

On the Kernel and the Nuclei of 8-Dimensional Locally Compact Quasifields

By

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1. Introduction. In this note we determine the nuclei of 8-dimensional real non-associative division algebras and also study the corresponding substructures of 8-dimensional locally compact quasifields.

1.1. The *left*, *middle* or *right nucleus* of a (not necessarily associative) division algebra D is defined to be the (skew) subfield consisting of those elements which associate with all elements of D from the left, middle or right respectively (cf. 1.3 below). A consequence of the theorems of this paper is the

Corollary. *If D is an 8-dimensional real division algebra with a unit, then the left nucleus, the middle nucleus and the right nucleus of D consist solely of the real scalar multiples of the unit element. In particular, all nuclei of D are isomorphic to the field \mathbb{R} of real numbers.*

In contrast, there exist real division algebras of dimension four with nuclei isomorphic to the complex field \mathbb{C} — examples are contained in [9] — and, of course, all three nuclei of the quaternion field \mathbb{H} are \mathbb{H} itself.

Actually, the corollary itself can be established directly and easily by applying a result in algebraic topology due to A. Borel and J.-P. Serre [2, 17.1]. (See the proof of Theorem 1 and the discussion in section 1.4 below for details.) The Borel-Serre result, however, cannot be directly applied to determine the nuclei of more general structures. Moreover, the proof of the Borel-Serre result as given in [2] makes use of heavy machinery of algebraic topology, and we can establish our corollary with more elementary techniques which involve only the homotopy theory of classical groups as expounded in Steenrod [18]. This is essentially done in the proof of Theorem 2 below.

Our motivation for the study of the kernel and the nuclei in division algebras and quasifields was the fact that these substructures are related to certain groups of axial collineations in the affine planes coordinatized by these domains. For the details of this relationship we refer the reader to Dembowski [5, 3.1.28 and 3.1.30].

1.2. In [14, § 7] H. Salzmann describes some of the basic properties of a *locally compact quasifield Q of dimension n ($n \geq 1$)*. The structure of Q can be characterized

as being the real vector space \mathbb{R}^n together with a continuous multiplication

$$\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n: (x, y) \mapsto x \circ y$$

which satisfies the following axioms:

- (Q 1) The zero vector $0 \in \mathbb{R}^n$ is a zero for \circ .
- (Q 2) There exists a two-sided unit 1 for \circ .
- (Q 3) For every $a, b \in \mathbb{R}^n$ with $a \neq b$ the map $\mathbb{R}^n \rightarrow \mathbb{R}^n: x \mapsto a \circ x - b \circ x$ is linear and nonsingular.
- (Q 4) For every $a, c \in \mathbb{R}^n$ with $a \neq 0$ there exists a solution to the equation $y \circ a = c$ (which by axiom (Q 3) is unique); this solution is required to depend continuously on the pair (a, c) .

As noted in [14, 7.12], J. F. Adams's theorem on the Hopf invariant implies that the only possible dimensions of such quasifields are 1, 2, 4 and 8. An n -dimensional real division algebra is just an n -dimensional locally compact quasifield with both distributive laws. Section 2 below contains examples of 8-dimensional locally compact quasifields which are not division algebras.

The kernel K of Q is defined to be the set of elements which associate and distribute with all elements of Q from the right:

$$K = \left\{ a \in \mathbb{R}^n \mid \wedge x, y \in \mathbb{R}^n: \begin{array}{l} (x \circ y) \circ a = x \circ (y \circ a) \quad \text{and} \\ (x + y) \circ a = x \circ a + y \circ a \end{array} \right\}.$$

In the special case of a division algebra the kernel coincides with the right nucleus.

It is easily verified that the kernel is a topological field with respect to vector space addition and the multiplication \circ . By continuity the real scalar multiples of the unit element form a central subfield of K isomorphic to \mathbb{R} . Thus K may be considered as a real associative division algebra. By a well-known theorem of Frobenius K is isomorphic to one of the topological fields \mathbb{R}, \mathbb{C} or \mathbb{H} . In the 8-dimensional case we prove more specifically:

Theorem 1. *The kernel of an 8-dimensional locally compact quasifield is isomorphic to \mathbb{R} .*

1.3. The left nucleus N_λ and the middle nucleus N_μ of Q are defined as in a division algebra, namely as

$$\begin{aligned} N_\lambda &= \{ a \in \mathbb{R}^n \mid \wedge x, y \in \mathbb{R}^n: (a \circ x) \circ y = a \circ (x \circ y) \}, \\ N_\mu &= \{ a \in \mathbb{R}^n \mid \wedge x, y \in \mathbb{R}^n: (x \circ a) \circ y = x \circ (a \circ y) \}. \end{aligned}$$

The nuclei N_λ and N_μ are not necessarily closed under addition. (See the examples in section 2 below.) However, $N_\lambda^\times = N_\lambda \setminus \{0\}$ and $N_\mu^\times = N_\mu \setminus \{0\}$ are topological groups with respect to the multiplication \circ (the verification of this fact is left to the reader). By identifying each element $a \in N_\lambda^\times$ with the linear map $\mathbb{R}^n \rightarrow \mathbb{R}^n: x \mapsto a \circ x$ we obtain an isomorphism of topological groups mapping N_λ^\times onto a subgroup of $GL(n, \mathbb{R})$; this subgroup is closed because of axiom (Q 2) and because of the continuity

of \circ . Analogous statements hold for the multiplicative group N_μ^\times of the middle nucleus. Consequently, N_λ^\times and N_μ^\times are both Lie groups. We shall prove

Theorem 2. *If N_λ^\times is the multiplicative group of the left nucleus of an 8-dimensional locally compact quasifield, then N_λ^\times is either 1) finite, 2) discrete and isomorphic to a semidirect product of a finite normal subgroup with \mathbb{Z} , or 3) isomorphic as a topological group to the direct product of a finite group and the multiplicative group \mathbb{R}_{pos} of the positive real numbers. The elements $a \in N_\lambda^\times$ of infinite order induce linear maps $x \mapsto a \circ x$ with determinant different from 1. These assertions hold for the multiplicative group N_μ^\times of the middle nucleus as well.*

In all cases N_λ^\times (and N_μ^\times) contains a largest compact subgroup F , and this subgroup is actually finite. By left multiplication F can be considered as a finite fixed point free linear group on \mathbb{R}^8 . Up to equivalence a complete list of all such groups can be found in Wolf [20, p. 225]. Further restrictions on F can be obtained by applying Folkman's results [6] on equivariant maps of spheres into the classical groups (cf. [3]). However, a complete determination of F using this approach seems to be inextricably tied up with the unsolved problem of determining the homotopy class of the map $\lambda: \mathbb{R}^8 \setminus \{0\} \rightarrow GL(8, \mathbb{R}): a \mapsto \lambda_a$ with $\lambda_a(x) = a \circ x$ which is associated with the quasifield multiplication \circ . If one assumes that λ is homotopic to the corresponding map of the classical Cayley algebra \mathbb{O} or its opposite algebra \mathbb{O}^{op} as is generally conjectured, Folkman's results can be used to show that F has at most two elements.

1.4. It is immediate that the Corollary in 1.1 above is a consequence of Theorems 1 and 2. Actually, this Corollary follows from each of them separately. In fact, by reversing the order of the factors in the multiplication of a division algebra D (that is by passing from D to the opposite algebra D^{op}) the roles of the left nucleus and the kernel are interchanged; and a similar switch involving the left and middle nuclei is obtained by transposition (see [4, § 2] and [13, Prop. 4 and Prop. 5(3)]).

1.5. A *weak nucleus* W of a division algebra D is a (skew) subfield of D such that the equation $a \circ (b \circ c) = (a \circ b) \circ c$ holds provided two of the three elements a, b, c of D lie in W . For example, any two-dimensional subfield of the Cayley algebra \mathbb{O} is a maximal weak nucleus of \mathbb{O} .

If D is a real n -dimensional division algebra, then the topological closure in D ($= \mathbb{R}^n$) of a weak nucleus is obviously a weak nucleus too. The following theorem may be of algebraic interest:

Theorem 3. *A topologically closed weak nucleus of an 8-dimensional real division algebra is isomorphic either to \mathbb{R} or to \mathbb{C} .*

2. Examples. The following examples illustrate some of the diverse forms taken on by nuclei of 8-dimensional locally compact quasifields.

2.1. In what follows, we identify the Cayley numbers \mathbb{O} with the real vector space \mathbb{R}^8 . Ordinary multiplication and inversion are understood to be those of \mathbb{O} .

Let $\varphi: \mathbb{R}_{\text{pos}} \rightarrow \mathbb{O}$ be a fixed continuous map from the positive reals into \mathbb{O} such that $\varphi(1)$ is the unit of \mathbb{O} and $|\varphi(r)| = r$ for all $r \in \mathbb{R}_{\text{pos}}$. We define a new multiplication \circ on \mathbb{R}^8 which depends on φ by

$$a \circ x = \begin{cases} ((a \cdot \varphi(|a|)^{-1})x) \cdot \varphi(|a|) & \text{for } a \neq 0, \\ 0 & \text{for } a = 0. \end{cases}$$

The reader can easily verify that \circ satisfies the axioms (Q 1) through (Q 4). Note that \circ is also norm preserving, i.e. $|a \circ x| = |a| |x|$. In general with respect to \circ and vector addition only one distributive law holds. The definition of \circ is analogous to the multiplication in 4-dimensional near fields ([11], [19]).

To describe the nuclei N_λ and N_μ of the quasifields with these multiplications we distinguish three cases.

Case 1: φ is a multiplicative homomorphism.

Then

$$N_\lambda^\times = \pm \varphi(\mathbb{R}_{\text{pos}}) \quad \text{and} \quad N_\mu^\times = \left\{ \pm \frac{\varphi(r)^2}{r} \mid r \in \mathbb{R}_{\text{pos}} \right\}.$$

For example, if φ is the homomorphism $\varphi(r) = r e^{i\alpha \cdot \log r}$ with a fixed real $\alpha \neq 0$, then neither N_λ nor N_μ are the one-dimensional subspace spanned by the unit, nor is $N_\lambda = N_\mu$.

Case 2: φ is not multiplicative, but the image of φ is contained in a two-dimensional subfield of \mathbb{O} .

Then either the set

$$J = \{r > 1 \mid \wedge s \in \mathbb{R}_{\text{pos}}: \varphi(rs) = \varphi(r)\varphi(s)\}$$

is empty, in which case

$$N_\lambda^\times = N_\mu^\times = \{1, -1\},$$

or J has a smallest element r_0 ; then

$$N_\lambda^\times = \{\pm \varphi(r_0)^n \mid n \in \mathbb{Z}\} \quad \text{and} \quad N_\mu^\times = \left\{ \pm \left(\frac{\varphi(r_0)^2}{r_0} \right)^n \mid n \in \mathbb{Z} \right\}.$$

Case 3: The image of φ is not contained in a two-dimensional subfield of \mathbb{O} .

Then either the set

$$J \cap \varphi^{-1}(\mathbb{R}) = \{r > 1 \mid \wedge s \in \mathbb{R}_{\text{pos}}: \varphi(rs) = \pm r \cdot \varphi(s)\}$$

is empty, in which case

$$N_\lambda^\times = N_\mu^\times = \{1, -1\},$$

or $J \cap \varphi^{-1}(\mathbb{R})$ has a smallest element r_1 ; then

$$N_\lambda^\times = N_\mu^\times = \{\pm r_1^n \mid n \in \mathbb{Z}\}.$$

The proofs consist of arithmetical calculations in \mathbb{O} using the Moufang identities. We shall not reproduce them here.

2.2. Further examples of 8-dimensional locally compact quasifields can be obtained as follows. Consider the affine plane over a real 8-dimensional division algebra different from the alternative field \mathbb{O} . Choose a frame of reference for this plane with coordinate axes nonparallel to the original ones. The coordinate domain with respect to this frame is an 8-dimensional locally compact quasifield ([14, § 7]). Explicit formulae for this change of coordinates are given in Skornyakov [16]. Both the left and middle nuclei of a quasifield obtained in this fashion consist of 0 and 1 only, since any nontrivial element of either nucleus would induce a collineation which would move the shears direction, contradicting a well-known theorem of Skornyakov ([17], see also [1, Hilfssatz 4 and 5] and [10, Theorem 6.16 p. 140]) together with the fact that \mathbb{O} is the only 8-dimensional alternative field over \mathbb{R} ([14, 7.27]).

Note that the examples of 2.1 above always have -1 in their nuclei and thus are not of the type we have just described.

3. Proofs of the theorems.

Proof of Theorem 1. Let us assume the theorem is false. Then there is an 8-dimensional locally compact quasifield Q whose kernel K is isomorphic to \mathbb{C} or to \mathbb{H} . In both cases \mathbb{C} can be identified with a two-dimensional subfield of K , and thus the vector space underlying Q may be regarded as a right complex 4-dimensional vector space (with respect to operations induced from Q). By definition of the kernel, for any fixed $a \in Q \setminus \{0\}$ ($= \mathbb{C}^4 \setminus \{0\}$) the map

$$\lambda_a: Q \rightarrow Q: x \mapsto a \circ x$$

is linear with respect to this vector space structure, and λ_a can be regarded as an element of $GL(4, \mathbb{C})$. We thus obtain a map

$$\lambda: \mathbb{C}^4 \setminus \{0\} \rightarrow GL(4, \mathbb{C}): a \mapsto \lambda_a.$$

We shall assume that the unit element 1 of Q is contained in the unit sphere S^7 of \mathbb{C}^4 . Evaluation at the unit gives $\lambda_a(1) = a \circ 1 = a$ for all a . In other words, the restriction of λ to S^7 would be a cross section for the map

$$\pi: GL(4, \mathbb{C}) \rightarrow S^7: \pi(\gamma) = \frac{\gamma(1)}{|\gamma(1)|}.$$

The theorem is thus proved by contradiction since actually (as a consequence of [2, 17.1]) this map does not, in fact, have a cross section. Indeed, choose an ordered orthonormal basis of \mathbb{C}^4 having 1 as its first vector, and represent elements of $GL(4, \mathbb{C})$ by matrices with respect to this ordered basis. The Gram-Schmidt orthogonalization process yields for every $A \in GL(4, \mathbb{C})$ a decomposition $A = U(A) \cdot D(A)$ with $U(A)$ unitary and $D(A)$ upper triangular with positive real diagonal entries so that, in particular, $D(A) \in \pi^{-1}(1)$. Also $U(A)$ and $D(A)$ depend continuously on A . The map $A \mapsto U(A)$ is then a retraction $\varrho: GL(4, \mathbb{C}) \rightarrow U(4, \mathbb{C})$ which respects

the fibers of π . Any cross section of π would by composition with ϱ yield a cross section of the fibration $\pi|U(4, \mathbb{C}): U(4, \mathbb{C}) \rightarrow S^7$. But by [2, 17.1] such a cross section does not exist. \square

Proof of Theorem 2. a) It suffices to prove the statement for N_λ^\times , since for any 8-dimensional locally compact quasifield Q there is another 8-dimensional locally compact quasifield whose left nucleus is isomorphic to the middle nucleus of Q ; this may be proved with [4] analogously as in [13, Prop. 4] using the transpose of the translation plane \mathcal{T} coordinatized by Q (cf. [4, § 2]). Another quasifield with this property is obtained by re-coordinatizing \mathcal{T} with the roles of the first and second coordinate axis interchanged ([5, 3.1.30], [16], [14, § 7]).

b) To prove the statement for N_λ^\times , consider the map

$$\lambda: \mathbb{R}^8 \setminus \{0\} \rightarrow GL(8, \mathbb{R}) : a \mapsto \lambda_a$$

where

$$\lambda_a(x) = a \circ x,$$

and its composition

$$\Delta: \mathbb{R}^8 \setminus \{0\} \rightarrow \mathbb{R}_{\text{pos}} : a \mapsto \det \lambda_a$$

with the determinant, where \mathbb{R}_{pos} denotes the multiplicative group of the positive reals. From the definition of N_λ it is immediate that by restriction Δ induces a continuous group homomorphism

$$\delta: N_\lambda^\times \rightarrow \mathbb{R}_{\text{pos}}.$$

By [8, 2.3] and since N_λ is closed in \mathbb{R}^8 , we have that $\delta^{-1}[r_1, r_2]$ is compact for all $0 < r_1 < r_2$. In particular,

$$F = \delta^{-1}(1)$$

is a compact normal subgroup of N_λ^\times ; and since any nontrivial subgroup of \mathbb{R}_{pos} is cyclic or dense, the image

$$H = \delta(N_\lambda^\times)$$

of δ is either 1) all of \mathbb{R}_{pos} , 2) discrete and isomorphic to \mathbb{Z} , or 3) trivial.

Therefore all that remains to be shown is that F is finite. For then it is easily seen that there is a closed subgroup \tilde{H} of N_λ^\times which is mapped isomorphically onto H by δ . In fact, if $H \cong \mathbb{R}_{\text{pos}}$, take \tilde{H} to be the connected component of 1 in N_λ^\times , which by the finiteness of F is a covering group via δ of the simply connected group \mathbb{R}_{pos} and thus is isomorphic to \mathbb{R}_{pos} . If $H \cong \mathbb{Z}$, take \tilde{H} to be the subgroup of N_λ^\times generated by an element a such that $\delta(a)$ generates H . Thus N_λ^\times will be a semidirect product of the normal subgroup F and the closed subgroup \tilde{H} isomorphic to either \mathbb{R}_{pos} , \mathbb{Z} or $\{1\}$.

c) We now prove that the compact subgroup F is finite. Since N_λ^\times is a Lie group, it suffices to show that N_λ^\times does not contain a torus. Proceeding indirectly, we assume that N_λ^\times contains a one-dimensional torus group T . The restriction of λ to T defines a fixed point free linear representation of T on \mathbb{R}^8 . We may identify \mathbb{R}^8 as a real vector space with \mathbb{C}^4 in such a way that $\lambda(T)$ is the group Θ of dilatations

by complex numbers of norm 1. We let $\Theta \subseteq GL(8, \mathbb{R})$ act on $GL(8, \mathbb{R})$ by left translation. Then by the definition of $N_\lambda^\times (\cong T)$ the restriction of λ to the unit sphere S^7 is Θ -equivariant; and, of course,

$$\pi: GL(8, \mathbb{R}) \rightarrow S^7: \gamma \mapsto \frac{\gamma(1)}{|\gamma(1)|}$$

is Θ -equivariant too. Therefore there are induced maps λ' and π' between the orbit spaces $S^7/\Theta = P_3(\mathbb{C})$ and $GL(8, \mathbb{R})/\Theta$ as indicated in the commutative diagram

$$\begin{array}{ccccc} S^7 & \xrightarrow{\lambda} & GL(8, \mathbb{R}) & \xrightarrow{\pi} & S^7 \\ \downarrow v & & \downarrow \tau & & \downarrow v \\ P_3(\mathbb{C}) & \xrightarrow{\lambda'} & GL(8, \mathbb{R})/\Theta & \xrightarrow{\pi'} & P_3(\mathbb{C}) \end{array}$$

Here v and τ denote the canonical projections. Since $\pi \circ \lambda = \text{id}$, we have $\pi' \circ \lambda' = \text{id}$ as well; in other words, λ' is a cross section for π' . Parts d) and e) below are now devoted to showing that, in fact, a cross section for π' does not exist. This contradiction will then prove the theorem.

d) The nonexistence of a cross section for π' may be seen most quickly by looking at the cohomology groups over the rationals \mathbb{Q} of the spaces involved. In fact, we have $H^4(P_3(\mathbb{C}), \mathbb{Q}) \cong \mathbb{Q}$ but $H^4(GL(8, \mathbb{R})/\Theta, \mathbb{Q}) = 0$ ([7, p. 90] and [15], [12]).

e) A different proof of this fact making use of the lower homotopy groups of the orbit spaces is the following. We claim first that there exist maps η, ψ, χ making the diagram

$$(*) \quad \begin{array}{ccc} S^3 & \xrightarrow{\psi} & GL(8, \mathbb{R}) \\ \eta \downarrow & & \downarrow \tau \\ S^2 & \xrightarrow{\chi} & GL(8, \mathbb{R})/\Theta \end{array}$$

commutative and such that ψ is not null-homotopic.

Indeed, regard \mathbb{C} as a subfield of \mathbb{H} , and identify \mathbb{R}^8 and \mathbb{H}^2 as real vector spaces in such a way that Θ is the subgroup of $GL(2, \mathbb{H}) \subseteq GL(8, \mathbb{R})$ given by

$$\Theta = \left\{ \begin{pmatrix} z & \\ & z \end{pmatrix} \mid z \in \mathbb{C}, |z| = 1 \right\}.$$

Furthermore, identify S^3 with the quaternions of norm 1, and define

$$\psi: S^3 \rightarrow GL(2, \mathbb{H}) \subseteq GL(8, \mathbb{R}): u \mapsto \begin{pmatrix} u & \\ & u \end{pmatrix}.$$

Via

$$u \mapsto \begin{pmatrix} u & \\ & 1 \end{pmatrix} \begin{pmatrix} \cos \vartheta & \sin \vartheta \\ -\sin \vartheta & \cos \vartheta \end{pmatrix} \begin{pmatrix} 1 & \\ & u \end{pmatrix} \begin{pmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{pmatrix}$$

($0 \leq \vartheta \leq \pi/2$) where $\cos \vartheta$ and $\sin \vartheta$ are considered as quaternions, ψ is homotopic to

$$u \mapsto \begin{pmatrix} u^2 & \\ & 1 \end{pmatrix},$$

which is not null-homotopic because

$$u \mapsto \begin{pmatrix} u & \\ & 1 \end{pmatrix}$$

is a generator of $\pi_3(GL(8, \mathbb{R})) \cong \mathbb{Z}$ ([18, 23.6 and 16.7]). Now $\tau \circ \psi$ factors through the orbit space of S^3 under the group $\{\mathbb{H} \rightarrow \mathbb{H} : x \mapsto zx \mid z \in \mathbb{C}, |z| = 1\}$, and this orbit space is $P_1(\mathbb{C}) = S^2$. We define η and χ to be the canonical projection (the Hopf map) and the map induced by ψ respectively.

We now use the maps in diagram (*) to show that a cross section

$$\lambda' : P_3(\mathbb{C}) \rightarrow GL(8, \mathbb{R})/\Theta$$

for π' does not exist. From the exact homotopy sequence of $v : S^7 \rightarrow S^7/\Theta = P_3(\mathbb{C})$ we have $\pi_2(P_3(\mathbb{C})) \cong \mathbb{Z}$; from the exact homotopy sequence of $\tau : GL(8, \mathbb{R}) \rightarrow GL(8, \mathbb{R})/\Theta$ we have $\pi_2(GL(8, \mathbb{R})/\Theta) \cong \mathbb{Z}$, since $\pi_2(GL(8, \mathbb{R}))$ is trivial ([18, 22.10]). Consequently, a cross section λ' for π' would induce an isomorphism

$$\lambda'_\# : \pi_2(P_3(\mathbb{C})) \rightarrow \pi_2(GL(8, \mathbb{R})/\Theta).$$

Let $\sigma : S^2 \rightarrow P_3(\mathbb{C})$ be a representative of the homotopy class $\lambda'^{-1}_\#[\chi]$ with χ as in diagram (*). Then in $\pi_3(GL(8, \mathbb{R})/\Theta)$ we have

$$\tau_\#[\psi] = [\chi \circ \eta] = [\lambda' \circ \sigma \circ \eta] = \lambda'_\#[\sigma \circ \eta].$$

The left-hand side of this equation is nonzero since $[\psi] \in \pi_3(GL(8, \mathbb{R}))$ is nonzero by construction and since $\tau_\# : \pi_3(GL(8, \mathbb{R})) \rightarrow \pi_3(GL(8, \mathbb{R})/\Theta)$ is an isomorphism. The right-hand side, however, is zero since $[\sigma \circ \eta] \in \pi_3(P_3(\mathbb{C})) = 0$, a contradiction. \square

Proof of Theorem 3. A closed weak nucleus of an 8-dimensional real division algebra contains the scalar multiples of the unit (since these form the smallest closed subfield of the algebra) and is therefore isomorphic to \mathbb{R}, \mathbb{C} or \mathbb{H} by the Frobenius theorem. We must exclude the quaternions from this list. Proceeding indirectly, we assume that the 8-dimensional real division algebra D has \mathbb{H} as a closed weak nucleus. Pick any $d \in D \setminus \mathbb{H}$. Then the underlying vector space of D is the direct sum $\mathbb{H} + \mathbb{H} \circ d$; and for all $u, x, y \in \mathbb{H}$ we have

$$u \circ (x + y \circ d) = ux + (uy) \circ d.$$

Thus the restriction of the map

$$\lambda : D \setminus \{0\} \rightarrow GL(8, \mathbb{R}) : a \mapsto \lambda_a \quad \text{with} \quad \lambda_a(x) = a \circ x$$

to the 3-sphere S^3 consisting of the elements of $\mathbb{H} \subseteq D$ of unit norm is described by

$$\lambda_u = \begin{pmatrix} u & \\ & u \end{pmatrix} \in GL(2, \mathbb{H}) \subseteq GL(8, \mathbb{R}) \quad (u \in \mathbb{H}, |u| = 1)$$

and therefore represents twice a generator of $\pi_3(GL(8, \mathbb{R})) \cong \mathbb{Z}$ ([18, 23.6]), cf. part e of the proof of Theorem 2. In particular, $\lambda|_{S^3}$ would not be null-homotopic. On the other hand, $\lambda|_{S^3}$ must be null-homotopic as it extends to $D \setminus \{0\}$ and $\pi_3(D \setminus \{0\}) \cong \pi_3(S^7) = 0$. This contradiction proves the theorem. \square

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