Boussinesq’s problem of viscoelasticity

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ABSTRACT

We consider an isochemical, isentropic, incompressible fluid half-space and study quasistatic viscoelastic perturbations, induced by two-dimensional (2D) surface loads, of a hydrostatic initial state. In view of the regional or local scale required for deformations of planets to be amenable to the half-space approximation, the model is assumed to be externally gravitating. We derive analytic solutions for the displacement and incremental stress components and study several approximations to the expressions. Particular emphasis is placed on discriminating between the material and local incremental stresses. Based on this distinction, deeper insight is gained into the physical significance of the solution.

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1 INTRODUCTION

Detailed investigations into the elastostatic deformation of a plane half-space subject to either displacement or traction boundary conditions were carried out by Boussinesq over a century ago and are summarized in his 1885 monograph. Today, the basic problem is usually referred to as Boussinesq’s problem, although in part of his work Boussinesq was preceded by Lamé & Clapeyron (1831) and Cerruti (1882). Since the publication of Boussinesq’s monograph, numerous authors have written on particular aspects of Boussinesq’s problem (e.g. Lamb, 1902; Terazawa, 1916; Love, 1929; Harding and Sneddon, 1945; Sneddon, 1946; Farrell, 1972).

A common feature of most work on Boussinesq’s problem is that the unper- turbed half-space is regarded as un- stressed. As far as the model is applied to study the regional or local deformation of planets, this assumption clearly cannot be satisfied. Effects due to a planet’s initial stress are, however, small for elastic perturbations whose lateral wavelength is in sufficiently short for the application of the half-space approximation (Cathles, 1975, pp. 35–39).

More significant is the influence of the initial stress for viscoelastic perturba- tion stresses then allowed Wolf to reduce the incremental field equations formally to those valid in the absence of initial stress, which could be solved using elementary methods.

In retrospect, Wolf’s (1985a) method of accounting for the initial stress is seen to be very similar to that used by Biot (1959), although Wolf was not aware of Biot’s publication at that time. As in Biot’s study, the significance of the modifications associated with the initial stress was not fully recognized by Wolf (1985a,b). This, in particular, applies to the physical meaning of the two kinds of incremental stress employed in Wolf’s analysis, which was not adequately discussed.

Recently, the theory of viscoelastodynamics for fluids in a state of hydrostatic initial stress has been reviewed (Wolf, 1991). In particular, rigorous deductions were given for the incremental field equations and continuity conditions and of the asymptotic approximations of the equations for short and long times after the onset of the perturbations. Special emphasis was placed on the distinctions between the Lagrangian and Eulerian kinematic formulations of the equations and between the material and local increments of the field quantities. Based on this, it was possible to interpret the short- and long-time asymptotic equations as the incremental field equations and continuity conditions of elastodynamics and of fluid dynamics, respectively.

In view of the progress achieved in our understanding of the theory of viscoelastodynamics for fluids in a state of hydrostatic initial stress, a re-examination of the viscoelastic Boussinesq’s problem within this improved theoretical framework appears to be justified. In the first place, such a re-examination is intended to clarify the physical interpretation of previous solutions to the problem. However, it should also serve as a guide for the physically correct treatment of more complicated problems.
In agreement with the heuristic character of the present study, the model to be analyzed is kept as simple as possible. We therefore consider the viscoelastic Boussinesq's problem using an isochemical, isentropic, incompressible half-space deformed by a 2D harmonic surface load. Since effects due to the perturbation of the gravity field are small for local or regional deformations of planets (Cathles, 1975, pp. 72–83), the half-space is assumed to be externally gravitating. In section 2, the relevant incremental field equations and boundary conditions and their Laplace transforms are collected. The equations are solved in section 3 by means of Love’s strain function and inverse Laplace transformation. Section 4 gives a discussion of the solution. Special care will be taken to discriminate between material and local incremental stresses. This distinction will prove necessary for a physically correct interpretation of the solution and its approximations.

2 FIELD EQUATIONS AND BOUNDARY CONDITIONS

The present study is concerned with Cartesian tensor fields. For brevity, we use for the fields the indicial notation and summation convention stipulating that the index subscripts i, j, k range over 1, 2, 3 and repeated indices imply summation. Note that the summation convention will later be suspended for the label subscripts x, y, z. We also employ the differentiation convention, i.e. index and label subscripts preceded by a comma denote partial differentiation with respect to the coordinate direction indicated by the subscript.

We assume now that the current state of a fluid at the time \( t \in [0, \infty) \) represents a small increment with respect to a hydrostatic initial state at the time \( t = 0 \). Further, we take as the spatial argument the initial particle position, \( X \in \mathbb{V}^{(0)} \), with \( \mathbb{V}^{(0)} \) the (open) region initially occupied by the fluid. This is commonly referred to as the Lagrangian kinematic formulation. The perturbation of an arbitrary initial field, \( f^{(0)}(X,t) \), can then be alternatively described in terms of the material incremental field, \( f^{(1)}(X,t) \), observed at the particle initially at \( X \), or in terms of the local incremental field, \( f^{(2)}(X,t) \), observed at the initial position, \( X \). The material and local incremental fields are related by

\[
f^{(2)}(X,t) = f^{(1)}(X,t) + f^{(0)}(X)u_i(X,t),
\]

(1)

where \( u(X,t) \) is the particle displacement and \( f^{(0)}(X)u_i(X,t) \) the advective incremental field (for further details cf. Dahlen, 1974; Grafarend, 1982; Wolf, 1991).

For the present study, we assume that the fluid is isochemical, isentropic, non-rotating, externally gravitating and subject only to gravitational volume forces. On these assumptions, the equations governing the initial state (e.g. Wolf, 1991) reduce to

\[
-p_i^{(0)} + p^{(0)}g_i = 0,
\]

(2)

\[
p^{(0)} \rho^{(0)} = k\rho^{(0)},
\]

(3)

where \( g_i \) is the gravitational force per unit mass, \( k \) the (isentropic) fluid bulk modulus, \( \rho^{(0)} \) the initial mechanical pressure, \( \rho^{(0)} \) the initial mass density and the argument \( X \) has been suppressed. Equations (2) and (3) are referred to as the initial equilibrium equation and the initial state equation. With \( g_i \) and \( k \) prescribed fields, (2) and (3) constitute the system of initial field equations to be satisfied by \( p^{(0)} \) and \( \rho^{(0)} \) for all \( X \in \mathbb{V}^{(0)} \).

Assuming now that the fluid undergoes quasistatic viscoelastic perturbations, the following equations apply (e.g. Wolf, 1991):

\[
t^{(1)}_i + (p^{(0)}u_i)_j - g_j(p^{(0)}u_j)_i = 0,
\]

(4)

\[
t^{(0)}_i = \int_0^t \left[ m_i(t-t') - \frac{2}{3} m_j(t-t') \delta_{ij} u_j(t') \delta_{ij} d^t t' + \int_0^{m_i(t-t')} \delta_t [u_j(t') + u_j(t')] d^t t',
\]

(5)

where \( m_i(t-t') \) and \( m_j(t-t') \) are the bulk and shear relaxation functions, \( t^{(0)} \) is the material integral Cauchy stress, \( \delta_{ij} \) the Kronecker symbol, \( \delta_t \) the partial derivative operator with respect to \( t' \) and the arguments \( X \) and \( t \) have been suppressed. Equations (4) and (5) are referred to as the incremental equation of motion and the incremental constitutive equation of viscoelasticity. With \( m_i(t-t') \) and \( m_j(t-t') \) prescribed fields and \( p^{(0)} \) and \( \rho^{(0)} \) obtained by solving (2) and (3), equations (4) and (5) constitute the system of incremental field equations of quasistatic viscoelastodynamics to be satisfied by \( t^{(0)}_i \) and \( u_i \) for all \( X \in \mathbb{V}^{(0)} \) and \( t \in [0, \infty) \).

To simplify the problem further, we assume that the fluid is incompressible. In the initial state, we thus have

\[
k \to \infty,
\]

(6)

\[
\rho^{(0)} \to 0
\]

(7)

With (6) and (7), the initial field equations, (2) and (3), reduce to a single equation

\[
-p_i^{(0)} = 0,
\]

(8)

where \( p \geq 0 \) is now a parameter. On the boundary \( \mathbb{S}^{(0)} \) of the region \( \mathbb{V}^{(0)} \) initially occupied by the fluid, \( p^{(0)} \) must satisfy conditions to be prescribed. We assume here that the boundary of the fluid is initially a free surface and therefore normal to \( g_i \). With \( u_i \) the unit vector in the direction of \( g_i \) and the notation

\[
f^{(0)}_i \to \lim_{t \to 0} f^{(0)}_i(X+e_t n),
\]

(9)

the initial boundary condition takes the form (e.g. Wolf, 1991)

\[
[p^{(0)}]_i = 0,
\]

(10)

where \( X \in \mathbb{S}^{(0)} \). Note that, by (8) and (10), the conditions \( p^{(0)} \) and \( \rho^{(0)} \) are equivalent for all \( X \in \mathbb{V}^{(0)} \). The fluid we are concerned with is therefore unstressed in the initial state only if it is non-gravitating.

On the assumption of incompressibility, it follows for the incremental state that

\[
m_i(t-t') \to \infty,
\]

(11)

\[
u_i \to 0.
\]

(12)

We also assume that \( m_i(t-t') \) is spatially homogeneous. Introducing \( m = m_2 \) for brevity, we thus require

\[
m_i(t-t') = 0.
\]

(13)

In view of (7), (11) and (12), the incremental field equations, (4) and (5), are replaced by

\[
t^{(0)}_i + (p^{(0)}u_i)_j = 0,
\]

(14)

\[
u_i = 0,
\]

(15)

\[
t^{(0)}_i = -p^{(0)} \delta_{ij} \int_0^t m_i(t-t') \times \delta_t [u_j(t') + u_j(t')] d^t t',
\]

(16)
where we have introduced
\[ p^{(0)} = -\int \lim_{u_i \to 0} m_i(t-t') \delta_i u_i(t') \, dt'. \]
(17)

For all \( p \in \mathcal{A}^{(0)} \), the solution to the incremental field equations must satisfy prescribed boundary conditions. Here, we are only concerned with perturbations due to surface loads. Employing the notation introduced in (9), the incremental boundary conditions take the form (e.g., Wolf, 1991)
\[ [n, t_i] + [-n, q] = 0, \]
(18)

where \( q \) is the prescribed incremental load pressure and \( X \in \mathcal{A}^{(0)} \).

The general solution to (14)–(16) can be derived by means of elementary methods if \( p^{(0)} \) and \( t^{(0)} \) are expressed in terms of the associated local increments. Since \( p^{(0)} = -p^{(0)} \delta_i \), it follows from (1) that
\[ p^{(0)} = p^{(0)} + p^{(0)} u_i, \]
(19)

\[ t^{(0)} = t^{(0)} - p^{(0)} \delta_q \delta_i. \]
(20)

Substitution of (19) and (20) into (14)–(16) and (18) gives
\[ \delta_{i_1} = 0, \]
(21)

\[ u_{i_1} = 0, \]
(22)

\[ [n, t_i] - [-n, q] = 0. \]
(24)

Note that the incremental field equations, (21)–(23), no longer depend on \( p^{(0)} \), and thus in particular agree with the ordinary field equations valid in the absence of initial stress. However, effects due to the initial stress enter through the incremental boundary condition, (24), which explicitly involves \( p^{(0)} \).

Solutions to (21)–(23) subject to (24) will be obtained by means of the Laplace transforms of the equations. The Laplace transform, \( \mathcal{L} \{ f(t) \} \), of a function \( f(t) \) is defined by
\[ \mathcal{L} \{ f(t) \} = \mathcal{L} \{ f(t) e^{-s t} \} \]
(25)

where the transform variable, \( s \), will be restricted to real values. For the convergence of the integral in (25) for \( s \)

larger than some value \( s_0 \) it is sufficient that \( f(t) \) be piecewise continuous for all \( t \in [0, \infty) \) and of exponential order as \( t \to \infty \). Elementary consequences of (25) are (e.g., LePage, 1961, pp. 285–328)
\[ \mathcal{L} \{ [p f(t)] \} = [p, s] \mathcal{L} \{ f(t) \}, \]
(26)

\[ \mathcal{L} \{ f(t-t') \} = \mathcal{L} \{ f(t) \} e^{-st}. \]
(27)

We will also need
\[ \mathcal{L} \{ H(t) e^{-st} \} = \frac{1}{s + t}. \]
(28)

where \( H(t) \) is the unit step function, here formally defined by
\[ H(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon + t} = 0, \quad \varepsilon < t < \infty \]
(29)

Using (25)–(27) and assuming that \( u_{i_1} \) vanishes at \( t = +0 \), we obtain from (21)–(24) the Laplace-transformed incremental field equations and boundary condition:
\[ \mathcal{L} \{ \delta_{i_1} \} = 0, \]
(30)

\[ [n, t_{i_1}] - [-n, q] = 0. \]
(31)

\[ \mathcal{L} \{ \delta_{i_2} \} = -p^{(0)} \delta_q \delta_i, \]
(32)

\[ [n, t_{i_2}] - [-n, q] = 0. \]
(33)

As in (30)–(33), the argument \( s \) of arbitrary Laplace-transformed incremental fields will usually be suppressed in the following; for brevity, we refer to such fields simply as 'incremental fields'.

We may use (32) to eliminate \( \delta_{i_1} \) from (30) and (33). Observing (13) and (31), we then arrive at the following incremental field equations and boundary condition:
\[ -p^{(0)} + \sin \mu \delta_{i_2} = 0, \]
(34)

\[ \delta_{i_2} = 0, \]
(35)

\[ [n, t_{i_2}] - [-n, q] = n \delta_{i_2}. \]
(36)

As an elementary example, we consider perturbations, induced by surface loads, of a plane half-space in a spatially homogeneous gravity field. When applied to planets, this approximation is appropriate only to perturbations whose 'typical' lateral wavelength is short compared with the planet's radius. The symmetry of the half-space suggests to introduce Cartesian coordinates: \( X = (x, y, z) \). We stipulate that \( 0 < x < \infty \) for all \( X \in \mathcal{A}^{(0)} \) such that \( x = 0 \)

for all \( X \in \mathcal{A}^{(0)} \). Then, \( \delta \equiv (g, 0, 0) \) must hold, where \( g \equiv (g, 0, 0) \) and the non-vanishing components of (8) and (10) become
\[ -p^{(0)} + pg \delta_{i_2} = 0. \]
(37)

\[ p^{(0)} = 0. \]
(38)

Supposing in the following that the perturbing load is 2D, we orient the \( y \)-axis normal to the strike of the load. It then follows from symmetry considerations that \( u_{i_2} = 0, \quad p^{(0)} = 0, \quad \delta_{i_2} = 0, \quad t_{i_2} = 0 \). Hence, the non-vanishing scalar components of (32) are
\[ \tilde{t}_{i_2} = -p^{(0)}, \]
(39)

\[ \tilde{u}_{i_2} = \sin \mu \tilde{u}_{i_2} + \tilde{u}_{i_2}, \]
(40)

\[ \tilde{t}_{i_2} = \tilde{p}^{(0)} + \tilde{p}^{(0)} \tilde{u}_{i_2} + \tilde{u}_{i_2}, \]
(41)

\[ \tilde{u}_{i_2} = -p^{(0)}. \]
(42)

Similarly, we find for the non-vanishing scalar components of (34) and (35) the relations
\[ -p^{(0)} + \sin \mu \tilde{u}_{i_2} + \tilde{u}_{i_2} = 0, \]
(43)

\[ -p^{(0)} + \sin \mu \tilde{u}_{i_2} + \tilde{u}_{i_2} = 0, \]
(44)

\[ \tilde{u}_{i_2} + \tilde{u}_{i_2} = 0. \]
(45)

With (37) and \( n \equiv (1, 0, 0) \), the scalar components of (36) take the forms
\[ \tilde{u}_{i_2} + \tilde{u}_{i_2} = 0, \]
(46)

\[ \tilde{p}^{(0)} + \tilde{p}^{(0)} \tilde{u}_{i_2} + \tilde{u}_{i_2} = 0, \]
(47)

Equations (43)–(45) are three simultaneous second-order partial differential equations for \( \tilde{u}_{i_2} \) and \( \tilde{u}_{i_2} \), which must be solved subject to (46) and (47). These equations must be completed by conditions requiring that the incremental fields and their spatial derivatives remain bounded as \( x \to \infty \).

3 SOLUTION TO THE INCREMENTAL EQUATIONS

We obtain the general solution to the equations by means of Love's strain function, \( \lambda \), defined by (e.g., Malvern, 1969, pp. 552–554)
\[ \tilde{u}_x = \lambda_{xy}, \]
(48)

\[ \tilde{u}_y = -\lambda_{xy}, \]
(49)

\[ \tilde{p}^{(0)} = -\sin \lambda_{xx} \lambda_{yy}, \]
(50)

Using (48)–(50), equations (39)–(42) can also be expressed in terms of \( \lambda \):
\[ \tilde{t}_{i_2} = \sin \lambda_{xx} + 3 \lambda_{yy}, \]
(51)
In addition to (69), we require that \( A \) and its spatial derivatives remain bounded as \( x \to \infty \).

For the purposes of the following discussion it will be sufficient to assume \( \xi \geq 0 \). The solution to (67) satisfying the condition at infinity can then be written in the form

\[ A = \frac{1}{\xi^2} (A + B \xi \hat{s}) e^{-\xi x}, \tag{70} \]

where \( A \) and \( B \) are integration constants. Using (70) to substitute for \( A \) and its derivatives in (66), we obtain

\( \hat{U}_x = (A + B \xi \hat{s}) e^{-\xi x}, \tag{71} \)

\( \hat{U}_y = i[A - B(1 - \xi \hat{s})] e^{-\xi x}, \tag{72} \)

\( \hat{\mu}_{\alpha \beta} = -2\xi e^{-\xi x}, \tag{73} \)

\( \hat{T}_{\alpha \beta} = 2\xi e^{-\xi x}, \tag{74} \)

\( \hat{T}_{\alpha y} = j(A - B(1 - \xi \hat{s})] e^{-\xi x}, \tag{75} \)

\( \hat{T}_{\alpha y} = -2\xi e^{-\xi x}, \tag{76} \)

\( \hat{T}_{\alpha y} = 2\xi e^{-\xi x}. \tag{77} \)

The constants \( A \) and \( B \) can be determined by substitution of (70) and its derivatives into (68) and (69). We get

\[ A = \frac{1}{\xi^2} \left( A + B \xi \hat{s} \right) e^{-\xi x}, \tag{78} \]

whence (71)–(77) take the forms

\( \hat{U}_x = -(A + B \xi \hat{s}) e^{-\xi x}, \tag{79} \)

\( \hat{U}_y = i(A - B(1 - \xi \hat{s})] e^{-\xi x}, \tag{80} \)

\( \hat{\mu}_{\alpha \beta} = -2\xi e^{-\xi x}, \tag{81} \)

\( \hat{T}_{\alpha y} = 2\xi e^{-\xi x}, \tag{82} \)

\( \hat{T}_{\alpha y} = j(A - B(1 - \xi \hat{s})] e^{-\xi x}, \tag{83} \)

\( \hat{T}_{\alpha y} = -2\xi e^{-\xi x}, \tag{84} \)

\( \hat{T}_{\alpha y} = 2\xi e^{-\xi x}. \tag{85} \)

A useful incremental field quantity to consider is the maximum shear stress occurring in the half-space. Since the maximum shear stress can be related to the difference between the largest and smallest principal stresses, we must first determine the principal stresses. With \( \hat{P}_{\alpha \beta} = -\hat{\mu}_{\alpha \beta} \) as one of the principal stresses, the other two are obtained from the characteristic equation

\[ \det \begin{bmatrix} \hat{P}_{\alpha \beta} - \hat{P}_{\alpha \beta} & \hat{P}_{\alpha \beta} \\ \hat{P}_{\alpha \beta} & \hat{P}_{\alpha \beta} \end{bmatrix} = 0. \tag{86} \]

Upon expansion of the determinant and use of (50), (51) and (53), the principal stresses are found to be

\[ \hat{P}_{\alpha \beta} = -\frac{1}{4} \left( \hat{P}_{\alpha \beta}^2 + \left( \hat{P}_{\alpha \beta}^2 \right)^2 + \left( \hat{P}_{\alpha \beta}^2 \right)^2 \right). \tag{87} \]

Since \( \hat{P}_{\alpha \beta} \) and \( \hat{P}_{\alpha \beta}^2 \) are the largest and smallest principal stresses, respectively, the maximum shear stress, \( \hat{I}_x \), is given by

\[ \hat{I}_x = \frac{1}{2} (\hat{P}_{\alpha \beta}^2 + 4\hat{P}_{\alpha \beta}^2). \tag{88} \]

If we put

\[ \hat{I}_x = \int \hat{T}_x d\xi \tag{90} \]

and equate (89) with (90), we find, using (58) and (82)–(84), the relation

\[ \hat{T}_x = \left( 1 + i \right) \frac{2\xi \hat{Q}}{2\xi \hat{Q} + \hat{P}} e^{-\xi x}. \tag{91} \]

Note that, according to (90) and (91), \( \hat{T}_x \) is independent of \( y \).

To proceed beyond (79)–(85) and (91), we must specify \( \hat{m} \) and \( \hat{Q} \). As a simple example, we consider the shear relaxation function for Maxwellian viscoelasticity (e.g. Christensen, 1982, pp. 16–20):

\[ m = \mu H(t) e^{-\alpha t}. \tag{92} \]

Note that \( m \) is determined by two parameters: the inverse Maxwell time, \( \alpha \), and the shear modulus, \( \mu \). We further consider an ‘instantaneous’ loading event:

\[ Q = Q' H(t). \tag{93} \]

In view of (28), the Laplace transforms of (92) and (93) are

\[ \hat{m} = \frac{\mu}{\xi + \alpha}. \tag{94} \]
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\[ \dot{Q} = \frac{Q'}{s}. \]  

From (94) and (95), we get

\[ \frac{\dot{Q}}{2\xi m + p g} = \frac{Q'}{2\xi \mu + p g} \left[ 1 + \frac{2\xi \mu}{p g} \right]. \]  

We also consider the depth, \( x_m \), where \( T_t \) assumes a maximum. Since

\[ \frac{d}{d\xi}(\xi e^{-\xi}) = (1-\xi)e^{-\xi}, \]

we obtain

\[ x_m = \frac{1}{\xi}. \]

Substitution of (110) into (106) then gives

\[ T_{ts} = \frac{2\xi \mu Q' H(t)}{2\xi \mu + p g} e^{-\xi}. \]  

The short-time limits of (107), (108) and (111) are

\[ [U]_t = 0, \quad \dot{U} = \frac{Q'}{2\xi \mu + p g}, \]  

\[ [P]_t = 0, \quad \dot{P} = \frac{2\xi \mu Q' H(t)}{2\xi \mu + p g} e^{-\xi}, \]  

\[ [T]_t = \frac{2\xi \mu Q' H(t)}{2\xi \mu + p g} e^{-\xi}. \]

Equations (112)–(114) apply to elasto-static equilibrium, governed by the shear modulus \( \mu \), in a half-space subject to the hydrostatic initial stress gradient \( pg \) (e.g. Wolf, 1985a, 1985b). Using (98), the long-time limits of (107), (108) and (111) are found to be

\[ [U]_t = 0, \quad \dot{U} = \frac{Q'}{2\xi \mu + p g}, \]  

\[ [P]_t = 0, \quad \dot{P} = \frac{2\xi \mu Q' H(t)}{2\xi \mu + p g} e^{-\xi}, \]  

\[ [T]_t = \frac{2\xi \mu Q' H(t)}{2\xi \mu + p g} e^{-\xi}. \]

Equations (115) describe hydro-static equilibrium in a half-space subject to the hydrostatic initial stress gradient \( pg \). The transition from the instantaneous elastostatic to the final hydrostatic equilibrium state is seen from (107), (108) and (111) to be exponential in time, where the inverse relaxation time, \( \beta \), is related to the parameters of the half-space by (98).

If \( 2\xi \mu (pg) < 1 \), equations (98), (107), (108) and (111) become, correct to the first order in the small quantities,

\[ \beta = \frac{pg\alpha}{2\xi \mu}. \]  

\[ [U]_t = 0, \quad \dot{U} = \frac{Q'H(t)}{2\xi \mu (1+\alpha t)}, \]  

\[ [P]_t = 0, \quad \dot{P} = \frac{Q'H(t)}{2\xi \mu} e^{-\xi}(1-\alpha t), \]  

\[ [T]_t = \frac{Q'H(t)}{2\xi \mu} e^{-\xi}(1-\alpha t). \]

Equation (122) is the expression for the inverse relaxation time for Newtonianviscous perturbations, controlled by the shear viscosity modulus \( \mu \alpha \), of a half-space subject to the hydrostatic initial stress gradient \( pg \) (Haskell, 1935, 1936; Ranalli, 1987, pp. 192–199).

If \( pg(2\xi \mu) = 0 \), equations (122)–(125) further reduce to

\[ \beta = 0, \]  

\[ [U]_t = 0, \quad \dot{U} = \frac{Q'H(t)}{2\xi \mu} (1+\alpha t), \]  

\[ [P]_t = 0, \quad \dot{P} = \frac{Q'H(t)}{2\xi \mu} e^{-\xi}(1-\alpha t), \]  

\[ [T]_t = \frac{Q'H(t)}{2\xi \mu} e^{-\xi}(1-\alpha t). \]

Equations (127)–(129) apply to elastostatic equilibrium in an initially unstressed half-space (e.g. Jeffreys, 1976, pp. 265–267). Equation (126) shows that the subsequent relaxation proceeds infinitely slow. On the other hand, it follows with (115) that

\[ [U]_t = 0, \quad \dot{U} = \infty, \]

where the instability is due to the absence of the gravitational force necessary to balance the load in the final hydrostatic state.
More practically, we may use the condition \( p g(2\mu) \leq 10^{-2} \) to estimate the minimum value of \( \xi \) in order that effects due to the hydrostatic initial stress may be neglected. Considering the Earth as an example and taking \( \rho = 10^{3} \text{kg m}^{-3} \), \( g = 10 \text{m s}^{-2} \) and \( \mu = 10^{11} \text{Pa} \) as values typical of the Earth, we obtain \( \xi = 10^{-5} \text{m}^{-1} \) as the minimum wave number. This approximately corresponds to a maximum wavelength of \( 10^{9} \text{m} \). Within the limits of the half-space approximation, the influence of the initial stress is therefore negligible as far as elastic perturbations are concerned. However, if \( \mu < 10^{11} \text{Pa} \), the initial stress becomes noticeable at much shorter wavelengths. This condition in particular applies to viscoelastic perturbations, where the 'effective' shear modulus, \( \mu' \), may become arbitrarily small.

Additional insight is gained by considering also \( P^{(6)} \). In view of (19), (37) and (58), we have in particular

\[
[P^{(6)}]_{J} - \psi = [P^{(4)}]_{J} - \psi + \rho g[U_{J}]_{J} - \psi. \tag{131}
\]

Upon substitution of (107) and (108), this becomes

\[
[P^{(6)}]_{J} = \psi = Q'H(t). \tag{132}
\]

Since it is readily shown that \( [T^{(2)}]_{J} = \psi = -[P^{(6)}]_{J} = \psi \), equation (132) simply expresses the balance required by (18) between the material incremental stress component normal to the surface of the half-space and the incremental load pressure.

In view of (116), it follows from (131) that

\[
[P^{(6)}]_{J} = \psi_{J}, \quad I_{J} = \psi_{J}, \quad \rho g[U_{J}]_{J} = \psi_{J}, I_{J} = \psi_{J}. \tag{133}
\]

In the final hydrostatic equilibrium state, the material incremental pressure is thus completely maintained by the advective incremental pressure associated with the displacement component in the direction of the initial pressure gradient. Assuming that (116) remains valid for \( \rho g \rightarrow 0 \), equation (133) also applies in this case and we require

\[
\lim_{\rho g \rightarrow 0} \rho g[U_{J}]_{J} = \psi_{J}, I_{J} = \psi_{J}. \tag{134}
\]

Together, (133) and (134) ensure that the incremental boundary condition given by (132) is satisfied in the final hydrostatic state even in the absence of initial stress. This illustrates the physical significance of the singularity in the solution for the displacement noted in (130) from a different point of view.

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**REFERENCES**


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## BOUSSINESQ’S PROBLEM OF VISCOELASTICITY

### APPENDIX: LIST OF SYMBOLS

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<th>Greek symbols</th>
<th>Calligraphic symbols</th>
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<td>(i)</td>
<td>(\mathcal{L})</td>
</tr>
<tr>
<td>f</td>
<td>(k)</td>
<td>(x_m)</td>
</tr>
<tr>
<td>(F_{0..})</td>
<td>(m)</td>
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</tr>
<tr>
<td>(f_{k..})</td>
<td>(\alpha)</td>
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<tr>
<td>(f_{k..})</td>
<td>(\beta)</td>
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<tr>
<td>(f_{k..})</td>
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<td>(f_{k..})</td>
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<td>(g)</td>
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<td>(t)</td>
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</table>

- \(f\): scalar field
- \(F_{0..}\): spectral density of \(f_{0..}\)
- \(f_{k..}\): tensor field
- \(f_{k..}\): partial derivative of \(f_{k..}\) with respect to \(X_k\)
- \(f_{k..}\): Laplace transform of \(f_{k..}\)
- \(f_{k..}\): initial value of \(f_{k..}\)
- \(f_{k..}\): local increment of \(f_{k..}\)
- \(f_{k..}\): material increment of \(f_{k..}\)
- \(g\): magnitude of \(g\)
- \(g\): gravitational force per unit mass
- \(H\): unit step function
- \(i\): imaginary unit
- \(k\): fluid bulk modulus
- \(m\): bulk relaxation function
- \(m\): shear relaxation function
- \(\alpha\): inverse Maxwell time
- \(\beta\): inverse relaxation time
- \(\delta_{ij}\): Kronecker symbol
- \(\partial\): partial derivative operator with respect to \(t\)
- \(\lambda\): Love’s strain function
- \(\mu\): shear modulus
- \(\xi\): wave number
- \(\rho\): mass density
- \(\mathcal{L}\): Laplace transformation functional
- \(\mathcal{V}\): 3D region