Geophys. J. Int. (1994) **116**, 321–348

Lamé's problem of gravitational viscoelasticity: the isochemical, incompressible planet

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Accepted 1993 June 28. Received 1993 June 23; in original form 1992 July 25

SUMMARY

We consider a spherical, isochemical, incompressible, non-rotating fluid planet and study infinitesimal, quasi-static, gravitational–viscoelastic perturbations, induced by surface loads, of a hydrostatic initial state. The analytic solution to the incremental field equations and interface conditions governing the problem is derived using a formulation in terms of the *isopotential* incremental pressure measuring the increment of the hydrostatic initial pressure with respect to a particular level surface of the gravitational potential. This admits the *decoupling* of the incremental equilibrium equation from the incremental potential equation. As result, two mutually independent (4×4) and (2×2) first-order ordinary differential systems in terms of the *mechanical* and *gravitational* quantities, respectively, are obtained, whose integration is algebraically easier than that of the conventional (6×6) differential system. In support of various types of application, we provide *transfer* functions, *impulse-response* functions and *Green's* functions for the full range of incremental field quantities of interest in studies of planetary deformations. The functional forms in the different solution domains involve explicit expressions for the Legendre degrees $n = 0$, $n = 1$ and $n \geq 2$, apply to any location in the interior or exterior of the planet and are valid for any type of *generalized Maxwell* viscoelasticity and for arbitrary surface loads.

Key words: generalized Maxwell viscoelasticity, gravitational viscoelastodynamics, Green's functions, impulse-response functions, surface loading, transfer functions.

1 INTRODUCTION

The problem of the elastostatic deformation of a spherical body was first investigated by Lamé (1854), who considered a spherical shell subjected to given volume forces and prescribed conditions on its inner and outer surfaces. Lamé formulated the field equations in spherical coordinates and derived the solution for the displacement in terms of surface harmonics. Lamé's problem was independently solved by Thomson (1864). In contrast to Lamé, he employed Cartesian coordinates and expanded the solution into solid harmonics.

Applications of Lamé's problem to deformation studies of planetary bodies were initially connected with the problem of correctly accounting for gravitation. The modifications introduced by gravitation were discussed by Love (1908), who pointed out two basic effects. One of them is related to the presence of *initial stress* in planetary interiors and requires a modification of the ordinary momentum equation valid in the absence of initial stress. Love (1911, pp. 89–93) implemented the necessary adjustments to the theory and derived the incremental momentum equation for a *hydrostatic* initial state. The other effect only arises if the planet is taken as compressible. In that case, the incremental gravitational force associated with perturbations of the initial density introduces a tendency towards *instability*. Normally, this tendency is, however, compensated by the opposing force resulting from the compressibility of the material. The stability of planets was studied in detail by Love (1911, pp. 89–104, 111–125).

A simplification of the investigations by Lamé, Thomson and Love as far as applications to planets are concerned is the assumption of homogeneous distributions of density, bulk modulus and shear modulus in the initial state. This constraint was removed by Herglotz (1905) and Hoskins (1910, 1920), who gave analytic solutions for elastostatic and gravitational–elastostatic deformations of a sphere, due to tidal volume forces, for simple types of variation of density and elasticity with radial distance. Later, Takeuchi (1950) extended these studies in order to allow for more realistic radial variations. This generalization required the use of numerical integration techniques. Takeuchi's approach was also employed in studies of gravitational–elastostatic perturbations of planetary bodies due to arbitrary surface loads. The latter problem was first investigated by Slichter & Caputo (1960), Caputo (1961, 1962) and Longman (1962, 1963), who calculated the Green's functions for displacement and incremental gravity. Longman's theory and numerical results were reviewed and extended by Farrell (1972).

More recently, a number of studies were concerned with gravitational–elastostatic perturbations of a compressible planet with fluid core. The solution to this problem proved not to be straightforward and occupied the investigators involved for several years. The competing approaches were reviewed by Longman (1975) and reconciled by Dahlen & Fels (1978); in both works, comprehensive bibliographies of the relevant publications may be found.

The development of the theory governing quasi-static, gravitational–viscoelastic perturbations of initially hydrostatic planets has been intimately related to the study of the earth's response to glacial surface loads. The basic theoretical work was completed by Peltier (1974) and Wu & Peltier (1982); an alternative approach was taken by Sabadini, Yuen & Boschi (1982) and Spada *et al.* (1992). Essential to both approaches is the formal reduction of the problem to the corresponding elastostatic problem, which is then solved by recasting the incremental field equations into a (6×6) first-order ordinary differential system with respect to the radial coordinate, leading to six fundamental solutions.

The virtue of the theory presented in these publications is that perturbations of realistic spherically symmetric planets can be calculated. On the other hand, explicit solution functions for elementary models are indispensable for a *physical* interpretation of the perturbations (e.g. Wolf 1991b) or for tests of the accuracy of numerical solution methods (e.g. Gasperini & Sabadini 1989; Wu 1992). So far, only a limited number of explicit solutions have been obtained on the assumption of incompressible *Maxwell* viscoelasticity. Thus, Sabadini *et al.* (1982) stated the six fundamental solutions for a homogeneous spherical layer. Considering the particular case of a homogeneous sphere, Wu & Peltier (1982) derived the special solution for displacement due to surface loading. Dragoni, Yuen & Boschi (1983) gave the special solution for displacement induced by volume forces in a sphere consisting of a homogeneous elastic lithosphere overlying a homogeneous viscoelastic mantle. Wolf (1984) and Amelung & Wolf (1994) studied selected two-layer models and derived special solutions for surface loading. For the same type of forcing, Wu (1990) analysed gravitational–viscoelastic perturbations of a two-layer sphere with arbitrary contrasts of density, shear modulus and viscosity across the interface.

In this study, infinitesimal, quasi-static, gravitational–viscoelastic perturbations, due to surface loads, of a spherical, isochemical, incompressible, non-rotating, fluid planet initially in hydrostatic equilibrium are reconsidered. The distinctive features of our analysis are the following.

We show that the incremental field equations can be recast into a form in which the equation for the (mechanical) momentum is *decoupled* from the equation for the (gravitational) potential. The *coupling* between the mechanical and gravitational aspects of the problem is then restricted to density discontinuities and expressed by incremental interface conditions. Instrumental to the decoupling of the incremental field equations is the use of a field quantity referred to as *isopotential* incremental pressure measuring the increment of the hydrostatic initial pressure with respect to a (perturbed) level surface of the gravitational potential (Section 2).

Using the appropriate ansatz for the decoupled incremental field equations, we then establish two mutually independent (4×4) and (2×2) first-order ordinary differential systems for the mechanical and gravitational aspects of the problem, respectively. The deduction of the general solutions to these systems is algebraically simpler than the deduction of the general solution to the conventional (6×6) differential system; the special solution is obtained in the usual way by means of the incremental interface conditions (Section 3).

For the main portion of our study, we are concerned with the derivation of special solution functions. In contrast to previous studies, a comprehensive catalogue of formulae covering all field quantities of interest in studies of planetary deformations is provided. Furthermore, *transfer* functions, *impulse-response* functions and *Green's* functions for the incremental fields in the appropriate solution domains are collected. The solution functions given involve explicit expressions for the Legendre degrees $n = 0, n = 1$ and $n \geq 2$ and apply to any location in the interior or exterior of planetary bodies and for arbitrary surface loads. Of theoretical and practical interest is the consideration of *generalized Maxwell* viscoelasticity, which includes a stability analysis of the solution for this type of viscoelasticity (Section 4).

We conclude our study with an assessment of the results obtained and a brief outlook on possible consequences (Section 5).

2 FIELD EQUATIONS AND INTERFACE CONDITIONS

In this section, the basic equations governing infinitesimal, quasi-static, gravitational–viscoelastic perturbations, due to surface loads, of a spherical, isochemical, incompressible, non-rotating, fluid planet initially in hydrostatic equilibrium are compiled. Section 2.1 presents the Cartesian-tensor forms of the field equations and interface conditions for the initial fields (Section 2.1.1) and the incremental fields and their Laplace transforms (Section 2.1.2). Section 2.2 defines the geometry of the problem to be solved (Section 2.2.1) and gives the scalar forms in spherical coordinates of the equations for the initial fields (Section 2.2.2) and the Laplace-transformed incremental fields (Section 2.2.3). Key points of this section are (i) the *uniform* use of the Lagrangian kinematic formulation in the internal and external domains and (ii) the *decoupling* of the incremental field equations for the *mechanical* quantities from those for the *gravitational* quantities. Complete derivations from first principles of the equations in this section can be found elsewhere (Wolf 1993, pp. 14–22, 27–31, 48–50).

2.1 Tensor equations

We consider Cartesian-tensor fields in indicial notation and imply for them the usual summation and differentiation conventions. Throughout this study, we employ the *Lagrangian* formulation, $f_{ij\ldots} = f_{ij\ldots}(\mathbf{X}, t)$, where the field value refers to the position, $X_i + u_i(\mathbf{X}, t)$, at the *current* time, t, of a material point whose position, X_i, at the *initial* time, $t = 0$, is taken as the spatial argument. The spatial domains of definition for the field equations and interface conditions are $\mathscr{X}_- \cup \mathscr{X}_+$ and $\partial\mathscr{X}$, respectively, with $\mathscr{X}_-$ the *internal* domain, $\mathscr{X}_+$ the *external* domain and $\partial\mathscr{X}$ the *interface*; $\mathscr{E} = \mathscr{X}_- \cup \mathscr{X}_+ \cup \partial\mathscr{X}$ is the Euclidian space domain. We also regard the *total* field value, $f_{ij\ldots}$, as perturbation of the *initial* field value, $f^{(0)}_{ij\ldots}$, such that $f_{ij\ldots} = f^{(0)}_{ij\ldots} + f^{(\delta)}_{ij\ldots}$ applies, with $f^{(\delta)}_{ij\ldots}$ the *material incremental* value. Accordingly, the temporal argument is the initial time, $t = 0$, for the initial fields, the current time, $t \in \mathscr{T}$, for the total and the incremental fields and the inverse Laplace time, $s \in \mathscr{S}$, for the Laplace-transformed incremental fields, with $\mathscr{T}$ the time domain $[0, \infty)$ and $\mathscr{S}$ the (complex) inverse Laplace-time domain. For all $X_i \in \mathscr{X}_- \cup \mathscr{X}_+$ and $t \in \mathscr{T}$, the field values are taken as continuously differentiable with respect to the arguments as many times as required; jump discontinuities are admitted for $X_i \in \partial\mathscr{X}$. We begin on the assumption that the fluid completely fills $\mathscr{E}$ and is *isochemical* in $\mathscr{X}_-$ and $\mathscr{X}_+$, respectively. More details on the mathematical and notational concepts underlying this study can be found in Appendix A.

2.1.1 *Equations for the initial fields*

For a fluid with the properties specified above, the initial field equations take the forms

$$-p^{(0)}_{,i} + \rho\phi^{(0)}_{,i} = 0, \tag{1}$$

$$g^{(0)}_i = \phi^{(0)}_{,i}, \tag{2}$$

$$\phi^{(0)}_{,ii} = -4\pi G\rho, \tag{3}$$

where G is Newton's gravitational constant, g_i the gravity (gravitational force per unit mass), p the (mechanical) pressure, ρ the volume-mass density and ϕ the (gravitational) potential. With ρ prescribed in $\mathscr{X}_-$ and $\mathscr{X}_+$, respectively, (1)–(3) constitute a system of partial differential equations for $g^{(0)}_i, p^{(0)}$ and $\phi^{(0)}$. The associated initial interface conditions are

$$[p^{(0)}]^+_- = 0, \tag{4}$$

$$[\phi^{(0)}]^+_- = 0, \tag{5}$$

$$[\phi^{(0)}_{,i}]^+_- = 0. \tag{6}$$

2.1.2 *Equations for the incremental fields*

On the assumption of infinitesimal, quasi-static, gravitational–viscoelastic perturbations of the specified fluid, the material form of the incremental field equations is given by

$$u_{i,i} = 0, \tag{7}$$

$$t^{(\delta)}_{ij} = -\delta_{ij}p^{(\delta)} + \int_0^t \mu(t-t')\,\partial_{t'}[u_{i,j}(t') + u_{j,i}(t')]\,dt', \tag{8}$$

$$t^{(\delta)}_{ij,j} + p^{(0)}_{,j}u_{j,i} + \rho(\phi^{(\delta)}_{,i} - \phi^{(0)}_{,j}u_{j,i}) = 0, \tag{9}$$

$$g^{(\delta)}_i = \phi^{(\delta)}_{,i} - \phi^{(0)}_{,j}u_{j,i}, \tag{10}$$

$$\phi^{(\delta)}_{,ii} - 2\phi^{(0)}_{,ij}u_{i,j} - \phi^{(0)}_{,i}u_{i,jj} = 4\pi G\rho u_{i,i}, \tag{11}$$

where t_{ij} is the (Cauchy) stress, t' the excitation time, u_i the material displacement, δ_{ij} the Kronecker symbol and $\mu(t-t')$ the shear-relaxation function. With $\mu(t-t')$ and ρ prescribed parameters in $\mathscr{X}_-$ and $\mathscr{X}_+$, respectively, and $p^{(0)}$ and $\phi^{(0)}$ given as special solution to the equations for the initial fields, (7)–(11) constitute a system of partial differential equations for

$g_i^{(\delta)}, p^{(\delta)}, t_{ij}^{(\delta)}, u_i$ and $\phi^{(\delta)}$. Supposing a material sheet on the interface, the material form of the associated incremental interface conditions can be written as

$$[u_i]_-^+ = 0, \tag{12}$$

$$[n_j^{(0)} t_{ij}^{(\delta)}]_-^+ = \gamma n_i^{(0)} \sigma, \tag{13}$$

$$[\phi^{(\delta)}]_-^+ = 0, \tag{14}$$

$$[n_i^{(0)}(\phi_{,i}^{(\delta)} - \phi_{,j}^{(0)} u_{j,i})]_-^+ = -4\pi G \sigma, \tag{15}$$

with $n_i^{(0)}$ the outward unit normal on $\partial\mathscr{X}$, $\gamma = -n_i^{(0)} g_i^{(0)}$ the magnitude of $g_i^{(0)}$ on $\partial\mathscr{X}$ and σ the (incremental) interface-mass density. Note that the definition of γ implies that $g_i^{(0)}$ and $n_i^{(0)}$ are anti-parallel (e.g. Batchelor 1967, pp. 14–20).

The derivation of the solution simplifies if the incremental field equations and interface conditions are put into their isopotential–local form. The definitions of material, isopotential and local incremental fields, $f_{ij\ldots}^{(\delta)}, f_{ij\ldots}^{(\partial)}$ and $f_{ij\ldots}^{(\Delta)}$, respectively, are considered in Appendix A2. In particular, we use (309) to express $p^{(\delta)}$ and $t_{ij}^{(\delta)}$ in terms of $p^{(\partial)}$ and $t_{ij}^{(\partial)}$ and (310) to express $g_i^{(\delta)}$ and $\phi^{(\delta)}$ in terms of $g_i^{(\Delta)}$ and $\phi^{(\Delta)}$. Taking the Laplace transforms of the resulting equations (e.g. LePage 1980, pp. 285–328) and denoting the Laplace transform of $f(t)$ by $\tilde{f}(s)$, we arrive at the following system of partial differential equations and associated interface conditions for $\tilde{g}_i^{(\Delta)}, \tilde{p}^{(\partial)}, \tilde{t}_{ij}^{(\partial)}, \tilde{u}_i$ and $\tilde{\phi}^{(\Delta)}$:

$$\tilde{u}_{i,i} = 0, \tag{16}$$

$$\tilde{t}_{ij}^{(\partial)} = -\delta_{ij} \tilde{p}^{(\partial)} + s\tilde{\mu}(\tilde{u}_{i,j} + \tilde{u}_{j,i}), \tag{17}$$

$$\tilde{t}_{ij,j}^{(\partial)} = 0, \tag{18}$$

$$\tilde{g}_i^{(\Delta)} = \tilde{\phi}_{,i}^{(\Delta)}, \tag{19}$$

$$\tilde{\phi}_{,ii}^{(\Delta)} = 0, \tag{20}$$

$$[\tilde{u}_i]_-^+ = 0, \tag{21}$$

$$[n_j^{(0)} \tilde{t}_{ij}^{(\partial)} - \rho n_i^{(0)}(\tilde{\phi}^{(\Delta)} + \phi_{,j}^{(0)} \tilde{u}_j)]_-^+ = \gamma n_i^{(0)} \tilde{\sigma}, \tag{22}$$

$$[\tilde{\phi}^{(\Delta)}]_-^+ = 0, \tag{23}$$

$$[n_i^{(0)}(\tilde{\phi}_{,i}^{(\Delta)} - 4\pi G \rho \tilde{u}_i)]_-^+ = -4\pi G \tilde{\sigma}. \tag{24}$$

We note that the incremental equilibrium equation, (18), no longer includes terms accounting for effects due to hydrostatic initial stress and gravitational perturbations and thus *formally* agrees with the corresponding equation valid in the absence of gravitation. However, such effects now appear in the traction interface condition, (22), which explicitly involves the initial pressure gradient, $\rho\phi_{,i}^{(0)}$, and the local incremental potential, $\tilde{\phi}^{(\Delta)}$.

Next, we eliminate $\tilde{g}_i^{(\Delta)}$ and $\tilde{t}_{ij}^{(\partial)}$ from the above equations. For this purpose, we introduce the rotation, $\tilde{\omega}_i$, defined by

$$\tilde{\omega}_i = \tfrac{1}{2} \epsilon_{ijk} \tilde{u}_{k,j}, \tag{25}$$

with ϵ_{ijk} the Levi–Civita symbol. Using (16), (17), (25) and the identity $\epsilon_{ijk}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$, eq. (18) takes the form

$$\tilde{p}_{,i}^{(\partial)} + 2s\tilde{\mu}\, \epsilon_{ijk} \tilde{\omega}_{k,j} = 0. \tag{26}$$

Eqs (16), (20), (25) and (26) constitute an alternative system of partial differential equations in terms of $\tilde{p}^{(\partial)}, \tilde{u}_i, \tilde{\phi}^{(\Delta)}$ and $\tilde{\omega}_i$. The incremental interface condition for $\tilde{p}^{(\partial)}$ follows upon substitution of (17) into (22):

$$[n_i^{(0)} \tilde{p}^{(\partial)} - s\tilde{\mu} n_j^{(0)}(\tilde{u}_{i,j} + \tilde{u}_{j,i}) + \rho n_i^{(0)}(\tilde{\phi}^{(\Delta)} + \phi_{,j}^{(0)} \tilde{u}_j)]_-^+ = -\gamma n_i^{(0)} \tilde{\sigma}. \tag{27}$$

Eqs (21), (23), (24) and (27) constitute the incremental interface conditions associated with the alternative system of differential equations given above.

2.2 Scalar equations in spherical coordinates

2.2.1 *Geometrical considerations*

We proceed on the assumption that the fluid is initially confined to $\mathscr{X}_-$. In this case, $p^{(0)} = 0$, $\mu(t - t') = 0$ and $\rho = 0$ for $X_i \in \mathscr{X}_+$ and it follows from the incremental field equations and interface conditions that $\tilde{u}_i =$ continuous for $X_i \in \partial\mathscr{X}$, $\tilde{u}_i =$ indeterminate for $X_i \in \mathscr{X}_+$ and the remaining mechanical quantities vanish for $X_i \in \mathscr{X}_+$.

It can be shown that, for a hydrostatic initial state in a non-rotating fluid, the level surfaces of $p^{(0)}$ and $\phi^{(0)}$ must be parallel planes, co-axial cylinders or concentric spheres (e.g. Batchelor 1967, pp. 14–20). Here, we consider *spherically* *symmetric* level surfaces and take their common centre as the origin, O, of a Cartesian coordinate system, $OX_1X_2X_3$. The spherical coordinates, r, θ and λ, are related to the Cartesian coordinates, X_1, X_2 and X_3, by

$$\left.\begin{aligned} r &= (X_1^2+X_2^2+X_3^2)^{1/2} && (28)\\ \theta &= \tan^{-1}\frac{(X_1^2+X_2^2)^{1/2}}{X_3} && (29)\\ \lambda &= \tan^{-1}\frac{X_2}{X_3} && (30)\end{aligned}\right\} X_1 \in (0,\infty), \quad X_2 \in (0,\infty), \quad X_3 \in [0,\infty),$$

where $r \in (0, \infty)$ is the radial distance, $\theta \in (0, \pi)$ the colatitude and $\lambda \in [0, 2\pi)$ the longitude of the observation point. For brevity, we refer to the fluid initially occupying $\mathscr{X}_-$ as *planet* and to the material sheet initially occupying $\partial\mathscr{X}$ as *load.* With a the radius of the planet, we then have $\mathscr{X}_- = \{X_i \mid r \in (0, a)\}$, $\mathscr{X}_+ = \{X_i \mid r \in (a, \infty)\}$ and $\partial\mathscr{X} = \{X_i \mid r = a\}$. Henceforth, we append to symbols of tensors the label subscripts r, θ and λ to denote their appropriate scalar components in spherical coordinates; the summation convention no longer applies.

2.2.2 *Equations for the initial fields*

On account of the spherical symmetry, the relevant components of the initial field equations and interface conditions, (1)–(6), reduce to

$$p^{(0)}_{,r} - \rho\phi^{(0)}_{,r} = 0, \qquad r<a, \tag{31}$$

$$g^{(0)}_r = \phi^{(0)}_{,r}, \qquad r \neq a, \tag{32}$$

$$(r^2\phi^{(0)}_{,r})_{,r} = \begin{cases} -4\pi r^2 G\rho, & r<a \\ 0, & r>a \end{cases}, \tag{33}$$

$$\left.\begin{aligned} [p^{(0)}]_- &= 0 && (34)\\ [\phi^{(0)}]_-^+ &= 0 && (35)\\ [\phi^{(0)}_{,r}]_-^+ &= 0 && (36)\end{aligned}\right\} \quad r=a.$$

These equations are to be supplemented by appropriate conditions ensuring that the initial fields remain bounded as $r \to 0$ and $r \to \infty$.

2.2.3 *Equations for the incremental fields*

Since the solution functions in the (r, θ, λ, t) domain are to be expressed in terms of Green's functions representing the contributions from point loads (Section 4.3), it is sufficient to restrict the following analysis to *axisymmetric* perturbations. For convenience, we let the X_3 axis coincide with the symmetry axis. Then, the relevant components of (16), (20), (25) and (26) reduce to

$$\left.\begin{aligned} &\tilde{u}_{r,\theta} - (r\tilde{u}_\theta)_{,r} + 2r\tilde{\omega}_\lambda = 0 && (37)\\ &r\sin\theta\tilde{p}^{(\partial)}_{,r} + 2s\tilde{\mu}(\sin\theta\tilde{\omega}_\lambda)_{,\theta} = 0 && (38)\\ &\tilde{p}^{(\partial)}_{,\theta} - 2s\tilde{\mu}(r\tilde{\omega}_\lambda)_{,r} = 0 && (39)\\ &\sin\theta(r^2\tilde{u}_r)_{,r} + r(\sin\theta\tilde{u}_\theta)_{,\theta} = 0 && (40)\end{aligned}\right\} \quad r<a,$$

$$\sin\theta(r^2\tilde{\phi}^{(\Delta)}_{,r})_{,r} + (\sin\theta\tilde{\phi}^{(\Delta)}_{,\theta})_{,\theta} = 0, \qquad r \neq a. \tag{41}$$

Eqs (37)–(40) are four partial differential equations of first order for the mechanical quantities $\tilde{p}^{(\partial)}, \tilde{u}_r, \tilde{u}_\theta$ and $\tilde{\omega}_\lambda$; they are decoupled from (41), which is a second-order partial differential equation in terms of the gravitational quantity $\tilde{\phi}^{(\Delta)}$.

The solutions to these equations must satisfy the appropriate incremental interface conditions. In view of the supposed symmetries, we find for the relevant components of (21), (23), (24) and (27) the expressions

$$\left.\begin{aligned} &[\tilde{p}^{(\partial)} - 2s\tilde{\mu}\tilde{u}_{r,r} + \rho(\tilde{\phi}^{(\Delta)} + \phi^{(0)}_{,r}\tilde{u}_r)]_- = \gamma\tilde{\sigma} && (42)\\ &[\tilde{u}_{r,\theta} + r\tilde{u}_{\theta,r} - \tilde{u}_\theta]_- = 0 && (43)\\ &[\tilde{\phi}^{(\Delta)}]_-^+ = 0 && (44)\\ &[\tilde{\phi}^{(\Delta)}_{,r}]_-^+ + [4\pi G\rho\tilde{u}_r]_- = -4\pi G\tilde{\sigma} && (45)\end{aligned}\right\} \quad r=a.$$

These equations are to be supplemented by appropriate conditions ensuring the boundedness of the incremental fields as $r \to 0$ and $r \to \infty$.

Before proceeding to the integration of the incremental equations, we note that a related decoupling method was briefly commented on by Richards & Hager (1984). In contrast to the present study, their scheme is limited to viscous-gravitational perturbations. Hence, the coupled form of the incremental field equations and interface conditions basic to their study involves the local incremental stress, $t_{ij}^{(\Delta)}$, and therefore differs from (7)–(15) above, which are written in terms of the material incremental stress, $t_{ij}^{(\delta)}$ (*cf.* Wolf 1991a for details on this difference).

3 INTEGRATION OF THE EQUATIONS

In this section, the scalar equations in spherical coordinates compiled above are solved. In Section 3.1 the special solution to the initial field equations and interface conditions is given. In Section 3.2, fundamental solutions with respect to θ satisfying the incremental field equations and interface conditions are sought in terms of Legendre polynomials. Based on this assumption, we deduce the special solution with respect to r in three steps. First, we establish the general solution to the (4×4) differential system governing the *mechanical* quantities (Section 3.2.1). After that, we derive the general solution to the (2×2) differential system for the *gravitational* quantities (Section 3.2.2). Finally, we determine the integration coefficients using the incremental interface conditions (Section 3.2.3).

3.1 Solution for the initial fields

The special solution to (31)–(33) which satisfies (34)–(36) and remains bounded as $r \to 0$ and $r \to \infty$ is well known (e.g. Ramsey 1981, pp. 45–51). Upon introducing the non-dimensional radial distance, $R = r/a$, the following formulae are obtained:

$$p^{(0)} = \tfrac{1}{2} a\gamma\rho(1 - R^2), \qquad R < 1, \tag{46}$$

$$\phi^{(0)} = \begin{cases} \tfrac{1}{2} a\gamma(3 - R^2), & R < 1 \\ a\gamma R^{-1}, & R > 1 \end{cases}, \tag{47}$$

$$g_r^{(0)} = \begin{cases} -\gamma R, & R < 1 \\ -\gamma R^{-2}, & R > 1 \end{cases}, \tag{48}$$

where the additive constant in the potential function has been chosen such that $\lim_{r \to \infty} \phi^{(0)} = 0$. Since $\gamma = 4\pi G a \rho/3$, the initial state is completely determined if any two of the parameters a, γ and ρ are given.

3.2 Solution for the incremental fields: (r, n, s) domain

In the following, we seek solutions to (37)–(41) subject to (42)–(45) in terms of Legendre polynomials of the first kind, $P_n(\cos\theta)$, where $n \in \{0, 1, \ldots\}$ is the Legendre degree and $P_n(\cos\theta)$ satisfies Legendre's equation (e.g. Lebedev 1972, pp. 44–51).

3.2.1 *General solution for the mechanical quantities*

We suppose fundamental solutions with respect to θ of the forms

$$\tilde{u}_r(r, \theta, s) = \tilde{U}_{rn}(r, s)\tilde{\zeta}_n(s)P_n(\cos\theta), \tag{49}$$

$$\tilde{u}_\theta(r, \theta, s) = -\tilde{U}_{\theta n}(r, s)\tilde{\zeta}_n(s)P_{n,\theta}(\cos\theta), \tag{50}$$

$$\tilde{p}^{(\delta)}(r, \theta, s) = \tilde{P}_n^{(\delta)}(r, s)\tilde{\zeta}_n(s)P_n(\cos\theta), \tag{51}$$

$$\tilde{\omega}_\lambda(r, \theta, s) = -\tilde{\Omega}_{\lambda n}(r, s)\tilde{\zeta}_n(s)P_{n,\theta}(\cos\theta), \tag{52}$$

where $\tilde{\zeta}_n(s)$ is the *non-dimensional* Legendre coefficient of $\tilde{\sigma}(\theta, s)$ (Section 3.2.3) and $\tilde{F}_n(r, s)$ the *normalized* Legendre coefficient of $\tilde{f}(r, \theta, s)$. Note that $\tilde{F}_n(r, s)$ is assumed to be a Laplace transform, which will be confirmed below (Section 4.3.2). To proceed further, we distinguish the Legendre degrees $n = 0$ and $n \geq 1$. For brevity, the arguments of the functions will usually be suppressed.

Degree $n = 0$. With $P_{0,\theta} = 0$, we may put

$$\tilde{U}_{\theta 0} = \tilde{\Omega}_{\lambda 0} = 0, \tag{53}$$

whence (37)–(40) reduce to

$$\left.\begin{aligned} r\tilde{P}^{(a)}_{0,r} &= 0 \quad (54)\\ r\tilde{U}_{r0,r} + 2\tilde{U}_{r0} &= 0 \quad (55)\end{aligned}\right\} \quad r < a.$$

General solutions are

$$\left.\begin{aligned} \tilde{U}_{r0} &= A^{(2)}R^{-2} \quad (56)\\ \tilde{P}^{(a)}_0 &= A^{(1)} \quad (57)\end{aligned}\right\} \quad R < 1,$$

with $A^{(1)}$ and $A^{(2)}$ arbitrary integration coefficients.

Degrees $n \geq 1$. Upon substitution of (49)–(52) and use of Legendre's equation, (37)–(40) take the forms

$$\left.\begin{aligned} \tilde{U}_{rn} + r\tilde{U}_{\theta n,r} + \tilde{U}_{\theta n} - 2r\tilde{\Omega}_{\lambda n} &= 0 \quad (58)\\ r\tilde{P}^{(a)}_{n,r} + 2n(n+1)s\tilde{\mu}\tilde{\Omega}_{\lambda n} &= 0 \quad (59)\\ \tilde{P}^{(a)}_n + 2s\tilde{\mu}(r\tilde{\Omega}_{\lambda n,r} + \tilde{\Omega}_{\lambda n}) &= 0 \quad (60)\\ r\tilde{U}_{rn,r} + 2\tilde{U}_{rn} + n(n+1)\tilde{U}_{\theta n} &= 0 \quad (61)\end{aligned}\right\} \quad r < a.$$

We introduce the vector $[Y_i]$ by non-dimensional solution

$$\begin{bmatrix} \tilde{U}_{rn} \\ \tilde{U}_{\theta n} \\ \tilde{P}^{(a)}_n \\ \tilde{\Omega}_{\lambda n} \end{bmatrix} = \begin{bmatrix} a & & & \\ & a & & \\ & & 2s\tilde{\mu}R^{-1} & \\ & & & R^{-1} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} \tag{62}$$

and consider fundamental solutions with respect to R of the form

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = R^{\kappa^{(k)}} \begin{bmatrix} Y_1^{(k)} \\ Y_2^{(k)} \\ Y_3^{(k)} \\ Y_4^{(k)} \end{bmatrix}. \tag{63}$$

Upon successive substitution of (62) and (63), eqs (58)–(61) can be recast into the following matrix equation:

$$\begin{bmatrix} \kappa^{(k)}+2 & n(n+1) & 0 & 0 \\ 1 & \kappa^{(k)}+1 & 0 & -2 \\ 0 & 0 & \kappa^{(k)}-1 & n(n+1) \\ 0 & 0 & 1 & \kappa^{(k)} \end{bmatrix} \begin{bmatrix} Y_1^{(k)} \\ Y_2^{(k)} \\ Y_3^{(k)} \\ Y_4^{(k)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{64}$$

This has non-trivial solutions only if the system determinant vanishes:

$$[(\kappa^{(k)}+1)(\kappa^{(k)}+2) - n(n+1)][\kappa^{(k)}(\kappa^{(k)}-1) - n(n+1)] = 0, \tag{65}$$

whose roots are

$$\kappa^{(1)} = n-1, \tag{66}$$

$$\kappa^{(2)} = n+1, \tag{67}$$

$$\kappa^{(3)} = -(n+2), \tag{68}$$

$$\kappa^{(4)} = -n. \tag{69}$$

The determination of the eigenvectors, $[Y_i^{(k)}]$, associated with the eigenvalues, $\kappa^{(k)}$, follows standard procedures, which are explicitly shown elsewhere (Wolf 1993, pp. 83–84). Putting $Y_1^{(1)} = n$, $Y_1^{(2)} = n(n+1)$, $Y_1^{(3)} = n+1$, $Y_1^{(4)} = n(n+1)$, we obtain

$$[Y_i^{(1)}] = [n, -1, 0, 0]^{\mathrm{T}}, \tag{70}$$

$$[Y_i^{(2)}] = [n(n+1), -(n+3), (n+1)(2n+3), -(2n+3)]^{\mathrm{T}}, \tag{71}$$

$$[Y_i^{(3)}] = [n+1, 1, 0, 0]^{\mathrm{T}}, \tag{72}$$

$$[Y_i^{(4)}] = [n(n+1), n-2, n(2n-1), 2n-1]^{\mathrm{T}}. \tag{73}$$

Since we have four fundamental solutions, the general solution with respect to R for the mechanical quantities can be written as

$$[Y_i] = \sum_{k=1}^{4} A^{(k)} R^{\kappa^{(k)}} [Y_i^{(k)}], \qquad R < 1, \tag{74}$$

where the integration coefficients, $A^{(k)}$, are arbitrary.

3.2.2. *General solution for the gravitational quantities*

We consider the radial component of (19):

$$\tilde{g}_r^{(\Delta)} = \tilde{\phi}_{,r}^{(\Delta)}, \tag{75}$$

whose substitution into (41) yields

$$\sin\theta (r^2 \tilde{g}_r^{(\Delta)})_{,r} + (\sin\theta \tilde{\phi}_{,\theta}^{(\Delta)})_{,\theta} = 0, \qquad r \neq a. \tag{76}$$

As for the mechanical quantities, we suppose fundamental solutions with respect to θ of the forms

$$\tilde{\phi}^{(\Delta)}(r, \theta, s) = \tilde{\Phi}_n^{(\Delta)}(r, s) \tilde{\zeta}_n(s) P_n(\cos\theta), \tag{77}$$

$$\tilde{g}_r^{(\Delta)}(r, \theta, s) = \tilde{G}_{rn}^{(\Delta)}(r, s) \tilde{\zeta}_n(s) P_n(\cos\theta). \tag{78}$$

Upon substitution of these equations and use of Legendre's equation, (75) and (76) reduce to

$$\left.\begin{aligned} &\tilde{\Phi}_{n,r}^{(\Delta)} - \tilde{G}_{rn}^{(\Delta)} = 0 && (79)\\ &n(n+1)\tilde{\Phi}_n^{(\Delta)} - r^2 \tilde{G}_{rn,r}^{(\Delta)} - 2r\tilde{G}_{rn}^{(\Delta)} = 0 && (80) \end{aligned}\right\} r \neq a.$$

Next, we introduce the vector $[Z_i]$ by non-dimensional solution

$$\begin{bmatrix} \tilde{\Phi}_n^{(\Delta)} \\ \tilde{G}_{rn}^{(\Delta)} \end{bmatrix} = \begin{bmatrix} 3a\gamma & \\ & 3\gamma R^{-1} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \tag{81}$$

and consider fundamental solutions with respect to R of the form

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = R^{\lambda^{(\ell)}} \begin{bmatrix} Z_1^{(\ell)} \\ Z_2^{(\ell)} \end{bmatrix}. \tag{82}$$

Upon successive substitution of (81) and (82), eqs (79) and (80) can be recast into the following matrix equation:

$$\begin{bmatrix} \lambda^{(\ell)} & -1 \\ n(n+1) & -(\lambda^{(\ell)}+1) \end{bmatrix} \begin{bmatrix} Z_1^{(\ell)} \\ Z_2^{(\ell)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{83}$$

This has non-trivial solutions only if the system determinant vanishes:

$$\lambda^{(\ell)}(\lambda^{(\ell)} + 1) - n(n+1) = 0, \tag{84}$$

whose roots are

$$\lambda^{(1)} = n, \tag{85}$$

$$\lambda^{(2)} = -(n+1). \tag{86}$$

The elements of the eigenvectors, $[Z_i^{(\ell)}]$, associated with the eigenvalues, $\lambda^{(\ell)}$, are found to satisfy

$$\boxed{\begin{aligned} &Z_2^{(1)} = nZ_1^{(1)}, && (87)\\ &Z_2^{(2)} = -(n+1)Z_1^{(2)}. && (88) \end{aligned}}$$

Putting $Z_1^{(1)} = Z_1^{(2)} = 1$, the following eigenvectors result:

$$\boxed{\begin{aligned} &[Z_i^{(1)}] = [1, n]^{\mathrm{T}}, && (89)\\ &[Z_i^{(2)}] = [1, -(n+1)]^{\mathrm{T}}. && (90) \end{aligned}}$$

Since we have two fundamental solutions, the general solution with respect to R for the gravitational quantities takes the form

$$[Z_i] = \sum_{\ell=1}^{2} B^{(\ell)} R^{\lambda^{(\ell)}} [Z_i^{(\ell)}], \qquad R \neq 1, \tag{91}$$

with the integration coefficients, $B^{(\ell)}$, arbitrary.

3.2.3 Special solution

Next, we adjust the general solutions with respect to R deduced above to the incremental interface conditions. This requires that the interface-mass density is of the form

$$\tilde{\sigma}(\theta, s) = \tilde{\sigma}_n(s) P_n(\cos\theta), \tag{92}$$

where $\tilde{\sigma}_n(s)$ is the *ordinary* Legendre coefficient of $\tilde{\sigma}(\theta, s)$. Its relation to the *non-dimensional* Legendre coefficient, $\zeta_n(s)$, is given by

$$\tilde{\sigma}_n(s) = a\rho(2n+1)\tilde{\zeta}_n(s). \tag{93}$$

We note that, since $\int_0^\pi P_n(\cos\theta)\sin\theta\, d\theta$ is finite for $n = 0$ and vanishes otherwise, $\tilde{\sigma}_0(s)$ corresponds to an *accreted* load and $\tilde{\sigma}_n(s)$ for $n \geq 1$ to a *redistributed* load.

Degree $n = 0$. As for the initial fields, we seek solutions that remain bounded as $r \to 0$ and $r \to \infty$. According to (53)–(57), this requires

$$\left.\begin{aligned} &\tilde{U}_{r0} = \tilde{U}_{\theta 0} = \tilde{\Omega}_{\lambda 0} = 0 && (94)\\ &\tilde{P}_0^{(\partial)} = A^{(1)} && (95)\end{aligned}\right\} \quad R < 1.$$

Imposing the constraint $\lim_{r\to\infty} \tilde{\phi}^{(\Delta)} = 0$ and considering the relevant equations, we also get

$$\tilde{\Phi}_0^{(\Delta)} = \begin{cases} 3a\gamma B^{(1)}, & R < 1 \\ 3a\gamma B^{(2)} R^{-1}, & R > 1 \end{cases}, \tag{96}$$

$$\tilde{G}_{r0}^{(\Delta)} = \begin{cases} 0, & R < 1 \\ -3\gamma B^{(2)} R^{-2}, & R > 1 \end{cases}. \tag{97}$$

The three coefficients are determined using (42)–(45). Expressing the incremental fields in these equations using the appropriate relations, we obtain upon some manipulation

$$\left.\begin{aligned} &[\tilde{P}_0^{(\partial)} + \rho\tilde{\Phi}_0^{(\Delta)}]_- = a\gamma\rho && (98)\\ &[\tilde{\Phi}_0^{(\Delta)}]_-^+ = 0 && (99)\\ &[a\rho\tilde{G}_{r0}^{(\Delta)}]_-^+ = -3a\gamma\rho && (100)\end{aligned}\right\} \quad R = 1.$$

Substituting (95)–(97) yields

$$A^{(1)} = -2a\gamma\rho, \tag{101}$$

$$B^{(1)} = B^{(2)} = 1. \tag{102}$$

Degrees $n \geq 1$. Again, we require a bounded solution for $r \to 0$ and $r \to \infty$. Observing the signs of the eigenvalues $\kappa^{(k)}$ and $\lambda^{(\ell)}$, (74) and (91) reduce to

$$\boxed{\begin{aligned} &[Y_i] = A^{(1)} R^{n-1} [Y_i^{(1)}] + A^{(2)} R^{n+1} [Y_i^{(2)}], \qquad R < 1, && (103)\\ &[Z_i] = \begin{cases} B^{(1)} R^n [Z_i^{(1)}], & R < 1 \\ B^{(2)} R^{-(n+1)} [Z_i^{(2)}], & R > 1 \end{cases}. && (104)\end{aligned}}$$

The four coefficients are determined following steps similar to those taken for Legendre degree $n = 0$. We obtain

$$\left.\begin{aligned} &[\tilde{P}_n^{(\partial)} - 2s\tilde{\mu}\tilde{U}_{rn,r} + \rho(\tilde{\Phi}_n^{(\Delta)} - \gamma\tilde{U}_{rn})]_- = a\gamma\rho(2n+1) && (105)\\ &[\tilde{U}_{rn} - r\tilde{U}_{\theta n,r} + \tilde{U}_{\theta n}]_- = 0 && (106)\\ &[\tilde{\Phi}_n^{(\Delta)}]_-^+ = 0 && (107)\\ &[a\rho\tilde{G}_{rn}^{(\Delta)}]_-^+ + [3\gamma\rho\tilde{U}_{rn}]_- = -3a\gamma\rho(2n+1) && (108)\end{aligned}\right\} \quad R = 1.$$

Upon successive substitution of the appropriate relations expressing the field quantities in terms of the integration coefficients, we arrive at (Wolf 1993, pp. 57–58)

$$\begin{bmatrix} -n\left[\dfrac{2(n-1)}{a\gamma\rho}s\tilde{\mu}+1\right] & -(n+1)\left[\dfrac{2(n^2-n-3)}{a\gamma\rho}s\tilde{\mu}+n\right] & 3 \\ n-1 & n(n+2) & 0 \\ -n & -n(n+1) & 2n+1 \end{bmatrix} \begin{bmatrix} A^{(1)} \\ A^{(2)} \\ B^{(1)} \end{bmatrix} = \begin{bmatrix} 2n+1 \\ 0 \\ 2n+1 \end{bmatrix}, \tag{109}$$

$$B^{(2)} = B^{(1)}. \tag{110}$$

If $n \geq 2$, the solution is given by

$$A^{(1)} = -(n+2)\frac{\gamma\rho}{k_n s\tilde{\mu} + \gamma\rho}, \tag{111}$$

$$A^{(2)} = \frac{n-1}{n}\frac{\gamma\rho}{k_n s\tilde{\mu} + \gamma\rho}, \tag{112}$$

$$B^{(1)} = B^{(2)} = \frac{k_n s\tilde{\mu}}{k_n s\tilde{\mu} + \gamma\rho}, \tag{113}$$

with $k_n = (2n^2 + 4n + 3)/(an)$ the Legendre wave number.

If $n = 1$, eq. (109) reduces to

$$\begin{bmatrix} -1 & 2\left(\dfrac{6}{a\gamma\rho}s\tilde{\mu}-1\right) & 3 \\ 0 & 3 & 0 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} A^{(1)} \\ A^{(2)} \\ B^{(1)} \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}. \tag{114}$$

Since the system determinant vanishes, the system is underdetermined and no unique solution exists. Considering the 'reduced' system

$$\begin{bmatrix} 0 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} A^{(1)} \\ A^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 3(1-B^{(1)}) \end{bmatrix}, \tag{115}$$

we may put $B^{(1)} = B^{(2)} = C$, whence we find $A^{(1)} = 3(C-1)$ and $A^{(2)} = 0$. Constant C can be determined by bearing in mind that, for redistributed loads, planet and load constitute a closed system whose centre of mass remains unperturbed. Since $P_1 = \cos\theta$, this is satisfied for $n = 1$ only if the interface-mass density associated with the load is anulled by the effective interface-mass density resulting from the perturbation of the surface of the planet. The mathematical expression of this condition is $[\rho\tilde{u}_r]_- = -\tilde{\sigma}$, which can be shown to be equivalent to $[\tilde{U}_{r1}]_- = -3a$. However, we also derive $[\tilde{U}_{r1}]_- = -3a(1-C)$. Hence, $C = 0$ and

$$A^{(1)} = -3, \tag{116}$$

$$A^{(2)} = B^{(1)} = B^{(2)} = 0. \tag{117}$$

4 SOLUTION FUNCTIONS FOR THE INCREMENTAL FIELDS

We begin by compiling the special solution functions for the individual fields in the (r, n, s) domain, where *transfer* functions are introduced (Section 4.1). This is followed by the specification of the shear-relaxation function for *generalized Maxwell* viscoelasticity (Section 4.2). After that, the solution functions are transformed to the (r, n, t) domain, where *impulse-response* functions are established (Section 4.3). Finally, we consider the transformation to the (r, θ, λ, t) domain and provide the appropriate *Green's* functions (Section 4.4).

4.1 Functions in the (r, n, s) domain

First, we give closed-form solution functions for the following fields: material displacement, isopotential incremental pressure and local incremental potential (Section 4.1.1); isopotential height (Section 4.1.2); strain (Section 4.1.3); rotation (Section 4.1.4); material, isopotential and local incremental stresses (Section 4.1.5); and material, isopotential and local incremental gravity (Section 4.1.6). After that, the agreement between our solution and that derived from the conventional (6×6) differential system is discussed and the half-space approximation derived (Section 4.1.7). Finally, the general form of the individual solution functions is established and transfer functions are defined (Section 4.1.8). All formulae are written in terms of the non-dimensional radial distance, $R = r/a$, with the Legendre degrees $n = 0$, $n = 1$ and $n \geq 2$ considered explicitly for each field. We recall that, for redistributed loads, $\tilde{\sigma}_0 = 0$ so that $n = 0$ is without relevance in that case.

4.1.1 Material displacement, isopotential incremental pressure and local incremental potential

The incremental fields $\tilde{u}_r, \tilde{u}_\theta, \tilde{p}^{(\partial)}$ and $\tilde{\phi}^{(\Delta)}$ are basic in the sense that all other incremental fields can be expressed in terms of them or their spatial derivatives. For convenient reference, we recollect the following fundamental solutions with respect to θ (Sections 3.2.1 and 3.2.2):

$$\tilde{u}_r = \tilde{U}_{rn}\tilde{\zeta}_n P_n, \tag{118}$$

$$\tilde{u}_\theta = -\tilde{U}_{\theta n}\tilde{\zeta}_n P_{n,\theta}, \tag{119}$$

$$\tilde{p}^{(\partial)} = \tilde{P}_n^{(\partial)}\tilde{\zeta}_n P_n, \tag{120}$$

$$\tilde{\phi}^{(\Delta)} = \tilde{\Phi}_n^{(\Delta)}\tilde{\zeta}_n P_n. \tag{121}$$

Degree n = 0. Eqs (94)–(96), (101) and (102) yield

$$\left.\begin{aligned} &\tilde{U}_{r0} = 0 && (122)\\ &\tilde{U}_{\theta 0} = 0 && (123)\\ &\tilde{P}_0^{(\partial)} = -2a\gamma\rho && (124) \end{aligned}\right\} \quad R<1,$$

$$\tilde{\Phi}_0^{(\Delta)} = \begin{cases} 3a\gamma, & R<1 \\ 3a\gamma R^{-1}, & R>1 \end{cases}. \tag{125}$$

Degree n = 1. Using (62), (81), (103) and (104) and substituting the appropriate equations for the eigenvectors and integration coefficients, we obtain

$$\left.\begin{aligned} &\tilde{U}_{r1} = -3a && (126)\\ &\tilde{U}_{\theta 1} = -3a && (127)\\ &\tilde{P}_1^{(\partial)} = 0 && (128) \end{aligned}\right\} \quad R<1,$$

$$\tilde{\Phi}_1^{(\Delta)} = 0, \qquad R \neq 0. \tag{129}$$

Degrees n ≥ 2. As for Legendre degree $n = 1$, we find

$$\boxed{\begin{aligned} &\left.\begin{aligned} &\tilde{U}_{rn} = -a[n(n+2)R^{n-1} - (n^2-1)R^{n+1}]\frac{\gamma\rho}{k_n s\tilde{\mu} + \gamma\rho} && (130)\\ &\tilde{U}_{\theta n} = \frac{a}{n}[n(n+2)R^{n-1} - (n-1)(n+3)R^{n+1}]\frac{\gamma\rho}{k_n s\tilde{\mu} + \gamma\rho} && (131)\\ &\tilde{P}_n^{(\partial)} = 2a\gamma\rho\frac{(2n+3)(n^2-1)}{2n^2+4n+3}R^n\frac{k_n s\tilde{\mu}}{k_n s\tilde{\mu} + \gamma\rho} && (132) \end{aligned}\right\} \quad R<1, \\ &\tilde{\Phi}_n^{(\Delta)} = \begin{cases} 3a\gamma R^n \dfrac{k_n s\tilde{\mu}}{k_n s\tilde{\mu} + \gamma\rho}, & R<1 \\ 3a\gamma R^{-(n+1)} \dfrac{k_n s\tilde{\mu}}{k_n s\tilde{\mu} + \gamma\rho}, & R>1 \end{cases}. \qquad (133) \end{aligned}}$$

With formulae for the four basic incremental field components, $\tilde{u}_r, \tilde{u}_\theta, \tilde{p}^{(\partial)}$ and $\tilde{\phi}^{(\Delta)}$, for $n = 0, n = 1$ and $n \geq 2$, respectively, we can now give formulae for any other incremental field quantity of interest. In order to relate isopotential increments to local or material increments, we need the solution for the isopotential height, $\tilde{h}$, which is given in the following section.

4.1.2 Isopotential height

With d_i the isopotential displacement (Appendix A1) and $\tilde{\phi}^{(\partial)} = 0$ by definition of the isopotential increment (Appendix A2), (309) and (310) lead to $\phi_{,i}^{(0)}\tilde{d}_i = -\tilde{\phi}^{(\Delta)}$. Since $n_i^{(0)}$ is anti-parallel to $\phi_{,i}^{(0)}$, can be replaced by $n_i^{(0)}n_j^{(0)}\phi_{,i}^{(0)}\tilde{d}_j = -\tilde{\phi}^{(\Delta)}$. An elementary consideration shows that, on the assumption of infinitesimal perturbations, $n_i^{(0)}\tilde{d}_i$ equals the height of the current level surface, $\phi = \phi'$, passing through $X_i + d_i$ with respect to the associated initial level surface, $\phi^{(0)} = \phi'$, passing through X_i, as measured in the direction of $n_i^{(0)}$. With $\tilde{h} = n_i^{(0)}\tilde{d}_i$ the isopotential height and $g_i^{(0)} = \phi_{,i}^{(0)}$, we thus obtain

$$\tilde{h} = -\frac{1}{n_i^{(0)}g_i^{(0)}}\tilde{\phi}^{(\Delta)}, \qquad X_i \in \mathscr{X}_- \cup \mathscr{X}_+ \tag{134}$$

or, in spherical coordinates,

$$\tilde{h} = -\frac{1}{g_r^{(0)}}\tilde{\phi}^{(\Delta)}, \qquad r \neq a. \tag{135}$$

Using (121) and

$$\tilde{h} = \tilde{H}_n\tilde{\zeta}_n P_n, \tag{136}$$

we get

$$\tilde{H}_n = -\frac{1}{g_r^{(0)}}\tilde{\Phi}_n^{(\Delta)}, \qquad r \neq a. \tag{137}$$

Degree $n = 0$. Substituting (48) and (125), the preceding equation gives

$$\tilde{H}_0 = \begin{cases} 3aR^{-1}, & R<1 \\ 3aR, & R>1 \end{cases}. \tag{138}$$

We note that, with $g_r^{(0)}$ vanishing for $R = 0$ and $R \to \infty$, $\tilde{H}_0$ becomes singular at these points.

Degree $n = 1$. In view of (48) and (129), it follows that

$$\tilde{H}_1 = 0. \tag{139}$$

Degrees $n \geq 2$. Upon substitution of (48) and (133), we obtain

$$\boxed{\tilde{H}_n = \begin{cases} 3aR^{n-1}\dfrac{k_n s\tilde{\mu}}{k_n s\tilde{\mu} + \gamma\rho}, & R<1 \\ 3aR^{-(n-1)}\dfrac{k_n s\tilde{\mu}}{k_n s\tilde{\mu} + \gamma\rho}, & R>1 \end{cases}.} \tag{140}$$

4.1.3 Strain

In Cartesian-tensor notation, the strain, $\tilde{e}_{ij}$, is defined by

$$\tilde{e}_{ij} = \tfrac{1}{2}(\tilde{u}_{i,j} + \tilde{u}_{j,i}), \qquad X_i \in \mathscr{X}_-. \tag{141}$$

In spherical coordinates, the non-vanishing components of $\tilde{e}_{ij}$ are given by

$$\left.\begin{aligned} \tilde{e}_{rr} &= \tilde{u}_{r,r} && (142)\\ \tilde{e}_{r\theta} &= \frac{1}{2r}(\tilde{u}_{r,\theta} + r\tilde{u}_{\theta,r} - \tilde{u}_\theta) && (143)\\ \tilde{e}_{\theta\theta} &= \frac{1}{r}(\tilde{u}_r + \tilde{u}_{\theta,\theta}) && (144)\\ \tilde{e}_{\lambda\lambda} &= \frac{1}{r}(\tilde{u}_r + \cot\theta\,\tilde{u}_\theta) && (145) \end{aligned}\right\} \quad r<a.$$

If we introduce

$$\tilde{e}_{rr} = \tilde{E}_{rrn}\tilde{\zeta}_n P_n, \tag{146}$$

$$\tilde{e}_{r\theta} = -\tilde{E}_{r\theta n}\tilde{\zeta}_n P_{n,\theta}, \tag{147}$$

$$\tilde{e}_{\theta\theta} = \tilde{E}^{(1)}_{\theta\theta n}\tilde{\zeta}_n P_n - \tilde{E}^{(2)}_{\theta\theta n}\tilde{\zeta}_n \cot\theta P_{n,\theta}, \tag{148}$$

$$\tilde{e}_{\lambda\lambda} = \tilde{E}^{(1)}_{\lambda\lambda n}\tilde{\zeta}_n P_n - \tilde{E}^{(2)}_{\lambda\lambda n}\tilde{\zeta}_n \cot\theta P_{n,\theta} \tag{149}$$

and observe (118), (119) and Legendre's equation, we obtain the following formulae:

$$\left.\begin{aligned} \tilde{E}_{rrn} &= \tilde{U}_{rn,r} && (150)\\ \tilde{E}_{r\theta n} &= -\frac{1}{2r}(\tilde{U}_{rn} - r\tilde{U}_{\theta n,r} + \tilde{U}_{\theta n}) && (151)\\ \tilde{E}^{(1)}_{\theta\theta n} &= \frac{1}{r}[\tilde{U}_{rn} + n(n+1)\tilde{U}_{\theta n}] && (152)\\ \tilde{E}^{(2)}_{\theta\theta n} &= -\frac{1}{r}\tilde{U}_{\theta n} && (153)\\ \tilde{E}^{(1)}_{\lambda\lambda n} &= \frac{1}{r}\tilde{U}_{rn} && (154)\\ \tilde{E}^{(2)}_{\lambda\lambda n} &= \frac{1}{r}\tilde{U}_{\theta n} && (155) \end{aligned}\right\} \quad r<a.$$

Degree $n = 0$. In view of (122) and (123), it follows that

$$\tilde{E}_{rr0} = \tilde{E}_{r\theta 0} = \tilde{E}^{(1)}_{\theta\theta 0} = \tilde{E}^{(2)}_{\theta\theta 0} = \tilde{E}^{(1)}_{\lambda\lambda 0} = \tilde{E}^{(2)}_{\lambda\lambda 0} = 0, \qquad R < 1. \tag{156}$$

Degree $n = 1$. Substituting (126) and (127), we arrive at

$$\left.\begin{aligned}
&\tilde{E}_{rr1} = 0 && (157)\\
&\tilde{E}_{r\theta 1} = 3R^{-1} && (158)\\
&\tilde{E}^{(1)}_{\theta\theta 1} = -9R^{-1} && (159)\\
&\tilde{E}^{(2)}_{\theta\theta 1} = 3R^{-1} && (160)\\
&\tilde{E}^{(1)}_{\lambda\lambda 1} = -3R^{-1} && (161)\\
&\tilde{E}^{(2)}_{\lambda\lambda 1} = -3R^{-1} && (162)
\end{aligned}\right\} R < 1.$$

Degrees $n \geq 2$. Using (130) and (131), we obtain the following formulae:

$$\boxed{\left.\begin{aligned}
&\tilde{E}_{rrn} = -(n-1)[n(n+2)R^{n-2} - (n+1)^2 R^n]\frac{\gamma\rho}{k_n s\tilde{\mu} + \gamma\rho} && (163)\\
&\tilde{E}_{r\theta n} = (n-1)(n+2)(R^{n-2} - R^n)\frac{\gamma\rho}{k_n s\tilde{\mu} + \gamma\rho} && (164)\\
&\tilde{E}^{(1)}_{\theta\theta n} = (n+2)[n^2 R^{n-2} - (n^2-1)R^n]\frac{\gamma\rho}{k_n s\tilde{\mu} + \gamma\rho} && (165)\\
&\tilde{E}^{(2)}_{\theta\theta n} = -\frac{1}{n}[n(n+2)R^{n-2} - (n-1)(n+3)R^n]\frac{\gamma\rho}{k_n s\tilde{\mu} + \gamma\rho} && (166)\\
&\tilde{E}^{(1)}_{\lambda\lambda n} = -[n(n+2)R^{n-2} - (n^2-1)R^n]\frac{\gamma\rho}{k_n s\tilde{\mu} + \gamma\rho} && (167)\\
&\tilde{E}^{(2)}_{\lambda\lambda n} = \frac{1}{n}[n(n+2)R^{n-2} - (n-1)(n+3)R^n]\frac{\gamma\rho}{k_n s\tilde{\mu} + \gamma\rho} && (168)
\end{aligned}\right\} R < 1.}$$

4.1.4 Rotation

In spherical coordinates, the non-vanishing component of (25) is

$$\tilde{\omega}_\lambda = -\frac{1}{2r}(\tilde{u}_{r,\theta} - r\tilde{u}_{\theta,r} - \tilde{u}_\theta), \qquad r < a. \tag{169}$$

Using (118), (119) and

$$\tilde{\omega}_\lambda = -\tilde{\Omega}_{\lambda n}\tilde{\zeta}_n P_{n,\theta}, \tag{170}$$

this takes the form

$$\tilde{\Omega}_{\lambda n} = \frac{1}{2r}(\tilde{U}_{rn} + r\tilde{U}_{\theta n,r} + \tilde{U}_{\theta n}), \qquad r < a. \tag{171}$$

Degree $n = 0$. By (122) and (123), the preceding equation reduces to

$$\tilde{\Omega}_{\lambda 0} = 0, \qquad R < 1. \tag{172}$$

Degree $n = 1$. With (126) and (127), it follows that

$$\tilde{\Omega}_{\lambda 1} = -3R^{-1}, \qquad R < 1. \tag{173}$$

Degrees $n \geq 2$. Using (130) and (131), we arrive at

$$\tilde{\Omega}_{\wedge n} = -\frac{(n-1)(2n+3)}{n} R^n \frac{\gamma\rho}{k_n s\tilde{\mu} + \gamma\rho}, \qquad R < 1. \tag{174}$$

4.1.5 Material, isopotential and local incremental stresses

In the first instance, the isopotential incremental stress, $\tilde{t}_{ij}^{(\partial)}$, has been introduced to decouple the incremental field equations for the mechanical quantities from those for the gravitational quantities (Section 2.1.2). Apart from that, $\tilde{t}_{ij}^{(\partial)}$ serves as a measure of the deviation from a hydrostatic equilibrium state. In this respect, it resembles the local incremental stress, $\tilde{t}_{ij}^{(\Delta)}$, which provides such a measure in the absence of gravitational perturbations. On the other hand, observations of the incremental stress at or below the surface of planets commonly refer to material points and therefore yield values for the material incremental stress, $\tilde{t}_{ij}^{(\delta)}$.

The relations between $\tilde{t}_{ij}^{(\delta)}, \tilde{t}_{ij}^{(\partial)}$ and $\tilde{t}_{ij}^{(\Delta)}$ are most conveniently expressed upon decomposition of the stresses into *spherical* and *deviatoric* increments. Since $\tilde{p}^{(\delta)} = -\tilde{t}_{ii}^{(\delta)}/3$, $\tilde{p}^{(\partial)} = -\tilde{t}_{ii}^{(\partial)}/3$, $\tilde{p}^{(\Delta)} = -\tilde{t}_{ii}^{(\Delta)}/3$ and $\tilde{s}_{ij}^{(\delta)} = \tilde{s}_{ij}^{(\partial)} = \tilde{s}_{ij}^{(\Delta)} = \tilde{s}_{ij}$, we obtain

$$\left.\begin{aligned} \tilde{t}_{ij}^{(\delta)} &= -\delta_{ij}\tilde{p}^{(\delta)} + \tilde{s}_{ij} \quad (175)\\ \tilde{t}_{ij}^{(\partial)} &= -\delta_{ij}\tilde{p}^{(\partial)} + \tilde{s}_{ij} \quad (176)\\ \tilde{t}_{ij}^{(\Delta)} &= -\delta_{ij}\tilde{p}^{(\Delta)} + \tilde{s}_{ij} \quad (177)\end{aligned}\right\} \quad X_i \in \mathscr{X}_-.$$

Spherical increments

With the spherical incremental stress equal to the negative of the incremental pressure, we proceed by considering the incremental pressure. In view of (309) and (310), it follows for the three measures of incremental pressure in spherical coordinates

$$\left.\begin{aligned} \tilde{p}^{(\delta)} &= \tilde{p}^{(\partial)} - p_{,r}^{(0)}(\tilde{h} - \tilde{u}_r) \quad (178)\\ \tilde{p}^{(\Delta)} &= \tilde{p}^{(\partial)} - p_{,r}^{(0)}\tilde{h} \quad (179)\end{aligned}\right\} \quad r < a.$$

With (118), (120), (136) and

$$\tilde{p}^{(\delta)} = (\tilde{P}_n^{(\Delta)} + \tilde{P}_n^{(\alpha)})\tilde{\zeta}_n P_n, \tag{180}$$

$$\tilde{p}^{(\Delta)} = \tilde{P}_n^{(\Delta)}\tilde{\zeta}_n P_n, \tag{181}$$

we find the following equations:

$$\left.\begin{aligned} \tilde{P}_n^{(\Delta)} &= \tilde{P}_n^{(\partial)} - p_{,r}^{(0)}\tilde{H}_n \quad (182)\\ \tilde{P}_n^{(\alpha)} &= p_{,r}^{(0)}\tilde{U}_{rn} \quad (183)\end{aligned}\right\} \quad r < a.$$

Degree $n = 0$. Substitution of (46), (122), (124) and (138) results in

$$\left.\begin{aligned} \tilde{P}_0^{(\Delta)} &= a\gamma\rho \quad (184)\\ \tilde{P}_0^{(\alpha)} &= 0 \quad (185)\end{aligned}\right\} \quad R < 1.$$

Degree $n = 1$. In view of (46), (126), (128) and (139), it follows that

$$\left.\begin{aligned} \tilde{P}_1^{(\Delta)} &= 0 \quad (186)\\ \tilde{P}_1^{(\alpha)} &= 3a\gamma\rho R \quad (187)\end{aligned}\right\} \quad R < 1.$$

Degrees $n \geq 2$. Using (46), (130), (132) and (140), we get

$$\left.\begin{aligned} \tilde{P}_n^{(\Delta)} &= 3a\gamma\rho\left[1 + \frac{2}{3}\frac{(2n+3)(n^2-1)}{2n^2+4n+3}\right]R^n \frac{k_n s\tilde{\mu}}{k_n s\tilde{\mu} + \gamma\rho} \quad (188)\\ \tilde{P}_n^{(\alpha)} &= a\gamma\rho[n(n+2)R^n - (n^2-1)R^{n+2}]\frac{\gamma\rho}{k_n s\tilde{\mu} + \gamma\rho} \quad (189)\end{aligned}\right\} \quad R < 1.$$

Deviatoric increment

In Cartesian-tensor notation, comparison of (17), (141) and (176) yields

$$\tilde{s}_{ij} = 2s\tilde{\mu}\tilde{e}_{ij}, \qquad X_i \in \mathscr{X}_-. \tag{190}$$

In spherical coordinates, the non-vanishing components of $\tilde{s}_{ij}$ are given by

$$\left.\begin{aligned}
\tilde{s}_{rr} &= 2s\tilde{\mu}\tilde{e}_{rr} && (191)\\
\tilde{s}_{r\theta} &= 2s\tilde{\mu}\tilde{e}_{r\theta} && (192)\\
\tilde{s}_{\theta\theta} &= 2s\tilde{\mu}\tilde{e}_{\theta\theta} && (193)\\
\tilde{s}_{\lambda\lambda} &= 2s\tilde{\mu}\tilde{e}_{\lambda\lambda} && (194)
\end{aligned}\right\} \quad r < a.$$

Introducing

$$\tilde{s}_{rr} = S_{rrn}\tilde{\zeta}_n P_n, \tag{195}$$

$$\tilde{s}_{r\theta} = -\tilde{S}_{r\theta n}\tilde{\zeta}_n P_{n,\theta}, \tag{196}$$

$$\tilde{s}_{\theta\theta} = \tilde{S}^{(1)}_{\theta\theta n}\tilde{\zeta}_n P_n - \tilde{S}^{(2)}_{\theta\theta n}\tilde{\zeta}_n \cot\theta P_{n,\theta}, \tag{197}$$

$$\tilde{s}_{\lambda\lambda} = \tilde{S}^{(1)}_{\lambda\lambda n}\tilde{\zeta}_n P_n - \tilde{S}^{(2)}_{\lambda\lambda n}\tilde{\zeta}_n \cot\theta P_{n,\theta} \tag{198}$$

and observing (146)–(149), we obtain the formulae

$$\left.\begin{aligned}
\tilde{S}_{rrn} &= 2s\tilde{\mu}\tilde{E}_{rrn} && (199)\\
\tilde{S}_{r\theta n} &= 2s\tilde{\mu}\tilde{E}_{r\theta n} && (200)\\
\tilde{S}^{(1)}_{\theta\theta n} &= 2s\tilde{\mu}\tilde{E}^{(1)}_{\theta\theta n} && (201)\\
\tilde{S}^{(2)}_{\theta\theta n} &= 2s\tilde{\mu}\tilde{E}^{(2)}_{\theta\theta n} && (202)\\
\tilde{S}^{(1)}_{\lambda\lambda n} &= 2s\tilde{\mu}\tilde{E}^{(1)}_{\lambda\lambda n} && (203)\\
\tilde{S}^{(2)}_{\lambda\lambda n} &= 2s\tilde{\mu}\tilde{E}^{(2)}_{\lambda\lambda n} && (204)
\end{aligned}\right\} \quad r < a.$$

Degree $n = 0$. In view of (156), it follows that

$$\tilde{S}_{rr0} = \tilde{S}_{r\theta 0} = \tilde{S}^{(1)}_{\theta\theta 0} = \tilde{S}^{(2)}_{\theta\theta 0} = \tilde{S}^{(1)}_{\lambda\lambda 0} = \tilde{S}^{(2)}_{\lambda\lambda 0} = 0, \qquad R < 1. \tag{205}$$

Degree $n = 1$. Upon substitution of (157)–(162) and definition of $k_1 = 9/a$, the following formulae result:

$$\left.\begin{aligned}
\tilde{S}_{rr1} &= 0 && (206)\\
\tilde{S}_{r\theta 1} &= \tfrac{2}{3}aR^{-1}k_1 s\tilde{\mu} && (207)\\
\tilde{S}^{(1)}_{\theta\theta 1} &= -2aR^{-1}k_1 s\tilde{\mu} && (208)\\
\tilde{S}^{(2)}_{\theta\theta 1} &= \tfrac{2}{3}aR^{-1}k_1 s\tilde{\mu} && (209)\\
\tilde{S}^{(1)}_{\lambda\lambda 1} &= -\tfrac{2}{3}aR^{-1}k_1 s\tilde{\mu} && (210)\\
\tilde{S}^{(2)}_{\lambda\lambda 1} &= -\tfrac{2}{3}aR^{-1}k_1 s\tilde{\mu} && (211)
\end{aligned}\right\} \quad R < 1.$$

Degrees $n \geq 2$. Using the definition of k_n and substituting (163)–(168), we arrive at

$$\boxed{\left.\begin{aligned}
\tilde{S}_{rrn} &= -2a\gamma\rho\frac{n(n-1)}{2n^2+4n+3}[n(n+2)R^{n-2} - (n+1)^2R^n]\frac{k_n s\tilde{\mu}}{k_n s\tilde{\mu} + \gamma\rho} && (212)\\
\tilde{S}_{r\theta n} &= 2a\gamma\rho\frac{n(n-1)(n+2)}{2n^2+4n+3}(R^{n-2} - R^n)\frac{k_n s\tilde{\mu}}{k_n s\tilde{\mu} + \gamma\rho} && (213)\\
\tilde{S}^{(1)}_{\theta\theta n} &= 2a\gamma\rho\frac{n(n+2)}{2n^2+4n+3}[n^2R^{n-2} - (n^2-1)R^n]\frac{k_n s\tilde{\mu}}{k_n s\tilde{\mu} + \gamma\rho} && (214)\\
\tilde{S}^{(2)}_{\theta\theta n} &= -2a\gamma\rho\frac{1}{2n^2+4n+3}[n(n+2)R^{n-2} - (n-1)(n+3)R^n]\frac{k_n s\tilde{\mu}}{k_n s\tilde{\mu} + \gamma\rho} && (215)\\
\tilde{S}^{(1)}_{\lambda\lambda n} &= -2a\gamma\rho\frac{n}{2n^2+4n+3}[n(n+2)R^{n-2} - (n^2-1)R^n]\frac{k_n s\tilde{\mu}}{k_n s\tilde{\mu} + \gamma\rho} && (216)\\
\tilde{S}^{(2)}_{\lambda\lambda n} &= 2a\gamma\rho\frac{1}{2n^2+4n+3}[n(n+2)R^{n-2} - (n-1)(n+3)R^n]\frac{k_n s\tilde{\mu}}{k_n s\tilde{\mu} + \gamma\rho} && (217)
\end{aligned}\right\} \quad R < 1.}$$

4.1.6 Material, isopotential and local incremental gravity

With the isopotential height, $\tilde{h}$, the material displacement, $\tilde{u}_i$, and the local incremental potential, $\tilde{\phi}^{(\Delta)}$, given, any of the increments of gravity, $\tilde{g}_i^{(\delta)}$, $\tilde{g}_i^{(\partial)}$ and $\tilde{g}_i^{(\Delta)}$, may be calculated.

Local increment

We consider (19), whose non-vanishing components in spherical coordinates are

$$\left.\begin{aligned} \tilde{g}_r^{(\Delta)} &= \tilde{\phi}_{,r}^{(\Delta)} \qquad (218)\\ \tilde{g}_\theta^{(\Delta)} &= \frac{1}{r}\tilde{\phi}_{,\theta}^{(\Delta)} \qquad (219)\end{aligned}\right\} \quad r \neq a.$$

If we define

$$\tilde{g}_r^{(\Delta)} = \tilde{G}_{rn}^{(\Delta)}\tilde{\zeta}_n P_n, \tag{220}$$

$$\tilde{g}_\theta^{(\Delta)} = -\tilde{G}_{\theta n}^{(\Delta)}\tilde{\zeta}_n P_{n,\theta} \tag{221}$$

and observe (121), we obtain the following equations:

$$\left.\begin{aligned} \tilde{G}_{rn}^{(\Delta)} &= \tilde{\Phi}_{n,r}^{(\Delta)} \qquad (222)\\ \tilde{G}_{\theta n}^{(\Delta)} &= -\frac{1}{r}\tilde{\Phi}_n^{(\Delta)} \qquad (223)\end{aligned}\right\} \quad r \neq a.$$

Degree $n = 0$. Considering (125) and $P_{0,\theta} = 0$, it follows that

$$\tilde{G}_{r0}^{(\Delta)} = \begin{cases} 0, & R < 1 \\ -3\gamma R^{-2}, & R > 1 \end{cases}, \tag{224}$$

$$\tilde{G}_{\theta 0}^{(\Delta)} = 0, \qquad R \neq 1. \tag{225}$$

Degree $n = 1$. With (129), we arrive at

$$\tilde{G}_{r1}^{(\Delta)} = \tilde{G}_{\theta 1}^{(\Delta)} = 0, \qquad R \neq 1. \tag{226}$$

Degrees $n \geq 2$. Substitution of (133) yields the formulae

$$\boxed{\begin{aligned} \tilde{G}_{rn}^{(\Delta)} &= \begin{cases} 3\gamma n R^{n-1}\dfrac{k_n s\tilde{\mu}}{k_n s\tilde{\mu} + \gamma\rho}, & R < 1 \\ -3\gamma(n+1)R^{-(n+2)}\dfrac{k_n s\tilde{\mu}}{k_n s\tilde{\mu} + \gamma\rho}, & R > 1 \end{cases}, \qquad (227)\\ \tilde{G}_{\theta n}^{(\Delta)} &= \begin{cases} -3\gamma R^{n-1}\dfrac{k_n s\tilde{\mu}}{k_n s\tilde{\mu} + \gamma\rho}, & R < 1 \\ -3\gamma R^{-(n+2)}\dfrac{k_n s\tilde{\mu}}{k_n s\tilde{\mu} + \gamma\rho}, & R > 1 \end{cases}. \qquad (228)\end{aligned}}$$

Material and isopotential increments

Observations of gravity changes at or below the surface of planets usually refer to material points and are frequently reduced to the geoid. Therefore, it is necessary to relate $\tilde{g}_i^{(\Delta)}$ to the material incremental gravity, $\tilde{g}_i^{(\delta)}$, and the isopotential incremental gravity, $\tilde{g}_i^{(\partial)}$. In spherical coordinates and with $g_\theta^{(0)} = 0$, it follows from (309) and (310) that

$$\left.\begin{aligned} \tilde{g}_r^{(\delta)} &= \tilde{g}_r^{(\Delta)} + g_{r,r}^{(0)}\tilde{u}_r \qquad (229)\\ \tilde{g}_\theta^{(\delta)} &= \tilde{g}_\theta^{(\Delta)} \qquad (230)\end{aligned}\right\} \quad r < a,$$

$$\left.\begin{aligned} \tilde{g}_r^{(\partial)} &= \tilde{g}_r^{(\Delta)} + g_{r,r}^{(0)}\tilde{h} \qquad (231)\\ \tilde{g}_\theta^{(\partial)} &= \tilde{g}_\theta^{(\Delta)} \qquad (232)\end{aligned}\right\} \quad r \neq a.$$

Note that, by (230) and (232), the colatitudinal components of the different measures of incremental gravity are identical in the joint spatial domains. The following analysis is therefore limited to the radial components. We introduce

$$\tilde{g}_r^{(\delta)} = (\tilde{G}_{rn}^{(\Delta)} + \tilde{G}_{rn}^{(\alpha)})\tilde{\zeta}_n P_n, \tag{233}$$

$$\tilde{g}_r^{(\partial)} = \tilde{G}_{rn}^{(\partial)}\tilde{\zeta}_n P_n \tag{234}$$

and get with (118), (136) and (220) the formulae

$$\left.\begin{aligned} \tilde{G}_{rn}^{(\alpha)} &= g_{r,r}^{(0)}\tilde{U}_{rn} \\ \tilde{G}_{rn}^{(\partial)} &= \tilde{G}_{rn}^{(\Delta)} + g_{r,r}^{(0)}\tilde{H}_n \end{aligned}\right\} \quad r \neq a. \tag{235, 236}$$

Degree $n = 0$. Upon substitution of (48), (122), (138) and (224), the preceding equations become

$$\tilde{G}_{r0}^{(\alpha)} = 0, \qquad R < 1, \tag{237}$$

$$\tilde{G}_{r0}^{(\partial)} = \begin{cases} -3\gamma R^{-1}, & R < 1 \\ 3\gamma R^{-2}, & R > 1 \end{cases}. \tag{238}$$

Degree $n = 1$. In view of (48), (126), (139) and (226), the following expressions result:

$$\tilde{G}_{r1}^{(\alpha)} = 3\gamma, \qquad R < 1, \tag{239}$$

$$\tilde{G}_{r1}^{(\partial)} = 0, \qquad R \neq 1. \tag{240}$$

Degrees $n \geq 2$. Considering (48), (130), (140) and (227), we obtain

$$\tilde{G}_{rn}^{(\alpha)} = \gamma[n(n+2)R^{n-1} - (n^2-1)R^{n+1}]\frac{\gamma\rho}{k_n s\tilde{\mu} + \gamma\rho}, \qquad R < 1, \tag{241}$$

$$\tilde{G}_{rn}^{(\partial)} = \begin{cases} 3\gamma(n-1)R^{n-1}\dfrac{k_n s\tilde{\mu}}{k_n s\tilde{\mu} + \gamma\rho}, & R < 1 \\ -3\gamma(n-1)R^{-(n+2)}\dfrac{k_n s\tilde{\mu}}{k_n s\tilde{\mu} + \gamma\rho}, & R > 1 \end{cases}. \tag{242}$$

4.1.7 Relations to other studies

Provided the underlying field equations and interface conditions are identical, the special solution functions derived above must be identical with those obtained from the conventional (6×6) differential system. Special solution functions for models similar to ours have been published by Wu & Peltier (1982), Spada *et al.* (1992) and Amelung & Wolf (1994). Here, we inspect the formulae in the (r, n, s) domain given by the first authors, which apply to a model identical to ours. Considering the radial displacement as an example, we write (130) in the form $\tilde{u}_{rn} = \tilde{U}_n P_n$ with $\tilde{U}_n = C_1 n/[2(2n+3)]r^{n+1} + C_2 r^{n-1}$. Upon expressing C_1 and C_2 in terms of the model parameters and with the appropriate notational changes applied, some algebra establishes the identity of (130) with the corresponding formulae in Wu & Peltier (1982) [*cf.* in particular their eqs (5), (29) and (32)].

Of some interest is also the case where the wavelength of planetary deformations is sufficiently short compared to the radius of the planet that the sphericity becomes irrelevant and gravitational perturbations may be neglected. Then, the half-space theory developed elsewhere (e.g. Wolf 1991b, 1993, pp. 33–45) is profitably employed and yields the desired results more easily than the spherical theory considered here. The accuracy of the half-space approximation has been tested computationally for a number of earth models (Wolf 1984; Amelung & Wolf 1994); here, we show how the solution functions for the sphere can be formally reduced to those for the half-space.

Since the problem is primarily of theoretical interest, we restrict our analysis to the radial displacement. Using (93), (118), (130), $R = 1 + x/a$, $\tilde{u}_r = \tilde{u}_x$ and the definition of k_n, it is expressible as

$$\tilde{u}_x = -\frac{\left(1+\frac{2}{n}\right)\left(1+\frac{x}{a}\right)^{n-1} - \left(1-\frac{1}{n^2}\right)\left(1+\frac{x}{a}\right)^{n+1}}{\frac{1}{n}\left(2+\frac{1}{n}\right)\left[\left(2+\frac{4}{n}+\frac{3}{n^2}\right)\frac{n}{a}s\tilde{\mu} + \gamma\rho\right]}\gamma\tilde{\sigma}_n P_n. \tag{243}$$

If we put $n/a = k$, this can be recast into

$$\tilde{u}_x = -\Gamma_n\left(1 + \frac{1}{n}kx\right)^n \gamma\tilde{\sigma}_n P_n, \tag{244}$$

where

$$\Gamma_n = \frac{\left(1+\frac{2}{n}\right)\left(1+\frac{1}{n}kx\right)^{-1} - \left(1-\frac{1}{n^2}\right)\left(1+\frac{1}{n}kx\right)}{\frac{1}{n}\left(2+\frac{1}{n}\right)\left[\left(2+\frac{4}{n}+\frac{3}{n^2}\right)ks\tilde{\mu} + \gamma\rho\right]}, \tag{245}$$

$$\left(1+\frac{1}{n}kx\right)^n = 1 + kx + \left(1-\frac{1}{n}\right)\frac{(kx)^2}{2!} + \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{(kx)^3}{3!} + \cdots. \tag{246}$$

Supposing now that $x/a \to 0$ and $n \to \infty$ such that k = finite, some manipulation yields

$$\lim_{n\to\infty} \Gamma_n = \frac{1-kx}{2ks\tilde{\mu} + \gamma\rho}, \tag{247}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}kx\right)^n = e^{kx}. \tag{248}$$

According to Hobson (1931, p. 299), $\tilde{\sigma}_n$ can always be chosen such that, correct to the order $1/\sqrt{n}$, the relation $\tilde{\sigma}_n P_n = \tilde{S}\sin[(n+\frac{1}{2})\theta + \pi/4]$ applies in some neighbourhood of θ. Selecting $\theta = \pi/2$ and putting $n\theta = ky$, this simplifies to

$$\tilde{\sigma}_n P_n = \tilde{S}\cos(ky). \tag{249}$$

In view of (244) and (247)–(249), we obtain for $x/a \to 0$ and $1/n \to 0$ the limit

$$\boxed{\tilde{u}_x = -\frac{\gamma\tilde{S}}{2ks\tilde{\mu} + \gamma\rho}(1-kx)e^{kx}\cos(ky).} \tag{250}$$

Eqs (249) and (250) are fully consistent with the special solution functions in Wolf (1991b, 1993, pp. 33–45) [*cf.* in particular eqs (4.40), (4.41) and (4.61) in the second reference] and, accordingly, give the vertical displacement in a half-space subject to a fixed and homogeneous gravity field and a sinusoidal load of wavelength $2\pi/k$ and amplitude $\tilde{S}$.

4.1.8 Transfer functions

Inspection of the solution functions listed above shows that the ordinary Legendre coefficients, $\tilde{f}_n(r,s) = \tilde{F}_n(r,s)\tilde{\zeta}_n(s)$, of the field quantities analysed can be decomposed according to

$$\boxed{\tilde{f}_n(r,s) = \tilde{F}_n(r)\tilde{T}_n(s)\tilde{\zeta}_n(s),} \tag{251}$$

which is the general form of the solution functions in the (r, n, s) domain. Function $\tilde{F}_n(r) = \tilde{F}_n(r,s)/\tilde{T}_n(s)$ specifies the radial dependence of $\tilde{f}_n(r,s)$ and can be directly obtained from the individual solution functions. Function $\tilde{T}_n(s)$ is referred to as *transfer* function and found to be of either of two types:

$$\tilde{T}_n^{(A)}(s) = \begin{cases} 1, & n = 1 \\ \dfrac{\gamma\rho}{k_n s\tilde{\mu}(s) + \gamma\rho}, & n \geq 2 \end{cases}, \tag{252}$$

$$\tilde{T}_n^{(B)}(s) = \begin{cases} 1, & n = 0 \\ k_1 s\tilde{\mu}(s), & n = 1 \\ \dfrac{k_n s\tilde{\mu}(s)}{k_n s\tilde{\mu}(s) + \gamma\rho}, & n \geq 2 \end{cases}. \tag{253}$$

As in the foregoing equations, the arguments of the functions considered will henceforth be displayed for clarity.

4.2 Generalized Maxwell viscoelasticity

For the inversion of $\tilde{T}_n^{(A)}(s)$ for $n \geq 2$ and $\tilde{T}_n^{(B)}(s)$ for $n \geq 1$, it is necessary to specify $\tilde{\mu}(s)$. We start from the general formula

$$\mu(t-t') = \int_0^\infty \bar{\mu}(\alpha')e^{-\alpha'(t-t')}\, d\alpha', \tag{254}$$

which expresses $\mu(t-t')$ in terms of its spectrum, $\bar{\mu}(\alpha')$. We can approximate the latter to any degree of accuracy required by $\bar{\mu}(\alpha') = \sum_{q=1}^{Q} \mu^{(q)}\delta(\alpha' - \alpha^{(q)})$, where $\alpha^{(q)} > 0$ is the qth inverse elemental Maxwell time and $\mu^{(q)} > 0$ the qth elemental elastic shear modulus, both prescribed for $q \in \{1, 2, \ldots, Q\}$; as usual, $\delta(\alpha' - \alpha^{(q)})$ denotes the (shifted) Dirac delta function. Using this approximation, (254) reduces to

$$\mu(t-t') = \sum_{q=1}^{Q} \mu^{(q)} e^{-\alpha^{(q)}(t-t')}, \tag{255}$$

which is the shear-relaxation function for *generalized Maxwell* viscoelasticity (e.g. Christensen 1982, pp. 16–20; Müller 1986; Wang 1986). Defining $\mu_e = \lim_{t-t' \to 0} \mu(t-t')$, we obtain in particular $\mu_e = \sum_{q=1}^{Q} \mu^{(q)}$, which is the elastic shear modulus. The Laplace transform of (255) with respect to $t - t'$ is

$$\tilde{\mu}(s) = \sum_{q=1}^{Q} \frac{\mu^{(q)}}{s + \alpha^{(q)}}. \tag{256}$$

4.3 Functions in the (r, n, t) domain

4.3.1 *Impulse-response functions*

We proceed with the transformation of the solution functions specified in Sections 4.1.1–4.1.6 from the (r, n, s) to the (r, n, t) domain. This requires inverse Laplace transformation of (251)–(253). Details on the inversion of the regular and singular functions entering into these equations can be found elsewhere (e.g. LePage 1980, pp. 285–328; Wolf 1993, pp. 80–82). Inverse Laplace transformation of (251) gives

$$f_n(r, t) = \bar{F}_n(r) \int_0^t T_n(t-t')\zeta_n(t')\, dt' \tag{257}$$

as general form of the solution functions in the (r, n, t) domain. Function $T_n(t-t')$ is the *impulse-response* function associated with the transfer function $\tilde{T}_n(s)$, which is of type $\tilde{T}_n^{(A)}(s)$ or $\tilde{T}_n^{(B)}(s)$.

With $\tilde{\mu}(s)$ specified and the general functional form of $f_n(r, t)$ established, (252) and (253) can now be inverted. We consider the Legendre degrees $n = 0$, $n = 1$ and $n \geq 2$ individually.

Degree $n = 0$. The shifted inverse Laplace transform of (253) is

$$T_0^{(B)}(t-t') = \delta(t-t'). \tag{258}$$

Degree $n = 1$. The shifted inverse Laplace transform of (252) is

$$T_1^{(A)}(t-t') = \delta(t-t'). \tag{259}$$

Upon substitution of (256) into (253), it follows that

$$\tilde{T}_1^{(B)}(s) = k_1 \sum_{q=1}^{Q} \mu^{(q)}\left(1 - \frac{\alpha^{(q)}}{s + \alpha^{(q)}}\right), \tag{260}$$

whence the shifted inverse Laplace transform takes the form

$$T_1^{(B)}(t-t') = k_1 \sum_{q=1}^{Q} \mu^{(q)}[\delta(t-t') - \alpha^{(q)} e^{-\alpha^{(q)}(t-t')}]. \tag{261}$$

Degrees $n \geq 2$. Substituting (256), we obtain from (252) the equation

$$\tilde{T}_n^{(A)}(s) = \gamma\rho\left(k_n \sum_{q=1}^{Q} \frac{\mu^{(q)}s}{s+\alpha^{(q)}} + \gamma\rho\right)^{-1}. \tag{262}$$

After some algebraic manipulation, this can be recast into

$$\tilde{T}_n^{(A)}(s) = \frac{\gamma\rho}{k_n\mu_e + \gamma\rho}[1 + \tilde{W}_n(s)], \tag{263}$$

where

$$\tilde{W}_n(s) = \frac{U_n(s)}{V_n(s)}, \tag{264}$$

$$U_n(s) = k_n \sum_{q=1}^{Q} \frac{\mu^{(q)}\alpha^{(q)}}{s+\alpha^{(q)}}, \tag{265}$$

$$V_n(s) = k_n\left(\mu_e - \sum_{q=1}^{Q} \frac{\mu^{(q)}\alpha^{(q)}}{s+\alpha^{(q)}}\right) + \gamma\rho. \tag{266}$$

The shifted inverse Laplace transform of (263) can formally be written as

$$\boxed{T_n^{(A)}(t-t') = \frac{\gamma\rho}{k_n\mu_e + \gamma\rho}[\delta(t-t') + W_n(t-t')].} \tag{267}$$

Comparing (252) and (253), we get

$$\tilde{T}_n^{(B)}(s) = 1 - \tilde{T}_n^{(A)}(s), \tag{268}$$

whose shifted inverse Laplace transform is

$$\boxed{T_n^{(B)}(t-t') = \frac{\gamma\rho}{k_n\mu_e + \gamma\rho}\left[\frac{k_n\mu_e}{\gamma\rho}\delta(t-t') - W_n(t-t')\right].} \tag{269}$$

4.3.2 Stability analysis

It remains to establish the functional form of $W_n(t-t')$. Inspecting (264)–(266), we note that $\tilde{W}_n(s)$ can be rewritten as the quotient of two polynomials in s (without common roots) of degrees $L = Q - 1$ in the enumerator and $M = Q$ in the denominator. Hence, the inverse Laplace transform of $\tilde{W}(s)$ exist and, according to the complex inversion formula and the residue theorem, can be specified upon determination of the roots of $V_n(s)$. To prove that all roots are simple and negative, we assume that the M poles of $V_n(s)$ have been ordered such that $0 > -\alpha^{(1)} > -\alpha^{(2)} > \cdots > \alpha^{(M)}$. Considering the interval $\mathscr{I}^{(1)} = (-\alpha^{(1)}, 0)$ first, we note that $\lim_{s \to -\alpha^{(1)}+0} V_n(s) = -\infty$ and $V_n(0) = 1$, whence one root must lie in $\mathscr{I}^{(1)}$. The remaining roots are found by considering the interval $\mathscr{I}^{(m)} = (-\alpha^{(m)}, -\alpha^{(m-1)})$, where $m \in \{2, 3, \ldots, M\}$. Since $\lim_{s \to -\alpha^{(m)}+0} V_n(s) = -\infty$ and $\lim_{s \to -\alpha^{(m-1)}-0} V_n(s) = \infty$, one root must also lie in each of $\mathscr{I}^{(2)}, \mathscr{I}^{(3)}, \ldots, \mathscr{I}^{(M)}$. However, $V_n(s)$ can have either M roots if all are simple, or less than M roots if at least one is multiple. Taking this into account, it follows that there is *exactly* one root in each of $\mathscr{I}^{(1)}, \mathscr{I}^{(2)}, \ldots, \mathscr{I}^{(M)}$.

Having established that $\tilde{V}_n(s)$ has M simple and negative roots, the functional form of $W_n(t-t')$ can now be given. Denoting the pole in $\mathscr{I}^{(m)}$ by $-\beta_n^{(m)}$, evaluation by means of the complex inversion formula and the residue theorem yields

$$W_n(t-t') = \sum_{m=1}^{M} \frac{U_n^{(m)}}{V_n^{(m)}} e^{-\beta_n^{(m)}(t-t')}, \qquad M = Q, \tag{270}$$

$$U_n^{(m)} = U_n(-\beta_n^{(m)}) = k_n \sum_{q=1}^{Q} \frac{\mu^{(q)}\alpha^{(q)}}{\alpha^{(q)} - \beta_n^{(m)}}, \tag{271}$$

$$V_n^{(m)} = d_s V_n(-\beta_n^{(m)}) = k_n \sum_{q=1}^{Q} \frac{\mu^{(q)}\alpha^{(q)}}{(\alpha^{(q)} - \beta_n^{(m)})^2}. \tag{272}$$

The mth term of the sum in (270) is called mth relaxation mode, with $U_n^{(m)}/V_n^{(m)}$ the modal amplitude, $\beta_n^{(m)}$ the inverse modal relaxation time and M the total number of relaxation modes. Eqs (270)–(272) completely determine the impulse-response functions $T_n^{(A)}(t-t')$ and $T_n^{(B)}(t-t')$ for $n \geq 2$ and, thus, the solution functions in the (r, n, t) domain.

4.3.3 Maxwell and Burgers viscoelasticity

General methods of obtaining closed-form expressions of the roots, $-\beta_n^{(m)}$, of $V_n(s)$ exist only for $M \leq 4$. In all other cases, numerical methods must normally be applied. In practice, such methods are even used for $M=3$ or $M=4$. Here, we evaluate $W_n(t-t')$ exactly for $M=1$ and $M=2$. Since $M=Q$, this is equivalent to $Q=1$ and $Q=2$, which correspond to Maxwell viscoelasticity and Burgers viscoelasticity, respectively.

Case $M=1$. Eq. (266) takes the simplified form

$$V_n(s) = \frac{(k_n\mu^{(1)} + \gamma\rho)s + \gamma\rho\alpha^{(1)}}{s+\alpha^{(1)}}, \tag{273}$$

with the root

$$-\beta_n^{(1)} = -\frac{\gamma\rho}{k_n\mu^{(1)} + \gamma\rho}\alpha^{(1)}, \tag{274}$$

whereas (270)–(272) lead to

$$W_n(t-t') = \frac{k_n\mu^{(1)}\alpha^{(1)}}{k_n\mu^{(1)} + \gamma\rho} e^{-\beta_n^{(1)}(t-t')}. \tag{275}$$

The solution for Maxwell viscoelasticity was discussed by Wu & Peltier (1982) and, more recently, by Amelung & Wolf (1993). Its characteristic feature is the single exponential decay mode, with the modal amplitude and modal relaxation time being simple functions of the Legendre degree and the parameters of the planet.

Case $M=2$. Eq. (266) reduces to

$$V_n(s) = \frac{(k_n\mu_e + \gamma\rho)s^2 + [(k_n\mu^{(2)} + \gamma\rho)\alpha^{(1)} + (k_n\mu^{(1)} + \gamma\rho)\alpha^{(2)}]s + \gamma\rho\alpha^{(1)}\alpha^{(2)}}{(s+\alpha^{(1)})(s+\alpha^{(2)})}, \tag{276}$$

which has the following roots:

$$\left.\begin{matrix}-\beta_n^{(1)}\\ -\beta_n^{(2)}\end{matrix}\right\} = -\frac{(k_n\mu^{(2)} + \gamma\rho)\alpha^{(1)} + (k_n\mu^{(1)} + \gamma\rho)\alpha^{(2)}}{2(k_n\mu_e + \gamma\rho)}$$
$$\pm\frac{\{[(k_n\theta^{(2)} + \gamma\rho)\alpha^{(1)} - (k_n\mu^{(1)} + \gamma\rho)\alpha^{(2)}]^2 + 4k_n^2\mu^{(1)}\mu^{(2)}\alpha^{(1)}\alpha^{(2)}\}^{1/2}}{2(k_n\mu_e + \gamma\rho)}. \tag{277}$$

Eqs (270)–(272) yield

$$W_n(t-t') = \sum_{m=1}^{2} \frac{(k_n\mu^{(1)}\alpha^{(1)} + k_n\mu^{(2)}\alpha^{(2)})\beta_n^{(m)} - k_n\mu_e\alpha^{(1)}\alpha^{(2)}}{2(k_n\mu_e + \gamma\rho)\beta_n^{(m)} + (k_n\mu^{(2)} + \gamma\rho)\alpha^{(1)} + (k_n\mu^{(1)} + \gamma\rho)\alpha^{(2)}} e^{-\beta_n^{(m)}(t-t')}. \tag{278}$$

The solution is now characterized by the superposition of two exponential decay modes. Compared to Maxwell viscoelasticity, the complexity of the functional forms for the modal amplitudes and relaxation times is greatly enhanced. A detailed discussion of the response characteristics of elementary models with Burgers viscoelasticity can be found in Rümpker (1990).

4.4 Functions in the (r, θ, λ, t) domain

The final step is the transformation of the solution function (257) from the (r, n, t) to the (r, θ, λ, t) domain. In the following, we will distinguish *axisymmetric* and *non-symmetric* loads and calculate the respective *Green's* functions.

4.4.1 Axisymmetric Green's functions

We first consider the Green's function for axisymmetric loads whose distribution is given by $\sigma(\theta', t')$, where θ' is the colatitude of the excitation point. On the assumption that $\sigma(\theta', t')$ is twice continuously differentiable with respect to θ' in $(0, \pi)$ and that

$\int_0^\pi [\sigma(\theta', t')]^2 \sin\theta' \, d\theta'$ is finite, the distribution can be expanded into a convergent Legendre series (e.g. Lebedev 1972, pp. 53–60):

$$\sigma(\theta', t') = \sum_{n=n_0}^{\infty} \sigma_n(t') P_n(\cos\theta'), \tag{279}$$

with

$$\sigma_n(t') = (n + \tfrac{1}{2}) \int_0^\pi \sigma(\theta', t') \sin\theta' P_n(\cos\theta') \, d\theta'. \tag{280}$$

We recall that only the term $\sigma_0(t')$ corresponds to a net mass; accordingly, $n_0 = 0$ applies if $\sigma(\theta', t')$ specifies an *accreted* load, and $n_0 = 1$ if it specifies a *redistributed* load. In view of the linearity of the problem, the solution of the load prescribed by (279) can be expressed as

$$f(r, \theta, t) = \sum_{n=n_0}^{\infty} f_n(r, t) \left\{ \begin{array}{l} P_n(\cos\theta) \\ -P_{n,\theta}(\cos\theta) \\ -\cot\theta P_{n,\theta}(\cos\theta) \end{array} \right. , \tag{281}$$

which, upon use of (93) and (257), takes the form

$$f(r, \theta, t) = \frac{1}{ap} \sum_{n=n_0}^{\infty} \frac{\bar{F}_n(r)}{2n+1} \left\{ \begin{array}{c} P_n(\cos\theta) \\ -P_{n,\theta}(\cos\theta) \\ -\cot\theta P_{n,\theta}(\cos\theta) \end{array} \right\} \int_0^t T_n(t - t') \sigma_n(t') \, dt'. \tag{282}$$

Substituting (280) and changing the sequence of summation and integrations, this becomes

$$f(r, \theta, t) = \int_0^\pi \int_0^t f^{(\mathrm{as})}(r, \theta, \theta', t - t') \sigma(\theta', t') \sin\theta' \, d\theta' \, dt', \tag{283}$$

$$f^{(\mathrm{as})}(r, \theta, \theta', t - t') = \frac{1}{2ap} \sum_{n=n_0}^{\infty} \bar{F}_n(r) \left\{ \begin{array}{c} P_n(\cos\theta) \\ -P_{n,\theta}(\cos\theta) \\ -\cot\theta P_{n,\theta}(\cos\theta) \end{array} \right\} P_n(\cos\theta') T_n(t - t'), \tag{284}$$

where $f^{(\mathrm{as})}(r, \theta, \theta', t - t')$ denotes the axisymmetric Green's function in the (r, θ, t) domain.

4.4.2 *Non-symmetric Green's functions*

It is now straightforward to deduce the Green's function for non-symmetric loads described by the distribution $\sigma(\theta', \lambda', t')$, where θ' and λ' are the colatitude and longitude of the excitation point, respectively. For this, we take into account that $f^{(\mathrm{as})}(r, \theta, \theta', t - t')$ is the normalized contribution to $f(r, \theta, t)$ from an *annular* load at colatitude θ'. The contribution to $f(r, \theta, t)$ from a *point* load on the symmetry axis thus follows from

$$f^{(\mathrm{ns})}(r, \theta, t - t') = \lim_{\theta' \to 0} f^{(\mathrm{as})}(r, \theta, \theta', t - t'). \tag{285}$$

Noting that, for a non-symmetric load, $f(r, \theta, \lambda, t)$ can be obtained by superposing the contributions from the appropriate distribution of point loads, the generalizations of (283) and (284) are

$$f(r, \theta, \lambda, t) = \int_0^\pi \int_0^{2\pi} \int_0^t f^{(\mathrm{ns})}(r, \vartheta, t - t') \sigma(\theta', \lambda', t') \sin\theta' \, d\theta' \, d\lambda' \, dt', \tag{286}$$

$$f^{(\mathrm{ns})}(r, \vartheta, t - t') = \frac{1}{2ap} \sum_{n=n_0}^{\infty} \bar{F}_n(r) \left\{ \begin{array}{c} P_n(\cos\vartheta) \\ -P_{n,\vartheta}(\cos\vartheta) \\ -\cot\vartheta P_{n,\vartheta}(\cos\vartheta) \end{array} \right\} T_n(t - t'), \tag{287}$$

where $f^{(\mathrm{ns})}(r, \vartheta, t - t')$ denotes the non-symmetric Green's function in the (r, θ, λ, t) domain and ϑ the angle between the observation and excitation points, with $\cos\vartheta = \cos(\theta - \theta')\cos(\lambda - \lambda')$. Since $\bar{F}_n(r) = \bar{F}_n(r, s)/\tilde{T}_n(s)$ is implied by the solution functions listed in Sections 4.1.1–4.1.6, $T_n(t - t')$ of the types $T_n^{(\mathrm{A})}(t - t')$ or $T_n^{(\mathrm{B})}(t - t')$ given in Section 4.3.1 and $\sigma(\theta', \lambda', t')$ prescribed, the solution to Lamé's problem of gravitational viscoelasticity is completely specified.

5 CONCLUDING REMARKS

The main results of our study are the following.

(1) We have given a complete and rigorous solution describing infinitesimal, quasi-static, gravitational–viscoelastic perturbations, induced by surface loads, of a spherical, isochemical, incompressible, non-rotating, fluid planet initially in hydrostatic equilibrium. The kinematic formulation is uniformly Lagrangian in the internal and external domains of the planet.

(2) The solution method adopted differs from the methods conventionally employed for the type of problem studied. Its main characteristic is that the incremental field equations are recast into two mutually *decoupled* (4×4) and (2×2) first-order ordinary differential systems in terms of the mechanical and gravitational quantities of the problem, respectively, the coupling being restricted to the incremental interface conditions. A useful property of the decoupled differential systems is that the complexity of the algebraic manipulations necessary to solve them is significantly less than for the (6×6) differential system commonly used. It is suggested that this simplification is also of some consequence when considering perturbations of layered planets.

(3) The decoupling of the incremental field equations is contingent upon the use of the *isopotential* incremental pressure measuring the increment of the hydrostatic pressure with respect to a particular isopotential surface. The rigorous definition of isopotential increments and the Lagrangian expressions relating them to the material and local increments conventionally employed in the mechanics of continua subject to initial stress are given in Appendix A.

(4) The resulting solution functions are specified for the (r, n, s), (r, n, t), (r, θ, t) and (r, θ, λ, t) domains. They involve explicit expressions for the Legendre degrees $n = 0$, $n = 1$ and $n \geq 2$, are valid at any location in the interior or exterior of the planet and comprise all incremental field quantities of interest in the dynamics of planetary bodies. Of significance is that the solution functions apply to arbitrary types of *generalized Maxwell* viscoelasticity. The inverse relaxation times characterizing the particular type are given as the poles of the quotient of two polynomials in terms of the Laplace frequency, s. Since all poles are simple and negative, the planet is always *stable* and its impulse response involves a series of exponential decay modes.

Some interesting consequences of our study are the following.

(1) The (4×4) differential system obtained *formally* agrees with the system governing the corresponding non-gravitating problem. Available solutions to this simpler problem can therefore be generalized *a posteriori* in order to include gravitation. Since the numerical modelling of viscoelastic perturbations of planets has so far been based on techniques developed for non-gravitating continua, our solution method opens a way of accounting for initial stress and gravitational perturbations when using these techniques.

(2) Allowance for perturbations due to *non-gravitational* volume forces can be made by an additional term, $\rho^{(0)} f_i^{(\delta)}$, on the left-hand side of (9). Since $f_i^{(0)} = 0$, we then have instead of (18) the expression $\tilde{t}_{ij,j}^{(\partial)} + \rho^{(0)} \tilde{f}_i^{(\Delta)} = 0$ and therefore again formal agreement between the isopotential–local form of the incremental field equations and the corresponding equations valid in the absence of initial stress and gravitation.

(3) In the case of perturbations of a *compressible* planet, we have $u_{i,i} \neq 0$ and $\rho = \rho^{(0)} + \rho^{(\delta)}$. As a consequence, (16) no longer applies, $\tilde{t}_{ij,j}^{(\partial)} - g_i^{(0)} (\rho^{(0)} \tilde{u}_j)_{,j} = 0$ replaces (18) and $\rho^{(0)}$ is to be substituted for ρ in (16)–(24). Hence, no additional mechanical–gravitational coupling is introduced by compressibility, but the formal reduction to the non-gravitating case is not achieved. A special type of compressibility where $(\rho^{(0)} u_i)_{,i}$ vanishes has recently been studied by Li & Yuen (1987) and Wolf (1991a).

ACKNOWLEDGMENTS

Part of this research was carried out while the author was a visiting scientist at the Geophysics Division, Geological Survey of Canada, Ottawa; the generous support provided by A. Lambert and C. Thomson is gratefully acknowledged. Comments by R. Sabadini and two anonymous reviewers on a first version of this paper have been very helpful.

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APPENDIX A: MATHEMATICAL AND NOTATIONAL CONCEPTS

We modify and extend the concepts developed in Dahlen (1974), Grafarend (1982) and Wolf (1991a); a complete exposition of the following analysis can be found in Wolf (1993, pp. 4–10).

A1 Kinematic formulations

Suppose Cartesian-tensor fields and employ for them the usual indicial notation and summation convention. Denote by $\mathscr{E}$ the unbounded 3-D Euclidian space domain, by $\mathscr{T}$ the time domain $[0, \infty)$ and consider the mapping $r_i = r_i(\mathbf{X}, t)$, where $X_i \in \mathscr{E}$ and $t \in \mathscr{T}$. We stipulate that $r_i = r_i(\mathbf{X}, t)$ is a one-to-one mapping of $\mathscr{E}$ onto itself with the property $X_i = r_i(\mathbf{X}, 0)$ and continuous differentiability with respect to the arguments as many times as required. We call t current time, r_i current position, $t = 0$ initial time and X_i initial position. Assume now that $\mathscr{E}$ is completely filled by a gravitating fluid. A particular mapping satisfying our assumptions then is

$$r_i^{(\mathrm{L})} = r_i^{(\mathrm{L})}(\mathbf{X}, t) = X_i + u_i(\mathbf{X}, t), \qquad X_i \in \mathscr{E}, \qquad t \in \mathscr{T}, \tag{288}$$

which is the kinematic formulation used for *material* points (particles). The mapping identifies each material point by its initial position, X_i, and relates to the point its current position, $r_i^{(L)}$, in terms of the material displacement, u_i, from its initial position. In addition, we define *isopotential* points, which move in the direction of the gradient of the gravitational potential currently existing at their respective positions such that the potential experienced by each of them remains constant during their motion. A second mapping satisfying our assumptions thus is

$$r_i^{(N)} = r_i^{(N)}(\mathbf{X}, t) = X_i + d_i(\mathbf{X}, t), \qquad X_i \in \mathscr{E}, \qquad t \in \mathscr{T}, \tag{289}$$

which is the kinematic formulation used for isopotential points. The mapping identifies each isopotential point by its initial position, X_i, and relates to the point its current position, $r_i^{(N)}$, in terms of the isopotential displacement, d_i, from its initial position.

The inverse mappings to (288) and (289) are, respectively,

$$\left.\begin{aligned} X_i^{(L)} &= X_i^{(L)}(\mathbf{r}, t) = r_i - U_i(\mathbf{r}, t) \qquad (290)\\ X_i^{(N)} &= X_i^{(N)}(\mathbf{r}, t) = r_i - D_i(\mathbf{r}, t) \qquad (291)\end{aligned}\right\} r_i \in \mathscr{E}, \qquad t \in \mathscr{T}.$$

In contrast to (288) and (289), eqs (290) and (291), refer to *local* points (places) identified by their position, r_i. In particular, (290) relates to each local point the initial position, $X_i^{(L)}$, of the material point currently at r_i by means of the material displacement, U_i. Similarly, (291) relates to each local point the initial position, $X_i^{(N)}$, of the isopotential point currently at r_i by means of the isopotential displacement, D_i. In view of the general assumptions to be satisfied by the mappings $r_i^{(L)}(\mathbf{X}, t)$ and $r_i^{(N)}(\mathbf{X}, t)$, (290) and (291) define one-to-one mappings that are continuously differentiable with respect to the arguments as many times as required.

We proceed by specifying the domains of definition of the mappings (288)–(291) more closely. Beginning with (288), we decompose $\mathscr{E}$ into two open subdomains: the simply connected *internal* domain, $\mathscr{X}_-^{(L)}$, and the complementary *external* domain, $\mathscr{X}_+^{(L)}$. With $\partial\mathscr{X}^{(L)}$ the 2-D *interface* between the two domains, it then follows that $\mathscr{E} = \mathscr{X}_-^{(L)} \cup \mathscr{X}_+^{(L)} \cup \partial\mathscr{X}^{(L)}$. We now define

$$\mathscr{R}_\pm^{(L)}(t) = \{r_i^{(L)}(\mathbf{X}, t) \mid X_i \in \mathscr{X}_\pm^{(L)}, t \in \mathscr{T}\}, \tag{292}$$

$$\partial\mathscr{R}^{(L)}(t) = \{r_i^{(L)}(\mathbf{X}, t) \mid X_i \in \partial\mathscr{X}^{(L)}, t \in \mathscr{T}\}. \tag{293}$$

Considering the physical interpretation of (288), $\mathscr{R}_\pm^{(L)}(t)$ and $\partial\mathscr{R}^{(L)}(t)$ are the current domains of those material points initially occupying $\mathscr{X}_\pm^{(L)}$ and $\partial\mathscr{X}^{(L)}$, respectively. In this study, we suppose that $\mathscr{R}_-^{(L)}(t)$ and $\mathscr{R}_+^{(L)}(t)$ are domains of *continuity* for the parameters of the fluid and that $\partial\mathscr{R}^{(L)}(t)$ is an interface of *discontinuity* for these parameters. We therefore take $\mathscr{R}_\pm^{(L)}(t) = \mathscr{R}_\pm^{(N)}(t)$, $\partial\mathscr{R}^{(L)}(t) = \partial\mathscr{R}^{(N)}(t)$ and define

$$\mathscr{X}_\pm^{(N)}(t) = \{X_i^{(N)}(\mathbf{r}, t) \mid r_i \in \mathscr{R}_\pm^{(N)}(t), t \in \mathscr{T}\}, \tag{294}$$

$$\partial\mathscr{X}^{(N)}(t) = \{X_i^{(N)}(\mathbf{r}, t) \mid r_i \in \partial\mathscr{R}^{(N)}(t), t \in \mathscr{T}\}. \tag{295}$$

In view of the physical interpretation of (291), $\mathscr{X}_\pm^{(N)}(t)$ and $\partial\mathscr{X}^{(N)}(t)$ are the initial domains of those isopotential points currently occupying $\mathscr{R}_\pm^{(N)}(t)$ and $\partial\mathscr{R}^{(N)}(t)$, respectively. Since $\mathscr{R}_\pm^{(L)}(t) = \mathscr{R}_\pm^{(N)}(t)$ and $\partial\mathscr{R}^{(L)}(t) = \partial\mathscr{R}^{(N)}(t)$, no distinction is required and the symbols $\mathscr{R}_\pm(t)$ and $\partial\mathscr{R}(t)$ are used henceforth.

Next, we give formulations equivalent to (288)–(291) for arbitrary field quantities. Since we wish to allow for the possibility that the values of such fields or their gradients are discontinuous on $\partial\mathscr{R}(t)$, all material, isopotential and local points currently on this interface are excluded. We thus disregard material points for which $X_i \in \partial\mathscr{X}^{(L)}$, isopotential points for which $X_i \in \partial\mathscr{X}^{(N)}(t)$ and local points for which $r_i \in \partial\mathscr{R}(t)$. With this, the generalizations to (288) and (289) are

$$f_{ij\ldots}^{(L)} = f_{ij\ldots}^{(L)}(\mathbf{X}, t), \qquad X_i \in \mathscr{X}_-^{(L)} \cup \mathscr{X}_+^{(L)}, \qquad t \in \mathscr{T}, \tag{296}$$

$$f_{ij\ldots}^{(N)} = f_{ij\ldots}^{(N)}(\mathbf{X}, t), \qquad X_i \in \mathscr{X}_-^{(N)}(t) \cup \mathscr{X}_+^{(N)}(t), \qquad t \in \mathscr{T}. \tag{297}$$

The quantity $f_{ij\ldots}^{(L)}$ in (296) is the current value of an arbitrary field at the material point whose initial position is X_i. Similarly, $f_{ij\ldots}^{(N)}$ in (297) is the current value of that field at the isopotential point whose initial position is X_i. Eq. (296) is commonly referred to as *Lagrangian* formulation of the field. Eq. (297) is non-conventional and here referred to as *Newtonian* formulation. The generalization to (290) and (291) is

$$F_{ij\ldots} = F_{ij\ldots}(\mathbf{r}, t), \qquad r_i \in \mathscr{R}_-(t) \cup \mathscr{R}_+(t), \qquad t \in \mathscr{T}. \tag{298}$$

This equation relates to each local point identified by its position, r_i, the current value, $F_{ij\ldots}$, of an arbitrary field at this point; it is commonly called *Eulerian* formulation of the field. The mappings defined in (296)–(298) are assumed to be single-valued and continuously differentiable with respect to the arguments as many times as required. As in the preceding equations, we use lower case symbols for the Lagrangian and Newtonian formulations of fields and upper case symbols for the Eulerian formulation. The Lagrangian and Newtonian formulations are distinguished by the label superscripts L and N.

A2 Perturbation equations

We assume that the current value of an arbitrary field represents a perturbation of its initial value. Allowing for discontinuities of the field values on $\partial\mathscr{R}(t)$, the Newtonian and Eulerian formulations of the perturbation equation are then straightforward only for isopotential and local points that are initially in $\mathscr{R}_-(0)$ and currently in $\mathscr{R}_-(t)$ or that are initially in $\mathscr{R}_+(0)$ and currently in $\mathscr{R}_+(t)$. We call such points *strictly* internal or external. For conciseness, we define

$$\bar{\mathscr{X}}_\pm^{(\mathrm{N})}(t) = \mathscr{X}_\pm^{(\mathrm{N})}(0) \cap \mathscr{X}_\pm^{(\mathrm{N})}(t), \tag{299}$$

$$\bar{\mathscr{R}}_\pm(t) = \mathscr{R}_\pm(0) \cap \mathscr{R}_\pm(t). \tag{300}$$

On account of these equations, the necessary and sufficient conditions for strictly internal or external isopotential points and for strictly internal or external local points, respectively, are therefore $X_i \in \bar{\mathscr{X}}_-^{(\mathrm{N})}(t) \cup \bar{\mathscr{X}}_+^{(\mathrm{N})}(t)$ and $r_i \in \bar{\mathscr{R}}_-(t) \cup \bar{\mathscr{R}}_+(t)$. Using this, the Lagrangian, Newtonian and Eulerian forms of the perturbation equation can be written as follows:

$$f_{ij\cdots}^{(\mathrm{L})}(\mathbf{X},t) = f_{ij\cdots}^{(\mathrm{L})}(\mathbf{X},0) + \delta f_{ij\cdots}^{(\mathrm{L})}(\mathbf{X},t), \qquad X_i \in \mathscr{X}_-^{(\mathrm{L})} \cup \mathscr{X}_+^{(\mathrm{L})}, \qquad t \in \mathscr{T}, \tag{301}$$

$$f_{ij\cdots}^{(\mathrm{N})}(\mathbf{X},t) = f_{ij\cdots}^{(\mathrm{N})}(\mathbf{X},0) + \partial f_{ij\cdots}^{(\mathrm{N})}(\mathbf{X},t), \qquad X_i \in \bar{\mathscr{X}}_-^{(\mathrm{N})}(t) \cup \bar{\mathscr{X}}_+^{(\mathrm{N})}(t), \qquad t \in \mathscr{T}, \tag{302}$$

$$F_{ij\cdots}(\mathbf{r},t) = F_{ij\cdots}(\mathbf{r},0) + \Delta F_{ij\cdots}(\mathbf{r},t), \qquad r_i \in \bar{\mathscr{R}}_-(t) \cup \bar{\mathscr{R}}_+(t), \qquad t \in \mathscr{T}. \tag{303}$$

We refer to the left-hand sides of the equations as *total* fields, to the first terms on the right-hand sides as *initial* fields and to the second terms on the right-hand sides as *incremental* fields. In particular, $\delta f_{ij\cdots}^{(\mathrm{L})}(\mathbf{X},t)$ is called *material* increment, $\partial f_{ij\cdots}^{(\mathrm{N})}(\mathbf{X},t)$ *isopotential* increment and $\Delta F_{ij\cdots}(\mathbf{r},t)$ *local* increment.

In some neighbourhood of $\partial\mathscr{R}(t)$, isopotential and local points are initially in $\mathscr{R}_-(0)$ and currently in $\mathscr{R}_+(t)$ or vice versa. Since the field values are not necessarily continuous on $\partial\mathscr{R}(t)$, such hybrid points require special consideration. In order that this be avoided, we need the Lagrangian forms of (302) and (303). Using the abbreviation $f_{ij\cdots,k}^{(\mathrm{L})}(\mathbf{X},t)$ for the gradient of $f_{ij\cdots}^{(\mathrm{L})}(\mathbf{X},t)$ with respect to X_k and assuming infinitesimal perturbations, we obtain upon some algebraic manipulations (Wolf 1993, pp. 7–9)

$$f_{ij\cdots}^{(\mathrm{L})}(\mathbf{X},t) = f_{ij\cdots}^{(\mathrm{L})}(\mathbf{X},0) + \partial f_{ij\cdots}^{(\mathrm{L})}(\mathbf{X},t) + f_{ij\ldots,k}^{(\mathrm{L})}(\mathbf{X},0)[u_k(\mathbf{X},t) - d_k^{(\mathrm{L})}(\mathbf{X},t)], \qquad X_i \in \mathscr{X}_-^{(\mathrm{L})} \cup \mathscr{X}_+^{(\mathrm{L})}, \qquad t \in \mathscr{T}, \tag{304}$$

$$f_{ij\cdots}^{(\mathrm{L})}(\mathbf{X},t) = f_{ij\cdots}^{(\mathrm{L})}(\mathbf{X},0) + \Delta f_{ij\cdots}^{(\mathrm{L})}(\mathbf{X},t) + f_{ij\ldots,k}^{(\mathrm{L})}(\mathbf{X},0)u_k(\mathbf{X},t), \qquad X_i \in \mathscr{X}_-^{(\mathrm{L})} \cup \mathscr{X}_+^{(\mathrm{L})}, \qquad t \in \mathscr{T}. \tag{305}$$

For notational convenience, we adopt several simplifications: (i) the arguments X_i, r_i and t are suppressed; (ii) the argument $t=0$ is indicated by the label superscript 0 appended to the function symbols; (iii) the material, isopotential and local increments are indicated by the label superscripts δ, ∂ and Δ appended to the function symbols. With these conventions, the three alternative forms of the Lagrangian perturbation equation, (301), (304) and (305), reduce to

$$\left.\begin{aligned} f_{ij\cdots}^{(\mathrm{L})} &= f_{ij\cdots}^{(\mathrm{L0})} + f_{ij\cdots}^{(\mathrm{L}\delta)} && (306)\\ f_{ij\cdots}^{(\mathrm{L})} &= f_{ij\cdots}^{(\mathrm{L0})} + f_{ij\cdots}^{(\mathrm{L}\partial)} + f_{ij\ldots,k}^{(\mathrm{L0})}(u_k - d_k^{(\mathrm{L})}) && (307)\\ f_{ij\cdots}^{(\mathrm{L})} &= f_{ij\cdots}^{(\mathrm{L0})} + f_{ij\cdots}^{(\mathrm{L}\Delta)} + f_{ij\ldots,k}^{(\mathrm{L0})}u_k && (308) \end{aligned}\right\} \qquad X_i \in \mathscr{X}_-^{(\mathrm{L})} \cup \mathscr{X}_+^{(\mathrm{L})}, \qquad t \in \mathscr{T},$$

whence

$$\left.\begin{aligned} f_{ij\cdots}^{(\mathrm{L}\delta)} &= f_{ij\cdots}^{(\mathrm{L}\partial)} + f_{ij\ldots,k}^{(\mathrm{L0})}(u_k - d_k^{(\mathrm{L})}) && (309)\\ f_{ij\cdots}^{(\mathrm{L}\delta)} &= f_{ij\cdots}^{(\mathrm{L}\Delta)} + f_{ij\ldots,k}^{(\mathrm{L0})}u_k && (310) \end{aligned}\right\} \qquad X_i \in \mathscr{X}_-^{(\mathrm{L})} \cup \mathscr{X}_+^{(\mathrm{L})}, \qquad t \in \mathscr{T}.$$

The second terms on the right-hand sides of (309) and (310) are called *advective* increments. They account for those parts of the increments resulting from the component of the motion of material or isopotential points parallel to the gradient of the initial field. In the present study, only the Lagrangian formulation is employed, allowing us to suppress L.

A3 Interface conditions

We consider the behaviour of field values on $\partial\mathscr{R}$. In order to formulate a condition expressing this behaviour, we locally assign to $\partial\mathscr{R}$ (the Lagrangian form of) the unit normal directed outward into $\mathscr{R}_+$. Denoting this vector by n_i and assuming $\epsilon > 0$, we define

$$\left.\begin{aligned}[f_{ij\cdots}]_{\pm} &= \lim_{\epsilon\to 0} f_{ij\cdots}(\mathbf{X} \pm \epsilon\mathbf{n}^{(0)}) \\ [f_{ij\cdots}]_{-}^{+} &= [f_{ij\cdots}]_{+} - [f_{ij\cdots}]_{-}\end{aligned}\right\} \quad X_i \in \partial\mathscr{X}, \qquad t \in \mathscr{T}. \qquad \begin{aligned}(311)\\(312)\end{aligned}$$

The interface condition for $f_{ij\cdots}$ can then be written as

$$[f_{ij\cdots}]_{-}^{+} = f_{ij\cdots}^{\pm}, \qquad X_i \in \partial\mathscr{X}, \qquad t \in \mathscr{T}, \tag{313}$$

where $f_{ij\cdots}^{\pm}$ is the increase of $f_{ij\cdots}$ in the direction of n_i.

APPENDIX B: LIST OF IMPORTANT SYMBOLS

B1 Latin symbols

$A^{(k)}$	integration coefficient of fundamental solution to (4×4) system
a	radius of planet
$B^{(\ell)}$	integration coefficient of fundamental solution to (2×2) system
d_i	isopotential displacement
e	2.71828 . . .
e_{ij}	strain
F_n	normalized Legendre coefficient of f
$\bar{F}_n$	r-dependent part of F_n
f	scalar field or function
$\tilde{f}$	Laplace transform of f
f_i	non-gravitational force per unit mass
f_n	ordinary Legendre coefficient of f
$f^{(\mathrm{as})}$	axisymmetric Green's function for f
$f^{(\mathrm{ns})}$	non-symmetric Green's function for f
$f_{ij\cdots}$	Cartesian tensor field of arbitrary rank
$f_{ij\ldots,k}$	gradient of $f_{ij\cdots}$ with respect to X_k
$f_{ij\cdots}^{(0)}$	initial value of $f_{ij\cdots}$
$f_{ij\cdots}^{(\Delta)}$	local increment of $f_{ij\cdots}^{(0)}$
$f_{ij\cdots}^{(\delta)}$	material increment of $f_{ij\cdots}^{(0)}$
$f_{ij\cdots}^{(\partial)}$	isopotential increment of $f_{ij\cdots}^{(0)}$
G	Newton's gravitational constant
g_i	gravity (gravitational force per unit mass)
h	isopotential height
k	sequential number of fundamental solution to (4×4) system
k_n	Legendre wave number
ℓ	sequential number of fundamental solution to (2×2) system
M	total number of relaxation modes
m	sequential number of relaxation mode
n	Legendre degree
n_i	outward unit normal on $\partial\mathscr{R}$
O	origin of coordinate system
P_n	Legendre polynomial of the first kind
p	(mechanical) pressure
Q	total number of Maxwell elements
q	sequential number of Maxwell element
R	non-dimensional radial distance of observation point
r	radial distance of observation point
r_i	current position of material point
s	inverse Laplace time
s_{ij}	deviatoric incremental stress
T_n	impulse-response function
$\tilde{T}_n$	transfer function
t	current time
t'	excitation time
t_{ij}	(Cauchy) stress
u_i	material displacement
X_i	initial position of material point
$[Y_i]$	solution vector of (4×4) system
$Y_i^{(k)}$	eigenvector of (4×4) system
$[Z_i]$	solution vector of (2×2) system
$Z_i^{(\ell)}$	eigenvector of (2×2) system

B2 Greek symbols

α'	inverse spectral time
$\alpha^{(q)}$	inverse elemental Maxwell time
$\beta_n^{(m)}$	inverse modal relaxation time
γ	magnitude of $g_i^{(0)}$ on $\partial\mathscr{X}$
δ	Dirac delta function
δ_{ij}	Kronecker symbol
∂_t	partial-derivative operator with respect to t
ϵ_{ijk}	Levi–Civita symbol
ζ_n	non-dimensional Legendre coefficient of σ
θ	colatitude of observation point
θ'	colatitude of excitation point
ϑ	angle between observation and excitation points
$\kappa^{(k)}$	eigenvalue of (4×4) system
λ	longitude of observation point
λ'	longitude of excitation point
$\lambda^{(\ell)}$	eigenvalue of (2×2) system
μ	shear-relaxation function
$\bar{\mu}$	shear-relaxation spectrum
μ_e	elastic shear modulus
$\mu^{(q)}$	elemental elastic shear modulus
π	3.14159 . . .
ρ	volume-mass density
σ	(incremental) interface-mass density
ϕ	gravitational potential
ω_i	rotation

B3 Calligraphic symbols

$\mathscr{E}$	Euclidian space domain
$\mathscr{R}_-$	internal domain of r_i
$\mathscr{R}_+$	external domain of r_i
$\mathscr{S}$	domain of s
$\mathscr{T}$	domain of t
$\mathscr{X}_-$	internal domain of X_i
$\mathscr{X}_+$	external domain of X_i
$\partial\mathscr{R}$	interface between $\mathscr{R}_-$ and $\mathscr{R}_+$
$\partial\mathscr{X}$	interface between $\mathscr{X}_-$ and $\mathscr{X}_+$