General Interface Problems—I

Serge Nicaise

Université de Valenciennes et du Hainaut Cambrésis, LM²1 and URAD 751 CNRS “GAT”,
Institut des Sciences et Techniques de Valenciennes, B.P. 311, F-59304-Valenciennes Cedex, France

and

Anna-Margarete Sändig

Mathematisches Institut a Universität Stuttgart, Postfach 80 11 40, D-70511 Stuttgart, Germany

Communicated by E. Meister

We study transmission problems for elliptic operators of order $2m$ with general boundary and interface conditions, introducing new covering conditions. This allows to prove solvability, regularity and asymptotics of solutions in weighted Sobolev spaces. We give some numerical examples for the location of the singular exponents.

1. Introduction

Boundary value problems in non-smooth domains are extensively studied in the literature; but for the applications interface problems (also called transmission problems) often appear, for instance, in solid mechanics if a body consists of composite materials. Unfortunately, in the literature, transmission problems are only studied in some particular cases (essentially, for second-order operators, see e.g. [3, 10, 17, 12, 8, 24, 19] and the references cited therein). So our main goal is to extend the theory of boundary value problems in non-smooth domains to general interface problems for operators of arbitrary order with non-constant coefficients. We restrict ourselves to domains of the plane $\mathbb{R}^2$ or the space $\mathbb{R}^3$, since these are the realistic domains for the mechanical applications. Obviously, the results given here could be extended to domains of $\mathbb{R}^n$, for arbitrary $n \geq 2$.

Let us mention that some mechanical problems lead to transmission problems set on 2-D networks. Such problems were studied in [22, 23] for the Laplace operator and the biharmonic one. In these papers, we actually see that boundary value problems in non-smooth domains and transmission problems are in the same framework of networks. Therefore, we certainly could extend the results stated here to 2-D or 3-D networks.
Our work is divided into two parts: In Part I, we introduce the class of interface problems in which we shall consider and study the regularity results in weighted Sobolev spaces for homogeneous operators with constant coefficients. We also give some numerical results about the eigenvalue problems. In Part II, we shall extend the previous results to general operators in usual Sobolev spaces. Finally, we shall consider the stabilization procedure which is necessary when unstable decompositions appear near critical angles of the conical points.

The plan of Part I is the following: In section 2, we define the general class of interface problems which we shall study. Along the external boundary, we impose classical boundary conditions satisfying the so-called Shapiro-Lopatinskii conditions, while on the interfaces we define general transmission conditions and introduce a new covering condition. Roughly speaking, this condition means that at every point of the interfaces we transform locally our interface problem into a system of boundary value problems, which satisfies the Shapiro-Lopatinskii condition. These problems will be called regular elliptic transmission problems.

For the applications, it is important to consider a weak formulation of interface problems. With the help of Green's formula, we describe 'weak' transmission problems, and give a necessary and sufficient condition in order to be regular elliptic ones (based on Agmon's arguments [1, 27]).

In section 3, the Shapiro-Lopatinskii conditions allow us to use Agranovitch-Visik's results to get, as for boundary value problems, the solvability, regularity and asymptotics of solutions in weighted Sobolev spaces.

Using the previous results and a lifting trace theorem we prove in section 4 that the weak solution of a transmission problem for homogeneous operators with constant coefficients admits a decomposition into singular and regular parts in weighted Sobolev spaces. This is made in a more or less usual way, except in dimension 2 and data in $H^t$, $k$ a positive integer (due to the limit case of the Sobolev imbedding theorem), where we use an interpolation argument (see [22] for a particular situation). Let us also notice that our boundary and interface conditions are not preserved by multiplication by cut-off functions. This leads to some difficulties, which are not usual.

In section 5, we finally give some numerical results about the location of the eigenvalues for some practical transmission problems in dimension 2, for instance, we consider the Laplace operator, the biharmonic operator and the Lamé system with different boundary and interface conditions for variable angles. In a forthcoming work, we shall consider some three-dimensional examples.

### 2. Formulation of the problem

#### 2.1. The domains

We shall consider interface problems in the following two- or three-dimensional domains (which is a realistic case).

In dimension 2, we suppose that $\Omega$ is a bounded domain consisting of $N$ parts $\Omega_i$, $i = 1, \ldots, N$, such that

$$\Omega = \bigcup_{i=1}^{N} \Omega_i.$$
We suppose that the boundary of $\Omega_i$ is Lipschitz-continuous and is a smooth curvilinear polygons, i.e.,

$$\partial \Omega_i = \bigcup_{q=1}^{Q_i} \tilde{\gamma}_{iq},$$

where the sides $\gamma_{iq}$ are smooth curves. The vertices of $\Omega_i$ will simply be the intersection between two consecutive sides.

The compatibility conditions between the $\Omega_i$'s are the following ones (see Fig. 1): for all $i, j \in \{1, \ldots, N\}; i \neq j$, one of the following holds:

(i) $\tilde{\Omega}_i \cap \tilde{\Omega}_j = \emptyset,$

(ii) $\tilde{\Omega}_i \cap \tilde{\Omega}_j$ is a whole common side, denoted by $\Gamma_{ij}$ (it is more convenient to suppose that $\Gamma_{ij}$ is open),

(iii) $\tilde{\Omega}_i \cap \tilde{\Omega}_j$ is a common vertex.

This definition is in accordance with the notion of two-dimensional topological networks introduced in Ref. [8].

Remark. In the above definition, the case of an interface which is smooth except at a finite number of points is included, introducing then some new artificial interfaces (see Fig. 2).

In dimension 3, for simplicity, we suppose that $\Omega$ is the union of two domains $\Omega_1, \Omega_2$:

$$\Omega = \tilde{\Omega}_1 \cup \tilde{\Omega}_2,$$

where $\tilde{\Omega}_1$ is completely included into $\Omega$. The boundary of $\Omega_1$ (resp. $\Omega_2$) is supposed to be smooth except at one point $0$, where $\Omega_1$ (resp. $\Omega_2$) coincides with a smooth cone $K_1$ (resp. $K_2$) of $\mathbb{R}^3$ (i.e. the intersection of $K_1$ (resp. $K_2$) with the unit sphere is a domain $G_1$ (resp. $G_2$) with a smooth boundary) (see Fig. 3).

In this paper, we exclude domains with edges (see Fig. 4); nevertheless, it is possible to consider them using Dauge's techniques of [7], for instance.
Let us introduce some notations, which we shall use in the sequel:

1. \( S \) will be the set of the vertices of \( \Omega \) (i.e. the union of all vertices of the \( \Omega_i \)'s); obviously, in dimension 3, it is reduced to one point 0.

2. For a fixed \( S \in S \), \( \mathcal{N}_S \) will be the set of integers \( i \) such that the face \( \Omega_i \) contains \( S \), in other words:
   \[ \mathcal{N}_S = \{ i \in \{1, \ldots, N\} : S \in \Omega_i \} \]
   in dimension 2,
   \[ \mathcal{N}_S = \{1, 2\} \]
   in dimension 3.

3. The indices \( i, i' \) always run from 1 to \( N \) in dimension 2 and from 1 to 2 in dimension 3 without any comment.
For a function $u$ defined on $\Omega$, we denote by $u_i$ its restriction to $\Omega_i$.

(5) The dimension of $\Omega$ will be denoted by $n$.

(6) In dimension 3, in order to have coherent notations, we set:

\[ \gamma_{11} = \partial \Omega_1 \text{ is the boundary of } \Omega_1, \]
\[ \gamma_{21} = \gamma_{12} = \partial \Omega_2, \]

where $\gamma_{22}$ is the external boundary of $\Omega_2$ (or equivalently the boundary of $\Omega$) and $\gamma_{21}$ is the common boundary between $\Omega_1$ and $\Omega_2$.

We now define some Sobolev spaces in $\Omega$.

**Definition 2.1.** Let $k$ be a non-negative integer, $p$ a real number $\geq 1$. We set

\[ H^{k,p}(\Omega) = \{ u \in L^p(\Omega) : u_i \in W^{k,p}(\Omega_i) \}, \]

it is a Banach space with the norm

\[ \| u \|_{k,p,\Omega} := \left( \sum_i \| u_i \|_{W^{k,p}(\Omega_i)}^p \right)^{1/p}. \]

For $p = 2$, it is a Hilbert space and we simply write it as $H^k(\Omega)$.

We now define

\[ C^k_c(\Omega) = \{ u \in C^\infty(\Omega) : u = 0 \text{ in a neighbourhood of the vertices of } \Omega \}, \]
\[ H^k_0(\Omega) = C^\infty_0(\Omega)^{H^k(\Omega)}, \]
\[ H^k_0(\Omega) = \{ u \in H^k(\Omega) : u_i \in H^k(\Omega_i) \}, \]

this last one is clearly a closed subspace of $H^k(\Omega)$.

Let us remark that $u \in H^k_0(\Omega)$ satisfies

\[ D^\alpha u_i(S) = 0, \quad \forall \alpha : |\alpha| < k - 1, \quad S \in \mathcal{S}, \quad i \in \mathcal{N}_S. \quad (2.1) \]

Due to [7, Theorem A.7] in dimension 3 and (3.1) of [7] in dimension 2, we have

\[ H^k_0(\Omega) = \{ u \in H^k(\Omega) \text{ satisfying (2.1)} \}. \]

This space $H^k_0(\Omega)$ is introduced in order to cancel the corner effects in Green's formula, for instance.

### 2.2. The operators

**Condition I.** Let $A_i$ be an elliptic differential operator of order $2m$ on $\Omega_i$ (properly elliptic in dimension 2) given by

\[ A_i(x, D) = \sum_{|\alpha| \leq 2m} a^i_\alpha(x) D^\alpha, \quad (2.2) \]

with smooth coefficients, i.e.

\[ a^i_\alpha \in C^\infty(\bar{\Omega}_i). \]

We shall consider classical boundary conditions on the external boundary of $\Omega$, i.e. if $\gamma_{1q}$ is not a common side of two subdomains (in the sequel, we simply write $\gamma_{1q} \in \mathcal{S}$),
then we consider \( m \) boundary operators \( B_{iq_j} \), \( j = 0, \ldots, m-1 \), defined by
\[
B_{iq_j}(x, D) = \sum_{|x| \leq m_{iq_j}} b_{iq_j}(x) D^x,
\] (2.3)
with \( b_{iq_j} \in C^\infty(\gamma_{iq}) \), \( m_{iq_j} \) is the order of the operator \( B_{iq_j} \), supposed to be \( \leq 2m - 1 \).

**Condition II.** As usual, we suppose that the system \( B_{iq} = \{ B_{iq_j} \}_{j=0}^{m-1} \) covers the operators \( A_i \) on \( \gamma_{iq} \in \mathcal{S} \) (see [13, Definition 2.1.4] for the exact terminology).

Conversely, the interface conditions are defined on the 'internal boundary' of \( \Omega_j \), in other words, let
\[
\gamma_{iq} = \gamma_{i'q'}
\]
be the common side of \( \Omega_i \) and \( \Omega_{i'} \) (in the sequel, we write \( \gamma_{iq} = \gamma_{i'q'} \in \mathcal{S} \)). Then we suppose given \( 4m \) operators, \( B_{iq_j} \) and \( B_{i'q'j'} \), \( j = 0, \ldots, 2m-1 \) defined by
\[
B_{iq_j}(x, D) = \sum_{|x| \leq m_{iq_j}} b_{iq_j}(x) D^x,
\]
(2.4)
\[
B_{i'q'j'}(x, D) = \sum_{|x| \leq m_{iq_j}} b_{i'q'j'}(x) D^x,
\]
(2.5)
where \( b_{iq_j}, b_{i'q'j'} \in C^\infty(\gamma_{iq}) \), \( m_{iq_j} \) being the order of both operators \( B_{iq_j} \) and \( B_{i'q'j'} \).

The covering condition is now expressed as follows: denote by \( A_i^0 \) the principal part of \( A_i \) on \( \Omega_j \):
\[
A_i^0(x, D) = \sum_{|x| = 2m} a_i^0(x) D^x.
\]
Analogously, \( B_{iq_j}^0 \) and \( B_{i'q'j'}^0 \) are the principal parts of \( B_{iq_j} \) and \( B_{i'q'j'} \).

For every couple of linearly independent vectors \((\zeta, \zeta') \in \mathbb{R}^n \), the ellipticity condition implies that the polynomial (in \( \tau \))
\[
A_i^0(x, \zeta + \tau \zeta')
\]
has \( m \) roots with positive imaginary parts denoted by \( \tau_{ik}^+(x, \zeta, \zeta') \), \( k = 1, \ldots, m \).

**Condition III.** For all \( x \in \gamma_{iq} = \gamma_{i'q'} \in \mathcal{S} \), all vectors \( \zeta \in \mathbb{R}^n \) tangent to \( \gamma_{iq} \) at \( x \) and all vectors \( \zeta' \in \mathbb{R}^n \) normal to \( \gamma_{iq} \) at \( x \), the rows of the matrix
\[
((B_{iq_j}^0(x, \zeta + \tau \zeta'))_{j=0}^{2m-1}(B_{iq_j}^0(x, \zeta - \tau \zeta'))_{j=0}^{2m-1}) \\
x \left( \begin{array}{cc}
A_i^0(x, \zeta - \tau \zeta') & 0 \\
0 & A_i^0(x, \zeta + \tau \zeta')
\end{array} \right)
\]
are linearly independent modulo the polynomial
\[
\prod_{k=1}^{m} (\tau - \tau_{ik}^+(x, \zeta, \zeta'))(\tau - \tau_{ik}^+(x, \zeta, -\zeta')).
\]
In other words, sending a neighbourhood \( \theta_i \) of \( x \) in \( \Omega_i \) into a neighbourhood \( \theta_i \) of \( x \) in \( \Omega_j \), by a mirror technique (i.e. flattening the boundary and using the change of variable \((x', x_n) \rightarrow (x', -x_n))\), this covering condition III says that the system
\[
((B_{iq_j}(x, D))_{j=0}^{2m-1}(B_{i'q'j}(x, D))_{j=0}^{2m-1})
\]
covers the system
\[
\begin{pmatrix}
A_i(x, D) & 0 \\
0 & \tilde{A}_i(x, D)
\end{pmatrix}
\]
in the sense of Douglish-Nirenberg, where \( \tilde{B}_{i'q'j}(x, D) \) (resp. \( \tilde{A}_i(x, D) \)) is the operator obtained from \( B_{i'q'j}(x, D) \) (resp. \( A_i(x, D) \)) by the above change of variables.

In this paper, we study the following interface problem:
\[
\begin{align*}
A_i u_i &= f_i \quad \text{in } \Omega_i, \quad \forall i = 1, \ldots, N, \\
B_{i'q'j} u_i &= g_{i'j} \quad \text{on } \gamma_{iq'}, \quad \forall j = 0, \ldots, m - 1, \quad \gamma_{iq} \in \mathcal{E}, \\
B_{i'q'j} u_i - B_{i'q'j} u_i &= g_{i'j} \quad \text{on } \gamma_{iq'} = \gamma_{i'q'}, \quad \forall j = 0, \ldots, 2m - 1, \quad \gamma_{iq} \in \mathcal{F}.
\end{align*}
\]

Actually, in the forthcoming sections, we shall investigate the solvability, regularity and asymptotic expansion of solution of this problem in weighted Sobolev spaces and in standard ones.

Since conditions I--III guarantee that we can handle transmission problems as elliptic boundary value problems for systems, we make the following definition.

**Definition 2.2.** We shall say that problem (2.6)--(2.8) is a regular elliptic transmission problem if conditions I--III are satisfied.

### 2.3. Green's formula

Let \( A_i \) be elliptic partial differential operators of order \( 2m \) satisfying condition I and given in divergence form, i.e., let \( a_{i\beta} \in C^\infty(\bar{\Omega}_i) \), for \( |\alpha|, |\beta| \leq m \), be such that \( A_i \) given by (2.2) admits also the expression
\[
A_i = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\beta|} D^\beta (a_{i\beta} D^\alpha).
\]

We now define the sesquilinear form \( a_i \) on \( H^m(\Omega_i) \) associated with \( A_i \) as follows:
\[
a_i(u, v) = \sum_{|\alpha|, |\beta| \leq m} \int_{\Omega_i} a_{i\beta}(x) D^\alpha u D^\beta \bar{v} \, dx.
\]
so that for every \( u, v \in \mathcal{D}(\Omega_i) \), we have
\[
a_i(u, v) = \int_{\Omega_i} A_i u \bar{v} \, dx.
\]

On each side \( \gamma_{iq} \) of \( \Omega_i \), we fix a Dirichlet system \( \{ F_{ij} \}_{j=0}^{m-1} \) of order \( m \) (see [13, Definition 2.2.1]) with coefficients in \( C^\infty(\bar{\gamma}_{iq}) \) and, without loss of generality, we may suppose that the order of \( F_{ij} \) is \( j \).

**Lemma 2.3.** For all sides \( \gamma_{iq} \) of \( \Omega_i \), there exists a normal system \( \{ \Phi_{ij} \}_{j=0}^{m-1} \) with coefficients in \( C^\infty(\bar{\gamma}_{iq}) \), the order of \( \Phi_{ij} \) being \( 2m - 1 - j \), such that
\[
\int_{\Omega_i} A_i u \bar{v} \, dx = a_i(u, v) + \sum_{q=1}^{2i} \sum_{j=0}^{m-1} \int_{\gamma_{iq}} \Phi_{ij} u F_{ij} \bar{v} \, d\sigma
\]
for all \( u \in H^{2m}(\Omega_i), \bar{v} \in H^m(\Omega_i) \).
Proof. For \( u \in C^\infty(\bar{\Omega}) \) and \( v \in C^\infty_0(\hat{\Omega}) \), (2.12) follows from [13, section 2.2.4] since we may ignore the vertices of \( \Omega \) by rounding the corner singularities outside the support of \( v \). In view of the classical results of [13, section 2.2.4], we immediately see that the \( \Phi_{ij} \)'s have coefficients in \( C^\infty(\bar{\Omega}) \) so that (2.12) follows by density.

If \( \gamma_{ij} \in \mathcal{E} \), we fix a partition of \( \{0, 1, \ldots, m - 1\} \) in \( \mathcal{F}_{ij} \cup \mathcal{F}_{iq} \), while if \( \gamma_{ij} = \gamma_{i'q'} \in \mathcal{I} \), we take a partition of \( \{0, 1, \ldots, m - 1\} \) in \( \mathcal{F}_{i'i} \cup \mathcal{F}_{j''} \cup \mathcal{F}_{i''} \). As usual, \( \mathcal{I} \) is the set of stable boundary or interface conditions, while \( \mathcal{F} \) will be the set of transversal ones (they do not have sense for \( u \in \mathcal{H}_m^m \)). We are now able to set

\[
V = \{ u \in \mathcal{H}_m^m(\Omega) \text{ satisfying } (2.13)-(2.15) \text{ hereafter} \},
\]

\[
F_{ij} u_j = 0 \quad \text{on } \gamma_{ij}, \quad \forall \gamma_{ij} \in \mathcal{E}, \ j \in \mathcal{F}_{ij},
\]

(2.13)

\[
\begin{aligned}
F_{iq} u_i &= 0 \quad \text{on } \gamma_{iq}, \\
F_{iq} u_i &= 0 \quad \text{on } \gamma_{iq}, \\
F_{iq} u_i &= F_{i'q'} u_{i'} \quad \text{on } \gamma_{iq}, \quad \forall \gamma_{iq} = \gamma_{i'q'} \in \mathcal{I}, \ j \in \mathcal{F}_{i'}^1, \\
F_{iq} u_i &= F_{i'q'} u_{i'} \quad \text{on } \gamma_{iq}, \quad \forall \gamma_{iq} = \gamma_{i'q'} \in \mathcal{I}, \ j \in \mathcal{F}_{i'}^2.
\end{aligned}
\]

(2.14)

(2.15)

Within this setting, some problems of the form (2.6)-(2.8) admit a weak formulation.

Lemma 2.4. Define the sesquilinear form \( a(\cdot, \cdot) \) on \( V \) as follows:

\[
a(u, v) = \sum_i a_i(u_i, v_i), \quad \forall u, v \in V.
\]

(2.16)

Let \( f \in L^2(\Omega) \) and let \( u \in \mathcal{H}_m^m(\Omega) \) be a solution of

\[
a(u, v) = \sum_i \int_{\Omega} f_i v_i \, dx, \quad \forall v \in V.
\]

(2.17)

Then \( u \) satisfies (2.6)-(2.8) with \( g_{ij} = 0, g_{ij} = 0 \), when the boundary and interface operators are defined as follows:

(i) If \( \gamma_{ij} \in \mathcal{E} \), then

\[
B_{ij} = \begin{cases}
F_{ij} & \text{if } j \in \mathcal{F}_{ij}, \\
\Phi_{ij} & \text{if } j \in \mathcal{F}_{ij}.
\end{cases}
\]

(ii) If \( \gamma_{ij} = \gamma_{i'q'} \in \mathcal{I} \), then

\[
\begin{aligned}
B_{ij} &= F_{ij}, \\
B_{ij} &= F_{ij}, \\
B_{ij} + m &= 0, \\
B_{i'q'} &= 0, \\
B_{i'q'} + m &= 0, \\
B_{i'q'} &= F_{i'q'}, \quad \text{if } j \in \mathcal{F}_{i'}^1, \\
B_{i'q'} &= F_{i'q'}, \\
B_{i'q'} + m &= -\Phi_{i'q'}, \quad \text{if } j \in \mathcal{F}_{i'}^2, \\
B_{i'q'} &= \Phi_{i'q'}, \\
B_{i'q'} &= 0, \\
B_{i'q'} + m &= 0, \\
B_{i'q'} &= \Phi_{i'q'}, \quad \text{if } j \in \mathcal{F}_{i'}.
\end{aligned}
\]
Proof. Applying (2.17) with \( v \) such that \( v_i \in \mathcal{D}(\Omega_i) \), for all \( i \), we see that \( u \) satisfies (2.6). The remainder follows from Green’s formula (2.12).  

An interface problem (2.6)-(2.8), defined by the previous procedure, will be called a weak transmission problem. An interesting question is to know whether it is a regular elliptic one. Following Agmon’s ideas [1], we give a necessary and sufficient condition of the form \( a(\cdot, \cdot) \) to satisfy the covering conditions of section 2.2.

We first start with a technical result which leads to the sufficiency of the condition. Introduce the Hilbert space

\[
Y = \prod_{\gamma_i \in \mathcal{S}} \prod_{j \in \mathcal{I}_{\gamma_i}} H^{m-j-1/2}(\gamma_{ij}) \times \prod_{\gamma_i = \gamma_i', \exists j \in \mathcal{I}_{\gamma_i}} \left\{ \prod_{j \in \mathcal{I}_{\gamma_i}} (H^{m-j-1/2}(\gamma_{ij}))^2 \right\} \times \prod_{j \in \mathcal{I}_{\gamma_i}} H^{m-j-1/2}(\gamma_{ij})
\]

and the trace-transmission operator

\[
F: \mathcal{M}^m(\Omega) \to Y: u \mapsto \{ F_{ij} u_i \}_{\gamma_i \in \mathcal{S}, j \in \mathcal{I}_{\gamma_i}} \times \{ F_{ij} u_i - F_{ij'} u_{i'} \}_{\gamma_i = \gamma_i'', \exists j \in \mathcal{I}_{\gamma_i}} \times \{ F_{ij} u_i - F_{ij'} u_{i'} \}_{\gamma_i = \gamma_i', j \in \mathcal{I}_{\gamma_i}},
\]

which is a bounded operator due to classical trace theorems.

**Proposition 2.5.** Assume that there exists two positive constants \( C_1, C_2 \) such that

\[
\| u \|_{L^2, \Omega} \leq C_1 \text{Re} a(u, u) + C_2(\| u \|_{H^1, \Omega} + \| F u \|_{L^2}) \quad \forall u \in \mathcal{M}^m(\Omega).
\]

Then

(i) for all \( \gamma_{ij} \in \mathcal{S} \) and all \( x \in \gamma_{ij} \), there exists a positive constant \( C \) such that

\[
|u|_{H^m(\mathbb{R}^d_t)}^2 \leq C \text{Re} \sum_{|\alpha| = |\beta| = m} \int_{\mathbb{R}^d_t} a_{\alpha\beta}(x)(D^\alpha u)(y)D^\beta \tilde{u}(y) \, dy + \sum_{j \in \mathcal{I}_{\gamma_i}} \| F^0_{ij}(x, D) u \|_{H^{m-j-1/2}(\mathbb{R}^d_t)}^2 \quad \forall u \in H^m(\mathbb{R}^d_t)
\]

(ii) for all \( \gamma_{ij} = \gamma_{i'j'} \in \mathcal{S} \) and all \( x \in \gamma_{ij} \), there exists a positive constant \( C \) such that

\[
|u_1|_{H^m(\mathbb{R}^d_t)}^2 + |u_2|_{H^m(\mathbb{R}^d_t)}^2 \leq C \text{Re} \sum_{|\alpha| = |\beta| = m} \left\{ \int_{\mathbb{R}^d_t} a_{\alpha\beta}(x)(D^\alpha u_1)(y)D^\beta \tilde{u}_1(y) \, dy \right\} + \sum_{j \in \mathcal{I}_{\gamma_i}} \| F^0_{ij}(x, D) u_1 \|_{H^{m-j-1/2}(\mathbb{R}^d_t)}^2 + \| F^0_{ij'}(x, D) u_2 \|_{H^{m-j-1/2}(\mathbb{R}^d_t)}^2
\]

for all \( (u_1, u_2) \in H^m(\mathbb{R}^d_t) \times H^m(\mathbb{R}^d_t) \), where \( \| \cdot \|_{H^m(\mathbb{R}^d_t)} \) denotes the semi-norm of \( H^m(\mathbb{R}^d_t) \).

**Proof.** Analogous arguments as in [7, Proposition 8.1].
Theorem 2.6. The sesquilinear form \( a(\cdot, \cdot) \) fulfills (2.19) if and only if problem (2.6)–(2.8) associated with \( a(\cdot, \cdot) \) via Lemma 2.4 is a regular elliptic transmission problem.

Proof. \( \Rightarrow \): For the external boundary, the sufficiency is a direct consequence of \([1, Theorems 5.1 and 3.2]\) since (2.20) is condition (5.6) of \([1, Theorem 5.1]\). For internal boundary points, it suffices to extend the previous theorems of \([1]\) to a system.

\( \Leftarrow \): We follow \([27, Theorem 19.3]\) which holds for boundary value problems. The estimates (2.20) and (2.22) follow using the Fourier transform and the covering conditions. Using a covering of \( \Omega \) by balls of sufficiently small radius and a perturbation argument, we obtain (2.19).

To finish this section, we shall show that if the range of the operator \( F \), defined by (2.19), is closed, then condition (2.19) is equivalent to the weak coerciveness of \( a(\cdot, \cdot) \) in \( V \). Let us notice that this range is always closed in dimension 3, while in dimension 2, it is not (see \([9, Theorem 1.6.1.5]\)).

Proposition 2.7. Assume that the range of \( F, R(F) \) is closed in \( Y \). Then (2.19) is equivalent to (2.22):

\[
\| u \|_{\tilde{H}^2,\Omega} \leq C \Re a(u, u) + C_0 \| u \|_{\tilde{H}^3,\Omega}, \quad \forall u \in V
\]

for some positive constants \( C, C_0 \).

Proof. It suffices to show that (2.22) implies (2.19). Since \( F \) is a bounded operator, by the closed graph theorem, there exists a positive constant \( C_4 \) such that

\[
\| \hat{u} \|^2_{\tilde{H}^m(\Omega)/\ker F} \leq C_4 \| Fu \|_Y, \quad \forall u \in \tilde{H}^m(\Omega),
\]

as usual \( \hat{u} \) denotes the equivalence class of \( u \in \tilde{H}^m(\Omega) \) in the quotient space \( \tilde{H}^m(\Omega)/\ker F \).

Fix \( u \in \tilde{H}^m(\Omega) \), and take an arbitrary \( v \) in \( \hat{u} \), since \( w := u - v \) belongs to \( V \), by (2.22) and the triangular inequality, we get

\[
\| u \|_{\tilde{H}^2,\Omega} \leq 2C \Re a(w, w) + 2C_0 \| w \|^2_{\tilde{H}^3,\Omega} + 2 \| v \|^2_{\tilde{H}^2,\Omega}.
\]

By the continuity of the form \( a(\cdot, \cdot) \) and the interpolation inequality

\[
ab \leq \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2, \quad \forall a, b, \varepsilon > 0,
\]

the previous inequality becomes

\[
\| u \|_{\tilde{H}^2,\Omega} \leq C_5 \Re a(u, u) + C_6 \{ \| u \|^2_{\tilde{H}^3,\Omega} + \| v \|^2_{\tilde{H}^2,\Omega}\}
\]

for some positive constants \( C_5 \) and \( C_6 \) independent of \( u \) and \( v \). Therefore, taking the infimum over \( v \) in (2.24) and using (2.23), we obtain (2.19).

2.4. Examples

In the sequel, we shall often illustrate our general statements by some classical examples of regular elliptic transmission problems: the Laplace operator with different media, the plate equation and the Lamé system with two different media both in the plane. For simplicity, we shall only give the formulation near a conical point.
Example 1. We consider the Dirichlet problem with $q$ different media ($q \geq 2$). More precisely, for $q$ openings, $\omega_l > 0$ and $q$ positive material constants $p_l$, $l = 1, \ldots, q$, we set (see Fig. 5)

$$
\sigma_0 = 0, \quad \sigma_l = \sum_{j=1}^{l} \omega_j, \quad l = 1, \ldots, q,
$$

$$
C_l = \{re^{i\omega} : r > 0, \sigma_{l-1} < \omega < \sigma_l\}, \quad l = 1, \ldots, q,
$$

$$
\Gamma_l = \{re^{i\omega} : r > 0\}, \quad \forall l = 0, 1, \ldots, q.
$$

We obviously suppose that $\sigma_q \leq 2\pi$ and define

$$
\Omega_l = C_l \cap B(0, 1),
$$

$$
\Gamma^l_1 = \Gamma_l \cap B(0, 1),
$$

$$
\Gamma^l_r = \partial \Omega_l \cap \partial B(0, 1).
$$

The following interface problem is extensively studied in the literature (see [12, 17, 21, 8] and the references cited therein):

$$
p_l \Delta u_l = f_l \text{ in } \Omega_l, \quad \forall l = 1, \ldots, q, \quad (2.25)
$$

$$
\gamma^l_+ u_l = \gamma^l_- u_{l+1} \quad \text{on } \Gamma^l_1, \quad \forall l = 1, \ldots, q-1, \quad (2.26)
$$

$$
p_l \gamma^l_+ \frac{\partial u_l}{\partial \nu} = p_{l+1} \gamma^l_- \frac{\partial u_{l+1}}{\partial \nu} \quad \text{on } \Gamma^l_1, \quad \forall l = 1, \ldots, q-1, \quad (2.27)
$$

$$
\gamma^0_+ u_1 = 0 \quad \text{on } \Gamma^0_1, \quad (2.28)
$$

$$
\gamma^q_+ u_q = 0 \quad \text{on } \Gamma^q_1, \quad (2.29)
$$

$$
\gamma^l_+ u_l = 0 \quad \text{on } \Gamma^l_r, \quad \forall l = 1, \ldots, q, \quad (2.30)
$$

where $\gamma^l_+$ is the trace operator on $\Gamma^l_1$.

This transmission problem (2.25)–(2.30) is clearly a regular elliptic one since the sesquilinear form, associated with it, is given by (see [8])

$$
\alpha(u, v) = \sum_{l=1}^{q} p_l \int_{\Omega_l} \nabla u_l \cdot \overline{\nabla v_l} \, dx.
$$
Therefore, $a(u, u)$ is equivalent to the semi-norm of $\mathcal{H}^1$; so, (2.19) obviously holds.

Some analogous transmission problems where the Laplace operator is replaced by the Helmholtz equation were studied by Meister in [17] and are also included in our framework (see also [6]).

For the three examples below, we shall use the notations of example 1 and only take two media (i.e. $q = 2$).

**Example 2.** We consider the pure transmission problem for the Laplace operator, i.e. $\sigma_2 = \omega_1 + \omega_2 = 2\pi$ (see also Fig. 2). Then given two material constants $p_1$ and $p_2$, we take (counting modulo 2)

$$ p_i \Delta u_i = f_i \quad \text{in } \Omega_i, \quad (2.31) $$

$$ \gamma^I u_i = \gamma^I u_{i+1} \quad \text{on } \Gamma^I, \quad (2.32) $$

$$ p_i \gamma^I \frac{\partial u_i}{\partial \omega} - p_{i+1} \gamma^I \frac{\partial u_{i+1}}{\partial \omega} = 0 \quad \text{on } \Gamma^I, \quad (2.33) $$

$$ \gamma^I u_i = 0 \quad \text{on } \Gamma^I, \forall i = 1, 2. \quad (2.34) $$

For the same reason as in example 1, it is a regular elliptic transmission problem.

**Example 3.** For $l = 1, 2$, we fix $E_l > 0$, Young's modulus and $v_l \in [0, 1]$, Poisson's ratio of the constitutive material of $C_l$. We also set

$$ \rho_l = E_l / (1 - v_l^2), $$

$$ M_l u = \rho_l \gamma^I \left( \nu_l \Delta u + (1 - v_l) \frac{\partial^2 u}{\partial (v^I)^2} \right), $$

$$ N_l u = \rho_l \gamma^I \left( \frac{\partial \Delta u}{\partial \nu^I} + (1 - v_l) \frac{\partial^3 u}{\partial v^I \partial (v^I)^2} \right), $$

where $v^I = (v^I_1, v^I_2)$ is the outer normal vector on $\partial \Omega_i$ in $\Omega_i$, while $\tau^I$ is the tangent vector to $\partial \Omega_i$, so that $(v^I, \tau^I)$ is a direct basis.

From [23], we know that the following transmission problem is a regular elliptic one:

$$ \rho_l \Delta^2 u_l = f_l \quad \text{in } \Omega_l, \quad l = 1, 2, \quad (2.35) $$

$$ \gamma^I \nu_1 u_1 = \gamma^I \frac{\partial u_1}{\partial \omega} = 0 \quad \text{on } \Gamma^I_0, \quad (2.36) $$

$$ \gamma^I \nu_2 u_2 = \gamma^I \frac{\partial u_2}{\partial \omega} = 0 \quad \text{on } \Gamma^I_2, \quad (2.37) $$

$$ \gamma^I \nu_1 u_1 = \gamma^I \nu_2 u_2 \quad \text{on } \Gamma^I_1, \quad (2.38) $$

$$ \gamma^I \frac{\partial u_1}{\partial \nu^I} = - \gamma^I \frac{\partial u_2}{\partial v^I} \quad \text{on } \Gamma^I_1, \quad (2.39) $$

$$ \gamma^I M_1 u_1 - \gamma^I M_2 u_2 = 0 \quad \text{on } \Gamma^I_1, \quad (2.40) $$

$$ \gamma^I N_1 u_1 + \gamma^I N_2 u_2 = 0 \quad \text{on } \Gamma^I_1, \quad (2.41) $$

$$ \gamma^I u_i = \gamma^I \frac{\partial u_i}{\partial \nu^I} = 0 \quad \text{on } \Gamma^I_i, \forall i = 1, 2. \quad (2.42) $$
Example 4. For the elasticity system with two different media, we introduce the Lamé constants $\lambda^l, \mu^l, l = 1, 2$ and set
\[
\sigma^l_{ij}(u) = \mu^l \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda^l \text{div } u \delta_{ij}, \forall i, j = 1, 2,
\]
where $u = (u_1, u_2)$.

We shall study the eigenvalues of the following transmission problems (2.43), (2.44) and (2.45), (2.45) (see [12], for more details), they are clearly regular elliptic using the so-called Korn’s inequality:
\[
\begin{aligned}
\frac{\partial}{\partial x_j} (\sigma^l_{ij}(u^l)) &= f^l \text{ in } \Omega^l, \quad l = 1, 2, \\
\gamma^l_1 u^l &= \gamma^l_1 u^2 \text{ on } \Gamma_1^l, \\
\gamma^l_1 (\sigma^l_{ij}(u^1) \cdot v_j^1) + \gamma^l_1 (\sigma^l_{ij}(u^2) \cdot v_j^2) &= 0 \text{ on } \Gamma_1^l, \\
\gamma^l_2 u^l &= 0 \text{ on } \Gamma_1^2, \quad \forall l = 1, 2,
\end{aligned}
\]
(2.43)
\[
\begin{aligned}
\gamma_0^l u^1 &= 0 \text{ on } \Gamma_0^1, \\
\gamma_0^l u^2 &= 0 \text{ on } \Gamma_0^2, \\
\gamma_0^l (\sigma^l_{ij}(u^1) \cdot v_j^1) &= 0 \text{ on } \Gamma_0^1, \\
\gamma_0^l (\sigma^l_{ij}(u^2) \cdot v_j^2) &= 0 \text{ on } \Gamma_0^2.
\end{aligned}
\]
(2.44)
(2.45)

3. Regularity in weighted Sobolev spaces

We have introduced in section 2 the spaces $H^1_c(\Omega)$ in connection with Green’s formula. But the theory of elliptic boundary value problems in domains with conical points is especially well-worked out in the framework of certain weighted Sobolev spaces, introduced by Kondratiev [11]. We now define the corresponding weighted Sobolev spaces for our boundary-transmission problem.

**Definition 3.1.** $V^{k,p}_b(\Omega)$ is the closure of $C^\infty(\Omega_0)$ with respect to the norm
\[
\| u^l; V^{k,p}_b(\Omega_i) \| = \sum_{|\alpha| \leq k} \| r_{\alpha}^l \| D^\alpha u^l \|_{L^p(\Omega_i)},
\]
(3.1)
where $r_{\alpha} = r_{\alpha}(x) = \text{dist}(x, \partial \Omega_i)$.

$V^{k,p}_b(\Omega) = \{ u \in L^p(\Omega); u|_{\Omega_i} = u_i \in V^{k,p}_b(\Omega_i) \}$
(3.2)
equipped with the norm
\[
\| u; V^{k,p}_b(\Omega) \| = \sum_{i=1}^N \| u^l; V^{k,p}_b(\Omega_i) \|.
\]

The space of traces on the boundary pieces of the subdomains $\Omega_i, i = 1, \ldots, N$; $q = 1, \ldots, Q_i$, is the quotient space
\[
V^{k,p}_b(\Omega_i, \gamma_{iq}) = V^{k,p}_b(\Omega_i) / \hat{V}^{k,p}_b(\Omega_i, \gamma_{iq}),
\]
where $V^{k,p}_b(\Omega_i, \gamma_{iq})$ is the closure of $C^\infty(\Omega_i_0)$, supp $u \cap \gamma_{iq} = \emptyset$ with respect to the norm (3.1).
We now formulate for the regular elliptic transmission problems (2.6)–(2.8), results about the solvability, regularity and asymptotic expansion, which can be proved analogously to the results for pure elliptic boundary value problems. We denote by

\[ U(x, D_x) = \{ A_j(x, D_x), B_{iqj}(x, D_x) \}_{j=0, \ldots, m-1} \]

the operator, which is generated by problem (2.6)–(2.8) and which maps \( \mathcal{V}_p^m(\Omega) \times \prod_{j=0}^{m-1} V_{p+1-m+1}^m(\gamma_{iq}) \) into \( \mathcal{V}_p^m(\Omega) \times \prod_{j=0}^{m-1} V_{p+1-m+1}^m(\gamma_{iq}) = \mathcal{V}_p^m(\Omega). \)

Let \( S \in \mathcal{S} \). We assume, for simplicity, that the domain \( \Omega_i \), for all \( i \in \mathcal{N}_S \), coincides in a neighbourhood of \( S \) with an infinite cone \( C_i \). Let \( C_S = \bigcup_{i \in \mathcal{N}_S} C_i \); all the spaces defined in \( \Omega \) are defined analogously in \( C_S \) (replacing, if necessary, \( r_i \) by the distance to \( S \)).

First we consider a special boundary-transmission problem in \( C_S \), which is generated by the principal parts of the operators (2.6)–(2.8) with frozen coefficients in \( S \):

\[
\begin{align*}
A_i^S u_i &= \sum_{|\alpha|=2m} a_i^S(S) D^\alpha u_i = f_i \quad \text{in } C_i, \quad i \in \mathcal{N}_S, \\
B_{iqj}^S u_i &= \sum_{|\alpha|=m+1} b_{iqj}(S) D^\alpha u_i = g_{iqj} \quad \text{on } \Gamma_{iq} \in \mathcal{E}_S, \quad j = 0, \ldots, m-1. \\
B_{iqj}^S u_i - B_{iqj}^S u_i &= \sum_{|\alpha|=m+1} [b_{iqj}(S) D^\alpha u_i - b_{iqj}(S) D^\alpha u_i] \\
&= g_{iqj} \quad \text{on } \Gamma_{iq} = \Gamma_{iq} \in \mathcal{E}_S, \quad j = 0, \ldots, 2m-1.
\end{align*}
\]

The notation \( \Gamma_{iq} \in \mathcal{E}_S \) or \( \Gamma_{iq} \in \mathcal{F}_S \) means \( \Gamma_{iq} \) is a side \( (n=2) \) or a surface \( (n=3) \) of the cone \( C_i \) and coincides in a neighbourhood of \( S \) with \( \gamma_{iq} \in \mathcal{E} \) or \( \gamma_{iq} \in \mathcal{F} \), respectively. In other words, \( \mathcal{E}_S \) is the union of the external pieces of the boundary of \( C_i \) and \( \mathcal{F}_S \) is the union of the interface pieces of the boundary of \( C_i \) for \( i \in \mathcal{N}_S \).

Analogous to (3.3), let us denote by \( U^S(D_x) \) the operator which is generated by (3.4)–(3.6).

Introducing polar co-ordinates \( r = r(x) = |x - S|, \omega \in S^{n-1}(S) \), where \( S^{n-1}(S) \) is a unit sphere around \( S \), and using the Mellin transform

\[
\frac{1}{\sqrt{2\pi}} \int_0^\infty r^{\lambda-1} u(r, \omega) dr = \hat{u}(\lambda, \omega),
\]

which maps \( rD_x \) into the complex parameter \( \lambda \), we obtain a boundary-transmission problem with parameter \( \lambda \) in the domains \( G_{iq}^S = \Omega_i \cap S^{n-1}(S), \ i \in \mathcal{N}_S \):

\[
\begin{align*}
L^S_i(\omega, D_\omega, \lambda) \hat{u}_i(\lambda, \omega) &= \hat{f}_i(\lambda, \omega) \quad \text{for } \omega \in G_{iq}^S, \\
M_{iqj}^S(\omega, D_\omega, \lambda) \hat{u}_i(\lambda, \omega) &= \hat{g}_{iqj}(\lambda, \omega) \quad \text{for } \omega \in \omega_{iq} \in \mathcal{E}_S, \\
M_{iqj}^S(\omega, D_\omega, \lambda) \hat{u}_i(\lambda, \omega) - M_{iqj}^S(\omega, D_\omega, \lambda) \hat{u}_i(\lambda, \omega) &= \hat{g}_{iqj}(\lambda, \omega) \\
&= \hat{g}_{iqj}(\lambda, \omega) \quad \text{for } \omega \in \omega_{iq} = \omega_{iq} \in \mathcal{F}_S.
\end{align*}
\]
where \( \omega_{i\ell} = \Gamma_{i\ell} \cap S^{n-1}(S) \). We say \( \omega_{i\ell} \in \mathcal{E}_S \) if \( \gamma_{i\ell} \in \mathcal{E} \) and \( \omega_{i\ell} \in \mathcal{F}_S \), if \( \gamma_{i\ell} \in \mathcal{F} \). Furthermore, it holds

\[
A^s_i(S, D_x) = r^{-2m} L^s_i(\omega, D_x, rD_x),
\]

\[
B^s_{i\ell}(S, D_x) = r^{-m_{i\ell}} M^s_{i\ell}(\omega; D_x, rD_x).
\]

The operator

\[
\mathcal{A}^S_i(\lambda) = \{ L^s_i(\omega, D_x, \lambda), M^s_{i\ell}(\omega, D_x, \lambda) \},
\]

which is generated by (3.7)–(3.9) and which is defined in \( G_S = \bigcup_{i \in N_S} G^S_i \) maps continuously:

\[
\mathcal{W}^{-2m+1, p}(G_S) \cap \bigcap_{i \in N_S} \mathcal{W}^{-1, p}(G^S_i) \ni \prod_{i \in N_S} \mathcal{W}^{-1, p}(G^S_i) \times \prod_{\omega_{i\ell} \in \mathcal{E}_S} W^{l+2m-m_{i\ell}-1/p, p}(\omega_{i\ell})
\]

\[
\times \prod_{\omega_{i\ell} \in \mathcal{F}_S} W^{l+2m-m_{i\ell}-1/p, p}(\omega_{i\ell})
\]

Here it is

\[
\mathcal{W}^{-2m+1, p}(G_S) = \{ u : u \mid_{G^S_i} \in W^{2m+1, p}(G^S_i), \forall i \in N_S \}.
\]

The generalized eigenvalues of the operator \( \mathcal{A}^S_i(\lambda) \) play an important role for the solvability, the regularity and the asymptotic expansion of the solution near \( S \). Let us shortly introduce the necessary standard definitions.

**Definition 3.2.** The complex number \( \lambda = \lambda_0 \) is an eigenvalue of the operator \( \mathcal{A}^S_i(\lambda) \), if there is a non-trivial function \( \varphi^S, \lambda_0, 0 \in \mathcal{W}^{-2m+1, 2}(G_S) \) with \( \mathcal{A}^S_i(\lambda_0) \varphi^S, \lambda_0, 0 = 0 \). The function \( \varphi^S, \lambda_0, 0 \) is called an eigensolution of \( \mathcal{A}^S_i(\lambda) \) for \( \lambda = \lambda_0 \).

It follows from the ellipticity of the operator \( \mathcal{W}^S(x, D_x) \) in \( C_S \) (cf. conditions I–III), the ellipticity of the operator \( \mathcal{A}^S_i(\lambda) \) in the sense of Agranovic and Visik [2]. Therefore, \( I^S, \lambda_0 = \dim \ker \mathcal{A}^S_i(\lambda_0) \) is a finite number and in a finite strip \( h \leq \Re \lambda \leq h \), there are only a finite number of eigenvalues.

Besides the linearly independent eigensolutions \( \varphi^S, \lambda_0, \mu, 0, \mu = 1, \ldots, I^S, \lambda_0 \), there exist \( N^S, \lambda_0, \mu, k = 1, \ldots, N^S, \lambda_0, \mu \) associated eigensolutions \( \varphi^S, \lambda_0, \mu, k \) in general. We set \( N^S, \lambda_0, \mu = 0 \), if no associated eigensolution exist. The associated eigensolutions are defined by the following definition.

**Definition 3.3.** The system \( \{ \varphi^S, \lambda_0, \mu, k \}_{\mu = 1, \ldots, I^S, \lambda_0, k = 0, \ldots, N^S, \lambda_0, \mu} \) consists of eigensolutions and associated eigensolutions (usually called system of Jordan chains) if

\[
\sum_{q=0}^{k} \frac{1}{q!} (\partial/\partial \lambda)^q \mathcal{A}^S_i(\lambda) \varphi^S, \lambda_0, \mu, k-q |_{\lambda = \lambda_0} = 0
\]

for \( k = 0, \ldots, N^S, \lambda_0, \mu, N^S, \lambda_0, \mu \) being decreasing with respect to \( \mu \).

The following theorems can be proved analogously to the theorems for elliptic boundary value problems (see e.g. [11] for \( p = 2 \), [15] for \( p \neq 2 \)).

**Theorem 3.4.** The operator \( \mathcal{U}(x, D_x) \), defined by (3.3), is a Fredholm operator iff no eigenvalue of \( \mathcal{A}^S_i(\lambda) \) lies on the line \( \Re \lambda = -\beta - (n/p) + 2m + 1 \) for all \( S \in \mathcal{F} \).
Theorem 3.5. If for all \( S \in \mathcal{S} \) no eigenvalue of \( \mathcal{A}_S^S(\lambda) \) is situated in the strip
\[
- \beta - \frac{n}{p} + 2m + l \leq \text{Re}\lambda \leq - \beta_1 - \frac{n}{p_1} + 2m + l_1,
\]
then a solution \( u \in \mathcal{Y}^{-1,2m,p}(\Omega) \) of (2.6)–(2.8) is contained in \( \mathcal{Y}^{-1,2m,p}(\Omega) \) too, provided
\[
f \in \mathcal{Y}^{-1,2m,p}(\Omega) \cap \mathcal{Y}^{-1,2m,p}(\Omega), \quad g_{ij} \in \mathcal{Y}^{-2m-1,1,1,2m}(\gamma_{ij})
\]
for \( \gamma_{ij} \in \mathcal{S} \) and \( j = 0, \ldots, m-1 \), and \( \gamma_{ij} \in \mathcal{S} \) and \( j = 0, \ldots, 2m-1 \).

Here \( l \) and \( l_1 \) are non-negative integers and \( \beta \) and \( \beta_1 \) are real numbers.

Theorem 3.6. Assume that \( \mathcal{U}^S(D_\lambda) = \mathcal{U}(x, D_\lambda) \) in a neighbourhood of \( S \) and the lines
\[
\text{Re}\lambda = 2m + l - \frac{n}{p} - \beta = h, \quad \text{Re}\lambda = 2m + l_1 - \frac{n}{p_1} - \beta_1 = h_1
\]
have no eigenvalue of \( \mathcal{A}_S^S(\lambda) \) with \( h < h_1 \). Then the following asymptotic expansion of a solution \( u \in \mathcal{Y}^{-1,2m,p}(\Omega) \) of problem (2.6)–(2.8), with data as in Theorem 3.5, holds near \( S \):
\[
u = w + \eta^S \sum_{S,\mu,\nu,k} c_{S,\mu,\nu,k} \sigma^{S,\mu,\nu,k}, \tag{3.11}
\]
where \( w \in \mathcal{Y}^{-1,2m,p}(\Omega) \), \( c_{S,\mu,\nu,k} \in \mathcal{C} \), \( \eta^S \) is an appropriate cut-off function such that \( \eta^S \equiv 1 \) in a neighbourhood of \( S \), the sum extends to all eigenvalues \( \lambda \) of \( \mathcal{A}_S(\xi) \) in the strip \( \text{Re} \xi \in \ ]h, h_1[ \) and \( \mu = 1, \ldots, K^S, \ k = 0, \ldots, N^S, \mu \) and, finally,
\[
\sigma^{S,\mu,\nu,k} = \rho^k \sum_{q=0}^k \frac{(\ln r)^q}{q!} \phi^{S,\mu,\nu,k-q} \tag{3.12}
\]

There are formulae for the coefficients \( c_{S,\mu,\nu,k} \) as in [16], this shows that the coefficients depend continuously on the right-hand side.

4. Regularity results in usual Sobolev spaces

We only give these regularity results for weak transmission problems, which are regular and for homogeneous operators with constant coefficients. This means that we fix operators \( A_i \), Dirichlet systems \( \{F_{ij}\}_{j=0}^\infty \) on all \( \gamma_{ij} \) satisfying the assumptions of section 2.3. Moreover, we suppose that the corresponding transmission problem (2.6)–(2.8) is a regular elliptic one or equivalently we suppose that the associated sesquilinear form \( a(\cdot, \cdot) \) satisfies (2.19). We also assume that \( A_i \) and \( F_{ij} \) are homogeneous with constant coefficients.

For some technical reasons, we further suppose that the sesquilinear form \( a(\cdot, \cdot) \) is strictly coercive, i.e. there exists \( \alpha > 0 \) such that
\[
\text{Re} \ a(u, u) \geq \alpha \|u\|^2, \quad \forall u \in V. \tag{4.1}
\]
For simplicity, in dimension 2, we suppose that the \( \Omega_i \)'s coincide with a wedge in a neighbourhood of each vertex (this was already the case in dimension 3).
Lemma 4.1. Let \( u \in V \) be a solution of problem (2.17) with a right-hand side \( f \in L^p(\Omega) \), where \( 1 < p \leq 2 \) in dimension 2 and \( p = 2 \) in dimension 3. Then

\[ u \in \mathcal{V}^{-2m,p}(\Omega), \]

with \( \alpha = m + (n/2) - (n/p) + \gamma' \) for all \( \gamma' \in ]0, 2 - (n/2)[. \)

Proof. Classical regularity results for systems (see \([20]\)) and the covering conditions imply that \( u \) belongs to \( \mathcal{V}^{-2m,p} \) far from the vertices of \( \Omega \). Therefore, it remains to prove (4.2) near the vertices.

Applying \([7, \text{Proposition AA.29}]\) to \( u_i \) belonging to \( \mathcal{H}^m(\Omega_i) \), we deduce that

\[ u_i \in \mathcal{V}^{-m,2}(\Omega_i), \quad \forall \gamma \in \left[0, 2 - \frac{n}{2}\right]. \]

By Hölder's inequality when \( p < 2 \), (4.3) implies that

\[ u_i \in \mathcal{V}^{-0,p,2m}(\Omega_i) \]

for all \( \alpha \) given in the lemma.

We now fix a vertex \( S \) of \( \Omega \) and use spherical co-ordinates \((r, \omega)\) centred at \( S \). Let us consider the following dyadic cover of \( C^0_S = \{ x \in \Omega : r(x) < 2r_0 \} \) for \( r_0 > 0 \) sufficiently small such that \( C^0_S \subset \Omega \); for all \( v \in \mathbb{N}^* \), we set

\[ S_v = \left\{ x \in \Omega : \frac{r_0}{v} < r(x) < \frac{2r_0}{v} \right\}, \]

\[ S'_v = \left\{ x \in \Omega : \frac{r_0}{2v} < r(x) < \frac{3r_0}{v} \right\}. \]

Interior Agmon–Douglis–Nirenberg estimates for systems ensure the existence of \( C > 0 \) such that

\[ \| u \|_{2m,p,S_v} \leq C \{ \| Au \|_{0,p,S_v} + \| u \|_{0,p,S_v} \}. \]

By similarity, we get (since the operators have constant coefficients and are homogeneous)

\[ \sum_{|\beta| \leq 2m} v^{-|\beta|} \| D^\beta u \|_{0,p,S} \leq C \{ v^{-2mp} \| Au \|_{0,p,S_v} + \| u \|_{0,p,S_v} \}. \]

Multiplying (4.5) by \( v^{-a+2m} \), using the fact that \( r(x) \) is equivalent to \( v \) on \( S_v \) and \( S'_v \) and summing over \( v \in \mathbb{N}^* \), we obtain

\[ \| u; \mathcal{V}^{-2m,p}(C^0_S) \| \leq C \{ \| f \|_{0,p,\Omega} + \| u; \mathcal{V}^{-0,p,2m}(\Omega) \| \}. \]

This proves the lemma, owing to (4.4).

From now on, \( \Phi_S \) will denote a cut-off function fulfilling \( \Phi_S \in D(\mathbb{R}^n) \), \( \Phi_S = 1 \) in a neighbourhood of the vertex \( S \), and \( \Phi_S = 0 \) in a neighbourhood of the other vertices of \( \Omega \).

Theorem 4.2. Assume that \( k - \gamma \geq 0 \) and that the line \( \text{Re} \zeta = k + 2m - n/p - \gamma \) contains no eigenvalue of \( A_S(\zeta) \) for all \( S \in \mathcal{S} \). Then for all \( f \in \mathcal{V}^{-k,p}(\Omega) \), with \( p \in ]1, +\infty[ \) in
In dimension 2 and \( p = 2 \) in dimension 3, there exists a unique solution \( u \in V \) of the problem
\[
A_iu_i = f_i \quad \text{in} \quad \Omega_i, \quad \forall i = 1, \ldots, N, \tag{4.6}
\]
\[
B_{ij}u_j = 0 \quad \text{on} \quad \gamma_{ij}, \quad \forall j = 0, \ldots, m - 1, \quad \gamma_{ij} \in \mathcal{E}, \tag{4.7}
\]
\[
B_{ij}u_i - B_{i'j'}u_{i'} = 0 \quad \text{on} \quad \gamma_{ij} = \gamma_{i'j'}, \quad \forall j = 0, \ldots, 2m - 1, \quad \gamma_{ij} \in \mathcal{E}. \tag{4.8}
\]
This solution admits the following expansion:
\[
u = u_0 + \sum_{S \in \mathcal{E}} \sum_{(\lambda, \mu, k) \in \Lambda^J_0(k,p)} c_{S, \lambda, \mu, k} \Phi_{S\lambda} S^{\lambda, \mu, k}, \tag{4.9}
\]
where \( u_0 \in \mathcal{V}^{k+2m,p}(\Omega) \), \( c_{S, \lambda, \mu, k} \in \mathbb{C} \) and we set
\[
\Lambda^J_0(k,p) = \left\{ (\lambda, \mu, k): \lambda \text{ is an eigenvalue of } \mathcal{A}_S(\zeta) \text{ such that } \right\}
\]
\[
\Re \lambda \in \left[ m - \frac{n}{2}, k + 2m - \frac{n}{p} - \gamma \left[ m - \frac{n}{p} - \gamma \right], \mu = 1, \ldots, I_s^s, k = 0, \ldots, N_s^s \right]. \tag{4.10}
\]
Moreover, there exists a constant \( C > 0 \) independent of \( u \) such that
\[
\| u_0, \mathcal{V}^{k+2m,p}(\Omega) \| + \sum_{S \in \mathcal{E}} \| c_{S, \lambda, \mu, k} \| \leq C \| f, \mathcal{V}^{k+2m,p}(\Omega) \|. \tag{4.11}
\]
Proof. For \( n = 2 \), [22, Lemma 1.2] shows that there exists \( r \in [1, 2] \) such that \( f \in L^r(\Omega) \), while for \( n = 3 \), the assumption \( k - \gamma \geq 0 \) directly implies that \( f \in L^2(\Omega) \). By Lemma 4.1, \( u \) belongs to \( \mathcal{V}^{2m,r}(\Omega) \), with \( \alpha = m + (n/2) - (n/r) + \gamma \), for all \( \gamma \in [0, 2 - (n/2)] \), where \( r = 2 \) in dimension 3. We easily check that \( \alpha > 0 \), which implies that \( f \in \mathcal{V}^{0} \cap \mathcal{V}^{k+2m,p}(\Omega) \). Applying the comparison theorem in weighted Sobolev spaces (Theorem 3.6) in each cone \( C_S \) to \( \Phi_S u \in \mathcal{V}_g^{k+2m,p}(\Omega) \) and choosing \( \gamma' \) sufficiently close to 0, we conclude.

In the sequel, we shall need the following lifting trace theorem.

Lemma 4.3. For all \( g_{ij} \in \mathcal{V}_g^{k+2m-\mu,1/p}(\gamma_{ij}), \quad j \in \{0, \ldots, m - 1\}, \quad \gamma_{ij} \in \mathcal{E}, \) and all \( g_{ij} \in \mathcal{V}_g^{k+2m-\mu,1/p}(\gamma_{ij}), \quad j \in \{0, \ldots, 2m - 1\}, \quad \gamma_{ij} = \gamma_{ij} \in \mathcal{E}, \) there exists \( v \in \mathcal{V}_g^{k+2m,p}(\Omega) \) fulfilling
\[
B_{ij}v = g_{ij} \quad \text{on} \quad \gamma_{ij}, \quad \forall j = 0, \ldots, m - 1, \quad \gamma_{ij} \in \mathcal{E}, \tag{4.12}
\]
\[
B_{ij}v - B_{ij'}v = g_{ij} - g_{ij'} \quad \text{on} \quad \gamma_{ij}, \quad \forall j = 0, \ldots, 2m - 1, \quad \gamma_{ij} = \gamma_{ij} \in \mathcal{E}. \tag{4.13}
\]
Proof. We firstly build new functions
\[
h_{ij} \in \mathcal{V}_g^{k+2m-\mu,1/p}(\gamma_{ij}), \quad h_{ij} \in \mathcal{V}_g^{k+2m+\mu,1/p}(\gamma_{ij}),
\]
\( j = 0, 1, \ldots, m - 1 \) such that if \( v \in \mathcal{V}_g^{k+2m,p}(\Omega) \) fulfills
\[
F_{ij}v = h_{ij} \quad \text{on} \quad \gamma_{ij}, \quad \Phi_{ij}v = h_{ij} \quad \text{on} \quad \gamma_{ij}, \quad \forall j = 0, 1, \ldots, m - 1, \tag{4.14}
\]
then \( v \) satisfies (4.12) and (4.13). The construction is the following.
If $\gamma_{iq} \in \mathcal{G}$, then we set

$$h_{iq} = g_{iq}, h_{iq}^1 = 0 \quad \text{if} \quad j \in \mathcal{G}_{iq},$$

$$h_{iq} = 0, h_{iq}^1 = g_{iq} \quad \text{if} \quad j \in \mathcal{F}_{iq}.$$ 

If $\gamma_{iq} = \gamma_{i'q'} \in \mathcal{G}$, then we take

(a) for $j \in \mathcal{G}_{i'q'}$,

$$h_{iq} = g_{ii'j}, h_{iq}^1 = -g_{ii'j+m},$$

$$h_{iq}^1 = h_{iq}^1 = 0,$$

(b) for $j \in \mathcal{F}_{i'q'}$,

$$h_{iq} = g_{ii'j}, h_{iq}^1 = 0,$$

$$h_{iq}^1 = g_{ii'j+m}, h_{iq}^1 = 0,$$

(c) for $j \in \mathcal{F}_{i'q'}$,

$$h_{iq} = h_{iq}^1 = 0,$$

$$h_{iq}^1 = g_{ii'j}, h_{iq}^1 = -g_{ii'j+m}.$$

From the definition of the $B$'s with respect to the $F$'s and $\Phi$'s, one easily shows that (4.14), (4.15) imply (4.12), (4.13).

Since the system $\{F_{iqj}, \Phi_{iqj}\}_{j=0}^{m-1}$ is a Dirichlet system on $\gamma_{iq}$, using [14, Lemma 3.1] (which also holds in dimension 3) and local charts, problem (4.14), (4.15) has a solution $v_i \in V_{\gamma}^{k+2m} \mathcal{P}(\Omega_i)$. \hfill \square

**Corollary 4.4.** Let the assumptions of Theorem 4.2 be satisfied. Then for all $f \in \mathcal{F}_{i'q'}^{k+2m-\mu \beta}(\Omega), g_{iqj} \in V_{\gamma}^{k+2m-\mu \beta} \mathcal{P}(\gamma_{iq}), j \in \{0, \ldots, m-1\}, \gamma_{iq} \in \mathcal{G}, g_{ii'j} \in V_{\gamma}^{k+2m-\mu \beta} \mathcal{P}(\gamma_{iq}), j \in \{0, \ldots, 2m-1\}, \gamma_{iq} = \gamma_{i'q'} \in \mathcal{G}$, there exists a unique weak solution $u \in \mathcal{H}^\mu_\gamma(\Omega)$ of problem (2.6)–(2.8), in the following sense: for a fixed $v \in \mathcal{V}_{\gamma}^{k+2m} \mathcal{P}(\Omega)$ satisfying (4.12), (4.13), $v \in \mathcal{V}$ and is solution of

$$a(u, w) = \sum_i \int_{\Omega_i} f_i \bar{w}_i \, dx, \quad \forall w \in \mathcal{V}, \quad (4.16)$$

where we set $f_i = f_i - A_i v_i$. This is equivalent to say that $u$ belongs to the affine space $V_0 := v + \mathcal{V}$, and satisfies

$$a(u, w) = \sum_i \int_{\Omega_i} f_i \bar{w}_i \, dx - \sum_{\gamma_{iq} \in \mathcal{G}} \sum_{j \in \mathcal{F}_{i'q'}} \int_{\gamma_{iq}} g_{ij} F_{ij} \bar{w}_i \, d\sigma$$

$$- \sum_{\gamma_{iq} \in \mathcal{G}} \left[ \sum_{j \in \mathcal{F}_{i'q'}} \int_{\gamma_{iq}} g_{ij} F_{ij} \bar{w}_i \, d\sigma \right]$$

$$+ \sum_{j \in \mathcal{F}_{i'q'}} \int_{\gamma_{iq}} \left\{ g_{ii'j} F_{ij} \bar{w}_i + g_{ii'j+m} F_{i'q'} \bar{w}_i \right\} \, d\sigma, \quad \forall w \in \mathcal{V}. \quad (4.17)$$

Moreover, this solution $u$ admits the expansion (4.9), with the same coefficient $c_{\mu, \lambda, \beta}$ and an estimate analogous to (4.11), where the right-hand side is replaced by the sum of norms of the data.
**Proof.** From Theorem 4.2, it is clear that \( \bar{u} \) solution of (4.16) admits the following expansion:

\[
\bar{u} = \bar{u}_0 + \sum_{(\lambda, \mu, k) \in A_0(k, p, \gamma)} c_{S, \lambda, \mu, k}(\bar{u}) \Phi_{S, \lambda, \mu, k},
\]

where \( \bar{u}_0 \in \mathcal{V}^{k+2m,p}(\Omega) \). This directly implies the desired expansion (4.9) for \( u \), since the fact that \( v \in \mathcal{V}^{k+2m,p}(\Omega) \) leads to

\[
c_{S, \lambda, \mu, k}(v) = 0.
\]

The assumption \( k - \gamma \geq 0 \) and the Sobolev imbedding theorem allow to show that

\[
\mathcal{V}^{k+2m,p}(\Omega) \subset \mathcal{V}^{2m,p}(\Omega) \subset H^m_v(\Omega)
\]

for \( 1/p + 1/q = 1 \). These both injections imply that Green's formula (2.12) still holds for \( u \in \mathcal{V}^{k+2m,p}(\Omega) \) and \( v \in H^m_v(\Omega) \). This permits to transform \( a(w, v) \) in (4.16) and to prove the equivalence between (4.16) and (4.17), using (4.12), (4.13) and the definitions of the \( B_{igj}'s \) with respect to the \( F_{igj}'s \) and the \( \Phi_{igj}'s \).

---

5. **Numerical examples**

As we have seen in section 3, the distribution of the eigenvalues of the operator \( \mathcal{A}_S(\lambda) \) plays an important role for the solvability, regularity and asymptotic expansion of the solution of our interface problems in weighted Sobolev spaces. Our goal is now to show how to compute the eigenvalues for some examples and in which cases associated eigensolutions exist.

**Example 1.** We consider again the Dirichlet-interface problem for Poisson's equation in a plane domain of \( q \) different materials, which was formulated by the relations (2.25)–(2.30). We introduce for the origin point \( S \) the generalized eigenvalue problem for the operator \( \mathcal{A}_S(\lambda) \):

\[
\begin{align*}
\frac{\partial^2 \tilde{u}_i(\lambda, \omega)}{\partial \omega^2} + \lambda^2 \tilde{u}_i(\lambda, \omega) &= 0 \quad \text{for } \sigma_{i-1} < \omega < \sigma_i, \; i = 1, \ldots, q, \\
\tilde{u}_i(\lambda, 0) &= \tilde{u}_i(\lambda, \sigma_q), \\
\tilde{u}_i(\lambda, \sigma_i) - \tilde{u}_{i+1}(\lambda, \sigma_i) &= 0, \\
p_i \frac{\partial \tilde{u}_i}{\partial \omega}(\lambda, \sigma_i) - p_{i+1} \frac{\partial \tilde{u}_{i+1}}{\partial \omega}(\lambda, \sigma_i) &= 0 \quad \text{for } i = 1, \ldots, q - 1.
\end{align*}
\]

(5.1) (5.2) (5.3) (5.4)

The solutions of (5.1) have the form

\[
\tilde{u}_i(\lambda, \omega) = C_{1i} \cos \lambda \omega + C_{2i} \sin \lambda \omega, \quad i = 1, \ldots, q.
\]

Inserting these solutions into the boundary conditions (5.2) and the interface conditions (5.3), (5.4), we get non-trivial solutions \( \tilde{u}_i(\lambda, \omega) \) for such values \( \lambda \) for which the
determinant \( D_q^\lambda (\lambda) \) vanishes. The zeros of \( D_q^\lambda (\lambda) \) are the eigenvalues of \( \mathcal{A}_\lambda (\lambda) \). \( D_q^\lambda (\lambda) \) is defined by the following recurrence formula:

**First step:**
\[
D_1^P(\lambda) = \sin(\lambda \omega_1), \quad D_1^M(\lambda) = p_1 \cos(\lambda \omega_1).
\]

**Second step:**
\[
D_2^P(\lambda) = p_2 \cos(\lambda \omega_2) \sin(\lambda \omega_1) + p_1 \sin(\lambda \omega_2) \cos(\lambda \omega_1), \\
D_2^M(\lambda) = -p_2^2 \sin(\lambda \omega_1) \sin(\lambda \omega_2) + p_1 p_2 \cos(\lambda \omega_1) \cos(\lambda \omega_2).
\]

**ith step:**
\[
D_i^P(\lambda) = p_i \cos(\lambda \omega_i) D_{i-1}^P(\lambda) + \sin(\lambda \omega_i) D_{i-1}^M(\lambda), \\
D_i^M(\lambda) = -p_i^2 \sin(\lambda \omega_i) D_{i-1}^P(\lambda) + p_i \cos(\lambda \omega_i) D_{i-1}^M(\lambda).
\]

Let us notice that \( D_q^M(\lambda) \) gives the eigenvalues of the transmission problem with mixed boundary condition on the external boundary, i.e. we replace the Dirichlet condition (2.28) on \( \Gamma_0 \) by
\[
\gamma_i \frac{\partial u_i}{\partial \nu_i} = 0 \quad \text{on} \quad \Gamma_0.
\]

From [21, Theorem 2.2], we know that the eigenvalues of \( \mathcal{A}_\lambda (\lambda) \) are real, since they can be defined as eigenvalues of a self-adjoint operator. The eigenvalues are simple, i.e. no associated eigenfunction exists; moreover, for 2 materials, we can prove that the strip \((-1/4, 1/4)\) is free of eigenvalues of \( \mathcal{A}_\lambda (\lambda) \) for arbitrary materials and angles
\[
\sigma_2 = \omega_1 + \omega_2.
\]

Figures 6 and 7 show the distribution of the eigenvalues of \( \mathcal{A}_\lambda (\lambda) \) for 2 materials;
\[ p_1 = 0.25 \, (10^{11} \, \text{Nm}^{-2}) \] (glass) and \[ p_2 = 1.5 \, (10^{11} \, \text{Nm}^{-2}) \] (molybdenum).

Figure 6 describes the eigenvalues for a domain with the reentrant corner \( \omega_1 + \omega_2 = 270^\circ \), where \( \omega_1 \) runs from 0 to 270°; Fig. 7 describes a domain with a slit, namely \( \omega_1 + \omega_2 = 360^\circ \), where \( \omega_1 \) runs from 0° to 360°.

Theorem 3.6 yields the following asymptotic expansion for a solution \( u \in V^{1,2}_0(\Omega) \):
\[
\Phi_S u_1 = \Phi_S \sum_{0 < \lambda_i < 1} c_i r^{\lambda_i} \left[ \tan \lambda_i (\omega_1 + \omega_2) \frac{\cos \lambda_i \omega_1}{\sin \lambda_i \omega_1} - 1 \right] \sin \lambda_i \omega + w_1, \\
\Phi_S u_2 = \Phi_S \sum_{0 < \lambda_i < 1} c_i r^{\lambda_i} \left[ - \tan \lambda_i (\omega_1 + \omega_2) \cos \lambda_i \omega + \sin \lambda_i \omega \right] + w_2,
\]

provided \( \cos \lambda_i (\omega_1 + \omega_2) \neq 0, \sin \lambda_i \omega_1 \neq 0 \). The remainder \( w_i \) have the property that \( \Phi_S w_i \in V^{1,2}_0(\Omega), \quad i = 1, 2 \). Let us point out that the singular functions are not the same on each face.

The situation for 3 materials \( p_1 = 0.25 \, (10^{11} \, \text{Nm}^{-2}), \quad p_2 = 0.5 \, (10^{11} \, \text{Nm}^{-2}), \quad p_3 = 1.5 \, (10^{11} \, \text{Nm}^{-2}) \) is shown in Fig. 8. We have considered a slit domain \( \sigma_3 = \omega_1 + \omega_2 + \omega_3 = 360^\circ, \omega_2 = 0^\circ, \omega_2 = 90^\circ, \omega_2 = 180^\circ \) and \( \omega_2 = 270^\circ \) are fixed and \( \omega_1 \) runs from 0° to \( \omega_2 \).
Example 2. We consider an inclusion with a conical boundary point in a plane domain of a different material; see (2.31)–(2.34) and Fig. 2. We introduce for the origin \( S \) the generalized eigenvalue problem for the operator \( \mathcal{A}_S(\lambda) \):

\[
\frac{\partial^2 \hat{u}_i(\lambda, \omega)}{\partial \omega^2} + \lambda^2 \hat{u}_i(\lambda, \omega) = 0 \quad \text{for} \ 0 < \omega < \omega_i, \ i = 1, 2, \ \omega_2 = 2\pi - \omega_1, \quad (5.6)
\]

\[
\hat{u}_1(\lambda, 0) - \hat{u}_2(\lambda, 0) = 0. \quad (5.7)
\]

\[
\hat{u}_1(\lambda, \omega_1) - \hat{u}_2(\lambda, 2\pi - \omega_1) = 0, \quad (5.8)
\]

\[
p_1 \frac{\partial \hat{u}_1(\lambda, 0)}{\partial \omega} + p_2 \frac{\partial \hat{u}_2(\lambda, 0)}{\partial \omega} = 0, \quad (5.9)
\]

\[
p_1 \frac{\partial \hat{u}_1(\lambda, \omega_1)}{\partial \omega} + p_2 \frac{\partial \hat{u}_2(\lambda, 2\pi - \omega_1)}{\partial \omega} = 0. \quad (5.10)
\]

Inserting the solutions \( \hat{u}_i(\lambda, \omega) = C_{1i} \cos \lambda \omega + C_{2i} \sin \lambda \omega, \ i = 1, 2 \), into the interface...
conditions (5.7)–(5.10), we get non-trivial solutions \( \hat{u}_i(\lambda, \omega) \) for such values \( \lambda \) for which the determinant

\[
D_\lambda(\lambda) = (p_1 - p_2)^2 \sin^2 \lambda(\pi - \omega_1) - (p_1 + p_2)^2 \sin^2 \lambda \pi
\]

vanishes. The eigenvalues of \( \mathcal{A}_S(\lambda) \) are the zeros of \( D_\lambda(\lambda) \) and are real, as in example 1.

Figure 9 shows the distribution of the eigenvalues of \( \mathcal{A}_S(\lambda) \) if \( \omega_1 \) runs from 0 to 360°. The number \( \lambda = 0 \) is an eigenvalue for all angles \( \omega_1 \). It leads to constant solutions which describe the rigid-body motion.

As Fig. 9 suggests, we can show that the number \( \lambda = k, k = 1, 2, \ldots \) is an eigenvalue if \( \omega_1 = (\pi/k)i, i = 0, 1, \ldots, 2k \). In these cases, \( \dim \ker \mathcal{A}_S(k) = 2 \), i.e. two linearly independent solutions exist and

\[
\begin{align*}
    u_1 &= r^k (C_1 \cos k\omega + C_2 \sin k\omega), \\
    u_2 &= r^k \left( -\frac{C_1}{p_1} p_1 \cos k\omega + C_2 \sin k\omega \right),
\end{align*}
\]

are eigensolutions of \( \mathcal{U}^S(D_x) \), where \( C_1 \) and \( C_2 \) are arbitrary coefficients.
Theorem 3.6 yields the following asymptotic expansion for a solution 
ue \in \mathcal{V}^{2,2}(\Omega)

of problem (2.31)–(2.34), provided \( \omega_1 \notin \{0, \pi, 2\pi\} \):

\[
\Phi_S u_1 = cr_0^0 \Phi_S \left( -\frac{p_2}{p_1} \cos \lambda_0 \omega_1 + \frac{p_2}{p_1} \sin \lambda_0 \omega_1 - \sin \lambda_0 (2\pi - \omega_1) \right) + w_1,
\]

\[
\Phi_S u_2 = cr_0^0 \Phi_S \left( \cos \lambda_0 \omega_1 + \frac{p_2}{p_1} \sin \lambda_0 \omega_1 - \sin \lambda_0 (2\pi - \omega_1) \right) + w_2.
\]

Here \( \lambda_0 = \lambda_0 (\omega_1) \) is the eigenvalue of \( \mathcal{A}_S (\lambda) \) in the strip \( 0 < \lambda_0 < 1 \) and the remainders are so regular that \( \Phi_S w_i \in \mathcal{V}^{2,2}(\Omega) \), \( i = 1, 2 \).

Example 3.1. We investigate the behaviour of the solutions of the plate equation (2.35)–(2.42) near a boundary corner point \( S \), where two different materials meet; Dirichlet conditions are given by (2.36), (2.37) and interface conditions by (2.38)–(2.41). We formulate for the origin \( S \) the generalized eigenvalue problem for the
corresponding operator $\mathcal{A}_g(\lambda)$:

$$
\frac{\partial^4 u_i(\lambda, \omega)}{\partial \omega^4} + 2(\lambda^2 - 2\lambda + 2) \frac{\partial^2 u_i(\lambda, \omega)}{\partial \omega^2} + \lambda^2 (\lambda - 2)^2 u_i(\lambda, \omega) = 0
$$

for $0 < \omega < \omega_1$ if $i = 1$, and $\omega_1 < \omega < \omega_1 + \omega_2$ if $i = 2$, \hspace{1cm} (5.12)

$$
\dot{u}_1(\lambda, 0) = \frac{\partial u_1}{\partial \omega}(\lambda, 0) = 0, \hspace{1cm} (5.13)
$$

$$
\dot{u}_2(\lambda, \sigma_2) = \frac{\partial u_2}{\partial \omega}(\lambda, \sigma_2) = 0, \hspace{1cm} (5.14)
$$

$$
\ddot{u}_1(\lambda, \omega_1) = \ddot{u}_2(\lambda, \omega_1), \hspace{1cm} (5.15)
$$

$$
\frac{\partial^2 u_1}{\partial \omega^2}(\lambda, \omega_1) = \frac{\partial^2 u_2}{\partial \omega^2}(\lambda, \omega_1), \hspace{1cm} (5.16)
$$

Fig. 9. Inclusion-interface problem for Poisson's equation, $p_1 = 0.25, p_2 = 1.5$
\[ \begin{align*}
\rho_1 \left[ \frac{\partial^2 \hat{u}_1}{\partial \omega^2} (\lambda, \omega_1) + A_1 \hat{u}_1(\lambda, \omega_1) \right] &= \rho_2 \left[ \frac{\partial^2 \hat{u}_2}{\partial \omega^2} (\lambda, \omega_1) + A_2 \hat{u}_2(\lambda, \omega_1) \right], \\
\rho_1 \left[ \frac{\partial^3 \hat{u}_1}{\partial \omega^3} (\lambda, \omega_1) + B_1 \frac{\partial \hat{u}_1}{\partial \omega}(\lambda, \omega_1) \right] &= \rho_2 \left[ \frac{\partial^3 \hat{u}_2}{\partial \omega^3} (\lambda, \omega_1) + B_2 \frac{\partial \hat{u}_2}{\partial \omega}(\lambda, \omega_1) \right],
\end{align*} \] (5.17)

where

\[ A_i = \nu_i \lambda^2 + (1 - \nu_i) \lambda, \quad B_i = (2 - \nu_i) \lambda^2 - 3(1 - \nu_i) \lambda + 2(1 - \nu_i) \quad i = 1, 2. \]

For \( \lambda \notin \{0, 1, 2\} \) the general solutions of (5.12) have the form

\[ \hat{u}_i(\lambda, \omega) = C_{i1} \sin \lambda \omega + C_{i2} \cos \lambda \omega + C_{i3} \sin (\lambda - 2) \omega + C_{i4} \cos (\lambda - 2) \omega. \] (5.19)

Inserting these solutions in the boundary conditions (5.13), (5.14) and the interface conditions (5.15)–(5.18), we get 8 equations for the unknowns \( C_{ij}, i = 1, 2, \)

**Fig 10.** Dirichlet-interface problem for the biharmonic equation, \( \nu_1 = 0.17, \nu_2 = 0.29, E_2/E_1 = 20, \omega_1 + \omega_2 = 270^\circ. \)
$j = 1, 2, 3, 4$. There are non-trivial solutions $u_j(\lambda, \omega)$ if the corresponding determinant $D_{A_2}^R(\lambda)$ vanishes. The zeros of $D_{A_2}^R(\lambda)$ are the eigenvalues of $A_3(\lambda)$. Figures 10 and 11 show the distribution of the eigenvalues for the materials concrete ($v_1 = 0.17$) and steel ($v_2 = 0.29$); $E_2/E_1 = 20$, for $\omega_1 + \omega_2 = 270^\circ$, $\omega_1$ runs from 0 to 270$^\circ$ and for a slit domain $\omega_1 + \omega_2 = 360^\circ$, $\omega_1$ runs from 0 to 360$^\circ$. The line $\lambda = 1$ is the symmetry axis.

The solid lines indicate real eigenvalues, while the dotted lines indicate the real parts of the pair of the complex eigenvalues.

There are some angles $\omega_1 = \omega_1(0)$, where $D_{A_2}^R(\lambda(\omega_1,0)) = d/d\lambda D_{A_2}^R(\lambda(\omega_1,0)) = 0$. They indicate such graph-points in Figs. 10 and 11 where the dotted curve pieces of the real parts of the complex eigenvalues start or end on the solid lines of the real eigenvalues. We call these points branching points of the graph (see [5]). They indicate that associated eigensolutions exist. Moreover, there are angles $\omega_1$, where dotted or solid lines cross. In these cases there exist $j = j_1 + 2j_2$ linearly independent eigensolutions, where $j_1$ is the number of the crossing solid lines, $j_2$ is the number of the crossing dotted lines and no associated eigensolution exists. Let us remark that the above considerations can be proved analogously as in [26, 4].

![Graph showing the distribution of eigenvalues for concrete and steel materials](image-url)

**Fig 11.** Dirichlet-interface problem for the biharmonic equation, $v_1 = 0.17$, $v_2 = 0.29$, $E_2/E_1 = 20$, $\omega_1 + \omega_2 = 360^\circ$
Example 3.2. We consider an inclusion with a conical boundary point in a plate (see Fig. 2). We introduce for the origin $S$ a generalized eigenvalue problem for the operator $A_S(\lambda)$ analogously to (5.12)–(5.18).

Figure 12 describes the distribution of the eigenvalues for the material concrete ($\nu_1 = 0.17$) and steel ($\nu_2 = 0.29$); $E_2/E_1 = 20$. The angle $\omega_1$ runs from 0 to $360^\circ$. The numbers $\lambda = 0, 1$ and 2 are eigenvalues, for every angle $\omega_1$.

The interpretation of the graph leads to analogous results as in example 3.1.

Example 4. We deal with the Dirichlet interface, the Neumann interface and the pure transmission problem (inclusion) for the linear elasticity system in a plane domain, consisting of two different media. In all cases, we have chosen Poisson's ratios $\nu_1 = 0.17$ and $\nu_2 = 0.29$ and the shear modulus $\mu_2/\mu_1 = 4$.

Let $S \in \mathcal{S}$ be the origin. As usual, we introduce local spherical basic vectors

$$e_r = \begin{pmatrix} \cos \omega \\ \sin \omega \end{pmatrix}, \quad e_\omega = \begin{pmatrix} -\sin \omega \\ \cos \omega \end{pmatrix}$$

Fig. 12. Inclusion-interface problem for the biharmonic equation, $\nu_1 = 0.17, \nu_2 = 0.29, E_2/E_1 = 20$
and write the displacement vectors $\mathbf{u}_i$ as $\mathbf{u}_i = u_{r,i}(r, \omega) \mathbf{e}_r + u_{\omega,i}(r, \omega) \mathbf{e}_\omega$, or shortly

$$\mathbf{u}_i = \begin{pmatrix} u_{r,i} \\ u_{\omega,i} \end{pmatrix}.$$  

The generalized eigenvalue problems for the corresponding operators $A_S(\lambda)$ have the following form.

Differential equations:

$$C_i \frac{\partial^2 \hat{u}_{r,i}(\lambda, \omega)}{\partial \omega^2} + (\lambda^2 - 1) \hat{u}_{r,i}(\lambda, \omega) + [(\lambda - 1) - C_i(\lambda + 1)] \frac{\partial \hat{u}_{\omega,i}(\lambda, \omega)}{\partial \omega} = 0,$$

$$\frac{\partial^2 \hat{u}_{\omega,i}(\lambda, \omega)}{\partial \omega^2} + C_i(\lambda^2 - 1) \hat{u}_{\omega,i}(\lambda, \omega) + [\lambda + 1 - C_i(\lambda - 1)] \frac{\partial \hat{u}_{r,i}(\lambda, \omega)}{\partial \omega} = 0$$

(5.20)  

(5.21)
for $0 < \omega < \omega_1$ if $i = 1$ and $\omega_1 < \omega < \omega_1 + \omega_2$ if $i = 2$. The constants $C_i = (1 - 2v_i/2(1 - v_i))$ depend on the medias.

**Dirichlet conditions:**
\[ \hat{u}_i(\lambda, \omega) = 0 \text{ for } \omega = 0 \text{ if } i = 1 \text{ and for } \omega = \omega_1 + \omega_2 \text{ if } i = 2. \] (5.22)

**Neumann conditions:**
\[ \delta[\hat{u}_i] \cdot v = \begin{pmatrix} \delta_{ro,i}(\lambda, \omega) \\ \delta_{so,i}(\lambda, \omega) \end{pmatrix} = 0 \] (5.23)

for $\omega = 0$ if $i = 1$ and for $\omega = \omega_1 + \omega_2$ if $i = 2$, where
\[ \delta_{ro,i}(\lambda, \omega) = \mu_i \left[ \frac{\partial \hat{u}_{r,i}(\lambda, \omega)}{\partial \omega} + (\lambda - 1) \frac{\partial \hat{u}_{s,i}(\lambda, \omega)}{\partial \omega} \right], \]

---

**Fig. 14.** Dirichlet-interface problem for Lamé's equations, $v_1 = 0.17$, $v_2 = 0.29$, $\mu_2/\mu_1 = 4$, $\omega_1 + \omega_2 = 270^\circ$
\[
\sigma_{\alpha \omega_i}(\lambda, \omega) = 2\mu_i \left[ \frac{\partial \hat{u}_{\omega_i}(\lambda, \omega)}{\partial \omega} + \hat{u}_{r_i}(\lambda, \omega) \right] \\
+ \lambda_i \left[ (\lambda + 1)\hat{u}_{r_i}(\lambda, \omega) + \frac{\partial \hat{u}_{\omega_i}(\lambda, \omega)}{\partial \omega} \right],
\]

where \(\mu_i\) and \(\lambda_i\) are the corresponding Lamé coefficients.

**Interface conditions:**

\[
\hat{u}_1(\lambda, \omega_1) - \hat{u}_2(\lambda, \omega_1) = 0,
\]

\[
\hat{\sigma}[\hat{u}_1] \cdot \mathbf{v} - \hat{\sigma}[\hat{u}_2] \cdot \mathbf{v} = 0 \text{ for } \omega = \omega_1,
\]

and additionally, if we consider an inclusion-interface problem:

\[
\hat{u}_1(\lambda, 0) - \hat{u}_2(\lambda, 2\pi) = 0,
\]

\[
\hat{\sigma}[u_1] \cdot \mathbf{v}(0) - \hat{\sigma}[\hat{u}_2] \cdot \mathbf{v}(2\pi) = 0.
\]

Fig. 15. Dirichlet-interface problem for Lamé's equations, \(v_1 = 0.17, v_2 = 0.29, \mu_2/\mu_1 = 4, \omega_1 + \omega_2 = 360^\circ\)
The general solutions of (5.20), (5.21) have for \( \lambda \neq 0 \) and \( 3 - \lambda - 4v_i \neq 0 \) the form

\[
\hat{u}_i(\lambda, \omega) = C_{1i} \left( 2\lambda \mu_i \sin(1 + \lambda)\omega \right) + C_{2i} \left( 2\lambda \mu_i \cos(1 + \lambda)\omega \right) \\
+ C_{3i} \left( \mu_i(1 - \lambda)(1 - D_i) \sin(1 - \lambda)\omega \right) \\
+ C_{4i} \left( \mu_i(1 + \lambda)(1 - D_i) \cos(1 - \lambda)\omega \right),
\]

where \( D_i = (3 + \lambda - 4v_i)/(3 - \lambda - 4v_i) \).

We insert \( \hat{u}_i(\lambda, \omega) \) in the corresponding boundary and interface conditions (5.22)–(5.27) and get 8 equations for the unknowns \( C_{ji}, j = 1, \ldots, 4, i = 1, 2 \). resepc-
respectively. There are non-trivial solutions, if the corresponding determinants $D^P_2(\lambda)$, $D^N_2(\lambda)$, $D^I_2(\lambda)$ vanish. Figures 14 and 15 show the distribution of the zeros of $D^P_2(\lambda)$, that means the distribution of the eigenvalues of the corresponding operator $A_5(\lambda)$ in a domain with a reentrant right angle and in a slit-domain.

Figures 16 and 17 describe the zeros of $D^P_2(\lambda)$ in the same domains. Let us remark that the numbers $\lambda = 0$ and $\lambda = 1$ are eigenvalues for every angle $\omega = \omega_1$.

Figure 18 gives the zeros of $D^I_2(\lambda)$, that means the eigenvalues of the inclusion interface problem. The numbers $\lambda = 0$ and $\lambda = 1$ are again eigenvalues for every angle $\omega = \omega_1$. The meaning of the solid and dotted lines is the same as in the former examples. The interpretation of branching points or points, where cross dotted and solid lines leads to analogous results as in example 3.1.

Let us remark that some numerical calculations can also be found in [25].

---

Fig. 17. Neumann-interface problem for Lamé's equations, $\nu_1 = 0.17, \nu_2 = 0.29, \mu_2/\mu_1 = 4, \omega_1 + \omega_2 = 360^\circ$
Fig. 18. Inclusion-interface problem for Lamé's equations, $v_1 = 0.17, v_2 = 0.29, \mu_2/\mu_1 = 4$

Acknowledgements

We thank Rainer Sändig for his help in the numerical experiments given in this paper.

References


