# Spectral Asymptotics for Dirichlet Laplacians on Random Cantor-Like Sets and on their Complement

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# Abstract

We study the spectral asymptotics for Laplacians with Dirichlet boundary conditions on random Cantor-like sets and on their complement.

In the first part, we determine the leading order in the Weyl expansion of the eigenvalue counting function for measure theoretical Laplacians $\Delta^\mu = \frac{d}{d\mu}\frac{d}{dx}$ with respect to statistically self-similar and random $V$-variable Cantor measures $\mu$.

In the second part, we investigate the classical one dimensional Laplacian $\frac{d^2}{dx^2}$ on the complement of statistically self-similar Cantor sets, called statistically self-similar Cantor strings. We establish a Strong Law of Large Numbers for the error of the first term in the Weyl asymptotics of the eigenvalue counting function for the classical Laplacian $\frac{d^2}{dx^2}$ on these statistically self-similar Cantor strings. Afterwards, we discuss the random fluctuation of the normalized error of the first term in the Weyl asymptotics of the eigenvalue counting function around its limit by giving a Central Limit Theorem. Since the Central Limit Theorem only makes a statement about convergence in distribution, we also establish an almost sure error estimate of the random fluctuation using a Law of the Iterated Logarithm.

# Zusammenfassung

Wir untersuchen die Spektralasymptotik von Laplace-Operatoren mit Dirichlet Randbedingungen auf zufälligen Cantor-ähnlichen Mengen und auf deren Komplement.

Im ersten Teil der Arbeit bestimmen wir die führende Ordnung in der Weylasymptotik von maßtheoretischen Laplace-Operatoren $\Delta^\mu = \frac{d}{d\mu}\frac{d}{dx}$ bezüglich statistisch selbst-ähnlicher und zufälliger $V$-variabler Cantormaße $\mu$.

Im zweiten Teil betrachten wir den klassischen eindimensionalen Laplace-Operator $\frac{d^2}{dx^2}$ auf dem Komplement von statistisch selbst-ähnlichen Cantormengen. Zunächst etablieren wir ein Starkes Gesetz der Großen Zahlen für den Fehler des ersten Ordnungsterms in der Weylasymptotik der Eigenwertzählfunktion des klassischen Laplace-Operators $\frac{d^2}{dx^2}$ auf dem Komplement von statistisch selbst-ähnlichen Cantormengen. Darauffolgend beweisen wir einen Zentralen Grenzwertsatz für die zufällige Fluktuation des normalisierten Fehlers des ersten Ordnungsterms der Eigenwertzählfunktion um dessen Grenzwert. Da der Zentrale Grenzwertsatz nur eine Konvergenzaussage in Verteilung trifft, geben wir eine fast sichere Fehlerschranke der zufälligen Fluktuation unter Anwendung eines Gesetzes des Iterierten Logarithmus.

# Chapter 1

# Introduction

## 1.1. Weyl's law and Berry's conjecture

In the course of time, many seemingly separated areas arose in mathematics. Questioning connections between spectral and geometric properties of sets were emphasized with Kac' paper [54] entitled "Can one hear the shape of a drum?" in which he investigated the equation

$$\frac{1}{2}\Delta_{|X}U + \omega^2 U = 0 \tag{1.1}$$

on a membrane $X \subseteq \mathbb{R}^d$ with $\Delta_{|X}$ denoting the Dirichlet Laplacian on $X$. Considering this equation was motivated by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \tag{1.2}$$

with Dirichlet boundary conditions $u_{|\partial X}(\,\cdot\,, t) = 0$ for all $t$. Solutions

$$u(x,t) = U(x)e^{i\omega t}, \quad x \in X, t \in [0,\infty) \tag{1.3}$$

of the wave equation represent pure tones the membrane is capable of producing, cf. [54, page 2]. Substituting (1.3) into the wave equation yields (1.1).

The main question of Kac in [54] is if the spectrum $\Upsilon(X)$ of $-\Delta_{|X}$ determines $X$ up to isometry. As shown in [39, 72] the answer is "no" in general. However, some geometric properties of $X$ are saved in $\Upsilon(X)$. Weyl showed that the eigenvalue counting function $N(X;\,\cdot\,)$ of $-\Delta_{|X}$ satisfies

$$N(X;\,\lambda) = (2\pi)^{-d}B_d \operatorname{vol}_d(X)\,\lambda^{d/2} + o\left(\lambda^{d/2}\right), \tag{1.4}$$

as $\lambda \to \infty$, where $B_d$ denotes the volume of the unit ball in $\mathbb{R}^d$. The reader is referred to [16, 75, 82–84]. This formula is nowadays known as *Weyl's law*.

Furthermore, in [84] Weyl conjectured that (1.4) can be expanded to

$$N(X;\,\lambda) = (2\pi)^{-d} B_d\, \mathrm{vol}_d(X)\, \lambda^{d/2} - c_2(d)\, \mathrm{vol}_{d-1}(\partial X)\, \lambda^{(d-1)/2} + o\left(\lambda^{(d-1)/2}\right), \quad (1.5)$$

as $\lambda \to \infty$. Under a regularity condition and if the boundary of $X$ is smooth, his conjecture was proven by Duistermaat and Guillemin [25] and Ivrii [50]. For a historical overview see Ivrii [51].

*Berry conjectured* in [10,11] that if the boundary of $X$ is not smooth, the second term in (1.5) should be driven by the Hausdorff dimension of $\partial X$. More precisely, he conjectured

$$N(X;\,\lambda) = (2\pi)^{-d} B_d\, \mathrm{vol}_d(X)\, \lambda^{d/2} - c_2\,(d, d_{\mathcal{H}})\; \mathcal{H}^{d_{\mathcal{H}}}(\partial X)\, \lambda^{d_{\mathcal{H}}/2} + o\left(\lambda^{d_{\mathcal{H}}/2}\right), \quad (1.6)$$

as $\lambda \to \infty$, where $d_{\mathcal{H}}$ denotes the Hausdorff dimension of $\partial X$, $\mathcal{H}^{d_{\mathcal{H}}}$ the $d_{\mathcal{H}}$-dimensional Hausdorff measure and $c_2\,(d, d_{\mathcal{H}}) > 0$ is a constant only depending on $d$ and $d_{\mathcal{H}}$.

His conjecture was proven wrong by Brossard and Carmona [14]. Moreover, they indicated that the second term should be driven by the Minkowski dimension of $\partial X$. Lapidus [60, Theorem 1.1] showed that if the Minkowski dimension $d_M$ of $\partial X$ is in $(d-1, d]$ and if the upper Minkwoski content of $\partial X$ is finite, then the error of the first term in (1.4) is of the order $O(\lambda^{d_M/2})$. Furthermore, Lapidus and Pomerance [61] investigated the case when $d = 1$ and the boundary of $X$ is Minkowski measurable. They showed that in this case the Minkowski content is the right measurement to capture the fractal length of the boundary of $X$ in the Weyl asymptotics. More precisely, [61, Corollary 2.3] yields

**Theorem 1.1.1 (c.f. [61, Corollary 2.3]):** *Let $X \subseteq \mathbb{R}$ be bounded and open such that $\partial X$ is Minkowski measurable with Minkowski dimension $d_M \in (0,1)$. Then, it holds*

$$N(X;\,\lambda) = \pi^{-1} vol_1(X)\, \lambda^{1/2} - c_2\,(d_M)\; M^{d_M}(\partial X) \lambda^{d_M/2} + o\left(\lambda^{d_M/2}\right),$$

*as $\lambda \to \infty$, where $M^{d_M}$ denotes the $d_M$-dimensional Minkowski content and*

$$c_2\,(d_M) = 2^{-(1-d_M)} \pi^{-d_M} (1 - d_M)(-\zeta(d_M)),$$

*with $\zeta$ being the Riemann zeta function.*

For higher dimensional sets it is in general hard to discover the growth of the second order term. Fleckinger-Pellé and Vassiliev [30] constructed sets for which the second order term in the Weyl asymptotics is proportional to a periodic function of $\log \lambda$.

Besides investigation on (1.6) when $\partial X$ is fractal, the leading order term in the Weyl expansion got in focus for generalized Laplacians acting on fractals. Authors such as Fukushima and Shima [37] and Kigami and Lapidus [57] considered (1.4) for Laplacians on p.c.f. fractals, Hambly [41,42] and Freiberg, Hambly and Hutchinson [35] on random fractals, Alonso-Ruiz and Freiberg [1,2], Alonso-Ruiz, Kelleher and Teplyaev [3] and Hauser [45,46] investigated Laplacians on the Hanoi attractor and Hauser [47] on streched fractals. Fujita [36], Freiberg [33, 34] and Arzt [4] worked on (1.4) for measure theoretical Laplacians acting on and off Cantor-like sets.

## 1.2. Statement of the problem for measure theoretical Laplacians

We consider a finite non-atomic Borel measure on some interval $[a, b]$. Typically, $\mu$ is a singular measure, i.e. $\mu$ has no Radon-Nikodym density with respect to the Lebesgue measure. Common examples are measures supported on the Cantor set but also fully supported measures with asymmetric mass distribution. Measure theoretical Laplacians $\Delta^\mu$ we investigate are the composition of two measure theoretical first order derivatives.

For a mathematical motivation, consider a function $f \in C^0([a, b], \mathbb{R})$. $f$ is weakly differentiable in $L_2$ with $L_2$-weak derivative $g$ if and only if $g \in L_2(\lambda^1)$ and

$$f(x) = f(a) + \int_a^x g(y)\, dy, \quad x \in [a, b],$$

see e.g. [13, Theorem 8.2]. Replacing the one dimensional Lebesgue measure $\lambda^1$ with $\mu$ leads to a measure theoretical first order derivative. As in Freiberg [31,32], we say $f : [a, b] \longrightarrow \mathbb{R}$ possesses a *$\mu$-derivative* $f^\mu \in L_2(\mu)$ if and only if

$$f(x) = f(a) + \int_a^x f^\mu(y)\, \mu(dy), \quad x \in [a, b],$$

see also Arzt [4, Section 2.1]. Beside other analytic properties, in Freiberg [31] it is shown that the $\mu$-derivative operator

$$\frac{d}{d\mu} : D_1^\mu \longrightarrow L_2(\mu)$$
$$f \mapsto f^\mu$$

is well-defined, where $D_1^\mu$ is defined in (2.1). Composing the $\mu$-derivative with the classical first derivative $\frac{d}{dx}$ leads to the measure theoretical Laplacian we are interested in

$$\Delta^\mu : D_2^\mu \longrightarrow L_2(\mu),$$
$$f \mapsto \frac{d}{d\mu} f',$$

where $D_2^\mu$ is defined in (2.2).

For this operator we study Weyl's law (1.4), meaning we study the equation

$$\Delta^\mu f = -\lambda f \tag{1.7}$$

with Dirichlet boundary conditions

$$f(a) = f(b) = 0.$$

By Bird, Ngai and Teplyaev [12, Theorem 5] the spectrum of $-\Delta^\mu$ with Dirichlet boundary conditions is pure point with eigenvalues having finite multiplicities, accumulating only at infinity. For more general measure theoretical Laplacians defined as the composition of $\frac{d}{d\mu}$ and $\frac{d}{d\nu}$ with some suitable measure $\nu$, these properties are preserved as shown in Freiberg [31, Lemma 5.1 and Corollary 6.9]. In addition to the mentioned, $\Delta^\mu$ were considered in numerous papers. For properties of the associated stochastic process see for example Küchler [58,59] and Löbus [68].

## 1.3. Physical motivation for measure theoretical Laplacians

The following physical motivation is taken from Arzt [4, Section 1.2]. In that work, further physical motivations for this operator with Dirichlet and also with Neumann boundary conditions are given.

We consider a flexible string, clamped between two points $a$ and $b$. If we deflect the string, a tension force drives the string back towards its state of equilibrium, cf. [4, page 9]. A solution $u$ of the wave equation (1.2) with Dirichlet boundary conditions and $X = [a, b]$ models the deviation of the string under the assumption that the mass distribution of the string is homogeneous. Replacing the constant in (1.2) by a mass density $\varrho : [a, b] \longrightarrow \mathbb{R}$ leads to

$$\varrho(x)\, \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2}. \tag{1.8}$$

A solution $u$ of this generalized wave equation with Dirichlet boundary condition $u(a, t) = u(b, t) = 0$ for all $t$ models the same phenomenon as before with the difference that the string has a possibly inhomogeneous mass distribution $\mu(dx) = \varrho(x)\, dx$ (we set the other involved constants to 1).

The ansatz $u(x, t) = \phi(x)\, \psi(t)$ leads to

$$\frac{\psi''(t)}{\psi(t)} = \frac{\phi''(x)}{\phi(x)\varrho(x)}.$$

The left hand side no longer depends on $x$, the right hand side no longer on $t$, and thus there exists a $\lambda \in \mathbb{R}$ such that

$$\frac{\psi''(t)}{\psi(t)} = \frac{\phi''(x)}{\phi(x)\varrho(x)} = -\lambda$$

for all $t$ and $x$. We consider the equation for $\phi$ and write it as

$$\phi''(x) = -\lambda \phi(x)\varrho(x).$$

Integrating both sides with respect to the Lebesgue measure leads to

$$\phi'(x) - \phi'(a) = -\lambda \int_a^x \phi(y)\, \varrho(y)\, dy$$

and thus

$$\phi'(x) = \phi'(a) - \lambda \int_a^x \phi(y)\, \varrho(y)\, dy.$$

Hence,

$$\phi'(x) = \phi'(a) - \lambda \int_a^x \phi(y)\, \mu(dy).$$

By definition of the $\mu$-derivative this means

$$\frac{d}{d\mu}\phi' = -\lambda\,\phi, \tag{1.9}$$

and consequently

$$\Delta^{\mu}\phi = -\lambda\,\phi$$

with Dirichlet boundary conditions

$$\phi(a) = \phi(b) = 0.$$

The eigenvalue problem (1.9) no longer involves the density of the mass distribution of the string. Therefore, we can reformulate the problem for singular mass distributions $\mu$.

Up to a multiplicative constant, the square root of the eigenvalues are given as the natural frequencies of the sting.

## 1.4. The Cantor set, Cantor-like sets and Cantor strings

> "Als ein Beispiel einer perfecten Punctmenge, die in keinem noch so kleinen Intervall überall dicht ist, führe ich den Inbegriff aller reellen Zahlen an, die in der Formel:
>
> $$z = \frac{c_1}{3} + \frac{c_2}{3^2} + \cdots + \frac{c_\nu}{3^\nu} + \cdots$$
>
> enthalten sind, wo die Coefficienten $c_\nu$ nach Belieben die beiden Werthe 0 und 2 anzunehmen haben und die Reihe sowohl ans einer endlichen, wie aus einer unendlichen Anzahl von Gliedern bestehen kann." ( [15], page 590, footnote 11)

In 1883, Cantor introduced in the cited excerpt the set nowadays known as the *Cantor set*. His investigation was based on giving an example for a perfect set which is nowhere dense. Thereby, he defined the considered set as all real numbers expressible as

$$\sum_{i=1}^{\infty} \frac{c_i}{3^i}, \qquad c_i \in \{0, 2\}.$$

A fixed-point argument based definition would be the following: By Hutchinson [48] there exists to every iterated function system $\mathcal{S} = \{S_1, \ldots, S_N\}$ on a complete metric space $(\mathcal{X}, d)$ a unique non-empty compact set $K$ such that

$$K = \bigcup_{i=1}^{N} S_i(K). \tag{1.10}$$

With

$$(\mathcal{X}, d) = ([0,1], |\cdot|), \quad \mathcal{S} = \left\{ \frac{1}{3}\, x,\ \frac{1}{3}\, x + \frac{2}{3} \right\}, \tag{1.11}$$

the unique non-empty compact set $K$ satisfying (1.10) is the set introduced by Cantor.

The Cantor set can also be defined as the limit of the iterative procedure in which the open second third interval of every remaining interval is removed, starting with the unit interval.

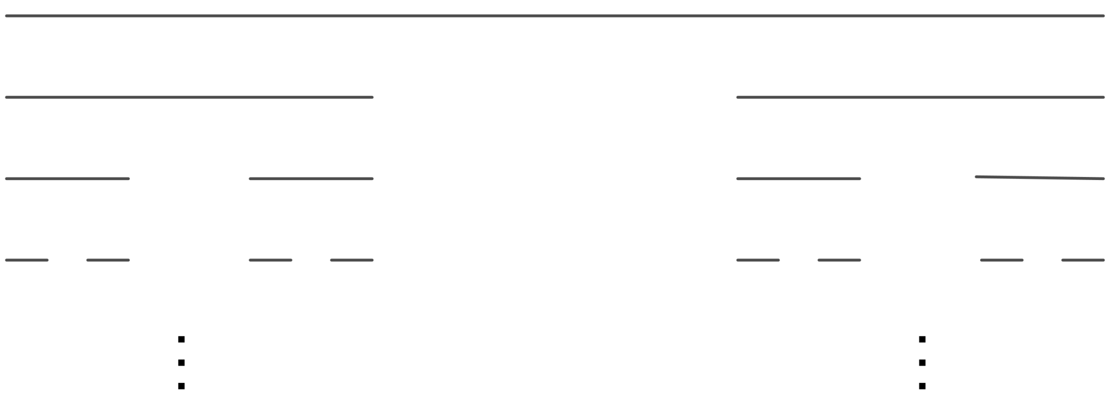

Figure 1.1: Iterative construction of the Cantor set

With (1.11) one can give an analytic definition of this iterative construction. Therefore, define for $x = (x_1, \ldots, x_n)$, $x_i \in \{1, 2\}$, $n \in \mathbb{N}$

$$S_x \coloneqq S_{x_1} \circ \cdots \circ S_{x_n}.$$

Then, the Cantor set $K$ is given as

$$K := \bigcap_{n=1}^{\infty} \bigcup_{|x|=n} S_x[0,1].$$

Furthermore, we call $\bigcup_{|x|=n} S_x[0,1]$ the *n-th approximation step* of the Cantor set.

Hausdorff introduced in [44] a new dimension concept, today known as the *Hausdorff dimension*. To give an example of a set whose Hausdorff dimension can take any value in $(0,1)$, he generalized the Cantor set by removing an interval with arbitrary fixed length ratio from the middle of each remaining interval.

Beardon [9] investigated the Hausdorff dimension of a more general construction which was introduced by Tsuji [80, 81]. Hereby, the considered set is defined by

$$K := \bigcap_{n=1}^{\infty} \bigcup_{x_1,\dots,x_n=1}^{N} C_{x_1,\dots,x_n},$$

where $N \in \mathbb{N}$, $N \geq 2$ and $C_{x_1,\dots,x_n}$ are connected compact sets (not necessarily one-dimensional) satisfying

- $C_{x_1,\dots,x_n} \supseteq C_{x_1,\dots,x_n,x_{n+1}}$,
- $C_1, \dots, C_N$ are pairwise disjoint ,
- there exists $A \in (0,1)$ such that

$$\left\| C_{x_1,\dots,x_n,x_{n+1}} \right\| \geq A \left\| C_{x_1,\dots,x_n} \right\|,$$

- there exists $B \in (0,1)$ such that for $x_{n+1} \neq y_{n+1}$

$$\rho(C_{x_1,\dots,x_n,x_{n+1}}, C_{x_1,\dots,x_n,y_{n+1}}) \geq B \left\| C_{x_1,\dots,x_n} \right\|,$$

where $\|\cdot\|$ denotes the diameter and

$$\rho(X,Y) := \inf \{|x-y| : x \in X, y \in Y\}.$$

On page 61 ff., Falconer [27] considered sets

$$K := \bigcap_{n=1}^{\infty} E_n, \tag{1.12}$$

with $E_n$ being the union of a finite number of disjoint closed intervals, $[0,1] \supset E_1 \supset E_2 \supset \cdots$ such that each interval in $E_n$ contains at least two intervals of $E_{n+1}$ and the maximum length of intervals in $E_n$ tends to 0 as $n$ tends to infinity. $E_n$ can be regarded as the $n$-th approximation step of $K$ in (1.12).

We call the above introduced generalized Cantor sets *Cantor-like*. Zähle [85], Falconer [28], Mauldin and Williams [70] and Graf [40] investigated the Hausdorff dimension of random Cantor-like sets. Arzt [4] investigated spectral asymptotics for measure theoretical Laplacians acting on random Cantor-like sets. In Mandelbrot [69] a random generalization of the Cantor set on the whole real line is treated.

The complement of such a one-dimensional Cantor-like set is a disjoint union of countably many open intervals those descending assorted lengths $\{l_n\}_{n\in\mathbb{N}}$ satisfy $\lim_{n\to\infty} l_n = 0$. Such a set is called *Cantor string*. Cantor strings and their randomizations are e.g. considered in Lapidus and Pomerance [61, 62], Hambly and Lapidus [43], Lapidus and van Frankenhuysen [64] and Charmoy, Croydon and Hambly [18].

## 1.5. Outline of the thesis

In the presented thesis we consider Weyl's law (1.4) and Berry's conjecture (1.6) on random Cantor sets and strings, respectively.

Firstly, we calculate the leading order in the asymptotics of the eigenvalue counting function for measure theoretical Laplacians $\Delta^\mu$ with respect to *statistically self-similar* and *random $V$-variable* Cantor measures.

Berry's conjecture is treated on the complement of statstically self-similar Cantor sets. For the classical Laplacian on these strings we make some statements about the error of the second term in the Weyl asymptotics.

We summarize in Chapter 2 the preliminary work on Weyl's law (1.4) for measure theoretical Laplacians $\Delta^\mu$ and Berry's conjecture (1.6) on statistically self-similar Cantor strings for the classical Laplacian.

In Section 3.1, C-M-J branching processes related to statistically self-similar Cantor sets are introduced. The Weak and Strong Law of Large Numbers and the Central Limit Theorem for these processes are given in Section 3.3. After these preliminary convergence results, we establish in Section 3.4 a Law of the Iterated Logarithm which is the first main result of the present thesis.

Figure 1.2: First two approximation steps of a recursive Cantor set

We need to control some moments of the almost sure limit of the normalized C-M-J branching process to apply the Central Limit Theorem and Law of the Iterated Logarithm. An investigation on the almost sure limit of the normalized C-M-J branching process terminates Chapter 3.

We shortly introduce recursive Cantor sets and corresponding Cantor measures, the precise definition follows in Section 4.2.1. Corresponding random recursive Cantor sets are also called statistically self-similar, e.g. in [18]. As explained, the Cantor set can be constructed by taking repeatedly the image of the remaining intervals under the IFS $\mathcal{S} = \{\frac{1}{3}x, \frac{1}{3}x + \frac{2}{3}\}$, starting with the unit interval. We generalize this procedure by considering a family of iterated function systems $\mathfrak{S}$ on $[0,1]$ (later more general on $[a,b]$) such that the image of the unit interval contains the boundary points. In the first approximation step we subdivide the unit interval according to an IFS $\mathcal{S}_\emptyset = \{S_1, \dots, S_{N_\emptyset}\} \in \mathfrak{S}$. Afterwards, we take the image of $S_1[0,1], \dots, S_{N_\emptyset}[0,1]$ under arbitrary iterated function systems $\mathcal{S}_{\emptyset,1}, \dots, \mathcal{S}_{\emptyset,N_\emptyset} \in \mathfrak{S}$. Subsequently, we continue analogously. By given family of iterated function systems $\mathfrak{S}$, this procedure can be encoded by a tree $I$. The limiting set $K^{(I)}$ is called *recursive Cantor set*. Such recursive structures for p.c.f. fractals are considered in Hambly [42].
In Figure 1.2, we see an example of a recursive Cantor set with $\mathfrak{S} = \{\mathcal{S}^{(1)}, \mathcal{S}^{(2)}\}$, $\mathcal{S}^{(1)}$ being the generator of the Cantor set and $\mathcal{S}^{(2)}$ splitting the unit interval into five subintervals where the open second and fourth fifth interval is removed. To each IFS we assign a *vector of weights* $m^{(j)} = (m_1^{(j)}, \dots, m_{N_j}^{(j)})$, $N_j = \left|\mathcal{S}^{(j)}\right|$, $m_i^{(j)} \in (0,1)$ for all $i$,

$$\sum_{i=1}^{N_j} m_i^{(j)} = 1, \quad j = 1, 2,$$


Figure 1.3: First two approximation steps of a recursive Cantor measure

with which we construct a Borel probability measure $\mu^{(I)}$ on $[0,1]$ with $\operatorname{supp}\mu^{(I)} = K^{(I)}$. $\mu^{(I)}$ weights an interval in an approximation step of $K^{(I)}$ by a product of entries of vectors of weights. This product is encoded in the same way as the composition of similarities of the used iterated function systems generating that particular interval. Figure 1.3 shows the mass distribution of $\mu^{(I)}$ for the intervals in Figure 1.2.

Afterwards, we construct a probability space in which every atomic event indicates a random tree. Such a random tree defines a statistically self-similar Cantor set $K^{(I)}$ and a corresponding random probability measure $\mu^{(I)}$. By using a scaling property of the eigenvalue counting function, similar to that of Arzt [4] who established his scaling property by following Kigami and Lapidus [57, Lemma 2.3], we are able to use the theory of C-M-J branching processes to determine in Section 4.2.3 the leading order in the Weyl asymptotics for the Dirichlet eigenvalue counting function $N^{\mu^{(I)}}$ of $-\Delta^{\mu^{(I)}}$. This leading order is characterized by an expectation equation, similar to the equation characterizing the leading order in the Weyl asymptotics of the Dirichlet eigenvalue counting function $N^{\mu}$ of $-\Delta^{\mu}$ for self-similar measures $\mu$.

An interpolation between random homogeneous and statistically self-similar structures can be done by trimming the tree $I$ in the manner that in every generation it is allowed to have at most a fixed number of different subtrees. We denote this fixed number by $V$. Corresponding sets under consideration in Section 4.3 are called *$V$-variable Cantor sets*. The case $V = 1$ leads to homogeneous Cantor sets and $V = \infty$ can be regarded as the recursive structure. This construction was introduced in [7,8]. Spectral asymptotics for the Laplacian on *$V$-variable Sierpinski Gaskets* are considered by Freiberg, Hambly and Hutchinson [35].

As an example for a $V$-variable construction, we consider four iterated function systems $\mathfrak{S} = \{S^{(1)}, S^{(2)}, S^{(3)}, S^{(4)}\}$ and set $V = 3$. We let $S^{(1)}$ and $S^{(2)}$ be the same IFSs as before, $S^{(3)}$ splitting the unit interval into three parts, where the open second quarter interval is removed and $S^{(4)}$ also splitting the unit interval into three parts, where the open third quarter interval is removed. The variable $V \in \mathbb{N}$ determines the number of *types*. We denote the types in our example by $\triangledown$, $\square$, $\Diamond$. In every approximation step, every type indicates an index of our index set $\{1, 2, 3, 4\}$. This indicated index can vary in different approximation steps. Figure 1.4 shows an example on how we construct a 3-Variable Cantor set in this setting. A $V$-variable Cantor set depends on a sequence of *environments* which determines in every step the indicated indices of each type and also the types of the intervals in the next step.

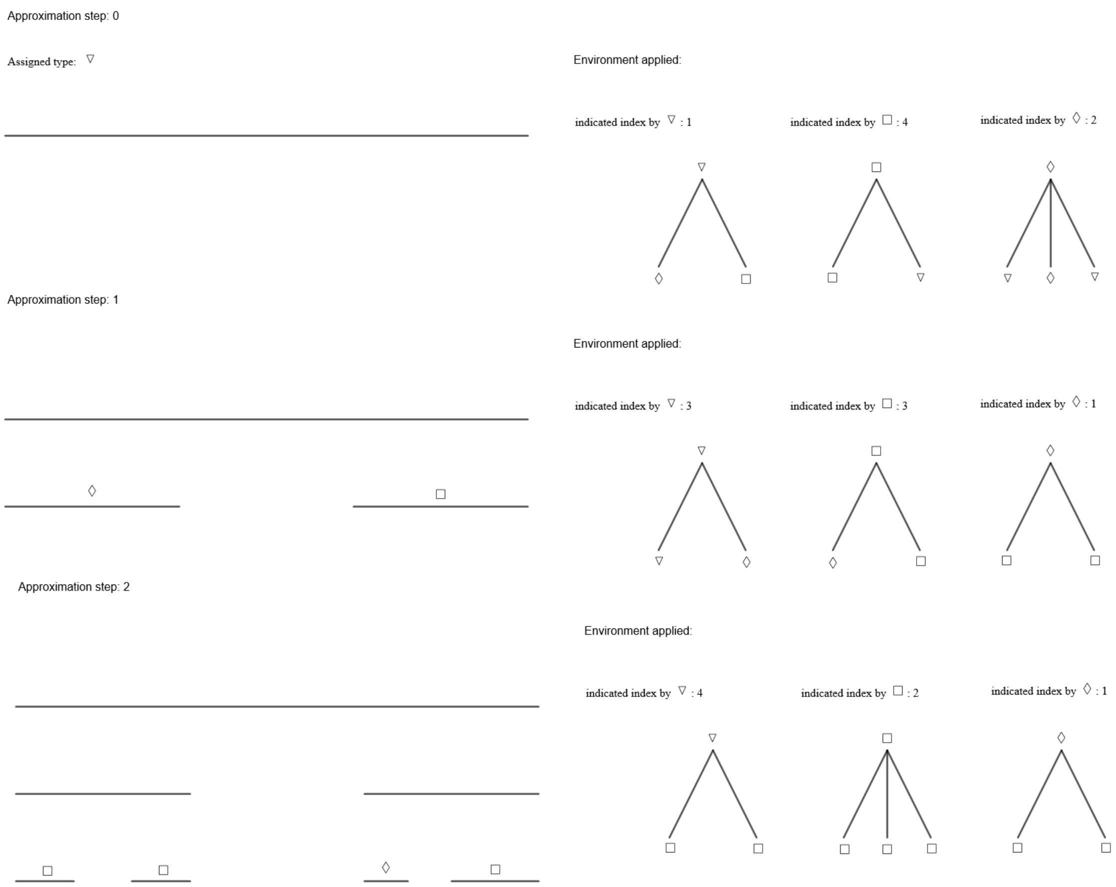


Figure 1.4: First approximation steps of one possible 3-variable Cantor set

After applying the environment in approximation step 2 of Figure 1.4 all assigned types are equal. In our random setting such levels occur infinitely often almost surely and are crucial for our consideration. We call such levels *neck levels* and discuss some properties in Section 4.3.2. Afterwards, we give in Section 4.3 the spectral asymptotics for random $V$-variable Cantor measures.

The main results of Chapter 4 are given in Theorem 4.2.9 and Theorem 4.3.18. These theorems are main results of the present thesis.

Taking the complement of a statistically self-similar Cantor set $K^{(I)}$ in $[0,1]$ leads to a set $U$, called *statistically self-similar Cantor string*. $U$ consists of a countable union of disjoint open intervals with descending lengths. We investigate the asymptotic behaviour of

$$\bar{N}(U;\,\lambda) := \frac{1}{\pi}\lambda^{1/2} - N(U;\,\lambda)$$

in Chapter 5. Therein, a crucial property of $\bar{N}(U;\,\cdot\,)$ is the decomposition

$$\bar{N}(U;\,\lambda) = \sum_{i=1}^{N_\emptyset-1} \bar{N}([0,1];\,L_i^2\lambda) + \sum_{i=1}^{N_\emptyset} \bar{N}(U_i;\,R_i^2\lambda)$$

with $N_\emptyset$ being the number of subintervals in the first approximation step, $L_i$ the length of the gap interval between $S_i[0,1]$ and $S_{i+1}[0,1]$, $R_i$ the scale factor of $S_i$ and $\{U_i\}_{i=1}^{N_\emptyset}$ i.i.d. copies of $U$. With this decomposition we can write $\bar{N}(U;\,e^{2\lambda})$ as a C-M-J branching process. In Section 5.2, we investigate the conditions for the Central Limit Theorem and the Law of the Iterated Logarithm. As shown in Section 5.3, controlling the error of the linear approximation of the underlying renewal function ensures the Central Limit Theorem and, if the normalized variance process does not converge to 0, the Law of the Iterated Logarithm to hold. We terminate Chapter 5 with an application, taken from [18], in which the normalized variance process does not converge to 0.

The main result of Chapter 5 is given in Theorem 5.3.2 which is a main result of the present thesis.

Finally, in Chapter 6, we give an outlook. For $\Delta^\mu$ this outlook concerns relations between the leading order term in Weyl's law for random $V$-variable and statistically self-similar Cantor measures. Also, we discuss the assumptions needed for the spectral asymptotics in Chapter 2.2 and Section 4.

A further investigation on the Central Limit Theorem and Law of the Iterated Logarithm for $\bar{N}(U;\,\cdot\,)$ is also mentioned.

**Notation.** For the length of a statement or proof $c_i$, $d_i$, $i \in \mathbb{N}$ are some non-negative numbers.

The present thesis is based on the following articles, for references of the first two articles, see [73] and [74]:

- L. A. Minorics, *Spectral Asymptotics for Krein-Feller-Operators with respect to Random Recursive Cantor Measures*. Preprint, 2017.
- L. A. Minorics, *Spectral Asymptotics for Krein-Feller-Operators with respect to $V$-variable Cantor Measures*. Preprint, 2018.
- B. M. Hambly and L. A. Minorics, *Some Limit Theorems for the Laplacian on Statistically Self-Similar Cantor Strings*. In preparation.

# Chapter 2

# Preliminaries

In this chapter we give preliminary results on Weyl's law for Laplacians under consideration concerning the first order term in the Weyl asymptotics for the eigenvalue counting function for measure theoretical Laplacians $\Delta^\mu$ with respect to certain measures $\mu$ and the second order term in the Weyl asymptotics for the eigenvalue counting function for the classical Laplacian on statistically self-similar Cantor strings.

## 2.1. Definition and basic properties of measure theoretical Laplacians

As Freiberg [31,32] we define a measure theoretical derivative. Therefore, let $\mu$ be a finite non-atomic Borel measure on $[a,b]$, $-\infty < a < b < \infty$ and

$$\mathfrak{D}_1^\mu := \Big\{ f : [a,b] \longrightarrow \mathbb{R} : \ \exists\, f^\mu \in L_2(\mu) : \\ f(x) = f(a) + \int_a^x f^\mu(y)\ \mu(dy), \quad x \in [a,b] \Big\}.$$

By Freiberg [31, Corollary 6.4], the *$\mu$-derivative*

$$\begin{aligned} \frac{d}{d\mu} : D_1^\mu := \mathfrak{D}_1^\mu / \sim_\mu &\longrightarrow L_2(\mu), \\ f &\mapsto \ f^\mu \end{aligned} \tag{2.1}$$

is well-defined, whereby $\sim_\mu$ denotes the equivalence relation in $L_2(\mu)$. Let

$$\mathfrak{D}_2^\mu := \mathfrak{D}_2^{\mu,\lambda^1} := \Big\{ f \in C^1([a,b],\mathbb{R}) : \exists\ (f')^\mu \in L_2\,(\mu) : \\ f'(x) = f'(a) + \int_a^x (f')^\mu(y)\ \mu(dy), \quad x \in [a,b] \Big\}.$$

The *measure theoretical Laplacian* $\Delta^\mu$ is given as

$$\begin{aligned}\Delta^\mu : D_2^\mu := \mathfrak{D}_2^\mu / \sim_\mu &\longrightarrow L_2(\mu) \\ f &\mapsto (f')^\mu,\end{aligned} \tag{2.2}$$

see also Arzt [4, Section 2.1]. Further, denote by $(\lambda_i^\mu)_{i\in\mathbb{N}}$ the ascending ordered sequence of Dirichlet eigenvalues of $-\Delta^\mu$ and, as in Chapter 1,

$$N^\mu(\lambda) := \#\left\{i \in \mathbb{N} : \lambda_i^\mu \leq \lambda\right\}.$$

We establish the following lemma to control the eigenvalue counting function of $\Delta^\mu$ in Section 4.

**Lemma 2.1.1:** *Let $\mu$ be a finite non-atomic Borel measure on $[a, b]$ with $a, b \in \operatorname{supp}\mu$. Then, there exists $c > 0$ such that*

$$N^\mu(s) \leq \mu([a, b])\, c\, s$$

*for all $s \geq 0$. Moreover, $c$ is independent of $\mu$.*

*Proof.* Let

$$g(x, y) := \frac{\min(x - a, y - a)\min(b - y, b - x)}{b - a}, \qquad x, y \in [a, b].$$

Then, with

$$\begin{aligned}T_g : L_2(\mu) &\longrightarrow L_2(\mu) \\ f &\mapsto \int_a^b g(\cdot, y) f(y)\, \mu(dy),\end{aligned}$$

we get

$$\begin{cases} -\frac{d}{d\mu}\frac{d}{dx} f = \lambda f \\ f(a) = f(b) = 0 \end{cases} \quad \text{if and only if} \quad T_g f = \frac{1}{\lambda} f, \tag{2.3}$$

cf. [31, Theorem 4.1]. In the following we want to use Mercer's Theorem, originally given in [71]. The version we use is taken from the book of Christmann and Steinwart [19].

By [19, Definition 4.1, Lemma 4.3, Lemma 4.6], $g$ is a continuous kernel and thus we can use Mercer's Theorem [19, Theorem 4.49] and therefore

$$g(x,y)=\sum_{i=1}^{\infty}\frac{1}{\lambda_i^\mu}\,f_i(x)\,f_i(y),$$

where $(f_i)_i$ are orthonormal eigenfunctions. Furthermore, the convergence is uniform. With

$$c:=\sup_{x\in[a,b]} g(x,x)=\frac{b-a}{4}$$

it follows

$$c\geq\sum_{i=1}^{\infty}\frac{1}{\lambda_i^\mu}\,f_i(x)\,f_i(x).$$

Integrating both sides with respect to $\mu$ yields for any $s\geq 0$

$$\mu([a,b])\,c\geq\sum_{i=1}^{\infty}\frac{1}{\lambda_i^\mu}=\int_0^\infty\frac{1}{u}\,N^\mu(du)\geq\int_0^s\frac{1}{u}\,N^\mu(du)\geq\frac{1}{s}\,N^\mu(s)$$

and thus the claim follows. □

In Chapter 4 we furthermore need relations between the Dirichlet and Neumann eigenvalue counting function of $-\Delta^\mu$. Therefore, we write $N_D^\mu$ and $N_N^\mu$ for the Dirichlet and Neumann eigenvalue counting function of $-\Delta^\mu$ respectively, where we call $\lambda$ a Neumann eigenvalue of $-\Delta^\mu$ if $\lambda$ is an eigenvalue of $-\Delta^\mu$ with Neumann boundary conditions

$$f'(a)=f'(b)=0.$$

For references of the following Lemma, see Freiberg [32, Proposition 5.5].

**Lemma 2.1.2 (c.f. [32, Proposition 5.5]):** *Let $\mu$ be a finite non-atomic Borel measure on $[a,b]$ with $a,b\in\operatorname{supp}\mu$. Then, for all $s\geq 0$ holds*

$$N_D^\mu(s)\leq N_N^\mu(s)\leq N_D^\mu(s)+2.$$

## 2.2. Spectral asymptotics for measure theoretical Laplacians on Cantor-like sets

### 2.2.1. Spectral asymptotics for self-similar Cantor measures

In this section we consider spectral asymptotics of $\Delta^\mu$ with respect to self-similar measures treated in Fujita [36]. Therefore, let $\mathcal{S} = \{S_1, ..., S_N\}$, $N \geq 2$ be an iterated function system given by

$$S_i(x) = r_i\, x + c_i, \quad x \in [a,b],$$

whereby $r_i \in (0,1)$ and $c_i \in \mathbb{R}$ are constants such that the open set condition is satisfied. As explained in Section 1.4, by Hutchinson [48] there exists a unique non-empty compact set $K = K(\mathcal{S}) \subseteq [a,b]$ such that

$$K = \bigcup_{i=1}^{N} S_i(K).$$

Furthermore, let $m = \{m_1, \dots, m_N\}$ be a vector of weights. Also shown in [48], there exists a unique Borel probability measure $\mu = \mu(\mathcal{S}, m)$ such that

$$\mu = \sum_{i=1}^{N} m_i\, \mu \circ S_i^{-1}$$

with $\operatorname{supp} \mu = K$. We call $K$ self-similar with respect to $\mathcal{S}$ and $\mu$ self-similar with respect to $\mathcal{S}$ and $m$ or simply *self-similar Cantor measure*. This measure $\mu$ is the weak limit of the sequence of Borel probability measures $(\mu_n)_{n\in\mathbb{N}_0}$ defined as

$$\mu_n \coloneqq \sum_{x\in\{1,\dots,N\}^n} m_{x_1} \cdots m_{x_n}\, \mu_0 \circ (S_{x_1} \circ S_{x_2} \circ \cdots \circ S_{x_n})^{-1}, \quad \mu_0 \coloneqq \frac{1}{b-a}\, \lambda^1_{|[a,b]}.$$

Hutchinson [48] furthermore showed that the Hausdorff and Minkowsi dimension of $K$ is given by the unique solution $d \in [0,1]$ of

$$\sum_{i=1}^{N} r_i^d = 1 \tag{2.4}$$

and it holds $\mathcal{H}^d(K) \in (0, \infty)$. Moreover, if $m_i = r_i^d$ for all $i$, it follows

$$\mu = \mathcal{H}^d(K)^{-1}\,\mathcal{H}^d(\,\cdot \cap K),$$

since for a Borel set $A \subseteq [a, b]$ follows

$$\begin{aligned}
\mathcal{H}^d(A \cap K) &= \mathcal{H}^d\left(\bigcup_{i=1}^N S_i\left(S_i^{-1}(A)\right) \cap S_i(K)\right) \\
&= \sum_{i=1}^N \mathcal{H}^d\left(S_i\left(S_i^{-1}(A)\right) \cap S_i(K)\right) \\
&= \sum_{i=1}^N \mathcal{H}^d\left(r_i\left(S_i^{-1}(A) \cap K\right)\right) \\
&= \sum_{i=1}^N r_i^d\,\mathcal{H}^d\left(S_i^{-1}(A) \cap K\right).
\end{aligned}$$

In this setting, the following theorem was established by Fujita [36, Theorem 3.6] and Freiberg [34, Theorem 3.2].

**Theorem 2.2.1 (c.f. [36, Theorem 3.6 and 34, Theorem 3.2]):** *Let $\gamma_s > 0$ be the unique solution of*

$$\sum_{i=1}^N (r_i m_i)^{\gamma_s} = 1.$$

*Then, it holds*

1. *If the additive group $\sum_{i=1}^N \mathbb{Z}\log(r_i m_i)$ is a dense subset of $\mathbb{R}$, then there exists $C > 0$ such that*

$$\lim_{\lambda\to\infty} N^\mu(\lambda)\,\lambda^{-\gamma_s} = C.$$

2. *If the additive group $\sum_{i=1}^N \mathbb{Z}\log(r_i m_i)$ is a discrete subgroup of $\mathbb{R}$ with period $T$, then there exists a $T$-periodic function $G$ such that*

$$N^\mu(\lambda) = (G(\log\lambda) + o(1))\,\lambda^{\gamma_s},$$

*as $\lambda \to \infty$.*

### 2.2.2. Spectral asymptotics for random homogeneous Cantor measures

The spectral asymptotics for $\Delta^{\mu}$ with respect to a generalization of self-similar Cantor measures is discussed in this section. We only consider a random version treated by Arzt [4, Section 3.5]. Therefore, let $J$ be a non-empty countable index set. To each $j \in J$ let $\mathcal{S}^{(j)} = \left\{S_1^{(j)}, ..., S_{N_j}^{(j)}\right\}$, $N_j \in \mathbb{N}$ be an IFS such that

$$S_i^{(j)}(x) = r_i^{(j)}\, x + c_i^{(j)}, \quad x \in [a,b],\ i = 1, ..., N_j,$$

where the constants $r_i^{(j)} \in (0,1)$, $c_i^{(j)} \in \mathbb{R}$ are chosen such that

$$a = S_1^{(j)}(a) < S_1^{(j)}(b) \leq S_2^{(j)}(a) < \cdots < S_{N_j}^{(j)}(b) = b. \tag{2.5}$$

Further, $\xi = (\xi_1, \xi_2, ...)$, $\xi_i \in J$ is called an environment sequence and define

$$W_n := \{1, ..., N_{\xi_1}\} \times \{1, ..., N_{\xi_2}\} \times \cdots \times \{1, ..., N_{\xi_n}\}, \quad n \in \mathbb{N}.$$

A generalization of self-similar, called *homogeneous Cantor set* is defined by

$$K^{(\xi)} := \bigcap_{n=1}^{\infty} \bigcup_{w \in W_n} \left(S_{w_1}^{(\xi_1)} \circ S_{w_2}^{(\xi_2)} \circ \cdots \circ S_{w_n}^{(\xi_n)}\right)([a,b]).$$

To this set, Arzt [4, Chapter 3] defined measures which are natural extensions of self-similar Cantor measures. Therefore, let $m^{(j)} = (m_1^{(j)}, ..., m_{N_j}^{(j)})$, $j \in J$ be a vector of weights. The Borel probability measure $\mu^{(\xi)}$ under consideration is defined as the weak limit of the sequence of Borel probability measures $\left(\mu_n^{(\xi)}\right)_{n \in \mathbb{N}}$,

$$\mu_n^{(\xi)} := \sum_{w \in W_n} m_{w_1}^{(\xi_1)} \cdots m_{w_n}^{(\xi_n)}\, \mu_0 \circ \left(S_{w_1}^{(\xi_1)} \circ \cdots \circ S_{w_n}^{(\xi_n)}\right)^{-1}, \qquad \mu_0 := \frac{1}{b-a}\, \lambda^1_{|[a,b]}. \tag{2.6}$$

$\mu^{(\xi)}$ is called *homogeneous Cantor measure*, corresponding to $K^{(\xi)}$. If $|J| = 1$, then the definition of self-similar Cantor sets and measures coincide with $K^{(\xi)}$ and $\mu^{(\xi)}$, respectively.

For the random set up, let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\xi = (\xi_1, \xi_2, ..)$ a sequence of i.i.d. $J$-valued random variables with $p_j := \mathbb{P}(\xi_i = j)$. In [4, Chapter 3] the following assumptions are discussed:

$$\sup_{j\in J} N_j < \infty, \tag{A1}$$

$$\inf_{j\in J} \min_{i=1,\dots,N_j} r_i^{(j)} m_i^{(j)} > 0, \tag{A2}$$

$$\sup_{j\in J} \max_{i=1,\dots,N_j} r_i^{(j)} m_i^{(j)} < 1, \tag{A3}$$

$$\prod_{\substack{j\in J,\\ \sum_{i=1}^{N_j} r_i^{(j)} m_i^{(j)} < 1}} \sum_{i=1}^{N_j} r_i^{(j)} m_i^{(j)} > 0, \tag{A4}$$

$$\prod_{\substack{j\in J,\\ \sum_{i=1}^{N_j} r_i^{(j)} m_i^{(j)} > 1}} \sum_{i=1}^{N_j} r_i^{(j)} m_i^{(j)} < \infty. \tag{A5}$$

With these assumptions, Arzt [4, Corollary 3.5.1] established the following theorem.

**Theorem 2.2.2 (c.f. [4, Corollary 3.5.1]):** *Let* (A1)-(A5) *be satisfied. Then, for the unique solution* $\gamma_h > 0$ *of*

$$\prod_{j\in J} \left( \sum_{i=1}^{N_j} \left( r_i^{(j)} m_i^{(j)} \right)^{\gamma_h} \right)^{p_j} = 1$$

*it holds that there exist* $C_1, C_2 > 0$, $\lambda_0 > 0$ *and* $c_1(\omega), c_2(\omega) > 0$ *for almost all* $\omega \in \Omega$ *such that*

$$C_1\, \lambda^{\gamma_h}\, e^{-c_1(\omega)\sqrt{\log\lambda \log\log\log\lambda}} \le N_D^{\mu^{(\xi(\omega))}}(\lambda) \le N_N^{\mu^{(\xi(\omega))}}(\lambda) \le C_2\, \lambda^{\gamma_h}\, e^{-c_2(\omega)\sqrt{\log\lambda \log\log\log\lambda}}$$

*for all* $\lambda \ge \lambda_0$.

See also Barlow and Hambly [6].

## 2.3. Spectral asymptotics for Laplacians on statistically self-similar Cantor strings

In this section we recap the results of Charmoy, Croydon and Hambly [18] about the Central Limit Theorem for statistically self-similar Cantor strings. Therefore, choose a deterministic number $\gamma \in (0,1)$ and a random vector $(T_1, \dots, T_n)$, where $n \geq 2$ is a deterministic natural number. Assume that $(T_1, \dots, T_n)$ satisfies

$$\sum_{i=1}^{n} T_i = 1, \qquad T_i \in (0,1) \qquad\quad a.s.$$

The construction of the random Cantor-like set under consideration is as follows. Replace the unit interval $[0,1]$ by $n$ equally spaced intervals with lengths $R_1 \coloneqq T_1^{1/\gamma}, \dots, R_n \coloneqq T_n^{1/\gamma}$. Afterwards, repeat this procedure for the remaining intervals independently, indefinitely. The limiting set $K$ is a statistically self-similar Cantor set. By construction, the measure $F$ defined by

$$F(dt) \coloneqq \mathbb{E} \sum_{i=1}^{n} e^{-\gamma t} \delta_{-\log R_i}(dt)$$

is a probability distribution on $[0, \infty)$, where $\delta$ denotes the Dirac delta function. By $H$ denote the corresponding renewal function

$$H \coloneqq 1 + \sum_{m=1}^{\infty} F^{*m},$$

where $F^{*m}$ denotes the $m$-fold convolution of $F$ with itself. Further, denote by $G$ the error of the linear approximation of the renewal function, i.e.

$$G(t) \coloneqq H(t) - t \left( \int_0^\infty u \, F(du) \right)^{-1}.$$

Under the following assumption of the speed of convergence of $G$, Charmoy, Croydon and Hambly [18, Theorem 4.3] established a Central Limit Theorem for $\bar{N}_U$. Therefore, it is assumed that $G$ converges to a finite constant (see Appendix A1).

**Assumption 2.3.1 (c.f. [18, Assumption 4.2]):** *There exist* $\beta_1 \in (\gamma/2, \infty)$ *and* $c, t_0 \in (0, \infty)$ *such that*

$$\left| G(t) - \lim_{u \to \infty} G(u) \right| \leq c\, e^{-\beta_1 t},$$

*for all* $t \geq t_0$.

**Theorem 2.3.2 (Central Limit Theorem, c.f. [18, Theorem 4.3]):** *Let* $N([0,1]\backslash K;\ \cdot\ )$ *be the Dirichlet eigenvalue counting function of the negative Laplacian on* $[0,1]\backslash K$,

$$\bar{N}([0,1]\backslash K; \lambda) := \frac{1}{\pi}\lambda^{1/2} - N(\lambda)$$

*and* $F$ *be non-lattice. Then, there exists a deterministic* $C > 0$ *such that*

$$\bar{N}([0,1]\backslash K; \lambda)\lambda^{-\gamma/2} \overset{\lambda\to\infty}{\longrightarrow} C \quad a.s.$$

*Further, if Assumption 2.3.1 holds, then there exists* $\sigma^2 > 0$ *such that*

$$\lambda^{\gamma/4}\left(\bar{N}([0,1]\backslash K; \lambda)\lambda^{-\gamma/2} - C\right) \overset{\lambda\to\infty}{\longrightarrow} N(0, \sigma^2),$$

*where* $N(0, \sigma^2)$ *denotes the normal distribution with expectation* $0$ *and variance* $\sigma^2$.

# Chapter 3

# C-M-J Branching Processes

Statistically self-similar Cantor sets can be encoded by random labelled trees $I$. Counting the individuals of $I$ according to some *characteristic* $\phi$ leads to C-M-J (Crump-Mode-Jagers) branching processes. With a suitable choice of $\phi$ it is possible to write the eigenvalue counting function as a C-M-J branching process.

The definition of these processes is given in Jagers [52], see also Crump and Mode [21, 22]. In Section 3.1, Section 3.2 and Section 3.3 we follow closely the structure of Charmoy, Croydon and Hambly [18, Chapter 2] and Hambly [42, Chapter 2 and 3].

## 3.1. Definition of C-M-J branching processes

For our investigation, define the address space as

$$T := \bigcup_{k=0}^{\infty} \mathbb{N}^k, \quad \mathbb{N}^0 := \emptyset. \tag{3.1}$$

Every $x \in T$ is a candidate for the random labelled tree (or population) $I$. $x \in T$ identifies a tuple $(\xi_x, L_x, \phi_x)$, defined on $(\Omega_x, \mathfrak{B}_x, \mathbb{P}_x)$, where $(\Omega_x, \mathfrak{B}_x, \mathbb{P}_x)$ are copies of some probability space $\left(\tilde{\Omega}, \tilde{\mathfrak{B}}, \tilde{\mathbb{P}}\right)$, consisting of the reproduction function $\xi_x$ which is a point process on $[0, \infty)$, the life length $L_x$ and a function $\phi_x$ on $\mathbb{R}$, where it is assumed that $\xi_x$, $x \in T$ are i.i.d. and that $\phi_x$ is càdlàg. We do not distinguish between the function $\xi_x$ and the indicated measure. Further, write $\xi_x(\infty)$ for $\xi_x([0, \infty))$. $(\xi_x, L_x, \phi_x)_x$ is defined on the product space $(\Omega, \mathfrak{B}, \mathbb{P})$ and is called *general branching process*. As explained in [76, Chapter 7], it is allowed that $\phi_x$ depends on the whole daughter process. In this case, $\phi_x$ is only defined on the product space. The random labelled tree $I$ is defined as follows.

**Definition 3.1.1 (Random Labelled Trees, c.f. [42, Chapter 2]):** *A random labelled tree $I$ is a subset of $T$ satisfying the following properties:*

- *It holds $\emptyset \in I$.*
- *If $(x_1, \dots, x_n) \in I$ then $(x_1, \dots, x_{n-1}) \in I$.*
- *For $(x_1, \dots, x_n) \in I$ it holds*

$$(x_1, \dots, x_{n+1}) \in I \quad \text{if and only if} \quad 1 \le x_{n+1} \le \xi_{(x_1,\dots,x_n)}(\infty).$$

Therefore, $I$ has a unique ancestor denoted by $\emptyset$. By $\sigma_x$, $x \in I$ denote the *birth time* of $x$ given by

$$\sigma_{(x_1,\dots,x_n)} = \sigma_{(x_1,\dots,x_{n-1})} + \inf\left\{t \in [0,\infty) : \ \xi_{(x_1,\dots,x_{n-1})}(t) \ge x_n\right\}$$

and assume that $\sigma_\emptyset = 0$. Therefore it holds

$$\xi_x = \sum_{i=1}^{\xi_x(\infty)} \delta_{\sigma_{x,i} - \sigma_x},$$

whereby $x, i$ denotes the $i$-th progeny of $x$.

We need some notation for individuals of $I$. For this notation, see also [35, Notation 2.2]. Define $\partial I$ as the boundary of $I$ that is the set of infinite paths through $I$ beginning at $\emptyset$. For $x = (x_1, \dots x_k) \in I$ we define the truncation of $x$ onto the first $n$ ancestors for $n \le k$ as $x_{|_n} := (x_1, \dots, x_n)$ and extend this definition onto $\partial I$ in the natural way. We use the relation $x \le y$ for $y \in I$ or $y \in \partial I$ if and only if ($|x| \le |y|$ and $y_{|_{|x|}} = x$). Hereby, $|y| = \infty$ for $y \in \partial I$. The composition of two individuals $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_n)$ is denoted by $xy := (x_1, \dots, x_k, y_1, \dots, y_n)$. Furthermore,

**Definition 3.1.2 (Cut-Set, c.f. [35, Notation 2.2]):** *A subset $C \subseteq I$ is called a cut set if and only if there exists for all $y \in \partial I$ exactly one $z \in C$ such that $z_{|_{|y|}} = y$.*

For $\alpha \in (0, \infty)$ define

$$\nu(dt) := \mathbb{E}\xi(dt), \quad \xi_\alpha(dt) := e^{-\alpha t}\xi(dt), \quad \nu_\alpha(dt) := \mathbb{E}\xi_\alpha(dt), \tag{3.2}$$

where we suppress the $x$ dependence if it does not cause confusion. We assume that the general branching process $(\xi_x, L_x, \phi_x)_x$ has a *Malthusian parameter*, i.e.

there exists a $\gamma \in (0, \infty)$ such that

$$\nu_\gamma(\infty) = \int_0^\infty e^{-\gamma t}\, \nu(dt) = 1.$$

By $\mu_k$ we denote the $k$-th moment

$$\mu_k := \int_0^\infty t^k\, \nu_\gamma(dt) \tag{3.3}$$

of $\nu_\gamma$.

The *C-M-J branching process* to given general branching process $(\xi_x, L_x, \phi_x)_x$ is defined as

$$Z^\phi(t) := \sum_{x \in I} \phi_x(t - \sigma_x).$$

Hereby, we assume that $\phi$ vanishes for negative times. In (3.10) it is explained how to obtain the asymptotics of $Z^\phi$ if this is not the case. The interpretation of $Z^\phi$ depends on the characteristic $\phi$. For $\phi(t) := \mathbb{1}_{\{t \geq 0\}}$ the corresponding C-M-J branching process

$$T_t := Z^\phi(t) = \sum_{x \in I} \mathbb{1}_{\{t \geq \sigma_x\}} \tag{3.4}$$

denotes the number of individuals born up to and including time $t$, c.f. Nerman [76].

## 3.2. Connection to statistically self-similar Cantor sets

Relations between C-M-J branching processes and statistically self-similar fractals were used e.g. in Hambly [42] and Charmoy, Croydon and Hambly [18]. For our investigation, we define to each $x \in T$ an iterated function system $(N_x, \Phi_{x,1}, \ldots, \Phi_{x,N_x})$ on some interval $[a, b]$ with $\Phi_{x,i}$ being a contraction similitude with random ratio $R_{x,i}$ and $N_x$ is a random natural number. We assume that $(N_x, \Phi_{x,1}, \ldots, \Phi_{x,N_x})$, $x \in T$ are i.i.d.

A random labelled tree $I \subseteq T$ is generated as in Definition 3.1.1 by the three properties

- It holds $\emptyset \in I$.
- If $(x_1, \dots, x_n) \in I$ then $(x_1, \dots, x_{n-1}) \in I$.
- For $(x_1, \dots, x_n) \in I$ it holds

$$(x_1, \dots, x_{n+1}) \in I \quad \text{if and only if} \quad 1 \leq x_{n+1} \leq N_{(x_1,\dots,x_n)}.$$

The *statistically self-similar Cantor set* $K$ is given by

$$K := \bigcap_{n=1}^{\infty} \bigcup_{\substack{|x|=n,\\ x\in I}} K_x, \qquad K_x := \Phi_{\emptyset,x_1} \circ \Phi_{(x_1),x_2} \circ \dots \circ \Phi_{(x_1,\dots,x_{n-1}),x_n}([a,b]).$$

This set satisfies

$$K = \bigcup_{i=1}^{N_\emptyset} \Phi_i(K_i),$$

with $(K_i)_{i=1,\dots,N_\emptyset}$ being i.i.d. and distributed like $K$. This property can be thought as a random version of self-similarity.

To obtain the Hausdorff dimension of $K$, the following condition is discussed in Falconer [28, Chapter 7].

**Condition 3.2.1 (c.f. [28, Chapter 7]):** *1. The collection* $(\mathit{int}\ K_x, x \in I)$ *forms a net, i.e.*

- *For* $x \leq y$ *it holds*

$$\mathit{int}\ K_x \subseteq \mathit{int}\ K_y.$$

- *If neither* $x \leq y$ *nor* $y \leq x$*, then*

$$\mathit{int}\ K_x \cap \mathit{int}\ K_y = \emptyset.$$

*This condition is a random analogue to the open set condition in the deterministic self-similar setting.*

2. *$K$ is proper, i.e. for every cut set $\mathcal{C} \subseteq I$ it holds that if $x \in \mathcal{C}$, there exists a point $y \in K_x$ such that for all $z \in \mathcal{C}$, $z \neq x$ it holds $y \notin K_z$.*

The Hausdorff dimension of $K$ were calculated by Falconer [28] and Mauldin and Williams [70].

**Theorem 3.2.2 (c.f. [28, Theorem 8.5 and 70, Theorem 3.6]):** *Let Condition 3.2.1 be satisfied. Then, on the event that $K$ is not empty, it holds that*

$$\dim_{\mathcal{H}} K = \inf\left\{ s \geq 0 : \mathbb{E}\left(\sum_{i=1}^{N_\emptyset} R_i^s\right) \leq 1 \right\} \quad a.s.$$

As explained in [18, Section 2.2], the reproduction function $\xi$ and life length $L$ of the general branching process which are related to $K$ are

$$\xi_x = \sum_{i=1}^{N_x} \delta_{-\log R_{x,i}}, \quad L_x = \sup_i \sigma_{x,i} - \sigma_x. \tag{3.5}$$

Hence, $K_x$ corresponds to an individual born at time $\sigma_x$ and $\mathrm{vol}_1(K_x) = e^{-\sigma_x}$. Further, by the definition of the Malthusian parameter $\gamma$, we infer

$$\mathbb{E}\left(\sum_{i=1}^{N_\emptyset} R_i^\gamma\right) = \int_0^\infty e^{-\gamma t}\, \nu(dt) = 1. \tag{3.6}$$

Thus, the Hausdorff dimension of $K$ coincides with the Malthusian parameter of $(\xi_x, L_x)_x$.

## 3.3. Strong Law of Large Numbers and Central Limit Theorem

By Jagers [52], the C-M-J branching process satisfies

$$Z^\phi(t) = \phi(t) + \sum_{i=1}^{\xi_\emptyset(\infty)} Z_i^\phi(t - \sigma_i), \tag{3.7}$$

whereby $\{Z_i^\phi\}_i$ are i.i.d. copies of $Z^\phi$. Multiplying this by $e^{-\gamma t}$, a random version of the renewal equation is obtained, since by taking the expectation

$$z^\phi(t) := \mathbb{E}e^{-\gamma t} Z^\phi(t), \qquad u^\phi(t) := \mathbb{E}e^{-\gamma t}\phi(t)$$

it follows

$$z^{\phi}(t) = u^{\phi}(t) + \int_0^{\infty} z^{\phi}(t-u)\,\nu_{\gamma}(du), \tag{3.8}$$

see [18], [42] and [76], whereby $\nu_\gamma$ is by definition a probability distribution, see Feller [29] for the renewal equation. Therefore, it is possible to obtain under certain regularity conditions the asymptotic behaviour of $z^\phi$. With the asymptotic behaviour of $z^\phi$ and the *fundamental martingale* $(W_t)_t$ corresponding to the general branching process, the asymptotics of $Z^\phi$ can be described. Thereby, the fundamental martingale is defined as

$$W_t \coloneqq \sum_{x \in A_t} e^{-\gamma \sigma_x}, \quad A_t \coloneqq \left\{ x \in I : \sigma_{x_{\mid_{|x|-1}}} \leq t < \sigma_x \right\},$$

see [18], [76]. The corresponding filtration $(\mathcal{F}_t)_t$ at time $t$ is given as the biography of the individuals born up to and including time $t$. To define this filtration formally, denote by $\boldsymbol{i}_n$ the $n$-th individual of the population, i.e. the individual of the population such that there are exactly $n-1$ individuals $x \in I$ such that their birth times satisfy

$$\sigma_x < \sigma_{\boldsymbol{i}_n}.$$

If there are several birth times simultaneously, sort the individuals by an arbitrary rule. Then, for $x \in T$ let $P_x$ be the projection map of the product space $(\Omega, \mathfrak{B})$ onto the $x$ component. As Nermann [76], define the $\sigma$-algebra $(E_n)_{n\in\mathbb{N}}$ as the smallest $\sigma$-algebra such that

$$\{\omega \in \Omega : \ \boldsymbol{i}_1(\omega) = z_1, ..., \boldsymbol{i}_n(\omega) = z_n\} \in E_n \quad \text{for all } z_1, ..., z_n \in T$$

and

$$A \cap \{\omega \in \Omega : \ x \in \{\boldsymbol{i}_1(\omega), ..., \boldsymbol{i}_n(\omega)\}\} \in E_n \quad \text{for all } A \in P_x^{-1}(\mathfrak{B}_x), \quad \text{for all } x \in T.$$

According to [76, Proposition 2.4], $(W_t)_t$ is a non-negative càdlàg $(\mathcal{F}_t)_t$-martingale, where

$$\mathcal{F}_t \coloneqq E_{T_t}, \tag{3.9}$$

with $T_t$ defined by (3.4) and further $\mathbb{E}W_t = 1$ for all $t$. Furthermore, for $x \in I$, we write $\mathcal{F}_x := E_{T_{\sigma_x}}$.

By Doob's martingale convergence theorem, $(W_t)_t$ converges almost surely to a random variable $W$. Moreover, as explained in [18, Theorem 2.1], Doney [23, 24] established the following theorem.

**Theorem 3.3.1 (c.f. [23, 24]):** *The following properties are equivalent:*

1. $\mathbb{E}\left(\xi_\gamma(\infty)(\log \xi_\gamma(\infty))_+\right) < \infty$
2. $\mathbb{E}W > 0$
3. $\mathbb{E}W = 1$
4. $W > 0$ *a.s. on the set where there is no extinction*
5. $(W_t)_t$ *is uniformly integrable*

*Otherwise,* $W = 0$ *a.s.*

Therefore, if the $x \log x$ property (typical in branching theory) of Theorem 3.3.1 holds, then the convergence of $(W_t)_t$ also takes place in $L_1$.

For the convergence theorems for $Z^\phi$ it is convenient to assume that $\phi$ vanishes for negative times. As explained in [18, Section 2.1], if this is not the case, we consider the C-M-J branching process $Z^{\chi^\phi}$ with

$$\chi_x^\phi(t) := \phi_x(t)\mathbb{1}_{t\geq 0} + \sum_{i=1}^{\xi_x(\infty)} Z_{x,i}^\phi(t - \sigma_{x,i})\mathbb{1}_{0\leq t<\sigma_i} \tag{3.10}$$

to get the asymptotic behaviour of $Z^\phi(t)$, as $t \to \infty$, since $Z^\phi \mathbb{1}_{[0,\infty)} = Z^{\chi^\phi}$.

The *Laws of Large Numbers* for C-M-J branching processes are originally from Nerman [76]. The versions we use are taken from Charmoy [17].

**Theorem 3.3.2 (Weak Law of Large Numbers, c.f. [17,76]):** *Let* $(\xi_x, L_x, \phi_x)_x$ *be a general branching process with Malthusian parameter* $\gamma$. *Assume that* $\nu_\gamma$ *is non-lattice,* $\phi \geq 0$ *and vanishes for negative times,* $u^\phi$ *is directly Riemann integrable and for all* $t$

$$\mathbb{E}\left(\sup_{u\leq t} \phi(u)\right) < \infty.$$

*Then, it holds*

$$z^{\phi}(t) \overset{t\to\infty}{\longrightarrow} z^{\phi}(\infty) := \mu_1^{-1} \int_0^{\infty} u^{\phi}(s)\, ds,$$

*where* $\mu_1$ *is defined in* (3.3), *and*

$$e^{-\gamma t} Z^{\phi}(t) \overset{t\to\infty}{\longrightarrow} z^{\phi}(\infty)\, W \quad \textit{in probability.}$$

The *Strong Law of Large Numbers* is given by [76] under the following condition on the general branching process.

**Condition 3.3.3 (c.f. [76, Condition 5.1 and Condition 5.2]):** *There exist non-increasing bounded positive integrable càdlàg functions* $g$ *and* $h$ *on* $[0, \infty)$ *such that*

$$\mathbb{E}\left(\sup_{t\geq 0} \frac{\xi_\gamma(\infty) - \xi_\gamma(t)}{g(t)}\right) < \infty, \qquad \mathbb{E}\left(\sup_{t\geq 0} \frac{e^{-\gamma t}\phi(t)}{h(t)}\right) < \infty.$$

As explained in [18, page 7], by choosing $g(t) := t^{-2} \wedge 1$ we thus obtain

$$\frac{\xi_\gamma(\infty) - \xi_\gamma(t)}{g(t)} \leq \int_t^{\infty} \frac{1}{g(s)}\, \xi_\gamma(ds) \leq \int_0^{\infty} \frac{1}{g(s)}\, \xi_\gamma(ds)$$

and assuming that the expected number of offsprings is finite, it thus follows

$$\mathbb{E}\left(\sup_{t\geq 0} \frac{\xi_\gamma(\infty) - \xi_\gamma(t)}{g(t)}\right) \leq \sup_{t\geq 0} \left\{(1 \vee t^2)e^{-\gamma t}\right\} \mathbb{E}\xi(\infty) < \infty.$$

The lattice case of the following Strong Law of Large numbers were treated in Gatzouras [38].

**Theorem 3.3.4 (Strong Law of Large Numbers, c.f. [17, 38, 76]):** *Let* $(\xi_x, L_x, \phi_x)_x$ *be a general branching process with Malthusian parameter* $\gamma$. *Assume that* $\phi \geq 0$ *and vanishes for negative times and that Condition 3.3.3 is satisfied. Then, it holds*

1. *If* $\nu_\gamma$ *is non-lattice then it holds*

$$z^{\phi}(t) \overset{t\to\infty}{\longrightarrow} z^{\phi}(\infty) = \mu_1^{-1} \int_0^{\infty} u^{\phi}(s)\, ds,$$

*where* $\mu_1$ *is defined in* (3.3)*, and*

$$e^{-\gamma t} Z^{\phi}(t) \overset{t\to\infty}{\longrightarrow} z^{\phi}(\infty)\, W \quad a.s.$$

2. *If* $\nu_\gamma$ *is lattice with period* $L$ *then there exists a* $L$*-periodic deterministic function* $G^{\phi}_{\gamma}$ *such that, as* $t \to \infty$*,*

$$Z^{\phi}(t) = (G^{\phi}_{\gamma}(t) + o(1))\, W\, e^{-\gamma t} \quad a.s.$$

*and* $G^{\phi}_{\gamma}$ *is defined as*

$$G^{\phi}_{\gamma}(t) \coloneqq L \cdot \frac{\sum_{j=-\infty}^{\infty} e^{-\gamma(t+jL)}\, \mathbb{E}(u^{\phi}(t+jL))}{\int_0^{\infty} s\, e^{-\gamma s}\, \nu(ds)}.$$

*Furthermore, if* $(W_t)_t$ *is uniformly integrable, the convergence is also in* $L_1$*.*

To justify the name of the last theorem, we consider an i.i.d. sequence $\{X_i\}_{i\in\mathbb{N}}$. Under certain regularity conditions, the classical Strong Law of Large Numbers reads

$$\frac{1}{n}\sum_{i=1}^{n} X_i \overset{n\to\infty}{\longrightarrow} \mathbb{E}X_1 \quad a.s.$$

With

$$Z^{\phi}(t) = \sum_{x\in I} \phi_x(t-\sigma_x) = \sum_{n=1}^{\infty} \sum_{\substack{x\in I,\\ |x|=n}} \phi_x(t-\sigma_x)$$

we obtain an interpretation of $Z^{\phi}$ as a sum $\sum_n Y_n(t)$ to which the right normalization factor is $e^{-\gamma t}$.

The following Central Limit Theorem is taken from [18]. Therefore, we consider a general branching process $(\xi_x, L_x, \bar{\zeta}_x)_x$, where $\bar{\zeta}$ is a characteristic such that the C-M-J branching process

$$\bar{Z}(t) \coloneqq Z^{\bar{\zeta}}(t)$$

is centered, i.e. $\mathbb{E}\bar{Z}(t) = 0$ for all $t$. The variance process $\bar{Z}^2$ has an important role in the Central Limit Theorem. By defining

$$q_\emptyset(t) := \bar{\zeta}_\emptyset(t)^2 + 2\,\bar{\zeta}_\emptyset(t) \sum_{i=1}^{\xi_\emptyset(\infty)} \bar{Z}_i(t-\sigma_i) + 2 \sum_{i=1}^{\xi_\emptyset(\infty)} \sum_{j<i} \bar{Z}_i(t-\sigma_i)\bar{Z}_j(t-\sigma_j),$$

it holds

$$\bar{Z}(t)^2 = Z^q(t). \tag{3.11}$$

Hence, $\bar{Z}^2$ can also be represented as a C-M-J branching process. Therefore, it also satisfies an equation of the form (3.7) and thus the renewal equation (3.8), i.e. by defining

$$v^{\bar{\zeta}}(t) := v(t) := e^{-\gamma t}\mathbb{E}\bar{Z}(t)^2, \quad r^{\bar{\zeta}}(t) := r(t) := e^{-\gamma t}\mathbb{E}q(t), \tag{3.12}$$

it holds

$$v(t) = r(t) + \int_0^\infty v(t-s)\,\nu_\gamma(ds). \tag{3.13}$$

Charmoy, Croydon and Hambly [18] assume for the Central Limit Theorem that $v$ converges to some finite constant.

Under the following two conditions, Charmoy, Croydon and Hambly [18] established a Central Limit Theorem for the C-M-J branching processes.

**Condition 3.3.5 (c.f. [18, Condition 2.6]):** *There exists $\epsilon \in (0, 1/2)$ such that*

$$e^{-\gamma t/2} \sum_{\sigma_x \le \epsilon t} \bar{\zeta}_x(t-\sigma_x) \xrightarrow{t\to\infty} 0, \quad \textit{in probability.}$$

As explained in [18, Condition 2.6], this condition can often be checked by using Nerman's Weak Law of Large Numbers Theorem 3.3.2.

**Condition 3.3.6 (c.f. [18, Condition 2.7]):** *There exists $\kappa \in (0, \infty)$ such that*

$$\sup_{t\in\mathbb{R}} \mathbb{E}\left(\left|e^{-\gamma t/2}\bar{Z}(t)\right|^{2+\kappa}\right) < \infty.$$

As explained in [18, Condition 2.7], it is convenient to show this condition for $\kappa = 1$ which can be done by writing $\bar{Z}^3$ as a C-M-J branching process.

**Theorem 3.3.7 (Central Limit Theorem, c.f. [18, Theorem 2.8]):** *Let $(\xi_x, L_x, \bar{\zeta}_x)_x$ be a general branching process with Malthusian parameter $\gamma$ such that $\mathbb{E}\bar{Z}(t) = 0$ for all $t$ and that $\nu_\gamma$ defined in (3.2) is non-lattice. Assume that Condition 3.3.5 and Condition 3.3.6 are satisfied and further that $v$ is bounded and*

$$v(t) \overset{t\to\infty}{\longrightarrow} v(\infty),$$

*where $v(\infty)$ is some finite constant. Then, it holds*

$$e^{-\gamma t/2}\bar{Z}(t) \overset{t\to\infty}{\longrightarrow} Z_\infty, \quad \textit{in distribution},$$

*where the distribution of $Z_\infty$ is characterised by*

$$\mathbb{E}\left(e^{i\theta Z_\infty}\right) = \mathbb{E}\left(e^{-\frac{1}{2}\theta^2 v(\infty) W}\right). \tag{3.14}$$

Again, to justify the name of this theorem, we consider an i.i.d. sequence $\{X_i\}_{i\in\mathbb{N}}$ with $\mathbb{E}X_1 = 0$. Under certain regularity conditions, the classical Central Limit Theorem reads

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \longrightarrow N(0, \mathrm{Var}X_1).$$

If $W = 1$ a.s., equation (3.14) characterizes the normal distribution with mean 0 and variance $v(\infty)$. Therefore, the distribution of $Z_\infty$ can be thought as a generalized normal distribution with random variance $v(\infty)W$. Further, $e^{-\gamma t/2}$ is given as the square root of the normalization factor of $\bar{Z}$ as $\frac{1}{\sqrt{n}}$ is of $\sum_{i=1}^{n} X_i$ in the classical Central Limit Theorem.

## 3.4. Law of the Iterated Logarithm

In this section, we proof the first main result (Theorem 3.4.3) of this thesis.

The Central Limit Theorem describes the random fluctuation of the normalized C-M-J branching process around its limit in distribution. However, we also want to have some almost sure properties. Therefore, we establish a Law of the Iterated Logarithm. For the Biggins martingale of supercritical branching random walks the Law of the Iterated Logarithm is treated by Iksanov and Kabluchko [49]. For our Law of the Iterated Logarithm, we need two technical assumptions on the general branching process which are related to Condition 3.3.5 and Condition 3.3.6.

**Condition 3.4.1:** *There exists* $\epsilon \in (0, 1/2)$ *and* $C(\omega) > 0$ *such that*

$$e^{-\gamma t/2} \sum_{\sigma_x \leq \epsilon t} \bar{\zeta}_x(t - \sigma_x) \leq C(\omega) \quad a.s.$$

**Condition 3.4.2:** *It holds that*

$$\sup_{t \in \mathbb{R}} \mathbb{E}\left( \left| e^{-\gamma t/2} \bar{Z}(t) \right|^3 \right) < \infty.$$

The Idea of the proof of the following theorem is taken from the proof of [18, Theorem 2.8] and [49, Theorem 1.6].

**Theorem 3.4.3 (Law of the Iterated Logarithm):** *Let* $(\xi_x, L_x, \bar{\zeta}_x)_x$ *be a general branching process with Malthusian parameter* $\gamma > 0$ *such that* $\bar{\zeta}$ *satisfies* $\mathbb{E}\bar{Z}(t) = 0$ *for all* $t$ *and* $\nu_\gamma$ *be non-lattice. Assume that*

$$v(t) \overset{t \to \infty}{\longrightarrow} v(\infty),$$

*where* $v(\infty) > 0$ *and further, assume that Condition 3.4.1 and Condition 3.4.2 hold and that*

$$\mathbb{E}\xi(\infty) \log \xi(\infty) < \infty.$$

*Then, for fixed* $h > 0$*,*

$$\limsup_{n \to \infty} \frac{e^{-\gamma h n/2} \bar{Z}(hn)}{\sqrt{2\, v(\infty) W \log hn}} \leq 1 \quad a.s.$$

*Proof.* We want to apply Theorem A.2.1 by using the Berry-Esseen estimate Theorem A.2.2. Therefore, define $\tilde{\mathrm{Z}}(t) := e^{-\gamma t/2} \bar{Z}(t)$ and $\tilde{\zeta}(t) := e^{-\gamma t/2}\, \bar{\zeta}(t)$. By (3.7) it follows

$$\tilde{\mathrm{Z}}(t) = \sum_{\sigma_x \leq \epsilon t} e^{-\sigma_x \gamma/2}\, \tilde{\zeta}(t - \sigma_x) + \sum_{x \in A_{\epsilon t}} e^{-\sigma_x \gamma/2}\, \tilde{\mathrm{Z}}_x(t - \sigma_x).$$

Let $\epsilon \in (0, 1/2)$ such that Condition 3.4.1 is satisfied. Using boundedness of the first term, it only remains to show that

$$\limsup_{n \to \infty} \frac{\sum_{x \in A_{\epsilon h n}} e^{-\sigma_x \gamma/2}\, \tilde{\mathrm{Z}}_x(hn - \sigma_x)}{\sqrt{2\, v(\infty) W \log hn}} \leq 1 \quad a.s.$$

Therefore, let $h = 1$. With analogous arguments the claim follows for arbitrary fixed $h > 0$.

$$T_n := \frac{\sum_{x\in A_{\epsilon n}} e^{-\sigma_x\gamma/2}\,\tilde{Z}_x(n-\sigma_x)}{\sqrt{\mathbb{E}\left(\sum_{x\in A_{\epsilon n}} e^{-\sigma_x\gamma}\,\tilde{Z}_x(n-\sigma_x)^2\big|\mathcal{G}_n\right)}}, \qquad \mathcal{G}_n := \mathcal{F}_{\epsilon n},$$

where $\mathcal{F}_{\epsilon n}$ is defined in (3.9). Theorem A.2.2 leads to

$$\sup_{y\in\mathbb{R}} |\mathbb{P}(T_n \le y|\mathcal{G}_n) - \Phi(y)| \le c_1 \frac{\sum_{x\in A_{\epsilon n}} \mathbb{E}\left(\left|e^{-\sigma_x\gamma/2}\,\tilde{Z}_x(n-\sigma_x)\right|^3\middle|\mathcal{G}_n\right)}{\left(\mathbb{E}\left(\sum_{x\in A_{\epsilon n}} e^{-\sigma_x\gamma}\,\tilde{Z}_x(n-\sigma_x)^2\big|\mathcal{G}_n\right)\right)^{3/2}}.$$

We now verify

$$\sum_{n=0}^{\infty} \frac{\sum_{x\in A_{\epsilon n}} \mathbb{E}\left(\left|e^{-\sigma_x\gamma/2}\,\tilde{Z}_x(n-\sigma_x)\right|^3\middle|\mathcal{G}_n\right)}{\left(\mathbb{E}\left(\sum_{x\in A_{\epsilon_n}} e^{-\sigma_x\gamma}\,\tilde{Z}_x(n-\sigma_x)^2\big|\mathcal{G}_n\right)\right)^{3/2}} < \infty$$

by using the following Lemma. Remark that by the $x\log x$ property of Theorem 3.3.1 it holds $W > 0$ a.s.

**Lemma 3.4.4:** *It holds that*

$$\mathbb{E}\left(\sum_{x\in A_{\epsilon t}} e^{-\sigma_x\gamma}\,\tilde{Z}_x(t-\sigma_x)^2\middle|\mathcal{G}_t\right) \overset{t\to\infty}{\longrightarrow} v(\infty)W \quad a.s.$$

Thus, it is enough to show that

$$\sum_{n=0}^{\infty}\sum_{x\in A_{\epsilon n}} \mathbb{E}\left(\left|e^{-\sigma_x\gamma/2}\,\tilde{Z}_x(n-\sigma_x)\right|^3\middle|\mathcal{G}_n\right) < \infty \quad a.s.$$

Since by Condition 3.4.2 $\sup_{t\in\mathbb{R}}\mathbb{E}|\,\tilde{Z}(t)|^3 < \infty$, it holds

$$\sum_{x\in A_{\epsilon n}} \mathbb{E}\left(\left|e^{-\sigma_x\gamma/2}\,\tilde{Z}_x(n-\sigma_x)\right|^3\middle|\mathcal{G}_n\right) \le c_2 \sum_{x\in A_{\epsilon n}} e^{-\frac{3}{2}\gamma\sigma_x} \le c_2 W_{\epsilon n} e^{-\frac{1}{2}\epsilon n\gamma}.$$

Since $W_n \overset{n\to\infty}{\longrightarrow} W$, $(W_n)_n$ is bounded and thus there exists a constant $d_1$ (depending on $\omega$) such that

$$\sum_{n=0}^{\infty}\sum_{x\in A_{\epsilon n}} \mathbb{E}\left(\left|e^{-\sigma_x\gamma/2}\,\tilde{Z}_x(n-\sigma_x)\right|^3\middle|\mathcal{G}_n\right) \le d_1(\omega)\sum_{n=0}^{\infty} e^{-\frac{1}{2}\epsilon n} < \infty \quad a.s.$$

Hence, the claim follows. It remains to show Lemma 3.4.4.

*Proof of Lemma 3.4.4.* Let $t_0 > 0$ be such that

$$|v(t/2) - v(\infty)| \leq \delta, \quad \text{for all } t \geq t_0.$$

Then, it follows

$$\left| \mathbb{E}\left( \sum_{x \in A_{\epsilon t}} e^{-\sigma_x \gamma} \tilde{Z}_x(t-\sigma_x)^2 - v(\infty) e^{-\gamma\sigma_x} \middle| \mathcal{G}_t \right) \right|$$
$$\leq \delta \sum_{x \in A_{\epsilon t} \setminus A_{t/2}} e^{-\gamma\sigma_x} + 2\|v\|_\infty \sum_{x \in A_{\epsilon t} \cap A_{t/2}} e^{-\gamma\sigma_x}$$

By [76, Corollary 5.9], there exists $K > 0$ for all $c > 0$ such that

$$\limsup_{t\to\infty} \sum_{x \in A_{\epsilon t} \cap A_{\epsilon t + c}} e^{-\gamma\sigma_x} \leq K \int_{c-K}^{\infty} g(t)\, dt\, W,$$

where $g(t) = t^{-2} \wedge 1$ and we can choose $c$ arbitrarily large. This ensures that $\sum_{x \in A_{\epsilon t} \cap A_{t/2}} e^{-\gamma\sigma_x}$ converges to zero almost surely and by adjusting $\delta$, the claim follows since $\sum_{x \in A_{\epsilon t} \setminus A_{t/2}} e^{-\gamma\sigma_x}$ is bounded by $W_{\epsilon t}$ which converges almost surely and is therefore bounded. □

This proofs Lemma 3.4.4 and hence Theorem 3.4.3 follows. □

Since $(\xi_x, L_x, -\bar{\zeta}_x)_x$ also satisfies the conditions of Theorem 3.4.3 if $(\xi_x, L_x, \bar{\zeta}_x)_x$ does, we obtain

**Corollary 3.4.5:** *Let $(\xi_x, L_x, \bar{\zeta}_x)_x$ be a general branching process with Malthusian parameter $\gamma > 0$ such that $\bar{\zeta}$ satisfies $\mathbb{E}\bar{Z}(t) = 0$ for all $t$ and that $\nu_\gamma$ is non-lattice. Assume that*

$$v(t) \xrightarrow{t\to\infty} v(\infty),$$

*where $v(\infty) > 0$ and further, assume that Condition 3.4.1 and Condition 3.4.2 hold and that*

$$\mathbb{E}\xi(\infty) \log \xi(\infty) < \infty.$$

*Then, for fixed $h > 0$, it holds*

$$-1 \leq \liminf_{n\to\infty} \frac{e^{-\gamma hn/2}\bar{Z}(hn)}{\sqrt{2\,v(\infty)W\log hn}} \leq \limsup_{n\to\infty} \frac{e^{-\gamma hn/2}\bar{Z}(hn)}{\sqrt{2\,v(\infty)W\log hn}} \leq 1 \quad a.s. \tag{3.15}$$

As for the Strong Law of Large Numbers and the Central Limit Theorem, we justify the name of this theorem by considering an i.i.d. sequence $\{X_i\}_{i\in\mathbb{N}}$. Under some regularity conditions and if the $X_i$ are centred, the classical Law of the Iterated Logarithm reads

$$-1 = \liminf_{n\to\infty} \frac{\sqrt{\frac{1}{n}}\sum_{i=1}^{n} X_i}{\sqrt{2\,\mathrm{Var}X_1\ \log\log n}} \leq \limsup_{n\to\infty} \frac{\sqrt{\frac{1}{n}}\sum_{i=1}^{n} X_i}{\sqrt{2\,\mathrm{Var}X_1\ \log\log n}} = 1.$$

In equation (3.15) $\mathrm{Var}X_1$ is replaced by $v(\infty)W$ which is the generalized variance of $Z_\infty$ in the Central Limit Theorem 2.3.2. Further, $e^{-\gamma t/2}$ is the square root of the normalization factor of $\bar{Z}$ and

$$\liminf_{n\to\infty} \setminus \limsup_{n\to\infty} \frac{e^{-\gamma n/2}\bar{Z}(n)}{\sqrt{2\,v(\infty)W\log n}} = \liminf_{n\to\infty} \setminus \limsup_{n\to\infty} \frac{e^{-\gamma n/2}\bar{Z}(n)}{\sqrt{2\,v(\infty)W\log\log e^{\gamma n}}}.$$

## 3.5. Some properties of W

For our applications of the Central Limit Theorem and the Law of the Iterated Logarithm, we need to control the moments of the almost sure limit of $Z^\phi(t)e^{-\gamma t}$. In this section we outline these properties.

Firstly, we investigate the equation (3.7). Multiplying with $e^{-\gamma t}$ and taking the limit $t \to \infty$, it follows by the Strong Law of Large Numbers Theorem 3.3.4 with $\phi(t) = 1_{\{t\geq 0\}}$

$$W = \sum_{i=1}^{\xi_\emptyset(\infty)} W_i e^{-\sigma_i \gamma} \quad a.s., \tag{3.16}$$

where we assume that $\mathbb{E}\xi(\infty) < \infty$ and $W_i$ is the limit of the fundamental martingale corresponding to the general branching process $(\xi_x, L_x, \phi_x)_{x\in\theta_i I}$, c.f. Hambly [42]. Furthermore, $\{W_i\}_i$ are i.i.d. distributed like $W$ and independent of $\xi_\emptyset$. For the moment bounds, we need the following lemma. The case $W = 1$ a.s. of

Lemma 3.5.1 were treated in [18, Lemma 3.5]. Therefore, let

$$\psi(\theta) := \mathbb{E} \sum_{i=1}^{\xi_\emptyset(\infty)} e^{-\theta\gamma\sigma_i}.$$

**Lemma 3.5.1 (c.f. [18, Lemma 3.5]):** *Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process. Then, it holds*

$$\mathbb{E} \sum_{x \in I} e^{-\theta\gamma\sigma_x} = \sum_{k=0}^{\infty} \psi(\theta)^k$$

*and the sum is finite for $\theta \in (1, \infty)$.*

*Proof.* Monotone convergence yields

$$\mathbb{E} \sum_{x \in I} e^{-\theta\gamma\sigma_x} = \sum_{k=0}^{\infty} \mathbb{E} \sum_{\substack{x \in I, \\ |x|=k}} e^{-\theta\gamma\sigma_x}$$

and therefore

$$\begin{aligned}
\mathbb{E} \sum_{\substack{x \in I, \\ |x|=k}} e^{-\theta\gamma\sigma_x} &= \mathbb{E} \sum_{\substack{x \in I, \\ |x|=k-1}} \sum_{i=1}^{\xi_x(\infty)} e^{-\theta\gamma\sigma_x} e^{-\theta\gamma(\sigma_{x,i}-\sigma_x)} \\
&= \mathbb{E} \sum_{\substack{x \in I, \\ |x|=k-1}} e^{-\theta\gamma\sigma_x} \mathbb{E}\left( \sum_{i=1}^{\xi_x(\infty)} e^{-\theta\gamma(\sigma_{x,i}-\sigma_x)} \middle| \mathcal{F}_{x|_{|x|-1}} \right) \\
&= \mathbb{E} \sum_{\substack{x \in I, \\ |x|=k-1}} e^{-\theta\gamma\sigma_x} \mathbb{E} \sum_{i=1}^{\xi_x(\infty)} e^{-\theta\gamma \inf\{t>0:\ \xi_x(t) \geq i\}} \\
&= \psi(\theta) \mathbb{E} \sum_{\substack{x \in I, \\ |x|=k-1}} e^{-\theta\gamma\sigma_x}.
\end{aligned}$$

Iterating over $k$, we get the desired equality and because $\psi(1) = 1$ and $\psi$ is strictly decreasing in $\theta$, the claim follows. □

To control the moments of C-M-J branching processes and the limit of the fundamental martingale $(W_t)_t$, we proof the following lemma. Similar results were also established by Mauldin and Williams [70].

**Lemma 3.5.2:** *Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process. Assume that $\phi \geq 0$ and vanishes for negative times and*

$$\phi(t) \leq c_1 \xi(\infty).$$

*Furthermore, assume that*

$$\mathbb{E}\xi(\infty)^6 < \infty$$

*and that Condition 3.3.3 is satisfied and $\nu_\gamma$ is non-lattice. Then, it holds*

$$\mathbb{E}Z^\phi(t)^6 \leq d_1 e^{6\gamma t} \quad \text{and} \quad \mathbb{E}W^6 < \infty.$$

*Proof.* By Fatou's lemma follows

$$\mathbb{E}W^6 \leq d_2 \liminf_{t\to\infty} \mathbb{E}(Z^\phi(t)e^{-\gamma t})^6.$$

Therefore, it is sufficient to show that $\mathbb{E}(Z^\phi(t)e^{-\gamma t})^6$ is bounded. We start by showing that $\mathbb{E}(Z^\phi(t)e^{-\gamma t})^2 < \infty$ and proceed iteratively. Using the idea of [18, Lemma 3.6], we show that the expectation of

$$e^{-2\gamma t}\Bigg(\sum_{x\in I}\phi_x(t-\sigma_x)^2 \tag{3.17}$$

$$+ 2\sum_{x\in I}\phi_x(t-\sigma_x)\sum_{i=1}^{\xi_x(\infty)} Z^\phi_{x,i}(t-\sigma_{x,i}) \tag{3.18}$$

$$+ 2\sum_{x\in I}\sum_{i=1}^{\xi_x(\infty)}\sum_{j<i} Z^\phi_{x,i}(t-\sigma_{x,i})\, Z^\phi_{x,j}(t-\sigma_{x,j})\Bigg) \tag{3.19}$$

is bounded, where we use (3.11). To deal with (3.17), we use $\phi(t) \leq c_1\xi(\infty)e^{\gamma t}$ to get

$$e^{-2\gamma t}\mathbb{E}\sum_{x\in I}\phi_x(t-\sigma_x)^2 \leq c_1^2\,\mathbb{E}\sum_{x\in I}\xi(\infty)^2 e^{-2\sigma_x\gamma} = c_1^2\,\mathbb{E}\xi(\infty)^2\sum_{k=0}^{\infty}\psi(2)^k < \infty.$$

For (3.18) and (3.19) we use the same strategy and remark that

$$\mathbb{E}Z^\phi(t) \leq c_2 e^{\gamma t}.$$

Thus, in (3.18) we get

$$
\begin{aligned}
&e^{-2\gamma t}\mathbb{E}\sum_{x\in I}\phi(t-\sigma_x)\sum_{i=1}^{\xi_x(\infty)} Z^{\phi}_{x,i}(t-\sigma_{x,i})\\
&\quad\le e^{-2\gamma t}c_1\mathbb{E}\sum_{x\in I}\xi_x(\infty)e^{\gamma(t-\sigma_x)}\sum_{i=1}^{\xi_x(\infty)}\mathbb{E}\left(Z^{\phi}_{x,i}(t-\sigma_{x,i})\middle|\mathcal{F}_x\right)\\
&\quad\le e^{-2\gamma t}c_1c_2\,\mathbb{E}\sum_{x\in I}\xi_x(\infty)e^{\gamma(t-\sigma_x)}\sum_{i=1}^{\xi_x(\infty)}e^{\gamma(t-\sigma_{x,i})}\\
&\quad\le c_1c_2\,\mathbb{E}\sum_{x\in I}e^{-2\gamma\sigma_x}\xi_x(\infty)^2\\
&\quad= c_1c_2\,\mathbb{E}\sum_{x\in I}e^{-2\gamma\sigma_x}\mathbb{E}\xi_x(\infty)^2\\
&\quad= c_1c_2\,\mathbb{E}\xi(\infty)^2\,\mathbb{E}\sum_{x\in I}e^{-2\gamma\sigma_x}\\
&\quad<\infty,
\end{aligned}
$$

where we used Lemma 3.5.1 and that $\sigma_{x,i}\ge\sigma_x$. For (3.19), remark that

$$
\mathbb{E}\left(Z^{\phi}_{x,i}(t-\sigma_{x,i})Z^{\phi}_{x,j}(t-\sigma_{x,j})\middle|\mathcal{F}_x\right)=\mathbb{E}\left(Z^{\phi}_{x,i}(t-\sigma_{x,i})\middle|\mathcal{F}_x\right)\mathbb{E}\left(Z^{\phi}_{x,j}(t-\sigma_{x,j})\middle|\mathcal{F}_x\right)
$$

and thus

$$
\begin{aligned}
&e^{-2\gamma t}\mathbb{E}\sum_{x\in I}\sum_{i=1}^{\xi_x(\infty)}\sum_{j<i}Z^{\phi}_{x,i}(t-\sigma_{x,i})\,Z^{\phi}_{x,j}(t-\sigma_{x,j})\\
&\quad\le c_2^2\,e^{-2\gamma t}\,\mathbb{E}\sum_{x\in I}\sum_{i=1}^{\xi_x(\infty)}\sum_{j<i}e^{\gamma(t-\sigma_{x,i})}e^{\gamma(t-\sigma_{x,j})}\\
&\quad\le c_2^2\,\mathbb{E}\sum_{x\in I}e^{-2\sigma_x\gamma}\xi_x(\infty)^3\\
&\quad<\infty.
\end{aligned}
$$

For the third moment, we define

$$
\begin{aligned}
Q_\emptyset(t):=&\phi_\emptyset(t)^3+3\phi_\emptyset(t)^2\sum_{i=1}^{\xi_\emptyset(\infty)}Z^{\phi}_i(t-\sigma_i)+3\phi_\emptyset(t)\sum_{i,j=1}^{\xi_\emptyset(\infty)}Z^{\phi}_i(t-\sigma_i)Z^{\phi}_j(t-\sigma_j)\\
&+\sum_{\substack{i,j,k=1,\\ \text{not all equal}}}^{\xi_\emptyset(\infty)}Z^{\phi}_i(t-\sigma_i)Z^{\phi}_j(t-\sigma_j)Z^{\phi}_k(t-\sigma_k).
\end{aligned}
$$

Therefore,

$$Z^{\phi}(t)^3 = Z^Q(t)$$

and we can proceed as for the second moment. Therefore, we only show how to estimate the third and fourth sum. By the estimates before it follows $\mathbb{E}e^{-2\gamma t}Z^{\phi}(t)^2 \leq c_3$ and thus

$$\begin{aligned}
&e^{-3\gamma t}\mathbb{E}\sum_{x\in I}\phi_x(t-\sigma_x)\sum_{i,j=1}^{\xi_x(\infty)} Z^{\phi}_{x,i}(t-\sigma_{x,i})Z^{\phi}_{x,j}(t-\sigma_{x,j}) \\
&\leq c_1 e^{-3\gamma t}\mathbb{E}\Bigg(\sum_{x\in I}\xi_x(\infty)e^{\gamma(t-\sigma_x)} \\
&\qquad \sum_{i,j=1}^{\xi_x(\infty)} \mathbb{E}\left(Z^{\phi}_{x,i}(t-\sigma_{x,i})^2\middle|\mathcal{F}_x\right)^{1/2}\mathbb{E}\left(Z^{\phi}_{x,j}(t-\sigma_{x,j})\middle|\mathcal{F}_x\right)^{1/2}\Bigg) \\
&\leq c_1\, c_3\, \mathbb{E}\sum_{x\in I} e^{-3\gamma\sigma_x}\xi_x(\infty)^3 \\
&< \infty.
\end{aligned}$$

For the last sum, assume without loss of generality that $i$ is different. Then,

$$\begin{aligned}
&\mathbb{E}\Big(Z^{\phi}_{x,i}(t-\sigma_{x,i})Z^{\phi}_{x,j}(t-\sigma_{x,j})Z^{\phi}_{x,k}(t-\sigma_{x,k})\Big|\mathcal{F}_x\Big) \\
&\quad = \mathbb{E}\left(Z^{\phi}_{x,i}(t-\sigma_{x,i})\middle|\mathcal{F}_x\right)\mathbb{E}\left(Z^{\phi}_{x,j}(t-\sigma_{x,j})Z^{\phi}_{x,k}(t-\sigma_{x,k})\middle|\mathcal{F}_x\right)
\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\mathbb{E}\left(Z^{\phi}_{x,j}(t-\sigma_{x,j})Z^{\phi}_{x,k}(t-\sigma_{x,k})\middle|\mathcal{F}_x\right) \\
&\quad \leq \mathbb{E}\left(Z^{\phi}_{x,j}(t-\sigma_{x,j})^2\middle|\mathcal{F}_x\right)^{1/2}\mathbb{E}\left(Z^{\phi}_{x,k}(t-\sigma_{x,k})^2\middle|\mathcal{F}_x\right)^{1/2}.
\end{aligned} \tag{3.20}$$

Afterwards, we can proceed iteratively to obtain the claim. □

# Chapter 4

# Spectral Asymptotics for Measure Theoretical Laplacians

In this chapter we study Weyl's law (1.4) for measure theoretical Laplacians $\Delta^\mu$ with respect to statistically self-similar and random $V$-variable Cantor measures. It may occur that the normalized eigenvalue counting function oscillates or is periodic rather than convergent. Therefore, we first give a generalization of the leading order in the Weyl asymptotics for $\Delta^\mu$. Afterwards, we investigate this generalization for $\Delta^\mu$ for the different Cantor measures.

The main results of this chapter are provided in Theorem 4.2.9 and Theorem 4.3.18. These theorems are results of the present thesis.

## 4.1. Spectral exponent

Equation (1.4) motivates the definition of the spectral dimension

$$\frac{d_s(X)}{2} := \lim_{\lambda\to\infty} \frac{\log N(\lambda)}{\log \lambda} \tag{4.1}$$

which leads to

$$d_s(X) = n$$

in (1.4). Many authors before studied the expression (4.1) for generalized Laplacians on fractals, e.g. [35, 42]. In this section, we investigate this expression for measure theoretical Laplacians $\Delta^\mu$ with respect to certain measures $\mu$. Therefore, we call the limit

$$\gamma := \gamma(\mu) := \lim_{\lambda\to\infty} \frac{\log N^\mu(\lambda)}{\log \lambda}$$

the *spectral exponent* of the corresponding measure theoretical Laplacian, c.f. Freiberg [34].

Hence, by Theorem 2.2.1, the spectral exponent of $\Delta^\mu$ with respect to self-similar Cantor measures $\mu$ is given as the unique solution $\gamma_s > 0$ of

$$\sum_{i=1}^{N} (r_i m_i)^{\gamma_s} = 1,$$

where we use the same notation as in Section 2.2.1.

By Theorem 2.2.2, the spectral exponent for random homogeneous Cantor measures is almost surely given by the unique solution $\gamma_h > 0$ of

$$\prod_{j\in J} \left( \sum_{i=1}^{N_j} \left( r_i^{(j)} m_i^{(j)} \right)^{\gamma_h} \right)^{p_j} = 1.$$

Hereby, we use the same notation as in Section 2.2.2.

## 4.2. Spectral exponent for statistically self-similar Cantor measures

### 4.2.1. Construction

The idea of the proofs, constructions and structure of Section 4.2 are taken from Arzt [4, Chapter 3] and Hambly [42]. Let $J$ be an index set. We define to each $j \in J$ an IFS $\mathcal{S}^{(j)}$. Therefore, let $N_j \in \mathbb{N}$, $N_j \geq 2$. Then $\mathcal{S}^{(j)} = \left( S_1^{(j)}, ..., S_{N_j}^{(j)} \right)$, where we define $S_i^{(j)} : [a, b] \longrightarrow [a, b]$ by

$$S_i^{(j)}(x) := r_i^{(j)} x + c_i^{(j)},$$

for some $r_i^{(j)} \in (0, 1)$, $c_i^{(j)} \in \mathbb{R}$, $i = 1, ..., N_j$ such that

$$a = S_1^{(j)}(a) < S_1^{(j)}(b) \leq S_2^{(j)}(a) < S_2^{(j)}(b) \leq \cdots \leq S_{N_j}^{(j)}(a) < S_{N_j}^{(j)}(b) = b.$$

Furthermore, let $m^{(j)} = \left( m_1^{(j)}, ..., m_{N_j}^{(j)} \right)$ be a vector of weights. Therefore, as in Section 2.2.2 an element of the index set $J$ identifies a tuple $\left( \mathcal{S}^{(j)}, m^{(j)} \right)$.

As in Chapter 3, we construct a (for now deterministic) tree $I$ with unique ancestor $\emptyset$. Every individual $x \in I$ indicates a index $j \in J$ which we also denote by $x$ for convenience. The number of children of $x$ is $N_x$ and by $I_n$ we denote the $n$-th generation of $I$, where $I_0 = \{\emptyset\}$. For $x \in I_n$, $x = (x_1, ..., x_n)$, we define

$$\begin{aligned} m_x &:= m_{x_1}^{(\emptyset)} \cdots m_{x_n}^{((x_1,\dots,x_{n-1}))}, \\ S_x &:= S_{x_1}^{(\emptyset)} \circ \dots \circ S_{x_n}^{((x_1,\dots,x_{n-1}))} \end{aligned}$$

and analogously $S_x^{-1}$ as the composition of the preimages of the contraction similitudes. For $n \in \mathbb{N}$ let

$$K_n^{(I)} := \bigcup_{x \in I_n} S_x([a,b]) = \bigcup_{\substack{x \in I, \\ |x|=n}} K_x, \quad K_x := S_x([a,b]).$$

The limiting set

$$K^{(I)} := \bigcap_{n=1}^{\infty} K_n^{(I)}$$

is called *recursive Cantor set.*

**Proposition 4.2.1:** *The set $K^{(I)}$ is compact and contains at least countably infinitely many elements, namely $S_{(x_1,\dots,x_n)}(a)$ and $S_{(x_1,\dots,x_n)}(b)$, $x_1 = 1, ..., N_\emptyset, ..., x_n = 1, ..., N_{(x_1,\dots,x_{n-1})}$, $n \in \mathbb{N}$.*

*Proof.* Let $x = (x_1, ..., x_n) \in I_n$. For $m \in \mathbb{N}$ let $x'$ and $x''$ be two individuals of the population such that $x' = x\mathbf{1}_m$, $\mathbf{1}_m := (1, ..., 1) \in \mathbb{R}^m$, $m \in \mathbb{N}$ and $x_1'', ..., x_n'' = x_1, ..., x_n$, $x_k'' = N_{\left(x_1,\dots,x_{k-1},N_{(x_1,\dots,x_{k-1})}\right)}$ for $k = n+1, .., n+m$. By definition

$$\begin{aligned} S_{x'}(a) &= S_x(a), \\ S_{x''}(b) &= S_x(b). \end{aligned}$$

Thus $S_x(a)$, $S_x(b) \in K_{n+m}^{(I)}$ for all $m \in \mathbb{N}$, which proofs the statement. □

By construction, it holds that

$$K^{(I)} = \bigcup_{i=1}^{N_\emptyset} S_i^{(\emptyset)} \left(K^{(\theta_i I)}\right), \tag{4.2}$$

where $\theta_i I$ denotes the subtree of $I$, rooted at $(i)$.

We define the recursive Cantor measures analogously to the homogeneous Cantor measures (2.6). Let

$$\mu_n^{(I)}(A) := \sum_{x \in I_n} m_x \, \mu_0 \left( S_x^{-1}(A) \right), \quad \mu_0(A) := \frac{1}{b-a} \lambda^1_{|[a,b]}(A)$$

for all $A \in \mathcal{B}([a,b])$, where $\mathcal{B}$ denotes the Borel-$\sigma$-algebra. The recursive Cantor measure $\mu^{(I)}$ for a Cantor set coded by $I$ is defined as the weak limit of $\left(\mu_n^{(I)}\right)_{n \in \mathbb{N}_0}$.

**Lemma 4.2.2:** *For all $x \in I$ holds*

$$\mu^{(I)}(S_x([a,b])) = m_x.$$

*Proof.* Let $x \in I_n$, $y \in I_{n+m}$, $n, m \in \mathbb{N}$. Because of

$$K_x \cap K_y = \begin{cases} K_y, & \text{if } y_{|n} = x \\ \emptyset, & \text{otherwise,} \end{cases}$$

we get

$$\begin{aligned} &\mu_{n+m}^{(I)}(K_x) \\ =& \sum_{y \in I_{n+m}} m_y \, \mu_0 \left( S_y^{-1}(K_x) \right) \\ =& \sum_{\substack{y \in I_{n+m}, \\ y_{|n} = x}} m_y \, \mu_0 \left( S_y^{-1}(K_x) \right). \end{aligned}$$

Since

$$\begin{aligned} &\left( S_{(x_1, \ldots, x_n, y_{n+1}, \ldots, y_{n+m})} \right)^{-1}(K_x) \\ =& \left( S_{y_{n+1}}^{((x_1, \ldots, x_n))} \circ \cdots \circ S_{y_{n+m}}^{((x_1, \ldots, x_n, y_{n+1}, \ldots, y_{n+m-1}))} \right)^{-1} \circ S_x^{-1}(K_x) \\ =& \left( S_{y_{n+1}}^{((x_1, \ldots, x_n))} \circ \cdots \circ S_{y_{n+m}}^{((x_1, \ldots, x_n, y_{n+1}, \ldots, y_{n+m-1}))} \right)^{-1} ([a,b]) \\ =& [a,b], \end{aligned}$$

we finally obtain

$$\mu_{n+m}^{(I)}(K_x) = \sum_{\substack{y \in I_{n+m}, \\ y_{|n} = x}} m_y = m_x.$$

□

**Lemma 4.2.3:** *It holds*

$$\mu^{(I)} = \sum_{i=1}^{N_\emptyset} m_i^{(\emptyset)} \, \mu^{(\theta_i I)} \circ S_i^{(\emptyset)^{-1}} \tag{4.3}$$

*and therefore for* $i \in \{1, ..., N_\emptyset\}$ *and* $A \in \mathcal{B}([a,b])$ *with* $A \subseteq S_i^{(\emptyset)}([a,b])$,

$$\mu^{(I)}(A) = m_i^{(\emptyset)} \left( \mu^{(\theta_i I)} \circ S_i^{(\emptyset)^{-1}} \right)(A).$$

*Proof.* Let $A \in \mathcal{B}([a,b])$. Then, we get

$$\begin{aligned}
&\sum_{i=1}^{N_\emptyset} m_i^{(\emptyset)} \, \mu_n^{(\theta_i I)} \left( \left( S_i^{(\emptyset)} \right)^{-1} (A) \right) \\
=&\sum_{i=1}^{N_\emptyset} \sum_{x_1=1}^{N_i} \cdots \sum_{x_n=1}^{N_{(i,x_1,\dots,x_{n-1})}} m_i^{(\emptyset)} \, m_{x_1}^{(i)} \cdots m_{x_n}^{((i,x_1,\dots,x_{n-1}))} \, \mu_0 \left( (S_{i,x_1,\dots,x_n})^{-1} (A) \right) \\
=&\sum_{x_1=1}^{N_\emptyset} \cdots \sum_{x_{n+1}=1}^{N_{(x_1,\dots,x_n)}} m_{x_1}^{(\emptyset)} \, m_{x_2}^{(x_1)} \cdots m_{x_{n+1}}^{((x_1,\dots,x_n))} \, \mu_0 \left( \left( S_{x_1,\dots,x_{n+1}} \right)^{-1} (A) \right) \\
=&\mu_{n+1}^{(I)}(A).
\end{aligned}$$

Taking the limit $n \to \infty$, we obtain the assertion. □

### 4.2.2. Dirichlet-Neumann bracketing

We establish a Dirichlet-Neumann bracketing with which we obtain the characteristic $\phi$ for the C-M-J branching process under consideration. To this end, we first give a scaling property for the $L_2$-Norm.

**Lemma 4.2.4:** *Let* $f, g \in L_2\left(\mu^{(I)}\right)$. *Then,*

$$\langle f, g \rangle_{L_2\left(\mu^{(I)}\right)} = \sum_{i=1}^{N_\emptyset} m_i^{(\emptyset)} \left\langle f \circ S_i^{(\emptyset)}, g \circ S_i^{(\emptyset)} \right\rangle_{L_2\left(\mu^{(\theta_i I)}\right)}.$$

*Proof.* With $\operatorname{supp} \mu^{(I)} = K^{(I)}$ and Lemma 4.2.3 we get

$$
\begin{aligned}
\langle f, g\rangle_{L_2(\mu^{(I)})} &= \int_{[a,b]} f\, g\, d\mu^{(I)} \\
&= \sum_{i=1}^{N_\emptyset} \int_{S_i^{(\emptyset)}([a,b])} f\, g\, d\mu^{(I)} \\
&= \sum_{i=1}^{N_\emptyset} \int_{[a,b]} f \circ S_i^{(\emptyset)}\, g \circ S_i^{(\emptyset)}\, d\left(\mu^{(I)} \circ S_i^{(\emptyset)}\right) \\
&= \sum_{i=1}^{N_\emptyset} m_i^{(\emptyset)} \int_{[a,b]} f \circ S_i^{(\emptyset)}\, g \circ S_i^{(\emptyset)}\, d\mu^{(\theta_i I)} \\
&= \sum_{i=1}^{N_\emptyset} m_i^{(\emptyset)} \langle f \circ S_i^{(\emptyset)}, g \circ S_i^{(\emptyset)}\rangle_{L_2\left(\mu^{(\theta_i I)}\right)}.
\end{aligned}
$$

□

Let $(\mathcal{E}^{(I)}, \mathcal{F})$ be the Dirichlet form on $L_2(\mu^{(I)})$ whose eigenvalues coincide with the Neumann eigenvalues of $-\Delta^{\mu^{(I)}}$. Namely,

$$
\begin{aligned}
\mathcal{E}(f, g) &= \int_a^b f'(x)\, g'(x)\, dx, \\
\mathcal{F} &= H^1\left([a, b]\right),
\end{aligned}
$$

see [32, Proposition 5.1], where $H^1([a, b])$ denotes the Sobolev space on $L_2\left(\lambda^1, [a, b]\right)$ of order 1. We write $N_N^{(I)}$ for $N_N^{\mu^{(I)}}$ and $N_D^{(I)}$ for $N_D^{\mu^{(I)}}$. To obtain the Dirichlet-Neumann bracketing, we follow Arzt [4, Section 3.2.2] and define a Dirichlet form $(\tilde{\mathcal{E}}^{(I)}, \tilde{\mathcal{F}}^{(I)})$. Let $\tilde{\mathcal{F}}^{(I)}$ be the set of all functions $f : [a, b] \longrightarrow \mathbb{R}$ with $f \circ S_i^{(\emptyset)} \in \mathcal{F}$ for all $i = 1, ..., N_\emptyset$ and

$$
f_{|\left(S_i^{(\emptyset)}(b), S_{i+1}^{(\emptyset)}(a)\right)} \in H^1\left(\left(S_i^{(\emptyset)}(b), S_{i+1}^{(\emptyset)}(a)\right)\right)
$$

for all $i = 1, ..., N_\emptyset - 1$. With [4, Proposition 3.2.1] follows $\mathcal{F} \subseteq \tilde{\mathcal{F}}^{(I)}$, but $\tilde{\mathcal{F}}^{(I)} \not\subseteq \mathcal{F}$, because $f \in \tilde{\mathcal{F}}^{(I)}$ has not to be continuous on the boundary points of $S_i^{(\emptyset)}([a, b])$. For all $f, g \in \tilde{\mathcal{F}}^{(I)}$, define

$$
\tilde{\mathcal{E}}^{(I)}(f, g) := \sum_{i=1}^{N_\emptyset} \frac{1}{r_i^{(\emptyset)}} \mathcal{E}\left(f \circ S_i^{(\emptyset)}, g \circ S_i^{(\emptyset)}\right) + \sum_{i=1}^{N_\emptyset - 1} \int_{S_i^{(\emptyset)}(b)}^{S_{i+1}^{(\emptyset)}(a)} f'(u)\, g'(u)\, du.
$$

By [4, Proposition 3.2.1] the following relation between $\tilde{\mathcal{E}}$ and $\mathcal{E}$ holds.

**Lemma 4.2.5 (c.f. [4, Proposition 3.2.1]):** *For all $f,g \in \mathcal{F}$ it holds, $f \circ S_i^{(\emptyset)}$, $g \circ S_i^{(\emptyset)} \in \mathcal{F}$ and*

$$\tilde{\mathcal{E}}^{(I)}(f,g) = \mathcal{E}(f,g).$$

Further, [4, Proposition 2.2.2] implies that the embedding $\tilde{F}^{(I)} \hookrightarrow L_2(\mu^{(I)})$ is a compact operator and thus we can refer to the eigenvalue counting function of the Dirichlet form $(\tilde{\mathcal{E}}^{(I)}, \tilde{\mathcal{F}}^{(I)})$ on $L_2\left(\mu^{(I)}\right)$. Let $N_{\left(\tilde{\mathcal{F}}^{(I)}, \tilde{\mathcal{E}}^{(I)}, \mu^{(I)}\right)}$ be the eigenvalue counting function of $(\tilde{\mathcal{E}}^{(I)}, \tilde{\mathcal{F}}^{(I)}, \mu^{(I)})$.

**Proposition 4.2.6:** *For all $t \geq 0$ holds*

$$N_{\left(\tilde{\mathcal{F}}^{(I)}, \tilde{\mathcal{E}}^{(I)}, \mu^{(I)}\right)}(t) = \sum_{i=1}^{N_\emptyset} N_N^{(\theta_i I)}\left(r_i^{(\emptyset)} m_i^{(\emptyset)} t\right).$$

*Proof.* Let $f$ be an eigenfunction of $\left(\tilde{\mathcal{E}}^{(I)}, \tilde{\mathcal{F}}^{(I)}, \mu^{(I)}\right)$ with eigenvalue $\lambda$, i.e.

$$\tilde{\mathcal{E}}^{(I)}(f,g) = \lambda \left\langle f, g\right\rangle_{L_2\left(\mu^{(I)}\right)} \quad \text{for all } g \in \tilde{\mathcal{F}}^{(I)}.$$

Because $f,g \in L_2\left(\mu^{(I)}\right)$ it follows by Lemma 4.2.4

$$\begin{aligned} &\sum_{i=1}^{N_\emptyset} \frac{1}{r_i^{(\emptyset)}} \mathcal{E}\left(f \circ S_i^{(\emptyset)}, g \circ S_i^{(\emptyset)}\right) + \sum_{i=1}^{N_\emptyset - 1} \int_{S_i^{(\emptyset)}(b)}^{S_{i+1}^{(\emptyset)}(a)} f'(u)\, g'(u)\, du \\ &= \lambda \sum_{i=1}^{N_\emptyset} m_i^{(\emptyset)} \left\langle f \circ S_i^{(\emptyset)}, g \circ S_i^{(\emptyset)} \right\rangle_{L_2\left(\mu^{(\theta_i I)}\right)}. \end{aligned} \tag{4.4}$$

We show that each summand on the left side equals each summand on the right side, respectively. Therefore, let $h \in \mathcal{F}$ and define for each $j \in \{1, ..., N_\emptyset\}$

$$\tilde{h}_j(t) := \begin{cases} h \circ S_j^{(\emptyset)^{-1}}(t), & \text{if } t \in S_j^{(\emptyset)}([a,b]), \\ 0, & \text{otherwise.} \end{cases}$$

Then $\tilde{h}_j \in \mathcal{F}, \tilde{h}_j \circ S_j^{(\emptyset)} = h$, for all $j \in \{1, ..., N_\emptyset\}$ and $\tilde{h}_j \circ S_i^{(\emptyset)} = 0$ for $i \neq j$. Moreover, $\tilde{h}_j'\big|_{\left(S_i^{(\emptyset)}(b), S_{i+1}^{(\emptyset)}(a)\right)} = 0$, $j = 1, ..., N_\emptyset$, $i = 1, .., N_\emptyset - 1$. With $g = \tilde{h}_j$, we then get in (4.4)

$$\frac{1}{r_j^{(\emptyset)}} \mathcal{E}\left(f \circ S_j^{(\emptyset)}, h\right) = \lambda\, m_j^{(\emptyset)} \left\langle f \circ S_j^{(\emptyset)}, h \right\rangle_{L_2\left(\mu^{(\theta_j I)}\right)}.$$

Because this equation holds for all $h \in \mathcal{F}$, $f \circ S_j^{(\emptyset)}$ is an eigenfunction of the Dirichletform $\left(\mathcal{E}, \mathcal{F}, \mu^{(\theta_j I)}\right)$ with eigenvalue $r_j^{(\emptyset)} m_j^{(\emptyset)} \lambda$ for all $j = 1, ..., N_\emptyset$.

Now, let $\lambda > 0$, such that for $i = 1, ..., N_\emptyset$, $r_i^{(\emptyset)} m_i^{(\emptyset)} \lambda$ is an eigenvalue of $\left(\mathcal{E}, \mathcal{F}, \mu^{(\theta_i I)}\right)$ with eigenfunction $f_i$. This means,

$$\mathcal{E}(f_i, g) = r_i^{(\emptyset)} m_i^{(\emptyset)} \lambda \left\langle f_i, g \right\rangle_{L_2\left(\mu^{(\theta_i I)}\right)},$$

for all $g \in \mathcal{F}$. Let

$$f(t) := \begin{cases} f_i \circ S_i^{(\emptyset)^{-1}}(t), & \text{if } t \in S_i^{(\emptyset)}([a,b]) \text{ for some } i \in \{1, ..., N_\emptyset\} \\ 0, & \text{otherwise.} \end{cases}$$

Then, $f \in \mathcal{F}$ and $f \circ S_i^{(\emptyset)} = f_i$, $i = 1, ..., N_\emptyset$ and therefore

$$\sum_{i=1}^{N_\emptyset} \frac{1}{r_i^{(\emptyset)}} \mathcal{E}\left(f \circ S_i^{(\emptyset)}, g\right) = \lambda \sum_{i=1}^{N_\emptyset} m_i^{(\emptyset)} \left\langle f \circ S_i^{(\emptyset)}, g \right\rangle_{L_2\left(\mu^{(\theta_i I)}\right)},$$

for all $g \in \mathcal{F}$. Since for $\tilde{g} \in \tilde{\mathcal{F}}^{(I)}$ it holds by definition of $\tilde{\mathcal{F}}^{(I)}$, $\tilde{g} \circ S_i^{(\emptyset)} \in \mathcal{F}$, $i = 1, ..., N_\emptyset$ and thus

$$\sum_{i=1}^{N_\emptyset} \frac{1}{r_i^{(\emptyset)}} \mathcal{E}(f \circ S_i^{(\emptyset)}, \tilde{g} \circ S_i^{(\emptyset)}) = \lambda \sum_{i=1}^{N_\emptyset} m_i^{(\emptyset)} \left\langle f \circ S_i^{(\emptyset)}, \tilde{g} \circ S_i^{(\emptyset)} \right\rangle_{L_2\left(\mu^{(\theta_i I)}\right)}.$$

But the left side of this equation is equal to $\tilde{\mathcal{E}}^{(I)}(f, \tilde{g})$, because $f'\big|_{\left(S_i^{(\emptyset)}(b), S_{i+1}^{(\emptyset)}(a)\right)} = 0$, for all $i = 1, ..., N_\emptyset - 1$. With Lemma 4.2.4 we then obtain

$$\tilde{\mathcal{E}}^{(I)}(f, \tilde{g}) = \lambda \left\langle f, \tilde{g} \right\rangle_{L_2\left(\mu^{(I)}\right)}, \tag{4.5}$$

for all $\tilde{g} \in \tilde{\mathcal{F}}^{(I)}$. Therefore, $\lambda$ is an eigenvalue of $(\tilde{\mathcal{E}}^{(I)}, \tilde{\mathcal{F}}^{(I)}, \mu^{(I)})$ with corresponding eigenfunction $f$. Using this, we can easily conclude the claim. □

Let $(\mathcal{F}_0, \mathcal{E})$ be the Dirichlet form on $L_2\left(\mu^{(I)}\right)$ whose eigenvalues coincide with the Dirichlet eigenvalues of $-\Delta^{\mu^{(I)}}$, i.e. $\mathcal{E}$ is defined as before and

$$\mathcal{F}_0 := \{f \in \mathcal{F} : \; f(a) = f(b) = 0\},$$

see [32, Proposition 5.3]. As for the Neumann eigenvalue counting function we follow Arzt [4, Section 3.2.3] and define a Dirichlet form $(\tilde{\mathcal{F}}_0^{(I)}, \tilde{\mathcal{E}}_0^{(I)})$ on $L_2\left(\mu^{(I)}\right)$

by

$$\tilde{\mathcal{F}}_0^{(I)} := \left\{ f \in \mathcal{F}_0 : \ f(x) = 0 \text{ for } x \in \left[ S_i^{(\emptyset)}(b), S_{i+1}^{(\emptyset)}(a) \right], \ i = 1, ..., N_\emptyset - 1 \right\}.$$

and

$$\tilde{\mathcal{E}}_0^{(I)} := \mathcal{E}\big|_{\tilde{\mathcal{F}}_0^{(I)} \times \tilde{\mathcal{F}}_0^{(I)}}.$$

**Proposition 4.2.7:** *For all $x \geq 0$ it holds*

$$N_{\left(\tilde{\mathcal{E}}_0^{(I)}, \tilde{\mathcal{F}}_0^{(I)}, \mu^{(I)}\right)}(x) = \sum_{i=1}^{N_\emptyset} N_D^{(\theta_i I)} \left( r_i^{(\emptyset)} m_i^{(\emptyset)} x \right).$$

*Proof.* Let $f$ be an eigenfunction of $(\tilde{\mathcal{E}}_0^{(I)}, \tilde{\mathcal{F}}_0^{(I)}, \mu^{(I)})$ with eigenvalue $\lambda$. Then

$$\tilde{\mathcal{E}}_0^{(I)}(f, g) = \lambda \left\langle f, g \right\rangle_{L_2\left(\mu^{(I)}\right)},$$

for all $g \in \mathcal{F}_0$. Therefore, by [4, Proposition 3.2.1] and Lemma 4.2.4,

$$\begin{aligned} &\sum_{i=1}^{N_\emptyset} \frac{1}{r_i^{(\emptyset)}} \mathcal{E} \left( f \circ S_i^{(\emptyset)}, g \circ S_i^{(\emptyset)} \right) + \sum_{i=1}^{N_\emptyset - 1} \int_{S_i^{(\emptyset)}(b)}^{S_{i+1}^{(\emptyset)}(a)} f'(u)\, g'(u)\, du \\ &= \lambda \sum_{i=1}^{N_\emptyset} m_i^{(\emptyset)} \left\langle f, g \right\rangle_{L_2\left(\mu^{(\theta_i I)}\right)}. \end{aligned}$$

For $h \in \mathcal{F}_0$ define

$$\tilde{h}_j(t) := \begin{cases} h \circ S_j^{(\emptyset)^{-1}}(t), & \text{if } t \in S_j^{(\emptyset)}([a, b]), \\ 0, & \text{otherwise.} \end{cases}$$

Because $h \in \mathcal{F}_0$, it follows $\tilde{h}_j \in \tilde{\mathcal{F}}_0^{(I)}$ and $\tilde{h}_j \circ S_j^{(\emptyset)} = h$ for $j = 1, ..., N_\emptyset$ and $\tilde{h}_j \circ S_i^{(\emptyset)} = 0$, if $i \neq j$. Hence,

$$\frac{1}{r_j^{(\emptyset)}} \mathcal{E} \left( f \circ S_j^{(\emptyset)}, h \right) = \lambda\, m_j^{(\emptyset)} \left\langle f \circ S_j^{(\emptyset)}, h \right\rangle_{L_2\left(\mu^{\left(\theta_j I\right)}\right)},$$

for all $j = 1, ..., N_\emptyset$. Therefore, $r_i^{(\emptyset)} m_i^{(\emptyset)} \lambda$ is an eigenvalue of $\left(\mathcal{E}, \mathcal{F}_0, \mu^{(\theta_i I)}\right)$ with eigenfunction $f \circ S_i^{(\emptyset)}$, $i = 1, ..., N_\emptyset$.

Now, let $r_i^{(\emptyset)} m_i^{(\emptyset)} \lambda$ be an eigenvalue of $\left(\mathcal{E}, \mathcal{F}_0, \mu^{(\theta_i I)}\right)$ for some $\lambda > 0$ with corresponding eigenfunction $f_i$, $i = 1, ..., N_\emptyset$. Therefore,

$$\mathcal{E}(f_i, g) = r_i^{(\emptyset)} m_i^{(\emptyset)} \lambda \left\langle f_i, g \right\rangle_{L_2\left(\mu^{(\theta_i I)}\right)},$$

for all $g \in \mathcal{F}_0$. Let

$$f(t) := \begin{cases} f_i \circ S_i^{(\emptyset)^{-1}}(t), & \text{if } t \in S_i^{(\emptyset)}([a,b]), \text{ for some } i \in \{1, ..., N_\emptyset\} \\ 0, & \text{otherwise.} \end{cases}$$

Since $f_i \in \mathcal{F}_0$, it follows $f \in \tilde{\mathcal{F}}_0^{(I)}$ and because of $f \circ S_i^{(\emptyset)} = f_i$, $i = 1, ..., N_\emptyset$,

$$\sum_{i=1}^{N_\emptyset} \frac{1}{r_i^{(\emptyset)}} \mathcal{E}\left(f \circ S_i^{(\emptyset)}, g\right) = \lambda \sum_{i=1}^{N_\emptyset} m_i^{(\emptyset)} \left\langle f \circ S_i^{(\emptyset)}, g \right\rangle_{L_2\left(\mu^{(\theta_i I)}\right)},$$

for all $g \in \mathcal{F}_0$. For $\tilde{g} \in \tilde{\mathcal{F}}_0^{(I)}$ it holds $\tilde{g} \circ S_i^{(\emptyset)} \in \mathcal{F}_0$, $i = 1, ..., N_\emptyset$. Analogously to (4.5) then follows

$$\tilde{\mathcal{E}}_0^{(I)}(f, \tilde{g}) = \lambda \left\langle f, \tilde{g} \right\rangle_{L_2\left(\mu^{(I)}\right)}.$$

Hence, $\lambda$ is an eigenvalue of $(\tilde{\mathcal{E}}_0^{(I)}, \tilde{\mathcal{F}}_0, \mu^{(I)})$ with eigenfunction $f$ and, as before, we can now easily conclude the claim. □

Since $(\tilde{\mathcal{E}}^{(I)}, \tilde{\mathcal{F}}^{(I)}, \mu^{(I)})$ is an extension of $\left(\mathcal{E}, \mathcal{F}, \mu^{(I)}\right)$ and $\left(\mathcal{E}, \mathcal{F}_0, \mu^{(I)}\right)$ is an extension of $(\tilde{\mathcal{E}}_0^{(I)}, \tilde{\mathcal{F}}_0^{(I)}, \mu^{(I)})$, we infer from the Max-Min principle, see e.g. Lapidus and Kigami [57, Theorem 4.5], the Dirichlet-Neumann bracketing:

**Corollary 4.2.8 (Dirichlet-Neumann bracketing):** *For all $t \geq 0$ holds*

$$\sum_{i=1}^{N_\emptyset} N_D^{(\theta_i I)}\left(r_i^{(\emptyset)} m_i^{(\emptyset)} t\right) \leq N_D^{(I)}(t) \leq N_N^{(I)}(t) \leq \sum_{i=1}^{N_\emptyset} N_N^{(\theta_i I)}\left(r_i^{(\emptyset)} m_i^{(\emptyset)} t\right).$$

### 4.2.3. Spectral asymptotics

We define a probability space $(\Omega, \mathfrak{B}, \mathbb{P})$ in which every atomic event indicates a random tree $I$. Let $(\tilde{\Omega}, \tilde{\mathfrak{B}}, \tilde{\mathbb{P}})$ be a probability space and $\tilde{U}_x$, $x \in T$ be i.i.d. $J$-valued random variables. The probability space we are interested in is defined as in Section 3.1, i.e. as the product space

$$(\Omega, \mathfrak{B}, \mathbb{P}) = \prod_{x \in T} (\Omega_x, \mathfrak{B}_x, \mathbb{P}_x),$$

whereby $(\Omega_x, \mathfrak{B}_x, \mathbb{P}_x)$ are copies of $(\tilde{\Omega}, \tilde{\mathfrak{B}}, \tilde{\mathbb{P}})$ and $T$ defined in (3.1). We set $U_x = \tilde{U}_x \circ P_x$, $x \in T$, where $P_x$ is the projection map onto the $x$-th component. $\omega \in \Omega$ indicates a random tree $I = I(\omega)$. This random tree is defined as in Section 3.2. The corresponding sets and measures $K^{(I)}$ and $\mu^{(I)}$ respectively are called *statistically self-similar*.

As explained in Section 3.2, there is a connection between statistically self-similar Cantor sets and branching processes. We use this connection to determine the spectral exponent in this section. The general branching process we use here not only depends on the scale factors as in (3.5), but also on the weights $m^{(j)}$. The considered reproduction function and life length are defined as

$$\xi_x := \sum_{i=1}^{N_x} \delta_{-\log r_i^{(x)} m_i^{(x)}}, \qquad L_x := \sup_i \sigma_{x,i} - \sigma_x. \tag{4.6}$$

With Corollary 4.2.8 for each $x \geq 0$ holds

$$\sum_{i=1}^{N_{U_\emptyset}} N_D^{(\theta_i I)} \left( r_i^{(U_\emptyset)} m_i^{(U_\emptyset)} t \right) \leq N_D^{(I)}(t) \leq N_N^{(I)}(t) \leq \sum_{i=1}^{N_{U_\emptyset}} N_N^{(\theta_i I)} \left( r_i^{(U_\emptyset)} m_i^{(U_\emptyset)} t \right).$$

For simplicity, we write in the following $N_\emptyset$, $r_i^{(\emptyset)}$ and $m_i^{(\emptyset)}$ for $N_{U_\emptyset}$, $r_i^{(U_\emptyset)}$ and $m_i^{(U_\emptyset)}$ and consider the scaling property

$$\sum_{i=1}^{\xi_\emptyset(\infty)} N_D^{(\theta_i I)} \left( r_i^{(\emptyset)} m_i^{(\emptyset)} t \right) \leq N_D^{(I)}(t).$$

Let

$$Z(t) := N_D^{(I)} \left( e^t \right), \quad Z_i(t) := N_D^{(\theta_i I)}(e^t)$$

and $(\xi_x, L_x, \eta_x)_x$ be the general branching process with $\xi$, $L$ defined in (4.6) and

$$\eta_\emptyset(t) := Z(t) - \sum_{i=1}^{\xi_\emptyset(\infty)} Z_i(t - \sigma_i).$$

It then follows that the C-M-J branching process $Z^\eta$ coincides with $Z$. As explained in (3.10) if $\eta$ does not vanish for negative times, we use

$$\chi^\eta_\emptyset(t) := \eta_\emptyset(t)\mathbb{1}_{\{t\geq 0\}} + \sum_{i=1}^{\xi_\emptyset(\infty)} Z_i^\eta(t-\sigma_i)\mathbb{1}_{\{0\leq t<\sigma_i\}}.$$

Thus, $Z^\eta(t)$ and $Z^{\chi^\eta}(t)$ have the same asymptotic behaviour, as $t \to \infty$.

In the following, we denote as in Section 3.2 by $W$ the limit of the underlying fundamental martingale corresponding to the general branching process $(\xi_x, L_x, \eta_x)_x$.

The following theorem is the main result of this section. The idea of the proof is taken from [42, Theorem 5.5].

**Theorem 4.2.9:** *Assume that*

$$\mathbb{E}N_\emptyset^2 < \infty. \tag{4.7}$$

*Then, the spectral exponent for statistically self-similar Cantor measures $\mu^{(I)}$ is given by the unique solution $\gamma_r > 0$ of*

$$\mathbb{E}\left(\sum_{i=1}^{N_\emptyset}\left(r_i^{(\emptyset)}m_i^{(\emptyset)}\right)^{\gamma_r}\right) = 1.$$

*Moreover,*

1. *If $\nu_{\gamma_r}$ defined in (3.2) is non-lattice, then*

$$\lim_{\lambda\to\infty} N_D^{(I)}(\lambda)\,\lambda^{-\gamma_r} = z^{\chi^\eta}(\infty)\,W \quad a.s.,$$

   *where*

$$z^{\chi^\eta}(\infty) := \frac{\int_0^\infty e^{-\gamma t}\,\mathbb{E}(\chi^\eta(t))\,dt}{\int_0^\infty t\,e^{-\gamma t}\,\nu(dt)}.$$

2. *If $\nu_{\gamma_r}$ is lattice with period $L$, then, as $\lambda \to \infty$,*

$$N_D^{(I)}(\lambda) = \left(G^{\chi^\eta}(\log(\lambda)) + o(1)\right)\,\lambda^{\gamma_r}\,W \quad a.s.,$$

*where $G$ is a periodic function with period $L$, given by*

$$G(\lambda) := L \cdot \frac{\sum_{j=-\infty}^{\infty} e^{-\gamma(\lambda+jL)}\, \mathbb{E}(\chi^\eta(\lambda+jL))}{\int_0^\infty t\, e^{-\gamma t}\, \nu(dt)}.$$

*Proof.* The proof relies on the following lemma.

**Lemma 4.2.10:** *Assume that*

$$\mathbb{E}N_{U_\emptyset}^2 < \infty.$$

*Then, $W > 0$ a.s. and the Malthusian parameter of the general branching process $(\xi_x, L_x, \eta_x)_x$ is the unique solution $\gamma_r > 0$ of*

$$\mathbb{E}\left(\sum_{i=1}^{N_\emptyset} \left(r_i^{(\emptyset)} m_i^{(\emptyset)}\right)^{\gamma_r}\right) = 1.$$

*Furthermore, it holds:*

1. *If $\nu_{\gamma_r}$ defined in (3.2) is non-lattice then*

$$\lim_{t\to\infty} Z(t)\, e^{-\gamma_r t} = z^{\chi^\eta}(\infty)\, W \quad a.s.,$$

   *where*

$$z^{\chi^\eta}(\infty) := \frac{\int_0^\infty e^{-\gamma_r t}\, \mathbb{E}(\chi^\eta(t))\, dt}{\int_0^\infty t\, e^{-\gamma_r t}\, \nu(dt)}.$$

2. *If $\nu_{\gamma_r}$ is lattice with period $L$, then, as $t \to \infty$,*

$$Z(t) = \left(G^{\chi^\eta}(t) + o(1)\right)\, e^{\gamma_r t}\, W \quad a.s.,$$

   *where $G^{\chi^\eta}$ is a periodic function with period $L$ given by*

$$G^{\chi^\eta}(t) = L \cdot \frac{\sum_{j=-\infty}^{\infty} e^{-\gamma_r(t+jL)}\, \mathbb{E}(\chi^\eta(t+jL))}{\int_0^\infty s\, e^{-\gamma_r s}\, \nu(ds)}.$$

We rescale Lemma 4.2.10 by $\lambda = \log(t)$ and therefore it remains to show Lemma 4.2.10.

*Proof of Lemma 4.2.10.* By Theorem 3.3.1 it follows $W > 0$ a.s. since

$$\mathbb{E}N_\emptyset \log N_\emptyset \leq \mathbb{E}N_\emptyset^2 < \infty.$$

Let

$$f(s) := \mathbb{E}\left(\sum_{i=1}^{N_\emptyset}\left(r_i^{(\emptyset)}m_i^{(\emptyset)}\right)^s\right).$$

By dominated convergence, $f : [0,\infty) \longrightarrow \mathbb{R}$ is continuous and because $r_i^{(j)}m_i^{(j)} < 1$ for all $j \in J$, $i = 1, ..., N_j$, $f$ is strictly decreasing. Since $N_j \geq 2$, $j \in J$ it holds

$$f(0) \geq 2$$

and

$$\lim_{s\to\infty} f(s) = 0.$$

By continuity, there exists $\gamma_r > 0$ such that $f(\gamma_r) = 1$. Furthermore, $\gamma_r$ is the unique solution strictly bigger than zero and also the Malthusian parameter of the considered general branching process. The first moment of $\nu_{\gamma_r}$ is finite since $\mathbb{E}N_\emptyset < \infty$. With $g(t) = t^{-2} \wedge 1$ the first part of Condition 3.3.3 is satisfied as explained in Section 3.3. By Lemma 2.1.1, there exists a deterministic constant $\tilde{c} > 0$ such that

$$Z(t) \leq \tilde{c}\, e^t. \tag{4.8}$$

Further, from the Dirichlet-Neumann bracketing follows that

$$0 \leq \eta(t) \leq \sum_{i=1}^{N_\emptyset}\left(N_N^{(\theta_i I)}\left(r_i^{(\emptyset)}m_i^{(\emptyset)}e^t\right) - N_D^{(\theta_i I)}\left(r_i^{(\emptyset)}m_i^{(\emptyset)}e^t\right)\right).$$

With Lemma 2.1.2 we thus obtain

$$\eta(t) \leq 2N_\emptyset. \tag{4.9}$$

Taking together (4.8) and (4.9) we obtain

$$\chi^\eta(t) \leq cN_\emptyset,$$

for some deterministic $c > 0$. Therefore, the second part of Condition 3.3.3 follows with $h(t) = e^{-\gamma t}$. The Lemma then follows from Theorem 3.3.4.

□

This proofs Lemma 4.2.10 and hence Theorem 4.2.9 follows. □

**Remark 4.2.11:** *By Lemma 2.1.2 the asymptotic behaviour of* $N_D^{(I)}(\lambda)\lambda^{-\gamma_r}$ *and* $N_N^{(I)}(\lambda)\lambda^{-\gamma_r}$ *coincide.*

### 4.2.4. Comparison between statistically self-similar and random homogeneous Cantor measures

We saw the construction of recursive Cantor sets and corresponding recursive Cantor measures. Then, we showed under a moment condition that the spectral exponent of $\Delta^\mu$ with respect to statistically self-similar Cantor measures is almost surely given by the unique solution $\gamma_r > 0$ of

$$\mathbb{E}\left(\sum_{i=1}^{N_\emptyset}\left(r_i^{(\emptyset)}m_i^{(\emptyset)}\right)^{\gamma_r}\right) = 1.$$

In Theorem 2.2.2 we recalled the results of [4] about the spectral asymptotics of $\Delta^\mu$ with respect to random homogeneous Cantor measures $\mu$. In particular, as explained in Section 4.1, the spectral exponent for random homogeneous Cantor measures is almost surely given by the unique solution $\gamma_h > 0$ of

$$\prod_{j\in J}\left(\sum_{i=1}^{N_j}\left(r_i^{(j)}m_i^{(j)}\right)^{\gamma_h}\right)^{p_j} = 1.$$

The next proposition relates $\gamma_r$ to $\gamma_h$, where we assume that (A1)-(A5) are satisfied.

**Proposition 4.2.12:** *With the notation above and in Theorem 2.2.2, it holds* $\gamma_h \le \gamma_r$ *and equality if and only if there exists* $\alpha > 0$ *such that*

$$\sum_{i=1}^{N_j}\left(r_i^{(j)}m_i^{(j)}\right)^{\alpha} = 1, \quad \textit{for all } j \in J. \tag{4.10}$$

*Proof.* Let $t_j(\alpha) := \sum_{i=1}^{N_j} \left(r_i^{(j)} m_i^{(j)}\right)^\alpha$, $j \in J$. With Jensen's inequality, we obtain

$$\sum_{j\in J} p_j \log\left(t_j(\alpha)\right) \leq \log\left(\sum_{j\in J} p_j\, t_j(\alpha)\right).$$

Since log is strictly increasing, equality holds if and only if $t_j(\alpha) = 1$ for all $j \in J$. Now, let (4.10) not be satisfied. Then,

$$0 = \sum_{j\in J} p_j \log\left(t_j(\gamma_h)\right) < \log\left(\sum_{j\in J} p_j\, t_j(\gamma_h)\right).$$

As $\log\left(\sum_{j\in J} p_j\, t_j(\alpha)\right)$ decreases as $\alpha$ increases, the assertion follows. □

**Remark 4.2.13:** *If $U_x = U_y$ for all $x, y \in I$ such that $|x| = |y|$, then the corresponding recursive Cantor measure is homogeneous. However, Theorem 4.2.9 makes no statement about the spectral exponent for random homogeneous Cantor measures, since the probability that $\mu^{(I)}$ is homogeneous is 0 in non-trivial cases.*

**Example 4.2.14 (c.f. [4, Section 4.5]):** *Let $J$ be countable and $p_j := \mathbb{P}(U_\emptyset = j) \in (0,1)$, $j \in J$. Further, assume that $r_1^{(j)} = ... = r_{N_j}^{(j)}$, $m_1^{(j)} = ... = m_{N_j}^{(j)}$ for all $j \in J$. Therefore, $m_i^{(j)} = \frac{1}{N_j}$ $i = 1, ..., N_j$ for all $j \in J$. Let $r := r_i^\emptyset$ and $N := N_\emptyset$. If conditions (A1)-(A5) are satisfied then the spectral exponent for random homogeneous Cantor measure is given by*

$$\gamma_h := \frac{\mathbb{E}\log N}{\mathbb{E}\log(N/r)},$$

*see [4, Page 64]. The spectral exponent for statistically self-similar Cantor measure is given by the unique solution $\gamma_r > 0$ of*

$$\mathbb{E}\left(N\left(r/N\right)^{\gamma_r}\right) = 1.$$

*If not $(r/N)^\alpha = 1/N$ for some $\alpha > 0$, for almost all $\omega \in \Omega$, we thus have*

$$0 = \log\left(\sum_{j\in J} p_j\, N_j \left(r_1^{(j)}/N_j\right)^{\gamma_r}\right) < \sum_{j\in J} p_j \log\left(N_j \left(r_1^{(j)}/N_j\right)^{\gamma_r}\right) = \mathbb{E}\log\left(N\left(r/N\right)^{\gamma_r}\right).$$

*Therefore,*

$$\gamma_h = \frac{\mathbb{E}\log N}{\mathbb{E}\log(N/r)} < \gamma_r.$$

## 4.3. Spectral exponent for random $V$-variable Cantor measures

### 4.3.1. Construction

The idea of the proofs, constructions and structure of Section 4.3 are taken from Arzt [4, Chapter 3] and Freiberg, Hambly and Hutchinson [35]. As in Section 2.2.2 and Section 4.2 we let $J$ be an index set and define to each $j \in J$ an IFS $\mathcal{S}^{(j)} = \left(S_1^{(j)}, ..., S_{N_j}^{(j)}\right)$ on $[a, b]$ with $N_j \in \mathbb{N}$, $N_j \geq 2$ by

$$S_i^{(j)}(x) \coloneqq r_i^{(j)} x + c_i^{(j)}$$

for some $r_i^{(j)} \in (0, 1)$, $c_i^{(j)} \in \mathbb{R}$, $i = 1, ..., N_j$ such that

$$a = S_1^{(j)}(a) < S_1^{(j)}(b) \leq S_2^{(j)}(a) < S_2^{(j)}(b) \leq \cdots \leq S_{N_j}^{(j)}(a) < S_{N_j}^{(j)}(b) = b$$

and further, a vector of weights $m^{(j)} = \left(m_1^{(j)}, ..., m_{N_j}^{(j)}\right)$.

For the construction of $V$-variable labelled trees we follow Freiberg, Hambly and Hutchinson [35, Section 2.4]. Therefore, we first need to define environments.

**Definition 4.3.1 (Environment, c.f. [35, Definition 2.7]):** *An environment $E$ is a matrix $E = (E(1), ..., E(V))$ which assigns to each $v \in \{1, ..., V\}$ both an index $j_v^E \in J$ and a sequence of types $\left(\tau_{v,i}^E\right)_{i=1}^{N_{j_v^E}}$, i.e.*

$$E(v) = \left(j_v^E, \tau_{v,1}^E \dots, \tau_{v,N_{j_v^E}}^E\right) \in J \times \{1, \dots, V\}^{N_{j_v^E}}, \quad v \in \{1, \dots, V\}.$$

A $V$-variable labelled tree depends on a sequence of environments $(E^k)_{k\geq 1}$, whereby the $n$-th generation (or level) of the tree is defined as follows (c.f. [35, Construction 2.8]):

Generation 0: Every $V$-variable tree has a unique ancestor, denoted by $\emptyset$. To this ancestor we assign a type $\tau^\emptyset \in \{1, \dots, V\}$.

Generation 1: Set $v \coloneqq \tau^\emptyset$ and $S_v \coloneqq S^{\left(j_v^{E^1}\right)}$. This determines the first IFS to be used. The number of children of the ancestor $\emptyset$ is the number of contractions of $S_v$. Assign to the $i$-th child of $\emptyset$ the type $\tau_{v,i}^{E^1}$.

Generation 2: Repeat the procedure used in generation 1 for every individual of the first generation, whereby $E^1$ is replaced by $E^2$.

⋮

We denote a $V$-variable tree by $I_V$ and its $n$-th generation by $I_{V,n}$. By construction, we assigned to each node $x \in I_{V,n}$ an index $j^{E^n}_{\tau^x}$ and therefore a tuple consisting of an IFS $S^{(j^{E^n}_{\tau^x})}$ and a vector of weights $m^{(j^{E^n}_{\tau^x})}$. For convenience, we denote this index also by $x$. As before, we define

$$m_x := m^{(\emptyset)}_{x_1} \cdots m^{((x_1,\ldots,x_{n-1}))}_{x_n},$$
$$S_x := S^{(\emptyset)}_{x_1} \circ \ldots \circ S^{((x_1,\ldots,x_{n-1}))}_{x_n}$$

and $S_x^{-1}$ as the composition of the preimages.

The *$V$-variable Cantor set* $K^{(I_V)}$ is then defined analogously to the recursive Cantor set $K^{(I)}$, namely

$$K^{(I_V)} := \bigcap_{n=1}^{\infty} \bigcup_{x \in I_{V,n}} S_x([a,b]), \quad K_x := S_x([a,b]).$$

The difference between $K^{(I_V)}$ and $K^{(I)}$ is that the number of different indices assigned to nodes in $I_{V,n}$ is uniformly bounded by $V$, whereas in Section 4.2 the number of different indices assigned to nodes in $I_n$ is in general not uniformly bounded.

The following result is transferred from Proposition 4.2.1.

**Proposition 4.3.2:** *The set $K^{(I_V)}$ is compact and contains at least countably infinitely many elements, namely $S_{(x_1,\ldots,x_n)}(a)$ and $S_{(x_1,\ldots,x_n)}(b)$, $x_1 = 1, \ldots, N_\emptyset, \ldots, x_n = 1, \ldots, N_{(x_1,\ldots,x_{n-1})}$.*

As for the recursive Cantor sets it follows

$$K^{(I_V)} = \bigcup_{i=1}^{N_\emptyset} S^{(\emptyset)}_i \left(K^{(\theta_i I_V)}\right). \tag{4.11}$$

The $V$-variable Cantor measure $\mu^{(I_V)}$ is also defined analogously to the recursive Cantor measure $\mu^{(I)}$, i.e. $\mu^{(I_V)}$ is the weak limit of $\left(\mu^{(I_V)}_n\right)_{n \in \mathbb{N}_0}$ defined by

$$\mu^{(I)}_n(A) := \sum_{x \in I_{V,n}} m_x \, \mu_0\left(S_x^{-1}(A)\right), \quad \mu_0(A) := \frac{1}{b-a} \lambda^1_{|_{[a,b]}}(A)$$

for all $A \in \mathcal{B}([a,b])$.

Lemma 4.3.3 and Lemma 4.3.4 are transferred from Section 4.2 to the $V$-variable setting.

**Lemma 4.3.3:** *For all $x \in I_V$ holds*

$$\mu^{(I_V)}(S_x([a,b])) = m_x$$

*and*

$$\mu^{(I_V)} = \sum_{i=1}^{N_\emptyset} m_i^{(\emptyset)} \, \mu^{(\theta_i I_V)} \circ S_i^{(\emptyset)^{-1}}. \tag{4.12}$$

**Lemma 4.3.4:** *Let $f, g \in L_2\left(\mu^{(I_V)}\right)$. Then,*

$$\langle f, g \rangle_{L_2\left(\mu^{(I_V)}\right)} = \sum_{i=1}^{N_\emptyset} m_i^{(\emptyset)} \left\langle f \circ S_i^{(\emptyset)}, g \circ S_i^{(\emptyset)} \right\rangle_{L_2\left(\mu^{(\theta_i I_V)}\right)}.$$

By definition of cut sets $\Lambda \subset I_V$ (see Definition 3.1.2), we obtain iteratively the following lemma.

**Lemma 4.3.5:** *Let $\Lambda \subset I_V$ be a cut set of $I_V$. Then it holds*

$$\langle f, g \rangle_{L_2\left(\mu^{(I_V)}\right)} = \sum_{x \in \Lambda} m_x \left\langle f \circ S_x, g \circ S_x \right\rangle_{L_2\left(\mu^{(\theta_x I_V)}\right)}$$

For the random set up, we also follow the construction of [35, Section 2.5]. Therefore, let $\mathbb{P}$ be a probability distribution on the index set $J$. From this probability distribution we obtain a probability distribution $\mathbb{P}_V$ on the set of environments by choosing $j_v^E$, $v \in \{1, ..., V\}$ independently according to $\mathbb{P}$ and choosing the types $\tau_{v,i}$ $1 \leq i \leq N_{j_v^E}$ i.i.d. according to the uniform distribution on $\{1, ..., V\}$ independently of the chosen indices.

Let $\Omega_V$ be the set of all $V$-variable trees to given index set $J$ and tuple $(\mathcal{S}^{(j)}, m^{(j)})$, $j \in J$. We choose $\tau^\emptyset \in \{1, ..., V\}$ according to the uniform distribution and independently the environments at each stage i.i.d. according to $\mathbb{P}_V$. This induces a probability distribution on $\Omega_V$ and on the set of $V$-variable fractals $K_V$. For convenience, we denote these probability distributions also by $\mathbb{P}_V$.

Throughout Section 4.3 we assume the following technical assumptions to hold:

$$\sup_{j\in J} N_j < \infty \tag{C1}$$

$$0 < m_{\inf} \coloneqq \inf_{j\in J} \min_{i=1,\dots,N_j} m_i^{(j)} \le \sup_{j\in J} \max_{i=1,\dots,N_j} m_i^{(j)} =: m_{\sup} < 1 \tag{C2}$$

$$0 < r_{\inf} \coloneqq \inf_{j\in J} \min_{i=1,\dots,N_j} r_i^{(j)} \le \sup_{j\in J} \max_{i=1,\dots,N_j} r_i^{(j)} =: r_{\sup} < 1 \tag{C3}$$

### 4.3.2. Neck levels

As mentioned in the introduction, an important tool to analyse the spectral asymptotics for $\Delta^{\mu^{(I_V)}}$ are neck levels which we define in this section. Further, we introduce a sequence of cut sets $(\Lambda_k)_k$, related to neck levels. In Section 4.3.3 we use this sequence to get a Dirichlet-Neumann bracketing. This Dirichlet-Neumann bracketing and Lemma 4.3.8 are then applied to obtain the spectral exponent. The idea to use this specific sequence of neck levels to determine the spectral exponent is taken from Freiberg, Hambly and Hutchinson [35].

**Definition 4.3.6 (c.f. [35, Definition 2.14]):** *Let $E$ be an environment. We call $E$ a neck if all $\tau_{v,i}^E$ are equal. Further, we call $n \in \mathbb{N}$ a neck level of a $V$-variable tree if the environment assigned to the $n$-th generation of the tree $E^n$ is a neck.*

These neck levels occur with probability one infinitely often and

$$\mathbb{E}_V n(1) < \infty,$$

where we denote by $n(k)$ the $k$-th neck level of the corresponding random $V$-variable tree. Remark that the sequence of times between neck levels is a sequence of geometric random variables, c.f. [35, Section 2.6]. We need the following property of sums of scale factors, included from [35, Lemma 2.16] to determine the spectral exponent.

**Lemma 4.3.7 (c.f. [35, Lemma 2.16]):** *Let* $s_i^{(j)} \in \mathbb{R}$ $i = 1, ..., N_j$, $j \in J$ *such that*

$$s_{\inf} := \inf_{j\in J} \min_{i=1,...,N_j} s_i^{(j)} > 0,$$
$$s_{\sup} := \sup_{j\in J} \max_{i=1,...,N_j} s_i^{(j)} < \infty.$$

*Then, with* $s_x := s_{x_1}^{(\emptyset)} \cdots s_{x_n}^{((x_1,...,x_{n-1}))}$, $x = (x_1, ..., x_n) \in I_V$ *it follows*

$$\lim_{k\to\infty} \frac{1}{k} \log \sum_{\substack{x\in I_V,\\ |x|=n(k)}} s_x = \mathbb{E}_V \log \sum_{\substack{x\in I_V,\\ |x|=n(1)}} s_x \quad a.s. \tag{4.13}$$

The sequence of cut sets we are interested in is defined as

$$\Lambda_0 := \emptyset,$$
$$\Lambda_k := \left\{ x \in I_V : \exists l \in \mathbb{N} : |x| = n(l) \quad \text{and} \quad m_x\, r_x \le e^{-k} < m_{x|_{n(l-1)}}\, r_{x|_{n(l-1)}} \right\}.$$

Next, we compare the asymptotical growth of objects, related to these cut sets. Therefore, we use the following notation. Let $f, g$ be real valued functions. We say $f$ is asymptotically dominated by $g$ and write

$$f \preceq g \quad \text{if and only if} \quad \limsup_{k\to\infty} \frac{f(k)}{g(k)} \le 1.$$

Let

$$M_k := |\Lambda_k|, \quad T_k := \frac{M_k}{\sum_{x\in\Lambda_k} r_x\, m_x},$$

$$y_k := \max_{x\in\Lambda_k} y_k(x), \quad y_k(x) := n(l) - n(l-1), \quad x \in \Lambda_k, \quad |x| = n(l).$$

The following lemma is a slight modification of [35, Lemma 3.8.(c)].

**Lemma 4.3.8 (c.f. [35, Lemma 3.8.(c)] ):** *There exists* $\alpha' > 0$ *such that*

$$k^{-\alpha'} e^{-k} \preceq (r_{\inf}\, m_{\inf})^{y_k}\, e^{-k} \le r_x\, m_x \le e^{-k}, \quad a.s. \quad \text{for } x \in \Lambda_k.$$

### 4.3.3. Dirichlet-Neumann bracketing

As in Section 4.2 we use a Dirichlet-Neumann bracketing to proof the spectral asymptotics for $\Delta^{\mu^{(I_V)}}$. Firstly, we give a scaling property for the Neumann eigenvalue counting function, which relies on the scaling property established by Arzt [4, Section 3.2.2]. Therefore, let $(\mathcal{E},\mathcal{F})$ be the Dirichlet form on $L_2(\mu^{(I_V)})$ whose eigenvalues coincide with the Neumann eigenvalues of $-\Delta^{\mu^{(I_V)}}$ which, as in Section 4.2, is given by

$$\begin{aligned}\mathcal{F} &= H^1\left([a,b]\right),\\ \mathcal{E}(f,g) &= \int_a^b f'(x)\,g'(x)\,dx.\end{aligned}$$

We write $N_N^{(I_V)}$ for the corresponding eigenvalue counting function. Our scaling property depends on the sequence $\Lambda_k$ defined in Section 4.3.2. Since $\Lambda_k$ is for all $k\in\mathbb{N}$ a cut set, there exists an $n\in\mathbb{N}$ such that $N_k := \left(N_\emptyset, N_{(N_\emptyset)}, N_{\left(N_\emptyset,N_{(N_\emptyset)}\right)},\dots\right)$, $|N_k| = n$ and $N_k\in\Lambda_k$. To each $x\in\Lambda_k\backslash\{N_k\}$ there exists a $x'\in\Lambda$ such that the right neighbour point in $K^{(I_V)}$ of $S_x(b)$ is $S_{x'}(a)$. We let $I_x$ be the gap interval between $S_x[a,b]$ and $S_{x'}[a,b]$ i.e. $I_x := (S_x(b), S_{x'}(a))$.
For the bracketing, we define a sequence of Dirichlet forms $\left((\mathcal{E}^k,\mathcal{F}^k)\right)_{k\in\mathbb{N}}$. Therefore, let

$$\begin{aligned}\mathcal{F}^k := \Big\{ f:[a,b]\longrightarrow\mathbb{R} \,:\, & f\circ S_x\in H^1\left([a,b]\right)\\ & \text{for all } x\in\Lambda_k \ \text{ and } \ f_{|I_x}\in H^1\left(I_x\right)\Big\}.\end{aligned}$$

Using Lemma 4.2.5 iteratively, we obtain:

**Lemma 4.3.9:** *Let $f,g\in\mathcal{F}$ and $k\in\mathbb{N}$. Then for all $x\in\Lambda_k$, $f\circ S_x, g\circ S_x\in\mathcal{F}$ and*

$$\mathcal{E}(f,g) = \sum_{x\in\Lambda_k}\frac{1}{r_x}\mathcal{E}\left(f\circ S_x, g\circ S_x\right) + \sum_{x\in\Lambda_k\backslash\{N_k\}}\int_{I_x} f'(u)\,g'(u)\,du.$$

Thus for

$$\mathcal{E}^k(f,g) := \sum_{x\in\Lambda_k}\frac{1}{r_x}\mathcal{E}\left(f\circ S_x, g\circ S_x\right) + \sum_{x\in\Lambda_k\backslash\{N_k\}}\int_{I_x} f'(u)\,g'(u)\,du, \quad f,g\in\mathcal{F}^k$$

it holds $(\mathcal{E},\mathcal{F})\subseteq(\mathcal{E}^k,\mathcal{F}^k)$. Analogously to [4, Section 3.2.2], we obtain that $(\mathcal{E}^k,\mathcal{F}^k)$ is a Dirichlet form on $L_2(\mu^{(I_V)})$ and that the embedding $\mathcal{F}^k\hookrightarrow L_2(\mu^{(I_V)})$

is a compact operator. Hence, we can refer to the eigenvalue counting function $N_N^k$ of $(\mathcal{E}^k, \mathcal{F}^k, \mu^{(I_V)})$.

**Proposition 4.3.10:** *For all $t \geq 0$, $k \in \mathbb{N}$ holds*

$$N_N^k(t) = \sum_{x \in \Lambda_k} N_N^{(\theta_x I_V)}(r_x\, m_x\, t)\,.$$

*Proof.* Let $f$ be an eigenfunction of $\left(\mathcal{E}^k, \mathcal{F}^k, \mu^{(I_V)}\right)$ with eigenvalue $\lambda$, i.e.

$$\mathcal{E}^k(f,g) = \lambda\,\langle f, g\rangle_{L_2\left(\mu^{(I_V)}\right)} \qquad \text{for all } g \in \mathcal{F}^k.$$

Because $f, g \in L_2\left(\mu^{(I_V)}\right)$ it holds by Lemma 4.3.5

$$\begin{aligned} &\sum_{x \in \Lambda_k} \frac{1}{r_x} \mathcal{E}\left(f \circ S_x, g \circ S_x\right) + \sum_{x \in \Lambda_k \setminus \{N_k\}} \int_{I_x} f'(u)\, g'(u)\, du \\ &= \lambda \sum_{x \in \Lambda_k} m_x\, \langle f \circ S_x, g \circ S_x\rangle_{L_2\left(\mu^{(\theta_x I_V)}\right)}\,. \end{aligned} \tag{4.14}$$

Now, we show that each summand on the left hand side equals each summand on the right hand side, respectively. Therefore, let $h \in \mathcal{F}$ and define for each $y \in \Lambda_k$

$$h_y^k(t) := \begin{cases} h \circ S_y^{-1}(t), & \text{if } t \in S_y([a,b]), \\ 0, & \text{otherwise.} \end{cases}$$

Then $h_y^k \in \mathcal{F}^k$, $h_y^k \circ S_y = h$ for all $y \in \Lambda_k$ and $h_y^k \circ S_x = 0$ for $y \neq x \in \Lambda_k$. Moreover, $h_y'\big|_{I_x} = 0$, for all $y \in \Lambda_k$, $x \in \Lambda_k \setminus \{N_k\}$. With $g = h_y$, we then get in (4.14)

$$\frac{1}{r_y}\,\mathcal{E}\left(f \circ S_y, h\right) = \lambda\, m_y\, \langle f \circ S_y, h\rangle_{L_2\left(\mu^{(\theta_y I_V)}\right)}\,.$$

Because this equation holds for all $h \in \mathcal{F}$, $f \circ S_y$ is an eigenfunction of the Dirichlet form $\left(\mathcal{E}, \mathcal{F}, \mu^{(\theta_y I_V)}\right)$ with eigenvalue $r_y\, m_y\, \lambda$ for all $y \in \Lambda_k$.

Now, let $\lambda > 0$ such that for $x \in \Lambda_k$ $r_x\, m_x\, \lambda$ is an eigenvalue of $\left(\mathcal{E}, \mathcal{F}, \mu^{(\theta_x I_V)}\right)$ with eigenfunction $f_x$, i.e.

$$\mathcal{E}(f_x, g) = r_x\, m_x\, \lambda\, \langle f_x, g\rangle_{L_2\left(\mu^{(\theta_x I_V)}\right)}$$

for all $g \in \mathcal{F}$. Let

$$f(t) := \begin{cases} f_x \circ S_x^{-1}(t), & \text{if } t \in S_x([a,b]) \text{ for some } x \in \Lambda_k, \\ 0, & \text{otherwise.} \end{cases}$$

Then $f \in \mathcal{F}^k$ and $f \circ S_x = f_x$, $x \in \Lambda_k$ and therefore

$$\sum_{x \in \Lambda_k} \frac{1}{r_x} \mathcal{E}\left(f \circ S_x, g\right) = \lambda \sum_{x \in \Lambda_k} m_x \left\langle f \circ S_x, g \right\rangle_{L_2\left(\mu^{(\theta_x I)}\right)}$$

for all $g \in \mathcal{F}$. Since for $g_k \in \mathcal{F}^k$ it holds by definition of $\mathcal{F}^k$, $g_k \circ S_x \in \mathcal{F}$, $x \in \Lambda_k$ and thus

$$\sum_{x \in \Lambda_k} \frac{1}{r_x} \mathcal{E}\left(f \circ S_x, g_k \circ S_x\right) = \lambda \sum_{x \in \Lambda_k} m_x \left\langle f \circ S_x, g_k \circ S_x \right\rangle_{L_2\left(\mu^{(\theta_x I_V)}\right)}. \tag{4.15}$$

But the left hand side of this equation is equal to $\mathcal{E}^k(f, g_k)$, because $f'\big|_{I_x} = 0$ for all $x \in \Lambda_k \backslash \{N_k\}$. With Proposition 4.3.5 we then obtain

$$\mathcal{E}^k(f, g_k) = \lambda \left\langle f, g_k \right\rangle_{L_2\left(\mu^{(I_V)}\right)}$$

for all $g_k \in \mathcal{F}^k$. Therefore, $\lambda$ is an eigenvalue of $\left(\mathcal{E}^k, \mathcal{F}^k, \mu^{(I_V)}\right)$ with corresponding eigenfunction $f$. Using this, we conclude the claim. □

Next, we give the scaling property for the Dirichlet eigenvalue counting function, which relies on the scaling property established by Arzt [4, Seciton 3.2.3]. Therefore, let $(\mathcal{E}, \mathcal{F}_0)$ be the Dirichletform on $L_2\left(\mu^{(I_V)}\right)$ whose eigenvalues coincide with the Dirichlet eigenvalues of $-\Delta^{\mu^{(I_V)}}$, i.e. $\mathcal{E}$ is defined as before and

$$\mathcal{F}_0 := \{f \in \mathcal{F} : \ f(a) = f(b) = 0\}.$$

As for the Neumann eigenvalue counting function, we define a sequence of Dirichlet forms $\left(\mathcal{E}^k, \mathcal{F}_0^k\right)$ on $L_2\left(\mu^{(I_V)}\right)$, where

$$\mathcal{F}_0^k := \{f \in \mathcal{F}_0 : \ f(t) = 0 \text{ for } t \in I_x, \ x \in \Lambda_k \backslash \{N_k\}\}, \quad k \in \mathbb{N}.$$

and

$$\mathcal{E}^k := \mathcal{E}_{|_{\mathcal{F}_0^k \times \mathcal{F}_0^k}}.$$

We denote the correspond eigenvalue counting function by $N_D^k$.

**Proposition 4.3.11:** *For all $t \geq 0$ holds*

$$N_D^k(t) = \sum_{x \in \Lambda_k} N_D^{(\theta_x I_V)}\left(r_x\, m_x\, t\right).$$

*Proof.* Let $f$ be an eigenfunction of $\left(\mathcal{E}^k, \mathcal{F}_0^k, \mu^{(I_V)}\right)$ with eigenvalue $\lambda$. Then,

$$\mathcal{E}^k(f,g) = \lambda \left\langle f, g\right\rangle_{L_2\left(\mu^{(I_V)}\right)},$$

for all $g \in \mathcal{F}_0^k$. Therefore, we get with Lemma 4.3.9 and Lemma 4.3.5,

$$\begin{aligned}
&\sum_{x \in \Lambda_k} \frac{1}{r_x} \mathcal{E}\left(f \circ S_x, g \circ S_x\right) + \sum_{x \in \Lambda_k \setminus \{N_k\}} \int_{I_x} f'(u)\, g'(u)\, du \\
&= \lambda \sum_{x \in \Lambda_k} m_x \left\langle f \circ S_x, g \circ S_x \right\rangle_{L_2\left(\mu^{(\theta_x I_V)}\right)}.
\end{aligned}$$

For $h \in \mathcal{F}_0$, we define

$$h_y^k(t) := \begin{cases} h \circ S_y^{-1}(t), & \text{if } t \in S_y([a,b]), \\ 0, & \text{otherwise.} \end{cases}$$

Because $h \in \mathcal{F}_0$, it follows $h_y^k \in \mathcal{F}_0^k$, $h_y^k \circ S_y = h$ for $y \in \Lambda_k$ and $h_y \circ S_x = 0$ for $y \neq x \in \Lambda_k$. Hence,

$$\frac{1}{r_y} \mathcal{E}\left(f \circ S_y, h\right) = \lambda\, m_y \left\langle f \circ S_y, h \right\rangle_{L_2\left(\mu^{(\theta_y I_V)}\right)},$$

for all $y \in \Lambda_k$. Therefore, $r_y\, m_y\, \lambda$ is an eigenvalue of $\left(\mathcal{E}, \mathcal{F}_0, \mu^{(\theta_y I_V)}\right)$ with eigenfunction $f \circ S_y$, $y \in \Lambda_k$.

Now, let $r_x\, m_x\, \lambda$ be an eigenvalue of $\left(\mathcal{E}, \mathcal{F}_0, \mu^{(\theta_x I_V)}\right)$ for some $\lambda > 0$ with corresponding eigenfunction $f_x$, $x \in \Lambda_k$, i.e.

$$\mathcal{E}(f_x, g) = r_x\, m_x\, \lambda \left\langle f_x, g\right\rangle_{L_2\left(\mu^{(\theta_x I_V)}\right)}$$

for all $g \in \mathcal{F}_0$. Let

$$f(t) := \begin{cases} f_x \circ S_x^{-1}(t), & \text{if } t \in S_x([a,b]) \text{ for some } x \in \Lambda_k, \\ 0, & \text{otherwise.} \end{cases}$$

Since $f_x \in \mathcal{F}_0$ it follows $f \in \mathcal{F}_0^k$ and because of $f \circ S_x = f_x$, $x \in \Lambda_k$, we obtain

$$\sum_{x \in \Lambda_k} \frac{1}{r_x} \mathcal{E}\left(f \circ S_x, g\right) = \lambda \sum_{x \in \Lambda_k} m_x \left\langle f \circ S_x, g \right\rangle_{L_2\left(\mu^{(\theta_x I_V)}\right)}$$

for all $g \in \mathcal{F}_0$. For $g_k \in \mathcal{F}_0^k$ holds $g_k \circ S_x \in \mathcal{F}_0$, $x \in \Lambda_k$ and therefore as in (4.15) follows

$$\mathcal{E}^k(f, g_k) = \lambda \left\langle f, g_k \right\rangle_{L_2\left(\mu^{(I_V)}\right)}$$

for all $g_k \in \mathcal{F}^k$. Hence, $\lambda$ is an eigenvalue of $\left(\mathcal{E}^k, \mathcal{F}_0^k, \mu^{(I_V)}\right)$ with eigenfunction $f$ and we conclude the claim. ☐

Since $\left(\mathcal{E}^k, \mathcal{F}^k, \mu^{(I_V)}\right)$ is an extension of $\left(\mathcal{E}, \mathcal{F}, \mu^{(I_V)}\right)$ and $\left(\mathcal{E}, \mathcal{F}_0, \mu^{(I_V)}\right)$ is an extension of $(\mathcal{E}^k, \mathcal{F}_0^k, \mu^{(I_V)})$ for all $k \in \mathbb{N}$, we infer from the Max-Min principle, see e.g. Lapidus and Kigami [57, Theorem 4.5], the Dirichlet-Neumann bracketing:

**Corollary 4.3.12 (Dirichlet-Neumann bracketing):** *For all $t \geq 0$ and $k \in \mathbb{N}$ holds*

$$\sum_{x \in \Lambda_k} N_D^{(\theta_x I_V)}\left(r_x\, m_x\, t\right) \leq N_D^{(I_V)}(t) \leq N_N^{(I_V)}(t) \leq \sum_{x \in \Lambda_k} N_N^{(\theta_x I_V)}\left(r_x\, m_x\, t\right).$$

### 4.3.4. Spectral asymptotics

In this section we investigate the spectral exponent for $V$-variable Cantor measures. Therefore, we first give some estimates for the first Dirichlet eigenvalue. Remark that we assume (C1)-(C3) to hold.

**Lemma 4.3.13:** *It holds*

$$\frac{1}{(b-a)} \leq \lambda_1^{\mu^{(I_V)}} \leq \frac{1 - r_{\inf}^2}{\left(r_{\inf}\, m_{\inf}(1 - r_{\sup})\right)^2 (b-a) r_{\sup}}.$$

*Proof.* For the first estimate, let $f$ be an eigenfunction of $\left(\mathcal{E}, \mathcal{F}_0, \mu^{(I_V)}\right)$ such that

$\|f\|_{L_2(\mu)} = 1$. By the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} f^2(x) &= (f(x) - f(a))^2 \\ &= \left(\int_a^x f'(y)dy\right)^2 \\ &\le \|f'\|^2_{L_2(\lambda^1,[a,x])}(x-a) \\ &\le \|f'\|^2_{L_2(\lambda^1,[a,b])}(b-a). \end{aligned}$$

Integration with respect to $\mu$ yields

$$1 \le \|f'\|^2_{L_2(\lambda^1,[a,b])}(b-a).$$

Since $f$ is an eigenfunction of $\left(\mathcal{E}, \mathcal{F}_0, \mu^{(I_V)}\right)$, we get

$$\|f'\|^2_{L_2(\lambda^1,[a,b])} = \langle f', f'\rangle_{L_2(\lambda^1,[a,b])} = \mathcal{E}(f,f) = \lambda_1^{\mu^{(I_V)}}.$$

Hence, the first estimate follows. For the second estimate, define $x_1 \coloneqq S_1^{(\emptyset)}\left(S^{(1)}_{N_{(1)}}(a)\right) = a + r_1^{(\emptyset)}\left(1 - r_{N_{(1)}}\right)(b-a)$, $x_2 \coloneqq S_1^{(\emptyset)}(b) = a + r_1^{(\emptyset)}(b-a)$ and

$$\hat{f}(x) \coloneqq \begin{cases} \frac{x-a}{x_1-a}, & \text{if } x \in [a, x_1] \\ 1, & \text{if } x \in (x_1, x_2] \\ \frac{b-x}{b-x_2}, & \text{if } x \in (x_2, b]. \end{cases}$$

Therefore, $\hat{f}$ is constant 1 on the very right second-level cell which remains from the very left first-level cell and linear interpolated from $a$ to $x_1$ and $b$ to $x_2$ such that $\hat{f} \in \mathcal{F}_0$. Hence,

$$\begin{aligned} \mathcal{E}(\hat{f},\hat{f}) &= \int_a^b \left(\hat{f}'\right)^2 dx \\ &= \frac{1}{x_1 - a} + \frac{1}{b - x_2} \\ &= \frac{1 - r_1^{(\emptyset)} + r_1^{(\emptyset)}\left(1 - r^{(1)}_{N_{(1)}}\right)}{r_1^{(\emptyset)}\left(1 - r^{(1)}_{N_{(1)}}\right)\left(1 - r_1^{(\emptyset)}\right)(b-a)}. \end{aligned}$$

Further,

$$\int_a^b \left(\hat{f}\right)^2 d\mu \ge m_1^{(\emptyset)} m^{(1)}_{N_{(1)}}.$$

As explained in [4, Section 3.3.2], it holds

$$\lambda_1^{\mu^{(I_V)}} = \inf_{f \in \mathcal{F}_0} \frac{\mathcal{E}(f,f)}{\|f\|^2_{L_2(\mu)}}.$$

For this result, see e.g. [26, Theorem 1.3]. Therefore,

$$\begin{aligned}
\lambda_1^{\mu^{(I_V)}} = \inf_{f \in \mathcal{F}_0} \frac{\mathcal{E}(f,f)}{\|f\|^2_{L_2(\mu)}} &\leq \frac{\mathcal{E}(\hat{f},\hat{f})}{\left\|\hat{f}\right\|^2_{L_2(\mu)}} \\
&\leq \frac{1 - r_1^{(\emptyset)} + r_1^{(\emptyset)}\left(1 - r_{N_{(1)}}^{(1)}\right)}{r_1^{(\emptyset)}\left(1 - r_{N_{(1)}}^{(1)}\right)\left(1 - r_1^{(\emptyset)}\right)(b-a)\, m_1^{(\emptyset)}\, m_{N_{(1)}}^{(1)}} \\
&\leq \frac{1 - r_{\inf}^2}{\left(r_{\inf}\, m_{\inf}(1 - r_{\sup})\right)^2 (b-a)}.
\end{aligned}$$

□

Together with the Dirichlet-Neumann bracketing, we can estimate $N_D^{\mu^{(I_V)}}$ by $M_k$. Therefore, let

$$\eta := r_{\inf}\, m_{\inf}.$$

**Lemma 4.3.14:** *There exist $c_1, c_2 > 0$ such that for almost all $\omega \in \Omega$*

$$N_D^{(I_V)}(T_k) \leq c_1 M_k, \quad M_k \leq N_D^{(I_V)}(c_2 T_k \eta^{-y_k})$$

*for all $k \geq 0$*

*Proof.* With Corollary 4.3.12 and Lemma 2.1.1, we get

$$\begin{aligned}
N_D^{(I_V)}(T_k) &\leq \sum_{x \in \Lambda_k} N_N^{(\theta_x I_V)}(m_x r_x T_k) \\
&\leq 2\, M_k + \sum_{x \in \Lambda_k} N_D^{(\theta_x I_V)}(m_x r_x T_k) \\
&\leq 2\, M_k + c\, T_k \sum_{x \in \Lambda_k} m_x r_x \\
&\leq c_1 M_k,
\end{aligned}$$

where the third estimate follows from Lemma 2.1.1. Together with

$$(r_x\, m_x)^{-1} \leq \eta^{-y_k} e^{+k} \leq \eta^{-y_k} T_k, \quad x \in \Lambda_k \quad \text{for all} \quad k$$

which follows from Lemma 4.3.8, and Lemma 4.3.12, Lemma 4.3.13, we get

$$M_k = \sum_{x\in\Lambda_k} N_D^{(\theta_x I_V)}\left(\lambda_1^{\mu^{(\theta_x I_V)}}\right) \le \sum_{x\in\Lambda_k} N_D^{(\theta_x I_V)}\left(c_2 r_x\, m_x\, (r_x\, m_x)^{-1}\right)$$
$$\le N_D^{(I_V)}(c_2 T_k \eta^{-y_k}),$$

thereby we also used that $\lambda_1^{\mu^{(I_V)}} < \lambda_2^{\mu^{(I_V)}}$, see [53, Theorem 4]. □

**Lemma 4.3.15:** *For almost all $\omega \in \Omega$ there exists $k_0(\omega) \in \mathbb{N}$ and $\alpha, c_1 > 0$ such that*

$$N_D^{(I_V)}(T_k) \le c_1 M_k, \quad M_k \le N_D^{(I_V)}(k^\alpha T_k), \quad \text{for } k > k_0(\omega).$$

*Proof.* The lemma follows from Lemma 4.3.14 and $\eta^{-y_k} \preceq k^{\alpha'}$ by Lemma 4.3.8. □

The spectral exponent is given as the unique zero strictly bigger than zero of the function defined in the next lemma. This lemma shows that this zero is indeed unique and exists. It is a slight modification of [35, Lemma 4.12].

**Lemma 4.3.16 (c.f. [35, Lemma 4.12]):** *Let*

$$f(t) := \mathbb{E}_V \log \sum_{\substack{x\in I_V,\\ |x|=n(1)}} (m_x r_x)^t, \quad t \ge 0.$$

*Then, there exists a unique $\gamma_V > 0$ such that $f(\gamma_V) = 0$.*

*Proof.* $f$ is strictly decreasing and continuous. Since $f(0) > 0$ and $\lim_{t\to\infty} f(t) = -\infty$, the claim follows. □

The following corollary is a slight modification of [35, Proposition 4.13].

**Corollary 4.3.17 (c.f. [35, Proposition 4.13] ):** *It holds almost surely*

$$\lim_{k\to\infty} \frac{1}{k} \log \sum_{\substack{x\in I_V,\\ |x|=n(k)}} (m_x r_x)^t = f(t), \quad t \ge 0.$$

*Proof.* This corollary follows from (4.13). □

The following theorem is the main result of this section. It is a slight modification of [35, Theorem 4.14].

**Theorem 4.3.18 (c.f. [35, Theorem 4.14]):** *The spectral exponent is almost surely given by the unique solution $\gamma_V > 0$ of*

$$f(\gamma_V) = 0,$$

*where $f$ is defined in Lemma 4.3.16.*

*Proof.* By Corollary 4.3.16, the solution exists and is unique. Therefore we have to show that

$$\lim_{s\to\infty} \frac{\log N_D^{(I_V)}(s)}{\log s} = \gamma_V \quad a.s.$$

To this end, we define for $|x| = n(k)$

$$\tau_x(t) := \frac{(r_x m_x)^t}{\sum\limits_{\substack{y\in I_V,\\ |y|=n(k)}} (r_y m_y)^t}.$$

By Proposition 4.3.17 we get for $t > \gamma_V$ (i.e. $f(x) < 0$) for $\epsilon = \epsilon(\omega) > 0$ small enough that for all $c > 0$ there exists $k_0 = k_0(\omega) \in \mathbb{N}$ such that

$$\tau_x(t) \geq (r_x m_x)^t e^{-k(f(t)+\epsilon)} \geq c\,(r_x m_x)^t, \qquad \text{for all } k \geq k_0. \tag{4.16}$$

Since for $|x| = n(k)$ it holds

$$\begin{aligned}\tau_x(t) &= \frac{\sum\limits_{\substack{l\in\theta_x I,\\ |l|=n(k+1)-n(k)}} (r_{xl} m_{xl})^t}{\sum\limits_{\substack{y\in I_V,\\ |y|=n(k)}} (r_y m_y)^t \sum\limits_{\substack{l\in\theta_x I,\\ |l|=n(k+1)-n(k)}} (r_l m_l)^t} \\ &= \frac{\sum\limits_{\substack{l\in\theta_x I,\\ |l|=n(k+1)-n(k)}} (r_{xl} m_{xl})^t}{\sum\limits_{\substack{y\in I_V,\\ |y|=n(k+1)}} (r_y m_y)^t},\end{aligned}$$

where the second equality holds because $\theta_{x_1} I_V = \theta_{x_2} I_V$ for all $|x_1| = |x_2| = n(k)$, we get for every $k \in \mathbb{N}$

$$\sum_{x\in\Lambda_k} \tau_x(t) = 1$$

and thus we obtain from Lemma 4.3.8 for some $t' > 0$ and all $k \geq k_0$,

$$1 = \sum_{x \in \Lambda_k} \tau_x(t) \geq \sum_{x \in \Lambda_k} c\,(r_x m_x)^t \succeq c M_k k^{-tt'} e^{-kt}.$$

Therefore,

$$M_k \preceq c k^{tt'} e^{kt} \quad a.s. \tag{4.17}$$

For $s > 1$ large enough let $k$ be such that $s \in (e^{k-1}, e^k]$. By Lemma 4.3.8 we then have $s \leq T_k$. Together with (4.17) and Lemma 4.3.15,

$$\frac{\log N_D^{(I_V)}(s)}{\log s} \leq \frac{\log N_D^{(I_V)}(T_k)}{\log s} \leq \frac{\log(cM_k)}{k-1} \preceq t \quad a.s.$$

Since this holds for all $t > \gamma_V$, it follows

$$\frac{\log N_D^{(I_V)}(s)}{\log s} \preceq \gamma_V \quad a.s.$$

Now, let $t < \gamma_V$ (i.e. $f(t) > 0$). For $\epsilon > 0$ small enough we get for some $k_0 \in \mathbb{N}$, analogously to the estimates in (4.16),

$$1 = \sum_{x \in \Lambda_k} \tau_x(t) \leq \sum_{x \in \Lambda_k} c\,(r_x m_x)^t \leq c M_k e^{-kt}, \qquad \text{for all } k \geq k_0$$

and thus

$$M_k \geq c e^{kt}, \qquad \text{for all } k \geq k_0.$$

From Lemma 4.3.15 it follows

$$\frac{\log N_D^{(I_V)}(k^\alpha T_k)}{k} \geq \frac{\log M_k}{k} \succeq t \quad a.s. \tag{4.18}$$

for some $\alpha > 0$. For $s > 1$ large enough and $k$ such that $s \in (e^{k-1}, e^k]$, we get again from Lemma 4.3.8 for some $\alpha' > 0$

$$k^\alpha T_k \preceq k^{\alpha'} e^k \leq e(1 + \log s)^{\alpha'} s \quad a.s.$$

and thus

$$\liminf_{k\to\infty}\frac{\log N_D^{(I_V)}(k^\alpha T_k)}{k}\leq\liminf_{t\to\infty}\frac{\log N_D^{(I_V)}(e(1+\log s)^{\alpha'}s)}{\log s}\quad a.s.$$

Since

$$\lim_{s\to\infty}\frac{\log e(1+\log s)^{\alpha'}s}{\log s}=1,\quad\lim_{s\to\infty}e(1+\log s)^{\alpha'}s=\infty$$

it follows

$$\liminf_{k\to\infty}\frac{\log N_D^{(I_V)}(k^\alpha T_k)}{k}\leq\liminf_{s\to\infty}\frac{\log N_D^{(I_V)}(s)}{\log s}\quad a.s.$$

Since (4.18) holds for all $t<\gamma_V$ we then obtain

$$\frac{\log N_D^{(I_V)}(s)}{\log s}\succeq\gamma\quad a.s. \tag{4.19}$$

Combining (4.18) and (4.19), the claim follows. □

**Remark 4.3.19:** *With Lemma 2.1.2 we also obtain*

$$\lim_{t\to\infty}\frac{\log N_N^{(I_V)}(t)}{\log t}=\gamma\quad a.s.$$

# Chapter 5

# Spectral Asymptotics for the Laplacian on Statistically Self-Similar Cantor Strings

As in [18, Chapter 3], we investigate applications of the Central Limit Theorem 3.3.7 in this chapter. Also, we consider the Law of the Iterated Logarithm 3.4.3 for the same applications, which is not treated in [18]. The reproduction function and life length of the considered general branching process are defined by (3.5).

Charmoy, Croydon and Hambly [18, Chapter 3] assumed (for the CLT) that $\xi(\infty) = n$ for all individuals for some $n \in \mathbb{N}$ and that the birth times $\sigma_1, \dots, \sigma_n$ are distributed such that for some fixed $\gamma \in (0, \infty)$ it holds

$$\sum_{i=1}^{n} e^{-\gamma\sigma_i} = 1 \qquad a.s.$$

With the latter assumption the limit $W$ of the fundamental martingale of the underlying general branching process is almost surely constant 1. We would like to give a Central Limit Theorem without these conditions.
For simplicity, we assume that $\nu_\gamma$, defined in (3.2), is non-lattice. By the discussion of the Central Limit Theorem in the lattice case (see [18, Section 2.5]), we expect similar results to hold if $\nu_\gamma$ is lattice.

The idea of the proofs, constructions and structure we use in this chapter are taken from [18], where the case $W = 1$ a.s. is treated.

In our setting, the general branching process $(\xi_x, L_x, \phi_x)_x$ with Malthusian parameter $\gamma$ satisfies Nerman's Strong Law of Large Numbers Theorem 3.3.4 which yields

$$e^{-\gamma t} Z^{\phi}(t) \stackrel{t\to\infty}{\longrightarrow} z^{\phi}(\infty)\, W \quad \text{a.s.}$$

in the non-lattice case. The focus of our investigation is the random fluctuation of $e^{-\gamma t} Z^{\phi}(t)$ around its limit. Therefore, as in [18, Chapter 3], we decompose $e^{\gamma t/2}(e^{-\gamma t} Z^{\phi}(t) - z^{\phi}(\infty)\, W)$ into two parts

$$\begin{aligned} &e^{\gamma t/2}\left(e^{-\gamma t} Z^{\phi}(t) - z^{\phi}(\infty)\, W\right) = \\ &\qquad e^{-\gamma t/2}\big(Z^{\phi}(t) - e^{\gamma t} z^{\phi}(t)\, W\big) + e^{\gamma t/2}\left(z^{\phi}(t) - z^{\phi}(\infty)\right) W. \end{aligned} \tag{5.1}$$

Firstly, the aim is to apply the Central Limit Theorem on the first summand. The second will converge to 0. Together with Slutsky's lemma we then obtain a result on the random fluctuation. Secondly, to describe the random fluctuation of $e^{-\gamma t} Z^{\phi}(t)$ around its limit almost surely, we want to apply the Law of the Iterated Logarithm Theorem 3.4.3 which can be done with the same decomposition.

The main result of this chapter is provided in Theorem 5.3.2 which is a result of the present thesis.

## 5.1. Centering the process

We start by defining a characteristic $\bar{\zeta}$ such that

$$Z^{\phi}(t) - e^{\gamma t} z^{\phi}(t)\, W = Z^{\bar{\zeta}}(t).$$

Then, as required for the Central Limit Theorem 3.3.7 and the Law of the Iterated Logarithm Theorem 3.4.3, $\bar{\zeta}$ is a centred characteristic. Therefore, let

$$\bar{\zeta}^{\phi}_{\emptyset}(t) \coloneqq \bar{\zeta}_{\emptyset}(t) \coloneqq \phi_{\emptyset}(t) + \sum_{i=1}^{\xi_{\emptyset}(\infty)} e^{\gamma(t-\sigma_i)} z^{\phi}(t-\sigma_i)\, W_i - e^{\gamma t} z^{\phi}(t)\, W. \tag{5.2}$$

Thanks to (3.16), we get

$$\begin{aligned} \bar{\zeta}_{\emptyset}(t) &= \phi_{\emptyset}(t) + \sum_{i=1}^{\xi_{\emptyset}(\infty)} e^{\gamma(t-\sigma_i)} z^{\phi}(t-\sigma_i)\, W_i - e^{\gamma t} z^{\phi}(t)\, W \\ &= \phi_{\emptyset}(t) + \sum_{i=1}^{\xi_{\emptyset}(\infty)} e^{\gamma(t-\sigma_i)} W_i \left(z^{\phi}(t-\sigma_i) - z^{\phi}(t)\right). \end{aligned}$$

With the last equality we can control the second term by controlling $|z^{\phi}(t-\sigma_i) - z^{\phi}(t)|$ as the following lemma shows.

**Lemma 5.1.1:** *Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process. Assume that*

$$|z^{\phi}(t) - z^{\phi}(\infty)| \leq c_1 e^{-\beta_1 t} \wedge c_2,$$

*for some $\beta_1 \in [0, \infty)$. Then,*

$$|\bar{\zeta}_{\emptyset}(t)| \leq |\phi_{\emptyset}(t)| + 2 \sum_{i=1}^{\xi_{\emptyset}(\infty)} W_i e^{-\gamma\sigma_i} \left( c_1 e^{(\gamma-\beta_1)t} e^{\beta_1 \sigma_i} \wedge c_2 e^{\gamma t} \right).$$

*Proof.* Remark that $W, W_i > 0$ a.s. and thus

$$\begin{aligned}
|\bar{\zeta}_{\emptyset}(t)| &\leq |\phi_{\emptyset}(t)| + \sum_{i=1}^{\xi_{\emptyset}(\infty)} e^{\gamma(t-\sigma_i)} W_i \left( |z^{\phi}(t-\sigma_i) - z^{\phi}(\infty)| + |z^{\phi}(t) - z^{\phi}(\infty) \right) \\
&\leq |\phi_{\emptyset}(t)| + \sum_{i=1}^{\xi_{\emptyset}(\infty)} e^{\gamma(t-\sigma_i)} W_i \left( \left( c_1 e^{-\beta_1(t-\sigma_i)} \wedge c_2 \right) + \left( c_1 e^{-\beta_1 t} \wedge c_2 \right) \right) \\
&\leq |\phi_{\emptyset}(t)| + 2 \sum_{i=1}^{\xi_{\emptyset}(\infty)} W_i e^{-\gamma\sigma_i} \left( c_1 e^{(\gamma-\beta_1)t} e^{\beta_1\sigma_i} \wedge c_2 e^{\gamma t} \right)
\end{aligned}$$

□

**Remark 5.1.2:** *If $\beta_1 \leq \gamma$ in Lemma 5.1.1, we infer*

$$|\bar{\zeta}_{\emptyset}(t)| \leq |\phi_{\emptyset}(t)| + 2c_1 e^{(\gamma-\beta_1)t} \sum_{i=1}^{\xi_{\emptyset}(\infty)} W_i$$

## 5.2. Investigation on the conditions of the Central Limit Theorem and the Law of the Iterated Logarithm

With Lemma 5.1.1, we are able to bound $\bar{\zeta}$ by a constant (depending on $\omega$) if $\beta_1 = \gamma$ and if we are able to bound $\phi$. In our investigation, $|\phi_{\emptyset}(t)| \leq \xi_{\emptyset}(\infty)\, c_1$ and hence for $\beta_1 = \gamma$,

$$|\bar{\zeta}_{\emptyset}(t)| \leq c_1 \left( \xi_{\emptyset}(\infty) + \sum_{i=1}^{\xi_{\emptyset}(\infty)} W_i \right).$$

We assume that $\phi$ and therefore also $\bar{\zeta}$ vanishes for negative times. As explained in (3.10), we can use other characteristics if $\phi$ does not vanish for negative times. We start our investigation by considering Condition 3.3.5.

**Lemma 5.2.1:** *Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process with Malthusian parameter $\gamma \in [0,1]$, $\mathbb{E}\xi(\infty) < \infty$ and $\bar{\zeta}$ be defined as in (5.2). Assume that $\phi \geq 0$ and vanishes for negative times, $\nu_\gamma$ defined in (3.2) is non-lattice and that*

$$|\bar{\zeta}_\emptyset(t)| \leq c_1 \left( \xi_\emptyset(\infty) + \sum_{i=1}^{\xi_\emptyset(\infty)} W_i \right).$$

*Then Condition 3.3.5 and Condition 3.4.1 are satisfied.*

*Proof.* Let $\epsilon < \gamma/2$. Then,

$$\left| e^{-\gamma t/2} \sum_{\sigma_x \leq \epsilon t} \bar{\zeta}_x(t - \sigma_x) \right| \leq e^{(\epsilon - \gamma/2)t} e^{-\epsilon t} c_1 \sum_{\sigma_x \leq \epsilon t} \left( \xi_x(\infty) + \sum_{i=1}^{\xi_x(\infty)} W_{x,i} \right).$$

From Theorem 3.3.4 follows

$$e^{-\gamma \epsilon t} \sum_{\sigma_x \leq \epsilon t} \left( \xi_x(\infty) + \sum_{i=1}^{\xi_x(\infty)} W_{x,i} \right) \xrightarrow{t \to \infty} c_2, \quad \text{a.s.}$$

Since $\epsilon < \gamma/2$, the claim follows. □

Next, we show that for $\beta_1 = \gamma$ in Lemma 5.1.1, Condition 3.3.6 is satisfied with $\kappa = 1$ and therefore Condition 3.4.2 is also satisfied. We firstly recall the definition of $\psi$ in Lemma 3.5.1, i.e.

$$\psi(\theta) := \mathbb{E} \sum_{i=1}^{\xi_\emptyset(\infty)} e^{-\theta \gamma \sigma_i}.$$

To check Condition 3.3.6 and Condition 3.4.2 we have to assume that the sixth moment of $\xi(\infty)$ is finite, i.e.

$$\mathbb{E}\xi(\infty)^6 < \infty.$$

One would expect that we need $\mathbb{E}\xi(\infty)^{\kappa+2} < \infty$ to get Condition 3.3.6 and therefore if we take $\kappa = 1$ it should be enough to have finiteness of the third moment.

However, since we use the Cauchy-Schwarz inequality to get Condition 3.3.6, we need boundedness of the $(3 \cdot 2 = 6)$-th moment of $\xi(\infty)$.

**Lemma 5.2.2:** *Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process with Malthusian parameter $\gamma \in [0,1]$ and $\bar{\zeta}$ defined as in (5.2). Assume that $\phi \geq 0$ and vanishes for negative $t$, $\nu_\gamma$ defined in (3.2) is non-lattice, the conditions of Lemma 3.5.2 are satisfied, i.e.*

$$\phi(t) \leq c_1 \, \xi(\infty), \qquad \mathbb{E}\xi(\infty)^6 < \infty$$

*and for all $t$*

$$|\,\bar{\zeta}_\emptyset(t)| \leq c_1 \left( \xi_\emptyset(\infty) + \sum_{i=1}^{\xi_\emptyset(\infty)} W_{x,i} \right).$$

*Furthermore, assume that $v$ defined in (3.12) is bounded. Then it holds*

$$\sup_{t \geq 0} \mathbb{E}|e^{-\gamma t/2} \bar{Z}(t)|^3 < \infty.$$

*Proof.* We define $\bar{Q}$ analogously to $Q$ in the proof of Lemma 3.5.2, that is

$$\begin{aligned} \bar{Q}_\emptyset(t) &:= \bar{\zeta}_\emptyset(t)^3 + 3\,\bar{\zeta}_\emptyset(t)^2 \sum_{i=1}^{\xi_\emptyset(\infty)} \bar{Z}_i(t-\sigma_i) + 3\,\bar{\zeta}_\emptyset(t) \sum_{i,j=1}^{\xi_\emptyset(\infty)} \bar{Z}_i(t-\sigma_i)\bar{Z}_j(t-\sigma_j) \\ &\quad + \sum_{\substack{i,j,k=1, \\ \text{not all equal}}}^{\xi_\emptyset(\infty)} \bar{Z}_i(t-\sigma_i)\bar{Z}_j(t-\sigma_j)\bar{Z}_k(t-\sigma_k). \end{aligned}$$

Therefore

$$\bar{Z}(t)^3 = Z^{\bar{Q}}(t)$$

and thus $|\bar{Z}(t)|^3$ is bounded by

$$\sum_{x\in I}|\,\bar{\zeta}_x(t-\sigma_x)|^3 \tag{5.3}$$

$$+\,3\sum_{x\in I}|\,\bar{\zeta}_x(t-\sigma_x)|^2\sum_{i=1}^{\xi_x(\infty)}|\bar{Z}_{x,i}(t-\sigma_{x,i})| \tag{5.4}$$

$$+\,3\sum_{x\in I}|\,\bar{\zeta}_x(t-\sigma_x)|\sum_{i,j=1}^{\xi_x(\infty)}|\bar{Z}_{x,i}(t-\sigma_{x,i})||\bar{Z}_{x,j}(t-\sigma_{x,j})| \tag{5.5}$$

$$+\sum_{x\in I}\sum_{\substack{i,j,k=1,\\ \text{not all equal}}}^{\xi_x(\infty)}|\bar{Z}_{x,i}(t-\sigma_{x,i})||\bar{Z}_{x,j}(t-\sigma_{x,j})||\bar{Z}_{x,k}(t-\sigma_{x,k})|. \tag{5.6}$$

Multiplying (5.3) with $e^{-3\gamma t/2}$ and using the estimate of $\bar{\zeta}$, we get

$$\begin{aligned}e^{-3\gamma t/2}\mathbb{E}\sum_{x\in I}|\,\bar{\zeta}_x(t-\sigma_x)|^3 &\le c_1^3\mathbb{E}\sum_{x\in I}e^{-\frac{3}{2}\gamma\sigma_x}\left(\xi_x(\infty)+\sum_{i=1}^{\xi_x(\infty)}W_{x,i}\right)^3\\ &= c_1^3\mathbb{E}\sum_{x\in I}e^{-\frac{3}{2}\gamma\sigma_x}\mathbb{E}\left(\left(\xi_x(\infty)+\sum_{i=1}^{\xi_x(\infty)}W_{x,i}\right)^3\middle|\mathcal{F}_x\right).\end{aligned}$$

Since $W_{x,i}$ is independent of $\mathcal{F}_x$ and $W_{x,j}$ for $j\neq i$, distributed like $W$, $\sigma_x$ is $\mathcal{F}_{||x|-1}$ measurable, $\xi_x$ is independent of $\mathcal{F}_{||x|-1}$ and the third moment is finite, there exists a $c_2$ such that

$$c_1^3\mathbb{E}\sum_{x\in I}e^{-\frac{3}{2}\gamma\sigma_x}\mathbb{E}\left(\left(\xi_x(\infty)+\sum_{i=1}^{\xi_x(\infty)}W_{x,i}\right)^3\middle|\mathcal{F}_x\right)\le c_1^3c_2\mathbb{E}\sum_{x\in I}e^{-\frac{3}{2}\gamma\sigma_x}$$

and thus

$$e^{-3\gamma t/2}\sum_{x\in I}|\,\bar{\zeta}_x(t-\sigma_x)|^3\le c_1^3c_2\mathbb{E}\sum_{x\in I}e^{-\frac{3}{2}\gamma\sigma_x}=c_1^3c_2\sum_{k=0}^{\infty}\psi(3/2)^k<\infty.$$

To estimate (5.4) we use the same arguments as for (5.3) and the boundedness of

$v$ to obtain

$$
\begin{aligned}
&3\, e^{-3\gamma t/2}\mathbb{E}\sum_{x\in I} |\,\bar{\zeta}_x(t-\sigma_x)|^2 \sum_{i=1}^{\xi_x(\infty)} |\bar{Z}_{x,i}(t-\sigma_{x,i})| \\
&\leq 3\, c_1^2\, e^{-3\gamma t/2}\mathbb{E}\sum_{x\in I} e^{\gamma(t-\sigma_x)} \left(\xi_x(\infty) + \sum_{i=1}^{\xi_x(\infty)} W_x\right)^2 \sum_{i=1}^{\xi_x(\infty)} |\bar{Z}_{x,i}(t-\sigma_{x,i})| \\
&\leq 3\, c_1^2\, e^{-3\gamma t/2}\mathbb{E}\sum_{x\in I} e^{\gamma(t-\sigma_x)}\mathbb{E}\left(\left(\xi_x(\infty) + \sum_{i=1}^{\xi_x(\infty)} W_{x,i}\right)^4 \middle| \mathcal{F}_x\right)^{1/2} \cdot \\
&\qquad\qquad \sum_{i=1}^{\xi_x(\infty)} \mathbb{E}(|\bar{Z}_{x,i}(t-\sigma_{x,i})|^2|\mathcal{F}_x)^{1/2} \\
&\leq 3\, c_1^2\, c_3^{1/2}\, e^{-3\gamma t/2}\mathbb{E}\sum_{x\in I} e^{3\gamma(t-\sigma_x)/2}\mathbb{E}\left(\left(\xi_x(\infty) + \sum_{i=1}^{\xi_x(\infty)} W_{x,i}\right)^4 \middle| \mathcal{F}_x\right)^{1/2} \xi_x(\infty) \\
&< \infty,
\end{aligned}
$$

where we get $\mathbb{E}\bar{Z}(t)^2 \leq c_3 e^{\gamma t}$ since $v$ is bounded and get the boundedness of $\mathbb{E}\left((\xi_x(\infty)+W_x)^4 \middle| \mathcal{F}_x\right)$ since the fourth moment of $\xi(\infty)$ is bounded.

To estimate (5.5), we need the following Lemma.

**Lemma 5.2.3:** *It holds for all $t$*

$$\mathbb{E}\bar{Z}(t)^4 \leq d_1 e^{3\gamma t}.$$

Therefore, we get

$$
\begin{aligned}
&e^{-3\gamma t/2}\mathbb{E}\sum_{x\in I}|\,\bar{\zeta}_x(t-\sigma_x)|\sum_{i,j=1}^{\xi_x(\infty)}|\bar{Z}_{x,i}(t-\sigma_{x,i})||\bar{Z}_{x,j}(t-\sigma_{x,j})| \\
&\le c_1\,e^{-3\gamma t/2}\mathbb{E}\sum_{x\in I}\left(\xi_x(\infty)+\sum_{i=1}^{\xi_x(\infty)}W_{x,i}\right)\sum_{i,j=1}^{\xi_x(\infty)}|\bar{Z}_{x,i}(t-\sigma_{x,i})||\bar{Z}_{x,j}(t-\sigma_{x,j})| \\
&\le c_1\,e^{-3\gamma t/2}\mathbb{E}\sum_{x\in I}\mathbb{E}\left(\left(\xi_x(\infty)+\sum_{i=1}^{\xi_x(\infty)}W_{x,i}\right)^2\middle|\mathcal{F}_x\right)^{1/2}\cdot \\
&\qquad\qquad \sum_{i,j=1}^{\xi_x(\infty)}\mathbb{E}\left((|\bar{Z}_{x,i}(t-\sigma_{x,i})||\bar{Z}_{x,j}(t-\sigma_{x,j})|)^2\middle|\mathcal{F}_x\right)^{1/2} \\
&\le d_1^{1/2}c_1e^{-3\gamma t/2}\mathbb{E}\sum_{x\in I}\mathbb{E}\left(\left(\xi_x(\infty)+\sum_{i=1}^{\xi_x(\infty)}W_{x,i}\right)^2\middle|\mathcal{F}_x\right)^{1/2}e^{\frac{3}{2}\gamma(t-\sigma_x)}\xi_x(\infty)^2 \\
&<\infty.
\end{aligned}
$$

For the last term, we argue as in (3.20) and use again the boundedness of $v$ to get

$$
\begin{aligned}
e^{-3\gamma t/2}\mathbb{E}\sum_{x\in I}\sum_{\substack{i,j,k=1,\\ \text{not all equal}}}^{\xi_\emptyset(\infty)}&|\bar{Z}_{x,i}(t-\sigma_{x,i})||\bar{Z}_{x,j}(t-\sigma_{x,j})||\bar{Z}_{x,k}(t-\sigma_{x,k})| \\
&\le c_3^{3/2}\,\mathbb{E}\xi(\infty)^3\,\mathbb{E}\sum_{x\in I}e^{-\frac{3}{2}\gamma\sigma_x} \\
&<\infty.
\end{aligned}
$$

It remains to show Lemma 5.2.3.

*Proof of Lemma 5.2.3.* We decompose $\bar{Z}^4$ as we did for $\bar{Z}^3$ to get a characteristic $\bar{f}$ such that

$$\bar{Z}^4 = Z^{\bar{f}}.$$

Thus

$$\begin{aligned}\bar{f}_\emptyset(t) = \bar{\zeta}_\emptyset(t)^4 + d_2\,\bar{\zeta}_\emptyset(t)^3 &\sum_{i=1}^{\xi_\emptyset(\infty)} \bar{Z}_i(t-\sigma_i) \\ &+ d_3\,\bar{\zeta}_\emptyset(t)^2 \sum_{i,j=1}^{\xi_\emptyset(\infty)} \bar{Z}_i(t-\sigma_i)\bar{Z}_j(t-\sigma_j) \\ &+ d_4\,\bar{\zeta}_\emptyset(t) \sum_{i,j,k=1}^{\xi_\emptyset(\infty)} \bar{Z}_i(t-\sigma_i)\bar{Z}_j(t-\sigma_j)\bar{Z}_k(t-\sigma_k) \\ &+ \sum_{\substack{i,j,k,l=1,\\ \text{not all equal}}} \bar{Z}_i(t-\sigma_i)\bar{Z}_j(t-\sigma_j)\bar{Z}_k(t-\sigma_k)\bar{Z}_l(t-\sigma_l).\end{aligned}$$

Therefore, we can bound $\bar{Z}^4$ by using the same arguments as for $\bar{Z}^3$. We start with the first term and get

$$e^{-3\gamma t}\mathbb{E}\sum_{x\in I}\bar{\zeta}_x(t-\sigma_x)^4 \le c_1^4 e^{-3\gamma t}\mathbb{E}\sum_{x\in I} e^{2\gamma(t-\sigma_x)}\left(\xi_x(\infty)+\sum_{i=1}^{\xi_x(\infty)} W_{x,i}\right)^4 < \infty.$$

For the second, we get

$$\begin{aligned}&e^{-3\gamma t}\mathbb{E}\sum_{x\in I}\bar{\zeta}_x(t-\sigma_x)^3\sum_{i=1}^{\xi_x(\infty)}|\bar{Z}_{x,i}(t-\sigma_{x,i})| \\ &\le c_1^3 e^{-3\gamma t}\mathbb{E}\sum_{x\in I} e^{3\gamma(t-\sigma_x)/2}\mathbb{E}\left((\xi_x(\infty)+W_x)^6\,\middle|\mathcal{F}_x\right)^{1/2}\sum_{i=1}^{\xi_x(\infty)}\mathbb{E}(\bar{Z}_{x,i}(t-\sigma_{x,i})^2|\mathcal{F}_x)^{1/2} \\ &< \infty,\end{aligned}$$

where we use that the sixth moment of $\xi(\infty)$ is bounded.

For the third term, we use that

$$\mathbb{E}Z^\phi(t)^4 \le d_5 e^{4\gamma t}$$

by the proof of Lemma 3.5.2. Therefore

$$
\begin{aligned}
&e^{-3\gamma t}\mathbb{E}\sum_{x\in I}\bar{\zeta}_x(t-\sigma_x)^2\sum_{i,j=1}^{\xi_x(\infty)}|\bar{Z}_{x,i}(t-\sigma_{x,i})||\bar{Z}_{x,j}(t-\sigma_{x,j})|\\
&\leq c_1^2\, e^{-3\gamma t}\mathbb{E}\sum_{x\in I}\mathbb{E}\left(\left(\xi(\infty)+\sum_{i=1}^{\xi_x(\infty)}W_{x,i}\right)^4\middle|\mathcal{F}_x\right)\cdot\\
&\qquad\qquad \sum_{i,j=1}^{\xi_x(\infty)}\mathbb{E}((|\bar{Z}_{x,i}(t-\sigma_{x,i})||\bar{Z}_{x,j}(t-\sigma_{x,j})|)^2|\mathcal{F}_x)^{1/2}\\
&\leq c_1^2\, d_5^{1/2}\, e^{-3\gamma t}\mathbb{E}\sum_{x\in I}\mathbb{E}\left(\left(\xi_x(\infty)+\sum_{i=1}^{\xi_x(\infty)}W_{x,i}\right)^4\middle|\mathcal{F}_x\right)^{1/2}\xi_x(\infty)^3 e^{2\gamma(t-\sigma_x)}\\
&<\infty.
\end{aligned}
$$

For the fourth term, we use

$$\mathbb{E}Z^{\phi}(t)^6\leq d_6 e^{6\gamma t}$$

and proceed as before. We estimate the last term as in (3.20) and hence the claim follows. □

This proofs Lemma 5.2.3 and hence Lemma 5.2.2 follows. □

## 5.3. Spectral asymptotics for statistically self-similar Cantor strings

The main result of our investigation in this chapter are included in this section. We first define the domain on which we consider the Laplacian and move on to the behaviour of its spectrum. The set we investigate is a subset of $[0,1]$ whose boundary is a statistically self-similar Cantor set. Many authors investigated spectral properties on the complement of Cantor-like sets. For references see [18, 43, 63].

We consider a random vector $(N, R_1, \ldots, R_N)$ with $R_i \in (0,1)$, $\sum_{i=1}^N R_i < 1$, $N\in\mathbb{N}$, $N\geq 2$ a.s. Then, we construct an iterated function system $(\Phi_1,\ldots,\Phi_N)$ on $[0,1]$ which splits the unit interval in $N$ equally spaced intervals with length ratios $(R_1,\ldots,R_N)$. The set under consideration is the statistically self-similar Cantor set

$$K \coloneqq \bigcap_{n=1}^{\infty} \bigcup_{\substack{x \in I, \\ |x|=n}} \Phi_x([0,1]).$$

We assume that $K$ forms a net and is proper and therefore by Theorem 3.2.2 on the event that $K$ is not empty, the Hausdorff dimension of $K$ is almost surely given by the unique solution $\gamma \in [0,1]$ of

$$\mathbb{E} \sum_{i=1}^{N} R_i^{\gamma} = 1.$$

As in [18, Theorem 4.1], it follows that Hausdorff and Minkowski dimension coincide a.s. and

$$\gamma = d_{\mathcal{H}} = d_M \quad a.s.$$

The set we investigate is $U \coloneqq [0,1] \backslash K$. It is a countable union of open intervals and by construction, $\partial U = K$. We call $U$ a *statistically self-similar Cantor string.* For references, see [18, 63].
In the following, we assume that the sixth moment of $N$ is bounded, i.e. we assume

$$\mathbb{E}N^6 = \mathbb{E}\xi(\infty)^6 < \infty.$$

Let $X \subseteq \mathbb{R}$ be a countable union of domains. The Dirichlet eigenvalue counting function of the Laplacian $\Delta_{|X}$ on $X$ is given by

$$N(X;\ \lambda) \coloneqq \# \left\{ \text{Dirichlet eigenvalues of } -\Delta_{|X} \leq \lambda \right\}.$$

We follow [18, 66] and define

$$\bar{N}(X;\ \lambda) \coloneqq \frac{1}{\pi} \mathrm{vol}_1(X) \lambda^{1/2} - N(X;\ \lambda).$$

If $X_1$ and $X_2$ are disjoint countable unions of domains it holds

$$\bar{N}(X_1 \cup X_2;\ \lambda) = \bar{N}(X_1;\ \lambda) + \bar{N}(X_2;\ \lambda). \tag{5.7}$$

Moreover, for $r \in (0, \infty)$ holds

$$\bar{N}(rX;\ \lambda) = \bar{N}\left(X;\ r^2\lambda\right). \tag{5.8}$$

We use an assumption on the convergence rate of the error of the linear approximation $G$ of the renewal function, see Appendix A.1. The assumption is related to Assumption 2.3.1.

**Assumption 5.3.1:** *There exist $\beta_1 \in (\gamma, \infty)$ and $c, t_0 \in (0, \infty)$ such that the error of the linear approximation $G$ of the renewal function of $\nu_\gamma$ defined in (3.2) satisfies*

$$\left| G(t) - \lim_{u \to \infty} G(u) \right| \leq c_1 e^{-\beta_1 t}$$

*for all $t \geq t_0$.*

In Section 2.3, the assumption is that $\beta_1 \in (\gamma/2, \infty)$. However, since we do not assume $W = 1$ a.s., we use a faster rate of convergence in the renewal theorem to obtain the strong law of large number for the square of the centred C-M-J branching process.

The following theorem is the main result of this chapter. The idea of the proof is taken from the proof of [18, Theorem 4.4].

**Theorem 5.3.2:** *Let $K$ be a statistically self-similar Cantor set with dimension $\gamma$, $\nu_\gamma$ defined in (3.2) be non-lattice, $\mathbb{E}N^6 < \infty$ and $U := [0,1] \backslash K$. Then, it holds*

$$\lambda^{-\gamma/2} \bar{N}(U;\ \lambda) \xrightarrow{\lambda \to \infty} CW \quad \text{a.s. and in } L^1,$$

*where $W$ is the almost sure and $L^1$ limit of the underlying fundamental martingale $(W_t)_t$ and $C$ is some strictly positive constant. Furthermore, $W > 0$ a.s.*

*If Assumption 5.3.1 holds, then*

$$\lambda^{\gamma/4} \left(\lambda^{-\gamma/2} \bar{N}(U;\ \lambda) - CW\right) \xrightarrow{t \to \infty} Z_\infty \quad \text{in distribution,}$$

*where the distribution of $Z_\infty$ is characterized by*

$$\mathbb{E}e^{i\theta Z_\infty} = \mathbb{E}e^{-\frac{1}{2}\theta^2 v(\infty) W},$$

*whereby* $v(\infty) := \lim_{t\to\infty} v(t)$ *with* $v$ *defined in* (3.12) *and, if* $v(\infty) > 0$ *it holds*

$$-1 \leq \liminf_{n\to\infty} \frac{e^{-\gamma n/4}\left(e^{-\gamma n/2}\bar{N}(U;\ e^n) - CW\right)}{\sqrt{2\,v(\infty)W\log n}}$$
$$\leq \limsup_{n\to\infty} \frac{e^{-\gamma n/4}\left(e^{-\gamma n/2}\bar{N}(U;\ e^n) - CW\right)}{\sqrt{2\,v(\infty)W\log n}} \leq 1$$

*almost surely.*

*Proof.* We follow the proof of [18, Theorem 4.3]. Therefore, let $P_i$ be the scale factor of the open gap interval between the $i$-th and $(i+1)$-th interval in the first approximation step of $K$. By the construction of $K$ and the properties (5.7) and (5.8) of $\bar{N}$ it follows

$$\begin{aligned}\bar{N}(U;\ \lambda) &= \sum_{i=1}^{\xi_\emptyset(\infty)-1} \bar{N}(P_i[0,1];\ \lambda) + \sum_{i=1}^{\xi_\emptyset(\infty)} \bar{N}(R_i U_i;\ \lambda)\\ &= \sum_{i=1}^{\xi_\emptyset(\infty)-1} \bar{N}\left([0,1];\ P_i^2\lambda\right) + \sum_{i=1}^{\xi_\emptyset(\infty)} \bar{N}\left(U_i;\ R_i^2\lambda\right),\end{aligned}$$

where $U_i$ are i.i.d. copies of $U$.

It is well known that the eigenvalues of $-\Delta_{|_{[0,1]}}$ are $(n\pi)^2$. Therefore

$$\bar{N}([0,1];\ \lambda) = \pi^{-1}\lambda^{1/2} - \left\lfloor \pi^{-1}\lambda^{1/2}\right\rfloor.$$

Thus, $\bar{N}([0,1];\ \lambda)$ is bounded by $1 \wedge \left(\pi^{-1}\lambda^{1/2}\right)$.

Next, we define the characteristic which we use to write $\bar{N}$ as a C-M-J branching process. Therefore,

$$\phi_\emptyset(t) := \sum_{i=1}^{\xi_\emptyset(\infty)-1} \bar{N}\left([0,1];\ P_i^2 e^{2t}\right).$$

Since

$$\bar{N}\left(U;\ e^{2t}\right) = \phi_\emptyset(t) + \sum_{i=1}^{\xi_\emptyset(\infty)} \bar{N}\left(U_i;\ e^{2(t-\sigma_i)}\right),$$

it holds

$$Z^{\phi}(t) = \bar{N}\left(U;\ e^{2t}\right).$$

Furthermore,

$$0 \le \phi(t) \le (\xi(\infty) - 1)(e^t \mathbb{1}_{t<0} + \mathbb{1}_{t\ge 0}), \quad Z^{\phi}(t)\mathbb{1}_{t<0} \le e^t \mathbb{1}_{t<0}. \tag{5.9}$$

To establish Nerman's Strong Law of Large Numbers to obtain the first part of the theorem, we use (3.10) and set

$$\chi_{\emptyset}(t) := \phi_{\emptyset}(t)\mathbb{1}_{t\ge 0} + \sum_{i=1}^{\xi_{\emptyset}(\infty)} Z_i^{\phi}(t-\sigma_i)\mathbb{1}_{0\le t<\sigma_i},$$

which is bounded by a constant times $\xi_{\emptyset}(\infty)$ because of (5.9). Therefore, we can use Theorem 3.3.4 and obtain

$$e^{-\gamma t} Z^{\chi}(t) \stackrel{t\to\infty}{\longrightarrow} \mu_1^{-1} \int_0^{\infty} e^{-\gamma s}\mathbb{E}\chi(s)\, ds\, W \quad \text{a.s. and in } L^1.$$

This means,

$$\lambda^{-\gamma/2}\bar{N}(U;\ \lambda) \stackrel{\lambda\to\infty}{\longrightarrow} \mu^{-1} \int_0^{\infty} e^{-\gamma s}\mathbb{E}\chi(s)\, ds\, W \quad \text{a.s. and in } L^1,$$

which yields the first part of the theorem by defining $C := \mu^{-1}\int_0^{\infty} e^{-\gamma s}\mathbb{E}\chi(s)\, ds$. For the second part, we define $\bar{\zeta}^{\phi}$ and $\bar{\zeta}^{\chi}$ as in (5.2) and let

$$\bar{Z}^{\phi} := Z^{\bar{\zeta}^{\phi}}, \quad \bar{Z}^{\chi} := Z^{\bar{\zeta}^{\chi}}.$$

By definition of $\chi$, these two C-M-J branching processes are equal for $t \ge 0$ and therefore, the variance functions $v^{\phi}(t) := v^{\bar{\zeta}^{\phi}}(t)$ and $v^{\chi}(t) := v^{\bar{\zeta}^{\chi}}(t)$ defined in (3.12) are equal for $t \ge 0$. Since $(\bar{Z}_i)_i$ are independent, we get

$$r^{\chi}(t) = e^{-\gamma t}\left(\mathbb{E}\,\bar{\zeta}_{\emptyset}^{\chi}(t)^2 + 2\mathbb{E}\,\bar{\zeta}_{\emptyset}^{\chi}(t)\sum_{i=1}^{\xi_{\emptyset}(\infty)} \bar{Z}_i^{\chi}(t-\sigma_i)\right),$$

where $r^{\chi} := r^{\bar{\zeta}^{\chi}}$ is defined as in (3.12). Since $\chi$ is bounded by a constant times $\xi(\infty)$ and

$$|z^{\chi}(t) - z^{\chi}(\infty)| \le c_3 e^{-\gamma t} \tag{5.10}$$

by Assumption 5.3.1 and Lemma A.1.2, we get by Lemma 5.1.1

$$\bar{\zeta}_\emptyset^\chi(t)^2 \leq c_1 \left( \xi_\emptyset(\infty) + \sum_{i=1}^{\xi_\emptyset(\infty)} W_i \right)^2 \tag{5.11}$$

and

**Lemma 5.3.3:** *It holds for all $\epsilon > 0$*

$$\mathbb{E}\bar{Z}^\chi(t)^2 \leq c_2(\epsilon) e^{\epsilon t + \gamma t} \quad t \geq 0.$$

Therefore,

$$\begin{aligned} &\mathbb{E}\,\bar{\zeta}_\emptyset^\chi(t) \sum_{i=1}^{\xi_\emptyset(\infty)} \bar{Z}_i^\chi(t-\sigma_i) \\ &\leq \mathbb{E}\left( \mathbb{E}\big(\bar{\zeta}_\emptyset^\chi(t)^2 \big| F_\emptyset\big)^{1/2} \sum_{i=1}^{\xi_\emptyset(\infty)} \mathbb{E}\left( \bar{Z}_i^\chi(t-\sigma_i)^2 \middle| F_\emptyset \right)^{1/2} \right) \\ &\leq c_3(\epsilon)\, e^{\gamma t/2 + \epsilon t/2}. \end{aligned} \tag{5.12}$$

Fix $\epsilon > 0$ sucht that $\gamma/2 + \epsilon < \gamma$ and thus there exist $c, \tau > 0$ such that

$$r^\phi(t) \leq c e^{-\tau|t|}.$$

and we can use renewal theorem of [66] to obtain

$$\lim_{t\to\infty} v^\chi(t) = \mu_1^{-1} \int_0^\infty e^{-\gamma s}\mathbb{E}\,\bar{\zeta}^\chi(s)^2 + 2e^{-\gamma s}\mathbb{E}\,\bar{\zeta}_\emptyset^\chi(s) \sum_{i=1}^{\xi_\emptyset(\infty)} \bar{Z}_i^\chi(s-\sigma_i)\, ds.$$

Therefore, the conditions of the Central Limit Theorem 3.3.7 are satisfied and thus we get by using the decomposition of $Z^\chi$ as in (5.1), the rate of convergence (5.17) and Slutsky's lemma

$$e^{\gamma t/2} \left( e^{-\gamma t} Z^\chi(t) - CW \right) \overset{t\to\infty}{\longrightarrow} Z_\infty \quad \text{in distribution.}$$

With analogous arguments we show that the conditions of the Law of the Iterated Logarithm 3.4.3 are satisfied and thus the claim follows. It remains to show Lemma 5.3.3.

*Proof of Lemma 5.3.3.* Since

$$\bar{Z}^{\chi}(t)^2 = \bar{Z}^{q^{\chi}}(t),$$

we get

$$\bar{Z}^{\chi}(t)^2 = \sum_{x\in I}\Bigg(\bar{\zeta}_x^{\chi}(t-\sigma_x)^2 + 2\,\bar{\zeta}_x^{\chi}(t-\sigma_x)\sum_{i=1}^{\xi_x(\infty)}\bar{Z}_{x,i}^{\chi}(t-\sigma_{x,i}) + \sum_{i=1}^{\xi_x(\infty)}\sum_{j<i}\bar{Z}_{x,i}^{\chi}(t-\sigma_{x,i})\bar{Z}_{x,j}^{\chi}(t-\sigma_{x,j})\Bigg)$$

Since $(\bar{Z}_{x,i}^{\chi})_i$ are i.i.d. and centred, it follows

$$\mathbb{E}\bar{Z}^{\chi}(t)^2 = \mathbb{E}\sum_{x\in I}\left(\bar{\zeta}_x^{\chi}(t-\sigma_x)^2 + 2\,\bar{\zeta}_x^{\chi}(t-\sigma_x)\sum_{i=1}^{\xi_x(\infty)}\bar{Z}_{x,i}^{\chi}(t-\sigma_{x,i})\right).$$

By Lemma 3.5.1 and Lemma 5.1.1 we control the first summand by $d_1 e^{\gamma t+\epsilon t}$ and since

$$\mathbb{E}\bar{Z}^{\chi}(t)^2 = \mathbb{E}(Z^{\chi}(t) - e^{\gamma t}\mathbb{E}e^{-\gamma t}Z^{\chi}(t))^2 \le d_2 e^{2\gamma t},$$

by using Cauchy-Schwarz as in (5.12), it follows

$$e^{-\gamma t-\epsilon t}\,\mathbb{E}\bar{Z}^{\chi}(t)^2 \le d_4\,\mathbb{E}\sum_{x\in I}e^{-\sigma_x(\gamma+\epsilon)} < \infty,$$

where the last estimate follows also by Lemma 3.5.1. □

□

**Example 5.3.4:** *Let the distribution of $(N, R_1, \dots, R_N)$ be defined by*

$$\begin{aligned}\mathbb{P}\Big((N, R_1, \dots, R_N) = (3, 1/5, 1/5, 1/5)\Big) &= \frac{1}{2},\\ \mathbb{P}\Big((N, R_1, \dots, R_N) = (2, 1/3, 1/3)\Big) &= \frac{1}{2}.\end{aligned} \tag{5.13}$$

Figure 5.1: First three approximation steps of a statistically self-similar Cantor set corresponding to (5.13)

*To check Assumption 5.3.1 we use Stone's Theorem A.1.1 where*

$$\begin{aligned} f(\omega) &= \int_0^\infty e^{\omega y} F(dy) \\ &= \mathbb{E} \sum_{i=1}^{N} R_i^{\gamma-\omega} \\ &= \frac{1}{2} \cdot 3 \cdot \left(\frac{1}{5}\right)^{\gamma-\omega} + \frac{1}{2} \cdot 2 \cdot \left(\frac{1}{3}\right)^{\gamma-\omega}. \end{aligned}$$

*Therefore, $f$ is analytic and it holds*

$$f(\omega) = 1 \quad \textit{if and only if} \quad \omega = 0.$$

*Hence, Assumption 5.3.1 is satisfied and thus it holds*

$$\lambda^{\gamma/4} \left( \lambda^{-\gamma/2} \bar{N}(U;\ \lambda) - CW \right) \stackrel{t\to\infty}{\longrightarrow} Z_\infty \quad \textit{in distribution}$$

*with the notation of Theorem 5.3.2.*

In this example we verified the conditions of the Central Limit Theorem in the case $W \neq 1$. To give an application of the Law of the Iterated Logarithm where we can verify $v(\infty) > 0$, we investigate the case $W = 1$ a.s. which is considered in [18] for the Central Limit Theorem.

## 5.4. Law of the Iterated Logarithm in the case $W = 1$

In this section we consider the same statistically self-similar Cantor strings as [18], those results on statistically self-similar Cantor strings we recapped in Section 2.3, i.e. we choose a deterministic number $\gamma \in (0,1)$ and a random vector $(T_1, \ldots, T_n)$ with a deterministic natural number $n \geq 2$. We further assume that

$$\sum_{i=1}^{n} T_i = 1, \quad T_i \in (0,1) \qquad a.s.$$

We replace the unit interval by $n$ equally spaced intervals with lengths $R_1 := T_1^{1/\gamma}, \ldots, R_n := T_n^{1/\gamma}$ and repeat this procedure for the remaining intervals independently and indefinitely. The limiting set $K$ is a statistically self-similar Cantor set those complement $U$ in $[0,1]$ is the statistically self-similar Cantor string under consideration in this section.

As explained in Section 5.3, on the event that $K$ is not empty Hausdorff and Minkowski dimension are almost surely given by $\gamma$.

Reproduction rate and life length under consideration are defined by (3.5). As in [18], we call a general branching process which satisfies $\xi(\infty) = n$ and

$$\sum_{i=1}^{n} e^{-\gamma\sigma_i} = 1 \qquad a.s.$$

a $\Delta_n$-general branching process and the corresponding statistically self-similar Cantor string $\Delta_n$-random Cantor string. We will see that in this setting Assumption 2.3.1 is enough to ensure the Law of the Iterated Logarithm for $\bar{N}(U;\,\cdot\,)$ to hold.

For the Law of the Iterated Logarithm we again split the considered C-M-J branching process as in (5.1) which leads to

$$\begin{aligned} &e^{\gamma t/2}\left(e^{-\gamma t} Z^{\phi}(t) - z^{\phi}(\infty)\right) = \\ &\quad e^{-\gamma t/2}\big(Z^{\phi}(t) - e^{\gamma t} z^{\phi}(t)\,\big) + e^{\gamma t/2}\left(z^{\phi}(t) - z^{\phi}(\infty)\right). \end{aligned} \tag{5.14}$$

Moreover, because $W = 1$ and $\xi(\infty) = n$ a.s., the centred characteristic $\bar{\zeta}$ defined by (5.2) will satisfy

$$\bar{\zeta}(t) \leq c_1 e^{\beta t}$$

for some $\beta < \gamma/2$. As in Lemma 5.2.1, we get the following lemma which is a slight modification of [18, Lemma 3.4].

**Lemma 5.4.1 (c.f. [18, Lemma 3.4]):** *Let $(\xi_x, L_x, \phi_x)_x$ be a $\Delta_n$-general branching process with Malthusian parameter $\gamma \in [0,1]$ and $\bar{\zeta}$ being the corresponding centred characteristic defined by (5.2). Assume that $\phi \geq 0$ and vanishes for negative times, $\nu_\gamma$ defined in (3.2) is non-lattice and that*

$$|\,\bar{\zeta}(t)| \leq c_1 e^{\beta t},$$

*for some $\beta < \gamma/2$. Then, Condition 3.4.1 is satisfied*

*Proof.* Let $0 < \epsilon < \gamma/2 - \beta$. Then,

$$\left| e^{-\gamma t/2} \sum_{\sigma_x \leq \epsilon t} \bar{\zeta}_x(t - \sigma_x) \right| \leq e^{(\epsilon - (\gamma/2 - \beta))t} e^{-\epsilon t} c_1 \sum_{\sigma_x \leq \epsilon t} 1.$$

From Theorem 3.3.4 follows

$$e^{-\gamma \epsilon t} \sum_{\sigma_x \leq \epsilon t} 1 \overset{t \to \infty}{\longrightarrow} c_2, \quad \text{a.s.}$$

Since $\epsilon < \gamma/2 - \beta$, the claim follows. □

The following lemma is taken from [18, Lemma 3.6].

**Lemma 5.4.2 (c.f. [18, Lemma 3.6]):** *Let $(\xi_x, L_x, \phi_x)_x$ be a $\Delta_n$-general branching process with Malthusian parameter $\gamma$ and $\bar{\zeta}$ being the corresponding centred characteristic defined by (5.2). Assume that $\phi \geq 0$ and vanishes for negative times and that*

$$|\,\bar{\zeta}(t)| \leq c_1 e^{\gamma t/2}.$$

*Then, Condition 3.4.2 is satisfied.*

With these two lemmas we are able to proof the Law of the Iterated Logarithm of this section. Before we give the theorem, we recall the assumption on the speed of convergence in Section 2.3.

**Assumption 5.4.3 (c.f. [18, Assumption 4.2]):** *There exist* $\beta_1 \in (\gamma/2, \infty)$ *and* $c, t_0 \in (0, \infty)$ *such that the error of the linear approximation* $G$ *of the renewal function of* $\nu_\gamma$ *defined in* (3.2) *satisfies*

$$\left| G(t) - \lim_{u \to \infty} G(u) \right| \leq c\, e^{-\beta_1 t},$$

*for all* $t \geq t_0$.

The proof of the following theorem is a slight modification of the proof of [18, Theorem 4.3].

**Theorem 5.4.4 (c.f. [18, Theorem 4.3]):** *Let* $U$ *be a* $\Delta_n$*-random Cantor string with dimension* $\gamma$ *and* $\nu_\gamma$ *defined in* (3.2) *be non-lattice. Then, it holds*

$$\lambda^{-\gamma/2} \bar{N}(U;\ \lambda) \overset{\lambda \to \infty}{\longrightarrow} C > 0 \quad a.s.\ and\ in\ L^1,$$

*Furthermore, if Assumption 5.4.3 holds, then*

$$\begin{aligned} -1 &\leq \liminf_{n \to \infty} \frac{e^{-\gamma n/4} \left( e^{-\gamma n/2} \bar{N}(U;\ e^n) - C \right)}{\sqrt{2\, v(\infty) \log n}} \\ &\leq \limsup_{n \to \infty} \frac{e^{-\gamma n/4} \left( e^{-\gamma n/2} \bar{N}(U;\ e^n) - C \right)}{\sqrt{2\, v(\infty) \log n}} \leq 1 \end{aligned}$$

*almost surely, whereby* $v(\infty) := \lim_{t \to \infty} v(t) \in (0, \infty)$ *with* $v$ *defined in* (3.12).

*Proof.* The first part follows from [18, Theorem 4.3]. For the second part we follow the proof of Theorem 5.3.2. Define

$$P := (1 - R_1 - \cdots - R_n)/(n-1).$$

By the construction of $K$ and the properties (5.7) and (5.8) of $\bar{N}$ it follows

$$\bar{N}(U;\ \lambda) = (n-1)\bar{N}\left([0,1];\ P^2 \lambda\right) + \sum_{i=1}^{n} \bar{N}\left(U_i;\ R_i^2 \lambda\right),$$

where $U_i$ are i.i.d. copies of $U$.

With the characteristic

$$\phi_\emptyset(t) := (n-1)\bar{N}\left([0,1];\ P^2 e^{2t}\right),$$

we get

$$\bar{N}\left(U;\ e^{2t}\right) = \phi_{\emptyset}(t) + \sum_{i=1}^{n} \bar{N}\left(U_i;\ e^{2(t-\sigma_i)}\right),$$

and hence

$$Z^{\phi}(t) = \bar{N}\left(U;\ e^{2t}\right)$$

with

$$0 \le \phi(t) \le (n-1)(e^t \mathbb{1}_{t<0} + \mathbb{1}_{t\ge 0}), \quad Z^{\phi}(t)\mathbb{1}_{t<0} \le e^t \mathbb{1}_{t<0}. \tag{5.15}$$

We define $\bar{\zeta}^{\phi}$ and $\bar{\zeta}^{\chi}$ as in (5.2) and let

$$\bar{Z}^{\phi} := Z^{\bar{\zeta}^{\phi}}, \quad \bar{Z}^{\chi} := Z^{\bar{\zeta}^{\chi}}.$$

By definition of $\chi$, these two C-M-J branching processes are equal for $t \ge 0$ and therefore, the variance functions $v^{\phi}(t) := v^{\bar{\zeta}^{\phi}}(t)$ and $v^{\chi}(t) := v^{\bar{\zeta}^{\chi}}(t)$ defined in (3.12) are equal for $t \ge 0$. Since $(\bar{Z}_i)_i$ are independent and $(\bar{\zeta}_x)_x$ are i.i.d., we get

$$r^{\phi}(t) = e^{-\gamma t}\mathbb{E}\,\bar{\zeta}^{\phi}_{\emptyset}(t)^2,$$

where $r^{\phi} := r^{\bar{\zeta}^{\phi}}$ is defined as in (3.12). By Assumption 5.4.3 and Lemma A.1.2 it holds

$$|z^{\phi}(t) - z^{\phi}(\infty)| \le c_1 e^{-\beta t}, \qquad \text{for some } \beta > \gamma/2.$$

Lemma 5.1.1 and (5.15) imply

$$|\,\bar{\zeta}^{\phi}(t)| \le c_2 e^{(\gamma-\beta)t}$$

and thus there exist $c, \tau > 0$ such that

$$r^{\phi}(t) \le c e^{-\tau|t|}.$$

and we can use renewal theorem of [66] to obtain

$$\lim_{t\to\infty} v^{\chi}(t) = \lim_{t\to\infty} v^{\phi}(t) = \mu_1^{-1}\int_0^{\infty} e^{-\gamma s}\mathbb{E}\,\bar{\zeta}^{\phi}(s)^2 ds \in (0,\infty) \tag{5.16}$$

Since $\chi$ is bounded and also

$$|z^{\chi}(t) - z^{\chi}(\infty)| \leq c_3 e^{-\beta t} \tag{5.17}$$

by Assumption 5.3.1 and Lemma A.1.2, we see that by Lemma 5.4.1 and Lemma 5.4.2 together with (5.16) the conditions of the Law of the Iterated Logarithm 3.4.3 are satisfied and hence by Corollary 3.15 it follows

$$-1 \leq \liminf_{n\to\infty} \frac{e^{-\gamma n/4}\bar{Z}^{\chi}(n/2)}{\sqrt{2\,v(\infty)\log n}} \leq \limsup_{n\to\infty} \frac{e^{-\gamma n/4}\bar{Z}^{\chi}(n/2)}{\sqrt{2\,v(\infty)\log n}} \leq 1 \quad a.s.$$

Together with the decomposition (5.14) the claim follows. □

The following example of $\Delta_n$-random Cantor strings is taken from [18, Chapter 5].

**Example 5.4.5 (c.f. [18, Chapter 5]):** *Consider a vector* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in (0,\infty)^n$. *Let* $\gamma \in (0,1)$ *and the distribution of* $(e^{-\sigma_1\gamma}, \ldots, e^{-\sigma_n\gamma})$ *given by*

$$(e^{-\sigma_1\gamma}, \ldots, e^{-\sigma_n\gamma}) \sim Dir(\boldsymbol{\alpha}),$$

*where* $Dir(\boldsymbol{\alpha})$ *denotes the Dirichlet distribution with weights* $\alpha_1, \ldots, \alpha_n > 0$.

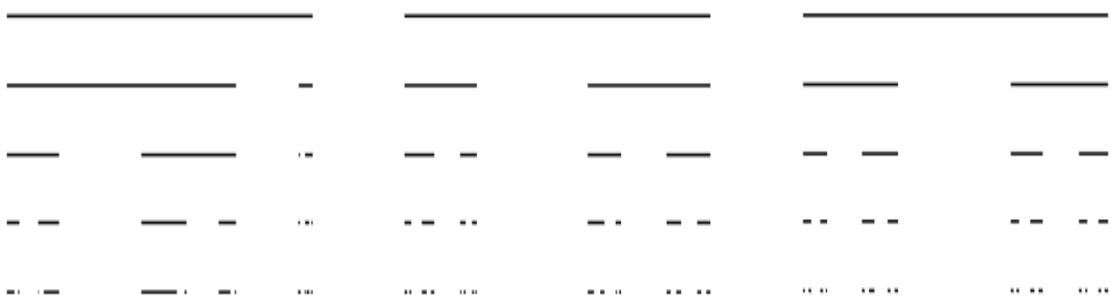

Figure 5.2: First four approximation steps of $\Delta_n$-random Cantor sets with $(e^{-\sigma_1\gamma}, \ldots, e^{-\sigma_n\gamma}) \sim Dir((\alpha,\alpha))$ for $\alpha = 1, 30, 80$ and $\gamma = 0.6$, [18, Figure 5].

*By [18, Lemma 5.2] it holds*

**Lemma 5.4.6 (c.f. [18, Lemma 5.2]):** *Assume that*

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = \frac{k}{n-1}, \quad k \in \{1,2,3,4\}, \quad n \geq 2. \tag{5.18}$$

*Then the Fourier transformation $f(\omega)$ of $\nu_\gamma$ defined in (3.2) is analytic and $\neq 1$ for all $\omega \in \mathbb{C}$ with $Re\,\omega \in (0,\gamma]$ and therefore by Stone's Theorem A.1.1*

$$G(t) - \frac{\mu_1}{2\mu_2^2} = O(e^{-\gamma t}).$$

*Hence, if $\boldsymbol{\alpha}$ satisfies (5.18) then the Law of the Iterated Logarithm 5.4.4 holds.*

In [18, Remark 4.4] it is explained that if

$$\left|z^\phi(\infty) - z^\phi(t)\right| \leq c_1 e^{-\beta t/2} \quad \text{for some } \beta_1 > \gamma/2 \tag{5.19}$$

is **not** satisfied, we should not expect the Central Limit Theorem to hold because then $v(\infty)$ is not finite and thus we should also not expect the Law of the Iterated Logarithm to hold if (5.19) is not satisfied.

# Chapter 6

# Outlook

This chapter outlines some open questions and conjectures regarding the present work.

## 6.1. Law of the Iterated Logarithm for C-M-J branching processes

In Corollary 3.4.5, we saw that if the general branching process $(\xi_x, L_x, \bar{\zeta}_x)_x$ with centred characteristic $\bar{\zeta}$ satisfies some regularity conditions and if $\nu_\gamma$ is non-lattice, then the corresponding C-M-J branching process $\bar{Z}$ satisfies for fixed $h > 0$,

$$-1 \leq \liminf_{n\to\infty} \frac{e^{-\gamma hn/2}\bar{Z}(hn)}{\sqrt{2\,v(\infty)W\log hn}} \leq \limsup_{n\to\infty} \frac{e^{-\gamma hn/2}\bar{Z}(hn)}{\sqrt{2\,v(\infty)W\log hn}} \leq 1 \quad a.s.$$

with $v(\infty)$ being the limit of the normalized variance process $\mathbb{E}e^{-\gamma t}\bar{Z}(t)^2$ and $W$ the limit of the underlying fundamental martingale. It seems natural to extend this Law of the Iterated Logarithm to all $t$ and further ask if the upper and lower bound for the lim inf and lim sup respectively also holds. In [49] a decomposition, with which the reverse inequality chain in the Law of the Iterated Logarithm for the biggins martingale was proven, is provided. It should be possible to get a similar decomposition for $\bar{Z}$. Therefore, we make the following conjecture.

**Conjecture 6.1.1:** *Let $(\xi_x, L_x, \bar{\zeta}_x)_x$ be a general branching process with Malthusian parameter $\gamma > 0$ such that $\bar{\zeta}$ satisfies $\mathbb{E}\bar{Z}(t) = 0$ for all $t$. Assume that*

$$v(t) \overset{t\to\infty}{\longrightarrow} v(\infty),$$

*where $v(\infty) > 0$ and further, assume that Condition 3.4.1 and Condition 3.4.2,*

$$\mathbb{E}\xi(\infty)\log\xi(\infty) < \infty$$

*hold and $\nu_\gamma$ is non-lattice. Then,*

$$-1 = \liminf_{t\to\infty} \frac{e^{-\gamma t/2}\bar{Z}(t)}{\sqrt{2\,v(\infty)W\log t}} < \limsup_{t\to\infty} \frac{e^{-\gamma t/2}\bar{Z}(t)}{\sqrt{2\,v(\infty)W\log t}} = 1 \quad a.s.$$

## 6.2. The spectral exponents $\gamma_h$, $\gamma_V$ and $\gamma_r$

In Section 1.4 we explained the connection between homogeneous, $V$-variable and recursive structures and showed in Section 4.2.4 that if the conditions of Theorem 2.2.2 are satisfied, it holds

$$\gamma_h \leq \gamma_r. \tag{6.1}$$

The first related question is if it is possible to relax the conditions of Theorem 2.2.2 to get the spectral exponent for random homogeneous Cantor measures. Assumptions (A1) and (A2) in particular seem to be very strong conditions in a random setting. However, it seems natural to ask if assumptions (A1)-(A5) can be replaced by (4.7) with which we would have the spectral exponent for random homogeneous and statistically self-similar Cantor measures under the same assumption. Related to that are similar questions for the $V$-variable setting.

By increasing $V$, it is allowed to use more different iterated function systems in the approximation steps of the corresponding $V$-variable Cantor set. Therefore, by increasing $V$, we get more thicker and thinner parts in the fractal and since we expect that the thicker parts dominate the thinner parts in the spectral asymptotics, we assume $(\gamma_V)_{V\in\mathbb{N}}$ to be an increasing sequence. Thus, we conjecture that

$$\gamma_h = \gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \cdots \leq \gamma_r$$

holds. Furthermore, one could ask if

$$\gamma_V \overset{V\to\infty}{\longrightarrow} \gamma_h$$

holds, cf. [35].

## 6.3. Applications of the CLT and LIL for C-M-J branching processes

Firstly, we consider the main theorem of Chapter 5. By Conjecture 6.1.1 immediately follows

**Conjecture 6.3.1:** *Let $K$ be a statistically self-similar Cantor set with dimension $\gamma$, $\nu_\gamma$ be non-lattice and $U := [0,1]\backslash K$. Assume that Assumption 5.3.1 is satisfied and $v(\infty) > 0$. Then it holds almost surely*

$$-1 = \liminf_{\lambda\to\infty} \frac{\lambda^{\gamma/4}\left(\lambda^{-\gamma/2}\bar{N}(U;\ \lambda) - CW\right)}{\sqrt{2Wv(\infty)\log\log\lambda}}$$
$$< \limsup_{\lambda\to\infty} \frac{\lambda^{\gamma/4}\left(\lambda^{-\gamma/2}\bar{N}(U;\ \lambda) - CW\right)}{\sqrt{2Wv(\infty)\log\log\lambda}} = 1,$$

*where we use the same notation as in Theorem 5.3.2.*

In Theorem 5.3.2 an explicit expression for $v(\infty)$ is given by

$$v(\infty) = \mu_1^{-1}\int_0^\infty e^{-\gamma s}\mathbb{E}\,\bar{\zeta}^\phi(s)^2 + 2e^{-\gamma s}\mathbb{E}\,\bar{\zeta}^\phi_\emptyset(s)\sum_{i=1}^{\xi_\emptyset(\infty)}\bar{Z}^\phi_i(s-\sigma_i)\,ds.$$

Therefore, $v(\infty) = 0$ if and only if

$$\mu_1^{-1}\int_0^\infty e^{-\gamma s}\mathbb{E}\,\bar{\zeta}^\phi(s)^2\,ds = -\int_0^\infty 2e^{-\gamma s}\mathbb{E}\,\bar{\zeta}^\phi_\emptyset(s)\sum_{i=1}^{\xi_\emptyset(\infty)}\bar{Z}^\phi_i(s-\sigma_i)\,ds,$$

which seems to be very unlikely. However, Graf [40] showed that $\mathcal{H}^\gamma(K) \in (0,\infty)$ if and only if $\sum_{i=1}^M R_i^\gamma = 1$ a.s., where we use the same notation as in Chapter 5. Therefore, it could be possible that $v(\infty) > 0$ if and only if $\sum_{i=1}^M R_i^\gamma = 1$ a.s. Even if this is the case $\lambda^{\gamma/4}$ could be the right factor to capture the random fluctuation of $\bar{N}(U;\ \lambda)\lambda^{-\gamma/2}$ around its limit, i.e. one could ask if

$$\lambda^{(\gamma+\epsilon)/4}\left(\lambda^{-\gamma/2}\bar{N}(U;\ \lambda) - CW\right) \xrightarrow{\lambda\to\infty} \infty \quad a.s.$$

for all $\epsilon > 0$.

Moreover, it should be possible to relax the assumption $\mathbb{E}N^6 < \infty$.

In Section 5.4 we saw that in the case $W = 1$ a.s. it holds $v(\infty) > 0$ and, in particular, the Law of the Iterated Logarithm 5.4.4 holds if Assumption 5.4.3 is satisfied which is

$$G(t) - \lim_{u\to\infty} G(u) = O\left(e^{-\beta\gamma t}\right), \quad \beta > 1/2, \tag{6.2}$$

whereas in the case $W \neq 1$ a.s. we assume that $G$ converges faster, namely

$$G(t) - \lim_{u\to\infty} G(u) = O\left(e^{-\tilde{\beta}\gamma t}\right), \quad \tilde{\beta} > 1.$$

The reason is that we use Cauchy-Schwarz to separate $\mathbb{E}W\bar{Z}(t-\sigma_x)^2$ in Lemma 5.2.2. If

$$\mathbb{E}W\bar{Z}(t-\sigma_x)^2 \leq c_1 e^{\gamma t}$$

it should be enough to require $\tilde{\beta} > 1/2$ for the Central Limit Theorem and Law of the Iterated Logarithm in the case where $W$ is not almost surely equal to 1.

As a second application, we consider the measure theoretical Laplacian $\Delta^{\mu^{(I)}}$ with respect to statistically self-similar Cantor measures $\mu^{(I)}$. Therefore define

$$\bar{N}_D^{(I)}(\lambda) := z^{\chi^\eta}(\infty)W - N_D^{(I)}(\lambda)\lambda^{-\gamma_r},$$

where we use the same notation as in Theorem 4.2.9. It could be possible to get the normalization factor of $\bar{N}_D^{(I)}$ and also a CLT and LIL for the random fluctuation.

## 6.4. Further convergence theorems for branching processes

Another tool to analyse the growth of sums of random variables is the *Large Deviation Theory*. We can also investigate this theory for C-M-J branching processes and ask if $Z^\phi$ satisfies

$$\mathbb{P}(Z^\phi(t)e^{-\gamma t} > s) \approx e^{-A(t)B(s)}$$

for some increasing functions $A$ and $B$.

Besides growth properties of C-M-J branching processes, we can consider the random fluctuation of the fundamental martingale $(W_t)_t$ around its limit. As pointed out in Section 3.4, Iksanov and Kabluchko [49] gave a Central Limit Theorem and Law of the Iterated Logarithm for the biggins martingale. Following this, one could investigate also a CLT and LIL for $(W_t)_t$, i.e. consider the asymptotics of

$$e^{\gamma t/2}(W - W_t),$$

where we conjecture that $e^{\gamma t/2}$ is the right normalization factor.

# Appendix

## A.1 Speed of Convergence in the Renewal Theorem

The assumptions in Chapter 5.2.2 are strongly related to the convergence rate of the renewal function which we introduce in this section. This assumption can be checked by using a Theorem of Stone [79], also included in this section. This theorem were used in [18] for the Central Limit Theorem in the case $W = 1$ a.s. The following investigation is taken from [18, Section 3.2]. We consider the renewal equation

$$z(t) = u(t) + \int_0^\infty z(t-s)\, F(ds), \tag{A.1.1}$$

where $F$ is a non-lattice probability distribution function on $[0, \infty)$. Typically, $z$ is given as

$$z(t) = \int_0^\infty u(t-y)\, H(dy),$$

where $H$ is given as the renewal measure

$$H = \sum_{n=0}^\infty F^{*n}.$$

Hereby, $F^{*n}$ denotes the $n$-fold convolution of $F$ with itself, see [29, 55]. If $F$ has finite mean $\mu_1$, then the renewal theorem of [29] shows that

$$\frac{H(t)}{\mu_1^{-1} t} \overset{t\to\infty}{\longrightarrow} 1.$$

Further, if $u$ is smooth enough, e.g. directly Riemann integrable, then by [29] it holds that

$$z(t) \overset{t\to\infty}{\longrightarrow} z(\infty) := \mu_1^{-1} \int_{-\infty}^\infty u(s)\, ds.$$

The error of the linear approximation of the renewal function is given by

$$G(t) := H(t) - \mu_1^{-1} t,$$

in whose convergence rate we are interested. When $F$ has a finite second moment $\mu_2$, then

$$G(t) \overset{t\to\infty}{\longrightarrow} \frac{\mu_2}{2\mu_1^2},$$

see [18] and references therein. The rate of convergence here can be studied by considering the Fourier transform $f$ of $F$ defined by

$$f(\omega) := \int_0^\infty e^{\omega s} F(ds), \quad \omega \in \mathbb{C}.$$

For references see e.g. [29, 65, 78, 79] and for the lattice case see also the Appendix B of [56]. The Theorem of Stone, pointed out in the introduction of this section is as follows.

**Theorem A.1.1 (c.f. [79 and 18, Theorem 3.2]):** *Suppose that there exists $r_1 \in (0,\infty)$ such that $f(\omega)$ is analytic and $\neq 1$ when $\mathrm{Re}\,\omega \in (0, r_1)$. Then, for every $r \in (0, r_1)$ it holds*

$$G(t) - \frac{\mu_2}{2\mu_1^2} = O(e^{-rt}).$$

We are rather interested in the rate of convergence of $z$ than of $G$. The following lemma relates the rate of convergence of $z^\phi$ to the rate of convergence of $G$. It is included from [18]. There, it is adapted from [20].

**Lemma A.1.2 (c.f. [18, Lemma 3.3]):** *Let $z$, $u$ and $F$ satisfy the renewal equation (A.1.1) and suppose that*

$$z(t) = \int_0^\infty u(t-y)\, H(dy) \overset{t\to\infty}{\longrightarrow} \mu_1^{-1} \int_{-\infty}^\infty u(y)\, dy.$$

*Then,*

$$z(\infty) - z(t) = \mu_1^{-1} \int_0^\infty u(t+y)\, dy - \int_0^\infty u(t-y)\, G(dy).$$

# A.2 Berry-Esseen Theorem

To proof our Law of the Iterated Logarithm we use the following theorem adapted from Asmussen and Hering [5, Proposition 7.2, page 436] which relies on the conditional Borel-Cantelli lemma. The idea to use [5, Proposition 7.2, page 436] to proof our Law of the Iterated Logarithm is taken from the proof of [49, Theorem 1.6].

**Theorem A.2.1 (c.f. [5, Proposition 7.2, page 436]):** *Let $(\mathcal{G}_n)_{n\in\mathbb{N}_0}$ be a filtration and $(X_n)_{n\in\mathbb{N}_0}$ be a sequence of random variables such that*

$$\sum_{n=0}^{\infty} \sup_{y\in\mathbb{R}} \left|\mathbb{P}\left(X_n \leq y|\mathcal{G}_n\right) - \Phi(y)\right| < \infty \quad a.s.,$$

*where $\Phi(y) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-x^2/2}\,dx$. Then,*

$$\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} \leq 1 \quad a.s.$$

*Furthermore, if there exists $m \in \mathbb{N}$ such that $X_n$ is $\mathcal{G}_{n+m}$ measurable for all $n$, then it holds*

$$\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = 1 \quad a.s.$$

We study the expression

$$\sum_{n=0}^{\infty} \sup_{y\in\mathbb{R}} \left|\mathbb{P}\left(X_n \leq y|\mathcal{G}_n\right) - \Phi(y)\right|$$

by using the well known Berry-Esseen estimate. The version we give here is taken from [49, Lemma 4.2].

**Theorem A.2.2 (Berry-Esseen, c.f. [49, Lemma 4.2]):** *Let $(Y_n)_{n\in\mathbb{N}}$ be independent random variables with $\mathbb{E}Y_i = 0$, $\sigma_i^2 := \mathbb{E}Y_i^2 < \infty$, $\rho_i := \mathbb{E}|Y_i|^3 < \infty$ and $\sum_{i=1}^{\infty}\sigma_i^2 < \infty$. Then,*

$$\sup_{y\in\mathbb{R}} \left|\mathbb{P}\left(\frac{\sum_{i=1}^{\infty} Y_i}{\sqrt{\sum_{i=1}^{\infty}\sigma_i^2}} \leq y\right) - \Phi(y)\right| \leq c_1 \frac{\sum_{i=1}^{\infty}\rho_i}{\left(\sum_{i=1}^{\infty}\sigma_i^2\right)^{3/2}},$$

*with $\Phi(y) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-x^2/2}\,dx$.*

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