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A method of calculating lithosphere thickness from observations of deglacial land uplift and tilt

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Abstract. The study proposes an inversion method applicable to observations of deglacial land uplift and tilt near the margin of a major Pleistocene ice-sheet. This information allows the determination of the cross-section of the ice-sheet and the calculation of the thickness of the lithosphere. The method is applied to the Pleistocene glaciations in Fennoscandia and Laurentia and yields similar lithosphere thicknesses of 110 ± 30 km and 130 ± 35 km, respectively. Since the method is based on a static theoretical model, the thicknesses must be interpreted as upper bounds.

Key words: Elastic plate - Ice-sheet - Inversion - Isostasy - Lithosphere

Introduction

The idea of using glacio-isostatic data to infer the elastic resistance of the earth's lithosphere is usually credited to McConnell (1968), McGinnis (1968), Brotchie and Silvester (1969). Walcott (1970a) and others, although these studies are predated by Niskanen's (1943, 1949) contributions to the subject.

Rather well known is the investigation of Walcott (1970a), who used the model of a thin elastic plate superimposed on an inviscid half-space and flexed by an external load of specified cross-section. Based on the tilted strandline of pro-glacial Lake Algonquin (Canada), Walcott inferred a flexural rigidity of the lithosphere of about 6×10^{24} N m, corresponding to a thickness of about 75 km.

Walcott's (1970a) model is characterized by several simplifications, viz. using (a) the plane half-space approximation and (b) the thin-plate approximation for the earth model and using (c) a two-dimensional approximation of the roughly axisymmetric ice-sheet for the load model. Simplifications (a), (b) and (c) were shown to be justified for predictions of surface displacement and slope close to the margins of loads of large radius (Comer, 1983; Wolf, 1984a, b, 1985a). Since the model is static, it also presupposes (d) that the earth's surface was in equilibrium with the Laurentide ice-sheet during the existence of Lake Algonquin and (e) that it is in equilibrium at present. Clearly, assumptions (d) and (e) do not hold exactly, in which case the inferred lithosphere thickness must be interpreted as an upper bound (Walcott, 1970b).

In previous studies of glacio-isostatic adjustment in Fennoscandia, no use has yet been made of observations near the margin of the Pleistocene ice-sheet. In the present study, an upper bound on the thickness of the Fennoscandian lithosphere is obtained by *inverting* Walcott's (1970a) method using uplift *and* tilt data from the ice front. The inclusion of uplift data eliminates the dependence of the estimate on the cross-section of the load model, which is a limitation of Walcott's approach. The method shows the thickness of the Fennoscandian lithosphere to be less than 110 ± 30 km. The same method is applied to the uplift and tilt data associated with Lake Algonquin and yields an upper bound of 130 ± 35 km for the Laurentian lithosphere.

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Theoretical model

The theoretical model considered is that of a thin elastic plate superimposed on an inviscid half-space and flexed by an external load. The analysis is confined to two dimensions, in which case the (symmetric) Green's function for downward displacement is (Gunn, 1943)

$$w_G(x) = \frac{a}{2\rho_2 g} \exp(-ax)[\sin(ax) + \cos(ax)], \qquad x \geqq 0, \quad (1)$$

where the load is at $x=0$. Parameter ρ_2 is the density of the inviscid half-space and $g = 9.81$ m s^{-2} the gravitational acceleration. Parameter a^{-1} is the flexural scale-length defined by

$$a^{-1} = \left(\frac{4D}{\rho_2 g}\right)^{1/4}, \quad (2)$$

with

$$D = \frac{\mu_1 h_1^3}{6(1-\nu_1)} \quad (3)$$

the flexural rigidity. Parameters h_1, μ_1 and ν_1 are the thickness, shear modulus and Poisson's ratio of the elastic plate, respectively.

The pressure exerted by a load of exponential cross-section is

$$q(x)=\rho_0 g h_0 \begin{cases} 0, & x \leqq 0 \\ 1-\exp(-bx), & x>0 \end{cases}, \tag{4}$$

with h_0 the maximum load thickness, ρ_0 the load density and b the load steepness. The displacement caused by q is calculated by convolution with the Green's function, Eq. (1), yielding

$$w(x)=\frac{\rho_0 h_0}{2\rho_2}\frac{b\exp(ax)}{(b+a)^2+a^2} \cdot [a\sin(ax)+(b+a)\cos(ax)], \quad x\leqq 0, \tag{5a}$$

$$w(x)=\frac{\rho_0 h_0}{2\rho_2}\left\{2-\frac{8a^4\exp(-bx)}{b^4+4a^4}-\frac{b\exp(-ax)}{(b-a)^2+a^2} \cdot [a\sin(ax)+(b-a)\cos(ax)]\right\}, \quad x>0. \tag{5b}$$

If $b\to\infty$, the solution degenerates to that of a load of rectangular cross-section (e.g. Jeffreys, 1976, pp. 270-272). The mathematically convenient (but physically unfounded) case $a=b$ was discussed previously (Walcott, 1970a).

At the load margin, $x=0$, the displacement and slope are

$$w=\frac{\rho_0 h_0}{2\rho_2}\frac{b(b+a)}{(b+a)^2+a^2} \tag{6}$$

and

$$w'=\frac{\rho_0 h_0 a}{2\rho_2}\frac{(b+a)^2-a^2}{(b+a)^2+a^2}. \tag{7}$$

Solving Eqs. (6) and (7) for b yields

$$b=a\frac{w_r-2w}{2(w_r-w)}\left[\left(1+\frac{8w}{w_r-2w}\frac{w_r-w}{w_r-2w}\right)^{1/2}-1\right] \tag{8}$$

and

$$b=a\left[\left(\frac{w_r'+w'}{w_r'-w'}\right)^{1/2}-1\right], \tag{9}$$

respectively, where $w_r=\lim_{b\to\infty} w=\rho_0 h_0/(2\rho_2)$ and $w_r'=\lim_{b\to\infty} w'=\rho_0 h_0 a/(2\rho_2)$. Parameter b may be eliminated from Eqs. (8) and (9), yielding

$$a=w'\frac{2w-3w_r-S^{1/2}}{2(w_r-2w)(w_r-w)-w_r(w_r+S^{1/2})}, \tag{10}$$

where $S=(w_r-2w)^2+8w(w_r-w)$. If w and w' are known at the load margin and h_0 and ρ_2 are assumed, Eq. (10) has a unique solution which, by Eqs. (2) and (3), may be converted into an estimate of h_1.

Data analysis

To estimate an upper bound on lithosphere thickness in Fennoscandia, data from the Helsinki region (Finland) are used. During the Late Weichselian (10-12 ka b.p.), this region was close to the ice front, which deposited the Salpausselkä terminal moraines in southern Finland (cf. Eronen, 1983). Niskanen's (1939) compilation of emergence data shows that Helsinki has emerged by at least 67 m since deglaciation. The land uplift w is obtained after corrections for eustatic sea-level rise (e.g. Andrews, 1970, pp. 22-24) have been applied and is estimated to be 100 ± 10 m. The oldest strandline near Helsinki is associated with an early stage of the Baltic Ice Lake, which formed at about 10.6 ka b.p. The strandline is only mapped over a short distance; its tilt w' can be inferred from strandline diagrams (e.g. Donner, 1980; Eronen, 1983) and is estimated to be $0.70\pm0.10\times10^{-3}$.

In Fig. 1, $b^{-1}=f(h_1, w)$ and $b^{-1}=g(h_1, w')$ for $w=100$ m and $w'=0.70\times10^{-3}$ are shown. The results apply to $h_0=2.45$ km and the parameter values listed in Table 1. Clearly, if either w or w' is given, the solution is non-unique. In agreement with Eqs. (2) and (3), incompressibility reduces the lithosphere thicknesses slightly; similarly small variations would result from uncertainties in μ_1 or ρ_2. If, however, both $w=100\pm10$ m and $w'=0.70\pm0.10\times10^{-3}$ are assumed, a unique solution, $h_1=110\pm30$ km and $b^{-1}=330\pm50$ km, is obtained (Appendix). Table 2 shows that the value inferred for h_1 is nearly insensitive to the value assumed for h_0. This suggests 110 ± 30 km as an upper bound on the Fennoscandian lithosphere thickness.

In previous studies, estimates of the Laurentian lithosphere thickness from strandlines of pro-glacial lakes were based on strandline tilt alone (Walcott, 1970a, b; Wolf, 1985b). The present study supplements that by taking uplift into account. Land uplift and tilt can be estimated from Fig. 2, which is adapted from Chapman's (1954) strandline diagram for Lake Algonquin. The diagram assumes that no vertical movement has occurred in the "region of horizontality" of the strandline at $s<-300$ km since its formation at 10-12 ka b.p. (Farrand, 1962; Broecker, 1966). Then, from the portion of the strandline near $s=-75$ km, $w=160\pm15$ m and $w'=1.00\pm0.15\times10^{-3}$ are estimated.

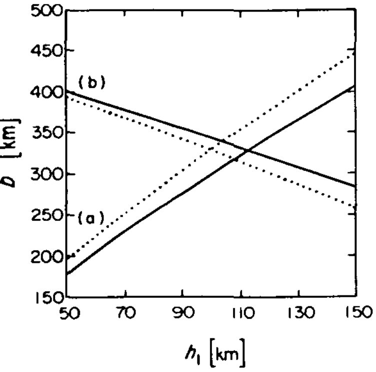
In Fig. 3, $b^{-1}=f(h_1, w)$ and $b^{-1}=g(h_1, w')$ for w

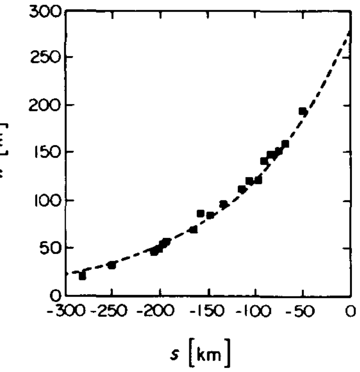
Table 1. Parameters of earth model

Layer l	h_l [km]	ρ_l [kg m^{-3}]	μ_l [N m^{-2}]	ν_l	η_l [Pa s]
1	h_1	0	0.67×10^{11}	ν_1	∞
2	∞	3380	0	0.5	0

Table 2. Results of inversion for Fennoscandia ($w=100$ m, $w'=0.70\times10^{-3}$, $\rho_0=910$ kg m^{-3} and $\nu_1=0.272$)

h_0 [km]	h_1 [km]	b^{-1} [km]
2.0	107	242
2.2	110	280
2.4	112	319
2.6	115	357
2.8	117	396
3.0	118	434

1


2


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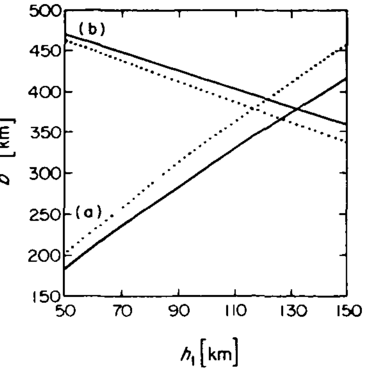

Fig. 1. Inverse steepness b^{-1} as function of lithosphere thickness h_1 for (a) $w = 100$ m or (b) $w' = 0.70 \times 10^{-3}$; inversion applies to $h_0 = 2.45$ km, $\rho_0 = 910$ kg m^{-3} and earth model of Table 1 with $\nu_1 = 0.272$ (*solid*) or $\nu_1 = 0.5$ (*dotted*)

Fig. 2. Observed land uplift (*squares*) and exponential approximation (*dashed*) as functions of horizontal distance s from location of ice front at time of existence of Lake Algonquin (adapted from Chapman, 1954; Broecker, 1966)

Fig. 3. Inverse steepness b^{-1} as function of lithosphere thickness h_1 for (a) $w = 160$ m or (b) $w' = 1.00 \times 10^{-3}$; inversion applies to $h_0 = 4.00$ km, $\rho_0 = 910$ kg m^{-3} and earth model of Table 1 with $\nu_1 = 0.272$ (*solid*) or $\nu_1 = 0.5$ (*dotted*)

Table 3. Results of inversion for Laurentia ($h_0 = 4.0$ km, $\rho_0 = 910$ kg m^{-3} and $\nu_1 = 0.272$)

s [km]	w [m]	w' [10^{-3}]	h_1 [km]	b^{-1} [km]
− 50	185	1.54	87	230
−100	122	1.01	96	411
−150	80	0.67	103	686
−200	53	0.44	108	1,102
−250	35	0.29	112	1,733
−300	23	0.19	115	2,691

$= 160$ m and $w' = 1.00 \times 10^{-3}$ are shown. The results apply to $h_0 = 4.0$ km and the parameter values listed in Table 1. Again, with $w = 160 \pm 15$ m and $w' = 1.00 \pm 0.15 \times 10^{-3}$, a unique solution, $h_1 = 130 \pm 35$ km and $b^{-1} = 380 \pm 60$ km, results (Appendix).

A possible problem with the proposed inversion method is that displacement and slope must be known at the location of the *equilibrium* load margin, $x = 0$, which, for isostatic disequilibrium, is not necessarily identical with the assumed location of the *actual* load margin, $s = 0$, at the time of strandline formation. The location of the equilibrium margin can be estimated from the second derivative of Eq. (5b), which shows that the inflection point of the displacement curve is always below the equilibrium load. Since the data do not suggest a sign change of the curvature (Fig. 2), any point s may, in principle, coincide with $x = 0$.

Table 3 shows the sensitivity of the estimate of h_1 to the location of the equilibrium load margin assumed. The Lake Algonquin strandline has been approximated by the function

$$w = w_0 \exp(cs), \qquad -300 \text{ km} \leqq s \leqq 0, \tag{11}$$

where $w_0 = 280$ m and $c^{-1} = 120$ km (Fig. 2). Although the sensitivity of h_1 to s is obvious, $h_1 < 130$ km always holds, which suggests 130 ± 35 km as an upper bound on the Laurentian lithosphere thickness.

Discussion and conclusion

Some idea of possible limitations of the inversion method may be gained from a discussion of the results obtained for the cross-sections of the ice-sheets. Nye's (1952) theoretical analysis shows that the cross-section of an (axisymmetric) ice-sheet of radius R is approximately given by the parabola

$$h(x) = h_0 \left(\frac{x}{R}\right)^{1/2}, \qquad 0 \leqq x \leqq R. \tag{12}$$

This can be replaced by the (two-dimensional) exponential cross-section

$$h(x) = h_0[1 - \exp(-bx)], \qquad 0 \leqq x < \infty \tag{13}$$

provided that

$$\int_0^R [1 - \exp(-bx)]\, dx = \int_0^R \left(\frac{x}{R}\right)^{1/2} dx. \tag{14}$$

Usually $\exp(-bR) \ll 1$, which reduces Eq. (14) to the condition

$$R = 3b^{-1}. \tag{15}$$

The abscissa b^{-1} of the intersection point in Fig. 1 is about 330 km. Equation (15) therefore suggests $R \cong 1{,}000$ km, which is close to the radius of the Fennoscandian ice-sheet at glacial maximum. Similarly, from Fig. 3, $b^{-1} \cong 380$ km and therefore $R \cong 1{,}150$ km; a closer estimate of R for the Laurentide ice-sheet at equilibrium would be 1,600 km.

Orowan (1949) and Nye (1952) showed that maximum ice thickness h_0 and basal shear stress τ are related by

$$h_0^2 = \frac{2\tau R}{\rho_0 g}. \tag{16}$$

Substituting for R from Eq. (15) and solving for τ yields

$$\tau = \frac{\rho_0 g h_0^2}{6b^{-1}}. \tag{17}$$

For the Fennoscandian ice-sheet, $h_0 = 2.45$ km has been used and $\tau = 0.27 \times 10^5$ N m^{-2} therefore results. For the Laurentide ice-sheet, $h_0 = 4.0$ km applies and $\tau = 0.63 \times 10^5$ N m^{-2} is obtained. Both estimates are similar to other estimates of τ for Pleistocene ice-sheets (cf. Paterson, 1981, pp. 162–164). However, if the inversion method is applied to the Lake Algonquin data for locations $s < -150$ km, b^{-1} becomes very large (Table 3) and Eqs. (15) and (17) no longer yield reasonable estimates of R and τ. The results for $s < -150$ km may therefore be disregarded.

A more important limitation of the proposed inversion method is that it only provides an *upper bound* on lithosphere thickness. The bound of 110 ± 30 km inferred from the Fennoscandian data is notable, however, because it is reasonably close to thickness estimates by McConnell (1968) and Cathles (1975). McConnell's thickness value was based on the short-wavelength part of an estimate of the relaxation-time spectrum of the Fennoscandian uplift. However, that part of the spectrum is controlled by kinks of the strandlines, most of which appear to be spurious (cf. Hyvärinen and Eronen, 1979; Walcott, 1980). Cathles, on the other hand, proposed that the strength of the lithosphere is insufficient to modify glacio-isostatic uplift in central Fennoscandia and therefore merely *adopted* a conventional thickness value.

In fact, the strength of the lithosphere may be such that the sensitivity of uplift to this feature is marked in central Fennoscandia. A similar sensitivity is, however, indicated to the presence of a low-viscosity layer below the lithosphere, which prevents a unique inference of lithosphere thickness from the observed central uplift. This was discussed recently (Wolf, 1985c, 1986a), where it was suggested that data from the ice front may help resolve the non-uniqueness of interpretations of uplift in central Fennoscandia. The present study serves as a first step and suggests that solutions $h_1 \geqq 110 \pm 30$ km can be eliminated. One of the objectives of future studies should be to improve this heuristic estimate using a dynamic earth model and a more accurate load model.

For the Laurentian lithosphere the study has suggested $h_1 \leqq 130 \pm 35$ km. Considering the geological and tectonic similarities between Fennoscandia and Laurentia, the closeness of the two bounds is encouraging. The Laurentian estimate also overlaps with previous results obtained using a dynamic earth model and a more accurate load model (Wolf, 1985b, 1986b). It thus reinforces the conclusion that the Lake Algonquin data are incompatible with thicknesses significantly larger than the value proposed. The present study, therefore, does not support Peltier's (1984) interpretation of relative-sea-level data from the North American east coast, which led him to suggest that the North American lithosphere is about 200 km in thickness. The conflict with Peltier's interpretation was discussed previously (Wolf 1985b).

Appendix

Sensitivity of estimates to uncertainties in observations

To estimate the sensitivity of h_1 to the uncertainties in w and w', D is eliminated from Eqs. (2) and (3) and Eq. (10) is substituted for a, yielding

$$h_1 = \left(\frac{w'}{w}\right)^{-4/3} F(w). \tag{18}$$

Substituting in Eq. (9) for a from Eq. (10) yields

$$b^{-1} = \frac{G(w)}{w' F(w)}. \tag{19}$$

Functions $F(w)$ and $G(w)$ are readily calculated and found to be only weakly sensitive to w for the values of interest. With $dF/dw = dG/dw = 0$,

$$\Delta h_1 = [(\Delta_w h_1)^2 + (\Delta_{w'} h_1)^2]^{1/2} \tag{20}$$

and

$$\Delta b^{-1} = \Delta_{w'} b^{-1} \tag{21}$$

obtains, where

$$\Delta_w h_1 = \frac{4}{3} h_1 \frac{\Delta w}{w}, \tag{22}$$

$$\Delta_{w'} h_1 = \frac{4}{3} h_1 \frac{\Delta w'}{w'} \tag{23}$$

and

$$\Delta_{w'} b^{-1} = b^{-1} \frac{\Delta w'}{w'}. \tag{24}$$

To estimate the uncertainties Δw and $\Delta w'$, random and systematic errors must be taken into account. Random errors in w are mainly caused by erroneous correlations of diachronous beach levels and appear as local kinks in strandline diagrams. Systematic errors are caused by (a) the uncertainty of the eustatic correction (Fennoscandia) or of the assumed zero-uplift level (Laurentia) and (b) the neglect of geoid perturbations.

Uncertainty (a) affects only w and, for Fennoscandia, can be estimated by comparing different eustatic corrections (e.g. Andrews, 1970, pp. 22–24); an estimate of uncertainty (b) can be obtained by calculating the deglaciation-induced geoid perturbation (e.g. Wolf, 1986b). In the present study, $\Delta w/w = 10\%$ and $\Delta w'/w' = 15\%$ are assumed, yielding $\Delta h_1/h_1 = 27\%$ and $\Delta b^{-1}/b^{-1} = 15\%$.

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References

Andrews, J.T.: A geomorphological study of post-glacial uplift with particular reference to Arctic Canada. London: Institute of British Geographers 1970

Broecker, W.S.: Glacial rebound and the deformation of the shorelines of proglacial lakes. J. Geophys. Res. **71**, 4777–4783, 1966

Brotchie, J.F., Silvester, R.: On crustal flexure. J. Geophys. Res. **74**, 5240–5252, 1969

Cathles, L.M.: The viscosity of the earth's mantle. Princeton: Princeton University Press 1975

Chapman, L.J.: An outlet of Lake Algonquin at Fossmill, Ontario. Proc. Geol. Assoc. Can. **6**, 61–68, 1954

Comer, R.P.: Thick plate flexure. Geophys. J.R. Astron. Soc. **72**, 101–113, 1983

Donner, J.: The determination and dating of synchronous Late Quaternary shorelines in Fennoscandia. In: Earth rheology, isostasy, and eustasy, N.-A. Mörner, ed.: pp. 285-293. Chichester: John Wiley 1980

Eronen, M.: Late Weichselian and Holocene shore displacement in Finland. In: Shorelines and isostasy, D.E. Smith, A.G. Dawson, eds.: pp. 183-207. New York: Academic Press 1983

Farrand, W.R.: Postglacial uplift in North America. Am. J. Sci. **260**, 181-199, 1962

Gunn, R.: A quantitative evaluation of the influence of the lithosphere on the anomalies of gravity. J. Franklin Inst. **236**, 47-65, 1943

Hyvärinen, H., Eronen, M.: The Quaternary history of the Baltic, the northern part. Acta Univ. Ups. Symp. Univ. Ups. Ann. Quing. Cel. **1**, 7-27, 1979

Jeffreys, H.: The earth, its origin, history and physical constitution, 6th edn. Cambridge: Cambridge University Press 1976

McConnell, R.K., Jr.: Viscosity of the mantle from relaxation time spectra of isostatic adjustment. J. Geophys. Res. **73**, 7089-7105, 1968

McGinnis, L.D.: Glacial crustal bending. Bull. Geol. Soc. Am. **79**, 769-775, 1968

Niskanen, E.: On the upheaval of land in Fennoscandia. Ann. Acad. Sci. Fenn. Ser. A **53**, 1-30, 1939

Niskanen, E.: On the deformation of the earth's crust under the weight of a glacial ice-load and related phenomena. Ann. Acad. Sci. Fenn. Ser. A3 **7**, 1-59, 1943

Niskanen, E.: On the elastic resistance of the earth's crust. Ann. Acad. Sci. Fenn. Ser. A3 **21**, 1-23, 1949

Nye, J.F.: A method of calculating the thickness of the ice-sheets. Nature **169**, 529-530, 1952

Orowan, E.: Contribution to discussion at joint meeting of the British Glaciological Society, the Rheologists' Club and the Institute of Metals. J. Glaciol. **1**, 231-236, 1949

Paterson, W.S.B.: The physics of glaciers, 2nd edn. Oxford: Pergamon Press 1981

Peltier, W.R.: The thickness of the continental lithosphere. J. Geophys. Res. **89**, 11303-11316, 1984

Walcott, R.I.: Isostatic response to loading of the crust in Canada. Can. J. Earth Sci. **7**, 716-727, 1970a

Walcott, R.I.: Flexural rigidity, thickness, and viscosity of the lithosphere. J. Geophys. Res. **75**, 3941-3954, 1970b

Walcott, R.I.: Rheological models and observational data of glacio-isostatic rebound. In: Earth rheology, isostasy, and eustasy, N.-A. Mörner, ed.: pp. 3-10. Chichester: John Wiley 1980

Wolf, D.: On the relation between two-dimensional and axisymmetric loads in plate flexure problems. J. Geophys. **54**, 232-235, 1984a

Wolf, D.: The relaxation of spherical and flat Maxwell earth models and effects due to the presence of the lithosphere. J. Geophys. **56**, 24-33, 1984b

Wolf, D.: Thick-plate flexure re-examined. Geophys. J. R. Astron. Soc. **80**, 265-273, 1985a

Wolf, D.: An improved estimate of lithospheric thickness based on a reinterpretation of tilt data from Pleistocene Lake Algonquin. Can. J. Earth Sci. **22**, 768-773, 1985b

Wolf, D.: The normal modes of a layered, incompressible Maxwell half-space. J. Geophys. **57**, 106-117, 1985c

Wolf, D.: Glacio-isostatic adjustment in Fennoscandia revisited. J. Geophys. **59**, 42-48, 1986a

Wolf, D.: On deglaciation-induced perturbations of the geoid. Can. J. Earth Sci. **23**, 269-272, 1986b

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