\})\beta_0$$

we have

$$\begin{aligned} f(z) = f(\mu_z) + \tfrac{1}{1!}f'(\mu_z)(z_0 - E\{z_0\}) + \tfrac{1}{1!}f'(\mu_z)\beta_0 + \\ + \tfrac{1}{2!}f''(\mu_z)(z_0 - E\{z_0\})^2 + \tfrac{1}{2!}f''(\mu_z)\beta_0^2 + f''(\mu_z)(z_0 - E\{z_0\})\beta_0 + O(3) \end{aligned}$$

$$E\{y\} = f(\mu_z) + f'(\mu_z)\beta_0 + \frac{1}{2}f''(\mu_z)\sigma_{z_0}^2 + \frac{1}{2}f''(\mu_z)\beta_0^2 + O(3)$$

leading to $E\{[y-E\{y\}][y-E\{y\}]\}$as

$$\begin{array}{c}\sigma_y^2 = f'^2(\mu_z)\sigma_{z_0}^2 + f'f''(\mu_z)E\{(z_0-E\{z_0\})^3\}+\\ 2f'f''(\mu_z)\sigma_{z_0}^2\beta_0 + \frac{1}{4}f''^2(\mu_z)E\{(z_0-E\{z_0\})^4\}+\\ f''^2(\mu_z)E\{(z_0-E\{z_0\})^3\}\beta_0 + f''^2(\mu_z)\sigma_{z_0}^2\beta_0^2+\\ +\frac{1}{4}f''^2(\mu_z)\sigma_{z_0}^4 - \frac{1}{2}f''^2(\mu_z)E\{(z_0-E\{z_0\})^2\}\sigma_{z_0}^2 + O(3)\end{array}$$

and with z_0 being *quasi-normally distributed*, thus $\pi_3 = E\{(z_0-E\{z_0\})^3\} = 0$ and $\pi_4 = E\{(z_0 - E\{z_0\})^4\} = 3\pi_2^2 = 3\sigma_{z_0}^4$, we have

$$\boxed{\sigma_y^2 = f'^2(\mu_z)\sigma_{z_0}^2 + 2f'f''(\mu_z)\sigma_{z_0}^2\beta_0 + \frac{1}{2}f''^2(\mu_z)\sigma_{z_0}^4 + f''^2(\mu_{z_0})\sigma_{z_0}^2\beta_0^2 + O(3)}$$

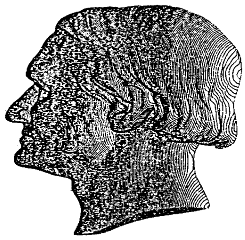
with the *first* and *third* terms (on the right hand side) being the right hand side terms of case 1 (cf. *E. Grafarend* and *B. Schaffrin* 1983, p.470).

Appendix A-4: C. F. Gauss combinatorial approach

CARL FRIEDRICH GAUSS

WERKE

NEUNTER BAND.



HERAUSGEGEBEN

VON DER

KÖNIGLICHEN GESELLSCHAFT DER WISSENSCHAFTEN

ZU

GÖTTINGEN.

IN COMMISSION BEI B. G. TEUBNER IN LEIPZIG.

1903.

BESTIMMUNG

DES

BREITENUNTERSCHIEDES

ZWISCHEN DEN

STERNWARTEN VON GÖTTINGEN UND ALTONA

DURCH

BEOBACHTUNGEN AM RAMSDENSCHEN ZENITHSECTOR

VON

CARL FRIEDRICH GAUSS,

RITTER DES GUELPHEN- UND DANNEBROG-ORDENS; K. GROSSBR. HANNOVERSCHEM HOFRATH;
PROFESSOR DER ASTRONOMIE UND DIRECTOR DER STERNWARTE IN GÖTTINGEN;
MITGLIED DER AKADEMIEN UND SOCIETÄTEN VON BERLIN, COPENHAGEN, EDINBURG, GÖTTINGEN,
LONDON, MÜNCHEN, NEAPEL, PARIS, PETERSBURG, STOCKHOLM,
DER AMERIKANISCHEN, ITALIENISCHEN, KURLÄNDISCHEN, LONDONER ASTRONOMISCHEN U. A.

GÖTTINGEN,

BEI VANDENHOECK UND RUPRECHT.

1828.

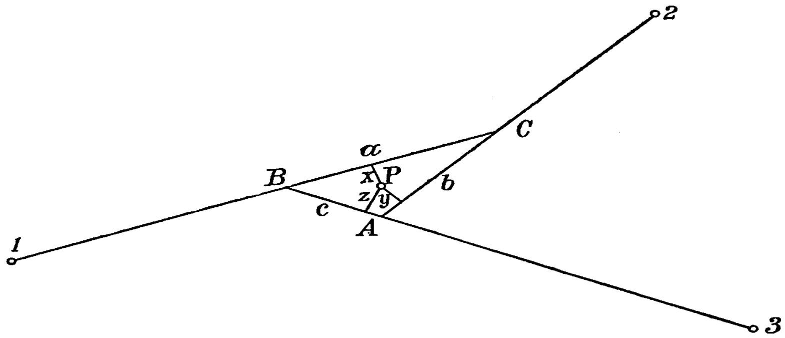
NACHLASS.

[1.]

Endresultat für den Ort eines Punktes in einer Ebene, der von drei bekannten aus angeschnitten ist.

Es bedeuten 10, 20, 30 die drei beobachteten Richtungen [nach P] und α, β, γ die entsprechenden Entfernungen.

Die drei einzelnen Resultate aus den Combinationen 2—3, 1—3, 1—2 seien A, B, C, zugleich die Winkel des durch jene gebildeten Dreiecks; die ihnen gegenüber stehenden Seiten a, b, c.



Perpendikel von dem gesuchten Orte auf a, b, c seien x, y, z. S doppelter Flächeninhalt des Dreiecks.

Es sind dann

$$\frac{x}{\alpha},\quad \frac{y}{\beta},\quad \frac{z}{\gamma}$$

die übrig bleibenden Fehler, also

$$\frac{xx}{\alpha\alpha}+\frac{yy}{\beta\beta}+\frac{zz}{\gamma\gamma}\quad \text{Minimum}$$

und

$$ax+by+cz = S.$$

Also werden x, y, z proportional den Grössen $\alpha\alpha a$, $\beta\beta b$, $\gamma\gamma c$;

$$x = \frac{\alpha\alpha a S}{\alpha\alpha aa+\beta\beta bb+\gamma\gamma cc},$$

etc.

[Bezeichnet (ABC) die Fläche des Dreiecks ABC, u. s. f., so ist

$$\begin{aligned} S = 2(ABC) &= (\alpha\alpha aa+\beta\beta bb+\gamma\gamma cc)k \\ 2(BPC) &= \alpha\alpha aak \\ 2(APC) &= \beta\beta bbk \\ 2(APB) &= \gamma\gamma cck, \end{aligned}$$

wo k die Correlate der Bedingungsgleichung ist. P ist der durch die Perpendikel x, y, z bestimmte Punkt.

Folglich wird, wenn A, B, C, P die complexen Grössen bedeuten, denen die Eckpunkte des Dreiecks ABC und der Punkt P entsprechen:

$$(\alpha\alpha aa+\beta\beta bb+\gamma\gamma cc)P = \alpha\alpha aaA+\beta\beta bbB+\gamma\gamma ccC.]$$

Es folgt hieraus, dass das Endresultat*)

$$\frac{\alpha\alpha aaA+\beta\beta bbB+\gamma\gamma ccC}{\alpha\alpha aa+\beta\beta bb+\gamma\gamma cc}$$

also ein Mittel aus den drei partiellen Resultaten A, B, C ist, indem man diesen die Gewichte

$$\alpha\alpha aa,\qquad \beta\beta bb,\qquad \gamma\gamma cc$$

beilegt, oder

$$\alpha\alpha \sin A^2,\qquad \beta\beta \sin B^2,\qquad \gamma\gamma \sin C^2.$$

Offenbar ist hier A zugleich der Winkel zwischen 20 und 30, u. s. f.

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