Mathematical Methods in the Applied Sciences. Vol. 15, 453–468 (1992)
MOS subject classification : 35 L 05, 35 L 30, 35 L 35, 35 G 15

Wave Equations with Time-dependent Spatial Operators of Higher Order

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Communicated by R. Leis

We study the initial-boundary value problem for $\partial_t^2 u(t,x) + A(t)u(t,x) + B(t)\partial_t u(t,x) = f(t,x)$ on $[0,T] \times \Omega$ ($\Omega \subset \mathbb{R}^n$) with a homogeneous Dirichlet boundary condition; here $A(t)$ denotes a family of uniformly strongly elliptic operators of order $2m$, $B(t)$ denotes a family of spatial differential operators of order less than or equal to m, and u is a scalar function. We prove the existence of a unique strong solution u. Furthermore, an energy estimate for u is given.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ and $m \in \mathbb{N}$ be given. We consider the problem

$$\left.\begin{array}{lll} \partial_t^2 u(t,x) + A(t)u(t,x) + B(t)\partial_t u(t,x) = f(t,x) & & \text{for } t\in[0,T],\ x\in\Omega, \\ u(t,.)\in \mathring{H}^m(\Omega) & & \text{for } t\in[0,T], \\ u(0,x) = u^0(x), & \partial_t u(0,x) = u^1(x) & \text{for } x\in\Omega. \end{array}\right\} \tag{1.1}$$

Here $\mathring{H}^m(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in the mth Sobolev space $H^m(\Omega)$, and $A(t)$, $B(t)$ denote families of spatial differential operators of order $2m$ and less than or equal to m, respectively. Problems of this kind appear in the study of fully non-linear wave equations (compare [9]). We make the following assumptions.

Assumption 1.1. (1) The operators A and B are given by

$$A(t)\varphi := \sum_{|\alpha|\leq 2m} [a_\alpha(t,.) + \tilde{a}_\alpha(t,.)]\partial_x^\alpha \varphi \quad \text{for } \varphi\in \mathring{H}^m(\Omega)\cap H^{2m}(\Omega), \tag{1.2}$$

$$B(t)\varphi := \sum_{|\beta|\leq m} [b_\beta(t,.) + \tilde{b}_\beta(t,.)]\partial_x^\beta \varphi \quad \text{for } \varphi\in H^m(\Omega), \tag{1.3}$$

where $a_\alpha, b_\beta \in C_b^{(k-1)m}([0,T]\times\bar{\Omega})$ and

$$\tilde{a}_\alpha, \tilde{b}_\beta \in \bigcap_{j=1}^{k-1} C^j([0,T], H^{(k-1-j)m}(\Omega)) \tag{1.4}$$

0170–4214/92/070453–16$13.00
Received 23 July 1991

for $|\alpha| \leqslant 2m$, $|\beta| \leqslant m$ and some $k \geqslant [n/2m] + 3$ ($[r] := \max\{j \in \mathbb{N} : j \leqslant r\}$).

(2) There exist constants $c_1, c_2 > 0$ such that

$$(-1)^m \operatorname{Re} \sum_{|\alpha|=2m} [a_\alpha(t,x) + \tilde{a}_\alpha(t,x)]\xi^\alpha \geqslant c_1 |\xi|^{2m}$$

$$\text{for } \xi \in \mathbb{R}^n, t \in [0,T], \quad x \in \bar{\Omega}, \tag{1.5}$$

$$\|[A(t) - A^*(t)]\varphi\| \leqslant c_2 \|\varphi\|_m \quad \text{for } \varphi \in \mathring{H}^m(\Omega) \cap H^{2m}(\Omega), t \in [0,T], \tag{1.6}$$

where $A^*(t)$ denotes the formal adjoint to the operator $A(t)$, and $\|\,.\,\|$, $\|\,.\,\|_m$ denote the norms in $L_2(\Omega)$ and $H^m(\Omega)$, respectively.

(3) There exists a constant $c_3 \geqslant 0$ such that

$$-\operatorname{Re}\langle B(t)\varphi, \varphi\rangle \leqslant c_3 \|\varphi\|^2 \quad \text{for } \varphi \in \mathring{H}^m(\Omega), t \in [0,T] \tag{1.7}$$

(here $\langle .,. \rangle$ denotes the inner product in $L_2(\Omega)$).

Remarks. (1) We admit that the coefficients of A and B are divided into two different parts, one being continuously differentiable with bounded derivatives, and the other lying in some spatial Sobolev space for every $t \in [0, T]$. This is essential for the application to non-linear problems.

(2) Condition (1.6) means that the part of $A(t)$ containing the derivatives $\partial_x^\alpha \varphi$ with $m + 1 \leqslant |\alpha| \leqslant 2m$ is symmetric. This condition is also used in [2]. An equivalent formulation of (1.6) is used in [3], (3.5), (3.6).

(3) Condition (1.7) is needed for the energy estimate. In the case $m = 1$, if $b_\beta + \tilde{b}_\beta$ is real valued for $|\beta| = 1$, (1.7) holds automatically if $k \geqslant [n/2m] + 4$ in (1.4). This can be shown by integrating by parts (compare [9]). More practical conditions for $B(t)$ guaranteeing (1.7) are given in [9].

(4) By Sobolev's lemma it follows from (1.4) and $k \geqslant [n/2m] + 3$ that $\tilde{a}_\alpha(t) \in C_b(\bar{\Omega})$ for $t \in [0, T]$. Hence (1.5) is well defined.

The aim of this paper is to prove the existence of a unique solution

$$u \in \mathscr{C}_T^k := \bigcap_{j=0}^{k} C^j([0,T], H^{(k-j)m}(\Omega)) \tag{1.8}$$

of (1.1). More precisely, we prove the following theorem.

Theorem 1.1. *Let Assumption* 1.1 *be satisfied for some* $k \geqslant k_0 := [n/2m] + 4$ *and let* $2 \leqslant j \leqslant k$. *If*

$$f \in \mathscr{C}_T^{j-2} \cap C^{j-1}([0,T], L_2(\Omega)) \tag{1.9}$$

and $u^0 \in H^{jm}(\Omega)$, $u^1 \in H^{(j-1)m}(\Omega)$ *such that* (u^0, u^1, f) *satisfies the compatibility condition (defined in section* 2) *of order* j, *then* (1.1) *has a unique solution* $u \in \mathscr{C}_T^j$. *Furthermore*

$$|u(t)|_j := \sum_{\nu=0}^{j} \|\partial_t^\nu u(t)\|_{(j-\nu)m}$$

$$\leqslant \exp(C_1 t)\left(D_1 |u(0)|_j + C_2 \int_0^t [\|\partial_t^{j-1} f(\tau)\| + |f(\tau)|_{j-2}]\,d\tau\right)$$

$$+ D_2 |f(t)|_{j-2} \quad \textit{for } t \in [0,T], \tag{1.10}$$

where the constants $C_1, C_2 > 0$ *depend only on* c_1, c_2, c_3 *(of Assumption* 1.1), *j, and*

$$\sup_{|\alpha|\leqslant 2m, |\beta|\leqslant m} \sup_{|\gamma|+jm\leqslant (k-1)m} \sup_{[0,T]\times\bar{\Omega}} (|\partial_t^j \partial_x^\gamma a_\alpha(t,x)| + |\partial_t^j \partial_x^\gamma b_\beta(t,x)|), \tag{1.11}$$

$$\sup_{|\alpha|\leqslant 2m, |\beta|\leqslant m} \sup_{[0,T]} (|\tilde{a}_\alpha(t)|_{k-1}^{\sim} + |\tilde{b}_\beta(t)|_{k-1}^{\sim}) \tag{1.12}$$

($|\tilde{a}_\alpha(t)|_{k-1}^{\sim} := |\partial_t \tilde{a}_\alpha(t)|_{k-2} + \|\tilde{a}_\alpha(t)\|_{(k-2)m}$), *whereas* $D_1, D_2 > 0$ *depend only on* j, c_1, c_2, c_3, (1.11), *and*

$$\sup_{|\alpha|\leqslant 2m, |\beta|\leqslant m} \sup_{[0,T]} (|\tilde{a}_\alpha(t)|_{k-2} + |\tilde{b}_\beta(t)|_{k-2}). \tag{1.13}$$

Remarks. (1) The fact that D_1, D_2 depend only on (1.13) and not on (1.12) is essential for the iteration procedure in [9], where this theorem is used. If D_1, D_2 are allowed to depend on (1.12), then the condition $k \geqslant [n/2m] + 4$ in Theorem 1.1 could be relaxed to $k \geqslant [n/2m] + 3$. But this would require a more complicated proof.

(2) The term $|u(0)|_j$ can be estimated by $\|u^0\|_{jm}$, $\|u^1\|_{(j-1)m}$ and $|f(0)|_{j-2}$ (compare (2.8)).

(3) Condition (1.9) can be relaxed to the assumption that $f \in \mathscr{C}_T^{j-2}$ and $\partial_t^{j-1} f \in L_2$ ($[0, T]$, $L_2(\Omega)$). Even (1.4) can be slightly relaxed.

There are many papers dealing with problems of the type (1.1) with $B = 0$. We only mention Kato [7], Lions and Magenes [10], and Dafermos and Hrusa [3]. In these papers the existence of the solution is proved in two different ways: by the aid of semigroups ([7] and [10]) and by energy methods ([10], §3.8.2, and [3]). In the case $m = 1$, if

$$|\mathrm{Re}\langle B(t)\varphi, \varphi\rangle| \leqslant c\|\varphi\|^2 \quad \text{for } \varphi \in \mathring{H}^m(\Omega), t \in [0, T],$$

Ikawa [5] proved the existence of a solution using semigroups. He considers operators $A(t)$ and $B(t)$ with vanishing $\tilde{a}_\alpha$ and $\tilde{b}_\beta$. In addition to the Dirichlet boundary condition he studies the Neumann boundary condition. Recently Koch proved in [8] an existence theorem for systems of hyperbolic equations with real-valued coefficients. He assumes that Ω is bounded and studies the more complicated case of time-dependent boundary conditions. Furthermore, he considers a problem similar to (1.1) with $m = 1$ (compare (2.12) in [8]). He uses energy methods.

The proof in this paper is also based on energy estimates but differs from the proof in [8]. Section 3 deals with elliptic equations of order $2m$. In particular, the regularity of a solution of elliptic equations is studied. In section 4 the existence of a unique strong solution $u \in \mathscr{C}_T^2$ of (1.1) is proved by the method of Faedo–Galerkin, which uses an approximation in finite-dimensional function-spaces. A higher degree of regularity is obtained by induction in section 5, by solving a system of a simple integral equations and an equation of the type of (1.1) (compare (5.3), (5.4)).

2. Notation. The compatibility condition

By Ω we denote a domain $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in C^\infty$ such that $\partial\Omega$ is bounded (or empty) or such that

$$\Omega = \mathbb{R}^{n'} \times \Omega' \tag{2.1}$$

with $n' < n$ and bounded $\Omega' \subset \mathbb{R}^{n-n'}$. Let

$$\partial_x^\alpha \varphi := \frac{\partial^{|\alpha|}\varphi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \quad \text{for } \alpha \in \mathbb{N}_0^n,$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We set

$$C_b^k(\Omega) := \{\varphi \in C^k(\Omega) : \partial_x^\alpha \varphi \text{ is bounded in } \Omega \text{ for } |\alpha| \leqslant k\}.$$

We use the abbreviation $u' = \partial_t u$, $u'' = \partial_t^2 u$ for time-dependent functions. Furthermore we write $u(t) \in \mathring{H}^m(\Omega)$ instead of $u(t, .) \in \mathring{H}^m(\Omega)$. Recalling the definition of $\mathscr{C}_T^k$ in (1.8) we set

$$|u(t)|_k := \sum_{j=0}^{k} \|\partial_t^j u(t)\|_{(k-j)m} \quad \text{for } t \in [0, T] \tag{2.2}$$

for $u \in \mathscr{C}_T^k$. Besides $\mathscr{C}_T^k$ we use the linear space

$$\tilde{\mathscr{C}}_T^k := \bigcap_{j=1}^{k} C^j([0, T], H^{(k-j)m}(\Omega)). \tag{2.3}$$

Note that $C([0, T], H^{km}(\Omega))$ contains $\mathscr{C}_T^k$ but not $\tilde{\mathscr{C}}_T^k$. For $v \in \tilde{\mathscr{C}}_T^k$ we set

$$\begin{aligned} |v(t)|_{\tilde{k}} &:= \sum_{j=1}^{k} \|\partial_t^j v(t)\|_{(k-j)m} + \|v(t)\|_{(k-1)m} \\ &= |v'(t)|_{k-1} + \|v(t)\|_{(k-1)m} \quad \text{for } t \in [0, T]. \end{aligned} \tag{2.4}$$

In order to give the compatibility condition we assume that $u \in \mathscr{C}_T^k$ is a solution of (1.1). From $u(t) \in \mathring{H}^m(\Omega)$ for $t \in [0, T]$ it follows that $\partial_t^j u(0) \in \mathring{H}^m(\Omega)$ for $j = 0, \ldots, k-1$ (compare [9]). Differentiating (1.1) formally $(j-2)$-times, we obtain

$$\begin{aligned} \partial_t^j u(0) = \partial_t^{j-2} f(0) - \sum_{\nu=0}^{j-2} \binom{j-2}{\nu} \{ &[\partial_t^\nu A(0)] \partial_t^{j-2-\nu} u(0) \\ &+ [\partial_t^\nu B(0)] \partial_t^{j-1-\nu} u(0)\} \end{aligned} \tag{2.5}$$

where

$$\partial_t^\nu A(0) = \sum_{|\alpha| \leqslant 2m} [\partial_t^\nu a_\alpha(0, .) + \partial_t^\nu \tilde{a}_\alpha(0, .)] \partial_x^\alpha \tag{2.6}$$

and $\partial_t^\nu B(0)$ is given analogously. We make the following definition.

Definition 2.1. *We say that* (u^0, u^1, f) *satisfies the compatibility condition of order* $k \in \mathbb{N}$ *if* $u^j \in \mathring{H}^m(\Omega)$ *for* $j = 0, \ldots, k-1$, *where* u^j *is recursively defined by*

$$u^j := \partial_t^{j-2} f(0) - \sum_{\nu=0}^{j-2} \binom{j-2}{\nu} \{[\partial_t^\nu A(0)] u^{j-2-\nu} + [\partial_t^\nu B(0)] u^{j-1-\nu}\} \tag{2.7}$$

for $j \geqslant 2$.

Remark. Let Assumption 1.1 be satisfied for some $k \geqslant [n/2m] + 3$ and let $2 \leqslant j \leqslant k$. If

$$u^0 \in H^{jm}(\Omega),\ u^1 \in H^{(j-1)m}(\Omega),\ \partial_t^{\nu-2} f(0) \in H^{(j-\nu)m}(\Omega) \text{ for } \nu = 2, \ldots, j,$$

then u^ν is well defined by (2.7) for $\nu = 2, \ldots, j$. Furthermore, it follows from Lemma

8.2 of [9] and induction that

$$|u(0)|_j = \sum_{\nu=0}^{j} \|u^\nu\|_{(j-\nu)m} \leq D[\|u^0\|_{jm} + \|u^1\|_{(j-1)m} + |f(0)|_{j-2}], \tag{2.8}$$

where $D > 0$ depends only on j, (1.11) and (1.13).

3. Elliptic equations

Consider the elliptic differential equations

$$A(t)u(t) = f(t) \quad \text{for } t \in [0, T], \tag{3.1}$$

where A satisfies Assumption 1.1 for some $k \geq k_0 = [n/2m] + 4$. We choose

$$\left.\begin{array}{l} c_{\alpha\beta} \in C_b^{(k-2)m+|\alpha|}([0, T] \times \bar{\Omega}), \\ \tilde{c}_{\alpha\beta} \in C([0, T], H^{(k-3)m+|\alpha|}(\Omega)) \end{array}\right\} \tag{3.2}$$

such that

$$A(t)u(t) = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\alpha|} \partial_x^\alpha \{[c_{\alpha\beta}(t) + \tilde{c}_{\alpha\beta}(t)] \partial_x^\beta u(t)\} \tag{3.3}$$

(compare, e.g., [1], section 8). It follows from (1.5) that

$$(-1)^m \operatorname{Re} \sum_{|\alpha| = |\beta| = m} [c_{\alpha\beta}(t, x) + \tilde{c}_{\alpha\beta}(t, x)] \xi^{\alpha+\beta} \geq c_1 |\xi|^{2m} \tag{3.4}$$

for $\xi \in \mathbb{R}^n$, $t \in [0, T]$ and $x \in \bar{\Omega}$. We set

$$a(t, \varphi, \psi) := \sum_{|\alpha| = |\beta| \leq m} \langle [c_{\alpha\beta}(t) + \tilde{c}_{\alpha\beta}(t)] \partial_x^\beta \varphi, \partial_x^\alpha \psi \rangle$$
$$\text{for } \varphi, \psi \in H^m(\Omega), t \in [0, T] \tag{3.5}$$

and study instead of (3.1) the equation

$$a(t, u(t), \varphi) = \langle f(t), \varphi \rangle \quad \text{for every } \varphi \in C_0^\infty(\Omega), t \in [0, T]. \tag{3.6}$$

We prove the following lemma.

Lemma 3.1. *Let $k \geq [n/2m] + 4$ and let Assumption 1.1 be satisfied. If $f \in C([0, T], H^j(\Omega))$ for some $j \leq (k-2)m$ and if $u \in C([0, T], \mathring{H}^m(\Omega))$ such that (3.6) holds, then $u \in C([0, T], H^{j+2m}(\Omega))$. Furthermore*

$$\|u(t)\|_{j+2m} \leq d(\|f(t)\|_j + \|u(t)\|) \quad \textit{for } t \in [0, T], \tag{3.7}$$

where $d > 0$ depends only on j and on

$$\sup_{|\alpha| \leq 2m} \sup_{[0,T]} \left(\sup_{|\gamma| \leq (k-2)m} \sup_{x \in \Omega} |\partial_x^\gamma a_\alpha(t, x)| + \|\tilde{a}_\alpha(t)\|_{(k-2)m} \right) \tag{3.8}$$

(and on Ω).

Proof. Let Ω be of the form (2.1). In the other cases Lemma 3.1 can be shown analogously. At first we prove the assertion in the case $j = 0$. We set

$$\Omega_R := \{x \in \Omega : |x| < R\}$$

and choose $R > 0$ so large that

$$\{x \in \Omega = \mathbb{R}^{n'} \times \Omega' : |x_i| \leqslant 1 \quad \text{for } i = 1, \ldots, n'\} \subset \Omega_R. \tag{3.9}$$

Let

$$T_z(\Omega_R) := \{x + z : x \in \Omega_R\} \quad \text{for } z = (z_1, \ldots, z_{n'}, 0, \ldots, 0). \tag{3.10}$$

Note that $(k - 3)m \geqslant [n/2] + 1$ and that therefore

$$c_{\alpha\beta}(t) + \tilde{c}_{\alpha\beta}(t) \in C_b^{|\alpha|}(\bar{\Omega}) \quad \text{for } |\alpha|, |\beta| \leqslant m,\ t \in [0, T]$$

by the lemma of Sobolev. Hence $a(t, \varphi, \psi)$ is right m-smooth for every $t \in [0, T]$ in the sense of Definition 9.1 of [1]. By the proof of Theorem 9.8 in [1] it follows from (3.6) that

$$\| u(t) \|_{H^{2m}(T_z(\Omega_R))} \leqslant d_1(\| f(t) \|_{L_2(T_z(\Omega_{R+1}))} + \| u(t) \|_{H^m(T_z(\Omega_{R+1}))}). \tag{3.11}$$

for every $z \in S := \mathbb{Z}^{n'} \times \{(0, \ldots, 0)\}$, where $d_1 > 0$ can be chosen to be independent of z. Note that

$$\Omega = \bigcup_{z \in S} T_z(\Omega_R) = \bigcup_{z \in S} T_z(\Omega_{R+1}), \tag{3.12}$$

and that every $x \in \Omega$ is contained only in a fixed finite number of sets $T_z(\Omega_{R+1})$. Hence summation over all $z \in S$ yields $u(t) \in H^{2m}(\Omega)$ and

$$\| u(t) \|_{2m} \leqslant d_2(\| f(t) \| + \| u(t) \|_m) \tag{3.13}$$

for $t \in [0, T]$. In view of (3.4) there exist constants $d_3, d_4 > 0$ such that

$$\operatorname{Re} a(t, \varphi, \varphi) \geqslant d_3 \| \varphi \|_m^2 - d_4 \| \varphi \|^2 \quad \text{for } \varphi \in \mathring{H}^m(\Omega),\ t \in [0, T]. \tag{3.14}$$

Hence it follows from (3.6) (with $\varphi := u(t)$) and (3.13) that (3.7) holds in the case $j = 0$. In particular, we have that $\| u(t) \|_{2m}$ is bounded on $[0, T]$.

If $t, t_1 \in [0, T]$, we obtain from (3.6) that

$$a(t, u(t) - u(t_1), \varphi) = \langle g_1(t, t_1), \varphi \rangle \quad \text{for every } \varphi \in C_0^\infty(\Omega), t \in [0, T] \tag{3.15}$$

with

$$g_1(t, t_1) := f(t) - f(t_1) + [A(t_1) - A(t)]u(t_1). \tag{3.16}$$

With Lemma 8.2 in [9] we conclude that $\| g_1(t, t_1) \| \to 0$ as $t \to t_1$. Hence (3.7) applied to (3.15) yields $\| u(t) - u(t_1) \|_{2m} \to 0$ as $t \to t_1$ and therefore $u \in C([0, T], H^{2m}(\Omega))$. This proves the assertion for $j = 0$.

Now let Lemma 3.1 be proved for $j = 0, \ldots, J \leqslant (k - 2)m - 1$. By the induction hypothesis we have $u \in C([0, T], H^{J+2m}(\Omega))$ and (3.7) with $j = J$. Let $\psi \in C_0^\infty(\Omega)$. From (3.6) and (3.3) we conclude that

$$a(t, \psi u(t), \varphi) = \langle g_2(t), \varphi \rangle \quad \text{for } \varphi \in C_0^\infty(\Omega),\ t \in [0, T] \tag{3.17}$$

with

$$g_2(t) := \psi f(t) + A(t)(\psi u(t)) - \psi A(t) u(t). \tag{3.18}$$

Note that the derivatives of u of order $2m$ vanish on the right-hand side of (3.18). From $\tilde{a}_\alpha \in C(0, T], H^{(k-2)m}(\Omega))$ for $|\alpha| \leqslant 2m$ and Lemma 8.2 in [9] it follows that $g_2 \in$

$C([0, T], H^{J+1}(\Omega))$. Setting $\varphi = \partial\Phi/\partial x_i$ in (3.17) with $\Phi \in C_0^\infty(\Omega)$, we conclude that

$$a\left(t, \frac{\partial(\psi u(t))}{\partial x_i}, \Phi\right) = \langle g_3(t), \Phi\rangle \quad \text{for } \Phi \in C_0^\infty(\Omega),\ t \in [0, T], \tag{3.19}$$

where

$$g_3(t) := \frac{\partial g_2(t)}{\partial x_i} - \sum_{|\alpha| \leqslant 2m} \left(\frac{\partial}{\partial x_i}[a_\alpha(t) + \tilde{a}_\alpha(t)]\right)\partial_x^\alpha(\psi u(t)). \tag{3.20}$$

We apply (3.7) with $j = J$ to (3.19) and obtain

$$\begin{aligned}\left\|\frac{\partial \psi u(t)}{\partial x_i}\right\|_{J+2m} &\leqslant d(\|g_3(t)\|_J + \|\psi u(t)\|) \\ &\leqslant d_1\left(\left\|\frac{\partial(\psi f(t))}{\partial x_i}\right\|_J + \|u(t)\|_{H^{J+2m}(\operatorname{supp}\psi)}\right)\end{aligned} \tag{3.21}$$

for $t \in [0, T]$ and $i = 1, \ldots, n$, where $d_1 > 0$ depends only on (3.8), J and ψ.

An analogous procedure can be performed around the boundary points. In fact, let $x_0 \in \partial\Omega$ and let U be an open neighbourhood of x_0 such that there exists a C^∞-mapping transforming $U \cap \Omega$ in $K_R^+ := \{x \in \mathbb{R}^n : |x| < R, x_n > 0\}$ and $\bar{U} \cap \partial\Omega$ in $\Gamma_R := \{x \in \mathbb{R}^n : |x| \leqslant R, x_n = 0\}$. Since such a mapping preserves the properties of our elliptic equation; we can assume that $U \cap \Omega = K_R^+$, $\bar{U} \cap \partial\Omega = \Gamma_R$.

For given $R' \in (0, R)$ we choose $\psi \in C_0^\infty(K_R)$ with $\psi = 1$ on $K_{R'} := \{x \in \mathbb{R}^n : |x| < R'\}$. Then (3.21) follows as above for $i = 1, \ldots, n-1$, since $\partial(\psi u(t))/\partial x_i \in \mathring{H}^m(K_R^+)$ for $i = 1, \ldots, n-1$.

Since $u \in H^{2m}(\Omega)$, we conclude from (3.17) that

$$A(t)[\psi u(t)] = g_2(t) \quad \text{for } t \in [0, T]$$

and hence

$$[a_{(0,\ldots,0,2m)}(t) + \tilde{a}_{(0,\ldots,0,2m)}(t)]\,\partial_x^{(0,\ldots,0,2m)}(\psi u(t)) = g_4(t) \tag{3.22}$$

with a suitable g_4, where according to (3.18) and (3.21)

$$\|g_4(t)\|_{J+1} \leqslant d_2[\|\psi f(t)\|_{J+1} + \|u(t)\|_{H^{J+2m}(\operatorname{supp}\psi)}].$$

Since

$$|a_{(0,\ldots,0,2m)}(t, x) + \tilde{a}_{(0,\ldots,0,2m)}(t, x)| \geqslant c_1 > 0$$

by (1.5), we conclude from (3.22) and (1.4) that

$$|\partial_x^{(0,\ldots,0,2m)}\psi u(t)\|_{J+1} \leqslant d_3(\|\psi f(t)\|_{J+1} + \|u(t)\|_{H^{J+2m}(\operatorname{supp}\psi)}). \tag{3.23}$$

We choose a finite number of suitable functions $\psi_1, \ldots, \psi_\nu$. Then we conclude from (3.21) and (3.23) that

$$\|u(t)\|_{H^{J+1+2m}(T_z(\Omega_R))} \leqslant d_4(\|f(t)\|_{H^{J+1}(T_z(\Omega_{R+1}))} + \|u(t)\|_{H^{J+2m}(T_z(\Omega_{R+1}))}) \tag{3.24}$$

for $t \in [0, T]$ and $z = (0, \ldots, 0)$, where $d_4 > 0$ depends only on (3.8) and J. Note that we can prove (3.24) for arbitrary $z \in S = \mathbb{Z}^n \times \{(0, \ldots, 0)\}$ by the same argument using the functions $\psi_1(x - z), \ldots, \psi_\nu(x - z)$. Hence d_4 can be chosen such that (3.24) holds for every $z \in S$. Using the argument leading to (3.13) we conclude with the induction hypothesis that (3.7) holds for $j = J + 1$. Finally, $u \in C([0, T], H^{J+1+2m}(\Omega))$ follows applying (3.7) with $j = J + 1$ to (3.15), since $\|g_1(t, t_1)\|_{J+1} \to 0$ as $t \to t_1$ by Lemma 8.2 of [9].

4. Existence and uniqueness of the solution

In this section we suppose that Assumption 1.1 is satisfied for $k = k_0 = [n/2m] + 4$. Let $a(t, \varphi, \psi)$ be defined by (3.2)–(3.5). Note that (1.6) implies that we can choose the coefficients of $a(t, \varphi, \psi)$ such that

$$c_{\alpha\beta} + \tilde{c}_{\alpha\beta} = \overline{(c_{\beta\alpha} + \tilde{c}_{\beta\alpha})} \quad \text{for } m + 1 \leqslant |\alpha| + |\beta| \leqslant 2m. \tag{4.1}$$

Furthermore it follows from (1.5) that there exist constants $d, d_1 > 0$ such that for

$$a_d(t, \varphi, \psi) := \sum_{\substack{|\alpha|, |\beta| \leqslant m \\ |\alpha| + |\beta| \geqslant m+1}} \langle [c_{\alpha\beta}(t) + \tilde{c}_{\alpha\beta}(t)] \partial_x^\beta \varphi, \partial_x^\alpha \psi \rangle + d \langle \varphi, \psi \rangle \tag{4.2}$$

it holds that

$$a_d(t, \varphi, \varphi) = \operatorname{Re} a_d(t, \varphi, \varphi) \geqslant d_1 \|\varphi\|_m^2 \quad \text{for } \varphi \in \mathring{H}^m(\Omega),\ t \in [0, T]. \tag{4.3}$$

Here and in the following we denote by $d, d_1, d_2, \ldots$ positive constants depending only on c_1, c_2, c_3 (of Assumption 1.1) and on (1.11), (1.13).

In order to prove the existence of a solution of (1.1) we use the method of Faedo–Galerkin and follow the considerations in [3]. We suppose that

$$u^0 \in \mathring{H}^m(\Omega) \cap H^{2m}(\Omega), \quad u^1 \in \mathring{H}^m(\Omega), \quad f \in H^1([0, T], L_2(\Omega)), \tag{4.4}$$

where the last condition means that $f, f' \in L_2([0, T], L_2(\Omega))$. Note that this implies that $f \in C([0, T], L_2(\Omega))$. Let

$$\{\xi_1, \xi_2, \ldots\} \subset \mathring{H}^m(\Omega) \cap H^{2m}(\Omega)$$

such that every finite subset is linearly independent and span $\{\xi_1, \xi_2, \ldots\}$ is dense in $L_2(\Omega)$. We seek an approximate solution

$$u_j(t) = \sum_{\nu=1}^{j} u_{j\nu}(t) \xi_\nu \tag{4.5}$$

of

$$\left. \begin{aligned} &\langle u_j''(t) + A(t) u_j(t) + B(t) u_j'(t) - f(t), \xi_\nu \rangle = 0 \quad \text{for } t \in [0, T], \nu = 1, \ldots, j, \\ &u_j(0) = u_j^0, \quad u_j'(0) = u_j^1, \end{aligned} \right\} \tag{4.6}$$

where $u_j^0, u_j^1 \in \operatorname{span}\{\xi_1, \ldots, \xi_j\}$ are chosen so that $\|u_j^\nu - u^\nu\|_{(2-\nu)m} \to 0$ as $j \to \infty$ $(\nu = 0, 1)$. Since (4.6) is a system of ordinary differential equations for the coefficients $u_{j\nu}$ $(\nu = 1, \ldots, j)$, it follows from standard classical theory that a solution $(u_{j1}, \ldots, u_{jj}) \in C^2([0, T])$ exists. From (4.6) we conclude that $(u_{j1}, \ldots, u_{jj}) \in H^3([0, T])$. Hence $u_j \in H^3([0, T], H^{2m}(\Omega))$.

We set

$$|u_j(t)|_E := [a_d(t, u_j(t), u_j(t)) + \|u_j'(t)\|^2]^{1/2} \quad \text{for } t \in [0, T]. \tag{4.7}$$

It holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_j(t)|_E^2 = 2\mathrm{Re}[a_d(t, u_j(t), u_j'(t)) + \langle u_j''(t), u_j'(t)\rangle] + a_d'(t, u_j(t), u_j(t)) \tag{4.8}$$

for $t \in [0, T]$, where

$$a_d'(t, \varphi, \psi) = \sum_{m+1 \leqslant |\alpha|+|\beta| \leqslant 2m} \langle [c_{\alpha\beta}'(t) + \tilde{c}_{\alpha\beta}'(t)]\partial_x^\beta \varphi, \partial_x^\alpha \psi\rangle. \tag{4.9}$$

Replacing ξ_v in (4.6) by $u_j'(t) \in \operatorname{span}\{\xi_1, \ldots, \xi_j\}$, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_j(t)|_E^2$$

$$= 2\mathrm{Re}\Big\langle f(t) - B(t)u_j'(t) - \sum_{|\alpha|+|\beta| \leqslant m} \partial_x^\alpha\{[c_{\alpha\beta}(t) + \tilde{c}_{\alpha\beta}(t)]\partial_x^\beta u_j(t)\} + du_j(t), u_j'(t)\Big\rangle$$

$$+ a_d'(t, u_j(t), u_j(t)).$$

With (1.7) and (4.3) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_j(t)|_E^2 \leqslant 2[\|f(t)\| + d_2|u_j(t)|_E]|u_j(t)|_E$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_j(t)|_E \leqslant \|f(t)\| + d_2|u_j(t)|_E \quad \text{for } t \in [0, T]. \tag{4.10}$$

With Gronwall's Lemma it follows that there is a constant $C > 0$ such that $|u_j(t)|_E \leqslant C$ for $t \in [0, T]$ and $j \in \mathbb{N}$.

In a similar way we prove that $|u_j'(t)|_E$ is bounded. We differentiate (4.6) with respect to t and obtain

$$\langle u_j'''(t) + A(t)u_j'(t) + B(t)u_j''(t) + A'(t)u_j(t) + B'(t)u_j'(t) - f'(t), \xi_v\rangle = 0 \tag{4.11}$$

for $t \in [0, T]$, $v = 1, \ldots, j$. We replace ξ_v by $u_j''(t)$ and conclude by the same argument as above that

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_j'(t)|_E^2 = -\langle A'(t)u_j(t), u_j''(t)\rangle + r(t) \tag{4.12}$$

with

$$|r(t)| \leqslant 2\|f'(t)\|\,|u_j'(t)|_E + d_3|u_j'(t)|_E^2 \leqslant \|f'(t)\|^2 + (d_3 + 1)|u_j'(t)|_E^2 \tag{4.13}$$

a.e. in $[0, T]$. Let $t_0 \in [0, T]$. We use

$$\int_0^t \langle A'(\tau)u_j(\tau), u_j''(\tau)\rangle\,\mathrm{d}\tau = \langle A'(\tau)u_j(\tau), u_j'(\tau)\rangle|_{\tau=0}^t - \int_0^t \langle \partial_\tau(A'(\tau)u_j(\tau)), u_j'(\tau)\rangle\,\mathrm{d}\tau$$

and

$$|\langle A'(t)u_j(t), u_j'(t)\rangle| = |a'(t, u_j(t), u_j'(t)| \leqslant \varepsilon|u_j'(t)|_E^2 + \frac{d_4}{\varepsilon}|u_j(t)|_E^2,$$

$$\langle \partial_\tau(A'(\tau)u_j(\tau)), u_j'(\tau)\rangle \leqslant d_1'(\|u_j(\tau)\|_m^2 + \|u_j'(\tau)\|_m^2)$$

with $\varepsilon > 0$. Here and in the following we denote by $d_1', d_2', \ldots$ positive constants depending on c_1, c_2 and c_3, and on (1.11) and (1.12). Integration of (4.12) yields

$$|u_j'(t)|_E^2 \leqslant d_5[|u_j'(0)|_E^2 + |u_j(0)|_E^2] + \varepsilon|u_j'(t)|_E^2 + \frac{d_4}{\varepsilon}|u_j(t)|_E^2$$
$$+ d_2' \int_0^t [|u_j'(\tau)|_E^2 + |u_j(\tau)|_E^2 + \|f'(\tau)\|^2]\,d\tau. \tag{4.14}$$

Note that $|u_j'(0)|_E^2$ is bounded as $j \to \infty$, since $\|\partial_t^\nu u_j(0) - u^\nu\|_{(2-\nu)m} \to 0$ as $j \to \infty$ for $\nu = 0, 1, 2$, where $u^2 = f(0) - A(0)u^0 - B(0)u^1$. In fact, (4.6) implies that $u_j''(0)$ converges weakly to u^2 in $L_2(\Omega)$ and

$$\overline{\lim_{j\to\infty}} \|u_j''(0)\|^2 = \overline{\lim_{j\to\infty}} \langle f(0) - A(0)u_j^0 - B(0)u_j^1, u_j''(0)\rangle \leqslant \|u^2\| \overline{\lim_{j\to\infty}} \|u_j''(0)\|.$$

Hence $\overline{\lim}_{j\to\infty} \|u_j''(0)\| \leqslant \|u^2\|$ and it follows that $u_j''(0) \to u^2$ in $L_2(\Omega)$ as $j \to \infty$.

With Gronwall's lemma we conclude from (4.14) and the boundedness of $|u_j(t)|_E^2$ that there exists a $C > 0$ such that $|u_j'(t)|_E^2 \leqslant C$ for $t \in [0, T]$, $j \in \mathbb{N}$. Hence we can extract a subsequence converging weakly to a

$$u \in H^2([0, T], L_2(\Omega)) \cap H^1([0, T], \mathring{H}^m(\Omega)).$$

By a standard argument (compare §3.8.2 in [10]) it follows that

$$u''(t) + A(t)u(t) + B(t)u'(t) = f(t) \quad \text{a.e. in } [0, T], \tag{4.15a}$$
$$u(t) \in \mathring{H}^m(\Omega) \quad \text{for } t \in [0, T], \tag{4.15b}$$
$$u(0) = u^0, \quad u'(0) = u^1. \tag{4.15c}$$

From (4.15) we obtain by the argument leading to (4.10) that

$$\frac{d}{dt}|u(t)|_E \leqslant \|f(t)\| + d_2|u(t)|_E \quad \text{a.e. in } [0, T]. \tag{4.16}$$

Using Gronwall's lemma we conclude that

$$|u(t)|_E \leqslant e^{d_2 t}\left(|u(0)|_E + \int_0^t \|f(\tau)\|\,d\tau\right) \quad \text{for } t \in [0, T], \tag{4.17}$$

and with (4.3) and (4.7) we obtain

$$|u(t)|_1 \leqslant d_5 e^{d_2 t}\left(|u(0)|_1 + \int_0^t \|f(\tau)\|\,d\tau\right) \quad \text{for } t \in [0, T]. \tag{4.18}$$

In the rest of this section we prove that $u \in \mathscr{C}_T^2$ and derive an estimate for $|u(t)|_2$. To this end we introduce a convenient concept of a weak solution.

Definition 4.1. *Let $u^0 \in \mathring{H}^m(\Omega)$, $u^1 \in L_2(\Omega)$ and $f \in L_1([0, T], L_2(\Omega))$. We say that*

$$u \in H^1([0, T], L_2(\Omega)) \cap L_2([0, T], \mathring{H}^m(\Omega))$$

is a weak solution of (1.1) *if $u(0) = u^0$ and*

$$\int_0^T [-a(t, u(t), v(t)) + \langle B(t)u(t) + u'(t), v'(t)\rangle + \langle B'(t)u(t) + f(t), v(t)\rangle]\,dt$$
$$= \langle u^1 + Bu^0, v(0)\rangle \tag{4.19}$$

for every $v \in H^1([0, T], L_2(\Omega)) \cap L_2([0, T], \mathring{H}^m(\Omega))$ *with* $v(T) = 0$ (*note that* $a(t, u(t), v(t))$ *is defined by* (3.5)).

Remarks. (1) Since $\partial_t v \in L_2([0, T], L_2(\Omega))$ it holds $v \in C([0, T], L_2(\Omega))$. Hence $v(0)$, $v(T)$ are well defined.

(2) Integration by parts shows that $u \in \mathscr{C}_T^2$ is a solution of (1.1) if and only if it is a weak solution.

In a first step we prove the uniqueness of a weak solution.

Lemma 4.1. *Let Assumption* 1.1 *be satisfied for* $k = k_0$. *If* $u^0 = u^1 = 0$ *and* $f(t) = 0$ *on* $[0, T]$, *and if*

$$u \in H^1([0, T], L_2(\Omega)) \cap L_2([0, T], \mathring{H}^m(\Omega))$$

is a weak solution of (1.1), *then* $u(t) = 0$ *on* $[0, T]$.

Proof. We proceed analogously to the proof of the uniqueness in §3.8.2 of [10]. Let $s \in (0, T)$ and

$$v(t) := \begin{cases} -\int_t^s u(\sigma)\,\mathrm{d}\sigma & \text{for } t < s, \\ 0 & \text{for } t \geqslant s. \end{cases} \tag{4.20}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}[a_d(t, v(t), v(t)) + 2\,\mathrm{Re}\langle B'(t)v(t), v(t)\rangle + \|u(t)\|^2]$$

$$= 2\,\mathrm{Re}[a_d(t, u(t), v(t)) + \langle B'(t)u(t), v(t)\rangle + \langle u'(t), v'(t)\rangle] + r(t)$$

with

$$|r(t)| \leqslant d_3'[\|v(t)\|_m^2 + \|u(t)\|^2]$$

a.e. in $[0, s)$. We conclude from (4.19) that

$$a_d(0, v(0), v(0)) + 2\,\mathrm{Re}\langle B'(0)v(0), v(0)\rangle + \|u(s)\|^2 \leqslant d_4' \int_0^s [\|v(t)\|_m^2 + \|u(t)\|^2]\,\mathrm{d}t. \tag{4.21}$$

There exists a $d_5' > 0$ such that

$$a_d(0, v(0), v(0)) + 2\,\mathrm{Re}\langle B'(0)v(0), v(0)\rangle \geqslant \frac{d_1}{2}\|v(0)\|_m^2 - d_5'\|v(0)\|^2.$$

Hence we obtain from (4.21) with the argument proving the uniqueness in §3.8.2 of [10] that $u(t) = 0$ for $t \in [0, T]$.

In the next step we prove the existence of a weak solution $u \in \mathscr{C}_T^1$ of (1.1).

Lemma 4.2. *Let Assumption* 1.1 *be satisfied for* $k = k_0$. *Furthermore let* $u^0 \in \mathring{H}^m(\Omega)$, $u^1 \in L_2(\Omega)$ *and* $f \in L_1([0, T], L_2(\Omega))$. *Then there exists a weak solution* $u \in \mathscr{C}_T^1$ *of* (1.1).

Proof. We approximate u^0, u^1 and f by sequences $\{u_j^0\}$ in $H^{2m}(\Omega) \cap \mathring{H}^m(\Omega)$, $\{u_j^1\}$ in $\mathring{H}^m(\Omega)$ and $\{f_j\}$ in $H^1([0, T], L_2(\Omega))$ such that $\|u^\nu - u_j^\nu\|_{(1-\nu)m} \to 0$ as $j \to \infty$

($\nu = 0, 1$) and

$$\int_0^T \|f(\tau) - f_j(\tau)\| \, d\tau \to 0 \quad \text{as } j \to \infty. \tag{4.22}$$

Let $u_j \in H^2([0, T], L_2(\Omega)) \cap H^1([0, T], \mathring{H}^m(\Omega)) \subset \mathscr{C}_T^1$ be the solution of (4.15) with data u_j^0, u_j^1 and f_j, which we have constructed above. We apply (4.18) to

$$u_j''(t) - u_\ell''(t) + A(t)[u_j(t) - u_\ell(t)] + B(t)[u_j'(t) - u_\ell'(t)] = f_j(t) - f_\ell(t) \quad \text{a.e. in } [0, T] \tag{4.23}$$

and obtain that $\sup_{[0,T]} |u_j(t) - u_\ell(t)|_1 \to 0$ as $j, \ell \to \infty$. Let $u \in \mathscr{C}_T^1$ be the limit of $\{u_j\}$. Since every u_j is a weak solution of (1.1) with data u_j^0, u_j^1 and f_j, it follows that u is a weak solution of (1.1).

We note that (4.16) implies

$$|u(t)|_E \leq |u(0)|_E + \int_0^t [\|f(\tau)\| + d_2 |u(\tau)|_E] \, d\tau.$$

With (4.3) it follows that

$$|u(t)|_1 \leq d_6 \left(|u(0)|_1 + \int_0^t [\|f(\tau)\| + |u(\tau)|_1] \, d\tau \right) \quad \text{for } t \in [0, T]. \tag{4.24}$$

This estimate holds for every u_j used in the above proof and therefore even for the weak solution $u \in \mathscr{C}_T^1$ of (1.1).

Consider the solution $u \in H^2([0, T], L_2(\Omega)) \cap H^1([0, T], \mathring{H}^m(\Omega))$ of (4.15) constructed above. It holds that

$$A(t)u(t) = f(t) - u''(t) - B(t)u'(t) \tag{4.25}$$

a.e. in $[0, T]$. We consider a fixed $t \in [0, T]$ and conclude by Lemma 3.1 (applied to functions being constant in t) that

$$\|u(t)\|_{2m} \leq d_7 [\|f(t)\| + \|u''(t)\| + \|u'(t)\|_m + \|u(t)\|]. \tag{4.26}$$

Since this inequality holds a.e. in $[0, T]$, we obtain that $u \in L_2([0, T], H^{2m}(\Omega))$.

From (4.11), $u_j'(0) \to u^1$, $u_j''(0) \to u^2$ and the construction of the solution u of (4.15) it follows by a standard argument (compare §3.8.2 of [10]) that $v := u'$ is a weak solution of

$$v''(t) + A(t)v(t) + B(t)v'(t) = f'(t) - A'(t)u(t) - B'(t)u'(t) \quad \text{for } t \in [0, T], \tag{4.27a}$$

$$v(t) \in \mathring{H}^m(\Omega) \quad \text{for } t \in [0, T], \tag{4.27b}$$

$$v(0) = u^1, \quad v'(0) = u^2. \tag{4.27c}$$

Since $f' - A'u - B'u' \in L_2([0, T], L_2(\Omega))$, Lemma 4.2 yields the existence of a weak solution $v \in \mathscr{C}_T^1$ of (4.27). From the uniqueness of v we obtain $u' = v$ and hence $u' \in \mathscr{C}_T^1$. With $u \in \mathscr{C}_T^1$, (4.25) and Lemma 3.1 we conclude that $u \in \mathscr{C}_T^2$. We apply (4.24) to (4.27) and obtain

$$|u'(t)|_1 \leq d_6 \left(|u'(0)|_1 + \int_0^t \|f'(t) - A'(t)u(t) - B'(t)u'(t)\| \, dt \right). \tag{4.28}$$

With (4.24) and (4.26) it follows that

$$|u(t)|_2 \leqslant d_8\left(|u(0)|_2 + \int_0^t [\|f'(\tau)\| + \|f(\tau)\| + |u(\tau)|_2]\,d\tau + \|f(t)\|\right) \tag{4.29}$$

for $t\in[0, T]$. Thus we have proved the following lemma.

Lemma 4.3. *Let Assumption* 1.1 *be satisfied for* $k = k_0$. *If* $u^0 \in H^{2m}(\Omega)\cap \mathring{H}^m(\Omega)$, $u^1 \in \mathring{H}^m(\Omega)$ *and* $f\in H^1([0, T], L_2(\Omega))$, *then* (1.1) *has a unique solution* $u\in\mathscr{C}_T^2$. *Furthermore* (4.29) *holds, where* $d_8 > 0$ *depends only on* c_1, c_2, c_3 *of Assumption* 1.1 *and on* (1.11), (1.13).

5. Higher regularity

We prove Theorem 1.1 by induction with respect to j. Instead of (1.10) we prove

$$|u(t)|_j \leqslant D_3\left(|u(0)|_j + \int_0^t [\|\partial_t^{j-1}f(\tau)\| + |f(\tau)|_{j-2} + C_3|u(\tau)|_j]\,d\tau\right)$$
$$+ D_4|f(t)|_{j-2} \quad \text{for } t\in[0, T], \tag{5.1}$$

where $C_3 > 0$ depends only on j, c_1, c_2 and c_3 (of Assumption 1.1) and on (1.11), (1.12), while $D_3, D_4 > 0$ depend only on j, c_1, c_2, c_3 and on (1.11), (1.13). Note that it follows from (5.1) by Gronwall's lemma that

$$|u(t)|_j \leqslant e^{D_3C_3t}\left(2D_3|u(0)|_j + D_3(2 + C_3D_4)\int_0^t [\|\partial_t^{j-1}f(\tau)\| + |f(\tau)|_{j-2}]\,d\tau\right)$$
$$+ D_4|f(t)|_{j-2} \quad \text{for } t\in[0, T]. \tag{5.2}$$

Hence (5.1) implies (1.10).

For $j = 2$, Theorem 1.1 is proved by Lemma 4.3. In order to prove Theorem 1.1 for $j \geqslant 3$, we differentiate (1.1) with respect to t (formally) and set $v := u'$. This yields

$$v''(t) + A(t)v(t) + B(t)v'(t) = f'(t) - A'(t)u(t) - B'(t)u'(t) \quad \text{for } t\in[0, T], \tag{5.3a}$$

$$v(t)\in \mathring{H}^m(\Omega) \quad \text{for } t\in[0, T], \tag{5.3b}$$

$$v(0) = u^1, \quad v'(0) = u^2, \tag{5.3c}$$

$$u(t) = u^0 + \int_0^t v(\tau)\,d\tau \quad \text{for } t\in[0, T] \tag{5.4}$$

with $u^2 = f(0) - A(0)u^0 - B(0)u^1$. On the other hand, if $(u, v)\in\tilde{\mathscr{C}}_T^{j+1}\times\mathscr{C}_T^j$ is a solution of (5.3), (5.4) (with $j \geqslant 2$), then $u\in\mathscr{C}_T^{j+1}$ and u solves (1.1). In fact, it follows from $u^0 \in \mathring{H}^m(\Omega)$, $v(t)\in \mathring{H}^m(\Omega)$ for $t\in[0, T]$ and (5.4) that $u(t)\in \mathring{H}^m(\Omega)$ for $t\in[0, T]$. Furthermore (5.4) implies $u' = v$ and $u(0) = u^0$, $u'(0) = u^1$. Integrating (5.3) with respect to t we obtain that (1.1) holds. Finally, it follows from

$$A(t)u(t) = f(t) - u''(t) - B(t)u'(t) \quad \text{for } t\in[0, T] \tag{5.5}$$

by Lemma 3.1 that $u\in C([0, T], H^{(j+1)m}(\Omega))$. This and $u\in\tilde{\mathscr{C}}_T^{j+1}$ imply $u\in\mathscr{C}_T^{j+1}$.

Let Theorem 1.1 be proved for $j = 2, \ldots, J \leqslant k - 1$. We prove the existence of a solution $(u, v)\in\tilde{\mathscr{C}}_T^{J+1}\times\mathscr{C}_T^J$ of (5.3), (5.4) by the method of successive approximations (as in [5]). Let $u_0 \in \tilde{\mathscr{C}}_T^{J+1}$ such that $\partial_t^\nu u_0(0) = u^\nu$ for $\nu = 0, \ldots, J$ (compare Lemma 8.8 of [9]); here u^ν is defined by (2.7) for $\nu \geq 2$. We define u_μ for $\mu \geq 1$ by

$$v_\mu''(t) + A(t)v_\mu(t) + B(t)v_\mu'(t) = f'(t) - A'(t)u_{\mu-1}(t) - B'(t)u_{\mu-1}'(t) \quad \text{for } t \in [0, T], \tag{5.6a}$$

$$v_\mu(t) \in \overset{\circ}{H}{}^m(\Omega) \quad \text{for } t \in [0, T], \tag{5.6b}$$

$$v_\mu(0) = u^1, \quad v_\mu'(0) = u^2, \tag{5.6c}$$

$$u_\mu(t) := u^0 + \int_0^t v_\mu(\tau)\,d\tau \quad \text{for } t \in [0, T]. \tag{5.7}$$

Then $u_\mu \in \tilde{\mathscr{C}}_T^{J+1}$ for $\mu = 1, 2, \ldots$, as can be seen in the following way: assume that $u_{\mu-1} \in \tilde{\mathscr{C}}_T^{J+1}$ and $\partial_t^\nu u_{\mu-1}(0) = u^\nu$ for $\nu = 0, \ldots, J$. Then $(u^1, u^2, f' - A'u_{\mu-1} - B'u_{\mu-1}')$ satisfies the compatibility condition for (5.6) of order J. Corollary 8.3 of [9] yields

$$f' - A'u_{\mu-1} - B'u_{\mu-1}' \in \tilde{\mathscr{C}}_T^{J-1} \subset \mathscr{C}_T^{J-2} \cap C^{J-1}([0, T], L_2(\Omega)). \tag{5.8}$$

By the induction hypothesis we obtain the existence of a unique solution $v_\mu \in \mathscr{C}_T^J$ of (5.6), and (5.7) yields $u_\mu \in \tilde{\mathscr{C}}_T^{J+1}$. Finally it follows from (5.6) and (5.7) that $\partial_t^\nu u_\mu(0) = u^\nu$ for $\nu = 0, \ldots, J$.

Consider $w_\mu := u_{\mu+1} - u_\mu$, $\tilde{w}_\mu := v_{\mu+1} - v_\mu$. It holds

$$\tilde{w}_\mu''(t) + A(t)\tilde{w}_\mu(t) + B(t)\tilde{w}_\mu'(t) = -A'(t)w_{\mu-1}(t) - B'(t)w_{\mu-1}'(t)$$
$$\text{for } t \in [0, T], \tag{5.9a}$$

$$\tilde{w}_\mu(t) \in \overset{\circ}{H}{}^m(\Omega) \quad \text{for } t \in [0, T], \tag{5.9b}$$

$$\partial_t^\nu \tilde{w}_\mu(0) = 0 \qquad \text{for } \nu = 0, 1, \ldots, J, \tag{5.9c}$$

$$w_\mu(t) = \int_0^t \tilde{w}_\mu(\tau)\,d\tau \quad \text{for } t \in [0, T] \tag{5.10}$$

for $\mu \geq 1$. By Corollary 8.3 in [9] we obtain

$$|A'(t)w_{\mu-1}(t) + B'(t)w_{\mu-1}'(t)|_{J-1}^{\sim} \leq d_1'|w_{\mu-1}(t)|_{J+1}^{\sim}, \tag{5.11}$$

$$|A'(t)w_{\mu-1}(t) + B'(t)w_{\mu-1}'(t)|_{J-2} \leq d_2'|w_{\mu-1}(t)|_J \tag{5.12}$$

for $t \in [0, T]$. Here and in the following we denote by $d_1', d_2', \ldots$, positive constants that may depend on c_1, c_2, c_3 (of Assumption 1.1), J and on (1.11), (1.12). We apply the induction hypothesis and (5.2) to (5.9). This yields

$$|\tilde{w}_\mu(t)|_J \leq d_3' e^{d_4' t} \int_0^t |w_{\mu-1}(\tau)|_{J+1}^{\sim}\,d\tau + d_5'|w_{\mu-1}(t)|_J$$
$$\leq (d_3' + d_5')e^{d_4' T} \int_0^t |w_{\mu-1}(\tau)|_{J+1}^{\sim}\,d\tau, \tag{5.13}$$

since $\partial_t^\nu w_\mu(0) = 0$ for $\nu = 0, \ldots, J$ and therefore

$$|w_{\mu-1}(t)|_J \leq \int_0^t |w_{\mu-1}'(\tau)|_J\,d\tau \leq \int_0^t |w_{\mu-1}(\tau)|_{J+1}^{\sim}\,d\tau \tag{5.14}$$

for $t \in [0, T]$.

From (5.10) and (5.13) we conclude that

$$\begin{aligned}|w_\mu(t)|_{J+1}^{\sim} &= \|w_\mu(t)\|_{Jm} + |w'_\mu(t)|_J \\ &\leqslant \int_0^t \|\tilde{w}_\mu(\tau)\|_{Jm}\mathrm{d}\tau + |\tilde{w}_\mu(t)|_J \\ &\leqslant (d'_3 + d'_5)(1+T)\mathrm{e}^{d'_4 T}\int_0^t |w_{\mu-1}(\tau)|_{J+1}^{\sim}\,\mathrm{d}\tau. \end{aligned} \tag{5.15}$$

Let $K(T) := (d'_3 + d'_5)(1+T)\mathrm{e}^{d'_4 T}$. By induction it follows that

$$|w_\mu(t)|_{J+1}^{\sim} \leqslant K(T)^\mu \frac{t^{\mu-1}}{(\mu-1)!}\int_0^T |w_0(\tau)|_{J+1}^{\sim}\,\mathrm{d}\tau \quad \text{for } t\in[0,T]. \tag{5.16}$$

Note that $w_\mu = u_{\mu+1} - u_\mu$, $w'_\mu = v_{\mu+1} - v_\mu$. Hence we obtain from (5.16) that

$$\begin{aligned}|v_\mu(t) - v_\nu(t)|_J &\leqslant |u_\mu(t) - u_\nu(t)|_{J+1}^{\sim} \\ &\leqslant C\sum_{l=\nu}^{\mu-1} K(T)^l \frac{T^{l-1}}{(l-1)!} \quad \text{for } \mu > \nu,\ t\in[0,T]\end{aligned} \tag{5.17}$$

with some $C > 0$. Hence $\{(u_\mu, v_\mu)\}$ converges in $\tilde{\mathscr{C}}_T^{J+1}\times\mathscr{C}_T^J$. From (5.6), (5.7) it follows that the limit $(u, v)\in\tilde{\mathscr{C}}_T^{J+1}\times\mathscr{C}_T^J$ is a solution of (5.3), (5.4). Thus $u\in\mathscr{C}_T^{J+1}$ and u is a solution of (1.1) by the considerations following (5.4).

It remains to prove (5.1) for $j = J+1$. In order to apply (5.1) with $j = J$ to (5.3) we note that

$$\begin{aligned}&\|\partial_t^{J-1}[A'(t)u(t) + B'(t)u(t)]\| + |A'(t)u(t) + B'(t)u(t)|_{J-2} \\ &\leqslant |A'(t)u(t) + B'(t)u(t)|_{J-1}^{\sim} \leqslant d'_6|u(t)|_{J+1}^{\sim},\end{aligned} \tag{5.18}$$

$$\begin{aligned}|A'(t)u(t) + B'(t)u(t)|_{J-2} &\leqslant d_1|u(t)|_J \\ &\leqslant d_1\left(|u(0)|_J + \int_0^t |u(\tau)|_{J+1}^{\sim}\,\mathrm{d}\tau\right)\end{aligned} \tag{5.19}$$

by Corollary 8.3 of [9]; here d_1 depends on J, c_1, c_2, c_3 and on (1.11), (1.13). Note that $v = u'$ in (5.3), (5.4). Applying (5.1) with $j = J$ to (5.3) and using (5.18), (5.19), we conclude that

$$\begin{aligned}|u'(t)|_J &\leqslant (D_3 + D_4 d_1)|u(0)|_J + D_3\int_0^t [\|\partial_t^J f(\tau)\| + |f'(\tau)|_{J-2}]\,\mathrm{d}\tau \\ &\quad + [D_3(C_3 + d'_6) + D_4 d_1]\int_0^t |u(\tau)|_{J+1}^{\sim}\,\mathrm{d}\tau + D_4|f'(t)|_{J-2} \\ &\quad \text{for } t\in[0,T].\end{aligned} \tag{5.20}$$

From (5.5) we obtain by Lemma 8.2 of [9] and by Lemma 3.1 that

$$\|u(t)\|_{(J+1)m} \leqslant d_2[\|f(t)\|_{(J-1)m} + |u'(t)|_{J-1} + \|u(t)\|] \tag{5.21}$$

for $t\in[0,T]$, where $d_2 > 0$ depends on J, c_1, c_2, c_3 and (1.11), (1.13). Combining (5.1) (with $j = J$), (5.20) and (5.21) we conclude that (5.1) holds with $j = J+1$.

Acknowledgements

The author is grateful to Professor R. Leis for encouraging this research and to H. Koch for intensive discussions.

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