Possibilistic Calculus as a Conservative Counterpart to Probabilistic Calculus

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Abstract

In this contribution, we revisit Zadeh's Extension Principle in the context of imprecise probabilities and present two simple modifications to obtain meaningful results when using possibilistic calculus to propagate credal sets of probability distributions through models. It is demonstrated how these results facilitate the possibilistic solution of two benchmark problems in uncertainty quantification.

Keywords: Possibility Theory, Fuzzy Arithmetic, Extension Principle, Uncertainty Propagation, Imprecise Probabilities, Probability-Possibility Consistency, Joint Possibility Distribution Aggregation

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1 Introduction

In the recent past, several authors, e.g. Oberkampf et al. [1] and Petryna and Drieschner [2], have challenged the scientific community to solve a selection of benchmark problems related to the propagation of non-probabilistic uncertainties in the hope of inspiring discourse, exchange and progress in the related fields.

One well-known theory for the treatment of imprecise probabilities [3] is possibility theory [4] based on Zadeh's theory of fuzzy sets [5]. See [6] for a comprehensive review of the use of imprecise probabilities in the engineering sciences. In possibility theory, the extension principle provides the general rule for the propagation of fuzzy variables. In this paper, the authors pick up the argument that this principle in its standard form is deficient for a consistent propagation of imprecise probabilities [7]. What is missing are general instructions on how to perform a consistent uncertainty propagation of marginal possibility distributions within the framework of imprecise probabilities. The intent of this contribution is to provide an intuitive modification of the extension principle in order to fix this issue and lay a basis for such a general theory.

When approaching this problem by looking for analogies between probability theory and possibility theory, it may seem critically important to first discuss the problem of (possibilistic) independence as done by many, e.g. [8, 9, 10, 11]. An exceptional discussion on possibilistic independence and the different concepts thereof is readily available in [12, 13, 14]. Here, it is argued that this concept is not actually necessary. Instead, it suffices to rely on the well-established definition of stochastic (in-)dependence when trying to find consistent aggregation operations in the framework of imprecise probabilities as a necessary prerequisite for their consistent propagation.

Only few scholars have so far contemplated this question. Baudrit et al. investigate a number of methods for the joint propagation of probability and possibility distributions [7], which may be seen as a complementary technique to the one considered here. However, as they investigate a hybrid case as opposed to the purely possibilistic case considered here, the techniques are not so easy to compare. In [15], a special case of the results in this paper is investigated, namely the aggregation of independent triangular fuzzy numbers. The results therein coincide with the more general formulation that is presented in the following. Finally, Jamison et al. [16] demonstrate how to compute with possibility densities derived from cumulative probability distribution functions of independent random variables and arrive at very similar results to the ones presented below. In a sense, the present paper deals with a generalization of their results towards arbitrary possibility distributions.

The remainder of this paper is organized as follows: Section 2 provides a brief overview of the necessary prerequisites to understand the main results in this paper and introduces the notation. The forward and inverse propagation of possibilistic uncertainties is covered in Section 3 in order to enable an understanding of the relevance of this contribution. The issue of consistent possibility propagation is further discussed and the related problems are highlighted in Section 4 where, ultimately, the necessary modifications to the extension principle are suggested and a comprehensive discussion of the implications is provided. In Section 5, the theoretical instructions are illustrated by the exemplary solution of two selected problems from [1] and [2]. Some concluding remarks are given in Section 6. Theorems 24 and 28 form the core of this paper.

2 Probability and Possibility Spaces

Below, $\mathscr{D}$ usually denotes the universe of discourse and $\mathscr{S}$ a σ-field thereon. Hence, they form a measurable space $(\mathscr{D},\mathscr{S})$. A probability and a possibility measure thereon may be defined in a similar manner.

Definition 1 (Probability Measure)**.** A function $\mathrm{P}:\mathscr{S}\to[0,1]$ is a *probability measure* if it possesses the following properties:

a. $\mathrm{P}(\emptyset)=0$,
b. $\mathrm{P}(\mathscr{D})=1$ and
c. $\mathrm{P}(U_1\cup U_2)=\mathrm{P}(U_1)+\mathrm{P}(U_2)$ for disjunct sets $U_1,U_2\in\mathscr{S}$.

Notice that, contrary to a probability measure, a possibility measure does not rely on the introduction of a σ-field. Yet, it is certainly possible to consider only the restriction $\Pi|_{\mathscr{S}}$ for comparison.

Definition 2 (Possibility Measure)**.** A function $\Pi:2^{\mathscr{D}}\to[0,1]$ is a *possibility measure* if it possesses the following properties:

a. $\Pi(\emptyset)=0$,
b. $\Pi(\mathscr{D})=1$ and
c. $\Pi(U_1\cup U_2)=\max(\Pi(U_1),\Pi(U_2))$ for disjunct sets $U_1,U_2\in 2^{\mathscr{D}}$.

As the possibility measure is not self-dual, one can also define the complementary necessity measure.

Definition 3 (Necessity Measure)**.** The *necessity measure* N corresponding to a possibility measure Π is defined through $\mathrm{N}(U)=1-\Pi(\mathscr{D}\setminus U)$ for all $U\in\mathscr{S}$.

The concept of a probability density is well-known, but not of major importance here, contrary to the analogue possibility density.

Definition 4 (Possibility Density)**.** The *possibility density* π of a possibility measure Π is defined by $\pi(\omega)=\Pi(\{\omega\})$ for all $\omega\in\mathscr{D}$ and, therefore, satisfies $\Pi(U)=\sup_{\omega\in U}\pi(\omega)$ for all $U\in\mathscr{S}$.

Definition 5 (Level Sets)**.** The *sublevel set* of a possibility measure Π for a given level $\alpha\in[0,1]$ is the set of all elements with a possibility density below this level, i.e. $S_\Pi^\alpha=\{\omega\in\mathscr{D}:\pi(\omega)\le\alpha\}$. Its *superlevel set* or *α-cut* is the set of all elements with possibility density equal to or higher than this level, i.e. $C_\Pi^\alpha=\{\omega\in\mathscr{D}:\pi(\omega)\ge\alpha\}$.

Possibility theory [4] can serve as a general framework for descriptions of uncertainty by imprecise probabilities [17]. Dubois and Prade [18] first pointed out that a possibility measure may be viewed as an upper probability measure. It is a special case of a 2-monotone chain measure [19]. The crucial property of consistency then links probability and possibility measures.

Definition 6 (Consistency)**.** A probability measure P and a possibility measure Π are said to be *consistent* if the probabilities of all events $U\in\mathscr{S}$ are bounded from above by their possibility $\mathrm{P}(U)\le\Pi(U)$.

Consistency may be verified in several ways.

Proposition 7. *Let* P *be a probability measure and* Π *a possibility measure, then the following statements are equivalent:*

a. P *and* Π *are consistent.*
b. $\mathrm{P}(U)\le\Pi(U)$ *for all* $U\in\mathscr{S}$.
c. $\mathrm{N}(U)\le\mathrm{P}(U)$ *for all* $U\in\mathscr{S}$.
d. $\mathrm{P}(S_\Pi^\alpha)\le\alpha$ *for all* $\alpha\in[0,1]$.
e. $\mathrm{P}(C_\Pi^\alpha)\ge 1-\alpha$ *for all* $\alpha\in[0,1]$.

Proof. See [20]. □

As a consequence of the second and third statement, necessity and possibility measures may be viewed as upper and lower probabilities [4]. The two latter statements, of course, require π to be $\mathscr{S}$-measurable, which is usually the case in engineering applications.

In conclusion, a possibility measure is just an equivalent representation of its credal set of consistent probability measures [19].

Definition 8 (Credal Set)**.** The *credal set* $\mathscr{C}$ induced by a possibility measure Π is the set of all probability measures consistent with Π, i.e. $\mathscr{C}(\Pi) = \{\mathrm{P} : \mathrm{P} \text{ and } \Pi \text{ are consistent}\}$.

Fuzzy set inclusion is replaced by the concept of specificity in possibility theory.

Definition 9 (Specificity)**.** Given two possibility measures Π_1 and Π_2, if $\Pi_1(U) \leq \Pi_2(U)$ for all $U \in \mathscr{S}$, then Π_1 is called *more specific* than Π_2, denoted by $\Pi_1 \preceq \Pi_2$.

It allows to compare the credal sets of two possibility measures in the following way.

Proposition 10. *Given two possibility measures Π_1 and Π_2 with $\Pi_1 \preceq \Pi_2$. Then it follows that their induced credal sets satisfy $\mathscr{C}(\Pi_1) \subseteq \mathscr{C}(\Pi_2)$.*

Proof. See [4]. □

Corollary 11. *Given two possibility measures Π_1 and Π_2, then $\Pi_1 \preceq \Pi_2$ if and only if their possibility densities satisfy $\pi_1(x) \leq \pi_2(x)$ for all $x \in \mathscr{D}$.*

According to Dubois and Prade, the *Principle of Minimum Specificity* is the guiding principle in possibility theory. Viewing possibility measures as a mere representation of the corresponding credal set, and being confronted with a set of alternative possibility measures, it "states that any hypothesis not known to be impossible cannot be ruled out" [21]. It may be seen as the analogon to the Principle of Maximum Entropy [22] in statistics and has e.g. been applied to measurement theory [23], evidential reasoning [24], or inverse problems [25].

It may be shown that e.g. triangular fuzzy numbers are possibility measures whose credal sets contain a plethora of probability measures. For instance, every unimodal probability measure sharing the same support and whose mode coincides with the nominal value of the triangular fuzzy number [26] is a consistent member. Hence, it would be desirable to be able to consistently propagate combinations of triangular fuzzy numbers through a model and be able to preserve these powerful properties.

3 Uncertainty Propagation

Uncertainty propagation investigates the relation of distributions of uncertain variables which are connected through some form of mapping.

Definition 12 (Uncertain Variable)**.** A $(\mathscr{D}, \mathscr{D}_X)$-measurable function $X : \mathscr{D} \to \mathscr{D}_X$ is called a $\mathscr{D}_X$-valued *uncertain variable*.

Below, a measure/distribution which may be either a possibility or a probability measure/distribution is denoted by μ.

Definition 13 (Pushforward Distribution)**.** Let μ be a possibility (probability) distribution on $(\mathscr{D}, \mathscr{S})$. The *pushforward possibility (probability) distribution* μ_X on $(\mathscr{D}_X, \mathscr{S}_X)$ of the uncertain variable X is defined by $\mu_X(U_X) = \mu\left(X^{-1}(U_X)\right)$ for all $U_X \in \mathscr{S}_X$.

The possibility density π_X of an uncertain fuzzy variable is known as *fuzzy membership function* in classical fuzzy set theory.

Below, the connection of the $\mathscr{D}_X$-valued uncertain variable X and the $\mathscr{D}_Y$-valued uncertain variable $Y = \phi(X)$ through a surjective and $(\mathscr{S}_X, \mathscr{S}_Y)$-measurable function $\phi : \mathscr{D}_X \to \mathscr{D}_Y$ is investigated.

Corollary 14. *Given an $\mathscr{D}_X$-valued uncertain variable X with possibility (probability) distribution μ_X, then the pushforward possibility (probability) distribution of Y is given through $\mu_Y(U_Y) = \mu_X\left(\phi^{-1}(U_Y)\right)$ for all $U_Y \in \mathscr{S}_Y$.*

As is well-known, the forward propagation in the respective frameworks preserves consistency.

Proposition 15 (Forward Uncertainty Propagation Consistency)**.** *Given an $\mathscr{D}_X$-valued uncertain variable X with possibility distribution Π_X and probability distribution $\mathrm{P}_X \in \mathscr{C}(\Pi_X)$ and the uncertain variable $Y = \phi(X)$, then $\mathrm{P}_Y \in \mathscr{C}(\Pi_Y)$.*

Proof. See [19] or [27]. □

The inverse direction is investigated in [25], for which the authors manage to show similar, yet slightly more involved, results.

Definition 16 (Set of Inverse Distributions)**.** Suppose Y is an $\mathscr{D}_Y$-valued uncertain variable with known possibility (probability) distribution μ_Y. In general there exists an infinite number of possibility (probability) distributions μ_X yielding this pushforward distribution under ϕ. These extensions may be gathered in the *set of inverse possibility (probability) distributions* $\mathscr{I}^{\phi}_{\mu_Y} = \left\{\mu_X : \mu_Y(U_Y) = \mu_X\left(\phi^{-1}(U_Y)\right) \; \forall U_Y \in \mathscr{S}_Y\right\}$.

Within this set, the minimum specific inverse possibility distribution plays a special role.

Definition 17 (Minimum Specific Inverse Possibility Distribution)**.** Suppose $Y = \phi(X)$ is an $\mathscr{D}_Y$-valued uncertain variable with known possibility distribution Π_Y. The *minimum specific inverse possibility distribution* of X is given by $\Pi_X^{\text{inv.}}(U_X) = \Pi_Y(\phi(U_X))$ for all $U_X \in \mathscr{S}_X$.

In particular, it is the best solution for the corresponding inverse problem as the following proposition suggests.

Proposition 18. *The minimum specific inverse possibility distribution*

a. is contained in the set of inverse possibility distributions $\mathscr{I}^{\phi}_{\Pi_Y}$*,*

b. is less specific than all $\Pi'_X \in \mathscr{I}^{\phi}_{\Pi_Y}$ *and*

c. if $\mathrm{P}_Y \in \mathscr{C}(\Pi_Y)$*, it follows that* $\mathscr{I}^{\phi}_{\mathrm{P}_Y} \subseteq \mathscr{C}\left(\Pi_X^{\text{inv.}}\right)$*.*

Proof. See [25]. □

4 Aggregation of Joint Possibility Distributions

The above results suggest that possibility theory is, in fact, a suitable conservative alternative for a stochastic propagation of uncertainties. However, one has take into account that the propositions recapitulated above hold for joint distributions only, if the multivariate case is to be considered. That is, if one wishes to consider multiple uncertain input variables X_k defined on $\mathscr{D}_{X_k}$, respectively for $k = 1, \ldots, N$, knowledge about their joint distribution $\Pi_{X_1,\ldots,X_N}$ on $\mathscr{D}_{X_1,\ldots,X_N} = \mathscr{D}_{X_1} \times \ldots \times \mathscr{D}_{X_N}$ is required. Yet, in most cases, only the marginal possibility (probability) distributions Π_{X_k} are known.

Definition 19 (Marginal Distribution)**.** The N *marginal possibility (probability) distributions* μ_{X_k} of the multivariate joint distribution $\mu_{X_1,\ldots,X_N}$ are given by $\mu_{X_k}(U_k) = \mu_{X_1,\ldots,X_N}\left(\mathscr{D}_{X_1} \times \ldots \times \mathscr{D}_{X_{k-1}} \times U_k \times \mathscr{D}_{X_{k+1}} \times \ldots \times \mathscr{D}_{X_N}\right)$ for all $U_k \in \mathscr{S}_{X_k}$ where $k = 1, \ldots, N$.

In probability theory, random variables are either independent or they are not. This is typically expressed by means of the covariance matrix which is diagonal in the case of independence. In that case, the joint probability distribution $\mathrm{P}_{X_1,\ldots,X_N}$ of the N independent random variables with marginal probability distributions P_{X_k} is given by the product $\mathrm{P}_{X_1,\ldots,X_N}(U_1 \times \ldots \times U_N) = \mathrm{P}_{X_1}(U_1) \cdot \ldots \cdot \mathrm{P}_{X_N}(U_N)$ for all $U_k \in \mathscr{S}_{X_k}$ where $k = 1, \ldots, N$.

However, the authors are not aware of any publications containing practical instructions on how to construct joint possibility distributions from marginal ones in order to facilitate their consistent propagation within the framework of imprecise probabilities.

Below, three possible aggregation operations are discussed. The first is the commonly employed non-interactive aggregation from fuzzy set theory. Its deficiencies are pointed out and two alternatives covering both the general aggregation and an aggregation under the assumption of stochastic independence are presented.

4.1 Zadeh's Aggregation

In the fuzzy community, Zadeh's Extension Principle [28] is the standard formulation for propagating fuzzy sets through a model, providing the basis for fuzzy arithmetic [29]. Comparing the Extension Principle to Proposition 15 yields the aggregation operator which Zadeh implicitly assumes.

Definition 20 (Zadeh's Aggregation)**.** Given N marginal possibility distributions Π_{X_k} for $k = 1, \ldots, N$, the joint possibility distribution $\Pi^{\text{Zadeh}}_{X_1,\ldots,X_N} = \mathscr{J}^{\text{Zadeh}}(\Pi_{X_1}, \ldots, \Pi_{X_N})$ produced by *Zadeh's aggregation* operator $\mathscr{J}^{\text{Zadeh}}$ is defined by $\Pi^{\text{Zadeh}}_{X_1,\ldots,X_N}(U_1 \times \ldots \times U_N) = \min_{k=1,\ldots,N} \Pi_{X_k}(U_k)$ for all $U_k \in \mathscr{S}_{X_k}$ where $k = 1, \ldots, N$.

Indeed, in [7] it is argued that Zadeh's aggregation yields the minimum specific joint possibility distribution whose marginals coincide with the original ones. This is sometimes also called *non-interactivity*. The example below, which is adapted from Baudrit et al., is intended to show that this aggregation operator is unsuitable within the framework of imprecise probabilities.

Example 21. Let $\mathscr{D}_{X_1} = \mathscr{D}_{X_2} = [0,1]$ and let X_1 and X_2 be two independent $[0,1]$-valued uncertain variables with the corresponding Borel σ-fields $\mathscr{S}_{X_1} = \mathscr{S}_{X_2} = \mathscr{B}([0,1])$, respectively. If $\mathrm{P}_{X_1} = \mathrm{P}_{X_2}$ are uniform probability distributions on $[0,1]$, it is easy to see that they belong to the credal sets of the possibility distributions $\Pi_{X_1} = \Pi_{X_2}$ with the one-sided triangular possibility densities $\pi_{X_1}(x) = \pi_{X_2}(x) = 1 - x$ for $x \in [0,1]$. Consequently, the cumulative probability distribution function of the uncertain variable $Y = X_1 + X_2$ is given by $\mathrm{P}_Y(Y \leq y) = \frac{1}{2}y^2$ for $y \in [0,1]$ and the cumulative possibility distribution function employing Zadeh's aggregation operator $\Pi_Y(Y \leq y) = 1$ is always greater than or equal to the former. However, the necessity $\mathrm{N}_Y((-\infty, y]) = \frac{1}{2}y$ for $y \in [0,1]$ fails to bound the probability from below. Refer to Figure 1. Baudrit et al. conclude that

> "[as] a consequence of the dependence between the choice of confidence levels, one cannot interpret the calculus of possibilistic variables as a conservative counterpart to the calculus of probabilistic variables under stochastic independence." – [7]

Below, two alternative aggregation operators are presented that aim at constructing a joint possibility distribution $\Pi_{X_1,\ldots,X_N}$ that is consistent with all probability $\mathrm{P}_{X_1,\ldots,X_N}$ distributions obtained from consistent marginal distributions.

4.2 General Aggregation

In the general case, the only requirement for the joint possibility distribution $\Pi_{X_1,\ldots,X_N}$ is that all joint probability distributions $\mathrm{P}_{X_1,\ldots,X_N}$ with marginal probabilities $\mathrm{P}_{X_k} \in \mathscr{C}\left(\Pi_{X_k}\right)$ for $k = 1,\ldots,N$ ought to be contained in $\mathscr{C}\left(\Pi_{X_1,\ldots,X_N}\right)$.

Definition 22 (General Aggregation)**.** Given N marginal possibility distributions Π_{X_k} for $k = 1,\ldots,N$, the joint possibility distribution $\Pi^{\text{gen.}}_{X_1,\ldots,X_N} = \mathscr{J}^{\text{gen.}}(\Pi_{X_1},\ldots,\Pi_{X_N})$ produced by the *general aggregation* operator $\mathscr{J}^{\text{gen.}}$ is defined by $\Pi^{\text{gen.}}_{X_1,\ldots,X_N}(U_1 \times \ldots \times U_N) = \min\limits_{k=1,\ldots,N} \min\left(1,\, N \cdot \Pi_{X_k}(U_k)\right)$ for all $U_k \in \mathscr{S}_{X_k}$ where $k = 1,\ldots,N$.

Corollary 23. *The possibility density $\pi^{\text{gen.}}_{X_1,\ldots,X_N}$ of $\Pi^{\text{gen.}}_{X_1,\ldots,X_N}$ obtained from the independent aggregation operator is given by $\pi^{\text{gen.}}_{X_1,\ldots,X_N}(x_1,\ldots,x_N) = \min\limits_{k=1,\ldots,N} \min\left(1, N \cdot \pi_{X_k}(x_k)\right)$ for all $x_k \in \mathscr{D}_{X_k}$ where $k = 1,\ldots,N$.*

Theorem 24. *The possibility distribution $\Pi^{\text{gen.}}_{X_1,\ldots,X_N}$ obtained from the general aggregation operator is consistent with all multivariate probability distributions $\mathrm{P}_{X_1,\ldots,X_N}$ whose marginal probability distributions P_{X_k} are consistent with Π_{X_k} for all $k = 1,\ldots,N$.*

Proof. Let $\Pi_{X_1},\ldots,\Pi_{X_N}$ be marginal possibility distributions, let $\Pi^{\text{gen.}}_{X_1,\ldots,X_N} = \mathscr{J}^{\text{gen.}}(\Pi_{X_1},\ldots,\Pi_{X_N})$ and let $\mathrm{P}_{X_1,\ldots,X_N}$ be a probability distribution with marginal probability distributions $\mathrm{P}_{X_k} \in \mathscr{C}\left(\Pi_{X_k}\right)$ for all $k = 1,\ldots,N$. The sublevel set for $\alpha \in [0,1]$ of $\Pi^{\text{gen.}}_{X_1,\ldots,X_N}$ consists of all $(x_1,\ldots,x_N)$ where there exists at least one $k = 1,\ldots,N$ with $N \cdot \pi_{X_k}(x_k) \leq \alpha$, i.e. x_k has a possibility density below $\frac{\alpha}{N}$. Hence,

$$S^{\alpha}_{\Pi^{\text{gen.}}_{X_1,\ldots,X_N}} = \bigcup_{k=1}^{N} \mathscr{D}_{X_1} \times \ldots \times \mathscr{D}_{X_{k-1}} \times S^{\frac{\alpha}{N}}_{\Pi_{X_k}} \times \mathscr{D}_{X_{k+1}} \times \ldots \times \mathscr{D}_{X_N} \tag{1}$$

and the sublevel set's probability may be bounded by

$$\mathrm{P}_{X_1,\ldots,X_N}\left(S^{\alpha}_{\Pi^{\text{gen.}}_{X_1,\ldots,X_N}}\right) \leq \sum_{k=1}^{N} \underbrace{\mathrm{P}_{X_1,\ldots,X_N}\left(\mathscr{D}_{X_1} \times \ldots \times \mathscr{D}_{X_{k-1}} \times S^{\frac{\alpha}{N}}_{\Pi_{X_k}} \times \mathscr{D}_{X_{k+1}} \times \ldots \times \mathscr{D}_{X_N}\right)}_{\overset{\text{Def. 19}}{=} \mathrm{P}_{X_k}\left(S^{\frac{\alpha}{N}}_{\Pi_{X_k}}\right) \overset{\text{Prop. 7}}{\leq} \frac{\alpha}{N}} \leq N \cdot \frac{\alpha}{N} = \alpha. \tag{2}$$

Therefore, Proposition 7 is fulfilled and $\mathrm{P}_{X_1,\ldots,X_N} \in \mathscr{C}\left(\Pi^{\text{gen.}}_{X_1,\ldots,X_N}\right)$. □

Example 25. Reconsidering Example 21, the joint possibility distribution $\Pi^{\text{gen.}}_{X_1,X_2} = \mathscr{J}^{\text{gen.}}(\Pi_{X_1},\Pi_{X_2})$ obtained from the general aggregation operator yields $\mathrm{N}^{\text{gen.}}_Y(Y \leq y) = 0 \leq \mathrm{P}_Y(Y \leq y) = \frac{1}{2}y^2 \leq \Pi^{\text{gen.}}_Y(Y \leq y) = 1$ for $y \in [0,1]$. Refer to Figure 1.

4.3 Independent Aggregation

Additional information about stochastic independence can greatly influence the quality of the results by making the joint possibility distribution more specific and, hence, the resulting credal set smaller.

Definition 26 (Independent Aggregation)**.** Given N marginal possibility distributions Π_{X_k} for $k = 1, \ldots, N$, the joint possibility distribution $\Pi^{\text{ind.}}_{X_1,\ldots,X_N} = \mathscr{J}^{\text{ind.}}(\Pi_{X_1}, \ldots, \Pi_{X_N})$ produced by the *independent aggregation* operator $\mathscr{J}^{\text{ind.}}$ is defined by $\Pi^{\text{ind.}}_{X_1,\ldots,X_N}(U_1 \times \ldots \times U_N) = \min_{k=1,\ldots,N} 1 - \left(1 - \Pi_{X_k}(U_k)\right)^N$ for all $U_k \in \mathscr{S}_{X_k}$ where $k = 1, \ldots, N$.

Corollary 27. *The possibility density $\pi^{\text{ind.}}_{X_1,\ldots,X_N}$ of $\Pi^{\text{ind.}}_{X_1,\ldots,X_N}$ obtained from the independent aggregation operator is given by $\pi^{\text{ind.}}_{X_1,\ldots,X_N}(x_1, \ldots, x_N) = \min_{k=1,\ldots,N} 1 - \left(1 - \pi_{X_k}(x_k)\right)^N$ for all $x_k \in \mathscr{D}_{X_k}$ where $k = 1, \ldots, N$.*

Theorem 28. *The possibility distribution $\Pi^{\text{ind.}}_{X_1,\ldots,X_N}$ obtained from the independent aggregation operator is consistent with all multivariate probability distributions $\mathrm{P}_{X_1,\ldots,X_N}$ that are constructed of the independent marginal probability distributions P_{X_k} which are consistent with Π_{X_k} for all $k = 1, \ldots, N$.*

Proof. Let $\Pi_{X_1}, \ldots, \Pi_{X_N}$ be marginal possibility distributions, let $\Pi^{\text{ind.}}_{X_1,\ldots,X_N} = \mathscr{J}^{\text{ind.}}(\Pi_{X_1}, \ldots, \Pi_{X_N})$ and let $\mathrm{P}_{X_1,\ldots,X_N}$ be a probability distribution with independent marginal probability distributions $\mathrm{P}_{X_k} \in \mathscr{C}\left(\Pi_{X_k}\right)$ for all $k = 1, \ldots, N$. The superlevel set for $\alpha \in [0, 1]$ of $\Pi^{\text{ind.}}_{X_1,\ldots,X_N}$ consists of all $(x_1, \ldots, x_N)$ where $1 - \left(1 - \pi_{X_k}(x_k)\right)^N \leq \alpha$, i.e. x_k has at least possibility density $1 - \sqrt[N]{1-\alpha}$, for all $k = 1, \ldots, N$. Hence,

$$C^{\alpha}_{\Pi^{\text{ind.}}_{X_1,\ldots,X_N}} = \bigcap_{k=1}^{N} \mathscr{D}_{X_1} \times \ldots \times \mathscr{D}_{X_{k-1}} \times C^{1-\sqrt[N]{1-\alpha}}_{\Pi_{X_k}} \times \mathscr{D}_{X_{k+1}} \times \ldots \times \mathscr{D}_{X_N} = C^{1-\sqrt[N]{1-\alpha}}_{\Pi_{X_1}} \times \ldots \times C^{1-\sqrt[N]{1-\alpha}}_{\Pi_{X_N}} \tag{3}$$

and using the marginal distributions' independence the superlevel set's probability may be bounded by

$$\begin{aligned}\mathrm{P}_{X_1,\ldots,X_N}\left(C^{\alpha}_{\Pi^{\text{ind.}}_{X_1,\ldots,X_N}}\right) &= \mathrm{P}_{X_1,\ldots,X_N}\left(C^{1-\sqrt[N]{1-\alpha}}_{\Pi_{X_1}} \times \ldots \times C^{1-\sqrt[N]{1-\alpha}}_{\Pi_{X_N}}\right) \\ &\overset{\text{ind.}}{=} \underbrace{\mathrm{P}_{X_1}\left(C^{1-\sqrt[N]{1-\alpha}}_{\Pi_{X_1}}\right)}_{\overset{\text{Prop. 7}}{\geq} \sqrt[N]{1-\alpha}} \cdot \ldots \cdot \underbrace{\mathrm{P}_{X_N}\left(C^{1-\sqrt[N]{1-\alpha}}_{\Pi_{X_N}}\right)}_{\overset{\text{Prop. 7}}{\geq} \sqrt[N]{1-\alpha}} \geq \left(\sqrt[N]{1-\alpha}\right)^N = 1 - \alpha.\end{aligned} \tag{4}$$

Therefore, Proposition 7 is fulfilled and $\mathrm{P}_{X_1,\ldots,X_N} \in \mathscr{C}\left(\Pi^{\text{ind.}}_{X_1,\ldots,X_N}\right)$. $\square$

Example 29. Reconsidering Example 21, the joint possibility distribution $\Pi^{\text{ind.}}_{X_1,X_2} = \mathscr{J}^{\text{ind.}}(\Pi_{X_1}, \Pi_{X_2})$ obtained from the independent aggregation operator yields $\mathrm{N}^{\text{ind.}}_Y(Y \leq y) = \frac{1}{4}y^2 \leq \mathrm{P}_Y(Y \leq y) = \frac{1}{2}y^2 \leq \Pi^{\text{ind.}}_Y(Y \leq y) = 1$ for $y \in [0, 1]$. Refer to Figure 1.

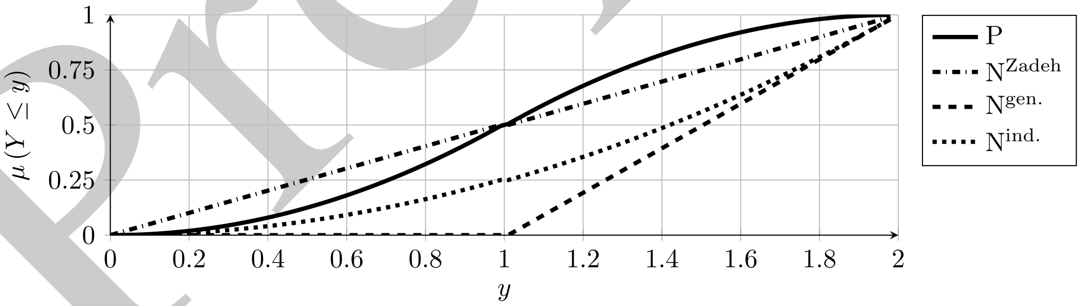


Figure 1: Cumulative probability and necessity distributions in Examples 21, 25 and 29.

4.4 Discussion

Some remarks concerning the results presented above may be given.

4.4.1 Minimal Specificity of Aggregated Distributions

The aggregation operators $\mathscr{J}^{\text{gen.}}$ and $\mathscr{J}^{\text{ind.}}$ produce minimal specific joint possibility distributions in the sense that all probability distributions, which cannot be ruled out, are included in their credal sets. After all, this is the idea behind their construction. Furthermore, they provide tight approximations of this credal set of joint probability distributions and are, therefore, maximally specific in the sense of not including more than what is needed.

4.4.2 Interpretation

The two aggregation operators may be interpreted as a nonlinear monotone rescaling operation of a possibility distribution by the scaling functions

$$\gamma_N^{\text{gen.}}(\pi) = \min(1, N \cdot \pi) \qquad \text{or} \qquad \gamma_N^{\text{ind.}}(\pi) = 1 - (1-\pi)^N \tag{5}$$

in three distinct ways:

a. The rescaling of Zadeh's joint possibility distribution, i.e. $\Pi_{X_1,\ldots,X_N}^{\text{gen./ind.}} = \gamma_N^{\text{gen./ind.}} \circ \Pi_{X_1,\ldots,X_N}^{\text{Zadeh}}$,

b. the rescaling of the original marginal possibility distributions, i.e. $\Pi_{X_k}^{\text{gen./ind.}} = \gamma_N^{\text{gen./ind.}} \circ \Pi_{X_k}$ for $k = 1, \ldots, N$ and subsequent application of the standard extension principle, the computation of the minimum specific inverse solution, etc. or

c. the rescaling of the output possibility distribution, i.e. $\Pi_Y^{\text{gen./ind.}} = \gamma_N^{\text{gen./ind.}} \circ \Pi_Y^{\text{Zadeh}}$, after the application of the standard extension principle, the computation of the minimum specific inverse solution, etc.

The advantage of the proposed strategy of rescaling is that it enables a simple modification of existing software for the automated evaluation of the extension principle, such as FAMOUS [30], to incorporate the proposed changes. Adding the appropriate rescaling operation in one of the three places suggested above, should require only minor effort.

4.4.3 Ordering of Rescaling Functions

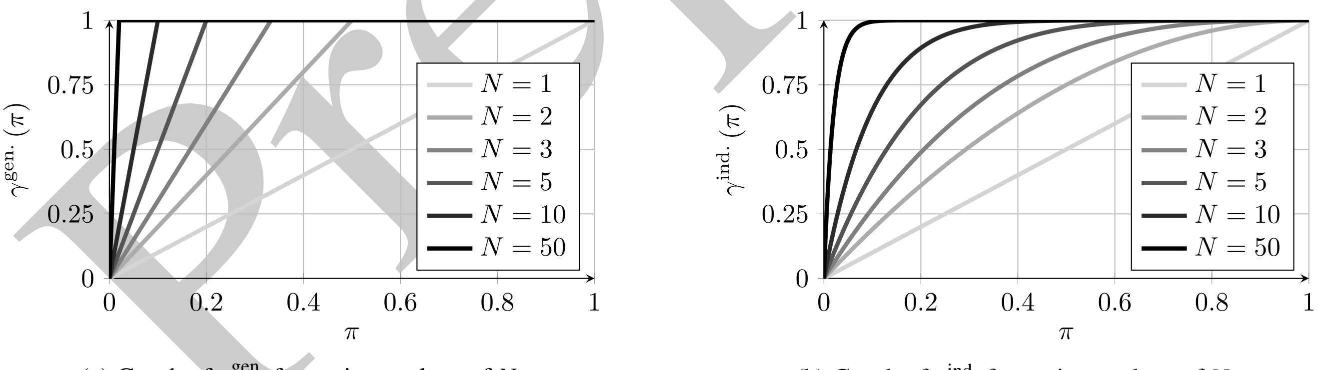
From the fact that $\pi \leq \gamma_N^{\text{ind.}}(\pi) \leq \gamma_N^{\text{gen.}}(\pi)$ for all $N \geq 1$ and $\pi \in [0,1]$, it follows immediately that

$$\Pi_{X_1,\ldots,X_N}^{\text{Zadeh}} \preceq \Pi_{X_1,\ldots,X_N}^{\text{ind.}} \preceq \Pi_{X_1,\ldots,X_N}^{\text{gen.}}. \tag{6}$$

This shows that the amount of information required by the respective aggregation operators is decreasing from left to right and is a further indicator that assumptions in the modeling process that cannot be justified will quickly yield improper results by arbitrarily excluding distributions which should not be neglected.

4.4.4 Interval Analysis as the Limit Case

For an increasing dimension N, the rescaling functions $\gamma_N^{\text{gen.}}$ and $\gamma_N^{\text{ind.}}$ converge to the unit function over $(0,1]$ as depicted in Figs. 2a and 2b. Thus, for many uncertain parameters, this becomes an argument for switching from a possibilistic uncertainty assessment to interval calculus.



(a) Graph of $\gamma_N^{\text{gen.}}$ for various values of N.

(b) Graph of $\gamma_N^{\text{ind.}}$ for various values of N.

Figure 2: Visualization of the limit behavior of the rescaling functions $\gamma_N^{\text{gen.}}$ and $\gamma_N^{\text{ind.}}$.

4.4.5 Non-Uniqueness of Consistent Aggregation Operators

The choice of the aggregation operators is to some extent arbitrary. They have been chosen such that the order produced by Zadeh's aggregation is preserved with the obvious advantage of an easy implementation. However, these are by no means the only possible choices, and depending on the application, other aggregation operations are imaginable. Since the aggregation operators $\mathscr{J} \in \{\mathscr{J}^{\text{gen.}}, \mathscr{J}^{\text{ind.}}\}$ are not associative, one could e.g. also arrive at different, yet equally valid, minimally specific and consistent, joint possibility distributions

$$\Pi_{X_1,\ldots,X_N} = \mathscr{J}\left(\Pi_{X_1}, \mathscr{J}\left(\Pi_{X_2}, \mathscr{J}\left(\Pi_{X_3}, \ldots\right)\right)\right). \tag{7}$$

4.4.6 Relation to Other Work

Similar results have been presented for the special case of triangular fuzzy numbers in [15]. Applying the independent aggregation operator to them yields exactly the same results. Considering that any (complementary) cumulative probability distribution function is a valid density of a consistent possibility distribution, the obtained results also coincide with [16].

4.4.7 Consequences for the Extension Principle

As pointed out above, the extension principle in its standard form should be applied cautiously. When using possibility theory for a qualitative description of uncertainty, e.g. for the imprecision in human speech, its use is well-justified, just as for the computation with intervals with fuzzy boundaries. However, when using possibility theory as a description of imprecise probabilities, e.g. when the uncertainties are of ill-known aleatory nature, the presented modifications need to be incorporated.

In the general case, the modified extension principle expressed in its usual form with possibility densities then reads

$$\pi_Y^{\text{gen.}}(y) = \max_{(x_1,\dots,x_N)\in\mathscr{D}_{X_1}\times\dots\times\mathscr{D}_{X_N}:y=\phi(x_1,\dots,x_N)} \min_{k=1,\dots,N} \min\left(1, N\cdot\pi_{X_k}(x_k)\right) \qquad \forall\, y\in\mathscr{D}_Y \tag{8}$$

and in the case of stochastic independence, one obtains

$$\pi_Y^{\text{ind.}}(y) = \max_{(x_1,\dots,x_N)\in\mathscr{D}_{X_1}\times\dots\times\mathscr{D}_{X_N}:y=\phi(x_1,\dots,x_N)} \min_{k=1,\dots,N} 1-\left(1-\pi_{X_k}(x_k)\right)^N \qquad \forall\, y\in\mathscr{D}_Y. \tag{9}$$

From the authors experience, the independent aggregation operation should prove most useful as independence is often a valid assumption and the general aggregation operator quickly leads to very conservative results even for relatively small values of N, cf. Figure 2a.

5 Benchmarks

This section will present solutions to a simple as well as a more involved benchmark problem in order to demonstrate the general applicability and usefulness of the theoretical results derived above. Bearing in mind that possibilistic uncertainty descriptions are typically rather coarse, it is not to be expected to arrive at tight bounds for the upper and lower probabilities. Instead, conservative bounds can be computed very efficiently.

5.1 Simple Problem

In [1], several challenge problems are presented in order to foster scientific exchange about how to solve them. Many solution approaches to these benchmarks have been computed, relying on p-boxes, coherent upper and lower probabilities, Distribution Envelope Determination, and many more (refer to Reliability Engineering and System Safety (2004), Vol. 85). For instance, Fetz and Oberguggenberger [9] provide an in-depth solution in a more general framework of imprecise probabilities. Yet, to the authors knowledge, no solutions in a purely possibilistic framework have been proposed. Problem 4 poses the following question:

> "Let the model of [a] physical process be given by $y=(a+b)^a$. The parameters a and b are independent. [...] What can be ascertained about the response of the system y, given only the stated information concerning a and b? [...] a is contained in the interval A, and b is given by a log-normal probability distribution. One has
>
> $$A=[0.1,1.0] \qquad \text{and} \qquad \log b\sim\mathscr{N}(\mu,\sigma).$$
>
> The value of the mean, μ, and the standard deviation, σ, are given, respectively, by the closed intervals
>
> $$M=[0.0,1.0] \qquad \text{and} \qquad S=[0.1,0.5]."$$
>
> – [1]

Here, all information is interpreted possibilistically. First of all, according to the principle of minimum specificity, the possibility density of a is chosen as the unit function over A and zero elsewhere, i.e.

$$\pi_a(u) = \begin{cases} 1 & \text{if } u\in A \\ 0 & \text{else.} \end{cases} \tag{10}$$

Viewed from the Dempster-Shafer Theory of Evidence [31], this corresponds to assigning all belief mass $m(A)=1$ to this interval, which is perfectly reasonable if one cannot further specify where a lies. All one knows about the probability distribution of a is its support A.

The variable $c = \log b$ possesses a probability distribution P_c that is contained in the set

$$\mathscr{P}_c = \{\mathscr{N}(\mu, \sigma) : \mu \in M, \sigma \in S\}. \tag{11}$$

According to [32], a maximum specific possibility distribution Π_c whose credal set $\mathscr{C}(\Pi_c)$ is a superset of $\mathscr{P}_c$ is e.g. found by defining the possibility density

$$\pi_c(s) = \max_{\mathrm{P}_c \in \mathscr{P}_c} \mathrm{P}_c(\{t \in \mathbb{R} : |t - \bar{c}| \geq |s - \bar{c}|\}) \qquad \forall s \in \mathbb{R} \tag{12}$$

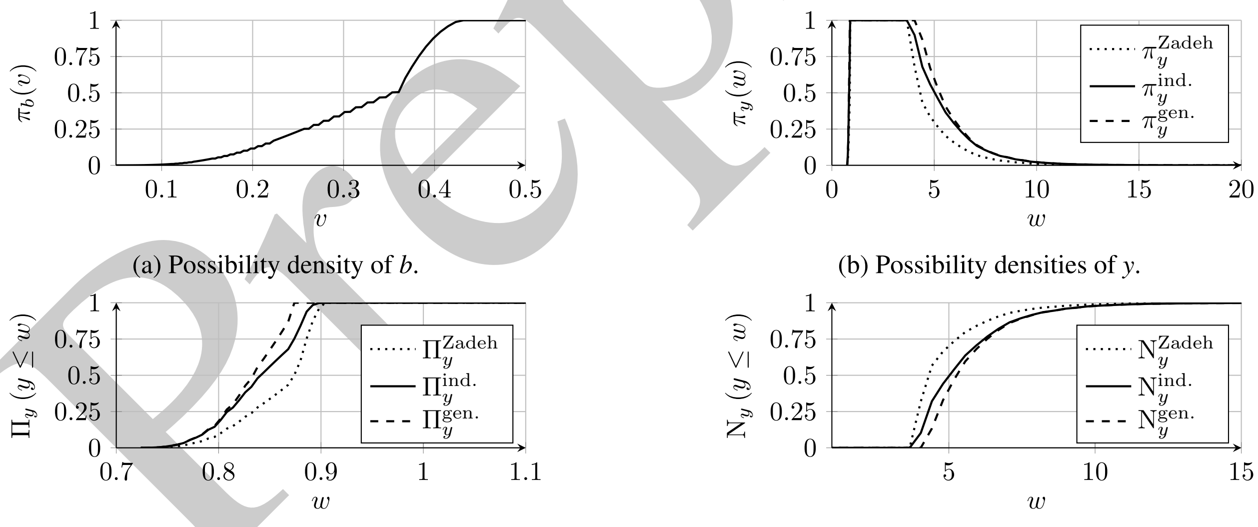
for any user-defined center point $\bar{c} \in \mathbb{R} \cup \{\pm\infty\}$. Notice that the quality of the bounds may strongly depend on the choice of $\bar{c}$. Here, $\bar{c} = 0$ is chosen. According to Definition 17, the possibility density of the minimum specific inverse possibility distribution of b, shown in Figure 3a, is then given by

$$\pi_b(v) = \pi_c(\log v) \qquad \forall v > 0. \tag{13}$$

Having found the adequate possibility distributions of a and b, one can then compute the corresponding possibility densities of y, i.e.

$$\begin{aligned} \pi_y^{\text{Zadeh}}(w) &= \max_{(u,v):w=(u+v)^u} \min[\pi_a(u), \pi_b(v)], \\ \pi_y^{\text{ind.}}(w) &= \max_{(u,v):w=(u+v)^u} \min\left[1-(1-\pi_a(u))^2, 1-(1-\pi_b(v))^2\right] \text{ and} \qquad \forall w \in \mathbb{R} \\ \pi_y^{\text{gen.}}(w) &= \max_{(u,v):w=(u+v)^u} \min[\min(1, 2\cdot\pi_a(u)), \min(1, 2\cdot\pi_b(v))] \end{aligned} \tag{14}$$

as shown in Figure 3b. Zadeh's solution π_y^{Zadeh} is given for reference, and $\pi_y^{\text{gen.}}$ would be the correct solution if one did not know about the independence of the two uncertain variables a and b. However, it is explicitly stated that they are indeed independent, and consequently, $\pi_y^{\text{ind.}}$ yields the appropriate solution. The unknown true cumulative probability $\mathrm{P}_y(y \leq w)$ can then e.g. be bounded from above by the cumulative possibility shown in Figure 3c, and from below by the cumulative necessity shown in Figure 3d.



(a) Possibility density of b. (b) Possibility densities of y.

(c) Cumulative possibilities of y. (d) Cumulative necessities of y.

Figure 3: Possibility distributions computed in the simple problem.

5.2 Complex Problem

In order not to undercut the publication of the complete results in response to the benchmark proposal by Petryna and Drieschner in [2], a simplified version of Challenge 1 posed therein is investigated here. Consider the portal crane structure depicted in Figure 4. It is subjected to an extreme load case consisting of three external forces, namely the wind load $F_1 = 15.7\,\mathrm{kN}$, the brake load $F_2 = 2.7\,\mathrm{kN}$ and the weight $F_3 = 2700\,\mathrm{kN}$ of the object to be lifted, and two possible failure modes are investigated, namely material failure caused by an exceedance of the maximum internal stress and stability failure caused by buckling. For a more detailed explanation refer to [2].

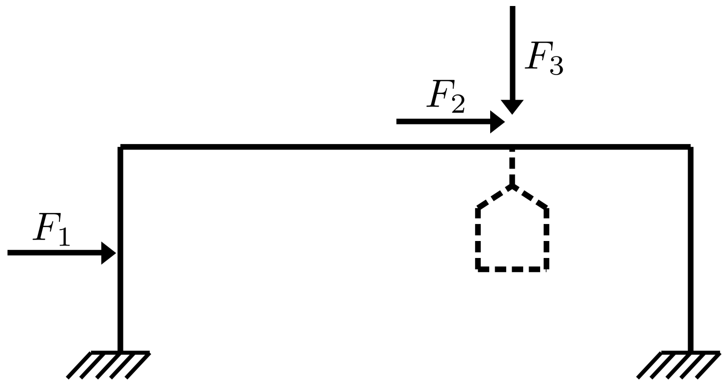


Figure 4: Load case on portal frame structure to be investigated.

The original problem considers a total of at least 19 uncertain parameters depending on the modeling approach and account for uncertainties about the geometry, loads and material. Here, only the Young's modulus E and the yield strength σ_y of the portal frame are considered as uncertain (material) parameters. The information about them is given as follows:

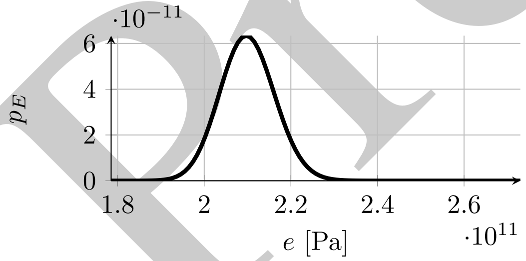
> "It is known that the frame is built of steel S355 [...] with the characteristic value of the yield strength of 275 MPa, that is the 5%-quantile value of the statistical distribution. The mean value of the Young's modulus E can be taken as 210000 MPa. According to the Probabilistic Model Code [...], log-normal distributions are recommended to be applied with a coefficient of variation of 7% for the yield strength σ_y and with a coefficient of variation of 3% for the Young's Modulus E." – [2]

These instruction leave little room for uncertainty about the actual probability distributions P_E and P_{σ_y}; their probability densities p_E and p_{σ_y} are shown in Figures 5a and 5b. One now may assume that a possibilistic analysis is unnecessary since the probabilities are known very precisely. Yet, information about their correlation is not provided. Assuming stochastic independence of these two variables would certainly not be in accordance with the experience of mechanical and civil engineers. Hence, a possibilistic analysis is still recommended.

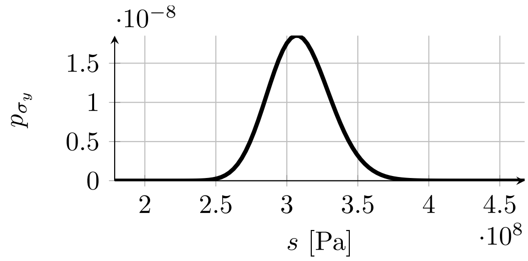
As pointed out in [20], the cumulative probability distribution functions are possibility densities that are consistent with the original probability distributions, i.e. for

$$\pi_E(e) = \mathrm{P}_E(E \le e) \qquad \text{and} \qquad \pi_{\sigma_y}(s) = \mathrm{P}_{\sigma_y}(\sigma_y \le s) \qquad \forall\, e, s \ge 0 \tag{15}$$

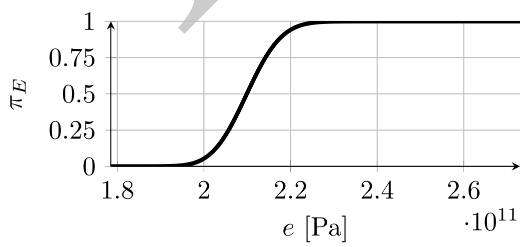
it holds that $\mathrm{P}_E \in \mathscr{C}(\Pi_E)$ and $\mathrm{P}_{\sigma_y} \in \mathscr{C}(\Pi_{\sigma_y})$. The possibility densities are shown in Figures 5c and 5d.



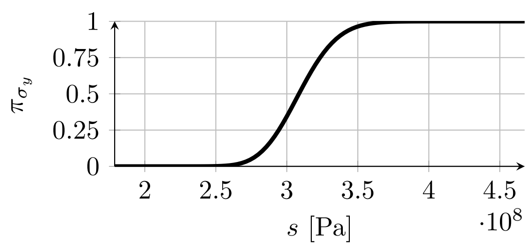
(a) Probability density of the Young's modulus E.



(b) Probability density of the yield strength σ_y.



(c) Possibility density of the Young's modulus E.



(d) Possibility density of the yield strength σ_y.

Figure 5: Probability and possibility distributions computed in the complex problem.

Failing configurations may be identified by checking negativity of a highly non-linear function $g = g(E, \sigma_y)$ involving, among other things, a matrix inversion and an eigenvalue analysis. The possibility

$$\Pi_{\text{fail}} = \max_{e,s \geq 0 \,:\, g(e,s) \leq 0} \pi_{E,\sigma_y}(e,s) \tag{16}$$

provides the upper bound for the true failure probability. Depending on the choice of the aggregation operator, one obtains

$$\Pi_{\text{fail}}^{\text{Zadeh}} = 0.0715, \qquad \Pi_{\text{fail}}^{\text{ind.}} = 0.1378 \qquad \text{and} \qquad \Pi_{\text{fail}}^{\text{gen.}} = 0.1429. \tag{17}$$

Accepting that independence cannot be assumed, the conclusion is that, under the given information about the material, the extreme load situation investigated will lead to a system failure in up to 14.29% of the cases.

6 Conclusion

This contribution presents precise instructions and examples on how to modify Zadeh's extension principle if possibility theory is used as a means of representing imprecise probabilities. It facilitates a consistent propagation of marginal possibility distributions through any model and is at the same time conceptually simple. Furthermore, it does not rely on the notion of possibilistic independence for which a final definition has not yet been agreed upon by the scientific community. Instead, it is solely based on well-defined terms from probability theory.

Among others, one important conclusion is certainly that for large numbers of uncertain variables with ill-known probability distributions, a possibilistic analysis may be replaced by a computationally less-demanding interval-valued analysis without loosing much information.

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