Representations of Hecke Algebras of Weyl Groups of Type A and B

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Abstract

The knowledge of the decomposition numbers of Hecke algebras associated to Weyl groups is very useful in the representation theory of finite groups of Lie type since the decomposition matrix of such an algebra embeds into that of the corresponding group. In the investigation of the Hecke algebras themselves, generic constructions — that is, constructions independent of the coefficient ring and the parameters — are a helpful tool. This thesis contributes to those two aspects of the theory of Hecke algebras.

The first part of this thesis is concerned with decomposition numbers of blocks of Hecke algebras of type A. In particular, we consider blocks having core (0) and weight 3. First, we derive an upper bound for the decomposition numbers of an arbitrary block. This is used to show that all the decomposition numbers of a block having core (0) and weight 3 are 0 or 1. That result in turn enables us to describe a combinatorial algorithm for their calculation. Furthermore, we show that the decomposition numbers of a block having core (0) and weight 3 depend only on the ordinary and the quantized characteristic of the coefficient field. Moreover, if the ordinary characteristic is neither 2 nor 3 then they are already determined by the quantized characteristic alone.

In the second part of this thesis, we construct generic Specht series for Hecke algebras of type A and generic bi-Specht series for Hecke algebras of type B. These are series of right ideals in those algebras such that all subquotients are Specht modules respectively bi-Specht modules. The construction of the Specht series generalizes ideas from Dipper and James for symmetric groups and Hecke algebras of type A. In particular, generic bases for the so-called PK-modules are introduced. The derivation of the bi-Specht series makes use of the Specht series and general methods from Dipper and James for the investigation of Hecke algebras of type B.

Zusammenfassung

Die Kenntnis der Zerlegungszahlen von mit Weyl-Gruppen assoziierten Hecke-Algebren ist sehr nützlich in der Darstellungstheorie endlicher Gruppen vom Lie-Typ, da die Zerlegungsmatrix einer solchen Algebra in die der entsprechenden Gruppe eingebettet ist. Zur Untersuchung der Hecke-Algebren selbst sind generische — das heißt vom Koeffizientenring und den Parametern unabhängige — Konstruktionen hilfreich. Die vorliegende Arbeit trägt zu diesen beiden Aspekten der Theorie der Hecke-Algebren bei.

Der erste Teil dieser Arbeit beschäftigt sich mit Zerlegungszahlen von Blöcken von Hecke-Algebren vom Typ A. Insbesondere werden Blöcke mit Kern (0) und Gewicht 3 betrachtet. Zunächst wird eine obere Schranke für die Zerlegungszahlen eines beliebigen Blocks hergeleitet. Damit wird gezeigt, daß die Zerlegungszahlen eines Blocks mit Kern (0) und Gewicht 3 nur die Werte 0 und 1 annehmen. Dies ermöglicht die Beschreibung eines kombinatorischen Algorithmus zu ihrer Berechnung. Weiter wird gezeigt, daß die Zerlegungszahlen eines Blocks mit Kern (0) und Gewicht 3 nur von der gewöhnlichen und der quantisierten Charakteristik des Koeffizientenkörpers abhängen. Wenn die gewöhnliche Charakteristik weder 2 noch 3 ist, sind sie sogar bereits durch die quantisierte Charakteristik bestimmt.

Im zweiten Teil dieser Arbeit werden generische Specht-Serien für Hecke-Algebren vom Typ A und generische Bi-Specht-Serien für Hecke-Algebren vom Typ B konstruiert. Dabei handelt es sich um Reihen von Rechtsidealen, bei denen alle Subquotienten Specht-Moduln beziehungsweise Bi-Specht-Moduln sind. Die Konstruktion der Specht-Serien verallgemeinert Ideen von Dipper und James für symmetrische Gruppen und Hecke-Algebren vom Typ A, insbesondere werden generische Basen für die sogenannten PK-Moduln bestimmt. Die Herleitung der Bi-Specht-Serien benutzt die Specht-Serien und allgemeine Methoden von Dipper und James zur Untersuchung von Hecke-Algebren vom Typ B.

Contents

Introduction

Hecke algebras associated to Weyl groups are involved in various branches of mathematics and physics. Hecke algebras of type A are employed in knot theory for the construction of topological invariants (see [JON]) and also occur in statistical mechanics (see [JIM1], [JIM2]). In the theory of quantum groups, they play the part of the symmetric group algebras in the quantized version of classical Schur-Weyl reciprocity between general linear groups and symmetric groups (see again [JIM2]). Furthermore, the Hecke algebras associated to Weyl groups are a valuable tool in the representation theory of algebraic groups and finite groups of Lie type (see [IWA] and [KL]). In particular, they are very useful for the determination of the decomposition numbers of finite groups of Lie type, namely their decomposition matrices embed into the decomposition matrices of the corresponding groups (see [DIP] and [DJ3]).

The first part of this thesis contributes to that latter application of Hecke algebras. Here, the decomposition matrices of certain blocks of Hecke algebras of type A are investigated. For a more detailed description, we fix coefficient rings and parameters. Let Q be a field and ψ be a discrete valuation on Q such that Q is complete with respect to ψ. With that, denote by S the valuation ring of ψ in Q, by I the valuation ideal of ψ in S, and write $F = S/I$ for the residue class field. Then we have the inclusion $S \hookrightarrow Q$ and the reduction modulo I $\bar{\cdot} : S \to F$. Next, fix a unit $a \in S$. Then a also is a unit in Q and $\bar{a}$ is a unit in F. We assume that there are natural numbers m satisfying $\sum_{i=0}^{m-1} \bar{a}^i = 0$ in F — the opposite not being interesting — and denote by $e_F(\bar{a})$ the minimum of these numbers. With that, we put $n = 3e_F(\bar{a})$ and build the Hecke algebras of type A_{n-1} over the coefficient rings Q, S, and F with the respective parameters a, a, and $\bar{a}$. We denote them by $\mathcal{H}_{A_{n-1}}^{(Q,a)}$, $\mathcal{H}_{A_{n-1}}^{(S,a)}$, and $\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}$ (see Section 1.2). We also assume that $\mathcal{H}_{A_{n-1}}^{(Q,a)}$ is semisimple.

Similar to the special case of symmetric groups, the blocks of these algebras are indexed by $e_F(\bar{a})$-cores of partitions of n and are divided into families according to the $e_F(\bar{a})$-weights of the indexing cores. The block under consideration in this thesis is indexed by the partition (0) of 0 and has $e_F(\bar{a})$-weight 3. Our choice of n

ensures that the algebras $\mathcal{H}^{(Q,a)}_{A_{n-1}}$, $\mathcal{H}^{(S,a)}_{A_{n-1}}$, and $\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}$ indeed have such a block. This block can be considered as the principal block of those algebras. We denote it by $B^{(0)}(n)$ and its decomposition matrix by $\Delta^{\mathcal{H}}_n((0))$ (see Section 1.8).

The results obtained for the block decomposition matrix $\Delta^{\mathcal{H}}_n((0))$ are as follows. First, it is shown that all of its entries are 0 or 1. Then, this fact and the quantized version of the Theorem of Schaper from [JM] are used to describe a purely combinatorial algorithm for the calculation of $\Delta^{\mathcal{H}}_n((0))$. Finally, it is shown that the matrix $\Delta^{\mathcal{H}}_n((0))$ depends only on the characteristic of F and the number $e_F(\bar{a})$, and moreover, if the characteristic of F is neither 2 nor 3 then it is already completely determined by $e_F(\bar{a})$. This proves a conjecture of James (see [JAM2, Section 4]) in the special case of the submatrix $\Delta^{\mathcal{H}}_n((0))$ of the decomposition matrix of the algebras $\mathcal{H}^{(Q,a)}_{A_{n-1}}$, $\mathcal{H}^{(S,a)}_{A_{n-1}}$, and $\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}$.

The second part of this thesis is concerned with generic features of Hecke algebras of type A and B. Generic means that these features are independent of the choice of the coefficient ring and the parameters for the Hecke algebra. Here, the coefficient ring can be an arbitrary integral domain. The features in question are generic Specht series for Hecke algebras of type A and generic bi-Specht series for Hecke algebras of type B.

A generic Specht series for a Hecke algebra of type A means a series of right ideals in that algebra such that all the quotients of successive ideals are isomorphic to Specht modules (see Section 1.3) and moreover all algebra elements, ideals, and homomorphisms occurring in the construction of this series are stable when changing the coefficient ring. Here, the Specht modules for Hecke algebras of type A are those from [DJ1].

Similarly, a generic bi-Specht series for a Hecke algebra of type B means a series of right ideals in that algebra such that all the quotients of successive ideals are isomorphic to bi-Specht modules (see Section 4.4) and moreover all algebra elements, ideals, and homomorphisms occurring in the construction of this series are stable when changing the coefficient ring. Here, the bi-Specht modules for Hecke algebras of type B are a generalization of those from [DJ3].

In this thesis, generic Specht series for Hecke algebras of type A are constructed by generalizing ideas from [JAM1, Section 16] and [DJ1, Section 7]. In particular, new generic bases of the intermediary modules $S^{\mu^{\#}\mu}$ from there are introduced (see Section 3.10). These generic Specht series for Hecke algebras of type A are then used to obtain generic bi-Specht series for Hecke algebras of type B by transferring the algebra elements, ideals, and homomorphisms employed in their construction to Hecke algebras of type B with methods from [DJ3].

The organization of this thesis is as follows. Chapter 1 collects background material and known facts about Hecke algebras of type A which will be used in later chapters. It starts with an overview of the combinatorics required for the representation theory of the Weyl groups of type A, that is, the symmetric groups. Here, we recall — amongst other things — compositions and partitions, hooks and rim hooks, β-sequences and abaci, cores, tableaux, Young subgroups and shortest representatives, and finally row number lists.

The next section reviews Hecke algebras of type A. It gives generators and relations for these algebras and lists some basic facts and notions related to coefficient rings. Section 1.3 describes the derivation of the irreducible representations of Hecke algebras of type A over fields as carried out in [DJ1]. It also recalls the generic permutation modules and Specht modules from there. The following section adapts the account on modular reduction and decomposition numbers for group algebras given in [CR1, §16] to the situation at hand. It fixes the notation for Grothendieck groups, modular systems, decomposition maps, and decomposition numbers for Hecke algebras of type A. Next, Section 1.5 describes the behavior of Specht modules with respect to modular reduction and states some consequences thereof. Then, in Section 1.6, new modular systems with nice properties are derived from a given one and are used to examine the dependency of the decomposition numbers of Hecke algebras of type A on the employed modular system. Section 1.7 translates the treatment of projective indecomposable modules and the Cartan-Brauer triangle for group algebras in [CR1, §18] to Hecke algebras of type A.

The following section collects some basic facts and notions from the block theory of Hecke algebras of type A. It recalls such things as block idempotents and block ideals, the parameterization of the blocks of Hecke algebras of type A by cores of partitions, and the block decomposition of modules, Grothendieck groups, projective class groups, and decomposition matrices. The next section treats induction of modules from a Hecke algebra of type A_{n-2} to the corresponding one of type A_{n-1}. The induction of Specht modules and projective indecomposable modules, both considered as elements of Grothendieck groups and projective class groups, is described in more detail. This is then used to derive an upper bound for the decomposition numbers of a block of the Hecke algebra of type A_{n-1} provided the decomposition numbers of the Hecke algebra of type A_{n-2} are known. Finally, Section 1.10 gives an account on Schaper's Theorem for Hecke algebras of type A proved in [JM]. It introduces the required notation and states the Theorem of Schaper. Then, it describes how this theorem can be used to obtain valuable information on decomposition numbers.

In Chapter 2, we investigate the decomposition matrices of blocks of Hecke algebras of type A having core (0) and weight 3. Here and in the following, we use the terminology and notation from above. In Section 2.1, it is shown that all entries of $\Delta_n^{\mathcal{H}}((0))$ are 0 or 1. This is done by using the upper bound for the decomposition numbers of an arbitrary block of the algebras $\mathcal{H}_{A_{n-1}}^{(Q,a)}$, $\mathcal{H}_{A_{n-1}}^{(S,a)}$, and $\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}$ from Section 1.9. In the case of the block $B^{(0)}(n)$, this upper bound depends on an upper bound U_{n-1} for the entries of the decomposition matrix $\Delta_{n-1}^{\mathcal{H}}$ of the algebras $\mathcal{H}_{A_{n-2}}^{(Q,a)}$, $\mathcal{H}_{A_{n-2}}^{(S,a)}$, and $\mathcal{H}_{A_{n-2}}^{(F,\bar{a})}$ (see Section 1.4) and a certain integer $J_{(0)}$ determined by combinatorics involving the partitions belonging to $B^{(0)}(n)$. Since every block of $\mathcal{H}_{A_{n-2}}^{(Q,a)}$, $\mathcal{H}_{A_{n-2}}^{(S,a)}$, and $\mathcal{H}_{A_{n-2}}^{(F,\bar{a})}$ has weight less than 3, the upper bound U_{n-1} can be obtained from results on such blocks in [RIC] and [JAM2].

For the evaluation of $J_{(0)}$, we proceed as follows. First, the partitions belonging to $B^{(0)}(n)$, that is, those having $e_F(\bar{a})$-core (0), are divided into families according to the shapes of the corresponding abaci as described in [MR2]. Then, the definition of $J_{(0)}$ in Theorem 1.9.18 is adapted to abacus notation. Finally, $J_{(0)}$ is determined through a case by case analysis of the various families of partitions lying in the block $B^{(0)}(n)$. The values obtained for U_{n-1} and $J_{(0)}$ now establish the upper bound 1 for all entries of $\Delta_n^{\mathcal{H}}((0))$.

Section 2.2 first shows how the matrix $\Delta_n^{\mathcal{H}}((0))$ can be calculated explicitly and then investigates its dependency on the employed coefficient rings and parameters. The explicit calculation of $\Delta_n^{\mathcal{H}}((0))$ is based on the quantized version of Schaper's Theorem from [JM]. This theorem reveals for every entry in a row of $\Delta_n^{\mathcal{H}}((0))$ if it vanishes or not provided the earlier rows — with respect to an appropriate ordering — are known (see Remark 1.10.9.(ii)). This fact and the upper bound 1 for all decomposition numbers of the block $B^{(0)}(n)$ established in Section 2.1 allow the explicit calculation of its decomposition matrix $\Delta_n^{\mathcal{H}}((0))$ in a straightforward inductive manner.

The second topic in Section 2.2 is the dependency of $\Delta_n^{\mathcal{H}}((0))$ on the coefficient rings and parameters underlying the algebras $\mathcal{H}_{A_{n-1}}^{(Q,a)}$, $\mathcal{H}_{A_{n-1}}^{(S,a)}$, and $\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}$. It turns out that the determining values are $e_F(\bar{a})$ and the characteristic of F. First, the relations between these values and the parameters a and $\bar{a}$ of the Hecke algebras are described (see Lemma 2.2.3 to Lemma 2.2.5). We see that there are three distinct cases to be considered. Using this distinction, the values of the valuation ψ on quantized integers (see Definition 1.2.2.(i)) are determined. Such expressions occur in the Theorem of Schaper. We see that these values are completely determined by $e_F(\bar{a})$ and the characteristic of F. This fact and the previously described method for calculating the matrix $\Delta_n^{\mathcal{H}}((0))$ using Schaper's Theorem now show that the decomposition numbers of $B^{(0)}(n)$ also depend only on the characteristic of F and $e_F(\bar{a})$.

Next, this dependency is reduced even further provided the characteristic of F is neither 2 nor 3. With this restriction, we get a factorization of the values of ψ on the relevant quantized integers where one factor is completely determined by $e_F(\bar{a})$ and the characteristic of F while the other one depends only on $e_F(\bar{a})$ and the ordinary integer behind the respective quantized integer. Combining this with the Theorem of Schaper and the method for calculating the decomposition numbers of the block $B^{(0)}(n)$ described above, we obtain that $\Delta_n^{\mathcal{H}}((0))$ is completely determined by $e_F(\bar{a})$ provided the characteristic of F is neither 2 nor 3.

Chapter 3 is concerned with the construction of generic Specht series for Hecke algebras of type A (see above). To be more specific, let us fix a degree n, an integral domain R, a unit $q \in R$, and with that the Hecke algebra $\mathcal{H}_{A_{n-1}}^{(R,q)}$. Sections 3.1 to 3.3 provide the combinatorics required for the construction of a generic Specht series for this algebra. Section 3.1 reviews ordering relations for row standard tableaux and the corresponding shortest representatives. Section 3.2 discusses PK_n-pairs $\mu^{\#}\mu$ and the operators A_c and R_c for them as described in [JAM1] and some related tableaux and permutations. In short, a PK_n-pair consists of a partition $\mu^{\#}$ and a composition μ of n (see Section 1.1) satisfying certain conditions. Moreover, if we have an index $c > 1$ such that the c-th and $(c-1)$-th entries of $\mu^{\#}$ and μ meet further requirements then we can apply the operator A_c to get another partition $\mu^{\#}A_c$ and a PK_n-pair $\mu^{\#}A_c\mu$ and also the operator R_c which gives us another composition μR_c and a PK_n-pair $\mu^{\#}\mu R_c$. Section 3.3 treats the aspects of row number lists employed in the construction of the Specht series. These are the connection to PK_n-pairs via good and bad entries, the ensuing organization of row number lists into sets $\mathcal{Z}^{\mu^{\#}\mu}$, and maps between such sets from [JAM1], but also ordering relations between related permutations.

Sections 3.4 and 3.5 review known modules and homomorphisms which are used in the construction of the Specht series. Section 3.4 recalls the definition of PK_n-modules $S_{(R,q)}^{\mu^{\#}\mu}$ indexed by PK_n-pairs from [DJ1] and collects some elementary facts about them. Section 3.5 gives an account on the construction of PK_n-homomorphisms

$$\Psi_{\mu^{\#}\mu c}^{(R,q)} : M_{(R,q)}^{\mu} \to M_{(R,q)}^{\mu R_c}$$

and lists some basic properties of them. Such a homomorphism is indexed by a PK_n-pair $\mu^{\#}\mu$ and an integer c which allows the application of the corresponding pair of operators A_c and R_c to $\mu^{\#}\mu$ (see above). It maps the permutation module $M_{(R,q)}^{\mu}$ (see Section 1.3) into $M_{(R,q)}^{\mu R_c}$ and the PK_n-module $S_{(R,q)}^{\mu^{\#}\mu}$ into $S_{(R,q)}^{\mu^{\#}\mu R_c}$. All this is drawn from [DJ1].

The next two sections introduce ZNL-elements and describe the effect of PK_n-homomorphisms on them. In Section 3.6, we define for a composition λ of n and every row number list ζ in the set $\mathcal{Z}^\lambda$ (see Section 1.1) the corresponding ZNL-element $z(\zeta)_{(R,q)}$ in $M^\lambda_{(R,q)}$. We also derive some basic facts about these elements. We determine, for example, their representations with respect to the row standard basis $\mathbf{B}^{M^\lambda}_{\mathrm{row\,std}}(R,q)$ of $M^\lambda_{(R,q)}$ (see Section 1.3). From this in turn we obtain their linear independence. In Section 3.7, we examine for a given PK_n-pair $\mu^\#\mu$ the images of the ZNL-elements $z(\zeta)_{(R,q)}$ indexed by row number lists $\zeta \in \mathcal{Z}^{\mu^\#\mu}$ under a PK_n-homomorphism $\Psi^{(R,q)}_{\mu^\#\mu c} : M^\mu_{(R,q)} \to M^{\mu R_c}_{(R,q)}$ and derive their representations with respect to the basis $\mathbf{B}^{M^{\mu R_c}}_{\mathrm{row\,std}}(R,q)$ of $M^{\mu R_c}_{(R,q)}$. We find that the $z(\zeta)_{(R,q)}$ with $\zeta \in \mathcal{Z}^{\mu^\# A_c\mu} \subseteq \mathcal{Z}^{\mu^\#\mu}$ are contained in $\operatorname{Ker}\Psi^{(R,q)}_{\mu^\#\mu c}$ and that the $z(\zeta)_{(R,q)}\Psi^{(R,q)}_{\mu^\#\mu c}$ with $\zeta \in \mathcal{Z}^{\mu^\#\mu} \setminus \mathcal{Z}^{\mu^\# A_c\mu}$ are linearly independent.

The following three sections establish bases of PK_n-modules consisting of ZNL-elements. In Section 3.8, we show that, given a PK_n-pair $\mu^\#\mu$ and a pair of operators A_c and R_c for it, the set $\left\{ z(\eta)_{(R,q)} \,\middle|\, \eta \in \mathcal{Z}^{\mu^\#\mu R_c} \right\}$ is an R-basis of the PK_n-module $S^{\mu^\#\mu R_c}_{(R,q)}$ provided $\left\{ z(\zeta)_{(R,q)} \,\middle|\, \zeta \in \mathcal{Z}^{\mu^\#\mu} \right\}$ is an R-basis of $S^{\mu^\#\mu}_{(R,q)}$. This is done by comparing the representations of the elements $z(\zeta)_{(R,q)}\Psi^{(R,q)}_{\mu^\#\mu c}$ for $\zeta \in \mathcal{Z}^{\mu^\#\mu} \setminus \mathcal{Z}^{\mu^\# A_c\mu}$ with respect to $\mathbf{B}^{M^{\mu R_c}}_{\mathrm{row\,std}}(R,q)$ to those of the elements $z(\eta)_{(R,q)}$ for $\eta \in \mathcal{Z}^{\mu^\#\mu R_c}$. In Section 3.9, we consider again a PK_n-pair $\mu^\#\mu$ and a pair of operators A_c and R_c for it and we also assume that $\left\{ z(\zeta)_{(R,q)} \,\middle|\, \zeta \in \mathcal{Z}^{\mu^\#\mu} \right\}$ is an R-basis of $S^{\mu^\#\mu}_{(R,q)}$. Given this, we show that $\left\{ z(\zeta)_{(R,q)} \,\middle|\, \zeta \in \mathcal{Z}^{\mu^\# A_c\mu} \right\}$ is an R-basis of $\operatorname{Ker}\left(\Psi^{(R,q)}_{\mu^\#\mu c} \Big\downarrow^{M^\mu}_{S^{\mu^\#\mu}} \right)$ by using the result of the preceding section and basic properties of ZNL-elements and PK_n-homomorphisms. From this in turn we easily obtain $\operatorname{Ker}\left(\Psi^{(R,q)}_{\mu^\#\mu c} \Big\downarrow^{M^\mu}_{S^{\mu^\#\mu}} \right) = S^{\mu^\# A_c\mu}_{(R,q)}$. In Section 3.10, we remove the basis assumption of the preceding two sections by induction along sequences of operators A_c and R_c applied to PK_n-pairs. The induction always starts with a PK_n-pair $\nu^\#\nu$ of a particular kind for which we have $S^{\nu^\#\nu}_{(R,q)} = M^\nu_{(R,q)}$. The main results of this section are that, given a PK_n-pair $\mu^\#\mu$, $\left\{ z(\zeta)_{(R,q)} \,\middle|\, \zeta \in \mathcal{Z}^{\mu^\#\mu} \right\}$ is an R-basis of $S^{\mu^\#\mu}_{(R,q)}$ and, given a pair of operators A_c and R_c applicable to $\mu^\#\mu$,

$$\operatorname{Ker}\left(\Psi^{(R,q)}_{\mu^\#\mu c} \Big\downarrow^{M^\mu}_{S^{\mu^\#\mu}} \right) = S^{\mu^\# A_c\mu}_{(R,q)}.$$

In the final section of this chapter, we construct generic Specht series for PK_n-modules by induction in binary trees from the leaves to the respective root. The vertices of these trees are labelled with PK_n-pairs, their edges with pairs of operators A_c and R_c. Moreover, the labels of the leaves correspond to certain PK_n-modules which have an obvious generic Specht series. Given a PK_n-pair $\mu^\#\mu$, we construct such a tree from it via repeated application of pairs of operators A_c and R_c. The root of this tree is labelled $\mu^\#\mu$, and the labels of its direct successors are $\mu^\# A_c\mu$ and $\mu^\#\mu R_c$ with an appropriate pair of operators A_c and R_c. Corresponding to this, we have the short exact sequence

$$0 \to S_{(R,q)}^{\mu^\# A_c\mu} \to S_{(R,q)}^{\mu^\#\mu} \to S_{(R,q)}^{\mu^\#\mu R_c} \to 0$$

which was established in the preceding sections. Here, the left map is the natural inclusion and the right map is $\Psi_{\mu^\#\mu c}^{(R,q)}\Big\downarrow_{S^{\mu^\#\mu}}^{M^\mu}$. Using this sequence, we can combine the inductively existing generic Specht series for $S_{(R,q)}^{\mu^\# A_c\mu}$ and $S_{(R,q)}^{\mu^\#\mu R_c}$ into a generic Specht series for $S_{(R,q)}^{\mu^\#\mu}$. This method also is used in [DJ1]. Since every permutation module is a PK_n-module and the right regular $\mathcal{H}_{A_{n-1}}^{(R,q)}$-module is a permutation module, this result gives us generic Specht series for PK_n-modules, permutation modules, and $\mathcal{H}_{A_{n-1}}^{(R,q)}$.

In Chapter 4, we construct generic bi-Specht series for Hecke algebras of type B (see above) by translating the constructions from the preceding chapter to Hecke algebras of type B. Section 4.1 provides the combinatorics required for Hecke algebras of type B and the bi-Specht series. Let us fix a degree n for the following. With that, the first part of this section describes the Weyl group of type B_n and introduces so-called left inclusions and right inclusions of Weyl groups of type A into other Weyl groups of type A and into the Weyl group of type B_n. The second part of this section recalls bi-compositions and bi-partitions of n and then introduces bi-PK_n-pairs and operators ${}^{(c)}A$, $A^{(c)}$, ${}^{(c)}R$, and $R^{(c)}$ for them. Bi-compositions, bi-partitions, and bi-PK_n-pairs all depend on an additional parameter $a \in \{0, \dots, n\}$. A bi-composition is a pair (λ, μ) where λ is a composition of a and μ is a composition of $n - a$. A bi-partition is a bi-composition where both parts are partitions. A bi-PK_n-pair $(\lambda^\#\lambda, \mu^\#\mu)$ consists of a PK_a-pair $\lambda^\#\lambda$ and a PK_{n-a}-pair $\mu^\#\mu$. The operators ${}^{(c)}A$, $A^{(c)}$, ${}^{(c)}R$, and $R^{(c)}$, indexed by integers $c > 1$, act on a bi-PK_n-pair $(\lambda^\#\lambda, \mu^\#\mu)$ via application of the operator A_c respectively R_c to $\lambda^\#\lambda$ respectively $\mu^\#\mu$, if possible, to get another bi-PK_n-pair.

Section 4.2 collects some general facts about Hecke algebras of type B. First, the construction of the Hecke algebra of type B_n over an integral domain R with a unit $q \in R$ and an arbitrary element $Q \in R$ via generators and relations is reviewed. This algebra is denoted by $\mathcal{H}_{B_n}^{(R,q,Q)}$. Then, the left inclusions and right inclusions for Weyl groups of type A and B from the preceding section are adapted to Hecke algebras of type A and B.

Section 4.3 introduces bi-permutation modules for $\mathcal{H}_{B_n}^{(R,q,Q)}$ and in the course of this describes a general method for the construction of "nice" right ideals in Hecke algebras of type B from right ideals in Hecke algebras of type A. This method is taken from [DJ3]. It is a translation to Hecke algebras of the corresponding method from the well known derivation of the representation theory of Weyl groups of type B from that of Weyl groups of type A (see, for example, [KER]). First, this section reviews the definition of certain elements $v_{a,n-a}^{(R,q,Q)}$ of $\mathcal{H}_{B_n}^{(R,q,Q)}$ with $a \in \{0, \ldots, n\}$. These elements are Hecke algebra analogues of certain sums over the base group $C_2 \times \cdots \times C_2$ (n times) of the Weyl group of type B_n when considered as a wreath product $C_2 \wr \mathfrak{S}_n$ where C_2 is the cyclic group of order 2. Then, we describe a filtration of $\mathcal{H}_{B_n}^{(R,q,Q)}$ with right ideals such that all the subquotients are of the form $v_{a,n-a}^{(R,q,Q)}\mathcal{H}_{B_n}^{(R,q,Q)}$. Next, suppose we have a right ideal M in $\mathcal{H}_{A_{a-1}}^{(R,q)}$ and a right ideal N in $\mathcal{H}_{A_{n-a-1}}^{(R,q)}$ with an integer $a \in \{1, \ldots, n-1\}$ and they both have R-bases. Denote the right inclusion of M into $\mathcal{H}_{B_n}^{(R,q,Q)}$ (see above) by $M^{\overset{a}{\rightarrow}}$ and the left inclusion of N into $\mathcal{H}_{B_n}^{(R,q,Q)}$ (see above) by $N^{\overset{n-a}{\leftarrow}}$. With that, an R-basis for the right ideal $v_{a,n-a}^{(R,q,Q)}\left(M^{\overset{a}{\rightarrow}}\right)\left(N^{\overset{n-a}{\leftarrow}}\right)\mathcal{H}_{B_n}^{(R,q,Q)}$ can be obtained from the R-bases of M and N. Finally, we define bi-permutation modules for $\mathcal{H}_{B_n}^{(R,q,Q)}$ by applying this to permutation modules for Hecke algebras of type A. Bi-permutation modules are indexed by bi-compositions, and the bi-permutation module corresponding to a bi-composition (λ, μ) is denoted by $M_{(R,q,Q)}^{(\lambda,\mu)}$.

In Section 4.4, we define bi-Specht modules for $\mathcal{H}_{B_n}^{(R,q,Q)}$ and exhibit R-bases for them. This is done along the lines from the preceding section using Specht modules for Hecke algebras of type A. Bi-Specht modules are indexed by bi-partitions, and the bi-Specht module corresponding to a bi-partition (λ, μ) is denoted by $S_{(R,q,Q)}^{(\lambda,\mu)}$.

Section 4.5 introduces bi-PK$_n$-modules for $\mathcal{H}_{B_n}^{(R,q,Q)}$, describes R-bases thereof, and discusses elementary relations between them and bi-permutation modules and bi-Specht modules. The construction of the bi-PK$_n$-modules employs PK-modules for Hecke algebras of type A and is based again on the method described in Section 4.3. Bi-PK$_n$-modules are indexed by bi-PK$_n$-pairs, and the bi-PK$_n$-module corresponding to a bi-PK$_n$-pair $(\lambda^{\#}\lambda, \mu^{\#}\mu)$ is denoted by $S_{(R,q,Q)}^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}$.

In Section 4.6, we define bi-PK$_n$-homomorphisms and describe their effect on bi-PK$_n$-modules. Such a homomorphism is indexed by a bi-PK$_n$-pair $(\lambda^{\#}\lambda, \mu^{\#}\mu)$ and an integer c which allows the application of the corresponding pair of operators ${}^{(c)}A$ and ${}^{(c)}R$ respectively $A^{(c)}$ and $R^{(c)}$ to $(\lambda^{\#}\lambda, \mu^{\#}\mu)$ (see above). By definition, bi-PK$_n$-homomorphisms map bi-permutation modules into bi-permutation modules. They are denoted by

$$^{(c)}\Psi_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R, q, Q) : M_{(R,q,Q)}^{(\lambda,\mu)} \to M_{(R,q,Q)}^{(\lambda R_c,\mu)}$$

and

$$\Psi^{(c)}_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q) : M^{(\lambda,\mu)}_{(R,q,Q)} \to M^{(\lambda,\mu R_c)}_{(R,q,Q)}.$$

These homomorphisms are derived from PK-homomorphisms for Hecke algebras of type A in a way compatible with the construction of bi-permutation modules from permutation modules for Hecke algebras of type A. This enables us to determine the images and kernels of bi-PK$_n$-homomorphisms when applied to bi-PK$_n$-modules. We get

$$\left(S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}_{(R,q,Q)}\right){}^{(c)}\Psi_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q) = S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)^{(c)}R}_{(R,q,Q)}$$

and

$$\operatorname{Ker}\left({}^{(c)}\Psi_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q)\big\downarrow^{M^{(\lambda,\mu)}}_{S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}}\right) = S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)^{(c)}A}_{(R,q,Q)}$$

and analogous statements for homomorphisms $\Psi^{(c)}_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q)$.

In Section 4.7, we use the definitions and results from the preceding sections to construct generic bi-Specht series for bi-PK$_n$-modules. These comprise generic bi-Specht series for bi-permutation modules as special cases which in turn lead to a generic bi-Specht series for the right regular $\mathcal{H}^{(R,q,Q)}_{B_n}$-module. The construction of generic bi-Specht series for bi-PK$_n$-modules is an adaption of the construction of generic Specht series for PK-modules from Section 3.11 to the situation at hand. Given a bi-PK$_n$-module $S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}_{(R,q,Q)}$, we build a binary tree from the bi-PK$_n$-pair $(\lambda^{\#}\lambda,\mu^{\#}\mu)$ via repeated application of pairs of operators ${}^{(c)}A$ and ${}^{(c)}R$ and also $A^{(c)}$ and $R^{(c)}$. We use induction on the vertices of this tree from the leaves to the root which is labelled $(\lambda^{\#}\lambda,\mu^{\#}\mu)$ and finally employ the exact sequence

$$0 \to S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)^{(c)}A}_{(R,q,Q)} \to S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}_{(R,q,Q)} \to S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)^{(c)}R}_{(R,q,Q)} \to 0$$

respectively

$$0 \to S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)A^{(c)}}_{(R,q,Q)} \to S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}_{(R,q,Q)} \to S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)R^{(c)}}_{(R,q,Q)} \to 0$$

(see above) where the left map is the natural inclusion and the right map is the restricted bi-PK$_n$-homomorphism ${}^{(c)}\Psi_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q)\big\downarrow^{M^{(\lambda,\mu)}}_{S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}}$ respectively $\Psi^{(c)}_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q)\big\downarrow^{M^{(\lambda,\mu)}}_{S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}}$ to combine the inductively existing generic bi-Specht series for $S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)^{(c)}A}_{(R,q,Q)}$ and $S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)^{(c)}R}_{(R,q,Q)}$ respectively $S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)A^{(c)}}_{(R,q,Q)}$ and $S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)R^{(c)}}_{(R,q,Q)}$ into a generic bi-Specht series for $S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}_{(R,q,Q)}$. This completes the derivation of generic bi-Specht series for Hecke algebras of type B. Using the constructions and results from [PAL], this chapter can easily be translated to Hecke algebras of type D, thus providing generic bi-Specht series for them as well.

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Deutschsprachige Übersicht

Mit Weyl-Gruppen assoziierte Hecke-Algebren tauchen in verschiedenen Zweigen der Mathematik und der Physik auf. Hecke-Algebren vom Typ A dienen in der Knotentheorie zur Konstruktion von topologischen Invarianten (siehe [JON]), außerdem werden sie auch in der statistischen Mechanik verwendet (siehe [JIM1], [JIM2]). In der Theorie der Quantengruppen spielen sie in der quantisierten Version der klassischen Schur-Weyl-Reziprozität zwischen generellen linearen Gruppen und symmetrischen Gruppen die Rolle der Gruppenalgebren symmetrischer Gruppen (siehe wiederum [JIM2]). Weiter sind die mit Weyl-Gruppen assoziierten Hecke-Algebren ein wertvolles Werkzeug in der Darstellungstheorie von algebraischen Gruppen und endlichen Gruppen vom Lie-Typ (siehe [IWA] und [KL]). Insbesondere sind sie sehr nützlich bei der Bestimmung der Zerlegungszahlen von endlichen Gruppen vom Lie-Typ, da ihre Zerlegungsmatrizen in die der entsprechenden Gruppen eingebettet sind (siehe [DIP] und [DJ3]).

Der erste Teil der vorliegenden Dissertation ist ein Beitrag zu dieser letzteren Anwendung von Hecke-Algebren. Hier werden die Zerlegungsmatrizen gewisser Blöcke von Hecke-Algebren vom Typ A untersucht. Um dies genauer zu beschreiben, wählen wir die folgenden Koeffizientenringe und Parameter. Sei Q ein Körper und sei ψ eine diskrete Bewertung auf Q, so daß Q vollständig bezüglich ψ ist. Damit bezeichne S den Bewertungsring von ψ in Q, I das Bewertungsideal von ψ in S und F den Restklassenkörper S/I. Dann hat man die Inklusion $S \hookrightarrow Q$ und die Reduktion modulo I $\bar{\cdot} : S \to F$. Weiter wird eine Einheit $a \in S$ fest gewählt. Dann ist a auch eine Einheit in Q und $\bar{a}$ ist eine Einheit in F. Wir nehmen an, daß es natürliche Zahlen m gibt, für die in F $\sum_{i=0}^{m-1} \bar{a}^i = 0$ gilt — das Gegenteil ist hier nicht von Interesse — und notieren das Minimum dieser Zahlen als $e_F(\bar{a})$. Mit alledem setzen wir $n = 3e_F(\bar{a})$ und bilden die Hecke-Algebren vom Typ A_{n-1} über den Koeffizientenringen Q, S und F mit den jeweiligen Parametern a, a und $\bar{a}$. Diese Algebren werden als $\mathcal{H}_{A_{n-1}}^{(Q,a)}$, $\mathcal{H}_{A_{n-1}}^{(S,a)}$ und $\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}$ notiert (siehe Abschnitt 1.2). Wir nehmen auch an, daß $\mathcal{H}_{A_{n-1}}^{(Q,a)}$ halbeinfach ist.

Ähnlich wie im Spezialfall der symmetrischen Gruppen werden die Blöcke dieser

Algebren durch $e_F(\bar{a})$-Kerne von Partitionen von n indiziert und durch die $e_F(\bar{a})$-Gewichte dieser Kerne in Familien eingeteilt. Der hier betrachtete Block wird durch die Partition (0) von 0 indiziert und hat $e_F(\bar{a})$-Gewicht 3. Die obige Wahl von n stellt sicher, daß die Algebren $\mathcal{H}^{(Q,a)}_{A_{n-1}}$, $\mathcal{H}^{(S,a)}_{A_{n-1}}$ und $\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}$ tatsächlich solch einen Block haben. Er kann als ihr Hauptblock angesehen werden. Wir bezeichnen diesen Block mit $B^{(0)}(n)$ und seine Zerlegungsmatrix mit $\Delta^{\mathcal{H}}_n((0))$ (siehe Abschnitt 1.8).

Die für die Blockzerlegungsmatrix $\Delta^{\mathcal{H}}_n((0))$ erhaltenen Resultate sind wie folgt. Zuerst wird gezeigt, daß alle ihre Einträge 0 oder 1 sind. Dann wird mit Hilfe dieser Tatsache und der quantisierten Version des Satzes von Schaper aus [JM] ein rein kombinatorischer Algorithmus zur Berechnung von $\Delta^{\mathcal{H}}_n((0))$ beschrieben. Schließlich wird gezeigt, daß die Matrix $\Delta^{\mathcal{H}}_n((0))$ nur von der Charakteristik von F und der Zahl $e_F(\bar{a})$ abhängt. Wenn die Charakteristik von F weder 2 noch 3 ist, kann die Abhängigkeit von dieser sogar noch eliminiert werden, so daß $\Delta^{\mathcal{H}}_n((0))$ vollständig durch $e_F(\bar{a})$ bestimmt ist. Dies beweist eine Vermutung von James (siehe [JAM2, Section 4]) für den Spezialfall der Teilmatrix $\Delta^{\mathcal{H}}_n((0))$ der Zerlegungsmatrix der Algebren $\mathcal{H}^{(Q,a)}_{A_{n-1}}$, $\mathcal{H}^{(S,a)}_{A_{n-1}}$ und $\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}$.

Der zweite Teil der vorliegenden Dissertation befaßt sich mit generischen Eigenschaften von Hecke-Algebren der Typen A und B. Generisch bedeutet, daß diese Eigenschaften unabhängig von der Wahl des Koeffizientenrings und der Parameter für die Hecke-Algebra sind. Dabei kann der Koeffizientenring ein beliebiger Integritätsbereich sein. Die betrachteten Eigenschaften sind generische Specht-Serien für Hecke-Algebren vom Typ A und generische Bi-Specht-Serien für Hecke-Algebren vom Typ B.

Eine generische Specht-Serie für eine Hecke-Algebra vom Typ A ist eine Reihe von Rechtsidealen in dieser Algebra, so daß alle Quotienten aufeinanderfolgender Ideale isomorph zu Specht-Moduln (siehe Abschnitt 1.3) sind und sich außerdem alle in der Konstruktion dieser Reihe auftretenden Algebra-Elemente, Ideale und Homomorphismen bei einem Wechsel des Koeffizientenrings stabil verhalten. Dabei werden die Specht-Moduln für Hecke-Algebren vom Typ A aus [DJ1] verwendet.

Analog ist eine generische Bi-Specht-Serie für eine Hecke-Algebra vom Typ B eine Reihe von Rechtsidealen in dieser Algebra, so daß alle Quotienten aufeinanderfolgender Ideale isomorph zu Bi-Specht-Moduln (siehe Abschnitt 4.4) sind und sich außerdem alle in der Konstruktion dieser Reihe auftretenden Algebra-Elemente, Ideale und Homomorphismen bei einem Wechsel des Koeffizientenrings stabil verhalten. Die dabei verwendeten Bi-Specht-Moduln sind eine Verallgemeinerung der Bi-Specht-Moduln aus [DJ3].

In der vorliegenden Dissertation werden Ideen aus [JAM1, Section 16] und

[DJ1, Section 7] auf Hecke-Algebren vom Typ A verallgemeinert, um generische Specht-Serien für diese zu erhalten. Insbesondere werden neue generische Basen der dort eingeführten “Zwischenmoduln” $S^{\mu^{\#}\mu}$ konstruiert (siehe Abschnitt 3.10). Aus diesen generischen Specht-Serien für Hecke-Algebren vom Typ A werden dann durch Übertragung der bei ihrer Konstruktion verwendeten Algebra-Elemente, Ideale und Homomorphismen auf Hecke-Algebren vom Typ B mit Methoden aus [DJ3] generische Bi-Specht-Serien für Hecke-Algebren vom Typ B gewonnen.

Der Aufbau der vorliegenden Dissertation ist wie folgt. In Kapitel 1 werden Hintergrundmaterial und bekannte Tatsachen über Hecke-Algebren vom Typ A, die später benötigt werden, zusammengestellt. Das Kapitel beginnt mit einer Übersicht über die für die Darstellungstheorie der Weyl-Gruppen vom Typ A — sprich der symmetrischen Gruppen — benötigte Kombinatorik. Dabei wird unter anderem an Kompositionen und Partitionen, Haken und Randhaken, β-Sequenzen und Rechenschieber, Kerne, Tableaux, Young-Untergruppen und kürzeste Repräsentanten und auch an Zeilennummernlisten erinnert.

Der nächste Abschnitt enthält einige elementare Dinge über Hecke-Algebren vom Typ A. Hier werden etwa Erzeuger und Relationen für diese Algebren angegeben und grundlegende Begriffe in Bezug auf die verwendeten Koeffizientenringe eingeführt. Abschnitt 1.3 beschreibt die Herleitung der irreduziblen Darstellungen von Hecke-Algebren vom Typ A über Körpern wie sie in [DJ1] durchgeführt wird. Dabei wird auch an die generischen Permutationsmoduln und Specht-Moduln von dort erinnert. Der nachfolgende Abschnitt überträgt die Ausführungen zur modularen Reduktion und zu Zerlegungszahlen für Gruppenalgebren aus [CR1, §16] auf die hier vorliegende Situation. Hier werden die Notationen für Grothendieck-Gruppen, modulare Systeme, Zerlegungsabbildungen und Zerlegungszahlen für Hecke-Algebren vom Typ A festgelegt. Als nächstes wird in Abschnitt 1.5 das Verhalten von Specht-Moduln bei modularer Reduktion zusammen mit einigen Konsequenzen davon beschrieben. Dann werden in Abschnitt 1.6 neue modulare Systeme mit besonderen Eigenschaften aus einem gegebenen modularen System abgeleitet und zur Untersuchung der Abhängigkeit der Zerlegungszahlen von Hecke-Algebren vom Typ A vom verwendeten modularen System benutzt. In Abschnitt 1.7 wird die Behandlung projektiv unzerlegbarer Moduln und des Cartan-Brauer-Dreiecks für Gruppenalgebren aus [CR1, §18] an Hecke-Algebren vom Typ A angepaßt.

Im nachfolgenden Abschnitt werden einige elementare Tatsachen und Begriffe aus der Blocktheorie von Hecke-Algebren vom Typ A zusammengestellt. Hier wird an solche Dinge wie Block-Idempotente und Block-Ideale, die Parametrisierung der Blöcke von Hecke-Algebren vom Typ A durch Kerne von Partitionen und die

Blockzerlegung von Moduln, Grothendieck-Gruppen, projektiven Klassengruppen und Zerlegungsmatrizen erinnert. Der nächste Abschnitt behandelt Induktion von Moduln von einer Hecke-Algebra vom Typ A_{n-2} zur entsprechenden Algebra vom Typ A_{n-1}. Die Induktion von Specht-Moduln und projektiv unzerlegbaren Moduln, betrachtet als Elemente sowohl von Grothendieck-Gruppen als auch von projektiven Klassengruppen, wird genauer beschrieben. Dies wird dann benutzt, um eine obere Schranke für die Zerlegungszahlen eines Blocks der Hecke-Algebra vom Typ A_{n-1} herzuleiten, wobei auch noch vorausgesetzt wird, daß die Zerlegungszahlen der Hecke-Algebra vom Typ A_{n-2} bekannt sind. Abschnitt 1.10 beschreibt schließlich den Satz von Schaper für Hecke-Algebren vom Typ A wie er in [JM] bewiesen wird. Zuerst wird die benötigte Notation eingeführt und der Satz von Schaper formuliert. Dann wird beschrieben, wie man mit Hilfe dieses Satzes wertvolle Informationen über Zerlegungszahlen gewinnen kann.

In Kapitel 2 untersuchen wir die Zerlegungsmatrizen von Blöcken von Hecke-Algebren vom Typ A mit Kern (0) und Gewicht 3. Hier und im folgenden werden die obigen Bezeichnungen und Notationen verwendet. In Abschnitt 2.1 wird gezeigt, daß alle Einträge von $\Delta_n^{\mathcal{H}}((0))$ 0 oder 1 sind. Dies wird ermöglicht durch die obere Schranke für die Zerlegungszahlen eines beliebigen Blocks der Algebren $\mathcal{H}_{A_{n-1}}^{(Q,a)}$, $\mathcal{H}_{A_{n-1}}^{(S,a)}$ und $\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}$ aus Abschnitt 1.9. Im Fall des Blocks $B^{(0)}(n)$ hängt diese obere Schranke ab von einer oberen Schranke U_{n-1} für die Einträge der Zerlegungsmatrix $\Delta_{n-1}^{\mathcal{H}}$ der Algebren $\mathcal{H}_{A_{n-2}}^{(Q,a)}$, $\mathcal{H}_{A_{n-2}}^{(S,a)}$ und $\mathcal{H}_{A_{n-2}}^{(F,\bar{a})}$ (siehe Abschnitt 1.4) und einer gewissen ganzen Zahl $J_{(0)}$, die durch kombinatorische Manipulationen der zu $B^{(0)}(n)$ gehörigen Partitionen bestimmt ist. Da das Gewicht jedes Blocks von $\mathcal{H}_{A_{n-2}}^{(Q,a)}$, $\mathcal{H}_{A_{n-2}}^{(S,a)}$ und $\mathcal{H}_{A_{n-2}}^{(F,\bar{a})}$ kleiner als 3 ist, kann die obere Schranke U_{n-1} aus Resultaten über solche Blöcke in [RIC] und [JAM2] gewonnen werden.

Zur Auswertung von $J_{(0)}$ gehen wir wie folgt vor. Zuerst werden die zu $B^{(0)}(n)$ gehörigen Partitionen — das heißt die mit $e_F(\bar{a})$-Kern (0) — entsprechend der Formen der ihnen zugeordneten Rechenschieber in Familien eingeteilt wie in [MR2] beschrieben. Dann wird die Definition von $J_{(0)}$ aus Satz 1.9.18 in Rechenschiebernotation übersetzt. Damit wird $J_{(0)}$ schließlich durch explizite Betrachtung jeder einzelnen Familie der in dem Block $B^{(0)}(n)$ liegenden Partitionen bestimmt. Die für U_{n-1} und $J_{(0)}$ erhaltenen Werte liefern nun die obere Schranke 1 für alle Einträge von $\Delta_n^{\mathcal{H}}((0))$.

Abschnitt 2.2 zeigt zuerst, wie die Matrix $\Delta_n^{\mathcal{H}}((0))$ explizit berechnet werden kann, und untersucht dann ihre Abhängigkeit von den verwendeten Koeffizientenringen und Parametern. Die explizite Berechnung von $\Delta_n^{\mathcal{H}}((0))$ beruht auf der quantisierten Version des Satzes von Schaper aus [JM]. Mit Hilfe dieses Satzes kann man für jeden Eintrag in einer Zeile von $\Delta_n^{\mathcal{H}}((0))$ entscheiden, ob er verschwindet oder nicht, vorausgesetzt die vorhergehenden Zeilen — bezüglich einer geeigneten Anordnung — sind bekannt (siehe Bemerkung 1.10.9.(ii)). Diese Tatsache und die in Abschnitt 2.1 hergeleitete obere Schranke 1 für alle Zerlegungszahlen des Blocks $B^{(0)}(n)$ ermöglichen die explizite Berechnung seiner Zerlegungsmatrix $\Delta_n^{\mathcal{H}}((0))$ durch einen einfachen Induktionsprozeß.

Der zweite Themenbereich in Abschnitt 2.2 ist die Abhängigkeit der Blockzerlegungsmatrix $\Delta_n^{\mathcal{H}}((0))$ von den Koeffizientenringen und Parametern der Algebren $\mathcal{H}_{A_{n-1}}^{(Q,a)}$, $\mathcal{H}_{A_{n-1}}^{(S,a)}$ und $\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}$. Es stellt sich heraus, daß diese Matrix durch den Wert $e_F(\bar{a})$ und die Charakteristik von F bestimmt ist. Zu diesem Ergebnis gelangt man folgendermaßen. Zunächst werden die Beziehungen zwischen $e_F(\bar{a})$ und der Charakteristik von F einerseits und den Parametern a und $\bar{a}$ der Hecke-Algebren andererseits beschrieben (siehe Lemma 2.2.3 bis Lemma 2.2.5). Es zeigt sich, daß dabei drei unterschiedliche Fälle betrachtet werden müssen. Mit Hilfe der Charakterisierungen dieser Fälle werden die Werte der Bewertung ψ auf quantisierten ganzen Zahlen (siehe Definition 1.2.2.(i)) ermittelt. Solche Ausdrücke tauchen im Satz von Schaper auf. Wir erhalten, daß diese Werte vollständig durch $e_F(\bar{a})$ und die Charakteristik von F bestimmt sind. Diese Tatsache und die im vorhergehenden Absatz beschriebene Methode zur Berechnung der Matrix $\Delta_n^{\mathcal{H}}((0))$ mit Hilfe des Satzes von Schaper zeigen schließlich, daß die Zerlegungszahlen von $B^{(0)}(n)$ wie behauptet nur von der Charakteristik von F und $e_F(\bar{a})$ abhängen.

Diese Abhängigkeit kann noch weiter reduziert werden, wenn man annimmt, daß die Charakteristik von F weder 2 noch 3 ist. Diese Voraussetzung ermöglicht eine genauere Aussage über die Werte von ψ auf den für die oben beschriebene Berechnung von $\Delta_n^{\mathcal{H}}((0))$ relevanten quantisierten ganzen Zahlen. Man erhält eine Faktorisierung dieser Werte, wobei der eine Faktor vollständig durch $e_F(\bar{a})$ und die Charakteristik von F bestimmt ist, während der andere nur von $e_F(\bar{a})$ und der der quantisierten ganzen Zahl zugrunde liegenden gewöhnlichen ganzen Zahl abhängt. Die Kombination dieser Faktorisierung mit dem Satz von Schaper und der oben beschriebenen Berechnung der Zerlegungszahlen des Blocks $B^{(0)}(n)$ ergibt, daß $\Delta_n^{\mathcal{H}}((0))$ bereits allein durch $e_F(\bar{a})$ bestimmt ist, vorausgesetzt die Charakteristik von F ist weder 2 noch 3.

Kapitel 3 befaßt sich mit der Konstruktion von generischen Specht-Serien für Hecke-Algebren vom Typ A (siehe oben). Dazu seien im folgenden ein Grad n, ein Integritätsbereich R, eine Einheit $q \in R$ und damit die Hecke-Algebra $\mathcal{H}_{A_{n-1}}^{(R,q)}$ fest gewählt. Die Abschnitte 3.1 bis 3.3 stellen die zur Konstruktion einer generischen Specht-Serie für diese Algebra benötigte Kombinatorik zur Verfügung. Abschnitt 3.1 erinnert an Ordnungsrelationen für zeilenstandard Tableaux und die entsprechenden kürzesten Repräsentanten. Abschnitt 3.2 diskutiert PK_n-Paare $\mu^{\#}\mu$ und die für sie erklärten Operatoren A_c und R_c wie in [JAM1] beschrieben und außerdem einige damit verbundene Tableaux und Permutationen. Kurz gesagt besteht ein PK_n-Paar aus einer Partition $\mu^{\#}$ und einer Komposition μ von n (siehe Abschnitt 1.1), die gewissen Bedingungen genügen. Wenn man weiter einen Index $c > 1$ hat, so daß die $(c-1)$. und c. Einträge von $\mu^{\#}$ und μ zusätzliche Anforderungen erfüllen, dann kann man die zugehörigen Operatoren A_c und R_c anwenden. Der Operator A_c liefert eine Partition $\mu^{\#}A_c$ und damit ein PK_n-Paar $\mu^{\#}A_c\mu$. Der Operator R_c liefert eine Komposition μR_c und weiter ein PK_n-Paar $\mu^{\#}\mu R_c$. Abschnitt 3.3 behandelt die verschiedenen Aspekte der Verwendung von Zeilennummernlisten bei der Konstruktion der Specht-Serien. Dies sind etwa die Verbindung zu PK_n-Paaren mittels guter und schlechter Einträge, die daraus resultierende Einteilung von Zeilennummernlisten in Mengen $\mathcal{Z}^{\mu^{\#}\mu}$ und Abbildungen zwischen solchen Mengen aus [JAM1], aber auch Ordnungsrelationen zwischen aus den Zeilennummernlisten abgeleiteten Permutationen.

Die Abschnitte 3.4 und 3.5 erinnern an bekannte Moduln und Homomorphismen, die zur Konstruktion der Specht-Serien benutzt werden. Abschnitt 3.4 wiederholt die Definition der durch PK_n-Paare indizierten PK_n-Moduln $S^{\mu^{\#}\mu}_{(R,q)}$ aus [DJ1] und stellt einige elementare Tatsachen über sie zusammen. Abschnitt 3.5 erklärt die Konstruktion von PK_n-Homomorphismen

$$\Psi^{(R,q)}_{\mu^{\#}\mu c} : M^{\mu}_{(R,q)} \to M^{\mu R_c}_{(R,q)}$$

und zählt einige grundlegende Eigenschaften von ihnen auf. Ein solcher Homomorphismus wird indiziert durch ein PK_n-Paar $\mu^{\#}\mu$ und eine ganze Zahl c, die die Anwendung des entsprechenden Paares von Operatoren A_c und R_c auf $\mu^{\#}\mu$ ermöglicht (siehe oben). Er bildet den Permutationsmodul $M^{\mu}_{(R,q)}$ (siehe Abschnitt 1.3) in den Permutationsmodul $M^{\mu R_c}_{(R,q)}$ und den PK_n-Modul $S^{\mu^{\#}\mu}_{(R,q)}$ in den PK_n-Modul $S^{\mu^{\#}\mu R_c}_{(R,q)}$ ab. Dies alles stammt aus [DJ1].

Die nächsten beiden Abschnitte führen ZNL-Elemente ein und beschreiben die Wirkung von PK_n-Homomorphismen auf ihnen. In Abschnitt 3.6 definieren wir für eine Komposition λ von n und jede Zeilennummernliste ζ in der Menge $\mathcal{Z}^{\lambda}$ (siehe Abschnitt 1.1) das entsprechende ZNL-Element $z(\zeta)_{(R,q)}$ in $M^{\lambda}_{(R,q)}$. Wir leiten auch einige elementare Tatsachen über diese Elemente her. So bestimmen wir etwa ihre Darstellungen bezüglich der Zeilenstandard-Basis $\mathbf{B}^{M^{\lambda}}_{\mathrm{row\,std}}(R,q)$ von $M^{\lambda}_{(R,q)}$ (siehe Abschnitt 1.3). Daraus wiederum erhalten wir ihre lineare Unabhängigkeit. In Abschnitt 3.7 untersuchen wir für ein gegebenes PK_n-Paar $\mu^{\#}\mu$ die Bilder der durch Zeilennummernlisten $\zeta \in \mathcal{Z}^{\mu^{\#}\mu}$ indizierten ZNL-Elemente $z(\zeta)_{(R,q)}$ unter einem PK_n-Homomorphismus $\Psi^{(R,q)}_{\mu^\#\mu c} : M^{\mu}_{(R,q)} \to M^{\mu R_c}_{(R,q)}$ und leiten deren Darstellungen bezüglich der Basis $\mathbf{B}^{M^{\mu R_c}}_{\text{row std}}(R,q)$ von $M^{\mu R_c}_{(R,q)}$ her. Es zeigt sich, daß die $z(\zeta)_{(R,q)}$ mit $\zeta \in \mathcal{Z}^{\mu^\# A_c \mu} \subseteq \mathcal{Z}^{\mu^\#\mu}$ in $\operatorname{Ker}\Psi^{(R,q)}_{\mu^\#\mu c}$ liegen und daß die $z(\zeta)_{(R,q)}\Psi^{(R,q)}_{\mu^\#\mu c}$ mit $\zeta \in \mathcal{Z}^{\mu^\#\mu} \setminus \mathcal{Z}^{\mu^\# A_c \mu}$ linear unabhängig sind.

In den folgenden drei Abschnitten werden aus ZNL-Elementen bestehende Basen von PK_n-Moduln hergeleitet. In Abschnitt 3.8 zeigen wir, daß für ein PK_n-Paar $\mu^\#\mu$ mit einem Paar von Operatoren A_c und R_c dafür die Menge $\left\{ z(\eta)_{(R,q)} \,\middle|\, \eta \in \mathcal{Z}^{\mu^\#\mu R_c} \right\}$ eine R-Basis des PK_n-Moduls $S^{\mu^\#\mu R_c}_{(R,q)}$ bildet, vorausgesetzt $\left\{ z(\zeta)_{(R,q)} \,\middle|\, \zeta \in \mathcal{Z}^{\mu^\#\mu} \right\}$ ist eine R-Basis von $S^{\mu^\#\mu}_{(R,q)}$. Diese Aussage erhalten wir durch einen Vergleich der Darstellungen der Elemente $z(\zeta)_{(R,q)}\Psi^{(R,q)}_{\mu^\#\mu c}$ für $\zeta \in \mathcal{Z}^{\mu^\#\mu} \setminus \mathcal{Z}^{\mu^\# A_c \mu}$ bezüglich $\mathbf{B}^{M^{\mu R_c}}_{\text{row std}}(R,q)$ mit denen der Elemente $z(\eta)_{(R,q)}$ für $\eta \in \mathcal{Z}^{\mu^\#\mu R_c}$. In Abschnitt 3.9 betrachten wir wiederum ein PK_n-Paar $\mu^\#\mu$ mit einem Paar von Operatoren A_c und R_c dafür und nehmen auch wieder an, daß $\left\{ z(\zeta)_{(R,q)} \,\middle|\, \zeta \in \mathcal{Z}^{\mu^\#\mu} \right\}$ eine R-Basis von $S^{\mu^\#\mu}_{(R,q)}$ ist. Mit diesen Voraussetzungen zeigen wir, daß $\left\{ z(\zeta)_{(R,q)} \,\middle|\, \zeta \in \mathcal{Z}^{\mu^\# A_c \mu} \right\}$ eine R-Basis von $\operatorname{Ker}\left(\Psi^{(R,q)}_{\mu^\#\mu c}\Big\downarrow^{M^\mu}_{S^{\mu^\#\mu}} \right)$ bildet. Dazu verwenden wir das Ergebnis des vorhergehenden Abschnitts und grundlegende Eigenschaften von ZNL-Elementen und PK_n-Homomorphismen. Mit Hilfe dieser Basis von $\operatorname{Ker}\left(\Psi^{(R,q)}_{\mu^\#\mu c}\Big\downarrow^{M^\mu}_{S^{\mu^\#\mu}} \right)$ ergibt sich leicht $\operatorname{Ker}\left(\Psi^{(R,q)}_{\mu^\#\mu c}\Big\downarrow^{M^\mu}_{S^{\mu^\#\mu}} \right) = S^{\mu^\# A_c \mu}_{(R,q)}$. In Abschnitt 3.10 entfernen wir die Annahme über die R-Basis von $S^{\mu^\#\mu}_{(R,q)}$ aus den beiden vorhergehenden Abschnitten mittels Induktion entlang Sequenzen von auf PK_n-Paare angewandten Operatoren A_c und R_c. Die Induktion beginnt immer mit einem PK_n-Paar $\nu^\#\nu$ spezieller Bauart, für das $S^{\nu^\#\nu}_{(R,q)} = M^{\nu}_{(R,q)}$ gilt. Dieser Abschnitt hat die folgenden beiden Hautpergebnisse. Zum einen ist für ein PK_n-Paar $\mu^\#\mu$ die Menge $\left\{ z(\zeta)_{(R,q)} \,\middle|\, \zeta \in \mathcal{Z}^{\mu^\#\mu} \right\}$ eine R-Basis von $S^{\mu^\#\mu}_{(R,q)}$. Und zum anderen gilt für ein PK_n-Paar $\mu^\#\mu$ mit einem Paar darauf anwendbarer Operatoren A_c und R_c

$$\operatorname{Ker}\left(\Psi^{(R,q)}_{\mu^\#\mu c}\Big\downarrow^{M^\mu}_{S^{\mu^\#\mu}} \right) = S^{\mu^\# A_c \mu}_{(R,q)}.$$

Im abschließenden Abschnitt dieses Kapitels konstruieren wir generische Specht-Serien für PK_n-Moduln mittels Induktion in Binärbäumen von den Blättern zur jeweiligen Wurzel. In diesen Bäumen sind die Knoten mit PK_n-Paaren beschriftet und die Kanten mit Paaren von Operatoren A_c und R_c. Außerdem entsprechen die Beschriftungen der Blätter gewissen PK_n-Moduln, die eine offensichtliche generische Specht-Serie besitzen. Aus einem gegebenen PK_n-Paar $\mu^\#\mu$ konstruieren wir solch einen Baum durch wiederholte Anwendung von Paaren von Operatoren A_c und R_c. Die Wurzel dieses Baumes ist mit $\mu^\#\mu$ beschriftet. Die Beschriftungen ihrer direkten

Nachfolger lauten $\mu^{\#}A_c\mu$ und $\mu^{\#}\mu R_c$ mit einem geeigneten Paar von Operatoren A_c und R_c. Entsprechend dazu haben wir die kurze exakte Sequenz

$$0 \to S_{(R,q)}^{\mu^{\#}A_c\mu} \to S_{(R,q)}^{\mu^{\#}\mu} \to S_{(R,q)}^{\mu^{\#}\mu R_c} \to 0,$$

die in den vorhergehenden Abschnitten hergeleitet wurde. Dabei ist die linke Abbildung die natürliche Inklusion und die rechte Abbildung die Einschränkung $\Psi_{\mu^{\#}\mu c}^{(R,q)}\Big\downarrow_{S^{\mu^{\#}\mu}}^{M^{\mu}}$. Mittels dieser Sequenz können wir aus den induktiv existierenden generischen Specht-Serien für $S_{(R,q)}^{\mu^{\#}A_c\mu}$ und $S_{(R,q)}^{\mu^{\#}\mu R_c}$ eine generische Specht-Serie für $S_{(R,q)}^{\mu^{\#}\mu}$ bilden. Diese Methode wird auch in [DJ1] benutzt. Da jeder Permutationsmodul ein PK_n-Modul ist und außerdem der rechtsreguläre $\mathcal{H}_{A_{n-1}}^{(R,q)}$-Modul ein Permutationsmodul, erhalten wir aus diesem Resultat generische Specht-Serien für PK_n-Moduln, Permutationsmoduln und $\mathcal{H}_{A_{n-1}}^{(R,q)}$.

In Kapitel 4 konstruieren wir generische Bi-Specht-Serien für Hecke-Algebren vom Typ B (siehe oben) durch Übertragung der Konstruktionen aus dem vorhergehenden Kapitel auf Hecke-Algebren vom Typ B. Abschnitt 4.1 stellt die für Hecke-Algebren vom Typ B und die Bi-Specht-Serien benötigte Kombinatorik bereit. Dazu sei im folgenden ein Grad n fest gewählt. Damit beschreibt der erste Teil dieses Abschnitts die Weyl-Gruppe vom Typ B_n und führt sogenannte Links- und Rechtsinklusionen von Weyl-Gruppen vom Typ A in andere Weyl-Gruppen vom Typ A und in die Weyl-Gruppe vom Typ B_n ein. Der zweite Teil dieses Abschnitts erinnert zunächst an Bi-Kompositionen und Bi-Partitionen von n und führt dann Bi-PK_n-Paare und Operatoren ${}^{(c)}A$, $A^{(c)}$, ${}^{(c)}R$ und $R^{(c)}$ für diese ein. Dabei hängen Bi-Kompositionen, Bi-Partitionen und Bi-PK_n-Paare noch von einem weiteren Parameter $a \in \{0, \ldots, n\}$ ab. Eine Bi-Komposition ist ein Paar (λ, μ) mit einer Komposition λ von a und einer Komposition μ von $n - a$. Eine Bi-Partition ist eine aus zwei Partitionen bestehende Bi-Komposition. Ein Bi-PK_n-Paar $(\lambda^{\#}\lambda, \mu^{\#}\mu)$ besteht aus einem PK_a-Paar $\lambda^{\#}\lambda$ und einem PK_{n-a}-Paar $\mu^{\#}\mu$. Die durch ganze Zahlen $c > 1$ indizierten Operatoren ${}^{(c)}A$, $A^{(c)}$, ${}^{(c)}R$ und $R^{(c)}$ wirken auf ein Bi-PK_n-Paar $(\lambda^{\#}\lambda, \mu^{\#}\mu)$ durch Anwendung des Operators A_c beziehungsweise R_c auf $\lambda^{\#}\lambda$ beziehungsweise $\mu^{\#}\mu$ — vorausgesetzt dies ist möglich — um wiederum ein Bi-PK_n-Paar zu erhalten.

Abschnitt 4.2 stellt einige allgemeine Tatsachen über Hecke-Algebren vom Typ B zusammen. Zunächst wird die Konstruktion der Hecke-Algebra vom Typ B_n über einem Integritätsbereich R mit einer Einheit $q \in R$ und einem beliebigen Element $Q \in R$ mittels Erzeugern und Relationen beschrieben. Diese Algebra wird als $\mathcal{H}_{B_n}^{(R,q,Q)}$ notiert. Dann werden die Links- und Rechtsinklusionen für Weyl-Gruppen der Typen A und B aus dem vorhergehenden Abschnitt auf Hecke-Algebren der

Typen A und B übertragen.

Abschnitt 4.3 führt Bi-Permutationsmoduln für $\mathcal{H}_{B_n}^{(R,q,Q)}$ ein und gibt dabei einen Überblick über eine allgemeine Methode zur Konstruktion von "interessanten" Rechtsidealen in Hecke-Algebren vom Typ B aus Rechtsidealen in Hecke-Algebren vom Typ A. Diese Methode stammt aus [DJ3]. Es handelt sich dabei um eine Übertragung der wohlbekannten Ableitung der Darstellungstheorie von Weyl-Gruppen vom Typ B aus der von Weyl-Gruppen vom Typ A (siehe etwa [KER]) auf Hecke-Algebren. Zunächst wiederholt dieser Abschnitt die Definition gewisser Elemente $v_{a,n-a}^{(R,q,Q)}$ von $\mathcal{H}_{B_n}^{(R,q,Q)}$ mit $a \in \{0,\ldots,n\}$. Wenn man die Weyl-Gruppe vom Typ B_n als Kranzprodukt $C_2 \wr \mathfrak{S}_n$ auffaßt, wobei C_2 die zyklische Gruppe der Ordnung 2 bezeichnet, dann entsprechen diese Elemente gewissen Summen über die Basisgruppe $C_2 \times \cdots \times C_2$ (n mal) des Kranzprodukts. Dann beschreiben wir eine Filtrierung von $\mathcal{H}_{B_n}^{(R,q,Q)}$ mit Rechtsidealen, bei der alle Subquotienten von der Form $v_{a,n-a}^{(R,q,Q)}\mathcal{H}_{B_n}^{(R,q,Q)}$ sind. Als nächstes werden Ideale mit R-Basen betrachtet. Dazu sei $a \in \{1,\ldots,n-1\}$ und damit M ein Rechtsideal in $\mathcal{H}_{A_{a-1}}^{(R,q)}$ und N ein Rechtsideal in $\mathcal{H}_{A_{n-a-1}}^{(R,q)}$, die beide R-Basen besitzen. Wenn man nun die Rechtsinklusion von M in $\mathcal{H}_{B_n}^{(R,q,Q)}$ (siehe oben) als $M^{\overset{a}{\rightarrow}}$ notiert und die Linksinklusion von N in $\mathcal{H}_{B_n}^{(R,q,Q)}$ (siehe oben) als $N^{\overset{n-a}{\leftarrow}}$, dann erhält man aus den R-Basen von M und N leicht eine R-Basis des Rechtsideals $v_{a,n-a}^{(R,q,Q)}\left(M^{\overset{a}{\rightarrow}}\right)\left(N^{\overset{n-a}{\leftarrow}}\right)\mathcal{H}_{B_n}^{(R,q,Q)}$ in $\mathcal{H}_{B_n}^{(R,q,Q)}$. Schließlich definieren wir Bi-Permutationsmoduln für Hecke-Algebren vom Typ B, indem wir diese Konstruktion auf Permutationsmoduln für Hecke-Algebren vom Typ A anwenden. Bi-Permutationsmoduln werden durch Bi-Kompositionen indiziert. Der einer Bi-Komposition (λ,μ) entsprechende Bi-Permutationsmodul wird als $M_{(R,q,Q)}^{(\lambda,\mu)}$ notiert.

In Abschnitt 4.4 definieren wir Bi-Specht-Moduln für Hecke-Algebren vom Typ B und leiten R-Basen für sie her. Das geschieht durch Anwendung der Methode aus dem vorhergehenden Abschnitt auf Specht-Moduln für Hecke-Algebren vom Typ A. Bi-Specht-Moduln werden durch Bi-Partitionen indiziert. Der einer Bi-Partition (λ,μ) entsprechende Bi-Specht-Modul wird als $S_{(R,q,Q)}^{(\lambda,\mu)}$ notiert.

Abschnitt 4.5 führt Bi-PK$_n$-Moduln für Hecke-Algebren vom Typ B ein, beschreibt R-Basen für sie und diskutiert elementare Beziehungen zwischen ihnen, Bi-Permutationsmoduln und Bi-Specht-Moduln. Die Konstruktion der Bi-PK$_n$-Moduln verwendet PK-Moduln für Hecke-Algebren vom Typ A und beruht wiederum auf der in Abschnitt 4.3 beschriebenen Methode. Bi-PK$_n$-Moduln werden durch Bi-PK$_n$-Paare indiziert. Der einem Bi-PK$_n$-Paar $(\lambda^{\#}\lambda,\mu^{\#}\mu)$ entsprechende Bi-PK$_n$-Modul wird als $S_{(R,q,Q)}^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}$ notiert.

In Abschnitt 4.6 definieren wir Bi-PK$_n$-Homomorphismen und beschreiben ihre

Wirkung auf Bi-PK$_n$-Moduln. Ein solcher Homomorphismus wird indiziert durch ein Bi-PK$_n$-Paar $(\lambda^{\#}\lambda, \mu^{\#}\mu)$ und eine ganze Zahl c, die die Anwendung des entsprechenden Paares von Operatoren ${}^{(c)}A$ und ${}^{(c)}R$ beziehungsweise $A^{(c)}$ und $R^{(c)}$ auf $(\lambda^{\#}\lambda, \mu^{\#}\mu)$ ermöglicht (siehe oben). Nach Definition bilden Bi-PK$_n$-Homomorphismen Bi-Permutationsmoduln in Bi-Permutationsmoduln ab. Sie werden als

$$ {}^{(c)}\Psi_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q) : M^{(\lambda,\mu)}_{(R,q,Q)} \to M^{(\lambda R_c,\mu)}_{(R,q,Q)} $$

und

$$ \Psi^{(c)}_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q) : M^{(\lambda,\mu)}_{(R,q,Q)} \to M^{(\lambda,\mu R_c)}_{(R,q,Q)} $$

notiert. Diese Homomorphismen werden aus PK-Homomorphismen für Hecke-Algebren vom Typ A abgeleitet. Das geschieht auf eine Art und Weise, die verträglich mit der Konstruktion von Bi-Permutationsmoduln aus Permutationsmoduln für Hecke-Algebren vom Typ A ist. So können wir die Bilder und Kerne von Bi-PK$_n$-Homomorphismen bestimmen, wenn diese auf Bi-PK$_n$-Moduln angewandt werden. Wir erhalten

$$ \left(S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}_{(R,q,Q)}\right) {}^{(c)}\Psi_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q) = S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)^{(c)}R}_{(R,q,Q)} $$

und

$$ \operatorname{Ker}\left({}^{(c)}\Psi_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q)\Big\downarrow^{M^{(\lambda,\mu)}}_{S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}} \right) = S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)^{(c)}A}_{(R,q,Q)} $$

und analoge Aussagen für Homomorphismen $\Psi^{(c)}_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q)$.

In Abschnitt 4.7 benutzen wir die Definitionen und Ergebnisse aus den vorhergehenden Abschnitten, um generische Bi-Specht-Serien für Bi-PK$_n$-Moduln zu konstruieren. Diese umfassen generische Bi-Specht-Serien für Bi-Permutationsmoduln, welche zu einer generischen Bi-Specht-Serie für den rechtsregulären $\mathcal{H}^{(R,q,Q)}_{B_n}$-Modul führen. Die Konstruktion von generischen Bi-Specht-Serien für Bi-PK$_n$-Moduln ist eine Anpassung der Konstruktion von generischen Specht-Serien für PK-Moduln aus Abschnitt 3.11 an die vorliegende Situation. Wir beginnen mit einem Bi-PK$_n$-Modul $S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}_{(R,q,Q)}$. Aus dem entsprechenden Bi-PK$_n$-Paar $(\lambda^{\#}\lambda, \mu^{\#}\mu)$ bilden wir einen Binärbaum durch wiederholte Anwendung von Paaren von Operatoren ${}^{(c)}A$ und ${}^{(c)}R$ und auch $A^{(c)}$ und $R^{(c)}$. Wir verwenden Induktion über die Knoten dieses Baumes von den Blättern zur Wurzel, die mit $(\lambda^{\#}\lambda, \mu^{\#}\mu)$ beschriftet ist. Schließlich benutzen wir eine der exakten Sequenzen

$$ 0 \to S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)^{(c)}A}_{(R,q,Q)} \to S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}_{(R,q,Q)} \to S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)^{(c)}R}_{(R,q,Q)} \to 0 $$

oder

$$ 0 \to S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)A^{(c)}}_{(R,q,Q)} \to S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}_{(R,q,Q)} \to S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)R^{(c)}}_{(R,q,Q)} \to 0. $$

Dabei ist die linke Abbildung beidesmal die natürliche Inklusion und die rechte Abbildung einmal die Einschränkung ${}^{(c)}\Psi_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q)\big\downarrow^{M^{(\lambda,\mu)}}_{S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}}$ und das andere mal die Einschränkung $\Psi^{(c)}_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q)\big\downarrow^{M^{(\lambda,\mu)}}_{S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}}$. Die Existenz und Exaktheit dieser Sequenzen wurde in den vorhergehenden Abschnitten nachgewiesen. Mit der geeigneten exakten Sequenz können wir aus den induktiv existierenden generischen Bi-Specht-Serien für $S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)^{(c)}A}_{(R,q,Q)}$ und $S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)^{(c)}R}_{(R,q,Q)}$ beziehungsweise für $S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)A^{(c)}}_{(R,q,Q)}$ und $S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)R^{(c)}}_{(R,q,Q)}$ eine generische Bi-Specht-Serie für $S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}_{(R,q,Q)}$ bilden. Damit ist die Herleitung von generischen Bi-Specht-Serien für Hecke-Algebren vom Typ B beendet. Mit Hilfe der Konstruktionen und Ergebnisse aus [PAL] kann dieses Kapitel fast wörtlich auf Hecke-Algebren vom Typ D übertragen werden, so daß generische Bi-Specht-Serien auch für diese zur Verfügung stehen.

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Chapter 1

Background

This chapter consists for the most part of descriptions of well known definitions, constructions, and results which are used in later chapters. Combinatorial notations are introduced and technical facts about Hecke algebras of type A are reviewed. Central results in this chapter are Theorem 1.8.23 and Theorem 1.9.18.

1.1 Combinatorics

This section introduces the combinatorics required for the definition of Hecke algebras and the description of their representation theory. Furthermore, notations for some elementary notions are fixed. References for the biggest part of the following material are [JAM1] and [HUM, Chapter 1, Chapter 2, Chapter 5].

The set $\{\dots,-2,-1,0,1,2,\dots\}$ of all integers is denoted by $\mathbb{Z}$. We denote the set $\{1,2,3,\dots\}$ of all positive integers by $\mathbb{N}$. The set $\{0,1,2,\dots\}$ of all nonnegative integers is denoted by $\mathbb{N}_0$. We denote the set $\{z/y \mid z \in \mathbb{Z}, y \in \mathbb{Z}\setminus\{0\}\}$ of all rational numbers by $\mathbb{Q}$.

For a finite set M, the number of its elements is denoted by $|M| \in \mathbb{N}_0$. For sets M, N and a map $f : M \to N$, the restriction of f to a subset $U \subseteq M$ is denoted by

$$f{\downarrow}_U^M : U \to N. \tag{1.1}$$

In everything that follows, $n \in \mathbb{N}$ denotes a fixed positive integer. The symmetric group $\mathfrak{S}_n$ is the group of all permutations on the set $\{1,\dots,n\}$. The parameter n is called the degree of the symmetric group $\mathfrak{S}_n$. Group elements operate from the right on these numbers and are written in cycle notation. Thus, for $u = (1,2,3)$ and $v = (1,2)(3,4)$, we have $2u = 3$ and $uv = (2,4,3)$. The neutral element of $\mathfrak{S}_n$ is denoted by $1_{\mathfrak{S}_n}$. For a set $M \subseteq \{1,\dots,n\}$, we put

$$\mathfrak{S}_M = \{w \in \mathfrak{S}_n \mid \forall j \in \{1,\dots,n\}\setminus M : jw = j\} \subseteq \mathfrak{S}_n. \tag{1.2}$$

$\mathfrak{S}_M$ is a subgroup of $\mathfrak{S}_n$.

$\mathfrak{S}_n$ is isomorphic to the Weyl group $W_{A_{n-1}}$ of type A_{n-1} which is generated by the elements

$$s_1, \ldots, s_{n-1} \tag{1.3}$$

together with the relations

$$\begin{aligned} \forall i \in \{1, \ldots, n-1\} : s_i^2 = 1_{W_A}, \\ \forall i \in \{1, \ldots, n-2\} : s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \\ \forall i, j \in \{1, \ldots, n-1\} \text{ with } |i-j| > 1 : s_i s_j = s_j s_i \end{aligned} \tag{1.4}$$

where 1_{W_A} denotes the neutral element of that group (see, for example, [HUP, Beispiel 19.7]). More generally, $W_{A_{n-1}}$ is called a Weyl group of type A and denoted by W_A. $W_{A_{n-1}}$ is the Weyl group of the root system of type A_{n-1} with the following Dynkin diagram.

Here, for each $j \in \{1, \ldots, n-1\}$, the vertex j corresponds to the generator s_j of $W_{A_{n-1}}$. These generators are called simple reflections. They correspond to the generating set

$$\mathfrak{B}_n = \{(1,2), \ldots, (n-1,n)\} \tag{1.5}$$

of $\mathfrak{S}_n$ consisting of transpositions of adjacent numbers. More precisely, one has

$$W_{A_{n-1}} \xrightarrow{\sim} \mathfrak{S}_n \quad \text{with} \quad s_j \mapsto (j, j+1) \text{ for } j \in \{1, \ldots, n-1\}. \tag{1.6}$$

In the following, $W_{A_{n-1}}$ and $\mathfrak{S}_n$ are identified by means of this isomorphism. The elements of $\mathfrak{B}_n$ also are called simple reflections.

With this, reduced expressions and the length function translate from $W_{A_{n-1}}$ to $\mathfrak{S}_n$. According to (1.3) and (1.4), each $w \in W_{A_{n-1}}$ can be expressed as a product of simple reflections. A reduced expression for w is such a representation with the smallest possible number of factors. The length $\ell(w) = \ell_A(w) = \ell_{A_{n-1}}(w)$ of w in $W_{A_{n-1}}$ is defined as the number of factors in a reduced expression of w. Thus, a reduced expression of w has the form

$$w = s_{i_1} \cdots s_{i_{\ell(w)}} \tag{1.7}$$

with certain $i_1, \ldots, i_{\ell(w)} \in \{1, \ldots, n-1\}$. For the length function

$$\ell_{A_{n-1}} = \ell_A = \ell : W_{A_{n-1}} \to \mathbb{N}_0, \quad w \mapsto \ell_{A_{n-1}}(w) = \ell_A(w) = \ell(w), \tag{1.8}$$

the following statements hold.

$$\begin{aligned}
&\text{(i)}\ \forall w \in W_{A_{n-1}} : \ell(w) = 0 \Leftrightarrow w = 1_{W_A}.\\
&\text{(ii)}\ \forall w \in W_{A_{n-1}} : \ell(w) = 1 \Leftrightarrow w \in \{s_1, \dots, s_{n-1}\}.\\
&\text{(iii)}\ \forall w \in W_{A_{n-1}}, s \in \{s_1, \dots, s_{n-1}\} : \ell(ws) \in \{\ell(w) - 1, \ell(w) + 1\}.\\
&\text{(iv)}\ \forall w \in W_{A_{n-1}} : \ell(w^{-1}) = \ell(w).
\end{aligned} \tag{1.9}$$

When interpreting a $w \in W_{A_{n-1}}$ as a permutation in $\mathfrak{S}_n$, its length can be determined as follows.

$$\ell(w) = |\{(i, j) \in \{1, \dots, n\} \times \{1, \dots, n\} \mid i < j \text{ and } iw > jw\}| \tag{1.10}$$

Furthermore, a reduced expression of $w \in \mathfrak{S}_n$ has the form

$$w = v_1 \cdots v_{\ell(w)} \tag{1.11}$$

with certain factors $v_1, \dots, v_{\ell(w)} \in \mathfrak{B}_n$.

In what follows, some combinatorial constructions related to the representation theory of symmetric groups are described. These will be generalized to Hecke algebras. Until further notice, let

$$m \in \mathbb{N}_0$$

be an arbitrarily chosen nonnegative integer.

Definition 1.1.1 *(i) A composition of m is a sequence*

$$\lambda = (\lambda_1, \lambda_2, \dots)$$

with entries $\lambda_j \in \mathbb{N}_0$ for $j \in \mathbb{N}$ such that

$$\sum_{j \in \mathbb{N}} \lambda_j = m$$

holds. This is denoted by

$$\lambda \vDash m.$$

(ii) For a $\lambda = (\lambda_1, \lambda_2, \dots) \vDash m$ and a $j \in \mathbb{N}_0$,

$$\lambda_j^+ = \sum_{i=1}^{j} \lambda_i$$

denotes the partial sum of the first j entries of λ.

(iii) The set of all compositions of m is denoted by

$$\Xi_m = \{\lambda \mid \lambda \vDash m\}.$$

Our notation of compositions uses the following simplifications. For every composition $\lambda = (\lambda_1, \lambda_2, \ldots)$, all entries with large enough indices are 0, these are omitted. Furthermore, successive entries of the same value are written in power notation. In doing so, the exponent 0 with the obvious interpretation will also be used. With these conventions, one has for example

$$\begin{aligned} \lambda &= (8,0,3,3,3,3,6,6,0,0,0,5,1,0,0,0,\ldots) \\ &= (8,0,3,3,3,3,6,6,0,0,0,5,1) \\ &= (8,0,3^4,6^2,0^3,5,1) \vDash 38. \end{aligned}$$

Definition 1.1.2 *(i) A partition of m is a $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash m$ such that the entries $\lambda_j \in \mathbb{N}_0$ with $j \in \mathbb{N}$ satisfy*

$$\lambda_1 \geq \lambda_2 \geq \cdots.$$

This is denoted by

$$\lambda \vdash m.$$

(ii) For an $e \in \mathbb{N} \cup \{\infty\}$, a $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash m$ is called e-singular if there is an index $j \in \mathbb{N}$ such that $\lambda_j = \lambda_{j+1} = \cdots = \lambda_{j+e-1} > 0$ holds. If λ is not e-singular, λ is called e-regular.

(iii) The set of all partitions of m is denoted by

$$\Pi_m = \{\lambda \mid \lambda \vdash m\}.$$

Furthermore, we put for an $e \in \mathbb{N} \cup \{\infty\}$

$$\Pi_{m,e} = \{\lambda \vdash m \mid \lambda \text{ } e\text{-regular}\}.$$

Remark 1.1.3 *Obviously, for an*

$$e \in \{m+1, m+2, \ldots\} \cup \{\infty\}$$

every $\lambda \in \Pi_m$ is e-regular, and we have

$$\Pi_{m,e} = \Pi_m.$$

The ordering relations described in the following definition also are considered in [MUR, Section 3, especially Definition 3.1] and elsewhere.

Definition 1.1.4 *Let* $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash m$ *and let* $\mu = (\mu_1, \mu_2, \ldots) \vDash m$.

(i) We write

$$\lambda < \mu$$

if there is an $i \in \mathbb{N}$ *such that both*

$$\lambda_i < \mu_i \qquad \text{and} \qquad \forall j \in \{1, \ldots, i-1\} : \lambda_j = \mu_j$$

hold. Furthermore, we write

$$\lambda \leq \mu$$

if

$$(\lambda < \mu) \vee (\lambda = \mu)$$

holds.

(ii) We write

$$\lambda \trianglelefteq \mu$$

if

$$\forall i \in \mathbb{N} : \lambda_i^+ \leq \mu_i^+$$

holds. Furthermore, we write

$$\lambda \vartriangleleft \mu$$

if

$$(\lambda \trianglelefteq \mu) \wedge (\lambda \neq \mu)$$

holds.

Lemma 1.1.5 *(i) The relation* $\leq$ *on the set* Ξ_m *is a total ordering relation.*

(ii) The relation $\trianglelefteq$ *on the set* Ξ_m *is a partial ordering relation.*

(iii) Let $\lambda, \mu \vDash m$. *Then*

$$\lambda \trianglelefteq \mu \Rightarrow \lambda \leq \mu.$$

Proof. See elsewhere, for example [MUR, Section 3]. ■

Definition 1.1.6 *Let* $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash m$.

(i) The diagram of λ *is the set of lattice points*

$$[\lambda] = \{(i, j) \mid i \in \mathbb{N} \text{ and } j \in \{1, \ldots, \lambda_i\}\} \subseteq \mathbb{N} \times \mathbb{N}.$$

(ii) For $k \in \mathbb{N}$, the k-th row of $[\lambda]$ is the set of lattice points

$$\{(k,j) \mid j \in \mathbb{N} \text{ and } (k,j) \in [\lambda]\} = \{(k,j) \mid j \in \{1,\ldots,\lambda_k\}\}.$$

The length of the k-th row of $[\lambda]$ is given by

$$|\{(k,j) \mid j \in \mathbb{N} \text{ and } (k,j) \in [\lambda]\}| = \lambda_k \in \mathbb{N}_0.$$

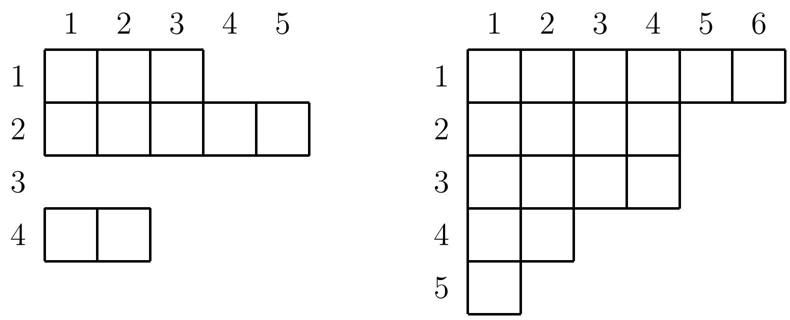
(iii) For $k \in \mathbb{N}$, the k-th column of $[\lambda]$ is the set of lattice points

$$\{(j,k) \mid j \in \mathbb{N} \text{ and } (j,k) \in [\lambda]\}.$$

The length of the k-th column of $[\lambda]$ is given by

$$|\{(j,k) \mid j \in \mathbb{N} \text{ and } (j,k) \in [\lambda]\}| \in \mathbb{N}_0.$$

The diagram $[\lambda]$ of a composition $\lambda = (\lambda_1, \lambda_2, \ldots)$ of m is represented by squares in the plane ordered by rows. A square is placed at each lattice point in $[\lambda]$. Then, for every $j \in \mathbb{N}$, the j-th row contains exactly λ_j squares, and the left ends of all rows are aligned one under another. The following picture shows on the left hand side the representation of the diagram $[\lambda]$ of $\lambda = (3,5,0,2) \vDash 10$ and on the right hand side the representation of the diagram $[\mu]$ of $\mu = (6,4^2,2,1) \vdash 17$.



The rows of these arrangements correspond exactly to the rows of $[\lambda]$ and $[\mu]$ as defined in Definition 1.1.6.(ii) and are numbered in ascending order from top to bottom. The columns of these arrangements correspond exactly to the columns of $[\lambda]$ and $[\mu]$ as defined in Definition 1.1.6.(iii) and are numbered in ascending order from left to right.

The following pair of statements gives some elementary properties of compositions, partitions, and their respective diagrams.

Lemma 1.1.7 *Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash m$.*

(i) For $i \in \mathbb{N}$ we have

$$\lambda_i = |\{(i,j) \mid j \in \mathbb{N} \text{ and } (i,j) \in [\lambda]\}| .$$

This shows that λ can be reconstructed from $[\lambda]$.

(ii) We have

$$|[\lambda]| = m.$$

Proof. (i) This follows immediately from Definition 1.1.6.(i).

(ii) This follows from statement (i) and Definition 1.1.1.(i). ■

Lemma 1.1.8 *Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash m$.*

(i) Let $(i,j) \in \lambda$ and let $\tilde{i} \in \{1, \ldots, i\}$. Then we have

$$(\tilde{i}, j) \in [\lambda] .$$

(ii) For $j \in \mathbb{N}$ we have

$$\begin{aligned} &|\{(i,j) \mid i \in \mathbb{N} \text{ and } (i,j) \in [\lambda]\}| \\ &\quad \geq \ |\{(i,j+1) \mid i \in \mathbb{N} \text{ and } (i,j+1) \in [\lambda]\}| . \end{aligned}$$

Proof. (i) This follows from Definition 1.1.2.(i) and Definition 1.1.6.(i).

(ii) This is immediate from Definition 1.1.6.(i). ■

Next, an important map on the set Π_m will be introduced.

Definition 1.1.9 *The map*

$$(\cdot)' : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$$

is defined by

$$(i,j)' = (j,i) \quad \text{for} \quad (i,j) \in \mathbb{N} \times \mathbb{N}.$$

Lemma 1.1.10 *(i) We have*

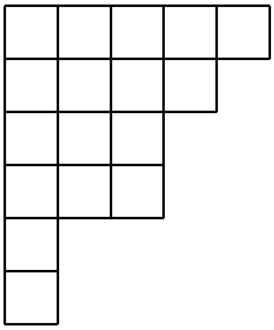
$$(\cdot)'(\cdot)' = \mathrm{id}_{\mathbb{N}\times\mathbb{N}}.$$

(ii) Let $\lambda \vdash m$ with associated diagram $[\lambda] \subseteq \mathbb{N} \times \mathbb{N}$. Then $[\lambda]'$ also is the diagram of a partition of m.

Proof. (i) This follows immediately from Definition 1.1.9.

(ii) This can be obtained from the statements in Lemma 1.1.7 and Lemma 1.1.8. ■

The following is the set $[\mu]'$ for the diagram of $\mu = (6, 4^2, 2, 1) \vdash 17$ shown after Definition 1.1.6.



According to Lemma 1.1.10, the following definition is meaningful.

Definition 1.1.11 *Let* $\lambda \vdash m$*. Then the partition* λ' *is defined by*

$$[\lambda'] = [\lambda]'.$$

λ' *is said to be transposed to* λ*. Furthermore, this defines the transposition map*

$$(\cdot)' : \Pi_m \to \Pi_m.$$

Remark 1.1.12 *More generally, for* $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash m$ *and every* $k \in \mathbb{N}$*, the entry* λ'_k *of* $\lambda' = (\lambda'_1, \lambda'_2, \ldots) \vdash m$ *is, according to Definition 1.1.6.(iii) and Lemma 1.1.7.(i), equal to the number of lattice points in the* k*-th column of* $[\lambda]$ *respectively the number of squares in the* k*-th column of the representation of* $[\lambda]$*. Thus, the* k*-th column of* $[\lambda]$ *can be written as*

$$\{(j, k) \mid j \in \{1, \ldots, \lambda'_k\}\}.$$

Since the diagram $[\lambda]$ *is the disjoint union of its columns, this shows for* $[\lambda]$ *and* $[\lambda']$

$$\begin{aligned}
[\lambda'] &= \{(i, j) \mid i \in \mathbb{N}, j \in \{1, \ldots, \lambda'_i\}\} \\
&= \bigcup_{i \in \mathbb{N}} \{(i, j) \mid j \in \{1, \ldots, \lambda'_i\}\} \\
&= \bigcup_{i \in \mathbb{N}} \{(i, j) \mid (j, i) \text{ in the } i\text{-th column of } [\lambda]\} \\
&= \{(i, j) \mid (j, i) \in [\lambda]\}.
\end{aligned}$$

Thus, $[\lambda']$ *is obtained from* $[\lambda]$ *by means of a "reflection" about the "diagonal"* $\{(j, j) \mid j \in \mathbb{N}\}$ *in* $\mathbb{N} \times \mathbb{N}$*. This maps the rows of* $[\lambda]$ *onto the columns of* $[\lambda']$ *and the columns of* $[\lambda]$ *onto the rows of* $[\lambda']$*. Furthermore, this shows that the first column of* $[\lambda']$ *contains* λ_1 *lattice points. Thus, one can write* $\lambda' = (\lambda'_1, \ldots, \lambda'_{\lambda_1})$ *with* $\lambda'_{\lambda_1} > 0$*.*

The objects and constructions described in the following are related to the block theory of Hecke algebras of type A. They stem from the block theory of group algebras of symmetric groups.

Definition 1.1.13 *Let* $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash m$ *with* $\lambda' = (\lambda'_1, \lambda'_2, \ldots) \vdash m$ *and let* $(i,j) \in [\lambda]$*. Then the* (i,j)*-hook in* λ *is defined as*

$$h^\lambda_{(i,j)} = \{(i,k) \in [\lambda] \mid k \in \{j, \ldots, \lambda_i\}\} \cup \left\{(k,j) \in [\lambda] \mid k \in \left\{i, \ldots, \lambda'_j\right\}\right\}.$$

$h^\lambda_{(i,j)}$ *also is called* (i,j)*-hook or hook in* λ *or just hook. We use the notation*

$$h^\lambda_{(i,j)} = h_{(i,j)} = h^\lambda = h.$$

The size of the set $h^\lambda_{(i,j)}$ *is called the hook length of* $h^\lambda_{(i,j)}$ *or, for short, the length of* $h^\lambda_{(i,j)}$*. This value is denoted by*

$$\left|h^\lambda_{(i,j)}\right| = \left|h_{(i,j)}\right| = \left|h^\lambda\right| = |h|.$$

With this, $h^\lambda_{(i,j)}$ *also is called a* $\left|h^\lambda_{(i,j)}\right|$*-hook.*

Definition 1.1.14 *Let* $\lambda \vdash m$ *and let* $(i,j) \in [\lambda]$*. Then the* (i,j)*-rim hook in* λ *is defined as*

$$r^\lambda_{(i,j)} = \left\{(\tilde{i}, \tilde{j}) \in [\lambda] \mid \tilde{i} \geq i,\ \tilde{j} \geq j \text{ and } (\tilde{i}+1, \tilde{j}+1) \notin [\lambda]\right\}.$$

$r^\lambda_{(i,j)}$ *also is called* (i,j)*-rim hook or rim hook in* λ *or just rim hook. We use the notation*

$$r^\lambda_{(i,j)} = r_{(i,j)} = r^\lambda = r.$$

The size of the set $r^\lambda_{(i,j)}$ *is called the rim hook length of* $r^\lambda_{(i,j)}$ *or, for short, the length of* $r^\lambda_{(i,j)}$*. This value is denoted by*

$$\left|r^\lambda_{(i,j)}\right| = \left|r_{(i,j)}\right| = \left|r^\lambda\right| = |r|.$$

With this, $r^\lambda_{(i,j)}$ *also is called a* $\left|r^\lambda_{(i,j)}\right|$*-rim hook.*

Lemma 1.1.15 *Let* $\lambda \vdash m$ *and let* $(i,j) \in \lambda$*. Then*

$$\left|h^\lambda_{(i,j)}\right| = \left|r^\lambda_{(i,j)}\right|.$$

Proof. See [JK, Seite 56]. ■

Now, rim hooks can be removed from and added to partitions such that the resulting objects are again partitions. This is described in more detail in the following.

Definition 1.1.16 *Let* $\lambda \vdash m$ *and let* $\mu \vdash \tilde{m}$ *with* $\tilde{m} \in \mathbb{N}_0$ *such that* $[\mu] \subseteq [\lambda]$ *holds and such that* $[\lambda] \setminus [\mu]$ *is a rim hook* r^λ *in* λ*. Then we write*

$$\mu = \lambda \setminus r^\lambda$$

and say that μ *is* λ *without the rim hook* r^λ*.*

Definition 1.1.17 *Let $\lambda \vdash m$ and let $\mu \vdash \tilde{m}$ with $\tilde{m} \in \mathbb{N}_0$ such that $[\lambda] \subseteq [\mu]$ holds and such that $[\mu] \setminus [\lambda]$ is a rim hook r^μ in μ. Then we write*

$$\mu = \lambda \cup r^\mu$$

and say that μ is λ together with the rim hook r^μ.

Later on, we also will require the removal and addition of single lattice points from and to the diagram of a partition. These operations are obtained from the removal and addition of rim hooks by considering rim hooks of length 1.

Definition 1.1.18 *Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash m$ and suppose that for an $i \in \mathbb{N}$ we have*

$$\lambda_i > \lambda_{i+1}.$$

Then the partition

$$\lambda \setminus \{(i, \lambda_i)\} = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots)$$

of $m-1$ is called λ without (i, λ_i). This is denoted by

$$\lambda \setminus (i, \lambda_i).$$

Definition 1.1.19 *Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash m$ and suppose that for an $i \in \mathbb{N}$ we have*

$$(i = 1) \vee ((i > 1) \wedge (\lambda_i < \lambda_{i-1})).$$

Then the partition

$$\lambda \cup \{(i, \lambda_i + 1)\} = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \ldots)$$

of $m+1$ is called λ together with $(i, \lambda_i + 1)$. This is denoted by

$$\lambda \cup (i, \lambda_i + 1).$$

The sets introduced in the following definition are used in Section 1.9.

Definition 1.1.20 *Let $k \in \mathbb{N}$.*

(i) Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash k-1$. Then the set $\lambda \uparrow \subseteq \Pi_k$ is defined as

$$\lambda \uparrow = \{\lambda \cup (1, \lambda_1 + 1)\} \cup \{\lambda \cup (i, \lambda_i + 1) \mid i \in \mathbb{N} \setminus \{1\} \text{ and } \lambda_i < \lambda_{i-1}\}.$$

(ii) Let $\mu = (\mu_1, \mu_2, \ldots) \vdash k$. Then the set $\mu \downarrow \subseteq \Pi_{k-1}$ is defined as

$$\mu \downarrow = \{\mu \setminus (i, \mu_i) \mid i \in \mathbb{N} \text{ and } \mu_i > \mu_{i+1}\}.$$

Lemma 1.1.21 *Let $k \in \mathbb{N}$, $\lambda \vdash k-1$, and $\mu \vdash k$. Then we have*

$$\mu \in \lambda \uparrow \Leftrightarrow \lambda \in \mu \downarrow .$$

Proof. This is obvious from Definition 1.1.18 and Definition 1.1.19. ■

When the removal and addition of rim hooks to a partition is executed by using its diagram, the result is not always easy to determine. In the following, different representations of partitions are described that simplify this operation. A thorough description of the following material can be found in [JK, Section 2.7].

Definition 1.1.22 *Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash m$. Then a β-sequence for λ is defined as a finite sequence*

$$\beta = (\beta_1, \ldots, \beta_c)$$

with a nonnegative integer

$$c \geq \begin{cases} 0 & \text{for} \quad m = 0 \\ \max\{k \in \mathbb{N} \mid \lambda_k > 0\} & \text{for} \quad m > 0 \end{cases} \tag{1.12}$$

and entries

$$\beta_j = \lambda_j + c - j \tag{1.13}$$

for $j \in \{1, \ldots, c\}$. The value $c \in \mathbb{N}_0$ is called the length of the β-sequence β.

Remark 1.1.23 *(i) A substantial difference between partitions and β-sequences is that a partition can have several positive entries of the same value whereas the entries of a β-sequence are always pairwise distinct. β-sequences are strictly decreasing.*

(ii) Obviously, a partition λ can be reconstructed from every β-sequence for λ by means of the relation (1.13). More generally, this relation shows that every strictly decreasing sequence of nonnegative integers is, in fact, a β-sequence for a unique partition.

(iii) From a given β-sequence for a partition, all other β-sequences for this partition can be easily obtained. Let, for example, $\beta = (\beta_1, \ldots, \beta_c)$ and $\tilde{\beta} = (\tilde{\beta}_1, \ldots, \tilde{\beta}_{\tilde{c}})$ with $c < \tilde{c}$ both be β-sequences for the same partition λ. Then we get from Definition 1.1.22

$$\beta = (\tilde{\beta}_1 + c - \tilde{c}, \ldots, \tilde{\beta}_c + c - \tilde{c})$$

and

$$\tilde{\beta} = (\beta_1 + \tilde{c} - c, \ldots, \beta_c + \tilde{c} - c, \tilde{c} - c - 1, \ldots, 0).$$

(iv) *In the following constructions, β-sequences are used for the representation of partitions. Some of these constructions impose a lower bound on the lengths of the β-sequences employed. This is no real restriction since, according to statement (iii), one can always pass from a given β-sequence for a partition to a longer β-sequence for the same partition.*

(v) *With the notation from Definition 1.1.22, formula (1.13) shows that for a given partition λ and a given length c as in (1.12), there is exactly one β-sequence for λ of length c.*

We have, for example, the β-sequences $\beta = (7, 5, 4, 2)$ and $\tilde{\beta} = (10, 8, 7, 5, 2, 1, 0)$ for the same partition $\lambda = (4, 3^2, 2)$. β has length 4, $\tilde{\beta}$ has length 7. β is the shortest β-sequence for λ.

Next, we describe how the removal and addition of rim hooks from and to partitions can be executed by using β-sequences.

Lemma 1.1.24 *Let $\lambda = (\lambda_1, \dots, \lambda_k) \vdash m$ with $k \in \mathbb{N}$. Furthermore, choose an $e \in \mathbb{N}$ and a β-sequence $\beta = (\beta_1, \dots, \beta_c)$ for λ with $c \geq k$. Finally, choose an $i \in \{1, \dots, c\}$. Then the following statements (i) and (ii) are equivalent.*

(i) *We have $m \geq e$ and there is a uniquely determined $\mu \vdash m - e$ with $[\mu] \subseteq [\lambda]$ such that $[\lambda] \setminus [\mu]$ is a rim hook r^λ in λ which satisfies $\left|r^\lambda\right| = e$ and $\min\left\{\tilde{i} \,|\, (\tilde{i}, \tilde{j}) \in r^\lambda\right\} = i$.*

(ii) *We have $\beta_i \geq e$ and $\beta_i - e \notin \{\beta_1, \dots, \beta_c\}$.*

If one of these equivalent conditions holds, a β-sequence for $\mu = \lambda \setminus r^\lambda$ is obtained from β by arranging the elements of the set

$$\{\beta_1, \dots, \beta_{i-1}, \beta_i - e, \beta_{i+1}, \dots, \beta_c\}$$

in descending order.

Proof. See [JK, Lemma 2.7.13]. ∎

Lemma 1.1.25 *Let $\lambda = (\lambda_1, \dots, \lambda_k) \vdash m$ with $k \in \mathbb{N}$. Furthermore, choose an $e \in \mathbb{N}$ and a β-sequence $\beta = (\beta_1, \dots, \beta_c)$ for λ with $c \geq k + e$. Finally, choose an $i \in \{1, \dots, c\}$. Then the following statements (i) and (ii) are equivalent.*

(i) *There is a uniquely determined $\mu \vdash m + e$ with $[\lambda] \subseteq [\mu]$ such that $[\mu] \setminus [\lambda]$ is a rim hook r^μ in μ which satisfies $|r^\mu| = e$ and $\max\left\{\tilde{i} \,|\, (\tilde{i}, \tilde{j}) \in r^\mu\right\} = i$.*

(ii) *We have $\beta_i + e \notin \{\beta_1, \dots, \beta_c\}$.*

If one of these equivalent conditions holds, a β-sequence for $\mu = \lambda \cup r^\mu$ is obtained from β by arranging the elements of the set

$$\{\beta_1, \ldots, \beta_{i-1}, \beta_i + e, \beta_{i+1}, \ldots, \beta_c\}$$

in descending order.

Proof. This follows easily from Lemma 1.1.24. ■

The removal and addition of single lattice points from and to partitions also can be described easily in terms of β-sequences.

Corollary 1.1.26 *Let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash m$ with $k \in \mathbb{N}$. Furthermore, choose a β-sequence $\beta = (\beta_1, \ldots, \beta_c)$ for λ with $c \geq k$. Finally, choose an $i \in \{1, \ldots, c\}$. Then the following statements (i) and (ii) are equivalent.*

(i) We have $\lambda_i > 0$ and (i, λ_i) can be removed from $[\lambda]$ to obtain $\lambda \setminus (i, \lambda_i)$.

(ii) We have
$$((i = c) \wedge (\beta_c > 0)) \vee ((i < c) \wedge (\beta_i - 1 > \beta_{i+1})).$$

If one of these equivalent conditions holds,

$$(\beta_1, \ldots, \beta_{i-1}, \beta_i - 1, \beta_{i+1}, \ldots, \beta_c)$$

is a β-sequence for $\lambda \setminus (i, \lambda_i)$.

Proof. This follows from Definition 1.1.18, Definition 1.1.22, and Lemma 1.1.24. ■

Corollary 1.1.27 *Let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash m$ with $k \in \mathbb{N}$. Furthermore, choose a β-sequence $\beta = (\beta_1, \ldots, \beta_c)$ for λ with $c \geq k + 1$. Finally, choose an $i \in \{1, \ldots, c\}$. Then the following statements (i) and (ii) are equivalent.*

(i) $(i, \lambda_i + 1)$ can be added to $[\lambda]$ to obtain $\lambda \cup (i, \lambda_i + 1)$.

(ii) We have
$$(i = 1) \vee ((i > 1) \wedge (\beta_i + 1 < \beta_{i-1})).$$

If one of these equivalent conditions holds,

$$(\beta_1, \ldots, \beta_{i-1}, \beta_i + 1, \beta_{i+1}, \ldots, \beta_c)$$

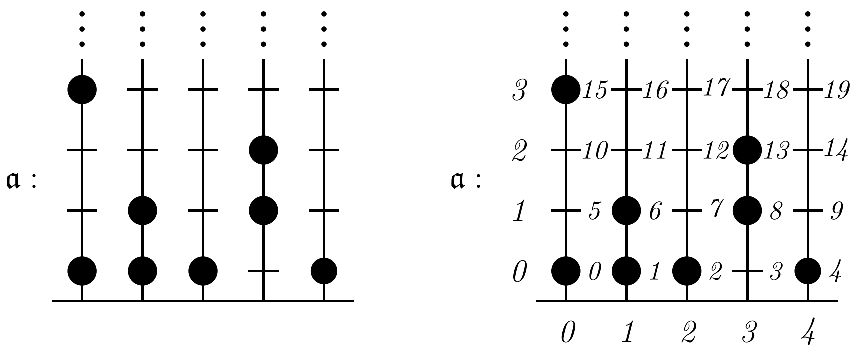
is a β-sequence for $\lambda \cup (i, \lambda_i + 1)$.

Proof. This follows from Definition 1.1.19, Definition 1.1.22, and Lemma 1.1.25. ∎

Now, a graphical representation of β-sequences is described. This representation makes it very easy to determine if and where a rim hook of a certain length can be removed from or added to a partition and also to execute this operation immediately.

Definition 1.1.28 *Let $e \in \mathbb{N}$. Then an e-abacus is defined as an arrangement $\mathfrak{a}$ of e parallel runners in a plane which contain a finite number of movable beads. The following picture shows on the left hand side such an arrangement for $e = 5$. The runners are bounded in one direction — downwards — and unbounded in the opposite direction — upwards. They are numbered from left to right in ascending order, the leftmost runner receives the number 0.*

The beads on the abacus are arranged in rows perpendicular to the runners. These rows are numbered bottom up in ascending order, the lowermost row receives the number 0. The possible places for beads on the abacus are numbered within each row from left to right and across the rows bottom up in ascending order, the place in the lower left corner receives the number 0. In the 5-abacus $\mathfrak{a}$ on the left hand side of the following picture, the places not occupied by beads are marked by horizontal bars. The right hand side of the picture shows the same abacus with explicitly numbered runners, rows, and places.



In the following, e-abaci will be represented like the 5-abacus on the left hand side of the previous picture and denoted by $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$.

In the following, the beads on an e-abacus with an $e \in \mathbb{N}$ will be moved along their respective runners and also within their respective rows. Movement within a column means translation along a runner up or down. Movement within a row means displacement along consecutive places. To be more specific, movement within a row in the upward direction means displacement to the right within a row and displacement from the rightmost place of a row to the leftmost place of the row above, movement within a row in the downward direction means displacement to the left within a row and displacement from the leftmost place of a row to the rightmost place of the row below.

Definition 1.1.29 *Let $\lambda \vdash m$, choose a β-sequence $\beta = (\beta_1, \dots, \beta_c)$ for λ, and fix an $e \in \mathbb{N}$. Then the e-abacus for β is defined as the particular e-abacus that contains beads exactly on the places with numbers $\beta_1, \dots, \beta_c$.*

More generally, an e-abacus is called an e-abacus for λ if the numbers of those places in that abacus which contain beads form a β-sequence for λ when arranged in descending order.

Remark 1.1.30 *Let $e \in \mathbb{N}$.*

(i) Obviously, the e-abacus for a β-sequence is uniquely determined, and the β-sequence can be reconstructed from it. Conversely, according to Definition 1.1.29, every e-abacus is the e-abacus of a uniquely determined β-sequence.

(ii) The relation between partitions and associated e-abaci is the same as that between partitions and associated β-sequences. A partition can be reconstructed from every e-abacus for it. Conversely, every e-abacus is an e-abacus for a uniquely determined partition.

If one has two e-abaci for the same partition, the one containing more beads is obtained from the one containing fewer beads through multiple successive movement within a row of all beads one place in the upward direction and simultaneous addition of a new bead in the place 0. Conversely, the e-abacus with fewer beads is obtained from the e-abacus with more beads through multiple successive removal of the bead in the place 0 (if there is one) and simultaneous movement within a row of all beads one place in the downward direction. These operations correspond exactly to the transitions between different β-sequences for the same partition.

(iii) In particular, Remark 1.1.23.(v) shows that for any given partition there are no two different abaci with the same number of beads.

The behavior of an e-abacus associated to a partition on removal and addition of rim hooks and single lattice points from and to that partition is obtained via Definition 1.1.29 directly from the corresponding results for a β-sequence associated to the partition.

Lemma 1.1.31 *Let* $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash m$ *with* $k \in \mathbb{N}$*. Furthermore, choose an* $e \in \mathbb{N}$ *and an* e*-abacus for* λ *containing* c *beads in the places* $\beta_1, \ldots, \beta_c$ *with* $c \geq k$ *and* $\beta_1 > \cdots > \beta_c \geq 0$*. Finally, choose an* $i \in \{1, \ldots, c\}$*. Then the following statements (i) and (ii) are equivalent.*

(i) We have $m \geq e$ *and there is a uniquely determined* $\mu \vdash m - e$ *with* $[\mu] \subseteq [\lambda]$ *such that* $[\lambda] \setminus [\mu]$ *is a rim hook* r^λ *in* λ *satisfying*

$$\left|r^\lambda\right| = e \qquad \textit{and} \qquad \min\left\{\tilde{i} \,\middle|\, (\tilde{i}, \tilde{j}) \in r^\lambda\right\} = i.$$

(ii) In the e*-abacus for* λ*, the place* β_i *is not contained in the lowermost row, and the place* $\beta_i - e$ *located one row below the place* β_i *is not occupied by a bead.*

If one of these equivalent conditions holds, an e*-abacus for* $\mu = \lambda \setminus r^\lambda$ *is obtained from the* e*-abacus for* λ *through movement within a column of the bead in the place* β_i *in the downward direction to the place* $\beta_i - e$*.*

Proof. This follows from Definition 1.1.29 and Lemma 1.1.24. ■

Lemma 1.1.32 *Let* $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash m$ *with* $k \in \mathbb{N}$*. Furthermore, choose an* $e \in \mathbb{N}$ *and an* e*-abacus for* λ *containing* c *beads in the places* $\beta_1, \ldots, \beta_c$ *with* $c \geq k+e$ *and* $\beta_1 > \cdots > \beta_c \geq 0$*. Finally, choose an* $i \in \{1, \ldots, c\}$*. Then the following statements (i) and (ii) are equivalent.*

(i) There is a uniquely determined $\mu \vdash m + e$ *with* $[\lambda] \subseteq [\mu]$ *such that* $[\mu] \setminus [\lambda]$ *is a rim hook* r^μ *in* μ *satisfying*

$$|r^\mu| = e \qquad \textit{and} \qquad \max\left\{\tilde{i} \,\middle|\, (\tilde{i}, \tilde{j}) \in r^\mu\right\} = i.$$

(ii) In the e*-abacus for* λ*, the place* $\beta_i + e$ *located one row above the place* β_i *is not occupied by a bead.*

If one of these equivalent conditions holds, an e*-abacus for* $\mu = \lambda \cup r^\mu$ *is obtained from the* e*-abacus for* λ *through movement within a column of the bead in the place* β_i *in the upward direction to the place* $\beta_i + e$*.*

Proof. This follows from Definition 1.1.29 and Lemma 1.1.25. ■

Corollary 1.1.33 *Let* $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash m$ *with* $k \in \mathbb{N}$*. Furthermore, choose an* $e \in \mathbb{N}$ *and an* e*-abacus for* λ *containing* c *beads in the places* $\beta_1, \ldots, \beta_c$ *with* $c \geq k$ *and* $\beta_1 > \cdots > \beta_c \geq 0$*. Finally, choose an* $i \in \{1, \ldots, c\}$*. Then the following statements (i) and (ii) are equivalent.*

(i) We have $\lambda_i > 0$ and (i, λ_i) can be removed from $[\lambda]$ to obtain $\lambda \setminus (i, \lambda_i)$.

(ii) In the e-abacus for λ, the place β_i is not coincident with the place 0, and the place $\beta_i - 1$ immediately preceding the place β_i is not occupied by a bead.

If one of these equivalent conditions holds, an e-abacus for $\lambda \setminus (i, \lambda_i)$ is obtained from the e-abacus for λ through movement within a row of the bead in the place β_i in the downward direction to the place $\beta_i - 1$.

Proof. This follows from Definition 1.1.29 and Corollary 1.1.26. ∎

Corollary 1.1.34 *Let $\lambda = (\lambda_1, \dots, \lambda_k) \vdash m$ with $k \in \mathbb{N}$. Furthermore, choose an $e \in \mathbb{N}$ and an e-abacus for λ containing c beads in the places $\beta_1, \dots, \beta_c$ with $c \geq k + 1$ and $\beta_1 > \dots > \beta_c \geq 0$. Finally, choose an $i \in \{1, \dots, c\}$. Then the following statements (i) and (ii) are equivalent.*

(i) $(i, \lambda_i + 1)$ can be added to $[\lambda]$ to obtain $\lambda \cup (i, \lambda_i + 1)$.

(ii) In the e-abacus for λ, the place $\beta_i + 1$ immediately preceding the place β_i is not occupied by a bead.

If one of these equivalent conditions holds, an e-abacus for $\lambda \cup (i, \lambda_i + 1)$ is obtained from the e-abacus for λ through movement within a row of the bead in the place β_i in the upward direction to the place $\beta_i + 1$.

Proof. This follows from Definition 1.1.29 and Corollary 1.1.27. ∎

Later on, certain partitions will occur, from which for a given $e \in \mathbb{N}$ no e-rim hooks can be removed. Such partitions are investigated in the following. All this is described in more detail in [JK, Section 2.7].

Definition 1.1.35 *Let $e \in \mathbb{N} \cup \{\infty\}$.*

(i) For $e < \infty$, a partition λ is called an e-core, if λ doesn't contain any e-rim hooks.

(ii) For $e = \infty$, every partition λ is called an e-core.

(iii) The set of all e-cores is denoted by

$$\Gamma_e = \{\lambda \vdash k \mid k \in \mathbb{N}_0 \text{ and } \lambda \text{ is an } e\text{-core}\}.$$

Remark 1.1.36 *Let $e \in \mathbb{N} \cup \{\infty\}$ and choose an e-core λ. Then it follows easily from Definition 1.1.2.(ii) that λ is e-regular.*

Lemma 1.1.37 *Let $e \in \mathbb{N}$. Furthermore, let $\lambda \vdash k$ with $k \in \mathbb{N}_0$. Then e-rim hooks can be removed successively from λ until the partition μ obtained in this process doesn't contain any e-rim hooks any more. This partition μ of $k - je$ with an appropriate $j \in \mathbb{N}_0$ is independent of the selection of the rim hooks removed from λ to obtain μ, it depends only on λ and e.*

Proof. See [JK, Theorem 2.7.16] ■

Definition 1.1.38 *(i) Let $e \in \mathbb{N}$ and $\lambda \vdash k$ with $k \in \mathbb{N}_0$. Then the partition μ constructed in Lemma 1.1.37 is called the e-core of λ. This is denoted by*

$$\mu = \gamma_e(\lambda). \tag{1.14}$$

Write $\mu \vdash k - je$ with $j \in \mathbb{N}_0$ as in Lemma 1.1.37. Then j is called the e-weight of λ. This is denoted by

$$j = g_e(\lambda).$$

(1.14) defines the map

$$\gamma_e : \bigcup_{i \in \mathbb{N}_0} \Pi_i \to \Gamma_e, \quad \lambda \mapsto \gamma_e(\lambda).$$

(ii) Let $e = \infty$ and choose a partition λ. Then the partition $\gamma_\infty(\lambda)$ is defined by

$$\gamma_\infty(\lambda) = \lambda. \tag{1.15}$$

$\gamma_\infty(\lambda)$ is called the ∞-core of λ. Furthermore, the integer $g_\infty(\lambda)$ is defined by

$$g_\infty(\lambda) = 0.$$

$g_\infty(\lambda)$ is called the ∞-weight of λ. (1.15) defines the map

$$\gamma_\infty : \bigcup_{i \in \mathbb{N}_0} \Pi_i \to \Gamma_\infty, \quad \lambda \mapsto \gamma_\infty(\lambda).$$

Definition 1.1.39 *Let $e \in \mathbb{N} \cup \{\infty\}$ and $k \in \mathbb{N}_0$.*

(i) The set of the e-cores of all partitions of k is denoted by

$$\Gamma_e(k) = \gamma_e(\Pi_k).$$

(ii) The set of all partitions of k having a given e-core $\mu \in \Gamma_e(k)$ is denoted by

$$\Pi_k^{\mu,e} = \gamma_e^{-1}(\{\mu\}) \cap \Pi_k.$$

(iii) *Let $\tilde{e} \in \mathbb{N} \cup \{\infty\}$. Then the set of all $\tilde{e}$-regular partitions of k having a given e-core $\nu \in \Gamma_e(k)$ is denoted by*

$$\Pi_{k,\tilde{e}}^{\nu,e} = \gamma_e^{-1}(\{\nu\}) \cap \Pi_{k,\tilde{e}}.$$

Remark 1.1.40 (i) *With the notations from Definition 1.1.38.(i), $g_e(\lambda)$ is the number of e-rim hooks which have to be removed from λ to obtain the core $\gamma_e(\lambda)$.*

(ii) *If, with the notations from Definition 1.1.39,*

$$\tilde{e} \in \{k+1, k+2, \ldots\} \cup \{\infty\}$$

holds then, according to Remark 1.1.3, we have

$$\Pi_{k,\tilde{e}}^{\nu,e} = \Pi_k^{\nu,e}.$$

For an $e \in \mathbb{N}$, the explicit determination of e-cores of given partitions can be executed easily by means of e-abaci.

Lemma 1.1.41 *Let $e \in \mathbb{N}$ and $\lambda \vdash m$. Then an e-abacus for $\gamma_e(\lambda)$ is obtained from an e-abacus for λ by moving all beads in the e-abacus for λ along their respective runners — that is, within a column — as far down as possible.*

Proof. This follows from Definition 1.1.38.(i), Lemma 1.1.37, and Lemma 1.1.31. ■

Next, some relations between partitions and their respective e-cores for different values of the parameter $e \in \mathbb{N}$ are described.

Lemma 1.1.42 *Let $e, \tilde{e} \in \mathbb{N}$ with $\tilde{e} \mid e$. Furthermore, let $\lambda \vdash m$ such that an e-rim hook can be removed from λ. Then this effect can be achieved through successive removal of several $\tilde{e}$-rim hooks.*

Proof. This can be seen by considering an e-abacus for λ and an $\tilde{e}$-abacus for lambda with the same number of beads, and by using the fact that $\tilde{e} \mid e$. ■

Lemma 1.1.43 *Let $e, \tilde{e} \in \mathbb{N}$ with $\tilde{e} \mid e$. Furthermore, let $\lambda \vdash m$ and $\mu = \gamma_e(\lambda)$. Then we have*

$$\gamma_{\tilde{e}}(\lambda) = \gamma_{\tilde{e}}(\mu).$$

Proof. This follows from Lemma 1.1.42. ■

Lemma 1.1.44 *Let $e, \tilde{e} \in \mathbb{N}$ with $\tilde{e} \mid e$. Furthermore, let $\nu \in \Gamma_{\tilde{e}}(m)$. Then we have* *(i)* $\Pi_m^{\nu,\tilde{e}} = \bigcup_{\substack{\mu\in\Gamma_e(m)\\ \gamma_{\tilde{e}}(\mu)=\nu}} \Pi_m^{\mu,e}$,

(ii) $\Pi_{m,e}^{\nu,\tilde{e}} = \bigcup_{\substack{\mu\in\Gamma_e(m)\\ \gamma_{\tilde{e}}(\mu)=\nu}} \Pi_{m,e}^{\mu,e}$.

Proof. This follows from Lemma 1.1.43 and Definition 1.1.39. ■

From now on, the number $m \in \mathbb{N}_0$ is no more required. In the following, the variable m will be used for arbitrary purposes. The next definition makes use of Definition 1.1.6.

Definition 1.1.45 *Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash n$.*

(i) A tableau $\mathbf{t}$ of λ is a bijection

$$\mathbf{t} : [\lambda] \to \{1, \ldots, n\}.$$

$\mathbf{t}$ also is called a λ-tableau or just a tableau. For a lattice point $(i,j) \in [\lambda]$, its image $(i,j)\mathbf{t}$ is called the entry at position (i,j) in the tableau $\mathbf{t}$ or just the (i,j)-entry in $\mathbf{t}$. Here, the map is written to the right of its argument. For $k \in \mathbb{N}$, the k-th row of $\mathbf{t}$ is defined as the restriction of $\mathbf{t}$ on the k-th row of $[\lambda]$, the k-th column of $\mathbf{t}$ is defined as the restriction of $\mathbf{t}$ on the k-th column of $[\lambda]$.

(ii) For a λ-tableau $\mathbf{t}$ and a $k \in \{1, \ldots, n\}$, let

$$(i,j) = (k)\mathbf{t}^{-1}.$$

Then the row number of k in $\mathbf{t}$ is defined by

$$(k)\zeta_{\mathbf{t}} = i.$$

Furthermore, the column number of k in $\mathbf{t}$ is defined by

$$(k)\sigma_{\mathbf{t}} = j.$$

(iii) A λ-tableau $\mathbf{t}$ is called row standard if, in every row of $\mathbf{t}$, the entries are arranged from left to right in ascending order, or equivalently, if

$$\forall i \in \mathbb{N}\ \forall j \in \{2, \ldots, \lambda_i\} : (i, j-1)\mathbf{t} < (i,j)\mathbf{t}$$

holds.

A λ-tableau $\mathbf{t}$ is called column standard if, in every column of $\mathbf{t}$, the entries are arranged top down in ascending order, or equivalently, if, for any column of $[\lambda]$ consisting of, say, the lattice points

$$(i_1, j), \ldots, (i_k, j)$$

with $j \in \mathbb{N}$, $k \in \mathbb{N}_0$ and indices $i_1, \ldots, i_k \in \mathbb{N}$ satisfying

$$i_1 < \cdots < i_k,$$

the relation

$$(i_1, j)\mathbf{t} < \cdots < (i_k, j)\mathbf{t}$$

holds.

A λ-tableau $\mathbf{t}$ is called standard if $\mathbf{t}$ is both row standard and column standard.

The next definition makes use of Definition 1.1.1.(iii).

Definition 1.1.46 *(i) Let $\lambda \vDash n$. Then the set of all λ-tableaux is denoted by*

$$\mathcal{T}^\lambda = \{\mathbf{t} : [\lambda] \to \{1, \ldots, n\} \mid \mathbf{t} \textit{ bijective}\}.$$

Furthermore, the set of all row standard λ-tableaux is denoted by

$$\mathcal{T}^\lambda_{\text{row std}} = \left\{\mathbf{t} \in \mathcal{T}^\lambda \,\middle|\, \mathbf{t} \textit{ row standard}\right\}.$$

Finally, the set of all standard λ-tableaux is denoted by

$$\mathcal{T}^\lambda_{\text{std}} = \left\{\mathbf{t} \in \mathcal{T}^\lambda \,\middle|\, \mathbf{t} \textit{ standard}\right\}.$$

(ii) The set of all tableaux of compositions of n is denoted by

$$\mathcal{T}^{\Xi_n} = \bigcup_{\lambda \in \Xi_n} \mathcal{T}^\lambda.$$

The set of all row standard tableaux of compositions of n is denoted by

$$\mathcal{T}^{\Xi_n}_{\text{row std}} = \bigcup_{\lambda \in \Xi_n} \mathcal{T}^\lambda_{\text{row std}}.$$

A λ-tableau $\mathbf{t}$ with $\lambda \vDash n$ is represented by writing for each lattice point $(i, j) \in [\lambda]$ the number $(i, j)\mathbf{t}$ in the corresponding square in the representation of $[\lambda]$. In the following picture, four tableaux $\mathbf{t}_1, \ldots, \mathbf{t}_4$ are represented.

$$
\begin{array}{ccc}
\begin{array}{|c|c|c|}
\hline
8 & 7 & 6 \\
\hline
\multicolumn{3}{c}{} \\
\hline
5 & 3 & \multicolumn{1}{|c}{} \\
\hline
4 & 2 & 1 \\
\hline
\end{array}
& \qquad &
\begin{array}{|c|c|c|c|}
\hline
5 & 7 & 8 & 9 \\
\hline
4 & 6 & \multicolumn{2}{|c}{} \\
\cline{1-2}
3 & \multicolumn{3}{|c}{} \\
\cline{1-1}
2 & \multicolumn{3}{|c}{} \\
\cline{1-1}
1 & \multicolumn{3}{|c}{} \\
\cline{1-1}
\end{array}
\\
\mathbf{t}_1 & & \mathbf{t}_2
\\[2em]
\begin{array}{|c|c|c|c|}
\hline
9 & 5 & 2 & \multicolumn{1}{|c}{} \\
\cline{1-3}
10 & 6 & 3 & \multicolumn{1}{|c}{} \\
\hline
11 & 7 & 4 & 1 \\
\hline
12 & 8 & \multicolumn{2}{|c}{} \\
\cline{1-2}
\end{array}
& \qquad &
\begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & 6 & 7 & 10 \\
\hline
3 & 5 & 8 & \multicolumn{2}{|c}{} \\
\cline{1-3}
4 & 9 & \multicolumn{3}{|c}{} \\
\cline{1-2}
\end{array}
\\
\mathbf{t}_3 & & \mathbf{t}_4
\end{array}
\tag{1.16}
$$

$\mathbf{t}_1$ is a non-row standard non-column standard $(3, 0, 2, 3)$-tableau, $\mathbf{t}_2$ is a row standard non-column standard $(4, 2, 1^3)$-tableau, $\mathbf{t}_3$ is a non-row standard column standard $(3^2, 4, 2)$-tableau, and $\mathbf{t}_4$ is a standard $(5, 3, 2)$-tableau. The rows and columns of these arrangements correspond exactly to the rows and columns of the tableaux $\mathbf{t}_1, \ldots, \mathbf{t}_4$ as defined in Definition 1.1.45.(i). The entry 7, for example, occurs in $\mathbf{t}_1$ and $\mathbf{t}_2$ at position $(1, 2)$, in $\mathbf{t}_3$ at position $(3, 2)$, and in $\mathbf{t}_4$ at position $(1, 4)$. Thus, the row numbers and column numbers of the entry 7 in $\mathbf{t}_1, \ldots, \mathbf{t}_4$ are

$$
\begin{aligned}
(7)\zeta_{\mathbf{t}_1} &= 1, \quad (7)\sigma_{\mathbf{t}_1} = 2, \\
(7)\zeta_{\mathbf{t}_2} &= 1, \quad (7)\sigma_{\mathbf{t}_2} = 2, \\
(7)\zeta_{\mathbf{t}_3} &= 3, \quad (7)\sigma_{\mathbf{t}_3} = 2, \\
(7)\zeta_{\mathbf{t}_4} &= 1, \quad (7)\sigma_{\mathbf{t}_4} = 4.
\end{aligned}
$$

Definition 1.1.47 *Let $\lambda \vDash n$. Then the map*

$$\mathcal{T}^\lambda \times \mathfrak{S}_n \to \mathcal{T}^\lambda, \quad (\mathbf{t}, w) \mapsto \mathbf{t}w,$$

where $\mathbf{t}w$ is the concatenation of the bijections $\mathbf{t} : [\lambda] \to \{1, \ldots, n\}$ and $w : \{1, \ldots, n\} \to \{1, \ldots, n\}$, is an operation of $\mathfrak{S}_n$ on $\mathcal{T}^\lambda$.

For a λ-tableau $\mathbf{t}$ with $\lambda \vDash n$ and a $w \in \mathfrak{S}_n$, the representation of the tableau $\mathbf{t}w$ is obtained from the representation of the tableau $\mathbf{t}$ through replacing every entry of the latter by its image under w. The following picture shows on the left hand side a $(2,3)$-tableau $\mathbf{t}$ and on the right hand side its image $\mathbf{t}w$ under $w = (1,2)(4,5) \in \mathfrak{S}_5$.

$$\begin{array}{cc} \begin{array}{|c|c|c} \hline 5 & 2 & \multicolumn{1}{|c}{} \\ \hline 1 & 3 & \multicolumn{1}{c|}{4} \\ \hline \end{array} & \begin{array}{|c|c|c} \hline 4 & 1 & \multicolumn{1}{|c}{} \\ \hline 2 & 3 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \\ \\ \mathbf{t} & \mathbf{t}w \end{array} \tag{1.17}$$

We say that a $w \in \mathfrak{S}_n$, when operating on a λ-tableau $\mathbf{t}$ with $\lambda \vDash n$, moves an entry $j \in \{1, \ldots, n\}$ in $\mathbf{t}$ downwards if the position of the entry j in the representation of $\mathbf{t}$ is higher than the position of the entry j in the representation of $\mathbf{t}w$. More formally, this means

$$(j)\zeta_{\mathbf{t}} < (j)\zeta_{\mathbf{t}w}.$$

In the example (1.17) above with $n = 5$, w moves the entry 2 in $\mathbf{t}$ downwards. Similarly, we say that a $w \in \mathfrak{S}_n$, when operating on a λ-tableau $\mathbf{t}$ with $\lambda \vDash n$, moves an entry $j \in \{1, \ldots, n\}$ in $\mathbf{t}$ upwards if the position of the entry j in the representation of $\mathbf{t}$ is lower than the position of the entry j in the representation of $\mathbf{t}w$. More formally, this means

$$(j)\zeta_{\mathbf{t}} > (j)\zeta_{\mathbf{t}w}.$$

In the example (1.17) above with $n = 5$, w moves the entry 4 in $\mathbf{t}$ upwards.

Now, the transposition of partitions and their diagrams will be generalized to the tableaux constructed from them. According to Lemma 1.1.10, the next definition is meaningful.

Definition 1.1.48 *Let $\lambda \vdash n$ and choose a λ-tableau $\mathbf{t} : [\lambda] \to \{1, \ldots, n\}$. Then the tableau $\mathbf{t}'$ transposed to $\mathbf{t}$ is defined by the following concatenation of bijections*

$$\mathbf{t}' : [\lambda'] \xrightarrow{(\cdot)'} [\lambda] \xrightarrow{\mathbf{t}} \{1, \ldots, n\}\,.$$

Lemma 1.1.49 *Let $\lambda \vdash n$, and choose a λ-tableau $\mathbf{t}$ and a $w \in \mathfrak{S}_n$. Then the following statements hold.*

(i) $\mathbf{t}'' = \mathbf{t}$.

(ii) $\forall j \in \{1, \ldots, n\} : ((j)\zeta_{\mathbf{t}} = (j)\sigma_{\mathbf{t}'}) \wedge ((j)\sigma_{\mathbf{t}} = (j)\zeta_{\mathbf{t}'})$.

(iii) For each $j \in \mathbb{N}$, transposition maps the entries in the j-th row of $\mathbf{t}$ from left to right on the entries in the j-th column of $\mathbf{t}'$ from top to bottom.

(iv) $\mathbf{t}$ is row standard $\Leftrightarrow$ $\mathbf{t}'$ is column standard.

(v) $\mathbf{t}$ *is standard* $\Leftrightarrow$ $\mathbf{t}'$ *is standard.*

(vi) $(\mathbf{t}w)' = (\mathbf{t}')\, w$.

Proof. (i) This can be obtained from Definition 1.1.48 and Lemma 1.1.10.(i).

(ii) This follows from Definition 1.1.9, Definition 1.1.11, Definition 1.1.45.(ii), and Definition 1.1.48.

(iii) This follows easily from Definition 1.1.45.(i), Definition 1.1.6.(ii), Definition 1.1.6.(iii), Definition 1.1.48, Remark 1.1.12, and statement (ii).

(iv) This is a consequence of Definition 1.1.45.(iii) and statement (iii).

(v) This follows from Definition 1.1.45.(iii) and statements (iv) and (i).

(vi) This is an easy consequence of Definition 1.1.48 and Definition 1.1.47. ■

Definition 1.1.50 *Let* $\mathbf{t} \in \mathcal{T}^{\Xi_n}$.

(i) The row stabilizer $\mathfrak{R}_{\mathbf{t}}$ *of* $\mathbf{t}$ *is defined as*

$$\mathfrak{R}_{\mathbf{t}} = \{w \in \mathfrak{S}_n \mid \forall j \in \{1, \ldots, n\} : (j)\zeta_{\mathbf{t}} = (j)\zeta_{\mathbf{t}w}\}.$$

(ii) The column stabilizer $\mathfrak{C}_{\mathbf{t}}$ *of* $\mathbf{t}$ *is defined as*

$$\mathfrak{C}_{\mathbf{t}} = \{w \in \mathfrak{S}_n \mid \forall j \in \{1, \ldots, n\} : (j)\sigma_{\mathbf{t}} = (j)\sigma_{\mathbf{t}w}\}.$$

Remark 1.1.51 *Definition 1.1.45 provides the following less formal description of the row stabilizer and the column stabilizer of a tableau* $\mathbf{t} \in \mathcal{T}^{\Xi_n}$.

(i) $\mathfrak{R}_{\mathbf{t}} = \left\{ w \in \mathfrak{S}_n \;\middle|\; \begin{array}{c} \textit{in every row of } \mathbf{t}, \\ w \textit{ permutes the entries amongst themselves} \end{array} \right\}.$

(ii) $\mathfrak{C}_{\mathbf{t}} = \left\{ w \in \mathfrak{S}_n \;\middle|\; \begin{array}{c} \textit{in every column of } \mathbf{t}, \\ w \textit{ permutes the entries amongst themselves} \end{array} \right\}.$

For every tableau $\mathbf{t} \in \mathcal{T}^{\Xi_n}$, $\mathfrak{R}_{\mathbf{t}}$ and $\mathfrak{C}_{\mathbf{t}}$ are subgroups of $\mathfrak{S}_n$. We have, for example, for the tableaux $\mathbf{r}$ and $\mathbf{s}$ as shown in the following picture

$$\begin{aligned} \mathfrak{R}_{\mathbf{r}} &= \mathfrak{S}_{\{2,6,7\}} \times \mathfrak{S}_{\{5,8\}} \times \mathfrak{S}_{\{1,3,4,9\}}, \\ \mathfrak{C}_{\mathbf{r}} &= \mathfrak{S}_{\{7,8,9\}} \times \mathfrak{S}_{\{4,5,6\}} \times \mathfrak{S}_{\{2,3\}} \end{aligned}$$

and

$$\begin{aligned} \mathfrak{R}_{\mathbf{s}} &= \mathfrak{S}_{\{4,5,9,10\}} \times \mathfrak{S}_{\{3,6,8\}} \times \mathfrak{S}_{\{2,7\}}, \\ \mathfrak{C}_{\mathbf{s}} &= \mathfrak{S}_{\{1,7,8,10\}} \times \mathfrak{S}_{\{2,6,9\}} \times \mathfrak{S}_{\{3,5\}}, \end{aligned}$$

where $\mathfrak{R}_{\mathbf{r}}$ and $\mathfrak{C}_{\mathbf{r}}$ are subgroups of $\mathfrak{S}_9$, and $\mathfrak{R}_{\mathbf{s}}$ and $\mathfrak{C}_{\mathbf{s}}$ are subgroups of $\mathfrak{S}_{10}$.

$$
\begin{array}{|c|c|c|c|}
\hline
7 & 6 & 2 & \\
\hline
8 & 5 & & \\
\hline
 & & & \\
\hline
9 & 4 & 3 & 1 \\
\hline
\end{array}
\qquad\qquad
\begin{array}{|c|c|c|c|}
\hline
10 & 9 & 5 & 4 \\
\hline
8 & 6 & 3 & \\
\hline
7 & 2 & & \\
\hline
1 & & & \\
\hline
\end{array}
$$

$$\mathbf{r} \qquad\qquad\qquad \mathbf{s}$$

Lemma 1.1.52 *Let $\lambda \vdash n$ and $t \in \mathcal{T}^\lambda$. Then*

$$\mathfrak{C}_{\mathbf{t}} = \mathfrak{R}_{\mathbf{t}'}.$$

Proof. This follows easily from Definition 1.1.50, Lemma 1.1.49.(ii), and Lemma 1.1.49.(vi). ■

Definition 1.1.53 *Let $\lambda \vDash n$. Then the lattice points in $[\lambda]$ can be ordered by rows from top to bottom and within the rows from left to right. This means that for $(i,j), (\tilde{i},\tilde{j}) \in [\lambda]$ we have*

$$(i,j) < (\tilde{i},\tilde{j}) \Leftrightarrow \left(i < \tilde{i}\right) \vee \left(\left(i = \tilde{i}\right) \wedge \left(j < \tilde{j}\right)\right).$$

Then $\mathbf{t}^\lambda \in \mathcal{T}^\lambda$ is defined as the order preserving map from the set $[\lambda]$ ordered by $<$ to the set $\{1,\ldots,n\}$ arranged in its natural ascending order.

In the following picture, the tableau on the left hand side is $\mathbf{t}^\lambda$ with $\lambda = (5,4,3^2,1) \vdash 16$, the tableau on the right hand side is $\mathbf{t}^\mu$ with $\mu = (4,3,0,2^2) \vDash 11$.

$$
\begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & 5 \\
\hline
6 & 7 & 8 & 9 & \\
\hline
10 & 11 & 12 & & \\
\hline
13 & 14 & 15 & & \\
\hline
16 & & & & \\
\hline
\end{array}
\qquad\qquad
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & 4 \\
\hline
5 & 6 & 7 & \\
\hline
 & & & \\
\hline
8 & 9 & & \\
\hline
10 & 11 & & \\
\hline
\end{array}
$$

Remark 1.1.54 *Obviously, for a $\lambda \vDash n$, the tableau $\mathbf{t}^\lambda$ from Definition 1.1.53 is row standard (see Definition 1.1.45.(iii)). It also is easy to see that $\mathbf{t}^\lambda$ is column standard and thus standard.*

Definition 1.1.55 *Let $\lambda \vDash n$. Then the row stabilizer $\mathfrak{R}_{\mathbf{t}^\lambda} \subseteq \mathfrak{S}_n$ of $\mathbf{t}^\lambda$ is called the Young subgroup of $\mathfrak{S}_n$ associated with λ and is denoted by $\mathfrak{S}_\lambda$.*

For every $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash n$, we have

$$\mathfrak{S}_\lambda = \times_{\substack{i\in\mathbb{N} \\ \lambda_i>0}} \mathfrak{S}_{\left\{\lambda^+_{i-1}+1,\ldots,\lambda^+_i\right\}} \subseteq \mathfrak{S}_n. \tag{1.18}$$

Lemma 1.1.56 *Let $\lambda \vDash n$. Then the set*

$$\mathcal{D}_\lambda = \left\{ w \in \mathfrak{S}_n \,\middle|\, \mathbf{t}^\lambda w \text{ is row standard} \right\}$$

has the following properties.

(i) *$\mathcal{D}_\lambda$ is a system of representatives for the right cosets of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$. This means that*

$$\mathfrak{S}_n = \bigcup_{d \in \mathcal{D}_\lambda} \mathfrak{S}_\lambda d,$$

where the union is disjoint.

(ii) *For every $w \in \mathfrak{S}_\lambda$ and every $d \in \mathcal{D}_\lambda$, we have*

$$\ell(wd) = \ell(w) + \ell(d).$$

(iii) *Every $d \in \mathcal{D}_\lambda$ is the unique shortest element in the coset $\mathfrak{S}_\lambda d$.*

Proof. See [DJ1, Lemma 1.1]. ■

Corollary 1.1.57 *Let $\lambda \vDash n$ and let $\mathcal{D}_\lambda$ as in Lemma 1.1.56. Then the set $\mathcal{D}_\lambda^{-1}$ has the following properties.*

(i) *$\mathcal{D}_\lambda^{-1}$ is a system of representatives for the left cosets of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$. This means that*

$$\mathfrak{S}_n = \bigcup_{f \in \mathcal{D}_\lambda^{-1}} f\mathfrak{S}_\lambda,$$

where the union is disjoint.

(ii) *For every $w \in \mathfrak{S}_\lambda$ and every $f \in \mathcal{D}_\lambda^{-1}$, we have*

$$\ell(fw) = \ell(f) + \ell(w).$$

Proof. This follows from Lemma 1.1.56 and (1.9) on page 3. ■

Lemma 1.1.56 shows that the following definition is meaningful.

Definition 1.1.58 *Let $\lambda \vDash n$.*

(i) *The set*

$$\mathcal{D}_\lambda = \left\{ w \in \mathfrak{S}_n \,\middle|\, \mathbf{t}^\lambda w \text{ is row standard} \right\}$$

is called the set of the shortest representatives of the right cosets of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$, or, for short, the set of the shortest representatives associated with λ.

(ii) *To an arbitrary $w \in \mathfrak{S}_n$, we can assign the uniquely determined representative $[w]^\lambda \in \mathcal{D}_\lambda$ satisfying*
$$\mathfrak{S}_\lambda w = \mathfrak{S}_\lambda [w]^\lambda.$$
We call this representative the shortest representative of w associated with λ.

Lemma 1.1.59 *Let $\lambda \models n$.*

(i) *The map*
$$\mathcal{D}_\lambda \to \mathcal{T}^\lambda_{\mathrm{row\,std}}, \quad d \mapsto \mathbf{t}^\lambda d$$
is a bijection.

(ii) *For $x, y \in \mathfrak{S}_n$, we have*
$$\left[[x]^\lambda y\right]^\lambda = [xy]^\lambda.$$

Proof. (i) This is an immediate consequence of Definition 1.1.45.(i) and Definition 1.1.47.

(ii) According to Definition 1.1.58.(ii), we have $\mathfrak{S}_\lambda [x]^\lambda = \mathfrak{S}_\lambda x$. This implies
$$\mathfrak{S}_\lambda [x]^\lambda y = \mathfrak{S}_\lambda xy.$$
In turn, this shows, again according to Definition 1.1.58.(ii), $\left[[x]^\lambda y\right]^\lambda = [xy]^\lambda$, as desired. ■

Next, some useful properties of standard tableaux and associated permutations are described.

Definition 1.1.60 *Let $\lambda \models n$. Then the set*
$$\mathcal{E}_\lambda = \left\{ w \in \mathfrak{S}_n \,\middle|\, \mathbf{t}^\lambda w \text{ is standard} \right\}$$
is called the set of the standard representatives of the right cosets of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$, or, for short, the set of the standard representatives associated with λ.

Remark 1.1.61 *According to Definition 1.1.45.(iii), we have for $\lambda \models n$*
$$\mathcal{E}_\lambda \subseteq \mathcal{D}_\lambda.$$

Lemma 1.1.62 *Let $\lambda \models n$, $\mathbf{t} \in \mathcal{T}^\lambda_{\mathrm{std}}$, and $s = (j, j+1) \in \mathfrak{B}_n$ with $j \in \{1, \ldots, n-1\}$. Then*
$$\mathbf{t}s \in \mathcal{T}^\lambda_{\mathrm{std}} \Leftrightarrow ((j)\zeta_{\mathbf{t}} \neq (j+1)\zeta_{\mathbf{t}}) \wedge ((j)\sigma_{\mathbf{t}} \neq (j+1)\sigma_{\mathbf{t}}).$$

Proof. This is immediate from Definition 1.1.45 and Definition 1.1.47. ■

Lemma 1.1.63 *Let $\lambda \vDash n$ and $f \in \mathcal{E}_\lambda \setminus \{1_{\mathfrak{S}_n}\}$. Then there is an $s \in \mathfrak{B}_n$ such that both $fs \in \mathcal{E}_\lambda$ and $\ell(fs) = \ell(f) - 1$ hold.*

Proof. This follows from [DJ1, Lemma 1.5]. ■

Definition 1.1.64 *The partition $\omega^{(n)} \in \Pi_n$ is defined by*

$$\omega^{(n)} = (1^n).$$

Lemma 1.1.65 *For the partition $\omega^{(n)}$ from Definition 1.1.64, the following statements hold.*

(i) The tableau $\mathbf{t}^{\omega^{(n)}}$ looks as follows.

$$\begin{array}{c} \boxed{1} \\ \vdots \\ \boxed{n} \end{array}$$

(ii) We have $\mathfrak{S}_{\omega^{(n)}} = \{1_{\mathfrak{S}_n}\}$.

(iii) We have $\mathcal{D}_{\omega^{(n)}} = \mathfrak{S}_n$.

Proof. (i) This is immediate from Definition 1.1.53.

(ii) This follows from statement (i) and Definition 1.1.55.

(iii) This can be obtained from statement (ii) and Lemma 1.1.56. ■

Definition 1.1.66 *Let $\lambda \vDash n$. Then the lattice points in $[\lambda]$ can be ordered by columns from left to right and within the columns from top to bottom. This means that for $(i,j), (\tilde{i},\tilde{j}) \in [\lambda]$ we have*

$$(i,j) < (\tilde{i},\tilde{j}) \Leftrightarrow \left(j < \tilde{j}\right) \vee \left(\left(j = \tilde{j}\right) \wedge \left(i < \tilde{i}\right)\right).$$

Then $\mathbf{t}_\lambda \in \mathcal{T}^\lambda$ is defined as the order preserving map from the set $[\lambda]$ ordered by $<$ to the set $\{1, \ldots, n\}$ arranged in its natural ascending order.

In the following picture, the tableau on the left hand side is $\mathbf{t}_\lambda$ with $\lambda = (5,4,3^2,1) \vdash 16$, the tableau on the right hand side is $\mathbf{t}_\mu$ with $\mu = (4,3,0,2^2) \vDash 11$.

1	6	10	14	16
2	7	11	15	
3	8	12		
4	9	13		
5				

1	5	9	11
2	6	10	

3	7
4	8

Definition 1.1.67 *Let $\lambda \vDash n$. Then the permutation w_λ is defined by*

$$w_\lambda = \left(\mathbf{t}^\lambda\right)^{-1} \mathbf{t}_\lambda \qquad \text{or equivalently} \qquad \mathbf{t}^\lambda w_\lambda = \mathbf{t}_\lambda.$$

Here, maps are written to the right of their respective arguments.

Lemma 1.1.68 *Let $\lambda \vDash n$. Then we have*

(i) $\mathbf{t}_\lambda \in \mathcal{T}^\lambda_{\mathrm{std}} \subseteq \mathcal{T}^\lambda_{\mathrm{row\,std}}$,

(ii) $w_\lambda \in \mathcal{D}_\lambda$.

Proof. (i) This is a consequence of Definition 1.1.66 and Definition 1.1.45.(iii).

(ii) This follows from Definition 1.1.58.(i), Definition 1.1.67, and statement (i).
■

Lemma 1.1.69 *Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$. Then we have*

(i) $(\mathbf{t}_\lambda)' = \mathbf{t}^{(\lambda')}$,

(ii) $w_\lambda^{-1} = w_{\lambda'}$,

(iii) $w_\lambda \in \mathcal{D}_{\lambda'}^{-1}$,

(iv) $w_\lambda \mathfrak{S}_{\lambda'} \subseteq \mathcal{D}_\lambda$.

Proof. (i) This is immediate from Definition 1.1.66, Definition 1.1.53, Definition 1.1.48, and Definition 1.1.9.

(ii) This follows from the calculation

$$\mathbf{t}^{(\lambda')} w_\lambda^{-1} = (\mathbf{t}_\lambda)' \, w_\lambda^{-1} = \left(\mathbf{t}_\lambda w_\lambda^{-1}\right)' = \left(\mathbf{t}^\lambda\right)' = \mathbf{t}_{\lambda'}.$$

(iii) This follows from the identity

$$\mathbf{t}^{(\lambda')} w_\lambda^{-1} = \mathbf{t}^{(\lambda')} w_{\lambda'} = \mathbf{t}_{\lambda'},$$

Lemma 1.1.68.(i), and Definition 1.1.58.(i).

(iv) This follows easily from the fact that for any two indices $i, j \in \mathbb{N}$ with $i < j$, any entry in the i-th row of $\mathbf{t}_\lambda$ is smaller than any entry in the j-th row of $\mathbf{t}_\lambda$. ■

According to Lemma 1.1.56, for a given $\lambda \vDash n$, the right cosets of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$ are parameterized by row standard λ-tableaux. However, this index set $\mathcal{T}^\lambda_{\text{row std}}$ is not closed under the operation of $\mathfrak{S}_n$ on $\mathcal{T}^\lambda$. Thus, the operation of $\mathfrak{S}_n$ on $\mathcal{T}^\lambda$ is not compatible with the operation of $\mathfrak{S}_n$ on the right cosets of $\mathfrak{S}_\lambda$. In the following, a different representation of row standard λ-tableaux will be constructed such that the obtained set, corresponding to $\mathcal{T}^\lambda_{\text{row std}}$, has a natural $\mathfrak{S}_n$-operation compatible with the $\mathfrak{S}_n$-operation on right cosets of $\mathfrak{S}_\lambda$.

Definition 1.1.70 *Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash n$.*

(i) A sequence

$$\zeta = ((1)\zeta, \ldots, (n)\zeta) \in \mathbb{N}^n$$

satisfying

$$\forall j \in \mathbb{N} : |\{k \in \{1, \ldots, n\} \mid (k)\zeta = j\}| = \lambda_j$$

is called a λ-row number list.

(ii) A λ-row number list ζ can be written in the power notation

$$\zeta = (b_1^{e_1}, b_2^{e_2}, \ldots)$$

with $b_j \in \mathbb{N}$ and $e_j \in \mathbb{N}_0$ for $j \in \mathbb{N}$, a power $b_j^{e_j}$ denoting e_j successive entries of the sequence $\zeta \in \mathbb{N}^n$ with value b_j.

(iii) The set of all λ-row number lists is denoted by $\mathcal{Z}^\lambda$.

Remark 1.1.71 *Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash n$.*

(i) Choose a λ-row number list $\zeta \in \mathcal{Z}^\lambda$ and write

$$\zeta = (b_1^{e_1}, b_2^{e_2}, \ldots).$$

Then it's easy to see that

$$\forall k \in \mathbb{N} : \sum_{\substack{j \in \mathbb{N} \\ b_j = k}} e_j = \lambda_k. \tag{1.19}$$

This shows together with Definition 1.1.1.(i)

$$\sum_{j \in \mathbb{N}} e_j = n.$$

Conversely, every sequence $(b_1^{e_1}, b_2^{e_2}, \ldots)$ with the property (1.19) is a λ-row number list.

(ii) We have

$$\mathcal{Z}^\lambda \neq \varnothing,$$

since, according to (i), $\mathcal{Z}^\lambda$ contains for example the λ-row number list with the power notation $(1^{\lambda_1}, 2^{\lambda_2}, \ldots)$.

For a given row standard λ-tableau $\mathbf{t}$ with $\lambda \vDash n$, $((1)\zeta_\mathbf{t}, \ldots, (n)\zeta_\mathbf{t})$ (see Definition 1.1.45.(ii)) obviously is a λ-row number list. This motivates the name "row number list". Furthermore, it shows that the following definition is meaningful.

Definition 1.1.72 *Let $\lambda \vDash n$. Then the map*

$$\zeta[\lambda] : \mathcal{T}^\lambda_{\text{row std}} \to \mathcal{Z}^\lambda$$

is defined by

$$\mathbf{t} \mapsto (\mathbf{t})\zeta[\lambda] = \zeta_\mathbf{t} = ((1)\zeta_\mathbf{t}, \ldots, (n)\zeta_\mathbf{t}).$$

Conversely, for any given $\lambda \vDash n$, a row standard λ-tableau can be assigned to every λ-row number list.

Definition 1.1.73 *Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash n$ and $\zeta \in \mathcal{Z}^\lambda$. With this data, the λ-tableau $\mathbf{t}_\zeta$ is defined as follows.*

According to Definition 1.1.70.(i), for every $i \in \mathbb{N}$, certain numbers $k_1, \ldots, k_{\lambda_i} \in \{1, \ldots, n\}$ are uniquely defined by the conditions

$$k_1 < \cdots < k_{\lambda_i} \qquad \textit{and} \qquad (k_1)\zeta = \cdots = (k_{\lambda_i})\zeta = i.$$

With this, we put for every $j \in \{1, \ldots, \lambda_i\}$

$$(i, j)\mathbf{t}_\zeta = k_j.$$

Obviously, we have for every $\lambda \vDash n$ and every $\zeta \in \mathcal{Z}^\lambda$

$$\mathbf{t}_\zeta \in \mathcal{T}^\lambda_{\text{row std}}.$$

This shows that the following definition is meaningful.

Definition 1.1.74 *Let $\lambda \vDash n$. Then the map*

$$\mathbf{t}[\lambda] : \mathcal{Z}^\lambda \to \mathcal{T}^\lambda_{\text{row std}}$$

is defined by

$$\zeta \mapsto (\zeta)\mathbf{t}[\lambda] = \mathbf{t}_\zeta.$$

Lemma 1.1.75 *Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash n$. Then the maps $\zeta[\lambda]$ and $\mathbf{t}[\lambda]$ are bijections, and one is the inverse of the other.*

Proof. This can be obtained from Definition 1.1.72, Definition 1.1.74, and Definition 1.1.45.(ii). ■

For a $\lambda \vDash n$, a sequence $\zeta = ((1)\zeta, \ldots, (n)\zeta) \in \mathcal{Z}^\lambda$ can be considered as a map $\zeta : \{1, \ldots, n\} \to \mathbb{N}$, $j \mapsto (j)\zeta$. With this, the operation of $\mathfrak{S}_n$ on the arguments of that map naturally induces an operation of $\mathfrak{S}_n$ on $\mathcal{Z}^\lambda$.

Definition 1.1.76 *Let $\lambda \vDash n$. Then the map*

$$\mathcal{Z}^\lambda \times \mathfrak{S}_n \to \mathcal{Z}^\lambda, \quad (\zeta, w) \mapsto \zeta w$$

with

$$\zeta w = ((1w^{-1})\zeta, \ldots, (nw^{-1})\zeta)$$

for $\zeta = ((1)\zeta, \ldots, (n)\zeta)$ is an operation of $\mathfrak{S}_n$ on $\mathcal{Z}^\lambda$.

That, for a $\zeta \in \mathcal{Z}^\lambda$ with $\lambda \vDash n$ and a $w \in \mathfrak{S}_n$, we actually have $\zeta w \in \mathcal{Z}^\lambda$, follows easily from Definition 1.1.70.(i). Later on, we will need the following properties of this operation of $\mathfrak{S}_n$ on $\mathcal{Z}^\lambda$.

Lemma 1.1.77 *Let $\lambda \vDash n$, $s = (j, j+1) \in \mathfrak{B}_n$ with $j \in \{1, \ldots, n-1\}$, and $d \in \mathcal{D}_\lambda$. Then the following statements hold.*

(i) Let $(j)\zeta_{\mathbf{t}^\lambda d} < (j+1)\zeta_{\mathbf{t}^\lambda d}$. Then we have $\ell(ds) = \ell(d) + 1$, $ds \in \mathcal{D}_\lambda$, and $\zeta_{\mathbf{t}^\lambda d} s = \zeta_{\mathbf{t}^\lambda ds}$.

(ii) Let $(j)\zeta_{\mathbf{t}^\lambda d} = (j+1)\zeta_{\mathbf{t}^\lambda d}$. Then we have $\ell(ds) = \ell(d) + 1$ and $\zeta_{\mathbf{t}^\lambda d} s = \zeta_{\mathbf{t}^\lambda d}$.

(iii) Let $(j)\zeta_{\mathbf{t}^\lambda d} > (j+1)\zeta_{\mathbf{t}^\lambda d}$. Then we have $\ell(ds) = \ell(d) - 1$, $ds \in \mathcal{D}_\lambda$, and $\zeta_{\mathbf{t}^\lambda d} s = \zeta_{\mathbf{t}^\lambda ds}$.

Proof. This is an easy consequence of Definition 1.1.53, (1.9).(iii), Definition 1.1.76, and Definition 1.1.47. ■

Remark 1.1.78 *Lemma 1.1.65 shows that for any $w \in \mathfrak{S}_n = \mathcal{D}_{\omega^{(n)}}$ we have*

$$\zeta_{\mathbf{t}^{\omega^{(n)}} w} = (1w^{-1}, \ldots, nw^{-1}).$$

From this sequence, w can be reconstructed. Thus, every $w \in \mathfrak{S}_n$ can be identified with its row number list $\zeta_{\mathbf{t}^{\omega^{(n)}} w} \in \mathcal{Z}^{\omega^{(n)}}$. This sequence $\zeta_{\mathbf{t}^{\omega^{(n)}} w}$ is called the permutation list of w. The injective map

$$\mathfrak{S}_n \to \mathcal{Z}^{\omega^{(n)}}, \quad w \mapsto \zeta_{\mathbf{t}^{\omega^{(n)}} w}$$

is, because of $\left|\mathcal{Z}^{\omega^{(n)}}\right| = \left|\mathcal{T}^{\omega^{(n)}}_{\text{row std}}\right| = \left|\mathcal{D}_{\omega^{(n)}}\right| = |\mathfrak{S}_n|$*, actually bijective, and furthermore compatible with the operations from the right of* $\mathfrak{S}_n$ *on these sets — on* $\mathfrak{S}_n$ *through multiplication from the right and on* $\mathcal{Z}^{\omega^{(n)}}$ *as in Definition 1.1.76. This shows that this map is an isomorphism of right* $\mathfrak{S}_n$*-sets. Thus, given the permutation list of any* $w \in \mathfrak{S}_n$*, Lemma 1.1.77 can be used to determine the behavior of the length when multiplying* w *from the right with a simple reflection* $s \in \mathfrak{B}_n$ *and furthermore to construct the permutation list of* ws *from that of* w*.*

1.2 Hecke algebras of type A

Now, Hecke algebras of type A, as also considered in [DJ1], will be introduced. There, further references on the background of these algebras can be found. Another good reference is [HUM, Chapter 7]. There, Hecke algebras of arbitrary type are constructed in a very general way. For the following, fix an $n \in \mathbb{N}$.

Next, several notions connected to the underlying coefficient ring are introduced.

Definition 1.2.1 *Let* R *be an integral domain and* $q \in R$ *be a unit. Then the pair* (R, q) *is called a coefficient pair.*

In the following, R is always an arbitrary but fixed integral domain with the additive neutral element 0_R and the multiplicative neutral element 1_R. Furthermore, $q \in R$ is always a unit.

Definition 1.2.2 *(i) Let* $j \in \mathbb{Z}$*. Then the element* $[j]_q$ *of* R *is defined as*

$$[j]_q = \begin{cases} \sum_{i=0}^{j-1} q^i & \text{for } j > 0 \\ 0_R & \text{for } j = 0 \\ -\sum_{i=j}^{-1} q^i & \text{for } j < 0 \end{cases}.$$

$[j]_q \in R$ *is called a q-number.*

(ii) The value $e_R(q) \in \{2, 3, \dots\} \cup \{\infty\}$ *is defined as*

$$e_R(q) = \inf\left\{ j \in \mathbb{N} \,\middle|\, [j]_q = 0_R \right\}.$$

Here, we use $\inf \varnothing = \infty$*.* $e_R(q)$ *is called the q-characteristic of* R*.*

Now we introduce the Hecke algebras. The Hecke algebra

$$\mathcal{H} = \mathcal{H}_A = \mathcal{H}_n = \mathcal{H}_A^{(R,q)} = \mathcal{H}_n^{(R,q)} = \mathcal{H}_{A_{n-1}}^{(R,q)} = \mathcal{H}_{A_{n-1}}(R, q)$$

of type A, or, more precisely, of type A_{n-1}, over the coefficient pair (R,q) is defined as the free R-module with basis $\{T_w \mid w \in \mathfrak{S}_n\}$ on which the rules

$$T_{1_{\mathfrak{S}_n}} = 1_{\mathcal{H}_A^{(R,q)}}$$

and

$$T_w T_s = \begin{cases} T_{ws} & \text{if} \quad \ell(ws) = \ell(w) + 1 \\ qT_{ws} + (q-1)T_w & \text{if} \quad \ell(ws) = \ell(w) - 1 \end{cases} \tag{1.20}$$

for $w \in \mathfrak{S}_n$ and $s \in \mathfrak{B}_n$ induce an associative multiplication. Here, $1_{\mathcal{H}_A^{(R,q)}}$ denotes the multiplicative neutral element of the algebra $\mathcal{H}_{A_{n-1}}^{(R,q)}$. Furthermore, the additive neutral element of $\mathcal{H}_{A_{n-1}}^{(R,q)}$ is denoted by $0_{\mathcal{H}_A^{(R,q)}}$. The parameter n is called the degree of the Hecke algebra $\mathcal{H}_{A_{n-1}}^{(R,q)}$. From the rules (1.20), the following properties of the multiplication can be derived.

For a $w \in \mathfrak{S}_n$ with a reduced expression $w = v_1 \cdots v_{\ell(w)}$ with $v_j \in \mathfrak{B}_n$ for $j \in \{1, \ldots, \ell(w)\}$, we have $T_w = T_{v_1} \cdots T_{v_{\ell(w)}}$. (1.21)

For $u \in \mathfrak{S}_n$ and $w \in \mathfrak{S}_n$ with $\ell(uw) = \ell(u) + \ell(w)$, we have $T_{uw} = T_u T_w$. (1.22)

For every $w \in \mathfrak{S}_n$, T_w is invertible in $\mathcal{H}_{A_{n-1}}^{(R,q)}$. (1.23)

If we put $q = 1_R$ then both cases in the rule (1.20) produce the same result, and we get

$$\mathcal{H}_{A_{n-1}}^{(R,1_R)} = R\mathfrak{S}_n. \tag{1.24}$$

Thus, for arbitrary units $q \in R$, we can consider $\mathcal{H}_{A_{n-1}}^{(R,q)}$ as a deformation of the group algebra $R\mathfrak{S}_n$.

The notations introduced in the next definition are useful in later constructions. They also are used in [MUR] and elsewhere.

Definition 1.2.3 *Let $X \subseteq \mathfrak{S}_n$.*

(i) $\iota_{(R,q)}^{(n)}(X) \in \mathcal{H}_{A_{n-1}}^{(R,q)}$ *is defined as*

$$\iota_{(R,q)}^{(n)}(X) = \sum_{w \in X} T_w.$$

As abbreviations, we write

$$\iota_{(R,q)}^{(n)}(X) = \iota_{(R,q)}(X) = \iota^{(n)}(X) = \iota(X).$$

(ii) $\varepsilon^{(n)}_{(R,q)}(X) \in \mathcal{H}^{(R,q)}_{A_{n-1}}$ *is defined as*

$$\varepsilon^{(n)}_{(R,q)}(X) = \sum_{w\in X}(-q)^{-\ell(w)}T_w.$$

As abbreviations, we write

$$\varepsilon^{(n)}_{(R,q)}(X) = \varepsilon_{(R,q)}(X) = \varepsilon^{(n)}(X) = \varepsilon(X).$$

The following anti-involution on $\mathcal{H}^{(R,q)}_{A_{n-1}}$ generalizes the anti-involution on $R\mathfrak{S}_n$ induced by the inversion on $\mathfrak{S}_n$.

Definition 1.2.4 *The R-linear map*

$$* : \mathcal{H}^{(R,q)}_{A_{n-1}} \to \mathcal{H}^{(R,q)}_{A_{n-1}}$$

is defined by

$$T^*_w = T_{w^{-1}} \quad \textit{for} \quad w \in \mathfrak{S}_n$$

and R-linear extension.

Lemma 1.2.5 *The map $*$ from Definition 1.2.4 is an anti-involution on $\mathcal{H}^{(R,q)}_{A_{n-1}}$. This means that for $x, y \in \mathcal{H}^{(R,q)}_{A_{n-1}}$, we have*

$$(xy)^* = y^*x^* \qquad \textit{and} \qquad x^{**} = x.$$

Proof. See [MUR, Lemma 2.3]. ■

By using $*$, dual modules of $\mathcal{H}^{(R,q)}_{A_{n-1}}$-modules can be constructed as in the case of group algebras (see also [CR1, §10D]).

Definition 1.2.6 *Let M be a right $\mathcal{H}^{(R,q)}_{A_{n-1}}$-module. Then the dual $\mathcal{H}^{(R,q)}_{A_{n-1}}$-module M^* is defined as*

$$M^* = \mathrm{Hom}_R(M, R)$$

with the operation

$$f \cdot x = x^* f : M \to R, \quad m \mapsto (m \cdot x^*)f$$

for $f \in \mathrm{Hom}_R(M, R)$ and $x \in \mathcal{H}^{(R,q)}_{A_{n-1}}$. Here, maps are written to the right of their respective arguments.

Next, we show that Hecke algebras are stable when changing the coefficient ring. So, let $\xi : R \to \tilde{R}$ be a ring homomorphism from R to another integral domain $\tilde{R}$. Then $\xi(q) \in \tilde{R}$ is a unit. Furthermore, $\tilde{R}$ can be considered a left R-module with the operation $a \cdot x = \xi(a)x \in \tilde{R}$ for $x \in \tilde{R}$ and $a \in R$. With this, the functor $- \otimes_R \tilde{R}$ can be constructed.

Lemma 1.2.7 *We have* $\mathcal{H}_{A_{n-1}}^{(R,q)} \otimes_R \tilde{R} \simeq \mathcal{H}_{A_{n-1}}^{(\tilde{R},\xi(q))}$ *as* $\tilde{R}$*-algebras.*

Proof. This follows easily from the construction of the Hecke algebra of type A_{n-1}. ■

In the following, the $\tilde{R}$-algebras $\mathcal{H}_{A_{n-1}}^{(R,q)} \otimes_R \tilde{R}$ and $\mathcal{H}_{A_{n-1}}^{(\tilde{R},\xi(q))}$ will be identified by means of the preceding statement. Then we have a map

$$- \otimes_R \tilde{R} : \mathcal{H}_{A_{n-1}}^{(R,q)} \to \mathcal{H}_{A_{n-1}}^{(\tilde{R},\xi(q))}, \quad h \mapsto h \otimes_R 1_{\tilde{R}}. \tag{1.25}$$

This map is compatible with the multiplicative structures on $\mathcal{H}_{A_{n-1}}^{(R,q)}$ and $\mathcal{H}_{A_{n-1}}^{(\tilde{R},\xi(q))}$. To be more specific, for $x, y \in \mathcal{H}_{A_{n-1}}^{(R,q)}$, we have

$$(xy) \otimes_R 1_{\tilde{R}} = (x \otimes_R 1_{\tilde{R}})\,(y \otimes_R 1_{\tilde{R}})\,. \tag{1.26}$$

Now, the general behavior of modules of Hecke algebras when changing the coefficient ring is examined.

Lemma 1.2.8 *(i) Let* M *be a right* $\mathcal{H}_{A_{n-1}}^{(R,q)}$*-module with the structure map*

$$\varrho_M : M \otimes_R \mathcal{H}_{A_{n-1}}^{(R,q)} \to M, \quad x \otimes_R y \mapsto xy.$$

Then ϱ_M *induces on* $M \otimes_R \tilde{R}$ *in a natural way an* $\mathcal{H}_{A_{n-1}}^{(\tilde{R},\xi(q))}$*-module structure. In particular, for every* $x \in M$ *and every* $y \in \mathcal{H}_{A_{n-1}}^{(R,q)}$*, we have*

$$(xy) \otimes_R 1_{\tilde{R}} = (x \otimes_R 1_{\tilde{R}})\,(y \otimes_R 1_{\tilde{R}})\,. \tag{1.27}$$

(ii) Let M *and* N *be right* $\mathcal{H}_{A_{n-1}}^{(R,q)}$*-modules and let*

$$f : M \to N$$

be an $\mathcal{H}_{A_{n-1}}^{(R,q)}$*-module homomorphism. Then*

$$f \otimes_R \mathrm{id}_{\tilde{R}} : M \otimes_R \tilde{R} \to N \otimes_R \tilde{R}$$

is an $\mathcal{H}_{A_{n-1}}^{(\tilde{R},\xi(q))}$*-module homomorphism.*

(iii) Let M *and* N *be right* $\mathcal{H}_{A_{n-1}}^{(R,q)}$*-modules. Then we have*

$$\Big(M \otimes_R \tilde{R}\Big) \oplus \Big(N \otimes_R \tilde{R}\Big) \simeq \Big(M \oplus N\Big) \otimes_R \tilde{R}$$

as right $\mathcal{H}_{A_{n-1}}^{(\tilde{R},\xi(q))}$*-modules.*

Proof. All this is clear from general facts on rings and modules. ■

Remark 1.2.9 *In the following, there will occur certain ideals in Hecke algebras, homomorphisms between such ideals, and several other objects whose constructions are independent of the underlying coefficient pair. This means that these objects are defined over arbitrary coefficient pairs, and the construction of such an object over the coefficient pair (R, q) is mapped exactly on the construction of the corresponding object over the coefficient pair $(\tilde{R}, \xi(q))$ when changing the coefficient ring as described above by applying the functor $- \otimes_R \tilde{R}$ to all algebra elements and module elements occurring in the construction of the object over the coefficient pair (R, q). Objects with this property are called generic.*

For example, according to Lemma 1.2.7, the Hecke algebras of type A themselves are generic.

1.3 Irreducible representations of Hecke algebras of type A

In this section, the construction of the irreducible modules of Hecke algebras of type A over various coefficient fields and parameters is described. This can be generalized from the special case of the group algebra of a symmetric group (see, for example, [JAM1, Section 11]) to the more general case of a Hecke algebra over a field. In this process, the combinatorial objects and constructions known from group algebras of symmetric groups are for the most part preserved. All modules considered in the following are finitely generated right modules, furthermore we fix an $n \in \mathbb{N}$ and a coefficient pair (R, q) as described in Definition 1.2.1.

First, permutation modules on right cosets of Young subgroups are generalized. This can be done over the coefficient ring R, a field is not required.

Definition 1.3.1 *Let $\lambda \vDash n$.*

(i) $x_\lambda^{(R,q)} \in \mathcal{H}_{A_{n-1}}^{(R,q)}$ *is defined as*

$$x_\lambda^{(R,q)} = \iota_{(R,q)}^{(n)}(\mathfrak{S}_\lambda).$$

As an abbreviation, we write

$$x_\lambda^{(R,q)} = x_\lambda.$$

(ii) The right ideal $M_{(R,q)}^\lambda$ *in* $\mathcal{H}_{A_{n-1}}^{(R,q)}$ *is defined as*

$$M_{(R,q)}^\lambda = x_\lambda^{(R,q)} \mathcal{H}_{A_{n-1}}^{(R,q)}.$$

As an abbreviation, we write

$$M^\lambda_{(R,q)} = M^\lambda.$$

$M^\lambda_{(R,q)}$ *is called the permutation module of* $\mathcal{H}^{(R,q)}_{A_{n-1}}$ *associated to* λ.

The next statement shows the close relation between a permutation module M^λ with $\lambda \vDash n$ and the corresponding permutation module of $\mathfrak{S}_n$ on the right cosets of the Young subgroup $\mathfrak{S}_\lambda$.

Theorem 1.3.2 *Let* $\lambda \vDash n$. *Then* $M^\lambda_{(R,q)}$ *has the* R*-basis*

$$\left\{ x_\lambda^{(R,q)} T_d \,\middle|\, d \in \mathcal{D}_\lambda \right\} = \left\{ x_\lambda^{(R,q)} T_d \,\middle|\, d \in \mathfrak{S}_n \text{ such that } \mathbf{t}^\lambda d \text{ is row standard} \right\}.$$

Proof. See [DJ1, Lemma 3.2]. ■

Definition 1.3.3 *Let* $\lambda \vDash n$. *Then the* R*-basis*

$$\left\{ x_\lambda^{(R,q)} T_d \,\middle|\, d \in \mathcal{D}_\lambda \right\} = \left\{ x_\lambda^{(R,q)} T_d \,\middle|\, d \in \mathfrak{S}_n \text{ such that } \mathbf{t}^\lambda d \text{ is row standard} \right\}$$

of $M^\lambda_{(R,q)}$ *from Theorem 1.3.2 is called the row standard basis of* $M^\lambda_{(R,q)}$ *and denoted by*

$$\mathbf{B}^{M^\lambda}_{\text{row std}}(R,q) \qquad or \qquad \mathbf{B}^{M^\lambda}_{\text{row std}}.$$

The operation of $\mathcal{H}^{(R,q)}_{A_{n-1}}$ on the basis elements of $M^\lambda_{(R,q)}$ from Theorem 1.3.2 can be described by using row standard λ-tableaux.

Lemma 1.3.4 *Let* $\lambda \vDash n, d \in \mathcal{D}_\lambda$, *and* $s = (j, j+1) \in \mathfrak{B}_n$ *with* $j \in \{1, \ldots, n-1\}$. *Then we have*

$$x_\lambda^{(R,q)} T_d T_s = \begin{cases} x_\lambda^{(R,q)} T_{ds} & \text{for } (j)\zeta_{\mathbf{t}^\lambda d} < (j+1)\zeta_{\mathbf{t}^\lambda d} \\ q x_\lambda^{(R,q)} T_d & \text{for } (j)\zeta_{\mathbf{t}^\lambda d} = (j+1)\zeta_{\mathbf{t}^\lambda d} \\ q x_\lambda^{(R,q)} T_{ds} + (q-1) x_\lambda^{(R,q)} T_d & \text{for } (j)\zeta_{\mathbf{t}^\lambda d} > (j+1)\zeta_{\mathbf{t}^\lambda d} \end{cases}.$$

Proof. See [DJ1, Lemma 3.2]. ■

The following lemma also describes an aspect of the operation of $\mathcal{H}^{(R,q)}_{A_{n-1}}$ on $M^\lambda_{(R,q)}$.

Lemma 1.3.5 *Let* $\lambda \vDash n$ *and* $w \in \mathfrak{S}_\lambda$. *Then*

$$x_\lambda^{(R,q)} T_w = q^{\ell(w)} x_\lambda^{(R,q)}.$$

Proof. This can be seen by using (1.10) on page 3, (1.18) on page 25, (1.21) on page 34, Lemma 1.3.4, Definition 1.1.53, and Definition 1.1.55. ■

Next, we will show that the module $M^{\lambda}_{(R,q)}$ with a $\lambda \vDash n$ is stable when changing the coefficient ring, similarly to the algebra $\mathcal{H}^{(R,q)}_{A_{n-1}}$. To this end, let $\xi : R \to \tilde{R}$ be a ring homomorphism from R to another integral domain $\tilde{R}$, as in Lemma 1.2.7 and Lemma 1.2.8.

Lemma 1.3.6 *Let $\lambda \vDash n$. Then we have $M^{\lambda}_{(R,q)} \otimes_R \tilde{R} \simeq M^{\lambda}_{(\tilde{R},\xi(q))}$ as $\mathcal{H}^{(\tilde{R},\xi(q))}_{A_{n-1}}$-modules.*

Proof. This follows easily from Theorem 1.3.2 and Lemma 1.3.4. ■

Remark 1.3.7 *(i) Lemma 1.3.6 shows that the permutation modules $M^{\lambda}_{(R,q)}$ with $\lambda \vDash n$ from Definition 1.3.1.(ii) are generic in the sense of Remark 1.2.9.*

(ii) Lemma 1.3.6 and Theorem 1.3.2 show that the row standard bases of permutation modules from Definition 1.3.3 are generic in the sense of Remark 1.2.9.

In the following constructions, permutation modules defined by means of partitions are considered. Some of these constructions can more generally be executed by using compositions (see [DJ1, Section 4]). However, this is not required in the following. The next definition makes use of Theorem 1.3.2.

Definition 1.3.8 *Let $\lambda \vdash n$. The symmetric bilinear form*

$$\beta^{\lambda} = \beta^{\lambda}_{(R,q)} : M^{\lambda}_{(R,q)} \times M^{\lambda}_{(R,q)} \to R$$

is defined by

$$\beta^{\lambda}_{(R,q)}(x^{(R,q)}_{\lambda} T_d, x^{(R,q)}_{\lambda} T_{\tilde{d}}) = \begin{cases} q^{\ell(d)} & \text{if } d = \tilde{d} \\ 0_R & \text{if } d \neq \tilde{d} \end{cases}$$

for basis elements $x^{(R,q)}_{\lambda} T_d$ and $x^{(R,q)}_{\lambda} T_{\tilde{d}}$ with $d, \tilde{d} \in \mathcal{D}_{\lambda}$, and bilinear extension to arbitrary elements of $M^{\lambda}_{(R,q)}$.

Remark 1.3.9 *Remark 1.3.7 shows that the bilinear form introduced in Definition 1.3.8 is generic in the sense of Remark 1.2.9.*

Next, the Specht modules known from the representation theory of $\mathfrak{S}_n$ will be generalized to Hecke algebras.

Definition 1.3.10 *Let $\lambda \vdash n$.*

(i) $y_\lambda^{(R,q)} \in \mathcal{H}_{A_{n-1}}^{(R,q)}$ *is defined as*

$$y_\lambda^{(R,q)} = \varepsilon_{(R,q)}^{(n)}(\mathfrak{S}_\lambda).$$

As an abbreviation, we write

$$y_\lambda^{(R,q)} = y_\lambda.$$

(ii) $z_\lambda^{(R,q)} \in \mathcal{H}_{A_{n-1}}^{(R,q)}$ *is defined as*

$$z_\lambda^{(R,q)} = x_\lambda^{(R,q)} T_{w_\lambda} y_{\lambda'}^{(R,q)}.$$

As an abbreviation, we write

$$z_\lambda^{(R,q)} = z_\lambda.$$

(iii) The right ideal $S_{(R,q)}^\lambda$ *in* $\mathcal{H}_{A_{n-1}}^{(R,q)}$ *is defined as*

$$S_{(R,q)}^\lambda = z_\lambda^{(R,q)} \mathcal{H}_{A_{n-1}}^{(R,q)}.$$

As an abbreviation, we write

$$S_{(R,q)}^\lambda = S^\lambda.$$

$S_{(R,q)}^\lambda$ *is called a q-Specht module or just a Specht module.*

In the following theorem, the construction of the standard basis for Specht modules of symmetric groups is generalized to q-Specht modules of Hecke algebras.

Theorem 1.3.11 *Let* $\lambda \vdash n$. *Then the set*

$$\left\{ z_\lambda^{(R,q)} T_f \;\middle|\; f \in \mathcal{E}_{\lambda'} \right\} = \left\{ z_\lambda^{(R,q)} T_f \;\middle|\; f \in \mathfrak{S}_n \text{ such that } \mathbf{t}^\lambda w_\lambda f \text{ is standard} \right\}$$

is an R-basis of the Specht module $S_{(R,q)}^\lambda$. *If* X *is an indeterminate over* $\mathbb{Z}$ *then we have for every* $f \in \mathcal{E}_{\lambda'}$

$$z_\lambda^{(R,q)} T_f = q^{a_f} T_{w_\lambda f} + \sum_{\substack{w \in \mathfrak{S}_n \\ \ell(w) > \ell(w_\lambda f)}} g_{f,w}(q) \cdot 1_R T_w$$

with an appropriate exponent $a_f \in \mathbb{Z}$ *and appropriate Laurent polynomials* $g_{f,w} \in \mathbb{Z}[X, X^{-1}]$ *for* $w \in \mathfrak{S}_n$ *with* $\ell(w) > \ell(w_\lambda f)$ *independent of* (R, q).

Proof. See [DJ1, Theorem 5.6 and Lemma 5.1]. The particular form of the coefficients follows from Definition 1.3.10.(ii), Definition 1.3.1.(i), Definition 1.3.10.(i), Definition 1.2.3, and the construction of the multiplication of $\mathcal{H}_n^{(R,q)}$ with the formulas (1.20). ■

Since, for every $\lambda \vdash n$, the tableau $\mathbf{t}^\lambda$ is standard, the preceding theorem shows that all Specht modules $S^\lambda_{(R,q)}$ are different from the null module.

Definition 1.3.12 *Let $\lambda \vdash n$. Then the R-basis*

$$\left\{ z_\lambda^{(R,q)} T_f \,\middle|\, f \in \mathcal{E}_{\lambda'} \right\} = \left\{ z_\lambda^{(R,q)} T_f \,\middle|\, f \in \mathfrak{S}_n \text{ such that } \mathbf{t}^\lambda w_\lambda f \text{ is standard} \right\}$$

of $S^\lambda_{(R,q)}$ from Theorem 1.3.11 is called the standard basis of $S^\lambda_{(R,q)}$ and denoted by

$$\mathbf{B}^{S^\lambda}_{\mathrm{std}}(R,q) \quad or \quad \mathbf{B}^{S^\lambda}_{\mathrm{std}}.$$

The following statement describes the operation of $\mathcal{H}^{(R,q)}_{A_{n-1}}$ on the standard basis of a Specht module $S^\lambda_{(R,q)}$.

Lemma 1.3.13 *Let $\lambda \vdash n, f \in \mathcal{E}_{\lambda'}, s \in \mathfrak{B}_n$, and fix an indeterminate X over $\mathbb{Z}$. Then we have*

$$z_\lambda^{(R,q)} T_f T_s = \sum_{d \in \mathcal{E}_{\lambda'}} g_{f,d}(q) \cdot 1_R z_\lambda^{(R,q)} T_d,$$

where the coefficient of every $z_\lambda^{(R,q)} T_d$ with $d \in \mathcal{E}_{\lambda'}$ is of the form $g_{f,d}(q) \cdot 1_R \in R$ with an appropriate Laurent polynomial $g_{f,d} \in \mathbb{Z}[X, X^{-1}]$ independent of (R,q).

Proof. From Theorem 1.3.11 and (1.20) on page 34, we get

$$z_\lambda T_f T_s = \sum_{w \in \mathfrak{S}_n} \tilde{g}_{f,w}(q) \cdot 1_R T_w$$

with appropriate Laurent polynomials $\tilde{g}_{f,w} \in \mathbb{Z}[X, X^{-1}]$ independent of (R,q). Since $z_\lambda T_f T_s \in S^\lambda_{(R,q)} \subseteq \mathcal{H}_n^{(R,q)}$ and by using Theorem 1.3.11 once again and also induction on the length of the $w \in \mathfrak{S}_n$, this expression can be rewritten as a linear combination of basis elements $z_\lambda T_d$ with $d \in \mathcal{E}_{\lambda'}$ where the coefficients indeed have the required form. ■

Next, we will show that the Specht module $S^\lambda_{(R,q)}$ with a $\lambda \vdash n$ is stable when changing the coefficient ring, similarly to the algebra $\mathcal{H}^{(R,q)}_{A_{n-1}}$ and the module $M^\lambda_{(R,q)}$. To this end, fix a ring homomorphism $\xi : R \to \tilde{R}$ from R into another integral domain $\tilde{R}$.

Lemma 1.3.14 *Let $\lambda \vdash n$. Then we have $S^\lambda_{(R,q)} \otimes_R \tilde{R} \simeq S^\lambda_{(\tilde{R},\xi(q))}$ as $\mathcal{H}^{(\tilde{R},\xi(q))}_{A_{n-1}}$-modules.*

Proof. This follows from Theorem 1.3.11 and Lemma 1.3.13. ■

Remark 1.3.15 *(i) Lemma 1.3.14 shows that the Specht modules $S^\lambda_{(R,q)}$ with $\lambda \vdash n$ from Definition 1.3.10.(iii) are generic in the sense of Remark 1.2.9.*

(ii) Lemma 1.3.14 and Theorem 1.3.11 show that the standard bases of Specht modules from Definition 1.3.12 are generic in the sense of Remark 1.2.9.

Now, the irreducible modules of Hecke algebras over fields will be constructed as quotients of Specht modules. Appropriate submodules of Specht modules S^λ with $\lambda \vdash n$ are obtained by means of the bilinear forms β^λ on the corresponding permutation modules M^λ. This procedure also is a generalization of the well known methods for symmetric groups (see [JAM1, Section 11]).

The following definition makes use of the notation (1.1) on page 1 and dual modules as introduced in Definition 1.2.6.

Definition 1.3.16 *Let $\lambda \vdash n$.*

(i) The symmetric bilinear form

$$\gamma^\lambda = \gamma^\lambda_{(R,q)} : S^\lambda_{(R,q)} \times S^\lambda_{(R,q)} \to R$$

is defined as

$$\gamma^\lambda_{(R,q)} = \beta^\lambda_{(R,q)} \Big\downarrow^{M^\lambda_{(R,q)} \times M^\lambda_{(R,q)}}_{S^\lambda_{(R,q)} \times S^\lambda_{(R,q)}} .$$

(ii) Through $\gamma^\lambda_{(R,q)}$, every $x \in S^\lambda_{(R,q)}$ induces an R-linear homomorphism

$$\gamma^\lambda_{(R,q)}(x,-) : S^\lambda_{(R,q)} \to R, \quad y \mapsto \gamma^\lambda_{(R,q)}(x,y).$$

This, in turn, induces the R-linear homomorphism

$$\begin{array}{rcccl} \varphi[\gamma^\lambda_{(R,q)}] : S^\lambda_{(R,q)} & \to & \left(S^\lambda_{(R,q)}\right)^* & = & \mathrm{Hom}_R(S^\lambda_{(R,q)}, R), \\ x & \mapsto & \varphi[\gamma^\lambda_{(R,q)}](x) & = & \gamma^\lambda_{(R,q)}(x,-). \end{array}$$

(iii) The radical $\mathrm{rad}\,\gamma^\lambda = \mathrm{rad}\,\gamma^\lambda_{(R,q)}$ of the symmetric bilinear form $\gamma^\lambda_{(R,q)}$ is defined as

$$\mathrm{rad}\,\gamma^\lambda_{(R,q)} = \mathrm{Ker}\,\varphi[\gamma^\lambda_{(R,q)}] = \left\{ x \in S^\lambda_{(R,q)} \,\middle|\, \forall y \in S^\lambda_{(R,q)} : \gamma^\lambda_{(R,q)}(x,y) = 0_R \right\}.$$

Remark 1.3.17 *Remark 1.3.7.(i), Remark 1.3.9, and Remark 1.3.15.(i) show that the bilinear form introduced in Definition 1.3.16.(i) is generic in the sense of Remark 1.2.9.*

Lemma 1.3.18 *Let $\lambda \vdash n$. Then* $\operatorname{rad}\gamma^{\lambda}_{(R,q)}$ *is an* $\mathcal{H}^{(R,q)}_{A_{n-1}}$*-submodule of* $S^{\lambda}_{(R,q)}$.

Proof. This follows from Definition 1.3.16, Definition 1.3.8, and [DJ1, Lemma 4.4]. ■

Definition 1.3.19 *Let $\lambda \vdash n$. Then the* $\mathcal{H}^{(R,q)}_{A_{n-1}}$*-module* $D^{\lambda} = D^{\lambda}_{(R,q)}$ *is defined as*

$$D^{\lambda}_{(R,q)} = S^{\lambda}_{(R,q)}/\operatorname{rad}\gamma^{\lambda}_{(R,q)}.$$

In the following, K always denotes a field with additive neutral element 0_K and multiplicative neutral element 1_K, furthermore let $r \in K \setminus \{0_K\}$ be an arbitrarily chosen but fixed element. In the next two statements, the role of the number $e_K(r)$ from Definition 1.2.2.(ii) for the algebra $\mathcal{H}^{(K,r)}_{A_{n-1}}$ is similar to that of the characteristic of K for the algebra $K\mathfrak{S}_n$.

Theorem 1.3.20 *(i) Let $\lambda \vdash n$. If λ is $e_K(r)$-regular, we have* $D^{\lambda}_{(K,r)} \neq 0_{\mathcal{H}^{(K,r)}_A}$ *and* $D^{\lambda}_{(K,r)}$ *is an absolutely irreducible* $\mathcal{H}^{(K,r)}_{A_{n-1}}$*-module. Here,* $0_{\mathcal{H}^{(K,r)}_A}$ *denotes the null ideal in* $\mathcal{H}^{(K,r)}_{A_{n-1}}$*. If λ is $e_K(r)$-singular, we have* $D^{\lambda}_{(K,r)} = 0_{\mathcal{H}^{(K,r)}_A}$.

(ii) Let $\lambda, \mu \in \Pi_{n,e_K(r)}$ with $\lambda \neq \mu$. Then we have $D^{\lambda}_{(K,r)} \not\cong D^{\mu}_{(K,r)}$.

(iii) The set

$$\left\{ D^{\lambda}_{(K,r)} \,\middle|\, \lambda \in \Pi_{n,e_K(r)} \right\}$$

is a complete system of representatives of the isomorphism classes of irreducible $\mathcal{H}^{(K,r)}_{A_{n-1}}$*-modules. It is parameterized by the set* $\Pi_{n,e_K(r)}$.

(iv) K is a splitting field for $\mathcal{H}^{(K,r)}_{A_{n-1}}$.

Proof. (i) See [DJ1, Theorem 4.9, Theorem 6.3.(i), Theorem 6.8.(i)].

(ii) See [DJ1, Corollary 4.13].

(iii) See [DJ1, Theorem 7.6].

(iv) This follows from statements (i) and (iii). ■

Theorem 1.3.21 *(i) The algebra* $\mathcal{H}^{(K,r)}_{A_{n-1}}$ *is semisimple if and only if*

$$e_K(r) > n$$

holds.

(ii) Suppose that $\mathcal{H}^{(K,r)}_{A_{n-1}}$ *is semisimple. Then the set*

$$\left\{ S^{\lambda}_{(K,r)} \,\middle|\, \lambda \in \Pi_n \right\}$$

is a complete system of representatives of the isomorphism classes of irreducible $\mathcal{H}^{(K,r)}_{A_{n-1}}$*-modules. It is parameterized by the set* Π_n.

(iii) Suppose that $\mathcal{H}_{A_{n-1}}^{(K,r)}$ *is semisimple. Then we have for every* $\lambda \vdash n$

$$D_{(K,r)}^{\lambda} = S_{(K,r)}^{\lambda}.$$

Proof. (i) See [DJ2, Theorem 4.3].

(ii) See [DJ2, Theorem 4.3].

(iii) This follows from Definition 1.3.19, Theorem 1.3.20, and statement (ii). ■

1.4 Modular reduction and decomposition numbers for Hecke algebras of type A

Now we consider modular reductions of Hecke algebras over various coefficient fields and related objects like Grothendieck groups, decomposition maps, and decomposition numbers. A good reference for the following material is [CR1, Chapter 2]. This book also provides general facts about projective modules, short exact sequences, and similar things which are used here without specific references. For the following, we fix an $n \in \mathbb{N}$.

Definition 1.4.1 *Let* R *be an integral domain and* $q \in R$ *be a unit.*

(i) The isomorphism class of an $\mathcal{H}_{A_{n-1}}^{(R,q)}$*-module* M *is denoted by* $[M]$*. The set of all isomorphism classes of all finitely generated right* $\mathcal{H}_{A_{n-1}}^{(R,q)}$*-modules is denoted by*

$$\mathcal{M}(\mathcal{H}_{A_{n-1}}^{(R,q)}) = \left\{ [M] \,\middle|\, M \text{ is a finitely generated right } \mathcal{H}_{A_{n-1}}^{(R,q)}\text{-module} \right\}.$$

(ii) The $\mathbb{Z}$*-submodule* $U_0(\mathcal{H}_{A_{n-1}}^{(R,q)})$ *of the free module* $\oplus_{[M]\in\mathcal{M}(\mathcal{H}_n^{(R,q)})}\mathbb{Z}[M]$ *is defined as the* $\mathbb{Z}$*-span of the set*

$$\left\{ [M]-[M']-[M''] \,\middle|\, \begin{array}{c} [M],[M'],[M''] \in \mathcal{M}(\mathcal{H}_{A_{n-1}}^{(R,q)}) \\ \text{such that there is a short exact sequence} \\ 0_{\mathcal{H}_A^{(R,q)}} \to M' \to M \to M'' \to 0_{\mathcal{H}_A^{(R,q)}} \\ \text{of } \mathcal{H}_{A_{n-1}}^{(R,q)}\text{-modules} \end{array} \right\}.$$

Here, $0_{\mathcal{H}_A}$ *denotes the null ideal in* $\mathcal{H}_{A_{n-1}}^{(R,q)}$*. With this, the Grothendieck group of* $\mathcal{H}_{A_{n-1}}^{(R,q)}$ *is defined as*

$$G_0(\mathcal{H}_{A_{n-1}}^{(R,q)}) = \left(\bigoplus_{[M]\in\mathcal{M}(\mathcal{H}_n^{(R,q)})} \mathbb{Z}[M] \right) / U_0(\mathcal{H}_{A_{n-1}}^{(R,q)}).$$

The structure of the Grothendieck groups of Hecke algebras over fields is described in the following two statements.

Lemma 1.4.2 *Let (K, r) be a coefficient pair with a field K.*

(i) We have

$$G_0(\mathcal{H}_{A_{n-1}}^{(K,r)}) = \bigoplus_{\mu \in \Pi_{n,e_K(r)}} \mathbb{Z}[D^{\mu}_{(K,r)}].$$

(ii) Let $[M] \in \mathcal{M}(\mathcal{H}_{A_{n-1}}^{(K,r)})$. Then we have in $G_0(\mathcal{H}_{A_{n-1}}^{(K,r)})$

$$[M] = \sum_{\mu \in \Pi_{n,e_K(r)}} x_{\mu}^{[M]}[D^{\mu}_{(K,r)}]$$

with coefficients $x_{\mu}^{[M]} \in \mathbb{N}_0$ for $\mu \in \Pi_{n,e_K(r)}$.

Proof. (i) This follows from Theorem 1.3.20.(iii) and [CR1, Proposition 16.6].

(ii) This follows from Theorem 1.3.20.(iii) and [CR1, proof of Proposition 16.6]. ∎

Lemma 1.4.3 *Let (K, r) be a coefficient pair with a field K such that $\mathcal{H}_{A_{n-1}}^{(K,r)}$ is semisimple.*

(i) We have

$$G_0(\mathcal{H}_{A_{n-1}}^{(K,r)}) = \bigoplus_{\mu \in \Pi_n} \mathbb{Z}[S^{\mu}_{(K,r)}].$$

(ii) Let $[M] \in \mathcal{M}(\mathcal{H}_{A_{n-1}}^{(K,r)})$. Then we have in $G_0(\mathcal{H}_{A_{n-1}}^{(K,r)})$

$$[M] = \sum_{\mu \in \Pi_n} y_{\mu}^{[M]}[S^{\mu}_{(K,r)}]$$

with coefficients $y_{\mu}^{[M]} \in \mathbb{N}_0$ for $\mu \in \Pi_n$.

Proof. (i) This follows from Theorem 1.3.21.(ii) and [CR1, Proposition 16.6].

(ii) This follows from Theorem 1.3.21.(ii) and [CR1, proof of Proposition 16.6]. ∎

According to Lemma 1.4.3.(i), the Grothendieck groups of all finitely generated modules of semisimple Hecke algebras of type A_{n-1} over fields are isomorphic. The following definition fixes such an isomorphism.

Definition 1.4.4 *Let (K, r) and $(\tilde{K}, \tilde{r})$ be two coefficient pairs with fields K and $\tilde{K}$ such that the Hecke algebras $\mathcal{H}_{A_{n-1}}^{(K,r)}$ and $\mathcal{H}_{A_{n-1}}^{(\tilde{K},\tilde{r})}$ are semisimple. Then the isomorphism*

$$\alpha^{\mathcal{H}_n}_{(K,r)(\tilde{K},\tilde{r})} : G_0(\mathcal{H}_{A_{n-1}}^{(K,r)}) \to G_0(\mathcal{H}_{A_{n-1}}^{(\tilde{K},\tilde{r})})$$

is defined by

$$[S^{\lambda}_{(K,r)}] \mapsto [S^{\lambda}_{(\tilde{K},\tilde{r})}] \quad \text{for} \quad \lambda \vdash n$$

and $\mathbb{Z}$-linear extension.

Next, systems of coefficient rings for the construction of decomposition maps of Hecke algebras are described. We proceed as in [CR1, §4C].

Definition 1.4.5 *Let K be a field. A discrete additive valuation on K is defined as a map*

$$\psi : K \setminus \{0_K\} \to \mathbb{Z}$$

with the following properties.

(i) ψ is an epimorphism of the multiplicative group $K \setminus \{0_K\}$ onto the additive group $\mathbb{Z}$. In particular, we have for $x, y \in K \setminus \{0_K\}$

$$\psi(xy) = \psi(x) + \psi(y).$$

(ii) For $x, y \in K \setminus \{0_K\}$ with $x + y \neq 0_K$, we have

$$\psi(x + y) \geq \min\{\psi(x), \psi(y)\}.$$

The pair (K, ψ) is called a valuated field.

Definition 1.4.6 *Let (K, ψ) be a valuated field. Then the discrete valuation ring of ψ is defined as*

$$S_\psi = \{x \in K \setminus \{0_K\} \mid \psi(x) \geq 0\} \cup \{0_K\} \subseteq K.$$

Furthermore, the valuation ideal of ψ is defined as

$$I_\psi = \{x \in K \setminus \{0_K\} \mid \psi(x) > 0\} \cup \{0_K\} \subseteq S_\psi \subseteq K.$$

Remark 1.4.7 *Let (K, ψ) be a valuated field. Then the properties of ψ stated in Definition 1.4.5 show that S_ψ is an integral domain and furthermore a local ring with the unique maximal ideal I_ψ and the group of multiplicative units $S_\psi \setminus I_\psi$.*

Definition 1.4.8 *A modular system is defined as a tuple*

$$\mathcal{K} = (Q, \psi, S, I, a, F)$$

of objects with the following specifications.

(i) (Q, ψ) *is a valuated field.*

(ii) S *is the discrete valuation ring of* ψ*.*

(iii) I *is the valuation ideal of* ψ*.*

(iv) $a \in S \setminus I$ *is a unit in* S*.*

(v) F *is the residue class field* S/I*.*

The natural projection from S *onto* F *is denoted by*

$$\bar{\cdot} : S \to F, \quad x \mapsto \bar{x} = x + I.$$

$\mathcal{K}$ *determines three coefficient pairs* (Q, a)*,* (S, a)*, and* $(F, \bar{a})$ *as in Definition 1.2.1. These are called the coefficient pairs associated to the modular system* $\mathcal{K}$*.*

For the following, we fix a modular system

$$\mathcal{K} = (Q, \psi, S, I, a, F)\,.$$

The next statement makes use of Definition 1.2.2.(ii).

Lemma 1.4.9 *(i) We have* $e_F(\bar{a}) \leq e_Q(a)$*.*

(ii) Let $e_Q(a) < \infty$*. Then* $e_F(\bar{a})$ *divides* $e_Q(a)$*.*

Proof. (i) If $e_Q(a) = \infty$, there is nothing to show. So let $e_Q(a) < \infty$. Then we have in Q according to Definition 1.2.2

$$[e_Q(a)]_a = \sum_{i=0}^{e_Q(a)-1} a^i = 0_Q.$$

From this, we get in F

$$[e_Q(a)]_{\bar{a}} = \sum_{i=0}^{e_Q(a)-1} \bar{a}^i = 0_F. \tag{1.28}$$

With that, the claim follows from Definition 1.2.2.(ii).

(ii) According to Definition 1.2.2.(ii), statement (i), and the assumption, we have

$$2 \leq e_F(\bar{a}) \leq e_Q(a) < \infty.$$

Thus, we can write

$$e_Q(a) = \alpha e_F(\bar{a}) + \beta \quad \text{with} \quad \alpha \in \mathbb{N}_0 \quad \text{and} \quad \beta \in \{0, \ldots, e_F(\bar{a}) - 1\}. \tag{1.29}$$

Then, to prove the claim, we must show $\beta = 0$. Now, from (1.29) and Definition 1.2.2.(i), we get in F the relation

$$[e_Q(a)]_{\bar{a}} = [\alpha]_{\bar{a}^{e_F(\bar{a})}} [e_F(\bar{a})]_{\bar{a}} + \bar{a}^{\alpha e_F(\bar{a})} [\beta]_{\bar{a}}.$$

This shows together with (1.28) from the proof of statement (i), Definition 1.2.2.(ii), and Definition 1.4.8

$$[\beta]_{\bar{a}} = 0_F.$$

From this, we get together with (1.29) and Definition 1.2.2

$$\beta = 0.$$

Thus, $e_F(\bar{a})$ divides $e_Q(a)$, as desired. ■

Now, some relations between the Hecke algebras over the coefficient pairs (Q, a), (S, a), and $(F, \bar{a})$ associated to $\mathcal{K}$ are described. The natural inclusion $\iota_{S,Q} : S \hookrightarrow Q$ allows the construction of the functor $- \otimes_S Q$, and one gets the following result.

Lemma 1.4.10 *We have $\mathcal{H}^{(S,a)}_{A_{n-1}} \otimes_S Q \simeq \mathcal{H}^{(Q,a)}_{A_{n-1}}$ as Q-algebras.*

Proof. This follows from Lemma 1.2.7. ■

In what follows, the algebras $\mathcal{H}^{(S,a)}_{A_{n-1}} \otimes_S Q$ and $\mathcal{H}^{(Q,a)}_{A_{n-1}}$ are identified by means of the preceding lemma. Now, the coefficient pairs (S, a) and $(F, \bar{a})$ are considered. The natural projection $\bar{\cdot} : S \to F,\ x \mapsto \bar{x} = x + I$ allows the construction of the functor $- \otimes_S F$, and one gets the following result.

Lemma 1.4.11 *We have $\mathcal{H}^{(S,a)}_{A_{n-1}} \otimes_S F \simeq \mathcal{H}^{(F,\bar{a})}_{A_{n-1}}$ as F-algebras.*

Proof. This follows from Lemma 1.2.7. ■

In what follows, the algebras $\mathcal{H}^{(S,a)}_{A_{n-1}} \otimes_S F$ and $\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}$ are identified by means of the preceding lemma.

The next definition relates $\mathcal{H}^{(Q,a)}_{A_{n-1}}$-modules and $\mathcal{H}^{(S,a)}_{A_{n-1}}$-modules. It makes use of Lemma 1.2.8.(i) and the functor $- \otimes_S Q$. See also [CR1, Definition 16.11].

Definition 1.4.12 *Let M be a finitely generated $\mathcal{H}_{A_{n-1}}^{(Q,a)}$-module. Then a full $\mathcal{H}_{A_{n-1}}^{(S,a)}$-lattice of M is defined as a $\mathcal{H}_{A_{n-1}}^{(S,a)}$-module N with the following two properties.*

(i) N is free over S with finite rank.

(ii) We have $N \otimes_S Q \simeq M$ as $\mathcal{H}_{A_{n-1}}^{(Q,a)}$-modules.

For example, according to Lemma 1.2.7, $\mathcal{H}_{A_{n-1}}^{(S,a)}$ is a full $\mathcal{H}_{A_{n-1}}^{(S,a)}$-lattice of $\mathcal{H}_{A_{n-1}}^{(Q,a)}$ if the algebras are considered as right modules for themselves. For $\mathcal{H}_{A_{n-1}}^{(Q,a)}$-modules and full $\mathcal{H}_{A_{n-1}}^{(S,a)}$-lattices thereof, the following statements hold.

Lemma 1.4.13 *(i) Every $\mathcal{H}_{A_{n-1}}^{(Q,a)}$-module has full $\mathcal{H}_{A_{n-1}}^{(S,a)}$-lattices.*

(ii) Let N be a full $\mathcal{H}_{A_{n-1}}^{(S,a)}$-lattice in an $\mathcal{H}_{A_{n-1}}^{(Q,a)}$-module M. Then we have $\operatorname{Rnk}_S N = \dim_Q M$.

Proof. (i) See [CR1, Proposition 16.15].

(ii) This follows immediately from Definition 1.4.12. ■

Next, we relate $\mathcal{H}_{A_{n-1}}^{(S,a)}$-modules and $\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}$-modules. The following definition makes use of Lemma 1.2.8.(i) and the functor $- \otimes_S F$.

Definition 1.4.14 *Let M be an $\mathcal{H}_{A_{n-1}}^{(S,a)}$-module. Then the $\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}$-module $M \otimes_S F$ is called the reduction of M modulo I. For this we write $\bar{M}$. The map $- \otimes_S F : M \to \bar{M}$ is, for short, denoted by*

$$\bar{\cdot} : M \to \bar{M}, \quad x \mapsto \bar{x} = x \otimes_S 1_F.$$

For example, according to Lemma 1.2.7, $\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}$ is the reduction modulo I of $\mathcal{H}_{A_{n-1}}^{(S,a)}$ if the algebras are considered as right modules for themselves. Furthermore, this gives together with the relation (1.27) from Lemma 1.2.8 the following property of the reduction modulo I of an $\mathcal{H}_{A_{n-1}}^{(S,a)}$-module M.

$$\forall x \in M, y \in \mathcal{H}_{A_{n-1}}^{(S,a)} : \overline{xy} = \bar{x}\bar{y} \tag{1.30}$$

Now we compare for a given $\mathcal{H}_{A_{n-1}}^{(Q,a)}$-module M the reductions modulo I of various full $\mathcal{H}_{A_{n-1}}^{(S,a)}$-lattices therein. For two full $\mathcal{H}_{A_{n-1}}^{(S,a)}$-lattices N_1 and N_2 in M, we will have in general $\bar{N}_1 \not\simeq \bar{N}_2$ as $\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}$-modules. However, the following important result holds.

Lemma 1.4.15 *Let M be an $\mathcal{H}_{A_{n-1}}^{(Q,a)}$-module, choose two full $\mathcal{H}_{A_{n-1}}^{(S,a)}$-lattices N_1 and N_2 thereof, and consider their reductions modulo I $\bar{N}_1$ and $\bar{N}_2$. Then we have in $G_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$*

$$[\bar{N}_1] = [\bar{N}_2].$$

Proof. See [CR1, Proposition 16.16]. ■

With the preceding result, we can assign to an isomorphism class of $\mathcal{H}_{A_{n-1}}^{(Q,a)}$-modules $[M] \in \mathcal{M}(\mathcal{H}_{A_{n-1}}^{(Q,a)})$ by means of a representative $M \in [M]$ and a full $\mathcal{H}_{A_{n-1}}^{(S,a)}$-lattice N of M (which exists according to Lemma 1.4.13.(i)) the uniquely determined element $[\bar{N}] \in G_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$. This assignment is compatible with the relations between the isomorphism classes of $\mathcal{H}_{A_{n-1}}^{(Q,a)}$-modules in $G_0(\mathcal{H}_{A_{n-1}}^{(Q,a)})$ (see [CR1, Proposition 16.17]) and thus induces a homomorphism from this Grothendieck group into $G_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$.

Definition 1.4.16 *The homomorphism*

$$D^{\mathcal{H}} = D_n^{\mathcal{H}} = D_{\mathcal{K}}^{\mathcal{H}} = D_{n,\mathcal{K}}^{\mathcal{H}} : G_0(\mathcal{H}_{A_{n-1}}^{(Q,a)}) \to G_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}), \quad [M] \mapsto [\bar{N}]$$

for $[M] \in \mathcal{M}(\mathcal{H}_{A_{n-1}}^{(Q,a)})$ *with a representative* $M \in [M]$ *and a full* $\mathcal{H}_{A_{n-1}}^{(S,a)}$*-lattice* N *of* M *is called the decomposition map for Hecke algebras associated with the degree* n *and the modular system* $\mathcal{K}$*.*

Finally, decomposition numbers of Hecke algebras are introduced as coefficients in matrix representations of decomposition maps.

Definition 1.4.17 *According to Lemma 1.4.2, the formulas*

$$D_{n,\mathcal{K}}^{\mathcal{H}}([D_{(Q,a)}^{\lambda}]) = \sum_{\mu \in \Pi_{n,e_F(\bar{a})}} d_{\lambda\mu}^{n,\mathcal{K}} [D_{(F,\bar{a})}^{\mu}]$$

with $\lambda \in \Pi_{n,e_Q(a)}$ *define uniquely determined numbers* $d_{\lambda\mu}^{n,\mathcal{K}} \in \mathbb{N}_0$ *for* $\lambda \in \Pi_{n,e_Q(a)}$ *and* $\mu \in \Pi_{n,e_F(\bar{a})}$*. These numbers are called the decomposition numbers for Hecke algebras associated with the degree* n *and the modular system* $\mathcal{K}$*. The matrix*

$$\Delta^{\mathcal{H}} = \Delta_n^{\mathcal{H}} = \Delta_{\mathcal{K}}^{\mathcal{H}} = \Delta_{n,\mathcal{K}}^{\mathcal{H}} = \left(d_{\lambda\mu}^{n,\mathcal{K}}\right)_{\substack{\lambda \in \Pi_{n,e_Q(a)} \\ \mu \in \Pi_{n,e_F(\bar{a})}}}$$

representing the map $D_{n,\mathcal{K}}^{\mathcal{H}}$ *with respect to the basis* $\left\{[D_{(Q,a)}^{\lambda}] \,\middle|\, \lambda \in \Pi_{n,e_Q(a)}\right\}$ *of the* $\mathbb{Z}$*-module* $G_0(\mathcal{H}_{A_{n-1}}^{(Q,a)})$ *and the basis* $\left\{[D_{(F,\bar{a})}^{\mu}] \,\middle|\, \mu \in \Pi_{n,e_F(\bar{a})}\right\}$ *of the* $\mathbb{Z}$*-module* $G_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$ *from Lemma 1.4.2.(i) is called the decomposition matrix for Hecke algebras associated with the degree* n *and the modular system* $\mathcal{K}$*.*

1.5 Modular reduction and Specht modules of Hecke algebras of type A

For this section, we fix an $n \in \mathbb{N}$, as before. Furthermore, $\mathcal{K}$ denotes a modular system as introduced in the previous section. This section describes the effect of the decomposition map $D^{\mathcal{H}}_{n,\mathcal{K}}$ on Specht modules. This is particularly important if the Hecke algebra $\mathcal{H}^{(Q,a)}_{A_{n-1}}$ is semisimple.

Lemma 1.5.1 *Let $\lambda \vdash n$. Then $S^\lambda_{(S,a)}$ is a full $\mathcal{H}^{(S,a)}_{A_{n-1}}$-lattice of $S^\lambda_{(Q,a)}$.*

Proof. This follows from Definition 1.4.12, Theorem 1.3.11, and Lemma 1.3.14. ■

Lemma 1.5.2 *Let $\lambda \vdash n$. Then $S^\lambda_{(F,\bar{a})}$ is the reduction modulo I of the full $\mathcal{H}^{(S,a)}_{A_{n-1}}$-lattice $S^\lambda_{(S,a)}$ of $S^\lambda_{(Q,a)}$.*

Proof. This follows from Lemma 1.5.1, Definition 1.4.14, and Lemma 1.3.14. ■

Corollary 1.5.3 *Let $\lambda \vdash n$. Then we have*

$$D^{\mathcal{H}}_{n,\mathcal{K}}([S^\lambda_{(Q,a)}]) = [S^\lambda_{(F,\bar{a})}].$$

Proof. This follows from Definition 1.4.16, Lemma 1.5.1, and Lemma 1.5.2. ■

Corollary 1.5.4 *Suppose that $\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}$ is semisimple. Then the decomposition matrix $\Delta^{\mathcal{H}}_{n,\mathcal{K}}$ is, if the row and column index sets are ordered in the same way, an identity matrix (i.e. it has ones on the diagonal and zeroes elsewhere).*

Proof. This follows from Theorem 1.3.21, Lemma 1.4.9.(i), Definition 1.4.17, and Corollary 1.5.3. ■

Lemma 1.5.5 *Suppose that $\mathcal{H}^{(Q,a)}_{A_{n-1}}$ is semisimple. Then, for every $\lambda \in \Pi_n$ and every $\mu \in \Pi_{n,e_F(\bar{a})}$, the decomposition number $d^{n,\mathcal{K}}_{\lambda\mu}$ is equal to the multiplicity of $D^\mu_{(F,\bar{a})}$ as a composition factor in $S^\lambda_{(F,\bar{a})}$.*

Proof. According to Theorem 1.3.20.(iii), Theorem 1.3.21.(ii), Definition 1.4.16, and Definition 1.4.17, the decomposition number $d^{n,\mathcal{K}}_{\lambda\mu}$ is obtained by choosing a full $\mathcal{H}^{(S,a)}_n$-lattice N of $S^\lambda_{(Q,a)}$, constructing its reduction modulo I $\bar{N}$, and determining the multiplicity of the irreducible module $D^\mu_{(F,\bar{a})}$ as a composition factor in $\bar{N}$. According to Lemma 1.5.1, one can choose $N = S^\lambda_{(S,a)}$. Then, according to Lemma 1.5.2, one has $\bar{N} = S^\lambda_{(F,\bar{a})}$. This shows the claim. ■

Corollary 1.5.6 *Suppose that $\mathcal{H}^{(Q,a)}_{A_{n-1}}$ is semisimple. Then, for every $\lambda \in \Pi_n$ and every $\mu \in \Pi_{n,e_F(\bar{a})}$, the decomposition number $d^{n,\mathcal{K}}_{\lambda\mu}$ is uniquely determined by the data λ, μ, and $(F,\bar{a})$.*

Proof. This follows immediately from Lemma 1.5.5. ■

The next lemma makes use of Definition 1.1.2 and Definition 1.1.4.(ii).

Lemma 1.5.7 *Suppose that $\mathcal{H}^{(Q,a)}_{A_{n-1}}$ is semisimple and let $\lambda \vdash n$. Then the following statements hold in $G_0(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})$.*

(i) We have

$$[S^{\lambda}_{(F,\bar{a})}] = \sum_{\mu \in \Pi_{n,e_F(\bar{a})}} d^{n,\mathcal{K}}_{\lambda\mu} [D^{\mu}_{(F,\bar{a})}].$$

(ii) If λ is $e_F(\bar{a})$-regular, we have

$$[S^{\lambda}_{(F,\bar{a})}] = [D^{\lambda}_{(F,\bar{a})}] + \sum_{\substack{\mu \in \Pi_{n,e_F(\bar{a})} \\ \mu \rhd \lambda}} d^{n,\mathcal{K}}_{\lambda\mu} [D^{\mu}_{(F,\bar{a})}].$$

(iii) If λ is $e_F(\bar{a})$-singular, we have

$$[S^{\lambda}_{(F,\bar{a})}] = \sum_{\substack{\mu \in \Pi_{n,e_F(\bar{a})} \\ \mu \rhd \lambda}} d^{n,\mathcal{K}}_{\lambda\mu} [D^{\mu}_{(F,\bar{a})}].$$

Proof. The identity in statement (i) is obtained from Theorem 1.3.21, Definition 1.4.17, and Corollary 1.5.3. This in turn, together with Lemma 1.5.5, the inclusion $S^{\lambda} \subseteq M^{\lambda}$ (see Definition 1.3.1 and Definition 1.3.10), [DJ1, Corollary 4.12.(i)], Theorem 1.3.20.(i), and [DJ1, Corollary 4.14], implies the identities in statements (ii) and (iii). ■

The next statement is required in the following considerations. It is a generalization of the corresponding property of group algebras (see [CR1, Corollary 18.14]).

Lemma 1.5.8 *Suppose that $\mathcal{H}^{(Q,a)}_{A_{n-1}}$ is semisimple. Then the following statements hold.*

(i) The decomposition map

$$D^{\mathcal{H}}_{n,\mathcal{K}} : G_0(\mathcal{H}^{(Q,a)}_{A_{n-1}}) \to G_0(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})$$

is surjective.

(ii) For the decomposition matrix

$$\Delta^{\mathcal{H}}_{n,\mathcal{K}} = \left(d^{n,\mathcal{K}}_{\lambda\mu} \right)_{\substack{\lambda \in \Pi_n \\ \mu \in \Pi_{n,e_F(\bar{a})}}},$$

we have

$$\mathrm{Rnk}_{\mathbb{Q}}\, \Delta^{\mathcal{H}}_{n,\mathcal{K}} = \left| \Pi_{n,e_F(\bar{a})} \right|.$$

Proof. (i) According to Lemma 1.4.3.(i) and Corollary 1.5.3, we must show that for every $\mu \in \Pi_{n,e_F(\bar{a})}$ the basis element $[D^{\mu}_{(F,\bar{a})}] \in G_0(\mathcal{H}_n^{(F,\bar{a})})$ can be written as a $\mathbb{Z}$-linear combination of the elements $[S^{\lambda}_{(F,\bar{a})}] \in G_0(\mathcal{H}_n^{(F,\bar{a})})$ with $\lambda \in \Pi_n$. This follows from [DJ1, Corollary 4.12, Corollary 4.14]. There, a certain order on the set Π_n is used, by means of which the required representations of the $[D^{\mu}_{(F,\bar{a})}]$ with $\mu \in \Pi_{n,e_F(\bar{a})}$ can be constructed inductively.

(ii) In the following sequence of $\mathbb{Z}$-modules, $0_{\mathbb{Z}}$ denotes the null module over $\mathbb{Z}$.

$$G_0(\mathcal{H}_n^{(Q,a)}) \xrightarrow{D^{\mathcal{H}}_{n,\mathcal{K}}} G_0(\mathcal{H}_n^{(F,\bar{a})}) \longrightarrow 0_{\mathbb{Z}}$$

According to statement (i), this sequence is exact. By tensoring over $\mathbb{Z}$ with $\mathbb{Q}$, one obtains an exact sequence of $\mathbb{Q}$-vector spaces, since $- \otimes_{\mathbb{Z}} \mathbb{Q}$ is right exact (see, for example, [CR1, §2B]). Now, according to Definition 1.4.17 and Theorem 1.3.21.(ii), the matrix representing the surjective map $D^{\mathcal{H}}_{n,\mathcal{K}} \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{Q}}$ with respect to the bases of $G_0(\mathcal{H}_n^{(Q,a)})$ and $G_0(\mathcal{H}_n^{(F,\bar{a})})$ from Lemma 1.4.3.(i) and Lemma 1.4.2.(i) tensored with $\mathbb{Q}$ over $\mathbb{Z}$ is just $\Delta^{\mathcal{H}}_{n,\mathcal{K}}$. Now the claim follows from general facts of linear algebra and the relation

$$\left|\Pi_{n,e_F(\bar{a})}\right| \leq |\Pi_n|$$

(see Definition 1.1.2.(iii)). ■

1.6 Dependence of the decomposition matrices of Hecke algebras of type A on the employed modular system

We continue to use the integer $n \in \mathbb{N}$ and the modular system $\mathcal{K}$ fixed in the previous section. This section will show that the decomposition numbers $d^{n,\mathcal{K}}_{\lambda\mu}$ with $\lambda \in \Pi_{n,e_Q(a)}$ and $\mu \in \Pi_{n,e_F(\bar{a})}$ are independent of the coefficient ring S in the modular system $\mathcal{K}$. To this end, we require, in addition to $\mathcal{K}$, two other modular systems. These are introduced next.

Let K be an arbitrary field, fix an element $r \in K \setminus \{0_K\}$, and consider an indeterminate X over K. Then the polynomial $f = X - r \in K[X]$ is irreducible.

With this, we define for every element $\frac{g}{h}$ of $K(X) \setminus \{0_{K(X)}\}$ with $g, h \in K[X]$ the integer

$$\psi_f(\frac{g}{h}) \in \mathbb{Z}$$

by means of the decomposition

$$\frac{g}{h} = \frac{\tilde{g}}{\tilde{h}} f^{\psi_f(\frac{g}{h})}$$

with $\tilde{g}, \tilde{h} \in K[X]$ such that both $f \nmid \tilde{g}$ and $f \nmid \tilde{h}$ hold. The map

$$\psi_f : K(X) \setminus \{0_{K(X)}\} \to \mathbb{Z}, \quad z \mapsto \psi_f(z)$$

defined in this way has the following properties.

(i) For arbitrary $y, z \in K(X) \setminus \{0_{K(X)}\}$ we have $\psi_f(yz) = \psi_f(y) + \psi_f(z)$.

(ii) For $y, z \in K(X) \setminus \{0_{K(X)}\}$ satisfying $y + z \neq 0_{K(X)}$ we have $\psi_f(y + z) \geq \min\{\psi_f(y), \psi_f(z)\}$.

Thus, ψ_f is a discrete additive valuation on $K(X)$ (see [CR1, §4C]). Associated to ψ_f is the discrete valuation ring

$$S_{\psi_f} = \left\{z \in K(X) \setminus \left\{0_{K(X)}\right\} \mid \psi_f(z) \geq 0\right\} \cup \left\{0_{K(X)}\right\}$$

with the unique maximal ideal

$$I_{\psi_f} = \left\{z \in K(X) \setminus \left\{0_{K(X)}\right\} \mid \psi_f(z) > 0\right\} \cup \left\{0_{K(X)}\right\}.$$

These also can be described as follows.

$$\begin{aligned} S_{\psi_f} &= K[X]_{fK[X]} \\ &= \left\{\frac{g}{h} \in K(X) \,\middle|\, g \in K[X],\, h \in K[X] \setminus \left\{0_{K[X]}\right\} \text{ such that } f \nmid h\right\} \end{aligned}$$

is the localization of $K[X]$ at the ideal $fK[X]$ (see [CR1, §4A]). With this,

$$\begin{aligned} I_{\psi_f} &= fS_{\psi_f} \\ &= f \cdot K[X]_{fK[X]} \\ &= \left\{\frac{g}{h} \in K(X) \,\middle|\, g \in K[X],\, h \in K[X] \setminus \left\{0_{K[X]}\right\} \text{ such that } f \mid g \text{ and } f \nmid h\right\} \end{aligned}$$

is the ideal generated by f in S_{ψ_f}. Because of $\psi_f(X) = 0 = \psi_f(\frac{1_K}{X})$, X is a unit in S_{ψ_f}. Furthermore, every element of S_{ψ_f} is congruent modulo I_{ψ_f} to an element of $K \subseteq S_{\psi_f} \subseteq K(X)$. Thus, we have

$$S_{\psi_f}/I_{\psi_f} = K,$$

and the natural projection

$$\bar{\cdot} : S_{\psi_f} \to K, \quad z \mapsto \bar{z} = z + I_{\psi_f}$$

maps X to

$$\bar{X} = r.$$

All in all, this construction provides for a given field K and a fixed $r \in K \setminus \{0_K\}$ a modular system with (K, r) as an associated coefficient pair such that K is the residue class field of the discrete valuation ring.

Definition 1.6.1 *Let K be an arbitrary field, fix an $r \in K \setminus \{0_K\}$, and choose an indeterminate X over K. Then the modular system $\mathcal{K}_{(K,r)}$ is defined by means of the discrete additive valuation ψ_{X-r} on $K(X)$ as*

$$\begin{aligned}\mathcal{K}_{(K,r)} &= \left(K(X), \psi_{X-r}, S_{\psi_{X-r}}, I_{\psi_{X-r}}, X, K\right) \\ &= \left(K(X), \psi_{X-r}, K[X]_{(X-r)K[X]}, (X-r) \cdot K[X]_{(X-r)K[X]}, X, K\right).\end{aligned}$$

Later on, we will require modular systems with a complete discrete valuation ring. Such a modular system is obtained from a given field K and a unit $r \in K \setminus \{0_K\}$ by means of the construction of the modular system $\mathcal{K}_{(K,r)}$ just introduced if, in addition, the field $K(X)$ is completed with respect to the valuation ψ_{X-r}. This is described in more detail in [CR1, §4C] and the further references given there.

Definition 1.6.2 *Let K be a field, fix an $r \in K \setminus \{0_K\}$, and choose an indeterminate X over K.*

(i) The completion of $K(X)$ with respect to the additive valuation ψ_{X-r} is denoted by

$$K\hat{(X)}.$$

$K(X)$ is considered a subset of $K\hat{(X)}$.

(ii) The discrete additive valuation on $K\hat{(X)}$ defined by continuous extension of ψ_{X-r} from $K(X) \setminus \{0_{K(X)}\}$ to $K\hat{(X)} \setminus \left\{0_{K\hat{(X)}}\right\}$ is denoted by

$$\hat{\psi}_{X-r} : K\hat{(X)} \setminus \left\{0_{K\hat{(X)}}\right\} \to \mathbb{Z}.$$

(iii) The discrete valuation ring in $K\hat{(X)}$ associated to $\hat{\psi}_{X-r}$ is denoted by

$$S_{\hat{\psi}_{X-r}} = \left\{z \in K\hat{(X)} \setminus \left\{0_{K\hat{(X)}}\right\} \,\middle|\, \hat{\psi}_{X-r}(z) \geq 0\right\} \cup \left\{0_{K\hat{(X)}}\right\}.$$

The unique maximal ideal in $S_{\hat{\psi}_{X-r}}$ is denoted by

$$I_{\hat{\psi}_{X-r}} = \left\{z \in K\hat{(X)} \setminus \left\{0_{K\hat{(X)}}\right\} \,\middle|\, \hat{\psi}_{X-r}(z) > 0\right\} \cup \left\{0_{K\hat{(X)}}\right\}.$$

Lemma 1.6.3 *Let K be a field, fix an $r \in K \setminus \{0_K\}$, and choose an indeterminate X over K. Then the following statements hold.*

(i) $S_{\hat{\psi}_{X-r}}$ is a complete discrete valuation ring.

(ii) We have $S_{\hat{\psi}_{X-r}}/I_{\hat{\psi}_{X-r}} = K$.

(iii) For the natural projection $\bar{\cdot}: S_{\hat{\psi}_{X-r}} \to S_{\hat{\psi}_{X-r}}/I_{\hat{\psi}_{X-r}} = K$, *we have the relation* $X \mapsto \bar{X} = r$.

Proof. (i) As a continuous extension of the discrete additive valuation ψ_{X-r}, $\hat{\psi}_{X-r}$ is discrete. The completeness of $S_{\hat{\psi}_{X-r}}$ with respect to $\hat{\psi}_{X-r}$ follows from the completeness of $K\hat{(X)}$ with respect to $\hat{\psi}_{X-r}$ and the continuity of $\hat{\psi}_{X-r}$ with respect to the topology on $K\hat{(X)}$ induced by $\hat{\psi}_{X-r}$.

(ii) See [CR1, §4C].

(iii) According to the construction of $S_{\hat{\psi}_{X-r}}$ and $\hat{\psi}_{X-r}$ in Definition 1.6.2, we have $X \in S_{\hat{\psi}_{X-r}}$. Now, $\bar{X} = r$ follows from the construction of $I_{\hat{\psi}_{X-r}}$, $\hat{\psi}_{X-r}$, and ψ_{X-r}. ■

According to the preceding lemma, one gets from Definition 1.6.2 for a given field K and an $r \in K \setminus \{0_K\}$ a modular system with a complete discrete valuation ring and an associated coefficient pair (K, r) such that K is the residue class field of the discrete valuation ring.

Definition 1.6.4 *Let K be a field, fix an $r \in K \setminus \{0_K\}$, and choose an indeterminate X over K. Then the modular system $\hat{\mathcal{K}}_{(K,r)}$ is defined by means of the discrete additive valuation $\hat{\psi}_{X-r}$ on $K\hat{(X)}$ as*

$$\hat{\mathcal{K}}_{(K,r)} = \left(K\hat{(X)}, \hat{\psi}_{X-r}, S_{\hat{\psi}_{X-r}}, I_{\hat{\psi}_{X-r}}, X, K\right).$$

Lemma 1.6.5 *Let K be an arbitrary field and choose an indeterminate X over K. Then the Hecke algebras $\mathcal{H}_{A_{n-1}}^{(K(X),X)}$ and $\mathcal{H}_{A_{n-1}}^{(K\hat{(X)},X)}$ are semisimple.*

Proof. X is transcendent over K. Thus, the claim follows from Theorem 1.3.21 and Definition 1.2.2. ■

Now we consider in addition to a given modular system

$$\mathcal{K} = (Q, \psi, S, I, a, F)$$

also the modular systems

$$\mathcal{K}_{(Q,a)} = \left(Q(Y), \psi_{Y-a}, Q[Y]_{(Y-a)Q[Y]}, (Y-a) \cdot Q[Y]_{(Y-a)Q[Y]}, Y, Q\right)$$

and

$$\mathcal{K}_{(F,\bar{a})} = \left(F(Z), \psi_{Z-\bar{a}}, F[Z]_{(Z-\bar{a})F[Z]}, (Z-\bar{a}) \cdot F[Z]_{(Z-\bar{a})F[Z]}, Z, F\right)$$

where Y is an indeterminate over Q and Z is an indeterminate over F. In the remainder of this section, we could use, instead of $\mathcal{K}_{(Q,a)}$ and $\mathcal{K}_{(F,\bar{a})}$, the modular systems $\hat{\mathcal{K}}_{(Q,a)}$ and $\hat{\mathcal{K}}_{(F,\bar{a})}$ as well. From the three modular systems $\mathcal{K}$, $\mathcal{K}_{(Q,a)}$, and

$\mathcal{K}_{(F,\bar{a})}$, we get four Hecke algebras over fields and furthermore four Grothendieck groups of categories of finitely generated modules. With these, we can construct the following diagram.

$$\begin{tikzcd}
G_0(\mathcal{H}_{A_{n-1}}^{(Q(Y),Y)}) \arrow[r, "\alpha^{\mathcal{H}_n}_{(Q(Y),Y)(F(Z),Z)}"] \arrow[d, "D^{\mathcal{H}}_{n,\mathcal{K}_{(Q,a)}}"'] & G_0(\mathcal{H}_{A_{n-1}}^{(F(Z),Z)}) \arrow[d, "D^{\mathcal{H}}_{n,\mathcal{K}_{(F,\bar{a})}}"] \\
G_0(\mathcal{H}_{A_{n-1}}^{(Q,a)}) \arrow[r, "D^{\mathcal{H}}_{n,\mathcal{K}}"'] & G_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})
\end{tikzcd} \tag{1.31}$$

Here, $D^{\mathcal{H}}_{n,\mathcal{K}_{(Q,a)}}$, $D^{\mathcal{H}}_{n,\mathcal{K}_{(F,\bar{a})}}$, and $D^{\mathcal{H}}_{n,\mathcal{K}}$ are decomposition maps as in Definition 1.4.16. Furthermore, $\alpha^{\mathcal{H}_n}_{(Q(Y),Y)(F(Z),Z)}$ is an isomorphism as in Definition 1.4.4, it exists according to Lemma 1.6.5. Similar diagrams are considered more generally in [GEC, Section 4, Section 5] for Hecke algebras of arbitrary type. The following lemma is proved in [GEC, Section 4] for Hecke algebras of arbitrary type.

Lemma 1.6.6 *The diagram (1.31) is commutative.*

Proof. According to Lemma 1.6.5 and Lemma 1.4.3.(i), it suffices to show the commutativity for every $[S^{\lambda}_{(Q(Y),Y)}] \in G_0(\mathcal{H}_n^{(Q(Y),Y)})$ with $\lambda \vdash n$. According to Definition 1.4.4, Definition 1.4.16, and Lemma 1.5.2, we have

$$\begin{aligned}
D^{\mathcal{H}}_{n,\mathcal{K}_{(F,\bar{a})}}(\alpha^{\mathcal{H}_n}_{(Q(Y),Y)(F(Z),Z)}([S^{\lambda}_{(Q(Y),Y)}])) &= D^{\mathcal{H}}_{n,\mathcal{K}_{(F,\bar{a})}}([S^{\lambda}_{(F(Z),Z)}]) \\
&= [S^{\lambda}_{(F,\bar{a})}] \\
&= D^{\mathcal{H}}_{n,\mathcal{K}}([S^{\lambda}_{(Q,a)}]) \\
&= D^{\mathcal{H}}_{n,\mathcal{K}}(D^{\mathcal{H}}_{n,\mathcal{K}_{(Q,a)}}([S^{\lambda}_{(Q(Y),Y)}])).
\end{aligned}$$

This proves the claim. ■

Corollary 1.6.7 *The decomposition matrices $\Delta^{\mathcal{H}}_{n,\mathcal{K}}$, $\Delta^{\mathcal{H}}_{n,\mathcal{K}_{(Q,a)}}$, and $\Delta^{\mathcal{H}}_{n,\mathcal{K}_{(F,\bar{a})}}$ satisfy*

$$\Delta^{\mathcal{H}}_{n,\mathcal{K}_{(Q,a)}} \cdot \Delta^{\mathcal{H}}_{n,\mathcal{K}} = \Delta^{\mathcal{H}}_{n,\mathcal{K}_{(F,\bar{a})}}. \tag{1.32}$$

Proof. This follows from Lemma 1.6.6 by considering the matrices representing the maps in the diagram (1.31) with respect to the bases of the Grothendieck groups in that diagram described in Lemma 1.4.2.(i) and Lemma 1.4.3.(i). See also Definition 1.4.4, Definition 1.4.17, and Theorem 1.3.21. ■

Now, the independence of the decomposition matrix of the discrete valuation ring S in the modular system $\mathcal{K}$ can be shown.

Theorem 1.6.8 *Let $\mathcal{K} = (Q, \psi, S, I, a, F)$ be an arbitrary modular system. Then, for every $\lambda \in \Pi_{n,e_Q(a)}$ and every $\mu \in \Pi_{n,e_F(\bar{a})}$, the decomposition number $d_{\lambda\mu}^{n,\mathcal{K}}$ is independent of the discrete valuation ring S in $\mathcal{K}$.*

Proof. The coefficient pairs (Q, a) and $(F, \bar{a})$ provide the modular systems $\mathcal{K}_{(Q,a)}$ and $\mathcal{K}_{(F,\bar{a})}$ as in Definition 1.6.1. With these, we can build a rectangle (1.31) in which Lemma 1.6.5, Lemma 1.6.6, Lemma 1.5.8, and Corollary 1.6.7 hold. Now Corollary 1.6.7, Lemma 1.5.8.(ii), and general facts from linear algebra show that the matrix $\Delta_{n,\mathcal{K}}^{\mathcal{H}}$ is completely determined by the matrices $\Delta_{n,\mathcal{K}_{(Q,a)}}^{\mathcal{H}}$ and $\Delta_{n,\mathcal{K}_{(F,\bar{a})}}^{\mathcal{H}}$. But these latter two matrices are independent of S, since S doesn't occur as a coefficient ring in the modular systems $\mathcal{K}_{(Q,a)}$ and $\mathcal{K}_{(F,\bar{a})}$. Thus, the matrix $\Delta_{n,\mathcal{K}}^{\mathcal{H}} = \left(d_{\lambda\mu}^{n,\mathcal{K}}\right)_{\substack{\lambda\in\Pi_{n,e_Q(a)}\\ \mu\in\Pi_{n,e_F(\bar{a})}}}$ also is independent of S. ■

1.7 Modular reduction and projective modules of Hecke algebras of type A

In the following, further properties of decomposition maps under certain assumptions on the employed modular systems are shown. We proceed as in [CR1, §18]. $n \in \mathbb{N}$ is still fixed.

First, we introduce projective class groups (see [CR1, §16B]).

Definition 1.7.1 *Let R be an integral domain and fix a unit $q \in R$.*

(i) The isomorphism class of an $\mathcal{H}_{A_{n-1}}^{(R,q)}$-module M is denoted by $[M]$. The set of all isomorphism classes of all finitely generated projective right $\mathcal{H}_{A_{n-1}}^{(R,q)}$-modules is denoted by

$$\mathcal{P}(\mathcal{H}_{A_{n-1}}^{(R,q)}) = \left\{[P] \,\middle|\, P \text{ is a finitely generated projective right } \mathcal{H}_{A_{n-1}}^{(R,q)}\text{-module}\right\}.$$

(ii) The $\mathbb{Z}$-submodule $V_0(\mathcal{H}_{A_{n-1}}^{(R,q)})$ of the free module $\oplus_{[P]\in\mathcal{P}(\mathcal{H}_n^{(R,q)})}\mathbb{Z}[P]$ is defined as the $\mathbb{Z}$-span of the set

$$\left\{ [P]-[P']-[P''] \;\middle|\; \begin{array}{c} [P],[P'],[P''] \in \mathcal{P}(\mathcal{H}_{A_{n-1}}^{(R,q)}) \\ \text{such that there is an isomorphism} \\ P \simeq P' \oplus P'' \\ \text{of } \mathcal{H}_{A_{n-1}}^{(R,q)}\text{-modules} \end{array} \right\}.$$

With this, the projective class group of $\mathcal{H}_{A_{n-1}}^{(R,q)}$ is defined as

$$K_0(\mathcal{H}_{A_{n-1}}^{(R,q)}) = \left(\bigoplus_{[P]\in\mathcal{P}(\mathcal{H}_n^{(R,q)})} \mathbb{Z}[P] \right) / V_0(\mathcal{H}_{A_{n-1}}^{(R,q)}).$$

For the remainder of this section, we fix a modular system

$$\mathcal{K} = (Q, \psi, S, I, a, F)$$

as in Definition 1.4.8 which satisfies the following conditions.

(i) The algebra $\mathcal{H}_{A_{n-1}}^{(Q,a)}$ is semisimple.

(ii) The discrete valuation ring S is complete with respect to the valuation ψ. (1.33)

Lemma 1.7.2 *(i) Let $\mu \in \Pi_{n,e_F(\bar{a})}$. Then the irreducible module $D_{(F,\bar{a})}^{\mu}$ has an indecomposable projective cover*

$$P^{\mu} = P_{(F,\bar{a})}^{\mu}.$$

$P_{(F,\bar{a})}^{\mu}$ is a finitely generated projective indecomposable right $\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}$-module and isomorphic to the right ideal $f_{\mathcal{H}_n^{(F,\bar{a})}}^{\mu}\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}$ generated by an appropriate indecomposable idempotent $f_{\mathcal{H}_n^{(F,\bar{a})}}^{\mu} \in \mathcal{H}_{A_{n-1}}^{(F,\bar{a})}$. $D_{(F,\bar{a})}^{\mu}$ is the only irreducible quotient of $P_{(F,\bar{a})}^{\mu}$.

(ii) The set

$$\left\{ P_{(F,\bar{a})}^{\mu} \,\middle|\, \mu \in \Pi_{n,e_F(\bar{a})} \right\}$$

is a complete system of representatives of the isomorphism classes of finitely generated projective indecomposable right $\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}$-modules. It is parameterized by the set $\Pi_{n,e_F(\bar{a})}$.

(iii) We have

$$K_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}) = \bigoplus_{\mu\in\Pi_{n,e_F(\bar{a})}} \mathbb{Z}[P_{(F,\bar{a})}^{\mu}].$$

(iv) Let $[P] \in \mathcal{P}(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$. Then we have in $K_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$

$$[P] = \sum_{\mu\in\Pi_{n,e_F(\bar{a})}} x_{\mu}^{[P]}[P_{(F,\bar{a})}^{\mu}]$$

with coefficients $x_{\mu}^{[P]} \in \mathbb{N}_0$ for $\mu \in \Pi_{n,e_F(\bar{a})}$.

Proof. The statements referred to in the following can be applied here because of the properties (1.33) of the modular system $\mathcal{K}$. See also [CR1, introductory remarks to Theorem 6.23].

(i) This follows from [CR1, Summary 18.1.(i)] and Theorem 1.3.20.(iii).

(ii) This follows from [CR1, Summary 18.1.(ii), Summary 18.1.(i)] and statement (i).

(iii) This follows from [CR1, Proposition 16.7] and statement (ii).

(iv) This follows from [CR1, proof of Proposition 16.7] and statement (ii). ■

Lemma 1.7.3 *(i) For every $\mu \in \Pi_{n,e_F(\bar{a})}$, there is a finitely generated projective indecomposable right $\mathcal{H}^{(S,a)}_{A_{n-1}}$-module*

$$P^{\mu} = P^{\mu}_{(S,a)}$$

such that

$$\overline{P^{\mu}_{(S,a)}} = P^{\mu}_{(F,\bar{a})}$$

holds. $P^{\mu}_{(S,a)}$ is isomorphic to the right ideal $f^{\mu}_{\mathcal{H}^{(S,a)}_n}\mathcal{H}^{(S,a)}_{A_{n-1}}$ generated by an appropriate indecomposable idempotent $f^{\mu}_{\mathcal{H}^{(S,a)}_n} \in \mathcal{H}^{(S,a)}_{A_{n-1}}$. For this idempotent, we have

$$\overline{f^{\mu}_{\mathcal{H}^{(S,a)}_n}}\mathcal{H}^{(F,\bar{a})}_{A_{n-1}} \simeq P^{\mu}_{(F,\bar{a})}.$$

(ii) The set

$$\left\{ P^{\mu}_{(S,a)} \,\middle|\, \mu \in \Pi_{n,e_F(\bar{a})} \right\}$$

is a complete system of representatives of the isomorphism classes of finitely generated projective indecomposable right $\mathcal{H}^{(S,a)}_{A_{n-1}}$-modules. It is parameterized by the set $\Pi_{n,e_F(\bar{a})}$.

(iii) We have

$$K_0(\mathcal{H}^{(S,a)}_{A_{n-1}}) = \bigoplus_{\mu \in \Pi_{n,e_F(\bar{a})}} \mathbb{Z}[P^{\mu}_{(S,a)}].$$

(iv) Let $[P] \in \mathcal{P}(\mathcal{H}^{(S,a)}_{A_{n-1}})$. Then we have in $K_0(\mathcal{H}^{(S,a)}_{A_{n-1}})$

$$[P] = \sum_{\mu \in \Pi_{n,e_F(\bar{a})}} x^{[P]}_{\mu}[P^{\mu}_{(S,a)}]$$

with coefficients $x^{[P]}_{\mu} \in \mathbb{N}_0$ for $\mu \in \Pi_{n,e_F(\bar{a})}$.

Proof. The statements referred to in the following can be applied here because of the properties (1.33) of the modular system $\mathcal{K}$. See also [CR1, introductory remarks to Theorem 6.23].

(i) This follows from [CR1, Summary 18.1.(iii), Summary 18.1.(i)] and Theorem 1.3.20.(iii).

(ii) This follows from [CR1, Summary 18.1.(iv), Summary 18.1.(iii)] and statement (i).

(iii) This follows from [CR1, Proposition 16.7] and statement (ii).

(iv) This follows from [CR1, proof of Proposition 16.7] and statement (ii). ∎

Now, certain homomorphisms between the Grothendieck groups $G_0(\mathcal{H}_{A_{n-1}}^{(Q,a)})$ and $G_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$ and the projective class group $K_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$ will be considered. The decomposition map

$$D_{n,\mathcal{K}}^{\mathcal{H}} : G_0(\mathcal{H}_{A_{n-1}}^{(Q,a)}) \to G_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$$

was introduced in Definition 1.4.16. Furthermore, the inclusion $\eta_{n,(F,\bar{a})}^{\mathcal{H}}$ of the category of the finitely generated projective $\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}$-modules in the category of all finitely generated $\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}$-modules induces a homomorphism from $K_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$ to $G_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$.

Definition 1.7.4 *The homomorphism*

$$C_{n,\mathcal{K}}^{\mathcal{H}} : K_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}) \to G_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}), \quad [P] \mapsto [\eta_{n,(F,\bar{a})}^{\mathcal{H}}(P)] = [P] \in G_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$$

induced by $\eta_{n,(F,\bar{a})}^{\mathcal{H}}$ *is called the Cartan map for Hecke algebras associated with the degree* n *and the modular system* $\mathcal{K}$*. The matrix*

$$\mathcal{C}_{n,\mathcal{K}}^{\mathcal{H}} = \left(\mathcal{C}_{n,\mathcal{K}}^{\mathcal{H}}(\lambda,\mu)\right)_{\substack{\lambda\in\Pi_{n,e_F(\bar{a})} \\ \mu\in\Pi_{n,e_F(\bar{a})}}}$$

representing the map $C_{n,\mathcal{K}}^{\mathcal{H}}$ *with respect to the basis* $\left\{[P_{(F,\bar{a})}^{\lambda}] \,\middle|\, \lambda \in \Pi_{n,e_F(\bar{a})}\right\}$ *of the* $\mathbb{Z}$*-module* $K_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$ *from Lemma 1.7.2.(iii) and the basis* $\left\{[D_{(F,\bar{a})}^{\lambda}] \,\middle|\, \lambda \in \Pi_{n,e_F(\bar{a})}\right\}$ *of the* $\mathbb{Z}$*-module* $G_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$ *from Lemma 1.4.2.(i), whose integer entries* $\mathcal{C}_{n,\mathcal{K}}^{\mathcal{H}}(\lambda,\mu)$ *for* $\lambda,\mu \in \Pi_{n,e_F(\bar{a})}$ *are uniquely determined by*

$$C_{n,\mathcal{K}}^{\mathcal{H}}([P_{(F,\bar{a})}^{\lambda}]) = \sum_{\mu\in\Pi_{n,e_F(\bar{a})}} \mathcal{C}_{n,\mathcal{K}}^{\mathcal{H}}(\lambda,\mu)[D_{(F,\bar{a})}^{\mu}],$$

is called the Cartan matrix for Hecke algebras associated with the degree n *and the modular system* $\mathcal{K}$*.*

Remark 1.7.5 *The Cartan map exists more generally for Hecke algebras over integral domains. However, in order to define the Cartan matrix as in Definition 1.7.4,* *the coefficient ring should be a field. The Cartan map and also the Cartan matrix both depend only on the coefficient pair directly involved, a whole modular system is not required. However, this degree of generality is not needed here. In what follows, Cartan maps and Cartan matrices will always occur in connection with whole modular systems, as in Definition 1.7.4. This motivates the notation chosen here.*

Finally, we introduce a homomorphism from $K_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$ to $G_0(\mathcal{H}_{A_{n-1}}^{(Q,a)})$. This is done in two steps. First, Lemma 1.2.8.(iii), Definition 1.4.14, and Definition 1.7.1 show that the following construction is meaningful.

Definition 1.7.6 *The homomorphism*

$$\bar{\cdot} : K_0(\mathcal{H}_{A_{n-1}}^{(S,a)}) \to K_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$$

is defined by

$$\overline{[P]} = [\bar{P}] \in K_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}) \quad \textit{for} \quad [P] \in \mathcal{P}(\mathcal{H}_{A_{n-1}}^{(S,a)})$$

and $\mathbb{Z}$-linear extension.

Lemma 1.7.7 *The homomorphism*

$$\bar{\cdot} : K_0(\mathcal{H}_{A_{n-1}}^{(S,a)}) \to K_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$$

from Definition 1.7.6 is an isomorphism.

Proof. See [CR1, Theorem 18.2]. ■

Furthermore, it follows from Lemma 1.2.8, Definition 1.4.1, and Definition 1.7.1 that the next definition is meaningful.

Definition 1.7.8 *The homomorphism*

$$- \otimes_S Q : K_0(\mathcal{H}_{A_{n-1}}^{(S,a)}) \to G_0(\mathcal{H}_{A_{n-1}}^{(Q,a)})$$

is defined by

$$[P] \otimes_S Q = [P \otimes_S Q] \in G_0(\mathcal{H}_{A_{n-1}}^{(Q,a)}) \quad \textit{for} \quad [P] \in \mathcal{P}(\mathcal{H}_{A_{n-1}}^{(S,a)})$$

and $\mathbb{Z}$-linear extension.

Through appropriate composition of the homomorphisms from Definition 1.7.6 and Definition 1.7.8, one obtains the desired map. See also [CR1, (18.3)].

Definition 1.7.9 *The composition of the maps*

$$\bar{(\cdot)}^{-1} : K_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}) \to K_0(\mathcal{H}_{A_{n-1}}^{(S,a)})$$

and

$$- \otimes_S Q : K_0(\mathcal{H}_{A_{n-1}}^{(S,a)}) \to G_0(\mathcal{H}_{A_{n-1}}^{(Q,a)})$$

defines the homomorphism

$$B_{n,\mathcal{K}}^{\mathcal{H}} : K_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}) \to G_0(\mathcal{H}_{A_{n-1}}^{(Q,a)}), \quad x \mapsto B_{n,\mathcal{K}}^{\mathcal{H}}(x) = (- \otimes_S Q)(\bar{(\cdot)}^{-1}(x)).$$

This map is called the Brauer map for Hecke algebras associated with the degree n and the modular system $\mathcal{K}$. The matrix representing $B_{n,\mathcal{K}}^{\mathcal{H}}$ with respect to the basis $\left\{[P_{(F,\bar{a})}^{\lambda}] \,\middle|\, \lambda \in \Pi_{n,e_F(\bar{a})}\right\}$ of the $\mathbb{Z}$-module $K_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$ from Lemma 1.7.2.(iii) and the basis $\left\{[S_{(Q,a)}^{\lambda}] \,\middle|\, \lambda \in \Pi_n\right\}$ of the $\mathbb{Z}$-module $G_0(\mathcal{H}_{A_{n-1}}^{(Q,a)})$ from Lemma 1.4.3.(i), whose integer entries $\mathcal{B}_{n,\mathcal{K}}^{\mathcal{H}}(\lambda, \mu)$ for $\lambda \in \Pi_{n,e_F(\bar{a})}$ and $\mu \in \Pi_n$ are uniquely determined by

$$B_{n,\mathcal{K}}^{\mathcal{H}}([P_{(F,\bar{a})}^{\lambda}]) = \sum_{\mu \in \Pi_n} \mathcal{B}_{n,\mathcal{K}}^{\mathcal{H}}(\lambda, \mu)[S_{(Q,a)}^{\mu}],$$

is denoted by

$$\mathcal{B}_{n,\mathcal{K}}^{\mathcal{H}} = \left(\mathcal{B}_{n,\mathcal{K}}^{\mathcal{H}}(\lambda, \mu)\right)_{\substack{\lambda \in \Pi_{n,e_F(\bar{a})} \\ \mu \in \Pi_n}} .$$

Now, all maps between the considered Grothendieck groups and projective class groups required in the following are available.

Definition 1.7.10 *The diagram*

$$\begin{tikzcd} & G_0(\mathcal{H}_{A_{n-1}}^{(Q,a)}) \arrow[d, "D_{n,\mathcal{K}}^{\mathcal{H}}"] \\ K_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}) \arrow[ur, "B_{n,\mathcal{K}}^{\mathcal{H}}"] \arrow[r, "C_{n,\mathcal{K}}^{\mathcal{H}}"'] & G_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}) \end{tikzcd} \tag{1.34}$$

is called the Cartan-Brauer triangle for Hecke algebras associated with the degree n and the modular system $\mathcal{K}$ or, for short, Cartan-Brauer triangle.

Lemma 1.7.11 *The Cartan-Brauer triangle from Definition 1.7.10 is commutative.*

Proof. According to Lemma 1.7.2.(iii), it suffices to show the commutativity for basis elements $[P^\lambda_{(F,\bar{a})}]$ of $K_0(\mathcal{H}_n^{(F,\bar{a})})$ with $\lambda \in \Pi_{n,e_F(\bar{a})}$. This follows easily from the construction of the maps $D^{\mathcal{H}}_{n,\mathcal{K}}$, $C^{\mathcal{H}}_{n,\mathcal{K}}$, and $B^{\mathcal{H}}_{n,\mathcal{K}}$ in Definition 1.4.16, Definition 1.7.4, and Definition 1.7.9. See also [CR1, Proposition 18.5]. ■

In order to describe further relations between the maps occurring in a Cartan-Brauer triangle, we now introduce bilinear forms on some products of the $\mathbb{Z}$-modules involved. It is shown in [CR1, §18B] that the following bilinear form is well defined.

Definition 1.7.12 *The bilinear form*

$$i^{\mathcal{H}}_{n,(Q,a)} : G_0(\mathcal{H}^{(Q,a)}_{A_{n-1}}) \times G_0(\mathcal{H}^{(Q,a)}_{A_{n-1}}) \to \mathbb{Z}$$

is defined by

$$i^{\mathcal{H}}_{n,(Q,a)}([M],[N]) = \dim_Q \operatorname{Hom}_{\mathcal{H}_n^{(Q,a)}}(M,N)$$

for $[M],[N] \in \mathcal{M}(\mathcal{H}^{(Q,a)}_{A_{n-1}})$ *and bilinear extension.*

According to condition (1.33).(i) and Lemma 1.4.3.(i), the Grothendieck group $G_0(\mathcal{H}^{(Q,a)}_{A_{n-1}})$ has the basis $\left\{[S^\lambda_{(Q,a)}] \,\middle|\, \lambda \in \Pi_n\right\}$. The next lemma states the values of the bilinear form $i^{\mathcal{H}}_{n,(Q,a)}$ on such basis elements.

Lemma 1.7.13 *For* $[S^\lambda_{(Q,a)}], [S^\mu_{(Q,a)}] \in G_0(\mathcal{H}^{(Q,a)}_{A_{n-1}})$ *with* $\lambda, \mu \in \Pi_n$ *we have*

$$i^{\mathcal{H}}_{n,(Q,a)}([S^\lambda_{(Q,a)}],[S^\mu_{(Q,a)}]) = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu \end{cases}.$$

Proof. This follows from Theorem 1.3.21.(ii) and Theorem 1.3.20.(iv). See also [CR1, §18B]. ■

It is shown in [CR1, Proposition 18.8] that the following bilinear form is well defined.

Definition 1.7.14 *The bilinear form*

$$j^{\mathcal{H}}_{n,(F,\bar{a})} : K_0(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}) \times G_0(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}) \to \mathbb{Z}$$

is defined by

$$j^{\mathcal{H}}_{n,(F,\bar{a})}([P],[M]) = \dim_F \operatorname{Hom}_{\mathcal{H}_n^{(F,\bar{a})}}(P,M)$$

for $[P] \in \mathcal{P}(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})$ *and* $[M] \in \mathcal{M}(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})$ *and bilinear extension.*

According to Lemma 1.7.2.(iii) and Lemma 1.4.2.(i), $K_0(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})$ and $G_0(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})$ have the bases $\left\{[P^\lambda_{(F,\bar{a})}] \,\middle|\, \lambda \in \Pi_{n,e_F(\bar{a})}\right\}$ and $\left\{[D^\lambda_{(F,\bar{a})}] \,\middle|\, \lambda \in \Pi_{n,e_F(\bar{a})}\right\}$, respectively. The following lemma states the values of the bilinear form $j^{\mathcal{H}}_{n,(F,\bar{a})}$ on such basis elements.

Lemma 1.7.15 *For* $[P^{\lambda}_{(F,\bar{a})}] \in K_0(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})$ *and* $[D^{\mu}_{(F,\bar{a})}] \in G_0(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})$ *with* $\lambda, \mu \in \Pi_{n,e_F(\bar{a})}$ *we have*

$$j^{\mathcal{H}}_{n,(F,\bar{a})}([P^{\lambda}_{(F,\bar{a})}],[D^{\mu}_{(F,\bar{a})}]) = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu \end{cases} .$$

Proof. This follows from Lemma 1.7.2, Theorem 1.3.20, and [CR1, Proposition 18.8]. ■

Now, the bilinear forms $i^{\mathcal{H}}_{n,(Q,a)}$ and $j^{\mathcal{H}}_{n,(F,\bar{a})}$ provide further relations between the maps $D^{\mathcal{H}}_{n,\mathcal{K}}$, $B^{\mathcal{H}}_{n,\mathcal{K}}$, and $C^{\mathcal{H}}_{n,\mathcal{K}}$.

Lemma 1.7.16 *Let* $\mathcal{K} = (Q, \psi, S, I, a, F)$ *be a modular system satisfying the conditions (1.33). Then the following statements hold.*

(i) The maps $B^{\mathcal{H}}_{n,\mathcal{K}}$ *and* $D^{\mathcal{H}}_{n,\mathcal{K}}$ *are transposes of one another with respect to the bilinear forms* $i^{\mathcal{H}}_{n,(Q,a)}$ *and* $j^{\mathcal{H}}_{n,(F,\bar{a})}$*. This means that for arbitrary elements* $f \in K_0(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})$ *and* $g \in G_0(\mathcal{H}^{(Q,a)}_{A_{n-1}})$ *we have*

$$i^{\mathcal{H}}_{n,(Q,a)}(B^{\mathcal{H}}_{n,\mathcal{K}}(f), g) = j^{\mathcal{H}}_{n,(F,\bar{a})}(f, D^{\mathcal{H}}_{n,\mathcal{K}}(g)).$$

(ii) The representing matrices $\Delta^{\mathcal{H}}_{n,\mathcal{K}}$ *of* $D^{\mathcal{H}}_{n,\mathcal{K}}$ *from Definition 1.4.17,* $\mathcal{B}^{\mathcal{H}}_{n,\mathcal{K}}$ *of* $B^{\mathcal{H}}_{n,\mathcal{K}}$ *from Definition 1.7.9, and* $\mathcal{C}^{\mathcal{H}}_{n,\mathcal{K}}$ *of* $C^{\mathcal{H}}_{n,\mathcal{K}}$ *from Definition 1.7.4 satisfy*

$$\mathcal{B}^{\mathcal{H}}_{n,\mathcal{K}} = \left(\Delta^{\mathcal{H}}_{n,\mathcal{K}}\right)^T$$

and

$$\mathcal{C}^{\mathcal{H}}_{n,\mathcal{K}} = \mathcal{B}^{\mathcal{H}}_{n,\mathcal{K}} \cdot \Delta^{\mathcal{H}}_{n,\mathcal{K}} = \left(\Delta^{\mathcal{H}}_{n,\mathcal{K}}\right)^T \cdot \left(\Delta^{\mathcal{H}}_{n,\mathcal{K}}\right).$$

(iii) Let $\mu \in \Pi_{n,e_F(\bar{a})}$*. Then we have in* $G_0(\mathcal{H}^{(Q,a)}_{A_{n-1}})$

$$B^{\mathcal{H}}_{n,\mathcal{K}}([P^{\mu}_{(F,\bar{a})}]) = \sum_{\lambda \in \Pi_n} d^{n,\mathcal{K}}_{\lambda\mu} [S^{\lambda}_{(Q,a)}]$$

with uniquely determined coefficients $d^{n,\mathcal{K}}_{\lambda\mu}$ *for* $\lambda \vdash n$.

Proof. (i) See [CR1, Theorem 18.9].

(ii) See [CR1, Corollary 18.10 and its proof].

(iii) This follows immediately from Definition 1.7.9 and statement (ii). ■

Remark 1.7.17 *The maps* $B^{\mathcal{H}}_{n,\mathcal{K}}$ *and* $C^{\mathcal{H}}_{n,\mathcal{K}}$ *also can be defined in the more general case that* $\mathcal{H}^{(Q,a)}_{A_{n-1}}$ *is not semisimple, and relations between them and the decomposition map can be investigated as well. In that case, the Grothendieck group* $G_0(\mathcal{H}^{(Q,a)}_{A_{n-1}})$ *and the projective class group* $K_0(\mathcal{H}^{(Q,a)}_{A_{n-1}})$ *don't coincide, and, instead of the Cartan-Brauer triangle from Definition 1.7.10, one obtains a rectangle. For such considerations see [GR, Section 2]. They will not be required in the following.*

Corollary 1.7.18 *Let $\mathcal{K} = (Q, \psi, S, I, a, F)$ be a modular system satisfying the conditions (1.33). Then the Brauer map*

$$B^{\mathcal{H}}_{n,\mathcal{K}} : K_0(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}) \to G_0(\mathcal{H}^{(Q,a)}_{A_{n-1}})$$

is injective.

Proof. This follows easily from Lemma 1.7.16.(i), Lemma 1.5.8.(i), and Lemma 1.7.15. ■

1.8 Block theory of Hecke algebras of type A

Now, the decomposition of Hecke algebras of type A and their module categories in blocks is described. We proceed as in [CR2, §56]. The central results in this section are from [DJ2] and [JM].

As before, we fix an $n \in \mathbb{N}$. Furthermore, let

$$\mathcal{K} = (Q, \psi, S, I, a, F)$$

be a modular system as in Definition 1.4.8 with the following additional properties.

(i) The algebra $\mathcal{H}^{(Q,a)}_{A_{n-1}}$ is semisimple.

(ii) The discrete valuation ring S is complete with respect to the valuation ψ. (1.35)

Most of the following statements hold under weaker assumptions on $\mathcal{K}$, but this level of generality is not required here.

Definition 1.8.1 *Let R be an integral domain and choose a unit $q \in R$. Then the center of the Hecke algebra $\mathcal{H}^{(R,q)}_{A_{n-1}}$ is denoted by*

$$Z(\mathcal{H}^{(R,q)}_{A_{n-1}}).$$

Lemma 1.8.2 *$Z(\mathcal{H}^{(S,a)}_{A_{n-1}})$ is free over S with rank $|\Pi_n|$. It has an S-basis*

$$\{c_\lambda(S, a) \mid \lambda \in \Pi_n\}$$

with the following properties.

(i) $\{c_\lambda(S, a) \otimes_S 1_Q \mid \lambda \in \Pi_n\} \subseteq \mathcal{H}^{(S,a)}_{A_{n-1}} \otimes_S Q = \mathcal{H}^{(Q,a)}_{A_{n-1}}$ *is a Q-basis of* $Z(\mathcal{H}^{(Q,a)}_{A_{n-1}})$.

(ii) $\{c_\lambda(S, a) \otimes_S 1_F \mid \lambda \in \Pi_n\} = \left\{\overline{c_\lambda(S, a)} \,\middle|\, \lambda \in \Pi_n\right\} \subseteq \overline{\mathcal{H}^{(S,a)}_{A_{n-1}}} = \mathcal{H}^{(F,\bar{a})}_{A_{n-1}}$ *is an F-basis of* $Z(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})$.

From this, we get

$$Z(\mathcal{H}_{A_{n-1}}^{(Q,a)}) = Z(\mathcal{H}_{A_{n-1}}^{(S,a)}) \otimes_S Q$$

and

$$Z(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}) = Z(\mathcal{H}_{A_{n-1}}^{(S,a)}) \otimes_S F.$$

This means that $Z(\mathcal{H}_{A_{n-1}}^{(S,a)})$ is a full $Z(\mathcal{H}_{A_{n-1}}^{(S,a)})$-lattice in $Z(\mathcal{H}_{A_{n-1}}^{(Q,a)})$ and $Z(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$ is the reduction of $Z(\mathcal{H}_{A_{n-1}}^{(S,a)})$ modulo I.

Proof. This follows from [GR, Theorem 5.2]. See also [DJ2, Section 2]. ■

Lemma 1.8.3 *(i) In $\mathcal{H}_{A_{n-1}}^{(S,a)}$, there is a decomposition of $1_{\mathcal{H}_A^{(S,a)}}$ in central primitive orthogonal idempotents*

$$1_{\mathcal{H}_A^{(S,a)}} = \sum_{i=1}^{m} b_{\mathrm{Idemp}}^{(i)}(\mathcal{H}_{A_{n-1}}^{(S,a)})$$

with an appropriate $m \in \mathbb{N}$. The summands $b_{\mathrm{Idemp}}^{(i)}(\mathcal{H}_{A_{n-1}}^{(S,a)})$ with $i \in \{1, \ldots, m\}$ are uniquely determined up to ordering in the sum.

(ii) The application of the functor $- \otimes_S F$ to the decomposition of $1_{\mathcal{H}_A^{(S,a)}}$ from statement (i) produces a decomposition of $1_{\mathcal{H}_A^{(F,\bar{a})}}$ in central primitive orthogonal idempotents

$$1_{\mathcal{H}_A^{(F,\bar{a})}} = 1_{\mathcal{H}_A^{(S,a)}} \otimes_S 1_F = \overline{1_{\mathcal{H}_A^{(S,a)}}} = \sum_{i=1}^{m} b_{\mathrm{Idemp}}^{(i)}(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$$

with the same $m \in \mathbb{N}$ as in statement (i). The summands $b_{\mathrm{Idemp}}^{(i)}(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$ with $i \in \{1, \ldots, m\}$ are uniquely determined up to ordering in the sum. This ordering is determined by the ordering of the elements $b_{\mathrm{Idemp}}^{(i)}(\mathcal{H}_{A_{n-1}}^{(S,a)})$ chosen in statement (i) and the relations

$$b_{\mathrm{Idemp}}^{(i)}(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}) = b_{\mathrm{Idemp}}^{(i)}(\mathcal{H}_{A_{n-1}}^{(S,a)}) \otimes_S 1_F = \overline{b_{\mathrm{Idemp}}^{(i)}(\mathcal{H}_{A_{n-1}}^{(S,a)})}$$

for $i \in \{1, \ldots, m\}$.

(iii) The application of the functor $- \otimes_S Q$ to the decomposition of $1_{\mathcal{H}_A^{(S,a)}}$ from statement (i) produces a decomposition of $1_{\mathcal{H}_A^{(Q,a)}}$ in central orthogonal idempotents

$$1_{\mathcal{H}_A^{(Q,a)}} = 1_{\mathcal{H}_A^{(S,a)}} \otimes_S 1_Q = \sum_{i=1}^{m} b_{\mathrm{Idemp}}^{(i)}(\mathcal{H}_{A_{n-1}}^{(Q,a)})$$

with the same $m \in \mathbb{N}$ *as in statement (i). The ordering of the summands* $b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(Q,a)}_{A_{n-1}})$ *with* $i \in \{1, \ldots, m\}$ *is determined by the ordering of the elements* $b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(S,a)}_{A_{n-1}})$ *chosen in statement (i) and the relations*

$$b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(Q,a)}_{A_{n-1}}) = b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(S,a)}_{A_{n-1}}) \otimes_S 1_Q$$

for $i \in \{1, \ldots, m\}$.

(iv) *The decomposition of* $1_{\mathcal{H}^{(S,a)}_A}$ *from statement (i) induces a decomposition of* $\mathcal{H}^{(S,a)}_{A_{n-1}}$ *in indecomposable two-sided ideals*

$$\mathcal{H}^{(S,a)}_{A_{n-1}} = \bigoplus_{i=1}^{m} B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(S,a)}_{A_{n-1}})$$

with the same $m \in \mathbb{N}$ *as in statement (i). The summands* $B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(S,a)}_{A_{n-1}})$ *with* $i \in \{1, \ldots, m\}$ *are uniquely determined up to ordering in the sum. This ordering is determined by the ordering of the* $b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(S,a)}_{A_{n-1}})$ *chosen in statement (i) and the relations*

$$b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(S,a)}_{A_{n-1}}) \in B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(S,a)}_{A_{n-1}})$$

for $i \in \{1, \ldots, m\}$. *For every* $i \in \{1, \ldots, m\}$, *we have*

$$\begin{aligned} B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(S,a)}_{A_{n-1}}) &= \mathcal{H}^{(S,a)}_{A_{n-1}} b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(S,a)}_{A_{n-1}}) \\ &= b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(S,a)}_{A_{n-1}})\mathcal{H}^{(S,a)}_{A_{n-1}} \\ &= \mathcal{H}^{(S,a)}_{A_{n-1}} b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(S,a)}_{A_{n-1}})\mathcal{H}^{(S,a)}_{A_{n-1}} \\ &= b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(S,a)}_{A_{n-1}})\mathcal{H}^{(S,a)}_{A_{n-1}} b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(S,a)}_{A_{n-1}}). \end{aligned}$$

Every ideal $B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(S,a)}_{A_{n-1}})$ *is an* S*-algebra with multiplicative neutral element* $b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(S,a)}_{A_{n-1}})$.

(v) *The application of the functor* $- \otimes_S F$ *to the decomposition of* $\mathcal{H}^{(S,a)}_{A_{n-1}}$ *from statement (iv) produces a decomposition of* $\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}$ *in indecomposable two-sided ideals*

$$\mathcal{H}^{(F,\bar{a})}_{A_{n-1}} = \mathcal{H}^{(S,a)}_{A_{n-1}} \otimes_S F = \overline{\mathcal{H}^{(S,a)}_{A_{n-1}}} = \bigoplus_{i=1}^{m} B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})$$

with the same $m \in \mathbb{N}$ *as in statement (i). The summands* $B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})$ *with* $i \in \{1, \ldots, m\}$ *are uniquely determined up to ordering in the sum. This ordering is determined by the ordering of the ideals* $B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(S,a)}_{A_{n-1}})$ *chosen in statement (iv) and the relations*

$$B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}) = B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(S,a)}_{A_{n-1}}) \otimes_S F = \overline{B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(S,a)}_{A_{n-1}})}$$

for $i \in \{1, \ldots, m\}$, *or equivalently by the ordering of the* $b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})$ *chosen in statement (ii) and the relations*

$$b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}) \in B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})$$

for $i \in \{1, \ldots, m\}$. *For every* $i \in \{1, \ldots, m\}$, *we have*

$$\begin{aligned} B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}) &= \mathcal{H}^{(F,\bar{a})}_{A_{n-1}} b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}) \\ &= b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})\mathcal{H}^{(F,\bar{a})}_{A_{n-1}} \\ &= \mathcal{H}^{(F,\bar{a})}_{A_{n-1}} b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})\mathcal{H}^{(F,\bar{a})}_{A_{n-1}} \\ &= b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})\mathcal{H}^{(F,\bar{a})}_{A_{n-1}} b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}). \end{aligned}$$

Every ideal $B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})$ *is an* F*-algebra with multiplicative neutral element* $b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})$.

(vi) The application of the functor $- \otimes_S Q$ *to the decomposition of* $\mathcal{H}^{(S,a)}_{A_{n-1}}$ *from statement (iv) produces a decomposition of* $\mathcal{H}^{(Q,a)}_{A_{n-1}}$ *in two-sided ideals*

$$\mathcal{H}^{(Q,a)}_{A_{n-1}} = \mathcal{H}^{(S,a)}_{A_{n-1}} \otimes_S Q = \bigoplus_{i=1}^{m} B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(Q,a)}_{A_{n-1}})$$

with the same $m \in \mathbb{N}$ *as in statement (i). The ordering of the summands* $B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(Q,a)}_{A_{n-1}})$ *with* $i \in \{1, \ldots, m\}$ *is determined by the ordering of the ideals* $B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(S,a)}_{A_{n-1}})$ *chosen in statement (iv) and the relations*

$$B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(Q,a)}_{A_{n-1}}) = B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(S,a)}_{A_{n-1}}) \otimes_S Q$$

for $i \in \{1, \ldots, m\}$, *or equivalently by the ordering of the* $b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(Q,a)}_{A_{n-1}})$ *chosen in statement (iii) and the relations*

$$b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(Q,a)}_{A_{n-1}}) \in B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(Q,a)}_{A_{n-1}})$$

for $i \in \{1, \ldots, m\}$. *For every* $i \in \{1, \ldots, m\}$, *we have*

$$\begin{aligned} B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(Q,a)}_{A_{n-1}}) &= \mathcal{H}^{(Q,a)}_{A_{n-1}} b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(Q,a)}_{A_{n-1}}) \\ &= b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(Q,a)}_{A_{n-1}})\mathcal{H}^{(Q,a)}_{A_{n-1}} \\ &= \mathcal{H}^{(Q,a)}_{A_{n-1}} b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(Q,a)}_{A_{n-1}})\mathcal{H}^{(Q,a)}_{A_{n-1}} \\ &= b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(Q,a)}_{A_{n-1}})\mathcal{H}^{(Q,a)}_{A_{n-1}} b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(Q,a)}_{A_{n-1}}). \end{aligned}$$

Every ideal $B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(Q,a)}_{A_{n-1}})$ *is a* Q*-algebra with multiplicative neutral element* $b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(Q,a)}_{A_{n-1}})$.

Proof. According to the remark following Definition 1.4.12, the remark following Definition 1.4.14, and Lemma 1.8.2, Hecke algebras of type A and their centers behave like group algebras and their centers with respect to the elementary constructions of modular representation theory (see [CR2, §56A and §56B]). Furthermore, because of property (1.35).(ii), the modular system $\mathcal{K}$ satisfies condition (a) in [CR2, Definition 56.3]. This shows that the arguments used in [CR2, §56A] for group algebras can be directly translated to Hecke algebras of type A to prove the various claims of the lemma.

(i) This follows from [CR2, Proposition 56.5.(i)].

(ii) This follows from [CR2, Proposition 56.5].

(iii) This follows immediately from statement (i).

(iv) This follows immediately from statement (i).

(v) This follows immediately from statements (ii) and (iv).

(vi) This follows immediately from statements (iii) and (iv). ■

Definition 1.8.4 *Let* $(R,q) \in \{(S,a),(F,\bar{a}),(Q,a)\}$.

(i) We call the central idempotents $b^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(R,q)}_{A_{n-1}})$ *with* $i \in \{1,\dots,m\}$ *from Lemma 1.8.3 the block idempotents of* $\mathcal{H}^{(R,q)}_{A_{n-1}}$.

(ii) The two-sided ideals $B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(R,q)}_{A_{n-1}})$ *with* $i \in \{1,\dots,m\}$ *from Lemma 1.8.3 are called the block ideals of* $\mathcal{H}^{(R,q)}_{A_{n-1}}$.

(iii) We call the categories of finitely generated right modules of the R*-algebras* $B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(R,q)}_{A_{n-1}})$ *with* $i \in \{1,\dots,m\}$ *from Lemma 1.8.3 the block categories of* $\mathcal{H}^{(R,q)}_{A_{n-1}}$. *For every* $i \in \{1,\dots,m\}$, *the block category of* $B^{(i)}_{\mathrm{Ideal}}(\mathcal{H}^{(R,q)}_{A_{n-1}})$*-modules is denoted by*

$$B^{(i)}_{\mathrm{Kat}}(\mathcal{H}^{(R,q)}_{A_{n-1}}).$$

Definition 1.8.5 *Let* $(R,q) \in \{(S,a),(F,\bar{a}),(Q,a)\}$ *and let* M *be a finitely generated right* $\mathcal{H}^{(R,q)}_{A_{n-1}}$*-module. If we have for an* $i \in \{1,\dots,m\}$ *with* $m \in \mathbb{N}$ *from Lemma 1.8.3.(i)*

$$Mb^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(R,q)}_{A_{n-1}}) = M$$

then we say that M *belongs to the block category* $B^{(i)}_{\mathrm{Kat}}(\mathcal{H}^{(R,q)}_{A_{n-1}})$.

Lemma 1.8.6 *Let* $(R,q) \in \{(S,a),(F,\bar{a}),(Q,a)\}$. *Furthermore, let* M *be a finitely generated right* $\mathcal{H}^{(R,q)}_{A_{n-1}}$*-module. Then we have with the notation from Definition 1.8.4*

$$M = \bigoplus_{i=1}^{m} \Big(Mb^{(i)}_{\mathrm{Idemp}}(\mathcal{H}^{(R,q)}_{A_{n-1}}) \Big) \tag{1.36}$$

where the right hand side is a direct sum of finitely generated right $\mathcal{H}_{A_{n-1}}^{(R,q)}$*-modules. Furthermore, for every* $i \in \{1,\dots,m\}$*, the summand* $Mb_{\text{Idemp}}^{(i)}(\mathcal{H}_{A_{n-1}}^{(R,q)})$ *belongs to the block category* $B_{\text{Kat}}^{(i)}(\mathcal{H}_{A_{n-1}}^{(R,q)})$.

Proof. This follows from the decomposition of $1_{\mathcal{H}_A^{(R,q)}}$ in block idempotents in Lemma 1.8.3. Since the block idempotents are orthogonal, (1.36) is a decomposition of M in an R-direct sum. Since the block idempotents are central, the summands on the right hand side of (1.36) are $\mathcal{H}_n^{(R,q)}$-modules. Finally, for every $i \in \{1,\dots,m\}$, the summand $Mb_{\text{Idemp}}^{(i)}(\mathcal{H}_n^{(R,q)})$ belongs to the block category $B_{\text{Kat}}^{(i)}(\mathcal{H}_n^{(R,q)})$ since $b_{\text{Idemp}}^{(i)}(\mathcal{H}_n^{(R,q)})$ is idempotent. ■

Definition 1.8.7 *Let* $(R,q) \in \{(S,a),(F,\bar{a}),(Q,a)\}$ *and let* M *be a finitely generated right* $\mathcal{H}_{A_{n-1}}^{(R,q)}$*-module. Then the decomposition (1.36) of* M *from Lemma 1.8.6 is called the block decomposition of* M.

Lemma 1.8.8 *Let* $(R,q) \in \{(S,a),(F,\bar{a}),(Q,a)\}$*. Furthermore, let* M *be a finitely generated right* $\mathcal{H}_{A_{n-1}}^{(R,q)}$*-module which, for an* $i \in \{1,\dots,m\}$ *with* $m \in \mathbb{N}$ *from Lemma 1.8.3, belongs to the block category* $B_{\text{Kat}}^{(i)}(\mathcal{H}_{A_{n-1}}^{(R,q)})$*. Then the following statements hold.*

(i) For $j \in \{1,\dots,m\}$ *with* $j \neq i$*, we have*

$$Mb_{\text{Idemp}}^{(j)}(\mathcal{H}_{A_{n-1}}^{(R,q)}) = 0.$$

Here, 0 denotes the null module of $\mathcal{H}_{A_{n-1}}^{(R,q)}$.

(ii) If $M \neq 0$ *then* M *belongs to exactly one block category of* $\mathcal{H}_{A_{n-1}}^{(R,q)}$.

(iii) For every $x \in M$ *we have*

$$xb_{\text{Idemp}}^{(i)}(\mathcal{H}_{A_{n-1}}^{(R,q)}) = x.$$

(iv) Submodules and homomorphic images of M *also belong to the block category* $B_{\text{Kat}}^{(i)}(\mathcal{H}_{A_{n-1}}^{(R,q)})$.

Proof. (i) From Definition 1.8.5 and the orthogonality of the block idempotents, we get for $j \neq i$

$$Mb_{\text{Idemp}}^{(j)}(\mathcal{H}_n^{(R,q)}) = Mb_{\text{Idemp}}^{(i)}(\mathcal{H}_n^{(R,q)})b_{\text{Idemp}}^{(j)}(\mathcal{H}_n^{(R,q)}) = 0.$$

(ii) This follows immediately from statement (i) and Definition 1.8.5.

(iii) This follows from the decomposition of $1_{\mathcal{H}_A^{(R,q)}}$ in block idempotents in Lemma 1.8.3 and statement (i).

(iv) This follows immediately from statement (iii) and Definition 1.8.5. ■

Remark 1.8.9 *Let $(R,q) \in \{(S,a),(F,\bar{a}),(Q,a)\}$. Furthermore, let M be an $\mathcal{H}_{A_{n-1}}^{(R,q)}$-module which, for an $i \in \{1,\ldots,m\}$ with $m \in \mathbb{N}$ from Lemma 1.8.3, belongs to the block category $B_{\mathrm{Kat}}^{(i)}(\mathcal{H}_{A_{n-1}}^{(R,q)})$. Then, according to Lemma 1.8.8.(iii), $b_{\mathrm{Idemp}}^{(i)}(\mathcal{H}_{A_{n-1}}^{(R,q)})$ operates as identity on M. Thus, M can be considered as a module for the algebra $B_{\mathrm{Ideal}}^{(i)}(\mathcal{H}_{A_{n-1}}^{(R,q)})$. This shows that M is in fact an object of the category $B_{\mathrm{Kat}}^{(i)}(\mathcal{H}_{A_{n-1}}^{(R,q)})$.*

The following statement shows that certain modules of Hecke algebras belong to a block category. These modules include the projective indecomposable modules from Lemma 1.7.2.(i).

Lemma 1.8.10 *(i) For every $\lambda \in \Pi_n$, $S_{(Q,a)}^{\lambda}$ belongs to a block category of $\mathcal{H}_{A_{n-1}}^{(Q,a)}$.*

(ii) For every $\lambda \in \Pi_n$, $S_{(S,a)}^{\lambda}$ belongs to a block category of $\mathcal{H}_{A_{n-1}}^{(S,a)}$.

(iii) For every $\lambda \in \Pi_n$, $S_{(F,\bar{a})}^{\lambda}$ belongs to a block category of $\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}$. For every $\mu \in \Pi_{n,e_F(\bar{a})}$, $D_{(F,\bar{a})}^{\mu}$ belongs to a block category of $\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}$. For every $\mu \in \Pi_{n,e_F(\bar{a})}$, $P_{(F,\bar{a})}^{\mu}$ belongs to a block category of $\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}$.

Proof. (i) Let $\lambda \vdash n$. According to condition (1.35).(i), $\mathcal{H}_n^{(Q,a)}$ is semisimple. Thus, according to Theorem 1.3.21.(ii), $S_{(Q,a)}^{\lambda}$ is irreducible. This shows that the block decomposition of $S_{(Q,a)}^{\lambda}$ (see Lemma 1.8.6) contains exactly one summand different from the null module. Thus, $S_{(Q,a)}^{\lambda}$ belongs to a block category of $\mathcal{H}_n^{(Q,a)}$.

(ii) Let $\lambda \vdash n$. Suppose that $S_{(S,a)}^{\lambda}$ doesn't belong to a block category of $\mathcal{H}_n^{(S,a)}$. Then the block decomposition of $S_{(S,a)}^{\lambda}$ contains at least two summands different from the null module. By tensoring this decomposition over S with Q, we get a decomposition of $S_{(Q,a)}^{\lambda}$ with at least two summands different from the null module (none of the nontrivial summands vanishes in the process since Q is the quotient field of S, see [CR1, §4A]; see also Lemma 1.3.14 and Lemma 1.2.8.(i)). This is a contradiction to the irreducibility of $S_{(Q,a)}^{\lambda}$. Thus, $S_{(S,a)}^{\lambda}$ belongs to a block category of $\mathcal{H}_n^{(S,a)}$.

(iii) Let $\lambda \vdash n$. Then the block decomposition of $S_{(F,\bar{a})}^{\lambda}$ is obtained from the block decomposition of $S_{(S,a)}^{\lambda}$ by tensoring over S with F (see Lemma 1.3.14 and Lemma 1.8.3.(ii)). According to statement (i), this decomposition of $S_{(S,a)}^{\lambda}$ contains exactly one summand different from the null module. Thus, the same is true for the decomposition of $S_{(F,\bar{a})}^{\lambda}$, and $S_{(F,\bar{a})}^{\lambda}$ belongs to a block category of $\mathcal{H}_n^{(F,\bar{a})}$.

Let $\mu \in \Pi_{n,e_F(\bar{a})}$. Then the fact that $D_{(F,\bar{a})}^{\mu}$ belongs to a block category of $\mathcal{H}_n^{(F,\bar{a})}$ follows from the irreducibility of $D_{(F,\bar{a})}^{\mu}$ as in the proof of statement (i).

Let $\mu \in \Pi_{n,e_F(\bar{a})}$. Since $P^{\mu}_{(F,\bar{a})}$ is projective indecomposable, the block decomposition of $P^{\mu}_{(F,\bar{a})}$ must contain exactly one summand different from the null module. Thus, $P^{\mu}_{(F,\bar{a})}$ belongs to a block category of $\mathcal{H}_n^{(F,\bar{a})}$. ■

The block categories of Hecke algebras and also the other objects from Definition 1.8.4 corresponding to them can be indexed in such a way that, for a module considered in the preceding lemma, the block category to which it belongs can be easily read off from its indexing partition.

Theorem 1.8.11 *The block idempotents, block ideals, and block categories of the Hecke algebras $\mathcal{H}^{(S,a)}_{A_{n-1}}$, $\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}$, and $\mathcal{H}^{(Q,a)}_{A_{n-1}}$ from Definition 1.8.4 can be indexed by the elements of the set $\Gamma_{e_F(\bar{a})}(n)$. More precisely, the indexing scheme can be chosen in such a way that each of the modules $S^{\lambda}_{(Q,a)}$, $S^{\lambda}_{(S,a)}$, $S^{\lambda}_{(F,\bar{a})}$, $D^{\lambda}_{(F,\bar{a})}$, and $P^{\lambda}_{(F,\bar{a})}$ indexed by an appropriate $\lambda \vdash n$ belongs to the block category of the appropriate Hecke algebra indexed by $\gamma_{e_F(\bar{a})}(\lambda) \in \Gamma_{e_F(\bar{a})}(n)$.*

Proof. [JM, Theorem 4.29] provides the desired indexing scheme with respect to the modules $S^{\lambda}_{(F,\bar{a})}$ with $\lambda \vdash n$ (see also [DJ2, Theorem 4.13]). It follows from Definition 1.8.5, Lemma 1.3.14, Lemma 1.8.3.(ii), and Lemma 1.8.3.(iii) that this indexing scheme also has the desired properties with respect to the modules $S^{\lambda}_{(S,a)}$ and $S^{\lambda}_{(Q,a)}$ with $\lambda \vdash n$. Finally, the desired properties of the indexing scheme with respect to the modules $D^{\mu}_{(F,\bar{a})}$ and $P^{\mu}_{(F,\bar{a})}$ with $\mu \in \Pi_{n,e_F(\bar{a})}$ are obtained by using Lemma 1.8.8.(iv), Definition 1.3.19, and Lemma 1.7.2.(i). ■

Now we can fix a better notation.

Definition 1.8.12 *(i) In the following, the block idempotents, block ideals, and block categories introduced in Definition 1.8.4 are no more indexed by numbers, but instead by the elements of the set $\Gamma_{e_F(\bar{a})}(n)$ as described in Theorem 1.8.11. For a given core $\mu \in \Gamma_{e_F(\bar{a})}(n)$ and a given coefficient pair $(R,q) \in \{(S,a),(F,\bar{a}),(Q,a)\}$, we write*

$$b^{\mu}_{\mathrm{Idemp}}(\mathcal{H}^{(R,q)}_{A_{n-1}}), \quad B^{\mu}_{\mathrm{Ideal}}(\mathcal{H}^{(R,q)}_{A_{n-1}}), \quad B^{\mu}_{\mathrm{Kat}}(\mathcal{H}^{(R,q)}_{A_{n-1}})$$

or, for short,

$$b^{\mu}_{\mathrm{Idemp}}, \quad B^{\mu}_{\mathrm{Ideal}}, \quad B^{\mu}_{\mathrm{Kat}}.$$

(ii) Let $\mu \in \Gamma_{e_F(\bar{a})}(n)$ and $(R,q) \in \{(S,a),(F,\bar{a}),(Q,a)\}$. Then the block ideal $B^{\mu}_{\mathrm{Ideal}}(\mathcal{H}^{(R,q)}_{A_{n-1}})$ and the block category $B^{\mu}_{\mathrm{Kat}}(\mathcal{H}^{(R,q)}_{A_{n-1}})$ are called μ-block or block of $\mathcal{H}^{(R,q)}_{A_{n-1}}$ for short. This is denoted by

$$B^{\mu}(\mathcal{H}^{(R,q)}_{A_{n-1}}) = B^{\mu}(n) = B^{\mu}.$$

The set $\Pi_n^{\mu,e_F(\bar{a})}$*, whose elements index Specht modules, irreducible modules, and projective indecomposable modules as in Lemma 1.8.10, also is called* μ*-block or block. This is denoted by*

$$\Pi_n^{\mu,e_F(\bar{a})} = B^\mu(n) = B^\mu.$$

The $e_F(\bar{a})$*-weight* $g_{e_F(\bar{a})}(B^\mu(n))$ *of the block* $B^\mu(n)$ *is defined as*

$$g_{e_F(\bar{a})}(B^\mu(n)) = \frac{n - |[\mu]|}{e_F(\bar{a})}.$$

In the case of $e_F(\bar{a}) = \infty$ *we get, by using the usual rules for calculations with* ∞*,* $g_{e_F(\bar{a})}(B^\mu(n)) = 0$*.*

(iii) Let $\mu \in \Gamma_{e_F(\bar{a})}(n)$*,* $(R,q) \in \{(S,a),(F,\bar{a}),(Q,a)\}$*, and choose a finitely generated right* $\mathcal{H}_{A_{n-1}}^{(R,q)}$*-module* M*. If* M *belongs to the block category* $B^\mu_{\mathrm{Kat}}(\mathcal{H}_{A_{n-1}}^{(R,q)})$*, we say for short that* M *belongs to the block* $B^\mu(\mathcal{H}_{A_{n-1}}^{(R,q)})$ *or lies in the block* $B^\mu(\mathcal{H}_{A_{n-1}}^{(R,q)})$*.*

(iv) Let $\lambda \in \Pi_n$ *with* $\mu = \gamma_{e_F(\bar{a})}(\lambda) \in \Gamma_{e_F(\bar{a})}(n)$*. Then we say that* λ *belongs to the block* $B^\mu(n)$ *or lies in the block* $B^\mu(n)$*.*

Remark 1.8.13 *Let* $\mu \in \Gamma_{e_F(\bar{a})}(n)$ *and* $\lambda \in \Pi_n^{\mu,e_F(\bar{a})}$*. With that, we see from Lemma 1.1.37, Definition 1.1.38, and Definition 1.8.12.(ii)*

$$g_{e_F(\bar{a})}(B^\mu(n)) = g_{e_F(\bar{a})}(\lambda).$$

From this, we also get

$$g_{e_F(\bar{a})}(B^\mu(n)) \in \mathbb{N}_0.$$

Next, the block decomposition of modules from Lemma 1.8.6 is translated to Grothendieck groups and projective class groups.

Lemma 1.8.14 *Let* $(R,q) \in \{(S,a),(F,\bar{a}),(Q,a)\}$ *and* $\mu \in \Gamma_{e_F(\bar{a})}(n)$*.*

(i) Let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of right $\mathcal{H}_{A_{n-1}}^{(R,q)}$*-modules. Then*

$$0 \to M' b^\mu_{\mathrm{Idemp}}(\mathcal{H}_{A_{n-1}}^{(R,q)}) \to M b^\mu_{\mathrm{Idemp}}(\mathcal{H}_{A_{n-1}}^{(R,q)}) \to M'' b^\mu_{\mathrm{Idemp}}(\mathcal{H}_{A_{n-1}}^{(R,q)}) \to 0$$

also is an exact sequence of right $\mathcal{H}_{A_{n-1}}^{(R,q)}$*-modules.*

(ii) *Let P be a projective right $\mathcal{H}_{A_{n-1}}^{(R,q)}$-module. Then $Pb_{\mathrm{Idemp}}^{\mu}(\mathcal{H}_{A_{n-1}}^{(R,q)})$ also is a projective right $\mathcal{H}_{A_{n-1}}^{(R,q)}$-module.*

Proof. (i) The homomorphisms in the sequence $0 \to M' \to M \to M'' \to 0$ are compatible with the operations of the block idempotents of $\mathcal{H}_n^{(R,q)}$ on the modules M', M, and M'' and thus also with the block decompositions of these modules (see Lemma 1.8.6). This shows the claim.

(ii) Since the idempotent $b_{\mathrm{Idemp}}^{\mu}(\mathcal{H}_n^{(R,q)})$ is central in $\mathcal{H}_n^{(R,q)}$, $Pb_{\mathrm{Idemp}}^{\mu}(\mathcal{H}_n^{(R,q)})$ is an $\mathcal{H}_n^{(R,q)}$-direct summand of P and thus projective. ■

The preceding lemma shows that the following definition is meaningful.

Definition 1.8.15 *Let $(R,q) \in \{(S,a),(F,\bar{a}),(Q,a)\}$ and $\mu \in \Gamma_{e_F(\bar{a})}(n)$.*

(i) *The endomorphism*

$$b_{\mathrm{Proj}}^{\mu} = b_{\mathrm{Proj}}^{\mu}(\mathcal{H}_{A_{n-1}}^{(R,q)}) : G_0(\mathcal{H}_{A_{n-1}}^{(R,q)}) \to G_0(\mathcal{H}_{A_{n-1}}^{(R,q)})$$

is defined by

$$b_{\mathrm{Proj}}^{\mu}(\mathcal{H}_{A_{n-1}}^{(R,q)})([M]) = [Mb_{\mathrm{Idemp}}^{\mu}(\mathcal{H}_{A_{n-1}}^{(R,q)})] \in G_0(\mathcal{H}_{A_{n-1}}^{(R,q)})$$

for $[M] \in \mathcal{M}(\mathcal{H}_{A_{n-1}}^{(R,q)})$ and $\mathbb{Z}$-linear extension.

(ii) *The endomorphism*

$$b_{\mathrm{Proj}}^{\mu} = b_{\mathrm{Proj}}^{\mu}(\mathcal{H}_{A_{n-1}}^{(R,q)}) : K_0(\mathcal{H}_{A_{n-1}}^{(R,q)}) \to K_0(\mathcal{H}_{A_{n-1}}^{(R,q)})$$

is defined by

$$b_{\mathrm{Proj}}^{\mu}(\mathcal{H}_{A_{n-1}}^{(R,q)})([P]) = [Pb_{\mathrm{Idemp}}^{\mu}(\mathcal{H}_{A_{n-1}}^{(R,q)})] \in K_0(\mathcal{H}_{A_{n-1}}^{(R,q)})$$

for $[P] \in \mathcal{P}(\mathcal{H}_{A_{n-1}}^{(R,q)})$ and $\mathbb{Z}$-linear extension.

For simplicity, the two endomorphisms of $G_0(\mathcal{H}_{A_{n-1}}^{(R,q)})$ and $K_0(\mathcal{H}_{A_{n-1}}^{(R,q)})$ introduced in the preceding definition are denoted by the same symbol. Now, the effect of the map $b_{\mathrm{Proj}}^{\mu}(\mathcal{H}_{A_{n-1}}^{(Q,a)})$ with a $\mu \in \Gamma_{e_F(\bar{a})}(n)$ on $G_0(\mathcal{H}_{A_{n-1}}^{(Q,a)})$ is described. This will be required in Section 1.9.

Lemma 1.8.16 *Let*

$$x = \sum_{\lambda \in \Pi_n} \xi_\lambda [S_{(Q,a)}^{\lambda}] \in G_0(\mathcal{H}_{A_{n-1}}^{(Q,a)})$$

with coefficients $\xi_\lambda \in \mathbb{Z}$ for $\lambda \in \Pi_n$ and let $\mu \in \Gamma_{e_F(\bar{a})}(n)$. Then we have

$$b_{\mathrm{Proj}}^{\mu}(\mathcal{H}_{A_{n-1}}^{(Q,a)})(x) = \sum_{\lambda \in \Pi_n^{\mu, e_F(\bar{a})}} \xi_\lambda [S_{(Q,a)}^{\lambda}].$$

Proof. This follows from Definition 1.8.15.(i), Definition 1.8.12.(i), Theorem 1.8.11, Definition 1.8.5, and Lemma 1.8.8.(i). ■

Next, the compatibility of the endomorphisms from Definition 1.8.15 and the Brauer map from Definition 1.7.9 is shown. This also will be required in Section 1.9.

Lemma 1.8.17 *Let $\mu \in \Gamma_{e_F(\bar{a})}(n)$. Then the diagram*

$$\begin{tikzcd}
K_0(\mathcal{H}^{(S,a)}_{A_{n-1}}) \arrow[r, "b^{\mu}_{\mathrm{Proj}}(\mathcal{H}^{(S,a)}_{A_{n-1}})"] \arrow[d, "\bar{\cdot}"'] & K_0(\mathcal{H}^{(S,a)}_{A_{n-1}}) \arrow[d, "\bar{\cdot}"] \\
K_0(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}) \arrow[r, "b^{\mu}_{\mathrm{Proj}}(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})"'] & K_0(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})
\end{tikzcd}$$

with maps from Definition 1.7.6 and Definition 1.8.15.(ii) is commutative.

Proof. This follows from the relation

$$b^{\mu}_{\mathrm{Idemp}}(\mathcal{H}^{(F,\bar{a})}_n) = \overline{b^{\mu}_{\mathrm{Idemp}}(\mathcal{H}^{(S,a)}_n)}$$

(see Lemma 1.8.3.(ii)) and the property (1.30) of the reduction modulo I of arbitrary $\mathcal{H}^{(S,a)}_n$-modules on page 49. ■

Lemma 1.8.18 *Let $\mu \in \Gamma_{e_F(\bar{a})}(n)$. Then the diagram*

$$\begin{tikzcd}
K_0(\mathcal{H}^{(S,a)}_{A_{n-1}}) \arrow[r, "b^{\mu}_{\mathrm{Proj}}(\mathcal{H}^{(S,a)}_{A_{n-1}})"] \arrow[d, "- \otimes_S Q"'] & K_0(\mathcal{H}^{(S,a)}_{A_{n-1}}) \arrow[d, "- \otimes_S Q"] \\
G_0(\mathcal{H}^{(Q,a)}_{A_{n-1}}) \arrow[r, "b^{\mu}_{\mathrm{Proj}}(\mathcal{H}^{(Q,a)}_{A_{n-1}})"'] & G_0(\mathcal{H}^{(Q,a)}_{A_{n-1}})
\end{tikzcd}$$

with maps from Definition 1.7.8 and Definition 1.8.15 is commutative.

Proof. This follows from the equation

$$b^{\mu}_{\mathrm{Idemp}}(\mathcal{H}^{(Q,a)}_n) = b^{\mu}_{\mathrm{Idemp}}(\mathcal{H}^{(S,a)}_n) \otimes_S 1_Q$$

in Lemma 1.8.3.(iii) and the formula (1.27) in Lemma 1.2.8. ■

Lemma 1.8.19 *Let $\mu \in \Gamma_{e_F(\bar{a})}(n)$. Then the diagram*

$$\begin{tikzcd}
K_0(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}) \arrow[rr, "b^{\mu}_{\mathrm{Proj}}(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})"] \arrow[d, "B^{\mathcal{H}}_{n,\mathcal{K}}"'] & & K_0(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}) \arrow[d, "B^{\mathcal{H}}_{n,\mathcal{K}}"] \\
G_0(\mathcal{H}^{(Q,a)}_{A_{n-1}}) \arrow[rr, "b^{\mu}_{\mathrm{Proj}}(\mathcal{H}^{(Q,a)}_{A_{n-1}})"'] & & G_0(\mathcal{H}^{(Q,a)}_{A_{n-1}})
\end{tikzcd}$$

with maps from Definition 1.7.9 and Definition 1.8.15 is commutative.

Proof. This follows from Lemma 1.8.17 and Lemma 1.8.18. ∎

Now the decomposition matrix

$$\Delta^{\mathcal{H}}_{n,\mathcal{K}} = \left(d^{n,\mathcal{K}}_{\lambda\mu}\right)_{\substack{\lambda\in\Pi_n \\ \mu\in\Pi_{n,e_F(\bar{a})}}}$$

will be subdivided into (matrix) blocks by making use of the classification of Specht modules and irreducible modules of the Hecke algebra $\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}$ into blocks (see Lemma 1.8.10.(iii)). Since, according to condition (1.35).(i), $\mathcal{H}^{(Q,a)}_{A_{n-1}}$ is semisimple, Definition 1.4.17 and Theorem 1.3.21 show that the rows of this matrix are indexed by the elements of the set Π_n. The second statement of the following theorem also can be found in [JAM2, Rule 5.2].

Theorem 1.8.20 *(i) Let $D^{\mu}_{(F,\bar{a})}$ be a composition factor of $S^{\lambda}_{(F,\bar{a})}$ with $\lambda \in \Pi_n$ and $\mu \in \Pi_{n,e_F(\bar{a})}$. Then $S^{\lambda}_{(F,\bar{a})}$ and $D^{\mu}_{(F,\bar{a})}$ belong to the same block of $\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}$.*

(ii) Let $\lambda \in \Pi_n$ and $\mu \in \Pi_{n,e_F(\bar{a})}$. Then we have

$$d^{n,\mathcal{K}}_{\lambda\mu} \neq 0 \Rightarrow \gamma_{e_F(\bar{a})}(\lambda) = \gamma_{e_F(\bar{a})}(\mu).$$

Proof. (i) According to Lemma 1.8.10.(iii), $S^{\lambda}_{(F,\bar{a})}$ and $D^{\mu}_{(F,\bar{a})}$ both belong to a respective block of $\mathcal{H}^{(F,\bar{a})}_n$. That both modules lie in the same block follows from Lemma 1.8.8.(iv).

(ii) This follows from Lemma 1.5.5, statement (i), and Theorem 1.8.11. ∎

The desired subdivision of decomposition matrices into matrix blocks can be obtained not only for modular systems as hitherto considered, but can be generalized to others as well. To this end, let

$$\tilde{\mathcal{K}} = \left(\tilde{Q}, \tilde{\psi}, \tilde{S}, \tilde{I}, \tilde{a}, \tilde{F}\right)$$

be a modular system as in Definition 1.4.8. $\tilde{\mathcal{K}}$ is not required to satisfy the conditions (1.35). Since, on the other hand, the modular system $\mathcal{K}$ has all the properties required from the modular system $\tilde{\mathcal{K}}$, the following definition also is valid for $\mathcal{K}$.

Definition 1.8.21 *Let $\nu \in \Gamma_{e_{\tilde{F}}(\bar{\tilde{a}})}(n)$. Then the submatrix*

$$\Delta^{\mathcal{H}}(\nu) = \Delta^{\mathcal{H}}_{n}(\nu) = \Delta^{\mathcal{H}}_{\tilde{\mathcal{K}}}(\nu) = \Delta^{\mathcal{H}}_{n,\tilde{\mathcal{K}}}(\nu) = \left(d^{n,\tilde{\mathcal{K}}}_{\lambda\mu}\right)_{\substack{\lambda\in\Pi^{\nu,e_{\tilde{F}}(\bar{a})}_{n,e_{\tilde{Q}}(\bar{a})}\\ \mu\in\Pi^{\nu,e_{\tilde{F}}(\bar{a})}_{n,e_{\tilde{F}}(\bar{a})}}}$$

of $\Delta^{\mathcal{H}}_{n,\tilde{\mathcal{K}}}$ is called ν-block of $\Delta^{\mathcal{H}}_{n,\tilde{\mathcal{K}}}$.

The following theorem is stated only for $\mathcal{K}$, it will later be generalized to $\tilde{\mathcal{K}}$.

Theorem 1.8.22 *Let $m = \left|\Gamma_{e_F(\bar{a})}(n)\right|$ and fix an enumeration*

$$\nu^{(1)}, \ldots, \nu^{(m)}$$

of the elements of $\Gamma_{e_F(\bar{a})}(n)$. Furthermore, order Π_n in such a way that for all $i, j \in \{1, \ldots, m\}$ with $i < j$ the elements of $\Pi_n^{\nu^{(i)},e_F(\bar{a})}$ precede the elements of $\Pi_n^{\nu^{(j)},e_F(\bar{a})}$. Finally, order $\Pi_{n,e_F(\bar{a})}$ in such a way that for all $i, j \in \{1, \ldots, m\}$ with $i < j$ the elements of $\Pi^{\nu^{(i)},e_F(\bar{a})}_{n,e_F(\bar{a})}$ precede the elements of $\Pi^{\nu^{(j)},e_F(\bar{a})}_{n,e_F(\bar{a})}$. With these orderings of the row and column index sets of the decomposition matrix $\Delta^{\mathcal{H}}_{n,\mathcal{K}} = \left(d^{n,\mathcal{K}}_{\lambda\mu}\right)_{\substack{\lambda\in\Pi_n\\ \mu\in\Pi_{n,e_F(\bar{a})}}}$, we have

$$\Delta^{\mathcal{H}}_{n,\mathcal{K}} = \begin{pmatrix} \Delta^{\mathcal{H}}_{n,\mathcal{K}}(\nu^{(1)}) & & 0 \\ & \ddots & \\ 0 & & \Delta^{\mathcal{H}}_{n,\mathcal{K}}(\nu^{(m)}) \end{pmatrix}.$$

Proof. This follows from Theorem 1.8.20.(ii) ■

Now, fix indeterminates Y over $\tilde{Q}$ and Z over $\tilde{F}$. Then we have the modular systems

$$\hat{\mathcal{K}}_{(\tilde{Q},\tilde{a})} = \left(\hat{\tilde{Q}(Y)}, \hat{\psi}_{Y-\tilde{a}}, S_{\hat{\psi}_{Y-\tilde{a}}}, I_{\hat{\psi}_{Y-\tilde{a}}}, Y, \tilde{Q}\right)$$

and

$$\hat{\mathcal{K}}_{(\tilde{F},\bar{\tilde{a}})} = \left(\hat{\tilde{F}(Z)}, \hat{\psi}_{Z-\bar{\tilde{a}}}, S_{\hat{\psi}_{Z-\bar{\tilde{a}}}}, I_{\hat{\psi}_{Z-\bar{\tilde{a}}}}, Z, \tilde{F}\right)$$

as in Definition 1.6.4, and, according to Lemma 1.6.6, also the following commutative diagram.

$$\begin{tikzcd}[column sep=huge, row sep=large]
G_0(\mathcal{H}_{A_{n-1}}^{(\tilde{Q}(\hat{Y}),Y)}) \arrow[r, "\alpha^{\mathcal{H}_n}_{(\tilde{Q}(\hat{Y}),Y)(\tilde{F}(\hat{Z}),Z)}"] \arrow[d, "D^{\mathcal{H}}_{n,\hat{\mathcal{K}}_{(\tilde{Q},\tilde{a})}}"'] & G_0(\mathcal{H}_{A_{n-1}}^{(\tilde{F}(\hat{Z}),Z)}) \arrow[d, "D^{\mathcal{H}}_{n,\hat{\mathcal{K}}_{(\tilde{F},\bar{\tilde{a}})}}"] \\
G_0(\mathcal{H}_{A_{n-1}}^{(\tilde{Q},\tilde{a})}) \arrow[r, "D^{\mathcal{H}}_{n,\tilde{\mathcal{K}}}"'] & G_0(\mathcal{H}_{A_{n-1}}^{(\tilde{F},\bar{\tilde{a}})})
\end{tikzcd}$$

Furthermore, according to Lemma 1.6.5 and Lemma 1.6.3.(i), the modular systems $\hat{\mathcal{K}}_{(\tilde{Q},\tilde{a})}$ and $\hat{\mathcal{K}}_{(\tilde{F},\bar{\tilde{a}})}$ satisfy the conditions (1.35). With this, Theorem 1.8.22 can be translated to the decomposition matrix

$$\Delta^{\mathcal{H}}_{n,\tilde{\mathcal{K}}} = \left(d^{n,\tilde{\mathcal{K}}}_{\lambda\mu}\right)_{\substack{\lambda\in\Pi_{n,e_{\tilde{Q}}(\tilde{a})} \\ \mu\in\Pi_{n,e_{\tilde{F}}(\bar{\tilde{a}})}}}$$

associated with the modular system $\tilde{\mathcal{K}}$.

Theorem 1.8.23 *Let $m = \left|\Gamma_{e_{\tilde{F}}(\bar{\tilde{a}})}(n)\right|$ and fix an enumeration*

$$\nu^{(1)}, \ldots, \nu^{(m)}$$

of the elements of $\Gamma_{e_{\tilde{F}}(\bar{\tilde{a}})}(n)$. Furthermore, order $\Pi_{n,e_{\tilde{Q}}(\tilde{a})}$ in such a way that for all $i,j \in \{1,\ldots,m\}$ with $i<j$ the elements of $\Pi^{\nu^{(i)},e_{\tilde{F}}(\bar{\tilde{a}})}_{n,e_{\tilde{Q}}(\tilde{a})}$ precede the elements of $\Pi^{\nu^{(j)},e_{\tilde{F}}(\bar{\tilde{a}})}_{n,e_{\tilde{Q}}(\tilde{a})}$. Finally, order $\Pi_{n,e_{\tilde{F}}(\bar{\tilde{a}})}$ in such a way that for all $i,j\in\{1,\ldots,m\}$ with $i<j$ the elements of $\Pi^{\nu^{(i)},e_{\tilde{F}}(\bar{\tilde{a}})}_{n,e_{\tilde{F}}(\bar{\tilde{a}})}$ precede the elements of $\Pi^{\nu^{(j)},e_{\tilde{F}}(\bar{\tilde{a}})}_{n,e_{\tilde{F}}(\bar{\tilde{a}})}$. With these orderings of the row and column index sets of the decomposition matrix $\Delta^{\mathcal{H}}_{n,\tilde{\mathcal{K}}} = \left(d^{n,\tilde{\mathcal{K}}}_{\lambda\mu}\right)_{\substack{\lambda\in\Pi_{n,e_{\tilde{Q}}(\tilde{a})} \\ \mu\in\Pi_{n,e_{\tilde{F}}(\bar{\tilde{a}})}}}$, we have

$$\Delta^{\mathcal{H}}_{n,\tilde{\mathcal{K}}} = \begin{pmatrix} \Delta^{\mathcal{H}}_{n,\tilde{\mathcal{K}}}(\nu^{(1)}) & & 0 \\ & \ddots & \\ 0 & & \Delta^{\mathcal{H}}_{n,\tilde{\mathcal{K}}}(\nu^{(m)}) \end{pmatrix}.$$

Proof. From Corollary 1.6.7 we get

$$\Delta^{\mathcal{H}}_{n,\hat{\mathcal{K}}_{(\tilde{Q},\tilde{a})}} \Delta^{\mathcal{H}}_{n,\tilde{\mathcal{K}}} = \Delta^{\mathcal{H}}_{n,\hat{\mathcal{K}}_{(\tilde{F},\bar{\tilde{a}})}}.$$

Furthermore, according to Lemma 1.6.5 and Lemma 1.6.3.(i), the modular systems $\hat{\mathcal{K}}_{(\tilde{Q},\tilde{a})}$ and $\hat{\mathcal{K}}_{(\tilde{F},\bar{\tilde{a}})}$ satisfy the conditions (1.35). Thus, the matrices

$$\Delta^{\mathcal{H}}_{n,\hat{\mathcal{K}}_{(\tilde{Q},\tilde{a})}} = \left(d^{n,\hat{\mathcal{K}}_{(\tilde{Q},\tilde{a})}}_{\eta\theta}\right)_{\substack{\eta\in\Pi_n\\ \theta\in\Pi_{n,e_{\tilde{Q}}(\tilde{a})}}} \qquad \text{and} \qquad \Delta^{\mathcal{H}}_{n,\hat{\mathcal{K}}_{(\tilde{F},\bar{\tilde{a}})}} = \left(d^{n,\hat{\mathcal{K}}_{(\tilde{F},\bar{\tilde{a}})}}_{\eta\theta}\right)_{\substack{\eta\in\Pi_n\\ \theta\in\Pi_{n,e_{\tilde{F}}(\bar{\tilde{a}})}}}$$

have the form described in Theorem 1.8.22.

If we have $e_{\tilde{Q}}(\tilde{a}) = \infty$ or $e_{\tilde{F}}(\bar{\tilde{a}}) = \infty$ then the claim follows easily from Lemma 1.4.9.(i), Theorem 1.3.21.(i), and Corollary 1.5.4. Thus, we assume in the following

$$e_{\tilde{Q}}(\tilde{a}) \in \mathbb{N} \quad \text{and} \quad e_{\tilde{F}}(\bar{\tilde{a}}) \in \mathbb{N}.$$

Then we get from Lemma 1.4.9.(ii)

$$e_{\tilde{F}}(\bar{\tilde{a}}) \mid e_{\tilde{Q}}(\tilde{a}).$$

This, together with Lemma 1.1.44, enables us to combine for every $\nu \in \Gamma_{e_{\tilde{F}}(\bar{\tilde{a}})}(n)$ all the blocks

$$\Delta^{\mathcal{H}}_{n,\hat{\mathcal{K}}_{(\tilde{Q},\tilde{a})}}(\mu) = \left(d^{n,\hat{\mathcal{K}}_{(\tilde{Q},\tilde{a})}}_{\eta\theta}\right)_{\substack{\eta\in\Pi_n^{\mu,e_{\tilde{Q}}(\tilde{a})}\\ \theta\in\Pi^{\mu,e_{\tilde{Q}}(\tilde{a})}_{n,e_{\tilde{Q}}(\tilde{a})}}}$$

indexed by a $\mu \in \Gamma_{e_{\tilde{Q}}(\tilde{a})}(n)$ with $\gamma_{e_{\tilde{F}}(\bar{\tilde{a}})}(\mu) = \nu$ to get a bigger block with row index set

$$\bigcup_{\substack{\mu\in\Gamma_{e_{\tilde{Q}}(\tilde{a})}(n)\\ \gamma_{e_{\tilde{F}}(\bar{\tilde{a}})}(\mu)=\nu}} \Pi_n^{\mu,e_{\tilde{Q}}(\tilde{a})} = \Pi_n^{\nu,e_{\tilde{F}}(\bar{\tilde{a}})}$$

and column index set

$$\bigcup_{\substack{\mu\in\Gamma_{e_{\tilde{Q}}(\tilde{a})}(n)\\ \gamma_{e_{\tilde{F}}(\bar{\tilde{a}})}(\mu)=\nu}} \Pi^{\mu,e_{\tilde{Q}}(\tilde{a})}_{n,e_{\tilde{Q}}(\tilde{a})} = \Pi^{\nu,e_{\tilde{F}}(\bar{\tilde{a}})}_{n,e_{\tilde{Q}}(\tilde{a})}.$$

The row index set of this combined block coincides with the row index set of the block

$$\Delta^{\mathcal{H}}_{n,\hat{\mathcal{K}}_{(\tilde{F},\bar{\tilde{a}})}}(\nu) = \left(d^{n,\hat{\mathcal{K}}_{(\tilde{F},\bar{\tilde{a}})}}_{\eta\theta}\right)_{\substack{\eta\in\Pi_n^{\nu,e_{\tilde{F}}(\bar{\tilde{a}})}\\ \theta\in\Pi^{\nu,e_{\tilde{F}}(\bar{\tilde{a}})}_{n,e_{\tilde{F}}(\bar{\tilde{a}})}}}$$

of $\Delta^{\mathcal{H}}_{n,\hat{\mathcal{K}}_{(\tilde{F},\bar{\tilde{a}})}}$ indexed by ν. Furthermore, the column index set of the combined block coincides with the row index set of the — at this point only formally defined — block

$$\Delta^{\mathcal{H}}_{n,\tilde{\mathcal{K}}}(\nu) = \left(d^{n,\tilde{\mathcal{K}}}_{\eta\theta}\right)_{\substack{\eta\in\Pi^{\nu,e_{\tilde{F}}(\bar{\tilde{a}})}_{n,e_{\tilde{Q}}(\tilde{a})}\\ \theta\in\Pi^{\nu,e_{\tilde{F}}(\bar{\tilde{a}})}_{n,e_{\tilde{F}}(\bar{\tilde{a}})}}}$$

of $\Delta^{\mathcal{H}}_{n,\tilde{\mathcal{K}}}$ indexed by ν. We also note that the column index sets of $\Delta^{\mathcal{H}}_{n,\hat{\mathcal{K}}_{(\tilde{F},\bar{\tilde{a}})}}(\nu)$ and $\Delta^{\mathcal{H}}_{n,\tilde{\mathcal{K}}}(\nu)$ coincide. Finally, we have according to Lemma 1.6.5 and Lemma 1.5.8.(ii)

$$\mathrm{Rnk}_{\mathbb{Q}}\, \Delta^{\mathcal{H}}_{n,\hat{\mathcal{K}}_{(\tilde{Q},\tilde{a})}} = \left|\Pi_{n,e_{\tilde{Q}}(\tilde{a})}\right|.$$

From all these properties and considerations, the claim also follows, by using elementary arguments from linear algebra, in the case $e_{\tilde{Q}}(\tilde{a}), e_{\tilde{F}}(\bar{\tilde{a}}) \in \mathbb{N}$. ■

1.9 Induction of Hecke algebra modules

In this section, we first introduce the notion of induction of modules of a Hecke algebra of type A_m with $m \in \mathbb{N}_0$ over a given coefficient pair to modules of the Hecke algebra of type A_{m+1} over the same coefficient pair. Then, we describe the behavior of Specht modules with respect to induction. Finally, we use induced projective modules to construct upper bounds for decomposition numbers of Hecke algebras of type A.

In the following, we fix an $n \in \mathbb{N} \setminus \{1\}$. Furthermore, R denotes always an integral domain and q is always a unit in R.

Definition 1.9.1 *For $m \in \mathbb{N} \setminus \{1\}$, $\chi^{(m)} \in \Pi_m$ is defined as*

$$\chi^{(m)} = (m-1, 1).$$

Lemma 1.9.2 *Let $m \in \mathbb{N} \setminus \{1\}$. Then we have in $\mathfrak{S}_m$*

$$\mathfrak{S}_{\chi^{(m)}} = \mathfrak{S}_{\{1,\dots,m-1\}} = \{w \in \mathfrak{S}_m \mid mw = m\}.$$

Proof. This follows immediately from Definition 1.1.55. See also (1.2) on page 1. ■

According to Lemma 1.9.2, the inclusion

$$\mathfrak{S}_{n-1} \hookrightarrow \mathfrak{S}_n, \quad w \mapsto w \in \mathfrak{S}_n \text{ with } nw = n$$

maps $\mathfrak{S}_{n-1}$ isomorphically onto the subgroup $\mathfrak{S}_{\chi^{(n)}}$ of $\mathfrak{S}_n$. This inclusion induces an inclusion of algebras

$$\mathcal{H}^{(R,q)}_{A_{n-2}} \hookrightarrow \mathcal{H}^{(R,q)}_{A_{n-1}}$$

defined by

$$T_w \mapsto T_w \in \mathcal{H}^{(R,q)}_{A_{n-1}}$$

for $T_w \in \mathcal{H}^{(R,q)}_{A_{n-2}}$ with $w \in \mathfrak{S}_{n-1}$. This follows from the construction of the Hecke algebras of type A in Section 1.2.

Definition 1.9.3 *The inclusion of algebras*

$$i_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} = i_{\mathcal{H}_{n-1}^{(R,q)}}^{\mathcal{H}_n^{(R,q)}} : \mathcal{H}_{A_{n-2}}^{(R,q)} \hookrightarrow \mathcal{H}_{A_{n-1}}^{(R,q)}$$

defined by

$$T_w \mapsto T_w \in \mathcal{H}_{A_{n-1}}^{(R,q)}$$

for $T_w \in \mathcal{H}_{A_{n-2}}^{(R,q)}$ *with* $w \in \mathfrak{S}_{n-1}$ *is called the standard inclusion of* $\mathcal{H}_{A_{n-2}}^{(R,q)}$ *into* $\mathcal{H}_{A_{n-1}}^{(R,q)}$ *or, for short, the standard inclusion.*

From now on, $\mathcal{H}_{A_{n-2}}^{(R,q)}$ is considered a subalgebra of $\mathcal{H}_{A_{n-1}}^{(R,q)}$ by means of the standard inclusion. Thus, $\mathcal{H}_{A_{n-1}}^{(R,q)}$ can be considered a left $\mathcal{H}_{A_{n-2}}^{(R,q)}$-module and a right $\mathcal{H}_{A_{n-1}}^{(R,q)}$-module at the same time, and the functor $- \otimes_{\mathcal{H}_{n-1}^{(R,q)}} \mathcal{H}_{A_{n-1}}^{(R,q)}$ can be constructed. This functor can be applied to right $\mathcal{H}_{A_{n-2}}^{(R,q)}$-modules to obtain right $\mathcal{H}_{A_{n-1}}^{(R,q)}$-modules.

Definition 1.9.4 *Let* M *be a finitely generated right* $\mathcal{H}_{A_{n-2}}^{(R,q)}$*-module. Then the right* $\mathcal{H}_{A_{n-1}}^{(R,q)}$*-module* $M \otimes_{\mathcal{H}_{n-1}^{(R,q)}} \mathcal{H}_{A_{n-1}}^{(R,q)}$ *is called the* $\mathcal{H}_{A_{n-1}}^{(R,q)}$*-module induced from* M*. This is denoted by*

$$M \otimes_{\mathcal{H}_{n-1}^{(R,q)}} \mathcal{H}_{A_{n-1}}^{(R,q)} = M \Big\uparrow_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} = M \Big\uparrow_{\mathcal{H}_{n-1}^{(R,q)}}^{\mathcal{H}_n^{(R,q)}}.$$

As in the case of group algebras, the induction of Hecke algebra modules has the following useful property.

Lemma 1.9.5 *The map*

$$\mathcal{H}_{A_{n-2}}^{(R,q)} \otimes_{\mathcal{H}_{n-1}^{(R,q)}} \mathcal{H}_{A_{n-1}}^{(R,q)} \to \mathcal{H}_{A_{n-1}}^{(R,q)}, \quad x \otimes_{\mathcal{H}_{n-1}^{(R,q)}} y \mapsto xy \tag{1.37}$$

is an isomorphism of right $\mathcal{H}_{A_{n-1}}^{(R,q)}$*-modules.*

Proof. According to [CR1, (2.16)], (1.37) is an isomorphism of left $\mathcal{H}_{n-1}$-modules. Furthermore, it follows directly from (1.37) that this map is compatible with the natural right $\mathcal{H}_n$-module structure on $\mathcal{H}_n$ given by multiplication. This shows the claim. ■

According to Lemma 1.1.56 and the property (1.22) on page 34, $\mathcal{H}_{A_{n-1}}^{(R,q)}$, if considered as a left $\mathcal{H}_{A_{n-2}}^{(R,q)}$-module, is free over $\mathcal{H}_{A_{n-2}}^{(R,q)}$ with the basis $\left\{ T_d \,\middle|\, d \in \mathcal{D}_{\chi^{(n)}} \right\}$. This shows that the functor $- \otimes_{\mathcal{H}_{n-1}^{(R,q)}} \mathcal{H}_{A_{n-1}}^{(R,q)}$ is exact (see [CR1, §2D]) and thus induces a homomorphism between the Grothendieck groups of the algebras under consideration.

Definition 1.9.6 *We call the homomorphism*

$$\cdot\Big\uparrow_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} = \cdot\Big\uparrow_{\mathcal{H}_{n-1}^{(R,q)}}^{\mathcal{H}_n^{(R,q)}} : G_0(\mathcal{H}_{A_{n-2}}^{(R,q)}) \to G_0(\mathcal{H}_{A_{n-1}}^{(R,q)})$$

determined by

$$[M]\Big\uparrow_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} = [M]\Big\uparrow_{\mathcal{H}_{n-1}^{(R,q)}}^{\mathcal{H}_n^{(R,q)}} = [M\Big\uparrow_{\mathcal{H}_{n-1}^{(R,q)}}^{\mathcal{H}_n^{(R,q)}}]$$

for $[M] \in \mathcal{M}(\mathcal{H}_{A_{n-2}}^{(R,q)})$ *the induction homomorphism for Grothendieck groups.*

There also is an induction homomorphism for projective class groups. We see from Lemma 1.9.5 that induction of free right $\mathcal{H}_{A_{n-2}}^{(R,q)}$-modules gives free right $\mathcal{H}_{A_{n-1}}^{(R,q)}$-modules. Thus, induction of a projective right $\mathcal{H}_{A_{n-2}}^{(R,q)}$-module gives a projective right $\mathcal{H}_{A_{n-1}}^{(R,q)}$-module (see [CR1, §2D]). This fact, the compatibility of the functor $-\otimes_{\mathcal{H}_{n-1}^{(R,q)}} \mathcal{H}_{A_{n-1}}^{(R,q)}$ with direct sums of right $\mathcal{H}_{A_{n-2}}^{(R,q)}$-modules, and Definition 1.7.1 show that the following definition is meaningful.

Definition 1.9.7 *We call the homomorphism*

$$\cdot\Big\uparrow_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} = \cdot\Big\uparrow_{\mathcal{H}_{n-1}^{(R,q)}}^{\mathcal{H}_n^{(R,q)}} : K_0(\mathcal{H}_{A_{n-2}}^{(R,q)}) \to K_0(\mathcal{H}_{A_{n-1}}^{(R,q)})$$

determined by

$$[P]\Big\uparrow_{\mathcal{H}_{n-1}}^{\mathcal{H}_n} = [P]\Big\uparrow_{\mathcal{H}_{n-1}^{(R,q)}}^{\mathcal{H}_n^{(R,q)}} = [P\Big\uparrow_{\mathcal{H}_{n-1}^{(R,q)}}^{\mathcal{H}_n^{(R,q)}}]$$

for $[P] \in \mathcal{P}(\mathcal{H}_{A_{n-2}}^{(R,q)})$ *the induction homomorphism for projective class groups.*

Next, the behavior of Specht modules with respect to induction is described. To this end, the following lemma uses Definition 1.1.19 and Definition 1.1.20.(i).

Lemma 1.9.8 *Let* $\lambda \vdash n-1$. *Furthermore, let* K *be a field and choose an* $r \in K \setminus \{0_K\}$. *Then we have in* $G_0(\mathcal{H}_{A_{n-1}}^{(K,r)})$

$$[S_{(K,r)}^{\lambda}]\Big\uparrow_{\mathcal{H}_{n-1}^{(K,r)}}^{\mathcal{H}_n^{(K,r)}} = \sum_{\mu \in \lambda\uparrow} [S_{(K,r)}^{\mu}].$$

Proof. This follows from [DJ1, Section 7]. If we write

$$\lambda = (\lambda_1, \ldots, \lambda_k)$$

with $\lambda_k > 0$ for an appropriate $k \in \mathbb{N}$ and put

$$\hat{\lambda} = (\lambda_1, \ldots, \lambda_k, 1) = \lambda \cup (k+1, 1) \vdash n$$

then we have, with the notation from there,

$$S^{\lambda}_{(K,r)}\Big\uparrow^{\mathcal{H}_n}_{\mathcal{H}_{n-1}} = S^{\lambda\hat{\lambda}}.$$

Now, in [DJ1, Section 7], a series of submodules of the module $S^{\lambda\hat{\lambda}}$ is constructed such that, with the notation from there, the subquotients are certain modules $S^{\mu\mu}$ for every $\mu \in \lambda\uparrow$. Each of these modules occurs with multiplicity 1. The argumentation in [DJ1, Section 7] makes use of the fact that the coefficient ring is a field (see in particular [DJ1, Theorem 7.4]). Furthermore, one has for every $\mu \in \lambda\uparrow$

$$S^{\mu\mu} = S^{\mu}_{(K,r)}.$$

This shows the claim. ■

Remark 1.9.9 *The result used in the proof of the preceding lemma is generalized in Chapter 3 to arbitrary integral domains as coefficient rings (see in particular Theorem 3.11.2 and its proof). Thus, the preceding lemma also holds for Hecke algebras and Specht modules over such coefficient rings.*

Now the behavior of projective indecomposable modules with respect to induction is considered. To this end, let

$$\mathcal{K} = (Q, \psi, S, I, a, F)$$

be a modular system as in Definition 1.4.8 with the following properties.

(i) The algebra $\mathcal{H}^{(Q,a)}_{A_{n-1}}$ is semisimple.

(ii) The discrete valuation ring S is complete with respect to the valuation ψ. (1.38)

Property (i) and Theorem 1.3.21.(i) show that the algebra $\mathcal{H}^{(Q,a)}_{A_{n-2}}$ also is semisimple. Furthermore, this choice of $\mathcal{K}$ makes available the results from Section 1.7 and Section 1.8. By combining these and the preceding considerations on the induction of modules, an upper bound for the entries in a block of the decomposition matrix $\Delta^{\mathcal{H}}_{n,\mathcal{K}}$ (see Definition 1.8.21) will be derived from the entries in the decomposition matrix $\Delta^{\mathcal{H}}_{n-1,\mathcal{K}}$ in the following.

First, the compatibility of the Brauer map from Definition 1.7.9 and the induction homomorphisms from Definition 1.9.6 and Definition 1.9.7 is shown.

Lemma 1.9.10 *Let R be an integral domain and choose a unit $q \in R$. Let $\tilde{R}$ be another integral domain and let $\xi : R \to \tilde{R}$ be a ring homomorphism. Then the diagram*

$$\begin{tikzcd}
\mathcal{H}_{A_{n-2}}^{(R,q)} \arrow[rr, "i_{\mathcal{H}_{n-1}^{(R,q)}}^{\mathcal{H}_n^{(R,q)}}"] \arrow[d, "- \otimes_R \tilde{R}"'] & & \mathcal{H}_{A_{n-1}}^{(R,q)} \arrow[d, "- \otimes_R \tilde{R}"] \\
\mathcal{H}_{A_{n-2}}^{(\tilde{R},\xi(q))} \arrow[rr, "i_{\mathcal{H}_{n-1}^{(\tilde{R},\xi(q))}}^{\mathcal{H}_n^{(\tilde{R},\xi(q))}}"'] & & \mathcal{H}_{A_{n-1}}^{(\tilde{R},\xi(q))}
\end{tikzcd}$$

with maps as in Definition 1.9.3 and (1.25) on page 36 is commutative.

Proof. This follows by considering the bases $\{T_w \mid w \in \mathfrak{S}_{n-1}\}$ and $\{T_w \mid w \in \mathfrak{S}_n\}$ of the algebras occurring in the diagram. Every homomorphism occurring in the diagram maps such a basis element of one algebra to the corresponding element of another algebra. ■

Lemma 1.9.11 *Let R be an integral domain and choose a unit $q \in R$. Let $\tilde{R}$ be another integral domain and let $\xi : R \to \tilde{R}$ be a ring homomorphism. Finally, let $m \in \mathbb{N}$ and choose an idempotent $f \in \mathcal{H}_{A_{m-1}}^{(R,q)}$. Then we have*

$$\left(f\mathcal{H}_{A_{m-1}}^{(R,q)}\right) \otimes_R \tilde{R} \simeq (f \otimes_R 1_{\tilde{R}})\, \mathcal{H}_{A_{m-1}}^{(\tilde{R},\xi(q))}$$

as right $\mathcal{H}_{A_{m-1}}^{(\tilde{R},\xi(q))}$-modules.

Proof. If we put

$$M_f = f\mathcal{H}_m^{(R,q)}$$

then we have the natural inclusion of right $\mathcal{H}_m^{(R,q)}$-modules

$$i_f : M_f \hookrightarrow \mathcal{H}_m^{(R,q)}.$$

An application of $- \otimes_R \tilde{R}$ to this map together with Lemma 1.2.7, Lemma 1.2.8.(i), and Lemma 1.2.8.(ii) produces the homomorphism of right $\mathcal{H}_m^{(\tilde{R},\xi(q))}$-modules

$$i_f \otimes_R \mathrm{id}_{\tilde{R}} : M_f \otimes_R \tilde{R} \to \mathcal{H}_m^{(\tilde{R},\xi(q))}.$$

Because of the compatibility of $- \otimes_R \tilde{R}$ with the multiplicative structures on $\mathcal{H}_m^{(R,q)}$ and $\mathcal{H}_m^{(\tilde{R},\xi(q))}$ (see equation (1.26) on page 36), we have for this latter homomorphism

$$(i_f \otimes_R \mathrm{id}_{\tilde{R}})(M_f \otimes_R \tilde{R}) = (f \otimes_R 1_{\tilde{R}})\, \mathcal{H}_m^{(\tilde{R},\xi(q))}.$$

Furthermore, an application of $-\otimes_R \tilde{R}$ to the decomposition

$$\mathcal{H}_m^{(R,q)} = i_f(M_f) \oplus \left(\left(1_{\mathcal{H}_A^{(R,q)}} - f\right) \mathcal{H}_m^{(R,q)}\right)$$

together with Lemma 1.2.7 and Lemma 1.2.8.(iii) shows that $i_f \otimes_R \mathrm{id}_{\tilde{R}}$ is again an inclusion. Thus, $i_f \otimes_R \mathrm{id}_{\tilde{R}}$ maps the module $\left(f\mathcal{H}_m^{(R,q)}\right) \otimes_R \tilde{R}$ isomorphically onto the submodule $(f \otimes_R 1_{\tilde{R}})\, \mathcal{H}_m^{(\tilde{R},\xi(q))}$ of $\mathcal{H}_m^{(\tilde{R},\xi(q))}$. This proves the claim. ■

Lemma 1.9.12 *Let R be an integral domain and choose a unit $q \in R$. Furthermore, let $f \in \mathcal{H}_{A_{n-2}}^{(R,q)} \subseteq \mathcal{H}_{A_{n-1}}^{(R,q)}$ be an idempotent. Then we have*

$$\left(f\mathcal{H}_{A_{n-2}}^{(R,q)}\right) \otimes_{\mathcal{H}_{n-1}^{(R,q)}} \mathcal{H}_{A_{n-1}}^{(R,q)} \simeq f\mathcal{H}_{A_{n-1}}^{(R,q)}$$

as right $\mathcal{H}_{A_{n-1}}^{(R,q)}$-modules.

Proof. The proof is analogous to that of Lemma 1.9.11. We have the natural inclusion of right $\mathcal{H}_{n-1}^{(R,q)}$-modules

$$j_f : f\mathcal{H}_{n-1}^{(R,q)} \hookrightarrow \mathcal{H}_{n-1}^{(R,q)}.$$

If we identify $\mathcal{H}_{n-1}^{(R,q)} \otimes_{\mathcal{H}_{n-1}^{(R,q)}} \mathcal{H}_n^{(R,q)}$ and $\mathcal{H}_n^{(R,q)}$ using the isomorphism (1.37) from Lemma 1.9.5, an application of $-\otimes_{\mathcal{H}_{n-1}^{(R,q)}} \mathcal{H}_n^{(R,q)}$ to j_f produces the homomorphism of right $\mathcal{H}_n^{(R,q)}$-modules

$$j_f \otimes_{\mathcal{H}_{n-1}^{(R,q)}} \mathrm{id}_{\mathcal{H}_n^{(R,q)}} : \left(f\mathcal{H}_{n-1}^{(R,q)}\right) \otimes_{\mathcal{H}_{n-1}^{(R,q)}} \mathcal{H}_n^{(R,q)} \to \mathcal{H}_n^{(R,q)}.$$

By using (1.37), we get for this map

$$(j_f \otimes_{\mathcal{H}_{n-1}^{(R,q)}} \mathrm{id}_{\mathcal{H}_n^{(R,q)}})(\left(f\mathcal{H}_{n-1}^{(R,q)}\right) \otimes_{\mathcal{H}_{n-1}^{(R,q)}} \mathcal{H}_n^{(R,q)}) = f\mathcal{H}_n^{(R,q)}.$$

Furthermore, an application of $-\otimes_{\mathcal{H}_{n-1}^{(R,q)}} \mathcal{H}_n^{(R,q)}$ to the decomposition

$$\mathcal{H}_{n-1}^{(R,q)} = j_f(f\mathcal{H}_{n-1}^{(R,q)}) \oplus \left(\left(1_{\mathcal{H}_A^{(R,q)}} - f\right) \mathcal{H}_{n-1}^{(R,q)}\right)$$

together with Lemma 1.9.5 and the compatibility of $-\otimes_{\mathcal{H}_{n-1}^{(R,q)}} \mathcal{H}_n^{(R,q)}$ with direct sums (see [CR1, (2.17)]) shows that $j_f \otimes_{\mathcal{H}_{n-1}^{(R,q)}} \mathrm{id}_{\mathcal{H}_n^{(R,q)}}$ is again an inclusion. Thus, $j_f \otimes_{\mathcal{H}_{n-1}^{(R,q)}} \mathrm{id}_{\mathcal{H}_n^{(R,q)}}$ maps the module $\left(f\mathcal{H}_{n-1}^{(R,q)}\right) \otimes_{\mathcal{H}_{n-1}^{(R,q)}} \mathcal{H}_n^{(R,q)}$ isomorphically onto the submodule $f\mathcal{H}_n^{(R,q)}$ of $\mathcal{H}_n^{(R,q)}$. This proves the claim. ■

Lemma 1.9.13 *The diagram*

$$\begin{tikzcd}
K_0(\mathcal{H}^{(S,a)}_{A_{n-2}}) \arrow[r, "\cdot\left\uparrow^{\mathcal{H}^{(S,a)}_{n}}_{\mathcal{H}^{(S,a)}_{n-1}}\right."] \arrow[d, "\overline{\,\cdot\,}"'] & K_0(\mathcal{H}^{(S,a)}_{A_{n-1}}) \arrow[d, "\overline{\,\cdot\,}"] \\
K_0(\mathcal{H}^{(F,\bar{a})}_{A_{n-2}}) \arrow[r, "\cdot\left\uparrow^{\mathcal{H}^{(F,\bar{a})}_{n}}_{\mathcal{H}^{(F,\bar{a})}_{n-1}}\right."'] & K_0(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})
\end{tikzcd}$$

is commutative.

Proof. From Lemma 1.7.3.(iii) we see that it suffices to show the commutativity for the elements of the basis $\left\{[P^\lambda_{(S,a)}]\,\middle|\,\lambda \in \Pi_{n-1,e_F(\bar{a})}\right\}$ of $K_0(\mathcal{H}^{(S,a)}_{n-1})$. Fix a basis element $[P^\mu_{(S,a)}]$ with a projective indecomposable module $P^\mu_{(S,a)}$ for a $\mu \in \Pi_{n-1,e_F(\bar{a})}$. According to Lemma 1.7.3.(i), there is an idempotent $f^\mu \in \mathcal{H}^{(S,a)}_{n-1}$ such that $P^\mu_{(S,a)} \simeq f^\mu\mathcal{H}^{(S,a)}_{n-1}$ and thus also $[P^\mu_{(S,a)}] = [f^\mu\mathcal{H}^{(S,a)}_{n-1}]$ hold. Now the claim follows from the calculation of the images $\overline{[P^\mu_{(S,a)}]\left\uparrow^{\mathcal{H}^{(S,a)}_{n}}_{\mathcal{H}^{(S,a)}_{n-1}}\right.}$ and $\left(\overline{[P^\mu_{(S,a)}]}\right)\left\uparrow^{\mathcal{H}^{(F,\bar{a})}_{n}}_{\mathcal{H}^{(F,\bar{a})}_{n-1}}\right.$ of the isomorphism class $[P^\mu_{(S,a)}]$ by using the representative $f^\mu\mathcal{H}^{(S,a)}_{n-1}$, Definition 1.7.6, Definition 1.9.7, Lemma 1.9.10, Lemma 1.9.11, and Lemma 1.9.12. ■

Lemma 1.9.14 *The diagram*

$$\begin{tikzcd}
K_0(\mathcal{H}^{(S,a)}_{A_{n-2}}) \arrow[r, "\cdot\left\uparrow^{\mathcal{H}^{(S,a)}_{n}}_{\mathcal{H}^{(S,a)}_{n-1}}\right."] \arrow[d, "-\otimes_S Q"'] & K_0(\mathcal{H}^{(S,a)}_{A_{n-1}}) \arrow[d, "-\otimes_S Q"] \\
G_0(\mathcal{H}^{(Q,a)}_{A_{n-2}}) \arrow[r, "\cdot\left\uparrow^{\mathcal{H}^{(Q,a)}_{n}}_{\mathcal{H}^{(Q,a)}_{n-1}}\right."'] & G_0(\mathcal{H}^{(Q,a)}_{A_{n-1}})
\end{tikzcd}$$

is commutative.

Proof. The proof is analogous to that of Lemma 1.9.13. From Lemma 1.7.3.(iii) we see that it suffices to show the commutativity for the elements of the basis $\left\{[P^\lambda_{(S,a)}]\,\middle|\,\lambda \in \Pi_{n-1,e_F(\bar{a})}\right\}$ of $K_0(\mathcal{H}^{(S,a)}_{n-1})$. Fix a basis element $[P^\mu_{(S,a)}]$ with a projective indecomposable module $P^{\mu}_{(S,a)}$ for a $\mu \in \Pi_{n-1,e_F(\bar{a})}$. According to Lemma 1.7.3.(i), there is an idempotent $f^{\mu} \in \mathcal{H}^{(S,a)}_{n-1}$ such that $P^{\mu}_{(S,a)} \simeq f^{\mu}\mathcal{H}^{(S,a)}_{n-1}$ and thus also $[P^{\mu}_{(S,a)}] = [f^{\mu}\mathcal{H}^{(S,a)}_{n-1}]$ hold. Now the claim follows from the calculation of the images $\left([P^{\mu}_{(S,a)}]\Big\uparrow^{\mathcal{H}^{(S,a)}_n}_{\mathcal{H}^{(S,a)}_{n-1}}\right) \otimes_S Q$ and $\left([P^{\mu}_{(S,a)}] \otimes_S Q\right)\Big\uparrow^{\mathcal{H}^{(Q,a)}_n}_{\mathcal{H}^{(Q,a)}_{n-1}}$ of the isomorphism class $[P^{\mu}_{(S,a)}]$ by using the representative $f^{\mu}\mathcal{H}^{(S,a)}_{n-1}$, Definition 1.7.8, Definition 1.9.6, Definition 1.9.7, Lemma 1.9.10, Lemma 1.9.11, and Lemma 1.9.12. ■

Lemma 1.9.15 *The diagram*

$$\begin{tikzcd}
K_0(\mathcal{H}^{(F,\bar{a})}_{A_{n-2}}) \arrow[r, "\cdot\big\uparrow^{\mathcal{H}^{(F,\bar{a})}_n}_{\mathcal{H}^{(F,\bar{a})}_{n-1}}"] \arrow[d, "B^{\mathcal{H}}_{n-1,\mathcal{K}}"'] & K_0(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}) \arrow[d, "B^{\mathcal{H}}_{n,\mathcal{K}}"] \\
G_0(\mathcal{H}^{(Q,a)}_{A_{n-2}}) \arrow[r, "\cdot\big\uparrow^{\mathcal{H}^{(Q,a)}_n}_{\mathcal{H}^{(Q,a)}_{n-1}}"'] & G_0(\mathcal{H}^{(Q,a)}_{A_{n-1}})
\end{tikzcd}$$

is commutative.

Proof. This follows from Definition 1.7.9, Lemma 1.9.13, and Lemma 1.9.14. ■

Now the decomposition of induced projective modules in projective indecomposable modules is described in more detail.

Lemma 1.9.16 *For an $m \in \mathbb{N}$, $[\mathcal{H}^{(S,a)}_{A_{m-1}}] \in K_0(\mathcal{H}^{(S,a)}_{A_{m-1}})$ has a decomposition*

$$[\mathcal{H}^{(S,a)}_{A_{m-1}}] = [U_1] + \cdots + [U_z]$$

in isomorphism classes of projective irreducible right $\mathcal{H}^{(S,a)}_{A_{m-1}}$-modules $U_1, \ldots, U_z$ with a $z \in \mathbb{N}$. The summands are uniquely determined up to ordering. For any given decomposition

$$[\mathcal{H}^{(S,a)}_{A_{m-1}}] = [V_1] + \cdots + [V_y]$$

of $[\mathcal{H}^{(S,a)}_{A_{m-1}}]$ in isomorphism classes of projective right $\mathcal{H}^{(S,a)}_{A_{m-1}}$-modules $V_1, \ldots, V_y$ with a $y \in \mathbb{N}$, there is a decomposition

$$\{1, \ldots, z\} = \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_y$$

of the index set $\{1, \ldots, z\}$ in pairwise disjoint subsets $\mathcal{J}_1, \ldots, \mathcal{J}_y$ such that, for every $i \in \{1, \ldots, y\}$, we have in $K_0(\mathcal{H}^{(S,a)}_{A_{m-1}})$

$$[V_i] = \sum_{j \in \mathcal{J}_i} [U_j].$$

Proof. According to property (1.38).(ii), $\mathcal{H}_m^{(S,a)}$ satisfies condition (ii) in [CR1, Theorem 6.12]. Thus, that theorem can be applied to finitely generated $\mathcal{H}_m^{(S,a)}$-modules. Now this theorem, the fact that $\mathcal{H}_m^{(S,a)}$-direct summands of $\mathcal{H}_m^{(S,a)}$ are projective (see [CR1, §2D]), and Definition 1.7.1.(ii) prove the claim. ■

Lemma 1.9.17 *Let $\nu \in \Gamma_{e_F(\bar{a})}(n)$.*

(i) Let $\mu \in \Pi_{n-1,e_F(\bar{a})}$. Then $b^\nu_{\mathrm{Proj}}(\mathcal{H}^{(Q,a)}_{A_{n-1}})(B^{\mathcal{H}}_{n,\mathcal{K}}([P^\mu_{(F,\bar{a})}]\Big\uparrow^{\mathcal{H}_n^{(F,\bar{a})}}_{\mathcal{H}_{n-1}^{(F,\bar{a})}})) \in G_0(\mathcal{H}^{(Q,a)}_{A_{n-1}})$ can be written as

$$b^\nu_{\mathrm{Proj}}(\mathcal{H}^{(Q,a)}_{A_{n-1}})(B^{\mathcal{H}}_{n,\mathcal{K}}([P^\mu_{(F,\bar{a})}]\Big\uparrow^{\mathcal{H}_n^{(F,\bar{a})}}_{\mathcal{H}_{n-1}^{(F,\bar{a})}})) = \sum_{\eta \in \Pi^{\nu,e_F(\bar{a})}_{n,e_F(\bar{a})}} f_{\mu\eta} B^{\mathcal{H}}_{n,\mathcal{K}}([P^\eta_{(F,\bar{a})}]) \tag{1.39}$$

with uniquely determined coefficients $f_{\mu\eta} \in \mathbb{N}_0$ for $\eta \in \Pi^{\nu,e_F(\bar{a})}_{n,e_F(\bar{a})}$.

(ii) Let $\eta \in \Pi^{\nu,e_F(\bar{a})}_{n,e_F(\bar{a})}$. Then there is a $\mu \in \Pi_{n-1,e_F(\bar{a})}$ such that we have for the corresponding coefficient $f_{\mu\eta}$ in (1.39)

$$f_{\mu\eta} > 0.$$

Proof. (i) According to Lemma 1.7.2.(iii), Lemma 1.7.2.(iv), and Definition 1.9.7, the element $[P^\mu_{(F,\bar{a})}]\Big\uparrow^{\mathcal{H}_n}_{\mathcal{H}_{n-1}} \in K_0(\mathcal{H}_n^{(F,\bar{a})})$ can be written as

$$[P^\mu_{(F,\bar{a})}]\Big\uparrow^{\mathcal{H}_n}_{\mathcal{H}_{n-1}} = \sum_{\eta \in \Pi_{n,e_F(\bar{a})}} f_{\mu\eta}[P^\eta_{(F,\bar{a})}]$$

with uniquely determined coefficients $f_{\mu\eta} \in \mathbb{N}_0$. Now an application of $b^\nu_{\mathrm{Proj}}(\mathcal{H}_n^{(F,\bar{a})})$ from Definition 1.8.15.(ii) together with Lemma 1.8.10.(iii), Definition 1.8.5, Lemma 1.8.8.(i), and Theorem 1.8.11 leads to

$$b^\nu_{\mathrm{Proj}}(\mathcal{H}_n^{(F,\bar{a})})([P^\mu_{(F,\bar{a})}]\Big\uparrow^{\mathcal{H}_n}_{\mathcal{H}_{n-1}}) = \sum_{\eta \in \Pi^{\nu,e_F(\bar{a})}_{n,e_F(\bar{a})}} f_{\mu\eta}[P^\eta_{(F,\bar{a})}].$$

Lemma 1.7.2.(iii) shows that the coefficients on the right hand side of the preceding equation also are uniquely determined. Now an application of the Brauer map $B^{\mathcal{H}}_{n,\mathcal{K}}$ from Definition 1.7.9 gives

$$B^{\mathcal{H}}_{n,\mathcal{K}}(b^\nu_{\mathrm{Proj}}(\mathcal{H}_n^{(F,\bar{a})})([P^\mu_{(F,\bar{a})}]\Big\uparrow^{\mathcal{H}_n}_{\mathcal{H}_{n-1}})) = \sum_{\eta \in \Pi^{\nu,e_F(\bar{a})}_{n,e_F(\bar{a})}} f_{\mu\eta} B^{\mathcal{H}}_{n,\mathcal{K}}([P^\eta_{(F,\bar{a})}]).$$

Because of the injectivity of $B^{\mathcal{H}}_{n,\mathcal{K}}$ (see Corollary 1.7.18), the coefficients on the right hand side are again uniquely determined. Furthermore, we have according to Lemma 1.8.19

$$B^{\mathcal{H}}_{n,\mathcal{K}}(b^{\nu}_{\mathrm{Proj}}(\mathcal{H}^{(F,\bar{a})}_n)([P^{\mu}_{(F,\bar{a})}]\Big\uparrow^{\mathcal{H}_n}_{\mathcal{H}_{n-1}})) = b^{\nu}_{\mathrm{Proj}}(\mathcal{H}^{(Q,a)}_n)(B^{\mathcal{H}}_{n,\mathcal{K}}([P^{\mu}_{(F,\bar{a})}]\Big\uparrow^{\mathcal{H}_n}_{\mathcal{H}_{n-1}})).$$

All this proves the claim.

(ii) According to Lemma 1.7.2.(i), there is an idempotent $f^{\eta} \in \mathcal{H}^{(F,\bar{a})}_n$ such that

$$P^{\eta}_{(F,\bar{a})} \simeq f^{\eta}\mathcal{H}^{(F,\bar{a})}_n$$

holds. Since $f^{\eta}\mathcal{H}^{(F,\bar{a})}_n$ is a direct summand of $\mathcal{H}^{(F,\bar{a})}_n$ (see [CR1, §6A]), it follows from Definition 1.7.1.(ii) and Lemma 1.9.16 that $[P^{\eta}_{(F,\bar{a})}]$ occurs in the representation of $[\mathcal{H}^{(F,\bar{a})}_n] \in K_0(\mathcal{H}^{(F,\bar{a})}_n)$ with respect to the basis $\left\{[P^{\lambda}_{(F,\bar{a})}] \,\middle|\, \lambda \in \Pi_{n,e_F(\bar{a})}\right\}$ from Lemma 1.7.2.(iii) with positive multiplicity. Furthermore, we get from the decomposition

$$[\mathcal{H}^{(F,\bar{a})}_{n-1}] = [U_1] + \cdots + [U_z]$$

of $[\mathcal{H}^{(F,\bar{a})}_{n-1}] \in K_0(\mathcal{H}^{(F,\bar{a})}_{n-1})$ in isomorphism classes of projective indecomposable modules $U_1, \ldots, U_z$ with a $z \in \mathbb{N}$ as in Lemma 1.9.16 by applying the map $\cdot\Big\uparrow^{\mathcal{H}_n}_{\mathcal{H}_{n-1}}$ from Definition 1.9.7 and using the relation

$$\mathcal{H}^{(F,\bar{a})}_{n-1}\Big\uparrow^{\mathcal{H}_n}_{\mathcal{H}_{n-1}} \simeq \mathcal{H}^{(F,\bar{a})}_n$$

(see Definition 1.9.4 and Lemma 1.9.5) the decomposition

$$[\mathcal{H}^{(F,\bar{a})}_n] = [U_1\Big\uparrow^{\mathcal{H}_n}_{\mathcal{H}_{n-1}}] + \cdots + [U_z\Big\uparrow^{\mathcal{H}_n}_{\mathcal{H}_{n-1}}]$$

of $[\mathcal{H}^{(F,\bar{a})}_n] \in K_0(\mathcal{H}^{(F,\bar{a})}_n)$ in isomorphism classes of projective modules. Thus, according to Lemma 1.9.16 and Lemma 1.7.2.(iii), there are a $j \in \{1, \ldots, z\}$ and a $\mu \in \Pi_{n-1,e_F(\bar{a})}$ such that

$$U_j \simeq P^{\mu}_{(F,\bar{a})}$$

holds and furthermore $[P^{\eta}_{(F,\bar{a})}]$ occurs in the representation of

$$[P^{\mu}_{(F,\bar{a})}]\Big\uparrow^{\mathcal{H}_n}_{\mathcal{H}_{n-1}} = [U_j\Big\uparrow^{\mathcal{H}_n}_{\mathcal{H}_{n-1}}] \in K_0(\mathcal{H}^{(F,\bar{a})}_n)$$

with respect to the basis $\left\{[P^{\lambda}_{(F,\bar{a})}] \,\middle|\, \lambda \in \Pi_{n,e_F(\bar{a})}\right\}$ with positive multiplicity. Now the claim is proved by applying $b^{\nu}_{\mathrm{Proj}}(\mathcal{H}^{(F,\bar{a})}_n)$ followed by $B^{\mathcal{H}}_{n,\mathcal{K}}$ to $[P^{\eta}_{(F,\bar{a})}]$ and $[P^{\mu}_{(F,\bar{a})}]\Big\uparrow^{\mathcal{H}_n}_{\mathcal{H}_{n-1}}$ and making use of Definition 1.8.15.(ii), Lemma 1.8.10.(iii), Definition 1.8.5, Lemma 1.8.8.(i), Theorem 1.8.11, Lemma 1.8.19, and the fact that the coefficients occurring on the right hand side of (1.39) are uniquely determined. ■

Next, an upper bound for the decomposition numbers in a block of the decomposition matrix $\Delta^{\mathcal{H}}_{n,\mathcal{K}}$ (see Definition 1.8.21) is derived from the entries in the decomposition matrix $\Delta^{\mathcal{H}}_{n-1,\mathcal{K}}$ by means of induced projective indecomposable modules.

Theorem 1.9.18 *Let $\nu \in \Gamma_{e_F(\bar{a})}(n)$. With this, put for every $\lambda \in \Pi_n^{\nu,e_F(\bar{a})}$ and every $\eta \in \Gamma_{e_F(\bar{a})}(n-1)$*

$$E_{\lambda,\eta} = \left| \lambda \downarrow \cap \Pi_{n-1}^{\eta,e_F(\bar{a})} \right|.$$

Furthermore, set for every $\lambda \in \Pi_n^{\nu,e_F(\bar{a})}$

$$H_\lambda = \max\left\{ E_{\lambda,\eta} \,|\, \eta \in \Gamma_{e_F(\bar{a})}(n-1) \right\}.$$

In addition let

$$J_\nu = \max\left\{ H_\lambda \,\middle|\, \lambda \in \Pi_n^{\nu,e_F(\bar{a})} \right\}.$$

Finally put

$$U_{n-1} = \max\left\{ d^{n-1,\mathcal{K}}_{\lambda\mu} \,\middle|\, \lambda \in \Pi_{n-1}, \mu \in \Pi_{n-1,e_F(\bar{a})} \right\}.$$

Then we have for every $\lambda \in \Pi_n^{\nu,e_F(\bar{a})}$ and every $\mu \in \Pi^{\nu,e_F(\bar{a})}_{n,e_F(\bar{a})}$

$$d^{n,\mathcal{K}}_{\lambda\mu} \leq J_\nu U_{n-1}.$$

This is equivalent to say that $J_\nu U_{n-1}$ is an upper bound for the entries in the ν-block $\Delta^{\mathcal{H}}_{n,\mathcal{K}}(\nu) = \left(d^{n,\mathcal{K}}_{\lambda\mu} \right)_{\substack{\lambda \in \Pi_n^{\nu,e_F(\bar{a})} \\ \mu \in \Pi^{\nu,e_F(\bar{a})}_{n,e_F(\bar{a})}}}$ of $\Delta^{\mathcal{H}}_{n,\mathcal{K}}$.

Proof. Fix a $\theta \in \Pi^{\nu,e_F(\bar{a})}_{n,e_F(\bar{a})}$. We see from Lemma 1.7.16.(iii) and Theorem 1.8.20.(ii) that $B^{\mathcal{H}}_{n,\mathcal{K}}([P^\theta_{(F,\bar{a})}]) \in G_0(\mathcal{H}_n^{(Q,a)})$ has the representation

$$B^{\mathcal{H}}_{n,\mathcal{K}}([P^\theta_{(F,\bar{a})}]) = \sum_{\lambda \in \Pi_n^{\nu,e_F(\bar{a})}} d^{n,\mathcal{K}}_{\lambda\theta} [S^\lambda_{(Q,a)}] \tag{1.40}$$

with respect to the basis $\left\{ [S^\lambda_{(Q,a)}] \,\middle|\, \lambda \in \Pi_n \right\}$ from Lemma 1.4.3.(i). Furthermore, according to Lemma 1.9.17.(ii), there is a $\mu \in \Pi_{n-1,e_F(\bar{a})}$ such that $B^{\mathcal{H}}_{n,\mathcal{K}}([P^\theta_{(F,\bar{a})}])$ occurs in the decomposition (1.39) of $b^\nu_{\mathrm{Proj}}(\mathcal{H}_n^{(Q,a)})(B^{\mathcal{H}}_{n,\mathcal{K}}([P^\mu_{(F,\bar{a})}] \Big\uparrow^{\mathcal{H}_n}_{\mathcal{H}_{n-1}})) \in G_0(\mathcal{H}_n^{(Q,a)})$ from Lemma 1.9.17.(i) with positive multiplicity. Let

$$b^\nu_{\mathrm{Proj}}(\mathcal{H}_n^{(Q,a)})(B^{\mathcal{H}}_{n,\mathcal{K}}([P^\mu_{(F,\bar{a})}] \Big\uparrow^{\mathcal{H}_n}_{\mathcal{H}_{n-1}})) = \sum_{\lambda \in \Pi_n^{\nu,e_F(\bar{a})}} g_{\lambda\mu} [S^\lambda_{(Q,a)}] \tag{1.41}$$

be the representation of the element $b^\nu_{\mathrm{Proj}}(\mathcal{H}_n^{(Q,a)})(B^{\mathcal{H}}_{n,\mathcal{K}}([P^\mu_{(F,\bar{a})}] \Big\uparrow^{\mathcal{H}_n}_{\mathcal{H}_{n-1}})) \in G_0(\mathcal{H}_n^{(Q,a)})$ with respect to the basis $\left\{ [S^\lambda_{(Q,a)}] \,\middle|\, \lambda \in \Pi_n \right\}$ with uniquely determined coefficients

$g_{\lambda\mu} \in \mathbb{N}_0$. It follows from Lemma 1.8.16 that only indices $\lambda \in \Pi_n^{\nu,e_F(\bar{a})}$ occur on the right hand side of (1.41). Furthermore, it follows from Definition 1.9.7, Lemma 1.7.2, Definition 1.7.9, Definition 1.7.6, Lemma 1.7.3, Definition 1.7.8, and Definition 1.8.15 that the coefficients on the right hand side of (1.41) are nonnegative. The decomposition (1.41) is obtained from the representation (1.39) in Lemma 1.9.17 by substituting decompositions of the form (1.40) for the summands $B^{\mathcal{H}}_{n,\mathcal{K}}([P^{\kappa}_{(F,\bar{a})}])$ with $\kappa \in \Pi^{\nu,e_F(\bar{a})}_{n,e_F(\bar{a})}$. Since all coefficients involved in this process are nonnegative (see Definition 1.4.17) and $B^{\mathcal{H}}_{n,\mathcal{K}}([P^{\theta}_{(F,\bar{a})}])$ in fact occurs as a summand, we get from this

$$\forall \lambda \in \Pi_n^{\nu,e_F(\bar{a})} : d^{n,\mathcal{K}}_{\lambda\theta} \leq g_{\lambda\mu}.$$

Since $\theta \in \Pi^{\nu,e_F(\bar{a})}_{n,e_F(\bar{a})}$ was arbitrarily chosen, it suffices for the proof of the claim to show that

$$\forall \lambda \in \Pi_n^{\nu,e_F(\bar{a})}, \mu \in \Pi_{n-1,e_F(\bar{a})} : g_{\lambda\mu} \leq J_\nu U_{n-1}. \tag{1.42}$$

In order to prove (1.42), we now express the coefficients $g_{\lambda\mu}$ in terms of the entries of the decomposition matrix $\Delta^{\mathcal{H}}_{n-1,\mathcal{K}}$. Fix a $\mu \in \Pi_{n-1,e_F(\bar{a})}$. According to Theorem 1.8.11 and Definition 1.8.12 — applied to Hecke algebras of type A_{n-2} — the projective indecomposable module $P^{\mu}_{(F,\bar{a})}$ lies in the block of $\mathcal{H}^{(F,\bar{a})}_{n-1}$ indexed by the core

$$\eta = \gamma_{e_F(\bar{a})}(\mu),$$

and according to Lemma 1.7.16.(iii) and Theorem 1.8.20.(ii) — applied again to Hecke algebras of type A_{n-2} — we have in $G_0(\mathcal{H}^{(Q,a)}_{n-1})$

$$B^{\mathcal{H}}_{n-1,\mathcal{K}}([P^{\mu}_{(F,\bar{a})}]) = \sum_{\kappa \in \Pi^{\eta,e_F(\bar{a})}_{n-1}} d^{n-1,\mathcal{K}}_{\kappa\mu} [S^{\kappa}_{(Q,a)}].$$

By using Lemma 1.9.15 and Lemma 1.9.8, we get from this

$$\begin{aligned} B^{\mathcal{H}}_{n,\mathcal{K}}([P^{\mu}_{(F,\bar{a})}]\Big\uparrow^{\mathcal{H}^{(F,\bar{a})}_n}_{\mathcal{H}^{(F,\bar{a})}_{n-1}}) &= \left(B^{\mathcal{H}}_{n-1,\mathcal{K}}([P^{\mu}_{(F,\bar{a})}])\right)\Big\uparrow^{\mathcal{H}^{(Q,a)}_n}_{\mathcal{H}^{(Q,a)}_{n-1}} \\ &= \sum_{\kappa \in \Pi^{\eta,e_F(\bar{a})}_{n-1}} d^{n-1,\mathcal{K}}_{\kappa\mu} \sum_{\lambda \in \kappa\uparrow} [S^{\lambda}_{(Q,a)}]. \end{aligned}$$

By applying $b^{\nu}_{\mathrm{Proj}}(\mathcal{H}^{(Q,a)}_n)$ and using Lemma 1.8.16, we obtain now

$$b^{\nu}_{\mathrm{Proj}}(\mathcal{H}^{(Q,a)}_n)(B^{\mathcal{H}}_{n,\mathcal{K}}([P^{\mu}_{(F,\bar{a})}]\Big\uparrow^{\mathcal{H}_n}_{\mathcal{H}_{n-1}})) = \sum_{\kappa \in \Pi^{\eta,e_F(\bar{a})}_{n-1}} d^{n-1,\mathcal{K}}_{\kappa\mu} \sum_{\lambda \in \kappa\uparrow \cap \Pi^{\nu,e_F(\bar{a})}_n} [S^{\lambda}_{(Q,a)}].$$

By using Definition 1.1.20 and Lemma 1.1.21, this double sum can be rewritten such that we get

$$\begin{aligned} & b^{\nu}_{\mathrm{Proj}}(\mathcal{H}_n^{(Q,a)})(B^{\mathcal{H}}_{n,\mathcal{K}}([P^{\mu}_{(F,\bar{a})}]\big\uparrow^{\mathcal{H}_n}_{\mathcal{H}_{n-1}})) \\ = & \sum_{\lambda\in\Pi_n^{\nu,e_F(\bar{a})}} \left(\sum_{\kappa\in\lambda\downarrow\cap\Pi_{n-1}^{\eta,e_F(\bar{a})}} d^{n-1,\mathcal{K}}_{\kappa\mu} \right) [S^{\lambda}_{(Q,a)}]. \end{aligned} \tag{1.43}$$

Because the coefficients in the decompositions (1.41) and (1.43) are uniquely determined (see Lemma 1.4.3.(i)), we have now for every $\lambda \in \Pi_n^{\nu,e_F(\bar{a})}$

$$\begin{aligned} g_{\lambda\mu} &= \sum_{\kappa\in\lambda\downarrow\cap\Pi_{n-1}^{\eta,e_F(\bar{a})}} d^{n-1,\mathcal{K}}_{\kappa\mu} \\ &\leq U_{n-1}\left|\lambda\downarrow\cap\Pi_{n-1}^{\eta,e_F(\bar{a})}\right| \\ &= U_{n-1}E_{\lambda,\eta} \\ &\leq U_{n-1}H_{\lambda} \\ &\leq U_{n-1}J_{\nu}. \end{aligned}$$

Since $\mu \in \Pi_{n-1,e_F(\bar{a})}$ was arbitrarily chosen, this proves (1.42) and thus the claim of the theorem. ■

In Section 2.1, the preceding theorem will be applied to the blocks of decomposition matrices indexed by the core (0) and having $e_F(\bar{a})$-weight 3.

1.10 The Theorem of Schaper for Hecke algebras of type A

This section describes the generalization of the Theorem of Schaper from group algebras of symmetric groups to Hecke algebras of type A. The result for group algebras of symmetric groups can be found in [SCH]. The generalization to Hecke algebras of type A has been done by James and Mathas in [JM] where the following material is presented in more detail.

The Theorem of Schaper is a useful tool for determining decomposition numbers of Hecke algebras of type A in an inductive manner. It involves the bilinear form from Definition 1.3.16 and rim hooks in partitions. For the following we fix an $n \in \mathbb{N}$. Furthermore, we choose a modular system

$$\mathcal{K} = (Q, \psi, S, I, a, F)$$

as in Definition 1.4.8 such that

$$\mathcal{H}_{A_{n-1}}^{(Q,a)} \text{ is semisimple.} \tag{1.44}$$

Part of the following holds under weaker assumptions on $\mathcal{K}$, but this degree of generality will not be required here.

First we describe the Jantzen filtration of Specht modules. Filtrations of this kind have been investigated by Jantzen in [JAN].

Definition 1.10.1 *(i) For $j \in \mathbb{N}_0$ we define*

$$I^{(j)} = \{x \in Q \setminus \{0_Q\} \mid \psi(x) \geq j\} \cup \{0_Q\}\,.$$

(ii) For $\lambda \vdash n$ and $j \in \mathbb{N}_0$ we define

$$S_{(S,a)}^{\lambda}(j) = \left\{x \in S_{(S,a)}^{\lambda} \,\middle|\, \forall y \in S_{(S,a)}^{\lambda} : \gamma_{(S,a)}^{\lambda}(x,y) \in I^{(j)}\right\}.$$

Lemma 1.10.2 *(i) We have $I^{(0)} = S$, $I^{(1)} = I$, and*

$$I^{(0)} \supseteq I^{(1)} \supseteq I^{(2)} \supseteq \cdots .$$

For every $j \in \mathbb{N}_0$, $I^{(j)}$ is an ideal in S.

(ii) Let $\lambda \vdash n$. Then we have $S_{(S,a)}^{\lambda}(0) = S_{(S,a)}^{\lambda}$ and

$$S_{(S,a)}^{\lambda}(0) \supseteq S_{(S,a)}^{\lambda}(1) \supseteq S_{(S,a)}^{\lambda}(2) \supseteq \cdots .$$

For every $j \in \mathbb{N}_0$, $S_{(S,a)}^{\lambda}(j)$ is an S-submodule and also an $\mathcal{H}_{A_{n-1}}^{(S,a)}$-submodule of $S_{(S,a)}^{\lambda}$.

(iii) Let $\lambda \vdash n$. Then we have $\overline{S_{(S,a)}^{\lambda}(0)} = S_{(F,\bar{a})}^{\lambda}$, $\overline{S_{(S,a)}^{\lambda}(1)} = \operatorname{rad} \gamma_{(F,\bar{a})}^{\lambda}$, and

$$\overline{S_{(S,a)}^{\lambda}(0)} \supseteq \overline{S_{(S,a)}^{\lambda}(1)} \supseteq \overline{S_{(S,a)}^{\lambda}(2)} \supseteq \cdots .$$

For every $j \in \mathbb{N}_0$, $\overline{S_{(S,a)}^{\lambda}(j)}$ is an F-subvectorspace and also an $\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}$-submodule of $S_{(F,\bar{a})}^{\lambda}$.

Proof. (i) This follows from Definition 1.4.5, Definition 1.4.6, Definition 1.4.8, and Definition 1.10.1.(i).

(ii) The first claim $S_{(S,a)}^{\lambda}(0) = S_{(S,a)}^{\lambda}$ is obtained from the relation $I^{(0)} = S$ in statement (i). The chain of inclusions follows from the chain of inclusions in statement (i). Furthermore, the fact that $S_{(S,a)}^{\lambda}(j)$ is an S-submodule of $S_{(S,a)}^{\lambda}$ follows from the fact that $I^{(j)}$ is an ideal in S. Finally, the fact that $S^\lambda_{(S,a)}(j)$ is an $\mathcal{H}^{(S,a)}_{A_{n-1}}$-submodule of $S^\lambda_{(S,a)}$ is obtained from Definition 1.10.1.(ii), Definition 1.3.16.(i), Definition 1.3.8, and [DJ1, Lemma 4.4].

(iii) The first claim $\overline{S^\lambda_{(S,a)}(0)} = S^\lambda_{(F,\bar{a})}$ follows from Lemma 1.5.2 and the relation $S^\lambda_{(S,a)}(0) = S^\lambda_{(S,a)}$ in statement (ii). The second claim $\overline{S^\lambda_{(S,a)}(1)} = \operatorname{rad}\gamma^\lambda_{(F,\bar{a})}$ is obtained from the relation $I^{(1)} = I$ in statement (i), the relation $F = \bar{S} = S/I$ in Definition 1.4.8, Definition 1.3.16.(iii), and the following compatibility property of the bilinear form γ^λ and the reduction modulo I (see Definition 1.4.14).

$$\forall x, y \in S^\lambda_{(S,a)} : \overline{\gamma^\lambda_{(S,a)}(x,y)} = \gamma^\lambda_{(F,\bar{a})}(\bar{x}, \bar{y})$$

This in turn is obtained from Definition 1.3.16.(i), Theorem 1.3.2, Remark 1.3.7.(i), and the following analogous property of the bilinear form β^λ.

$$\forall d, \tilde{d} \in \mathcal{D}_\lambda : \overline{\beta^\lambda_{(S,a)}(x^{(S,a)}_\lambda T_d, x^{(S,a)}_\lambda T_{\tilde{d}})} = \beta^\lambda_{(F,\bar{a})}(x^{(F,\bar{a})}_\lambda T_d, x^{(F,\bar{a})}_\lambda T_{\tilde{d}})$$

This finally follows from Definition 1.3.8. The chain of inclusions is obtained from the corresponding chain of inclusions in statement (ii) by applying reduction modulo I. Similarly, the fact that $\overline{S^\lambda_{(S,a)}(j)}$ is an F-subvectorspace and an $\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}$-submodule of $S^\lambda_{(F,\bar{a})}$ is obtained from the analogous claim in statement (ii). ■

Definition 1.10.3 *Let $\lambda \vdash n$. Then the filtration*

$$S^\lambda_{(F,\bar{a})} = \overline{S^\lambda_{(S,a)}(0)} \supseteq \overline{S^\lambda_{(S,a)}(1)} \supseteq \overline{S^\lambda_{(S,a)}(2)} \supseteq \cdots$$

of $S^\lambda_{(F,\bar{a})}$ with the $\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}$-submodules from Lemma 1.10.2.(iii) is called the Jantzen filtration of $S^\lambda_{(F,\bar{a})}$.

In order to state the Theorem of Schaper, we next introduce an indexing scheme for the isomorphism classes of Specht modules in the Grothendieck groups $G_0(\mathcal{H}^{(F,\bar{a})}_{A_{m-1}})$ with $m \in \mathbb{N}$ by means of β-sequences.

Definition 1.10.4 *Let $(\beta_1, \ldots, \beta_c)$ be a finite sequence of length $c \in \mathbb{N}$ with entries $\beta_j \in \mathbb{Z}$ for $j \in \{1, \ldots, c\}$ such that $\{\beta_1, \ldots, \beta_c\} \neq \{0, \ldots, c-1\}$ holds.*

(i) If the numbers $\beta_1, \ldots, \beta_c$ are all nonnegative and pairwise distinct, there is a $w \in \mathfrak{S}_c$ such that the sequence $(\beta_{1w}, \ldots, \beta_{cw})$ is strictly decreasing and thus a β-sequence for an appropriate partition λ of m with an $m \in \mathbb{N}$. With this, the element $S(\beta_1, \ldots, \beta_c)$ of $G_0(\mathcal{H}^{(F,\bar{a})}_{A_{m-1}})$ is defined as

$$S(\beta_1, \ldots, \beta_c) = (-1)^{\ell(w)} \cdot [S^\lambda_{(F,\bar{a})}].$$

(ii) If some of the numbers $\beta_1, \ldots, \beta_c$ are negative or equal to one another, then for every $m \in \mathbb{N}$ the element $S(\beta_1, \ldots, \beta_c)$ of $G_0(\mathcal{H}_{A_{m-1}}^{(F,\bar{a})})$ is defined as

$$S(\beta_1, \ldots, \beta_c) = 0.$$

Remark 1.10.5 *The condition $\{\beta_1, \ldots, \beta_c\} \neq \{0, \ldots, c-1\}$ in Definition 1.10.4 excludes the possibility that rearranging the sequence $(\beta_1, \ldots, \beta_c)$ produces a β-sequence for the partition (0).*

Now the Theorem of Schaper for Hecke algebras of type A can be formulated. The formulation makes use of Definition 1.2.2.(i).

Theorem 1.10.6 *Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$ with $\lambda' = (\lambda_1', \lambda_2', \ldots)$ and let $\beta = (\beta_1, \ldots, \beta_c)$ be a β-sequence for λ. Then we have in $G_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$*

$$\begin{aligned}\sum_{j \in \mathbb{N}} [\overline{S_{(S,a)}^{\lambda}(j)}] \; = \; & \sum_{k=1}^{\lambda_1} \sum_{i=1}^{\lambda_k'-1} \sum_{j=i+1}^{\lambda_k'} \Big(\psi([|r_{(i,k)}^{\lambda}|]_a) - \psi([|r_{(j,k)}^{\lambda}|]_a) \Big) \cdot \\ & \qquad S(\beta_1, \ldots, \beta_{i-1}, \beta_i + \left|r_{(j,k)}^{\lambda}\right|, \beta_{i+1}, \ldots \\ & \qquad\qquad \ldots, \beta_{j-1}, \beta_j - \left|r_{(j,k)}^{\lambda}\right|, \beta_{j+1}, \ldots, \beta_c).\end{aligned} \tag{1.45}$$

Proof. From the semisimplicity of $\mathcal{H}_n^{(Q,a)}$ (see (1.44)) and Theorem 1.3.21.(i) we get

$$e_Q(a) > n.$$

This shows together with Definition 1.2.2

$$\forall k \in \{1, \ldots, n\} : [k]_a \neq 0_Q.$$

From this and Definition 1.1.14 we see that the differences involving the valuation ψ in formula (1.45) are well defined. Now the claim is obtained from [JM, Theorem 4.7 and Theorem 4.13]. ■

Remark 1.10.7 *The right hand side of the identity (1.45) is a finite sum representing an element of $G_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$. This shows that the sum on the left hand side of that formula contains only finitely many summands different from 0.*

The following corollary restricts the identity (1.45) to a block. It makes use of Definition 1.1.38 and Definition 1.8.15.(i).

Corollary 1.10.8 *Suppose that $\mathcal{H}_{A_{n-1}}^{(Q,a)}$ is semisimple and S is complete with respect to the valuation ψ. With the notation from Theorem 1.10.6, let furthermore $\mu = \gamma_{e_F(\bar{a})}(\lambda)$. Then we have in $G_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$, again with the notation from Theorem 1.10.6,*

$$\begin{aligned}
&\sum_{j\in\mathbb{N}} [\overline{S_{(S,a)}^{\lambda}(j)}] \\
&= \sum_{k=1}^{\lambda_1} \sum_{i=1}^{\lambda'_k-1} \sum_{j=i+1}^{\lambda'_k} \left(\psi(\left[\left|r_{(i,k)}^{\lambda}\right|\right]_a) - \psi(\left[\left|r_{(j,k)}^{\lambda}\right|\right]_a) \right) \cdot \\
&\qquad b_{\mathrm{Proj}}^{\mu}(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}) S(\beta_1, \ldots, \beta_{i-1}, \beta_i + \left|r_{(j,k)}^{\lambda}\right|, \beta_{i+1}, \ldots \\
&\qquad\qquad \ldots, \beta_{j-1}, \beta_j - \left|r_{(j,k)}^{\lambda}\right|, \beta_{j+1}, \ldots, \beta_c).
\end{aligned} \tag{1.46}$$

Proof. If we apply the homomorphism $b_{\mathrm{Proj}}^{\mu}(\mathcal{H}_n^{(F,\bar{a})}) : G_0(\mathcal{H}_n^{(F,\bar{a})}) \to G_0(\mathcal{H}_n^{(F,\bar{a})})$ from Definition 1.8.15.(i) to both sides of the formula (1.45), we obtain for the left hand side by using Definition 1.8.5, Lemma 1.8.8.(iv), Lemma 1.8.10.(iii), Theorem 1.8.11, and Definition 1.10.3

$$\begin{aligned}
b_{\mathrm{Proj}}^{\mu}(\mathcal{H}_n^{(F,\bar{a})})(\sum_{j\in\mathbb{N}} [\overline{S_{(S,a)}^{\lambda}(j)}]) &= \sum_{j\in\mathbb{N}} \left(b_{\mathrm{Proj}}^{\mu}(\mathcal{H}_n^{(F,\bar{a})})([\overline{S_{(S,a)}^{\lambda}(j)}]) \right) \\
&= \sum_{j\in\mathbb{N}} [\overline{S_{(S,a)}^{\lambda}(j)} b_{\mathrm{Idemp}}^{\mu}(\mathcal{H}_n^{(F,\bar{a})})] \\
&= \sum_{j\in\mathbb{N}} [\overline{S_{(S,a)}^{\lambda}(j)}].
\end{aligned}$$

This proves the claim. ■

Remark 1.10.9 *(i) According to Lemma 1.1.24 and Lemma 1.1.25, the construction of the sequence*

$$(\beta_1, \ldots, \beta_{i-1}, \beta_i + \left|r_{(j,k)}^{\lambda}\right|, \beta_{i+1}, \ldots, \beta_{j-1}, \beta_j - \left|r_{(j,k)}^{\lambda}\right|, \beta_{j+1}, \ldots, \beta_c) \tag{1.47}$$

from the β-sequence $(\beta_1, \ldots, \beta_c)$ in the identities (1.45) from Theorem 1.10.6 and (1.46) from Corollary 1.10.8 can be interpreted in such a way that one removes the rim hook $r_{(j,k)}^{\lambda}$ from λ and tries to add a rim hook r with

$$|r| = \left|r_{(j,k)}^{\lambda}\right| \qquad \text{and} \qquad \max\left\{ \tilde{i} \,\middle|\, (\tilde{i}, \tilde{k}) \in r \right\} = i$$

to λ. If this is possible, one obtains because of

$$\min\left\{ \tilde{j} \,\middle|\, (\tilde{j}, \tilde{k}) \in r_{(j,k)}^{\lambda} \right\} = j > i$$

by rearranging (1.47) a β-sequence for a $\mu \vdash n$ with $\mu \rhd \lambda$ (see Definition 1.1.4.(ii)).

(ii) Suppose that $\mathcal{H}_{A_{n-1}}^{(Q,a)}$ is semisimple and S is complete with respect to the valuation ψ. Then Corollary 1.10.8 can be used to determine decomposition numbers $d_{\eta\theta}^{n,\mathcal{K}}$ with $\eta \in \Pi_n$ and $\theta \in \Pi_{n,e_F(\bar{a})}$ in the following way.

Let $\lambda \vdash n$ and $\mu = \gamma_{e_F(\bar{a})}(\lambda)$. With this let

$$[\operatorname{rad} \gamma_{(F,\bar{a})}^{\lambda}] = \sum_{\nu \in \Pi_{n,e_F(\bar{a})}^{\mu,e_F(\bar{a})}} a_\nu [D_{(F,\bar{a})}^{\nu}]$$

and

$$\sum_{j\in\mathbb{N}} [\overline{S_{(S,a)}^{\lambda}(j)}] = \sum_{\nu \in \Pi_{n,e_F(\bar{a})}^{\mu,e_F(\bar{a})}} b_\nu [D_{(F,\bar{a})}^{\nu}] \tag{1.48}$$

be the representations of $[\operatorname{rad} \gamma_{(F,\bar{a})}^{\lambda}]$ and $\sum_{j\in\mathbb{N}} [\overline{S_{(S,a)}^{\lambda}(j)}]$ in $G_0(\mathcal{H}_{A_{n-1}}^{(F,\bar{a})})$ with respect to the basis $\left\{ [D_{(F,\bar{a})}^{\kappa}] \,\middle|\, \kappa \in \Pi_{n,e_F(\bar{a})} \right\}$ from Lemma 1.4.2.(i) with coefficients $a_\nu, b_\nu \in \mathbb{N}_0$ for $\nu \in \Pi_{n,e_F(\bar{a})}^{\mu,e_F(\bar{a})}$. From Definition 1.3.16.(iii), Definition 1.10.3, Lemma 1.8.8.(iv), and Theorem 1.8.20 we see that the indices in the sums on the right hand sides of these decompositions can be restricted to the μ-block. Now we get from Definition 1.3.19, Theorem 1.3.20.(i), and Lemma 1.5.7 for every $\nu \in \Pi_{n,e_F(\bar{a})}^{\mu,e_F(\bar{a})}$

$$\begin{aligned} a_\nu = d_{\lambda\nu}^{n,\mathcal{K}} &= 0 && \textit{if} \quad \nu \not\trianglerighteq \lambda, \\ a_\nu = 0 \textit{ and } d_{\lambda\nu}^{n,\mathcal{K}} &= 1 && \textit{if} \quad \nu = \lambda, \\ a_\nu &= d_{\lambda\nu}^{n,\mathcal{K}} && \textit{if} \quad \nu \triangleright \lambda, \end{aligned} \tag{1.49}$$

the case $\nu = \lambda$ occurring only for an $e_F(\bar{a})$-regular λ. We also get from Lemma 1.10.2.(iii) for every $\nu \in \Pi_{n,e_F(\bar{a})}^{\mu,e_F(\bar{a})}$

$$a_\nu \leq b_\nu \qquad \textit{and} \qquad a_\nu = 0 \Leftrightarrow b_\nu = 0. \tag{1.50}$$

Now if the coefficients in the decompositions

$$[S_{(F,\bar{a})}^{\kappa}] = \sum_{\nu \in \Pi_{n,e_F(\bar{a})}^{\mu,e_F(\bar{a})}} d_{\kappa\nu}^{n,\mathcal{K}} [D_{(F,\bar{a})}^{\nu}]$$

(see Lemma 1.5.5) for $\kappa \in \Pi_{n,e_F(\bar{a})}^{\mu,e_F(\bar{a})}$ with $\kappa \triangleright \lambda$ are known, they enable us to calculate the coefficients in the decomposition (1.48) by using statement (i) and the formula (1.46). In turn, from these coefficients b_ν with $\nu \in \Pi_{n,e_F(\bar{a})}^{\mu,e_F(\bar{a})}$ we get by means of (1.49) and (1.50) conditions on the decomposition numbers $d_{\lambda\nu}^{n,\mathcal{K}}$ with $\nu \in \Pi_{n,e_F(\bar{a})}^{\mu,e_F(\bar{a})}$.

If these decomposition numbers $d_{\lambda\nu}^{n,\mathcal{K}}$ *with* $\nu \in \Pi_{n,e_F(\bar{a})}^{\mu,e_F(\bar{a})}$ *can be completely determined, the decomposition*

$$[S_{(F,\bar{a})}^{\lambda}] = \sum_{\nu \in \Pi_{n,e_F(\bar{a})}^{\mu,e_F(\bar{a})}} d_{\lambda\nu}^{n,\mathcal{K}} [D_{(F,\bar{a})}^{\nu}]$$

is available and can be used as above on the right hand side of the identity (1.46) to derive conditions on the decomposition numbers $d_{\kappa\nu}^{n,\mathcal{K}}$ *for* $\kappa \in \Pi_n^{\mu,e_F(\bar{a})}$ *with* $\lambda \rhd \kappa$ *and* $\nu \in \Pi_{n,e_F(\bar{a})}^{\mu,e_F(\bar{a})}$*. This shows how Corollary 1.10.8 and induction on the partial ordering* $\trianglerighteq$ *on the set* $\Pi_n^{\mu,e_F(\bar{a})}$ *can be used to get information on the decomposition numbers* $d_{\kappa\nu}^{n,\mathcal{K}}$ *with* $\kappa \in \Pi_n^{\mu,e_F(\bar{a})}$ *and* $\nu \in \Pi_{n,e_F(\bar{a})}^{\mu,e_F(\bar{a})}$*.*

Chapter 2

Blocks of Hecke algebras of type A having weight 3 and empty core

This chapter investigates the decomposition numbers belonging to certain blocks of Hecke algebras of type A. The central results describe an algorithm for the explicit calculation of these decomposition numbers (see Theorem 2.2.1) and their dependence of the underlying modular system (see Theorem 2.2.10). Further important results are Theorem 2.1.8 and Theorem 2.1.11.

2.1 Bounds for the decomposition numbers

In this section we consider blocks of various Hecke algebras of type A whose associated core is the partition (0) of 0, that is, the empty partition. The degree of the Hecke algebras under consideration is such that the weight of these blocks (see Definition 1.8.12.(ii)) is 3. We will determine bounds for the values of the entries of the associated (0)-blocks of the decomposition matrices (see Definition 1.8.21) of these Hecke algebras. These bounds will be used in the next section to show how these matrix blocks can be calculated explicitly.

In this section, $n \in \mathbb{N}$ is a positive integer. Furthermore,

$$\mathcal{K} = (Q, \psi, S, I, a, F)$$

denotes a modular system as in Definition 1.4.8 satisfying the following conditions.

$$\begin{aligned}&\text{(i) The algebra } \mathcal{H}_{A_{n-1}}^{(Q,a)} \text{ is semisimple.}\\ &\text{(ii) The discrete valuation ring } S \text{ is complete with respect to the valuation } \psi.\end{aligned} \tag{2.1}$$

The main result first will be proved for modular systems of this kind and then will be generalized to arbitrary modular systems by means of Corollary 1.6.7.

Next we describe some properties of blocks having $e_F(\bar{a})$-weight 0, 1, or 2. These properties will be required later. The following three statements make use of Definition 1.8.21. The next lemma covers blocks having $e_F(\bar{a})$-weight 0.

Lemma 2.1.1 *Let $\nu \in \Pi_n \cap \Gamma_{e_F(\bar{a})}(n)$. Then we have for every entry $d_{\eta\theta}^{n,\mathcal{K}}$ of $\Delta_{n,\mathcal{K}}^{\mathcal{H}}(\nu) = \left(d_{\lambda\mu}^{n,\mathcal{K}}\right)_{\substack{\lambda \in \Pi_n^{\nu,e_F(\bar{a})} \\ \mu \in \Pi_{n,e_F(\bar{a})}^{\nu,e_F(\bar{a})}}}$ with $\eta \in \Pi_n^{\nu,e_F(\bar{a})}$ and $\theta \in \Pi_{n,e_F(\bar{a})}^{\nu,e_F(\bar{a})}$*

$$d_{\eta\theta}^{n,\mathcal{K}} \in \{0,1\}\,.$$

Proof. Since the $e_F(\bar{a})$-core ν is itself a partition of n and because of Remark 1.1.36, we have

$$\Pi_n^{\nu,e_F(\bar{a})} = \Pi_{n,e_F(\bar{a})}^{\nu,e_F(\bar{a})} = \{\nu\}\,.$$

With that, we get from Lemma 1.5.7 and Lemma 1.4.2.(i)

$$\Delta_{n,\mathcal{K}}^{\mathcal{H}}(\nu) = \left(d_{\nu\nu}^{n,\mathcal{K}}\right) = (1)\,.$$

This proves the claim. ■

The following lemma covers blocks having $e_F(\bar{a})$-weight 1.

Lemma 2.1.2 *Let $n \geq e_F(\bar{a})$ and let $\nu \in \Pi_{n-e_F(\bar{a})} \cap \Gamma_{e_F(\bar{a})}(n)$. Then we have for every entry $d_{\eta\theta}^{n,\mathcal{K}}$ of $\Delta_{n,\mathcal{K}}^{\mathcal{H}}(\nu) = \left(d_{\lambda\mu}^{n,\mathcal{K}}\right)_{\substack{\lambda \in \Pi_n^{\nu,e_F(\bar{a})} \\ \mu \in \Pi_{n,e_F(\bar{a})}^{\nu,e_F(\bar{a})}}}$ with $\eta \in \Pi_n^{\nu,e_F(\bar{a})}$ and $\theta \in \Pi_{n,e_F(\bar{a})}^{\nu,e_F(\bar{a})}$*

$$d_{\eta\theta}^{n,\mathcal{K}} \in \{0,1\}\,.$$

Proof. See [JAM2, (3.12) and Theorem 6.5]. ■

The next lemma covers blocks having $e_F(\bar{a})$-weight 2.

Lemma 2.1.3 *Let $n \geq 2e_F(\bar{a})$ and let $\nu \in \Pi_{n-2e_F(\bar{a})} \cap \Gamma_{e_F(\bar{a})}(n)$. Then we have for every entry $d_{\eta\theta}^{n,\mathcal{K}}$ of $\Delta_{n,\mathcal{K}}^{\mathcal{H}}(\nu) = \left(d_{\lambda\mu}^{n,\mathcal{K}}\right)_{\substack{\lambda \in \Pi_n^{\nu,e_F(\bar{a})} \\ \mu \in \Pi_{n,e_F(\bar{a})}^{\nu,e_F(\bar{a})}}}$ with $\eta \in \Pi_n^{\nu,e_F(\bar{a})}$ and $\theta \in \Pi_{n,e_F(\bar{a})}^{\nu,e_F(\bar{a})}$*

$$d_{\eta\theta}^{n,\mathcal{K}} \in \{0,1\}\,.$$

Proof. This follows from [RIC, Conjecture 4.7]. That conjecture in turn is proved by Theorem 1.10.6 (see also [JM]). ■

Now we are in a position to investigate blocks having $e_F(\bar{a})$-weight 3. To this end, we assume

$$e_F(\bar{a}) < \infty$$

until further notice. Furthermore note that, according to Definition 1.2.2.(ii), we have $e_F(\bar{a}) \geq 2$. With this, we first introduce a useful notation for the partitions in such a block. This notation comes from [MR2]. According to Lemma 1.1.41, given an abacus $\mathfrak{a}$ for the $e_F(\bar{a})$-core indexing a block having $e_F(\bar{a})$-weight 3, one obtains an abacus for a partition in that block by moving three (not necessarily pairwise distinct) beads in $\mathfrak{a}$ within their respective columns one place in the upward direction. Doing this in all possible ways produces abaci for all partitions in the considered block. This shows together with Remark 1.1.30.(iii) that the notations introduced in the following definition indeed represent the partitions in that block.

Definition 2.1.4 *Let $n \geq 3e_F(\bar{a})$ and $\nu \in \Pi_{n-3e_F(\bar{a})} \cap \Gamma_{e_F(\bar{a})}(n)$ and choose an abacus $\mathfrak{a}$ for ν having at least three beads on each runner.*

(i) For $i \in \{0, \ldots, e_F(\bar{a}) - 1\}$,

$$\langle i \rangle_{\mathfrak{a}}$$

denotes the partition corresponding to the abacus obtained from $\mathfrak{a}$ through movement of the uppermost bead on runner i within its column by three places in the upward direction.

(ii) For $i \in \{0, \ldots, e_F(\bar{a}) - 1\}$,

$$\langle i, i \rangle_{\mathfrak{a}}$$

denotes the partition corresponding to the abacus obtained from $\mathfrak{a}$ through movement of the uppermost bead on runner i within its column by two places in the upward direction and movement of the next lower bead on runner i within its column by one place in the upward direction.

(iii) For $i, j \in \{0, \ldots, e_F(\bar{a}) - 1\}$ with $i \neq j$,

$$\langle i, j \rangle_{\mathfrak{a}}$$

denotes the partition corresponding to the abacus obtained from $\mathfrak{a}$ through movement of the uppermost bead on runner i within its column by two places in the upward direction and movement of the uppermost bead on runner j within its column by one place in the upward direction.

(iv) For $i \in \{0, \ldots, e_F(\bar{a}) - 1\}$,

$$\langle i, i, i \rangle_{\mathfrak{a}}$$

denotes the partition corresponding to the abacus obtained from $\mathfrak{a}$ through movement of the three uppermost beads on runner i within their column by one place in the upward direction.

(v) *For* $i, j \in \{0, \ldots, e_F(\bar{a}) - 1\}$ *with* $i \neq j$,

$$\langle i, i, j \rangle_{\mathfrak{a}}$$

denotes the partition corresponding to the abacus obtained from $\mathfrak{a}$ *through movement of the two uppermost beads on runner* i *and the uppermost bead on runner* j *within their columns by one place in the upward direction.*

(vi) *For pairwise distinct* $i, j, k \in \{0, \ldots, e_F(\bar{a}) - 1\}$,

$$\langle i, j, k \rangle_{\mathfrak{a}}$$

denotes the partition corresponding to the abacus obtained from $\mathfrak{a}$ *through movement of the uppermost beads on the runners* i, j, k *within their columns by one place in the upward direction.*

Next, we show how the constants H_λ from Theorem 1.9.18 can be easily determined by using abaci. The following statement makes use of Definition 1.1.20.(ii) and Definition 1.1.38.(i).

Lemma 2.1.5 *Let* $n > 1$ *and* $e_F(\bar{a}) < \infty$. *Furthermore, let* $\lambda \vdash n$ *and* $\mu, \tilde{\mu} \in \lambda \downarrow$. *Finally, let* $\mathfrak{a}$ *be an* $e_F(\bar{a})$*-abacus for* λ. *Then the following two statements are equivalent.*

(i) *For the* $e_F(\bar{a})$*-cores of* μ *and* $\tilde{\mu}$, *we have*

$$\gamma_{e_F(\bar{a})}(\mu) = \gamma_{e_F(\bar{a})}(\tilde{\mu}).$$

(ii) *There is an* $i \in \{0, \ldots, e_F(\bar{a}) - 1\}$ *such that abaci for* μ *and* $\tilde{\mu}$ *can be obtained from* $\mathfrak{a}$ *through movement of respectively one appropriate bead on runner* i *within its row by one place in the downward direction.*

Proof. According to Definition 1.1.20.(ii) and Corollary 1.1.33, there is a uniquely determined

$$j \in \{0, \ldots, e_F(\bar{a}) - 1\}$$

such that an abacus $\mathfrak{b}$ for μ is obtained from $\mathfrak{a}$ through movement of an appropriate bead on runner j within its row by one place in the downward direction. Similarly, there is a uniquely determined

$$\tilde{j} \in \{0, \ldots, e_F(\bar{a}) - 1\}$$

such that an abacus $\tilde{\mathfrak{b}}$ for $\tilde{\mu}$ is obtained from $\mathfrak{a}$ through movement of an appropriate bead on runner $\tilde{j}$ within its row by one place in the downward direction. Thus we have

$$(ii) \Leftrightarrow j = \tilde{j}.$$

Furthermore, according to Lemma 1.1.41, an abacus $\mathfrak{c}$ for $\gamma_{e_F(\bar{a})}(\lambda)$ is obtained from $\mathfrak{a}$ through movement of all beads in $\mathfrak{a}$ within their columns as far in the downward direction as possible. Similarly, an abacus $\mathfrak{d}$ for $\gamma_{e_F(\bar{a})}(\mu)$ is obtained from $\mathfrak{b}$ and an abacus $\tilde{\mathfrak{d}}$ for $\gamma_{e_F(\bar{a})}(\tilde{\mu})$ is obtained from $\tilde{\mathfrak{b}}$ through movement of all beads in $\mathfrak{b}$ and $\tilde{\mathfrak{b}}$ within their columns as far in the downward direction as possible. The construction of all these abaci from $\mathfrak{a}$ shows that $\mathfrak{c}$ and $\mathfrak{d}$ differ only on runners j and k with

$$k = \begin{cases} j-1 & \text{if} \quad j > 0 \\ e_F(\bar{a}) - 1 & \text{if} \quad j = 0 \end{cases}.$$

More specifically, $\mathfrak{d}$ contains on runner j one bead less than $\mathfrak{c}$ and on runner k one bead more than $\mathfrak{c}$. Similarly, $\mathfrak{c}$ and $\tilde{\mathfrak{d}}$ differ only on runners $\tilde{j}$ and $\tilde{k}$ with

$$\tilde{k} = \begin{cases} \tilde{j}-1 & \text{if} \quad \tilde{j} > 0 \\ e_F(\bar{a}) - 1 & \text{if} \quad \tilde{j} = 0 \end{cases}.$$

More specifically, $\tilde{\mathfrak{d}}$ contains on runner $\tilde{j}$ one bead less than $\mathfrak{c}$ and on runner $\tilde{k}$ one bead more than $\mathfrak{c}$. From this we get

$$j = \tilde{j} \Leftrightarrow \mathfrak{d} = \tilde{\mathfrak{d}}.$$

Furthermore, the construction of the abaci $\mathfrak{d}$ and $\tilde{\mathfrak{d}}$ shows that they contain the same number of beads. Thus we have according to Remark 1.1.30.(iii)

$$\mathfrak{d} = \tilde{\mathfrak{d}} \Leftrightarrow (i).$$

This proves the claim. ■

Lemma 2.1.6 *Let $n > 1$, $e_F(\bar{a}) < \infty$, and $\lambda \vdash n$ and choose an $e_F(\bar{a})$-abacus $\mathfrak{a}$ for λ. With that, let for every $i \in \{0, \ldots, e_F(\bar{a}) - 1\}$*

$$L_i = \left| \left\{ j \in \mathbb{N}_0 \;\middle|\; \begin{array}{c} \textit{runner } i \textit{ in } \mathfrak{a} \\ \textit{contains a bead in row } j \\ \textit{that can be moved within its row} \\ \textit{by one place in the downward direction} \end{array} \right\} \right|.$$

Then we have with the notations from Theorem 1.9.18

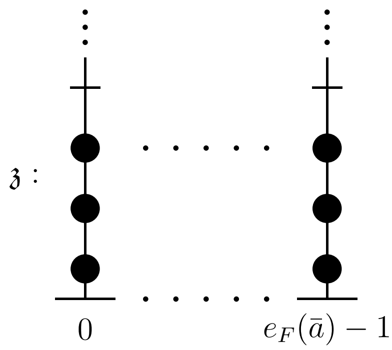
$$H_\lambda = \max \{ L_i \mid i \in \{0, \ldots, e_F(\bar{a}) - 1\} \}.$$

Proof. This follows immediately from the Definition of H_λ in Theorem 1.9.18 and Lemma 2.1.5. ■

From now on we assume

$$n = 3e_F(\bar{a})$$

until further notice. Then we have $(0) \in \Gamma_{e_F(\bar{a})}(n)$, and thus the Hecke algebras $\mathcal{H}_{A_{n-1}}^{(Q,a)}$, $\mathcal{H}_{A_{n-1}}^{(S,a)}$, and $\mathcal{H}_{A_{n-1}}^{(F,\bar{a})}$ have a block of weight 3 with core (0). In order to manipulate partitions in this block by means of abaci, we fix the following abacus $\mathfrak{z}$ for the partition (0).



(2.2)

This abacus contains enough beads for all required manipulations (see Remark 1.1.30.(ii)).

The next theorem also makes use of the notations from Theorem 1.9.18.

Theorem 2.1.7 *Let $n = 3e_F(\bar{a})$ and $\lambda \in \Pi_n^{(0),e_F(\bar{a})}$. Then we have*

$$H_\lambda \leq 1.$$

Proof. Let $\mathfrak{a}$ be the $e_F(\bar{a})$-abacus for λ constructed from the abacus $\mathfrak{z}$ for (0) as in Definition 2.1.4. The claim is proved by applying Lemma 2.1.6 to λ with the abacus $\mathfrak{a}$ and explicitly considering all cases in Definition 2.1.4.

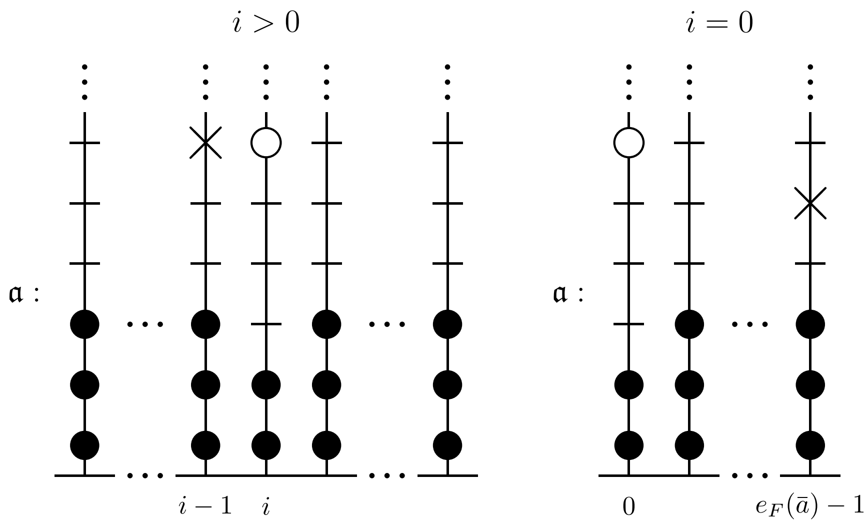
As an example, we consider the case (i). Then we have

$$\lambda = \langle i \rangle_{\mathfrak{z}}$$

with an $i \in \{0, \ldots, e_F(\bar{a}) - 1\}$, and $\mathfrak{a}$ is obtained from $\mathfrak{z}$ through movement of the uppermost bead on runner i within its column in the upward direction from row 2 to row 5. Since there is no bead in $\mathfrak{z}$ that can be moved within its row by one place in the downward direction (see (2.2)), any bead in $\mathfrak{a}$ that can be moved within its row by one place in the downward direction must sit on runner i or runner j with

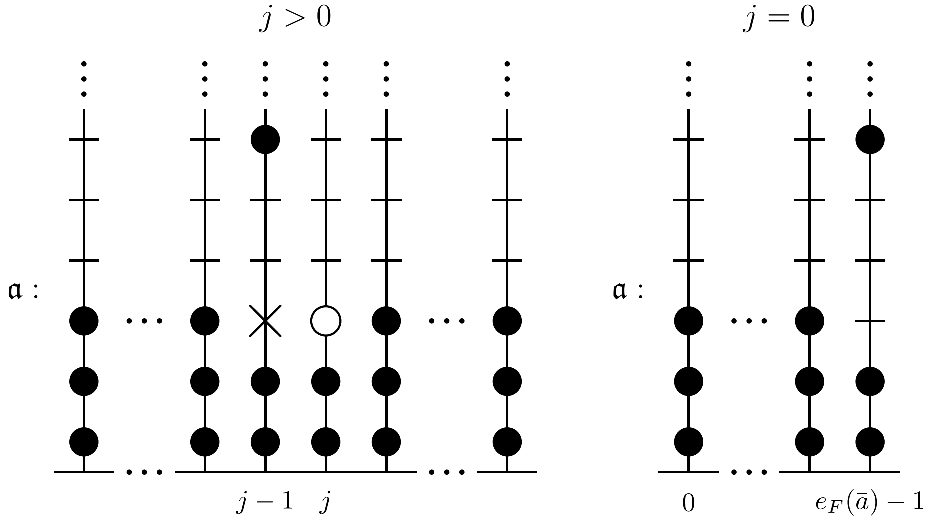
$$j = \begin{cases} i+1 & \text{if} \quad i < e_F(\bar{a}) - 1 \\ 0 & \text{if} \quad i = e_F(\bar{a}) - 1 \end{cases} .$$

For the beads on runner i, we have two different situations depending on $i > 0$ or $i = 0$. In the case $i > 0$, if we have a bead movable within its row by one place in the downward direction, it will end up in the same row on runner $i - 1$. In the case $i = 0$, if we have a bead movable within its row by one place in the downward direction, it will end up one row below on runner $e_F(\bar{a}) - 1$. These two situations are displayed in the following picture.



In both cases, runner i contains only one bead movable within its row by one place in the downward direction. This bead is depicted as $\bigcirc$. The position occupied by it after the movement is marked with a $\times$.

Similarly, for the beads on runner j, we have the following two different situations depending on $j > 0$ or $j = 0$.



In the case $j > 0$, runner j contains one bead movable within its row by one place in the downward direction. This bead is depicted as $\bigcirc$. The position occupied by it after the movement is marked with a $\times$. In the case $j = 0$, there is no bead on runner j that can be moved within its row by one place in the downward direction.

With the notations from Lemma 2.1.6, we now have shown all in all

$$\forall k \in \{0, \dots, e_F(\bar{a}) - 1\} : L_k \leq 1$$

and thus

$$H_{\langle i \rangle_{\mathfrak{z}}} \leq 1.$$

This proves the claim of the theorem in the case $\lambda = \langle i \rangle_{\mathfrak{z}}$.

Similar considerations prove the claim of the theorem in the remaining cases of Definition 2.1.4. ■

Now we are able to easily determine all constants required for the application of Theorem 1.9.18 to the $e_F(\bar{a})$-core (0) and the associated block $\Delta^{\mathcal{H}}_{n,\mathcal{K}}((0))$ of the decomposition matrix $\Delta^{\mathcal{H}}_{n,\mathcal{K}}$, thus obtaining an upper bound for the entries of $\Delta^{\mathcal{H}}_{n,\mathcal{K}}((0))$.

Theorem 2.1.8 *Let $n = 3e_F(\bar{a})$. Then we have for every entry $d^{n,\mathcal{K}}_{\eta\theta}$ of the matrix $\Delta^{\mathcal{H}}_{n,\mathcal{K}}((0)) = \left(d^{n,\mathcal{K}}_{\lambda\mu}\right)_{\substack{\lambda \in \Pi^{(0),e_F(\bar{a})}_n \\ \mu \in \Pi^{(0),e_F(\bar{a})}_{n,e_F(\bar{a})}}}$ with $\eta \in \Pi^{(0),e_F(\bar{a})}_n$ and $\theta \in \Pi^{(0),e_F(\bar{a})}_{n,e_F(\bar{a})}$*

$$d^{n,\mathcal{K}}_{\eta\theta} \in \{0, 1\}.$$

Proof. According to Definition 1.4.17, we have

$$d^{n,\mathcal{K}}_{\eta\theta} \in \mathbb{N}_0.$$

Thus, it remains to show $d_{\eta\theta}^{n,\mathcal{K}} \leq 1$. This can be derived as follows from Theorem 1.9.18. With the notation from there, we get from Theorem 2.1.7

$$J_{(0)} \leq 1.$$

Furthermore, it follows from the condition $n = 3e_F(\bar{a})$, Definition 1.8.12.(ii), Remark 1.8.13, and Definition 1.1.38.(i) that the Hecke algebras of degree $n-1$ over the coefficient pairs associated to $\mathcal{K}$ only have blocks of $e_F(\bar{a})$-weight 0, 1, or 2. From this together with Theorem 1.8.22, Lemma 2.1.1, Lemma 2.1.2, and Lemma 2.1.3 we get

$$U_{n-1} \leq 1.$$

Now we obtain from Theorem 1.9.18

$$d_{\eta\theta}^{n,\mathcal{K}} \leq J_{(0)} U_{n-1} \leq 1,$$

as desired. ■

Remark 2.1.9 *The preceding theorem is proved by Martin and Russell in [MR1] for the special case of group algebras of symmetric groups (see (1.24) on page 34).*

Corollary 2.1.10 *Let $n \in \mathbb{N}$ and $e_F(\bar{a}) \in \{2, 3, \ldots\} \cup \{\infty\}$ with $n \leq 3e_F(\bar{a})$. Then we have for every entry $d_{\eta\theta}^{n,\mathcal{K}}$ of the matrix $\Delta_{n,\mathcal{K}}^{\mathcal{H}} = \left(d_{\lambda\mu}^{n,\mathcal{K}}\right)_{\substack{\lambda\in\Pi_n \\ \mu\in\Pi_{n,e_F(\bar{a})}}}$ with $\eta \in \Pi_n$ and $\theta \in \Pi_{n,e_F(\bar{a})}$*

$$d_{\eta\theta}^{n,\mathcal{K}} \in \{0, 1\}.$$

Proof. Because of the condition $n \leq 3e_F(\bar{a})$, the algebras $\mathcal{H}_n^{(Q,a)}$, $\mathcal{H}_n^{(S,a)}$, and $\mathcal{H}_n^{(F,\bar{a})}$ only have blocks of weight 0, 1, or 2 and possibly a block of weight 3 with associated core (0). Thus, the claim follows from Theorem 1.8.22, Lemma 2.1.1, Lemma 2.1.2, Lemma 2.1.3, and Theorem 2.1.8. ■

Finally, the preceding corollary is generalized to arbitrary modular systems. To this end, let

$$\tilde{\mathcal{K}} = \left(\tilde{Q}, \tilde{\psi}, \tilde{S}, \tilde{I}, \tilde{a}, \tilde{F}\right)$$

be a modular system as in Definition 1.4.8. Then we get from Definition 1.6.4 with the coefficient pairs $(\tilde{Q}, \tilde{a})$ and $(\tilde{F}, \bar{\tilde{a}})$ and indeterminates Y over $\tilde{Q}$ and Z over $\tilde{F}$ the modular systems

$$\hat{\mathcal{K}}_{(\tilde{Q},\tilde{a})} = \left(\hat{\tilde{Q}(Y)}, \hat{\psi}_{Y-\tilde{a}}, S_{\hat{\psi}_{Y-\tilde{a}}}, I_{\hat{\psi}_{Y-\tilde{a}}}, Y, \tilde{Q}\right)$$

and

$$\hat{\mathcal{K}}_{(\tilde{F},\bar{\tilde{a}})} = \left(\hat{\tilde{F}(Z)}, \hat{\psi}_{Z-\bar{\tilde{a}}}, S_{\hat{\psi}_{Z-\bar{\tilde{a}}}}, I_{\hat{\psi}_{Z-\bar{\tilde{a}}}}, Z, \tilde{F}\right).$$

With these, Corollary 2.1.10 can be generalized to the decomposition numbers of Hecke algebras associated with the degree n and the modular system $\tilde{\mathcal{K}}$.

Theorem 2.1.11 *Let $n \in \mathbb{N}$ and $e_{\tilde{F}}(\bar{\bar{a}}) \in \{2,3,\ldots\} \cup \{\infty\}$ with $n \le 3e_{\tilde{F}}(\bar{\bar{a}})$. Then we have for every entry $d_{\eta\theta}^{n,\tilde{\mathcal{K}}}$ of the matrix $\Delta_{n,\tilde{\mathcal{K}}}^{\mathcal{H}} = \left(d_{\lambda\mu}^{n,\tilde{\mathcal{K}}}\right)_{\substack{\lambda\in\Pi_{n,e_{\tilde{Q}}(\tilde{a})} \\ \mu\in\Pi_{n,e_{\tilde{F}}(\bar{\bar{a}})}}}$ with $\eta \in \Pi_{n,e_{\tilde{Q}}(\tilde{a})}$ and $\theta \in \Pi_{n,e_{\tilde{F}}(\bar{\bar{a}})}$*

$$d_{\eta\theta}^{n,\tilde{\mathcal{K}}} \in \{0,1\}.$$

Proof. By applying Corollary 1.6.7 to the situation at hand with the modular systems $\tilde{\mathcal{K}}$, $\hat{\mathcal{K}}_{(\tilde{Q},\tilde{a})}$, and $\hat{\mathcal{K}}_{(\tilde{F},\bar{\bar{a}})}$ we get

$$\Delta_{n,\hat{\mathcal{K}}_{(\tilde{F},\bar{\bar{a}})}}^{\mathcal{H}} = \Delta_{n,\hat{\mathcal{K}}_{(\tilde{Q},\tilde{a})}}^{\mathcal{H}} \Delta_{n,\tilde{\mathcal{K}}}^{\mathcal{H}}. \tag{2.3}$$

Furthermore, it follows from Definition 1.6.4, Lemma 1.6.3.(i), and Lemma 1.6.5 that the modular systems $\hat{\mathcal{K}}_{(\tilde{F},\bar{\bar{a}})}$ and $\hat{\mathcal{K}}_{(\tilde{Q},\tilde{a})}$ satisfy the conditions (2.1). In addition, Lemma 1.6.3.(iii) shows that for the q-characteristics of the coefficient pair $(\tilde{F},\ \bar{Z}) = (\tilde{F},\ \bar{\bar{a}})$ associated to $\hat{\mathcal{K}}_{(\tilde{F},\bar{\bar{a}})}$ and the coefficient pair $(\tilde{Q},\ \bar{Y}) = (\tilde{Q},\ \tilde{a})$ associated to $\hat{\mathcal{K}}_{(\tilde{Q},\tilde{a})}$ we have

$$e_{\tilde{F}}(\bar{Z}) = e_{\tilde{F}}(\bar{\bar{a}}) \qquad \text{and} \qquad e_{\tilde{Q}}(\bar{Y}) = e_{\tilde{Q}}(\tilde{a}).$$

Now the preceding considerations together with the assumption $n \le 3e_{\tilde{F}}(\bar{\bar{a}})$ and Lemma 1.4.9.(i) show that Corollary 2.1.10 holds for the decomposition matrices $\Delta_{n,\hat{\mathcal{K}}_{(\tilde{F},\bar{\bar{a}})}}^{\mathcal{H}}$ and $\Delta_{n,\hat{\mathcal{K}}_{(\tilde{Q},\tilde{a})}}^{\mathcal{H}}$. Thus, each of their entries is either 0 or 1.

According to Lemma 1.6.5 and Lemma 1.5.8.(ii), we also have for the matrix $\Delta_{n,\hat{\mathcal{K}}_{(\tilde{Q},\tilde{a})}}^{\mathcal{H}} = \left(d_{\lambda\mu}^{n,\hat{\mathcal{K}}_{(\tilde{Q},\tilde{a})}}\right)_{\substack{\lambda\in\Pi_n \\ \mu\in\Pi_{n,e_{\tilde{Q}}(\tilde{a})}}}$

$$\operatorname{Rnk}_{\mathbb{Q}} \Delta_{n,\hat{\mathcal{K}}_{(\tilde{Q},\tilde{a})}}^{\mathcal{H}} = \left|\Pi_{n,e_{\tilde{Q}}(\tilde{a})}\right|.$$

This shows that every column of that matrix contains at least one entry different from 0.

Finally, according to Definition 1.4.17, the entries of the decomposition matrix $\Delta_{n,\tilde{\mathcal{K}}}^{\mathcal{H}}$ are nonnegative.

The claim follows from all these properties of the matrices $\Delta_{n,\tilde{\mathcal{K}}}^{\mathcal{H}}$, $\Delta_{n,\hat{\mathcal{K}}_{(\tilde{Q},\tilde{a})}}^{\mathcal{H}}$, and $\Delta_{n,\hat{\mathcal{K}}_{(\tilde{F},\bar{\bar{a}})}}^{\mathcal{H}}$ by explicitly considering the relations between their entries given in (2.3). ■

2.2 Calculation of the decomposition numbers depending on the modular system

Now we will show how the decomposition numbers of blocks having weight 3 and empty core, as considered in the preceding section, can be calculated explicitly. In the course of that, we also will obtain results on the dependence of these decomposition numbers on the underlying modular system.

In what follows,

$$\mathcal{K} = (Q, \psi, S, I, a, F)$$

denotes a modular system as in Definition 1.4.8 such that

$$\mathcal{H}_{A_{n-1}}^{(Q,a)} \text{ is semisimple} \tag{2.4}$$

and furthermore

$$e_F(\bar{a}) < \infty \tag{2.5}$$

holds. With this, we put

$$n = 3e_F(\bar{a}). \tag{2.6}$$

These conditions on n and $e_F(\bar{a})$ ensure that the Hecke algebras over the coefficient pairs associated to $\mathcal{K}$ have a block of weight 3 with core (0).

Because of (2.4) and according to Corollary 1.5.6, the decomposition matrix $\Delta_{n,\mathcal{K}}^{\mathcal{H}}$ is uniquely determined by the data n and $(F, \bar{a})$. Thus we can modify the modular system $\mathcal{K}$ as follows without changing the associated decomposition numbers. First, we put

$$(F, \bar{a}) = (K, r).$$

Then we can assume without loss of generality

$$\mathcal{K} = \hat{\mathcal{K}}_{(K,r)} = \left(K(\hat{X}), \hat{\psi}_{X-r}, S_{\hat{\psi}_{X-r}}, I_{\hat{\psi}_{X-r}}, X, K \right) \tag{2.7}$$

where X is an indeterminate over K (see Definition 1.6.4). This modular system satisfies the conditions (2.1) on page 101 (see Lemma 1.6.5).

With these assumptions, the (0)-block $\Delta_{n,\mathcal{K}}^{\mathcal{H}}((0))$ of the decomposition matrix $\Delta_{n,\mathcal{K}}^{\mathcal{H}}$ (see Definition 1.8.21) can be determined explicitly. To this end, the following theorem makes use of the results from Section 1.10.

Theorem 2.2.1 *The entries of the matrix*

$$\Delta_{n,\mathcal{K}}^{\mathcal{H}}((0)) = \left(d_{\eta\theta}^{n,\mathcal{K}} \right)_{\substack{\eta \in \Pi_n^{(0),e_K(r)} \\ \theta \in \Pi_{n,e_K(r)}^{(0),e_K(r)}}}$$

can be calculated explicitly by using the Theorem of Schaper.

Proof. The matrix $\Delta_{n,\mathcal{K}}^{\mathcal{H}}((0))$ will be calculated by induction on the elements of the row index set $\Pi_n^{(0),e_K(r)}$ using the partial ordering $\trianglerighteq$ (see Definition 1.1.39 and Definition 1.1.4.(ii)), Remark 1.10.9.(ii), and Theorem 2.1.8. To this end, fix a

$$\lambda \in \Pi_n^{(0),e_K(r)}$$

and inductively suppose that all matrix rows indexed by partitions

$$\kappa \in \Pi_n^{(0),e_K(r)} \quad \text{with} \quad \kappa \rhd \lambda$$

are already known. In order to determine the matrix row indexed by λ, we must, according to (1.49) on page 98, calculate the decomposition numbers

$$d_{\lambda\nu}^{n,\mathcal{K}} \quad \text{for} \quad \nu \in \Pi_{n,e_K(r)}^{(0),e_K(r)} \quad \text{with} \quad \nu \rhd \lambda. \tag{2.8}$$

To do this, we first use the induction hypothesis to calculate the coefficients

$$b_\nu \in \mathbb{N}_0 \quad \text{with} \quad \nu \in \Pi_{n,e_K(r)}^{(0),e_K(r)}$$

introduced in Remark 1.10.9.(ii) as described there. With these values b_ν and the relations (1.50) on page 98 we can decide for each of the coefficients

$$a_\nu \in \mathbb{N}_0 \quad \text{with} \quad \nu \in \Pi_{n,e_K(r)}^{(0),e_K(r)}$$

also introduced in Remark 1.10.9.(ii) whether it is 0 or not. Furthermore, we have according to (1.49) on page 98 and Theorem 2.1.8

$$\forall \nu \in \Pi_{n,e_K(r)}^{(0),e_K(r)} \text{ with } \nu \rhd \lambda : a_\nu = d_{\lambda\nu}^{n,\mathcal{K}} \in \{0,1\}\,.$$

Since we know for every $\nu \in \Pi_{n,e_K(r)}^{(0),e_K(r)}$ whether a_ν is 0 or not, we also know the exact values of all decomposition numbers (2.8). Again according to (1.49) on page 98, we thus have determined all decomposition numbers

$$d_{\lambda\nu}^{n,\mathcal{K}} \quad \text{with} \quad \nu \in \Pi_{n,e_K(r)}^{(0),e_K(r)}.$$

Now we know the row of the matrix $\Delta_{n,\mathcal{K}}^{\mathcal{H}}((0))$ indexed by λ and inductively the whole matrix $\Delta_{n,\mathcal{K}}^{\mathcal{H}}((0))$. ∎

The preceding theorem enables us to further investigate the dependence of $\Delta_{n,\mathcal{K}}^{\mathcal{H}}((0))$ on the underlying modular system. To this end, we first examine the behavior of the valuation $\hat{\psi}_{X-r}$ associated with the modular system $\mathcal{K}$ under consideration (see (2.7) on page 111) when applied to the q-numbers occurring in the formula (1.46) on page 97.

Definition 2.2.2 *Let $\tilde{K}$ be a field. Then the value*

$$p_{\tilde{K}} \in \mathbb{N} \cup \{\infty\}$$

is defined as follows. If $\tilde{K}$ has positive characteristic, we define $p_{\tilde{K}}$ to be that characteristic. If $\tilde{K}$ has characteristic 0, we define $p_{\tilde{K}}$ to be ∞. In other words, $p_{\tilde{K}}$ is the additive order of $1_{\tilde{K}}$ in $\tilde{K}$.

Lemma 2.2.3 *For the modular system $\mathcal{K}$ under consideration, one of the following alternatives holds.*

(i) We have $r = 1_K$ and $p_K \in \mathbb{N}$.

(ii) r is a root of unity in K distinct from 1_K and we have $p_K = \infty$. (2.9)

(iii) r is a root of unity in K distinct from 1_K and we have $p_K \in \mathbb{N}$.

Proof. If r is not a root of unity in K, we get from Definition 1.2.2 $e_K(r) = e_F(\bar{a}) = \infty$. This is a contradiction to the assumption (2.5) on page 111. Thus, r is a root of unity in K. Similarly, we get from the combination $r = 1_K$ and $p_K = \infty$, by using Definition 2.2.2, $e_K(r) = e_F(\bar{a}) = \infty$ and again a contradiction to the assumption (2.5). This shows the claim. ■

Lemma 2.2.4 *For the modular system $\mathcal{K}$ under consideration, the following statements hold.*

(i) If $r = 1_K$ then we have $p_K = e_K(1_K)$.

(ii) If r is a root of unity in K distinct from 1_K then $e_K(r)$ is the multiplicative order of r in K.

(iii) If r is a root of unity in K distinct from 1_K and if $p_K \in \mathbb{N}$ then $e_K(r)$ and p_K have no nontrivial common divisors, that is, they are relatively prime.

Proof. (i) From the assumptions and Definition 1.2.2.(i), we get

$$\forall j \in \mathbb{N} : [j]_r = [j]_{1_K} = j \cdot 1_K.$$

This, Definition 1.2.2.(ii), and Definition 2.2.2 show the claim.

(ii) From the assumptions and Definition 1.2.2.(i), we get

$$\forall j \in \mathbb{N} : [j]_r = \sum_{m=0}^{j-1} r^m = \frac{r^j - 1_K}{r - 1_K}.$$

This and Definition 1.2.2.(ii) show the claim.

(iii) This follows from statement (ii), Definition 2.2.2, and general facts from field theory. ■

Lemma 2.2.5 *For the modular system $\mathcal{K}$ under consideration, the following statements hold.*

(i) Alternative (2.9).(i) is equivalent to

$$p_K = e_K(r). \tag{2.10}$$

(ii) Alternative (2.9).(ii) is equivalent to

$$p_K = \infty. \tag{2.11}$$

(iii) Alternative (2.9).(iii) is equivalent to

$$\begin{array}{c} p_K \in \mathbb{N} \\ \textit{and} \\ p_K \textit{ and } e_K(r) \textit{ have no nontrivial common divisors.} \end{array} \tag{2.12}$$

Proof. This follows easily from Lemma 2.2.4. ■

The next lemma makes use of the considerations from the beginning of Section 1.6, especially Definition 1.6.2.(ii).

Lemma 2.2.6 *Let $i \in \mathbb{N}$. Then the following statements on the modular system $\mathcal{K}$ under consideration hold.*

(i) Assume (2.10) and let

$$i = j\,(p_K)^a$$

with uniquely determined values $j \in \mathbb{N}$ and $a \in \mathbb{N}_0$ such that $p_K \nmid j$ holds. Then we have

$$\hat{\psi}_{X-r}([i]_X) = \begin{cases} 0 & \textit{if } \ e_K(r) \nmid i \\ (p_K)^a - 1 & \textit{if } \ e_K(r) \mid i \end{cases}.$$

(ii) Assume (2.11). Then we have

$$\hat{\psi}_{X-r}([i]_X) = \begin{cases} 0 & \textit{if } \ e_K(r) \nmid i \\ 1 & \textit{if } \ e_K(r) \mid i \end{cases}.$$

(iii) Assume (2.12) and let

$$i = j\,(p_K)^a \tag{2.13}$$

with uniquely determined values $j \in \mathbb{N}$ and $a \in \mathbb{N}_0$ such that $p_K \nmid j$ holds. Then we have

$$\hat{\psi}_{X-r}([i]_X) = \begin{cases} 0 & \textit{if } \ e_K(r) \nmid i \\ (p_K)^a & \textit{if } \ e_K(r) \mid i \end{cases}.$$

Proof. (i) According to Definition 1.6.2.(ii), the considerations from the beginning of Section 1.6, Definition 1.2.2.(i), and Lemma 2.2.5.(i), $\hat{\psi}_{X-r}([i]_X)$ is equal to the multiplicity of $X - r = X - 1_K$ in

$$[i]_X = \sum_{m=0}^{i-1} X^m = \frac{X^i - 1_K}{X - 1_K}.$$

Furthermore we have, according to Definition 2.2.2 and general facts from field theory,

$$X^i - 1_K = \big(X^j - 1_K\big)^{(p_K)^a}$$

where the polynomial $X^j - 1_K$ is separable (that is, it has no nonconstant divisors with multiplicity greater than 1) and contains the factor $X - 1_K$. This shows

$$\hat{\psi}_{X-r}([i]_X) = (p_K)^a - 1$$

which in turn, together with the assumption (2.10), proves the claim.

(ii) In order to determine $\hat{\psi}_{X-r}([i]_X)$, we proceed as in the proof of statement (i). However, we see from Definition 2.2.2 and general facts from field theory that, in the situation at hand, the polynomial $X^i - 1_K$ is separable (that is, it has no nonconstant divisors with multiplicity greater than 1) and furthermore contains, according to assumption (2.11), Lemma 2.2.5.(ii), and Lemma 2.2.4.(ii), the factor $X - r$ if and only if $e_K(r) \mid i$. Because of $r \neq 1_K$ (see again Lemma 2.2.5.(ii)), these properties translate to the polynomial

$$\frac{X^i - 1_K}{X - 1_K} = \sum_{m=0}^{i-1} X^m = [i]_X.$$

This shows the claim.

(iii) Again, we proceed as in the proof of statement (i). Just like there, we obtain

$$X^i - 1_K = \big(X^j - 1_K\big)^{(p_K)^a}$$

where the polynomial $X^j - 1_K$ is separable (that is, it has no nonconstant divisors with multiplicity greater than 1) and furthermore contains, according to assumption (2.12), Lemma 2.2.5.(iii), and Lemma 2.2.4.(ii), the factor $X - r$ if and only if $e_K(r) \mid j$. In addition, we get from (2.13) and the assumption (2.12)

$$e_K(r) \mid j \Leftrightarrow e_K(r) \mid i.$$

All this together with the fact $r \neq 1_K$ (see again Lemma 2.2.5.(iii)) shows that the multiplicity of the factor $X - r$ in the polynomial

$$[i]_X = \frac{X^i - 1_K}{X - 1_K},$$

or equivalently the value $\hat{\psi}_{X-r}([i]_X)$, is given by the expression in statement (iii). ■

The following corollary improves on Corollary 1.5.6 in the situation at hand. It makes use of Definition 1.8.21 and Definition 1.1.39.

Corollary 2.2.7 *The entries of the matrix*

$$\Delta^{\mathcal{H}}_{n,\mathcal{K}}((0)) = \left(d^{n,\mathcal{K}}_{\eta\theta}\right)_{\substack{\eta\in\Pi_n^{(0),e_K(r)} \\ \theta\in\Pi_{n,e_K(r)}^{(0),e_K(r)}}}$$

are uniquely determined by the data $e_K(r)$ and p_K.

Proof. The inductive calculation of the entries of $\Delta^{\mathcal{H}}_{n,\mathcal{K}}((0))$ described in the proof of Theorem 2.2.1 is based on the evaluation and manipulation of the formula (1.46) on page 97. This involves combinatorial manipulations with partitions and the application of the valuation $\hat{\psi}_{X-r}$ to certain q-numbers. The occurring partitions and the combinatorial manipulations applied to them depend only on $e_K(r)$ (see (2.6)). According to Lemma 2.2.6, Lemma 2.2.5, and Lemma 2.2.3, the behavior of $\hat{\psi}_{X-r}$ when applied to the occurring q-numbers is completely determined by $e_K(r)$ and p_K. This completes the proof. ■

The result of the preceding corollary can be further improved for arbitrary values of $e_K(r)$ and sufficiently large values of p_K. This is described in the following.

Lemma 2.2.8 *Assume $p_K > 3$. Then there are a constant*

$$A_{e_K(r),p_K} \in \mathbb{N},$$

depending only on $e_K(r)$ and p_K, and a map

$$B_{e_K(r)} : \{1,\ldots,n\} \to \mathbb{N}_0, \quad i \mapsto B_{e_K(r)}(i),$$

depending only on $e_K(r)$, such that

$$\forall i \in \{1,\ldots,n\} : \hat{\psi}_{X-r}([i]_X) = A_{e_K(r),p_K} B_{e_K(r)}(i) \tag{2.14}$$

holds.

Proof. In order to show the claim, we distinguish the various cases (2.9) and use their characterizations (2.10), (2.11), and (2.11). We define the constant $A_{e_K(r),p_K} \in \mathbb{N}$ as

$$A_{e_K(r),p_K} = \begin{cases} p_K - 1 & \text{if} \quad (2.10) \quad \text{holds} \\ 1 & \text{if} \quad (2.11) \quad \text{holds} \\ 1 & \text{if} \quad (2.12) \quad \text{holds} \end{cases}.$$

Furthermore, we define the map $B_{e_K(r)} : \{1, \ldots, n\} \to \mathbb{N}_0,\ i \mapsto B_{e_K(r)}(i)$ as

$$B_{e_K(r)}(i) = \begin{cases} 0 & \text{if} \quad e_K(r) \nmid i \\ 1 & \text{if} \quad e_K(r) \mid i \end{cases}.$$

Now we establish the factorization (2.14). First we consider an

$$i \in \{1, \ldots, n\} \quad \text{such that} \quad e_K(r) \nmid i.$$

Here we get from Lemma 2.2.6 and the construction of $B_{e_K(r)}$

$$\hat{\psi}_{X-r}([i]_X) = 0 = A_{e_K(r), p_K} B_{e_K(r)}(i).$$

Now we consider an

$$i \in \{1, \ldots, n\} \quad \text{such that} \quad e_K(r) \mid i.$$

If (2.10) holds, we get from (2.6) and the assumption $p_K > 3$

$$(p_K)^2 > n \geq i$$

and also

$$i = j p_K$$

with an appropriate $j \in \mathbb{N}$ not divisible by p_K. With that we get from Lemma 2.2.6.(i) and the construction of $A_{e_K(r), p_K}$ and $B_{e_K(r)}$

$$\hat{\psi}_{X-r}([i]_X) = p_K - 1 = A_{e_K(r), p_K} B_{e_K(r)}(i).$$

If (2.11) holds, we obtain from Lemma 2.2.6.(ii) and the construction of $A_{e_K(r), p_K}$ and $B_{e_K(r)}$

$$\hat{\psi}_{X-r}([i]_X) = 1 = A_{e_K(r), p_K} B_{e_K(r)}(i).$$

Finally if (2.12) holds, we get from (2.6) and the assumption $p_K > 3$

$$e_K(r) p_K > n \geq i. \tag{2.15}$$

Now suppose that $p_K \mid i$ holds. Then we get from that together with the relation $e_K(r) \mid i$ and (2.12) the relation $e_K(r) p_K \mid i$ and thus $e_K(r) p_K \leq i$. This is a contradiction to (2.15). So we must have

$$p_K \nmid i.$$

From this, Lemma 2.2.6.(iii), and the construction of $A_{e_K(r), p_K}$ and $B_{e_K(r)}$, we get

$$\hat{\psi}_{X-r}([i]_X) = 1 = A_{e_K(r), p_K} B_{e_K(r)}(i).$$

This completes the proof. ■

The next statement makes use of Definition 1.1.39.(ii).

Corollary 2.2.9 *Assume $p_K > 3$ and let $\lambda \in \Pi_n^{(0),e_K(r)}$. Then we have in the Grothendieck group $G_0(\mathcal{H}_{A_{n-1}}^{(K,r)})$ with the notation from Corollary 1.10.8 and Lemma 2.2.8*

$$\begin{aligned}
&\sum_{j\in\mathbb{N}} [\overline{S^{\lambda}_{(S_{\hat{\psi}_{X-r}},X)}(j)}] \\
&\quad = A_{e_K(r),p_K} \sum_{k=1}^{\lambda_1} \sum_{i=1}^{\lambda'_k-1} \sum_{j=i+1}^{\lambda'_k} \left(B_{e_K(r)}(\left|r^{\lambda}_{(i,k)}\right|) - B_{e_K(r)}(\left|r^{\lambda}_{(j,k)}\right|)\right) \cdot \qquad (2.16)\\
&\qquad\qquad b^{(0)}_{\mathrm{Proj}}(\mathcal{H}^{(K,r)}_{A_{n-1}}) S(\beta_1, \ldots, \beta_{i-1}, \beta_i + \left|r^{\lambda}_{(j,k)}\right|, \beta_{i+1}, \ldots \\
&\qquad\qquad\qquad \ldots, \beta_{j-1}, \beta_j - \left|r^{\lambda}_{(j,k)}\right|, \beta_{j+1}, \ldots, \beta_c).
\end{aligned}$$

Proof. According to the considerations at the beginning of this section, the modular system under consideration satisfies the assumptions of corollary 1.10.8. Furthermore, according to Definition 1.1.14 and Lemma 1.1.7.(ii), the rim hook lengths occurring in the identity (1.46) on page 97 are not smaller than 1 and not bigger than n. With that, the claim follows from the substitution of the factorization (2.14) into (1.46). ■

Theorem 2.2.10 *Assume $p_K > 3$. Then the entries of the matrix*

$$\Delta^{\mathcal{H}}_{n,\mathcal{K}}((0)) = \left(d^{n,\mathcal{K}}_{\eta\theta}\right)_{\substack{\eta\in\Pi_n^{(0),e_K(r)} \\ \theta\in\Pi_{n,e_K(r)}^{(0),e_K(r)}}}$$

are uniquely determined by the datum $e_K(r)$.

Proof. The partitions to be considered are determined by $e_K(r)$ and the condition (2.6) on page 111. The matrix entries are obtained from the inductive calculation of $\Delta^{\mathcal{H}}_{n,\mathcal{K}}((0))$ described in the proof of Theorem 2.2.1 combined with Corollary 2.2.9. In the course of an induction step, decomposition numbers already calculated — and by induction hypothesis uniquely determined by $e_K(r)$ — are substituted into formula (1.46) on page 97 as described in Remark 1.10.9.(ii) to determine the coefficients

$$b_\nu \in \mathbb{N}_0 \quad \text{with} \quad \nu \in \Pi_{n,e_K(r)}^{(0),e_K(r)}$$

introduced in that remark.

But in the further course of the proof of Theorem 2.2.1, we don't need to know the exact value of such a coefficient, but only whether it is 0 or not. Now, in the situation at hand, the expression (2.16) which is equivalent to (1.46) on page 97 can be used for the calculation of the b_ν. In that expression (2.16), only the constant factor $A_{e_K(r),p_K}$ depends on p_K. The sum in this expression is composed for the one part of combinatorial manipulations depending only on $e_K(r)$ and for the other part of terms which are, according to Lemma 2.2.8, uniquely determined by $e_K(r)$. The calculation of the b_ν is done by first substituting expressions which, by induction hypothesis, are completely determined by $e_K(r)$ and then applying some elementary algebraic manipulations. This shows that whether a coefficient b_ν is 0 or not depends only on $e_K(r)$.

If we now complete the induction step exactly as in the proof of Theorem 2.2.1, the preceding considerations show that the decomposition numbers just obtained are completely determined by $e_K(r)$. Thus, induction shows that, in the situation at hand, the whole matrix $\Delta^{\mathcal{H}}_{n,\mathcal{K}}((0))$ is uniquely determined by $e_K(r)$, as desired. ■

Remark 2.2.11 *(i) [LLT, Conjecture 6.6], [GRO], [ARI, Paragraph 4.7], and [MAT, Theorem 4.3] show that in the case $p_K = \infty$ the decomposition matrix $\Delta^{\mathcal{H}}_{n,\mathcal{K}}$ can be calculated using a combinatorial algorithm. According to Theorem 2.2.10, this procedure also produces the matrix $\Delta^{\mathcal{H}}_{n,\mathcal{K}}((0))$ in the more general case $p_K > 3$.*

(ii) In [JAM2, Section 4], it is conjectured that for $p_K \in \mathbb{N}$ with

$$e_K(r)p_K > n,$$

the decomposition matrix $\Delta^{\mathcal{H}}_{n,\mathcal{K}}$ coincides with the decomposition matrix $\Delta^{\mathcal{H}}_{n,\mathcal{K}}$ in the case $p_K = \infty$. Theorem 2.2.10 and (2.6) on page 111 show that this conjecture is true for the submatrix $\Delta^{\mathcal{H}}_{n,\mathcal{K}}((0))$.

Chapter 3

Generic Specht series for Hecke algebras of type A

In this chapter we construct certain series of submodules of Hecke algebras of type A and more generally of permutation modules of such algebras (see Definition 1.3.1.(ii)). The property of interest of these series is that all quotients of adjacent submodules are Specht modules. Because of that, these series are called Specht series. They are generic in the sense of Remark 1.2.9 and generalize the Specht series from [DJ1, Section 7] over fields to arbitrary integral domains as coefficient rings. This is done by explicitly constructing appropriate bases of the modules involved.

The first three sections of this chapter provide the required combinatorial statements and objects, the derivation of the generic Specht series follows in the subsequent eight sections.

3.1 Ordering relations for shortest representatives of right cosets of Young subgroups in $\mathfrak{S}_n$

This section introduces and compares, for an arbitrary composition λ, various orderings on the set $\mathcal{D}_\lambda$ (see Definition 1.1.58.(i)). To this end, we first introduce and investigate ordering relations on related sets. The ordering relations on the sets $\mathcal{D}_\lambda$ for various compositions λ will be required later on in the construction of generic bases of the subquotients occurring in Specht series. For all the following we fix an $n \in \mathbb{N}$.

First we introduce and compare three ordering relations for tableaux (see Definition 1.1.45). This also is described elsewhere, for example in [MUR, Section 3, especially Definition 3.1]. The second part of the following definition makes use of Definition 1.1.46.(ii) and (1.1) on page 1.

Definition 3.1.1 *(i) For a given tableau* $\mathbf{t}$ *of a composition of* n*, this composition is said to be associated to* $\mathbf{t}$ *and denoted by* $\lambda^{\mathbf{t}}$*. The diagram* $[\lambda^{\mathbf{t}}]$ *also is said to be associated to* $\mathbf{t}$*.*

(ii) Let $\mathbf{t} \in \mathcal{T}^{\Xi_n}_{\text{row std}}$ *and* $m \in \{1, \ldots, n\}$*. Then we define the row standard tableau* $\mathbf{t} \Downarrow^n_m$ *as*

$$\mathbf{t} \Downarrow^n_m \;=\; \left(\mathbf{t}^{-1} \Big\downarrow^{\{1,\ldots,n\}}_{\{1,\ldots,m\}}\right)^{-1} \;:\; \begin{array}{ccc} (\{1,\ldots,m\})\mathbf{t}^{-1} & \to & \{1,\ldots,m\}\,, \\ (i,j) & \mapsto & (i,j)\mathbf{t}. \end{array}$$

$\mathbf{t} \Downarrow^n_m$ *is called the target restriction of* $\mathbf{t}$ *to* m *or, for short, the target restriction of* $\mathbf{t}$*.*

(iii) Let $\mathbf{s}$ *and* $\mathbf{t}$ *be* λ*-tableaux with* $\lambda \vDash n$*. Then a chain of length* k *from* $\mathbf{s}$ *to* $\mathbf{t}$ *with* $k \in \mathbb{N}_0$ *is defined as a sequence*

$$\mathbf{r}_0 = \mathbf{s}, \mathbf{r}_1, \ldots, \mathbf{r}_{k-1}, \mathbf{r}_k = \mathbf{t}$$

of λ*-tableaux such that for every* $j \in \{1, \ldots, k\}$ *we have*

$$\mathbf{r}_j = \mathbf{r}_{j-1} v_j$$

with an appropriate $v_j \in \mathfrak{B}_n \cup \{1_{\mathfrak{S}_n}\}$*.*

(iv) Using the notation from (iii), a chain of length k *from* $\mathbf{s}$ *to* $\mathbf{t}$ *is called descending, if for every* $j \in \{1, \ldots, k\}$ *we have*

$$v_j = (i_j, i_j + 1) \in \mathfrak{B}_n$$

with $i_j \in \{1, \ldots, n-1\}$ *and furthermore*

$$(i_j)\zeta_{\mathbf{r}_{j-1}} < (i_j + 1)\zeta_{\mathbf{r}_{j-1}}.$$

Remark 3.1.2 *(i) For a given tableau* $\mathbf{t}$ *of a composition of* n*, the associated diagram* $[\lambda^{\mathbf{t}}]$ *is obtained from the representation of* $\mathbf{t}$ *(see picture (1.16) on page 22) by removing all the entries.*

(ii) Choose a row standard λ*-tableau* $\mathbf{t}$ *with* $\lambda \vDash n$ *and let* $m \in \{1, \ldots, n\}$*. Then the representation of* $\mathbf{t} \Downarrow^n_m$ *is obtained from the representation of* $\mathbf{t}$ *by removing all squares with entries greater than* m*. Since* $\mathbf{t}$ *is row standard, this procedure only eliminates squares from the ends of the rows in the representation of* $\mathbf{t}$*, thus leaving indeed a row standard tableau with associated composition* $\lambda^{\mathbf{t}\Downarrow^n_m}$*.*

(iii) *Let* $\mathbf{s}$ *and* $\mathbf{t}$ *be* λ*-tableaux with* $\lambda \vDash n$ *and let*

$$\mathbf{r}_0 = \mathbf{s}, \mathbf{r}_1, \ldots, \mathbf{r}_{k-1}, \mathbf{r}_k = \mathbf{t}$$

be a chain from $\mathbf{s}$ *to* $\mathbf{t}$ *with* $k \in \mathbb{N}_0$*. Then for every* $j \in \{1, \ldots, k\}$ *the tableaux* $\mathbf{r}_j$ *and* $\mathbf{r}_{j-1}$ *differ at most by the application of a simple reflection.*

(iv) *Using the notation from (iii), suppose that the chain from* $\mathbf{s}$ *to* $\mathbf{t}$ *is descending and consider the transition from* $\mathbf{r}_{j-1}$ *to* $\mathbf{r}_j = \mathbf{r}_{j-1} v_j$ *for a* $j \in \{1, \ldots, k\}$*. In the course of this, the simple reflection* $v_j = (i_j, i_j + 1) \in \mathfrak{B}_n$ *moves the entry* i_j *in the representation of* $\mathbf{r}_{j-1}$ *downwards and the entry* $i_j + 1$ *in the same representation upwards.*

The following lemma is a useful observation regarding the compositions associated to the target restrictions of a row standard tableau.

Lemma 3.1.3 *Let* $\mathbf{t} \in \mathcal{T}^{\Xi_n}_{\text{row std}}$*. Then* $\mathbf{t}$ *can be recovered from the sequence of compositions*

$$\lambda^{\mathbf{t}\Downarrow^n_1}, \lambda^{\mathbf{t}\Downarrow^n_2}, \ldots, \lambda^{\mathbf{t}\Downarrow^n_{n-1}}, \lambda^{\mathbf{t}\Downarrow^n_n}.$$

Proof. According to the construction of the compositions $\lambda^{\mathbf{t}\Downarrow^n_m}$ with $m \in \{1, \ldots, n\}$ in Definition 3.1.1, $\lambda^{\mathbf{t}\Downarrow^n_1}$ differs from the composition (0) by exactly one entry having the value 1. If the index of that entry is denoted by i_1 then we have

$$(1)\zeta_{\mathbf{t}} = i_1.$$

Similarly, for a $j \in \{2, \ldots, n\}$, the compositions $\lambda^{\mathbf{t}\Downarrow^n_j}$ and $\lambda^{\mathbf{t}\Downarrow^n_{j-1}}$ differ only in their entries at one particular index i_j. For this index we have

$$(j)\zeta_{\mathbf{t}} = i_j.$$

Thus we know the row numbers of all entries in the tableau $\mathbf{t}$ and, since $\mathbf{t}$ is row standard (see Definition 1.1.45.(iii)), also the tableau itself. ∎

The next definition makes use of the ordering relations for compositions from Definition 1.1.4 and Lemma 1.1.5 and also of the set from Definition 1.1.46.(ii).

Definition 3.1.4 *(i)* *Let* $\mathbf{s}, \mathbf{t} \in \mathcal{T}^{\Xi_n}_{\text{row std}}$*. Then we write*

$$\mathbf{s} < \mathbf{t}$$

if there is an $m \in \{1, \ldots, n\}$ *such that both*

$$\lambda^{\mathbf{s}\Downarrow^n_m} < \lambda^{\mathbf{t}\Downarrow^n_m} \qquad \textit{and} \qquad \forall k \in \{m+1, \ldots, n\} : \lambda^{\mathbf{s}\Downarrow^n_k} = \lambda^{\mathbf{t}\Downarrow^n_k}$$

hold. Furthermore we write

$$\mathbf{s} \leq \mathbf{t}$$

if

$$(\mathbf{s} < \mathbf{t}) \vee (\mathbf{s} = \mathbf{t})$$

holds.

(ii) Let $\mathbf{s}, \mathbf{t} \in \mathcal{T}^{\Xi_n}_{\text{row std}}$. *Then we write*

$$\mathbf{s} \trianglelefteq \mathbf{t}$$

if

$$\forall m \in \{1, \ldots, n\} : \lambda^{\mathbf{s}\Downarrow^n_m} \trianglelefteq \lambda^{\mathbf{t}\Downarrow^n_m}$$

holds. Furthermore we write

$$\mathbf{s} \triangleleft \mathbf{t}$$

if

$$(\mathbf{s} \trianglelefteq \mathbf{t}) \wedge (\mathbf{s} \neq \mathbf{t})$$

holds.

(iii) Let $\lambda \vDash n$ *and* $\mathbf{s}, \mathbf{t} \in \mathcal{T}^{\lambda}_{\text{row std}}$. *Then we write*

$$\mathbf{s} \preceq \mathbf{t}$$

if there is a descending chain from $\mathbf{t}$ *to* $\mathbf{s}$. *Furthermore we write*

$$\mathbf{s} \prec \mathbf{t}$$

if

$$(\mathbf{s} \preceq \mathbf{t}) \wedge (\mathbf{s} \neq \mathbf{t})$$

holds.

Lemma 3.1.5 *(i) The relation* $\leq$ *on the set* $\mathcal{T}^{\Xi_n}_{\text{row std}}$ *is a total ordering relation.*

(ii) The relation $\trianglelefteq$ *on the set* $\mathcal{T}^{\Xi_n}_{\text{row std}}$ *is a partial ordering relation.*

(iii) Let $\lambda \vDash n$. *Then the relation* $\preceq$ *on the set* $\mathcal{T}^{\lambda}_{\text{row std}}$ *is a partial ordering relation.*

(iv) Let $\lambda \vDash n$ *and* $\mathbf{s}, \mathbf{t} \in \mathcal{T}^{\lambda}_{\text{row std}}$. *Then we have*

$$\mathbf{s} \preceq \mathbf{t} \Rightarrow \mathbf{s} \trianglelefteq \mathbf{t}.$$

(v) Let $\mathbf{s}, \mathbf{t} \in \mathcal{T}^{\Xi_n}_{\text{row std}}$*. Then we have*

$$\mathbf{s} \trianglelefteq \mathbf{t} \Rightarrow \mathbf{s} \leq \mathbf{t}.$$

(vi) Let $\mathbf{s}, \mathbf{t} \in \mathcal{T}^{\Xi_n}_{\text{row std}}$*. Then we have*

$$\mathbf{s} \triangleleft \mathbf{t} \Rightarrow \mathbf{s} < \mathbf{t}.$$

Proof. (i) The reflexivity of the relation $\leq$ on the set $\mathcal{T}^{\Xi_n}_{\text{row std}}$ follows immediately from Definition 3.1.4.(i).

Now choose $\mathbf{s}, \mathbf{t} \in \mathcal{T}^{\Xi_n}_{\text{row std}}$. Then if we have

$$\forall m \in \{1, \ldots, n\} : \lambda^{\mathbf{s}\Downarrow^n_m} = \lambda^{\mathbf{t}\Downarrow^n_m},$$

we also have, according to Lemma 3.1.3,

$$\mathbf{s} = \mathbf{t}.$$

If this is not the case then there is an $m \in \{1, \ldots, n\}$ such that

$$\lambda^{\mathbf{s}\Downarrow^n_m} \neq \lambda^{\mathbf{t}\Downarrow^n_m} \qquad \text{and} \qquad \forall k \in \{m+1, \ldots, n\} : \lambda^{\mathbf{s}\Downarrow^n_k} = \lambda^{\mathbf{t}\Downarrow^n_k}$$

hold. Here we have $\mathbf{s} \neq \mathbf{t}$ and, according to Lemma 1.1.5.(i), either $\lambda^{\mathbf{s}\Downarrow^n_m} < \lambda^{\mathbf{t}\Downarrow^n_m}$ or $\lambda^{\mathbf{s}\Downarrow^n_m} > \lambda^{\mathbf{t}\Downarrow^n_m}$. From this we get with Definition 3.1.4.(i) that

$$\text{either} \quad \mathbf{s} < \mathbf{t} \quad \text{or} \quad \mathbf{s} > \mathbf{t}$$

holds. All in all we see that exactly one of the relations $\mathbf{s} = \mathbf{t}$, $\mathbf{s} < \mathbf{t}$, or $\mathbf{s} > \mathbf{t}$ holds. This shows that the relation $\leq$ on the set $\mathcal{T}^{\Xi_n}_{\text{row std}}$ is total and antisymmetric.

Now choose $\mathbf{s}, \mathbf{t}, \mathbf{u} \in \mathcal{T}^{\Xi_n}_{\text{row std}}$ such that $\mathbf{s} < \mathbf{t} < \mathbf{u}$ holds. According to Definition 3.1.4.(i), we then have an $i \in \{1, \ldots, n\}$ satisfying

$$\lambda^{\mathbf{s}\Downarrow^n_i} < \lambda^{\mathbf{t}\Downarrow^n_i} \qquad \text{and} \qquad \forall h \in \{i+1, \ldots, n\} : \lambda^{\mathbf{s}\Downarrow^n_h} = \lambda^{\mathbf{t}\Downarrow^n_h}$$

and similarly a $j \in \{1, \ldots, n\}$ satisfying

$$\lambda^{\mathbf{t}\Downarrow^n_j} < \lambda^{\mathbf{u}\Downarrow^n_j} \qquad \text{and} \qquad \forall h \in \{j+1, \ldots, n\} : \lambda^{\mathbf{t}\Downarrow^n_h} = \lambda^{\mathbf{u}\Downarrow^n_h}.$$

If we put

$$k = \max\{i, j\}$$

then we get for this index

$$\lambda^{\mathbf{s}\Downarrow^n_k} < \lambda^{\mathbf{t}\Downarrow^n_k} \leq \lambda^{\mathbf{u}\Downarrow^n_k} \qquad \text{or} \qquad \lambda^{\mathbf{s}\Downarrow^n_k} \leq \lambda^{\mathbf{t}\Downarrow^n_k} < \lambda^{\mathbf{u}\Downarrow^n_k}$$

and also
$$\forall h \in \{k+1, \ldots, n\} : \lambda^{\mathbf{s}\Downarrow_h^n} = \lambda^{\mathbf{t}\Downarrow_h^n} = \lambda^{\mathbf{u}\Downarrow_h^n}.$$

Again according to Definition 3.1.4.(i), this shows $\mathbf{s} < \mathbf{u}$. Thus the relation $\leq$ on the set $\mathcal{T}^{\Xi_n}_{\text{row std}}$ is transitive.

(ii) The reflexivity, antisymmetry, and transitivity of the relation $\trianglelefteq$ on the set $\mathcal{T}^{\Xi_n}_{\text{row std}}$ follows easily from Definition 3.1.4.(ii), the corresponding properties of the relation $\trianglelefteq$ on the set Ξ_n shown in Lemma 1.1.5.(ii), and Lemma 3.1.3.

(iii) The reflexivity of the relation $\preceq$ on the set $\mathcal{T}^{\Xi_n}_{\text{row std}}$ follows immediately from Definition 3.1.4.(iii).

Choose $\mathbf{s}, \mathbf{t} \in \mathcal{T}^{\lambda}_{\text{row std}}$ such that $\mathbf{s} \preceq \mathbf{t} \preceq \mathbf{s}$ holds. Then we have, according to Definition 3.1.4.(iii), a descending chain from $\mathbf{t}$ to $\mathbf{s}$ and a descending chain from $\mathbf{s}$ to $\mathbf{t}$. Let $a \in \mathbb{N}_0$ be the length of the descending chain from $\mathbf{t}$ to $\mathbf{s}$ and $b \in \mathbb{N}_0$ be the length of the descending chain from $\mathbf{s}$ to $\mathbf{t}$. Concatenation of these chains gives a descending chain
$$\mathbf{r}_0 = \mathbf{t}, \mathbf{r}_1, \ldots, \mathbf{r}_{a+b-1}, \mathbf{r}_{a+b} = \mathbf{t}$$
from $\mathbf{t}$ to $\mathbf{t}$. Let $v_1, \ldots, v_{a+b} \in \mathfrak{B}_n$ be the associated simple reflections with
$$\mathbf{r}_i = \mathbf{r}_{i-1} v_i \quad \text{for} \quad i \in \{1, \ldots, a+b\}$$
(see Definition 3.1.1). Furthermore, fix the one $d \in \mathcal{D}_\lambda$ satisfying
$$\mathbf{t} = \mathbf{t}^\lambda d.$$
Then we inductively get, by using Definition 3.1.1.(iv) and Lemma 1.1.77.(i), for every $i \in \{0, \ldots, a+b\}$
$$\mathbf{r}_i = \mathbf{t}^\lambda d v_1 \cdots v_i \qquad \text{and} \qquad \ell(d v_1 \cdots v_i) = \ell(d) + i.$$
Now this and the fact $\mathbf{r}_0 = \mathbf{t} = \mathbf{r}_{a+b}$ imply
$$\mathbf{t} = \mathbf{r}_0 = \mathbf{t}^\lambda d = \mathbf{r}_{a+b} = \mathbf{t}^\lambda d v_1 \cdots v_{a+b}$$
and furthermore
$$d = \left(\mathbf{t}^\lambda\right)^{-1} \mathbf{t} = d v_1 \cdots v_{a+b}$$
which finally leads to
$$\ell(d) = \ell(d v_1 \cdots v_{a+b}) = \ell(d) + a + b.$$
From this we get $a + b = 0$ and, because of $a, b \in \mathbb{N}_0$, also
$$a = b = 0.$$

Thus the descending chains from $\mathbf{t}$ to $\mathbf{s}$ and from $\mathbf{s}$ to $\mathbf{t}$ considered above both have length 0, and we get from Definition 3.1.1.(iii)

$$\mathbf{s} = \mathbf{t}.$$

This shows that the relation $\preceq$ on the set $\mathcal{T}^{\lambda}_{\text{row std}}$ is antisymmetric.

Now choose $\mathbf{s}, \mathbf{t}, \mathbf{u} \in \mathcal{T}^{\lambda}_{\text{row std}}$ satisfying $\mathbf{s} \preceq \mathbf{t} \preceq \mathbf{u}$. Then we have as above in the proof of the antisymmetry property a descending chain from $\mathbf{t}$ to $\mathbf{s}$ and a descending chain from $\mathbf{u}$ to $\mathbf{t}$. Concatenation of these chains gives a descending chain from $\mathbf{u}$ to $\mathbf{s}$. According to Definition 3.1.4.(iii), this means $\mathbf{s} \preceq \mathbf{u}$. Thus the relation $\preceq$ on the set $\mathcal{T}^{\lambda}_{\text{row std}}$ is transitive.

(iv) According to Definition 3.1.4.(iii), we have a descending chain

$$\mathbf{r}_0 = \mathbf{t}, \mathbf{r}_1, \ldots, \mathbf{r}_{k-1}, \mathbf{r}_k = \mathbf{s}$$

from $\mathbf{t}$ to $\mathbf{s}$ with a certain length $k \in \mathbb{N}_0$, tableaux $\mathbf{r}_1, \ldots, \mathbf{r}_{k-1} \in \mathcal{T}^{\lambda}$, and simple reflections $v_1, \ldots, v_k \in \mathfrak{B}_n$ satisfying

$$\forall i \in \{1, \ldots, k\} : \mathbf{r}_i = \mathbf{r}_{i-1} v_i.$$

From Definition 3.1.1.(iv), Lemma 1.1.77.(i), and Definition 1.1.58.(i) we inductively get, as in the proof of statement (iii),

$$\forall i \in \{0, \ldots, k\} : \mathbf{r}_i \in \mathcal{T}^{\lambda}_{\text{row std}}.$$

Thus we have, according to Definition 3.1.4.(iii),

$$\forall i \in \{1, \ldots, k\} : \mathbf{r}_{i-1} \succeq \mathbf{r}_i.$$

Using the transitivity of the relation $\trianglelefteq$ on the set $\mathcal{T}^{\lambda}_{\text{row std}} \subseteq \mathcal{T}^{\Xi_n}_{\text{row std}}$, it suffices now to show $\forall i \in \{1, \ldots, k\} : \mathbf{r}_{i-1} \trianglerighteq \mathbf{r}_i$. To this end, we fix an $i \in \{1, \ldots, k\}$. Furthermore, let

$$v_i = (j, j+1)$$

with an appropriate $j \in \{1, \ldots, n-1\}$. According to Definition 3.1.1.(iv) and Remark 3.1.2.(iv), $\mathbf{r}_{i-1}$ and $\mathbf{r}_i$ differ by the transposition of j and $j+1$ such that in $\mathbf{r}_{i-1}$ the entry j is located above the entry $j+1$ and in $\mathbf{r}_i$ the entry j is located below the entry $j+1$. In other words, we can write

$$y = (j)\zeta_{\mathbf{r}_{i-1}} = (j+1)\zeta_{\mathbf{r}_i} \qquad \text{and} \qquad z = (j+1)\zeta_{\mathbf{r}_{i-1}} = (j)\zeta_{\mathbf{r}_i}$$

and then have

$$y < z.$$

Every other entry $h \in \{1, \dots, n\} \setminus \{j, j+1\}$ occupies the same place in $\mathbf{r}_{i-1}$ and $\mathbf{r}_i$ and we have

$$(h)\zeta_{\mathbf{r}_{i-1}} = (h)\zeta_{\mathbf{r}_i}.$$

In order to prove the desired relation $\mathbf{r}_{i-1} \trianglerighteq \mathbf{r}_i$, we now consider restrictions $\mathbf{r}_{i-1} \Downarrow_h^n$ and $\mathbf{r}_i \Downarrow_h^n$ of the tableaux $\mathbf{r}_{i-1}$ and $\mathbf{r}_i$ with $h \in \{1, \dots, n\}$ as in Definition 3.1.1.(ii). We distinguish three cases.

First let

$$h \in \{1, \dots, j-1\}.$$

Then $\mathbf{r}_{i-1} \Downarrow_h^n$ and $\mathbf{r}_i \Downarrow_h^n$ only contain entries smaller than j, and the preceding considerations show that each of these entries occupies the same place in $\mathbf{r}_{i-1} \Downarrow_h^n$ and $\mathbf{r}_i \Downarrow_h^n$. Thus we have

$$\mathbf{r}_{i-1} \Downarrow_h^n = \mathbf{r}_i \Downarrow_h^n$$

and, according to Definition 3.1.1.(i), furthermore

$$\lambda^{\mathbf{r}_{i-1} \Downarrow_h^n} = \lambda^{\mathbf{r}_i \Downarrow_h^n}.$$

Now let

$$h \in \{j+1, \dots, n\}.$$

Then each of $\mathbf{r}_{i-1} \Downarrow_h^n$ and $\mathbf{r}_i \Downarrow_h^n$ contains the entry j as well as the entry $j+1$ and, according to the considerations above, these two tableaux differ only by the transposition of j and $j+1$, as do already $\mathbf{r}_{i-1}$ and $\mathbf{r}_i$. In other words, corresponding positions in $\mathbf{r}_{i-1} \Downarrow_h^n$ and $\mathbf{r}_i \Downarrow_h^n$ are occupied either by the same number or in one tableau by j and in the other one by $j+1$. By deleting the entries of these tableaux we get (see Remark 3.1.2.(i))

$$\left[\lambda^{\mathbf{r}_{i-1} \Downarrow_h^n}\right] = \left[\lambda^{\mathbf{r}_i \Downarrow_h^n}\right]$$

and furthermore (see Lemma 1.1.7.(i))

$$\lambda^{\mathbf{r}_{i-1} \Downarrow_h^n} = \lambda^{\mathbf{r}_i \Downarrow_h^n}.$$

Finally let

$$h = j.$$

Then both $\mathbf{r}_{i-1} \Downarrow_j^n$ and $\mathbf{r}_i \Downarrow_j^n$ contain besides the entry j only entries smaller than j and, according to the considerations above, every entry different from j occupies the same position in $\mathbf{r}_{i-1} \Downarrow_j^n$ and $\mathbf{r}_i \Downarrow_j^n$. Since j is the biggest entry in both $\mathbf{r}_{i-1} \Downarrow_j^n$ and $\mathbf{r}_i \Downarrow_j^n$ and both $\mathbf{r}_{i-1}$ and $\mathbf{r}_i$ are row standard, we obtain $\mathbf{r}_i \Downarrow_j^n$ from $\mathbf{r}_{i-1} \Downarrow_j^n$ by moving j from the end of row y to the end of row z, using the notation from above. This shows that if we put

$$\mu = (\mu_1, \mu_2, \ldots) = \lambda^{\mathbf{r}_{i-1} \Downarrow_j^n} \qquad \text{and} \qquad \nu = (\nu_1, \nu_2, \ldots) = \lambda^{\mathbf{r}_i \Downarrow_j^n}$$

then we have

$$\mu_y = \nu_y + 1, \qquad \mu_z = \nu_z - 1, \qquad \text{and} \qquad \forall x \in \mathbb{N} \setminus \{y, z\} : \mu_x = \nu_x.$$

Because of $y < z$ (see above), this immediately implies (see Definition 1.1.4.(ii))

$$\lambda^{\mathbf{r}_{i-1} \Downarrow_j^n} = \mu \trianglerighteq \nu = \lambda^{\mathbf{r}_i \Downarrow_j^n}.$$

Altogether, we get from these relations between $\lambda^{\mathbf{r}_{i-1} \Downarrow_h^n}$ and $\lambda^{\mathbf{r}_i \Downarrow_h^n}$ for all values of $h \in \{1, \ldots, n\}$ according to Definition 3.1.4.(ii)

$$\mathbf{r}_{i-1} \trianglerighteq \mathbf{r}_i,$$

as desired.

(v) We can assume $\mathbf{s} \neq \mathbf{t}$. Then Lemma 3.1.3 provides us with an $i \in \{1, \ldots, n\}$ such that

$$\lambda^{\mathbf{s} \Downarrow_i^n} \neq \lambda^{\mathbf{t} \Downarrow_i^n} \qquad \text{and} \qquad \forall j \in \{i+1, \ldots, n\} : \lambda^{\mathbf{s} \Downarrow_j^n} = \lambda^{\mathbf{t} \Downarrow_j^n} \tag{3.1}$$

hold. Because of $\mathbf{s} \lhd \mathbf{t}$ (see Definition 3.1.4.(ii)), this implies

$$\lambda^{\mathbf{s} \Downarrow_i^n} \lhd \lambda^{\mathbf{t} \Downarrow_i^n}$$

and, according to Lemma 1.1.5.(iii), furthermore

$$\lambda^{\mathbf{s} \Downarrow_i^n} < \lambda^{\mathbf{t} \Downarrow_i^n}.$$

From this together with (3.1) we get, according to Definition 3.1.4.(i),

$$\mathbf{s} < \mathbf{t},$$

as desired.

(vi) This follows from Definition 3.1.4.(i), Definition 3.1.4.(ii), and statement (v). ■

Next, we introduce two ordering relations for permutations. These relations can be defined not only on symmetric groups, but also on arbitrary Coxeter groups (see for example [HUM, Section 5.9]). The following definition makes use of reduced expressions of permutations as introduced in (1.11) on page 3.

Definition 3.1.6 *For $u, w \in \mathfrak{S}_n$ we write*

$$u \preceq w$$

if there are a reduced expression

$$u = v_1 \cdots v_{\ell(u)}$$

for u with $v_1, \ldots, v_{\ell(u)} \in \mathfrak{B}_n$ and a $k \in \{0, \ldots, \ell(u)\}$ satisfying

$$w = v_1 \cdots v_k.$$

Furthermore we write

$$u \prec w$$

if

$$(u \preceq w) \wedge (u \neq w)$$

holds. $\preceq$ is called the weak Bruhat ordering on the symmetric group $\mathfrak{S}_n$.

Lemma 3.1.7 *The relation $\preceq$ on $\mathfrak{S}_n$ is a partial ordering relation.*

Proof. See [HUM, Section 5.9]. ■

The weak Bruhat ordering on $\mathfrak{S}_n$ provides a useful characterization of the sets of standard representatives associated to compositions of n (see Definition 1.1.60). The following lemma also makes use of Definition 1.1.67.

Lemma 3.1.8 *Let $\lambda \vDash n$. Then we have*

$$\mathcal{E}_\lambda = \{w \in \mathfrak{S}_n \mid w \succeq w_\lambda\}.$$

Proof. See [DJ1, Lemma 1.5]. ■

The next definition makes use of the set of general reflections in $\mathfrak{S}_n$. This set is defined as

$$\mathfrak{C}_n = \{(i,j) \mid i,j \in \{1,\ldots,n\} \text{ such that } i \neq j\}.$$

The notion of general reflections comes from the general theory of Coxeter groups (see [HUM, Section 5.7]).

Definition 3.1.9 *For $u, w \in \mathfrak{S}_n$ we write*

$$u \trianglelefteq w$$

if there are an $m \in \mathbb{N}_0$ and general reflections $t_1, \ldots, t_m \in \mathfrak{C}_n$ satisfying

$$u = wt_1 \cdots t_m$$

and

$$\forall j \in \{1, \ldots, m\} : \ell(wt_1 \cdots t_{j-1}) < \ell(wt_1 \cdots t_j).$$

Furthermore we write

$$u \lhd w$$

if

$$(u \unlhd w) \wedge (u \neq w)$$

holds. $\unlhd$ is called the strong Bruhat ordering or just the Bruhat ordering on the symmetric group $\mathfrak{S}_n$.

Lemma 3.1.10 *The relation $\unlhd$ on $\mathfrak{S}_n$ is a partial ordering relation.*

Proof. See [HUM, Section 5.9]. ■

Now we describe a useful characterization of the Bruhat ordering.

Definition 3.1.11 *Consider a product*

$$v_1 \cdots v_m$$

with $m \in \mathbb{N}_0$ and $v_1, \ldots, v_m \in \mathfrak{B}_n$.

(i) A subexpression of $v_1 \cdots v_m$ is defined as a product

$$v_{i_1} \cdots v_{i_j}$$

with $j \in \{0, \ldots, m\}$ and indices $i_1, \ldots, i_j \in \{1, \ldots, m\}$ satisfying

$$1 \leq i_1 < i_2 < \cdots < i_{j-1} < i_j \leq m.$$

(ii) We say that a $w \in \mathfrak{S}_n$ can be represented as a subexpression of the given product $v_1 \cdots v_m$ if w can be written as a product of simple reflections such that this product is a subexpression of $v_1 \cdots v_m$.

Theorem 3.1.12 *Let $u, w \in \mathfrak{S}_n$. Fix a reduced expression*

$$u = v_1 \cdots v_{\ell(u)} \tag{3.2}$$

for u with $v_1, \ldots, v_{\ell(u)} \in \mathfrak{B}_n$. Then the following statements are equivalent.

(i) We have $u \trianglelefteq w$.

(ii) The permutation w can be represented as a subexpression of the reduced expression (3.2) for u.

Proof. See [HUM, Section 5.10]. ■

The previously described ordering relations will now be related to shortest representatives of right cosets of Young subgroups in $\mathfrak{S}_n$.

Lemma 3.1.13 *Let $\lambda \vDash n$ and $d, f \in \mathcal{D}_\lambda$. Then the following statements hold.*

(i) $\mathbf{t}^\lambda d \preceq \mathbf{t}^\lambda f \Leftrightarrow d \preceq f$.

(ii) $\mathbf{t}^\lambda d \trianglelefteq \mathbf{t}^\lambda f \Leftrightarrow d \trianglelefteq f$.

(iii) $\mathbf{t}^\lambda d \triangleleft \mathbf{t}^\lambda f \Leftrightarrow d \triangleleft f$.

Proof. (i) See [MUR, Lemma 3.8.(i)].

(ii) See [MUR, Lemma 3.8.(ii)].

(iii) This follows easily from Definition 3.1.4.(ii), Definition 3.1.9, and statement (ii). ■

The following definition makes use of Definition 3.1.4.(i).

Definition 3.1.14 *Let $\lambda \vDash n$ and $d, f \in \mathcal{D}_\lambda$. Then we write*

$$d < f$$

if

$$\mathbf{t}^\lambda d < \mathbf{t}^\lambda f$$

holds. Furthermore we write

$$d \leq f$$

if

$$\mathbf{t}^\lambda d \leq \mathbf{t}^\lambda f$$

holds.

The next statement makes use of the ordering relations $\preceq$ from Definition 3.1.6 and $\trianglelefteq$ from Definition 3.1.9.

Lemma 3.1.15 *Let $\lambda \vDash n$. Then the following statements hold.*

(i) The relation $\leq$ on the set $\mathcal{D}_\lambda$ is a total ordering relation.

(ii) Let $d, f \in \mathcal{D}_\lambda$. Then we have

$$d \preceq f \Rightarrow d \trianglelefteq f.$$

(iii) Let $d, f \in \mathcal{D}_\lambda$. Then we have

$$d \trianglelefteq f \Rightarrow d \leq f.$$

(iv) Let $d, f \in \mathcal{D}_\lambda$. Then we have

$$d \triangleleft f \Rightarrow d < f.$$

Proof. (i) This is obtained from Definition 3.1.14, Lemma 1.1.59.(i), and Lemma 3.1.5.(i).

(ii) This follows from Lemma 3.1.5.(iv) and Lemma 3.1.13.

(iii) This follows from Lemma 3.1.13.(ii), Lemma 3.1.5.(v), and Definition 3.1.14.

(iv) This is obtained from Lemma 3.1.13.(iii), Lemma 3.1.5.(vi), and Definition 3.1.14. ■

3.2 PK-pairs

The combinatorial objects introduced in this section are used to construct and index the generic modules, homomorphisms, and basis elements occurring in the derivation of the Specht series. These combinatorial objects also are considered in [JAM1, Section 15, Section 16, Section 17]. As always, n denotes a positive integer.

The next definition makes use of Definition 1.1.6.(i).

Definition 3.2.1 *A pair $\mu^\# \mu$ with $\mu = (\mu_1, \mu_2, \ldots) \vDash n$ and $\mu^\# = (\mu_1^\#, \mu_2^\#, \ldots) \vdash k$ with a $k \in \{1, \ldots, n\}$ satisfying*

$$\mu_1^\# = \mu_1$$

and

$$\left[\mu^\#\right] \subseteq [\mu] \qquad \text{or equivalently} \qquad \forall i \in \mathbb{N} : \mu_i^\# \leq \mu_i$$

is called a partition-composition-pair of degree n or just a partition-composition-pair. This is abbreviated as PK$_n$-pair or just as PK-pair.

The symbol 00 without any relation to partitions and compositions also is called a PK-pair.

If we consider for example

$$\begin{aligned} \lambda^{\#} &= (3, 2^2, 1) \vdash 8, & \mu^{\#} &= (2^2, 1^4) \vdash 8, \\ \lambda &= (3^2, 2^2, 1^2) \vdash 12, & \mu &= (2, 3^2, 2, 1^3) \vDash 13 \end{aligned} \tag{3.3}$$

then we can build the PK$_{12}$-pair $\lambda^{\#}\lambda$ and the PK$_{13}$-pair $\mu^{\#}\mu$. Both $\lambda^{\#}$ and μ and also $\mu^{\#}$ and λ cannot be combined into PK-pairs. The abbreviation PK stands for partition-composition.

Remark 3.2.2 *Since, using the notation from Definition 3.2.1, $\mu^{\#}$ is a partition of a positive integer, we have*

$$\mu_1^{\#} > 0.$$

This and the condition $\mu_1^{\#} = \mu_1$ show that the composition μ necessarily satisfies

$$\mu_1 > 0.$$

Definition 3.2.3 *Let $\mu^{\#}\mu$ be a PK_n-pair with $\mu^{\#}\mu \neq 00$, $\mu = (\mu_1, \mu_2, \ldots) \vDash n$, and $\mu^{\#} = (\mu_1^{\#}, \mu_2^{\#}, \ldots) \vdash k$ for a $k \in \{1, \ldots, n\}$. Then a*

$$c \in \mathbb{N} \setminus \{1\}$$

satisfying

$$\mu_{c-1}^{\#} = \mu_{c-1} \qquad \text{and} \qquad \mu_c^{\#} < \mu_c$$

is called an AR-index for $\mu^{\#}\mu$.

In the example (3.3) above, 4 is an AR-index for $\lambda^{\#}\lambda$ and 2 is an AR-index for both $\lambda^{\#}\lambda$ and $\mu^{\#}\mu$. The abbreviation AR stands for add-raise. This notation will make sense after Definition 3.2.5.

Remark 3.2.4 *Consider a PK_n-pair $\mu^{\#}\mu$. If there is an AR-index c for $\mu^{\#}\mu$ as in Definition 3.2.3 then we have, using the notation from there and Remark 3.2.2, $\mu_1 > 0$ and also $\mu_c > \mu_c^{\#} \geq 0$ with $c > 1$. Thus at least two entries of $\mu \vDash n$ are positive and we must have*

$$n \geq 2.$$

Now we consider two operators which construct new PK-pairs from a given one.

Definition 3.2.5 *Let $\mu^{\#}\mu$ be a PK_n-pair with $\mu^{\#}\mu \neq 00$, $\mu = (\mu_1, \mu_2, \ldots) \vDash n$, and $\mu^{\#} = (\mu_1^{\#}, \mu_2^{\#}, \ldots) \vdash k$ for a $k \in \{1, \ldots, n\}$. Furthermore, let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^{\#}\mu$.*

(i) We put

$$\mu^{\#} A_c = (\mu_1^{\#}, \dots, \mu_{c-1}^{\#}, \mu_c^{\#} + 1, \mu_{c+1}^{\#}, \dots) \vDash k + 1.$$

$\mu^{\#} A_c$ is called the image of the partition $\mu^{\#}$ under the operator A_c.

(ii) We put

$$\mu R_c = (\mu_1, \dots, \mu_{c-2}, \mu_{c-1} + \mu_c - \mu_c^{\#}, \mu_c^{\#}, \mu_{c+1}, \dots) \vDash n.$$

μR_c is called the image of the composition μ under the operator R_c.

(iii) In the case $c > 2$, we combine the compositions $\mu^{\#}$ and μR_c into the pair $\mu^{\#}\mu R_c$. In the case $c = 2$, we define the pair $\mu^{\#}\mu R_c$ as

$$(\mu_1 + \mu_2 - \mu_2^{\#}, \mu_2^{\#}, \dots)\mu R_c,$$

that is, we modify the first entry of $\mu^{\#}$. $\mu^{\#}\mu R_c$ is called the image of the PK$_n$-pair $\mu^{\#}\mu$ under the operator R_c.

(iv) If $\mu^{\#} A_c \vdash k + 1$, we combine the partition $\mu^{\#} A_c$ and the composition μ into the pair $\mu^{\#} A_c \mu$. If $\mu^{\#} A_c$ is not a partition or equivalently if $\mu_{c-1}^{\#} = \mu_c^{\#}$, we declare $\mu^{\#} A_c \mu$ to be 00. $\mu^{\#} A_c \mu$ is called the image of the PK$_n$-pair $\mu^{\#}\mu$ under the operator A_c.

If we consider for example the PK$_{15}$-pair $\mu^{\#}\mu$ with

$$\mu^{\#} = (5, 3, 2, 1) \vdash 11 \qquad \text{and} \qquad \mu = (5, 3, 4, 1, 2) \vDash 15$$

and the AR-index 3 for it, we can apply the operators A_3 and R_3 to obtain first

$$\mu^{\#} A_3 = (5, 3^2, 1) \vdash 12 \qquad \text{and} \qquad \mu R_3 = (5^2, 2, 1, 2) \vDash 15$$

and with that the PK$_{15}$-pairs $\mu^{\#} A_3 \mu$ and $\mu^{\#}\mu R_3$.

Remark 3.2.6 *(i) The pairs $\mu^{\#}\mu R_c$ and $\mu^{\#} A_c \mu$ introduced in Definition 3.2.5 are always distinct from the initial pair $\mu^{\#}\mu$. In the case of $\mu^{\#}\mu R_c$, this follows from the condition*

$$\mu_c^{\#} < \mu_c$$

in Definition 3.2.3.

(ii) The operators A_c and R_c from Definition 3.2.5 also can be constructed without the condition

$$\mu_{c-1}^{\#} = \mu_{c-1}$$

in Definition 3.2.3. However, this condition is crucial to their application (see proof of Lemma 3.3.9).

The first part of the following lemma makes use of Definition 1.1.6.(i).

Lemma 3.2.7 *(i) Let $\mu^{\#}\mu$ be a PK$_n$-pair with $\mu^{\#}\mu \neq 00$ and choose an AR-index $c \in \mathbb{N} \setminus \{1\}$ for $\mu^{\#}\mu$. Then we have*

$$\left[\mu^{\#}\right] \subseteq \left[\mu^{\#}A_c\right].$$

(ii) Let $\mu^{\#}\mu$ be a PK$_n$-pair with $\mu^{\#}\mu \neq 00$ and choose an AR-index $c \in \mathbb{N} \setminus \{1\}$ for $\mu^{\#}\mu$. Then $\mu^{\#}\mu R_c$ and $\mu^{\#}A_c\mu$ are PK$_n$-pairs. Furthermore, we have $\mu^{\#}\mu R_c \neq 00$.

(iii) Consider a PK$_n$-pair $\mu^{\#}\mu$. Then there is an AR-index $c \in \mathbb{N} \setminus \{1\}$ for $\mu^{\#}\mu$ if and only if

$$\mu^{\#}\mu \neq 00 \qquad \textit{and} \qquad \mu^{\#} \neq \mu$$

hold.

(iv) For every PK$_n$-pair $\mu^{\#}\mu$ with $\mu^{\#}\mu \neq 00$, there is a

$$\nu = (\nu_1, \nu_2, \ldots) \vDash n$$

such that $(\nu_1)\nu$ is a PK$_n$-pair from which the PK$_n$-pair $\mu^{\#}\mu$ can be obtained by iterated application of appropriate operators A_c and R_c with AR-indices $c \in \mathbb{N} \setminus \{1\}$.

Proof. (i) This follows immediately from Definition 3.2.5.(i) and Definition 1.1.6.(i).

(ii) First we consider $\mu^{\#}A_c\mu$. In the case $\mu^{\#}A_c\mu = 00$, there is nothing to prove. In the case $\mu^{\#}A_c\mu \neq 00$, $\mu^{\#}A_c$ is a partition, and we get from Definition 3.2.1, the conditions in Definition 3.2.3, and the construction of $\mu^{\#}A_c$ in Definition 3.2.5 with

$$\mu^{\#} = (\mu_1^{\#}, \mu_2^{\#}, \ldots), \quad \mu = (\mu_1, \mu_2, \ldots), \quad \mu^{\#}A_c = ((\mu^{\#}A_c)_1, (\mu^{\#}A_c)_2, \ldots)$$

the fact

$$\forall i \in \mathbb{N} : (\mu^{\#}A_c)_i \leq \mu_i$$

and because of $c > 1$ in particular

$$(\mu^{\#}A_c)_1 = \mu_1^{\#} = \mu_1.$$

Thus $\mu^{\#}A_c\mu$ is a PK$_n$-pair (see Definition 3.2.1).

Now we consider $\mu^{\#}\mu R_c$. $\mu^{\#}\mu R_c \neq 00$ follows immediately from the definition, since this pair contains the composition μR_c of n.

In the case $c > 2$, the given partition $\mu^\#$ is not modified in the course of the construction of the pair $\mu^\#\mu R_c$. Now we get from Definition 3.2.1 and the construction of μR_c in Definition 3.2.5 with

$$\mu^\# = (\mu_1^\#, \mu_2^\#, \ldots), \quad \mu = (\mu_1, \mu_2, \ldots), \quad \mu R_c = ((\mu R_c)_1, (\mu R_c)_2, \ldots)$$

the fact

$$\forall i \in \mathbb{N} : \mu_i^\# \leq (\mu R_c)_i$$

and because of $c > 2$ in particular

$$\mu_1^\# = \mu_1 = (\mu R_c)_1.$$

Thus $\mu^\#\mu R_c$ is a PK_n-pair.

In the case $c = 2$, we write

$$\mu R_2 = ((\mu R_2)_1, (\mu R_2)_2, \ldots).$$

With that, the first entry of the given partition $\mu^\#$ is changed to

$$\mu_1 + \mu_2 - \mu_2^\# = (\mu R_2)_1$$

in the course of the construction of $\mu^\#\mu R_2$. According to Definition 3.2.1, we have $\mu_1^\# = \mu_1$ and, because of $c = 2$ and the conditions in Definition 3.2.3, furthermore $\mu_2^\# < \mu_2$. This implies (see also Definition 1.1.2)

$$\mu_1 + \mu_2 - \mu_2^\# > \mu_1^\# \geq \mu_2^\# \geq \cdots.$$

Thus the modified composition $\mu^\#$ is a partition of a positive integer. This partition has the same first entry as μR_2. In addition, we have according to Definition 3.2.5 and Definition 3.2.1

$$\mu_2^\# = (\mu R_2)_2 \qquad \text{and} \qquad \forall j \in \mathbb{N} \setminus \{1, 2\} : \mu_j^\# \leq \mu_j = (\mu R_2)_j.$$

Thus $\mu^\#\mu R_2$ also is a PK_n-pair.

(iii) Let $\mu^\#\mu$ be a PK_n-pair. If we have $\mu^\#\mu = 00$, there is no AR-index according to Definition 3.2.3. Now suppose $\mu^\#\mu \neq 00$ with

$$\mu^\# = (\mu_1^\#, \mu_2^\#, \ldots) \qquad \text{and} \qquad \mu = (\mu_1, \mu_2, \ldots).$$

In the case $\mu^\# = \mu$, there is no $c \in \mathbb{N} \setminus \{1\}$ with $\mu_c^\# < \mu_c$ and thus no AR-index. In the case $\mu^\# \neq \mu$, we get from the facts $\left[\mu^\#\right] \subseteq [\mu]$ and $\mu_1^\# = \mu_1$ a minimal $c_0 \in \mathbb{N} \setminus \{1\}$ satisfying

$$\mu_{c_0}^\# < \mu_{c_0}.$$

This c_0 necessarily also satisfies

$$\mu^{\#}_{c_0-1} = \mu_{c_0-1}.$$

Thus c_0 is an AR-index for $\mu^{\#}\mu$.

(iv) See [JAM1, Section 15, especially 15.12]. ■

Now we introduce certain tableaux and permutations based on PK-pairs. The following definition makes use of Definition 1.1.6 and Definition 1.1.45.(i).

Definition 3.2.8 *Let $\mu^{\#}\mu$ be a PK_n-pair as in Definition 3.2.1 with $\mu^{\#}\mu \neq 00$.*

(i) The lattice points in $[\mu]$ can be ordered in such a way that the lattice points in $\left[\mu^{\#}\right] \subseteq [\mu]$ precede the lattice points in $[\mu] \setminus \left[\mu^{\#}\right]$ and furthermore the ascending order for the lattice points in $\left[\mu^{\#}\right]$ is by columns from left to right and within each column from top to bottom and finally the ascending order for the lattice points in $[\mu] \setminus \left[\mu^{\#}\right]$ is by rows from top to bottom and within each row from left to right. This is equivalent to say that for $(i,j), (\tilde{i},\tilde{j}) \in [\mu]$ we have

$$\begin{aligned}(i,j) < (\tilde{i},\tilde{j}) \quad \Leftrightarrow \quad & \Big(\left((i,j) \in \left[\mu^{\#}\right]\right) \wedge \left((\tilde{i},\tilde{j}) \in \left[\mu^{\#}\right]\right) \wedge \\ & \qquad \left(\left(j < \tilde{j}\right) \vee \left(\left(j = \tilde{j}\right) \wedge \left(i < \tilde{i}\right)\right)\right) \Big) \vee \\ & \left(\left((i,j) \in \left[\mu^{\#}\right]\right) \wedge \left((\tilde{i},\tilde{j}) \notin \left[\mu^{\#}\right]\right)\right) \vee \\ & \Big(\left((i,j) \notin \left[\mu^{\#}\right]\right) \wedge \left((\tilde{i},\tilde{j}) \notin \left[\mu^{\#}\right]\right) \wedge \\ & \qquad \left(\left(i < \tilde{i}\right) \vee \left(\left(i = \tilde{i}\right) \wedge \left(j < \tilde{j}\right)\right)\right) \Big).\end{aligned}$$

With that we define

$$\mathbf{t}^{\mu^{\#}\mu} \in \mathcal{T}^{\mu}$$

as the order preserving map from the set $[\mu]$ ordered by $<$ to the set $\{1,\ldots,n\}$ arranged in its natural ascending order.

(ii) The permutation $w_{\mu^{\#}\mu} \in \mathfrak{S}_n$ is defined as

$$w_{\mu^{\#}\mu} = \left(\mathbf{t}^{\mu}\right)^{-1} \mathbf{t}^{\mu^{\#}\mu}$$

or equivalently by the condition

$$\mathbf{t}^{\mu} w_{\mu^{\#}\mu} = \mathbf{t}^{\mu^{\#}\mu}.$$

(iii) The permutation $g_{\mu^\#\mu} \in \mathfrak{S}_n$ *is defined as*

$$g_{\mu^\#\mu} = w_{\mu^\#\mu}^{-1} w_\mu$$

or equivalently by the condition

$$w_{\mu^\#\mu} g_{\mu^\#\mu} = w_\mu .$$

The following picture shows the μ-tableau $\mathbf{t}^{\mu^\#\mu}$ for the PK_{22}-pair $\mu^\#\mu$ with

$$\mu^\# = (7, 3^2, 2) \vdash 15 \qquad \text{and} \qquad \mu = (7, 5, 6, 4) \vDash 22.$$

1	5	9	12	13	14	15
2	6	10	16	17		
3	7	11	18	19	20	
4	8	21	22			

Furthermore, we have for this PK_{22}-pair

$$w_{\mu^\#\mu} = (2, 5, 13, 3, 9, 6, 14, 7, 15, 11, 16, 18, 20, 8)(4, 12, 17, 19)$$

and

$$g_{\mu^\#\mu} = (12, 13, 17, 18, 15, 22, 16, 14, 20, 21).$$

The notions introduced in the following serve the better description of properties of and constructions with the tableaux and permutations introduced in the preceding definition. The next definition makes use of the notation (1.1) on page 1, Definition 1.1.6, and Definition 1.1.45.(i).

Definition 3.2.9 *Let* $\lambda \vDash n$*. Moreover let* $\nu \vDash k$ *with a* $k \in \{1, \ldots, n\}$ *satisfying* $[\nu] \subseteq [\lambda]$*. Finally choose a* λ*-tableau* $\mathbf{t}$*. Then the map*

$$\mathbf{t}\Big\downarrow^{[\lambda]}_{[\nu]} : [\nu] \to \{1, \ldots, n\}, \quad (i, j) \mapsto (i, j)\mathbf{t}\Big\downarrow^{[\lambda]}_{[\nu]} = (i, j)\mathbf{t}$$

is called the source restriction of $\mathbf{t}$ *from* $[\lambda]$ *to* $[\nu]$ *or the source restriction of* $\mathbf{t}$ *to* $[\nu]$ *or just the source restriction of* $\mathbf{t}$*. For a lattice point* $(i, j) \in [\nu]$*, its image* $(i, j)\mathbf{t}\Big\downarrow^{[\lambda]}_{[\nu]} = (i, j)\mathbf{t}$ *is called the entry at the position* (i, j) *in* $\mathbf{t}\Big\downarrow^{[\lambda]}_{[\nu]}$ *or just the* (i, j)*-entry in* $\mathbf{t}\Big\downarrow^{[\lambda]}_{[\nu]}$*. The rows of* $\mathbf{t}\Big\downarrow^{[\lambda]}_{[\nu]}$ *are defined as the restrictions of* $\mathbf{t}\Big\downarrow^{[\lambda]}_{[\nu]}$ *to the rows of* $[\nu]$*. The columns of* $\mathbf{t}\Big\downarrow^{[\lambda]}_{[\nu]}$ *are defined as the restrictions of* $\mathbf{t}\Big\downarrow^{[\lambda]}_{[\nu]}$ *to the columns of* $[\nu]$*.* $\mathbf{t}\Big\downarrow^{[\lambda]}_{[\nu]}$ *is represented by labelling for every lattice point* $(i, j) \in [\nu]$ *the corresponding square in the representation of* $[\nu]$ *with the* (i, j)*-entry in* $\mathbf{t}\Big\downarrow^{[\lambda]}_{[\nu]}$*.*

Remark 3.2.10 *The notions introduced in the preceding definition can be applied in particular to PK_n-pairs $\mu^\#\mu$ different from* 00 *and the restrictions* $\mathbf{t}\Big\downarrow^{[\mu]}_{[\mu^\#]}$ *of μ-tableaux* $\mathbf{t}$.

The next statement makes use of Definition 1.1.58.(i), Definition 1.1.60, and Definition 1.1.66.

Lemma 3.2.11 *Let $\mu^\#\mu$ be a PK_n-pair with $\mu^\#\mu \neq 00$. Then we have*

(i) $\mathbf{t}^{\mu^\#\mu}\Big\downarrow^{[\mu]}_{[\mu^\#]} = \mathbf{t}_{\mu^\#}$,

(ii) $\mathbf{t}^{\mu^\#\mu} \in \mathcal{T}^{\mu}_{\mathrm{std}}$,

(iii) $w_{\mu^\#\mu} \in \mathcal{E}_\mu \subseteq \mathcal{D}_\mu$,

(iv) $\ell(w_\mu) = \ell(w_{\mu^\#\mu}) + \ell(g_{\mu^\#\mu})$,

(v) $\mathbf{t}^{\mu^\#\mu} g_{\mu^\#\mu} = \mathbf{t}_\mu$.

Proof. (i) This follows immediately from Definition 3.2.8.(i), Definition 3.2.9, Remark 3.2.10, and Definition 1.1.66.

(ii) From $\left[\mu^\#\right] \subseteq [\mu]$ we see that for every $j \in \mathbb{N}$ the j-th row of $[\mu]$ consists of the j-th row of $\left[\mu^\#\right]$ on the left and some lattice points in $[\mu] \setminus \left[\mu^\#\right]$ on the right. According to Definition 3.2.8.(i), statement (i), and because of $\mathbf{t}_{\mu^\#} \in \mathcal{T}^{\mu^\#}_{\mathrm{std}}$ (see Lemma 1.1.68.(i)), the entries in each of these two parts of the j-th row of $\mathbf{t}^{\mu^\#\mu}$ are arranged in ascending order from left to right and furthermore every entry in the left part is smaller than every entry in the right part. This shows that for every $j \in \mathbb{N}$ all entries in the j-th row of $\mathbf{t}^{\mu^\#\mu}$ are arranged in ascending order from left to right. Thus we have

$$\mathbf{t}^{\mu^\#\mu} \in \mathcal{T}^{\mu}_{\mathrm{row\,std}}.$$

Since $\mu^\# = (\mu^\#_1, \mu^\#_2, \ldots)$ is a partition (see Definition 1.1.2.(i)), we also have that for every $j \in \mathbb{N}$ the j-th column of $[\mu]$ consists of the j-th column of $\left[\mu^\#\right]$ as upper part and some lattice points in $[\mu] \setminus \left[\mu^\#\right]$ as lower part. To see this, suppose that the j-th column of $\left[\mu^\#\right]$ is nonempty and put

$$k = \max\left\{ i \in \mathbb{N} \,\middle|\, (i,j) \in \left[\mu^\#\right] \right\}.$$

Because of $(k,j) \in \left[\mu^\#\right]$ we then have $j \leq \mu^\#_k$ and get

$$\forall i \in \{1, \ldots, k-1\} : j \leq \mu^\#_k \leq \mu^\#_i.$$

Thus for every $i \in \{1, \ldots, k-1\}$ the lattice point (i,j) is contained in the j-th column of $\left[\mu^{\#}\right]$. This shows that the j-th column of $[\mu]$ consists above (k,j) only of lattice points in $\left[\mu^{\#}\right]$. Again according to the construction of $\mathbf{t}^{\mu^{\#}\mu}$, statement (i), and because $\mathbf{t}_{\mu^{\#}}$ is standard, the entries of $\mathbf{t}^{\mu^{\#}\mu}$ in this upper part of the j-th column of $[\mu]$ are arranged in ascending order from top to bottom and furthermore each one of these entries is smaller than every entry in the lower part. In addition, we get from the construction of the map $\mathbf{t}^{\mu^{\#}\mu}$ on the set $[\mu] \setminus \left[\mu^{\#}\right]$ in Definition 3.2.8.(i) that the entries of $\mathbf{t}^{\mu^{\#}\mu}$ in the lower part of the j-th column of $[\mu]$ — consisting entirely of lattice points in $[\mu] \setminus \left[\mu^{\#}\right]$ — also are arranged in ascending order from top to bottom. Thus all entries in the j-th column of $\mathbf{t}^{\mu^{\#}\mu}$ are arranged in ascending order from top to bottom. This shows that $\mathbf{t}^{\mu^{\#}\mu}$ is column standard.

All in all we now have

$$\mathbf{t}^{\mu^{\#}\mu} \in \mathcal{T}^{\mu}_{\mathrm{std}},$$

as desired.

(iii) This follows immediately from statement (ii). See also Remark 1.1.61.

(iv) This is obtained from statement (iii), Lemma 3.1.8, Definition 3.1.6, and finally Definition 3.2.8.(iii).

(v) This follows immediately from Definition 3.2.8.(ii), Definition 3.2.8.(iii), and Definition 1.1.67. ■

The following definition makes use of Definition 1.1.45.(ii) and Definition 1.1.66.

Definition 3.2.12 *For a given PK_n-pair $\mu^{\#}\mu$ with $\mu^{\#}\mu \neq 00$ we put*

$$U_{\mu^{\#}\mu} = \left\{ w \in \mathfrak{S}_n \;\middle|\; \begin{array}{r} \forall (i,j) \in [\mu] \setminus \left[\mu^{\#}\right] : (i,j)\mathbf{t}^{\mu^{\#}\mu} w = (i,j)\mathbf{t}^{\mu^{\#}\mu}; \\ \forall (i,j) \in \left[\mu^{\#}\right] : \Big((i,j)\mathbf{t}^{\mu^{\#}\mu} (\mathbf{t}^{\mu^{\#}\mu} w)^{-1} \in \left[\mu^{\#}\right] \wedge \\ ((i,j)\mathbf{t}^{\mu^{\#}\mu})\sigma_{\mathbf{t}^{\mu^{\#}\mu} w} = j \Big) \end{array} \right\}$$

and

$$V_{\mu^{\#}\mu} = \left\{ w \in \mathfrak{S}_n \;\middle|\; \begin{array}{r} \forall (i,j) \in [\mu] \setminus \left[\mu^{\#}\right] : (i,j)\mathbf{t}_{\mu} w = (i,j)\mathbf{t}_{\mu}; \\ \forall (i,j) \in \left[\mu^{\#}\right] : \Big((i,j)\mathbf{t}_{\mu} (\mathbf{t}_{\mu} w)^{-1} \in \left[\mu^{\#}\right] \wedge \\ ((i,j)\mathbf{t}_{\mu})\sigma_{\mathbf{t}_{\mu} w} = j \Big) \end{array} \right\}.$$

Remark 3.2.13 *Let $\mu^{\#}\mu$ be a PK_n-pair with $\mu^{\#}\mu \neq 00$. Then $U_{\mu^{\#}\mu}$ and $V_{\mu^{\#}\mu}$ can* *be described less formally as follows.*

$$U_{\mu^{\#}\mu} = \left\{ w \in \mathfrak{S}_n \;\middle|\; \begin{array}{c} \textit{w fixes the entries in } \mathbf{t}^{\mu^{\#}\mu} \\ \textit{at positions in } [\mu] \setminus \left[\mu^{\#}\right] \\ \textit{and permutes the entries in each column} \\ \textit{of } \mathbf{t}^{\mu^{\#}\mu}\Big\downarrow^{[\mu]}_{\left[\mu^{\#}\right]} \textit{ amongst themselves} \end{array} \right\}$$

$$V_{\mu^{\#}\mu} = \left\{ w \in \mathfrak{S}_n \;\middle|\; \begin{array}{c} \textit{w fixes the entries in } \mathbf{t}_{\mu} \\ \textit{at positions in } [\mu] \setminus \left[\mu^{\#}\right] \\ \textit{and permutes the entries in each column} \\ \textit{of } \mathbf{t}_{\mu}\Big\downarrow^{[\mu]}_{\left[\mu^{\#}\right]} \textit{ amongst themselves} \end{array} \right\}$$

The next statement uses (1.2) on page 1, Definition 1.1.1.(ii), Remark 1.1.12, Definition 1.1.55, Definition 1.1.58.(i), and Definition 1.1.67.

Lemma 3.2.14 *Let $\mu^{\#}\mu$ be a PK_n-pair with $\mu^{\#}\mu \neq 00$, $\mu = (\mu_1, \mu_2, \ldots) \vDash n$, and $\mu^{\#} = (\mu^{\#}_1, \mu^{\#}_2, \ldots) \vdash k$ for a $k \in \{1, \ldots, n\}$. Then we also have $\mu^{\#}_1 = \mu_1$ and furthermore $\mu^{\#\prime} = (\mu^{\#\prime}_1, \ldots, \mu^{\#\prime}_{\mu_1}) \vdash k$ with $\mu^{\#\prime}_{\mu_1} > 0$. With that, the following statements hold.*

(i) Put for $j \in \{1, \ldots, \mu_1\}$

$$U^{(j)}_{\mu^{\#}\mu} = \mathfrak{S}_{\left\{\mu^{\#\prime+}_{j-1}+1, \ldots, \mu^{\#\prime+}_{j}\right\}} \subseteq \mathfrak{S}_n. \tag{3.4}$$

Then we have

$$U_{\mu^{\#}\mu} = U^{(1)}_{\mu^{\#}\mu} \times \cdots \times U^{(\mu_1)}_{\mu^{\#}\mu} \subseteq \mathfrak{S}_n. \tag{3.5}$$

The length function on $\mathfrak{S}_n$ is additive with respect to this decomposition of $U_{\mu^{\#}\mu}$ into a direct product. This means that for an $x \in U_{\mu^{\#}\mu}$ and its uniquely determined decomposition

$$x = x_1 \cdots x_{\mu_1}$$

with $x_j \in U^{(j)}_{\mu^{\#}\mu}$ for $j \in \{1, \ldots, \mu_1\}$ we have

$$\ell(x) = \ell(x_1) + \cdots + \ell(x_{\mu_1}).$$

Furthermore, if we put

$$\eta = (\mu^{\#\prime}_1, \ldots, \mu^{\#\prime}_{\mu_1}, 1^{n-k}) \vDash n \tag{3.6}$$

then we have

$$U_{\mu^{\#}\mu} = \mathfrak{S}_{\eta}. \tag{3.7}$$

(ii) *For $i \in \mathbb{N}$ we denote by $m_i \in \mathbb{N}_0$ the number of lattice points contained in the i-th column of $[\mu]$. With that we put for $k \in \mathbb{N}_0$*

$$m_k^+ = \sum_{j=1}^{k} m_j.$$

Finally we put for $j \in \{1, \dots, \mu_1\}$

$$V_{\mu^\# \mu}^{(j)} = \mathfrak{S}_{\left\{m_{j-1}^+ + 1, \dots, m_{j-1}^+ + \mu_j^{\#\prime}\right\}} \subseteq \mathfrak{S}_n. \tag{3.8}$$

Then we have

$$V_{\mu^\# \mu} = V_{\mu^\# \mu}^{(1)} \times \cdots \times V_{\mu^\# \mu}^{(\mu_1)} \subseteq \mathfrak{S}_n. \tag{3.9}$$

The length function on $\mathfrak{S}_n$ is additive with respect to this decomposition of $V_{\mu^\# \mu}$ into a direct product. This means that for a $y \in V_{\mu^\# \mu}$ and its uniquely determined decomposition

$$y = y_1 \cdots y_{\mu_1}$$

with $y_j \in V_{\mu^\# \mu}^{(j)}$ for $j \in \{1, \dots, \mu_1\}$ we have

$$\ell(y) = \ell(y_1) + \cdots + \ell(y_{\mu_1}).$$

Furthermore, if we put

$$\theta = (\mu_1^{\#\prime}, 1^{m_1 - \mu_1^{\#\prime}}, \dots, \mu_{\mu_1}^{\#\prime}, 1^{m_{\mu_1} - \mu_{\mu_1}^{\#\prime}}, 1^{n - m_{\mu_1}^+}) \vDash n \tag{3.10}$$

then we have

$$V_{\mu^\# \mu} = \mathfrak{S}_\theta. \tag{3.11}$$

(iii) *We have*

$$V_{\mu^\# \mu} = g_{\mu^\# \mu}^{-1} U_{\mu^\# \mu} g_{\mu^\# \mu}$$

or equivalently

$$g_{\mu^\# \mu} V_{\mu^\# \mu} = U_{\mu^\# \mu} g_{\mu^\# \mu}.$$

(iv) *Let $w \in U_{\mu^\# \mu}$ and put $\tilde{w} = g_{\mu^\# \mu}^{-1} w g_{\mu^\# \mu}$. Then we have $\tilde{w} \in V_{\mu^\# \mu}$, $g_{\mu^\# \mu} \tilde{w} = w g_{\mu^\# \mu}$, and*

$$\ell(g_{\mu^\# \mu} \tilde{w}) = \ell(g_{\mu^\# \mu}) + \ell(\tilde{w}) = \ell(w g_{\mu^\# \mu}) = \ell(w) + \ell(g_{\mu^\# \mu}).$$

(v) *We have*

$$w_{\mu^\# \mu} U_{\mu^\# \mu} \subseteq \mathcal{D}_\mu.$$

(vi) *We have*

$$w_\mu V_{\mu^\#\mu} \subseteq \mathcal{D}_\mu.$$

(vii) $w_{\mu^\#\mu}^{-1}$ *is the shortest representative of the right coset* $U_{\mu^\#\mu} w_{\mu^\#\mu}^{-1}$ *of the Young subgroup* $U_{\mu^\#\mu}$ *in* $\mathfrak{S}_n$*. In other words, we have, using (3.6) and (3.7) from statement (i),*

$$w_{\mu^\#\mu}^{-1} \in \mathcal{D}_\eta.$$

(viii) w_μ^{-1} *is the shortest representative of the right coset* $V_{\mu^\#\mu} w_\mu^{-1}$ *of the Young subgroup* $V_{\mu^\#\mu}$ *in* $\mathfrak{S}_n$*. In other words, we have, using (3.10) and (3.11) from statement (ii),*

$$w_\mu^{-1} \in \mathcal{D}_\theta.$$

(ix) *Let* $c \in \mathbb{N} \setminus \{1\}$ *be an AR-index for* $\mu^\#\mu$*. Then we have*

$$U_{\mu^\#\mu} = U_{\mu^\#\mu R_c}.$$

(x) *Let* $\lambda^\#\mu$ *be another* PK_n*-pair satisfying* $\lambda^\#\mu \neq 00$ *and*

$$\left[\mu^\#\right] \subseteq \left[\lambda^\#\right].$$

Then there is a set $\mathcal{F} \subseteq \mathfrak{S}_n$ *such that*

$$V_{\lambda^\#\mu} = V_{\mu^\#\mu}\mathcal{F}$$

holds and moreover each $w \in V_{\lambda^\#\mu}$ *has a uniquely determined decomposition*

$$w = uf$$

with $u \in V_{\mu^\#\mu}$ *and* $f \in \mathcal{F}$ *and finally arbitrary* $u \in V_{\mu^\#\mu}$ *and* $f \in \mathcal{F}$ *satisfy*

$$\ell(uf) = \ell(u) + \ell(f).$$

Proof. (i) Since $\mu^\#$ is a partition, $\left[\mu^\#\right]$ has exactly $\mu_1^\# = \mu_1$ nonempty columns (see Definition 1.1.6 and Definition 3.2.1). Definition 3.2.8.(i), Definition 3.2.9, and Remark 3.2.10 show that for every $j \in \{1, \ldots, \mu_1\}$ the j-th column of $\mathbf{t}^{\mu^\#\mu}\Big\downarrow^{[\mu]}_{[\mu^\#]}$ contains the entries

$$\mu_{j-1}^{\#\prime+} + 1, \ldots, \mu_j^{\#\prime+}$$

in that order from top to bottom. Thus, according to Definition 3.2.12 and Remark 3.2.13, the permutations in $U_{\mu^\#\mu}$ can be characterized as those permuting for every $j \in \{1, \ldots, \mu_1\}$ the elements of the set $\left\{\mu_{j-1}^{\#\prime+} + 1, \ldots, \mu_j^{\#\prime+}\right\}$ amongst themselves and fixing all numbers not contained in any one of these pairwise disjoint sets. From this we get the decomposition (3.5) of $U_{\mu^\#\mu}$ into a direct product.

The additivity of the length function with respect to this decomposition is easily obtained from (1.10) on page 3 and the fact that the sets on which the various factors of the direct product operate are pairwise disjoint intervals of integers.

Finally, the relation

$$U_{\mu^\#\mu} = \mathfrak{S}_\eta$$

is obtained by comparing (1.18) on page 25 — applied with the composition η — and (3.4).

(ii) The proof of this statement is for the most part analogous to that of statement (i).

Since $\mu^\#$ is a partition, $\left[\mu^\#\right]$ is composed of μ_1 nonempty columns. Definition 1.1.66, Definition 3.2.9, Remark 3.2.10, and the construction of the m_i with $i \in \mathbb{N}$ and the m_k^+ with $k \in \mathbb{N}_0$ show that for every $j \in \{1, \ldots, \mu_1\}$ the j-th column of $\mathbf{t}_\mu\Big\downarrow_{\left[\mu^\#\right]}^{[\mu]}$ contains the entries

$$m_{j-1}^+ + 1, \ldots, m_{j-1}^+ + \mu_j^{\#\prime}$$

in that order from top to bottom. Thus, according to Definition 3.2.12 and Remark 3.2.13, the permutations in $V_{\mu^\#\mu}$ can be characterized as those permuting for every $j \in \{1, \ldots, \mu_1\}$ the elements of the set $\left\{m_{j-1}^+ + 1, \ldots, m_{j-1}^+ + \mu_j^{\#\prime}\right\}$ amongst themselves and fixing all numbers not contained in any one of these pairwise disjoint sets. From this we get the decomposition (3.9) of $V_{\mu^\#\mu}$ into a direct product.

The additivity of the length function with respect to this decomposition is obtained as in the proof of statement (i).

Finally, we get from the inclusion $\left[\mu^\#\right] \subseteq [\mu]$, Remark 1.1.12, and the construction of the m_i with $i \in \mathbb{N}$ and the m_k^+ with $k \in \mathbb{N}_0$ the relations

$$\forall j \in \{1, \ldots, \mu_1\} : m_j \geq \mu_j^{\#\prime}$$

and also

$$n \geq m_{\mu_1}^+.$$

This shows that θ is indeed a well defined composition of n. Now we get

$$V_{\mu^\#\mu} = \mathfrak{S}_\theta$$

as in the proof of statement (i) by comparing (1.18) on page 25 — applied with the composition θ — and (3.9).

(iii) According to Lemma 3.2.11.(v), we have $\mathbf{t}^{\mu^{\#}\mu} g_{\mu^{\#}\mu} = \mathbf{t}_\mu$. This shows that $g_{\mu^{\#}\mu}$ maps for every $j \in \mathbb{N}$ the entries of the j-th column of $\mathbf{t}^{\mu^{\#}\mu}\Big\downarrow^{[\mu]}_{[\mu^{\#}]}$ bijectively onto the entries of the j-th column of $\mathbf{t}_\mu\Big\downarrow^{[\mu]}_{[\mu^{\#}]}$ and furthermore the entries in $\mathbf{t}^{\mu^{\#}\mu}$ at positions from $[\mu] \setminus [\mu^{\#}]$ bijectively onto the entries in $\mathbf{t}_\mu$ at positions from $[\mu] \setminus [\mu^{\#}]$. The claim follows from these properties of $g_{\mu^{\#}\mu}$ and Remark 3.2.13.

(iv) $\tilde{w} \in V_{\mu^{\#}\mu}$ follows from statement (iii). $g_{\mu^{\#}\mu}\tilde{w} = w g_{\mu^{\#}\mu}$ follows immediately from the construction of $\tilde{w}$. Moreover, we get from $\mathbf{t}^{\mu^{\#}\mu} g_{\mu^{\#}\mu} = \mathbf{t}_\mu$ (see Lemma 3.2.11.(v)), using the notions and considerations from statements (i) and (ii) and their proofs, that $g_{\mu^{\#}\mu}$ maps for every $j \in \{1, \ldots, \mu_1\}$ the entries

$$\mu^{\#\prime+}_{j-1} + 1, \ldots, \mu^{\#\prime+}_{j}$$

of the j-th column of $\mathbf{t}^{\mu^{\#}\mu}\Big\downarrow^{[\mu]}_{[\mu^{\#}]}$ in this order — that is, order preserving — to the entries

$$m^{+}_{j-1} + 1, \ldots, m^{+}_{j-1} + \mu^{\#\prime}_{j}$$

of the j-th column of $\mathbf{t}_\mu\Big\downarrow^{[\mu]}_{[\mu^{\#}]}$. Now the remainder of the claim follows from this property of $g_{\mu^{\#}\mu}$, Remark 3.2.13, and (1.10) on page 3.

(v) Fix a $u \in U_{\mu^{\#}\mu}$. According to Definition 1.1.58.(i) and Definition 3.2.8.(ii), we must show that $\mathbf{t}^{\mu^{\#}\mu}u$ is row standard.

First, Lemma 3.2.11.(ii) and Definition 1.1.45.(iii) show that $\mathbf{t}^{\mu^{\#}\mu}$ is row standard. Moreover, Definition 3.2.8.(i) shows that every entry in $\mathbf{t}^{\mu^{\#}\mu}$ at a position from $[\mu^{\#}]$ is smaller than every entry in $\mathbf{t}^{\mu^{\#}\mu}$ at a position from $[\mu] \setminus [\mu^{\#}]$. Finally, Lemma 3.2.11.(i), Definition 1.1.66, Definition 3.2.9, and Remark 3.2.10 show that for arbitrary $i, j \in \mathbb{N}$ with $i < j$ every entry in the i-th column of $\mathbf{t}^{\mu^{\#}\mu}\Big\downarrow^{[\mu]}_{[\mu^{\#}]}$ is smaller than every entry in the j-th column of $\mathbf{t}^{\mu^{\#}\mu}\Big\downarrow^{[\mu]}_{[\mu^{\#}]}$.

From these properties of $\mathbf{t}^{\mu^{\#}\mu}$ and Remark 3.2.13 we see that $\mathbf{t}^{\mu^{\#}\mu}u$ is row standard. This completes the proof as explained above.

(vi) The proof of this statement is similar to the proof of statement (v).

Fix a $v \in V_{\mu^{\#}\mu}$. According to Definition 1.1.58.(i) and Definition 1.1.67, we must show that $\mathbf{t}_\mu v$ is row standard.

Now we see from Definition 1.1.66 that for arbitrary $i, j \in \mathbb{N}$ with $i < j$ every entry in the i-th column of $\mathbf{t}_\mu$ is smaller than every entry in the j-th column of $\mathbf{t}_\mu$. From this property of $\mathbf{t}_\mu$ and Remark 3.2.13 we see that $\mathbf{t}_\mu v$ is row standard. This completes the proof as explained above.

(vii) According to (3.6) and (3.7) from statement (i), we must show that $\mathbf{t}^{\eta} w_{\mu^{\#}\mu}^{-1}$ is row standard.

To this end, we first compare $\mathbf{t}^{\eta}$ from Definition 1.1.53 and $\mathbf{t}^{\mu^{\#}\mu}$ from Definition 3.2.8.(i). We easily get from the definitions of these tableaux that for every $j \in \{1, \ldots, \mu_1\}$ the sequence of the entries in the j-th row of $\mathbf{t}^{\eta}$ when considered from left to right coincides with the sequence of the entries in the j-th column of $\mathbf{t}^{\mu^{\#}\mu}\Big\downarrow^{[\mu]}_{[\mu^{\#}]}$ when considered from top to bottom (see also Definition 3.2.9 and Remark 3.2.10).

Now an application of $w_{\mu^{\#}\mu}^{-1}$ (see Definition 3.2.8.(ii)) to the relations described in the preceding paragraph shows that for every $j \in \{1, \ldots, \mu_1\}$ the sequence of the entries in the j-th row of $\mathbf{t}^{\eta} w_{\mu^{\#}\mu}^{-1}$ when considered from left to right coincides with the sequence of the entries in the j-th column of

$$\left(\mathbf{t}^{\mu^{\#}\mu}\Big\downarrow^{[\mu]}_{[\mu^{\#}]}\right) w_{\mu^{\#}\mu}^{-1} = \left(\mathbf{t}^{\mu^{\#}\mu} w_{\mu^{\#}\mu}^{-1}\right)\Big\downarrow^{[\mu]}_{[\mu^{\#}]} = \mathbf{t}^{\mu}\Big\downarrow^{[\mu]}_{[\mu^{\#}]}$$

when considered from top to bottom. Since $\mathbf{t}^{\mu}$ is column standard, we get from this that for every $j \in \{1, \ldots, \mu_1\}$ the entries in the j-th row of $\mathbf{t}^{\eta} w_{\mu^{\#}\mu}^{-1}$ when considered from left to right are arranged in ascending order.

Furthermore, we see from (3.6) that every other row of $\mathbf{t}^{\eta} w_{\mu^{\#}\mu}^{-1}$ contains at most one entry.

All in all, we get that $\mathbf{t}^{\eta} w_{\mu^{\#}\mu}^{-1}$ is row standard. This proves the claim as described above.

(viii) The proof of this statement is similar to the proof of statement (vii).

According to (3.10) and (3.11) from statement (ii), we must show that $\mathbf{t}^{\theta} w_{\mu}^{-1}$ is row standard.

To this end, we decompose θ in $\mu_1 + 1$ successive subsequences. Using the notation from statement (ii), we define for every $j \in \{1, \ldots, \mu_1\}$ the j-th subsequence as

$$(\mu_j^{\#\prime}, 1^{m_j - \mu_j^{\#\prime}}).$$

The $(\mu_1 + 1)$-th subsequence is defined as

$$(1^{n - m^{+}_{\mu_1}}, 0, 0, 0, \ldots).$$

The concatenation of these sequences in ascending order according to their numbering results in θ (see (3.10)).

Now we compare $\mathbf{t}^{\theta}$ from Definition 1.1.53 and $\mathbf{t}_{\mu}$ from Definition 1.1.66. We easily get from the definitions of these tableaux and the construction of θ that for every $j \in \{1, \ldots, \mu_1\}$ the sequence of the entries in the row of $\mathbf{t}^{\theta}$ corresponding to the first entry in the j-th subsequence of θ when considered from left to right coincides with the sequence of the entries in the j-th column of $\mathbf{t}_\mu \big\downarrow^{[\mu]}_{[\mu^\#]}$ when considered from top to bottom (see also Definition 3.2.9 and Remark 3.2.10).

Now an application of w_μ^{-1} (see Definition 1.1.67) to the relations described in the preceding paragraph shows that for every $j \in \{1, \ldots, \mu_1\}$ the sequence of the entries in the row of $\mathbf{t}^\theta w_\mu^{-1}$ corresponding to the first entry in the j-th subsequence of θ when considered from left to right coincides with the sequence of the entries in the j-th column of

$$\left(\mathbf{t}_\mu \big\downarrow^{[\mu]}_{[\mu^\#]}\right) w_\mu^{-1} = \left(\mathbf{t}_\mu w_\mu^{-1}\right) \big\downarrow^{[\mu]}_{[\mu^\#]} = \mathbf{t}^\mu \big\downarrow^{[\mu]}_{[\mu^\#]}$$

when considered from top to bottom. Since $\mathbf{t}^\mu$ is column standard, we get from this that for every $j \in \{1, \ldots, \mu_1\}$ the entries in the row of $\mathbf{t}^\theta w_\mu^{-1}$ corresponding to the first entry in the j-th subsequence of θ when considered from left to right are arranged in ascending order.

Furthermore, we see from (3.10) that every other row of $\mathbf{t}^\theta w_\mu^{-1}$ contains at most one entry.

All in all, we get that $\mathbf{t}^\theta w_\mu^{-1}$ is row standard. This proves the claim as described above.

(ix) In order to prove this statement, we distinguish the cases $c > 2$ and $c = 2$.

First we consider the case $c > 2$. Then we see from Definition 3.2.5.(iii) that the PK$_n$-pairs $\mu^\#\mu$ and $\mu^\#\mu R_c$ contain the same partition. This partition is denoted by ν in the following. With that, we obtain from Definition 3.2.8.(i), Definition 3.2.9, and Lemma 3.2.11.(i)

$$\mathbf{t}^{\mu^\#\mu} \big\downarrow^{[\mu]}_{[\nu]} = \mathbf{t}_\nu = \mathbf{t}^{\mu^\#\mu R_c} \big\downarrow^{[\mu R_c]}_{[\nu]}.$$

This shows together with Definition 3.2.12 and Remark 3.2.13

$$U_{\mu^\#\mu} = U_{\mu^\#\mu R_c} \quad \text{for} \quad c > 2.$$

Now we consider the case $c = 2$. Then we see from Definition 3.2.5.(iii) that the PK$_n$-pairs $\mu^\#\mu$ and $\mu^\#\mu R_2$ contain different partitions. If we denote the partition contained in $\mu^\#\mu$ by α and the partition contained in $\mu^\#\mu R_2$ by β then β is obtained from α by increasing the first entry. According to Remark 1.1.12, this implies that the transposed partition β' is obtained from α' by appending some entries with the value 1. These new entries of β' contribute only trivial factors to the decomposition of $U_{\mu^\#\mu R_2}$ into a direct product as described in statement (i). Since the remaining entries of β' are identical to the entries of α', the remaining factors in the decomposition of $U_{\mu^{\#}\mu R_2}$ into a direct product as described in statement (i) are identical to the factors in the analogous decomposition of $U_{\mu^{\#}\mu}$. Thus we have

$$U_{\mu^{\#}\mu} = U_{\mu^{\#}\mu R_2}.$$

Now the claim is proved for all possible values of c.

(x) If we write $\lambda^{\#} = (\lambda_1^{\#}, \lambda_2^{\#}, \ldots)$, we get from Definition 3.2.1

$$\lambda_1^{\#} = \mu_1 = \mu_1^{\#}.$$

According to Remark 1.1.12, we also have $\lambda^{\#\prime} = (\lambda_1^{\#\prime}, \ldots, \lambda_{\mu_1}^{\#\prime})$ with $\lambda_{\mu_1}^{\#\prime} > 0$. In particular, $\lambda^{\#\prime}$ and $\mu^{\#\prime}$ have the same number of positive entries. Furthermore, we get from the condition $\left[\mu^{\#}\right] \subseteq \left[\lambda^{\#}\right]$, Definition 1.1.9, and Definition 1.1.11

$$\left[\mu^{\#\prime}\right] \subseteq \left[\lambda^{\#\prime}\right].$$

According to Definition 1.1.6.(i), this implies

$$\forall j \in \{1, \ldots, \mu_1\} : \mu_j^{\#\prime} \leq \lambda_j^{\#\prime}. \tag{3.12}$$

If we now consider the decompositions

$$V_{\lambda^{\#}\mu} = V_{\lambda^{\#}\mu}^{(1)} \times \cdots \times V_{\lambda^{\#}\mu}^{(\mu_1)} \subseteq \mathfrak{S}_n \tag{3.13}$$

and

$$V_{\mu^{\#}\mu} = V_{\mu^{\#}\mu}^{(1)} \times \cdots \times V_{\mu^{\#}\mu}^{(\mu_1)} \subseteq \mathfrak{S}_n \tag{3.14}$$

of $V_{\lambda^{\#}\mu}$ and $V_{\mu^{\#}\mu}$ with factors $V_{\lambda^{\#}\mu}^{(j)}$ and $V_{\mu^{\#}\mu}^{(j)}$ for $j \in \{1, \ldots, \mu_1\}$ from statement (ii), we get from (3.12)

$$\forall j \in \{1, \ldots, \mu_1\} : V_{\mu^{\#}\mu}^{(j)} \subseteq V_{\lambda^{\#}\mu}^{(j)}$$

and thus

$$V_{\mu^{\#}\mu} \subseteq V_{\lambda^{\#}\mu}.$$

More specifically, we have that for every $j \in \{1, \ldots, \mu_1\}$ the group $V_{\mu^{\#}\mu}^{(j)}$ is the Young subgroup associated with $\nu^{(j)} = (\mu_j^{\#\prime}, 1^{\lambda_j^{\#\prime} - \mu_j^{\#\prime}}) \vdash \lambda_j^{\#\prime}$ of the symmetric group $V_{\lambda^{\#}\mu}^{(j)}$ of degree $\lambda_j^{\#\prime}$ (see (1.18) on page 25). Thus there is for every $j \in \{1, \ldots, \mu_1\}$ the set of shortest representatives of all right cosets of $V_{\mu^{\#}\mu}^{(j)}$ in $V_{\lambda^{\#}\mu}^{(j)}$. If we denote this set by $\mathcal{F}^{(j)}$, we have

$$\forall j \in \{1, \ldots, \mu_1\} : V_{\lambda^{\#}\mu}^{(j)} = V_{\mu^{\#}\mu}^{(j)} \mathcal{F}^{(j)}. \tag{3.15}$$

Since the factors in the decomposition (3.13) of $V_{\lambda\#\mu}$ commute, substitution of (3.15) leads to

$$\begin{aligned} V_{\lambda\#\mu} &= \left(V^{(1)}_{\mu\#\mu}\mathcal{F}^{(1)}\right)\cdots\left(V^{(\mu_1)}_{\mu\#\mu}\mathcal{F}^{(\mu_1)}\right) \\ &= \left(V^{(1)}_{\mu\#\mu}\cdots V^{(\mu_1)}_{\mu\#\mu}\right)\left(\mathcal{F}^{(1)}\cdots\mathcal{F}^{(\mu_1)}\right). \end{aligned} \tag{3.16}$$

Thus if we put

$$\mathcal{F} = \mathcal{F}^{(1)} \times \cdots \times \mathcal{F}^{(\mu_1)} \subseteq V_{\lambda\#\mu} \subseteq \mathfrak{S}_n \tag{3.17}$$

then we get from (3.16) and (3.14)

$$V_{\lambda\#\mu} = V_{\mu\#\mu}\mathcal{F}. \tag{3.18}$$

Being a product of the systems of representatives $\mathcal{F}^{(j)}$ for the right cosets of the direct factors $V^{(j)}_{\mu\#\mu}$ of $V_{\mu\#\mu}$ in the corresponding direct factors $V^{(j)}_{\lambda\#\mu}$ of $V_{\lambda\#\mu}$ for $j \in \{1,\ldots,\mu_1\}$, $\mathcal{F}$ is a system of representatives for the right cosets of $V_{\mu\#\mu}$ in $V_{\lambda\#\mu}$. This implies that if we have a decomposition

$$w = uf$$

of an arbitrary $w \in V_{\lambda\#\mu}$ into a product of an appropriate $u \in V_{\mu\#\mu}$ and an appropriate $f \in \mathcal{F}$ then the factors u and f are uniquely determined.

Finally, choose an arbitrary $u \in V_{\mu\#\mu}$ and an arbitrary $f \in \mathcal{F} \subseteq V_{\lambda\#\mu}$. Then we have, according to (3.13), (3.14), and (3.17), uniquely determined decompositions

$$u = u_1 \cdots u_{\mu_1} \qquad \text{and} \qquad f = f_1 \cdots f_{\mu_1}$$

with $u_j \in V^{(j)}_{\mu\#\mu}$ and $f_j \in \mathcal{F}^{(j)} \subseteq V^{(j)}_{\lambda\#\mu}$ for $j \in \{1,\ldots,\mu_1\}$. As shown in statement (ii), these decompositions satisfy

$$\ell(u) = \ell(u_1) + \cdots + \ell(u_{\mu_1}) \quad \text{and} \quad \ell(f) = \ell(f_1) + \cdots + \ell(f_{\mu_1}). \tag{3.19}$$

From these decompositions we get

$$uf = (u_1 \cdots u_{\mu_1})(f_1 \cdots f_{\mu_1}) = (u_1 f_1) \cdots (u_{\mu_1} f_{\mu_1}) \tag{3.20}$$

(see (3.16)). Now (3.18) and (3.15) show that both

$$uf \in V_{\lambda\#\mu} \qquad \text{and} \qquad \forall j \in \{1,\ldots,\mu_1\} : u_j f_j \in V^{(j)}_{\lambda\#\mu}$$

hold. Thus (3.20) is the uniquely determined decomposition of $uf \in V_{\lambda\#\mu}$ into a product of elements of the factors $V^{(j)}_{\lambda\#\mu}$ of $V_{\lambda\#\mu}$ for $j \in \{1,\ldots,\mu_1\}$. From this fact, statement (ii), the construction of the $\mathcal{F}^{(j)}$ for $j \in \{1, \ldots, \mu_1\}$, and (3.19) we get

$$\begin{aligned} \ell(uf) &= \ell(u_1 f_1) + \cdots + \ell(u_{\mu_1} f_{\mu_1}) \\ &= (\ell(u_1) + \ell(f_1)) + \cdots + (\ell(u_{\mu_1}) + \ell(f_{\mu_1})) \\ &= (\ell(u_1) + \cdots + \ell(u_{\mu_1})) + (\ell(f_1) + \cdots + \ell(f_{\mu_1})) \\ &= \ell(u) + \ell(f). \end{aligned}$$

Now all claims in statement (x) are proved. ■

The Young subgroups $U_{\mu^\# \mu}$ and $V_{\mu^\# \mu}$ of $\mathfrak{S}_n$ with a PK$_n$-pair $\mu^\# \mu \neq 00$ are used in Section 3.4 to construct new Hecke algebra modules.

The following definitions and statements will be used in Section 3.5 to construct module homomorphisms. The next definition makes use of (1.2) on page 1 and Definition 1.1.1.(ii).

Definition 3.2.15 *Let $\mu^\# \mu$ be a PK$_n$-pair with $\mu^\# \mu \neq 00$, $\mu = (\mu_1, \mu_2, \ldots) \vDash n$, and $\mu^\# = (\mu_1^\#, \mu_2^\#, \ldots) \vdash k$ for a $k \in \{1, \ldots, n\}$. Furthermore, let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^\# \mu$. Then we define the set $I_{\mu^\# \mu c} \subseteq \mathfrak{S}_n$ as*

$$I_{\mu^\# \mu c} = \left\{ w \in \mathfrak{S}_n \;\middle|\; \begin{array}{c} w \in \mathfrak{S}_{\left\{\mu_{c-1}^+ + 1, \ldots, \mu_c^+\right\}}; \\ \left(\mu_{c-1}^+ + 1\right) w < \cdots < \left(\mu_{c-1}^+ + \mu_c - \mu_c^\#\right) w; \\ \left(\mu_{c-1}^+ + \mu_c - \mu_c^\# + 1\right) w < \cdots < \mu_c^+ w \end{array} \right\}.$$

Remark 3.2.16 *Let $\mu^\# \mu$ and c be as in Definition 3.2.15. Then the set $I_{\mu^\# \mu c}$ introduced therein can be characterized less formally as follows.*

$$I_{\mu^\# \mu c} = \left\{ w \in \mathfrak{S}_n \;\middle|\; \begin{array}{c} \textit{w only permutes the entries in the} \\ \textit{c-th row of } \mathbf{t}^\mu \textit{ amongst themselves such that} \\ \textit{in the c-th row of } \mathbf{t}^\mu w \textit{ the left } \mu_c - \mu_c^\# \textit{ entries} \\ \textit{and the right } \mu_c^\# \textit{ entries respectively are} \\ \textit{arranged in ascending order from left to right} \end{array} \right\}$$

Lemma 3.2.17 *Let $\mu^\# \mu$ be a PK$_n$-pair with $\mu^\# \mu \neq 00$ and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for it. Then the set $I_{\mu^\# \mu c}$ introduced in Definition 3.2.15 satisfies*

(i) $I_{\mu^\# \mu c} \subseteq \mathfrak{S}_\mu$,

(ii) $I_{\mu^\# \mu c} = (\mathfrak{S}_{\mu R_c} \mathfrak{S}_\mu) \cap \mathcal{D}_{\mu R_c}$.

Proof. (i) This follows from Definition 3.2.15 and the product decomposition (1.18) of Young subgroups on page 25.

(ii) First we consider the inclusion $I_{\mu^\# \mu c} \subseteq (\mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu) \cap \mathcal{D}_{\mu R_c}$. From statement (i) we get

$$I_{\mu^\# \mu c} \subseteq \mathfrak{S}_\mu \subseteq \mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu.$$

In order to obtain the inclusion $I_{\mu^\# \mu c} \subseteq \mathcal{D}_{\mu R_c}$, we consider the operation of an arbitrarily chosen $f \in I_{\mu^\# \mu c}$ on the tableau $\mathbf{t}^{\mu R_c}$. If we write $\mu = (\mu_1, \mu_2, \ldots)$ and $\mu^\# = (\mu_1^\#, \mu_2^\#, \ldots)$ then we get from Definition 3.2.5.(ii) $\mu R_c = ((\mu R_c)_1, (\mu R_c)_2, \ldots) \vDash n$ with

$$(\mu R_c)_j = \begin{cases} \mu_{c-1} + \mu_c - \mu_c^\# & \text{for} \quad j = c-1 \\ \mu_c^\# & \text{for} \quad j = c \\ \mu_j & \text{for} \quad j \in \mathbb{N} \setminus \{c-1, c\} \end{cases}.$$

Thus the $(c-1)$-th row and the c-th row of $\mathbf{t}^{\mu R_c}$ have the following form (see also Definition 1.1.1.(ii)).

$$\begin{array}{rl} (c-1)\text{-th row}: & \mu_{c-2}^+ + 1, \ldots, \mu_{c-1}^+, \mu_{c-1}^+ + 1, \ldots, \mu_{c-1}^+ + \mu_c - \mu_c^\# \\ c\text{-th row}: & \mu_{c-1}^+ + \mu_c - \mu_c^\# + 1, \ldots, \mu_c^+ \end{array} \tag{3.21}$$

All elements of $\{1, \ldots, n\}$ not fixed by f are contained in these two rows of $\mathbf{t}^{\mu R_c}$. From the conditions on the arrangement of the images of these elements under f we easily get that $\mathbf{t}^{\mu R_c} f$ is row standard. This shows $f \in \mathcal{D}_{\mu R_c}$ and furthermore

$$I_{\mu^\# \mu c} \subseteq \mathcal{D}_{\mu R_c}$$

and finally

$$I_{\mu^\# \mu c} \subseteq (\mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu) \cap \mathcal{D}_{\mu R_c}.$$

Now we will show the inclusion $(\mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu) \cap \mathcal{D}_{\mu R_c} \subseteq I_{\mu^\# \mu c}$. To this end, we consider the operation of an arbitrarily chosen $f \in (\mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu) \cap \mathcal{D}_{\mu R_c}$ with a product decomposition

$$f = xy \tag{3.22}$$

with $x \in \mathfrak{S}_{\mu R_c}$ and $y \in \mathfrak{S}_\mu$ on $\mathbf{t}^{\mu R_c}$ and $\mathbf{t}^\mu$. Since μR_c and μ differ only in the entries with indices $c-1$ and c and these entries satisfy

$$(\mu R_c)_{c-1} + (\mu R_c)_c = \left(\mu_{c-1} + \mu_c - \mu_c^\#\right) + \left(\mu_c^\#\right) = \mu_{c-1} + \mu_c,$$

we see that $[\mu R_c]$ and $[\mu]$ and thus also $\mathbf{t}^{\mu R_c}$ and $\mathbf{t}^\mu$ are identical except for the rows $c-1$ and c. This shows that first x operating on $\mathbf{t}^{\mu R_c}$ and then y operating on $\mathbf{t}^{\mu R_c} x$ leaves every entry not contained in row $c-1$ or row c within its row. Now (3.22) shows that this also is true for f operating on $\mathbf{t}^{\mu R_c}$. Moreover, both $\mathbf{t}^{\mu R_c}$ and $\mathbf{t}^{\mu R_c} f$ are row standard, since $f \in \mathcal{D}_{\mu R_c}$. All this shows that every entry in $\mathbf{t}^{\mu R_c} f$

not contained in row c or row $c-1$ occupies the same position as it does in $\mathbf{t}^{\mu R_c}$. This means that f fixes every one of these entries. Thus we get from (3.21) and (1.2) on page 1

$$f \in \mathfrak{S}_{\{\mu^+_{c-2}+1,\ldots,\mu^+_c\}}. \tag{3.23}$$

Next we put

$$\begin{aligned} L &= \left\{\mu^+_{c-2}+1,\ldots,\mu^+_{c-1}\right\}, \\ M &= \left\{\mu^+_{c-1}+1,\ldots,\mu^+_{c-1}+\mu_c-\mu^\#_c\right\}, \\ N &= \left\{\mu^+_{c-1}+\mu_c-\mu^\#_c+1,\ldots,\mu^+_c\right\}. \end{aligned}$$

With that, we see from (3.21) that $x \in \mathfrak{S}_{\mu R_c} = \mathfrak{R}_{\mathbf{t}^{\mu R_c}}$ permutes the elements of the sets $L \cup M$ and N respectively amongst themselves and leaves them in their respective rows in $\mathbf{t}^{\mu R_c}$. According to Definition 1.1.53, the elements of L, M, and N are exactly the entries in the $(c-1)$-th row and the c-th row of $\mathbf{t}^\mu$, which are displayed in the following diagram.

$$\begin{aligned} (c-1)\text{-th row}: &\quad \mu^+_{c-2}+1,\ldots,\mu^+_{c-1} \\ c\text{-th row}: &\quad \mu^+_{c-1}+1,\ldots,\mu^+_{c-1}+\mu_c-\mu^\#_c,\mu^+_{c-1}+\mu_c-\mu^\#_c+1,\ldots,\mu^+_c \end{aligned}$$

From this we see that $y \in \mathfrak{S}_\mu = \mathfrak{R}_{\mathbf{t}^\mu}$ permutes the elements of the sets L and $M \cup N$ respectively amongst themselves. All this shows together with (3.22) and (3.23) that the application of f to $\mathbf{t}^{\mu R_c}$ leaves the elements of L in the $(c-1)$-th row and rearranges the elements of $M \cup N$ within the $(c-1)$-th row and the c-th row. Thus the entries in the $(c-1)$-th row of $\mathbf{t}^{\mu R_c} f$ are composed of all elements of the set L and $\mu_c - \mu^\#_c$ elements of the set $M \cup N$, and moreover the c-th row of $\mathbf{t}^{\mu R_c} f$ consists of the remaining $\mu^\#_c$ elements of the set $M \cup N$. Now since $\mathbf{t}^{\mu R_c} f$ is row standard and every element of L is smaller than every element of $M \cup N$, the elements of the set L in the $(c-1)$-th row of $\mathbf{t}^{\mu R_c} f$ must occupy the leftmost positions in this row and must be arranged in ascending order from left to right, exactly as they are in the $(c-1)$-th row of $\mathbf{t}^{\mu R_c}$. This implies that f fixes each of these elements individually. From that and (3.23) we get

$$f \in \mathfrak{S}_{\{\mu^+_{c-1}+1,\ldots,\mu^+_c\}}. \tag{3.24}$$

Again since $\mathbf{t}^{\mu R_c} f$ is row standard, the rightmost $\mu_c - \mu^\#_c$ entries in the $(c-1)$-th row of $\mathbf{t}^{\mu R_c} f$ and also all entries in the c-th row of $\mathbf{t}^{\mu R_c} f$ respectively must be arranged in ascending order from left to right. This fact and (3.21) together imply

$$\left(\mu^+_{c-1}+1\right) f < \cdots < \left(\mu^+_{c-1}+\mu_c-\mu^\#_c\right) f \tag{3.25}$$

and

$$\left(\mu_{c-1}^{+} + \mu_c - \mu_c^{\#} + 1\right) f < \cdots < \mu_c^{+} f. \tag{3.26}$$

(3.24), (3.25), and (3.26) now show $f \in I_{\mu^{\#}\mu c}$ and furthermore

$$(\mathfrak{S}_{\mu R_c}\mathfrak{S}_{\mu}) \cap \mathcal{D}_{\mu R_c} \subseteq I_{\mu^{\#}\mu c}.$$

Thus we have all in all

$$I_{\mu^{\#}\mu c} = (\mathfrak{S}_{\mu R_c}\mathfrak{S}_{\mu}) \cap \mathcal{D}_{\mu R_c},$$

as desired. ■

The next lemma makes use of representations of tableaux and compositions as described in Section 1.1. It constructs from a given representation of a tableau the representation of another tableau by moving around entries or equivalently the squares containing them.

Lemma 3.2.18 *Let $\mu^{\#}\mu$ be a PK_n-pair with $\mu^{\#}\mu \neq 00$ and write $\mu = (\mu_1, \mu_2, \ldots)$ and $\mu^{\#} = (\mu_1^{\#}, \mu_2^{\#}, \ldots)$. Furthermore, let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^{\#}\mu$. Finally, choose a $w \in \mathfrak{S}_n$. Then the tableau $\mathbf{t}^{\mu R_c}w$ is obtained from the tableau $\mathbf{t}^{\mu}w$ by moving the leftmost $\mu_c - \mu_c^{\#}$ entries of the c-th row in the given order to the end of the $(c-1)$-th row and moving the remaining $\mu_c^{\#}$ entries of the c-th row in the given order to the beginning of the c-th row.*

Proof. From Definition 1.1.53 and Definition 3.2.5.(ii) we get the following construction of $\mathbf{t}^{\mu R_c}$ from $\mathbf{t}^{\mu}$.

> $\mathbf{t}^{\mu R_c}$ is obtained from $\mathbf{t}^{\mu}$ by moving the leftmost $\mu_c - \mu_c^{\#}$ entries of the c-th row in the given order to the end of the $(c-1)$-th row and moving the remaining $\mu_c^{\#}$ entries of the c-th row in the given order to the beginning of the c-th row. (3.27)

The displacement of squares labelled with certain entries is compatible with the operation of permutations on these entries. Thus an application of w to (3.27) yields the following construction of $\mathbf{t}^{\mu R_c}w$ from $\mathbf{t}^{\mu}w$.

> $\mathbf{t}^{\mu R_c}w$ is obtained from $\mathbf{t}^{\mu}w$ by moving the leftmost $\mu_c - \mu_c^{\#}$ entries of the c-th row in the given order to the end of the $(c-1)$-th row and moving the remaining $\mu_c^{\#}$ entries of the c-th row in the given order to the beginning of the c-th row.

This completes the proof. ■

The following lemma is similar to the preceding one. It also makes use of Definition 3.2.5.(ii).

Lemma 3.2.19 *Let $\mu^{\#}\mu$ be a PK_n-pair with $\mu^{\#}\mu \neq 00$ and write $\mu = (\mu_1, \mu_2, \ldots)$ and $\mu^{\#} = (\mu_1^{\#}, \mu_2^{\#}, \ldots)$. Furthermore, let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^{\#}\mu$. Finally, choose an $f \in I_{\mu^{\#}\mu c}$ and a $w \in \mathfrak{S}_n$. With that, write*

$$fw = ud$$

with uniquely determined permutations $u \in \mathfrak{S}_{\mu R_c}$ and $d \in \mathcal{D}_{\mu R_c}$. Then the tableau $\mathbf{t}^{\mu R_c} d$ is obtained from the tableau $\mathbf{t}^{\mu} w$ by moving $\mu_c - \mu_c^{\#}$ appropriate entries of the c-th row to the end of the $(c-1)$-th row, moving the remaining $\mu_c^{\#}$ entries of the c-th row to the beginning of the c-th row, and arranging the entries of the various rows in ascending order from left to right.

Proof. According to Definition 3.2.15 and Remark 3.2.16, an application of $f \in I_{\mu^{\#}\mu c}$ to $\mathbf{t}^{\mu}$ permutes only the entries in the c-th row amongst themselves and fixes the entries in all other rows. This implies that $\mathbf{t}^{\mu} fw$ and $\mathbf{t}^{\mu} w$ differ only by a permutation of the entries in the c-th row. From this fact and Lemma 3.2.18 — applied to $\mathbf{t}^{\mu} fw$ in order to obtain $\mathbf{t}^{\mu R_c} fw$ — we get the following construction of $\mathbf{t}^{\mu R_c} fw$ from $\mathbf{t}^{\mu} w$.

> $\mathbf{t}^{\mu R_c} fw$ is obtained from $\mathbf{t}^{\mu} w$ by moving $\mu_c - \mu_c^{\#}$ appropriate entries of the c-th row to the end of the $(c-1)$-th row and moving the remaining $\mu_c^{\#}$ entries of the c-th row to the beginning of the c-th row. (3.28)

Furthermore, an application of $u \in \mathfrak{S}_{\mu R_c}$ to $\mathbf{t}^{\mu R_c}$ leaves each entry in its respective row. This implies that $\mathbf{t}^{\mu R_c} ud = \mathbf{t}^{\mu R_c} fw$ and $\mathbf{t}^{\mu R_c} d$ differ only by a permutation of the entries in the various rows respectively amongst themselves. Moreover, because of $d \in \mathcal{D}_{\mu R_c}$, the entries in the various rows of $\mathbf{t}^{\mu R_c} d$ are arranged in ascending order from left to right. From all this and (3.28) we get the following construction of $\mathbf{t}^{\mu R_c} d$ from $\mathbf{t}^{\mu} w$.

> $\mathbf{t}^{\mu R_c} d$ is obtained from $\mathbf{t}^{\mu} w$ by moving $\mu_c - \mu_c^{\#}$ appropriate entries of the c-th row to the end of the $(c-1)$-th row, moving the remaining $\mu_c^{\#}$ entries of the c-th row to the beginning of the c-th row, and arranging the entries of the various rows in ascending order from left to right.

This completes the proof. ■

The next Lemma makes use of Lemma 1.1.8.(i), Remark 1.1.12, Definition 1.1.45.(i), Definition 1.1.45.(ii), and in particular Definition 3.2.5.(iv) and Lemma 3.2.7.(ii).

Lemma 3.2.20 *Let $\mu^{\#}\mu$ be a PK_n-pair with $\mu^{\#}\mu \neq 00$ and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for it such that $\mu^{\#}A_c$ is a partition or equivalently $\mu^{\#}A_c\mu \neq 00$ holds. Moreover, suppose that we have a $w \in \mathfrak{S}_n$ such that the entries of every column of $(\mathbf{t}^{\mu}w)\Big\downarrow^{[\mu]}_{[\mu^{\#}A_c]}$ when considered from top to bottom form an ascending sequence of successive integers. More formally, this means that if we write $\mu^{\#}A_c = ((\mu^{\#}A_c)_1, (\mu^{\#}A_c)_2, \ldots)$ and $\mu^{\#}A_c{}' = ((\mu^{\#}A_c{}')_1, \ldots, (\mu^{\#}A_c{}')_{(\mu^{\#}A_c)_1})$ then*

$$\forall j \in \left\{1, \ldots, (\mu^{\#}A_c)_1\right\} \ \forall i \in \left\{1, \ldots, (\mu^{\#}A_c{}')_j\right\} : \qquad (3.29)$$
$$(i,j)\mathbf{t}^{\mu}w = (1,j)\mathbf{t}^{\mu}w + i - 1$$

holds. Finally, let $f \in I_{\mu^{\#}\mu c}$ and write

$$fw = ud$$

with uniquely determined permutations $u \in \mathfrak{S}_{\mu R_c}$ and $d \in \mathcal{D}_{\mu R_c}$. Then there is an $m \in \{2, \ldots, n\}$ with the following properties.

$$\begin{aligned}
(m)\left(\mathbf{t}^{\mu}w\right)^{-1}, (m-1)\left(\mathbf{t}^{\mu}w\right)^{-1} &\in \left[\mu^{\#}A_c\right] \\
(m)\sigma_{\mathbf{t}^{\mu}w} &= (m-1)\sigma_{\mathbf{t}^{\mu}w} \\
(m)\zeta_{\mathbf{t}^{\mu R_c}d} &= (m-1)\zeta_{\mathbf{t}^{\mu R_c}d}
\end{aligned}$$

Proof. According to Remark 3.2.4, the assumptions of the lemma imply $n \geq 2$. Thus the claim is meaningful.

First we note that, according to the assumptions, $\mu^{\#}A_c\mu$ is a PK-pair with $\mu^{\#}A_c\mu \neq 00$ (see Lemma 3.2.7.(ii)). Thus we get from Definition 3.2.1

$$\left[\mu^{\#}A_c\right] \subseteq [\mu]\,.$$

Next, if we write $\mu = (\mu_1, \mu_2, \ldots) \vDash n$, $\mu^{\#} = (\mu_1^{\#}, \mu_2^{\#}, \ldots) \vdash k$, and $\mu^{\#}A_c = (\mu_1^{\#}, \ldots, \mu_{c-1}^{\#}, \mu_c^{\#} + 1, \mu_{c+1}^{\#}, \ldots) \vdash k+1$ as in Definition 3.2.5, we see that the left $\mu_c^{\#} + 1$ entries in the c-th row of the representation of $\mathbf{t}^{\mu}w$ all occupy squares contained in the representation of $\left[\mu^{\#}A_c\right]$.

Furthermore, we can apply Lemma 3.2.19 to the situation at hand and construct $\mathbf{t}^{\mu R_c}d$ from $\mathbf{t}^{\mu}w$ as described there. In the course of this construction, $\mu_c - \mu_c^{\#}$ squares from the c-th row of $\mathbf{t}^{\mu}w$ are moved to the $(c-1)$-th row, and only $\mu_c^{\#}$ squares remain in the c-th row. This implies that a square contained in $\left[\mu^{\#}A_c\right]$ with a certain entry $m \in \{1, \ldots, n\}$ must be moved from the c-th row to the $(c-1)$-th row. More formally, we have for this m

$$(m)\left(\mathbf{t}^{\mu}w\right)^{-1} \in \left[\mu^{\#}A_c\right], \qquad (m)\zeta_{\mathbf{t}^{\mu}w} = c, \qquad (m)\zeta_{\mathbf{t}^{\mu R_c}d} = c - 1. \qquad (3.30)$$

From this, the fact $c > 1$ (see Definition 3.2.3), and the property (3.29) of the tableau $\mathbf{t}^\mu w$ we get on the one hand

$$m \in \{2, \ldots, n\} \tag{3.31}$$

and on the other hand that, in the representation of $\mathbf{t}^\mu w$, the square containing the entry $m-1$ is located directly above the square containing the entry m and also is contained in the representation of $\left[\mu^\# A_c\right]$ (see Lemma 1.1.8.(i)). More formally, we have

$$\begin{gathered}(m-1)\left(\mathbf{t}^\mu w\right)^{-1} \in \left[\mu^\# A_c\right], \\ (m-1)\sigma_{\mathbf{t}^\mu w} = (m)\sigma_{\mathbf{t}^\mu w}, \qquad (m-1)\zeta_{\mathbf{t}^\mu w} = (m)\zeta_{\mathbf{t}^\mu w} - 1 = c - 1.\end{gathered} \tag{3.32}$$

From $(m-1)\zeta_{\mathbf{t}^\mu w} = c-1$ we get in turn that, in the course of the construction of $\mathbf{t}^{\mu R_c} d$ from $\mathbf{t}^\mu w$ described in Lemma 3.2.19, the entry $m-1$ remains in the $(c-1)$-th row. Thus we have

$$(m-1)\zeta_{\mathbf{t}^{\mu R_c} d} = (m-1)\zeta_{\mathbf{t}^\mu w} = c - 1. \tag{3.33}$$

Now the claim follows directly from (3.30), (3.31), (3.32), and (3.33). ■

Remark 3.2.21 *Definition 1.1.67 and Definition 1.1.66 show that, using the notation from Lemma 3.2.20, condition (3.29) is satisfied in particular by the permutation* $w_\mu \in \mathfrak{S}_n$ *and the tableau* $\mathbf{t}^\mu w_\mu = \mathbf{t}_\mu$.

Now a distinguished element of the set from Definition 3.2.15 is introduced. The next definition makes use of (1.2) on page 1 and Definition 1.1.1.(ii).

Definition 3.2.22 *Let* $\mu^\#\mu$ *be a* PK_n*-pair with* $\mu^\#\mu \neq 00$, $\mu = (\mu_1, \mu_2, \ldots)$, *and* $\mu^\# = (\mu_1^\#, \mu_2^\#, \ldots)$. *Furthermore, let* $c \in \mathbb{N} \setminus \{1\}$ *be an AR-index for* $\mu^\#\mu$. *Then we define the permutation*

$$f_{\mu^\#\mu c} \in \mathfrak{S}_{\left\{\mu_{c-1}^+ + 1, \ldots, \mu_c^+\right\}} \subseteq \mathfrak{S}_n$$

by

$$\begin{aligned}&(\mu_{c-1}^+ + 1) f_{\mu^\#\mu c} = \mu_{c-1}^+ + \mu_c^\# + 1, \ldots, (\mu_{c-1}^+ + \mu_c - \mu_c^\#) f_{\mu^\#\mu c} = \mu_{c-1}^+ + \mu_c, \\ &(\mu_{c-1}^+ + \mu_c - \mu_c^\# + 1) f_{\mu^\#\mu c} = \mu_{c-1}^+ + 1, \ldots, \mu_c^+ f_{\mu^\#\mu c} = \mu_{c-1}^+ + \mu_c^\#.\end{aligned}$$

Remark 3.2.23 *If we apply, using the notation from Definition 3.2.22, the permutation* $f_{\mu^\#\mu c}$ *to the tableau* $\mathbf{t}^\mu$, *we see that* $f_{\mu^\#\mu c}$ *moves the leftmost* $\mu_c^\#$ *entries in the c-th row of the representation of this tableau in the given order to the rightmost* $\mu_c^\#$ *places of the c-th row and the rightmost* $\mu_c - \mu_c^\#$ *entries in the c-th row in the given order to the leftmost* $\mu_c - \mu_c^\#$ *places of the c-th row. All entries in other rows remain fixed.*

Now some properties of the permutation introduced in Definition 3.2.22 are described.

Lemma 3.2.24 *Let $\mu^{\#}\mu$ be a PK_n-pair with $\mu^{\#}\mu \neq 00$ and write $\mu = (\mu_1, \mu_2, \ldots)$ and $\mu^{\#} = (\mu_1^{\#}, \mu_2^{\#}, \ldots)$. Furthermore, let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^{\#}\mu$. Finally, choose a $w \in \mathfrak{S}_n$. Then the tableau $\mathbf{t}^{\mu R_c} f_{\mu^{\#}\mu c} w$ is obtained from the tableau $\mathbf{t}^{\mu} w$ by moving the rightmost $\mu_c - \mu_c^{\#}$ entries in the c-th row of $\mathbf{t}^{\mu} w$ — that is, the entries in the c-th row of $\mathbf{t}^{\mu} w$ occupying positions not contained in $\left[\mu^{\#}\right]$ — in the given order to the end of the $(c-1)$-th row of $\mathbf{t}^{\mu} w$.*

Proof. From Remark 3.2.23 we get the following construction of $\mathbf{t}^{\mu} f_{\mu^{\#}\mu c} w$ from $\mathbf{t}^{\mu} w$.

> $\mathbf{t}^{\mu} f_{\mu^{\#}\mu c} w$ is obtained from $\mathbf{t}^{\mu} w$ by moving the leftmost $\mu_c^{\#}$ entries in the c-th row of $\mathbf{t}^{\mu} w$ — that is, the entries in the c-th row of $\mathbf{t}^{\mu} w$ occupying positions contained in $\left[\mu^{\#}\right] \subseteq [\mu]$ — in the given order to the end of the c-th row of $\mathbf{t}^{\mu} w$ and moving the rightmost $\mu_c - \mu_c^{\#}$ entries in the c-th row of $\mathbf{t}^{\mu} w$ — that is, the entries in the c-th row of $\mathbf{t}^{\mu} w$ occupying positions not contained in $\left[\mu^{\#}\right] \subseteq [\mu]$ — in the given order to the beginning of the c-th row of $\mathbf{t}^{\mu} w$. (3.34)

Moreover, Lemma 3.2.18 provides a method for the construction of $\mathbf{t}^{\mu R_c} f_{\mu^{\#}\mu c} w$ from $\mathbf{t}^{\mu} f_{\mu^{\#}\mu c} w$. Now the proof is completed by appending this method to (3.34). ■

The next statement makes use of Definition 3.2.5.(ii), Lemma 3.2.7.(ii), and Definition 3.2.8.

Lemma 3.2.25 *Let $\mu^{\#}\mu$ be a PK_n-pair with $\mu^{\#}\mu \neq 00$ and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for it. Then the permutation $f_{\mu^{\#}\mu c}$ from Definition 3.2.22 satisfies*

(i) $f_{\mu^{\#}\mu c} \in I_{\mu^{\#}\mu c}$,

(ii) $\mathbf{t}^{\mu R_c} f_{\mu^{\#}\mu c} w_{\mu^{\#}\mu} = \mathbf{t}^{\mu^{\#}\mu R_c}$.

Proof. (i) This follows immediately from Definition 3.2.15 and Definition 3.2.22.

(ii) In the following we write $\mu \vDash n$ and $\mu^{\#} \vdash k$ with a $k \in \{1, \ldots, n\}$ as in Definition 3.2.1.

Lemma 3.2.24 provides a method for the construction of $\mathbf{t}^{\mu R_c} f_{\mu^{\#}\mu c} w_{\mu^{\#}\mu}$ from $\mathbf{t}^{\mu} w_{\mu^{\#}\mu} = \mathbf{t}^{\mu^{\#}\mu}$. This method doesn't move entries occupying positions contained in $\left[\mu^{\#}\right]$. From this fact, Definition 3.2.9, Remark 3.2.10, Lemma 3.2.7.(ii), Definition 3.2.5, and Lemma 3.2.11.(i) we get

$$\left(\mathbf{t}^{\mu R_c} f_{\mu^{\#}\mu c} w_{\mu^{\#}\mu}\right)\Big\downarrow_{\left[\mu^{\#}\right]}^{\left[\mu R_c\right]} = \mathbf{t}^{\mu^{\#}\mu}\Big\downarrow_{\left[\mu^{\#}\right]}^{[\mu]} = \mathbf{t}_{\mu^{\#}}.$$

Here, $\mu^{\#}$ and μR_c are not considered a PK_n-pair. In turn, the preceding equation and Definition 1.1.66 show that the entries $1, \dots, k$ in $\mathbf{t}^{\mu R_c} f_{\mu^{\#}\mu c} w_{\mu^{\#}\mu}$ occupy the positions contained in $\left[\mu^{\#}\right]$ and are arranged in ascending order by columns from left to right and within each column from top to bottom.

Furthermore, Definition 3.2.8.(i) shows that the entries $k+1, \dots, n$ in $\mathbf{t}^{\mu^{\#}\mu}$ occupy the positions not contained in $\left[\mu^{\#}\right]$ and are arranged in ascending order by rows from top to bottom and within each row from left to right. This type of arrangement is preserved by the above-mentioned construction of $\mathbf{t}^{\mu R_c} f_{\mu^{\#}\mu c} w_{\mu^{\#}\mu}$ from $\mathbf{t}^{\mu^{\#}\mu}$ which moves all entries in the c-th row of $\mathbf{t}^{\mu^{\#}\mu}$ occupying positions not contained in $\left[\mu^{\#}\right]$ in the given order to the end of the $(c-1)$-th row of $\mathbf{t}^{\mu^{\#}\mu}$.

From all this we get the following description of $\mathbf{t}^{\mu R_c} f_{\mu^{\#}\mu c} w_{\mu^{\#}\mu}$.

> In $\mathbf{t}^{\mu R_c} f_{\mu^{\#}\mu c} w_{\mu^{\#}\mu}$, the positions contained in $\left[\mu^{\#}\right]$ are occupied by the entries $1, \dots, k$ arranged in ascending order by columns from left to right and within each column from top to bottom, and furthermore the positions not contained in $\left[\mu^{\#}\right]$ are occupied by the entries $k+1, \dots, n$ arranged in ascending order by rows from top to bottom and within each row from left to right. (3.35)

Here, $\mu^{\#}$ and μR_c are not considered a PK_n-pair. This description is now employed to compare $\mathbf{t}^{\mu R_c} f_{\mu^{\#}\mu c} w_{\mu^{\#}\mu}$ and $\mathbf{t}^{\mu^{\#}\mu R_c}$. We distinguish the cases $c > 2$ and $c = 2$.

First we consider the case $c > 2$. Then, according to Definition 3.2.5.(iii), the PK_n-pairs $\mu^{\#}\mu$ and $\mu^{\#}\mu R_c$ contain the same partition. From this we get by using Definition 3.2.8.(i) — applied to the PK_n-pair $\mu^{\#}\mu R_c$ — and (3.35)

$$\mathbf{t}^{\mu R_c} f_{\mu^{\#}\mu c} w_{\mu^{\#}\mu} = \mathbf{t}^{\mu^{\#}\mu R_c} \quad \text{for} \quad c > 2.$$

Now we consider the case $c = 2$. Here, according to Definition 3.2.5.(iii), the PK_n-pairs $\mu^{\#}\mu$ and $\mu^{\#}\mu R_2$ contain different partitions. In the following we denote the partition contained in $\mu^{\#}\mu R_2$ by $\nu \vdash m$ with an appropriate $m \in \{1, \dots, n\}$, the partition contained in $\mu^{\#}\mu$ is still denoted by $\mu^{\#}$. Definition 3.2.3, Definition 3.2.5.(ii), and Definition 3.2.5.(iii) show that ν is obtained from $\mu^{\#}$ by increasing the first entry of $\mu^{\#}$ to the value of the first entry of μR_2. This implies $m > k$, and furthermore $[\nu]$ is obtained from $\left[\mu^{\#}\right]$ by appending a certain number of lattice points to the end of the first row such that the length of the first row of $[\mu R_2]$ is reached. Since $\mu^{\#}$ is a partition, each of these added lattice points constitutes a column of $[\nu]$.

According to Definition 3.2.8.(i), the entries $1, \dots, m$ in $\mathbf{t}^{\mu^{\#}\mu R_2}$ occupy the positions contained in $[\nu]$ and are arranged in ascending order by columns from left to right and within each column from top to bottom. This and the preceding considerations show that, on the one hand, the entries $1, \ldots, k$ occupy the positions contained in $\left[\mu^{\#}\right] \subseteq [\nu]$ and are arranged in ascending order by columns from left to right and within each column from top to bottom, and, on the other hand, the entries $k+1, \ldots, m$ occupy the added positions in $[\nu]$ — that is, the not yet occupied positions in the first row of $[\mu R_2]$ — and also are arranged in ascending order by columns from left to right and within each column from top to bottom. In other words, the entries $k+1, \ldots, m$ occupy the positions in the first row of $[\mu R_2]$ not contained in $\left[\mu^{\#}\right]$ and are arranged in ascending order from left to right.

Moreover, the entries $m+1, \ldots, n$ in $\mathbf{t}^{\mu^{\#}\mu R_2}$ occupy the positions contained in $[\mu R_2] \setminus [\nu]$ and are arranged in ascending order by rows from top to bottom and within each row from left to right. Again according to the preceding considerations, the positions contained in $[\mu R_2] \setminus [\nu]$ are exactly the positions in $[\mu R_2]$ contained in neither $\left[\mu^{\#}\right]$ nor the first row of $[\mu R_2]$.

Now a comparison of this description of $\mathbf{t}^{\mu^{\#}\mu R_2}$ and (3.35) shows

$$\mathbf{t}^{\mu R_2} f_{\mu^{\#}\mu 2} w_{\mu^{\#}\mu} = \mathbf{t}^{\mu^{\#}\mu R_2}.$$

This completes the proof. ■

The permutation introduced in Definition 3.2.22 is uniquely determined by the properties shown in Lemma 3.2.25. The following lemma provides another useful characterization.

Lemma 3.2.26 *Let $\mu^{\#}\mu$ be a PK_n-pair with $\mu^{\#}\mu \neq 00$ and write $\mu = (\mu_1, \mu_2, \ldots)$ and $\mu^{\#} = (\mu_1^{\#}, \mu_2^{\#}, \ldots)$. Furthermore let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^{\#}\mu$, and finally let $f \in I_{\mu^{\#}\mu c} \setminus \left\{f_{\mu^{\#}\mu c}\right\}$ and $w \in \mathfrak{S}_n$. Then there are column numbers*

$$j \in \left\{1, \ldots, \mu_c^{\#}\right\} \qquad \textit{and} \qquad \tilde{j} \in \left\{1, \ldots, \mu_c - \mu_c^{\#}\right\}$$

satisfying

$$(c, j)\mathbf{t}^{\mu} w = (c, \tilde{j})\mathbf{t}^{\mu} f w.$$

In other words, one of the leftmost $\mu_c^{\#}$ entries in the c-th row of $\mathbf{t}^{\mu} w$ coincides with one of the leftmost $\mu_c - \mu_c^{\#}$ entries in the c-th row of $\mathbf{t}^{\mu} f w$.

Proof. According to Definition 3.2.15, Definition 1.1.1.(ii), and Definition 1.1.53, $\mathbf{t}^{\mu}$ and $\mathbf{t}^{\mu} f$ and thus also $\mathbf{t}^{\mu} w$ and $\mathbf{t}^{\mu} f w$ differ only by a permutation of the entries in the c-th row amongst themselves. Now suppose that none of the leftmost $\mu_c^{\#}$ entries in the c-th row of $\mathbf{t}^{\mu} w$ occupies one of the leftmost $\mu_c - \mu_c^{\#}$ positions in the c-th row of $\mathbf{t}^{\mu} f w$. This implies that an application of f to $\mathbf{t}^{\mu}$ moves the leftmost

$\mu_c^\#$ entries in the c-th row to the rightmost $\mu_c^\#$ positions in this row. But this in turn implies that f moves the rightmost $\mu_c - \mu_c^\#$ entries in the c-th row of $\mathbf{t}^\mu$ to the leftmost $\mu_c - \mu_c^\#$ positions in this row. With all this we get from Definition 3.2.15 and Definition 3.2.22 (see also Remark 3.2.16 and Remark 3.2.23)

$$f = f_{\mu^\# \mu c},$$

a contradiction. Thus our assumption is wrong, and one of the leftmost $\mu_c^\#$ entries in the c-th row of $\mathbf{t}^\mu w$ coincides with one of the leftmost $\mu_c - \mu_c^\#$ entries in the c-th row of $\mathbf{t}^\mu f w$. More formally, there are column indices $j \in \{1, \ldots, \mu_c^\#\}$ and $\tilde{j} \in \{1, \ldots, \mu_c - \mu_c^\#\}$ satisfying

$$(c, j)\mathbf{t}^\mu w = (c, \tilde{j})\mathbf{t}^\mu f w,$$

as desired. ∎

The next lemma is similar to Lemma 3.2.20. It makes use of Definition 1.1.6, Lemma 1.1.8.(i), Remark 1.1.12, Definition 1.1.45.(ii), Definition 1.1.53, Definition 1.1.55, Lemma 1.1.56.(i), Definition 3.2.5.(ii), Definition 3.2.9, and finally Remark 3.2.10.

Lemma 3.2.27 *Let $\mu^\#\mu$ be a PK_n-pair with $\mu^\#\mu \neq 00$ and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^\#\mu$. Moreover, suppose that we have a $w \in \mathfrak{S}_n$ such that the entries of every column of $(\mathbf{t}^\mu w)\Big\downarrow^{[\mu]}_{[\mu^\#]}$ when considered from top to bottom form an ascending sequence of successive integers. More formally, this means that if we write $\mu^\# = (\mu_1^\#, \mu_2^\#, \ldots)$ and $\mu^{\#\prime} = (\mu_1^{\#\prime}, \ldots, \mu_{\mu_1^\#}^{\#\prime})$ then*

$$\forall j \in \left\{1, \ldots, \mu_1^\#\right\} \; \forall i \in \left\{1, \ldots, \mu_j^{\#\prime}\right\} : \quad (i, j)\mathbf{t}^\mu w = (1, j)\mathbf{t}^\mu w + i - 1 \tag{3.36}$$

holds. Finally, let $f \in I_{\mu^\#\mu c} \setminus \{f_{\mu^\#\mu c}\}$ and write

$$f w = u d$$

with uniquely determined permutations $u \in \mathfrak{S}_{\mu R_c}$ and $d \in \mathcal{D}_{\mu R_c}$. Then there is an $m \in \{2, \ldots, n\}$ with the following properties.

$$\begin{aligned} (m)\left(\mathbf{t}^\mu w\right)^{-1}, (m-1)\left(\mathbf{t}^\mu w\right)^{-1} &\in \left[\mu^\#\right] \\ (m)\sigma_{\mathbf{t}^\mu w} &= (m-1)\sigma_{\mathbf{t}^\mu w} \\ (m)\zeta_{\mathbf{t}^{\mu R_c} d} &= (m-1)\zeta_{\mathbf{t}^{\mu R_c} d} \end{aligned}$$

Proof. According to Remark 3.2.4, the assumptions of the lemma imply $n \geq 2$. Thus the claim is meaningful.

In the following we write $\mu = (\mu_1, \mu_2, \ldots)$ and denote $\mu^\#$ as in the formulation of the statement. Then an application of Lemma 3.2.26 to the given data supplies us with an $m \in \{1, \ldots, n\}$ satisfying

$$(m)\sigma_{\mathbf{t}^\mu w} \in \left\{1, \ldots, \mu_c^\#\right\}, \qquad (m)\zeta_{\mathbf{t}^\mu w} = c \tag{3.37}$$

and

$$(m)\sigma_{\mathbf{t}^\mu fw} \in \left\{1, \ldots, \mu_c - \mu_c^\#\right\}, \qquad (m)\zeta_{\mathbf{t}^\mu fw} = c. \tag{3.38}$$

From (3.37), the fact $c > 1$, and (3.36) we see

$$m \in \{2, \ldots, n\}. \tag{3.39}$$

Finally, (3.37) and Definition 1.1.6.(i) show

$$(m)\left(\mathbf{t}^\mu w\right)^{-1} \in \left[\mu^\#\right]. \tag{3.40}$$

Now because of (3.40) and again the fact $c > 1$, and since $\mu^\#$ is a partition, it follows from condition (3.36) and Lemma 1.1.8 that, in the representation of $\mathbf{t}^\mu w$, the square containing the entry $m - 1$ is located immediately above the square occupied by the entry m and thus also is located in the representation of $\left[\mu^\#\right]$. More formally, we have

$$\begin{aligned} (m-1)\left(\mathbf{t}^\mu w\right)^{-1} &\in \left[\mu^\#\right], \\ (m-1)\sigma_{\mathbf{t}^\mu w} &= (m)\sigma_{\mathbf{t}^\mu w}, \\ (m-1)\zeta_{\mathbf{t}^\mu w} &= (m)\zeta_{\mathbf{t}^\mu w} - 1 = c - 1. \end{aligned} \tag{3.41}$$

By using Definition 3.2.15 and Remark 3.2.16, we get from this

$$(m-1)\zeta_{\mathbf{t}^\mu fw} = (m-1)\zeta_{\mathbf{t}^\mu w} = c - 1.$$

All this shows together with (3.38) and Lemma 3.2.18, applied with the permutation $fw \in \mathfrak{S}_n$, the following relation concerning the positions of m and $m-1$ in $\mathbf{t}^{\mu R_c} fw$.

$$(m)\zeta_{\mathbf{t}^{\mu R_c} fw} = (m-1)\zeta_{\mathbf{t}^{\mu R_c} fw} = c - 1$$

Now since $u \in \mathfrak{S}_{\mu R_c} = \mathfrak{R}_{\mathbf{t}^{\mu R_c}}$ (see Definition 1.1.55, Definition 1.1.50, and Remark 1.1.51), the tableaux $\mathbf{t}^{\mu R_c} fw = \mathbf{t}^{\mu R_c} ud$ and $\mathbf{t}^{\mu R_c} d$ differ only by a permutation of the entries in the various rows respectively amongst themselves. This and the preceding relation imply

$$(m)\zeta_{\mathbf{t}^{\mu R_c} d} = (m-1)\zeta_{\mathbf{t}^{\mu R_c} d}. \tag{3.42}$$

The claim now follows from (3.39), (3.40), (3.41), and (3.42). ■

Remark 3.2.28 *Definition 1.1.67, Definition 1.1.66, Definition 3.2.8, and Lemma 3.2.11 show that, using the notation from Lemma 3.2.27, condition (3.36) is satisfied in particular by the permutation* $w_\mu \in \mathfrak{S}_n$ *and the tableau* $\mathbf{t}^\mu w_\mu = \mathbf{t}_\mu$ *as well as the permutation* $w_{\mu\#\mu} \in \mathfrak{S}_n$ *and the tableau* $\mathbf{t}^\mu w_{\mu\#\mu} = \mathbf{t}^{\mu\#\mu}$.

3.3 Row number lists

This section describes some constructions with the row number lists introduced in Section 1.1 (see in particular Definition 1.1.70). These constructions also are considered in [JAM1, Section 15]. Here they are employed in the derivation of the generic bases of the modules occurring in the construction of the Specht series. In the following we use the notations for row number lists and associated objects introduced in Section 1.1. n continues to denote a positive integer.

First we associate certain tableaux, compositions, and permutations to row number lists.

Definition 3.3.1 *Fix a* $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash n$ *with* $\lambda_1 > 0$ *and choose a* $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathcal{Z}^\lambda$.

(i) The λ*-tableau* $\mathbf{t}(\zeta)$ *is defined by the following construction in the course of which the diagram* $[\lambda]$ *is filled with numbers corresponding to the entries of* ζ. *Also, these numbers and the corresponding entries of* ζ *are divided into good ones and bad ones.*

The construction starts with the diagram

$$\mathbf{t}(\zeta)^0 = [\lambda]\,.$$

Now fix a $j \in \{1, \ldots, n\}$ *and suppose that* $\mathbf{t}(\zeta)^{j-1}$ *is already defined. Then the diagram* $\mathbf{t}(\zeta)^j$ *is derived from the diagram* $\mathbf{t}(\zeta)^{j-1}$ *by means of the entry* ζ_j *of* ζ *as follows.*

If we have $\zeta_j = 1$ *then* $\mathbf{t}(\zeta)^j$ *is obtained from* $\mathbf{t}(\zeta)^{j-1}$ *by entering* j *into the leftmost empty square of the first row of* $\mathbf{t}(\zeta)^{j-1}$. *Furthermore,* j *is called a good entry of the diagram and* ζ_j *is called a good entry of* ζ.

If we have $\zeta_j > 1$ *and the* $(\zeta_j - 1)$*-th row of* $\mathbf{t}(\zeta)^{j-1}$ *contains more good entries than the* ζ_j*-th row then* $\mathbf{t}(\zeta)^j$ *is obtained from* $\mathbf{t}(\zeta)^{j-1}$ *by entering* j *into the leftmost empty square of the* ζ_j*-th row of* $\mathbf{t}(\zeta)^{j-1}$. *Furthermore,* j *is called a good entry of the diagram and* ζ_j *is called a good entry of* ζ.

If we have $\zeta_j > 1$ *and the* $(\zeta_j - 1)$*-th row of* $\mathbf{t}(\zeta)^{j-1}$ *contains exactly as many good entries as the* ζ_j*-th row then* $\mathbf{t}(\zeta)^j$ *is obtained from* $\mathbf{t}(\zeta)^{j-1}$ *by entering*

j into the rightmost empty square of the ζ_j-th row of $\mathbf{t}(\zeta)^{j-1}$. Furthermore, j is called a bad entry of the diagram and ζ_j is called a bad entry of ζ.

With this, $\mathbf{t}(\zeta)$ is defined as

$$\mathbf{t}(\zeta) = \mathbf{t}(\zeta)^n.$$

(ii) For $j \in \mathbb{N}$ we define $\nu(\zeta)_j \in \mathbb{N}_0$ to be the number of good entries in the j-th row of $\mathbf{t}(\zeta)$. With this, we define the composition $\nu(\zeta)$ as

$$\nu(\zeta) = (\nu(\zeta)_1, \nu(\zeta)_2, \ldots).$$

(iii) The permutation $g(\zeta) \in \mathfrak{S}_n$ is defined as

$$g(\zeta) = \mathbf{t}_\lambda^{-1}\mathbf{t}(\zeta)$$

or equivalently by the condition

$$\mathbf{t}_\lambda g(\zeta) = \mathbf{t}(\zeta).$$

Remark 3.3.2 *(i) The condition $\lambda_1 > 0$ in Definition 3.3.1 is imposed here and in the following mostly for simplicity without really being necessary. It is required where PK-pairs occur in the following constructions (see Remark 3.2.2). Since later on these constructions are used only in conjunction with PK-pairs, the above condition on λ is not a substantial restriction.*

(ii) The construction of $\mathbf{t}(\zeta)$ for a $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathcal{Z}^\lambda$ with $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash n$ in Definition 3.3.1.(i) does indeed produce a λ-tableau since for every $j \in \{1, \ldots, n\}$ in the course of the transition from $\mathbf{t}(\zeta)^{j-1}$ to $\mathbf{t}(\zeta)^j$ the value j is entered into the ζ_j-th row of the diagram λ. And since ζ is a λ-row number list (see Definition 1.1.70.(i)), for every $k \in \mathbb{N}$ exactly λ_k entries are inserted in the k-th row in the course of the entire construction. This shows that in the j-th step of the construction an empty square is available in the ζ_j-th row and at the end of the construction the diagram $[\lambda]$ is filled with the numbers $1, \ldots, n$.

The statements (iv) and (v) in the next lemma make use of Definition 3.2.9.

Lemma 3.3.3 *Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash n$ with $\lambda_1 > 0$ and let $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathcal{Z}^\lambda$. Then the following statements hold.*

(i) We have

$$\nu(\zeta) \vdash k$$

with a $k \in \{1, \ldots, n\}$*.*

(ii) The diagrams $[\nu(\zeta)]$ *and* $[\lambda]$ *satisfy*

$$[\nu(\zeta)] \subseteq [\lambda] .$$

(iii) $\nu(\zeta)\lambda$ *is a* PK_n*-pair with* $\nu(\zeta)\lambda \neq 00$*.*

(iv) In every row of $\mathbf{t}(\zeta)\big\downarrow^{[\lambda]}_{[\nu(\zeta)]}$ *the entries are arranged in ascending order from left to right.*

(v) In every column of $\mathbf{t}(\zeta)\big\downarrow^{[\lambda]}_{[\nu(\zeta)]}$ *the entries are arranged in ascending order from top to bottom.*

Proof. (i) Definition 3.3.1.(i) shows that $\mathbf{t}(\zeta)$ contains at most n good entries. This definition and the condition $\lambda_1 > 0$ also imply that $\mathbf{t}(\zeta)$ contains at least one good entry. Thus we have, according to Definition 3.3.1.(ii),

$$\nu(\zeta) \vDash k$$

with a $k \in \{1, \ldots, n\}$. Furthermore, for every $j \in \{0, \ldots, n\}$ and every $m \in \mathbb{N}$ the m-th row of the diagram $\mathbf{t}(\zeta)^j$ from Definition 3.3.1.(i) contains at least as many good entries as the $(m+1)$-th row. This is true for $j = 0$ and follows inductively from the construction of the diagrams for $j > 0$. From this fact applied with $j = n$, Definition 3.3.1.(i), and Definition 3.3.1.(ii) we get

$$\nu(\zeta)_1 \geq \nu(\zeta)_2 \geq \cdots .$$

This shows

$$\nu(\zeta) \vdash k$$

with a $k \in \{1, \ldots, n\}$, as desired.

(ii) From the construction of $\mathbf{t}(\zeta)$ in Definition 3.3.1.(i) we get (see Definition 3.1.1.(i))

$$\left[\lambda^{\mathbf{t}(\zeta)}\right] = [\lambda] .$$

Thus for every $j \in \mathbb{N}$ the j-th row of $\mathbf{t}(\zeta)$ contains at most λ_j good entries. According to Definition 3.3.1.(ii), this means

$$\forall j \in \mathbb{N} : \nu(\zeta)_j \leq \lambda_j ,$$

or equivalently

$$[\nu(\zeta)] \subseteq [\lambda].$$

(iii) From Definition 3.3.1.(i) and Definition 3.3.1.(ii) we get

$$\nu(\zeta)_1 = \lambda_1.$$

This shows together with statements (i) and (ii) that $\nu(\zeta)\lambda$ is a PK$_n$-pair as in Definition 3.2.1. Obviously, $\nu(\zeta)\lambda$ is different from 00.

(iv) According to statement (ii), we can build $\mathbf{t}(\zeta)\big\downarrow^{[\lambda]}_{[\nu(\zeta)]}$. According to Definition 3.3.1.(i) and Definition 3.3.1.(ii), $\mathbf{t}(\zeta)\big\downarrow^{[\lambda]}_{[\nu(\zeta)]}$ contains exactly the good entries of $\mathbf{t}(\zeta)$. In the construction of $\mathbf{t}(\zeta)$ these entries are entered into the various rows of $[\nu(\zeta)] \subseteq [\lambda]$ in ascending order from left to right. This proves the claim.

(v) If in the course of the construction of $\mathbf{t}(\zeta)$ in Definition 3.3.1.(i) a good entry is entered into the diagram $[\nu(\zeta)] \subseteq [\lambda]$ below the first row then a good entry with a smaller value is already located immediately above this new entry. The proof of the claim is now completed as in the proof of statement (iv). ■

The following lemma makes use of Definition 1.1.70.(iii), Definition 1.1.58.(ii), and Definition 1.1.67.

Lemma 3.3.4 *Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash n$ with $\lambda_1 > 0$. Then the map*

$$\mathscr{Z}^\lambda \to \mathcal{D}_\lambda, \quad \zeta \mapsto [w_\lambda g(\zeta)]^\lambda$$

is a bijection.

Proof. Fix a $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathscr{Z}^\lambda$. Then we see from the construction of $\mathbf{t}(\zeta)$ and the construction of $\mathbf{t}_\zeta$ (see Definition 1.1.73) that for every $j \in \{1, \ldots, n\}$ the entry j is located in the (ζ_j)-th row of each of the tableaux $\mathbf{t}(\zeta)$ and $\mathbf{t}_\zeta$. This means that for every $i \in \mathbb{N}$ the set of the entries in the i-th row of $\mathbf{t}(\zeta)$ and the set of the entries in the i-th row of $\mathbf{t}_\zeta$ coincide. Moreover, Definition 1.1.67, Definition 1.1.58.(ii), and Definition 1.1.55 show that the tableau $\mathbf{t}^\lambda[w_\lambda g(\zeta)]^\lambda$ is obtained from the tableau

$$\mathbf{t}^\lambda w_\lambda g(\zeta) = \mathbf{t}_\lambda g(\zeta) = \mathbf{t}(\zeta)$$

by a permutation of the entries in the various rows of $\mathbf{t}(\zeta)$ respectively amongst themselves. Furthermore, according to Definition 1.1.58 again and also Definition 1.1.73, both $\mathbf{t}^\lambda[w_\lambda g(\zeta)]^\lambda$ and $\mathbf{t}_\zeta$ are row standard. All this implies

$$\mathbf{t}^\lambda[w_\lambda g(\zeta)]^\lambda = \mathbf{t}_\zeta.$$

Now the desired bijection is obtained as the concatenation of the bijection from Definition 1.1.74 (see also Lemma 1.1.75) and the inverse of the bijection from Lemma 1.1.59.(i). ■

The sets of row number lists introduced next are employed in the construction and indexing of the generic bases of the modules occurring in the derivation of the Specht series.

Definition 3.3.5 *For a PK_n-pair $\mu^{\#}\mu$ with $\mu^{\#}\mu \neq 00$ we define $\mathcal{Z}^{\mu^{\#}\mu} \subseteq \mathcal{Z}^{\mu}$ as*

$$\mathcal{Z}^{\mu^{\#}\mu} = \left\{ \zeta \in \mathcal{Z}^{\mu} \,\middle|\, \left[\mu^{\#}\right] \subseteq [\nu(\zeta)] \right\}.$$

Furthermore we put

$$\mathcal{Z}^{00} = \varnothing.$$

The statements (iii) and (iv) in the following lemma make use of Definition 3.2.3 and Definition 3.2.5.(iv).

Lemma 3.3.6 *Let $\mu^{\#}\mu$ be a PK_n-pair with $\mu^{\#}\mu \neq 00$. Then the following statements hold.*

(i) We have

$$\mathcal{Z}^{\mu^{\#}\mu} \neq \varnothing.$$

(ii) For a PK_n-pair $\lambda^{\#}\mu$ with $\lambda^{\#}\mu \neq 00$ and $\left[\mu^{\#}\right] \subseteq \left[\lambda^{\#}\right]$ we have

$$\mathcal{Z}^{\lambda^{\#}\mu} \subseteq \mathcal{Z}^{\mu^{\#}\mu}.$$

(iii) Let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^{\#}\mu$. Then we have

$$\mathcal{Z}^{\mu^{\#} A_c \mu} \subseteq \mathcal{Z}^{\mu^{\#}\mu}.$$

(iv) Let $\mu^{\#} = (\mu^{\#}_1, \mu^{\#}_2, \ldots) \vdash k$ with a $k \in \{1, \ldots, n\}$ and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^{\#}\mu$. Then we have

$$\mathcal{Z}^{\mu^{\#}\mu} \setminus \mathcal{Z}^{\mu^{\#} A_c \mu} = \left\{ \zeta \in \mathcal{Z}^{\mu^{\#}\mu} \,\middle|\, \mu^{\#}_c = \nu(\zeta)_c \right\}.$$

Proof. (i) Let $\mu = (\mu_1, \mu_2, \ldots)$ and consider

$$\zeta = (1^{\mu_1}, 2^{\mu_2}, \ldots) \in \mathcal{Z}^{\mu}$$

(see Definition 1.1.70 and Remark 1.1.71). Then we see from Definition 3.3.1.(i) that for every $j \in \mathbb{N}$ the j-th row of $\mathbf{t}(\zeta)$ contains exactly $\min\{\mu_1, \ldots, \mu_j\}$ good entries. According to Definition 3.3.1.(ii), this means

$$\forall j \in \mathbb{N} : \nu(\zeta)_j = \min\{\mu_1, \ldots, \mu_j\}.$$

If we furthermore write $\mu^\# = (\mu_1^\#, \mu_2^\#, \ldots)$ then we get from Definition 3.2.1 and Definition 1.1.2.(i)

$$\forall j \in \mathbb{N} : \mu_j^\# \leq \min\{\mu_1, \ldots, \mu_j\}.$$

All in all we now have

$$\forall j \in \mathbb{N} : \mu_j^\# \leq \nu(\zeta)_j.$$

This is equivalent to

$$\left[\mu^\#\right] \subseteq [\nu(\zeta)].$$

Now Definition 3.3.5 shows

$$\zeta \in \mathcal{Z}^{\mu^\# \mu}$$

and furthermore

$$\mathcal{Z}^{\mu^\# \mu} \neq \varnothing,$$

as desired.

(ii) According to Definition 3.3.5, we have for every $\zeta \in \mathcal{Z}^{\lambda^\# \mu}$

$$\left[\lambda^\#\right] \subseteq [\nu(\zeta)].$$

From this and the assumptions we get

$$\left[\mu^\#\right] \subseteq [\nu(\zeta)],$$

that is,

$$\zeta \in \mathcal{Z}^{\mu^\# \mu}.$$

Since $\zeta \in \mathcal{Z}^{\lambda^\# \mu}$ is arbitrarily chosen, the claim follows.

(iii) In the case $\mu^\# A_c \mu = 00$ we have $\mathcal{Z}^{\mu^\# A_c \mu} = \varnothing$ (see Definition 3.3.5) and there is nothing to prove. In the case $\mu^\# A_c \mu \neq 00$ the claim follows from Lemma 3.2.7.(i) and statement (ii).

(iv) In order to prove this statement, we distinguish the cases $\mu^\# A_c \mu \neq 00$ and $\mu^\# A_c \mu = 00$.

First we consider the case $\mu^\# A_c \mu \neq 00$. Here statement (iii) shows $\mathcal{Z}^{\mu^\# A_c \mu} \subseteq \mathcal{Z}^{\mu^\# \mu}$ and we must prove $\forall \zeta \in \mathcal{Z}^{\mu^\# \mu} : \zeta \notin \mathcal{Z}^{\mu^\# A_c \mu} \Leftrightarrow \mu_c^\# = \nu(\zeta)_c$. To this end, fix a $\zeta \in \mathcal{Z}^{\mu^\# \mu}$. Then Definition 3.3.5 shows $\left[\mu^\#\right] \subseteq [\nu(\zeta)]$ (see also Definition 3.3.1.(ii)). This is equivalent to

$$\forall j \in \mathbb{N} : \mu_j^\# \leq \nu(\zeta)_j. \tag{3.43}$$

Because of $\mu^\# A_c \mu \neq 00$ we furthermore have $\mu^\# A_c = ((\mu^\# A_c)_1, (\mu^\# A_c)_2, \ldots) \vdash k{+}1$ (see Definition 3.2.5). From this, Definition 3.3.5, and Definition 1.1.6.(i) we get

$$\begin{aligned} \zeta \notin \mathcal{Z}^{\mu^\# A_c \mu} \quad &\Leftrightarrow \quad \left[\mu^\# A_c\right] \not\subseteq [\nu(\zeta)] \\ &\Leftrightarrow \quad \exists j \in \mathbb{N} \text{ such that } (\mu^\# A_c)_j \not\leq \nu(\zeta)_j. \end{aligned} \tag{3.44}$$

Now we have according to Definition 3.2.5.(i)

$$(\mu^{\#}A_c)_j = \begin{cases} \mu_c^{\#}+1 & \text{for} \quad j=c \\ \mu_j^{\#} & \text{for} \quad j \in \mathbb{N}\setminus\{c\} \end{cases}.$$

This implies together with (3.44) and (3.43)

$$\zeta \notin \mathcal{Z}^{\mu^{\#}A_c\mu} \Leftrightarrow \mu_c^{\#}+1 \not\leq \nu(\zeta)_c$$

and furthermore because of $\mu_c^{\#} \leq \nu(\zeta)_c$ (see (3.43))

$$\zeta \notin \mathcal{Z}^{\mu^{\#}A_c\mu} \Leftrightarrow \mu_c^{\#} = \nu(\zeta)_c.$$

This proves the claim in the case $\mu^{\#}A_c\mu \neq 00$.

Now we consider the case $\mu^{\#}A_c\mu = 00$. Here we have $\mathcal{Z}^{\mu^{\#}A_c\mu} = \varnothing$ according to Definition 3.3.5 and we must show $\forall \zeta \in \mathcal{Z}^{\mu^{\#}\mu} : \mu_c^{\#} = \nu(\zeta)_c$. To this end, fix a $\zeta \in \mathcal{Z}^{\mu^{\#}\mu} \subseteq \mathcal{Z}^{\mu}$. According to Definition 3.3.5 and Definition 1.1.6.(i), we then have

$$\mu_c^{\#} \leq \nu(\zeta)_c$$

(see (3.43)). Moreover, we see from Lemma 3.3.3.(i) and Definition 1.1.2.(i)

$$\nu(\zeta)_c \leq \nu(\zeta)_{c-1}.$$

Furthermore, we get from Lemma 3.3.3.(ii) and Definition 1.1.6.(i) with μ as in the proof of statement (i)

$$\nu(\zeta)_{c-1} \leq \mu_{c-1}.$$

According to Definition 3.2.3, we also have

$$\mu_{c-1} = \mu_{c-1}^{\#},$$

and finally the fact $\mu^{\#}A_c\mu = 00$ and Definition 3.2.5.(iv) imply

$$\mu_{c-1}^{\#} = \mu_c^{\#}.$$

From all these relations we obtain

$$\mu_c^{\#} \leq \nu(\zeta)_c \leq \nu(\zeta)_{c-1} \leq \mu_{c-1} = \mu_{c-1}^{\#} = \mu_c^{\#},$$

that is,

$$\mu_c^{\#} = \nu(\zeta)_c.$$

This also proves the claim in the case $\mu^{\#}A_c\mu = 00$. ■

Lemma 3.3.7 *Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash n$ with $\lambda_1 > 0$. Then $(\lambda_1)\lambda$ is a PK$_n$-pair with $(\lambda_1)\lambda \neq 00$ and we have*

$$\mathcal{Z}^{(\lambda_1)\lambda} = \mathcal{Z}^{\lambda}.$$

Proof. Because of the assumption $\lambda_1 > 0$ and according to Definition 1.1.1.(i), we have

$$1 \leq \lambda_1 \leq n.$$

Now Definition 3.2.1 shows that $(\lambda_1)\lambda$ is a PK-pair with $(\lambda_1)\lambda \neq 00$.

Furthermore, we have according to Definition 3.3.5

$$\mathcal{Z}^{(\lambda_1)\lambda} \subseteq \mathcal{Z}^{\lambda}.$$

In order to prove the reverse inclusion, fix a

$$\zeta \in \mathcal{Z}^{\lambda}.$$

Then, according to Lemma 3.3.3, $\nu(\zeta)\lambda$ with

$$\nu(\zeta) = (\nu(\zeta)_1, \nu(\zeta)_2, \ldots)$$

as in Definition 3.3.1 is a PK-pair satisfying $\nu(\zeta)\lambda \neq 00$. Thus, according to Definition 3.2.1, we have the relation

$$\nu(\zeta)_1 = \lambda_1.$$

This implies

$$[(\lambda_1)] \subseteq [\nu(\zeta)]$$

and with Definition 3.3.5 furthermore

$$\zeta \in \mathcal{Z}^{(\lambda_1)\lambda}.$$

Because $\zeta \in \mathcal{Z}^{\lambda}$ is arbitrarily chosen we now have

$$\mathcal{Z}^{\lambda} \subseteq \mathcal{Z}^{(\lambda_1)\lambda}$$

and all in all

$$\mathcal{Z}^{(\lambda_1)\lambda} = \mathcal{Z}^{\lambda},$$

as desired. ■

The construction introduced in the following also is considered in [JAM1, Section 15]. Here it is used in the investigation of homomorphisms between the modules occurring in the derivation of the Specht series.

Definition 3.3.8 *Let $\mu^{\#}\mu$ be a PK_n-pair with $\mu^{\#}\mu \neq 00$ and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^{\#}\mu$. Then for*

$$\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathcal{Z}^{\mu^{\#}\mu} \setminus \mathcal{Z}^{\mu^{\#}A_c\mu}$$

the sequence

$$\mathcal{J}_{\mu^{\#}\mu c}(\zeta) = (\eta_1, \ldots, \eta_n) \in \mathbb{N}^n$$

is defined by

$$\eta_j = \begin{cases} c-1 & \textit{if we have } \zeta_j = c \textit{ and } \zeta_j \textit{ is a bad entry of } \zeta \\ \zeta_j & \textit{otherwise} \end{cases}$$

for $j \in \{1, \ldots, n\}$.

Lemma 3.3.9 *Let $\mu^{\#}\mu$ be a PK_n-pair with $\mu^{\#}\mu \neq 00$ and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^{\#}\mu$. Then the map*

$$\mathcal{J}_{\mu^{\#}\mu c} : \mathcal{Z}^{\mu^{\#}\mu} \setminus \mathcal{Z}^{\mu^{\#}A_c\mu} \to \mathbb{N}^n, \quad \zeta \mapsto \mathcal{J}_{\mu^{\#}\mu c}(\zeta)$$

maps the set $\mathcal{Z}^{\mu^{\#}\mu} \setminus \mathcal{Z}^{\mu^{\#}A_c\mu}$ bijectively onto the set $\mathcal{Z}^{\mu^{\#}\mu R_c} \subseteq \mathbb{N}^n$.

Proof. See [JAM1, Theorem 15.14]. The notion of a sequence of type μ used there corresponds to the notion of a μ-row number list used here. Also, the notions of good and bad entries in a sequence of type μ and its corresponding row number list from [JAM1, Definition 15.2] and Definition 3.3.1.(i) coincide. The condition $\mu^{\#}_{c-1} = \mu_{c-1}$ imposed in Definition 3.2.3 is used in the proof of [JAM1, Theorem 15.14] (see also Remark 3.2.6.(ii)). ■

Corollary 3.3.10 *Let $\mu^{\#}\mu$ be a PK_n-pair with $\mu^{\#}\mu \neq 00$ and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^{\#}\mu$. Then we have*

$$\mathcal{Z}^{\mu^{\#}\mu} \setminus \mathcal{Z}^{\mu^{\#}A_c\mu} \neq \varnothing.$$

Proof. According to Lemma 3.2.7.(ii), $\mu^{\#}\mu R_c$ is a PK-pair satisfying $\mu^{\#}\mu R_c \neq 00$. With that we get from Lemma 3.3.6.(i)

$$\mathcal{Z}^{\mu^{\#}\mu R_c} \neq \varnothing.$$

The claim now follows from Lemma 3.3.9. ■

Lemma 3.3.11 *Let $\mu^{\#}\mu$ be a PK_n-pair with $\mu^{\#}\mu \neq 00$ and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^{\#}\mu$. Then we have*

$$\forall \zeta \in \mathcal{Z}^{\mu^{\#}\mu} \setminus \mathcal{Z}^{\mu^{\#}A_c\mu} : \left[w_{\mu R_c} g(\mathcal{J}_{\mu^{\#}\mu c}(\zeta))\right]^{\mu R_c} = \left[f_{\mu^{\#}\mu c} w_\mu g(\zeta)\right]^{\mu R_c}.$$

Proof. According to Corollary 3.3.10, we have $\mathcal{Z}^{\mu^\# \mu} \setminus \mathcal{Z}^{\mu^\# A_c \mu} \neq \varnothing$. So fix a $\zeta \in \mathcal{Z}^{\mu^\# \mu} \setminus \mathcal{Z}^{\mu^\# A_c \mu}$. Then Definition 1.1.58 and Lemma 1.1.59.(i) show that to prove the desired identity $\big[w_{\mu R_c} g(\mathcal{J}_{\mu^\# \mu c}(\zeta))\big]^{\mu R_c} = \big[f_{\mu^\# \mu c} w_\mu g(\zeta)\big]^{\mu R_c}$ it suffices to prove $\mathbf{t}^{\mu R_c}\big[w_{\mu R_c} g(\mathcal{J}_{\mu^\# \mu c}(\zeta))\big]^{\mu R_c} = \mathbf{t}^{\mu R_c}\big[f_{\mu^\# \mu c} w_\mu g(\zeta)\big]^{\mu R_c}$.

Now we have according to Definition 1.1.58, Definition 1.1.55, Definition 1.1.45, Definition 1.1.66, Definition 1.1.67, and Definition 3.3.1

$$\begin{aligned}
\mathbf{t}^{\mu R_c}\big[w_{\mu R_c} g(\mathcal{J}_{\mu^\# \mu c}(\zeta))\big]^{\mu R_c} &= \mathbf{t}^{\mu R_c} w_{\mu R_c} g(\mathcal{J}_{\mu^\# \mu c}(\zeta)) \quad \text{with entries} \\
&\quad\ \text{in each row arranged in ascending} \\
&\quad\ \text{order from left to right} \\
&= \mathbf{t}_{\mu R_c} g(\mathcal{J}_{\mu^\# \mu c}(\zeta)) \quad \text{with entries} \\
&\quad\ \text{in each row arranged in ascending} \\
&\quad\ \text{order from left to right} \\
&= \mathbf{t}(\mathcal{J}_{\mu^\# \mu c}(\zeta)) \quad \text{with entries} \\
&\quad\ \text{in each row arranged in ascending} \\
&\quad\ \text{order from left to right.}
\end{aligned}$$

Moreover, if we write $\mu = (\mu_1, \mu_2, \ldots)$ and $\mu^\# = (\mu_1^\#, \mu_2^\#, \ldots)$, we get from Definition 1.1.58.(ii), Definition 1.1.55, Definition 1.1.45.(iii), Lemma 3.2.24, Definition 1.1.66, Definition 1.1.67, Definition 3.3.1, Lemma 3.3.6.(iv), and Definition 3.3.8

$$\begin{aligned}
\mathbf{t}^{\mu R_c}\big[f_{\mu^\# \mu c} w_\mu g(\zeta)\big]^{\mu R_c} &= \mathbf{t}^{\mu R_c} f_{\mu^\# \mu c} w_\mu g(\zeta) \quad \text{with entries} \\
&\quad\ \text{in each row arranged in ascending} \\
&\quad\ \text{order from left to right} \\
&= (\mathbf{t}^{\mu} w_\mu g(\zeta) \quad \text{with the rightmost} \\
&\quad\ \mu_c - \mu_c^\# \text{ entries of the } c\text{-th row} \\
&\quad\ \text{moved to the end of the} \\
&\quad\ (c-1)\text{-th row}) \quad \text{with entries} \\
&\quad\ \text{in each row arranged in ascending} \\
&\quad\ \text{order from left to right} \\
&= (\mathbf{t}_{\mu} g(\zeta) \quad \text{with the rightmost} \\
&\quad\ \mu_c - \mu_c^\# \text{ entries of the } c\text{-th row} \\
&\quad\ \text{moved to the end of the} \\
&\quad\ (c-1)\text{-th row}) \quad \text{with entries} \\
&\quad\ \text{in each row arranged in ascending}
\end{aligned}$$

$$\begin{aligned}
&\quad \text{order from left to right} \\
&= (\mathbf{t}(\zeta) \quad \text{with the bad entries of the} \\
&\quad c\text{-th row moved to the end of the} \\
&\quad (c-1)\text{-th row)} \quad \text{with entries} \\
&\quad \text{in each row arranged in ascending} \\
&\quad \text{order from left to right} \\
&= \mathbf{t}(\mathcal{J}_{\mu^\# \mu c}(\zeta)) \quad \text{with entries} \\
&\quad \text{in each row arranged in ascending} \\
&\quad \text{order from left to right.}
\end{aligned}$$

Now the claim follows from the preceding two calculations as explained above. ■

Corollary 3.3.12 *Let $\mu^\#\mu$ be a PK_n-pair with $\mu^\#\mu \neq 00$ and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^\#\mu$. Then the map*

$$\mathcal{Z}^{\mu^\#\mu} \setminus \mathcal{Z}^{\mu^\# A_c \mu} \to \mathcal{D}_{\mu R_c}, \quad \zeta \mapsto \left[f_{\mu^\#\mu c} w_\mu g(\zeta)\right]^{\mu R_c}$$

is injective.

Proof. Lemma 3.3.9, the inclusion $\mathcal{Z}^{\mu^\# \mu R_c} \subseteq \mathcal{Z}^{\mu R_c}$ (see Definition 3.3.5), and Lemma 3.3.4 — applied with the composition μR_c — show that the map

$$\mathcal{Z}^{\mu^\#\mu} \setminus \mathcal{Z}^{\mu^\# A_c \mu} \to \mathcal{D}_{\mu R_c}, \quad \zeta \mapsto \left[w_{\mu R_c} g(\mathcal{J}_{\mu^\#\mu c}(\zeta))\right]^{\mu R_c}$$

is injective. Now the claim follows from this and Lemma 3.3.11. ■

Now we introduce certain sets of permutations associated to row number lists. These sets are used in the construction of the generic basis elements of the modules occurring in the derivation of the Specht series. The following definition makes use of Definition 3.2.12.

Definition 3.3.13 *Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash n$ with $\lambda_1 > 0$ and let $\zeta \in \mathcal{Z}^\lambda$. Then $Y(\zeta) \subseteq \mathfrak{S}_n$ is defined as*

$$Y(\zeta) = V_{\nu(\zeta)\lambda}.$$

The next statement uses notation as in Lemma 3.2.14.(ii) and furthermore makes use of Definition 3.1.14 and Lemma 3.1.15.(i).

Lemma 3.3.14 *Let $\mu = (\mu_1, \mu_2, \ldots) \vDash n$ with $\mu_1 > 0$ and let $\eta \in \mathcal{Z}^\mu$. Then the following statements hold.*

(i) *For $i \in \mathbb{N}$ we denote by $m_i \in \mathbb{N}_0$ the number of lattice points contained in the i-th column of $[\mu]$. With that we put for $k \in \mathbb{N}_0$*

$$m_k^+ = \sum_{j=1}^{k} m_j.$$

Furthermore, we write $\nu(\eta) = (\nu(\eta)_1, \nu(\eta)_2, \ldots) \vdash k$ with a $k \in \{1, \ldots, n\}$. Then we also have $\nu(\eta)_1 = \mu_1$ and $\nu(\eta)' = (\nu(\eta)'_1, \ldots, \nu(\eta)'_{\mu_1}) \vdash k$ with $\nu(\eta)'_{\mu_1} > 0$. Now if we put

$$\kappa = (\nu(\eta)'_1, 1^{m_1 - \nu(\eta)'_1}, \ldots, \nu(\eta)'_{\mu_1}, 1^{m_{\mu_1} - \nu(\eta)'_{\mu_1}}, 1^{n - m^+_{\mu_1}}) \vDash n$$

then

$$Y(\eta) = \mathfrak{S}_\kappa$$

holds.

(ii) *$[w_\mu g(\eta)]^\mu \in \mathcal{D}_\mu$ is the unique maximal element of $\{[w_\mu y g(\eta)]^\mu \mid y \in Y(\eta)\} \subseteq \mathcal{D}_\mu$ with respect to the ordering relation $\leq$ on $\mathcal{D}_\mu$. In other words, we have*

$$[w_\mu g(\eta)]^\mu \in \{[w_\mu y g(\eta)]^\mu \mid y \in Y(\eta)\}$$

and

$$\forall y \in Y(\eta) \setminus \{1_{\mathfrak{S}_n}\} : [w_\mu y g(\eta)]^\mu < [w_\mu g(\eta)]^\mu.$$

Proof. (i) From Lemma 3.3.3.(i) we get $\nu(\eta) \vdash k$ with a $k \in \{1, \ldots, n\}$. According to Lemma 3.3.3.(iii), $\nu(\eta)\mu$ is a PK-pair satisfying $\nu(\eta)\mu \neq 00$. With this we obtain from Definition 3.2.1 $\nu(\eta)_1 = \mu_1$. This fact and Remark 1.1.12 prove the form of $\nu(\eta)'$ described in the statement. Now the remaining claims follow from Definition 3.3.13 and an application of Lemma 3.2.14.(ii) to the PK-pair $\nu(\eta)\mu$.

(ii) According to statement (i), we have $1_{\mathfrak{S}_n} \in Y(\eta)$ and thus

$$[w_\mu g(\eta)]^\mu \in \{[w_\mu y g(\eta)]^\mu \mid y \in Y(\eta)\}.$$

Now fix a

$$y \in Y(\eta) \setminus \{1_{\mathfrak{S}_n}\}.$$

In order to prove the relation $[w_\mu y g(\eta)]^\mu < [w_\mu g(\eta)]^\mu$, we first compare the tableaux

$$\mathbf{t}^\mu w_\mu g(\eta) = \mathbf{t}_\mu g(\eta) = \mathbf{t}(\eta) \qquad \text{and} \qquad \mathbf{t}^\mu w_\mu y g(\eta) = \mathbf{t}_\mu y g(\eta)$$

(see Definition 3.3.1). According to Definition 3.3.13, Definition 3.2.12, and Remark 3.2.13, $\mathbf{t}_\mu y$ is obtained from $\mathbf{t}_\mu$ by a permutation of the entries in the various columns of $\mathbf{t}_\mu\big\downarrow^{[\mu]}_{[\nu(\eta)]}$ respectively amongst themselves (see Definition 3.2.9). With that, an application of $g(\eta)$ to these tableaux shows that $\mathbf{t}_\mu y g(\eta)$ is obtained from $\mathbf{t}_\mu g(\eta)$ by a permutation of the entries in the various columns of

$$\left(\mathbf{t}_\mu\big\downarrow^{[\mu]}_{[\nu(\eta)]}\right) g(\eta) = (\mathbf{t}_\mu g(\eta))\big\downarrow^{[\mu]}_{[\nu(\eta)]}$$

respectively amongst themselves. Since $y \neq 1_{\mathfrak{S}_n}$, we also have

$$\mathbf{t}_\mu g(\eta) \neq \mathbf{t}_\mu y g(\eta).$$

Thus there is an entry

$$m \in \{1, \dots, n\}$$

such that on the one hand m occupies different positions in $\mathbf{t}_\mu g(\eta)$ and $\mathbf{t}_\mu y g(\eta)$ and on the other hand every $k \in \{m+1, \dots, n\}$ occupies the same position in $\mathbf{t}_\mu g(\eta)$ and $\mathbf{t}_\mu y g(\eta)$. In other words, using Definition 1.1.45.(ii), we can write

$$\forall k \in \{m+1, \dots, n\} : (k)\zeta_{\mathbf{t}_\mu y g(\eta)} = (k)\zeta_{\mathbf{t}_\mu g(\eta)}. \tag{3.45}$$

Furthermore, in the construction of $\mathbf{t}_\mu y g(\eta)$ from $\mathbf{t}_\mu g(\eta)$ described above not all entries different from m can remain fixed, in addition to m there must be another — by choice of m necessarily smaller — entry which is moved. This shows

$$m > 1. \tag{3.46}$$

Moreover, the construction of $\mathbf{t}_\mu y g(\eta)$ from $\mathbf{t}_\mu g(\eta)$ described above shows that the positions occupied by m in both these tableaux are located in the same column within $[\nu(\eta)]$. Finally, Lemma 3.3.3.(v) and the choice of m show that the positions in the column of $(\mathbf{t}_\mu g(\eta))\big\downarrow^{[\mu]}_{[\nu(\eta)]}$ containing m which are located below m only contain entries bigger than m. These entries are not moved in the transition from $\mathbf{t}_\mu g(\eta)$ to $\mathbf{t}_\mu y g(\eta)$. Since in the course of this transition m itself is moved within its column in $(\mathbf{t}_\mu g(\eta))\big\downarrow^{[\mu]}_{[\nu(\eta)]}$, the position occupied by m in $\mathbf{t}_\mu y g(\eta)$ must be located above the position occupied by m in $\mathbf{t}_\mu g(\eta)$. More formally, we have

$$(m)\zeta_{\mathbf{t}_\mu y g(\eta)} < (m)\zeta_{\mathbf{t}_\mu g(\eta)}. \tag{3.47}$$

Now we move from the tableaux $\mathbf{t}_\mu g(\eta) = \mathbf{t}^\mu w_\mu g(\eta)$ and $\mathbf{t}_\mu y g(\eta) = \mathbf{t}^\mu w_\mu y g(\eta)$ to the tableaux

$$\mathbf{t}^\mu [w_\mu g(\eta)]^\mu \qquad \text{and} \qquad \mathbf{t}^\mu [w_\mu y g(\eta)]^\mu.$$

According to Definition 1.1.58 and Definition 1.1.55, both $\mathbf{t}^\mu w_\mu g(\eta)$ and $\mathbf{t}^\mu [w_\mu g(\eta)]^\mu$ as well as $\mathbf{t}^\mu w_\mu y g(\eta)$ and $\mathbf{t}^\mu [w_\mu y g(\eta)]^\mu$ respectively differ by a permutation of the entries in the various rows amongst themselves. With this we get from (3.45) and (3.47)

$$\forall k \in \{m+1,\ldots,n\} : (k)\zeta_{\mathbf{t}^\mu[w_\mu y g(\eta)]^\mu} = (k)\zeta_{\mathbf{t}^\mu[w_\mu g(\eta)]^\mu} \tag{3.48}$$

and

$$(m)\zeta_{\mathbf{t}^\mu[w_\mu y g(\eta)]^\mu} < (m)\zeta_{\mathbf{t}^\mu[w_\mu g(\eta)]^\mu}. \tag{3.49}$$

From (3.48), Definition 3.1.1.(ii), Remark 3.1.2.(ii), Definition 3.1.1.(i), and Remark 3.1.2.(i) we easily obtain

$$\forall j \in \{m,\ldots,n\} : \lambda^{(\mathbf{t}^\mu[w_\mu y g(\eta)]^\mu)\Downarrow_j^n} = \lambda^{(\mathbf{t}^\mu[w_\mu g(\eta)]^\mu)\Downarrow_j^n}. \tag{3.50}$$

Similarly, we get from (3.46), (3.48), (3.49), and Definition 1.1.4.(i)

$$\lambda^{(\mathbf{t}^\mu[w_\mu y g(\eta)]^\mu)\Downarrow_{m-1}^n} < \lambda^{(\mathbf{t}^\mu[w_\mu g(\eta)]^\mu)\Downarrow_{m-1}^n}. \tag{3.51}$$

(3.50) and (3.51) together now show, according to Definition 3.1.4.(i),

$$\mathbf{t}^\mu[w_\mu y g(\eta)]^\mu < \mathbf{t}^\mu[w_\mu g(\eta)]^\mu$$

which, according to Definition 3.1.14, implies

$$[w_\mu y g(\eta)]^\mu < [w_\mu g(\eta)]^\mu.$$

This proves the claim. ■

The following lemma states additional facts about Young subgroups $Y(\zeta)$ associated to certain row number lists ζ. It makes use of Definition 3.2.1, Definition 3.2.3, Definition 3.2.5, Definition 3.2.22, Definition 1.1.58, Definition 3.1.14, and Lemma 3.1.15.(i).

Lemma 3.3.15 *Let $\mu^\#\mu$ be a PK_n-pair with $\mu^\#\mu \neq 00$ and let $c \in \mathbb{N}\setminus\{1\}$ be an AR-index for $\mu^\#\mu$. Furthermore let $\eta \in \mathcal{Z}^{\mu^\#\mu} \setminus \mathcal{Z}^{\mu^\# A_c \mu} \subseteq \mathcal{Z}^\mu$. Then the following statements hold.*

(i) $\left(f_{\mu^\#\mu c} w_\mu\right)^{-1}$ *is the shortest representative of the right coset* $Y(\eta)\left(f_{\mu^\#\mu c} w_\mu\right)^{-1}$ *of the Young subgroup* $Y(\eta)$ *in* $\mathfrak{S}_n$.

(ii) The permutation $\left[f_{\mu^\#\mu c} w_\mu g(\eta)\right]^{\mu R_c} \in \mathcal{D}_{\mu R_c}$ *is the uniquely determined maximal element of the set* $\left\{\left[f_{\mu^\#\mu c} w_\mu y g(\eta)\right]^{\mu R_c} \,\middle|\, y \in Y(\eta)\right\} \subseteq \mathcal{D}_{\mu R_c}$ *with respect to the ordering relation* $\leq$ *on* $\mathcal{D}_{\mu R_c}$. *In other words, we have*

$$\left[f_{\mu^\#\mu c} w_\mu g(\eta)\right]^{\mu R_c} \in \left\{\left[f_{\mu^\#\mu c} w_\mu y g(\eta)\right]^{\mu R_c} \,\middle|\, y \in Y(\eta)\right\}$$

and

$$\forall y \in Y(\eta)\setminus\{1_{\mathfrak{S}_n}\} : \left[f_{\mu^\#\mu c} w_\mu y g(\eta)\right]^{\mu R_c} < \left[f_{\mu^\#\mu c} w_\mu g(\eta)\right]^{\mu R_c}.$$

Proof. (i) The proof of this statement is similar to the proof of statement (viii) in Lemma 3.2.14.

Write $\mu = (\mu_1,\ \mu_2, \ldots) \vDash n$. Then we get from Lemma 3.3.14 with the notation from there (see also Remark 3.2.2)

$$\kappa = (\nu(\eta)'_1, 1^{m_1-\nu(\eta)'_1}, \ldots, \nu(\eta)'_{\mu_1}, 1^{m_{\mu_1}-\nu(\eta)'_{\mu_1}}, 1^{n-m^+_{\mu_1}}) \vDash n$$

and

$$Y(\eta) = \mathfrak{S}_\kappa.$$

With this and Definition 1.1.58.(i), we must show that $\mathbf{t}^\kappa \left(f_{\mu^\# \mu c} w_\mu\right)^{-1}$ is row standard.

To this end, we divide κ into $\mu_1 + 1$ successive subsequences. Using the notation from Lemma 3.3.14, we define for every $j \in \{1, \ldots, \mu_1\}$ the j-th subsequence as

$$(\nu(\eta)'_j, 1^{m_j-\nu(\eta)'_j}).$$

The $(\mu_1 + 1)$-th subsequence is defined as

$$(1^{n-m^+_{\mu_1}}, 0, 0, 0, \ldots).$$

The result of the concatenation of these sequences in the order implied by their numbering is exactly κ.

Now we compare the tableaux $\mathbf{t}^\kappa$ and $\mathbf{t}_\mu$. Since for every $j \in \{1, \ldots, \mu_1\}$ the j-th subsequence of κ is a composition of m_j, we obtain from the definitions of these tableaux and the particular form of the subsequences of κ the following statement (see also Definition 3.2.9, Remark 3.2.10, and Lemma 3.3.3.(iii)).

> For every $j \in \{1, \ldots, \mu_1\}$ the sequence of the entries in the row of $\mathbf{t}^\kappa$ corresponding to the first entry in the j-th subsequence of κ when considered from left to right coincides with the sequence of the entries in the j-th column of $\mathbf{t}_\mu\big\downarrow^{[\mu]}_{[\nu(\eta)]}$ when considered from top to bottom. (3.52)

Next we compare the tableaux $\mathbf{t}_\mu$ and $\mathbf{t}^{\mu R_c} f_{\mu^\# \mu c} w_\mu$. To this end, we write $\mu^\# = (\mu^\#_1, \mu^\#_2, \ldots)$ in the following. Then we get from the assumption $\eta \in \mathcal{Z}^{\mu^\# \mu} \setminus \mathcal{Z}^{\mu^\# A_c \mu}$ and Lemma 3.3.6.(iv)

$$\mu^\#_c = \nu(\eta)_c. \tag{3.53}$$

Moreover, we have according to Lemma 3.3.3.(ii)

$$[\nu(\eta)] \subseteq [\mu]\,. \tag{3.54}$$

This and the construction of μR_c from μ in Definition 3.2.5.(ii) lead to

$$[\nu(\eta)] \subseteq [\mu R_c]\,. \tag{3.55}$$

Now, according to Lemma 3.2.24, the tableau $\mathbf{t}^{\mu R_c} f_{\mu^\# \mu c} w_\mu$ is obtained from the tableau $\mathbf{t}^\mu w_\mu = \mathbf{t}_\mu$ by moving the rightmost $\mu_c - \mu_c^\#$ entries in the c-th row of $\mathbf{t}_\mu$ in the given order to the end of the $(c-1)$-th row of $\mathbf{t}_\mu$. This process doesn't move the leftmost $\nu(\eta)_c = \mu_c^\#$ (see (3.53)) entries in the c-th row of $\mathbf{t}_\mu$ — that is, exactly the entries in the c-th row occupying positions contained in $[\nu(\eta)]$ — and the entries in all other rows of $\mathbf{t}_\mu$. This implies

$$\mathbf{t}_\mu \Big\downarrow^{[\mu]}_{[\nu(\eta)]} = \left(\mathbf{t}^{\mu R_c} f_{\mu^\# \mu c} w_\mu\right) \Big\downarrow^{[\mu R_c]}_{[\nu(\eta)]}. \tag{3.56}$$

Here, $\nu(\eta)$ and μR_c are not considered a PK-pair.

Now we get from (3.52) and (3.56) the following relation between rows of $\mathbf{t}^\kappa$ and columns of $\mathbf{t}^{\mu R_c} f_{\mu^\# \mu c} w_\mu$.

> For every $j \in \{1, \ldots, \mu_1\}$ the sequence of the entries in the row of $\mathbf{t}^\kappa$ corresponding to the first entry in the j-th subsequence of κ when considered from left to right coincides with the sequence of the entries in the j-th column of $\left(\mathbf{t}^{\mu R_c} f_{\mu^\# \mu c} w_\mu\right) \Big\downarrow^{[\mu R_c]}_{[\nu(\eta)]}$ when considered from top to bottom.

As before, $\nu(\eta)$ and μR_c are not considered a PK-pair.

The application of $\left(f_{\mu^\# \mu c} w_\mu\right)^{-1}$ to the tableaux $\mathbf{t}^\kappa$ and $\mathbf{t}^{\mu R_c} f_{\mu^\# \mu c} w_\mu$ occurring in the preceding relation leads to the following statement.

> For every $j \in \{1, \ldots, \mu_1\}$ the sequence of the entries in the row of $\mathbf{t}^\kappa \left(f_{\mu^\# \mu c} w_\mu\right)^{-1}$ corresponding to the first entry in the j-th subsequence of κ when considered from left to right coincides with the sequence of the entries in the j-th column of $\mathbf{t}^{\mu R_c} \Big\downarrow^{[\mu R_c]}_{[\nu(\eta)]}$ when considered from top to bottom.

Again, $\nu(\eta)$ and μR_c are not considered a PK-pair. From the preceding relation and the fact that $\mathbf{t}^{\mu R_c}$ is column standard, we see that for every $j \in \{1, \ldots, \mu_1\}$ the entries in the row of $\mathbf{t}^\kappa \left(f_{\mu^\# \mu c} w_\mu\right)^{-1}$ corresponding to the first entry in the j-th subsequence of κ are arranged in ascending order from left to right. Furthermore, the construction of the subsequences of κ and the particular form of κ show that every other row of $\mathbf{t}^\kappa \left(f_{\mu^\# \mu c} w_\mu\right)^{-1}$ contains at most one entry.

All this implies that $\mathbf{t}^\kappa \left(f_{\mu^\# \mu c} w_\mu\right)^{-1}$ is row standard. From this the claim follows as explained above.

(ii) The proof of this statement is similar to the proof of statement (ii) in Lemma 3.3.14.

According to Lemma 3.3.14.(i), we have $1_{\mathfrak{S}_n} \in Y(\eta)$ and thus

$$\left[f_{\mu^\# \mu c} w_\mu g(\eta)\right]^{\mu R_c} \in \left\{ \left[f_{\mu^\# \mu c} w_\mu y g(\eta)\right]^{\mu R_c} \,\middle|\, y \in Y(\eta) \right\}.$$

Now fix a

$$y \in Y(\eta) \setminus \{1_{\mathfrak{S}_n}\}.$$

In order to establish the desired relation $\left[f_{\mu^\# \mu c} w_\mu y g(\eta)\right]^{\mu R_c} < \left[f_{\mu^\# \mu c} w_\mu g(\eta)\right]^{\mu R_c}$, we compare various tableaux. To this end, we write $\mu = (\mu_1, \mu_2, \ldots)$, $\mu^\# = (\mu_1^\#, \mu_2^\#, \ldots)$, and $\nu(\eta) = (\nu(\eta)_1, \nu(\eta)_2, \ldots)$ as in the proof of statement (i). With that we have, again as in the proof of statement (i),

$$\mu_c^\# = \nu(\eta)_c, \qquad [\nu(\eta)] \subseteq [\mu], \qquad [\nu(\eta)] \subseteq [\mu R_c]$$

(see (3.53), (3.54), (3.55), and Definition 1.1.6).

Now, according to Lemma 3.2.24, the tableau $\mathbf{t}^{\mu R_c} f_{\mu^\# \mu c} w_\mu g(\eta)$ is obtained from the tableau $\mathbf{t}^\mu w_\mu g(\eta) = \mathbf{t}_\mu g(\eta) = \mathbf{t}(\eta)$ by moving the rightmost $\mu_c - \mu_c^\#$ entries in the c-th row of $\mathbf{t}^\mu w_\mu g(\eta)$ in the given order to the end of the $(c-1)$-th row of $\mathbf{t}^\mu w_\mu g(\eta)$. This process doesn't move the leftmost $\nu(\eta)_c = \mu_c^\#$ (see above) entries in the c-th row of $\mathbf{t}^\mu w_\mu g(\eta)$ — that is, exactly the entries in the c-th row of $\mathbf{t}^\mu w_\mu g(\eta)$ occupying positions contained in $[\nu(\eta)]$ — and the entries in all other rows of $\mathbf{t}^\mu w_\mu g(\eta)$. This implies (see Definition 3.2.9)

$$\left(\mathbf{t}^{\mu R_c} f_{\mu^\# \mu c} w_\mu g(\eta)\right)\Big\downarrow^{[\mu R_c]}_{[\nu(\eta)]} = \left(\mathbf{t}^\mu w_\mu g(\eta)\right)\Big\downarrow^{[\mu]}_{[\nu(\eta)]} = \mathbf{t}(\eta)\Big\downarrow^{[\mu]}_{[\nu(\eta)]}.$$

Here, $\nu(\eta)$ and μR_c are not considered a PK-pair. Now we get the following statement from the preceding relation and Lemma 3.3.3.(v).

The entries in every column of $\left(\mathbf{t}^{\mu R_c} f_{\mu^\# \mu c} w_\mu g(\eta)\right)\Big\downarrow^{[\mu R_c]}_{[\nu(\eta)]}$ are arranged in ascending order from top to bottom. (3.57)

As before, $\nu(\eta)$ and μR_c are not considered a PK-pair.

In the same way, again according to Lemma 3.2.24, the tableau $\mathbf{t}^{\mu R_c} f_{\mu^\# \mu c} w_\mu$ is obtained from the tableau $\mathbf{t}^\mu w_\mu = \mathbf{t}_\mu$ by moving the rightmost $\mu_c - \mu_c^\#$ entries in the c-th row of $\mathbf{t}^\mu w_\mu$ in the given order to the end of the $(c-1)$-th row of $\mathbf{t}^\mu w_\mu$ and not touching the leftmost $\nu(\eta)_c = \mu_c^\#$ entries in the c-th row and the entries in all other rows of $\mathbf{t}^\mu w_\mu$. This implies

$$\left(\mathbf{t}^{\mu R_c} f_{\mu^\# \mu c} w_\mu\right)\Big\downarrow^{[\mu R_c]}_{[\nu(\eta)]} = \left(\mathbf{t}^\mu w_\mu\right)\Big\downarrow^{[\mu]}_{[\nu(\eta)]} = \mathbf{t}_\mu\Big\downarrow^{[\mu]}_{[\nu(\eta)]}. \tag{3.58}$$

Again, $\nu(\eta)$ and μR_c are not considered a PK-pair. Furthermore, Definition 3.3.13, Definition 3.2.12, and Remark 3.2.13 show that the tableau $\mathbf{t}^{\mu} w_{\mu} y$ is obtained from the tableau $\mathbf{t}^{\mu} w_{\mu}$ by an application of $y \in Y(\eta)$ which only permutes the entries in the various columns of $(\mathbf{t}^{\mu} w_{\mu})\big\downarrow^{[\mu]}_{[\nu(\eta)]}$ respectively amongst themselves. From this and (3.58) we see that the tableau $\mathbf{t}^{\mu R_c} f_{\mu\#\mu c} w_{\mu} y$ is obtained from the tableau $\mathbf{t}^{\mu R_c} f_{\mu\#\mu c} w_{\mu}$ by a permutation of the entries in the various columns of $\left(\mathbf{t}^{\mu R_c} f_{\mu\#\mu c} w_{\mu}\right)\big\downarrow^{[\mu R_c]}_{[\nu(\eta)]}$ respectively amongst themselves. As before, $\nu(\eta)$ and μR_c are not considered a PK-pair. Finally, the application of $g(\eta)$ to the tableaux occurring in this construction leads to the following relation between $\mathbf{t}^{\mu R_c} f_{\mu\#\mu c} w_{\mu} g(\eta)$ and $\mathbf{t}^{\mu R_c} f_{\mu\#\mu c} w_{\mu} y g(\eta)$.

> The tableau $\mathbf{t}^{\mu R_c} f_{\mu\#\mu c} w_{\mu} y g(\eta)$ is obtained from the tableau $\mathbf{t}^{\mu R_c} f_{\mu\#\mu c} w_{\mu} g(\eta)$ by a permutation of the entries in the various columns of $\left(\mathbf{t}^{\mu R_c} f_{\mu\#\mu c} w_{\mu} g(\eta)\right)\big\downarrow^{[\mu R_c]}_{[\nu(\eta)]}$ respectively amongst themselves. (3.59)

Again, $\nu(\eta)$ and μR_c are not considered a PK-pair.

Now statements (3.57) and (3.59) enable us to establish the desired relation $\left[f_{\mu\#\mu c} w_{\mu} y g(\eta)\right]^{\mu R_c} < \left[f_{\mu\#\mu c} w_{\mu} g(\eta)\right]^{\mu R_c}$. Because of $y \neq 1_{\mathfrak{S}_n}$ we have

$$\mathbf{t}^{\mu R_c} f_{\mu\#\mu c} w_{\mu} g(\eta) \neq \mathbf{t}^{\mu R_c} f_{\mu\#\mu c} w_{\mu} y g(\eta).$$

So we must have an entry

$$m \in \{1, \ldots, n\}$$

such that on the one hand m occupies different positions in $\mathbf{t}^{\mu R_c} f_{\mu\#\mu c} w_{\mu} g(\eta)$ and $\mathbf{t}^{\mu R_c} f_{\mu\#\mu c} w_{\mu} y g(\eta)$ and on the other hand every $k \in \{m+1, \ldots, n\}$ occupies the same position in $\mathbf{t}^{\mu R_c} f_{\mu\#\mu c} w_{\mu} g(\eta)$ and $\mathbf{t}^{\mu R_c} f_{\mu\#\mu c} w_{\mu} y g(\eta)$. Thus we can write, using Definition 1.1.45.(ii),

$$\forall k \in \{m+1, \ldots, n\} : (k)\zeta_{\mathbf{t}^{\mu R_c} f_{\mu\#\mu c} w_{\mu} y g(\eta)} = (k)\zeta_{\mathbf{t}^{\mu R_c} f_{\mu\#\mu c} w_{\mu} g(\eta)}. \tag{3.60}$$

Moreover, the construction of $\mathbf{t}^{\mu R_c} f_{\mu\#\mu c} w_{\mu} y g(\eta)$ from $\mathbf{t}^{\mu R_c} f_{\mu\#\mu c} w_{\mu} g(\eta)$ described in (3.59) cannot fix all entries different from m, in addition to m there must be another — by choice of m necessarily smaller — entry which is moved. This shows

$$m > 1. \tag{3.61}$$

Furthermore, again according to (3.59), the positions occupied by m in the tableaux $\mathbf{t}^{\mu R_c} f_{\mu\#\mu c} w_{\mu} g(\eta)$ and $\mathbf{t}^{\mu R_c} f_{\mu\#\mu c} w_{\mu} y g(\eta)$ are located in the same column but in different rows of $[\nu(\eta)]$. Finally, (3.57) and the choice of m show that the positions in the column of $\left(\mathbf{t}^{\mu R_c} f_{\mu\#\mu c} w_{\mu} g(\eta)\right)\big\downarrow^{[\mu R_c]}_{[\nu(\eta)]}$ containing m which are located below m

only contain entries bigger than m. These entries are not moved in the transition from $\mathbf{t}^{\mu R_c} f_{\mu \# \mu c} w_\mu g(\eta)$ to $\mathbf{t}^{\mu R_c} f_{\mu \# \mu c} w_\mu y g(\eta)$. Since in the course of this transition m itself is moved within its column in $\left(\mathbf{t}^{\mu R_c} f_{\mu \# \mu c} w_\mu g(\eta)\right) \Big\downarrow^{[\mu R_c]}_{[\nu(\eta)]}$, the position occupied by m in $\mathbf{t}^{\mu R_c} f_{\mu \# \mu c} w_\mu y g(\eta)$ must be located above the position occupied by m in $\mathbf{t}^{\mu R_c} f_{\mu \# \mu c} w_\mu g(\eta)$. More formally, we have

$$(m)\zeta_{\mathbf{t}^{\mu R_c} f_{\mu \# \mu c} w_\mu y g(\eta)} < (m)\zeta_{\mathbf{t}^{\mu R_c} f_{\mu \# \mu c} w_\mu g(\eta)}. \tag{3.62}$$

Now we move from the tableaux $\mathbf{t}^{\mu R_c} f_{\mu \# \mu c} w_\mu g(\eta)$ and $\mathbf{t}^{\mu R_c} f_{\mu \# \mu c} w_\mu y g(\eta)$ to the tableaux

$$\mathbf{t}^{\mu R_c}\left[f_{\mu \# \mu c} w_\mu g(\eta)\right]^{\mu R_c} \qquad \text{and} \qquad \mathbf{t}^{\mu R_c}\left[f_{\mu \# \mu c} w_\mu y g(\eta)\right]^{\mu R_c}.$$

From Definition 1.1.58.(ii) and Definition 1.1.55 we see that $\mathbf{t}^{\mu R_c} f_{\mu \# \mu c} w_\mu g(\eta)$ and $\mathbf{t}^{\mu R_c}\left[f_{\mu \# \mu c} w_\mu g(\eta)\right]^{\mu R_c}$ and also $\mathbf{t}^{\mu R_c} f_{\mu \# \mu c} w_\mu y g(\eta)$ and $\mathbf{t}^{\mu R_c}\left[f_{\mu \# \mu c} w_\mu y g(\eta)\right]^{\mu R_c}$ respectively differ by a permutation of the entries in the various rows amongst themselves. This means that the row numbers of entries remain fixed in the transition from $\mathbf{t}^{\mu R_c} f_{\mu \# \mu c} w_\mu g(\eta)$ to $\mathbf{t}^{\mu R_c}\left[f_{\mu \# \mu c} w_\mu g(\eta)\right]^{\mu R_c}$ and in the transition from $\mathbf{t}^{\mu R_c} f_{\mu \# \mu c} w_\mu y g(\eta)$ to $\mathbf{t}^{\mu R_c}\left[f_{\mu \# \mu c} w_\mu y g(\eta)\right]^{\mu R_c}$. With that we get from (3.60) and (3.62)

$$\forall k \in \{m+1, \ldots, n\} : (k)\zeta_{\mathbf{t}^{\mu R_c}\left[f_{\mu \# \mu c} w_\mu y g(\eta)\right]^{\mu R_c}} = (k)\zeta_{\mathbf{t}^{\mu R_c}\left[f_{\mu \# \mu c} w_\mu g(\eta)\right]^{\mu R_c}} \tag{3.63}$$

and

$$(m)\zeta_{\mathbf{t}^{\mu R_c}\left[f_{\mu \# \mu c} w_\mu y g(\eta)\right]^{\mu R_c}} < (m)\zeta_{\mathbf{t}^{\mu R_c}\left[f_{\mu \# \mu c} w_\mu g(\eta)\right]^{\mu R_c}}. \tag{3.64}$$

From (3.63), Definition 3.1.1, and Remark 3.1.2.(ii) we easily obtain

$$\forall j \in \{m, \ldots, n\} : \lambda^{\left(\mathbf{t}^{\mu R_c}\left[f_{\mu \# \mu c} w_\mu y g(\eta)\right]^{\mu R_c}\right) \Downarrow^n_j} = \lambda^{\left(\mathbf{t}^{\mu R_c}\left[f_{\mu \# \mu c} w_\mu g(\eta)\right]^{\mu R_c}\right) \Downarrow^n_j}. \tag{3.65}$$

Similarly, we get from (3.61), (3.63), (3.64), and Definition 1.1.4.(i)

$$\lambda^{\left(\mathbf{t}^{\mu R_c}\left[f_{\mu \# \mu c} w_\mu y g(\eta)\right]^{\mu R_c}\right) \Downarrow^n_{m-1}} < \lambda^{\left(\mathbf{t}^{\mu R_c}\left[f_{\mu \# \mu c} w_\mu g(\eta)\right]^{\mu R_c}\right) \Downarrow^n_{m-1}}. \tag{3.66}$$

(3.65) and (3.66) together now show, according to Definition 3.1.4.(i),

$$\mathbf{t}^{\mu R_c}\left[f_{\mu \# \mu c} w_\mu y g(\eta)\right]^{\mu R_c} < \mathbf{t}^{\mu R_c}\left[f_{\mu \# \mu c} w_\mu g(\eta)\right]^{\mu R_c}$$

which in turn, according to Definition 3.1.14, leads to

$$\left[f_{\mu \# \mu c} w_\mu y g(\eta)\right]^{\mu R_c} < \left[f_{\mu \# \mu c} w_\mu g(\eta)\right]^{\mu R_c}.$$

This proves the claim of the statement. ■

Now we are in possession of all the combinatorial objects and statements required in the derivation of the generic Specht series.

3.4 PK-modules for Hecke algebras of type A

In this and the following sections of this chapter the construction of Specht series for Hecke algebras of type A and certain modules of these algebras is carried out. As always, $n \in \mathbb{N}$ denotes a positive integer. Furthermore, we fix a coefficient pair (R,q) as in Definition 1.2.1 for all that follows.

Next we introduce a family of modules for $\mathcal{H}_{A_{n-1}}^{(R,q)}$ indexed by PK$_n$-pairs. Module families of this kind also are considered in [DJ1, Section 7] and [JAM1, Section 17, in particular Definition 17.4]. The following definition makes use of Definition 1.2.3.(ii).

Definition 3.4.1 *Let $\mu^{\#}\mu$ be a PK$_n$-pair. If we have $\mu^{\#}\mu \neq 00$ then the right ideal $S_{(R,q)}^{\mu^{\#}\mu}$ in $\mathcal{H}_{A_{n-1}}^{(R,q)}$ is defined as*

$$S_{(R,q)}^{\mu^{\#}\mu} = x_{\mu}^{(R,q)} T_{w_\mu} \varepsilon_{(R,q)}^{(n)}(V_{\mu^{\#}\mu}) \mathcal{H}_{A_{n-1}}^{(R,q)}.$$

If we have $\mu^{\#}\mu = 00$ then the right ideal $S_{(R,q)}^{\mu^{\#}\mu} = S_{(R,q)}^{00}$ in $\mathcal{H}_{A_{n-1}}^{(R,q)}$ is defined as

$$S_{(R,q)}^{00} = 0_{\mathcal{H}_A^{(R,q)}}.$$

Here, $0_{\mathcal{H}_A^{(R,q)}}$ denotes the null ideal in $\mathcal{H}_{A_{n-1}}^{(R,q)}$. We write

$$S_{(R,q)}^{\mu^{\#}\mu} = S^{\mu^{\#}\mu} \qquad \text{and} \qquad S_{(R,q)}^{00} = S^{00}.$$

In any case, $S_{(R,q)}^{\mu^{\#}\mu}$ is called a PK-module of degree n or a PK$_n$-module or just a PK-module.

Remark 3.4.2 *The PK$_n$-modules introduced in Definition 3.4.1 are generic in the sense of Remark 1.2.9.*

The next statement is a simple consequence of Lemma 3.2.14.(x), it makes use of Definition 1.2.3.(ii).

Lemma 3.4.3 *Let $\mu^{\#}\mu$ and $\lambda^{\#}\mu$ be PK$_n$-pairs with $\mu^{\#}\mu \neq 00 \neq \lambda^{\#}\mu$ and*

$$[\mu^{\#}] \subseteq [\lambda^{\#}]. \tag{3.67}$$

Then we have

$$\varepsilon_{(R,q)}^{(n)}(V_{\lambda^{\#}\mu}) = \varepsilon_{(R,q)}^{(n)}(V_{\mu^{\#}\mu}) h$$

with an appropriate right factor $h \in \mathcal{H}_{A_{n-1}}^{(R,q)}$.

Proof. From the assumption (3.67) and Lemma 3.2.14.(x) we get a set $\mathcal{F} \subseteq \mathfrak{S}_n$ with the property

$$V_{\lambda^{\#}\mu} = V_{\mu^{\#}\mu}\mathcal{F}$$

such that every $w \in V_{\lambda^{\#}\mu}$ has a unique decomposition $w = uf$ with $u \in V_{\mu^{\#}\mu}$ and $f \in \mathcal{F}$ and furthermore arbitrary $u \in V_{\mu^{\#}\mu}$ and $f \in \mathcal{F}$ satisfy

$$\ell(uf) = \ell(u) + \ell(f).$$

From this, Definition 1.2.3.(ii), and (1.22) on page 34 we obtain

$$\varepsilon(V_{\lambda^{\#}\mu}) = \varepsilon(V_{\mu^{\#}\mu})\varepsilon(\mathcal{F}).$$

This proves the claim if we put $h = \varepsilon(\mathcal{F})$. ■

The following lemma makes use of Definition 1.2.3.(ii), Definition 3.2.3, and Definition 3.2.5.(iv).

Lemma 3.4.4 *Let $\mu^{\#}\mu$ be a PK$_n$-pair with $\mu^{\#}\mu \neq 00$ and $\mu = (\mu_1, \mu_2, \ldots) \vDash n$. Then the following statements hold.*

(i) We have

$$S^{\mu^{\#}\mu}_{(R,q)} = x^{(R,q)}_{\mu} T_{w_{\mu^{\#}\mu}} \varepsilon^{(n)}_{(R,q)}(U_{\mu^{\#}\mu}) \mathcal{H}^{(R,q)}_{A_{n-1}}.$$

(ii) For $\mu^{\#} = (\mu_1) \vdash \mu_1$ we have

$$S^{\mu^{\#}\mu}_{(R,q)} = S^{(\mu_1)\mu}_{(R,q)} = M^{\mu}_{(R,q)}.$$

(iii) For $\mu^{\#} = \mu$ we have

$$S^{\mu^{\#}\mu}_{(R,q)} = S^{\mu\mu}_{(R,q)} = S^{\mu}_{(R,q)}.$$

(iv) Let $\lambda^{\#}\mu \neq 00$ be another PK$_n$-pair satisfying $\left[\mu^{\#}\right] \subseteq \left[\lambda^{\#}\right]$. Then we have

$$S^{\lambda^{\#}\mu}_{(R,q)} \subseteq S^{\mu^{\#}\mu}_{(R,q)}.$$

(v) Let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^{\#}\mu$. Then we have

$$S^{\mu^{\#}A_c\mu}_{(R,q)} \subseteq S^{\mu^{\#}\mu}_{(R,q)}.$$

(vi) We have

$$S^{\mu^{\#}\mu}_{(R,q)} \subseteq M^{\mu}_{(R,q)}.$$

Proof. (i) From Definition 3.2.8.(iii), Lemma 3.2.11.(iv), and (1.22) on page 34 we obtain

$$T_{w_\mu} = T_{w_{\mu^\#\mu}} T_{g_{\mu^\#\mu}}. \tag{3.68}$$

Furthermore we get from Lemma 3.2.14, Definition 1.2.3, and again (1.22)

$$T_{g_{\mu^\#\mu}} \varepsilon(V_{\mu^\#\mu}) = \varepsilon(U_{\mu^\#\mu}) T_{g_{\mu^\#\mu}}. \tag{3.69}$$

Now the substitution of (3.68) and (3.69) into Definition 3.4.1 and the invertibility of $T_{g_{\mu^\#\mu}}$ in $\mathcal{H}_n$ (see (1.23) on page 34) show the claim.

(ii) Since $\left[\mu^\#\right] = [(\mu_1)]$ consists of a single row, every column of $\left[\mu^\#\right]$ contains at most one square. This shows together with Definition 3.2.12 and Remark 3.2.13

$$V_{\mu^\#\mu} = V_{(\mu_1)\mu} = \left\{1_{\mathfrak{S}_n}\right\}.$$

According to Definition 1.2.3.(ii), this means

$$\varepsilon(V_{\mu^\#\mu}) = \varepsilon(V_{(\mu_1)\mu}) = 1_{\mathcal{H}_A}. \tag{3.70}$$

Now the substitution of (3.70) into Definition 3.4.1, (1.23) on page 34, and Definition 1.3.1.(ii) show the claim.

(iii) The condition $\mu^\# = \mu$ implies

$$[\mu] \setminus \left[\mu^\#\right] = \varnothing$$

and furthermore with Definition 3.2.9 for the tableau $\mathbf{t}_\mu$

$$\mathbf{t}_\mu \Big\downarrow^{[\mu]}_{[\mu^\#]} = \mathbf{t}_\mu \Big\downarrow^{[\mu]}_{[\mu]} = \mathbf{t}_\mu.$$

From all this we see, using Remark 3.2.13 and Remark 1.1.51.(ii),

$$V_{\mu^\#\mu} = V_{\mu\mu} = \mathfrak{C}_{\mathbf{t}_\mu}.$$

Now the assumption $\mu^\# = \mu$ and Definition 3.2.1 show that μ is a partition, which enables us to apply Lemma 1.1.52 and Lemma 1.1.69.(i) from which we get

$$V_{\mu^\#\mu} = V_{\mu\mu} = \mathfrak{R}_{\mathbf{t}(\mu')} = \mathfrak{S}_{\mu'}.$$

This implies

$$\varepsilon(V_{\mu^\#\mu}) = \varepsilon(V_{\mu\mu}) = \varepsilon(\mathfrak{S}_{\mu'}) = y_{\mu'}. \tag{3.71}$$

Now the substitution of (3.71) into Definition 3.4.1 and Definition 1.3.10 prove the claim.

(iv) Lemma 3.4.3 can be applied to the situation at hand and shows that we have

$$\varepsilon(V_{\lambda^\# \mu}) = \varepsilon(V_{\mu^\# \mu})h$$

with an appropriate $h \in \mathcal{H}_n$. If we substitute this into Definition 3.4.1 then we get

$$\begin{aligned} S^{\lambda^\# \mu} &= x_\mu T_{w_\mu} \varepsilon(V_{\lambda^\# \mu})\mathcal{H}_n \\ &= x_\mu T_{w_\mu} \varepsilon(V_{\mu^\# \mu}) h \mathcal{H}_n \\ &\subseteq x_\mu T_{w_\mu} \varepsilon(V_{\mu^\# \mu})\mathcal{H}_n \\ &= S^{\mu^\# \mu}, \end{aligned}$$

as desired.

(v) In the case $\mu^\# A_c \mu = 00$, there is nothing to prove (see Definition 3.4.1). In the case $\mu^\# A_c \mu \neq 00$, the claim follows from Lemma 3.2.7.(i), Lemma 3.2.7.(ii), and statement (iv).

(vi) This follows immediately from Definition 1.3.1.(ii) and Definition 3.4.1. ■

3.5 PK-homomorphisms for Hecke algebras of type A

Now we describe certain generic homomorphisms between PK-modules. Such homomorphisms also are considered in [DJ1, Section 7]. As before, n denotes a positive integer and (R, q) denotes a coefficient pair.

The following two statements are of a more technical nature. The first one of them makes use of Lemma 1.1.56, Corollary 1.1.57, Definition 1.2.3.(i), and Definition 3.2.5.(ii).

Lemma 3.5.1 *Let $\mu^\# \mu$ be a PK$_n$-pair with $\mu^\# \mu \neq 00$ and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^\# \mu$. Then we have in $\mathcal{H}^{(R,q)}_{A_{n-1}}$*

$$\begin{aligned} \iota^{(n)}_{(R,q)}(\mathcal{D}^{-1}_\mu \cap (\mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu)) x^{(R,q)}_\mu &= \iota^{(n)}_{(R,q)}(\mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu) \\ &= x^{(R,q)}_{\mu R_c} \iota^{(n)}_{(R,q)}((\mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu) \cap \mathcal{D}_{\mu R_c}). \end{aligned}$$

Proof. According to Corollary 1.1.57.(i), the set of the shortest representatives of cosets occurring in the decomposition of $\mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu$ into left cosets of $\mathfrak{S}_\mu$ is given by

$$\mathcal{D}^{-1}_\mu \cap (\mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu).$$

Similarly, according to Lemma 1.1.56.(i), the set of the shortest representatives of cosets occurring in the decomposition of $\mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu$ into right cosets of $\mathfrak{S}_{\mu R_c}$ is given by

$$(\mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu) \cap \mathcal{D}_{\mu R_c}.$$

Thus we have

$$\left(\mathcal{D}_\mu^{-1} \cap (\mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu)\right)\mathfrak{S}_\mu = \mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu = \mathfrak{S}_{\mu R_c}\left((\mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu) \cap \mathcal{D}_{\mu R_c}\right).$$

The claim now follows easily from Corollary 1.1.57.(ii), Lemma 1.1.56.(ii), (1.22) on page 34, and Definition 1.2.3.(i). ■

The next lemma makes use of Definition 3.2.8.(i) and Definition 1.2.3.(ii).

Lemma 3.5.2 *Let $m \in \{2, \ldots, n\}$, $\lambda \vDash n$, and $d \in \mathcal{D}_\lambda$ satisfying*

$$(m)\zeta_{\mathbf{t}^\lambda d} = (m-1)\zeta_{\mathbf{t}^\lambda d}.$$

(i) Let $\mu^\# \mu$ be a PK_n-pair with $\mu^\#\mu \neq 00$ and the following properties.

$$\begin{array}{rcl} (m)\mathbf{t}_\mu^{-1}, (m-1)\mathbf{t}_\mu^{-1} & \in & \left[\mu^\#\right] \\ (m)\sigma_{\mathbf{t}_\mu} & = & (m-1)\sigma_{\mathbf{t}_\mu} \end{array}$$

Then we have in $\mathcal{H}_{A_{n-1}}^{(R,q)}$

$$x_\lambda^{(R,q)} T_d \varepsilon_{(R,q)}^{(n)}(V_{\mu^\#\mu}) = 0_{\mathcal{H}_A^{(R,q)}}.$$

(ii) Let $\mu^\# \mu$ be a PK_n-pair with $\mu^\#\mu \neq 00$ and the following properties.

$$\begin{array}{rcl} (m)\left(\mathbf{t}^{\mu^\#\mu}\right)^{-1}, (m-1)\left(\mathbf{t}^{\mu^\#\mu}\right)^{-1} & \in & \left[\mu^\#\right] \\ (m)\sigma_{\mathbf{t}^{\mu^\#\mu}} & = & (m-1)\sigma_{\mathbf{t}^{\mu^\#\mu}} \end{array}$$

Then we have in $\mathcal{H}_{A_{n-1}}^{(R,q)}$

$$x_\lambda^{(R,q)} T_d \varepsilon_{(R,q)}^{(n)}(U_{\mu^\#\mu}) = 0_{\mathcal{H}_A^{(R,q)}}.$$

Proof. (i) In the following we will consider the transposition

$$s = (m-1, m) \in \mathfrak{B}_n \subseteq \mathfrak{S}_n$$

(see (1.5) on page 2). If we put

$$(m)\sigma_{\mathbf{t}_\mu} = (m-1)\sigma_{\mathbf{t}_\mu} = k \in \mathbb{N}$$

then we get from the assumptions $(m)\mathbf{t}_\mu^{-1}, (m-1)\mathbf{t}_\mu^{-1} \in \left[\mu^\#\right]$ and $(m)\sigma_{\mathbf{t}_\mu} = (m-1)\sigma_{\mathbf{t}_\mu}$ together with Definition 3.2.12, Remark 3.2.13, and the notation from Lemma 3.2.14.(ii)

$$s \in V_{\mu^\#\mu}^{(k)} \subseteq V_{\mu^\#\mu}.$$

Furthermore, we see from (1.18) on page 25, using the notation from Lemma 3.2.14 and in particular from statement (ii) thereof, that the group $\{1_{\mathfrak{S}_n}, s\}$ is the Young subgroup of the symmetric group $V^{(k)}_{\mu^\#\mu}$ of degree $\mu^{\#\prime}_k$ associated with the composition $(1^{(m-2)-m^+_{k-1}}, 2, 1^{\mu^{\#\prime}_k-(m-m^+_{k-1})})$ of $\mu^{\#\prime}_k$. Thus we have the set of shortest representatives of the right cosets of $\{1_{\mathfrak{S}_n}, s\}$ in $V^{(k)}_{\mu^\#\mu}$. In the following this set is denoted by $\mathcal{F}$.

Using the preceding considerations, we now derive a product decomposition of $\varepsilon(V_{\mu^\#\mu}) \in \mathcal{H}_n$. First, we get from Definition 1.2.3.(ii), the additivity of the length function with respect to the decomposition (3.9) of $V_{\mu^\#\mu}$ in Lemma 3.2.14.(ii), and (1.22) on page 34, using the notation from Lemma 3.2.14,

$$\varepsilon(V_{\mu^\#\mu}) = \varepsilon(V^{(1)}_{\mu^\#\mu}) \cdots \varepsilon(V^{(\mu_1)}_{\mu^\#\mu}).$$

Furthermore, we obtain from the arguments just employed together with the commutativity of elements of different factors $V^{(h)}_{\mu^\#\mu}$ with $h \in \{1, \ldots, \mu_1\}$

$$\forall i, j \in \{1, \ldots, \mu_1\} : \varepsilon(V^{(i)}_{\mu^\#\mu})\varepsilon(V^{(j)}_{\mu^\#\mu}) = \varepsilon(V^{(j)}_{\mu^\#\mu})\varepsilon(V^{(i)}_{\mu^\#\mu}).$$

Finally, we see from the construction of the set $\mathcal{F} \subseteq \mathfrak{S}_n$ described above and Lemma 1.1.56

$$\varepsilon(V^{(k)}_{\mu^\#\mu}) = \varepsilon(\{1_{\mathfrak{S}_n}, s\})\varepsilon(\mathcal{F}).$$

All this shows that $\varepsilon(V_{\mu^\#\mu})$ has the left factor

$$\varepsilon(\{1_{\mathfrak{S}_n}, s\}) = T_{1_{\mathfrak{S}_n}} - q^{-1}T_s = 1_{\mathcal{H}_A} - q^{-1}T_s.$$

The assumption $(m)\zeta_{\mathbf{t}^\lambda d} = (m-1)\zeta_{\mathbf{t}^\lambda d}$ and Lemma 1.3.4 now imply

$$x_\lambda T_d \varepsilon(\{1_{\mathfrak{S}_n}, s\}) = 0_{\mathcal{H}_A}.$$

This proves the claim.

(ii) The proof of this statement is completely analogous to that of statement (i). If we consider the transposition

$$s = (m-1, m) \in \mathfrak{B}_n \subseteq \mathfrak{S}_n$$

and put

$$(m)\sigma_{\mathbf{t}^{\mu^\#\mu}} = (m-1)\sigma_{\mathbf{t}^{\mu^\#\mu}} = k \in \mathbb{N}$$

then we get from the assumptions of the statement, Definition 3.2.12, and Remark 3.2.13, using the notation from Lemma 3.2.14.(i),

$$s \in U^{(k)}_{\mu^\#\mu} \subseteq U_{\mu^\#\mu}.$$

Furthermore, we see, using the notation from Lemma 3.2.14 and in particular from statement (i) thereof, that the group $\{1_{\mathfrak{S}_n}, s\}$ is the Young subgroup of the symmetric group $U^{(k)}_{\mu^\#\mu}$ of degree $\mu^{\#\prime}_k$ (see (1.18) on page 25) associated with $(1^{(m-2)-\mu^{\#\prime+}_{k-1}}, 2, 1^{\mu^{\#\prime}_k-(m-\mu^{\#\prime+}_{k-1})}) \models \mu^{\#\prime}_k$. Thus we have the set of shortest representatives of the right cosets of $\{1_{\mathfrak{S}_n}, s\}$ in $U^{(k)}_{\mu^\#\mu}$. In the following this set is denoted by $\mathcal{F}$.

Using the preceding considerations, we now derive a product decomposition of $\varepsilon(U_{\mu^\#\mu}) \in \mathcal{H}_n$. From Definition 1.2.3.(ii), the additivity of the length function with respect to the decomposition (3.5) of $U_{\mu^\#\mu}$ in Lemma 3.2.14.(i), and (1.22) on page 34 we get, using the notation from Lemma 3.2.14,

$$\varepsilon(U_{\mu^\#\mu}) = \varepsilon(U^{(1)}_{\mu^\#\mu}) \cdots \varepsilon(U^{(\mu_1)}_{\mu^\#\mu}).$$

Furthermore, we obtain from the arguments just employed together with the commutativity of elements of different factors $U^{(h)}_{\mu^\#\mu}$ with $h \in \{1, \ldots, \mu_1\}$

$$\forall i, j \in \{1, \ldots, \mu_1\} : \varepsilon(U^{(i)}_{\mu^\#\mu})\varepsilon(U^{(j)}_{\mu^\#\mu}) = \varepsilon(U^{(j)}_{\mu^\#\mu})\varepsilon(U^{(i)}_{\mu^\#\mu}).$$

Finally, we see from the construction of the set $\mathcal{F} \subseteq \mathfrak{S}_n$ described above and Lemma 1.1.56

$$\varepsilon(U^{(k)}_{\mu^\#\mu}) = \varepsilon(\{1_{\mathfrak{S}_n}, s\})\varepsilon(\mathcal{F}).$$

Thus $\varepsilon(U_{\mu^\#\mu})$ has the left factor

$$\varepsilon(\{1_{\mathfrak{S}_n}, s\}) = T_{1_{\mathfrak{S}_n}} - q^{-1}T_s = 1_{\mathcal{H}_A} - q^{-1}T_s.$$

Now we get from Lemma 1.3.4 and the assumption $(m)\zeta_{\mathbf{t}^\lambda d} = (m-1)\zeta_{\mathbf{t}^\lambda d}$

$$x_\lambda T_d \varepsilon(\{1_{\mathfrak{S}_n}, s\}) = 0_{\mathcal{H}_A}.$$

This proves the claim. ■

Definition 1.3.1 and Lemma 3.5.1 show that the following definition is meaningful.

Definition 3.5.3 *Let $\mu^\#\mu$ be a PK_n-pair with $\mu^\#\mu \neq 00$ and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^\#\mu$. Then the $\mathcal{H}^{(R,q)}_{A_{n-1}}$-homomorphism*

$$\Psi^{(R,q)}_{\mu^\#\mu c} : M^{\mu}_{(R,q)} \to M^{\mu R_c}_{(R,q)}$$

is defined by

$$\begin{aligned} x^{(R,q)}_\mu \Psi^{(R,q)}_{\mu^\#\mu c} &= \iota^{(n)}_{(R,q)}(\mathcal{D}^{-1}_\mu \cap (\mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu))x^{(R,q)}_\mu \\ &= x^{(R,q)}_{\mu R_c}\iota^{(n)}_{(R,q)}((\mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu) \cap \mathcal{D}_{\mu R_c}) \end{aligned}$$

and $\mathcal{H}^{(R,q)}_{A_{n-1}}$-linear extension. We write

$$\Psi^{(R,q)}_{\mu^{\#}\mu c} = \Psi_{\mu^{\#}\mu c}.$$

$\Psi^{(R,q)}_{\mu^{\#}\mu c}$ is called a PK-homomorphism of degree n or a PK_n-homomorphism or just a PK-homomorphism.

Remark 3.5.4 *The PK_n-homomorphisms from Definition 3.5.3 are generic in the sense of Remark 1.2.9. In fact, they are particular elements of the generic bases of the sets of homomorphisms between permutation modules of a Hecke algebra of type A constructed by Dipper and James in [DJ1, Section 3] (see especially [DJ1, Theorem 3.4]).*

Next we derive some properties of PK_n-homomorphisms which are fundamental to the construction of Specht series. The following statement corresponds to [DJ1, Lemma 7.1]. It makes use of Definition 3.2.15, Lemma 3.2.7.(ii), Definition 3.4.1, and the notation (1.1) on page 1.

Lemma 3.5.5 *Let $\mu^{\#}\mu$ be a PK_n-pair with $\mu^{\#}\mu \neq 00$ and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^{\#}\mu$. Then we have for the homomorphism $\Psi^{(R,q)}_{\mu^{\#}\mu c} : M^{\mu}_{(R,q)} \to M^{\mu R_c}_{(R,q)}$*

(i) $x^{(R,q)}_{\mu}\Psi^{(R,q)}_{\mu^{\#}\mu c} = x^{(R,q)}_{\mu R_c} \sum_{f \in I_{\mu^{\#}\mu c}} T_f$,

(ii) $S^{\mu^{\#}A_c\mu}_{(R,q)} \subseteq \operatorname{Ker}\left(\Psi^{(R,q)}_{\mu^{\#}\mu c}\Big\downarrow^{M^{\mu}_{(R,q)}}_{S^{\mu^{\#}\mu}_{(R,q)}}\right)$,

(iii) $S^{\mu^{\#}\mu}_{(R,q)}\Psi^{(R,q)}_{\mu^{\#}\mu c} = S^{\mu^{\#}\mu R_c}_{(R,q)}$.

Proof. (i) This is obtained from Definition 3.5.3, Definition 1.2.3.(i), and Lemma 3.2.17.(ii).

(ii) According to Lemma 3.4.4.(vi), we can build $\Psi_{\mu^{\#}\mu c}\Big\downarrow^{M^{\mu}}_{S^{\mu^{\#}\mu}}$. Moreover, we get from Lemma 3.4.4.(v)

$$S^{\mu^{\#}A_c\mu} \subseteq S^{\mu^{\#}\mu}.$$

This shows that the claim makes sense. Now in the case $\mu^{\#}A_c\mu = 00$ we have, according to Definition 3.4.1, $S^{00} = 0_{\mathcal{H}_A}$ (the null ideal in $\mathcal{H}_n$) and there is nothing to prove. Thus we can assume

$$\mu^{\#}A_c\mu \neq 00$$

in the following. Then we see from Definition 3.4.1 that it is enough to prove $x_{\mu}T_{w_{\mu}}\varepsilon(V_{\mu^{\#}A_c\mu})\Psi_{\mu^{\#}\mu c} = 0_{\mathcal{H}_A}$ (the additive neutral element of $\mathcal{H}_n$). Now we get from Definition 3.5.3, statement (i), Lemma 3.2.17.(i), and Lemma 1.1.68.(ii)

$$\begin{aligned} x_\mu T_{w_\mu} \varepsilon(V_{\mu^\# A_c \mu}) \Psi_{\mu^\# \mu c} &= x_\mu \Psi_{\mu^\# \mu c} T_{w_\mu} \varepsilon(V_{\mu^\# A_c \mu}) \\ &= x_{\mu R_c} \left(\sum_{f \in I_{\mu^\# \mu c}} T_f \right) T_{w_\mu} \varepsilon(V_{\mu^\# A_c \mu}) \\ &= x_{\mu R_c} \left(\sum_{f \in I_{\mu^\# \mu c}} T_{f w_\mu} \right) \varepsilon(V_{\mu^\# A_c \mu}). \end{aligned} \tag{3.72}$$

Now we fix an arbitrary

$$f_0 \in I_{\mu^\# \mu c}$$

and consider the corresponding summand

$$x_{\mu R_c} T_{f_0 w_\mu} \varepsilon(V_{\mu^\# A_c \mu})$$

on the right hand side of (3.72). According to Lemma 1.1.56.(i), we can write

$$f_0 w_\mu = ud$$

with uniquely determined permutations

$$u \in \mathfrak{S}_{\mu R_c} \qquad \text{and} \qquad d \in \mathcal{D}_{\mu R_c}.$$

From that we obtain with Lemma 1.1.56.(ii) and Lemma 1.3.5

$$x_{\mu R_c} T_{f_0 w_\mu} = x_{\mu R_c} T_{ud} = x_{\mu R_c} T_u T_d = q^{\ell(u)} x_{\mu R_c} T_d. \tag{3.73}$$

Now Lemma 3.2.20, applied with $w_\mu \in \mathfrak{S}_n$ and $f_0 \in I_{\mu^\# \mu c}$ (see also Remark 3.2.21), and Lemma 3.5.2.(i), applied with the composition μR_c and the PK-pair $\mu^\# A_c \mu$, show that we have

$$x_{\mu R_c} T_d \varepsilon(V_{\mu^\# A_c \mu}) = 0_{\mathcal{H}_A}$$

and thus

$$x_{\mu R_c} T_{f_0 w_\mu} \varepsilon(V_{\mu^\# A_c \mu}) = 0_{\mathcal{H}_A}.$$

This in turn implies together with (3.72) and the choice of $f_0 \in I_{\mu^\# \mu c}$

$$x_\mu T_{w_\mu} \varepsilon(V_{\mu^\# A_c \mu}) \Psi_{\mu^\# \mu c} = 0_{\mathcal{H}_A},$$

as desired. The claim now follows from this and Definition 3.4.1 as explained above.

(iii) According to Definition 3.5.3 and Lemma 3.4.4.(vi), the claim of the statement makes sense. Moreover, Lemma 3.4.4.(i) shows that it suffices to prove

$x_\mu T_{w_{\mu\#\mu}} \varepsilon(U_{\mu\#\mu})\Psi_{\mu\#\mu c} = x_{\mu R_c} T_{w_{\mu\#\mu R_c}} \varepsilon(U_{\mu\#\mu R_c})$. By using Definition 3.5.3, statement (i), Lemma 3.2.17.(i), and Lemma 3.2.11.(iii), we get as in the calculation (3.72) in the proof of statement (ii)

$$\begin{aligned} x_\mu T_{w_{\mu\#\mu}} \varepsilon(U_{\mu\#\mu})\Psi_{\mu\#\mu c} &= x_\mu \Psi_{\mu\#\mu c} T_{w_{\mu\#\mu}} \varepsilon(U_{\mu\#\mu}) \\ &= x_{\mu R_c} \left(\sum_{f \in I_{\mu\#\mu c}} T_f \right) T_{w_{\mu\#\mu}} \varepsilon(U_{\mu\#\mu}) \qquad (3.74) \\ &= x_{\mu R_c} \left(\sum_{f \in I_{\mu\#\mu c}} T_{f w_{\mu\#\mu}} \right) \varepsilon(U_{\mu\#\mu}). \end{aligned}$$

Now we fix an arbitrary

$$f_0 \in I_{\mu\#\mu c} \setminus \{f_{\mu\#\mu c}\}$$

(see Definition 3.2.22 and Lemma 3.2.25.(i)) and consider the corresponding summand

$$x_{\mu R_c} T_{f_0 w_{\mu\#\mu}} \varepsilon(U_{\mu\#\mu})$$

on the right hand side of (3.74). According to Lemma 1.1.56.(i), we can write

$$f_0 w_{\mu\#\mu} = ud$$

with uniquely determined permutations

$$u \in \mathfrak{S}_{\mu R_c} \qquad \text{and} \qquad d \in \mathcal{D}_{\mu R_c}.$$

From that we obtain as in the calculation (3.73) in the proof of statement (ii)

$$x_{\mu R_c} T_{f_0 w_{\mu\#\mu}} = x_{\mu R_c} T_{ud} = x_{\mu R_c} T_u T_d = q^{\ell(u)} x_{\mu R_c} T_d.$$

Now Lemma 3.2.27, applied with $w_{\mu\#\mu} \in \mathfrak{S}_n$ and $f_0 \in I_{\mu\#\mu c} \setminus \{f_{\mu\#\mu c}\}$ (see also Remark 3.2.28), and Lemma 3.5.2.(ii), applied with the composition μR_c, show that we have

$$x_{\mu R_c} T_d \varepsilon(U_{\mu\#\mu}) = 0_{\mathcal{H}_A}$$

and thus

$$x_{\mu R_c} T_{f_0 w_{\mu\#\mu}} \varepsilon(U_{\mu\#\mu}) = 0_{\mathcal{H}_A}.$$

This in turn implies together with (3.74), Lemma 3.2.25.(i), and the choice of $f_0 \in I_{\mu\#\mu c} \setminus \{f_{\mu\#\mu c}\}$

$$x_\mu T_{w_{\mu\#\mu}} \varepsilon(U_{\mu\#\mu})\Psi_{\mu\#\mu c} = x_{\mu R_c} T_{f_{\mu\#\mu c} w_{\mu\#\mu}} \varepsilon(U_{\mu\#\mu}).$$

Using Lemma 3.2.25.(ii), Definition 3.2.8.(ii), and Lemma 3.2.14.(ix), this can be rewritten as

$$x_\mu T_{w_{\mu\#_\mu}} \varepsilon(U_{\mu\#\mu})\Psi_{\mu\#\mu c} = x_{\mu R_c} T_{w_{\mu\#\mu R_c}} \varepsilon(U_{\mu\#\mu R_c}),$$

as desired. The claim now follows from this and Lemma 3.4.4.(i) as explained above. ∎

3.6 ZNL-elements of Hecke algebras of type A

In this section the row number lists introduced in Section 1.1 (see Definition 1.1.70) and the associated constructions described in Section 3.3 are employed to define useful generic elements of PK-modules and to describe their representations with respect to the row standard bases of permutation modules (see Definition 1.3.3). Later on, it will be shown that appropriate sets of such elements are bases of PK-modules and that they can be used to determine the kernels of the restrictions of PK-homomorphisms to PK-modules. This is essential for the construction of the Specht series. As always, n denotes a positive integer and (R, q) denotes a coefficient pair.

The following two statements are of a rather general and technical nature. The first one of them makes use of Definition 3.1.9.

Lemma 3.6.1 *For arbitrary $u, v \in \mathfrak{S}_n$ we have in $\mathcal{H}^{(R,q)}_{A_{n-1}}$*

$$T_u T_v = q^j T_{uv} + \sum_{\substack{w\in\mathfrak{S}_n \\ w \lhd uv}} c_w T_w$$

with an appropriate exponent $j \in \mathbb{Z}$ and appropriate coefficients $c_w \in R$ for all $w \in \mathfrak{S}_n$ satisfying $w \lhd uv$.

Proof. This follows from [DJ1, Lemma 2.1.(iii)], Definition 3.1.9, and Lemma 3.1.10. ∎

The next Lemma makes use of Definition 1.1.58, Definition 1.3.1, Theorem 1.3.2, and Definition 3.1.9.

Lemma 3.6.2 *Let $\lambda \vDash n$, $f \in \mathcal{D}_\lambda$, and $w \in \mathfrak{S}_n$. Then we have in $M^\lambda_{(R,q)}$*

$$x^{(R,q)}_\lambda T_f T_w = q^j x^{(R,q)}_\lambda T_{[fw]^\lambda} + \sum_{\substack{d\in\mathcal{D}_\lambda \\ d \lhd [fw]^\lambda}} c_d x^{(R,q)}_\lambda T_d$$

with an appropriate exponent $j \in \mathbb{Z}$ and appropriate coefficients $c_d \in R$ for all $d \in \mathcal{D}_\lambda$ satisfying $d \lhd [fw]^\lambda$.

Proof. This follows from [DJ1, Lemma 3.2.(iv)], Definition 3.1.9, and Lemma 3.1.10. ■

Now we employ row number lists to define certain elements of permutation modules. It will turn out that these elements have the useful properties described at the beginning of this section. The following definition makes use of Definition 1.1.70, Definition 1.3.1, Definition 1.1.67, Definition 1.2.3.(ii), Definition 3.3.13, and Definition 3.3.1.(iii). Furthermore we remind the reader of Remark 3.3.2.(i).

Definition 3.6.3 *For a $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash n$ with $\lambda_1 > 0$ and a $\zeta \in \mathcal{Z}^\lambda$ we define $z(\zeta)_{(R,q)} \in M^\lambda_{(R,q)}$ as*

$$z(\zeta)_{(R,q)} = x_\lambda^{(R,q)} T_{w_\lambda} \varepsilon^{(n)}_{(R,q)}(Y(\zeta)) T_{g(\zeta)}.$$

$z(\zeta)_{(R,q)}$ is called the row number list element associated to ζ or the ZNL-element associated to ζ or just a row number list element or a ZNL-element. We write

$$z(\zeta)_{(R,q)} = z(\zeta).$$

The abbreviation ZNL stands for row number list.

Remark 3.6.4 *(i) The ZNL-elements from Definition 3.6.3 are generic in the sense of Remark 1.2.9.*

(ii) The ZNL-elements from Definition 3.6.3 are very similar to the elements of permutation modules of symmetric groups introduced in [JAM1, Definition 17.2]. However, the former ones are not a direct generalization of the latter ones.

In the following we investigate ZNL-elements. The next Lemma makes use of Definition 3.2.1, Remark 3.2.2, Definition 3.3.5, and Definition 3.4.1.

Lemma 3.6.5 *Let $\mu^\# \mu$ be a PK$_n$-pair with $\mu^\# \mu \neq 00$. Then we have*

$$\left\{ z(\zeta)_{(R,q)} \,\middle|\, \zeta \in \mathcal{Z}^{\mu^\# \mu} \right\} \subseteq S^{\mu^\# \mu}_{(R,q)}.$$

Proof. Write $\mu = (\mu_1, \mu_2, \ldots) \vDash n$ and fix an arbitrary $\zeta \in \mathcal{Z}^{\mu^\# \mu} \subseteq \mathcal{Z}^\mu$. According to Remark 3.2.2, we then have $\mu_1 > 0$ and thus we can build the ZNL-element $z(\zeta) \in M^\mu$ associated to ζ as in Definition 3.6.3. Moreover, Definition 3.3.1.(ii), Lemma 3.3.3.(iii), and Definition 3.3.5 show that $\nu(\zeta)\mu$ is a PK-pair with the properties

$$\nu(\zeta)\mu \neq 00 \qquad \text{and} \qquad \left[\mu^\#\right] \subseteq \left[\nu(\zeta)\right].$$

With that we get from Definition 3.6.3, Definition 3.3.13, Definition 3.4.1, and Lemma 3.4.4.(iv)

$$z(\zeta) \in S^{\nu(\zeta)\mu} \subseteq S^{\mu^{\#}\mu},$$

as desired. ■

The following statement makes use of Definition 1.1.70, Definition 1.3.1.(ii), Theorem 1.3.2, and Definition 3.1.14.

Lemma 3.6.6 *Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash n$ with $\lambda_1 > 0$.*

(i) Let $\zeta \in \mathcal{Z}^\lambda$. Then we have in $M^\lambda_{(R,q)}$

$$z(\zeta)_{(R,q)} = q^{j^{(\zeta)}} x_\lambda^{(R,q)} T_{[w_\lambda g(\zeta)]^\lambda} + \sum_{\substack{d \in \mathcal{D}_\lambda \\ d < [w_\lambda g(\zeta)]^\lambda}} c_d^{(\zeta)} x_\lambda^{(R,q)} T_d \tag{3.75}$$

with an appropriate exponent $j^{(\zeta)} \in \mathbb{Z}$ and appropriate coefficients $c_d^{(\zeta)} \in R$ for all $d \in \mathcal{D}_\lambda$ satisfying $d < [w_\lambda g(\zeta)]^\lambda$.

(ii) The set

$$\left\{ z(\zeta)_{(R,q)} \,\middle|\, \zeta \in \mathcal{Z}^\lambda \right\} \subseteq M^\lambda_{(R,q)}$$

is linearly independent over R.

(iii) The set

$$\left\{ z(\zeta)_{(R,q)} \,\middle|\, \zeta \in \mathcal{Z}^\lambda \right\} \subseteq M^\lambda_{(R,q)}$$

is an R-basis of $M^\lambda_{(R,q)}$.

Proof. (i) First we consider the product $x_\lambda T_{w_\lambda} \varepsilon(Y(\zeta))$ occurring in the definition of $z(\zeta)$. Using Definition 1.2.3.(ii), Definition 3.3.13, Lemma 3.2.14.(viii), and Corollary 1.1.57, this product can be rewritten as follows.

$$\begin{aligned} x_\lambda T_{w_\lambda} \varepsilon(Y(\zeta)) &= x_\lambda T_{w_\lambda} \sum_{y \in Y(\zeta)} (-q)^{-\ell(y)} T_y \\ &= x_\lambda \sum_{y \in Y(\zeta)} (-q)^{-\ell(y)} T_{w_\lambda y} \end{aligned}$$

Postmultiplying with $T_{g(\zeta)}$, we further obtain according to Definition 3.6.3

$$z(\zeta) = \sum_{y \in Y(\zeta)} (-q)^{-\ell(y)} x_\lambda T_{w_\lambda y} T_{g(\zeta)}. \tag{3.76}$$

Now since, according to Definition 3.3.13 and Lemma 3.2.14.(vi), we have

$$\forall y \in Y(\zeta) : w_\lambda y \in \mathcal{D}_\lambda,$$

we can employ Lemma 3.6.2 to express the summands on the right hand side of (3.76) as follows.

$$\forall y \in Y(\zeta) : x_\lambda T_{w_\lambda y} T_{g(\zeta)} = q^{j^{(y)}} x_\lambda T_{[w_\lambda y g(\zeta)]^\lambda} + \sum_{\substack{d \in \mathcal{D}_\lambda \\ d \lhd [w_\lambda y g(\zeta)]^\lambda}} c_d^{(y)} x_\lambda T_d \tag{3.77}$$

The preceding expression contains exponents $j^{(y)} \in \mathbb{Z}$ depending on $y \in Y(\zeta)$ and coefficients $c_d^{(y)} \in R$ depending on $y \in Y(\zeta)$ and $d \in \mathcal{D}_\lambda$ satisfying $d \lhd [w_\lambda y g(\zeta)]^\lambda$. By substituting (3.77) into (3.76) and rearranging terms we obtain, using Lemma 3.3.14.(ii) and Lemma 3.1.15.(iv),

$$z(\zeta) = q^{j^{(\zeta)}} x_\lambda T_{[w_\lambda g(\zeta)]^\lambda} + \sum_{\substack{d \in \mathcal{D}_\lambda \\ d < [w_\lambda g(\zeta)]^\lambda}} c_d^{(\zeta)} x_\lambda T_d$$

with an appropriate exponent $j^{(\zeta)} \in \mathbb{Z}$ and appropriate coefficients $c_d^{(\zeta)} \in R$ for all $d \in \mathcal{D}_\lambda$ satisfying $d < [w_\lambda g(\zeta)]^\lambda$. This representation of $z(\zeta)$ as an R-linear combination of elements of the basis $\mathbf{B}^{M^\lambda}_{\text{row std}}$ of M^λ (see Definition 1.3.3 and Definition 1.1.58.(ii)) hat the desired form.

(ii) According to Remark 1.1.71.(ii), we have

$$\left\{ z(\zeta) \,\middle|\, \zeta \in \mathcal{Z}^\lambda \right\} \neq \varnothing.$$

Now consider an equation

$$\sum_{\zeta \in \mathcal{Z}^\lambda} r_\zeta z(\zeta) = 0_{\mathcal{H}_A} \tag{3.78}$$

with certain coefficients $r_\zeta \in R$ for $\zeta \in \mathcal{Z}^\lambda$. Here, $0_{\mathcal{H}_A}$ denotes the additive neutral element of $\mathcal{H}_n$. With that, put

$$\mathcal{Y} = \left\{ \zeta \in \mathcal{Z}^\lambda \,\middle|\, r_\zeta \neq 0_R \right\} \subseteq \mathcal{Z}^\lambda. \tag{3.79}$$

Here, 0_R denotes the additive neutral element of R. Then, in order to establish the linear independence of $\{z(\zeta) \mid \zeta \in \mathcal{Z}^\lambda\}$, we must show $\mathcal{Y} = \varnothing$.

Now if we have $\mathcal{Y} \neq \varnothing$, there is an

$$\eta \in \mathcal{Y} \qquad \text{with} \qquad \forall \zeta \in \mathcal{Y} \setminus \{\eta\} : [w_\lambda g(\zeta)]^\lambda < [w_\lambda g(\eta)]^\lambda \tag{3.80}$$

(see Definition 3.3.1.(iii), Lemma 3.3.4, Definition 3.1.14, and Lemma 3.1.15.(i)). With these properties of η, we obtain by substituting (3.75) from statement (i) into the left hand side of (3.78) and rearranging terms

$$q^{j^{(\eta)}} r_\eta x_\lambda T_{[w_\lambda g(\eta)]^\lambda} + \sum_{\substack{d \in \mathcal{D}_\lambda \\ d < [w_\lambda g(\eta)]^\lambda}} \tilde{r}_d x_\lambda T_d = 0_{\mathcal{H}_A}$$

with an appropriate exponent $j^{(\eta)} \in \mathbb{Z}$ and appropriate coefficients $\tilde{r}_d \in R$ for all $d \in \mathcal{D}_\lambda$ satisfying $d < [w_\lambda g(\eta)]^\lambda$. From this and Theorem 1.3.2 we get

$$q^{j^{(\eta)}} r_\eta = 0_R$$

and thus, according to Definition 1.2.1,

$$r_\eta = 0_R.$$

This is a contradiction to (3.79) and (3.80).

So we must have

$$\mathcal{Y} = \varnothing$$

and the set $\{z(\zeta) \mid \zeta \in \mathcal{Z}^\lambda\}$ is linearly independent over R.

(iii) The linear independence of the set $\{z(\zeta) \mid \zeta \in \mathcal{Z}^\lambda\} \subseteq M^\lambda$ over R follows from statement (ii). In order to show that this set generates M^λ over R, fix a $y \in M^\lambda \setminus \{0_{\mathcal{H}_A}\}$. Then we have according to Theorem 1.3.2

$$y = \sum_{d \in \mathcal{D}_\lambda} c_d x_\lambda T_d$$

with uniquely determined coefficients $c_d \in R$ for all $d \in \mathcal{D}_\lambda$. Because of $y \neq 0_{\mathcal{H}_A}$ there is a $d_1 \in \mathcal{D}_\lambda$ satisfying

$$c_{d_1} \neq 0_R \qquad \text{and} \qquad \forall d \in \mathcal{D}_\lambda \setminus \{d_1\} : c_d \neq 0_R \Rightarrow d < d_1$$

(see Definition 3.1.14 and Lemma 3.1.15.(i)). Thus we can write

$$y = c_{d_1} x_\lambda T_{d_1} + \sum_{\substack{d \in \mathcal{D}_\lambda \\ d < d_1}} c_d x_\lambda T_d. \tag{3.81}$$

Furthermore we get from Lemma 3.3.4 a $\zeta_1 \in \mathcal{Z}^\lambda$ with the property

$$d_1 = [w_\lambda g(\zeta_1)]^\lambda.$$

This, (3.75) from statement (i), and (3.81) imply

$$y - c_{d_1} q^{-j^{(\zeta_1)}} z(\zeta_1) = \sum_{\substack{d \in \mathcal{D}_\lambda \\ d < d_1}} \tilde{c}_d x_\lambda T_d \tag{3.82}$$

with $j^{(\zeta_1)} \in \mathbb{Z}$ as in statement (i) and appropriate coefficients $\tilde{c}_d \in R$ for all $d \in \mathcal{D}_\lambda$ satisfying $d < d_1$. This elimination of a summand from the row standard basis of M^λ on the right hand side in the course of the transition from (3.81) to (3.82) can be repeated inductively downwards along the ordering $<$ on the set $\mathcal{D}_\lambda$ until none of these summands are left. Then we have an expression

$$y - \hat{c}_1 z(\zeta_1) - \cdots - \hat{c}_m z(\zeta_m) = 0_{\mathcal{H}_A}$$

with an appropriate $m \in \mathbb{N}$ and appropriate coefficients $\hat{c}_j \in R$ and row number lists $\zeta_j \in \mathcal{Z}^\lambda$ for all $j \in \{1, \dots, m\}$. Now since $y \in M^\lambda \setminus \{0_{\mathcal{H}_A}\}$ was arbitrarily chosen, this shows that M^λ is generated over R by the set $\{z(\zeta) \mid \zeta \in \mathcal{Z}^\lambda\}$. All in all we see that this set is indeed an R-basis of M^λ, as desired. ■

Corollary 3.6.7 *Let $\mu^\# \mu$ be a PK$_n$-pair with $\mu^\# \mu \neq 00$. Then we have*

$$S_{(R,q)}^{\mu^\# \mu} \neq 0_{\mathcal{H}_A}.$$

Here, $0_{\mathcal{H}_A}$ denotes the null ideal in $\mathcal{H}_{A_{n-1}}^{(R,q)}$.

Proof. According to Lemma 3.3.6.(i), there is a

$$\zeta \in \mathcal{Z}^{\mu^\# \mu}.$$

With that we get from Lemma 3.6.5

$$z(\zeta) \in S^{\mu^\# \mu}.$$

Lemma 3.6.6.(i), Definition 1.1.58.(ii), and Theorem 1.3.2 show furthermore

$$z(\zeta) \neq 0_{\mathcal{H}_A}.$$

Here, $0_{\mathcal{H}_A}$ denotes the additive neutral element of $\mathcal{H}_n$. This proves the claim. ■

3.7 Homomorphic images of ZNL-elements of Hecke algebras of type A

Now we investigate the effect of the PK-homomorphisms introduced in Section 3.5 (see Definition 3.5.3) on the ZNL-elements introduced in the preceding section (see Definition 3.6.3). As before, n denotes a positive integer and (R, q) denotes a coefficient pair as in Definition 1.2.1. Furthermore we fix for this section a PK$_n$-pair $\mu^\# \mu \neq 00$ and an AR-index $c \in \mathbb{N} \setminus \{1\}$ for $\mu^\# \mu$ (see Definition 3.2.1 and Definition 3.2.3).

The next statement is a simple consequence of Lemma 3.5.5.(ii), it makes use of Definition 3.2.5.(iv), Definition 3.3.5, Lemma 3.3.6.(iii), Definition 3.6.3, and Definition 3.5.3.

Corollary 3.7.1 *Let $\zeta \in \mathcal{Z}^{\mu^\# A_c \mu} \subseteq \mathcal{Z}^{\mu^\# \mu}$. Then we have*

$$z(\zeta)_{(R,q)} \Psi^{(R,q)}_{\mu^\# \mu c} = 0_{\mathcal{H}_A}.$$

Here, $0_{\mathcal{H}_A}$ denotes the additive neutral element of $\mathcal{H}^{(R,q)}_{A_{n-1}}$.

Proof. In the case $\mu^\# A_c \mu = 00$ we have $\mathcal{Z}^{\mu^\# A_c \mu} = \varnothing$ and there is nothing to show. In the case $\mu^\# A_c \mu \neq 00$ the claim follows from Lemma 3.6.5 and Lemma 3.5.5.(ii). ∎

In the following lemma a useful representation of the images of ZNL-elements under PK-homomorphisms is derived. It makes use of Definition 3.2.5.(ii), Definition 3.2.22, Definition 3.3.13, Definition 1.2.3.(ii), Definition 3.3.1.(iii), and Definition 1.1.58.(ii).

Lemma 3.7.2 *Let $\zeta \in \mathcal{Z}^{\mu^\# \mu}$. Then we have*

$$\begin{aligned} z(\zeta)_{(R,q)} \Psi^{(R,q)}_{\mu^\# \mu c} &= x^{(R,q)}_{\mu R_c} T_{f_{\mu^\# \mu c} w_\mu} \varepsilon^{(n)}_{(R,q)}(Y(\zeta)) T_{g(\zeta)} \\ &= q^j x^{(R,q)}_{\mu R_c} T_{\left[f_{\mu^\# \mu c} w_\mu\right]^{\mu R_c}} \varepsilon^{(n)}_{(R,q)}(Y(\zeta)) T_{g(\zeta)} \end{aligned}$$

with an appropriate exponent $j \in \mathbb{Z}$.

Proof. First we get from Definition 3.6.3, Definition 3.5.3, and Lemma 3.5.5.(i)

$$\begin{aligned} z(\zeta)\Psi_{\mu^\# \mu c} &= x_\mu T_{w_\mu} \varepsilon(Y(\zeta)) T_{g(\zeta)} \Psi_{\mu^\# \mu c} \\ &= x_\mu \Psi_{\mu^\# \mu c} T_{w_\mu} \varepsilon(Y(\zeta)) T_{g(\zeta)} \\ &= x_{\mu R_c} \sum_{f \in I_{\mu^\# \mu c}} T_f T_{w_\mu} \varepsilon(Y(\zeta)) T_{g(\zeta)}. \end{aligned}$$

Using Lemma 3.2.17.(i) and Lemma 1.1.68.(ii), this can be rewritten as

$$\begin{aligned} z(\zeta)\Psi_{\mu^\# \mu c} &= x_{\mu R_c} \sum_{f \in I_{\mu^\# \mu c}} T_{f w_\mu} \varepsilon(Y(\zeta)) T_{g(\zeta)} \\ &= \sum_{f \in I_{\mu^\# \mu c}} x_{\mu R_c} T_{f w_\mu} \varepsilon(Y(\zeta)) T_{g(\zeta)}. \end{aligned} \tag{3.83}$$

Now we fix an

$$f_0 \in I_{\mu^\# \mu c} \setminus \left\{ f_{\mu^\# \mu c} \right\}$$

(see Definition 3.2.22) and consider the summand $x_{\mu R_c} T_{f_0 w_\mu} \varepsilon(Y(\zeta)) T_{g(\zeta)}$ on the right hand side of (3.83) corresponding to it. Because of $\zeta \in \mathcal{Z}^{\mu^\# \mu}$, Definition 3.3.5, Definition 3.3.13, and Lemma 3.4.3 we have

$$\varepsilon(Y(\zeta)) = \varepsilon(V_{\mu^\# \mu}) h$$

with an appropriate $h \in \mathcal{H}_n$. By substituting this into the summand corresponding to f_0 on the right hand side of (3.83) we obtain

$$x_{\mu R_c} T_{f_0 w_\mu} \varepsilon(Y(\zeta)) T_{g(\zeta)} = x_{\mu R_c} T_{f_0 w_\mu} \varepsilon(V_{\mu \# \mu}) \tilde{h}$$

with an appropriate $\tilde{h} \in \mathcal{H}_n$. Furthermore we can write

$$f_0 w_\mu = ud$$

with uniquely determined permutations

$$u \in \mathfrak{S}_{\mu R_c} \qquad \text{and} \qquad d \in \mathcal{D}_{\mu R_c}.$$

From this and Lemma 1.3.5 we get

$$x_{\mu R_c} T_{f_0 w_\mu} = x_{\mu R_c} T_{ud} = x_{\mu R_c} T_u T_d = q^{\ell(u)} x_{\mu R_c} T_d. \tag{3.84}$$

Now Lemma 3.2.27, applied with $w_\mu \in \mathfrak{S}_n$ and $f_0 \in I_{\mu \# \mu c} \setminus \{f_{\mu \# \mu c}\}$ (see also Remark 3.2.28), and Lemma 3.5.2.(i), applied with the composition μR_c, show

$$x_{\mu R_c} T_d \varepsilon(V_{\mu \# \mu}) = 0_{\mathcal{H}_A}.$$

Here, $0_{\mathcal{H}_A}$ denotes the additive neutral element of $\mathcal{H}_n$. From all this we get

$$x_{\mu R_c} T_{f_0 w_\mu} \varepsilon(V_{\mu \# \mu}) = 0_{\mathcal{H}_A}$$

and furthermore

$$x_{\mu R_c} T_{f_0 w_\mu} \varepsilon(Y(\zeta)) T_{g(\zeta)} = 0_{\mathcal{H}_A}.$$

By substituting the preceding equation into (3.83) and taking into account the choice of f_0 we obtain

$$z(\zeta) \Psi_{\mu \# \mu c} = x_{\mu R_c} T_{f_{\mu \# \mu c} w_\mu} \varepsilon(Y(\zeta)) T_{g(\zeta)}, \tag{3.85}$$

which is the first identity in the claim.

In order to prove the second identity in the claim, we decompose $f_{\mu \# \mu c} w_\mu$ with respect to the right cosets of $\mathfrak{S}_{\mu R_c}$ in $\mathfrak{S}_n$. Using Definition 1.1.58.(ii), we can write

$$f_{\mu \# \mu c} w_\mu = \tilde{u} \big[f_{\mu \# \mu c} w_\mu \big]^{\mu R_c} \quad \text{with} \quad \tilde{u} \in \mathfrak{S}_{\mu R_c}.$$

From this we get as in the calculation (3.84)

$$\begin{aligned} x_{\mu R_c} T_{f_{\mu \# \mu c} w_\mu} &= x_{\mu R_c} T_{\tilde{u} [f_{\mu \# \mu c} w_\mu]^{\mu R_c}} \\ &= x_{\mu R_c} T_{\tilde{u}} T_{[f_{\mu \# \mu c} w_\mu]^{\mu R_c}} \\ &= q^{\ell(\tilde{u})} x_{\mu R_c} T_{[f_{\mu \# \mu c} w_\mu]^{\mu R_c}}. \end{aligned}$$

By substituting this into the right hand side of (3.85) we obtain the second identity in the claim with $j = \ell(\tilde{u}) \in \mathbb{Z}$. ■

The next theorem describes the representations of the images of ZNL-elements under PK-homomorphisms with respect to the row standard bases of permutation modules (see Definition 1.3.1.(ii) and Definition 1.3.3). It makes use of Definition 3.3.5, Definition 3.2.5, Definition 3.6.3, Definition 3.5.3, Definition 3.2.22, Definition 3.3.1, Definition 1.1.58, and Definition 3.1.14.

Theorem 3.7.3 *Let $\zeta \in \mathcal{Z}^{\mu^\# \mu} \setminus \mathcal{Z}^{\mu^\# A_c \mu}$. Then we have in $M^{\mu R_c}_{(R,q)}$*

$$z(\zeta)_{(R,q)} \Psi^{(R,q)}_{\mu^\# \mu c} = q^j x^{(R,q)}_{\mu R_c} T_{\left[f_{\mu^\# \mu c} w_\mu g(\zeta)\right]^{\mu R_c}} + \sum_{\substack{d \in \mathcal{D}_{\mu R_c} \\ d < \left[f_{\mu^\# \mu c} w_\mu g(\zeta)\right]^{\mu R_c}}} c_d x^{(R,q)}_{\mu R_c} T_d \tag{3.86}$$

with an appropriate exponent $j \in \mathbb{Z}$ and appropriate coefficients $c_d \in R$ for all $d \in \mathcal{D}_{\mu R_c}$ satisfying $d < \left[f_{\mu^\# \mu c} w_\mu g(\zeta)\right]^{\mu R_c}$.

Proof. The proof of this claim is similar to the proof of Lemma 3.6.6.(i).

According to Lemma 3.7.2, we have

$$z(\zeta) \Psi_{\mu^\# \mu c} = x_{\mu R_c} T_{f_{\mu^\# \mu c} w_\mu} \varepsilon(Y(\zeta)) T_{g(\zeta)}.$$

The left factor $x_{\mu R_c} T_{f_{\mu^\# \mu c} w_\mu} \varepsilon(Y(\zeta))$ of the right hand side of the preceding identity can be rewritten as follows by using Definition 1.2.3.(ii), Lemma 3.3.14.(i), and Lemma 3.3.15.(i).

$$\begin{aligned} x_{\mu R_c} T_{f_{\mu^\# \mu c} w_\mu} \varepsilon(Y(\zeta)) &= x_{\mu R_c} T_{f_{\mu^\# \mu c} w_\mu} \sum_{y \in Y(\zeta)} (-q)^{-\ell(y)} T_y \\ &= x_{\mu R_c} \sum_{y \in Y(\zeta)} (-q)^{-\ell(y)} T_{f_{\mu^\# \mu c} w_\mu y} \end{aligned}$$

By postmultiplying this with the factor $T_{g(\zeta)}$ we obtain

$$z(\zeta) \Psi_{\mu^\# \mu c} = \sum_{y \in Y(\zeta)} (-q)^{-\ell(y)} x_{\mu R_c} T_{f_{\mu^\# \mu c} w_\mu y} T_{g(\zeta)}. \tag{3.87}$$

Now we fix a

$$y \in Y(\zeta)$$

and consider the corresponding summand $x_{\mu R_c} T_{f_{\mu^\# \mu c} w_\mu y} T_{g(\zeta)}$ on the right hand side of (3.87). Using Definition 1.1.58.(ii), we can write

$$f_{\mu^\# \mu c} w_\mu y = u^{(y)} \left[f_{\mu^\# \mu c} w_\mu y\right]^{\mu R_c} \quad \text{with an appropriate} \quad u^{(y)} \in \mathfrak{S}_{\mu R_c}.$$

From this decomposition, Lemma 1.3.5, and Lemma 3.6.2 we get

$$\begin{aligned}
x_{\mu R_c} T_{f_{\mu\#\mu c} w_\mu y} T_{g(\zeta)} &= x_{\mu R_c} T_{u^{(y)} \left[f_{\mu\#\mu c} w_\mu y\right]^{\mu R_c}} T_{g(\zeta)} \\
&= x_{\mu R_c} T_{u^{(y)}} T_{\left[f_{\mu\#\mu c} w_\mu y\right]^{\mu R_c}} T_{g(\zeta)} \\
&= q^{\ell(u^{(y)})} x_{\mu R_c} T_{\left[f_{\mu\#\mu c} w_\mu y\right]^{\mu R_c}} T_{g(\zeta)} \\
&= q^{j^{(y)}} x_{\mu R_c} T_{\left[\left[f_{\mu\#\mu c} w_\mu y\right]^{\mu R_c} g(\zeta)\right]^{\mu R_c}} + \\
&\qquad \sum_{\substack{d \in \mathcal{D}_{\mu R_c} \\ d \lhd \left[\left[f_{\mu\#\mu c} w_\mu y\right]^{\mu R_c} g(\zeta)\right]^{\mu R_c}}} c_d^{(y)} x_{\mu R_c} T_d \\
&= q^{j^{(y)}} x_{\mu R_c} T_{\left[f_{\mu\#\mu c} w_\mu y g(\zeta)\right]^{\mu R_c}} + \\
&\qquad \sum_{\substack{d \in \mathcal{D}_{\mu R_c} \\ d \lhd \left[f_{\mu\#\mu c} w_\mu y g(\zeta)\right]^{\mu R_c}}} c_d^{(y)} x_{\mu R_c} T_d
\end{aligned}$$

with an exponent $j^{(y)} \in \mathbb{Z}$ depending on y and coefficients $c_d^{(y)} \in R$ depending on y and $d \in \mathcal{D}_{\mu R_c}$ satisfying $d \lhd \left[f_{\mu\#\mu c} w_\mu y g(\zeta)\right]^{\mu R_c}$.

Now we substitute the preceding identity for each $y \in Y(\zeta)$ into the right hand side of (3.87) and rearrange terms using Lemma 3.3.15.(ii) and Lemma 3.1.15.(iv). Then we obtain

$$z(\zeta)\Psi_{\mu\#\mu c} = q^j x_{\mu R_c} T_{\left[f_{\mu\#\mu c} w_\mu g(\zeta)\right]^{\mu R_c}} + \sum_{\substack{d \in \mathcal{D}_{\mu R_c} \\ d < \left[f_{\mu\#\mu c} w_\mu g(\zeta)\right]^{\mu R_c}}} c_d x_{\mu R_c} T_d$$

with an appropriate exponent $j \in \mathbb{Z}$ and appropriate coefficients $c_d \in R$ for all $d \in \mathcal{D}_{\mu R_c}$ satisfying $d < \left[f_{\mu\#\mu c} w_\mu g(\zeta)\right]^{\mu R_c}$. This representation of $z(\zeta)\Psi_{\mu\#\mu c}$ as an R-linear combination of elements of the basis $\mathbf{B}_{\text{row std}}^{M^{\mu R_c}}$ of $M^{\mu R_c}$ (see Definition 1.3.3) has the desired form. ■

The row number lists not considered in Theorem 3.7.3 — that is, those in the set $\mathcal{Z}^{\mu\# A_c \mu} \subseteq \mathcal{Z}^{\mu\#\mu}$ — are covered in Corollary 3.7.1.

The following corollary uses Lemma 3.2.7.(ii), Lemma 3.6.5, Lemma 3.5.5.(iii), and Lemma 3.4.4.(vi).

Corollary 3.7.4 *The set*

$$\left\{ z(\zeta)_{(R,q)} \Psi_{\mu\#\mu c}^{(R,q)} \,\middle|\, \zeta \in \mathcal{Z}^{\mu\#\mu} \setminus \mathcal{Z}^{\mu\# A_c \mu} \right\} \subseteq S_{(R,q)}^{\mu\#\mu R_c} \subseteq M_{(R,q)}^{\mu R_c}$$

is linearly independent over R.

Proof. The proof of this statement is analogous to that of Lemma 3.6.6.(ii).

According to Corollary 3.3.10, we have

$$\left\{ z(\zeta)\Psi_{\mu^{\#}\mu c} \,\middle|\, \zeta \in \mathcal{Z}^{\mu^{\#}\mu} \setminus \mathcal{Z}^{\mu^{\#} A_c \mu} \right\} \neq \varnothing.$$

Now we consider a linear combination

$$\sum_{\zeta \in \mathcal{Z}^{\mu^{\#}\mu} \setminus \mathcal{Z}^{\mu^{\#} A_c \mu}} r_\zeta z(\zeta) \Psi_{\mu^{\#}\mu c} = 0_{\mathcal{H}_A} \tag{3.88}$$

with coefficients $r_\zeta \in R$ for all $\zeta \in \mathcal{Z}^{\mu^{\#}\mu} \setminus \mathcal{Z}^{\mu^{\#} A_c \mu}$. Here, $0_{\mathcal{H}_A}$ denotes the additive neutral element of $\mathcal{H}_n$. With that we put

$$\mathcal{Y} = \left\{ \zeta \in \mathcal{Z}^{\mu^{\#}\mu} \setminus \mathcal{Z}^{\mu^{\#} A_c \mu} \,\middle|\, r_\zeta \neq 0_R \right\} \subseteq \mathcal{Z}^{\mu^{\#}\mu} \setminus \mathcal{Z}^{\mu^{\#} A_c \mu}. \tag{3.89}$$

Here, 0_R denotes the additive neutral element of R. In order to prove the claim, we must show $\mathcal{Y} = \varnothing$.

Suppose that we have $\mathcal{Y} \neq \varnothing$. Then we get from Corollary 3.3.12, Definition 3.1.14, and Lemma 3.1.15.(i) an

$$\eta \in \mathcal{Y} \quad \text{with} \quad \forall \zeta \in \mathcal{Y} \setminus \{\eta\} : \left[f_{\mu^{\#}\mu c} w_\mu g(\zeta) \right]^{\mu R_c} < \left[f_{\mu^{\#}\mu c} w_\mu g(\eta) \right]^{\mu R_c} \tag{3.90}$$

(see also Definition 3.2.22 and Definition 3.3.1.(iii)). Using this property of η, we substitute (3.86) from Theorem 3.7.3 into the left hand side of (3.88) and rearrange terms in order to obtain

$$q^j r_\eta x_{\mu R_c} T_{\left[f_{\mu^{\#}\mu c} w_\mu g(\eta) \right]^{\mu R_c}} + \sum_{\substack{d \in \mathcal{D}_{\mu R_c} \\ d < \left[f_{\mu^{\#}\mu c} w_\mu g(\eta) \right]^{\mu R_c}}} \tilde{r}_d x_{\mu R_c} T_d = 0_{\mathcal{H}_A}$$

with an appropriate exponent $j \in \mathbb{Z}$ and appropriate coefficients $\tilde{r}_d \in R$ for all $d \in \mathcal{D}_{\mu R_c}$ satisfying $d < \left[f_{\mu^{\#}\mu c} w_\mu g(\eta) \right]^{\mu R_c}$. From this equation and Theorem 1.3.2, we get

$$q^j r_\eta = 0_R$$

and, using Definition 1.2.1, furthermore

$$r_\eta = 0_R$$

in contradiction to (3.89) and (3.90).

Thus we must have

$$\mathcal{Y} = \varnothing$$

and the set $\left\{ z(\zeta)\Psi_{\mu^{\#}\mu c} \,\middle|\, \zeta \in \mathcal{Z}^{\mu^{\#}\mu} \setminus \mathcal{Z}^{\mu^{\#} A_c \mu} \right\}$ is linearly independent over R. ■

3.8 Statements on bases of images of PK-homomorphisms for Hecke algebras of type A

This section describes how the existence of a certain basis of a PK-module leads to the existence of an analogous basis of the image of that PK-module under a PK-homomorphism (see Definition 3.4.1, Definition 3.5.3, and Lemma 3.5.5.(iii)). As always, n denotes a positive integer and (R, q) denotes a coefficient pair.

The following lemma makes use of Definition 3.2.1, Definition 3.2.3, Definition 3.6.3, Definition 3.3.5, Lemma 3.6.5, Definition 3.2.5, and Lemma 3.2.7.(ii).

Lemma 3.8.1 *Let $\mu^\#\mu$ be a PK_n-pair with $\mu^\#\mu \neq 00$ and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^\#\mu$. Suppose that*

$$\left\{ z(\zeta)_{(R,q)} \,\middle|\, \zeta \in \mathcal{Z}^{\mu^\#\mu} \right\} \subseteq S^{\mu^\#\mu}_{(R,q)}$$

is an R-basis of $S^{\mu^\#\mu}_{(R,q)}$. Then

$$\left\{ z(\eta)_{(R,q)} \,\middle|\, \eta \in \mathcal{Z}^{\mu^\#\mu R_c} \right\} \subseteq S^{\mu^\#\mu R_c}_{(R,q)}$$

is an R-basis of $S^{\mu^\#\mu R_c}_{(R,q)}$.

Proof. In this proof, we write

$$\mathbf{B}^{\mu^\#\mu} = \left\{ z(\zeta) \,\middle|\, \zeta \in \mathcal{Z}^{\mu^\#\mu} \right\} \qquad \text{and} \qquad \mathbf{B}^{\mu^\#\mu R_c} = \left\{ z(\eta) \,\middle|\, \eta \in \mathcal{Z}^{\mu^\#\mu R_c} \right\}.$$

According to Lemma 3.3.6.(i) and Lemma 3.2.7.(ii), these sets are nonempty.

First we show that, given the assumptions of the claim, the set

$$\mathbf{C}^{\mu^\#\mu c} = \left\{ z(\zeta)\Psi_{\mu^\#\mu c} \,\middle|\, \zeta \in \mathcal{Z}^{\mu^\#\mu} \setminus \mathcal{Z}^{\mu^\# A_c \mu} \right\} \tag{3.91}$$

(see Definition 3.5.3 and Definition 3.2.5.(iv)) is an R-basis of $S^{\mu^\#\mu R_c}$. According to Corollary 3.3.10, this set is nonempty. According to Corollary 3.7.4, it is contained in $S^{\mu^\#\mu R_c}$ and also linearly independent over R. The assumption that $\mathbf{B}^{\mu^\#\mu}$ forms an R-basis of $S^{\mu^\#\mu}$, Lemma 3.5.5.(iii), and Corollary 3.7.1 show that the set $\mathbf{C}^{\mu^\#\mu c}$ generates $S^{\mu^\#\mu R_c}$ over R. Thus, $\mathbf{C}^{\mu^\#\mu c}$ is in fact an R-Basis of $S^{\mu^\#\mu R_c}$.

Now we consider the set $\mathbf{B}^{\mu^\#\mu R_c} \subseteq S^{\mu^\#\mu R_c}$. Its linear independence over R follows from Lemma 3.6.6.(ii). In order to prove the claim, we must show that it generates $S^{\mu^\#\mu R_c}$ over R. This is done in the remainder of the proof.

We will require a distinguished numbering of the elements of $\mathcal{Z}^{\mu^\#\mu R_c}$. Let

$$m = \left| \mathcal{Z}^{\mu^\#\mu R_c} \right| \in \mathbb{N}$$

(see Lemma 3.3.6.(i) and Lemma 3.2.7.(ii)). With that, let

$$\eta_1, \ldots, \eta_m$$

be an enumeration of the elements of $\mathcal{Z}^{\mu^\# \mu R_c} \subseteq \mathcal{Z}^{\mu R_c}$ satisfying

$$\forall i, j \in \{1, \ldots, m\} : i < j \Rightarrow [w_{\mu R_c} g(\eta_i)]^{\mu R_c} > [w_{\mu R_c} g(\eta_j)]^{\mu R_c} \tag{3.92}$$

(see Definition 3.3.5, Lemma 3.3.4, Definition 3.1.14, and Lemma 3.1.15.(i)). Using this and Lemma 3.3.9 and putting

$$\eta_j = \mathcal{J}_{\mu^\# \mu c}(\zeta_j) \quad \text{for} \quad j \in \{1, \ldots, m\}, \tag{3.93}$$

we obtain an enumeration

$$\zeta_1, \ldots, \zeta_m \tag{3.94}$$

of the elements of $\mathcal{Z}^{\mu^\# \mu} \setminus \mathcal{Z}^{\mu^\# A_c \mu}$ satisfying

$$\begin{aligned} &\forall i, j \in \{1, \ldots, m\} : i < j \Rightarrow \\ &\qquad \left[w_{\mu R_c} g(\mathcal{J}_{\mu^\# \mu c}(\zeta_i))\right]^{\mu R_c} > \left[w_{\mu R_c} g(\mathcal{J}_{\mu^\# \mu c}(\zeta_j))\right]^{\mu R_c} \end{aligned}$$

or, according to Lemma 3.3.11, equivalently

$$\forall i, j \in \{1, \ldots, m\} : i < j \Rightarrow \left[f_{\mu^\# \mu c} w_\mu g(\zeta_i)\right]^{\mu R_c} > \left[f_{\mu^\# \mu c} w_\mu g(\zeta_j)\right]^{\mu R_c}. \tag{3.95}$$

Now we fix an

$$h \in S^{\mu^\# \mu R_c}.$$

For this element we will inductively construct certain coefficients

$$\alpha_j \in R \quad \text{with} \quad j \in \{1, \ldots, m\}$$

and R-linear combinations of ZNL-elements from $\mathbf{B}^{\mu^\# \mu R_c} \subseteq S^{\mu^\# \mu R_c}$

$$a_j = \sum_{i=1}^{j} \alpha_i z(\eta_i) \in S^{\mu^\# \mu R_c} \quad \text{with} \quad j \in \{0, \ldots, m\}$$

such that for every $j \in \{0, \ldots, m\}$ in the representation of $h - a_j \in S^{\mu^\# \mu R_c} \subseteq M^{\mu R_c}$ with respect to the R-basis $\mathbf{B}^{M^{\mu R_c}}_{\text{row std}}$ of $M^{\mu R_c}$ the coefficients of all the basis elements

$$x_{\mu R_c} T_{\left[w_{\mu R_c} g(\eta_i)\right]^{\mu R_c}} \in \mathbf{B}^{M^{\mu R_c}}_{\text{row std}} \quad \text{with} \quad i \in \{1, \ldots, j\}$$

vanish (see Lemma 3.4.4.(vi), Definition 1.3.3, and Definition 1.1.58.(ii)). The induction process starts with

$$a_0 = 0_{\mathcal{H}_A} \in S^{\mu^\# \mu R_c} \subseteq \mathcal{H}_n.$$

Here, $0_{\mathcal{H}_A}$ denotes the additive neutral element of $\mathcal{H}_n$ (see also Definition 3.4.1). If for a $j \in \{1, \ldots, m\}$ the coefficients $\alpha_i \in R$ with $i \in \{1, \ldots, j-1\}$ and the R-linear combinations $a_i \in S^{\mu^\# \mu R_c}$ with $i \in \{0, \ldots, j-1\}$ are already constructed then we choose $\alpha_j \in R$ and

$$a_j = \sum_{i=1}^{j} \alpha_i z(\eta_i) = a_{j-1} + \alpha_j z(\eta_j) \in S^{\mu^\# \mu R_c}$$

such that in the representation of $h - a_j \in S^{\mu^\# \mu R_c} \subseteq M^{\mu R_c}$ with respect to the R-basis $\mathbf{B}^{M^{\mu R_c}}_{\text{row std}}$ of $M^{\mu R_c}$ the coefficient of the basis element

$$x_{\mu R_c} T_{\left[w_{\mu R_c} g(\eta_j)\right]^{\mu R_c}} \in \mathbf{B}^{M^{\mu R_c}}_{\text{row std}}$$

vanishes. The formula (3.75) in Lemma 3.6.6.(i) and Definition 1.2.1 show that this is possible. Now the particular choice of α_j and a_j, formula (3.75) from Lemma 3.6.6.(i), the relations (3.92), and the induction hypothesis, which states that in the representation of $h - a_{j-1}$ with respect to the R-basis $\mathbf{B}^{M^{\mu R_c}}_{\text{row std}}$ of $M^{\mu R_c}$ the coefficients of all the basis elements

$$x_{\mu R_c} T_{\left[w_{\mu R_c} g(\eta_i)\right]^{\mu R_c}} \in \mathbf{B}^{M^{\mu R_c}}_{\text{row std}} \quad \text{with} \quad i \in \{1, \ldots, j-1\}$$

vanish, all together show that in the representation of

$$h - a_j = h - a_{j-1} - \alpha_j z(\eta_j)$$

with respect to the R-basis $\mathbf{B}^{M^{\mu R_c}}_{\text{row std}}$ of $M^{\mu R_c}$ the coefficients of all the basis elements

$$x_{\mu R_c} T_{\left[w_{\mu R_c} g(\eta_i)\right]^{\mu R_c}} \in \mathbf{B}^{M^{\mu R_c}}_{\text{row std}} \quad \text{with} \quad i \in \{1, \ldots, j\}$$

vanish. This in turn shows that the induction hypothesis also holds for α_j and a_j and the induction can be continued. Proceeding in this way, we obtain for $j = m$ an R-linear combination $a_m \in S^{\mu^\# \mu R_c}$ of ZNL-elements from $\mathbf{B}^{\mu^\# \mu R_c}$ such that in the representation of $h - a_m$ with respect to the basis $\mathbf{B}^{M^{\mu R_c}}_{\text{row std}}$ of $M^{\mu R_c}$ the coefficients of all the basis elements

$$x_{\mu R_c} T_{\left[w_{\mu R_c} g(\eta)\right]^{\mu R_c}} \in \mathbf{B}^{M^{\mu R_c}}_{\text{row std}} \quad \text{with} \quad \eta \in \mathcal{Z}^{\mu^\# \mu R_c}$$

vanish.

Now we show $h = a_m$. To this end, we write $h - a_m$ as an R-linear combination of the R-basis $\mathbf{C}^{\mu^\# \mu c}$ of $S^{\mu^\# \mu R_c}$ (see the beginning of the proof). Let

$$h - a_m = \sum_{i=1}^{m} \xi_i z(\zeta_i) \Psi_{\mu^\# \mu c} \quad \text{with} \quad \xi_i \in R \quad \text{for} \quad i \in \{1, \ldots, m\}$$

(see (3.91) and (3.94)). Suppose that we have $\{i \in \{1, \ldots, m\} \mid \xi_i \neq 0_R\} \neq \varnothing$. With this assumption, put

$$j = \min \{i \in \{1, \ldots, m\} \mid \xi_i \neq 0_R\} \in \{1, \ldots, m\}. \tag{3.96}$$

Here, 0_R denotes the additive neutral element of R. Now we can write

$$h - a_m = \xi_j z(\zeta_j)\Psi_{\mu^\#\mu c} + \sum_{i=j+1}^{m} \xi_i z(\zeta_i)\Psi_{\mu^\#\mu c}. \tag{3.97}$$

Next, we substitute the representations of the elements $z(\zeta)\Psi_{\mu^\#\mu c} \in M^{\mu R_c}$ for $\zeta \in \mathcal{Z}^{\mu^\#\mu} \setminus \mathcal{Z}^{\mu^\# A_c \mu}$ with respect to the R-basis $\mathbf{B}^{M^{\mu R_c}}_{\text{row std}}$ of $M^{\mu R_c}$ from Theorem 3.7.3 into the right hand side of the preceding equation and rearrange terms, taking into account (3.95) from above. Thus we obtain

$$h - a_m = \xi_j q^k x_{\mu R_c} T_{\left[f_{\mu^\#\mu c} w_\mu g(\zeta_j)\right]^{\mu R_c}} + \sum_{\substack{d \in \mathcal{D}_{\mu R_c} \\ d < \left[f_{\mu^\#\mu c} w_\mu g(\zeta_j)\right]^{\mu R_c}}} \tilde{\xi}_d x_{\mu R_c} T_d$$

with an appropriate exponent $k \in \mathbb{Z}$ and appropriate coefficients $\tilde{\xi}_d \in R$ for all $d \in \mathcal{D}_{\mu R_c}$ satisfying $d < \left[f_{\mu^\#\mu c} w_\mu g(\zeta_j)\right]^{\mu R_c}$. Now, according to Definition 1.2.1 and the choice of j, we have for the coefficient of $x_{\mu R_c} T_{\left[f_{\mu^\#\mu c} w_\mu g(\zeta_j)\right]^{\mu R_c}}$

$$\xi_j q^k \neq 0_R. \tag{3.98}$$

Furthermore, we see from Lemma 3.3.11 and (3.93)

$$\left[f_{\mu^\#\mu c} w_\mu g(\zeta_j)\right]^{\mu R_c} = \left[w_{\mu R_c} g(\mathcal{J}_{\mu^\#\mu c}(\zeta_j))\right]^{\mu R_c} = \left[w_{\mu R_c} g(\eta_j)\right]^{\mu R_c}. \tag{3.99}$$

Now (3.98) and (3.99) together lead to a contradiction to the construction of a_m which ensures that the coefficient $\xi_j q^k$ of the basis element $x_{\mu R_c} T_{\left[w_{\mu R_c} g(\eta_j)\right]^{\mu R_c}}$ vanishes (see the preceding paragraph). Thus j cannot be chosen as in (3.96) and we have

$$\forall i \in \{1, \ldots, m\} : \xi_i = 0_R$$

and furthermore

$$h - a_m = 0_{\mathcal{H}_A} \qquad \text{or equivalently} \qquad h = a_m.$$

This shows that $\mathbf{B}^{\mu^\#\mu R_c}$ generates $S^{\mu^\#\mu R_c}$ over R and all in all is an R-basis of $S^{\mu^\#\mu R_c}$, as desired. ■

3.9 Statements on bases of kernels of PK-homomorphisms for Hecke algebras of type A

This section describes how the existence of a certain basis of a PK-module leads to the existence of an analogous basis of the kernel of a PK-homomorphism restricted to that PK-module (see Definition 3.4.1, Definition 3.5.3, Lemma 3.4.4.(vi), and Lemma 3.5.5.(ii)). As before, n denotes a positive integer and (R,q) denotes a coefficient pair.

The next lemma uses Definition 3.2.1, Definition 3.2.3, Definition 3.2.5.(iv), Definition 3.6.3, Definition 3.3.5, Lemma 3.6.5, and Lemma 3.3.6.(iii).

Lemma 3.9.1 *Let $\mu^{\#}\mu$ be a PK$_n$-pair with $\mu^{\#}\mu \neq 00$ and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^{\#}\mu$ such that we have $\mu^{\#} A_c \mu \neq 00$. Suppose that*

$$\left\{ z(\zeta)_{(R,q)} \,\middle|\, \zeta \in \mathcal{Z}^{\mu^{\#}\mu} \right\} \subseteq S^{\mu^{\#}\mu}_{(R,q)}$$

is an R-basis of $S^{\mu^{\#}\mu}_{(R,q)}$. Then

$$\left\{ z(\eta)_{(R,q)} \,\middle|\, \eta \in \mathcal{Z}^{\mu^{\#}A_c\mu} \right\} \subseteq S^{\mu^{\#}\mu}_{(R,q)}$$

is an R-basis of $\operatorname{Ker}\left(\Psi^{(R,q)}_{\mu^{\#}\mu c} \Big\downarrow^{M^{\mu}_{(R,q)}}_{S^{\mu^{\#}\mu}_{(R,q)}} \right) \subseteq S^{\mu^{\#}\mu}_{(R,q)}$.

Proof. In this proof, we write

$$\begin{aligned} \mathbf{B}^{\mu^{\#}\mu} &= \left\{ z(\zeta) \,\middle|\, \zeta \in \mathcal{Z}^{\mu^{\#}\mu} \right\}, \\ \mathbf{B}^{\mu^{\#}A_c\mu} &= \left\{ z(\eta) \,\middle|\, \eta \in \mathcal{Z}^{\mu^{\#}A_c\mu} \right\}, \\ K^{\mu^{\#}\mu c} &= \operatorname{Ker}\left(\Psi_{\mu^{\#}\mu c} \Big\downarrow^{M^{\mu}}_{S^{\mu^{\#}\mu}} \right). \end{aligned}$$

According to Lemma 3.3.6.(iii) and Lemma 3.6.5, we have

$$\mathbf{B}^{\mu^{\#}A_c\mu} \subseteq \mathbf{B}^{\mu^{\#}\mu} \subseteq S^{\mu^{\#}\mu}.$$

This and the assumption that $\mathbf{B}^{\mu^{\#}\mu}$ is an R-basis of $S^{\mu^{\#}\mu}$ together imply that $\mathbf{B}^{\mu^{\#}A_c\mu}$ is linearly independent over R. Furthermore, we get from Corollary 3.7.1

$$\mathbf{B}^{\mu^{\#}A_c\mu} \subseteq K^{\mu^{\#}\mu c}.$$

Now it remains to show that $\mathbf{B}^{\mu^{\#}A_c\mu}$ generates $K^{\mu^{\#}\mu c}$ over R. To this end, we fix an

$$h \in K^{\mu^{\#}\mu c} \subseteq S^{\mu^{\#}\mu}.$$

According to the assumptions of the claim, we can write

$$h = \sum_{\zeta \in \mathcal{Z}^{\mu^\# \mu}} r_\zeta z(\zeta)$$

with uniquely determined coefficients $r_\zeta \in R$ for all $\zeta \in \mathcal{Z}^{\mu^\# \mu}$. By applying $\Psi_{\mu^\# \mu c}$ and taking into account Corollary 3.7.1 we get from this

$$0_{\mathcal{H}_A} = h\Psi_{\mu^\# \mu c} = \sum_{\zeta \in \mathcal{Z}^{\mu^\# \mu} \setminus \mathcal{Z}^{\mu^\# A_c \mu}} r_\zeta z(\zeta) \Psi_{\mu^\# \mu c}.$$

Corollary 3.7.4 now shows

$$\forall \zeta \in \mathcal{Z}^{\mu^\# \mu} \setminus \mathcal{Z}^{\mu^\# A_c \mu} : r_\zeta = 0_R$$

which implies

$$h = \sum_{\eta \in \mathcal{Z}^{\mu^\# A_c \mu}} r_\eta z(\eta).$$

Thus, $\mathbf{B}^{\mu^\# A_c \mu}$ generates $K^{\mu^\# \mu c}$ over R and all in all is an R-basis of $K^{\mu^\# \mu c}$, as desired. ■

The following statement makes use of Definition 3.2.1, Definition 3.2.3, Definition 3.6.3, Definition 3.3.5, Lemma 3.6.5, and Definition 3.2.5.(iv).

Lemma 3.9.2 *Let $\mu^\# \mu$ be a PK$_n$-pair with $\mu^\# \mu \neq 00$ and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^\# \mu$. Suppose that*

$$\left\{ z(\zeta)_{(R,q)} \,\middle|\, \zeta \in \mathcal{Z}^{\mu^\# \mu} \right\} \subseteq S^{\mu^\# \mu}_{(R,q)}$$

is an R-basis of $S^{\mu^\# \mu}_{(R,q)}$. Then the following statements hold.

(i) Let $\mu^\# A_c \mu \neq 00$. Then

$$\left\{ z(\eta)_{(R,q)} \,\middle|\, \eta \in \mathcal{Z}^{\mu^\# A_c \mu} \right\} \subseteq S^{\mu^\# A_c \mu}_{(R,q)}$$

is an R-basis of $S^{\mu^\# A_c \mu}_{(R,q)}$.

(ii) We have

$$\operatorname{Ker}\left(\Psi^{(R,q)}_{\mu^\# \mu c} \Big\downarrow^{M^\mu_{(R,q)}}_{S^{\mu^\# \mu}_{(R,q)}} \right) = S^{\mu^\# A_c \mu}_{(R,q)}.$$

Proof. First we consider the case $\mu^\# A_c \mu \neq 00$. Here we get from Lemma 3.6.5 — applied to the PK-pair $\mu^\# A_c \mu$ — and Lemma 3.5.5.(ii)

$$\left\{ z(\eta) \,\middle|\, \eta \in \mathcal{Z}^{\mu^\# A_c \mu} \right\} \subseteq S^{\mu^\# A_c \mu} \subseteq \operatorname{Ker}\left(\Psi_{\mu^\# \mu c} \Big\downarrow^{M^\mu}_{S^{\mu^\# \mu}} \right).$$

Moreover, the assumptions of the claim and Lemma 3.9.1 show that the set on the left hand side of the preceding chain of inclusions is an R-basis of the module on the right hand side. This proves statement (i) and also statement (ii) for $\mu^{\#}A_c\mu \neq 00$.

Now we consider the case $\mu^{\#}A_c\mu = 00$. Here we only must prove statement (ii). Because of $\mu^{\#}A_c\mu = 00$ and Definition 3.4.1, the claim is equivalent to the injectivity of $\Psi_{\mu^{\#}\mu c}\Big\downarrow^{M^{\mu}}_{S^{\mu^{\#}\mu}}$. This property of $\Psi_{\mu^{\#}\mu c}\Big\downarrow^{M^{\mu}}_{S^{\mu^{\#}\mu}}$ follows from the assumptions of the claim, Definition 3.3.5, and Corollary 3.7.4.

Now the claim is completely proved. ■

3.10 ZNL-bases for PK-modules and kernels of PK-homomorphisms for Hecke algebras of type A

In this section we first derive generic bases of PK-modules (see Remark 1.2.9 and Definition 3.4.1), then we derive a useful description of the kernels of PK-homomorphisms restricted to PK-modules (see Definition 3.5.3), and finally we derive a representation of Specht modules as intersections of kernels of PK-homomorphisms. As always, n denotes a positive integer and (R, q) denotes a coefficient pair.

The next theorem makes use of Definition 3.2.1, Definition 3.6.3, Definition 3.3.5, Definition 3.4.1, and Lemma 3.6.5.

Theorem 3.10.1 *Let $\mu^{\#}\mu$ be a PK_n-pair with $\mu^{\#}\mu \neq 00$. Then the set*

$$\left\{ z(\zeta)_{(R,q)} \,\middle|\, \zeta \in \mathcal{Z}^{\mu^{\#}\mu} \right\} \subseteq S^{\mu^{\#}\mu}_{(R,q)}$$

is an R-basis of $S^{\mu^{\#}\mu}_{(R,q)}$.

Proof. According to Lemma 3.2.7.(iv), there is a $\nu = (\nu_1, \nu_2, \ldots) \vDash n$ such that $(\nu_1)\nu$ forms a PK-pair from which the given PK-pair $\mu^{\#}\mu$ can be reached by an application of an appropriate chain of operators A_c and R_c with AR-indices $c \in \mathbb{N} \setminus \{1\}$. More specifically, we have an index $m \in \mathbb{N}_0$ and PK-pairs $\lambda^{\#(j)}\lambda^{(j)} \neq 00$ for $j \in \{0, \ldots, m\}$ with

$$\lambda^{\#(0)}\lambda^{(0)} = (\nu_1)\nu \qquad \text{and} \qquad \lambda^{\#(m)}\lambda^{(m)} = \mu^{\#}\mu.$$

Moreover, for every $j \in \{1, \ldots, m\}$ we have an AR-index $c_j \in \mathbb{N} \setminus \{1\}$ for the PK-pair $\lambda^{\#(j-1)}\lambda^{(j-1)}$ such that

$$\lambda^{\#(j)}\lambda^{(j)} = \lambda^{\#(j-1)}A_{c_j}\lambda^{(j-1)} \qquad \text{or} \qquad \lambda^{\#(j)}\lambda^{(j)} = \lambda^{\#(j-1)}\lambda^{(j-1)}R_{c_j}$$

holds. The claim will now be proved by induction on $j \in \{0, \dots, m\}$.

The induction starts with $j = 0$ and the PK-pair $\lambda^{\#(0)}\lambda^{(0)} = (\nu_1)\nu$. Here we have, according to Lemma 3.4.4.(ii) and Lemma 3.3.7, $S^{\lambda^{\#(0)}\lambda^{(0)}} = M^{\lambda^{(0)}}$ and $\mathcal{Z}^{\lambda^{\#(0)}\lambda^{(0)}} = \mathcal{Z}^{\lambda^{(0)}}$. With that, the claim follows from Lemma 3.6.6.(iii).

Now we consider a $j \in \{1, \dots, m\}$ and assume that $\left\{ z(\zeta) \,\middle|\, \zeta \in \mathcal{Z}^{\lambda^{\#(j-1)}\lambda^{(j-1)}} \right\}$ is an R-basis of $S^{\lambda^{\#(j-1)}\lambda^{(j-1)}}$. Then it follows in the case $\lambda^{\#(j)}\lambda^{(j)} = \lambda^{\#(j-1)}\lambda^{(j-1)}R_{c_j}$ from Lemma 3.8.1 and in the case $\lambda^{\#(j)}\lambda^{(j)} = \lambda^{\#(j-1)}A_{c_j}\lambda^{(j-1)}$ from Lemma 3.9.2.(i) that $\left\{ z(\zeta) \,\middle|\, \zeta \in \mathcal{Z}^{\lambda^{\#(j)}\lambda^{(j)}} \right\}$ is an R-basis of the module $S^{\lambda^{\#(j)}\lambda^{(j)}}$.

Thus, for every $j \in \{0, \dots, m\}$ the set $\left\{ z(\zeta) \,\middle|\, \zeta \in \mathcal{Z}^{\lambda^{\#(j)}\lambda^{(j)}} \right\}$ forms an R-basis of $S^{\lambda^{\#(j)}\lambda^{(j)}}$, and the case $j = m$ with $\lambda^{\#(m)}\lambda^{(m)} = \mu^{\#}\mu$ shows that $\left\{ z(\zeta) \,\middle|\, \zeta \in \mathcal{Z}^{\mu^{\#}\mu} \right\}$ is an R-basis of $S^{\mu^{\#}\mu}$, as desired. ■

Definition 3.10.2 *Let $\mu^{\#}\mu$ be a PK_n-pair with $\mu^{\#}\mu \neq 00$. Then the R-basis*

$$\left\{ z(\zeta)_{(R,q)} \,\middle|\, \zeta \in \mathcal{Z}^{\mu^{\#}\mu} \right\} \subseteq S^{\mu^{\#}\mu}_{(R,q)}$$

of $S^{\mu^{\#}\mu}_{(R,q)}$ from Theorem 3.10.1 is called the row number list basis of $S^{\mu^{\#}\mu}_{(R,q)}$ or just the ZNL-basis of $S^{\mu^{\#}\mu}_{(R,q)}$. We denote this basis by

$$\mathbf{B}^{S^{\mu^{\#}\mu}}_{\mathrm{ZNL}}(R,q) \qquad or \qquad \mathbf{B}^{S^{\mu^{\#}\mu}}_{\mathrm{ZNL}}.$$

Remark 3.10.3 *Remark 3.4.2, Remark 3.6.4.(i), and Theorem 3.10.1 show that the ZNL-bases of PK-modules from Definition 3.10.2 are generic in the sense of Remark 1.2.9.*

The next corollary records another fact on bases of PK-modules. It makes use of Definition 3.2.1, Definition 3.2.3, Definition 3.6.3, Definition 3.5.3, Definition 3.3.5, and Definition 3.2.5.

Corollary 3.10.4 *Let $\mu^{\#}\mu$ be a PK_n-pair with $\mu^{\#}\mu \neq 00$ and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^{\#}\mu$. Then the set*

$$\left\{ z(\zeta)_{(R,q)} \Psi^{(R,q)}_{\mu^{\#}\mu c} \,\middle|\, \zeta \in \mathcal{Z}^{\mu^{\#}\mu} \setminus \mathcal{Z}^{\mu^{\#}A_c\mu} \right\} \subseteq S^{\mu^{\#}\mu R_c}_{(R,q)}$$

is an R-Basis of $S^{\mu^{\#}\mu R_c}_{(R,q)}$.

Proof. This is obtained from Theorem 3.10.1 and the beginning of the proof of Lemma 3.8.1. ■

The following statement makes use of Definition 3.2.1, Definition 3.2.3, Definition 3.5.3, Lemma 3.4.4.(vi), and Definition 3.2.5.(iv).

Theorem 3.10.5 *Let $\mu^{\#}\mu$ be a PK_n-pair with $\mu^{\#}\mu \neq 00$ and let $c \in \mathbb{N}\setminus\{1\}$ be an AR-index for $\mu^{\#}\mu$. Then we have*

$$\operatorname{Ker}\left(\Psi^{(R,q)}_{\mu^{\#}\mu c}\Big\downarrow^{M^{\mu}_{(R,q)}}_{S^{\mu^{\#}\mu}_{(R,q)}}\right) = S^{\mu^{\#}A_c\mu}_{(R,q)}.$$

Proof. This follows from Theorem 3.10.1 and Lemma 3.9.2.(ii). ■

Remark 3.10.6 *(i) Remark 1.3.7.(i), Remark 3.4.2, Remark 3.5.4, and Theorem 3.10.5 show that the kernels of PK-homomorphisms restricted to PK-modules as considered in Theorem 3.10.5 are generic in the sense of Remark 1.2.9.*

(ii) In [DJ1, Section 7] Theorem 3.10.5 is proved under the assumption that the coefficient ring R is a field (see especially [DJ1, Lemma 7.3]).

The next corollary makes use of Definition 1.1.1.(ii) and Definition 3.5.3.

Corollary 3.10.7 *Let $\lambda = (\lambda_1, \ldots, \lambda_m) \vdash n$ with $m \in \mathbb{N}\setminus\{1\}$ and $\lambda_m > 0$. With that, put for every $i \in \{2, \ldots, m\}$ and every $j \in \{0, \ldots, \lambda_i - 1\}$*

$$\lambda^{\#(i,j)} = (\lambda_1, \ldots, \lambda_{i-1}, j) \vdash \lambda^{+}_{i-1} + j.$$

Then we have in $M^{\lambda}_{(R,q)}$

$$S^{\lambda}_{(R,q)} = \bigcap_{i\in\{2,\ldots,m\}} \bigcap_{j\in\{0,\ldots,\lambda_i-1\}} \operatorname{Ker}\Psi^{(R,q)}_{\lambda^{\#(i,j)}\lambda i}.$$

Proof. In this proof, we write

$$\mathcal{I} = \{(i,j) \mid i \in \{2, \ldots, m\} \text{ and } j \in \{0, \ldots, \lambda_i - 1\}\}.$$

Since $\lambda \vdash n$, we have in particular $\lambda_1 > 0$. For every $(i,j) \in \mathcal{I}$ this fact and the choice of j for a given i show that the composition $\lambda^{\#(i,j)}$ is a partition of a positive integer. The construction of $\lambda^{\#(i,j)}$ also implies $\left[\lambda^{\#(i,j)}\right] \subseteq [\lambda]$. From all this we see that for every $(i,j) \in \mathcal{I}$ the partitions $\lambda^{\#(i,j)}$ and λ form a PK-pair with $\lambda^{\#(i,j)}\lambda \neq 00$. Furthermore, it easily follows from the construction of $\lambda^{\#(i,j)}$ with an $(i,j) \in \mathcal{I}$ that i satisfies the conditions imposed on an AR-index for $\lambda^{\#(i,j)}\lambda$ from Definition 3.2.3. These considerations show that the claim of the corollary is meaningful.

Next, we order the set $\mathcal{I}$ lexicographically. More formally, we define for $i \in \{2, \ldots, m\}$, $j \in \{0, \ldots, \lambda_i - 1\}$ and $\tilde{i} \in \{2, \ldots, m\}$, $\tilde{j} \in \{0, \ldots, \lambda_{\tilde{i}} - 1\}$

$$(i,j) < (\tilde{i},\tilde{j}) \Leftrightarrow \left(i < \tilde{i}\right) \vee \left(\left(i = \tilde{i}\right) \wedge \left(j < \tilde{j}\right)\right).$$

This is a total ordering on the set $\mathcal{I}$ with the smallest pair $(2,0)$ and the biggest pair $(m, \lambda_m - 1)$.

Furthermore, we define for $(i,j) \in \mathcal{I}$ with $(i,j) \neq (m, \lambda_m - 1)$ the pair $(i,j)^\wedge$ as

$$(i,j)^\wedge = \begin{cases} (i, j+1) & \text{if} \quad j < \lambda_i - 1 \\ (i+1, 0) & \text{if} \quad j = \lambda_i - 1 \end{cases} .$$

Then we have $(i,j)^\wedge \in \mathcal{I}$. Moreover, $(i,j)^\wedge$ is the immediate successor of (i,j) in the lexicographic ordering on $\mathcal{I}$, that is, we have $(i,j) < (i,j)^\wedge$ and there is no $(\tilde{i}, \tilde{j}) \in \mathcal{I}$ satisfying $(i,j) < (\tilde{i}, \tilde{j}) < (i,j)^\wedge$. From Definition 3.2.5.(i) we also obtain

$$\forall (i,j) \in \mathcal{I} \setminus \{(m, \lambda_m - 1)\} : \lambda^{\#(i,j)} A_i = \lambda^{\#(i,j)^\wedge}. \tag{3.100}$$

Finally, we note the relation

$$\lambda^{\#(m, \lambda_m - 1)} A_m = \lambda \tag{3.101}$$

which again follows directly from Definition 3.2.5.(i).

For the remainder of the proof, we put

$$K = \bigcap_{(i,j) \in \mathcal{I}} \operatorname{Ker} \Psi_{\lambda^{\#(i,j)} \lambda i} \subseteq M^\lambda \tag{3.102}$$

(see Definition 3.5.3). In order to prove the claim, we must show $S^\lambda \subseteq K$ and $K \subseteq S^\lambda$.

First we verify the inclusion $S^\lambda \subseteq K$. Because of (3.102), it suffices to show for an arbitrary $(i,j) \in \mathcal{I}$ the inclusion $S^\lambda \subseteq \operatorname{Ker} \Psi_{\lambda^{\#(i,j)} \lambda i}$. To this end, we fix such an (i,j). Then we have according to Theorem 3.10.5

$$\operatorname{Ker} \left(\Psi_{\lambda^{\#(i,j)} \lambda i} \Big\downarrow^{M^\lambda}_{S^{\lambda^{\#(i,j)} \lambda}} \right) = S^{\lambda^{\#(i,j)} A_i \lambda}. \tag{3.103}$$

(3.100), (3.101), the considerations concerning the plausibility of the claim of the corollary, and Definition 3.2.1 show that $\lambda^{\#(i,j)} A_i \lambda$ is a PK-pair with $\lambda^{\#(i,j)} A_i \lambda \neq 00$. In particular, we have $\left[\lambda^{\#(i,j)} A_i\right] \subseteq [\lambda]$. From this fact, Lemma 3.4.4.(iii), and Lemma 3.4.4.(iv) we get

$$S^\lambda = S^{\lambda\lambda} \subseteq S^{\lambda^{\#(i,j)} A_i \lambda}.$$

This relation and (3.103) in turn show

$$S^\lambda \subseteq \operatorname{Ker} \left(\Psi_{\lambda^{\#(i,j)} \lambda i} \Big\downarrow^{M^\lambda}_{S^{\lambda^{\#(i,j)} \lambda}} \right) \subseteq \operatorname{Ker} \Psi_{\lambda^{\#(i,j)} \lambda i},$$

as desired. From the choice of $(i,j) \in \mathcal{I}$ and (3.102) we now obtain

$$S^\lambda \subseteq K.$$

Next we consider the reverse inclusion $K \subseteq S^\lambda$. In order to verify it, we fix a $y \in K$. We now show for every $(i,j) \in \mathcal{I}$ the relation $y \in S^{\lambda^{\#(i,j)}\lambda}$. We do this by induction on the elements of $\mathcal{I}$ using the lexicographic ordering. The induction starts with the smallest pair $(2,0)$. In this case we have $\lambda^{\#(2,0)} = (\lambda_1)$ and thus, according to Lemma 3.4.4.(ii), $S^{\lambda^{\#(2,0)}\lambda} = M^\lambda$. This shows the induction hypothesis for the pair $(2,0)$. For the induction step we consider an arbitrary $(i,j) \in \mathcal{I} \setminus \{(m, \lambda_m - 1)\}$. Suppose that we have for this pair $y \in S^{\lambda^{\#(i,j)}\lambda}$. With that, we get from the fact $y \in K$, (3.102), Theorem 3.10.5, and (3.100)

$$\begin{aligned} y &\in K \cap S^{\lambda^{\#(i,j)}\lambda} \\ &\subseteq \operatorname{Ker} \Psi_{\lambda^{\#(i,j)}\lambda i} \cap S^{\lambda^{\#(i,j)}\lambda} \\ &= \operatorname{Ker}\left(\Psi_{\lambda^{\#(i,j)}\lambda i}\Big\downarrow^{M^\lambda}_{S^{\lambda^{\#(i,j)}\lambda}}\right) \\ &= S^{\lambda^{\#(i,j)}A_i\lambda} \\ &= S^{\lambda^{\#(i,j)^\wedge}\lambda}. \end{aligned}$$

This also shows the induction hypothesis for the pair $(i,j)^\wedge$ and thus inductively for all pairs $(i,j) \in \mathcal{I}$. Now we consider in particular the biggest pair $(m, \lambda_m - 1) \in \mathcal{I}$. For this pair a calculation analogous to the preceding one but using (3.101) instead of (3.100) and furthermore Lemma 3.4.4.(iii) shows

$$\begin{aligned} y &\in K \cap S^{\lambda^{\#(m,\lambda_m-1)}\lambda} \\ &\subseteq \operatorname{Ker} \Psi_{\lambda^{\#(m,\lambda_m-1)}\lambda m} \cap S^{\lambda^{\#(m,\lambda_m-1)}\lambda} \\ &= \operatorname{Ker}\left(\Psi_{\lambda^{\#(m,\lambda_m-1)}\lambda m}\Big\downarrow^{M^\lambda}_{S^{\lambda^{\#(m,\lambda_m-1)}\lambda}}\right) \\ &= S^{\lambda^{\#(m,\lambda_m-1)}A_m\lambda} \\ &= S^{\lambda\lambda} \\ &= S^\lambda. \end{aligned}$$

This and the arbitrary choice of $y \in K$ now imply

$$K \subseteq S^\lambda.$$

All in all, we have $S^\lambda \subseteq K$ and $K \subseteq S^\lambda$ and thus

$$S^\lambda = K,$$

which proves the claim of the corollary. ■

Remark 3.10.8 *(i) The case left out in the statement of Corollary 3.10.7, that is, $m = 1$, is trivial. Using the notation from Corollary 3.10.7, we have for*

$m = 1$ the relation $\lambda = (n)$. Now the permutation module $M^{(n)}_{(R,q)}$ is free over R of rank 1 with the basis element $x^{(R,q)}_{(n)}$. Furthermore we get from Definition 1.1.64 and Lemma 1.1.65.(ii) the identities $(n)' = (1^n) = \omega^{(n)}$, $\mathfrak{S}_{(n)'} = \{1_{\mathfrak{S}_n}\}$, $\mathbf{t}_{(n)} = \mathbf{t}^{(n)}$, $w_{(n)} = 1_{\mathfrak{S}_n}$, $y^{(R,q)}_{(n)'} = 1_{\mathcal{H}^{(R,q)}_A}$, $z^{(R,q)}_{(n)} = x^{(R,q)}_{(n)}$, and finally $S^{(n)}_{(R,q)} = M^{(n)}_{(R,q)}$.

(ii) *In [DJ1, Section 7] Corollary 3.10.7 is proved under the assumption that the coefficient ring R is a field (see especially [DJ1, Theorem 7.5]).*

3.11 Construction of generic Specht series for Hecke algebras of type A and associated permutation modules and PK-modules

Now we complete the derivation of the generic Specht series for Hecke algebras of type A. To this end, we first give a formal definition of Specht series for modules of Hecke algebras of type A. Then we construct generic Specht series for PK-modules and permutation modules and finally also for Hecke algebras of type A. As before, n denotes a positive integer and (R, q) denotes a coefficient pair as in Definition 1.2.1.

Definition 3.11.1 *Let M be a right $\mathcal{H}^{(R,q)}_{A_{n-1}}$-module. Then a series of submodules*

$$0_{\mathcal{H}_A} = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{m-1} \subseteq M_m = M$$

with an $m \in \mathbb{N}_0$ and the property

$$\forall j \in \{1, \ldots, m\} : M_j/M_{j-1} \simeq S^{\lambda^{(j)}}_{(R,q)} \text{ as } \mathcal{H}^{(R,q)}_{A_{n-1}}\text{-modules for an appropriate } \lambda^{(j)} \vdash n$$

is called a Specht series for M. Here, $0_{\mathcal{H}_A}$ denotes the trivial $\mathcal{H}^{(R,q)}_{A_{n-1}}$-submodule of M. The number m is called the length of this Specht series.

The following theorem makes use of Definition 3.2.1 and Definition 3.4.1.

Theorem 3.11.2 *Let $\mu^{\#}\mu$ be a PK_n-pair. Then there is a Specht series for the PK_n-module $S^{\mu^{\#}\mu}_{(R,q)}$.*

Proof. In the case $\mu^{\#}\mu = 00$ we have, according to Definition 3.4.1, $S^{\mu^{\#}\mu} = 0_{\mathcal{H}_A}$. Here, $0_{\mathcal{H}_A}$ denotes the null ideal in $\mathcal{H}_n$. This shows the claim with the Specht series

$$0_{\mathcal{H}_A} = S^{\mu^{\#}\mu}$$

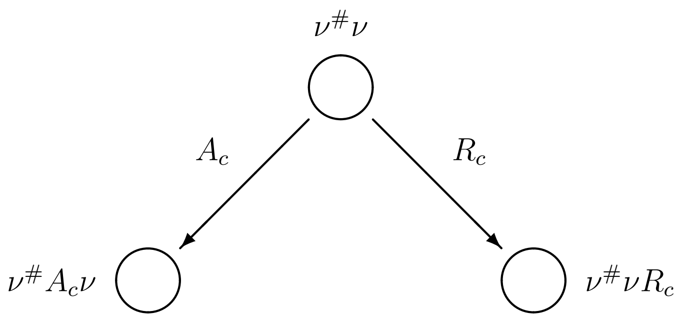
of length 0.

In the case $\mu^\#\mu \neq 00$ and $\mu^\# = \mu$ we have, according to Lemma 3.4.4.(iii), $S^{\mu^\#\mu} = S^\mu$. This shows the claim with the Specht series

$$0_{\mathcal{H}_A} \subseteq S^\mu = S^{\mu^\#\mu}$$

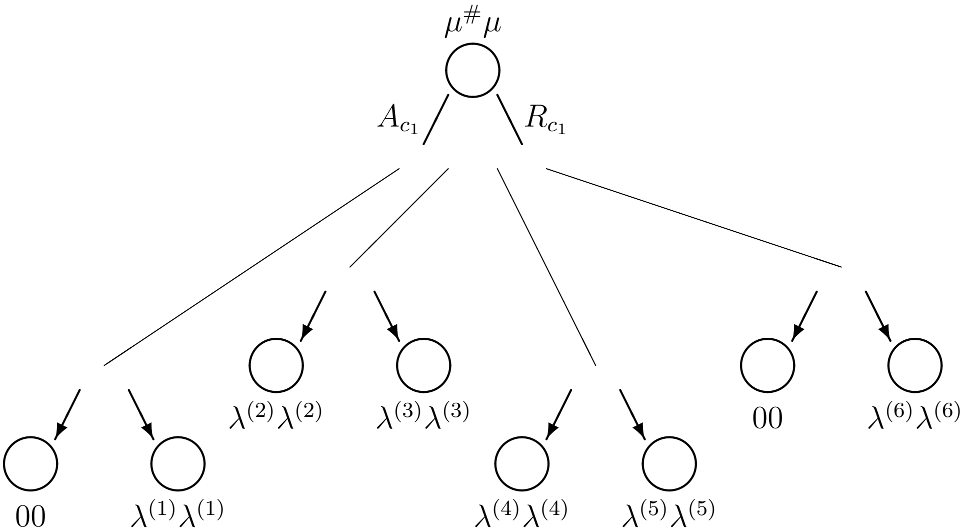
of length 1.

In the remainder of the proof, we assume $\mu^\#\mu \neq 00$ and $\mu^\# \neq \mu$. Then we get from Lemma 3.2.7.(iii) an AR-index $c_1 \in \mathbb{N} \setminus \{1\}$ for the PK-pair $\mu^\#\mu$ which enables us to construct the PK-pairs $\mu^\# A_{c_1}\mu$ and $\mu^\#\mu R_{c_1}$ as in Definition 3.2.5. Now there might be an AR-index $c_2 \in \mathbb{N} \setminus \{1\}$ for $\mu^\# A_{c_1}\mu$ respectively an AR-index $c_3 \in \mathbb{N} \setminus \{1\}$ for $\mu^\#\mu R_{c_1}$ such that the application of the operators A_{c_2} and R_{c_2} to $\mu^\# A_{c_1}\mu$ respectively the operators A_{c_3} and R_{c_3} to $\mu^\#\mu R_{c_1}$ leads to further PK-pairs. The iteration of this process as long as possible produces a binary tree (that is, every vertex in the tree has zero or two successors) whose vertices are labelled with PK-pairs and whose edges are labelled with operators A_c and R_c with appropriate AR-indices $c \in \mathbb{N} \setminus \{1\}$. More specifically, the root of the tree (that is, the vertex without predecessor) is labelled $\mu^\#\mu$, and if a vertex of the tree has two successors then the label of this vertex is a PK-pair $\nu^\#\nu \neq 00$, the labels of the edges leading to its successors are A_c and R_c with an AR-index $c \in \mathbb{N} \setminus \{1\}$ for $\nu^\#\nu$, the label of the vertex at the other end of the edge labelled A_c is $\nu^\# A_c\nu$, and the label of the vertex at the other end of the edge labelled R_c is $\nu^\#\nu R_c$. This part of the tree is displayed in the following picture.



(3.104)

Now let $\nu^{\#}\nu \neq 00$ be an arbitrary PK-pair occurring in the tree and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\nu^{\#}\nu$. Then the application of the operator A_c to $\nu^{\#}\nu$ either produces 00 or increases the number of lattice points in the diagram $\left[\nu^{\#}\right] \subseteq [\nu]$ but leaves unchanged the number of lattice points in the diagram $[\nu]$ (see Definition 3.2.1 and Definition 3.2.5). Furthermore, the application of the operator R_c to $\nu^{\#}\nu$ moves lattice points from the c-th row of the diagram $[\nu]$ to the $(c-1)$-th row of $[\nu]$ and possibly increases the number of lattice points in the diagram $\left[\nu^{\#}\right] \subseteq [\nu]$ (see Definition 1.1.6, Definition 3.2.1, and Definition 3.2.5). This shows that any iterative application of operators A_c and R_c with appropriate AR-indices $c \in \mathbb{N} \setminus \{1\}$ to $\mu^{\#}\mu$ inevitably produces after a finite number of steps a PK-pair to which no such operators can be applied any more. From this we see that the binary tree constructed in this way from $\mu^{\#}\mu$ only contains a finite number of vertices. Moreover, Lemma 3.2.7.(iii) implies that the labels of the leaves of this tree (that is, the vertices without successors) are of the form 00 or $\lambda\lambda$ with an appropriate $\lambda \vdash n$. Thus the complete binary tree has the form



(3.105)

with appropriate $\lambda^{(1)}, \ldots, \lambda^{(6)} \in \Pi_n$ (see Definition 1.1.2.(iii)) and possibly more such partitions.

The claim of the theorem in the case $\mu^{\#}\mu \neq 00$ and $\mu^{\#} \neq \mu$ is now proved by induction on the labels of the vertices of this tree along the edges from the leaves to the root. The induction start is provided by the two special cases $\mu^{\#}\mu = 00$ on the one hand and $\mu^{\#}\mu \neq 00$ and $\mu^{\#} = \mu$ on the other hand considered at the beginning of the proof together with the above considerations concerning the labelling of the leaves of the binary tree. For the induction step we consider a vertex of the tree which is not a leaf. This vertex is then, as shown in picture (3.104), labelled with a PK-pair $\nu^{\#}\nu \neq 00$ and the labels of its successors are $\nu^{\#}A_c\nu$ and $\nu^{\#}\nu R_c$ with an

AR-index $c \in \mathbb{N} \setminus \{1\}$ for $\nu^{\#}\nu$. With this data we get from Lemma 3.4.4.(v)

$$0_{\mathcal{H}_A} \subseteq S^{\nu^{\#} A_c \nu} \subseteq S^{\nu^{\#}\nu} \tag{3.106}$$

and furthermore from Lemma 3.5.5.(iii) and Theorem 3.10.5

$$S^{\nu^{\#}\nu} / S^{\nu^{\#} A_c \nu} \simeq S^{\nu^{\#}\nu R_c} \quad \text{as } \mathcal{H}_n\text{-modules,} \tag{3.107}$$

the isomorphism being induced by the map $\Psi_{\nu^{\#}\nu c}\Big\downarrow^{M^{\nu}}_{S^{\nu^{\#}\nu}}$. Now if the claim of the theorem holds for $\nu^{\#} A_c \nu$ and $\nu^{\#}\nu R_c$ then we can use the Specht series for $S^{\nu^{\#} A_c \nu}$ and $S^{\nu^{\#}\nu R_c}$ and the isomorphism (3.107) to refine the series (3.106) for $S^{\nu^{\#}\nu}$ to a Specht series for $S^{\nu^{\#}\nu}$. Thus the claim of the theorem also holds for $\nu^{\#}\nu$ and inductively for all PK-pairs occurring as labels of vertices in the binary tree (3.105). In particular, the claim of the theorem holds for the label of the root of the tree, namely $\mu^{\#}\mu$. ■

Remark 3.11.3 *(i) Remark 1.3.15.(i), Remark 3.4.2, Remark 3.5.4, and Remark 3.10.6.(i) show that the Specht series for PK-modules constructed in the proof of Theorem 3.11.2 are generic in the sense of Remark 1.2.9.*

(ii) In [DJ1, Section 7] Theorem 3.11.2 is proved under the assumption that the coefficient ring R is a field (see especially [DJ1, Theorem 7.4]).

Corollary 3.11.4 *Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash n$ with $\lambda_1 > 0$. Then there is a Specht series for the permutation module $M^{\lambda}_{(R,q)}$.*

Proof. The assumption $\lambda_1 > 0$ allows us to build the PK-pair $(\lambda_1)\lambda$ (see Definition 3.2.1). Now the claim follows from Lemma 3.4.4.(ii) and Theorem 3.11.2. ■

Remark 3.11.5 *(i) Remark 3.11.3.(i) shows that the Specht series for permutation modules constructed in the proof of Corollary 3.11.4 are generic in the sense of Remark 1.2.9.*

(ii) Consider $\lambda, \mu \vDash n$ which differ only by a permutation of their entries. Then we have according to [DJ1, Lemma 4.3]

$$M^{\lambda}_{(R,q)} \simeq M^{\mu}_{(R,q)}$$

as $\mathcal{H}^{(R,q)}_{A_{n-1}}$-modules. The particular isomorphism that is constructed in [DJ1, Lemma 4.3] also is generic in the sense of Remark 1.2.9. All this shows that the condition imposed on the composition in the statement of Corollary 3.11.4 is not essential.

Corollary 3.11.6 *If we consider* $\mathcal{H}_{A_{n-1}}^{(R,q)}$ *as a right* $\mathcal{H}_{A_{n-1}}^{(R,q)}$*-module then there is a Specht series for this module.*

Proof. From Definition 1.1.64, Lemma 1.1.65.(ii), and Definition 1.3.1 we see

$$M^{\omega^{(n)}} = \mathcal{H}_n.$$

Now the claim follows from Corollary 3.11.4. ■

Remark 3.11.7 *Remark 3.11.5.(i) shows that the Specht series for* $\mathcal{H}_{A_{n-1}}^{(R,q)}$ *constructed in the proof of Corollary 3.11.6 is generic in the sense of Remark 1.2.9.*

This completes the derivation of the generic Specht series for Hecke algebras of type A.

Chapter 4

Generic bi-Specht series for Hecke algebras of type B

This chapter describes the generalization of the generic Specht series for Hecke algebras of type A from the preceding chapter to Hecke algebras of type B.

The first and the second section describe the combinatorial background required for the treatment of Hecke algebras of type B and the algebras themselves. The next three sections introduce and investigate the modules employed in the construction of the bi-Specht series. The following section describes certain homomorphisms between these modules. The final section carries out the construction of the bi-Specht series for Hecke algebras of type B and describes how this construction can be adapted to Hecke algebras of type D. The central results are Theorem 4.7.4 and Theorem 4.7.6.

4.1 Combinatorics for Hecke algebras of type B

This section provides the combinatorial objects and constructions relevant to Hecke algebras of type B. References for the following material are [DJ3, Section 2] and [DJM, Section 3]. As always, $n \in \mathbb{N}$ denotes a positive integer.

The first part of this section introduces the Weyl groups underlying the Hecke algebras of type B (see also [DJ3, Section 2] and [HUM, Chapter 1, Chapter 2, Chapter 5]).

Definition 4.1.1 *The group W_{B_n} is defined to be generated by the elements*

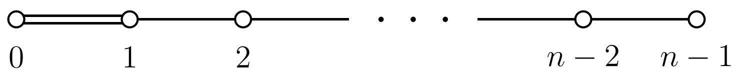
$$t, s_1, \ldots, s_{n-1} \tag{4.1}$$

subject to the relations

$$t^2 = 1_{W_B},$$
$$\forall i \in \{1, \ldots, n-1\} : s_i^2 = 1_{W_B},$$
$$ts_1ts_1 = s_1ts_1t,$$
$$\forall i \in \{2, \ldots, n-1\} : ts_i = s_it,$$
$$\forall i \in \{1, \ldots, n-2\} : s_is_{i+1}s_i = s_{i+1}s_is_{i+1},$$
$$\forall i, j \in \{1, \ldots, n-1\} \text{ with } |i-j| > 1 : s_is_j = s_js_i$$

where 1_{W_B} *denotes the neutral element of this group.* W_{B_n} *is the Weyl group of type* B_n. W_{B_n} *also is called a Weyl group of type* B *and denoted by* W_B. *The generators (4.1) are called the simple reflections in* W_{B_n}.

W_{B_n} is the Weyl group of the root system of type B_n with the following Dynkin diagram.



Here, the simple reflection t corresponds to the vertex 0 and for every index $j \in \{1, \ldots, n-1\}$ the simple reflection s_j corresponds to the vertex j. Moreover, the element $t \in W_{B_n}$ generates a subgroup isomorphic to C_2 (the cyclic group of order 2) and the set $\{s_1, \ldots, s_{n-1}\} \subseteq W_{B_n}$ generates a subgroup isomorphic to $\mathfrak{S}_n \simeq W_{A_{n-1}}$ (see (1.6) on page 2). With these two subgroups of W_{B_n} we have the following realization of W_{B_n} as a wreath product (see for example [DJ3, Section 2] or [HUM, Section 1.1]).

$$W_{B_n} \simeq C_2 \wr \mathfrak{S}_n \tag{4.2}$$

The notions of reduced expressions and length for the elements of the Weyl group W_{B_n} are defined as in (1.7) on page 2 and (1.8) on page 2. The length function

$$\ell_{B_n} = \ell_B = \ell : W_{B_n} \to \mathbb{N}_0, \quad w \mapsto \ell_{B_n}(w) = \ell_B(w) = \ell(w)$$

has properties analogous to those in (1.9) on page 3 (see also [DJ3, Section 2]).

Next we describe certain embeddings of Weyl groups of type A into other Weyl groups of type A and also into Weyl groups of type B. From the construction of the Weyl groups of the types A and B in (1.3) on page 2, (1.4) on page 2, and Definition 4.1.1 we see that the next definition is meaningful.

Definition 4.1.2 *(i) Let $m \in \{1, \dots, n\}$. Then the assignments*

$$W_{A_{m-1}} \ni s_i \mapsto s_i^{\overset{m}{\leftarrow}} = s_i \in W_{A_{n-1}} \quad \textit{for} \quad i \in \{1, \dots, m-1\}$$

define by multiplicative extension a length preserving and injective group homomorphism

$$\cdot^{\overset{m}{\leftarrow}} : W_{A_{m-1}} \hookrightarrow W_{A_{n-1}}, \quad w \mapsto w^{\overset{m}{\leftarrow}}.$$

This homomorphism is called the left inclusion of $W_{A_{m-1}}$ into $W_{A_{n-1}}$. As an abbreviation, we write

$$\cdot^{\overset{m}{\leftarrow}} = \cdot^{\leftarrow}.$$

(ii) Let $m \in \{1, \dots, n\}$. Then the assignments

$$W_{A_{m-1}} \ni s_i \mapsto s_i^{\overset{m}{\rightarrow}} = s_{i+n-m} \in W_{A_{n-1}} \quad \textit{for} \quad i \in \{1, \dots, m-1\}$$

define by multiplicative extension a length preserving and injective group homomorphism

$$\cdot^{\overset{m}{\rightarrow}} : W_{A_{m-1}} \hookrightarrow W_{A_{n-1}}, \quad w \mapsto w^{\overset{m}{\rightarrow}}.$$

This homomorphism is called the right inclusion of $W_{A_{m-1}}$ into $W_{A_{n-1}}$. As an abbreviation, we write

$$\cdot^{\overset{m}{\rightarrow}} = \cdot^{\rightarrow}.$$

(iii) The assignments

$$W_{A_{n-1}} \ni s_i \mapsto s_i \in W_{B_n} \quad \textit{for} \quad i \in \{1, \dots, n-1\}$$

define by multiplicative extension a length preserving and injective group homomorphism

$$W_{A_{n-1}} \hookrightarrow W_{B_n}, \quad W_{A_{n-1}} \ni w \mapsto w \in W_{B_n}.$$

We identify the group $W_{A_{n-1}}$ with its image in W_{B_n} under this embedding.

Remark 4.1.3 *(i) The embedding of $W_{A_{n-1}}$ into W_{B_n} from Definition 4.1.2.(iii) has already been used in the derivation of (4.2).*

(ii) The embedding of $W_{A_{n-1}}$ into W_{B_n} from Definition 4.1.2.(iii) also allows us to apply constructions for Weyl groups of type A from the preceding chapters to Weyl groups of type B.

Lemma 4.1.4 *Let $a \in \{1, \dots, n-1\}$. Choose $u \in W_{A_{n-a-1}}$ and $v \in W_{A_{a-1}}$. Then we have in $W_{A_{n-1}}$*

(i) $u^{\overset{n-a}{\leftarrow}} v^{\overset{a}{\rightarrow}} = v^{\overset{a}{\rightarrow}} u^{\overset{n-a}{\leftarrow}}$,

(ii) $\ell_{A_{n-1}}(u^{\overset{n-a}{\leftarrow}} v^{\overset{a}{\rightarrow}}) = \ell_{A_{n-1}}(u^{\overset{n-a}{\leftarrow}}) + \ell_{A_{n-1}}(v^{\overset{a}{\rightarrow}})$.

Proof. (i) According to Definition 4.1.2.(i), $u^{\leftarrow} \in W_{A_{n-1}}$ is a product of factors from the set $\{s_1, \ldots, s_{n-a-1}\} \subseteq W_{A_{n-1}}$. Similarly, according to Definition 4.1.2.(ii), $v^{\rightarrow} \in W_{A_{n-1}}$ is a product of factors from the set $\{s_{n-a+1}, \ldots, s_{n-1}\} \subseteq W_{A_{n-1}}$. According to (1.4) on page 2, every element of the set $\{s_1, \ldots, s_{n-a-1}\}$ commutes with every element of the set $\{s_{n-a+1}, \ldots, s_{n-1}\}$. This shows the claim.

(ii) This follows from the considerations in the proof of statement (i), (1.6) on page 2, and (1.10) on page 3. ■

The following lemma makes use of the isomorphism (1.6) on page 2 and the notation (1.2) on page 1.

Lemma 4.1.5 *Let $a \in \{1, \ldots, n-1\}$. Then we have in $W_{A_{n-1}} \simeq \mathfrak{S}_n$ the following identities.*

(i) $W_{A_{n-a-1}}{}^{\overset{n-a}{\leftarrow}} = \mathfrak{S}_{\{1,\ldots,n-a\}}$.

(ii) $W_{A_{a-1}}{}^{\overset{a}{\rightarrow}} = \mathfrak{S}_{\{n-a+1,\ldots,n\}}$.

(iii) $\left(W_{A_{n-a-1}}{}^{\overset{n-a}{\leftarrow}}\right) \cap \left(W_{A_{a-1}}{}^{\overset{a}{\rightarrow}}\right) = \{1_{W_A}\}$.

(iv) $\left(W_{A_{n-a-1}}{}^{\overset{n-a}{\leftarrow}}\right) \left(W_{A_{a-1}}{}^{\overset{a}{\rightarrow}}\right) = \mathfrak{S}_{(n-a,a)}$.

Proof. Statements (i) and (ii) follow from the consideration of the sets of simple reflections generating the respective groups and (1.6) on page 2. Statement (iii) follows from (1.6) and statements (i) and (ii). Furthermore, Lemma 4.1.4.(i) and statement (iii) show that the product $\left(W_{A_{n-a-1}}{}^{\overset{n-a}{\leftarrow}}\right) \left(W_{A_{a-1}}{}^{\overset{a}{\rightarrow}}\right)$ in $W_{A_{n-1}}$ is direct. Now statement (iv) follows from a comparison of statements (i) and (ii) with the decomposition (1.18) of Young subgroups on page 25. ■

Now we describe some constructions relevant to the representation theory of Weyl groups of type B. They generalize constructions known from the representation theory of symmetric groups employed in the preceding chapters. The notions introduced in the next definition can also be found, for example, in [DJM, Section 3].

Definition 4.1.6 *Let $a \in \{0, \ldots, n\}$.*

(i) A pair (λ, μ) consisting of $\lambda \vDash a$ and $\mu \vDash n-a$ is called an a-bi-composition of n or a bi-composition of n or just an a-bi-composition or a bi-composition.

(ii) *A pair* (λ, μ) *consisting of* $\lambda \vdash a$ *and* $\mu \vdash n - a$ *is called an a-bi-partition of* n *or a bi-partition of* n *or just an a-bi-partition or a bi-partition.*

The following definition makes use of Definition 3.2.1.

Definition 4.1.7 (i) *Let* $a \in \{1, \dots, n-1\}$. *Furthermore, let* $\lambda^\# \lambda$ *be a* PK_a*-pair with* $\lambda^\# \lambda \neq 00$ *and let* $\mu^\# \mu$ *be a* PK_{n-a}*-pair with* $\mu^\# \mu \neq 00$. *Then the pair*

$$(\lambda^\# \lambda, \mu^\# \mu)$$

is called an a-bi-PK_n*-pair or an a-bi-PK-pair or just a bi-*PK_n*-pair or a bi-PK-pair.*

(ii) *Let* $\mu^\# \mu$ *be a* PK_n*-pair with* $\mu^\# \mu \neq 00$. *Then the pair*

$$(00, \mu^\# \mu)$$

is called a 0-bi-PK_n*-pair or a 0-bi-PK-pair or just a bi-*PK_n*-pair or a bi-PK-pair.*

(iii) *Let* $\lambda^\# \lambda$ *be a* PK_n*-pair with* $\lambda^\# \lambda \neq 00$. *Then the pair*

$$(\lambda^\# \lambda, 00)$$

is called an n-bi-PK_n*-pair or an n-bi-PK-pair or just a bi-*PK_n*-pair or a bi-PK-pair.*

(iv) *The pair*

$$(00, 00)$$

also is called a bi-PK-pair.

In the preceding definition, the abbreviation PK stands for partition-composition. The following remark makes use of Definition 4.1.6.(i).

Remark 4.1.8 (i) *Let* $a \in \{1, \dots, n-1\}$ *and fix an a-bi-*PK_n*-pair* $(\lambda^\# \lambda, \mu^\# \mu)$. *Then* (λ, μ) *is an a-bi-composition.*

(ii) *Let* $(00, \mu^\# \mu)$ *be a 0-bi-*PK_n*-pair. Then* $((0), \mu)$ *is a 0-bi-composition.*

(iii) *Let* $(\lambda^\# \lambda, 00)$ *be an n-bi-*PK_n*-pair. Then* $(\lambda, (0))$ *is an n-bi-composition.*

The next definition uses Definition 3.2.3 and Definition 3.2.5. Lemma 3.2.7.(ii) shows that it is meaningful.

Definition 4.1.9 *Let $(\lambda^\#\lambda, \mu^\#\mu)$ be a bi-PK$_n$-pair.*

(i) Suppose $\lambda^\#\lambda \neq 00$ and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\lambda^\#\lambda$. Then the bi-PK-pair $(\lambda^\#\lambda, \mu^\#\mu)^{(c)}A$ is defined as

$$(\lambda^\#\lambda, \mu^\#\mu)^{(c)}A = \begin{cases} (\lambda^\# A_c\lambda, \mu^\#\mu) & \text{for } \lambda^\# A_c\lambda \neq 00 \\ (00, 00) & \text{for } \lambda^\# A_c\lambda = 00 \end{cases}.$$

$(\lambda^\#\lambda, \mu^\#\mu)^{(c)}A$ is called the image of the bi-PK$_n$-pair $(\lambda^\#\lambda, \mu^\#\mu)$ under the operator ${}^{(c)}A$.

(ii) Suppose $\mu^\#\mu \neq 00$ and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^\#\mu$. Then the bi-PK-pair $(\lambda^\#\lambda, \mu^\#\mu)A^{(c)}$ is defined as

$$(\lambda^\#\lambda, \mu^\#\mu)A^{(c)} = \begin{cases} (\lambda^\#\lambda, \mu^\# A_c\mu) & \text{for } \mu^\# A_c\mu \neq 00 \\ (00, 00) & \text{for } \mu^\# A_c\mu = 00 \end{cases}.$$

$(\lambda^\#\lambda, \mu^\#\mu)A^{(c)}$ is called the image of the bi-PK$_n$-pair $(\lambda^\#\lambda, \mu^\#\mu)$ under the operator $A^{(c)}$.

(iii) Suppose $\lambda^\#\lambda \neq 00$ and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\lambda^\#\lambda$. Then the bi-PK-pair $(\lambda^\#\lambda, \mu^\#\mu)^{(c)}R$ is defined as

$$(\lambda^\#\lambda, \mu^\#\mu)^{(c)}R = (\lambda^\#\lambda R_c, \mu^\#\mu).$$

$(\lambda^\#\lambda, \mu^\#\mu)^{(c)}R$ is called the image of the bi-PK$_n$-pair $(\lambda^\#\lambda, \mu^\#\mu)$ under the operator ${}^{(c)}R$.

(iv) Suppose $\mu^\#\mu \neq 00$ and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^\#\mu$. Then the bi-PK-pair $(\lambda^\#\lambda, \mu^\#\mu)R^{(c)}$ is defined as

$$(\lambda^\#\lambda, \mu^\#\mu)R^{(c)} = (\lambda^\#\lambda, \mu^\#\mu R_c).$$

$(\lambda^\#\lambda, \mu^\#\mu)R^{(c)}$ is called the image of the bi-PK$_n$-pair $(\lambda^\#\lambda, \mu^\#\mu)$ under the operator $R^{(c)}$.

Now we introduce some useful elements of W_{B_n}. The next definition makes use of Definition 4.1.2.(iii), it is modelled on [DJ3, Definition 2.3].

Definition 4.1.10 *Let $a \in \{0, \ldots, n\}$. Then the element $w_{a,n-a} \in W_{A_{n-1}} \subseteq W_{B_n}$ is defined as*

$$w_{a,n-a} = (s_{n-1}s_{n-2}\cdots s_2 s_1)^{n-a}.$$

Remark 4.1.11 *Let $a \in \{0, \ldots, n\}$. Then we get from (1.6) on page 2*

$$s_{n-1}s_{n-2}\cdots s_2 s_1 = (1, 2, \cdots, n-1, n).$$

From this in turn we see that for $a \in \{1, \ldots, n-1\}$ the element $w_{a,n-a}$ from Definition 4.1.10 maps the numbers $1, \ldots, a$ in the given order to the numbers $n-a+1, \ldots, n$ and the numbers $a+1, \ldots, n$ in the given order to the numbers $1, \ldots, n-a$. Moreover, for $a \in \{0, n\}$ we obtain $w_{a,n-a} = 1_{\mathfrak{S}_n}$.

4.2 Hecke algebras of type B

In this section we describe Hecke algebras of type B, as also considered in [DJ3]. Further information on the history and background of these algebras also can be found there. Moreover, we refer the reader to [HUM, Chapter 7] where Hecke algebras of arbitrary type are constructed in a very general way. As before, $n \in \mathbb{N}$ denotes a positive integer.

The next definition is analogous to Definition 1.2.1.

Definition 4.2.1 *Let R be an integral domain. Furthermore, let $q \in R$ be a unit and $Q \in R$ be an arbitrary element. Then the triple (R, q, Q) is called a coefficient triple.*

Remark 4.2.2 *Let (R, q, Q) be a coefficient triple as in Definition 4.2.1. Then (R, q) is a coefficient pair as in Definition 1.2.1.*

The following description of the Hecke algebra of type B_n is from [DJ3, Section 3], it makes use of Definition 4.1.1.

Definition 4.2.3 *Let (R, q, Q) be a coefficient triple. Then the Hecke algebra*

$$\mathcal{H}_{B_n}^{(R,q,Q)} = \mathcal{H}_B^{(R,q,Q)} = \mathcal{H}_{B_n} = \mathcal{H}_B$$

of type B_n — or more generally of type B — over the coefficient triple (R, q, Q) is defined as the free R-module with basis $\{T_w \mid w \in W_{B_n}\}$ on which the rules

(i) $T_{1_{W_B}} = 1_{\mathcal{H}_B^{(R,q,Q)}}$,

(ii) $T_t^2 = QT_{1_{W_B}} + (Q-1)T_t$,

(iii) $\forall i \in \{1, \ldots, n-1\} : T_{s_i}^2 = qT_{1_{W_B}} + (q-1)T_{s_i}$,

(iv) $T_w = T_{v_1} \cdots T_{v_{\ell(w)}}$ *for every* $w \in W_{B_n}$ *having a reduced expression* $w = v_1 \cdots v_{\ell_B(w)}$ *with factors* $v_1, \ldots, v_{\ell_B(w)} \in \{t, s_1, \ldots, s_{n-1}\}$

induce an associative multiplication. Here, $1_{\mathcal{H}_B^{(R,q,Q)}}$ *denotes the multiplicative neutral element of the algebra* $\mathcal{H}_{B_n}^{(R,q,Q)}$. *Furthermore, the additive neutral element of* $\mathcal{H}_{B_n}^{(R,q,Q)}$ *is denoted by* $0_{\mathcal{H}_B^{(R,q,Q)}}$. *The parameter* n *is called the degree of the Hecke algebra* $\mathcal{H}_{B_n}^{(R,q,Q)}$.

For all the following we fix a coefficient triple (R, q, Q). From the rules for the multiplication on $\mathcal{H}_{B_n}$ in Definition 4.2.3 we get the following useful facts (see Definition 4.1.2.(iii), (1.22) on page 34, (1.23) on page 34, and [DJ3, (3.1)]).

$$\text{For } u, v \in W_{B_n} \text{ satisfying } \ell_{B_n}(uv) = \ell_{B_n}(u) + \ell_{B_n}(v), \text{ we have } T_{uv} = T_u T_v. \tag{4.3}$$

$$\text{For every } w \in W_{A_{n-1}} \subseteq W_{B_n},\ T_w \text{ is invertible in } \mathcal{H}_{B_n}^{(R,q,Q)}. \tag{4.4}$$

Now we investigate the behavior of $\mathcal{H}_{B_n}$ when changing the coefficient ring. This is done as in the case of type A in Section 1.2. Let $\tilde{R}$ be an integral domain and let $\xi : R \to \tilde{R}$ be a ring homomorphism. Then $(\tilde{R}, \xi(q), \xi(Q))$ is a coefficient triple as in Definition 4.2.1. Furthermore, using ξ, $\tilde{R}$ can be considered a left R-module. This allows us to build the functor $- \otimes_R \tilde{R}$.

Lemma 4.2.4 *We have* $\mathcal{H}_{B_n}^{(R,q,Q)} \otimes_R \tilde{R} \simeq \mathcal{H}_{B_n}^{(\tilde{R},\xi(q),\xi(Q))}$ *as* $\tilde{R}$*-algebras.*

Proof. This proof is completely analogous to that of Lemma 1.2.7. ∎

Remark 4.2.5 *Lemma 4.2.4 and its proof show that Hecke algebras of type B are generic in the sense of Remark 1.2.9.*

Next we generalize Definition 4.1.2 from Weyl groups to Hecke algebras. This can be done because the maps considered there are length preserving, further because of the similar constructions of the Hecke algebras of types A and B in Section 1.2 and Definition 4.2.3, and finally because of Remark 4.2.2.

Definition 4.2.6 *(i) Let* $m \in \{1, \ldots, n\}$. *Then the assignments*

$$\mathcal{H}_{A_{m-1}}^{(R,q)} \ni T_w \mapsto T_w^{\overset{m}{\leftarrow}} = T_{w^{\overset{m}{\leftarrow}}} \in \mathcal{H}_{A_{n-1}}^{(R,q)} \quad \textit{for} \quad w \in W_{A_{m-1}}$$

define by R*-linear extension an injective algebra homomorphism*

$$\cdot^{\overset{m}{\leftarrow}} : \mathcal{H}_{A_{m-1}}^{(R,q)} \hookrightarrow \mathcal{H}_{A_{n-1}}^{(R,q)}, \quad h \mapsto h^{\overset{m}{\leftarrow}}.$$

This homomorphism is called the left inclusion of $\mathcal{H}_{A_{m-1}}^{(R,q)}$ *into* $\mathcal{H}_{A_{n-1}}^{(R,q)}$. *As an abbreviation, we write*

$$\cdot^{\overset{m}{\leftarrow}} = \cdot^{\leftarrow}.$$

(ii) *Let $m \in \{1, \dots, n\}$. Then the assignments*

$$\mathcal{H}_{A_{m-1}}^{(R,q)} \ni T_w \mapsto T_w^{\overset{m}{\to}} = T_{w^{\overset{m}{\to}}} \in \mathcal{H}_{A_{n-1}}^{(R,q)} \quad \textit{for} \quad w \in W_{A_{m-1}}$$

define by R-linear extension an injective algebra homomorphism

$$\cdot^{\overset{m}{\to}} : \mathcal{H}_{A_{m-1}}^{(R,q)} \hookrightarrow \mathcal{H}_{A_{n-1}}^{(R,q)}, \quad h \mapsto h^{\overset{m}{\to}}.$$

This homomorphism is called the right inclusion of $\mathcal{H}_{A_{m-1}}^{(R,q)}$ into $\mathcal{H}_{A_{n-1}}^{(R,q)}$. As an abbreviation, we write

$$\cdot^{\overset{m}{\to}} = \cdot^{\to}.$$

(iii) *The assignments*

$$\mathcal{H}_{A_{n-1}}^{(R,q)} \ni T_w \mapsto T_w \in \mathcal{H}_{B_n}^{(R,q,Q)} \quad \textit{for} \quad w \in W_{A_{n-1}}$$

define by R-linear extension an injective algebra homomorphism

$$\mathcal{H}_{A_{n-1}}^{(R,q)} \hookrightarrow \mathcal{H}_{B_n}^{(R,q,Q)}, \quad \mathcal{H}_{A_{n-1}}^{(R,q)} \ni h \mapsto h \in \mathcal{H}_{B_n}^{(R,q,Q)}.$$

We identify the algebra $\mathcal{H}_{A_{n-1}}^{(R,q)}$ with its image in $\mathcal{H}_{B_n}^{(R,q,Q)}$ under this embedding.

Remark 4.2.7 *(i) Lemma 1.2.7 and Lemma 4.2.4 together with their proofs show that the embeddings from Definition 4.2.6 are generic in the sense of Remark 1.2.9.*

(ii) *The embedding from Definition 4.2.6.(iii) also allows us to apply constructions for Hecke algebras of type A from the preceding chapters to Hecke algebras of type B. According to statement (i), generic constructions for Hecke algebras of type A remain generic when considered in Hecke algebras of type B.*

Lemma 4.2.8 *Fix an $a \in \{1, \dots, n-1\}$. Then all the following statements hold in $\mathcal{H}_{A_{n-1}}^{(R,q)}$.*

(i) *Let $x \in \mathcal{H}_{A_{n-a-1}}^{(R,q)}$ and $y \in \mathcal{H}_{A_{a-1}}^{(R,q)}$. Then we have in $\mathcal{H}_{A_{n-1}}^{(R,q)}$*

$$x^{\overset{n-a}{\leftarrow}} y^{\overset{a}{\to}} = y^{\overset{a}{\to}} x^{\overset{n-a}{\leftarrow}}.$$

(ii) $\left(\mathcal{H}_{A_{n-a-1}}^{(R,q)}{}^{\overset{n-a}{\leftarrow}}\right)\left(\mathcal{H}_{A_{a-1}}^{(R,q)}{}^{\overset{a}{\to}}\right) \subseteq \mathcal{H}_{A_{n-1}}^{(R,q)}$ *is an R-subalgebra.*

(iii) *We have*

$$\left(\mathcal{H}_{A_{n-a-1}}^{(R,q)}{}^{\overset{n-a}{\leftarrow}}\right)\left(\mathcal{H}_{A_{a-1}}^{(R,q)}{}^{\overset{a}{\to}}\right) \simeq \mathcal{H}_{A_{n-a-1}}^{(R,q)} \otimes_R \mathcal{H}_{A_{a-1}}^{(R,q)}$$

as R-algebras.

(iv) $\mathcal{H}_{A_{n-1}}^{(R,q)}$ *is a free left* $\left(\mathcal{H}_{A_{n-a-1}}^{(R,q)}{}^{\overset{n-a}{\leftarrow}}\right)\left(\mathcal{H}_{A_{a-1}}^{(R,q)}{}^{\overset{a}{\rightarrow}}\right)$*-module with basis*

$$\left\{T_g \,|\, g \in \mathcal{D}_{(n-a,a)}\right\}.$$

In other words, we have

$$\mathcal{H}_{A_{n-1}}^{(R,q)} = \bigoplus_{g\in\mathcal{D}_{(n-a,a)}} \left(\mathcal{H}_{A_{n-a-1}}^{(R,q)}{}^{\overset{n-a}{\leftarrow}}\right)\left(\mathcal{H}_{A_{a-1}}^{(R,q)}{}^{\overset{a}{\rightarrow}}\right) T_g,$$

the sum being direct over $\left(\mathcal{H}_{A_{n-a-1}}^{(R,q)}{}^{\overset{n-a}{\leftarrow}}\right)\left(\mathcal{H}_{A_{a-1}}^{(R,q)}{}^{\overset{a}{\rightarrow}}\right)$.

(v) Let

$$M \subseteq \mathcal{H}_{A_{a-1}}^{(R,q)} \qquad \text{and} \qquad N \subseteq \mathcal{H}_{A_{n-a-1}}^{(R,q)}$$

be right ideals. Furthermore, let

$$\{x_i \mid i \in \mathcal{I}\} \subseteq M$$

be an R*-basis of* M *with a certain index set* $\mathcal{I}$*. Finally, let*

$$\{y_j \mid j \in \mathcal{J}\} \subseteq N$$

be an R*-basis of* N *with a certain index set* $\mathcal{J}$*. Then the set*

$$\begin{aligned}
&\left(\{x_i \mid i \in \mathcal{I}\}^{\overset{a}{\rightarrow}}\right)\left(\{y_j \mid j \in \mathcal{J}\}^{\overset{n-a}{\leftarrow}}\right)\left\{T_g \,|\, g \in \mathcal{D}_{(n-a,a)}\right\} \\
&= \left\{\left(x_i{}^{\overset{a}{\rightarrow}}\right)\left(y_j{}^{\overset{n-a}{\leftarrow}}\right)T_g \,\middle|\, i \in \mathcal{I},\ j \in \mathcal{J},\ g \in \mathcal{D}_{(n-a,a)}\right\} \\
&\subseteq \mathcal{H}_{A_{n-1}}^{(R,q)}
\end{aligned}$$

is an R*-basis of the right ideal*

$$\left(M^{\overset{a}{\rightarrow}}\right)\left(N^{\overset{n-a}{\leftarrow}}\right)\mathcal{H}_{A_{n-1}}^{(R,q)} \subseteq \mathcal{H}_{A_{n-1}}^{(R,q)}.$$

Proof. (i) This follows from the construction of the Hecke algebras of type A in Section 1.2 — in particular formula (1.22) on page 34 — and Lemma 4.1.4.

(ii) This follows easily from statement (i).

(iii) We see from the construction of the Hecke algebras of type A in Section 1.2 and Lemma 4.1.4.(ii) that $\left(\mathcal{H}_{A_{n-a-1}}{}^{\leftarrow}\right)\left(\mathcal{H}_{A_{a-1}}{}^{\rightarrow}\right)$ is generated over R by the set

$$\begin{aligned}
&\left\{T_{u^{\leftarrow}} \,|\, u \in W_{A_{n-a-1}}\right\}\left\{T_{v^{\rightarrow}} \,|\, v \in W_{A_{a-1}}\right\} \\
&= \left\{T_{u^{\leftarrow}v^{\rightarrow}} \,|\, u \in W_{A_{n-a-1}},\ v \in W_{A_{a-1}}\right\} \\
&\subseteq \mathcal{H}_{A_{n-1}}.
\end{aligned} \tag{4.5}$$

Now Lemma 4.1.5.(iii) shows that the elements of this set are indexed by pairs (u, v) with $u \in W_{A_{n-a-1}}$ and $v \in W_{A_{a-1}}$. Again according to the construction of the Hecke algebras of type A, this set also is linearly independent over R. Thus it is an R-basis of the R-algebra $\left(\mathcal{H}_{A_{n-a-1}}{}^{\leftarrow}\right)\left(\mathcal{H}_{A_{a-1}}{}^{\rightarrow}\right)$. Furthermore, $\mathcal{H}_{A_{n-a-1}} \otimes_R \mathcal{H}_{A_{a-1}}$ has the R-basis

$$\left\{T_u \otimes_R T_v \mid u \in W_{A_{n-a-1}},\, v \in W_{A_{a-1}}\right\}. \tag{4.6}$$

The elements of this set also are indexed by pairs (u, v) with $u \in W_{A_{n-a-1}}$ and $v \in W_{A_{a-1}}$. The desired isomorphism now is obtained by identifying elements of the bases (4.5) and (4.6) having the same index pair.

(iv) From the construction of the Hecke algebras of type A in Section 1.2, the isomorphism (1.6) on page 2, Lemma 1.1.56, Lemma 4.1.4, and Lemma 4.1.5 we get

$$\begin{aligned}
\mathcal{H}_{A_{n-1}} &= \bigoplus_{x\in\mathfrak{S}_n} RT_x \\
&= \bigoplus_{g\in\mathcal{D}_{(n-a,a)}} \left(\bigoplus_{w\in\mathfrak{S}_{(n-a,a)}} RT_w\right) T_g \\
&= \bigoplus_{g\in\mathcal{D}_{(n-a,a)}} \left(\bigoplus_{u\in W_{A_{n-a-1}}} \bigoplus_{v\in W_{A_{a-1}}} RT_{u^{\leftarrow}} T_{v^{\rightarrow}}\right) T_g \\
&= \bigoplus_{g\in\mathcal{D}_{(n-a,a)}} \left(\bigoplus_{u\in W_{A_{n-a-1}}} RT_{u^{\leftarrow}}\right)\left(\bigoplus_{v\in W_{A_{a-1}}} RT_{v^{\rightarrow}}\right) T_g \\
&= \bigoplus_{g\in\mathcal{D}_{(n-a,a)}} \left(\mathcal{H}_{A_{n-a-1}}{}^{\leftarrow}\right)\left(\mathcal{H}_{A_{a-1}}{}^{\rightarrow}\right) T_g,
\end{aligned}$$

all occurring sums being direct over R. This proves the claim.

(v) From statements (i), (iii), and (iv) we get

$$\begin{aligned}
\left(M^{\rightarrow}\right)\left(N^{\leftarrow}\right)\mathcal{H}_{A_{n-1}} &= \left(M^{\rightarrow}\right)\left(N^{\leftarrow}\right)\left(\bigoplus_{g\in\mathcal{D}_{(n-a,a)}} \left(\mathcal{H}_{A_{n-a-1}}{}^{\leftarrow}\right)\left(\mathcal{H}_{A_{a-1}}{}^{\rightarrow}\right) T_g\right) \\
&= \bigoplus_{g\in\mathcal{D}_{(n-a,a)}} \left(\left(M^{\rightarrow}\right)\left(N^{\leftarrow}\right)\right)\left(\left(\mathcal{H}_{A_{n-a-1}}{}^{\leftarrow}\right)\left(\mathcal{H}_{A_{a-1}}{}^{\rightarrow}\right)\right) T_g \\
&= \bigoplus_{g\in\mathcal{D}_{(n-a,a)}} \left(\left(M\mathcal{H}_{A_{a-1}}\right)^{\rightarrow}\right)\left(\left(N\mathcal{H}_{A_{n-a-1}}\right)^{\leftarrow}\right) T_g \\
&= \bigoplus_{g\in\mathcal{D}_{(n-a,a)}} \left(M^{\rightarrow}\right)\left(N^{\leftarrow}\right) T_g \\
&= \bigoplus_{g\in\mathcal{D}_{(n-a,a)}} \left(\left(\bigoplus_{i\in\mathcal{I}} Rx_i\right)^{\rightarrow}\right)\left(\left(\bigoplus_{j\in\mathcal{J}} Ry_j\right)^{\leftarrow}\right) T_g
\end{aligned}$$

$$= \bigoplus_{g \in \mathcal{D}_{(n-a,a)}} \bigoplus_{i \in \mathcal{I}} \bigoplus_{j \in \mathcal{J}} R\,(x_i{}^{\rightarrow})\,(y_j{}^{\leftarrow})\,T_g,$$

all occurring sums being direct at least over R. This proves the claim. ■

Now we record some identities in $\mathcal{H}_{B_n}$ which will be useful later on. The following statement makes use of Definition 4.1.1.

Lemma 4.2.9 *(i) Let $i \in \{1, \ldots, n\}$. Then we have in $\mathcal{H}_{B_n}^{(R,q,Q)}$*

$$T_{s_{i-1}\cdots s_1} T_t T_{s_1 \cdots s_{i-1}} = T_{s_{i-1}\cdots s_1 t s_1 \cdots s_{i-1}}.$$

(ii) Let $i, j \in \{1, \ldots, n\}$. Then we have in $\mathcal{H}_{B_n}^{(R,q,Q)}$

$$T_{s_{i-1}\cdots s_1 t s_1 \cdots s_{i-1}} T_{s_{j-1}\cdots s_1 t s_1 \cdots s_{j-1}} = T_{s_{j-1}\cdots s_1 t s_1 \cdots s_{j-1}} T_{s_{i-1}\cdots s_1 t s_1 \cdots s_{i-1}}.$$

Proof. (i) See [DJ3, considerations after (3.1)].

(ii) See [DJ3, considerations after Definition 3.2]. ■

4.3 Bi-permutation modules for Hecke algebras of type B

In this section we introduce a family of modules for Hecke algebras of type B which is based on the permutation modules for Hecke algebras of type A from Definition 1.3.1. This procedure is similar to the investigation of the representation theory of Weyl groups of type B by means of the representation theory of Weyl groups of type A (see, for example, [KER]). We keep the notation from the preceding section, that is, n denotes a positive integer and (R, q, Q) denotes a coefficient triple as in Definition 4.2.1.

First, we provide the elements of $\mathcal{H}_{B_n}$ required for the construction of the modules. The following definition makes use of Definition 4.1.10, Definition 4.2.6.(iii), and Lemma 4.2.9. The latter shows that the arrangement of the factors in the products occurring in part (ii) is not important. This definition follows [DJ3, Definition 3.2 and Definition 3.8].

Definition 4.3.1 *(i) Let $a \in \{0, \ldots, n\}$. Then we define the element $h_{a,n-a}^{(R,q)} \in \mathcal{H}_{A_{n-1}}^{(R,q)} \subseteq \mathcal{H}_{B_n}^{(R,q,Q)}$ as*

$$h_{a,n-a}^{(R,q)} = T_{w_{a,n-a}}.$$

As an abbreviation, we write

$$h_{a,n-a}^{(R,q)} = h_{a,n-a}.$$

(ii) Let $m \in \{0, \ldots, n\}$. Then we define the elements $u_m^+(R,q,Q), u_m^-(R,q,Q) \in \mathcal{H}_{B_n}^{(R,q,Q)}$ as

$$u_m^+(R,q,Q) = \prod_{i=1}^{m} \left(q^{i-1} + T_{s_{i-1}\cdots s_1} T_t T_{s_1 \cdots s_{i-1}}\right)$$

and

$$u_m^-(R,q,Q) = \prod_{i=1}^{m} \left(Qq^{i-1} - T_{s_{i-1}\cdots s_1} T_t T_{s_1 \cdots s_{i-1}}\right).$$

As an abbreviation, we write

$$u_m^+(R,q,Q) = u_m^+$$

and

$$u_m^-(R,q,Q) = u_m^-.$$

(iii) Let $a \in \{0, \ldots, n\}$. Then we define the element $v_{a,n-a}^{(R,q,Q)} \in \mathcal{H}_{B_n}^{(R,q,Q)}$ as

$$v_{a,n-a}^{(R,q,Q)} = u_a^+(R,q,Q) h_{a,n-a}^{(R,q)} u_{n-a}^-(R,q,Q).$$

As an abbreviation, we write

$$v_{a,n-a}^{(R,q,Q)} = v_{a,n-a}.$$

Remark 4.3.2 *(i) From Remark 4.1.11 we see*

$$h_{0,n}^{(R,q)} = h_{n,0}^{(R,q)} = 1_{\mathcal{H}_B^{(R,q,Q)}}.$$

(ii) From Definition 4.3.1 and statement (i) we see

$$v_{0,n}^{(R,q,Q)} = u_n^-(R,q,Q) \qquad \text{and} \qquad v_{n,0}^{(R,q,Q)} = u_n^+(R,q,Q).$$

The next lemma makes use of Definition 4.2.6 and Remark 4.2.7.(ii). Statements (v) and (vi) also are proved in [DJ3, Lemma 3.10].

Lemma 4.3.3 *(i) Let $a \in \{1, \ldots, n\}$ and $x \in \mathcal{H}_{A_{a-1}}^{(R,q)}$. Then we have in $\mathcal{H}_{A_{n-1}}^{(R,q)}$*

$$\left(x^{\overset{a}{\leftarrow}}\right) h_{a,n-a}^{(R,q)} = h_{a,n-a}^{(R,q)} \left(x^{\overset{a}{\rightarrow}}\right).$$

(ii) Let $a \in \{0, \ldots, n-1\}$ and $y \in \mathcal{H}_{A_{n-a-1}}^{(R,q)}$. Then we have in $\mathcal{H}_{A_{n-1}}^{(R,q)}$

$$\left(y^{\overset{n-a}{\rightarrow}}\right) h_{a,n-a}^{(R,q)} = h_{a,n-a}^{(R,q)} \left(y^{\overset{n-a}{\leftarrow}}\right).$$

(iii) Let $m \in \{1,\ldots,n\}$ and $x \in \mathcal{H}^{(R,q)}_{A_{m-1}}$. Then we have in $\mathcal{H}^{(R,q,Q)}_{B_n}$

$$\left(x^{\overset{m}{\leftarrow}}\right) u^+_m(R,q,Q) = u^+_m(R,q,Q)\left(x^{\overset{m}{\leftarrow}}\right)$$

and

$$\left(x^{\overset{m}{\leftarrow}}\right) u^-_m(R,q,Q) = u^-_m(R,q,Q)\left(x^{\overset{m}{\leftarrow}}\right).$$

(iv) Let $m \in \{0,\ldots,n-1\}$ and $y \in \mathcal{H}^{(R,q)}_{A_{n-m-1}}$. Then we have in $\mathcal{H}^{(R,q,Q)}_{B_n}$

$$\left(y^{\overset{n-m}{\rightarrow}}\right) u^+_m(R,q,Q) = u^+_m(R,q,Q)\left(y^{\overset{n-m}{\rightarrow}}\right)$$

and

$$\left(y^{\overset{n-m}{\rightarrow}}\right) u^-_m(R,q,Q) = u^-_m(R,q,Q)\left(y^{\overset{n-m}{\rightarrow}}\right).$$

(v) Let $a \in \{1,\ldots,n\}$ and $x \in \mathcal{H}^{(R,q)}_{A_{a-1}}$. Then we have in $\mathcal{H}^{(R,q,Q)}_{B_n}$

$$\left(x^{\overset{a}{\leftarrow}}\right) v^{(R,q,Q)}_{a,n-a} = v^{(R,q,Q)}_{a,n-a}\left(x^{\overset{a}{\rightarrow}}\right).$$

(vi) Let $a \in \{0,\ldots,n-1\}$ and $y \in \mathcal{H}^{(R,q)}_{A_{n-a-1}}$. Then we have in $\mathcal{H}^{(R,q,Q)}_{B_n}$

$$\left(y^{\overset{n-a}{\rightarrow}}\right) v^{(R,q,Q)}_{a,n-a} = v^{(R,q,Q)}_{a,n-a}\left(y^{\overset{n-a}{\leftarrow}}\right).$$

Proof. Statements (i) and (ii) follow from [DJ3, (2.5) and (2.7)], formula (1.22) on page 34, and Definition 4.2.6. Furthermore, statements (iii) and (iv) follow from [DJ3, Proposition 3.4], the product representation (1.7) on page 2, and again formula (1.22). Finally, statements (v) and (vi) follow from Definition 4.3.1 and statements (i) to (iv). ■

Now we employ the elements introduced in Definition 4.3.1.(iii) to construct a useful series of right ideals in $\mathcal{H}_{B_n}$. The next theorem makes use of Definition 1.9.3 and Definition 4.2.6.(iii), it is derived from [DJ3, Theorem 3.17].

Theorem 4.3.4 *If we put, in addition to Definition 4.3.1.(iii),*

$$v^{(R,q,Q)}_{0,0} = v_{0,0} = 1_{\mathcal{H}^{(R,q,Q)}_B} \tag{4.7}$$

then we have for arbitrary

$$a, b \in \mathbb{N}_0 \quad \textit{with} \quad a+b<n$$

a short exact sequence of right ideals in $\mathcal{H}^{(R,q,Q)}_{B_n}$

$$0_{\mathcal{H}^{(R,q,Q)}_B} \to v^{(R,q,Q)}_{a,b+1}\mathcal{H}^{(R,q,Q)}_{B_n} \to v^{(R,q,Q)}_{a,b}\mathcal{H}^{(R,q,Q)}_{B_n} \to v^{(R,q,Q)}_{a+1,b}\mathcal{H}^{(R,q,Q)}_{B_n} \to 0_{\mathcal{H}^{(R,q,Q)}_B},$$

the homomorphism $v^{(R,q,Q)}_{a,b+1}\mathcal{H}^{(R,q,Q)}_{B_n} \to v^{(R,q,Q)}_{a,b}\mathcal{H}^{(R,q,Q)}_{B_n}$ being the natural inclusion and the homomorphism $v^{(R,q,Q)}_{a,b}\mathcal{H}^{(R,q,Q)}_{B_n} \to v^{(R,q,Q)}_{a+1,b}\mathcal{H}^{(R,q,Q)}_{B_n}$ being induced by premultiplication with a certain element of $\mathcal{H}^{(R,q,Q)}_{B_n}$.

Proof. In the case $a + b > 0$ we see from Definition 4.3.1, Definition 1.9.3, and Definition 4.2.6.(iii)

$$u_a^+ \in \mathcal{H}_{B_n}, \qquad h_{a,b} \in \mathcal{H}_{A_{a+b-1}} \subseteq \mathcal{H}_{A_{n-1}} \subseteq \mathcal{H}_{B_n}, \qquad u_b^- \in \mathcal{H}_{B_n}$$

and thus

$$v_{a,b} \in \mathcal{H}_{B_n}.$$

According to (4.7), this also holds in the case $a + b = 0$. Similarly we get for arbitrary values of $a + b$

$$v_{a,b+1} \in \mathcal{H}_{B_n} \qquad \text{and} \qquad v_{a+1,b} \in \mathcal{H}_{B_n}.$$

This shows that the claim of the theorem is meaningful. The short exact sequence and the particular forms of the homomorphisms occurring therein now follow from [DJ3, Theorem 3.17 and its proof]. ■

Remark 4.3.5 *Definition 4.3.1, Remark 4.2.5, and [DJ3, proof of Theorem 3.17] show that the short exact sequences from Theorem 4.3.4 are generic in the sense of Remark 1.2.9.*

Corollary 4.3.6 *There is a series of right ideals in* $\mathcal{H}_{B_n}^{(R,q,Q)}$

$$0_{\mathcal{H}_B^{(R,q,Q)}} = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{m-1} \subseteq M_m = \mathcal{H}_{B_n}^{(R,q,Q)} \tag{4.8}$$

with a certain $m \in \mathbb{N}$ *such that we have for every* $j \in \{1, \ldots, m\}$

$$M_j / M_{j-1} \simeq v_{a_j, n-a_j}^{(R,q,Q)} \mathcal{H}_{B_n}^{(R,q,Q)} \quad \textit{as } \mathcal{H}_{B_n}^{(R,q,Q)}\textit{-modules}$$

with an appropriate $a_j \in \{0, \ldots, n\}$.

Proof. This follows from repeated applications of Theorem 4.3.4 with various values of the parameters a and b occurring there. The starting point is $a = b = 0$. Then we have according to Theorem 4.3.4

$$v_{0,0}\mathcal{H}_{B_n} = \mathcal{H}_{B_n}$$

and

$$v_{0,0}\mathcal{H}_{B_n} / v_{0,1}\mathcal{H}_{B_n} \simeq v_{1,0}\mathcal{H}_{B_n} \quad \text{as } \mathcal{H}_{B_n}\text{-modules.}$$

From this we see that in the case $n = 1$ the series

$$0_{\mathcal{H}_B} \subseteq v_{0,1}\mathcal{H}_{B_n} \subseteq v_{0,0}\mathcal{H}_{B_n} = \mathcal{H}_{B_n}$$

has the desired properties. In the case $n > 1$ this series can be refined by applying Theorem 4.3.4 to the quotients of adjacent right ideals occurring in it. Here, the particular values of the parameters a and b involved always satisfy $a + b = 1$. If we iterate this procedure until we reach the upper bound for $a + b$ in Theorem 4.3.4 then we obtain a series of right ideals in $\mathcal{H}_{B_n}$ with the desired properties. ■

Remark 4.3.7 *Remark 4.3.5 and the proof of Corollary 4.3.6 show that the series of right ideals constructed in Corollary 4.3.6 is generic in the sense of Remark 1.2.9.*

Next, we investigate the behavior of ideals in Hecke algebras of type A when multiplying them up to ideals in Hecke algebras of type B with the elements introduced in Definition 4.3.1. The following statement makes use of Definition 4.2.6.(iii) and Definition 4.1.2.(iii).

Theorem 4.3.8 *Choose an $a \in \{0, \ldots, n\}$. Then the following two statements hold in $\mathcal{H}_{B_n}^{(R,q,Q)}$.*

(i) We have

$$v_{a,n-a}^{(R,q,Q)} \mathcal{H}_{B_n}^{(R,q,Q)} = v_{a,n-a}^{(R,q,Q)} \mathcal{H}_{A_{n-1}}^{(R,q)}.$$

(ii) The right ideal $v_{a,n-a}^{(R,q,Q)} \mathcal{H}_{B_n}^{(R,q,Q)}$ in $\mathcal{H}_{B_n}^{(R,q,Q)}$ has the R-basis

$$\left\{ v_{a,n-a}^{(R,q,Q)} T_w \,\middle|\, w \in W_{A_{n-1}} \subseteq W_{B_n} \right\}.$$

Proof. (i) See [DJ3, Theorem 3.13].

(ii) See [DJ3, Theorem 3.15]. ■

The next corollary makes use of Definition 4.2.6 and Definition 1.1.58.

Corollary 4.3.9 *Let $a \in \{1, \ldots, n-1\}$. With that, let $M \subseteq \mathcal{H}_{A_{a-1}}^{(R,q)}$ and $N \subseteq \mathcal{H}_{A_{n-a-1}}^{(R,q)}$ be right ideals. Then the following statements hold.*

(i) For the right $\mathcal{H}_{B_n}^{(R,q,Q)}$-ideal $v_{a,n-a}^{(R,q,Q)} \left(M^{\overset{a}{\to}} \right) \left(N^{\overset{n-a}{\leftarrow}} \right) \mathcal{H}_{B_n}^{(R,q,Q)}$ we have

$$v_{a,n-a}^{(R,q,Q)} \left(M^{\overset{a}{\to}} \right) \left(N^{\overset{n-a}{\leftarrow}} \right) \mathcal{H}_{B_n}^{(R,q,Q)} = v_{a,n-a}^{(R,q,Q)} \left(M^{\overset{a}{\to}} \right) \left(N^{\overset{n-a}{\leftarrow}} \right) \mathcal{H}_{A_{n-1}}^{(R,q)}.$$

(ii) Let $\{x_i \mid i \in \mathcal{I}\} \subseteq M$ be an R-basis of M with a certain index set $\mathcal{I}$ and let $\{y_j \mid j \in \mathcal{J}\} \subseteq N$ be an R-basis of N with a certain index set $\mathcal{J}$. Then the set

$$\begin{aligned} & v_{a,n-a}^{(R,q,Q)} \left(\{x_i \mid i \in \mathcal{I}\}^{\overset{a}{\to}} \right) \left(\{y_j \mid j \in \mathcal{J}\}^{\overset{n-a}{\leftarrow}} \right) \left\{ T_g \,|\, g \in \mathcal{D}_{(n-a,a)} \right\} \\ = & \left\{ v_{a,n-a}^{(R,q,Q)} \left(x_i^{\overset{a}{\to}} \right) \left(y_j^{\overset{n-a}{\leftarrow}} \right) T_g \,\middle|\, i \in \mathcal{I},\, j \in \mathcal{J},\, g \in \mathcal{D}_{(n-a,a)} \right\} \\ \subseteq & \mathcal{H}_{B_n}^{(R,q,Q)} \end{aligned}$$

is an R-basis of the right ideal $v_{a,n-a}^{(R,q,Q)} \left(M^{\overset{a}{\to}} \right) \left(N^{\overset{n-a}{\leftarrow}} \right) \mathcal{H}_{B_n}^{(R,q,Q)}$ in $\mathcal{H}_{B_n}^{(R,q,Q)}$.

Proof. (i) From Lemma 4.3.3.(v), Lemma 4.3.3.(vi), and Theorem 4.3.8.(i) we get

$$\begin{aligned} v_{a,n-a}\left(M^{\rightarrow}\right)\left(N^{\leftarrow}\right)\mathcal{H}_{B_n} &= \left(M^{\leftarrow}\right)\left(N^{\rightarrow}\right)v_{a,n-a}\mathcal{H}_{B_n} \\ &= \left(M^{\leftarrow}\right)\left(N^{\rightarrow}\right)v_{a,n-a}\mathcal{H}_{A_{n-1}} \\ &= v_{a,n-a}\left(M^{\rightarrow}\right)\left(N^{\leftarrow}\right)\mathcal{H}_{A_{n-1}}. \end{aligned}$$

This shows the claim.

(ii) Theorem 4.3.8 shows that we obtain an R-basis for $v_{a,n-a}\left(M^{\rightarrow}\right)\left(N^{\leftarrow}\right)\mathcal{H}_{A_{n-1}}$ by premultiplication of an R-basis for $\left(M^{\rightarrow}\right)\left(N^{\leftarrow}\right)\mathcal{H}_{A_{n-1}}$ with $v_{a,n-a}$. Now the claim follows from statement (i) and Lemma 4.2.8.(v). ■

The next statement makes use of Definition 4.2.6.

Corollary 4.3.10 *Let $M \subseteq \mathcal{H}_{A_{n-1}}^{(R,q)}$ be a right ideal. Then the following statements hold in $\mathcal{H}_{B_n}^{(R,q,Q)}$.*

(i) We have

$$\begin{aligned} v_{0,n}^{(R,q,Q)}\left(M^{\overset{n}{\leftarrow}}\right)\mathcal{H}_{B_n}^{(R,q,Q)} &= v_{0,n}^{(R,q,Q)}M\mathcal{H}_{B_n}^{(R,q,Q)} \\ &= v_{0,n}^{(R,q,Q)}\left(M^{\overset{n}{\leftarrow}}\right)\mathcal{H}_{A_{n-1}}^{(R,q)} \\ &= v_{0,n}^{(R,q,Q)}M\mathcal{H}_{A_{n-1}}^{(R,q)} \\ &= v_{0,n}^{(R,q,Q)}\left(M^{\overset{n}{\leftarrow}}\right) \\ &= v_{0,n}^{(R,q,Q)}M. \end{aligned}$$

(ii) We have

$$\begin{aligned} v_{n,0}^{(R,q,Q)}\left(M^{\overset{n}{\rightarrow}}\right)\mathcal{H}_{B_n}^{(R,q,Q)} &= v_{n,0}^{(R,q,Q)}M\mathcal{H}_{B_n}^{(R,q,Q)} \\ &= v_{n,0}^{(R,q,Q)}\left(M^{\overset{n}{\rightarrow}}\right)\mathcal{H}_{A_{n-1}}^{(R,q)} \\ &= v_{n,0}^{(R,q,Q)}M\mathcal{H}_{A_{n-1}}^{(R,q)} \\ &= v_{n,0}^{(R,q,Q)}\left(M^{\overset{n}{\rightarrow}}\right) \\ &= v_{n,0}^{(R,q,Q)}M. \end{aligned}$$

Proof. (i) This follows from Definition 4.2.6.(i), Lemma 4.3.3.(vi), and Theorem 4.3.8.(i).

(ii) This is obtained from Definition 4.2.6.(ii), Lemma 4.3.3.(v), and Theorem 4.3.8.(i). ■

The next corollary makes use of Definition 4.2.6.

Corollary 4.3.11 *Let $M \subseteq \mathcal{H}_{A_{n-1}}^{(R,q)}$ be a right ideal. Furthermore, let $\{x_i \mid i \in \mathcal{I}\} \subseteq M$ be an R-basis of M with a certain index set $\mathcal{I}$. Then the following statements hold.*

(i) The set

$$\begin{aligned} v_{0,n}^{(R,q,Q)}\left(\{x_i \mid i \in \mathcal{I}\}^{\overset{n}{\leftarrow}}\right) &= v_{0,n}^{(R,q,Q)}\left(\{x_i \mid i \in \mathcal{I}\}\right) \\ &= \left\{ v_{0,n}^{(R,q,Q)}\left(x_i^{\overset{n}{\leftarrow}}\right) \,\middle|\, i \in \mathcal{I} \right\} \\ &= \left\{ v_{0,n}^{(R,q,Q)} x_i \,\middle|\, i \in \mathcal{I} \right\} \end{aligned}$$

is an R-basis of the right ideal $v_{0,n}^{(R,q,Q)}\left(M^{\overset{n}{\leftarrow}}\right)\mathcal{H}_{B_n}^{(R,q,Q)} = v_{0,n}^{(R,q,Q)} M \mathcal{H}_{B_n}^{(R,q,Q)}$ contained in $\mathcal{H}_{B_n}^{(R,q,Q)}$.

(ii) The set

$$\begin{aligned} v_{n,0}^{(R,q,Q)}\left(\{x_i \mid i \in \mathcal{I}\}^{\overset{n}{\rightarrow}}\right) &= v_{n,0}^{(R,q,Q)}\left(\{x_i \mid i \in \mathcal{I}\}\right) \\ &= \left\{ v_{n,0}^{(R,q,Q)}\left(x_i^{\overset{n}{\rightarrow}}\right) \,\middle|\, i \in \mathcal{I} \right\} \\ &= \left\{ v_{n,0}^{(R,q,Q)} x_i \,\middle|\, i \in \mathcal{I} \right\} \end{aligned}$$

is an R-basis of the right ideal $v_{n,0}^{(R,q,Q)}\left(M^{\overset{n}{\rightarrow}}\right)\mathcal{H}_{B_n}^{(R,q,Q)} = v_{n,0}^{(R,q,Q)} M \mathcal{H}_{B_n}^{(R,q,Q)}$ contained in $\mathcal{H}_{B_n}^{(R,q,Q)}$.

Proof. (i) From Definition 4.2.6.(i) we see that the sets occurring in the claim are equal. Moreover, according to Theorem 4.3.8, we obtain an R-basis for $v_{0,n}M$ by premultiplication of an R-basis for M with $v_{0,n}$. Now an application of Corollary 4.3.10.(i) completes the proof of the claim.

(ii) The proof of this statement makes use of Definition 4.2.6.(ii) and Corollary 4.3.10.(ii) and is otherwise completely analogous to the proof of statement (i). ■

Now we employ the algebra elements introduced in Definition 4.3.1 to define modules for Hecke algebras of type B in the form of right ideals. In addition, the next definition makes use of Definition 4.1.6.(i), Definition 4.2.6, and Lemma 4.2.8.(i).

Definition 4.3.12 *(i) Let $a \in \{1, \ldots, n-1\}$ and let (λ, μ) be an a-bi-composition of n. Then the right ideal $M_{(R,q,Q)}^{(\lambda,\mu)}$ in $\mathcal{H}_{B_n}^{(R,q,Q)}$ is defined as*

$$\begin{aligned} M_{(R,q,Q)}^{(\lambda,\mu)} &= v_{a,n-a}^{(R,q,Q)}\left(M_{(R,q)}^{\lambda}{}^{\overset{a}{\rightarrow}}\right)\left(M_{(R,q)}^{\mu}{}^{\overset{n-a}{\leftarrow}}\right)\mathcal{H}_{B_n}^{(R,q,Q)} \\ &= v_{a,n-a}^{(R,q,Q)}\left(x_{\lambda}^{(R,q)}{}^{\overset{a}{\rightarrow}}\right)\left(x_{\mu}^{(R,q)}{}^{\overset{n-a}{\leftarrow}}\right)\mathcal{H}_{B_n}^{(R,q,Q)}. \end{aligned}$$

(ii) *Let $((0),\mu)$ be a 0-bi-composition of n. Then the right ideal $M^{((0),\mu)}_{(R,q,Q)}$ in $\mathcal{H}^{(R,q,Q)}_{B_n}$ is defined as*

$$\begin{aligned} M^{((0),\mu)}_{(R,q,Q)} &= v^{(R,q,Q)}_{0,n} M^{\mu}_{(R,q)} \mathcal{H}^{(R,q,Q)}_{B_n} \\ &= v^{(R,q,Q)}_{0,n} \left(M^{\mu}_{(R,q)}{}^{\overset{n}{\leftarrow}} \right) \mathcal{H}^{(R,q,Q)}_{B_n} \\ &= v^{(R,q,Q)}_{0,n} x^{(R,q)}_{\mu} \mathcal{H}^{(R,q,Q)}_{B_n} \\ &= v^{(R,q,Q)}_{0,n} \left(x^{(R,q)}_{\mu}{}^{\overset{n}{\leftarrow}} \right) \mathcal{H}^{(R,q,Q)}_{B_n}. \end{aligned}$$

(iii) *Let $(\lambda,(0))$ be an n-bi-composition of n. Then the right ideal $M^{(\lambda,(0))}_{(R,q,Q)}$ in $\mathcal{H}^{(R,q,Q)}_{B_n}$ is defined as*

$$\begin{aligned} M^{(\lambda,(0))}_{(R,q,Q)} &= v^{(R,q,Q)}_{n,0} M^{\lambda}_{(R,q)} \mathcal{H}^{(R,q,Q)}_{B_n} \\ &= v^{(R,q,Q)}_{n,0} \left(M^{\lambda}_{(R,q)}{}^{\overset{n}{\rightarrow}} \right) \mathcal{H}^{(R,q,Q)}_{B_n} \\ &= v^{(R,q,Q)}_{n,0} x^{(R,q)}_{\lambda} \mathcal{H}^{(R,q,Q)}_{B_n} \\ &= v^{(R,q,Q)}_{n,0} \left(x^{(R,q)}_{\lambda}{}^{\overset{n}{\rightarrow}} \right) \mathcal{H}^{(R,q,Q)}_{B_n}. \end{aligned}$$

For an a-bi-composition (λ,μ) of n with $a \in \{0,\ldots,n\}$, the right ideal $M^{(\lambda,\mu)}_{(R,q,Q)}$ in $\mathcal{H}^{(R,q,Q)}_{B_n}$ is called the bi-permutation module of $\mathcal{H}^{(R,q,Q)}_{B_n}$ associated to (λ,μ). As an abbreviation, we write

$$M^{(\lambda,\mu)}_{(R,q,Q)} = M^{(\lambda,\mu)}.$$

Remark 4.3.13 *(i) Definition 4.3.1, Remark 1.3.7.(i), Remark 4.2.5, and Remark 4.2.7 show that the bi-permutation modules of Hecke algebras of type B introduced in Definition 4.3.12 are generic in the sense of Remark 1.2.9.*

(ii) In [DJM, Definition 4.19], using the notation from there, certain right ideals M^{λ} for $\mathcal{H}^{(R,q,Q)}_{B_n}$ are defined. These right ideals are indexed by bi-partitions of n and, at a superficial glance, are similar to the correspondingly indexed bi-permutation modules from Definition 4.3.12. In fact, every right ideal M^{λ} contains the bi-permutation module indexed by the same bi-partition, in general as a strict subset.

Next, we derive generic bases of bi-permutation modules for Hecke algebras of type B. The following statement makes use of Definition 4.1.6.(i), Definition 1.3.3, and Definition 4.2.6.

Theorem 4.3.14 *(i) Let $a \in \{1, \ldots, n-1\}$ and let (λ, μ) be an a-bi-composition of n. Then the set*

$$\begin{aligned}
& v_{a,n-a}^{(R,q,Q)} \left(\mathbf{B}_{\text{row std}}^{M^\lambda}(R,q)^{\overset{a}{\rightarrow}} \right) \left(\mathbf{B}_{\text{row std}}^{M^\mu}(R,q)^{\overset{n-a}{\leftarrow}} \right) \{T_g \mid g \in \mathcal{D}_{(n-a,a)}\} \\
= & \left\{ v_{a,n-a}^{(R,q,Q)} \left(\left(x_\lambda^{(R,q)} T_d \right)^{\overset{a}{\rightarrow}} \right) \left(\left(x_\mu^{(R,q)} T_f \right)^{\overset{n-a}{\leftarrow}} \right) T_g \;\middle|\; \begin{array}{c} d \in \mathcal{D}_\lambda, \\ f \in \mathcal{D}_\mu, \\ g \in \mathcal{D}_{(n-a,a)} \end{array} \right\} \\
\subseteq & \ \mathcal{H}_{B_n}^{(R,q,Q)}
\end{aligned}$$

is an R-basis of $M_{(R,q,Q)}^{(\lambda,\mu)}$.

(ii) Let $((0), \mu)$ be a 0-bi-composition of n. Then the set

$$\begin{aligned}
v_{0,n}^{(R,q,Q)} \mathbf{B}_{\text{row std}}^{M^\mu}(R,q) &= v_{0,n}^{(R,q,Q)} \left(\mathbf{B}_{\text{row std}}^{M^\mu}(R,q)^{\overset{n}{\leftarrow}} \right) \\
&= \left\{ v_{0,n}^{(R,q,Q)} x_\mu^{(R,q)} T_f \;\middle|\; f \in \mathcal{D}_\mu \right\} \\
&= \left\{ v_{0,n}^{(R,q,Q)} \left(\left(x_\mu^{(R,q)} T_f \right)^{\overset{n}{\leftarrow}} \right) \;\middle|\; f \in \mathcal{D}_\mu \right\} \\
&\subseteq \mathcal{H}_{B_n}^{(R,q,Q)}
\end{aligned}$$

is an R-basis of $M_{(R,q,Q)}^{((0),\mu)}$.

(iii) Let $(\lambda, (0))$ be an n-bi-composition of n. Then the set

$$\begin{aligned}
v_{n,0}^{(R,q,Q)} \mathbf{B}_{\text{row std}}^{M^\lambda}(R,q) &= v_{n,0}^{(R,q,Q)} \left(\mathbf{B}_{\text{row std}}^{M^\lambda}(R,q)^{\overset{n}{\rightarrow}} \right) \\
&= \left\{ v_{n,0}^{(R,q,Q)} x_\lambda^{(R,q)} T_d \;\middle|\; d \in \mathcal{D}_\lambda \right\} \\
&= \left\{ v_{n,0}^{(R,q,Q)} \left(\left(x_\lambda^{(R,q)} T_d \right)^{\overset{n}{\rightarrow}} \right) \;\middle|\; d \in \mathcal{D}_\lambda \right\} \\
&\subseteq \mathcal{H}_{B_n}^{(R,q,Q)}
\end{aligned}$$

is an R-basis of $M_{(R,q,Q)}^{(\lambda,(0))}$.

Proof. (i) The claim follows from Definition 4.3.12.(i), Theorem 1.3.2, Definition 1.3.3, and Corollary 4.3.9.(ii).

(ii) From Definition 4.2.6 we see that the sets occurring in the claim are equal. The remainder of the claim now follows from Definition 4.3.12.(ii), Theorem 1.3.2, Definition 1.3.3, and Corollary 4.3.11.(i).

(iii) From Definition 4.2.6 we see that the sets occurring in the claim are equal. The remainder of the claim now follows from Definition 4.3.12.(iii), Theorem 1.3.2, Definition 1.3.3, and Corollary 4.3.11.(ii). ■

Remark 4.3.15 *From Definition 4.3.1, Remark 1.3.7.(ii), Remark 4.2.5, and Remark 4.2.7 we see that the bases of bi-permutation modules of Hecke algebras of type B from Theorem 4.3.14 are generic in the sense of Remark 1.2.9.*

Finally, we record a special property of certain bi-permutation modules. The next lemma makes use of Definition 1.1.64.

Lemma 4.3.16 *In $\mathcal{H}_{B_n}^{(R,q,Q)}$ we have*

(i) $\forall a \in \{1, \dots, n-1\} : M_{(R,q,Q)}^{(\omega^{(a)}, \omega^{(n-a)})} = v_{a,n-a}^{(R,q,Q)} \mathcal{H}_{B_n}^{(R,q,Q)}$,

(ii) $M_{(R,q,Q)}^{((0), \omega^{(n)})} = v_{0,n}^{(R,q,Q)} \mathcal{H}_{B_n}^{(R,q,Q)}$,

(iii) $M_{(R,q,Q)}^{(\omega^{(n)}, (0))} = v_{n,0}^{(R,q,Q)} \mathcal{H}_{B_n}^{(R,q,Q)}$.

Proof. From Lemma 1.1.65.(ii) and Definition 1.3.1 we obtain

$$\forall j \in \{1, \dots, n\} : M^{\omega^{(j)}} = \mathcal{H}_{A_{j-1}}.$$

Now all claims follow from Definition 4.3.12. ■

Remark 4.3.17 *Lemma 4.3.16 shows that the quotients of adjacent right ideals in the series (4.8) from Corollary 4.3.6 are in fact bi-permutation modules.*

4.4 Bi-Specht modules for Hecke algebras of type B

Here, we construct a family of modules for Hecke algebras of type B which is based on the Specht modules for Hecke algebras of type A from Definition 1.3.10. This procedure is completely analogous to the construction of the bi-permutation modules for Hecke algebras of type B in the preceding section. As always, $n \in \mathbb{N}$ denotes a positive integer and (R, q, Q) denotes a coefficient triple as in Definition 4.2.1.

The following definition makes use of Definition 4.1.6.(ii), Definition 4.3.1.(iii), Definition 4.2.6, and Lemma 4.2.8.(i).

Definition 4.4.1 *(i) Let $a \in \{1, \dots, n-1\}$ and let (λ, μ) be an a-bi-partition of n. Then the right ideal $S_{(R,q,Q)}^{(\lambda,\mu)}$ in $\mathcal{H}_{B_n}^{(R,q,Q)}$ is defined as*

$$\begin{aligned} S_{(R,q,Q)}^{(\lambda,\mu)} &= v_{a,n-a}^{(R,q,Q)} \left(S_{(R,q)}^{\lambda} \overset{a}{\rightarrow} \right) \left(S_{(R,q)}^{\mu} \overset{n-a}{\leftarrow} \right) \mathcal{H}_{B_n}^{(R,q,Q)} \\ &= v_{a,n-a}^{(R,q,Q)} \left(z_{\lambda}^{(R,q)} \overset{a}{\rightarrow} \right) \left(z_{\mu}^{(R,q)} \overset{n-a}{\leftarrow} \right) \mathcal{H}_{B_n}^{(R,q,Q)}. \end{aligned}$$

(ii) Let $((0),\mu)$ be a 0-bi-partition of n. Then the right ideal $S^{((0),\mu)}_{(R,q,Q)}$ in $\mathcal{H}^{(R,q,Q)}_{B_n}$ is defined as

$$\begin{aligned} S^{((0),\mu)}_{(R,q,Q)} &= v^{(R,q,Q)}_{0,n} S^{\mu}_{(R,q)} \mathcal{H}^{(R,q,Q)}_{B_n} \\ &= v^{(R,q,Q)}_{0,n} \left(S^{\mu}_{(R,q)} {}^{\overset{n}{\leftarrow}} \right) \mathcal{H}^{(R,q,Q)}_{B_n} \\ &= v^{(R,q,Q)}_{0,n} z^{(R,q)}_{\mu} \mathcal{H}^{(R,q,Q)}_{B_n} \\ &= v^{(R,q,Q)}_{0,n} \left(z^{(R,q)}_{\mu} {}^{\overset{n}{\leftarrow}} \right) \mathcal{H}^{(R,q,Q)}_{B_n}. \end{aligned}$$

(iii) Let $(\lambda,(0))$ be an n-bi-partition of n. Then the right ideal $S^{(\lambda,(0))}_{(R,q,Q)}$ in $\mathcal{H}^{(R,q,Q)}_{B_n}$ is defined as

$$\begin{aligned} S^{(\lambda,(0))}_{(R,q,Q)} &= v^{(R,q,Q)}_{n,0} S^{\lambda}_{(R,q)} \mathcal{H}^{(R,q,Q)}_{B_n} \\ &= v^{(R,q,Q)}_{n,0} \left(S^{\lambda}_{(R,q)} {}^{\overset{n}{\rightarrow}} \right) \mathcal{H}^{(R,q,Q)}_{B_n} \\ &= v^{(R,q,Q)}_{n,0} z^{(R,q)}_{\lambda} \mathcal{H}^{(R,q,Q)}_{B_n} \\ &= v^{(R,q,Q)}_{n,0} \left(z^{(R,q)}_{\lambda} {}^{\overset{n}{\rightarrow}} \right) \mathcal{H}^{(R,q,Q)}_{B_n}. \end{aligned}$$

For an a-bi-partition (λ,μ) of n with $a \in \{0,\ldots,n\}$, the right ideal $S^{(\lambda,\mu)}_{(R,q,Q)}$ in $\mathcal{H}^{(R,q,Q)}_{B_n}$ is called the bi-Specht module of $\mathcal{H}^{(R,q,Q)}_{B_n}$ associated to (λ,μ). As an abbreviation, we write

$$S^{(\lambda,\mu)}_{(R,q,Q)} = S^{(\lambda,\mu)}.$$

Remark 4.4.2 *(i) Definition 4.3.1, Remark 1.3.15.(i), Remark 4.2.5, and Remark 4.2.7 show that the bi-Specht modules of Hecke algebras of type B from Definition 4.4.1 are generic in the sense of Remark 1.2.9.*

(ii) In [DJ3, Definition 5.2.(i)], using the notation from there, right ideals $S^{\lambda,\mu}$ indexed by bi-partitions of n are defined under certain assumptions on the coefficient triple (R,q,Q). If these right ideals exist then every one of them physically coincides with the bi-Specht module from Definition 4.4.1 indexed by the same bi-partition.

Now we derive generic bases of bi-Specht modules for Hecke algebras of type B. The following statement makes use of Definition 4.1.6.(ii), Definition 4.3.1.(iii), Definition 1.3.12, Definition 4.2.6, and Definition 1.1.60.

Theorem 4.4.3 *(i) Let $a \in \{1, \dots, n-1\}$ and let (λ, μ) be an a-bi-partition of n. Then the set*

$$\begin{aligned} & v_{a,n-a}^{(R,q,Q)} \left(\mathbf{B}_{\mathrm{std}}^{S^\lambda}(R,q)^{\overset{a}{\rightarrow}} \right) \left(\mathbf{B}_{\mathrm{std}}^{S^\mu}(R,q)^{\overset{n-a}{\leftarrow}} \right) \{ T_c \,|\, c \in \mathcal{D}_{(n-a,a)} \} \\ = & \left\{ v_{a,n-a}^{(R,q,Q)} \left(\left(z_\lambda^{(R,q)} T_f \right)^{\overset{a}{\rightarrow}} \right) \left(\left(z_\mu^{(R,q)} T_g \right)^{\overset{n-a}{\leftarrow}} \right) T_c \;\middle|\; \begin{array}{c} f \in \mathcal{E}_{\lambda'}, \\ g \in \mathcal{E}_{\mu'}, \\ c \in \mathcal{D}_{(n-a,a)} \end{array} \right\} \\ \subseteq & \ \mathcal{H}_{B_n}^{(R,q,Q)} \end{aligned}$$

is an R-basis of $S_{(R,q,Q)}^{(\lambda,\mu)}$.

(ii) Let $((0), \mu)$ be a 0-bi-partition of n. Then the set

$$\begin{aligned} v_{0,n}^{(R,q,Q)} \mathbf{B}_{\mathrm{std}}^{S^\mu}(R,q) &= v_{0,n}^{(R,q,Q)} \left(\mathbf{B}_{\mathrm{std}}^{S^\mu}(R,q)^{\overset{n}{\leftarrow}} \right) \\ &= \left\{ v_{0,n}^{(R,q,Q)} z_\mu^{(R,q)} T_g \;\middle|\; g \in \mathcal{E}_{\mu'} \right\} \\ &= \left\{ v_{0,n}^{(R,q,Q)} \left(\left(z_\mu^{(R,q)} T_g \right)^{\overset{n}{\leftarrow}} \right) \;\middle|\; g \in \mathcal{E}_{\mu'} \right\} \\ &\subseteq \mathcal{H}_{B_n}^{(R,q,Q)} \end{aligned}$$

is an R-basis of $S_{(R,q,Q)}^{((0),\mu)}$.

(iii) Let $(\lambda, (0))$ be an n-bi-partition of n. Then the set

$$\begin{aligned} v_{n,0}^{(R,q,Q)} \mathbf{B}_{\mathrm{std}}^{S^\lambda}(R,q) &= v_{n,0}^{(R,q,Q)} \left(\mathbf{B}_{\mathrm{std}}^{S^\lambda}(R,q)^{\overset{n}{\rightarrow}} \right) \\ &= \left\{ v_{n,0}^{(R,q,Q)} z_\lambda^{(R,q)} T_f \;\middle|\; f \in \mathcal{E}_{\lambda'} \right\} \\ &= \left\{ v_{n,0}^{(R,q,Q)} \left(\left(z_\lambda^{(R,q)} T_f \right)^{\overset{n}{\rightarrow}} \right) \;\middle|\; f \in \mathcal{E}_{\lambda'} \right\} \\ &\subseteq \mathcal{H}_{B_n}^{(R,q,Q)} \end{aligned}$$

is an R-basis of $S_{(R,q,Q)}^{(\lambda,(0))}$.

Proof. (i) The claim follows from Definition 4.4.1.(i), Theorem 1.3.11, Definition 1.3.12, and Corollary 4.3.9.(ii).

(ii) From Definition 4.2.6 we see that the sets occurring in the claim are equal. The remainder of the claim now follows from Definition 4.4.1.(ii), Theorem 1.3.11, Definition 1.3.12, and Corollary 4.3.11.(i).

(iii) From Definition 4.2.6 we see that the sets occurring in the claim are equal. The remainder of the claim now follows from Definition 4.4.1.(iii), Theorem 1.3.11, Definition 1.3.12, and Corollary 4.3.11.(ii). ■

Remark 4.4.4 *From Definition 4.3.1, Remark 1.3.15.(ii), Remark 4.2.5, and Remark 4.2.7 we see that the bases of bi-Specht modules of Hecke algebras of type B from Theorem 4.4.3 are generic in the sense of Remark 1.2.9.*

4.5 Bi-PK-modules for Hecke algebras of type B

In this section we introduce a family of modules for Hecke algebras of type B which is based on the PK-modules for Hecke algebras of type A from Definition 3.4.1. This procedure is completely analogous to the construction of the bi-permutation modules and the bi-Specht modules in the preceding sections. As before, $n \in \mathbb{N}$ denotes a positive integer and (R, q, Q) denotes a coefficient triple as in Definition 4.2.1.

The next definition makes use of Definition 4.1.7, Definition 4.3.1.(iii), Definition 3.4.1, Definition 4.2.6, and Lemma 4.2.8.(i).

Definition 4.5.1 *(i) Let $a \in \{1, \ldots, n-1\}$ and let $(\lambda^{\#}\lambda, \mu^{\#}\mu)$ be an a-bi-PK$_n$-pair. Then the right ideal $S_{(R,q,Q)}^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}$ in $\mathcal{H}_{B_n}^{(R,q,Q)}$ is defined as*

$$S_{(R,q,Q)}^{(\lambda^{\#}\lambda,\mu^{\#}\mu)} = v_{a,n-a}^{(R,q,Q)} \left(S_{(R,q)}^{\lambda^{\#}\lambda} \stackrel{a}{\rightarrow} \right) \left(S_{(R,q)}^{\mu^{\#}\mu} \stackrel{n-a}{\leftarrow} \right) \mathcal{H}_{B_n}^{(R,q,Q)}.$$

(ii) Let $(00, \mu^{\#}\mu)$ be a 0-bi-PK$_n$-pair. Then the right ideal $S_{(R,q,Q)}^{(00,\mu^{\#}\mu)}$ in $\mathcal{H}_{B_n}^{(R,q,Q)}$ is defined as

$$\begin{aligned} S_{(R,q,Q)}^{(00,\mu^{\#}\mu)} &= v_{0,n}^{(R,q,Q)} S_{(R,q)}^{\mu^{\#}\mu} \mathcal{H}_{B_n}^{(R,q,Q)} \\ &= v_{0,n}^{(R,q,Q)} \left(S_{(R,q)}^{\mu^{\#}\mu} \stackrel{n}{\leftarrow} \right) \mathcal{H}_{B_n}^{(R,q,Q)}. \end{aligned}$$

(iii) Let $(\lambda^{\#}\lambda, 00)$ be an n-bi-PK$_n$-pair. Then the right ideal $S_{(R,q,Q)}^{(\lambda^{\#}\lambda,00)}$ in $\mathcal{H}_{B_n}^{(R,q,Q)}$ is defined as

$$\begin{aligned} S_{(R,q,Q)}^{(\lambda^{\#}\lambda,00)} &= v_{n,0}^{(R,q,Q)} S_{(R,q)}^{\lambda^{\#}\lambda} \mathcal{H}_{B_n}^{(R,q,Q)} \\ &= v_{n,0}^{(R,q,Q)} \left(S_{(R,q)}^{\lambda^{\#}\lambda} \stackrel{n}{\rightarrow} \right) \mathcal{H}_{B_n}^{(R,q,Q)}. \end{aligned}$$

(iv) The right ideal $S_{(R,q,Q)}^{(00,00)}$ in $\mathcal{H}_{B_n}^{(R,q,Q)}$ is defined as

$$S_{(R,q,Q)}^{(00,00)} = 0_{\mathcal{H}_B^{(R,q,Q)}}.$$

Here, $0_{\mathcal{H}_B^{(R,q,Q)}}$ denotes the null ideal in $\mathcal{H}_{B_n}^{(R,q,Q)}$.

For an a-bi-PK$_n$-pair $(\lambda^{\#}\lambda,\mu^{\#}\mu)$ with $a\in\{0,\ldots,n\}$, the right ideal $S_{(R,q,Q)}^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}$ in $\mathcal{H}_{B_n}^{(R,q,Q)}$ is called an a-bi-PK$_n$-module or an a-bi-PK-module or just a bi-PK$_n$-module or a bi-PK-module. The right ideal $S_{(R,q,Q)}^{(00,00)}$ also is called a bi-PK-module. As abbreviations, we write

$$S_{(R,q,Q)}^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}=S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}\qquad\text{and}\qquad S_{(R,q,Q)}^{(00,00)}=S^{(00,00)}.$$

Remark 4.5.2 *Definition 4.3.1, Remark 3.4.2, Remark 4.2.5, and Remark 4.2.7 show that the bi-PK-modules of Hecke algebras of type B from Definition 4.5.1 are generic in the sense of Remark 1.2.9.*

Next, we derive generic bases of bi-PK-modules for Hecke algebras of type B. The following statement makes use of Definition 4.1.7, Definition 4.3.1.(iii), Definition 3.6.3, Remark 3.2.2, Definition 4.2.6, and Definition 3.3.5.

Theorem 4.5.3 *(i) Let $a\in\{1,\ldots,n-1\}$ and let $(\lambda^{\#}\lambda,\mu^{\#}\mu)$ be an a-bi-PK$_n$-pair. Then the set*

$$\left\{ v_{a,n-a}^{(R,q,Q)}\left(z(\zeta)_{(R,q)}{}^{\overset{a}{\rightarrow}}\right)\left(z(\eta)_{(R,q)}{}^{\overset{n-a}{\leftarrow}}\right)T_d \;\middle|\; \begin{array}{c}\zeta\in\mathcal{Z}^{\lambda^{\#}\lambda},\\ \eta\in\mathcal{Z}^{\mu^{\#}\mu},\\ d\in\mathcal{D}_{(n-a,a)}\end{array}\right\}\subseteq\mathcal{H}_{B_n}^{(R,q,Q)}$$

is an R-basis of $S_{(R,q,Q)}^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}$.

(ii) Let $(00,\mu^{\#}\mu)$ be a 0-bi-PK$_n$-pair. Then the set

$$\begin{aligned}&\left\{v_{0,n}^{(R,q,Q)}z(\eta)_{(R,q)}\;\middle|\;\eta\in\mathcal{Z}^{\mu^{\#}\mu}\right\}\\ &\quad=\left\{v_{0,n}^{(R,q,Q)}\left(z(\eta)_{(R,q)}{}^{\overset{n}{\leftarrow}}\right)\;\middle|\;\eta\in\mathcal{Z}^{\mu^{\#}\mu}\right\}\\ &\quad\subseteq\mathcal{H}_{B_n}^{(R,q,Q)}\end{aligned}$$

is an R-basis of $S_{(R,q,Q)}^{(00,\mu^{\#}\mu)}$.

(iii) Let $(\lambda^{\#}\lambda,00)$ be an n-bi-PK$_n$-pair. Then the set

$$\begin{aligned}&\left\{v_{n,0}^{(R,q,Q)}z(\zeta)_{(R,q)}\;\middle|\;\zeta\in\mathcal{Z}^{\lambda^{\#}\lambda}\right\}\\ &\quad=\left\{v_{n,0}^{(R,q,Q)}\left(z(\zeta)_{(R,q)}{}^{\overset{n}{\rightarrow}}\right)\;\middle|\;\zeta\in\mathcal{Z}^{\lambda^{\#}\lambda}\right\}\\ &\quad\subseteq\mathcal{H}_{B_n}^{(R,q,Q)}\end{aligned}$$

is an R-basis of $S_{(R,q,Q)}^{(\lambda^{\#}\lambda,00)}$.

Proof. (i) The claim follows from Definition 4.5.1.(i), Theorem 3.10.1, and Corollary 4.3.9.(ii).

(ii) From Definition 4.2.6 we see that the sets occurring in the claim are equal. The remainder of the claim now follows from Definition 4.5.1.(ii), Theorem 3.10.1, and Corollary 4.3.11.(i).

(iii) From Definition 4.2.6 we see that the sets occurring in the claim are equal. The remainder of the claim now follows from Definition 4.5.1.(iii), Theorem 3.10.1, and Corollary 4.3.11.(ii). ∎

Remark 4.5.4 *Definition 4.3.1, Remark 3.10.3, Remark 4.2.5, and Remark 4.2.7 show that the bases of bi-PK-modules of Hecke algebras of type B from Theorem 4.5.3 are generic in the sense of Remark 1.2.9.*

Now we record some properties of the bi-PK-modules for Hecke algebras of type B which will be required later. The next statement makes use of Definition 4.1.7, Remark 4.1.8, and Definition 4.3.12.

Lemma 4.5.5 *Let $a \in \{0, \dots, n\}$ and let $(\lambda^{\#}\lambda, \mu^{\#}\mu)$ be an a-bi-PK$_n$-pair with $(\lambda^{\#}\lambda, \mu^{\#}\mu) \neq (00, 00)$. Then we have in $\mathcal{H}_{B_n}^{(R,q,Q)}$*

$$S_{(R,q,Q)}^{(\lambda^{\#}\lambda,\mu^{\#}\mu)} \subseteq M_{(R,q,Q)}^{(\lambda,\mu)}.$$

Proof. This follows from Definition 4.5.1, Definition 4.3.12, and Lemma 3.4.4.(vi). ∎

The following lemma uses Definition 4.1.7, Definition 3.2.1, Remark 4.1.8, Definition 4.1.6, and Definition 4.4.1.

Lemma 4.5.6 *(i) Let $a \in \{1, \dots, n-1\}$, $\lambda \vdash a$, and $\mu \vdash n-a$. Then we have for the a-bi-PK$_n$-pair $(\lambda\lambda, \mu\mu)$*

$$S_{(R,q,Q)}^{(\lambda\lambda,\mu\mu)} = S_{(R,q,Q)}^{(\lambda,\mu)}.$$

(ii) Let $\mu \vdash n$. Then we have for the 0-bi-PK$_n$-pair $(00, \mu\mu)$

$$S_{(R,q,Q)}^{(00,\mu\mu)} = S_{(R,q,Q)}^{((0),\mu)}.$$

(iii) Let $\lambda \vdash n$. Then we have for the n-bi-PK$_n$-pair $(\lambda\lambda, 00)$

$$S_{(R,q,Q)}^{(\lambda\lambda,00)} = S_{(R,q,Q)}^{(\lambda,(0))}.$$

Proof. Everything is obtained from Definition 4.5.1, Lemma 3.4.4.(iii), and Definition 4.4.1. ■

The next statement makes use of Definition 3.2.1, Remark 3.2.2, Definition 4.1.7, Definition 4.3.12, and Definition 4.1.6.(i).

Lemma 4.5.7 *The following statements hold in $\mathcal{H}_{B_n}^{(R,q,Q)}$.*

(i) Let $a \in \{1, \ldots, n-1\}$. With that, choose a $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash a$ having $\lambda_1 > 0$ and a $\mu = (\mu_1, \mu_2, \ldots) \vDash n-a$ with $\mu_1 > 0$. Then we have

$$S_{(R,q,Q)}^{((\lambda_1)\lambda,(\mu_1)\mu)} = M_{(R,q,Q)}^{(\lambda,\mu)}.$$

(ii) Let $\mu = (\mu_1, \mu_2, \ldots) \vDash n$ with $\mu_1 > 0$. Then we have

$$S_{(R,q,Q)}^{(00,(\mu_1)\mu)} = M_{(R,q,Q)}^{((0),\mu)}.$$

(iii) Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash n$ with $\lambda_1 > 0$. Then we have

$$S_{(R,q,Q)}^{((\lambda_1)\lambda,00)} = M_{(R,q,Q)}^{(\lambda,(0))}.$$

Proof. (i) The assumptions on λ and μ ensure that the PK_a-pair $(\lambda_1)\lambda$, the PK_{n-a}-pair $(\mu_1)\mu$, and the a-bi-PK_n-pair $((\lambda_1)\lambda, (\mu_1)\mu)$ are all well defined. Furthermore, we can use λ and μ to build the a-bi-composition (λ, μ). This shows that the claim is meaningful. The desired identity now follows from Definition 4.5.1.(i), Lemma 3.4.4.(ii), and Definition 4.3.12.(i).

(ii) The proof of this statement makes use of Definition 4.5.1.(ii) and Definition 4.3.12.(ii) and is otherwise completely analogous to the proof of statement (i).

(iii) The proof of this statement makes use of Definition 4.5.1.(iii) and Definition 4.3.12.(iii) and is otherwise completely analogous to the proof of statement (i). ■

4.6 Bi-PK-homomorphisms for Hecke algebras of type B

Now we introduce and investigate generic homomorphisms between bi-PK-modules. These homomorphisms are based on the PK-homomorphisms for Hecke algebras of type A from Definition 3.5.3. As always, $n \in \mathbb{N}$ denotes a positive integer and (R, q, Q) denotes a coefficient triple as in Definition 4.2.1.

The following technical statement makes use of Definition 4.1.7, Definition 3.2.3, Definition 1.2.3.(i), Definition 3.2.5.(ii), Definition 4.2.6, Remark 4.2.7.(ii), and Definition 4.3.1.(iii).

Lemma 4.6.1 *(i) Let $a \in \{1, \dots, n-1\}$. With that, let $(\lambda^\# \lambda, \mu^\# \mu)$ be an a-bi-PK_n-pair and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\lambda^\# \lambda$. Then we have in the algebra $\mathcal{H}_{B_n}^{(R,q,Q)}$*

$$\begin{aligned}&\left(\iota^{(a)}_{(R,q)}(\mathcal{D}_\lambda{}^{-1}\cap(\mathfrak{S}_{\lambda R_c}\mathfrak{S}_\lambda))^{\overset{a}{\leftarrow}}\right) v^{(R,q,Q)}_{a,n-a}\left(x_\lambda^{(R,q)\overset{a}{\rightarrow}}\right)\left(x_\mu^{(R,q)\overset{n-a}{\leftarrow}}\right)\\ &\quad= v^{(R,q,Q)}_{a,n-a}\left(x_{\lambda R_c}^{(R,q)\overset{a}{\rightarrow}}\right)\left(x_\mu^{(R,q)\overset{n-a}{\leftarrow}}\right)\left(\iota^{(a)}_{(R,q)}((\mathfrak{S}_{\lambda R_c}\mathfrak{S}_\lambda)\cap\mathcal{D}_{\lambda R_c})^{\overset{a}{\rightarrow}}\right).\end{aligned}$$

(ii) Let $a \in \{1, \dots, n-1\}$. With that, let $(\lambda^\# \lambda, \mu^\# \mu)$ be an a-bi-PK_n-pair and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^\# \mu$. Then we have in the algebra $\mathcal{H}_{B_n}^{(R,q,Q)}$

$$\begin{aligned}&\left(\iota^{(n-a)}_{(R,q)}(\mathcal{D}_\mu{}^{-1}\cap(\mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu))^{\overset{n-a}{\rightarrow}}\right) v^{(R,q,Q)}_{a,n-a}\left(x_\lambda^{(R,q)\overset{a}{\rightarrow}}\right)\left(x_\mu^{(R,q)\overset{n-a}{\leftarrow}}\right)\\ &\quad= v^{(R,q,Q)}_{a,n-a}\left(x_{\lambda}^{(R,q)\overset{a}{\rightarrow}}\right)\left(x_{\mu R_c}^{(R,q)\overset{n-a}{\leftarrow}}\right)\left(\iota^{(n-a)}_{(R,q)}((\mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu)\cap\mathcal{D}_{\mu R_c})^{\overset{n-a}{\leftarrow}}\right).\end{aligned}$$

(iii) Let $(00, \mu^\# \mu)$ be a 0-bi-PK_n-pair and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^\# \mu$. Then we have in the algebra $\mathcal{H}_{B_n}^{(R,q,Q)}$

$$\begin{aligned}&\left(\iota^{(n)}_{(R,q)}(\mathcal{D}_\mu{}^{-1}\cap(\mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu))^{\overset{n}{\rightarrow}}\right) v^{(R,q,Q)}_{0,n}\left(x_\mu^{(R,q)\overset{n}{\leftarrow}}\right)\\ &\quad= v^{(R,q,Q)}_{0,n}\left(x_{\mu R_c}^{(R,q)\overset{n}{\leftarrow}}\right)\left(\iota^{(n)}_{(R,q)}((\mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu)\cap\mathcal{D}_{\mu R_c})^{\overset{n}{\leftarrow}}\right).\end{aligned}$$

(iv) Let $(\lambda^\# \lambda, 00)$ be an n-bi-PK_n-pair and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\lambda^\# \lambda$. Then we have in $\mathcal{H}_{B_n}^{(R,q,Q)}$

$$\begin{aligned}&\left(\iota^{(n)}_{(R,q)}(\mathcal{D}_\lambda{}^{-1}\cap(\mathfrak{S}_{\lambda R_c}\mathfrak{S}_\lambda))^{\overset{n}{\leftarrow}}\right) v^{(R,q,Q)}_{n,0}\left(x_\lambda^{(R,q)\overset{n}{\rightarrow}}\right)\\ &\quad= v^{(R,q,Q)}_{n,0}\left(x_{\lambda R_c}^{(R,q)\overset{n}{\rightarrow}}\right)\left(\iota^{(n)}_{(R,q)}((\mathfrak{S}_{\lambda R_c}\mathfrak{S}_\lambda)\cap\mathcal{D}_{\lambda R_c})^{\overset{n}{\rightarrow}}\right).\end{aligned}$$

Proof. All claims follow from Lemma 4.3.3.(v), Lemma 4.3.3.(vi), Lemma 3.5.1, and Lemma 4.2.8.(i). ■

The preceding statement and Definition 4.3.12 show that the next two definitions are meaningful. In addition, these two definitions make use of Definition 4.1.7, Definition 3.2.3, Remark 4.1.8, Definition 4.3.1.(iii), Definition 4.2.6, Definition 1.2.3.(i), and Definition 3.2.5.(ii).

Definition 4.6.2 *Let $a \in \{1, \dots, n-1\}$ and let $(\lambda^\# \lambda, \mu^\# \mu)$ be an a-bi-PK_n-pair.*

(i) Let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\lambda^{\#}\lambda$. Then the $\mathcal{H}_{B_n}^{(R,q,Q)}$-homomorphism

$$^{(c)}\Psi_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q) : M_{(R,q,Q)}^{(\lambda,\mu)} \to M_{(R,q,Q)}^{(\lambda R_c,\mu)}$$

is defined by

$$\begin{aligned} & v_{a,n-a}^{(R,q,Q)} \left(x_\lambda^{(R,q)} {}^{\overset{a}{\rightarrow}} \right) \left(x_\mu^{(R,q)} {}^{\overset{n-a}{\leftarrow}} \right) {}^{(c)}\Psi_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q) \\ = & \left(\iota_{(R,q)}^{(a)} (\mathcal{D}_\lambda{}^{-1} \cap (\mathfrak{S}_{\lambda R_c}\mathfrak{S}_\lambda)) {}^{\overset{a}{\leftarrow}} \right) v_{a,n-a}^{(R,q,Q)} \left(x_\lambda^{(R,q)} {}^{\overset{a}{\rightarrow}} \right) \left(x_\mu^{(R,q)} {}^{\overset{n-a}{\leftarrow}} \right) \\ = & v_{a,n-a}^{(R,q,Q)} \left(x_{\lambda R_c}^{(R,q)} {}^{\overset{a}{\rightarrow}} \right) \left(x_\mu^{(R,q)} {}^{\overset{n-a}{\leftarrow}} \right) \left(\iota_{(R,q)}^{(a)} ((\mathfrak{S}_{\lambda R_c}\mathfrak{S}_\lambda) \cap \mathcal{D}_{\lambda R_c}) {}^{\overset{a}{\rightarrow}} \right) \end{aligned}$$

and $\mathcal{H}_{B_n}^{(R,q,Q)}$-linear extension.

(ii) Let $c' \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^{\#}\mu$. Then the $\mathcal{H}_{B_n}^{(R,q,Q)}$-homomorphism

$$\Psi^{(c')}_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q) : M_{(R,q,Q)}^{(\lambda,\mu)} \to M_{(R,q,Q)}^{(\lambda,\mu R_{c'})}$$

is defined by

$$\begin{aligned} & v_{a,n-a}^{(R,q,Q)} \left(x_\lambda^{(R,q)} {}^{\overset{a}{\rightarrow}} \right) \left(x_\mu^{(R,q)} {}^{\overset{n-a}{\leftarrow}} \right) \Psi^{(c')}_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q) \\ = & \left(\iota_{(R,q)}^{(n-a)} (\mathcal{D}_\mu{}^{-1} \cap (\mathfrak{S}_{\mu R_{c'}}\mathfrak{S}_\mu)) {}^{\overset{n-a}{\rightarrow}} \right) v_{a,n-a}^{(R,q,Q)} \left(x_\lambda^{(R,q)} {}^{\overset{a}{\rightarrow}} \right) \left(x_\mu^{(R,q)} {}^{\overset{n-a}{\leftarrow}} \right) \\ = & v_{a,n-a}^{(R,q,Q)} \left(x_\lambda^{(R,q)} {}^{\overset{a}{\rightarrow}} \right) \left(x_{\mu R_{c'}}^{(R,q)} {}^{\overset{n-a}{\leftarrow}} \right) \left(\iota_{(R,q)}^{(n-a)} ((\mathfrak{S}_{\mu R_{c'}}\mathfrak{S}_\mu) \cap \mathcal{D}_{\mu R_{c'}}) {}^{\overset{n-a}{\leftarrow}} \right) \end{aligned}$$

and $\mathcal{H}_{B_n}^{(R,q,Q)}$-linear extension.

$^{(c)}\Psi_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q)$ and $\Psi^{(c')}_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q)$ are called a-bi-PK$_n$-homomorphisms or a-bi-PK-homomorphisms or just bi-PK$_n$-homomorphisms or bi-PK-homomorphisms. As abbreviations, we write

$$^{(c)}\Psi_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q) = {}^{(c)}\Psi_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}$$

and

$$\Psi^{(c')}_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q) = \Psi^{(c')}_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}.$$

Definition 4.6.3 *(i) Let $(00,\mu^{\#}\mu)$ be a 0-bi-PK$_n$-pair and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^{\#}\mu$. Then the $\mathcal{H}_{B_n}^{(R,q,Q)}$-homomorphism*

$$\Psi^{(c)}_{(00,\mu^{\#}\mu)}(R,q,Q) : M_{(R,q,Q)}^{((0),\mu)} \to M_{(R,q,Q)}^{((0),\mu R_c)}$$

is defined by

$$\begin{aligned} & v_{0,n}^{(R,q,Q)}\left(x_\mu^{(R,q)\overset{n}{\leftarrow}}\right)\Psi^{(c)}_{(00,\mu^\#\mu)}(R,q,Q) \\ = & \left(\iota^{(n)}_{(R,q)}(\mathcal{D}_\mu{}^{-1}\cap(\mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu))^{\overset{n}{\rightarrow}}\right)v_{0,n}^{(R,q,Q)}\left(x_\mu^{(R,q)\overset{n}{\leftarrow}}\right) \\ = & v_{0,n}^{(R,q,Q)}\left(x_{\mu R_c}^{(R,q)\overset{n}{\leftarrow}}\right)\left(\iota^{(n)}_{(R,q)}((\mathfrak{S}_{\mu R_c}\mathfrak{S}_\mu)\cap\mathcal{D}_{\mu R_c})^{\overset{n}{\leftarrow}}\right) \end{aligned}$$

and $\mathcal{H}_{B_n}^{(R,q,Q)}$*-linear extension.* $\Psi^{(c)}_{(00,\mu^\#\mu)}(R,q,Q)$ *is called a 0-bi-PK*$_n$*-homomorphism or just a 0-bi-PK-homomorphism.*

(ii) Let $(\lambda^\#\lambda,00)$ *be an n-bi-PK*$_n$*-pair and let* $c'\in\mathbb{N}\setminus\{1\}$ *be an AR-index for* $\lambda^\#\lambda$*. Then the* $\mathcal{H}_{B_n}^{(R,q,Q)}$*-homomorphism*

$$^{(c')}\Psi_{(\lambda^\#\lambda,00)}(R,q,Q): M^{(\lambda,(0))}_{(R,q,Q)}\to M^{(\lambda R_{c'},(0))}_{(R,q,Q)}$$

is defined by

$$\begin{aligned} & v_{n,0}^{(R,q,Q)}\left(x_\lambda^{(R,q)\overset{n}{\rightarrow}}\right){}^{(c')}\Psi_{(\lambda^\#\lambda,00)}(R,q,Q) \\ = & \left(\iota^{(n)}_{(R,q)}(\mathcal{D}_\lambda{}^{-1}\cap(\mathfrak{S}_{\lambda R_{c'}}\mathfrak{S}_\lambda))^{\overset{n}{\leftarrow}}\right)v_{n,0}^{(R,q,Q)}\left(x_\lambda^{(R,q)\overset{n}{\rightarrow}}\right) \\ = & v_{n,0}^{(R,q,Q)}\left(x_{\lambda R_{c'}}^{(R,q)\overset{n}{\rightarrow}}\right)\left(\iota^{(n)}_{(R,q)}((\mathfrak{S}_{\lambda R_{c'}}\mathfrak{S}_\lambda)\cap\mathcal{D}_{\lambda R_{c'}})^{\overset{n}{\rightarrow}}\right) \end{aligned}$$

and $\mathcal{H}_{B_n}^{(R,q,Q)}$*-linear extension.* ${}^{(c')}\Psi_{(\lambda^\#\lambda,00)}(R,q,Q)$ *is called an n-bi-PK*$_n$*-homomorphism or just an n-bi-PK-homomorphism.*

$\Psi^{(c)}_{(00,\mu^\#\mu)}(R,q,Q)$ *and* ${}^{(c')}\Psi_{(\lambda^\#\lambda,00)}(R,q,Q)$ *also are called bi-PK*$_n$*-homomorphisms or just bi-PK-homomorphisms. As abbreviations, we write*

$$\Psi^{(c)}_{(00,\mu^\#\mu)}(R,q,Q)=\Psi^{(c)}_{(00,\mu^\#\mu)}$$

and

$${}^{(c')}\Psi_{(\lambda^\#\lambda,00)}(R,q,Q)={}^{(c')}\Psi_{(\lambda^\#\lambda,00)}.$$

Remark 4.6.4 *Remark 4.2.5, Remark 4.3.13, Definition 4.3.1, Definition 1.2.3, and Remark 4.2.7 show that the bi-PK-homomorphisms for Hecke algebras of type B from the preceding two definitions are generic in the sense of Remark 1.2.9.*

The following two lemmata relate the bi-PK-homomorphisms for Hecke algebras of type B from Definition 4.6.2 and Definition 4.6.3 to the PK-homomorphisms for Hecke algebras of type A from Definition 3.5.3. They make use of Definition 4.1.7, Definition 4.3.1.(iii), Definition 4.2.6, Definition 4.3.12, Remark 4.1.8, Definition 3.2.3, and Definition 3.2.5.(ii).

Lemma 4.6.5 *Let $a \in \{1, \dots, n-1\}$ and let $(\lambda^\# \lambda, \mu^\# \mu)$ be an a-bi-PK$_n$-pair. Furthermore choose $x \in M^\lambda_{(R,q)}$, $y \in M^\mu_{(R,q)}$, and $h \in \mathcal{H}_{B_n}^{(R,q,Q)}$ and consider the element*

$$v_{a,n-a}^{(R,q,Q)} \left(x^{\overset{a}{\rightarrow}}\right) \left(y^{\overset{n-a}{\leftarrow}}\right) h \in M^{(\lambda,\mu)}_{(R,q,Q)}.$$

(i) Let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\lambda^\# \lambda$. Then we have in $M^{(\lambda R_c,\mu)}_{(R,q,Q)}$

$$\begin{aligned} &\left(v_{a,n-a}^{(R,q,Q)} \left(x^{\overset{a}{\rightarrow}}\right) \left(y^{\overset{n-a}{\leftarrow}}\right) h\right) {}^{(c)}\Psi_{(\lambda^\#\lambda,\mu^\#\mu)}(R,q,Q) \\ &\qquad = v_{a,n-a}^{(R,q,Q)} \left(\left(x\Psi^{(R,q)}_{\lambda^\#\lambda c}\right)^{\overset{a}{\rightarrow}}\right) \left(y^{\overset{n-a}{\leftarrow}}\right) h. \end{aligned}$$

(ii) Let $c' \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^\# \mu$. Then we have in $M^{(\lambda,\mu R_{c'})}_{(R,q,Q)}$

$$\begin{aligned} &\left(v_{a,n-a}^{(R,q,Q)} \left(x^{\overset{a}{\rightarrow}}\right) \left(y^{\overset{n-a}{\leftarrow}}\right) h\right) \Psi^{(c')}_{(\lambda^\#\lambda,\mu^\#\mu)}(R,q,Q) \\ &\qquad = v_{a,n-a}^{(R,q,Q)} \left(x^{\overset{a}{\rightarrow}}\right) \left(\left(y\Psi^{(R,q)}_{\mu^\#\mu c'}\right)^{\overset{n-a}{\leftarrow}}\right) h. \end{aligned}$$

Proof. This follows from Definition 4.6.2, Lemma 4.3.3.(v), Lemma 4.3.3.(vi), Lemma 4.2.8.(i), and Definition 3.5.3. ∎

Lemma 4.6.6 *(i) Let $(00, \mu^\# \mu)$ be a 0-bi-PK$_n$-pair. Furthermore choose $x \in M^\mu_{(R,q)}$ and $h \in \mathcal{H}_{B_n}^{(R,q,Q)}$ and consider the element*

$$v_{0,n}^{(R,q,Q)} \left(x^{\overset{n}{\leftarrow}}\right) h \in M^{((0),\mu)}_{(R,q,Q)}.$$

Finally let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^\# \mu$. Then we have in $M^{((0),\mu R_c)}_{(R,q,Q)}$

$$\left(v_{0,n}^{(R,q,Q)} \left(x^{\overset{n}{\leftarrow}}\right) h\right) \Psi^{(c)}_{(00,\mu^\#\mu)}(R,q,Q) = v_{0,n}^{(R,q,Q)} \left(\left(x\Psi^{(R,q)}_{\mu^\#\mu c}\right)^{\overset{n}{\leftarrow}}\right) h.$$

(ii) Let $(\lambda^\# \lambda, 00)$ be an n-bi-PK$_n$-pair. Furthermore choose $x \in M^\lambda_{(R,q)}$ and $h \in \mathcal{H}_{B_n}^{(R,q,Q)}$ and consider the element

$$v_{n,0}^{(R,q,Q)} \left(x^{\overset{n}{\rightarrow}}\right) h \in M^{(\lambda,(0))}_{(R,q,Q)}.$$

Finally let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\lambda^\# \lambda$. Then we have in $M^{(\lambda R_c,(0))}_{(R,q,Q)}$

$$\left(v_{n,0}^{(R,q,Q)} \left(x^{\overset{n}{\rightarrow}}\right) h\right) {}^{(c)}\Psi_{(\lambda^\#\lambda,00)}(R,q,Q) = v_{n,0}^{(R,q,Q)} \left(\left(x\Psi^{(R,q)}_{\lambda^\#\lambda c}\right)^{\overset{n}{\rightarrow}}\right) h.$$

Proof. This follows easily from Definition 4.6.3, Lemma 4.3.3.(v), Lemma 4.3.3.(vi), and Definition 3.5.3. ■

Now we describe the effect of bi-PK-homomorphisms on bi-PK-modules. The next statement makes use of Definition 4.1.7, Definition 3.2.3, Definition 4.3.12, Remark 4.1.8, Definition 3.2.5, Definition 4.5.1, Lemma 4.5.5, and Definition 4.1.9.

Lemma 4.6.7 *(i) Let $a \in \{1, \ldots, n-1\}$. With that, let $(\lambda^{\#}\lambda, \mu^{\#}\mu)$ be an a-bi-PK$_n$-pair and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\lambda^{\#}\lambda$. Then we have in the module $M^{(\lambda R_c, \mu)}_{(R,q,Q)}$*

$$\left(S^{(\lambda^{\#}\lambda, \mu^{\#}\mu)}_{(R,q,Q)}\right) {}^{(c)}\Psi_{(\lambda^{\#}\lambda, \mu^{\#}\mu)}(R,q,Q) = S^{(\lambda^{\#}\lambda R_c, \mu^{\#}\mu)}_{(R,q,Q)} = S^{(\lambda^{\#}\lambda, \mu^{\#}\mu)^{(c)}R}_{(R,q,Q)}.$$

(ii) Let $a \in \{1, \ldots, n-1\}$. With that, let $(\lambda^{\#}\lambda, \mu^{\#}\mu)$ be an a-bi-PK$_n$-pair and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^{\#}\mu$. Then we have in the module $M^{(\lambda, \mu R_c)}_{(R,q,Q)}$

$$\left(S^{(\lambda^{\#}\lambda, \mu^{\#}\mu)}_{(R,q,Q)}\right) \Psi^{(c)}_{(\lambda^{\#}\lambda, \mu^{\#}\mu)}(R,q,Q) = S^{(\lambda^{\#}\lambda, \mu^{\#}\mu R_c)}_{(R,q,Q)} = S^{(\lambda^{\#}\lambda, \mu^{\#}\mu)R^{(c)}}_{(R,q,Q)}.$$

(iii) Let $(00, \mu^{\#}\mu)$ be a 0-bi-PK$_n$-pair and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^{\#}\mu$. Then we have in the module $M^{((0), \mu R_c)}_{(R,q,Q)}$

$$\left(S^{(00, \mu^{\#}\mu)}_{(R,q,Q)}\right) \Psi^{(c)}_{(00, \mu^{\#}\mu)}(R,q,Q) = S^{(00, \mu^{\#}\mu R_c)}_{(R,q,Q)} = S^{(00, \mu^{\#}\mu)R^{(c)}}_{(R,q,Q)}.$$

(iv) Let $(\lambda^{\#}\lambda, 00)$ be an n-bi-PK$_n$-pair and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\lambda^{\#}\lambda$. Then we have in the module $M^{(\lambda R_c, (0))}_{(R,q,Q)}$

$$\left(S^{(\lambda^{\#}\lambda, 00)}_{(R,q,Q)}\right) {}^{(c)}\Psi_{(\lambda^{\#}\lambda, 00)}(R,q,Q) = S^{(\lambda^{\#}\lambda R_c, 00)}_{(R,q,Q)} = S^{(\lambda^{\#}\lambda, 00)^{(c)}R}_{(R,q,Q)}.$$

Proof. (i) We get from Lemma 4.5.5, Definition 4.5.1.(i), Lemma 4.6.5.(i), Lemma 3.5.5.(iii), and Definition 4.1.9.(iii)

$$\begin{aligned}
\left(S^{(\lambda^{\#}\lambda, \mu^{\#}\mu)}\right) {}^{(c)}\Psi_{(\lambda^{\#}\lambda, \mu^{\#}\mu)} &= \left(v_{a,n-a}\left(S^{\lambda^{\#}\lambda^{\rightarrow}}\right)\left(S^{\mu^{\#}\mu^{\leftarrow}}\right)\mathcal{H}_{B_n}\right) {}^{(c)}\Psi_{(\lambda^{\#}\lambda, \mu^{\#}\mu)} \\
&= v_{a,n-a}\left(\left(S^{\lambda^{\#}\lambda}\Psi_{\lambda^{\#}\lambda c}\right)^{\rightarrow}\right)\left(S^{\mu^{\#}\mu^{\leftarrow}}\right)\mathcal{H}_{B_n} \\
&= v_{a,n-a}\left(S^{\lambda^{\#}\lambda R_c{}^{\rightarrow}}\right)\left(S^{\mu^{\#}\mu^{\leftarrow}}\right)\mathcal{H}_{B_n} \\
&= S^{(\lambda^{\#}\lambda R_c, \mu^{\#}\mu)} \\
&= S^{(\lambda^{\#}\lambda, \mu^{\#}\mu)^{(c)}R}.
\end{aligned}$$

This shows the claim.

(ii) The proof of this statement uses Lemma 4.6.5.(ii) and Definition 4.1.9.(iv) and is otherwise completely analogous to the proof of statement (i).

(iii) The proof of this statement uses Definition 4.5.1.(ii), Lemma 4.6.6.(i), and Definition 4.1.9.(iv) and is otherwise completely analogous to the proof of statement (i).

(iv) The proof of this statement uses Definition 4.5.1.(iii) and Lemma 4.6.6.(ii) and is otherwise completely analogous to the proof of statement (i). ■

The following statement makes use of Definition 4.1.7, Definition 3.2.3, Definition 4.3.12, Remark 4.1.8, Definition 4.5.1, Lemma 4.5.5, and Definition 4.1.9.

Lemma 4.6.8 *(i) Let $a \in \{1, \dots, n-1\}$. With that, let $(\lambda^{\#}\lambda, \mu^{\#}\mu)$ be an a-bi-PK$_n$-pair and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\lambda^{\#}\lambda$. Then we have in the module $M^{(\lambda,\mu)}_{(R,q,Q)}$*

$$\operatorname{Ker}\left({}^{(c)}\Psi_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q)\Big\downarrow^{M^{(\lambda,\mu)}_{(R,q,Q)}}_{S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}_{(R,q,Q)}} \right) = S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)^{(c)}A}_{(R,q,Q)}.$$

(ii) Let $a \in \{1, \dots, n-1\}$. With that, let $(\lambda^{\#}\lambda, \mu^{\#}\mu)$ be an a-bi-PK$_n$-pair and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^{\#}\mu$. Then we have in the module $M^{(\lambda,\mu)}_{(R,q,Q)}$

$$\operatorname{Ker}\left(\Psi^{(c)}_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q)\Big\downarrow^{M^{(\lambda,\mu)}_{(R,q,Q)}}_{S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}_{(R,q,Q)}} \right) = S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)A^{(c)}}_{(R,q,Q)}.$$

(iii) Let $(00, \mu^{\#}\mu)$ be a 0-bi-PK$_n$-pair and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\mu^{\#}\mu$. Then we have in the module $M^{((0),\mu)}_{(R,q,Q)}$

$$\operatorname{Ker}\left(\Psi^{(c)}_{(00,\mu^{\#}\mu)}(R,q,Q)\Big\downarrow^{M^{((0),\mu)}_{(R,q,Q)}}_{S^{(00,\mu^{\#}\mu)}_{(R,q,Q)}} \right) = S^{(00,\mu^{\#}\mu)A^{(c)}}_{(R,q,Q)}.$$

(iv) Let $(\lambda^{\#}\lambda, 00)$ be an n-bi-PK$_n$-pair and let $c \in \mathbb{N} \setminus \{1\}$ be an AR-index for $\lambda^{\#}\lambda$. Then we have in the module $M^{(\lambda,(0))}_{(R,q,Q)}$

$$\operatorname{Ker}\left({}^{(c)}\Psi_{(\lambda^{\#}\lambda,00)}(R,q,Q)\Big\downarrow^{M^{(\lambda,(0))}_{(R,q,Q)}}_{S^{(\lambda^{\#}\lambda,00)}_{(R,q,Q)}} \right) = S^{(\lambda^{\#}\lambda,00)^{(c)}A}_{(R,q,Q)}.$$

Proof. (i) For this proof, we put

$${}_{(c)}\mathbf{A}^{(\lambda^{\#}\lambda,\mu^{\#}\mu)} = \left\{ v_{a,n-a}\left(z(\zeta)^{\rightarrow}\right)\left(z(\eta)^{\leftarrow}\right)T_d \;\middle|\; \begin{array}{c} \zeta \in \mathcal{Z}^{\lambda^{\#}\lambda} \setminus \mathcal{Z}^{\lambda^{\#}A_c\lambda}, \\ \eta \in \mathcal{Z}^{\mu^{\#}\mu}, \\ d \in \mathcal{D}_{(n-a,a)} \end{array} \right\}$$

and

$${}_{(c)}\mathbf{B}^{(\lambda^{\#}\lambda,\mu^{\#}\mu)} = \left\{ v_{a,n-a}\left(z(\zeta)^{\rightarrow}\right)\left(z(\eta)^{\leftarrow}\right)T_d \;\middle|\; \begin{array}{c} \zeta \in \mathcal{Z}^{\lambda^{\#}A_c\lambda}, \\ \eta \in \mathcal{Z}^{\mu^{\#}\mu}, \\ d \in \mathcal{D}_{(n-a,a)} \end{array} \right\}.$$

Obviously, ${}_{(c)}\mathbf{A}^{(\lambda^\#\lambda,\mu^\#\mu)}$ and ${}_{(c)}\mathbf{B}^{(\lambda^\#\lambda,\mu^\#\mu)}$ are disjoint. Moreover, Theorem 4.5.3.(i) shows that $\left({}_{(c)}\mathbf{A}^{(\lambda^\#\lambda,\mu^\#\mu)}\right) \cup \left({}_{(c)}\mathbf{B}^{(\lambda^\#\lambda,\mu^\#\mu)}\right)$ is an R-basis of $S^{(\lambda^\#\lambda,\mu^\#\mu)}$.

Now we get from Lemma 4.6.5.(i)

$$\begin{aligned}
&\left({}_{(c)}\mathbf{A}^{(\lambda^\#\lambda,\mu^\#\mu)}\right){}^{(c)}\Psi_{(\lambda^\#\lambda,\mu^\#\mu)} \\
&= \left\{ v_{a,n-a}\left(\left(z(\zeta)\Psi_{\lambda^\#\lambda c}\right)^{\rightarrow}\right)\left(z(\eta)^{\leftarrow}\right)T_d \;\middle|\; \begin{array}{c}\zeta \in \mathcal{Z}^{\lambda^\#\lambda} \setminus \mathcal{Z}^{\lambda^\# A_c\lambda},\\ \eta \in \mathcal{Z}^{\mu^\#\mu},\\ d \in \mathcal{D}_{(n-a,a)}\end{array} \right\} \\
&= v_{a,n-a}\left(\left\{ z(\zeta)\Psi_{\lambda^\#\lambda c} \;\middle|\; \zeta \in \mathcal{Z}^{\lambda^\#\lambda} \setminus \mathcal{Z}^{\lambda^\# A_c\lambda}\right\}^{\rightarrow}\right)\cdot \\
&\qquad \left(\left\{ z(\eta) \;\middle|\; \eta \in \mathcal{Z}^{\mu^\#\mu}\right\}^{\leftarrow}\right)\left\{T_d \,\middle|\, d \in \mathcal{D}_{(n-a,a)}\right\}.
\end{aligned}$$

From this, Corollary 3.10.4, Theorem 3.10.1, Corollary 4.3.9.(ii), Definition 4.5.1.(i), and Lemma 4.6.7.(i) we see that the set $\left({}_{(c)}\mathbf{A}^{(\lambda^\#\lambda,\mu^\#\mu)}\right){}^{(c)}\Psi_{(\lambda^\#\lambda,\mu^\#\mu)}$ is an R-basis of

$$v_{a,n-a}\left(S^{\lambda^\#\lambda R_c \rightarrow}\right)\left(S^{\mu^\#\mu\leftarrow}\right)\mathcal{H}_{B_n} = S^{(\lambda^\#\lambda R_c,\mu^\#\mu)} = \left(S^{(\lambda^\#\lambda,\mu^\#\mu)}\right){}^{(c)}\Psi_{(\lambda^\#\lambda,\mu^\#\mu)}$$

and in particular linearly independent over R. Furthermore, we get from Lemma 4.6.5.(i) and Corollary 3.7.1 for every $v_{a,n-a}\left(z(\zeta)^{\rightarrow}\right)\left(z(\eta)^{\leftarrow}\right)T_d \in {}_{(c)}\mathbf{B}^{(\lambda^\#\lambda,\mu^\#\mu)}$ with $\zeta \in \mathcal{Z}^{\lambda^\# A_c\lambda}$, $\eta \in \mathcal{Z}^{\mu^\#\mu}$, and $d \in \mathcal{D}_{(n-a,a)}$

$$\begin{aligned}
\left(v_{a,n-a}\left(z(\zeta)^{\rightarrow}\right)\left(z(\eta)^{\leftarrow}\right)T_d\right){}^{(c)}\Psi_{(\lambda^\#\lambda,\mu^\#\mu)} &= v_{a,n-a}\left(\left(z(\zeta)\Psi_{\lambda^\#\lambda c}\right)^{\rightarrow}\right)\left(z(\eta)^{\leftarrow}\right)T_d \\
&= v_{a,n-a}0_{\mathcal{H}_A}\left(z(\eta)^{\leftarrow}\right)T_d \\
&= 0_{\mathcal{H}_B}.
\end{aligned}$$

Now we distinguish the cases $\lambda^\# A_c\lambda = 00$ and $\lambda^\# A_c\lambda \neq 00$. First we consider the case

$$\lambda^\# A_c\lambda = 00.$$

On the one hand, according to Definition 3.3.5, we have in this case $\mathcal{Z}^{\lambda^\# A_c\lambda} = \varnothing$ and thus ${}_{(c)}\mathbf{B}^{(\lambda^\#\lambda,\mu^\#\mu)} = \varnothing$. From this and the preceding considerations we obtain

$$\operatorname{Ker}\left({}^{(c)}\Psi_{(\lambda^\#\lambda,\mu^\#\mu)}\Big\downarrow^{M^{(\lambda,\mu)}}_{S^{(\lambda^\#\lambda,\mu^\#\mu)}}\right) = 0_{\mathcal{H}_B}.$$

On the other hand, according to Definition 4.1.9.(i) and Definition 4.5.1.(iv), we have in this case $(\lambda^\#\lambda,\mu^\#\mu)^{(c)}A = (00,00)$ and thus

$$S^{(\lambda^\#\lambda,\mu^\#\mu)^{(c)}A} = 0_{\mathcal{H}_B}.$$

All this proves the claim in the case $\lambda^\# A_c\lambda = 00$.

Next we consider the case

$$\lambda^{\#}A_c\lambda \neq 00.$$

On the one hand, we get from Lemma 3.3.6.(i) and Definition 1.1.58.(i) in this case ${}_{(c)}\mathbf{B}^{(\lambda^{\#}\lambda,\mu^{\#}\mu)} \neq \varnothing$. This and the preceding considerations imply that the set ${}_{(c)}\mathbf{B}^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}$ is an R-basis of $\operatorname{Ker}\left({}^{(c)}\Psi_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}\big\downarrow^{M^{(\lambda,\mu)}}_{S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}}\right)$. On the other hand, we get from Definition 4.1.9.(i) and Theorem 4.5.3.(i) in this case that the set ${}_{(c)}\mathbf{B}^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}$ is an R-basis of $S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)(c)A}$. All this also proves the claim in the case $\lambda^{\#}A_c\lambda \neq 00$.

(ii) The proof of this statement uses Lemma 4.6.5.(ii), Lemma 4.6.7.(ii), and Definition 4.1.9.(ii) and is otherwise completely analogous to the proof of statement (i).

(iii) The proof of this statement uses Theorem 4.5.3.(ii), Lemma 4.6.6.(i), Corollary 4.3.11.(i), Definition 4.5.1.(ii), Lemma 4.6.7.(iii), and Definition 4.1.9.(ii) and is otherwise completely analogous to the proof of statement (i).

(iv) The proof of this statement uses Theorem 4.5.3.(iii), Lemma 4.6.6.(ii), Corollary 4.3.11.(ii), Definition 4.5.1.(iii), and Lemma 4.6.7.(iv) and is otherwise completely analogous to the proof of statement (i). ■

4.7 Construction of generic bi-Specht series for Hecke algebras of type B and associated bi-permutation modules and bi-PK-modules

Now we complete the derivation of the generic bi-Specht series for Hecke algebras of type B. This procedure is analogous to the construction of generic Specht series for Hecke algebras of type A in Section 3.11. First we give a formal definition of bi-Specht series for modules of Hecke algebras of type B. Then we construct generic bi-Specht series for bi-PK-modules and bi-permutation modules and finally also for Hecke algebras of type B. As before, $n \in \mathbb{N}$ denotes a positive integer and (R, q, Q) denotes a coefficient triple as in Definition 4.2.1.

The next definition makes use of Definition 4.1.6.(ii) and Definition 4.4.1.

Definition 4.7.1 *Let M be a right $\mathcal{H}_{B_n}^{(R,q,Q)}$-module. Then a series of submodules*

$$0_{\mathcal{H}_B} = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{m-1} \subseteq M_m = M$$

with an $m \in \mathbb{N}_0$ is called a bi-Specht series for M if there are for every $j \in \{1, \ldots, m\}$ an $a_j \in \{0, \ldots, n\}$ and an a_j-bi-partition $(\lambda, \mu)^{(j)}$ such that

$$M_j/M_{j-1} \simeq S_{(R,q,Q)}^{(\lambda,\mu)^{(j)}} \quad \text{as } \mathcal{H}_{B_n}^{(R,q,Q)}\text{-modules}$$

holds. The number m is called the length of the bi-Specht series.

The following lemma makes use of Definition 4.1.7, Definition 3.2.1, and Definition 4.5.1.

Lemma 4.7.2 *(i) Let $a \in \{1, \ldots, n-1\}$, $\lambda \vdash a$, $\mu \vdash n-a$ and consider the a-bi-PK_n-pair $(\lambda\lambda, \mu\mu)$. Then the a-bi-PK_n-module $S^{(\lambda\lambda,\mu\mu)}_{(R,q,Q)}$ has a bi-Specht series.*

(ii) Let $\mu \vdash n$ and consider the 0-bi-PK_n-pair $(00, \mu\mu)$. Then the 0-bi-PK_n-module $S^{(00,\mu\mu)}_{(R,q,Q)}$ has a bi-Specht series.

(iii) Let $\lambda \vdash n$ and consider the n-bi-PK_n-pair $(\lambda\lambda, 00)$. Then the n-bi-PK_n-module $S^{(\lambda\lambda,00)}_{(R,q,Q)}$ has a bi-Specht series.

(iv) The bi-PK-module $S^{(00,00)}_{(R,q,Q)}$ has a bi-Specht series.

Proof. (i) Lemma 4.5.6.(i) shows that $S^{(\lambda\lambda,\mu\mu)}$ has the bi-Specht series

$$0_{\mathcal{H}_B} \subseteq S^{(\lambda,\mu)} = S^{(\lambda\lambda,\mu\mu)}$$

of length 1.

(ii) Lemma 4.5.6.(ii) shows that $S^{(00,\mu\mu)}$ has the bi-Specht series

$$0_{\mathcal{H}_B} \subseteq S^{((0),\mu)} = S^{(00,\mu\mu)}$$

of length 1.

(iii) Lemma 4.5.6.(iii) shows that $S^{(\lambda\lambda,00)}$ has the bi-Specht series

$$0_{\mathcal{H}_B} \subseteq S^{(\lambda,(0))} = S^{(\lambda\lambda,00)}$$

of length 1.

(iv) Obviously, $S^{(00,00)}$ has the bi-Specht series

$$0_{\mathcal{H}_B} = S^{(00,00)}$$

of length 0. ■

Remark 4.7.3 *Remark 4.5.2 shows that the bi-Specht series from the proof of Lemma 4.7.2 are generic in the sense of Remark 1.2.9.*

The next theorem also makes use of Definition 4.1.7, Definition 3.2.1, and Definition 4.5.1.

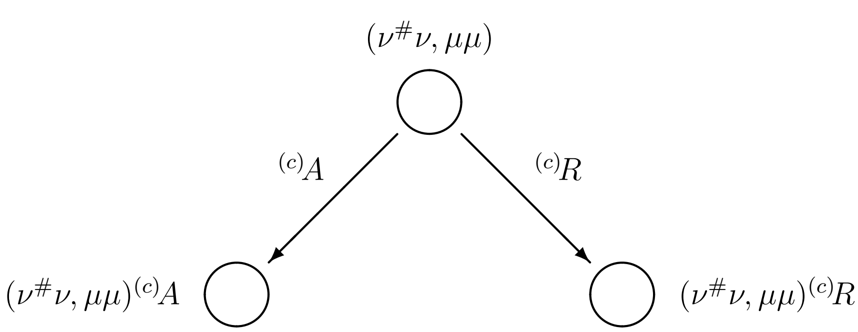
Theorem 4.7.4 *(i) Let $a \in \{1, \dots, n-1\}$. With that, let $\lambda^{\#}\lambda$ be a PK$_a$-pair satisfying $\lambda^{\#}\lambda \neq 00$, let $\mu \vdash n-a$, and consider the a-bi-PK$_n$-pair $(\lambda^{\#}\lambda, \mu\mu)$. Then there is a bi-Specht series for the a-bi-PK$_n$-module $S_{(R,q,Q)}^{(\lambda^{\#}\lambda,\mu\mu)}$.*

(ii) Let $a \in \{1, \dots, n-1\}$. With that, let $\mu^{\#}\mu$ be a PK$_{n-a}$-pair satisfying $\mu^{\#}\mu \neq 00$, let $\lambda \vdash a$, and consider the a-bi-PK$_n$-pair $(\lambda\lambda, \mu^{\#}\mu)$. Then there is a bi-Specht series for the a-bi-PK$_n$-module $S_{(R,q,Q)}^{(\lambda\lambda,\mu^{\#}\mu)}$.

(iii) Let $(00, \mu^{\#}\mu)$ be a 0-bi-PK$_n$-pair. Then there is a bi-Specht series for the 0-bi-PK$_n$-module $S_{(R,q,Q)}^{(00,\mu^{\#}\mu)}$.

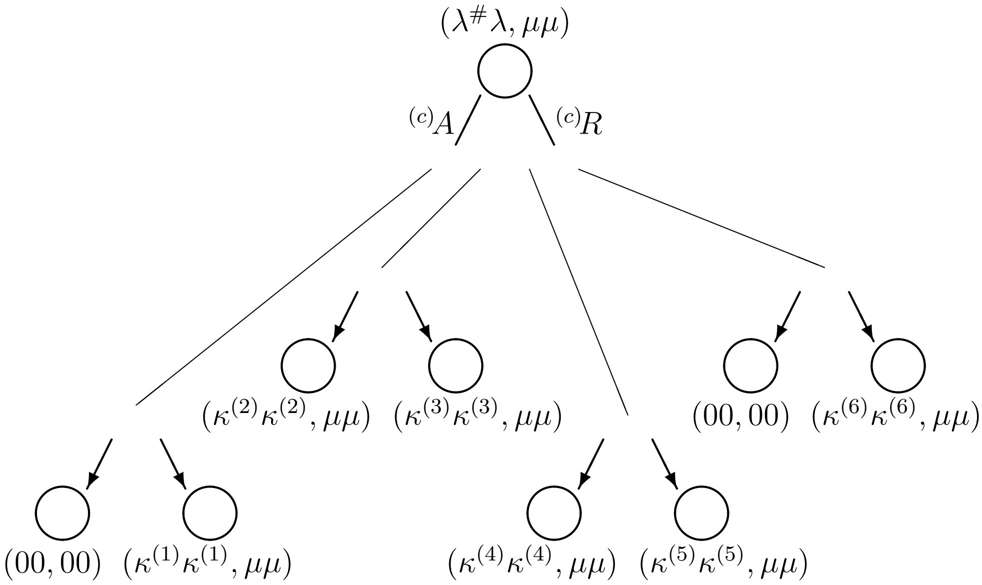
(iv) Let $(\lambda^{\#}\lambda, 00)$ be an n-bi-PK$_n$-pair. Then there is a bi-Specht series for the n-bi-PK$_n$-module $S_{(R,q,Q)}^{(\lambda^{\#}\lambda,00)}$.

Proof. (i) If we apply the construction of the binary tree (3.105) on page 216 described in the proof of Theorem 3.11.2 to the left entry $\lambda^{\#}\lambda$ of the a-bi-PK$_n$-pair $(\lambda^{\#}\lambda, \mu\mu)$, taking into account Definition 4.1.9, then we obtain a binary tree (that is, every vertex in the tree has zero or two successors) whose vertices are labelled with bi-PK-pairs and whose edges are labelled with operators ${}^{(c)}A$ and ${}^{(c)}R$ with appropriate AR-indices $c \in \mathbb{N} \setminus \{1\}$. More specifically, the root of the tree (that is, the vertex without predecessor) is labelled $(\lambda^{\#}\lambda, \mu\mu)$, and if a vertex of the tree has two successors then the label of this vertex is an a-bi-PK$_n$-pair $(\nu^{\#}\nu, \mu\mu)$, the labels of the edges leading to its successors are ${}^{(c)}A$ and ${}^{(c)}R$ with an AR-index $c \in \mathbb{N} \setminus \{1\}$ for $\nu^{\#}\nu$, the label of the vertex at the other end of the edge labelled ${}^{(c)}A$ is $(\nu^{\#}\nu, \mu\mu)^{(c)}A$, and the label of the vertex at the other end of the edge labelled ${}^{(c)}R$ is $(\nu^{\#}\nu, \mu\mu)^{(c)}R$. This part of the tree is displayed in the following picture.



(4.9)

The further considerations in the proof of Theorem 3.11.2 show together with Definition 4.1.9 that this binary tree contains only a finite number of vertices and that the label of a leaf of this tree (that is, a vertex without successors) is either $(00, 00)$ or an a-bi-PK$_n$-pair $(\kappa\kappa, \mu\mu)$ with an appropriate $\kappa \vdash a$. Thus the complete binary tree has the form



(4.10)

with appropriate $\kappa^{(1)}, \ldots, \kappa^{(6)} \in \Pi_a$ (see Definition 1.1.2.(iii)) and possibly more such partitions.

The claim is now proved by induction on the labels of the vertices of this tree along the edges from the leaves to the root. The induction start is provided by Lemma 4.7.2.(i) and Lemma 4.7.2.(iv) together with the above considerations concerning the labelling of the leaves of the binary tree. For the induction step we consider a vertex of the tree which is not a leaf. This vertex is then, as shown in picture (4.9), labelled with an a-bi-PK$_n$-pair $(\nu^{\#}\nu, \mu\mu)$ and the labels of its successors are $(\nu^{\#}\nu, \mu\mu)^{(c)}A$ and $(\nu^{\#}\nu, \mu\mu)^{(c)}R$ with an AR-index $c \in \mathbb{N} \setminus \{1\}$ for $\nu^{\#}\nu$. With this data we obtain from Lemma 4.6.7.(i) and Lemma 4.6.8.(i) the series

$$0_{\mathcal{H}_B} \subseteq S^{(\nu^{\#}\nu,\mu\mu)^{(c)}A} \subseteq S^{(\nu^{\#}\nu,\mu\mu)} \tag{4.11}$$

with

$$S^{(\nu^{\#}\nu,\mu\mu)} / S^{(\nu^{\#}\nu,\mu\mu)^{(c)}A} \simeq S^{(\nu^{\#}\nu,\mu\mu)^{(c)}R} \quad \text{as } \mathcal{H}_{B_n}\text{-modules,} \tag{4.12}$$

the isomorphism being induced by the map ${}^{(c)}\Psi_{(\nu^{\#}\nu,\mu\mu)}\Big\downarrow^{M^{(\nu,\mu)}}_{S^{(\nu^{\#}\nu,\mu\mu)}}$. Now if the claim of the theorem holds for $(\nu^{\#}\nu, \mu\mu)^{(c)}A$ and $(\nu^{\#}\nu, \mu\mu)^{(c)}R$ then we can use the bi-Specht series for $S^{(\nu^{\#}\nu,\mu\mu)^{(c)}A}$ and $S^{(\nu^{\#}\nu,\mu\mu)^{(c)}R}$ and the isomorphism (4.12) to refine the series (4.11) for $S^{(\nu^{\#}\nu,\mu\mu)}$ to a bi-Specht series for $S^{(\nu^{\#}\nu,\mu\mu)}$. Proceeding inductively in this way, we finally obtain a bi-Specht series for $S^{(\lambda^{\#}\lambda,\mu\mu)}$, as desired.

(ii) The proof of this statement uses Definition 4.1.9.(ii), Definition 4.1.9.(iv), Lemma 4.6.7.(ii), and Lemma 4.6.8.(ii) and is otherwise completely analogous to the proof of statement (i).

(iii) The proof of this statement uses Definition 4.1.9.(ii), Definition 4.1.9.(iv), Lemma 4.7.2.(ii), Lemma 4.6.7.(iii), and Lemma 4.6.8.(iii) and is otherwise completely analogous to the proof of statement (i).

(iv) The proof of this statement uses Lemma 4.7.2.(iii), Lemma 4.6.7.(iv), and Lemma 4.6.8.(iv) and is otherwise completely analogous to the proof of statement (i). ■

Remark 4.7.5 *Remark 4.5.2, Remark 4.7.3, and Remark 4.6.4 show that the bi-Specht series constructed in the proof of Theorem 4.7.4 are generic in the sense of Remark 1.2.9.*

The following statement makes use of Definition 4.1.7.(i) and Definition 4.5.1.(i).

Theorem 4.7.6 *Let $a \in \{1, \ldots, n-1\}$ and let $(\lambda^\#\lambda, \mu^\#\mu)$ be an a-bi-PK$_n$-pair. Then there is a bi-Specht series for the a-bi-PK$_n$-module $S_{(R,q,Q)}^{(\lambda^\#\lambda,\mu^\#\mu)}$.*

Proof. The proof of this theorem is similar to the proof of Theorem 4.7.4.(i).

First we construct, as in the proofs of Theorem 4.7.4.(i) and Theorem 3.11.2, starting from $(\lambda^\#\lambda, \mu^\#\mu)$ and making repeated use of Definition 4.1.9, a binary tree with a finite number of vertices. The vertices of the tree are labelled with bi-PK-pairs and the labels of the edges of the tree are operators ${}^{(c)}A$, ${}^{(c)}R$, $A^{(c)}$, and $R^{(c)}$ with appropriate AR-indices $c \in \mathbb{N}\backslash\{1\}$. More specifically, the root of the tree (that is, the vertex without predecessor) is labelled $(\lambda^\#\lambda, \mu^\#\mu)$, and if a vertex of the tree has two successors then the label of this vertex is an a-bi-PK$_n$-pair $(\kappa^\#\kappa, \nu^\#\nu)$ and the part of the binary tree consisting of this vertex, its two successors, and the connecting edges has one of the following two forms.

$$\begin{tikzcd} & (\kappa^\#\kappa, \nu^\#\nu) \arrow[dl, "{}^{(c)}A"'] \arrow[dr, "{}^{(c)}R"] & \\ (\kappa^\#\kappa, \nu^\#\nu)^{(c)}A & & (\kappa^\#\kappa, \nu^\#\nu)^{(c)}R \end{tikzcd} \tag{4.13}$$

$$\begin{tikzcd} & (\kappa^\#\kappa, \nu^\#\nu) \arrow[dl, "A^{(c')}"'] \arrow[dr, "R^{(c')}"] & \\ (\kappa^\#\kappa, \nu^\#\nu)A^{(c')} & & (\kappa^\#\kappa, \nu^\#\nu)R^{(c')} \end{tikzcd} \tag{4.14}$$

Here, $c \in \mathbb{N} \setminus \{1\}$ denotes an AR-index for the PK$_a$-pair $\kappa^{\#}\kappa$ and $c' \in \mathbb{N} \setminus \{1\}$ denotes an AR-index for the PK$_{n-a}$-pair $\nu^{\#}\nu$. Moreover, the leaves of the binary tree are labelled with bi-PK-pairs to which none of the operators ${}^{(c)}A$, ${}^{(c)}R$, $A^{(c)}$, or $R^{(c)}$ with an appropriate AR-index $c \in \mathbb{N} \setminus \{1\}$ can be applied. In the situation at hand, such a bi-PK-pair must be, according to Definition 4.1.9, Definition 3.2.5, and Lemma 3.2.7.(iii), either $(00, 00)$ or of the form $(\kappa\kappa, \nu\nu)$ with appropriate $\kappa \in \Pi_a$ and $\nu \in \Pi_{n-a}$ (see Definition 1.1.2.(iii)).

The claim of the theorem is now proved by induction on the labels of the vertices of this binary tree along the edges from the leaves to the root. The induction start is provided by Lemma 4.7.2.(i) and Lemma 4.7.2.(iv) together with the above considerations concerning the labelling of the leaves of the tree. The induction step makes use of picture (4.13), picture (4.14), Lemma 4.6.7.(i), Lemma 4.6.7.(ii), Lemma 4.6.8.(i), and Lemma 4.6.8.(ii) and is otherwise completely analogous to the induction step in the proof of Theorem 4.7.4.(i). Proceeding inductively in this way, we finally obtain a bi-Specht series for $S^{(\lambda^{\#}\lambda, \mu^{\#}\mu)}$, as desired. ■

Remark 4.7.7 *Remark 4.5.2, Remark 4.7.3, and Remark 4.6.4 show that be bi-Specht series constructed in the proof of Theorem 4.7.6 are generic in the sense of Remark 1.2.9.*

From Lemma 4.7.2, Remark 4.7.3, Theorem 4.7.4, Remark 4.7.5, Theorem 4.7.6, and Remark 4.7.7 we see that every bi-PK$_n$-module for $\mathcal{H}_{B_n}$ has a generic bi-Specht series.

The next corollary makes use of Definition 4.3.12 and Definition 4.1.6.(i).

Corollary 4.7.8 *(i) Let $a \in \{1, \ldots, n-1\}$, $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash a$ with $\lambda_1 > 0$, and $\mu = (\mu_1, \mu_2, \ldots) \vDash n - a$ with $\mu_1 > 0$. Then there is a bi-Specht series for the bi-permutation module $M^{(\lambda,\mu)}_{(R,q,Q)}$.*

(ii) Let $\mu = (\mu_1, \mu_2, \ldots) \vDash n$ with $\mu_1 > 0$. Then there is a bi-Specht series for the bi-permutation module $M^{((0),\mu)}_{(R,q,Q)}$.

(iii) Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vDash n$ with $\lambda_1 > 0$. Then there is a bi-Specht series for the bi-permutation module $M^{(\lambda,(0))}_{(R,q,Q)}$.

Proof. (i) The assumptions $\lambda_1 > 0$ and $\mu_1 > 0$ allow us, according to Definition 3.2.1, to build the PK$_a$-pair $(\lambda_1)\lambda$ and the PK$_{n-a}$-pair $(\mu_1)\mu$. According to Definition 4.1.7.(i), these can be combined into the a-bi-PK$_n$-pair $((\lambda_1)\lambda, (\mu_1)\mu)$. Now the desired bi-Specht series for $M^{(\lambda,\mu)}$ is obtained from Lemma 4.5.7.(i) and Theorem 4.7.6.

(ii) The proof of this statement uses Definition 4.1.7.(ii), Lemma 4.5.7.(ii), and Theorem 4.7.4.(iii) and is otherwise completely analogous to the proof of statement (i).

(iii) The proof of this statement uses Definition 4.1.7.(iii), Lemma 4.5.7.(iii), and Theorem 4.7.4.(iv) and is otherwise completely analogous to the proof of statement (i). ■

Remark 4.7.9 *Remark 4.7.5 and Remark 4.7.7 show that the bi-Specht series from the proof of Corollary 4.7.8 are generic in the sense of Remark 1.2.9.*

Corollary 4.7.10 *There is a bi-Specht series for $\mathcal{H}_{B_n}^{(R,q,Q)}$.*

Proof. This follows from Corollary 4.3.6, Lemma 4.3.16, Remark 4.3.17, and Corollary 4.7.8. ■

Remark 4.7.11 *Remark 4.3.7 and Remark 4.7.9 show that the bi-Specht series from the proof of Corollary 4.7.10 is generic in the sense of Remark 1.2.9.*

This completes the derivation of the generic bi-Specht series for Hecke algebras of type B.

Remark 4.7.12 *Using the definitions, constructions, statements, and results in [PAL], this chapter carries over almost word for word from Hecke algebras of type B to Hecke algebras of type D, thus also providing generic bi-Specht series for the latter.*

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Notation

The following notations are arranged in order of their first appearance.

$\mathbb{Z}$	set $\{\ldots,-2,-1,0,1,2,\ldots\}$ of all integers (see page 1)
$\mathbb{N}$	set $\{1,2,3,\ldots\}$ of all positive integers (see page 1)
$\mathbb{N}_0$	set $\{0,1,2,\ldots\}$ of all nonnegative integers (see page 1)
$\mathbb{Q}$	set $\{z/y \mid z \in \mathbb{Z}, y \in \mathbb{Z} \setminus \{0\}\}$ of all rational numbers (see page 1)
$\lvert M \rvert$	cardinality of a set (see page 1)
$f\downarrow^M_U$	restriction of a map (see page 1)
$\mathfrak{S}_n$	symmetric group on $\{1,\ldots,n\}$ (see page 1)
$1_{\mathfrak{S}_n}$	neutral element of $\mathfrak{S}_n$ (see page 1)
$\mathfrak{S}_M$	symmetric group on a set (see page 2)
$W_{A_{n-1}}$	Weyl group of type A_{n-1} (see page 2)
$\mathfrak{B}_n$	generating set of $\mathfrak{S}_n$ consisting of transpositions of adjacent numbers (see page 2)
$X \xrightarrow{\sim} Y$	isomorphism from one algebraic structure to another (see page 2)
$\ell_{A_{n-1}}(w)$	length of $w \in W_{A_{n-1}}$ (see page 2)
$\lambda \vDash m$	composition of $m \in \mathbb{N}_0$ (see page 3)
λ_j^+	partial sum of the entries of $\lambda \vDash m$ (see page 3)
Ξ_m	set $\{\lambda \mid \lambda \vDash m\}$ of all compositions of $m \in \mathbb{N}_0$ (see page 4)
$\lambda \vdash m$	partition of $m \in \mathbb{N}_0$ (see page 4)
Π_m	set $\{\lambda \mid \lambda \vdash m\}$ of all partitions of $m \in \mathbb{N}_0$ (see page 4)
$\Pi_{m,e}$	set $\Pi_{m,e} = \{\lambda \vdash m \mid \lambda\ e\text{-regular}\}$ of all e-regular partitions of $m \in \mathbb{N}_0$ (see page 4)

$\lambda \leq \mu$	total ordering relation between elements of Ξ_m (see page 5)
$X \vee Y$	OR of boolean values (see page 5)
$\lambda \trianglelefteq \mu$	partial ordering relation between elements of Ξ_m (see page 5)
$X \wedge Y$	AND of boolean values (see page 5)
$[\lambda]$	diagram of $\lambda \models m$ (see page 5)
$M \times N$	direct product of sets (see page 5)
$(\cdot)'$	transposition map on $\mathbb{N} \times \mathbb{N}$ (see page 7); also denotes the transposition map on Π_m (see page 8)
id_M	identity map on a set (see page 7)
$[\lambda]'$	transpose of $[\lambda]$ with $\lambda \vdash m$ (see page 7)
λ'	transpose of $\lambda \vdash m$ (see page 8)
$h^\lambda_{(i,j)}$	(i,j)-hook in $\lambda \vdash m$ (see page 9)
$\left\lvert h^\lambda_{(i,j)} \right\rvert$	hook length of $h^\lambda_{(i,j)}$ (see page 9)
$r^\lambda_{(i,j)}$	(i,j)-rim hook in $\lambda \vdash m$ (see page 9)
$\left\lvert r^\lambda_{(i,j)} \right\rvert$	rim hook length of $r^\lambda_{(i,j)}$ (see page 9)
$\lambda \setminus r$	$\lambda \vdash m$ without a rim hook (see page 9)
$\lambda \cup r$	$\lambda \vdash m$ together with a rim hook (see page 10)
$\lambda \setminus (i, \lambda_i)$	$\lambda \vdash m$ without a lattice point (see page 10)
$\lambda \cup (i, \lambda_i + 1)$	$\lambda \vdash m$ together with a lattice point (see page 10)
$\lambda \uparrow$	set of partitions obtained from $\lambda \vdash k \in \mathbb{N}_0$ by adding a lattice point in every possible way (see page 10)
$\mu \downarrow$	set of partitions obtained from $\mu \vdash k \in \mathbb{N}$ by removing a lattice point in every possible way (see page 10)
Γ_e	set $\{\lambda \vdash k \mid k \in \mathbb{N}_0$ and λ is an e-core$\}$ of all e-cores (see page 17)
$\gamma_e(\lambda)$	e-core of $\lambda \vdash k$ (see page 18)
$g_e(\lambda)$	e-weight of $\lambda \vdash k$ (see page 18)
$\Gamma_e(k)$	set of the e-cores of all partitions of $k \in \mathbb{N}_0$ (see page 18)
$\Pi_k^{\mu,e}$	set of all partitions of $k \in \mathbb{N}_0$ having a given e-core μ (see page 18)

$\Pi_{k,\tilde{e}}^{\nu,e}$	set of all $\tilde{e}$-regular partitions of $k \in \mathbb{N}_0$ having a given e-core ν (see page 19)
$x \mid y$	x divides y for integers or polynomials (see page 19)
$(i,j)\mathbf{t}$	entry in a tableau (see page 20)
$(k)\zeta_{\mathbf{t}}$	row number of an entry in a tableau (see page 20)
$(k)\sigma_{\mathbf{t}}$	column number of an entry in a tableau (see page 20)
$\mathcal{T}^{\lambda}$	set $\{\mathbf{t} : [\lambda] \to \{1,\dots,n\} \mid \mathbf{t} \text{ bijective}\}$ of all λ-tableaux (see page 21)
$\mathcal{T}^{\lambda}_{\text{row std}}$	set $\left\{\mathbf{t} \in \mathcal{T}^{\lambda} \,\middle|\, \mathbf{t} \text{ row standard}\right\}$ of all row standard λ-tableaux (see page 21)
$\mathcal{T}^{\lambda}_{\text{std}}$	set $\left\{\mathbf{t} \in \mathcal{T}^{\lambda} \,\middle|\, \mathbf{t} \text{ standard}\right\}$ of all standard λ-tableaux (see page 21)
$\mathcal{T}^{\Xi_n}$	set of all tableaux of compositions of $n \in \mathbb{N}$ (see page 21)
$\mathcal{T}^{\Xi_n}_{\text{row std}}$	set of all row standard tableaux of compositions of $n \in \mathbb{N}$ (see page 21)
$\mathbf{t}'$	transpose of a tableau (see page 23)
$\mathfrak{R}_{\mathbf{t}}$	row stabilizer of a tableau (see page 24)
$\mathfrak{C}_{\mathbf{t}}$	column stabilizer of a tableau (see page 24)
$G \times H$	direct product of groups (see page 24)
$\mathbf{t}^{\lambda}$	λ-tableau whose entries are arranged in ascending order by rows from top to bottom and within the rows from left to right (see page 25)
$\mathfrak{S}_{\lambda}$	Young subgroup of $\mathfrak{S}_n$ associated with $\lambda \vDash n$ (see page 25)
$\mathcal{D}_{\lambda}$	set of the shortest representatives associated with $\lambda \vDash n$ (see page 26)
$[w]^{\lambda}$	shortest representative of $w \in \mathfrak{S}_n$ associated with $\lambda \vDash n$ (see page 27)
$\mathcal{E}_{\lambda}$	set of the standard representatives associated with $\lambda \vDash n$ (see page 27)
$\omega^{(n)}$	partition (1^n) (see page 28)

$\mathbf{t}_\lambda$	λ-tableau whose entries are arranged in ascending order by columns from left to right and within the columns from top to bottom (see page 28)
w_λ	permutation mapping t^λ to t_λ (see page 29)
$(j)\zeta$	entry of a row number list (see page 30)
$\mathcal{Z}^\lambda$	set of all λ-row number lists (see page 30)
$\varnothing$	empty set (see page 31)
$\zeta[\lambda]$	map assigning to a row standard λ-tableau its λ-row number list (see page 31)
$\zeta_{\mathbf{t}}$	row number list of a tableau (see page 31)
$\mathbf{t}_\zeta$	row standard tableau of a row number list (see page 31)
$\mathbf{t}[\lambda]$	map assigning to a λ-row number list its row standard λ-tableau (see page 31)
0_R	additive neutral element of a ring (see page 33)
1_R	multiplicative neutral element of a ring (see page 33)
$[j]_q$	q-number associated to a coefficient pair (see page 33)
$e_R(q)$	q-characteristic of a coefficient pair (see page 33)
$\mathcal{H}_{A_{n-1}}^{(R,q)}$	Hecke algebra of type A_{n-1} over the coefficient pair (R,q) (see page 34)
T_w	defining basis element of a Hecke algebra of type A indexed by an element of the underlying Weyl group (see page 34); also denotes a defining basis element of a Hecke algebra of type B indexed by an element of the underlying Weyl group (see page 226)
$\iota_{(R,q)}^{(n)}(X)$	"unsigned" sum $\sum_{w\in X} T_w$ over some defining basis elements in a Hecke algebra of type A (see page 34)
$\varepsilon_{(R,q)}^{(n)}(X)$	"signed" sum $\sum_{w\in X}(-q)^{-\ell(w)}T_w$ over some defining basis elements in a Hecke algebra of type A (see page 35)
h^*	image of $h \in \mathcal{H}_{A_{n-1}}^{(R,q)}$ under the anti-involution induced by inversion on $\mathfrak{S}_n$ (see page 35)
$\mathrm{Hom}_R(M,N)$	set of all R-homomorphisms from one R-module to another (see page 35)

M^*	dual module of an $\mathcal{H}_{A_{n-1}}^{(R,q)}$-module (see page 35)
$M \otimes_R N$	tensor product over R of two R-modules (see page 35)
$X \simeq Y$	isomorphism relation between algebraic structures (see page 36)
$M \oplus N$	direct sum of two modules (see page 36)
$x_\lambda^{(R,q)}$	"unsigned" sum $\iota_{(R,q)}^{(n)}(\mathfrak{S}_\lambda) \in \mathcal{H}_{A_{n-1}}^{(R,q)}$ (see page 37)
$M_{(R,q)}^\lambda$	permutation module of $\mathcal{H}_{A_{n-1}}^{(R,q)}$ associated to $\lambda \vDash n$ (see page 38)
$\mathbf{B}_{\text{row std}}^{M^\lambda}(R,q)$	row standard basis $\left\{x_\lambda^{(R,q)} T_d \,\middle|\, d \in \mathcal{D}_\lambda\right\}$ of $M_{(R,q)}^\lambda$ (see page 38)
$\beta_{(R,q)}^\lambda$	symmetric bilinear form on $M_{(R,q)}^\lambda$ (see page 39)
$y_\lambda^{(R,q)}$	"signed" sum $\varepsilon_{(R,q)}^{(n)}(\mathfrak{S}_\lambda) \in \mathcal{H}_{A_{n-1}}^{(R,q)}$ (see page 40)
$z_\lambda^{(R,q)}$	generator of $S_{(R,q)}^\lambda$ (see page 40)
$S_{(R,q)}^\lambda$	Specht module of $\mathcal{H}_{A_{n-1}}^{(R,q)}$ associated to $\lambda \vdash n$ (see page 40)
$\mathbf{B}_{\text{std}}^{S^\lambda}(R,q)$	standard basis $\left\{z_\lambda^{(R,q)} T_f \,\middle|\, f \in \mathcal{E}_{\lambda'}\right\}$ of $S_{(R,q)}^\lambda$ (see page 41)
$\gamma_{(R,q)}^\lambda$	symmetric bilinear form on $S_{(R,q)}^\lambda$ (see page 42)
$\varphi[\gamma_{(R,q)}^\lambda]$	R-homomorphism from $S_{(R,q)}^\lambda$ to $\left(S_{(R,q)}^\lambda\right)^*$ induced by $\gamma_{(R,q)}^\lambda$ (see page 42)
$\operatorname{rad} \gamma_{(R,q)}^\lambda$	radical of $\gamma_{(R,q)}^\lambda$ (see page 42)
$D_{(K,r)}^\lambda$	irreducible $\mathcal{H}_{A_{n-1}}^{(K,r)}$-module associated to $\lambda \in \Pi_{n,e_K(r)}$ (see page 43)
$[M]$	isomorphism class of an $\mathcal{H}_{A_{n-1}}^{(R,q)}$-module (see page 44)
$\mathcal{M}(\mathcal{H}_{A_{n-1}}^{(R,q)})$	set of the isomorphism classes of $\mathcal{H}_{A_{n-1}}^{(R,q)}$-modules (see page 44)
$G_0(\mathcal{H}_{A_{n-1}}^{(R,q)})$	Grothendieck group of $\mathcal{H}_{A_{n-1}}^{(R,q)}$ (see page 44)
$\alpha_{(K,r)(\tilde{K},\tilde{r})}^{\mathcal{H}_n}$	isomorphism from $G_0(\mathcal{H}_{A_{n-1}}^{(K,r)})$ to $G_0(\mathcal{H}_{A_{n-1}}^{(\tilde{K},\tilde{r})})$ with semisimple Hecke algebras (see page 46)
S_ψ	discrete valuation ring of a discrete additive valuation (see page 46)
I_ψ	valuation ideal of a discrete additive valuation (see page 46)

$\bar{\cdot}$	natural projection from a discrete valuation ring S_ψ onto its residue class field S_ψ/I_ψ (see page 47); also denotes the reduction map from an $\mathcal{H}^{(S,a)}_{A_{n-1}}$-module M to its reduction $\bar{M}$ (see page 49); furthermore denotes the reduction map from $K_0(\mathcal{H}^{(S,a)}_{A_{n-1}})$ to $K_0(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})$ (see page 62)
$f : U \hookrightarrow M$	inclusion map (see page 48)
$\mathrm{Rnk}_R\, M$	rank of a free module over a ring (see page 49)
$\dim_K V$	dimension of a vector space over a field (see page 49)
$\bar{M}$	reduction of an $\mathcal{H}^{(S,a)}_{A_{n-1}}$-module (see page 49)
$D^{\mathcal{H}}_{n,\mathcal{K}}$	decomposition map for Hecke algebras of type A associated with the degree n and the modular system $\mathcal{K}$ (see page 50)
$d^{n,\mathcal{K}}_{\lambda\mu}$	decomposition number for Hecke algebras of type A associated with the degree n and the modular system $\mathcal{K}$ (see page 50)
$\Delta^{\mathcal{H}}_{n,\mathcal{K}}$	decomposition matrix for Hecke algebras of type A associated with the degree n and the modular system $\mathcal{K}$ (see page 50)
$\mathrm{Rnk}_{\mathbb{Q}}\, \Delta^{\mathcal{H}}_{n,\mathcal{K}}$	$\mathbb{Q}$-rank of a decomposition matrix for Hecke algebras of type A (see page 52)
$R[X]$	polynomial ring over the coefficient ring R in the indeterminate X (see page 53)
$K(X)$	rational function field over the coefficient field K in the indeterminate X (see page 54)
$x \nmid y$	x doesn't divide y for integers or polynomials (see page 54)
ψ_f	discrete additive valuation on a rational function field associated to an irreducible polynomial f (see page 54)
R_I	localization of a ring at an ideal (see page 54)
$\mathcal{K}_{(K,r)}$	modular system associated to a field K and a unit $r \in K$ (see page 55)
$\hat{K}$	completion of a valuated field (see page 55)

$\hat{\psi}$	continuous extension of a valuation ψ to the completion $\hat{K}$ of the underlying valuated field K (see page 55)
$\hat{\mathcal{K}}_{(K,r)}$	complete modular system associated to a field K and a unit $r \in K$ (see page 56)
$\mathcal{P}(\mathcal{H}^{(R,q)}_{A_{n-1}})$	set of the isomorphism classes of projective $\mathcal{H}^{(R,q)}_{A_{n-1}}$-modules (see page 58)
$K_0(\mathcal{H}^{(R,q)}_{A_{n-1}})$	projective class group of $\mathcal{H}^{(R,q)}_{A_{n-1}}$ (see page 59)
$P^{\mu}_{(F,\bar{a})}$	indecomposable projective cover of $D^{\mu}_{(F,\bar{a})}$ with $\mu \in \Pi_{n,e_F(\bar{a})}$ (see page 59)
$P^{\mu}_{(S,a)}$	projective indecomposable $\mathcal{H}^{(S,a)}_{A_{n-1}}$-module associated to $\mu \in \Pi_{n,e_F(\bar{a})}$ (see page 60)
$\eta^{\mathcal{H}}_{n,(F,\bar{a})}$	inclusion of the category of projective $\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}$-modules into the category of all $\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}$-modules (see page 61)
$C^{\mathcal{H}}_{n,\mathcal{K}}$	Cartan map for Hecke algebras of type A associated with the degree n and the modular system $\mathcal{K}$ (see page 61)
$\mathcal{C}^{\mathcal{H}}_{n,\mathcal{K}}$	Cartan matrix for Hecke algebras of type A associated with the degree n and the modular system $\mathcal{K}$ (see page 61)
$\mathcal{C}^{\mathcal{H}}_{n,\mathcal{K}}(\lambda,\mu)$	entry of $\mathcal{C}^{\mathcal{H}}_{n,\mathcal{K}}$ (see page 61)
$-\otimes_S Q$	extension of coefficients from $K_0(\mathcal{H}^{(S,a)}_{A_{n-1}})$ to $G_0(\mathcal{H}^{(Q,a)}_{A_{n-1}})$ (see page 62)
$B^{\mathcal{H}}_{n,\mathcal{K}}$	Brauer map for Hecke algebras of type A associated with the degree n and the modular system $\mathcal{K}$ (see page 63)
$\mathcal{B}^{\mathcal{H}}_{n,\mathcal{K}}$	matrix representing $B^{\mathcal{H}}_{n,\mathcal{K}}$ (see page 63)
$\mathcal{B}^{\mathcal{H}}_{n,\mathcal{K}}(\lambda,\mu)$	entry of $\mathcal{B}^{\mathcal{H}}_{n,\mathcal{K}}$ (see page 63)
$i^{\mathcal{H}}_{n,(Q,a)}$	bilinear form of intertwining numbers on $G_0(\mathcal{H}^{(Q,a)}_{A_{n-1}}) \times G_0(\mathcal{H}^{(Q,a)}_{A_{n-1}})$ (see page 64)
$j^{\mathcal{H}}_{n,(F,\bar{a})}$	bilinear form of intertwining numbers on $K_0(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}}) \times G_0(\mathcal{H}^{(F,\bar{a})}_{A_{n-1}})$ (see page 64)
M^T	transpose of a matrix (see page 65)
$Z(\mathcal{H}^{(R,q)}_{A_{n-1}})$	center of $\mathcal{H}^{(R,q)}_{A_{n-1}}$ (see page 66)
$c_\lambda(S,a)$	element of an S-basis of $Z(\mathcal{H}^{(S,a)}_{A_{n-1}})$ indexed by a $\lambda \vdash n$ (see page 66)

$b_{\mathrm{Idemp}}^{(i)}(\mathcal{H}_{A_{n-1}}^{(R,q)})$	block idempotent of $\mathcal{H}_{A_{n-1}}^{(R,q)}$ indexed by an arbitrary number (see page 70)
$B_{\mathrm{Ideal}}^{(i)}(\mathcal{H}_{A_{n-1}}^{(R,q)})$	block ideal of $\mathcal{H}_{A_{n-1}}^{(R,q)}$ indexed by an arbitrary number (see page 70)
$B_{\mathrm{Kat}}^{(i)}(\mathcal{H}_{A_{n-1}}^{(R,q)})$	block category of $\mathcal{H}_{A_{n-1}}^{(R,q)}$ indexed by an arbitrary number (see page 70)
$b_{\mathrm{Idemp}}^{\mu}(\mathcal{H}_{A_{n-1}}^{(R,q)})$	block idempotent of $\mathcal{H}_{A_{n-1}}^{(R,q)}$ indexed by a core (see page 73)
$B_{\mathrm{Ideal}}^{\mu}(\mathcal{H}_{A_{n-1}}^{(R,q)})$	block ideal of $\mathcal{H}_{A_{n-1}}^{(R,q)}$ indexed by a core (see page 73)
$B_{\mathrm{Kat}}^{\mu}(\mathcal{H}_{A_{n-1}}^{(R,q)})$	block category of $\mathcal{H}_{A_{n-1}}^{(R,q)}$ indexed by a core (see page 73)
$B^{\mu}(\mathcal{H}_{A_{n-1}}^{(R,q)})$	μ-block of $\mathcal{H}_{A_{n-1}}^{(R,q)}$ (see page 73)
$B^{\mu}(n)$	μ-block of $\mathcal{H}_{A_{n-1}}^{(R,q)}$ (see page 74)
$g_{e_F(\bar{a})}(B^{\mu}(n))$	$e_F(\bar{a})$-weight of $B^{\mu}(n)$ (see page 74)
$b_{\mathrm{Proj}}^{\mu}(\mathcal{H}_{A_{n-1}}^{(R,q)})$	block projection on $G_0(\mathcal{H}_{A_{n-1}}^{(R,q)})$ (see page 75); also denotes the block projection on $K_0(\mathcal{H}_{A_{n-1}}^{(R,q)})$ (see page 75)
$\Delta_{n,\mathcal{K}}^{\mathcal{H}}(\nu)$	ν-block of $\Delta_{n,\mathcal{K}}^{\mathcal{H}}$ (see page 78)
$\chi^{(m)}$	partition $(m-1,1)$ (see page 81)
$i_{\mathcal{H}_{n-1}^{(R,q)}}^{\mathcal{H}_{n}^{(R,q)}}$	standard inclusion of $\mathcal{H}_{A_{n-2}}^{(R,q)}$ into $\mathcal{H}_{A_{n-1}}^{(R,q)}$ (see page 82)
$M\Big\uparrow_{\mathcal{H}_{n-1}^{(R,q)}}^{\mathcal{H}_{n}^{(R,q)}}$	$\mathcal{H}_{A_{n-1}}^{(R,q)}$-module induced from an $\mathcal{H}_{A_{n-2}}^{(R,q)}$-module (see page 82)
$\cdot\Big\uparrow_{\mathcal{H}_{n-1}^{(R,q)}}^{\mathcal{H}_{n}^{(R,q)}}$	induction homomorphism from $G_0(\mathcal{H}_{A_{n-2}}^{(R,q)})$ to $G_0(\mathcal{H}_{A_{n-1}}^{(R,q)})$ (see page 83); also denotes the induction homomorphism from $K_0(\mathcal{H}_{A_{n-2}}^{(R,q)})$ to $K_0(\mathcal{H}_{A_{n-1}}^{(R,q)})$ (see page 83)
$E_{\lambda,\eta}$	constant used in the derivation of an upper bound for decomposition numbers of Hecke algebras of type A (see page 91)
H_{λ}	constant used in the derivation of an upper bound for decomposition numbers of Hecke algebras of type A (see page 91)

J_ν	constant used in the derivation of an upper bound for decomposition numbers of Hecke algebras of type A (see page 91)
U_{n-1}	constant used in the derivation of an upper bound for decomposition numbers of Hecke algebras of type A (see page 91)
$I^{(j)}$	ideal in a discrete valuation ring used in the construction of Jantzen filtrations for Specht modules (see page 94)
$S^\lambda_{(S,a)}(j)$	$\mathcal{H}^{(S,a)}_{A_{n-1}}$-submodule of $S^\lambda_{(S,a)}$ used in the construction of the Jantzen filtration for $S^\lambda_{(F,\bar{a})}$ (see page 94)
$\overline{S^\lambda_{(S,a)}(j)}$	term in the Jantzen filtration of $S^\lambda_{(F,\bar{a})}$ (see page 95)
a_ν	coefficient used in the calculation of decomposition numbers for Hecke algebras of type A with the Theorem of Schaper (see page 98)
b_ν	coefficient used in the calculation of decomposition numbers for Hecke algebras of type A with the Theorem of Schaper (see page 98)
$\langle i\rangle_{\mathfrak{a}}$	angle notation for a partition in a block of weight 3 (see page 103)
$\langle i,i\rangle_{\mathfrak{a}}$	angle notation for a partition in a block of weight 3 (see page 103)
$\langle i,j\rangle_{\mathfrak{a}}$	angle notation for a partition in a block of weight 3 (see page 103)
$\langle i,i,i\rangle_{\mathfrak{a}}$	angle notation for a partition in a block of weight 3 (see page 103)
$\langle i,i,j\rangle_{\mathfrak{a}}$	angle notation for a partition in a block of weight 3 (see page 104)
$\langle i,j,k\rangle_{\mathfrak{a}}$	angle notation for a partition in a block of weight 3 (see page 104)
L_i	constant used in the derivation of an upper bound for decomposition numbers of Hecke algebras of type A (see page 105)
$\mathfrak{z}$	particular abacus for the partition (0) (see page 106)

p_K	additive order of 1_K in a field K (see page 112)
$\lambda^{\mathbf{t}}$	composition associated to a tableau (see page 122)
$[\lambda^{\mathbf{t}}]$	diagram associated to a tableau (see page 122)
$\mathbf{t}\Downarrow_m^n$	target restriction of a tableau (see page 122)
$\mathbf{s} \leq \mathbf{t}$	total ordering relation between elements of $\mathcal{T}^{\Xi_n}_{\text{row std}}$ (see page 124)
$\mathbf{s} \trianglelefteq \mathbf{t}$	partial ordering relation between elements of $\mathcal{T}^{\Xi_n}_{\text{row std}}$ (see page 124)
$\mathbf{s} \preceq \mathbf{t}$	partial ordering relation between elements of $\mathcal{T}^{\lambda}_{\text{row std}}$ with a $\lambda \vDash n$ (see page 124)
$u \preceq w$	weak Bruhat ordering between elements of $\mathfrak{S}_n$ (see page 130)
$\mathfrak{C}_n$	set of general reflections in $\mathfrak{S}_n$ (see page 130)
$u \trianglelefteq w$	(strong) Bruhat ordering between elements of $\mathfrak{S}_n$ (see page 131)
$d \leq f$	total ordering relation between elements of $\mathcal{D}_\lambda$ with a $\lambda \vDash n$ (see page 132)
$\mu^{\#}\mu$	PK_n-pair (see page 133)
00	special PK_n-pair (see page 133)
$\mu^{\#}A_c$	image of the partition $\mu^{\#}$ from the PK_n-pair $\mu^{\#}\mu$ under the operator A_c (see page 135)
μR_c	image of the composition μ from the PK_n-pair $\mu^{\#}\mu$ under the operator R_c (see page 135)
$\mu^{\#}\mu R_c$	image of the PK_n-pair $\mu^{\#}\mu$ under the operator R_c (see page 135)
$\mu^{\#}A_c\mu$	image of the PK_n-pair $\mu^{\#}\mu$ under the operator A_c (see page 135)
$\mathbf{t}^{\mu^{\#}\mu}$	μ-tableau whose entries are arranged in ascending order first by columns within $\left[\mu^{\#}\right]$ and then by rows within $[\mu] \setminus \left[\mu^{\#}\right]$ (see page 138)
$w_{\mu^{\#}\mu}$	permutation mapping $\mathbf{t}^{\mu}$ to $\mathbf{t}^{\mu^{\#}\mu}$ (see page 138)
$g_{\mu^{\#}\mu}$	permutation mapping $\mathbf{t}^{\mu^{\#}\mu}$ to $\mathbf{t}_{\mu}$ (see page 139)

$\mathbf{t}\Big\downarrow_{[\nu]}^{[\lambda]}$	source restriction of a tableau (see page 139)
$(i,j)\mathbf{t}\Big\downarrow_{[\nu]}^{[\lambda]}$	entry in the source restriction of a tableau (see page 139)
$U_{\mu^\#\mu}$	subgroup of $\mathfrak{S}_n$ permuting only the entries in each column of $\mathbf{t}^{\mu^\#\mu}\Big\downarrow_{[\mu^\#]}^{[\mu]}$ amongst themselves (see page 141)
$V_{\mu^\#\mu}$	subgroup of $\mathfrak{S}_n$ permuting only the entries in each column of $\mathbf{t}_\mu\Big\downarrow_{[\mu^\#]}^{[\mu]}$ amongst themselves (see page 141)
$U_{\mu^\#\mu}^{(j)}$	direct factor of $U_{\mu^\#\mu}$ (see page 142)
m_i	number of lattice points in the i-th column of a composition (see page 143)
m_k^+	number of lattice points in the first k columns of a composition (see page 143)
$V_{\mu^\#\mu}^{(j)}$	direct factor of $V_{\mu^\#\mu}^{(j)}$ (see page 143)
$I_{\mu^\#\mu c}$	subset of $\mathfrak{S}_n$ permuting only the entries in the c-th row of $\mathbf{t}^\mu$ such that the images of the left $\mu_c - \mu_c^\#$ entries and those of the the right $\mu_c^\#$ entries respectively are arranged in ascending order from left to right (see page 151)
$f_{\mu^\#\mu c}$	element of $I_{\mu^\#\mu c}$ moving the left $\mu_c^\#$ entries in the c-th row of $\mathbf{t}^\mu$ to the right end of that row and the right $\mu_c - \mu_c^\#$ entries to the left end (see page 157)
$\mathbf{t}(\zeta)$	λ-tableau associated to $\zeta \in \mathcal{Z}^\lambda$ via good and bad entries (see page 163)
$\mathbf{t}(\zeta)^j$	partially filled diagram of a composition used in the construction of $t(\zeta)$ (see page 163)
$\nu(\zeta)_j$	number of good entries in the j-th row of $\mathbf{t}(\zeta)$ (see page 164)
$\nu(\zeta)$	partition associated to $\zeta \in \mathcal{Z}^\lambda$ via good and bad entries (see page 164)
$g(\zeta)$	permutation associated to $\zeta \in \mathcal{Z}^\lambda$ via good and bad entries (see page 164)
$\mathcal{Z}^{\mu^\#\mu}$	set of all $\zeta \in \mathcal{Z}^\mu$ satisfying $\left[\mu^\#\right] \subseteq [\nu(\zeta)]$ (see page 167)
$\mathcal{Z}^{00}$	empty set — this is a special case of $\mathcal{Z}^{\mu^\#\mu}$ (see page 167)

$\mathcal{J}_{\mu^\#\mu c}(\zeta)$	image of $\zeta \in \mathcal{Z}^{\mu^\#\mu} \setminus \mathcal{Z}^{\mu^\# A_c \mu}$ under $\mathcal{J}_{\mu^\#\mu c}$ (see page 171)
$\mathcal{J}_{\mu^\#\mu c}$	bijection from $\mathcal{Z}^{\mu^\#\mu} \setminus \mathcal{Z}^{\mu^\# A_c \mu}$ to $\mathcal{Z}^{\mu^\#\mu R_c}$ (see page 171)
$Y(\zeta)$	ZNL-subgroup of $\mathfrak{S}_n$ with $\zeta \in \mathcal{Z}^\lambda$ for certain $\lambda \vDash n$ (see page 173)
$S^{\mu^\#\mu}_{(R,q)}$	PK$_n$-module of $\mathcal{H}^{(R,q)}_{A_{n-1}}$ associated to the PK$_n$-pair $\mu^\#\mu$ (see page 182)
$S^{00}_{(R,q)}$	null ideal in $\mathcal{H}^{(R,q)}_{A_{n-1}}$ — this is a special case of $S^{\mu^\#\mu}_{(R,q)}$ (see page 182)
$\Psi^{(R,q)}_{\mu^\#\mu c}$	PK$_n$-homomorphism associated to a PK$_n$-pair and an AR-index for it (see page 189)
$z(\zeta)_{(R,q)}$	ZNL-element in $M^\lambda_{(R,q)}$ associated to $\zeta \in \mathcal{Z}^\lambda$ for certain $\lambda \vDash n$ (see page 193)
$\mathbf{B}^{S^{\mu^\#\mu}}_{\text{ZNL}}(R,q)$	ZNL-basis of $S^{\mu^\#\mu}_{(R,q)}$ for $\mu^\#\mu \neq 00$ (see page 210)
W_{B_n}	Weyl group of type B_n (see page 220)
C_2	cyclic group of order 2 (see page 220)
$G \wr H$	wreath product of groups (see page 220)
$\ell_{B_n}(w)$	length of $w \in W_{B_n}$ (see page 220)
$\cdot^{\overset{m}{\leftarrow}}$	left inclusion of $W_{A_{m-1}}$ into $W_{A_{n-1}}$ (see page 221); also denotes the left inclusion of $\mathcal{H}^{(R,q)}_{A_{m-1}}$ into $\mathcal{H}^{(R,q)}_{A_{n-1}}$ (see page 226)
$\cdot^{\overset{m}{\rightarrow}}$	right inclusion of $W_{A_{m-1}}$ into $W_{A_{n-1}}$ (see page 221); also denotes the right inclusion of $\mathcal{H}^{(R,q)}_{A_{m-1}}$ into $\mathcal{H}^{(R,q)}_{A_{n-1}}$ (see page 227)
$W_{A_{n-1}} \hookrightarrow W_{B_n}$	inclusion of $W_{A_{n-1}}$ into W_{B_n} through identification of simple reflections (see page 221)
(λ, μ)	a-bi-composition of n with $\lambda \vDash a$ and $\mu \vDash n-a$ (see page 222); also denotes an a-bi-partition of n with $\lambda \vdash a$ and $\mu \vdash n-a$ (see page 223)
$(\lambda^\#\lambda, \mu^\#\mu)$	a-bi-PK$_n$-pair with a PK$_a$-pair $\lambda^\#\lambda$ and a PK$_{n-a}$-pair $\mu^\#\mu$ (see page 223)
$(00, \mu^\#\mu)$	0-bi-PK$_n$-pair with a PK$_n$-pair $\mu^\#\mu$ (see page 223)
$(\lambda^\#\lambda, 00)$	n-bi-PK$_n$-pair with a PK$_n$-pair $\lambda^\#\lambda$ (see page 223)

$(00, 00)$	special bi-PK-pair (see page 223)
$(\lambda^{\#}\lambda, \mu^{\#}\mu)^{(c)}A$	image of the bi-PK$_n$-pair $(\lambda^{\#}\lambda, \mu^{\#}\mu)$ with $\lambda^{\#}\lambda \neq 00$ under the operator $^{(c)}A$ (see page 224)
$(\lambda^{\#}\lambda, \mu^{\#}\mu)A^{(c)}$	image of the bi-PK$_n$-pair $(\lambda^{\#}\lambda, \mu^{\#}\mu)$ with $\mu^{\#}\mu \neq 00$ under the operator $A^{(c)}$ (see page 224)
$(\lambda^{\#}\lambda, \mu^{\#}\mu)^{(c)}R$	image of the bi-PK$_n$-pair $(\lambda^{\#}\lambda, \mu^{\#}\mu)$ with $\lambda^{\#}\lambda \neq 00$ under the operator $^{(c)}R$ (see page 224)
$(\lambda^{\#}\lambda, \mu^{\#}\mu)R^{(c)}$	image of the bi-PK$_n$-pair $(\lambda^{\#}\lambda, \mu^{\#}\mu)$ with $\mu^{\#}\mu \neq 00$ under the operator $R^{(c)}$ (see page 224)
$w_{a,n-a}$	a times shift permutation in $\mathfrak{S}_n$ (see page 224)
$\mathcal{H}_{B_n}^{(R,q,Q)}$	Hecke algebra of type B_n over the coefficient triple (R, q, Q) (see page 226)
$\mathcal{H}_{A_{n-1}}^{(R,q)} \hookrightarrow \mathcal{H}_{B_n}^{(R,q,Q)}$	inclusion of $\mathcal{H}_{A_{n-1}}^{(R,q)}$ into $\mathcal{H}_{B_n}^{(R,q,Q)}$ through identification of defining basis elements (see page 227)
$h_{a,n-a}^{(R,q)}$	a times shift permutation in $\mathcal{H}_{A_{n-1}}^{(R,q)}$ (see page 230)
$u_m^{+}(R, q, Q)$	"unsigned" sum over the base group of W_{B_n} in $\mathcal{H}_{B_n}^{(R,q,Q)}$ (see page 231)
$u_m^{-}(R, q, Q)$	"signed" sum over the base group of W_{B_n} in $\mathcal{H}_{B_n}^{(R,q,Q)}$ (see page 231)
$v_{a,n-a}^{(R,q,Q)}$	"partially signed" sum over the base group of W_{B_n} in $\mathcal{H}_{B_n}^{(R,q,Q)}$ (see page 231)
$v_{0,0}^{(R,q,Q)}$	additive neutral element of $\mathcal{H}_{B_n}^{(R,q,Q)}$ — this is a special case of $v_{a,n-a}^{(R,q,Q)}$ (see page 232)
$M_{(R,q,Q)}^{(\lambda,\mu)}$	bi-permutation module of $\mathcal{H}_{B_n}^{(R,q,Q)}$ associated to the a-bi-composition (λ, μ) with $a \in \{1, \ldots, n-1\}$ (see page 236)
$M_{(R,q,Q)}^{((0),\mu)}$	bi-permutation module of $\mathcal{H}_{B_n}^{(R,q,Q)}$ associated to the 0-bi-composition $((0), \mu)$ (see page 237)
$M_{(R,q,Q)}^{(\lambda,(0))}$	bi-permutation module of $\mathcal{H}_{B_n}^{(R,q,Q)}$ associated to the n-bi-composition $(\lambda, (0))$ (see page 237)
$S_{(R,q,Q)}^{(\lambda,\mu)}$	bi-Specht module of $\mathcal{H}_{B_n}^{(R,q,Q)}$ associated to the a-bi-partition (λ, μ) with $a \in \{1, \ldots, n-1\}$ (see page 239)
$S_{(R,q,Q)}^{((0),\mu)}$	bi-Specht module of $\mathcal{H}_{B_n}^{(R,q,Q)}$ associated to the 0-bi-partition $((0), \mu)$ (see page 240)

$S^{(\lambda,(0))}_{(R,q,Q)}$	bi-Specht module of $\mathcal{H}^{(R,q,Q)}_{B_n}$ associated to the n-bi-partition $(\lambda,(0))$ (see page 240)
$S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}_{(R,q,Q)}$	a-bi-PK$_n$-module of $\mathcal{H}^{(R,q,Q)}_{B_n}$ associated to the a-bi-PK$_n$-pair $(\lambda^{\#}\lambda,\mu^{\#}\mu)$ with $a\in\{0,\ldots,n\}$ (see page 242)
$S^{(00,\mu^{\#}\mu)}_{(R,q,Q)}$	0-bi-PK$_n$-module of $\mathcal{H}^{(R,q,Q)}_{B_n}$ associated to the 0-bi-PK$_n$-pair $(00,\mu^{\#}\mu)$ (see page 242)
$S^{(\lambda^{\#}\lambda,00)}_{(R,q,Q)}$	n-bi-PK$_n$-module of $\mathcal{H}^{(R,q,Q)}_{B_n}$ associated to the n-bi-PK$_n$-pair $(\lambda^{\#}\lambda,00)$ (see page 242)
$S^{(00,00)}_{(R,q,Q)}$	null ideal in $\mathcal{H}^{(R,q,Q)}_{B_n}$ — this is a special case of $S^{(\lambda^{\#}\lambda,\mu^{\#}\mu)}_{(R,q,Q)}$ (see page 242)
${}^{(c)}\Psi_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q)$	a-bi-PK$_n$-homomorphism associated to the a-bi-PK$_n$-pair $(\lambda^{\#}\lambda,\mu^{\#}\mu)$ with $a\in\{1,\ldots,n-1\}$ and the AR-index c for $\lambda^{\#}\lambda$ (see page 247)
$\Psi^{(c')}_{(\lambda^{\#}\lambda,\mu^{\#}\mu)}(R,q,Q)$	a-bi-PK$_n$-homomorphism associated to the a-bi-PK$_n$-pair $(\lambda^{\#}\lambda,\mu^{\#}\mu)$ with $a\in\{1,\ldots,n-1\}$ and the AR-index c' for $\mu^{\#}\mu$ (see page 247)
$\Psi^{(c)}_{(00,\mu^{\#}\mu)}(R,q,Q)$	0-bi-PK$_n$-homomorphism associated to the 0-bi-PK$_n$-pair $(00,\mu^{\#}\mu)$ and the AR-index c for $\mu^{\#}\mu$ (see page 248)
${}^{(c')}\Psi_{(\lambda^{\#}\lambda,00)}(R,q,Q)$	n-bi-PK$_n$-homomorphism associated to the n-bi-PK$_n$-pair $(\lambda^{\#}\lambda,00)$ and the AR-index c' for $\lambda^{\#}\lambda$ (see page 248)

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