The KdV and Whitham limit for a spatially periodic Boussinesq model

Von der Fakultät Mathematik und Physik der Universität Stuttgart zur Erlangung der Würde eines Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigte Abhandlung

Vorgelegt von
Roman Bauer
aus Leningrad (ehemalige UdSSR)

Hauptberichter:	Prof. Dr. Guido Schneider
Mitberichter:	Prof. Dr. Ben Schweizer
	Priv.-Doz. Dr. Wolf-Patrick Düll

Tag der mündlichen Prüfung: 12. Juli 2016

Institut für Analysis, Dynamik und Modellierung der Universität Stuttgart

2016

The KdV and Whitham limit for a spatially periodic Boussinesq model

Roman Bauer

IADM, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart Germany,
email: roman.bauer@mathematik.uni-stuttgart.de

Zusammenfassung

Wir betrachten die KdV-Approximation und die Whitham-Approximation für ein räumlich periodisches Boussinesq-Modell. Wir zeigen Abschätzungen der Differenz zwischen der KdV- beziehungsweise der Whitham-Approximation und echten Lösungen des ursprünglichen Modells, welche garantieren, dass diese Amplitudengleichungen korrekte Vorhersagen über die Dynamik des räumlich periodischen Boussinesq-Modells über die natürlichen Zeitskalen machen. Der Beweis basiert auf Blochwellenanalysis und Energieabschätzungen.

Abstract

We consider the KdV-approximation and the Whitham approximation for a spatially periodic Boussinesq model. We prove estimates of the difference between the approximations and true solutions of the original model, which guarantee that these amplitude equations make correct predictions about the dynamics of the spatially periodic Boussinesq model over the natural time scales. The proof is based on Bloch wave analysis and energy estimates. It is the first justification result of the KdV or Whitham approximation for a dispersive PDE posed in a spatially periodic medium of non-small contrast.

Contents

Chapter 1

Introduction

There exists a zoo of amplitude equations which can be derived via multiple scaling analysis in the long wave limit for various dispersive wave systems possessing conserved quantities. Among these amplitude equations, we consider two which are independent of the small perturbation parameter, namely the KdV equation and the Whitham equation. It is the purpose of this thesis to discuss the validity of these approximations for a spatially periodic Boussinesq model with non-small contrast.

1.1 The KdV approximation

The KdV equation occurs as an approximation equation in the description of small temporal and spacial modulations of long waves in various dispersive wave systems. Examples are the water wave problem or the equations from plasma physics, cf. [CeSa98]. For instance for the Boussinesq equation

$$\partial_t^2 u(x,t) = \partial_x^2 u(x,t) - \partial_x^4 u(x,t) + \partial_x^2 (u(x,t)^2) \tag{1.1}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$, and $u(x,t) \in \mathbb{R}$ with the ansatz

$$u(x,t) = \varepsilon^2 A(X,T), \tag{1.2}$$

where $X = \varepsilon(x+t)$, $T = \varepsilon^3 t$, $A(X,T) \in \mathbb{R}$, and $0 < \varepsilon \ll 1$ a small perturbation parameter, the KdV equation

$$\partial_T A = -\frac{1}{2}\partial_X^3 A + \frac{1}{2}\partial_X (A^2) \tag{1.3}$$

can be derived by inserting (1.2) into (1.1) and equating the coefficients in front of ε^6 to zero. There exist various justification results. Estimates that the formal KdV approximation and true solutions of the various formulations of the water wave problem stay close together over the natural KdV time scale have been shown for instance in [Cra85, SW00, SW02, BCL05, Du12]. Such results are a non-trivial task since solutions of order $\mathcal{O}(\varepsilon^2)$ have shown to be existent on an $\mathcal{O}(1/\varepsilon^3)$ time scale. For (1.1) a possible approximation result is formulated as follows:

Theorem 1.1.1 *Let $T_0 > 0$ and let $A \in C([0,T_0], H^5(\mathbb{R}))$ be a solution to the KdV equation* (1.3)*. Then there exists an $\varepsilon_0 > 0$ and a constant $C > 0$, such that for every $0 < \varepsilon < \varepsilon_0$ we have solutions $u \in C([0,T_0], H^1(\mathbb{R}))$ of the Boussinesq equation* (1.1) *with*

$$\sup_{t\in[0,T_0/\varepsilon^3]} \|u(\cdot_x, t) - \varepsilon^2 A(\varepsilon(\cdot_x + t), \varepsilon^3 t)\|_{H^1(\mathbb{R})} \leq C\varepsilon^{7/2}.$$

For this special problem, the proof is rather short and very instructive for the subsequent analysis. Therefore, we recall it in Section 2.

The last years have seen some attempts to justify the KdV equation in more complicated geometrical situations, including periodic media. It has been justified in [Igu07] for the water wave problem over a periodic bottom with long wave oscillations of the bottom of magnitude $\mathcal{O}(\varepsilon^2)$. The result is also included in [Cha09], where general bottom topographies of small amplitude have been handled. The result is based on [Cha07], where other amplitude systems have been justified.

In case of oscillations of the bottom of magnitude $\mathcal{O}(1)$, no approximation result has been given in existing literature. The situation in this case is much more complicated due to feared resonances which may occur in the spatially periodic case and which are not present in the spatially homogeneous case or in case of small amplitude oscillations of the bottom. As a first attempt to solve this question for the water wave problem, we consider a spatially periodic Boussinesq equation

$$\partial_t^2 u(x,t) = \partial_x(a(x)\partial_x u(x,t)) - \partial_x^2(b(x)\partial_x^2 u(x,t)) + \partial_x(c(x)\partial_x(u(x,t)^2)) \qquad (1.4)$$

with $x \in \mathbb{R}$, $t \geq 0$, $u(x,t) \in \mathbb{R}$, and spatially 2π-periodic coefficient functions $a, c \in C^1(S^1)$ and $b \in C^2(S^1)$ satisfying

$$\min\left\{\inf_{x\in\mathbb{R}} a(x), \inf_{x\in\mathbb{R}} b(x)\right\} \geq C > 0.$$

Moreover, a, b and c need to satisfy another one-dimensional condition given later on. It is satisfied, e.g. if c is also a positive function and a, b and c are even. For this equation, we derive the KdV equation and prove an approximation result, which is formulated in Theorem 3.4.1. It guarantees that the KdV equation makes correct predictions about the dynamics of the spatially periodic Boussinesq model (1.4) over the natural KdV time scale. A common way to prove this would be to

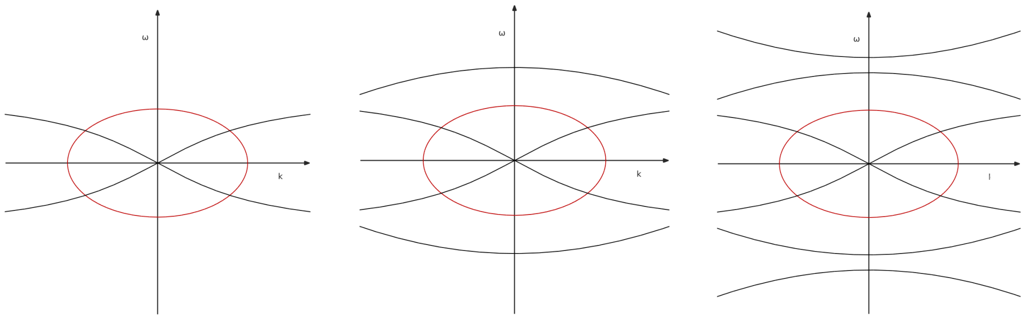

Figure 1.1: *The left panel shows the curves of eigenvalues over the Fourier wave numbers as it appears for the water wave problem [Cra85, SW00, SW02, BCL05, Igu07, Du12]. The middle panel shows the finitely many curves of eigenvalues as it appears for the poly-atomic FPU system [CS11, CCPS12]. The right panel shows the infinitely many curves of eigenvalues over the Bloch wave numbers as it appears for the spatially periodic Boussinesq model* (1.4)*. The KdV equations describe the modes at the wave numbers* $k = 0$ *with vanishing eigenvalues. One of the two curves describes wave packets moving to the left, the other curve wave packets moving to the right.*

generalize a method which has been developed in [CS11] and which has already been applied to the poly-atomic FPU problem in [CCPS12]. However, for fixed Bloch/Fourier wave number the present problem is infinite-dimensional in contrast to the systems considered in [CS11] and [CCPS12]. As a consequence, the normal form transform, which is a major part of the proof, would be much more demanding from an analytic point of view.

Interestingly, for the spatially periodic Boussinesq equation we were able to find an energy which allowed us to incorporate the normal form transform into the energy estimates. Using these coordinates the dangerous terms can be shown to be sufficiently small. The presented result is the first justification result of the KdV approximation for a dispersive PDE posed in a spatially periodic medium of non-small contrast.

1.2 The Whitham approximation

There is another long wave limit which leads to a non-trivial amplitude system. With the ansatz

$$u(x,t) = A(X,T), \tag{1.5}$$

where $X = \varepsilon x$, $T = \varepsilon t$, $A(X,T) \in \mathbb{R}$, and $0 < \varepsilon \ll 1$ a small perturbation parameter, we obtain

$$\partial_T^2 A = \partial_X^2 A + \partial_X^2 F(A), \tag{1.6}$$

where $F(A) = A^2$ which can be written as a system of conservation laws

$$\partial_T A = \partial_X B, \qquad \partial_T B = \partial_X A + \partial_X F(A), \tag{1.7}$$

in the following called the Whitham system. The only rigorous approximation result we are aware of is [DS09], where the periodic wave trains of the NLS equation are approximated over the natural Whitham time scale. As mentioned above, such results are a non-trivial task, since solutions of order $\mathcal{O}(1)$ have shown to be existent on an $\mathcal{O}(1/\varepsilon)$ time scale. For (1.1) a possible approximation result is formulated as follows:

Theorem 1.2.1 *Let $T_0 > 0$ and let $A \in C([0,T_0], H^3(\mathbb{R}))$ be a solution of (1.6) such that*

$$\sup_{T\in[0,T_0]} \|A(\cdot,T)\|_{L^\infty} < \frac{1}{2}.$$

Then there exists a $C > 0$, such that for all $\varepsilon > 0$ we have solutions $u \in C([0,T_0/\varepsilon], H^1(\mathbb{R}))$ of (1.1), such that

$$\sup_{t\in[0,T_0/\varepsilon]} \|u(\cdot,t) - A(\varepsilon\cdot,\varepsilon t)\|_{H^1(\mathbb{R})} \leq C\varepsilon^{3/2}.$$

As already mentioned, for this special problem the proof is rather short and very instructive for the subsequent analysis. Therefore, it will be given in Chapter 2, where we examine the situation for constant coefficient functions.

In the beginning of Chapter 3, we give main definitions and statements of the one-dimensional Bloch wave theory before we prove existence and uniqueness of local solutions for our spatially periodic Boussinesq system (1.4). The Bloch wave theory will be used to derive the dispersion relation, which we will use in order to construct our approximations. We formulate and prove our main approximation statement for the periodic Boussinesq model. Instead of using the Bloch wave theory we go back to physical space and construct bounded quantities which give us the control of the $H^2(\mathbb{R})$-norm of the error term.

Chapter 2

The Spatially Homogeneous Case

This chapter contains first definitions and a sketch of a proof of Theorem 1.1.1 and Theorem 1.2.1. As already mentioned, it is rather short and very instructive for the subsequent analysis. For the Boussinesq model (1.1) we have local existence and uniqueness of solutions in every Sobolev space H^s with $s > 1/2$. In detail, $(u(\cdot_x, t)|_{t=0}, \partial_t u(\cdot_x, t)|_{t=0}) \in H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})$ implies $(u(\cdot_x, t), \partial_t u(\cdot_x, t)) \in H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})$ as long as the solution exists. The local existence and uniqueness of solutions combined with the subsequent estimates yields the existence and uniqueness of solutions for all $t \in [0, T_0/\varepsilon^3]$ in the KdV case and all $t \in [0, T_0/\varepsilon]$ in the Whitham case. The residual

$$\mathrm{Res}(u)(x,t) = -\partial_t^2 u(x,t) + \partial_x^2 u(x,t) - \partial_x^4 u(x,t) + \partial_x^2 (u(x,t)^2)$$

measures how much a function u fails to satisfy the Boussinesq model (1.1) and appears later on as part of the inhomogeneity in the equation for the error.

In the beginning we give some fundamental definitions.

2.1 Fourier transform

The spaces $L^2(\mathbb{R})$ and $L^2(S^1)$, are given by the sets of all measurable functions u on $\mathbb{R}$ and S^1 such that

$$\|u\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} |u(x)|^2 \, dx < \infty, \text{ and } \|u\|^2_{L^2(S^1)} = \int_{S^1} |u(x)|^2 \, dx < \infty.$$

These spaces are Hilbert spaces with the inner products

$$\langle u, v \rangle_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} u(x)\overline{v(x)} \, dx \,, \text{ and } \langle u, v \rangle_{L^2(S^1)} = \int_{S^1} u(x)\overline{v(x)} \, dx.$$

The Schwartz space on $\mathbb{R}$ is the function space

$$S(\mathbb{R}) = \{u \in C^\infty(\mathbb{R}) : \quad \forall \alpha, \beta \in \mathbb{N}_0 : \sup_{x\in\mathbb{R}} \left| x^\alpha \partial_x^\beta u(x) \right| < \infty\}.$$

The Schwartz space $S(\mathbb{R})$ is a dense subspace of $L^2(\mathbb{R})$. In this work, the continuous Fourier transform of a function $u \in S(\mathbb{R})$ is given by the function

$$(\mathcal{F}_{\mathbb{R}} u)(k) = \widehat{u}(k) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} u(x) dx.$$

Since this defines an automorphism on $S(\mathbb{R})$ it is possible to extend the Fourier transform on $L^2(\mathbb{R})$. The discrete Fourier transform of a 2π-periodic function u is given by the sequence $(\widehat{u}_k)_{k\in\mathbb{N}}$ where

$$(\mathcal{F}_{S^1} u)_k = \widehat{u}_k := \frac{1}{2\pi} \int_{S^1} e^{-ikx} u(x) dx$$

is called the k-th Fourier coefficient of u. Products of elements $u, v \in L^2(\mathbb{R})$ or $u, v \in L^2(S^1)$ are being mapped to the convolution terms

$$(\widehat{u} \star \widehat{v})(k) = \int_{-\infty}^{\infty} \widehat{u}(m) \widehat{v}(k-m) dm,$$

or $(\widehat{u} \star \widehat{v})_k = \sum_{k\in\mathbb{Z}} \widehat{u}_m \widehat{v}_{k-m}$, respectively. For an $s \in \mathbb{R}_0^+$ the spaces $H^s(\mathbb{R})$ and $H^s(S^1)$ are given by

$$H^s(\mathbb{R}) = \{u \in L^2(\mathbb{R}) : \quad \|u\|^2_{H^s(\mathbb{R})} = \int_{-\infty}^{\infty} (1+k^2)^s |\widehat{u}(k)|^2 \, dk < \infty\}$$

and

$$H^s(S^1) = \{u \in L^2(S^1) : \quad \|u\|^2_{H^s(S^1)} = \sum_{k\in\mathbb{Z}} (1+k^2)^s |\widehat{u}_k|^2 < \infty\}$$

and are Hilbert spaces with the inner products

$$\langle u, v \rangle_{H^s(\mathbb{R})} = \int_{\infty}^{\infty} (1+k^2)^s \, \widehat{u}(k) \overline{\widehat{v}(k)} \, dk,$$

and

$$\langle u, v \rangle_{H^s(S^1)} = \sum_{k\in\mathbb{Z}} (1+k^2)^s \, \widehat{u}_k \overline{\widehat{v}_k}.$$

For a fixed $s > 1/2$ these spaces are Banach algebras, i.e., products of elements in $H^s(\mathbb{R})$ or in $H^s(S^1)$, remain in $H^s(\mathbb{R})$ or in $H^s(S^1)$, and the embeddings $H^s(\mathbb{R}) \hookrightarrow$

$C(\mathbb{R})$, and $H^s(S^1) \hookrightarrow C(S^1)$ are continuous mappings. This means that there exists a $C > 0$ such that for every element u of $H^s(\mathbb{R})$ or of $H^s(S^1)$, there exists a continuous representative $\widetilde{u} \in C(\mathbb{R})$ or in $\widetilde{u} \in C(S^1)$, respectively, such that

$$\|\widetilde{u}\|_{C(\mathbb{R})} \leq C\|u\|_{H^s(\mathbb{R})}, \quad \text{or} \quad \|\widetilde{u}\|_{C(S^1)} \leq C\|u\|_{H^s(S^1)}.$$

The images of theses spaces under the (continuous or discrete) Fourier transform are denoted by $L^2(s)$ and $\ell^2(s)$. For convolutions of functions defined on the real line we have the estimates

$$\|\widehat{u} \star \widehat{v}\|_{L^2(s)} \leq \|\widehat{u}\|_{L^1(s)}\|\widehat{v}\|_{L^2(s)}, \quad \text{and} \quad \|\widehat{u} \star \widehat{v}\|_{L^2(s)} \leq C\|\widehat{u}\|_{L^2(s)}\|\widehat{v}\|_{L^2(s)},$$

where the first estimate holds for every $s \geq 0$, while the last estimate requires $s > 1/2$. Similarly, with the same conditions to s, on the unit circle we have the estimates

$$\|\widehat{u} \star \widehat{v}\|_{\ell^2(s)} \leq \|\widehat{u}\|_{\ell^1(s)}\|\widehat{v}\|_{\ell^2(s)} \quad \text{and} \quad \|\widehat{u} \star \widehat{v}\|_{\ell^2(s)} \leq C\|\widehat{u}\|_{\ell^2(s)}\|\widehat{v}\|_{\ell^2(s)}.$$

For natural s the norms on $H^s(S^1)$ and on $H^s(\mathbb{R})$ defined above are equivalent to the norms

$$\left(\sum_{n\leq s} \|\partial_x^n u\|^2_{L^2(\mathbb{R})}\right)^{1/2} \quad \text{and} \quad \left(\sum_{n\leq s} \|\partial_x^n u\|^2_{L^2(S^1)}\right)^{1/2}$$

which are easier to handle and hence will be used instead in the appropriate cases.

2.2 Local existence and uniqueness of solutions

In this subsection we show local existence and uniqueness for solutions to the Boussinesq equation (1.1) for initial conditions $(u(0, \cdot_x), v(0, \cdot_x)) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s > 1/2$ where

$$v(t, x) := (-\partial_x^2 + \partial_x^4)^{-1/2}\partial_t u(t, \cdot_x).$$

In order to apply a fixed point argument we write (1.1) as

$$\partial_t^2 u = \underbrace{(\partial_x^2 - \partial_x^4)}_{-\Lambda^2} u + \underbrace{(\partial_x^2(u^2))}_{\widetilde{F}(u)}$$

where $\Lambda : H^{s+2}(\mathbb{R}) \to H^s(\mathbb{R})$ is given in Fourier-space by its symbol $\widetilde{\omega}(k) = |k|\sqrt{1+k^2}$ and is a positive self-adjoint operator in $L^2(\mathbb{R})$. We remark that, since $\mathcal{F}\Lambda$ and $\mathcal{F}\partial_x$ are multiplication operators, they commute and so Λ is self-adjoint in every $H^s(\mathbb{R})$. The non-linear mapping $\widetilde{F} : H^{s+2}(\mathbb{R}) \to H^s(\mathbb{R})$ is continuous and locally Lipschitz continuous (for $s > 1/2$). Using the new coordinates we have the first order system

$$\begin{aligned}\partial_t u &= \Lambda v,\\ \partial_t v &= -\Lambda u + \Lambda^{-1}\partial_x^2(u^2)\end{aligned}$$

in the phase space $H^s(\mathbb{R}) \times H^s(\mathbb{R})$. We can write this system as

$$\partial_t w \; = \; i\Omega w + F(w) \tag{2.1}$$

where $w = (u, v)$ and

$$\Omega = \begin{pmatrix} 0 & -i\Lambda \\ i\Lambda & 0 \end{pmatrix} : H^2(\mathbb{R}, \mathbb{C}) \times H^2(\mathbb{R}, \mathbb{C}) \to L^2(\mathbb{R}, \mathbb{C}) \times L^2(\mathbb{R}, \mathbb{C})$$

is continuous and again self-adjoint in every Sobolev space $H^s(\mathbb{R}, \mathbb{C}) \times H^s(\mathbb{R}, \mathbb{C})$. The nonlinear mapping $F(w) = F(u, v)$ is given by

$$F(u, v) = \begin{pmatrix} 0 \\ \Lambda^{-1}\partial_x^2(u^2) \end{pmatrix}.$$

We recall Stone's Theorem [P83, Theorem 1.10.8]:

Theorem 2.2.1 (Stone) *Let X be a complex Hilbert space. Then the linear operator*

$$i\Omega : dom(\Omega) \subset X \to X$$

is the generator of a strongly continuous one-parameter group of unitary operators $T(t) : X \to X$ if and only if Ω is a self-adjoint operator in X.

For $H^s(\mathbb{R}, \mathbb{C}) \times H^s(\mathbb{R}, \mathbb{C})$ we define the subspace of real valued functions

$$Y^s := H^s(\mathbb{R}, \mathbb{R}) \times H^s(\mathbb{R}, \mathbb{R}).$$

By Stone's Theorem $i\Omega$ generates a strongly continuous group of unitary operators $T(t)$ in $H^s(\mathbb{R}, \mathbb{C}) \times H^s(\mathbb{R}, \mathbb{C})$ which in the following is denoted by $T(t) = e^{i\Omega t}$. Note that $i\Omega$ is Y^s-valued on Y^{s+2}. Hence Y^s is an invariant subspace with respect to the flow $T(t)$ such that the restriction $T(t)|_{Y^s} : Y^s \to Y^s$ is a strongly continuous group of contractions on Y^s. The non-linear mapping F is a locally Lipschitz continuous mapping $Y^s \to Y^s$ if Y^s is a Banach algebra. Hence, for our arguments $s > 1/2$ is needed. Next, we show local existence and uniqueness of solutions in Y^s. In detail, for an arbitrary initial condition $w(0) = (u(0), v(0)) \in Y^s$ this argument uses the existence of a unique fixed point of the mapping

$$G(w)(t) \;=\; e^{i\Omega t} w(0) + \int_0^t e^{i\Omega(t-s)} F(w(s))\, ds$$

in the complete metric space

$$M := \left\{ w \in C(I, Y^s) : w(0) = (u_0, v_0)\ ,\ \|w\|_{C(I,Y^s)} \leq 2\|w(0)\|_{Y^s} \right\}$$

where $I = [0, t_0]$ with $t_0 < 0$ suitably chosen below and where the metric in M is being induced by the canonical norm in $C^0(I, Y^s)$. The non-linear mapping F can be expressed by a continuous linear mapping N such that we have $F(u, v) = N(0, u^2)$. In a first step we show that G maps M to M. We have

$$\|G(w)\|_{C(I,Y^s)} \leq \|w(0)\|_{Y^s} + t_0\|N\|_{Lin(Y^s)} C_s \|w\|^2_{C(I,Y^s)} \;\leq\; 2\|w(0)\|_{Y^s}$$

if $t_0 \leq (C_s\|N\|_{Lin(Y^s)}\|(u_0, v_0)\|_{Y^s})^{-1}$. Next we show the contraction property:

$$\begin{aligned}\|G(w) - G(\widetilde{w})\|_{C(I,Y^s)} \leq & t_0\|N\|_{Lin(Y^s)} C_s \|w - \widetilde{w}\|_{C(I,Y^s)} 2\|(u_0, v_0)\|_{Y^s} \\ \leq & \frac{1}{2}\|w - \widetilde{w}\|_{C(I,Y^s)}\end{aligned}$$

if $t_0 \leq (4C_s\|N\|_{Lin(Y^s)}\|(u_0, v_0)\|_{Y^s})^{-1}$. Hence we have the following local existence and uniqueness statement:

Lemma 2.2.2 *For every $s > 1/2$ and every initial condition*

$$(u(0), \Lambda^{-1}\partial_t u(0)) \in Y^s$$

there exists a $t_0 > 0$, a finite constant $C > 0$ and a solution $u \in C([0, t_0], Y^s)$ for the spatially homogeneous Boussinesq equation (1.1) such that

$$\sup_{t\in[0,t_0]} \|(u(t), \Lambda^{-1}\partial_t u(t))\|_{Y^s} \leq C.$$

2.3 The KdV case

For the KdV approximation

$$\varepsilon^2\psi(x, t, \varepsilon) = \varepsilon^2 A(\varepsilon(x + ct), \varepsilon^3 t)$$

we find for the residual

$$\mathrm{Res}(\varepsilon^2\psi) = -\varepsilon^4 c^2 \partial_X^2 A - 2c\varepsilon^6 \partial_T \partial_X A - \varepsilon^8 \partial_T^2 A + \varepsilon^4 \partial_X^2 A - \varepsilon^6 \partial_X^4 A + \varepsilon^6 \partial_X^2 (A^2) = -\varepsilon^8 \partial_T^2 A$$

if we choose $c^2 = 1$ and A to satisfy the KdV equation (1.3) for $c = 1$, and the KdV equation

$$\partial_T A = \frac{1}{2}\partial_X^3 A - \frac{1}{2}\partial_X(A^2)$$

for the reverse direction $c = -1$. Using the KdV equation the residual term $\partial_T^2 A$ can be expressed as a spacial derivative

$$\begin{aligned}\partial_T^2 A &= \frac{1}{2}\partial_T(-\partial_X^3 A + \partial_x(A^2)) = \frac{1}{2}\partial_X\partial_T(-\partial_X^2 A + A^2)\\ &= \frac{1}{4}\partial_X\bigg(\partial_X^5 A - \partial_X^3(A^2) - A(\partial_X^3 A) + A\partial_X(A^2)\bigg).\end{aligned}$$

Hence ∂_x^{-1} can be applied to the residual and we find

$$\sup_{t\in[0,T_0/\varepsilon^3]} \|\partial_x^{-1}\mathrm{Res}(\varepsilon^2\psi(\cdot,t))\|_{H^s(\mathbb{R})} = \sup_{t\in[0,T_0/\varepsilon^3]} \|\partial_x^{-1}\varepsilon^8\partial_T^2 A(\varepsilon(\cdot_x+ct),\varepsilon^3 t)\|_{H^s(\mathbb{R})} \leq C\varepsilon^{13/2}.$$

Therefore $\partial_X^5 A \in H^s(\mathbb{R})$ is necessary. For this $A \in H^{s+5}(\mathbb{R})$ is sufficient. The formal error of order $\mathcal{O}(\varepsilon^7)$ is reduced by a factor $\varepsilon^{-1/2}$ due to the scaling properties of the L^2-norm. The difference $\varepsilon^{7/2}R = u - \varepsilon^2\psi$ satisfies the error equation

$$\partial_t^2 R = \partial_x^2 R - \partial_x^4 R + 2\varepsilon^2\partial_x^2(\psi R) + \varepsilon^{7/2}\partial_x^2(R^2) + \varepsilon^{-7/2}\mathrm{Res}(\varepsilon^2\psi).$$

Obviously $2\partial_x^2(\varepsilon^2\psi R)$ is the dangerous term since it seems to lack one ε-power in order to stay bounded. One possible idea is to try to eliminate this term using an appropriate transformation

$$\mathcal{U} = R + \mathcal{M}(\varepsilon^2\psi R)$$

where $\mathcal{M}$ is a linear mapping. Unfortunately, as we will see in Chapter 4, using this ansatz the dangerous terms can not be transformed away completely.

Yet in our case another method can be applied. For simplicity reasons we restrict ourselves to the case $R(0,\cdot_x) = 0$. We multiply the error equation with $-\partial_t\partial_x^{-2}R$ and integrate it w.r.t. x and find the energy

$$E(t) = \frac{1}{2}\int_{\mathbb{R}}(\partial_t\partial_x^{-1}R)^2 + (1+2\varepsilon^2\psi)(R)^2 + (\partial_x R)^2 dx. \tag{2.2}$$

For the time derivative we find

$$\partial_t E = \int(\partial_t\varepsilon^2\psi)R^2 + (2\varepsilon^{7/2}R\partial_x R - \partial_x^{-1}Res(\varepsilon^2\psi))\cdot\partial_x^{-1}\partial_t R dx = \mathcal{O}(\varepsilon^3)$$

where we use $\partial_t\psi = \mathcal{O}(\varepsilon)$ and $\partial_x\psi = \mathcal{O}(\varepsilon)$ in $C([0,T_0/\varepsilon^3],C(\mathbb{R}))$. In detail the functional E is an upper bound for the squared H^1-norm for

$$0 < \varepsilon < \varepsilon_0 := \left(2 \sup_{T \leq T_0} \|A(\cdot_X, T)\|_{C^0}\right)^{-1/2}$$

and satisfies

$$\partial_t E \leq C\varepsilon^3 E + C\varepsilon^{7/2} E^{3/2} + C\varepsilon^3 E^{1/2} \leq \frac{1}{2} C\varepsilon^3 (2 + E + E(\varepsilon E)^{1/2})$$

where we used the trivial estimate $\sqrt{x} \leq 1 + x$. Note, that if $E(t)$ can be bounded over the long time interval $[0, T_0/\varepsilon^3]$ by Sobolev's embedding theorem we have a $\mathcal{O}(1)$-bound for R in $C(\mathbb{R})$. Gronwall's inequality yields in this situation the required bound.

Lemma 2.3.1 *Let E be a non-negative function and $C, T_0, \varepsilon > 0$, such that*

$$\partial_t E \leq \frac{1}{2} C\varepsilon^3 (2 + E + E(\varepsilon E)^{1/2})$$

is satisfied. Then we have $E(t) \leq e^{CT_0} - 1$ for all $t \leq T_0/\varepsilon^3$.

Proof: For $T_0 > 0$ choose $M := e^{CT_0} - 1$. Then while $E \leq M$ we have $\varepsilon E \leq 1$ for all $\varepsilon < \frac{1}{M}$ such that E satisfies

$$\partial_t E \leq C\varepsilon^3 + C\varepsilon^3 E = C\varepsilon^3(1 + E)$$

and $E(t) \leq e^{C\varepsilon^3 t} - 1 \leq e^{CT_0} - 1 = M$ for all $t \leq T_0/\varepsilon^3$ follows.

2.4 The Whitham case

For the Whitham approximation $\psi(x, t) = A(\varepsilon x, \varepsilon t)$ we find for the residual $\mathrm{Res}(\psi) = -\varepsilon^4 \partial_X^4 A$ if we choose A to satisfy the Whitham equation (1.6). Therefore

$$\sup_{t \in [0, T_0/\varepsilon^3]} \|\partial_x^{-1} \mathrm{Res}(\psi(\cdot, t, \varepsilon))\|_{H^s} = \sup_{t \in [0, T_0/\varepsilon^3]} \|\partial_x^{-1} \varepsilon^4 \partial_X^4 A(\varepsilon \cdot, \varepsilon t)\|_{H^s} \leq C\varepsilon^{5/2},$$

such that $A \in H^{s+3}$ is necessary. Again, the formal error of order $\mathcal{O}(\varepsilon^3)$ is reduced by a factor $\varepsilon^{-1/2}$ due to the scaling properties of the L^2-norm. The difference $\varepsilon^{3/2} R = u - \psi$ satisfies

$$\partial_t^2 R = \partial_x^2 R - \partial_x^4 R + 2\partial_x^2(\psi R) + \varepsilon^{3/2} \partial_x^2 (R^2) + \varepsilon^{-3/2} \mathrm{Res}(\psi).$$

We multiply the error equation with $\partial_t \partial_x^{-2} R$, integrate it w.r.t. x, and find

$$\partial_t \int_{\mathbb{R}} (\partial_t \partial_x^{-1} R)^2 + (1+2\psi) R^2 + (\partial_x R)^2 \, dx = \mathcal{O}(\varepsilon) \tag{2.3}$$

where again we used $\partial_t \psi = \mathcal{O}(\varepsilon)$. The energy

$$E = \int_{\mathbb{R}} (\partial_t \partial_x^{-1} R)^2 + (1+2\psi) R^2 + (\partial_x R)^2 \, dx$$

is an upper bound for the squared H^1-norm for ψ sufficiently small. In detail

$$\sup_{t \in [0, T_0/\varepsilon^3]} \sup_{x \in \mathbb{R}} |\psi(x,t)| \, < \, 1/2$$

is needed. The energy E satisfies

$$\partial_t E \leq C\varepsilon E + C\varepsilon^{3/2} E^{3/2} + C\varepsilon E^{1/2} \leq \frac{1}{2} \varepsilon C (2 + E + E(\varepsilon E)^{1/2}).$$

From now on the argumentation from Lemma 2.3.1 from the KdV case can be applied:

Lemma 2.4.1 *Let E be a non-negative function and $C, T_0, \varepsilon > 0$, such that*

$$\partial_t E \leq \frac{1}{2} C\varepsilon (2 + E + E(\varepsilon E)^{1/2})$$

is satisfied. Then we have $E(t) \leq e^{CT_0} - 1$ for all $t \leq T_0/\varepsilon$.

Proof: For $T_0 > 0$ choose $M := e^{CT_0} - 1$. Then while $E \leq M$ we have $\varepsilon E \leq 1$ for all $\varepsilon < \frac{1}{M}$ such that E satisfies

$$\partial_t E \leq C\varepsilon + C\varepsilon E = C\varepsilon^3 (1 + E)$$

and $E(t) \leq e^{C\varepsilon t} - 1 \leq e^{CT_0} - 1 = M$ for all $t \leq T_0/\varepsilon$ follows.

In the spatially homogeneous case, justification of the KdV approximation as well as justification of the Whitham approximation can be done with one single energy estimate. In both cases multiplying the error equation by $\partial_t \partial_x^{-2} R$ and integrating it with respect to x yields an energy which gives the control of the $\| \cdot \|_{H^1}$-norm of the error. For the spatially periodic case we will need to use another energy, which then will give us the control of the H^2-norm of the error.

Chapter 3

The Spatially Periodic Boussinesq Model

3.1 Preliminaries

In this section we introduce the Bloch transform. It generalizes the standard Fourier theory and will be used for the derivation of the dispersion relation.

Definition 3.1.1 *The Bloch transform of a function $u \in L^2(\mathbb{R}, \mathbb{C})$ is the function*

$$\mathcal{B}u \in L^2([-1/2, 1/2], L^2(S^1, \mathbb{C}))$$

given by

$$\mathcal{B}u(\ell)(x) = \sum_{k \in \mathbb{Z}} e^{ikx} \widehat{u}(k+\ell). \tag{3.1}$$

Note that by construction we have $\mathcal{B}u(x, \ell+1) = e^{-ix}\mathcal{B}u(x, \ell)$. The embedding

$$\mathrm{Id} : L^2(S^1 \times [-1/2, 1/2]) \to L^2([-1/2, 1/2], L^2(S^1))$$

is an isometric isomorphism such that we can define $\widetilde{u}(x, \ell) := \mathcal{B}u(\ell)(x)$ for simplicity reasons. For fixed ℓ the function $\widetilde{u}(\cdot_x, \ell)$ is in $L^2(S^1)$ and the k-th Fourier coefficient of $\widetilde{u}(\cdot_x, \ell)$ is given by $\widehat{u}(k+\ell)$ such that we have

$$(\mathcal{F}_{S^1}(\mathcal{B}u)(\ell)(\cdot_x))_k = (\mathcal{F}_{\mathbb{R}}u)(k+\ell).$$

Therefore, we write $\widehat{u}_k(\ell) := \widehat{u}(k+\ell)$. Bloch transform maps $L^2(\mathbb{R}, \mathbb{C})$-functions to families of $L^2(S^1, \mathbb{C})$-functions. Definition (3.1.1) is motivated by the representation

$$
\begin{aligned}
u(x) &= \int_{\mathbb{R}} e^{i\ell x}\widehat{u}(\ell)\,d\ell = \sum_{k\in\mathbb{Z}}\int_{-1/2+k}^{1/2+k} e^{i\ell x}\widehat{u}(\ell)\,d\ell \\
&= \sum_{k\in\mathbb{Z}}\int_{-1/2}^{1/2} e^{ikx}e^{i\ell x}\widehat{u}(k+\ell)\,d\ell = \int_{-1/2}^{1/2} e^{i\ell x}\underbrace{\sum_{k\in\mathbb{Z}} e^{ikx}\widehat{u}(k+\ell)}_{=\widetilde{u}(x,\ell)}\,d\ell.
\end{aligned}
$$

Lemma 3.1.2 *Bloch transform is an isometric isomorphism between* $L^2(\mathbb{R},\mathbb{C})$ *and*

$$
\widetilde{L}^2 \;:=\; \{\widetilde{u} \in L^2([-1/2,1/2], L^2(S^1,\mathbb{C}))\}
$$

where the inner product is given by

$$
\langle \widetilde{u}, \widetilde{v}\rangle_{\widetilde{L}^2} := \int_{-1/2}^{1/2} \langle \widetilde{u}(\cdot_x,\ell), \widetilde{v}(\cdot_x,\ell)\rangle_{L^2(S^1,\mathbb{C})} d\ell.
$$

The inverse mapping is given by

$$
\left(\mathcal{B}^{-1}\widetilde{u}\right)(x) \;=\; \int_{-1/2}^{1/2} e^{i\ell x}\widetilde{u}(x,\ell)\,d\ell. \tag{3.2}
$$

Proof: Let u, v be Schwartz functions. Then we have

$$
\begin{aligned}
\langle \widetilde{u}, \widetilde{v}\rangle_{\widetilde{L}^2} \;&=\; \int_{-1/2}^{1/2} \langle \widetilde{u}(\cdot,\ell), \widetilde{v}(\cdot,\ell)\rangle_{L^2(S^1,\mathbb{C})}\,d\ell = 2\pi \int_{-1/2}^{1/2} \sum_{k\in\mathbb{Z}} \widehat{u}_k(\ell)\overline{\widehat{v}_k(\ell)}\,d\ell \\
&= 2\pi \sum_{k\in\mathbb{Z}} \int_{-1/2}^{1/2} \widehat{u}(k+\ell)\overline{\widehat{v}(k+\ell)}\,d\ell = 2\pi \int_{-\infty}^{\infty} \widehat{u}(\ell)\overline{\widehat{v}(\ell)}\,d\ell \;=\; \langle u, v\rangle_{L^2(\mathbb{R})}.
\end{aligned}
$$

Using this isometric character the continuity and bijectivity follow from the denseness of $S(\mathbb{R})$ in $L^2(\mathbb{R})$. $\square$

Bloch transform maps the derivative of a function u to the shifted derivative $(\partial_x + i\ell)\widetilde{u}$ what motivates the definition

$$
\partial := \partial_x + i\ell.
$$

Due to the isomorphism $H^s(S^1,\mathbb{C}) \longrightarrow \ell^2(s)$ the spaces $H^s(\mathbb{R})$ and

$$
\{\,\{\widehat{u}_k(\cdot_\ell)\}_{k\in\mathbb{Z}} \in L^2(\mathbb{R},\ell^2(s)) : \quad \widehat{u}_k(\ell+1) \;=\; \widehat{u}_{k+1}(\ell)\}
$$

are isomorphic under $\mathcal{B}$. For every $s \geq 0$ the Bloch transform is an isomorphism between $H^s(\mathbb{R})$ and

$$\widetilde{H}^s := \{\widetilde{u} \in L^2([-1/2, 1/2], H^s(S^1, \mathbb{C}))\}. \tag{3.3}$$

Lemma 3.1.3 *Bloch transform of a product $u \cdot v$ is given by*

$$\mathcal{B}(u \cdot v)(\ell)(x) = (\widetilde{u} \star \widetilde{v})(x, \ell) = \int_{-1/2}^{1/2} \widetilde{u}(x, m)\widetilde{v}(x, \ell - m)\, d\ell. \tag{3.4}$$

Proof: By construction we have $\widetilde{u}(\ell + 1, x) = e^{-ix}\widetilde{u}(\ell, x)$. Hence

$$\begin{aligned}
\mathcal{B}(u \cdot v)(\ell)(x) &= \sum_{k\in\mathbb{Z}} e^{ikx}(\widehat{u} \star \widehat{v})(k + \ell) = \sum_{k\in\mathbb{Z}} e^{ikx} \int_{-\infty}^{\infty} \widehat{u}(m)\widehat{v}(k + \ell - m)\, dm \\
&= \sum_{k\in\mathbb{Z}} e^{ikx} \sum_{n\in\mathbb{Z}} \int_{-1/2}^{1/2} \widehat{u}(n + \widetilde{m})\widehat{v}(k + \ell - n - \widetilde{m})\, d\widetilde{m} \\
&= \int_{-1/2}^{1/2} \sum_{n\in\mathbb{Z}} \widehat{u}(n + \widetilde{m}) \underbrace{\sum_{k\in\mathbb{Z}} e^{ikx}\widehat{v}(k + \ell - n - \widetilde{m})}_{=\widetilde{v}(\ell-n-\widetilde{m},x)=e^{inx}\widetilde{v}(\ell-\widetilde{m},x)}\, d\widetilde{m} \\
&= \int_{-1/2}^{1/2} \sum_{n\in\mathbb{Z}} e^{inx}\widehat{u}(n + \widetilde{m}) \cdot \widetilde{v}(\ell - \widetilde{m}, x)\, d\widetilde{m} \\
&= \int_{-1/2}^{1/2} \widetilde{u}(\widetilde{m}, x)\widetilde{v}(\ell - \widetilde{m}, x)\, d\widetilde{m}.
\end{aligned}$$

□

Bloch transform is an invariant mapping with respect to multiplication with 2π-periodic functions:

Lemma 3.1.4 *Let $u \in L^2(\mathbb{R})$. If $\chi \in L^\infty(S^1)$ then $\mathcal{B}$ maps $\chi \cdot u$ on $\chi \cdot \mathcal{B}(u)$.*

Proof: The statement follows from

$$\mathcal{B}^{-1}(\chi \cdot \widetilde{u})(x) = \int_{-1/2}^{1/2} e^{i\ell x}(\chi(x) \cdot \widetilde{u}(x, \ell))\, d\ell = \chi(x) \int_{-1/2}^{1/2} e^{i\ell x} \cdot \widetilde{u}(x, \ell)\, d\ell = \chi(x) \cdot u(x).$$

□

For justification it is essential that we give a bound not only for the so called residual $Res(\varepsilon^2\psi)$ of the approximation but also for $\partial_x^{-1} Res(\varepsilon^2\psi)$. Therefore we will need useful characterizations for the existence of $\partial_x^{-1} Res(\varepsilon^2\psi)$.

Lemma 3.1.5 *Let $\widetilde{v} \in \widetilde{L}^2$. Then the equation $\widetilde{v} = \partial\widetilde{w}$ has a solution $\widetilde{w} \in \widetilde{L}^2$ if and only if $\widetilde{g} \in L^2((-1/2, 1/2), \mathbb{C})$ where*

$$\widetilde{g}(\ell) = \frac{1}{\ell} \int_{S^1} \widetilde{v}(\ell, x)\, dx.$$

Proof: Let $\widetilde{w}$ be a solution to $\widetilde{v} = \partial\widetilde{w}$. Then we have $\widehat{w}_k(\ell) = \frac{1}{i(k+\ell)}\widehat{v}_k(\ell)$ and

$$\|\widetilde{w}\|^2_{\widetilde{L}^2} = \sum_{k\in\mathbb{Z}} \int_{-1/2}^{1/2} \frac{1}{(k+\ell)^2} |\widehat{v}_k(\ell)|^2\, d\ell = \int_{-1/2}^{1/2} \frac{1}{\ell^2} |\widehat{v}_0(\ell)|^2\, d\ell + \sum_{k\neq 0} \int_{-1/2}^{1/2} \frac{1}{(k+\ell)^2} |\widehat{v}_k(\ell)|^2\, d\ell.$$

Since $\widetilde{v} \in \widetilde{L}^2$, the last term on the right hand side is bounded, e.g. by $4\|\widetilde{v}\|^2_{\widetilde{L}^2}$. Hence the left hand side of the equation from above is finite if and only if

$$\int_{-1/2}^{1/2} \frac{1}{\ell^2} |\widehat{v}_0(\ell)|^2\, d\ell < \infty.$$

This is equivalent to $\frac{1}{(\cdot_\ell)}\widehat{v}_0(\cdot_\ell) = g(\cdot_\ell) \in L^2(-1/2, 1/2)$. □

In Bloch space the periodic Boussinesq equation (1.4) is given by

$$\partial_t^2 \widetilde{u}(x, \ell) = \partial(a\partial)\widetilde{u}(x, \ell) - \partial^2(b(x)\partial^2)\widetilde{u}(x, \ell) + \partial(c\partial)(\widetilde{u}(x, \cdot_\ell) \star \widetilde{u}(x, \cdot_\ell)). \quad (3.5)$$

Before we address the derivation of the approximation for the periodic Boussinesq equation (1.4), we prove local existence and uniqueness of solutions in physical x-space. For simplicity reasons, we will restrict ourselfves to the case $s \in \mathbb{N}$.

3.2 Local existence and uniqueness

In this subsection we show local existence and uniquiness for the first order system corresponding to the periodic Boussinesq equation. Therefore we rewrite the spatially periodic Boussinesq equation into

$$\partial_t^2 u = \partial_x(\overbrace{a - \partial_x(b\partial_x)}^{\mathcal{D}^2})\partial_x u + \partial_x(c\partial_x)u^2.$$

Here $\mathcal{D}^2 = a - \partial_x(b\partial_x)$ is a positive definite self-adjoint operator and possesses a positive self-adjoint square root $\mathcal{D} : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ which is a continuous operator and the corresponding quadratic form is equivalent to the square of the $H^1(\mathbb{R})-$norm. Hence $\mathcal{D}^{-1}$ is a continuous mapping from $L^2(\mathbb{R})$ to $H^1(\mathbb{R})$. Setting

$v := \mathcal{D}^{-1}\partial_x^{-1}\partial_t u$ we find the first order system

$$\partial_t w \;=\; i\Omega w + F(w)$$

where

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \Omega := \begin{pmatrix} 0 & -i\partial_x \mathcal{D} \\ -i\mathcal{D}\partial_x & 0 \end{pmatrix}, \qquad F(u,v) \;=\; \begin{pmatrix} 0 \\ \mathcal{D}^{-1}(c\partial_x)u^2 \end{pmatrix}.$$

For this equation we consider the phase space $Y^s = (H^s(\mathbb{R},\mathbb{R}))^2$ where the inner product in Y^s is defined by

$$\langle w_1, w_2\rangle_s \;:=\; \langle (\mathrm{I}+\Omega^2)^{s/2} w_1, (\mathrm{I}+\Omega^2)^{s/2} w_2\rangle_{L^2(\mathbb{R})\times L^2(\mathbb{R})}.$$

Note that, in analogy to the case of constant coefficients, since Ω is self-adjoint in $Y^s + iY^s = H^s(\mathbb{R},\mathbb{C}) \times H^s(\mathbb{R},\mathbb{C})$ by Stone's Theorem $i\Omega$ generates a strongly continuous semigroup of contractions $T(t) =: e^{i\Omega t}$ in $Y^s + iY^s$, and, with the same arguments as for the proof for the case of constant coefficients, Y^s is invariant with respect to $T(t)$. In the next step we need the continuity of the non-linear mapping F:

Lemma 3.2.1 *For $s \in \mathbb{N}$ the non-linear mapping F is locally Lipschitz-continuous as a mapping $Y^s \to Y^s$ if $c \in C^{s-1}(S^1)$.*

Proof: We have

$$\begin{aligned}\|\mathcal{D}^{-1} c\partial_x(u^2 - \widetilde{u}^2)\|_{H^s} &\le \|\mathcal{D}^{-1}\|_{Lin(H^{s-1},H^s)} \|c\partial_x\|_{Lin(H^s,H^{s-1})} (C\|u-\widetilde{u}\|_{H^s}\|u+\widetilde{u}\|_{H^s}) \\ &\le C\|\mathcal{D}^{-1}\|_{Lin(H^{s-1},H^s)} \|c\|_{C^{s-1}(S^1)} \|u-\widetilde{u}\|_{H^s}\|u+\widetilde{u}\|_{H^s}. \qquad \square\end{aligned}$$

Lemma 3.2.2 *For every $s \in \mathbb{N}$ and every initial condition $(u(0), \mathcal{D}^{-1}\partial_x^{-1}\partial_t u(0)) \in Y^s$ there exists a $t_0 > 0$, a $C > 0$ and a solution $u \in C([0,t_0], H^s(\mathbb{R},\mathbb{R})$ for the spatially periodic Boussinesq equation (1.4) where $c \in C^1(S^1)$ such that*

$$\sup_{t\in[0,t_0]} \|(u(t), \mathcal{D}^{-1}\partial_x^{-1}\partial_t u(t))\|_{Y^s} \le C.$$

Proof: We consider the mapping

$$G(w) \;=\; e^{i\Omega t} w_0 + \int_0^t e^{i\Omega(t-s)} F(w)(s) ds \tag{3.6}$$

in the complete metric space

$$M := \left\{ w \in C(I, Y^s) : w(0) = (u_0, v_0)\ ,\ \|w\|_{C(I,Y^s)} \le 2\|w(0)\|_{Y^s} \right\}$$

where $I = [0, t_0]$ and where the metric in M is being induced by the canonical norm in $C(I, Y^s)$. Again the non-linear mapping F can be expressed by a continuous linear mapping such that we have $F(u,v) = (0, \mathcal{D}^{-1}(c\partial_x)u^2) = N(0, u^2)$ with a continuous linear mapping $N = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{D}^{-1}(c\partial_x) \end{pmatrix} \in Lin(Y^s, Y^s)$ if $\widehat{c} \in \ell^1(s-1)$. G maps M to M:

$$\|G(w)\|_{C(I,Y^s)} \leq \|w(0)\|_{Y^s} + t_0 \|N\|_{Lin(Y^s)} C_s \|w\|^2_{C(I,Y^s)} \;\leq\; 2\|w(0)\|_{Y^s}$$

if $t_0 \leq (C_s \|N\|_{Lin(Y^s)} \|(u_0, v_0)\|_{Y^s})^{-1}$. Next we show the contraction property:

$$\begin{aligned} \|G(w) - G(\widetilde{w})\|_{C(I,\mathcal{X}^s)} \leq & t_0 \|N\|_{Lin(Y^s)} C_s \|w - \widetilde{w}\|_{C(I,Y^s)} 2\|(u_0, v_0)\|_{Y^s} \\ \leq & \frac{1}{2} \|w - \widetilde{w}\|_{C(I,Y^s)} \end{aligned}$$

if $t_0 \leq (4C_s \|N\|_{Lin(Y^s)} \|(u_0, v_0)\|_{Y^s})^{-1}$. □

3.2.1 Spectral properties of the linear operator

For the derivation of the KdV and the Whitham equation, we need the dispersion relation. We consider the linear operator

$$L(\partial_x) \;=\; \partial_x(a\partial_x) - \partial_x^2(b\partial_x^2) \tag{3.7}$$

in $L^2(\mathbb{R})$ and domain $H^4(\mathbb{R})$ where $a \in C^1(\mathbb{R})$ and $b \in C^2(\mathbb{R})$ are even positive periodic functions such that

$$\min \left\{ \inf_{S^1} a \, , \, \inf_{S^1} b \right\} > 0.$$

$L(\partial_x)$ is self-adjoint and negative semidefinite, which implies that the spectrum is a subset of $\mathbb{R}_0^-$. In the case of $a = b = 1$ the linear operator can be studied using Fourier transform and is given by a multiplication operator in Fourier space, uniquely determined by its symbol $-\omega(k)^2 = -k^2(1+k^2)$. In case of non-constant but periodic a and b the linear operator can be studied using Bloch transform in the Bloch space

$$\widetilde{L}^2 = \{\widetilde{u} \in L^2([-1/2, 1/2], L^2(S^1, \mathbb{C}))\}$$

with the inner product

$$\langle \widetilde{u}, \widetilde{v} \rangle_{\widetilde{L}^2} = \int_{-1/2}^{1/2} \langle \widetilde{u}(\ell), \widetilde{v}(\ell) \rangle_{L^2(S^1,\mathbb{C})} \, d\ell.$$

Since the non-linearity and the spectrum possess a more complex structure in the case of non-constant periodic coefficients the derivation of the corresponding approximations is much more demanding. The Bloch transform $\mathcal{B}$ maps $L(\partial_x)$ to the linear operator

$$\widetilde{L}(\partial) := \mathcal{B}L(\partial_x) = \partial(a\partial) - \partial^2(b\partial^2) \tag{3.8}$$

which defines a linear continuous mapping $\widetilde{H}^4 \to \widetilde{L}^2$, where

$$\partial = \partial_x + i\ell$$

and where $\widetilde{H}^s$ is defined in (3.3). We also refer to [DAS15] where the multi-dimensional generalization of the operator $\partial(a\partial)$ is being observed as the right hand side of a second order system. For fixed $\ell \in [-1/2, 1/2]$ the operator $\widetilde{L}(\partial)$ is self-adjoint and negative semidefinite in $L^2(S^1)$ with domain $H^4(S^1)$. For each pair $(\lambda, \widetilde{f})$ where λ is an eigenvalue of $\widetilde{L}(\partial_x)$ and $\widetilde{f}$ is an eigenfunction corresponding to λ there exist analytic mappings

$$\lambda : [-1/2, 1/2] \to \mathbb{R}, \qquad \widetilde{f} : [-1/2, 1/2] \to H^4(S^1),$$

such that $\lambda(0) = \lambda$ and $\widetilde{f}(0) = \widetilde{f}$. The curves of eigenvalues λ can intersect, the number of intersecting curves in a given ℓ is equal to the dimension of the eigenspace of $\lambda(\ell)$. An introduction to general Bloch wave theory is given in [DLPSW11]. In case of constant $a(x) = a_0$ and $b(x) = b_0$ the eigenfunctions are given by $f_k(\ell)(x) = \frac{1}{\sqrt{2\pi}}e^{ikx}$ and the curves of eigenvalues are given by

$$\lambda_k(\ell) = -\omega_k(\ell)^2 = -(k+\ell)^2(a_0 + b_0(k+\ell)^2).$$

In case of non-constant coefficient functions a and b these curves can split up (Fig. 3.1). The eigenspace of $\widetilde{L}(\partial_x)$ corresponding to the zero eigenvalue is one-dimensional since the corresponding equation $\widetilde{L}(\partial_x)\widetilde{f}(x) = 0$ implies the equation

$$\int_{S^1} a(x)|\partial_x \widetilde{f}(x)|^2 + b(x)|\partial_x^2 \widetilde{f}(x)|^2 \, dx = 0$$

such that $\partial_x \widetilde{f} = 0$ follows. In order to satisfy the claim $\|\widetilde{f}\|_{L^2(S^1,\mathbb{C})} = 1$ we choose $\widetilde{f}(x) = (2\pi)^{-1/2}$. Hence there exist one unique smooth curve

$$\lambda(\cdot_\ell) = -\omega(\cdot_\ell)^2 \in C^\infty([-1/2, 1/2], \mathbb{R})$$

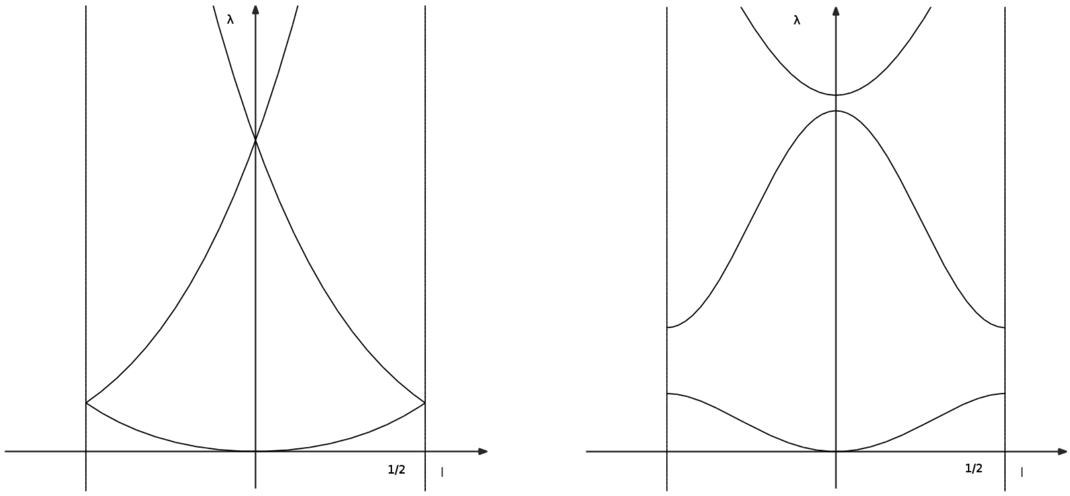

Figure 3.1: The left panel shows a sketch of the curves of eigenvalues over the Bloch wave numbers as they appear for the case of constant coefficient functions. The curves cross in $|\ell| = 1/2$ and $\ell = 0$. The right panel shows a possible configuration of the spectral situation for the spatially periodic Boussinesq model over Bloch wave numbers. The curves can split up and cross in different ℓ. Spectral gaps can occur.

of eigenvalues and one unique curve of corresponding eigenfunctions $\widetilde{f}(\ell)(\cdot_x) \in C^\infty([-1/2, 1/2], C^\infty(S^1, \mathbb{C}))$ satisfying

$$\forall \ell \in [-1/2, 1/2]: \qquad \widetilde{L}(\partial)\widetilde{f}(\ell) = -\omega(\ell)^2 \widetilde{f}(\ell), \qquad (\omega(0), \widetilde{f}(0)) = \left(0, \frac{1}{\sqrt{2\pi}}\right).$$

Due to $\widetilde{L}(\overline{\partial}) = \overline{\widetilde{L}(\partial)}$ we have $-\omega(-\ell)^2 \widetilde{f}(-\ell) = \widetilde{L}(\overline{\partial})\widetilde{f}(-\ell) = \overline{\widetilde{L}(\partial)}\widetilde{f}(-\ell)$ such that obviously $\overline{\widetilde{f}(-\ell)}$ is the curve of eigenvalues corresponding to $-\omega(-\ell)^2$. The unique character of these curves then implies $-\omega(-\ell)^2 = -\omega(\ell)^2$ and $\widetilde{f}(-\ell) = \overline{\widetilde{f}(\ell)}$.

Definition 3.2.3 *We define the corresponding spaces and generalized projections:*

i) Let $\lambda(\ell) = -\omega(\ell)^2$ be the unique curve of eigenvalues satisfying $\lambda(0) = 0$ and let $\widetilde{f}(\ell)$ be the corresponding curve of eigenfunctions satisfying $\|\widetilde{f}(\ell)\|_{L^2(S^1, \mathbb{C})} = 1$. The coordinate function $P(\cdot_\ell) \in C^\infty([-1/2, 1/2], (L^2(S^1, \mathbb{C})')$ is given by

$$P(\ell)\widetilde{u} = \langle \widetilde{u}, \widetilde{f}(\ell) \rangle_{L^2(S^1, \mathbb{C})}.$$

The function $Q(\cdot_\ell) \in C^\infty([-1/2, 1/2], Lin(L^2(S^1, \mathbb{C})))$ is given by

$$(Q(\cdot_\ell)\widetilde{u})(x) = \widetilde{u}(x) - (P(\ell)\widetilde{u})\widetilde{f}(\ell)(x).$$

ii) For fixed ℓ the subspace $\widetilde{V}(\ell) \subset \widetilde{L}^2$ is given by

$$\widetilde{V}(\ell) := \{\widetilde{v} \in L^2(S^1,\mathbb{C}) : P(\ell)\widetilde{v} = 0\}$$

and the subspace $\widetilde{U}(\ell) \subset L^2(S^1,\mathbb{C})$ is defined by $\widetilde{U}(\ell) = \widetilde{V}(\ell)^{\perp_{L^2(S^1,\mathbb{C})}}$.

iii) The subspace $\widetilde{V} \subset \widetilde{L}^2$ is defined by

$$\widetilde{V} := \{\widetilde{v} \in \widetilde{L}^2 : \widetilde{v}(\cdot_x,\ell) \in \widetilde{V}(\ell)\}$$

and $\widetilde{U} := \widetilde{V}^{\perp_{\widetilde{L}^2}}$.

Note: We have $\widetilde{U} = \{\widetilde{u} \in \widetilde{L}^2 : \ \widetilde{u}(\cdot_x,\ell) \in \widetilde{U}(\ell)\}$. Moreover $\widetilde{L}(\partial)|_{\widetilde{V}(\ell)} : \widetilde{V}(\ell) \to \widetilde{V}(\ell)$ is an invertible operator. Inserting the second order Taylor polynomial $\widetilde{f}(0) + \ell\widetilde{f}'(0) + \frac{1}{2}\ell^2\widetilde{f}''(0)$ of $\widetilde{f}(\ell)$ with respect to ℓ where

$$\widetilde{f}^{(n)}(0)(\cdot_x) := \partial_\ell^n \widetilde{f}(\ell)(\cdot_x)|_{\ell=0}$$

into the eigenvalue equation $\widetilde{L}(\partial)\widetilde{f}(\ell) = -\omega^2(\ell)\widetilde{f}(\ell)$ and using $\omega(0) = 0$, $\omega(-\ell)^2 = \omega(\ell)^2$, $\widetilde{f}(0)(x) = (2\pi)^{-1/2}$ and $\widetilde{f}(-\ell) = \overline{\widetilde{f}(\ell)}$, which implies that the even ℓ-derivatives of $\widetilde{f}(\ell)$ in $\ell = 0$ are purely real functions w.r.t. x, while the odd $\ell-$derivatives are purely imaginary functions w.r.t. x, we obtain the equations

$$\begin{aligned} 0 =& i\partial_x a\widetilde{f}(0) + \widetilde{L}(\partial_x)\widetilde{f}'(0) \\ 0 =& (-a + \partial_x^2 b)\widetilde{f}(0) + i\Big(\{a,\partial_x\} - 2\{\partial_x(b\partial_x),\partial_x\}\Big)\widetilde{f}'(0) + \frac{1}{2}\widetilde{L}(\partial_x)\widetilde{f}''(0) \end{aligned}$$

where $\{A, B\} := AB + BA$ is the so called anti-commutator of A and B. Since $\partial_x a \in \widetilde{V}(0)$ we can rewrite these equations to

$$\widetilde{f}'(0) = -i\widetilde{f}(0)\widetilde{L}(\partial_x)^{-1}(\partial_x a), \tag{3.9}$$

$$\begin{aligned} \frac{1}{2}\widetilde{L}(\partial_x)\widetilde{f}''(0) =& (a - b'')\widetilde{f}(0) - i(\partial_x(a\cdot) + a\partial_x\cdot)\widetilde{f}'(0) \\ &- 2i(\partial_x(b\partial_x^2\cdot) + \partial_x^2(b\partial_x)\cdot)\widetilde{f}'(0). \end{aligned} \tag{3.10}$$

The linear operator $\widetilde{L}(\partial_x)$ maps odd functions to odd functions and even functions to even functions. Since a is even, the right hand side of (3.9) is an odd function and we find $\widetilde{f}'(0)$ to be an odd and, as already expected, purely imaginary function. Using this information we find from (3.10) that $\widetilde{f}''(0)$ is even and, hence, that $\partial_x\widetilde{f}''(0)$ is odd. This will be needed for the derivation of the KdV equation and the Whitham equation. Since $\omega(\ell)^2$ is smooth and even, there exists a constant $C > 0$ such that

$$|\omega(\ell)^2 - \omega'(0)^2\ell^2 - \frac{1}{3}\omega'(0)\omega'''(0)\ell^4| \,\leq\, C|\ell|^6. \tag{3.11}$$

Next, consider

$$\beta(\ell, m, \ell - m) \,=\, -\int_{S^1} c\partial(\widetilde{f}(m)\widetilde{f}(\ell - m))\overline{\partial\widetilde{f}(\ell)}\, dx. \tag{3.12}$$

This term appears in the non-linear part since we have

$$P(\ell)\partial(c\partial)(P\widetilde{u}(\cdot_\ell) \star P\widetilde{u}(\cdot_\ell)) = \int_{-1/2}^{1/2} \beta(\ell, m, \ell - m)\widetilde{u}(m)\widetilde{u}(\ell - m)\, dm.$$

Since $\partial\widetilde{f}(\ell) = i\ell\widetilde{f}(0) + \partial(\widetilde{f}(\ell) - \widetilde{f}(0))$ and since $\beta(\cdot, \cdot, \cdot)$ is bounded, we find $\beta(\ell, m, \ell - m) = \mathcal{O}(\ell)$. This means that there exists a $C > 0$ such that

$$|\beta(\ell, m, \ell - m)| \leq C|\ell|$$

uniformly in m. Plugging the second order Taylor polynomial into (3.12) after a straight forward computation we find that there exist smooth functions $h_{j_1,j_2}(\ell, m)$ such that

$$\frac{\beta(\ell, m, \ell - m)}{\ell} \,=\, -\,\ell\mu_1 + im(\ell - m)\mu_2 + \sum_{j_1+j_2=3} h_{j_1,j_2}(\ell, m)\ell^{j_1}m^{j_2} \tag{3.13}$$

where $\mu_1 = \frac{1}{\sqrt{2\pi}}\int_{S^1} c(x)\alpha^2(x)dx$ and $\mu_2 = \int_{S^1} c(x)\alpha(x)\widetilde{\alpha}(x)dx$ and where

$$\alpha(x) := \widetilde{f}(0) - i\partial_x\widetilde{f}'(0) \quad \text{and} \quad \widetilde{\alpha}(x) := \partial_x(\widetilde{f}'(0)^2 - \widetilde{f}(0)\widetilde{f}''(0)). \tag{3.14}$$

The integrand of μ_2 is an odd function such that this term vanishes and so we have the estimate

$$|\ell^{-1}(\beta(\ell, m, \ell - m) + \mu_1\ell^2)| \,\leq\, C\,|\ell^3 + m\ell^2 + m^2\ell + m^3|. \tag{3.15}$$

Further computations need the following lemma, which uses the elementary scaling properties of the L^2-norm:

Lemma 3.2.4 *Let $\widehat{A} \in L^2(n)(\mathbb{R})$.*

i) Let $\widetilde{u}(x, \ell) = \ell^n\widehat{A}(\varepsilon^{-1}\ell)\widetilde{g}(\ell)(x)$ where $\widetilde{g} \in C([-1/2, 1/2], L^2(S^1, \mathbb{C}))$. Then *we have*

$$\|\widetilde{u}\|_{\widetilde{L}^2} \leq \varepsilon^{n+1/2} \cdot \sup_{\ell \in [-1/2,1/2]} \|g(\ell)(\cdot)\|_{L^2(S^1,\mathbb{C})} \cdot \|\widehat{A}\|_{L^2(n)(\mathbb{R})}.$$

ii) Let $\widetilde{v}(x,\ell) = \widehat{A}(\varepsilon^{-1}\ell)\widetilde{g}(\ell)(x)$. *For a given* $\gamma \in C^n([-1/2,1/2], L^2(S^1,\mathbb{C}))$ *we have a* C_γ *and an* $\varepsilon_0 > 0$ *such that for all* $\varepsilon < \varepsilon_0$ *we have*

$$\| \left(\gamma(\cdot_\ell) - \sum_{m<n} \frac{(\cdot_\ell)^m}{m!} \gamma^{(m)}(0) \right) \widetilde{v}\|_{\widetilde{L}^2} \leq C_\gamma \varepsilon^{n+1/2} \|\widehat{A}\|_{L^2(n)(\mathbb{R})}.$$

3.3 Derivation of the approximations

3.3.1 Derivation of the KdV equation

The residual measures how much a function fails to satisfy the periodic Boussinesq equation (1.4).

Definition 3.3.1 *The (Boussinesq) residual of a function*

$$u \in C([0,t_0], H^4(\mathbb{R},\mathbb{C})) \cap C^2([0,t_0], L^2(\mathbb{R},\mathbb{C}))$$

is the function $Res_{Bous}(u) \in C([0,t_0], L^2(\mathbb{R},\mathbb{C}))$, *in the following just* $Res(u)$, *given by*

$$Res_{Bous}(u)(x,t) := -\partial_t^2 u(x,t) + L(\partial_x)u(x,t) + \partial_x(c\partial_x)u^2(x,t). \tag{3.16}$$

For

$$\widetilde{u} \in C([0,t_0], \widetilde{H}^4) \cap C^2([0,t_0], \widetilde{L}^2)$$

the residual is the function $Res_{\mathcal{B}}(\widetilde{u}) \in C^0([0,t_0], \widetilde{L}^2)$, *in the following just* $Res(\widetilde{u})$, *given by*

$$Res_{\mathcal{B}}(\widetilde{u})(x,\ell) := \mathcal{B}\, Res(\mathcal{B}^{-1}\widetilde{u})(x,\ell). \tag{3.17}$$

The residual of the approximation appears in the equation for the difference between the true solution and its approximation as an inhomogeneity. Hence, the aim is to construct an approximation such that the residual is small in a certain sence. In detail, for our justification the estimates

$$\|Res(\varepsilon^2\psi)\|_{L^2} \leq C\varepsilon^{13/2} \qquad \text{and} \qquad \|\partial_x^{-1} Res(\varepsilon^2\psi)\|_{L^2} \leq C\varepsilon^{13/2}$$

are needed. In case of constant a and b the ansatz for the KdV-equation is

$$\varepsilon^2\widehat{\psi}(k,t) \;=\; \varepsilon\widehat{A}\left(\frac{k}{\varepsilon},\varepsilon^3 t\right)e^{i\omega'(0)kt}$$

where $-\omega^2(k)$ is the symbol of $L(\partial_x)$ in Fourier space. In the spatially periodic case a first ansatz is given by

$$\epsilon^2\widetilde{\psi}(x,\ell,t) \;=\; \varepsilon\widehat{A}\left(\frac{\ell}{\varepsilon},\varepsilon^3 t\right)e^{i\omega'(0)\ell t}\widetilde{f}(\ell)(x)\chi(\ell) \tag{3.18}$$

where $\chi(\cdot_\ell)$ is the characteristic function on $[-\delta/2,\delta/2]$ and where $|\delta|\le 1/2$ is so small that the first curve of eigenvalues is separated from the rest. Plugging (3.18) into (3.17) gives the equation

$$\begin{aligned}
Res_{\widetilde{u}}(x,\ell) = \Big(& -\epsilon^7\partial_T^2\widehat{A}(\varepsilon^{-1}\ell,\varepsilon^3 t) - 2i\omega'(0)\varepsilon^5(\varepsilon^{-1}\ell)\partial_T\widehat{A}(\varepsilon^{-1}\ell,\varepsilon^3 t) \qquad (3.19)\\
& -\frac{1}{3}\omega'(0)\omega'''(0)\varepsilon^5(\varepsilon^{-1}\ell)^4\widehat{A}(\varepsilon^{-1}\ell,\varepsilon^3 t)\\
& +(\omega'(0)^2\ell^2+\frac{1}{3}\omega'(0)\omega'''(0)\ell^4-\omega(\ell)^2)\varepsilon\widehat{A}(\varepsilon^{-1}\ell,\varepsilon^3 t)\Big)e^{i\omega'(0)\ell t}\widetilde{f}(\ell)\chi(\ell)\\
& +\varepsilon^2\underbrace{\int_{-1/2}^{1/2}\beta(\ell,m,\ell-m)\widehat{A}\Big(\frac{m}{\varepsilon}\Big)\widehat{A}\Big(\frac{\ell-m}{\varepsilon}\Big)\chi(m)\chi(\ell-m)\,dm}_{(\star)}\;e^{i\omega'(0)\ell t}\widetilde{f}(\ell)
\end{aligned}$$

where $Res_{\widetilde{u}}(x,\ell) := \langle\widetilde{u}(\cdot_x,\ell),\widetilde{f}(\ell)(\cdot_x)\rangle_{L^2(S^1,\mathbb{C})}$ and $(\star)$ is given by

$$\begin{aligned}
(\star) &= \int_{-\delta/2}^{\delta/2}\beta(\ell,m,\ell-m)\widehat{A}\Big(\frac{m}{\varepsilon}\Big)\widehat{A}\Big(\frac{\ell-m}{\varepsilon}\Big)\chi(\ell-m)dm\\
&= \int_{-\delta/2}^{\delta/2}-\ell^2\mu_1\widehat{A}\Big(\frac{m}{\varepsilon}\Big)\widehat{A}\Big(\frac{\ell-m}{\varepsilon}\Big)\chi(\ell-m)dm\\
&\quad+\int_{-\delta/2}^{\delta/2}\big(\beta(\ell,m,\ell-m)+\ell^2\mu_1\big)\,\widehat{A}\Big(\frac{m}{\varepsilon}\Big)\widehat{A}\Big(\frac{\ell-m}{\varepsilon}\Big)\chi(\ell-m)dm\\
&= \int_{-\delta/2}^{\delta/2}-\ell^2\mu_1\widehat{A}\Big(\frac{m}{\varepsilon}\Big)\widehat{A}\Big(\frac{\ell-m}{\varepsilon}\Big)dm\,\chi(\ell)\\
&\quad+\mu_1\ell^2\int_{-\delta/2}^{\delta/2}\widehat{A}\Big(\frac{m}{\varepsilon}\Big)\widehat{A}\Big(\frac{\ell-m}{\varepsilon}\Big)(\chi(\ell)-\chi(\ell-m))dm \qquad (3.20)\\
&\quad+\int_{-\delta/2}^{\delta/2}\big(\beta(\ell,m,\ell-m)+\ell^2\mu_1\big)\,\widehat{A}\Big(\frac{m}{\varepsilon}\Big)\widehat{A}\Big(\frac{\ell-m}{\varepsilon}\Big)\chi(\ell-m)dm. \qquad (3.21)
\end{aligned}$$

Lemma 3.3.2 *Let $s > 1/2$ and $\widehat{A} \in L^2(s)$. Then we have*

$$\left\| \int_{\delta/\varepsilon}^{\infty} \widehat{A}(M)\widehat{A}\Big(\frac{\cdot\ell}{\varepsilon} - M\Big) dM \right\|_{L^2([-1/2,1/2])} \leq \varepsilon^{s+1/2}\delta^{-s}\|\widehat{A}\|^2_{L^2(s)}.$$

Proof: For $M \geq \delta/\varepsilon$ we have $1 \leq \frac{M\varepsilon}{\delta}$ such that we can estimate

$$\begin{aligned}\left\| \int_{\delta/\varepsilon}^{\infty} \widehat{A}(M)\widehat{A}\Big(\frac{\cdot\ell}{\varepsilon} - M\Big) dM \right\|_{L^2([-1/2,1/2])} &\leq \varepsilon^{1/2} \left\| \int_{\delta/\varepsilon}^{\infty} \widehat{A}(M)\widehat{A}(\cdot_L - M)\Big(\frac{\varepsilon^s M^s}{\delta^s}\Big) dM \right\|_{L^2(\mathbb{R})} \\ &\leq \varepsilon^{s+1/2}\delta^{-s}\|\widehat{A}\|^2_{L^2(s)(\mathbb{R})}. \qquad \square\end{aligned}$$

For (3.21) using (3.15) we find

$$\begin{aligned}|(3.21)| &\leq \int_{-1/2}^{1/2} \Big|\beta(\ell, m, \ell-m) + \ell^2\mu_1\Big| \cdot \Big|\widehat{A}\Big(\frac{m}{\varepsilon}\Big)\widehat{A}\Big(\frac{\ell-m}{\varepsilon}\Big)\Big| dm \\ &\leq C\varepsilon^5 \int_{-1/2\varepsilon}^{1/2\varepsilon} \Big| \Big(\frac{\ell}{\varepsilon}\Big)^4 + \Big(\frac{\ell}{\varepsilon}\Big)^3 M + \Big(\frac{\ell}{\varepsilon}\Big)^2 M^2 + \Big(\frac{\ell}{\varepsilon}\Big) M^3\Big| \cdot \Big|\widehat{A}\Big(M)\widehat{A}\Big(\frac{\ell}{\varepsilon} - M\Big)\Big| dM\end{aligned}$$

and so using Lemma (3.2.4) the $L^2([-1/2, 1/2])-$ norm of (3.21) can be estimated by $C\varepsilon^{11/2}\|\partial_X^4 A^2\|_{L^2(\mathbb{R})} \leq C\varepsilon^{11/2}\|A\|^2_{H^4}$. For (3.20) we have $g(\ell, m) = \chi(\ell) - \chi(\ell - m) = 0$ for $|\ell - m| \leq \delta/2$. Hence the $L^2([-1/2, 1/2])-$norm can be estimated by

$$\begin{aligned}&\int_{-\delta}^{\delta} \mu_1^2\ell^4 \left| \int_{-\delta/2}^{\delta/2} \widehat{A}\Big(\frac{m}{\varepsilon}\Big)\widehat{A}\Big(\frac{\ell-m}{\varepsilon}\Big) g(\ell,m) dm \right|^2 d\ell \\ &\leq \int_{-\delta}^{\delta} \mu_1^2\ell^4 \left(\int_{-\delta/2}^{\delta/2} |\widehat{A}\Big(\frac{m}{\varepsilon}\Big)| \cdot \Big(\frac{|\ell-m|}{\delta/2}\Big)^{n_1} \cdot \Big(\frac{|m|}{\delta/2}\Big)^{n_2} |\widehat{A}\Big(\frac{\ell-m}{\varepsilon}\Big)| dm \right)^2 d\ell \\ &\leq C\varepsilon^{2n_1+2n_2+7} \int_{-\delta/\varepsilon}^{\delta/\varepsilon} L^4 \left(\int_{-\delta/(2\varepsilon)}^{\delta/(2\varepsilon)} |\widehat{A}(M)| \cdot |L-M|^{n_1}|M|^{n_2} \cdot |\widehat{A}(L-M)| dM \right)^2 dL \\ &\leq C\varepsilon^{2n_1+2n_2+7} \int_{-\infty}^{\infty} L^4 \left(\int_{-\infty}^{\infty} |\widehat{A}(M)| \cdot |L-M|^{n_1}|M|^{n_2} \cdot |\widehat{A}(L-M)| dM \right)^2 dL \\ &= C\varepsilon^{2n_1+2n_2+7}\| \, |\widehat{B}| \star |\widehat{C}| \, \|^2_{L^2(2)} \quad \leq \quad C'\varepsilon^{2n_1+2n_2+5}\|\widehat{B}\|^2_{L^2(2)}\|\widehat{C}\|^2_{L^2(2)} \\ &\leq C\,\varepsilon^{2n_1+2n_2+7}\|\widehat{A}\|_{L^2(n_1+2)}\|\widehat{A}\|_{L^2(n_2+2)}\end{aligned}$$

where $\widehat{B}(K) = K^{n_1}\widehat{A}(K)$ and $\widehat{C}(K) = K^{n_2}\widehat{A}(K)$. Putting all together if we choose $n_1 = n_2 = 2$ and $A \in C([0,T_0], H^6(\mathbb{R}))$ to satisfy the KdV equation

$$\partial_T A(X,T) = -\frac{\omega'''(0)}{6}\partial_X^3 A(X,T) + \frac{\mu_1}{2\omega'(0)}\partial_X(A^2(X,T)) \tag{3.22}$$

we find the estimate $\|Res_u\|_{\widetilde{L}^2} \le C\varepsilon^{15/2}$. Since we have that $P(\ell)Res(\widetilde{u}) = \mathcal{O}(\ell)$ we also find the estimate $\|\partial^{-1}Res_u\|_{\widetilde{L}^2} \le C'\varepsilon^{13/2}$. For the corresponding complementary part we find

$$\begin{aligned}
&\|Res_v\|^2_{\widetilde{L}^2} = \|Q\partial(c\partial)((\varepsilon^2\widetilde{\psi}) \star (\varepsilon^2\widetilde{\psi}))\|^2_{\widetilde{L}^2} \le \|\partial(c\partial)(\varepsilon^2\widetilde{\psi}) \star (\varepsilon^2\widetilde{\psi})\|^2_{\widetilde{L}^2} \qquad (3.23)\\
&= \int_{-1/2}^{1/2}\int_{S^1}\left|\int_{-1/2}^{1/2}\varepsilon^2\widehat{A}\left(\frac{m}{\varepsilon}\right)\widehat{A}\left(\frac{\ell-m}{\varepsilon}\right)\partial(c\partial)(\widetilde{f}(m)\widetilde{f}(\ell-m))\chi(m)\chi(\ell-m)dm\right|^2 dx\,d\ell\\
&\le \int_{-\delta}^{\delta}\int_{S^1}\left|\int_{-\delta/2}^{\delta/2}\varepsilon^2\widehat{A}\left(\frac{m}{\varepsilon}\right)\widehat{A}\left(\frac{\ell-m}{\varepsilon}\right)\partial(c\partial)(\widetilde{f}(m)\widetilde{f}(\ell-m))dm\right|^2 dx\,d\ell\\
&\le \varepsilon^7\int_{-\delta/\varepsilon}^{\delta/\varepsilon}\int_{S^1}\Bigg|\int_{-\delta/(2\varepsilon)}^{\delta/(2\varepsilon)}\widehat{A}(M)\,\widehat{A}(L-M)\times\\
&\qquad\qquad (\partial_x + i\varepsilon L)(c(\partial_x + iL))(\widetilde{f}(\varepsilon M)\widetilde{f}(\varepsilon(L-M)))dM\Bigg|^2 dx\,dL\\
&\le \varepsilon^9\int_{-\infty}^{\infty}\int_{S^1}\left(\int_{-\infty}^{\infty}|\widehat{A}(M)\,\widehat{A}(L-M)|\,|L|\,|\widetilde{f}(0)\partial_x(\widetilde{f}(0) - i\partial_x\widetilde{f}'(0))|dM\right)^2 dx\,dL + \mathcal{O}(\varepsilon^{11})\\
&\le C'\varepsilon^9\|\,|\widehat{A}| \star |\widehat{A}|\,\|^2_{L^2(1)(\mathbb{R})} + \mathcal{O}(\varepsilon^{11}) \le C\varepsilon^9
\end{aligned}$$

where we used $\widetilde{f}(m)\widetilde{f}(\ell-m) = \widetilde{f}(0)^2 + i\ell\widetilde{f}(0)\widetilde{f}'(0) +$ higher order terms. In order to estimate the $\widetilde{L}^2-$norm of $\partial^{-1}Res_v$ we can use the following lemma:

Lemma 3.3.3 *$\partial(\cdot_\ell)^{-1} : \widetilde{V} \to \widetilde{L}^2$ is a continuous operator.*

Proof: Let $\widetilde{v} \in \widetilde{V}$. We have

$$\begin{aligned}
|\frac{1}{\ell}\int_{S^1}\widetilde{v}(\ell,x)\,dx| &= |\frac{\sqrt{2\pi}}{\ell}\int_{S^1}\widetilde{v}(\ell,x)\overline{\widetilde{f}(0)}\,dx|\\
&= |\frac{\sqrt{2\pi}}{\ell}\int_{S^1}\widetilde{v}(\ell,x)\overline{(\widetilde{f}(\ell) - (\widetilde{f}(\ell) - \widetilde{f}(0))}\,dx|\\
&= |\sqrt{2\pi}\int_{S^1}\widetilde{v}(\ell,x)\overline{\frac{\widetilde{f}(\ell)-\widetilde{f}(0)}{\ell}})\,dx|\\
&\le \sqrt{2\pi}\sup_{\xi\in[-1/2,1/2]}\|\widetilde{f}'(\xi)\|_{L^2(S^1,\mathbb{C})}\|\widetilde{v}(\ell,\cdot)\|_{L^2(S^1,\mathbb{C})}
\end{aligned}$$

where we used $P(\ell)\widetilde{v}(x,\ell) = 0$ in $L^2(S^1,\mathbb{C})$. Hence using Lemma (3.1.5) and the triangle inequality we find

$$\|\partial^{-1}\widetilde{v}\|_{\widetilde{L}^2} \leq \left(\sqrt{2\pi}\sup_{\xi\in[-1/2,1/2]}\|\widetilde{f}'(\xi)\|_{L^2(S^1,\mathbb{C})} + 2\right)\|\widetilde{v}\|_{\widetilde{L}^2}. \qquad \square$$

We have $\|\partial^{-1}Res_v\|_{L^2} \leq C\varepsilon^{9/2}$ which is too large for our purposes. A better approximation is needed and will be given in the following subsection.

Improved approximation

In order to find a better approximation we extend our first ansatz to

$$\varepsilon^2\widetilde{\psi}(x,\ell,t) = (\widetilde{u}(x,\ell,t) + \widetilde{v}(x,\ell,t))e^{i\omega'(0)\ell t}$$

where $\widetilde{u}(x,\ell,t) = \left(\varepsilon\widehat{A}\left(\frac{\ell}{\varepsilon},\varepsilon^3 t\right)\widetilde{f}(\ell)(x)\right)\chi(\ell)$ and $\widetilde{v}(\cdot_x,\ell) \in V(\ell)$ for $|\ell| < \delta$ and $= 0$ otherwise. Since we have that

$$Res(\widetilde{u}+\widetilde{v}) = Res(\widetilde{u}) + Res(\widetilde{v}) + 2\partial(c\partial)(\widetilde{u}\star\widetilde{v})$$

we find

$$\begin{aligned}\partial^{-1}Res(\varepsilon^2\widetilde{\psi})(x,\ell,t) \quad = \quad & \partial^{-1}\left(Q(\ell)Res(\widetilde{u}e^{i\omega'(0)\ell t}) - (\omega(\ell)^2 + \widetilde{L}(\partial))\widetilde{v}e^{i\omega'(0)\ell t}\right) \\ & + 2Q(\ell)c\partial(\widetilde{u}\star\widetilde{v})e^{i\omega'(0)\ell t} + Q(\ell)c\partial(\widetilde{v}\star\widetilde{v})e^{i\omega'(0)\ell t} \\ & + \partial^{-1}\left(\omega(\ell)^2 - \ell^2\omega'(0)^2\right)\widetilde{v}e^{i\omega'(0)\ell t} + \partial^{-1}P(\ell)Res(\widetilde{u}e^{i\omega'(0)\ell t}) \\ & - 2i\omega'(0)\ell\partial^{-1}\partial_t\widetilde{v}e^{i\omega'(0)\ell t} - \partial^{-1}\partial_t^2\widetilde{v}e^{i\omega'(0)\ell t}.\end{aligned}$$

If we choose

$$\widetilde{v}(x,\ell,t) = \left(\omega^2(\ell) + \widetilde{L}(\partial)\right)^{-1} Q\partial(c\partial)(\widetilde{u}\star\widetilde{u})(x,\ell,t)\chi(\ell)$$

we find the estimate $\|\widetilde{v}\|_{\widetilde{H}^s} \leq \|\partial(c\partial)(\widetilde{u}\star\widetilde{u})\|_{\widetilde{H}^{s-2}} \leq C\varepsilon^{9/2}$ for $s > 1/2$. The approximation is well defined since the $\widetilde{L}^2$-adjoint of the operator $(\omega(\ell)^2+\widetilde{L}(\partial))^{-1}\partial(c\partial) \in Lin(\widetilde{H}^2,\widetilde{H}^4)$ possesses a continuous linear extension $\widetilde{L}^2 \to \widetilde{L}^2$.

Lemma 3.3.4 *Let $s > 1/2$. Then for $\widetilde{u},\widetilde{v} \in \widetilde{H}^s$ we have $\|\widetilde{u}\star_\ell\widetilde{v}\|_{\widetilde{H}^s} \leq C\|\widetilde{u}\|_{\widetilde{H}^s}\|\widetilde{v}\|_{\widetilde{H}^s}$ and $\|\widetilde{u}\star_\ell\widetilde{v}\|_{\widetilde{H}^s} \leq C\|\widetilde{u}\|_{L^1([-1/2,1/2],H^s(S^1,\mathbb{C}))}\|\widetilde{v}\|_{\widetilde{H}^s}$.*

Proof: The first estimate follows directly from the corresponding estimate $\|uv\|_{H^s} \leq C\|u\|_{H^s}\|v\|_{H^s}$ in physical space. For the second one we have

$$\begin{aligned}\|\widetilde{u} \star_\ell \widetilde{v}\|_{\widetilde{H}^s} &= \| \, \|(\widetilde{u} \star_\ell \widetilde{v})\|_{H^s(S^1,\mathbb{C})} \, \|_{L^2([-1/2,1/2])} \\ &\leq C\| \, \|\widetilde{u}\|_{H^s(S^1,\mathbb{C})} \star_\ell \|\widetilde{v}\|_{H^s(S^1,\mathbb{C})} \, \|_{L^2([-1/2,1/2])} \\ &\leq C\| \, \|\widetilde{u}\|_{H^s(S^1,\mathbb{C})} \, \|_{L^1([-1/2,1/2])} \cdot \| \, \|\widetilde{v}\|_{H^s(S^1,\mathbb{C})} \, \|_{L^2([-1/2,1/2])} \\ &= C\|\widetilde{u}\|_{L^1([-1/2,1/2],H^s(S^1,\mathbb{C}))}\|\widetilde{v}\|_{\widetilde{H}^s}. \end{aligned}$$

□

Using these inequalities we finally find the estimates

$$\begin{aligned}\|(Qc\partial)(\widetilde{u} \star \widetilde{v})\|_{\widetilde{L}^2} &\leq C\|\widetilde{u}\|_{L^1([-1/2,1/2],H^1(S^1,\mathbb{C}))}\|\widetilde{v}\|_{\widetilde{H}^1} \leq C\varepsilon^{13/2}, \\ \|(Qc\partial)(\widetilde{v} \star \widetilde{v})\|_{\widetilde{L}^2} &\leq C\|\widetilde{v}\|_{L^1([-1/2,1/2],H^1(S^1,\mathbb{C}))}\|\widetilde{v}\|_{\widetilde{H}^1} \leq C\varepsilon^{19/2}, \\ \|\partial^{-1}\partial_t^2\widetilde{v}\|_{\widetilde{L}^2} &\leq C\|\partial_t^2(\widetilde{u} \star \widetilde{u})\|_{\widetilde{L}^2} \leq C\varepsilon^{19/2}, \\ \|\partial^{-1}(\omega(\ell)^2 - \ell^2\omega'(0)^2)\widetilde{v}\|_{\widetilde{L}^2} &\leq C\|(\cdot_\ell)^4(\widetilde{u} \star \widetilde{u})\|_{\widetilde{L}^2} \leq \varepsilon^{17/2}, \\ \|(\cdot_\ell)\partial^{-1}\partial_t\widetilde{v}\|_{\widetilde{L}^2} &\leq C\varepsilon^{17/2}\end{aligned}$$

and so, taken all together, we can estimate the residual terms in physical space by

$$\|Res(\varepsilon^2\psi)\|_{L^2} \leq C\varepsilon^{13/2} \text{ and } \|\partial_x^{-1}Res(\varepsilon^2\psi)\|_{L^2} \leq C\varepsilon^{13/2}.$$

Remark 3.3.5 *Note that for the derivation of the KdV equation we only need local spectral properties, more precisely the values of $\omega'(0)$ and $\omega'''(0)$ and the quadratic structure of the non-linearity. The global spectral behavior will be involved in the justification part, where we will need the equivalence of the quadratic form corresponding to the linear operator and the squared $H^2(\mathbb{R})-$norm.*

3.3.2 Derivation of the Whitham equation

In order to derive a Whitham equation

$$\partial_T^2 U = \partial_X^2 U + \partial_X^2 F(U) \tag{3.24}$$

with a continuous mapping $F : H^2 \longrightarrow H^2$ we make the ansatz

$$\widetilde{\psi}(x,\ell,t) \;=\; \left(\varepsilon^{-1}\widehat{A} + \widehat{B}\right)\left(\frac{\ell}{\varepsilon}, \varepsilon t\right)\widetilde{f}(\ell)\chi(\ell) \tag{3.25}$$

$$+ \, (\widehat{U} + \varepsilon\widehat{V})\left(x, \frac{\ell}{\varepsilon}, \varepsilon t\right) \tag{3.26}$$

where $\widehat{U}(\cdot_x,\ell)\in V(0)$ and where

$$\begin{aligned}\widehat{A}(L) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iLy}A(y)dy, & \widehat{B}(L) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iLy}B(y)dy,\\ \widehat{U}(x,L) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iLy}U(x,y)dy, & \widehat{V}(x,L) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iLy}V(x,y)dy.\end{aligned}$$

Solving the residual equations

$$\begin{aligned} Res_{\widetilde{u}}(x,\ell,t) &= \langle Res(\widetilde{\psi})(\cdot_x,\ell,t),\widetilde{f}(\ell)(\cdot_x)\rangle_{L^2(S^1,\mathbb{C})}\\ &= (-\varepsilon\partial_T^2\widehat{A}-\varepsilon^2\widehat{B})\left(\frac{\ell}{\varepsilon},\varepsilon t\right)\\ &\quad -\omega'(0)^2L^2(\varepsilon\widehat{A}+\varepsilon^2\widehat{B})\left(\frac{\ell}{\varepsilon},\varepsilon t\right)\\ &\quad +\langle\partial(c\partial)(\widetilde{\psi}\star\widetilde{\psi})(\cdot_x,\ell,t),\widetilde{f}(\ell)(\cdot_x)\rangle_{L^2(S^1,\mathbb{C})}+\mathcal{O}(\varepsilon^{5/2})\end{aligned}$$

and going back to the physical y-space we find the equations

$$\begin{aligned}\partial_T^2A(y) &= \omega'(0)^2\partial_y^2A(y)+\mu\partial_y^2(A^2(y))-\sqrt{\frac{2}{\pi}}\partial_y\left(A(y)\int_{S^1}U(x,y)\gamma(x)dx\right)\\ \partial_T^2B(y) &= \omega'(0)^2\partial_y^2B(y)+2\mu\partial_y^2(AB)(y)-\sqrt{\frac{2}{\pi}}\partial_y\left(B(y)\int_{S^1}U(x,y)\gamma(x)dx\right)\\ &\quad -\sqrt{\frac{2}{\pi}}\partial_y^2\left(A(y)\int_{S^1}V(x,y)c(\widetilde{f}(0)-i\partial_x\widetilde{f}'(0))dx\right)\\ &\quad +2\partial_y\left(\partial_yA(y)\int_{S^1}i\widetilde{f}'(0)U(x,y)\gamma(x)dx\right)-\partial_y\int_{S^1}U^2(x,y)\gamma(x)dx\end{aligned} \tag{3.27}$$

where $\gamma(x)=\partial_x(c(x)\alpha(x))$ and where α is defined in (3.14). The complementary equations for the residual give for the correctors the equations

$$0 = \widetilde{L}(\partial_x)\widehat{U}+G_1(\widehat{A},\widehat{B},\widehat{U},\widehat{V}) \tag{3.28}$$

$$0 = \widetilde{L}(\partial_x)\widehat{V}+L\widetilde{L}'(0)\widehat{U}+\frac{1}{2}G_2(\widehat{A},\widehat{B},\widehat{U},\widehat{V}) \tag{3.29}$$

where $G_i(\widehat{A},\widehat{B},\widehat{U},\widehat{V})=\partial_\epsilon^iG(\varepsilon)(\widehat{A},\widehat{B},\widehat{U},\widehat{V})_{|\varepsilon=0}$ and

$$G(\varepsilon)(\widehat{A},\widehat{B},\widehat{U},\widehat{V})(T,x,L)=Q(\varepsilon L)(\partial_x+i\varepsilon L)c(\partial_x+i\varepsilon L)(\widetilde{\psi}\star\widetilde{\psi})(\varepsilon T,x,\varepsilon L).$$

For (3.28) we find in physical space

$$0 \;=\; \widetilde{L}(\partial_x)U(x,y) + \sqrt{\frac{2}{\pi}}A(y)Q(0)\partial_x(c\partial_x)U(x,y) + \sqrt{\frac{2}{\pi}}\partial_y(A^2(y))Q(0)\gamma(x).$$

Using $Q(0)|_{V(0)} = \mathrm{Id}$ we find the solution

$$\begin{aligned} U(x,y) &= -2\sqrt{\frac{2}{\pi}}A(y)\partial_y A(y)\left(\mathrm{Id} + \sqrt{\frac{2}{\pi}}A(y)\widetilde{L}(\partial_x)^{-1}\partial_x(c\partial_x\cdot)\right)^{-1}\left(\widetilde{L}(\partial_x)^{-1}\gamma\right)(x) \\ &= -2\sqrt{\frac{2}{\pi}}A(y)\partial_y A(y)\sum_{n=0}^{\infty}\left(-\sqrt{\frac{2}{\pi}}A(y)\widetilde{L}(\partial_x)^{-1}\partial_x(c\partial_x\cdot)\right)^{n}\left(\widetilde{L}(\partial_x)^{-1}\gamma\right)(x). \end{aligned}$$

Here we used that we have $\sqrt{\frac{2}{\pi}}A(y)\partial_x(c\partial_x\widetilde{L}(\partial_x)^{-1}\cdot) \in Lin(L^2)$. Hence its L^2-adjoint $\sqrt{\frac{2}{\pi}}A(y)\widetilde{L}(\partial_x)^{-1}\partial_x(c\partial_x\cdot)$ is a linear continuous operator such that for sufficiently small A in $C([0,T_0],C(\mathbb{R}))$ we have

$$\|\sqrt{\frac{2}{\pi}}A(y)\widetilde{L}(\partial_x)^{-1}\partial_x(c\partial_x\cdot)\|_{Lin(L^2(S^1))} = \|\sqrt{\frac{2}{\pi}}A(y)\partial_x(c\partial_x\widetilde{L}(\partial_x)^{-1})\|_{Lin(L^2(S^1))} \le q < 1.$$

For the last non-linear term in the equation for A we find

$$A(y)\int_{S^1}U(x,y)\gamma(x)dx = -2\sqrt{\frac{2}{\pi}}\partial_y\left(A^3(y)\sum_{n=0}^{\infty}\frac{c_n(y)}{n+3}\right)$$

where

$$\begin{aligned} c_n(y) &= \int_{S^1}\left(-\sqrt{\frac{2}{\pi}}A(y)\widetilde{L}(\partial_x)^{-1}\partial_x(c\partial_x)\right)^{n}\widetilde{L}(\partial_x)^{-1}\gamma(x)\cdot\gamma(x)dx \\ &\le q^n\,\|\widetilde{L}(\partial_x)^{-1}\|_{Lin(\widetilde{V}(0),L^2(S^1))}\|\gamma\|^2_{L^2(S^1)}. \end{aligned}$$

Using this we finally find the Whitham equation

$$\partial_T^2 A(y) \quad = \quad \omega'(0)^2\partial_y^2 A(y) + \partial_y^2\left(\mu_1 A^2(y) + \frac{4}{\pi}A^3(y)\sum_{n=0}^{\infty}\frac{c_n(y)}{n+3}\right). \tag{3.30}$$

The remaining equation (3.29) for V can be solved with respect to V for small A and B (in $C([0,T_0],\mathbb{R})$). In detail for

$$H(\widehat{A},\widehat{B},\widehat{U},\widehat{V}) = L\widetilde{L}'(\partial_x)\widehat{U} + \widetilde{L}(\partial_x)\widehat{V} + G_1(\widehat{A},\widehat{B},\widehat{V})$$

we have $H(0,0,0,0)=0$ and

$$H(0,0,\delta\widetilde{V}) \;=\; \delta\widetilde{L}(\partial_x)\widetilde{V} + \mathcal{O}(\delta^2) = \delta\partial_{\widehat{V}}H(0,0,0,0)(\widetilde{V}) + \mathcal{O}(\delta^2).$$

Since $\widetilde{L}(\partial_x)|_{V(0)} : V(0) \to V(0)$ is an invertible operator, for sufficiently small A and B (in L^2) we find $\widehat{V} = \widehat{V}(\widehat{A},\widehat{B})$ such that $H(\widehat{A},\widehat{B},\widehat{V}(\widehat{A},\widehat{B}))=0$. Using this information, for B, we find a linear inhomogeneous equation.

Lemma 3.3.6 *Let $T_1 > 0$ and let $A \in C([0,T_1],H^4)$ be a solution to the Whitham equation (3.30) such that*

$$\sup_{T\le T_1, y\in\mathbb{R}} |A(y)|\sqrt{\frac{2}{\pi}}\|\widetilde{L}(\partial_x)^{-1}\partial_x(c\partial_x\cdot)\|_{Lin(V(0),V(0))} \le q < 1. \tag{3.31}$$

Let $T_0 \le T_1$ and let $B \in C([0,T_0],H^4)$ be a solution to (3.27) and let T_0, μ and B be chosen so small that the equations from above can be solved for $\widehat{U}$ and $\widehat{V}$. Then there exists a $C>0$ such that for all eps >0 we have

$$\sup_{t\in[0,T_0/\varepsilon^3]} \|Res(\psi)(\cdot_x,t)\|_{L^2(\mathbb{R})} \;\le\; C\varepsilon^{3/2} \;\; \textit{and} \;\; \sup_{t\in[0,T_0/\varepsilon^3]} \|\partial_x^{-1}Res(\psi)(\cdot_x,t)\|_{L^2(\mathbb{R})} \;\le\; C\varepsilon^{3/2}.$$

Remark 3.3.7 *Note that, in contrast to the KdV case, in the Whitham case we have additional conditions $\|A\|_{C([0,T_1],C(\mathbb{R}))} < C$ and $\|B\|_{C([0,T_0],C(\mathbb{R}))} < C$ where C is so small that we can solve the equations for A and B.*

3.4 Justification of the KdV approximation and the Whitham approximation

3.4.1 The main result

Our approximation result for the KdV approximation and the Whitham approximation is as follows:

Theorem 3.4.1 *Consider the spatially periodic Boussinesq equation (3.32) where $a \in C^1(S^1)$ and $b \in C^2(S^1)$ are positive even functions and where $c \in C^1(S^1,\mathbb{C})$ satisfies the condition*

$$\int_{S^1} c(x)\alpha(x)\widetilde{\alpha}(x)\,dx = 0$$

where α and $\widetilde{\alpha}$ are defined in (3.13).

i) Let $T_0 > 0$ and let $A \in C([0,T_0], H^6(\mathbb{R}))$ be a solution to the KdV equation (3.22). *Then there exists an $\varepsilon_0 > 0$ and a $C > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ we have solutions $u \in C([0,T_0/\varepsilon^3], H^2(\mathbb{R}))$ of our spatially periodic Boussinesq equation* (1.4) *with*

$$\sup_{t\in[0,T_0/\varepsilon^3]} \|u(\cdot_x,t) - \varepsilon^2 \widetilde{f}(0)A(\varepsilon(\cdot_x + \omega'(0)t), \varepsilon^3 t)\|_{H^2} \leq C\varepsilon^{5/2}.$$

ii) There exists a $C_1 > 0$ such that for all $T_0 > 0$ and every solution $A \in C([0,T_0], H^4(\mathbb{R}))$ of the Whitham equation (3.30) *with $\|A(\cdot_X,T)\|_{L^\infty} \leq C_1$ we have for all $\varepsilon \geq 0$ solutions $u \in C([0,T_0/\varepsilon], H^2(\mathbb{R}))$ of* (1.4) *with*

$$\sup_{t\in[0,T_0/\varepsilon^3]} \|u(\cdot_x,t) - \widetilde{f}(0)A(\varepsilon\cdot_x, \varepsilon t)\| \leq C\varepsilon^{3/2}.$$

Note, that the condition on c is satisfied, if e.g. c is a positive and even function. Mind also the difference of the formulations with respect for the dependencies on the perturbation parameter ε.

In the previous section we constructed the KdV approximation and improved approximations, which allow us to make the residual small enough for our purposes.

3.4.2 Justification of the KdV approximation

We write the solution u of the spatially periodic Boussinesq equation

$$\partial_t^2 u(x,t) = \partial_x(a(x)\partial_x u(x,t)) - \partial_x^2(b(x)\partial_x^2 u(x,t)) + \partial_x(c(x)\partial_x(u(x,t)^2)) \quad (3.32)$$

as a sum of the approximation $\epsilon^2\psi$ plus some error $\varepsilon^{7/2}R$, i.e., $u = \epsilon^2\psi + \epsilon^{7/2}R$ where R satisfies the equation

$$\partial_t^2 R = L(\partial_x)R + 2\underline{\partial_x(c\partial_x(\varepsilon^2\psi R))} + \epsilon^{7/2}\partial_x(c\partial_x(R^2)) + \varepsilon^{-7/2}\mathrm{Res}(\varepsilon^2\psi). \quad (3.33)$$

Obviously, the underlined term is the dangerous one and may be the generator of unbounded growth with respect to the perturbation parameter ε since it is of order ε^2 on a $\mathcal{O}(\varepsilon^{-3})-$timescale. The energy from the spatially constant case cannot be applied directly and needs to be modified in a non-trivial way. In order to adapt the arguments we need to write the equation for the error as a first order system $\partial_t Z = \Lambda Z + \mathcal{O}(\varepsilon^3)$ such that we can define the energy using the linear skew symmetric operator Λ.

Normal form transform

First, we want to discuss a possible normal form transform in order to choose appropriate coordinates, using which the dangerous terms can be shown to be small. This is a common way to eliminate disturbing terms, e.g. in the justification of the non-linear Schrödinger equation for a spatially periodic wave equation considered in [BSTU06]. We don't give an exact proof for the provided statement since the method using the normal form transform already fails in the much more simple case of constant coefficients.

As before, for every $(-\omega^2, \widetilde{f}) \subset \mathbb{C} \times (\widetilde{L}^2 \backslash \{0\})$ where $\widetilde{L}(\partial_x)\widetilde{f} = -\omega^2 \widetilde{f}$ we have $C^\infty-$curves of eigenvalues $-\omega(\ell)^2$ and eigenfunctions $\widetilde{f}(\ell)(\cdot_x)$ with

$$\widetilde{L}(\partial)\widetilde{f}(\ell)(\cdot_x) = -\omega^2(\ell)\widetilde{f}(\ell)(\cdot_x) \text{ and } (\omega(0), \widetilde{f}(0)(\cdot_x)) = (\omega, \widetilde{f}).$$

Let J be an index set to any complete set of eigenfunctions to $\widetilde{L}(\partial_x)$ such that every $\widetilde{u} \in \widetilde{L}^2$ possesses a unique representation

$$\widetilde{u}(t, x, \ell) = \sum_{i \in J} \underbrace{\langle \widetilde{u}(t, \cdot_x, \ell), \widetilde{f}_i(\ell)(\cdot_x)\rangle_{L^2(S^1,\mathbb{C})}}_{=\widetilde{u}_i(t,\ell)} \widetilde{f}_i(\ell)(x).$$

In these coordinates the Boussinesq equation is given by

$$\partial_t^2 \widetilde{u}_i(x, \ell) = -\omega_i(\ell)^2 \widetilde{u}_i(x, \ell) + \int_{-1/2}^{1/2} \sum_{j_1, j_2 \in J} \beta^i_{j_1,j_2}(\ell, m, \ell - m)\widetilde{u}_{j_1}(m)\widetilde{u}_{j_2}(\ell - m)\, d\ell$$

where

$$\beta^i_{j_1,j_2}(\ell, m, \ell - m) := \int_{S^1} \widetilde{f}_{j_1}(m)\widetilde{f}_{j_2}(\ell - m)\overline{\partial(c\partial)\widetilde{f}_i(\ell)} dx.$$

Every linear mapping $M : \widetilde{L}^2 \to \widetilde{L}^2$ is uniquely determined by the values on the corresponding eigenspaces such that we have the generalized matrix representation

$$M(\ell)\widetilde{f}_i(\ell) = \sum_{i,j} m_{i,j}(\ell)\widetilde{f}_j(\ell).$$

For the normal form transform we choose the ansatz

$$R = \mathcal{R} + M(2\varepsilon^2\psi\mathcal{R}).$$

The coordinate functions

$$\widetilde{r}_i(\ell) = \langle \widetilde{R}(\cdot_x, \ell, t), \widetilde{f}_i(\ell)(\cdot_x)\rangle_{L^2(S^1,\mathbb{C})} \text{ resp. } \widetilde{\tau}_i(\ell) = \langle \widetilde{\mathcal{R}}(\cdot_x, \ell, t), \widetilde{f}_i(\ell)(\cdot_x)\rangle_{L^2(S^1,\mathbb{C})}$$

satisfy the relations

$$\widetilde{r}_i(\ell) \quad = \quad \widetilde{\tau}_i(\ell) + \sqrt{\frac{2}{\pi}} \sum_{k \in J} m_{ki}(\ell) \int_{-1/2}^{1/2} \varepsilon \widehat{A}\left(\frac{m}{\varepsilon}, 0\right) \widetilde{\tau}_k(\ell - m) dm + \mathcal{O}(\varepsilon^3)$$

in $L^1([-1/2, 1/2])$ for $i \in J$. We have

$$\begin{aligned}
\widetilde{r}_i(\ell) &= \widetilde{\tau}_i(\ell) + 2\langle M(\varepsilon^2 \widetilde{\psi} \star_\ell \widetilde{R}(\cdot_x, \cdot_\ell, t)), \widetilde{f}_i(\ell)\rangle_{L^2(S^1,\mathbb{C})} \\
&= \widetilde{\tau}_i(\ell) + 2\langle M\left(\sum_{k \in J} \langle \varepsilon^2 \widetilde{\psi} \star_\ell \widetilde{R}(\cdot_x, \cdot_\ell, t), \widetilde{f}_k(\ell)\rangle_{L^2(S^1,\mathbb{C})} \widetilde{f}_k(\ell)\right), \widetilde{f}_i(\ell)\rangle_{L^2(S^1,\mathbb{C})} \\
&= \widetilde{\tau}_i(\ell) + 2\langle \sum_{k \in J} \langle \varepsilon^2 \widetilde{\psi} \star_\ell \widetilde{R}(\cdot_x, \cdot_\ell, t), \widetilde{f}_k(\ell)\rangle_{L^2(S^1,\mathbb{C})} \sum_{j \in J} m_{k,j}(\ell) \widetilde{f}_j(\ell), \widetilde{f}_i(\ell)\rangle_{L^2(S^1,\mathbb{C})} \\
&= \widetilde{\tau}_i(\ell) + 2 \sum_{k \in J} \sum_{j \in J} \langle \varepsilon^2 \widetilde{\psi} \star_\ell \widetilde{R}(\cdot_x, \ell, t), \widetilde{f}_k(\ell)\rangle_{L^2(S^1,\mathbb{C})} m_{k,j}(\ell) \langle \widetilde{f}_j(\ell), \widetilde{f}_i(\ell)\rangle_{L^2(S^1,\mathbb{C})} \\
&= \widetilde{\tau}_i(\ell) + 2 \sum_{k \in J} \langle \varepsilon^2 \widetilde{\psi} \star_\ell \widetilde{R}(\cdot_x, \ell, t), \widetilde{f}_k(\ell)\rangle_{L^2(S^1,\mathbb{C})} m_{k,i}(\ell)
\end{aligned}$$

and

$$\begin{aligned}
&\langle \varepsilon^2 \widetilde{\psi} \star_\ell \widetilde{R}(\cdot_x, \ell, t), \widetilde{f}_k(\ell)\rangle_{L^2(S^1,\mathbb{C})} \\
&\qquad = \langle \int_{-1/2}^{1/2} (\varepsilon \widehat{A}(\varepsilon^{-1} m, \varepsilon^3 t) \widetilde{f}(m) e^{i\omega'(0)mt} + \mathcal{O}(\varepsilon^3)) \times \\
&\qquad\qquad \sum_j \widetilde{r}_j(\ell - m) \widetilde{f}_j(\ell - m), \widetilde{f}_k(\ell)\rangle_{L^2(S^1,\mathbb{C})} dm \\
&\qquad = \int_{-1/2}^{1/2} \varepsilon \widehat{A}(\varepsilon^{-1} m, \varepsilon^3 t) \widetilde{f}(0) \widetilde{r}_j(\ell - m) dm \times \\
&\qquad\qquad \langle \widetilde{f}_j(\ell), \widetilde{f}_k(\ell)\rangle_{L^2(S^1,\mathbb{C})} + \mathcal{O}(\varepsilon^3) \\
&\qquad = \varepsilon \widetilde{f}(0) \int_{-1/2}^{1/2} \widehat{A}(\varepsilon^{-1} m, 0) \widetilde{r}_j(\ell - m) dm + \mathcal{O}(\varepsilon^3).
\end{aligned}$$

Hence using $\varepsilon \partial_t \widehat{A} = \varepsilon^4 \partial_T \widehat{A}$, $\omega_j(\ell - m) = \omega_j(\ell) + \mathcal{O}(m)$ and $\widetilde{f}_j(\ell) = \widetilde{f}_j(0) + \mathcal{O}(\ell)$ we find in lowest order the set of equations

$$\partial_t^2 \widetilde{\tau}_j(\ell) \quad = \quad -\omega_j^2(\ell) \widetilde{\tau}_j(\ell) + \sqrt{\frac{2}{\pi}} \sum_{j,k \in J} s_{j,k}(\ell) \int_{-1/2}^{1/2} \varepsilon \widehat{A}\left(\frac{m}{\varepsilon}, 0\right) \widetilde{\tau}_k(\ell - m) dm + \mathcal{O}(\varepsilon^3)$$

where

$$s_{j,k}(\ell) = m_{jk}(\ell)\Big(\omega_k^2(\ell) - \omega_j^2(\ell)\Big) - \int_{S^1} c\partial\widetilde{f}_j(\ell)\overline{\partial\widetilde{f}_k(\ell)}dx.$$

Here $s_{j,j}(\ell) = 0$ is needed to eliminate the diagonal elements, but this implies

$$\int_{S^1} c(x)\,|\partial\widetilde{f}_j(\ell)(x)|^2\,dx \;=\; 0$$

and hence cannot be satisfied if $\inf_{S^1} c \;>\; 0$. As we see, the diagonal elements cannot be eliminated with our ansatz.

Energy estimates

In this section we give the proof to the main theorem 3.4.1. First we note that the two terms on the right hand side of (3.33) can be written as

$$\partial_x(a\partial_x R) - \partial_x^2(b\partial_x^2 R) + 2\partial_x(c\varepsilon^2\psi\partial_x R) + 2\partial_x(c(\partial_x\varepsilon^2\psi)R). \tag{3.34}$$

The last one of these terms is of order $\mathcal{O}(\varepsilon^3)$ in $L^2(\mathbb{R})$ due to the long wave character of the approximation $\varepsilon^2\psi$. If we choose $a + 2c\varepsilon^2\psi > C > 0$ uniformly in time the first three terms can be written as $\partial_x\mathcal{B}\partial_x R$ where $\mathcal{B}$ is the self-adjoint positive definite operator

$$\mathcal{B} = (a + 2c\varepsilon^2\psi) - \partial_x(b\partial_x).$$

Hence there exists a positive definite self-adjoint square root $\mathcal{A}$, that means a positive-definite self-adjoint operator $\mathcal{A}$ with $\mathcal{A}^2 = \mathcal{B}$. The associated operator norm

$$\|\cdot\|_{\widetilde{H}^n} = \|\mathcal{A}^n\cdot\|_{L^2}$$

is equivalent to the H^n-norm and $\mathcal{A}^{-1}$ is a continuous linear mapping from L^2 to H^1. In fact for fixed time $\mathcal{A}^2$ is a sectorial operator and we have the estimate

$$\|(\lambda + \mathcal{A}^2)^{-1}\|_{Lin(L^2,L^2)} \leq \frac{1}{\lambda + C^\star}, \qquad 2C^\star = \inf_{S^1} a(\cdot_x)$$

for $\lambda > 0$ if we again choose ε to be sufficiently small. We can show even more:

Lemma 3.4.2 *For $\mathcal{A} = (a + 2c\varepsilon^2\psi - \partial_x(b\partial_x))^{1/2}$ there exists an $\alpha \in \mathbb{R}$ and an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ we have the estimate*

$$\forall t \in [0, T_0/\varepsilon^3]: \qquad \|(\omega^2 + \mathcal{A}^2)^{-1}\|_{Lin(\widetilde{H}^2,\widetilde{H}^2)} \leq \frac{1}{\omega^2 + \alpha^2}.$$

Proof: Consider the equation $u = (\omega^2 + \mathcal{A}^2)v$. We have

$$\begin{aligned}\|u\|_{\widetilde{H}^2}\|v\|_{\widetilde{H}^2} \;\geq\; \langle u, v\rangle_{\widetilde{H}^2} \;=\; \langle(\omega^2 + \mathcal{A}^2)v, v\rangle_{\widetilde{H}^2} \;\geq\; \omega^2\|v\|^2_{\widetilde{H}^2} + \langle \mathcal{A}^2 v, v\rangle_{\widetilde{H}^2} \\ \geq (\omega^2 + \alpha^2)\|v\|^2_{\widetilde{H}^2}\end{aligned}$$

where we used

$$\begin{aligned}|\langle \mathcal{A}^2 v, v\rangle_{\widetilde{H}^2}| &= \langle (a + 2c\varepsilon^2\psi - \partial_x(b\partial_x))\mathcal{A}^2 v, \mathcal{A}^2 v\rangle_{L^2} \\ &= \langle (a + 2c\varepsilon^2\psi)\mathcal{A}^2 v, \mathcal{A}^2 v\rangle_{L^2} + \underbrace{\langle b\partial_x \mathcal{A}^2 v, \partial_x A^2 v\rangle_{L^2}}_{\geq 0} \\ &\geq \left(\inf_{x\in S^1} a(x) - 2\sup_{x\in S^1} c(x)\cdot \sup_{x\in\mathbb{R}}\sup_{t\in[0,T_0/\varepsilon^3]} \varepsilon^2|\psi(x,t)|\right)\cdot \|\mathcal{A}^2 v\|^2_{L^2} \\ &\geq \frac{1}{2}\inf_{x\in S^1} a(x)\cdot\|v\|^2_{\widetilde{H}^2} > 0\end{aligned}$$

if ε_0 is chosen such that

$$2\|\varepsilon_0^2\psi(t,\cdot)\|_{C^0(\mathbb{R})} \leq \frac{\inf_{S^1} a(x)}{2\sup_{S^1} c(x)} \tag{3.35}$$

such that $\|v\|_{\widetilde{H}^2} = \|(\omega^2 + \mathcal{A}^2)^{-1}u\|_{\widetilde{H}^2} \leq (\omega^2 + \alpha^2)^{-1}\|u\|_{\widetilde{H}^2}$ follows. □

The equation for the error is given by

$$\partial_t^2 R = \partial_x(\mathcal{A}^2\partial_x R) + 2\partial_x(c(\partial_x\varepsilon^2\psi)R) + \epsilon^{7/2}\partial_x(c\partial_x(R^2)) + \varepsilon^{-7/2}\mathrm{Res}(\varepsilon^2\psi). \tag{3.36}$$

In order to show the boundedness of solutions to (3.36) with respect to the perturbation parameter ε we use energy estimates. Multiplication of (3.36) by $\partial_t R$ and integration with respect to x gives

$$\begin{aligned}\frac{1}{2}\partial_t\int(\partial_t R)^2 dx = -\int \mathcal{A}\partial_x R\cdot\mathcal{A}\partial_t\partial_x R dx - 2\int c(\partial_x\varepsilon^2\psi)R\cdot\partial_t\partial_x R dx \\ + \varepsilon^{7/2}\int\partial_x(c\partial_x)R^2\cdot\partial_t R dx + \varepsilon^{-7/2}\int Res(\varepsilon^3\psi)\cdot\partial_t R dx.\end{aligned}$$

Here for the first term on the right hand site we have

$$\begin{aligned}\int \mathcal{A}\partial_x R\cdot\mathcal{A}\partial_t\partial_x R dx =& \partial_t\frac{1}{2}\int(\mathcal{A}\partial_x R)^2 dx - \frac{1}{2}\int \partial_t\mathcal{A}\partial_x R\cdot\mathcal{A}\partial_x R + \mathcal{A}\partial_x R\cdot\partial_t\mathcal{A}\partial_x R dx \\ =& \partial_t\frac{1}{2}\int(\mathcal{A}\partial_x R)^2 dx - \frac{1}{2}\int(\partial_t\mathcal{A}\mathcal{A} + \mathcal{A}\partial_t\mathcal{A})\partial_x R\cdot\partial_x R dx\end{aligned}$$

$$=\partial_t \frac{1}{2}\int (\mathcal{A}\partial_x R)^2 dx - \frac{1}{2}\int (2c\partial_t(\varepsilon^2\psi))\partial_x R\cdot \partial_x R dx$$

such that we have

$$\begin{aligned}
\frac{1}{2}\partial_t \int (\partial_t R)^2 + (\mathcal{A}\partial_x R)^2 dx = & \int (2c\partial_t(\varepsilon^2\psi))\partial_x R\cdot\partial_x R dx - 2\int \partial_x(c(\partial_x\varepsilon^2\psi)R)\cdot\partial_t R dx \\
& + \varepsilon^{7/2}\int \partial_x(c\partial_x)R^2\cdot\partial_t R dx + \varepsilon^{-7/2}\int Res(\varepsilon^3\psi)\cdot\partial_t R dx \\
\leq & \underbrace{2\|c\partial_t\varepsilon^2\psi\|_{L^\infty}}_{\leq C\varepsilon^3}\|R\|_{H^2}^2 + 2\underbrace{\|\partial_x(c(\partial_x\varepsilon^2\psi)R)\|_{L^2}}_{\leq C\varepsilon^3\|c\|_{C^1}\|R\|_{H^1}}\|\partial_t R\|_{L^2} \\
& + \varepsilon^{7/2}\|c\|_{C^1}\|R^2\|_{H^2}\|\partial_t R\|_{L^2} + \underbrace{\varepsilon^{-7/2}\|Res(\varepsilon^2\psi)\|_{L^2}}_{C\varepsilon^4}\|\partial_t R\|_{L^2} \\
\leq & C\varepsilon^3(\|R\|_{H^2}^2 + \|R\|_{H^2}\|\partial_t R\|_{L^2} \\
& + \varepsilon^{1/2}\|R\|_{H^2}^2\|\partial_t R\|_{L^2} + \varepsilon\|\partial_t R\|_{L^2}).
\end{aligned}$$

The quadratic form on the left hand side of this inequality can not bound every term on the right hand side, we additionally need control of $\|R\|_{L^2}$ and of $\|\partial_t R\|_{L^2}$. Multiplication of both sides of (3.36) with $-\partial_x^{-1}\mathcal{A}^{-2}\partial_x^{-1}(\partial_t R)$ and integration with respect to x gives

$$\begin{aligned}
\frac{1}{2}\partial_t \int (\mathcal{A}^{-1}\partial_x^{-1}\partial_t R)^2 dx = & \int ((\partial_t\mathcal{A}^{-1})\partial_x^{-1}\partial_t R)\cdot(\mathcal{A}^{-1}\partial_x^{-1}\partial_t R)dx \\
& + \int (\mathcal{A}^{-1}\partial_x^{-1}\partial_t^2 R)\cdot(\mathcal{A}^{-1}\partial_x^{-1}\partial_t R)dx \\
= & \frac{1}{2}\int ((\partial_t\mathcal{A}^{-1})\partial_x^{-1}\partial_t R)\cdot(\mathcal{A}^{-1}\partial_x^{-1}\partial_t R) + (\mathcal{A}^{-1}\partial_x^{-1}\partial_t R)\cdot((\partial_t\mathcal{A}^{-1})\partial_x^{-1}\partial_t R)dx \\
& - \underbrace{\int R\partial_t R dx}_{=\frac{1}{2}\partial_t\|R\|_{L^2}^2} - \underbrace{2\cdot\int (c\partial_x(\varepsilon^2\psi)R)\cdot(\mathcal{A}^{-2}\partial_x^{-1}\partial_t R)dx}_{\leq 2\sup_{t\leq T_0/\varepsilon^3}\|c\partial_x(\varepsilon^2\psi)\|_{L^\infty}\|R\|_{L^2}\|\mathcal{A}^{-2}\partial_x^{-1}\partial_t R\|_{L^2}} \\
& - 2\varepsilon^{7/2}\underbrace{\int cR\partial_x R\cdot(\mathcal{A}^{-2}\partial_x^{-1}\partial_t R)dx}_{\leq\frac{1}{2}\|c\|_{L^\infty}\|\partial_x R\|_{L^2}^2\|\mathcal{A}^{-1}\partial_x^{-1}\partial_t R\|_{L^2}} - \underbrace{\varepsilon^{-7/2}\int (\partial_x^{-1}Res(\varepsilon^2\psi))\cdot(\mathcal{A}^{-2}\partial_x^{-1}\partial_t R)dx.}_{\leq C\varepsilon^3\|\mathcal{A}^{-2}\partial_x^{-1}\partial_t R\|_{L^2}}
\end{aligned}$$

Using the relation $\partial_t(\mathcal{A}^{-1}) = -\mathcal{A}^{-1}(\partial_t\mathcal{A})\mathcal{A}^{-1}$ and the estimate

$$\|\mathcal{A}^{-2}\partial_x^{-1}\partial_t R\|_{H^2} \leq \|\mathcal{A}^{-1}\partial_x^{-1}\partial_t R\|_{H^2}$$

we find

$$\int ((\partial_t \mathcal{A}^{-1})\partial_x^{-1}\partial_t R)(\mathcal{A}^{-1}\partial_x^{-1}\partial_t R)dx = -\frac{1}{2}\int \underbrace{((\partial_t\mathcal{A})\mathcal{A}+\mathcal{A}(\partial_t\mathcal{A}))}_{=2c\partial_t(\varepsilon^2\psi)\cdot}(\mathcal{A}^{-2}\partial_x^{-1}\partial_t R)$$
$$\times(\mathcal{A}^{-2}\partial_x^{-1}\partial_t R)dx$$
$$\leq \sup_{t\leq T_0/\varepsilon^3}\|c\partial_t(\varepsilon^2\psi)\|_{L^\infty}\|\mathcal{A}^{-2}\partial_x^{-1}\partial_t R\|_{L^2}^2$$
$$\leq C\varepsilon^3\|\mathcal{A}^{-1}\partial_x^{-1}\partial_t R\|_{L^2}^2.$$

Finally we find

$$\frac{1}{2}\partial_t\int(\mathcal{A}^{-1}\partial_x^{-1}\partial_t R)^2+R^2\,dx \;\leq\; C\varepsilon^3(1+\|R\|_{L^2}+\|\partial_x R\|_{L^2}^2)\cdot\|\mathcal{A}^{-2}\partial_x^{-1}\partial_t R\|_{L^2}.$$

Again, the quadratic form on the left hand side of this inequality can not bound every term on the right side, we additionally need control of $\|\partial_x R\|_{L^2}$. But, for the sum of both quadratic forms

$$E(t):=\frac{1}{2}\int(\mathcal{A}^{-1}\partial_x^{-1}\partial_t R)^2+R^2+(\partial_t R)^2+(\mathcal{A}\partial_x R)^2dx$$

we find

$$\partial_t E(t)\leq\widetilde{C}\varepsilon^3(E+\varepsilon^{1/2}E^{3/2}+\varepsilon E^{1/2})\leq\frac{1}{2}C\varepsilon^3(2+E+\varepsilon^{1/2}E^{3/2}).$$

Now we can apply Lemma 2.3.1: For a given $T_0>0$ let $M:=e^{CT_0}-1$. Then for all $\varepsilon<\varepsilon_0:=\frac{1}{M}$ while $E<M$ we have $\varepsilon E<1$ such that E satisfies $\partial_t E\leq C\varepsilon^3(1+E)$. Hence we have $E(t)\leq e^{C\varepsilon^3 t}-1\leq M$ and

$$\sup_{t\leq T_0/\varepsilon^3}E(t)\leq e^{CT_0}-1=M.$$

We can extend Theorem 3.4.1:

Lemma 3.4.3 *With the assumptions from Theorem 3.4.1 we have*

$$\sup_{t\leq T_0/\varepsilon^3}\|u(t,\cdot_x)-\varepsilon^2\widetilde{f}(0)A(\varepsilon(\cdot_x+\omega'(0)t),\varepsilon^3 t)$$
$$-\varepsilon^3\widetilde{f}(0)(\widetilde{L}(\partial_x)^{-1}a')\partial_X A(\varepsilon(\cdot_x+\omega'(0)t),\varepsilon^3 t)\|_{C(\mathbb{R})}\leq C\varepsilon^{7/2}.$$

Proof: We have

$$\varepsilon^2\psi(t,x)=\int_{-1/2}^{1/2}e^{i\ell x}\varepsilon\widehat{A}(\ell\varepsilon^{-1},\varepsilon^3 t)e^{i\omega'(0)\ell t}\widetilde{f}(\ell)\chi(\ell)dl+\mathcal{O}(\varepsilon^{7/2})$$

$$
\begin{aligned}
&= \int_{-1/2}^{1/2} e^{i\ell x} \varepsilon \widehat{A}(\ell\varepsilon^{-1}, \varepsilon^3 t) e^{i\omega'(0)\ell t} (\widetilde{f}(0) + \ell \widetilde{f}'(0) + \dots) \chi(\ell) dl + \mathcal{O}(\varepsilon^{7/2}) \\
&= \int_{-1/2}^{1/2} \varepsilon \widehat{A}(\ell\varepsilon^{-1}, \varepsilon^3 t) e^{i\omega'(0)\ell t + i\ell x} \widetilde{f}(0) d\ell \\
&\qquad - i\widetilde{f}'(0)\varepsilon^2 \int_{-1/2}^{1/2} i(\ell\varepsilon^{-1}) \widehat{A}(\ell\varepsilon^{-1}, \varepsilon^3 t) d\ell + \mathcal{O}(\varepsilon^{7/2}) \\
&= \widetilde{f}(0)\varepsilon^2 A(\varepsilon(x + \omega'(0)t), \varepsilon^3 t) \\
&\qquad + \widetilde{f}(0)(\widetilde{L}(\partial_x)^{-1} a')\varepsilon^3 \partial_X A(\varepsilon(x + \omega'(0)t), \varepsilon^3 t) + \mathcal{O}(\varepsilon^{7/2})
\end{aligned}
$$

where we used $\widetilde{f}'(0) = -i\widetilde{f}(0)(\widetilde{L}(\partial_x)^{-1}a')(x)$. □

3.4.3 Justification of the Whitham approximation

In case of Whitham approximation we consider the energy

$$
E(t) := \frac{1}{2}\left(\|\partial_t R\|_{L^2}^2 + \|\widetilde{A}^{-1}\partial_x^{-1}\partial_t R\|_{L^2}^2 + \|R\|_{L^2}^2 + \|\widetilde{A}\partial_x R\|_{L^2}^2 \right) \tag{3.37}
$$

where $\widetilde{A}^2 := a + 2\psi c - \partial_x(b\partial_x)$ for sufficiently small approximation ψ is a positive self-adjoint operator. Note the relationship to the energy used for the KdV case. Using the estimates from the justification of the KdV equation we find

$$
\partial_t E = \varepsilon(E + \varepsilon^{1/2}E^{3/2} + \varepsilon E^{1/2}) \leq \frac{1}{2}C\varepsilon(2 + E + \varepsilon^{1/2}E^{3/2})
$$

such that an application of Lemma 2.4.1 gives the boundedness for E on the corresponding $\mathcal{O}(\epsilon^{-1})-$timescale. Also note, that we only need estimates we already proved in the justification part for the KdV equation.

3.5 Straightforward justification using a two parameter semigroup

Instead of multiplying the error equation by seemingly random terms and integrating with respect to x, the energy from above can be derived from a skew symmetric first order representation in a straightforward way using the variation of constants formula. Consider again equation (3.34) and rewrite this to

$$
\partial_t Z = \Lambda Z + \varepsilon^3 F(t, Z) \tag{3.38}
$$

where $Z = (R, P = \mathcal{A}^{-1}\partial_x^{-1}\partial_t R)$, $\Lambda = \begin{pmatrix} 0 & \partial_x\mathcal{A} \\ \mathcal{A}\partial_x & 0 \end{pmatrix}$ and

$$F(t,Z) = G(t,R,P) = (0, H(t,R,P))$$

where

$$H(t,R,P) = \underbrace{2\mathcal{A}^{-1}(c(\partial_x\varepsilon^{-1}\psi)R)}_{I} - \varepsilon^{-3}\underbrace{\mathcal{A}^{-1}\partial_t\mathcal{A}P}_{II}$$

$$+\underbrace{\varepsilon^{1/2}\mathcal{A}^{-1}(2cR\partial_x R)}_{III} + \underbrace{\varepsilon^{-13/2}\mathcal{A}^{-1}\partial_x^{-1}\mathrm{Res}(\varepsilon^2\psi)}_{IV}$$

and

$$\Lambda = \begin{pmatrix} 0 & \partial_x\mathcal{A} \\ \mathcal{A}\partial_x & 0 \end{pmatrix}.$$

Lemma 3.5.1 *Let $A \in C([0,T_0], H^6(\mathbb{R}))$ satisfy the KdV equation (3.22) and $c \in C^1(S^1)$. Then we have*

$$\|H(t,Z)\|_{H^2} \leq C(\|Z\|_{H^2\times H^2} + \varepsilon^{1/2}\|Z\|^2_{H^2\times H^2} + 1).$$

Proof: In $H^n(\mathbb{R}) \times H^n(\mathbb{R})$ we use the equivalent inner product

$$\langle u, v\rangle_{\widetilde{H}^n} := \langle \mathcal{A}^n u, \mathcal{A}^n v\rangle_{L^2}.$$

The advantage of this inner product, of course, is, that $\mathcal{A}$ is self-adjoint with respect to it. First of all note that we have the equivalence of the norms uniformly in time. In detail fix $T_0 > 0$ and let $A \in C^2([0,T_0], H^6(\mathbb{R}))$ be a solution to the corresponding KdV equation. Next fix $\varepsilon_0 > 0$ given in 3.35. Then we have

$$\forall t \leq T_0/\varepsilon^3: \quad \min\{\frac{1}{2}\inf_{S^1} a, \inf_{S^1} b\}\cdot\|u\|^2_{H^1} \leq \|\mathcal{A}u\|^2_{L^2} \leq \max\{\frac{3}{2}\sup_{S^1} a, \sup_{S^1} b\}\cdot\|u\|^2_{H^1}$$

and the main statement follows by iteration. First note that for (I) we have

$$\begin{aligned}\|\mathcal{A}^{-1}(c\partial_x\varepsilon^{-1}\psi)R\|_{\widetilde{H}^2} &= \|c(\partial_x\varepsilon^{-1}\psi)R\|_{\widetilde{H}^1} \leq C\|c\|_{C^1(S^1)}\|\partial_x\varepsilon^{-1}\psi\|_{C^1(\mathbb{R})}\|R\|_{H^1} \\ &\leq C\|c\|_{C^1(S^1)}\|\partial_x\varepsilon^{-1}\psi\|_{C^1(\mathbb{R})}\|R\|_{H^2} \leq C\|R\|_{H^2} \leq C\|R\|_{\widetilde{H}^2}\end{aligned}$$

since by construction we have $\|\partial_x\varepsilon^{-1}\psi\|_{C^1(\mathbb{R})} \leq C$. For (III) we find

$$\|\varepsilon^{1/2}\mathcal{A}^{-1}(2cR\partial_x R)\|_{\widetilde{H}^2} = \|\varepsilon^{1/2}c\partial_x R^2\|_{\widetilde{H}^1} \leq \varepsilon^{1/2}C\|c\|_{C^1(S^1)}\|R\|^2_{\widetilde{H}^2}.$$

For the last residual term (IV) the required estimates have already been made in the previous section. In order to show the estimate for (II) we recall the formula for fractional powers of closed operators: For a $\gamma \in (0,1)$ the operator $B^{-\gamma}$ is defined by

$$B^{-\gamma}U = \frac{\sin(\gamma\pi)}{\pi}\int_0^\infty \lambda^{-\gamma}(\lambda+B)^{-1}U d\lambda. \tag{3.39}$$

Choosing $B=\mathcal{A}^2$ and $\gamma = 1/2$ we find the formulas

$$\mathcal{A}^{-1}U = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(\lambda+\mathcal{A}^2)^{-1}U d\lambda, \quad \text{and} \quad \mathcal{A}^{-1}U = \frac{1}{\pi}\int_{-\infty}^\infty (\omega^2+\mathcal{A}^2)^{-1}U d\omega,$$

respectively. Hence we can express $\mathcal{A}$ in terms of $\mathcal{A}^2$. This is useful since we have control over the time derivative of $\mathcal{A}^2$ in H^2. In order to estimate (II) we need to deal with the time derivative of $\mathcal{A}^{-1}$. We recall that from $\mathrm{Id} = \mathcal{A}\mathcal{A}^{-1}$ we have the formula $\partial_t\mathcal{A} = -\mathcal{A}\partial_t(\mathcal{A}^{-1})\mathcal{A}$. Applying this principle twice, first to $\mathcal{A}^{-1}$ and then to $(\omega^2+\mathcal{A}^2)^{-1}$, we find

$$\begin{aligned}\|\mathcal{A}^{-1}\partial_t\mathcal{A}P\|_{\widetilde{H}^2} &= \|\partial_t(\mathcal{A}^{-1})\mathcal{A}P\|_{\widetilde{H}^2}\\ &= \frac{1}{\pi}\left\|\int_{-\infty}^\infty (\omega^2+A^2)^{-1}\left(2c\partial_t(\varepsilon^2\psi)(\omega^2+\mathcal{A}^2)^{-1}\mathcal{A}P)\right) d\omega\right\|_{\widetilde{H}^2}\\ &\le \frac{1}{\pi}\int_{-\infty}^\infty \|(\omega^2+A^2)^{-1}\left(2c\partial_t(\varepsilon^2\psi)(\omega^2+\mathcal{A}^2)^{-1}\mathcal{A}P)\right)\|_{\widetilde{H}^2}\, d\omega \end{aligned} \tag{3.40}$$

These terms can be estimated using the following lemmas.

Lemma 3.5.2 *For every positive definite self-adjoint operator X in a Hilbert space H such that*

$$\|(\lambda+X)^{-1}\|_{Lin(H)} \le (\lambda+1)^{-1}.$$

we have $\|X^\mu(\lambda+X)^{-1}\|_{Lin(H)} \le \frac{4}{\pi}(\lambda+1)^{\mu-1}$ for $\mu\in[0,1]$, $\lambda\ge 0$.

Proof: Let $Xu = \lambda v + Xv$. Since X is positive and self-adjoint and due to the assumed estimate, X^{-1} is a continuous and also positive definite and self-adjoint operator in H and possesses a positive definite self-adjoint square root $X^{-1/2}$ which is also a continuous operator in H. Then we have

$$\|v\|_H^2 = \langle u-\lambda X^{-1}v, v\rangle_H = \langle u,v\rangle_H - \lambda\|X^{-1/2}v\|_H^2 \le \|u\|_{L^2}\|v\|_H$$

which proves the case $\mu = 1$. Using formula (3.39) in the case $0 < \mu < 1$ we find

$$\begin{aligned} X^\mu(\lambda+X)^{-1} =& X^{\mu-1}X(\lambda+X)^{-1} && (3.41)\\ =& \frac{\sin(\pi(\mu-1))}{\pi}\int_0^\infty \widetilde{\lambda}^{\mu-1}X(\widetilde{\lambda}+X)^{-1}(\lambda+X)^{-1}d\widetilde{\lambda} \\ =& \frac{\sin(\pi(1-\mu))}{\pi}\int_0^{\lambda+1} \widetilde{\lambda}^{\mu-1}\Big(X(\widetilde{\lambda}+X)^{-1}\Big)(\lambda+X)^{-1}d\widetilde{\lambda} && (3.42)\\ &+\frac{\sin(\pi(1-\mu))}{\pi}\int_{\lambda+1}^\infty \widetilde{\lambda}^{\mu-1}(\widetilde{\lambda}+X)^{-1}\Big(X(\lambda+X)^{-1}\Big)d\widetilde{\lambda} && (3.43) \end{aligned}$$

where we can now use the first result in order to estimate the operator norm of (3.42) by

$$\frac{|\sin(\pi\mu)|}{\pi}\int_0^{\lambda+1}\widetilde{\lambda}^{\mu-1}(\lambda+1)^{-1}d\widetilde{\lambda} = \frac{|\sin(\pi\mu)|}{\pi\mu}\frac{\lambda^\mu}{(1+\lambda)} \le \frac{|\sin(\pi\mu)|}{\pi\mu}(1+\lambda)^{\mu-1}.$$

The operator norm of (3.43) can be estimated by

$$\frac{|\sin(\pi\mu)|}{\pi}\int_{\lambda+1}^\infty \widetilde{\lambda}^{\mu-1}(\widetilde{\lambda}+1)^{-1}d\widetilde{\lambda} \le \frac{\sin(\pi\mu)}{\pi}\int_{\lambda+1}^\infty \widetilde{\lambda}^{\mu-2}d\widetilde{\lambda} \le \frac{|\sin(\pi\mu)|}{\pi(1-\mu)}(1+\lambda)^{\mu-1}.$$

Altogether we have

$$\|X^\mu(\lambda+X)^{-1}\|_{Lin(H)} \le \frac{|\sin(\pi\mu)|}{\pi\mu(1-\mu)}(\lambda+1)^{\mu-1} \le C(\lambda+1)^{\mu-1} \qquad (3.44)$$

where $C = \frac{4}{\pi}$ is an optimal choice. □
Generalization of this lemma by scaling of X gives the following lemma:

Lemma 3.5.3 *For every positive definite self-adjoint operator X in a Hilbert space H there exists an $\alpha > 0$ such that*

$$\|(\lambda+X)^{-1}\|_{Lin(H)} \le (\lambda+\alpha)^{-1}$$

and we have a $C > 0$ such that $\|X^\mu(\lambda+X)^{-1}\|_{Lin(H)} \le \frac{4}{\pi}(\lambda+\alpha)^{\mu-1}$ for $\mu \in [0,1]$, $\lambda \ge 0$.

Proof: Choose

$$\alpha := \inf_{v\neq 0}\frac{\langle Xv, v\rangle_H}{\|v\|_H^2} > 0.$$

The operator $Y := \frac{1}{\alpha}X$ satisfies the estimate

$$\|(\lambda+Y)^{-1}\|_{Lin(H)} \le (\lambda+1)^{-1}$$

such that we can use Lemma (3.5.2) and find

$$\begin{aligned}\|X^\mu(\lambda+X)^{-1}\|_{Lin(H)} &= \alpha^{\mu-1}\|Y^\mu(\lambda\alpha^{-1}+Y)^{-1}\|_{Lin(H)}\\ &\le C\alpha^{\mu-1}(\lambda\alpha^{-1}+1)^{\mu-1} = C(\lambda+\alpha)^{\mu-1}.\end{aligned} \qquad \square$$

Finally, we can estimate the right hand side of (3.40) by

$$\begin{aligned}(3.40) =& \frac{1}{\pi}\int_{-\infty}^{\infty} \|\mathcal{A}^2(\omega^2+\mathcal{A}^2)^{-1}\left(2c\partial_t(\varepsilon^2\psi)(\omega^2+\mathcal{A}^2)^{-1}\mathcal{A}R\right)\|_{L^2}d\omega\\ \le& \frac{1}{\pi}\int_{-\infty}^{\infty} \|\mathcal{A}^2(\omega^2+\mathcal{A}^2)^{-1}\|_{Lin(L^2)}2\|c\partial_t(\varepsilon^2\psi)\|_{L^\infty(\mathbb{R})}\|(\omega^2+\mathcal{A}^2)^{-1}\mathcal{A}R\|_{L^2}d\omega\\ \le& C\varepsilon^3\int_{-\infty}^{\infty}\|(\omega^2+\mathcal{A}^2)^{-1}\mathcal{A}R\|_{L^2}d\omega \;=\; C\varepsilon^3\int_{-\infty}^{\infty}\|(\omega^2+\mathcal{A}^2)^{-1}\mathcal{A}^{-1}R\|_{\widetilde{H}^2}d\omega\\ \le& C\varepsilon^3\int_{-\infty}^{\infty}(\omega^2+\alpha^2)^{-1}d\omega\,\|R\|_{\widetilde{H}^2} \;=\; C\varepsilon^3\|R\|_{\widetilde{H}^2} \;\le\; C\varepsilon^3\|Z\|_{H^2\times H^2}.\end{aligned} \qquad \square$$

In order to apply the variation of constants formula we need an estimate for the evolution system. Consider the linear non-autonomous equation

$$\partial_t Z(t) = \Lambda(t)Z(t) \tag{3.45}$$

where $\lambda(t) = \begin{pmatrix} 0 & \partial_x\mathcal{A}(t)\\ \mathcal{A}(t)\partial_x & 0\end{pmatrix}$. For a fixed t_0 by Stone's theorem $\Lambda(t_0)$ is the generator of a strongly continuous group $T(t,t_0)$ of unitary operators on $H^2(\mathbb{R},\mathbb{C})\times H^2(\mathbb{R},\mathbb{C})$. In the terminology from [P83] (p. 134 and following) we have the stability constants $\omega = 0$ and $M = 1$. Moreover we have $\|Z\|_{H^2(\mathbb{R},\mathbb{C})\times H^2(\mathbb{R},\mathbb{C})} \le \|Z\|_{H^4(\mathbb{R},\mathbb{C})\times H^4(\mathbb{R},\mathbb{C})}$ and $H^4(\mathbb{R},\mathbb{C})$ is $\Lambda(t_0)$-admissible, i.e. $H^4(\mathbb{R},\mathbb{C})$ is an invariant subspace of $T(t,t_0)$ such that the restriction $\widetilde{T}(t,t_0)$ of $T(t,t_0)$ to $H^4(\mathbb{R},\mathbb{C})$ is a strongly continuous group of unitary operators on $H^4(\mathbb{R},\mathbb{C})$.

Lemma 3.5.4 *The solution to (3.45) is given by $Z(t) = T(t,t_0)Z(t_0)$ where the evolution system $T(t,s)$ is a mapping $\mathbb{R}_0^+\times\mathbb{R}_0^+ \to Lin(H^2(\mathbb{R},\mathbb{C})\times H^2(\mathbb{R},\mathbb{C}))$ which satisfies the following conditions:*

i) $\forall Z\in H^2\times H^2:\qquad \lim_{t\to s,+}T(t,s)Z = Z,$

ii) $\forall t_0\le t_1\le t_2:\qquad T(t_2,t_0) = T(t_2,t_1)T(t_1,t_0).$

iii) We have $\partial_t T(t,s) = \Lambda(t)T(t,s)$, $\partial_s T(t,s) = -T(t,s)\Lambda(s)$ *and*

$$\|T(t,s)Z\|_{H^2(\mathbb{R},\mathbb{C})\times H^2(\mathbb{R},\mathbb{C})} \le e^{C\varepsilon^3(t-s)}\|Z\|_{H^2(\mathbb{R},\mathbb{C})\times H^2(\mathbb{R},\mathbb{C})}.$$

Proof: The first two points and the rules for the derivatives follow directly from [P83, Theorem 3.1]. In order to show the boundedness of the evolution system $T(t,s)$ with respect to the $\mathcal{O}(\varepsilon^3)-$time scale, first, we define $X^2 = H^2(\mathbb{R},\mathbb{C}) \times H^2(\mathbb{R},\mathbb{C})$ with the inner product $\langle Z, \widetilde{Z}\rangle_{X^2} = \langle (\mathrm{Id} - \Lambda(t)^2)Z, \widetilde{Z}\rangle_{L^2(\mathbb{R},\mathbb{C})\times L^2(\mathbb{R},\mathbb{C})}$. Again we have equivalence of the norms uniformly in time on the $\mathcal{O}(\varepsilon^3)-$time scale. Using $[\Lambda(t), \mathrm{Id} - \Lambda(t)^2] = 0$ in a straight forward computation we find

$$\begin{aligned}
\partial_t \|Z(t)\|_{X^2}^2 &= \langle \partial_t(\mathrm{Id} - \Lambda(t)^2)Z(t), Z(t)\rangle_{L^2\times L^2} \\
&= \langle \partial_t(\mathrm{Id} - \partial_x\mathcal{A}^2\partial_x)R, R\rangle_{L^2} + \langle \partial_t(\mathrm{Id} - \mathcal{A}\partial_x^2\mathcal{A})\widetilde{P}, \widetilde{P}\rangle_{L^2} \\
&= -2\langle c(\partial_t\varepsilon^2\psi)\partial_x R, \partial_x R\rangle_{L^2} + \langle \partial_x\mathcal{A}P, \partial_x(\partial_t\mathcal{A})P\rangle_{L^2} + \langle \partial_x(\partial_t\mathcal{A})P, \partial_x\mathcal{A}P\rangle_{L^2} \\
&\leq 2\|c\|_{C(S^1)}\|\partial_t\varepsilon^2\psi\|_{C(\mathbb{R})}\|R\|_{H^1}^2 + 2\|\partial_x\mathcal{A}P\|_{L^2}\|\partial_x(\partial_t\mathcal{A})P\|_{L^2} \\
&\leq C\varepsilon^3\|Z(t)\|_{X^2}^2 + C\|Z(t)\|_{X^2}\|\mathcal{A}^{-1}\partial_t\mathcal{A}P\|_{H^2} \quad \leq \quad C\varepsilon^3\|Z(t)\|_{X^2}^2
\end{aligned}$$

where the last estimate has been derived in the proof for the boundedness of the non-linearity. □

Mild solutions of (3.38) with $Z(t_0) = Z_0$ are given by

$$Z(t) = T(t,t_0)Z_0 + \varepsilon^3 \int_{t_0}^t T(t,s)F(Z(s),s)ds.$$

For the rest of the proof, for simplicity reasons, we will assume $t_0 = 0$ and $Z(0) = Z_0 = 0$. Let $z(t) = \|Z(t)\|_{\widetilde{H}^2\times\widetilde{H}^2}$. In order to apply a Gronwall argument we estimate $z(t)$ using the formula from above by

$$\begin{aligned}
z(t) &= \left\| \int_0^t T(t,s)F(Z(s),s)ds \right\|_{H^s\times H^s} \leq \int_0^t e^{C\varepsilon^3(t-s)}\|F(Z(s),s)\|_{H^s\times H^s}ds \\
&\leq \int_0^t \frac{1}{2}e^{C\varepsilon^3(t-s)}\varepsilon^3(2 + z(s) + \varepsilon^{1/2}z^2(s))ds \leq C\varepsilon^3 t + \frac{1}{2}C\varepsilon^3\int_0^t (z(s) + \varepsilon^{1/2}z(s)^2)ds
\end{aligned}$$

where we used $\exp(C\varepsilon^3(t-s)) \leq e^{CT_0} \leq C$ for $t,s \in [0, T_0/\varepsilon^3]$. Choose $M := e^{CT_0} - 1$ and $\varepsilon < \varepsilon_0 := M^{-2}$. Then z satisfies the inequality

$$z(t) \quad \leq \quad C\varepsilon^3 t + C\varepsilon^3 \int_0^t z(s)ds.$$

as long as $z(t) < M$. Solutions to this can be estimated using Gronwall's inequality by

$$\begin{aligned} z(t) \leq\ & C\varepsilon^3 t + C^2\varepsilon^6 \int_0^t s e^{C\varepsilon^3(t-s)} ds \\ &= C\varepsilon^3 t + C^2\varepsilon^6 \left(\frac{-s}{C\varepsilon^3} e^{C\varepsilon^3(t-s)} \bigg|_0^t + \frac{1}{C\varepsilon^3} \int_0^t e^{C\varepsilon^3(t-s)} ds \right) \\ &= e^{C\varepsilon^3 t} - 1 \leq M \end{aligned}$$

for all $[0, T_0/\varepsilon^3]$. Hence we have shown the boundedness with respect to the long time interval $[0, T_0/\varepsilon^3]$.

3.6 Discussion

Differences between the KdV and the Whitham case arise already in the case of constant coefficients, when, in the KdV case, for any $T_0 > 0$ and every solution A to the corresponding KdV-equation the needed conditions for justification can be satisfied by choosing a sufficiently small ε. This is not the case for the Whitham approximation. Here we need to choose T_0 and solutions A such that

$$\sup_{t<T_0} \|A(\cdot_X, T)\|_{C^0(\mathbb{R})} < 1/2.$$

Essential differences between the constant coefficient case and the case of periodic coefficients come from the increased complexity of the underlying analysis induced by the use of the Bloch transform. As a result of this need the equations for the correction term become much more demanding. Moreover, while in the KdV case the equation for the approximation is uncoupled from the equations for the correctors, in case of Whitham's approximation, at first glance, we have nonlinear coupling of these equations. Nevertheless, the equations can be reduced to one autonomous equation for the approximation and a set of autonomous differential-algebraic equations for the correctors.

List of Figures

Bibliography

[BCCLSU08] Carsten Blank, Martina Chirilus Bruckner, Christopher Chong, Vincent Lescarret, Guido Schneider, Hannes Uecker. A remark about the justification of the nonlinear Schrödinger equation in quadratic spatially periodic media. *Z. Angew. Math. Phys.* 59(3):554-557, 2008.

[BCL05] Jerry LLoyd Bona, Thierry Colin, David Lannes. Long wave approximations for water waves. *Arch. Ration. Mech. Anal.* 178(3):373–410, 2005.

[BSTU06] Kurt Busch, Guido Schneider, Lasha Tkeshelashvili, Hannes Uecker. Justification of the Nonlinear Schrödinger equation in spatially periodic media. *ZAMP*, 57:1–35, 2006.

[CCPS12] Martina Chirilus-Bruckner, Christopher Chong, Oskar Prill, Guido Schneider. Rigorous description of macroscopic wave packets in infinite periodic chains of coupled oscillators by modulation equations. *Discrete Contin. Dyn. Syst. Ser.* S 5(5): 879–901, 2012.

[CeSa98] Carlo Cercignani, David Sattinger. Scaling limits and models in physical processes. *DMV Seminar, 28. Birkhäuser Verlag, Basel, 1998.*

[Cha07] Florent Chazel. Influence of bottom topography on long water waves. *M2AN Math. Model. Numer. Anal.*, 41(4):771–799, 2007.

[Cha09] Florent Chazel. On the Korteweg-de Vries approximation for uneven bottoms. *Eur. J. Mech. B Fluids* 28(2): 234–252, 2009.

[Cra85] Walter Craig. An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits. *Comm. Partial Differential Equations*, 10(8):787–1003, 1985.

[CS11] Christopher Chong, Guido Schneider. The validity of the KdV approximation in case of resonances arising from periodic media. *J. Math. Anal. Appl.* 383(2): 330–336, 2011.

[DAS15] Tomas Dohnal, Agnes Lamacz, Ben Schweizer. Dispersive homogenized models and coefficient formulas for waves in general periodic media. *Asymptotic Analysis, Vol. 93, no. 1-2, preprint* 2015.

[DLPSW11] Willy Dörfler, Armin Lechleiter, Michael Plum, Guido Schneider, Christian Wieners. On the spectra of periodic differential operators. *Photonic Chrystals: Methematical Analysis and Numerical Approximation* 64-76, 2011.

[DS09] Wolf-Patrick Düll, Guido Schneider. Validity of Whitham's Equations for the Modulation of Periodic Traveling Waves in the NLS Equation. *J. Nonlinear Science* 19(5): 453-466, 2009.

[Du12] Wolf-Patrick Düll. Validity of the Korteweg-de Vries approximation for the two-dimensional water wave problem in the arc length formulation. *Comm. Pure Appl. Math.* 65(3):381–429, 2012.

[Igu07] Tatsuo Iguchi. A long wave approximation for capillary-gravity waves and an effect of the bottom. *Comm. Partial Differential Equations* 32(1-3):37–85, 2007.

[P83] Ammon Pazy *Semigroups of Linear Operators and Applications to Partial Differential Equations (Applied Mathematical Sciences Volume 44)* 1983.

[Schn95] Guido Schneider. Validity and limitation of the Newell-Whitehead equation. *Math. Nachr.*, 176:249–263, 1995.

[Schn98] Guido Schneider. The long wave limit for a Boussinesq equation. *SIAM J. Appl. Math.* 58(4):1237–1245, 1998.

[SW00] Guido Schneider, Clarence Eugene Wayne. The long-wave limit for the water wave problem. I. The case of zero surface tension. *Comm. Pure Appl. Math.*, 53(12):1475–1535, 2000.

[SW02] Guido Schneider, Clarence Eugene Wayne. The rigorous approximation of long-wavelength capillary-gravity waves. *Arch. Ration. Mech. Anal.*, 162(3):247–285, 2002.

Erklärung über die Eigenständigkeit der Dissertation

Ich versichere, dass ich die vorliegende Arbeit mit dem Titel

„The KdV and Whitham limit for a spatially periodic Boussinesq model“

selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe; aus fremden Quellen entnommene Passagen und Gedanken sind als solche kenntlich gemacht.

Ein Promotionsverfahren wurde zu keinem früheren Zeitpunkt an einer anderen Hochschule oder bei einem anderen Fachbereich beantragt.

Name/Name: ________________________________

Unterschrift/Signed: ________________________________

Datum/Date: ________________________________