Spectral Asymptotics for Stretched Fractals

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Contents

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Abstract

In this thesis we conduct analysis of spectral asymptotics for some Laplacians on non-self-similar fractals.

In the first part we consider the stretched Sierpiński gasket. It is constructed by stretching the copies of the self-similar Sierpiński gasket apart from each other and reconnecting them by one-dimensional lines. Therefore, it consists of a one-dimensional line part and a higher dimensional fractal part. In [5] Alonso-Ruiz, Freiberg and Kigami introduced resistance forms on this set. A resistance form induces a metric on the underlying space which is called the resistance metric. We would like to calculate the Hausdorff dimension of the stretched Sierpiński gasket with respect to this metric, but to be able to do so we need to introduce some conditions. Depending on the choice of the resistance form this dimension takes values between 1 and $\frac{\ln 3}{\ln 5 - \ln 3}$. These values correspond to the dimensions of the one-dimensional part and the self-similar Sierpiński gasket. Both limiting cases can be realized.

If we equip the stretched Sierpiński gasket with a measure we obtain Dirichlet forms and thus self-adjoint operators. They have non-negative discrete spectrum, which we analyze by studying the asymptotic growing of the eigenvalue counting function. Under the same conditions as before we can calculate the order of the leading term of these asymptotics, which turns out to interpolate between $\frac{1}{2}$ and $\frac{\ln 9}{\ln 5}$. Again, these values belong to the one-dimensional part and the self-similar Sierpiński gasket, respectively, and both border cases can be achieved. As it turns out, these values, Hausdorff dimension and order of the leading term, satisfy the same relation as one known for p.c.f. self-similar sets. The next question that arises is if there are oscillations in the leading term which are typical for highly symmetrical self-similar fractals. The stretched Sierpiński gasket is not self-similar but it still exhibits very high symmetry. We have to distinguish between the appearance of a periodic function in the leading term and oscillations in general. The first one is unlikely as we will see, whereas the second one still holds. This means there are oscillations in the leading term, but these will not have this very strict periodic behavior that we know of the self-similar Sierpiński gasket. We will show that there exist localized eigenfunctions which have eigenvalues with very high multiplicities, and this property yields the oscillations in the spectral asymptotics.

In the second part we would like to generalize the concept of stretching to as many fractals as possible and introduce the notion of stretched fractals. We introduce resistance forms on these fractals and answer the same questions as in the first part for the stretched Sierpiński gasket. After introducing conditions on the resistance forms we can calculate the Hausdorff dimension with respect to the induced resistance metric. Furthermore, equipping the fractal with a measure, we get self-adjoint operators, for which we study the asymptotic behavior of the eigenvalue counting function by calculating the order of the leading term.

Zusammenfassung

Diese Arbeit behandelt Spektralasymptotiken für Operatoren auf einigen nicht selbstähnlichen Fraktalen.

Im ersten Teil behandeln wir das gestreckte Sierpiński Dreieck. Diese Menge entsteht aus dem selbstähnlichen Sierpiński Dreieck, indem wir die Kontraktionsfaktoren der Ähnlichkeiten verringern, was dazu führt, dass sich die Kopien nicht mehr berühren. Daher werden diese mit eindimensionalen Linien erneut verbunden. Damit besteht das gestreckte Sierpiński Dreieck aus einem eindimensionalen Teil sowie einem höherdimensionalen fraktalen Teil. In [5] wurden von Alonso-Ruiz, Freiberg und Kigami sogenannte Resistance Formen auf diesem Fraktal konstruiert. Resistance Formen induzieren eine Metrik auf dem zugrundeliegenden Raum, die sogenannte Resistance Metrik. Wir wollen die Hausdorff Dimension des gestreckten Sierpiński Dreiecks in Bezug zu dieser Metrik berechnen. Um das zu erreichen, müssen wir jedoch Bedingungen an die Resistance Formen einführen. Unter diesen Bedingungen kann die Hausdorff Dimension alle Werte zwischen 1 und $\frac{\ln 3}{\ln 5 - \ln 3}$ annehmen. Diese Werte entsprechen den Dimensionen des eindimensionalen Teils sowie der Dimension des selbstähnlichen Sierpiński Dreiecks. Auch die Grenzfälle sind beide möglich.

Fügen wir nun noch ein Maß hinzu, so erhalten wir Dirichlet Formen und daher selbstadjungierte Operatoren. Diese besitzen nichtnegatives diskretes Spektrum, welches wir analysieren indem wir das asymptotische Verhalten der Eigenwertzählfunktion untersuchen. Unter den gleichen Bedingungen wie zuvor können wir die Ordnung des führenden Terms dieser Aysmptotiken bestimmen, welcher sich zwischen $\frac{1}{2}$ und $\frac{\ln 9}{\ln 5}$ befindet. Das entspricht erneut den Werten des eindimensionalen Falles sowie des selbstähnlichen Sierpiński Dreiecks und wiederum können beide Grenzfälle angenommen werden. Es zeigt sich, dass die beiden berechneten Werte - Hausdorff Dimension und Ordnung des führenden Terms - eine Gleichung erfüllen, die bisher für p.c.f. selbstähnliche Fraktale bekannt war. Die nächste Frage, die sich stellt, ist die nach Oszillationen im führenden Term. Diese sind typisch für sehr symmetrische selbstähnliche Fraktale. Das gestreckte Sierpiński Dreieck ist jedoch nicht selbstähnlich, besitzt aber dennoch sehr viel Symmetrie. Wir müssen nun zwischen der Existenz einer periodischen Funktion und Oszillationen an sich unterscheiden. Ersteres ist unwahrscheinlich, das Zweite hingegen lässt sich zeigen. Das heißt die Oszillationen, die wir im führenden Term finden, sind nicht so regulär wie im selbstähnlichen Fall. Wir finden lokalisierte Eigenfunktionen, deren Eigenwerte sehr hohe Multiplizitäten besitzen.

Im zweiten Teil wollen wir das Konzept des Streckens auf so viele weitere Fraktale wie möglich erweitern. Wir konstruieren Resistance Formen auf diesen gestrecktenen Fraktalen und beantworten die gleichen Fragen, die wir schon im ersten Teil für das gestreckte Sierpiński Dreieck behandelt haben. Nachdem wir Bedingungen an die Resistance Formen gestellt haben, können wir die Hausdorff Dimension bezüglich der induzierten Resistance Metrik berechnen. Wir führen erneut ein Maß ein, um selbstadjungierte Operatoren zu erhalten. Für diese Operatoren können wir die Ordnung des führenden Terms der Asymptotik der Eigenwertzählfunktion bestimmen.

1 Introduction

Calculating spectral properties is an important tool in physics, for example, to calculate how heat or waves propagate through media. To gather information about the solutions of the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0, \tag{1.1}$$

or the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0, \tag{1.2}$$

we need to gain information about the eigenvalues of the Laplacian Δ, which is very crucial in these equations. For bounded domains $\Omega \subset \mathbb{R}^n$ the Dirichlet Laplacian Δ_D has negative discrete spectrum accumulating only at negative infinity. One way to analyze this spectrum is to calculate the asymptotic growing of the eigenvalues. The eigenvalue counting function $N_D^\Omega(x)$ counts the eigenvalues of $-\Delta_D$ smaller than x and it has the asymptotic behavior

$$N_D^\Omega(x) = \frac{\tau_n}{(2\pi)^n} \operatorname{Vol}_n(\Omega) x^{\frac{n}{2}} + o(x^{\frac{n}{2}}) \quad \text{as } x \to \infty, \tag{1.3}$$

where τ_n denotes the volume of the unit ball in $\mathbb{R}^n$. This result is originally due to Weyl [56]. As a consequence of this result, we see that we can get information about the set Ω just from the knowledge of the eigenvalues, namely the dimension n, as well as the volume of the set. There is a one-to-one correspondence between the eigenvalues and the eigenfrequencies of Ω. Therefore, the problem of what information of Ω is saved in the spectrum of Δ, is also known through the famous question "Can one hear the shape of a drum?" by Kac [34]. As it turns out, there are examples of non-isometric sets that have exactly the same eigenvalues. The first counterexamples were 16-dimensional tori by Milnor [51]. A very illustrative construction of such domains with the same spectrum in $\mathbb{R}^2$ can be found in [15]. This means we cannot expect to get all information of Ω, but it is still a very challenging and pursued matter what information is possible to collect.

In the 70s the interest in fractals grew, mainly thanks to Mandelbrot; see for example [48, 49]. Fractals are sets whose geometric size cannot be described sufficiently by the classical Lebesgue measures, which means there are other measures needed. In 1918 Hausdorff introduced the *Hausdorff measures* in [29], which are much better fitted for this task. Closely connected is the notion of *Hausdorff dimension* which is a concept of dimension that allows non-integer values; a property that is often satisfied by fractals. Since nature is not built up of smooth structures, but rather porous, disordered and finely structured material, fractals are much better suited to describe natural patterns and phenomena than smooth manifolds from classical analysis. When we use fractals as a model we would like to solve equations like (1.1) and (1.2). This means we need the notion of a Laplacian. However, on fractals we are not able to define a limit of difference quotients leading to the classical derivative, since we lack any smoothness that is needed to do so. Therefore, another way to define a Laplacian is needed.

Basically, there are two approaches. One way to define a Laplacian is the probabilistic approach which was first used for the Sierpiński gasket by Goldstein [22] and Kusuoka [46] and a little bit later by Barlow and Perkins [6]. These ideas were extended by Lindstrøm to nested fractals in [47]. Here the idea is to define a Brownian motion as a limit of discrete random walks on the approximating graphs. The Laplacian is then defined as the infinitesimal generator of the Brownian motion. In this thesis, however, we will use the second, so-called analytical approach which is due to Kigami. There the Laplacian is defined as a renormalized limit of discrete Laplacians on the approximating graphs. This was first done by Kigami for the Sierpiński gasket in [36] and later for the so-called *p.c.f. self-similar sets* in [37]. These, roughly spoken, are sets that consist of scaled copies of themselves which are connected only via a finite number of points. How does the spectrum of these operators behave? The naive generalization of (1.3) to Laplacians on fractals would be

$$N_D^F(x) = C_d \mathcal{H}^d(F) x^{\frac{d}{2}} + o(x^{\frac{d}{2}}) \quad \text{as } x \to \infty, \tag{1.4}$$

where $d = \dim_H(F)$ is the Hausdorff dimension of F, $\mathcal{H}^d(F)$ the d-dimensional Hausdorff measure of F and C_d a constant that only depends on d. This was conjectured by Berry in [11] and [12]. However, this turned out to be false. Shima [53] and Fukushima-Shima [19] calculated the eigenvalues of the Laplacian on the Sierpiński gasket S by the *eigenvalue decimation method*. This is a method that allows us to calculate the eigenvalues of the Laplacian on S by calculating the eigenvalues of the approximating discrete Laplacians, although it is only applicable to a very restricted class of fractals. Using this, they showed

$$0 < \liminf_{x\to\infty} N_D(x) x^{-\frac{1}{2}d_S} \leq \limsup_{x\to\infty} N_D(x) x^{-\frac{1}{2}d_S} < \infty \tag{1.5}$$

with $d_S = \frac{\ln 9}{\ln 5}$ which does not coincide with $\dim_H(F) = \frac{\ln 3}{\ln 2}$. This disproved (1.4). Furthermore, they showed

$$\liminf_{x\to\infty} N_D(x) x^{-\frac{1}{2}d_S} < \limsup_{x\to\infty} N_D(x) x^{-\frac{1}{2}d_S}. \tag{1.6}$$

Hence there is no constant in front of the leading term as in (1.3). Instead, there are oscillations. Later, Kigami and Lapidus in [38] calculated the leading term of the eigenvalue counting function in (1.5) for p.c.f. self-similar sets, provided there is a so-called regular harmonic structure, by using a technique called *Dirichlet-Neumann bracketing.* They also found an interesting relation between the exponent d_S and the Hausdorff dimension $\dim_{H,R}$ calculated with respect to the resistance metric, given by

$$d_S = \frac{2\dim_{H,R}}{\dim_{H,R}+1}. \tag{1.7}$$

In the words of Kac, this means even though we were not able to "hear" the Hausdorff dimension with respect to the Euclidean metric, we can "hear" the value $\dim_{H,R}$. Furthermore, Kigami and Lapidus proved the existence of a log-periodic function in front of the leading term for the case of arithmetic weights, using renewal theory. However, this does not help to answer whether or not (1.6) holds, since this log-periodic function could still be constant. Later, Barlow and Kigami showed (1.6) for very symmetric cases in [8] by using arguments that utilize the symmetry of the set. They showed the existence of localized eigenfunctions. These are eigenfunctions that are only supported on a proper subset, a very interesting phenomenon that does not appear in classical analysis. Such localized eigenfunctions give us solutions of the heat and wave equation where the heat and energy stay in the support. This represents, for example, perfectly heat insulated or soundproofed rooms. The associated eigenvalues have very high multiplicities leading to high jumps in the eigenvalue counting function. We can use this to show (1.6).

Another generalization is due to Kajino [35] where the order of the leading term in (1.5) was calculated for more general self-similar sets.

In this thesis we examine some non-self-similar sets for the properties we have just introduced. We construct operators and calculate the Hausdorff dimension with respect to the resistance metric. Furthermore, we investigate the spectrum of these operators by calculating the order of the leading term and by looking for oscillations and periodicity. This thesis mainly consists of two parts.

First, we consider the *stretched Sierpiński gasket* in Chapter 2.

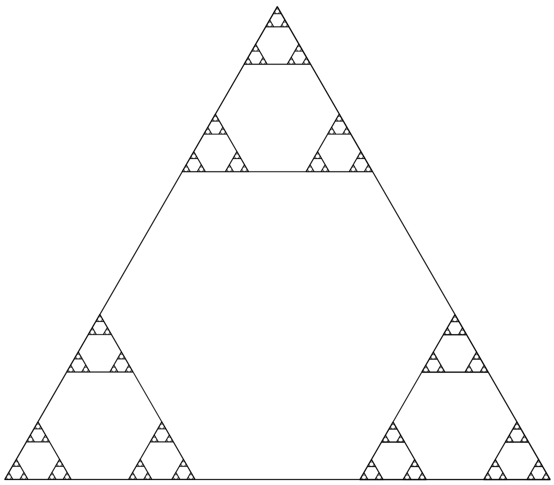

Figure 1.1: The stretched Sierpiński gasket.

This is a non-self-similar set which still exhibits a lot of symmetry and has been analyzed geometrically in [2] by Alonso-Ruiz and Freiberg. We can decompose this set into two parts, namely into the one-dimensional lines which we call the line part, and the fractal dust which we call the fractal part. For the exact definition of these terms we refer to Chapter 2. By lowering the contraction ratio of the self-similar Sierpiński gasket, we sort of "stretch" the copies apart from each other, justifying the name. It is also known by the name "Hanoi attractor". This is due to its connection to the game "The Tower of Hanoi". In this game we have n disks stacked on one of three rods and ordered by their size.

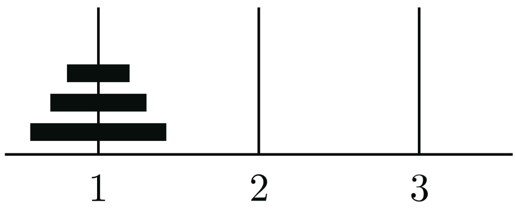

Figure 1.2: The Tower of Hanoi with 3 disks.

The goal is to move the whole tower to one of the other two rods, but it is only allowed to move one disk at a time and only to move smaller disks on top of bigger disks. We denote by the word $w = w_1 \cdots w_n \in \{1, 2, 3\}^n$ of length n the state of the game where $w_i = j$ means that disk i is on rod j. Disk 1 is the biggest, whereas disk n is the smallest disk. We then see that the n-th graph approximation of the stretched Sierpiński gasket, or Hanoi attractor, corresponds to all possible states of the game with n disks where the edges correspond to the allowed moves in this game.

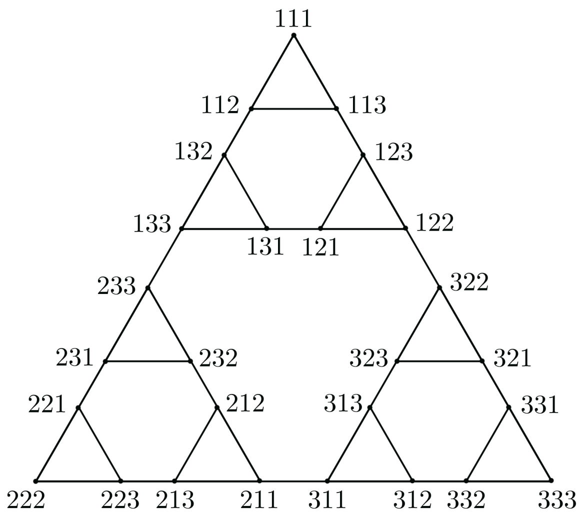

Figure 1.3: Allowed moves with 3 disks.

This connection between the game and the Hanoi attractor can be found in [2] and an extensive overview of mathematical facts about the Tower of Hanoi in [33].

In [5] Alonso-Ruiz, Freiberg and Kigami introduced resistance forms on the stretched Sierpiński gasket that consist of two parts. One belongs to the higher dimensional part of the stretched Sierpiński gasket and is very similar to the resistance form on the Sierpiński gasket. The other one belongs to the one-dimensional edges. The choice of the resistances in the approximating electrical networks is not unique. We can choose sequences of these resistances which are also called *sequences of matching pairs*, and therefore, there exists a big variety of resistance forms on the stretched Sierpiński gasket. In [5] the authors treat the *completely symmetric* ones that exhibit the intuitive symmetries and show that each of the completely symmetric resistance forms is obtained in this way.

There are some prior works concerning spectral asymptotics on the stretched Sierpiński gasket. The first is by Alonso-Ruiz and Freiberg [3]. There they constructed a Dirichlet form and calculated the order of the leading term in the asymptotics of the eigenvalue counting function of the associated operator. This leading term turns out to be of the same order as for the Sierpiński gasket, namely $\frac{\ln 3}{\ln 5}$. The resistance form used corresponds to one coming from a fixed sequence of matching pairs of [5] and the measure is the sum of the normalized Hausdorff measure on the self-similar part and a scaled Lebesgue measure on the line part; see also Section 2.4. However, the scaling parameter is chosen in such a way that the influence of the line part is not too big.

Another work is by Alonso-Ruiz, Kelleher and Teplyaev [4]. The approach in this article is by the use of quantum graphs. The stretched Sierpiński gasket is viewed as a so-called *fractal quantum graph*. The measure used on the one-dimensional edges is more general than the one in [3], while there is no mass on the higher dimensional fractal part. Furthermore, the quadratic forms only consist of the line part and do not consider the fractal dust inside the stretched Sierpiński gasket. Therefore, the calculated leading order in the asymptotics of the eigenvalue counting function turns out to be smaller than $\frac{\ln 3}{\ln 5}$. There are also some numerical results by Alonso-Ruiz, Chen, Gu, Strichartz and Zhou in [1]. The resistance forms considered are special cases of [5] and the measures are the same as in [4].

In Chapter 2 we combine the ideas of [3] and [4] and generalize them to the resistance forms of [5]. We use these resistance forms and choose a suitable measure to get regular Dirichlet forms and thus self-adjoint operators with non-negative discrete spectrum. This spectrum can be analyzed in terms of the eigenvalue counting function and its asymptotic behavior. We calculate the leading order and furthermore, recover the relation (1.7) between spectral dimension and Hausdorff dimension with respect to the resistance metric. We also show that there are oscillations in the leading term leading to (1.6). However, it is very unlikely to achieve the appearance of a log-periodic function in the leading term which we know of the self-similar case. We lack a certain symmetry that is needed, in all but some special cases.

The second part, Chapter 3, is a generalization of Chapter 2. We would like to generalize the idea of "stretching" to as many fractals as possible. For the Sierpiński gasket it is very important that the copies only intersect at finitely many points. Therefore, we are able to stretch the copies apart and connect these points again by a one-dimensional line. We generalize the "stretching" to p.c.f. self-similar fractals that satisfy a certain connectedness condition which ensures that we know how to connect the copies after stretching them apart. We answer the same questions that we addressed in Chapter 2 for the stretched Sierpiński gasket for these *stretched fractals*. A similar idea of mixing fractals was introduced in [1], where the authors introduced the so-called *hybrid fractals*.

Lastly, in Chapter 4, we address some open problems and further research.

This thesis is based on the following three articles, cf. references [30, 31, 32].

- E. Hauser, *Spectral asymptotics on the Hanoi attractor.* To appear in the volume *Analysis, Probability and Mathematical Physics on Fractals: Proceedings of the 6th Cornell Fractals Conference*, to be published by World Scientific.
- E. Hauser, *Oscillations on the Stretched Sierpinski Gasket.* Preprint, 2018.
- E. Hauser, *Spectral Asymptotics for Stretched Fractals.* Preprint, 2018.

2 Stretched Sierpiński gasket

The stretched Sierpiński gasket is a non-self-similar set that still exhibits a lot of symmetry. Let p_1, p_2, p_3 be the vertices of an equilateral triangle with side length 1 and for $\alpha \in (0,1)$ define $G_i : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$G_i(x) := \frac{1-\alpha}{2}(x - p_i) + p_i, \ i \in \{1,2,3\}$$

and

$$e_{ij} := \{\lambda G_i(p_j) + (1-\lambda) G_j(p_i) \mid \lambda \in [0,1]\}$$

for $(i,j) \in \{(1,2),(2,3),(3,1)\} =: B$. Then there exists a unique non-empty compact set $K_\alpha \subset \mathbb{R}^2$ with

$$K_\alpha = G_1(K_\alpha) \cup G_2(K_\alpha) \cup G_3(K_\alpha) \cup e_{12} \cup e_{23} \cup e_{31}.$$

This fact can be found in [5, Proposition 3.3] and is due to [28, Theorem 1]. K_α is called the *stretched Sierpiński gasket* because the contraction ratios are smaller than the ones for the Sierpiński gasket and the gaps are filled with one-dimensional lines. Also define Σ_α as the unique non-empty compact subset of $\mathbb{R}^2$ with

$$\Sigma_\alpha = G_1(\Sigma_\alpha) \cup G_2(\Sigma_\alpha) \cup G_3(\Sigma_\alpha),$$

which is a subset of K_α. The sets K_α for $\alpha \in (0,1)$ are pairwise homeomorphic [5, Proposition 3.4], and since the resistance forms will only depend on the topology on K_α we can omit the parameter α in the notation for K_α and Σ_α.

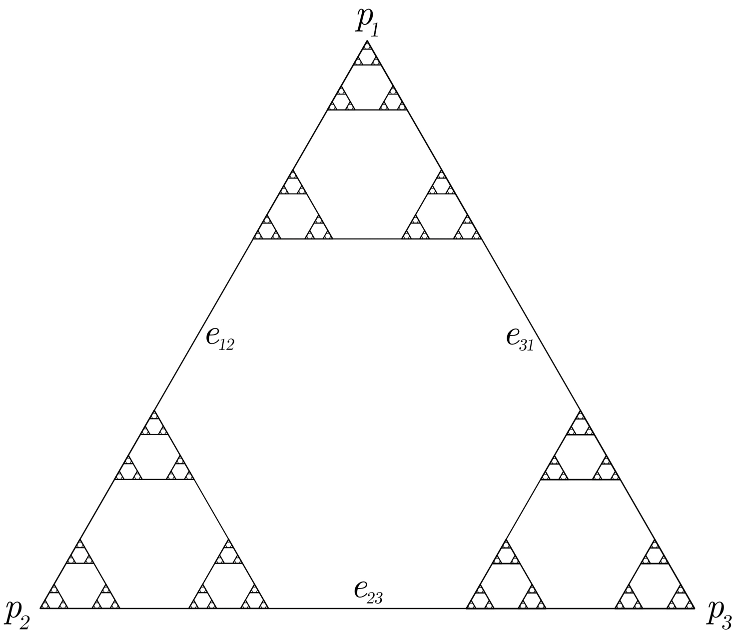

Figure 2.1: The stretched Sierpiński gasket.

We need to introduce some notation that is commonly used. We denote the alphabet by $\mathcal{A} := \{1,2,3\}$ and the set of all words of finite length by $\mathcal{A}_0^* := \bigcup_{k\geq 0} \mathcal{A}^k$ which also includes the empty word. For $w \in \mathcal{A}_0^*$ define

$G_w = G_{w_1} \circ \ldots \circ G_{w_n}$ for $w = w_1 \cdots w_n$ and $G_w = \mathrm{id}$ for the empty word $w \in \mathcal{A}^0$,

$V_0 := \{p_1, p_2, p_3\}$, $V_n := \bigcup_{w\in\mathcal{A}^n} G_w(V_0)$,

$e_{ij}^w := G_w(e_{ij})$,

$K_w := G_w(K)$, $K_n := \bigcup_{w\in\mathcal{A}^n} K_w$,

$J_n := \overline{K \backslash K_n}$,

$\Sigma_w := G_w(\Sigma)$, $\Sigma_n := \bigcup_{w\in\mathcal{A}^n} \Sigma_w$.

We refer to Σ as the *fractal part* and to $J = \bigcup_{k\geq 1} J_k$ as the *line part* of K. We call K_w an *n-cell* if $w \in \mathcal{A}^n$.

This chapter is organized as follows. In Section 2.1 the construction of the resistance forms from [5] is briefly discussed. To be able to do the calculations we need to introduce a condition on the resistances. This condition is introduced in Section 2.2 following some important estimates for the resistances. In Section 2.3 the Hausdorff dimension of the stretched Sierpiński gasket is calculated with respect to the resistance metric coming from a resistance form that fulfills this condition. This value is more useful for the analysis on a set, than the one calculated with respect to the Euclidean metric. In Section 2.4 the measures are introduced that are used to get self-adjoint operators. In Section 2.5 we calculate the leading order of the eigenvalue counting functions, i.e., we show (1.5). Together with the results about the Hausdorff dimension we show that (1.7) still holds. Afterwards, we weaken the condition for a moment to get more general results in Section 2.6. In Section 2.7 we would like to refine (1.5) and answer the questions we addressed previously: Are there oscillations and periodicity? As it turns out we are still able to show (1.6) even though it is most likely that there is no strict periodicity. This means the oscillations are not as regular as in the self-similar case, since we lack some kind of symmetry. In some special cases we are able to get back the symmetry that is needed to show strict periodicity.

2.1 Recapitulation of resistance forms on the stretched Sierpiński gasket

To be able to study analysis on the stretched Sierpiński gasket we need to introduce a resistance form on K. The definition of resistance forms can be found in Definition B.1. The choice of these forms is not unique and thus we get different operators and different spectral asymptotics. This construction was carried out in [5] and this section includes a brief recapitulation of this work.

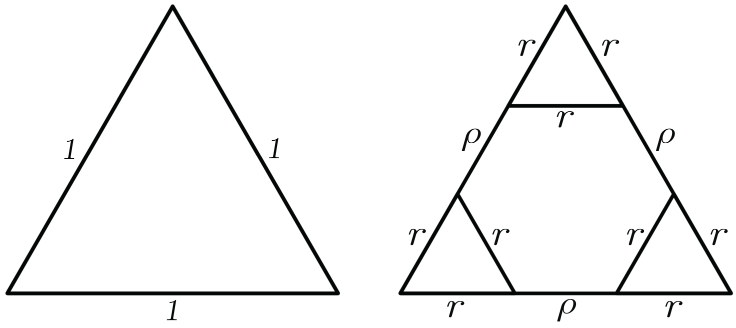

Figure 2.2: Resistances on the first graph approximation.

In Figure 2.2 you can see the first graph approximation of K beside the graph that just contains the vertices p_1, p_2 and p_3. Due to symmetry we would like to have the resistances on the smaller triangles all equal to r and also all equal to ρ on the edges joining them. This electrical network should be equivalent to the one on the left with all resistances equal to 1.

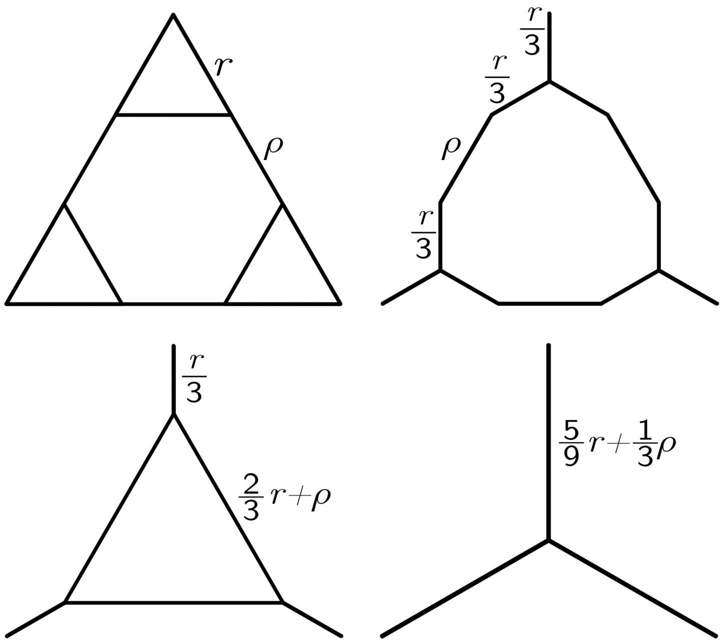

Figure 2.3: Transformations of equivalent electrical networks.

Applying the Δ-Y-transformation repeatedly and combining resistors in series as in Figure 2.3 leads to

$$\frac{5}{3}r + \rho = 1.$$

Such a pair $(r, \rho) \in (0, \infty)^2$ is called a *matching pair*; see [5, Definition 7.4]. In the next graph approximation the smaller triangles get divided further in the same fashion.

In general in the $(m+1)$-th graph approximation the right triangle in Figure 2.4 is required to be equivalent to the left one with all resistances δ_m. The same calculation as for

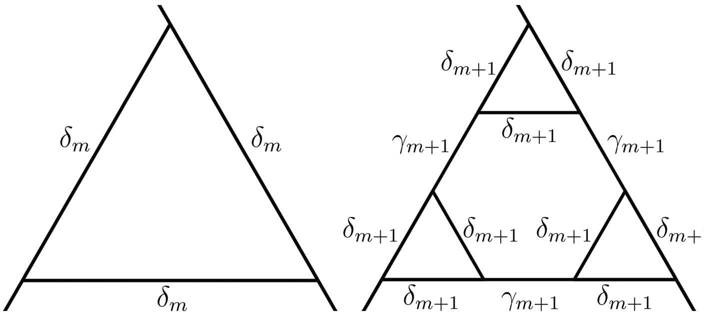

Figure 2.4: Resistances in the $m+1$ graph approximation.

the first graph approximation shows it has to hold that

$$\delta_{m+1} = \delta_m \cdot r_{m+1} \quad \text{and} \quad \gamma_{m+1} = \delta_m \cdot \rho_{m+1}$$

with a matching pair (r_{m+1}, ρ_{m+1}), i.e., $\frac{5}{3}r_{m+1} + \rho_{m+1} = 1$. Notice that the resistances of the edges connecting adjoining cells from the previous graph approximations do not change. We thus obtain

$$\delta_m = r_1 \cdots r_m \quad \text{and} \quad \gamma_m = r_1 \cdots r_{m-1}\rho_m$$

with $\frac{5}{3}r_i + \rho_i = 1$ for all i. Such a sequence $\mathcal{R} = \{(r_i, \rho_i)\}_{i\geq 1}$ of matching pairs is also called a *compatible sequence* because each of those sequences will lead to a resistance form on K; see [5, Theorem 7.16].

With these resistances we can define a quadratic form which will consist of two parts. One part is very similar to the usual resistance form on the Sierpiński gasket. For $u \in \ell(K)$ define

$$\begin{aligned}
Q_0^{\Sigma}(u,u) &:= (u(p_1) - u(p_2))^2 + (u(p_2) - u(p_3))^2 + (u(p_3) - u(p_1))^2, \\
Q_m^{\Sigma}(u,u) &:= \sum_{w \in \mathcal{A}^m} Q_0^{\Sigma}(u \circ G_w, u \circ G_w), \\
\mathcal{E}_{\mathcal{R}}^{\Sigma}(u,u) &:= \lim_{m\to\infty} \frac{1}{\delta_m} Q_m^{\Sigma}(u,u).
\end{aligned}$$

However, this form ignores the edges that connect m-cells. To get a form on all of K we need a second part. This can be achieved with the usual one-dimensional Dirichlet energy summed over all such lines. We denote the Sobolev space on $[0,1]$ by

$$H^1([0,1]) := \left\{ u \;\middle|\; u \in \ell([0,1]), \frac{du}{dx} \in L^2([0,1], \lambda^1) \right\},$$

where $\frac{du}{dx}$ is the weak derivative of u with respect to the Lebesgue measure λ^1. Denote by $(e_{ij}^w)_-$ and $(e_{ij}^w)^+$ the endpoints of e_{ij}^w. Then with $\xi_{e_{ij}^w}(t) = (1-t)(e_{ij}^w)_- + t(e_{ij}^w)^+$, $t \in [0,1]$, we can define the Sobolev space on the one-dimensional lines e_{ij}^w by

$$H^1(e_{ij}^w) = \{u \mid u \in \ell(e_{ij}^w), u \circ \xi_{e_{ij}^w} \in H^1([0,1])\}.$$

Now for $k \geq 1$ define

$$\mathcal{D}_k^I(u,u) := \sum_{\substack{w\in\mathcal{A}^{k-1}\\(i,j)\in B}} \underbrace{\int_0^1 \left(\frac{d(u\circ\xi_{e_{ij}^w})}{dx}\right)^2 dx}_{\mathcal{D}_{e_{ij}^w}(u,u):=}$$

as well as

$$\mathcal{E}_{\mathcal{R}}^I(u,u) := \sum_{k=1}^{\infty} \frac{1}{\gamma_k}\mathcal{D}_k^I(u,u)$$

and the sum of those two parts as our final quadratic form

$$\mathcal{E}_{\mathcal{R}}(u,u) := \mathcal{E}_{\mathcal{R}}^{\Sigma}(u,u) + \mathcal{E}_{\mathcal{R}}^I(u,u),$$

which is defined on

$$\mathcal{F}_{\mathcal{R}} = \left\{ u \;\middle|\; u \in C(K) : \mathcal{E}_{\mathcal{R}}(u,u) < \infty,\ u|_{e_{ij}^w} \in H^1(e_{ij}^w), \forall w \in \mathcal{A}_0^*, (i,j) \in B \right\}.$$

In [5, Theorem 7.16] the authors showed that for any sequence of matching pairs $\mathcal{R} = \{(r_i,\rho_i)\}_{i\geq 1}$ the form $(\mathcal{E}_{\mathcal{R}},\mathcal{F}_{\mathcal{R}})$ is indeed a resistance form on K whose resistance metric is compatible with the Euclidean topology on K. Resistance forms are also called energies since the approximating resistance forms correspond to the energy of the electrical network.

This construction immediately induces a bilinear form on $\mathcal{F}_{\mathcal{R}}$ given by

$$\begin{aligned}
Q_0^{\Sigma}(u,v) &:= (u(p_1)-u(p_2))(v(p_1)-v(p_2)) + (u(p_2)-u(p_3))(v(p_2)-v(p_3))\\
&\quad + (u(p_3)-u(p_1))(v(p_3)-v(p_1)),\\
Q_m^{\Sigma}(u,v) &:= \sum_{w\in\mathcal{A}^m} Q_0^{\Sigma}(u\circ G_w, v\circ G_w),\\
\mathcal{E}_{\mathcal{R}}^{\Sigma}(u,v) &:= \lim_{m\to\infty} \frac{1}{\delta_m} Q_m^{\Sigma}(u,v),
\end{aligned}$$

as well as

$$\begin{aligned}
\mathcal{D}_k^I(u,v) &:= \sum_{\substack{w\in\mathcal{A}^{k-1}\\(i,j)\in B}} \int_0^1 \left(\frac{d(u\circ\xi_{e_{ij}^w})}{dx}\right)\left(\frac{d(v\circ\xi_{e_{ij}^w})}{dx}\right) dx,\\
\mathcal{E}_{\mathcal{R}}^I(u,v) &:= \sum_{k=1}^{\infty} \frac{1}{\gamma_k}\mathcal{D}_k^I(u,v),
\end{aligned}$$

and the sum

$$\mathcal{E}_{\mathcal{R}}(u,v) := \mathcal{E}_{\mathcal{R}}^{\Sigma}(u,v) + \mathcal{E}_{\mathcal{R}}^I(u,v).$$

The construction of these resistance forms is presented in much greater detail in [5].

2.2 Conditions

The goal is to study the stretched Sierpiński gasket. One way to analyze a set is to study its geometric properties. Probably the most significant geometric value is the Hausdorff dimension. To calculate it we have to choose a metric. The stretched Sierpiński gasket can be embedded in $\mathbb{R}^2$, meaning we could choose the Euclidean metric d_E. This value was calculated in [2]. However, it depends on α. Since the resistance forms do not depend on α we would like to choose another metric that shares this characteristic. The resistance metric only depends on the resistances, therefore, only on the sequence of matching pairs $\mathcal{R}$ and furthermore, it reflects the analysis on the set much better.

We also would like to study the analysis on the stretched Sierpiński gasket, in particular the spectral asymptotics. In the self-similar case, these calculations can be done with the help of the self-similar scaling properties of the resistance forms. We have something similar on the stretched Sierpiński gasket but it is not quite as nice as in the self-similar case.

Lemma 2.1. *Let $\mathcal{R} = \{(r_i, \rho_i)\}_{i\geq 1}$ be a sequence of matching pairs. With $n \geq 0$ and $\mathcal{R}^{(n)} = \{(r_{n+k}, \rho_{n+k})\}_{k\geq 1}$ we have for $u \in \mathcal{F}_\mathcal{R}$ and $w \in \mathcal{A}^n$ that $u \circ G_w \in \mathcal{F}_{\mathcal{R}^{(n)}}$ and*

$$\mathcal{E}_\mathcal{R}(u,u) = \sum_{w\in\mathcal{A}^n} \frac{1}{\delta_n} \mathcal{E}_{\mathcal{R}^{(n)}}(u \circ G_w, u \circ G_w) + \sum_{k=1}^{n} \frac{1}{\gamma_k} \mathcal{D}_k^I(u,u).$$

This rescaling was shown in [5, Lemma 7.18]. We see that there is an additional term on the right-hand side. This comes from the one-dimensional lines that connect the n-cells. The other difference is that the quadratic forms $\mathcal{E}_\mathcal{R}$ on the left and $\mathcal{E}_{\mathcal{R}^{(n)}}$ on the right-hand side differ. As the sequences of matching pairs $\mathcal{R}$ and $\mathcal{R}^{(n)}$ are different ones, the quadratic forms are different. This also does not happen in the self-similar case. We denote the resistances of the electrical networks according to $\mathcal{R}^{(n)}$ by

$$\delta_m^{(n)} := r_{n+1} \cdots r_{n+m},$$
$$\gamma_m^{(n)} := r_{n+1} \cdots r_{n+m-1}\rho_{n+m}.$$

Since we still would like to use this rescaling we need to compare the forms $\mathcal{E}_\mathcal{R}$ and $\mathcal{E}_{\mathcal{R}^{(n)}}$. To be able to do so we need to introduce some conditions on the sequence $\mathcal{R}$.

Condition 2.1. *We consider a sequence of matching pairs $\mathcal{R} = \{(r_i, \rho_i)\}_{i\geq 1}$ such that there exists an $r \in (0, \frac{3}{5}]$ and constants $\tilde{\kappa}_1, \tilde{\kappa}_2 > 0$ with*

$$\tilde{\kappa}_1 r^m \leq \delta_m \leq \tilde{\kappa}_2 r^m$$

for all $m \geq 0$.

With $\delta_m^{(n)} = r_{n+1} \cdots r_{n+m} = \frac{\delta_{n+m}}{\delta_n}$ we obtain

$$\frac{\tilde{\kappa}_1}{\tilde{\kappa}_2} r^m \leq \delta_m^{(n)} \leq \frac{\tilde{\kappa}_2}{\tilde{\kappa}_1} r^m.$$

Without loss of generality we can assume that $\tilde{\kappa}_1 \leq 1 \leq \tilde{\kappa}_2$, meaning with $\kappa_1 := \frac{\tilde{\kappa}_1}{\tilde{\kappa}_2}$ and $\kappa_2 := \frac{\tilde{\kappa}_2}{\tilde{\kappa}_1}$ we get the following behavior of the resistances.

Lemma 2.2. *Let $\mathcal{R} = \{(r_i, \rho_i)\}_{i\geq 1}$ be a sequence of matching pairs that fulfills Condition 2.1. Then*

$$\kappa_1 r^m \leq \delta_m^{(n)} \leq \kappa_2 r^m$$

for all $m, n \geq 0$.

Remark. If for $r \in (0, \frac{3}{5}]$ we consider a sequence $\mathcal{R} = \{(r_i, \rho_i)\}_{i\geq 1}$ such that

$$\sum_{i=1}^{\infty} |r - r_i| < \infty,$$

then Condition 2.1 is fulfilled. To see this we notice that by the limit comparison test the series $\sum_{i=1}^{\infty} |\ln(r^{-1} r_i)|$ converges and thus

$$\prod_{i=1}^{\infty} r^{-1} r_i \in (0, \infty).$$

This means the sequence $a_m := \prod_{i=1}^{m} r^{-1} r_i$ is bounded from above and below and therefore, there are constants $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ such that

$$\tilde{\kappa}_1 r^m \leq \delta_m \leq \tilde{\kappa}_2 r^m$$

for all $m \geq 0$. Actually, this summability condition has a very nice meaning in the case of $r = \frac{3}{5}$. This is the only case where the fractal part of the quadratic form really exists; see [5].

2.3 Hausdorff dimension in resistance metric

In this section we calculate the Hausdorff dimension of K with respect to the resistance metric coming from one of the resistance forms described in Section 2.1. The resistance metric is defined in (RF4) in Definition B.1. It is a metric on K due to Proposition B.2. We can calculate the distance between points $x, y \in K$ by

$$R_{\mathcal{R}}(x, y)^{-1} = \inf\{\mathcal{E}_{\mathcal{R}}(u, u) \mid u \in \mathcal{F}_{\mathcal{R}},\ u(x) = 0, u(y) = 1\}.$$

The topology on K is the same with either the Euclidean metric d_E or the resistance metric $R_{\mathcal{R}}$; see [5, Theorem 7.16]. That means the closure of

$$J = \bigcup_{\substack{w \in \mathcal{A}_0^* \\ (i,j) \in B}} e_{ij}^w$$

is the same with either one.

Considering $K = \Sigma \cup J$, we have

$$\dim_{H,R_\mathcal{R}} K = \max\{\dim_{H,R_\mathcal{R}} \Sigma, \dim_{H,R_\mathcal{R}} J\},$$

where $\dim_{H,R_\mathcal{R}}$ denotes the Hausdorff dimension calculated with respect to the resistance metric $R_\mathcal{R}$. By $\operatorname{diam}(X,d) := \sup\{d(x,y) \mid x,y \in X\}$ we denote the diameter of a set X with metric d.

Lemma 2.3. *Let $\mathcal{R} = \{(r_i,\rho_i)\}_{i\geq 1}$ be a sequence of matching pairs. Then*

$$\dim_{H,R_\mathcal{R}} J = 1.$$

Proof. To see this, we show $\dim_{H,R_\mathcal{R}}(e_{ij}^w) = 1$ for each $(i,j) \in B$ and $w \in \mathcal{A}_0^*$. The result then follows by the σ-stability of the Hausdorff dimension.

First divide e_{ij}^w into two parts of the same size by cutting it in the middle and denote the two parts by $(e_{ij}^w)_1$ and $(e_{ij}^w)_2$. We show that we can find positive constants a, b such that

$$a \cdot d_E(x,y) \leq R_\mathcal{R}(x,y) \leq b \cdot d_E(x,y)$$

for all $x, y \in (e_{ij}^w)_1$. For $u \in \mathcal{F}_\mathcal{R}$ with $u(x) = 1$ and $u(y) = 0$ we have

$$\mathcal{E}_\mathcal{R}(u,u) \geq \frac{1}{\gamma_{|w|+1}} \mathcal{D}_{e_{ij}^w}(u,u),$$

since the expression on the right-hand side is just one part of the whole resistance form. We also obtain

$$\mathcal{D}_{e_{ij}^w}(u,u) \geq \frac{\operatorname{diam}(e_{ij}^w, d_E)}{d_E(x,y)},$$

because the last term is the inverse of the resistance metric associated with $\mathcal{D}_{e_{ij}^w}$. We, therefore, get for the resistance metric

$$R_\mathcal{R}(x,y) \leq \frac{\gamma_{|w|+1}}{\operatorname{diam}(e_{ij}^w, d_E)} d_E(x,y).$$

Now, for the other estimate we assume without loss of generality that x is closer to the endpoint of e_{ij}^w in $(e_{ij}^w)_1$ than y. Then define u as follows. We choose $u(x) = 0$, $u(y) = 1$ and the linear interpolation between them. Also continue u with 0 from x to the endpoint of e_{ij}^w and with 1 from y to the middle point. On the second part of e_{ij}^w we copy the behavior of u by reflecting on the middle point. We illustrate this construction of u which is mapped to $[0,1]$ in Figure 2.5. This is a function in $\mathcal{F}_\mathcal{R}$ and its energy is calculated as

$$\mathcal{E}_\mathcal{R}(u,u) = 2 \cdot \frac{1}{\gamma_{|w|+1}} \frac{\operatorname{diam}(e_{ij}^w, d_E)}{d_E(x,y)},$$

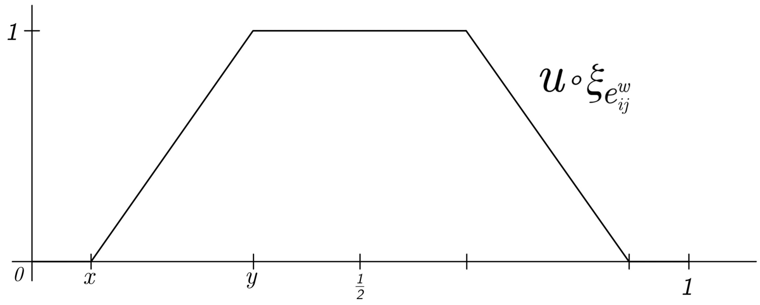

Figure 2.5: Construction of u.

leading to the desired lower bound. The same estimates hold for $(e_{ij}^w)_2$. We thus know that

$$\begin{aligned}\dim_{H,R_{\mathcal{R}}}(e_{ij}^w) &= \max\{\dim_{H,R_{\mathcal{R}}}(e_{ij}^w)_1, \dim_{H,R_{\mathcal{R}}}(e_{ij}^w)_2\} \\ &= \max\{\dim_{H,d_E}(e_{ij}^w)_1, \dim_{H,d_E}(e_{ij}^w)_2\} \\ &= 1.\end{aligned}$$

The result follows by σ-stability. □

Lemma 2.4. *Let $\mathcal{R} = \{(r_i, \rho_i)\}_{i\geq 1}$ be a sequence of matching pairs that fulfills Condition 2.1. Then there exists a constant $c_0 > 0$, such that*

$$\operatorname{diam}(\Sigma_w, R_{\mathcal{R}}) \leq c_0 \cdot r^m$$

for all $w \in \mathcal{A}^m$, $m \in \mathbb{N}$.

Proof. Let $w \in A^m$. Then from [5, Lemma 7.19] we know that

$$|u(x) - u(y)|^2 \leq 16\delta_m \mathcal{E}_{\mathcal{R}}(u, u)$$

for all $u \in \mathcal{F}_{\mathcal{R}}$ and $x, y \in G_w(K) = K_w$. This means

$$\operatorname{diam}(K_w, R_{\mathcal{R}}) \leq 16\delta_m.$$

Since $\Sigma_w \subset K_w$ we get with Lemma 2.2 that

$$\operatorname{diam}(\Sigma_w, R_{\mathcal{R}}) \leq 16\delta_m \leq 16\kappa_2 r^m.$$

□

The next lemma shows how far the m-cells of Σ are apart from each other. We need the distance of a point $x \in K$ to a set $A \subset K$, which is defined by

$$R_{\mathcal{R}}(x, A) := \inf\{R_{\mathcal{R}}(x, y) \mid y \in A\}.$$

Lemma 2.5. *Let $\mathcal{R} = \{(r_i, \rho_i)\}_{i\geq 1}$ be a sequence of matching pairs that fulfills Condition 2.1. Then there is a constant $c_0 > 0$, such that*

$$\#\{w \in \mathcal{A}^m \mid R_{\mathcal{R}}(x, \Sigma_w) \leq c_0 r^m\} \leq 4$$

for all $x \in \Sigma$ and $m \in \mathbb{N}$.

Proof. For a fixed $u \in \mathcal{F}_{\mathcal{R}}$ with $u(x) = 0$ and $u(y) = 1$ we have

$$R_{\mathcal{R}}(x, y) \geq \frac{1}{\mathcal{E}_{\mathcal{R}}(u, u)}.$$

We are looking for a u, such that this estimate is good enough. Let $w \in \mathcal{A}^m$, $y \in \Sigma_w$ and $x \in \Sigma \backslash \Sigma_w$. Define $\tilde{u}_m$ on V_m as

$$\begin{aligned} &\tilde{u}_m(l) = 1, \text{ for all } l \in G_w(V_0) \text{ and any } l \in V_m \text{ adjoining } G_w(V_0),\\ &\tilde{u}_m(l) = 0, \text{ otherwise.} \end{aligned}$$

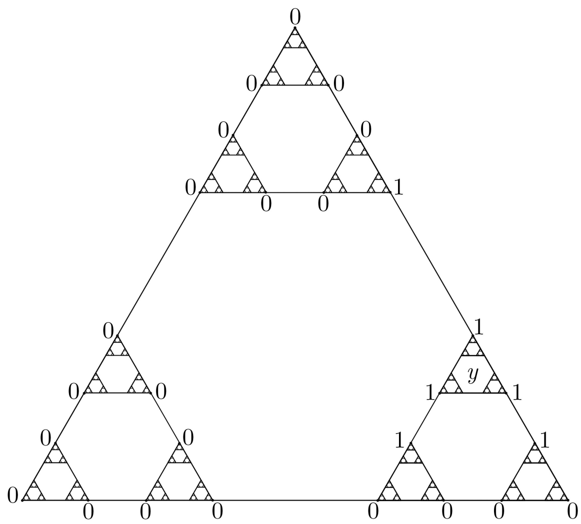

Figure 2.6: Construction of $\tilde{u}_m$ with $m = 2$.

In Figure 2.6 the construction of this function is illustrated, where y lies anywhere in the 2-cell which is marked with "y". Then define $u_m \in \mathcal{F}_{\mathcal{R}}$ as the harmonic extension of $\tilde{u}_m$. The extension u_m is constant 1 on K_w (and hence, on Σ_w) and constant 0 in all but at most three m-cells differing from K_w. For x in these m-cells, where u_m is constant 0, we can use this function to get an estimate of $R_{\mathcal{R}}(x, y)$ for all $y \in \Sigma_w$ by

$$\mathcal{E}_{\mathcal{R}}(u_m, u_m) \leq 3 \cdot 2 \frac{1}{r_1 \cdots r_m} = 6 \frac{1}{\delta_m}.$$

By the estimates of δ_m from Lemma 2.2 we get

$$\mathcal{E}_{\mathcal{R}}(u_m, u_m) \leq \frac{6}{\kappa_1} r^{-m}$$

and thus

$$R_{\mathcal{R}}(x,y) \geq \frac{\kappa_1}{6} r^m.$$

For fixed x there are at most four m-cells for which this construction does not work. We, therefore, have the desired result. □

We can now put these lemmata together to get a result about the Hausdorff dimension of K with respect to the resistance metric.

Theorem 2.6. *Let $\mathcal{R} = \{(r_i, \rho_i)\}_{i\geq 1}$ be a sequence of matching pairs that fulfills Condition 2.1. Then*

$$\dim_{H,R_{\mathcal{R}}}(K) = \max\left\{1, \frac{\ln 3}{-\ln r}\right\}.$$

Proof. By Lemmata 2.4 and 2.5 it follows from [40, Theorem 2.4, i.p. Corollary 1.3] that

$$\dim_{H,R_{\mathcal{R}}}(\Sigma) = \frac{\ln 3}{-\ln r}.$$

From Lemma 2.3 we know that

$$\dim_{H,R_{\mathcal{R}}}(J) = 1$$

and thus the result follows by

$$\dim_{H,R_{\mathcal{R}}}(K) = \max\{\dim_{H,R_{\mathcal{R}}}(J), \dim_{H,R_{\mathcal{R}}}(\Sigma)\}.$$

□

Remark. For $r = \frac{3}{5}$ this is the same value as for the Sierpiński gasket with the usual self-similar resistance form; see [39, Example 4.4].

2.4 Measures and operators

Until now we have just resistance forms. To get Dirichlet forms and thus operators on the stretched Sierpiński gasket we need to introduce measures. These measures have to be locally finite Borel measures with full support; see Appendix B.3 or [45, Chapter 9]. Since we are working on a compact set we are looking for finite measures.

We would like to describe the measures as the sum of two parts. These parts represent the fractal and the line part in accordance with the geometric appearance of the stretched Sierpiński gasket. It is clear how the measure on the fractal part should look like. We use the normalized Hausdorff measure on K, which distributes mass equally to the cells:

$$\mu_\Sigma(K_w) := \mu_\Sigma(\Sigma_w) := \left(\frac{1}{3}\right)^{|w|}.$$

This is a finite measure but it does not have full support. It is only supported on a proper subset of K, namely Σ, which is the attractor of the similitudes G_1, G_2, G_3 alone. The measure μ_Σ is too rough to measure the one-dimensional lines. Therefore, we need a second part on these lines. We would like the line part of the measure to fulfill the same symmetries as K. We start by assigning some value $a > 0$ to each of the edges e_{12}, e_{23} and e_{31}.

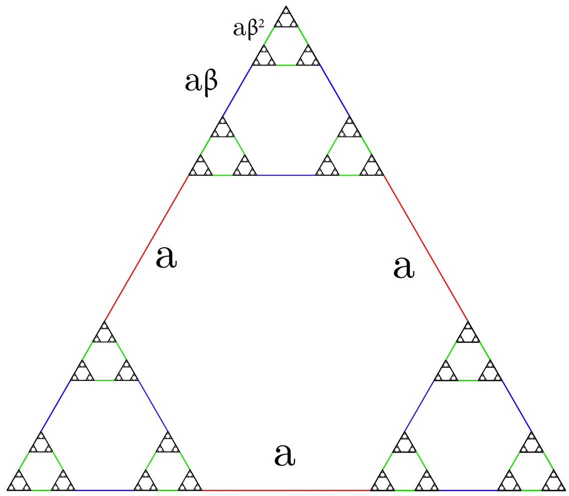

Figure 2.7: Line part of the measure.

Now, each time the lines get a level deeper the measure gets scaled by some value $\beta > 0$. In general we define

$$\mu_I|_{e_{ij}^w} := a\beta^{|w|} \frac{\lambda^1}{\lambda^1(e_{ij}^w)}.$$

This means it behaves like the one-dimensional Lebesgue measure on e_{ij}^w but it is normalized and then scaled by $a\beta^{|w|}$. Hence, it does not depend on the one-dimensional Lebesgue measure of e_{ij}^w. The construction is illustrated in Figure 2.7. We would like μ_I to be normalized and finite. We obtain

$$\begin{aligned} \mu_I(J) &= 3a \sum_{k=0}^{\infty} (3\beta)^k \\ &= \frac{a}{\frac{1}{3} - \beta} \overset{!}{=} 1 \end{aligned}$$

and therefore, for μ_I to be finite we have to choose

$$\beta \in \left(0, \frac{1}{3}\right)$$

and to be normalized

$$a = \frac{1}{3} - \beta.$$

This measure fulfills the requirements since the closure of the union of all one-dimensional lines is K by [5, Proposition 3.3] and $\bigcup_{m \geq 0} V_m \backslash V_0 \subset J$. Thus, we can use μ_I to get Dirichlet forms. The fractal part μ_Σ cannot be used alone, but we can use any convex combination of these measures.

Definition 2.7. Let $\beta \in (0, \frac{1}{3})$ and μ_I and μ_Σ as before. Then we define for $\eta \in (0,1]$ the measures μ_η as

$$\mu_\eta := \eta\mu_I + (1-\eta)\mu_\Sigma.$$

Note that $\eta = 1$ is allowed. In this case we do not have a fractal part. This will be noticeable in the spectral asymptotics. How do these measures scale for smaller cells? This is addressed in the following propositions.

Proposition 2.8. *Let $w \in \mathcal{A}_0^*$. Then*

$$\mu_\Sigma(K_w) = \frac{1}{3^{|w|}} \quad \text{and} \quad \mu_I(K_w) = \beta^{|w|}.$$

Proof. This is immediate since

$$\mu_\Sigma \circ G_w = 3^{-|w|} \quad \text{and} \quad \mu_I \circ G_w = \beta^{|w|}\mu_I$$

by the definition of μ_Σ and μ_I. □

By this we get an estimate for μ_η with $\eta \in (0,1)$ which will be useful later on. For $w \in \mathcal{A}_0^*$ we have

$$\beta^{|w|} \leq \mu_\eta(K_w) \leq \left(\frac{1}{3}\right)^{|w|}.$$

However, we can improve the lower bound to order $(\frac{1}{3})^{|w|}$ if we use the scaling of the fractal part of the measure.

Proposition 2.9. *Let $w \in \mathcal{A}_0^*$ and $\eta \in (0,1)$. Then*

$$(1-\eta)\left(\frac{1}{3}\right)^{|w|} \leq \mu_\eta(K_w) \leq \left(\frac{1}{3}\right)^{|w|}.$$

With these measures we can define Dirichlet forms and therefore, operators on $L^2(K,\mu_\eta)$. From Lemma B.12 we know that $\mathcal{D}_\mathcal{R} := \overline{\mathcal{F}_\mathcal{R} \cap C_0(K)}^{\mathcal{E}_{\mathcal{R},1}^{\frac{1}{2}}} = \mathcal{F}_\mathcal{R}$ since $(K, R_\mathcal{R})$ is compact.

Lemma 2.10. *$(\mathcal{E}_\mathcal{R}, \mathcal{D}_\mathcal{R})$ is a regular Dirichlet form on $L^2(K,\mu_\eta)$.*

Proof. From Theorem B.15 and [5, Theorem 7.16] it follows that $(\mathcal{E}_\mathcal{R}, \mathcal{D}_\mathcal{R})$ is a regular Dirichlet form. □

By introducing Dirichlet boundary conditions on V_0 we get another Dirichlet form with $\mathcal{D}_\mathcal{R}^0 := \{u \mid u \in \mathcal{D}_\mathcal{R},\ u|_{V_0} \equiv 0\}$.

Lemma 2.11. *$(\mathcal{E}_\mathcal{R}|_{\mathcal{D}_\mathcal{R}^0 \times \mathcal{D}_\mathcal{R}^0}, \mathcal{D}_\mathcal{R}^0)$ is a regular Dirichlet form on $L^2(K\backslash V_0, \mu_\eta|_{K\backslash V_0})$.*

Proof. This follows from [45, Theorem 10.3]. □

We denote the associated self-adjoint operators with dense domains by $-\Delta_N^{\mu_\eta,\mathcal{R}}$ resp. $-\Delta_D^{\mu_\eta,\mathcal{R}}$. We refer to [13, Chapter 10] for the construction of the associated operators.

Lemma 2.12. *$-\Delta_N^{\mu_\eta,\mathcal{R}}$ and $-\Delta_D^{\mu_\eta,\mathcal{R}}$ have discrete non-negative spectrum.*

Proof. From [5, Theorem 7.16] we know that $(K, R_{\mathcal{R}})$ is compact and therefore, it follows from [45, Lemma 9.7] that the inclusion map $\iota : \mathcal{D}_{\mathcal{R}} \hookrightarrow C(K)$ with the norms $\mathcal{E}_{\mathcal{R},1}^{\frac{1}{2}}$ resp. $||\cdot||_\infty$ is a compact operator. Since the inclusion map from $C(K)$ to $L^2(K, \mu_\eta)$ is continuous, the inclusion from $\mathcal{D}_{\mathcal{R}}$ to $L^2(K, \mu_\eta)$ is a compact operator and thus, by [13, Theorem 5, Chapter 10.1] the spectrum of $-\Delta_N^{\mu_\eta,\mathcal{R}}$ is discrete and by $\mathcal{E}_{\mathcal{R}}(u, u) \geq 0$ for all $u \in \mathcal{D}_{\mathcal{R}}$ non-negative. Since $\mathcal{D}_{\mathcal{R}}^0 \subset \mathcal{D}_{\mathcal{R}}$ the same follows for $-\Delta_D^{\mu_\eta,\mathcal{R}}$ by [13, Theorem 4, Chapter 10.2]. □

2.5 Spectral asymptotics

Due to Lemma 2.12 we can write the eigenvalues in non-decreasing order and study the eigenvalue counting functions. Denote by $\lambda_k^{N,\mu_\eta,\mathcal{R}}$ the k-th eigenvalue of $-\Delta_N^{\mu_\eta,\mathcal{R}}$ and $\lambda_k^{D,\mu_\eta,\mathcal{R}}$ that of $-\Delta_D^{\mu_\eta,\mathcal{R}}$ for $k \geq 1$. Now, define

$$\begin{aligned} N_N^{\mu_\eta,\mathcal{R}}(x) &:= \#\{k \geq 1 \mid \lambda_k^{N,\mu_\eta,\mathcal{R}} \leq x\}, \\ N_D^{\mu_\eta,\mathcal{R}}(x) &:= \#\{k \geq 1 \mid \lambda_k^{D,\mu_\eta,\mathcal{R}} \leq x\}. \end{aligned}$$

The homomorphism theorem applied to $\mathcal{D}_{\mathcal{R}} \ni u \mapsto u|_{V_0} \in \ell(V_0)$ yields a linear isomorphism from $\mathcal{D}_{\mathcal{R}}/\mathcal{D}_{\mathcal{R}}^0$ to $\ell(V_0)$. We thus have $\dim \mathcal{D}_{\mathcal{R}}/\mathcal{D}_{\mathcal{R}}^0 = 3$. From this we immediately get by [13, Theorem 5, Chapter 10.2]

$$N_D^{\mu_\eta,\mathcal{R}}(x) \leq N_N^{\mu_\eta,\mathcal{R}}(x) \leq N_D^{\mu_\eta,\mathcal{R}}(x) + 3, \ \forall x \geq 0.$$

Therefore, both the Dirichlet and Neumann eigenvalue counting functions, have the same asymptotic growing.

2.5.1 Results

We would like to study the asymptotic behavior of the eigenvalue counting functions as $x \to \infty$. In the next theorem we obtain the order of the leading term.

Theorem 2.13. *Let $\mathcal{R}$ be a sequence of matching pairs that fulfills Condition 2.1 and let $\eta \in (0, 1]$. Then there exist constants $0 < C_1, C_2 < \infty$ and $x_0 \geq 0$ such that for all $x \geq x_0$*

$$C_1 x^{\frac{1}{2} d_S^{\mu_\eta,\mathcal{R}}} \leq N_D^{\mu_\eta,\mathcal{R}}(x) \leq N_N^{\mu_\eta,\mathcal{R}}(x) \leq C_2 x^{\frac{1}{2} d_S^{\mu_\eta,\mathcal{R}}}$$

with

$$d_S^{\mu_\eta,\mathcal{R}} = \begin{cases} \max\{1, \frac{\ln 9}{\ln 3 - \ln r}\}, & \text{for } \eta \in (0,1), \\ \max\{1, \frac{\ln 9}{-\ln(\beta r)}\}, & \text{for } \eta = 1 \text{ with } \beta \neq \frac{1}{9r}. \end{cases}$$

We see that if r is too small, namely $r \leq \frac{1}{3}$, this leading order is always 1 and thus dominated by the one-dimensional lines. If we, on the other hand, have $\eta \in (0,1)$ and $r = \frac{3}{5}$ this is the same value as for the self-similar Sierpiński gasket, which is $\frac{\ln 9}{\ln 5}$; see [38, Example 2]. Another observation is that the measure scaling parameter β of the line part does not appear in the leading order if our measure includes the fractal part.

Therefore, the choice of the sequence of matching pairs as well as the measure has a big influence on the spectral asymptotics on the stretched Sierpiński gasket.

Let us denote $d_S^{\mathcal{R}}(K) := d_S^{\mu_{0.5},\mathcal{R}}$, which is the biggest possible value for a fixed sequence $\mathcal{R}$. We call this value the *spectral dimension* of K with respect to $\mathcal{R}$.

Remark 2.14. From Theorem 2.6 we know that $\dim_{H,R_{\mathcal{R}}}(K) = \max\{1, \frac{\ln 3}{-\ln r}\}$. Together with Theorem 2.13 we obtain

$$d_S^{\mathcal{R}}(K) = \frac{2\dim_{H,R_{\mathcal{R}}}(K)}{\dim_{H,R_{\mathcal{R}}}(K)+1}.$$

This relation was shown to hold for p.c.f. self-similar sets in [38, Theorem A.2] and is now also valid for a non-self-similar set.

2.5.2 Proof of Theorem 2.13

We will give the proof for $\mu := \mu_\eta$ with $\eta \in (0,1)$ and only at the end show what is different in the case of a measure μ_I without a fractal part. The main technique of the proof is the Dirichlet-Neumann bracketing as in [35], where it was applied to self-similar sets. Here we apply it to a non-self-similar set, which is possible since the stretched Sierpiński gasket is still finitely ramified. We split the proof into the upper and lower estimate.

U: Upper estimate

The upper bound is obtained by successively adding new Neumann boundary conditions at the points $V_m \backslash V_0$, thus making the domain bigger and therefore, increasing the eigenvalue counting function. This is done by defining the domains

$$\begin{aligned}
\mathcal{D}_{\mathcal{R},K_m} : &= \{u \mid u \in L^2(K_m, \mu|_{K_m}), \exists f \in \mathcal{D}_{\mathcal{R}} : f|_{K_m} = u\} \\
&= \{u \mid u \in L^2(K,\mu), u|_{K_m^c} = 0, \exists f \in \mathcal{D}_{\mathcal{R}} : f|_{K_m} = u\}, \\
\mathcal{D}_{\mathcal{R},J_m} : &= \{u \mid u \in L^2(J_m, \mu|_{J_m}), \exists f \in \mathcal{D}_{\mathcal{R}} : f|_{J_m} = u\} \\
&= \{u \mid u \in L^2(K,\mu), u|_{J_m^c} = 0, \exists f \in \mathcal{D}_{\mathcal{R}} : f|_{J_m} = u\}.
\end{aligned}$$

Considering $\mathcal{D}_{\mathcal{R},J_m}$ we see that if we take f to be harmonic on all of the m-cells, we get

$$\mathcal{D}_{\mathcal{R},J_m} = \bigoplus_{\substack{w\in\mathcal{A}^n, n<m \\ (i,j)\in B}} H^1(e_{ij}^w).$$

It is obvious that $\mathcal{D}_{\mathcal{R},K_m}$ is orthogonal to $\mathcal{D}_{\mathcal{R},J_m}$ in $L^2(K,\mu)$ and

$$\mathcal{D}_{\mathcal{R}} \subset \mathcal{D}_{\mathcal{R},K_m} \oplus \mathcal{D}_{\mathcal{R},J_m}.$$

On this bigger domain we define the form $\tilde{\mathcal{E}}_{\mathcal{R}}$ for $f = g + h$, with $g \in \mathcal{D}_{\mathcal{R},K_m}, h \in \mathcal{D}_{\mathcal{R},J_m}$ by

$$\tilde{\mathcal{E}}_{\mathcal{R}}(f,f) := \mathcal{E}_{\mathcal{R}}^{\Sigma}(g,g) + \sum_{k=m+1}^{\infty} \frac{1}{\gamma_k} \mathcal{D}_k^I(g,g) + \sum_{k=1}^{m} \frac{1}{\gamma_k} \mathcal{D}_k^I(h,h)$$

and

$$\begin{aligned} \mathcal{E}_{\mathcal{R},K_m}(g,g) &= \mathcal{E}_{\mathcal{R}}^{\Sigma}(g,g) + \sum_{k=m+1}^{\infty} \frac{1}{\gamma_k} \mathcal{D}_k^I(g,g), \\ \mathcal{E}_{\mathcal{R},J_m}(h,h) &= \sum_{k=1}^{m} \frac{1}{\gamma_k} \mathcal{D}_k^I(h,h). \end{aligned}$$

Lemma 2.15. *$(\tilde{\mathcal{E}}_{\mathcal{R}}, \mathcal{D}_{\mathcal{R},K_m} \oplus \mathcal{D}_{\mathcal{R},J_m})$, $(\mathcal{E}_{\mathcal{R},K_m}, \mathcal{D}_{\mathcal{R},K_m})$ and $(\mathcal{E}_{\mathcal{R},J_m}, \mathcal{D}_{\mathcal{R},J_m})$ are regular Dirichlet forms with discrete non-negative spectrum on $L^2(K_m \cup J_m, \mu)$, $L^2(K_m, \mu|_{K_m})$ and $L^2(J_m, \mu|_{J_m})$ respectively, where $K_m \cup J_m$ is equipped with the direct sum topology and $\tilde{\mathcal{E}}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R},K_m} \oplus \mathcal{E}_{\mathcal{R},J_m}$.*

Proof. $(\mathcal{E}_{\mathcal{R},J_m}, \mathcal{D}_{\mathcal{R},J_m})$ is just the sum of scaled Dirichlet energies on one-dimensional edges, and hence it is a regular Dirichlet form on $L^2(J_m, \mu|_{J_m})$ with discrete non-negative spectrum. We note that

$$(\mathcal{E}_{\mathcal{R},K_m}, \mathcal{D}_{\mathcal{R},K_m}) = \bigoplus_{w \in \mathcal{A}^m} \left(\delta_m^{-1} \mathcal{E}_{\mathcal{R}^{(m)}}\left((\cdot) \circ G_w, (\cdot) \circ G_w\right), \mathcal{D}_{\mathcal{R}^{(m)},w}\right)$$

where

$$\mathcal{D}_{\mathcal{R}^{(m)},w} := \{u \in C(K_w) \mid u \circ G_w \in \mathcal{D}_{\mathcal{R}^{(m)}}\}.$$

Therefore, $(\mathcal{E}_{\mathcal{R},K_m}, \mathcal{D}_{\mathcal{R},K_m})$ is itself a Dirichlet form with non-negative spectrum as an orthogonal sum of such Dirichlet forms. The results for $\tilde{\mathcal{E}}_{\mathcal{R}}$ follow immediately. △

We denote by $N(\mathcal{E}, \mathcal{D}, \mu, x)$ the eigenvalue counting function of a regular Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(X, \mu)$ evaluated at $x \geq 0$, which is the same as the one for the associated self-adjoint operator. This can easily be seen by the definition of the associated operator in [13, Theorem 2, Chapter 10.1]. Since μ is always the same measure, or its restriction to the different parts, we omit it in the notation. For the eigenvalue counting functions of the mentioned forms this means

$$\begin{aligned} N_N^{\mu,\mathcal{R}}(x) &\leq N(\tilde{\mathcal{E}}_{\mathcal{R}}, \mathcal{D}_{\mathcal{R},K_m} \oplus \mathcal{D}_{\mathcal{R},J_m}, x) \\ &= N(\mathcal{E}_{\mathcal{R},K_m}, \mathcal{D}_{\mathcal{R},K_m}, x) + N(\mathcal{E}_{\mathcal{R},J_m}, \mathcal{D}_{\mathcal{R},J_m}, x), \ \forall x \geq 0. \end{aligned}$$

The first line follows from the fact that the domain gets bigger and thus the eigenvalue counting function gets bigger; see [13, Theorem 4, Chapter 10.2]. The second line is due to the orthogonality of the domains and therefore, eigenfunctions on the common domain split into the sum of eigenfunctions on the smaller domains.

The introduction of the Neumann boundary conditions on $V_m \backslash V_0$ leads to the decoupling of the m-cells and the edges joining them. This means the calculations can be done separately.

U.1: Fractal part $(\mathcal{E}_{\mathcal{R},K_m}, \mathcal{D}_{\mathcal{R},K_m})$

Define a set of measures on K as follows:

$$\mu^w := \mu(K_w)^{-1} \mu \circ G_w.$$

This is a measure on all of K but it only reflects the features of μ on K_w. We notice that

$$\mu^w(K) = \mu(K_w)^{-1} \mu(K_w) = 1, \ \forall w$$

as well as

$$\int_K u \circ G_w d\mu^w = \mu(K_w)^{-1} \int_{K_w} u d\mu.$$

In the following we use the so-called *uniform Poincaré inequality*; see [35, Definition 4.2]. For measures ν on K we define $\overline{u}^\nu := \int_K u d\nu$.

Proposition 2.16. $(\mathcal{E}_\mathcal{R}, \mathcal{D}_\mathcal{R})$ *satisfies a uniform Poincaré inequality, i.e., for all* $u \in \mathcal{D}_\mathcal{R}$ *and* $w \in \mathcal{A}_0^*$

$$\mathcal{E}_\mathcal{R}(u,u) \geq C_{PI} \int_K |u - \bar{u}^{\mu^w}|^2 d\mu^w$$

for any $C_{PI} \leq \frac{1}{M}$*, where* $0 < M = \sup_{p,q \in K} R_\mathcal{R}(p,q) < \infty$.

For our purpose we can choose $C_{PI} = \frac{1}{M}$. Then, since $M \leq 16$ by [5, Lemma 7.19], the constant is independent of w as well as $\mathcal{R}$. Therefore, it holds for all Dirichlet forms $(\mathcal{E}_\mathcal{R}, \mathcal{D}_\mathcal{R})$ and all measures μ^w.

Proof. By the definition of the resistance metric we get

$$M\mathcal{E}_\mathcal{R}(u,u) \geq R_\mathcal{R}(p,q)\mathcal{E}_\mathcal{R}(u,u) \geq |u(p) - u(q)|^2.$$

Integrating twice over K with respect to μ^w leads to

$$\begin{aligned}
\int_K \int_K M\mathcal{E}_\mathcal{R}(u,u) d\mu^w(q) d\mu^w(p) &\geq \int_K \int_K |u(p) - u(q)|^2 d\mu^w(q) d\mu^w(p) \\
&\geq \int_K \left(u(p) - \int_K u(q) d\mu^w(q) \right)^2 d\mu^w(p) \\
&= \int_K |u(p) - \bar{u}^{\mu^w}|^2 d\mu^w(p).
\end{aligned}$$

In the second line we used the Jensen's inequality, leading to the desired inequality

$$\mathcal{E}_{\mathcal{R}}(u,u) \geq \frac{1}{M\mu^w(K)^2}\int_K |u-\bar{u}^{\mu^w}|^2 d\mu^w = \frac{1}{M}\int_K |u-\bar{u}^{\mu^w}|^2 d\mu^w. \qquad \triangle$$

Since there are 3^m independent cells in $\mathcal{D}_{\mathcal{R},K_m}$ the 3^m first eigenvalues are all 0, where the eigenfunctions are the functions that are constant on each m-cell. We are interested in the first non-zero eigenvalue $\lambda^m_{3^m+1}$.

Let $u \in \mathcal{D}_{\mathcal{R},K_m}$ be a normalized eigenfunction to the eigenvalue $\lambda^m_{3^m+1}$. Then u is orthogonal to every v that is constant on the m-cells, as this is a linear combination of eigenfunctions to lower eigenvalues. We consider this eigenvalue

$$\begin{aligned}
\lambda^m_{3^m+1} &= \mathcal{E}_{\mathcal{R},K_m}(u,u) \\
&= \mathcal{E}^{\Sigma}_{\mathcal{R}}(u,u) + \sum_{k=m+1}^{\infty}\frac{1}{\gamma_k}\mathcal{D}^I_k(u,u) \\
&= \frac{1}{\delta_m}\sum_{w\in\mathcal{A}^m}\mathcal{E}_{\mathcal{R}^{(m)}}(u\circ G_w, u\circ G_w) \\
&\overset{PI}{\geq} \kappa_2^{-1} r^{-m}\sum_{w\in\mathcal{A}^m} C_{PI}\int_K |u\circ G_w - \overline{u\circ G_w}^{\mu^w}|^2 d\mu^w.
\end{aligned}$$

In the third line we applied the rescaling from Lemma 2.1 and in the following line the Poincaré inequality from Proposition 2.16 to the function $u\circ G_w$ and the Dirichlet form $(\mathcal{E}_{\mathcal{R}^{(m)}}, \mathcal{D}_{\mathcal{R}^{(m)}})$ with measure μ^w. Note that

$$\begin{aligned}
\overline{u\circ G_w}^{\mu^w} &= \int_K u\circ G_w d\mu^w \\
&= \mu(K_w)\int_K u\circ \mathbb{1}_{K_w} d\mu \\
&= 0,
\end{aligned}$$

since u is orthogonal to functions that are constant on m-cells. We obtain

$$\begin{aligned}
\lambda^m_{3^m+1} &\geq \kappa_2^{-1} r^{-m}\sum_{w\in\mathcal{A}^m} C_{PI}\frac{1}{\mu(K_w)}\int_{K_w} u^2 d\mu \\
&\geq r^{-m}\frac{C_{PI}}{\kappa_2 \max\limits_{w\in\mathcal{A}^m}\mu(K_w)}\int_{K_m} u^2 d\mu \\
&\geq \frac{r^{-m}}{3^{-m}}\cdot\frac{C_{PI}}{\kappa_2} = C_u\left(\frac{3}{r}\right)^m.
\end{aligned}$$

We used the upper bound of Proposition 2.9 for the measure. The constant C_u is independent of m. We have $\lambda^m_{3^m+1} \geq C_u(3/r)^m$ that means for $x < C_u(3/r)^m$ we get

$$N(\mathcal{E}_{\mathcal{R},K_m}, \mathcal{D}_{\mathcal{R},K_m}, x) \leq 3^m.$$

For $x \geq C_u$ take $m \in \mathbb{N}$ such that $C_u(3/r)^{m-1} \leq x < C_u(3/r)^m$. Then

$$\begin{aligned} N(\mathcal{E}_{\mathcal{R},K_m}, \mathcal{D}_{\mathcal{R},K_m}, x) \leq 3^m = 3 \cdot 3^{m-1} = 3 \left(\left(\frac{3}{r} \right)^{\frac{\ln(3)}{\ln(3/r)}} \right)^{m-1} \\ = 3 \left(\left(\frac{3}{r} \right)^{m-1} \right)^{\frac{\ln(3)}{\ln(3/r)}} \leq 3 \left(\frac{x}{C_u} \right)^{\frac{\ln(3)}{\ln(3/r)}} \\ = \underbrace{3 C_u^{-\frac{\ln(3)}{\ln(3/r)}}}_{C_2' :=} x^{\frac{\ln(3)}{\ln(3/r)}}. \end{aligned}$$

This is an asymptotic growing that is independent of m.

U.2: Line part $(\mathcal{E}_{\mathcal{R},J_m}, \mathcal{D}_{\mathcal{R},J_m})$

Due to the decoupling through the Neumann boundary conditions the domain and form split into

$$\begin{aligned} \mathcal{E}_{\mathcal{R},J_m} &= \bigoplus_{\substack{w \in \mathcal{A}^n, n < m \\ (i,j) \in B}} \frac{1}{\gamma_{|w|+1}} \int_0^1 \left(\frac{d(\cdot \circ \xi_{e_{ij}^w})}{dx} \right)^2 dx, \\ \mathcal{D}_{\mathcal{R},J_m} &= \bigoplus_{\substack{w \in \mathcal{A}^n, n < m \\ (i,j) \in B}} H^1(e_{ij}^w). \end{aligned}$$

Then it holds for the eigenvalue counting function that

$$N(\mathcal{E}_{\mathcal{R},J_m}, \mathcal{D}_{\mathcal{R},J_m}, x) = \sum_{\substack{w \in \mathcal{A}^n, n < m \\ (i,j) \in B}} N \left(\frac{1}{\gamma_{|w|+1}} \int_0^1 \left(\frac{d(\cdot \circ \xi_{e_{ij}^w})}{dx} \right)^2 dx, H^1(e_{ij}^w), x \right).$$

For any $u \in L^2(e_{ij}^w, \mu|_{e_{ij}^w})$ we have $\int_{e_{ij}^w} u^2 d\mu = a\beta^{|w|}\eta \int_0^1 (u \circ \xi_{e_{ij}^w})^2 dx$. Therefore, there is a one-to-one correspondence of the eigenvalues between the standard Neumann Laplacian on $(0,1)$ and the restriction of the energy to one edge, which yields

$$N \left(\frac{1}{\gamma_{|w|+1}} \int_0^1 \left(\frac{d(\cdot \circ \xi_{e_{ij}^w})}{dx} \right)^2 dx, H^1(e_{ij}^w), x \right) = N(-\Delta_N|_{(0,1)}, \eta a \beta^{|w|} \gamma_{|w|+1} x).$$

It is well known that the Neumann Laplacian on $(0,1)$ has the eigenvalues

$$(\pi k)^2, \text{ for } k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$$

and therefore, the eigenvalue counting function is given by

$$N(-\Delta_N|_{(0,1)}, x) = \left\lfloor \frac{\sqrt{x}}{\pi} \right\rfloor + 1 := \max \left\{ k \in \mathbb{Z} \mid k \leq \frac{\sqrt{x}}{\pi} \right\} + 1,$$

which leads to

$$N(-\Delta_N|_{(0,1)}, x) \leq \frac{1}{\pi}\sqrt{x} + 1$$

for all $x \geq 0$. We obtain

$$\begin{aligned}
N(\mathcal{E}_{\mathcal{R},J_m}, \mathcal{D}_{\mathcal{R},J_m}, x) &\leq \sum_{k=1}^{m}\sum_{j=1}^{3^k} N(-\Delta_N|_{(0,1)}, \eta a \beta^{k-1}\gamma_k x) \\
&\leq \sum_{k=1}^{m}\sum_{j=1}^{3^k}\left(\frac{1}{\pi}\sqrt{\eta a \beta^{k-1}\gamma_k x} + 1\right) \\
&\leq \sum_{k=1}^{m}\left(\frac{3^k}{\pi}\sqrt{a\beta^{k-1}\gamma_k x} + 3^k\right) \\
&\leq \frac{3}{2}(3^m - 1) + \frac{\sqrt{a}}{\pi}\sqrt{x}\sum_{k=1}^{m}\sqrt{9^k\beta^{k-1}\kappa_2 r^{k-1}} \\
&\leq \frac{3}{2}3^m + \frac{3\sqrt{a\kappa_2}}{\pi}\sqrt{x}\sum_{k=0}^{m-1}\sqrt{9\beta r}^{\,k}.
\end{aligned}$$

From here on we have to distinguish a few cases:

(1) $r > \frac{1}{3}$ and $\frac{1}{9r} \leq \beta$.

(2) ($r > \frac{1}{3}$ and $0 < \beta < \frac{1}{9r}$) or $r \leq \frac{1}{3}$.

We start with the first case and additionally assume $\frac{1}{9r} < \beta$. Then

$$N(\mathcal{E}_{\mathcal{R},J_m}, \mathcal{D}_{\mathcal{R},J_m}, x) \leq \frac{3}{2}3^m + \frac{3\sqrt{a\kappa_2}}{\pi(\sqrt{9\beta r} - 1)}\sqrt{9\beta r}^{\,m}\sqrt{x}.$$

For the fractal part we have used the m for which $C_u(3/r)^{m-1} \leq x < C_u(3/r)^m$. Therefore,

$$\begin{aligned}
N(\mathcal{E}_{\mathcal{R},J_m}, \mathcal{D}_{\mathcal{R},J_m}, x) &\leq \frac{3}{2}3^m + \frac{3\sqrt{a\kappa_2}}{\pi(\sqrt{9\beta r} - 1)}\sqrt{9\beta r}^{\,m}\sqrt{C_u\left(\frac{3}{r}\right)^m} \\
&= \frac{3}{2}3^m + \frac{3\sqrt{a\kappa_2 C_u}}{\pi(\sqrt{9\beta r} - 1)}\sqrt{9\beta 3}^{\,m}.
\end{aligned}$$

Since $\beta < \frac{1}{3}$ we get a constant $\widetilde{C}_2''$, such that for x with $C_u(3/r)^{m-1} \leq x < C_u(3/r)^m$ we have

$$N(\mathcal{E}_{\mathcal{R},J_m}, \mathcal{D}_{\mathcal{R},J_m}, x) \leq \widetilde{C}_2'' \cdot 3^m.$$

For $\beta = \frac{1}{9r}$, we can change to $\tilde{\beta} = \beta + \epsilon$ with $\frac{1}{9r} < \tilde{\beta} < \frac{1}{3}$ and get the same results.

With the same calculations as for the fractal part we get the same order for the upper bound with a constant C_2''. For $x \geq C_u$ we have

$$N(\mathcal{E}_{\mathcal{R},J_m}, \mathcal{D}_{\mathcal{R},J_m}, x) \leq C_2'' x^{\frac{\ln(3)}{\ln(3/r)}}.$$

Now let us look at the second case. Here we always have $9\beta r < 1$. We obtain

$$\begin{aligned} N(\mathcal{E}_{\mathcal{R},J_m}, \mathcal{D}_{\mathcal{R},J_m}, x) &\leq \frac{3}{2}3^m + \frac{3\sqrt{a\kappa_2}}{\pi}\sqrt{x}\sum_{k=0}^{m-1}\sqrt{9\beta r}^k \\ &\leq \frac{3}{2}3^m + \frac{3\sqrt{a\kappa_2}}{\pi}\sqrt{x}\sum_{k=0}^{\infty}\sqrt{9\beta r}^k \\ &= \frac{3}{2}3^m + \frac{3\sqrt{a\kappa_2}}{\pi}\frac{1}{1-\sqrt{9\beta r}}\sqrt{x}. \end{aligned}$$

The first part is handled exactly as before to give the same order as for the fractal part and the latter part is of order $x^{\frac{1}{2}}$. This means if $r \geq \frac{1}{3}$ the leading order is $\frac{\ln(3)}{\ln(3/r)}$, while if $r < \frac{1}{3}$ the leading order is $\frac{1}{2}$. This is the upper estimate of the desired result.

L: Lower estimate

The idea here is to successively add new Dirichlet boundary conditions on the points V_m, thus lowering the eigenvalue counting function. We introduce

$$\begin{aligned} \mathcal{D}^0_{\mathcal{R},K_m} &:= \{u \mid u \in \mathcal{D}^0_{\mathcal{R}}, u|_{V_m} \equiv 0\}, \\ \mathcal{D}^0_{\mathcal{R},K_w} &:= \{u \mid u \in \mathcal{D}^0_{\mathcal{R},K_m}, u|_{K_w^c} \equiv 0\},\ w \in \mathcal{A}^m, \\ \mathcal{D}^0_{\mathcal{R},e^w_{ij}} &:= \{u \mid u \in \mathcal{D}^0_{\mathcal{R},K_m}, u|_{(e^w_{ij})^c} \equiv 0\},\ (i,j) \in B, w \in \mathcal{A}^k, k < m. \end{aligned}$$

For the quadratic forms we omit the restriction in the notation; for example, we write $(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R},K_m})$ instead of $(\mathcal{E}_{\mathcal{R}}|_{\mathcal{D}^0_{\mathcal{R},K_m}\times\mathcal{D}^0_{\mathcal{R},K_m}}, \mathcal{D}^0_{\mathcal{R},K_m})$.

Lemma 2.17. *$(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R},K_m})$, $(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R},K_w})$ and $(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R},e^w_{ij}})$ are regular Dirichlet forms on $L^2(K\backslash V_m, \mu|_{K\backslash V_m})$, $L^2(K_w\backslash G_w(V_0), \mu|_{K_w\backslash G_w(V_0)})$ and $L^2(\mathrm{int}_K(e^w_{ij}), \mu|_{\mathrm{int}_K(e^w_{ij})})$ respectively, with discrete non-negative spectrum.*

Proof. The proof works just like the one of Lemma 2.11. △

We also get the following estimate by [13, Theorem 4, Chapter 10.2]:

$$N(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R},K_m}, x) \leq N_D^{\mu,\mathcal{R}}(x),\ \forall x \geq 0.$$

Due to the finite ramification and the condition that the functions in $\mathcal{D}^0_{\mathcal{R},K_m}$ have to be zero on V_m, this domain splits into the domain restricted to the different parts:

$$\mathcal{D}^0_{\mathcal{R},K_m} = \left(\bigoplus_{w\in\mathcal{A}^m}\mathcal{D}^0_{\mathcal{R},K_w}\right) \bigoplus \left(\bigoplus_{\substack{w\in\mathcal{A}^n, n<m \\ (i,j)\in B}}\mathcal{D}^0_{\mathcal{R},e^w_{ij}}\right).$$

That implies for the eigenvalue counting function, for all $x \geq 0$

$$\sum_{w \in \mathcal{A}^m} N(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R},K_w}, x) + \sum_{\substack{w \in \mathcal{A}^n, n<m \\ (i,j) \in B}} N(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R},e^w_{ij}}, x) \leq N_D^{\mu,\mathcal{R}}(x).$$

Again due to the decoupling, the individual eigenvalue counting functions can be calculated separately.

L.1: Fractal part $(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R},K_w})$

We would like to get an upper estimate of the first eigenvalue of $(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R},K_w})$, which is positive due to the Dirichlet boundary conditions. This estimate gives us a lower estimate for $N(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R},K_w}, x)$. The first eigenvalue can be calculated via the following Rayleigh quotient, which can be found in [13, Theorem 1, Chapter 10.2]:

$$\lambda_1^w = \inf_{\substack{u \in \mathcal{D}^0_{\mathcal{R},K_w} \\ u \not\equiv 0}} \frac{\mathcal{E}_{\mathcal{R}}(u,u)}{||u||^2},$$

where $||u|| := \left(\int_K u^2 d\mu\right)^{\frac{1}{2}}$. This means we get upper bounds for all $u \in \mathcal{D}^0_{\mathcal{R},K_w}$ with $u \not\equiv 0$ by

$$\lambda_1^w \leq \frac{\mathcal{E}_{\mathcal{R}}(u,u)}{||u||^2}.$$

The idea is to find a $u \in \mathcal{D}^0_{\mathcal{R},K_w}$ which is “good enough” in the sense that this estimate leads to a good upper bound.

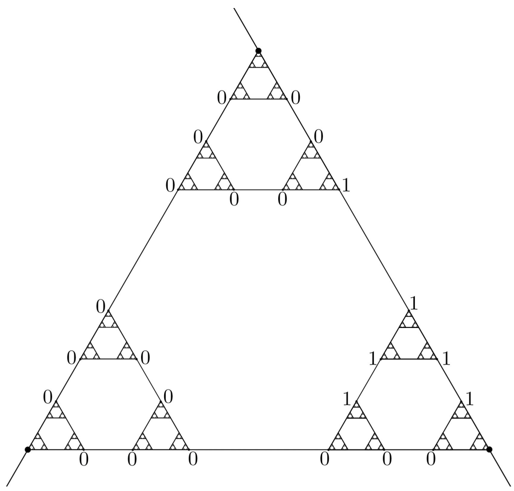

Figure 2.8: Construction of u_m.

In K_w we look for the biggest cell where there are no Dirichlet boundary conditions, which are indicated by black dots in Figure 2.8. This is an $(m+2)$-cell; choose any of those. Set

$$\tilde{u}_m|_{V_{m+2}} := \begin{cases} 1, & \text{on this cell and any adjoined vertices,} \\ 0, & \text{anywhere else.} \end{cases}$$

Then extend $\tilde{u}_m$ harmonically to $u_m \in \mathcal{D}^0_{\mathcal{R},K_w}$. The energy of this function is calculated as

$$\begin{aligned}\mathcal{E}_{\mathcal{R}}(u_m, u_m) &= 6 \cdot \delta_{m+2}^{-1} \\ &\leq \frac{6}{\kappa_1} r^{-(m+2)},\end{aligned}$$

where we applied the estimate of δ_{m+2} from Lemma 2.2.

We need a lower estimate for the L^2-norm of u_m to get an upper estimate of λ_1^w. There is an $(m+2)$-cell $K_{\tilde{w}}$ in K_w where u_m is constant 1. Therefore,

$$\begin{aligned}||u_m||^2 &= \int_{K_w} |u_m|^2 d\mu \\ &\geq \int_{K_{\tilde{w}}} \underbrace{|u_m|^2}_{=1} d\mu \\ &= \mu(K_{\tilde{w}}).\end{aligned}$$

Combining the estimates for the energy and the L^2-norm leads to

$$\lambda_1^w \leq \frac{6}{\kappa_1} \frac{r^{-(m+2)}}{\mu(K_{\tilde{w}})}.$$

If we include the estimates for the measures from Proposition 2.9 we obtain

$$\begin{aligned}\lambda_1^w &\leq \frac{6}{\kappa_1(1-\eta)} \left(\frac{3}{r}\right)^{m+2} \\ &= \underbrace{\frac{6(3r^{-1})^2}{\kappa_1(1-\eta)}}_{C_l:=} \left(\frac{3}{r}\right)^m.\end{aligned}$$

For $x \geq C_l(3/r)$ choose $m \in \mathbb{N}$ such that

$$C_l(3/r)^m \leq x < C_l(3/r)^{m+1}.$$

For these x it holds that there is at least one eigenvalue smaller than x from $(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R},K_w})$ and thus

$$N(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R},K_w}, x) \geq 1.$$

Summing over all m-cells leads to

$$\begin{aligned}\sum_{w \in \mathcal{A}^m} N(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R},K_w}, x) \geq 3^m &= \frac{1}{3}\left((3r^{-1})^{m+1}\right)^{\frac{\ln(3)}{\ln(3/r)}} \\ &\geq \underbrace{\frac{1}{3} C_l^{\frac{-\ln(3)}{\ln(3/r)}}}_{C_1':=} \cdot x^{\frac{\ln(3)}{\ln(3/r)}}.\end{aligned}$$

L.2: Line part $(\mathcal{E}_\mathcal{R}, \mathcal{D}^0_{\mathcal{R},e^w_{ij}})$

Let $r \geq \frac{1}{3}$. In the previous calculations we saw that the fractal part already gives a lower bound with the same order as the upper bound. Therefore, the influence of the line part cannot be bigger than the fractal part. Here we can use the trivial estimate

$$\sum_{\substack{w\in\mathcal{A}^n,n<m\\(i,j)\in B}} N(\mathcal{E}_\mathcal{R}, \mathcal{D}^0_{\mathcal{R},e^w_{ij}}, x) \geq 0.$$

However, if $r < \frac{1}{3}$ this order $\frac{\ln(3)}{\ln(3/r)}$ is strictly smaller than $\frac{1}{2}$. This means we need a better lower estimate. Therefore, let us choose just one one-dimensional line, say e_{12}. Then

$$\begin{aligned}\sum_{\substack{w\in\mathcal{A}^n,n<m\\(i,j)\in B}} N(\mathcal{E}_\mathcal{R}, \mathcal{D}^0_{\mathcal{R},e^w_{ij}}, x) &\geq N(\mathcal{E}_\mathcal{R}, \mathcal{D}^0_{\mathcal{R},e_{12}}, x)\\ &= N(-\Delta_D|_{(0,1)}, \eta a\rho_1 x)\\ &\geq \underbrace{\frac{c_0}{\pi}\sqrt{\eta a\rho_1}}_{C_1'':=} \cdot x^{\frac{1}{2}}\end{aligned}$$

for x big enough with a constant $0 < c_0 < \infty$.

Now we have the desired estimates if our measure includes the fractal part. It remains to show the result when we only choose the line part of the measure.

Use only the line part of the measure: $\mu = \mu_1 = \mu_I$

We know that this is a self-similar measure in the sense that

$$\mu_I(K_w) = \beta^{|w|}.$$

In the proof we used the following estimates for the measure:

$$(1-\eta)\left(\frac{1}{3}\right)^{|w|} \leq \mu_\eta(K_w) \leq \left(\frac{1}{3}\right)^{|w|}$$

with $\eta \in (0,1)$. Whenever we used these estimates we can replace this by $\beta^{|w|}$. The proof works exactly the same as before if $\beta \neq \frac{1}{9r}$, which leads to the asymptotic growing of order

$$\max\left\{\frac{1}{2}, \frac{\ln 3}{-\ln(\beta r)}\right\}.$$

If $\beta = \frac{1}{9r}$ we are not able to change β to $\tilde{\beta} = \beta + \epsilon$ as in U.2 since we are dependent on the exact value of β. Therefore, we have to exclude this value. $\square$

2.6 Generalization

If we no longer demand that our sequences of matching pairs $\mathcal{R} = \{(r_i, \rho_i)\}_{i\geq 1}$ fulfill Condition 2.1, we can consider weaker conditions.

Condition 2.2. *We consider a sequence of matching pairs $\mathcal{R} = \{(r_i, \rho_i)\}_{i\geq 1}$, such that there exist $r_*, r^* \in (0, \frac{3}{5}]$ and $\kappa_1, \kappa_2 > 0$ with*

$$\kappa_1 (r_*)^m \leq \delta_m^{(n)} \leq \kappa_2 (r^*)^m$$

for all $m, n \geq 0$.

Remark. We would like to give an example for which Condition 2.2 is satisfied. Let

$$r^* := \limsup_{m\to\infty} r_m, \qquad r_* := \liminf_{m\to\infty} r_m$$

and assume the sequence of matching pairs $\mathcal{R} = \{(r_i, \rho_i)\}_{i\geq 1}$ satisfies

$$\sum_{m: r_m > r^*} (r_m - r^*) < \infty,$$
$$\sum_{m: r_m < r_*} (r_* - r_m) < \infty.$$

These summability conditions express that the elements of the sequence which are above r^* and below r_* behave nicely. They are equivalent to

$$0 < \prod_{m: r_m > r^*} (r^*)^{-1} r_m < \infty,$$
$$0 < \prod_{m: r_m < r_*} (r_*)^{-1} r_m < \infty.$$

Then we get constants κ^*, κ_* with

$$1 \leq \prod_{k\leq m: r_k > r^*} (r^*)^{-1} r_k \leq \kappa^*, \ \forall m,$$
$$\kappa_* \leq \prod_{k\leq m: r_k < r_*} (r_*)^{-1} r_k \leq 1, \quad \forall m.$$

Since $r_k > r^*$ in the first product we have that $\kappa^* > 1$ and analogously $\kappa_* < 1$. With that we get estimates for δ_m by

$$\begin{aligned}
\delta_m &= r_1 \cdots r_m \\
&= \left(\prod_{k\leq m: r_k \leq r^*} r_k \right) \cdot \left(\prod_{k\leq m: r_k > r^*} r_k \right) \\
&\leq (r^*)^{\#\{k\leq m | r_k \leq r^*\}} \cdot \kappa^* (r^*)^{\#\{k\leq m | r_k > r^*\}} \\
&= \kappa^* (r^*)^m.
\end{aligned}$$

There is also a bound from below by

$$\begin{aligned}\delta_m &= r_1 \cdots r_m \\ &= \left(\prod_{k \le m | r_k \ge r_*} r_k \right) \cdot \left(\prod_{k \le m | r_k < r_*} r_k \right) \\ &\ge (r_*)^{\#\{k \le m | r_k \ge r_*\}} \cdot \kappa_* (r_*)^{\#\{k \le m | r_k < r_*\}} \\ &= \kappa_* (r_*)^m.\end{aligned}$$

We can easily show that the same estimates also hold for $\delta_m^{(n)}$ with the same constants κ^* and κ_*.

For sequences $\mathcal{R}$ that fulfill Condition 2.2 we can apply the same proofs as before to get more general results. For the Hausdorff dimension in the resistance metric we get the following result.

Theorem 2.18. *Let $\mathcal{R} = \{(r_i, \rho_i)\}_{i \ge 1}$ be a sequence of matching pairs that fulfills Condition 2.2. Then*

$$\max\left\{1, \frac{\ln(3)}{-\ln(r_*)}\right\} \le \dim_{H,R_{\mathcal{R}}}(K) \le \max\left\{1, \frac{\ln(3)}{-\ln(r^*)}\right\}.$$

Proof. The proofs of Lemma 2.3, Lemma 2.4 and Lemma 2.5 work exactly the same with r^* resp. r_* in place of r. The latter two lemmata are responsible for the upper and lower bound for the fractal part in the proof of [40, Theorem 2.4]. □

The results of Section 2.5 can also be generalized to sequences of matching pairs with these weaker conditions.

Theorem 2.19. *Let $\mathcal{R} = \{(r_i, \rho_i)\}_{i \ge 1}$ be a sequence of matching pairs that fulfills Condition 2.2. Then there exist constants $0 < C_1, C_2 < \infty$ and $x_0 > 0$, such that for all $x \ge x_0$*

$$C_1 x^{\frac{1}{2} d_{S,1}^{\mu_\eta,\mathcal{R}}} \le N_D^{\mu_\eta,\mathcal{R}}(x) \le N_N^{\mu_\eta,\mathcal{R}}(x) \le C_2 x^{\frac{1}{2} d_{S,2}^{\mu_\eta,\mathcal{R}}}$$

with

$$d_{S,1}^{\mu_\eta,\mathcal{R}} = \begin{cases} \max\{1, \frac{\ln 9}{\ln 3 - \ln r_*}\}, & \eta \in (0,1), \\ \max\{1, \frac{\ln 9}{-\ln(\beta r_*)}\}, & \eta = 1, \end{cases}$$

$$d_{S,2}^{\mu_\eta,\mathcal{R}} = \begin{cases} \max\{1, \frac{\ln 9}{\ln 3 - \ln r^*}\}, & \eta \in (0,1), \\ \max\{1, \frac{\ln 9}{-\ln(\beta r^*)}\}, & \eta = 1, \beta \ne \frac{1}{9 r^*}. \end{cases}$$

Proof. The proof in Section 2.5 works again if we use the estimates of $\delta_m^{(n)}$ given by Condition 2.2 as well as Lemma 2.1 and change r to r^* resp. r_* for the upper resp. lower bound. □

2.7 Refinements

Until now we have the leading order of the asymptotic behavior of the eigenvalue counting functions for sequences $\mathcal{R}$ that fulfill Condition 2.1. We would like to refine these results. In particular we are interested in what is in front of the leading term. In the smooth case, e.g., for the Laplacian on bounded domains $\Omega \subset \mathbb{R}^d$, we know from Weyl [56] that

$$N_D^{\Omega}(x) = \frac{\tau_d}{(2\pi)^d} \operatorname{Vol}_d(\Omega) x^{\frac{d}{2}} + o(x^{\frac{d}{2}}) \quad \text{as } x \to \infty,$$

where τ_d denotes the volume of the unit ball in $\mathbb{R}^d$. In this case we can get information about the volume of the set Ω just by knowing the eigenvalues of the Laplacian. If we consider Laplacians on fractals, the first question that arises is if we even have a constant in front of the leading term; see Chapter 1. That means if lim inf and lim sup of the eigenvalue counting function scaled by its leading order coincide or differ,

$$0 < \liminf_{x\to\infty} N_D(x) x^{-\frac{1}{2}d_S} \overset{=}{\underset{<}{?}} \limsup_{x\to\infty} N_D(x) x^{-\frac{1}{2}d_S} < \infty.$$

Roughly speaking, if our set, form and measure have a lot of symmetry, then these terms will not coincide but instead there will be a log-periodic function in place of the constant; see [38]. We would like to answer this for the stretched Sierpiński gasket in this section for $\mathcal{R}$ that fulfill Condition 2.1 with $r \in [\frac{1}{3}, \frac{3}{5}]$, since then the leading term is driven by the fractal part. In Section 2.7.1 we review these questions on the self-similar Sierpiński gasket and then try applying the ideas in the stretched case. However, in Section 2.7.3 we will see that this strict periodic behavior of the self-similar case is most likely not present. Nonetheless, there are oscillations that are responsible for the strict inequality between lim inf and lim sup, which is shown in Section 2.7.4. This means the oscillations are present even without strict periodicity and thus are not as regular as in the self-similar case. Lastly in Section 2.7.5 we consider some special cases that give us back the symmetry needed to show periodicity. Additionally, we achieve some remainder estimates for these cases.

2.7.1 Review on oscillations on the Sierpiński gasket

Before we start to look for oscillations in the leading term for the stretched Sierpiński gasket we quickly review how we can tackle this problem for the usual Sierpiński gasket.

The Sierpiński gasket S, which is drawn in Figure 2.9, is the attractor of the IFS (F_1, F_2, F_3) consisting of three similitudes with contraction ratios $\frac{1}{2}$. On the Sierpiński

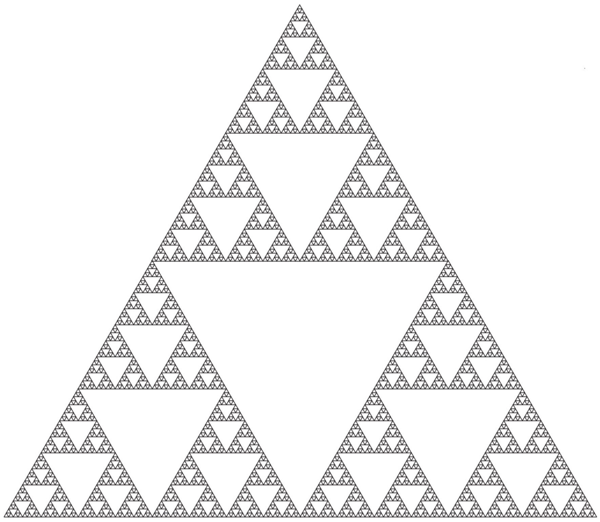

Figure 2.9: The Sierpiński gasket.

gasket we have the following useful rescaling property of the symmetric energy:

$$\mathcal{E}(u,v) = \sum_{i=1}^{3} \left(\frac{3}{5}\right)^{-1} \mathcal{E}(u \circ F_i, v \circ F_i);$$

see [38, Lemma 6.1]. If we use the normalized Hausdorff measure to get a Dirichlet form, it is well known that the eigenvalue counting functions (both Dirichlet and Neumann) have the asymptotic growing with spectral exponent $d_S = \frac{\ln 9}{\ln 5}$; see [19, 38]. This means

$$0 < \liminf_{x\to\infty} N_D(x) x^{-\frac{1}{2}d_S} \le \limsup_{x\to\infty} N_D(x) x^{-\frac{1}{2}d_S} < \infty.$$

We are in particular interested in the $\le$ sign. Do the two values coincide or do they differ? To answer this we introduce Dirichlet and Neumann boundary conditions on $V_1 \backslash V_0$ to get the following inequalities:

$$\sum_{i=1}^{3} N_D(\tfrac{1}{5}x) \le N_D(x) \le N_N(x) \le \sum_{i=1}^{3} N_N(\tfrac{1}{5}x) \le \sum_{i=1}^{3} N_D(\tfrac{1}{5}x) + 9; \tag{2.1}$$

see [38]. Now we have the Dirichlet eigenvalue counting function wedged in between some scaled version of itself. The scaling $\frac{1}{5}$ consists of two factors. One is the scaling of the energy $\frac{3}{5}$ and the other is the scaling of the measure $\frac{1}{3}$.

We can use these inequalities to apply renewal theory (see [38, 42]) and get a positive periodic function G with period $\frac{1}{2}\ln 5$ such that

$$N_D(x) = G\left(\frac{\ln x}{2}\right) x^{\frac{1}{2}d_S} + \mathcal{O}(1) \quad \text{as } x \to \infty.$$

The boundedness of the remainder term is strongly connected to the boundedness of the error term in the inequalities (2.1).

However, this does not answer the question of convergence since the periodic function G could still be constant. To get an answer we look at *localized eigenfunctions*. These are eigenfunctions that are supported on a proper subset of the Sierpiński gasket. The existence of such eigenfunctions is strongly connected to the existence of so-called *Dirichlet-Neumann eigenfunctions*, which are eigenfunctions that are simultaneously eigenfunctions to Dirichlet and Neumann boundary conditions. As it turns out, such eigenfunctions exist on the Sierpiński gasket; see [8, Theorem 6.6]. We also call these *pre-localized eigenfunctions* since we can construct localized eigenfunctions by the use of them. Let u be such a pre-localized eigenfunction with eigenvalue λ. We then define for $w \in \mathcal{A}^n$

$$u_w(x) := \begin{cases} u \circ F_w^{-1}(x), & x \in S_w, \\ 0, & \text{otherwise}, \end{cases}$$

where $S_w = F_w(S)$ is an n-cell of the Sierpiński gasket. Using the rescaling properties of the energy and the measure we can show that u_w itself is again an eigenfunction (Dirichlet and Neumann) with eigenvalue $\lambda 5^n$; see [8, Lemma 4.2]. We can do this construction for all n-cells and there are 3^n many. This means the eigenvalue $\lambda 5^n$ has multiplicity at least 3^n. Therefore, we have a sequence of growing eigenvalues with very high multiplicities. High multiplicities lead to very big jumps in the eigenvalue counting function. This sequence of eigenvalues is enough to show that $\liminf$ and $\limsup$ cannot coincide; see [8, Theorem 4.4] or compare to the calculations in Section 2.7.4.

2.7.2 How the measure scales

We consider μ_η for $\eta \in (0, 1]$. In Propositions 2.8 and 2.9 from Section 2.4 we started to examine the scaling behavior of these measures. We would like to further investigate the behavior of μ_η on n-cells. In the proof of Theorem 2.13 we used a set of measures μ_η^w. These measures were defined for $w \in \mathcal{A}_0^*$ by

$$\mu_\eta^w := \mu_\eta(K_w)^{-1} \mu_\eta \circ G_w.$$

The measure μ_η^w projects the properties of μ_η in K_w to all of K. For self-similar measures ν we have $\nu^w = \nu$ but we are not in that case. The two parts μ_I and μ_Σ both fulfill the equality on their own but not the sum of them. We have

$$\mu_I^w = \mu_I \quad \text{and} \quad \mu_\Sigma^w = \mu_\Sigma \quad \text{but} \quad \mu_\eta^w \neq \mu_\eta \quad \text{for} \quad \eta \in (0, 1).$$

This is the case since the scaling constants $\mu_I(K_w)$ and $\mu_\Sigma(K_w)$ are not equal, i.e., $\beta \neq \frac{1}{3}$.

So equality is too much to ask for, but they still have a lot in common. We have

$$\begin{aligned}
\mu_\eta^w &= \mu_\eta(K_w)^{-1}\mu_\eta \circ G_w \\
&= \mu_\eta(K_w)^{-1}\left((\eta\mu_I + (1-\eta)\mu_\Sigma)\circ G_w\right) \\
&= \frac{\eta}{\mu_\eta(K_w)}\mu_I \circ G_w + \frac{1-\eta}{\mu_\eta(K_w)}\mu_\Sigma \circ G_w \\
&= \frac{\eta\mu_I(K_w)}{\mu_\eta(K_w)}\mu_I^w + \frac{(1-\eta)\mu_\Sigma(K_w)}{\mu_\eta(K_w)}\mu_\Sigma^w \\
&= \underbrace{\frac{\eta\mu_I(K_w)}{\mu_\eta(K_w)}}_{=:\eta^w}\mu_I + \frac{(1-\eta)\mu_\Sigma(K_w)}{\mu_\eta(K_w)}\mu_\Sigma \\
&= \eta^w\mu_I + (1-\eta^w)\mu_\Sigma,
\end{aligned}$$

which means we have obtained the following result.

Lemma 2.20. *For $\eta \in (0,1)$ and $w \in \mathcal{A}_0^*$ with $\eta^w := \frac{\eta\mu_I(K_w)}{\mu_\eta(K_w)}$ we have*

$$\begin{aligned}
\mu_\eta^w &= \mu_\eta(K_w)^{-1}\mu_\eta \circ G_w \\
&= \mu_{\eta^w}.
\end{aligned}$$

Thus μ_η^w is again a convex combination of the two parts of the measure but with different parameter η^w. The measures μ_η^w are all the same for words of the same length. This is due to the fact that μ_η is very symmetric and behaves the same on all n-cells. So we could also write

$$\mu_\eta^{(n)} := \mu_\eta^w, \text{ for any } w \in \mathcal{A}^n.$$

We also get estimates for the L^2-norms. We have

$$\begin{aligned}
||f||^2_{\mu_\eta^w} &:= \int_K f^2 d\mu_\eta^w \\
&= \eta^w \int_K f^2 d\mu_I + (1-\eta^w)\int_K f^2 d\mu_\Sigma, \\
||f||^2_{\mu_\eta} &= \eta \int_K f^2 d\mu_I + (1-\eta)\int_K f^2 d\mu_\Sigma.
\end{aligned}$$

Now, since $\mu_I(K_w) \leq \mu_\Sigma(K_w)$ we get the following lemma.

Lemma 2.21. *For $\eta \in (0,1)$ and $w \in \mathcal{A}_0^*$, with $\eta^w := \frac{\eta\mu_I(K_w)}{\mu_\eta(K_w)}$ we have*

$$\frac{\mu_I(K_w)}{\mu_\eta(K_w)} \cdot ||f||^2_{\mu_\eta} \leq ||f||^2_{\mu_\eta^w} \leq \frac{\mu_\Sigma(K_w)}{\mu_\eta(K_w)} \cdot ||f||^2_{\mu_\eta}.$$

for all $f \in L^2(K, \mu_\eta)$.

It is easy to see that these estimates are sharp.

2.7.3 How the energy scales and why renewal theory is not applicable

In this section we look at the energy and see how it rescales. We would like to obtain estimates for the eigenvalue counting functions similar to ones for the Sierpiński gasket in Section 2.7.1. We have to be careful since our Dirichlet form, and hence the eigenvalue counting function, have many dependencies that we should include in the notation whenever it is necessary. As in the proof of Theorem 2.13 we denote a Dirichlet form $\mathcal{E}$ with domain $\mathcal{D}$ in the Hilbert space $L^2(X, \mu)$ by

$$(\mathcal{E}, \mathcal{D}, \mu)$$

and the corresponding eigenvalue counting function evaluated at $x \geq 0$ by

$$N(\mathcal{E}, \mathcal{D}, \mu, x).$$

In Lemma 2.1 we saw that we have the following rescaling:

$$\mathcal{E}_{\mathcal{R}}(u, v) = \sum_{w \in \mathcal{A}^n} \frac{1}{\delta_n} \mathcal{E}_{\mathcal{R}^{(n)}}(u \circ G_w, v \circ G_w) + \sum_{k=1}^{n} \frac{1}{\gamma_k} \mathcal{D}_k^I(u, v).$$

Again we would like to point out the differences from the self-similar case. There is an additional term in the right-hand side which comes from the one-dimensional lines connecting the n-cells. As it turns out we will be able to work with this as it is somehow of lower order. Also the quadratic forms $\mathcal{E}_{\mathcal{R}}$ on the left and $\mathcal{E}_{\mathcal{R}^{(n)}}$ on the right-hand side are different ones. Nonetheless, we try to get the same results concerning periodicity. We introduce Neumann boundary conditions on $V_n \backslash V_0$ as in the proof of Theorem 2.13. With

$$\mathcal{D}_{\mathcal{R}, K_w} := \{u \mid u \in L^2(K, \mu_\eta), u|_{K_w^c} \equiv 0, \exists f \in \mathcal{D}_{\mathcal{R}} : f|_{K_w} = u\}$$

and

$$\mathcal{E}_{\mathcal{R}, K_w}(u, v) := \frac{1}{\delta_n} \mathcal{E}_{\mathcal{R}^{(n)}}(u \circ G_w, v \circ G_w)$$

for all $u, v \in \mathcal{D}_{\mathcal{R}, K_w}$ we have the following result.

Lemma 2.22. *For $\eta \in (0, 1]$ and $w \in \mathcal{A}^n$ we have*

$$N(\mathcal{E}_{\mathcal{R}, K_w}, \mathcal{D}_{\mathcal{R}, K_w}, \mu_\eta, x) = N(\mathcal{E}_{\mathcal{R}^{(n)}}, \mathcal{D}_{\mathcal{R}^{(n)}}, \mu_\eta^w, \mu_\eta(K_w)\delta_n x)$$

for all $x \geq 0$.

Proof. Let u be an eigenfunction of $(\mathcal{E}_{\mathcal{R}, K_w}, \mathcal{D}_{\mathcal{R}, K_w}, \mu_\eta)$ with eigenvalue λ. That means for all $v \in \mathcal{D}_{\mathcal{R}, K_w}$ we have

$$\mathcal{E}_{\mathcal{R}, K_w}(u, v) = \lambda(u, v)_{\mu_\eta} := \lambda \int_K uv \, d\mu_\eta.$$

Since $\mathcal{E}_{\mathcal{R},K_w}(u,v) = \frac{1}{\delta_n}\mathcal{E}_{\mathcal{R}^{(n)}}(u \circ G_w, v \circ G_w)$ and

$$\begin{aligned} \lambda(u,v)_{\mu_\eta} &= \lambda \int_K uv d\mu_\eta \\ &= \lambda \int_{K_w} uv d\mu_\eta \\ &= \lambda \mu_\eta(K_w) \int_K u \circ G_w \cdot v \circ G_w d\mu_\eta^w, \end{aligned}$$

we obtain

$$\mathcal{E}_{\mathcal{R}^{(n)}}(u \circ G_w, v \circ G_w) = \lambda \delta_n \mu_\eta(K_w)(u \circ G_w, v \circ G_w)_{\mu_\eta^w}.$$

This holds for all $v \in \mathcal{D}_{\mathcal{R},K_w}$, but due to the construction of $\mathcal{D}_{\mathcal{R},K_w}$ and Lemma 2.1 we have $\mathcal{D}_{\mathcal{R}^{(n)}} = \{v \circ G_w \mid v \in \mathcal{D}_{\mathcal{R},K_w}\}$. This implies that $u \circ G_w$ is an eigenfunction of $(\mathcal{E}_{\mathcal{R}^{(n)}}, \mathcal{D}_{\mathcal{R}^{(n)}}, \mu_\eta^w)$ with eigenvalue $\lambda \mu_\eta(K_w)\delta_n$.

We can obtain the other direction analogously with $\tilde{u} := u \circ G_w^{-1}$ as an eigenfunction of $(\mathcal{E}_{\mathcal{R},K_w}, \mathcal{D}_{\mathcal{R},K_w}, \mu_\eta)$ if u is an eigenfunction of $(\mathcal{E}_{\mathcal{R}^{(n)}}, \mathcal{D}_{\mathcal{R}^{(n)}}, \mu_\eta^w)$. We, therefore, have the desired result. □

With an analogous proof we get the same result for the Dirichlet case.

Lemma 2.23. *For $\eta \in (0,1]$ and $w \in \mathcal{A}^n$ we have*

$$N(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R},K_w}, \mu_\eta, x) = N(\mathcal{E}_{\mathcal{R}^{(n)}}, \mathcal{D}^0_{\mathcal{R}^{(n)}}, \mu_\eta^w, \mu_\eta(K_w)\delta_n x)$$

for all $x \geq 0$.

There are many differences from the self-similar case. We have different forms, domains and measures. To be able to apply renewal theory we need to get rid of these differences. We already have estimates for the measures. Now we need estimates for the quadratic forms. To be able to do this we need to tighten our conditions. As a quick reminder the quadratic forms are of the following structure:

$$\mathcal{E}_{\mathcal{R}}(u,u) = \lim_{k\to\infty} \frac{1}{\delta_k} Q_k^\Sigma(u,u) + \sum_{k\geq 1} \frac{1}{\gamma_k} \mathcal{D}_k^I(u,u),$$

with

$$\delta_k = r_1 \cdots r_k \quad \text{and} \quad \gamma_k = r_1 \cdots r_{k-1}\rho_k,$$

as well as

$$\mathcal{E}_{\mathcal{R}^{(n)}}(u,u) = \lim_{k\to\infty} \frac{1}{\delta_k^{(n)}} Q_k^\Sigma(u,u) + \sum_{k\geq 1} \frac{1}{\gamma_k^{(n)}} \mathcal{D}_k^I(u,u),$$

with

$$\delta_k^{(n)} = r_{n+1} \cdots r_{n+k} \quad \text{and} \quad \gamma_k^{(n)} = r_{n+1} \cdots r_{n+k-1}\rho_{n+k}.$$

To compare these forms we need estimates between δ_k and $\delta_k^{(n)}$, as well as γ_k and $\gamma_k^{(n)}$. We have

$$\delta_k^{(n)} = \delta_k \frac{r_{k+1} \cdots r_{k+n}}{\delta_n}.$$

From Lemma 2.2 we get

$$\kappa_1 \frac{r^n}{\delta_n} \delta_k \le \delta_k^{(n)} \le \kappa_2 \frac{r^n}{\delta_n} \delta_k.$$

We also need to compare the resistances γ_k and $\gamma_k^{(n)}$. We can express $\gamma_k^{(n)}$ as

$$\begin{aligned} \gamma_k^{(n)} &= \frac{\gamma_{n+k}}{\delta_n} \\ &= \gamma_k \frac{r_k \cdots r_{n+k-1}}{\delta_n} \frac{\rho_{n+k}}{\rho_k}. \end{aligned}$$

Again by Lemma 2.2 we obtain

$$\kappa_1 \frac{r^n}{\delta_n} \frac{\rho_{n+k}}{\rho_k} \gamma_k \le \gamma_k^{(n)} \le \kappa_2 \frac{r^n}{\delta_n} \frac{\rho_{n+k}}{\rho_k} \gamma_k.$$

To obtain similar estimates as for $\delta_k^{(n)}$ we need constants $\kappa_3, \kappa_4 > 0$ such that $\kappa_3 \le \frac{\rho_{n+k}}{\rho_k} \le \kappa_4$ for all $n, k \ge 1$. E.g. this is satisfied if Condition 2.1 is fulfilled with $r < \frac{3}{5}$.

Lemma 2.24. *Let $\mathcal{R}$ be a sequence of matching pairs that fulfills Condition 2.1 and assume there are constants $\kappa_3, \kappa_4 > 0$ such that $\kappa_3 \le \frac{\rho_{n+k}}{\rho_k} \le \kappa_4$ for all $n, k \ge 1$. Then we have constants $0 < c_1 \le c_2 < \infty$ such that*

$$c_1 \frac{r^n}{\delta_n} \mathcal{E}_{\mathcal{R}^{(n)}}(u,u) \le \mathcal{E}_{\mathcal{R}}(u,u) \le c_2 \frac{r^n}{\delta_n} \mathcal{E}_{\mathcal{R}^{(n)}}(u,u)$$

for all $u \in \mathcal{D}_{\mathcal{R}} = \mathcal{D}_{\mathcal{R}^{(n)}}$ and all $n \ge 0$.

The estimates immediately imply that the domains coincide. With the help of this result we can get estimates on the eigenvalue counting functions. We would like to compare the eigenvalues and the eigenvalue counting functions of $(\mathcal{E}_{\mathcal{R}}, \mathcal{D}_{\mathcal{R}}, \mu_\eta)$ and $(\mathcal{E}_{\mathcal{R}^{(n)}}, \mathcal{D}_{\mathcal{R}^{(n)}}, \mu_\eta^w)$. To do this we use the Max-Min principle; see [13, Theorem 2, Chapter 10.2]:

$$\lambda_k(\mathcal{E}, \mathcal{D}, \nu) = \max_{\substack{\Phi \subset \mathcal{D}_{\mathcal{R}} \\ \dim \mathcal{D}_{\mathcal{R}}/\Phi \le k-1}} \inf_{\substack{u \in \Phi \\ ||u||_\nu = 1}} \mathcal{E}(u,u),$$

where the maximum is taken over all subspaces Φ with co-dimension equal to or less than $k-1$. The eigenvalues depend on the resistance form as well as on the measure ν. The occurring norm $||u||_\nu$ is the L^2-norm in $L^2(K, \nu)$. The condition in the infimum can be changed to $||u||_\nu \ge 1$ since it takes its lowest value at $||u||_\nu = 1$. In the following we make use of estimates between the L^2-norms from Lemma 2.21 and between the quadratic forms from Lemma 2.24 to relate the k-th eigenvalues

$$\lambda_k(\mathcal{E}_{\mathcal{R}}, \mathcal{D}_{\mathcal{R}}, \mu_\eta) \quad \text{and} \quad \lambda_k(\mathcal{E}_{\mathcal{R}^{(n)}}, \mathcal{D}_{\mathcal{R}^{(n)}}, \mu_\eta^{(n)}).$$

First we notice that the parameter range for the maximum does not change as $\mathcal{D}_{\mathcal{R}} = \mathcal{D}_{\mathcal{R}^{(n)}}$. However, we have different Hilbert spaces $L^2(K, \mu_\eta)$ and $L^2(K, \mu_\eta^w)$, which means the norm changes. By Lemma 2.24,

$$\begin{aligned}
\lambda_k(\mathcal{E}_{\mathcal{R}}, \mathcal{D}_{\mathcal{R}}, \mu_\eta) &= \max_{\substack{\Phi\subset\mathcal{D}_{\mathcal{R}} \\ \dim \mathcal{D}_{\mathcal{R}}/\Phi\leq k-1}} \inf_{\substack{u\in\Phi \\ ||u||_{\mu_\eta}\geq 1}} \mathcal{E}_{\mathcal{R}}(u,u) \\
&\leq \max_{\substack{\Phi\subset\mathcal{D}_{\mathcal{R}} \\ \dim \mathcal{D}_{\mathcal{R}}/\Phi\leq k-1}} \inf_{\substack{u\in\Phi \\ ||u||_{\mu_\eta}\geq 1}} c_2 \frac{r^n}{\delta_n} \mathcal{E}_{\mathcal{R}^{(n)}}(u,u).
\end{aligned}$$

In Lemma 2.21 we saw

$$||u||_{\mu_\eta^w} \leq \sqrt{\frac{\mu_\Sigma(K_w)}{\mu_\eta(K_w)}} ||u||_{\mu_\eta},$$

which means the condition

$$||u||_{\mu_\eta^w} \geq \sqrt{\tfrac{\mu_\Sigma(K_w)}{\mu_\eta(K_w)}}$$

is stronger than

$$||u||_{\mu_\eta} \geq 1.$$

Therefore, the set over which the infimum is taken gets smaller and thus the infimum gets bigger. This means

$$\begin{aligned}
\lambda_k(\mathcal{E}_{\mathcal{R}}, \mathcal{D}_{\mathcal{R}}, \mu_\eta) &\leq c_2 \frac{r^n}{\delta_n} \max_{\substack{\Phi\subset\mathcal{D}_{\mathcal{R}} \\ \dim \mathcal{D}_{\mathcal{R}}/\Phi\leq k-1}} \inf_{\substack{u\in\Phi \\ ||u||_{\mu_\eta^w}\geq \sqrt{\frac{\mu_\Sigma(K_w)}{\mu_\eta(K_w)}}}} \mathcal{E}_{\mathcal{R}^{(n)}}(u,u) \\
&= c_2 \frac{r^n}{\delta_n} \max_{\substack{\Phi\subset\mathcal{D}_{\mathcal{R}} \\ \dim \mathcal{D}_{\mathcal{R}}/\Phi\leq k-1}} \inf_{\substack{u\in\Phi \\ ||u||_{\mu_\eta^w}\geq 1}} \mathcal{E}_{\mathcal{R}^{(n)}}\left(\sqrt{\frac{\mu_\Sigma(K_w)}{\mu_\eta(K_w)}}u, \sqrt{\frac{\mu_\Sigma(K_w)}{\mu_\eta(K_w)}}u\right) \\
&= c_2 \frac{r^n}{\delta_n} \frac{\mu_\Sigma(K_w)}{\mu_\eta(K_w)} \max_{\substack{\Phi\subset\mathcal{D}_{\mathcal{R}} \\ \dim \mathcal{D}_{\mathcal{R}}/\Phi\leq k-1}} \inf_{\substack{u\in\Phi \\ ||u||_{\mu_\eta^w}\geq 1}} \mathcal{E}_{\mathcal{R}^{(n)}}(u,u) \\
&= c_2 \frac{r^n}{\delta_n} \frac{\mu_\Sigma(K_w)}{\mu_\eta(K_w)} \lambda_k(\mathcal{E}_{\mathcal{R}^{(n)}}, \mathcal{D}_{\mathcal{R}}, \mu_\eta^w).
\end{aligned}$$

The other direction works the same, so we get

$$c_1 \tfrac{r^n}{\delta_n} \tfrac{\mu_I(K_w)}{\mu_\eta(K_w)} \lambda_k(\mathcal{E}_{\mathcal{R}^{(n)}}, \mathcal{D}_{\mathcal{R}}, \mu_\eta^w) \leq \lambda_k(\mathcal{E}_{\mathcal{R}}, \mathcal{D}_{\mathcal{R}}, \mu_\eta) \leq c_2 \tfrac{r^n}{\delta_n} \tfrac{\mu_\Sigma(K_w)}{\mu_\eta(K_w)} \lambda_k(\mathcal{E}_{\mathcal{R}^{(n)}}, \mathcal{D}_{\mathcal{R}}, \mu_\eta^w).$$

For the eigenvalue counting functions we obtain

$$N\left(\mathcal{E}_{\mathcal{R}^{(n)}}, \mathcal{D}_{\mathcal{R}}, \mu_\eta^w, \tfrac{x}{c_2 \frac{r^n}{\delta_n} \frac{\mu_\Sigma(K_w)}{\mu_\eta(K_w)}}\right) \leq N(\mathcal{E}_{\mathcal{R}}, \mathcal{D}_{\mathcal{R}}, \mu_\eta, x) \leq N\left(\mathcal{E}_{\mathcal{R}^{(n)}}, \mathcal{D}_{\mathcal{R}}, \mu_\eta^w, \tfrac{x}{c_1 \frac{r^n}{\delta_n} \frac{\mu_I(K_w)}{\mu_\eta(K_w)}}\right).$$

If we change $\mathcal{D}_{\mathcal{R}}$ to $\mathcal{D}_{\mathcal{R}}^0$ we get the same results for the Dirichlet eigenvalues and counting functions. Combining this with the estimates from before with

$$\mathcal{D}_{\mathcal{R},J_n} = \{u \mid u \in L^2(K,\mu_\eta), u|_{J_n^c} \equiv 0, \exists f \in \mathcal{D}_{\mathcal{R}} : f|_{J_n} = u\}$$

and

$$\mathcal{E}_{\mathcal{R},J_n}(u,v) = \sum_{k=1}^{n} \frac{1}{\gamma_k} \mathcal{D}_k^I(u,v)$$

for all $u, v \in \mathcal{D}_{\mathcal{R},J_n}$ as in the proof of Theorem 2.13 leads to the following estimates of the eigenvalue counting functions for all $x \geq 0$:

$$\begin{aligned}
N_D^{\mu_\eta,\mathcal{R}}(x) &\leq N_N^{\mu_\eta,\mathcal{R}}(x) \\
&\leq \sum_{w\in\mathcal{A}^n} N(\mathcal{E}_{\mathcal{R},K_w}, \mathcal{D}_{\mathcal{R},K_w}, \mu_\eta, x) + N(\mathcal{E}_{\mathcal{R},J_n}, \mathcal{D}_{\mathcal{R},J_n}, \mu_\eta, x) \\
&= \sum_{w\in\mathcal{A}^n} N(\mathcal{E}_{\mathcal{R}^{(n)}}, \mathcal{D}_{\mathcal{R}^{(n)}}, \mu_\eta^w, \mu_\eta(K_w)\delta_n x) + N(\mathcal{E}_{\mathcal{R},J_n}, \mathcal{D}_{\mathcal{R},J_n}, \mu_\eta, x) \\
&\leq \sum_{w\in\mathcal{A}^n} N(\mathcal{E}_{\mathcal{R}}, \mathcal{D}_{\mathcal{R}}, \mu_\eta, \mu_\eta(K_w)\delta_n x c_2 \tfrac{r^n}{\delta_n} \tfrac{\mu_\Sigma(K_w)}{\mu_\eta(K_w)}) + N(\mathcal{E}_{\mathcal{R},J_n}, \mathcal{D}_{\mathcal{R},J_n}, \mu_\eta, x) \\
&= \sum_{w\in\mathcal{A}^n} N_N^{\mu_\eta,\mathcal{R}}(c_2 r^n \mu_\Sigma(K_w)x) + N(\mathcal{E}_{\mathcal{R},J_n}, \mathcal{D}_{\mathcal{R},J_n}, \mu_\eta, x) \\
&\leq \sum_{w\in\mathcal{A}^n} (N_D^{\mu_\eta,\mathcal{R}}(c_2 r^n \mu_\Sigma(K_w)x) + 3) + N(\mathcal{E}_{\mathcal{R},J_n}, \mathcal{D}_{\mathcal{R},J_n}, \mu_\eta, x).
\end{aligned}$$

The lower bound works analogously. We therefore have the following estimates which correspond to an iterated version of (2.1) for the self-similar Sierpiński gasket.

Lemma 2.25. *Let $\mathcal{R}$ be a sequence of matching pairs that fulfills Condition 2.1 and assume there are constants $\kappa_3, \kappa_4 > 0$ such that $\kappa_3 \leq \frac{\rho_{n+k}}{\rho_k} \leq \kappa_4$ for all $n, k \geq 1$. Then there are constants $0 < c_1 \leq c_2 < \infty$ such that for all $n \geq 1$ and $x \geq 0$, if $\eta \in (0,1)$,*

$$\begin{aligned}
\sum_{w\in\mathcal{A}^n} N_D^{\mu_\eta,\mathcal{R}}(c_1(\beta r)^n x) &\leq N_D^{\mu_\eta,\mathcal{R}}(x) \leq N_N^{\mu_\eta,\mathcal{R}}(x) \\
&\leq \sum_{w\in\mathcal{A}^n} N_D^{\mu_\eta,\mathcal{R}}(c_2(\tfrac{r}{3})^n x) + 3^{n+1} + N(\mathcal{E}_{\mathcal{R},J_n}, \mathcal{D}_{\mathcal{R},J_n}, \mu_\eta, x),
\end{aligned}$$

and if $\eta = 1$,

$$\begin{aligned}
\sum_{w\in\mathcal{A}^n} N_D^{\mu_I,\mathcal{R}}(c_1(\beta r)^n x) &\leq N_D^{\mu_I,\mathcal{R}}(x) \leq N_N^{\mu_I,\mathcal{R}}(x) \\
&\leq \sum_{w\in\mathcal{A}^n} N_D^{\mu_I,\mathcal{R}}(c_2(\beta r)^n x) + 3^{n+1} + N(\mathcal{E}_{\mathcal{R},J_n}, \mathcal{D}_{\mathcal{R},J_n}, \mu_I, x).
\end{aligned}$$

The constants c_1 and c_2 are the ones from the estimates between the quadratic forms in Lemma 2.24.

So for $\eta \in (0,1)$ we do not even get the same scaling. We have βr for the lower and $\frac{r}{3}$ for the upper bound. This is due to the two different scalings of the measure. If we only use the line part μ_I we do not need the estimates of the L^2-norms from Lemma 2.21 since $\mu_I^{(n)} = \mu_I$ and thus, we have the same scaling βr on both sides, but we still have the constants c_1 and c_2. In general they do not coincide, even asymptotically as $n \to \infty$. We cannot apply renewal theory here because we need exactly the same value on both sides.

Furthermore, since we cannot get rid of the constants it is very unlikely to get strict periodic behavior even with other methods. We lack some kind of symmetry which is necessary to do that. There are some special cases where we are able to get strict periodicity and we will handle them in Section 2.7.5.

2.7.4 Existence of localized eigenfunctions and non-convergence

So we saw that strict periodic behavior as for the Sierpiński gasket is not very likely. However, we still would like to answer the question of convergence. Meaning, even if there is no strict periodic behavior, there could still be oscillations. To answer this question we look at localized eigenfunctions just as for the Sierpiński gasket. As it turns out, these still exist on the stretched Sierpiński gasket. To show this we first show the existence of Dirichlet-Neumann eigenfunctions, i.e., functions that are simultaneously eigenfunctions of both $-\Delta_D^{\mu_\eta,\mathcal{R}}$ and $-\Delta_N^{\mu_\eta,\mathcal{R}}$. Since we can construct localized eigenfunctions by the use of them we also call them pre-localized eigenfunctions.

Lemma 2.26. *Let $\mathcal{R}$ be any sequence of matching pairs and $\eta \in (0,1]$. Then there exists a Dirichlet-Neumann eigenfunction u with eigenvalue λ of $(\mathcal{E}_\mathcal{R}, \mathcal{D}_\mathcal{R}, \mu_\eta)$. This means $\exists u \in \mathcal{D}_\mathcal{R}^0$ with $u \not\equiv 0$ and $\mathcal{E}_\mathcal{R}(u,v) = \lambda(u,v)_{\mu_\eta}$, $\forall v \in \mathcal{D}_\mathcal{R}$.*

Proof. The proof of the existence of localized eigenfunctions follows the arguments in [8] where the existence was shown for Laplacians on p.c.f. self-similar sets under certain conditions on the symmetry of the set. The stretched Sierpiński gasket is not self-similar but the strong symmetry suffices to apply the ideas. We modify the ideas slightly to get more information about the eigenvalue.

On the stretched Sierpiński gasket we have the following symmetries, which are the same as for the Sierpiński gasket. These symmetries are fulfilled by the geometry of the set, the resistance forms and the measures μ_η.

- Three Rotations: $0°$, $120°$ and $240°$.
- Three Reflections: One in each bisecting line.

By σ we denote the $120°$ rotation and by τ the reflection in the bisecting line trough p_1 as illustrated in Figure 2.10.

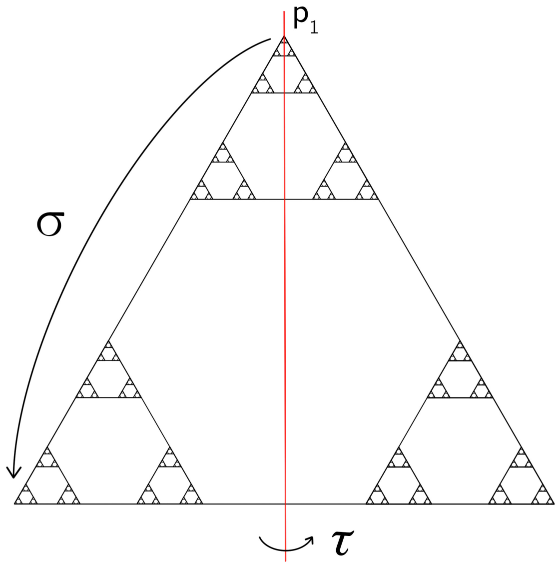

Figure 2.10: Symmetries of the stretched Sierpiński gasket.

We divide K into six parts in the following manner illustrated in Figure 2.11.

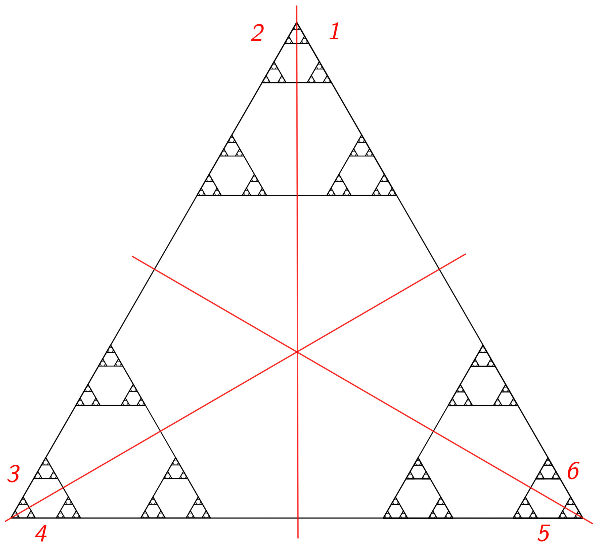

Figure 2.11: Dividing the stretched Sierpiński gasket into six parts.

The individual parts are denoted by $\tilde{K}_i$ and the intersections $\tilde{V}_i := \tilde{K}_i \cap (\tilde{K}_{i+1} \cup \tilde{K}_{i-1})$ where i is taken modulo 6. That means $\tilde{V}_i$ is the intersection of $\tilde{K}_i$ with the bisecting lines indicated in Figure 2.11. On $\tilde{V}_i$ we introduce Dirichlet boundary conditions. There are countably infinitely many of those points and all but one lie in the middle of one-dimensional lines. We define the corresponding domains by

$$\mathcal{D}^0_{\mathcal{R},i} := \{u \mid u \in \mathcal{D}_{\mathcal{R}},\ u|_{\tilde{V}_i} \equiv 0,\ u|_{\tilde{K}_i^c} \equiv 0\}.$$

Denote the parts of the quadratic forms by $\mathcal{E}_{\mathcal{R},i} := \mathcal{E}_{\mathcal{R}}|_{\mathcal{D}^0_{\mathcal{R},i} \times \mathcal{D}^0_{\mathcal{R},i}}$.

Lemma 2.27. *$(\mathcal{E}_{\mathcal{R},i}, \mathcal{D}^0_{\mathcal{R},i})$ is a regular Dirichlet form on $L^2(\tilde{K}_i \backslash \tilde{V}_i, \mu_\eta|_{\tilde{K}_i \backslash \tilde{V}_i})$ with discrete non-negative spectrum, $\mathcal{D}^0_{\mathcal{R},1} \oplus \cdots \oplus \mathcal{D}^0_{\mathcal{R},6} = \{u \in \mathcal{D}_{\mathcal{R}} \mid u|_{\bigcup_{i=1}^6 \tilde{V}_i} \equiv 0\}$ and for $u, v \in \mathcal{D}^0_{\mathcal{R},1} \oplus \cdots \oplus \mathcal{D}^0_{\mathcal{R},6}$ we have*

$$\mathcal{E}_{\mathcal{R}}(u, v) = \sum_{i=1}^{6} \mathcal{E}_{\mathcal{R},i}(u|_{\tilde{K}_i}, v|_{\tilde{K}_i}).$$

Proof. From [45, Theorem 10.3] (see also [20, Theorem 4.4.3]) we know that $(\mathcal{E}_{\mathcal{R},i}, \mathcal{D}^0_{\mathcal{R},i})$ is a regular Dirichlet form on $L^2(\tilde{K}_i \backslash \tilde{V}_i, \mu_\eta|_{\tilde{K}_i \backslash \tilde{V}_i})$. The spectrum is discrete since $\mathcal{D}^0_{\mathcal{R},i} \subset \mathcal{D}_{\mathcal{R}}$; see [13, Theorem 4, Chapter 10.2]. The $\mathcal{D}^0_{\mathcal{R},i}$ are orthogonal to each other with respect to $\mathcal{E}_{\mathcal{R}}$ as well as the inner product of $L^2(K, \mu_\eta)$. Therefore, we have the desired equality. △

Let φ be any eigenfunction of $(\mathcal{E}_{\mathcal{R},1}, \mathcal{D}^0_{\mathcal{R},1})$ with measure μ_η and $\eta \in (0, 1]$ with eigenvalue λ. We can use this φ to construct a Dirichlet-Neumann eigenfunction on the stretched Sierpiński gasket. By $\tilde{\varphi}$ we denote the reflection of φ along the bisecting line through p_1. I.e., $\tilde{\varphi} := \varphi \circ \tau$. We glue these functions together in the following fashion illustrated in Figure 2.12. Let

$$\begin{aligned} \varphi_1 &:= \varphi, & \varphi_4 &:= -\tilde{\varphi} \circ \sigma^2, \\ \varphi_2 &:= -\tilde{\varphi}, & \varphi_5 &:= \varphi \circ \sigma, \\ \varphi_3 &:= \varphi \circ \sigma^2, & \varphi_6 &:= -\tilde{\varphi} \circ \sigma. \end{aligned}$$

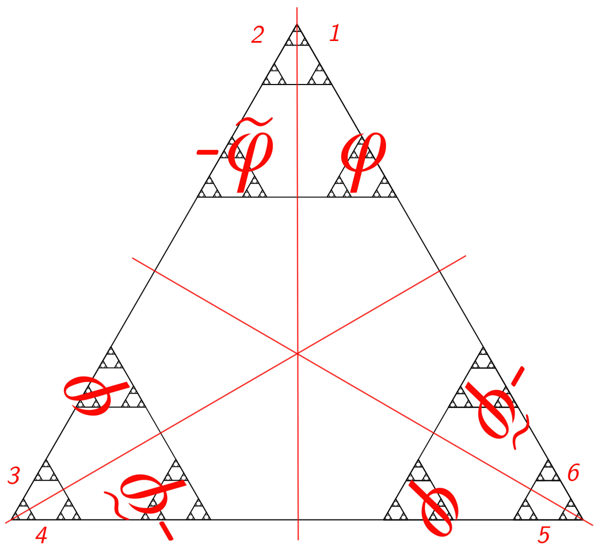

Figure 2.12: Gluing φ.

We denote the resulting function on K by $\Phi := \sum_{i=1}^{6} \varphi_i$. Thanks to symmetry, φ_i is an eigenfunction of $(\mathcal{E}_{\mathcal{R},i}, \mathcal{D}^0_{\mathcal{R},i})$ with measure $\mu_\eta|_{\tilde{K}_i}$ and eigenvalue λ. The Dirichlet conditions and Lemma 2.27 ensure that $\Phi \in \mathcal{D}^0_{\mathcal{R}}$. We now show that Φ itself is a Dirichlet-Neumann eigenfunction, i.e., $\mathcal{E}_{\mathcal{R}}(\Phi, v) = \lambda(\Phi, v)_{\mu_\eta}$ for all $v \in \mathcal{D}_{\mathcal{R}}$. Due to the symmetry we have the following equations:

$$\mathcal{E}_{\mathcal{R}}(\Phi, v) = \mathcal{E}_{\mathcal{R}}(\Phi \circ \tau, v \circ \tau) = \mathcal{E}_{\mathcal{R}}(\Phi, -v \circ \tau), \tag{2.2}$$

$$\mathcal{E}_{\mathcal{R}}(\Phi, v) = \mathcal{E}_{\mathcal{R}}(\Phi \circ \sigma, v \circ \sigma) = \mathcal{E}_{\mathcal{R}}(\Phi, v \circ \sigma), \tag{2.3}$$

$$\mathcal{E}_{\mathcal{R}}(\Phi, v) = \mathcal{E}_{\mathcal{R}}(\Phi \circ \sigma^2, v \circ \sigma^2) = \mathcal{E}_{\mathcal{R}}(\Phi, v \circ \sigma^2). \tag{2.4}$$

From (2.2) we obtain

$$\mathcal{E}_{\mathcal{R}}(\Phi, v) = \mathcal{E}_{\mathcal{R}}(\Phi, \underbrace{\tfrac{v - v\circ\tau}{2}}_{\omega :=}).$$

Now, ω is anti-symmetric with respect to τ and therefore, vanishes on the bisecting line through p_1. If we apply (2.3) and (2.4) to ω we get

$$\mathcal{E}_{\mathcal{R}}(\Phi, v) = \mathcal{E}_{\mathcal{R}}(\Phi, \omega) = \mathcal{E}_{\mathcal{R}}(\Phi, \underbrace{\tfrac{\omega + \omega\circ\sigma + \omega\circ\sigma^2}{3}}_{f :=}).$$

Since $f = -f \circ \tau = f \circ \sigma$ we know that f vanishes on $\bigcup_{i=1}^6 \tilde{V}_i$. That means $f \in \bigoplus_{i=1}^6 \mathcal{D}^0_{\mathcal{R},i}$ by Lemma 2.27 and thus

$$\begin{aligned}
\mathcal{E}_{\mathcal{R}}(\Phi, v) &= \mathcal{E}_{\mathcal{R}}(\Phi, f) \\
&\overset{(i)}{=} \sum_{i=1}^{6} \mathcal{E}_{\mathcal{R},i}(\Phi|_{\tilde{K}_i}, f|_{\tilde{K}_i}) \\
&\overset{(ii)}{=} \sum_{i=1}^{6} \lambda(\Phi|_{\tilde{K}_i}, f|_{\tilde{K}_i})_{\mu_\eta|_{\tilde{K}_i}} \\
&\overset{(iii)}{=} \lambda(\Phi, f)_{\mu_\eta} \\
&\overset{(iv)}{=} \lambda(\Phi, v)_{\mu_\eta}.
\end{aligned}$$

(i) holds due to Lemma 2.27, (ii) since the parts of Φ are eigenfunctions, (iii) is clear and (iv) is true as μ_η fulfills the same symmetries as $\mathcal{E}_{\mathcal{R}}$. Hence, Φ is a pre-localized eigenfunction. □

We can use the same idea as in the case of the Sierpiński gasket to get localized eigenfunctions from the pre-localized ones. Recall that $\mu_\eta^{(n)} = \mu_\eta^w$ for $|w| = n$. In Lemma 2.20 we showed that this was again a convex combination of μ_I and μ_Σ. Therefore, Lemma 2.26 is applicable. Let $u^{(n)}$ be a pre-localized eigenfunction with eigenvalue λ_n of $(\mathcal{E}_{\mathcal{R}^{(n)}}, \mathcal{D}_{\mathcal{R}^{(n)}}, \mu_\eta^{(n)})$. Now, for $w \in \mathcal{A}^n$ define

$$u_w^{(n)}(x) := \begin{cases} u^{(n)} \circ G_w^{-1}(x), & x \in K_w, \\ 0, & \text{otherwise.} \end{cases}$$

Then $u_w^{(n)}$ is a localized eigenfunction of $(\mathcal{E}_{\mathcal{R}}, \mathcal{D}_{\mathcal{R}})$ and $(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R}})$ with measure μ_η and eigenvalue $\frac{\lambda_n}{\delta_n \mu_\eta(K_w)}$, localized in the sense that $\operatorname{supp}(u_w^{(n)}) \subset K_w$.

To show this we notice that $u_w^{(n)} \in C(K)$. Since $u^{(n)} \in \mathcal{D}^0_{\mathcal{R}^{(n)}}$ we have $u_w^{(n)}|_{V_0} \equiv 0$ and $u_w^{(n)} \in \mathcal{D}^0_{\mathcal{R}}$. Here the finiteness of $\mathcal{E}_{\mathcal{R}}(u_w^{(n)}, u_w^{(n)})$ follows from Lemma 2.1 by

$$\begin{aligned}\mathcal{E}_{\mathcal{R}}(u_w^{(n)}, u_w^{(n)}) &= \frac{1}{\delta_n}\mathcal{E}_{\mathcal{R}^{(n)}}(u^{(n)} \circ G_w^{-1} \circ G_w, u^{(n)} \circ G_w^{-1} \circ G_w)\\ &= \frac{1}{\delta_n}\mathcal{E}_{\mathcal{R}^{(n)}}(u^{(n)}, u^{(n)})\\ &< \infty,\end{aligned}$$

as $u^{(n)} \in \mathcal{D}_{\mathcal{R}^{(n)}}$. Now, for all $v \in \mathcal{D}_{\mathcal{R}}$ we have

$$\begin{aligned}\mathcal{E}_{\mathcal{R}}(u_w^{(n)}, v) &= \frac{1}{\delta_n}\mathcal{E}_{\mathcal{R}^{(n)}}(u^{(n)}, \underbrace{v \circ G_w}_{\in \mathcal{D}_{\mathcal{R}^{(n)}}})\\ &= \frac{1}{\delta_n}\lambda_n(u^{(n)}, v \circ G_w)_{\mu_\eta^w}\\ &= \frac{\lambda_n}{\delta_n}\int_K u^{(n)} \cdot v \circ G_w \; d\mu_\eta^w\\ &= \frac{\lambda_n}{\delta_n}\frac{1}{\mu_\eta(K_w)}\int_{K_w} u^{(n)} \circ G_w^{-1} \cdot v \; d\mu_\eta\\ &= \frac{\lambda_n}{\delta_n \mu_\eta(K_w)}(u_w^{(n)}, v)_{\mu_\eta}.\end{aligned}$$

Therefore, $u_w^{(n)}$ is an eigenfunction of $(\mathcal{E}_{\mathcal{R}}, \mathcal{D}_{\mathcal{R}}, \mu_\eta)$ and since $u_w^{(n)} \in \mathcal{D}^0_{\mathcal{R}}$, it is also one of $(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R}}, \mu_\eta)$. We can do this for any n-cell, which implies the multiplicity of the eigenvalue $\frac{\lambda_n}{\delta_n \mu_\eta(K_w)}$ is at least 3^n because there are that many n-cells. We define $\mu_\eta(n) := \mu_\eta(K_w)$.

We thus have shown the following result.

Lemma 2.28. *Let $\mathcal{R}$ be any sequence of matching pairs and $\eta \in (0,1]$. Then for all $n \in \mathbb{N}$ and $w \in \mathcal{A}^n$ there exists an eigenfunction u_w of $-\Delta_D^{\mu_\eta,\mathcal{R}}$ with $\operatorname{supp}(u_w) \subset K_w$ and multiplicity of the corresponding eigenvalue at least 3^n.*

However, the pre-localized eigenfunction $u^{(n)}$ with eigenvalue λ_n depend on $\mathcal{R}^{(n)}$ and $\mu_\eta^{(n)}$ meaning it is a different one for every n. In particular λ_n may not be the same for all n. This is a different situation in comparison to the self-similar case. In the case of the Sierpiński gasket there was only one pre-localized eigenfunction necessary.

The other scaling parameters δ_n and $\mu_\eta(n)$ are the right ones but to be able to calculate the growing rate of the eigenvalues of localized eigenfunctions we need further information about λ_n such as if it is bounded. Since our proof of Lemma 2.26 was slightly different than the one in [8] we can use this to get estimates on λ_n.

In the proof of Lemma 2.26 we saw that the eigenvalue of Φ is the same as the one of φ

with respect to $(\mathcal{E}_{\mathcal{R},1}, \mathcal{D}^0_{\mathcal{R},1}, \mu_\eta)$. We are able to get estimates on the first eigenvalue there by

$$\lambda_1 = \inf_{u \in \mathcal{D}^0_{\mathcal{R},1}} \frac{\mathcal{E}_{\mathcal{R},1}(u,u)}{||u||^2_{\mu_\eta}}.$$

We would like to find a function $u \in \mathcal{D}^0_{\mathcal{R},1}$ such that this value is bounded uniformly from above for all $\mathcal{R}^{(n)}$ and all μ_η with $\eta \in (0,1]$. To do this we choose the values of u on $V_3 \cap \tilde{K}_1$ as indicated in Figure 2.13 and extend harmonically. We only consider sequences $\mathcal{R}$ that fulfill Condition 2.1.

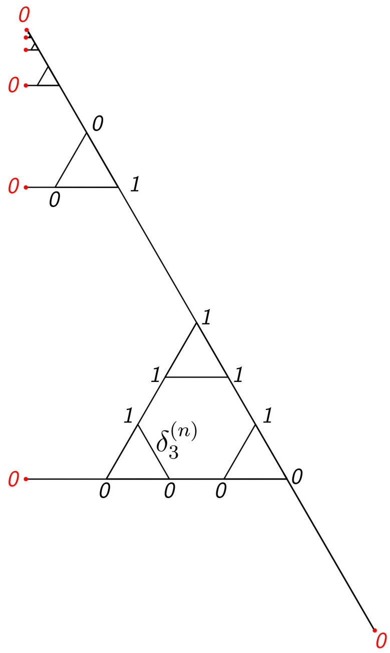

Figure 2.13: Construction of u on $\tilde{K}_1$.

Then the energy of u is

$$\mathcal{E}_{\mathcal{R}^{(n)},1}(u,u) = 6\frac{1}{\delta_3^{(n)}} \leq \frac{6}{\kappa_1 r^3}.$$

Also since we have a 3-cell $K_{\tilde{w}}$ where u is constant 1 we get an estimate on the L^2-norm by

$$||u||^2_{\mu_\eta} = \int_{\tilde{K}_1} u^2 d\mu_\eta \geq \int_{K_{\tilde{w}}} 1 d\mu_\eta = \mu_\eta(K_{\tilde{w}}) = \eta\beta^3 + (1-\eta)(\tfrac{1}{3})^3 \geq \beta^3.$$

Therefore, the first eigenvalue λ_1 of $(\mathcal{E}_{\mathcal{R}^{(n)},1}, \mathcal{D}^0_{\mathcal{R}^{(n)},1})$ with measure $\mu_\eta^{(n)}|_{\tilde{K}_1 \setminus \tilde{V}_1}$ is bounded as

$$\lambda_1 \leq \frac{6}{\kappa_1 r^3 \beta^3}.$$

This constant is independent of n as well as η.

Choose as λ_n the Dirichlet-Neumann eigenvalue which corresponds to the first Dirichlet eigenfunction φ of $(\mathcal{E}_{\mathcal{R}^{(n)},1}, \mathcal{D}^0_{\mathcal{R}^{(n)},1})$ with measure $\mu_\eta^{(n)}$. Thus, the resulting localized eigenfunctions on the stretched Sierpiński gasket give us a sequence of eigenvalues

$$\nu_n = \frac{\lambda_n}{\delta_n \mu_\eta(n)} \leq \begin{cases} \tilde{c}\left(\frac{3}{r}\right)^n, & \eta \in (0,1), \\ \tilde{c}(\beta r)^{-n}, & \eta = 1, \end{cases}$$

with multiplicities at least 3^n.

For completeness we also give a lower bound for λ_n. We do not really need it for the argument, but nonetheless, it shows that the localized eigenfunctions are indeed responsible for the asymptotic growing of the eigenvalue counting function. From [5, Lemma 7.19] we know that

$$|u(p) - u(q)|^2 \leq 16\mathcal{E}_{\mathcal{R}}(u,u),$$

for all $u \in \mathcal{D}_{\mathcal{R}}$. Essentially this means that the diameter of K with respect to the resistance metric is bounded from above by 16, which is in particular independent of $\mathcal{R}$. For $u \in \mathcal{D}^0_{\mathcal{R},1}$ and $p_1 \in V_0$ we have

$$\begin{aligned} \mathcal{E}_{\mathcal{R},1}(u,u) &\geq \frac{1}{16}|u(x) - u(p_1)|^2 \\ &= \frac{1}{16}|u(x)|^2. \end{aligned}$$

Integrating this over K with respect to μ_η leads to

$$\begin{aligned} \mathcal{E}_{\mathcal{R},1}(u,u) &\geq \frac{1}{16}\int_K |u(x)|^2 d\mu_\eta \\ &= \frac{1}{16}||u||^2_{\mu_\eta}. \end{aligned}$$

Therefore, the first Dirichlet eigenvalue of $(\mathcal{E}_{\mathcal{R},1}, \mathcal{D}^0_{\mathcal{R},1})$ with measure μ_η is at least $\frac{1}{16}$. This is independent of the measure μ_η and the sequence of matching pairs $\mathcal{R}$. Thus we have

$$\lambda_n \geq \frac{1}{16}.$$

All together we have found a sequence of eigenvalues ν_n with multiplicities at least 3^n such that

$$\left.\begin{array}{l} \tilde{c}_1\left(\frac{3}{r}\right)^n \\ \tilde{c}_1\left(\beta r\right)^{-n} \end{array}\right\} \leq \nu_n \leq \begin{cases} \tilde{c}_2\left(\frac{3}{r}\right)^n, & \eta \in (0,1), \\ \tilde{c}_2\left(\beta r\right)^{-n}, & \eta = 1, \end{cases}$$

with constants $0 < \tilde{c}_1 \leq \tilde{c}_2 < \infty$ independent of n.

We are now able to show that we cannot have convergence if $r \in [\frac{1}{3}, \frac{3}{5}]$. Again we emphasize that the upper estimate is the one we need. The lower estimate was actually already implied by Theorem 2.13. Let $\eta \in (0,1)$. From Theorem 2.13 we know that

$$\limsup_{x\to\infty} N_D^{\mu_\eta,\mathcal{R}}(x)x^{-\frac{1}{2}d_S^{\mu_\eta,\mathcal{R}}} - \liminf_{x\to\infty} N_D^{\mu_\eta,\mathcal{R}}(x)x^{-\frac{1}{2}d_S^{\mu_\eta,\mathcal{R}}} < \infty.$$

With $N_D^{\mu_\eta,\mathcal{R}}(x)_- := \lim_{\epsilon\searrow 0} N_D^{\mu_\eta,\mathcal{R}}(x-\epsilon)$ we have

$$\begin{aligned}
\limsup_{x\to\infty} N_D^{\mu_\eta,\mathcal{R}}(x)x^{-\frac{1}{2}d_S^{\mu_\eta,\mathcal{R}}} &- \liminf_{x\to\infty} N_D^{\mu_\eta,\mathcal{R}}(x)x^{-\frac{1}{2}d_S^{\mu_\eta,\mathcal{R}}} \\
&\geq \limsup_{n\to\infty} N_D^{\mu_\eta,\mathcal{R}}(\nu_n)\cdot \nu_n^{-\frac{1}{2}d_S^{\mu_\eta,\mathcal{R}}} - \liminf_{n\to\infty} N_D^{\mu_\eta,\mathcal{R}}(\nu_n)_- \cdot \nu_n^{-\frac{1}{2}d_S^{\mu_\eta,\mathcal{R}}} \\
&\geq \limsup_{n\to\infty} \left(N_D^{\mu_\eta,\mathcal{R}}(\nu_n) - N_D^{\mu_\eta,\mathcal{R}}(\nu_n)_-\right)\cdot \nu_n^{-\frac{1}{2}d_S^{\mu_\eta,\mathcal{R}}} \\
&\geq \limsup_{n\to\infty} 3^n \cdot \nu_n^{-\frac{1}{2}d_S^{\mu_\eta,\mathcal{R}}} \\
&\geq \lim_{n\to\infty} 3^n \cdot \left(\tilde{c}_2(\tfrac{3}{r})^n\right)^{-\frac{1}{2}d_S^{\mu_\eta,\mathcal{R}}} \\
&= \lim_{n\to\infty} 3^n \cdot \tilde{c}_2^{-\frac{1}{2}d_S^{\mu_\eta,\mathcal{R}}} \cdot 3^{-n} \\
&= \tilde{c}_2^{-\frac{1}{2}d_S^{\mu_\eta,\mathcal{R}}} > 0.
\end{aligned}$$

The calculation for $\eta = 1$ is analogous. This closes the proof of the following theorem.

Theorem 2.29 (Non-convergence on the stretched Sierpiński gasket).
Let $\mathcal{R}$ be a sequence of matching pairs that fulfills Condition 2.1 with $r \in [\frac{1}{3}, \frac{3}{5}]$ and let $\eta \in (0,1]$. If $\eta = 1$ let $\beta > \frac{1}{9r}$. Then we have

$$\liminf_{x\to\infty} N_D^{\mu_\eta,\mathcal{R}}(x)x^{-\frac{1}{2}d_S^{\mu_\eta,\mathcal{R}}} < \limsup_{x\to\infty} N_D^{\mu_\eta,\mathcal{R}}(x)x^{-\frac{1}{2}d_S^{\mu_\eta,\mathcal{R}}}.$$

2.7.5 Special cases with more symmetry

We were not able to show periodicity in the general setting. This is due to the broad option of choosing the sequences of matching pairs, which destroys the symmetry needed to show periodicity. But we can look at some special sequences to give us back the symmetry.

One problem was that the measures μ_η were not self-similar for $\eta \in (0,1)$. We need to get estimates on the L^2-norms. To avoid this we could choose the measure $\mu_1 = \mu_I$ which by itself is self-similar in the sense that $\mu_I^{(n)} = \mu_I$ for all n. In Lemma 2.25 we reached the following estimates for sequences of matching pairs $\mathcal{R}$ that fulfill Condition 2.1 with constants $\kappa_3, \kappa_4 > 0$ such that $\kappa_3 \leq \frac{\rho_{n+k}}{\rho_k} \leq \kappa_4$ for all $n, k \geq 1$:

$$3^n N_D^{\mu_I,\mathcal{R}}(c_1(\beta r)^n x) \leq N_D^{\mu_I,\mathcal{R}}(x) \leq 3^n N_D^{\mu_I,\mathcal{R}}(c_2(\beta r)^n x) + 3^{n+1} + N(\mathcal{E}_{\mathcal{R},J_n}, \mathcal{D}_{\mathcal{R},J_n}, \mu_I, x)$$

with $c_1 \leq 1 \leq c_2$ for all n. In general these constants c_1 and c_2 are not 1, even asymptotically as $n \to \infty$. However, there is a situation where they are 1. This is the case if the quadratic forms $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}^{(n)}}$ coincide, which can be achieved by choosing constant sequences of matching pairs:

$$(r_i, \rho_i) = (r, \rho), \ \forall i.$$

If the sequence is constant we have $\mathcal{R} = \mathcal{R}^{(n)}$ for all $n \geq 0$ and thus $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}^{(n)}}$. This means we get the following rescaling of the eigenvalue counting function:

$$3N_D^{\mu_I,\mathcal{R}}(\beta r x) \leq N_D^{\mu_I,\mathcal{R}}(x) \leq 3N_D^{\mu_I,\mathcal{R}}(\beta r x) + 9 + N(\mathcal{E}_{\mathcal{R},J_1}, \mathcal{D}_{\mathcal{R},J_1}, \mu_I, x).$$

We would like to apply renewal theory to the Dirichlet eigenvalue counting function to show the existence of log-periodic behavior. The version of the renewal theorem we use can be found in [42] and it is a refinement of the version from [38], which itself is a modification of the one from Feller [17].

Theorem 2.30 (Renewal theorem, Kigami [42, Theorem A.1],[43, Theorem B.4.3])**.** *Let f be a real-valued function on $\mathbb{R}$ with $f(t) \to 0$ as $t \to -\infty$. Suppose f satisfies a renewal equation*

$$f(t) = \sum_{j=1}^{N} f(t - m_j T) p_j + u(t),$$

where $T \in (0, \infty)$, $m_1, m_2, \ldots, m_N$ are positive integers whose greatest common divider is 1, $\sum_{j=1}^N p_j = 1$ and $p_j > 0$ for all j. Also assume that $u : \mathbb{R} \to \mathbb{R}$ is bounded and that $\sum_{j=-\infty}^{\infty} |u_j(t)|$ converges uniformly on $[0, T]$, where $u_j(t) := u(t + jT)$ for $t \in \mathbb{R}$. Set $G(t) := (\sum_{j=1}^N m_j p_j)^{-1} \sum_{j=-\infty}^{\infty} u_j(t)$. Then as $t \to \infty$, $|f(t) - G(t)| \to 0$.

Moreover, set $Q(z) := (1 - \sum_{j=1}^N p_j z^{m_j})/(1 - z)$ and define $\beta := \min\{|z| : Q(z) = 0\}$ and $m := \max\{$multiplicity of $Q(z) = 0$ at $w : |w| = \beta, Q(w) = 0\}$. If there exist $C > 0$ and $\alpha > 1$ such that $|u(t)| \leq C\alpha^{-t}$ for all $t \in \mathbb{R}$, then, as $t \to \infty$,

$$|G(t) - f(t)| = \begin{cases} \mathcal{O}(t^{m-1}\beta^{-t/T}) & \text{if } \alpha^T > \beta, \\ \mathcal{O}(t^m \alpha^{-t}) & \text{if } \alpha^T = \beta, \\ \mathcal{O}(\alpha^{-t}) & \text{if } \alpha^T < \beta. \end{cases}$$

For the sake of simplicity of the notation we omit μ_I and $\mathcal{R}$ in the notation $N_D^{\mu_I,\mathcal{R}}(x)$, since they are always fixed, and we only write $N_D(x)$. We do the same for $d_S^{\mu_I,\mathcal{R}}$ and define

$$\begin{aligned} R(x) &:= N_D(x) - 3N_D(\beta r x), \\ f(t) &:= e^{-t d_S} N_D(e^{2t}), \\ u(t) &:= e^{-t d_S} R(e^{2t}), \\ T &:= -\ln \sqrt{\beta r}. \end{aligned}$$

We see that f and u are right-continuous and $f(t) \to 0$ as $t \to -\infty$. With $N = 3$, $m_j = 1$

and $p_j = \frac{1}{3}$ for all j, and assuming $r > \frac{1}{9\beta}$, we have

$$\begin{aligned}\sum_{j=1}^{N} f(t+\ln\sqrt{\beta r})\frac{1}{3} &= f(t+\ln\sqrt{\beta r}) \\ &= e^{-(t+\ln\sqrt{\beta r})d_S} N_D(e^{2(t+\ln\sqrt{\beta r})}) \\ &= e^{-td_S}\,(\beta r)^{-\frac{d_S}{2}}\, N_D(e^{2t}\beta r) \\ &= e^{-td_S} 3 N_D(e^{2t}\beta r) \\ &= e^{-td_S}(N_D(e^{2t}) - R(e^{2t})) \\ &= f(t) - u(t).\end{aligned}$$

Therefore, they fulfill the renewal equation. We need to show that $\sum_{j=-\infty}^{\infty} |u(t+jT)|$ converges uniformly in $t \in [0,T]$. The first Dirichlet eigenvalue λ_1^D is positive and thus $R(x) = 0$ for all $x < \lambda_1^D$. This means there is a j_0 such that

$$\sum_{j=-\infty}^{\infty} |u(t+jT)| = \sum_{j=j_0}^{\infty} |u(t+jT)|.$$

Furthermore, since $R(x) = 0$ for $x < \lambda_1^D$ and $0 \leq R(x) \leq 9 + N(\mathcal{E}_{\mathcal{R},J_1}, \mathcal{D}_{\mathcal{R},J_1}, \mu_I, x)$ we find a constant $c_0 \geq 0$ such that

$$0 \leq R(x) \leq c_0 x^{\frac{1}{2}}$$

for all $x \geq 0$, leading to

$$0 \leq u(t) = e^{-td_S} R(e^{2t}) \leq c_0 e^{-t(d_S-1)} \tag{2.5}$$

for all $t \in \mathbb{R}$, and thus for all $t \in [0,T]$ we have

$$0 \leq u(t+jT) \leq c_0 e^{-jT(d_S-1)}$$

and

$$\begin{aligned}\sum_{j=-\infty}^{\infty} |u(t+jT)| &= \sum_{j=j_0}^{\infty} |u(t+jT)| \\ &\leq \sum_{j=j_0}^{\infty} c_0 e^{-(t+jT)(d_S-1)} \\ &= c_0 e^{-t(d_S-1)} \sum_{j=j_0}^{\infty} e^{-jT(d_S-1)}.\end{aligned}$$

To make sure that this sum converges we need $d_S > 1$. In this case the sum converges independently of t and therefore, we have uniform convergence. We have $Q(z) = 1$ and thus $\beta = \infty$ meaning we are in the third case of Theorem 2.30. It follows from (2.5) that

$$|G(t) - f(t)| \leq \tilde{c}_0 e^{-t(d_S-1)}.$$

Substituting $x = e^{2t}$ leads to

$$|G(\tfrac{\ln x}{2}) - x^{-\frac{1}{2}d_S} N_D(x)| \leq \tilde{c}_0 x^{-\frac{1}{2}(d_S - 1)}.$$

Theorem 2.31. *Let $\mathcal{R} = \{(r_i, \rho_i)\}_{i\geq 1}$ be a sequence of matching pairs with $r_i = r$ for all i and $\mu = \mu_I$ with $\frac{1}{9\beta} < r < \frac{3}{5}$. Then there is a right-continuous positive non-constant periodic function G with period $T = -\ln\sqrt{\beta r}$ such that*

$$N_D^{\mu_I,\mathcal{R}}(x) = G\left(\frac{\ln x}{2}\right) x^{\frac{1}{2}d_S^{\mu_I,\mathcal{R}}} + \mathcal{O}(x^{\frac{1}{2}}) \quad \text{as } x \to \infty$$

with $d_S^{\mu_I,\mathcal{R}} = \frac{\ln 9}{-\ln(\beta r)}$.

Proof. We can apply Theorem 2.13 to get the leading order $d_S^{\mu_I,\mathcal{R}}$, for which $d_S^{\mu_I,\mathcal{R}} > 1$ holds due to the conditions on β and r. From the calculations in this section and the Renewal theorem 2.30 we get the right-continuous positive periodic function G with period $T = -\ln\sqrt{\beta r}$ as well as the remainder estimate. From Theorem 2.29 we know that G is non-constant. $\square$

Another special case is where $\mathcal{R}$ is periodic. Let $\mathcal{R}$ be a sequence of matching pairs, such that there is an $n \in \mathbb{N}$ with

$$\mathcal{R}^{(n)} = \mathcal{R}.$$

Then $\mathcal{R}$ fulfills Condition 2.1 with $r = \delta_n^{\frac{1}{n}}$ and we get the asymptotics by applying Theorem 2.13 with the rescaling

$$\mathcal{E}_{\mathcal{R}}(u,u) = \sum_{w\in\mathcal{A}^{n\cdot k}} \frac{1}{(\delta_n)^k} \mathcal{E}_{\mathcal{R}^{(n\cdot k)}}(u \circ G_w, u \circ G_w) + \sum_{j=1}^{n\cdot k} \frac{1}{\gamma_j} \mathcal{D}_j^I(u,u)$$

for all $k \geq 1$. With $\mu = \mu_I$ we get the following estimates for the eigenvalue counting functions:

$$3^n N_D^{\mu_I,\mathcal{R}}(\beta^n \delta_n x) \leq N_D^{\mu_I,\mathcal{R}}(x) \leq 3^n N_D^{\mu_I,\mathcal{R}}(\beta^n \delta_n x) + 3^{n+1} + N(\mathcal{E}_{\mathcal{R},J_n}, \mathcal{D}_{\mathcal{R},J_n}, \mu_I, x).$$

If $\frac{1}{9\beta} < \delta_n^{\frac{1}{n}} < \frac{3}{5}$ we get the asymptotic growing with

$$d_S^{\mu_I,\mathcal{R}} = \frac{\ln 9}{-\ln(\beta\delta_n^{\frac{1}{n}})}.$$

Also in this case we can apply the renewal theorem in Theorem 2.30 and the existence of localized eigenfunctions from Section 2.7.4 and obtain that $N_D^{\mu_I,\mathcal{R}}(x) x^{-\frac{1}{2}d_S^{\mu_I,\mathcal{R}}}$ does not converge. In particular there exists a right-continuous, positive and non-constant periodic function G with period $-\frac{1}{2}\ln(\beta^n\delta_n)$ such that

$$N_D^{\mu_I,\mathcal{R}}(x) = G\left(\frac{\ln x}{2}\right) x^{\frac{1}{2}d_S^{\mu_I,\mathcal{R}}} + \mathcal{O}(x^{\frac{1}{2}}) \quad \text{as } x \to \infty.$$

It is easy to see that this is the only case with $\mathcal{E}_{\mathcal{R}^{(n)}} = \mathcal{E}_{\mathcal{R}}$. That means, in all other cases the constants c_1, c_2 from the estimates in Lemma 2.24 are not equal.

3 Generalization to stretched fractals

In Chapter 2 we treated the stretched Sierpiński gasket. Now, in this chapter we generalize the idea of *stretching* to more self-similar sets. We will call the resulting sets *stretched fractals*.

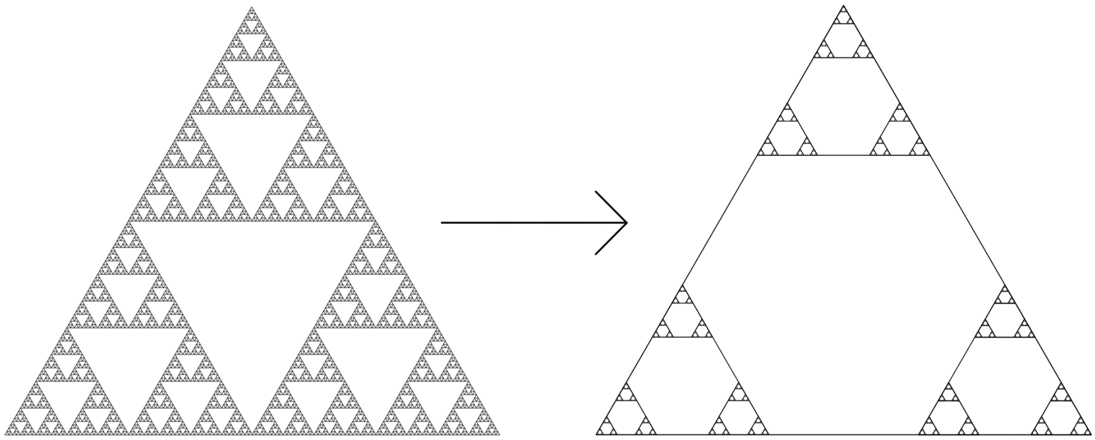

Figure 3.1: Stretching the Sierpiński gasket.

Chapter 3 is structured as follows. In Section 3.1 we construct stretched fractals and include some examples. In Section 3.2 we build a sequence of approximating graphs and introduce the notion of *regular sequences of harmonic structures*, which is a generalization of *regular harmonic structures* in the self-similar case. Afterwards, we construct resistance forms on stretched fractals in Section 3.3, presuming we have a regular sequence of harmonic structures. In Section 3.4 we describe measures on stretched fractals, which allows us to get Dirichlet forms from the resistance forms and thus the self-adjoint operators that we would like to study. In Section 3.5 we introduce a condition that is necessary to calculate both Hausdorff dimension in the resistance metric and the leading order of the eigenvalue counting function, which is done in Section 3.6 resp. Section 3.7. Lastly, in Section 3.8 we mention the refinements done for the stretched Sierpiński gasket in Section 2.7 in the general setting and give some ideas how to solve the questions of periodicity and non-convergence. We can reuse most of the ideas from Chapter 2, but we have to refine some arguments for greater generality.

3.1 Stretched fractals

The stretched Sierpiński gasket is constructed by lowering the contraction ratios of the similitudes of the self-similar Sierpiński gasket and filling the arising holes with one-dimensional lines; see Figure 3.1.

We would like to generalize this concept of stretching to more self-similar fractals. For the Sierpiński gasket S it was essential that two copies $F_i(S)$ and $F_j(S)$ only intersect at a single point. Therefore, it is clear how we should connect these copies if we stretch them apart. In general, we need the fractal that we would like to stretch to be finitely ramified. In this case we can connect the copies that get stretched away from each other by one-dimensional lines. These are the so-called *p.c.f. self-similar fractals* introduced by Kigami in [37]. We have included a short survey about p.c.f. self-similar sets in Appendix A. We introduce three conditions (C1),(C2) and (C3), which we call *connectedness conditions* and which we need to define stretched fractals.

3.1.1 Definition of stretched fractals

Let $(F_1, \ldots, F_N)$ be the *Iterated Function System* (IFS for short) of a connected p.c.f. self-similar set $F \subset \mathbb{R}^d$. That means F is a non-empty compact subset of $\mathbb{R}^d$ such that

$$F = \bigcup_{i=1}^{N} F_i(F),$$

where F_i are contracting similitudes with distinct unique fixed points q_i.

We will introduce some notation that is commonly used. We denote the alphabet by $\mathcal{A} := \{1, \ldots, N\}$ and the set of all words of finite length $\mathcal{A}^* := \bigcup_{n\geq 1} \mathcal{A}^n$ and $\mathcal{A}_0^* := \bigcup_{n\geq 0} \mathcal{A}^n$ if we also would like to include the empty word. For $w = w_1 \cdots w_n \in \mathcal{A}^n$ we denote by $F_w := F_{w_1} \circ \ldots \circ F_{w_n}$ the composition of the similitudes and by $F_w := \mathrm{id}$ the identity if $w = \emptyset$ is the empty word. Furthermore $F_w(F)$ is called an *n-cell* of F if $|w| = n$.

Let us define the *critical set* $\mathcal{C}$. This set plays an important role in the construction of stretched fractals. Let

$$\mathcal{C} := \bigcup_{\substack{i,j\in\mathcal{A}\\ i\neq j}} F_i(F) \cap F_j(F).$$

That means $\mathcal{C}$ is the set of the points where 1-cells of F meet. In contrast to Appendix A our critical set $\mathcal{C}$ is a subset of F, whereas in the literature the critical set $\tilde{\mathcal{C}}$ is a subset of the shift space $\mathcal{A}^{\mathbb{N}}$. With this we define the so-called *post critical set* $\mathcal{P}$ by

$$\mathcal{P} := \{x \in F \mid \exists w \in \mathcal{A}^* : F_w(x) \in \mathcal{C}\}.$$

Again, these are also points of F instead of the shift space $\mathcal{A}^{\mathbb{N}}$. The post critical set $\mathcal{P}$ consists of all points that get mapped to the critical set $\mathcal{C}$ by some finite composition of the similitudes $F_1, \ldots, F_N$. For p.c.f. self-similar sets we know that $\#\mathcal{P} < \infty$. For nested fractals $\mathcal{P}$ is made up of the essential fixed points; see [37, Example 8.5]. In general $\mathcal{P}$ can have elements that are no fixed points; for example, see Hata's tree in Section 3.1.2. To be still able to stretch these fractals we need to make an assumption on $\mathcal{P}$. We only consider connected p.c.f. self-similar sets, such that

$$\forall p \in \mathcal{P}\ \exists w \in \mathcal{A}_0^*\ \exists q \in \{q_i \mid i \in \mathcal{A}\} \text{ such that } p = F_w(q). \tag{C1}$$

That means, each post critical point is the image of a fixed point under some finite composition of the similitudes or one itself. This is obviously true for nested fractals and it is also true for Hata's tree which is not a nested fractal.

We would like to introduce a quantity that describes the *level of connectedness* at points of $\mathcal{C}$. This value is called the *multiplicity* of a point $c \in \mathcal{C}$ and it counts how many 1-cells of F meet at c. We can define this value for all $x \in F$ by

$$\rho(x) := \#\{i \in \mathcal{A} \mid x \in F_i(F)\}.$$

We can also count the n-cells of F that meet at x:

$$\rho_n(x) := \#\{w \in \mathcal{A}^n \mid x \in F_w(F)\}.$$

For nested fractals it was proved by Lindstrøm in [47, Proposition IV.16] that $\rho(c) = \rho_n(c)$ for all $c \in \mathcal{C}$ and all $n \geq 1$. This tells us that the n-cells are connected in the same way as the 1-cells. This is something which we are going to use throughout the construction of resistance forms and thus we require our p.c.f. self-similar set to fulfill this second connectedness condition. We only consider connected p.c.f. self-similar sets, such that

$$\rho(c) = \rho_n(c), \text{ for all } c \in \mathcal{C} \text{ and } n \geq 1. \tag{C2}$$

Actually, the proof for nested fractals in [47, Proposition IV.16] mainly uses the nesting property, which is also true for all p.c.f. self-similar sets; see [37]. Thus it is possible that condition (C2) holds true in general for p.c.f. self-similar sets.

Now the critical set $\mathcal{C}$ together with $\rho(c)$ for all $c \in \mathcal{C}$ describes how the fractal is connected. In particular we have $\rho(c) \geq 2$ for all $c \in \mathcal{C}$. Next we wish to be able to say which post critical points get mapped to $c \in \mathcal{C}$. For all $c \in \mathcal{C}$ there are

$$w^{c,1}, \dots, w^{c,\rho(c)} \in \mathcal{A}^*$$

and fixed points

$$q_1^c, \dots, q_{\rho(c)}^c \in \mathcal{P},$$

such that

$$F_{w^{c,l}}(q_l^c) = c, \ \forall l \in \{1, \dots, \rho(c)\},$$

where $w_1^{c,l}$ are pairwise distinct. We can do this since we consider p.c.f. self-similar sets that fulfill (C1). The first letters $w_1^{c,l}$ are different, which indicates that c belongs to $\rho(c)$ many different 1-cells.

We can now define a new IFS $(G_1, \ldots, G_N)$ with $0 < \alpha < 1$ by

$$G_i := \alpha(F_i - q_i) + q_i.$$

This procedure lowers the contraction ratio of F_i by multiplying it by α and it does this in a way that preserves the fixed point. It sort of compresses the image of F_i linearly into its fixed point. We define Σ_α as the unique non-empty compact solution of

$$\Sigma_\alpha = \bigcup_{i=1}^{N} G_i(\Sigma_\alpha).$$

By changing the similitudes it could happen that new intersections of the 1-cells of Σ_α appear. This is something we would like to prevent. Rather we would like to ensure that Σ_α is a totally disconnected set. By Theorem A.5 we know that every p.c.f. self-similar set satisfies the *open set condition*. We only consider connected p.c.f. self-similar sets such that

$$F \text{ satisfies the } \textit{convex open set condition}, \tag{C3}$$

i.e. F satisfies the open set condition with an convex open set O.

Lemma 3.1. *Let $(F_1, \ldots, F_N)$ be the IFS of a connected p.c.f. self-similar set that fulfills condition (C3). Then Σ_α is totally disconnected.*

Proof. We have a convex open set $O \subset \mathbb{R}^d$ such that

$$F_i(O) \cap F_j(O) = \emptyset,\ \forall i \neq j$$

and

$$F_i(O) \subset O,\ \forall i \in \mathcal{A}.$$

Since O is convex we know that $G_i(O) \subset F_i(O)$ for all $i \in \mathcal{A}$. Thus Σ_α with the IFS $(G_1, \ldots, G_N)$ also satisfies the open set condition with the same open set O. Since $q_i \notin \mathcal{C}$, which can easily be seen using [43, Lemma 1.3.14], we have

$$\operatorname{dist}(G_i(O), F_i(O)^c) = \inf\{d(x,y) \mid x \in G_i(O), y \in F_i(O)^c\} > 0$$

for all $i \in \mathcal{A}$. Therefore, all copies $G_i(\Sigma_\alpha)$ are positively separated and thus Σ_α is totally disconnected. $\square$

The copies were connected at the critical set $\mathcal{C}$. We would like to save the degree of connectedness by introducing one-dimensional lines reconnecting the copies. For all $c \in \mathcal{C}$ we define

$$e_{c,l} := \{\lambda G_{w^{c,l}}(q_l^c) + (1-\lambda)c \mid \lambda \in [0,1]\},\ \forall l \in \{1, \ldots, \rho(c)\}.$$

Note that $G_{w^{c,l}}(q_l^c)$ is the point that got stretched away from c. Since the fixed points $q_i \notin \mathcal{C}$ for all $i \in \mathcal{A}$, by [43, Lemma 1.3.14], we know that $e_{c,l}$ is a one-dimensional object for all c, l.

For $w \in \mathcal{A}^*$ we denote $e^w_{c,l} := G_w(e_{c,l})$. Now we can define the stretched fractal associated to the p.c.f. self-similar set F.

Definition 3.2. Let $(F_1, \dots, F_N)$ be the IFS of a connected p.c.f. self-similar set F that fulfills the connectedness conditions (C1), (C2) and (C3). The unique non-empty compact set K_α that fulfills the equation

$$K_\alpha = \bigcup_{i=1}^{N} G_i(K_\alpha) \cup \bigcup_{c \in \mathcal{C}} \bigcup_{l=1}^{\rho(c)} e_{c,l}$$

is called the *stretched fractal* associated to F. The unique non-empty compact set Σ_α that satisfies

$$\Sigma_\alpha = \bigcup_{i=1}^{N} G_i(\Sigma_\alpha)$$

is called the *fractal part*, whereas

$$J_\alpha = \bigcup_{\substack{c \in \mathcal{C}, w \in \mathcal{A}_0^* \\ l \in \{1, \dots, \rho(c)\}}} e^w_{c,l}$$

is the *line part* of K_α.

We can imagine the construction by fixing the points of $\mathcal{C}$, stretching the copies away from each $c \in \mathcal{C}$ and then adding lines connecting the copies with c like a "spider's web". Since the fixed points of G_i and F_i are the same we ensure that q^c_l and thus $G_{w^{c,l}}(q^c_l)$ are elements of Σ_α. By this K_α is a connected set. Therefore, (C1) ensures connectedness. We will include a few examples of stretched fractals at the end of this section.

Solutions of equations like the one in Definition 3.2 are already known. Barnsley denoted such a setting in [9, Chapter 3.4] by *IFS with condensation* where $\bigcup_{c \in \mathcal{C}} \bigcup_{l=1}^{\rho(c)} e_{c,l}$ is called the *condensation set*. Since this is compact, so is the unique solution K_α. In [21] Fraser called such a solution an *inhomogeneous self-similar set* and calculated its box dimension. This is much harder than to calculate the Hausdorff dimension since the box dimension is not countably stable. In particular the lower box dimension is not even finitely stable. The Hausdorff dimension with respect to the Euclidean metric d_E is calculated very easily due to its countable stability. From [52, Lemma 3.9] we know with the so-called *orbital set*

$$\mathcal{O} = \bigcup_{w \in \mathcal{A}_0^*} G_w \left(\bigcup_{c \in \mathcal{C}} \bigcup_{l=1}^{\rho(c)} e_{c,l} \right)$$

that

$$K_\alpha = \Sigma_\alpha \cup \mathcal{O} = \overline{\mathcal{O}}.$$

Proposition 3.3. *Let K_α be a stretched fractal from Definition 3.2. Then*

$$\dim_{H,d_E}(K_\alpha) = \max\{\dim_{H,d_E}(\Sigma_\alpha), 1\}.$$

This value strongly depends on the stretching parameter α. The resistance forms, however, will only depend on the topology which does not depend on α.

Proposition 3.4. *The K_α are pairwise homeomorphic for different α.*

Proof. We denote by G_w^α the similitudes which correspond to K_α as well as $e_{c,l}^{\alpha,w}$ for $w \in \mathcal{A}_0^*$. We know that Σ_α is homeomorphic to $\mathcal{A}^\mathbb{N}$ by the coding map ι^α which maps $\mathcal{A}^\mathbb{N}$ to Σ_α by

$$\iota^\alpha(w) = \bigcap_{n\geq 1} G_{w_1\cdots w_n}^\alpha(K_\alpha).$$

For $\alpha_1, \alpha_2 \in (0,1)$ we thus know that Σ_{α_1} and Σ_{α_2} are homeomorphic by the homeomorphism

$$\varphi_{\alpha_1,\alpha_2} := \iota_{\alpha_2} \circ (\iota_{\alpha_1})^{-1}.$$

Also, we know that $e_{c,l}^\alpha$ is homeomorphic to $[0,1]$ for all $c \in \mathcal{C}$ and $l \in \{1,\ldots,\rho(c)\}$. We denote the homeomorphism by $\iota_{c,l}^\alpha$. We can extend $\varphi_{\alpha_1,\alpha_2}$ to all $e_{c,l}^{\alpha_1,w}$ with $w \in \mathcal{A}_0^*$ by

$$\varphi_{\alpha_1,\alpha_2}|_{e_{c,l}^{\alpha_1,w}} := G_w^{\alpha_2} \circ \iota_{c,l}^{\alpha_2} \circ (\iota_{c,l}^{\alpha_1})^{-1} \circ (G_w^{\alpha_1})^{-1}.$$

We see that the extended $\varphi_{\alpha_1,\alpha_2}$ is a homeomorphism between K_{α_1} and K_{α_2}. □

We, therefore, omit the parameter α in the notation and only write K for the stretched fractal. Similarly we only write Σ for Σ_α and J for J_α. This also means that the Hausdorff dimension with respect to the Euclidean metric is not a very good quantity to describe the analysis on K which, as we will see, does not depend on α. In Section 3.6 we will calculate the Hausdorff dimension with respect to the resistance metric, which is much better suited for this job.

We give some further notation. For $w \in \mathcal{A}_0^*$ let

$K_w := G_w(K),\ K_n := \bigcup_{w\in\mathcal{A}^n} K_w,$

$J_n := \overline{K\backslash K_n}\ ,$

$\Sigma_w := G_w(\Sigma),\ \Sigma_n := \bigcup_{w\in\mathcal{A}^n} \Sigma_w.$

We take the closure of $K\backslash K_n$ in defining J_n to include the endpoints of the one-dimensional lines. K_w is called an n-cell of K if $|w| = n$.

3.1.2 Examples

In this section we include some examples of p.c.f. self-similar fractals we can stretch.

Stretched Sierpiński gasket

The stretched Sierpiński gasket was the subject of prior work. It was analyzed geometrically in [2] and analytically in [3] and [4]. In [5] the authors introduced so-called completely symmetric resistance forms which satisfy the same symmetries as the set. In Chapter 2 the leading order of the eigenvalue counting function of the associated operators was calculated.

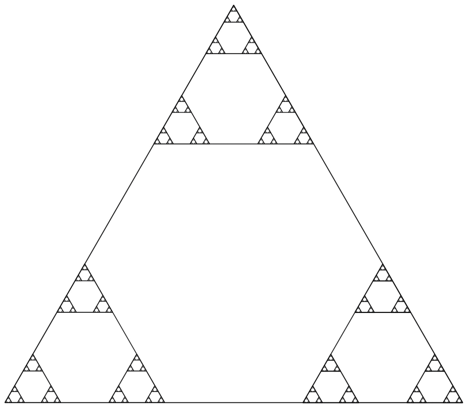

Figure 3.2: The stretched Sierpiński gasket.

The Sierpiński gasket has three similitudes and three critical points which all have multiplicity 2. The post critical set consists of all three fixed points of the similitudes, which are the corner points of the big triangle; see [37, Example 8.2]. The words $w^{c,l}$ all have length one since the Sierpiński gasket is nested.

Stretched level-3 Sierpiński gasket

The *level-3 Sierpiński gasket* (see [50]) has six similitudes and seven critical points. Six of

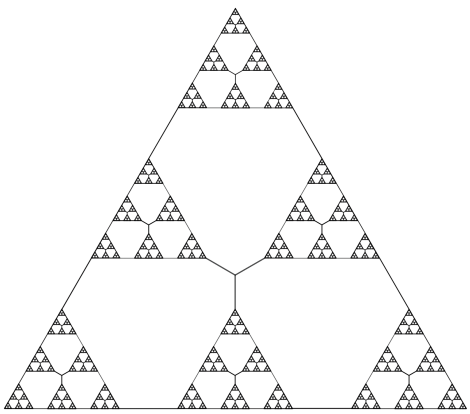

Figure 3.3: The stretched level-3 Sierpiński gasket.

the critical points have multiplicity 2, whereas the inner critical point has multiplicity 3. By connecting the copies of all three 1-cells that got stretched away from this point to it, we keep the level of connectedness. The post critical set consists of the essential fixed points, which are again the corner points of the outer triangle.

Stretched Sierpiński gasket in higher dimensions

There is a generalization of the Sierpiński gasket to higher dimensions [16]. These are nested fractals that can be stretched.

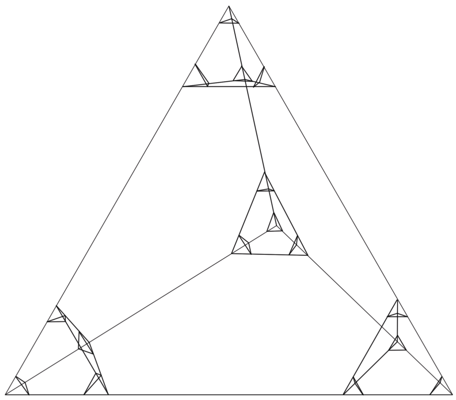

Figure 3.4: The stretched Sierpiński gasket in $\mathbb{R}^3$.

In $\mathbb{R}^d$ we have $d+1$ similitudes, where each copy of the stretched fractal is connected to all other copies by connecting lines over the critical points. We have $\frac{d(d+1)}{2}$ many critical points, which all have multiplicity 2. The post critical points are all the fixed points of the similitudes.

Stretched Lindstrøm snowflake

The *Lindstrøm snowflake* was introduced by Lindstrøm in [47] as an example of nested fractals. It has seven similitudes and 12 critical points, which all have multiplicity 2. The

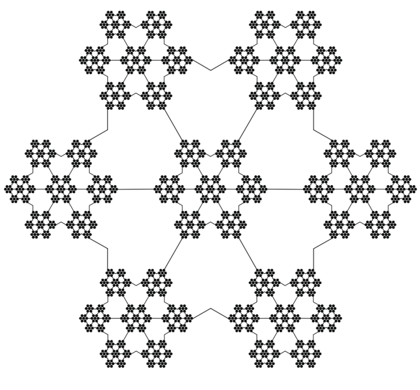

Figure 3.5: The stretched Lindstrøm snowflake.

post critical set consists of the essential fixed points, which are the fixed points of the outer six similitudes.

Stretched Vicsek set

The *Vicsek set*, introduced in [54], consists of five similitudes and four critical points with multiplicity 2. The post critical points are the fixed points of the outer four similitudes.

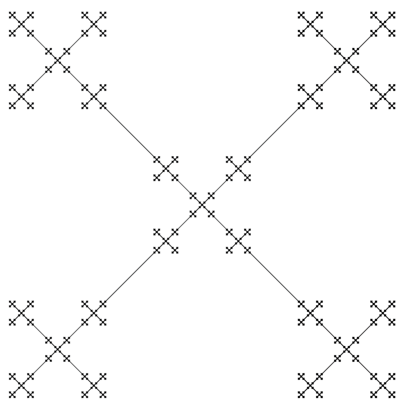

Figure 3.6: The stretched Vicsek set.

Stretched Hata's tree

Hata's tree is a p.c.f. self-similar set which is not a nested fractal and was constructed by Hata in [27]. We have the following similitudes:

$$F_1(x) = \frac{1}{\sqrt{12}} \begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$
$$F_2(x) = \frac{2}{3} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}.$$

This self-similar set is called Hata's tree.

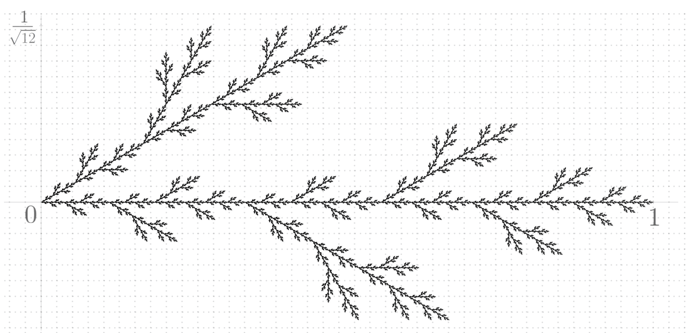

Figure 3.7: Hata's tree.

There is exactly one critical point $c = \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}$ with multiplicity 2 and the post critical set is

$$\mathcal{P} := \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/\sqrt{12} \end{pmatrix} \right\}.$$

It is $\begin{pmatrix} 1/2 \\ 1/\sqrt{12} \end{pmatrix} = F_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $c = F_1 \begin{pmatrix} 1/2 \\ 1/\sqrt{12} \end{pmatrix} = F_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which means we have the words $w^{c,1} = 2$ and $w^{c,2} = 11$. Therefore, even though Hata's tree is not a nested fractal, since it lacks the symmetry axiom, it fulfills the conditions (C1), (C2) and (C3) and thus we are able to stretch it. Stretching this set with $\alpha = \frac{9}{10}$ gives us the following two similitudes:

$$G_1(x) = \frac{9}{10\sqrt{12}} \begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$
$$G_2(x) = \frac{3}{5} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{2}{5} \\ 0 \end{pmatrix}.$$

According to the construction we need to connect the points $G_1^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $G_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ with c. This leads to the connecting lines

$$e_{c,1} = \left\{ \lambda (\tfrac{9}{10})^2 \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} + (1-\lambda) \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} \,\middle|\, \lambda \in [0,1] \right\},$$
$$e_{c,2} = \left\{ \lambda \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} + (1-\lambda) \begin{pmatrix} \frac{2}{5} \\ 0 \end{pmatrix} \,\middle|\, \lambda \in [0,1] \right\}.$$

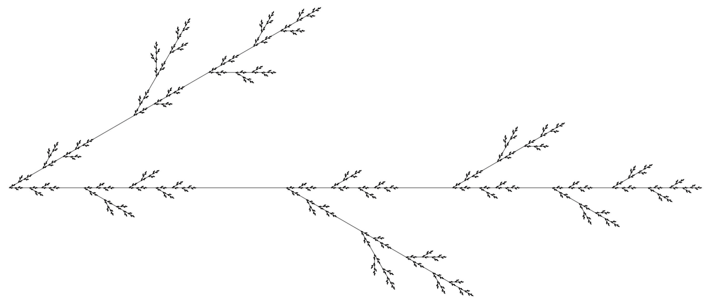

Figure 3.8: Stretched Hata's tree.

Remark. There is another way to define stretched fractals. Instead of connecting all of the copies with the critical point we could directly connect the copies with themselves.

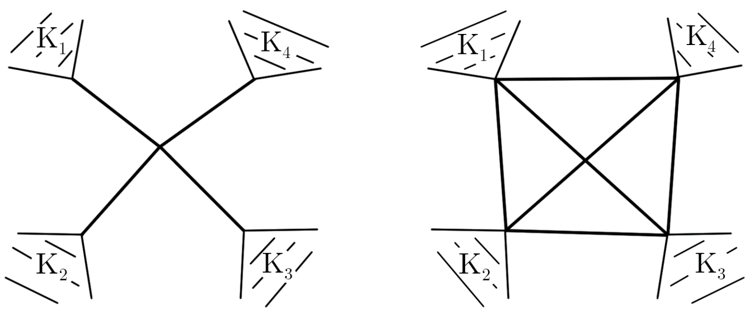

Figure 3.9: Another way to connect the copies.

However, since our stretched fractals can be embedded in $\mathbb{R}^d$ this would cause problems when we have four or more copies meeting at one critical point. As you can see in Figure 3.9 the diagonal lines have another intersection that changes the level of connectedness and thus we would like to avoid this scenario.

3.2 Graph approximation and harmonic structures

To be able to introduce Dirichlet forms on stretched fractals we need to approximate K by a sequence of finite graphs and choose resistances on the graph edges. This is the goal of this section.

3.2.1 Graph approximation

For the first graph approximation we would like to define the set of vertices V_0 similar as for p.c.f self-similar sets, cf. Definition A.3. In our notation this would mean to define V_0 as $\mathcal{P}$. However, in general we do not have $\mathcal{P} \subset \Sigma$; see Hata's tree in Section 3.1.2. The connectedness condition (C1) enables us to define a set in Σ which corresponds to $\mathcal{P}$. From (C1) we know that for each $p \in \mathcal{P}$ we have a finite word $w^p \in \mathcal{A}_0^*$ and a fixed point $q_p \in \mathcal{P}$ such that $p = F_{w^p}(q_p)$. Since the fixed points of the similitudes G_i and F_i are the same and thus $q_i \in \Sigma$ for all i, we can define

$$V_0 := \{G_{w^p}(q_p) \mid p \in \mathcal{P}\},$$

as a subset of Σ. Note that for stretched nested fractals we have $V_0 = \mathcal{P}$. We take this set as our vertices and connect all of them pairwise. Let

$$E_0 := \{\{x, y\} \mid x, y \in V_0,\ x \neq y\}.$$

In the next graph approximation we have two kinds of vertices. One originate by applying the similitudes G_j to the points of V_0 and are denoted by

$$P_1 := \bigcup_{j=1}^{N} G_j(V_0).$$

We see that $V_0 \subset P_1$. This is the case since $p = G_{w^p}(q_p)$. If $w^p = \emptyset$ then p is a fixed point and thus $p \in P_1$. For $w^p = w_1 \cdots w_n$ we know that $\tilde{p} := G_{w_2 \cdots w_n}(q_p)$ is in V_0 since $F_{w_2 \cdots w_n}(q_p) \in \mathcal{P}$ and thus $p = G_{w_1}(\tilde{p}) \in P_1$.

The other kind of vertices are the critical points describing how the cells $G_j(V_0)$ are connected. Let

$$C_1 := \mathcal{C}.$$

The union of these two parts gives us the set of vertices

$$V_1 := P_1 \cup C_1.$$

Now we describe how these vertices are connected. The points of $G_j(V_0)$ should be connected in the same way as V_0 was. This gives us the edge relation on P_1

$$E_1^{\Sigma} := \{\{G_i x, G_i y\} \mid \{x, y\} \in E_0,\ i \in \mathcal{A}\}.$$

The points that got stretched away from points in $\mathcal{C}$ should again be connected with these points to reflect the geometry of K. We define

$$E_1^I := E_{1,1}^I := \{\{c, G_{w^{c,l}}(q_l^c)\} \mid c \in \mathcal{C}, l \in \{1, \dots, \rho(c)\}\}.$$

We know that $G_{w^{c,l}}(q_l^c)$ is an element of P_1. This gives us the graph (V_1, E_1) with vertices $V_1 = P_1 \cup C_1$ and edge set $E_1 := E_1^{\Sigma} \cup E_1^I$.

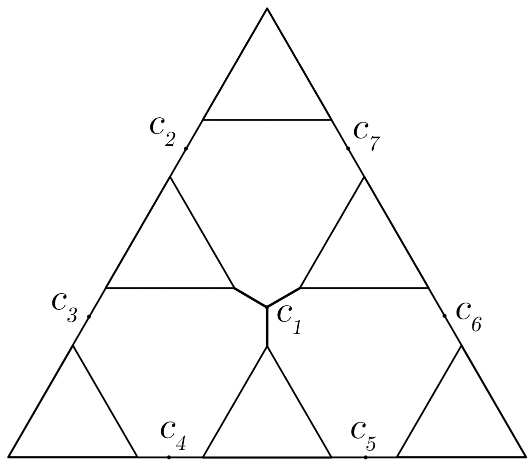

Figure 3.10: (V_1, E_1) for the stretched level-3 Sierpiński gasket.

We are now ready to define the whole sequence of graphs. In general the vertices will consist of two different kinds of points. Let

$$\begin{aligned} P_n &:= \bigcup_{w \in \mathcal{A}^n} G_w(V_0), \\ C_{k,k} &:= \bigcup_{w \in \mathcal{A}^{k-1}} G_w(\mathcal{C}), \\ C_n &:= \bigcup_{k=1}^{n} C_{k,k}, \\ V_n &:= P_n \cup C_n. \end{aligned}$$

We notice, that $V_n \subset V_{n+1}$ for all $n \geq 0$.

Similar we define the edge set by

$$\begin{aligned}
E_n^\Sigma &:= \{\{G_w x, G_w y\} \mid \{x,y\} \in E_0,\ w \in \mathcal{A}^n\},\\
E_{k,k}^I &:= \{\{G_w x, G_w y\} \mid \{x,y\} \in E_{1,1}^I,\ w \in \mathcal{A}^{k-1}\},\\
E_n^I &:= \bigcup_{k=1}^n E_{k,k}^I,\\
E_n &:= E_n^\Sigma \cup E_n^I.
\end{aligned}$$

We call the edges in E_n^I *connecting edges* and the ones in E_n^Σ *fractal edges*. This leads to a sequence of graphs $\Gamma_n := (V_n, E_n)$ for $n \geq 0$. We introduce some further notation by

$$\begin{aligned}
V_* &:= \bigcup_{n\geq 0} V_n,\\
P_* &:= \bigcup_{n\geq 1} P_n,\\
C_* &:= \bigcup_{n\geq 1} C_n.
\end{aligned}$$

It follows from general theory, e.g. [43, Theorem 1.1.7], that P_* is dense in Σ. Therefore,

$$\overline{V}_* = \Sigma \cup C_*,$$

which can be seen with [52, Lemma 3.9] if we choose $\mathcal{C}$ as the condensation set.

3.2.2 Harmonic structures

Until now we have the approximating graphs. We need resistances on the edges to define quadratic forms and thus operators. Define resistance functions

$$r_n : E_n \to [0, \infty]$$

that assign each edge in E_n a resistance.

We would like to choose resistances on E_n in such a way that the electrical networks (V_n, E_n, r_n) are all equivalent and thus a compatible sequence, compare to Definition B.5. Similar to the self-similar case it suffices to have the existence of r_0 and r_1 such that (V_0, E_0, r_0) and (V_1, E_1, r_1) are equivalent. Such values, or functions, will be called a *harmonic structure* in analogy to the self-similar case, compare to [44, Definition 9.5]. For an edge $e = \{x, y\}$ we write $G_i(e) := \{G_i(x), G_i(y)\}$. Choose values

$$\begin{aligned}
r_e &:= r_0(e) \in (0, \infty],\ \forall e \in E_0,\\
\rho_e &:= r_1(e) \in (0, \infty),\ \forall e \in E_1^I,
\end{aligned}$$

and $0 < \lambda < 1$.

Then define r_1 on the remaining edges in E_1^Σ by

$$r_1(G_i(e)) := \lambda r_0(e), \ \forall e \in E_0, i \in \mathcal{A}.$$

With this we have chosen all values for r_0 and r_1. Since we allow that $r_0(e) = \infty$ we need to make sure that the network is connected.

Definition 3.5. Let (V, E) be a finite graph and $r : E \to [0, \infty]$. We call an electrical network (V, E, r) *connected* if for all $p, \tilde{p} \in V$ there exist $\{p_0, p_1\}, \ldots, \{p_{n-1}, p_n\} \in E$ with $p_0 = p$ and $p_n = \tilde{p}$ such that $r(\{p_i, p_{i+1}\}) < \infty$ for all $i \in \{0, \ldots, n-1\}$.

If the electrical networks (V_0, E_0, r_0) and (V_1, E_1, r_1) are equivalent and the network (V_0, E_0, r_0) is connected we call $(r_0, \lambda, \{\rho_e\}_{e \in E_1^I})$ a harmonic structure on K. *Electrically equivalent* can also be expressed in terms of quadratic forms. We use $r_0(x, y) := r_0(\{x, y\})$ and

$$\begin{aligned} E_0(g) &:= \sum_{\{x,y\} \in E_0} \frac{1}{r_0(x, y)} (g(x) - g(y))^2, \\ E_1(f) &:= \sum_{\{x,y\} \in E_1} \frac{1}{r_1(x, y)} (f(x) - f(y))^2, \end{aligned}$$

for $g : V_0 \to \mathbb{R}$ and $f : V_1 \to \mathbb{R}$ with r_0 and r_1 chosen as before. The trace of $E_1(\cdot)$ on V_0 is

$$E_1|_{V_0}(g) = \inf\{E_1(f) \mid f : V_1 \to \mathbb{R}, \ f|_{V_0} = g\}.$$

We can now give the exact definition of harmonic structures.

Definition 3.6 (Harmonic structure)**.** $(r_0, \lambda, \{\rho_e\}_{e \in E_1^I})$ is a *harmonic structure* on K if and only if

1. (V_0, E_0, r_0) is connected,
2. $E_1|_{V_0}(g) = E_0(g) \quad$ for all $\quad g : V_0 \to \mathbb{R}$.

For fixed r_0 we cannot expect that λ and $\{\rho_e\}_{e \in E_1^I}$ are unique. In fact, this is a major feature of stretched fractals; see [5] or Section 3.2.3. Since the edges in E_1^I correspond to the connecting lines $e_{c,l}$ with $c \in \mathcal{C}$ and $l \in \{1, \ldots, \rho(c)\}$ we denote $\rho_{c,l} := \rho_e$ for the edge e which corresponds to $e_{c,l}$. For the stretched level-3 Sierpiński gasket you can see the resistances in Figure 3.11.

In the next graph approximation the electrical network (V_2, E_2, r_2) has to be equivalent to (V_1, E_1, r_1) and thus to (V_0, E_0, r_0). The edges in E_1^I are still part of E_2 and are not transformed in any way, so they will have the same resistance as in (V_1, E_1, r_1). The subgraphs with vertices $G_i(V_0)$ get divided in the same fashion as V_0 was in the first step but the resistances are now scaled by λ compared to the values on (V_0, E_0, r_0). We can, therefore,

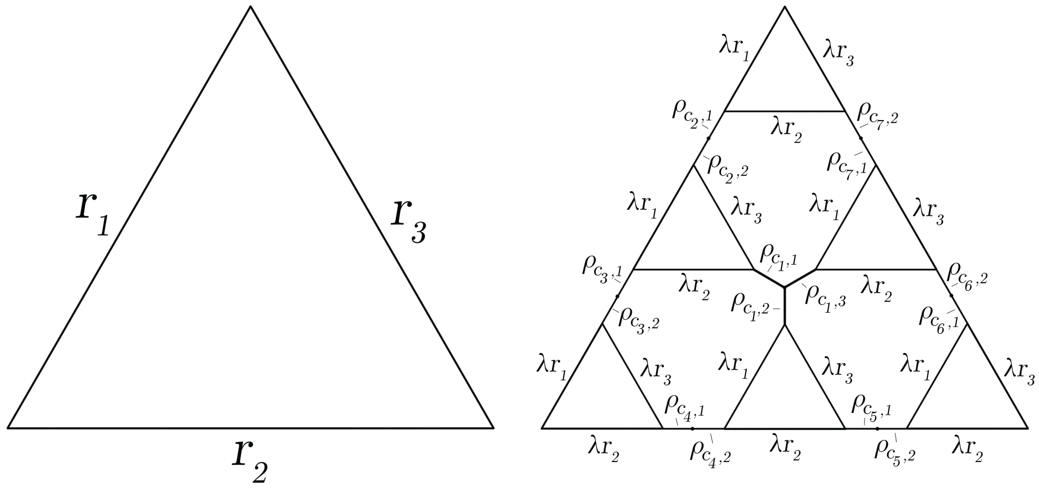

Figure 3.11: Resistances on (V_0, E_0) and (V_1, E_1).

choose another harmonic structure to get electrically equivalent networks. We choose the same resistances for all subgraphs with vertices $G_i(V_0)$.

The second graph approximation of the stretched level-3 Sierpiński gasket is pictured in Figure 3.12. The dotted lines indicate that the problem of choosing resistances is exactly the same as before in the first graph approximation.

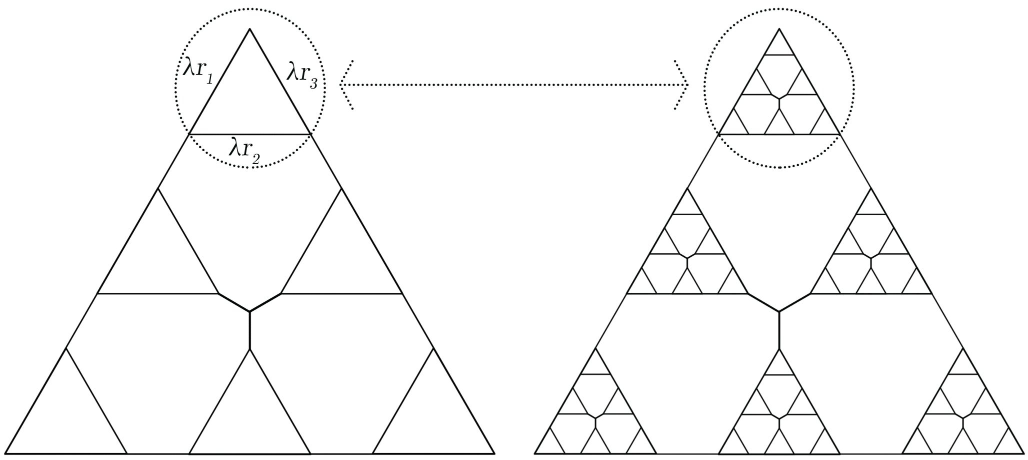

Figure 3.12: Resistances on second graph approximation.

We can follow this procedure in each step and thus we have to choose a sequence of harmonic structures

$$\mathcal{R} := \{(r_0, \lambda_i, \{\rho_e^i\}_{e \in E_1^I})\}_{i \geq 1},$$

such that $(r_0, \lambda_i, \{\rho_e^i\}_{e \in E_1^I})$ is a harmonic structure for all i. Notice that r_0 has to be the same for all harmonic structures.

With this sequence we can define the values for r_n by

1. r_n on E_n^Σ by
$$r_n(G_w e) := \lambda_1 \cdots \lambda_n r_0(e)$$
for all $e \in E_0$ and $w \in \mathcal{A}^n$.

2. r_n on E_n^I by
$$r_1(e) = \rho_e^1$$
for $e \in E_1^I$ and
$$r_n(G_w e) = \lambda_1 \cdots \lambda_{|w|} \rho_e^{|w|+1}$$
for all $w \in \bigcup_{k=1}^{n-1} \mathcal{A}^k,\ e \in E_1^I$ and $n \geq 2$.

By the definition of harmonic structures $\{(V_n, E_n, r_n)\}_{n\geq 0}$ is a sequence of equivalent electrical networks.

Since r_0 is fixed for the whole sequence of harmonic structures we omit it in the notation of $\mathcal{R}$ whenever we do not explicitly need it. Additionally for the sake of notation we denote $\boldsymbol{\rho}^i := \{\rho_e^i\}_{e\in E_1^I}$.

Definition 3.7 (Regular sequence of harmonic structures)**.** Let $\mathcal{R} = \{(\lambda_i, \boldsymbol{\rho}^i)\}_{i\geq 1}$ be a sequence of harmonic structures (with fixed r_0). We call $\mathcal{R}$ a *regular sequence of harmonic structures* if it fulfills the following two conditions:

(1) $\exists \lambda^* < 1$ such that $\lambda_i \leq \lambda^*$ for all i,

(2) $\rho^* := \sup\{\rho \mid \rho \in \boldsymbol{\rho}^i,\ i \geq 1\} < \infty$.

The condition (1) is an immediate generalization of regular harmonic structures from [44, Definition 9.5]. Condition (2) is a technical condition that we need to show the existence of resistance forms on K.

3.2.3 Examples

We only have to consider the graphs (V_0, E_0) and (V_1, E_1) and choose resistances on the edges in accordance with the way described this section such that the electrical networks are equivalent.

Stretched Sierpiński gasket

This has been handled extensively in Chapter 2. However, the edge set E_1^I was slightly different since the copies got connected by only one end-to-end edge. Let us choose the resistances as in Figure 3.13.

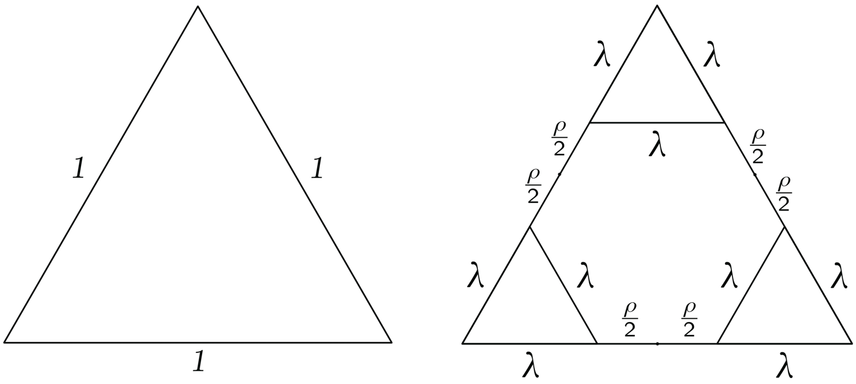

Figure 3.13: Harmonic structure on the stretched Sierpiński gasket.

That means $r_0 \equiv 1$. With this choice we are in the framework of Chapter 2. From Section 2.1 we know that

$$\frac{5}{3}\lambda + \rho = 1. \tag{3.1}$$

All sequences $\mathcal{R} = \{(\lambda_i, \boldsymbol{\rho}^i)\}_{i\geq 1}$ with $\boldsymbol{\rho}^i \equiv \frac{\rho_i}{2}$ and $\frac{5}{3}\lambda_i + \rho_i = 1$ for all i are regular sequences of harmonic structures. From (3.1) we know that $0 < \lambda_i < \frac{3}{5}$ where the upper bound $\frac{3}{5}$ is exactly the renormalization factor in the self-similar case [37, Example 8.2].

Stretched level-3 Sierpiński gasket

We choose the resistances on (V_0, E_0) and (V_1, E_1) as in Figure 3.14.

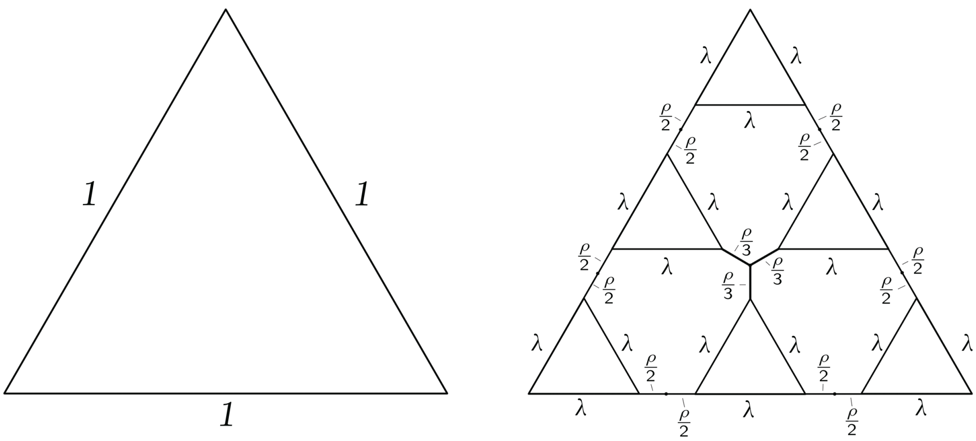

Figure 3.14: Harmonic structure on the stretched level-3 Sierpiński gasket.

We apply some equivalent electrical network transformations which are illustrated in Figure 3.15.

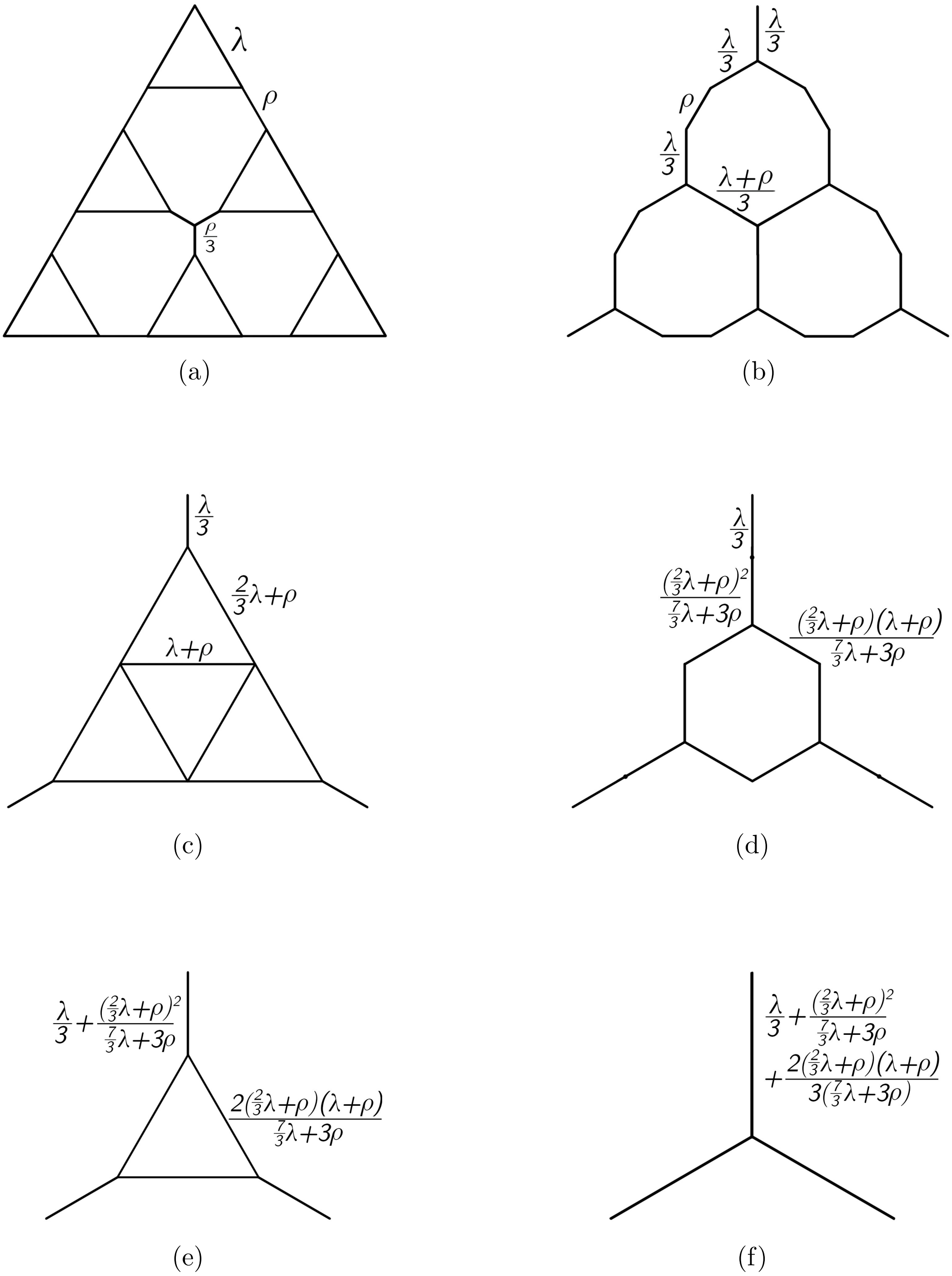

Figure 3.15: Equivalent transformations of the electrical networks.

If we apply the Δ-Y-transformation one last time to the last network we see that 3 times the resistance from Figure 3.15f has to be equal to 1. This means we can choose λ and ρ that fulfill

$$5\lambda^2 + \lambda\left(\frac{31}{3}\rho - \frac{7}{3}\right) + 5\rho^2 - 3\rho = 0.$$

This relation also appeared in a different setting in [1], where the same electrical network was used. We can easily show that this allows pairs (λ, ρ) for all $\lambda \in (0, \frac{7}{15})$. If we choose such a harmonic structure in each step we get regular sequences of harmonic structures. The upper limit $\frac{7}{15}$ for λ is exactly the renormalization in the self-similar case; see [50].

Stretched Sierpiński gasket in higher dimensions

The graph (V_0, E_0) consists of the complete graph with $d+1$ nodes where all edges have resistance 1. In the first graph approximation we have $d+1$ complete graphs which are all connected over a critical point to all other d complete graphs. The remaining nodes are the fixed points of the similitudes. The resistances of the edges in the complete graphs are λ and the ones on the connecting edges are $\frac{\rho}{2}$.

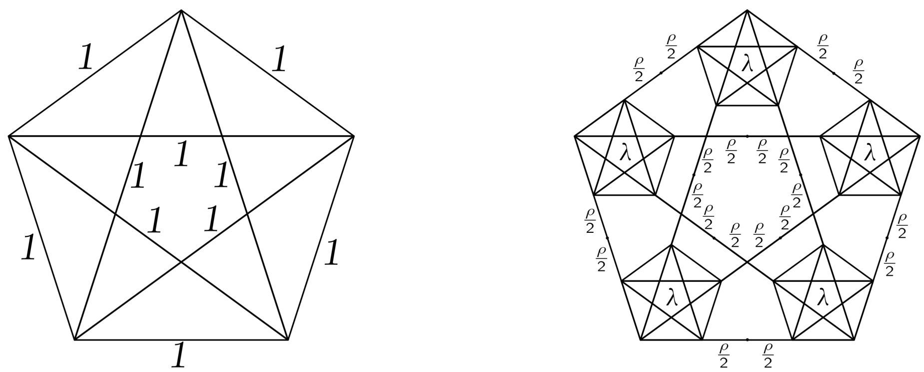

Figure 3.16: Harmonic structure on the stretched Sierpiński gasket in $\mathbb{R}^4$.

We use the star-mesh-transformation, which is a generalization of the Δ-Y-transformation and originally due to Campbell [14], to see in which cases these networks are equivalent.

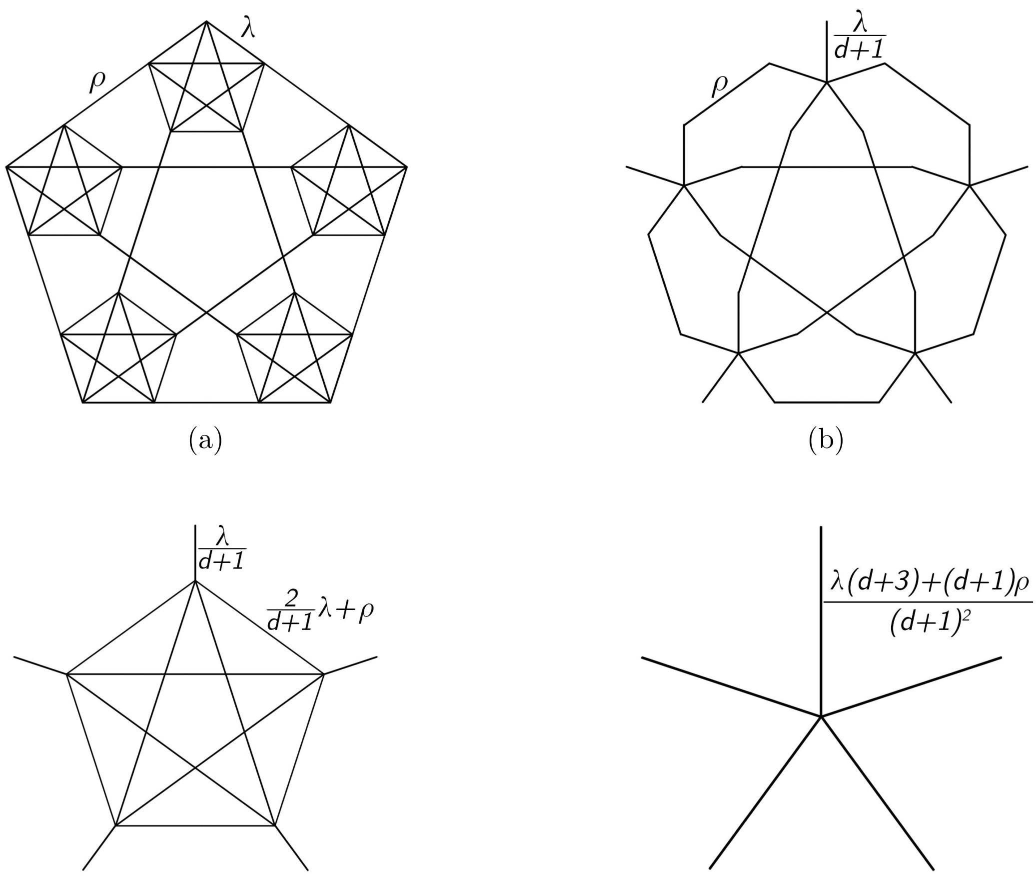

Figure 3.17: Equivalent transformations of the electrical networks.

We obtain

$$\lambda \cdot \frac{d+3}{d+1} + \rho = 1,$$

which means that we can reach every $\lambda \in (0, \frac{d+1}{d+3})$ by a pair (λ, ρ). The upper limit $\frac{d+1}{d+3}$ is the renormalization in the self-similar case; see [16].

Stretched Vicsek set

Let us choose the resistances as in Figure 3.18.

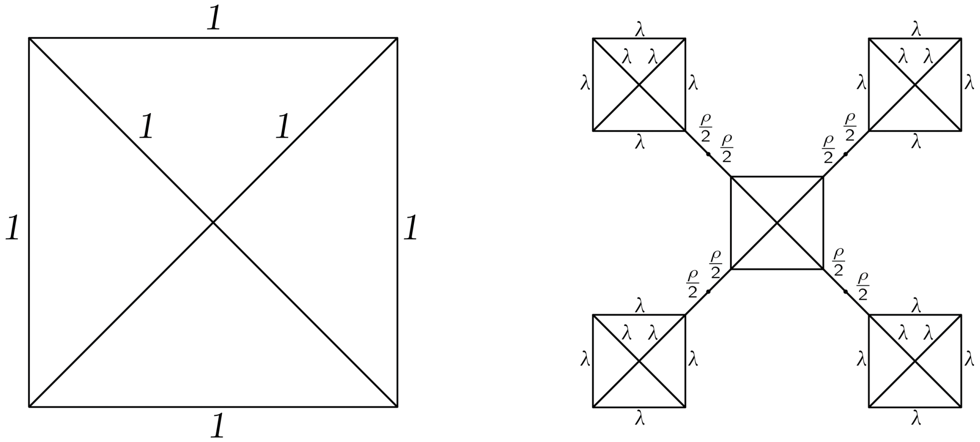

Figure 3.18: Harmonic structure on the stretched Vicsek set.

We again use the star-mesh-transformation on all 5 small squares.

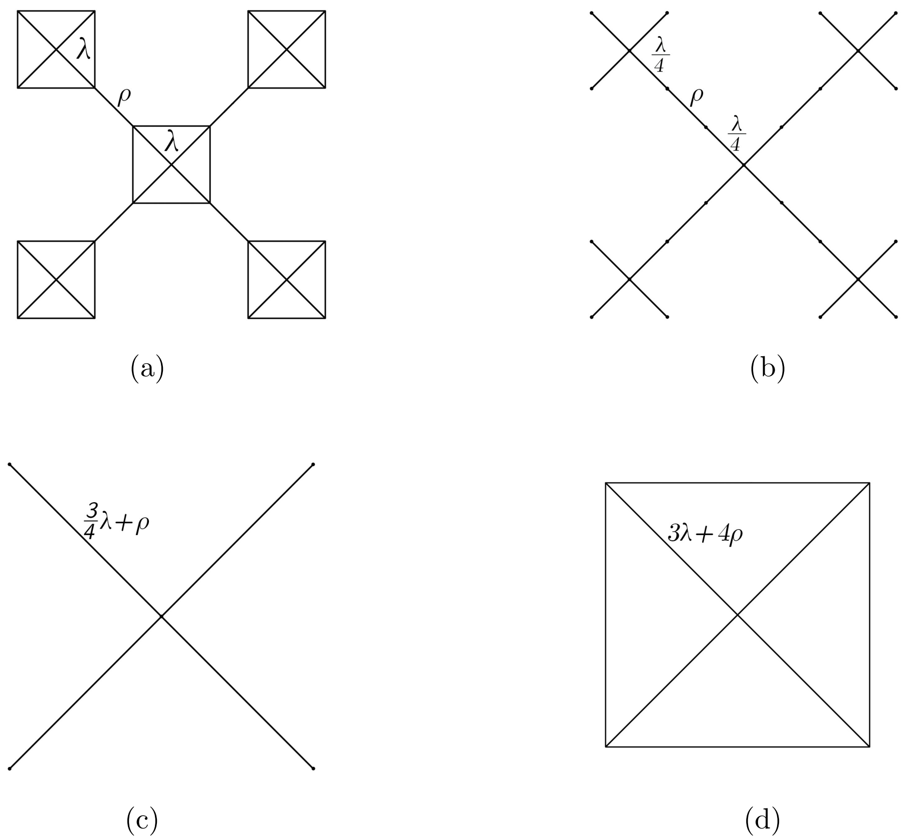

Figure 3.19: Equivalent transformations of the electrical networks.

These calculations show that the networks are equivalent if for (λ, ρ) it holds that

$$3\lambda + 4\rho = 1.$$

The choice of $r_0 \equiv 1$ comes from the resistances in the self-similar case. In this case this is the only symmetric choice which gives us a non-degenerate harmonic structure; see [7, Example 6.13]. So we use this information and generalize it to the stretched case. We can reach all $\lambda \in (0, \frac{1}{3})$, where, again, the upper bound $\frac{1}{3}$ is the renormalization in the self-similar case; see [7].

Stretched Hata's tree

We view the graphs (V_0, E_0) and (V_1, E_1) with the resistances from Figure 3.20.

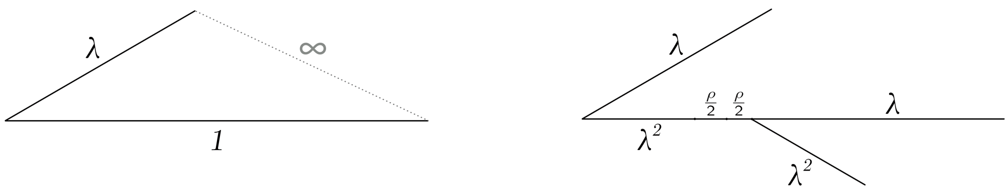

Figure 3.20: Harmonic structure on stretched Hata's tree.

Note that the resistance on the dotted edge is ∞ but the graph is still connected. The networks are equivalent if

$$\lambda^2 + \lambda + \rho = 1.$$

This is solvable for all $\lambda \in (0, \frac{\sqrt{5}-1}{2})$ where the upper bound is the renormalization in the self-similar case; see [37, Example 8.4]. Note, however, that r_0 explicitly depends on λ. Since r_0 has to stay the same when we choose a sequence of harmonic structures we see that here in this case $(\lambda_i)_{i\geq 1}$ has to be constant.

3.3 Resistance forms

In this section we construct resistance forms on stretched fractals. The definition can be found in Appendix B. We will be able to get such forms on K for all regular sequences of harmonic structures.

The resistance form will consist of two parts that represent the fractal and the line part that is present in these stretched fractals. The fractal part is very similar to the usual resistance form on the self-similar set, i.e., the attractor of the F_i. We will first construct a resistance form on V_* which does not consider the one-dimensional lines. This can be extended to the closure of V_* with respect to the resistance metric. Next we show that the Euclidean and resistance metric introduce the same topology, that means the closure of V_*

is the same with either one. We thus have a resistance form on $\Sigma \cup C_*$. The next step is to substitute parts of the resistance form and introduce Dirichlet energies on the one-dimensional lines. This is then shown to be a resistance form on the whole set K. Again, the resistance metric on K induces the same topology as the Euclidean metric d_E. The construction follows the ideas of [5] where this was done for the stretched Sierpiński gasket.

3.3.1 Resistance form on V_*

First we define a quadratic form on the approximating graphs that is associated to the energy of the electrical network.

Definition 3.8 (Fractal part)**.** Let $\mathcal{R} = \{(r_0, \lambda_i, \boldsymbol{\rho}^i)\}_{i\geq 1}$ be a sequence of harmonic structures and $u : V_0 \to \mathbb{R}$. Then define

$$\hat{\mathcal{E}}_{\mathcal{R},0}(u) := Q^{\Sigma}_{r_0}(u) := \sum_{\{x,y\}\in E_0} \frac{1}{r_0(x,y)}(u(x) - u(y))^2.$$

With this define a quadratic form for $u : V_n \to \mathbb{R}$ by

$$\hat{\mathcal{E}}^{\Sigma}_{\mathcal{R},n}(u) := \sum_{w\in\mathcal{A}^n} \frac{1}{\delta_n} Q^{\Sigma}_{r_0}(u \circ G_w),$$

where we use the abbreviation $\delta_n := \lambda_1 \cdots \lambda_n$.

This, however, ignores the connecting edges. Therefore, we introduce a second quadratic form.

Definition 3.9 (Line part)**.** For $\boldsymbol{\rho} = \{\rho_e\}_{e\in E^I_1}$ and $u : V_1 \to \mathbb{R}$ define

$$Q^I_{\boldsymbol{\rho}}(u) := \sum_{\{x,y\}\in E^I_1} \frac{1}{\rho_{\{x,y\}}}(u(x) - u(y))^2.$$

Now for a sequence of harmonic structures $\mathcal{R} = \{(\lambda_i, \boldsymbol{\rho}^i)\}_{i\geq 1}$ and $u : V_n \to \mathbb{R}$ we define

$$\hat{\mathcal{E}}^I_{\mathcal{R},n}(u) := Q^I_{\boldsymbol{\rho}^1}(u) + \sum_{k=2}^{n} \frac{1}{\lambda_1 \cdots \lambda_{k-1}} \underbrace{\sum_{w\in\mathcal{A}^{k-1}} Q^I_{\boldsymbol{\rho}^k}(u \circ G_w)}_{Q^I_{\boldsymbol{\rho}^k,k}(u):=}.$$

We denote $\gamma_1 := 1$ and $\gamma_k := \delta_{k-1} = \lambda_1 \cdots \lambda_{k-1}$ for $k \geq 2$. Then this writes as follows for $n \geq 1$:

$$\hat{\mathcal{E}}^I_{\mathcal{R},n}(u) := \sum_{k=1}^{n} \frac{1}{\gamma_k} Q^I_{\boldsymbol{\rho}^k,k}(u).$$

With the sum of line and fractal part we can define quadratic forms for functions in $\ell(V_n)$ that correspond to the energy of the electrical network (V_n, E_n, r_n).

Definition 3.10 (Quadratic form on V_n)**.** Let $\mathcal{R} = \{(\lambda_i, \boldsymbol{\rho}^i)\}_{i\geq 1}$ be a sequence of harmonic structures and $u \in \ell(V_n) = \{u \mid u : V_n \to \mathbb{R}\}$ for $n \geq 1$. Then let

$$\hat{\mathcal{E}}_{\mathcal{R},n}(u) := \hat{\mathcal{E}}^{\Sigma}_{\mathcal{R},n}(u) + \hat{\mathcal{E}}^{I}_{\mathcal{R},n}(u).$$

Since V_n is finite, these quadratic forms are resistance forms and since the graphs form a sequence of equivalent electrical networks, the sequence of resistance forms $\{(\hat{\mathcal{E}}_{\mathcal{R},n}, \ell(V_n))\}_{n\geq 0}$ builds a sequence of compatible resistance forms. That means $(\hat{\mathcal{E}}_{\mathcal{R},n}(u|_{V_n}))_{n\geq 0}$ is a non-decreasing sequence for all $u \in \ell(V_*)$ and therefore, a limit exists in $[0, \infty]$.

Definition 3.11 (Resistance form on V_*)**.** Let $\mathcal{R} = \{(\lambda_i, \boldsymbol{\rho}^i)\}_{i\geq 1}$ be a sequence of harmonic structures. Then let

$$\hat{\mathcal{E}}_{\mathcal{R}}(u) := \lim_{n\to\infty} \hat{\mathcal{E}}_{\mathcal{R},n}(u|_{V_n}),$$

which is defined on

$$\hat{\mathcal{F}}_{\mathcal{R}} := \{u \mid u \in \ell(V_*),\ \lim_{n\to\infty} \hat{\mathcal{E}}_{\mathcal{R},n}(u|_{V_n}) < \infty\}.$$

From general theory it follows that $(\hat{\mathcal{E}}_{\mathcal{R}}, \hat{\mathcal{F}}_{\mathcal{R}})$ is a resistance form on V_*; see Theorem B.6.

Remark 3.12. These quadratic forms immediately induce symmetric bilinear forms if we replace $Q^{\Sigma}_{r_0}(u)$ from above by

$$Q^{\Sigma}_{r_0}(u, v) := \sum_{\{x,y\}\in E_0} \frac{1}{r_0(x, y)}(u(x) - u(y))(v(x) - v(y))$$

and $Q^{I}_{\boldsymbol{\rho}}(u)$ by

$$Q^{I}_{\boldsymbol{\rho}}(u, v) := \sum_{\{x,y\}\in E^I_1} \frac{1}{\rho_{\{x,y\}}}(u(x) - u(y))(v(x) - v(y)),$$

which leads to a bilinear form $\hat{\mathcal{E}}_{\mathcal{R}}(u, v)$ on V_* for $u, v \in \hat{\mathcal{F}}_{\mathcal{R}}$. This is the same symmetric bilinear form which we would get by the polarization identity.

3.3.2 Resistance form on $\Sigma \cup C_*$

By general theory (again Theorem B.6) we know that this can be extended to a resistance form on $\overline{V}_*$ where the closure is taken with respect to the resistance metric of $(\hat{\mathcal{E}}_{\mathcal{R}}, \hat{\mathcal{F}}_{\mathcal{R}})$. We denote this metric by $\hat{R}_{\mathcal{R}}(\cdot, \cdot)$. We are, however, interested in a resistance form on $\Sigma \cup C_*$. We show that the resistance metric $\hat{R}_{\mathcal{R}}$ and the Euclidean metric d_E are inducing the same topology and therefore, we can get a resistance form on $\overline{V}_* = \Sigma \cup C_*$ where the closure is taken with either metric.

Denote by $R_{\mathcal{R},n}(\cdot, \cdot)$ the resistance metric on V_n from $(\hat{\mathcal{E}}_{\mathcal{R},n}, \ell(V_n))$. The diameter of a set X with respect to a metric d is denoted by $\operatorname{diam}(X, d) := \sup\{d(x, y) \mid x, y \in X\}$.

Lemma 3.13. *Let $\mathcal{R} = \{(r_0, \lambda_i, \boldsymbol{\rho}^i)\}_{i\geq 1}$ be a regular sequence of harmonic structures. Then*

$$\operatorname{diam}(V_n, R_{\mathcal{R},n}) \leq c_0 < \infty$$

for all $n \geq 0$ with a constant $c_0 > 0$ that only depends on λ^*, ρ^* and r_0.

Proof. We define a constant

$$C := \left(\sum_{c\in\mathcal{C}} \rho(c)\right) \rho^* + N \sum_{\substack{e\in E_0 \\ r_0(e)<\infty}} r_0(e).$$

The first sum is exactly the number of connecting edges in (V_1, E_1), which means we could also write $\# E_1^I$. We then multiply it by an upper bound for all ρ_e^i. The second part is the sum of all finite resistances in E_0 and then multiplied by the number of similitudes. Note that $\lambda_i \leq 1$. That means C is an upper bound for the sum of all finite resistances on (V_1, E_1) independent of the choice of $(\lambda, \boldsymbol{\rho})$ from $\{(\lambda_i, \boldsymbol{\rho}^i)\}_{i\geq 1}$.

Now let q be any point of V_1. Then it holds that

$$R_{\mathcal{R},1}(q, p) \leq C$$

for all $p \in V_0$. Since (V_1, E_1, r_1) is connected there is a path from q to p where each edge has finite resistance and is only used once. In C we count each edge of E_1 with finite resistance and therefore, get an upper bound of the summed up resistances along this path. Due to the triangle inequality and the fact that the effective resistance is always less than or equal to the direct resistance we get the desired inequality.

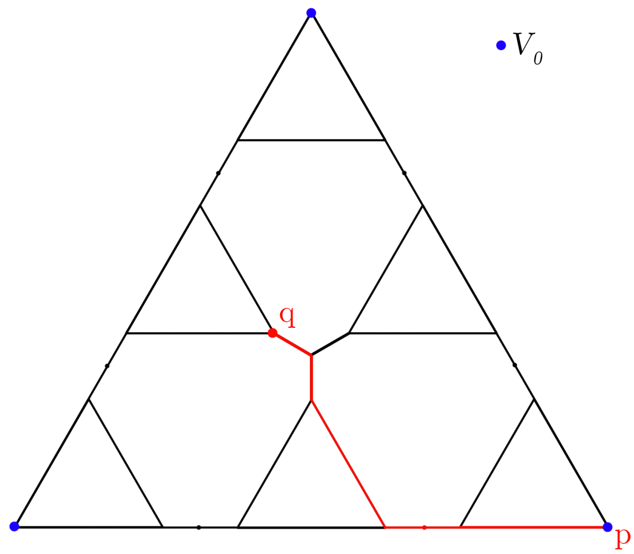

Figure 3.21: Connect V_1 with V_0 on the stretched level-3 Sierpiński gasket.

Next let q_1 be any point of V_2 and look for a path to the next point in V_1 and denote it by p_1. The problem is the same as from V_1 to V_0 but the resistances are multiplied by λ_1. That means

$$R_{\mathcal{R},2}(q_1, p_1) \leq \lambda_1 C.$$

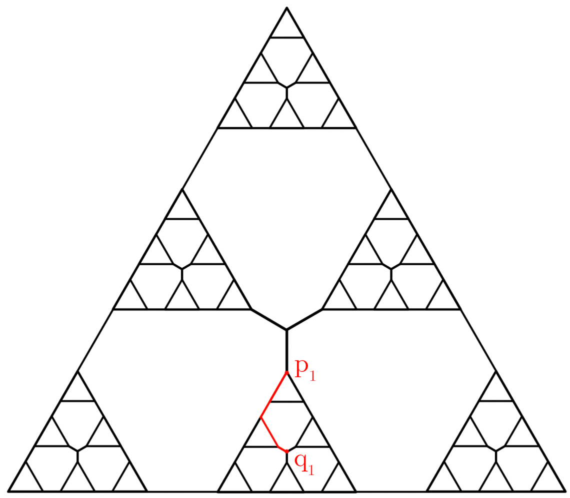

Figure 3.22: Connect V_2 with V_1 on the stretched level-3 Sierpiński gasket.

Now if $q \in V_n$, we would like to define a sequence of points in V_k from some $p \in V_0$ to q. First assume that $q \in P_n$, which means $q = G_{w_1 \cdots w_n}(\tilde{p})$ for some $\tilde{p} \in V_0$. Then define

$$\begin{aligned} q_n &:= q, \\ q_k &:= G_{w_1 \cdots w_k}(\tilde{p}), \ \ k = 1, \dots, n-1, \\ q_0 &:= p \in V_0, \end{aligned}$$

where p can be chosen arbitrarily.

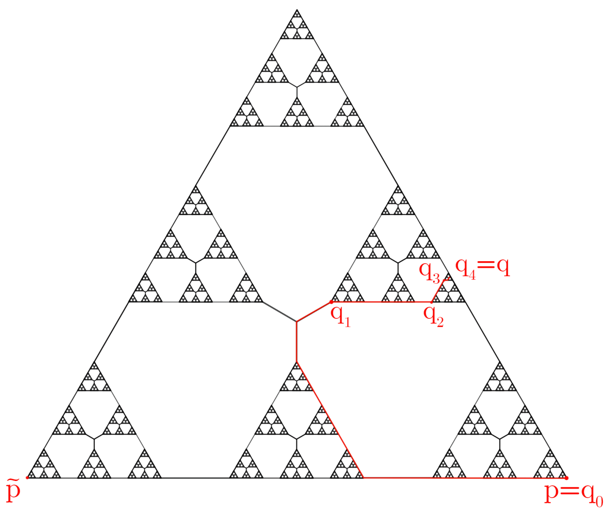

Figure 3.23: Path from q to p on the stretched level-3 Sierpiński gasket.

Actually we can choose any point $\tilde{p} \in V_0$ for the definition of q_k; it is only important that q_k and q_{k+1} are in the same k-cell. If q is not in P_n, that means $q \in C_n$, we have to add an additional point $q_n \in P_n$. Choose one that is connected to q in Γ_n and define $q_{n+1} = q$. This is always possible and the resistance is always $\leq \rho^*$. Thus we obtain

$$\begin{aligned} R_{\mathcal{R},n}(q,p) &\leq \underbrace{\rho^*}_{\text{if } q \text{ is not in } P_n} + \sum_{k=1}^{n} R_{\mathcal{R},n}(q_k, q_{k-1}) \\ &\leq \rho^* + \sum_{k=1}^{n} R_{\mathcal{R},k}(q_k, q_{k-1}) \\ &\leq \rho^* + \sum_{k=1}^{n} \underbrace{\lambda_1 \cdots \lambda_{k-1}}_{:=1 \text{ for } k=1} C \\ &\leq \rho^* + C \sum_{k=1}^{n} (\lambda^*)^{k-1} \\ &\leq \rho^* + C \sum_{k=0}^{\infty} (\lambda^*)^{k} =: \tilde{C} < \infty. \end{aligned}$$

This holds, since the sequence of harmonic structures is regular and therefore, $\lambda^* < 1$.

Now if $q, \tilde{q} \in V_n$ choose any point $p \in V_0$. Then

$$\begin{aligned} R_{\mathcal{R},n}(q,\tilde{q}) &\leq R_{\mathcal{R},n}(q,p) + R_{\mathcal{R},n}(p,\tilde{q}) \\ &\leq 2\tilde{C}, \end{aligned}$$

and therefore

$$\operatorname{diam}(V_n, R_{\mathcal{R},n}) \leq 2\tilde{C}, \ \forall n.$$

The points $q_0, \dots, q_n$ can be chosen very arbitrarily; the only condition is that q_{k-1} and q_k are in the same $(k-1)$-cell. Because of this we are allowed to choose the same point in V_0 for q and $\tilde{q}$. □

In the self-similar case some rescaling property of the resistance form was very important. We have something similar here, but not quite as nice.

Lemma 3.14 (Rescaling of $\hat{\mathcal{E}}_{\mathcal{R}}$)**.** *Let $\mathcal{R} = \{(\lambda_i, \boldsymbol{\rho}^i)\}_{i\geq 1}$ be a sequence of harmonic structures and let $\mathcal{R}^{(n)} := \{(\lambda_{n+i}, \boldsymbol{\rho}^{n+i})\}_{i\geq 1}$ be the sequence that starts at $n+1$. Then it holds for $u, v \in \hat{\mathcal{F}}_{\mathcal{R}}$ that $u \circ G_w, v \circ G_w \in \hat{\mathcal{F}}_{\mathcal{R}^{(n)}}$ for all $w \in \mathcal{A}^n$ and*

$$\hat{\mathcal{E}}_{\mathcal{R}}(u,v) = \sum_{w\in\mathcal{A}^n} \frac{1}{\delta_n} \hat{\mathcal{E}}_{\mathcal{R}^{(n)}}(u \circ G_w, v \circ G_w) + \sum_{k=1}^{n} \frac{1}{\gamma_k} Q^I_{\boldsymbol{\rho}^k,k}(u,v).$$

Proof. We show the result for the quadratic form. The result for the bilinear form immediately follows by the polarization identity. We have

$$\begin{aligned}
\hat{\mathcal{E}}_{\mathcal{R},n+m}(u) &= \hat{\mathcal{E}}^{\Sigma}_{\mathcal{R},n+m}(u) + \hat{\mathcal{E}}^{I}_{\mathcal{R},n+m}(u) \\
&= \sum_{w\in\mathcal{A}^{n+m}} \frac{1}{\delta_{n+m}} Q^{\Sigma}_{r_0}(u\circ G_w) + \sum_{k=1}^{n+m} \frac{1}{\gamma_k} \sum_{w\in\mathcal{A}^{k-1}} Q^{I}_{\boldsymbol{\rho}^k}(u\circ G_w) \\
&= \sum_{w\in\mathcal{A}^{n}} \frac{1}{\delta_n} \sum_{\tilde{w}\in\mathcal{A}^{m}} \frac{1}{\lambda_{n+1}\cdots\lambda_{n+m}} Q^{\Sigma}_{r_0}(u\circ G_w\circ G_{\tilde{w}}) + \sum_{k=1}^{n} \frac{1}{\gamma_k} Q_{\boldsymbol{\rho}^k,k}(u) \\
&\quad + \sum_{w\in\mathcal{A}^{n}} \frac{1}{\delta_n} \sum_{k=1}^{m} \underbrace{\frac{1}{\lambda_{n+1}\cdots\lambda_{n+k-1}}}_{:=1 \text{ for } k=1} \sum_{\tilde{w}\in\mathcal{A}^{k-1}} Q^{I}_{\boldsymbol{\rho}^{n+k}}(u\circ G_w\circ G_{\tilde{w}}) \\
&= \sum_{w\in\mathcal{A}^{n}} \frac{1}{\delta_n} \Big(\hat{\mathcal{E}}^{\Sigma}_{\mathcal{R}^{(n)},m}(u\circ G_w) + \hat{\mathcal{E}}^{I}_{\mathcal{R}^{(n)},m}(u\circ G_w) \Big) + \sum_{k=1}^{n} \frac{1}{\gamma_k} Q_{\boldsymbol{\rho}^k,k}(u) \\
&= \sum_{w\in\mathcal{A}^{n}} \frac{1}{\delta_n} \hat{\mathcal{E}}_{\mathcal{R}^{(n)},m}(u\circ G_w) + \sum_{k=1}^{n} \frac{1}{\gamma_k} Q_{\boldsymbol{\rho}^k,k}(u).
\end{aligned}$$

By taking the limit as $m\to\infty$ we get the desired result. □

Lemma 3.15. *Let $\mathcal{R} = \{(r_0,\lambda_i,\boldsymbol{\rho}^i)\}_{i\geq 1}$ be a regular sequence of harmonic structures. Then*

$$\operatorname{diam}(G_w V_*, \hat{R}_{\mathcal{R}}) \leq c_0 \delta_n$$

for all $w\in\mathcal{A}^n$ with a constant $c_0>0$ only depending on λ^*, ρ^* and r_0.

Proof. From the rescaling in Lemma 3.14 we immediately get for all $w\in\mathcal{A}^n$ that

$$\frac{1}{\delta_n}\hat{\mathcal{E}}_{\mathcal{R}^{(n)}}(u\circ G_w) \leq \hat{\mathcal{E}}_{\mathcal{R}}(u).$$

Let $p,q\in G_w(V_*)$, which means there exist $x,y\in V_*$ such that $p=G_w(x)$ and $q=G_w(y)$. For $u\in\hat{\mathcal{F}}_{\mathcal{R}}$ we have

$$\frac{|u(p)-u(q)|^2}{\hat{\mathcal{E}}_{\mathcal{R}}(u)} \leq \delta_n \frac{|u(G_w(x))-u(G_w(y))|^2}{\hat{\mathcal{E}}_{\mathcal{R}^{(n)}}(u\circ G_w)} \leq \delta_n \hat{R}_{\mathcal{R}^{(n)}}(x,y).$$

Since $x,y\in V_*$ there exists a $k\in\mathbb{N}$ with $x,y\in V_k$. Then the effective resistance between x and y can be calculated with the effective resistance on the graph (V_k,E_k) with the resistance function that belongs to the sequence $\mathcal{R}^{(n)}$. From Lemma 3.13 we know that there is a constant c_0 that only depends on λ^*, ρ^* and r_0, and thus it is valid for $\mathcal{R}^{(n)}$ for all n. This means

$$\hat{R}_{\mathcal{R}^{(n)}}(x,y) = R_{\mathcal{R}^{(n)},k}(x,y) \leq c_0,$$

which leads to

$$\frac{|u(p)-u(q)|^2}{\hat{\mathcal{E}}_{\mathcal{R}}(u)} \leq \delta_n c_0,\ \forall u\in\hat{\mathcal{F}}_{\mathcal{R}}.$$

Taking the supremum over all $u \in \hat{\mathcal{F}}_{\mathcal{R}}$ leads to $\hat{R}_{\mathcal{R}}(p,q) \leq \delta_n c_0$. This holds for all $p, q \in G_w(V_*)$, which gives us the desired result. $\square$

This means the diameter of n-cells goes to 0 for smaller cells (small in terms of big n). This is very important to compare Cauchy sequences. Roughly: Cauchy sequences have to be in smaller getting cells, or in some fixed C_k. The diameter of small n-cells (i.e., big n) goes to zero for both resistance and Euclidean metric. We now give an exact proof of this fact.

Lemma 3.16. *Let $\mathcal{R}$ be a regular sequence of harmonic structures. Then, the completion of V_* is the same with either the resistance metric $\hat{R}_{\mathcal{R}}$ or the Euclidean metric d_E.*

Proof. We show that $(V_*, \hat{R}_{\mathcal{R}})$ and (V_*, d_E) have the same Cauchy sequences.

First let $(x_i)_{i\geq 1}$ be a Cauchy sequence with respect to the Euclidean metric d_E in V_*. Then there are two possibilities.

Since P_n and C_n are positively separated with respect to d_E, there is either an $i_0 \geq 1$ with $x_i = x \in C_n$ for all $i \geq i_0$ or there is a $w = w_1 w_2 \cdots \in \mathcal{A}^{\mathbb{N}}$ with $\forall m \exists i_m \geq 1$ such that $\forall i \geq i_m : x_i \in G_{w_1 \cdots w_m}(V_*)$.

In fact, this is only true since the n-cells are also positively separated. This is due to the stretching and it is not true in the self-similar case. In the first case it is obviously also a Cauchy sequence with respect to the resistance metric. Let us, therefore, look at the second case.

From Lemma 3.15 we know that $\operatorname{diam}(G_{w_1 \cdots w_m}(V_*), \hat{R}_{\mathcal{R}}) \leq \delta_m c_0 \to 0$. Therefore, we have for all $k, l \geq i_m$ that $\hat{R}_{\mathcal{R}}(x_k, x_l) \leq \delta_m c_0$, which makes $(x_i)_{i\geq 1}$ a Cauchy sequence with respect to the resistance metric.

Now take any Cauchy sequence $(x_i)_{i\geq 1}$ with respect to the resistance metric $\hat{R}_{\mathcal{R}}$. We have

$$V_* = \sum_{w \in \mathcal{A}^n} G_w(V_*) \dot{\cup} C_n.$$

For $w \in \mathcal{A}^n$ define

$$\begin{aligned} u &\equiv 1, \text{ on } G_w(V_*), \\ u &\equiv 0, \text{ on } G_w(V_*)^c. \end{aligned}$$

We can easily see that $u \in \hat{\mathcal{F}}_{\mathcal{R}}$, and thus

$$\hat{R}_{\mathcal{R}}(x,y) \geq \frac{|u(x) - u(y)|^2}{\hat{\mathcal{E}}_{\mathcal{R}}(u)} = \frac{1}{\hat{\mathcal{E}}_{\mathcal{R}}(u)} > 0$$

for all $x \in G_w(V_*)$ and $y \in G_w(V_*)^c$.

Therefore

$$\inf\{\hat{R}_{\mathcal{R}}(x,y) \mid x \in G_w(V_*), y \in C_n\} > 0$$

and also for $\tilde{w} \in \mathcal{A}^n \backslash \{w\}$

$$\inf\{\hat{R}_{\mathcal{R}}(x,y) \mid x \in G_w(V_*), y \in G_{\tilde{w}}(V_*)\} > 0.$$

Since we have only finitely many n-cells we can even find a common bound for all n-cells. This means, that the n-cells are positively separated with respect to $\hat{R}_{\mathcal{R}}$ and also positively separated away from C_n. We can, therefore, use the same argument as before: There is either an x in some C_n such that $(x_i)_{i\geq 1}$ gets trapped at x or we have smaller getting cells where all but finitely many x_i lie. In either case $(x_i)_{i\geq 1}$ is also a Cauchy sequence with respect to the Euclidean metric. □

Due to this lemma we know that $\overline{V}_* = \Sigma \cup C_*$ where the closure is taken with respect to the resistance metric $\hat{R}_{\mathcal{R}}$ if $\mathcal{R}$ is a regular sequence of harmonic structures. If $\mathcal{R}$ is not regular we are not able to prove this result. In this case it could happen that $\overline{V}_*$ is a proper subset of $\Sigma \cup C_*$ and thus we do not get a resistance form on $\Sigma \cup C_*$.

This is an analogy to the self-similar case; compare [45, Proposition 20.7]. Therefore, the choice of the terms *regular* and *harmonic structure* is justified.

Remark 3.17. We denote the extension of $(\hat{\mathcal{E}}_{\mathcal{R}}, \hat{\mathcal{F}}_{\mathcal{R}})$ to $\Sigma \cup C_*$ again by $(\hat{\mathcal{E}}_{\mathcal{R}}, \hat{\mathcal{F}}_{\mathcal{R}})$ as no confusion can occur. $(\hat{\mathcal{E}}_{\mathcal{R}}, \hat{\mathcal{F}}_{\mathcal{R}})$ is a resistance form on $\Sigma \cup C_*$ due to [43, Theorem 2.3.10].

3.3.3 Resistance form on K

Until now we have resistance forms on $\Sigma \cup C_*$. However, we would like to have resistance forms on K, which means we need to replace the squared differences along the edges that represent connecting lines with some form that considers all values of u along this line and not just the endpoints. For these one-dimensional lines we can use the usual Dirichlet energy.

Consider the edges in E_n^I. These have a one-to-one correspondence with the connecting lines $e_{c,l}^w$ with $w \in \mathcal{A}^k$, $k \leq n-1$. For $\{x,y\} \in E_n^I$ we know that x and y are the endpoints of $e_{c,l}^w$ and thus define $\xi_{e_{c,l}^w}(t) := \xi_{xy}(t) := tx + (1-t)y$, for $t \in [0,1]$. That means $\xi_{e_{c,l}^w}$ maps $u|_{e_{c,l}^w}$ to a function $u \circ \xi_{e_{c,l}^w}$ on $[0,1]$. Look at the Dirichlet energy on this line:

$$\mathcal{D}_{e_{c,l}^w}(u) := \mathcal{D}_{xy}(u) := \int_0^1 \left(\frac{d(u\circ\xi_{xy})}{dz}\right)^2 dz.$$

This can be defined if $u|_{e_{c,l}^w} \circ \xi_{e_{c,l}^w}$ is in $H^1([0,1])$. We see that this does not depend on the orientation of ξ_{xy}, and therefore, the choice of endpoints of $e_{c,l}^w$ is not important. We introduce some further notation concerning the Sobolev spaces on $e_{c,l}^w$ by

$$H^1(e_{c,l}^w) := \{u \mid u : e_{c,l}^w \to \mathbb{R},\ u \circ \xi_{e_{c,l}^w} \in H^1([0,1])\}.$$

Definition 3.18 (Approximating quadratic forms on K)**.** For $\boldsymbol{\rho} = \{\rho_e\}_{e\in E_1^I}$ and $u \in \ell(J_1)$ with $u|_{e_{c,l}} \in H^1(e_{c,l})$ for all $c \in \mathcal{C}$ and $l \in \{1, \dots, \rho(c)\}$ define

$$\mathcal{D}_{\boldsymbol{\rho}}(u) := \sum_{\{x,y\}\in E_1^I} \frac{1}{\rho_{\{x,y\}}} \mathcal{D}_{xy}(u).$$

Now for a sequence of harmonic structures $\mathcal{R} = \{(\lambda_i, \boldsymbol{\rho}^i)\}_{i\geq 1}$ and $n \geq 1$ we define

$$\mathcal{E}^I_{\mathcal{R},n}(u) := \sum_{k=1}^{n} \frac{1}{\gamma_k} \underbrace{\sum_{w\in\mathcal{A}^{k-1}} \mathcal{D}_{\boldsymbol{\rho}^k}(u \circ G_w)}_{\mathcal{D}_{\boldsymbol{\rho}^k,k}(u):=}$$

and the whole form with $\mathcal{E}^\Sigma_{\mathcal{R},n} = \hat{\mathcal{E}}^\Sigma_{\mathcal{R},n}$ by

$$\mathcal{E}_{\mathcal{R},n}(u) := \mathcal{E}^\Sigma_{\mathcal{R},n}(u) + \mathcal{E}^I_{\mathcal{R},n}(u)$$

for $u \in \ell(J_n)$ with $u|_{e^w_{c,l}} \in H^1(e^w_{c,l})$ for all $w \in \mathcal{A}^k$, $k \leq n-1$, $c \in \mathcal{C}$ and $l \in \{1, \dots, \rho(c)\}$.

The definition of $\mathcal{E}^I_{\mathcal{R},n}$ is analogous to the definition of $\hat{\mathcal{E}}^I_{\mathcal{R},n}$ in Definition 3.9 after replacing squared differences with Dirichlet energies. Again, we would like to take the limit of these quadratic forms but we need to verify that this is well-defined.

Lemma 3.19. *Let $\mathcal{R}$ be a sequence of harmonic structures. Then $(\mathcal{E}_{\mathcal{R},n}(u))_{n\geq 0}$ is non-decreasing for $u \in C(K)$ with $u|_{e^w_{c,l}} \in H^1(e^w_{c,l})$ for all $c \in \mathcal{C}, l \in \{1, \dots, \rho(c)\}$ and $w \in \mathcal{A}_0^*$.*

Proof. (1.) First we notice that

$$\hat{\mathcal{E}}^I_{\mathcal{R},n}(u) \leq \mathcal{E}^I_{\mathcal{R},n}(u), \ \forall u.$$

This is true since $(f(1) - f(0))^2 \leq \int_0^1 \left(\frac{df}{dx}\right)^2 dx$ for all $f \in H^1([0,1])$. This is the only difference between $\hat{\mathcal{E}}^I_{\mathcal{R},n}$ and $\mathcal{E}^I_{\mathcal{R},n}$. The range of sums and the prefactors are the same. This holds in general for $Q^I_{\boldsymbol{\rho}}(u) \leq \mathcal{D}_{\boldsymbol{\rho}}(u)$.

(2.) Next we show

$$\mathcal{E}_{\mathcal{R},n}(u) \leq \mathcal{E}_{\mathcal{R},n+1}(u), \ \forall u.$$

Since $(\lambda_{n+1}, \boldsymbol{\rho}^{n+1})$ is a harmonic structure we have

$$Q^\Sigma_{r_0}(u) \leq \frac{1}{\lambda_{n+1}} \sum_{i\in\mathcal{A}} Q^\Sigma_{r_0}(u \circ G_i) + Q^I_{\boldsymbol{\rho}^{n+1}}(u).$$

Summing over all n-cells leads to

$$\sum_{w\in\mathcal{A}^n} Q^\Sigma_{r_0}(u \circ G_w) \leq \frac{1}{\lambda_{n+1}} \sum_{w\in\mathcal{A}^{n+1}} Q^\Sigma_{r_0}(u \circ G_w) + \sum_{w\in\mathcal{A}^n} Q^I_{\boldsymbol{\rho}^{n+1}}(u \circ G_w).$$

By applying (1.) for $Q^I_{\boldsymbol{\rho}^{n+1}}$ and multiplying both sides with $\frac{1}{\delta_n} = \frac{1}{\gamma_{n+1}}$ we get

$$\mathcal{E}^{\Sigma}_{\mathcal{R},n}(u) \leq \mathcal{E}^{\Sigma}_{\mathcal{R},n+1}(u) + \frac{1}{\gamma_{n+1}} \sum_{w\in\mathcal{A}^n} \mathcal{D}_{\boldsymbol{\rho}^{n+1}}(u \circ G_w).$$

Now if we add $\sum_{k=1}^n \frac{1}{\gamma_k}\mathcal{D}_{\boldsymbol{\rho}^k,k}(u)$ on both sides we get the desired result. □

We see that the limit is a well-defined object in $[0,\infty]$.

Definition 3.20 (Resistance form on K)**.** Let $\mathcal{R} = \{(\lambda_i, \boldsymbol{\rho}^i)\}_{i\geq 1}$ be a sequence of harmonic structures. We denote

$$\mathcal{E}_{\mathcal{R}}(u) := \lim_{n\to\infty} \mathcal{E}_{\mathcal{R},n}(u),$$

which is defined on

$$\mathcal{F}_{\mathcal{R}} := \left\{ u \;\middle|\; u \in C(K), \quad \begin{array}{l} u|_{e^w_{c,l}} \in H^1(e^w_{c,l}) \ \forall c \in \mathcal{C}, l \in \{1,\dots,\rho(c)\}, w \in \mathcal{A}^*_0, \\ \lim_{n\to\infty} \mathcal{E}_{\mathcal{R},n}(u) < \infty \end{array} \right\}.$$

Remark 3.21. Again, if we replace $\mathcal{D}_{e^w_{c,l}}(u)$ by

$$\mathcal{D}_{e^w_{c,l}}(u,v) := \mathcal{D}_{xy}(u,v) := \int_0^1 \left(\frac{d(u\circ\xi_{xy})}{dz}\right)\left(\frac{d(v\circ\xi_{xy})}{dz}\right) dz,$$

we obtain a bilinear form $\mathcal{E}_{\mathcal{R}}(u,v)$ for $u, v \in \mathcal{F}_{\mathcal{R}}$.

These forms fulfill the same rescaling as $\hat{\mathcal{E}}_{\mathcal{R}}$.

Lemma 3.22 (Rescaling of $\mathcal{E}_{\mathcal{R}}$)**.** *Let $\mathcal{R}$ be a sequence of harmonic structures. Then for all $u, v \in \mathcal{F}_{\mathcal{R}}$ and $w \in \mathcal{A}^n$ we have $u \circ G_w, v \circ G_w \in \mathcal{F}_{\mathcal{R}^{(n)}}$ and*

$$\mathcal{E}_{\mathcal{R}}(u,v) = \sum_{w\in\mathcal{A}^n} \frac{1}{\delta_n}\mathcal{E}_{\mathcal{R}^{(n)}}(u\circ G_w, v\circ G_w) + \sum_{k=1}^n \frac{1}{\gamma_k}\mathcal{D}_{\boldsymbol{\rho}^k,k}(u,v).$$

Proof. This works exactly the same as for $\hat{\mathcal{E}}_{\mathcal{R}}$ in Lemma 3.14. □

We now show that $(\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}})$ is indeed a resistance form on K.

Theorem 3.23. *Let $\mathcal{R}$ be a regular sequence of harmonic structures on K. Then $(\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}})$ is a regular resistance form on K where the associated resistance metric $R_{\mathcal{R}}$ is inducing the same topology as the Euclidean metric.*

In order to show Theorem 3.23 we have to show (RF1)–(RF5) of Definition B.1.

Lemma 3.24. *Let $\mathcal{R} = \{(r_0, \lambda_i, \boldsymbol{\rho}^i)\}_{i\geq 1}$ be a regular sequence of harmonic structures. Then there is a constant $c_0 > 0$ only depending on λ^*, ρ^* and r_0 such that we have for all $u \in \mathcal{F}_{\mathcal{R}}$ and $x, y \in K$*

$$|u(x) - u(y)|^2 \leq c_0 \mathcal{E}_{\mathcal{R}}(u).$$

Proof. We have three distinct cases:

(1) $x, y \in \overline{V}_*$,

(2) $x \in \overline{V}_*$, $y \notin \overline{V}_*$,

(3) $x, y \notin \overline{V}_*$.

For case (1) notice: If $u \in \mathcal{F}_{\mathcal{R}}$ we have that $u|_{\overline{V}_*} \in \hat{\mathcal{F}}_{\mathcal{R}}$ since

$$\hat{\mathcal{E}}_{\mathcal{R}}(u|_{\overline{V}_*}) \leq \mathcal{E}_{\mathcal{R}}(u).$$

As $(\hat{\mathcal{E}}_{\mathcal{R}}, \hat{\mathcal{F}}_{\mathcal{R}})$ is the extended resistance form on $\overline{V}_*$ Lemma 3.15 still holds and thus

$$|u(x) - u(y)|^2 \leq c_1 \hat{\mathcal{E}}_{\mathcal{R}}(u|_{\overline{V}_*}) \leq c_1 \mathcal{E}_{\mathcal{R}}(u)$$

with a constant c_1 only depending on λ^*, ρ^* and r_0.

Now consider case (2): $x \in \overline{V}_*$ and $y \notin \overline{V}_*$: That means y is in some $e^w_{c,l}$ with $c \in \mathcal{C}$, $l \in \{1, \dots, \rho(c)\}$, $w \in \mathcal{A}_0^*$ and in particular it is not one of the endpoints. Let p be one of the endpoints of $e^w_{c,l}$; we may choose $p := G_w(c)$. Then $p \in \overline{V}_*$, which implies

$$|u(p) - u(x)|^2 \leq c_1 \mathcal{E}_{\mathcal{R}}(u).$$

Now $y, p \in e^w_{c,l}$, the resistance of $e^w_{c,l}$ is $\gamma_{|w|}\rho_{c,l}^{|w|+1}$. We thus have

$$|u(y) - u(p)|^2 \leq \underbrace{\gamma_{|w|}\rho_{c,l}^{|w|+1}}_{\leq \rho^*} \frac{1}{\gamma_{|w|}\rho_{c,l}^{|w|+1}} \mathcal{D}_{e^w_{c,l}}(u) \leq \rho^* \mathcal{E}_{\mathcal{R}}(u).$$

The last inequality holds since the Dirichlet energy on $e^w_{c,l}$ is only one part of the whole energy. Since $(a+b)^2 \leq 2a^2 + 2b^2$ we get

$$\begin{aligned} |u(x) - u(y)|^2 &= |u(x) - u(p) + u(p) - u(y)|^2 \\ &\leq 2|u(x) - u(p)|^2 + 2|u(p) - u(y)|^2 \\ &\leq 2(c_1 + \rho^*)\mathcal{E}_{\mathcal{R}}(u). \end{aligned}$$

We can use these two cases to handle the last one (3): $x, y \notin \overline{V}_*$.
Choose any $p \in \overline{V}_*$. Then

$$|u(x) - u(p)|^2 \leq 2(c_1 + \rho^*)\mathcal{E}_{\mathcal{R}}(u)$$

as well as

$$|u(y) - u(p)|^2 \leq 2(c_1 + \rho^*)\mathcal{E}_{\mathcal{R}}(u)$$

and thus

$$|u(x) - u(y)|^2 \leq 2|u(x) - u(p)|^2 + 2|u(y) - u(p)|^2$$
$$\leq \underbrace{8(c_1 + \rho^*)}_{c_0 :=} \mathcal{E}_{\mathcal{R}}(u).$$

Therefore, it holds for all $x, y \in K$ and $u \in \mathcal{F}_{\mathcal{R}}$ that

$$|u(x) - u(y)|^2 \leq c_0 \mathcal{E}_{\mathcal{R}}(u).$$

□

In analogy to Lemma 3.15 we can refine these results with the help of the rescaling property.

Corollary 3.25. *Let $\mathcal{R} = \{(r_0, \lambda_i, \boldsymbol{\rho}^i)\}_{i \geq 1}$ be a regular sequence of harmonic structures. Then for all $x, y \in K_w$ with $w \in \mathcal{A}^n$ and $u \in \mathcal{F}_{\mathcal{R}}$ we have*

$$|u(x) - u(y)|^2 \leq c_0 \delta_n \mathcal{E}_{\mathcal{R}}(u)$$

with a constant $c_0 > 0$ only depending on λ^*, ρ^* and r_0.

Proof. Since the constant c_0 from Lemma 3.24 only depends on λ^*, ρ^* and r_0 it holds also for $(\mathcal{E}_{\mathcal{R}^{(n)}}, \mathcal{F}_{\mathcal{R}^{(n)}})$. There are $x', y' \in K$ with $x = G_w(x')$ and $y = G_w(y')$. From the rescaling we know $u \circ G_w \in \mathcal{F}_{\mathcal{R}^{(n)}}$ and thus

$$|u(x) - u(y)|^2 = |u(G_w(x')) - u(G_w(y'))|^2$$
$$\leq c_0 \mathcal{E}_{\mathcal{R}^{(n)}}(u \circ G_w)$$
$$\leq c_0 \delta_n \mathcal{E}_{\mathcal{R}}(u).$$

□

Proof of Theorem 3.23:
(RF1): $\mathcal{F}_{\mathcal{R}}$ is a linear space and $\mathcal{E}_{\mathcal{R}}(u) \geq 0$ is obviously satisfied. If $\mathcal{E}_{\mathcal{R}}(u) = 0$, then $\hat{\mathcal{E}}_{\mathcal{R}}(u|_{\overline{V}_*}) = 0$. Since this is a resistance form on $\overline{V}_*$ we know that u is constant on $\overline{V}_*$. Also we know that $\mathcal{D}_{e^w_{c,l}}(u) = 0$ for all $e^w_{c,l}$ and thus u is constant on all of them. Since $G_w(c) \in e^w_{c,l} \cap \overline{V}_*$ the constants have to be the same on all parts and therefore, u is constant on K.

(RF2): Fix any $p \in V_0$. Then it is enough to show that $\mathcal{F}_{\mathcal{R},0} := \{u \mid u \in \mathcal{F}_{\mathcal{R}},\ u(p) = 0\}$ is complete with respect to $\mathcal{E}_{\mathcal{R}}$. Let $(u_n)_{n \geq 1}$ be a Cauchy sequence in $\mathcal{F}_{\mathcal{R},0}$ with respect to $\mathcal{E}_{\mathcal{R}}$. I.e.,

$$\mathcal{E}_{\mathcal{R}}(u_n - u_m) \to 0, \text{ for } m \geq n,\ n \to \infty.$$

Since $u_n(p) = u_m(p) = 0$ we have by Lemma 3.24

$$\begin{aligned}|u_n(x) - u_m(x)|^2 &= |(u_n - u_m)(x) - (u_n - u_m)(p)|^2 \\ &\leq c_0 \mathcal{E}_{\mathcal{R}}(u_n - u_m).\end{aligned}$$

That means we have the uniform convergence of $(u_n)_{n\geq 1}$ and therefore, there is a $u \in C(K)$ with $u_n \to u$. Since $\mathcal{D}_{e^w_{c,l}}$ is a resistance form itself and $\mathcal{D}_{e^w_{c,l}}(u_n - u_m) \to 0$ we get that $u|_{e^w_{c,l}} \in H^1(e^w_{c,l})$.

It remains to show that $u_n \to u$ with respect to $\mathcal{E}_{\mathcal{R}}$ and that $\mathcal{E}_{\mathcal{R}}(u) < \infty$. Let $m \geq n$. Then

$$\mathcal{E}_{\mathcal{R},k}(u_n - u_m) \leq \mathcal{E}_{\mathcal{R}}(u_n - u_m) \leq \underbrace{\sup_{l\geq n} \mathcal{E}_{\mathcal{R}}(u_n - u_l)}_{<\infty}.$$

If we let m go to infinity we get

$$\mathcal{E}_{\mathcal{R},k}(u_n - u) \leq \sup_{l\geq n} \mathcal{E}_{\mathcal{R}}(u_n - u_l).$$

We were able to substitute u for u_m in the limit since in $\mathcal{E}_{\mathcal{R},k}$ only squared differences of u and Dirichlet energies appear. We already know that u_m converges to u with respect to them. Next we let $k \to \infty$ and obtain

$$\mathcal{E}_{\mathcal{R}}(u_n - u) \leq \sup_{l\geq n} \mathcal{E}_{\mathcal{R}}(u_n - u_l) < \infty.$$

That means $u_n - u \in \mathcal{F}_{\mathcal{R}}$ and by $u_n \in \mathcal{F}_{\mathcal{R}}$ this implies $u \in \mathcal{F}_{\mathcal{R}}$ because this is a linear space. Also for $n \to \infty$ we get

$$\mathcal{E}_{\mathcal{R}}(u_n - u) \to 0 \quad \text{and thus} \quad u_n \xrightarrow{\mathcal{E}_{\mathcal{R}}} u.$$

(RF3): (1) x or $y \notin \overline{V}_*$

Without loss of generality let x be this point. Then there exists an $e^w_{c,l}$ with $x \in e^w_{c,l}$ but $x \notin \overline{V}_*$. We look for a function $u \in \mathcal{F}_{\mathcal{R}}$ such that

$$u(x) = 1,\ u|_{(e^w_{c,l})^c} \equiv 0.$$

For example we can use linear interpolation between x and the endpoints of $e^w_{c,l}$. Then $u \in \mathcal{F}_{\mathcal{R}}$ and $u(\tilde{y}) < 1$ for all $\tilde{y} \neq x$.

(2) $x, y \in \overline{V}_*$:

We find a u in the extended domain of $\hat{\mathcal{F}}_{\mathcal{R}}$ with $u(x) \neq u(y)$. We can extend u continuously to a function $\tilde{u}$ by linear interpolation on all $e^w_{c,l}$. Then

$$\mathcal{E}_{\mathcal{R}}(\tilde{u}) = \hat{\mathcal{E}}_{\mathcal{R}}(u)$$

and thus $\tilde{u} \in \mathcal{F}_{\mathcal{R}}$ with $\tilde{u}(x) = u(x) \neq u(y) = \tilde{u}(y)$.

(RF4): This follows from Lemma 3.24.

(RF5): We have $\overline{u} = (0 \vee u) \wedge 1$. It is clear that $\overline{u} \in C(K)$, and also $\overline{u}|_{e^w_{c,l}} \in H^1(e^w_{c,l})$. We see that

$$|\overline{u}(x) - \overline{u}(y)|^2 \leq |u(x) - u(y)|^2, \ \forall u, x, y$$

as well as

$$\mathcal{D}_{e^w_{c,l}}(\overline{u}) \leq \mathcal{D}_{e^w_{c,l}}(u)$$

for all $e^w_{c,l}$. Furthermore

$$\mathcal{E}^{\Sigma}_{\mathcal{R},n}(\overline{u}) \leq \mathcal{E}^{\Sigma}_{\mathcal{R},n}(u)$$

and thus

$$\mathcal{E}_{\mathcal{R},n}(\overline{u}) \leq \mathcal{E}_{\mathcal{R},n}(u).$$

This leads to

$$\mathcal{E}_{\mathcal{R}}(\overline{u}) \leq \mathcal{E}_{\mathcal{R}}(u)$$

meaning $\overline{u} \in \mathcal{F}_{\mathcal{R}}$.

So far we have shown that $(\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}})$ is a resistance form on K. It remains to show that the topologies with respect to the resistance and Euclidean metrics are the same. Let $\iota : (K, d_E) \to (K, R_{\mathcal{R}})$ be the identity mapping and $(x_n)_{n\geq 1}$ a sequence in K with $x_n \xrightarrow{d_E} x$. We have to show that $(x_n)_{n\geq 1}$ also converges to x with the resistance metric to show that ι is continuous.

We have three cases:

(1) x lies in the interior of some $e^w_{c,l}$.

(2) $x \in C_*$.

(3) $x \in \overline{V}_* \backslash C_* = \Sigma$.

Consider (1). In this case we find an $n_0 \geq 0$ such that for all $n \geq n_0$ we have $x_n \in e^w_{c,l}$. Let $u \in \mathcal{F}_{\mathcal{R}}$. Then

$$\frac{|u(x_n) - u(x)|^2}{\mathcal{E}_{\mathcal{R}}(u)} \leq \frac{|u(x_n) - u(x)|^2}{(\gamma_{|w|}\rho_{c,l}^{|w|+1})^{-1}\mathcal{D}_{e^w_{c,l}}(u)}.$$

Now $\mathcal{D}_{e^w_{c,l}}$ itself is a resistance form and its associated resistance metric $\frac{d_E(x,y)}{\operatorname{diam}(e^w_{c,l}, d_E)}$. Therefore, we obtain

$$\frac{|u(x_n) - u(x)|^2}{\mathcal{E}_{\mathcal{R}}(u)} \leq \gamma_{|w|}\rho_{c,l}^{|w|+1} \frac{d_E(x_n, x)}{\operatorname{diam}(e^w_{c,l}, d_E)},$$

which leads to

$$R_{\mathcal{R}}(x_n, x) \leq \frac{\gamma_{|w|}\rho_{c,l}^{|w|+1}}{\operatorname{diam}(e^w_{c,l}, d_E)} d_E(x_n, x) \xrightarrow{n\to\infty} 0.$$

(2) $x \in C_*$, i.e., $x = G_w(c)$ for some $c \in \mathcal{C}$ and $w \in \mathcal{A}_0^*$. Then there is an $n_0 \geq 0$ such that for all $n \geq n_0$ we have

$$x_n \in \bigcup_{l \in \{1,\dots,\rho(c)\}} e_{c,l}^w.$$

That means the elements of the sequence may jump around the various lines that are connected to $x = G_w(c)$ in this "spider's web". We decompose the sequence $(x_n)_{n\geq 1}$ into various subsequences $\{x_n \mid x_n \in e_{c,l}^w\}$, $\forall l$ and $\{x_n \mid x_n = x\}$. For the latter it is clear that $R_{\mathcal{R}}(x_n, x) \to 0$ and for the first we can apply (1). We thus have $R_{\mathcal{R}}(x_n, x) \to x$ for all subsequences and thus for the whole sequence $(x_n)_{n\geq 1}$.

(3) There is a word $w \in \mathcal{A}^{\mathbb{N}}$ such that $x = \lim_{m\to\infty} G_{w_1\cdots w_m}(p)$.
Now either (I) $\forall m$ we have $x_n \in G_{w_1\cdots w_m}(K)$ for n big enough
or (II) the sequence $(x_n)_{n\geq 1}$ can be divided into two parts where the first contains all points that behave like (I) and the second contains all points that do not, i.e., are in some edges $e_{c,l}^w$.
For the first case (I) we know that the diameter of n-cells goes to 0 by Corollary 3.25 and thus it converges in the resistance metric. For the second case (II) we can apply the ideas we already introduced.

That means the identity map $\iota : (K, d_E) \to (K, R)$ is continuous. Since (K, d_E) is compact, so is (K, R) and thus ι^{-1} is also continuous. Therefore, the topologies are the same and the resistance forms are regular by Lemma B.14. $\square$

3.4 Measures and operators

Until now we have just resistance forms. To get Dirichlet forms and thus self-adjoint operators we need to introduce Borel measures. These measures have to fulfill some requirements. They have to be locally finite, i.p. finite due to the compactness of K, and to be supported on the whole set K; see also Appendix B.3 or [45, Chapter 9].

3.4.1 Measures

We would like to describe the measures on K as the sum of a fractal and a line part in accordance with the geometric appearance of K.

It is clear how the fractal part of the measure should look like. We would like as much symmetry as possible. Therefore, we use the normalized self-similar measure on K which distributes mass equally onto the n-cells. Choose

$$\mu_\Sigma(K_w) = \mu_\Sigma(\Sigma_w) = \left(\frac{1}{N}\right)^{|w|},$$

which gives us a measure on K that fulfills

$$\mu_\Sigma = \sum_{i=1}^{N} \frac{1}{N} \cdot \mu_\Sigma \circ G_i^{-1}.$$

We see, however, that μ_Σ is only supported on the fractal dust Σ. This is a proper subset of K and therefore, μ_Σ does not have full support. That means we cannot use μ_Σ to get Dirichlet forms. This measure is too rough to detect the one-dimensional lines. We, therefore, need another measure that is able to do so.

For this line part we ignore the length of $e^w_{c,l}$ according to the one-dimensional Lebesgue measure λ^1. Since we are analyzing K only topologically, this value is not giving us much information. We need a measure that assigns these lines some weights which are finite when summed up. For the initial lines $e_{c,l}$ we set

$$\mu_I(e_{c,l}) := a_{c,l}$$

with $a_{c,l} > 0$ for $c \in \mathcal{C}$ and $l \in \{1, \ldots, \rho(c)\}$.

How should this measure scale for lines $e^w_{c,l}$. For symmetry reasons we would like that the scaling is independent of the n-cell that we consider. We thus define

$$\mu_I(e^w_{c,l}) := \beta^{|w|} a_{c,l}$$

with some $\beta > 0$. We easily see that we need $\beta < \frac{1}{N}$ to get a finite measure on $J = \bigcup_{n \geq 1} J_n$. On the lines we define the measure by

$$\mu_I|_{e^w_{c,l}} := \beta^{|w|} a_{c,l} \cdot \frac{\lambda^1}{\lambda^1(e^w_{c,l})}, \; w \in \mathcal{A}_0^*.$$

This means it behaves like the one-dimensional Lebesgue measure on $e^w_{c,l}$ but it is normalized and then scaled by $\beta^{|w|} a_{c,l}$. Therefore, it does not depend on the value of $\lambda^1(e^w_{c,l})$. If $\beta < \frac{1}{N}$ we have $a := \mu_I(J) < \infty$. We choose the $a_{c,l}$ such that $\mu_I(J) = 1$, by dividing by a. Calculating $\mu_I(J)$ gives us

$$\sum_{\substack{c \in \mathcal{C} \\ l \in \{1,\ldots,\rho(c)\}}} a_{c,l} = 1 - \beta N.$$

If $\beta \to 0$ then more mass is distributed to bigger edges, big in the sense of short words w, and if $\beta \to \frac{1}{N}$ the mass is distributed more equally which displays the geometry better. As a matter of fact $\beta = \frac{1}{N}$ is not possible, that means the real geometry of K is distorted by the measure μ_I.

Since $V_*\backslash V_0 \subset J$ and $K = \Sigma \cup J$ we know that J is dense in K. Therefore, μ_I has full support and can be used to get Dirichlet forms. The measures that we consider will be convex combinations of the two measures.

Definition 3.26. Let $\beta \in (0, \frac{1}{N})$ and μ_I and μ_Σ as above. Then for $\eta \in (0,1]$ define

$$\mu_\eta := \eta\mu_I + (1-\eta)\mu_\Sigma.$$

Note that $\eta = 0$ is not allowed, since μ_Σ does not have full support. The line part $\mu_1 = \mu_I$, however, can be used alone. In this case we do not have any fractal part in the measure and this will reflect in the spectral asymptotics. We now study how the measure of n-cells behaves.

Proposition 3.27. *Let $w \in \mathcal{A}_0^*$. Then*

$$\mu_\Sigma(K_w) = \left(\frac{1}{N}\right)^{|w|} \qquad \textit{and} \qquad \mu_I(K_w) = \beta^{|w|}.$$

Proof. This is immediate since

$$\mu_\Sigma \circ G_w = N^{-|w|} \text{ and } \mu_I \circ G_w = \beta^{|w|}\mu_I$$

by the definition of μ_Σ and μ_I. □

For μ_η this leads to

$$\beta^{|w|} \leq \mu_\eta(K_w) \leq \left(\frac{1}{N}\right)^{|w|}.$$

We can improve the lower bound to order $\left(\frac{1}{N}\right)^{|w|}$ as in Section 2.4.

Proposition 3.28. *Let $w \in \mathcal{A}_0^*$ and $\eta \in (0,1)$. Then*

$$(1-\eta)\left(\frac{1}{N}\right)^{|w|} \leq \mu_\eta(K_w) \leq \left(\frac{1}{N}\right)^{|w|}.$$

3.4.2 Operators

With these measures we can define Dirichlet forms and therefore, self-adjoint operators on $L^2(K, \mu_\eta)$. Let $\mathcal{R}$ be a regular sequence of harmonic structures.

Now since $(K, R_\mathcal{R})$ is compact we have $\mathcal{D}_\mathcal{R} := \overline{\mathcal{F}_\mathcal{R} \cap C_0(K)}^{\mathcal{E}_{\mathcal{R},1}^{\frac{1}{2}}} = \mathcal{F}_\mathcal{R}$ due to Lemma B.12.

Lemma 3.29. *Let $\mathcal{R}$ be a regular sequence of harmonic structures. Then $(\mathcal{E}_\mathcal{R}, \mathcal{D}_\mathcal{R})$ is a regular Dirichlet form on $L^2(K, \mu_\eta)$.*

Proof. From Theorem 3.23 we know that $(\mathcal{E}_\mathcal{R}, \mathcal{F}_\mathcal{R})$ is a regular resistance form on K. The statement follows from Theorem B.15. □

Introducing Dirichlet boundary conditions we get another Dirichlet form with the domain $\mathcal{D}^0_\mathcal{R} := \{u \mid u \in \mathcal{D}_\mathcal{R},\ u|_{V_0} \equiv 0\}$.

Lemma 3.30. *Let $\mathcal{R}$ be a regular sequence of harmonic structures. Then $(\mathcal{E}_\mathcal{R}|_{\mathcal{D}^0_\mathcal{R}\times\mathcal{D}^0_\mathcal{R}}, \mathcal{D}^0_\mathcal{R})$ is a regular Dirichlet form on $L^2(K\backslash V_0, \mu_\eta|_{K\backslash V_0})$.*

Proof. This follows from Lemma 3.29 and [45, Theorem 10.3] or [20, Theorem 4.4.3]. □

We denote the associated self-adjoint operators with dense domains by $-\Delta_N^{\mu_\eta,\mathcal{R}}$ resp. $-\Delta_D^{\mu_\eta,\mathcal{R}}$. For the construction of the associated operator we refer to [13, Chapter 10].

Lemma 3.31. $-\Delta_N^{\mu_\eta,\mathcal{R}}$ *and* $-\Delta_D^{\mu_\eta,\mathcal{R}}$ *have discrete non-negative spectrum.*

Proof. Since $(K, R_\mathcal{R})$ is compact it follows from [45, Lemma 9.7] that the inclusion map $\iota : \mathcal{D}_\mathcal{R} \hookrightarrow C(K)$ with the norms $\mathcal{E}^{\frac{1}{2}}_{\mathcal{R},1}$ resp. $||\cdot||_\infty$ is a compact operator. Since the inclusion map from $C(K)$ to $L^2(K, \mu_\eta)$ is continuous, the inclusion from $\mathcal{D}_\mathcal{R}$ to $L^2(K, \mu_\eta)$ is a compact operator and therefore, the spectrum of $-\Delta_N^{\mu_\eta,\mathcal{R}}$ is discrete and by [13, Theorem 5, Chapter 10.1] and non-negative as $\mathcal{E}_\mathcal{R}(u) \geq 0$ for all $u \in \mathcal{D}_\mathcal{R}$. Since $\mathcal{D}^0_\mathcal{R} \subset \mathcal{D}_\mathcal{R}$ the same follows for $-\Delta_D^{\mu_\eta,\mathcal{R}}$ by [13, Theorem 4, Chapter 10.2]. □

3.5 Conditions

In Section 3.4 we constructed Dirichlet forms and thus self-adjoint operators on stretched fractals. We needed regular sequences of harmonic structures to do so. We now analyze these operators by calculating some values that give a further description of the underlying fractal. These values are the Hausdorff dimension calculated with respect to the resistance metric and the asymptotic growing of the eigenvalue counting function. But to be able to do this we need to introduce a condition on the sequences of harmonic structures.

Condition 3.1. *We consider a regular sequence of harmonic structures $\mathcal{R} = \{(\lambda_i, \boldsymbol{\rho}^i)\}_{i\geq 1}$ such that there exists a $\lambda \in (0, \lambda^*]$ and constants $\tilde{\kappa}_1, \tilde{\kappa}_2 > 0$ with*

$$\tilde{\kappa}_1\lambda^m \leq \delta_m \leq \tilde{\kappa}_2\lambda^m$$

for all $m \geq 0$.

Without loss of generality we can assume that $\tilde{\kappa}_1 \leq 1 \leq \tilde{\kappa}_2$ and thus with $\kappa_1 := \frac{\tilde{\kappa}_1}{\tilde{\kappa}_2}$ and $\kappa_2 := \frac{\tilde{\kappa}_2}{\tilde{\kappa}_1}$ we obtain the following estimates for $\delta_m^{(n)} = \lambda_{n+1}\cdots\lambda_{n+m} = \frac{\delta_{n+m}}{\delta_n}$.

Lemma 3.32. *Let $\mathcal{R} = \{(\lambda_i, \boldsymbol{\rho}^i)\}_{i\geq 1}$ be a regular sequence of harmonic structures that fulfills Condition 3.1. Then there exist constants $0 < \kappa_1 \leq \kappa_2 < \infty$ with*

$$\kappa_1 \lambda^m \leq \delta_m^{(n)} \leq \kappa_2 \lambda^m$$

for all $m, n \geq 0$.

This means if we have a regular sequence of harmonic structures that fulfills Condition 3.1 we have control over the resistances that appear in the rescaling of the quadratic form in Lemma 3.22. And we have this control for all sequences $\mathcal{R}^{(n)}$ with the same constants κ_1 and κ_2.

Remark. Let us consider a regular sequence of harmonic structures $\mathcal{R} = \{(\lambda_i, \boldsymbol{\rho}^i)\}_{i\geq 1}$ such that there exists a $\lambda \in (0, \lambda^*]$ with

$$\sum_{i=1}^{\infty} |\lambda_i - \lambda| < \infty.$$

Then Condition 3.1 is satisfied.

We can see this by using the limit comparison test to easily show that in this case $\sum_{k=1}^{\infty} |\ln(\lambda^{-1}\lambda_k)|$ converges and thus

$$\prod_{i=1}^{\infty} \lambda^{-1}\lambda_i \in (0, \infty).$$

This means the sequence $a_m := \prod_{i=1}^{m} \lambda^{-1}\lambda_i$ is bounded from above and below by constants $0 < \tilde{\kappa}_1 \leq \tilde{\kappa}_2 < \infty$, which leads to

$$\tilde{\kappa}_1 \lambda^m \leq \delta_m \leq \tilde{\kappa}_2 \lambda^m$$

for all $m \geq 0$.

3.6 Hausdorff dimension in resistance metric

The Hausdorff dimension is a value which describes the geometric size of a set. It strongly depends on the metric that we choose to calculate it. In Proposition 3.3 we calculated the Hausdorff dimension of stretched fractals with respect to the Euclidean metric. This, however, is not a very meaningful value to describe the analysis on a set. We saw that the resistance forms do not depend on the stretching parameter but only on the topology on K. The resistance metric is a better choice to describe the analytic structure of the stretched fractal, so we would like to calculate the Hausdorff dimension with respect to this metric.

The line part of K is defined by

$$J = \bigcup_{\substack{c\in\mathcal{C}, w\in\mathcal{A}_0^* \\ l\in\{1,\dots,\rho(c)\}}} e_{c,l}^w = \bigcup_{n\geq 1} J_n,$$

then $K = \Sigma \cup J$; note this is not disjoint. We calculate the dimension of the two parts and due to the stability the Hausdorff dimension of the union will be the bigger of the two.

Lemma 3.33. *For any sequence of harmonic structures $\mathcal{R}$ we have*

$$\dim_{H,R_{\mathcal{R}}} J = 1.$$

Proof. We show that $\dim_{H,R_{\mathcal{R}}}(e_{c,l}^w) = 1$ for all $e_{c,l}^w$. The result follows from σ-stability. To show this look for constants $0 < a \leq b < \infty$ with

$$a \cdot d_E(x,y) \leq R_{\mathcal{R}}(x,y) \leq b \cdot d_E(x,y)$$

for all $x, y \in e_{c,l}^w$.

(1.) $R_{\mathcal{R}}(x,y) \leq b \cdot d_E(x,y)$

For this we consider $u \in \mathcal{F}_{\mathcal{R}}$ with $u(x) = 1$ and $u(y) = 0$. We have

$$\mathcal{E}_{\mathcal{R}}(u) \geq \frac{1}{\gamma_{|w|}\rho_{c,l}^{|w|+1}} \mathcal{D}_{e_{c,l}^w}(u)$$

for all such u, since this is just one part of the energy. The resistance metric of a one-dimensional Dirichlet energy $\mathcal{D}_{e_{c,l}^w}$ is $\frac{d_E(x,y)}{\operatorname{diam}(e_{c,l}^w, d_E)}$. Thus we obtain

$$\mathcal{E}_{\mathcal{R}}(u) \geq \frac{1}{\gamma_{|w|}\rho_{c,l}^{|w|+1}} \frac{\operatorname{diam}(e_{c,l}^w, d_E)}{d_E(x,y)}.$$

This means for the resistance metric

$$R_{\mathcal{R}}(x,y) \leq \frac{\gamma_{|w|}\rho_{c,l}^{|w|+1}}{\operatorname{diam}(e_{c,l}^w, d_E)} d_E(x,y).$$

(2.) $R_{\mathcal{R}}(x,y) \geq a \cdot d_E(x,y)$

Without loss of generality let $d_E(x, G_w(c)) > d_E(y, G_w(c))$. Then define u as follows by

$$u(x) = 0,\ u(y) = 1$$

and the linear interpolation between them. Also continue u constant 0 from x to the endpoint which is not $G_w(c)$ and from y to $G_w(c)$ with 1. Now we copy this behavior onto the other lines $e^w_{c,\tilde{l}}$ with $\tilde{l} \in \{1, \ldots, \rho(c)\}$ and $\tilde{l} \neq l$. That means we would like that

$$u \circ \xi_{e^w_{c,\tilde{l}}}(t) = u \circ \xi_{e^w_{c,l}}(t), \ \forall t \in [0,1].$$

Outside of these edges, we set the function constant 0. The construction is illustrated in Figure 3.24. Then $u \in \mathcal{F}_\mathcal{R}$ and we can calculate the energy of u as

$$\mathcal{E}_\mathcal{R}(u) = \left(\sum_{\tilde{l} \in \{1,\ldots,\rho(c)\}} \frac{1}{\gamma_{|w|} \rho_{c,\tilde{l}}^{|w|+1}} \right) \cdot \frac{\operatorname{diam}(e^w_{c,l}, d_E)}{d_E(x,y)}.$$

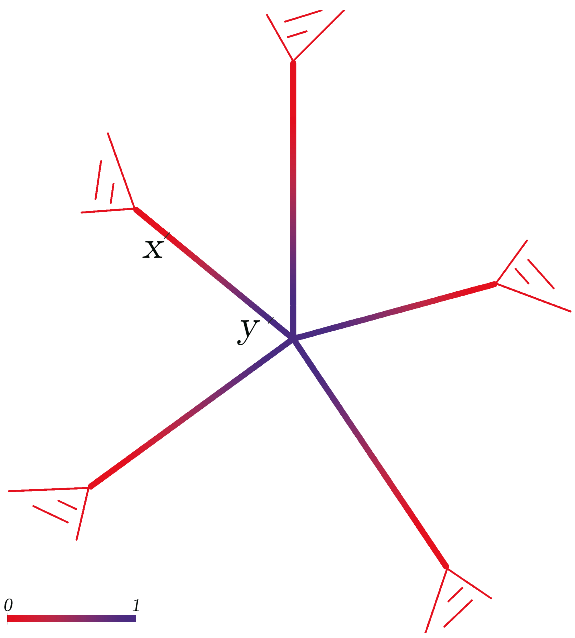

Figure 3.24: Construction of u on connecting lines.

Note that the different lines $e^w_{c,\tilde{l}}$ can have different length, with respect to d_E, but since we stretched the function such that the proportion of the different parts of u stays the same, the energy is calculated in this way. Since u is one of the functions for which the supremum is taken at

$$R_\mathcal{R}(x,y) = \sup \left\{ \frac{|u(x) - u(y)|^2}{\mathcal{E}_\mathcal{R}(u)} \;\middle|\; u \in \mathcal{F}_\mathcal{R}, \ \mathcal{E}_\mathcal{R}(u) > 0 \right\}$$

we get

$$R_\mathcal{R}(x,y) \geq a \cdot d_E(x,y).$$

The constants a, b depend on various things, but they are constant for fixed $e^w_{c,l}$ and hold for all $x, y \in e^w_{c,l}$. □

Next we calculate the Hausdorff dimension of Σ. This is a self-similar set and $R_\mathcal{R}|_\Sigma$ is a metric on Σ. We can apply the ideas of [40] to calculate this value.

Lemma 3.34. *Let $\mathcal{R} = \{(\lambda_i, \boldsymbol{\rho}^i)\}_{i\geq 1}$ be a regular sequence of harmonic structures that fulfills Condition 3.1. Then there exists a constant $c_0 > 0$ such that*

$$\operatorname{diam}(\Sigma_w, R_{\mathcal{R}}) \leq c_0 \lambda^n$$

for all $w \in \mathcal{A}^n$ and $n \in \mathbb{N}$.

Proof. We know from Corollary 3.25 that there is a constant $\tilde{c}_0$ such that

$$\operatorname{diam}(K_w, R_{\mathcal{R}}) \leq \tilde{c}_0 \delta_n.$$

Since $\Sigma_w \subset K_w$ we get with Lemma 3.32

$$\operatorname{diam}(\Sigma_w, R_{\mathcal{R}}) \leq \operatorname{diam}(K_w, R_{\mathcal{R}}) \leq \tilde{c}_0 \delta_n \leq \tilde{c}_0 \kappa_2 \lambda^n.$$

□

Lemma 3.35. *Let $\mathcal{R} = \{(\lambda_i, \boldsymbol{\rho}^i)\}_{i\geq 1}$ be a regular sequence of harmonic structures that fulfills Condition 3.1. Then there is an $M \geq 0$ and $c_0 > 0$ such that for all $x \in \Sigma$ we have*

$$\#\{w \in \mathcal{A}^n \mid R_{\mathcal{R}}(x, \Sigma_w) \leq c_0 \lambda^n\} \leq M + 1$$

for all $n \in \mathbb{N}$.

Proof. Since

$$R_{\mathcal{R}}(x, y) = \sup\left\{ \frac{|u(x) - u(y)|^2}{\mathcal{E}_{\mathcal{R}}(u)} \;\middle|\; u \in \mathcal{F}_{\mathcal{R}},\ \mathcal{E}_{\mathcal{R}}(u) > 0 \right\}$$

we get for a fixed $u \in \mathcal{F}_{\mathcal{R}}$ with $u(x) = 0$ and $u(y) = 1$

$$R_{\mathcal{R}}(x, y) \geq \frac{1}{\mathcal{E}_{\mathcal{R}}(u)}.$$

We are looking for a u such that this estimate is good enough. Let $w \in \mathcal{A}^n$, $y \in \Sigma_w$ and $x \in \Sigma \backslash \Sigma_w$. We define a function $\tilde{u}_n$ on V_n and then extend it harmonically to $u_n \in \mathcal{F}_{\mathcal{R}}$. Under harmonic extension the energy does not change, so we are able to calculate $\mathcal{E}_{\mathcal{R}}(u_n)$. Define

$$\tilde{u}_n := 1, \quad \text{on } G_w(V_0).$$

Now search for all n-cells that are connected to $G_w(V_0)$ over some $c \in C_*$. There are at most $M := \#\mathcal{C}\#V_0$ many of those; see [40, Lemma 3.3]. Set $\tilde{u}_n = 1$ on all $c \in C_*$ that are connected to $G_w(V_0)$ by some line in J and also 1 on the endpoints that intersect with the other n-cells. Set $\tilde{u}_n = 0$ on all remaining points of V_n. This procedure is illustrated in Figure 3.25 for the stretched level-3 Sierpiński gasket.

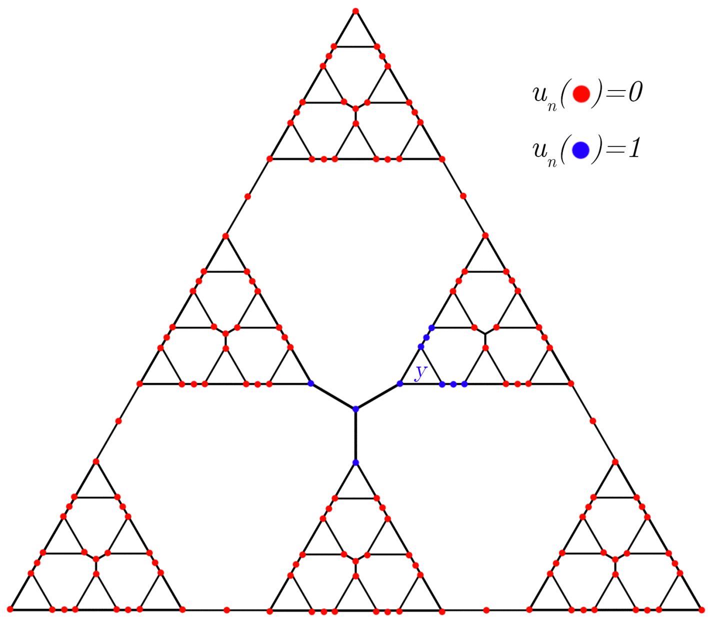

Figure 3.25: Construction of u_n.

Next, we extend $\tilde{u}_n$ harmonically to $u_n \in \mathcal{F}_{\mathcal{R}}$. We can bound the energy by

$$\mathcal{E}_{\mathcal{R}}(u_n) = \hat{\mathcal{E}}_{\mathcal{R},n}(\tilde{u}_n) \leq M \cdot \#E_0 \cdot \frac{1}{\delta_n \min_{e\in E_0} r_0(e)}$$
$$\leq \frac{M\#E_0}{\kappa_1 \min_{e\in E_0} r_0(e)} \cdot \lambda^{-n},$$

where we applied Lemma 3.32. This leads to

$$R_{\mathcal{R}}(x,y) \geq \frac{\kappa_1 \min_{e\in E_0} r_0(e)}{M\#E_0} \cdot \lambda^n.$$

This procedure can be done for all $y \in \Sigma_w$ and x such that $u_n(x) = 0$, that means all x that are not in Σ_w and all other connected n-cells. There are, therefore, at most $M+1$ many n-cells, including Σ_w itself, for which this construction does not work. This gives us the desired result. □

Now we are able to calculate the Hausdorff dimension of K.

Theorem 3.36. *Let $\mathcal{R} = \{(\lambda_i, \boldsymbol{\rho}^i)\}_{i\geq 1}$ be a regular sequence of harmonic structures that fulfills Condition 3.1. Then*

$$\dim_{H,R_{\mathcal{R}}}(K) = \max\left\{1, \frac{\ln N}{-\ln \lambda}\right\}.$$

Proof. By Lemmata 3.34 and 3.35 it follows from [40, Theorem 2.4 or Corollary 1.3] that

$$\dim_{H,R_{\mathcal{R}}}(\Sigma) = \frac{\ln N}{-\ln \lambda}.$$

From Lemma 3.33 and $K = \Sigma \cup J$ we get the result. □

3.6.1 Examples

With this result and the harmonic structures that we calculated in Section 3.2.3 we are now able to calculate the values of the Hausdorff dimension with respect to the resistance metric of these stretched fractals for different choices of regular sequences of harmonic structures that fulfill Condition 3.1. For comparison we also list the values in the self-similar case. In the second column we list all possible values in the stretched case.

	$\dim_{H,R_{\mathcal{R}}}$	
	self-similar	stretched
Sierpiński gasket	$\frac{\ln 3}{-\ln \frac{3}{5}}$	$\max\{1, \frac{\ln 3}{-\ln \lambda}\}, \quad \lambda \in (0, \frac{3}{5}]$
Level-3 Sierpiński gasket	$\frac{\ln 6}{-\ln \frac{7}{15}}$	$\max\{1, \frac{\ln 6}{-\ln \lambda}\}, \quad \lambda \in (0, \frac{7}{15}]$
Sierpiński gasket in $\mathbb{R}^d$	$\frac{\ln(d+1)}{-\ln(\frac{d+1}{d+3})}$	$\max\{1, \frac{\ln(d+1)}{-\ln \lambda}\}, \lambda \in (0, \frac{d+1}{d+3}]$
Vicsek set	$\frac{\ln 5}{-\ln \frac{1}{3}}$	$\max\{1, \frac{\ln 5}{-\ln \lambda}\}, \quad \lambda \in (0, \frac{1}{3}]$
Hata's tree	$\frac{\ln 2}{\ln 2 - \ln(\sqrt{5}-1)}$	$\max\{1, \frac{\ln 2}{-\ln \lambda}\}, \quad \lambda \in (0, \frac{\sqrt{5}-1}{2})$

The values of the self-similar case were calculated in general in [40]. With the renormalization factors we get the according values. In general the value in the stretched case is less than or equal to the one in the self-similar case. We, however, are able to get the same value in all but one case. For stretched Hata's tree we saw that we can only choose constant sequences of λ_i. Therefore, they cannot converge to the upper bound and thus we are not able reach the same value as in the self-similar case.

3.7 Spectral asymptotics

Let μ_η be any of the measures from Definition 3.26 and $\mathcal{R}$ a regular sequence of harmonic structures. Due to Lemma 3.31 we can write the eigenvalues in non-decreasing order and study the eigenvalue counting function. We denote by $\sigma_k^{N,\mu_\eta,\mathcal{R}}$ the k-th eigenvalue of $-\Delta_N^{\mu_\eta,\mathcal{R}}$ and $\sigma_k^{D,\mu_\eta,\mathcal{R}}$ that of $-\Delta_D^{\mu_\eta,\mathcal{R}}$ for $k \geq 1$. Now we can define the eigenvalue counting functions

$$\begin{aligned} N_N^{\mu_\eta,\mathcal{R}}(x) &:= \#\{k \geq 1 \mid \sigma_k^{N,\mu_\eta,\mathcal{R}} \leq x\}, \\ N_D^{\mu_\eta,\mathcal{R}}(x) &:= \#\{k \geq 1 \mid \sigma_k^{D,\mu_\eta,\mathcal{R}} \leq x\}. \end{aligned}$$

The homomorphism theorem applied to $\mathcal{D}_\mathcal{R} \ni u \mapsto u|_{V_0} \in \ell(V_0)$ yields a linear isomorphism from $\mathcal{D}_\mathcal{R}/\mathcal{D}_\mathcal{R}^0$ to $\ell(V_0)$. We thus have $\dim \mathcal{D}_\mathcal{R}/\mathcal{D}_\mathcal{R}^0 = N$. From this it follows by [13, Theorem 5, Chapter 10.2] that

$$N_D^{\mu_\eta,\mathcal{R}}(x) \leq N_N^{\mu_\eta,\mathcal{R}}(x) \leq N_D^{\mu_\eta,\mathcal{R}}(x) + N, \ \forall x \geq 0.$$

We would like to study the asymptotic behavior of the eigenvalue counting functions. However, we can only calculate the leading order of the eigenvalue counting functions for regular sequences of harmonic structures that fulfill Condition 3.1 of Section 3.5. In the following section we will state the results for such sequences.

3.7.1 Results

The next theorem summarizes the results for the order of the leading term for various regular sequences of harmonic structures and measures.

Theorem 3.37. *Let $\mathcal{R}$ be a regular sequence of harmonic structures that fulfills Condition 3.1 and let $\eta \in (0,1]$. Then there exist constants $0 < C_1, C_2 < \infty$ and $x_0 \geq 0$ such that for all $x \geq x_0$*

$$C_1 x^{\frac{1}{2} d_S^{\mu_\eta,\mathcal{R}}} \leq N_D^{\mu_\eta,\mathcal{R}}(x) \leq N_N^{\mu_\eta,\mathcal{R}}(x) \leq C_2 x^{\frac{1}{2} d_S^{\mu_\eta,\mathcal{R}}}$$

with

$$d_S^{\mu_\eta,\mathcal{R}} = \begin{cases} \max\{1, \frac{\ln N^2}{\ln N - \ln\lambda}\}, & \text{for } \eta \in (0,1), \\ \max\{1, \frac{\ln N^2}{-\ln(\beta\lambda)}\}, & \text{for } \eta = 1, \text{ with } \beta \neq \frac{1}{N^2\lambda}. \end{cases}$$

The order of the leading term is the maximum of the two values. One value corresponds to the fractal part inside the stretched fractal. However, if λ gets too small the one-dimensional lines become the dominant part and the leading order becomes 1.

The constants C_1 and C_2 depend on $\mathcal{R}$ and μ_η. We see that the scaling parameter β of the line part of the measure does not appear in the leading order if the fractal part of the measure exists. The choice of the regular sequence of harmonic structures as well as the choice of the measure has a big influence on the spectral asymptotics on K. We call the value $d_S^{\mu_{0.5},\mathcal{R}}(K) =: d_S^{\mathcal{R}}(K)$ the spectral dimension of the stretched fractal K with respect to $\mathcal{R}$.

Remark 3.38. In Theorem 3.36 we calculated $\dim_{H,R_\mathcal{R}}(K) = \max\{1, \frac{\ln N}{-\ln\lambda}\}$. We thus obtain

$$d_S^{\mathcal{R}}(K) = \frac{2\dim_{H,R_\mathcal{R}}(K)}{\dim_{H,R_\mathcal{R}}(K) + 1}.$$

This relation was shown to hold for p.c.f. self-similar sets in [38, Theorem A.2] and in Chapter 2 for the stretched Sierpiński gasket. It is thus valid in general for stretched fractals.

3.7.2 Examples

We now list the values for the examples for which we calculated the harmonic structures in Section 3.2.3 and compare them to the self-similar case. The measure $\tilde{\mu}_\Sigma$ that we use in the self-similar case is the self-similar measure that assigns each n-cell the same weight.

	$d_S^{\mu,\mathcal{R}}$		
	self-similar	stretched	
Measure	$\tilde{\mu}_\Sigma$	μ_η, $\eta \in (0,1)$	μ_1
Sierpiński gasket	$\frac{\ln 9}{\ln 5}$	$\max\{1, \frac{\ln 9}{\ln 3 - \ln \lambda}\}$	$\max\{1, \frac{\ln 9}{-\ln \beta\lambda}\}$
		$\lambda \in (0, \frac{3}{5}]$	
		$\beta \in (0, \frac{1}{3})$	$\beta \in (0, \frac{1}{3}), \beta \neq \frac{1}{9\lambda}$
Level-3 Sierpiński gasket	$\frac{2\ln 6}{\ln 6 - \ln \frac{7}{15}}$	$\max\{1, \frac{2\ln 6}{\ln 6 - \ln \lambda}\}$	$\max\{1, \frac{2\ln 6}{-\ln \beta\lambda}\}$
		$\lambda \in (0, \frac{7}{15}]$	
		$\beta \in (0, \frac{1}{6})$	$\beta \in (0, \frac{1}{6}), \beta \neq \frac{1}{6^2\lambda}$
Sierpiński gasket in $\mathbb{R}^d$	$\frac{2\ln(d+1)}{\ln(d+3)}$	$\max\{1, \frac{2\ln(d+1)}{\ln(d+1) - \ln \lambda}\}$	$\max\{1, \frac{2\ln(d+1)}{-\ln \beta\lambda}\}$
		$\lambda \in (0, \frac{d+1}{d+3}]$	
		$\beta \in (0, \frac{1}{d+1})$	$\beta \in (0, \frac{1}{d+1})$, $\beta \neq \frac{1}{(d+1)^2\lambda}$
Vicsek set	$\frac{2\ln 5}{\ln 15}$	$\max\{1, \frac{2\ln 5}{\ln 5 - \ln \lambda}\}$	$\max\{1, \frac{2\ln 5}{-\ln \beta\lambda}\}$
		$\lambda \in (0, \frac{1}{3}]$	
		$\beta \in (0, \frac{1}{5})$	$\beta \in (0, \frac{1}{5}), \beta \neq \frac{1}{5^2\lambda}$
Hata's tree	$\frac{\ln 4}{\ln 4 - \ln(\sqrt{5}-1)}$	$\max\{1, \frac{\ln 4}{\ln 2 - \ln \lambda}\}$	$\max\{1, \frac{\ln 4}{-\ln \beta\lambda}\}$
		$\lambda \in (0, \frac{\sqrt{5}-1}{2})$	
		$\beta \in (0, \frac{1}{2})$	$\beta \in (0, \frac{1}{2}), \beta \neq \frac{1}{4\lambda}$

The values for the self-similar column come from the results of [38] together with the renormalization factors for these examples. As for the Hausdorff dimension the values for $d_S^{\mu_\eta,\mathcal{R}}$ are less than or equal to the corresponding values in the self-similar case. They can reach the same value in all examples but stretched Hata's tree.

3.7.3 Proof of Theorem 3.37

The proof follows the same ideas as the proof of Theorem 2.13 in Chapter 2. We will carry out the argument for $\mu = \mu_\eta$ with $\eta \in (0,1)$ and at the end show what happens for $\mu = \mu_1$. We split the proof into the upper and lower estimate.

U: Upper estimate

We obtain the upper estimate by successively adding new Neumann boundary conditions at the points $V_m \backslash V_0$, thus making the domain bigger and therefore, increasing the eigenvalue counting function. We can introduce the Neumann conditions by defining the domains

$$\begin{aligned}
\mathcal{D}_{\mathcal{R},K_m} &:= \{u \mid u \in L^2(K_m, \mu|_{K_m}),\ \exists f \in \mathcal{D}_{\mathcal{R}} : f|_{K_m} = u\},\\
\mathcal{D}_{\mathcal{R},J_m} &:= \{u \mid u \in L^2(J_m, \mu|_{J_m}),\ \forall e_{c,l}^w \subset J_m \exists f \in \mathcal{D}_{\mathcal{R}} : f|_{e_{c,l}^w} = u|_{e_{c,l}^w}\}.
\end{aligned}$$

Since the lines $e_{c,l}^w$ in J_m are decoupled by the new Neumann boundary conditions we can see that

$$\mathcal{D}_{\mathcal{R},J_m} = \bigoplus_{\substack{c\in\mathcal{C}, l\in\{1,\ldots,\rho(c)\}\\ w\in A^n, n<m}} H^1(e_{c,l}^w).$$

We also notice that $\mathcal{D}_{\mathcal{R},K_m}$ is orthogonal to $\mathcal{D}_{\mathcal{R},J_m}$ in $L^2(K,\mu)$ and

$$\mathcal{D}_{\mathcal{R}} \subset \mathcal{D}_{\mathcal{R},K_m} \oplus \mathcal{D}_{\mathcal{R},J_m}.$$

We can define a new quadratic form $\tilde{\mathcal{E}}_{\mathcal{R}}$ on this bigger domain for $f = g + h$ with $g \in \mathcal{D}_{\mathcal{R},K_m}$ and $h \in \mathcal{D}_{\mathcal{R},J_m}$ by

$$\tilde{\mathcal{E}}_{\mathcal{R}}(f) := \mathcal{E}_{\mathcal{R}}^{\Sigma}(g) + \sum_{k=m+1}^{\infty} \frac{1}{\gamma_k} \mathcal{D}_{\boldsymbol{\rho}^k,k}(g) + \sum_{k=1}^{m} \frac{1}{\gamma_k} \mathcal{D}_{\boldsymbol{\rho}^k,k}(h)$$

and

$$\begin{aligned}
\mathcal{E}_{\mathcal{R},K_m}(g) &:= \mathcal{E}_{\mathcal{R}}^{\Sigma}(g) + \sum_{k=m+1}^{\infty} \frac{1}{\gamma_k} \mathcal{D}_{\boldsymbol{\rho}^k,k}(g),\\
\mathcal{E}_{\mathcal{R},J_m}(h) &:= \sum_{k=1}^{m} \frac{1}{\gamma_k} \mathcal{D}_{\boldsymbol{\rho}^k,k}(h).
\end{aligned}$$

Lemma 3.39. *$(\tilde{\mathcal{E}}_{\mathcal{R}}, \mathcal{D}_{\mathcal{R},K_m} \oplus \mathcal{D}_{\mathcal{R},J_m})$, $(\mathcal{E}_{\mathcal{R},K_m}, \mathcal{D}_{\mathcal{R},K_m})$ and $(\mathcal{E}_{\mathcal{R},J_m}, \mathcal{D}_{\mathcal{R},J_m})$ are regular Dirichlet forms with discrete non-negative spectrum and $\tilde{\mathcal{E}}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R},K_m} \oplus \mathcal{E}_{\mathcal{R},J_m}$.*

Proof. $(\mathcal{E}_{\mathcal{R},J_m},\mathcal{D}_{\mathcal{R},J_m})$ is just the sum of scaled Dirichlet energies on one-dimensional edges, hence it is a regular Dirichlet form on $L^2(J_m,\mu|_{J_m})$ with discrete non-negative spectrum. We note that

$$(\mathcal{E}_{\mathcal{R},K_m},\mathcal{D}_{\mathcal{R},K_m}) = \bigoplus_{w\in\mathcal{A}^m} \left(\delta_m^{-1}\mathcal{E}_{\mathcal{R}^{(m)}}\left((\cdot)\circ G_w,(\cdot)\circ G_w\right),\mathcal{D}_{\mathcal{R}^{(m)},w}\right)$$

where

$$\mathcal{D}_{\mathcal{R}^{(m)},w} := \{u\in C(K_w) \mid u\circ G_w \in \mathcal{D}_{\mathcal{R}^{(m)}}\}.$$

Therefore, $(\mathcal{E}_{\mathcal{R},K_m},\mathcal{D}_{\mathcal{R},K_m})$ is itself a Dirichlet form with non-negative spectrum as an orthogonal sum of such Dirichlet forms. The results for $\tilde{\mathcal{E}}_{\mathcal{R}}$ follow immediately. △

The eigenvalue counting function has many dependencies. We use the same notation as in Chapter 2. For a Dirichlet form $\mathcal{E}$ with domain $\mathcal{D}$ on the Hilbert space $L^2(X,\mu)$ we denote the eigenvalue counting function evaluated at $x\geq 0$ by $N(\mathcal{E},\mathcal{D},\mu,x)$. This is the same as the eigenvalue counting function of the self-adjoint operator associated to the Dirichlet form; see the definition of the associated operator [13, Theorem 2, Chapter 10.1]. In our case the measure is always μ or its restriction to the particular part. We will, therefore, omit it in the notation. For the eigenvalue counting functions of the newly introduced Dirichlet forms this means that

$$\begin{aligned}N_N^{\mu,\mathcal{R}}(x) &\leq N(\tilde{\mathcal{E}}_{\mathcal{R}},\mathcal{D}_{\mathcal{R},K_m}\oplus\mathcal{D}_{\mathcal{R},J_m},x)\\ &= N(\mathcal{E}_{\mathcal{R},K_m},\mathcal{D}_{\mathcal{R},K_m},x)+N(\mathcal{E}_{\mathcal{R},J_m},\mathcal{D}_{\mathcal{R},J_m},x),\ \forall x\geq 0.\end{aligned}$$

This is due to [13, Theorem 4, Chapter 10.2] and the fact that $\mathcal{D}_{\mathcal{R},K_m}$ is orthogonal to $\mathcal{D}_{\mathcal{R},J_m}$. The introduction of the new Neumann boundary conditions leads to the decoupling of the m-cells and the lines joining them. Therefore, the calculations can be done separately. We start with $(\mathcal{E}_{\mathcal{R},K_m},\mathcal{D}_{\mathcal{R},K_m})$ which we call the fractal part.

U.1: Fractal part $(\mathcal{E}_{\mathcal{R},K_m},\mathcal{D}_{\mathcal{R},K_m})$

Define new measures on K for $w\in\mathcal{A}_0^*$ by

$$\mu^w := \mu(K_w)^{-1}\mu\circ G_w.$$

This is a measure on all of K but it only reflects the features of μ on K_w. We notice a few immediate properties. We have

$$\mu^w(K) = \mu(K_w)^{-1}\mu(K_w) = 1$$

as well as

$$\int_K u\circ G_w d\mu^w = \mu(K_w)^{-1}\int_{K_w} u d\mu$$

for all $w \in \mathcal{A}_0^*$. Now, for the upper estimate of the fractal part we use the so-called *uniform Poincaré inequality*; see [35, Definition 4.2]. We define $\mathcal{R}^{(0)} := \mathcal{R}$ as well as $\overline{u}^\nu := \int_K u d\nu$ for measures ν on K.

Proposition 3.40. $(\mathcal{E}_{\mathcal{R}^{(n)}}, \mathcal{D}_{\mathcal{R}^{(n)}})$ *satisfies a uniform Poincaré inequality for all* $n \geq 0$. *I.e., for all* $n \geq 0$, $u \in \mathcal{D}_{\mathcal{R}^{(n)}}$ *and* $w \in \mathcal{A}_0^*$ *it holds that*

$$\mathcal{E}_{\mathcal{R}^{(n)}}(u) \geq C_{PI} \int_K |u - \overline{u}^{\mu^w}|^2 d\mu^w$$

with a constant $0 < C_{PI} < \infty$.

Proof. Due to Lemma 3.24 we know that there is a constant $\mathcal{M} \in (0, \infty)$ only depending on λ^*, ρ^* and r_0 with

$$R_{\mathcal{R}^{(n)}}(p, q) \leq \mathcal{M}, \ \forall p, q \in K, \ \forall n.$$

By the definition of the resistance metric we obtain

$$\mathcal{M}\mathcal{E}_{\mathcal{R}^{(n)}}(u) \geq R_{\mathcal{R}^{(n)}}(p, q)\mathcal{E}_{\mathcal{R}^{(n)}}(u) \geq |u(p) - u(q)|^2.$$

Integrating twice over K with respect to μ^w leads to

$$\begin{aligned}
\int_K \int_K \mathcal{M}\mathcal{E}_{\mathcal{R}^{(n)}}(u) d\mu^w(q) d\mu^w(p) &\geq \int_K \int_K |u(p) - u(q)|^2 d\mu^w(q) d\mu^w(p) \\
&\geq \int_K \left(u(p) - \int_K u(q) d\mu^w(q) \right)^2 d\mu^w(p) \\
&= \int_K |u(p) - \overline{u}^{\mu^w}|^2 d\mu^w(p).
\end{aligned}$$

We thus have

$$\mathcal{E}_{\mathcal{R}^{(n)}}(u) \geq \frac{1}{\mathcal{M}\mu^w(K)^2} \int_K |u - \overline{u}^{\mu^w}|^2 d\mu^w = \frac{1}{\mathcal{M}} \int_K |u - \overline{u}^{\mu^w}|^2 d\mu^w.$$

That means we have $C_{PI} = \frac{1}{\mathcal{M}}$ which holds for all $\mathcal{R}^{(n)}$ and all measures μ^w. $\triangle$

We have N^m independent cells in K_m that means the first N^m eigenvalues are all 0, where the eigenfunctions are the functions that are constant on each m-cell. We are interested in the first non-zero eigenvalue which we call $\sigma^m_{N^m+1}$.

Let $u \in \mathcal{D}_{\mathcal{R},K_m}$ be the normalized eigenfunction to this eigenvalue $\sigma^m_{N^m+1}$. Then u is orthogonal to every v that is constant on the m-cells, since this is a linear combination of eigenfunctions to lower eigenvalues. As u is normalized we get

$$\begin{aligned}
\sigma^m_{N^m+1} &= \mathcal{E}_{\mathcal{R},K_m}(u) \\
&\overset{(i)}{=} \frac{1}{\delta_m} \sum_{w\in\mathcal{A}^m} \mathcal{E}_{\mathcal{R}^{(n)}}(u\circ G_w) \\
&\overset{(ii)}{\geq} \frac{1}{\kappa_2\lambda^m} \sum_{w\in\mathcal{A}^m} C_{PI} \int_K |u\circ G_w - \overline{u\circ G_w}^{\mu^w}|^2 d\mu^w.
\end{aligned}$$

In (i) we used the rescaling of the energy from Lemma 3.22 and in (ii) the Poincaré inequality, i.e., Proposition 3.40. Note that

$$\begin{aligned}
\overline{u\circ G_w}^{\mu^w} &= \int_K u\circ G_w d\mu^w \\
&= \mu(K_w) \int_K u \circ \mathbb{1}_{K_w} d\mu \\
&= 0,
\end{aligned}$$

since u is orthogonal to functions that are constant on m-cells. We obtain

$$\begin{aligned}
\sigma^m_{N^m+1} &\geq \frac{1}{\kappa_2\lambda^m} \sum_{w\in\mathcal{A}^m} C_{PI} \frac{1}{\mu(K_w)} \int_{K_w} u^2 d\mu \\
&\geq \lambda^{-m} \frac{C_{PI}}{\kappa_2 \max\limits_{w\in\mathcal{A}^m} \mu(K_w)} \int_{K_m} u^2 d\mu \\
&\geq \frac{\lambda^{-m}}{N^{-m}} \frac{C_{PI}}{\kappa_2} = C_u \left(\frac{N}{\lambda}\right)^m.
\end{aligned}$$

We used the upper estimate for the measure from Proposition 3.28. So now we know that $\sigma^m_{N^m+1} \geq C_u(N/\lambda)^m$. That means for $x < C_u(N/\lambda)^m$ we have

$$N(\mathcal{E}_{\mathcal{R},K_m}, \mathcal{D}_{\mathcal{R},K_m}, x) \leq N^m.$$

For $x \geq C_u$ take $m \in \mathbb{N}$ such that $C_u(N/\lambda)^{m-1} \leq x < C_u(N/\lambda)^m$. We then know by the previous calculations that

$$\begin{aligned}
N(\mathcal{E}_{\mathcal{R},K_m}, \mathcal{D}_{\mathcal{R},K_m}, x) \leq N^m = N\cdot N^{m-1} &= N\left(\left(\frac{N}{\lambda}\right)^{\frac{\ln(N)}{\ln(N/\lambda)}}\right)^{m-1} \\
&= N\left(\left(\frac{N}{\lambda}\right)^{m-1}\right)^{\frac{\ln(N)}{\ln(N/\lambda)}} \leq N\left(\frac{x}{C_u}\right)^{\frac{\ln(N)}{\ln(N/\lambda)}} \\
&= \underbrace{N C_u^{-\frac{\ln(N)}{\ln(N/\lambda)}}}_{C_2':=} x^{\frac{\ln(N)}{\ln(N/\lambda)}},
\end{aligned}$$

which is independent of m.

U.2: Line part $(\mathcal{E}_{\mathcal{R},J_m}, \mathcal{D}_{\mathcal{R},J_m})$

Due to the decoupling through the Neumann boundary conditions the domain and form split into

$$\mathcal{E}_{\mathcal{R},J_m} = \bigoplus_{\substack{c\in\mathcal{C},l\in\{1,\dots,\rho(c)\}\\ w\in\mathcal{A}^n,n<m}} \frac{1}{\gamma_{|w|+1}\rho_{c,l}^{|w|+1}} \int_0^1 \left(\frac{d(\cdot\circ\xi_{e_{c,l}^w})}{dx}\right)^2 dx,$$

$$\mathcal{D}_{\mathcal{R},J_m} = \bigoplus_{\substack{c\in\mathcal{C},l\in\{1,\dots,\rho(c)\}\\ w\in\mathcal{A}^n,n<m}} H^1(e_{c,l}^w).$$

Then due to the orthogonality it holds for the eigenvalue counting function that

$$N(\mathcal{E}_{\mathcal{R},J_m}, \mathcal{D}_{\mathcal{R},J_m}, x) = \sum_{\substack{c\in\mathcal{C},l\in\{1,\dots,\rho(c)\}\\ w\in\mathcal{A}^n,n<m}} N\left(\frac{1}{\gamma_{|w|+1}\rho_{c,l}^{|w|+1}} \int_0^1 \left(\frac{d(\cdot\circ\xi_{e_{c,l}^w})}{dx}\right)^2 dx, H^1(e_{c,l}^w), x\right).$$

For any $u \in L^2(e_{c,l}^w, \mu|_{e_{c,l}^w})$ we have $\int_{e_{c,l}^w} u^2 d\mu = a\beta^{|w|}\eta \int_0^1 (u\circ\xi_{e_{c,l}^w})^2 dx$. Therefore, there is a one-to-one correspondence of the eigenvalues between the standard Neumann Laplacian on $(0,1)$ and the restriction of the energy to one edge. This means for the eigenvalue counting functions from above that

$$N\left(\frac{1}{\gamma_{|w|+1}\rho_{c,l}^{|w|+1}} \int_0^1 \left(\frac{d(\cdot\circ\xi_{e_{c,l}^w})}{dx}\right)^2 dx, H^1(e_{c,l}^w), x\right) = N(-\Delta_N|_{(0,1)}, \eta a_{c,l}\beta^{|w|}\gamma_{|w|+1}\rho_{c,l}^{|w|+1}x).$$

By the same estimate we already used in the proof of Theorem 2.13 in Chapter 2,

$$N(-\Delta_N|_{(0,1)}, x) \le \frac{1}{\pi}\sqrt{x} + 1,\ \forall x \ge 0,$$

we get

$$\begin{aligned} N(\mathcal{E}_{\mathcal{R},J_m}, \mathcal{D}_{\mathcal{R},J_m}, x) &= \sum_{\substack{c\in\mathcal{C},l\in\{1,\dots,\rho(c)\}\\ w\in\mathcal{A}^n,n<m}} N(-\Delta_N|_{(0,1)}, \eta a_{c,l}\beta^{|w|}\gamma_{|w|+1}\rho_{c,l}^{|w|+1}x) \\ &\le \sum_{\substack{c\in\mathcal{C},l\in\{1,\dots,\rho(c)\}\\ w\in\mathcal{A}^n,n<m}} \frac{1}{\pi}\sqrt{\eta a_{c,l}\beta^{|w|}\gamma_{|w|+1}\rho_{c,l}^{|w|+1}x} + 1. \end{aligned}$$

We include upper bounds for the various terms.

Since $\eta \leq 1$ as well as $a_{c,l} \leq 1$ and $\rho_{c,l}^k \leq \rho^*$ we have

$$\begin{aligned}
N(\mathcal{E}_{\mathcal{R},J_m}, \mathcal{D}_{\mathcal{R},J_m}, x) &\leq \sum_{\substack{c\in\mathcal{C}, l\in\{1,\dots,\rho(c)\}\\ w\in\mathcal{A}^n, n<m}} \frac{1}{\pi}\sqrt{\beta^{|w|}\gamma_{|w|+1}\rho^* x} + 1 \\
&= \sum_{\substack{w\in\mathcal{A}^n\\ n<m}} \#E_I^1 \left(\frac{\rho^*}{\pi}\sqrt{\beta^{|w|}\gamma_{|w|+1}x} + 1\right) \\
&= \sum_{k=0}^{m-1} N^k \#E_I^1 \left(\frac{\rho^*}{\pi}\sqrt{\beta^k\gamma_{k+1}x} + 1\right) \\
&= \sum_{k=0}^{m-1} N^k \#E_I^1 + \sum_{k=0}^{m-1} \#E_I^1 \frac{\rho^*}{\pi}\sqrt{N^{2k}\beta^k\gamma_{k+1}x} \\
&\leq \#E_I^1 \frac{N^m - 1}{N-1} + \sum_{k=0}^{m-1} \#E_I^1 \frac{\rho^*\sqrt{\kappa_2}}{\pi}\sqrt{N^{2k}\beta^k\lambda^k x} \\
&\leq \frac{\#E_I^1}{N-1}N^m + \frac{\#E_I^1\rho^*\sqrt{\kappa_2 x}}{\pi}\sum_{k=0}^{m-1}\sqrt{N^2\beta\lambda}^k. \qquad (3.2)
\end{aligned}$$

From here on we have to distinguish two cases:

(1) $\lambda > \frac{1}{N}$ and $\frac{1}{N^2\lambda} \leq \beta < \frac{1}{N}$.

(2) ($\lambda > \frac{1}{N}$ and $0 < \beta < \frac{1}{N^2\lambda}$) or $\lambda \leq \frac{1}{N}$.

Let us consider the first case and additionally assume that $\beta \neq \frac{1}{N^2\lambda}$. Then $N^2\beta\lambda > 1$ and we get from (3.2)

$$N(\mathcal{E}_{\mathcal{R},J_m}, \mathcal{D}_{\mathcal{R},J_m}, x) \leq \frac{\#E_I^1}{N-1}N^m + \frac{\#E_I^1\rho^*\sqrt{\kappa_2}}{\pi(\sqrt{N^2\beta\lambda}-1)}\sqrt{N^2\beta\lambda}^m\sqrt{x}.$$

For the fractal part we chose m according to x by $C_u(N/\lambda)^{m-1} \leq x < C_u(N/\lambda)^m$. Therefore,

$$\begin{aligned}
N(\mathcal{E}_{\mathcal{R},J_m}, \mathcal{D}_{\mathcal{R},J_m}, x) &\leq \frac{\#E_I^1}{N-1}N^m + \frac{\#E_I^1\rho^*\sqrt{\kappa_2}}{\pi(\sqrt{N^2\beta\lambda}-1)}\sqrt{N^2\beta\lambda}^m\sqrt{C_u(N\lambda^{-1})^m} \\
&= \frac{\#E_I^1}{N-1}N^m + \frac{\#E_I^1\rho^*\sqrt{\kappa_2 C_u}}{\pi(\sqrt{N^2\beta\lambda}-1)}\sqrt{N^3\beta}^m.
\end{aligned}$$

Since $\beta < \frac{1}{N}$ we get

$$N(\mathcal{E}_{\mathcal{R},J_m}, \mathcal{D}_{\mathcal{R},J_m}, x) \leq \frac{\#E_I^1}{N-1}N^m + \frac{\#E_I^1\rho^*\sqrt{\kappa_2 C_u}}{\pi(\sqrt{N^2\beta\lambda}-1)}N^m.$$

Now if $\beta = \frac{1}{N^2\lambda}$ we can change to $\tilde{\beta} := \beta + \epsilon$ with $\frac{1}{N^2\lambda} < \tilde{\beta} < \frac{1}{N}$ and still get the result.

This means we get a constant $\tilde{C}_2''$ such that for x with $C_u(N/\lambda)^{m-1} \leq x < C_u(N/\lambda)^m$ we have

$$N(\mathcal{E}_{\mathcal{R},J_m},\mathcal{D}_{\mathcal{R},J_m},x)\leq \tilde{C}_2'' N^m.$$

With the same calculations as for the fractal part we get the same order $\frac{\ln(N)}{\ln(N/\lambda)}$ for the upper bound. That means for $x\geq C_u$ there is a constant C_2, such that

$$N_N^{\mu,\mathcal{R}}(x)\leq C_2 x^{\frac{\ln(N)}{\ln(N/\lambda)}}.$$

We still have to show the second case. Here we always have $N^2\beta\lambda<1$. This means we get from (3.2)

$$\begin{aligned}N(\mathcal{E}_{\mathcal{R},J_m},\mathcal{D}_{\mathcal{R},J_m},x)&\leq \frac{\#E_I^1}{N-1}N^m+\frac{\#E_I^1\rho^*\sqrt{\kappa_2 x}}{\pi}\sum_{k=0}^{\infty}\sqrt{N^2\beta\lambda}^k\\ &=\leq \frac{\#E_I^1}{N-1}N^m+\frac{\#E_I^1\rho^*\sqrt{\kappa_2}}{\pi}\frac{1}{1-\sqrt{N^2\beta\lambda}}\cdot x^{\frac{1}{2}}.\end{aligned}$$

For the first term with N^m the calculation from before gives us the upper bound with order $\frac{\ln(N)}{\ln(N/\lambda)}$. Now if $\lambda>\frac{1}{N}$ this is bigger than $\frac{1}{2}$ and thus it is the bigger order of asymptotic growing.

However, if $\lambda\leq\frac{1}{N}$ we have $\frac{\ln(N)}{\ln(N/\lambda)}\leq\frac{1}{2}$ and thus $x^{\frac{1}{2}}$ is the order of the leading term.

These estimates give us the desired upper bounds.

L: Lower estimate

The idea to get a lower bound is to add new Dirichlet boundary conditions on V_m which decreases the domain and thus lowers the eigenvalue counting function. Define

$$\begin{aligned}\mathcal{D}^0_{\mathcal{R},K_m}&:=\{u\mid u\in\mathcal{D}^0_{\mathcal{R}},\ u|_{V_m}\equiv 0\},\\ \mathcal{D}^0_{\mathcal{R},K_w}&:=\{u\mid u\in\mathcal{D}^0_{\mathcal{R},K_m},\ u|_{K_w^c}\equiv 0\},\ w\in\mathcal{A}^m,\\ \mathcal{D}^0_{\mathcal{R},e^w_{c,l}}&:=\{u\mid u\in\mathcal{D}^0_{\mathcal{R},K_m},\ u|_{(e^w_{c,l})^c}\equiv 0\},\ w\in\mathcal{A}^k,k<m.\end{aligned}$$

As in Chapter 2 we omit the restrictions for the quadratic forms and, for example, write $(\mathcal{E}_{\mathcal{R}},\mathcal{D}^0_{\mathcal{R},K_m})$ instead of $(\mathcal{E}_{\mathcal{R}}|_{\mathcal{D}^0_{\mathcal{R},K_m}\times\mathcal{D}^0_{\mathcal{R},K_m}},\mathcal{D}^0_{\mathcal{R},K_m})$.

Lemma 3.41. *$(\mathcal{E}_{\mathcal{R}},\mathcal{D}^0_{\mathcal{R},K_m})$, $(\mathcal{E}_{\mathcal{R}},\mathcal{D}^0_{\mathcal{R},K_w})$ and $(\mathcal{E}_{\mathcal{R}},\mathcal{D}^0_{\mathcal{R},e^w_{c,l}})$ are regular Dirichlet forms on $L^2(K\backslash V_m,\mu|_{K\backslash V_m})$, $L^2(K_w\backslash G_w(V_0),\mu|_{K_w\backslash G_w(V_0)})$ and $L^2(\mathrm{int}_K(e^w_{c,l}),\mu|_{\mathrm{int}_K(e^w_{c,l})})$, respectively, with discrete non-negative spectrum.*

Proof. Since $K\backslash V_m$ is open, $(\mathcal{E}_{\mathcal{R}},\mathcal{D}^0_{\mathcal{R},K_m})$ is a regular Dirichlet form with [45, Theorem 10.3] or [20, Theorem 4.4.3]. Since $\mathcal{D}^0_{\mathcal{R},K_m}\subset\mathcal{D}^0_{\mathcal{R}}$ the spectrum is discrete and non-negative with [13, Theorem 4, Chapter 10.2]. Since $K_w\backslash V_m$ for $w\in A^m$ and $e^w_{c,l}\backslash V_m$ for $w\in A^k$ with $k<m$ are also open the rest of the statement follows analogously. △

Again, due to the fact that $\mathcal{D}^0_{\mathcal{R},K_m} \subset \mathcal{D}^0_{\mathcal{R}}$ and [13, Theorem 4, Chapter 10.2] we get an estimate on the eigenvalue counting function by

$$N(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R},K_m}, x) \leq N_D^{\mu,\mathcal{R}}(x).$$

Because of the finite ramification and the fact that functions in $\mathcal{D}^0_{\mathcal{R},K_m}$ have to be zero on V_m, this domain splits into the domain restricted to the different parts, meaning

$$\mathcal{D}^0_{\mathcal{R},K_m} = \left(\bigoplus_{w \in \mathcal{A}^m} \mathcal{D}^0_{\mathcal{R},K_w} \right) \bigoplus \left(\bigoplus_{\substack{c \in \mathcal{C}, l \in \{1,\dots,\rho(c)\} \\ w \in \mathcal{A}^n, n<m}} \mathcal{D}^0_{\mathcal{R},e^w_{c,l}} \right).$$

For the eigenvalue counting function this leads to

$$\sum_{w \in \mathcal{A}^m} N(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R},K_w}, x) + \sum_{\substack{c \in \mathcal{C}, l \in \{1,\dots,\rho(c)\} \\ w \in \mathcal{A}^n, n<m}} N(\mathcal{E}_{\mathcal{R}}, D^0_{\mathcal{R},e^w_{c,l}}, x) \leq N_D^{\mu,\mathcal{R}}(x)$$

for all $x \geq 0$ by orthogonality. Again, due to the decoupling, the individual eigenvalue counting functions can be calculated separately.

L.1: Fractal part $(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R},K_w})$

This time we would like an upper estimate of the first eigenvalue σ_1^w of $(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R},K_w})$, which is positive due to the Dirichlet boundary conditions. This gives us a lower estimate for $N(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R},K_w}, x)$. The first eigenvalue can be calculated via the following Rayleigh quotient; see [13, Theorem 1, Chapter 10.2]. It is

$$\sigma_1^w = \inf_{\substack{u \in \mathcal{D}^0_{\mathcal{R},K_w} \\ u \not\equiv 0}} \frac{\mathcal{E}_{\mathcal{R}}(u)}{||u||_\mu^2},$$

where $||u||_\mu$ denotes the L^2-norm with respect to μ. This leads to

$$\sigma_1^w \leq \frac{\mathcal{E}_{\mathcal{R}}(u)}{||u||_\mu^2}$$

for each $0 \not\equiv u \in \mathcal{D}^0_{\mathcal{R},K_w}$. We would like to find a $u \in \mathcal{D}^0_{\mathcal{R},K_w}$ such that this estimate is good enough.

Let us consider the fixed m-cell K_w. We have Dirichlet boundary conditions on V_m. There are $\#V_0$ many points of V_m in K_w. Take the smallest $j \in \mathbb{N}$ such that $N^j > \#V_0$. There are N^j many $(m+j)$-cells inside K_w which means that there is at least one that does not include any points of V_m. Therefore, there are no Dirichlet boundary conditions anywhere in this $(m+j)$-cell $K_{\hat{w}}$ with $|\hat{w}| = m+j$.

We, however, have to look for an even smaller cell. We do the same procedure again and look for a cell that has no common points of V_{m+j} with $K_{\hat{w}}$. With the same arguments there is an $(m+2j)$-cell $K_{\tilde{w}}$ with $|\tilde{w}| = m+2j$ inside $K_{\hat{w}}$ that fulfills this requirement.

We now construct a function on K_w that is in $\mathcal{D}^0_{\mathcal{R},K_w}$ with the help of $K_{\tilde{w}}$. The construction is very similar to the one in the proof of Lemma 3.35 where we calculated the Hausdorff dimension of K with respect to the resistance metric. Define $\tilde{u}_m$ on $G_{\tilde{w}}(V_0)$ to be constant 1. Now search for all $(m+2j)$-cells that are connected to $K_{\tilde{w}}$ over some $c \in \mathcal{C}_*$. There are at most $M = \#\mathcal{C}\#V_0$ many of those. Set $\tilde{u}_m = 1$ on all $c \in C_*$ that are connected to $G_{\tilde{w}}(V_0)$ in E_{m+2j} and also 1 on all other points that are connected to these c. By the way we chose $K_{\tilde{w}}$ and $K_{\hat{w}}$ we made sure that all points where we set $\tilde{u}_m$ to be 1 are not in V_m. On all other points of V_{m+2j} we choose $\tilde{u}_m$ to be 0. Then extend $\tilde{u}_m$ harmonically to a function u_m in $\mathcal{D}^0_{\mathcal{R},K_w}$. Note that the Dirichlet conditions on V_m are satisfied. Again, similar to Lemma 3.35 we can calculate the energy of u_m by

$$\begin{aligned}\mathcal{E}_{\mathcal{R}}(u_m) = \hat{\mathcal{E}}_{\mathcal{R},m+2j}(\tilde{u}_m) &\leq M \cdot \#E_0 \cdot \frac{1}{\delta_{m+2j} \min_{e\in E_0} r_0(e)} \\ &\leq \frac{M\#E_0}{\kappa_1 \min_{e\in E_0} r_0(e)} \cdot \lambda^{-(m+2j)},\end{aligned}$$

where we applied Lemma 3.32. We also need a lower estimate for the L^2-norm of u_m to get an upper estimate on σ_1^w. But we know that u_m is constant 1 on $K_{\tilde{w}}$, meaning

$$\begin{aligned}||u_m||_\mu^2 &= \int_{K_w} |u_m|^2 d\mu \\ &\geq \int_{K_{\tilde{w}}} \underbrace{|u_m|^2}_{=1} d\mu \\ &= \mu(K_{\tilde{w}}).\end{aligned}$$

Bringing those two estimates together leads to

$$\sigma_1^w \leq \frac{M\#E_0}{\kappa_1 \min_{e\in E_0} r_0(e)} \frac{\lambda^{-(m+2j)}}{\mu(K_{\tilde{w}})}.$$

With the estimates for the measures from Proposition 3.28 we obtain

$$\begin{aligned}\sigma_1^w &\leq \frac{M\#E_0}{\kappa_1 \min_{e\in E_0} r_0(e)(1-\eta)} (N\lambda^{-1})^{m+2j} \\ &= \underbrace{\frac{M\#E_0(N\lambda^{-1})^{2j}}{\kappa_1 \min_{e\in E_0} r_0(e)(1-\eta)}}_{C_l :=} \cdot \left(\frac{N}{\lambda}\right)^m.\end{aligned}$$

Note that j is independent of m. For $x \geq C_l(N/\lambda)$ choose $m \in \mathbb{N}$ such that

$$C_l \left(\frac{N}{\lambda}\right)^m \leq x < C_l \left(\frac{N}{\lambda}\right)^{m+1}.$$

For these x it holds that at least one eigenvalue of $(\mathcal{E}_\mathcal{R}, \mathcal{D}^0_{\mathcal{R},K_w})$ is smaller than x, i.e.,

$$N(\mathcal{E}_\mathcal{R}, \mathcal{D}^0_{\mathcal{R},K_w}, x) \geq 1.$$

Summing over all m-cells leads to

$$\begin{aligned}\sum_{w\in\mathcal{A}^m} N(\mathcal{E}_\mathcal{R}, \mathcal{D}^0_{\mathcal{R},K_w}, x) \geq N^m &= \frac{1}{N}((N\lambda^{-1})^{m+1})^{\frac{\ln N}{\ln(N/\lambda)}} \\ &\geq \underbrace{\frac{1}{N} C_l^{-\frac{\ln N}{\ln(N/\lambda)}}}_{C_1' :=} x^{\frac{\ln N}{\ln(N/\lambda)}}.\end{aligned}$$

L.2: Line part $(\mathcal{E}_\mathcal{R}, \mathcal{D}^0_{\mathcal{R},e^w_{c,l}})$

In the previous calculations we saw that the fractal part already gives a lower bound with the same order as the upper bound for $\lambda > \frac{1}{N}$. Therefore, the influence of the line part cannot be bigger than the fractal part. We can use the trivial estimate

$$\sum_{\substack{c\in\mathcal{C}, l\in\{1,\dots,\rho(c)\} \\ w\in\mathcal{A}^n, n<m}} N(\mathcal{E}_\mathcal{R}, \mathcal{D}^0_{\mathcal{R},e^w_{c,l}}, x) \geq 0.$$

If, however, $\lambda \leq \frac{1}{N}$ this order of $\frac{\ln(N)}{\ln(N/\lambda)}$ is at most $\frac{1}{2}$, so it is not the one we would like. To achieve the right one, we can use just one of the one-dimensional lines, say $e_{c,l}$. We obtain

$$\begin{aligned}\sum_{\substack{c\in\mathcal{C}, l\in\{1,\dots,\rho(c)\} \\ w\in\mathcal{A}^n, n<m}} N(\mathcal{E}_\mathcal{R}, \mathcal{D}^0_{\mathcal{R},e^w_{c,l}}, x) &\geq N(\mathcal{E}_\mathcal{R}, \mathcal{D}^0_{\mathcal{R},e_{c,l}}, x) \\ &= N(-\Delta_D|_{(0,1)}, \eta a_{c,l}\rho^1_{c,l} x) \\ &\geq \underbrace{\frac{c_0}{\pi}\sqrt{\eta a_{c,l}\rho^1_{c,l}}}_{C_1'' :=} \cdot x^{\frac{1}{2}}.\end{aligned}$$

with a constant $c_0 > 0$ for x big enough. This suffices to show the desired result when our measure includes the fractal part.

Remaining: $\mu = \mu_1 = \mu_I$

We still need to show the case if $\mu = \mu_1 = \mu_I$. Here we know that

$$\mu_I(K_w) = \beta^{|w|}.$$

Whenever we used $(1-\eta)\left(\frac{1}{N}\right)^{|w|} \leq \mu_\eta(K_w) \leq \left(\frac{1}{N}\right)^{|w|}$ in the proof for μ_η with $\eta \in (0,1)$ we can exchange this estimate with

$$\mu_I(K_w) = \beta^{|w|}.$$

For $\beta \neq \frac{1}{N^2\lambda}$ the rest of the proof works exactly the same as in the case of $\eta \in (0,1)$ and this leads to the asymptotic growing

$$\max\left\{\frac{1}{2}, \frac{\ln N}{-\ln(\beta\lambda)}\right\}.$$

However, if $\beta = \frac{1}{N^2\lambda}$, i.e., $N^2\beta\lambda = 1$, we cannot change β to $\tilde{\beta} = \beta + \epsilon$ as in the case of $\eta \in (0,1)$ since we need the exact value β for the following calculation. This leads to an additional $log(x)$ term in the upper bound. We will not include this result in the theorem since it does not fit to the other cases. $\square$

3.8 Refinements

In Section 2.7 we achieved some refinements of the asymptotics for the stretched Sierpiński gasket. We saw that strict periodicity as in the self-similar case is very unlikely in this setting. We still achieved the existence of oscillations in the leading term. With some additional symmetry we were able to show periodicity for some special cases and also got some remainder estimates. We would like to generalize these results to the general setting of stretched fractals.

3.8.1 Periodicity

In Section 2.7.3 we obtained the following estimates on the eigenvalue counting function for sequences $\mathcal{R}$ that fulfill Condition 2.1 with constants $\kappa_3, \kappa_4 > 0$ such that $\kappa_3 \leq \frac{\rho_{n+k}}{\rho_k} \leq \kappa_4$ for all $n, k \geq 1$ and measures μ_η with $\beta < \frac{1}{3}$ and $\eta \in (0,1)$. In this case

$$\begin{aligned}\sum_{w\in\mathcal{A}^n} N_D^{\mu_\eta,\mathcal{R}}(c_1(\beta r)^n x) &\leq N_D^{\mu_\eta,\mathcal{R}}(x) \leq N_N^{\mu_\eta,\mathcal{R}}(x)\\ &\leq \sum_{w\in\mathcal{A}^n} N_D^{\mu_\eta,\mathcal{R}}(c_2(\tfrac{r}{3})^n x) + 3^{n+1} + N(\mathcal{E}_{\mathcal{R},J_n}, \mathcal{D}_{\mathcal{R},J_n}, \mu_\eta, x).\end{aligned}$$

We can also only use the line part $\mu_1 = \mu_I$ to get the same scalings $(\beta r)^n$ on both sides, meaning

$$\begin{aligned}\sum_{w\in\mathcal{A}^n} N_D^{\mu_I,\mathcal{R}}(c_1(\beta r)^n x) &\leq N_D^{\mu_I,\mathcal{R}}(x) \leq N_N^{\mu_I,\mathcal{R}}(x)\\ &\leq \sum_{w\in\mathcal{A}^n} N_D^{\mu_I,\mathcal{R}}(c_2(\beta r)^n x) + 3^{n+1} + N(\mathcal{E}_{\mathcal{R},J_n}, \mathcal{D}_{\mathcal{R},J_n}, \mu_I, x).\end{aligned}$$

To be able to achieve these inequalities we had to compare domains $\mathcal{D}_\mathcal{R}$ and $\mathcal{D}_{\mathcal{R}^{(n)}}$, quadratic forms $\mathcal{E}_\mathcal{R}$ and $\mathcal{E}_{\mathcal{R}^{(n)}}$ and measures μ_η and μ_η^w. This was only possible since we tightened the condition from Condition 2.1 to those sequences $\mathcal{R}$ that additionally have constants $\kappa_3, \kappa_4 > 0$ such that $\kappa_3 \leq \frac{\rho_{n+k}}{\rho_k} \leq \kappa_4$ for all $n, k \geq 1$.

In the general setting we have the same problems. If we introduce a stronger condition on the regular sequences of harmonic structures in analogy to the refined condition from Section 2.7.3, we can get estimates of the following kind with other constants c_1, c_2. For $\eta \in (0,1)$ we have

$$\begin{aligned}\sum_{w\in\mathcal{A}^n} N_D^{\mu_\eta,\mathcal{R}}(c_1(\beta\lambda)^n x) \le N_D^{\mu_\eta,\mathcal{R}}(x) \le N_N^{\mu_\eta,\mathcal{R}}(x)& \\ \le \sum_{w\in\mathcal{A}^n} N_D^{\mu_\eta,\mathcal{R}}(c_2(\tfrac{\lambda}{N})^n x) + N^{n+1} + N(\mathcal{E}_{\mathcal{R},J_n}, \mathcal{D}_{\mathcal{R},J_n}, \mu_\eta, x)&\end{aligned}$$

and for $\eta = 1$, i.e., $\mu = \mu_I$

$$\begin{aligned}\sum_{w\in\mathcal{A}^n} N_D^{\mu_I,\mathcal{R}}(c_1(\beta\lambda)^n x) \le N_D^{\mu_I,\mathcal{R}}(x) \le N_N^{\mu_I,\mathcal{R}}(x)& \\ \le \sum_{w\in\mathcal{A}^n} N_D^{\mu_I,\mathcal{R}}(c_2(\beta\lambda)^n x) + N^{n+1} + N(\mathcal{E}_{\mathcal{R},J_n}, \mathcal{D}_{\mathcal{R},J_n}, \mu_I, x).&\end{aligned}$$

The differences to the self-similar case are for one the additional term $N(\mathcal{E}_{\mathcal{R},J_n}, \mathcal{D}_{\mathcal{R},J_n}, \mu_I, x)$ on the right-hand side. However, this is of lower order than the leading term and therefore, we can manage this term. The main problem, however, are the constants c_1 and c_2. These are due to the estimates between $\mathcal{E}_\mathcal{R}$ and $\mathcal{E}_{\mathcal{R}^{(n)}}$. As we saw in Section 2.7.3 we are not able to apply renewal theory in this case and furthermore, this shows that we lack some kind of symmetry to have periodicity. Therefore, it is very unlikely to get this very repetitive behavior of the eigenvalues that we know from the self-similar case.

We can get rid of the constants in the case of constant sequences of resistances and we will handle this in Section 3.8.3.

3.8.2 Non-convergence

In Section 2.7.4 we showed for the stretched Sierpiński gasket that even though we did not find a periodic function we still have these oscillations. To show this we used a series of localized eigenfunctions whose eigenvalues have very big multiplicities. The existence of localized eigenfunctions is closely related to the existence of Dirichlet-Neumann eigenfunctions. We were able to show the existence of these eigenfunctions due to the symmetries of the stretched Sierpiński gasket. The idea of the proof is from Barlow and Kigami [8] where it was applied to self-similar sets in a much more general setting. Actually, the self-similarity is not that crucial but rather the symmetries that are satisfied by the set, the energy and the measure. Therefore, we can use the same ideas to get Dirichlet-Neumann eigenfunctions provided we have enough symmetry. In this section we discuss the symmetries that we need to apply the ideas of [8]. In this work Barlow and Kigami introduced the so-called *p.c.f. morphisms.* This notion still makes sense on stretched fractals. For a bijection $g : K \to K$, and $f : K \to \mathbb{R}$ define $T_g f : K \to \mathbb{R}$ by $T_g f(x) = f(g^{-1}(x))$.

Definition 3.42 ([8, Definition 5.1])**.** A function $g : K \to K$ is a p.c.f. morphism for a Dirichlet form $(\mathcal{E}, \mathcal{D})$ and a measure μ if

1. g is bijective,
2. g is a homeomorphism of K,
3. $g : V_0 \to V_0$,
4. $\mu \circ g^{-1} = \mu$,
5. if $\phi \in \mathcal{D}$ then $T_g\phi, T_{g^{-1}}\phi \in \mathcal{D}$ and $\mathcal{E}(\phi, \psi) = \mathcal{E}(T_g\phi, T_g\psi)$ for all $\psi \in \mathcal{D}$.

We denote the group of p.c.f. morphisms for $(\mathcal{E}, \mathcal{D})$ and μ on K by $\mathcal{G}$.

Note that this means the symmetries have to be fulfilled by the set itself, the energy and the measure. For the energy this means that also the resistances of the one-dimensional lines have to fulfill this symmetry, namely $\boldsymbol{\rho}^i$. For the stretched Sierpiński gasket in Chapter 2 this was true since the resistances are the same for all connecting edges of the same level. On the stretched level-3 Sierpiński gasket we could use the same resistances on the one-dimensional lines that belong to a critical point with multiplicity 2 and another resistance for those that are connected to the critical point in the middle with multiplicity 3. The same holds for the measure. We have to choose the measure $a_{c,l}$ on the one-dimensional lines in such a way that they fulfill the symmetry. In this case we would have the same symmetry on the stretched level-3 Sierpiński gasket as on the stretched Sierpiński gasket.

We now give a condition on the symmetry group $\mathcal{G}$ of K which allows us to apply the ideas of [8] to get Dirichlet-Neumann eigenfunctions. We call a finite subgroup of p.c.f. morphisms *vertex transitive* on V_0 if its group action on V_0 is transitive.

Theorem 3.43 ([8, Theorem 5.4])**.** *Suppose that G is a finite subgroup of $\mathcal{G}$ which is vertex transitive on V_0, and that there exists $h \in \mathcal{G}$ with $h \notin G$, such that*

$$\bigcup_{g \in G} \{x \in K \mid h^{-1}(g(x)) = x\} \neq K.$$

Then there exists a Dirichlet-Neumann eigenfunction.
I.e., $\exists u \in \mathcal{D}^0_{\mathcal{R}}$ with $\mathcal{E}_{\mathcal{R}}(u, v) = \sigma \mathcal{E}_{\mathcal{R}}(u, v)$ for all $v \in \mathcal{D}_{\mathcal{R}}$.

If we have such Dirichlet-Neumann eigenfunctions we can construct localized eigenfunctions by using them. Recall the measures $\mu^w_\eta := \mu_\eta(K_w)^{-1}\mu_\eta \circ G_w$ for $w \in \mathcal{A}^*_0$. Due to symmetry this only depends on $|w| = n$ and therefore, we can denote it by $\mu^{(n)}_\eta$.

Lemma 3.44. *Let $u^{(n)}$ be a Dirichlet-Neumann eigenfunction of $(\mathcal{E}_{\mathcal{R}^{(n)}}, \mathcal{D}_{\mathcal{R}^{(n)}})$ and $\mu^{(n)}_\eta$ with eigenvalue σ_n. Then*

$$u^{(n)}_w(x) := \begin{cases} u^{(n)} \circ G^{-1}_w(x), & x \in K_w, \\ 0, & \textit{otherwise}, \end{cases}$$

is itself a Dirichlet-Neumann eigenfunction of $(\mathcal{E}_{\mathcal{R}}, \mathcal{D}_{\mathcal{R}})$ and $(\mathcal{E}_{\mathcal{R}}, \mathcal{D}^0_{\mathcal{R}})$ with measure μ_η and eigenvalue $\frac{\sigma_n}{\mu_\eta(K_w)\delta_n}$.

Proof. First we notice that $u_w^{(n)} \in \mathcal{D}^0_{\mathcal{R}}$. Since $u^{(n)} \in \mathcal{D}^0_{\mathcal{R}^{(n)}}$ we have $u_w^{(n)}|_{V_0} \equiv 0$ and $u_w^{(n)} \in C(K)$ and for the finiteness of the quadratic form $\mathcal{E}_{\mathcal{R}}$

$$\begin{aligned}\mathcal{E}_{\mathcal{R}}(u_w^{(n)}) &= \frac{1}{\delta_n}\mathcal{E}_{\mathcal{R}^{(n)}}(u^{(n)} \circ G_w^{-1} \circ G_w) \\ &= \frac{1}{\delta_n}\mathcal{E}_{\mathcal{R}^{(n)}}(u^{(n)}) \\ &< \infty.\end{aligned}$$

Now for all $v \in \mathcal{D}_{\mathcal{R}}$ we obtain

$$\begin{aligned}\mathcal{E}_{\mathcal{R}}(u_w^{(n)}, v) &= \frac{1}{\delta_n}\mathcal{E}_{\mathcal{R}^{(n)}}(u^{(n)}, \underbrace{v \circ G_w}_{\in \mathcal{D}_{\mathcal{R}^{(n)}}}) \\ &= \frac{1}{\delta_n}\sigma_n(u^{(n)}, v \circ G_w)_{\mu_\eta^w} \\ &= \frac{\sigma_n}{\delta_n}\int_K u^{(n)} \cdot v \circ G_w \, d\mu_\eta^w \\ &= \frac{\sigma_n}{\delta_n}\frac{1}{\mu_\eta(K_w)}\int_{K_w} u^{(n)} \circ G_w^{-1} \cdot v \, d\mu_\eta \\ &= \frac{\sigma_n}{\delta_n \mu_\eta(K_w)}(u_w^{(n)}, v)_{\mu_\eta}.\end{aligned}$$

□

The new eigenfunction $u_w^{(n)}$ is obviously localized. $\mu_\eta(K_w)$ only depends on $|w| = n$, and therefore, the multiplicity of the eigenvalue $\frac{\sigma_n}{\mu_\eta(K_w)\delta_n}$ is at least N^n, since this is the number of n-cells.

Now if $\lambda \geq \frac{1}{N}$ and we assume that our stretched fractal together with $(\mathcal{E}_{\mathcal{R}^{(n)}}, \mathcal{D}_{\mathcal{R}^{(n)}})$ and $\mu_\eta^{(n)}$ have enough symmetry to fulfill the conditions of Theorem 3.43 for all n, we get a sequence of eigenvalues ν_n with constants $0 < c_1 \leq c_2 < \infty$ and

$$\left.\begin{matrix} c_1\sigma_n\left(\frac{N}{\lambda}\right)^n \\ c_1\sigma_n\left(\beta\lambda\right)^{-n}\end{matrix}\right\} \leq \nu_n \leq \begin{cases} c_2\sigma_n\left(\frac{N}{\lambda}\right)^n, & \mu = \mu_\eta \text{ with } \eta \in (0,1), \\ c_2\sigma_n\left(\beta\lambda\right)^{-n}, & \mu = \mu_I,\end{cases}$$

and multiplicities at least N^n. However, as in Section 2.7.4 we have this dependency on σ_n. We need to make sure to get a similar decomposition of K as for the stretched Sierpiński gasket. If we have enough symmetry to get such a decomposition and thus estimates for the σ_n uniformly in n we can use the same calculation as in Section 2.7.4 to show that we have non-convergence by the use of this sequence of eigenvalues with very high multiplicities.

3.8.3 Special cases

As we mentioned in Section 3.8.1 we have different domains $\mathcal{D}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}^{(n)}}$, different quadratic forms $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}^{(n)}}$ and different measures μ_η and μ_η^w. We now look at some special cases where these are actually already all the same.

We know that μ_I by itself is self-similar in the sense that

$$\begin{aligned}\mu_I^w &= \mu_I(K_w)^{-1}\mu_I \circ G_w\\ &= \mu_I,\end{aligned}$$

and therefore, we choose $\mu = \mu_I$ as our measure.

Next we only consider constant sequences of harmonic structures $\mathcal{R} = \{(r_0, \lambda_i, \boldsymbol{\rho}^i)\}_{i\geq 1}$ with $\frac{1}{N} < \lambda_i = \lambda < 1$ and $\boldsymbol{\rho}^i = \boldsymbol{\rho} = \{\rho_e\}_{e\in E_1^I}$ for all $i \geq 1$. This is obviously a regular sequence of harmonic structures that fulfills Condition 3.1. In this case we have

$$\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}^{(n)}},\ \forall n$$

which immediately implies that the domains $\mathcal{D}_{\mathcal{R}} = \mathcal{D}_{\mathcal{R}^{(n)}}$ coincide for all n. We can now apply the ideas of Section 2.7.5 to this more general setting on stretched fractals. We define

$$\begin{aligned}R(x) &:= N_D^{\mu_I,\mathcal{R}}(x) - N \cdot N_D^{\mu_I,\mathcal{R}}(\beta\lambda x),\\ f(t) &:= e^{-td_S^{\mu_I,\mathcal{R}}} N_D^{\mu_I,\mathcal{R}}(e^{2t}),\\ u(t) &:= e^{-td_S^{\mu_I,\mathcal{R}}} R(e^{2t}),\\ T &:= -\ln\sqrt{\beta\lambda}.\end{aligned}$$

We would like to apply Theorem 2.30 [42, Theorem A.1]. We see that f is real-valued and $f(t) \to 0$ as $t \to -\infty$. With $m_j = 1$ and $p_j = \frac{1}{N}$ for all j we have

$$\begin{aligned}\sum_{j=1}^{N} f(t + \ln\sqrt{\beta\lambda})\frac{1}{N} &= f(t + \ln\sqrt{\beta\lambda})\\ &= e^{-(t+\ln\sqrt{\beta\lambda})d_S^{\mu_I,\mathcal{R}}} N_D^{\mu_I,\mathcal{R}}(e^{2(t+\ln\sqrt{\beta\lambda})})\\ &= e^{-td_S^{\mu_I,\mathcal{R}}} (\beta\lambda)^{-\frac{d_S^{\mu_I,\mathcal{R}}}{2}} N_D^{\mu_I,\mathcal{R}}(e^{2t}\beta\lambda)\\ &= e^{-td_S^{\mu_I,\mathcal{R}}} N \cdot N_D^{\mu_I,\mathcal{R}}(e^{2t}\beta\lambda)\\ &= e^{-td_S^{\mu_I,\mathcal{R}}} (N_D^{\mu_I,\mathcal{R}}(e^{2t}) - R(e^{2t}))\\ &= f(t) - u(t),\end{aligned}$$

which means they satisfy the renewal equation. It remains to show, that u is bounded and $\sum_{j=-\infty}^{\infty} |u(t + jT)|$ converges uniformly for all $t \in [0, T]$ but this works analogously as for the stretched Sierpiński gasket in Section 2.7.5. Therefore, Theorem 2.30 is applicable and we get a periodic function G with period $T = -\ln\sqrt{\beta\lambda}$ such that

$$N_D^{\mu_I,\mathcal{R}}(x) = G\left(\frac{\ln x}{2}\right) x^{\frac{1}{2}d_S^{\mu_I,\mathcal{R}}} + \mathcal{O}(x^{\frac{1}{2}}).$$

Furthermore, if the geometry of the stretched fractal, the energy $\mathcal{E}_{\mathcal{R}}$ (i.e., the distribution of the resistances $\boldsymbol{\rho}$ on the line part edges E_1^I) and the measure μ_I (i.e., the $a_{c,l}$) fulfill enough symmetry and allow the existence of a Dirichlet-Neumann eigenfunction as well as estimates on the Dirichlet-Neumann eigenvalues we know from Section 3.8.2 how to prove non-convergence.

4 Outlook and further research

Existence of regular harmonic structures

The idea of this work was very similar to [38]. Namely, if we have a regular harmonic structure we can choose a sequence and thus get resistance forms. After choosing a measure we get Dirichlet forms and thus operators. We showed the existence of regular harmonic structures for a few examples by explicitly calculating them. The question remains, in which cases such a regular harmonic structure exists. One possible approach is to show that if we have a harmonic structure in the self-similar case, this also induces one in the stretched case. In all our examples this was the case, since we always used the same resistances on (V_0, E_0) as in the self-similar case. This means, the choice of r_0 was influenced by the existence of a regular harmonic structure on the self-similar set.

If we have no way to compare it to the self-similar case we would still like to prove existence for as many sets as possible. The first set of fractals for which we would like to try this would be stretched nested fractals. As in [47] this could mean getting the existence without knowing the value of λ.

Comparison of d_S in the self-similar and the stretched case

We saw in the examples that the values for Hausdorff dimension and leading order for the asymptotics in the stretched case are less than or equal to those in the self-similar case. We believe this is always true.

We can give heuristic arguments for this conjecture. If we set the resistances $\rho = 0$ on all connecting edges in E_1^I this would mean that points that were connected by this edge get identified with each other. This gives us back the first graph approximation in the self-similar case. (V_0, E_0) is the same for both self-similar and stretched cases. By increasing $\rho > 0$ on the edges in E_1^I we still would like to have an equivalent network for (V_0, E_0). This means that the effective resistance between those points in V_0 has to stay the same. However, we know from general electrical theory that if we increase the resistances on the connecting edges, the resistances on the fractal edges in E_1^Σ have to decrease in order to keep the effective resistances at the same level.

For Hata's tree we saw that the same values as for the self-similar case was not possible. This gives rise to the question in which cases this is possible and to find criteria to characterize stretched fractals.

More general harmonic structures and measures

The harmonic structures that we used are very symmetric. We have the same renormalization in each cell. This is a big restriction and there will likely be stretched fractals for which there is no regular harmonic structure that fulfills this symmetry. This means we need to generalize our notion of harmonic structure to allow different scaling in different cells.

We, however, believe that there is no new difficulty in obtaining Hausdorff dimension and the leading order in the asymptotics. We need to introduce a few more indices and to make sure the scalings for different cells converge on their own to a limit. With such conditions we should be able to prove the results with a combination of the proof in this work and the ideas from [38] or [35] concerning partitions of the shift space.

The same holds for the measures that we used. These were very symmetric and we should replace them by more general ones. We would like to allow different scaling in different cells for both fractal- and line-part of the measure. But again, there should be no new difficulties in obtaining Hausdorff dimension and leading order of the spectral asymptotics by connecting the ideas of this work and [38, 35].

Does the fractal part of the resistance form really exist

Besides the construction of resistance forms on the stretched Sierpiński gasket, the main result of [5] examined the resistance forms $\mathcal{E}_{\mathcal{R}}$. In particular the authors studied the fractal part $\mathcal{E}_{\mathcal{R}}^{\Sigma}$ and showed that it only survives in a special case. In our notion from Chapter 3 this is the case $\sum_{i\geq 1} |\lambda_i - \frac{3}{5}| < \infty$. In all other cases we have $f \in \mathcal{F}_{\mathcal{R}} \Rightarrow \mathcal{E}_{\mathcal{R}}^{\Sigma}(f) = 0$.

This question can be generalized to stretched fractals. When does the fractal part $\mathcal{E}_{\mathcal{R}}^{\Sigma}$ in the resistance form of a stretched fractal survive? The immediate conjecture is: If λ_{ss} is the renormalization in the self-similar case we conjecture that $\mathcal{E}_{\mathcal{R}}^{\Sigma}$ survives if and only if we have $\sum_{i\geq 1} |\lambda_i - \lambda_{ss}| < \infty$ for the sequence of regular harmonic structures.

However, it is not possible to apply the same proof as in [5] since it strongly depends on the value of $\lambda_{ss} = \frac{3}{5}$. These values are not known in general.

More stretching

We were able to stretch p.c.f. self-similar fractals that fulfilled a certain connectedness condition (C1),(C2) and (C3). We did this by introducing one-dimensional lines. We could fill the holes with other objects than just lines. For example for each $c \in \mathcal{C}$ we could fill the hole with a fractal that has $\rho(c)$ many boundary points.

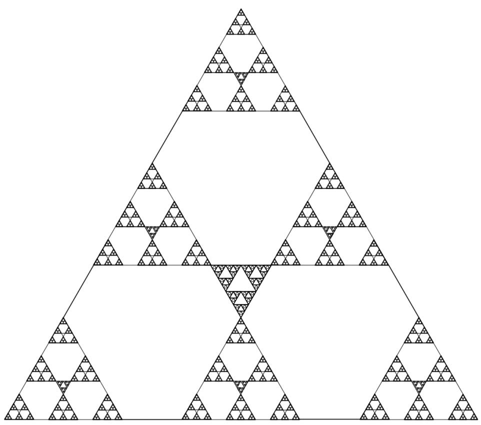

Figure 4.1: Filling the hole with the self-similar Sierpiński gasket.

We saw that the one-dimensionality of the lines influenced the dimension as well as the leading order of the spectral asymptotics. It would be interesting to see how other objects influence these values.

The set from Figure 4.1 was treated in [1]. The authors calculated the leading order of the spectral asymptotics; although the measure did not have a fractal part, we can still see that the properties of all three parts, namely line part, Sierpiński gasket and level-3 Sierpiński gasket, appear in the asymptotics.

We can also stretch sets that are not p.c.f., for example, the unit square $[0,1]^2$. It is the attractor of four similitudes

$$F_1\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0.5&0\\0&0.5\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix},$$

$$F_2\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0.5&0\\0&0.5\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}+\begin{pmatrix}0.5\\0\end{pmatrix},$$

$$F_3\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0.5&0\\0&0.5\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}+\begin{pmatrix}0\\0.5\end{pmatrix},$$

$$F_4\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0.5&0\\0&0.5\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}+\begin{pmatrix}0.5\\0.5\end{pmatrix}.$$

However, there is not an obvious way to connect the copies if we lower the contraction ratios. We could still be using one-dimensional lines. It is also not obvious how we should place these lines. This procedure, however, changes the connectedness of the fractal and gives us a completely new fractal which has to be analyzed geometrically and analytically. We have to place the lines in such a way that it connects the 1-cells Σ_i to ensure connectedness.

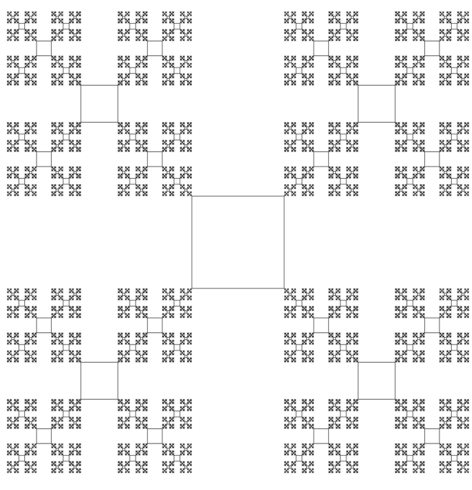

Figure 4.2: The stretched unit square - version 1.

We can also place the lines somewhere else.

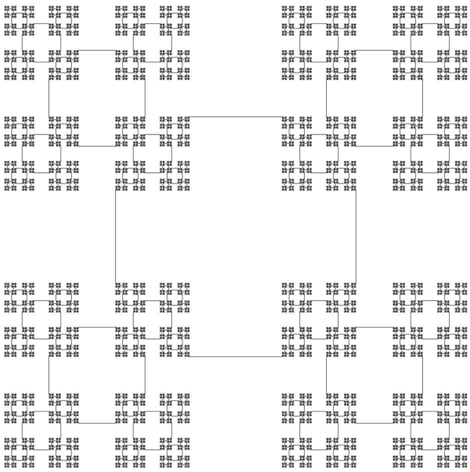

Figure 4.3: The stretched unit square - version 2.

Another way to connect the copies is to use two-dimensional regions.

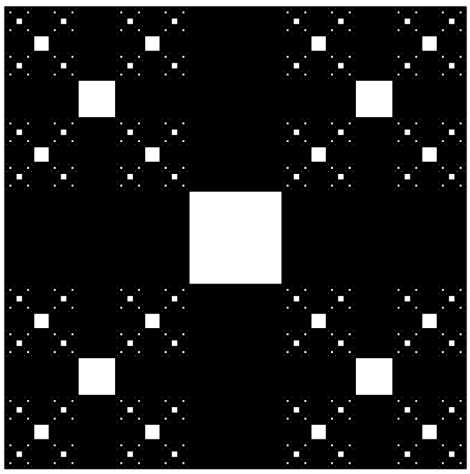

Figure 4.4: The stretched unit square - version 3.

This gives us a completely different fractal. The two-dimensional part will dominate the geometric and analytic appearance.

There are many ways to connect the copies between one- and two-dimensional objects. This gives rise to many new and interesting fractals.

Introduce randomness

The construction of stretched fractals was purely deterministic. We could also consider random fractals. There are various ways to define such random sets. For example, mixing two fractals randomly. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two Iterated Function Systems. Now if in each step we toss a coin to choose which IFS we apply to all of the copies we obtain the *homogeneous* *random* fractals; see [23]. If we allow in the construction to apply a different IFS to each copy in each construction step we have much more diversity, which leads to the *random recursive* fractals; see [24]. There is also a concept of random fractals which interpolates between these two constructions. These are the V*-variable* fractals; see [9, 10].

These ideas can be extended to stretched fractals.

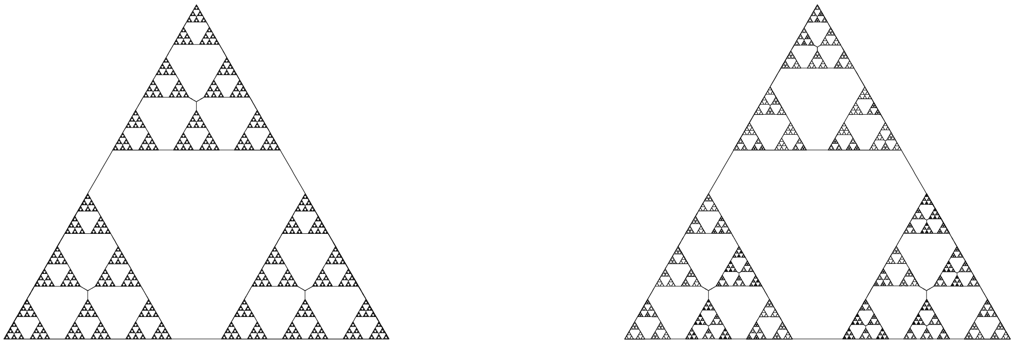

Figure 4.5: Homogeneous random and random recursive stretched fractals.

To calculate the asymptotic growing for Laplacians on such random sets we will have to combine the ideas from this thesis with the results for homogeneous random and random recursive fractals in [25, 26] and for V-variable fractals in [18].

Stochastic process

The existence of Dirichlet forms is closely connected to the existence of stochastic processes on the underlying space. Namely, given a regular Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(K, \mu)$, there exists a Hunt process $(X_t)_{t\geq 0}$ on K, i.e. a strong Markov process which is quasi-left-continuous with respect to the minimum admissible filtration $(F_t)_{t\geq 0}$; see [20, Theorem 7.2.1]. In particular, if $(\mathcal{E}, \mathcal{D})$ is a local regular Dirichlet form then this process $(X_t)_{t\geq 0}$ turns out to be a diffusion, i.e. a Hunt process with almost surely continuous sample paths; see [20, Theorem 7.2.2].

This means we know that such stochastic processes exist on stretched fractals. However, the proof of existence is non-constructive, which means we do not know how these processes behave on stretched fractals.

For the self-similar Sierpiński gasket the diffusion can be obtained as a limit of renormalized discrete random walks; see [22, 46, 6]. Due to the lack of self-similarity this approach does not work for the stretched case.

Another way to study a stochastic process is to analyze its heat kernel $p_t(x, y)$. By the study of this transition density we can obtain properties of the stochastic process, such as how fast it moves. The first step in this direction would be to obtain heat kernel estimates.

Appendices

A P.C.F. Self-Similar Sets

In this chapter we include a very brief introduction to the notion of p.c.f. self-similar sets, which is a very abstract description of finitely ramified self-similar sets. These sets were first introduced by Kigami in [37]. The introduction in this section is mainly from a later work [44, Section 9]. A very detailed overview can be found in [43, Chapters 1 and 3].

Definition A.1. Let K be a compact metrizable topological space and let $\mathcal{A}$ be a finite set. Also, let F_i, for $i \in \mathcal{A}$, be a continuous injection from K to itself. Then $(K, \mathcal{A}, \{F_i\}_{i\in\mathcal{A}})$ is called a self-similar structure if there exists a continuous surjection $\pi : \mathcal{A}^{\mathbb{N}} \to K$ such that $F_i \circ \pi = \pi \circ i$ for every $i \in \mathcal{A}$, where $\mathcal{A}^{\mathbb{N}}$ is the one-sided shift space and $\sigma_i : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ is defined by $\sigma_i(w_1w_2w_3\cdots) = iw_1w_2w_3\cdots$ for each $w_1w_2w_3\cdots \in \mathcal{A}^{\mathbb{N}}$.

Note that if $(K, \mathcal{A}, \{F_i\}_{i\in\mathcal{A}})$ is a self-similar structure, then K is self-similar in the following sense:

$$K = \bigcup_{i\in\mathcal{A}} F_i(K).$$

We introduce some notation. $\mathcal{A}^m$ is the collection of words of length m. For $w = w_1\cdots w_m \in \mathcal{A}^m$, we define $F_w : K \to K$ by $F_w = F_{w_1} \circ \cdots \circ F_{w_m}$ and $K_w = F_w(K)$. In particular $\mathcal{A}^0 = \{\emptyset\}$ and $F_\emptyset$ is the identity map. Also we define $\mathcal{A}^*_0 = \bigcup_{m\geq 0} \mathcal{A}^m$.

Definition A.2. Let $(K, \mathcal{A}, \{F_i\}_{i\in\mathcal{A}})$ be a self-similar structure. We define the critical set $\tilde{\mathcal{C}} \subset \mathcal{A}^{\mathbb{N}}$ and the post critical set $\tilde{\mathcal{P}} \subset \mathcal{A}^{\mathbb{N}}$ by

$$\tilde{\mathcal{C}} = \pi^{-1}\left(\bigcup_{i\neq j}(K_i \cap K_j)\right) \text{ and } \tilde{\mathcal{P}} = \bigcup_{n\geq 1} \sigma^n(\tilde{\mathcal{C}}),$$

where σ is the shift map from $\mathcal{A}^{\mathbb{N}}$ to itself defined by $\sigma(w_1w_2\cdots) = w_2w_3\cdots$. A self-similar structure is called post critically finite (p.c.f. for short) if and only if $\#(\tilde{\mathcal{P}})$ is finite, where $\#(X)$ denotes the number of elements of a set X.

Now, we fix a connected p.c.f. self-similar structure $(K, \mathcal{A}, \{F_i\}_{i\in\mathcal{A}})$ with $\#(\mathcal{A}) \geq 2$.

Definition A.3. Let $V_0 = \pi(\tilde{\mathcal{P}})$. For $m \geq 1$ set

$$V_m = \bigcup_{w\in\mathcal{A}^m} F_w(V_0) \text{ and } V_* = \bigcup_{m\geq 0} V_m.$$

It is easy to see that $\emptyset \neq V_m \subsetneq V_{m+1}$ and that K is the closure of V_*.

We can develop a theory of a sequence of discrete Laplacians on V_m which in the limit leads to a Laplacian on K. The details of this procedure can be found in the mentioned sources [37, 43, 44].

Definition A.4 (Open set condition)**.** A self-similar structure $(K, \mathcal{A}, \{F_i\}_{i\in\mathcal{A}})$ is said to satisfy the *open set condition* (*OSC* for short) if there exists an open set $O \subset K$ such that

(i) $F_i(O) \cap F_j(O) = \emptyset, \ \forall i \neq j$,

(ii) $F_i(O) \subset O, \ \forall i \in \mathcal{A}$.

This is a separation condition which is very useful in fractal geometry. It states that the overlaps are small in some sense. As it turns out, every p.c.f. self-similar set satisfies the open set condition, which was proved by Wen and Ni in [55].

Theorem A.5. *Let* $(K, \mathcal{A}, \{F_i\}_{i\in\mathcal{A}})$ *be a p.c.f. self-similar structure. Then* $(K, \mathcal{A}, \{F_i\}_{i\in\mathcal{A}})$ *satisfies the open set condition.*

Proof. See [55, Corollary 1]. □

B Resistance and Dirichlet forms

In this chapter we introduce the notion of resistance forms and present the most important facts that are used throughout this work. Later, we introduce the notion of Dirichlet forms and show the connection between the two. The theory of resistance forms was introduced by Kigami in [41]; see also his monograph [43]. Section B.3, however, comes from a more recent work [45]. Section B.2 is mainly from [20].

B.1 Resistance forms

To define a resistance form we only need a set, which is canonically equipped with a metric but not with any measure.

Definition B.1. Let X be a set. A pair $(\mathcal{E}, \mathcal{F})$ is called a resistance form on X if it satisfies the following conditions (RF1) through (RF5):

(RF1) $\mathcal{F}$ is a linear subspace of $\ell(X) = \{u \mid u : X \to \mathbb{R}\}$ containing constants and $\mathcal{E}$ is a non-negative definite symmetric quadratic form on $\mathcal{F}$. $\mathcal{E}(u) = 0$ if and only if u is constant on X.

(RF2) Let $\sim$ be the equivalence relation on $\mathcal{F}$ defined by declaring that $u \sim v$ if and only if $u - v$ is constant on X. Then $(\mathcal{F}/\sim, \mathcal{E})$ is a Hilbert space.

(RF3) If $x, y \in X$ and $x \neq y$, then there exists $u \in \mathcal{F}$ such that $u(x) \neq u(y)$.

(RF4) For any $p, q \in X$,

$$\sup \left\{ \frac{|u(p) - u(q)|^2}{\mathcal{E}(u)} \;\middle|\; u \in \mathcal{F}, \mathcal{E}(u) > 0 \right\}$$

is finite. The above supremum is denoted by $R_{(\mathcal{E},\mathcal{F})}(p, q)$.

(RF5) For any $u \in \mathcal{F}, \overline{u} \in \mathcal{F}$ and $\mathcal{E}(\overline{u}) \leq \mathcal{E}(u)$, where $\overline{u}$ is defined by

$$\overline{u}(p) = \begin{cases} 1 & \text{if } u(p) \geq 1, \\ u(p) & \text{if } 0 < u(p) < 1, \\ 0 & \text{if } u(p) \leq 0. \end{cases}$$

Proposition B.2. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X. The supremum in (RF4) is the maximum and $R_{(\mathcal{E},\mathcal{F})}$ is a metric on X.*

Proof. See [43, Theorem 2.3.4]. □

Definition B.3. Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X. $R_{(\mathcal{E},\mathcal{F})}$ is called the resistance metric on X associated with the resistance form $(\mathcal{E}, \mathcal{F})$.

Proposition B.4. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X and let R be the associated resistance metric. For any $x, y \in X$ and any $u \in \mathcal{F}$,*

$$|u(x) - u(y)|^2 \leq R(x, y)\mathcal{E}(u, u).$$

In particular, $u \in \mathcal{F}$ is continuous with respect to the resistance metric.

Proof. This is immediate from (RF4). □

Definition B.5. Let $\{V_m\}_{m \geq 0}$ be a sequence of finite sets and let $\mathcal{E}_m$ be a resistance form on V_m for $m \geq 0$. $\{(V_m, \mathcal{E}_m)\}_{m \geq 0}$ is called a compatible sequence if and only if $V_m \subset V_{m+1}$ and

$$\mathcal{E}_m(u, u) = \min\{\mathcal{E}_{m+1}(v, v) \mid v \in \ell(V_{m+1}), u = v|_{V_m}\}$$

for any $m \geq 0$.

Theorem B.6. *Let $\{V_m\}_{m \geq 0}$ be a sequence of finite sets and let $\mathcal{E}_m$ be a resistance form on V_m for $m \geq 0$. Assume that $\mathcal{S} = \{(V_m, \mathcal{E}_m)\}_{m \geq 0}$ is a compatible sequence. Let $V_* = \bigcup_{m \geq 0} V_m$. Define*

$$\mathcal{F}_{\mathcal{S}} = \{u \mid u \in \ell(V_*), \lim_{m \to \infty} \mathcal{E}_m(u|_{V_m}, u|_{V_m}) < \infty\}$$

and

$$\mathcal{E}_{\mathcal{S}}(u, v) = \lim_{m \to \infty} \mathcal{E}_m(u|_{V_m}, v|_{V_m})$$

for any $u, v \in \mathcal{F}_{\mathcal{S}}$. Then $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}})$ is a resistance form on V_* and

$$\mathcal{E}_m(u, u) = \min\{\mathcal{E}_{\mathcal{S}}(v, v) \mid v \in \mathcal{F}_{\mathcal{S}}, u = v|_{V_m}\}$$

for any $m \geq 0$ and $u \in \ell(V_m)$. Moreover, let $R_{\mathcal{S}}$ be the resistance metric associated with $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}})$ and let (X, R) be the completion of $(V_*, R_{\mathcal{S}})$. Then there exists a unique resistance form $(\mathcal{E}, \mathcal{F})$ on X such that, for any $u \in \mathcal{F}$, u is a continuous function on X, $u|_{V_*} \in \mathcal{F}_{\mathcal{S}}$ and $\mathcal{E}(u, u) = \mathcal{E}_{\mathcal{S}}(u|_{V_*}, u|_{V_*})$. In particular, R coincides with the resistance metric associated with $(\mathcal{E}, \mathcal{F})$.

Proof. See [43, Theorems 2.2.6 and 2.3.10]. □

B.2 Dirichlet forms

In this section we briefly introduce the notion of Dirichlet forms. See [20, Chapter 1] for basics on Dirichlet forms.

Let X be a locally compact separable metric space and μ a σ-finite Borel measure on X that satisfies $\mu(A) < \infty$ for any compact set A and $\mu(O) > 0$ for any non-empty open set O. We define $C_0(X)$ by

$$C_0(X) = \{f : X \to \mathbb{R} \mid f \text{ is continuous and } \operatorname{supp}(f) \text{ is compact}\},$$

where supp(f) is defined as the closure in X of $f^{-1}(\mathbb{R}\backslash\{0\})$ and called the support of f. Let $\mathcal{E}$ be a non-negative definite symmetric bilinear form defined on $\mathcal{D}$ which is a dense linear subspace of $L^2(X,\mu)$. Define a new non-negative definite symmetric bilinear form for all $\alpha > 0$ by

$$\mathcal{E}_\alpha(u,v) = \mathcal{E}(u,v) + \alpha \int_X uv d\mu, \; u,v \in \mathcal{D}.$$

Then $\mathcal{E}_\alpha$ induces a metric on $\mathcal{D}$. All these metrics $\mathcal{E}_\alpha^{\frac{1}{2}}$ and $\mathcal{E}_\beta^{\frac{1}{2}}$ are equivalent for $\alpha, \beta > 0$.

Definition B.7. A non-negative definite symmetric bilinear form $\mathcal{E}$ with dense domain $\mathcal{D}$ is called closed if $(\mathcal{D}, \mathcal{E}_1^{\frac{1}{2}})$ is complete.

Definition B.8. A non-negative definite symmetric bilinear form $\mathcal{E}$ with dense domain $\mathcal{D}$ is called Markovian if for all $u \in \mathcal{D}$ we have

$$\overline{u} \in \mathcal{D} \text{ and } \mathcal{E}(\overline{u}, \overline{u}) \leq \mathcal{E}(u,u).$$

Definition B.9. A closed Markovian form $(\mathcal{E}, \mathcal{D})$ is called a Dirichlet form on $L^2(X,\mu)$.

Definition B.10. A Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(X,\mu)$ is called regular if and only if $\mathcal{D} \cap C_0(X)$ is dense in $\mathcal{D}$ with respect to $\mathcal{E}_1^{\frac{1}{2}}$ and dense in $C_0(X)$ with respect to the supremum norm $||\cdot||_\infty$ defined by $||u||_\infty := \sup_{x\in X} |u(x)|$.

B.3 Resistance forms as Dirichlet forms

We now connect resistance forms with Dirichlet forms. Roughly speaking, if we have a resistance form and equip (X,R) with a measure we get a Dirichlet form. In the following paragraph we describe this fact a little more precisely.

Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X and R the associated resistance metric on X. We require our metric space (X,R) to be separable, complete and locally compact. Let μ be a Borel measure on X with $0 < \mu(B_R(x,r)) < \infty$ for any $x \in X$ and $r > 0$. Here $B_R(x,r)$ denotes the ball in resistance metric R with radius r centered at x. With these assumptions we know that $C_0(X)$ is dense in $L^2(X,\mu)$, where $C_0(X)$ denotes the space of continuous functions on X with compact support.

Lemma B.11. *$(\mathcal{F} \cap L^2(X,\mu), \mathcal{E}_1)$ is a Hilbert space.*

Proof. See [43, Theorem 2.4.1]. □

We denote by $\mathcal{D}$ the closure of $\mathcal{F} \cap C_0(X)$ in $(\mathcal{F} \cap L^2(X,\mu), \mathcal{E}_1)$.

Lemma B.12. *If (X,R) is compact, then $\mathcal{F} = \mathcal{D}$.*

Proof. In this case we know that $C(X) := \{f : X \to \mathbb{R} \mid f \text{ is continuous}\} = C_0(X)$ and thus with Proposition B.4 we have $\mathcal{F} \subset C(X) = C_0(X)$. Therefore, $\mathcal{F} \cap C_0(X) = \mathcal{F} = \mathcal{F} \cap L^2(X,\mu)$, which is closed with respect to $\mathcal{E}_1^{\frac{1}{2}}$. □

Definition B.13. A resistance from $(\mathcal{E}, \mathcal{F})$ on X is called regular if and only if $\mathcal{F} \cap C_0(X)$ is dense in $C_0(X)$ with respect to the supremum norm $|| \cdot ||_\infty$.

The following lemma provides a very useful criterion to check regularity.

Lemma B.14. *If (X, R) is compact, then $(\mathcal{E}, \mathcal{F})$ is regular.*

Proof. See [45, Corollary 6.4]. □

Theorem B.15. *Let $(\mathcal{E}, \mathcal{F})$ be a regular resistance form on X. Then $(\mathcal{E}, \mathcal{D})$ is a regular Dirichlet form on $L^2(X, \mu)$.*

Proof. See [45, Theorem 9.4]. □

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