Sketched Stable Planes

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Abstract

Standard objects in classical (topological) geometry are the real affine and hyperbolic planes. Both of them can be seen as (open) subplanes of the real projective plane (endowed with the standard topology) and thus share a common theory. This may serve as a brief illustration of the importance of the notion of *embeddability*.

One particularly nice class of topological planes are the so called *stable planes* – in fact, the above examples are stable planes; as well as the projective planes over the real and complex numbers, Hamilton quaternions and Cayley octaves, the so called *classical planes*. Moreover, every open subplane of a stable plane again is a stable plane. Consequently, one way of understanding a given stable plane is trying to embed it into one of more profound acquaintanceship, preferredly one of the classical planes.

An elegant way of constructing stable planes uses *stable partitions* of Lie groups. Planes of that type can be treated more efficiently studying these groups along with certain stabilisers, the so called *sketches*, rather than the original geometries. This method has so far yielded results in several cases where intrinsic methods had not been gratifying.

Maier in his dissertation gives a classification of all 4-dimensional connected Lie groups which allow for a stable partition. Only one of them, the Frobenius group $\Gamma = \mathbb{R} \ltimes \mathrm{Hei}_3\mathbb{R}$, had not been expected, and it hosts an infinite number of stable partitions. Our objective is whether or not the resulting stable planes $\mathcal{P}$ are embeddable into an already well known plane. Using sketches, it can be proved that none of these planes $\mathcal{P}$ is embeddable into the classical projective plane $\mathcal{P}_2\mathbb{C}$. As an interesting counterpoint, those planes — hostile as they are towards being embedded into classical planes — do contain an abundance of both, affine and non-affine 2-dimensional classical subplanes.

The full automorphism group Σ of such a plane $\mathcal{P}$ does *not* contain a certain selection of classical groups. Some conclusions can be drawn as to how soluble Σ is : either it is soluble or it contains one copy of a subgroup with Lie algebra $\mathfrak{sl}_2\mathbb{R}$. The normaliser $\mathrm{N}_\Sigma(\Gamma)$ of Γ in Σ turns out to be soluble, after all.

On a more general basis, the interplay of being a sketched geometry and a stable plane is studied : Is there any particular reason why all the examples of sketched stable planes so far have been point homogeneous geometries ? And indeed, any line homogeneous sketched stable plane is necessarily flag homogeneous.

Zusammenfassung

Der Begriff der *stabilen Ebenen* verallgemeinert alltägliche klassische Ebenen wie die reelle affine Ebene oder die reelle hyperbolische Ebene. Besonders schöne Exemplare lassen sich aus Gruppen mit gewissen Partitionen konstruieren, die sogenannten *skizzierten* stabilen Ebenen. Die Gruppen, die solche *stabilen Partitionen* zulassen, sind sehr häufig Liegruppen und haben nach einem Satz von Löwen die Dimension 2, 4, 8 oder 16.

Maier klassifizierte alle vierdimensionalen Liegruppen mit stabilen Partitionen. Die stabilen Ebenen, die sich aus den vier Kandidaten ergeben, sind wohlbekannt — mit Ausnahme derer, die aus der Frobeniusgruppe $\Gamma = \mathbb{R} \ltimes \mathrm{Hei}_3\mathbb{R}$ entstehen. Diese Familie von Ebenen wird hier näher beleuchtet.

Neben dem erwähnten Konstruktionsmechanismus spielt der Begriff der *Einbettung* eine tragende Rolle. Beispielsweise lassen sich die affine und auch die hyperbolische reelle Ebene als offene Unterebenen einbetten in die reelle projektive Ebene, erschließen sich mithin dem gemeinsamen Zugriff mit Hilfe nur einer Theorie. Umgekehrt ist jede offene Unterebene einer stabilen Ebene wiederum eine stabile Ebene. Auf diesem Wege kann man sich also mit einer fremden stabilen Ebene vertraut machen — indem man nämlich eine bekannte Ebene findet, in die sie einbettbar wäre. Die begehrtesten "Betten" sind natürlich die *klassischen stabilen Ebenen*, also die projektiven Ebenen über den reellen Zahlen, den komplexen Zahlen, den Hamiltonschen Quaternionen und den Cayleyschen Oktaven.

Es wird nachgewiesen, daß keine der aus Γ konstruierten stabilen Ebenen auf irgendeinem Wege in die vierdimensionale komplexe projektive Ebene eingebettet werden kann. Dieses Ergebnis schränkt die Suche nach der vollen Automorphismengruppe Σ einer solchen Ebene deutlich ein : gewisse klassische Gruppen können nicht als Automorphismengruppen auftauchen, und mithin ist Σ entweder selbst auflösbar oder enthält genau ein Exemplar einer Untergruppe mit Liealgebra $\mathfrak{sl}_2\mathbb{R}$. Ihr Normalisator $\mathrm{N}_\Sigma(\Gamma)$ ist auflösbar.

Umgekehrt ergibt sich, daß diese Ebenen selber eine Vielzahl von zweidimensionalen Unterebenen enthalten, die affine oder nichtaffine Unterebenen der reellen affinen Ebene sind.

In allgemeinerem Kontext wird ausgeleuchtet, weshalb bislang keine anderen als punkthomogene skizzierte stabile Ebenen bekannt sind: jede geradenhomogene skizzierte stabile Ebene ist notwendigerweise bereits fahnenhomogen.

Contents

Preface

Mathematics has a long and fruitful tradition of translating problems from one of its areas into another one which has already been understood more deeply and which might shed a new light on the subject at hand. The translation mechanism which will be the central thread running through the present thesis is called *sketching*, and it helps to understand problems dealing with geometries by studying their transformation groups along with suitable stabilisers.

It is as early as 1927 that Young in [77] introduces group partitions of possibly non-abelian groups. In 1954, this notion is taken up by André [2] who uses them for the construction of certain point homogeneous geometries, his so-called "translation structures". (In our notation, thus, he treats incidence structures of the form $\mathbb{P}(\Gamma; \{1\}, \mathcal{F})$, where $\mathcal{F}$ is a group partition of the abstract group Γ.) André characterises translation planes as being precisely those translation structures which arise from planar partitions ("congruences") of a necessarily abelian group. In 1951, Freudenthal in a little aside in [13] hints at the possibility of constructing flag homogeneous geometries on planes from a group along with two of its subgroups. The same train of thought is developed in 1961 by Higman and McLaughlin in [22].

Stroppel [59] in 1992 gives a useful generalisation of the method to geometries on planes which are not flag homogeneous. Finally in 1993, Stroppel [60] takes a categorical point of view and introduces a reconstruction method for geometries with an arbitrary number of types (point, lines, ...). Moreover, he establishes that the method is a reconstruction method, indeed : any geometry satisfying suitable homogeneity conditions, a so-called *sketched geometry*, is fully determined by its transformation group along with certain stabilisers, its so-called *sketch*.

Applications of this translation technique have been highly rewarded in quite a number of cases where mere study of the geometrical problems had not been successful. A beautiful recent example is Grundhöfer, Kramer and Knarr's classification [16]+[17] of flag homogeneous compact connected polygons. Further success could be noted in [57], [61], [63], [65] and [72].

The present thesis wishes to add to the applications of Stroppel's reconstruction method to certain "topological planes". Topological geometry studies incidence geometries which are endowed with a topology that in a certain sense is compatible with the geometric operations. Very little can be said about topological planes in general, though, and they may escape far from the scope of classical planes. It is thus desirable to impose further topological or homogeneity conditions.

Salzmann and his school have been primarily concerned with the classification of compact connected projective planes, where the key is given by the dimension of their automorphism group. Yet, other types of topological planes arose to the left and right of their way, one of which will be studied here : In 1976, Löwen [31] coined the notion of *stable planes*, i.e., topological linear spaces where planarity was modelled by an additional "stability axiom". It in particular covers the classical projective, affine and hyperbolic planes. Additional topological hypotheses – local compactness and finite positive topological dimension – stimulate surprisingly strong results on their actual dimension and also on their automorphism groups; see [39], [32].

Every open subplane of a stable plane is a stable plane on its own. For instance, the real affine plane $\mathcal{A}_2\mathbb{R}$ and the real hyperbolic plane $\mathrm{IH}\,\mathbb{R}$ are both open subplanes of the real projective plane $\mathcal{P}_2\mathbb{R}$ and thus share a common theory. This is why, given an unknown example of the species, it is an obvious question if it could be established as an open subplane of some well-known, preferredly classical stable plane. In that way, *embeddability problems* mark their appearance on stage. And it is here that things come to full circle : Stroppel's mechanism of translating difficult problems on geometries into a corresponding question on their sketches has also lead the way towards quite rewarding results on embeddability. Stroppel [61], for instance, achieves an embedding of Strambach's 2-dimensional $\mathrm{SL}_2\mathbb{R}$-plane into Löwen's 4-dimensional $\mathrm{SL}_2\mathbb{C}$-plane.

We will join a brief guide to the actual parts of the present thesis. The general structure is thus that Chapter 1 provides for the fundamental notions and results on sketched geometries and stable planes. Chapter 2 is independent of all the other parts and deals with the general theory of sketched stable planes, whereas the remaining chapters treat one particular family of sketched stable planes. Chapters 3 and 4 are parallel studies of embeddability problems. Chapter 5 relies upon the results from chapter 3 and interprets their relevance for the study of the automorphism group of the planes under review.

Chapter 2 asks for the interplay of being a stable plane and being a sketched geometry. A sketched linear space is necessarily point *or* line homogeneous. Yet a huge dominance of the point homogeneous species has been observed, and for a good reason : it can be proved that a line homogeneous sketched stable plane must be flag homogeneous. The proof is based on Löwen's classification [38] of stable planes with at least two isotropic points, which yields a list of candidates. In fact, a line homogeneous sketched stable plane $(\Gamma, \mathcal{P})$ entirely consists of Γ-isotropic points. Some of Löwen's candidates will be disqualified due to possession of non-isotropic points. For the remaining ones flag homogeneity can be established.

Chapters 3 through 5 take up the main issue of the present thesis : *Peter planes*. Such were baptised those stable planes which arise from stable partitions of the Lie group $\Gamma = \mathbb{R} \ltimes \mathrm{Hei}_3\mathbb{R}$, stemming from Maier's classification [44] of 4-dimensional connected Lie groups allowing for stable partitions. Maier proves that there are four such groups, and the stable planes three of them give rise to are well-known. It is the fourth group Γ and the corresponding stable planes which have so far evaded any firm grip.

After a brief introduction to the subject, Chapter 3 deals with the question whether or not Peter planes can be embedded as open subplanes into the classical 4-dimensional projective plane $\mathcal{P}_2\mathbb{C}$. The answer may be anticipated straight away : *No*, there is no embedding of stable planes. The clue here is the access via STROPPEL's translation mechanism, which will be followed from a question on stable planes towards a corresponding question on subgroups of $\operatorname{Aut}\mathcal{P}_2\mathbb{C}$. The whole chapter is one huge proof by contradiction, setting out with the assumption of an embedding $\mathsf{H} : \mathcal{P} \to \mathcal{P}_2\mathbb{C}$ of stable planes and culminating in largely misbehaved line stabilisers in $\mathcal{P}_2\mathbb{C}$. Moreover, a brief note will explain why the same sort of argument establishes that there is no canonical Γ-equivariant embedding of $\mathcal{P}$ into the translation planes $\mathcal{T}_k$ that arise from the BETTEN spreads $\mathcal{S}_k$ of $\mathbb{C}^2$.

Chapter 4 then adds a counterpoint by producing an abundance of both, affine and non-affine, 2-dimensional subplanes of the real plane any Peter plane does contain, whilst on the other hand refusing an embedding of its precious self. More precisely, every 2-dimensional Lie subalgebra of $\mathfrak{g} = \ell\Gamma = \mathbb{R} \propto \mathrm{hei}_3\mathbb{R}$ which does not happen to be an element of the stable partition $\mathcal{S}$ gives rise to such a subplane of $\mathcal{P}$. Exemplarily, concrete embeddings of two standard 2-dimensional planes, one affine, the other non-affine, into the original Peter planes $\mathcal{P}_k = \mathbb{P}(\Gamma; \{1\}, \mathcal{S}_k^{\exp})$ are presented. The abelian and non-abelian 2-dimensional Lie subalgebras of $\mathfrak{g}$ are classified; as a matter of fact, the abelian ones are those contained in a certain hyperplane. This observation is helpful in proving that any stable partition of $\mathfrak{g}$ contains precisely one abelian fibre.

Chapter 5 is dedicated to the full automorphism group Σ of a Peter plane. The non-embeddability results from chapter 3 imply that Σ does *not* contain a selection of classical groups. Some conclusions can be drawn as to how soluble the connected component Σ^1 is : either it is soluble or it contains precisely one copy of a group with Lie algebra $\mathfrak{sl}_2\mathbb{R}$. After all, the normaliser $\mathrm{N}_\Sigma(\Gamma)$ of Γ in Σ turns out to be soluble. Consequently, if ever there is a subgroup whose Lie algebra is $\mathfrak{sl}_2\mathbb{R}$, it cannot be hidden in $\mathrm{N}_\Sigma(\Gamma)$.

I would like to express my gratitude towards all those who have helped and supported me. First of all, there are my parents who paved the way for my doing as much mathematics as I ever wished to, and then of course there are those who accompanied my path through mathematics. My warmest thanks goes out towards my supervisor, Markus Stroppel, who has always had a word of advice or encouragement, and who devotes far more than the usual share of time and energy to his students. I wish to thank Hermann Hähl for co-refereeing this thesis and for providing a helping hand whenever required. I deeply appreciate their style of doing and communicating mathematics.

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The commutative diagrams are drawn using PAUL TAYLOR's `diagrams` package.

Kurzfassung in deutscher Sprache

Es ist ein probates Mittel, Sachverhalte und Fragestellungen aus einem Teilgebiet der Mathematik in ein anderes zu übersetzen, zumal dann, wenn letzteres bereits genauer erforscht ist und die Hoffnung besteht, auf diesem Wege einen anderen Blickwinkel auf Problem und mögliche Lösungswege zu erhaschen. Der Mechanismus, um den sich die vorliegende Arbeit drehen wird, leistet die Übersetzung zwischen Inzidenzgeometrien auf der einen Seite und deren Automorphismengruppen samt einigen ausgewählten Stabilisatoren auf der anderen Seite. Bereits an dieser Stelle sei darauf hingewiesen, daß wir in Übereinstimmung mit Felix KLEINS Erlangener Programm unter einer Geometrie $(\Gamma; \mathcal{P})$ stets eine Inzidenzstruktur $\mathcal{P} = (P, \mathcal{L})$ mit Punktraum P und Geradenraum $\mathcal{L}$ zusammen mit der Wirkung einer Automorphismengruppe Γ verstehen werden.

Im Jahre 1927 taucht bei YOUNG [77] erstmalig der Begriff der Gruppenpartition möglicherweise nicht-abelscher Gruppen auf. Er wird von ANDRÉ [2] im Jahre 1954 aufgegriffen, um aus einer Gruppe Γ mit ihren Nebenklassenräumen bezüglich der Partitionselemente punkthomogene Inzidenzgeometrien $(\Gamma; (\Gamma, \Gamma/\mathcal{F}))$ zu gewinnen, die er als "Translationsstrukturen" bezeichnet. ANDRÉ erkennt, daß die Translationsebenen genau diejenigen sind, die auf diesem Wege aus einer planaren Gruppenpartition hervorgehen, und daß die Gruppe in diesem Fall notwendigerweise abelsch ist. Im Jahre 1951 deutet FREUDENTHAL in [13] die Möglichkeit an, aus einer Gruppe Γ und zwei ihrer Untergruppen, Λ und M, eine fahnenhomogene Geometrie $(\Gamma; (\Gamma/\Lambda, \Gamma/\mathsf{M}))$ zu gewinnen. Derselbe Gedanke wird auch 1961 von HIGMAN AND MCLAUGHLIN [22] verfolgt.

STROPPEL [59] arbeitete 1992 eine Verallgemeinerung auch für nicht-fahnenhomogene Geometrien aus, die er 1993 in [60] noch auf kategorientheoretische Füße stellt. In der letztgenannten Arbeit wird der Mechanismus ganz allgemein für Geometrien mit beliebig vielen Typen (Punkten, Geraden, ...) eingeführt; da für unser Anliegen der zweitypige Fall ausreicht, wollen wir es auch in der Notation dabei belassen: Sind eine Gruppe Γ und zwei Mengen $\mathcal{R}_P$ und $\mathcal{R}_\mathcal{L}$ von Untergruppen von Γ gegeben, so definiert

$$\mathbb{P}\,(\Gamma; \mathcal{R}_P, \mathcal{R}_\mathcal{L}) := (\Gamma; (\Gamma/\mathcal{R}_P, \Gamma/\mathcal{R}_\mathcal{L}))$$

eine Inzidenzgeometrie. STROPPEL weist nach, daß das ganze als Rekonstruktionsmechanismus geeigneter Geometrien tauglich ist: Eine Inzidenzgeometrie $(\Gamma; \mathcal{P})$ mit einem System R_P von Repräsentanten der Punktbahnen unter Γ und einem System $R_\mathcal{L}$ von Repräsentanten der Geradenbahnen unter Γ mit der Eigenschaft, daß $R_P \times R_\mathcal{L}$ ein Repräsentantensytem der Fahnenbahnen unter Γ ist, nennt man eine *skizzierte Geometrie*. Liegt eine solche skizzierte Geometrie vor, so bezeichnen wir

$$\mathbb{S}\,(\Gamma; \mathcal{P}) := (\Gamma; \{\Gamma_p \mid p \in R_P\}, \{\Gamma_L \mid L \in R_\mathcal{L}\})$$

als ihre *Skizze*. Stets ist $\mathbb{P}\,\mathbb{S}\,(\Gamma;\mathcal{P})$ eine zu $(\Gamma;\mathcal{P})$ isomorphe Geometrie. Umgekehrt gilt für jedes Paar $(\mathcal{R}_P,\mathcal{R}_\mathcal{L})$ wüster Haufen von Untergruppen einer Gruppe Γ, daß $\mathbb{S}\,\mathbb{P}\,(\Gamma;\mathcal{R}_P,\mathcal{R}_\mathcal{L}) = (\Gamma;\mathcal{R}_P,\mathcal{R}_\mathcal{L})$. Darüber hinaus kann man $\mathbb{P}$ und $\mathbb{S}$ mit passenden Morphismenabbildungen versehen, um zu einem Paar adjungierter Funktoren zwischen einer Kategorie SGeo^* skizzierter Geometrien und der Kategorie Sk der Skizzen zu gelangen. Wichtig in unserem Kontext ist auch, daß sowohl $\mathbb{P}$ als auch $\mathbb{S}$ Monomorphismen erhalten.

In Situationen, in denen eine rein geometrische Betrachtungsweise nicht von Erfolg gekrönt war, hat der skizzierte Übersetzungsmechanismus schon gute Dienste geleistet. Ein sehr schönes Beispiel der jüngsten Geschichte ist Grundhöfer, Knarr und Kramers Klassifikation [16]+[17] aller fahnenhomogenen kompakten zusammenhängenden verallgemeinerten Polygone $\mathcal{P}$. Dabei trat das Problem auf, daß die maximal kompakte Untergruppe K einer fahnentransitiv wirkenden Gruppe nicht notwendigerweise wieder fahnentransitiv wirken muß. Wohl aber stellte sich heraus, daß $(\mathsf{K};\mathcal{P})$ stets skizziert ist, und davon ausgehend sicherte die Rekonstruktionsmethode die erfolgreiche Weiterführung des Programms. Charakterisierungen spezieller Geometrien mit Hilfe der Rekonstruktionsmethode sind zu finden in [67], [15], [63], [65], [70], [7] und [72]. Weiteres über die Theorie punkthomogener skizzierter Geometrien findet sich in [69].

Die vorliegende Arbeit möchte einen kleinen Beitrag leisten zur Anwendung der Rekonstruktionsmethode auf spezielle topologische Ebenen. Topologische Geometrie im allgemeinen beschäftigt sich mit Geometrien, deren Punkt- und Geradenraum dergestalt topologisiert werden, daß die Topologien mit den geometrischen Operationen verträglich sind: eine *topologische Ebene* $(P,\mathcal{L})$ ist eine Ebene bestehend aus topologischen Räumen P und $\mathcal{L}$ mit der Eigenschaft, daß Verbinden und Schneiden stetig sind. Nun kann man über topologische Ebenen generell recht wenige Aussagen treffen, und auch können sie sich sehr weit entfernen von dem, was wir uns klassischerweise unter Ebenen vorstellen. Daher ist es zweckmäßig im Sinne einer schönen Theorie, weitere Bedingungen topologischer oder inzidenzgeometrischer Natur zu stellen.

Löwen [31] prägte 1976 den Begriff der *stabilen Ebene*, d.h. einer topologischen Ebene $(P,\mathcal{L})$, bei der zudem der Definitionsbereich des Schneidens offen ist in $\mathcal{L}\times\mathcal{L}$. Durchweg seien hier als stabile Ebenen solche mit lokalkompaktem Punkt- und Geradenraum sowie positiver endlicher Dimension bezeichnet, den Bezeichnungen in [31] zufolge also *stabile lp-Ebenen*. Klassische Beispiele sind die reelle projektive, affine und hyperbolische Ebene sowie deren Verwandte über $\mathbb{C}$, $\mathbb{H}$ und $\mathbb{O}$. Die topologischen Zusatzbedingungen führen zu überraschend starken Einschränkungen: nach Löwen [39] treten stabile (lp-) Ebenen ausschließlich in den Dimensionen 2, 4, 8 und 16 auf.

Jede offene Unterebene einer stabilen Ebene ist wieder eine stabile Ebene. Daher scheint der Gedanke naheliegend, angesichts eines unbekannten Exemplars der Gattung zunächst einmal zu fragen, ob es sich womöglich um eine offene Unterebene bekannter Exemplare, am liebsten gleich der klassischen Ebenen, handelt. Auch und gerade im Zusammenhang mit derartigen *Einbettungsproblemen* erwies sich der eingangs dargestellt Rekonstruktionsmechanismus als außerordentlich hilfreich. Beispielsweise gelang in Stroppel [61] die Einbettung der zweidimensionalen STRAMBACH'schen $\mathrm{SL}_2\mathbb{R}$-Ebene $\mathcal{S}_\mathbb{R}$ in ihr vierdimensionales Analogon, LÖWENS $\mathrm{SL}_2\mathbb{C}$-Ebene $\mathcal{S}_\mathbb{C}$, als stabile Ebene; seit deren Auftritt in STRAMBACHS Klassifikation [56] der $\mathbb{R}^2$-Ebenen mit dreidimensionaler Automorphismengruppe — übrigens auch unter Zuhilfenahme der Rekonstruktionsmethode für fahnenhomogene Geometrien — stand fest, daß eine Einbettung der Geometrie $(\mathrm{SL}_2\mathbb{R}, \mathcal{S}_\mathbb{R})$ in die Geometrie $(\mathrm{SL}_2\mathbb{C}, \mathcal{S}_\mathbb{C})$ nicht möglich sein würde. Weitere mit Hilfe von Skizzen gelöste Einbettungsprobleme finden sich in [57], [68] und auch in [76]; in [62] und [68] wurden Einbettungsprobleme stabiler Ebenen auch auf anderem Wege angegangen.

Eine tragende Rolle bei der Klassifikation topologischer Ebenen spielen oft deren volle Automorphismengruppen. SALZMANN und seine Schule streben stets eine Bestimmung der Ebene aus der Dimension ihrer Automorphismengruppe an. Hierbei tritt der Begriff der *kritischen Dimension* gewisser Klassen von Ebenen auf, das heißt, jene Dimension c, für die noch nicht-klassische Beispiele $\mathcal{P}$ mit $\dim \mathrm{Aut}\,\mathcal{P} = c$ existieren, jedoch $\dim \mathrm{Aut}\,\mathcal{P} > c$ bereits erzwingt, daß $\mathcal{P}$ eine klassische Ebene ist. Für vierdimensionale stabile Ebenen weist STROPPEL 1993 nach, daß die kritische Dimension $c_4 \leq 12$ ist [64, 16.4]. BICKEL [5] klassifiziert 1995 die vierdimensionalen stabilen Ebenen mit mindestens neundimensionaler Automorphismengruppe; sie sind sämtlich offene Unterebenen der komplexen projektiven Ebene $\mathcal{P}_2\mathbb{C}$. Da die BETTENschen Translationsebenen $\mathcal{T}_k$, die ein kurzes Gastspiel in Abschnitt 3.8 geben, nicht-klassische Beispiele mit achtdimensionaler Automorphismengruppe sind, ist damit $c_4 = 8$ besiegelt.

Wir wollen hier einen kurzen Überblick über die einzelnen Kapitel geben. Im Hinblick auf die Gesamtstruktur ist zu bemerken, daß Kapitel 2 eigenständig ist, wohingegen Kapitel 3 bis 5 sich mit dem gemeinsamen Hauptthema der vorliegenden Arbeit – den Peter-Ebenen – beschäftigen. Dabei sind Kapitel 3 und 4 parallelen Überlegungen zur Einbettbarkeit gewidmet, während Kapitel 5 einzufangen versucht, welches Licht die Ergebnisse aus Kapitel 3 auf die Kenntnis der Automorphismengruppen der Peter-Ebenen werfen.

Geradenhomogene skizzierte stabile Ebenen

Es ist unmittelbar einsichtig, daß ein skizzierter linearer Raum punkt- oder geradenhomogen sein muß, denn widrigenfalls würde die Skizziertheit für die Existenz nichteindeutiger Verbindungsgeraden sorgen. Nun wurden bei den skizzierten stabilen Ebenen bislang nur punkthomogene Exemplare beobachtet und daraus die Frage abgeleitet, ob dahinter System steckt. Dieser Frage wird hier nachgespürt und dargelegt, weshalb sie mit ja zu beantworten ist. Deus ex machina ist hierbei LÖWENS Klassifikation [38] der stabilen Ebenen mit mindestens zwei isotropen Punkten. In der Tat ist es einfach einzusehen, daß in einer skizzierten geradenhomogenen stabilen Ebene $(\Gamma, \mathcal{P})$ jeder Punkt p bereits Γ-isotrop sein muß, d.h. der Stabilisator Γ_p transitiv auf dem Punktbüschel in p operiert. LÖWENS Klassifikation liefert in dieser Situation eine Liste von Kandidatinnen (2.3.3), von denen allerdings noch einige ob des Besitzes nicht-isotroper Punkte disqualifiziert werden müssen. Die verbleibenden Kandidatinnen sind die affinen, projektiven und (inneren) hyperbolischen Ebenen über den reellen Zahlen $\mathbb{R}$, den komplexen

Zahlen $\mathbb{C}$, den Hamiltonschen Quaternionen $\mathbb{H}$ oder den Cayleyschen Oktaven $\mathbb{O}$. Weitere Ergebnisse aus demselben Artikel [38] ermöglichen es, die Geometrien $(\overline{\Gamma}, \mathcal{P})$ als bereits fahnenhomogen zu erkennen (2.3.7). Ein wenig Arbeit muß noch investiert werden, um dies auch für die ursprüngliche Geometrie $(\Gamma, \mathcal{P})$ nachzuweisen. Insgesamt kann somit gezeigt werden, daß jede geradenhomogene skizzierte stabile Ebene bereits fahnenhomogen ist, es in der Tat also keine rein geradenhomogenen Exemplare geben kann (2.3.28). Mit diesem Ergebnis ist in unserem Kontext auch eine Verallgemeinerung eines Ergebnisses aus Block [6] erreicht, das die Frage aufwarf, ob auch bei nicht-endlichen Geometrien die Anzahl der Geradenbahnen stets die der Punktbahnen überschreitet.

Ein Nichteinbettungssatz für Peter-Ebenen

Gegenstand des überwiegenden Teiles der vorliegenden Arbeit ist eine Familie von stabilen Ebenen, die aus stabilen Faserungen der Frobenius-Gruppe

$$\Gamma = \mathbb{R} \ltimes \mathrm{Hei}_3\mathbb{R} \cong \left\{ \begin{pmatrix} a^2 & x & z \\ & a & y \\ & & 1 \end{pmatrix} \mid a, x, y, z \in \mathbb{R},\ a > 0 \right\}$$

mit nichtabelschem Frobenius-Kern $\mathrm{Hei}_3\mathbb{R}$ hervorgeht. In [44] beschreibt Maier, welche Eigenschaften eine Gruppenpartition einer Liegruppe zu einer *stabilen Partition* machen: eine planare Partition $\mathcal{F}$ einer Liegruppe Υ erzeugt in $\mathbb{P}(\Upsilon; \{1\}, \mathcal{F})$ genau dann eine stabile Ebene, wenn sie kompakt ist bezüglich der Graßmann-Topologie. Maier klassifiziert alle vierdimensionalen zusammenhängenden Liegruppen, die eine stabile Partition zulassen, und erhält genau vier Gruppen (1.5.1 ff). Drei davon sind samt den zugehörigen stabilen Ebenen wohlbekannt: es handelt sich um Translationsebenen, die in [76] beschriebenen halben Translationsebenen und komplexe Minkowski-Ebenen. Lediglich die vierte Gruppe, eben unser Γ, gibt noch Rätsel auf. Ziel war es nun, ein wenig zur Erhellung der Lage beizutragen.

Sei hierzu $\mathcal{P}$ eine Ebene der Gestalt $\mathbb{P}(\Gamma; \{1\}, \mathcal{F})$, wobei $\mathcal{F}$ eine stabile Partition der Liegruppe $\Gamma = \mathbb{R} \ltimes \mathrm{Hei}_3\mathbb{R}$ ist. Einfachheitshalber seien diese Ebenen von nun an als *Peter-Ebenen* angesprochen. Sehr erhellend wäre es sicherlich, könnte $\mathcal{P}$ als offene Unterebene einer der klassischen Ebenen, in unserem Fall der vierdimensionalen projektiven Ebene $\mathcal{P}_2\mathbb{C}$, erkannt werden. Fragestellung des dritten Kapitels ist somit: Gibt es eine Einbettung einer beliebigen Peter-Ebene in die komplexe projektive Ebene ? Technischer gefragt: Gibt es einen Morphismus $\mathsf{H} : \mathcal{P} \to \mathcal{P}_2\mathbb{C}$ von stabilen Ebenen ? Dies ist nun eines jener Probleme auf der Ebene stabiler Ebenen, die wir lieber auf die Ebene der Skizzen transferieren. Es folgt eine ausführliche Erörterung, wie dies möglich ist, kulminierend in folgendem Kochrezept: Angenommen, es gäbe einen solchen Morphismus H. Dann existierte auch ein stetiger Gruppenmorphismus $\varepsilon : \Gamma \to \mathrm{PSL}_3\mathbb{C}$ und ein Punkt p der projektiven Ebene mit trivialem Stabilisator $\Gamma_p^\varepsilon = 1$ derart, daß der Stabilisator Γ_L^ε jeder Gerade L des Büschels in p die Dimension 2 hat (3.3.37). Umgekehrt bedeutet dies: *Will man nachweisen, daß es keinen Morphismus $\mathsf{H} : \mathcal{P} \to \mathcal{P}_2\mathbb{C}$ stabiler Ebenen gibt, muß man (1) alle möglichen stetigen injektiven Gruppenmorphismen ε von Γ nach $\mathrm{PSL}_3\mathbb{C}$*

finden, (2) für alle diese ε alle Punkte p mit trivialem Stabilisator Γ_p^ε finden und (3) für jeden dieser Punkte eine Gerade L des Büschels in p finden, für die die Dimension des Stabilisators Γ_L^ε von 2 verschieden ist. Nun verraten wir hier schon soviel, daß es am Ende keine Einbettung von $\mathcal{P}$ in $\mathcal{P}_2\mathbb{C}$ geben wird. Um dies nachzuweisen, muß also nur das Kochrezept abgearbeitet werden.

Zunächst besorgt man sich mit Hilfe der zugehörigen (auflösbaren) Liealgebren und deren Kommutatorreihen alle Kandidaten für injektive Morphismen von Liealgebren (3.4.11). Eminent hilfreich ist hier wie im späteren Verlauf die Tatsache, daß es sich bei der Exponentialfunktion $\exp : \ell\Gamma \to \Gamma$ um einen Homöomorphismus handelt. Man erfährt am Ende, daß das Bild von Γ unter ε (bis auf Konjugation) beschrieben wird durch zwei komplexe Parameter u und v:

$$\Gamma^\varepsilon = \left\{ \left[\begin{array}{ccc} a & a(tu+r) & a(s+rt(1+uv)+\frac{t^2}{2}u(1+uv)+\frac{r^2}{2}v) \\ & 1 & t(1+uv)+rv \\ & & a^{-1} \end{array} \right] \middle| \begin{array}{c} a,r,s,t \in \mathbb{R} \\ a > 0 \end{array} \right\}.$$

Um die Punktstabilisatoren zu studieren, zerlegt man den Punktraum von $\mathcal{P}_2\mathbb{C}$ zunächst einmal in die Bahnen unter Γ^ε, wobei diese Zerlegung abhängig ist von der Wahl der Parameter u und v. Untersuchung von Repräsentanten der Punktbahnen liefert dann, daß nur in vier Fällen triviale Stabilisatoren auftreten (3.6.10), nämlich bei reellem u für alle Punkte der Bahnen von $\mathbb{C}(1,i,0)$ und $\mathbb{C}(1,-i,0)$, und bei nicht-reellem u für alle Punkte der Bahnen von $\mathbb{C}(1,0,i)$ und $\mathbb{C}(1,0,-i)$.

Im Anschluß müssen die Geradenbüschel in diesen Punkten untersucht werden. Hierbei vereinfacht sich die Rechnung dadurch, daß die reellen Punkte eine Baer-Unterebene der komplexen projektiven Ebene bilden, es also ausreicht, lediglich Verbindungsgeraden des Punktes p mit reellen Punkten zu untersuchen. Hierbei finden sich in der Tat in allen vier Fällen Geraden L durch p mit $\dim \Gamma_L^\varepsilon \neq 2$. Daraus erhält man den gewünschten Widerspruch und folglich die Nichteinbettbarkeit der Peter-Ebenen in die komplexe projektive Ebene (3.7.8).

Klassische Unterebenen in Peter-Ebenen

Kontrapunktisch zum vorangehenden Kapitel befaßt sich dieses mit der Frage, ob die Peter-Ebenen selber klassische zweidimensionale Unterebenen enthalten. Vorgestellt werden in Abschnitt 4.1 als archetypische zweidimensionale skizzierte Ebenen die affine Ebene $\mathcal{A}_2\mathbb{R}$ unter der punkttransitiven Wirkung der abelschen zweidimensionalen Liegruppe $\mathsf{A} \cong \mathrm{Aff}\,\mathbb{R}$ und eine nicht-affine offene Halbebene in $\mathcal{A}_2\mathbb{R}$ unter der ebenfalls punkttransitiven Wirkung der nicht-abelschen zweidimensionalen Liegruppe $\Delta \cong \mathrm{Dil}_1\mathbb{R}$. Mit Hilfe ihrer Skizzen und der zugehörigen stabilen Faserungen ihrer Liealgebren können diese als Unterebenen der ursprünglichen Peter-Ebenen entlarvt werden (4.3). Unter *ursprünglichen Peter-Ebenen* verstehen wir hierbei jene, die aus einer Betten-Partition $\mathcal{S}_k$ der Liealgebra

$$\mathfrak{g} = \ell\Gamma \cong \left\{ \begin{pmatrix} 2t & a & c \\ & t & b \\ & & 0 \end{pmatrix} \mid t,a,b,c \in \mathbb{R} \right\}$$

hervorgehen. Nicht nur enthält generell jede beliebige Peter-Ebene solche *Baer-Unterebenen* — abgeschlossene zweidimensionale Unterebenen, die offene Unterebenen der reellen affinen Ebene sind —, sondern jeder ihrer Punkte ist in einer Vielzahl solcher Unterebenen enthalten:

Sei $\mathfrak{d}$ eine zweidimensionale Lie-Unteralgebra von $\mathfrak{g}$, die keine Faser der Partition $\mathcal{S}$ von $\mathfrak{g}$ ist. Dann induziert $\mathcal{S}$ eine stabile Partition $\mathcal{F}$ auf $\mathfrak{d}$, und die zweidimensionale stabile Ebene $\mathcal{U} := \mathbb{P}(\mathfrak{d}^{\exp}; \{1\}, \mathcal{F}^{\exp})$ ist isomorph zu einem der beiden Prototypen. Insbesondere ist $\mathcal{U}$ genau dann affin, wenn $\mathfrak{d}$ abelsch ist. Über die Skizze läßt sich $\mathcal{U}$ einbetten in die Peter-Ebene $\mathcal{P} = \mathbb{P}(\mathfrak{g}^{\exp}; \{1\}, \mathcal{S}^{\exp})$ (4.2.10).

Entscheidend für die Existenz solcher skizzierter Baer-Unterebenen, in denen ein Punkt $p = X^{\exp}$ von $\mathcal{P}$ liegt, ist also die Frage, ob das Element $X \in \mathfrak{g}$ in einer zweidimensionalen Unteralgebra liegt, die keine Faser der stabilen Partition ist. Dies leitete über zur Klassifikation der zweidimensionalen Lie-Unteralgebren von $\mathfrak{g}$ und der Frage, wie sich eine stabile Partition von $\mathfrak{g}$ überhaupt zusammensetzt. Hierzu betrachten wir zunächst die Bahnen unter $\mathrm{Aut}\,\mathfrak{g}$: die Liealgebra $\mathfrak{g}$ zerfällt in drei Typen von Bahnen, die wir der Einfachheit halber rot, gelb und grün anstreichen (4.4.4). Es stellt sich heraus, daß jeder beliebige Punkt in beliebig vielen nicht-affinen skizzierten Baer-Unterebenen enthalten ist. Bei den affinen skizzierten Baer-Unterebenen hingegen entscheidet die Farbzugehörigkeit, denn die zweidimensionalen abelschen Lie-Unteralgebren liegen komplett innerhalb der rot-gelben Hyperebene. Daher können die Punkte der grünen Bahnen überhaupt nicht in einer solchen affinen skizzierten Baer-Unterebene enthalten sein. Im Gegenzug sind rote Punkte in beliebig vielen affinen skizzierten Baer-Unterebenen enthalten (4.6.10). Für gelbe Punkte gilt es, zwei Fälle zu unterscheiden: Jedes Element $X \in \mathfrak{g}$ der gelben Bahn unter $\mathrm{Aut}\,\mathfrak{g}$ ist in genau einer abelschen zweidimensionalen Lie-Unteralgebra $\mathfrak{d}$ enthalten. Ist $\mathfrak{d}$ kein Partitionselement, können wir unverzagt eine affine skizzierte Baer-Unterebene um $X^{\exp}$ konstruieren. Ist $\mathfrak{d}$ aber just die eindeutige Faser von $\mathcal{S}$, in der das Element X liegt, ist es auf dem beschriebenen Wege nicht möglich, eine affine skizzierte Baer-Unterebene zu konstruieren, die $X^{\exp}$ enthielte. Wohl aber kann man sich in diesem Fall damit trösten, daß der Punkt $X^{\exp}$ immerhin in einer affinen Gerade enthalten ist, sprich, in einer solchen Gerade, zu der es durch jeden Punkt der Peter-Ebene genau eine Parallele gibt (4.6.5). Darüber hinaus mag es beruhigen, daß jede stabile Partition von $\mathfrak{g}$ lediglich eine abelsche Faser enthält (4.5.5).

Über die Automorphismengruppe von Peter-Ebenen

Das Nichteinbettbarkeitsresultat aus Kapitel 3 birgt Informationen über die volle Automorphismengruppe $\Sigma := \mathrm{Aut}\,\mathcal{P}$ einer beliebigen Peter-Ebene $\mathcal{P} = \mathbb{P}(\Gamma; \{1\}, \mathcal{S}^{\exp})$. Zunächst einmal wissen wir über Σ lediglich, daß sie mindestens vierdimensional ist, muß sie doch $\Gamma = \mathbb{R} \ltimes \mathrm{Hei}_3\mathbb{R}$ enthalten. Die oben erwähnte Klassifikation von Bickel [5] verrät darüber hinaus, daß $4 \leq \dim \Sigma \leq 8$, denn alle vierdimensionalen stabilen (lp-) Ebenen mit mindestens neundimensionaler Automorphismengruppe sind als offene Unterebenen der komplexen projektiven Ebene $\mathcal{P}_2\mathbb{C}$ erkannt, und eine solche ist $\mathcal{P}$ nun gerade nicht.

Eine wesentliche Rolle spielt die Tatsache, daß $\Gamma = \mathbb{R} \ltimes \mathrm{Hei}_3\mathbb{R}$ keine echten kompakten

Untergruppen enthält (5.1.31). Um dies einzusehen, zerlegt man mit Hilfe des Satzes von MAL'VEC-IWASAWA die lokalkompakte zusammenhängende Gruppe Γ in eine maximal kompakte Untergruppe M und einen Vektorgruppenanteil. Da Γ homöomorph ist zu $\mathbb{R}^4$, müssen auch alle höheren Homotopiegruppen von M trivial sein. Gelingt es zu zeigen, daß M selbst trivial ist, folgt bereits, daß Γ kompaktfrei ist. Dieses Resultat erhielte man recht schnell, jedoch teuer, mit Hilfe eines Satzes von TODA [75] (in [47]). Wir ziehen es vor, keine solch großen Geschütze aufzufahren, und betreiben ein wenig Theorie kompakter Liegruppen, um das gewünschte Ziel zu erreichen. Aufgrund der Auflösbarkeit von Γ sind wir hier in der glücklichen Lage, unsere kompakte, einfach zusammenhängende Liegruppe $\mathsf{M} \leq \Gamma$ ohne weiterführende Referenzen als trivial zu erkennen. Dennoch wäre dies auch ohne die Auflösbarkeit von M möglich, wenn zusätzlich Trivialität von $\pi_3(\mathsf{M})$ bekannt ist und man sich auf einen Satz von BOTT [9] beruft, der in der Tat TODAS Resultat zugrunde liegt: Die dritte Homotopiegruppe jeder kompakten, einfach zusammenhängenden, fasteinfachen Liegruppe Υ ist $\pi_3(\Upsilon) = \mathbb{Z}$. Diese Überlegungen wurden der eigentlichen Untersuchung der Automorphismengruppe Σ am Stück vorangestellt.

Getreu der Devise, daß diejenigen Aussagen über Peter-Ebenen am schönsten sind, die andeuten, was sie alles nicht tun, wenden wir uns der Erkenntnis zu, daß ihre Automorphismengruppen Σ gewisse Gruppen *nicht* enthalten: Von LÖWEN bereits gründlich untersucht wurden solche vierdimensionalen stabilen (lp-) Ebenen, deren Automorphismengruppe leidlich groß ist und $\mathrm{SO}_3\mathbb{R}$ oder $\mathrm{SU}_2\mathbb{C}$ enthält. Da auch solche Ebenen als offene Unterebenen von $\mathcal{P}_2\mathbb{C}$ erkannt sind, liefert unser Nichteinbettbarkeitssatz, daß die Automorphismengruppe Σ einer Peter-Ebene weder $\mathrm{SO}_3\mathbb{R}$ noch $\mathrm{SU}_2\mathbb{C}$ enthält (5.2).

Diese Erkenntnis bedingt wiederum Aussagen über die Auflösbarkeit der Zusammenhangskomponente Σ^1 der Automorphismengruppe: Da weder $\mathrm{SU}_2\mathbb{C}$ noch $\mathrm{SO}_3\mathbb{R}$ Untergruppen von Σ sind, kann auch deren Liealgebra $\ell\,\Sigma$ die dreidimensionale Liealgebra $\mathfrak{so}_3\mathbb{R} = \ell(\mathrm{SO}_3\mathbb{R}) = \ell(\mathrm{SU}_2\mathbb{C})$ nicht enthalten. Betrachtet man die Levi-Zerlegung $\ell\,\Sigma = \mathfrak{s} \propto \mathfrak{r}$ in den halbeinfachen Anteil $\mathfrak{s}$ und das auflösbare Radikal $\mathfrak{r}$ und zerlegt $\mathfrak{s} = \bigoplus_{j\in J} \mathfrak{e}_j$ in einfache Bestandteile $\mathfrak{e}_j$, stellt man fest, daß keines der Ideale $\mathfrak{e}_j$ die einfache Liealgebra $\mathfrak{so}_3\mathbb{R}$ enthalten darf. Es stellt sich also die Frage, welche reellen einfachen Liealgebren es gäbe, in denen $\mathfrak{so}_3\mathbb{R}$ nicht zu finden wäre. Wir stellen für den nichteingeweihten Leser noch kurz die nötigen Vokabeln und Zusammenhänge bereit, um der Klassifikation der (komplexen) einfachen Liealgebren, beispielsweise TITS' Tabellen [74], entnehmen zu können, daß es darauf nur eine Antwort gibt: $\mathfrak{e}_j = \mathfrak{sl}_2\mathbb{R}$. Da nach einem Resultat von LÖWEN [32] jede halbeinfache Automorphismengruppe einer vierdimensionalen (lp-) Ebene bereits fasteinfach ist, folgt nun zwingend, daß sich in $\mathfrak{s} = \bigoplus_{j\in J} \mathfrak{e}_j = (\mathfrak{sl}_2\mathbb{R})^n$ höchstens ein Exemplar $\mathfrak{sl}_2\mathbb{R}$ verbirgt (5.3.26): *Die Liealgebra $\ell\,\Sigma$ der vollen Automorphismengruppe einer Peter-Ebene ist entweder auflösbar, oder es gilt $\ell\,\Sigma = \mathfrak{sl}_2\mathbb{R} \propto \mathfrak{r}$, wobei $\mathfrak{r}$ das auflösbare Radikal bezeichne.*

Leider ist bislang ungeklärt, welcher der beiden Fälle nun tatsächlich vorliegt. Einen Schritt kann man sich der Wahrheit nähern, wenn man statt der vollen Automorphismengruppe den Normalisator $\mathsf{N} := \mathrm{N}_\Sigma(\Gamma)$ von Γ in Σ betrachtet: Die stabile Partition $\mathcal{S}$ von $\mathfrak{g} = \ell\,\Gamma$ ist invariant unter der Wirkung des Stabilisators N_{p_o} von $p_o := 1 \in \Gamma$ auf $\mathfrak{g}$. Daher wirkt N_{p_o} auf der Translationsebene $\mathbb{P}(\mathfrak{g}; \{\mathbf{0}\}, \mathcal{S})$. Da aus Kapitel 4 einiges über die Zusammensetzung der stabilen Partition $\mathcal{S}$ bekannt ist, kann man schließen, daß

N_{p_o} eine maximale Fahne fixiert und mithin auflösbar ist. Nach dem Frattini-Argument ist $\mathsf{N} = \mathsf{N}_{p_o} \ltimes \mathsf{\Gamma}$ sowie $\mathsf{N}_{p_o} \cong \mathsf{N}/\mathsf{\Gamma}$, und daraus folgt, daß der Normalisator $\mathsf{N} = \mathrm{N}_{\Sigma}(\mathsf{\Gamma})$ auflösbar ist. Folglich enthält die Liealgebra $\ell\,\mathsf{N}$ keinen $\mathfrak{sl}_2\mathbb{R}$-Anteil, und N selbst enthält keinerlei Liegruppe Υ, deren Liealgebra $\ell\,\Upsilon = \mathfrak{sl}_2\mathbb{R}$ wäre (5.3.37).

1. Foundations

1.1. Sketched geometries

1.1.1. Categories and sketched geometries

Sketched geometries as such were introduced by Stroppel in [59] and under a category theoretical point of view in [60]. The following will give a brief resumé drawn from these articles, along with some additions from chapters 1 and 2 of [76]. The present thesis will be concerned with planes only, and this is why we restrict the theory to two-typed geometries. Moreover, morphisms are expected to map points to points and lines to lines. This fortunately enables us to skip most of the distracting formalism.

It will be convenient to adopt a slightly category theoretical point of view and, accordingly, present objects along with morphisms straight away.

The idea behind "sketched geometries" is to replace the study of incidence structures by the study of transformation groups and certain subgroups. We will hence start out with various categories of planes and end up with presenting those "piles" of groups, together with the translation mechanism and some of its properties.

1.1.1 Definition. *Incidence Structures* (Inc). An **incidence structure** $(\mathcal{A}, I) = (A_P, A_\mathcal{L}, I)$ consists of a point space A_P, a line space $A_\mathcal{L}$ and an incidence relation $I \subseteq A_P \times A_\mathcal{L}$. A point $p \in A_P$ and a line $L \in \mathcal{L}$ are **incident** if $(p, L) \in I$; an incident pair (p, L) is called a **flag**.

A **morphism of incidence structures** is a pair $\mathsf{H} = (\mathsf{H}_P, \mathsf{H}_\mathcal{L}) : \mathcal{A} \to \mathcal{B}$ consisting of a point map $\mathsf{H}_P : A_P \to B_P$ and a line map $\mathsf{H}_\mathcal{L} : A_\mathcal{L} \to B_\mathcal{L}$ mapping each flag $(p, L) \in I$ again to a flag $(p^{\mathsf{H}_P}, L^{\mathsf{H}_\mathcal{L}}) \in I$.

1.1.2 Remark. By laxness of notation, we will mostly write an incidence structure as $\mathcal{A} = (A_P, A_\mathcal{L})$ whenever no special emphasis is placed on the incidence relation I. The point map H_P of a morphism H in Inc is usually referred to as **lineation**, whereas point maps of isomorphisms in Inc are called **collineations**.

The heart of the theory is the (convenient) action of groups on incidence structures. In accordance with Felix Klein's Erlangen Programme, a *geometry* in our sense will thus be an incidence structure endowed with a transformation group.

1.1.3 Definition. *Geometries* (Geo). A **geometry** is a triple $(\Gamma, \gamma; \mathcal{A})$ consisting of an incidence structure $\mathcal{A}$ and a group Γ with

point action $\quad \gamma_P : A_P \times \Gamma \to A_P$ and
line action $\quad \gamma_\mathcal{L} : A_\mathcal{L} \times \Gamma \to A_\mathcal{L}$

such that an element $\alpha \in \Gamma$ maps each flag (p, L) to a flag $((p,\alpha)^{\gamma_P}, (L,\alpha)^{\gamma_\mathcal{L}}) =: ((p, L), \alpha)^\gamma \in I$.

A **morphism of geometries** $(\varepsilon, \mathsf{H}) : (\Gamma, \gamma; \mathcal{A}) \to (\Delta, \delta; \mathcal{B})$ consists of a morphism $\mathsf{H} : \mathcal{A} \to \mathcal{B}$ of incidence structures and a group morphism $\varepsilon : \Gamma \to \Delta$ such that for both types, points $t = P$ and lines $t = \mathcal{L}$, the following diagram commutes :

$$\begin{tikzcd}
A_t \times \Gamma \arrow[r, "\gamma_t"] \arrow[d, "\mathsf{H}_t \times \varepsilon"'] & A_t \arrow[d, "\mathsf{H}_t"] \\
B_T \times \Delta \arrow[r, "\delta_t"'] & B_t
\end{tikzcd}$$

○

Now those geometries will have to be singled out which are accessible to STROPPEL's translation mechanism.

1.1.4 Definition. *Sketched Geometries* (SGeo). A **sketched geometry** $(R; \Gamma, \gamma; \mathcal{A}, I)$ is a geometry $(\Gamma, \gamma; \mathcal{A}, I)$ along with a pair $R = (R_P, R_\mathcal{L})$ of sets such that

- for each type $t \in \{P, \mathcal{L}\}$, the subset $R_t \subseteq A_t$ is a system of representatives for the action γ_t of Γ on A_t
- $R_P \times R_\mathcal{L} \subseteq I$ consists of flags, and it is a system of representatives for the flag action of Γ on I

We also say that the family R **sketches** the geometry $(\Gamma; \mathcal{A})$.

For technical reasons which will be clarified later on, we impose another condition on the objects of SGeo :

$$(*) \qquad \forall t \in \{P, \mathcal{L}\}\, \forall x, y \in R_t. \quad \Gamma_x = \Gamma_y \Longrightarrow x = y$$

It states that different representatives should have different stabilisers. The sketched geometries satisfying $(*)$ will be collected in the full subcategory SGeo^* of SGeo.

A morphism $(\varepsilon, \mathsf{H}) : (R; \Gamma; \mathcal{A}) \to (Q; \Delta; \mathcal{B})$ of geometries is called a **morphism of sketched geometries** if it maps each representative to another representative; that is, if $R_t^{\mathsf{H}_t} \subseteq Q_t$ for both types $t \in \{P, \mathcal{L}\}$. ○

1.1.5 Remark. Again, if the extra information is not emphasised we denote a sketched geometry simply by $(R; \Gamma; \mathcal{A})$. Basic examples of sketched geometries, such as affine planes and half planes, are treated later on, for instance in 4.1.10 and 4.1.5.

1.1.6 Definition. Denote by $\mathcal{U}_{\mathsf{Inc}} : \mathsf{SGeo} \to \mathsf{Inc}$ the corresponding forgetful functor which forgets all the information on transformation groups and representatives.

The promise was to extract purely group theoretical information from such a sketched geometry which enables us to reconstruct the original geometry. How to do this ? The trick is to collect the transformation group along with the stabilisers of the representatives — and then to know how to regain the information dropped.

1.1.7 Definition. *Sketches* (Sk). A **sketch** $(\Gamma;\mathcal{R})$ consists of some group Γ and arbitrary sets $\mathcal{R}_P$ and $\mathcal{R}_\mathcal{L}$ of subgroups of Γ. A **morphism** $(\varepsilon,\mathsf{E}) : (\Gamma;\mathcal{R}) \to (\Delta;\mathcal{Q})$ of sketches consists of a group morphism $\varepsilon : \Gamma \to \Delta$ and maps $\mathsf{E}_t : \mathcal{R}_t \to \mathcal{Q}_t$ for each type $t \in \{P,\mathcal{L}\}$, such that

$$\forall \Lambda \in \mathcal{R}_t. \quad \Lambda^\varepsilon \le \Lambda^{\mathsf{H}_t}.$$

○

All alone, these sketches look rather bloodless. They come to life, though, when linked with sketched geometries.

1.1.8 Definition. *The functor* $\mathbb{S}$. Consider the map

$$\begin{array}{rccc} \mathbb{S}: & \mathrm{ob}\,\mathsf{SGeo} & \to & \mathrm{ob}\,\mathsf{Sk} \\ & (R;\Gamma;\mathcal{A}) & \mapsto & (\Gamma;\mathcal{R}) \end{array}$$

where, for both types $t \in \{P,\mathcal{L}\}$, we define

$$\mathcal{R}_t := \{\Gamma_x \mid x \in R_t\}$$

as the sets of stabilisers of the chosen representatives. It can be complemented by a morphism map

$$\begin{array}{rccc} \mathbb{S}: & \mathrm{morph}\,\mathsf{SGeo} & \to & \mathrm{morph}\,\mathsf{Sk} \\ & (\varepsilon;\mathsf{H}) & \mapsto & (\varepsilon,\mathsf{E}) \end{array}$$

where

$$\begin{array}{rccc} \mathsf{E}_t: & \mathcal{R}_t & \to & \mathcal{Q}_t \\ & \Gamma_x & \mapsto & \Delta_{x^{\mathsf{H}_t}} \end{array}$$

for $t \in \{P,\mathcal{L}\}$. These definitions make up for a functor $\mathbb{S} : \mathsf{SGeo}^* \to \mathsf{Sk}$. ○

1.1.9 Remark. Note that here, the subcategory SGeo^* appears on stage : the morphism map would not be well-defined on SGeo; cf. [76, Bsp. 2.15].

The opposite direction shall be explored by another functor :

1.1.10 Definition. Let $(\Gamma;\mathcal{R})$ be a sketch. Define for each type $t \in \{P,\mathcal{L}\}$

- point and line spaces

$$A_t := \bigcup_{\Lambda \in \mathcal{R}_t} \Gamma/\Lambda = \{\Lambda\alpha \mid \Lambda \in \mathcal{R}_t,\ \alpha \in \Gamma\}$$

- an incidence relation

$$I := \{(\Lambda\alpha, \mathsf{M}\alpha) \mid \alpha \in \Gamma,\ \Lambda \in \mathcal{R}_P,\ \mathsf{M} \in \mathcal{R}_{\mathcal{L}}\}$$

- a family $\gamma = (\gamma_P, \gamma_{\mathcal{L}})$ of actions

$$\begin{array}{rccc} \gamma_t : & A_t \times \Gamma & \to & A_t \\ & (\Lambda\alpha, \beta) & \mapsto & \Lambda \cdot \alpha\beta \end{array}$$

and define the object map

$$\begin{array}{rccc} \mathbb{P} : & \operatorname{ob} \mathsf{Sk} & \to & \operatorname{ob} \mathsf{SGeo}^* \\ & (\Gamma; \mathcal{R}) & \mapsto & (\mathcal{R}; \Gamma, \gamma; \mathcal{A}, I) \end{array}.$$

A morphism $(\varepsilon, \mathsf{E}) : (\Gamma, \mathcal{R}) \to (\Delta; \mathcal{Q})$ of sketches is mapped to

$$(\varepsilon, \mathsf{H}) : \mathbb{P}(\Gamma; \mathcal{R}) \to \mathbb{P}(\Delta; \mathcal{Q}),$$

where

$$\begin{array}{rccc} \mathsf{H}_t : & \bigcup\limits_{\Lambda \in \mathcal{R}_t} \Gamma/\Lambda & \to & \bigcup\limits_{\Lambda \in \mathcal{Q}_t} \Delta/\Lambda \\ & \Lambda\alpha & \mapsto & \Lambda^{\mathsf{E}_t} \cdot \alpha^{\varepsilon} \end{array}$$

for $t \in \{P, \mathcal{L}\}$. This defines a functor $\mathbb{P} : \mathsf{Sk} \to \mathsf{SGeo}^*$. ○

1.1.11 Theorem.

a) $\mathbb{S} \circ \mathbb{P} = \mathrm{id}_{\mathsf{Sk}}$

b) $\forall \mathcal{P} \in \operatorname{ob} \mathsf{SGeo}^*.\quad \mathbb{P}\,\mathbb{S}\,\mathcal{P} \cong \mathcal{P}$

c) $\mathbb{S}$ *and* $\mathbb{P}$ *are adjoint functors.*

d) $\mathbb{S}$ *and* $\mathbb{P}$ *are full and faithful.*

e) $\mathbb{S}$ *and* $\mathbb{P}$ *preserve monomorphisms and epimorphisms.*

Proof. [60] and also [76, Chapter 2]. □

We will be frequently dealing with the search for embeddings. As a consequence, a characterisation of monomorphisms (and epimorphisms) in the various categories will be helpful. Recall that a morphism $\varphi \in \operatorname{morph}_{\mathcal{C}}(A, B)$ of a category $\mathcal{C}$ is called a **monomorphism** if for any object $X \in \operatorname{ob} \mathcal{C}$ and arbitrary morphisms $\alpha, \beta \in \operatorname{morph}_{\mathcal{C}}(X, A)$ equality $\alpha\varphi = \beta\varphi$ implies $\alpha = \beta$. It is called an **epimorphism** if for arbitrary $X \in \operatorname{ob} \mathcal{C}$ and $\alpha, \beta \in \operatorname{morph}_{\mathcal{C}}(B, X)$ equality $\varphi\alpha = \varphi\beta$ implies $\alpha = \beta$.

$$X \underset{\beta}{\overset{\alpha}{\rightrightarrows}} A \overset{\varphi}{\rightarrowtail} B \qquad\qquad A \overset{\varphi}{\twoheadrightarrow} B \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X$$

1.1.12 Lemma. *Let* $\mathsf{H} \in \operatorname{morph} \mathsf{Inc}$. *Then*

a) H *is an epimorphism if and only if* H_P *and* $\mathsf{H}_{\mathcal{L}}$ *are epimorphisms each.*

b) H *is a monomorphism if and only if* H_P *and* $\mathsf{H}_{\mathcal{L}}$ *are monomorphisms each.*

1.1.13 Lemma. *Let* $(\varepsilon, \mathsf{H}) \in \operatorname{morph} \mathsf{Geo}$. *Then*

a) $(\varepsilon, \mathsf{H})$ *is an epimorphism if and only if* ε, H_P *and* $\mathsf{H}_{\mathcal{L}}$ *are epimorphisms each.*

b) $(\varepsilon, \mathsf{H})$ *is a monomorphism if and only if* ε, H_P *and* $\mathsf{H}_{\mathcal{L}}$ *are monomorphisms each.*

1.1.14 Lemma. *Let* $(\varepsilon, \mathsf{H})$ *be a morphism in* SGeo *or* SGeo^*. *Then*

a) $(\varepsilon, \mathsf{H})$ *is an epimorphism if and only if* ε, H_P *and* $\mathsf{H}_{\mathcal{L}}$ *are epimorphisms each.*

b) $(\varepsilon, \mathsf{H})$ *is a monomorphism if* ε, H_P *and* $\mathsf{H}_{\mathcal{L}}$ *are monomorphisms each. Conversely, if* $(\varepsilon, \mathsf{H})$ *is monomorphism, then so is* ε.

c) **Warning.** *In* SGeo *and* SGeo^*, *the morphism* $(\varepsilon, \mathsf{H})$ *being monic does not necessarily imply that* H_P *and* $\mathsf{H}_{\mathcal{L}}$ *are monomorphisms. For a counterexample, consult [76, 2.7].*

1.1.15 Lemma. *Let* $(\varepsilon, \mathsf{E}) \in \operatorname{morph} \mathsf{Sk}$. *Then*

- $(\varepsilon, \mathsf{E})$ *is an epimorphism if and only if* ε, E_P *and* $\mathsf{E}_{\mathcal{L}}$ *are epimorphisms each.*
- $(\varepsilon, \mathsf{E})$ *is a monomorphism if and only if* ε, E_P *and* $\mathsf{E}_{\mathcal{L}}$ *are monomorphisms each.*

1.1.2. Homogeneity and sketched geometries

Certain homogeneity conditions interact quite smoothly with sketchedness. What follows is a loose collection of observations that will prove useful.

1.1.16 Definition. A geometry $(\Gamma, \mathcal{P}, I)$ is called **point homogeneous** if Γ acts transitively on the point space P. It is **line homogeneous** if Γ acts transitively on the line space $\mathcal{L}$. And it is **flag homogeneous** if Γ acts transitively on the flag space I.

1.1.17 Lemma. *Let* $(\Gamma, \mathcal{P})$ *be a point and line homogeneous geometry. It is sketched if and only if it is flag homogeneous.*

1.1.18 Definition. A **linear space** is an incidence structure $(P, \mathcal{L})$ with the property that any two distinct points can be joined by a unique line, every line has at least two points and there is a quadrangle.

1.1.19 Lemma. *A sketched linear space is point or line homogeneous.*

Proof. Assume a sketched geometry that is neither point nor line homogeneous. Then the systems of point and line representatives would contain at least two elements each, and by the crucial axiom of sketched geometries they would form a bigon — which is forbidden in a linear space. □

1.1.20 Lemma. *Some point homogeneous sketched geometry $\mathcal{P} = \mathbb{P}(\Gamma; \{1\}, \mathcal{F})$ is a linear space if and only if $\mathcal{F}$ is a group partition of Γ.*

Proof. (For a definition of group partitions, see 1.4.1.) If $\mathcal{P}$ is a linear space, then any point $\alpha \in \Gamma$ can be joined to the origin by a unique line $\Lambda \in \mathcal{F}$. This is as much as saying that any element of Γ is contained in a unique fibre of $\mathcal{F}$. Conversely, for any two distinct points $\alpha, \beta \in \Gamma$, a group partition $\mathcal{F}$ contains a unique fibre Λ such that $\alpha^{-1}\beta \in \Lambda$. Then α and β are joined by the unique line $\Lambda\alpha = \Lambda\beta$. □

1.1.21 Definition. Let $(\Gamma, \mathcal{P})$ be a geometry. A line $L \in \mathcal{L}$ is called **Γ-isotropic** if the (set-wise) stabiliser Γ_L acts transitively on the point row P_L. Γ-isotropy of a point is defined dually. For $\Gamma = \operatorname{Aut}\mathcal{P}$ one briefly speaks about **isotropy**.

1.1.22 Lemma. *Let $x \in P \mathbin{\dot\cup} \mathcal{L}$ be a point or a line in an arbitrary plane $(P, \mathcal{L})$, and let $\Upsilon \leq \operatorname{Aut}_{\mathsf{Inc}}(P, \mathcal{L})$ be an automorphism group of the plane. If x is Υ-isotropic then every element in its orbit x^Υ is Υ-isotropic.*

Proof. Consider some element $y = x^\gamma \in x^\Upsilon$ of the orbit, and let u and v be distinct items incident with y. Then $u^{\gamma^{-1}}$ and $v^{\gamma^{-1}}$ are distinct items incident with x. By the hypothesis, there is an element $\alpha \in \Upsilon_x$ mapping $u^{\gamma^{-1}}$ to $v^{\gamma^{-1}}$. Therefore, $u^{\gamma^{-1}\alpha\gamma} = v^{\gamma^{-1}\gamma} = v$, and $\gamma^{-1}\alpha\gamma \in \gamma^{-1}\Upsilon_x\gamma = \Upsilon_{x^\gamma} = \Upsilon_y$. Hence, y is Υ-isotropic. □

1.1.23 Lemma. *A point homogeneous geometry $(\Gamma, \mathcal{P})$ is sketched if and only if every line is Γ-isotropic. Dually, a line homogeneous geometry $(\Gamma, \mathcal{P})$ is sketched if and only if every point is Γ-isotropic.*

Proof. We will prove the first assertion; cf. [69, Lemma 4]. The proof for the dual one can be achieved by dualising. $\Longrightarrow$: Because of point homogeneity there is a point representative $p_o \in P$ and a set $R_\mathcal{L} \subseteq \mathcal{L}_{p_0}$ of line representatives. Consider a flag (p, L). There is a representative $L_0 \in R_\mathcal{L}$ and a group element $\alpha \in \Gamma$ which maps L_0 to L. Then $p_0^\alpha \in L$. Due to Γ-isotropy there is an element $\beta \in \Gamma_L$ such that $p_0^{\alpha\beta} = p$. Moreover, $L_0^{\alpha\beta} = L^\beta = L$.

$\Longleftarrow$: Let L be a line, and let p and q be points on L. Due to point homogeneity, $(\Gamma, \mathcal{P})$ can be sketched by systems R_P and $R_\mathcal{L}$ of representatives such that R_P consists of one point p_0 only. Then there are line representatives $L_0, K_0 \in R_\mathcal{L}$ and elements $\alpha, \beta \in \Gamma$ such that $(p_0, L_0)^\alpha = (p, L)$ and $(p_0, K_0)^\beta = (q, L)$; thus $L_0 = K_0$. But then $\lambda := \alpha^{-1}\beta$ moves α to β and satisfies $L^\lambda = L_0^\beta = L$, hence is contained in the stabiliser Γ_L. Thus L is Γ-isotropic. □

Most of the present thesis deals with point homogeneous sketched geometries. Yet, we will not a priori restrict our attention to them and adapt all the definitions accordingly — as do André [2] and Maier [44]. On the contrary, the entire second chapter will be dedicated to the question of line homogeneity for sketched stable (lp)-planes.

1.2. Stable planes

A topological linear space is a linear space whose point and line spaces are endowed with topologies such that the intersection of lines and the joining of points become continuous operations in their respective domains. In order to get a stronger hold on these planes, though, it is desirable to impose further conditions, be it on the incidence structure or on the topologies. The notion of "stable planes" now follows the latter concept and tries to make up for some sort of "planarity" by "stability" of the intersection operation.

1.2.1 Definition. A linear space $(P, \mathcal{L})$ is called a **stable plane** if the following holds.

(i) P and $\mathcal{L}$ are locally compact Hausdorff spaces of finite and positive (covering) dimension.

(ii) Joining of points is a continuous operation $\vee : P^2 \setminus \{(p,p) \mid p \in P\} \to \mathcal{L}$.

(iii) Intersection of lines is a continuous operation $\wedge : \operatorname{dom} \wedge \to P$.

(iv) *Stability.* The domain $\operatorname{dom} \wedge$ of intersection is an open subset of $\mathcal{L} \times \mathcal{L}$.

The concept of stable planes was coined in 1976 by Löwen [31] — yet in a more general way : P and $\mathcal{L}$ could well be arbitrary topological spaces. The objects we defined here are often addressed as **stable lp-planes**. In our context though, this confinement will be both, sufficient and handy. For a survey on stable planes we refer the reader to [31], [66] and [18]. Stable planes are a generalisation of many a classical plane.

1.2.2 Example. Stable planes.

a) *The real affine plane* $\mathcal{A}_2\mathbb{R}$
with point space $\mathbb{R}^2$ and line space $\{\mathbb{R}x + a \mid a \in \mathbb{R}^2,\ x \in \mathbb{R}^2 \setminus \mathbf{0}\}$

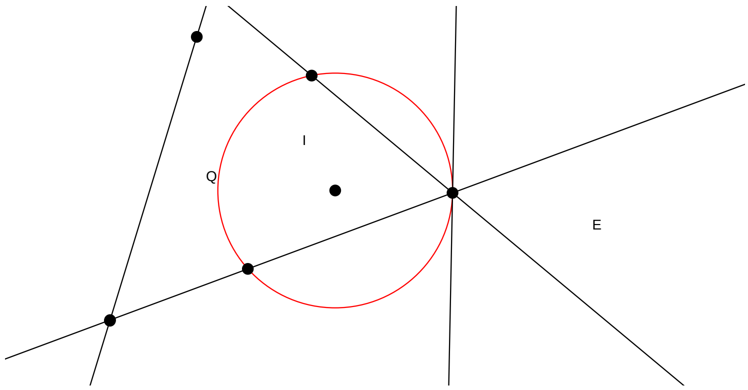
b) *The real projective plane* $\mathcal{P}_2\mathbb{R}$
with point space $\mathrm{P}_2\mathbb{R} = \{\mathbb{R}x \mid x \in \mathbb{R}^3 \setminus \mathbf{0}\} = \mathfrak{u}_1(\mathbb{R}^3)$ and line space $\mathcal{L}_2\mathbb{R} = \mathfrak{u}_2(\mathbb{R}^3)$

c) *Real hyperbolic planes.* Consider the bilinear form

$$\begin{array}{rccl} h: & \mathbb{R}^3 \times \mathbb{R}^3 & \to & \mathbb{R} \\ & (x,y) & \mapsto & xAy^{\mathsf{T}} = x_1y_1 + x_2y_2 - x_3y_3 \end{array}$$

described by the diagonal matrix $\operatorname{Diag}(1,1,-1)$. It provides three prominent open subplanes of $\mathcal{P}_2\mathbb{R}$, namely

the *interior hyperbolic plane*	$\mathrm{IH}\,\mathbb{R}$	induced on $I := \{\mathbb{R}x \mid h(x,x) < 0\}$
the *exterior hyperbolic plane*	$\mathrm{EH}\,\mathbb{R}$	induced on $E := \{\mathbb{R}x \mid h(x,x) > 0\}$
the *united hyperbolic plane*	$\mathrm{UH}\,\mathbb{R}$	induced on $U \cup E$



d) Analogues $\mathcal{A}_2\mathbb{F}$, $\mathcal{P}_2\mathbb{F}$, $\mathrm{IH}\,\mathbb{F}$, $\mathrm{EH}\,\mathbb{F}$ and $\mathrm{UH}\,\mathbb{F}$ over the skew fields $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$. Here, $h : \mathbb{F}^3 \times \mathbb{F}^3 \to \mathbb{F} : (x, y) \mapsto xAy^{\alpha\mathsf{T}}$ is an α-sesquilinear form with respect to conjugation $\alpha : \mathbb{F} \to \mathbb{F} : x \mapsto \overline{x}$.

1.2.3 *Every **open subplane** — that is, a non-empty open subset of the point space along with the induced system of lines — of a stable plane is a stable plane.* In that sense, all the examples above are open subplanes of the classical projective planes $\mathcal{P}_2\mathbb{F}$ for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. As a matter of fact, a strong similarity between arbitrary stable planes and open subplanes of the classical planes is observable. This phenomenon is referred to as *domination by classical planes*. Yet, there are examples which are not of that type; cf. [34, §5], [71, 6.2] and [20].

1.2.4 According to deep results by Löwen, every stable (lp-) plane $\mathcal{P} = (P, \mathcal{L})$ is of dimension $2l := \dim P \in \{2, 4, 8, 16\}$. The dimensions of the point space P and line space $\mathcal{L}$ coincide. Every line pencil is homotopy equivalent to the sphere $\mathbb{S}_l$; see [39]. If lines are known to be manifolds, then line pencils are homeomorphic to the sphere $\mathbb{S}_l$, and lines are open submanifolds of $\mathbb{S}_l$; cf. [31, 1.19f]. This in particular applies to stable (lp-) planes of dimension $2l \leq 4$, by [31, 1.13]. The full automorphism group $\mathrm{Aut}\,\mathcal{P}$, endowed with the compact-open topology, is a topological group with a countable basis, and it acts on $\mathcal{P}$ as a topological transformation group; see [31, §2]. Moreover, $\mathrm{Aut}\,\mathcal{P}$ has a strong tendency to be a Lie group; see [31, 2.10+2.11], [32, Satz A], and also Szenthe's theorem [54, 96.14].

It may thus be pardonable that the major part of the present thesis is concerned with 4-dimensional stable planes which are (re-)constructed from the action of locally compact Lie groups.

1.3. Morphisms and embeddings of stable planes

Our main topic will be the recognition of stable subplanes in stable planes. It may thus be worthwhile to have a glimpse at the notion of embeddings of stable planes. On our way, we will introduce a category of stable planes, following Stroppel [58].

1.3.1 Definition. A lineation $\pi : (P, \mathcal{L}) \to (P', \mathcal{L}')$ is **collapsed** if there is a line $L' \in \mathcal{L}'$ such that $P^\pi \subseteq P_{L'}$.

Every continuous lineation between stable planes which is not injective is collapsed or locally constant [58, Thm. 6]. So it does make sense to cling to injective lineations when talking about stable planes.

1.3.2 Lemma. *Let $\pi : (P, \mathcal{L}) \to (P', \mathcal{L}')$ be an injective lineation between stable planes. Then the following is true :*

a) *There is a unique map $\lambda : \mathcal{L} \to \mathcal{L}'$ such that (π, λ) becomes a morphism of incidence structures. In fact, λ is the map*

$$\begin{array}{rccc} \lambda : & \mathcal{L} & \to & \mathcal{L}' \\ & p \vee q & \mapsto & p^\pi \vee q^\pi . \end{array}$$

b) *π non-collapsed* $\implies$ *λ injective*

c) *π continuous* $\implies$ *λ continuous*

Proof. Lemma 3 in [58]. □

This is corroborated by work of Dörfner who in [12] generalises Stroppel's results on endomorphisms :

1.3.3 Theorem. *Let $\mathcal{P}$ and $\mathcal{P}'$ be stable planes of equal dimension $\dim P = \dim P'$. Let $\pi : \mathcal{P} \to \mathcal{P}'$ be a continuous and injective lineation. Then π is open and non-collapsed.*

We will cast all this into a category.

1.3.4 Definition. The category StP of stable planes. The objects of StP shall be stable (lp-) planes. A map $\mathsf{H} : \mathcal{P} \to \mathcal{P}'$ shall be called a **morphism of stable planes** if $\mathsf{H} \in \operatorname{morph} \mathsf{Inc}$ and the point map H_P is continuous, injective and non-collapsed.

1.3.5 Proposition. *Let $\mathsf{H} : \mathcal{P} \to \mathcal{P}'$ be a morphism in StP. Then the following holds :*

a) *The line map $\mathsf{H}_\mathcal{L}$ is injective and continuous.*

b) *If the point map $\mathsf{H}_P : P \to P'$ is a homeomorphism then $\mathsf{H} : \mathcal{P} \to \mathcal{P}'$ is an isomorphism of stable planes.*

Now assume further that the planes are of equal dimension $\dim P = \dim P'$. Then

c) *H_P is open.*

d) *$\mathsf{H}_P : P \to P^{\mathsf{H}_P}$ is a homeomorphism.*

e) *$\mathsf{H} : \mathcal{P} \to \mathcal{P}^{\mathsf{H}}$ is an isomorphism.*

Proof. *ad (a).* Parts (b) and (c) of 1.3.2. *ad (b).* The inverse $\pi^{-1} : P' \to P$ of the point map is continuous, injective and open. By (a), there also is an inverse $\lambda^{-1} : \mathcal{L}' \to \mathcal{L}$, and we easily check that $(\pi^{-1}, \lambda^{-1}) \in \operatorname{morph} \mathsf{Inc}$. Thus λ^{-1} is the unique line map corresponding to π^{-1} promised in part (a) of 1.3.2; hence, $(\pi^{-1}, \lambda^{-1}) = (\pi, \lambda)^{-1} \in \operatorname{morph} \mathsf{StP}$. Part (c) follows from 1.3.3, and parts (d) and (e) are immediate consequences of the preceding parts. □

Due to the definition, every morphism in the category StP is monic. We want to coin a notion of embedding different from monomorphisms, though, which takes account of the topological aspect. Note that we do not wish to restrict the definition to open embeddings because we will also consider embeddings of planes of smaller dimension into higher dimensional ones, for instance the sketched Baer subplanes in chapter 4.

1.3.6 Definition. A morphism $\mathsf{H} : \mathcal{P} \to \mathcal{P}'$ of stable planes is an **embedding of stable planes** if the co-restriction $\mathsf{H}_P : P \to P^{\mathsf{H}_P}$ is a homeomorphism.

1.3.7 Corollary. **a)** *Every morphism between stable planes of equal dimension is an embedding.*

b) *Every embedding $\mathsf{H} : \mathcal{P} \to \mathcal{P}'$ of stable planes is an isomorphism from $\mathcal{P}$ onto $\mathcal{P}^{\mathsf{H}}$.*

Note that an isomorphism in StP is, very straightforwardly, an isomorphism in Inc whose point and line maps are homeomorphisms [1.3.2].

1.4. Construction of stable planes from stable partitions

Stable planes as well as sketched geometries have been introduced. This, of course, calls for the concept of *sketched stable planes*. Given a topological group Υ along with a group partition $\mathcal{F}$ of Υ, what properties of $\mathcal{F}$ and Υ will grant a stable plane $\mathbb{P}(\Upsilon; \{\mathsf{1}\}, \mathcal{F})$? This subject has been exploited by Maier in his dissertation [44], using results from Plaumann and Strambach [50].

The notion of partitions in several classes of objects will be crucial.

1.4.1 Definition. Let $\mathcal{C}$ be one of the categories VecSp of real vector spaces, LieAlg of real Lie algebras, Gp of groups and LieGp of Lie groups. Denote by $\mathbf{0}$ the null object in $\mathcal{C}$. Let X be an object in $\mathcal{C}$. Then a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is called a **$\mathcal{C}$-partition** of X if

- $X = \bigcup_{\Lambda \in \mathcal{F}} \Lambda$
- $\forall \Lambda, \mathsf{M} \in \mathcal{F}. \quad \Lambda \neq \mathsf{M} \Longrightarrow \Lambda \cap \mathsf{M} = \mathbf{0}$
- Each fibre $\Lambda \in \mathcal{F}$ is a $\mathcal{C}$-substructure of X.

By "$\mathcal{C}$-substructure', we mean a vector subspace, Lie subalgebra or subgroup in VecSp, LieAlg and Gp, respectively, and a closed subgroup in LieGp.

1.4.2 Definition. Let $\mathcal{C} \in \{\mathsf{VecSp}, \mathsf{Gp}, \mathsf{LieGp}\}$ and let $X \in \operatorname{ob}\mathcal{C}$. Denote by $\diamond$ the binary operation $\diamond : X \times X \to X$. A $\mathcal{C}$-partition $\mathcal{F}$ of X will be called a **planar $\mathcal{C}$-partition** of X if for any two distinct fibres $\mathsf{\Lambda}, \mathsf{M} \in \mathcal{F}$ we get $\mathsf{\Lambda} \diamond \mathsf{M} = X$. A LieAlg-partition is called planar if it is planar as a VecSp-partition.

Planar VecSp-partitions are often referred to as **spreads**. They can also be characterised via the dimension of their fibres.

1.4.3 Lemma. *Let V be a real vector space of finite dimension. Then a VecSp-partition of V is planar if and only if $\dim V = 2k$ for some $k \in \mathbb{N}^+$ and the fibres in $\mathcal{F}$ are of dimension k. This clearly remains true for LieAlg-partitions.*

There are results on translation planes gained from partitions. André in [2] algebraically characterises those partitions :

1.4.4 Theorem (André 1954). *Let $\mathcal{F}$ be a Gp-partition of a group Υ. Then $\mathbb{P}(\Upsilon; \{\mathbb{1}\}, \mathcal{F})$ is a translation plane if and only if $\mathcal{F}$ is planar. In that case, Υ is an abelian group, carries a vector space structure of even dimension, and $\mathcal{F}$ is a (planar) VecSp-partition.*

Proof. §3 (Sätze 7 and 9) and §4 in [2]. □

Topological properties of these spreads leading towards translation planes were studied in Löwen [43]. In [42], in a more general context, he introduces the so-called **disc topology** on the line space of a stable plane; see also Löwen [43] and Maier [44, 2.1.5]. Note that, indeed, the line space of a stable plane whose lines are manifolds carries the disc topology with respect to a suitable metric on its point space. For a spread of $\mathbb{R}^{2n}$, the disc topology equals the Graßmann topology; see [44, p. 21]. The following result on topological translation planes remains true for vector spaces over arbitrary locally compact non-discrete fields, and in this general version can be found in [43].

1.4.5 Theorem (Löwen 1989). *Let V be a real vector space of dimension $2n \in \{2, 4, 8, 16\}$, and let $\mathcal{S}$ be a planar VecSp-partition of V. Consider the translation plane $\mathcal{T} := \mathbb{P}(V; \{\mathbf{0}\}, \mathcal{S})$ as a topological plane, where the line space is endowed with the disc topology. Then $\mathcal{T}$ is locally compact and connected if and only if $\mathcal{S}$ is compact in the Graßmann manifold $\mathfrak{u}_n(V)$ of n-dimensional subspaces of V.*

As a planar partition in the usual sense can only occur on abelian groups and, moreover, the associated translation structure $\mathbb{P}(\Upsilon; \{\mathtt{1}\}, \mathcal{F})$ invariably is a translation plane, it seems appropriate to use a slightly modified notion of planarity for Lie groups. For the moment, we will not simply refer to it as "planarity", but indicate that it was introduced by Maier in [44, p. 6] :

1.4.6 Definition. A LieGp-partition $\mathcal{F}$ of a Lie group Υ of dimension $2n$ will be called **Peter-planar** if every fibre $\Lambda \in \mathcal{F}$ is n-dimensional.

Our aim will be the construction of stable planes from Lie algebras along with suitable LieAlg-partitions, making use of associated LieGp-partitions. Thus, the interplay between properties of LieAlg- and LieGp-partitions will be of importance. Starting out with Lie groups, the situation is fairly straightforward :

1.4.7 Lemma. *Let Υ be a Lie group of dimension $\dim \Upsilon = 2n$. Let $\mathcal{F}$ be a LieGp-partition of Υ and put $\ell\,\mathcal{F} := \{\ell\,\Lambda \mid \Lambda \in \mathcal{F}\}$. Then the following statements are true :*

a) *$\ell\,\mathcal{F}$ is a LieAlg-partition of $\ell\,\Upsilon$.*

b) *If $\mathcal{F}$ is Peter-planar then $\ell\,\mathcal{F}$ is planar.*

The converse only holds for specials scenarios. The later parts of this thesis will treat 4-dimensional Lie groups with bijective exponential functions, which will justify a restriction to that case. Note that in general, we could not even be sure whether or not the exponential image of a partition of the Lie algebra would be a Set-partition of the Lie group.

1.4.8 Lemma. *Let $\mathcal{S}$ be a planar LieAlg-partition of some 4-dimensional real Lie algebra $\mathfrak{g}$. Assume a bijective exponential map $\exp : \mathfrak{g} \to \Upsilon$. Then $\mathcal{S}^{\exp} := \{\Lambda^{\exp} \mid \Lambda \in \mathcal{S}\}$ is a Peter-planar LieGp-partition of Υ.*

Proof. The essential point in the 4-dimensional situation is that the fibres of a planar partition are 2-dimensional Lie algebras, and there are only two isomorphism types of those : abelian and non-abelian; cf. 4.2.1. Let $\mathfrak{d}$ be a fibre in $\mathcal{S}$.

The abelian case. If $\mathfrak{d}$ is abelian, then $\exp_{\mathfrak{d}}$ obeys the exponential law, hence is a group morphism. Consequently, $\mathfrak{d}^{\exp}$ is a subgroup of Υ.

The non-abelian case. If $\mathfrak{d}$ is non-abelian, then there is a LieAlg-isomorphism $\eta : \mathfrak{n} \to \mathfrak{d}$, mapping "the" non-abelian 2-dimensional prototype $\mathfrak{n}$ to $\mathfrak{d}$. Moreover, there is a homeomorphic exponential function $\exp_{\mathfrak{n}} : \mathfrak{n} \to \Delta$ mapping $\mathfrak{n}$ to our "standard" non-abelian 2-dimensional Lie group Δ; see 4.1.1 for a concrete version.

$$\begin{tikzcd}
\Delta \arrow[rr, "\varepsilon"] \arrow[dr, dotted, "\varepsilon/_{\Delta^{\varepsilon}}"] & & \Upsilon \\
& \mathfrak{d}^{\exp_{\mathfrak{g}}} \arrow[ur] & \\
& \mathfrak{d} \arrow[u, "\exp_{\mathfrak{g}}|_{\mathfrak{d}^{\exp}}"] & \\
\mathfrak{n} \arrow[uuu, "\exp_{\mathfrak{n}}"] \arrow[ur, "\eta"] \arrow[rr, "\hat{\eta}"] & & \mathfrak{g} \arrow[uuu, "\exp_{\mathfrak{g}}"'] \arrow[ul, "\mathrm{id}"']
\end{tikzcd}$$

The isomorphism η can be read as an injective LieAlg-morphism $\hat{\eta} : \mathfrak{n} \to \mathfrak{g}$, and thus there is a LieGp-morphism $\varepsilon : \Delta \to \Upsilon$ such that the outer diagram commutes. But then,

$$\mathfrak{d}^{\exp_{\mathfrak{g}}} = \mathfrak{n}^{\eta \cdot \exp_{\mathfrak{g}}} = \mathfrak{n}^{\hat{\eta} \cdot \exp_{\mathfrak{g}}} = \mathfrak{n}^{\exp_{\mathfrak{n}} \cdot \varepsilon} = \Delta^{\varepsilon} \leq \Upsilon,$$

which is a subgroup of Υ.

All in all, $\dim \mathfrak{g} = 4$ implies that every $\mathfrak{d}^{\exp}$ for $\mathfrak{d} \in \mathcal{S}$ is a subgroup of Υ, indeed. Moreover, such a $\mathfrak{d}^{\exp}$ is a closed subgroup of dimension 2, as exp is a homeomorphism. Finally, $\mathcal{S}^{\exp}$ is a Set-partition because $\exp_{\mathfrak{g}}$ was assumed to be a bijection. Thus, $\mathcal{S}^{\exp}$ is a Peter-planar LieGp-partition of Υ. □

1.4.9 Definition. Let $\mathcal{F}$ be a LieGp-partition of a Lie group Υ. $\mathcal{F}$ is called a **stable LieGp-partition** of Υ if the plane $\mathbb{P}(\Upsilon; \{1\}, \mathcal{F})$ is a stable plane. Here, the line space shall be endowed with the disc topology.

Trying to model an analogue for partitions of Lie algebras, several possibilities meet the eye. We will pick the one which is the closest in spirit.

1.4.10 Definition. Let $\mathcal{S}$ be a LieAlg-partition of a real Lie algebra $\mathfrak{g}$. $\mathcal{S}$ is called a **stable LieAlg-partition** of $\mathfrak{g}$ if the plane $\mathbb{P}(\mathfrak{g}; \{0\}, \mathcal{S})$ is a locally compact translation plane. Here again, the line space is thought of as endowed with the disc topology.

As the plane associated with a stable LieAlg-partition has to be a translation plane, it is an immediate consequence of 1.4.4 that

1.4.11 Lemma. *Any stable* LieAlg*-partition is planar.*

Maier in the second chapter of his dissertation [44] clarified the interplay between stability on the Lie algebra versus the Lie group levels. In order to formulate his results, it is necessary to introduce some more vocabulary.

1.4.12 Definition. A linear space $(P, \mathcal{L})$ is called a **homogeneous plane** if

- P is the coset space of a connected Lie group Υ modulo some subgroup, endowed with the quotient topology; i.e., a **homogeneous space**.
- $\mathcal{L}$ is an Υ-invariant subset of the space $\mathcal{A}_n(P)$ of all closed n-dimensional submanifolds of P, where $n := \frac{1}{2}\dim P$. Endow $\mathcal{L}$ with the disc topology.

1.4.13 Definition. Let P be a topological manifold of dimension $2n$. Two closed n-dimensional submanifolds L and K are said to **intersect regularly** in some point $p \in P$ if there is a neighbourhood U of p and a homeomorphism $\varphi : U \to \mathbb{R}^{2n}$ which sends $K \cap U$ to $\mathbb{R}^n \times \mathbf{0}$ and $L \cap U$ to $\mathbf{0} \times \mathbb{R}^n$.

1.4.14 Lemma. **a)** *Any two n-dimensional Lie subalgebras of a $2n$-dimensional Lie algebra which intersect trivially intersect regularly.*

b) *Any two n-dimensional Lie subgroups of a $2n$-dimensional Lie group intersect regularly, if they intersect trivially.*

The nucleus of the whole theory is

1.4.15 Theorem (Maier 1999). *A homogeneous plane $(P, \mathcal{L})$ is a stable plane if and only if there is a point $p \in P$ such that $\mathcal{L}_p$ is a compact subset of $\mathcal{L}$ and any pair of lines in $\mathcal{L}_p$ intersects regularly..*

1.4.16 Theorem (Maier 1999). *Let $\mathcal{P} = (P, \mathcal{L})$ be a homogeneous plane with respect to some connected Lie group Υ. Assume that*

- *any pair of intersecting lines in $\mathcal{L}$ intersects regularly*
- *all lines are connected*
- *for each line $L \in \mathcal{L}$, the stabiliser Υ_L acts transitively on the point row P_L; i.e., each line is* ***Υ-isotropic***
- $\forall p \in P\ \forall L \in \mathcal{L}_p. \quad \Upsilon_p \leq \Upsilon_L$

Then $\mathcal{P}$ is a stable plane if and only if there is a point $p \in P$ such that the tangent plane $T_p(\mathcal{P}) := (T_p(P), \{T_p(L) \mid L \in \mathcal{L}\})$ is a topological translation plane, or equivalently, a stable plane.

Proof. [44], theorems 2.3.10 and 2.4.6. □

This can be drastically shortened talking about point-homogeneous sketched planes gained from partitions of connected Lie groups.

1.4.17 Corollary. *Let Υ be a connected Lie group along with a Peter-planar* LieGp*-partition $\mathcal{F}$ of Υ. Consider the plane*

$$\mathcal{P} = (P, \mathcal{L}) := \mathbb{P}(\Upsilon; \{1\}, \mathcal{F})$$

where the point space is $P \approx \Upsilon$ and the line space is endowed with the disc topology. Then the following statements are equivalent :

$$\begin{array}{ll} & \mathcal{P} \text{ is a stable plane} \\ \Longleftrightarrow & T_1\mathcal{P} = \mathbb{P}(\ell\,\Upsilon; \{\mathbf{0}\}, \ell\,\mathcal{F}) \text{ is a stable plane} \\ \Longleftrightarrow & \text{the planar partition } \ell\,\mathcal{F} \text{ of } \ell\,\Upsilon \text{ is compact in the Graßmann manifold} \end{array}$$

In other words, $\mathcal{F}$ is a stable LieGp*-partition of Υ if and only if $\ell\,\mathcal{F}$ is a stable* LieAlg*-partition of $\ell\,\Upsilon$.*

Proof. We have to check all the hypothesis of 1.4.16. Let us have a glimpse at some of them. Put $n := \frac{1}{2}\dim\Upsilon$.

$\mathcal{L}$ is an Υ-invariant subset of $\mathcal{A}_n(\Upsilon)$. Every fibre of $\mathcal{F}$ is a closed n-dimensional subgroup of Υ because $\mathcal{F}$ is a Peter-planar LieGp-partition. And Υ-invariance comes with being a sketched geometry.

Line stabilisers act transitively on point rows. (This is what, conversely, would force $\mathcal{P}$ to be sketched; see 1.1.23.) Note that for any $\Lambda \in \mathcal{F}$ and $\gamma \in \Upsilon$, the line stabiliser of $\Lambda\gamma$ is $\Upsilon_{\Lambda\gamma} = \gamma^{-1}\Lambda\gamma$. Consider α and β in $\Lambda\gamma$. Then $\lambda := \alpha^{-1}\beta \in \Upsilon_{\Lambda\gamma}$ is an element which maps α to β.

Connectivity of lines can be found as Satz 6.4 in Plaumann and Strambach [50]. *Regular intersection of intersecting lines* was established in 1.4.14.

All hypotheses verified, theorem 1.4.16 yields that $\mathcal{P}$ is a stable plane if and only if there is some point $p \in P$ such that $T_p(\mathcal{P})$ is a (locally compact and connected) topological translation plane. The plane $\mathcal{P}$ being homogeneous, this is equivalent to saying that the translation plane $T_0(\mathcal{P}) = \mathbb{P}(\ell\,\Upsilon; \{\mathbf{0}\}, \ell\,\mathcal{F})$ is a (locally compact and connected) topological translation plane, in other words, equivalent to $\ell\,\mathcal{F}$ being a stable LieAlg-partition of $\ell\,\Upsilon$. □

Getting back to 4-dimensional Lie groups, this encourages the construction of stable planes – which are not translation planes – from stable LieAlg-partitions. Lemma 1.4.8 and corollary 1.4.17 combine as follows.

1.4.18 Corollary. *Let $\mathfrak{g}$ be a 4-dimensional Lie algebra along with a planar* LieAlg*-partition $\mathcal{S}$. Assume a bijective exponential function $\exp : \mathfrak{g} \to \Upsilon$. Then the planar* LieGp*-partition $\mathcal{S}^{\exp}$ of Υ is stable if and only if $\mathcal{S}$ is stable.*

In the latter situation, therefore, all presumable notions of "stability" do coincide :

$$\begin{array}{ll} & \mathcal{S} \text{ is a stable partition of } \mathfrak{g} \\ \Longleftrightarrow & \text{the associated translation plane } \mathbb{P}(\mathfrak{g}; \{\mathbf{0}\}, \mathcal{S}) \text{ is a stable plane} \\ \Longleftrightarrow & \mathcal{S}^{\exp} \text{ is a stable partition of } \mathfrak{g}^{\exp} = \Upsilon. \end{array}$$

1.5. Stable partitions of 4-dimensional Lie groups

1.5.1 Theorem (Maier 1999). *Let Υ be a 4-dimensional connected Lie group with a stable* LieGp*-partition. Then Υ is isomorphic to one of the following groups :*

$$\mathbb{R}^4, \quad \mathrm{Dil}_1\mathbb{C}, \quad \mathrm{Dil}_3^1\mathbb{R}, \quad \mathbb{R} \ltimes_\delta \mathrm{Hei}_3\mathbb{R}.$$

Here, for a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, the ***dilatation group*** *is defined as*

$$\mathrm{Dil}_n\mathbb{F} := \left\{ \left(\begin{array}{c|c} a\mathbb{1} & u \\ \hline & 1 \end{array} \right) \middle| \, a \in \mathbb{F}^\times, \, u \in \mathbb{F}^n \right\},$$

and $\mathrm{Dil}_n^1\mathbb{F}$ denotes its connected component. The semidirect product $\mathbb{R} \ltimes_\delta \mathrm{Hei}_3\mathbb{R}$ is given via conjugation $\delta : \mathbb{R} \to \mathrm{Aut}(\mathrm{Hei}_3\mathbb{R}) : t \mapsto (tD)^{\exp}$, where $D := \mathrm{diag}(2, 1, 0)$.

Proof. [44, 4.3.2]; for a matrix description of $\mathbb{R} \ltimes_\delta \mathrm{Hei}_3\mathbb{R}$ see chapter 3. □

These were obtained via a classification of the corresponding Lie algebras allowing for stable partitions :

1.5.2 Theorem (Maier 1999). *Let $\mathfrak{g}$ be a 4-dimensional Lie algebra with a stable* LieAlg*-partition. Then $\mathfrak{g}$ is isomorphic to one of the following Lie algebras :*

$$\mathbb{R}^4, \quad \mathrm{dil}_1\mathbb{C}, \quad \mathrm{dil}_3\mathbb{R}, \quad \mathbb{R} \propto_\alpha \mathrm{hei}_3\mathbb{R}.$$

Here, a ***dilatation algebra*** *is defined as*

$$\mathrm{dil}_n\mathbb{F} := \left\{ \left(\begin{array}{c|c} a\mathbb{1} & u \\ \hline & 0 \end{array} \right) \middle| \, a \in \mathbb{F}, \, u \in \mathbb{F}^n \right\},$$

and the semidirect sum $\mathbb{R} \propto_\alpha \mathrm{hei}_3\mathbb{R}$ is given by the adjoint action $\alpha : \mathbb{R} \to \mathrm{Der}(\mathrm{hei}_3\mathbb{R}) : t \mapsto tD$, where $D := \mathrm{diag}(2, 1, 0)$.

Proof. [44, 4.1.7] □

In chapter 4.2 of [44], Maier exposes a construction for examples of stable partitions of these Lie algebras. Let us briefly put on record that

- for $\mathrm{dil}_1\mathbb{C}$, there is a unique stable partition
- for the abelian Lie algebra $\mathbb{R}^4$ and the almost abelian Lie algebra $\mathrm{dil}_3\mathbb{R}$, any pair (f_1, f_2) of a decreasing function $f_1 : \mathbb{R} \to \mathbb{R}$ and an increasing function $f_2 : \mathbb{R} \to \mathbb{R}$ gives rise to a stable LieAlg-partition
- for $\mathfrak{g} = \mathbb{R} \propto_\delta \mathrm{hei}_3\mathbb{R}$, any decreasing function $g : \mathbb{R} \to \mathbb{R}$ gives rise to a stable LieAlg-partition.

This is not to be understood as a classification. For $\mathbb{R}^4$, $\mathrm{dil}_3\mathbb{R}$ and $\mathfrak{g} = \mathbb{R} \propto_\alpha \mathrm{hei}_3\mathbb{R}$ it is likely that many more can be found. Furthermore, the isomorphism problem has not yet been attacked. Note though, that the results in chapter 3 do not depend on the concrete stable partition.

The stable planes $\mathcal{P} := \mathbb{P}(\Upsilon; \{\mathbb{1}\}, \mathcal{F})$ constructed from these stable partitions $\mathcal{F}$ of $\ell\,\Upsilon$ are well-known for the first three groups :

- for $\Upsilon = \mathbb{R}^4$, the plane $\mathcal{P}$ is translation plane; cf. 1.4.4
- for $\Upsilon = \mathrm{Dil}_3^1\mathbb{R}$, the plane $\mathcal{P}$ is half a translation plane; cf. [76, ch. 4] or 1.5.3.
- for $\Upsilon = \mathrm{Dil}_1\mathbb{C}$, the plane $\mathcal{P}$ is a certain open subplane of $\mathcal{A}_2\mathbb{C}$; cf. 1.5.4

Last but not least, the planes corresponding to the Frobenius group $\Gamma = \mathbb{R} \ltimes_\delta \mathrm{Hei}_3\mathbb{R}$ have not yet revealed their true faces. They will provide a startling subject for chapters 3 through 5, and for that purpose be baptised **Peter planes**.

1.5.3 Real dilatation groups and open halves of translation planes. Let $n \in \mathbb{N}$, and let $\mathcal{S}$ be a planar VecSp-partition of the dilatation algebra $\mathrm{dil}_{2n-1}\mathbb{R}$. There is an isomorphism

$$\begin{array}{rccc} \alpha : & \mathrm{dil}_{2n-1}\mathbb{R} & \to & \mathbb{R}^{2n} \\ & \left(\begin{array}{c|c} s\mathbb{1} & x \\ \hline & 0 \end{array}\right) & \mapsto & (x|s) \end{array}$$

of vector spaces, which translates $\mathcal{S}$ into a planar VecSp-partition $\mathcal{S}^\alpha$ of $\mathbb{R}^{2n}$. By 1.4.4, the plane $\mathcal{T} := \mathcal{U}_{\mathsf{Inc}}\mathbb{P}(\mathbb{R}^{2n}; \{\mathbf{0}\}, \mathcal{S}^\alpha)$ is a translation plane.

As $\mathrm{dil}_{2n-1}\mathbb{R}$ is an almost abelian Lie algebra, $\mathcal{S}$ is a planar LieAlg-partition. Its exponential image under the bijection $\exp : \mathrm{dil}_{2n-1}\mathbb{R} \to \mathrm{Dil}^+_{2n-1}\mathbb{R}$ is a LieGp-partition of the connected component

$$\mathrm{Dil}^1_{2n-1}\mathbb{R} \;=\; \mathrm{Dil}^+_{2n-1}\mathbb{R} := \left\{ \left(\begin{array}{c|c} a\mathbb{1} & u \\ \hline & 1 \end{array}\right) \middle|\, a > 0,\ u \in \mathbb{R}^{2n-1} \right\}$$

of the dilatation group. The plane $\mathcal{H} := \mathcal{U}_{\mathsf{Inc}}\mathbb{P}(\mathrm{Dil}^+_{2n-1}\mathbb{R}; \{\mathbb{1}\}, \mathcal{S}^{\exp})$ is isomorphic to half a translation plane, namely the open subplane of $\mathcal{T}$ induced by $H := \{x \in \mathbb{R}^{2n} \mid x_{2n} > 0\}$. For a proof, consult [76, Ch. 4].

1.5.4 $\mathrm{Dil}_1\mathbb{C}$ **and the complex Minkowski plane.** Consider the open subplane $\mathcal{M}(\mathbb{C})$ of $\mathcal{A}_2\mathbb{C}$ induced by the point space $M(\mathbb{C}) := \{(x_1, x_2) \in \mathbb{C}^2 \mid x_1 \neq 0\}$; cf. Löwen [33, 2.11] or [35, 2.5]. There is a point transitive action of $\mathrm{Dil}_1\mathbb{C}$ on $\mathcal{M}(\mathbb{C})$ which describes a point homogeneous sketched stable plane $(\mathrm{Dil}_1\mathbb{C}, \mathcal{M}(\mathbb{C}))$. (In order to verify this, one can readily copy down the calculations elaborated in paragraph 4.1.) The sketch $\mathbb{S}(\mathrm{Dil}_1\mathbb{C}, \mathcal{M}(\mathbb{C}))$ then yields a stable LieAlg-partition of $\mathrm{dil}_1\mathbb{C}$. As there is a unique stable LieAlg-partition of $\mathrm{dil}_1\mathbb{C}$, every stable plane gained from such a partition must be isomorphic to the complex Minkowski plane $\mathcal{M}(\mathbb{C})$.

2. Line homogeneous sketched stable planes

The study of planes along with their collineation groups naturally provokes the study of possible constellations concerning the numbers of point and line orbits. For finite linear spaces, the point space always seems to be "privileged" in a way, as can be deduced from theorem 2.1 in Block [6] :

2.0.5 Theorem (Block 1967). *Let $\mathcal{P} = (P, \mathcal{L})$ be a linear space, and let $\Gamma \leq \operatorname{Aut} \mathcal{P}$ be an arbitrary group of collineations. If P is finite then the following is true :*

a) $|P| \leq |\mathcal{L}|$

b) *The number $|P/\Gamma|$ of point orbits does not exceed the number $|\mathcal{L}/\Gamma|$ of line orbits.*

Following the principle that counting in finite geometry might be replaced by compactness arguments in topological geometry, one could ask for a generalisation of Block's result on (locally) compact planes, stable planes for instance. A direct transfer will not be possible, as can be seen in the light of the following example (which will be elaborated below): The real hyperbolic motion group $\mathrm{PO}_3\mathbb{R}(1)$ acts on the closed hyperbolic plane $\overline{\mathrm{IH}\mathbb{R}}$, producing two point orbits but only one line orbit. Hence, more restrictions are to be expected.

We will tackle a generalisation for sketched stable planes. Recall from 1.1.19 that a sketched linear space automatically is point or line homogeneous, such that for our purposes we are left with the line homogeneous case. It will be dealt with largely exploiting results by Löwen, after an extensive introduction to the dramatis personae.

2.1. Euclidean, hyperbolic and skew hyperbolic geometries

Some typical stable planes will be presented in detail — incidentally, we pick those that will later on play their role within the classification of line homogeneous sketched stable planes. They will be presented as under their classical motion groups, and the existence of sketches will be studied. The reader familiar with the folklore of the material may happily skip those parts and go to the classification straight away.

Throughout this chapter, $\mathbb{F}$ denotes one of the (skew) fields $\mathbb{R}$ of real numbers, $\mathbb{C}$ of complex numbers or the Hamilton quaternions $\mathbb{H}$, unless otherwise stated. For the octonions $\mathbb{O}$, the relevant results remain true, whereas the technical approach may differ. Corresponding citations are given in due place, mainly referring to chapters 16 through 18 in SALZMANN ET AL. [54]. Different models of $\mathcal{P}_2\mathbb{O}$ can be found in ASLAKSEN [3] or ALLCOCK [1].

2.1.1 Unitary groups. Let $V = \mathbb{F}^n$ be a right vector space over some skew field $\mathbb{F}$ of characteristic $\operatorname{char}\mathbb{F} \neq 2$ and α an involutory antiautomorphism of $\mathbb{F}$. Let $h : V \times V \to \mathbb{F}$ be some non-degenerate, α-hermitian form on V. The **unitary group** of V with respect to h is defined as

$$\mathrm{U}(V,h) := \{\varphi \in \mathrm{GL}(V) \mid \forall v,w \in V \,.\, h(v^\varphi, w^\varphi) = h(v,w)\} \quad .$$

Given some basis, h can be described as $h(v,w) = vHw^*$ for some matrix $H \in \operatorname{Mat}_n \mathbb{F}$, where $v^* := (v^\alpha)^\mathsf{T}$. Then the unitary group turns out to be

$$\mathrm{U}(V,h) = \{A \in \mathrm{GL}_n\mathbb{F} \mid AHA^* = H\} \quad .$$

We talk about **orthogonality** whenever $\mathbb{F}$ is a (commutative) field of characteristic $\operatorname{char}\mathbb{F} \neq 2$ and h is symmetric, i.e., $\alpha = \mathrm{id}_\mathbb{F}$. Quite prominent examples are

$\mathrm{O}_n\mathbb{F} := \{A \in \mathrm{GL}_n\mathbb{F} \mid AA^\mathsf{T} = \mathbb{1}\}$	for $H = \mathbb{1}$, $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$ and $\alpha = \mathrm{id}_\mathbb{F}$
$\mathrm{U}_n\mathbb{C} := \{A \in \mathrm{GL}_n\mathbb{C} \mid A\overline{A}^\mathsf{T} = \mathbb{1}\}$	for $H = \mathbb{1}$, $\mathbb{F} = \mathbb{C}$ and complex conjugation

Affine planes

2.1.2 The real example. Consider the semidirect product $\Psi := \mathrm{SO}_2\mathbb{R} \ltimes_\alpha \mathbb{R}^2$, where $\alpha = \mathrm{id} : \mathrm{SO}_2\mathbb{R} \to \mathrm{GL}_2\mathbb{R}$ is the natural representation. Note that Ψ is isomorphic to the matrix group

$$\Psi \cong \begin{array}{|c|c|}\hline \mathrm{SO}_2\mathbb{R} & \\ \hline \mathbb{R}^2 & 1 \\ \hline \end{array} \quad ,$$

and as such comes with a canonical action on the real affine plane $\mathcal{A}_2\mathbb{R}$:

$$\begin{array}{rcccrcr} \mathbb{R}^2 \times \Psi & \to & \mathbb{R}^2 & : & (x,(A,v)) & \mapsto & xA + v \\ \mathcal{L} \times \Psi & \to & \mathcal{L} & : & (\mathbb{R}x + y,(A,v)) & \mapsto & \mathbb{R}xA + yA + v \end{array}$$

The normal subgroup $\mathbb{R}^2 \trianglelefteq \Psi$ acts as the translation group. Therefore, the point action is transitive, and so is the line action, as a matter of fact. The geometry $(\Psi, \mathcal{A}_2\mathbb{R})$ is a flag transitive sketched geometry; for a system of representatives any flag may be chosen.

This real affine geometry can be transferred to any of the affine planes $\mathcal{A}_2\mathbb{F}$ over one of the skew fields $\mathbb{C}$ or $\mathbb{H}$.

2.1.3 The plane. The **affine plane** $\mathcal{A}_2\mathbb{F}$ is defined by the point space $\mathbb{F}^2$ and the line space $\{\mathbb{F}x + y \mid x,y \in \mathbb{F}^2 \wedge x \neq 0\}$; incidence is given by the relation $\in$.

2.1.4 The Euclidean motion group. Consider the semidirect product

$$\Psi := \mathrm{Spin}_{n+1} \ltimes \mathbb{F}^2 \cong \begin{array}{|c|c|}\hline \mathrm{Spin}_{n+1} & \\ \hline \mathbb{F}^2 & 1 \\ \hline\end{array} \quad ,$$

where $n := \dim_{\mathbb{R}} \mathbb{F} \in \{1,2,4,8\}$. For $m \geq 3$, the **spinor group** Spin_m denotes the universal covering group of $\mathrm{SO}_m\mathbb{R}$. By definition, Spin_2 is the two-fold covering of $\mathrm{SO}_2\mathbb{R}$. Note that $\mathrm{Spin}_3 \cong \mathrm{SU}_2\mathbb{C}$ and $\mathrm{Spin}_5 \cong \mathrm{U}_2\mathbb{H}$; see 11.26 and 18.9 in [54]. For another description of Spin_9, see [54, 18.8]. Except for the abelian $\mathrm{SO}_2\mathbb{R}$, the spinor groups are almost simple Lie groups.

2.1.5 The geometry. The semidirect product Ψ acts on the affine plane $\mathcal{A}_2\mathbb{F}$ via

$$\begin{array}{rcccrcl} \mathbb{F}^2 \times \Psi & \to & \mathbb{F}^2 & : & (x,(A,v)) & \mapsto & xA+v \\ \mathcal{L} \times \Psi & \to & \mathcal{L} & : & (\mathbb{F}x+y,(A,v)) & \mapsto & \mathbb{F}xA+yA+v \end{array}$$

Again, the translation group $\mathbb{F}^2 \trianglelefteq \Psi$ ensures point transitivity. Flag transitivity follows from transitivity of $\mathrm{Spin}_{n+1} \leq \Psi$ on the "directions" $\mathrm{P}_2\mathbb{F}$; cf. [54, 13.15]. All in all, $(\Psi, \mathcal{A}_2\mathbb{F})$ is a flag homogeneous sketched geometry with representatives $R_P = \{0\}$ and $R_{\mathcal{L}} = \{\mathbb{F}e_1\}$, for instance. Flag homogeneity is also given for $\mathbb{F} = \mathbb{O}$; cf. [18, 5.4].

Projective planes

2.1.6 The plane. Consider the **projective plane** $\mathcal{P}_2\mathbb{F} = (\mathrm{P}_2\mathbb{F}, \mathcal{L}_2\mathbb{F})$ over $\mathbb{F}$, which is given by

the point space $\mathrm{P}_2\mathbb{F} = \mathfrak{u}_1(\mathbb{F}^3) = \{\mathbb{F}x \mid x \in \mathbb{F}^3 \setminus \mathbf{0}\}$
the line space $\mathcal{L}_2\mathbb{F} = \mathfrak{u}_2(\mathbb{F}^3) = \{\mathrm{Ker}\, a \mid a^{\mathsf{T}} \in \mathbb{F}^3 \setminus \mathbf{0}\}$,

where a is a column vector defining a hyperplane

$$\mathrm{Ker}\, a := \{x \in \mathbb{F}^3 \mid x \cdot a = 0\}.$$

Incidence is given by the subspace relation.

2.1.7 Proposition. *Consider $\mathcal{P}_2\mathbb{F}$ as a topological plane, endowed with the quotient topology on $\mathrm{P}_2\mathbb{F}$ which is induced by the canonical projection $\mathbb{F}^3 \setminus \mathbf{0} \twoheadrightarrow \mathrm{P}_2\mathbb{F}$. Denote $n := \dim_{\mathbb{R}} \mathbb{F}$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.*

a) *$\mathrm{P}_2\mathbb{F}$ and $\mathcal{L}_2\mathbb{F}$ are compact Hausdorff spaces.*

b) $\forall L \in \mathcal{L}_2\mathbb{F}.\quad L \approx \mathbb{S}_n$
In particular, every line is (path-)connected and compact.

c) *The flag space $\{(p,L) \in \mathrm{P}_2\mathbb{F} \times \mathcal{L}_2\mathbb{F} \mid p \leq L\}$ is closed in $\mathrm{P}_2\mathbb{F} \times \mathcal{L}_2\mathbb{F}$.*

d) *$\mathcal{P}_2\mathbb{F}$ is a topological projective plane.*

The same assertions hold for $\mathbb{F} = \mathbb{O}$.

Proof. *ad (a).* 14.4 and 16.9 in [54]. *ad (b).* 14.7 and 16.13 in [54]. *ad (c).* This is proved in [54, 14.6] and within the proof of [54, 16.11]. *ad (d).* 14.4 and 16.11 in [54]. □

2.1.8 The elliptic motion group. Let $\alpha : \mathbb{F} \to \mathbb{F}$ denote conjugation in $\mathbb{F}$, that is,

$$\begin{array}{rcrcl} \alpha : \mathbb{C} \to \mathbb{C} & : & z = x + iy & \mapsto & x - iy = \overline{z} \\ \alpha : \mathbb{H} \to \mathbb{H} & : & u + jv & \mapsto & \overline{u} - jv \quad . \end{array}$$

Along with the unit matrix $\mathbb{1}$, this defines an α-sesquilinear form

$$\begin{array}{rrcl} e: & \mathbb{F}^3 \times \mathbb{F}^3 & \to & \mathbb{F} \\ & (x, y) & \mapsto & x\mathbb{1}y^\mathsf{T} = x_1 y_1^\alpha + x_2 y_2^\alpha + x_3 y_3^\alpha \end{array}$$

The corresponding unitary group is

$$\mathrm{U}_3\mathbb{F} := \mathrm{U}(\mathbb{F}^3, e) = \{A \in \mathrm{GL}_3\mathbb{F} \mid A \cdot A^{\alpha\mathsf{T}} = \mathbb{1}\}.$$

Note that the elliptic motion groups $\mathrm{PU}_3\mathbb{F}$ below are simple Lie groups for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$; see [54, 18.27]. The elliptic motion group over the octonions is isomorphic to the (simple) real exceptional group $F_{4(-52)}$; see [54, 18.15].

2.1.9 The geometry. The unitary group $\mathrm{U}_3\mathbb{F}$ acts on $\mathcal{P}_2\mathbb{F}$ via the

$$\begin{array}{lccl} \text{point action} & \mathrm{P}_2\mathbb{F} \times \mathrm{U}_3\mathbb{F} & \to & \mathrm{P}_2\mathbb{F} \\ & (\mathbb{F}x, A) & \mapsto & \mathbb{F}xA \\ \text{line action} & \mathcal{L}_2\mathbb{F} \times \mathrm{U}_3\mathbb{F} & \to & \mathcal{L}_2\mathbb{F} \\ & (\operatorname{Ker} a, A) & \mapsto & \operatorname{Ker}(A^{-1}a). \end{array}$$

For each matrix $A \in \mathrm{GL}_3\mathbb{F}$ define the map

$$[A] : \mathrm{P}_2\mathbb{F} \to \mathrm{P}_2\mathbb{F} : \quad \mathbb{F} \mapsto \mathbb{F}xA.$$

The projective group then is

$$\mathrm{PGL}_3\mathbb{F} := \{[A] \mid A \in \mathrm{GL}_3\mathbb{F}\},$$

and the projective unitary group consequently is

$$\mathrm{PU}_3\mathbb{F} = \{[A] \mid A \in \mathrm{U}_3\mathbb{F}\}.$$

We will consider the effective action

$$\begin{array}{rclcrcl} \mathrm{P}_2\mathbb{F} \times \mathrm{PU}_3\mathbb{F} & \to & \mathrm{P}_2\mathbb{F} & : & (\mathbb{F}x, [A]) & \mapsto & \mathbb{F}xA \\ \mathcal{L}_2\mathbb{F} \times \mathrm{PU}_3\mathbb{F} & \to & \mathcal{L}_2\mathbb{F} & : & (\operatorname{Ker} a, [A]) & \mapsto & \operatorname{Ker}(A^{-1}a) \end{array}$$

of $\mathrm{PU}_3\mathbb{F}$ on $\mathcal{P}_2\mathbb{F}$.

2.1.10 Lemma. $(\mathrm{PU}_3\mathbb{F}, \mathcal{P}_2\mathbb{F})$ *is a flag homogeneous sketched geometry with point representative* $R_P = \{\mathbb{F}e_1\}$ *and line representative* $R_\mathcal{L} = \{\operatorname{Ker}\begin{pmatrix}0\\0\\1\end{pmatrix}\}$, *for instance. Flag transitivity is also given for* $\mathbb{F} = \mathbb{O}$.

Proof. Flag transitivity of the elliptic motion group is checked by hand, or by opening [54] at 13.15 and 18.10. For the set of representatives we may pick our favourite flag. ∎

Hyperbolic planes

The hyperbolic planes emerge from considering $\mathcal{P}_2\mathbb{F}$ under the action of a second motion group, which is gained from an α-sesquilinear form of Witt index 1.

2.1.11 The hyperbolic motion group. Again, denote by $\alpha : \mathbb{F} \to \mathbb{F}$ conjugation in $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Along with the diagonal matrix

$$H := \operatorname{diag}(1, 1, -1) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

this defines the hyperbolic α-sesquilinear form

$$\begin{array}{rccl} h: & \mathbb{F}^3 \times \mathbb{F}^3 & \to & \mathbb{F} \\ & (x, y) & \mapsto & xHy^{\alpha\mathsf{T}} = x_1 y_1^\alpha + x_2 y_2^\alpha - x_3 y_3^\alpha \end{array} .$$

The corresponding unitary group is

$$\mathrm{U}_3\mathbb{F}(1) := \mathrm{U}(\mathbb{F}^3, h) = \{A \in \mathrm{GL}_3\mathbb{F} \mid AHA^{\alpha\mathsf{T}} = H\},$$

and the projective group is

$$\mathrm{PU}_3\mathbb{F}(1) := \{[A] \mid A \in \mathrm{U}_3\mathbb{F}(1)\}.$$

Note that the hyperbolic motion groups $\mathrm{PU}_3\mathbb{F}(1)$ are simple Lie groups for $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$. For the real case, its connected component $\mathrm{PO}_3^1\mathbb{R}(1)$ is simple; see [54, 18.27]. The hyperbolic motion group over the octonions is isomorphic to the (simple) real exceptional Lie group $F_{4(-20)}$; see [54, 18.26].

2.1.12 The geometry. The point action

$$\mathrm{P}_2\mathbb{F} \times \mathrm{PU}_3\mathbb{F}(1) \to \mathrm{P}_2\mathbb{F} : (\mathbb{F}x, [A]) \mapsto \mathbb{F}xA$$

along with the line action

$$\mathcal{L}_2\mathbb{F} \times \mathrm{PU}_3\mathbb{F}(1) \to \mathcal{L}_2\mathbb{F} : (\operatorname{Ker} a, [A]) \mapsto \operatorname{Ker}(A^{-1}a) = \operatorname{Ker}(HA^{\alpha\mathsf{T}}Ha)$$

then turn $(\mathrm{PU}_3\mathbb{F}(1), \mathcal{P}_2\mathbb{F})$ into a geometry.

In order to obtain the point orbits one could, of course, use Witt's theorem. But when dealing with skew hyperbolic planes it is convenient to re-use some of the stabilisers we will compute here.

2.1.13 Lemma.

a) *The stabiliser of the point* $o := \mathbb{F}(0, 0, 1)$ *is*

$$\Phi_o := \mathrm{PU}_3\mathbb{F}(1)_{\mathbb{F}e_3} = \left\{ \left[\begin{array}{c|c} B & \\ \hline & w \end{array}\right] \;\middle|\; \begin{array}{l} B \in \mathrm{U}_2\mathbb{F} \\ ww^\alpha = 1 \end{array} \right\}.$$

b) Φ_o *leaves* $W := \operatorname{Ker}\begin{pmatrix}0\\0\\1\end{pmatrix}$ *invariant.*

c) Φ_o *acts transitively on the point row of* W.

Proof. [54, 13.14] □

2.1.14 Lemma. *The point orbits of* $\mathcal{P}_2\mathbb{F}$ *under* Φ_o *are*

$$\mathbb{F}(0,0,1)^{\Phi_o} = \{\mathbb{F}(0,0,1)\} =: \mathbf{S_0}$$

$$\mathbb{F}(0,1,0)^{\Phi_o} = \{\mathbb{F}(x,y,0) \mid (x,y) \neq (0,0)\}$$

$$\mathbb{F}(\lambda,0,1)^{\Phi_o} = \{\mathbb{F}(x,y,1) \mid xx^\alpha + yy^\alpha = \lambda\lambda^\alpha =: r\} =: \mathbf{S_r}$$
for $\lambda \in \mathbb{R}^+$, *hence* $r \in \mathbb{R}^+$

2.1.15 Lemma. ***Point orbits under*** $\mathrm{PU}_3\mathbb{F}(1)$**.**

$$\begin{array}{lcccccl}
\mathbb{F}e_3^{\mathrm{PU}_3\mathbb{F}(1)} & = & \bigcup\limits_{0\le r<1} S_r & = & \{\mathbb{F}x \mid h(x,x)<0\} & =: & \mathbf{I}\\
\mathbb{F}e_2^{\mathrm{PU}_3\mathbb{F}(1)} & = & W \cup \bigcup\limits_{r>1} S_r & = & \{\mathbb{F}x \mid h(x,x)>0\} & =: & \mathbf{E}\\
\mathbb{F}(1,0,1)^{\mathrm{PU}_3\mathbb{F}(1)} & = & S_1 & = & \{\mathbb{F}x \mid h(x,x)=0\} & =: & \mathbf{Q}
\end{array}$$

Proof. $\mathbb{F}e_3^{\mathrm{PU}_3\mathbb{F}(1)}$ and $\mathbb{F}e_2^{\mathrm{PU}_3\mathbb{F}(1)}$ contain elements of every S_r for $0 \le r < 1$ and $r > 1$, respectively. This, in fact, is achieved by the one parameter group

$$\left\{ \left[\begin{array}{c|cc} 1 & & \\ \hline & \sqrt{1+t^2} & t \\ & t & \sqrt{1+t^2} \end{array}\right] \,\middle|\, t \in \mathbb{R} \right\} \le \mathrm{PU}_3\mathbb{F}(1)_{\mathbb{F}e_1} \quad .$$

Transitivity of Φ_o on the sets S_r as well as on W then yields the assertion. □

Finding the line orbits of $\mathcal{P}_2\mathbb{F}$ under the action of the hyperbolic motion group can be simplified a lot by the exploitation of the duality principle for projective planes. To that end, we will first translate the current situation into new notions, and afterwards, with great relish, milk the conception.

2.1.16 The standard hyperbolic polarity.

$$\begin{array}{rlcccc}
\pi: & \mathrm{P}_2\mathbb{F} \to \mathcal{L}_2\mathbb{F} & : & U & \mapsto & U^{\perp_h}\\
 & \mathcal{L}_2\mathbb{F} \to \mathrm{P}_2\mathbb{F} & : & U & \mapsto & U^{\perp_h}
\end{array}$$

is a polarity of $\mathcal{P}_2\mathbb{F}$, i.e., an involutory duality. In homogeneous coordinates, it will be written as

$$\begin{array}{rlcccc}
\pi: & \mathrm{P}_2\mathbb{F} \to \mathcal{L}_2\mathbb{F} & : & \mathbb{F}x & \mapsto & \operatorname{Ker}(Hx^{\alpha\mathsf{T}})\\
 & \mathcal{L}_2\mathbb{F} \to \mathrm{P}_2\mathbb{F} & : & \operatorname{Ker} a & \mapsto & \mathbb{F}(a^{\alpha\mathsf{T}}H)
\end{array} \quad .$$

A collineation γ of $\mathcal{P}_2\mathbb{F}$ is said to **commute** with a duality π if $\gamma\pi = \pi\gamma$. Each element of $\mathrm{PU}_3\mathbb{F}(1)$ commutes with the standard hyperbolic polarity π. A point p is called an **absolute point** if it is incident with its **polar** p^π. Conversely, an **absolute line** L is by definition incident with its **pole** L^π. The set of absolute points/lines is left invariant by every collineation which commutes with π.

2.1.17 Definition. A point $p \in \mathcal{P}_2\mathbb{F}$ is called an **exterior point** if it is not absolute, yet incident with an absolute line. It is an **interior point** if it is not absolute and not incident with any absolute line. Let us collect the different types of points into

$$\begin{array}{lcl} \mathsf{Abs} & := & \text{the set of all absolute points} \\ \mathsf{Int} & := & \text{the set of all interior points} \\ \mathsf{Ext} & := & \text{the set of all exterior points .} \end{array}$$

The fact that any element of $\mathrm{PU}_3\mathbb{F}(1)$ commutes with π and therefore maps absolute items to absolute items, immediately yields

2.1.18 Lemma. $\mathrm{PU}_3\mathbb{F}(1)$ *leaves* Abs, Int *and* Ext *invariant.*

2.1.19 Lemma. $\mathsf{Abs} = Q = \mathbb{F}(1,0,1)^{\mathrm{PU}_3\mathbb{F}(1)}$

Proof. $\mathbb{F}x$ and $\mathbb{F}x^\pi$ are incident if and only if $0 = x \cdot Hx^{\alpha\mathsf{T}}$, that is, if and only if $x \in Q$. Transitivity was proved in 2.1.15. □

2.1.20 Lemma. $\mathrm{Ker}\, a$ *is an absolute line if and only if* $\mathbb{F}a^\mathsf{T} \in Q = \mathsf{Abs}$. □

2.1.21 Lemma. $\mathbb{F}e_3$ *is an interior point, and* $\mathbb{F}e_2$ *is an exterior point.*

Proof. *ad (a).* $\mathbb{F}e_3$ is not absolute, since it is not contained in Q. Let $L \in \mathcal{L}_{\mathbb{F}e_3}$. Then $L = \mathrm{Ker}\begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix}$ for $(a_1, a_2) \neq (0,0)$. Its pole is $L_\pi = \mathbb{F}(a_1^\alpha, a_2^\alpha, 0)$, and L is not incident with its pole because $(a_1^\alpha, a_2^\alpha, 0) \cdot \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} = a_1a_1^\alpha + a_2a_2^\alpha > 0$. Therefore L is not absolute, and p is an interior point. *ad (b).* Clearly, $\mathbb{F}e_2$ is not absolute, either. Let $L \in \mathcal{L}_{\mathbb{F}e_2}$. Then $L = \mathrm{Ker}\begin{pmatrix} a_1 \\ 0 \\ a_3 \end{pmatrix}$ with $(a_1, a_3) \neq (0,0)$. L is incident with its pole $L^\pi = \mathbb{F}(a_1^\alpha, 0, -a_3^\alpha)$ if and only if $a_1a_1^\alpha = a_3a_3^\alpha$. This is achievable, and hence $\mathbb{F}e_2$ is an exterior point. □

2.1.22 Corollary.
$$\begin{array}{lll} I & = \mathbb{F}e_3^{\mathrm{PU}_3\mathbb{F}(1)} & = \mathsf{Int} \\ E & = \mathbb{F}e_2^{\mathrm{PU}_3\mathbb{F}(1)} & = \mathsf{Ext} \end{array}$$

Proof. Lemma 2.1.21 along with lemma 2.1.18 and the orbits known from 2.1.15. □

Equipped with that knowledge the line orbits are going to be tackled.

2.1.23 Definition. For the moment, a line L will be contained in

$$\begin{array}{lcl} \mathsf{TypeT} & :\Leftrightarrow & L \text{ is absolute} \\ \mathsf{TypeP} & :\Leftrightarrow & L \text{ is not absolute, yet incident with an absolute point} \\ \mathsf{TypeS} & :\Leftrightarrow & L \text{ is not absolute, and it is not incident with any absolute point.} \end{array}$$

These definitions are dual to those of Abs, Int and Ext, which accounts for

2.1.24 Lemma. *The polarity π is a bijection interchanging the following sets of points and lines :*

$$\begin{array}{rcl} Q = \mathsf{Abs} & \longleftrightarrow & \mathsf{TypeT} \\ I = \mathsf{Int} & \longleftrightarrow & \mathsf{TypeP} \\ E = \mathsf{Ext} & \longleftrightarrow & \mathsf{TypeS} \quad . \end{array}$$

2.1.25 Corollary.
$$\begin{array}{lclcl} \mathsf{TypeT} & = & Q^\pi & = & \{\mathrm{Ker}\, a \mid h(a^\mathsf{T}, a^\mathsf{T}) = 0\} \\ \mathsf{TypeP} & = & I^\pi & = & \{\mathrm{Ker}\, a \mid h(a^\mathsf{T}, a^\mathsf{T}) < 0\} \\ \mathsf{TypeS} & = & E^\pi & = & \{\mathrm{Ker}\, a \mid h(a^\mathsf{T}, a^\mathsf{T}) > 0\} \end{array}$$

Moreover, talking about orbits, we harvest

2.1.26 Corollary. *Putting*

$$\begin{array}{lclcl} T_o & := & \mathbb{F}(1,0,1)^\pi & = & \mathrm{Ker}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ P_o & := & \mathbb{F}e_3^\pi & = & \mathrm{Ker}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ S_o & := & \mathbb{F}e_2^\pi & = & \mathrm{Ker}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{array}$$

we get

$$\begin{array}{lcl} \mathsf{TypeT} & = & T_o^{\mathrm{PU}_3\mathbb{F}(1)} \\ \mathsf{TypeP} & = & P_o^{\mathrm{PU}_3\mathbb{F}(1)} \\ \mathsf{TypeS} & = & S_o^{\mathrm{PU}_3\mathbb{F}(1)} \quad . \end{array}$$

Proof. As an example, let us consider the lines of type T :

$$\mathsf{TypeT} = Q^\pi = \mathbb{F}(1,0,1)^{\mathrm{PU}_3\mathbb{F}(1)\cdot\pi} = \mathbb{F}(1,0,1)^{\pi\cdot\mathrm{PU}_3\mathbb{F}(1)} = T_o^{\mathrm{PU}_3\mathbb{F}(1)},$$

making use of the above, 2.1.15 and the fact that $\mathrm{PU}_3\mathbb{F}(1)$ commutes with π. $\square$

It remains to link those abstract types of lines to our intuition : indeed, they are what we want them to be, namely the tangents, passing lines and secants with respect to the unital Q.

2.1.27 Definition. A line $L \in \mathcal{L}_2\mathbb{F}$ is called a

passing line	$:\Leftrightarrow$	$\lvert L \cap Q\rvert = 0$
tangent	$:\Leftrightarrow$	$\lvert L \cap Q\rvert = 1$
secant	$:\Leftrightarrow$	$\lvert L \cap Q\rvert \geq 2$

2.1.28 Lemma. *T_o is a tangent, P_o is a passing line, and S_o is a secant.*

Proof. *T_o is a tangent :* Any point q contained in T_o is of the form $q = \mathbb{F}(a,b,a)$. Additionally, it is a point of the unital if and only if $0 = h(q,q) = aa^\alpha + bb^\alpha - aa^\alpha = bb^\alpha$, which is true if and only if $q = \mathbb{F}(1,0,1)$. Thus, $|T_o \cap Q| = 1$.

S_o is a secant : Any point p contained in S_o is of the form $q = \mathbb{F}(a,0,b)$ with $(a,b) \neq (0,0)$. It is absolute if and only if $0 = h(q,q) = aa^\alpha - bb^\alpha$. As $b \neq 0$, this is equivalent to $q = \mathbb{F}(ab^{-1},0,1)$, where $(ab^{-1})(ab^{-1})^\alpha = (aa^\alpha)(bb^\alpha)^{-1} = 1$, in other words, $q = \mathbb{F}(x,0,1)$ satisfying $xx^\alpha = 1$. Hence,

$$S_o \cap Q = \{\mathbb{F}(x,0,1) \mid xx^\alpha = 1\},$$

which corresponds to $\mathbb{S}_{n-1}$ for $n := \dim_{\mathbb{R}} \mathbb{F}$. Thus, $|S_o \cap Q| = |\mathbb{S}_{n-1}| \geq 2$.

P_o is a passing line : A point $q = \mathbb{F}(a,b,0) \in S_o$, where $(a,b) \neq (0,0)$, is absolute if and only if $0 = h(q,q) = aa^\alpha + bb^\alpha > 0$, which is to say : never. Hence, $|P_o \cap Q| = 0$. $\square$

2.1.29 Corollary. TypeT *is the set of tangents,*
TypeP *is the set of passing lines, and*
TypeS *is the set of secants.*

Proof. Let $L_o \in \{T_o, P_o, S_o\}$ be one of the standard lines, let $\alpha \in \mathrm{PU}_3\mathbb{F}(1)$. Due to 2.1.18 and the fact that α is an isomorphism in Inc, we get $|L_o^\alpha \cap Q| = |L_o \cap Q|$. $\square$

Three little details long to be mentioned : To start with, from the proof of 2.1.28 we extract

2.1.30 Lemma. *Let S be a secant in $\mathcal{P}_2\mathbb{F}$. Then the intersection of S and the unital Q is homeomorphic to the sphere $\mathbb{S}_{n-1}$, where $n := \dim_{\mathbb{R}} \mathbb{F}$.* $\square$

2.1.31 Lemma.

a) *S_o contains interior points.*

b) *Neither T_o nor P_o contain interior points.*

c) *A line S in $\mathcal{P}_2\mathbb{F}$ is a secant if and only if it contains interior points.*

Proof. $\mathbb{F}e_3 \in I \cap S_o$, and by 2.1.15 and 2.1.26, every line of type S contains interior points. Little standard computations reveal that neither T_o nor P_o contain interior points, and again, this implies that none of the lines of type T or P does. $\square$

2.1.32 Lemma. *The line pencil of any exterior point in $\mathcal{P}_2\mathbb{F}$ comprises secants, tangents and passing lines.*

Proof. The line pencil $\mathcal{L}_{\mathbb{F}e_2}$ of $\mathbb{F}e_2 \in E$ contains $\operatorname{Ker}\begin{pmatrix}0\\0\\1\end{pmatrix} = P_o \in \mathsf{TypeP}$, $\operatorname{Ker}\begin{pmatrix}1\\0\\-1\end{pmatrix} = T_o \in \mathsf{TypeT}$ as well as $\operatorname{Ker}\begin{pmatrix}1\\0\\0\end{pmatrix} \in \mathsf{TypeS}$. This implies that every exterior point $\mathbb{F}e_2^\alpha$, for $\alpha \in \mathrm{PU}_3\mathbb{F}(1)$, is incident with their respective images under α. $\square$

2.1.33 The interior, exterior and united hyperbolic planes. Motivated by the point orbits of the hyperbolic motion group on $\mathcal{P}_2\mathbb{F}$, the corresponding open subplanes of $\mathcal{P}_2\mathbb{F}$ have been singled out. They generalise the real stable planes presented in the introductory chapter :

the **interior hyperbolic plane**	$\mathrm{IH}\,\mathbb{F}$	is induced by the point set	I
the **exterior hyperbolic plane**	$\mathrm{EH}\,\mathbb{F}$	is induced by the point set	E
the **united hyperbolic plane**	$\mathrm{UH}\,\mathbb{F}$	is induced by the point set	$I \cup E$

Note that by 2.1.32, the line spaces of the exterior and united hyperbolic planes equal $\mathcal{L}_2\mathbb{F}$, whereas the line space of the interior hyperbolic plane consists of the secants only.

Let us summarise the results on orbits and sketchability so far :

2.1.34 Proposition.

a) $(\mathrm{PU}_3\mathbb{F}(1), \mathcal{P}_2\mathbb{F})$ *consists of the three point orbits I, E and Q. The line orbits are the sets of secants, tangents and passing lines. It cannot be sketched.*

b) $(\mathrm{PU}_3\mathbb{F}(1), \mathrm{UH}\,\mathbb{F})$ *consists of the two point orbits I and E along with the three line orbits. It cannot be sketched, either.*

Proof. The orbits have been provided for in 2.1.15 and 2.1.26. As to sketchability, recall that by 1.1.19, a sketched linear space has to be point homogeneous or line homogeneous, which both candidates are not. $\square$

It remains to show that the action of $\mathrm{PU}_3\mathbb{F}(1)$ on both, $\mathrm{IH}\,\mathbb{F}$ and $\mathrm{EH}\,\mathbb{F}$ ensures sketched geometries. As they are point homogeneous, criterion 1.1.23 states that equivalently, we may show that each line is $\mathrm{PU}_3\mathbb{F}(1)$-isotropic. For convenience's sake, while talking about stabilisers, we will again abbreviate $\Phi := \mathrm{PU}_3\mathbb{F}(1)$.

2.1.35 Lemma. *Let $x \in \mathrm{P}_2\mathbb{F} \,\dot\cup\, \mathcal{L}_2\mathbb{F}$ be a point or a line in $\mathcal{P}_2\mathbb{F}$. Let $\Upsilon \leq \Phi$ be a subgroup of $\mathrm{PU}_3\mathbb{F}(1)$. Then $\Upsilon_x = \Upsilon_{x^\pi}$, that is, the stabiliser of a point equals the stabiliser of its polar, and vice versa.*

Proof. Let $\gamma \in \Upsilon_x$. Then $x^{\pi\gamma} = x^{\gamma\pi} = x^\pi$, because every element of the hyperbolic motion group $\mathrm{PU}_3\mathbb{F}(1)$ commutes with the hyperbolic polarity. Conversely, let $\gamma \in \Upsilon_{x^\pi}$. Then $x^\gamma = x^{\pi\cdot\pi\gamma} = x^{\pi\gamma\cdot\pi} = x^{\pi\pi} = x$, since π is an involution. $\square$

As the line space of $\mathrm{IH}\,\mathbb{F}$ consists of the secants only, the interior plane can be dealt with fairly quickly :

2.1.36 Lemma.

a) *$S_o \cap I$ is Φ-isotropic.*

b) *Every secant in $\mathrm{IH}\,\mathbb{F}$ is Φ-isotropic.*

Proof. *ad (a).* The group

$$\mathsf{K} := \left\{ K_x := \begin{bmatrix} \sqrt{1+xx^\alpha} & 0 & x \\ 0 & 1 & 0 \\ x^\alpha & 0 & \sqrt{1+xx^\alpha} \end{bmatrix} \middle| \, x \in \mathbb{F} \right\}$$

is contained in $\mathrm{PU}_3\mathbb{F}(1)_{\mathbb{F}e_2} \leq \Phi_{\mathbb{F}e_2} = \Phi_{S_o}$. Therefore,

$$\mathbb{F}e_3^{\Phi_{S_o}} \supseteq \{\mathbb{F}(x, 0, \sqrt{1+xx^\alpha}) \mid x \in \mathbb{F}\} = \{\mathbb{F}(x,0,1) \mid x \in \mathbb{F},\ |x| < 1\},$$

which equals $S_o \cap I$. *Part (b)* follows from part (a), 1.1.22 and transitivity of Φ on secants. □

2.1.37 Proposition. $(\mathrm{PU}_3\mathbb{F}(1), \mathrm{IH}\,\mathbb{F})$ *is a flag homogeneous sketched geometry. A system of representatives is, for instance,*

$$R_P = \{\mathbb{F}e_3\} \text{ and } R_{\mathcal{L}} = \{\mathrm{Ker} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \}.$$

Flag homogeneity also holds for $\mathbb{F} = \mathbb{O}$.

Proof. This emerges from the criterion in 1.1.23, feeding it with point transitivity 2.1.15, Φ-isotropy of the secants, and the fact that the secants provide for all lines in $\mathrm{IH}\,\mathbb{F}$, which is implied by 2.1.31. For the octonion case, flag transitivity is proved in [54, 18.23]. □

Applying the same criterion to the exterior plane $\mathrm{EH}\,\mathbb{F}$ means asking all types of lines for Φ-isotropy.

2.1.38 Lemma.

a) *$S_o \cap E$ is Φ-isotropic.*

b) *P_o is Φ-isotropic.*

c) *$T_o \cap E$ is Φ-isotropic.*

d) *Every line in $\mathrm{EH}\,\mathbb{F}$ is Φ-isotropic.*

Proof. *ad (a).* Using the elements $K_x \in \mathsf{K}$ from 2.1.36, one gets

$$\mathbb{F}e_1^{\Phi_{S_o}} \supseteq \{\mathbb{F}(\sqrt{1+xx^\alpha}, 0, x) \mid x \in \mathbb{F}\} = \{\mathbb{F}e_1\} \cup \{\mathbb{F}(x,0,1) \mid x \in \mathbb{F},\ |x| > 1\},$$

which equals $S_o \cap E$. *ad (b).* By 2.1.13, the stabiliser $\Phi_{P_o} = \Phi_{\mathbb{F}e_3} = \Phi_o$ acts transitively on $W = P_o$. *ad (c).* The group

$$\mathsf{N} := \left\{ \begin{bmatrix} 1 - \frac{ss^\alpha}{2} & -s^\alpha & -\frac{ss^\alpha}{2} \\ s & 1 & s \\ \frac{ss^\alpha}{2} & s^\alpha & 1 + \frac{ss^\alpha}{2} \end{bmatrix} \middle| \ s \in \mathbb{F} \right\}$$

is contained in $\mathrm{PU}_3\mathbb{F}(1)_{\mathbb{F}(1,0,1)} = \Phi_{\mathbb{F}(1,0,1)} = \Phi_{T_o}$. Therefore, $\mathbb{F}e_2^{\Phi_{T_o}} = \{\mathbb{F}(s,1,s) \mid s \in \mathbb{F}\}$, which equals $T_o \cap E$. *ad (d).* We have seen that the standard lines S_o, T_o and P_o are Φ-isotropic. By 1.1.22, then so is each line within their orbits under Φ. By 2.1.26 and 2.1.32, these orbits constitute the line space of $\mathrm{EH}\,\mathbb{F}$. □

Again, point homogeneity and criterion 1.1.23 yield

2.1.39 Proposition. $(\mathrm{PU}_3\mathbb{F}(1), \mathrm{EH}\,\mathbb{F})$ *is a point homogeneous sketched geometry. A system of representatives is, for instance,*

$$R_P = \{\mathbb{F}e_2\} \quad \text{and} \quad R_{\mathcal{L}} = \{\mathrm{Ker}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mathrm{Ker}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathrm{Ker}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\}.$$

Skew hyperbolic planes

For the real case $\mathbb{F} = \mathbb{R}$ an interesting series of non-desarguesian compact projective planes is obtained from $\mathcal{P}_2\mathbb{R}$ by keeping the tangents and passing lines as they are, but modifying the secants once they leave the interior of the quadric.

2.1.40 The automorphism group. Consider the connected component

$$\Omega := (\mathrm{PO}_3\mathbb{R}(1))^1$$

of $\mathrm{PO}_3\mathbb{R}(1)$. Intuitively, it consists of those elements from $\mathrm{PO}_3\mathbb{R}(1) = \mathrm{PSO}_3\mathbb{R}(1)$ that respect the orientation on the quadric Q. The reflection $\rho : \mathrm{P}_2\mathbb{R} \to \mathrm{P}_2\mathbb{R} : \mathbb{R}(x,y,z) \mapsto \mathbb{R}(x,-y,z)$ is not contained in Ω. As a matter of fact, $\mathrm{PO}_3\mathbb{R}(1) = \Omega \cdot \langle\rho\rangle$. It is moreover true that $\Omega = (\mathrm{PO}_3\mathbb{R}(1))^1 = (\mathrm{PO}_3\mathbb{R}(1))' \cong \mathrm{PSL}_2\mathbb{R}$; see [54, 15]. The hyperbolic group Ω contains the (path connected) one parameter groups

$$\mathsf{K} := \left\{ K_r := \begin{bmatrix} \sqrt{1+r^2} & 0 & r \\ 0 & 1 & 0 \\ r & 0 & \sqrt{1+r^2} \end{bmatrix} \middle| \ r \in \mathbb{R} \right\}$$

$$\mathsf{N} := \left\{ \left[\begin{array}{ccc} 1-\frac{s^2}{2} & -s & -\frac{s^2}{2} \\ s & 1 & s \\ \frac{s^2}{2} & s & 1+\frac{s^2}{2} \end{array} \right] \middle| \quad s \in \mathbb{R} \quad \right\}$$

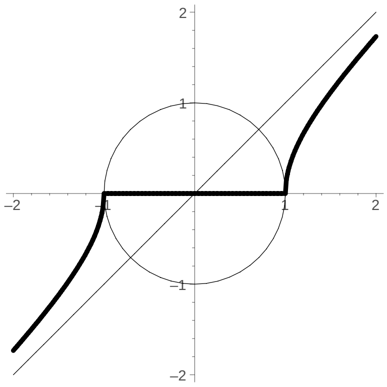
required earlier on, as well as the one parameter group

$$\mathsf{\Delta} := \left\{ \left[\begin{array}{c|c} B & \\ \hline & 1 \end{array} \right] \middle| B \in \mathrm{SO}_2\mathbb{R} \right\} \cong \mathrm{SO}_2\mathbb{R}.$$

As a matter of fact, Ω is generated by K, N and $\mathsf{\Delta}$; see [4, p. 32ff].

2.1.41 Skew hyperbolic planes. Let t be a real parameter. The new "standard" secant is defined as

$$L_t := \{\mathbb{R}(x,0,1) \mid |x| < 1\} \,\dot\cup\, \{\mathbb{R}(x,y,z) \mid y^2 = t^2(x^2 - z^2) \,\wedge\, txy \geq 0\}.$$



The **skew hyperbolic plane** (or *modified hyperbolic plane*) $\mathrm{H}_t\,\mathbb{R}$ is defined by

the point space $\mathrm{P}_2\mathbb{R}$
the line space $\mathcal{L}_t := \{L \in \mathcal{L}_2\mathbb{R} \mid |L \cap Q| \leq 1\} \,\dot\cup\, \{L_t^\omega \mid \omega \in \Omega\}$;

in other words, the set of secants in $\mathcal{P}_2\mathbb{R}$ is replaced by the orbit L_t^Ω of "modified" secants.

Note that the characterisation of lines in $\mathrm{H}_t\,\mathbb{R}$ – via the number of points it has in common with Q – remains the same; a line is a (modified) secant if and only if it intersects with the interior I. The open subplanes of $\mathrm{H}_t\,\mathbb{R}$ induced by E and $I \cup E$ are called the **exterior** and **united skew hyperbolic planes** $\mathrm{EH}_t\,\mathbb{R}$ and $\mathrm{UH}_t\,\mathbb{R}$, respectively; the interior hyperbolic plane induced by I equals the non-modified hyperbolic plane $\mathrm{IH}\,\mathbb{R}$, because the secants have not been modified inside I. The simple group Ω is the full automorphism group of $\mathrm{H}_t\,\mathbb{R}$, as well as of the interior, exterior and united skew hyperbolic planes. For further information on skew hyperbolic planes, the reader is referred to chapter 35 in Salzmann et al. [54] and Salzmann [53, 5.3], [52].

2.1.42 Lemma. **a)** *The stabiliser of the point* $o := \mathbb{R}(0,0,1)$ *is* $\Omega_o = \mathsf{\Delta} \cong \mathrm{SO}_2\mathbb{R}$.

b) *Ω_o acts transitively on Q.*

c) *Ω_o leaves $W = \operatorname{Ker}\begin{pmatrix}0\\0\\1\end{pmatrix}$ invariant.*

d) *Ω_o acts transitively on the point row of W.*

e) *The sets $S_r := \{\mathbb{R}(x,y,1) \mid x^2+y^2 = r\}$ for $r > 0$ are point orbits under the action of Ω_o.*

Proof. *ad (a).* Computation reveals that

$$\mathrm{SO}_3\mathbb{R}(1)_o = \left\{ \left(\begin{array}{c|c} B & \\ \hline & b \end{array} \right) \middle| B \in \mathrm{SO}_2\mathbb{R} \wedge b^2 = 1 \right\}.$$

ad (b). The action of Ω_o on $Q = \{\mathbb{R}(x,y,1) \mid x^2+y^2 = 1\}$ is therefore equivalent to the action of $\mathrm{SO}_2\mathbb{R}$ on $\mathrm{P}_1\mathbb{R} = \mathbb{S}_1$. *ad (c+d).* Part (a) implies that the point row $\{\mathbb{R}(x,y,0) \mid (x,y) \neq (0,0)\}$ of the line W at infinity is invariant under Ω_o. This action again is equivalent to the transitive action of $\mathrm{SO}_2\mathbb{R}$ on a circle. *ad (e).* The sets S_r are left invariant due to orthogonality of $\mathrm{SO}_2\mathbb{R} \cong \Omega_o$. The action is transitive because it is, once more, equivalent to the action of $\mathrm{SO}_2\mathbb{R}$ on a circle. □

2.1.43 Lemma. *Consider the points of intersection $a := \mathbb{R}(1,0,1)$ and $b := \mathbb{R}(-1,0,1)$ of the standard secant L_t and the quadric Q.*

a) $\Omega_{L_t} = \Omega_{\{a,b\}} = \Omega_{\operatorname{Ker}\begin{pmatrix}0\\1\\0\end{pmatrix}} = \Omega_{\mathbb{R}e_2}$

b) $\Omega_{\{a,b\}} \geq \Omega_{a,b} \geq \mathsf{K}$

N.B.: As a matter of fact, $\Omega_{\{a,b\}} \cong \Omega_{a,b} \rtimes <\sigma>$, where $\sigma : \mathrm{P}_2\mathbb{R} \to \mathrm{P}_2\mathbb{R} : \mathbb{R}x \to \mathbb{R}xH$ is the half-turn with axis $\mathbb{R}e_3$.

c) *Put $p_t := \mathbb{R}(1,t,0) \in L_t \cap E$. The orbit $o^{\Omega_{a,b}}$ contains elements of each S_r with $0 \leq r < 1$. The orbit $p_t^{\Omega_{a,b}}$ contains elements of each S_r with $r > 1$.*

d) $L_t \setminus \{a,b\} = (L_t \cap I) \cup (L_t \cap E) = p_t^{\Omega_{a,b}} \cup o^{\Omega_{a,b}}$.

Proof. *ad (a).* Every automorphism fixing $\{a,b\}$ fixes $L_t = a \vee b$. Conversely, every element of Ω_{L_t} has to leave $\{a,b\} = L_t \cap Q$ invariant. Therefore, $\Omega_{L_t} = \Omega_{\{a,b\}}$. Switching our point of view towards $\mathcal{P}_2\mathbb{R}$, we see that $\{a,b\} = S_o \cap Q$, and hence $\Omega_{\{a,b\}} = \Omega_{S_o} = \Omega_{\mathbb{R}e_2}$. *ad (b).* These are the elements $K_x \in \mathsf{K}$ already called in in 2.1.38, which are actually contained in the connected component $\Omega_{\mathbb{R}e_2} \leq \Phi_{\mathbb{R}e_2}$. *ad (c).* Using these elements, we get $o\,K_s = \mathbb{R}(s,0,\sqrt{1+s^2}) = \mathbb{R}(\frac{s}{1+s^2},0,1)$ with $r := \frac{s^2}{1+s^2}$ reaching any real value $0 \leq r < 1$. For $s \neq 0$, the images of p_t are $p_tK_s = \mathbb{R}(\sqrt{1+s^2},t,s) = \mathbb{R}(\frac{\sqrt{1+s^2}}{s},\frac{t}{s},1)$ with $r := \frac{1}{s^2}(1+s^2+t^2)$ reaching any real value $r > 1$. *ad (d).* $o^{\Omega_{a,b}} \supseteq \{\mathbb{R}(r,0,\sqrt{1+r^2} \mid r \in \mathbb{R}\} = L_t \cap I$ as before, and $p_t^{\Omega_{a,b}} \supseteq \{\mathbb{R}(\sqrt{1+r^2},t,r) \mid r \in \mathbb{R}\} = \{p_t\} \cup \{\mathbb{R}(x,y,1) \mid y^2 = t^2(x^2-1) \wedge txy \geq 0\} = L_t \cap E$. □

2.1.44 Corollary. *Every (modified) secant in* $\mathrm{IH}\,\mathbb{R}$ *and* $\mathrm{EH}_t\,\mathbb{R}$ *is* Ω*-isotropic.*

Proof. By part (c) of 2.1.43, the line stabiliser Ω_{L_t} is both, transitive on $L_t \cap I$ and $L_t \cap E$. The (modified) secants in $\mathrm{IH}\,\mathbb{R}$ and $\mathrm{EH}_t\,\mathbb{R}$, respectively, by definition form an orbit, and therefore 1.1.22 ensures Ω-isotropy for each of the secants. □

2.1.45 Corollary. *The point orbits of* Ω *on* $\mathrm{H}_t\,\mathbb{R}$ *are*

$$\begin{aligned} o^\Omega &= I \\ p_t^\Omega &= E \\ a^\Omega &= Q. \end{aligned}$$

Proof. Part (d) of 2.1.43 along with parts (e) and (d) of 2.1.42. □

2.1.46 Lemma. *The line orbits of* Ω *on* $\mathrm{H}_t\,\mathbb{R}$ *are the sets of passing lines, tangents and modified secants.*

Proof. Passing lines and tangents are conservative, i.e., directly imported from $\mathcal{P}_2\mathbb{R}$ in a non-modified way. Therefore, given transitivity of Ω on I and Q, we may once more profit from the hyperbolic polarity : the set of tangents equals $Q^\pi = a^{\Omega\cdot\pi} = a^{\pi\cdot\Omega} = T_o^\Omega$. Analogically, the set of passing lines equals P_o^Ω. Hence, there are at most these three orbits, and there are precisely these three orbits because $\Omega \le \mathrm{PO}_3\mathbb{R}(1)$. □

2.1.47 Corollary. *The line pencil of any exterior point in* $\mathrm{H}_t\,\mathbb{R}$ *contains passing lines, tangents and (modified) secants.*

Proof. The line pencil of our favourite exterior point $p_t = \mathbb{R}(1, t, 0)$ contains the secant L_t, the passing line $P_o = \mathrm{Ker}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and the tangent $T_t := \mathrm{Ker}\begin{pmatrix} -t \\ 1 \\ \sqrt{1+t^2} \end{pmatrix}$. Point transitivity of Ω on E (proved in 2.1.45) sees for the same situation in every exterior point. □

2.1.48 Corollary.

a) $(\Omega,\, \mathrm{H}_t\,\mathbb{R})$ *consists of three point orbits and three line orbits. It cannot be sketched.*

b) $(\Omega,\, \mathrm{UH}_t\,\mathbb{R})$ *consists of two point orbits and three line orbits. It cannot be sketched, either.*

c) $(\Omega,\, \mathrm{IH}\,\mathbb{R})$ *is a flag homogeneous sketched geometry.*

Proof. Non-sketchability is gained from lemmas 2.1.45 and 2.1.46, fed into criterion 1.1.19. The information on the types of lines required for $\mathrm{UH}_t\,\mathbb{R}$ stems from 2.1.47. As to $\mathrm{IH}\,\mathbb{R}$, by 2.1.45 and 2.1.44 it is a point homogeneous plane with Ω-isotropic lines and as a such is sketched by criterion 1.1.23. □

A discussion of the exterior skew hyperbolic plane requires knowledge on the Ω-isotropy of every single type of lines in $\mathrm{H}_t\,\mathbb{R}$.

2.1.49 Lemma. **a)** *Ω_{P_o} acts transitively on P_o.*

b) *Every passing line in $\mathrm{EH}_t\,\mathbb{R}$ is Ω-isotropic.*

Proof. Transitivity of $\Omega_{P_o} = \Omega_{\mathbb{R}e_3}$ on $P_o \cap E = P_o = W$ is precisely part (d) of 2.1.42. Lemma 1.1.22 then establishes Ω-isotropy of every line in the orbit P_o^Ω, by 2.1.46 thus on every passing line. □

2.1.50 Lemma. **a)** *Ω_{T_o} acts transitively on $T_o \cap E$.*

b) *Every tangent in $\mathrm{EH}_t\,\mathbb{R}$ is Ω-isotropic.*

Proof. Recall from 2.1.35 and 2.1.38 that $\Omega_{T_o} = \Omega_{a^\pi} = \Omega_a \geq \mathsf{N}$. Therefore, lemma 2.1.38 already states that $\mathbb{R}e_2^{\Omega_{T_o}} \supseteq \{\mathbb{R}(x,1,x) \mid x \in \mathbb{R}\} = T_o \setminus \{a\} = T_o \cap E$. Again, 1.1.22 and 2.1.46 establish the assertion. □

2.1.51 Proposition. $(\Omega,\, \mathrm{EH}_t\,\mathbb{R})$ *is a point homogeneous sketched geometry. A system of representatives is, for instance, given by*

$$R_P = \{\mathbb{R}(1,t,0)\} \quad \text{and} \quad R_{\mathcal{L}} = \{L_t, \operatorname{Ker}\begin{pmatrix}0\\0\\1\end{pmatrix}, \operatorname{Ker}\begin{pmatrix}-t\\1\\\sqrt{1+t^2}\end{pmatrix}\}.$$

2.2. Non-isotropic points therein

For the classification of line homogeneous sketched stable planes it will turn out to be of importance (in 2.3.3) that the planes $\mathrm{UH}\,\mathbb{F}$, $\mathrm{H}_t\,\mathbb{R}$ and $\mathrm{UH}_t\,\mathbb{R}$ with $t \neq 0$ contain non-isotropic points. This information will be provided beforehand, profiting by the particular topological or geometrical structures of these candidates.

Non-isotropic points in united hyperbolic planes

2.2.1 Lemma. *For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, any passing line in $\mathrm{UH}\,\mathbb{F}$ is compact and connected.*

Proof. By 2.1.7, any line in $\mathcal{P}_2\mathbb{F}$ is compact and connected. Passing lines in $\mathrm{UH}\,\mathbb{F}$ are by definition imported from $\mathcal{P}_2\mathbb{F}$ without doing them any harm. □

2.2.2 Lemma. *For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, neither tangents nor secants in $\mathrm{UH}\,\mathbb{F}$ are compact.*

Proof. *The unital Q is closed in* $\mathrm{P}_2\mathbb{F}$. The set $\mathrm{Flags} := \{(p, L) \in \mathrm{P}_2\mathbb{F} \times \mathcal{L}_2\mathbb{F} \mid p \text{ incident } L\}$ of flags is closed in $\mathrm{P}_2\mathbb{F} \times \mathcal{L}_2\mathbb{F}$; cf. 2.1.7. Hence so is $\mathrm{Fix}\,\pi \cap \mathrm{Flags} = \{(p, p^\pi) \mid p \in P \wedge p \text{ incident } p^\pi\}$. The canonical projection $\mathrm{pr} : \mathrm{P}_2\mathbb{F} \times \mathcal{L}_2\mathbb{F} \to \mathrm{P}_2\mathbb{F}$ is a closed map, because $\mathrm{P}_2\mathbb{F}$ and $\mathcal{L}_2\mathbb{F}$ are compact Hausdorff spaces [2.1.7]. Consequently, the set $(\mathrm{Fix}\,\pi \cap \mathrm{Flags})^{\mathrm{pr}} = \{p \in P \mid p \text{ incident } p^\pi\} = Q$ of absolute points is a closed subset of $\mathrm{P}_2\mathbb{F}$. Furthermore, *Q is compact.*

Now let L be a tangent or secant in $\mathrm{UH}\,\mathbb{F}$, and let $L' \in \mathcal{L}_2\mathbb{F}$ be the line in $\mathcal{P}_2\mathbb{F}$ it stems from. Then $L' \cap Q = L' \setminus L \neq \emptyset$. The point set of L is not closed in $\mathrm{P}_2\mathbb{F}$: otherwise, the connected set $L' = L \,\dot\cup\, (L' \cap Q)$ were the disjoint union of two proper closed subsets. Consequently L is not compact, because $\mathrm{P}_2\mathbb{F}$ is a compact Hausdorff space. □

2.2.3 Lemma. *For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, tangents in $\mathrm{UH}\,\mathbb{F}$ are connected, whereas secants are not.*

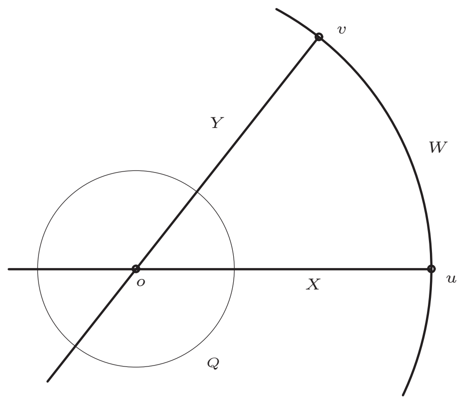
Proof. Let $L' \in \mathcal{L}_2\mathbb{F}$ be a line in $\mathcal{P}_2\mathbb{F}$, and let L be its trace in $\mathrm{UH}\,\mathbb{F}$. By 2.1.7, the point set of L' is homeomorphic to the n-sphere $\mathbb{S}_n$, where $n := \dim_{\mathbb{R}} \mathbb{F}$. If L' is a tangent, then $|L' \setminus L| = 1$, and therefore L is homeomorphic to the n-sphere minus one point, which still is (path-)connected. If L' is a secant, consider the decomposition $L = L' \setminus Q = (L' \cap I) \,\dot\cup\, (L' \cap E)$. I and E are open subsets of $\mathrm{P}_2\mathbb{F}$, and therefore $L' \cap I$ as well as $L' \cap E$ are open subsets of L. Moreover, the standard secant S_o from 2.1.26 comprises interior as well as exterior points. As the secants form an orbit under $\mathrm{PU}_3\mathbb{F}(1)$, so does any other secant $L' \in \mathcal{L}_2\mathbb{F}$, and consequently both, $L' \cap I$ and $L' \cap E$ are non-empty. Therefore, the secant L in $\mathrm{UH}\,\mathbb{F}$ is not connected. □

2.2.4 Corollary. *For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, every exterior point in $\mathrm{UH}\,\mathbb{F}$ is non-isotropic.*

Proof. This follows from 2.1.32, 2.2.1 and 2.2.2 along with the fact that we are considering continuous automorphisms of $\mathrm{UH}\,\mathbb{F}$. As a matter of fact, the sets of passing lines, tangents and secants, respectively, are invariant under $\mathrm{Aut}\,\mathrm{UH}\,\mathbb{F}$. □

The explicit constructions behind these arguments are valid for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. For the octonions $\mathbb{O}$, technique becomes more elaborate whereas the structural arguments remain the same. We will content ourselves with a brief guide line as to how to browse through the results in chapter 18 of Salzmann et al. [54] in order to obtain the desired result.

2.2.5 The octonion case. The octonion projective plane $\mathcal{P}_2\mathbb{O}$ allows for polarities; denote by π^- the standard hyperbolic polarity presented in [54, 18.0]. The hyperbolic motion group Φ^- consists of those automorphisms of $\mathcal{P}_2\mathbb{O}$ which commute with π^-. There is a "polar triangle" offered by [54, 18.20], consisting of an interior point o and exterior points u and v along with lines $o^{\pi^-} = uv =: W'$, $u^{\pi^-} = ov =: Y'$ and $v^{\pi^-} = ou =: X'$.



By [54, 18.21], the line W' at infinity is a passing line with respect to the unital Q of absolute points. X' on the other hand, by [54, 18.24], is a secant. Now consider their traces in the united hyperbolic plane $\mathrm{UH}\,\mathbb{O}$, which is induced by the point set $\mathrm{P}_2\mathbb{O} \setminus Q$.

The line $W := W' \setminus Q = W'$ *is compact*, by 2.2.1, being a passing line. *The line* $X := X' \setminus Q$ *is not compact*, by 2.2.2. *The exterior point* u *is non-isotropic.* We have seen that u is incident with the compact line W and the non-compact line X, and they cannot be transferred into one another using a continuous automorphism of $\mathcal{P}_2\mathbb{O}$.

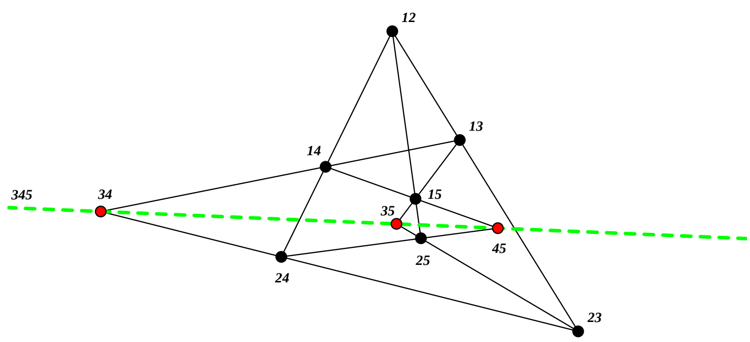
2.2.6 Corollary. *The exterior points in* $\mathrm{UH}\,\mathbb{O}$ *are non-isotropic.*

Proof. Let p be an exterior point. By [54, 18.23], the hyperbolic motion group acts transitively on the set of exterior points, which provides for an element $\alpha \in \Phi^-$ mapping u to p. Then $W^\alpha \in \mathcal{L}_p$ is a compact line and $X^\alpha \in \mathcal{L}_p$ is a non-compact one. Again, there cannot be any automorphism of $\mathcal{P}_2\mathbb{O}$ mapping one to the other. $\square$

Non-isotropic points in skew hyperbolic planes

Again, our aim will be non-isotropy of the exterior points. Nevertheless, as the point space of $\mathrm{H}_t\,\mathbb{R}$ equals that of the projective plane $\mathcal{P}_2\mathbb{R}$, the topological arguments used for $\mathrm{UH}\,\mathbb{F}$ cannot be a handle here. Instead, a geometrical notion will be exploited.

2.2.7 Definition. Desargues configuration.



The configuration is said to **close** if the points 34, 35 and 45 are collinear. Let $\mathcal{P} = (P, \mathcal{L})$ be a topological plane. A point p in $\mathcal{P}$ is called **desarguesian** if it possesses a desarguesian convex neighbourhood, i.e., a neighbourhood U of p which is convex with respect to the line system $\mathcal{L}$ and which has the the property that every Desargues configuration contained in U closes. If every point in $\mathcal{P}$ is desarguesian the plane will be called **locally desarguesian**. It is said to be **desarguesian** if simply any Desargues configuration closes.

2.2.8 Lemma. *The sets of desarguesian and non-desarguesian points, respectively, are invariant under the action of* $\operatorname{Aut} \mathrm{H}_t \mathbb{R}$.

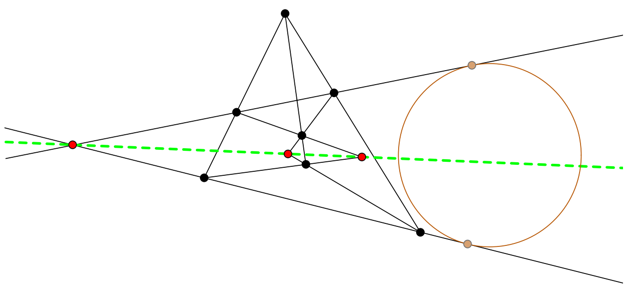
Proof. Let $p \in \mathrm{P}_2\mathbb{R}$ be a desarguesian point, and let U be one of its desarguesian neighbourhoods. Let α be an automorphism of $\mathrm{H}_t \mathbb{R}$. As α is homeomorphic and an isomorphism in Inc, it maps U onto a desarguesian neighbourhood of p^α. $\square$

Clearly, every desarguesian plane is locally desarguesian. The converse is true for special cases only, as studied by Polley. The exterior points in $\mathrm{H}_t \mathbb{R}$, for instance, can be dealt with due to [51, Satz 1], bearing in mind that the exterior E is homeomorphic to a Möbius strip.

2.2.9 Theorem (Polley 1972). *Every locally desarguesian topological plane whose point space is a Möbius strip is desarguesian.* $\square$

2.2.10 Corollary. *For $t \neq 0$, none of the exterior points of* $\mathrm{H}_t \mathbb{R}$ *is desarguesian.*

Proof. *The exterior skew hyperbolic plane* $\mathrm{EH}_t\ \mathbb{R}$ *is non-desarguesian.* In fact, a Desargues configuration in $\mathcal{P}_2\mathbb{R}$ can be constructed which consists of ten exterior points and nine passing lines or tangents, and whose tenth line is a (conservative) secant. Switching the point of view towards $\mathrm{H}_t \mathbb{R}$ reveals a Desargues configuration whose tenth line does not exist. In fact, there is no modified secant which could join the same three exterior points the old secant used to join, because there is no (proper) conic which contains three points that lie on a "straight" line.



Then Polley's theorem 2.2.9 yields that $\mathrm{EH}_t\ \mathbb{R}$ is not locally desarguesian. — An elementary proof of this fact, avoiding the use of Polley's theorem, will be found in Dörfner [12]. $\square$

2.2.11 Lemma.

a) *The interior points are desarguesian in* $\mathrm{H}_t\,\mathbb{R}$.

b) *For* $t \neq 0$*, the points on the quadric are non-desarguesian in* $\mathrm{H}_t\,\mathbb{R}$.

Proof. *ad (a).* The interior plane of $\mathrm{H}_t\,\mathbb{R}$ is isomorphic to the customary (desarguesian) hyperbolic plane $\mathrm{IH}\,\mathbb{R}$. *ad (b).* Assume the existence of a desarguesian point $q \in Q$ on the quadric. Let U be a desarguesian neighbourhood of q. Then U contains an exterior point p, along with a neighbourhood $p \in V \subseteq U$ of p. As V is contained in U, it must be desarguesian, and therefore p is a desarguesian point. Yet, lemma 2.2.10 has just established that all the exterior points are non-desarguesian. By this contradiction, the point q on the quadric is necessarily non-desarguesian. □

2.2.12 Corollary. *Consider a point* p *in* $\mathrm{H}_t\,\mathbb{R}$*, where* $t \neq 0$*. Then*

$p \in I$	$\iff$	*p is desarguesian*
$p \in Q$	$\iff$	*p is non-desarguesian, yet every neighbourhood contains desarguesian points*
$p \in E$	$\iff$	*p is non-desarguesian, and there is a neighbourhood which does not contain desarguesian points.*

□

2.2.13 Corollary. *For* $t \neq 0$*, the automorphism group* $\mathrm{Aut}\,\mathrm{H}_t\,\mathbb{R}$ *leaves* I*,* Q *and* E *invariant.*

Proof. This follows from 2.2.8 and from the fact that automorphisms of $\mathrm{H}_t\,\mathbb{R}$ are homeomorphisms. □

2.2.14 Corollary. *The sets of tangents, passing lines and modified secants, respectively, are invariant under the action of* $\mathrm{Aut}\,\mathrm{H}_t\,\mathbb{R}$.

Proof. By 2.1.41, the secants are the only lines to intersect with the interior, and by definition, the passing lines are the ones not to intersect with the quadric. □

This finally suffices to establish the exterior points as being non-isotropic.

2.2.15 Corollary. *For* $t \neq 0$*, none of the exterior points of* $\mathrm{H}_t\,\mathbb{R}$ *is isotropic.*

Proof. Let $p \in E$. By 2.1.47, the line pencil $\mathcal{L}_p$ contains lines of each type. Assuming that p were isotropic would imply the existence of an automorphism which could map tangents to secants, say. Yet, this is ruled out by 2.2.14. □

Non-isotropic points in united skew hyperbolic planes

As the modified secants are still homeomorphic to the old ones, we can transport the arguments from the non-modified plane $\mathrm{UH}\,\mathbb{R}$ to $\mathrm{UH}_t\,\mathbb{R}$ and receive the same result :

2.2.16 Lemma. *For any $t \in \mathbb{R}$, none of the exterior points of $\mathrm{UH}_t\,\mathbb{R}$ is isotropic.* □

2.3. Classification of line homogeneous sketched stable planes

The present aim is a complete classification of line homogeneous sketched stable planes in the sense that – as opposed to point homogeneous ones – they have no life of their own : line homogeneity entails flag homogeneity; as a consequence, every sketched stable plane is point homogeneous. The passepartout for this project is LÖWEN's article [38] on *Stable planes with isotropic points*. In a first step, the stable planes admitting line homogeneous sketched collineation groups will be detected. This is fairly close at hand, given LÖWEN's results and some slight amount of zoology, which has been provided for on the preceding pages. A second step then strives for enlightenment on the question of the transformation groups themselves, still under suitable restrictions on the transformation group. A third step finally rids the result of these restrictions.

The planes

Let $\mathcal{P}$ be a stable (lp-)plane. Let $\Gamma \leq \mathrm{Aut}_{\mathsf{StP}}\,\mathcal{P}$ be an arbitrary collineation group with the property that $(\Gamma, \mathcal{P})$ is a line homogeneous sketched geometry. Lemma 1.1.23 immediately yields isotropy of points.

2.3.1 Corollary. *In the above situation, every point in $\mathcal{P}$ is Γ-isotropic.*

The zoological key to the following classification has been collected in the preceding sections :

2.3.2 Lemma. *The united hyperbolic planes $\mathrm{UH}\,\mathbb{F}$ for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ as well as the skew hyperbolic planes $\mathrm{H}_t\,\mathbb{R}$ and united skew hyperbolic planes $\mathrm{UH}_t\,\mathbb{R}$, for $t \neq 0$, contain non-isotropic points.*

Proof. 2.2.4 and 2.2.6, 2.2.15 and 2.2.16. □

The operating ingredient, though, is theorem 1 from [38] :

2.3.3 Theorem (Löwen 1983). *The stable (lp-) planes containing two isotropic points are precisely the following :*

$\mathcal{A}_2\mathbb{F}$	$\mathrm{UH}\,\mathbb{F}$
$\mathcal{P}_2\mathbb{F}$	$\mathrm{H}_t\,\mathbb{R}$
$\mathrm{IH}\,\mathbb{F}$	$\mathrm{UH}_t\,\mathbb{R}$

where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$.

Assembling all this we obtain our choice of planes :

2.3.4 Proposition. *Let $\mathcal{P}$ be a stable (lp-) plane, and let $\Gamma \leq \mathrm{Aut}_{\mathsf{StP}}\,\mathcal{P}$ be an arbitrary group of collineations such that $(\Gamma, \mathcal{P})$ is a line homogeneous sketched geometry. Then $\mathcal{P}$ is one of the planes $\mathcal{A}_2\mathbb{F}$, $\mathcal{P}_2\mathbb{F}$ or $\mathrm{IH}\,\mathbb{F}$ for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$.*

Proof. Corollary 2.3.1 states that every point in $\mathcal{P}$ is Γ-isotropic, hence a fortiori isotropic. This justifies the application of LÖWEN's theorem 2.3.3, which leaves us with the above list of candidates, whose second column will be eliminated by 2.3.2, due to possession of non-isotropic points. □

The geometries

In order to earmark the group, further results from LÖWEN [38] can be profited from.

2.3.5 Proposition. *Let $\mathcal{P}$ be a stable (lp-) plane and consider some* closed *subgroup $\Upsilon \leq \mathrm{Aut}_{\mathsf{StP}}\,\mathcal{P}$. Denote by U the set of all Υ-isotropic points. If $|U| \geq 2$, then the following holds.*

a) *U is open in P.*

b) *The connected component Υ^1 acts flag transitively on every open subplane induced by a connected component of U.*

Proof. Proposition 3.14 in [38] states a slightly weaker assertion, in as much as it considers the entire automorphism group $\Upsilon = \mathrm{Aut}\,\mathcal{P}$. Nevertheless, LÖWEN's proof can be directly copied down for any closed subgroup $\Upsilon \leq \mathrm{Aut}\,\mathcal{P}$. □

2.3.6 Corollary. *In the above situation, $(\overline{\Gamma}^1, \mathcal{P})$ is flag homogeneous.*

Proof. By 2.3.1, every point in $\mathcal{P}$ is $\overline{\Gamma}$-isotropic. The planes singled out in 2.3.4 are connected. Therefore, an application of 2.3.5 deals with the connected set $U = P$ and thus yields flag transitivity of $\overline{\Gamma}^1$ on the whole of $\mathcal{P}$. □

Given flag homogeneity, corollary 1.4 from [38] matches the situation and gives a classification of the groups involved : in the elliptic and hyperbolic cases, $\overline{\Gamma}^1$ contains a copy of the (simple, connected component of the) elliptic and hyperbolic motion groups, respectively. In the euclidean case, it contains a copy of both, the translation group and the spinor group Spin_{n+1} for $n := \dim_{\mathbb{R}} \mathbb{F}$, where Spin_2 is defined to be $\mathrm{SO}_2\mathbb{R}$ itself.

2.3.7 Theorem. *Let $\mathcal{P}$ be a stable (lp-) plane and let $\Gamma \leq \mathrm{Aut}_{\mathsf{StP}}\,\mathcal{P}$ be an arbitrary group of collineations. If $(\Gamma, \mathcal{P})$ is a line homogeneous sketched geometry then the following holds :*

a) *The plane $\mathcal{P}$ is one of the affine planes $\mathcal{A}_2\mathbb{F}$, the projective planes $\mathcal{P}_2\mathbb{F}$ or the (inner) hyperbolic planes $\mathrm{IH}\,\mathbb{F}$ for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$.*

b) *The geometry* $(\overline{\Gamma}^1, \mathcal{P})$ *is flag homogeneous.*

c) *The connected component* $\overline{\Gamma}^1$ *contains a subgroup* Ψ *isomorphic to one of the following groups :*

	$\mathcal{P}_2\mathbb{F}$ *elliptic*	$\mathrm{IH}\,\mathbb{F}$ *hyperbolic*	$\mathcal{A}_2\mathbb{F}$ *euclidean*
$\mathbb{R}$	$\mathrm{PO}_3\mathbb{R}$	$\mathrm{PO}_3^1\mathbb{R}(1)$	$\mathrm{Spin}_2 \ltimes \Theta_{\mathbb{R}} \cong \mathrm{SO}_2\mathbb{R} \ltimes \mathbb{R}^2$
$\mathbb{C}$	$\mathrm{PU}_3\mathbb{C}$	$\mathrm{PU}_3\mathbb{C}(1)$	$\mathrm{Spin}_3 \ltimes \Theta_{\mathbb{C}} \cong \mathrm{SU}_2\mathbb{C} \ltimes \mathbb{C}^2$
$\mathbb{H}$	$\mathrm{PU}_3\mathbb{H}$	$\mathrm{PU}_3\mathbb{H}(1)$	$\mathrm{Spin}_5 \ltimes \Theta_{\mathbb{H}} \cong \mathrm{U}_2\mathbb{H} \ltimes \mathbb{H}^2$
$\mathbb{O}$	$\mathrm{F}_{4(-52)}$	$\mathrm{F}_{4(-20)}$	$\mathrm{Spin}_9 \ltimes \mathbb{O}^2$

Generalisation to arbitrary subgroups

The defect of the previous result is that it yields flag transitivity for closed transformation groups only. Fortunately, this can be cured. The formal key here can be the notion of a Mal'cev closure of Lie algebras.

2.3.8 Definition. Let Υ be a Lie group and $\mathfrak{g} := \ell\,\Upsilon$ its Lie algebra. Let $\mathfrak{h} \leq \mathfrak{g}$ be a Lie subalgebra of $\mathfrak{g}$, and denote by $\Phi := \langle \mathfrak{h}^{\exp} \rangle$ the Lie subgroup of Υ corresponding to $\mathfrak{h}$. Then $\overline{\Phi}$ is the least one among the closed Lie subgroups of Υ containing Φ. Its Lie algebra $\mathfrak{h}^M := \ell\,\overline{\Phi} \leq \mathfrak{g}$ contains $\mathfrak{h}$ and is called the **Mal'cev closure** of $\mathfrak{h}$ in $\mathfrak{g}$.

By a theorem by Mal'cev, the commutator subalgebras of $\mathfrak{h}^M$ and $\mathfrak{h}$ coincide; see [48, 5.3]. Using this closure one obtains the operating lemma which in most cases makes the motion group a subgroup of Γ.

2.3.9 Lemma. *Consider* $(\Gamma, \mathcal{P})$ *and* $\Psi \leq \overline{\Gamma}$ *as in* 2.3.7*. Then the following assertions are true :*

a) $\ell\,\Psi' \leq \mathfrak{g}'$*, hence* $\Psi' \leq \Gamma' \leq \Gamma$*.*

b) *If* Ψ *is* ***perfect****, that is, if* $\Psi' = \Psi$*, then* $\Psi \leq \Gamma' \leq \Gamma$*.*

Proof. By hypothesis, Ψ is contained in the closure $\overline{\Gamma}$, which implies $\ell\,\Psi' \leq (\ell\,\overline{\Gamma})' = (\mathfrak{g}^M)' = \mathfrak{g}'$ by Mal'cev's theorem. Therefore, $\Psi' \leq \Gamma' \leq \Gamma$. □

This will be applied to the motion groups Ψ contained in $\overline{\Gamma}$, which are listed in 2.3.7. The hyperbolic and elliptic motion groups $\mathrm{Ell}\,\mathbb{R} = \mathrm{PO}_3\mathbb{R}$, $\mathrm{Ell}\,\mathbb{C} = \mathrm{PU}_3\mathbb{C}$, $\mathrm{Ell}\,\mathbb{H} = \mathrm{U}_3\mathbb{H}$, $\mathrm{Ell}\,\mathbb{O} = \mathrm{F}_{4(-52)}$ and $\mathrm{Hyp}\,\mathbb{R} = \mathrm{PO}_3'\mathbb{R}(1)$, $\mathrm{Hyp}\,\mathbb{C} = \mathrm{PU}_3\mathbb{C}(1)$, $\mathrm{Hyp}\,\mathbb{H} = \mathrm{PU}_3\mathbb{H}(1)$, $\mathrm{Hyp}\,\mathbb{O} = \mathrm{F}_{4(-20)}$ are simple connected Lie groups, and as such are perfect groups; cf. 2.1.8 and 2.1.11.

2.3.10 Lemma. *Every almost simple Lie group is perfect.*

Proof. The normal subgroup Υ' translates to an ideal $\ell\,\Upsilon' \trianglelefteq \ell\,\Upsilon$ of the simple (non-abelian) Lie algebra $\ell\,\Upsilon$. ☐

As to the euclidean motion groups from 2.1.4, we should have a closer look. Moreover, the necessity arises to distinguish between the real case and the non-real ones.

2.3.11 Lemma.

a) *The commutator subgroup of* $\operatorname{Euc}\mathbb{F} = \operatorname{Spin}_{n+1} \ltimes \mathbb{F}^2$ *is* $(\operatorname{Euc}\mathbb{F})' = \operatorname{Spin}'_{n+1} \ltimes \mathbb{F}^2$.

b) *The euclidean motion groups* $\operatorname{Euc}\mathbb{F}$ *over* $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$ *are perfect, whereas* $(\operatorname{Euc}\mathbb{R})' = \mathsf{1} \ltimes \mathbb{R}^2$.

Proof. *ad (a).* Let $n := \dim_{\mathbb{R}} \mathbb{F} \in \{1, 2, 4, 8\}$. Note that $-\mathbb{1} \in \operatorname{Spin}_{n+1}$. In each case, $[\langle -\mathbb{1} \rangle \ltimes \mathbf{0}, \mathsf{1} \ltimes \mathbb{F}^2] = \mathsf{1} \ltimes \mathbb{F}^2$; in fact, for $v \in \mathbb{F}^2$ we compute

$$\begin{aligned}
&[(-\mathbb{1}, 0), (\mathbb{1}, v)] = (-\mathbb{1}, 0)(\mathbb{1}, v)(-\mathbb{1}, 0)^{-1}(\mathbb{1}, v)^{-1} \\
&= (-\mathbb{1}, v)(-\mathbb{1}, -v) = (\mathbb{1}, -v\mathbb{1} - v) = (\mathbb{1}, -2v).
\end{aligned}$$

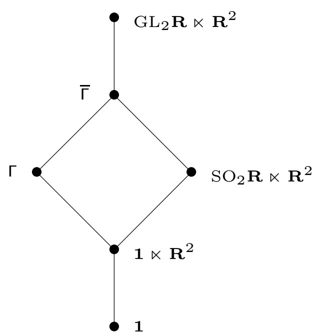
Moreover, for $\mathsf{T} := \operatorname{Spin}_{n+1}$, we get $[\mathsf{T} \ltimes \mathbf{0}, \mathsf{T} \ltimes \mathbf{0}] = \mathsf{T}' \ltimes \mathbf{0}$, and therefore $(\operatorname{Euc}\mathbb{F})' \geq \mathsf{T}' \ltimes \mathbb{F}^2$. The converse inclusion is immediate, given the description in 2.1.4. *ad (b).* The groups $\operatorname{Spin}_3 \cong \mathrm{SU}_2\mathbb{C}$, $\operatorname{Spin}_5 \cong \mathrm{U}_2\mathbb{H}$ and Spin_9 are almost simple Lie groups and, by 2.3.10, are perfect. Part (a) yields the assertion. As to $\operatorname{Euc}\mathbb{R}$, the abelian part $\operatorname{Spin}_2 = \mathrm{SO}_2\mathbb{R}$ accounts for $(\operatorname{Euc}\mathbb{R})' = \mathrm{SO}_2'\mathbb{R} \ltimes \mathbb{R}^2 = \mathsf{1} \ltimes \mathbb{R}^2$. ☐

2.3.12 Corollary. *For planes* $\mathcal{P}$ *and groups* $\mathsf{\Gamma}$ *and* $\mathsf{\Psi} \leq \overline{\mathsf{\Gamma}}$ *as in 2.3.7, with exception of* $(\mathsf{\Psi}, \mathcal{P}) = (\operatorname{Euc}\mathbb{R}, \mathcal{A}_2\mathbb{R})$, *the motion group* $\mathsf{\Psi}$ *is contained in* $\mathsf{\Gamma}$.

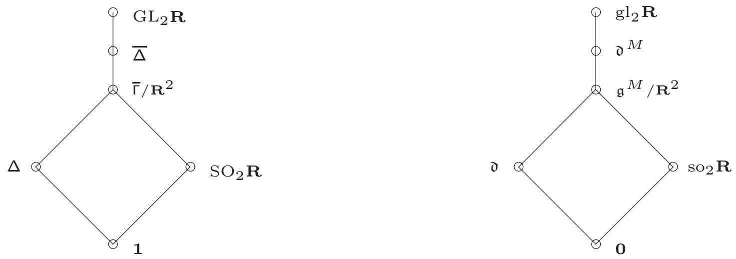
The problematic case hence is the real affine plane, whose (euclidean) motion group $\mathsf{\Psi} = \mathrm{SO}_2\mathbb{R} \ltimes \mathbb{R}^2$ is not perfect. Nevertheless it will be possible to establish the desired inclusion.

The real affine case

2.3.13 Recall that $\operatorname{Aut}\mathcal{A}_2\mathbb{R} = \mathrm{GL}_2\mathbb{R} \ltimes \mathbb{R}^2$; see [54, 12.10]. We are facing the following situation : just like in 2.3.9, we understand that the commutator group $\ell\,\mathsf{\Psi}' = \mathsf{1} \ltimes \mathbb{R}^2$ is contained in $\mathsf{\Gamma}$. Then the inclusion can be visualised as follows:



Denote by $\rho : \mathrm{GL}_2\mathbb{R} \ltimes \mathbb{R}^2 \twoheadrightarrow \mathrm{GL}_2\mathbb{R}$ the canonical quotient map. Put $\Delta := \Gamma^\rho$ and observe that $\overline{\Gamma}^\rho \leq \overline{\Gamma^\rho} = \overline{\Delta}$. Therefore, the situation modulo $\mathbb{R}^2$ can be caught in the following lattice:



Switching the point of view to the corresponding Lie algebras, abbreviate $\mathfrak{g} := \ell\,\Gamma$ and $\mathfrak{d} := \ell\,\Delta$. Then their Mal'cev closures satisfy

$$\ell(\overline{\Gamma}^\rho) = \ell(\overline{\Gamma})^{\ell\,\rho} = \mathfrak{g}^M/\mathbb{R}^2 \quad \text{and} \quad \ell(\overline{\Gamma^\rho}) = \ell(\overline{\Delta}) = \mathfrak{d}^M,$$

thus also $\mathfrak{g}^M/\mathbb{R}^2 \leq \mathfrak{d}^M$. This legitimises the lattice to the right hand as being the proper translation of the situation to the world of Lie algebras.

Given that very lattice, the general plan for the remainder is as follows : all Lie algebras $\mathfrak{d}^M$ satisfying $\mathfrak{so}_2\mathbb{R} \leq \mathfrak{d}^M \leq \mathfrak{gl}_2\mathbb{R}$ can be determined. These candidates will be checked and proved to be Mal'cev closed, i.e., $\mathfrak{d}^M = \mathfrak{d}$ (corollary 2.3.26). As a consequence, $\mathfrak{so}_2\mathbb{R} \leq \mathfrak{d}$ and hence $\mathrm{SO}_2\mathbb{R} \leq \Delta$ (corollary 2.3.27).

To that end pick a basis

$$\mathbb{1}, \quad I := \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad H := \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad T := \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

of $\mathfrak{gl}_2\mathbb{R}$. The corresponding Lie brackets are $[I, H] = -2T$, $[I, T] = 2H$ and $[T, H] = -2I$.

2.3.14 Lemma. *Every Lie subalgebra $\mathfrak{so}_2\mathbb{R} \leq \mathfrak{h} \leq \mathfrak{gl}_2\mathbb{R}$ must be one of the Lie algebras $\mathfrak{so}_2\mathbb{R}$, $\mathfrak{a} := \mathbb{R}\mathbb{1} \oplus \mathbb{R}I$, $\mathfrak{sl}_2\mathbb{R}$ or $\mathfrak{gl}_2\mathbb{R}$. The corresponding Lie groups are the rotations $\mathrm{SO}_2\mathbb{R}$, the rotation-stretchings $\mathbb{R}^+ \times \mathrm{SO}_2\mathbb{R}$, the special linear group $\mathrm{SL}_2\mathbb{R}$ and $\mathrm{GL}_2\mathbb{R}$.*

Proof. Consider the representation $\mathrm{ad}\,|_{\mathfrak{so}_2\mathbb{R}} : \mathfrak{so}_2\mathbb{R} \to \mathrm{Der}(\mathfrak{gl}_2\mathbb{R})$ of the 1-dimensional abelian Lie algebra $\mathfrak{so}_2\mathbb{R} = \mathbb{R}I$. With respect to the basis above, the linear map $\mathrm{ad}\,I$ corresponds to the matrix

$$\left(\begin{array}{cc|cc} 0 & & & \\ & 0 & & \\ \hline & & & -2 \\ & & 2 & \end{array}\right),$$

with eigenvalues 0, $2i$ and $-2i$. The reduction of $\mathfrak{gl}_2\mathbb{R}$ modulo $\mathfrak{so}_2\mathbb{R}$ into irreducible real $\mathfrak{so}_2\mathbb{R}$-submodules therefore is $\mathfrak{gl}_2\mathbb{R}/\mathfrak{so}_2\mathbb{R} \cong \mathbb{R}\mathbb{1} \oplus (\mathbb{R}T \oplus \mathbb{R}U)$.

Any Lie algebra $\mathfrak{h}$ of $\mathfrak{g}$ containing $\mathfrak{so}_2\mathbb{R}$ clearly is $\mathfrak{so}_2\mathbb{R}$-invariant and has to be composed of these components. Thus, $\mathfrak{h}$ must be one of the algebras $\mathbb{R}I = \mathfrak{so}_2\mathbb{R}$, $\mathfrak{a} = \mathbb{R}I + \mathbb{R}\mathbb{1}$, $\mathbb{R}I + \mathbb{R}H + \mathbb{R}U = \mathfrak{sl}_2\mathbb{R}$ or $\mathfrak{gl}_2\mathbb{R}$. □

In order to scrutinize these candidates $\overline{\Delta}$ for dense subgroups Δ it is worthwhile to make sure the topologies behave just like they are expected to.

2.3.15 Lemma. *The compact-open topology on* $\operatorname{Aut}\mathcal{A}_2\mathbb{R} = \mathrm{GL}_2\mathbb{R} \ltimes \mathbb{R}^2$ *coincides with the matrix topology on*

$$\mathrm{GL}_2\mathbb{R} \ltimes \mathbb{R}^2 = \begin{array}{|c|c|}\hline \mathrm{GL}_2\mathbb{R} & \\ \hline \mathbb{R}^2 & 1 \\ \hline \end{array} \subseteq \mathrm{Mat}_3\,\mathbb{R}.$$

Proof. Apply lemma 3.3.19 to $\Upsilon = \mathrm{GL}_2\mathbb{R} \ltimes \mathbb{R}^2$, $X = \mathbb{R}^2$ and the injection $f : \Upsilon \to X : \left(\begin{array}{c|c} A & \\ \hline v & 1 \end{array}\right) \mapsto (e_1 A + v, e_2 A + v)$. □

2.3.16 Corollary. *Consider some subgroup* $\Upsilon \leq \operatorname{Aut}\mathcal{A}_2\mathbb{R}$. *The following topologies on* $\Phi := \Upsilon/\mathbb{R}^2$ *coincide :*

- *the matrix topology induced from* $\mathrm{GL}_2\mathbb{R}$
- *the quotient topology on* $\Upsilon/\mathbb{R}^2$*, where* Υ *carries the compact-open topology with respect to the action of* Υ *on* $\mathbb{R}^2$

Proof.

$$\begin{tikzcd}
\operatorname{Aut}\mathcal{A}_2\mathbb{R} \arrow[rrr, "\cong"] \arrow[ddd] & & & \mathrm{GL}_2\mathbb{R} \ltimes \mathbb{R}^2 \arrow[ddd] \\
& \Psi_{\mathrm{co}} \arrow[ul, hook] \arrow[r] \arrow[d] & \Psi_{\mathrm{mat}} \arrow[ur, hook] \arrow[d] & \\
& \Phi_{\mathrm{co}} \arrow[r] \arrow[dl, hook] & \Phi_{\mathrm{mat}} \arrow[dr, hook] & \\
(\operatorname{Aut}\mathcal{A}_2\mathbb{R})/\mathbb{R}^2 \arrow[rrr, "\cong"] & & & \mathrm{GL}_2\mathbb{R}
\end{tikzcd}$$

By the lemma above, the topological groups $\operatorname{Aut}\mathcal{A}_2\mathbb{R}$ and $\mathrm{GL}_2\mathbb{R} \ltimes \mathbb{R}^2$ are isomorphic, and so are their respective quotients modulo $\mathbb{R}^2$. Now, the concatenation $\Phi_{\mathrm{co}} \to \mathrm{GL}_2\mathbb{R}$ of continuous maps is continuous, and by the universal property of the embedding $\Phi_{\mathrm{mat}} \to \mathrm{GL}_2\mathbb{R}$, so is $\Phi_{\mathrm{co}} \to \Phi_{\mathrm{mat}}$. By analogy, $\Phi_{\mathrm{co}} \to (\operatorname{Aut}\mathcal{A}_2\mathbb{R})/\mathbb{R}^2$ being an embedding yields continuity of $\Phi_{\mathrm{mat}} \to \Phi_{\mathrm{co}}$. Hence, these topologies on Φ coincide, indeed. □

In our situation then, we may consider the groups Δ and $\overline{\Delta}$ as endowed with their "usual" matrix topology.

2.3.17 Lemma. *If the Mal'cev closure is* $\mathfrak{d}^M = \mathfrak{gl}_2\mathbb{R}$, *then* $\mathfrak{d} = \mathfrak{d}^M = \mathfrak{gl}_2\mathbb{R}$ *and* $\Delta = \overline{\Delta} = \mathrm{GL}_2\mathbb{R}$.

Proof. From Mal'cev's theorem we know that $\mathfrak{d}' = (\mathfrak{d}^M)' = \mathfrak{sl}_2\mathbb{R} \leq \mathfrak{d} \leq \mathfrak{gl}_2\mathbb{R}$. Hence $\mathfrak{d} = \mathfrak{sl}_2\mathbb{R}$ or $\mathfrak{d} = \mathfrak{gl}_2\mathbb{R}$. But $\mathrm{SL}_2\mathbb{R}$ is not dense in $\mathrm{GL}_2\mathbb{R}$. Therefore $\mathfrak{d} = \mathfrak{d}^M = \mathfrak{gl}_2\mathbb{R}$. ☐

2.3.18 Lemma. *If* $\mathfrak{d}^M = \mathfrak{so}_2\mathbb{R}$, *then* $\mathfrak{d} = \mathfrak{d}^M = \mathfrak{so}_2\mathbb{R}$ *and* $\Delta = \overline{\Delta} = \mathrm{SO}_2\mathbb{R}$.

Proof. The only Lie algebra properly contained in the 1-dimensional algebra $\mathfrak{so}_2\mathbb{R}$ is the trivial one, whose exponential image $\mathtt{1}$ is certainly not dense in $\mathrm{SO}_2\mathbb{R}$. ☐

2.3.19 Lemma. *Consider the case* $\mathfrak{d}^M = \mathfrak{a} = \mathbb{R}\mathbb{1} + \mathbb{R}I$. *Then the following is true :*

a) *The proper Lie subalgebras of* $\mathfrak{a}$ *are of the form* $\mathbb{R}X$ *for* $X \in \mathfrak{a}$. *The proper Lie subgroups of the group* $\mathsf{A} := \langle \mathfrak{a}^{\exp} \rangle$ *of rotation-stretchings are precisely the one parameter groups* $\mathbb{R}X^{\exp}$.

b) *The one parameter groups are closed in* $\mathrm{GL}_2\mathbb{R}$.

c) *If* $\mathfrak{d}^M = \mathfrak{a}$, *then* $\mathfrak{d} = \mathfrak{d}^M = \mathfrak{a}$ *and* $\Delta = \overline{\Delta} = \mathsf{A}$. *Every Lie subalgebra of* $\mathfrak{a}$ *is Mal'cev closed.*

Proof. Note that $\mathsf{A} = \mathbb{R}^+ \times \mathrm{SO}_2\mathbb{R} \approx \mathbb{R}^+ \times \mathbb{S}_1 \approx \mathbb{C}^\times \approx \mathbb{R}^2 \backslash \mathbf{0}$. For $X = a\mathbb{1} + bI \in \mathfrak{a}$ we may without loss assume $a = 1$, and the restriction of the exponential function corresponds to $\mathbb{R}X \to \mathbb{C}^\times : t(\mathbb{1} + bI) \mapsto e^t e^{tbi}$. Then $\mathbb{R}X^{\exp} = \{e^t e^{tbi} \mid t \in \mathbb{R}\}$, which is closed in $\mathbb{C}^\times$. (Note that visualised in $\mathbb{R}^2 \setminus \mathbf{0}$, the one parameter group $\mathbb{R}\mathbb{1}^{\exp}$ corresponds to an open half axis, $\mathbb{R}I^{\exp}$ corresponds to the unit circle, and any other one parameter group corresponds to a logarithmic spiral.) We have thus established that every Lie subalgebra of $\mathfrak{a}$ is Mal'cev closed; there is no proper $\mathfrak{h} < \mathfrak{a}$ satisfying $\mathfrak{h}^M = \mathfrak{a}$; A possesses no proper dense Lie subgroups. ☐

What remains to be done is to examine the Lie subalgebras of the fourth candidate, $\mathfrak{sl}_2\mathbb{R}$. Following Hilgert and Hofmann [23], we provide some preparatory presentations.

2.3.20 A basis of $\mathfrak{sl}_2\mathbb{R}$ is given by

$$H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad T = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad U := \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = I;$$

recall the Lie brackets $[H,T] = 2U$, $[H,U] = 2T$ and $[T,U] = -2H$. With respect to this basis the **Killing form** $\kappa(X,Y) = \mathrm{tr}(\mathrm{ad}\,X \cdot \mathrm{ad}\,Y)$ corresponds to the matrix $8 \cdot \mathrm{Diag}(1,1,-1)$, i.e., $\kappa(x_1H + x_2T + x_3U, y_1H + y_2T + y_3U) = 8(x_1y_1 + x_2y_2 - x_3y_3)$. Consider the adjoint action $\mathrm{Ad} : \mathrm{SL}_2\mathbb{R} \to \mathrm{Aut}(\mathfrak{sl}_2\mathbb{R}) : g \mapsto (A \mapsto g^{-1}Ag)$. The orbits of $\mathrm{SL}_2\mathbb{R}$ on $\{\mathbb{R}X \mid X \in \mathfrak{sl}_2\mathbb{R} \setminus \mathbf{0}\}$ are the unital $\{\mathbb{R}X \mid \kappa(X,X) = 0\}$, its interior $\{\mathbb{R}X \mid \kappa(X,X) < 0\} = \mathrm{SL}_2\mathbb{R}.\mathbb{R}U$ and its exterior $\{\mathbb{R}X \mid \kappa(X,X) > 0\} = \mathrm{SL}_2\mathbb{R}.\mathbb{R}T$; see [23, 1.1].

2.3.21 Lemma. *Every one parameter group* $\mathbb{R}X^{\exp} \leq \mathrm{SL}_2\mathbb{R}$ *is conjugate to*

$$\begin{array}{rclcl} \mathbb{R}U^{\exp} & = & \mathrm{SO}_2\mathbb{R} & \iff & \kappa(X,X) < 0 \\ \mathbb{R}H^{\exp} & = & \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \mid a > 0 \right\} & \iff & \kappa(X,X) > 0 \\ \mathbb{R}P^{\exp} & = & \mathbb{1} + \mathbb{R}P & \iff & \kappa(X,X) = 0, \end{array}$$

where $P := \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} \in \mathfrak{sl}_2\mathbb{R}$, *satisfying* $\kappa(P,P) = 0$.

Proof. [23, 1.2] □

2.3.22 Corollary. *Every one parameter group in* $\mathrm{SL}_2\mathbb{R}$ *is closed in* $\mathrm{GL}_2\mathbb{R}$.

Proof. By 2.3.21, every one parameter group $\mathbb{R}X^{\exp}$ is conjugate to one of the above. Being defined by equations, these three standard groups are closed in $\mathrm{GL}_2\mathbb{R}$. As conjugation is a homeomorphism in topological groups, the one parameter group $\mathbb{R}X^{\exp}$ is also closed in $\mathrm{GL}_2\mathbb{R}$. □

The two-dimensional Lie subgroups in $\mathrm{SL}_2\mathbb{R}$ are described in proposition 1.1 of [23] :

2.3.23 Lemma. *Some 2-dimensional vector subspace* $\mathfrak{h}$ *of* $\mathfrak{sl}_2\mathbb{R}$ *is a Lie subalgebra if and only if it is the orthogonal space* $\mathfrak{h} = X^{\perp_\kappa}$ *of some element* $X \in \mathfrak{sl}_2\mathbb{R}$ *satisfying* $\kappa(X,X) = 0$. □

2.3.24 Lemma. *Every 2-dimensional Lie subgroup of* $\mathrm{SL}_2\mathbb{R}$ *is closed in* $\mathrm{GL}_2\mathbb{R}$.

Proof. *Step 1.* We elect $P^\perp = \mathbb{R}H + \mathbb{R}P =: \mathfrak{t}$ for our favourite 2-dimensional Lie subalgebra. Its exponential image

$$\mathsf{T} := \langle \mathfrak{t}^{\exp} \rangle = \left\{ \begin{pmatrix} a & x \\ & a^{-1} \end{pmatrix} \middle| \begin{array}{l} a > 0 \\ x \in \mathbb{R} \end{array} \right\}$$

is closed in $\mathrm{GL}_2\mathbb{R}$. *Step 2.* By 2.3.23 there is some $X \in \mathfrak{sl}_2\mathbb{R}$ with $\kappa(X,X) = 0$ such that $\mathfrak{h} = X^\perp$. By 2.3.20 there is an element $g \in \mathrm{SL}_2\mathbb{R}$ such that $\mathrm{Ad}\, g.P = g^{-1}Pg = \pm X$. Then $\mathfrak{h} = X^\perp = (g^{-1}Xg)^\perp = g^{-1}X^\perp g$, because the Killing form is invariant under Ad; that is, for every $X, Y \in \mathfrak{sl}_2\mathbb{R}$ and $g \in \mathrm{SL}_2\mathbb{R}$ the equation $\kappa(\mathrm{Ad}\, g.X, Y) = \kappa(X, \mathrm{Ad}\, g^{-1}.Y)$ holds (see [21, p.220]). *Step 3.* This implies that the exponential image $\mathsf{H} := \langle \mathfrak{h}^{\exp} \rangle = \langle (g^{-1}\mathfrak{t}\, g)^{\exp} \rangle = g^{-1}\mathsf{T}g$ is conjugate to the closed subgroup T of $\mathrm{GL}_2\mathbb{R}$. And as conjugation is homeomorphic, H is closed in $\mathrm{GL}_2\mathbb{R}$, too. □

2.3.25 Corollary.

a) *Every Lie subgroup of* $\mathrm{SL}_2\mathbb{R}$ *is closed in* $\mathrm{GL}_2\mathbb{R}$. *Every Lie subalgebra of* $\mathfrak{sl}_2\mathbb{R}$ *is Mal'cev closed.*

b) *If* $\mathfrak{d}^M = \mathfrak{sl}_2\mathbb{R}$, *then* $\mathfrak{d} = \mathfrak{d}^M = \mathfrak{sl}_2$ *and* $\Delta = \overline{\Delta} = \mathrm{SL}_2\mathbb{R}$.

Proof. 2.3.22 and 2.3.24 □

2.3.26 Corollary. *Every Lie subalgebra $\mathfrak{so}_2\mathbb{R} \leq \mathfrak{h} \leq \mathfrak{gl}_2\mathbb{R}$ is Mal'cev closed.*

Proof. The candidates from 2.3.14 were one by one dealt with in 2.3.17, 2.3.18, 2.3.19 and 2.3.25. □

2.3.27 Corollary. *Getting back to our situation in 2.3.13, we have established $\mathfrak{d} = \mathfrak{d}^M$ and $\Delta = \overline{\Delta}$. Therefore, $\mathfrak{so}_2\mathbb{R} \leq \mathfrak{d}$ and $\mathrm{SO}_2\mathbb{R} \leq \Delta$, which finally means that $\ell\,\Psi \leq \mathfrak{g}$ and $\Psi \leq \Gamma$ also hold for the real affine case.*

Proof. We know that the Mal'cev closure $\mathfrak{d}^M$ is a Lie subalgebra of $\mathfrak{gl}_2\mathbb{R}$ containing $\mathfrak{so}_2\mathbb{R}$. Hence we have just found out that then $\mathfrak{d} = \mathfrak{d}^M$ as well as $\Delta = \overline{\Delta}$, by 2.3.26. This implies $\mathfrak{g} = \mathfrak{d} \propto \mathbb{R}^2 = \mathfrak{d}^M \propto \mathbb{R}^2 = \mathfrak{g}^M$ and $\Gamma = \Delta \ltimes \mathbb{R}^2 = \overline{\Delta} \ltimes \mathbb{R}^2 = \overline{\Gamma}$, and therefore $\Psi \leq \overline{\Gamma} \leq \Gamma$. □

These are all the ingredients we need for a generalisation of 2.3.7 to arbitrary Lie subgroups.

2.3.28 Theorem. *Let $\mathcal{P}$ be a stable (lp-) plane and let $\Gamma \leq \mathrm{Aut}_{\mathsf{StP}}\mathcal{P}$ be an arbitrary group of collineations. If $(\Gamma, \mathcal{P})$ is a line homogeneous sketched geometry then the following holds :*

a) *The plane $\mathcal{P}$ is one of the affine planes $\mathcal{A}_2\mathbb{F}$, the projective planes $\mathcal{P}_2\mathbb{F}$ or the (inner) hyperbolic planes $\mathrm{IH}\,\mathbb{F}$ for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$.*

b) *The connected component Γ^1 contains a subgroup isomorphic to one of the following motion groups :*

	$\mathcal{P}_2\mathbb{F}$ *elliptic*	$\mathrm{IH}\,\mathbb{F}$ *hyperbolic*	$\mathcal{A}_2\mathbb{F}$ *euclidean*
$\mathbb{R}$	$\mathrm{PO}_3\mathbb{R}$	$\mathrm{PO}_3^1\mathbb{R}(1)$	$\mathrm{Spin}_2 \ltimes \Theta_\mathbb{R} \cong \mathrm{SO}_2\mathbb{R} \ltimes \mathbb{R}^2$
$\mathbb{C}$	$\mathrm{PU}_3\mathbb{C}$	$\mathrm{PU}_3\mathbb{C}(1)$	$\mathrm{Spin}_3 \ltimes \Theta_\mathbb{C} \cong \mathrm{SU}_2\mathbb{C} \ltimes \mathbb{C}^2$
$\mathbb{H}$	$\mathrm{PU}_3\mathbb{H}$	$\mathrm{PU}_3\mathbb{H}(1)$	$\mathrm{Spin}_5 \ltimes \Theta_\mathbb{H} \cong \mathrm{U}_2\mathbb{H} \ltimes \mathbb{H}^2$
$\mathbb{O}$	$\mathrm{F}_{4(-52)}$	$\mathrm{F}_{4(-20)}$	$\mathrm{Spin}_9 \ltimes \mathbb{O}^2$

c) *The geometry $(\Gamma^1, \mathcal{P})$ is flag homogeneous.*

Proof. *ad (a).* Nothing needs to be added to part (a) of 2.3.7. *ad (b).* Part (b) of 2.3.7 states that the motion group Ψ is contained in the topological closure $\overline{\Gamma}^1$. As a matter of fact, Ψ is already contained in Γ^1, which has been proved in 2.3.12 for all the cases except the real affine one and in 2.3.27 for the real affine case. *ad (c).* The zoological part of the present chapter established that the geometries $(\Psi, \mathcal{P})$ are flag homogeneous; cf. 2.1.5, 2.1.10 and 2.1.37. □

Consequences

Some immediate conclusions can be drawn from the classification. First of all, indeed it yields a generalisation of Block's result in a very trivial way :

2.3.29 Corollary. *Let $\Delta \leq \mathrm{Aut}_{\mathsf{StP}} \mathcal{P}$ be a collineation group of a stable (lp-) plane. If $(\Delta, \mathcal{P})$ is sketched then it is point homogeneous.*

Proof. By 1.1.19, the geometry is point homogeneous or line homogeneous. In case of line homogeneity, theorem 2.3.7 yields flag homogeneity. □

Moreover, there are not a great many chances of drawing sketched geometries from proper skew hyperbolic planes :

2.3.30 Corollary. *Consider a skew hyperbolic plane $\mathrm{H}_t\,\mathbb{R}$ with $t \in \mathbb{R}$. If there is a collineation group $\Delta \leq \mathrm{Aut}_{\mathsf{StP}}\, \mathrm{H}_t\,\mathbb{R}$ such that $(\Delta, \mathrm{H}_t\,\mathbb{R})$ is sketched, then $\mathrm{H}_t\,\mathbb{R} = \mathrm{H}_0\,\mathbb{R} = \mathcal{P}_2\mathbb{R}$ is the real projective plane and $(\Delta, \mathcal{P}_2\mathbb{R})$ is flag homogeneous.*

Proof. Being sketched, the geometry is point or line homogeneous [1.1.19]. *In case of line homogeneity*, the classification 2.3.7 yields that $\mathrm{H}_t\,\mathbb{R} = \mathcal{P}_2\mathbb{R}$ and that $\overline{\Delta} = \Delta$ acts flag transitively on $\mathcal{P}_2\mathbb{R}$. *In case of point homogeneity*, recall from [54, 35.2] that the skew hyperbolic planes are connected compact projective planes. Therefore Löwen [36] proved that $\mathrm{H}_t\,\mathbb{R}$ is classical, thus $\mathrm{H}_t\,\mathbb{R} = \mathrm{H}_0\,\mathbb{R} = \mathcal{P}_2\mathbb{R}$, and moreover that Δ contains the elliptic motion group. As by 2.1.10, the geometry $(\mathrm{PO}_3\mathbb{R}, \mathcal{P}_2\mathbb{R})$ is flag homogeneous, so is $(\Delta, \mathcal{P}_2\mathbb{R})$. □

3. A non-embeddability theorem for Peter planes

3.1. The planes ...

Throughout the remains of this thesis, any group called Γ shall be the the 4-dimensional Lie group

$$\Gamma := \left\{ \begin{pmatrix} a^2 & x & z \\ & a & y \\ & & 1 \end{pmatrix} \mid a, x, y, z \in \mathbb{R},\ a > 0 \right\}.$$

This is the Frobenius group mentioned by Plaumann-Strambach in [50, I §4], whose non-abelian Frobenius kernel is the 3-dimensional Heisenberg group

$$\mathrm{Hei}_3\mathbb{R} := \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

Γ can be interpreted as a semidirect product in several related ways :

$$\Gamma = \{\mathrm{Diag}(a^2, a, 1) \mid a > 0\} \ltimes_\kappa \mathrm{Hei}_3\mathbb{R} \quad \cong \quad \mathbb{R}^+ \ltimes_\delta \mathrm{Hei}_3\mathbb{R} \quad \cong \quad \mathbb{R} \ltimes_{\tilde{\delta}} \mathrm{Hei}_3\mathbb{R},$$

where the group morphisms constituting the semidirect products should be thought of as

conjugation κ for the inner product

$$\begin{array}{rrcl} \delta : & \mathbb{R}^+ & \to & \mathrm{Aut}(\mathrm{Hei}_3\mathbb{R}) \\ & a & \mapsto & (M \mapsto \mathrm{Diag}(a^2, a, 1)^{-1} \cdot M \cdot \mathrm{Diag}(a^2, a, 1)) \\ \tilde{\delta} : & \mathbb{R} & \to & \mathrm{Aut}(\mathrm{Hei}_3\mathbb{R}) \\ & t & \mapsto & (M \mapsto \mathrm{diag}(-2t, -t, 0)^{\exp} \cdot M \cdot \mathrm{diag}(2t, t, 0)^{\exp}). \end{array}$$

The isomorphisms are given by multiplication

$$\begin{array}{rccl} i : & \mathbb{R}^+ \ltimes_\delta \mathrm{Hei}_3\mathbb{R} & \to & \Gamma \\ & (a, M) & \mapsto & \mathrm{diag}(a^2, a, 1) \cdot M \\ j : & \mathbb{R} \ltimes_{\tilde{\delta}} \mathrm{Hei}_3\mathbb{R} & \to & \Gamma \\ & (t, M) & \mapsto & \mathrm{diag}(2t, t, 0)^{\exp} \cdot M \end{array}$$

In order to describe a stable partition of Γ it turns out to be more convenient to find an appropriate stable partition of the corresponding Lie algebra

$$\mathfrak{g} := \left\{ \begin{pmatrix} 2t & a & c \\ & t & b \\ & & 0 \end{pmatrix} \mid t, a, b, c \in \mathbb{R} \right\}.$$

This Lie algebra can also be described as a semidirect sum

$$\mathfrak{g} = \mathbb{R} \cdot \operatorname{diag}(2,1,0) \underset{\mathrm{ad}}{\propto} \mathrm{hei}_3\mathbb{R} \quad \cong \quad \mathbb{R} \propto_\omega \mathrm{hei}_3\mathbb{R},$$

where the constituting action is

$$\omega : \mathbb{R} \to \operatorname{Der}(\mathrm{hei}_3\mathbb{R}) : t \mapsto \operatorname{ad}\operatorname{diag}(2t, t, 0).$$

The isomorphism is given by addition

$$\mathbb{R} \propto_\omega \mathrm{hei}_3\mathbb{R} \to \mathfrak{g} : (t, M) \mapsto \operatorname{diag}(2t, t, 0) + M.$$

In order to see that the corresponding exponential function is a homeomorphism, we describe it as a restriction of a homeomorphic exponential function $\exp : \mathfrak{r} \to \mathsf{P}$ for a larger group

$$\mathsf{P} := \left\{ A \in \mathrm{GL}_3\mathbb{C} \;\middle|\; \begin{array}{l} A \text{ is upper triangular matrix} \\ \text{with positive real diagonal entries} \end{array} \right\} = \boxed{\begin{array}{ccc} \mathbb{R}^+ & \mathbb{C} & \mathbb{C} \\ & \mathbb{R}^+ & \mathbb{C} \\ & & \mathbb{R}^+ \end{array}}.$$

Consider the Lie algebra

$$\mathfrak{r} := \left\{ X \in \mathfrak{gl}_3\mathbb{C} \;\middle|\; \begin{array}{l} X \text{ is upper triangular matrix} \\ \text{with real diagonal entries} \end{array} \right\} = \boxed{\begin{array}{ccc} \mathbb{R} & \mathbb{C} & \mathbb{C} \\ & \mathbb{R} & \mathbb{C} \\ & & \mathbb{R} \end{array}}.$$

3.1.1 Lemma. $\exp_{\mathfrak{r}} : \mathfrak{r} \to \mathsf{P} : A \mapsto \sum_{\nu=0}^{\infty} \frac{1}{\nu!} A^\nu$ *is a homeomorphism.*

Proof. As to being a bijection, one can brutally compute its inverse function. This would involve five different cases and become rather awkward. This is why we prefer to give a brief outline of a more conceptual approach :

A **Cartan subalgebra** of a Lie algebra is a nilpotent Lie subalgebra which equals its own normaliser. Given a Lie algebra $\mathfrak{a}$ along with a Cartan subalgebra $\mathfrak{h}$, an element α of the dual space $\mathfrak{h}^*$ is called a **root** of $\mathfrak{a}$ with respect to $\mathfrak{h}$ if $\mathbf{0} \neq \mathfrak{a}_\alpha := \{X \in \mathfrak{a} \mid \forall H \in \mathfrak{h}. \quad \operatorname{ad} H. X = H^\alpha \cdot X\}$. A soluble real Lie algebra $\mathfrak{a}$ is called **exponential** if there is a Cartan subalgebra $\mathfrak{h}$ with the property that for any root α of $\mathfrak{a}_\mathbb{C}$ with respect to $\mathfrak{h}_\mathbb{C}$, we get $\mathfrak{h}^\alpha \cap i\mathbb{R} = \mathbf{0}$. By [24, III 7.29] and the theorem of Dixmier [24, III 7.30], bijectivity of the exponential function $\exp : \ell\,\Upsilon \to \Upsilon$ of a simply connected Lie group Υ follows from exponentiality of its Lie algebra $\ell\,\Upsilon$. Now,

$$\mathfrak{t} := \boxed{\begin{array}{ccc} \mathbb{R} & & \\ & \mathbb{R} & \\ & & \mathbb{R} \end{array}}$$

is a Cartan subalgebra of $\mathfrak{r}$, and the roots of $\mathfrak{r}$ with respect to $\mathfrak{t}$ are

$$\begin{aligned} \alpha_0 &= (0,0,0) && \text{with} && \mathfrak{r}_0 = \mathfrak{t} \\ \alpha_1 &= (1,-1,0) && \text{with} && \mathfrak{r}_{\alpha_1} = \mathbb{C}\begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \\ \alpha_2 &= (0,1,-1) && \text{with} && \mathfrak{r}_{\alpha_2} = \mathbb{C}\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} \\ \alpha_3 &= (1,0,-1) && \text{with} && \mathfrak{r}_{\alpha_3} = \mathbb{C}\begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}. \end{aligned}$$

Hence no purely imaginary ad-eigenvalues occur in $\mathfrak{t}$, and therefore $\mathfrak{r}$ is exponential. Moreover, $\mathsf{P} \approx \mathbb{R}^9$ is simply connected, and thus the exponential function is bijective.

P as well as $\mathfrak{r}$ being endowed with the matrix topology induced from $\operatorname{Mat}_3 \mathbb{C} \approx (\mathbb{C}^3)^3$, the exponential function is continuous. It remains to show that exp is open. This is true because P, being a manifold, has the domain invariance property; cf. notes in A.1.5. In fact, consider a neighbourhood U of 0 in $\mathfrak{r}$. Then U contains a (compact) closed ball B around 0. Now the restriction $\exp|_B^{B^{\exp}} : B \to B^{\exp}$ is a homeomorphism, as exp is continuous and bijective. Therefore, the interior I of B is homeomorphic to its image $I^{\exp} \subseteq B^{\exp} \subseteq \mathsf{P}$. Due to the domain invariance property for P this proves that $I^{\exp}$ is open in P, and thus $U^{\exp}$ is a neighbourhood of 1 in P, indeed. Hence exp is an open map. □

Now $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{r}$ and Γ is a subgroup of P, and one can verify that $\mathfrak{g}^{\exp} = \Gamma$. Therefore, $\exp_{\mathfrak{g}} : \mathfrak{g} \to \Gamma$ is the co-restriction $\exp_{\mathfrak{g}} = \exp_{\mathfrak{r}}|_{\mathfrak{g}}^{\Gamma}$ of the homeomorphism $\exp_{\mathfrak{r}}$, and as such is a homeomorphism itself; cf. A.1.4.

3.1.2 Corollary. *The exponential function* $\exp_{\mathfrak{g}} : \mathfrak{g} \to \Gamma$ *is a homeomorphism.*

3.1.3 The spreads. We are now looking for stable LieAlg-partitions of $\mathfrak{g}$. Maier exposes a whole family of them in chapter 4.2 of [44]. One prominent series, dependent on a parameter $k \geq 1$, are the so-called **Betten spreads** $\mathcal{S}_k$: Consider the following 2-dimensional Lie subalgebras of $\mathfrak{g}$ as described by Löwe in [29] :

$$\mathfrak{s} := \left\{ \begin{pmatrix} 0 & 0 & c \\ & 0 & b \\ & & 0 \end{pmatrix} \mid b, c \in \mathbb{R} \right\}$$

and

$$\mathfrak{u}(a,b,c) := \left\langle \begin{pmatrix} 2 & 0 & b \\ & 1 & a \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & a \\ & 0 & c \\ & & 0 \end{pmatrix} \right\rangle_{\mathrm{LA}} \qquad \text{for } a,b,c \in \mathbb{R};$$

note that it suffices to consider the vector subspace generated by these two matrices, as it automatically is a Lie subalgebra. For any real parameter $k \geq 1$ the spread

$$\mathcal{S}_k := \{\mathfrak{s}\} \cup \{\mathfrak{u}(a,b,-kb) \mid a,b \in \mathbb{R},\ b \leq 0\} \cup \{\mathfrak{u}(a,b,-b) \mid a,b \in \mathbb{R},\ b \geq 0\}$$

is a stable LieAlg-partition of $\mathfrak{g}$. It gives rise to a stable plane

$$\mathcal{P}_k := \mathcal{U}_{\mathsf{Inc}}\mathbb{P}\left(\Gamma; \{\mathbb{1}\}, \mathcal{S}_k^{\exp}\right).$$

3.1.4 Definition. More generally, let $\mathcal{S}$ be an arbitrary stable LieAlg-partition of $\mathfrak{g} = \mathbb{R} \propto_\omega \mathrm{hei}_3\mathbb{R}$. Then by 1.4.18, its exponential image $\mathcal{S}^{\exp}$ is a stable LieGp-partition of $\Gamma = \mathbb{R} \ltimes_{\tilde{\delta}} \mathrm{Hei}_3\mathbb{R}$. The stable plane

$$\mathcal{P} := \mathcal{U}_{\mathsf{Inc}}\mathbb{P}\left(\Gamma; \{\mathbb{1}\}, \mathcal{S}^{\exp}\right)$$

will be called a **Peter plane**. The planes $\mathcal{P}_k$ gained from a Betten spread $\mathcal{S}_k$ will eventually be referred to as the **original Peter planes**.

3.1.5 The basis. For later convenience we will introduce a basis of $\mathfrak{g}$:

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, d = \begin{pmatrix} 2 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}.$$

The Lie bracket on $\mathfrak{g}$ behaves as follows :

$$\begin{array}{rclcrcl} [d,e_1] & = & e_1 & \qquad & [e_1,e_2] & = & e_3 \\ [d,e_2] & = & e_2 & & [e_1,e_3] & = & 0 \\ [d,e_3] & = & 2e_3 & & [e_2,e_3] & = & 0. \end{array}$$

The Betten spreads are not invariant under conjugation in Γ. In order to lay hands on a concrete example, consider the fibre $\mathfrak{w} := \mathfrak{u}(0,0,0) = \mathbb{R}d + \mathbb{R}e_1 \in \mathcal{S}_k$ and an element $A = \mathbb{1} + ze_3 \in \Gamma'' \setminus \mathbb{1}$. The conjugate of the fibre is $\mathfrak{w}^A = \mathbb{R}e_1 + \mathbb{R}(d + ze_3)$, which intersects the fibre non-trivially in $\mathfrak{w} \cap \mathfrak{w}^A = \mathbb{R}e_1$. Thus, $\mathfrak{w}^A$ cannot be a fibre in $\mathcal{S}_k$.

3.2. ... and their bed

The question arose whether or not the stable planes $\mathcal{P}$ presented as Peter planes could possibly be (open) subplanes of one of the classical planes, the most immediate candidate being the 4-dimensional complex projective plane $\mathcal{P}_2\mathbb{C}$. This problem will keep us occupied throughout the remains of the present chapter. Hence, it might be worthwhile to first of all introduce the projective plane along with some useful features.

3.2.1 The **complex projective plane** $\mathcal{P}_2\mathbb{C}$ is given by

$$\begin{array}{lcll} \text{the point space} & \mathrm{P}_2\mathbb{C} & := & \mathfrak{u}_1\mathbb{C}^3 \\ \text{the line space} & \mathcal{L}_2\mathbb{C} & := & \mathfrak{u}_2\mathbb{C}^3 \end{array}$$

along with the incidence relation $\subseteq$. Endowed with the quotient topology from $\mathbb{C}^3 \setminus \{\mathbf{0}\}$, it becomes a compact connected 4-dimensional stable plane.

3.2.2 Action of Γ on $\mathcal{P}_2\mathbb{C}$. Let $\pi : \Gamma \to \mathrm{PGL}_3\mathbb{C}$ denote the canonical quotient map of $\Gamma \leq \mathrm{GL}_3\mathbb{C}$ into the projective general linear group $\mathrm{PGL}_3\mathbb{C}$. (Due to the "1" in the lower right corner of the matrices, π is an injection; Γ can be identified with the subgroup Γ^π of the projective group.) Clearly, $\Gamma^\pi \leq \mathrm{PGL}_3\mathbb{C}$ acts on $\mathcal{P}_2\mathbb{C}$. Whereas $\mathrm{PGL}_3\mathbb{C}$ acts transitively on the point space, its subgroup Γ^π will not. Some light computation reveals the point orbits :

3.2.3 Lemma. *Γ^π acts on the point space $\mathrm{P}_2\mathbb{C}$, inducing the following nine orbits :*

$$\begin{array}{lcll} \mathbb{C}(0,0,1)^{\Gamma^\pi} & = & \{\mathbb{C}(0,0,1)\} & \text{darkblue} \\ \mathbb{C}(0,1,0)^{\Gamma^\pi} & = & \{\mathbb{C}(0,1,y) \mid y \in \mathbb{R}\} & \text{yellow} \\ \mathbb{C}(1,0,0)^{\Gamma^\pi} & = & \{\mathbb{C}(1,x,y) \mid x,y \in \mathbb{R}\} & \text{red} \\ \mathbb{C}(0,1,\pm i)^{\Gamma^\pi} & = & \{\mathbb{C}(0,1,z) \mid \operatorname{Im} z \gtrless 0\} & \text{lightblue} \\ \mathbb{C}(1,\pm i,0)^{\Gamma^\pi} & = & \{\mathbb{C}(1,w,z) \mid w,z \in \mathbb{C}, \operatorname{Im} w \gtrless 0\} & \text{lilac} \\ \mathbb{C}(1,0,\pm i)^{\Gamma^\pi} & = & \{\mathbb{C}(1,x,z) \mid x \in \mathbb{R}, \operatorname{Im} z \gtrless 0\} & \text{green} \end{array}$$

For reasons of intuitive reference later on the different orbits are colour coded. It will be extremely helpful to simply refer to points from the last orbit as "green points", et cetera. Note that the "lilac", "green" and "light blue" code comprises two orbits each, though.

Some Pythia tells us that in the sequel we will have to study pencils of lines incident with points of the four (!) orbits $\mathbb{C}(1,\pm i,0)^{\Gamma^\pi}$ and $\mathbb{C}(1,0,\pm i)^{\Gamma^\pi}$, typically represented by the pencils $\mathcal{L}_{p_0}$ and $\mathcal{L}_{q_0}$, where $p_0 := \mathbb{C}(1,\pm i,0)$ and $q_0 := \mathbb{C}(1,0,\pm i)$. In order to prevent us from working too much it seems appropriate to introduce the notion of a Baer subplane of a projective plane.

3.2.4 Definition. Let $\mathcal{B}$ be a proper subplane of some projective plane $\mathcal{P}$. Then $\mathcal{B}$ is called a **Baer subplane** of $\mathcal{P}$ if every point of $\mathcal{P}$ is incident with some line of $\mathcal{B}$ and every line of $\mathcal{P}$ carries a point of $\mathcal{B}$.

3.2.5 The lilac and green pencils. The real projective plane $\mathcal{P}_2\mathbb{R}$ is a Baer subplane of the complex projective plane $\mathcal{P}_2\mathbb{C}$. This comes in handy, because $\mathrm{P}_2\mathbb{R} = \mathbb{C}(0,0,1)^{\Gamma^\pi} \cup \mathbb{C}(0,1,0)^{\Gamma^\pi} \cup \mathbb{C}(1,0,0)^{\Gamma^\pi}$ consists of the three real orbits above, exactly. As a consequence, any treatment of a pencil $\mathcal{L}_p$ reduces to considering lines which are obtained by joining the point p to any point from a real orbit. Doing this to $p_0 = \mathbb{C}(1,\pm i,0)$ we will find the following types of lines in the lilac pencil $\mathcal{L}_{p_0}$:

L	parameter form	homogeneous coordinates	
	$p_0 \oplus \mathbb{C}(0,0,1)$	$\ker(1 \quad \pm i \quad 0)^{\mathsf{T}}$	complex
	$p_0 \oplus \mathbb{C}(0,1,0)$	$\ker(0 \quad 0 \quad 1)^{\mathsf{T}}$	real
	$p_0 \oplus \mathbb{C}(0,1,c), c \in \mathbb{R}, c \neq 0$	$\ker(1 \quad \pm i \quad \pm ic^{-1})^{\mathsf{T}}$	complex
	$p_0 \oplus \mathbb{C}(1,d,c), c,d \in \mathbb{R}, c \neq 0$	$\ker(1 \quad \pm i \quad -c^{-1}(1 \pm id))^{\mathsf{T}}$	complex

Note that $\mathcal{L}_{p_0}$ contains exactly one real line, namely $\ker(0 \quad 0 \quad 1)^{\mathsf{T}} = p_0 \oplus \mathbb{C}(0,1,0) = p_0 \oplus \mathbb{C}(1,0,0) \in \mathcal{L}_2\mathbb{R}$. In fact, any real line $L \in \mathcal{L}$ must have real homogeneous coordinates $L = \ker \mathbf{a}$, $\mathbf{a} = (a_1, a_2, a_3)^{\mathsf{T}} \in \mathbb{R}^3 \setminus \{(0,0,0)^{\mathsf{T}}\}$. This forces $0 = a_1 \pm a_2 i$, thus $a_1 = a_2 = 0$ and therefore $L = \ker(0 \quad 0 \quad a_3)^{\mathsf{T}} = \ker(0 \quad 0 \quad 1)^{\mathsf{T}}$. This line, of course, also contains all the points $\mathbb{C}(1,d,0)$ with $d \in \mathbb{R}$; this is why they do not appear in the table above.

The same procedure applied to $q_0 = \mathbb{C}(1,0,\pm i)$ figures out the green pencil $\mathcal{L}_{q_0}$ as

L	parameter form	homogeneous coordinates	
	$q_0 \oplus \mathbb{C}(0,0,1)$	$\ker(0 \quad 1 \quad 0)^{\mathsf{T}}$	real
	$q_0 \oplus \mathbb{C}(0,1,c), c \in \mathbb{R}$	$\ker(\pm i \quad -c \quad 1)^{\mathsf{T}}$	complex
	$q_0 \oplus \mathbb{C}(1,d,c), c,d \in \mathbb{R}, d \neq 0$	$\ker(\pm id(c^2+1)^{-1}(c \pm i) \quad 1 \quad -d(c^2+1)^{-1}(c \pm i))^{\mathsf{T}}$	complex

Here the (!) real line also contains all the points $\mathbb{C}(1,0,c)$ with $c \in \mathbb{R}$, such that the option $d = 0$ has been suppressed.

3.2.6 $\mathcal{P}_2\mathbb{C}$ as a sketched geometry. The group $\mathrm{PGL}_3\mathbb{C}$ acts flag transitively on $\mathcal{P}_2\mathbb{C}$. The resulting geometry $(\mathrm{PGL}_3\mathbb{C}, \mathcal{P}_2\mathbb{C})$ is flag homogeneous and sketched by, for instance, the representatives $\mathrm{R}_P = \{\mathbb{C}e_1\}$ and $\mathrm{R}_{\mathcal{L}} = \{\mathbb{C}e_1 + \mathbb{C}e_2\}$. A sketch then would be given by $\mathbb{S}(\Pi; \mathcal{P}_2\mathbb{C}) = (\Pi; \{\Pi_{\mathbb{C}e_1}\}, \{\Pi_{\mathbb{C}e_1+\mathbb{C}e_2}\})$, where we abbreviate $\Pi := \mathrm{PGL}_3\mathbb{C}$. Note that the geometry $(\Gamma^\pi, \mathcal{P}_2\mathbb{C})$, on the other hand, cannot be sketched because it is neither point nor line homogeneous; see 1.1.19.

3.3. A categorical user's manual for the embedding of planes

As already mentioned in 1.5.1, one would like to know more on the nature of Peter planes, because they are the remaining mystery in Maier's classification of stable planes gained from stable (LieGp-) partitions of 4-dimensional connected Lie groups. The most homely information here would be the knowledge that in some way or other, a Peter plane $\mathcal{P}$ could be found as a subplane of a classical plane; and the candidate at hand is the complex projective plane. "In some way or other" should read, more precisely, as the existence of a (mono-) morphism $\mathsf{H} : \mathcal{P} \to \mathcal{P}_2\mathbb{C}$ of stable planes; that is, a continuous, injective and non-collapsed lineation. Let us for a moment, for a longer one in fact, assume the existence of such a morphism

$$\mathsf{H} : \mathcal{P} \hookrightarrow \mathcal{P}_2\mathbb{C}$$

of stable planes. What should it look like ? What could one possibly do to get hold of it ? Remembering that the plane $\mathcal{P}$ stems from a sketched geometry $(\mathsf{\Gamma}, \mathcal{P}) = \mathbb{P}(\mathsf{\Gamma}, \mathcal{Q})$ with sketch $(\mathsf{\Gamma}, \mathcal{Q}) := (\mathsf{\Gamma}; \{\mathsf{1}\}, \mathcal{S}^{\exp})$, could there be a way of translating the problem of finding an embedding of (topological) incidence structures into one of finding an embedding of sketches ? After all, there might be a slightly better chance to handle groups than nasty looking planes.

Such a translation from the category Inc into Sk is possible, indeed — yet not without touching upon several other categories on our way. Let us lay open our intentions in order to motivate the reader to follow through more tedious details afterwards :

3.3.1 Aim. Assume a morphism $\mathsf{H} : \mathcal{P} \hookrightarrow \mathcal{P}_2\mathbb{C} \in \operatorname{morph} \mathsf{StP}$ of stable planes.

(StP) As a matter of fact, $\mathsf{H} : \mathcal{P} \to \mathcal{P}_2\mathbb{C}$ is an open embedding of stable planes.

(Geo) There is a continuous group monomorphism

$$\varepsilon : \mathsf{\Gamma} \to \mathrm{PGL}_3\mathbb{C} \in \operatorname{morph} \mathsf{TopGp}$$

such that

$$(\varepsilon, \mathsf{H}) : (\mathsf{\Gamma}; \mathcal{P}) \to (\mathrm{PGL}_3\mathbb{C}; \mathcal{P}_2\mathbb{C}) \in \operatorname{morph} \mathsf{Geo}$$

is a (topological) embedding of geometries.

(SGeo) This induces an isomorphism

$$(\varepsilon, \mathsf{H}) : (\mathsf{\Gamma}; \mathcal{Q}; \mathcal{P}) \to (\mathsf{\Gamma}^{\varepsilon}; \mathcal{Q}^{\mathsf{H}}; \mathcal{P}^{\mathsf{H}}) \in \operatorname{morph} \mathsf{SGeo}^{*}$$

with

- $R_P := \mathcal{Q}^{\mathsf{H}}_P = \{p\}$ for some $p \in \mathrm{P}_2\mathbb{C}$ satisfying $\mathsf{\Gamma}^{\varepsilon}_p = \mathsf{1}$
- $R_{\mathcal{L}} := \mathcal{Q}^{\mathsf{H}}_{\mathcal{L}} = (\mathcal{L}_2\mathbb{C})_p$.

(Sk) Via $\mathbb{S}$, the above translates to the existence of an isomorphism

$$(\varepsilon, \mathsf{E}) : (\mathsf{\Gamma}; \mathcal{Q}) \to (\mathsf{\Gamma}^{\varepsilon}; \mathcal{R}) \in \operatorname{morph} \mathsf{Sk}$$

of sketches, where

- $\varepsilon : \mathsf{\Gamma} \to \mathrm{PGL}_3\mathbb{C}$ is a continuous group monomorphism
- $\mathcal{R}_P = \{\mathsf{\Gamma}^{\varepsilon}_p\}$ for some $p \in \mathrm{P}_2\mathbb{C}$ satisfying $\mathsf{\Gamma}^{\varepsilon}_p = \mathsf{1}$
- $\mathcal{R}_{\mathcal{L}} = \{\mathsf{\Gamma}^{\varepsilon}_L \mid L \in (\mathcal{L}_2\mathbb{C})_p\}$.

The remains of the present section will cover the stepwise transitions from category to category, till finally reaching Sk.

3.3.1. Transition from incidence structures to geometries

We will start by establishing the morphism H as an embedding of stable planes. Doing this, we will give more detailed arguments for the concrete incarnation of the general results in the introductory chapter, notably proposition 1.3.5. A first glimpse will reveal that by 1.3.2, the line map $\mathsf{H}_{\mathcal{L}}$ is injective and continuous; this implies moreover, that H is a monomorphism of incidence structures (see 1.1.12).

3.3.2 A topological space X is said to have the **domain invariance property** if every subset homeomorphic to some open subset is open itself. Euclidean spaces as well as n-manifolds possess the domain invariance property; for a list of detailed literature, consult [54, 51.18]. This fact will play a vital role in the following technical lemma.

3.3.3 Lemma. *Let $f : X \to Y$ be a continuous injection, Y an n-manifold.*

(i) If $x \in X$ and U is an open neighbourhood of x which is homeomorphic to $\mathbb{R}^n$ and whose closure $\overline{U}$ is compact, then its image U^f is also homeomorphic to $\mathbb{R}^n$ and thus open in Y.

(ii) If X is moreover an n-manifold then f is an open map.

Proof. Note that we could refer to [54, 51.19] right away, but prefer to give an outline of the arguments involved.

ad (i). Consider a closed subset A of $\overline{U}$. As $\overline{U}$ is compact A is compact as well, and so is its image A^f in $\overline{U}^f$. Due to Y being a T_2-space, this implies that A^f is closed in Y. Thus we understand that $f|_{\overline{U}} : \overline{U} \to \overline{U}^f$ is a homeomorphism. Then so is the restriction $f|_U : U \to U^f$. Therefore $\mathbb{R}^n \approx U \approx U^f$. Now domain invariance of Y yields that U^f is open in Y; in fact, every n-manifold contains open subsets homeomorphic to $\mathbb{R}^n$.

ad (ii). Let V be an open subset of X. As X is a (locally compact) n-manifold, for every $x \in X$ there is some open neighbourhood $U_x \approx \mathbb{R}^n$ whose closure $\overline{U_x}$ is compact and contained in V. Now $V = \bigcup_{x \in V} U_x$, and *(i)* reveals $V^f = \bigcup_{x \in X} U_x^f$ as open in Y. Therefore f is an open map. $\square$

3.3.4 Corollary. *$\mathsf{H} : \mathcal{P} \to \mathcal{P}_2\mathbb{C}$ is an open embedding of stable planes. In particular, P^{H} is open in $\mathrm{P}_2\mathbb{C}$.*

Consequently, via H, the Peter plane $\mathcal{P}$ is an open subplane of the compact projective plane $\mathcal{P}_2\mathbb{C}$, which allows us to apply LÖWEN's

3.3.5 Corollary from the **Local Fundamental Theorem** (1982). Let $\mathcal{P}$ be an open subspace of one of the classical projective spaces $\mathcal{P}_n\mathbb{F}$ over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ with $n \geq 2$. Then every automorphism α of $\mathcal{P}$ is induced by a (unique !) automorphism $\tilde{\alpha}$ of $\mathcal{P}_n\mathbb{F}$ which leaves $\mathcal{P}$ invariant.

Proof. [37, Corollary 2] $\square$

This is fairly convenient, for we can directly use it to construct the desired group morphism $\varepsilon : \Gamma \to \mathrm{PGL}_3\mathbb{C}$ — with some obstacles to come, yet.

3.3.6 Definition.

$$\begin{array}{cccc} \varepsilon : & \Gamma & \to & \operatorname{Aut}\mathcal{P}_2\mathbb{C} \\ & \alpha & \mapsto & \tilde{\alpha} \end{array}$$

The diagram is to be read in StP.

$$\begin{tikzcd} \mathcal{P}_2\mathbb{C} \arrow[r, tail, two heads, "\exists!\ \tilde{\alpha}"] & \mathcal{P}_2\mathbb{C} \\ \mathcal{P} \arrow[u, "\mathsf{H}"] \arrow[r, tail, two heads, "\forall\alpha"'] & \mathcal{P} \arrow[u, "\mathsf{H}"'] \end{tikzcd}$$

3.3.7 Lemma. *ε is a group monomorphism.*

Proof. First of all note that due to uniqueness of $\tilde{\alpha}$ the map is well defined. In order to verify $(\alpha\beta)^\varepsilon = \alpha^\varepsilon\beta^\varepsilon$ for every $\alpha, \beta \in \Gamma$, chase the following diagram, again using uniqueness of $\widetilde{\alpha\beta}$.

$$\begin{tikzcd} \mathcal{P}_2\mathbb{C} \arrow[rr, tail, two heads, "\widetilde{\alpha\beta}"] \arrow[dr, "\tilde{\alpha}"] & & \mathcal{P}_2\mathbb{C} \\ & \mathcal{P}_2\mathbb{C} \arrow[ur, "\tilde{\beta}"'] & \\ \mathcal{P} \arrow[uu, "\mathsf{H}"] \arrow[r, tail, two heads, "\alpha"'] & \mathcal{P} \arrow[u, "\mathsf{H}"'] \arrow[r, tail, two heads, "\beta"'] & \mathcal{P} \arrow[uu, "\mathsf{H}"'] \end{tikzcd}$$

Injectivity of ε amounts to the statement that H is monic in the category of incidence structures; in fact, for $\alpha, \beta \in \Gamma$ equality $\tilde{\alpha} = \tilde{\beta}$ implies $\alpha\mathsf{H} = \mathsf{H}\tilde{\alpha} = \mathsf{H}\tilde{\beta} = \beta\mathsf{H}$, which forces $\alpha = \beta$. $\square$

By construction of ε, the action of Γ on P and the action of Γ^ε on P^{H_P} are equivalent. The same is true for the line actions, and, in fact, the flag actions. This is just another way of saying that we are dealing with a morphism of geometries. Moreover, by 1.1.13, it is monic (epic) in Geo because ε, H_P and $\mathsf{H}_{\mathcal{L}}$ are injective (surjective).

3.3.8 Corollary. $(\varepsilon, \mathsf{H}) : (\Gamma, \mathcal{P}) \to (\Gamma^\varepsilon, \mathcal{P}^{\mathsf{H}})$ *is an isomorphism in* Geo.

In order to verify continuity of ε it occurs mandatory to understand the topologies involved. What topologies on Γ can we think of ? (For convenience's sake, we will here simply write H, meaning the point map H_P.)

$\mathcal{T}_0$	$:=$	topology induced by the compact-open topology on $\operatorname{Aut}\mathcal{P}$
$\mathcal{T}_1$	$:=$	topology induced by the matrix topology on $\mathrm{Mat}_3\mathbb{C} \approx \mathbb{C}^9$
$\mathcal{T}_2$	$:=$	compact-open topology with respect to the action of Γ on P
$\mathcal{T}_3$	$:=$	topology on Γ^ε induced by the compact-open topology on $\operatorname{Aut}\mathcal{P}_2\mathbb{C}$, translated to Γ via the bijection ε
$\mathcal{T}_4$	$:=$	compact open topology on Γ^ε with respect to the action of Γ^ε on P^{H}, translated via ε

The question remains which of these happen to coincide. At a first glimpse, we notice that $\mathcal{T}_0 = \mathcal{T}_2$, by the definition of the compact-open topology.

3.3.9 Lemma. *On Γ^ε, there is a coincidence of the compact-open topology with respect to the action of Γ^ε on P^{H}, the compact-open topology with respect to the action of Γ^ε on $\mathrm{P}_2\mathbb{C}$, and the topology induced from $\operatorname{Aut}\mathcal{P}_2\mathbb{C}$ by the compact-open topology with respect to the action of $\operatorname{Aut}\mathcal{P}_2\mathbb{C}$ on $\mathrm{P}_2\mathbb{C}$.*

Proof. As P^{H} is an open subset of $\mathrm{P}_2\mathbb{C}$, theorem 4b in [19] states the first coincidence. The second one is immediate, as by definition $\lceil C, U\rceil_{\Gamma^\varepsilon} = \lceil C, U\rceil_{\mathrm{PGL}_3\mathbb{C}} \cap \Gamma^\varepsilon$ for compact subsets $C \subseteq \mathrm{P}_2\mathbb{C}$ and open subsets $U \subseteq \mathrm{P}_2\mathbb{C}$. □

3.3.10 Corollary. $\mathcal{T}_3 = \mathcal{T}_4$.

3.3.11 Lemma. $\mathcal{T}_2 = \mathcal{T}_4$.

Proof. Use the fact that $\mathsf{H}|_P^{P^{\mathsf{H}}}$ is a homeomorphism and $\Gamma : P$ and $\Gamma^\varepsilon : P^{\mathsf{H}}$ are equivalent actions.

$$\begin{tikzcd} P^{\mathsf{H}} \times \Gamma^\varepsilon \arrow[r] & P^{\mathsf{H}} \\ P \times \Gamma \arrow[r] \arrow[u, "\mathsf{H} \times \varepsilon"] & P \arrow[u, "\mathsf{H}"'] \end{tikzcd}$$

Concretely, let U be an open subset of P and C a compact one. We want to check whether the image $\lceil C, U\rceil^\varepsilon$ of the subbasis element $\lceil C, U\rceil \in \mathcal{T}_2$ describes an open set in $\mathcal{T}_4$. In fact, the continuous image C^{H} is compact. Moreover, U^{H} is open in P^{H}. Verify

$$\lceil C^{\mathsf{H}}, U^{\mathsf{H}}\rceil = \left\{\tilde{\alpha} \in \Gamma^\varepsilon \mid C^{\mathsf{H}\tilde{\alpha}} = C^{\alpha\mathsf{H}} \subseteq U^{\mathsf{H}}\right\} = \{\tilde{\alpha} \in \Gamma^\varepsilon \mid C^\alpha \subseteq U\}$$

and accept $\lceil C, U\rceil^\varepsilon = \lceil C^{\mathsf{H}}, U^{\mathsf{H}}\rceil$ as open in Γ^ε with respect to the compact open topology induced by $\Gamma^\varepsilon : P^{\mathsf{H}}$. Consequently $\lceil C, U\rceil \in \mathcal{T}_4$. The converse inclusion can be obtained analogically. □

3.3.12 It remains to compare $\mathcal{T}_1$ to $\mathcal{T}_2$. Note that by construction $\Gamma = P$, and the action $\Gamma : P$ is right multiplication $\Gamma \times \Gamma \to \Gamma : (\alpha, \beta) \mapsto \alpha\beta$. As a consequence — the topological group $\Gamma \leq \mathrm{GL}_3\mathbb{C}$ being endowed with the matrix topology $\mathcal{T}_1$ — this action is continuous. Now the compact open topology is the coarsest topology which allows the action to be continuous [55, I §7.9, Hilfssatz 3]. Thus $\mathcal{T}_2$ is coarser than $\mathcal{T}_1$, in other words, $\mathrm{id} : \Gamma \to \Gamma_{\mathrm{co}}$ is continuous.

$$\begin{tikzcd} \Gamma \arrow[d, "\mathrm{id}"'] & \\ \Gamma_{\mathrm{co}} \arrow[r, "\mathrm{id}"] & \Gamma_{\mathrm{po}} \arrow[ul, "\mathrm{id}"'] \end{tikzcd}$$

Let us furthermore consider the product topology on $\mathcal{C}(P,P) \subseteq P^P$, which as a matter of fact is the "point-open" topology on Γ with respect to the action by multiplication as described in 3.3.18. Certainly, every one of the subbasis elements $\lceil\{p\}, U\rceil$ is open in Γ_{co} as well, such that $\mathrm{id} : \Gamma_{\mathrm{co}} \to \Gamma_{\mathrm{po}}$ is continuous. On the other hand, with our particular action by multiplication, we get $\lceil\{1\}, U\rceil = \{\alpha \in \Gamma \mid 1 \cdot \alpha \in U\} = U$, such that $\mathrm{id} : \Gamma_{\mathrm{po}} \to \Gamma$ is continuous. Thus we have established $\Gamma = \Gamma_{\mathrm{co}} = \Gamma_{\mathrm{po}}$, meaning $\mathcal{T}_1 = \mathcal{T}_2$.

As a matter of fact, we have just proved the following special case of lemma 3.3.19.

3.3.13 Lemma. *Let Υ be a topological group. Consider the action $\rho : \Upsilon \times \Upsilon \to \Upsilon$ by right multiplication. Denote by Υ_{co} the group endowed with the compact-open topology with respect to ρ, and by Υ_{po} the group endowed with the* point-open topology. *Then $\Upsilon \approx \Upsilon_{\mathrm{co}} \approx \Upsilon_{\mathrm{po}}$.*

In our particular case again, we harvest

3.3.14 Lemma. $\mathcal{T}_0 = \mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_3 = \mathcal{T}_4$

3.3.15 Corollary. *$\varepsilon : \Gamma \to \mathrm{Aut}\,\mathcal{P}_2\mathbb{C}$ is an embedding of topological groups.*

Proof. $\mathrm{Aut}\,\mathcal{P}_2\mathbb{C}$ is endowed with the compact-open topology with respect to its action on $\mathrm{P}_2\mathbb{C}$. Thus by the previous lemma, the co-restriction of ε to its image Γ^ε, which is endowed with the topology induced from $\mathrm{Aut}\,\mathcal{P}_2\mathbb{C}$, is a homeomorphism. □

At that stage it can only be claimed that $\Gamma^\varepsilon \leq \mathrm{Aut}\,\mathcal{P}_2\mathbb{C}$. Nevertheless, we are confident that Γ^ε after all is a subgroup of $\mathrm{PGL}_3\mathbb{C}$. What can be said about that ? First of all, talking about continuous collineations, the automorphism group is $\mathrm{Aut}\,\mathcal{P}_2\mathbb{C} = \mathrm{P\Gamma L}_3\mathbb{C} = \mathrm{Aut}\,\mathbb{C} \ltimes \mathrm{PGL}_3\mathbb{C}$; cf. [54, 13.6.]. Here the group $\mathrm{Aut}\,\mathbb{C}$ of continuous field automorphisms consists of two elements only, namely identity and complex conjugation κ. Moreover, its connected component is $(\mathrm{Aut}\,\mathcal{P}_2\mathbb{C})^1 = \mathbb{1} \ltimes \mathrm{PGL}_3\mathbb{C}$. In fact, $\mathrm{PGL}_3\mathbb{C}$ is connected — use connectedness of $\mathrm{GL}_3\mathbb{C}$ [21, Kapitel I, §2, Satz 14] and continuity of the canonical quotient map $\mathrm{GL}_3\mathbb{C} \twoheadrightarrow \mathrm{PGL}_3\mathbb{C}$ — and $\mathbb{1} \ltimes \mathrm{PGL}_3\mathbb{C}$ and $\{\kappa\} \ltimes \mathrm{PGL}_3\mathbb{C}$ are two non-trivial open and closed subsets of $\mathrm{Aut}\,\mathcal{P}_2\mathbb{C}$. By 3.3.15 the morphism ε is continuous, with the consequence that $\Gamma^\varepsilon \leq (\mathrm{Aut}\,\mathcal{P}_2\mathbb{C})^1 = \mathbb{1} \ltimes \mathrm{PGL}_3\mathbb{C}$. Thus we have established

3.3.16 Lemma. $\Gamma^{\varepsilon} \leq \mathrm{PGL}_3\mathbb{C} \trianglelefteq \mathrm{Aut}\,\mathcal{P}_2\mathbb{C}$

From now on we will consider the continuous group monomorphism $\varepsilon : \Gamma \to \mathrm{PGL}_3\mathbb{C}$. All in all, from 3.3.8 we get

3.3.17 Corollary. $(\varepsilon, \mathsf{H}) : (\Gamma, \mathcal{P}) \to (\mathrm{PGL}_3\mathbb{C}, \mathcal{P}_2\mathbb{C}) \in \mathrm{morph}\,\mathsf{Geo}$ *is a monomorphism of geometries. All of its components are embeddings.*

Note that up to further notice, the matrix group $\mathrm{PGL}_3\mathbb{C} = \mathrm{PSL}_3\mathbb{C}$ is thought of as endowed with the topology induced from $\mathrm{Aut}\,\mathcal{P}_2\mathbb{C}$. This mishap will be cured in the following aside.

3.3.2. Excursus on the topologies involved

One may obtain a wider choice of topologies on the groups involved here. The main tool will be a more general version of the above argument on the action of a topological group on itself by right multiplication.

3.3.18 Definition. Let Υ be a group which acts on a topological space X. The **point-open topology** on Υ is given by the subbasis $\{\lceil\{p\}, U\rceil \mid p \in X \wedge \text{ open } U \subseteq X\}$, where $\lceil\{p\}, U\rceil := \{\alpha \in \Upsilon \mid p^{\alpha} \in U\}$.

3.3.19 Lemma. *Let Υ be a topological group which acts continuously on some topological space X. Denote by Υ_{co} the group endowed with the compact-open topology with respect to the action, denote by Υ_{po} the group endowed with the corresponding point-open topology.*

a) $\mathrm{id} : \Upsilon \to \Upsilon_{\mathrm{co}}$ *is continuous.*

b) $\mathrm{id} : \Upsilon_{\mathrm{co}} \to \Upsilon_{\mathrm{po}}$ *is continuous.*

Now assume the existence of a natural number n and points $p_1, \ldots, p_n \in X$ such that the map

$$\begin{array}{rccl} f: & \Upsilon & \to & X^n \\ & \alpha & \mapsto & (p_j^{\alpha})_{j=1,\ldots,n} \end{array}$$

is injective and $f : \Upsilon \to X^n$ (where Υ is considered with its original topology) is an embedding. Then furthermore

c) $f : \Upsilon_{\mathrm{po}} \to X^n$ *is continuous.*

d) $\mathrm{id} : \Upsilon_{\mathrm{po}} \to \Upsilon$ *is continuous.*

e) *The three topologies coincide, i.e., $\Upsilon = \Upsilon_{\mathrm{co}} = \Upsilon_{\mathrm{po}}$.*

Proof. *ad (a).* The compact-open topology is the coarsest topology on Υ which allows the action to be continuous [55, I §7.9, Hilfssatz 3]. As by hypothesis, the action is continuous, the original topology on Υ is finer than the compact-open topology. *ad (b).* For every point $p \in X$, the set $\{p\}$ is compact. *ad (c).* A basis element of the product topology on X^n is of the form $U = U_1 \times \ldots \times U_n$ for open subsets U_j of X. Its pre-image

$$U^{f^{\leftarrow}} = \{\alpha \in \Upsilon \mid \forall j \leq n. \quad p_j^\alpha \in U_j\} = \bigcap_{j=1}^{n} \lceil \{p\}, U_j \rceil$$

is open with respect to the point open topology on Υ_{co}.

$$\begin{tikzcd}
\Upsilon \arrow[r, hook, "f"] \arrow[d, "\mathrm{id}"'] & X^n \\
\Upsilon_{\mathrm{co}} \arrow[r, "\mathrm{id}"] & \Upsilon_{\mathrm{po}} \arrow[u, "f"'] \arrow[ul, dotted, "\mathrm{id}"']
\end{tikzcd}$$

ad (d). This is due to the universal property of embeddings : as $f : \Upsilon_{\mathrm{po}} \to X^n$ is continuous, so is $\mathrm{id} : \Upsilon_{\mathrm{po}} \to \Upsilon$; see A.1.2. *ad (e).* Immediate consequence of (a), (b) and (d). □

3.3.20 Remark. Note that for any group Υ acting on some topological space X and any subgroup $\Psi \leq \Upsilon$, the compact-open topology on Ψ with respect to the action of Ψ on X and the topology induced from the compact-open topology on Υ trivially coincide. This is why lemmata 3.3.21 and 3.3.28 will be proved for $\mathrm{GL}_n\mathbb{C}$ and $\mathrm{PGL}_n\mathbb{C}$, respectively, yet hold for any of their subgroups.

3.3.21 Corollary. *Consider an arbitrary subgroup $\Upsilon \leq \mathrm{GL}_n\mathbb{C}$ endowed with the matrix topology induced from $(\mathbb{C}^n)^n$. Denote by Υ_{co} and Υ_{po} the group endowed with the compact-open and point-open topologies, respectively, with respect to the standard action of Υ on $\mathbb{C}^n$. Then these three topologies coincide, i.e., $\Upsilon = \Upsilon_{\mathrm{co}} = \Upsilon_{\mathrm{po}}$.*

Proof. With respect to the matrix topology on Υ, its action on $\mathbb{C}^n$ is continuous. Pick the standard basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$ and put

$$\begin{array}{rccl} h: & \Upsilon & \to & (\mathbb{C}^n)^n \\ & A & \mapsto & (e_j A)_{j=1,\ldots,n}. \end{array}$$

Then h is an injection. In fact, $h : \Upsilon \to (\mathbb{C}^n)^n$ is an embedding. Hence we may apply lemma 3.3.19, which yields equality of the topologies. □

The analogical result will be obtained for the projective situation, yet after some more preparations.

3.3.22 Lemma. *The action of* $\mathrm{PSL}_3\mathbb{C}$ *on* $\mathrm{P}_2\mathbb{C}$ *is continuous.*

Proof.

$$\begin{tikzcd}
\mathbb{C}^3\setminus\mathbf{0}\times \mathrm{SL}_3\mathbb{C} \arrow[r, "\omega_1"'] \arrow[d, "\lambda\times\mathrm{id}"'] \arrow[ddr] & \mathbb{C}^3\setminus\mathbf{0} \arrow[d, "\lambda"] \\
\mathrm{P}_2\mathbb{C}\times \mathrm{SL}_3\mathbb{C} \arrow[d, "\mathrm{id}\times\pi"'] & \mathrm{P}_2\mathbb{C} \arrow[d, equal] \\
\mathrm{P}_2\mathbb{C}\times \mathrm{PSL}_3\mathbb{C} \arrow[r, dotted, "\omega_3"] & \mathrm{P}_2\mathbb{C}
\end{tikzcd}$$

We may use the universal property of the quotient map $(\lambda\times\mathrm{id})\cdot(\mathrm{id}\times\pi)$, see A.1.2. In fact, the diagonal $(\lambda\times\mathrm{id})\cdot(\mathrm{id}\times\pi)\cdot\omega_3=\omega_1\cdot\lambda\cdot\mathrm{id}$ is continuous because λ and the action ω_2 of $\mathrm{PSL}_3\mathbb{C}$ on $\mathbb{C}^3\setminus\mathbf{0}$ are continuous. By the universal property then so is the action ω_3 of $\mathrm{PSL}_3\mathbb{C}$ on $\mathrm{P}_2\mathbb{C}$. □

Our candidate for an embedding of $\mathrm{PSL}_3\mathbb{C}$ into a product of projective spaces $\mathrm{P}_2\mathbb{C}$ will be

$$\begin{array}{rccc} I: & \mathrm{PSL}_3\mathbb{C} & \to & (\mathrm{P}_2\mathbb{C})^4 \\ & [A] & \mapsto & ([q_jA])_{j=1,\dots,4} \end{array}$$

with $q_1:=[e_1], q_2:=[e_2], q_3:=[e_3]$ and $q_4:=[e_1+e_2+e_3]$.

3.3.23 Lemma. $I:\mathrm{PSL}_3\mathbb{C}\to(\mathrm{P}_2\mathbb{C})^4$ *is an injection.*

Proof. Consider $[A],[B]\in\mathrm{PSL}_3\mathbb{C}$ with the property that $[A]^I=[B]^I$. Then there are factors $\lambda_1,\lambda_2,\lambda_3\in\mathbb{C}^\times$ such that $b_j=\lambda_ja_j$ holds for the rows of the matrices. The fourth component ensures another coefficient $\lambda_4\in\mathbb{C}^\times$ satisfying $\lambda_4(a_1+a_2+a_3)=\lambda_4(e_1+e_2+e_3)A=(e_1+e_2+e_3)B=b_1+b_2+b_3=\lambda_1a_1+\lambda_2a_2+\lambda_2a_3$, thus $\lambda_1=\lambda_2=\lambda_3=\lambda_4$ and consequently $[A]=[B]$. □

3.3.24 Lemma. *The image of* $\mathrm{PSL}_3\mathbb{C}$ *under* I *is the set*

$$\begin{array}{rcl} F & := & \{([p_j])_{j=1,\dots,4}\in(\mathrm{P}_2\mathbb{C})^4 \mid \textit{each triple of projective points } [p_j] \textit{ is non-collinear}\} \\ & = & \{([p_j])_{j=1,\dots,4}\in(\mathrm{P}_2\mathbb{C})^4 \mid \textit{each triple of affine points } p_j \textit{ forms a basis of } \mathbb{C}^3\} \end{array}$$

of (ordinary) four-gons in $\mathcal{P}_2\mathbb{C}$.

Proof. For $[A]\in\mathrm{PSL}_3\mathbb{C}=\mathrm{PGL}_3\mathbb{C}$ the lines of A form a basis of $\mathbb{C}^3$. Thus the inclusion $(\mathrm{PSL}_3\mathbb{C})^I\subseteq F$ is immediate. Now let p be a four-gon in $\mathcal{P}_2\mathbb{C}$, with $p_j=[a_j]$, $a_j\in\mathbb{C}^3\setminus\mathbf{0}$, for $j\leq 4$. As a_1,a_2,a_3 form a basis of $\mathbb{C}^3$, there are unique coefficients $\mu_1,\mu_2,\mu_3\in\mathbb{C}$ such that $a_4=\sum_{j=1}^3\mu_ja_j$. Due to $p\in F$, these coefficients μ_j are non-zero. Put

$A := (a_1\ a_2\ a_3)^\intercal$. Our aim is to find coefficients $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}^\times$ satisfying

$$\begin{array}{lrcl} (1) & \lambda_4 a_4 & = & (1,1,1)\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} A \\ (2) & 1 & = & \lambda_1\lambda_2\lambda_3 \cdot \det A. \end{array}$$

From (2) we get $\lambda_3 = (\lambda_1\lambda_2 \cdot \det A)^{-1}$. From (1) then, we understand that we are looking for solutions of the equation

$$\lambda_4(\mu_1, \mu_2, \mu_3) = (\lambda_1, \lambda_2, (\lambda_1\lambda_2 \det A)^{-1}).$$

As a matter of fact, there is a solution, namely

$$\begin{array}{rcl} \lambda_4 & = & (\mu_1\mu_2\mu_3 \cdot \det A)^{-\frac{1}{3}} \neq 0 \\ \lambda_1 & = & \mu_1\lambda_4 \neq 0 \\ \lambda_2 & = & \mu_2\lambda_4 \neq 0 \\ \lambda_3 & = & (\lambda_1\lambda_2 \det A)^{-1} \neq 0. \end{array}$$

Thus, $I|^F : \mathrm{PSL}_3\mathbb{C} \to F$ is surjective, bijective even. □

In order to establish $I : \mathrm{PSL}_3\mathbb{C} \to (\mathrm{P}_2\mathbb{C})^4$ as an embedding we will appeal to the following result on the action of locally compact groups.

3.3.25 Theorem. *Let $\omega : X \times \Upsilon \to X$ be a continuous action of a topological group Υ on a topological space X. Pick $x \in X$, and consider the bijection*

$$\begin{array}{rccl} \Omega : & \Upsilon/\Upsilon_x & \to & x^\Upsilon \\ & \Upsilon_x \cdot \alpha & \mapsto & x^\alpha\,. \end{array}$$

Then

a) *Ω is continuous.*

b) *If the group Υ is σ-compact and locally compact, and if the orbit x^Υ is a locally compact Hausdorff space, then Ω is a homeomorphism.*

Proof. The theorem is due to Freudenthal [13]. A complete proof can be found in [73, 8.8]. For more references, see [54, 96.8]. □

Consider the action

$$\begin{array}{rccl} \omega : & F \times \mathrm{PSL}_3\mathbb{C} & \to & F \\ & (p, M) & \mapsto & (p_j^M)_{j=1,\dots,4}, \end{array}$$

where p_j^M refers to the action of $M \in \mathrm{PSL}_3\mathbb{C}$ on $p_j \in \mathrm{P}_2\mathbb{C}$. As by 3.3.22 the latter action is continuous, so is ω. With respect to this action, the situation so far reformulates into the following orbit and stabiliser and therefore allows for an application of the theorem above :

3.3.26 Corollary. *Pick* $\mathbf{q_o} := (q_j)_{j=1\ldots,4} = ([e_1],[e_2],[e_3],[e_1+e_2+e_3]) \in (\mathrm{P}_2\mathbb{C})^4$.

a) $I = \omega(\mathbf{q_o}, \cdot\,)$

b) ${\mathbf{q_o}}^{\mathrm{PSL}_3\mathbb{C}} = F$

c) $(\mathrm{PSL}_3\mathbb{C})_{\mathbf{q_o}} = 1$

Proof. Let $M = [A] \in \mathrm{PSL}_3\mathbb{C}$. Then $M^I = ([e_1A],[e_2A],[e_3A],[(e_1+e_2+e_3)A]) = (q_j^M)_{j=1,\ldots,4} = (\mathbf{q_o}, M)^\omega$. As to the orbit, for $p \in F$ put $M := p^{I^{-1}}$. Then $p = M^I = (\mathbf{q_o}, M)^\omega$. As to the stabiliser, note that I is injective. □

3.3.27 Corollary. *The co-restriction* $I|^F : \mathrm{PSL}_3\mathbb{C} \to F$ *is a homeomorphism, and* $I : \mathrm{PSL}_3\mathbb{C} \to (\mathrm{P}_2\mathbb{C})^4$ *is an embedding.*

Proof. As ${\mathbf{q_o}}^{\mathrm{PSL}_3\mathbb{C}} = F$ and point stabilisers are trivial, we can identify the map "Ω" from theorem 3.3.25 with $\Omega = I|^F : \mathrm{PSL}_3\mathbb{C} \to F$. The spaces $\mathrm{SL}_3\mathbb{C}$ and $\mathbb{C}^3 \setminus \mathbf{0}$ are connected and locally compact. As both quotient maps are open, both, $\mathrm{PSL}_3\mathbb{C}$ and $\mathrm{P}_2\mathbb{C}$ are locally compact, too. So $\mathrm{PSL}_3\mathbb{C}$ is connected and locally compact, hence σ-compact, and the closed subset $F \subseteq (\mathrm{P}_2\mathbb{C})^4$ is a locally compact Hausdorff space; see A.1.3. Thus the theorem is applicable and yields that $I|^F$ is homeomorphic. □

Now we are in a position to conclude coincidence of several topologies on $\mathrm{PSL}_3\mathbb{C}$.

3.3.28 Corollary. *Let* Υ *be an arbitrary subgroup of* $\mathrm{PSL}_3\mathbb{C}$*, endowed with the topology induced by the quotient topology on* $\mathrm{SL}_3\mathbb{C}$ *with respect to the matrix topology on* $\mathrm{SL}_3\mathbb{C}$*. Denote by* Υ_{co} *and* Υ_{po} *the group endowed with the compact-open and point-open topologies, respectively, with respect to the action of* Υ *on* $\mathrm{P}_2\mathbb{C}$*. Then* $\Upsilon = \Upsilon_{\mathrm{co}} = \Upsilon_{\mathrm{po}}$.

Proof. By 3.3.22, the action of $\mathrm{PSL}_3\mathbb{C}$ on $\mathrm{P}_2\mathbb{C}$ is continuous, and 3.3.27 provides for a suitable embedding $I : \mathrm{PSL}_3\mathbb{C} \to (\mathrm{P}_2\mathbb{C})^4$. Thus 3.3.19 establishes the desired result. □

This finally enables us to make use of different descriptions of the topologies in question when dealing with our mappings ε and φ.

$$\begin{tikzcd}
\Gamma \arrow[rr, hook, "\varepsilon"] \arrow[dr, "\varepsilon|^{\Gamma^\varepsilon}"] \arrow[ddd, equal] & & \mathrm{PSL}_3\mathbb{C} \\
& \Gamma^\varepsilon \arrow[ur, hook] & \\
& \Gamma^\varphi \arrow[u, "\varphi|^{\Gamma^\varepsilon}_{\Gamma^\varphi}"] \arrow[dr, hook] & \\
\Gamma \arrow[ur, "\varphi|^{\Gamma^\varphi}"] \arrow[rr, tail, "\varphi"] & & \mathrm{SL}_3\mathbb{C} \arrow[uuu, two heads, "\pi"']
\end{tikzcd}$$

3.3.29 We have by now seen that the following topologies on Γ^ε coincide :

- the compact-open topology with respect to the action of Γ^ε on $\mathrm{P}_2\mathbb{C}$
- the topologies induced from $\mathrm{PSL}_3\mathbb{C}$, where $\mathrm{PSL}_3\mathbb{C}$ may be endowed with any of the topologies described in 3.3.28.

3.3.30 Corollary. $\varepsilon|^{\Gamma^\varepsilon} : \Gamma \to \Gamma^\varepsilon \leq \mathrm{PSL}_3\mathbb{C}$ *is a homeomorphism.*

Proof. This follows from 3.3.29 and 3.3.14, using $\mathcal{T}_3$ on Γ along with the compact-open topology with respect to the action $\Gamma^\varepsilon : \mathrm{P}_2\mathbb{C}$ on Γ^ε. $\square$

3.3.3. Transition from geometries to sketched geometries

Remember that the assumed morphism $\mathsf{H} : \mathcal{P} \to \mathcal{P}_2\mathbb{C}$ has been established as an open embedding of stable planes. Henceforth, we will concentrate on its co-restriction and switch towards talking about isomorphisms of all sorts : the open subplane

$$\mathcal{P}^{\mathsf{H}} := (P^{\mathsf{H}_P}, \mathcal{L}^{\mathsf{H}_\mathcal{L}}) \subset \mathcal{P}_2\mathbb{C}$$

is a stable plane isomorphic to $\mathcal{P}$ via H.

3.3.31 A morphism H of incidence structures is said to **preserve pencils** if the pencil of every point p is mapped onto the pencil of the image of p; i.e., if $(\mathcal{L}_p)^{\mathsf{H}_\mathcal{L}} = (\mathcal{L}')_{p^{\mathsf{H}_P}}$. Lemma 1.1 in Stroppel [66] affirms that our open embedding $\mathsf{H} : \mathcal{P} \to \mathcal{P}_2\mathbb{C}$ preserves pencils.

3.3.32 Lemma. $(\varepsilon, \mathsf{H}) : (\Gamma; \mathcal{Q}; \mathcal{P}) \to (\Gamma^\varepsilon, R; \mathcal{P}^\mathsf{H}) \in \operatorname{morph} \mathsf{SGeo}^*$, *where*

- $\mathcal{Q} := (\{1\}, \mathcal{S}^{\exp})$
- $R := (\mathcal{Q}_P^{\mathsf{H}_P}, \mathcal{Q}_\mathcal{L}^{\mathsf{H}_\mathcal{L}})$

is an isomorphism of sketched geometries. All of its components are homeomorphisms.

Proof. As to the objects, first of all $\mathbb{P}(\Gamma; \mathcal{Q}) = (\Gamma; \mathcal{Q}; \mathcal{P})$ certainly is a sketched geometry, by construction. As $(\varepsilon, \mathsf{H})$ is an isomorphism in Geo, this implies that $R = \mathcal{Q}^\mathsf{H}$ sketches $(\Gamma^\varepsilon, \mathcal{P}^\mathsf{H})$, and therefore $(\Gamma^\varepsilon; R; \mathcal{P}^\mathsf{H}) \in \operatorname{ob} \mathsf{SGeo}^*$.

Talking about morphisms, $(\varepsilon, \mathsf{H})$ is already recognized as an isomorphism in Geo, and by 1.1.13 and 1.1.14 so it is in SGeo^*, for all its components are isomorphisms in their respective categories. □

There is more precious information on the set of line representatives to be drawn from the fact that H is an open map. This requires a short glimpse at the set of point representatives.

3.3.33 Lemma. $R_P = \{p\}$ *for some* $p \in P^{\mathsf{H}_P} \subseteq \mathrm{P}_2\mathbb{C}$ *satisfying* $\Gamma^\varepsilon_p = 1$.

Proof. First observe that Γ acts sharply transitive on $P = \Gamma$. But then Γ^ε acts transitively on P^{H_P} because the actions $\Gamma : P$ and $\Gamma^\varepsilon : P^{\mathsf{H}_P}$ are equivalent. Thus there is some $p \in P^{\mathsf{H}_P} \subseteq \mathrm{P}_2\mathbb{C}$ with $P^{\mathsf{H}_P} = p^{\Gamma^\varepsilon}$; as a matter of fact, we may pick $p = 1^{\mathsf{H}_P}$. Moreover, due to equivalence of actions, the point stabiliser of p turns out to be $\Gamma^\varepsilon_p = (\Gamma_1)^\varepsilon = 1$. □

3.3.34 Lemma. $R_\mathcal{L} = (\mathcal{L}_2\mathbb{C})_p$ *for* $p \in \mathrm{P}_2\mathbb{C}$ *as in 3.3.33.*

Proof. Observe that $R_\mathcal{L} = \mathcal{S}^{\exp \cdot \mathsf{H}_\mathcal{L}} = (\mathcal{L}_1)^{\mathsf{H}_\mathcal{L}} = (\mathcal{L}_2\mathbb{C})_{1^{\mathsf{H}_P}} = (\mathcal{L}_2\mathbb{C})_p$. The third equation is due the fact that H preserves pencils (3.3.31), thus due to openness of H. □

A résumé of the preceding three lemmata arrives at

3.3.35 Corollary. $(\varepsilon, \mathsf{H}) : (\Gamma; \mathcal{Q}; \mathcal{P}) \to (\Gamma^\varepsilon; R; \mathcal{P}^\mathsf{H}) \in \operatorname{morph} \mathsf{SGeo}^*$ *with*

- $R_P = \{p\}$ *for some* $p \in \mathrm{P}_2\mathbb{C}$ *satisfying* $\Gamma^\varepsilon_p = 1$
- $R_\mathcal{L} = (\mathcal{L}_2\mathbb{C})_p$

is a continuous isomorphism of sketched geometries.

3.3.4. Transition from sketched geometries to sketches

3.3.36 At that stage not much work is left to do. All there is, is to apply the functor $\mathbb{S} : \mathsf{SGeo}^* \to \mathsf{Sk}$ to the situation in 3.3.35. A glimpse at the objects involved immediately yields

$$\mathbb{S}\,(\Gamma; \mathcal{Q}; \mathcal{P}) = \mathbb{S}\,\mathbb{P}\,(\Gamma; \mathcal{Q}) = (\Gamma; \mathcal{Q}) = (\Gamma; \{1\}, \mathcal{S}^{\exp})$$

as well as

$$\mathbb{S}\,(\Gamma^{\varepsilon}; R; \mathcal{P}^{\mathsf{H}}) = (\Gamma^{\varepsilon}; \{1\}, \{\Gamma^{\varepsilon}_{L} \mid L \in (\mathcal{L}_2\mathbb{C})_p\}) =: (\Gamma^{\varepsilon}; \mathcal{R}).$$

As any functor preserves isomorphisms,

$$(\varepsilon, \mathsf{E}) := \mathbb{S}\,(\varepsilon, \mathsf{H}) \in \operatorname{morph} \mathsf{Sk}$$

is an isomorphism of sketches. To be more concrete about E, note that a line stabiliser $\Lambda = \Gamma_{\Lambda} \in \mathcal{Q}_{\mathcal{L}} = \mathcal{S}^{\exp}$ is mapped to $\Lambda^{\mathsf{E}_{\mathcal{L}}} = (\Gamma^{\varepsilon})_{\Lambda^{\mathsf{H}_{\mathcal{L}}}} = (\Gamma_{\Lambda})^{\varepsilon} = \Lambda^{\varepsilon}$.

3.3.37 Proposition. *Assume the existence of a continuous monomorphism*

$$\mathsf{H} : \mathcal{P} \to \mathcal{P}_2\mathbb{C} \in \operatorname{morph} \mathsf{Inc}.$$

Then there is an isomorphism

$$(\varepsilon, \mathsf{E}) : (\Gamma; \{1\}, \mathcal{S}^{\exp}) \to (\Gamma^{\varepsilon}; \mathcal{R}) \in \operatorname{morph} \mathsf{Sk}$$

where

(1) $\varepsilon : \Gamma \to \mathrm{PGL}_3\mathbb{C}$ *is a continuous monomorphism of groups*

(2) $\mathcal{R}_P = \{\Gamma^{\varepsilon}_{p}\}$ *for some* $p \in \mathrm{P}_2\mathbb{C}$ *satisfying* $\Gamma^{\varepsilon}_{p} = 1$

(3) $\mathcal{R}_{\mathcal{L}} = \{\Gamma^{\varepsilon}_{L} \mid L \in (\mathcal{L}_2\mathbb{C})_p\}$.

In particular, every line stabiliser $\mathsf{M} \in \mathcal{R}_{\mathcal{L}}$ *is of dimension* 2.

Proof. For *(1)*, *(2)* and *(3)* recollect 3.3.7, 3.3.15, 3.3.35 and 3.3.36 above. In order to agree with the remark on the dimensions of the line stabilisers, take into account that

- $(\varepsilon, \mathsf{E})$ being epic implies that in particular $\mathsf{E}_{\mathcal{L}}$ is a surjection [1.1.15]
- every fibre in $\mathcal{S}^{\exp}$ is of dimension 2, as $\mathcal{S}^{\exp}$ is Peter-planar
- the co-restriction $\varepsilon : \Gamma \to \Gamma^{\varepsilon}$ is a homeomorphism [3.3.30].

Let $\mathsf{M} \in \mathcal{R}_{\mathcal{L}}$. Due to surjectivity of $\mathsf{E}_{\mathcal{L}}$, there is a fibre $\Lambda \in \mathcal{S}^{\exp}$ such that $\Lambda^{\mathsf{E}_{\mathcal{L}}} = \mathsf{M}$. Then $2 = \dim \Lambda = \dim \Lambda^{\varepsilon} = \Lambda^{\mathsf{E}_{\mathcal{L}}} = \dim \mathsf{M}$. □

3.3.38 Recipe. What have we learnt in the meantime, revisiting our original question ? If ever there is a morphism $\mathsf{H} : \mathcal{P} \hookrightarrow \mathcal{P}_2\mathbb{C}$ of stable planes, there is a continuous injective group morphism $\varepsilon : \Gamma \to \mathrm{PGL}_3\mathbb{C}$ and some point $p \in \mathrm{P}_2\mathbb{C}$ with trivial stabiliser $\Gamma_p^\varepsilon = \mathtt{1}$ such that for every line L in the pencil $(\mathcal{L}_2\mathbb{C})_p$ in p the stabiliser Γ_L^ε is of dimension 2. As a consequence, the non-existence of such a morphism H of stable planes can be established by proving that for every monomorphism $\varepsilon : \Gamma \to \mathrm{PGL}_3\mathbb{C}$ of topological groups and for every point $p \in \mathrm{P}_2\mathbb{C}$ the point stabiliser Γ_p^ε is non-trivial or there is some line $L \in (\mathcal{L}_2\mathbb{C})_p$ whose stabiliser is of dimension $\dim \Gamma_L^\varepsilon \neq 2$. This describes the itinerary for the remains of the chapter :

1. Find all continuous injective morphisms ε of the group Γ into $\mathrm{PGL}_3\mathbb{C}$.
2. Single out all the points $p \in \mathrm{P}_2\mathbb{C}$ satisfying $\Gamma_p^\varepsilon = \mathtt{1}$.
3. For those points p check the line pencils $(\mathcal{L}_2\mathbb{C})_p$ for lines L with $\dim \Gamma_L^\varepsilon \neq 2$.

3.4. Hunting down group monics

For a start, we will now be searching for *all* continuous group monomorphisms

$$\varepsilon : \Gamma \hookrightarrow \mathrm{PGL}_3\mathbb{C}.$$

At first glimpse, this looks hopeless. Can we do anything to make it look more hospitable ? To that end, let us have a look at the constellation of groups. Note that $\mathrm{PGL}_3\mathbb{C} = \mathrm{PSL}_3\mathbb{C}$; in fact, for any $A \in \mathrm{GL}_3\mathbb{C}$ with $a := \det A \neq 0$ we have $\det(a^{-\frac{1}{3}}A) = 1$ and $[A] = [a^{-\frac{1}{3}}A] \in \mathrm{PSL}_3\mathbb{C}$. Moreover, $\pi : \mathrm{SL}_3\mathbb{C} \twoheadrightarrow \mathrm{PSL}_3\mathbb{C} : A \mapsto [A]$ is a universal covering with kernel $\ker \pi = \{\lambda \mathbb{1} \mid \lambda \in \mathbb{C},\ \lambda^3 = 1\}$.

3.4.1 Lemma. **a)** *For any morphism $\varepsilon : \Gamma \to \mathrm{PSL}_3$ there is a unique lifting $\varphi : \Gamma \to \mathrm{SL}_3\mathbb{C}$ satisfying $\varphi\pi = \varepsilon$. In other words, the following diagram commutes :*

$$\begin{tikzcd}
\Gamma \arrow[r, "\varepsilon"] \arrow[d, equal] & \mathrm{PSL}_3\mathbb{C} \\
\Gamma \arrow[r, "\varphi"] & \mathrm{SL}_3\mathbb{C} \arrow[u, two heads, "\pi"]
\end{tikzcd}$$

b) *If ε is monic then so is φ.*

Proof. *ad (a).* This is true because Γ is simply connected; see [26, A2.6] or [49, 3.7.6]. *ad (b).* Let $\alpha, \beta \in \operatorname{morph} \mathsf{Gp}$ such that $\alpha\varphi = \beta\varphi$. Then $\alpha\varphi\pi = \alpha\varepsilon = \beta\varepsilon = \beta\varphi\pi$, and as ε is monic, we get $\alpha = \beta$. $\square$

The consequence is that given a group embedding $\varepsilon : \Gamma \to \mathrm{PGL}_3\mathbb{C}$ we should be able to find its lifting $\varphi : \Gamma \to \mathrm{SL}_3\mathbb{C}$. Note that different embeddings ε_1 and ε_2 produce different liftings φ_1 and φ_2. In fact, $\varphi_1 = \varphi_2$ implies $\varepsilon_1 = \varphi_1\pi = \varphi_2\pi = \varepsilon_2$.

Thus, in the sequel we will be hunting for any possible group monomorphism $\varphi : \Gamma \to \mathrm{SL}_3\mathbb{C}$. As Γ will turn out to be a soluble group we will be able to further embellish the task by applying Lie's theorem A.3.4. To that end we will briefly give the commutator series of Γ and $\mathfrak{g}$.

3.4.2 Lemma.

a)
$$\begin{aligned}
\Gamma &= \left\{ \begin{pmatrix} a^2 & x & z \\ & a & y \\ & & 1 \end{pmatrix} \mid a, x, y, z \in \mathbb{R},\ a > 0 \right\} \\
\Gamma' &= \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\} \\
\Gamma'' &= \left\{ \begin{pmatrix} 1 & 0 & z \\ & 1 & 0 \\ & & 1 \end{pmatrix} \mid z \in \mathbb{R} \right\} \\
\Gamma''' &= 1
\end{aligned}$$

b)
$$\begin{aligned}
\mathfrak{g} &= \left\{ \begin{pmatrix} 2t & x & z \\ & t & y \\ & & 0 \end{pmatrix} \mid t, x, y, z \in \mathbb{R} \right\} \\
\mathfrak{g}' &= \left\{ \begin{pmatrix} 0 & x & z \\ & 0 & y \\ & & 0 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\} \\
\mathfrak{g}'' &= \left\{ \begin{pmatrix} 0 & 0 & z \\ & 0 & 0 \\ & & 0 \end{pmatrix} \mid z \in \mathbb{R} \right\} \\
\mathfrak{g}''' &= \mathbf{0}
\end{aligned}$$

3.4.3 Corollary. Γ *is a soluble group and* $\mathfrak{g}$ *is a soluble Lie algebra.* □

Denote by $\eta : \mathfrak{g} \to \mathfrak{sl}_3\mathbb{C}$ the morphism $\eta := \ell\varphi : \ell\Gamma \to \ell(\mathrm{SL}_3\mathbb{C})$ of Lie algebras.

$$\begin{tikzcd}
\Gamma \arrow[r, "\varphi"] & \mathrm{SL}_3\mathbb{C} \\
\mathfrak{g} \arrow[u, "\exp_{\mathfrak{g}}"] \arrow[r, dashed, "\eta"] & \mathfrak{sl}_3\mathbb{C} \arrow[u, "\exp"']
\end{tikzcd}$$

3.4.4 Corollary. *The homomorphic images* Γ^{φ} *and* $\mathfrak{g}^{\eta}$ *are soluble.*

Proof. Homomorphic images of soluble Lie algebras are soluble; see A.3.5. □

Now we are allowed to apply LIE's theorem A.3.4 to the soluble Lie algebra $\mathfrak{g}^\eta$ and get

3.4.5 Lemma. *$\mathfrak{g}^\eta$ stabilises some flag in $\mathbb{C}^3$. In other words, up to conjugation in* $\mathrm{PGL}_3\mathbb{C}$, *the image $\mathfrak{g}^\eta$ consists of upper triangular matrices :*

$$\mathfrak{g}^\eta \leq \mathfrak{d} := \left\{ \begin{pmatrix} a & x & z \\ & b & y \\ & & c \end{pmatrix} \mid a,b,c,x,y,z \in \mathbb{C}, \quad a+b+c \neq 0 \right\} \leq \mathfrak{sl}_3\mathbb{C} \quad .$$

As an immediate consequence

$$\Gamma^\varphi = \mathfrak{g}^{\eta \cdot \exp} \leq \mathfrak{d}^{\exp} \leq \Delta,$$

where

$$\Delta := \left\{ \begin{pmatrix} a & x & z \\ & b & y \\ & & c \end{pmatrix} \mid a,b,c,x,y,z \in \mathbb{C}, \quad abc = 1 \right\} \leq \mathrm{SL}_3\mathbb{C} \quad .$$

3.4.6 Lemma.

Any monic $\varepsilon : \Gamma \hookrightarrow \mathrm{PGL}_3\mathbb{C}$ induces a monomorphism $\eta : \mathfrak{g} \to \mathfrak{d}$.

$$\begin{tikzcd} \Gamma \arrow[r, "\varphi"] & \Delta \\ \mathfrak{g} \arrow[u, "\exp_{\mathfrak{g}}"] \arrow[r, "\eta"'] & \mathfrak{d} \arrow[u, "\exp"'] \end{tikzcd}$$

Proof. This follows from lemma 3.4.1 and lemma 3.4.5. In fact, $\exp_{\mathfrak{g}} \cdot \varphi$ is injective, and hence so is η. □

Instead of hunting down group monomorphisms, we may be tempted to first of all search for monomorphisms $\eta : \mathfrak{g} \hookrightarrow \mathfrak{d}$ of the corresponding Lie algebras : if there is any monic $\varepsilon : \Gamma \hookrightarrow \mathrm{PGL}_3\mathbb{C}$ then we should be able to find some monomorphism $\eta : \mathfrak{g} \to \mathfrak{d}$ of Lie algebras. A closer look at the commutator series of Δ and $\mathfrak{d}$ will turn out to be convenient for finding monomorphisms.

3.4.7 Lemma.

a)
$$\begin{aligned} \Delta &= \left\{ \begin{pmatrix} a & x & z \\ & b & y \\ & & c \end{pmatrix} \mid a,b,c,x,y,z \in \mathbb{C},\, abc = 1 \right\} \\ \Delta' &= \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \mid x,y,z \in \mathbb{C} \right\} \\ \Delta'' &= \left\{ \begin{pmatrix} 1 & 0 & z \\ & 1 & 0 \\ & & 1 \end{pmatrix} \mid z \in \mathbb{C} \right\} \\ \Delta''' &= 1 \end{aligned}$$

b)
$$\begin{aligned}
\mathfrak{d} &= \left\{ \begin{pmatrix} a & x & z \\ & b & y \\ & & c \end{pmatrix} \mid a, b, c, x, y, z \in \mathbb{C},\ a + b + c = 0 \right\} \\
\mathfrak{d}' &= \left\{ \begin{pmatrix} 0 & x & z \\ & 0 & y \\ & & 0 \end{pmatrix} \mid x, y, z \in \mathbb{C} \right\} \\
\mathfrak{d}'' &= \left\{ \begin{pmatrix} 0 & 0 & z \\ & 0 & 0 \\ & & 0 \end{pmatrix} \mid z \in \mathbb{C} \right\} \\
\mathfrak{d}''' &= \mathbf{0}
\end{aligned}$$

3.4.8 Lemma. *Let $\eta : \mathfrak{g} \to \mathfrak{d}$ be a monomorphism. Then so are $\eta|_{\mathfrak{g}'} : \mathfrak{g}' \to \mathfrak{d}'$ and $\eta|_{\mathfrak{g}''} : \mathfrak{g}'' \to \mathfrak{d}''$.*

Smaller Lie algebras are easier to handle. So we will try and find possible embeddings of $\mathfrak{g}'' = \ell(\Gamma'')$, then of $\mathfrak{g}' = \ell(\Gamma')$ and so forth, gradually climbing our way up towards embeddings of $\mathfrak{g} = \ell\,\Gamma$.

Step 1. The bottom-most step will hence determine $\eta|_{\mathfrak{g}''}$ up to conjugation in Δ.

3.4.9 Lemma. *Any monic $\eta : \mathfrak{g}'' \to \mathfrak{d}''$ can be described by $e_3 \mapsto we_3$ for some $w \in \mathbb{C}\backslash\mathbf{0}$. Without loss we can assume $w = 1$.*

Proof. η is a monomorphism if and only if $w \neq 0$. For $w \neq 0$ we can apply the inner automorphism $e_3 \mapsto \operatorname{Diag}(1, w, w^{-1})^{-1} \cdot e_3 \cdot \operatorname{Diag}(1, w, w^{-1})$, which transforms we_3 into e_3. Thus, without loss of generality $\eta = \mathrm{id}$. □

Next we will determine the extensions of $\eta|_{\mathfrak{g}''}$ to $\mathfrak{g}'$. All we know at first glimpse is that $\eta|_{\mathfrak{g}'}$ must be of the form

$$\begin{aligned}
\eta \ :\ \mathfrak{g}' &\to \mathfrak{d}' \\
e_3 &\mapsto e_3 \\
e_1 &\mapsto \begin{pmatrix} 0 & u & w \\ & 0 & v \\ & & 0 \end{pmatrix} \\
e_2 &\mapsto \begin{pmatrix} 0 & u' & w' \\ & 0 & v' \\ & & 0 \end{pmatrix}
\end{aligned}$$

where e_3, e_1^η, e_2^η form a basis of $\mathfrak{d}'$. We can learn more about $u, v, w, u', v', w' \in \mathbb{C}$ taking into account that any morphism of Lie algebras had better respect Lie brackets. Recall from 3.1.5 that the Lie bracket in $\mathfrak{g}$ is given by

$$\begin{aligned}
[d, e_1] &= e_1 & \qquad [e_1, e_2] &= e_3 \\
[d, e_2] &= e_2 & \qquad [e_1, e_3] &= 0 \\
[d, e_3] &= 2e_3 & \qquad [e_2, e_3] &= 0\,.
\end{aligned}$$

At that stage, the only important condition is $e_3 = e_3^\eta = [e_1, e_2]^\eta = [e_1^\eta, e_2^\eta]$, which directly translates to $1 = uv' - u'v$. In other words, *a linear mapping η of the form above is a morphism of Lie algebras if and only if $1 = uv' - u'v$.* Note that in that case $e_3^\eta, e_1^\eta, e_2^\eta$ automatically form a basis of $\mathfrak{g}'$. Moreover, $1 = uv' - u'v$ implies that $\neg(u = v = 0) \wedge \neg(u' = v' = 0) \wedge \neg(u = u' = 0)$.

Step 2. We can now try and find automorphisms of $\mathfrak{d}'$ in order to make the image of $\mathfrak{g}'$ look more pleasant. One thing we are allowed to do is conjugate e_1^η with matrices that will not destroy what we have already achieved; more explicitly, we may use elements of the centraliser

$$C_3 := C_\Delta(e_3) = \{A \in \Delta \mid a = c\} \quad .$$

There are two cases to be distinguished between :

Case $u \neq 0$. Using

$$A := \begin{pmatrix} u^{\frac{1}{3}} & 0 & 0 \\ & u^{-\frac{2}{3}} & -u^{-\frac{2}{3}} w \\ & & u^{\frac{1}{3}} \end{pmatrix} \in C_3$$

we get

$$\begin{pmatrix} 0 & u & w \\ & 0 & v \\ & & 0 \end{pmatrix}^A = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & uv \\ & & 0 \end{pmatrix} .$$

Case $u = 0$. Then $e_1^\eta \in \langle e_2, e_3 \rangle$, and any conjugate $e_1^{\eta \cdot A}$ with $A \in C_3$ is also contained in $\langle e_2, e_3 \rangle$. Therefore, the intended embellishment cannot be achieved by conjugation alone. But we may first apply some left twist by an automorphism of $\mathfrak{g}$ satisfying $e_1^\alpha = e_2$ and $e_2^\alpha = -e_1$, and fixing e_3 and d; cf. 4.4.2. (Note that every $\alpha \in \operatorname{Aut} \mathfrak{g}$ fixes d.) Then $e_1^{\alpha\eta} = u'e_1 + v'e_2 + w'e_3$. As $u = u' = 0$ would imply that η were not monic u' has to be non-zero, and we are transferred to the first case. Consequently, there is some $A \in C_3$ with the property that

$$e_1^{\alpha\eta \cdot A} = 1 \cdot e_1 + u'v' \cdot e_2 .$$

Summarising both cases and pointing out that in the end we will be merely interested in the image $\mathfrak{g}^\eta = \mathfrak{g}^{\alpha\eta}$, we may thus assume without loss of generality that $e_3^\eta = e_3$ and $u = 1$, i.e.,

$$e_1^\eta = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & v \\ & & 0 \end{pmatrix} \qquad \text{for some } v \in \mathbb{C}.$$

Step 3. In order to simplify e_2^η, we may use any element of

$$C_{13} := C_3 \cap C_\Delta\left(\begin{pmatrix} 0 & 1 & 0 \\ & 0 & v \\ & & 0 \end{pmatrix}\right) = \left\{ \begin{pmatrix} a & x & z \\ & a & xv \\ & & a \end{pmatrix} \mid a, x, z \in \mathbb{C} \, , \, a^3 = 1 \right\} \quad .$$

Conjugation by such an element

$$A = \begin{pmatrix} a & x & z \\ & a & xv \\ & & a \end{pmatrix}$$

maps $u'e_1 + v'e_2 + w'e_3$ to

$$\begin{pmatrix} 0 & u' & w' \\ & 0 & v' \\ & & 0 \end{pmatrix}^A = \begin{pmatrix} 0 & u' & w' - a^{-1}x \\ & 0 & v' \\ & & 0 \end{pmatrix} \quad .$$

For instance, conjugation by

$$A := \begin{pmatrix} 1 & w' & 0 \\ & 1 & w'v \\ & & 1 \end{pmatrix} \in C_{13}$$

yields

$$\begin{pmatrix} 0 & u' & w' \\ & 0 & v' \\ & & 0 \end{pmatrix}^A = \begin{pmatrix} 0 & u' & 0 \\ & 0 & v' \\ & & 0 \end{pmatrix} \quad .$$

This is as good as it gets.

3.4.10 Lemma. *Any monic $\eta|_{\mathfrak{g}'} : \mathfrak{g}' \to \mathfrak{d}'$ can be described by*

$$\begin{aligned} e_3 &\mapsto e_3 \\ e_1 &\mapsto e_1 + ve_2 \\ e_2 &\mapsto u'e_1 + v'e_2 \end{aligned}$$

where $u', v' \in \mathbb{C}$ and $1 = v' - u'v$.

Step 4. All that is left to do now is find an appropriate image of $d \in \mathfrak{g}$. Consider

$$d^\eta = \begin{pmatrix} r & u'' & w'' \\ & s & v'' \\ & & t \end{pmatrix}$$

for some $r, s, t, u'', v'', w'' \in \mathbb{C}$. The fact that η is a Lie algebra morphism implies $r = t+2$, $s = t + 1$ and $v'' = u'' = 0$. Moreover, $d^\eta \in \mathfrak{sl}_3\mathbb{C}$ requires $0 = \operatorname{tr} d^\eta = 3(t + 1)$, hence $t = -1$. Thus we get

$$d^\eta = \begin{pmatrix} 1 & 0 & w'' \\ & 0 & 0 \\ & & -1 \end{pmatrix}$$

for a start. The stabiliser we are interested in is

$$C_{123} := C_{13} \cap C_{\Delta}(\begin{pmatrix} 0 & u' & 0 \\ & 0 & v' \\ & & 0 \end{pmatrix}) = \left\{ \begin{pmatrix} a & 0 & z \\ & a & 0 \\ & & a \end{pmatrix} \mid a, z \in \mathbb{C}\,,\, a^3 = 1 \right\} \quad .$$

Indeed, conjugation by

$$A := \begin{pmatrix} 1 & 0 & -\frac{w''}{2} \\ & 1 & 0 \\ & & 1 \end{pmatrix} \in C_{123}$$

transforms d^η into $\mathrm{Diag}(1, 0, -1) = d - \mathbb{1}$. All in all we have at that stage proved

3.4.11 Proposition. *Let $\eta : \mathfrak{g} \to \mathfrak{d}$ a monomorphism of Lie algebras. Then, with $\alpha \in \operatorname{Aut} \mathfrak{g}$ as in Step 2, there are parameters $u, v \in \mathbb{C}$ such that up to conjugation in Δ*

$$\begin{array}{rrlcl}
\alpha\eta : & e_3 & \mapsto & \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix} & = & e_3 \\[2ex]
& e_1 & \mapsto & \begin{pmatrix} 0 & 1 & 0 \\ & 0 & v \\ & & 0 \end{pmatrix} & = & e_1 + v e_2 \\[2ex]
& e_2 & \mapsto & \begin{pmatrix} 0 & u & 0 \\ & 0 & 1 + uv \\ & & 0 \end{pmatrix} & = & u e_1 + (1 + uv) e_2 \\[2ex]
& d & \mapsto & \begin{pmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & -1 \end{pmatrix} & = & d - \mathbb{1} \quad .
\end{array}$$

The homomorphic image of $\mathfrak{g}$ then is

$$\mathfrak{g}^{\alpha\eta} = \mathfrak{g}^\eta = \ell\,\Gamma^{\ell\varphi} = \left\{ \begin{pmatrix} r & xu + y & z \\ & 0 & x(1 + uv) + yv \\ & & -r \end{pmatrix} \mid r, x, y, z \in \mathbb{R} \right\}.$$

□

Conversely, such a morphism η is monic for every choice of $u, v, \in \mathbb{C}$ indeed : The image $\mathfrak{g}^\eta$ is 4-dimensional if and only if

$$0 \neq \det \begin{pmatrix} u & 1 + uv \\ 1 & v \end{pmatrix} = uv - 1 - uv = -1,$$

that is, always.

3.4.12 In order to handle the stabilisers from 3.3.38, the explicit image Γ^φ is required. Direct computing leaves us with some utterly ugly matrices. More elegantly, the Lie algebra can be written as the (inner) semidirect sum

$$\mathfrak{g}^\eta = \mathbb{R}d^\eta \propto (\mathfrak{g}')^\eta = \mathbb{R}d^\eta \propto \mathrm{hei}_3\mathbb{R}^\eta = \mathbb{R}d^\eta \propto (\mathbb{R}e_1^\eta \propto (\mathbb{R}e_2^\eta \oplus \mathbb{R}e_3^\eta)).$$

By A.3.3, the corresponding Lie group $\mathfrak{g}^{\eta\cdot\exp} = \Gamma^\varphi$ is the (inner) semidirect product

$$\Gamma^\varphi = (e^{\mathbb{R}d})^\varphi \ltimes \Gamma'^\varphi = (e^{\mathbb{R}d})^\varphi \ltimes \mathrm{Hei}_3\mathbb{R}^\varphi = (e^{\mathbb{R}d})^\varphi \ltimes ((e^{\mathbb{R}e_1})^\varphi \ltimes (e^{\mathbb{R}e_2+\mathbb{R}e_3})^\varphi),$$

which computes as

$$\Gamma^\varphi = \left\{ \left(\begin{matrix} a & a(tu+r) & a(s+rt(1+uv)+\frac{t^2}{2}u(1+uv)+\frac{r^2}{2}v) \\ & 1 & t(1+uv)+rv \\ & & a^{-1} \end{matrix} \right) \middle| \begin{matrix} a,r,s,t \in \mathbb{R} \\ a>0 \end{matrix} \right\}.$$

Consequently,

$$\Gamma^\varepsilon = \Gamma^{\varphi\pi} = \{[A] \mid A \in \Gamma^\varphi\} \cong \Gamma^\varphi.$$

3.4.13 Remark. We have already seen in 3.3.30 that under the hypothesis of the existence of an embedding $\mathsf{H} : \mathcal{P} \to \mathcal{P}_2\mathbb{C}$, the group morphism ε constructed on our way is an embedding. Now that we actually know its image, the fact that all diagonal entries are positive real numbers enables us to add that φ is an embedding, too. Recall from 3.1.1 that there is a homeomorphic exponential function between

$$\mathfrak{r} := \boxed{\begin{matrix} \mathbb{R} & \mathbb{C} & \mathbb{C} \\ & \mathbb{R} & \mathbb{C} \\ & & \mathbb{R} \end{matrix}} \quad \text{and} \quad \mathsf{P} := \boxed{\begin{matrix} \mathbb{R}^+ & \mathbb{C} & \mathbb{C} \\ & \mathbb{R}^+ & \mathbb{C} \\ & & \mathbb{R}^+ \end{matrix}}.$$

a) $\mathfrak{g}^\eta \leq \mathfrak{r} \cap \mathfrak{sl}_3\mathbb{C}$ and $\Gamma^\varphi \leq \mathsf{P} \cap \mathrm{SL}_3\mathbb{C}$

b) $\exp|_{\mathfrak{r}\cap\mathfrak{sl}_3\mathbb{C}}^{\mathsf{P}\cap\mathrm{SL}_3\mathbb{C}} : \mathfrak{r} \cap \mathfrak{sl}_3\mathbb{C} \to \mathsf{P} \cap \mathrm{SL}_3\mathbb{C}$ is a homeomorphism.

c) $\exp|_{\mathfrak{g}^\eta}^{\Gamma^\varphi} : \mathfrak{g}^\eta \to \Gamma^\varphi$ is a homeomorphism.

d) $\eta|^{\mathfrak{g}^\eta} : \mathfrak{g} \to \mathfrak{g}^\eta$ is a homeomorphism.

e) $\varphi|^{\Gamma^\varphi} : \Gamma \to \Gamma^\varphi$ is a homeomorphism.

f) $\varphi : \Gamma \to \mathrm{SL}_3\mathbb{C}$ is an embedding.

Proof. *ad (a).* 3.4.12 *ad (b).* Let $A \in \mathsf{P}\cap\mathrm{SL}_3\mathbb{C}$. As by 3.1.1, the exponential map $\exp_\mathfrak{r}$ is surjective, there is an element $X \in \mathfrak{r}$ such that $X^{\exp} = A$. Now $1 = \det A = (\operatorname{tr} X)^{\exp}$, and as the diagonal elements of X are real numbers, this implies $\operatorname{tr} X = 0$. Hence, $X \in \mathfrak{r} \cap \mathfrak{sl}_3\mathbb{C}$. Therefore, $\exp|_{\mathfrak{r}\cap\mathfrak{sl}_3\mathbb{C}}^{\mathsf{P}\cap\mathrm{SL}_3\mathbb{C}} : \mathfrak{r} \cap \mathfrak{sl}_3\mathbb{C} \to \mathsf{P} \cap \mathrm{SL}_3\mathbb{C}$ is a co-restriction of a homeomorphism and as such is homeomorphic itself; cf. A.1.4. *ad (c).* Note that $\mathfrak{g}^{\eta\cdot\exp} = \mathfrak{g}^{\exp\cdot\varphi} = \Gamma^\varphi$, as $\exp_\mathfrak{g}$ is surjective. Hence again, the surjection $\exp|_{\mathfrak{g}^\eta}^{\Gamma^\varphi}$ is a co-restriction of a homeomorphism and thus is homeomorphic. *ad (d).* $\eta : \mathfrak{g} \to \mathfrak{sl}_3\mathbb{C}$ is an injective $\mathbb{R}$-linear map. Therefore, the co-restriction $\eta|^{\mathfrak{g}^\eta} : \mathfrak{g} \to \mathfrak{g}^\eta$ is a continuous $\mathbb{R}$-linear bijection; and then so is its inverse. This establishes that $\eta|^{\mathfrak{g}^\eta}$ is an open map. *ad (e).* This follows from (c) and (d), and from $\exp_\mathfrak{g}$ being a homeomorphism, too. $\square$

$$
\begin{tikzcd}
 & & & & \mathrm{PSL}_3\mathbb{C} \\
 & & & & \mathrm{SL}_3\mathbb{C} \arrow[u, twoheadrightarrow, "\pi"'] \\
\Gamma \arrow[uurrrr, "\varepsilon"] \arrow[urrrr, "\varphi"] \arrow[rr, "\varphi|^{\mathsf{P}\cap\mathrm{SL}_3\mathbb{C}}"'] \arrow[dr, "\varphi|^{\Gamma^\varphi}"'] & & \mathsf{P}\cap\mathrm{SL}_3\mathbb{C} \arrow[r, hook] \arrow[urr, hook] & \mathsf{P} & \\
 & \Gamma^\varphi \arrow[ur, hook] & & & \\
 & \mathfrak{g}^\eta \arrow[u, "\exp_{\mathfrak{r}}|^{:}_{:}"'] \arrow[dr, hook] & & & \\
\mathfrak{g} \arrow[uuu, "\exp_{\mathfrak{g}}"] \arrow[ur, "\eta|^{\mathfrak{g}^\eta}"] \arrow[rr, "\eta|^{\mathfrak{r}\cap\mathfrak{sl}_3\mathbb{C}}"] \arrow[drrrr, "\eta"'] & & \mathfrak{r}\cap\mathfrak{sl}_3\mathbb{C} \arrow[uuu, "\exp_{\mathfrak{r}}|^{:}_{:}"'] \arrow[r, hook] \arrow[drr, hook] & \mathfrak{r} \arrow[uuu, "\exp_{\mathfrak{r}}"'] & \\
 & & & & \mathfrak{sl}_3\mathbb{C} \arrow[uuuuu, "\exp_{\mathfrak{sl}_3\mathbb{C}}"']
\end{tikzcd}
$$

After all these modifications our original recipe 3.3.38 can be rendered in a more explicit way :

3.4.14 Recipe II. Existence of a morphism $\mathsf{H} : \mathcal{P} \to \mathcal{P}_2\mathbb{C}$ of stable planes implies the existence of some continuous group monomorphism $\varphi : \Gamma \to \Delta$ as in 3.4.12, given by complex parameters u and v, and the existence of some point $p \in \mathrm{P}_2\mathbb{C}$ such that $\Gamma_p^\varphi = \mathbb{1}$ and such that the line stabiliser of every line $L \in (\mathcal{L}_2\mathbb{C})_L$ is 2-dimensional.

Non-existence of such an embedding H will hence be proved by proving that for any choice of u and v and every $p \in \mathrm{P}_2\mathbb{C}$ satisfying $\Gamma_p^\varphi = \mathbb{1}$ there is at least one line $L \in (\mathcal{L}_2\mathbb{C})_p$ whose line stabiliser is of dimension $\dim \Gamma_L^\varphi \neq 2$.

Proof. Let us for a moment distinguish between two different group monics φ and ψ, where $\varphi : \Gamma \to \mathrm{SL}_3\mathbb{C}$ purely denotes the (unique) lifting of ε, satisfying $\varphi\pi = \varepsilon$. On the other hand, $\psi : \Gamma \to \Delta$ denotes the "embellished" version of φ from 3.4.12. Note that the search for a favourable appearance of φ has been achieved by conjugation by some matrix $A \in \mathrm{SL}_3\mathbb{C}$; hence $\psi = \varphi\kappa_A : \Gamma \to \Delta : M \mapsto A^{-1}M^{\varphi}A$. It is essential that $\Gamma^{\psi} \cap \ker\pi = 1$ and therefore $\Gamma^{\varepsilon} \cong \Gamma^{\varepsilon\cdot\kappa_{[A]}} = \Gamma^{\varphi\cdot\kappa_A\cdot\pi} = \Gamma^{\psi\pi} \cong \Gamma^{\psi}$. Tracing our way backwards, we see that the following implications hold :

$$
\begin{aligned}
& \forall\psi : \Gamma \to \Delta \in \text{monic}\,\mathsf{TopGp} \text{ of "beautiful form" as in 3.4.12} \quad \forall q \in \mathrm{P}_2\mathbb{C}. \\
& \Gamma^{\psi}_{q} \neq 1 \quad \vee \quad (\exists K \in (\mathcal{L}_2\mathbb{C})_q.\ \dim \Gamma^{\psi}_{K} \neq 2) \\
\Longrightarrow\ & \forall\varphi : \Gamma \to \mathrm{SL}_3\mathbb{C} \in \text{monic}\,\mathsf{TopGp} \quad \forall p \in \mathrm{P}_2\mathbb{C}. \\
& \Gamma^{\varphi}_{p} \neq 1 \quad \vee \quad (\exists L \in (\mathcal{L}_2\mathbb{C})_p.\ \dim \Gamma^{\varphi}_{L} \neq 2) \\
\Longrightarrow\ & \forall\varepsilon : \Gamma \to \mathrm{PGL}_3\mathbb{C} \in \text{monic}\,\mathsf{TopGp} \quad \forall p \in \mathrm{P}_2\mathbb{C}. \\
& \Gamma^{\varepsilon}_{p} \neq 1 \quad \vee \quad (\exists L \in (\mathcal{L}_2\mathbb{C})_p.\ \dim \Gamma^{\varepsilon}_{L} \neq 2) \\
\Longrightarrow\ & \nexists \quad \mathsf{H} : \mathcal{P} \to \mathcal{P}_2\mathbb{C} \in \text{morph}\,\mathsf{StP}
\end{aligned}
$$

As a matter of fact, the first implication is simple linear algebra, the second one is due to $\Gamma^{\varepsilon} \cong \Gamma^{\psi}$ and hence equivalence of their actions on the projective plane, and the third step just restates recipe 3.3.38, backed up by the information that φ is an embedding, too.

By abuse of notation, throughout the remains of the chapter these distinctions will be ignored; ε and $\psi\pi$ will be identified. The group acting will be $\Gamma^{\varepsilon} \cong \Gamma^{\psi\pi}$. $\square$

3.5. The point orbits

Due to the reflections in 3.4.14 we will have to single out every point $p \in \mathrm{P}_2\mathbb{C}$ having a trivial stabiliser and then compute the line stabilisers for every single line through p. Being lazy and mortal, we will try and pool points with similar behaviour into point orbits with respect to the action of $\Gamma^{\varepsilon} \leq \mathrm{PGL}_3\mathbb{C}$: For any other point $q = p^{\alpha}$ with $\alpha \in \Gamma^{\varepsilon}$ contained in the same orbit the stabiliser $\Gamma^{\varepsilon}_{q} = \Gamma^{\varepsilon}_{p^{\alpha}} = (\Gamma^{\varepsilon}_{p})^{\alpha}$ is trivial if and only if Γ^{ε}_{p} is trivial. Thus, in the end we will be down to the task of scrutinising the stabilisers of only one representative per orbit.

The present chapter is therefore dedicated to the search of all those point orbits — the result of which will largely depend on the choice of u and v. We will first of all have a look at some particularly friendly orbits and gather a sufficient number of them. Then the point of view will be switched and a documentation of all point orbits for every "characteristic" constellation of the complex parameters u and v will be given.

Throughout the whole section consider an element

$$
A = \begin{bmatrix} a & a(tu+r) & a(s+tr(1+uv)+\frac{t^2}{2}u(1+uv)+\frac{r^2}{2}v \\ & 1 & t(1+uv)+rv \\ & & a^{-1} \end{bmatrix} \in \Gamma^{\varepsilon} \quad ,
$$

where $a, r, s, t \in \mathbb{R}$ and $a > 0$. Remember that $u, v \in \mathbb{C}$ are the parameters determining the group embedding $\varepsilon := \varepsilon_{(u,v)}$. (As a matter of fact, A will appear within the proofs only.)

3.5.1 Lemma. ***The dark blue orbit.*** $\mathbb{C}(0,0,1)^{\Gamma^\varepsilon} = \{\mathbb{C}(0,0,1)\}$.

3.5.2 Lemma. ***The yellow orbits.*** $\mathbb{C}(0,1,0)^{\Gamma^\varepsilon} =$

$$\begin{cases} \{\mathbb{C}(0,1,y) \mid y \in \mathbb{R}\} & \text{for} \quad u \notin \mathbb{R}, v = 0 \\ & \phantom{\text{for}} \quad u, v \in \mathbb{R} \\ \{\mathbb{C}(0,1,w) \mid w \in \mathbb{C}\} & \text{for} \quad u \notin \mathbb{R}, v \in \mathbb{R}^\times \\ & \phantom{\text{for}} \quad u \in \mathbb{R}, v \notin \mathbb{R} \\ & \phantom{\text{for}} \quad u, v \notin \mathbb{R}, 1 + uv \notin \mathbb{R}v \\ \{\mathbb{C}(0,1,rv) \mid r \in \mathbb{R}\} & \text{for} \quad u, v \notin \mathbb{R}, 1 + uv \in \mathbb{R}v \end{cases}$$

Proof. We get

$$\mathbb{C}(0,1,0) \cdot A = \mathbb{C}(0,1,t(1+uv) + rv) \quad ,$$

which immediately explains all cases except for $u, v \notin \mathbb{R}$. In the latter case it has to be taken into account that v and $1 + uv$, interpreted as $\mathbb{R}$-vectors, may or may not be linearly dependent. □

3.5.3 Lemma. ***The light blue orbits.*** $\mathbb{C}(0,1,\pm i)^{\Gamma^\varepsilon} =$

$$\begin{cases} \{\mathbb{C}(0,1,w) \mid w \in \mathbb{C}\} & \text{for} \quad u, v \in \mathbb{R} \\ & \phantom{\text{for}} \quad u \notin \mathbb{R}, v = 0 \\ & \phantom{\text{for}} \quad u, v \notin \mathbb{R}, 1 + uv \notin \mathbb{R}v \\ \{\mathbb{C}(0,1,w) \mid \operatorname{Im} w \gtrless 0\} & \text{for} \quad u \notin \mathbb{R}, v \in \mathbb{R}^\times \\ & \phantom{\text{for}} \quad u \in \mathbb{R}, v \notin \mathbb{R} \\ \{\mathbb{C}(0,1,rv \pm ai) \mid r \in \mathbb{R}, a > 0\} & \text{for} \quad u, v \notin \mathbb{R}, 1 + uv \in \mathbb{R}v \end{cases}$$

Note that for reasons of convenience we are handling **two** orbits at the same time, one for $+i$ and one for $-i$.

Proof. We compute

$$\mathbb{C}(0,1,\pm i) = \mathbb{C}(0,1,t(1+uv) + rv \pm ai) \quad ,$$

which automatically yields the statements on the (all) real and mixed cases. For $u, v \notin \mathbb{R}$ we will again have to pay attention to the linear independence of v and $1 + uv$ as $\mathbb{R}$-vectors. □

3.5.4 Lemma. ***The red orbits.*** $\mathbb{C}(1,0,0)^{\Gamma^\varepsilon} =$

$$\begin{cases} \{\mathbb{C}(1,x,y) \mid x, y \in \mathbb{R}\} & \text{for} \quad u, v \in \mathbb{R} \\ \{\mathbb{C}(1,x,w) \mid x \in \mathbb{R}, \operatorname{Im} w = \frac{x^2}{2} \operatorname{Im} v\} & \text{for} \quad u \in \mathbb{R}, v \notin \mathbb{R} \\ \{\mathbb{C}(1,z,w) \mid z \in \mathbb{C}, \operatorname{Im} w = \operatorname{term}(z)\} & \text{for} \quad u \notin \mathbb{R} \end{cases}$$

where $\operatorname{term}(z) := \operatorname{Im}(tr(1+uv) + \frac{t^2}{2}u(1+uv) + \frac{r^2}{2}v)$ *with* $t := \operatorname{Im} z \cdot (\operatorname{Im} u)^{-1}$ *and* $r := \operatorname{Re} z - \operatorname{Im} z \cdot \operatorname{Re} u \cdot (\operatorname{Im} u)^{-1}$.

Proof.

$$\mathbb{C}(1,0,0)\cdot A = \mathbb{C}(1, tu+r, s+tr(1+uv)+\frac{t^2}{2}u(1+uv)+\frac{r^2}{2}v) \quad .$$

The real case can be verified stante pedes. As to the case $u, v \notin \mathbb{R}$, for any $z \in \mathbb{C}$ there are unique $t, r \in \mathbb{R}$ such that $z = tu + r$; namely the ones given above. Now for an arbitrary $w \in \mathbb{C}$ we will try and find $a > 0$ as well as $s \in \mathbb{R}$ such that the third coordinate above equals w. Thanks to s we may happily ignore the real part and concentrate on the imaginary part. It is forced to be $\operatorname{Im} w = \operatorname{term}(z)$.

For a real parameter u the situation changes slightly in as far as the second coordinate $x = tu + r$ can be chosen freely from $\mathbb{R}$, as well as the real part of the third one. Its imaginary part turns out to be $\operatorname{Im} w = \frac{1}{2}x^2 \operatorname{Im} z$. □

3.5.5 Lemma. ***The green orbits.*** $\mathbb{C}(1,0,\pm i)^{\Gamma^\varepsilon} =$

$$\begin{cases} \{\mathbb{C}(1,x,y) \mid x,y \in \mathbb{R}\} & \text{for} \quad u,v \in \mathbb{R} \\ \{\mathbb{C}(1,x,w) \mid x \in \mathbb{R}, \operatorname{Im} w \gtrless \frac{1}{2}x^2 \operatorname{Im} v\} & \text{for} \quad u \in \mathbb{R}, v \notin \mathbb{R} \\ \{\mathbb{C}(1,z,w) \mid z \in \mathbb{C}, \operatorname{Im} w \gtrless \operatorname{term}(z)\} & \text{for} \quad u \notin \mathbb{R} \end{cases}$$

where $\operatorname{term}(z)$ *is defined as in 3.5.4.*

Proof.

$$\mathbb{C}(1,0,\pm i)\cdot A = \mathbb{C}(1, tu+r, s+tr(1+uv)+\frac{t^2}{2}u(1+uv)+\frac{r^2}{2}v \pm ai) \quad .$$

The important point is that a third component $w \in \mathbb{C}$ must satisfy $\operatorname{Im} w = \operatorname{Im}(tr(1+uv)+\frac{t^2}{2}u(1+uv)+\frac{r^2}{2}v) \pm a$, where $a > 0$. So, still considering the third component, these two orbits fill up the space the previous red ones have left out in the non-real cases. □

3.5.6 Lemma. ***The lilac orbits.*** *These two orbits are particularly nasty, such that the cases actually required are given only :*

$$\mathbb{C}(1,\pm i,0)^{\Gamma^\varepsilon} = \{\mathbb{C}(1,z,w) \mid \operatorname{Im} z \gtrless 0, w \in \mathbb{C}\} \text{ for } u,v \in \mathbb{R} \text{ or } u \in \mathbb{R}, v \notin \mathbb{R} \quad .$$

Proof.

$$\mathbb{C}(1,\pm i,0)\cdot A = \mathbb{C}(1, tu+r\pm ai, s+tr(1+uv)+\frac{t^2}{2}u(1+uv)+\frac{r^2}{2}v \pm i\cdot a(t(1+uv)+rv)).$$

Let $z \in \mathbb{C}$ with $\operatorname{Im} z \gtrless 0$ and $w \in \mathbb{C}$. In the real case the second coordinate z determines $a > 0$ uniquely, and for any t there is a unique r such that $z = tu + r \pm ai$. Due to s the real part of the third component can be chosen arbitrarily. The imaginary part is $\operatorname{Im} w = \pm a(t(1+uv)+rv)$. Thus we are looking for a solution (t, r) of

$$(\operatorname{Re} z, \operatorname{Im} w) = (t, r)\begin{pmatrix} u & a(1+uv) \\ 1 & av \end{pmatrix},$$

and because the according determinant equals $auv - a(1+uv) = -a < 0$ we are granted one, a unique one even.

Now consider the case $u \in \mathbb{R}, v \notin \mathbb{R}$: Nothing changes for the second coordinate $z = tu + r \pm ai$. The imaginary part of the third one computes as

$$\begin{aligned} \operatorname{Im} w &= \frac{1}{2} \operatorname{Im} v(tu+r)^2 \pm a(t + \operatorname{Re} v(tu+r)) \\ &= \frac{1}{2} \operatorname{Im} v \operatorname{Re} z \pm a(t + \operatorname{Re} v \operatorname{Re} z) \quad . \end{aligned}$$

Thus t is determined uniquely by $\operatorname{Im} w$; then $\operatorname{Re} z$ determines r, and finally $\operatorname{Re} w$ decides the choice of s. □

It may be enlightening to have a look at the actual orbit partition of $\mathrm{P}_2\mathbb{C}$, dependent on the constellations of parameters u and v.

3.5.7 Corollary. *There are five typical cases to be distinguished between. The following renders the complete orbit partitions for each of them.*

Case $u, v \in \mathbb{R}$***.***

$$\begin{aligned} \mathbb{C}(0,0,1)^{\Gamma^\varepsilon} &= \{\mathbb{C}(0,0,1)\} \\ \mathbb{C}(0,1,0)^{\Gamma^\varepsilon} &= \{\mathbb{C}(0,1,y) \mid y \in \mathbb{R}\} \\ \mathbb{C}(0,1,\pm i)^{\Gamma^\varepsilon} &= \{\mathbb{C}(0,1,w) \mid \operatorname{Im} w \gtrless 0\} \\ \mathbb{C}(1,0,0)^{\Gamma^\varepsilon} &= \{\mathbb{C}(1,x,y) \mid x, y \in \mathbb{R}\} \\ \mathbb{C}(1,0,\pm i)^{\Gamma^\varepsilon} &= \{\mathbb{C}(1,x,w) \mid x \in \mathbb{R}, \operatorname{Im} w \gtrless 0\} \\ \mathbb{C}(1,\pm i,0)^{\Gamma^\varepsilon} &= \{\mathbb{C}(1,z,w) \mid \operatorname{Im} z \gtrless 0, w \in \mathbb{C}\} \end{aligned}$$

Case $u \in \mathbb{R}$ ***and*** $v \notin \mathbb{R}$***.***

$$\begin{aligned} \mathbb{C}(0,0,1)^{\Gamma^\varepsilon} &= \{\mathbb{C}(0,0,1)\} \\ \mathbb{C}(0,1,0)^{\Gamma^\varepsilon} &= \{\mathbb{C}(0,1,w) \mid w \in \mathbb{C}\} \\ \mathbb{C}(1,0,0)^{\Gamma^\varepsilon} &= \{\mathbb{C}(1,x,w) \mid x \in \mathbb{R}, \operatorname{Im} w = \tfrac{x^2}{2} \operatorname{Im} v\} \\ \mathbb{C}(1,0,\pm i)^{\Gamma^\varepsilon} &= \{\mathbb{C}(1,x,w) \mid x \in \mathbb{R}, \operatorname{Im} w \gtrless \tfrac{x^2}{2} \operatorname{Im} v\} \\ \mathbb{C}(1,\pm i,0)^{\Gamma^\varepsilon} &= \{\mathbb{C}(1,z,w) \mid \operatorname{Im} z \gtrless 0, w \in \mathbb{C}\} \end{aligned}$$

Case $u \notin \mathbb{R}$ ***and*** $v = 0$***.***

$$\begin{aligned} \mathbb{C}(0,0,1)^{\Gamma^\varepsilon} &= \{\mathbb{C}(0,0,1)\} \\ \mathbb{C}(0,1,0)^{\Gamma^\varepsilon} &= \{\mathbb{C}(0,1,y) \mid y \in \mathbb{R}\} \\ \mathbb{C}(0,1,\pm i)^{\Gamma^\varepsilon} &= \{\mathbb{C}(0,1,w) \mid \operatorname{Im} w \gtrless 0\} \\ \mathbb{C}(1,0,0)^{\Gamma^\varepsilon} &= \{\mathbb{C}(1,z,w) \mid z \in \mathbb{C}, \operatorname{Im} w = \frac{\operatorname{Im} z^2}{2 \operatorname{Im} u}\} \\ \mathbb{C}(1,0,\pm i)^{\Gamma^\varepsilon} &= \{\mathbb{C}(1,z,w) \mid z \in \mathbb{C}, \operatorname{Im} w \gtrless \frac{\operatorname{Im} z^2}{2 \operatorname{Im} u}\} \end{aligned}$$

Case $u \notin \mathbb{R}$ ***and*** $v \in \mathbb{R}^\times$***.***

$$\begin{array}{lcl}
\mathbb{C}(0,0,1)^{\Gamma^\varepsilon} & = & \{\mathbb{C}(0,0,1)\} \\
\mathbb{C}(0,1,0)^{\Gamma^\varepsilon} & = & \{\mathbb{C}(0,1,w) \mid w \in \mathbb{C}\} \\
\mathbb{C}(1,0,0)^{\Gamma^\varepsilon} & = & \{\mathbb{C}(1,z,w) \mid z \in \mathbb{C},\ \operatorname{Im} w = \operatorname{term}(z)\} \\
\mathbb{C}(1,0,\pm i)^{\Gamma^\varepsilon} & = & \{\mathbb{C}(1,z,w) \mid z \in \mathbb{C},\ \operatorname{Im} w \gtrless \operatorname{term}(z)\}
\end{array}$$

Case $u, v \notin \mathbb{R}$***.***

$$\begin{array}{lcl}
\mathbb{C}(0,0,1)^{\Gamma^\varepsilon} & = & \{\mathbb{C}(0,0,1)\} \\
\mathbb{C}(0,1,0)^{\Gamma^\varepsilon} & = & \begin{cases} \{\mathbb{C}(0,1,w) \mid w \in \mathbb{C}\} & \textit{for } (1+uv) \notin \mathbb{R}v \\ \{\mathbb{C}(0,1,rv) \mid r \in \mathbb{R}\} & \textit{for } (1+uv) \in \mathbb{R}v \end{cases} \\
\mathbb{C}(0,1,\pm i)^{\Gamma^\varepsilon} & = & \begin{cases} \mathbb{C}(0,1,0)^{\Gamma^\varepsilon} & \textit{for } (1+uv) \notin \mathbb{R}v \\ \{\mathbb{C}(0,1,rv \pm ai) \mid a, r \in \mathbb{R},\ a > 0\} & \textit{for } (1+uv) \in \mathbb{R}v \end{cases} \\
\mathbb{C}(1,0,0)^{\Gamma^\varepsilon} & = & \{\mathbb{C}(1,z,w) \mid z \in \mathbb{C},\ \operatorname{Im} w = \operatorname{term}(z)\} \\
\mathbb{C}(1,0,\pm i)^{\Gamma^\varepsilon} & = & \{\mathbb{C}(1,z,w) \mid z \in \mathbb{C},\ \operatorname{Im} w \gtrless \operatorname{term}(z)\}
\end{array}$$

where $\operatorname{term}(z) := \operatorname{Im}\left(tr(1+uv) + \frac{t^2}{2}u(1+uv) + \frac{r^2}{2}v\right)$ *with* $t, r \in \mathbb{R}$ *uniquely such that* $z = tu + r$.

3.5.8 For a better idea of what is left to do the above will be summarized in a tabular. An orbit is marked

$\times$	iff	it is necessary and not covered by any other orbit marked $\times$
colour	iff	it is already covered by the orbit *colour* (which then had better be marked $\times$).

Thus, per row the orbits marked $\times$ make up a partition of $\mathrm{P}_2\mathbb{C}$.

	$\mathbb{C}(0,0,1)$ blue	$\mathbb{C}(0,1,0)$ yellow	$\mathbb{C}(0,1,\pm i)$ lightblue	$\mathbb{C}(1,0,0)$ red	$\mathbb{C}(1,0,\pm i)$ green	$\mathbb{C}(1,\pm i,0)$ lilac
$u, v \in \mathbb{R}$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
$u \in \mathbb{R}, v \notin \mathbb{R}$	$\times$	$\times$	yellow	$\times$	$\times$	$\times$
$u \notin \mathbb{R}, v = 0$	$\times$	$\times$	$\times$	$\times$	$\times$	red $\cup$ green
$u \notin \mathbb{R}, v \in \mathbb{R}^\times$	$\times$	$\times$	yellow	$\times$	$\times$	red $\cup$ green
$u, v \notin \mathbb{R}$	$\times$	$\times$	$\times$ yellow *	$\times$	$\times$	red $\cup$ green

* $\times$ for $1 + uv \notin \mathbb{R}v$ and *yellow* for $1 + uv \in \mathbb{R}v$

Having worked our way through all this, are we any better off than before ? Talking of point stabilisers, we do no longer have to compute one for each point in $\mathrm{P}_2\mathbb{C}$ but only one per "$\times$" in table 3.5.8 above. Which certainly is a slightly improved situation — this at least is a finite problem.

3.6. The point stabilisers

Remember that our task is to single out every point $p \in \mathrm{P}_2\mathbb{C}$ with trivial stabiliser Γ_p^ε. This still — despite the radical reduction from section 3.5 — looks like tedious work. We may again be tempted to search for an equivalent within the according Lie algebras. Indeed, there is humble aid : for a point $p = \mathbb{C}w \in \mathrm{P}_2\mathbb{C}$, the Lie subalgebra

$$S_w := \{M \in \mathfrak{g}^\eta \mid wM \in \mathbb{C}w\} \leq \mathfrak{g}^\eta$$

is apt to nominate candidates.

3.6.1 Lemma. *For any $w \in \mathbb{C}^3$ triviality of the point stabiliser $\Gamma^\varepsilon_{\mathbb{C}w}$ implies $S_w = \mathbf{0}$.*

Proof.

$$\begin{tikzcd} & \mathrm{PGL}_3\mathbb{C} \\ \Gamma \arrow[ur, hook, "\varepsilon"] \arrow[r, hook, "\varphi"] & \Delta \arrow[u, "\pi"'] \\ \mathfrak{g} \arrow[u, "\exp_{\mathfrak{g}}"] \arrow[r, hook, "\eta"'] & \ell\Delta \arrow[u, "\exp"'] \end{tikzcd}$$

Let $p = \mathbb{C}w \in \mathrm{P}_2\mathbb{C}$, and let $M \in \mathfrak{g}^\eta$ be an element of S_w, satisfying $wM = \lambda w$ for some $\lambda \in \mathbb{C}$. Then $w \cdot M^{\exp} = w \cdot \sum_{\nu=0}^{\infty} \frac{1}{\nu!} M^\nu = \sum_{\nu=0}^{\infty} \frac{1}{\nu!}\lambda^\nu w = e^\lambda \cdot w$, i.e., $p \cdot M^{\exp} = p$. Therefore, $S_w^{\exp} \subseteq \Gamma_p^\varphi$.
Now assume moreover that $\Gamma_p^\varepsilon = \mathtt{1}$. Then $M^{\exp\pi} \in \Gamma_p^{\varphi\pi} = \Gamma_p^\varepsilon = \mathtt{1}$, hence $M^{\exp\pi} = \mathbb{1}$. Injectivity of ε and $\exp_{\mathfrak{g}}$ implies $M = \mathbf{0}$.

$\square$

Presumably, the S_w are easier to produce than the group stabilisers. In the sequel these sets will be listed in order to come up with candidates for trivial point stabilisers. Throughout the proofs, let

$$M = \begin{pmatrix} r & xu+y & z \\ & 0 & x(1+uv)+yu \\ & & -r \end{pmatrix} \quad ,$$

with $r, x, y, z \in \mathbb{R}$, be an element of the Lie algebra $\mathfrak{g}^\eta$.

3.6.2 Lemma. $S_{(0,0,1)} = \mathfrak{g}^\eta$.

Some of the equations turn out to be rather awkward. This is why we occasionally just state whether or not $S_w = \mathbf{0}$. This suffices for all subsequent applications, by 3.6.1 and part (2) of 3.3.37.

3.6.3 Lemma. $S_{(0,1,0)} \neq \mathbf{0}$.

Proof. $(0,1,0) \cdot M = (0,0,x(1+uv)+yu)$. Note that r and z are not involved at all; as a consequence $\dim S_{(0,1,0)} \geq 2$. □

3.6.4 Lemma. $S_{(0,1,\pm i)} \neq \mathbf{0}$.

Proof. Application of the very same argument to z in $(0,1,\pm i) \cdot M = (0,1,0) \cdot M \mp (0,0,ri)$.

3.6.5 Lemma.

$$S_{(1,0,0)} = \left\{ \left(\begin{matrix} r & 0 & 0 \\ & 0 & x \\ & & -r \end{matrix} \right) \middle| r, x \in \mathbb{R} \right\} \text{ for } u \in \mathbb{R}$$

and

$$S_{(1,0,0)} = \left\{ \left(\begin{matrix} r & 0 & 0 \\ & 0 & 0 \\ & & -r \end{matrix} \right) \middle| r \in \mathbb{R} \right\} \text{ for } u \notin \mathbb{R}$$

Proof. $(1,0,0) \cdot M = (r, xu+y, z) \in \mathbb{C}(1,0,0)$ if and only if $z = 0$ and $y = -xu$. If $u \notin \mathbb{R}$ this forces $0 = y = x \operatorname{Re} u$ and $0 = x \operatorname{Im} u$, in other words $x = y = 0$. □

3.6.6 Lemma.

$$S_{(1,0,\pm i)} = \left\{ \left(\begin{matrix} 0 & 0 & 0 \\ & 0 & x \\ & & 0 \end{matrix} \right) \middle| x \in \mathbb{R} \right\} \text{ for } u \in \mathbb{R}$$

and

$$S_{(1,0,\pm i)} = \mathbf{0} \text{ for } u \notin \mathbb{R} \quad .$$

Proof. $\lambda(1,0,\pm i) = (r, xu+y, z \mp ri)$ if and only if $\lambda = r$ and $y = -xu \in \mathbb{R}$ as well as $0 = z \pm 2ri$. The latter implies $0 = z = r$. The penultimate equation can be solved non-trivially for $u \in \mathbb{R}$; otherwise it forces x and y to be 0. □

3.6.7 Lemma. $S_{(1,\pm i,0)} = \mathbf{0}$ *unless* $1 = \operatorname{Im} u \operatorname{Im} v$ *and* $0 = \operatorname{Re} v \operatorname{Im} u$.

Proof. We figure out that $\lambda(1,\pm i,0) = (r, xu+y, z \pm i(x(1+uv)+yu))$ if and only if

- $\lambda = r$
- $y = x \operatorname{Re} u$
- $r = \pm x \operatorname{Im} u$
- $0 = z \mp x \operatorname{Im} u \operatorname{Re} v$
- $0 = x(1 - \operatorname{Im} u \operatorname{Im} v) \quad .$

If $0 \neq 1 - \operatorname{Im} u \operatorname{Im} v$ — thus in particular if u or v is real — then we need $x = 0$, which again implies $r = y = z = 0$.

Otherwise we can freely choose x (which then uniquely determines r, y and z). Yet entry 23 equals $x(1 - \operatorname{Im} u \operatorname{Im} v + i \operatorname{Re} v \operatorname{Im} u) = xi \operatorname{Re} v \operatorname{Im} u$, which equals 0 if $\operatorname{Re} v \operatorname{Im} u = 0$; but then entry 13 equals $z = \pm x \operatorname{Im} u \operatorname{Re} v = 0$. So for $0 = 1 - \operatorname{Im} u \operatorname{Im} v$ and $\operatorname{Re} v \operatorname{Im} u = 0$ we end up with $S_w = \mathbf{0}$, nevertheless. Finally, for $0 = 1 - \operatorname{Im} u \operatorname{Im} v$ but $\operatorname{Re} v \operatorname{Im} u \neq 0$ the set S_w is not trivial (but 1-dimensional). □

Resuming the above five lemmas in combination with table 3.5.8, the only candidates left are $\mathbb{C}(1, \pm i, 0)$ for $u \in \mathbb{R}$ and $\mathbb{C}(1, 0, \pm i)$ for $u \notin \mathbb{R}$. In both cases the point stabilisers turn out to be trivial, indeed :

3.6.8 Lemma. $\Gamma^{\varepsilon}_{\mathbb{C}(1,\pm i,0)} = \mathbb{1}$ *for* $u \in \mathbb{R}$.

Proof. Some matrix $A \in \Gamma^{\varepsilon}_{\mathbb{C}(1,\pm i,0)}$ had better satisfy (exchanging a for a^{-1})

$$\mathbb{C}(1, \pm i, 0) = \mathbb{C}\left(1, tu + r \pm ai, s + tr(1 + uv) + \frac{t^2}{2} u(1 + uv) + \frac{r^2}{2} v \pm ai(t(1 + uv) + rv)\right) \quad .$$

Equality of the second component can be guaranteed for $r = -tu$ and $a = 1$ only. Using these, the third component is $s - \frac{t^2}{2} u(1 + uv) + \frac{t^2}{2} u^2 v \pm i(t(1 + uv) - tuv) = s - \frac{t^2}{2} u \pm it$, which equals 0 if an only if $t = 0$ and $s = \frac{t^2}{2} u = 0$. But then also $r = 0$; hence $A = \mathbb{1}$. □

3.6.9 Lemma. $\Gamma^{\varepsilon}_{\mathbb{C}(1,0,\pm i)} = \mathbb{1}$ *for* $u \notin \mathbb{R}$.

Proof. We must guarantee

$$\mathbb{C}(1, 0, \pm i) = \mathbb{C}(1, 0, \pm i) \cdot A = \mathbb{C}\left(1, tu + r, s + tr(1 + uv) + \frac{t^2}{2} u(1 + uv) + \frac{r^2}{2} v \pm ai\right) \quad .$$

Equality of the second component holds if and only if $0 = r + t \operatorname{Re} u$ and $0 = t \operatorname{Im} u$, that is $t = r = 0$. The third component requires a solution of $\pm i = s \pm a^{-1} i$, which is $s = 0$ and $a = 1$. □

What does this mean for the original question of embeddability of $\mathcal{P}$ into $\mathcal{P}_2\mathbb{C}$?

3.6.10 Corollary. *An embedding of* $(\Gamma; \{\mathbb{1}\}, \mathcal{S}_k^{\exp})$ *into* $(\Gamma^{\varepsilon}; \{\Gamma^{\varepsilon}_p\}, \{\Gamma^{\varepsilon}_L \mid L \in (\mathcal{L}_2\mathbb{C})_p\})$ *can only be achieved for a point p contained in one of the lilac orbits* $\mathbb{C}(1, \pm i, 0)^{\Gamma^{\varepsilon}}$ *with* $u \in \mathbb{R}$ *and the green orbits* $\mathbb{C}(1, 0, \pm i)^{\Gamma^{\varepsilon}}$ *with* $u \notin \mathbb{R}$.

3.7. The line stabilisers

So far, candidates for the group embedding $\varepsilon : \Gamma \to \mathrm{PGL}_3\mathbb{C}$ and for the points $p \in \mathrm{P}_2\mathbb{C}$ with trivial stabiliser have been determined. The time has come to talk about lines.

Following the recipe in 3.4.14, we will successively leaf through the five "typical" cases having crystallised in 3.5.8 and strive to find at least one line $L \in (\mathcal{L}_2\mathbb{C})_p$ with $\dim \Gamma^\varepsilon_L \neq 2$. By 3.6.10, each of the five cases determines candidates for the point representative p : $p_0 = \mathbb{C}(1, \pm i, 0)$ for $u \in \mathbb{R}$ and $q_0 = \mathbb{C}(1, 0, \pm i)$ for $u \notin \mathbb{R}$. In section 3.2 the fact that the real points form a Baer subplane in $\mathcal{P}_2\mathbb{C}$ was used to gain an explicit description of the line pencils through p_0 and q_0 [3.2.5]. Their homogeneous coordinates will be used here.

3.7.1 Lemma. *Let $A \in \mathrm{PGL}_3\mathbb{C}$ and $L = \operatorname{Ker} \mathbf{a}$ for $\mathbf{a} \in \mathbb{C}^3 \setminus \mathbf{0}$ be a projective line. Then*

a) $L^A = \operatorname{Ker}(A^{-1}\mathbf{a})$

b) *A stabilizes L if and only if $A\mathbf{a} \in \mathbb{C}\mathbf{a}$*

Proof. *ad (a).* Remember that ordinary vectors are row vectors, but $\mathbf{a}$ is a column vector. We get $L^A = \{\mathbb{C}xA \mid 0 = x \cdot \mathbf{a} = xA \cdot A^{-1}\mathbf{a}\} = \{\mathbb{C}y \mid 0 = y \cdot A^{-1}\mathbf{a}\} = \operatorname{Ker}(A^{-1}\mathbf{a})$.
ad (b). $L = L^A$ iff $\operatorname{Ker} \mathbf{a} = \operatorname{Ker}(A^{-1}\mathbf{a})$ iff $\lambda\mathbf{a} = A\mathbf{a}$ for some $\lambda \in \mathbb{C}^\times$ iff $A\mathbf{a} \in \mathbb{C}^\times\mathbf{a}$. $\square$

As usual let

$$A = \begin{bmatrix} a & a(tu+r) & a(s + tr(1+uv) + \frac{t^2}{2}u(1+uv) + \frac{r^2}{2}v \\ & 1 & t(1+uv) + rv \\ & & a^{-1} \end{bmatrix} \in \Gamma^\varepsilon \quad ,$$

with $a, r, s, t \in \mathbb{R}$ and $a > 0$, wherever an element $A \in \Gamma^\varepsilon \leq \mathrm{PGL}_3\mathbb{C}$ appears.

The case $u, v \in \mathbb{R}$. Then we need to check the line pencil of the standard lilac point $p_0 = \mathbb{C}(1, \pm i, 0)$ for adequate lines. For instance, consider a line L of type $L = p_0 \oplus \mathbb{C}(1, d, c)$ with $c, d \in \mathbb{R}$, $c \neq 0$.

3.7.2 Lemma. *If $u, v \in \mathbb{R}$ then for any red-lilac line $L = p_0 \oplus \mathbb{C}(1, d, c)$ with $c, d \in \mathbb{R}$, $c \neq 0$, the line stabiliser Γ^ε_L is trivial.*

Proof. In homogeneous coordinates, $L = \operatorname{Ker}(1 \quad \pm i \quad -c^{-1}(1 \pm id))^\intercal$. Let $A \in \Gamma^\varepsilon_L$; let $\lambda \in \mathbb{C}^\times$ with

$$\lambda \begin{pmatrix} 1 \\ \pm i \\ -c^{-1}(1 \pm id) \end{pmatrix} =$$

$$\begin{pmatrix} a \pm ia(tu+r) - ac^{-1}(1 \pm id)(s + tr(1+uv) + \frac{t^2}{2}u(1+uv) + \frac{r^2}{2}v) \\ \pm i - -c^{-1}(1 \pm id)(t(1+uv) + rv) \\ -a^{-1}c^{-1}(1 \pm id) \end{pmatrix} \quad .$$

The third component forces $\lambda = a^{-1}$. From the second component we learn that $0 = t(1+uv) + rv$ and $a = 1$. The first one then comes in with $0 = s + tr(1+uv) + \frac{t^2}{2}u(1+uv) + \frac{r^2}{2}v$ and $0 = tu + r$, and consequently $A = \mathbb{1}$. $\square$

The case $u \in \mathbb{R}, v \notin \mathbb{R}$. The corresponding candidate for a point representative is of the same type $p_0 = \mathbb{C}(1, \pm i, 0)$, but this time the lines of type $L = p_0 \oplus \mathbb{C}(0,1,c) = \operatorname{Ker}(1 \quad \pm i \quad ic^{-1})^{\mathsf{T}}$, $c \neq 0$, are by far easier to handle. Again we get

3.7.3 Lemma. *If $u \in \mathbb{R}$ and $v \notin \mathbb{R}$, then for any yellow-lilac line $L = p_0 \oplus \mathbb{C}(0,1,c)$ with $c \in \mathbb{R}^{\times}$ the line stabiliser Γ_L^{ε} is trivial.*

Proof. $A \in \Gamma_L^{\varepsilon}$ stabilizes L if and only if

$$\lambda \begin{pmatrix} 1 \\ \pm i \\ \pm ic \end{pmatrix} = \begin{pmatrix} a \pm ia(tu+r) \pm iac^{-1}(1 \pm id)(s + tr(1+uv) + \frac{t^2}{2}u(1+uv) + \frac{r^2}{2}v) \\ \pm i \pm ic(t(1+uv)+rv) \\ \pm ica^{-1} \end{pmatrix}$$

for some $\lambda \in \mathbb{C}^{\times}$. Again $\lambda = a^{-1}$ follows immediately. Moreover equality of the second components can only be guaranteed if $0 = \mp c^{-1} \operatorname{Im} v(tu+r)$, that is $r = -tu$ along with $t = -c(1-a^{-1})$. Then some juggling of the first components subsequently spits out $a = 1$ and $s = \frac{t^2}{2}u$. But putting $a = 1$ in the equation for t above forces $t = 0$ and finally $A = \mathbb{1}$. $\square$

The remaining three cases are those where the lilac orbits are subsumed by the green and red ones [see 3.5.8], and in fact 3.6.10 states that the candidate for a point representative p must be a green one : $q_0 = \mathbb{C}(1, 0, \pm i)$. As a consequence the line L must be drawn from a green pencil.

3.7.4 Lemma. *If $u \notin \mathbb{R}$ and $v = 0$, then for any red-green line $L = q_0 \oplus \mathbb{C}(1,d,c)$ with $c, d \in \mathbb{R}, d \neq 0$ the line stabiliser Γ_L^{ε} is trivial.*

Proof. In homogeneous coordinates, such a red-green line is of the form $L = \operatorname{Ker}(\pm id(c^2+1)^{-1}(c \pm i) \quad 1 \quad -d(c^2+1)^{-1}(c \pm i))^{\mathsf{T}}$. Let $A \in \Gamma_L^{\varepsilon}$ and $\lambda \in \mathbb{C}^{\times}$ with

$$\lambda \begin{pmatrix} \pm id(c^2+1)^{-1}(c \pm i) \\ 1 \\ -d(c^2+1)^{-1}(c \pm i) \end{pmatrix} =$$

$$\begin{pmatrix} \pm iad(c^2+1)^{-1}(c \pm i) + a(tu+r) - d(c^2+1)^{-1}(c \pm i) \cdot (s + tr(1+uv) + \frac{t^2}{2}u(1+uv) + \frac{r^2}{2}v) \\ 1 - d(c^2+1)^{-1}(c \pm i) \cdot (t(1+uv)+rv) \\ -a^{-1}d(c^2+1)^{-1}(c \pm i) \end{pmatrix}.$$

Then $\lambda = a^{-1}$. Furthermore, $t = 0$ and $a = 1$ follow from the real and imaginary parts, respectively, of the second component. Then the first one yields $s = 0$ as well as $r = 0$. $\square$

3.7.5 Lemma. *For $u \notin \mathbb{R}$ and $v \in \mathbb{R}^{\times}$ any yellow-green line $L = q_0 \oplus \mathbb{C}(0,1,c)$ with $c \in \mathbb{R}$ has trivial stabiliser Γ_L^{ε}.*

Proof. In homogeneous coordinates, $L = \operatorname{Ker}(\pm i \quad -c \quad 1)$. Any $\lambda \in \mathbb{C}^\times$ satisfying

$$\lambda \begin{pmatrix} \pm i \\ -c \\ 1 \end{pmatrix} = \begin{pmatrix} \pm ia - ac(tu + r) + (s + tr(1 + uv) + \frac{t^2}{2}u(1 + uv) + \frac{r^2}{2}v) \\ -c + (t(1 + uv) + rv) \\ -a^{-1} \end{pmatrix}$$

must be $\lambda = a^{-1}$. Then $t = 0$ and $r = cv^{-1}(1 - a^{-1})$ follow from the second component, and the first one then affirms $a = 1$ and $s = 0$, and there we are. $\square$

The fifth and final case requires some special care. Algebraically, we can prove that these line stabilisers are of dimension at most 1, which already suffices for our purposes.

Beyond that, certain line stabilisers turn out to be finite, and in that case they are even trivial : Γ^ε *does not contain non-trivial finite subgroups.* In fact, by 3.3.30 the topological groups Γ^ε and Γ are isomorphic, any finite subgroup is compact, and proposition 5.1.31 later on settles that Γ does not contain non-trivial compact subgroups. An ad hoc argument for the same claim could be given as follows : Consider a finite subgroup $\mathsf{E} \leq \Gamma$ and an element $1 \neq \alpha \in \mathsf{E}$. There is an element $0 \neq A \in \mathfrak{g}$ such that $A^{\exp} = \alpha$, and for any natural number n the exponential image of $nA \in \mathfrak{g}$ is $(nA)^{\exp} = \alpha^n$. As α is necessarily of finite order, this contradicts injectivity of $\exp_{\mathfrak{g}}$.

3.7.6 Lemma. *Let $u, v \notin \mathbb{R}$ and consider an arbitrary yellow-green line $L = q_0 \oplus \mathbb{C}(0, 1, c)$ with $c \in \mathbb{R}$. Then $\dim \Gamma^\varepsilon_L \leq 1$.*

Proof. We are talking about $L = \operatorname{Ker}(\pm i \quad -c \quad 1)$. Let $A \in \Gamma^\varepsilon_L$. As usual it can be figured out that

(1+2)
$$\begin{pmatrix} 0 \\ c(1 - a^{-1}) \end{pmatrix} = (t, r) \cdot \begin{pmatrix} \operatorname{Im}(uv) & 1 + \operatorname{Re}(uv) \\ \operatorname{Im} v & \operatorname{Re} v \end{pmatrix}$$

(3) $s = rc(1 + t \operatorname{Re} u) - tr(1 + \operatorname{Re}(uv)) - \frac{t^2}{2} \operatorname{Re}(u + u^2 v) - \frac{r^2}{2} \operatorname{Re} v$

(4) $\pm a^{-2} = \pm 1 - tc \operatorname{Im} u + tr \operatorname{Im}(uv) + \frac{t^2}{2} \operatorname{Im}(u + u^2 v) \frac{r^2}{2} \operatorname{Im} v.$

Equation (1) is equivalent to

(1') $r = -\frac{\operatorname{Im}(uv)}{\operatorname{Im} v} t.$

Squaring (2) and subtracting (4), we get some equation of the form

(5) $0 = t \cdot (m + t \cdot n),$

where m and n are terms in c, u and v. Instead of explicitly solving this equation observe that

- **if** $m = n = 0$ then t can be chosen arbitrarily and uniquely determines a, r and s. Then Γ^ε_L is (at most) 1-dimensional.

- **if** $m \neq 0$ **and** $n = 0$ then $t = 0$, and as a consequence $\Gamma_L^\varepsilon = \mathbb{1}$.
- **if** $m = 0$ **and** $n \neq 0$ then $t = 0$, and as a consequence $\Gamma_L^\varepsilon = \mathbb{1}$.
- **if** $m \cdot n \neq 0$ then there are two solutions for t (one of which is $t = 0$). Each of them determines unique solutions for a, r and s, such that $|\Gamma_L^\varepsilon| \leq 2$. By the remark above, though, Γ^ε is not supposed to have non-trivial finite subgroups, and therefore Γ_L^ε is trivial.

In either case, $\dim \Gamma_L^\varepsilon$ never exceeds 1. □

Summarising, for every group embedding $\varepsilon := \varepsilon(u, v)$ with $u, v \in \mathbb{C}$ and for every $p \in \mathrm{P}_2\mathbb{C}$ with $\Gamma_p^\varepsilon = 1$, there is a line $L \in (\mathcal{L}_2\mathbb{C})_p$ with $\dim \Gamma_L^\varepsilon < 2$. Expressed on the level of sketches, by proposition 3.3.37 the following has been proved :

3.7.7 Corollary. *There is no monomorphism* $(\varepsilon, \mathsf{E}) : (\Gamma; \{1\}, \mathcal{S}_k^{\exp}) \hookrightarrow (\Gamma^\varepsilon; \mathcal{R})$ *of sketches with the property that*

- $\varepsilon : \Gamma \to \mathrm{PGL}_3\mathbb{C}$ *is a continuous monomorphism in* Gp
- $\mathcal{R}_P = \{\Gamma_p^\varepsilon\}$ *for some* $p \in \mathrm{P}_2\mathbb{C}$ *with* $\Gamma_p^\varepsilon = 1$
- $\mathcal{R}_\mathcal{L} = \{\Gamma_L^\varepsilon \mid L \in (\mathcal{L}_2\mathbb{C})_p\}$.

Recalling all the way we have come, its essence summarised in 3.3.38, this finally proves the non-existence of the fictitious embedding of stable planes we started out with in section 3.3.

3.7.8 Theorem (Non-embeddability of Peter planes). *Let* $\mathcal{P} = \mathbb{P}(\Gamma; \{1\}, \mathcal{S}^{\exp})$ *be a Peter plane, given by a stable partition* $\mathcal{S}^{\exp}$ *of* $\Gamma = \mathbb{R} \ltimes \mathrm{Hei}_3\mathbb{R}$. *Such a plane* $\mathcal{P}$ *cannot be embedded into the complex projective plane* $\mathcal{P}_2\mathbb{C}$ *as a stable plane. As a matter of fact, there is no morphism* $\mathsf{H} : \mathcal{P} \to \mathcal{P}_2\mathbb{C} \in \mathrm{morph}\,\mathsf{StP}$.

3.8. One more way of not embedding Peter planes

So far we have obtained the result that Maier's planes $\mathcal{P}$ that arise from stable partitions of the 4-dimensional Frobenius group $\Gamma = \mathbb{R} \ltimes \mathrm{Hei}_3\mathbb{R}$ cannot be found as open subplanes of the classical plane $\mathcal{P}_2\mathbb{C}$. One possible next move would be to search for less classical planes that might contain $\mathcal{P}$ as an open subplane. Dealing with $\mathcal{P}_2\mathbb{C}$, we had the good fortune of Löwen's Local Fundamental Theorem 3.3.5 giving rise to a group morphism stemming from a possible embedding of stable planes. As there is no such deus ex machina to be seen for non-classical planes we will have to restrict the question of embeddability to embeddability of *geometries*.

The candidates to be considered here are the translation planes

$$\mathcal{T}_k := \mathcal{U}_{\mathsf{Inc}}\mathbb{P}(\mathbb{C}^2; \{0\}, \mathcal{S}_k)$$

that arise from the Betten spreads $\mathcal{S}_k$ ($k \geq 1$) of the Lie algebra $\mathfrak{g} = \mathbb{R} \propto \mathrm{hei}_3\mathbb{R}$, as introduced in 3.1.3. Their full automorphism group is the 8-dimensional group

$$\mathrm{Aut}\,\mathcal{T}_k = \langle \beta \rangle \ltimes (\mathrm{GL}_2\mathbb{R} \ltimes \mathbb{C}^2)\,,$$

where the (left) action of the general linear group is given by ordinary multiplication with column vectors, and where β denotes the Baer involution interchanging the halves of the plane and fixing $\mathcal{S}_k^+ \cap \mathcal{S}_k^-$. In that way, $\mathbb{C}^2$ describes the translation group of $\mathcal{T}_k$. For details, see Salzmann et al. [54, 73.13].

Any continuous group morphism $\varepsilon : \mathsf{\Gamma} \to \mathrm{Aut}\,\mathcal{T}_k$ would map the connected group $\mathsf{\Gamma}$ into the connected component

$$\mathsf{A} := \mathrm{GL}_2^+\mathbb{R} \ltimes \mathbb{C}^2 = \begin{array}{|c|c|}\hline \mathrm{GL}_2^+\mathbb{R} & \mathbb{C}^2 \\ \hline & 1 \\ \hline\end{array}$$

of $\mathrm{Aut}\,\mathcal{T}_k$. Assume there were an embedding $(\varepsilon, \mathsf{H}) : (\mathsf{\Gamma}, \mathcal{P}) \to (\mathsf{A}, \mathcal{T}_k)$ of geometries with the property that the co-restriction $\varepsilon|^{\mathsf{\Gamma}^\varepsilon}$ is homeomorphic and that $\mathsf{H} : \mathcal{P} \to \mathcal{T}_k$ is a morphism of stable planes. Again, H is an open embedding, and Stroppel [66, 1.1] ensures that it preserves pencils. Hence the same mechanism may be started that we have already seen at work, beginning with 3.3.32 and culminating in proposition 3.3.37. And again, the same recipe crystallises : *In order to prove the non-existence of such an embedding it suffices to make sure that for every continuous injective group morphism $\varepsilon : \mathsf{\Gamma} \to \mathsf{A}$ and for every point $p \in \mathbb{C}^2$ with trivial point stabiliser $\mathsf{\Gamma}_p^\varepsilon = 1$ the pencil $\mathcal{K}_p$ of p in $\mathcal{T}_k$ contains at least one line L whose line stabiliser has dimension* $\dim \mathsf{\Gamma}_L^\varepsilon \neq 2$.

Trying to come up with all the possibilities for the continuous group monomorphism $\varepsilon : \mathsf{\Gamma} \to \mathsf{A}$ terminated with ε being determined by quite large a number of parameters — which definitely determined not to tackle the point and line stabilisers. Therefore we just made sure the obvious candidate $\varepsilon = \mathrm{id} : \mathsf{\Gamma} \to \mathsf{A}$ does not allow for any such embedding $(\mathrm{id}, \mathsf{H}) : (\mathsf{\Gamma}, \mathcal{P}) \to (\mathsf{A}, \mathcal{T}_k)$ and then went on to study different matters.

3.8.1 Proposition. *Let $\mathcal{P}$ be a Peter plane as in 3.1.4, and for $k \geq 1$ consider the translation plane $(\mathsf{A}, \mathcal{T}_k)$ as introduced above. There is no morphism $\mathsf{H} : \mathcal{P} \to \mathcal{T} \in \mathrm{morph}\,\mathsf{StP}$ which would turn $(\mathrm{id}, \mathsf{H}) : (\mathsf{\Gamma}, \mathcal{P}) \to (\mathsf{A}, \mathcal{T}_k)$ into a monomorphism of geometries.*

Proof. The (five) point orbits of $\mathsf{\Gamma}$ on $\mathbb{C}^2$ are

$$\begin{aligned}
\mathsf{\Gamma} \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \mathbb{R}^2 \\
\mathsf{\Gamma} \cdot \begin{pmatrix} \pm i \\ 0 \end{pmatrix} &= \left\{ \begin{pmatrix} w \\ y \end{pmatrix} \middle| \, y \in \mathbb{R} \wedge \mathrm{Im}\, w \gtrless 0 \right\} \\
\mathsf{\Gamma} \cdot \begin{pmatrix} 0 \\ \pm i \end{pmatrix} &= \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \middle| \, u \in \mathbb{C} \wedge \mathrm{Im}\, v \gtrless 0 \right\}.
\end{aligned}$$

The only ones having a trivial point stabiliser are the lilac points, represented by $p_o := (0, \pm i)^\mathsf{T}$. Computing the line stabilisers of adjacent lines reveals that, for instance, $\mathsf{\Gamma}_L = 1$ where $L := \mathbb{C}(1, i)^\mathsf{T} + p_o \in \mathcal{K}_{p_o}$ is one of the non-modified lines through p_o. $\square$

4. Classical subplanes in Peter planes

Whereas in the previous chapters we found out that none of the Peter planes $\mathcal{P}$ is embeddable into 4-dimensional "classical" planes, we will now study 2-dimensional "classical" subplanes of $\mathcal{P}$. It will turn out that almost every point in $\mathcal{P}$ is contained in an abundance of both, affine and non-affine 2-dimensional closed subplanes of $\mathcal{P}$ which are open subplanes of the affine plane $\mathcal{A}_2\mathbb{R}$ – we will refer to these as *Baer subplanes.* There are two archetypes of those Baer subplanes appearing in our context, and we present them below.

4.1. Two prototypes

The standard non-affine plane

4.1.1 The group. We will consider the 2-dimensional simply connected non-abelian Lie group

$$\Delta := \left\{ \left(\begin{array}{cc|c} s & t & 0 \\ & 1 & 0 \\ \hline & & 1 \end{array} \right) \middle| \; s, t \in \mathbb{R},\ s > 0 \right\} \cong \mathrm{Dil}_1^1\mathbb{R} \quad .$$

Its Lie algebra is the 2-dimensional non-abelian Lie algebra

$$\mathfrak{n} := \left\{ \left(\begin{array}{cc|c} a & b & 0 \\ & 0 & 0 \\ \hline & & 0 \end{array} \right) \middle| \; a, b \in \mathbb{R} \right\} \cong \mathrm{dil}_1\mathbb{R},$$

and the exponential map

$$\begin{array}{cccll} \exp_{\mathfrak{n}} : & \mathfrak{n} & \to & \Delta & \\ & \left(\begin{array}{cc|c} a & b & 0 \\ & 0 & 0 \\ \hline & & 0 \end{array} \right) & \mapsto & \left(\begin{array}{cc|c} e^a & b \cdot \frac{e^a - 1}{a} & 0 \\ & 1 & 0 \\ \hline & & 1 \end{array} \right) & \text{for } a \neq 0 \\ & \left(\begin{array}{cc|c} 0 & b & 0 \\ & 0 & 0 \\ \hline & & 0 \end{array} \right) & \mapsto & \left(\begin{array}{cc|c} 1 & b & 0 \\ & 1 & 0 \\ \hline & & 1 \end{array} \right) & \end{array}$$

is a homeomorphism. Its inverse is given by

$$
\begin{array}{llll}
\ln_{\mathfrak{n}} : & \Delta & \longrightarrow & \mathfrak{n} \\
& \left(\begin{array}{cc|c} s & t & 0 \\ & 1 & 0 \\ \hline & & 1 \end{array}\right) & \mapsto & \left(\begin{array}{cc|c} \ln s & t \cdot \frac{\ln s}{s-1} & 0 \\ & 0 & 0 \\ \hline & & 0 \end{array}\right) \quad \text{for } s \neq 1 \\
& \left(\begin{array}{cc|c} 1 & t & 0 \\ & 1 & 0 \\ \hline & & 1 \end{array}\right) & \mapsto & \left(\begin{array}{cc|c} 0 & t & 0 \\ & 0 & 0 \\ \hline & & 0 \end{array}\right)
\end{array} .
$$

4.1.2 The geometry. Moreover, we will study the open subplane $\mathcal{H}$ of the real affine plane $\mathcal{A}_2\mathbb{R}$ induced by the "right" half plane $H := \{(x, y) \mid x, y, \in \mathbb{R},\ x > 0\}$. Its line space, consequently, is $\mathfrak{u}_1(\mathbb{R}^2) + H$. An action of Δ on $\mathcal{H}$ is given by

$$
\begin{array}{lll}
H \times \Delta & \longrightarrow & H \\
\left((x,y), \left(\begin{array}{cc|c} s & t & 0 \\ & 1 & 0 \\ \hline & & 1 \end{array}\right)\right) & \mapsto & (sx, tx + y) \leftrightsquigarrow (x, y, 1) \left(\begin{array}{cc|c} s & t & 0 \\ & 1 & 0 \\ \hline & & 1 \end{array}\right) .
\end{array}
$$

For typography's sake, let us identify Δ and H via

$$
(s, t) \longleftrightarrow \left(\begin{array}{cc|c} s & t & 0 \\ & 1 & 0 \\ \hline & & 1 \end{array}\right),
$$

which encourages the notation $(\mathbf{x}, \mathbf{y})^{(\mathbf{s},\mathbf{t})} := (xs, xt + y)$. Note that the set of 1-dimensional subspaces $\mathfrak{u}_1(\mathbb{R}^2)$ is left invariant under the action of Δ. The corresponding action on the lines of $\mathcal{H}$ then is

$$
\begin{array}{lclcl}
(\mathfrak{u}_1(\mathbb{R}^2) + H) & \times & \Delta & \longrightarrow & \mathfrak{u}_1(\mathbb{R}^2) + H \\
(\mathbb{R}x + y, & & (s, t)) & \mapsto & \mathbb{R}\, x^{(s,t)} + y^{(s,t)}
\end{array} .
$$

This action turns $(\Delta, \mathcal{H})$ into a geometry.

4.1.3 Lemma. ***Representatives and the sketched geometry.***

a) *Δ acts transitively on the point space H.*

b) $\mathbb{R}(0,1)^\Delta = \mathbb{R}(0,1)$
$\mathbb{R}(1,m)^\Delta = \{\mathbb{R}(1, \frac{t+m}{s}) \mid (s,t) \in H\}$ *for* $m \in \mathbb{R}$

c) *Representatives for the line orbits – incident with the point $p_0 := (1, 0)$ – are $R_{\mathcal{L}} := \{\mathbb{R}(0, 1) + p_0\} \cup \{\mathfrak{u}_1(\mathbb{R}^2) + p_0\}$.*

d) *$\{p_0\} \times R_{\mathcal{L}}$ is a system of representatives for the flag orbits.*

e) *Thus, $(\Delta; (\{p_0\}, R_{\mathcal{L}}); \mathcal{H})$ is a sketched geometry.*

Proof. *ad (a,b).* Simple verification. *ad (c).* $(\mathbb{R}(0,1)+p_0)^{\Delta} = \mathbb{R}(0,1)+H$ makes up for all the vertical lines. All the non-vertical lines $\mathbb{R}(1,b)+y$ for $b \in \mathbb{R}$ and $y \in H$ can be written in a unique way as $\mathbb{R}(1,b)+y = (\mathbb{R}(1,m)+p_0)^y$, where $m = by_1 - y_2$. *ad (d).* Any flag in $\mathcal{H}$ is of the form $(y, \mathbb{R}x+y)$, where, without loss of generality, $x = (0,1)$ or $x = (1,b)$ for some real number b. For $x = (0,1)$, we get $(y, \mathbb{R}(0,1)+y) = (p_0, \mathbb{R}(0,1)+p_0)^y$. For $x = (1,b)$, putting $m = by_1 - y_2$, we get $(y, \mathbb{R}(1,b)+y) = (p_0, \mathbb{R}(1,m)+p_0)^y$. *ad (e).* By definition, using the above. □

4.1.4 Lemma. ***Stabilisers and the sketch.***

a) $\Delta_{p_0} = \mathbb{1}$

b) $$\Delta_{\mathbb{R}(0,1)+p_0} = \boxed{\begin{array}{cc|c} 1 & \mathbb{R} & 0 \\ & 1 & 0 \\ \hline & & 1 \end{array}}$$

$$\Delta_{\mathbb{R}(1,m)+p_0} = \left\{ \left(\begin{array}{cc|c} s & (s-1)m & 0 \\ & 1 & 0 \\ \hline & & 1 \end{array} \right) \middle| \, s > 0 \right\} \quad \textit{for } m \in \mathbb{R}.$$

c) *The sketch corresponding to the sketched geometry* $(\Delta, \mathcal{H})$ *is* $\mathbb{S}(\Delta, \mathcal{H}) = (\Delta; \{\mathbb{1}\}, \mathcal{R}_{\mathcal{L}})$, *where*

$$\mathcal{R}_{\mathcal{L}} = \{ \boxed{\begin{array}{cc|c} 1 & \mathbb{R} & 0 \\ & 1 & 0 \\ \hline & & 1 \end{array}} \} \cup \{ \left\{ \left(\begin{array}{cc|c} s & (s-1)m & 0 \\ & 1 & 0 \\ \hline & & 1 \end{array} \right) \middle| \, s > 0 \right\} \mid m \in \mathbb{R} \}.$$

Proof. *ad (b).* For the vertical representative, note that $(s,t) \in \Delta$ fixes $\mathbb{R}(0,1) + (1,0) = (\mathbb{R}(0,1)+p_0)^{(s,t)} = \mathbb{R}(0,1)+(s,t)$ if and only if $(s-1,t) \in \mathbb{R}(0,1)$, thus $s = 1$. For non-vertical representatives $\mathbb{R}(1,m)+p_0$, note that $(s,t) \in \Delta_{\mathbb{R}(1,m)}$ if and only if $\mathbb{R}(1,m) = \mathbb{R}(1, \frac{m+t}{s})$, hence $\Delta_{\mathbb{R}(1,m)} = \{(s,(s-1)m) \mid s > 0\}$. Now, $(s,t) \in \Delta_{\mathbb{R}(1,m)+p_0}$ if and only if $(s,t) \in \Delta_{\mathbb{R}(1,m)}$ and $p_0^{(s,t)} - p_0 = (s-1,t) = (s-1)(1,m) \in \mathbb{R}(1,m)$, and this is true for every $(s,t) \in \Delta_{\mathbb{R}(1,m)}$. Thus, $\Delta_{\mathbb{R}(1,m)+p_0} = \Delta_{\mathbb{R}(1,m)}$. *ad (c).* This follows from (a) and (b), as sketching a geometry just means collecting the group acting, along with its point and line stabilisers for the respective representatives. □

Peter planes are constructed from partitions of a 4-dimensional Lie algebra $\mathfrak{g}$. So, as we are aiming at embeddings of the above 2-dimensional stable planes into Peter planes, it may be worthwhile studying the situation on a Lie algebra basis. Recall from 4.1.1 that $\exp_{\mathfrak{n}} : \mathfrak{n} \to \Delta$ is a homeomorphism.

4.1.5 Lemma. ***The stable partition of the Lie algebra $\mathfrak{n}$.***

a)
$$\left.\begin{array}{|cc|c|}\hline 1 & \mathbb{R} & 0 \\ & 1 & 0 \\ \hline & & 1 \\ \hline\end{array}\right.^{\ln_{\mathfrak{n}}} = \begin{array}{|cc|c|}\hline 0 & \mathbb{R} & 0 \\ & 0 & 0 \\ \hline & & 0 \\ \hline\end{array} =: \mathfrak{t}$$

$$\left\{ \left(\begin{array}{cc|c} s & (s-1)m & 0 \\ & 1 & 0 \\ \hline & & 1 \end{array} \right) \mid s > 0 \right\}^{\ln_{\mathfrak{n}}} = \mathbb{R} \cdot \left(\begin{array}{cc|c} 1 & m & 0 \\ & 0 & 0 \\ \hline & & 0 \end{array} \right) =: \mathfrak{v}(m)$$

for $m \in \mathbb{R}$

b) $\mathcal{T} := \{\mathfrak{t}\} \cup \{\mathfrak{v}(m) \mid m \in \mathbb{R}\} = \mathfrak{u}_1(\mathfrak{n})$ *is a stable partition of the Lie algebra $\mathfrak{n}$.*

c) *The 2-dimensional non-affine geometry we are studying can be obtained as*

$$(\Delta, \mathcal{H}) = \mathbb{P}\,(\Delta; \{\mathbb{1}\}, \mathcal{T}^{\exp}).$$

The standard affine plane

4.1.6 The group. Consider the 2-dimensional simply connected abelian Lie group

$$\mathsf{A} = \left\{ \begin{pmatrix} 1 & s & t \\ & 1 & 0 \\ & & 1 \end{pmatrix} \middle| \; s, t \in \mathbb{R} \right\}.$$

Its Lie algebra is the 2-dimensional commutative Lie algebra

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & a & b \\ & 0 & 0 \\ & & 0 \end{pmatrix} \middle| \; a, b \in \mathbb{R} \right\},$$

and the exponential map

$$\begin{array}{cccc} \exp_{\mathfrak{a}} : & \mathfrak{a} & \to & \mathsf{A} \\ & \begin{pmatrix} 0 & a & b \\ & 0 & 0 \\ & & 0 \end{pmatrix} & \mapsto & \begin{pmatrix} 1 & a & b \\ & 1 & 0 \\ & & 1 \end{pmatrix} \end{array}$$

is a homeomorphism. Of course it is, as by identifying

$$(a, b) \longleftrightarrow \begin{pmatrix} 0 & a & b \\ & 0 & 0 \\ & & 0 \end{pmatrix},$$

we are looking at naught but the standard abelian Lie algebra $\mathbb{R}^2 \cong \mathfrak{a}$, whose simply connected Lie group is $(\mathbb{R}^2, +) \cong \mathsf{A}$.

4.1.7 The geometry. The abelian Lie group A acts on the affine plane $\mathcal{A}_2\mathbb{R}$ by the point action

$$\begin{array}{ccc} \mathbb{R}^2 \times \mathsf{A} & \to & \mathbb{R}^2 \\ \left((x,y), \begin{pmatrix} 1 & s & t \\ & 1 & 0 \\ & & 1 \end{pmatrix}\right) & \mapsto & (x+s, y+t) \leftrightsquigarrow (1,x,y) \begin{pmatrix} 1 & s & t \\ & 1 & 0 \\ & & 1 \end{pmatrix} \end{array}$$

Again, we identify $\mathsf{A} \cong \mathbb{R}^2$ an write $(\mathbf{x},\mathbf{y})^{(\mathbf{s},\mathbf{t})} := (x+s, y+t)$. Correspondingly, a group element $(s,t) \in \mathsf{A}$ will map an affine line $\mathbb{R}x + y$ to the affine line $\mathbb{R}x + y^{(s,t)}$; A acts on $\mathcal{A}_2\mathbb{R}$ via translations. This action turns $(\mathsf{A}, \mathcal{A}_2\mathbb{R})$ into a geometry.

4.1.8 Lemma. ***Representatives and the sketched geometry.***

a) A *acts transitively on* $\mathbb{R}^2$.

b) *For every* $x \in \mathbb{R}^2 \setminus \mathbf{0}$, *the line orbit of* $\mathbb{R}x$ *is* $\mathbb{R}x^{\mathsf{A}} = \mathbb{R}x + \mathbb{R}^2$.

c) *Representatives for the line orbits – incident with the origin* $q_0 := (0,0)$ *– are just all lines through the origin, i.e.,* $R_{\mathcal{L}} = \mathfrak{u}_1(\mathbb{R}^2)$.

d) $\{q_0\} \times R_{\mathcal{L}}$ *is representative for flag orbits.*

e) *Thus,* $(\mathsf{A}; \{q_0\}, R_{\mathcal{L}}; \mathcal{A}_2\mathbb{R})$ *is a sketched geometry.*

Proof. Simple verification. $\square$

4.1.9 Lemma. ***Stabilisers and the sketch.***

a) $\mathsf{A}_{q_0} = \mathtt{1}$

b) $\forall \Lambda \in \mathfrak{u}_1(\mathbb{R}^2). \quad \mathsf{A}_\Lambda = \Lambda$

c) *The sketch associated with the geometry* $(\mathsf{A}; \mathcal{A}_2\mathbb{R})$ *is* $\mathbb{S}(\mathsf{A}; \mathcal{A}_2\mathbb{R}) = (\mathsf{A}; \{\mathtt{1}\}, \mathcal{R}_{\mathcal{L}})$, *where*

$$R_{\mathcal{L}} = \{ \boxed{\begin{matrix} 1 & 0 & \mathbb{R} \\ & 1 & 0 \\ & & 1 \end{matrix}} \} \cup \{ \left\{ \begin{pmatrix} 1 & s & sm \\ & 1 & 0 \\ & & 1 \end{pmatrix} \middle| s \in \mathbb{R} \right\} \mid m \in \mathbb{R} \}.$$

Proof. *ad (a).* $q_0^{(s,t)} = (0,0)^{(s,t)} = (s,t)$. *ad (b).* $\mathfrak{u}_1(\mathbb{R}^2) \ni \Lambda = \mathbb{R}x$ for $x \in \mathbb{R}^2 \setminus \mathbf{0}$. Then $\mathbb{R}x = (\mathbb{R}x)^{(s,t)} = \mathbb{R}x + (s,t)$ if and only if $(s,t) \in \mathbb{R}x$; hence $\mathsf{A}_\Lambda = \Lambda$, by abuse of notation. (Also, $\mathsf{A}_{\Lambda+y} = \mathsf{A}_\Lambda = \Lambda$.) *ad (c).* The set of line stabilisers for $\mathfrak{u}_1(\mathbb{R}^2)$ thus is identified with $\mathfrak{u}_1(\mathbb{R}^2) = \{\mathbb{R}(0,1)\} \cup \{\mathbb{R}(1,m) \mid m \in \mathbb{R}\}$ itself. "Dis-identification" then yields the set $\mathcal{R}_{\mathcal{L}}$ of line stabilisers above. $\square$

Again, this calls for a translation onto the Lie algebra level, which leads towards

4.1.10 Lemma. ***The stable partition of the Lie algebra $\mathfrak{a}$.***

a)
$$\boxed{\begin{matrix} 1 & 0 & \mathbb{R} \\ & 1 & 0 \\ & & 1 \end{matrix}}^{\ln_{\mathfrak{a}}} = \boxed{\begin{matrix} 0 & 0 & \mathbb{R} \\ & 0 & 0 \\ & & 0 \end{matrix}} =: \mathfrak{t}$$

$$\left\{ \begin{pmatrix} 1 & s & sm \\ & 1 & 0 \\ & & 1 \end{pmatrix} \mid s \in \mathbb{R} \right\}^{\ln_{\mathfrak{a}}} = \mathbb{R} \begin{pmatrix} 0 & 1 & m \\ & 0 & 0 \\ & & 0 \end{pmatrix} =: \mathfrak{v}(m) \quad \textit{for } m \in \mathbb{R}$$

b) $\mathcal{T} := \{\mathfrak{t}\} \cup \{\mathfrak{v}(m) \mid m \in \mathbb{R}\} = \mathfrak{u}_1(\mathfrak{a})$ *is a stable partition of the Lie algebra* $\mathfrak{a}$.

c) *The* 2*-dimensional affine geometry we are studying can be written as*

$$(\mathsf{A}; \mathcal{A}_2\mathbb{R}) = \mathbb{P}\,(\mathsf{A}; \{\mathbb{1}\}, \mathcal{T}^{\exp}).$$

In both cases, the abelian and the non-abelian one, we end up dealing with a 2-dimensional affine or non-affine sketched geometry of the form $\mathcal{U} = \mathbb{P}\,(\mathfrak{d}^{\exp}; \{\mathbb{1}\}, \mathcal{F}^{\exp})$, where $\mathfrak{d}$ is a 2-dimensional Lie subalgebra of $\mathfrak{g} = \mathbb{R} \propto \mathrm{hei}_3\mathbb{R}$ along with a planar partition $\mathcal{F}$. We now claim that in this very situation, $\mathcal{U}$ will always be embeddable into any Peter plane. The general situation will be studied. Yet, in section 4.3, we will exemplarily continue the explicit embedding process of the two 2-dimensional planes above.

4.2. Sketched Baer subplanes from 2-dimensional Lie subalgebras

Having studied the abelian and non-abelian standard examples, we are in a position to cope with any 2-dimensional Lie subalgebra, as there are only two isomorphism types of 2-dimensional Lie algebras; cf. [27, §1.4].

4.2.1 Lemma. *Let $\mathfrak{d}$ be a 2-dimensional Lie algebra. Then one of the following statements is true :*

a) $\mathfrak{d}$ *is abelian.*
Then $\mathfrak{d} \cong \mathfrak{a}$, and there is a homeomorphic exponential map $\exp_{\mathfrak{d}} : \mathfrak{d} \to \mathsf{A}$.

b) $\mathfrak{d}$ *is non-abelian.*
Then $\mathfrak{d} \cong \mathfrak{n}$, and there is a homeomorphic exponential map $\exp_{\mathfrak{d}} : \mathfrak{d} \to \Delta$.

From now on let $\mathcal{P}$ be a Peter plane $\mathcal{P} = \mathbb{P}(\Gamma; \{1\}, \mathcal{S}^{\exp})$, where $\mathcal{S}$ is one of the stable partitions of $\mathfrak{g} = \ell\,\Gamma = \mathbb{R} \propto \mathrm{hei}_3\mathbb{R}$ (see 3.1.3). Let us consider a 2-dimensional Lie algebra $\mathfrak{d}$ along with an injective Lie algebra morphism $\eta : \mathfrak{d} \to \mathfrak{g}$; in other words, let us study a 2-dimensional Lie subalgebra of $\mathfrak{g}$. By the lemma above we may assume a homeomorphic exponential map $\exp_{\mathfrak{d}} : \mathfrak{d} \to \Theta$, where $\Theta \in \{\mathsf{A}, \Delta\}$ is the simply connected Lie group satisfying $\ell\,\Theta = \mathfrak{d}$. Let us furthermore assume — and this is crucial — that $\mathfrak{d}^\eta \notin \mathcal{S}$ is none of the fibres in $\mathcal{S}$. Now, put

$$\mathcal{F} := \{\Lambda^{\eta^{\leftarrow}} \mid \Lambda \in \mathcal{S} \text{ and } \Lambda^{\eta^{\leftarrow}} \neq \mathbf{0}\}.$$

4.2.2 Lemma. *$\mathcal{F}$ is a planar partition of the Lie algebra $\mathfrak{d}$.*

Proof. Begin by considering the set $\mathcal{F}' := \{\Lambda \cap \mathfrak{d}^\eta \mid \Lambda \in \mathcal{S} \text{ and } \Lambda \cap \mathfrak{d}^\eta \neq \mathbf{0}\}$ of Lie subalgebras of $\mathfrak{g}$. We will verify that $\mathcal{F}'$ is a planar partition of $\mathfrak{d}^\eta$. Then, by injectivity of the morphism η, $\mathcal{F} = (\mathcal{F}')^{\eta^{\leftarrow}}$ is a planar partition of $\mathfrak{d}$.

In fact, it is easy to verify that $\mathcal{F}'$ is a Lie algebra partition of $\mathfrak{d}^\eta$. In order to understand planarity it is helpful to see that every element of $\mathcal{F}'$ is 1-dimensional : Let Λ be a fibre in $\mathcal{S}$. If $\Lambda \cap \mathfrak{d}^\eta$ were 2-dimensional, then $\Lambda \cap \mathfrak{d}^\eta = \mathfrak{d}^\eta$ were a fibre of $\mathcal{S}$, which we forbade. As we also excluded 0-dimensional intersections from being elements of $\mathcal{F}'$, the subalgebra $\Lambda \cap \mathfrak{d}^\eta$ must be of dimension 1. This now implies planarity of $\mathcal{F}'$: Let $\Lambda, \mathsf{M} \in \mathcal{S}$. As $(\Lambda \cap \mathfrak{d}^\eta) \cap (\mathsf{M} \cap \mathfrak{d}^\eta) = \Lambda \cap \mathsf{M} \cap \mathfrak{d}^\eta = \mathbf{0}$, the subalgebra $(\Lambda \cap \mathfrak{d}^\eta) \oplus (\mathsf{M} \cap \mathfrak{d}^\eta) \leq \mathfrak{d}^\eta$ has dimension 2, thus has to be $\mathfrak{d}^\eta$ itself. Hence, $\mathcal{F}'$ is a planar partition of $\mathfrak{d}^\eta$. □

Using this partition of $\mathfrak{d}$, we can construct a new sketched geometry

$$\mathcal{U} := \mathbb{P}(\Theta; \{1\}, \mathcal{F}^{\exp}).$$

This is the very candidate we would like to establish as a closed 2-dimensional subplane of every Peter plane. What do we know about $\mathcal{U}$ at that stage ? We do know precisely what it is, as there are only two choices for Θ.

4.2.3 Lemma. *Presentation of the candidates.*

a) $\mathcal{F} = \mathfrak{u}_1(\mathfrak{d})$

b) *$\mathcal{U}$ is one of the two standard planes presented in the previous section :*

$\mathfrak{d}$ *abelian*	$\Longrightarrow$	$\mathcal{U} \cong (\mathsf{A}, \mathcal{A}_2\mathbb{R})$	*as in 4.1.10*
$\mathfrak{d}$ *non-abelian*	$\Longrightarrow$	$\mathcal{U} \cong (\Delta, \mathcal{H})$	*as in 4.1.5*

c) *In particular, $\mathcal{U}$ is a stable plane and $\mathcal{F}$ is a stable partition of $\mathfrak{d}$.*

Proof. *ad (a).* By 4.2.2, $\mathcal{F}$ is a planar partition of the 2-dimensional vector space $\mathfrak{d}$, and there is only one of those. *ad (b).* From 4.2.1 we know that it suffices to distinguish between the abelian and the non-abelian cases. In both cases, part (a) ensures $\mathcal{F} = \mathfrak{u}_1(\mathfrak{d})$, such that $(\Theta; \{1\}, \mathcal{F}^{\exp})$ is the sketch of one of the stable planes presented in the previous section. More precisely, we can trace the stepwise translation of isomorphisms through the categories in the lines of the following table — with commutativity of $\mathfrak{d}$ triggering the choice between the third or fourth columns :

category	object	isomorphism type	
		$\mathfrak{d}$ abelian	$\mathfrak{d}$ non-abelian
Sk	$(\mathfrak{d}; \{\mathbf{0}\}, \mathcal{F})$	$(\mathfrak{a}; \{\mathbf{0}\}, \mathfrak{u}_1(\mathfrak{a}))$	$(\mathfrak{n}; \{\mathbf{0}\}, \mathfrak{u}_1(\mathfrak{n}))$
Sk	$(\Theta; \{\mathbf{1}\}, \mathcal{F}^{\exp})$	$(\mathsf{A}; \{\mathbf{1}\}, \mathfrak{u}_1(\mathfrak{a})^{\exp})$	$(\Delta; \{\mathbf{1}\}, \mathfrak{u}_1(\mathfrak{n})^{\exp})$
SGeo	$(\Theta; \mathcal{U})$	$(\mathsf{A}; \mathcal{A}_2\mathbb{R})$	$(\Delta; \mathcal{H})$

ad (c). Both candidates are open subplanes of the stable plane $\mathcal{A}_2\mathbb{R}$ and as such are stable, too. This, by definition, makes $\mathcal{F}$ a stable partition of the Lie algebra $\mathfrak{d}$. $\square$

The way of embedding these 2-dimensional stable planes into a 4-dimensional Peter plane is to have a closer look at their sketches. Our next aim, therefore, will be finding an embedding of sketches, and we will start our search at a Lie algebra level. Consider the map

$$\mathsf{H} : \begin{array}{ccc} \mathcal{F} & \to & \mathcal{S} \\ \Lambda^{\eta^{\leftarrow}} & \mapsto & \Lambda \end{array} .$$

4.2.4 Lemma.

a) H *is an injection.*

b) $\forall \mathsf{M} \in \mathcal{F}. \quad \mathsf{M}^{\eta} \leq \mathsf{M}^{\mathsf{H}}$

c) $(\eta; \mathsf{H}) : (\mathfrak{d}; \mathcal{F}) \to (\mathfrak{g}; \mathcal{S})$ *is a monomorphism in* Sk.

Proof. To start with, one might wonder why H should be well-defined at all. In fact, if Λ and M are fibres in $\mathcal{S}$ such that $\Lambda^{\eta^{\leftarrow}} = \mathsf{M}^{\eta^{\leftarrow}}$, then $\mathbf{0} \neq \Lambda^{\eta^{\leftarrow}\eta} = \mathsf{M}^{\eta^{\leftarrow}\eta} \subseteq \Lambda \cap \mathsf{M}$, which implies $\Lambda = \mathsf{M}$. Injectivity, on the other hand, is fairly immediate. *ad (b).* Consider the fibre $\mathsf{M} = \Lambda^{\eta^{\leftarrow}}$ of $\mathcal{F}$. Then $\mathsf{M}^{\eta} = \Lambda^{\eta^{\leftarrow}\eta} \leq \Lambda = \mathsf{M}^{\mathsf{H}}$. *ad (c).* Part (b) just says that (η, H) is a morphism of sketches, and being a monomorphism follows from injectivity of both components [1.1.15], hence from (a). $\square$

Our next task will be translating the current situation back to the level of Lie groups.

4.2.5 Lemma.

There is a continuous group morphism $\varepsilon : \Theta \to \Gamma$ *satisfying* $\varepsilon \cdot \exp_{\mathfrak{g}} = \exp_{\mathfrak{d}} \cdot \eta$. *Moreover,* ε *is injective, and the co-restriction* $\varepsilon|^{\Theta^{\varepsilon}} : \Theta \to \Theta^{\varepsilon}$ *is a homeomorphism. In other words,* ε *is an embedding of topological groups.*

$$\begin{tikzcd} \Theta \arrow[r, dashed, "\varepsilon"] & \Gamma \\ \mathfrak{d} \arrow[u, "\exp_{\mathfrak{d}}"] \arrow[r, hook, "\eta"'] & \mathfrak{g} \arrow[u, "\exp_{\mathfrak{g}}"'] \end{tikzcd}$$

Proof. By the Open Mapping Theorem A.2.3, the Lie algebra morphism $\eta : \mathfrak{d} \to \mathfrak{g}$ is an embedding of topological spaces. By 4.2.1 and 3.1.2, both exponential functions, $\exp_{\mathfrak{d}}$ and $\exp_{\mathfrak{g}}$, are homeomorphisms. Therefore the continuous group morphism $\varepsilon : \Theta \to \Gamma$ provided by Lie theory ([10, Chap. III, §6, no. 1, Thme. 1]) is not only locally injective, but globally. Moreover, the universal property of embeddings (see A.1.2) carries over from η to ε. $\square$

We can use H to define the mapping

$$\begin{array}{rccc} \mathsf{E}_{\mathcal{L}} : & \mathcal{F}^{\exp} & \to & \mathcal{S}^{\exp} \\ & \Lambda & \mapsto & \Lambda^{\exp_{\mathfrak{d}}^{\leftarrow} \cdot \mathsf{H} \cdot \exp_{\mathfrak{g}}} \end{array}$$

between the partitions of Θ and Γ. If one puts $\mathsf{E} := (\mathsf{E}_P, \mathsf{E}_{\mathcal{L}})$, where

$$\mathsf{E}_P : \{1\} \to \{1\} : 1 \mapsto 1,$$

then we end up with an embedding of sketches.

4.2.6 Lemma. $(\varepsilon, \mathsf{E}) : (\Theta; \{1\}, \mathcal{F}^{\exp}) \to (\Gamma; \{1\}, \mathcal{S}^{\exp})$ *is a monomorphism of sketches.*

Proof. The morphism property here reduces to Λ^{ε} being contained in $\Lambda^{\mathsf{E}_{\mathcal{L}}}$ for every fibre $\Lambda \in \mathcal{F}^{\exp}$. In fact, this is true due to 4.2.4 : Let $\Lambda = \mathsf{M}^{\exp} \in \mathcal{F}^{\exp}$. Then $\Lambda^{\varepsilon} = \mathsf{M}^{\exp \cdot \varepsilon} = \mathsf{M}^{\eta \cdot \exp} \le \mathsf{M}^{\mathsf{H} \cdot \exp} = \Lambda^{\mathsf{E}_{\mathcal{L}}}$. Moreover, $\mathsf{E}_{\mathcal{L}}$ is injective, as by 4.2.4, H is injective, and by 4.2.1 so is $\exp_{\mathfrak{d}}$. Summarised, all three components — ε, E_P and $\mathsf{E}_{\mathcal{L}}$ — are injections, which by 1.1.15 makes $(\varepsilon, \mathsf{E})$ a monomorphism in Sk. $\square$

Application of the functor $\mathbb{P}$ yields a morphism

$$(\varepsilon, \mathsf{N}) : \quad \mathbb{P}(\Theta; \{1\}, \mathcal{F}^{\exp}) \quad \to \quad \mathbb{P}(\Gamma; \{1\}, \mathcal{S}^{\exp})$$

of sketched geometries. Remember that written down in an explicit way, the point and line maps are

$$\begin{array}{lcccc} \mathsf{N}_P = \varepsilon & : & \Theta & \to & \Gamma \\ \mathsf{N}_{\mathcal{L}} & : & \bigcup_{\Lambda \in \mathcal{F}^{\exp}} \Theta/\Lambda & \to & \bigcup_{\Lambda \in \mathcal{S}^{\exp}} \Gamma/\Lambda \\ & & \Lambda\alpha & \mapsto & \Lambda^{\mathsf{E}_{\mathcal{L}}} \cdot \alpha^{\varepsilon} \end{array}.$$

As by 1.1.11, the functor $\mathbb{P}$ preserves monomorphisms, we get an embedding of sketched geometries.

4.2.7 Lemma. $(\varepsilon, \mathsf{N}) : (\Theta, \mathcal{U}) \to (\Gamma, \mathcal{P})$ *is a monomorphism of sketched geometries.*

In particular, the point map $\mathsf{N}_P = \varepsilon$ is quite well-behaved : it is continuous and injective and its co-restriction is a homeomorphism. Nevertheless, the crux is obvious : We cannot yet decide whether or not the line map $\mathsf{N}_{\mathcal{L}}$ is injective. By the example in [76, 2.7] we know that it does not necessarily have to. A means of — not only — overcoming that barrier is provided by the theory of morphisms of stable planes from the first chapter. As a matter of fact, we will apply proposition 1.3.5 in order to establish $\mathsf{N} : \mathcal{U} \to \mathcal{P}$ as an embedding of stable planes.

4.2.8 Corollary. *$\mathsf{N}_{\mathcal{L}}$ is continuous and injective, and $\mathsf{N} : \mathcal{U} \to \mathcal{P}$ is an embedding of stable planes.*

Proof. In 4.2.3, we saw that $\mathcal{U}$ is a stable plane. Already knowing that the point map N_P is continuous and injective, we need to understand that it is non-collapsed : Pick two different fibres Λ and M in $\mathcal{F} = \mathfrak{u}_1(\mathbb{R}^2)$. Due to injectivity of $\mathsf{E}_{\mathcal{L}}$ (from 4.2.6), $\Lambda^{\mathsf{E}_{\mathcal{L}}}$ and $\mathsf{M}^{\mathsf{E}_{\mathcal{L}}}$ are still distinct, and hence these two distinct lines $(\Lambda \cdot 1)^{\mathsf{N}_{\mathcal{L}}} = \Lambda^{\mathsf{E}_{\mathcal{L}}} \neq \mathsf{M}^{\mathsf{E}_{\mathcal{L}}} = (\mathsf{M} \cdot 1)^{\mathsf{N}_{\mathcal{L}}}$ are contained in the image of $\mathsf{N}_{\mathcal{L}}$. This completes all the hypothesis required, and theorem 1.3.5 terminates the proof. $\square$

Technically resuming here, we get the following

4.2.9 Corollary. *The morphism $(\varepsilon, \mathsf{N}) : (\Theta, \mathcal{U}) \to (\Gamma, \mathcal{P})$ of sketched geometries has the following properties :*

a) *$(\varepsilon, \mathsf{N})$ is a monomorphism of sketched geometries.*

b) *$\varepsilon : \Theta \to \Gamma$ is a continuous group monomorphism, and its co-restriction is an isomorphism of topological groups.*

c) *Both, the point map N_P and the line map $\mathsf{N}_{\mathcal{L}}$, are continuous and injective. In particular, $\mathsf{N} : \mathcal{U} \to \mathcal{P}$ is a monomorphism of incidence structures.*

d) *$\mathsf{N} : \mathcal{U} \to \mathcal{P}$ is an embedding of stable planes.*

Proof. *ad (a,b).* See 4.2.7. *ad (c).* See 4.2.7 and 4.2.8. By 1.1.12, the morphism N is a monomorphism in the category Inc if and only if both components are injective. $\square$

A rough overall résumé will be added before closing the general section.

4.2.10 Proposition. *Let $\mathcal{P} = \mathbb{P}(\Gamma; \{1\}, \mathcal{S}^{\exp})$ be a Peter plane, and let $\mathfrak{d}$ be a 2-dimensional Lie subalgebra with injective Lie algebra morphism $\eta : \mathfrak{d} \to \mathfrak{g}$. Assume that $\mathfrak{d}^{\exp} \notin \mathcal{S}$. Consider the 2-dimensional plane $\mathcal{U} := \mathbb{P}(\Theta; \{1\}, \mathcal{F}^{\exp})$, where*
Θ is the simply connected Lie group $\Theta \in \{\mathsf{A}, \Delta\}$ with $\ell\,\Theta = \mathfrak{d}$ and
$\mathcal{F} := \{\Lambda^{\overleftarrow{\eta}} \mid \Lambda \in \mathcal{S}$ and $\Lambda^{\overleftarrow{\eta}} \neq \mathbf{0}\}$.
Then $\mathcal{U}$ is an open subplane of the real affine plane $\mathcal{A}_2\mathbb{R}$, and $\mathcal{U}$ is embeddable into $\mathcal{P}$ as a closed subplane; the embedding being meant as an embedding of sketched geometries as well as of stable planes.

4.3. The prototypes as sketched Baer subplanes of the original Peter planes

Are there any tangible incarnations of the situation discussed above ? Remember our two standard examples, the affine plane $(\mathsf{A}, \mathcal{A}_2\mathbb{R})$ and its non-affine open subplane $(\Delta, \mathcal{H})$. As a little demonstration of what is happening in the previous section, let us trace the procedure for those two examples.

The standard non-affine plane revisited

The 2-dimensional non-affine plane we are considering is

$$\mathcal{U} := (\Delta; \mathcal{H}) = \mathbb{P}\left(\Delta; \{\mathbb{1}\}, \mathcal{T}^{\exp}\right)$$

as given in 4.1.5. First of all the embedding procedure requires an embedding of the Lie algebra $\mathfrak{n}$ into $\mathfrak{g}$. There are a huge number of choices — consult 4.4.10 for a classification of possible images — and we pick our favourite one.

4.3.1 Lemma. ***Lie algebra and Lie group embeddings.***

a) $\eta :$

$$\begin{array}{ccc} \mathfrak{n} & \to & \mathfrak{g} \\ \left(\begin{array}{cc|c} a & b & 0 \\ & 0 & 0 \\ \hline & & 0 \end{array}\right) & \mapsto & \begin{pmatrix} a & 0 & b \\ & \frac{a}{2} & 0 \\ & & 0 \end{pmatrix} \end{array}$$

is a monomorphism of Lie algebras.

b) *The corresponding monomorphism of Lie groups is*

$\varepsilon :$

$$\begin{array}{ccc} \Delta & \to & \Gamma \\ \left(\begin{array}{cc|c} s & t & 0 \\ & 1 & 0 \\ \hline & & 1 \end{array}\right) & \mapsto & \begin{pmatrix} s & 0 & t \\ & \sqrt{s} & 0 \\ & & 1 \end{pmatrix} \end{array}$$

Proof. *ad (a).* Simple verification. *ad (b).* Very much so. Yet, let us say that we request the verification of the equation $\exp_{\mathfrak{n}} \cdot \varepsilon = \eta \cdot \exp_{\mathfrak{g}}$. Let $X \in \mathfrak{n}$. Then, for $a \neq 0$, we get

$$X^{\exp_{\mathfrak{n}} \cdot \varepsilon} =: \left(\begin{array}{cc|c} a & b & 0 \\ & 0 & 0 \\ \hline & & 0 \end{array}\right)^{\exp_{\mathfrak{n}} \cdot \varepsilon} = \left(\begin{array}{cc|c} e^a & b \cdot \frac{e^a - 1}{a} & 0 \\ & 1 & 0 \\ \hline & & 1 \end{array}\right)^{\varepsilon} = \begin{pmatrix} e^a & 0 & b \cdot \frac{e^a - 1}{a} \\ & e^{\frac{a}{2}} & 0 \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 & b \\ & \frac{a}{2} & 0 \\ & & 0 \end{pmatrix}^{\exp_{\mathfrak{g}}} = \left(\begin{array}{cc|c} a & b & 0 \\ & 0 & 0 \\ \hline & & 0 \end{array}\right)^{\eta \cdot \exp_{\mathfrak{g}}} = X^{\eta \cdot \exp_{\mathfrak{g}}}.$$

For $a = 0$, the same is true. $\square$

In order to be capable of concrete calculations, let us try and embed $\mathcal{U}$ into the *original Peter planes* $\mathcal{P}_k := \mathbb{P}\left(\Gamma; \{\mathbb{1}\}, \mathcal{S}_k^{\exp}\right)$, for $k \geq 1$, stemming from the Betten spread

$$\mathcal{S}_k := \{\mathfrak{s}\} \cup \{\mathfrak{u}(a, b, -kb) \mid a, b \in \mathbb{R},\ b \leq 0\} \cup \{\mathfrak{u}(a, b, -b) \mid a, b \in \mathbb{R},\ b \geq 0\},$$

where we recall from 3.1.3 that

$$\begin{aligned} \mathfrak{s} &:= \left\{ \begin{pmatrix} 0 & 0 & y \\ & 0 & x \\ & & 0 \end{pmatrix} \middle|\, x, y \in \mathbb{R} \right\} \\ \mathfrak{u}(a, b, c) &:= \left\langle \begin{pmatrix} 2 & 0 & b \\ & 1 & a \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & a \\ & 0 & c \\ & & 0 \end{pmatrix} \right\rangle \end{aligned}$$

for $a, b, c \in \mathbb{R}$. Let us have a look at what will happen to the Lie subalgebras in $\mathcal{T}$ under our map η. Will they find a fibre in $\mathcal{S}_k$ each whose subspaces they are ? Will those even be unique ? We realise that

$$
\begin{array}{lclcl}
\mathfrak{t}^\eta & = & \left\lvert\begin{array}{cc|c} 0 & \mathbb{R} & 0 \\ & 0 & 0 \\ \hline & & 0 \end{array}\right\rvert^\eta & = & \left\lvert\begin{array}{ccc} 0 & 0 & \mathbb{R} \\ & 0 & 0 \\ & & 0 \end{array}\right\rvert & \leq & \mathfrak{s} \\
\mathfrak{v}(m)^\eta & = & \left(\mathbb{R}\begin{pmatrix} 1 & m & 0 \\ & 0 & 0 \\ \hline & & 0 \end{pmatrix}\right)^\eta & = & \mathbb{R}\begin{pmatrix} 2 & 0 & m \\ & 1 & 0 \\ & & 0 \end{pmatrix} & \leq & \begin{cases} \mathfrak{u}(0, m, -km) & \text{for } m \leq 0 \\ \mathfrak{u}(0, m, -m) & \text{for } m \geq 0 \end{cases}
\end{array}
$$

for $m \in \mathbb{R}$. As a matter of fact, the fibres to be inhabited are unique. Consequently, we may make use of the map

$$
\begin{array}{rrcl}
\mathsf{H}: & \mathcal{T} & \to & \mathcal{S}_k \\
& \mathfrak{t} & \mapsto & \mathfrak{s} \\
& \mathfrak{v}(m) & \mapsto & \begin{cases} \mathfrak{u}(0, m, -km) & \text{for } m \leq 0 \\ \mathfrak{u}(0, m, -m) & \text{for } m \geq 0. \end{cases}
\end{array}
$$

4.3.2 Lemma.

a) $\mathsf{H}: \mathcal{T} \to \mathcal{S}_k$ *is injective.*

b) $\forall \Lambda \in \mathcal{T}. \quad \Lambda^\eta \leq \Lambda^{\mathsf{H}}$

c) H *is the only map satisfying (a) and (b).*

d) $(\eta, \mathsf{H}): (\mathfrak{n}, \mathcal{T}) \to (\mathfrak{g}, \mathcal{S}_k)$ *is a monomorphism of sketches.*

Hence, (η, H) is the embedding of sketches on the Lie algebra level we may use in order to proceed with the embedding of $\mathcal{U} = (\Delta, \mathcal{H})$ into each of the planes $\mathcal{P}_k$ as described in the previous section.

4.3.3 Corollary. *In particular, the* 2*-dimensional non-affine plane* $\mathcal{H}$ *has been constructively established as a closed subplane of every original Peter plane* $\mathcal{P}_k$, $k \geq 1$.

The standard affine plane revisited

Our second example is provided by the 2-dimensional affine plane

$$\mathcal{U} := (\mathsf{A}, \mathcal{A}_2\mathbb{R}) = \mathbb{P}\left(\mathsf{A}; \{\mathtt{1}\}, \mathcal{T}^{\exp}\right)$$

as presented in 4.1.10. Again, we start by picking our favourite embedding of the Lie algebras; which is fairly easy. Consider

$$\eta = \mathrm{id}_{\mathfrak{a}}: \quad \mathfrak{a} = \mathbb{R}e_1 + \mathbb{R}e_3 \quad \to \quad \mathfrak{g} = \mathbb{R}d + \mathbb{R}e_1 + \mathbb{R}e_2 + \mathbb{R}e_3$$

as our choice of Lie algebra monomorphism. The corresponding monomorphism ε of Lie groups, of course, is

$$\varepsilon = \mathrm{id}_{\mathsf{A}} : \mathsf{A} \to \Gamma.$$

And again, there is a unique way of embedding $\mathcal{T}$ into $\mathcal{S}_k$ such that we end up with a monomorphism of sketches :

$$\begin{array}{rcll} \mathfrak{t} & = & \boxed{\begin{matrix} 0 & 0 & \mathbb{R} \\ & 0 & 0 \\ & & 0 \end{matrix}} & \leq \ \mathfrak{s} \\ \mathfrak{v}(m) & = & \mathbb{R}\begin{pmatrix} 0 & 1 & m \\ & 0 & 0 \\ & & 0 \end{pmatrix} & \leq \ \mathfrak{u}(m,0,0) \end{array}$$

for $m \in \mathbb{R}$. This inspires the map

$$\begin{array}{rrcll} \mathsf{H}: & \mathcal{T} & \to & \mathcal{S}_k & \\ & \mathfrak{t} & \mapsto & \mathfrak{s} & \\ & \mathfrak{v}(m) & \mapsto & \mathfrak{u}(m,0,0) & \text{for } m \in \mathbb{R}. \end{array}$$

4.3.4 Lemma.

a) $\mathsf{H} : \mathcal{T} \to \mathcal{S}_k$ *is injective.*

b) $\forall \Lambda \in \mathcal{T}. \quad \Lambda^\eta \leq \Lambda^{\mathsf{H}}$

c) H *is unique with properties (a) and (b).*

So here, $(\eta, \mathsf{H}) : (\mathfrak{a}, \mathcal{T}) \to (\mathfrak{g}, \mathcal{S}_k)$ is the monomorphism of sketches the embedding of $\mathcal{U} = (\mathsf{A}, \mathcal{A}_2\mathbb{R})$ into any $\mathcal{P}_k$ can be started with.

4.3.5 Corollary. *In particular, the* 2*-dimensional affine plane* $\mathcal{A}_2\mathbb{R}$ *has been actively established as a closed subplane of every original Peter plane* $\mathcal{P}_k$, $k \geq 1$.

4.4. Classification of 2-dimensional Lie subalgebras of $\mathfrak{g}$

So far, it has been said that more or less any 2-dimensional Lie subalgebra of $\mathfrak{g}$ may be used in order to construct a sketched Baer subplane (provided it is not a fibre of the stable partition involved), and two examples have been exhibited. Naturally, one of the arising questions is that for more examples. In other words, what 2-dimensional Lie subalgebras are there ? And when do they not qualify for the construction process, i.e., when are they fibres of the spread used in defining the Peter plane ? Is every point of a Peter plane contained in an affine or non-affine sketched Baer subplane of that sort ? And if so, is that sketched Baer subplane unique ?

4.4.1 For convenience's sake, let us recall the basis for $\mathfrak{g}$ we picked in 3.1.5 : the vector space $\mathfrak{g}$ is generated by the elements

$$d = \begin{pmatrix} 2 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}.$$

Their Lie brackets are

$$\begin{array}{ll} [d, e_1] = e_1 & [e_1, e_2] = e_3 \\ [d, e_2] = e_2 & [e_1, e_3] = 0 \\ [d, e_3] = 2e_3 & [e_2, e_3] = 0 \end{array}$$

The commutator algebra of $\mathfrak{g}$ is $\mathfrak{g}' = \mathbb{R}e_1 + \mathbb{R}e_2 + \mathbb{R}e_3$, and $\mathfrak{g}'' = \mathbb{R}e_3$. We will abbreviate an element of $\mathfrak{g}$ as

$$(a, v, x) := ad + v_1 e_1 + v_2 e_2 + x e_3$$

with $x, a \in \mathbb{R}$ and $v \in \mathbb{R}^2$. The Lie bracket in $\mathfrak{g}$ then computes as

$$[(a, v, x), (b, w, y)] = (0, aw - bv, 2 \cdot \det \begin{pmatrix} a & x \\ b & y \end{pmatrix} + \det \begin{pmatrix} v \\ w \end{pmatrix}).$$

The automorphism group and its orbits

It will be helpful to lay hands on the group of all automorphisms of $\mathfrak{g}$. What do we know about it ? In the first place, every automorphism is a linear map of the 4-dimensional real vector space $\mathfrak{g} \cong \mathbb{R}^4$. Hence, $\operatorname{Aut} \mathfrak{g}$ is a subgroup of $\mathrm{GL}_4\mathbb{R}$. Stepwise incorporation of their properties as Lie algebra morphisms reveals their true nature.

4.4.2 Proposition.

$$\operatorname{Aut} \mathfrak{g} = \left\{ \left(\begin{array}{c|c|c} 1 & \sigma^{\mathsf{T}} I A^{\mathsf{T}} & \tau \\ \hline & A & \sigma \\ \hline & & \det A \end{array} \right) \;\middle|\; A \in \mathrm{GL}_2\mathbb{R},\ \sigma^{\mathsf{T}} \in \mathbb{R}^2,\ \tau \in \mathbb{R} \right\}$$

where $I := \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. *The matrices refer to the basis* d, e_1, e_2, e_3 *of* $\mathfrak{g}$.

Proof. The commutator algebras $\mathfrak{g}'$ and $\mathfrak{g}''$ are invariant under the action of $\operatorname{Aut} \mathfrak{g}$. Consequently, with respect to the basis d, e_1, e_2, e_3 and vectors being understood as row vectors, an element φ of the automorphism group can be written as a matrix

$$\varphi = \left(\begin{array}{c|c|c} \rho & \pi & \tau \\ \hline & A & \sigma \\ \hline & & \alpha \end{array} \right)$$

with $A \in \mathrm{GL}_2\mathbb{R}$, $\sigma^\mathsf{T}, \pi \in \mathbb{R}^2$, $\alpha, \rho \in \mathbb{R}^\times$ and $\tau \in \mathbb{R}$. We start by detecting the automorphism group of $\mathfrak{g}'$: Consider $(0, v, x), (0, w, y) \in \mathfrak{g}'$. Then

$$[(0, v, x), (0, w, y)] = (0, 0, \det \begin{pmatrix} v \\ w \end{pmatrix}),$$

and φ being a morphism yields the equation

$$[(0, v, x)^\varphi, (0, w, y)^\varphi] = (0, 0, \det \begin{pmatrix} v \\ w \end{pmatrix})^\varphi,$$

in other words

$$(0, \mathbf{0}, \det A \cdot \det \begin{pmatrix} v \\ w \end{pmatrix}) = (0, \mathbf{0}, \alpha \cdot \det \begin{pmatrix} v \\ w \end{pmatrix})$$

and hence finally $\alpha = \det A$. Wanting to make the d-component enter the game, we now consider the morphism condition concerning

$$[(1, \mathbf{0}, 0), (0, w, y)] = (0, w, 2y),$$

which yields the equation

$$(0,\ \rho w A,\ 2\rho(w\sigma + y \det A) + \det \begin{pmatrix} \pi \\ wA \end{pmatrix}) = (0,\ wA,\ w\sigma + 2y \det A).$$

This implies $\rho = 1$, and moreover $0 = w\sigma + \det \begin{pmatrix} \pi \\ wA \end{pmatrix}$ for any $w \in \mathbb{R}^2$. Picking $w = (0, 1)$ and $w = (1, 0)$, successively, we get

$$\sigma = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \cdot A \cdot \pi^\mathsf{T} = I \cdot A \cdot \pi^\mathsf{T}.$$

□

4.4.3 Remark. As an aside, within the above proof we could have also deduced that

$$\mathrm{Aut}(\mathfrak{g}') = \left\{ \left(\begin{array}{c|c} A & \sigma \\ \hline & \det A \end{array} \right) \middle| \ A \in \mathrm{GL}_2\mathbb{R},\ \sigma^\mathsf{T} \in \mathbb{R}^2 \right\}.$$

4.4.4 Lemma. *Under the action of its automorphism group* $\mathrm{Aut}\,\mathfrak{g}$*, the Lie algebra* $\mathfrak{g}$ *splits into the following orbits :*

						colour code
$(0, \mathbf{0}, 1)^{\mathrm{Aut}\,\mathfrak{g}}$	$=$	$\mathbb{R}(0, \mathbf{0}, 1)$	$=$	$\mathfrak{g}''$		*red*
$(0, (0, 1), 0)^{\mathrm{Aut}\,\mathfrak{g}}$	$=$	$\{(0, w, x) \mid w \in \mathbb{R}^2 \setminus \mathbf{0},\ x \in \mathbb{R}\}$	$=$	$\mathfrak{g}' \setminus \mathfrak{g}''$		*yellow*
$(\lambda, \mathbf{0}, 0)^{\mathrm{Aut}\,\mathfrak{g}}$	$=$	$\{(\lambda, v, x) \mid v \in \mathbb{R}^2,\ x \in \mathbb{R}\}$		*for* $\lambda \in \mathbb{R}^\times$		*green*

The question to be answered here will be :

(PP) Given a point p in a Peter plane, how many affine or non-affine Baer subplanes $\mathcal{U}$, as constructed in 4.2.10, are there such that p is a point in $\mathcal{U}$?

Due to their construction from 2-dimensional subalgebras this problem is closely linked with the following question :

(LA) Given an element X of the Lie algebra $\mathfrak{g}$, how many 2-dimensional Lie subalgebras $\mathfrak{d}$, abelian or non-abelian, are there such that $X \in \mathfrak{d}$?

4.4.5 Definition. Denote by

$$\begin{aligned} \mathfrak{A} &:= \{\mathfrak{a} \leq \mathfrak{g} \mid \dim \mathfrak{a} = 2 \ \wedge \ \mathfrak{a} \text{ is abelian}\} \\ \mathfrak{N} &:= \{\mathfrak{n} \leq \mathfrak{g} \mid \dim \mathfrak{n} = 2 \ \wedge \ \mathfrak{n} \text{ is non-abelian}\} \end{aligned}$$

the sets of all abelian and non-abelian, respectively, 2-dimensional Lie subalgebras of $\mathfrak{g}$.

Now, given $X \in \mathfrak{g}$, there will not be more "different behaviours" as to being contained in subalgebras in $\mathfrak{A}$ or $\mathfrak{N}$ than there are different orbits in $\mathfrak{g}$. In fact, if $X \in \mathfrak{a} \in \mathfrak{A}$, then for any $\varphi \in \operatorname{Aut} \mathfrak{g}$ we will get $X^\varphi \in \mathfrak{a}^\varphi \in \mathfrak{A}$; and very much the same for non-abelian $\mathfrak{n} \in \mathfrak{N}$.

As a consequence, if we just want to learn about the numbers of abelian or non-abelian $\mathfrak{d} \in \mathfrak{A} \cup \mathfrak{N}$ some given point is contained in, it suffices to concentrate on representatives of the orbits above. In the table below, the points X are representatives of their respective orbits — explicitly, $X = (0, \mathbf{0}, 1)$ for red points, $X = (0, (0, 1), 0)$ for yellow points and $X = (\lambda, \mathbf{0}, 0)$ for green points — whereas $Y = (b, w, y)$ is an arbitrary point in $\mathfrak{g}$. The table describes, for a given representative X, what additional hypothesis concerning the choice of Y makes the Lie subalgebra $\mathfrak{d} := \langle X, Y \rangle_{\mathrm{LA}} \leq \mathfrak{g}$ an element of $\mathfrak{A} \cup \mathfrak{N}$, that is, 2-dimensional.

X	**Y**	additional hypothesis	$\mathfrak{d}$ is	remarks
red	yellow		abelian	
	green		non-abelian	$\mathfrak{d}' = \mathbb{R}X$
yellow	red		abelian	
	yellow	$w \in \mathbb{R}(0,1)$	abelian	= yellow+red
		$w \notin \mathbb{R}(0,1)$	$>$ 2-dim	
	green	$\det \begin{pmatrix} 0 & 1 \\ w_1 & w_2 \end{pmatrix} = 0$	non-abelian	$\mathfrak{d}' = \mathbb{R}X$
		$\det \begin{pmatrix} 0 & 1 \\ w_1 & w_2 \end{pmatrix} \neq 0$	$>$ 2-dim	
green	red		non-abelian	$\mathfrak{d}' = \mathbb{R}Y$
	yellow	$y = 0$	non-abelian	$\mathfrak{d}' = \mathbb{R}Y$
		$y \neq 0$	$>$ 2-dim	
	green	$w = 0$	non-abelian	= green+red
		$w \neq 0,\ y = 0$	non-abelian	= green+yellow
		$w \neq 0, y \neq 0$	$>$ 2-dim	

We can already realize that green points will never be contained in any abelian 2-dimensional Lie subalgebra. A yellow point will determine a unique abelian $\mathfrak{a} \in \mathfrak{A}$ to be contained in. In any other cases, there is a promise of an abundance of $\mathfrak{d} \in \mathfrak{A} \cup \mathfrak{N}$ containing the given point X.

It is just as well possible to straightforwardly compute a similar table for arbitrary X. Yet, there is a characterisation of elements $X \in \mathfrak{g}$ contained in subalgebras of $\mathfrak{A}$ and $\mathfrak{N}$, respectively, via eigenspaces of the corresponding adjoint maps $\operatorname{ad} X$, which may shed light on the situation.

4.4.6 Definition. Let φ be an $\mathbb{R}$-linear map on $\mathfrak{g}$. Then the eigenspace of φ with respect to an eigenvalue μ will be denoted by

$$T_\mu(\varphi) := \{X \in \mathfrak{g} \mid X^\varphi = \mu \cdot X\}.$$

4.4.7 Lemma. *Characterisation of habitation.* *Let $X \in \mathfrak{g}$.*

a) $\exists \mathfrak{a} \in \mathfrak{A}.\ X \in \mathfrak{a} \iff T_0(\operatorname{ad} X) > \mathbb{R}X$

b) $\exists \mathfrak{n} \in \mathfrak{N}.\ X \in \mathfrak{n} \setminus \mathfrak{n}' \iff \operatorname{ad} X$ *has an eigenvalue* $\mu \neq 0$

c) $\exists \mathfrak{n} \in \mathfrak{N}.\ X \in \mathfrak{n}' \iff$
there is some $Y \in \mathfrak{g} \setminus \mathbb{R}X$ *and an eigenvalue* $\mu \neq 0$ *of* $\operatorname{ad} Y$ *such that* $X \in T_\mu(\operatorname{ad} Y)$

Proof. *ad (a).* Here, $X \in \mathfrak{a} := \mathbb{R}X + \mathbb{R}Y$ for some $Y \in \mathfrak{g} \setminus \mathbb{R}X$ satisfying $[X, Y] = 0$. *ad (b).* $X \in \mathfrak{n} := \mathbb{R}X + \mathbb{R}Y$ for some $Y \in \mathfrak{g} \setminus \mathbb{R}X$ satisfying $[X, Y] \in \mathbb{R}Y$. Let us have a closer look at the forward implication : Starting out with a Lie subalgebra $\mathfrak{n} \in \mathfrak{N}$ such that $X \in \mathfrak{n} \setminus \mathfrak{n}'$, we can find a second basis element $Z \in \mathfrak{g} \setminus \mathbb{R}X$ of $\mathfrak{n}$. As $\mathfrak{n}$ is non-abelian, there are real numbers a and b such that $\operatorname{ad} X.Z = aX + bZ \neq 0$. Moreover, $b \neq 0$ because $X \notin \mathfrak{n}'$. Hence, $Y := aX + bZ$ is an eigenvector of $\operatorname{ad} X$ with respect to the eigenvalue $\mu := b \neq 0$. *ad (c).* $X \in \mathfrak{n} := \mathbb{R}X + \mathbb{R}Y$ for some $y \in \mathfrak{g} \setminus \mathbb{R}X$ satisfying $[X, Y] \in \mathbb{R}X$. ☐

Recall from 3.1.5 the eigenvalues and eigenspaces of the elements of our preferred basis of $\mathfrak{g}$:

	eigenvalue	basis of eigenspace
d	0	d
	1	e_1, e_2
	2	e_3
e_1	0	e_1, e_2
e_2	0	e_2, e_3
e_3	0	e_1, e_2, e_3

Thus we can get an idea of what may happen. As a matter of fact, the complete characterisation looks like this :

4.4.8 Lemma. *Let $X \in \mathfrak{g}$.*

a) $\exists \mathfrak{a} \in \mathfrak{A}.\ X \in \mathfrak{a}$
$\iff$ *one of the following is true :*

i) *X is red; then $T_0(\operatorname{ad} X) = \mathbb{R}e_1 + \mathbb{R}e_2 + \mathbb{R}e_3$*

ii) *X is yellow; then $T_0(\operatorname{ad} X) = \mathbb{R}X + \mathbb{R}e_3$*

b) $\exists \mathfrak{n} \in \mathfrak{N}.\ X \in \mathfrak{n} \setminus \mathfrak{n}'$
$\iff$
X is green. For $X = (\lambda, x, v) \in (\lambda, \mathbf{0}, 0)^{\operatorname{Aut}\mathfrak{g}}$, $\lambda \in \mathbb{R}^\times$, the eigenvalues are λ and 2λ with respective eigenspaces

$$T_\lambda(\operatorname{ad} X) = \{(0, w, -\lambda^{-1} \cdot \det \begin{pmatrix} v \\ w \end{pmatrix} \mid w \in \mathbb{R}^2\}$$
$$T_{2\lambda}(\operatorname{ad} X) = \mathbb{R}e_3$$

c) $\exists \mathfrak{n} \in \mathfrak{N}.\ X \in \mathfrak{n}'$
$\iff$ *one of the following is true :*

i) *X is red. Then for any green $Y = (\lambda, v, x)$ we get $X \in T_{2\lambda}(\operatorname{ad} Y) = \mathbb{R}e_3$.*

ii) *X is yellow. Then X can be written as $X = (0, w, -\lambda^{-1} \cdot \det \begin{pmatrix} v \\ w \end{pmatrix})$ with unique $w \in \mathbb{R}^2 \setminus \mathbf{0}$ and arbitrary $\lambda \in \mathbb{R}^\times$ or $v \in \mathbb{R}^2$. For green $Y \in (\lambda, v, 0) + \mathbb{R}e_3$, we get $X \in T_\lambda(\operatorname{ad} Y) = \{(0, z, -\lambda^{-1} \cdot \det \begin{pmatrix} z \\ w \end{pmatrix}) \mid z \in \mathbb{R}^2\}$. These are the only elements with ad-eigenvector X.*

Proof. Some straightforward but lengthy computation. □

4.4.9 Remark. The actual Lie subalgebras $\mathfrak{d} \in \mathfrak{A} \cup \mathfrak{N}$ can be deduced from the above lemma as follows :

in (a) : $\forall Y \in T_0(\operatorname{ad} X) \setminus \mathbb{R}X.\quad \mathfrak{a} := \mathbb{R}X + \mathbb{R}Y \in \mathfrak{A}$
in (b) : $\forall \mu \in \{\lambda, 2\lambda\} \quad \forall Y \in T_\mu(\operatorname{ad} X).\quad \mathfrak{n} := \mathbb{R}X + \mathbb{R}Y \in \mathfrak{N}$
in (c) : For each of the green Y mentioned, we get $X \in \mathfrak{n}' \leq \mathfrak{n} := \mathbb{R}X + \mathbb{R}Y \in \mathfrak{N}$.

4.4.10 Corollary. ***Classification of 2-dimensional Lie subalgebras of $\mathfrak{g}$.***

$$\begin{aligned}
\mathfrak{A} &= \{\mathbb{R}e_3 + \mathbb{R}X \mid X \text{ yellow}\} \\
&= \{T_0(\operatorname{ad} X) \mid X \text{ yellow}\} \\
\mathfrak{N} &= \{\mathbb{R}e_3 + \mathbb{R}X \mid X \text{ green}\} \\
&\quad \cup \{\mathbb{R}(1, v, x) + \mathbb{R}(0, w, -\det \begin{pmatrix} v \\ w \end{pmatrix}) \mid v \in \mathbb{R}^2,\ x \in \mathbb{R},\ w \in \mathbb{R}^2 \setminus \mathbf{0}\} \\
&= \{\mathbb{R}X + \mathbb{R}Y \mid \text{ green } X \in (1, \mathbf{0}, 0)^{\operatorname{Aut}\mathfrak{g}} \ \wedge\ Y \in T_2(\operatorname{ad} X)\} \\
&\quad \cup \{\mathbb{R}X + \mathbb{R}Y \mid \text{ green } X \in (1, \mathbf{0}, 0)^{\operatorname{Aut}\mathfrak{g}} \ \wedge\ Y \in T_1(\operatorname{ad} X)\}
\end{aligned}$$

This finally is an answer to question (LA), as well as the major part of an answer to question (PP) :

4.4.11 Corollary. ***Habitation within the Lie algebra.*** *"Number" of 2-dimensional Lie subalgebras $\mathfrak{d} \leq \mathfrak{g}$ containing some given $X \in \mathfrak{g}$:*

	$\mathfrak{d}$ abelian	$\mathfrak{d}$ non-abelian
X red	many	many
X yellow	one	many
X green	—	many+1

4.4.12 Proposition. ***Habitation within the Peter plane.*** *"Number" of sketched Baer subplanes $\mathcal{U} = \mathbb{P}\left(\mathfrak{d}^{\exp}; \{1\}, \mathcal{F}^{\exp}\right)$ with $\mathfrak{d} \leq \mathfrak{g}$ and $\mathcal{F} := \{\Lambda \cap \mathfrak{d} \mid \Lambda \in \mathcal{S} \wedge \Lambda \cap \mathfrak{d} \neq \mathbf{0}\}$ containing a given point $p = X^{\exp}$ of the Peter plane $\mathcal{P} = \mathbb{P}\left(\mathfrak{g}^{\exp}; \{1\}, \mathcal{S}^{\exp}\right)$:*

	$\mathcal{U}$ affine	$\mathcal{U}$ non-affine
X red	many	many
X yellow	≤ 1	many
X green	—	many

Proof. This can be directly copied down from the table above, yet, bearing in mind that the Lie subalgebra $\mathfrak{d} \in \mathfrak{A} \cup \mathfrak{N}$ we happen to pick must not be a fibre in $\mathcal{S}$. At most one of these Lie subalgebras is a fibre; in fact, two such fibres would intersect non-trivially in X and hence be equal. Consequently, this restriction only affects the (unique) abelian Lie subalgebra for yellow points X. $\square$

The existence of an affine sketched Baer subplane containing a given yellow point remains uncertain : When does it happen that the abelian Lie subalgebra $\mathfrak{d}$ is a fibre of $\mathcal{S}$? This question will be the subject of the following studies.

4.5. Abelian fibres in stable partitions of $\mathfrak{g}$

Studying the number of abelian fibres a stable partition of $\mathfrak{g}$ may contain, the following little observation — gained from 4.4.10 — turns out to be helpful :

4.5.1 Remark. **a)** The commutator algebra $\mathfrak{g}' = \mathbb{R}e_1 + \mathbb{R}e_2 + \mathbb{R}e_3$ forms a hyperplane in the vector space $\mathfrak{g} \cong \mathbb{R}^4$.

b) Any abelian fibre of a planar partition $\mathcal{S}$ of $\mathfrak{g}$ is entirely contained in the hyperplane $\mathfrak{g}'$.

c) Any non-abelian fibre Λ of a stable partition $\mathcal{S}$ of $\mathfrak{g}$ intersects $\mathfrak{g}'$ in a 1-dimensional subspace $\Lambda \cap \mathfrak{g}$.

d) As a matter of fact, (b) and (c) remain true for 2-dimensional Lie subalgebras instead of a fibres.

Proof. Recall that every fibre of a stable, hence planar, partition of a 4-dimensional Lie algebra is a 2-dimensional Lie subalgebra. $\square$

This already has a very immediate consequence :

4.5.2 Corollary. *Any planar partition of* $\mathfrak{g} = \mathbb{R} \propto \mathrm{hei}_3\mathbb{R}$ *contains at most one abelian fibre.*

Proof. Assume the existence of different abelian fibres $\mathsf{\Lambda}$ and M in $\mathcal{S}$. Then $\mathfrak{g}' \geq \mathsf{\Lambda} \oplus \mathsf{M} = \mathfrak{g}$ by part (b) of remark 4.5.1. This is a contradiction, though. □

Obtaining a lower bound for the number of abelian fibres in $\mathcal{S}$ requires, for instance, some additional knowledge of the topology of stable partitions. We will briefly indicate some of it :

4.5.3 Proposition. **a)** *A stable partition* $\mathcal{S}$ *of a* 4*-dimensional Lie algebra* $\mathfrak{g}$ *is homeomorphic to the* 2*-sphere* $\mathbb{S}_2$*.*

b) *The Graßmann manifold* $\mathfrak{u}_1(\mathbb{R}^3)$ *is homeomorphic to the cross-cap* $\mathrm{P}_2\mathbb{R}$*.*

c) $\mathbb{S}_2$ *and* $\mathrm{P}_2\mathbb{R}$ *are non-homeomorphic.* One way of legitimising this is by stating that $\mathbb{S}_2$ is the prototype of an orientable closed surface of genus 0 (with associated word $aa^{-1} = \square$), whereas the cross-cap represents the non-orientable closed surface of genus 1 (with associated word aa). Another hint is provided by their respective fundamental groups $\pi_1(\mathbb{S}_2) = \mathbb{1}$ and $\pi_1(\mathrm{P}_2\mathbb{R}) = \mathbb{Z}_2$.

Proof. For (a), see [31, Satz 1.19] or [54, 64.4b]. Parts (b) and (c) can be found in any book on topology or algebraic topology treating the classification of compact surfaces, for instance Ossa [49, p.104 ff] or Massey [45, chapter 1]. □

4.5.4 Lemma. *Any stable partition* $\mathcal{S}$ *of* $\mathfrak{g}$ *contains at least one abelian fibre.*

Proof. Assume that $\mathcal{S}$ consists of non-abelian fibres only. Part (c) of 4.5.1 then yields a map

$$\begin{array}{rccc} \pi : & \mathcal{S} & \to & \mathfrak{u}_1(\mathfrak{g}') \\ & \mathsf{\Lambda} & \mapsto & \mathsf{\Lambda} \cap \mathfrak{g}' . \end{array}$$

π *is a continuous bijection.* Consider fibres $\mathsf{\Lambda}$ and M satisfying $\mathsf{\Lambda}^\pi = \mathsf{M}^\pi$. Both, $\mathsf{\Lambda}^\pi = \mathsf{\Lambda} \cap \mathfrak{g}'$ and $\mathsf{M}^\pi = \mathsf{M} \cap \mathfrak{g}'$ are 1-dimensional. Consequently, the fibres $\mathsf{\Lambda}$ and M intersect non-trivially and hence are equal. Thus, π is injective. As to surjectivity, every 1-dimensional subspace of $\mathfrak{g}'$ is of the form $\mathbb{R}X$ for some $X \in \mathfrak{g}' \setminus \mathbf{0}$. As $\mathcal{S}$ is a vector space partition, there is some fibre $\mathsf{\Lambda} \in \mathcal{S}$ such that $X \in \mathsf{\Lambda}$, hence $\mathbb{R}X \subseteq \mathsf{\Lambda} \cap \mathfrak{g}'$.

π *is open.* Now we need to apply our secret knowledge on the fibrations we talk about. By 4.5.3, $\mathcal{S} \approx \mathbb{S}_2$ is compact, and the cross-cap $\mathfrak{u}_1(\mathfrak{g}') = \mathfrak{u}_1(\mathbb{R}^3) \approx \mathrm{P}_2\mathbb{R}$ is a Hausdorff space. This implies openness of π.

Here we are with the 2-sphere $\mathcal{S} \approx \mathbb{S}_2$ and the cross-cap $\mathfrak{u}_1(\mathfrak{g}') \approx \mathrm{P}_2\mathbb{R}$ being homeomorphic, which certainly is a lie; cf. part (c) of 4.5.3. Thus, we have established the existence of at least one abelian fibre in $\mathcal{S}$. □

4.5.5 Corollary. *Any stable partition of* $\mathfrak{g} = \mathbb{R} \propto \mathrm{hei}_3\mathbb{R}$ *contains exactly one abelian fibre.*

4.5.6 Remark. Coming back to question (PP) and its partial answer in 4.4.12, we can now add another part to the answer : Given a yellow point p and the unique abelian $\mathfrak{a} \in \mathfrak{A}$ containing p, there is *exactly one chance* of $\mathfrak{a}$ being a fibre. In any other case, $\mathfrak{a}$ may be used for the construction, and p happily settles down in the middle of an affine sketched Baer subplane of $\mathcal{P}$. Moreover, even if $\mathfrak{a}$ happens to be "the wrong one", something nice at least can be said about p, which will be exposed in the sequel.

4.6. Affine lines in Peter planes

A yellow point p induces a unique abelian Lie subalgebra $\mathfrak{a} \in \mathfrak{A}$. In case $\mathfrak{a}$ is a fibre in $\mathcal{S}$ we are not in a position to use it for the construction of a sketched Baer subplane of $\mathcal{P}$ containing p; simply because 4.2.2 could not be maintained. Nevertheless, we can be consoled by the fact that in this case "something affine" still happens : let us study *affine lines* in Peter planes. Our aim will be to establish a link between abelian fibres in $\mathcal{S}$ and "affine lines" in the corresponding Peter plane. To that end, before introducing the notion of "affine lines", let us study the structure of stable partitions of $\mathfrak{g}$.

4.6.1 Lemma. *A* 2*-dimensional Lie subalgebra of* $\mathfrak{g} = \mathbb{R} \propto \mathrm{hei}_3\mathbb{R}$ *is an ideal in* $\mathfrak{g}$ *if and only if it is abelian. In other words,* $\mathfrak{A}$ *is the set of all* 2*-dimensional ideals in* $\mathfrak{g}$.

Proof. Lemma 4.4.10 gives a list of all 2-dimensional Lie subalgebras in $\mathfrak{g}$. An effective, though highly non-elegant, way of establishing the claim hence is to run through them all and check the ideal property.

Any abelian $\mathfrak{a} \in \mathfrak{A}$ is of the form $\mathfrak{d} = \mathbb{R}X + \mathbb{R}Y$ for red X and yellow Y. Then for any red or yellow $Z \in \mathfrak{g}'$, we get $\mathrm{ad}\, Z.\mathfrak{a} \leq \mathfrak{g}'' = \mathbb{R}X \leq \mathfrak{a}$. For green $Z \in \mathfrak{g} \setminus \mathfrak{g}'$, we compute that $\mathrm{ad}\, Z.X \in \mathbb{R}X \leq \mathfrak{a}$ and $\mathrm{ad}\, Z.Y \in \mathfrak{a}$, and consequently $\mathrm{ad}\, Z.\mathfrak{a} \leq \mathfrak{a}$. Hence $\mathfrak{a} \trianglelefteq \mathfrak{g}$ is an ideal in $\mathfrak{g}$.

It remains to show that the non-abelian Lie subalgebras $\mathfrak{n} \in \mathfrak{N}$ are not ideals in $\mathfrak{g}$. One the one hand, by 4.5.1d, the intersection of such a non-abelian Lie subalgebra $\mathfrak{n}$ with the commutator algebra $\mathfrak{g}'$ is 1-dimensional. On the other hand, $\mathfrak{n}$ contains some green point $X = \lambda_0 d + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$, $\lambda_0 \neq 0$. Assume that $\mathfrak{n}$ is an ideal in $\mathfrak{g}$. Then so is the intersection $\mathfrak{n} \cap \mathfrak{g}'$, which then would contain $\mathbb{R}[X, e_1] + \mathbb{R}[X, e_2] + \mathbb{R}[X, e_3] = \mathbb{R}(\lambda_0 e_1 - \lambda_2 e_3) + \mathbb{R}(\lambda_0 e_2 + \lambda_1 e_3) + \mathbb{R}(2\lambda_0 e_3) = \mathbb{R}e_1 + \mathbb{R}e_2 + \mathbb{R}e_3 = \mathfrak{g}'$. This contradicts $\dim(\mathfrak{n} \cap \mathfrak{g}') = 1$, and hence $\mathfrak{n}$ cannot be an ideal. □

It is a known fact for connected Lie groups that ideals in the Lie algebra make up for normal subgroups in the Lie group, and vice versa (cf. [24, p. 215] or [26, Prop 5.49]).

4.6.2 Lemma. *Every stable partition* $\mathcal{S}$ *of* $\mathfrak{g}$ *contains exactly one fibre which is an ideal in* $\mathfrak{g}$*. The stable partition* $\mathcal{S}^{\exp}$ *of* Γ*, then, contains exactly one fibre which is a normal subgroup in* Γ.

Proof. 4.5.5 along with 4.6.1. The bit on Lie groups has been explained above, taking into account that $\exp_{\mathfrak{g}}$ is a bijection. □

The above observations will be crucial for the study of our actual topic, "affine lines" in Peter planes.

4.6.3 Definition. Let $(P, \mathcal{L})$ be an incidence structure. A line $L \in \mathcal{L}$ will be called an **affine line** if it satisfies the parallel axiom, i.e., if for every point $p \in P$ which is not contained in L there is a unique line $K \in \mathcal{L}$ containing p and having empty intersection $L \cap K = \emptyset$.

4.6.4 Definition. A fibre Λ of a group partition $\mathcal{R}$ of some group Ψ will be called an **affine fibre** in $\mathcal{R}$ if for any other fibre $\mathsf{M} \in \mathcal{R} \setminus \{\Lambda\}$ we get $\Lambda\mathsf{M} = \Psi = \mathsf{M}\Lambda$.

4.6.5 Lemma. *Let $\mathcal{P} = (P, \mathcal{L}) = \mathbb{P}(\Psi; \{1\}, \mathcal{R})$ be the sketched geometry gained from an arbitrary group Ψ along with a group partition $\mathcal{R}$. Consider a line $\Lambda\alpha$ in $\mathcal{P}$, with $\Lambda \in \mathcal{R}$ and $\alpha \in \Psi$. Then $\Lambda\alpha$ is an affine line in $\mathcal{P}$ if and only if Λ is an affine fibre in $\mathcal{R}$.*

Proof. Bear in mind that for any $\beta \in \Psi \setminus \{\alpha\}$ the line $\Lambda\beta$ is a parallel line to $\Lambda\alpha$ incident with β, anyway, as $\Lambda\alpha \cap \Lambda\beta = \emptyset$. This accounts for the third step in the following series of equivalences :

$$\begin{array}{llll}
 & \Lambda\alpha \text{ is an affine line in } \mathcal{P} & & \\
\iff & \forall \beta \in \Psi \setminus \Lambda\alpha & \exists!\, L = \mathsf{M}\gamma \in \mathcal{L}\,. & \beta \in \mathsf{M}\gamma \parallel \Lambda\alpha \\
\iff & \forall \beta \in \Psi \setminus \Lambda\alpha & \exists!\, \mathsf{M} \in \mathcal{R}\,. & \mathsf{M}\beta \cap \Lambda\alpha = \emptyset \\
\iff & \forall \beta \in \Psi \setminus \Lambda\alpha & \forall \mathsf{M} \in \mathcal{R} \setminus \{\Lambda\}\,. & \mathsf{M}\beta \cap \Lambda\alpha \neq \emptyset \\
\iff & \forall \mathsf{M} \in \mathcal{R} \setminus \Lambda\alpha & \forall \beta \in \Psi \setminus \Lambda\alpha\,. & \mathsf{M}\beta \cap \Lambda\alpha \neq \emptyset \\
\iff & \forall \mathsf{M} \in \mathcal{R} \setminus \{\Lambda\} & \forall \beta \in \Psi \setminus \Lambda\alpha\,. & \beta \in \mathsf{M}\Lambda\alpha \\
\iff & \forall \mathsf{M} \in \mathcal{R} \setminus \{\Lambda\}\,. & \Psi \setminus \Lambda\alpha \subseteq \mathsf{M}\Lambda\alpha & \\
\iff & \forall \mathsf{M} \in \mathcal{R} \setminus \{\Lambda\}\,. & \Psi = \mathsf{M}\Lambda\alpha & \\
\iff & \forall \mathsf{M} \in \mathcal{R} \setminus \{\Lambda\}\,. & \Psi = \mathsf{M}\Lambda & \\
\iff & \Lambda \text{ is an affine fibre in } \mathcal{R} & &
\end{array}$$

□

We can therefore consider affine lines as a purely algebraic phenomenon.

4.6.6 Lemma. *Let $\mathcal{R}$ be a group partition of a group Ψ. Every affine fibre $\Lambda \in \mathcal{R}$ is a normal subgroup in Ψ.*

Proof. *Assume that Λ is not normal in Ψ.* This means the existence of an element $\alpha \in \Psi$ with the property that $\Lambda^\alpha := \alpha^{-1}\Lambda\alpha \subset \Lambda$ or $\Lambda \subsetneq \Lambda^\alpha$. The second inclusion is equivalent to the inclusion $\Lambda \subset \Lambda^{\alpha^{-1}}$, such that we may replace α by α^{-1} in order to state that *there is some element $\alpha \in \Psi$ such that $\Lambda \subset \Lambda^\alpha$.*

There is a fibre $\mathsf{M} \in \mathcal{R} \setminus \{\Lambda\}$ such that $\mathsf{M} \cap \Lambda^\alpha \neq 1$. Pick an element $1 \neq \beta \in \Lambda^\alpha \setminus \Lambda$. As $\mathcal{R}$ is a partition of Ψ there is some fibre $\mathsf{M} \in \mathcal{R}^{\exp}$ such that $\beta \in \mathsf{M}$; quite obviously $\mathsf{M} \neq \Lambda$.

There is an element $\mu \in \mathsf{M}$ such that $\Lambda^\alpha = \Lambda^\mu$. By the hypothesis, Λ is an affine fibre, and therefore $\alpha \in \Upsilon = \Lambda\mathsf{M}$ can be written as $\alpha = \lambda\mu$ with $\lambda \in \Lambda$ and $\mu \in \mathsf{M}$. But then, $\Lambda^\alpha = \alpha^{-1}\Lambda\alpha = \mu^{-1}\Lambda\mu = \Lambda^\mu$.

Finally, this implies $1 \neq \mathsf{M} \cap \Lambda^\alpha = \mathsf{M} \cap \Lambda^\mu = \mathsf{M}^\mu \cap \Lambda^\mu = (\mathsf{M} \cap \Lambda)^\mu = 1$, which is a contradiction. As a consequence, Λ had better be a normal subgroup in Ψ. $\square$

So far, the results have not required any restrictions, neither on the group nor on its group partition. In order to obtain the converse of the above result, though, we will have to come back to our concrete situation.

4.6.7 Lemma. *Let $\mathcal{S}^{\exp}$ be a stable partition of $\Gamma = \mathbb{R} \ltimes \mathrm{Hei}_3\mathbb{R}$, and let Λ be a fibre of $\mathcal{S}^{\exp}$. If Λ is normal in Γ then it is an affine fibre in $\mathcal{S}^{\exp}$.*

Proof. Let $\Lambda \in \mathcal{S}^{\exp}$ be a normal subgroup in Γ. Let $\mathsf{M} \in \mathcal{S}^{\exp} \setminus \{\Lambda\}$. Show : $\Lambda\mathsf{M} = \Gamma$. First of all observe that *there is an isomorphism η mapping the semidirect sum $\ell\,\mathsf{M} \propto \ell\,\Lambda$ onto $\mathfrak{g}$*, because $\ell\,\mathsf{M}$ and $\ell\,\Lambda$ are distinct fibres in the planar partition $\mathcal{S}$ and $\ell\,\Lambda$ is an ideal in $\mathfrak{g}$. Denote by $\tilde{\mathsf{M}}$ and $\tilde{\Lambda}$ the simply connected Lie groups satisfying $\ell\,\tilde{\mathsf{M}} = \ell\,\mathsf{M}$ and $\ell\,\tilde{\Lambda} = \ell\,\Lambda$, respectively. Then *there is an isomorphism φ mapping the semidirect product $\tilde{\mathsf{M}} \ltimes \tilde{\Lambda}$ onto Γ;* see A.3.3.

$$\begin{tikzcd}
\tilde{\mathsf{M}} \ltimes \tilde{\Lambda} \arrow[r, "\varphi"] & \Gamma \\
\ell\,\mathsf{M} \propto \ell\,\Lambda \arrow[u, "\exp"] \arrow[r, "\eta"'] & \mathfrak{g} \arrow[u, "\exp_{\mathfrak{g}}"']
\end{tikzcd}$$

This isomorphism satisfies $(\tilde{\mathsf{M}} \times 1)^\varphi = \mathsf{M}$ and $(\tilde{\Lambda} \times 1)^\varphi = \Lambda$. In fact, $(\ell\,\mathsf{M} \oplus \mathbf{0})^\eta = \ell\,\mathsf{M}$, and $\left\langle (\ell\,\mathsf{M} \oplus \mathbf{0})^{\exp} \right\rangle = \tilde{\mathsf{M}} \times 1$ as $\tilde{\mathsf{M}} \times 1$ is connected. Therefore,

$$(\tilde{\mathsf{M}} \times 1)^\varphi = \left\langle (\ell\,\mathsf{M} \oplus \mathbf{0})^{\exp} \right\rangle^\varphi = (\ell\,\mathsf{M} \oplus \mathbf{0})^{\eta \exp_{\mathfrak{g}}} = (\ell\,\mathsf{M})^{\exp_{\mathfrak{g}}} = \mathsf{M}.$$

The same argument applies to Λ. These facts imply that

$$\Gamma = (\tilde{\mathsf{M}} \ltimes \tilde{\Lambda})^\varphi = \left\langle \tilde{\mathsf{M}} \times 1, 1 \times \tilde{\Lambda} \right\rangle^\varphi = \left\langle (\tilde{\mathsf{M}} \times 1)^\varphi, (1 \times \tilde{\Lambda})^\varphi \right\rangle = \langle \mathsf{M}, \Lambda \rangle\,,$$

which establishes $\Gamma = \mathsf{M}\Lambda$, as Λ is normal in Γ. $\square$

4.6.8 Corollary. *Let $\mathcal{S}^{\exp}$ be a stable partition of $\Gamma = \mathbb{R} \ltimes \mathrm{Hei}_3\mathbb{R}$, and let Λ be a fibre of $\mathcal{S}^{\exp}$. Then Λ is an affine fibre if and only if it is a normal subgroup in Γ.*

This yields a characterisation of affine lines in Peter planes :

4.6.9 Proposition. *Let $\mathcal{S}$ be a stable partition of $\mathfrak{g} = \mathbb{R} \propto \mathrm{hei}_3\mathbb{R}$. Consider a line $\Lambda^{\exp}\alpha$ ($\Lambda \in \mathcal{S}$, $\alpha \in \Gamma$) in the corresponding Peter plane $\mathcal{P} = \mathbb{P}(\Gamma; \{\mathtt{1}\}, \mathcal{S}^{\exp})$. Then $\Lambda^{\exp}\alpha$ is an affine line in $\mathcal{P}$ if and only if Λ is "the" abelian fibre in $\mathcal{S}$.*

Proof. 4.6.8, 4.6.5, 4.6.1 and 4.5.5. □

Let us try and technically resume the whole situation, answering question (PP) as well as the one left over in section 4.4 :

4.6.10 Proposition. *Let $\mathcal{P} = \mathbb{P}(\Gamma; \{\mathtt{1}\}, \mathcal{S}^{\exp})$ be a Peter plane. Let $p = X^{\exp} \in \Gamma$ be a point in $\mathcal{P}$. Recall that* "sketched Baer subplane" *will be meant to denote a closed 2-dimensional subplane $\mathcal{U} = \mathbb{P}(\Theta; \{\mathtt{1}\}, \mathcal{F}^{\exp})$ of $\mathcal{P}$ as constructed in 4.2.10.*

- **a)** *If p is a red point, then $\mathcal{P}$ hosts infinitely many affine sketched Baer subplanes as well as infinitely many non-affine sketched Baer subplanes containing p.*
- **b)** *If p is a green point, then there is an infinite number of non-affine sketched Baer subplanes of $\mathcal{P}$ containing p, but there is no such affine sketched Baer subplane.*
- **c)** *If p is a yellow point, it is contained in an infinite number of non-affine sketched Baer subplanes of $\mathcal{P}$.*

Now, let $\Lambda_0 = \mathbb{R}X_0 + \mathbb{R}e_3$ be the unique abelian fibre contained in the stable partition $\mathcal{S}$ of $\mathfrak{g}$.

- **d)** *If p is a yellow point not contained in $\Lambda_0^{\exp}$, then $\mathcal{P}$ hosts exactly one affine sketched Baer subplane containing p.*
- **e)** *If p is a yellow point contained in $\Lambda_0^{\exp}$, then we cannot construct an affine sketched Baer subplane containing p, but p is contained in the affine line $\Lambda_0^{\exp}$ in $\mathcal{P}$.*

Proof. *ad (a).* 4.4.12 *ad (b).* 4.4.12 *ad (c).* 4.4.12 *ad (d).* 4.4.12 and 4.4.11 *ad (e).* 4.4.12, 4.6.9 and 4.5.5 □

5. On the automorphism group of Peter planes

The present chapter will continue the study of those planes $\mathcal{P}$ that arise from a stable partition of the Frobenius group $\Gamma = \mathbb{R} \ltimes \mathrm{Hei}_3\mathbb{R}$. Chapter 3 establishes that such a Peter plane $\mathcal{P}$ cannot be found as an open subplane in $\mathcal{P}_2\mathbb{C}$. This non-embeddability result brings about consequences for the appearance of the full automorphism group $\Sigma = \mathrm{Aut}\,\mathcal{P}$ of a Peter plane.

5.1. Γ is compact-free

One fact that does play a major role in our reasoning is the fact that the group $\Gamma = \mathbb{R} \ltimes \mathrm{Hei}_3\mathbb{R}$ does not contain compact subgroups, which will be established in the sequel.

5.1.1 Definition. A topological group is said to be **compact-free** if the trivial group is its only compact subgroup.

One major tool to be exploited is the existence of a maximal compact subgroup stated in

5.1.2 Theorem (Mal'cev-Iwasawa). *Let Δ be some locally compact connected topological group. Then*

a) *Δ contains some maximal compact subgroup M. Moreover, each maximal compact subgroup is connected and conjugate to M.*

b) *There are a number k and one-parameter subgroups $\mathsf{P}_j \approx \mathbb{R}$, $j \leq k$ such that the multiplicative map*

$$\begin{array}{ccc} \mathsf{P}_1 \times \ldots \times \mathsf{P}_k \times \mathsf{M} & \to & \Delta \\ (\rho_1, \ldots, \rho_k, \varphi) & \mapsto & \rho_1 \cdots \rho_k \cdot \varphi \end{array}$$

is a homeomorphism. In particular,

$$\Delta \approx \mathbb{R}^k \times \mathsf{M}.$$

[54, 93.10] □

Consequently, the topological space $\Gamma = \mathbb{R} \ltimes \mathrm{Hei}_3\mathbb{R}$ can be written as Cartesian product

$$\Gamma \approx \mathsf{M} \times \mathbb{R}^k$$

in the category of topological spaces, where M is some maximal compact subgroup of Γ and $k \in \mathbb{N}$. Our aim will be to prove that $\mathsf{M} \leq \Gamma$ will have to be trivial; and thus $k = 4$. Let us first of all collect our knowledge on Γ :

5.1.3 Lemma. **a)** $\Gamma \approx \mathbb{R}^4$ *via the homeomorphism*

$$\begin{array}{ccc} \mathbb{R}^4 & \longrightarrow & \Gamma \\ (t,x,y,z) & \longmapsto & \begin{pmatrix} e^{2t} & x & z \\ & e^t & y \\ & & 1 \end{pmatrix} \end{array}$$

b) Γ *is a simply connected (locally compact) Lie group of dimension* $\dim \Gamma = 4$.

Proof. Everything follows, once you believe the homeomorphism to be a homeomorphism. And this is true by 3.1.2. $\square$

5.1.4 Lemma. $\mathbb{R}^k \times \mathsf{M}$ *is (simply) connected if and only if* M *is (simply) connected.*

Proof. For simple connectedness, this drops out of the fact that for any two path connected topological spaces A and B the fundamental group $\pi_1(\mathsf{A} \times \mathsf{B})$ is $\pi_1(\mathsf{A} \times \mathsf{B}) \cong \pi_1(\mathsf{A}) \times \pi_1(\mathsf{B})$. $\square$

5.1.5 Remark. Any locally path connected, connected topological space is path connected. In fact, every path connected component is an open subset; and if there were more than one of them, the space would no longer be connected. In particular, any connected Lie group is path connected, which allows us not to pay attention to the choice of the base point for the homotopy groups involved.

Thus, the M we are looking for will have to be a simply connected maximal compact subgroup of Γ. In fact, there is more to be known on its homotopy groups :

5.1.6 Lemma.

a) $\forall n \in \mathbb{N}.\quad \pi_n(\Gamma) \cong \pi_n(\mathbb{R}^4) = \mathsf{1}$

b) $\forall n \in \mathbb{N}.\quad \pi_n(\mathsf{M}) = \mathsf{1}$

Proof. Let n be some natural number. For the nth homotopy group of some contractible space A, thus in particular for $\mathbb{R}^4$, it is true that $\pi_n(\mathsf{A}) = \mathsf{1}$. Moreover, it is also true that $\pi_n(\mathsf{A} \times \mathsf{B}) \cong \pi_n(\mathsf{A}) \times \pi_n(\mathsf{B})$ for arbitrary path connected groups A and B. This yields the second assertion. – For bare facts on higher homotopy groups, listed in appendix A.1.6, see [14, 7.10+11], for instance. $\square$

Certainly the trivial group is one simply connected compact Lie group with all higher homotopy groups trivial. And in fact, it is the only one. In order to see this, we can rely upon a theorem by Toda; the underlying fact is that the simply connected compact Lie groups are all classified, and there is none, indeed, whose higher homotopy groups are all trivial.

5.1.7 Theorem (Toda 1976). *Two simply connected compact, and hence semisimple, Lie groups are isomorphic if and only if they have isomorphic homotopy groups for each dimension.* *[75] cited in [47, Thm 1.9], quoted as in [8]* □

From this we could easily read that Γ is a compact-free group. Toda's theorem, though, is a mighty theorem and it provides much more than we actually need. It occurs to be wise to fall back on some of the structure theory of compact Lie groups, which will lead us to the concept of *compact Lie algebras*. In the end the simple fact will remain that our maximal compact subgroup M has to be semisimple and soluble all at the same time, which does not leave it much choice : M is trivial, and thus Γ is compact-free.

5.1.1. The commutator subgroup of a compact connected Lie group

Consider an arbitrary compact connected Lie group M along with its Lie algebra $\mathfrak{m} := \ell\,\mathsf{M}$. Naturally, the one specimen of interest later on will be a maximal compact subgroup M of $\Gamma = \mathbb{R} \ltimes \mathrm{Hei}_3\mathbb{R}$, but for the moment the concrete appearance is of no importance.

5.1.8 Lemma. *Let Υ be a compact connected (linear) Lie group.*

a) $\ell(\Upsilon') = (\ell\,\Upsilon)'$*, and in particular* $(\ell\,\Upsilon')^{\exp} \subseteq \mathsf{M}'$.

b) *The connected component $Z(\Upsilon)^1$ of the centre is a compact closed subgroup of Υ.*

c) $\exp : \ell\,\Upsilon \to \Upsilon$ *is surjective.*

Proof. *ad (a).* [26, 5.60] — Note that compactness is not required for this part; cf. [54, 94.16]. *ad (b).* Centres and connected components are always closed. Therefore, compactness of M causes compactness of $\mathrm{Z}(\mathsf{M})^1$. *ad (c).* [26, 6.30] □

From now on consider the commutator subgroup M' of M and its Lie algebra $\ell(\mathsf{M}') = \mathfrak{m}'$. Note that $\mathfrak{m}'$ is an ideal in $\mathfrak{m}$. What we are looking for is a complement $\mathfrak{k}$ of $\mathfrak{m}'$ in $\mathfrak{m}$. It will turn out that $\mathfrak{m}$ can be endowed with an inner product, and the orthogonal space with respect to that inner product yields the desired complement. The motivation for that very construction will be briefly touched here.

5.1.9 Haar-measure and Weyl's trick. For any locally compact group Υ define

$$K(\Upsilon) := \{ f \in \mathcal{C}(\Upsilon, \mathbb{R}) \mid \operatorname{supp} f \text{ compact } \},$$

where the **support** of f is the topological closure of $\operatorname{supp} f := \{x \in \Upsilon \mid f(x) \neq 0\}$. Denote by $K_+(\Upsilon)$ all non-negative elements of $K(\Upsilon)$. Then

$$||f|| := \max\{|f(x)| \mid x \in \Upsilon\}$$

defines a norm on $K(\Upsilon)$, and moreover the group Υ operates on $K(\Upsilon)$ by

$$\begin{array}{rcccl} \mathsf{\Lambda}_g & : & f & \mapsto & \lambda_{g^{-1}} f \\ \mathsf{P}_g & : & f & \mapsto & \rho_{g^{-1}} f. \end{array}$$

A linear functional $\mu : K(\Upsilon) \to \mathbb{R}$ is called a **positive measure** if $\mu(f) \geq 0$ for all $f \in K_+(\Upsilon)$. We call a positive measure that satisfies $\mu(\lambda_g f) = \mu(f)$ or $\mu(\rho_g f) = \mu(f)$ for all $g \in \Upsilon$ and all $f \in K(\Upsilon)$ **left invariant** or **right invariant**, respectively. Some left or right invariant measure, finally, which is moreover positive definite is called a **Haar measure**. Every compact group can be endowed with a Haar measure; see [26, 2.8].

Now let Υ be a compact group with Haar measure μ and V some finite dimensional Hilbert space with scalar product $(\cdot \mid \cdot) : V \times V \to \mathbb{R}$. Given some continuous group morphism $\pi : \Upsilon \to \mathrm{GL}(V)$ there is a scalar product $\langle \cdot \mid \cdot \rangle$ on V such that for every $g \in \Upsilon$ the linear mappings g^π are orthogonal. In fact, this scalar product is given by

$$\langle x \mid y \rangle := \int_{\Upsilon} (g^\pi.x \mid g^\pi.y) \, \mathrm{d}\,\mu(g) := \mu(\chi_{x,y})$$

for $x, y \in V$, where

$$\begin{array}{rcccl} \chi_{x,y} & : & \Upsilon & \to & \mathbb{R} \\ & & g & \mapsto & (g^\pi.x \mid g^\pi.y) \end{array} \quad .$$

In our particular context, we choose π to be $\mathrm{Ad} : \mathsf{M} \to \mathrm{GL}(\mathfrak{g})$. Then the scalar product constructed thus on $\mathfrak{g}$ is Ad-invariant (or "M-invariant"); that is to say that for any $X, Y \in \mathfrak{g}$ and $g \in \mathsf{M}$ it is true that $\big\langle \mathrm{Ad}\, g.X \mid \mathrm{Ad}\, g.Y \big\rangle = \langle X \mid Y \rangle$. As it is moreover true that for all $X \in \mathfrak{g}$ the equation

$$\frac{\mathrm{d}}{\mathrm{d}\, t} e^{\mathrm{ad}\, tX} = e^{\mathrm{ad}\, tX} \cdot \mathrm{ad}\, X$$

holds, one can conclude that the scalar product is also **invariant** in the sense that $\langle Y \mid [X, Z] \rangle = \langle [Y, X] \mid Z \rangle$ for all $X, Y, Z \in \mathfrak{g}$.

A finite dimensional real Lie algebra endowed with an invariant scalar product is called a **compact Lie algebra** or (finite dimensional) **Hilbert Lie algebra**. (For the general notion of potentially infinite dimensional Hilbert Lie algebras, see chapter 6 in [26], along with the errata [25] for definition 6.3.) Using this vocabulary, the above may be interpreted as a motivation of the fact that the Lie algebra of a compact Lie group is a compact Lie algebra. As a matter of fact, the converse assertion is also true : *Let $\mathfrak{m}$ be a finite dimensional real Lie algebra. Then $\mathfrak{m}$ is compact if and only if there is some compact Lie group Υ such that $\mathfrak{m} = \ell\,\Upsilon$*; see [26, 6.6] or [24, III 5.4]. Note that this notion of a compact Lie algebra is misleading in as far as $\mathfrak{m}$ endowed with its natural topology never ever is a compact topological space (unless it is singleton). It is not true in general that any Lie group with a compact Lie algebra is compact. Nevertheless it is true for semisimple ones :

5.1.10 Theorem (Weyl). *Let $\mathfrak{m}$ be a semisimple compact (real) Lie algebra and Υ a connected Lie group with Lie algebra $\ell\,\Upsilon = \mathfrak{m}$. Then Υ is compact.* *[24, III 5.13]* $\square$

Remember that we were looking for an appropriate complement for $\mathfrak{m}'$ in $\mathfrak{m}$. We suggest that the orthogonal subspace

$$\mathfrak{k} := (\mathfrak{m}')^{\perp}$$

with respect to the above scalar product does the job. This is due to a general fact on compact Lie algebras :

5.1.11 Proposition. *Let $\mathfrak{m}$ be a compact (finite dimensional) Lie algebra.*

a) *Let $\mathfrak{a}$ be some ideal in $\mathfrak{m}$. Then its orthogonal space $\mathfrak{a}^{\perp}$ is an ideal in $\mathfrak{m}$ as well, and $\mathfrak{m} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$.*

b) *Any abelian ideal $\mathfrak{a}$ of $\mathfrak{m}$ is contained in the centre $\mathfrak{z}(\mathfrak{m})$ of $\mathfrak{m}$.*

c) *The ideal relation is transitive for compact Lie algebras.*

Proof. [26, 6.4] *ad (a).* The fact that $\langle \cdot \mid \cdot \rangle$ is invariant ensures that $\mathfrak{a}^{\perp}$ is an ideal in $\mathfrak{m}$; its positive definiteness causes $\mathfrak{a}$ and $\mathfrak{a}^{\perp}$ to be disjoint. *ad (b).* We write $\mathfrak{m}$ as $\mathfrak{m} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$ and note that $[\mathfrak{a}, \mathfrak{a}^{\perp}] \leq \mathfrak{a} \cap \mathfrak{a}^{\perp} = \mathbf{0}$, which induces the commutator of $\mathfrak{a}$ and $\mathfrak{m}$ to be $[\mathfrak{a}, \mathfrak{m}] = [\mathfrak{a}, \mathfrak{a} \oplus \mathfrak{a}^{\perp}] = [\mathfrak{a}, \mathfrak{a}] \oplus [\mathfrak{a}, \mathfrak{a}^{\perp}] = \mathbf{0} \oplus \mathbf{0} = \mathbf{0}$, due to the above and the fact that $\mathfrak{a}$ is abelian. *ad (c).* Let $\mathfrak{k} \trianglelefteq \mathfrak{i} \trianglelefteq \mathfrak{m}$. Then $[\mathfrak{k}, \mathfrak{i}^{\perp}] \subseteq [\mathfrak{i}, \mathfrak{i}^{\perp}] = \mathbf{0}$. We know from part (a) that $\mathfrak{i} \oplus \mathfrak{i}^{\perp} = \mathfrak{m}$. As a consequence, $[\mathfrak{k}, \mathfrak{m}] = [\mathfrak{k}, \mathfrak{i} \oplus \mathfrak{i}^{\perp}] \subseteq [\mathfrak{k}, \mathfrak{i}] + [\mathfrak{k}, \mathfrak{i}^{\perp}] \subseteq \mathfrak{k} + \mathbf{0} = \mathfrak{k}$, which means that $\mathfrak{k}$ is an ideal in $\mathfrak{m}$. $\square$

As a consequence of this decomposition to our situation, we get

$$\mathfrak{m} = \mathfrak{m}' \oplus \mathfrak{k}$$

in particular. Another important fact is due to the prominence of the commutator algebra $\mathfrak{m}'$.

5.1.12 Corollary. $\mathfrak{k} = \mathfrak{z}(\mathfrak{m})$

Proof. *ad "$\subseteq$".* Of course, $[\mathfrak{k}, \mathfrak{m}]$ is contained in the commutator subalgebra $\mathfrak{m}'$, as well as in $\mathfrak{k}$, because $\mathfrak{k}$ is an ideal in $\mathfrak{m}$. Thus $[\mathfrak{k}, \mathfrak{m}] \leq \mathfrak{k} \cap \mathfrak{m} = \mathbf{0}$, and hence $[\mathfrak{k}, \mathfrak{m}] = \mathbf{0}$. This just states that $\mathfrak{k}$ is contained in the centre $\mathfrak{z}(\mathfrak{m})$ of $\mathfrak{m}$.

ad "$\supseteq$". Let Z be an element of the centre $\mathfrak{z}(\mathfrak{m})$. Due to invariance of the scalar product, this implies that for any $X, Y \in \mathfrak{m}$ we get $0 = \langle [X, Y] \mid Z \rangle = \langle X \mid [Y, Z] \rangle$. Any element $W \in \mathfrak{m}'$ is a linear combination of such elements $[Y, Z]$, such that $\langle X, W \rangle = 0$, too. But this just means that $X \in (\mathfrak{m}')^{\perp} = \mathfrak{k}$. $\square$

The fact that $\mathfrak{k}$ and $\mathfrak{m}$ commute directly implies the exponential law :

5.1.13 Lemma. $\forall X \in \mathfrak{k}\, \forall Y \in \mathfrak{m}. \quad (X+Y)^{\exp} = X^{\exp} \cdot Y^{\exp}$

More is known on the nature of the commutator group of a compact connected Lie group.

5.1.14 Proposition. *Every (finite dimensional) compact Lie algebra has a semisimple commutator algebra.*

Proof. [26, 6.4 vi] Remember from 5.1.12 that the compact Lie algebra $\mathfrak{m}$ is the direct sum $\mathfrak{m} = \mathfrak{m}' \oplus \mathfrak{z}(\mathfrak{m})$ of its commutator algebra and its centre. We can invoke part (a) of 5.1.11 in order to decompose the finite dimensional Lie algebra $\mathfrak{m}$ into a direct sum $\mathfrak{m} = \mathfrak{a}_1 \oplus \ldots \oplus \mathfrak{a}_k$ of ideals $\mathfrak{a}_j$ of $\mathfrak{m}$, where each $\mathfrak{a}_j$ has no non-trivial proper ideals. Note that the $\mathfrak{a}_j$ are ideals in $\mathfrak{m}$, indeed, as being an ideal is transitive for finite dimensional compact Lie algebras, by part (c) of 5.1.11.

Considering the 1-dimensional ideals $\mathfrak{a}_j$, we see that they have to be abelian and thus, again by 5.1.11, contained in the centre $\mathfrak{z}(\mathfrak{m})$. Now consider some ideal $\mathfrak{a}_j$ with $\dim \mathfrak{a}_j > 1$. It is forcedly simple. Thus, $\mathfrak{a}_j'$ is either trivial or $\mathfrak{a}_j$ itself. It cannot be trivial, as $\mathfrak{a}_j$ is non-abelian. Consequently, $\mathfrak{a}_j' = \mathfrak{a}_j$, and in particular $\mathfrak{a}_j \leq \mathfrak{m}'$. Denote by $\mathfrak{c}$ the direct sum of all 1-dimensional ideals $\mathfrak{a}_j$ and by $\mathfrak{s}$ the direct sum of all simple ideals $\mathfrak{a}_j$. Then the above argument proves that $\mathfrak{c}$ is contained in the centre $\mathfrak{z}(\mathfrak{m})$ and $\mathfrak{s}$ is contained in the commutator algebra $\mathfrak{m}'$. But then $\mathfrak{m} = \bigoplus \mathfrak{a}_j = \mathfrak{c} \oplus \mathfrak{s} \leq \mathfrak{z}(\mathfrak{m}) \oplus \mathfrak{m}' = \mathfrak{m}$, which implies $\mathfrak{c} = \mathfrak{z}(\mathfrak{m})$ as well as $\mathfrak{s} = \mathfrak{m}'$. In particular, $\mathfrak{m}'$ is semisimple. □

5.1.15 Corollary. *The commutator subgroup* M' *of* M *is compact and connected.*

Proof. The commutator subgroup of any connected Lie group is connected. By 5.1.14 and 5.1.8(a), M' is semisimple, and being a subalgebra of the compact Lie algebra $\mathfrak{m}$, its Lie algebra $\ell(\mathsf{M}') = \mathfrak{m}'$ is compact. Therefore, Weyl's theorem 5.1.10 yields compactness of M' if endowed with its Lie topology gained from $\mathfrak{m}'$. The identity map from M' endowed with this Lie topology into M' endowed with the topology induced by M is a continuous bijective group morphism, mapping a compact space onto a Hausdorff space. Hence it is open, the two topologies coincide, and compactness of M' with respect to the topology induced by M is established. □

5.1.2. The centre of a compact connected Lie group

Consider the exponential image

$$\mathsf{K} := \mathfrak{k}^{\exp} = \mathfrak{z}(\mathfrak{m})^{\exp}$$

of $\mathfrak{k}$. First of all note that it already is a group in its own right.

5.1.16 Lemma.

a) $\exp : \mathfrak{k} \longrightarrow \mathsf{M}$ *is a continuous group morphism.*

b) $\mathsf{K} = \mathfrak{k}^{\exp} \leq \mathsf{M}$

c) K *is abelian.*

d) K *is connected.*

e) $\exp|_{\mathfrak{k}} : \mathfrak{k} \longrightarrow \mathsf{Z}(\mathsf{M})^1$ *is surjective, i.e.,* $\mathsf{K} = \mathfrak{z}(\mathfrak{k})^{\exp} = \mathsf{Z}(\mathsf{M})^1$.

Proof. *ad (a) and (b).* Let $X, Y \in \mathfrak{k}$. Then the exponential rule 5.1.13 states that $(X+Y)^{\exp} = X^{\exp} \cdot Y^{\exp}$. Furthermore, for any $X \in \mathfrak{k}$ we have $(X^{\exp})^{-1} = (-X)^{\exp}$. The exponential mapping has thus been exposed as a group morphism. Consequently, the image K of $\mathfrak{k}$, is a subgroup of M. Moreover, the restriction $\exp|_{\mathfrak{k}}$ of the continuous function $\exp : \mathfrak{m} \longrightarrow \mathsf{M}$ is still continuous. *Assertion (c)* is an immediate consequence of 5.1.12. *Assertion (d)* is due to the fact that $\mathsf{K} = \mathfrak{k}^{\exp}$ is a continuous homomorphic image of the connected set $\mathfrak{k}$. Finally, the inclusion $\mathsf{K} \leq \mathsf{Z}(\mathsf{M})^1$ in *part (e)* follows from the exponential law 5.1.13, along with parts (c) and (d). As to a proof of the inverse inclusion, the reader is referred to appendix A.3.8. $\square$

5.1.17 Corollary. $\exp : \mathfrak{k} \longrightarrow \mathsf{Z}(\mathsf{M})^1$ *is a quotient morphism.*

Proof. It suffices to show that exp is continuous and open, since any continuous open surjection $\varphi : X \longrightarrow Y$ is a quotient map. (In fact, any open subset T of Y has an open preimage $T^{\leftarrow} \subseteq X$, due to continuity of φ. Conversely, any set $T \subseteq Y$ with open $T^{\varphi^{\leftarrow}}$ satisfies $T = T^{\varphi^{\leftarrow}\varphi}$ due to surjectivity. But then T is open in Y by openness of φ.)

Continuity is stated in 5.1.16. *Openness* follows from the Open Mapping Theorem A.2.3 : the additive group $\mathfrak{k}$, being locally compact and connected, qualifies as a locally compact σ-compact group [73, 5.18], and $\mathsf{Z}(\mathsf{M})^1$ is a locally compact Hausdorff space. As moreover exp has just been proved to be a continuous group epimorphism, A.2.3 establishes that exp is open. $\square$

5.1.18 Lemma.

Tracing a homomorphism theorem for topological groups, one can verify (in order of appearance), that ι has to be continuous and open, in other words a homeomorphism, and moreover an isomorphism of topological groups.

$$\begin{tikzcd} & \mathfrak{k} \arrow[dl, "\pi"'] \arrow[dr, "\exp"] & \\ \mathfrak{k}/\ker\exp \arrow[rr, "\iota"'] & & \mathsf{Z}(\mathsf{M})^1 \end{tikzcd}$$

Proof. ι is continuous because exp is continuous and π is open. It is open because π is continuous and exp is open [5.1.17]. $\square$

Note that, considered as a topological group, $(\mathfrak{k}, +)$ is isomorphic to $(\mathbb{R}^n, +)$ for some natural number $n \in \mathbb{N}$. Our aim is to understand the nature of the connected component $\mathsf{Z}(\mathsf{M})^1$. To this end, let us further investigate the structure of $\ker \exp$ in $\mathbb{R}^n$.

5.1.19 Lemma. $\ker \exp$ *is a discrete subgroup of* $\mathbb{R}^n$*.*

Proof. As we are dealing with topological groups, it suffices to prove that $\mathbf{0}$ is open in $\ker \exp$. Let us first of all note that $\mathbf{0}$ is contained in the kernel, since $\mathbf{0}^{\exp} = \mathbb{1}$. Now all we need is a neighbourhood U of $\mathbf{0}$ in $\mathbb{R}^n$ such that $U \cap \ker \exp = \mathbf{0}$. But the exponential function $\exp : \mathfrak{k} \to \mathsf{M}$ is a local homeomorphism; that is to say, there is a neighbourhood U of $\mathbf{0}$ in $\mathfrak{k} \cong \mathbb{R}^n$ such that $\exp|_U$ is a homeomorphism. Then $U \cap \ker \exp = \mathbf{0}$, due to injectivity. □

5.1.20 Proposition. *Let D be a discrete and closed subgroup of $\mathbb{R}^n$. Then there is a linearly independent subset Z of $\mathbb{R}^n$ with the property that $D = \sum_{b \in Z} \mathbb{Z}b$. In particular, there is some $k \leq n$ such that $D \cong \mathbb{Z}^k$.*

Proof. [73, 19.4b]. □

5.1.21 Lemma. *For any $k \leq n$, $\mathbb{R}^n/\mathbb{Z}^k$ is compact if and only if $k = n$.*

Proof. A torus $\mathbb{T} \approx \mathbb{S}_1$ is compact, and by Tychonov's theorem, so is $\mathbb{T}^l$ for any $l \in \mathbb{N}$. Conversely, there is an isomorphism of topological spaces between $\mathbb{R}^n/\mathbb{Z}^k$ and $(\mathbb{R}/\mathbb{Z})^k \times \mathbb{R}^{n-k} \cong \mathbb{T}^k \times \mathbb{R}^{n-k}$. As $\mathbb{R}^l$ is non-compact for $l \geq 1$, $\mathbb{T}^k \times \mathbb{R}^{k-n}$ can only be compact if $\mathbb{R}^{n-k}$ entirely vanishes, that is, if $n = k$. □

5.1.22 Corollary. $\mathsf{Z}(\mathsf{M})^1 \cong \mathbb{R}^n/\mathbb{Z}^n \cong \mathbb{T}^n$

Proof. We have found out that the topological group $\mathsf{Z}(\mathsf{M})^1$ is isomorphic to $\mathfrak{k}/\ker \exp$, and the kernel of $\exp$ is a discrete and closed subgroup of $\mathfrak{k} \cong \mathbb{R}^n$ [5.1.18 and 5.1.19]. Hence by 5.1.20, without loss of generality, there is a basis $X_1, \ldots, X_n$ of $\mathbb{R}^n$ with the property that $\ker \exp \cong \mathbb{Z}X_1 + \ldots + \mathbb{Z}X_k$. Therefore, $\mathsf{Z}(\mathsf{M})^1 \cong \mathfrak{k}/\ker \exp \cong \mathbb{R}^n/\mathbb{Z}^k$. As by 5.1.8 the connected component of the centre is compact, lemma 5.1.21 forces $k = n$ and thus the assertion. □

5.1.3. Simply connected compact Lie groups

So far it has been sufficient to ask the Lie group M for compactness and connectedness. In the sequel the consequences of further hypotheses on the first and eventually the third homotopy groups will be studied. Recall that that the fundamental group of products of tori $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$ is $\pi_1(\mathbb{T}^n) = \mathbb{Z}^n$; cf. appendix A.1.6.

5.1.23 Lemma. $\mathsf{Z}(\mathsf{M})^1 \times \mathsf{M}'$ *is a covering space of* M*.*

Proof. Abbreviate $\mathsf{Z} := \mathrm{Z}(\mathsf{M})^1$. Consider the group morphism $p : \mathsf{Z}\times\mathsf{M}' \to \mathsf{M} : (a,b) \mapsto ab$. For topological groups it suffices to show that p is a continuous, open surjection and that its kernel is discrete; cf. [54, 94.2]. First of all, p is a group morphism indeed, because Z is contained in the centre of M. *Surjectivity.* Let $g \in \mathsf{M}$. By 5.1.8(c), the exponential function is surjective, and g is the exponential image of some element $X \in \mathfrak{m}$. By 5.1.12, there are (unique) elements $A \in \mathfrak{z}(\mathfrak{m})$ and $B \in \mathfrak{m}'$ such that $A + B = X \in \mathfrak{m} = \mathfrak{z}(\mathfrak{m}) \oplus \mathfrak{m}'$. Then $g = X^{\exp} = (A+B)^{\exp} = A^{\exp}\cdot B^{\exp} \in (\mathsf{Z}\times\mathsf{M}')^p$ by the exponential law 5.1.13, 5.1.16 and 5.1.8(a). *Openness.* Being a continuous surjection between a compact space and a Hausdorff space, p is a closed map. Thus it is a quotient map; cf. [73, 1.33(b)]. Quotient maps between topological groups are open; cf. [73, 5.2(a)]. *The kernel* of p is $\ker p = \{(a, a^{-1}) \mid a \in \mathsf{Z}\cap\mathsf{M}'\}$, and $\mathsf{Z}\cap\mathsf{M}'$ is discrete. In fact, the exponential map is a local isomorphism, and hence there are neighbourhoods U of 0 in $\mathfrak{m}$ and V of 1 in M such that $U \cong V$, and $\mathbb{1} = (U \cap (\mathfrak{z}(\mathfrak{m}) \cap \mathfrak{m}'))^{\exp} = V \cap (\mathsf{Z}\cap\mathsf{M}')$. For topological groups, the existence of such a neighbourhood V just means that $\mathsf{Z}\cap\mathsf{M}'$ is discrete. □

5.1.24 Lemma. *If the compact connected Lie group M is moreover simply connected then $Z(\mathsf{M})^1 = \mathbb{1}$ and $\mathsf{M}' = \mathsf{M}$ is simply connected.*

Proof. By A.1.6, 5.1.23 and 5.1.22 the fundamental group $\pi_1(\mathsf{M}) \geq \pi_1(\mathrm{Z}(\mathsf{M})^1 \times \mathsf{M}') = \pi_1(\mathrm{Z}(\mathsf{M})^1) \times \pi_1(\mathsf{M}') = \mathbb{Z}^n \times \pi_1(\mathsf{M}')$ can only be trivial if both, $\mathbb{Z}^n$ as well as $\pi_1(\mathsf{M}')$, are trivial, in other words, if $\mathbb{T}^n \cong \mathrm{Z}(\mathsf{M})^1 \cong \mathbb{1}$ and $\mathsf{M}' = \mathsf{M}$ is simply connected. □

5.1.25 Corollary. $\mathsf{M} = \mathsf{M}'$

Proof. $\mathsf{M} = \mathfrak{m}^{\exp} = (\mathfrak{k}\oplus\mathfrak{m}')^{\exp} = \mathfrak{k}^{\exp}\cdot(\mathfrak{m}')^{\exp} \leq \mathrm{Z}(\mathsf{M})^1\cdot\mathsf{M}' \leq \mathsf{M}$. The first equation holds because of surjectivity of the exponential function, by part (c) of 5.1.8. The third equation holds because $\mathfrak{k} = \mathfrak{z}(\mathfrak{m})$ and $\mathfrak{m}'$ commute. Moreover, $\mathfrak{m}'^{\exp}$ is contained in the commutator group M', by part (a) of 5.1.8. But then, necessarily equality holds everywhere, and $\mathsf{M} = \mathrm{Z}(\mathsf{M})^1\cdot\mathsf{M}' = \mathsf{M}'$, in particular. □

5.1.26 Corollary. M *is semisimple.*

Proof. $\mathsf{M} = \mathsf{M}'$ is a compact connected Lie group and thus semisimple by 5.1.14. □

Let us briefly summarise the theory so far :

5.1.27 Proposition. *For any compact connected Lie group M the following holds :*

a) *The corresponding Lie algebra $\mathfrak{m} := \ell\,\mathsf{M}$ is a compact Lie algebra, and it can be decomposed as $\mathfrak{m} = \mathfrak{m}'\oplus\mathfrak{z}(\mathfrak{m})$.*

b) M' *is semisimple.*

c) $\mathfrak{z}(\mathfrak{m})^{\exp} = \mathrm{Z}(\mathsf{M})^1 \cong \mathbb{T}^n$ *for some natural number n.*

d) *If $\pi_1(\mathsf{M}) = \mathbb{1}$, then $\mathrm{Z}(\mathsf{M})^1$ is trivial and $\mathsf{M} = \mathsf{M}'$ is semisimple.* □

Now there are two possible ways how to proceed. If our aim is an immediate statement on our particular problem, then we have gathered all we need. Indeed, we remember from 3.4.3 that $\mathfrak{g} = \mathbb{R} \propto \mathfrak{hei}_3\mathbb{R}$ is a soluble Lie algebra. Thus, its ideal $\mathfrak{m}$ has to be semisimple and soluble at the same time, which forces it to be $\mathfrak{m} = \mathbf{0}$. Hence our maximal compact subgroup M of $\Gamma = \mathbb{R} \ltimes \mathrm{Hei}_3\mathbb{R}$ is trivial.

5.1.28 Corollary. *If* M *is a soluble compact connected Lie group whose fundamental group is trivial, then* $\mathsf{M} = \mathsf{1}$.

Yet, there is an interesting general way to proceed whenever we are not in a position to say that our simply connected compact Lie group M is soluble but if we do know that its third homotopy group is trivial. To that end the semisimple Lie algebra $\mathfrak{m}$ is written as a sum $\mathfrak{m} = \oplus_{k=1}^{r} \mathfrak{e}_k$ of simple Lie subalgebras. For $k \leq r$ consider the simply connected Lie groups E_k satisfying $\ell\,\mathsf{E}_k = \mathfrak{e}_k$. These E_k turn out to be compact, connected, simply connected, almost simple Lie groups. In fact, compactness follows from Weyl's theorem 5.1.10. Moreover, $\mathsf{E}_1 \times \ldots \times \mathsf{E}_r$ is the simply connected universal covering group of M. The essential key is given by a result from Bott [9], which is actually at the bottom of Toda's theorem 5.1.7 dismissed earlier on, and which does not make use of the classification of simple Lie groups.

5.1.29 Theorem (Bott 1954). *Let* Υ *be a compact, simply connected, almost simple Lie group. Then its third homotopy group is* $\pi_3(\Upsilon) = \mathbb{Z}$.

This theorem guarantees that for all $k \leq r$ the homotopy groups $\pi_3(\mathsf{E}_k)$ are infinite and cyclic. If we assume $\pi_3(\mathsf{M})$ to be trivial, then $\mathsf{1} = \pi_3(\mathsf{M}) = \pi_3(\mathsf{E}_1 \times \ldots \times \mathsf{E}_r) = \pi_3(\mathsf{E}_1) \times \ldots \times \pi_3(\mathsf{E}_r) = \mathbb{Z}^r$, which is only possible for $r = 0$. This implies $\mathfrak{m} = \mathbf{0}$ and thus $\mathsf{M} = \mathsf{1}$. Summarising, the following has been proved :

5.1.30 Proposition. *Let* M *be a compact connected Lie group satisfying* $\pi_1(\mathsf{M}) = \mathsf{1}$ *and* $\pi_3(\mathsf{M}) = \mathsf{1}$. *Then* $\mathsf{M} = \mathsf{1}$.

By 5.1.6, any maximal compact subgroup M of $\Gamma = \mathbb{R} \ltimes \mathrm{Hei}_3\mathbb{R}$ satisfies these conditions. Hence this way one could also have established that M is trivial.

Either way, all this culminates in the insight that our particular Γ is compact-free :

5.1.31 Corollary. $\Gamma = \mathbb{R} \ltimes \mathrm{Hei}_3\mathbb{R}$ *is compact-free.*

Proof. Let X be some compact subgroup of Γ. Then X is contained in some maximal compact subgroup M of Γ, and we have just proved that $\mathsf{M} = \mathsf{1}$. Hence, also $\mathsf{X} = \mathsf{1}$. $\square$

5.2. Some groups the automorphism group does not contain

Consequences of the non-embeddability results from chapter 3 on the full automorphism group Σ of a Peter plane $\mathcal{P}$ will be studied. We will begin by proving that certain prominent groups, $\mathrm{SO}_3\mathbb{R}$ and $\mathrm{SU}_2\mathbb{C}$, do *not* qualify as automorphism groups of $\mathcal{P}$. The arguments ruling out these two candidates as subgroups of Σ are widely parallel, which is not too surprising as $\mathrm{SU}_2\mathbb{C}$ is the two-fold covering group of $\mathrm{SO}_3\mathbb{R}$. For that reason, we will start out with the *overall assumption* of some (topological) subgroup $\Phi \leq \Sigma$ which is assumed to be isomorphic to $\mathrm{SO}_3\mathbb{R}$ or $\mathrm{SU}_2\mathbb{C}$, alternatively. We will make an explicit choice as soon as it is necessary. As a byproduct of these considerations it will turn out that $\mathcal{P}$ cannot be LÖWEN's $\mathrm{SL}_2\mathbb{C}$-plane either, nor any of its open subplanes.

The phrases "Σ contains Φ" or "Φ is embeddable into Σ" shall refer to the existence of an embedding of topological groups, that means, a continuous injective group morphism $\varepsilon : \Phi \to \Sigma$ whose co-restriction $\varepsilon|^{\Phi^\varepsilon} : \Phi \twoheadrightarrow \Phi^\varepsilon$ is open. In our context, though, the embedded group Φ is usually compact and Σ is a Hausdorff space, with the consequence that any continuous injective group morphism $\varepsilon : \Phi \to \Sigma$ automatically is such an embedding of topological groups.

Later on, we will see that $\mathrm{SO}_3\mathbb{R}$ and $\mathrm{SU}_2\mathbb{C}$ not being automorphism groups of $\mathcal{P}$ allows for quite far reaching conclusions on the extent of solubility of Σ. In fact, it can be established that $\ell\,\Sigma$ contains at most one copy of $\mathfrak{sl}_2\mathbb{R}$ as a non-soluble factor, if any.

5.2.1 Lemma. ***Some facts on the classical groups involved***

a) $\mathrm{SU}_2\mathbb{C}$ *is the two-fold covering group of* $\mathrm{SO}_3\mathbb{R}$.

b) $\mathfrak{so}_3\mathbb{R} = \ell(\mathrm{SO}_3\mathbb{R}) = \ell(\mathrm{SU}_2\mathbb{C}) = \mathfrak{v}$, *where* $\mathfrak{v}$ *denotes the* 3*-dimensional* vector product algebra $\mathfrak{v} = (\mathbb{R}^3, +, \times)$.

c) $\mathfrak{so}_3\mathbb{R}$ does not have any 2-dimensional Lie subalgebras.

d) The Lie algebra $\mathfrak{so}_3\mathbb{R}$ is simple. In other words, $\mathrm{SO}_3\mathbb{R}$ and $\mathrm{SU}_2\mathbb{C}$ are almost simple Lie groups.

e) Neither $\mathrm{SO}_3\mathbb{R}$ nor $\mathrm{SU}_2\mathbb{C}$ contain compact subgroups of dimension 2.

Proof. *Part (b)* is a direct consequence of (a); for a proof of *part (a)*, we refer you to lemma 5.3.7. In order to verify *(c)*, assume the existence of some Lie subalgebra $\mathfrak{w} \leq \mathfrak{v}$ of dimension at least 2. Then $\mathfrak{w}$ contains two linearly independent vectors $a, b \in \mathbb{R}^3$. Their vector product $a \times b$ then is a third vector linearly independent of both, a and b, and moreover it is also contained in $\mathfrak{w}$. Consequently, $\mathfrak{w}$ is of dimension at least 3, thus equals $\mathfrak{v}$. In order to prove *part (d)*, it remains to be established that $\mathfrak{v}$ does not have any 1-dimensional ideals. This can be ruled out by very much the same argument : The Lie bracket, i.e., vector product, of any non-trivial vector in any 1-dimensional ideal and some other vector not contained in the ideal will be non-trivial and orthogonal to the plane spanned by them, thus never be contained in the ideal. As to *part (e)*, any 2-dimensional subgroup of Φ would give rise to some 2-dimensional subalgebra of $\ell\,\Phi = \mathfrak{v}$, and thus, by (c), dissolves into nonexistence. $\square$

5.2.2 Lemma. ***Some facts on*** $\Sigma = \operatorname{Aut}\mathcal{P}$

a) Σ *is a locally compact, metrizable topological group with a countable basis.*

b) Σ *is a Lie group.*

Proof. *ad (a).* Löwen [31, §2]. *ad (b).* Szenthe's theorem [54, 96.14]. — Note that, as the point space of a Peter plane is homeomorphic to $\mathbb{R}^4$, it can be obtained in a cheaper way that the connected component Σ^1 is a Lie group : see lemma 2.11 in Löwen [31], using [30]. Note moreover, that Σ is not necessarily a linear Lie group. $\square$

5.2.3 Lemma.

a) Φ *is a closed subgroup of* Σ.

b) Γ *is a closed subgroup of* Σ.

Proof. *ad (a).* Being a compact subgroup of the Hausdorff group Σ, Φ is automatically closed. *ad (b).* Γ comes endowed with the topology induced by the compact-open topology on Σ; cf. 3.3.14. Thus, Γ is a locally compact subgroup of the Hausdorff group Σ. In that case, it is forcedly closed [73, 4.8]. $\square$

5.2.4 Corollary. *If* Σ *contains some subgroup* Φ *isomorphic to either* $\mathrm{SO}_3\mathbb{R}$ *or* $\mathrm{SU}_2\mathbb{C}$ *then* $\dim\Sigma \geq 7$.

Proof. By the lemma above, $\Phi \cap \Gamma$ is a closed subgroup of the compact group Φ, thus compact itself. Here we are, looking for a compact subgroup of Γ. As we have seen in lemma 5.1.31 of the preceding section, Γ is compact-free; that is to say, there is only one choice, namely $\Phi \cap \Gamma = \mathbb{1}$. Trying to learn about the dimension of Σ becomes a lot easier once we have a look at the corresponding Lie algebras : $\dim\Sigma = \dim \ell\,\Sigma \geq \dim(\ell\,\Gamma + \ell\,\Phi) = \dim \ell\,\Gamma + \dim \ell\,\Phi - \dim(\ell\,\Phi \cap \ell\,\Gamma) = \dim\Gamma + \dim \mathrm{SO}_3\mathbb{R} - \dim(\Phi \cap \Gamma) = 4 + 3 - 0 = 7$. Thus, we have established the inequality $\dim\Sigma \geq 7$. $\square$

5.2.1. $\mathrm{SO}_3\mathbb{R}$ is not an automorphism group of $\mathcal{P}$

This is the point, where the itineraries part. Let us, for the moment, concentrate on $\Phi \cong \mathrm{SO}_3\mathbb{R}$. Here, the key is theorem 1 from Löwen [40], which states that in such a situation $\mathcal{P}$ is basically well-known :

5.2.5 Theorem (Löwen 1986). *If the automorphism group of a locally compact 4-dimensional stable plane has dimension at least 5 and contains a group $\Phi \cong \mathrm{SO}_3\mathbb{R}$, then the plane is the complex projective plane $\mathcal{P}_2\mathbb{C}$ or the open subplane obtained from $\mathcal{P}_2\mathbb{C}$ by removing the points of either a conic section ("complex oval plane") or the real projective plane $\mathcal{P}_2\mathbb{R}$.*

From our previous fundamental failure 3.7.8 of embedding any Peter plane $\mathcal{P}$ into $\mathcal{P}_2\mathbb{C}$ as an open subplane we can now deduce that the full automorphism group Σ of $\mathcal{P}$ cannot contain $\mathrm{SO}_3\mathbb{R}$.

5.2.6 Proposition. *There is no continuous group monomorphism mapping* $\mathrm{SO}_3\mathbb{R}$ *into* Σ.

5.2.2. $\mathrm{SL}_2\mathbb{C}$ is not an automorphism group of $\mathcal{P}$

As a short aside, which will turn out to be useful shortly, we can safely say that $\mathcal{P}$ does not allow an action of the (non-compact) group $\mathrm{SL}_2\mathbb{C}$. To that end, denote by $\mathcal{S}_\mathbb{C}$ the $\mathrm{SL}_2\mathbb{C}$-plane as introduced by Löwen in [41]. Note that $\mathcal{S}_\mathbb{C}$ is the unique stable plane with point set $\mathbb{R}^4$ and connected lines apart from $\mathcal{A}_2\mathbb{C}$ which allows an action of $\mathrm{SL}_2\mathbb{C}$. Its automorphism group is the extension of $\mathrm{SL}_2\mathbb{C}$ by a certain collineation, namely $\tau : \mathbb{C}^2 \to \mathbb{C}^2 : (u, v) \mapsto -i(\overline{u}, \overline{v})$, and thus 6-dimensional.

5.2.7 Proposition. *$\mathcal{P}$ is neither isomorphic to $\mathcal{S}_\mathbb{C}$ nor to $\mathcal{S}_\mathbb{C} \setminus \{0\}$. There is no embedding of topological groups mapping* $\mathrm{SL}_2\mathbb{C}$ *into* Σ.

Proof. As the automorphism group of the punctured $\mathrm{SL}_2\mathbb{C}$-plane seems not to be known to literature as yet, a separate study of the punctured and non-punctured planes will be called for : First of all, $\mathcal{P}$ cannot be the punctured plane $\mathcal{S}_\mathbb{C} \setminus \{0\}$ because the simply connected set $\mathbb{R}^4$ and $\mathbb{C}^2 \setminus \{0\}$ are clearly not homeomorphic. As to $\mathcal{P}$ not being isomorphic to $\mathcal{S}_\mathbb{C}$, note that a second proof will emerge as a corollary from Bickel's theorem 5.2.10. We will give *Version A* here : Assume $\mathcal{P} = \mathcal{S}_\mathbb{C}$. This means that $\mathcal{P}$ admits an action of $\mathrm{SL}_2\mathbb{C}$ and thus in particular an action of its (maximal compact) subgroup $\mathrm{SU}_2\mathbb{C}$. But then corollary 5.2.4 implies $\dim \Sigma \geq 7$, which contradicts the fact that $\dim \Sigma = \dim \mathrm{Aut}\, \mathcal{S}_\mathbb{C} = 6$. Hence, $\mathcal{P}$ is not the $\mathrm{SL}_2\mathbb{C}$-plane $\mathcal{S}_\mathbb{C}$, either. □

5.2.3. $\mathrm{SU}_2\mathbb{C}$ is not an automorphism group of $\mathcal{P}$

Let us again take up the thread at the point where we knew that Σ is of dimension at least 7 if it contains some subgroup Φ isomorphic to $\mathrm{SO}_3\mathbb{R}$ or $\mathrm{SU}_2\mathbb{C}$, i.e., corollary 5.2.4. This time dealing with $\Phi \cong \mathrm{SU}_2\mathbb{C}$, we will exploit a result on 4-dimensional stable planes that allow $\mathrm{SU}_2\mathbb{C}$ for an automorphism group [41, Thm. 5] :

5.2.8 Theorem (Löwen 1986). *Let $\mathcal{P}$ be a locally compact stable plane of dimension 4. Let moreover Δ be a connected Lie group that is a subgroup of $\mathrm{Aut}\,\mathcal{P}$, satisfying*

$\dim \Delta \geq 6$. *If Δ contains a subgroup $\Phi \cong \mathrm{SU}_2\mathbb{C}$, then either $\mathcal{P}$ belongs to a certain class of open subplanes of $\mathcal{P}_2\mathbb{C}$ or $\mathcal{P} \in \{\mathcal{S}_\mathbb{C}, \mathcal{S}_\mathbb{C} \setminus \{0\}\}$.*

Again, the non-embeddability result in 3.7.8 forbids any open subplane of the complex projective plane. The troublemakers are, of course, the $\mathrm{SL}_2\mathbb{C}$-plane and its punctured open subplane. But these two planes have been excluded as candidates in lemma 5.2.7. Hence, the overall assumption that $\mathcal{P}$ suffers an action of the group $\mathrm{SU}_2\mathbb{C}$ must necessarily be wrong : the full automorphism group Σ of $\mathcal{P}$ does not contain $\mathrm{SU}_2\mathbb{C}$.

5.2.9 Proposition. *There is no continuous group monomorphism mapping* $\mathrm{SU}_2\mathbb{C}$ *into* Σ.

5.2.4. Application of a result by Bickel

Although not ultimately necessary at that stage, we may pay attention to quite strong a result by Bickel [5, Korollar 4.3.2], which establishes the critical dimension c_4 of 4-dimensional stable (lp-) planes as being $c_4 = 8$. Here, the **critical dimension** is the natural number c_4 with the property that there are non-classical examples of 4-dimensional stable planes whose automorphism group is of dimension c_4, but any such $\mathcal{P}$ with $\dim \mathrm{Aut}\,\mathcal{P} > c_4$ has to be classical. As a matter of fact, Betten's translation planes $\mathcal{T}_k$ in 3.8 are non-classical stable planes with an 8-dimensional automorphism group; cf. [54, 73.13].

5.2.10 Theorem (Bickel 1995). *Any 4-dimensional stable plane with an automorphism group of dimension at least 9 is isomorphic to one of the seven planes $\mathcal{P}_2\mathbb{C}, \mathcal{A}_2\mathbb{C}, \mathcal{P}_2\mathbb{C} \setminus \{\infty\}, \mathrm{C}(\mathbb{C}), \mathrm{UC}(\mathbb{C}), \mathrm{DC}(\mathbb{C}), \mathrm{DUC}(\mathbb{C})$, i.e., the complex projective or affine planes, the cylinder plane, the united cylinder plane or their respective duals described in §3.6 of [5].*

This theorem gives rise to *version B* of the proof of proposition 5.2.7, proceeding as follows :

5.2.11 Lemma. *If any Peter plane $\mathcal{P}$ were the $\mathrm{SL}_2\mathbb{C}$-plane $\mathcal{S}_\mathbb{C}$ then the full automorphism group Σ were of dimension at least* 10.

Proof. Assume $\mathcal{P} = \mathcal{S}_\mathbb{C}$. Then $\mathrm{SL}_2\mathbb{C}$ acts on $\mathcal{P}$, and its action on the point space $P = \Gamma \approx \mathbb{C}^2$ is the standard action on $\mathbb{C}^2$ (see [41]). Thus, it fixes the origin o. In other words, $\Sigma_o \geq \Psi \cong \mathrm{SL}_2\mathbb{C}$. Moreover, $\Psi \cap \Gamma \leq \Sigma_o \cap \Gamma \leq \Gamma_o = \mathbb{1}$, because Γ acts sharply transitive on $P = \Gamma$. Therefore, $\Psi \cap \Gamma = \mathbb{1}$. But then we are in a position to say that $\dim \Sigma = \dim \ell\,\Sigma \geq \dim(\ell\,\Psi + \ell\,\Gamma) = \dim \ell\,\Psi + \dim \ell\,\Gamma - \dim(\ell\,\Psi \cap \ell\,\Gamma) = 6 + 4 - 0 = 10$, since $\ell\,\Psi \cap \ell\,\Gamma = \mathbf{0}$. □

All these seven planes above are by construction open subplanes of the complex projective plane $\mathcal{P}_2\mathbb{C}$. Bickel's result hence yields that if the arbitrary Peter plane $\mathcal{P}$ were isomorphic to $\mathcal{S}_\mathbb{C}$ it were an open subplane of $\mathcal{P}_2\mathbb{C}$, which by 3.7.8 it is not; hence none of the Peter planes is isomorphic to the $\mathrm{SL}_2\mathbb{C}$-plane. Consequently, the above is an alternative proof of 5.2.7 : *The Peter plane $\mathcal{P}$ is neither the $\mathrm{SL}_2\mathbb{C}$-plane nor contained therein as an open $\mathrm{SL}_2\mathbb{C}$-invariant subplane. Equivalently, Σ does not contain $\mathrm{SL}_2\mathbb{C}$.*

Apart from that, but by the same argument, Bickel's theorem gives us an upper bound on the dimension of Σ :

5.2.12 Corollary. $4 \le \dim \Sigma \le 8$

Proof. As the 4-dimensional group Γ is a subgroup of Σ, the lower bound is immediate. As to the upper bound, assume that $\dim \Sigma \ge 9$. Then once more, $\mathcal{P}$ would have to be one of the planes mentioned in Bickel's classification, and they are without exception open subplanes of $\mathcal{P}_2\mathbb{C}$. But this contradicts theorem 3.7.8. □

5.3. How soluble is the automorphism group ?

5.3.1. Zoological considerations concerning $\mathfrak{so}_3\mathbb{R}$

5.3.1 Unitary Lie algebras. Let $V = \mathbb{F}^n$ be a right vector space over some skew field $\mathbb{F}$ of characteristic $\operatorname{char}\mathbb{F} \ne 2$ and α an involutional antiautomorphism of $\mathbb{F}$. Let $h : V \times V \to \mathbb{F}$ be some non-degenerate, ε-α-hermitian linear form on V, where $\varepsilon \in \{1, -1\}$. Recall from chapter 2 that h can be described as $h(v,w) = vHw^*$ for some matrix $H \in \mathrm{Mat}_n\,\mathbb{F}$, where $w^* := (w^\alpha)^\mathsf{T}$. Then the **unitary group** of V with respect to h is

$$\begin{aligned} \mathrm{U}(V,h) &:= \{\varphi \in \mathrm{GL}(V) \mid \forall v,w \in V \,.\, h(v^\varphi, w^\varphi) = h(v,w)\} \\ &= \{A \in \mathrm{GL}_n\mathbb{F} \mid AHA^* = H\} \quad . \end{aligned}$$

Its Lie algebra, called the **unitary Lie algebra**, can be identified as

$$\mathfrak{u}(V,h) := \ell(\mathrm{U}(V,h)) = \left\{X \in \mathrm{Mat}_n\,\mathbb{F} \mid HX^* = -XH\right\} \quad .$$

We talk about **orthogonality** whenever $\mathbb{F}$ is a (commutative) field of characteristic $\operatorname{char}\mathbb{F} \ne 2$ and h is symmetric, i.e., $\varepsilon = 1$ and $\alpha = \mathrm{id}_\mathbb{F}$. Quite prominent examples are

$\mathfrak{o}_n\mathbb{F} := \{X \in \mathrm{Mat}_n\,\mathbb{F} \mid X^\mathsf{T} = -X\}$	for $H = \mathbb{1}$, $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$, $\varepsilon = 1$ and $\alpha = \mathrm{id}_\mathbb{F}$
$\mathfrak{u}_n\mathbb{C} := \{X \in \mathrm{Mat}_n\,\mathbb{C} \mid \overline{X}^\mathsf{T} = -X\}$	for $H = \mathbb{1}$, $\mathbb{F} = \mathbb{C}$, $\varepsilon = 1$ and complex conjugation
$\mathfrak{sp}_{2n}\mathbb{C} := \{X \in \mathrm{Mat}_{2n}\,\mathbb{C} \mid HX^* = -XH\}$	for $\mathbb{F} = \mathbb{C}$, $\varepsilon = -1$, complex conjugation and $H = \left(\begin{array}{c\|c} & \mathbb{1}_n \\ \hline -\mathbb{1}_n & \end{array}\right)$.

Note that all the elements in $\mathfrak{o}_n\mathbb{F}$ must have zero diagonal elements, which implies that $\mathfrak{o}_n\mathbb{F} = \mathfrak{so}_n\mathbb{F}$. As to other complex orthogonal Lie algebras, they are all isomorphic : for every symmetric non-degenerate bilinear form h on $\mathbb{C}^n$, we get

$$\mathfrak{o}_n(\mathbb{C}, h) \cong \mathfrak{o}_n(\mathbb{C}, \mathbb{1}) \quad .$$

For a survey on sesquilinear forms, see [11, §5ff].

5.3.2 Quaternions. Folklore — as exposed in Mäurer [46] — has it that Hamilton's quaternions can be considered as

$$\mathbb{H} = \mathbb{R} \cdot \mathrm{SU}_2\mathbb{C} = \left\{ \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix} \middle| \; u, v \in \mathbb{C} \right\} \quad .$$

Complex conjugation can be extended from $\mathbb{C} \leq \mathbb{H}$ to

$$\overline{h} := \begin{pmatrix} \overline{u} & -v \\ \overline{v} & u \end{pmatrix}$$

for $h \in \mathbb{H}$. The quadratic form $q : \mathbb{H} \to \mathbb{H} : h \mapsto h\overline{h} = (u\overline{u} + v\overline{v}) \cdot \mathbb{1}$ induces a positive definite symmetric bilinear form $\beta : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ by polarisation $\beta(h, g) := \frac{1}{2}(q(h+g) - q(h) - q(g))$. There is an isomorphism which yields the following identifications with the "usual" description $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$:

	$\mathbb{H} = \mathbb{R} \cdot \mathrm{SU}_2\mathbb{C}$	$\mathbb{H} = \mathbb{C} + \mathbb{C}j = \mathbb{R}^4$
element	$h = \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix}$	$a = u_1 + u_2 i + v_1 j + v_2 k$ $= (u_1, u_2, v_1, v_2)^\mathsf{T} \in \mathbb{R}^4$
conjugation	$\overline{h} = \begin{pmatrix} \overline{u} & -v \\ \overline{v} & u \end{pmatrix}$	$\overline{a} = u_1 - u_2 i - v_1 j - v_2 k$
quadratic form	$q(h) = h\overline{h} = (u\overline{u} + v\overline{v}) \cdot \mathbb{1}$	$q(a) = u_1^2 + u_2^2 + v_1^2 + v_2^2 = a^\mathsf{T} \mathbb{1}\, a$
bilinear form	$\beta(h, g)$	$\langle a \mid b \rangle = a^\mathsf{T} \mathbb{1}\, b$

Thus, the bilinear form is nothing but the "ordinary" scalar product on the vector space $\mathbb{R}^4 \cong \mathbb{H}$; the inner product spaces $(\mathbb{H}, \beta)$ and $(\mathbb{R}^4, \mathbb{1})$ are isomorphic. Consequently, so are their respective special orthogonal groups $\mathrm{SO}(\mathbb{H}, \beta)$ and $\mathrm{SO}_4\mathbb{R}$. We will make use of both descriptions in the sequel.

5.3.3 Actions of $\mathrm{SU}_2\mathbb{C}$. Consider the multiplicative group

$$S := \{h \in \mathbb{H} \mid |h| = 1\} = \mathrm{SU}_2\mathbb{C} \leq \mathbb{H} \quad .$$

Note that $S = \mathrm{SU}_2\mathbb{C} \approx \mathbb{S}_3$ is a simply connected compact group. For every $a \in S$, right and left multiplication

$$\begin{aligned} \rho_a &: \mathbb{H} \to \mathbb{H} &&: x \mapsto xa \\ \lambda_a &: \mathbb{H} \to \mathbb{H} &&: x \mapsto \overline{a}x \end{aligned}$$

are orthogonal $\mathbb{R}$-linear maps with respect to β. As a matter of fact, ρ_a and λ_a are even contained in the connected component $\mathrm{O}(\mathbb{H}, \beta)^1 = \mathrm{SO}(\mathbb{H}, \beta)$. This encourages treating the group morphism

$$\begin{array}{rccc} \varphi : & S \times S & \to & \mathrm{SO}(\mathbb{H}, \beta) \\ & (a, b) & \mapsto & \lambda_a \rho_b \end{array} \quad .$$

Its kernel is $\ker \varphi = \{(-1, -1), (1, 1)\} = \langle (-1, -1) \rangle$. In order to understand its surjectivity one can study actions of $S \times S$ on S :

(i) $S \times S$ acts transitively on S with stabiliser $(S \times S)_1 = \{(a,a) \mid a \in S\} \cong S$

(ii) $(S \times S)_1$ acts transitively on $1^\perp \cap S$ with stabiliser $(S \times S)_{1,i} = \{(a,a) \mid a \in S \cap \mathbb{C}\}$

Assertion (i) can be hand-crafted, whereas the proof of assertion (ii) requires some more zeal :

The stabiliser $(S \times S)_1$ acts on $\mathfrak{V} := 1^\perp = \operatorname{Pu}\mathbb{H}$. The group morphism φ can thus be restricted to $(S \times S)_1$ and maps into $\mathrm{SO}(\mathfrak{V}, \beta|_{\mathfrak{V}\times\mathfrak{V}})$:

$$\varphi|_{(S\times S)_1} : (S \times S)_1 \to \mathrm{SO}(\mathfrak{V}, \beta|_{\mathfrak{V}\times\mathfrak{V}})$$

But again, $\mathrm{SO}(\mathfrak{V}, \beta|_{\mathfrak{V}\times\mathfrak{V}}) = \mathrm{SO}_3\mathbb{R}$, such that $\varphi|_{(S\times S)_1}$ can be identified with a group morphism

$$\begin{array}{rccc} \psi : & S & \to & \mathrm{SO}_3\mathbb{R} \\ & a & \mapsto & \lambda_a \rho_a \end{array} .$$

Transitivity of $(S \times S)_1$ on $\mathfrak{V}$ would be implied by surjectivity of ψ, because $\mathrm{SO}_3\mathbb{R}$ acts transitively on the vector space $\mathbb{R}_3 = \mathfrak{V}$. As, moreover, $\mathfrak{V} \cap S$ is invariant under $S \cong (S \times S)_1$, this would account for transitivity of $(S \times S)_1$ on $\mathfrak{V} \cap S$.

5.3.4 As a matter of fact, $\psi : S \to \mathrm{SO}_3\mathbb{R}$ **is surjective :**

(I) *Every half turn with respect to an axis in $\mathbb{R}^3$ is contained in S^ψ.* For a proof, let $v \in \mathfrak{V} \setminus \mathbf{0}$. Then v is skew-symmetric, i.e., $\overline{v} = -v$. This implies $v^2 = -v\overline{v} = -(\det v) \cdot \mathbb{1}$; hence v^ψ has order 2. For $v \in \mathfrak{V} \cap S$, the linear map v^ψ pointwisely fixes $\mathbb{R}v$. Consequently, v^ψ is a half turn around $\mathbb{R}v$. Now, any axis in $\mathbb{R}^3$ can be written as $\mathbb{R}v$ for some $v \in \mathfrak{V} \cap S$, which completes the first part of the argument.

(II) Let $\delta \in \mathrm{SO}_3\mathbb{R}$. As every rotation in $\mathbb{R}^3$ is the product of two reflections, there are reflections σ_1 and σ_2 with respect to planes in $\mathbb{R}^3$ such that $\delta = \sigma_1\sigma_2$. For $\nu \in \{1,2\}$, the linear map $-\mathrm{id} \cdot \sigma_\nu = \sigma_\nu \cdot (-\mathrm{id})$ is of order 2 and has determinant $+1$, hence is a rotation by π. By (I), $\mathrm{SO}_3\mathbb{R} \ni (-\mathrm{id} \cdot \sigma_1) \cdot (-\mathrm{id} \cdot \sigma_2) = (-id)^2 \cdot \sigma_1\sigma_2 = \delta$. This proves $\mathrm{SO}_3\mathbb{R} \subseteq S^\psi$ and thus surjectivity of ψ.

5.3.5 Let $H := (S \times S)^\varphi \leq \mathrm{SO}(\mathbb{H}, \beta)$ and abbreviate $G := \mathrm{SO}(\mathbb{H}, \beta) = \mathrm{SO}_4\mathbb{R}$.

$$\begin{tikzcd}
S \times S \arrow[r, "\varphi"] & H \arrow[r, hook] & G \\
S \cong (S \times S)_1 \arrow[r] \arrow[u, hook] & H_1 \arrow[r, hook] \arrow[u, hook] & G_1 \arrow[u, hook] \\
(S \times S)_{1,i} \arrow[r] \arrow[u, hook] & H_{1,i} \arrow[r, hook] \arrow[u, hook] & G_{1,i} \arrow[u, hook]
\end{tikzcd}$$

Note, that $G_{1,i,j} = (\mathrm{SO}_4\mathbb{R})_{e_1,e_2,e_3} = \mathbb{1}$. Thus stepwise application of the Frattini-argument A.2.1 yields, from bottom to top :

(ii) by 5.3.3(ii), $H_1 \leq G_1$ acts transitively on $1^\perp \cap S \ni i$.
Thus $G_1 = H_{1,i} \cdot H_1 = H_1$.

(i) by 5.3.3(i), $H \leq G$ acts transitively on $S \ni 1$.
Thus $G = H_1 \cdot H = H$.

Assertion (i) above states that φ is surjective. Recall that $S = \mathrm{SU}_2\mathbb{C} \approx \mathbb{S}_2$ is simply connected, and hence so is $S \times S$; cf. A.1.6. As a surjective continuous morphism from the compact group $S \times S$ to the Hausdorff group $\mathrm{SO}_4\mathbb{R}$, it is a quotient map (cf. A.2.3), and having a discrete kernel finally makes it a universal covering :

5.3.6 Proposition.

a) $\varphi : \mathrm{SU}_2\mathbb{C} \times \mathrm{SU}_2\mathbb{C} \to \mathrm{SO}_4\mathbb{R}$ *is a universal covering.*

b) $\mathfrak{so}_4\mathbb{R} \cong \mathfrak{su}_2\mathbb{C} \oplus \mathfrak{su}_2\mathbb{C}$

c) $\mathfrak{so}_4\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C} \oplus \mathfrak{sl}_2\mathbb{C}$

Proof. *ad (b) :* by (a), $\mathfrak{so}_4\mathbb{R} = \ell(\mathrm{SO}_4\mathbb{R}) \cong \ell(S \times S) = \ell(\mathrm{SU}_2\mathbb{C}) \oplus \ell(\mathrm{SU}_2\mathbb{C}) = \mathfrak{su}_2\mathbb{C} \oplus \mathfrak{su}_2\mathbb{C}$. *ad (c) :* This follows from (b) by considering the respective complexifications : $\mathbb{C} \otimes \mathfrak{so}_4\mathbb{R} \cong (\mathbb{C} \otimes \mathfrak{su}_2\mathbb{C}) \oplus (\mathbb{C} \otimes \mathfrak{su}_2\mathbb{C})$, in other words, $\mathfrak{so}_4\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C} \oplus \mathfrak{sl}_2\mathbb{C}$. (For complexifications, see 5.3.13). □

Stopping to look at assertion (ii) above reveals $\psi : \mathrm{SU}_2\mathbb{C} \to \mathrm{SO}_3\mathbb{R}$ being a quotient map with discrete kernel $\ker\psi = \langle -\mathbb{1} \rangle$, thus a universal covering, too.

5.3.7 Proposition.

a) $\psi : \mathrm{SU}_2\mathbb{C} \to \mathrm{SO}_3\mathbb{R}$ *is a universal covering.*

b) $\mathrm{SU}_2\mathbb{C} / \langle -\mathbb{1} \rangle \cong \mathrm{SO}_3\mathbb{R}$

c) $\mathfrak{su}_2\mathbb{C} \cong \mathfrak{so}_3\mathbb{R}$

d) $\mathfrak{so}_4\mathbb{R} \cong \mathfrak{so}_3\mathbb{R} \oplus \mathfrak{so}_3\mathbb{R}$.

Proof. *ad (b) :* By the homomorphism theorem, $\mathrm{SU}_2\mathbb{C} / \langle -\mathbb{1} \rangle = S/\ker\psi \cong \mathrm{SO}_3\mathbb{R}$. *ad (c) :* By (a), we get $\mathfrak{so}_3\mathbb{R} = \ell(SO_3\mathbb{R}) \cong \ell(SU_2\mathbb{C}) = \mathfrak{su}_2\mathbb{C}$. *ad (d) :* This follows from (b) and 5.3.6(b). □

This now may suffice as a zoological equipment for the following paragraphs.

5.3.2. Consequences for the Levi decomposition

The previous results — more precisely $\mathrm{SO}_3\mathbb{R}$ and $\mathrm{SU}_2\mathbb{C}$ not being embeddable into the full automorphism group Σ of a Peter plane — imply that in the end, the Lie algebra $\mathfrak{so}_3\mathbb{R}$ is not a subalgebra of $\ell\,\Sigma$. We will explain the reason why, and continue by outlining the consequences for the solubility of Σ.

5.3.8 Proposition. *The Lie algebra $\mathfrak{so}_3\mathbb{R}$ is not embeddable into $\ell\,\Sigma$.*

Proof.

$$\begin{tikzcd}
& \mathrm{SU}_2\mathbb{C}/\ker\varepsilon \arrow[dr] & \\
\mathrm{SU}_2\mathbb{C} \arrow[ur] \arrow[rr, "\varepsilon"] & & \Sigma \\
\mathfrak{so}_3\mathbb{R} \arrow[u, "\exp"] \arrow[rr, "\varphi"] & & \ell\,\Sigma \arrow[u, "\exp"']
\end{tikzcd}$$

Assume an injective morphism $\varphi : \mathfrak{so}_3\mathbb{R} \hookrightarrow \ell\,\Sigma$ of Lie algebras. As $\mathrm{SU}_2\mathbb{C}$ is simply connected, there is a continuous group morphism $\varepsilon : \mathrm{SU}_2\mathbb{C} \to \Sigma$; cf. [24, I 9.11]. Yet we do not know whether or not ε is injective. What we do know is that *the kernel of ε is discrete.* In order to establish discreteness consider Campbell-Hausdorff neighbourhoods $U \in \mathcal{U}_0(\mathfrak{so}_3\mathbb{R})$ of 0 in $\mathfrak{so}_3\mathbb{R}$ and $V \in \mathcal{U}_1(\mathrm{SU}_2\mathbb{C})$ of 1 in $\mathrm{SU}_2\mathbb{C}$ such that $\exp : U \to V$ is a homeomorphism, and another pair $S \in \mathcal{U}_0(\ell\,\Sigma)$ and $T \in \mathcal{U}_1(\Sigma)$ of homeomorphic neighbourhoods. Now consider the neighbourhood $W := V \cap T^{\varepsilon^{\leftarrow}}$ of 1 in $\mathrm{SU}_2\mathbb{C}$. Let $g \in W \cap \ker\varepsilon$. There is a unique element $X \in U$ with $X^{\exp} = g$. Then $X^{\varphi\exp} = X^{\exp\varepsilon} = g^{\varepsilon} = 1$, thus $S \ni X^{\varphi} = 0$, and due to injectivity of φ we get $X = 0$. This implies $g = 1$ and thus establishes a neighbourhood W in $\mathrm{SU}_2\mathbb{C}$ satisfying $W \cap \ker\varepsilon = \mathtt{1}$. Therefore, $\mathtt{1}$ is open in $\ker\varepsilon$ and $\ker\varepsilon$ is discrete.

Its being discrete implies that *the kernel of ε is either trivial or equals $\langle -\mathbb{1}\rangle$*. In fact, $\ker\varepsilon$ is a (discrete) normal subgroup of $\mathrm{SU}_2\mathbb{C}$. It is generally true that the only normal subgroup of $\mathrm{SU}_2\mathbb{C}$ is its centre $\mathrm{Z}(\mathrm{SU}_2\mathbb{C}) = \langle -\mathbb{1}\rangle$; see [21, I §§4.4, 4.5] for instance. Yet for our purposes, it suffices to know its discrete normal subgroups, which are by far easier to obtain : The action of $\mathrm{SU}_2\mathbb{C}$ on $\ker\varepsilon$ by conjugation is continuous. Being discrete, $\ker\varepsilon$ is totally disconnected. Therefore, the (connected) orbit of each element in $\ker\varepsilon$ under conjugation in $\mathrm{SU}_2\mathbb{C}$ consists of one element only, which means that $\ker\varepsilon \leq \mathrm{Z}(\mathrm{SU}_2\mathbb{C}) = \langle -\mathbb{1}\rangle$.

Hence, the canonical decomposition of ε yields an injective continuous group morphism from $\mathrm{SU}_2\mathbb{C}/\ker\varepsilon$ to Σ. As a matter of fact, this morphism is an embedding of topological groups, since the co-restriction of a continuous injection from a compact space into a Hausdorff space is open. From 5.3.7 we know that this quotient is either $\mathrm{SO}_3\mathbb{R}$ or $\mathrm{SU}_2\mathbb{C}$

itself. But this harshly contradicts 5.2.6 and 5.2.9, for neither of these two embeddings exists. ☐

Now consider the Levi decomposition

$$\ell\,\Sigma = \mathfrak{s} \ltimes \mathfrak{r}$$

of $\ell\,\Sigma$, where $\mathfrak{r} = \sqrt{\ell\,\Sigma}$ is the soluble radical. The semisimple Lie algebra $\mathfrak{s}$ can be written as a direct sum $\mathfrak{s} = \bigoplus_{j \in J} \mathfrak{e}_j$ of simple Lie algebras $\mathfrak{e}_j$. Furthermore, 5.3.8 implies that $\mathfrak{so}_3\mathbb{R}$ is not embeddable into any of the $\mathfrak{e}_j$. Summarizing, *we are looking for simple real Lie algebras $\mathfrak{e}_j$ which do not admit $\mathfrak{so}_3\mathbb{R}$ as a subalgebra.*

5.3.3. Semisimple complex Lie algebras, real and compact forms

There is quite straight an answer to that problem : the only possible candidate is $\mathfrak{sl}_2\mathbb{R}$. In order to give a detailed argument, though, we should be familiar with the structure theory of semisimple complex Lie algebras, real forms and their maximal compact subalgebras. For our purposes it suffices to be aware of the summary given in 5.3.16. For the convenience of the reader we will nevertheless give a brief outline, mainly following [24] and [27].

Due to technical necessities, let us introduce the group

$$\operatorname{Inn}_{\mathfrak{g}} \mathfrak{a} := \left\langle e^{\operatorname{ad} a} \right\rangle \leq \operatorname{Aut} \mathfrak{g}$$

of inner automorphisms with respect to some Lie subalgebra $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$. For semisimple Lie algebras, $\operatorname{Inn}_{\mathfrak{g}} \mathfrak{g}$ coincides with the connected component $(\operatorname{Aut}\mathfrak{g})^1$ of the automorphism group of $\mathfrak{g}$; see proof of [24, III 6.4] along with [28, I 1.97+98].

5.3.9 Root space decomposition. Let $\mathfrak{g}$ denote a semisimple Lie algebra over $\mathbb{C}$. A Lie subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is called **maximal toral** if every element $\operatorname{ad} X$ is diagonalisable whenever $X \in \mathfrak{t}$. Every semisimple Lie algebra possesses maximal toral subalgebras, and they are necessarily abelian. Now, let $\mathfrak{t}$ denote such a maximal toral subalgebra. For every $\alpha \in \mathfrak{t}^*$ we define

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid \forall T \in \mathfrak{t}. \quad [T, X] = T^\alpha \cdot X\} \quad .$$

If $\mathfrak{g}_\alpha \neq \mathbf{0}$, then $\alpha \neq 0$ is called a **root** of $\mathfrak{g}$ relative to $\mathfrak{t}$. In that case, $\mathfrak{g}_\alpha$ is called a **root space**. We denote by Φ the set of all roots relative to $\mathfrak{t}$.

A partial order can be defined on $\mathfrak{t}^*$ with respect to which $\Phi = \Phi_+ \cup (-\Phi_+)$, where Φ_+ is the set of all positive elements of $\Phi \subseteq \mathfrak{t}^*$. The algebra $\mathfrak{g}$ can be written as the direct sum

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \quad ;$$

we call this the **root space decomposition** of $\mathfrak{g}$. Moreover, $\mathfrak{g}_0 = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{t}$.

5.3.10 Cartan decomposition. Let $\mathfrak{g}$ denote a semisimple real Lie algebra and κ its Killing form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} : (X, Y) \mapsto \operatorname{tr}(\operatorname{ad} X \cdot \operatorname{ad} Y)$. An automorphism τ of $\mathfrak{g}$ is called a **Cartan involution** if

- $\tau^2 = \operatorname{id}_{\mathfrak{g}}$
- κ is negative definite on $\mathfrak{k} := \{X \in \mathfrak{g} \mid X^\tau = X\} = \operatorname{Fix} \tau$
- κ is positive definite on $\mathfrak{p} := \{X \in \mathfrak{g} \mid X^\tau = -X\}$.

This yields the **Cartan decomposition** $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$. Note that $\mathfrak{k}$ is a Lie subalgebra, whereas $\mathfrak{p}$ is a vector subspace only; in fact, $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$.

The Cartan decomposition of a semisimple Lie algebra is unique up to isomorphism. More precisely, for Cartan decompositions $\mathfrak{k}_1 \oplus \mathfrak{p}_1 = \mathfrak{g} = \mathfrak{k}_2 \oplus \mathfrak{p}_2$ there is some automorphism $\gamma \in \operatorname{Inn}_{\mathfrak{g}} \mathfrak{g}$ such that $\mathfrak{k}_1^\gamma = \mathfrak{k}_2$ and $\mathfrak{p}_1^\gamma = \mathfrak{p}_2$.

5.3.11 Compact Lie algebras. Recall that a finite dimensional real Lie algebra $\mathfrak{g}$ is called **compact** if there is some positive definite symmetric bilinear form $\beta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ which is moreover **invariant**, in the sense that for any $X, Y, Z \in \mathfrak{g}$ we get $\beta(X, [Y, Z]) = \beta([X, Y], Z)$. Note that the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ deliberately offers a compact subalgebra of $\mathfrak{g}$: with $\beta := -\kappa|_{\mathfrak{k} \times \mathfrak{k}}$, the subalgebra $\mathfrak{k}$ is clearly compact.

In order to talk about uniqueness results another theorem has proven essential :

5.3.12 Theorem. *Let $\mathfrak{g}$ be a semisimple real Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then the following is true :*

a) *For any compact subgroup U of $(\operatorname{Aut} \mathfrak{g})^1 = \operatorname{Inn}_{\mathfrak{g}} \mathfrak{g}$, there is some $\gamma \in \operatorname{Inn}_{\mathfrak{g}} \mathfrak{g}$ such that $\gamma^{-1} U \gamma \subseteq K := \operatorname{Inn}_{\mathfrak{g}} \mathfrak{k}$.* [24, III.6.22]

b) *Let G be a connected Lie group with Lie algebra $\ell\, G = \mathfrak{g}$, and put $K := \langle \mathfrak{k}^{\exp} \rangle \leq G$. Then for any compact subgroup U of G there is an element $g \in G$ such that $g^{-1} U g \subseteq K$.* [24, III.6.25] $\square$

5.3.13 Complexification and real forms. Cartan decompositions and the notion of compactness make sense for real Lie algebras only, as they require the notion of positiveness. Yet, there are ways of using these concepts also when dealing with complex Lie algebras. What we need is a translation of real into complex Lie algebras, and vice versa : The **complexification** of a real Lie algebra $\mathfrak{g}$ is the tensor product $\mathfrak{g}_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$.

On the other hand, there are two different methods of gaining real Lie algebras from a complex Lie algebra $\mathfrak{g}$: First of all, we can simply read $\mathfrak{g}$ as a real Lie algebra, i.e., take $\mathbb{R}$ for the scalar field. The resulting real Lie algebra is called the **realification** of $\mathfrak{g}$ and is denoted by $\mathfrak{g}_{\mathbb{R}}$. A **real form** of a complex Lie algebra $\mathfrak{g}$ is a real Lie algebra $\mathfrak{e}$ with complexification $\mathfrak{e}_{\mathbb{C}} = \mathbb{C} \otimes \mathfrak{e} \cong \mathfrak{g}$. A real form which is moreover compact will be called a **compact form** of $\mathfrak{g}$.

5.3.14 The compact form of a semisimple complex Lie algebra. Let $\mathfrak{g}$ be a semisimple complex Lie algebra. Denote by $\mathfrak{t}$ a maximal toral subalgebra and consider its root space decomposition relative to $\mathfrak{t}$ as described in 5.3.9. Pick a suitable $\mathbb{C}$-basis $T_1, \dots, T_m$ of $\mathfrak{t}$ such that the Killing form κ is negative definite on the real subalgebra $i\tilde{\mathfrak{t}} := i \cdot \bigoplus_{j=1}^{m} \mathbb{R}T_j$ of $\mathfrak{t}_{\mathbb{R}}$. Moreover, we may choose suitable $X_\alpha \in \mathfrak{g}_\alpha$ for every root $\alpha \in \Phi$, with the property that $\kappa(X_\alpha, X_{-\alpha}) = 1$. Then

$$\mathfrak{g}_k := i\tilde{\mathfrak{t}} \oplus \bigoplus_{\alpha \in \Phi_+} \left(\quad \mathbb{R}(X_\alpha - X_{-\alpha}) \oplus i\mathbb{R}(X_\alpha + X_{-\alpha}) \quad \right)$$

is a compact form of $\mathfrak{g}$, and $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}_k \oplus i\mathfrak{g}_k$ is a Cartan decomposition of the realification of $\mathfrak{g}$. WEYL's theorem can be used to obtain a maximality result for $\mathfrak{g}_k$: for any semisimple compact Lie subalgebra $\mathfrak{c}$ of $\mathfrak{g}_{\mathbb{R}}$ there is an element $\gamma \in \operatorname{Aut}\mathfrak{g}$ such that $\mathfrak{c}^\gamma \subseteq \mathfrak{g}_k$.

Proof. Let G be the simply connected group with Lie algebra $\ell G = \mathfrak{g}_{\mathbb{R}}$. By WEYL's theorem 5.1.10, $C := \langle \mathfrak{c}^{\exp} \rangle \leq G$ is compact. By 5.3.12, there is an element $g \in G$ satisfying $g^{-1}Cg \subseteq \langle \mathfrak{k}^{\exp} \rangle$. Thus

$$\mathfrak{k} \geq \ell(g^{-1}Cg) = \ell(\mathfrak{c}^{I_g}) = \ell(\mathfrak{c}^{\exp \cdot I_g}) = \ell(\mathfrak{c}^{\operatorname{Ad} g \cdot \exp}) = \mathfrak{c}^{\operatorname{Ad} g} \quad ,$$

and $\operatorname{Ad} g$ is an element in $\operatorname{Aut}\mathfrak{g}$. □

5.3.15 Maximal compact subalgebras of real forms. Let $\mathfrak{g}$ be a semisimple complex Lie algebra with compact form $\mathfrak{g}_k$. Let $\mathfrak{e}$ be some real form of $\mathfrak{g}$. Then there is a certain automorphism $\tau \in \operatorname{Aut}\mathfrak{g}$ such that

$$\begin{aligned} \mathfrak{k} &:= \mathfrak{e} \cap \mathfrak{g}_k^\tau \\ \mathfrak{p} &:= \mathfrak{e} \cap i\mathfrak{g}_k^\tau \end{aligned}$$

form a Cartan decomposition $\mathfrak{e} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{e}$. As to uniqueness, again every semisimple compact Lie algebra $\mathfrak{c}$ of $\mathfrak{e}$ can be found as $\mathfrak{c} \cong \mathfrak{c}^\gamma \leq \mathfrak{k}$ for some $\gamma \in \operatorname{Aut}\mathfrak{e}$.

Finally, it is true that every simple real Lie algebra occurs in one of the two forms discussed above. Thus, we can summarize :

5.3.16 Theorem. ***Maximal compact subalgebras of semisimple Lie algebras.***

a) *Any simple real Lie algebra is either the realification or a real form of some simple complex Lie algebra.*

b) *Let $\mathfrak{g}$ be a simple complex Lie algebra. Then $\mathfrak{g}$ possesses a compact form $\mathfrak{g}_k$, and for any other semisimple compact subalgebra $\mathfrak{c}$ of $\mathfrak{g}_{\mathbb{R}}$ there is an automorphism $\gamma \in \operatorname{Aut}\mathfrak{g}$ such that $\mathfrak{c}^\gamma \leq \mathfrak{g}_k$.*

c) *Let $\mathfrak{g}$ be a simple complex Lie algebra. Then any real form $\mathfrak{e}$ of $\mathfrak{g}$ has a maximal compact subalgebra $\mathfrak{k}$, and for any other semisimple compact subalgebra $\mathfrak{c}$ of $\mathfrak{e}$ there is an automorphism $\gamma \in \operatorname{Aut}\mathfrak{e}$ such that $\mathfrak{c}^\gamma \leq \mathfrak{k}$.* □

5.3.4. What the classification of simple Lie algebras can do for us

Equipped with that knowledge, we can go out and open TITS' tables [74] : What we will find is a complete list of all simple complex Lie algebras along with their real forms and their respective maximal compact subalgebras.

Remember from 5.3.8 that we are looking for simple summands $\mathfrak{e}_j$ of the semisimple part $\mathfrak{s} = \oplus_{j \in J} \mathfrak{e}_j$ of $\ell\,\Sigma$. Not containing $\mathfrak{so}_3\mathbb{R}$ as a subalgebra is one necessary condition of being a candidate. As $\mathfrak{e} := \mathfrak{e}_j$ is a simple real Lie algebra, by 5.3.16(a) it can be found in TITS' tables [74] :

Case 1 : $\mathfrak{e}$ *is the realification of a simple complex Lie algebra* $\mathfrak{g}$. From 5.3.16(b), $\mathfrak{so}_3\mathbb{R}$ is a subalgebra of $\mathfrak{e}$ if and only if $\mathfrak{so}_3\mathbb{R}$ is a subalgebra of the compact form $\mathfrak{g}_k$ of $\mathfrak{g}$. And $\mathfrak{g}_k$ is also listed in TITS' tables.

Case 2 : $\mathfrak{e}$ *is a real form of a simple complex Lie algebra* $\mathfrak{g}$. From 5.3.16(c), $\mathfrak{so}_3\mathbb{R}$ is a subalgebra of $\mathfrak{e}$ if and only if $\mathfrak{so}_3\mathbb{R}$ is a subalgebra of the maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{e}$. And $\mathfrak{k}$ is also listed in TITS' tables.

By browsing the tables, then, and eliminating every simple real Lie algebra which does contain $\mathfrak{so}_3\mathbb{R}$, promising candidates will be singled out.

5.3.17 A word of warning. TITS uses the very same notation for two different items : the simple complex Lie algebra $\mathfrak{g}$ is denoted by the same chiffre as its (unique) compact form $\mathfrak{g}_k$. For the four series, for instance, this means :

dim	chiffre	$\mathfrak{g}$	$\mathfrak{g}_k$	
$n^2 + 2n$	A_n	$\mathfrak{sl}_{n+1}\mathbb{C}$	$A_n^{\mathbb{C},0} = \mathfrak{su}_{n+1}\mathbb{C}$	$n \geq 1$
$2n^2 + n$	B_n	$\mathfrak{so}_{2n+1}\mathbb{C}$	$B_n^{\mathbb{R},0} = \mathfrak{so}_{2n+1}\mathbb{R}$	$n \geq 2$
$2n^2 + n$	C_n	$\mathfrak{sp}_{2n}\mathbb{C}$	$C_n^{\mathbb{H},0} = \mathfrak{su}_n\mathbb{H}$	$n \geq 3$
$2n^2 - n$	D_n	$\mathfrak{so}_{2n}\mathbb{C}$	$D_n^{\mathbb{R},0} = \mathfrak{so}_{2n}\mathbb{R}$	$n \geq 4$

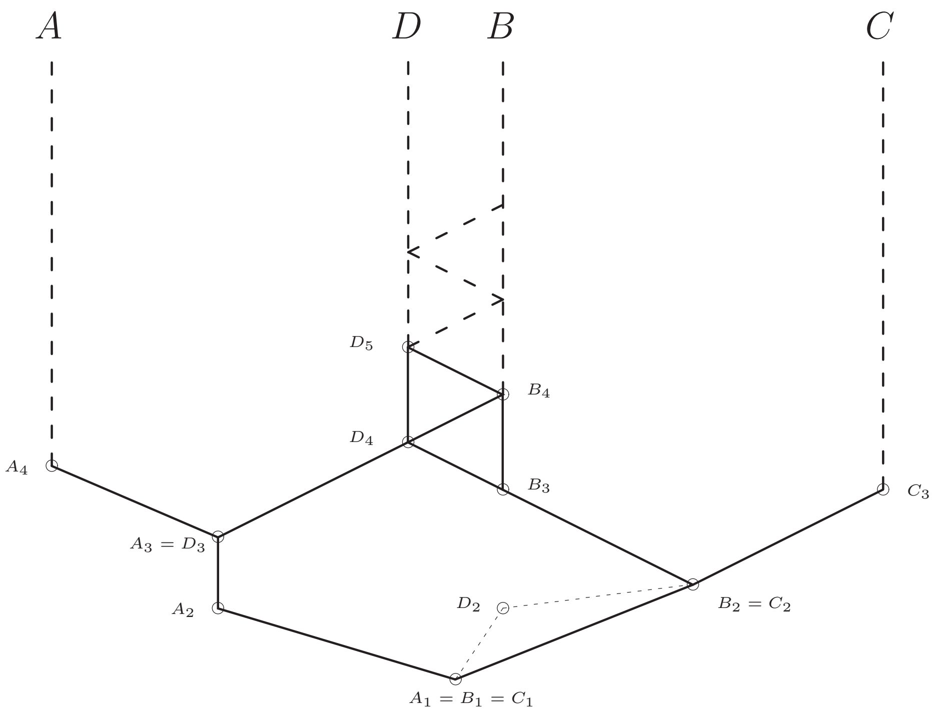
5.3.18 Inclusions and identities. Considering the series of simple complex Lie algebras, the elements within one series form a chain with respect to the inclusion order, the smallest element being the one with the smallest index. The series B and D are closely related, in as much as

$$B_n \hookrightarrow D_{n+1} \hookrightarrow B_{n+1} \qquad \text{for } n \geq 3 \quad ,$$

in other words $\mathfrak{so}_{2n+1}\mathbb{C} \hookrightarrow \mathfrak{so}_{2n+2}\mathbb{C} \hookrightarrow \mathfrak{so}_{2n+3}\mathbb{C}$. Furthermore, there are certain identities near the bottom element of the ordered set :

$$\begin{aligned} A_1 \cong B_1 \cong C_1 \\ B_2 \cong C_2 \\ A_3 \cong D_3 \end{aligned}$$

Note that $D_2 \cong A_1 \times A_1$ is properly semisimple (cf. 5.3.6(c)). The inclusion ordered set — at least the part of immediate interest here — looks like this :



The very same inclusions hold for the corresponding compact forms. This is due to

5.3.19 Lemma. *Let* $\mathfrak{a}, \mathfrak{b}$ *be simple complex Lie algebras and* $\mathfrak{a}_k, \mathfrak{b}_k$ *their respective compact forms. If* $\mathfrak{a} \hookrightarrow \mathfrak{b}$ *then* $\mathfrak{a}_k \hookrightarrow \mathfrak{b}_k$.

Proof. We will consider the simply connected Lie groups A and B with Lie algebras $\ell\,\mathsf{A} = \mathfrak{a}$ and $\ell\,\mathsf{B} = \mathfrak{b}$. Then there is an embedding $\varphi : \mathsf{A} \hookrightarrow \mathsf{B}$. Put $\mathsf{K} := \langle \mathfrak{a}_k^{\exp} \rangle \leq \mathsf{A}$ and $\mathsf{M} := \langle \mathfrak{b}_k^{\exp} \rangle \leq \mathsf{B}$. As both, $\mathfrak{a}$ and $\mathfrak{b}$, are simple compact Lie algebras, Weyl's theorem 5.1.10 ensures the compactness of K and M.

$$\begin{tikzcd}
\mathsf{A} \arrow[r, hook, "\varphi"] & \mathsf{B} \\
\mathsf{K} \arrow[u, hook, "\mathrm{id}_{\mathsf{A}}|_{\mathsf{K}}"] \arrow[r, hook, "\psi"] & \mathsf{M} \arrow[u, hook', "\mathrm{id}_{\mathsf{B}}|_{\mathsf{M}}"'] \\
\mathfrak{a}_k \arrow[u, "\exp"] \arrow[r, hook, "\ell\,\psi"] & \mathfrak{b}_k \arrow[u, "\exp"']
\end{tikzcd}$$

Now $\varphi|_{\mathsf{K}}$ is an embedding of K into B. By 5.3.12(b), there is some element $g \in \mathsf{B}$ such that $g^{-1}\mathsf{K}^{\varphi}g \leq \mathsf{M}$. In other words, $\psi := \varphi I_g$ is an embedding of K into the maximal compact subgroup M of B; here, I_g denotes conjugation by g. But then, $\ell\,\psi$ is an embedding of $\ell\,\mathsf{K} = \mathfrak{a}_k$ into $\ell\,\mathsf{M} = \mathfrak{b}_k$. Therefore, $\mathfrak{a}_k \hookrightarrow \mathfrak{b}_k$. $\square$

A theorem by Bickel implies that $\dim \Sigma = \dim \ell\Sigma \leq 8$ (cf. 5.2.10, 5.2.12). Thus we will here restrict our attention to Lie algebras of real dimension not more than 10. Yet, we could just as well exclude those of higher dimension using the same genre of arguments as before. The results will be collected in the following table, referring to single arguments given below.

5.3.20

Realifications

$\dim_{\mathbb{R}} \mathfrak{e}$	$\mathfrak{g}$	$\mathfrak{g}_k$	contains $\mathfrak{so}_3\mathbb{R}$	reference
6	$A_1 = \mathfrak{sl}_2\mathbb{C}$	$A_1 = \mathfrak{so}_3\mathbb{R}$	yes	5.3.21

Real forms

$\dim_{\mathbb{R}} \mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{k}$	contains $\mathfrak{so}_3\mathbb{R}$	reference
3	$A_1^{\mathbb{R}}$	$\mathbb{R}$	no	5.3.22
3	$A_1^{\mathbb{C},0}$	$A_1^{\mathbb{C},0} = A_1 = \mathfrak{so}_3\mathbb{R}$	yes	
8	$A_2^{\mathbb{R}}$	$B_1 = A_1$	yes	
8	$A_2^{\mathbb{C},0}$	$A_2^{\mathbb{C},0} = A_2$	yes	5.3.18
8	$A_2^{\mathbb{C},1}$	$A_1 \times \mathbb{R}$	yes	
10	$B_2^{\mathbb{R},0}$	$B_2^{\mathbb{R},0} = B_2$	yes	5.3.18
10	$B_2^{\mathbb{R},1}$	$D_2 = A_1 \times A_1$	yes	5.3.23
10	$B_2^{\mathbb{R},2}$	$D_1 \times B_1$	yes	5.3.18

5.3.21 Lemma. *$\mathfrak{so}_3\mathbb{R}$ is contained in every simple real Lie algebra which is a realification of a simple complex Lie algebra.*

Proof. By 5.3.18, the bottom element $A_1 = \mathfrak{so}_3\mathbb{R}$ is contained in every one of the compact forms of all possible simple complex Lie algebras, thus in their realifications. □

5.3.22 Indeed, $A_1^{\mathbb{R}} = \mathfrak{sl}_2\mathbb{R}$ *does not contain* $\mathfrak{so}_3\mathbb{R}$. For if it did, the 3-dimensional Lie algebra $\mathfrak{so}_3\mathbb{R}$ would have to be embeddable into the 1-dimensional Lie algebra $\mathbb{R}$, which is a contradiction. □

5.3.23 Lemma. *The compact form D_2 is the direct sum $D_2 \cong A_1 \oplus A_1$.*

Proof. Lemma 5.3.7(b); as the compact forms are $D_2 = \mathfrak{so}_4\mathbb{R}$ and $A_1 = \mathfrak{so}_3\mathbb{R}$. □

All in all, we have seen that the only remaining candidate is $\mathfrak{e}_j = \mathfrak{sl}_2\mathbb{R}$. As a matter of fact, $\mathfrak{sl}_2\mathbb{R}$ *is the only simple real Lie algebra which does not contain* $\mathfrak{so}_3\mathbb{R}$. Consequently, the Levi decomposition $\ell\Sigma = \mathfrak{s} \ltimes \mathfrak{r}$ must obey

5.3.24 Corollary. $\ell\Sigma = \mathfrak{s} \ltimes \mathfrak{r} = (\mathfrak{sl}_2\mathbb{R})^n \ltimes \sqrt{\ell\Sigma}$ *for some* $n \in \mathbb{N}_0$

5.3.5. Solubility revisited

Löwen has studied automorphism groups of 4-dimensional stable planes and come up with a result which offers further insight into the structure of Σ :

5.3.25 Theorem (Löwen 1978). *Every semisimple group of automorphisms of a 4-dimensional stable plane is almost simple.* [32] □

Applied to the semisimple automorphism group $\langle \mathfrak{s}^{\exp} \rangle \leq \Sigma$ of a Peter plane, Löwen's theorem yields that $\mathfrak{s}$ is a simple Lie algebra; thus, in the light of 5.3.24, $\mathfrak{s} = \mathbf{0}$ or $\mathfrak{s} = \mathfrak{sl}_2\mathbb{R}$. Summarising our knowledge on solubility of the full automorphism group of $\mathcal{P}$, we get

5.3.26 Theorem. *Let $\mathcal{P}$ be a Peter plane and denote by Σ its full automorphism group. Then its Lie algebra $\ell\,\Sigma$ is either soluble altogether, or $\ell\,\Sigma = \mathfrak{sl}_2\mathbb{R} \ltimes \mathfrak{r}$, where $\mathfrak{r} = \sqrt{\ell\,\Sigma}$ denotes its soluble radical.*

5.3.6. Solubility of a normaliser

The question still remains whether or not the full automorphism group $\Sigma = \operatorname{Aut}\mathcal{P}$ of a Peter plane is soluble, or, to put it another way, whether or not $\mathrm{SL}_2\mathbb{R}$ is an automorphism group of a Peter plane. A partial answer will be attempted, concluding that after all, if ever $\mathrm{SL}_2\mathbb{R}$ or one of its kins is a subgroup of Σ it will not be entirely found within the normaliser of Γ. This normaliser, at least, will turn out to be soluble.

$\Sigma = \operatorname{Aut}\mathcal{P}$ being the full automorphism group of a Peter plane $\mathcal{P}$, it comes naturally equipped with a (point) action

$$\omega : P \times \Sigma \to P$$

on $\mathcal{P} = (P, \mathcal{L}) = \mathbb{P}(\Gamma; \{1\}, \mathcal{S}^{\exp})$. As $\exp_{\mathfrak{g}}$ is a bijection, it allows the following construction of an action of Σ on the Lie algebra $\mathfrak{g} = \ell\,\Gamma$:

$$\begin{array}{rccc} \Omega : & \mathfrak{g} \times \Sigma & \to & \mathfrak{g} \\ & (X, \alpha) & \mapsto & (X^{\exp_{\mathfrak{g}}}, \alpha)^{\omega \cdot \exp_{\mathfrak{g}}^{-1}} . \end{array}$$

$$\begin{tikzcd} \Gamma \times \Sigma \arrow[r, "\omega"] & \gamma \\ \mathfrak{g} \times \Sigma \arrow[u, "\exp_{\mathfrak{g}} \times \mathrm{id}"] \arrow[r, dotted, "\Omega"] & \mathfrak{g} \arrow[u, "\exp_{\mathfrak{g}}"'] \end{tikzcd}$$

Consider the normaliser

$$\mathsf{N} := \mathrm{N}_{\Sigma}(\Gamma)$$

of Γ in Σ.

5.3.27 Lemma.

a) $\mathsf{\Gamma}$ *is closed in* Σ.

b) N *is closed in* Σ.

c) N *is a Lie group with Lie algebra* $\ell\,\mathsf{N} = \mathfrak{n}_{\ell\Sigma}(\mathfrak{g}) =: \mathfrak{n}$.

Proof. *ad (a).* By 3.3.14, the matrix topology τ_1 on $\mathsf{\Gamma}$ coincides with the one induced by the compact-open topology on Σ with respect to the action $\Sigma : P$. But endowed with the matrix topology, $\mathsf{\Gamma}$ is homeomorphic to $\mathbb{R}^4$, which is locally compact. Thus $\mathsf{\Gamma}$ is a locally compact subgroup of the locally compact Hausdorff group Σ; see [73, 4.8]. Therefore it is closed in Σ. *ad (b).* As the normaliser of a closed subgroup is closed (see [26, 5.53]), closedness of N is a consequence of part (a). *Part (c)* follows from [54, 94.12] along with the fact that the normaliser is the largest subgroup or Lie subalgebra in which $\mathsf{\Gamma}$ or $\mathfrak{g} = \ell\,\mathsf{\Gamma}$, respectively, is normal. Alternatively, see [10, chapitre III, § 9, no. 4, prop. 11]. □

5.3.28 Lemma. *The restriction of* ω *to the action of* $\mathsf{\Gamma}$ *on* $\mathsf{\Gamma}$ *is equivalent to right multiplication. In particular,* $\mathsf{\Gamma}$ *acts sharply transitive on* $\mathsf{\Gamma}$. *For any point* $p_o \in P = \mathsf{\Gamma}$, *the point evaluation*

$$\mathrm{eval}_\omega : \mathsf{\Gamma} \to \mathsf{\Gamma} : \gamma \mapsto p_o^\gamma = (p_o, \gamma)^\omega$$

is a bijection.

Proof. By the construction of a Peter plane $\mathbb{P}(\mathsf{\Gamma}; \{\mathbb{1}\}, \mathcal{S}^{\exp})$, there is a sharply transitive point action of $\mathsf{\Gamma} \leq \mathrm{Aut}\,\mathcal{P}$, given by right multiplication

$$\rho : P \times \mathsf{\Gamma} \to P : (\gamma, \alpha) \mapsto \gamma\alpha.$$

Therefore, for any point $p_o \in P = \mathsf{\Gamma}$, evaluation

$$\mathrm{eval}_\rho : \mathsf{\Gamma} \to P : \gamma \mapsto (p_o, \gamma)^\rho$$

is a bijection. Consider the group monomorphism $\sigma : \mathsf{\Gamma} \to \Sigma : \alpha \mapsto (\sigma_\alpha : (p_o, \gamma)^\rho \mapsto (p_o, \gamma\alpha)^\rho)$.

$$\begin{tikzcd}
\mathsf{\Gamma} \times \Sigma \arrow[r, "\omega"] & \mathsf{\Gamma} \\
\mathsf{\Gamma} \times \mathsf{\Gamma} \arrow[u, "\mathrm{eval}_\rho \times \sigma"] \arrow[r, "\rho"] & \mathsf{\Gamma} \arrow[u, "\mathrm{eval}_\rho"']
\end{tikzcd}$$

One can verify that this diagram commutes, meaning that ρ and the restriction $\omega|_{P\times\mathsf{\Gamma}}$ are equivalent. □

5.3.29 Lemma. *For any point $p_o \in P = \mathsf{\Gamma}$, consider the restriction of the actions to the point stabilisers in N.*

a) $\omega|_{\mathsf{\Gamma}\times\mathsf{N}_{p_o}} : \mathsf{\Gamma} \times \mathsf{N}_{p_o} \to \mathsf{\Gamma}$ *is equivalent to conjugation*

$$\kappa : \mathsf{\Gamma} \times \mathsf{N}_{p_o} \to \mathsf{\Gamma} : (\gamma, \alpha) \mapsto \alpha^{-1}\gamma\alpha\,.$$

b) $\Omega|_{\mathfrak{g}\times\mathsf{N}_{p_o}} : \mathfrak{g} \times \mathsf{N}_{p_o} \to \mathfrak{g}$ *is equivalent to the adjoint representation*

$$\mathrm{Ad} : \mathfrak{g} \times \mathsf{N}_{p_o} \to \mathfrak{g} : (X, \alpha) \mapsto \mathrm{Ad}\,\alpha.\,X\,.$$

Proof. *ad (a).* Note that κ is an action, indeed, as $\mathsf{\Gamma} \trianglelefteq \mathsf{N}$. For arbitrary group elements $\gamma \in \mathsf{\Gamma}$ and $\alpha \in \mathsf{N}_{p_o}$, we get $((p_o, \gamma)^\omega, \alpha)^\omega = (p_o, \gamma\alpha)^\omega = (p_o, \alpha^{-1}\gamma\alpha)^\omega$, as α^{-1} fixes p_o. Hence, the outer square of the diagram commutes and the actions are equivalent.

$$\begin{tikzcd}
\mathsf{\Gamma}\times\mathsf{N}_{p_o} \arrow[rrr, "\omega"] & & & \mathsf{\Gamma} \\
& \mathfrak{g}\times\mathsf{N}_{p_o} \arrow[ul, "\exp_{\mathfrak{g}}+\mathrm{id}"'] \arrow[r, "\Omega"] & \mathfrak{g} \arrow[ur, "\exp_{\mathfrak{g}}"] & \\
& \mathfrak{g}\times\mathsf{N}_{p_o} \arrow[u, dashed, "\mathrm{Eval}_\omega\times\mathrm{id}"] \arrow[r, "\mathrm{Ad}"] \arrow[dl, "\exp_{\mathfrak{g}}+\mathrm{id}"] & \mathfrak{g} \arrow[u, dashed, "\mathrm{Eval}_\omega"'] \arrow[dr, "\exp_{\mathfrak{g}}"] & \\
\mathsf{\Gamma}\times\mathsf{N}_{p_o} \arrow[uuu, "\mathrm{eval}_\omega\times\mathrm{id}"] \arrow[rrr, "\kappa"] & & & \mathsf{\Gamma} \arrow[uuu, "\mathrm{eval}_\omega"']
\end{tikzcd}$$

ad (b). Ad indeed describes an action of N on $\mathfrak{g}$, as N was chosen as the normaliser of $\mathsf{\Gamma}$. In fact, let $\alpha \in \mathsf{N}$, $X \in \mathfrak{g} = \{Y \in \ell\,\mathsf{N} \mid (\mathbb{R}X)^{\exp} \subseteq \mathsf{\Gamma}\}$. In order to show that $\mathrm{Ad}\,\alpha.X$ is contained in $\mathfrak{g}$ it suffices to show that the one-parameter group of $\mathrm{Ad}\,\alpha.X$ is contained in $\mathsf{\Gamma}$. Let $t \in \mathbb{R}$. Then $(t \cdot (\mathrm{Ad}\,\alpha.X))^{\exp} = (\mathrm{Ad}\,\alpha.(tX))^{\exp} = (tX)^{\exp\kappa_\alpha} \in \mathsf{\Gamma}$, because, in order of appearance, $\mathrm{Ad}\,\alpha$ is linear, $\mathrm{Ad}\,\alpha \cdot \exp = \exp \cdot \kappa_\alpha$, $(\mathbb{R}X)^{\exp} \in \mathsf{\Gamma}$ as $X \in \ell\,\mathsf{\Gamma}$, and finally $\mathsf{\Gamma} \trianglelefteq \mathsf{N}$. As to commutativity of the inner square, let $\alpha \in \mathsf{N}_{p_o}$. By definition, $\Omega(\cdot, \alpha) = \ell(\omega(\cdot, \alpha))$, which by part (a) is equivalent to $\ell(\kappa(\cdot, \alpha)) = \mathrm{Ad}\,\alpha$. If needs be, the diagram can be made commutative by the bijection $\mathrm{Eval}_\omega := \exp_{\mathfrak{g}} \cdot \mathrm{eval}_\omega \cdot \exp_{\mathfrak{g}}^{-1} : \mathfrak{g} \to \mathfrak{g}$. ◻

Now pick $p_o := 1 \in \mathsf{\Gamma}$.

5.3.30 Lemma.

a) *$\mathcal{S}^{\exp}$ is invariant under the action ω of N_{p_o}.*

b) *$\mathcal{S}$ is invariant under the action Ω of N_{p_o}.*

Proof. *ad (a).* $(\mathsf{N}, \mathcal{P})$ is a subgeometry of $(\Sigma, \mathcal{P})$. Therefore, N_{p_o} leaves the pencil $\mathcal{L}_{p_o} = \mathcal{L}_1 = \mathcal{S}^{\exp}$ invariant. *ad (b).* Let $\mathsf{\Lambda} \in \mathcal{S}$ and $\alpha \in \mathsf{N}_{p_o}$. Then $\mathsf{\Lambda}^{\exp\cdot\alpha} \in \mathcal{S}^{\exp}$, by (a), and $(\mathsf{\Lambda}, \alpha)^{\Omega} = (\mathsf{\Lambda}^{\exp\cdot\alpha})^{\exp^{-1}} \in \mathcal{S}^{\exp\cdot\exp^{-1}} = \mathcal{S}$, as $\exp_{\mathfrak{g}}$ is bijective. $\square$

Consider the translation plane

$$\mathcal{T} := \mathbb{P}\,(\mathfrak{g}; \{0\}, \mathcal{S})$$

constructed from the planar LieAlg-partition $\mathcal{S}$ of $\mathfrak{g}$. The linear point action Ω is complemented by the line action

$$\begin{array}{rcl} \mathfrak{g}/\mathcal{S} \times \mathsf{N}_{p_o} & \to & \mathfrak{g}/\mathcal{S} \\ (\mathsf{\Lambda} + X, \alpha) & \mapsto & \mathsf{\Lambda}^{\alpha} + X^{\alpha}\,. \end{array}$$

This is an action indeed, because the spread has proved invariant under the point action. Thus $(\mathsf{N}_{p_o}, \mathcal{T})$ becomes a geometry.

5.3.31 Lemma.

a) *N_{p_o} fixes $\mathfrak{g}'' = \mathbb{R}e_3$.*

b) *N_{p_o} fixes $\mathfrak{g}' = \mathbb{R}e_3 + \mathbb{R}e_2 + \mathbb{R}e_1$.*

Denote by $\mathsf{\Lambda}_o \in \mathcal{S}$ the unique fibre containing $\mathfrak{g}'' = \mathbb{R}e_3$. As a matter of fact, by 4.5.5 and 4.4.10, $\mathsf{\Lambda}_o$ is the (unique) abelian fibre in $\mathcal{S}$.

c) *N_{p_o} fixes $\mathsf{\Lambda}_o$.*

d) *N_{p_o} fixes the maximal flag $\mathfrak{g}'' \leq \mathsf{\Lambda}_o \leq \mathfrak{g}' \leq \mathfrak{g}$.*

Proof. *Parts (a) and (b)* are due to the fact that $(\mathfrak{g}')^{\exp} = \mathsf{\Gamma}'$ and $(\mathfrak{g}'')^{\exp} = \mathsf{\Gamma}''$ are normal subgroups of N; in fact, they are characteristic subgroups of the normal subgroup $\mathsf{\Gamma}$ of N. Let $\alpha \in \mathsf{N}$, $\Delta \in \{\mathsf{\Gamma}', \mathsf{\Gamma}''\}$ and $X \in \ell\,\Delta$. Then, as for $\Delta = \mathsf{\Gamma}$ above, $(\mathbb{R} \cdot (\mathrm{Ad}\,\alpha.X))^{\exp} = (\mathrm{Ad}\,\alpha.\mathbb{R}X)^{\exp} = (\mathbb{R}X)^{\exp\kappa_\alpha} \subseteq \Delta$, and thus $\mathrm{Ad}\,\alpha.X \in \ell\,\Delta$. *ad (c).* From 4.4.10 we know that $\mathsf{\Lambda}_o$ is a red-yellow linear combination and hence contained in $\mathfrak{g}'$. Now any $\alpha \in \mathsf{N}_{p_o}$ on the one hand fixes $\mathfrak{g}'' \leq \mathsf{\Lambda}_o$ and on the other hand maps $\mathsf{\Lambda}_o$ to some fibre $\mathsf{\Lambda}_o^{\alpha} \in \mathcal{S}$. Thus $\mathsf{\Lambda}_o^{\alpha} = \mathsf{\Lambda}_o$. *ad (d).* From the proof of (c) we recall that $\mathfrak{g}'' \leq \mathsf{\Lambda}_o \leq \mathfrak{g}' \leq \mathfrak{g}$ is a flag, indeed. Then (a) through (c) ensure that this flag is fixed by each $\alpha \in \mathsf{N}_{p_o}$. $\square$

5.3.32 Corollary. *N_{p_o} is isomorphic to a subgroup of the group of all upper triangular matrices in* $\mathrm{GL}_4\mathbb{R}$.

Proof. Pick an element $f_2 \in \Lambda_o$ such that $\Lambda_o = \mathbb{R}e_3 + \mathbb{R}f_2$, and pick another element $f_1 \in \mathfrak{g}'$ such that $\mathfrak{g}' = \Lambda_o + \mathbb{R}f_1$. As any element $\alpha \in \mathsf{N}_{p_o}$ acts linearly on $\mathfrak{g} = \mathbb{R}^4$, it can be described as a matrix in $\mathrm{GL}_4\mathbb{R}$. As by 5.3.31, α fixes the flag $\mathfrak{g}'' \leq \Lambda_o \leq \mathfrak{g}'$, with respect to the basis d, f_1, f_2, e_3 of $\mathfrak{g}$ it can be written as an upper triangular matrix. □

5.3.33 Corollary. *N_{p_o} is soluble.*

Proof. The group of all upper triangular matrices in $\mathrm{GL}_4\mathbb{R}$ is soluble. By 5.3.32, then so is its subgroup N_{p_o}. □

5.3.34 Lemma.

a) $\mathsf{N} = \mathsf{N}_{p_o} \ltimes \Gamma$

b) $\mathsf{N}_{p_o} \cong \mathsf{N}/\Gamma$

Proof. In our context, it suffices to treat the above as statements on (abstract) groups. *ad (a).* Γ acts transitively on $P = \Gamma$. Thus, the Frattini argument A.2.1 ensures that $\mathsf{N} = \mathsf{N}_{p_o}\Gamma$. Moreover, $\mathsf{N}_{p_o} \cap \Gamma \leq \Gamma_{p_o} = \mathbb{1}$, and finally Γ is a normal subgroup of its normaliser $\mathsf{N} = \mathsf{N}_\Sigma(\Gamma)$. Therefore, $\mathsf{N} = \mathsf{N}_{p_o} \ltimes \Gamma$ is a semidirect product. *ad (b).* From (a) we conclude that $\mathsf{N}/\Gamma = \mathsf{N}_{p_o}\Gamma/\Gamma \cong \mathsf{N}_{p_o}/(\Gamma \cap \mathsf{N}_{p_o}) \cong \mathsf{N}_{p_o}$ are isomorphic groups. □

5.3.35 Proposition.

a) *N is a soluble group.*

b) *$\mathfrak{n} = \ell\,\mathsf{N}$ is a soluble Lie algebra.*

Proof. *Part (a)* follows from the facts that $\mathsf{N}_{p_o} \cong \mathsf{N}/\Gamma$ as well as Γ are soluble, and from solubility being an extension property. As to *part (b)*, every connected Lie group is soluble if and only if its Lie algebra is soluble; cf. [54, 94.17]. Therefore, $\ell(\mathsf{N}^1) = \ell\,\mathsf{N}$ is soluble. □

5.3.36 Corollary. *There is no monomorphism of Lie algebras mapping $\mathfrak{sl}_2\mathbb{R}$ into $\mathfrak{n}$.*

Proof. Assume the existence of such a morphism $\eta : \mathfrak{sl}_2\mathbb{R} \to \mathfrak{n}$. Then the image $(\mathfrak{sl}_2\mathbb{R})^\eta$ is a simple Lie subalgebra of the soluble algebra $\mathfrak{n}$. Which is a contradiction; in fact, this would imply $\mathfrak{sl}_2\mathbb{R} = \mathfrak{sl}_2\mathbb{R}^{(3)} \leq \mathfrak{n}^{(3)} = \mathbf{0}$. □

5.3.37 Corollary. *Let Φ be a Lie group with Lie algebra $\ell\,\Phi \cong \mathfrak{sl}_2\mathbb{R}$. Then there is no continuous injective group morphism $\varphi : \Phi \rightarrowtail \mathsf{N}$.*

Proof.

$$
\begin{tikzcd}
\mathrm{SL}_2\mathbb{R} \arrow[r, tail, two heads, "\cong"] & \Phi \arrow[r, tail, "\varphi"] & \mathsf{N} \\
\mathfrak{sl}_2\mathbb{R} \arrow[u, "\exp"] \arrow[r, tail, two heads] & \ell\,\Phi \arrow[u] \arrow[r, tail, "\ell\,\varphi"] & \mathfrak{n} \arrow[u, "\exp"']
\end{tikzcd}
$$

Assuming the existence of such a monomorphism φ, the Lie functor yields a monomorphism $\ell\,\varphi$ of Lie algebras mapping $\mathfrak{sl}_2\mathbb{R} \cong \ell\,\Phi$ into $\mathfrak{n}$. Which is out-ruled by 5.3.36. $\square$

A. Appendix

The appendix is meant as a loose collection of definitions and results which might prove useful, yet would have interrupted the train of thought in the main part of the thesis.

Note that the application of a map $f : X \to Y$ to an element $x \in X$ is denoted by x^f. Similarly, the action of a group Γ on a set X is written as $(x, \alpha) \mapsto x^\alpha$. Matrices are usually applied to row vectors from the right; hence scalars operate from the left.

A.1. Topology

General topology

A.1.1 Definition. Let X and Y be topological spaces. A continuous injection $f : X \to Y$ is called an **embedding** if its co-restriction $f|^{X^f} : X \to X^f$ is an open map.

A.1.2 Lemma.

a) ***Universal property of quotient maps.*** *Let $q : X \to Y$ be a continuous surjection. Then q is a quotient map if and only if the following condition holds :*

(Q) *Consider an arbitrary topological space Z and arbitrary maps $f : X \to Z$ and $g : Y \to Z$ such that $qg = f$. Then continuity of f implies continuity of g.*

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[dr, "f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

b) ***Universal property of embeddings.*** *Let $e : X \to Y$ be a continuous injection. Then e is an embedding if and only if the following condition holds :*

(E) *Consider an arbitrary topological space Z and arbitrary maps $f : Z \to X$ and $g : Z \to Y$ with $fe = g$. Then continuity of g implies continuity of f.*

$$\begin{tikzcd} X \arrow[r, hook, "e"] & Y \\ & Z \arrow[ul, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Proof. *ad (a).* Consider a quotient map q, and let f and g be maps with $qg = f$. Assume that f is continuous. Let T be an open subset of Z. Then $T^{g^{\leftharpoonup} q^{\leftharpoonup}} = T^{f^{\leftharpoonup}}$ is open in X, by continuity of f, and therefore g is continuous, too. — Now assume that q satisfies the universal property **Q**. Consider the topological space $Z := Y_{\text{quot}}$ endowed with the quotient topology with respect to q, and consider the quotient map $\pi : X \to Z$. Being a quotient map, π is continuous, and therefore property **Q** for q ensures continuity of $\mathrm{id} : Y \to Y_{\text{quot}}$. On the other hand, π is a quotient map and as such satisfies **Q**. Therefore, continuity of q implies continuity of $\mathrm{id} : Y_{\text{quot}} \to Y$. All in all, $Y \approx Y_{\text{quot}}$ carries the quotient topology, in other words, q is a quotient map. See also [73, 1.33e]

ad (b). Consider an embedding e, and let f and g be maps with $fe = g$. Assume that g is continuous. Let T be an open subset of X. Then T^e is open in X^e, i.e., there is an open subset S of Y such that $S \cap X^e = T^e$. Then $T^{eg^{\leftharpoonup}} = (S \cap X^e)^{g^{\leftharpoonup}} = S^{g^{\leftharpoonup}} \cap X^{eg^{\leftharpoonup}} = S^{g^{\leftharpoonup}} \cap X^{ee^{\leftharpoonup} f^{\leftharpoonup}} = S^{g^{\leftharpoonup}} \cap X^{f^{\leftharpoonup}} = S^{g^{\leftharpoonup}} \cap Z = S^{g^{\leftharpoonup}}$, which is open in Z, as g is continuous. Thus, f is continuous. — Conversely, assume property **E**. Consider $Z := X^e$ endowed with the topology induced from Y. Then $g := \mathrm{id} : X^e \to Y$ is an embedding, and, in particular, continuous. Therefore by property **E** for g and e, the map $f := (e|^{X^e})^{-1} : X^e \to X$ is a homeomorphism.

□

A.1.3 Lemma. *Let $f : X \to Y$ be an open quotient map between Hausdorff spaces X and Y. If X is locally compact, then so is Y.*

Proof. Let $y \in Y$. Due to surjectivity, there is some $x \in X$ such that $x^f = y$. As X is locally compact, there is a compact neighbourhood $K \subseteq X$ of x. Let U be an open neighbourhood of x contained in K. Then $y = x^f \in U^f \subseteq K^f$. As f is continuous, K^f is compact. As it is open, U^f is open. Hence y possesses a compact neighbourhood. As we are dealing with Hausdorff spaces, this suffices for local compactness. □

A.1.4 Lemma. *Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous mapping.*

a) *Consider a subset Z of X. If f is an embedding, then its restriction $f|_Z : Z \to Y$ to Z is an embedding, too.*

b) *Let Z be a subset of X. If f is a homeomorphism, then $f|_Z^{Z^f} : Z \to Z^f$ is a homeomorphism, too.*

c) *Let W be a subset of $X^f \subseteq Y$. If f is an embedding, then its co-restriction $f|_{W^{f^{\leftharpoonup}}}^{W} : W^{f^{\leftharpoonup}} \to W$ is a homeomorphism.* □

A.1.5 A topological space X is said to have the **domain invariance property** if the following is true : If $U \subseteq X$ is open and $f : U \to V \subseteq X$ is a homeomorphism, then V is open in X as well. Euclidean spaces have the domain invariance property, and this carries over to all n-manifolds. For references consult [54, 51.19].

Homotopy Theory

A.1.6 Lemma. ***Higher homotopy groups.*** *[14, Chap. I 7] All topological spaces considered here are supposed to be path connected, such that we can abstain from the base point with respect to which homotopy groups are computed.*

- **a)** *For any star shaped topological space X and any natural number $n \geq 1$, $\pi_n(X)$ is trivial.*
- **b)** *For any two topological spaces X and Y and $n \geq 1$, the nth homotopy group of $X \times Y$ is $\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$.*
- **c)** *Let U be a covering space of X. Then $\pi_1(U) \leq \pi_1(X)$ and $\pi_n(U) \cong \pi_n(X)$ for $n \geq 2$.*
- **d)** $\forall n \in \mathbb{N}.\quad \pi_1(\mathbb{T}^n) = \mathbb{Z}^n$
- **e)** $\forall n \in \mathbb{N}\, \forall m \in \mathbb{N}^+.\quad \pi_m(\mathbb{T}^n) = \begin{cases} \mathbb{Z} & \text{for } m = n \\ 1 & \text{for } m < n \end{cases}$ □

A.2. Groups and topological groups

A.2.1 Lemma (Frattini argument). *Let G be a group which acts on a set X. Let H be a subgroup of G and a a point in X. Then transitiveness of H on X implies $G = G_a H$.*

Proof. Let $g \in G$. Because of transitivity of H, there is some $h \in H$ such that $a^g = a^h$, in other words $gh^{-1} \in G_a$. Hence $g = (gh^{-1})h \in G_a H$. □

A.2.2 A topological space is called **σ-compact** if it is the union of a countable family of compact spaces. Every connected, locally compact group is σ-compact.

A.2.3 Theorem (Open Mapping Theorem). *Let Υ be a locally compact, σ-compact group. Then every surjective continuous homomorphism from Υ onto a locally compact Hausdorff group is an open map.* *[73, 5.18]* □

A.2.4 Theorem (Dimension formula). *If Δ is a closed subgroup of the locally compact group Γ, then $\dim \Gamma = \dim \Delta + \dim \Gamma/\Delta$.* *[54, 93.7]* □

A.3. Lie algebras and Lie groups

A.3.1 Definition. Let $\mathfrak{n}$ and $\mathfrak{k}$ be Lie algebras and $\alpha \in \mathrm{Hom}(\mathfrak{k}, \mathrm{Der}(\mathfrak{n}))$. Define a Lie bracket on the direct sum $\mathfrak{k} \oplus \mathfrak{n}$ of vector spaces by

$$[(k, m), (l, n)] := ([k, l], [m, n] + k^\alpha.n - l^\alpha.m)$$

for any $n, m \in \mathfrak{n}$ and $k, l \in \mathfrak{h} = k$. Endowed with this bracket, $\mathfrak{k} \oplus \mathfrak{n}$ becomes a Lie algebra, usually called the **semidirect sum** $\mathfrak{k} \propto_\alpha \mathfrak{n}$ of the Lie algebras $\mathfrak{k}$ and $\mathfrak{n}$. Then $\mathfrak{k}$ is isomorphic to the Lie subalgebra $\mathfrak{k} \propto_\alpha \mathbf{0}$, and $\mathfrak{n}$ is isomorphic to the ideal $\mathbf{0} \propto_\alpha \mathfrak{n}$ in $\mathfrak{k} \propto_\alpha \mathfrak{n}$. For $\alpha \equiv 0$, we get the **direct sum** $\mathfrak{k} \propto_0 \mathfrak{n} = \mathfrak{k} \oplus \mathfrak{n}$ of the Lie algebras $\mathfrak{k}$ and $\mathfrak{n}$. By abuse of notation, we have often denoted the semidirect sum of Lie algebras by $\ltimes$, when dealing with Lie algebras only.

A.3.2 Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{k} \leq \mathfrak{g}$ a Lie subalgebra and $\mathfrak{n} \trianglelefteq \mathfrak{g}$ an ideal in $\mathfrak{g}$ such that $\mathfrak{k} \cap \mathfrak{n} = \mathbf{0}$. Then $\mathfrak{k} \propto_{\mathrm{ad}} \mathfrak{n}$ is isomorphic to a Lie subalgebra of $\mathfrak{g}$ via addition $\mathfrak{k} \propto_{\mathrm{ad}} \mathfrak{n} \to \mathfrak{g} : (k, n) \mapsto k + n$. It is usually identified with that subalgebra and called the **inner semidirect sum** of $\mathfrak{k}$ and $\mathfrak{n}$. If $\varphi : \mathfrak{g} \to \mathfrak{h}$ is an injective morphism of Lie algebras, then the image $(\mathfrak{k} \propto_{\mathrm{ad}} \mathfrak{n})^\varphi$ is isomorphic to the semidirect sum $\mathfrak{g}^\varphi \propto_{\mathrm{ad}} \mathfrak{n}^\varphi$.

A.3.3 Theorem. *Let Υ be a simply connected Lie group such that its Lie algebra is a semidirect sum $\ell\Upsilon \cong \mathfrak{a} \propto_\alpha \mathfrak{b}$. Denote by A and B the simply connected Lie groups satisfying $\ell\mathsf{A} = \mathfrak{a}$ and $\ell\mathsf{B} = \mathfrak{b}$. Then $\Upsilon \cong \mathsf{A} \ltimes_\beta \mathsf{B}$. (In particular, for inner semidirect sums $\alpha = \mathrm{ad}$, the action β corresponds to conjugation κ.)* *[24, III 3.16]* □

A.3.4 Theorem (Lie). *Let V be a finite dimensional complex vector space, and let $\mathfrak{g}$ be a soluble Lie subalgebra of $\mathfrak{gl}(V)$. Then $\mathfrak{g}$ stabilizes some flag in V; in other words, the matrices of $\mathfrak{g}$ relative to a suitable basis of V are upper triangular. [27, 4.1 Corollary A]* □

A.3.5 Lemma. *Homomorphic images of soluble Lie algebras are soluble.*

Proof. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras, $\varphi \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$. Show by induction that $(\mathfrak{g}^\varphi)^{(j)} = (\mathfrak{g}^{(j)})^\varphi$ for every $j \in \mathbb{N}$; then compare commutator series. □

A.3.6 Definition. A Lie algebra $\mathfrak{g}$ is called **simple** if $\mathfrak{g}$ is nonabelian and if moreover any ideal $\mathfrak{h}$ is either $\mathfrak{h} = \mathbf{0}$ or $\mathfrak{h} = \mathfrak{g}$. Note that the assertion "$\mathfrak{g}$ is nonabelian" may be replaced by the assertion that $\dim \mathfrak{g} > 1$. A Lie algebra $\mathfrak{g}$ is called **semisimple** if any abelian ideal $\mathfrak{a}$ in $\mathfrak{g}$ has to be $\mathfrak{a} = \mathbf{0}$. Equivalently, $\mathfrak{g}$ is the direct sum of simple ideals.

A.3.7 Definition. An abstract group is called **simple** if it has no normal subgroups but the trivial one. A Lie group Υ, however, is usually called **simple** if its Lie algebra $\ell\Upsilon$ is a simple Lie algebra. For the sake of clarity we will here use the notion of an **almost simple** Lie group. A Lie group shall be called **semisimple** if its Lie algebra is semisimple.

A.3.8 Proposition. *Let Υ be a Lie group with Lie algebra $\mathfrak{g}$. If Υ is compact, then $\mathfrak{z}(\mathfrak{g})^{\exp} = \mathrm{Z}(\Upsilon)^1$.*

Proof. The inclusion $\mathfrak{z}(\mathfrak{g})^{\exp} \subseteq \mathrm{Z}(\Upsilon)^1$ has already been proved in 5.1.16. Denote by $\mathfrak{l}$ the Lie algebra $\mathfrak{l} := \ell(\mathrm{Z}(\Upsilon))$. By parts (b) and (c) of 5.1.8, the exponential image of the compact Lie algebra $\mathfrak{l}$ in the compact Lie group $\mathrm{Z}(\mathsf{M})$ is the connected component of the

Lie group, and therefore $\mathfrak{l}^{\exp} = \langle \mathfrak{l}^{\exp} \rangle = \mathrm{Z}(\Upsilon)^1$. Let $A \in \mathfrak{l}$, which means that $(\mathbb{R}A)^{\exp} \subseteq \mathrm{Z}(\Upsilon)^1$. Then for all $s, t \in \mathbb{R}$ and for all $X \in \mathfrak{g}$ we get $(tA)^{\exp} \cdot (sX)^{\exp} = (sX)^{\exp} \cdot (tA)^{\exp}$. Differentiation of both sides of that equation with respect to t yields

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left((tA)^{\exp}(sX)^{\exp}\right) = A(tA)^{\exp}(sX)^{\exp} = (sX)^{\exp}A(tA)^{\exp} = \frac{\mathrm{d}}{\mathrm{d}\,t}\left((sX)^{\exp}(tA)^{\exp}\right).$$

Evaluation at $t = 0$ leaves us with

$$A \cdot (sX)^{\exp} = (sX)^{\exp} \cdot A.$$

If we repeat this procedure with respect to s, evaluation of the derivative at $s = 0$ yields $AX = XA$; and that for all $X \in \mathfrak{g}$, as we remember. Thus, $A \in \mathfrak{z}(\mathfrak{g})$, and $\mathfrak{l} \leq \mathfrak{z}(\mathfrak{g})$ is established.

Now, from the inverse inclusion in lemma 5.1.16(e), we get $\mathrm{Z}(\Upsilon)^1 = \mathfrak{l}^{\exp} \leq \mathfrak{z}(\mathfrak{g})^{\exp} \leq \mathrm{Z}(\Upsilon)^1$, and thus forcedly $\mathfrak{z}(\mathfrak{g})^{\exp} = \mathrm{Z}(\Upsilon)^1$. □

Bibliography

[1] D. Allcock, *Identifying models of the projective octonion plane*, Geom. Dedicata, 65 (1997), pp. 215 – 217. 20

[2] J. André, *Über nicht-desarguessche Ebenen mit transitiver Translationsgruppe*, Math. Z. 60 (1954), pp. 156 – 186. v, ix, 6, 11

[3] H. Aslaksen, *Restricted homogeneous coordinates for the Cayley projective planes*, Geom. Dedicata, 40 (1991), pp. 245 – 250. 20

[4] D. Betten. Vorlesungsmanuskript, Universität Tübingen, WS 1972/73. 31

[5] H. Bickel, *Vier-dimensionale stabile Ebenen mit großer Automorphismengruppe.* Dissertation, Technische Universität Braunschweig, 1995. xi, xiv, 128

[6] R. E. Block, *On the orbits of collineation groups*, Math. Z. 96 (1967), pp. 33 – 49. xii, 19

[7] A. Blunck and M. Stroppel, *Klingenberg chain spaces*, Abh. Math. Sem. Univ. Hamburg, 65 (1995), pp. 225 – 238. x

[8] S. Boekholt, *Compact Lie groups with isomorphic homotopy groups*, Journal of Lie Theory, 8 (1998), pp. 183 – 185. 117

[9] R. Bott, *On torsion in Lie groups*, Proc. Nat. Acad. Sciences USA, 40 (1954), pp. 586 – 588. xv, 124

[10] N. Bourbaki, *Groupes et algèbres de Lie, Chapitres 2 et 3*, Hermann, Paris, 1972. 99, 141

[11] J. Dieudonné, *La géométrie des groupes classiques*, Springer, troisième ed., 1970. 130

[12] T. Dörfner, *Wirkungen von Halbgruppen auf stabilen Ebenen.* Manuskript, Universität Stuttgart. 9, 37

[13] H. Freudenthal, *Einige Sätze über topologische Gruppen*, Ann. of Math., 37 (1936), pp. 217 – 226. v, ix, 63

[14] M. J. Greenberg and J. R. Harper, *Algebraic Topology*, Addison Wesley, 1981. 116, 149

[15] T. Grundhöfer, M. Joswig, and M. Stroppel, *Slanted sympletic quadrangles*, Geom. Dedicata, 49 (1994), pp. 143 – 154. x

[16] T. Grundhöfer, L. Kramer, and N. Knarr, *Flag-homogeneous compact connected polygons*, Geom. Dedicata, 55 (1995), pp. 95 – 114. v, x

[17] ——, *Flag-homogeneous compact connected polygons II. Special issue dedicated to Helmut R. Salzmann on the occasion of his 70th birthday*, Geom. Dedicata, 83 (2000), pp. 1 – 29. v, x

[18] T. Grundhöfer and R. Löwen, *Linear topological geometries*, in Handbook of Incidence Geometry, F. Buekenhout, ed., Elsevier, Amsterdam, 1995, ch. 23, pp. 1255 – 1324. 7, 21

[19] T. Grundhöfer and M. Stroppel, *On restrictions of automorphism groups of compact projective planes to subplanes*, Results in Math. 21 (1992), pp. 319 – 327. 58

[20] ——, *Direct limits and maximality of stable planes*, Arch. Math. 75 (2000), pp. 65 – 74. 8

[21] W. Hein, *Struktur- und Darstellungstheorie der klassischen Gruppen*, Springer, 1980. 46, 59, 133

[22] D. G. Higman and J. E. McLaughlin, *Geometric ABA-groups*, Illinois J. of Math., 5 (1961), pp. 382 – 397. v, ix

[23] J. Hilgert and K. H. Hofmann, *Old and new on* SL(2), manuscripta math. 54 (1985), pp. 17 – 52. 45, 46

[24] J. Hilgert and K.-H. Neeb, *Lie-Gruppen und Lie-Algebren*, Vieweg, Braunschweig, 1991. 50, 111, 118, 119, 133, 134, 135, 150

[25] K. H. Hofmann and S. A. Morris, *Errata in "The Structure of Compact Groups"*. http://www.ballarat.edu.au/~smorris/errata.pdf. 118

[26] ——, *The Structure of Compact Groups*, de Gruyter, Berlin, New York, 1998. 68, 111, 117, 118, 119, 120, 141

[27] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer, 1972. 96, 134, 150

[28] A. W. Knapp, *Lie Groups Beyond an Introduction*, Birkhäuser, Boston, 1996. 134

[29] H. Löwe, *Notizen über eine stabile Faserung der Frobeniusgruppe bei Plaumann und Strambach*, circa 1996. 51

[30] R. Löwen, *Locally compact connected groups acting on euclidean space with Lie isotropy groups are Lie*, Geom. Dedicata, 5 (1976), pp. 171 – 174. 126

[31] ——, *Vierdimensionale stabile Ebenen*, Geom. Dedicata, 5 (1976), pp. 239 – 294. vi, x, 7, 8, 110, 126

[32] ——, *Halbeinfache Automorphismengruppen von vierdimensionalen stabilen Ebenen sind quasi-einfach*, Math. Ann. 236 (1978), pp. 15 – 28. vi, xv, 8, 140

[33] ——, *Symmetric planes*, Pacific J. Math. 84 (1979), pp. 137 – 159. 17

[34] ——, *Central collineations and the parallel axiom in stable planes*, Geom. Dedicata, 10 (1981), pp. 283 – 315. 8

[35] ——, *Equivariant embeddings of low dimensional symmetric planes*, Mh. Math., 91 (1981), pp. 19 – 37. 17

[36] ——, *Homogeneous compact projective planes*, J. Reine Angew. Math. 321 (1981), pp. 217 – 220. 48

[37] ——, *A local "Fundamental Theorem" for classical topological projective planes*, Arch. Math. 38 (1982), pp. 286 – 288. 56

[38] ——, *Stable planes with isotropic points*, Math. Z. 182 (1983), pp. 49 – 61. vi, xi, xii, 39, 40

[39] ——, *Topology and dimension of stable planes : On a conjecture of H. Freudenthal*, J. reine und angewandte M. 343 (1983), pp. 108 – 122. vi, x, 8

[40] ——, *Actions of* $\mathrm{SO}_3\mathbb{R}$ *on 4-dimensional stable planes*, Aequationes math. 30 (1986), pp. 212 – 222. 126

[41] ——, *Actions of* Spin_3 *on 4-dimensional stable planes*, Geom. Dedicata, 21 (1986), pp. 1 – 12. 127, 128

[42] ——, *A criterion for stability of planes*, Arch. Math. 46 (1986), pp. 275 – 278. 11

[43] ——, *Compact spreads and compact translation planes over locally compact fields*, Journal of Geometry, 36 (1989), pp. 110 – 116. 11

[44] P. Maier, *Partitions of Lie groups and point-regular stable geometries.* Dissertation, Technische Universität Darmstadt, 1999. vi, xii, 6, 10, 11, 12, 14, 16, 51

[45] W. S. Massey, *A Basic Course in Algebraic Topology*, Springer, 1991. 110

[46] H. Mäurer, *Die Quaternionenschiefkörper*, Math. Semesterberichte, 46 (1999), pp. 93 – 96. 130

[47] M. Mimura, *Homotopy theory of Lie groups*, in Handbook of Algebraic Topology, F. Buekenhout, ed., Elsevier, Amsterdam, 199?, ch. 19, pp. – . xv, 117

[48] A. L. Onishchik, ed., *Lie Groups and Lie Algebras I*, Springer, 1991. 41

[49] E. Ossa, *Topologie*, Vieweg, 1992. 68, 110

[50] P. Plaumann and K. Strambach, *Partitionen Liescher und algebraischer Gruppen*, Forum Math. 2 (1990), pp. 523 – 578. 10, 15, 49

[51] C. Polley, *Lokal desarguessche Geometrien auf dem Möbiusband*, Arch. Math., 23 (1972), pp. 346 – 347. 37

[52] H. Salzmann, *Kompakte Ebenen mit einfacher Kollineationsgruppe*, Arch. Math. XIII (1962), pp. 98 – 109. 31

[53] ——, *Topological planes*, Advances in Math. 2 (1967), pp. 1 – 60. 31

[54] H. Salzmann, D. Betten, T. Grundhöfer, H. Hähl, R. Löwen, and M. Stroppel, *Compact Projective Planes*, de Gruyter, 1995. 8, 20, 21, 22, 23, 24, 29, 30, 31, 35, 36, 42, 48, 56, 59, 63, 89, 110, 115, 117, 123, 126, 128, 141, 144, 148, 149

[55] H. Schubert, *Topologie*, Teubner, Stuttgart, 1969. 59, 61

[56] K. Strambach, *Zur Klassifikation von Salzmann-Ebenen mit dreidimensionaler Kollineationsgruppe*, Math. Ann., 179 (1968), pp. 15 – 30. xi

[57] M. Stroppel, *A characterisation of quaternion planes*, Geom. Dedicata, 36 (1990), pp. 405 – 410. v, xi

[58] ——, *Endomorphisms of stable planes*, Seminar Sophus Lie, 2 (1992), pp. 75 – 81. 8, 9

[59] ——, *Reconstruction of incidence geometries from groups of automorphisms*, Arch. Math. 58 (1992), pp. 621 – 624. v, ix, 1

[60] ——, *A categorical glimpse at the reconstruction of geometries*, Geom. Dedicata, 46 (1993), pp. 47 – 60. v, ix, 1, 4

[61] ——, *Embedding a non-embeddable stable plane*, Geom. Dedicata, 45 (1993), pp. 93 – 99. v, vi, x

[62] ——, *A note on Hilbert and Beltrami systems*, Results in Math., 24 (1993), pp. 342 – 347. xi

[63] ——, *Quaternion hermitian planes*, Res. Math., 23 (1993), pp. 387 – 397. v, x

[64] ——, *Stable planes with large groups of automorphisms: the interplay of incidence, topology, and homogeneity*. Habilitationsschrift, Technische Hochschule Darmstadt, 1993. xi

[65] ——, *Locally compact Hughes planes*, Canad. Math. Bull. 37 (1994), pp. 112 – 123. v, x

[66] ——, *Stable planes*, Discrete Math. 129 (1994), pp. 181 – 189. 7, 65, 89

[67] ——, *Actions of almost simple groups on compact eight-dimensional projective planes are classicalm almost*, Geom. Dedicata, 58 (1995), pp. 117 – 125. x

[68] ——, *Actions of almost simple groups on eight-dimensional stable planes*, Math. Z., 226 (1997), pp. 1 – 9. xi

[69] ——, *Point-regular geometries*, J. Geom. 59 (1997), pp. 173 – 181. x, 6

[70] ——, *The skew-hyperbolic motion group of the quaternion plane*, Monatsh. Math., 123 (1997), pp. 253 – 273. x

[71] ——, *Bemerkungen zur ersten nicht desaguesschen ebenen Geometrie bei Hilbert*, J. geom. 63 (1998), pp. 183 – 195. 8

[72] ——, *Compact 3-dimensional elation quadrangles*, Geom. Dedicata, 83 (2000), pp. 149 – 167. v, x

[73] ——, *Locally compact groups*. Manuscript, Darmstadt/Stuttgart, version of Oct 1, 1999. 63, 121, 122, 123, 126, 141, 148, 149

[74] J. Tits, *Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen*, Springer, 1967. xv, 137

[75] Toda, *A note on compact semi-simple Lie groups*, Japanese J. of Math. 2 (1976), pp. 355 – 360. xv, 117

[76] A. Wich, *Skizzierte Geometrien*. Diplomarbeit, Technische Hochschule Darmstadt, 1996. xi, xii, 1, 3, 4, 5, 17, 99

[77] J. W. Young, *On the partitions of a group and the resulting classification*, Bull. Am. Math. Soc., 33 (1927), pp. 453 – 461. v, ix

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