# Research Notes

## Proof of Two Theorems Related to the Energy of Acoustic Bloch Waves in Periodically Inhomogeneous Media

**Beweis zweier Sätze für die Energie akustischer Blochwellen in periodisch inhomogenen Medien**

**Démonstration de deux théorèmes sur l'énergie des ondes acoustiques de Bloch dans des milieux inhomogènes périodiques**

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*Dedicated to Professor Dr. rer. nat. Dr. h.c. Alfred Seeger on the occasion of his 65th birthday*

### 1. Introduction

Two theorems which are applicable to plane waves in homogeneous elastic media are generalized to the case of periodic inhomogeneity. The first theorem is known as Rayleigh's principle and essentially says that, on average, kinetic energy equals potential energy. The second theorem states that group velocity and velocity of energy transport ("energy velocity") are the same. Both equalities are verified below for periodically inhomogeneous media. As distinguished from the homogeneous case one has to deal with Bloch waves instead of plane waves and has to perform spatial averages over an elementary cell of the lattice. By means of plausible arguments the theorems can also be applied to the modes of infinitely extended plates and beams.

### 2. Fundamental equations

The proofs are given within the framework of the linearized theory of elastodynamics starting from the Lagrangian density $L$

$$L(\dot{u}_i, u_{i,j}, x_i) = e_{\text{kin}} - e_{\text{pot}},$$
$$e_{\text{kin}} = \tfrac{1}{2}\varrho\, \dot{u}_i \dot{u}_i,$$
$$e_{\text{pot}} = \tfrac{1}{2}\boldsymbol{\sigma}\cdot\cdot\,\boldsymbol{\varepsilon} = \tfrac{1}{2}u_{i,j}C_{ijkl}u_{k,l}. \tag{1}$$

$\boldsymbol{u}$ is the displacement field, $\dot{\boldsymbol{u}}$ its time derivative; $e_{\text{kin}}$ and $e_{\text{pot}}$ are kinetic and potential energy densities, $\varrho$ and $\underline{C}$ are mass density and tensor of elastic constants, both of which are periodic functions of the space variable $\boldsymbol{x}$; $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ denote stress and strain fields. In the abbreviated, boldface notation tensor products which imply summations over two Cartesian subscripts are distinguished by two dots from the common products with a single dot. A fourth-order tensor is written as an underlined letter like $\underline{C}$. Einstein's summation convention is used and a subscript after a comma means differentiation with respect to the corresponding space coordinate. The equations of motion are derived from Hamilton's principle

$$\delta \int L\, \mathrm{d}t\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3 = 0. \tag{2}$$

In view of the dependencies in (1) the variation of $L$ is evaluated as

$$\delta L = \frac{\partial L}{\partial \dot{u}_i}\delta\dot{u}_i + \frac{\partial L}{\partial u_{i,j}}\delta u_{i,j} \tag{3}$$

and leads to the Lagrange-Euler equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{u}_i}\right) + \frac{\mathrm{d}}{\mathrm{d}x_j}\left(\frac{\partial L}{\partial u_{i,j}}\right) = 0, \tag{4}$$

explicitly,

$$\varrho\, \ddot{u}_i = \sigma_{ij,j} = C_{ijkl}u_{k,lj} + C_{ijkl,j}u_{k,l}. \tag{5}$$

A conservation equation for the total "acoustic energy",

$$e_{\text{tot}} = e_{\text{kin}} + e_{\text{pot}} = v_i\frac{\partial L}{\partial v_i} - L, \tag{6}$$

can be derived ($v_i = \dot{u}_i$),

$$\dot{e}_{\text{tot}} + \nabla\cdot\boldsymbol{S} = 0, \tag{7}$$

where

$$S_i = -\sigma_{ij}v_j = v_j\frac{\partial L}{\partial u_{j,i}} \tag{8}$$

is the energy flux density. By analogy with Poynting's vector in electrodynamics $\boldsymbol{S}$ may be called Kirchhoff's vector, because Kirchhoff was the first to derive (7) in 1876 ([1] cited after [2, p. 36]; the corresponding equation in electrodynamics [3] was derived by Poynting and Heaviside in 1884).

The time average of Kirchhoff's vector is usually called (acoustic) intensity:

$$\boldsymbol{I} = \langle \boldsymbol{S} \rangle. \tag{9}$$

The velocity of energy propagation is defined as

$$\boldsymbol{c}_e = \frac{\boldsymbol{I}}{w_{\text{tot}}} \tag{10}$$

with the time-averaged energy density:

$$w_{\text{tot}} = \langle e_{\text{tot}} \rangle. \tag{11}$$

On the other hand the group velocity of a time-harmonic wave with frequency $\omega$ and wave vector $\boldsymbol{k}$ is given by the gradient of $\omega$ with respect to $\boldsymbol{k}$:

$$\boldsymbol{C} = \nabla_k\, \omega. \tag{12}$$

The fundamental elastodynamic solutions for media with periodically varying material properties are known as Bloch waves. A Bloch wave

$$\boldsymbol{u}_k(\boldsymbol{r}, t) = \mathrm{Re}\{\boldsymbol{p}_k(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}\} \tag{13}$$

can be regarded as a plane wave which is modulated by the spatially periodic function $\boldsymbol{p}_k(\boldsymbol{r})$. In the present paper we consider only propagating Bloch waves and no damping, i.e. $\omega$ and $\boldsymbol{k}$ are real quantities.

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## 3. Rayleigh's principle for progressive waves

Rayleigh's principle dates back to 1877 [4] and has proved useful in numerous applications, especially as a variational principle. There is even a monograph on this subject [5]. Most applications deal with the eigenmodes of finite systems, however, the principle can also be formulated for systems which are of infinite extent in at least one spatial dimension. Following Pierce [6] we refer to this formulation as "Rayleigh's principle for progressive waves". In section 4.3 of [6] it is pointed out that usually some sort of averaging is necessary before the equality of potential and kinetic energies emerges.

Rayleigh's principle for progressive waves has been proved for homogeneous media by Lighthill in his impressive article on group velocity [7, p. 23–25]. In the following the proof is extended to periodically inhomogeneous media (with arbitrary elastic anisotropy). It is known [8] that the principle does not hold locally: a spatial average over an elementary cell of the lattice is required. In short-hand notation, we are going to prove

$$\langle\langle L \rangle\rangle = 0 \tag{14}$$

for Bloch waves, where the double brackets denote time averaging over one period $T = 2\pi/\omega$ and space averaging over one elementary cell (e.c.). An elementary but rather cumbersome proof of (14) is given in the appendix. However, it is much more elegant to generalize Lighthill's proof, which consists of two parts:

In the first part it is shown that Hamilton's principle (2) is also valid in the modified form

$$\delta \int_T \frac{dt}{T} \int_{\text{e.c.}} \frac{dV}{V_{\text{e.c.}}} L = \langle\langle \delta L \rangle\rangle = 0 \tag{15}$$

($V_{\text{e.c.}}$: volume of elementary cell), where a Bloch wave $\boldsymbol{u}$ is considered and the variation $\delta \boldsymbol{u}$ has also the form of a Bloch wave. This means as opposed to the original principle (2) that the variation $\delta \boldsymbol{u}$ does not vanish at the limits of integration. Explicitly written and after partial integration (15) becomes

$$\left\langle\!\!\left\langle \frac{\partial L}{\partial \dot{u}_i} \delta \dot{u}_i + \frac{\partial L}{\partial u_{i,j}} \delta u_{i,j} \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{u}_i}\delta u_i\right] + \frac{d}{dx_j}\left[\frac{\partial L}{\partial u_{i,j}}\delta u_i\right] - \left[\frac{d}{dt}\frac{\partial L}{\partial \dot{u}_i}\right]\delta u_i - \left[\frac{d}{dx_j}\frac{\partial L}{\partial \dot{u}_{i,j}}\right]\delta u_i \right\rangle\!\!\right\rangle = 0. \tag{16}$$

The last two terms cancel because of the equations of motion (4). After time integration of the first term and, with the second term, conversion of the integration over $V_{\text{e.c.}}$ to an integration over the surface of the elementary cell ($dF$: surface element) one obtains

$$\int_{\text{e.c.}} dV \left[\frac{\partial L}{\partial \dot{u}_i}\delta u_i\right]_t^{t+T} + \int_T dt \int_{\text{e.c.}} dF_j \frac{\partial L}{\partial u_{i,j}} \delta u_i = 0. \tag{17}$$

The expression in square brackets is the product of momentum and variation of displacement, both of which oscillate with frequency $\omega$. The product therefore exhibits frequency $2\omega$. Consequently, the first term in (17) vanishes. In the second term one has to integrate over the product of strain and variation of displacement. This product also oscillates with frequency $2\omega$; the time integration, however, has not yet been done and would lead in general to a nonzero result. In order to make the second term vanish the spatial periodicity must be used. Because of the phase $\boldsymbol{k} \cdot \boldsymbol{r}$ the product in question is not spatially periodic (like the Bloch wave (13) itself). Only after the time integration we get rid of this phase factor and obtain the desired periodicity: the time-averaged product is equal on "opposite" points of the surface of the elementary cell, "opposite" meaning "connected by a lattice vector". Since the surface elements $d\boldsymbol{F}$ point in opposite directions at opposite points, the surface integration yields zero, and the modified principle (15) is verified.

The second part of the proof of (14) considers a special variation, namely a simple change of amplitude of the Bloch wave:

$$\delta u_i = \alpha u_i, \quad 0 < \alpha \ll 1. \tag{18}$$

Because the Lagrangian is a homogeneous function of the second degree in its variables $\dot{u}_i$ and $u_{i,j}$, its variation reads

$$\delta L = (1 + \alpha)^2 L - L \approx 2\alpha L. \tag{19}$$

Application of (15) to (19) completes the proof of Rayleigh's principle for progressive waves (14) for the case of Bloch waves in periodic media. In order to emphasize its physical implications it may be written in the less abstract form

$$\langle\langle e_{\text{kin}} \rangle\rangle = \langle\langle e_{\text{pot}} \rangle\rangle. \tag{20}$$

## 4. Equivalence of energy velocity and group velocity

Like Rayleigh's principle the relation between energy transport and group velocity has a long history. According to Sommerfeld [9, p. 168–170] the beginnings originate with Stokes (definition and derivation of group velocity, 1876), Reynolds and Rayleigh (link to energy transport, both 1877). It was discovered that in numerous cases the group velocity $\boldsymbol{C}$ and the velocity of energy transport $\boldsymbol{c}_e$ of a propagating wave coincide:

$$\nabla_k \omega = \boldsymbol{C} = \boldsymbol{c}_e = \frac{\boldsymbol{I}}{w_{\text{tot}}}. \tag{21}$$

In 1957 Biot [10] dealt extensively with this subject and found the identity (21) confirmed in various systems: in electromagnetic systems, in compressible fluids, in elastically isotropic or anisotropic solids with and without pre-stress, even in inhomogeneous solids provided they are homogeneous along the propagation direction of the wave. Unfortunately, Biot's lines of argument sometimes appear to lack the rigour commonly expected from a convincing proof. We will follow again the reasoning of Lighthill's paper [7], which was published in 1965 (without any reference to [10]!). Lighthill proves (21) for a plane wave in a homogeneous conservative system with even nonlinear wave equations. The treatment of the nonlinear case is due to Whitham [11] and will not be persued further here.

The equivalence (21) of energy velocity and group velocity is proved below for Bloch waves in a medium with arbitrary anisotropy and with spatially periodic, but otherwise arbitrary inhomogeneity. The variational principle (15) tailored for Bloch waves and Rayleigh's principle for Bloch waves (14) constitute important elements of this proof.

In contrast with the "modest" variation (18) of the Bloch wave for the proof of Rayleigh's principle the variation now comprises frequency and wave vector of the Bloch wave. The wave (13) is compared to the varied Bloch wave

$$\boldsymbol{u}_K(\boldsymbol{r}, t') = \operatorname{Re}\{\boldsymbol{p}_K(\boldsymbol{r}) e^{i(\boldsymbol{K}\cdot\boldsymbol{r} - \Omega t')}\},$$

$$\delta \boldsymbol{p} = \boldsymbol{p}_K - \boldsymbol{p}_k, \quad \delta \boldsymbol{k} = \boldsymbol{K} - \boldsymbol{k}, \quad \delta\omega = \Omega - \omega. \tag{22}$$

Since both Bloch waves are by definition solutions to the equations of motion, the change in frequency $\delta\omega$ is a function of the change $\delta k$. The relation is established by means of the group velocity $C$:

$$\delta\omega = C \cdot \delta k. \tag{23}$$

Due to the difference in frequency the time periods, over which one has to integrate for the time averages, are different, too. With new variables

$$\tau = \omega t, \quad \tau' = \Omega t' \tag{24}$$

the time averaging can be written in a uniform way:

$$\int_{T^{(\prime)}} \frac{dt^{(\prime)}}{T^{(\prime)}} \ldots = \int_{2\pi} \frac{d\tau^{(\prime)}}{2\pi} \ldots. \tag{25}$$

For the evaluation of the variation (3) of the Lagrangian the following derivatives are needed:

$$\begin{aligned} \dot{u}_k &= +\omega \operatorname{Im}\{p_k e^{i(k\cdot r - \tau)}\}, \\ \dot{u}_K &= +\Omega \operatorname{Im}\{p_K e^{i(K\cdot r - \tau')}\}, \\ \nabla u_k &= -k \operatorname{Im}\{p_k e^{i(k\cdot r - \tau)}\}, \\ \nabla u_K &= -K \operatorname{Im}\{p_K e^{i(K\cdot r - \tau')}\} \end{aligned} \tag{26}$$

($d/dt = d/dt'$ leads to $\dot{\tau} = \omega$ and $\dot{\tau}' = \Omega$). Considering the time average to be performed later one may choose the time coordinate $\tau'$ at every point $r$ in such a way that the exponential factors of both Bloch waves become equal. This trick renders the variations simple:

$$\begin{aligned} \delta\dot{u} &= \operatorname{Im}\{(p_k \delta\omega + \omega\,\delta p) e^{i(k\cdot r - \tau)}\}, \\ \delta(\nabla u) &= -\operatorname{Im}\{(p_k \delta k + k\,\delta p) e^{i(k\cdot r - \tau)}\}. \end{aligned} \tag{27}$$

Now Hamilton's principle (15) says that the variation of the Lagrangian vanishes on average provided wave vector and frequency stay the same during the variation. For that reason the second terms in each line of (27) need no longer be considered. Using (26a) one is left with

$$\delta\dot{u} = \dot{u}_k \frac{\delta\omega}{\omega}, \quad \delta(\nabla u) = -\dot{u}_k \frac{\delta k}{\omega}. \tag{28}$$

With (28) substituted in (3) we obtain

$$\omega\,\delta L = \frac{\partial L}{\partial \dot{u}_i} \dot{u}_i \delta\omega - \frac{\partial L}{\partial u_{i,j}} \dot{u}_i \delta k_j, \tag{29}$$

which leads by means of (6) and (8) to

$$\omega\,\delta L = (e_{tot} + L)\,\delta\omega - S_j \delta k_j. \tag{30}$$

Upon averaging over time and space the Lagrangian density on the right side of (30) vanishes because of Rayleigh's principle (14). However, the latter is not only valid for the first Bloch wave (13) but also for the second (22), hence the left side of (30) will vanish, too. Finally,

$$\langle\langle e_{tot} \rangle\rangle\,\delta\omega + \langle\langle S \rangle\rangle \cdot \delta k \tag{31}$$

or

$$\nabla_k \omega = C = \frac{\langle\langle S \rangle\rangle}{\langle\langle e_{tot} \rangle\rangle} = \frac{\langle I \rangle}{\langle w_{tot} \rangle}. \tag{32}$$

Compared to (21) an additional spatial averaging is required in order to discover the familiar relation between group velocity, energy flux and energy density. Correspondingly the proof differs from that for homogeneous media essentially by this averaging over an elementary cell, without which Rayleigh's principle is not valid for periodically inhomogeneous media. From this observation one might conclude that an extension of the validity of (32) to the nonlinear case, which could be achieved by Whitham and Lighthill for homogeneous media, is also possible for periodically inhomogeneous media.

## 5. Applications

Theorems (20) and (32) have already been used for the derivation of analytical expressions for low-frequency Bloch waves in one-dimensional periodic media [12]. In particular, a simple formula was found for the phase velocity, which in this case is equal to the group velocity. It appears difficult to imagine how this result could have been obtained without making use of Rayleigh's principle for Bloch waves (20). Both theorems can of course be equally useful for numerical calculations like in [8], either for checking or avoiding extensive computations.

Other less obvious applications concern propagating modes in homogeneous (or even inhomogeneous) plates and beams of infinite extent. In a "gedankenexperiment" such an object can be repeated periodically in empty space with no interactions between the objects. Such a system of, for instance, plates is a periodic medium, which has Bloch-wave-type solutions with real wave vectors along directions of infinite extent of the plates. Consequently, theorems (20) and (32) hold, and for symmetry reasons they not only hold for the total system, but also for each member of the system, i.e. for each plate itself. For homogeneous plates and beams the space average can be replaced by an average over the cross section. The theorems corresponding to (20) and (32) for these cases might be derived without recourse to the artificial system of infinitely many objects, but their inference as special cases from the general case proved above is certainly theoretically appealing. Application of both theorems to free plate waves made it possible to arrive at manageable analytical expressions for the intensity of arbitrary plate modes [13].

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## Appendix: Elementary proof of Rayleigh's principle for Bloch waves in periodic media

The first part of the proof is similar to the derivation of the virial theorem of classical mechanics [14]. One consider the auxiliary quantity

$$g = \varrho\,\dot{u} \cdot u \tag{A.1}$$

with time derivative

$$\dot{g} = \varrho\,\ddot{u} \cdot u + \varrho\,\dot{u} \cdot \dot{u}, \tag{A.2}$$

which, because of the equation of motion

$$\varrho\,\ddot{u} = \nabla \cdot \sigma, \tag{A.3}$$

can be written as

$$\dot{g} = (\nabla \cdot \sigma) \cdot u + 2\,e_{kin}. \tag{A.4}$$

For time-harmonic processes with period $T$ the quantity $g$ is also periodic with $T$,

$$g(t+T) - g(t) = \int_t^{t+T} \dot{g}\,dt = 0, \tag{A.5}$$

and the time average of the kinetic energy density becomes

$$w_{kin} = \langle e_{kin} \rangle_t = -\tfrac{1}{2} \langle (\nabla \cdot \sigma) \cdot u \rangle_t. \tag{A.6}$$

The right side of this equation corresponds to "Clausius' virial".

In the next step the generalized Hooke's law

$$\sigma = \underline{C} \cdot\cdot\, \varepsilon \tag{A.7}$$

and

$$e_{pot} = \tfrac{1}{2} \sigma \cdot\cdot\, \varepsilon = \tfrac{1}{2} \varepsilon \cdot\cdot\, \underline{C} \cdot\cdot\, \varepsilon. \tag{A.8}$$

is used. Insertion of (A.7) into (A.6) gives

$$w_{kin} = -\tfrac{1}{2} \langle \sigma_{ij,i} u_j \rangle_t = -\tfrac{1}{2} \langle (C_{ijkl} \varepsilon_{kl})_{,i} u_j \rangle_t. \tag{A.9}$$

Because of $C_{ijkl} = C_{ijlk}$ we have

$$\begin{aligned} w_{kin} &= -\tfrac{1}{2} \langle (C_{ijkl} u_{l,k})_{,i} u_j \rangle_t \\ &= \tfrac{1}{2} \langle C_{ijkl,i} u_{l,k} u_j + C_{ijkl} u_{l,ik} u_j \rangle_t. \end{aligned} \tag{A.10}$$

This is to be compared to ($C_{ijkl} = C_{jikl}$)

$$w_{pot} = + \tfrac{1}{2} \langle C_{ijkl} u_{l,k} u_{j,i} \rangle_t. \tag{A.11}$$

The displacement field is assumed to be a Bloch wave

$$u(r,t) = p(r) \exp[i(k \cdot r - \omega t)] \tag{A.12}$$

in complex notation, which is conveniently used for the time averages

$$w_{kin} = -\tfrac{1}{4} \mathrm{Re} \{ C_{ijkl,i} u_{l,k} u_j^* + C_{ijkl} u_{l,ik} u_j^* \} \tag{A.13}$$

and

$$w_{pot} = + \tfrac{1}{4} \mathrm{Re} \{ C_{ijkl} u_{l,k} u_{j,i}^* \} \tag{A.14}$$

($\omega$ and $k$ are assumed to be real quantities). The derivatives of the displacement field are

$$\begin{aligned} u_{l,k} &= (p_{l,k} + i k_k p_l), \\ u_{l,ik} &= (p_{l,ik} + i [k_k p_{l,i} + k_i p_{l,k}] - k_k k_i p_l) \end{aligned} \tag{A.15}$$

and one gets

$$\begin{aligned} w_{kin} = \tfrac{1}{4} \mathrm{Re} \{ & C_{ijkl,i} (p_{l,k} + i k_k p_l) p_j^* \\ & + C_{ijkl} (p_{l,ki} + i [k_i p_{l,k} + k_k p_{l,i}] - k_i k_k p_l) p_j^* \}, \end{aligned} \tag{A.16}$$

$$\begin{aligned} w_{pot} = \tfrac{1}{4} \mathrm{Re} \{ & C_{ijkl} (p_{l,k} p_{j,i}^* + i [k_k p_l p_{j,i}^* - k_i p_{l,k} p_j^*] \\ & + k_i k_k p_l p_j^*) \}. \end{aligned} \tag{A.17}$$

Upon subtracting these two equations the number of terms is reduced by four:

$$\begin{aligned} w_{pot} - w_{kin} = \tfrac{1}{4} \mathrm{Re} \{ & C_{ijkl,i} (p_{l,k} + i k_k p_l) p_j^* \\ & + C_{ijkl} (p_{l,ki} p_j^* + p_{l,k} p_{j,i}^* + i k_k [p_{l,i} p_j^* + p_l p_{j,i}^*]) \}. \end{aligned} \tag{A.18}$$

In the last step we carry out a Fourier expansion of the spatially periodic functions $\underline{C}$ and $p$:

$$\begin{aligned} C_{ijkl}(r) &= \sum_\lambda C_{ijkl}^{(\lambda)} \exp[i G_\lambda \cdot r], \\ p_i(r) &= \sum_\mu p_i^{(\mu)} \exp[i G_\mu \cdot r]. \end{aligned} \tag{A.19}$$

The sums extend over all points of the reciprocal lattice (greek letters: vectorial indices; $G$: reciprocal lattice vectors). Now the differentiations in (A.18) can be carried out explicitly:

$$\begin{aligned} w_{pot} = w_{kin} = \tfrac{1}{4} \mathrm{Re} & \sum_{\lambda,\mu,\nu} C_{ijkl}^{(\lambda)} p_l^{(\mu)} p_j^{(\nu)*} \\ & \cdot \{ - G_i^{(\lambda)} G_k^{(\mu)} - G_i^{(\lambda)} k_k - G_k^{(\mu)} G_i^{(\mu)} + G_k^{(\mu)} G_i^{(\nu)} \\ & - k_k G_i^{(\mu)} + k_k G_i^{(\nu)} \} \exp[i (G^{(\lambda)} + G^{(\mu)} - G^{(\nu)}) \cdot r]. \end{aligned} \tag{A.20}$$

(Written out in detail there are 14 summations on the right side of this equation for the case of a three-dimensional medium: 4 sums over the cartesian indices $(i, j, k, l)$, 3 times 3 sums over the reciprocal lattice and one sum for the scalar product in the exponential factor!) This local difference of the time averages of kinetic and potential energy density can surely be different from zero, but it vanishes upon spatial averaging over an elementary cell of the periodic medium. This can be seen as follows: Only terms with

$$G^{(\lambda)} + G^{(\mu)} - G^{(\nu)} = 0 \tag{A.21}$$

lead to nonzero integrals, i.e. $\lambda$ can be substituted by $\nu - \mu$. Since

$$G^{(\nu-\mu)} = G^{(\nu)} - G^{(\mu)}, \tag{A.22}$$

the terms in curly brackets cancel. Thus,

$$\langle w_{pot} \rangle = \langle w_{kin} \rangle \tag{A.23}$$

is proved for a Bloch wave in a periodic medium.

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