# STABLE PLANES WITH LARGE GROUPS OF AUTOMORPHISMS:

# THE INTERPLAY OF INCIDENCE, TOPOLOGY, AND HOMOGENEITY

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## PREFACE

The theory of topological planes (or *stable planes*, to stress the importance of the stability axiom) originates from the foundations of geometry. In fact, a simultaneous axiomatic treatment of the "classical plane geometries" — the euclidean, hyperbolic and elliptic plane — has to combine incidence properties with topological (or ordering) properties as well as some assumptions that nowadays are conveniently stated by means of a group action (distance, or angles, among others). The use of topology instead of an ordering makes it also possible to include, e.g., the complex plane geometries. Of course, the theory will be substantial only if one imposes some conditions on the topologies involved. It turns out that the assumption of locally compactness in combination with connectedness singles out a very manageable class of topological planes. This class includes the planes whose point space is a two-dimensional manifold; i.e., the (topologically) nearest relatives of the classical plane geometries.

Two-dimensional planes with (simply) connected lines were studied systematically from the 1950's on; key papers to this part of the theory are **[31]** and [3]. Turning to the higher dimensional cases, it turns out, however, that connectedness of the lines is not as natural a condition as it seems for two-dimensional planes. While stable planes of dimension 2 and 4 were studied systematically by R. LÖWEN without assuming connectedness of lines, results on higher-dimensional planes were few. A major breakthrough was made by R. LÖWEN's deep result **[20]** on the possible dimensions of point spaces of stable planes; the only possibilities are 2, 4, 8, and 16. These values are attained by the planes over the field $\mathbf{R}$ of real numbers, the field $\mathbf{C}$ of complex numbers, the skew field $\mathbf{H}$ of Hamilton's quaternions, and the alternative division algebra $\mathbf{O}$ of Cayley's octonions, respectively. As a consequence of R. LÖWEN's result, it is possible to proceed in the theory of stable planes by induction on the dimension of the point space. Moreover, there is a "classical reference object" for each stable plane, namely the projective plane over $\mathbf{F} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{O}\}$ that has the same dimension.

The present treatment has two aims. On the one hand, we study stable planes of all possible dimensions with respect to possible actions of locally compact groups. Our main result is that the projective and the affine planes over $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, and $\mathbf{O}$ are the "most homogeneous stable planes" (in a sense to be made precise in **15.5**). On the other hand, we attempt to give an overview of the existing theory. In particular, we have included (without proofs) the results that are needed in the proofs of the new results.

In our overview of the existing theory, important special cases are excluded: We do not discuss the results that have been obtained under additional hypotheses on (simply) connectedness of the point space, or of the lines; see **[31]**, [3], and **[14]**. The elaborate classification of compact connected projective planes with large groups of automorphisms may be found in the key papers **[31]**, **[1]**, **[25]**, **[38]**, and **[37]**. This theory will find a comprehensive treatment in a monograph **[39]**. Finally, we do not give an account of the theory of symmetric planes, cf. **15.6**. This fascinating combination of the theories of stable planes and of symmetric spaces has proved to be very fruitful for the study of low dimensional planes, and looks promising for planes of dimension 8 and 16.

NOTATION. If $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ is one of the classical division algebras, we write $\mathrm{P}_2\mathbb{F}$ for the projective plane, and $\mathrm{A}_2\mathbb{F}$ for the affine plane over $\mathbb{F}$.

For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, we denote by $\mathrm{GL}_n\mathbb{F}$ the group of invertible $n \times n$ matrices over $\mathbb{F}$. For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, we write $\mathrm{O}_n\mathbb{F}$ for the group of orthogonal $n \times n$ matrices with respect to the bilinear form that is given by $f(u,v) = uv'$. If $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$, then $\mathrm{U}_n\mathbb{F}$ denotes the group of unitary $n \times n$ matrices over $\mathbb{F}$, with respect to the hermitian form that is given by $f(u,v) = u\bar{v}'$. More generally, we write $\mathrm{U}(J)$ for the group of all unitary $n \times n$ matrices, where the form is given by $f(u,v) = uJ\bar{v}'$ for a $n \times n$-matrix $J$. If $J$ is the diagonal matrix $(1)^{n-r} \times (-1)^r$, we also write $\mathrm{O}_n\mathbb{R}(r)$, $\mathrm{U}_n\mathbb{C}(r)$, or $\mathrm{U}_n\mathbb{H}(r)$ for the unitary groups with respect to the forms defined by $f(u,v) = uJ\bar{v}'$. The prefixes S and P denote the subgroup consisting of elements of determinant 1, and the factor group modulo the center, respectively. Note that in the case of the skew field $\mathbb{H}$ we have to use J. DIEUDONNÉ's determinant function, which takes its non-zero values in the commutator factor group of the multiplicative group of $\mathbb{H}$. Via polar coordinates in $\mathbb{H}$, however, this factor group may be identified with the multiplicative group of positive real numbers.

Group actions will always be from the right, and we shall use exponential notation. The stabilizer of $x$ in $\Gamma$ is denoted by $\Gamma_x$. If $X$ is a subset of the set that $\Gamma$ acts on, the pointwise stabilizer of $X$ is denoted by $\Gamma_{[X]}$. Centralizer and normalizer of a subgroup $\Delta$ in $\Gamma$ are denoted by $\mathrm{C}_\Gamma\Delta$ and $\mathrm{N}_\Gamma\Delta$, respectively. If $\Gamma$ is a topological group, we denote its connected component by $\Gamma^1$. If $\Gamma$ is locally compact and almost simple, we say that $\Gamma$ is of type $\mathsf{X}$, if the factor group $\Gamma/\mathrm{Z}$ modulo the center $\mathrm{Z}$ of $\Gamma$ is the simple Lie group of type $\mathsf{X}$, cf. **A6.2**. E.g., a group $\Gamma$ of type $\mathsf{A}_{n-1}^{\mathbb{R}}$ satisfies $\Gamma/\mathrm{Z} \cong \mathrm{PSL}_n\mathbb{R}$. We use the notation of J. TITS' tables **[55]** for the simple Lie algebras.

A reader who is not familiar with the basic structure theory of locally compact connected groups may find it useful to have a look at the Appendix before turning to the applications to geometry. In particular, the Appendix serves as a justification for the use of notions (and the intuition!) from the theory of Lie groups. Note that the Appendix has a separate bibliography ("References", numbered by **R1**...).

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## TABLE OF CONTENTS

CHAPTER I

# INTRODUCTION

The first chapter introduces basic notions. We quote fundamental results (some of them quite deep) from the literature. Most of the material is known, and included for the sake of easy reference.

## 1. Generalities

In this section, we define stable planes and their subplanes, and collect some of their fundamental properties. The main sources for the material are R. LÖWEN's fundamental papers [14], [20]. Inspiration has also been drawn from H. SALZMANN's papers about geometries on surfaces, and about compact projective planes.

**1.1** DEFINITION. Let $P$ and $\mathcal{L}$ be sets, and $I \subseteq P \times \mathcal{L}$. Then $(P, \mathcal{L}, I)$ is called an incidence structure. For $(p, L) \in I$, we shall say that $p$ and $L$ are incident, or that $p$ lies on $L$, or that $L$ passes through $p$. An incidence structure $(P, \mathcal{L}, I)$ is called a *linear space* (with point set $P$ and line set $\mathcal{L}$) if the following axioms (J) and (Q) are satisfied.

- (J) For any two elements $p, q$ of $P$, there exists exactly one line $L \in \mathcal{L}$ that is incident with both of them.
- (Q) Every line $L \in \mathcal{L}$ is incident with at least two points, and there exist four points such that no three of them are incident with one line.

**1.2** NOTATION. The line $L$ in (J) will be denoted by $p \vee q$, or $pq$ for short. Axiom (J) implies that for any two lines $K, L \in \mathcal{L}$, there exists at most one point that is incident with both of them. If $p$ is such a point, we say that $K$ and $L$ are *intersecting*, and write $K \wedge L := p$. The set of all pairs of intersecting lines is denoted by $\mathcal{D}_\wedge$, and the set of pairs of different points is denoted by $\mathcal{D}_\vee$. We obtain mappings

$$\vee : \mathcal{D}_\vee \to \mathcal{L} : (p, q) \mapsto pq$$
$$\wedge : \mathcal{D}_\wedge \to P : (K, L) \mapsto K \wedge L ,$$

the *geometric operations* (joining points and intersecting lines).

If $X$ and $Y$ are disjoint subsets of $P$, we write $XY := \{xy;\ x \in X, y \in Y\}$. If $\mathcal{X}$ and $\mathcal{Y}$ are subsets of $\mathcal{L}$ such that $\mathcal{X} \times \mathcal{Y} \subseteq \mathcal{D}_\wedge$, we write

$$\mathcal{X} \wedge \mathcal{Y} := \{X \wedge Y;\ X \in \mathcal{X}, Y \in \mathcal{Y}\} .$$

More general, we write $Z^\vee := \{pq;\ (p, q) \in Z\}$ for every subset $Z$ of $\mathcal{D}_\vee$, and $\mathcal{Z}^\wedge := \{K \wedge L;\ (K, L) \in \mathcal{Z}\}$ for every subset $\mathcal{Z}$ of $\mathcal{D}_\wedge$.

For every subset $X$ of $P$, we denote by $\mathcal{L}_X$ the set of all lines that are incident with at least one point in $X$. For every point $p \in P$, the set $\mathcal{L}_p := \mathcal{L}_{\{p\}}$ is called the *pencil* in $p$.

Quite often, we shall tacitly identify each line with the set of points that are incident with it. Thus "$\in$" denotes incidence.

Examples of linear spaces exist in abundance. In particular, every affine or projective space is a linear space. We shall study linear spaces that have nice topological properties; namely, we require that the geometric operations are continuous with respect to locally compact finite-dimensional topologies on the sets of points and of lines. By the topological dimension $\dim X$ of a space $X$, we mean its small inductive dimension, cf. the Appendix. To be precise:

**1.3** DEFINITION. A *stable plane* is a linear space $\mathbf{M} = (M, \mathcal{M})$, where the point space $M$ and the line space $\mathcal{M}$ are endowed with locally compact topologies such that

- The mappings $\vee$ (joining points) and $\wedge$ (intersecting lines) are continuous.
- The set $\mathcal{D}_\wedge$ of pairs of intersecting lines is open in $\mathcal{M} \times \mathcal{M}$ (axiom of stability).
- The point space $M$ has positive and finite (topological) dimension.

Examples of stable planes are the well known projective and affine planes over the locally compact connected (skew) fields $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$; their point spaces have dimension 2, 4, and 8, respectively. Possibly less well known are the planes over Cayley's octonions $\mathbb{O}$, these planes provide examples where the point space has dimension 16.

A wealth of examples is provided by the class of compact connected projective planes of finite dimension, and their open subplanes; see the definition of subplane below. Historically, the notion of stable plane originates from "line systems in the plane", as studied by L.A. SKORNJAKOV, H. SALZMANN, and others. These line systems were later also called "Salzmann planes" and "$\mathbb{R}^2$-planes". Note also that the first examples of non-desarguesian geometries were obtained as line systems in $\mathbb{R}^2$ by E. BELTRAMI, F. KLEIN, and D. HILBERT. Another early example of the sort is F.R. MOULTON's plane.

General information about stable planes can be found in the work of R. LÖWEN; in particular, see [14], [17], [20].

**1.4** THEOREM.
*In every stable plane $\mathbf{M} = (M, \mathcal{M})$, one has the following topological properties.*

- **a.** *The spaces $M$, $\mathcal{M}$ are separable metric spaces.*
- **b.** *For every line $L$ and every point $q \notin L$, the restriction of $\vee$ to $L \times \{q\}$ yields an open embedding of $L$ into $\mathcal{M}_q$. In particular, the spaces $L$ and $\mathcal{M}_p$ are locally homeomorphic for every $L \in \mathcal{M}$ and every $p \in P$.*
- **c.** *The spaces $M$, $\mathcal{M}$, $L \times L$ and $\mathcal{M}_p \times \mathcal{M}_p$ are locally homeomorphic.*
- **d.** *The spaces $\mathcal{M}_p$ and $\mathcal{M}$ are globally and locally arcwise connected, and the spaces $M$ and $L$ are locally arcwise connected.*

PROOF. Assertions **a**–**c** have been proved by R. LÖWEN in [14, 1.9, 1.2, 1.4]. In order to prove assertion **d**, we first remark that every line $L$ is locally arcwise connected [14, 1.12]. From **b** and **c** we infer that $M$, $\mathcal{M}$ and $\mathcal{M}_p$ are locally arcwise connected as well. Since each pencil is connected [14, 1.14], we infer that $\mathcal{M}_p$ is arcwise connected. For any two lines $K, L$ in $\mathcal{M}$, we choose points $p \in K$ and $q \in L$ such that $p \neq q$. Now there exist arcs from $K$ to $pq$ in $\mathcal{M}_p$ and from $pq$ to $L$ in $\mathcal{M}_q$. □

As an immediate consequence of **1.4d**, we obtain that *projective* stable planes (where $\mathcal{D}_\wedge$ consists of all pairs of different lines) are connected; just interchange the rôles of points and lines.

There are competing notions of dimension for topological spaces in general. However, from **1.4a** we infer that the most commonly used dimension functions (e.g., small and large inductive dimension, or covering dimension) coincide on these spaces, cf. [**10**]. The same applies to $\mathcal{M}$, to each line $L$ (considered as a subset of $M$), and to each pencil $\mathcal{M}_x$. If one of these spaces is a topological manifold, i.e., locally homeomorphic to $\mathbb{R}^n$ for some $n$, then its topological dimension equals $n$. An important feature of small inductive dimension is that it is locally defined and monotone (even if applied to subspaces that are not closed). This will be important if we study subplanes. The equality $\dim M = \dim \mathcal{M}$ allows us to define $\dim \mathbb{M} := \dim M$ for every stable plane $\mathbb{M} = (M, \mathcal{M})$.

For our purposes, the following deep results of R. LÖWEN [**20**] are fundamental.

**1.5** DIMENSION THEOREM.

- **a.** *The only possible values for* $\dim M$ *are the integers* 2, 4, 8, *and* 16.
- **b.** *For each point* $p \in M$, *the line pencil* $\mathcal{M}_p = \{L;\ p \in L \in \mathcal{M}\}$ *is a compact connected homotopy l-sphere, where* $l = \dim \mathcal{M}_p = \dim L = \frac{1}{2} \dim M$.
- **c.** *A closed subset* $Y$ *of* $X \in \{M, \mathcal{M}\} \cup \mathcal{M} \cup \{\mathcal{M}_p;\ p \in M\}$ *has nonempty interior if, and only if,* $\dim Y = \dim X$.
- **d.** *Every connected open subset* $Y$ *of* $X \in \{M, \mathcal{M}\} \cup \mathcal{M} \cup \{\mathcal{M}_p;\ p \in M\}$ *is a Cantor manifold; i.e.,* $Y \setminus Z$ *is connected for every closed subspace* $Z$ *of* $Y$ *with* $\dim Z < \dim X - 1$.

The restriction on the dimension of the point space suggests an inductive treatment. For this purpose, we need suitable notions of *subplanes.*

**1.6** DEFINITIONS. Let $\mathbb{M} = (M, \mathcal{M})$ be a linear space. For any subset $D$ of $M$, let $\mathcal{M}|_D$ be the set of lines that are incident with at least two points of $D$. Then $\mathbb{D} = (D, \mathcal{M}|_D)$ is called *the geometry induced on* $D$. If $D$ contains a quadrangle (i.e., four points such that no three of them are collinear), then $\mathbb{D}$ is called a *subplane* of $\mathbb{M}$. A subplane $\mathbb{D} = (D, \mathcal{M}|_D)$ is called *full* (*in* $\mathbb{M}$), if $D$ contains each point of $M$ that is the intersection of two lines of $\mathcal{M}|_D$. A subplane of a stable plane is called *open, closed, d-dimensional* etc., if its point space has the property in question.

There are two important cases where a subplane is a stable plane again: if the subplane is open, or if it is a closed full subplane of positive dimension. Passing to open subplanes will help to reduce the number of cases in proofs; e.g., one often deletes a closed subset of points with special properties. The open subplanes of compact connected projective planes form a large class of examples of stable planes. There are, however, stable planes that do not admit open embeddings into projective planes (see [**53**] for an easy example).

Note that, for each full subplane $\mathbb{D} = (D, \mathcal{D})$, the geometry induced on the closure of $D$ is a closed full subplane. If a subset $X$ of $M$ contains a quadrangle, then there is a smallest closed full subplane $\langle X \rangle$ of $M$, called *the subplane generated by* $X$.

We have the following maximum property for full closed subplanes of positive dimension:

**1.7** LEMMA. *Let* $\mathbb{E} = (E, \mathcal{E})$ *be a full closed subplane of* $\mathbb{M}$ *with* $\dim E > 0$. *If* $E \subset F$ *and* $\mathbb{F} = (F, \mathcal{F})$ *is a closed subplane, then* $\dim E < \dim F$.

PROOF. Assume that $\dim E = \dim F$. According to [20, Th. 11c)], $E$ has nonempty interior $E^\circ$ in $F$. For any point $z \in F$, choose points $x, y \in E^\circ$ such that $y \notin xz$. Each of the lines $xz$, $yz$ meets the open set $E^\circ$ in more than one point. Thus $xz$ and $yz$ belong to $\mathcal{E}$ and hence $z \in E$, since $\mathbb{E}$ is full. □

Of particular interest are proper full closed subplanes that have the maximal possible dimension.

**1.8** DEFINITION. A closed full subplane $\mathbb{B} = (B, \mathcal{B})$ of $\mathbb{M}$ is called a *Baer subplane*, if $\dim B = \frac{1}{2} \dim M$.

In the theory of projective planes, the notion of a Baer subplane is defined already; there it means a subplane with the property that every point of the larger plane lies on a line of the subplane. In order to prove that this notion is consistent with **1.8**, we need two lemmas. The first of them will also be fundamental for the study of straight actions.

**1.9** RESULT. [15, 1.1] *Let* $\mathbb{M} = (M, \mathcal{M})$ *be a stable plane, and assume that* $\mathcal{L}$ *is a subset of* $\mathcal{M}$ *such that each point* $x \in M$ *is incident with exactly one member* $L_x$ *of* $\mathcal{L}$. *Then the following hold.*

- **a.** *The line* $L_x$ *depends continuously on* $x$. *In particular, connectedness of* $M$ *implies that* $\mathcal{L}$ *is connected.*
- **b.** *The set* $\mathcal{L}$ *is closed in* $\mathcal{M}$, *and locally homeomorphic to a line in* $\mathcal{M}$. *In particular,* $\mathcal{L}$ *is a locally compact separable metric space, and* $\dim \mathcal{L} = \frac{1}{2} \dim M$.

Mutatis mutandis, the assertions of **1.9** about $\mathcal{L}$ still hold true if there only exists a neighborhood $U$ in $M$ such that every point of $U$ lies on a member of $\mathcal{L}$, as one easily sees after passing to the stable plane that is induced on the interior of $U$.

**1.10** LEMMA. *Let* $\mathbb{M} = (M, \mathcal{M})$ *be a stable plane, and assume that* $\mathbb{B} = (B, \mathcal{B})$ *is a full closed subplane such that* $\dim B = \frac{1}{2} \dim M$. *Then there exist arbitrarily small compact neighborhoods in* $B$ *that are homeomorphic to neighborhoods in a pencil* $\mathcal{M}_x$.

PROOF. Choose a compact neighborhood $V$ in $B$ and a point $x \in M \setminus B$. The mapping $\pi: V \to \mathcal{M}_x : b \mapsto bx$ is continuous and closed (since $V$ is compact). Since $\mathbb{B}$ is full, there is at most one line in $\mathcal{B} \cap \mathcal{M}_x$, and we may choose $V$ such that $\pi$ is injective. Then $V$ is homeomorphic to $\mathcal{W} = V^\pi$, hence $\dim \mathcal{W} = \dim \mathcal{M}_x$, and $\mathcal{W}$ is a neighborhood of some line in $\mathcal{M}_x$ by **1.5c**. □

Note that $B$ and $\mathcal{M}_x$ are in fact locally homeomorphic, since the spaces $M$, $\mathcal{M}$, $L$, and $\mathcal{M}_x$ have the domain invariance property [20, 11b)].

**1.11** COROLLARY. *Let $\mathbf{M} = (M, \mathcal{M})$ be a stable plane. If there is a Baer subplane $\mathbf{B} = (B, \mathcal{B})$ such that $B$ is a (topological) manifold, then each line pencil of $\mathbf{M}$ (and hence each line and the point space $M$) is a manifold.*

**1.12** THEOREM. *Let $\mathbf{M} = (M, \mathcal{M})$ be a stable plane, and assume that $\mathbf{B} = (B, \mathcal{B})$ is a proper full closed subplane. Then $\dim B = \frac{1}{2} \dim M$ if, and only if, the set of points that are incident with an element of $\mathcal{B}$ has nonempty interior in $M$.*

PROOF. If $X$ is a nonempty open set of points that are incident with an element of $\mathcal{B}$, then $X \setminus B$ is a nonempty open subset of $M$. Since $\mathbf{B}$ is a full subplane, every point of $X \setminus B$ is incident with exactly one element of $\mathcal{B}$. Applying **1.9b** to the stable plane $(X \setminus B, \mathcal{M}|_{X \setminus B})$, we obtain that $\dim \mathcal{B} = \frac{1}{2} \dim M$.

Now assume that $\dim B = \frac{1}{2} \dim M$. Let $K$ be a line in $\mathcal{B}$, and choose a point $x \in K \setminus B$ and a line $L \in \mathcal{M}_x \setminus \{K\}$. Since $\mathbf{B}$ is full, we know that $L \notin \mathcal{B}$. By stability, we find compact neighborhoods $\mathcal{U}$ of $K$ in $\mathcal{B}$ and $\mathcal{V}$ of $L$ in $\mathcal{M}_x$ such that $\mathcal{U} \times \mathcal{V} \subseteq \mathcal{D}_\wedge$. If we choose these neighborhoods small enough, we obtain in addition that $\mathcal{U} \cap \mathcal{V} \subseteq M \setminus B$ and that $\mathcal{U}$ is homeomorphic to a neighborhood in $\mathcal{M}_x$. In particular, we have that $\dim \mathcal{U} \times \mathcal{V} = \dim \mathcal{M}_x \times \mathcal{M}_x = \dim M$.

We claim that the restriction $\wedge|_{\mathcal{U} \times \mathcal{V}}$ is an injection. Assume that there exist $K_i \in \mathcal{U}$ and $L_i \in \mathcal{V}$ such that $K_1 \wedge L_1 = K_2 \wedge L_2$. Since $\mathcal{U} \wedge \mathcal{V}$ contains no point of the full subplane $\mathbf{B}$, we infer that $K_1 = K_2$. The equality $L_1 = L_2$ follows from the fact that $L_i \in \mathcal{M}_x$. We conclude that $\mathcal{U} \wedge \mathcal{V}$ is a homeomorphic image of the compact space $\mathcal{U} \times \mathcal{V}$. In particular, $\dim(\mathcal{U} \wedge \mathcal{V}) = \dim M$. According to **1.5c**, the set $\mathcal{U} \wedge \mathcal{V}$ has nonempty interior in $M$, and the theorem is proved. □

## 2. Morphisms

In this section, we introduce a category of stable planes and derive basic results on morphisms in this category. The main source is [46], cf. also [50]. The fundamental theorem on $\operatorname{Aut} \mathbf{M}$ is taken from [14, 2.9].

**2.1** DEFINITION. Let $\mathbf{E} = (E, \mathcal{E})$ and $\mathbf{F} = (F, \mathcal{F})$ be linear spaces. A mapping $\pi: E \to F$ is called a *lineation* from $\mathbf{E}$ to $\mathbf{F}$ if for each line $L \in \mathcal{E}$ there exists a line $L' \in \mathcal{F}$ (not necessarily unique) such that $L^\pi \subseteq L'$. A lineation $\pi$ is called *collapsed* if the image $E^\pi$ is contained in some line $H \in \mathcal{F}$.

Including a mapping of lines in the definition would be redundant for injective lineations.

**2.2** RESULT. [46, 3] *Let $\pi: E \to F$ be an injective lineation from a stable plane $(E, \mathcal{E})$ to a stable plane $(F, \mathcal{F})$.*

- **a.** *There exists a unique mapping $\lambda: \mathcal{E} \to \mathcal{F}$ such that $L^\pi \subseteq L^\lambda$ for each $L \in \mathcal{E}$.*
- **b.** *If $\pi$ is noncollapsed, then $\lambda$ is injective.*
- **c.** *If $\pi$ is continuous, then $\lambda$ is continuous.*

**2.3** RESULT. [46, 6] *If a continuous lineation $\pi: E \to F$ from a stable plane $(E, \mathcal{E})$ to a stable plane $(F, \mathcal{F})$ is not injective, then either $\pi$ is collapsed or $\pi$ is locally constant.*

The assertion of **2.3** has been proved in [46] for stable planes in a wider sense: The point space is only required to be nondiscrete, but need neither be locally compact nor of positive or finite dimension. For stable planes in the sense of **1.3**, the converse of **2.3** is also true.

**2.4** THEOREM. *If a continuous lineation $\pi: E \to F$ from a stable plane $(E, \mathcal{E})$ to a stable plane $(F, \mathcal{F})$ with $\dim E = \dim F$ is injective, then $\pi$ is neither collapsed nor locally constant.*

PROOF. Let $U$ be a compact neighborhood in $E$. Since $\pi$ is injective, the restriction $\pi|_U$ is a homeomorphism onto $U^\pi$. Consequently, $\dim U^\pi = \dim E = \dim F$, and $U$ cannot be contained in a line of $\mathbf{F}$. □

If $E$ is connected (in particular, if $\mathbf{E}$ is projective), then every noncollapsed lineation is injective. It is easy to produce examples of noncollapsed locally constant lineations of stable planes that are not connected.

**2.5** EXAMPLE. *For any stable plane $(E, \mathcal{E})$ and any triangle $x, y, z$ in $E$ there are neighborhoods $X, Y, Z$ of $x, y, z$, respectively, such that $XY, XZ, YZ$ are mutually disjoint neighborhoods of the lines $xy, xz, yz$ in $\mathcal{E}$, respectively. Hence the mapping*

$$\pi: X \cup Y \cup Z \to X \cup Y \cup Z: p \mapsto \begin{cases} x & \text{if } p \in X \\ y & \text{if } p \in Y \\ z & \text{if } p \in Z \end{cases}$$

*is a locally constant lineation of the stable plane that is induced on the point set $X \cup Y \cup Z$. The same procedure applies if we replace the triangle by any discrete subset $S$ of $E$ with the property that no line meets $S$ in more than two points.*

Theorem **2.3** and the examples motivate the introduction of a category **StP** of stable planes, whose morphisms are continuous injective noncollapsed lineations. In contrast, for a category of projective planes one would take as morphisms those lineations whose image contains a quadrangle (and hence is a projective plane).

Using the fact that $M$ has the domain invariance property [**20**, 11b)], one obtains that images of morphisms in **StP** are stable planes.

**2.6** RESULT. [**46**, 8] *Let $\mathbf{M} = (M, \mathcal{M})$ be a stable plane. Then each endomorphism $\pi$ of $\mathbf{M}$ is an open mapping. Moreover, $\pi$ is an isomorphism from $(M, \mathcal{M})$ onto $(M^\pi, \mathcal{M}|_{M^\pi})$.*

As a consequence of **2.6**, we have that each semigroup of endomorphisms of a stable plane is "almost a group":

**2.7** RESULT. [**46**, 10] *Assume that $\mathbf{M} = (M, \mathcal{M})$ is a stable plane, and let $\Sigma$ be a set of endomorphisms of $\mathbf{M}$, endowed with the compact-open topology derived from the action on $M$. If $\Sigma$ is closed under composition, then $\Sigma$ is a cancellative topological semigroup; i.e., for $\pi, \alpha, \beta \in \Sigma$, the implications $\pi\alpha = \pi\beta \Rightarrow \alpha = \beta$ and $\alpha\pi = \beta\pi \Rightarrow \alpha = \beta$ hold.*

The problem whether or not a cancellative nonabelian semigroup is embeddable in some group is a delicate one. Interesting results have been obtained for the case that the semigroup is locally euclidean, or even analytic. In particular, there are examples of analytic semigroups that are not embeddable in groups. See [**9**] or [**12**] for the state of the art.

**2.8** REMARK. The existence of proper endomorphisms sharply distinguishes the class of stable planes from the subclass of compact connected projective planes, cf. [**46**, 9]. In fact, there exist stable planes whose endomorphisms form a semigroup that is substantially bigger than the group of automorphisms, cf. [**46**, 14, 15]. There arises the interesting question whether or not it is possible to describe certain stable planes from a semigroup of endomorphisms, while the corresponding group of automorphisms is too small to determine the plane. However, this question will not be pursued here.

Another consequence of **2.3** and **2.6** is the fact that every continuous *surjective* lineation of stable planes is an isomorphism.

**2.9** RESULT. [**14**, 2.3, 2.9] *Endowed with the compact-open topology derived from the action on $M$, the group* $\operatorname{Aut}\mathbb{M}$ *of all automorphisms (i.e., continuous surjective lineations) of* $\mathbb{M}$ *is a locally compact separable transformation group both on $M$ and $\mathcal{M}$.*

From the fact that every compact subgroup of $\operatorname{Aut}\mathbb{M}$ has finite dimension [**51**], we infer that $\operatorname{Aut}\mathbb{M}$ has finite dimension. Thus $\operatorname{Aut}\mathbb{M}$ is "almost a Lie group", cf. the Appendix.

We close this section with the assertion that endomorphisms respect the process of generating subplanes.

**2.10** LEMMA. *Let $\mathbb{M} = (M, \mathcal{M})$ be a stable plane, and assume that $X \subseteq M$ contains a quadrangle. Then $\langle X \rangle$ is invariant under each endomorphism $\alpha$ of* $\mathbb{M}$ *that leaves $X$ invariant, and $\alpha$ acts trivially on $\langle X \rangle$ if the action of $\alpha$ on $X$ is trivial.*

PROOF. This follows from the fact that it is possible to describe $\langle X \rangle$ "from below"; see [**43**, 3.1]. □

## 3. Actions

In this section, we define actions of topological groups on stable planes. Thus we generalize the notion of "group of automorphisms". Restrictions of actions to invariant subplanes or to interesting subgroups (e.g., Lie subgroups) become more easily tractable.

**3.1** DEFINITION. By an *action* of a topological group $\Delta$ on $\mathbb{M}$ we mean a continuous group homomorphism $\alpha: \Delta \to \operatorname{Aut}\mathbb{M}$. If there is no danger of confusion, we just say that $\Delta$ acts on $\mathbb{M}$. An injective action will be called *effective*, and an action with totally disconnected kernel will be called *almost effective*.

If $\Delta$ acts effectively on $\mathbb{M}$, we shall also call $\Delta$ a *group of automorphisms of* $\mathbb{M}$, although the topology of $\Delta$ may be finer than that of $\Delta^\alpha$.

In view of the fact that $\operatorname{Aut}\mathbb{M}$ is locally compact (cf. **2.9**), actions of locally compact groups are of particular interest. Exactly the closed subgroups of $\operatorname{Aut}\mathbb{M}$ are locally compact (with respect to the induced topology). However, there may be continuous injective group homomorphisms from locally compact groups into $\operatorname{Aut}\mathbb{M}$ such that the image is not a closed subgroup[1]. In particular, such a situation might

[1] E.g., the mapping $t \mapsto \begin{pmatrix} e^{it} & \\ & e^{i\pi t} \end{pmatrix}$ is a continuous injective group homomorphism from $\mathbb{R}$ to the group $\mathrm{GL}_2\mathbb{C}$ with nonclosed image.

occur if we restrict a given action to some invariant subplane; see the next section for details.

Apart from the fact that we lose certain compactness criteria, there do not arise substantial additional problems from the change of the topology that occurs with actions whose image is not closed in Aut **M**. By arguments similar to those in [**43**, 3.2], even results that use compactness criteria may be transferred. Of course, any (continuous) action of a compact group has closed image in Aut **M**.

The structure theory of locally compact groups is well developed, see the Appendix. The following observation will be fundamental for our study of "large" groups of automorphisms, since it provides a "logarithmic" analog for counting arguments, as used in the study of finite geometries.

**3.2** RESULT. [8] *cf.* **A1.13**. *Let $\Delta$ be a topological transformation group on a separable metric space of finite dimension, and assume that there exists a countable covering of $\Delta$ by relatively compact open subsets. Then the dimensions of orbits and stabilizers are related in the following way:*

$$\dim x^\Delta = \dim \Delta/(\Delta_x) \text{ and } \dim \Delta = \dim x^\Delta + \dim \Delta_x.$$

*In general, the orbit $x^\Delta$ and the coset space $\Delta/(\Delta_x)$ are not homeomorphic. If $x^\Delta$ is locally compact (in particular, if $\Delta$ is compact), then the canonical bijection is a homeomorphism.*

Recall from **1.4** that the spaces $M$, $\mathcal{M}$, $L$ and $\mathcal{M}_p$ are separable metric for every stable plane $\mathbf{M} = (M, \mathcal{M})$ and every $L \in \mathcal{M}$, $p \in M$. If $\mathcal{L}$ is a subset of $\mathcal{M}$ such that every point of some neighborhood in $M$ lies on a unique member of $\mathcal{L}$, then $\mathcal{L}$ is separable metric as well, cf. **1.9**.

A locally compact group $\Delta$ has a countable covering by relatively compact open subsets if, and omly if, there exists an open subgroup $\Omega$ such that $\Omega/(\Delta^1)$ modulo the connected component $\Delta^1$ of $\Delta$ is compact and $\Delta/\Omega$ is countable. In particular, this is the case if $\Delta$ is connected, or if $\Delta/(\Delta^1)$ is countable, or if $\Delta/(\Delta^1)$ is compact.

**3.3** DEFINITIONS. Assume that the group $\Delta$ acts on a stable plane $(M, \mathcal{M})$.

- **a.** The (action of the) group $\Delta$ is called *planar*, if the set $\operatorname{Fix}\Delta$ of fixed points has positive dimension and contains a quadrangle.
- **b.** If each orbit $x^\Delta$ is contained in a line $L_x$, then (the action of) $\Delta$ is called *straight*. In this situation, the line $L_x$ is uniquely determined for each point $x$ that is moved by $\Delta$, and we shall write $\mathcal{M}_\Delta = \{L_x; x \in M \setminus \operatorname{Fix}\Delta\}$.
- **c.** If $\Delta$ acts trivially on $\mathcal{M}_z$ for some point $z \in M$, then $z$ is called the *center* of $\Delta$.
- **d.** If $\Delta$ acts trivially on some line $L \in \mathcal{M}$, then $L$ is called an *axis* of $\Delta$.
- **e.** If $\Delta$ acts trivially on some nonempty open subset $U$ of some line $L \in \mathcal{M}$, then $L_U := L$ is called a *semi-axis* of $\Delta$.

**3.4** REMARK. Straight actions have been called "quasi-perspective" in the literature. However, since the terms "perspectivity" and "projectivity" are nowadays reserved for mappings between lines (or pencils) that are obtained by (repeated) "projecting", the term "quasi-perspectivity" should no longer be used to denote a special type of collineation.

If $\Delta$ acts straightly and nontrivially, then every point of the open set $M \setminus \operatorname{Fix}\Delta$ lies on a member of $\mathcal{M}_\Delta$. According to **1.9b**, the set $\mathcal{M}_\Delta$ is locally homeomorphic to a line in $\mathcal{M}$, and $\dim \mathcal{M}_\Delta = \frac{1}{2}\dim M$.

It is easy to see that the geometry induced on the set of fixed points of a planar group is a full closed subplane $\mathbf{F} = (F, \mathcal{F})$, where $\dim F > 0$. In fact, planar groups are just the kernels of restrictions to (closed full) subplanes of positive dimension.

**3.5** REMARKS. It is not known whether or not the stabilizer of a quadrangle $Q$ in a stable plane is a planar group; i.e., whether or not the subplane $\langle Q \rangle$ has positive dimension. Even in the case of projective planes (of dimension 8 or 16), it is conceivable that $\dim\langle Q\rangle = 0$, cf. [**34**]. Groups that act trivial on some *nondiscrete* subplane were called *semi-planar* in [**43**].

It is easy to construct examples of (nonprojective) stable planes with a quadrangle $Q$ that generates a *finite* subplane; one has to remove some of the intersection points of lines that occur during the process of generating $\langle Q \rangle$.

In Chapter III, we shall heavily use information about planar groups, while subplanes of dimension 0 do not occur at all. This is due to the fact that planar groups occur quite naturally, if one studies centralizers.

**3.6** LEMMA. *Let $\Delta$ be a connected nontrivial group of automorphisms of a stable plane $\mathbf{M} = (M, \mathcal{M})$.*

- **a.** *If there is a point $x \in M$ such that the orbit $x^\Delta$ is not contained in any line, then $x^\Delta$ generates a subplane $\mathbb{E} = (E, \mathcal{E})$, where $\dim E \geq \dim x^\Delta > 0$. For each subgroup $\Psi$ of $\operatorname{Aut}\mathbb{M}$ that commutes with $\Delta$, the stabilizer $\Psi_x$ acts trivially on $\mathbb{E}$.*
- **b.** *If there is no such orbit, then $\Delta$ acts straightly. In this case, for any two points $x \in M \setminus \operatorname{Fix}\Delta$ and $y \in M \setminus (\operatorname{Fix}\Delta \cup L_x)$, the set $x^\Delta \cup y^\Delta$ generates a subplane $\mathbf{F} = (F, \mathcal{F})$, where $\dim F \geq 2\dim x^\Delta > 0$. For each subgroup $\Psi$ of $\operatorname{Aut}\mathbf{M}$ that commutes with $\Delta$, the stabilizer $\Psi_{x,y}$ acts trivially on $\mathbf{F}$.*

PROOF. Assume first that $x^\Delta$ is not contained in any line. Since $x^\Delta$ is connected, it follows that $x^\Delta$ contains a quadrangle. For the subplane $\mathbf{E} = (E, \mathcal{E})$ that is generated by $x^\Delta$, we have that $x^\Delta \subseteq E$ and therefore $\dim E \geq \dim x^\Delta$.

If $\Delta$ acts straightly, we choose a point $x$ that is moved by $\Delta$. Then $x^\Delta$ is contained in a line $L_x$, and $\Delta$ cannot fix all points outside $L_x$. For every point $y$ outside $L_x$ that is moved by $\Delta$, we infer that $x^\Delta \cup y^\Delta$ contains a quadrangle. The orbit $x^\Delta$ is contained in a line of the subplane $\mathbf{F} = (F, \mathcal{F})$ that is generated by $x^\Delta \cup y^\Delta$, whence $\dim F \geq 2\dim x^\Delta$. $\square$

A typical problem in the theory of stable planes with "much homogeneity" (i.e., with a "large group of automorphisms") is the following. Determine "all" actions $\alpha\colon \Delta \to \operatorname{Aut}\mathbf{M}$ of a given (locally compact) group $\Delta$, where $\mathbf{M}$ belongs to a given class of stable planes (e.g., the class of $m$-dimensional planes, or a class of planes with other special properties). In particular, we need a notion of *equivalent action*. The restriction of $\alpha$ to some open invariant subplane yields an action again. Finally, there is the problem whether or not the stable plane $\mathbf{M}$ can be embedded in a larger plane (in particular, a projective plane!) such that the action $\alpha$ extends.

The following definition (which amounts to the introduction of a *category of actions*) covers phenomena like "equivalent actions" or "equivariant embeddings".

**3.7 Definition.** Let $\alpha\colon \Gamma \to \operatorname{Aut}\mathbf{E}$ and $\beta\colon \Delta \to \operatorname{Aut}\mathbf{F}$ be actions of topological groups $\Gamma$, $\Delta$ on stable planes $\mathbf{E} = (E, \mathcal{E})$ and $\mathbf{F} = (F, \mathcal{F})$. A pair $(\pi, \mu)$ is called a *morphism of actions* if $\mu\colon \Gamma \to \Delta$ is a morphism of topological groups and $\pi\colon \mathbf{E} \to \mathbf{F}$ is a morphism in $\mathbf{StP}$ (i.e., $\pi\colon E \to F$ is a continuous injective noncollapsed lineation) such that the following diagram commutes.

$$\begin{tikzcd} E \times \Gamma \arrow[r, "(x,\gamma)\mapsto x^{(\gamma^\alpha)}"] \arrow[d, "\pi\times\mu"'] & E \arrow[d, "\pi"] \\ F \times \Delta \arrow[r, "(y,\delta)\mapsto y^{(\delta^\beta)}"'] & F \end{tikzcd}$$

If $\mu$ is injective, we say that $(\pi, \mu)$ is an *embedding of actions*; recall that $\pi$ is injective by definition, and open by **2.6.** If $\pi$ and $\mu$ are isomorphisms, we call $(\pi, \mu)$ a *quasi-equivalence of actions.* If moreover, $\Gamma = \Delta$ and $\mu = \mathrm{id}_\Gamma$, then $(\pi, \mathrm{id}_\Gamma)$ is called an *equivalence of actions.*

See [**50**] for a general discussion of categories of actions on incidence structures.

**3.8 Examples.**

a. *The action of the connected component $\Omega \cong \mathrm{PSL}_2\mathbf{R}$ of the real hyperbolic motion group on the (interior) hyperbolic plane is embedded in an action of $\Omega$ on the* modified hyperbolic planes *that were constructed by* H. Salzmann, *cf.* [**21**]. *However, there is no embedding of the action of the* full *hyperbolic motion group (including the reflections) in an action on a modified hyperbolic plane.*

b. K. Strambach *has constructed a 2-dimensional stable plane with an action $\alpha$ of the group $\Sigma$ of all real $2 \times 2$-matrices of determinant $\pm 1$, see* [**40**], *cf. also* [**29**]. *He proved that this action cannot be embedded in an action on a 2-dimensional projective plane. Later,* R. Löwen *proved that there is no proper embedding of $\alpha$ in any action on a 2-dimensional stable plane* [**21**]. *However, there exist embeddings of $\alpha$ in the action of the group of all* complex *$2 \times 2$-matrices of determinant $\pm 1$ on* R. Löwen*'s complex analog to* K. Strambach*'s plane* [**49**]. *Moreover, the restriction of $\alpha$ to the maximal compact subgroup $\mathrm{O}_2\mathbf{R}$ of $\Sigma$ embeds into an action of $\mathrm{O}_2\mathbf{R}$ on a 2-dimensional projective plane* [**54, 17**].

## 4. Restrictions of actions

In the study of geometries with large automorphism groups it is often convenient to restrict the action of the group to some subgeometry. In the case of topological geometries, there arises the question whether or not the topology of the group is affected seriously by the restriction. All the material in this section is taken from [**4**].

For each subset $X \subseteq M$, the restriction map $\delta \mapsto \delta|_X$ is continuous with respect to the compact-open topologies on $\mathcal{C}(M, M)$ and $\mathcal{C}(X, M)$. If $X$ is invariant, then the corestriction to $X$ does not change the topology. However, the restriction map need neither be open nor closed. If, in particular, the set $X$ is $\Delta$-invariant for some closed subgroup $\Delta$ of $\operatorname{Aut}\mathbf{M}$, then the group $\Delta|_X$ need not be closed in $\operatorname{Aut}\langle X\rangle$, and $\Delta|_X$ need not be topologically isomorphic to the factor group $\Delta/\mathrm{K}$, where K is the kernel of the restriction. The continuity of the restriction map, however, yields the following information about the closure $\Upsilon$ of $\Delta|_X$ in $\mathrm{Aut}\langle X\rangle$.

- **a.** If $\Delta/\mathrm{K}$ is compact, then $\Delta|_X$ is compact and hence closed in $\mathrm{Aut}\langle X\rangle$. Consequently, the groups $\Delta/\mathrm{K}$ and $\Delta|_X$ are topologically isomorphic.
- **b.** If $\Delta/\mathrm{K}$ is connected, then $\Upsilon$ is connected.
- **c.** $\dim \Delta/\mathrm{K} = \dim \Delta|_X \leq \dim \Upsilon$, see **A1.13**.
- **d.** The factor group $\Delta/\mathrm{K}$ is abelian, nilpotent, or solvable if, and only if, the group $\Upsilon$ has the property in question.

**4.1** Example. *Let $\Gamma$ be the group of all orientation-preserving similarities of the Euclidean plane that fix the origin. This group is isomorphic to $\Delta = \mathbf{R} \times (\mathbf{R}/\mathbf{Z})$. The element $(1, \pi+\mathbf{Z})$ generates a discrete subgroup $\mathrm{Z} \cong \mathbf{Z}$ of $\Delta$. The projection $pr_2$ to the second factor of $\Delta$, however, maps Z onto a dense subgroup. This projection corresponds to the restriction of $\Gamma$ to the pencil in the origin (or to the line at infinity). Hence $pr_2|_\mathrm{Z}$ is not a quotient mapping, although $pr_2$ is* (cf. [**7**, 3.8]).

We know of no example for a restriction to a subplane that is not a quotient mapping. It has been shown that restrictions to Baer subplanes in compact projective planes are quotient mappings [4]; the proof rests on the compactness of the point space. For actions on stable planes, we know at least that passing to open subplanes does not affect the topology.

**4.2** Result. [4, 4]
*Let $\mathbf{M} = (M, \mathcal{M})$ be a stable plane, and let $\Delta$ be a subgroup of* $\mathrm{Aut}\,\mathbf{M}$.

- **a.** *The compact-open topologies derived from the actions on $M$ and $\mathcal{M}$, respectively, coincide for $\Delta$.*
- **b.** *For each nonempty open subset $U$ of $M$, the compact-open topologies derived from the action on $M$ and the restriction to $U$, respectively, coincide for $\Delta$.*
- **c.** *For each nonempty open subset $\mathcal{U}$ of $\mathcal{M}$, the compact-open topologies derived from the action on $M$ and the restriction to $\mathcal{U}$, respectively, coincide for $\Delta$.*
- **d.** [**43**, 4.6] *If $\Delta$ is planar and $p \in \mathrm{Fix}\,\Delta$, then the compact-open topologies derived from the action on $M$ and the restriction to $\mathcal{M}_p$, respectively, coincide for $\Delta$.*

The following consequence of **4.2** generalizes a result of H. Hähl about actions on translation planes.

**4.3** Result. [4, 5] *Assume that $\Xi = \mathbf{R}^m$ and let $\Phi$ be a closed subgroup of $\mathrm{GL}_m\mathbf{R}$. The linear action of $\Phi$ on $\Xi$ gives rise to a semi-direct product $\Delta = \Phi \ltimes \Xi$. Let $\mathbf{M} = (M, \mathcal{M})$ be a stable plane such that $\dim M = m$. Then each effective action $\alpha : \Delta \to \mathrm{Aut}\,\mathbf{M}$ such that $\Phi^\alpha = (\Delta^\alpha)_p$ for some point $p \in M$ (or $\Phi^\alpha = (\Delta^\alpha)_L$ for some line $L \in \mathcal{M}$) is a topological embedding.*

CHAPTER II

# A COMPILATION OF KNOWN RESULTS

*The second chapter comprises more specialized results about actions of locally compact groups on stable planes.*

## 5. Planar actions

In this section, we collect some of the results on planar actions. Bounds for the dimension of a planar group (depending on the dimension of the plane and on the dimension of the set of fixed points) are of particular interest for the applications in subsequent sections. The material is taken from [**43**], which was inspired by H. SALZMANN's work on automorphism groups of locally compact ternary fields [**34**], [**35**].

**5.1** RESULT. [**43**, 6.1, 6.3, 6.8, 6.10] *Let* $\mathbf{M}$ *be a stable plane. Any closed subgroup* $\Delta$ *of* $\operatorname{Aut}\mathbb{M}$ *that acts trivially on a Baer subplane* $\mathbb{B} = (B, \mathcal{B})$ *is compact.*

This compactness criterion is useful since the structure of compact planar groups is strictly restrained.

**5.2** RESULTS. *Let* $\Delta$ *be a compact connected planar group on a stable plane* $\mathbf{M}$.

- **a.** [**15**, 1.6] *If* $\dim \mathbf{M} \leq 4$, *then* $\Delta = \mathbb{1}$.
- **b.** [**43**, 6.13, a)] *If* $\dim \mathbf{M} = 8$, *then either* $\Delta \cong \mathrm{SO}_3\mathbf{R}$, $\Delta \cong \mathrm{SO}_2\mathbf{R}$, $\Delta = \mathbb{1}$, *or* $\Delta$ *is an abelian non-Lie group. In any case,* $\dim \Delta \leq 3$.
- **c.** [**43**, 6.13, b)] *If* $\dim \mathbf{M} = 16$, *then either* $\Delta$ *is of type* $\mathsf{G}_2$, *or* $\dim \Delta \leq 8$ *and* $\Delta$ *is isomorphic to one of the groups* $\mathrm{SU}_3\mathbf{C}$, $\mathrm{SO}_4\mathbf{R}$, $\mathrm{U}_2\mathbf{C}$, $\mathrm{SU}_2\mathbf{C}$, $\mathrm{SO}_3\mathbf{R}$, $\mathrm{SO}_2\mathbf{R} \times \mathrm{SO}_2\mathbf{R}$, $\mathrm{SO}_2\mathbf{R}$, $\mathbb{1}$, *or* $\Delta$ *is a non-Lie group of dimension* $\leq 7$.
- **d.** [**43**, 6.3, 6.8] *If the set of fixed points of* $\Delta$ *carries a Baer subplane of* $\mathbf{M}$, *then either* $\dim \mathbf{M} \leq 4$ *and* $\Delta = \mathbb{1}$, *or* $\dim \mathbf{M} = 8$ *and* $\dim \Delta \leq 1$, *or* $\dim \mathbf{M} = 16$ *and* $\dim \Delta \leq 3$, *or* $\dim \mathbf{M} = 16$ *and* $\Delta$ *is a non-Lie group* (*and* $\dim \Delta \leq 7$).
- **e.** [**43**, 6.10] *If* $\Delta$ *is abelian, then* $\dim \Delta < \frac{1}{2} \dim \mathbf{M}$.

There is a conjecture that every closed planar group is compact. Even in the case of projective planes, however, this could be verified only under various additional assumptions, such as associativity of the addition in a corresponding ternary field, or existence of (a tower of) Baer subplanes, or differentiability of the geometric operations. As a concequence, we have to work with results that provide only weaker information about the structure and the dimension of planar groups. In the case of 16-dimensional planes, this has the consequence that our bounds for the dimension of solvable, or almost simple groups of automorphisms are probably not sharp.

**5.3** RESULT. Rigidity properties of eight-dimensional planes [**43**, 7.6]
*Let $\Lambda$ be a nontrivial planar group of an eight-dimensional stable plane. Then* $\dim \Lambda + \dim \operatorname{Fix} \Lambda \leq 5$.

**5.4** RESULT. Rigidity properties of 16-dimensional planes [**43**, 8.21]
*Let $\Lambda$ be a closed planar subgroup of* $\operatorname{Aut} \mathbf{M}$, *where* $\mathbf{M} = (M, \mathcal{M})$ *is a stable plane, and* $\dim M = 16$.

- **a.** *Either* $\dim \Lambda \leq 12$, *or the identity component of $\Lambda$ is isomorphic to the compact exceptional Lie group of type $\mathsf{G}_{2(-14)}$ or the semi-direct product of the groups* $\mathrm{SU}_3\mathbb{C}$ *and* $\mathbb{C}^3$. *In both cases,* $\dim \Lambda = 14$.
- **b.** *If* $\dim \operatorname{Fix} \Lambda = 8$, *then $\Lambda$ is compact, and* $\dim \Lambda \leq 7$.
- **c.** *If* $\dim \operatorname{Fix} \Lambda = 4$, *then* $\dim \Lambda \leq 11$.
- **d.** *If $\Phi$ is a nontrivial, connected subgroup of $\Lambda$, then either* $\dim \mathrm{C}_\Lambda \Phi \leq 11$, *or* $\dim \Lambda = 12$.
- **e.** *If $\Lambda$ is semi-simple, then $\Lambda \cong \mathsf{G}_{2(-14)}$, or* $\dim \Lambda \leq 10$.

## 6. Straight actions, involutions

In this section, we collect material on straight actions from the existing literature. Of particular interest are involutory automorphisms (these are always straight), and their centralizers. Information about involutions will be useful in the study of actions of semi-simple groups. The material is taken from [**47**].

**6.1** RESULT. [**47**, Th. 5] *Let $(M, \mathcal{M})$ be a stable plane, and let $\delta \neq \mathbf{1}$ be an automorphism of prime order. If $\delta$ is straight, in particular, if $\delta$ is an involution, then one (and only one) of the following holds.*

- **a.** *$\delta$ is free; i.e.,* $\operatorname{Fix} \delta = \emptyset$,
- **b.** *$\delta$ is a Baer collineation; i.e.,* $(\operatorname{Fix} \delta, \mathcal{M}|_{\operatorname{Fix} \delta})$ *is a Baer subplane.*
- **c.** *$\delta$ has a center or an axis (or both).*

Examples of free involutions are easily constructed by deleting the fixed points from $M$. Similarly, one obtains involutions with an axis, but no center, or vice versa. Note, however, that axial involutions always fix exactly two lines in the pencil $\mathcal{M}_a$ in a point $a$ of the axis, just as if they had some "imaginary center", cf. [**47**, 7]. The second fixed line through $a$ will be denoted by $C_a$.

**6.2** RESULT. [**47**, 8] *Let $\Delta$ be an elementary abelian group of automorphisms of a stable plane $(M, \mathcal{M})$ and assume that there is a line $A \in \mathcal{M}$ such that $A$ is the axis of each $\delta \in \Delta$. Then $\Delta$ is cyclic. In particular, commuting involutions with the same axis are equal.*

**6.3** RESULT. "Triangle Lemma" [**47**, 10]
*Let $\mathbf{M}$ be a stable plane, and let $\Phi$ and $\Delta$ be subgroups of* $\operatorname{Aut} \mathbf{M}$ *such that $\Phi$ fixes a point $x$ and $\Delta$ fixes a triangle pointwise.*

- **a.** *If there are three commuting involutions in $\Phi$, then at least one of them has no axis through $x$.*
- **b.** *If there are four commuting involutions in $\Delta$, then at least one of them is a Baer collineation.*

c. *If there are three commuting involutions in $\Phi$ such that each of them has center and axis, then the centers form a nondegenerate triangle.*

d. *Let $\alpha, \beta, \gamma$ be three commuting involutions in $\Delta$. If none of them is a Baer collineation, then $\gamma = \alpha\beta$, and $\langle \alpha, \beta \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$.*

In the study of stable planes with large automorphism groups, it will be useful to have upper bounds for the dimension of straight groups.

**6.4** RESULT. [47, 11] *Let $\Delta$ be a locally compact group of automorphisms of a stable plane $\mathbf{M} = (M, \mathcal{M})$ with $\dim M = 2l$.*

a. *If $\Delta$ has a semi-axis, then $\dim \Delta \leq 3l$.*

b. *If $\Delta$ acts straightly, then $\dim \Delta \leq 3l$.*

c. *If $\Delta$ acts straightly and has a semi-axis, then $\dim \Delta \leq l$.*

d. *If $\Delta$ has a semi-axis $U$ and centralizes an involution $\sigma \in \Delta$, then $\sigma$ has axis $L_U$. Consequently, $\dim \Delta \leq l$, and $\Delta$ acts freely on $\mathcal{M}_u \setminus \{L_U, C_u\}$ for each point $u \in U$.*

Subgroups of the classical groups give examples that show that the bounds in **6.4** are sharp. Much lower bounds can be proved for compact groups. To some extent, these bounds may replace the information that a compact connected straight group on a compact connected projective plane of finite dimension acts freely outside the set of fixed points.

**6.5** RESULT. [51, 2.14] *Let $\Theta$ be a compact connected straight group on a stable plane $\mathbb{M} = (M, \mathcal{M})$.*

a. *Each point orbit has dimension strictly less than $\frac{1}{2} \dim M$.*

b. *If $\dim M = 2$, then $\Theta = \mathbf{1}$.*

c. *If $\dim M = 4$, then $\Theta$ is abelian, and $\dim \Theta \leq 2$.*

d. *If $\Theta$ is almost simple, then either $\dim M = 8$ and $\dim \Theta = 3$, or $\dim M = 16$, and $\dim \Theta \leq 28$.*

## 7. Actions of compact groups

The possibilities for actions of compact groups are much more restrained than for actions of locally compact groups in general. The results that are collected in this section will also help to exclude actions of large almost simple groups, since almost simple groups tend to contain rather large compact subgroups. The material is taken from [**51**].

**7.1** RESULT. Main Theorem on compact groups of automorphisms [**51**]
*Let $\mathbb{M} = (M, \mathcal{M})$ be a stable plane, where $\dim M = 2l$, and let $\mathrm{P}_2\mathbb{F}$ be the projective plane over the $l$-dimensional real alternative division algebra $\mathbb{F}$ (i.e., $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, according to $l$).*

a. *If $\Phi$ is a compact connected group of automorphisms of $\mathbf{M}$, then $\Phi$ is isomorphic to the elliptic motion group $\mathrm{E}$ on $\mathrm{P}_2\mathbb{F}$, or $\dim \Phi \leq \dim \mathrm{E} - \dim M$ (i.e., the dimension of the stabilizer of a point).*

b. *If $\Phi$ is (locally) isomorphic to $\mathrm{E}$, then $\mathbf{M} \cong \mathrm{P}_2\mathbb{F}$, and the action of $\Phi$ on $\mathbf{M}$ is equivalent to the usual action of $\mathrm{E}$ on $\mathrm{P}_2\mathbb{F}$.*

Note that the elliptic motion group has dimension 3, 8, 21, or 52, according to the value of $l$. Hence the bound $\dim \mathrm{E} - \dim M$ equals 1, 4, 13, or 36, respectively.

There are only few isomorphism types of compact Lie groups of low dimension. Since $\mathrm{Aut}\,\mathbb{M}$ is a Lie group if $\dim \mathbf{M} = 2$ or if $\dim \mathbf{M} = 4$ and $\dim \mathrm{Aut}\,\mathbf{M} \geq 4$, we have precise information about the possibilities for compact groups of automorphisms of stable planes of low dimension.

**7.2** RESULT. [**51**, 3.1] *Let $\Phi$ be a nontrivial compact connected group of automorphisms of a 2-dimensional stable plane* $\mathbb{M}$. *Then $\Phi \cong \mathrm{SO}_2\mathbf{R}$, or $\Phi \cong \mathrm{SO}_3\mathbf{R}$. In the latter case, the plane* $\mathbf{M}$ *is isomorphic to the projective plane over* $\mathbf{R}$, *and the action of $\Phi$ is equivalent to the usual one.*

**7.3** RESULT. [**51**, 3.2] *Let $\Phi$ be a compact connected group of automorphisms of a 4-dimensional stable plane* $\mathbb{M}$. *Then either $\dim \Phi \leq 4$ and $\Phi$ is isomorphic to one of the groups $\mathrm{U}_2\mathbb{C}$, $\mathrm{SU}_2\mathbb{C}$, $\mathrm{SO}_3\mathbb{R}$, or $\Phi$ is an abelian group of dimension at most 3, or $\Phi \cong \mathrm{PSU}_3\mathbb{C}$. In the last case, the plane* $\mathbb{M}$ *is isomorphic to the projective plane over* $\mathbb{C}$, *and the action of $\Phi$ is equivalent to the usual one.*

For compact groups of automorphisms of stable planes of dimension 8 or 16, the picture is more complicated. The following results have been used in the proofs of **7.1**, **7.2**, and **7.3**, but are also of their own interest.

**7.4** RESULT. [**19**] *Let $\mathbb{M} = (M, \mathcal{M})$ be a stable plane, and let $\Gamma$ be a group of automorphisms of* $\mathbb{M}$. *If there are two points $x_1, x_2 \in M$ such that the stabilizer $\Gamma_{x_i}$ acts transitively on the line pencil $\mathcal{M}_{x_i}$ (for $i \in \{1,2\}$), then* $\mathbb{M}$ *contains a flag homogeneous open subplane* $\mathbb{E}$. *This subplane is isomorphic to the elliptic, hyperbolic or euclidean plane of the adequate dimension. In particular,* $\mathbf{E}$ *is isomorphic to an open subplane of the projective plane $\mathrm{P}_2\mathbf{F}$ (where $\mathbf{F} \in \{\mathbf{R}, \mathbb{C}, \mathbf{H}, \mathbb{O}\}$, according to $\dim M$), and $\Gamma$ is isomorphic to a subgroup of $\mathrm{Aut}\,\mathrm{P}_2\mathbf{F}$.*

**7.5** RESULT. [**51**, 2.10] *Let $\Delta$ be a compact connected group of automorphisms of a stable plane $\mathbf{M} = (M, \mathcal{M})$.*

- **a.** *If there is a point $x$ such that $\dim x^\Delta = \dim M$, then* $\mathbf{M}$ *is isomorphic to the projective plane over* $\mathbf{R}, \mathbb{C}, \mathbf{H}$, *or* $\mathbb{O}$, *according to $\dim M$, and $\Delta$ is isomorphic to the corresponding elliptic motion group* E *(i.e., $\mathrm{PSO}_3\mathbf{R}$, $\mathrm{PSU}_3\mathbb{C}$, $\mathrm{PSU}_3\mathbf{H}$, or $\mathsf{F}_{4(-52)}$, respectively). Moreover, the action of $\Delta$ on* $\mathbf{M}$ *is equivalent to the usual action of* E.
- **b.** *If $\Delta$ fixes a point $x \in M$ and there is a line $L$ through $x$ such that $l = \dim \mathcal{M}_x = \dim L^\Delta$, then $\mathcal{M}_x$ is homeomorphic to the sphere $\mathbf{S}_l$, and $\Delta$ is a two-fold covering group of $\mathrm{SO}_{l+1}\mathbf{R}$ (i.e., $\mathrm{SO}_2\mathbf{R}$ for $l = 1$, and $\mathrm{Spin}_{l+1}$ for $l > 1$). Moreover, the action of $\Delta$ on $\mathcal{M}_x$ is equivalent to the usual (linear, almost effective) action on $\mathbf{S}_l$. In particular, the central involution of $\Delta$ has center $x$.*
- **c.** *If $\Delta$ fixes a line $L$ and there is a point $x \in L$ such that $l = \dim L = \dim x^\Delta$, then $L$ is homeomorphic to the sphere $\mathbf{S}_l$, and $\Delta$ is a two-fold covering group of $\mathrm{SO}_{l+1}\mathbf{R}$. Moreover, the action of $\Delta$ on $L$ is equivalent to the usual action on $\mathbf{S}_l$. In particular, the line $L$ is a projective line (i.e., it meets each other line), and the central involution of $\Delta$ has axis $L$.*

## 8. Solvable groups

Before we turn to almost simple groups in the next section, we report on results about the other extreme.

**8.1** RESULT. [**45**, 2.11] *Assume that* $\Lambda$ *is a solvable group of automorphisms of a stable plane* $\mathbf{M} = (M, \mathcal{M})$, *where* $\dim M \leq 4$. *Then* $\dim \Lambda \leq \frac{5}{2} \dim M$.

**8.2** EXAMPLE. *The bound in* **8.1** *is sharp: For* $\mathbf{F} \in \{\mathbb{R}, \mathbb{C}\}$, *the solvable group*

$$\Lambda = \left\{ \begin{pmatrix} a & b & c \\ & d & e \\ & & 1 \end{pmatrix} ;\, a, b, c, d, e \in \mathbf{F}, ad \neq 0 \right\}$$

*acts effectively on the projective plane* $\mathrm{P}_2\mathbf{F}$, *and* $\dim \Lambda = \frac{5}{2} \dim \mathrm{P}_2\mathbf{F}$.

Turning to planes of higher dimension, the bound $\frac{5}{2} \dim M$ is no longer attained in the projective case, cf. [**26**]. The proof for the projective case heavily relies on the existence of fixed elements for solvable groups and the structure theory for compact connected projective planes. Thus it does not carry over to the general case. We have the following result (which might be improved).

**8.3** RESULT. [**45**, 2.13] *Assume that* $\Lambda$ *is a solvable group of automorphisms of a stable plane* $\mathbf{M} = (M, \mathcal{M})$.

**a.** *If* $\dim M = 8$, *then* $\dim \Lambda \leq 18$ $(< \frac{5}{2} \dim M)$.

**b.** *If* $\dim M = 16$, *then* $\dim \Lambda \leq 40$ $(= \frac{5}{2} \dim M)$.

**8.4** EXAMPLE. *The solvable group*

$$\Lambda = \left\{ \begin{pmatrix} a & b & c \\ & d & e \\ & & f \end{pmatrix} ;\, b, c, e \in \mathbb{H};\ a, d, f \in \mathbb{C};\ adf \neq 0;\ |f| = 1 \right\}$$

*acts almost effectively on the projective plane* $\mathrm{P}_2\mathbb{H}$, *and* $\dim \Lambda = 17$. *According to* [**26**], *this is the maximal possible dimension for a solvable group of automorphisms of an eight-dimensional compact projective plane. For the case of 16-dimensional compact projective planes,* M. LÜNEBURG *has proved that a solvable group of automorphisms can have dimension 30 at most* [**26**]. *He also gives examples that show that this bound is sharp.*

## 9. Actions of almost simple groups

In this section, we collect results about possible actions of almost simple groups on stable planes of dimension at most 8, and about possible actions of compact almost simple groups on stable planes of arbitrary dimension. The results on 2-dimensional planes are due to H. SALZMANN, K. STRAMBACH, and, in final form, to R. LÖWEN [**21**]. The 4-dimensional case has been solved completely by R. LÖWEN [**15**], while the results for 8-dimensional planes are taken from [**42**].

Most prominent among the (almost) simple groups of automorphisms of stable planes are the (connected components of) the full groups of automorphisms of the projective planes over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, and the corresponding elliptic and hyperbolic motion groups; these groups are

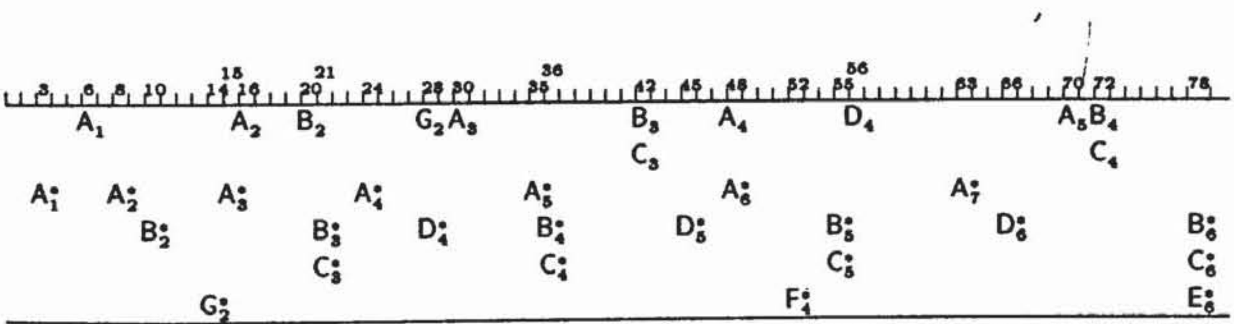
$$\mathrm{PSL}_3\mathbb{R},\ \mathrm{PSO}_3\mathbb{R},\ \mathrm{PSO}_3\mathbb{R}(1),$$
$$\mathrm{PSL}_3\mathbb{C},\ \mathrm{PSU}_3\mathbb{C},\ \mathrm{PSU}_3\mathbb{C}(1),$$
$$\mathrm{PSL}_3\mathbb{H},\ \mathrm{PSU}_3\mathbb{H},\ \mathrm{PSU}_3\mathbb{H}(1),$$

and the exceptional simple Lie groups of type $\mathsf{E}_{6(-26)}$, $\mathsf{F}_{4(-52)}$, $\mathsf{F}_{4(-20)}$, respectively. The possible actions have been completely determined by R. LÖWEN.

**9.1** RESULT. [22] *Let* $\mathbf{M}$ *be a stable plane of dimension* $m$, *and assume that* $\Delta$ *is a locally compact group that is locally isomorphic to the elliptic or hyperbolic motion group of the projective plane over* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, *or* $\mathbb{O}$, *according to* $m$. *Then every effective action of* $\Delta$ *is embedded into the usual action, except in the case of the real hyperbolic motion group. In this case, one has to assume that* $\Delta$ *is isomorphic to the (disconnected) hyperbolic motion group in order to exclude nonclassical actions.*

The actions of groups that are locally isomorphic to the real hyperbolic motion group on 2-dimensional stable planes have been determined in [21]; they are embedded in the actions on F.R. MOULTON's projective planes, on H. SALZMANN's projective modified hyperbolic planes, and on K. STRAMBACH's exceptional plane. See [21] for a description of these planes as well as further references.

**9.2** The following table attempts to visualize the distribution of simple Lie algebras in the range that is relevant for this paper. Recall that there are exceptional isomorphisms $\mathsf{A}_1 \cong \mathsf{B}_1 \cong \mathsf{C}_1$, $\mathsf{B}_2 \cong \mathsf{C}_2$, $\mathsf{A}_3 \cong \mathsf{D}_3$. Note also that $\mathsf{D}_2 \cong \mathsf{A}_1 \times \mathsf{A}_1$ is not simple. An asterisk (*) indicates the real forms.




Dimensions of simple Lie algebras.

**9.3** RESULT. [21] *Let* $\Delta$ *be a locally compact almost simple group of automorphisms of a 2-dimensional stable plane* $\mathbf{M}$. *Then* $\Delta$ *is isomorphic to* $\mathrm{PSL}_3\mathbf{R}$, $\mathrm{SL}_2\mathbf{R}$, $\mathrm{SO}_3\mathbf{R}$, *the connected component* $\Omega \cong \mathrm{PSL}_2\mathbf{R}$ *of* $\mathrm{SO}_3\mathbb{R}(1)$, *or the simply connected covering* $\tilde{\Omega}$ *of* $\mathrm{PSL}_2\mathbf{R}$. *The action of* $\Delta$ *on* $\mathbf{M}$ *is embedded in one of the following.*

- **a.** *The classical action of* $\mathrm{PSL}_3\mathbf{R}$ *on* $\mathrm{P}_2\mathbf{R}$, *or its dual.*
- **b.** *The action of* $\Omega$ *on a modified real hyperbolic plane.*
- **c.** *The action of* $\mathrm{SL}_2\mathbf{R}$ *on K. STRAMBACH's exceptional* $\mathrm{SL}_2\mathbf{R}$*-plane.*
- **d.** *The action of* $\tilde{\Omega}$ *on a Moulton plane.*

**9.4** RESULT. **[23]** *Let $\Delta$ be a locally compact almost simple group of automorphisms of a 4-dimensional stable plane $\mathbb{M}$. Then $\Delta$ is isomorphic to one of the following groups.*

$$\mathrm{PSL}_3\mathbb{C},\quad \mathrm{PSU}_3\mathbb{C},\quad \mathrm{PSU}_3\mathbb{C}(1),$$
$$\mathrm{PSL}_3\mathbb{R},\quad \mathrm{SL}_2\mathbb{C},\quad \mathrm{PSL}_2\mathbb{C}\cong \mathrm{SO}_3\mathbb{C},$$
$$\mathrm{SU}_2\mathbb{C},\quad \mathrm{SO}_3\mathbb{R},\quad \mathrm{SL}_2\mathbb{R},\quad \mathrm{PSL}_2\mathbb{R},\quad \text{a covering of } \mathrm{PSL}_2\mathbb{R},$$

*or $\Delta$ is a non-Lie group, and the factor group modulo the center is isomorphic to* $\mathrm{PSL}_2\mathbb{R}$.

*Apart from the higher coverings of $\mathrm{PSL}_2\mathbb{R}$ and the non-Lie groups, all these groups are subgroups of $\mathrm{PSL}_3\mathbb{C}$. In the remaining cases, we know of no effective action. If $\Delta$ is one of the groups $\mathrm{PSL}_3\mathbb{C}$, $\mathrm{PSU}_3\mathbb{C}$, $\mathrm{PSU}_3\mathbb{C}(1)$, or $\mathrm{PSL}_3\mathbb{R}$, then the action of $\Delta$ on $\mathbb{M}$ is embedded in the classical action of $\mathrm{PSL}_3\mathbb{C}$ on $\mathrm{P}_2\mathbb{C}$, or its dual. Actions of $\mathrm{SL}_2\mathbb{C}$ are either embedded in the classical action of $\mathrm{PSL}_3\mathbb{C}$ on $\mathrm{P}_2\mathbb{C}$, or embedded in the action of $\mathrm{SL}_2\mathbb{C}$ on R. LÖWEN's complex analog to K. STRAMBACH's exceptional $\mathrm{SL}_2\mathbb{R}$-plane.*

Note that the assertions of **9.3** and **9.4** remain true if we only assume that $\Delta$ is *semi-simple*, instead of almost simple.

**9.5** THEOREM. *Let $\Delta$ be a locally compact almost simple group of automorphisms of an 8-dimensional stable plane $\mathbb{M}$. Then either $\dim\Delta \le 16$, or $\Delta$ is isomorphic to one of the groups*

$$\mathrm{PSL}_3\mathbb{H},\ \mathrm{PSU}_3\mathbb{H},\ \mathrm{PSU}_3\mathbb{H}(1).$$

*Every nontrivial action of these groups is embedded in the classical action of $\mathrm{PSL}_3\mathbb{H}$ on $\mathrm{P}_2\mathbb{H}$, or its dual. If $\dim\Delta = 16$, then $\Delta$ is isomorphic to $\mathrm{SL}_3\mathbb{C}$ or to $\mathrm{PSL}_3\mathbb{C}$. Every effective action of $\mathrm{SL}_3\mathbb{C}$ on $\mathbb{M}$ is embedded in the action of $\mathrm{SL}_3\mathbb{C}$ on a Hughes plane (possibly $\mathrm{P}_2\mathbb{H}$).*

PROOF. For $\dim\Delta > 16$, the assertion has been proved in [**42**, Theorem B.]. If $\dim\Delta = 16$, then $\Delta$ is isomorphic either to $\mathrm{SL}_3\mathbb{C}$, or to $\mathrm{PSL}_3\mathbb{C}$. The effective actions of $\mathrm{SL}_3\mathbb{C}$ on 8-dimensional stable planes have been determined in [**52**]. □

We know of no example for an action of $\mathrm{PSL}_3\mathbb{C}$ on an 8-dimensional plane. Actions on *projective* planes are impossible by a result of H. SALZMANN [**36**]. However, the proof cannot be extended to actions on arbitrary stable planes.

**9.6** LEMMA. *No almost simple group of type $\mathsf{D}_6^{\mathbb{H}}$ can act nontrivially on a stable plane.*

PROOF. Let $\Delta$ be an almost simple group of type $\mathsf{D}_6^{\mathbb{H}}$. This means that the factor group $\Delta/\mathrm{Z}$ modulo the center $\mathrm{Z}$ of $\Delta$ is isomorphic to the simple Lie group $\mathrm{P}\alpha\mathrm{U}_6\mathbb{H}$. For our purposes, it will be convenient to describe the group $\mathrm{S}\alpha\mathrm{U}_6\mathbb{H}$ as the group of those elements of $\mathrm{GL}_6\mathbb{H}$ that leave invariant the skew hermitian form $\sum_{\mu=1}^{6} x_\mu i \bar{y}_\mu$. With respect to this description, it is easy to see that $\mathrm{S}\alpha\mathrm{U}_6\mathbb{H}\cap \mathrm{GL}_6\mathbb{C} = \mathrm{U}_6\mathbb{C}$. We may assume that $\Delta$ contains a subgroup $\Phi$ that is either equal to $\mathrm{SU}_6\mathbb{C}$ or equal to $\mathrm{PSU}_6\mathbb{C}$, cf. **A8.6a**. Moreover, we have that the connected component $\Psi$ of the centralizer of $\Phi$ in $\Delta$ has dimension 1.

It is easy to see that every involution in $\mathrm{SU}_6\mathbb{C}$ is either a conjugate of the diagonal matrix $\beta := \mathrm{diag}(-1,-1,1,1,1,1)$, or a conjugate of $-\beta$, or equal to the central involution $-\mathbb{1}$. Similarly, the conjugacy classes of involutions in $\mathrm{PSU}_6\mathbb{C}$ are represented by the matrices $\beta$, $-\beta$, and the diagonal matrices $\gamma := \mathrm{diag}(-i, i, i, i, i, i)$, $\delta := \mathrm{diag}(-i, -i, -i, i, i, i)$.

We infer that each involution $\varphi \in \Phi$ is centralized by a group $\Lambda$ that is locally isomorphic either to $\mathrm{SU}_4\mathbb{C}$ or to $\mathrm{SU}_3\mathbb{C} \times \mathrm{SU}_3\mathbb{C}$. From **5.2**, we know that a group that is locally isomorphic to $\mathrm{SU}_3\mathbb{C}$ cannot act trivially on a Baer subplane. If $\varphi$ is planar, this implies that $\Lambda$ acts almost effectively on the Baer subplane that is induced on the set of fixed points of $\varphi$, in contradiction to **7.1**.

We consider the stabilizer $\Phi_x$ of a point $x$ that is moved by $\Psi$. The connected component of $\Phi_x$ is a compact connected Lie group of dimension at least 19. In particular, $\Phi_x$ contains commuting involutions. None of these involutions is planar. Since $\Phi_x$ acts trivially on the connected orbit $x^\Psi$, we conclude that no involution in $\Phi$ has center $x$. Thus we obtain that all involutions in $\Phi_x$ have the same axis, in contradiction to **6.2**.

We have shown that an almost simple group of type $\mathsf{D}_6^{\mathbb{H}}$ cannot act almost effectively on an 8-dimensional stable plane. This implies that such a group cannot act nontrivially. □

**9.7** THEOREM. *Let $\Delta$ be a locally compact almost simple group of automorphisms of a 16-dimensional stable plane $\mathbb{M}$. Then either $\dim\Delta \leq 56$, or $\Delta$ is isomorphic to* $\mathrm{Aut}\,\mathrm{P}_2\mathbb{O}$ *(i.e., the 78-dimensional simple Lie group of type $\mathsf{E}_{6(-26)}$). Every nontrivial action of $\mathsf{E}_{6(-26)}$ on a stable plane is equivalent to the usual action on $\mathrm{P}_2\mathbb{O}$, or its dual.*

PROOF. According to **9.1**, every action of $\mathsf{F}_{4(-52)}$ embeds in the usual action of $\mathsf{E}_{6(-26)}$, or its dual. In particular, this entails the assertion about the action of $\mathsf{E}_{6(-26)}$. Assume that $\Delta$ is neither isomorphic to $\mathsf{E}_{6(-26)}$ nor to the elliptic motion group $\mathsf{F}_{4(-52)}$. According to **7.1**, the maximal compact subgroups of $\Delta$ have dimension at most 36. The dimension of a solvable subgroup of $\Delta$ is at most 40 by **8.3b**. The Iwasawa decomposition of $\Delta$ yields that $\dim\Delta \leq 36 + 40 + 1 = 77$, see **A6.3**.

If $66 < \dim\Delta \leq 77$, then $\Delta$ is a complex Lie group of type $\mathsf{A}_5$, $\mathsf{B}_4$, or $\mathsf{C}_4$, cf. **9.2** and **A7.6**. It is easy to see that each group of type $\mathsf{A}_5$ contains a subgroup of dimension 60 with a normal subgroup isomorphic to $\mathbb{C}^5 = \mathbb{R}^{10}$. According to **12.3c**, this subgroup cannot act on a 16-dimensional plane. In every group of type $\mathsf{B}_4$, we find a 2-dimensional (abelian) subgroup with a 44-dimensional centralizer. This case is excluded by **10.6c**. Finally, each group of type $\mathsf{C}_4$ contains a semi-simple subgroup of type $\mathsf{A}_1 \times \mathsf{C}_3$, and cannot act by **10.6d**.

If $56 < \dim\Delta \leq 66$, then $\Delta$ is a group of type $\mathsf{A}_7^*$ or of type $\mathsf{D}_6^*$, cf. **9.2**. It is easy to see that every group of type $\mathsf{A}_7^{\mathbb{R}}$ or $\mathsf{A}_7^{\mathbb{C},*}$ contains a closed subgroup $\Gamma$ of dimension 49 such that $\Gamma$ centralizes a nontrivial connected subgroup (e.g., choose $\mathrm{GL}_7\mathbb{R} \hookrightarrow \mathrm{SL}_8\mathbb{R}$, or $\mathrm{U}_7\mathbb{C}(r) \hookrightarrow \mathrm{SU}_8\mathbb{C}(r)$). Every group of type $\mathsf{A}_7^{\mathbb{H}}$ contains a subgroup of type $\mathsf{A}_1^*$ with a 36-dimensional centralizer (e.g., consider $\mathrm{SL}_1\mathbb{H}$ and $\mathrm{GL}_3\mathbb{H}$ in $\mathrm{SL}_4\mathbb{H}$). Thus all groups of type $\mathsf{A}_7^*$ are excluded by **10.6b** or **10.6d**.

Finally, every group of type $\mathsf{D}_6^{\mathbb{R},*}$ contains a subgroup of type $\mathsf{B}_4^* \times \mathsf{A}_1^*$, which cannot act by **10.6d**. In the remaining case, $\Delta$ is of type $\mathsf{D}_6^{\mathbb{H}}$, and cannot act by **9.6**. □

**9.8** Remark. In view of the results **9.3**, **9.4**, and **9.5**, it seems promising to study actions of almost simple groups of dimension, say, at least 36, on stable planes of dimension 16. However, it is reasonable to postpone the general case, at least until the special case of actions on projective planes is thoroughly understood. Here, work is still in progress.

We conclude this section with a report on results about actions of compact almost simple groups.

**9.9** Result. [**51**, 4.2] *Let* $\mathbf{M} = (M, \mathcal{M})$ *be a stable plane. If* $\mathrm{Aut}\,\mathbf{M}$ *has a compact connected almost simple subgroup of type* $\mathsf{G}_2$ *or* $\mathsf{A}_3$, *then* $\dim M = 16$.

**9.10** Result. [**51**, 4.3] *If* $\Phi$ *is a compact connected almost simple group of automorphisms of an 8-dimensional stable plane* $\mathbf{M}$, *then* $\dim \Phi \leq 10$, *or* $\Phi$ *is isomorphic to the elliptic motion group* $\mathrm{PSU}_3\mathbb{H}$.

**9.11** Result. [**51**, 5.10] *Let* $\Delta$ *be a compact almost simple group of automorphisms of a stable plane* $(M, \mathcal{M})$. *If* $\dim \Delta > 28$, *then* $\dim M = 16$, *and* $\Delta$ *is locally isomorphic to* $\mathrm{SU}_6\mathbb{C}$, $\mathrm{SO}_9\mathbb{R}$, $\mathrm{SU}_4\mathbb{H}$ *or the elliptic motion group* $\mathsf{F}_{4(-52)}$.

Probably, Result **9.11** is not sharp. We know of no example for an action of $\mathrm{SU}_6\mathbb{C}$, $\mathrm{SO}_9\mathbb{R}$, or $\mathrm{SU}_4\mathbb{H}$. The covering $\mathrm{Spin}_9$ of $\mathrm{SO}_9\mathbb{R}$ acts on $\mathrm{P}_2\mathbb{O}$. It is not known whether or not there are any nonclassical actions.

CHAPTER III

# ON GROUPS THAT ARE SEMI-SIMPLE, AND ON GROUPS THAT ARE NOT

Mostly, the results in this chapter are new; at least for stable planes of higher dimension. Starting with bounds for the dimensions of groups of automorphisms that centralize each other, we then derive bounds for the dimension of semi-simple groups that are not almost simple, as well as for groups that are not semi-simple.

## 10. Centralizers

In this section, we use results on straight and planar actions in order to derive dimension bounds for pairs of groups that centralize each other.

**10.1** LEMMA. *Let* $\mathbb{M}$ *be a stable plane, and assume that* $\Phi$ *and* $\Psi$ *are subgroups of* $\operatorname{Aut}\mathbb{M}$ *that centralize each other. If* $x$ *is a point such that* $\Phi_x$ *is nontrivial, then* $\dim x^\Psi \leq \frac{1}{2}\dim\mathbb{M}$.

PROOF. If $\dim x^\Psi > \frac{1}{2}\dim\mathbb{M}$, then $x^\Psi$ is neither contained in a line nor in a proper subplane, cf. **1.5a,b**. Consequently, $x^\Psi$ generates $\mathbb{M}$, and $\Phi_x$ is trivial. □

**10.2** THEOREM. *Assume that* $\mathbf{M} = (M, \mathcal{M})$ *is a stable plane of dimension* $m \in \{2,4\}$. *Let* $\Phi, \Psi$ *be locally compact groups, and assume that* $\alpha: \Phi \to \operatorname{Aut}\mathbf{M}$ *and* $\beta: \Psi \to \operatorname{Aut}\mathbf{M}$ *are almost effective actions such that* $[\Phi^\alpha, \Psi^\beta] = \mathbb{1}$. *Then the following hold.*

- **a.** *If* $\dim\Psi > m$, *then* $\Phi$ *acts straightly.*
- **b.** *If* $\dim\Phi \geq 1$, *then* $\dim\Psi \leq 2m$.
- **c.** *If* $\dim\Phi > \frac{m}{2}$, *then* $\dim\Psi \leq m$.
- **d.** *In any case,* $\dim\Phi + \dim\Psi \leq \frac{5}{2}m$.

PROOF. Recall from **5.2** that $m \in \{2,4\}$ implies that every planar group has dimension 0. Assume first that $\dim\Phi \geq 1$. If there exists a point $x \in M$ such that $x^\Phi$ is not contained in any line, then $\Psi_x$ is planar by **3.6**, whence $\dim\Psi_x = 0$. This implies that $\dim\Psi \leq m$, and **a** is proved. If $\Phi$ acts straightly, then we find points $x, y$ such that $\Psi_{x,y}$ is planar. This implies that $\dim\Psi \leq 2m$. We have proved assertion **b**. Now assume that $\dim\Phi > \frac{m}{2}$. If there exists a point $x \in M$ such that $\dim\Phi_x = 0$, then $x^\Phi$ cannot be contained in a line or a proper subplane, and we infer that $\dim\Psi \leq m$. If $\dim\Phi_x \geq 1$ for every point $x \in M$, then $\Psi$ acts straightly, and $\dim\Psi \leq m$ by **3.6** and **10.1**. This proves assertion **c**. The last assertion follows immediately (after interchanging the rôles of $\Phi$ and $\Psi$). □

**10.3** REMARK. The bounds in **10.2b,c** are sharp. E.g., consider the following subgroups of $\mathrm{SL}_3\mathbf{K}$, where $\mathbf{K} \in \{\mathbb{R}, \mathbb{C}\}$, according to the value of $m$.

**b.** $\Phi = \left\{ \begin{pmatrix} 1 & & z \\ & 1 & \\ & & 1 \end{pmatrix} ; z \in \mathbf{K} \right\}$, $\Psi = \left\{ \begin{pmatrix} a & x & z \\ & a^{-2} & y \\ & & a \end{pmatrix} ; a, x, y, z \in \mathbf{K}, a \neq 0 \right\}$

**c.** $\Phi = \left\{ \begin{pmatrix} a & & \\ & a & \\ & & a^{-2} \end{pmatrix} ; a \in \mathbf{K}, a \neq 0 \right\}$, $\Psi = \left\{ \begin{pmatrix} A & \\ & \det A^{-1} \end{pmatrix} ; A \in \mathrm{GL}_2\mathbf{K} \right\}$

**c'.** $\Phi = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ x & y & 1 \end{pmatrix} ; x, y \in \mathbf{K} \right\} = \Psi$

**10.4** THEOREM. *Assume that* $\mathbf{M} = (M, \mathcal{M})$ *is a stable plane of dimension* 8. *Let* $\Phi, \Psi$ *be locally compact groups, and assume that* $\alpha\colon \Phi \to \operatorname{Aut}\mathbf{M}$ *and* $\beta\colon \Psi \to \operatorname{Aut}\mathbf{M}$ *are almost effective actions such that* $[\Phi^\alpha, \Psi^\beta] = \mathbb{1}$. *Then the following hold.*

**a.** *If* $\dim \Psi > 11$, *then* $\Phi$ *acts straightly.*
**b.** *If* $\dim \Phi \geq 1$, *then* $\dim \Psi \leq 19$.
**c.** *If* $\dim \Phi \geq 2$, *then* $\dim \Psi \leq 17$.
**d.** *If* $\dim \Phi \geq 3$, *then* $\dim \Psi \leq 16$.
**e.** *If* $\dim \Phi \geq 5$, *then* $\dim \Psi \leq 8$.
**f.** *In any case,* $\dim \Phi^\alpha \Psi^\beta \leq 19$.

PROOF. Without loss, we may assume that $\Phi$ and $\Psi$ are connected. If there exists a point $x$ such that $x^\Phi$ is not contained in a line, then $\Psi_x$ is planar by **3.6a**, and $\dim \Psi_x \leq 3$ by **5.3**. Consequently, $\dim \Psi \leq 11$, and assertion **a** is proved.

If $\Phi$ acts straightly, we choose points $x$ and $y$ such that $x^\Phi \cup y^\Phi$ generates a subplane $\mathbb{E}$, cf. **3.6b**. Since $\Psi_{x,y}$ acts trivially on $\mathbb{E}$, we infer from **5.3** that $\dim \Psi_{x,y} \leq 3$, and $\dim \Psi \leq 8 + 8 + 3 = 19$. Thus assertion **b** is proved.

Moreover, $\dim \mathbb{E} = 4$ implies that $\dim \Psi \leq 8 + 8 + 1 = 17$, while $\dim \mathbb{E} = 8$ means that $\mathbb{E} = \mathbf{M}$, and $\dim \Psi \leq 16$. If $\dim \Phi \geq 2$ and $\dim \mathbb{E} = 2$, or if $\dim \Phi \geq 3$ and $\dim \mathbb{E} \leq 4$, then both $\Phi_x$ and $\Phi_y$ are nontrivial. From **10.1** and **5.3** we infer that $\dim \Psi \leq 4 + 4 + 3 = 11$. This completes the proof of assertions **c** and **d**.

Now assume that $\dim \Phi \geq 5$. If there exists a point $x$ such that $x^\Phi$ generates a subplane $\mathbb{E}$, then either $\mathbb{E} = \mathbf{M}$ and $\dim \Psi \leq 8$, or $\Phi_x$ is nontrivial, and $\dim \Psi \leq 4 + 3 = 7$ by **10.1** and **5.3**. There remains the case where $\Phi$ acts straightly. For every point $x$, we infer that $\Phi_x$ is nontrivial, and that $\dim x^\Psi \leq 4$. We can choose $x$ and $y$ in such a way that $x^\Phi \cup y^\Phi$ generates a subplane $\mathbb{E}$, and that $x^\Psi \cup y^\Psi$ generates a subplane $\mathbb{F}$. If $\Psi_{x,y}$ is trivial for some choice of $x, y$, we obtain that $\dim \Psi \leq 8$. If $\Psi_{x,y}$ is nontrivial for every choice of $x, y$, we have that $\mathbb{E}$ is a proper subplane, whence $\dim \Phi_z \geq 3$ for $z \in \{x, y\}$ and $\dim \Phi_{x,y} \geq 1$. We conclude that $\dim z^\Psi \leq 2$, and $\dim \Psi \leq 2 + 2 + 3 = 7$. This completes the proof of **e**.

Assertion **f** follows from **b**–**e**; note that in the situation of **b** we may assume that $\Phi \leq \Psi$. □

**10.5** REMARK. The bounds in **10.4b–e** are sharp. E.g., consider the following subgroups of $\mathrm{SL}_3\mathbb{H}$. For the description of subgroups of $\mathrm{GL}_3\mathbb{H}$, we employ J. DIEUDONNÉ's determinant $\mathrm{Det}\colon \mathrm{GL}_n\mathbb{H} \to (0, \infty)$.

**b.** $\Phi = \left\{ \begin{pmatrix} 1 & & t \\ & 1 & \\ & & 1 \end{pmatrix} ; t \in \mathbb{R} \right\}$, $\Psi = \left\{ \begin{pmatrix} a & x & z \\ & b & y \\ & & a \end{pmatrix} ; a, b, x, y, z \in \mathbb{H}, |a^2 b| = 1 \right\}$

**b'.** $\Phi = \left\{ \begin{pmatrix} r & & \\ & r & \\ & & r^{-2} \end{pmatrix} ; r \in \mathbb{R} \setminus \{0\} \right\}$,

$$\Psi = \left\{ \begin{pmatrix} A & \\ & a \end{pmatrix} ; A \in \mathrm{GL}_2\mathbb{H}, a \in \mathbb{H}, |a| \operatorname{Det} A = 1 \right\}$$

**c.** $\Phi = \left\{ \begin{pmatrix} 1 & & t \\ & 1 & \\ & & 1 \end{pmatrix} ; t \in \mathbb{C} \right\}$, $\Psi = \left\{ \begin{pmatrix} a & x & z \\ & b & y \\ & & a^i \end{pmatrix} ; a, b, x, y, z \in \mathbb{H}, |a^2 b| = 1 \right\}$

**c'.** $\Phi = \left\{ \begin{pmatrix} r & & \\ & r & \\ & & c \end{pmatrix} ; r \in \mathbb{R}, c \in \mathbb{C}, |r^2 c| = 1 \right\}$,

$$\Psi = \left\{ \begin{pmatrix} A & \\ & c \end{pmatrix} ; A \in \mathrm{GL}_2\mathbb{H}, c \in \mathbb{C}, |c| \operatorname{Det} A = 1 \right\}$$

**d.** $\Phi = \left\{ \begin{pmatrix} 1 & & t \\ & 1 & \\ & & 1 \end{pmatrix} ; t \in \mathbb{H} \right\}$, $\Psi = \left\{ \begin{pmatrix} r & x & z \\ & b & y \\ & & r \end{pmatrix} ; r \in \mathbb{R}, b, x, y, z \in \mathbb{H}, r^2 |b| = 1 \right\}$

**d'.** $\Phi = \left\{ \begin{pmatrix} r & & \\ & r & \\ & & a \end{pmatrix} ; r \in \mathbb{R}, a \in \mathbb{H}, r^2 |a| = 1 \right\}$,

$$\Psi = \left\{ \begin{pmatrix} A & \\ & \operatorname{Det} A^{-1} \end{pmatrix} ; A \in \mathrm{GL}_2\mathbb{H} \right\}$$

**e.** $\Phi = \left\{ \begin{pmatrix} c & & \\ & c & \\ & & a \end{pmatrix} ; c \in \mathbb{C}, a \in \mathbb{H}, |c^2 a| = 1 \right\}$,

$$\Psi = \left\{ \begin{pmatrix} A & \\ & \operatorname{Det} A^{-1} \end{pmatrix} ; A \in \mathrm{GL}_2\mathbb{C} \right\}$$

**e'.** $\Phi = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ x & y & 1 \end{pmatrix} ; x, y \in \mathbb{H} \right\} = \Psi$

**10.6 Theorem.** *Assume that $\mathbf{M} = (M, \mathcal{M})$ is a stable plane of dimension 16. Let $\Phi, \Psi$ be locally compact groups, and assume that $\alpha \colon \Phi \to \operatorname{Aut} \mathbf{M}$ and $\beta \colon \Psi \to \operatorname{Aut} \mathbf{M}$ are almost effective actions such that $[\Phi^\alpha, \Psi^\beta] = \mathbf{1}$. Then the following hold.*

- **a.** *If $\dim \Psi > 30$, then $\Phi$ acts straightly.*
- **b.** *If $\dim \Phi \geq 1$, then $\dim \Psi \leq 46$.*
- **c.** *If $\dim \Phi \geq 2$, then $\dim \Psi \leq 43$.*
- **d.** *If $\dim \Phi \geq 3$, then $\dim \Psi \leq 35$.*
- **e.** *If $\dim \Phi \geq 16$, then $\dim \Psi \leq 16$.*

Proof. If there exists a point $x$ such that $x^\Phi$ is not contained in a line, then $\Psi_x$ is planar, and $\dim \Psi_x \leq 14$ by **5.4**. Consequently, $\dim \Psi \leq 30$, and assertion **a** is proved.

If $\Phi$ acts straightly, we choose points $x$ and $y$ such that $x^\Phi \cup y^\Phi$ generates a subplane $\mathbf{E}$. From **5.4**, we infer that $\dim \Psi_{x,y} \leq 14$, and $\dim \Psi \leq 46$, as asserted in **b**.

If $\dim \Phi \geq 2$ and $\dim \mathbf{E} = 2$, then the stabilizers $\Phi_x$ and $\Phi_y$ are both nontrivial. According to **10.1**, this implies that $\dim \Psi \leq 8 + 8 + 14 = 30$. If $\dim \mathbf{E} \geq 4$, then $\dim \Psi \leq 16 + 16 + 11 = 43$, in view of **5.4b,c**.

If $\dim \Phi \geq 3$ and $\dim \mathbf{E} \leq 4$, then both $\Phi_x$ and $\Phi_y$ are nontrivial. Using **10.1**, we infer that $\dim \Psi \leq 8 + 8 + 14 = 30$. There remains the case where $\dim \mathbf{E} \geq 8$. We may assume that $\Psi$ is closed in $\operatorname{Aut} \mathbf{M}$, and that $\Psi$ is connected. Let $\mathrm{Z}$ be the center of $\Psi$. We know that $\Psi/\mathrm{Z}$ is a Lie group, cf. **A6.2a**. If $\mathrm{Z}_{x,y}$ is nontrivial, then $\dim \Psi \leq 8 + 8 + 7 = 23$, cf. **10.1**. If $\mathrm{Z}_{x,y}$ is trivial, then $\Psi_{x,y}$ is a (compact) Lie group, and $\dim \Psi_{x,y} \leq 3$, cf. **5.2d**. This implies that $\dim \Psi \leq 16 + 16 + 3 = 35$.

Now assume that $\dim \Phi \geq 16$. We consider first the case where there exists a point $x$ such that $x^\Phi$ generates a subplane $\mathbf{E}$. If $\mathbf{E} = \mathbf{M}$, then $\Psi_x$ is trivial, and $\dim \Psi \leq 16$. If $\mathbf{E}$ is a Baer subplane, then $\Phi_x$ is nontrivial, and $\dim \Psi \leq 8+7 = 15$ by **10.1** and **5.2d**. In the remaining cases, we choose a point $y$ of $\mathbf{E}$ such that $x^\Psi \cup y^\Psi$ generates a subplane $\mathbf{F}$. Then $\dim x^\Psi \leq \dim \mathbf{F}$. If $\dim \mathbf{E} = 4$, then $\dim \Phi_{x,y} \geq 8$, and $\dim \mathbf{F} \leq 4$ by **5.2d**. This implies that $\dim \Psi \leq 4+11 = 15$ by **5.4**. If $\dim \mathbf{E} = 2$, we obtain that $\dim \Phi_{x,y} \geq 12$, whence $\dim \mathbf{F} = 2$ and $\dim \Psi \leq 2 + 14 = 16$.

There remains the case where $\Phi$ acts straightly. Using **3.6b**, we find two points $x, y$ such that $x^\Phi \cup y^\Phi$ generates a subplane $\mathbf{E}$, and $x^\Psi \cup y^\Psi$ generates a subplane $\mathbf{F}$. If $\mathbf{E} = \mathbf{M}$, then $\dim \Psi \leq 8+8 = 16$, cf. **10.1**. If $\mathbf{E}$ is a Baer subplane, then $\dim \Phi_{x,y} \geq 8$, and $\dim \mathbf{F} \leq 4$. This implies that $\dim \Psi \leq 4+4+7 = 15$. If $\dim \mathbf{E} = 4$, then $\dim \Phi_{x,y} \geq 12$, and $\dim \mathbf{F} = 2$. Hence $\dim \Psi \leq 2+2+11 = 15$. Finally, assume that $\dim \mathbf{E} = 2$. Then $\dim \Psi \leq 1+1+14 = 16$, or $\mathbf{F}$ is generated by $z^\Psi$ for $z \in \{x, y\}$. In the latter case, $\dim \Phi_z \leq 14$, and we obtain the contradiction $\dim \Phi \leq 1 + 14 < 16$. □

**10.7** Remarks. The bound in **10.6b** is attained by the centralizer of a one-parameter group of translations in $\mathrm{Aut}\, \mathrm{P}_2\mathbb{O}$. The centralizer of a planar involution in $\mathrm{Aut}\, \mathrm{P}_2\mathbb{O}$ is the product of commuting almost simple groups of dimension 3 and 35. Thus the bound in **10.6d** is sharp. The bound in **10.6e** is attained by the full group of translations of the affine plane over the octonions.

As a byproduct, we have re-proved the fact that the dimension of an abelian group is bounded by the dimension of the point space [**48**, 3.3].

**10.8** Corollary. *If a locally compact abelian group $\Delta$ acts almost effectively on a stable plane $\mathbf{M}$, then $\dim \Delta \leq \dim \mathbf{M}$.*

## 11. Semi-simple groups

In this section, we derive dimension bounds for groups of automorphisms that are semi-simple, but not almost simple. We test our method with stable planes of low dimension (where the results are known) before we proceed to stable planes of dimension 8 and 16.

Let $\Delta$ be a locally compact connected group of finite dimension. If $\Delta$ is semi-simple, then it either is almost simple, or there are almost simple normal subgroups $\Sigma_1, \ldots, \Sigma_n$ such that $\Delta = \Sigma_1 \cdots \Sigma_n$, and $1 \leq i < j \leq n$ implies that $[\Sigma_i, \Sigma_j] = \mathbb{1}$. If we order these factors in such a way that $\dim \Sigma_i \leq \dim \Sigma_{i+1}$, then the $n$-tuple $(\dim \Sigma_1, \ldots, \dim \Sigma_n)$ depends only on the isomorphism type of $\Delta$.

**11.1** Definition. For every locally compact connected semi-simple group $\Delta = \Sigma_1 \cdots \Sigma_n$ with almost simple factors $\Sigma_i$ such that $\dim \Sigma_i \leq \dim \Sigma_{i+1}$, we define $d_\Delta := (\dim \Sigma_1, \ldots, \dim \Sigma_n)$.

Applying **10.2**, **10.4** and **10.6**, we obtain the following three theorems.

**11.2** Theorem. *Let $\Delta$ be a locally compact connected semi-simple group, and assume that $\Delta$ acts almost effectively on a stable plane of dimension $m \in \{2, 4\}$. Then $\Delta$ is almost simple.*

PROOF. Assume that $\Delta$ is not almost simple, and let $\Sigma$ be a factor of smallest dimension. Then $\dim \Sigma \geq 3$. If $m = 2$, we obtain that $\dim \Delta \geq 6 > 5$ in contradiction to **10.2d**. There remains the case where $m = 4$. Since $\dim \Sigma > \frac{m}{2}$, we infer from **10.2b** that $\dim \Delta - \dim \Sigma \leq 4$. Consequently, we have that $d_\Delta = (3,3)$. This last case is difficult, it has been excluded by R. LÖWEN in [**15**, pp. 21–26]. □

The case where $m = 4$ and $d_\Delta = (3,3)$ is difficult since *a priori* it is conceivable that the two factors either both act straightly or both act freely on $M$. In fact, this is what happens in the case of abelian groups whose dimension equals the dimension of the point space. The case where $m = 8$ and $d_\Delta = (8,8)$ presents similar problems, we did not yet succeed in proving that it does not occur.

Since the kernel of the restriction to an invariant Baer subplane of an 8-dimensional stable plane has dimension at most 1 by **5.3**, we have the following consequence of **11.2**.

**11.3** COROLLARY. *Assume that a locally compact group $\Delta$ acts effectively on an 8-dimensional plane $\mathbb{M}$. If an involution $\alpha \in \Delta$ is centralized by a semi-simple, not almost simple subgroup of* $\operatorname{Aut}\mathbb{M}$, *then $\alpha$ is not planar (viz., $\alpha$ either is free, or has an axis or a center).*

Before we study actions of semi-simple groups on 8-dimensional planes, we treat a special case.

**11.4** LEMMA. *Assume that $\Delta$ is locally isomorphic to* $\mathrm{SL}_3\mathbb{C}$. *If $\Delta$ acts effectively on a stable plane $\mathbb{M}$, then either* $\dim \mathbb{M} = 16$, *or the connected component $\Psi$ of the centralizer of $\Delta$ in* $\operatorname{Aut}\mathbb{M}$ *is a solvable group of dimension at most* 4.

PROOF. (i) According to **9.3**, the group $\Delta$ cannot act effectively on a 2-dimensional plane. If $\dim \mathbf{M} = 4$, then the centralizer of $\Delta$ in $\operatorname{Aut}\mathbf{M}$ is trivial by **9.4**. Therefore, it suffices to study the case where $\dim \mathbf{M} = 8$. From **10.4e** we infer that $\dim \Psi \leq 4$, and $\Psi$ is straight by **10.4a**. If $\Psi$ is not solvable, then there exists an almost simple subgroup $\Sigma$ of $\Psi$, and $\dim \Sigma = 3$. After removing a closed invariant set, we have that no point is fixed by $\Sigma$ or $\Delta$. We claim that $\Sigma$ acts freely on $M$. In fact, if $\Sigma_x$ is nontrivial, then we infer from **10.1** that $\dim \Delta \leq 4+8+3 < 16$, a contradiction. We conclude that for every point $x$ there exists a point $y$ such that $x^\Sigma \cup y^\Sigma$ generates $\mathbb{M}$. Consequently, $\Delta_{x,y}$ is trivial, and $\dim \Delta_x = 8$ for every point $x$.
(ii) The simply connected covering group $\tilde{\Delta}$ is isomorphic to $\mathrm{SL}_3\mathbb{C}$, and $\Delta$ is isomorphic either to $\mathrm{SL}_3\mathbf{C}$ or to $\mathrm{PSL}_3\mathbf{C}$. In both cases, it is easy to see that the involutions in $\Delta$ form a single conjugacy class, and that each involution is centralized by a subgroup of $\Delta$ that is isomorphic to $\mathrm{SL}_2\mathbb{C}$. Since every element of $\Delta$ is also centralized by $\Sigma$, we obtain from **11.3** that no involution in $\Delta$ is planar. Since $\Sigma$ fixes no point, the involutions cannot have centers. There remain the cases that either all involutions are axial, or all involutions are free. In particular, the stabilizer $\Delta_x$ contains no commuting involutions by **6.2**.
(iii) If an eight-dimensional closed connected subgroup of $\Delta$ is semi-simple, then it is even almost simple, and a real form of $\Delta$. This implies that every such group is isomorphic to $(\mathrm{P})\mathrm{SU}_3\mathbb{C}$, $(\mathrm{P})\mathrm{SU}_3\mathbb{C}(1)$, or $\mathrm{SL}_3\mathbf{R}$. Since each of these groups contains commuting involutions, we obtain that $\Delta_x^1$ is not semi-simple. Since $\Delta_x^1$ is a normal subgroup of $(\Sigma\Delta)_x$, this implies that the connected component of $(\Sigma\Delta)_x$ is not semi-simple.

(iv) Since $\Sigma$ acts freely, the restriction of the natural epimorphism $\Sigma\Delta \to \Delta/(\Sigma\cap\Delta)$ to the stabilizer $(\Sigma\Delta)_x$ induces a monomorphism onto a subgroup $\Lambda$ of $\Delta/(\Sigma\cap\Delta)$. Let $\tilde{\Lambda}$ be the connected component of the corresponding subgroup of $\tilde{\Delta} \cong \mathrm{SL}_3\mathbf{C}$. From the fact that $\dim x^{\Delta} = 8$ we infer that $\dim \tilde{\Lambda} = 11$. The normal subgroup $\Delta_x$ of $(\Sigma\Delta)_x$ gives an 8-dimensional normal subgroup $\Theta$ of $\tilde{\Lambda}$.
(v) We restrict the natural action of $\mathrm{SL}_3\mathbf{C}$ on $\mathbf{C}^3$ to $\tilde{\Lambda}$. Since the centralizer of $\mathrm{SL}_3\mathbf{C}$ in $\mathrm{GL}_3\mathbf{C}$ is a group isomorphic to $\mathbf{C}^*$ and almost disjoint to $\mathrm{SL}_3\mathbf{C}$, we infer that irreducibility of $\tilde{\Lambda}$ implies that $\tilde{\Lambda}$ is semi-simple (cf. **A6.5b**), in contradiction to step (iii). Therefore, $\tilde{\Lambda}$ is contained in the stabilizer of a proper subspace of $\mathbf{C}^3$; i.e., in a group that is the semi-direct product of $\Xi = \mathbf{C}^2$ and $\Omega = \mathrm{GL}_2\mathbf{C}$. Since $\dim \Xi\Omega = 12$, we know that $\tilde{\Lambda} \cap \Xi$ and $\tilde{\Lambda} \cap \Omega'$ are subgroups of co-dimension at most 1 in $\Xi$ and $\Omega' = \mathrm{SL}_2\mathbf{C}$, respectively. This implies that $\Omega'$ is contained in $\tilde{\Lambda}$, and $\Xi \leq \tilde{\Lambda}$ since $\Omega'$ acts irreducibly on $\Xi$. Now $\Theta \cap \Xi\Omega'$ is a normal subgroup of dimension 7 or 8, in contradiction to the fact that $\Xi$ is the only proper closed connected normal subgroup of $\Xi\Omega'$. □

**11.5** THEOREM. *Let $\Delta$ be a locally compact connected semi-simple group, and assume that $\Delta$ acts almost effectively on a stable plane of dimension* 8. *Then either $\Delta$ is almost simple, or* $\dim \Delta \leq 14$, *or* $d_\Delta \in \{(3,15),(8,8)\}$.

PROOF. Let $\Sigma$ be an almost simple factor of maximal dimension, and let $\Psi$ denote the connected component of $\mathrm{C}_\Delta\Sigma$. Recall that $\Psi$ is the product of all almost simple factors except $\Sigma$. From **9.5** we infer that $\dim \Sigma \leq 16$. If $\dim \Sigma = 14$, then $\Sigma$ is a real form of type $\mathsf{G}_2$, and has no subgroup of dimension greater than 9, cf. **A9.5**. Consequently, there are no nontrivial actions of $\Sigma$ on spaces of dimension 4. However, the centralizer $\Psi$ is a nontrivial straight group by **10.4a**, and $\Sigma$ leaves $\mathcal{M}_\Psi$ invariant. Consequently, $\Sigma$ acts trivially on $\mathcal{M}_\Psi$, and $\Delta$ is a straight group of dimension at least 17, in contradiction to **6.4**. Therefore $\dim \Sigma \in \{16,15,10,8,6,3\}$. However, the case $d_\Delta = (3,16)$ is excluded by **11.4**; note that every locally compact almost simple group of dimension 16 is covered by $\mathrm{SL}_3\mathbf{C}$, cf. **A8.6**.

If $\dim \Sigma \geq 10$, we infer from **10.4e** that $\dim \Psi \leq 4$; in fact $\dim \Psi = 3$, since there are no semi-simple groups of dimension 4. If $\dim \Sigma \geq 6$, we know that $\dim \Psi \leq 8$ by **10.4e**. Finally, there remains the case that all almost simple factors of $\Delta$ have dimension 3. Let $\Phi$ be the product of two of the factors. Then the product of the other factors has dimension at most 8 by **10.4e**, and we infer that $\dim \Delta \leq 12$. □

**11.6** REMARK. The case $d_\Delta = (3,15)$ occurs; in fact, the subgroup

$$\Delta = \left\{ \begin{pmatrix} A & \\ & a \end{pmatrix} ; A \in \mathrm{SL}_2\mathbf{H}, a \in \mathbf{H}, |a| = 1 \right\}$$

of $\mathrm{SL}_3\mathbf{H}$ acts almost effectively on $\mathrm{P}_2\mathbf{H}$. According to [41], every effective action of $\mathrm{SL}_2\mathbf{H}$ on an 8-dimensional stable plane embeds in this action. Probably, this is the only possibility for the case $m = 8$ and $d_\Delta = (3,15)$ (at least, there is no other almost effective action of a semi-simple group of type $(3,15)$ on a *projective* stable plane of dimension 8, see [36]). However, this conjecture will be hard to verify. One of the major problems seems to originate from the fact that the 2-dimensional projective geometry over the ring of real $2 \times 2$-matrices is a kind of 8-dimensional plane geometry (but not a stable plane!) which admits an almost effective action of the group $\mathrm{SL}_2\mathbb{R} \times \mathrm{SL}_4\mathbb{R}$.

We turn to planes of dimension 16 now.

**11.7** RESULT. **[51,** 5.3] *If a compact group $\Delta$ of automorphisms of a 16-dimensional stable plane is semi-simple, but not almost simple, then* $\dim \Delta \leq 35$.

**11.8** THEOREM. *Let $\Delta$ be a locally compact connected semi-simple group, and assume that $\Delta$ acts almost effectively on a stable plane of dimension* 16. *Then either $\Delta$ is almost simple, or* $\dim \Delta \leq 38$.

PROOF. Let $\Sigma$ be an almost simple factor of minimal dimension. From **10.6** we infer that the connected component $\Psi$ of $\mathrm{C}_\Delta\Sigma$ has dimension at most 35, and this bound reduces to 30 if $\Sigma$ is not straight.

Assume first that there exists a point $x$ such that $x^\Sigma$ is not contained in a line, and let $\mathbb{E}$ be the subplane that is generated by $x^\Sigma$. If $\dim \Sigma \leq 8$, then $\dim \Delta \leq 38$. So we may assume that $\dim \Sigma \geq 10$. If $\Sigma_x$ is trivial, then $\mathbb{E} = \mathbb{M}$, and $\dim \Psi \leq 16$. This implies that $\dim \Delta \leq 32$. If $\mathbb{E}$ is a Baer subplane, then $\Sigma_x$ is nontrivial, and $\dim \Psi \leq 8 + 7 = 15$ by **10.1** and **5.4b**. Thus $\dim \Delta \leq 30$ in this case. Since $\Sigma$ cannot act almost effectively on a 2-dimensional plane, there remains the case that $\dim \mathbb{E} = 4$. From **9.4**, we infer that $\Sigma$ induces the 16-dimensional group $\mathrm{PSL}_3\mathbb{C}$ on $\mathbb{E}$. According to **10.1** and **5.4c**, the dimension of $\Psi$ is bounded by $8 + 11 = 19$, and we obtain that $\dim \Delta \leq 35$.

Assume now that $\Sigma$ acts straightly, and let $x$ and $y$ be points such that $x^\Sigma \cup y^\Sigma$ generates a subplane $\mathbb{F}$. Then $\dim x^\Sigma \leq \frac{1}{2} \dim \mathbb{F}$. From **6.4** we infer that $\dim \mathbb{F} \geq \frac{2}{3} \dim \Sigma$.

If $\dim \Sigma = 3$, then $\dim \Psi \leq 35$ by **10.6**, and we infer that $\dim \Delta \leq 38$.

Now consider the case where $\dim \Sigma \in \{6, 8\}$. If both $\Sigma_x$ and $\Sigma_y$ are nontrivial, then $\dim \Delta \leq 8 + 8 + 8 + 11 = 35$. If one of the stabilizers is trivial, we infer that $\mathbb{F} = \mathbb{M}$, and $\dim \Delta \leq 8 + 16 + 16 = 40$. Of course, $\dim \Delta \geq 39$ implies that $\dim \Sigma = 8$. Since there are no almost simple groups of dimension 31 or 32, there exists an almost simple factor $\Phi$ of $\Psi$ such that the connected component $\Omega$ of the centralizer of $\Phi$ in $\Psi$ has dimension at least 8, recall that $\Sigma$ was chosen as a factor of minimal dimension. Since $\dim \Phi \geq 8$ as well, we infer from **10.6e** that $\dim \Omega \leq 16$, and that $\dim \Phi \leq 16$. If $\dim \Delta > 38$, then either $\dim \Omega = 16$ or $\dim \Phi = 16$. In both cases, we have reached a contradiction to **10.6e**.

If $\dim \Sigma = 10$, then $\dim \mathbb{F} \geq 8$, and both $\Sigma_x$ and $\Sigma_y$ are nontrivial. Hence $\dim \Delta \leq 10 + 8 + 8 + 7 = 33$. If $\dim \Sigma \geq 14$, then $\mathbb{F} = \mathbb{M}$, and both $\Sigma_x$ and $\Sigma_y$ are nontrivial. Hence $\dim \Psi \leq 8 + 8 = 16$, and $\dim \Delta \leq 32$. □

**11.9** REMARK. The centralizer $\Gamma$ of a planar involution in $\mathrm{Aut}\,\mathrm{P}_2\mathbb{O}$ is a semi-simple group, and $d_\Gamma = (3, 35)$. Thus the bound in **11.8** is sharp.

## 12. Groups that are not semi-simple

In this section, our results on centralizers are employed to derive bounds for the dimension of groups of automorphisms that are not semi-simple. These results are obtained almost simultaneously for stable planes of arbitrary (positive, finite) dimension.

For the sake of readability, the next three theorems are stated separately. However, we shall prove them all together by a single argument.

**12.1** THEOREM. *Let $\Delta$ be a locally compact connected group, and assume that $\Delta$ acts almost effectively on a stable plane of dimension $m \in \{2,4\}$.*

- **a.** *If $\Delta$ is not semi-simple, then $\dim \Delta \leq 3m$.*
- **b.** *If $\Delta$ contains a nontrivial compact connected abelian normal subgroup, then $\dim \Delta \leq 2m$.*
- **c.** *If $\Delta$ contains a normal subgroup $\Xi \cong \mathbf{R}^t$, then $t \leq m$, and $\dim \Delta \leq t + 2m$.*
- **d.** *If $\Delta$ contains a normal subgroup $\Xi \cong \mathbf{R}^t$ and $\dim \Delta > t + m$, then every one-dimensional closed connected subgroup of $\Xi$ acts straightly.*

**12.2** THEOREM. *Let $\Delta$ be a locally compact connected group, and assume that $\Delta$ acts almost effectively on a stable plane of dimension 8.*

- **a.** *If $\Delta$ is not semi-simple, then $\dim \Delta \leq 27$.*
- **b.** *If $\Delta$ contains a nontrivial compact connected abelian normal subgroup, then $\dim \Delta \leq 19$.*
- **c.** *If $\Delta$ contains a normal subgroup $\Xi \cong \mathbf{R}^t$, then $t \leq 8$, and $\dim \Delta \leq t + 19$.*
- **d.** *If $\Delta$ contains a normal subgroup $\Xi \cong \mathbf{R}^t$ and $\dim \Delta > t + 11$, then every one-dimensional closed connected subgroup of $\Xi$ acts straightly.*

**12.3** THEOREM. *Let $\Delta$ be a locally compact connected group, and assume that $\Delta$ acts almost effectively on a stable plane of dimension 16.*

- **a.** *If $\Delta$ is not semi-simple, then $\dim \Delta \leq 62$.*
- **b.** *If $\Delta$ contains a nontrivial compact connected abelian normal subgroup, then $\dim \Delta \leq 46$.*
- **c.** *If $\Delta$ contains a normal subgroup $\Xi \cong \mathbf{R}^t$, then $t \leq 16$, and $\dim \Delta \leq t + 46$.*
- **d.** *If $\Delta$ contains a normal subgroup $\Xi \cong \mathbf{R}^t$ and $\dim \Delta > t + 30$, then every one-dimensional closed connected subgroup of $\Xi$ acts straightly.*

PROOF of **12.1**, **12.2**, **12.3**. If $\Delta$ has a nontrivial compact connected abelian normal subgroup $\Theta$, then $\Theta$ lies in the center of $\Delta$ by **A6.4**, and the dimension bound for $\Delta$ follows from **10.2**, **10.4**, and **10.6**, respectively. If no such $\Theta$ exists, then every minimal connected abelian normal subgroup $\Xi$ of $\Delta$ is isomorphic to $\mathbf{R}^t$ for some $t \leq \dim M$. Via conjugation, the group $\Delta$ acts $\mathbf{R}$-linearly on $\Xi$. In particular, the centralizer of $\xi \in \Xi \setminus \{\mathbb{1}\}$ is already the centralizer of a closed connected subgroup. Thus the assertions follow immediately from the observation that $\dim \Delta \leq t + \dim \mathrm{C}_\Delta \xi$ and **10.2**, **10.4**, and **10.6**, respectively. □

## 13. A characterization of translation planes

Translation planes provide very homogeneous examples of stable (even affine or projective) planes. We report a characterization of translation planes in terms of stable planes, and indicate a family of stable planes that are closely related (so-called *shear planes*).

**13.1** RESULT. [48, 3.3] *Let $\Delta$ be a connected locally compact abelian group of automorphisms of a stable plane $\mathbf{M} = (M, \mathcal{M})$ with $\dim M = 2l$. Then we have* *one of the following (mutually exclusive) cases:*

a. *The group $\Delta$ acts straightly. In this case, the stabilizer of any line moved by $\Delta$ is trivial, and* $\dim\Delta \le \dim M = 2l$.
b. *There is a point $p \in M$ whose orbit $p^\Delta$ generates the whole plane. Consequently, the stabilizer $\Delta_p$ is trivial, and* $\dim\Delta \le \dim M = 2l$.
c. *There are points whose orbits are not contained in any line, but each of these orbits generates a proper subplane. In this case,* $\dim\Delta \le \frac{3}{2}l < \dim M$.

**13.2** RESULT. [**48**, 3.4] *Let $\Delta$ be a connected locally compact abelian group of automorphisms of a stable plane $\mathbf{M} = (M, \mathcal{M})$ and assume that $\dim\Delta = \dim M$. Then either $\Delta$ acts straightly with an open orbit in $\mathcal{M}$, or $\Delta$ has an open orbit in $M$. In both cases, the action of $\Delta$ on the open orbit is sharply transitive.*

**13.3** REMARKS. Prominent examples of abelian groups of the maximal possible dimension are translation groups in translation planes, and their duals. The latter obviously act straightly. In its natural (linear) action on the real affine plane, the group $\left\{\left(\begin{smallmatrix} a & \\ & b\end{smallmatrix}\right);\, a,b>0\right\} \cong \mathbb{R}^2$ is neither a translation group nor straight (indeed, the one-parameter subgroup $\left\{\left(\begin{smallmatrix} a & \\ & a^{-1}\end{smallmatrix}\right);\, a>0\right\}$ does not act straightly). There are also compact 4-dimensional projective planes admitting a group of automorphisms that is isomorphic to $\mathbb{R}^4$ but is not a translation group, see [**11**, Satz 1,3)].

If a stable plane is an affine translation plane, then there is a unique open embedding in a compact projective plane such that the complement of the stable plane is a line $W$. Every plane that is obtained by deleting a closed part of $W$ is called an *almost projective translation plane.*

**13.4** RESULT. [**48**, 3.6] *Let $\mathbf{M} = (M, \mathcal{M})$ be a stable plane of dimension $m$, and let $\Delta \cong \mathbb{R}^m$ be a group of automorphisms of $\mathbf{M}$ such that each one-parameter subgroup acts straightly. If $\Delta$ is not straight, then $\mathbf{M}$ is an almost projective translation plane with $\Delta$ acting as full group of translations.*

Straight effective actions of $\mathbb{R}^m$ on $m$-dimensional stable planes are also understood quite well. However, there is a greater variety of possibilities.

**13.5** RESULTS. [**48**, 3.6], [**13**, 3.2] *Let $\mathbf{M} = (M, \mathcal{M})$ be a stable plane of dimension $2l$, and let $\Delta \cong \mathbb{R}^{2l}$ be a straight group of automorphisms of $\mathbf{M}$.*

a. *The set* $\operatorname{Fix}\Delta$ *has at most one element. If a point $z \in \operatorname{Fix}\Delta$ exists, then each element $\delta \in \Delta$ has center $z$.*
b. *For each point $x \in M \setminus \operatorname{Fix}\Delta$, we have that $\Delta_x \cong \mathbb{R}^l$, and $\Delta_x$ acts sharply transitively on $\mathcal{M}_x \setminus \{L_x\}$, where $L_x$ is the line containing $x^\Delta$. In particular, the group $\Delta$ is transitive on $L_x \setminus \operatorname{Fix}\Delta$, and $\Delta_x = \Delta_{[L_x]}$.*
c. *Every line in $\mathcal{M}_\Delta$ meets every line outside $\mathcal{M}_\Delta$; i.e., $\mathcal{M}_\Delta \times (\mathcal{M} \setminus \mathcal{M}_\Delta) \subset \mathcal{D}_\wedge$.*
d. *The plane $\mathbf{M}$ is a dual affine (or projective) translation plane if, and only if, the set $\mathcal{M}_\Delta$ is compact. In particular, this is the case if* $\operatorname{Fix}\Delta$ *is nonempty.*

If $\Delta = \mathbb{R}^m$ is a straight group of automorphisms of a stable plane $\mathbf{M}$ of dimension $m$, H. LÖWE [**13**] noticed that the set $\mathcal{P} = \{\Delta_x;\, x \in M \setminus \operatorname{Fix}\Delta\}$ forms a *partial spread* in $\mathbb{R}^m$; i.e., $\mathcal{P}$ consist of subspaces of dimension $\frac{m}{2}$ such that the intersection of two of these subspaces is trivial. Conversely, H. LÖWE shows that a partial spread $\mathcal{P}$ yields a stable plane $S_2(\mathcal{P})$ if, and only if, $\mathcal{P}$ is a topological manifold of dimension $\frac{m}{2}$ (in the topology that is induced from the Grassmann manifold), see [**13**, Th. 1]. Every such plane is called a *shear plane.*

The next assertion indicates that translation planes and shear planes (including dual translation planes) belong to the most homogeneous planes. Note, however, that it is not yet clear whether or not there exist planes admitting a vector group that has almost the dimension of the point space, and a large normalizer.

**13.6** THEOREM. *Assume that a locally compact group $\Delta$ acts almost effectively on a stable plane* $\mathbf{M}$ *of dimension $m$. If $\Delta$ has a normal subgroup $\Xi \cong \mathbf{R}^m$, then* $\mathbf{M}$ *is either an almost projective translation plane, or a shear plane, in each of the following situations.*

**a.** $m \leq 4$ *and* $\dim \Delta > 2m$.
**b.** $m = 8$ *and* $\dim \Delta > 19$.
**c.** $m = 16$ *and* $\dim \Delta > 46$.

PROOF. The assertions follow immediately from **12.1d**, **12.2d**, and **12.3d**, combined with **13.4** and **13.5**. □

The following necessary condition for the embeddability of a shear plane in a dual translation plane is also due to H. LÖWE.

**13.7** RESULT. [**13**, Th. 2] *If a shear plane $S_2(\mathcal{P})$ of dimension $m$ is embeddable in a compact projective plane of dimension $m$, then the closure of $\mathcal{P}$ in the corresponding Grassmann manifold is a partial spread.*

While every partial spread in $\mathbf{R}^2$ is a part of the unique spread that consists of all one-dimensional subspaces, there exist examples of partial spreads in the higher dimensions that define shear planes but fail to fulfil the condition stated in **13.7**. These planes are examples of stable planes that are not open subplanes of projective planes.

**13.8** REMARKS. For $l = 1$, there are no translation planes except the desarguesian plane over $\mathbf{R}$. Translation planes admitting large automorphism groups have been studied by D. BETTEN in the case $l = 2$. The cases $l = 4, 8$ have been treated by H. HÄHL. See [**1**], [**7**], and the references given there.

Examples of shear planes are obtained as open subplanes of dual translation planes; there are also examples that cannot be embedded in a dual translation plane, cf. **13.7**.

The following assertions prepare grounds for a characterization of the classical planes by means of sufficiently large groups that are not semi-simple.

**13.9** THEOREM. *Assume that $\Xi = \mathbf{R}^m$ acts straightly and almost effectively on a stable plane $\mathbf{M} = (M, \mathcal{M})$ of dimension $m$. If $\Xi = \bigcup_{x \in M} \Xi_x$, then* $\mathbf{M}$ *is a dual affine or projective translation plane.*

PROOF. If there exists a point $z \in \operatorname{Fix}\Xi$, then the assertion follows from **13.5d**. So assume that $\operatorname{Fix}\Xi$ is empty. From **13.5b**, we know that $\Xi_x$ is a vector subspace of dimension $\frac{m}{2}$, and that $\Xi_x$ and $\Xi_y$ have nontrivial intersection only if they coincide. Since $\Xi = \bigcup_{x \in M} \Xi_x$, we have shown that $\{\Xi_x;\ x \in M \setminus \operatorname{Fix}\Xi\}$ forms a spread in $\Xi$. □

13.10 COROLLARY. *Let* $\mathbf{M}$ *be a stable plane of dimension* $m$*, and assume that a locally compact group* $\Delta$ *acts effectively on* $\mathbf{M}$*. Assume that* $\Delta$ *has a normal subgroup* $\Xi \cong \mathbf{R}^m$*, and that* $\Xi$ *acts straightly. If* $\Delta$ *acts (via conjugation) transitively on the set of one-dimensional subspaces of* $\Xi = \mathbf{R}^m$*, then* $\mathbf{M}$ *is a dual affine or projective translation plane. In particular, this assertion holds if* $\Delta$ *acts transitively on* $\Xi \setminus \{\mathbb{1}\}$.

PROOF. The assertion follows immediately from **13.9**; recall that $\Xi_x$ is a nontrivial vector subspace of $\Xi$ for every point $x$. □

## 14. Characterizations of the classical affine planes

In this section, we give characterizations of the classical planes over $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, and $\mathbf{O}$ in terms of actions of vector groups on stable planes.

**14.1** THEOREM. *Let* $\Delta$ *be a locally compact connected group of automorphisms of a stable plane* $\mathbb{M}$ *of dimension* $m$*, and let* $\mathrm{A}_2\mathbb{F}$ *be the classical affine plane of dimension* $m$*. Recall that* $\dim \operatorname{Aut} \mathrm{A}_2\mathbb{F}$ *equals* 6, 12, 27, *or* 62*, according to* $m = 2, 4, 8, 16$*. If* $\Delta$ *is not semi-simple, then* $\dim \Delta < \dim(\operatorname{Aut} \mathrm{A}_2\mathbb{F})$*, or* $\Delta \cong \operatorname{Aut} \mathrm{A}_2\mathbf{F}$*, and the action embeds in the usual action on* $\mathrm{P}_2\mathbf{F}$*, or its dual.*

PROOF. According to **12.1**, **12.2**, and **12.3**, we only have to show that $\dim \Delta = \dim(\operatorname{Aut} \mathrm{A}_2\mathbf{F})$ implies that the action of $\Delta$ on $\mathbf{M}$ embeds in the action of $\operatorname{Aut} \mathrm{A}_2\mathbb{F}$ on $\mathrm{P}_2\mathbb{F}$, or its dual. So assume that $\dim \Delta = \dim \operatorname{Aut} \mathrm{A}_2\mathbb{F}$. We know that $\Delta$ has a minimal normal subgroup $\Xi \cong \mathbb{R}^m$. From **13.6**, we infer that either $\Xi$ is not straight, and $\mathbb{M}$ is an almost projective translation plane, or $\Xi$ acts straightly, and $\mathbf{M}$ is a shear plane. For every nontrivial element $\xi$ of $\Xi$, we conclude from **10.2b**, **10.4b**, and **10.6b** that $\dim \xi^\Delta = \dim \Xi$. Thus $\xi^\Delta$ is open in $\Xi$. Since $\Xi \setminus \{\mathbb{1}\}$ is connected, this yields that $\Delta$ acts transitively on $\Xi \setminus \{\mathbb{1}\}$. If $\mathbf{M}$ is a translation plane, we infer that $\Delta$ acts transitively on the set of parallel classes of affine lines. Since $\Xi$ acts transitively on the set of (affine) points, we infer that $\mathbb{M}$ is classical by the result of H. SALZMANN [33] that the planes $\mathrm{A}_2\mathbf{F}$ are the only locally compact connected affine planes that admit a line-transitive group of automorphisms.

If $\Xi$ acts straightly, then $\mathbf{M}$ is a dual translation plane by **13.10**. The group $\Delta$ acts transitively on the pencil $\mathcal{M}_z$, where $z$ is the common center of all elements of $\Xi$. Since $x^\Xi = xz \setminus \{z\}$ for every point $x$ different from $z$, we infer that $\Delta$ acts transitively on $M \setminus z$. Thus $\mathbf{M}$ is classical by the dual of [33]. □

We close this section with examples of results that characterize the locally compact connected desarguesian planes by means of rather small (yet prominent) groups of automorphisms.

**14.2** RESULT. [48, 3.8] *Let* $\mathbf{F} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$*, and assume that a group* $\Delta \cong \mathbf{F}^2$ *of automorphisms of a stable plane* $\mathbf{M} = (M, \mathcal{M})$ *acts sharply transitively on* $M$*. Let* $\Phi$ *be a locally compact group of automorphisms of* $\mathbf{M}$ *such that* $\Phi$ *fixes a point* $p \in M$*, normalizes* $\Delta$ *and acts* $\mathbf{F}$*-linearly on* $\Delta = \mathbf{F}^2$ *(via conjugation). If* $\mathbf{F} = \mathbf{H}$ *and* $\dim \mathrm{C}_\Phi\delta > 1$ *for each* $\delta \in \Delta$*, or if* $\mathbf{F} \in \{\mathbf{R}, \mathbf{C}\}$ *and* $\dim \mathrm{C}_\Phi\delta \geq 1$ *for each* $\delta \in \Delta$*, then each one-parameter subgroup of* $\Delta$ *acts straightly, and the geometry induced on* $x^\Delta$ *is the desarguesian affine plane with* $\Delta$ *acting as full group of translations.*

**14.3** RESULT. [22] *Let* $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, *and assume that the euclidean motion group of* $A_2\mathbb{F}$ *acts almost effectively on a stable plane* $\mathbb{M}$. *Then* $\dim \mathbb{M} \geq \dim A_2\mathbb{F}$, *and* $\dim \mathbb{M} = \dim A_2\mathbb{F}$ *implies that the action embeds in the action of the euclidean group on* $P_2\mathbb{F}$, *or its dual.*

Recall that the euclidean motion group is the semi-direct product $\mathbb{R}^2 \rtimes SO_2\mathbb{R}$, $\mathbb{C}^2 \rtimes SU_2\mathbb{C}$, $\mathbb{H}^2 \rtimes SU_2\mathbb{H}$, or $\mathbb{O}^2 \times \mathrm{Spin}_9$, according to $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$.

**14.4** RESULT. [48, 4.4] *Let* $\Phi\Delta$ *be a group of automorphisms of an 8-dimensional stable plane, with* $\Delta \cong \mathbb{R}^8 = \mathbb{H}^2$ *and* $\Phi = U(J)$, *where* $J \in \left\{ \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} i & \\ & i \end{pmatrix} \right\}$ *describes a nondegenerate (skew) hermitian form on* $\mathbb{H}^2$. *Assume that* $\Phi$ *acts on* $\Delta$ *in the usual,* $\mathbb{H}$*-linear way.*

a. *If* $\Delta$ *is not straight, then* $\mathbb{M}$ *is isomorphic to the plane induced on the complement of some closed subset of a line in the projective quaternion plane, and the group* $\Phi\Delta$ *acts in the usual way.*
b. *If* $\Delta$ *acts straightly and* $J = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, *then the stable plane* $\mathbb{M}$ *is isomorphic to the dual euclidean quaternion plane, or its projective closure.*
c. *If* $\Delta$ *acts straightly and* $J = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, *then the stable plane* $\mathbb{M}$ *has an open* $\Phi\Delta$*-invariant subplane that is isomorphic to the cylinder plane* $\mathbb{M}_{(1,-1,0)}$.
d. *If* $\Delta$ *acts straightly,* $J = \begin{pmatrix} i & \\ & i \end{pmatrix}$, *and the involution* $\alpha = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ *is not planar, then the stable plane* $\mathbb{M}$ *has an open* $\Phi\Delta$*-invariant subplane that is isomorphic to the skew cylinder plane* $\mathbb{M}_{(i,i,0)}$.

The cylinder plane $\mathbb{M}_{(1,-1,0)}$ and the skew cylinder plane $\mathbb{M}_{(i,i,0)}$ are open subplanes of the projective planes over $\mathbb{H}$, defined by certain degenerate (skew) hermitian forms on $\mathbb{H}^3$. See [48] for a precise definition. In the situation of **14.4d**, the planarity of $\alpha$ may be forced by the assumption that the action of $\Delta$ embeds in the action of a group $\Gamma$ such that $C_\Gamma\alpha$ has special properties, e.g., is semi-simple, but not almost simple.

An analog of **14.4** should be true for hermitian planes over $\mathbb{R}$ and $\mathbb{C}$, but has not been proved yet.

CHAPTER IV

# STABLE PLANES WITH LARGE AUTOMORPHISM GROUPS

We collect our results and combine them to show that the classical projective and affine planes are the most homogeneous stable planes.

## 15. Notions of homogeneity

Our aim in this treatise is to prove that every "sufficiently homogeneous" stable plane is isomorphic to one of the projective and (dual) affine planes over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. If we want to establish such a statement, we have to make precise what "homogeneity" means for a stable plane, and how we can compare different planes with respect to some notion of homogeneity. Throughout this chapter, let $\mathbf{M} = (M, \mathcal{M})$ be a stable plane of dimension $m$. We shall describe homogeneity in terms of groups of automorphisms. Only at the end of this section, we shall briefly digress to a discussion of alternatives.

**15.1** The first homogeneity condition that comes to mind is

(**P**) *There exists a point-transitive group of automorphisms.*

This is a rather weak condition. In fact, if a group $\Delta$ of automorphisms has an open orbit $U$ in $M$, then $\Delta$ acts point-transitively on the stable plane $(U, \mathcal{M}|_U)$. Many point- homogeneous stable planes are known; apart from the classical projective, affine or hyperbolic planes we mention the large family of affine translation planes, and H. GROH's *arc planes*. In the special case of compact connected projective planes, however, condition (**P**) is very strong, it characterizes the classical projective planes over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, see [**30**], [**16**].

**15.2** A very strong homogeneity condition is

(**F**) *There exists a group of automorphisms that acts transitively on the set of flags (i.e., the set of incident point-line pairs).*

In fact, R. LÖWEN has proved in [**19**] that even the assumption that the stabilizer $\Delta_{x_i}$ acts transitively on $\mathcal{M}_{x_i}$ for some group $\Delta \leq \operatorname{Aut}\mathbf{M}$ and two (different!) points $x_1, x_2$ enforces that $\mathbf{M}$ contains an open $\Delta$-invariant classical plane. Moreover, the action of $\Delta$ on $\mathbb{M}$ embeds in the usual action of $\operatorname{Aut}\mathrm{P}_2\mathbf{F}$ on $\mathrm{P}_2\mathbf{F}$ or its dual, or $\Delta$ is isomorphic to $\mathrm{PSL}_2\mathbf{R}$, and the action embeds in the action on a modified hyperbolic plane.

**15.3** For the sake of completeness, we mention the following homogeneity condition.

(**L**) *There exists a group of automorphisms that acts transitively on the set of lines.*

This condition seems hard to deal with in general. Of course, it is just the dual of **(P)** in the special case of projective planes. For affine planes, condition **(L)** is very strong, it characterizes the classical affine planes over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, see [**33**].

**15.4** It would seem a reasonable weakening of condition **(F)** if one only requires the following.

**(f)** *There exists a group of automorphisms with an open orbit in the set of flags.*

Here the topology on the set of flags is induced from the product topology on $M \times \mathcal{M}$. Planes that satisfy condition **(f)** have been called *flexible*. From [**20**, Th. 1], one deduces that an orbit $\mathcal{U}$ in the flag space is open if, and only if, $\dim \mathcal{U} = \frac{3}{2} \dim M$. This observation is the key to classification results about flexible compact connected projective planes, since it implies that $\dim \operatorname{Aut} \mathbf{M} \geq \frac{3}{2} \dim M$.

**15.5** An important idea of H. SALZMANN is to require that $\operatorname{Aut} \mathbf{M}$ is "large" in the following sense.

**(D)** *There exists a group* $\Delta$ *of automorphisms such that* $\dim \Delta \geq b_m$, *where* $b_m$ *is a suitably chosen integer, depending on* $m = \dim M$.

Of course, the bound $b_m$ has to be chosen differently, if one considers special classes of actions; e.g., actions on projective planes, on translation planes, or on stable planes with additional topological properties. It will also be of interest to study groups with special structure. Most prominent, perhaps, are the (almost) simple groups. Interesting and useful results have also been obtained for compact groups, and for abelian groups. In the last case, one needs additional assumptions on the action of the group (or the one-dimensional subgroups). These additional properties may be enforced by the assumption that the normalizer in the full group of automorphisms is "big enough".

We close this section by briefly indicating notions of homogeneity that do not make use of *group actions*, at least not explicitly.

**15.6** Involutory automorphisms play a prominent rôle in the study of geometries. The following condition proved to be useful in the theory of stable planes.

**(S)** *There exists an open set of points that are centers of involutory automorphisms.*

In this case, one is lead to the study of *symmetric planes*, where the topological and the incidence structure are combined with the structure of a symmetric space whose symmetries are automorphisms of the plane, cf. [**18**]. Symmetric planes of dimension 2 and 4 have been studied by R. LÖWEN, H.-P. SEIDEL, and most recently by H. LÖWE. The theory of symmetric planes yields strong results about actions of almost simple groups (where many involutions exist), and the extension of this theory to the case of stable planes of higher dimension seems promising.

**15.7** Contrasting the situation for projective planes, the study of actions of semigroups on stable planes seems to be an interesting field. However, an investigation of the endomorphisms of the known examples of (nonclassical) stable planes will have to take place first. E.g., it would be interesting whether or not K. STRAMBACH's exceptional plane has endomorphisms that are not automorphisms.

**15.8** Since the theory of stable planes may be regarded as a "local theory of compact connected projective planes", it might also be fruitful to study (local) actions of local groups rather than actions of groups or semigroups. In particular, the study of semigroup actions might lead to an interest in local actions, since the known solution for the embeddability problem for semigroups on manifolds **[9]** is rather a local solution. Moreover, recent investigations of R. BÖDI and L. KRAMER about differentiable structures that are compatible with an incidence structure seem to indicate a need for a local theory.

**15.9** Last, but not least, there are several *geometrical* notions of homogeneity. Most prominent, perhaps, is Desargues' Theorem and its degenerations. For the case of (discrete) projective planes, R. BAER has shown how to translate the validity of Desargues' Theorem for certain centers and axes into the existence of groups of axial collineations. For stable planes (in fact, for topological planes in general), it is also interesting to study the consequences of *local* forms of Desargues' Theorem. For 2-dimensional stable planes with connected lines, C. POLLEY has obtained very satisfying results, in particular, every locally desarguesian 2-dimensional stable plane with connected lines is globally desarguesian **[27]**, **[28]**. This is no longer true if one drops the assumption that the lines are connected; e.g., consider the Moulton planes.

Generalizing R. BAER's point of view, there arises the question to what extent local geometric homogeneity conditions can be expressed in terms of actions of groups. Possibly, local actions are better suited for this purpose.

## 16. Every sufficiently homogeneous plane is classical

In this section, our investigations culminate. We briefly recall some of our results, and combine them to obtain our Main Theorem. Throughout this section, let $\Delta$ be a connected locally compact group, let $\mathbf{M}$ be a stable plane of dimension $m$, and let $\alpha\colon \Delta \to \operatorname{Aut}\mathbb{M}$ be an effective action.

**16.1** THEOREM. *Assume that $\Delta$ is semi-simple.*

- **a.** *If $m = 2$, then $\Delta$ is an almost simple Lie group of dimension 3 or 8. All possible actions are known.*
- **b.** *If $m = 4$, then $\Delta$ is almost simple. If $\dim\Delta \geq 6$, then $\Delta$ is a Lie group of dimension 6, 8, or 16, and all possible actions are known; in fact, they are embedded in the classical action of $\mathrm{PSL}_3\mathbb{C}$, or in one exceptional action of $\mathrm{SL}_2\mathbb{C}$.*
- **c.** *If $m = 8$ and $\dim\Delta > 16$, then $\Delta$ is a Lie group of dimension 18, 21, or 35, and all possible actions embed in the classical action of $\mathrm{PSL}_3\mathbb{H}$. There is some information about the cases where $\dim\Delta \in \{15, 16\}$; in particular, there do exist nonclassical actions.*
- **d.** *If $m = 16$, then either the action is equivalent to the classical action of $\mathsf{E}_{6(-26)}$ on $\mathrm{P}_2\mathbb{O}$, or $\Delta$ is almost simple, and $\dim\Delta \leq 56$, or $\dim\Delta \leq 38$.*

For details, see **9.3**, **9.4**, **9.5**, **9.7**, and **11.2**, **11.5**, **11.8**.

**16.2** THEOREM. *If $\Delta$ is not semi-simple, then either $\Delta$ contains a normal subgroup $\Xi \cong \mathbb{R}^t$, or there exists a connected nontrivial central subgroup. In the first case,* *we have that $t \leq m$, and* $\dim \Delta$ *is bounded by the dimension $a_m$ of the full group of automorphisms of the classical affine plane of dimension $m$. In the second case,* $\dim \Delta \leq a_m - m$. *If* $\dim \Delta = a_m$, *then the action embeds into the classical action.*

For details, see **12.1**, **12.2**, **12.3** and **14.1**.

**16.3** THEOREM. *If $\Delta$ is compact, then either the action is equivalent to the action of the elliptic motion group* E *on the classical projective plane of dimension $m$, or* $\dim \Delta \leq \dim \mathrm{E} - m$.

For details, see **7.1**.

**16.4** MAIN THEOREM. *Let $\mathbf{M}$ be a stable plane of dimension $m$, and let $\mathbf{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ be chosen such that* $\dim \mathrm{P}_2\mathbf{F} = m$. *Then one of the following cases occurs.*

a. *The plane $\mathbb{M}$ is isomorphic to $\mathrm{P}_2\mathbb{F}$, and* $\dim \mathrm{Aut}\,\mathbb{M}$ *equals* 8, 16, 35, *or* 78, *according to the value of $m$.*
b. *The plane $\mathbb{M}$ is either isomorphic to the affine plane $\mathrm{A}_2\mathbf{F}$, or isomorphic to $\mathrm{P}_2\mathbb{F}$ with one point deleted. In this case,* $\dim \mathrm{Aut}\,\mathbb{M}$ *equals* 6, 12, 27, *or* 62, *according to the value of $m$.*
c. *In any other case,* $\dim \mathrm{Aut}\,\mathbf{M}$ *is strictly less than* $\dim \mathrm{Aut}\,\mathrm{A}_2\mathbf{F}$.

PROOF. Combine the results on almost simple groups **9.3**, **9.4**, **9.5**, and **9.7** with the results on semi-simple groups **11.2**, **11.5**, and **11.8** and with the results on groups that are not semi-simple **12.1**, **12.2**, and **12.3**. Finally, apply **14.1**. □

**16.5** As in [**53**], we attempt to describe the state of the art in a table. Note that, in comparison with the table in [**53**], our new version has two more rows ("general", and "not semi-simple"), and that the ?'s in [**53**] have been replaced.

Let $\mathbf{M} = (M, \mathcal{M})$ be a stable plane, and assume that $\alpha\colon \Gamma \to \mathrm{Aut}\,\mathbf{M}$ is an effective action of a locally compact group. In the spirit of **15.5**, we are interested in the possibilities for $\alpha$, if $\dim \Gamma$ is "large enough" (depending on the structure of $\Gamma$). Except where marked by $*$, the given bounds are known to be sharp.

| structure of $\Gamma$ | $\dim M = 2$ | $\dim M = 4$ | $\dim M = 8$ | $\dim M = 16$ |
|---|---|---|---|---|
| general | $\dim\Gamma \le 8$; if $\dim\Gamma \ge 6$ then $\alpha$ is classical | $\dim\Gamma \le 16$; if $\dim\Gamma \ge 12$ then $\alpha$ is classical | $\dim\Gamma \le 35$; if $\dim\Gamma \ge 27$ then $\alpha$ is classical | $\dim\Gamma \le 78$; if $\dim\Gamma \ge 62$ then $\alpha$ is classical |
| almost simple | $\dim\Gamma \in \{3,8\}$, and $\alpha$ is known | $\dim\Gamma \in \{3,6,8,16\}$; if $\dim\Gamma > 3$ then $\alpha$ is known | $\dim\Gamma \le 35$; if $\dim\Gamma > 16$ then $\alpha$ is classical | $\dim\Gamma \le 78$; if $\dim\Gamma > 56$ then• $\alpha$ is classical |
| semi-simple, but not almost simple | impossible | impossible | $\dim\Gamma \le 18$ | $\dim\Gamma \le 38$ |
| not semi-simple | either $\Gamma$ has a center of positive dimension, and | | | |
| | $\dim\Gamma \le 4$ | $\dim\Gamma \le 8$ | $\dim\Gamma \le 19$ | $\dim\Gamma \le 46$ |
| | or $\Gamma$ has a normal subgroup $\Xi \cong \mathbb{R}^t$ with $t \le \dim M$, and | | | |
| | $\dim\Gamma \le 4 + t \le 6$ | $\dim\Gamma \le 8 + t \le 12$ | $\dim\Gamma \le 19 + t \le 27$ | $\dim\Gamma \le 46 + t \le 62$ |
| solvable | $\dim\Gamma \le \frac{5}{2}\dim M$ | | $\dim\Gamma \dot{\le} 18$ | $\dim\Gamma \dot{\le} 40$ |
| abelian | $\dim\Gamma \le \dim M$ | | | |
| compact | $\dim\Gamma = 3$, and $\alpha$ is classical, or $\dim\Gamma \le 1$ | $\dim\Gamma = 8$, and $\alpha$ is classical, or $\dim\Gamma \le 4$ | $\dim\Gamma = 21$, and $\alpha$ is classical, or $\dim\Gamma \le 13$ | $\dim\Gamma = 52$, and $\alpha$ is classical, or $\dim\Gamma \le 36$ |

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APPENDIX

# LIE STRUCTURE THEORY FOR NON-LIE GROUPS

This appendix is intended to serve as a reference to results on the structure of locally compact (connected, finite-dimensional) groups. Much of the material is known, some of it is new.

The class of locally compact groups admits a strong structure theory. This is due to the fact that—via approximation (projective limits)—important parts of the theory of Lie groups and Lie algebras carry over. This phenomenon becomes particularly striking if one assumes, in addition, that the groups under consideration are connected and of finite dimension. The aim of the present notes is to collect results and to show that Lie theory completely describes the rough structure (i.e., the lattice of closed connected subgroups) of locally compact finite-dimensional groups. Moreover, we shall describe the possibilities for locally compact connected non-Lie groups of finite dimension.

We shall only consider Hausdorff groups (and shall, therefore, form quotients only with respect to closed subgroups—except in Example **A1.7**).

## A1. Dimension

First of all, we need a notion of dimension. Mainly, we shall use the so-called *small inductive dimension*, denoted by dim.

**A1.1** DEFINITION. Let $X$ be a topological space. We say that $\dim X = -1$ if, and only if, $X$ is empty. If $X$ is non-empty, and $n$ is a natural number, then we say that $\dim X \leq n$ if, and only if, for every point $x \in X$ and every neighborhood $U$ of $x$ in $X$ there exists a neighborhood $V$ of $x$ such that $V \subseteq U$ and the boundary $\partial V$ satisfies $\dim \partial V \leq n-1$. Finally, let $\dim X$ denote the minimum of all $n$ such that $\dim X \leq n$; if no such $n$ exists, we say that $X$ has infinite dimension.

Obviously, $\dim X$ is a topological invariant. A non-empty space $X$ satisfies $\dim X = 0$ if, and only if, there exists a neighborhood base consisting of closed open sets. Consequently, a $T_1$ space of dimension 0 is totally disconnected.

See **[R25]** for a study of the properties of dim for separable metric spaces. Although it is quite intuitive, our dimension function does not work well for arbitrary spaces. Other dimension functions, notably covering dimension **[R39, 3.1.1]**, have turned out to be better suited for general spaces, while they coincide with dim for separable metric spaces. See **[R39]** for a comprehensive treatment. Note, however, that small inductive dimension coincides with large inductive dimension and covering dimension, if applied to locally compact groups **[R2]**, **[R37]**. The duality theory for compact abelian groups uses covering dimension rather than inductive dimension, cf. [**R33**, pp. 106–111], [**R15**, 3.11]. For this special case, we shall prove the equality in **A4.7** below.

We collect some important properties of small inductive dimension.

**A1.2** THEOREM. *For every natural number $n$, we have that $\dim \mathbf{R}^n = n$.*

PROOF. [**R25**, Th. IV 1], or [**R39**, 3.2.7] in combination with [**R39**, 4.5.10]. □

**A1.3** LEMMA. *Let $X$ be a non-empty Hausdorff space.*

- **a.** *For every subspace $Y$ of $X$, we have that $\dim Y \leq \dim X$.*
- **b.** $\dim X \leq n$ *if, and only if, every point $x \in X$ has some neighborhood $U_x$ such that* $\dim U_x \leq n$.
- **c.** $\dim X \geq n$ *if, and only if, there exists a point $x \in X$ with arbitrarily small neighborhoods of dimension at least $n$.*
- **d.** *If $X$ is locally homogeneous, then* $\dim X = \dim U$ *for every neighborhood $U$ in $X$.*
- **e.** *If $X$ is locally compact, then* $\dim X = 0$ *if, and only if, $X$ is totally disconnected.*
- **f.** *If $X$ is the product of a family of (non-empty) finite discrete spaces, then* $\dim X = 0$.
- **g.** *If* $\dim X = 0$, *then* $\dim(\mathbf{R}^n \times X) = n$ *for every natural number $n$.*

PROOF. An argument by induction on $\dim X$ yields **a**, compare [**R39**, 4.1.4]. Assertions **b** and **c** are immediate consequences of the definition, and they imply assertion **d**. By **b**, it suffices to prove assertion **e** for compact spaces. A compact (Hausdorff) space has at every point a neighborhood base of closed open sets if, and only if, it is totally disconnected [**R39**, 3.1.3]. Assertion **f** follows immediately from **e** since the product of a family of (non-empty) finite discrete spaces is compact and totally disconnected.

Finally, assume that $\dim X = 0$. We proceed by induction on $n$. If $n = 0$, then $\dim(\mathbb{R}^0 \times X) = \dim X = 0$. So assume that $n > 0$, and that $\dim \mathbf{R}^{n-1} \times X = n-1$. Let $U$ be a neighborhood of $(r, x)$ in $\mathbf{R}^n \times X$. Since $\dim X = 0$, there exists a closed open set $V$ in $X$ and a ball $B$ around $r$ in $\mathbf{R}^n$ such that $(r, x) \in B \times V \subseteq U$. Since the boundary $\partial(B \times V)$ is contained in $\partial B \times V$, we infer from our induction hypothesis that $\dim U \leq n$. The subspace $\mathbf{R}^n \times \{x\}$ of $\mathbf{R}^n \times X$ has dimension $n$. This completes the proof of assertion **g**. □

**A1.4** REMARK. Assertion **A1.3d** applies, in particular, to topological manifolds, and to topological groups. Note that, if a space $X$ is not locally homogeneous, it may happen that there exists a point $x \in X$ with the property that $\dim U < \dim X$ for every sufficiently small neighborhood $U$ of $x$. E.g., consider the topological sum of $\mathbf{R}$ and a single point.

Let $G$ be a locally compact group. Finiteness of $\dim G$ allows to obtain analogues of counting arguments, as used in the theory of finite groups. In particular, we have the following:

**A1.5** THEOREM. *Let $G$ be a locally compact group, and assume that $H$ is a closed subgroup of $G$. Then* $\dim G = \dim G/H + \dim H$.

PROOF. This follows from [**R34**, Sect. 5, Cor 2], since by **A1.2** and **A1.3** inductive dimension has the properties a)–e) that are required in [**R34**, p. 64f]. □

**A1.6** REMARK. The same conclusion holds if we replace small inductive dimension by large inductive dimension, or by covering dimension, see [**R38**].

**A1.7** EXAMPLE. *The closedness assumption on $H$ is indispensable in* **A1.5**. *E.g., consider the additive group* $\mathbf{R}$. *Since* $\mathbf{Q}$ *is dense in* $\mathbf{R}$, *the factor group* $\mathbf{R}/\mathbf{Q}$ *has the indiscrete topology. Hence* $\dim \mathbf{R}/\mathbf{Q} = 0 = \dim \mathbf{Q}$, *but* $\dim \mathbf{R} = 1$.

Denoting the connected component of $G$ by $G^1$ and using **A1.3e**, we obtain:

**A1.8** THEOREM.

- **a.** *If $G$ is a locally compact group, then* $\dim G = \dim G^1$.
- **b.** *If $H$ is a closed subgroup of a locally compact group $G$, and* $\dim H = \dim G < \infty$, *then* $G^1 \leq H$.

PROOF. Assertion a follows from **A1.5** and the fact that $G/G^1$ is totally disconnected [**R15**, 7.3]. If $\dim H = \dim G < \infty$ then $\dim G/H = 0$ by **A1.5**, and assertion **b** follows from the fact that the connected component can only act trivially on the totally disconnected space $G/H$.

**A1.9** THEOREM. *Let $G$ be a locally compact group. Assume that $N$ is a closed normal subgroup, and $C$ is a closed $\sigma$-compact subgroup such that* $\dim(C \cap N) = 0$ *and* $CN = G$. *Then* $\dim G = \dim C + \dim N$.

PROOF. Since $G/N = CN/N \cong C/(C \cap N)$ [**R15**, 5.33], the assertion follows from Theorem **A1.5**. □

**A1.10** DEFINITION.

- **a.** If, in the situation of **A1.9**, we have in addition that $C$ and $N$ are *connected*[2], we say that $G$ is an *almost semi-direct product* of $C$ and $N$.
- **b.** If, moreover, the subgroup $C$ is normal as well, we say that $G$ is an *almost direct product* of $C$ and $N$.

The terminology suggests that every almost (semi-)direct product is a proper (semi-)direct product, "up to a totally disconnected normal subgroup". This may be made precise in two different ways. Either the almost (semi-)simple product is obtained *from* a proper (semi-)direct product by forming the quotient modulo a totally disconnected subgroup, or one obtains a proper (semi-)direct product *after* passing to such a quotient. From the first of these viewpoints, our terminology is fully justified. In fact, every almost (semi-)direct product $G = CN$ is isomorphic to the quotient of the proper (semi-)direct product $C \ltimes N$ modulo $K$, where the action of $C$ on $N$ is given by conjugation in $G$, and $K = \{(g, g^{-1});\ g \in C \cap N\}$ is isomorphic to the totally disconnected group $C \cap N$.

From the second point of view, our terminology is adequate for almost direct products, but almost semi-direct products are more delicate.

**A1.11** THEOREM. *If $G$ is an almost direct product of (closed connected) subgroups $N_1$ and $N_2$, then $N_1 \cap N_2$ is contained in the center of $G$. The factor group*

[2] Note that a locally compact connected group is generated by any compact neighborhood [**R15**, 5.7], and is therefore $\sigma$-compact.

*$G/(N_1 \cap N_2)$ is the direct product of $N_1/(N_1 \cap N_2)$ and $N_2/(N_1 \cap N_2)$. Moreover, $\dim G = \dim(G/(N_1 \cap N_2))$, and $\dim N_i = \dim(N_i/(N_1 \cap N_2))$.*

PROOF. The assertions follow from the fact that the connected group $G$ acts trivially on the totally disconnected normal subgroup $N_1 \cap N_2$, and **A1.5**. □

**A1.12** EXAMPLE. *For almost semi-direct products $G = CN$, the intersection $C \cap N$ need not be a normal subgroup of $G$. E.g., let $N = \mathrm{SO}_3\mathbb{R}$, let $C = \mathbb{T}$, the circle group, and let $\gamma: C \to N$ be an embedding. Let $a$ be an element of order 4 in $C$. Now $(c, x)(d, y) := (cd, (d^{-1})^\gamma x d^\gamma y)$ defines a semi-direct product $G = C \ltimes N$. It is easy to see that $Z = \langle (a, (a^{-1})^\gamma) \rangle$ is contained in the center of $G$. We infer that $\bar{G} := G/Z$ is an almost semi-direct product of $\bar{C} := ZC/Z$ and $\bar{N} := ZN/Z$, and that $Z(a, 1) = Z(1, a^\gamma)$ belongs to $\bar{C} \cap \bar{N}$, but $Z(1, a)$ is not central in $\bar{G}$. Since $\bar{G}$ is connected and $\bar{C} \cap \bar{N}$ is totally disconnected, normality of $\bar{C} \cap \bar{N}$ would imply that $\bar{C} \cap \bar{N}$ is central.*

**A1.13** LEMMA. *Assume that the topological space $X$ is the countable union of relatively compact neighborhoods $U_n$ such that $\dim U_n = d$ for every $n$, and let $Y$ be a separable metric space. If $\varphi: X \to Y$ is a continuous injection, then $\dim X = \dim X^\varphi \leq \dim Y$.*

PROOF. We adapt the proof from [**R13**]. Small inductive dimension is defined locally, whence $\dim X = \dim U_n$ for every $n$. Since $U_n$ is compact, we obtain that $U_n$ and $U_n^\varphi$ are homeomorphic, and $\dim U_n = \dim U_n^\varphi$. Now $\dim X^\varphi = \dim U_n^\varphi$ by the sum theorem [**R35**, p. 14]. Finally, monotony of dim yields that $\dim X^\varphi \leq \dim Y$. □

Every locally compact connected group $G$ and every quotient space $G/H$, where $H$ is a closed subgroup of $G$, satisfies the assumptions on $X$ in **A1.13**. In fact, the group $G$ is algebraically generated by every compact neighborhood of $\mathbb{1}$. Therefore, we obtain the following applications.

**A1.14** COROLLARY. *Let $G$ be a locally compact connected group. If $G$ acts on a separable metric space $Y$, then $\dim(G/G_y) = \dim y^G \leq \dim Y$, where $y \in Y$ is any point, $G_y$ is its stabilizer, and $y^G$ its orbit under the given action. Important special cases are the following.*

- **a.** *If $L$ is a Lie group, and $\alpha: G \to L$ is a continuous homomorphism, then $\dim(G/\ker\alpha) = \dim G^\alpha \leq \dim L$.*
- **b.** *If $G$ acts linearly on $V \cong \mathbb{R}^n$, then $\dim(G/G_v) = \dim vG \leq \dim V = n$, where $v \in V$ is any vector, $G_v$ is its stabilizer, and $vG$ its orbit under the given action.*

In general, a bijective continuous homomorphism of topological groups need not be a topological isomorphism; e.g., consider the identity with respect to the discrete and some non-discrete group topology. Locally compact connected groups, however, behave well.

**A1.15** THEOREM. *Let $G$ be a locally compact group, and assume that $G$ is $\sigma$-compact. Then the following hold:*

- **a.** *If $X$ is a locally compact space, and $\alpha: (X, G) \to X$ is a continuous transitive action, then the mapping $g \mapsto \alpha(x, g)$ is open for every $x \in X$.*

**b.** *If $\mu: G \to H$ is a surjective continuous homomorphism onto a locally compact group $H$, then $\mu$ is in fact a topological isomorphism.*

PROOF. Assertion **a** is due to [**R10**], cf. also [**R23**]. Assertion **b** follows by an application of **a** to the regular action of $G$ on $H = G^\mu$. Compare also [**R15**, 3.29]. □

Recall that a locally compact group $G$ is $\sigma$-compact if it is compactly generated; in particular if $G/G^1$ is compact, or if $G/G^1$ is countable.

## A2. The Approximation Theorem

If $G$ is a locally compact group such that $G/G^1$ is compact, then there exist arbitrarily small compact normal subgroups such that the factor group is a Lie group. To be precise:

**A2.1** APPROXIMATION THEOREM. *Let $G$ be a locally compact group such that $G/G^1$ is compact.*

- **a.** *For every neighborhood $U$ of $\mathbb{1}$ in $G$ there exists a compact normal subgroup $N$ of $G$ such that $N \subseteq U$ and $G/N$ admits local analytic coordinates that render the group operations analytic.*
- **b.** *If, moreover, $\dim G < \infty$, then there exists a neighborhood $V$ of $\mathbf{1}$ such that every subgroup $H \subseteq V$ satisfies $\dim H = 0$. That is, there is a totally disconnected compact normal subgroup $N$ such that $G/N$ is a Lie group with $\dim G = \dim G/N$.*

PROOF. [**R32**, Chap. IV], [**R11**, Th. 9], see also [**R30**, II.10, Th.18]. □

**A2.2** REMARK. For locally compact groups in general, one knows that there always exists an open subgroup $G$ such that $G/G^1$ is compact, cf. [**R11**, 3.5].

We obtain a useful criterion.

**A2.3** THEOREM. *A locally compact group $G$ is a Lie group if, and only if, every compact subgroup of $G$ is a Lie group. If $G$ is a locally compact group such that $G/G^1$ is compact, then we can say even more: in this case, the group $G$ is a Lie group if, and only if, every compact normal subgroup is a Lie group.*

PROOF. Closed subgroups of Lie groups are Lie groups; see, e.g., [**R17**, VIII.1]. Conversely, assume that every compact subgroup of $G$ is a Lie group. According to **A2.2**, there exists an open subgroup $H$ of $G$ such that $H/H^1$ is compact. Let $N$ be a compact normal subgroup of $H$ such that $H/N$ is a Lie group. Then $N$ is a Lie group by our assumption, and has, therefore, no small subgroups. Consequently, there exists a neighborhood $U$ in $H$ such that every subgroup $M \subseteq N \cap U$ is trivial. Let $M$ be a compact normal subgroup of $H$ such that $M \subseteq U$ and $H/M$ is a Lie group. Then $H/(M \cap N)$ is a Lie group as well [**R11**, 1.5], but $M \cap N = \{\mathbf{1}\}$. Thus $H$ is a Lie group, and $G$ is a Lie group as well, since $H$ is open in $G$. If $G/G^1$ is compact, our proof works for $H = G$, yielding the second part of our assertion. □

For the case where $G/G^1$ is compact, the criterion **A2.3** can also be deduced from the fact that the class of Lie groups is closed with respect to extensions [**R27**, Th. 7].

Compact subgroups play an important rôle in the theory of locally compact groups. They are understood quite well (see also the chapter on compact groups), especially in the connected case.

**A2.4** THEOREM. *Let $G$ be a locally compact group such that $G/G^1$ is compact.*

- **a.** *Every compact subgroup of $G$ is contained in some maximal compact subgroup of $G$.*
- **b.** *The maximal compact subgroups of $G$ form a single conjugacy class.*
- **c.** *There exists some natural number $n$ such that the underlying topological space of $G$ is homeomorphic to $\mathbf{R}^n \times C$, where $C$ is one of the maximal compact subgroups of $G$.*
- **d.** *In particular, every maximal compact subgroup of a locally compact connected group is connected.*

PROOF. [**R27**, §4, Th. 13], cf. also [**R17**, Th. 3.1], and [**R21**]. □

Considering, for example, a discrete infinite torsion group, one easily sees that some connectedness assumption is essential for the mere existence of maximal compact subgroups. Note that **A2.4**, in combination with the solution of D. HILBERT's Fifth Problem, provides another proof for **A2.3**.

There is, in general, no natural choice of $N$ such that $G/N$ is a Lie group. However, we have:

**A2.5** THEOREM. *Let $G$ be a locally compact connected group of finite dimension. If both $N_1$ and $N_2$ are closed normal subgroups such that $\dim N_i = 0$ and $G/N_i$ is a Lie group, then $G/N_1$ is locally isomorphic to $G/N_2$.*

PROOF. The factor group $G/(N_1 \cap N_2)$ is also a Lie group, cf. [**R11**, 1.5]. Now $N_i/(N_1 \cap N_2)$ is a Lie group of dimension 0, and therefore discrete. This implies that $G/(N_1 \cap N_2)$ is a covering group for both $G/N_1$ and $G/N_2$. □

It is often more convenient to work with compact normal subgroups than with arbitrary closed normal subgroups. The general case may be reduced to the study of quotients with respect to compact kernels.

**A2.6** THEOREM. *Let $G$ be a locally compact connected group of finite dimension. If $N$ is a closed normal subgroup such that $\dim N = 0$ and $G/N$ is a Lie group, then there exists a compact normal subgroup $M$ of $G$ such that $M \leq N$ and $G/M$ is a Lie group. The natural mapping $\pi: G/M \to G/N$ is a covering.*

PROOF. Choose a compact neighborhood $U$ of $\mathbf{1}$ in $G$. According to **A2.1b**, there exists a compact normal subgroup $N'$ such that $N' \subseteq U$ and $G/N'$ is a Lie group. Now $M := N \cap N'$ has the required properties; in fact, $G/M$ is a Lie group by [**R11**, 1.5], and the kernel of the natural mapping $\pi: G/M \to G/N$ is a totally disconnected Lie group, hence discrete. □

A main reason why, in general, quotients with respect to compact subgroups behave better than quotients with respect to arbitrary closed subgroups is the following.

**A2.7** LEMMA. *Let $G$ be a topological group, and let $H$ be a compact subgroup of $G$. Then the natural mapping $\pi: G \to G/H$ is a perfect mapping, i.e., for every compact subset $C \subseteq G/H$ the preimage $C^{\pi^{-1}}$ is also compact.*

PROOF. Since $H$ is compact, the natural mapping $\pi$ is closed [**R15**, 5.18]. Now $\pi$ is a closed mapping with compact fibers, and therefore perfect [**R8**, XI.5]. $\square$

## A3. The rough structure

In this section, we introduce the lattice of closed connected subgroups of a locally compact group, with additional binary operations. We show that this structure is preserved under the forming of quotients modulo compact totally disconnected normal subgroups.

We start with a fundamental observation. For subsets $A, B$ of a topological group $G$, let $[A, B]$ be the *closed* subgroup that is generated by the set $\{a^{-1}b^{-1}ab;\ a \in A, b \in B\}$. With this notation, we have:

**A3.1** LEMMA. *Let $G$ be a topological group, and let $N$ be a totally disconnected closed normal subgroup of $G$. If $A, B$ are connected subgroups of $G$, then $[A, B] = \{\mathbb{1}\}$ if, and only if, $[AN/N, BN/N] = \{\mathbb{1}\}$.*

PROOF. This follows directly from the fact that $[A, B]$ is connected [**R27**, Lemma 2.1]. $\square$

Let $G$ be a locally compact group. We are interested in the lattice of closed connected subgroups; i.e., for closed connected subgroups $A, B$ of $G$, we consider the smallest *closed* (necessarily connected) subgroup $A \vee B$ that contains both $A$ and $B$, and the biggest *connected* (necessarily closed) subgroup $A \wedge B$ that is contained in both $A$ and $B$. Note that $A \wedge B = (A \cap B)^{\mathbf{1}}$. Moreover, we are interested in the connected components of the normalizer and the centralizer of $B$, taken in $A$ (denoted by $\mathrm{N}^{\mathbf{1}}_A B$ and $\mathrm{C}^{\mathbf{1}}_A B$, respectively). Finally, recall that the commutator subgroup $[A, B]$ is necessarily connected, while closedness is enforced by the very definition.

**A3.2** DEFINITION.

- **a.** For any locally compact group, let **Struc**$(G)$ be the algebra of all closed connected subgroups of $G$, endowed with the binary operations $\vee$, $\wedge$, $\mathrm{N}^{\mathbf{1}}$, $\mathrm{C}^{\mathbf{1}}$, $[\,,\,]$, as introduced above. We call **Struc**$(G)$ the *rough structure* of $G$.
- **b.** Let **Comp**$(G)$ be the set of *compact* connected subgroups of $G$, and let **Cpfree**$(G)$ be the set of *compact-free* closed connected subgroups of $G$.

**A3.3** REMARKS.

- **a.** Note that **Comp**$(G)$ and **Cpfree**$(G)$ are subsets but, in general, not subalgebras of **Struc**$(G)$.
- **b.** Of course, **Struc**$(G)$ = **Struc**$(G^{\mathbf{1}})$.

We are going to investigate the effect of continuous group homomorphisms on the rough structure. Our results will justify the vague feeling that the quotient of a locally compact group by a compact totally disconnected normal subgroup has "roughly the same structure".

**A3.4** PROPOSITION. *Let $G$ and $H$ be locally compact groups, and let $\alpha: G \to H$ be a continuous homomorphism. For every closed connected subgroup $A$ of $G$, let $A^{\bar\alpha}$ be the closure of $A^\alpha$ in $H$.*

- **a.** *The mapping $\bar\alpha$ maps* **Struc**$(G)$ *to* **Struc**$(H)$, *and* **Comp**$(G)$ *to* **Comp**$(H)$.
- **b.** *For $A \leq B \leq G$, we have that $A^{\bar\alpha} \leq B^{\bar\alpha}$.*
- **c.** *For every choice of $A, B \in$* **Struc**$(G)$, *we have that $A^{\bar\alpha} \vee B^{\bar\alpha} \leq (A \vee B)^{\bar\alpha}$ and $A^{\bar\alpha} \wedge B^{\bar\alpha} \geq (A \wedge B)^{\bar\alpha}$.*
- **d.** *For every choice of $A, B \in$* **Struc**$(G)$, *we have that $(\mathrm{N}^1_A B)^{\bar\alpha} \leq \mathrm{N}^1_{A^\alpha} B^{\bar\alpha}$, and that $(\mathrm{C}^1_A B)^{\bar\alpha} \leq \mathrm{C}^1_{A^\alpha} B^{\bar\alpha}$.*

PROOF. Assertions **a** and **b** are obvious from the definition of $\bar\alpha$ and the fact that every continuous mapping preserves compactness. From $A, B \leq C$ it follows that $A^{\bar\alpha}, B^{\bar\alpha} \leq C^{\bar\alpha}$. This implies that $A^{\bar\alpha} \vee B^{\bar\alpha} \leq (A \vee B)^{\bar\alpha}$. The second part of **c** follows analogously. Assertion **d** follows from the well-known inequalities $(\mathrm{N}_A B)^\alpha \leq \mathrm{N}_{A^\alpha} B^\alpha$ and $(\mathrm{C}_A B)^\alpha \leq \mathrm{C}_{A^\alpha} B^\alpha$, combined with the fact that continuous images of connected spaces are connected. □

Even if $\alpha$ is a quotient morphism with totally disconnected kernel, the mapping $\bar\alpha$ may be far from being injective. E.g., consider a quotient mapping from $\mathbf{R}^2$ onto $\mathbf{T}^2$. In fact, the rough structure **Struc**$(\mathbf{R}^2)$ has uncountably many elements, while **Struc**$(\mathbf{T}^2)$ is countable. However, we have:

**A3.5** THEOREM. *Let $G$ be a locally compact group, let $N$ be a compact totally disconnected normal subgroup, and let $\pi$ be the natural epimorphism from $G$ onto $G/N$. Then the following hold:*

- **a.** *For every $A \in$* **Struc**$(G)$, *we have that $A^\pi = A^{\bar\pi}$, and that $\dim A = \dim A^\pi$.*
- **b.** *The mapping $\bar\pi$ induces an isomorphism of* **Struc**$(G)$ *onto* **Struc**$(G/N)$.
- **c.** *The mapping $\bar\pi$ induces bijections of* **Comp**$(G)$ *onto* **Comp**$(G/N)$, *and of* **Cpfree**$(G)$ *onto* **Cpfree**$(G/N)$.

PROOF. According to **A3.3**, we may assume that $G$ is connected. Hence $G$ is $\sigma$-compact, and so is every closed subgroup $A$ of $G$. In particular, $AB/A \cong B/(A \cap B)$ for every closed subgroup $B$ of $\mathrm{N}_G A$, see [**R15**, 5.33]. Mutatis mutandis, the same assertion holds for the epimorphic images.
(i) Being an epimorphism with compact kernel, the mapping $\pi$ is closed [**R15**, 5.18]. Moreover, we infer that the restriction of $\pi$ to $A$ is a closed surjection onto $A^\pi$, hence a quotient mapping. Therefore, $\dim A = \dim A^\pi$ by **A1.5**, and assertion **a** is proved.
(ii) For every $H \in$ **Struc**$(G/N)$, let $H^{\pi^\leftarrow}$ be the connected component of the $\pi$-preimage $H^{\pi^{-1}}$. Since $\pi$ is continuous, we infer that $\pi^\leftarrow$ is a mapping from **Struc**$(G/N)$ to **Struc**$(G)$. For every $H$ in **Struc**$(G/N)$, the group $H/(H^{\pi^\leftarrow \pi}) \cong H^{\pi^{-1}}/(H^{\pi^\leftarrow} N)$ is totally disconnected. Hence $H^{\pi^\leftarrow \pi} \geq H^1$, and $H^{\pi^\leftarrow \pi} = H$ since $H$ is connected. For every $A$ in **Struc**$(G)$, we have that $A$ is a normal closed subgroup of $AN = A^{\pi\pi^{-1}}$; recall that $N$ centralizes $A$. The quotient $AN/A \cong N/(N \cap A)$ is totally disconnected. We infer that $A = (AN)^1 = A^{\pi\pi^\leftarrow}$.
(iii) The mapping $\pi^\leftarrow$ is monotone. In fact, let $H \leq K$ in **Struc**$(G/N)$. Then $H^{\pi^\leftarrow}$ is a connected subgroup of $H^{\pi^{-1}} \leq K^{\pi^{-1}}$, hence $H^{\pi^\leftarrow} \leq K^{\pi^\leftarrow}$. In view of **A3.4b**, this shows that $\bar\pi$ respects the binary operations $\vee$ and $\wedge$.

(iv) From **A3.1**, we infer that $\pi$ respects the operations $[,]$ and $\mathrm{C}^1$. Arguments similar to those in step (ii) show that $\pi$ respects the operation $\mathrm{N}^1$ as well; recall that every epimorphism of (discrete) groups maps normalizers to normalizers.
(v) The natural epimorphism $\pi$ is a proper continuous mapping, see **A2.7**. Thus $\pi$ is a bijection of $\mathbf{Comp}(G)$ onto $\mathbf{Comp}(G/N)$. For $A \in \mathbf{Cpfree}(G)$, we obtain that $A \cap N = \{\mathbf{1}\}$, hence $A \cong A^\pi \in \mathbf{Cpfree}(G/N)$. Conversely, assume that $A \in \mathbf{Struc}(G) \setminus \mathbf{Cpfree}(G)$. According to **A2.4d**, there exists a connected non-trivial compact subgroup $C$ of $A$. Now $C^\pi$ is a non-trivial compact subgroup of $A^\pi$. This completes the proof of assertion **c**. □

## A4. Compact groups

By the following result, the structure theory of compact connected groups is, essentially, reduced to the theory of compact almost simple Lie groups and the theory of compact abelian groups:

**A4.1** THEOREM.

**a.** *Let $G$ be a compact connected group. Then there exist a compact connected abelian group $C$, a family $(S_i)_{i\in I}$ of almost simple compact Lie groups $S_i$, and a surjective homomorphism $\eta : C \times \prod_{i\in I} S_i \to G$ with $\dim\ker\eta = 0$. The image $C^\eta$ is the connected component of the center of $G$, and the commutator group $G'$ equals $(\prod_{i\in I} S_i)^\eta$.*

**b.** *Conversely, every group of the form $C \times \prod_{i\in I} S_i$, as in* **a**, *is compact; hence also $(C \times \prod_{i\in I} S_i)^\eta$.*

PROOF. [**R29**, Th. 1, Th. 2], cf. [**R5**, App. I, no. 3, Prop. 2], [**R27**, Remark after Lemma 2.4], [**R45**, §25]. □

Note that, in general, the connected component $C^\eta$ of the center of $G$ is not a complement, but merely a supplement of the commutator group in $G$. Since the topology of the commutator group $(\prod_{i\in I} S_i)^\eta$ is well understood, a complement would be fine in order to show the more delicate topological features of $G$. The following result asserts the existence of a complement (which, in general, is not contained in the center of $G$).

**A4.2** THEOREM. *Every compact connected group is a semi-direct product of its commutator group and an abelian compact connected group.*

PROOF. [**R20**, 2.4]. A generalization to locally compact groups, involving rather technical assumptions, is given in [**R19**, Th. 6]. □

For a compact connected group $G$, let $\eta: C \times \prod_{i\in I} S_i \to G$ be an epimorphism as in **A4.1**. The possible factors $S_i$ are known from Lie Theory; see, e.g., [**R36**, Ch. 5]. In order to understand the structure of $C$, one employs the PONTRYAGIN-VAN KAMPEN duality for (locally) compact abelian groups. See [**R41**], [**R42**] for a treatment that stresses the functorial aspects of duality. The dual $\hat{C}$ is a discrete torsion-free abelian group of rank $c$, and $c$ equals the covering dimension of $C$ if one of the two is finite [**R33**, Th. 34, p. 108], [**R15**, 24.28]. Hence there are embeddings $\mathbf{Z}^{(c)} \to \hat{C}$ and $\hat{C} \to \mathbf{Q} \otimes \hat{C} \cong \mathbf{Q}^{(c)}$. Dualizing again, we obtain a convenient description of the class of compact connected abelian groups:

**A4.3 THEOREM.** *Let $C$ be a compact connected abelian group.*

a. *If $C$ has finite covering dimension $c$, then there are epimorphisms $\sigma: \hat{\mathbf{Q}}^c \to C$ and $\tau: C \to \mathbf{T}^c$, both with totally disconnected kernel.*
b. *If $C$ has infinite covering dimension, then there exists a cardinal number $c$ such that there are epimorphisms $\sigma: \hat{\mathbf{Q}}^c \to C$ and $\tau: C \to \mathbf{T}^c$, both with totally disconnected kernel.*

Sometimes, one needs a more detailed description, as supplied by

**A4.4 REMARK.** Dualizing the description of $\mathbf{Q}$ as inductive limit of the system $(\frac{1}{n}\mathbf{Z})_{n\in\mathbf{N}}$ (endowed with natural inclusions $\frac{1}{n}\mathbf{Z} \to \frac{1}{nd}\mathbf{Z}$), we obtain that the character group $\hat{\mathbf{Q}}$ is the projective limit of the system $(T_n)_{n\in\mathbf{N}}$, where $T_n = \mathbf{T}$ for each $n$, with epimorphisms $t \mapsto t^d: T_{nd} \to T_n$. Every one-dimensional compact connected group is an epimorphic image of $\hat{\mathbf{Q}}$.

Within the boundaries that are set up by the fact that locally compact connected abelian groups are divisible [**R15**, 24.25], we are free to prescribe the torsion subgroup of a one-dimensional compact connected group. In fact, let $\mathbb{P}$ be the set of all prime numbers, and let $P \subseteq \mathbb{P}$ be an arbitrary subset. In the multiplicative monoid of natural numbers, let $\mathbf{N}_P$ be the submonoid generated by $P$ (i.e., $\mathbf{N}_P$ consists of all natural numbers whose prime decomposition uses only factors from $P$). With this notation, we have:

**A4.5 THEOREM.** *For every subset $P \subseteq \mathbb{P}$, there exists a compact connected group $C$ with $\dim C = 1$ and the following properties: If $c \in C$ has finite order $n$, then $n \in \mathbf{N}_P$. Conversely, for every $n \in \mathbf{N}_P$ there exists some $c \in C$ of order $n$.*

PROOF. The limit $S_P$ of the subsystem $(T_n)_{n\in\mathbf{N}_P}$ of the projective system considered in **A4.4** has the required property. □

See [**R15**, 10.12–10.15] for alternate descriptions of the "solenoids" $S_P$.

**A4.6 EXAMPLES.** *Of course, $S_{\mathbb{P}} = \hat{\mathbf{Q}}$, and $S_\emptyset = \mathbf{T}$. The group $S_{\{p\}}$ is the dual of the group $\bigcup_{n=0}^{\infty} \frac{1}{p^n}\mathbf{Z}$, its torsion group has elements of orders that are not divisible by $p$.*

We conclude this chapter with an observation that relates **A4.3** to the inductive dimension function, as used in the rest of this paper.

**A4.7 THEOREM.** *Small inductive dimension and covering dimension coincide for compact connected abelian groups.*

PROOF. Let $A$ be a compact connected abelian group, and let $d$ denote its covering dimension. The dual group $\hat{A}$ is discrete [**R15**, 23.17] and torsion-free (since $A$ is connected, [**R15**, 24.25]). Assume first that $d$ is finite. According to [**R33**, Th. 34, p. 108], we have the equality $d = \operatorname{rank} \hat{A}$. For a maximal free subgroup $F$ of $\hat{A}$ we infer that $F \cong \mathbf{Z}^d$, and $\hat{A}/F$ is a torsion group. Consequently, the annihilator $F^\perp$ is totally disconnected, and has inductive dimension 0 by **A1.8a**. Now $\mathbf{T}^d \cong \hat{F} \cong A/(F^\perp)$, and we conclude from **A1.5** and **A1.8** that $\dim A = \dim \mathbf{T}^d = d$. If $d$ is infinite, then $\operatorname{rank} A$ is infinite, and we infer that $\dim A$ is infinite as well. □

## A5. The abelian case

In this section, we study connected locally compact abelian groups. Special attention will be given to decompositions of such groups, and their automorphisms.

For the structure theory of locally compact *abelian* groups, PONTRJAGIN-VAN KAMPEN duality is the strongest tool by far. See, e.g. [**R15**, Chap. VI], [**R3**], [**R33**], [**R40**, VI]. For the functorial aspects of duality theory, see [**R41**], [**R42**]. We give some results that are of interest for our special point of view. In particular, we concentrate on the connected case.

**A5.1** DECOMPOSITION THEOREM. *Let $A$ be a locally compact connected abelian group. Then there exist closed subgroups $R$ and $C$ of $A$ such that $A$ is the (interior) direct product $R \times C$, and $R \cong \mathbb{R}^a$ for some natural number $a$, while $C$ is compact and connected. The group $C$ is the maximal compact subgroup of $A$, hence it is a characteristic subgroup.*

PROOF. [**R15**, 9.14], [**R33**, Th. 26]. □

The Decomposition Theorem is a special case of the Theorem of MALCEV and IWASAWA **A2.4**. Using the decomposition $A = R \times C$, we shall gain information about the automorphisms of $A$. The following lemma, which is also of interest for its own sake, will be needed.

**A5.2** LEMMA. *Let $a, b$ be natural numbers. Every* continuous *group homomorphism from $\mathbb{R}^a$ to $\mathbb{R}^b$ is an $\mathbb{R}$-linear mapping.*

PROOF. For every $x \in \mathbb{R}^n$ and every integer $z \neq 0$, there exists exactly one element $y \in \mathbb{R}^n$ (namely, $\frac{1}{z}x$) such that $zy = x$. Therefore every additive mapping $\mu: \mathbb{R}^a \to \mathbb{R}^b$ is in fact $\mathbb{Q}$-linear. Continuity of $\mu$ implies that $\mu$ is even $\mathbb{R}$-linear, since $\mathbb{Q}x$ is dense in $\mathbb{R}x$ for every $x \in \mathbb{R}^a$. □

Note that, if $a, b \neq 0$, then there exist many discontinuous $\mathbb{Q}$-linear mappings from $\mathbb{R}^a$ to $\mathbb{R}^b$.

Given a decomposition $A = R \times C$ as in **A5.1**, the subgroup $R$ is not characteristic in $A = R \times C$, except if $R = A$. In fact, we have the following.

**A5.3** THEOREM. *Let $C$ be a compact group, $R \cong \mathbb{R}^a$, and $A = R \times C$.*

- **a.** *If $\alpha: R \to C$ is a continuous homomorphism, then $\Gamma_\alpha := \{(x, x^\alpha); x \in R\}$ is a closed subgroup of $A$, and $\Gamma_\alpha \cong R$. Moreover, the mapping $\mu_\alpha := ((r, c) \mapsto (r, r^\alpha c))$ is an automorphism of $A$.*
- **b.** *If $B$ is a closed subgroup of $A$ such that $B \cong \mathbb{R}^b$, then there exists some continuous homomorphism $\alpha: R \to C$ such that $B \leq \Gamma_\alpha$. In particular, $b \leq a$.*
- **c.** *If $\mu: \mathbb{R} \to A$ is a continuous homomorphism such that $\mathbb{R}^\mu$ is not closed in $A$, then $\mathbb{R}^\mu \subset C$.*

PROOF. Assertion **a** is straightforward, using the fact that the graph of a continuous function is closed, if the codomain is Hausdorff. Let $B \cong \mathbb{R}^b$ be a closed subgroup of $A$. We consider the projections $\pi_R: A \to R: (r, c) \mapsto r$ and $\pi_C: A \to C: (r, c) \mapsto c$. Since $B$ is compact-free, the restriction of $\pi_R$ to $B$ is injective. Hence there exists a section $\sigma: R \to B$, and for $\alpha := \sigma\pi_C$ we infer that $B \leq \Gamma_\alpha$. This proves **b**. Let $\mu: \mathbf{R} \to A$ be a continuous homomorphism, and assume that $\mathbf{R}^\mu \not\leq C$. Then $\mathbf{R}^{\mu\pi_R}$ is a non-trivial subgroup of $R \cong \mathbf{R}^a$. We infer that $\mu\pi_R$ is an $\mathbf{R}$-linear mapping, and that there exists a section $\sigma: R \to \mathbf{R}$. Now $\mathbf{R}^\mu \leq \Gamma_{\sigma\mu\pi_C}$ is closed in $A$. This proves assertion **c**. □

**A5.4** EXAMPLES.

- **a.** *Dense one-parameter subgroups (or dense analytical subgroups) are familiar from Lie theory; most prominent, perhaps, is the "dense wind" $\mathbf{R} \to \mathbf{T}^2$. In the realm of Lie groups, the closure of a non-closed analytical subgroup has larger dimension than the subgroup itself.*
- **b.** *The dual of the monomorphism $\mathbf{Q} \to \mathbf{R}$, where $\mathbf{Q}$ carries the discrete topology, yields a monomorphism $\mathbf{R} \to \hat{\mathbf{Q}}$ with dense image. Note that* $\dim \hat{\mathbf{Q}} = 1 = \dim \mathbf{R}$. *Equidimensional immersions are typical for non-Lie groups; see* [**R22**], *and* **A8.6** *below.*

Next, we study automorphism groups of locally compact connected abelian groups. We endow $\mathrm{Aut}(A)$ with the coarsest Hausdorff topology that makes $\mathrm{Aut}(A)$ a topological (not necessarily locally compact) transformation group on $A$ (see [**R1**], [**R15**, §26]). With respect to this topology, $\mathrm{Aut}(A)$ and $\mathrm{Aut}(\hat{A})$ are isomorphic as topological groups [**R15**, 26.9]. This has the following immediate consequences [**R15**, 26.8, 26.10]:

**A5.5** THEOREM.

- **a.** *The group of automorphisms of a compact abelian group is totally disconnected.*
- **b.** *Let $G$ be a connected group, and assume that $N$ is a compact abelian normal subgroup of $G$. Then $N$ lies in the center of $G$.*

**A5.6** Let $R$ and $C$ be arbitrary topological groups, but assume that $C$ is abelian[3]. It will be convenient to use additive notation. Let $\alpha$ be an endomorphism of the direct sum $R \oplus C$, and assume that $\alpha$ leaves $C$ invariant. Since $(r+c)^\alpha = r^\alpha + c^\alpha$, we can write $\alpha$ as the (pointwise) sum of the restrictions $\alpha|_R$ and $\alpha|_C$. Since $C^\alpha \leq C$, the restriction $\alpha|_C$ may be considered as an endomorphism of $C$. The restriction $\alpha|_R$ may be decomposed as the sum of the co-restrictions $\alpha|_R^R$ and $\alpha|_R^C$, i.e., we write $r^\alpha = r^{\alpha|_R^R} + r^{\alpha|_R^C}$, where $r^{\alpha|_R^R} \in R$ and $r^{\alpha|_R^C} \in C$. It is very convenient to use the matrix description

$$(r,c)^\alpha = \left(r^{\alpha|_R^R}, r^{\alpha|_R^C} + c^{\alpha|C}\right) = (r,c)\begin{pmatrix} \alpha|_R^R & \alpha|_R^C \\ 0 & \alpha|_C \end{pmatrix} \quad .$$

In fact, an easy computation shows that the usual matrix product describes the composition of endomorphisms of $R \oplus C$, namely

$$(r,c)^{\alpha\beta} = (r,c)\begin{pmatrix} \alpha|_R^R\beta|_R^R & \alpha|_R^R\beta|_R^C + \alpha|_R^C\beta|_C \\ 0 & \alpha|_C\beta|_C \end{pmatrix} \quad .$$

[3] If $C$ is not abelian, the following remarks remain valid if we consider $\mathrm{Hom}(R, Z)$ instead of $\mathrm{Hom}(R, C)$, where $Z$ is the center of $C$.

The group of all automorphisms of $R \oplus C$ that leave $C$ invariant is obtained as

$$\left\{ \begin{pmatrix} \rho & g \\ 0 & \gamma \end{pmatrix} ; \rho \in \mathrm{Aut}(R), \gamma \in \mathrm{Aut}(C), g \in \mathrm{Hom}(R, C) \right\} \quad .$$

Obviously, we have that

$$\left\{ \begin{pmatrix} \mathrm{id}_R & g \\ 0 & \mathrm{id}_C \end{pmatrix} ; g \in \mathrm{Hom}(R, C) \right\}$$

is a normal subgroup, and that

$$\left\{ \begin{pmatrix} \rho & 0 \\ 0 & \mathrm{id}_C \end{pmatrix} ; \rho \in \mathrm{Aut}(R) \right\} \text{ and } \left\{ \begin{pmatrix} \mathrm{id}_R & 0 \\ 0 & \gamma \end{pmatrix} ; \gamma \in \mathrm{Aut}(C) \right\}$$

are subgroups that centralize each other. That is, the group of those automorphisms of $R \oplus C$ that leave $C$ invariant can be written as a semi-direct product $\mathrm{Aut}(R) \ltimes \mathrm{Hom}(R, C) \rtimes \mathrm{Aut}(C)$. Note that parentheses are not necessary.

**A5.7** THEOREM. *Let $A$ be a locally compact connected abelian group, and write $A = RC$, where $R \cong \mathbf{R}^a$ and $C$ is compact and connected.*

- **a.** *The group of automorphisms of $A$ is isomorphic to the semi-direct product $\mathrm{Aut}(C) \ltimes \mathrm{Hom}(\mathbf{R}^a, C) \rtimes \mathrm{GL}_a\mathbf{R}$, the connected component $\mathrm{Aut}(A)^{\mathbf{1}}$ is isomorphic to $\mathrm{Hom}(\hat{C}, \mathbf{R}^a) \rtimes \mathrm{GL}_a\mathbf{R}$.*
- **b.** *If $\dim C = c < \infty$, then $\mathrm{Aut}(A)^{\mathbf{1}}$ is a linear Lie group; in fact, there is a monomorphism $\iota\colon \mathrm{Aut}(A) \to \mathrm{GL}_c\mathbb{Q} \ltimes \mathrm{Hom}(\mathbb{Q}^c, \mathbf{R}^a) \rtimes \mathrm{GL}_a\mathbf{R}$, where $\mathbb{Q}$ and $\mathrm{GL}_c\mathbb{Q}$ carry the discrete topologies.*

PROOF. The group $\mathrm{Aut}(A)$ leaves invariant the (unique) maximal compact subgroup $C$ of $A$. Together with the remarks in **A5.6**, this gives the first part of the assertion. From $\mathrm{Hom}(R, C) \cong \mathrm{Hom}(\hat{C}, \mathbf{R}^a) \leq \mathrm{Hom}(\mathbb{Q} \otimes \hat{C}, \mathbf{R}^a)$ we infer that there exists a monomorphism from $\mathrm{Aut}(A)$ to the group $L := \mathrm{Aut}(C) \ltimes \mathrm{Hom}(\mathbb{Q}^c, \mathbf{R}^a) \rtimes \mathrm{GL}_a\mathbf{R}$. Now assume that $\dim C < \infty$. According to [**R15**, 24.28], $\dim(\mathbb{Q} \otimes \hat{C}) = \mathrm{rank}\, \hat{C} = \dim C$. Hence $L$ is a (linear) Lie group. Since $L$ has no small subgroups, the same holds for $\mathrm{Aut}(A)$. Hence $\mathrm{Aut}(A)^{\mathbf{1}}$ is a (connected) Lie group [**R32**, Ch. III, 4.4], and the restriction of $\iota$ to $\mathrm{Aut}(A)^{\mathbf{1}}$ is analytic, see [**R17**, VII, Th. 4.2] or [**R44**, Sect. 2.11]. □

**A5.8** COROLLARY. *Let $G$ be a locally compact connected group, and assume that $A$ is a closed connected normal abelian subgroup of $G$. If $\dim A < \infty$, then $G/C_G A$ is an analytic subgroup of $\mathbf{R}^{ca} \rtimes \mathrm{GL}_a\mathbf{R}$, where $C$ is a compact group of dimension $c$, and $A \cong \mathbf{R}^a \times C$.*

An important application is the following.

**A5.9** THEOREM. *Let $G$ be a compact group, and assume that $a$ is a natural number, and that $C$ is a compact connected abelian group. If $\mu\colon G \to \mathrm{Aut}(\mathbf{R}^a \times C)$ is a continuous homomorphism of topological groups, then the following hold:*

- **a.** *Both $\mathbf{R}^a$ and $C$ are invariant under $G^\mu$.*
- **b.** *There exists a positive definite symmetric bilinear form on $\mathbf{R}^a$ that is invariant under $G^\mu$. Consequently, $\mu$ induces a completely reducible $\mathbf{R}$-linear action of $G$ on $\mathbf{R}^a$.*
- **c.** *If $G^\mu$ is connected, then $G^\mu$ acts trivially on $C$.*

PROOF. Assertion a follows from **A5.7** and the fact that $\mathrm{Hom}(\hat{C}, \mathbf{R})$ is compact-free. The group $G^\mu$ induces a compact subgroup of $\mathrm{GL}_a\mathbf{R}$. According to [**R15**, 22.23], or [**R36**, Chap. 3 §4], there exists a $G^\mu$-invariant positive definite symmetric bilinear form $q$ on $\mathbf{R}^a$. If $V$ is a $G^\mu$-invariant subspace of $\mathbf{R}^a$, then the orthogonal complement with respect to $q$ is $G^\mu$-invariant as well. This completes the proof of assertion **b**. The last assertion follows from **A5.5a**. □

An interesting feature of locally compact connected abelian groups is the fact that the lattice of closed *connected* subgroups is complemented:

**A5.10** THEOREM. *Let $A$ be a locally compact connected abelian group, and assume that $B$ is a closed connected subgroup of $A$. Then there exists a closed connected subgroup $K$ of $A$ such that $A = BK$ and* $\dim(B \cap K) = 0$ *(i.e., $B \cap K$ is totally disconnected).*

PROOF. It suffices to show the existence of a closed subgroup $S$ such that $BS = A$ and $\dim(B \cap S) = 0$; in fact, connectedness of $A$ implies that $BS^1 = A$ (consider the action of $A$ on the totally disconnected homogeneous space $A/(BS^1)$).

(i) Assume first that $A$ is compact. Then the dual group $\hat{A}$ is discrete [**R15**, 23.17] and torsion-free (since $A$ is connected, [**R15**, 24.25]). Consequently, $\hat{A}$ embeds in $Q := \mathbf{Q} \otimes \hat{A}$, taken with the discrete topology. Since $\hat{A}$ spans the $\mathbf{Q}$-vector space $Q$, there exists a basis $E \subset \hat{A}$ for $Q$. Moreover, we can choose $E$ in such a way that $E \cap B^\perp$ is a basis for the subspace $U$ spanned by $B^\perp$. Now $E \setminus B^\perp$ spans a complement $V$ of $U$ in $Q$. Writing $L := V \cap \hat{A}$, we infer that $B^\perp \cap L = \{\mathbf{1}\}$. Since $E \subset B^\perp \cup L$, the factor group $Q/(B^\perp L)$ is a torsion group, and so is $\hat{A}/(B^\perp L)$. We conclude that $BL^\perp = A$, and $\dim(B \cap L^\perp) = 0$.

(ii) In the general case, we write $A = R \times C$ and $B = S \times D$ with compact groups $C, D$, where $R \cong \mathbf{R}^a$ and $S \cong \mathbf{R}^b$. According to **A5.3b**, there exists a continuous homomorphism $\alpha\colon R \to C$ such that $S$ is contained in the graph $\Gamma_\alpha$, and $A = \Gamma_\alpha \times C$ by **A5.3a**. Therefore, we may assume that $S = \mathbf{R}^b \leq R = \mathbf{R}^a$. For any subgroup $Z \cong \mathbf{Z}^a$ of $\mathbf{R}^a$ such that $B \cap Z \cong \mathbf{Z}^b$, the group $A/Z$ is compact, and $BZ/Z$ is a compact, hence closed, subgroup. Now (i) applies, and we infer that there exists a closed subgroup $S$ of $A$ such that $A = BS$ and $\dim(B \cap S) = 0$. □

**A5.11** REMARKS.

- **a.** The example of a two-dimensional indecomposable group in [**R40**, Bsp. 68] shows that, in general, a complement for a closed connected subgroup need not exist.
- **b.** Complements do exist in abelian connected Lie groups; this can be derived from the fact that, in this case, the dual group is isomorphic to $\mathbf{R}^a \times \mathbf{Z}^c$.
- **c.** If $A$ is a locally compact abelian group, and $B$ is a closed connected subgroup of $A$ such that $B$ is a Lie group (i.e., $B$ is isomorphic to $\mathbf{R}^a \times \mathbf{T}^c$ for suitable cardinal numbers $a < \infty$ and $c$), then there exists a complement for $B$ in $A$, see [**R3**, 6.16].
- **d.** The assertion of **A5.10** can also be derived from **b** and **A3.5**.

## A6. Notions of simplicity

We are now going to introduce the concepts "almost simple", "semi-simple", "minimal closed connected abelian normal subgroup", "solvable radical" in the context of locally compact connected groups of finite dimension. See [**R17**, XII.3.1] for a comparison of the concepts of solvability and nilpotency in topological groups and in discrete groups.

A locally compact connected non-abelian group $G$ is called *semi-simple* if it has no non-trivial closed connected abelian normal subgroup; the group $G$ is called *almost simple* if it has no proper non-trivial closed connected normal subgroup.

Let $(G_i)_{i \in I}$ be a family of normal subgroups of a topological group $G$. Generalizing **A1.10b**, we call the group $G$ an *almost direct product* of the groups $G_i$, if $G$ is generated by $\bigcup_{i \in I} G_i$ and the intersection of $G_j$ with the subgroup generated by $\bigcup_{i \in I \setminus \{j\}} G_i$ is totally disconnected. Examples are given by compact connected groups **A4.1**, and also by semi-simple groups:

**A6.1** THEOREM. *A locally compact connected group of finite dimension is semi-simple if, and only if, it is the almost direct product of a finite family $(S_i)_{1 \le i \le n}$ of almost simple (closed connected) subgroups $S_i$.*

PROOF. This follows from the corresponding theorem on Lie groups [**R5**, III §9 no. 8 Prop. 26] via the Approximation Theorem **A2.1a** and **A3.5**. □

**A6.2** THEOREM. *Let $G$ be a locally compact connected group.*

a. *If $G$ is almost simple, then every proper closed normal subgroup is contained in the center $Z$ of $G$, and $Z$ is totally disconnected. In particular, $G/Z$ is a simple Lie group with $\dim G/Z = \dim G < \infty$.*
b. *If $G$ is semi-simple and of finite dimension, then every closed connected normal subgroup is of the form $S_{i_1} \cdots S_{i_k}$, where the $S_{i_j}$ are some of the almost simple factors from* **A6.1**.

PROOF. Let $N$ be a proper closed normal subgroup of $G$. The connected component $N^1$ is a proper closed connected normal subgroup of $G$. If $G$ is almost simple, we infer that $N^1 = \{\mathbb{1}\}$. Via conjugation, the connected group $G$ acts trivially on the totally disconnected group $N$. Therefore $N$ is contained in $Z$. Applying this reasoning to the case where $N = Z$, we obtain that $Z$ is totally disconnected. The rest of assertion **a** follows from **A1.5** and **A2.1**. Assertion **b** follows from **A3.5** and the corresponding theorem on Lie groups [**R5**, I, §6, no. 2, Cor. 1; III, §6, no. 6, Prop. 14]. □

Our next observation makes precise the intuition that an almost simple group either has large compact subgroups, or large solvable subgroups.

**A6.3** THEOREM. *Let $G$ be a locally compact connected almost simple group. Then there exist a compact subgroup $C$ and closed connected subgroups $T$ and $D$ of $G$ such that the following hold.*

a. *The group $C$ is compact and semi-simple, $T$ is a subgroup of dimension at most 1 that centralizes $C$, and $D$ is solvable.*
b. $G = TCD$, and $\dim G \le \dim C + \dim D + 1$.
c. *The group $D$ is a simply connected, compact-free linear Lie group.*

**d.** *The center $Z$ of $G$ is contained in $TC$, and $TC/Z$ is a maximal compact subgroup of $G/Z$, while $CZ/Z$ is the commutator group of $TC/Z$.*

PROOF. The centralizer of the commutator group of a maximal compact subgroup of a simple Lie group has dimension at most 1. The assertions follow immediately from the Iwasawa decomposition for simple real Lie groups [**R14**, VI, 5.3] by an application of **A3.5** and **A2.1**. □

**A6.4** THEOREM. *Let $G$ be a locally compact group, and assume that $A$ is a closed connected abelian normal subgroup such that $\dim A < \infty$. Then there exists a minimal closed connected abelian normal subgroup $M \leq A$, and $0 < \dim M \leq \dim A$. Moreover:*

**a.** *Either the group $M$ is compact, or it is isomorphic with $\mathbf{R}^m$, where $m = \dim M$.*

**b.** *If $M$ is compact, then $M$ lies in the center of the connected component $G^1$.*

PROOF. The set $\mathcal{A}$ of closed connected abelian normal subgroups of $G$ that are contained in $A$ is partially ordered by inclusion. Since $\dim X = \dim Y$ for $X, Y \in \mathcal{A}$ implies that $X = Y$ by **A1.8b**, there are only chains of finite length in $\mathcal{A}$. The maximal compact subgroup $C$ of a minimal element of $\mathcal{A}$ is a closed connected characteristic subgroup of $M$, hence either $M = C$ or $C = \{\mathbb{1}\}$ by minimality. In the latter case, $M \cong \mathbb{R}^m$ by **A5.1**. Assertion **b** is immediate from **A5.5b**. □

From **A6.4**, we infer that the class of locally compact connected groups of finite dimension splits into the class of semi-simple groups, and the class of groups with a minimal closed connected abelian normal subgroup $M$. The action of $G$ on $M$ via conjugation is well understood:

**A6.5** THEOREM. *Let $G$ be a locally compact group, and assume that there exists a minimal closed connected abelian normal subgroup $M \cong \mathbf{R}^m$.*

**a.** *The group $G$ acts (via conjugation) $\mathbf{R}$-linearly and irreducibly on $M$.*

**b.** *The factor group $L = G/\mathrm{C}_G M$ is a linear Lie group, in fact, a closed subgroup of $\mathrm{GL}_m\mathbf{R}$. The commutator group $S$ of $L$ is also closed in $\mathrm{GL}_m\mathbf{R}$, and we have that $L \cong SZ$, where $S$ is either trivial or semi-simple, and $Z$ is the connected component of the center of $L$. Moreover, $Z$ is isomorphic to a closed connected subgroup of the multiplicative group $\mathbf{C}^\bullet$.*

**c.** *For every one-parameter subgroup $R$ of $M$, we have that $\dim G/\mathrm{C}_G R \leq \dim M$.*

PROOF. The action via conjugation yields a continuous homomorphism $G \to \mathrm{GL}_m\mathbf{R}$, cf. **A5.2**. Every invariant subspace $V$ of $M \cong \mathbf{R}^m$ is a closed connected normal subgroup of $G$. Minimality of $M$ implies that $V = M$, or $V$ is trivial. This proves assertion **a**.

The factor group $G/\mathrm{C}_G M$ is a Lie group [**R17**, VIII.1.1], which acts effectively on $M \cong \mathbf{R}^m$. This action is a continuous homomorphism of Lie groups. From [**R5**, II.6.2, Cor. 1(ii)] we infer that the image $L$ of $G/\mathrm{C}_G M$ in $\mathrm{GL}_m\mathbf{R}$ is an analytic subgroup. Moreover, we know that $L$ is irreducible on $\mathbf{R}^m$. According to [**R7**], the group $L$ is closed in $\mathrm{GL}_m\mathbf{R}$. Hence we may identify $L$ and $G/\mathrm{C}_G M$, cf. **A1.15**. The commutator group $S$ of $L$ is closed, see [**R17**, XVIII.4.5].

From [**R44**, 3.16.2] we infer that the radical of the group $L$ is contained in the center $Z$ of $L$, whence $L = SZ$. According to Schur's Lemma [**R28**, p. 118, p. 257], the centralizer of $L$ in $\mathrm{End}_{\mathbf{R}}(M)$ is a skew field. Since this skew field is also a finite-dimensional algebra over $\mathbf{R}$, we infer that it is isomorphic to $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$, cf. [**R9**]. (See also [**R6**, 6.7].) Thus $Z$ generates a commutative subfield of $\mathbf{H}$, hence $Z \leq \mathbf{C}^*$. This completes the proof of assertion **b**.

Assertion **c** follows readily from **A1.14b**, since by linearity $\mathrm{C}_G R = \mathrm{C}_G r$ for every non-trivial element $r$ of $R$. □

**A6.6** THEOREM. *In every locally compact connected group $G$ of finite dimension, there exists a maximal closed connected solvable normal subgroup* (*called the* solvable radical $\sqrt{G}$ *of* $G$). *Of course, $G = \sqrt{G}$ iff $G$ is solvable, and $\sqrt{G}$ is non-trivial if $G$ is not semi-simple. The factor group $G/\sqrt{G}$ is semi-simple* (*or trivial*).

PROOF. Obviously, the radical is generated by the union of all closed connected normal solvable subgroups, cf. [**R27**, Th. 15]. □

If $G$ is a connected *linear* Lie group, or a *simply connected* Lie group, it is known [**R17**, XVIII.4], [**R44**, 3.18.13] that there exists a *closed* subgroup $S$ of $G$ such that $G = S\sqrt{G}$ and $\dim(S \cap \sqrt{G}) = 0$. Such a (necessarily semi-simple) subgroup is called a *Levi-complement* in $G$. Even for Lie groups, however, such an $S$ does not exist in general (see [**R44**, Ch. 3, Ex. 47] for an example). Apart from the fact that, in the Lie case, one has at least an *analytic* (possibly non-closed) Levi complement [**R44**, 3.18.13], one also has some information about the general case:

**A6.7** THEOREM.

- **a.** *Let $L$ be a Lie group, and let $S$ be a semi-simple analytic subgroup of $L$. Then the closure of $S$ in $L$ is an almost direct product of $S$ and an abelian closed connected subgroup of $L$.*
- **b.** *Let $G$ be a locally compact connected group of finite dimension, and let $\sqrt{G}$ be the solvable radical of $G$. Then there exists a closed subgroup $H$ of $G$ such that $G = H\sqrt{G}$ and $(H \cap \sqrt{G})^1 \leq \mathrm{C}_G H$.*

PROOF. Without loss, we may assume that $S$ is dense in $L$. The adjoint action of $S$ on the Lie algebra $\mathfrak{l}$ of $L$ is completely reducible, hence there exists a complement $\mathfrak{c}$ of the Lie algebra $\mathfrak{s}$ of $S$ such that $[\mathfrak{s}, \mathfrak{c}] \leq \mathfrak{c} \cap [\mathfrak{l}, \mathfrak{l}]$. According to [**R17**, XVI.2.1], we have that $[\mathfrak{l}, \mathfrak{l}] = [\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$. This implies that $[\mathfrak{s}, \mathfrak{c}] = 0$, and assertion **a** follows.

If $G$ is a Lie group, then assertion **b** can be obtained from **a**. In fact, the closure of the Levi complement $S$ of $G$ is of the form $SC$, where $C$ is a closed connected subgroup of $\mathrm{C}_G S$, and the connected component of $SC \cap \sqrt{G}$ is contained in the radical $C$ of $SC$. Applying **A3.5**, we obtain assertion **b** in general. □

**A6.8** REMARK. If $G$ is an algebraic group, then the decomposition in **A6.7b** is the so-called algebraic Levi decomposition into an almost semi-direct product of a reductive group and the unipotent radical, see [**R36**, Ch. 6].

## A7. On the existence of non-Lie groups of finite dimension

In this section, we construct some examples of non-Lie groups, and solve the problem whether or not a given simple Lie group is the quotient of some almost simple non-Lie group.

**A7.1 LEMMA.** *Assume that $(I,>)$ is a directed set, and let $(\pi_{ij}: G_i \to G_j)_{i>j}$ be a projective system of locally compact groups. If $\pi_{ij}$ has compact kernel for all $i, j$ such that $i > j$, then the projective limit is a locally compact group.*

PROOF. Let $G$ be the projective limit. For every $i \in I$, the natural mapping $\pi_i: G \to G_i$ has compact kernel, since this kernel is the projective limit of compact groups. Hence $\pi_i$ is a proper mapping by **A2.7**, and the preimage of a compact neighborhood in $G_i$ is a compact neighborhood in $G$. □

**A7.2 LEMMA.** *Assume that $(I,>)$ is a directed set, and let $(\pi_{ij}: G_i \to G_j)_{i>j}$ be a projective system of locally compact groups such that $\ker \pi_{ij}$ is finite for all $i > j$. Let $G$ denote the projective limit. If $I$ has a smallest element, then every projection $\pi_i: G \to G_i$ has compact totally disconnected kernel.*

PROOF. Assume that $a$ is the smallest element of $I$. For every $i \in I$, let $K_i$ denote the kernel of $\pi_{ia}$. The kernel of $\pi_a$ is the projective limit $K$ of the system $\left(\pi_{ij}|_{K_i}^{K_j}: K_i \to K_j\right)_{i>j}$. Since $\ker \pi_i \leq \ker \pi_a$, the assertion follows from the fact that $K$ is a closed subgroup of the compact totally disconnected group $\prod_{i \in I} K_i$. □

**A7.3 THEOREM.** *Let $G$ be a locally compact connected group of finite dimension.*

- **a.** *If $G$ is not a Lie group, and $N$ is a compact totally disconnected normal subgroup such that $G/N$ is a Lie group, then there exists an infinite sequence $\pi_n: L_{n+1} \to L_n$ of $c_n$-fold coverings of connected Lie groups such that $L_0 = G/N$ and $1 < c_n < \infty$ for every $n$.*
- **b.** *Conversely, let $L$ be a connected Lie group, and let $\pi_n: L_{n+1} \to L_n$ be an infinite sequence of $c_n$-fold coverings of connected Lie groups such that $L_0 = L$ and $1 < c_n < \infty$ for every $n$. Then there exists a locally compact connected non-Lie group $G$ with a compact totally disconnected normal subgroup $N$ such that $G/N \cong L$.*

PROOF. Assume that $G$ and $N$ satisfy the assumptions of **a**. If $N$ is discrete, then $G$ is a Lie group, in contradiction to our assumption. Hence there exists a neighborhood $U$ of $\mathbf{1}$ in $G$ such that $N \not\subseteq U$ and $N \cap U \neq \{\mathbf{1}\}$. According to **A2.1**, there exists some compact normal subgroup $M \subseteq U$ such that $G/M$ is a Lie group. The natural mapping $G/N \to G/(N \cap M)$ is a proper finite covering. Iterating this process, we obtain assertion **a**.

In the situation of **b**, consider the projective system $\pi_n: L_{n+1} \to L_n$. By **A7.1**, the limit is a locally compact group $G$. The projective limit $N$ of the kernels of the natural mappings $G \to G_i$ is a compact infinite group, and $N$ is totally disconnected by **A7.2**. Hence $G$ is not a Lie group. □

**A7.4** REMARKS.

a. Our technical assumption in **A7.2** that $I$ has a smallest element seems to be adequate for the application in **A7.3b**. The example [**R22**, bottom of page 260] of an infinite-dimensional projective limit of a system of one-dimensional groups shows that **A7.2** does not hold without some assumption of the sort.
b. Theorem **A7.3b** could also be derived from [**R22**, 2.2, 3.3]. Roughly speaking, the method of K.H. HOFMANN, T.S. WU and J.S. YANG [**R22**] consists of a dimension-preserving compactification of the center of a given group.

**A7.5** The fundamental group of a semi-simple compact Lie group is finite, cf. [**R26**, §7.12]. This implies that, for a connected Lie group $L$, the existence of a sequence of coverings as in Theorem **A7.3** is equivalent to the existence of a central torus in a maximal compact subgroup of $L$. The *simple* Lie groups with this property are sometimes called *hermitian groups*, they give rise to non-compact irreducible hermitian symmetric spaces [**R14**, VIII.6.1]. In the terminology of [**R43**], the corresponding simple Lie algebras are the real forms $\mathrm{A}_l^{\mathbb{C},p}$ $(1 \le p \le \frac{l+1}{2})$, $\mathrm{B}_l^{\mathbb{R},2}$ $(l \ge 2)$, $\mathrm{D}_l^{\mathbb{R},2}$ $(l \ge 3)$, $\mathrm{D}_{2p}^{\mathbb{H}}$ $(p \ge 2)$, $\mathrm{D}_{2p+1}^{\mathbb{H}}$ $(p \ge 2)$, $\mathrm{E}_{6(-14)}$, $\mathrm{E}_{7(-25)}$. In [**R36**], these algebras are denoted as $\mathfrak{su}_{p,l+1-p}$ (including $\mathfrak{su}_{p,p}$), $\mathfrak{so}_{2,l-1}$, $\mathfrak{so}_{2,l-2}$, $\mathfrak{u}_{2p}^*(\mathbb{H})$, $\mathfrak{u}_{2p+1}^*(\mathbb{H})$, $EIII$, and $EVII$, respectively.

Consequently, we know the locally compact almost simple non-Lie groups.

**A7.6** THEOREM. *Let $G$ be a locally compact connected group, and assume that $G$ is not a Lie group. Then $G$ is almost simple if, and only if, the center $Z$ of $G$ is totally disconnected and $G/Z$ is a hermitian group (cf.* **A7.5***).*

Of course, a similar result holds for semi-simple non-Lie groups: at least one of the almost simple factors in **A6.1** is not a Lie group.

## A8. Arcwise connected subgroups of locally compact groups

In the theory of Lie groups, arcwise connectedness plays an important rôle. In fact, according to a theorem of H. YAMABE [**R12**], the arcwise connected subgroups of a Lie group are in one-to-one correspondence with the subalgebras of the corresponding Lie algebra. Our aim in this section is to extend this to the case of locally compact groups of finite dimension. To this end, we shall refine the topology of the arc component, and show that we obtain a Lie group topology.

**A8.1** DEFINITION. Let $G$ be a topological group, and let $\mathcal{U}$ be a neighborhood base at $\mathbb{1}$. For $W \in \mathcal{U}$, let $\mathcal{U}_W = \{U \in \mathcal{U};\, U \subseteq W\}$, of course $\mathcal{U}_W$ is again a neighborhood base at $\mathbb{1}$. For every $U \in \mathcal{U}$, we denote by $U^{\mathrm{arc}}$ the arc component of $\mathbb{1}$ in $U$. For $W \in \mathcal{U}$, let $\mathcal{U}_W^{\mathrm{arc}} = \{U^{\mathrm{arc}};\, U \in \mathcal{U}_W\}$.

Easy verification shows that the system $\{Vg;\, V \in \mathcal{U}_W^{\mathrm{arc}}, g \in G\}$ forms a base for a group topology on $G$. For every $W \in \mathcal{U}$, we obtain the same group topology on $G$, this topology shall be denoted by $\mathcal{T}_{\mathcal{U}^{\mathrm{loc\,arc}}}$. Obviously, the topology $\mathcal{T}_{\mathcal{U}^{\mathrm{loc\,arc}}}$ is locally arcwise connected.

**A8.2** PROPOSITION.

**a.** *The topology $\mathcal{T}_{\mathcal{U}^{\mathrm{loc\,arc}}}$ is finer than the original topology on $G$.*

**b.** *A function $\alpha: [0,1] \to G$ is continuous with respect to the original topology if, and only if, it is continuous with respect to $\mathcal{T}_{\mathcal{U}^{\mathrm{loc\,arc}}}$.*

PROOF. For $U \in \mathcal{U}$ and $g \in U$, we find $V \in \mathcal{U}$ such that $Vg \subseteq U$. Now $V^{\mathrm{arc}}g \subseteq U$, and we infer that $U \in \mathcal{T}_{\mathcal{U}^{\mathrm{loc\,arc}}}$. The "only if"-part of assertion **b** follows immediately from **a**. So assume that $\alpha$ is continuous with respect to the original topology, let $r \in [0,1]$ and $U \in \mathcal{U}$. By continuity, there is a connected neighborhood $I$ of $r$ such that $I^\alpha \subseteq Ur^\alpha$. Now continuity of $\alpha$ with respect to $\mathcal{T}_{\mathcal{U}^{\mathrm{loc\,arc}}}$ follows from the fact that $I^\alpha \subseteq U^{\mathrm{arc}}r^\alpha$. □

**A8.3** COROLLARY.

**a.** *The arc component $G^{\mathrm{arc}}$ of $G$ coincides with the arc component of $G$ with respect to $\mathcal{T}_{\mathcal{U}^{\mathrm{loc\,arc}}}$.*

**b.** *Algebraically, $G^{\mathrm{arc}}$ is generated by $U^{\mathrm{arc}}$ for every $U \in \mathcal{U}$.*

**c.** *With respect to $\mathcal{T}_{\mathcal{U}^{\mathrm{loc\,arc}}}$, the arc component is again arcwise connected.*

While $G^{\mathrm{arc}}$ is understood to be endowed with the induced original topology, we shall write $G^{\mathrm{loc\,arc}}$ for the topological group $G^{\mathrm{arc}}$ with the topology induced from $\mathcal{T}_{\mathcal{U}^{\mathrm{loc\,arc}}}$. According to **A8.2a**, the inclusion $G^{\mathrm{arc}} \to G$ yields a continuous injection $\iota: G^{\mathrm{loc\,arc}} \to G$.

**A8.4** THEOREM. *Assume that $G$ is a locally compact group of finite dimension, and let $\mathcal{U}$ be a neighborhood base at $\mathbf{1}$. Then the following hold:*

**a.** *If $W \in \mathcal{U}$ is the direct product of a compact totally disconnected normal subgroup $C$ of $G$ and some local Lie group $\Lambda \subseteq G$, then $G^{\mathrm{arc}}$ is algebraically generated by the connected component $\Lambda^{\mathbf{1}}$. In particular, $G^{\mathbf{1}} \leq G^{\mathrm{arc}}C$.*

**b.** *The factor group $G/C$ is a Lie group, in fact, the natural mapping $\pi: G \to G/C$ restricts to a topological isomorphism of $\Lambda$ onto a neighborhood of $\mathbf{1}$ in $G/C$.*

**c.** *$G^{\mathrm{loc\,arc}}$ is a connected Lie group, and $\iota\pi: G^{\mathrm{loc\,arc}} \to (G/C)^{\mathbf{1}}$ is a covering.*

**d.** *The arc component $G^{\mathrm{arc}}$ is dense in $G^{\mathbf{1}}$.*

**e.** *The sets $\mathrm{Hom}(\mathbf{R}, G)$ and $\mathrm{Hom}(\mathbf{R}, G^{\mathrm{arc}})$ coincide. The mapping*

$$\alpha \mapsto \alpha\iota: \mathrm{Hom}(\mathbf{R}, G^{\mathrm{loc\,arc}}) \to \mathrm{Hom}(\mathbf{R}, G)$$

*is a bijection.*

PROOF. For every $U \in \mathcal{U}$, the connected component $G^{\mathbf{1}}$ is contained in the subgroup $\langle U \rangle$ that is algebraically generated by $U$. In particular, $G^{\mathbf{1}} \leq \langle W \rangle = C\langle \Lambda \rangle$; recall that $C$ is a normal subgroup of $G$. The connected component $\Lambda^{\mathbf{1}}$ is arcwise connected, therefore $\Lambda^{\mathbf{1}} = W^{\mathrm{arc}}$. This implies that $\Lambda^{\mathbf{1}}$ is open in $G^{\mathrm{loc\,arc}}$, whence $G^{\mathrm{loc\,arc}} = \langle \Lambda^{\mathbf{1}} \rangle$. This proves assertion **a**. From the fact that $W$ is the direct product of $C$ and $\Lambda$, we conclude that $\pi|_\Lambda$ is injective. The quotient mapping $\pi$ is open, hence $\Lambda^\pi = W^\pi$ is open in $G/C$. Therefore, the group $G/C$ is locally isomorphic to $\Lambda$, and **b** is proved. Since $V := (\Lambda^{\mathbf{1}})^\pi$ is open in $G/C$, we obtain that $(G/C)^{\mathbf{1}}$ is generated by $V$. Hence $\iota\pi: G^{\mathrm{loc\,arc}} \to (G/C)^{\mathbf{1}}$ is surjective, and assertion **c** holds. An application of **A3.5b** to the closure of $G^{\mathrm{arc}}$ and the restriction of $\pi$ to $G^{\mathbf{1}}$ yields assertion **d**. Finally, assertion **e** is an immediate consequence of **A8.2b**. □

**A8.5** REMARKS.

**a.** From K. IWASAWA's local product theorem [**R11**, Th. B] we know that in every locally compact group there exists a neighborhood $W$ with the properties that are required in **A8.4a**.

**b.** In view of **A8.4e**, we define the *Lie algebra* of $G$ as $\mathrm{Hom}(\mathbf{R}, G)$, cf. [**R30**, II.11.9, p. 140]. We then have the exponential function $\exp: \mathrm{Hom}(\mathbf{R}, G) \to G: \alpha \mapsto 1^\alpha$. For every subalgebra $\mathfrak{s}$ of $\mathrm{Hom}(\mathbf{R}, G)$, it seems reasonable to define the corresponding *arcwise connected* subgroup that is generated by $\exp \mathfrak{s}$. This is in contrast with R. LASHOF's definition [**R31**, 4.20], while our definition of the Lie algebra essentially amounts to the same as R. LASHOF's.

**c.** A source for further information on $G$ might be the epimorphism

$$\eta: G^{\mathrm{loc\,arc}} \times C \to G = G^{\mathrm{arc}} C: (x, c) \mapsto x^\iota c.$$

Note that $\eta$ is a local isomorphism, and therefore a quotient mapping.

**d.** As an immediate consequence of the local product theorem, we have that a locally compact group of finite dimension is a Lie group if, and only if, it is locally connected. However, it is not clear *a priori* that $G^{\mathrm{loc\,arc}}$ is locally compact.

We collect some consequences of **A8.4**.

**A8.6** THEOREM. *Let $G$ be a locally compact connected group of finite dimension.*

**a.** *Let $M$ be a compact normal subgroup such that $\dim M = 0$ and $G/M$ is a Lie group. For the natural mapping $\pi_M: G \to G/M$, we have that $\iota\pi_M: G^{\mathrm{loc\,arc}} \to G/M$ is a covering. In particular, $\dim G = \dim G/M = \dim G^{\mathrm{loc\,arc}}$.*

**b.** *The group $G$ is a Lie group if, and only if, the composite $\iota\pi_M$ is a finite covering.*

**c.** *If $H$ is a connected Lie group, and $\alpha: H \to G$ is a continuous homomorphism, then $\alpha$ factors through $\iota$.*

**d.** *The group $G$ is a Lie group if, and only if, the morphism $\iota$ is surjective.*

PROOF. The kernel $K = G^{\mathrm{arc}} \cap M$ of $\iota\pi$ is closed in $G^{\mathrm{loc\,arc}}$ and totally disconnected. Since $G^{\mathrm{loc\,arc}}$ is a Lie group, we infer that $K$ is discrete. Since $G^{\mathrm{loc\,arc}}$ and $G/M$ are connected Lie groups of the same dimension, we conclude that $\iota\pi$ is surjective, hence assertion **a** holds. If $G$ is a Lie group, then $G = G^{\mathrm{arc}} = G^{\mathrm{loc\,arc}}$. Being totally disconnected, the subgroup $M$ is discrete and compact, hence finite. Thus $\pi_M$ is a finite covering. Now assume that $\iota\pi_M$ has finite kernel $K = G^{\mathrm{arc}} \cap M$. Let $U$ be a neighborhood of $\mathbb{1}$ in $G$ such that $U \cap K = \{\mathbb{1}\}$. According to **A2.1b**, there exists an normal totally disconnected compact subgroup $N$ such that $N \subseteq U$ and $G/N$ is a Lie group. For every such $N$, we obtain that $\iota\pi_N$ is an isomorphism. If $N$ is non-trivial, we may pass to a neighborhood $V$ of $\mathbb{1}$ in $U$ such that $N$ is not contained in $V$. Then we find a normal compact subgroup $N' \subseteq N \cap V$, and obtain a proper covering $G/N' \to G/N$, in contradiction to the fact that $\iota\pi_{N'}$ is an isomorphism. This implies that $N = \{\mathbb{1}\}$, and $G$ is a Lie group. Thus assertion **b** is proved. In the situation of **c**, it suffices to show that $\alpha$ is continuous with respect to $\mathcal{T}_{\mathcal{U}^{\mathrm{loc\,arc}}}$; in fact $H^\alpha$ is arcwise connected, hence contained in $G^{\mathrm{arc}}$. For every

$U \in \mathcal{U}$, we find a neighborhood $V$ of $\mathbf{1}$ in $H$ such that $V^\alpha \subseteq U$. Since $H$ is locally arcwise connected, we may assume that $V$ is arcwise connected. This implies that $V^\alpha \subseteq U^{\text{arc}}$, whence $\alpha$ is continuous with respect to $\mathcal{T}_{\mathcal{U}^{\text{loc arc}}}$. In order to prove **d**, assume first that $\iota$ is surjective. Then $\iota$ is a homeomorphism by the open mapping theorem [**R15**, 5.29], hence $G$ is a Lie group. The proof of **d** is completed by the observation that every connected Lie group is arcwise connected. □

## A9. Algebraic groups

In this last section, we briefly indicate how certain results from the theory of complex algebraic groups yield results on the rough structure of locally compact groups of finite dimension.

Let $G$ be a locally compact group. If $\dim G < \infty$, and $A, B \in \mathbf{Struc}(G)$ such that $A < B$, then $\dim A < \dim B$ by **A1.8b**. Consequently, every chain in $\mathbf{Struc}(G)$ has a maximal and a minimal element. This corresponds to the fact that analytic (arcwise connected) subgroups of a Lie group are in one-to-one correspondence to the subalgebras of the Lie algebra, where the dimension function is obviously injective on every chain. Upper bounds for the dimension of subgroups of a given locally compact group $G$ yield lower bounds for the dimension of separable metric spaces that admit a non-trivial action of $G$, cf. **A1.14**. In order to gain information about the maximal elements in $\mathbf{Struc}(G)$, we shall try to employ information from the theory of algebraic groups. The maximal *algebraic* subgroups of a complex algebraic group are understood quite well. E.g., one has the following result, cf. [**R24**, 30.4].

**A9.1** THEOREM. *Let $G$ be a reductive complex algebraic group. Then every maximal algebraic subgroup of $G$ either is parabolic or has reductive Zariski-component.*

Parabolic subgroups are those that contain a Borel subgroup. Every parabolic subgroup is a conjugate of a so called standard parabolic subgroup, and these are easy to describe. In fact, they are in one-to-one correspondence to the subsets of a base for the lattice of roots of $G$ relative to a maximal torus. Cf. [**R24**, 30.1].

The reductive subgroups of reductive complex algebraic groups are known, see [**R4**].

There arises the question as to what extent these results are applicable in order to describe the maximal closed subgroups of a given locally compact group, or even a Lie group. First of all, we note that an important class of Lie groups consists in fact of algebraic groups, cf. [**R36**, Ch. 3, Th. 5].

**A9.2** THEOREM. *Let $G$ be a connected complex linear Lie group, and assume that $G$ equals its commutator group. Then $G$ admits a unique complex algebraic structure. In particular, every complex semi-simple linear Lie group is complex algebraic.*

While every algebraic subgroup of a complex algebraic group $G$ is closed in the Lie topology, the converse does not hold in general. However, the structure of the *algebraic closure* $H^{\text{alg}}$ of a connected analytic subgroup $H$ of $G$ (i.e., the smallest algebraic subgroup that contains $H$) is to some extent controlled by the structure of $H$. In particular, the commutator group of $H^{\text{alg}}$ equals that of $H$, cf. [**R18**, VIII.3.1]. This implies the following.

**A9.3** THEOREM. *Let $G$ be a complex semi-simple linear Lie group. Then every maximal closed connected subgroup is algebraic.*

Via complexification, we obtain an estimate for the possible dimensions of maximal closed subgroups of *real* semi-simple Lie groups (and thus of locally compact semi-simple groups).

**A9.4** THEOREM. *Let $G$ be a semi-simple (real) Lie group. If $H$ is a proper subgroup, then $\dim H \leq m_G$, where $m_G$ denotes the maximal (complex) dimension of proper subgroups of the complexification of $G$.*

Since, e.g., the parabolic subgroups have no counterpart in compact real forms, the estimate in **A9.4** may be quite rough. However, it is attained in the case of split real forms.

**A9.5** EXAMPLE. *Consider a complex simple Lie group of type $\mathsf{G}_2$. Then a reductive subgroups is either semi-simple of type $\mathsf{A}_2$, $\mathsf{A}_1 \times \mathsf{A}_1$, $\mathsf{A}_1$, or a product of $\mathsf{A}_1$ with a one-dimensional centralizer, or abelian of dimension at most two. The maximal parabolic subgroups are semi-direct products of a Levi factor of type $\mathsf{A}_1$ and a solvable radical of dimension 6. Consequently, if $G$ is a locally compact almost simple group such that the factor group modulo the center is a real form of $\mathsf{G}_2$, then the maximal elements in* $\mathbf{Struc}(G)$ *have dimension at most 9. Note that, if $G$ is the compact real form, then every subgroup is reductive, and the maximal elements in* $\mathbf{Struc}(G)$ *have dimension at most 8. Since $\dim G = 14$, we infer that if $G$ acts non-trivially on a separable metric space $X$, then $\dim X \geq 5$, and $\dim X \geq 6$ if $G$ is compact.*

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