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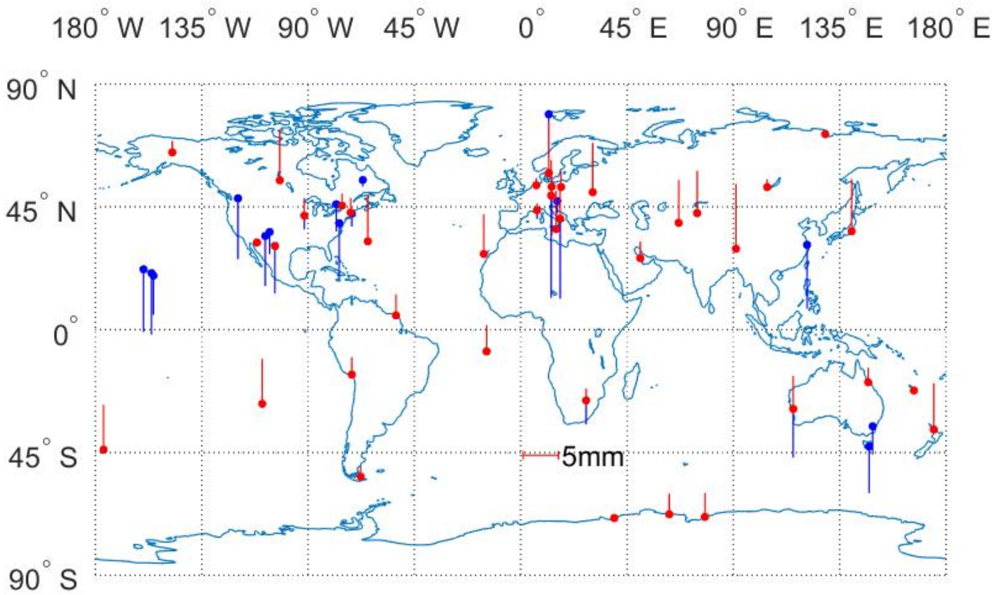
Converted Total Least Squares method and Gauss-Helmert model with applications to transformation among ITRF realizations



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Abstract

This thesis is an extension and improvement of the theory and applications of Converted Total Least Squares method(CTLS). Converted Total Least Squares (CTLS) dealing with the errors-in-variables (EIV) model take the stochastic design matrix elements as virtual observations, and the TLS problem can be transformed into a LS problem. In the coordinate transformation, the transformation model is always used after centering like it is published in most papers. This thesis directly use the transformation model to generate a new design matrix with CTLS method. The result will present the consistency of the transformation model with and without centering in coordinates transformation. Then the 3D Helmert-transformation in Gauss-Helmert and Gauss-Markoff model is introduced(Koch 2002). The study is to find that, the connections between CTLS and the Gauss-Helmert model. To prove their similarity is a strong support for the theory of the CTLS method. After that, this thesis gives a brief introduction to the International Terrestrial Reference System(ITRF). The CTLS has been proved itself with coordinate transformation in Baden-Württemberg with equa weight and large scale. The new application with more parameters and smaller scale together with the weight information in ITRF is presented. The comparison and accuracy assessment of the published parameters and the parameters estimated by CTLS are discussed in detail with the applications.

Key words: Converted Total Least Squares, Helmert-transformation, Gauss-Helmert model, International Terrestrial Reference System, accuracy assessment.

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Chapter 1

Introduction

1.1 Background and Motivation

Before the detailed introducion of this thesis, a brief review of the development of Total Least Squares is necessary. Total Least Squares (TLS) is a method of fitting that is appropriate when there are errors in both the observation vector and in the design matrix in computational mathematics and engineering, which is also referred as Errors-In-Variables (EIV) modelling or orthogonal regression in the statistical community. The TLS/EIV principle studied by Adcock (1878) already more than one century ago. Kendall and Stuart (1969) described this problem as structural relationship model models. In geodetic application this method was discussed by Koch (2002) and studied recently by Schaffrin (2005). How to obtain the best parameter estimation values and give the statistical information of parameters in the EIV model is not 'perfectly' solved. Nevertheless, the EIV model is still becming increasingly widespread in remote sensing (Felus and Schaffrin 2005) and geodetic datum transformation (Schaffrin and Felus 2006, 2008; Akyilmaz 2007; Cai and Grafarend 2009).

In 1980, the mathematical structure of TLS was completed by Golub and Van Loan(1980), who gave the first numerically stable algorithm based on matrix singular value decomposition. With the rapid development of numerical method over the last decade, various approach for TLS emerged. These include singular value decomposition (SVD), the completely orthogonal approach, the Cholesky decomposition approach, the iterative approach, and so on (Van Huffel et al., 1993,1997,2002; Schaffrin et al.,2003), the most representative of which are the SVD and iterative solution. However, there are some problems in the both methods. In the SVD method, some elements of design matrix may be non-stochastic, or some elements containing errors could appear more than once. To perform the minimum norm constraint without this consideration is inappropriate and may result in large deviations. By the Iteration method, since the iteration solutions can be a problem if there is a high degree of nonlinearity. In addition, this method has also the problem by the repetition of parameters in design matrix.

According to the research results by Yao Y., Kong,J., Cai, J., and Sneeuw N. (2010), one method called Converted Total Least Squares (CTLS) was developed since 2010. This method can perform the processs without iteration and at the same time solve the problem by the repetition of elements and the non-stochastic elements containin errors in design matrix. In the bachelor thesis by Dong, D.(2017), the CTLS method was systematically introduced and applicated with coordinates transformation in Baden-Württemberg together with other three estimators. The comparison among these estimators proved the advantages of CTLS.

Further study is required to complete the theory and extension the application cases. In the bachelor thesis by Dong, D.(2017), the transformation models (6-parameter affine transformation model and 7-parameter Helmert transformation model) were used after centering, which was pointed out by the host professor during the author's participation in InterGeo Berlin 2017. It is important to find out if CTLS can directly used in transformation model. Further more, the application by coordinates transformation in Badens-Württemberg is under the same weight condition, which is not persuasive for CTLS to deal with weight information in study cases. Moreover, to test the CTLS in new applications with more complex parameters and different scals is very significant.

1.2 Outline

This thesis contents five chapters. Chapter 1 gives a brief review and introduction for the development and history of Total Least Squares problem, as well as the motivation and outline of this work. Chapter 2 presents the CTLS dealing directly with transformation models and the comparison between the consequences with and without centering. Chapter 3 introduces the 3D Helmert-transformation in Gauss-Helmert and Gauss-Markoff model. The detailed derivations are given to compare the theory and methodology with CTLS. It also presents the accuracy assessment for both methods. Chapter 4 intruduces the information and history of International Terrestrial Reference System (ITRF). The knowlage and the resource of data preparation are explaned. The relationship between each ITRF frame and its transformation parameters are detailed explaned and derivated in equations. Chapter 5 summarizes the conclusions of the whole thesis.

Appendix: The CTLS's mainly derivation equation, which will not be introduced in detail in this thesis.

Firstly take classic Gauss-Markoff model of LS as basis equation.

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{\xi} + \boldsymbol{e}_y \tag{1.1}$$

Augmenting the observation equations that take design matrix elements as virtual observation on the basis of the original error equation.

$$\boldsymbol{y}_a = \boldsymbol{\xi}_a + \boldsymbol{e}_a \tag{1.2}$$

Where $\boldsymbol{y_a}$ is comprised of the design matrix elements that contain errors, and $\boldsymbol{\xi_a}$ is comprised of the new parameters. If (4.1)is combine with (4.2), a mathematical model under the new algorithm can be obtained.

$$\begin{aligned} \boldsymbol{y} &= \boldsymbol{A}\boldsymbol{\xi} + \boldsymbol{e}_y \\ \boldsymbol{y}_a &= \boldsymbol{\xi}_a + \boldsymbol{e}_a \end{aligned} \tag{1.3}$$

It should be clear that $\boldsymbol{y_a}$ contains only the observations of design matrix. To distinguish the design matrix in the original model, the symbol $\boldsymbol{A_\xi}$ is used to denote the design matrix in (4.1), which is formed by the initial value of parameters $\boldsymbol{\xi_a}$ and some elements without errors.
Based on the above model, we can get the following error equations

$$\begin{aligned} \boldsymbol{e}_y &= (\boldsymbol{A}_\xi^0 + \boldsymbol{E}_A)(\boldsymbol{\xi}^0 + \Delta\boldsymbol{\xi}) - \boldsymbol{y} \\ &= \boldsymbol{A}_\xi^0\Delta\boldsymbol{\xi} + \boldsymbol{E}_A\boldsymbol{\xi}^0 + \boldsymbol{A}_\xi^0\boldsymbol{\xi}^0 - \boldsymbol{y} + \Delta\boldsymbol{A}\Delta\boldsymbol{\xi} \qquad \rightarrow \boldsymbol{E}_A\Delta\boldsymbol{\xi} \approx \boldsymbol{0} \\ &= \boldsymbol{A}_\xi^0\Delta\boldsymbol{\xi} + \boldsymbol{B}\Delta\boldsymbol{a} + \boldsymbol{A}_\xi^0\boldsymbol{\xi}^0 - \boldsymbol{y} \\ \boldsymbol{e}_a &= \boldsymbol{a} - \boldsymbol{y}_a \end{aligned} \tag{1.4}$$

Where $\boldsymbol{E_A}$ is composed of $\boldsymbol{\Delta a}$, the corrections to the new parameters, and $\boldsymbol{B\Delta a}$ is the rewritten form of $\boldsymbol{E_A\xi^0}$. In converting $\boldsymbol{E_A\xi^0}$ to $\boldsymbol{B\Delta a}$, which is the key step for the approach. $\boldsymbol{A_\xi^0}$ is composed of non-stochastic elements in the design matrix and the initial value $\boldsymbol{a}$.

Define $\boldsymbol{z} = \begin{bmatrix} \boldsymbol{y} - \boldsymbol{A}_\xi^0\boldsymbol{\xi}^0 \\ \boldsymbol{a} - \boldsymbol{y}_a \end{bmatrix}$, $\boldsymbol{A}_z = \begin{bmatrix} \boldsymbol{A}_\xi^0 & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{E} \end{bmatrix}$, $\Delta\boldsymbol{\eta} = \begin{bmatrix} \Delta\boldsymbol{\xi} \\ \Delta\boldsymbol{a} \end{bmatrix}$, $\boldsymbol{e}_z = \begin{bmatrix} \boldsymbol{e}_y \\ \boldsymbol{e}_a \end{bmatrix}$, (4.5) can be reduced to:

$$\boldsymbol{z} = \boldsymbol{A}_\eta\Delta\boldsymbol{\eta} + \boldsymbol{e}_z \tag{1.5}$$

Where $\boldsymbol{e_z}$ is the residual vector of all observations, $\boldsymbol{A_\eta}$ is formed by the initial values of the parameters, and $\boldsymbol{\Delta\eta}$ is comprised of the corrections to all parameters. The estimation criterion is still $\boldsymbol{e}_z^T\boldsymbol{P}_z\boldsymbol{e}_z \rightarrow min$, which is the same as $\boldsymbol{e}_y^T\boldsymbol{P}_y\boldsymbol{e}_y + \boldsymbol{e}_a^T\boldsymbol{P}_a\boldsymbol{e}_a \rightarrow min$. Since the TLS problem is transformed into the classical LS problem, the adjustment can be completed by following the classical LS principle. The new weight matrix is $\boldsymbol{P}_z = \begin{bmatrix} \boldsymbol{P}_y & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}_a \end{bmatrix}$. And the TLS problem can be solved considering the weight of observations and stochastic design matrix by:

$$\Delta\hat{\boldsymbol{\eta}} = (\boldsymbol{A}_\eta^T\boldsymbol{P}_z\boldsymbol{A}_\eta)^{-1}\boldsymbol{A}_\eta^T\boldsymbol{P}_z\boldsymbol{z} \tag{1.6}$$

Chapter 2

Transformation models with and without centering

2.1 6-parameter affine transformation model(2D)

With the 2D affine transformation, where six parameters are to be determined, both coordinate directions are rotated with two different angles α and β. So that not only the distances and the angles are distorted, but usually also the original orthogonality of the axes of coordinates is lost. An affine transformation preserves collinearity and ratios of distances. While an affine transformation preserves proportions on lines, it does not necessarily preserve angles or lengths(Cai, J. and Grafarend, E. 2009).

The 6-parameter affine transformation model between any two plane coordinates systems, e.g. from Grauss-Krüger coordinate(H,R) in DHDN (G) directly to the UTM-Coordinate (N,E) in ETRS89 can be written as

$$\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{E} \end{bmatrix} = \begin{bmatrix} \lambda_H cos\alpha & -\lambda_R sin\beta \\ \lambda_H sin\alpha & \lambda_R cos\beta \end{bmatrix} \begin{bmatrix} \boldsymbol{H} \\ \boldsymbol{R} \end{bmatrix} + \begin{bmatrix} t_N \\ t_E \end{bmatrix} \tag{2.1}$$

Where t_N and t_E are translation parameters; α and β are rotation parameters; λ_H and λ_R are scale corrections.

2.1.1 Transformation with centering

When the coordinates are transformed, the 6-parameter affine transformation model is centralized in order to vanish the translation parameters.

$$\begin{aligned} \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{E} \end{bmatrix} &= \begin{bmatrix} \lambda_H cos\alpha & -\lambda_R sin\beta \\ \lambda_H sin\alpha & \lambda_R cos\beta \end{bmatrix} \begin{bmatrix} \boldsymbol{H} \\ \boldsymbol{R} \end{bmatrix} + \begin{bmatrix} t_N \\ t_E \end{bmatrix} \\ &=: \begin{bmatrix} \xi_{11} & \xi_{21} \\ \xi_{12} & \xi_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{H} \\ \boldsymbol{R} \end{bmatrix} + \begin{bmatrix} \xi_{31} \\ \xi_{32} \end{bmatrix} = \begin{bmatrix} \boldsymbol{H} & \boldsymbol{R} & 0 & 0 & 1 & 0 \\ 0 & 0 & \boldsymbol{H} & \boldsymbol{R} & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_{11} \\ \xi_{21} \\ \xi_{12} \\ \xi_{22} \\ \xi_{31} \\ \xi_{32} \end{bmatrix} \end{aligned} \tag{2.2}$$

Because the element '1' and '0' have no error, the translation parameters shall disappear by centering this equation. Thus, after the centering the coordinates in the mid point, the translation parameters t_N and t_E will be automatically vanished. Then the observation and old coordinates are centered on their average values in the form:

$$\begin{bmatrix} \underline{\boldsymbol{N}} \\ \underline{\boldsymbol{E}} \end{bmatrix} =: \begin{bmatrix} \xi_{11} & \xi_{21} \\ \xi_{12} & \xi_{22} \end{bmatrix} \begin{bmatrix} \underline{\boldsymbol{H}} \\ \underline{\boldsymbol{R}} \end{bmatrix} \tag{2.3}$$

with

$$\underline{\boldsymbol{N}} = \boldsymbol{N} - mean(\boldsymbol{N}), \underline{\boldsymbol{E}} = \boldsymbol{E} - mean(\boldsymbol{E})$$
$$\underline{\boldsymbol{H}} = \boldsymbol{H} - mean(\boldsymbol{H}), \underline{\boldsymbol{R}} = \boldsymbol{R} - mean(\boldsymbol{R})$$

In Converted Total Least Squares for the n couple of coordinates with the same transformation model, which is suited for the application of TLS solution.

$$\boldsymbol{E}\left\{\begin{bmatrix} \underline{N}_1 \\ \vdots \\ \underline{N}_n \\ \underline{E}_1 \\ \vdots \\ \underline{E}_n \end{bmatrix}\right\} = \boldsymbol{E}\left\{\begin{bmatrix} \underline{H}_1 & \underline{R}_1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \underline{H}_n & \underline{R}_n & 0 & 0 \\ 0 & 0 & \underline{H}_1 & \underline{R}_1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \underline{H}_n & \underline{R}_n \end{bmatrix}\right\} \begin{bmatrix} \xi_{11} \\ \xi_{21} \\ \xi_{12} \\ \xi_{22} \end{bmatrix}$$

Reform the EIV observation model from

$$\boldsymbol{y} - \boldsymbol{e_y} = (\boldsymbol{A} - \boldsymbol{E_A})\boldsymbol{\xi}$$

to

$$\boldsymbol{z} = \boldsymbol{A_\eta}\boldsymbol{\Delta\eta} + \boldsymbol{e_z} \tag{2.4}$$

Where $\boldsymbol{z} = \begin{bmatrix} \boldsymbol{y} - \boldsymbol{A^0_\xi \xi^0} \\ \boldsymbol{a} - \boldsymbol{y_a} \end{bmatrix}, \boldsymbol{A_z} = \begin{bmatrix} \boldsymbol{A^0_\xi} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{E} \end{bmatrix}, \boldsymbol{\Delta\eta} = \begin{bmatrix} \boldsymbol{\Delta\xi} \\ \boldsymbol{\Delta a} \end{bmatrix}, \boldsymbol{e_z} = \begin{bmatrix} \boldsymbol{e_y} \\ \boldsymbol{e_a} \end{bmatrix}$

$$\begin{aligned} \boldsymbol{e_y} &= (\boldsymbol{A^0_\xi} + \boldsymbol{E_A})(\boldsymbol{\xi^0} + \boldsymbol{\Delta\xi}) - \boldsymbol{y} \\ &= \boldsymbol{A^0_\xi \Delta\xi} + \boldsymbol{E_A \xi^0} + \boldsymbol{A^0_\xi \xi^0} - \boldsymbol{y} + \boldsymbol{\Delta A \Delta\xi} \qquad \rightarrow \boldsymbol{E_A \Delta\xi} \approx \boldsymbol{0} \\ &= \boldsymbol{A^0_\xi \Delta\xi} + \boldsymbol{B\Delta a} + \boldsymbol{A^0_\xi \xi^0} - \boldsymbol{y} \\ \boldsymbol{e_a} &= \boldsymbol{a} - \boldsymbol{y_a} \end{aligned}$$

The key step here is converting $\boldsymbol{E_A \xi^0}$ to $\boldsymbol{B\Delta a}$.
Create the correspond matrixs

$$\underset{(2n\times 4)}{\boldsymbol{E_A}} = \begin{bmatrix} \Delta\underline{H}_1 & \Delta\underline{R}_1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \Delta\underline{H}_n & \Delta\underline{R}_n & 0 & 0 \\ 0 & 0 & \Delta\underline{H}_1 & \Delta\underline{R}_1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \Delta\underline{H}_n & \Delta\underline{R}_n \end{bmatrix}, \quad \underset{(4\times 1)}{\boldsymbol{\xi^0}} = \begin{bmatrix} \xi^0_{11} \\ \xi^0_{21} \\ \xi^0_{12} \\ \xi^0_{22} \end{bmatrix}, \quad \underset{(2n\times 1)}{\boldsymbol{\Delta a}} = \begin{bmatrix} \Delta\underline{H}_1 \\ \vdots \\ \Delta\underline{H}_n \\ \Delta\underline{R}_1 \\ \vdots \\ \Delta\underline{R}_n \end{bmatrix}$$

$$E_A\boldsymbol{\xi}^0 = \boldsymbol{B}\boldsymbol{\Delta a} = (\begin{bmatrix}\xi_{11}^0 & \xi_{21}^0\\ \xi_{12}^0 & \xi_{22}^0\end{bmatrix} \otimes \boldsymbol{I_n})\boldsymbol{\Delta a}$$

$$\underset{(2n\times 2n)}{\boldsymbol{B}} = \begin{bmatrix}\xi_{11}^0 & \xi_{21}^0\\ \xi_{12}^0 & \xi_{22}^0\end{bmatrix} \otimes \boldsymbol{I_n} = \begin{bmatrix}\xi_{11}^0 & 0 & 0 & \xi_{21}^0 & 0 & 0\\ 0 & \ddots & 0 & 0 & \ddots & 0\\ 0 & 0 & \xi_{11}^0 & 0 & 0 & \xi_{21}^0\\ \xi_{12}^0 & 0 & 0 & \xi_{22}^0 & 0 & 0\\ 0 & \ddots & 0 & 0 & \ddots & 0\\ 0 & 0 & \xi_{12}^0 & 0 & 0 & \xi_{22}^0\end{bmatrix}$$

The solution of CTLS is

$$\boldsymbol{\Delta\hat{\eta}} = (\boldsymbol{A_\eta^T P_z A_\eta})^{-1}\boldsymbol{A_\eta^T P_z z}$$

The solution $\boldsymbol{\Delta\hat{\eta}}$ is a $(2n+4)\times 1$ vector. The first 4 elements of $\boldsymbol{\Delta\eta}$ are the corrections of $\boldsymbol{\xi}$ and the following $2n$ elements are the corrections of $\boldsymbol{y_a}$, which are the corrections for the initial design matrix $\boldsymbol{A}$. The final transformation parameters are $\boldsymbol{\hat{\xi}} = \boldsymbol{\Delta\xi} + \boldsymbol{\xi_0}$, with $\boldsymbol{\xi_0}$ calculated from the LS solution.

2.1.2 Transformation without centering

The original tansformation model can be written as follow:

$$\begin{aligned}\begin{bmatrix}\boldsymbol{N}\\ \boldsymbol{E}\end{bmatrix} &= \begin{bmatrix}\lambda_H cos\alpha & -\lambda_R sin\beta\\ \lambda_H sin\alpha & \lambda_R cos\beta\end{bmatrix}\begin{bmatrix}\boldsymbol{H}\\ \boldsymbol{R}\end{bmatrix} + \begin{bmatrix}t_N\\ t_E\end{bmatrix}\\ &=: \begin{bmatrix}\xi_{11} & \xi_{21}\\ \xi_{12} & \xi_{22}\end{bmatrix}\begin{bmatrix}\boldsymbol{H}\\ \boldsymbol{R}\end{bmatrix} + \begin{bmatrix}\xi_{31}\\ \xi_{32}\end{bmatrix} = \begin{bmatrix}\boldsymbol{H} & \boldsymbol{R} & 0 & 0 & 1 & 0\\ 0 & 0 & \boldsymbol{H} & \boldsymbol{R} & 0 & 1\end{bmatrix}\begin{bmatrix}\xi_{11}\\ \xi_{21}\\ \xi_{12}\\ \xi_{22}\\ \xi_{31}\\ \xi_{32}\end{bmatrix}\end{aligned} \tag{2.5}$$

In Converted Total Least Squares for the n couple of coordinates with the same transformation model, the observation equationas are witten as:

$$\boldsymbol{E}\left\{\begin{bmatrix}N_1\\ \vdots\\ N_n\\ E_1\\ \vdots\\ E_n\end{bmatrix}\right\} = \boldsymbol{E}\left\{\begin{bmatrix}H_1 & R_1 & 0 & 0 & 1 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ H_n & R_n & 0 & 0 & 1 & 0\\ 0 & 0 & H_1 & R_1 & 0 & 1\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & H_n & R_n & 0 & 1\end{bmatrix}\right\}\begin{bmatrix}\xi_{11}\\ \xi_{21}\\ \xi_{12}\\ \xi_{22}\\ \xi_{31}\\ \xi_{32}\end{bmatrix}$$

Reform the EIV observation model from

$$\boldsymbol{y} - \boldsymbol{e_y} = (\boldsymbol{A} - \boldsymbol{E_A})\boldsymbol{\xi}$$

to

$$z = A_\eta \Delta\eta + e_z \tag{2.6}$$

Convert $\boldsymbol{E_A \zeta^0}$ to $\boldsymbol{B\Delta a}$.
Create the correspond matrixs

$$\underset{(2n\times 6)}{\boldsymbol{E_A}} = \begin{bmatrix} \Delta H_1 & \Delta R_1 & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta H_n & \Delta R_n & 0 & 0 & 1 & 0 \\ 0 & 0 & \Delta H_1 & \Delta R_1 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \Delta H_n & \Delta R_n & 0 & 1 \end{bmatrix}, \quad \underset{(6\times 1)}{\boldsymbol{\zeta^0}} = \begin{bmatrix} \zeta_{11}^0 \\ \zeta_{21}^0 \\ \zeta_{12}^0 \\ \zeta_{22}^0 \\ \zeta_{31}^0 \\ \zeta_{32}^0 \end{bmatrix}, \quad \underset{(3n\times 1)}{\boldsymbol{\Delta a}} = \begin{bmatrix} \Delta H_1 \\ \vdots \\ \Delta H_n \\ \Delta R_1 \\ \vdots \\ \Delta R_n \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\boldsymbol{E_A \zeta^0} = \boldsymbol{B\Delta a} = \left(\begin{bmatrix} \zeta_{11}^0 & \zeta_{21}^0 & \zeta_{31}^0 \\ \zeta_{12}^0 & \zeta_{22}^0 & \zeta_{32}^0 \end{bmatrix} \otimes \boldsymbol{I_n} \right) \boldsymbol{\Delta a}$$

$$\underset{(2n\times 3n)}{\boldsymbol{B}} = \begin{bmatrix} \zeta_{11}^0 & \zeta_{21}^0 & \zeta_{31}^0 \\ \zeta_{12}^0 & \zeta_{22}^0 & \zeta_{32}^0 \end{bmatrix} \otimes \boldsymbol{I_n} = \begin{bmatrix} \zeta_{11}^0 & 0 & 0 & \zeta_{21}^0 & 0 & 0 & \zeta_{31}^0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & 0 & \zeta_{11}^0 & 0 & 0 & \zeta_{21}^0 & 0 & 0 & \zeta_{31}^0 \\ \zeta_{12}^0 & 0 & 0 & \zeta_{22}^0 & 0 & 0 & \zeta_{32}^0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & 0 & \zeta_{12}^0 & 0 & 0 & \zeta_{22}^0 & 0 & 0 & \zeta_{32}^0 \end{bmatrix}$$

The solution of CTLS is

$$\boldsymbol{\Delta\hat{\eta}} = (\boldsymbol{A_\eta^T P_z A_\eta})^{-1} \boldsymbol{A_\eta^T P_z z}$$

We can see from the equation element $\Delta\boldsymbol{a}$, it contains the corrections for the variables in design matrix and also contains the elements '1' due to the structure of transformation model. These '1' are necessary to bring the 2 translation parameters into estimation progress, which violats the original idea of CTLS. It's meaningless to discuss the solution of CTLS hier. Hence the solutions will be presented with LS methode.

2.1.3 Presentation and comparison of the results

Table 2.1: *Comparison of 6-parameter affine transformation parameters with LS*

Transformation	6-parameter affine transformation GK(DHDN)-UTM(ETRS89)					
Deformation	$t_N(m)$	$t_E(m)$	$\alpha('')$	$\beta('')$	$d\lambda_H(\times 10^{-4})$	$d\lambda_R(\times 10^{-4})$
With	437.194567	119.756709	0.165368	-0.196455	-3.996797	-3.988430
Without	437.194567	119.756709	0.165368	-0.196455	-3.996797	-3.988430

Table 2.2: *Numerical deviation of 6-parameter affine transformation with LS*

Transformation Deformation	Collocated sites	Absolute mean of Residuals (m) $[V_N]$	$[V_E]$	Max.absolute mean of Residuals(m) $[V_N]$	$[V_E]$	RMS (m)	Standard deviation of unit weight (m)
With	B-W 131	0.1049	0.0804	0.3288	0.3226	0.1187	0.1199
Without	B-W 131	0.1049	0.0804	0.3288	0.3226	0.1187	0.1199

As we can see from the statistical results above, the transfromation model with or without deformation has no affect for the transformation parameters in 6-parameter affine transformation model. And the CTLS is not adaptive for the no-centering model in coordinate transformation.

2.2 7-parameter Helmert transformation models(3D)

The 7-parameter Helmert transformation performs a conformal transformation, where the ratios of distances and the angles preserve invariantly. A 'local' non-geocentric X_L,Y_L,Z_L-system can be transformed into a 'global' geocentric X_G,Y_G,Z_G-system with the help of a 7-parameter Helmert transformation model.

$$\begin{aligned} \begin{bmatrix} \boldsymbol{X}_G \\ \boldsymbol{Y}_G \\ \boldsymbol{Z}_G \end{bmatrix} &= \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix} + (1 + d\lambda) \begin{bmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_L \\ \boldsymbol{Y}_L \\ \boldsymbol{Z}_L \end{bmatrix} \\ &\cong \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix} + \begin{bmatrix} d\lambda & \gamma & -\beta \\ -\gamma & d\lambda & \alpha \\ \beta & -\alpha & d\lambda \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_L \\ \boldsymbol{Y}_L \\ \boldsymbol{Z}_L \end{bmatrix} + \begin{bmatrix} \boldsymbol{X}_L \\ \boldsymbol{Y}_L \\ \boldsymbol{Z}_L \end{bmatrix} \end{aligned} \tag{2.7}$$

Where T_X,T_Y,T_Z are translate parameters; α , β and γ are differential rotation parameters; $d\lambda$ is scale correction.

2.2.1 Transformation with centering

The following formula has been used for the estimation of the parameters in seven-parameter Helmert transformation.

$$\begin{aligned} \begin{bmatrix} \boldsymbol{X}_G \\ \boldsymbol{Y}_G \\ \boldsymbol{Z}_G \end{bmatrix} &= (1 + d\lambda) \begin{bmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_L \\ \boldsymbol{Y}_L \\ \boldsymbol{Z}_L \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix} \\ \boldsymbol{X}_G &= \lambda \begin{bmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{bmatrix} \boldsymbol{X}_L + \boldsymbol{T}_L \end{aligned} \tag{2.8}$$

Where λ is scale factor, α, β, γ are rotation angles. The translation terms T_X, T_Y, T_Z are the coordinates of the origin of the 3-D network.

After the linearization, the formula is rewritten:

$$\begin{bmatrix} \boldsymbol{X}_G \\ \boldsymbol{Y}_G \\ \boldsymbol{Z}_G \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -\boldsymbol{Z}_L & \boldsymbol{Y}_L & \boldsymbol{X}_L \\ 0 & 1 & 0 & \boldsymbol{Z}_L & 0 & -\boldsymbol{X}_L & \boldsymbol{Y}_L \\ 0 & 0 & 1 & -\boldsymbol{Y}_L & \boldsymbol{X}_L & 0 & \boldsymbol{Z}_L \end{bmatrix} \begin{bmatrix} T_X \\ T_Y \\ T_Z \\ \delta\alpha \\ \delta\beta \\ \delta\gamma \\ \lambda \end{bmatrix} \tag{2.9}$$

After centering the coordinates in the midpoints, the translation parameter T_X, T_Y, T_Z will disappear, and then the observations and old coordinates are centered on their average values. This will be assumed in the following:

$$\begin{bmatrix} \boldsymbol{x}_g \\ \boldsymbol{y}_g \\ \boldsymbol{z}_g \end{bmatrix} = \begin{bmatrix} 0 & -\boldsymbol{z}_l & \boldsymbol{y}_l & \boldsymbol{x}_l \\ \boldsymbol{z}_l & 0 & -\boldsymbol{x}_l & \boldsymbol{y}_l \\ -\boldsymbol{y}_l & \boldsymbol{x}_l & 0 & \boldsymbol{z}_l \end{bmatrix} \begin{bmatrix} \delta\alpha \\ \delta\beta \\ \delta\gamma \\ \lambda \end{bmatrix} \tag{2.10}$$

with

$$\begin{bmatrix} \boldsymbol{x}_g \\ \boldsymbol{y}_g \\ \boldsymbol{z}_g \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_G \\ \boldsymbol{Y}_G \\ \boldsymbol{Z}_G \end{bmatrix} - mean \begin{bmatrix} \boldsymbol{X}_G \\ \boldsymbol{Y}_G \\ \boldsymbol{Z}_G \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{x}_l \\ \boldsymbol{y}_l \\ \boldsymbol{z}_l \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_L \\ \boldsymbol{Y}_L \\ \boldsymbol{Z}_L \end{bmatrix} - mean \begin{bmatrix} \boldsymbol{X}_L \\ \boldsymbol{Y}_L \\ \boldsymbol{Z}_L \end{bmatrix} \tag{2.11}$$

In Converted Total Least Squares for the n couple of coordinates with the same transformation model, which is suited for the application of TLS solution.

$$\boldsymbol{E}\left\{ \begin{bmatrix} x_{g1} \\ \vdots \\ x_{gn} \\ y_{g1} \\ \vdots \\ y_{gn} \\ z_{g1} \\ \vdots \\ z_{gn} \end{bmatrix} \right\} =: \boldsymbol{E}\left\{ \begin{bmatrix} 0 & -z_{l1} & y_{l1} & x_{l1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & -z_{ln} & y_{ln} & x_{ln} \\ z_{l1} & 0 & -x_{l1} & y_{l1} \\ \vdots & \vdots & \vdots & \vdots \\ z_{ln} & 0 & -x_{ln} & y_{ln} \\ -y_{l1} & x_{l1} & 0 & z_{l1} \\ \vdots & \vdots & \vdots & \vdots \\ -y_{ln} & x_{ln} & 0 & z_{ln} \end{bmatrix} \right\} \begin{bmatrix} \delta\alpha \\ \delta\beta \\ \delta\gamma \\ \lambda \end{bmatrix}$$

Reform the EIV observation model from

$$\boldsymbol{y} - \boldsymbol{e}_y = (\boldsymbol{A} - \boldsymbol{E}_A)\boldsymbol{\xi}$$

to

$$\boldsymbol{z} = \boldsymbol{A}_\eta \Delta\boldsymbol{\eta} + \boldsymbol{e}_z \tag{2.12}$$

Where $\boldsymbol{z} = \begin{bmatrix} \boldsymbol{y} - \boldsymbol{A}_\xi^0 \boldsymbol{\xi}^0 \\ \boldsymbol{a} - \boldsymbol{y}_a \end{bmatrix}, \boldsymbol{A}_z = \begin{bmatrix} \boldsymbol{A}_\xi^0 & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{E} \end{bmatrix}, \Delta\boldsymbol{\eta} = \begin{bmatrix} \Delta\boldsymbol{\xi} \\ \Delta\boldsymbol{a} \end{bmatrix}, \boldsymbol{e}_z = \begin{bmatrix} \boldsymbol{e}_y \\ \boldsymbol{e}_a \end{bmatrix}$

$$
\begin{aligned}
e_y &= (A_\xi^0 + E_A)(\xi^0 + \Delta\xi) - y \\
&= A_\xi^0 \Delta\xi + E_A \xi^0 + A_\xi^0 \xi^0 - y + \Delta A \Delta\xi \qquad \rightarrow E_A \Delta\xi \approx 0 \\
&= A_\xi^0 \Delta\xi + B\Delta a + A_\xi^0 \xi^0 - y \\
e_a &= a - y_a
\end{aligned}
$$

The key step here is converting $E_A\xi^0$ to $B\Delta a$.
Create the correspond matrixs

$$
\underset{(3n\times 4)}{E_A} = \begin{bmatrix} 0 & -\Delta z_{l1} & \Delta y_{l1} & \Delta x_{l1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & -\Delta z_{ln} & \Delta y_{ln} & \Delta x_{ln} \\ \Delta z_{l1} & 0 & -\Delta x_{l1} & \Delta y_{l1} \\ \vdots & \vdots & \vdots & \vdots \\ \Delta z_{ln} & 0 & -\Delta x_{ln} & \Delta y_{ln} \\ -\Delta y_{l1} & \Delta x_{l1} & 0 & \Delta z_{l1} \\ \vdots & \vdots & \vdots & \vdots \\ -\Delta y_{ln} & \Delta x_{ln} & 0 & \Delta z_{ln} \end{bmatrix}, \qquad \underset{(4\times 1)}{\xi^0} = \begin{bmatrix} \xi_{11}^0 \\ \xi_{21}^0 \\ \xi_{12}^0 \\ \xi_{22}^0 \end{bmatrix}, \qquad \underset{(3n\times 1)}{\Delta a} = \begin{bmatrix} \Delta x_{l1} \\ \vdots \\ \Delta x_{ln} \\ \Delta y_{l1} \\ \vdots \\ \Delta y_{ln} \\ \Delta z_{l1} \\ \vdots \\ \Delta z_{ln} \end{bmatrix}
$$

$$
E_A\xi^0 = B\Delta a = \left(\begin{bmatrix} \xi_{22}^0 & \xi_{12}^0 & -\xi_{21}^0 \\ -\xi_{12}^0 & \xi_{22}^0 & \xi_{11}^0 \\ \xi_{21}^0 & -\xi_{11}^0 & \xi_{22}^0 \end{bmatrix} \otimes I_n \right) \Delta a
$$

$$
\begin{aligned}
\underset{(3n\times 3n)}{B} &= \begin{bmatrix} \xi_{22}^0 & \xi_{12}^0 & -\xi_{21}^0 \\ -\xi_{12}^0 & \xi_{22}^0 & \xi_{11}^0 \\ \xi_{21}^0 & -\xi_{11}^0 & \xi_{22}^0 \end{bmatrix} \otimes I_n \\
&= \begin{bmatrix}
\xi_{22}^0 & 0 & 0 & \xi_{12}^0 & 0 & 0 & -\xi_{21}^0 & 0 & 0 \\
0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 \\
0 & 0 & \xi_{22}^0 & 0 & 0 & \xi_{12}^0 & 0 & 0 & -\xi_{21}^0 \\
-\xi_{12}^0 & 0 & 0 & \xi_{22}^0 & 0 & 0 & \xi_{11}^0 & 0 & 0 \\
0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 \\
0 & 0 & -\xi_{12}^0 & 0 & 0 & \xi_{22}^0 & 0 & 0 & \xi_{11}^0 \\
\xi_{21}^0 & 0 & 0 & -\xi_{11}^0 & 0 & 0 & \xi_{22}^0 & 0 & 0 \\
0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 \\
0 & 0 & \xi_{21}^0 & 0 & 0 & -\xi_{11}^0 & 0 & 0 & \xi_{22}^0
\end{bmatrix}
\end{aligned}
$$

The solution of CTLS is

$$
\Delta\hat{\eta} = (A_\eta^T P_z A_\eta)^{-1} A_\eta^T P_z z \tag{2.13}
$$

The solution $\Delta\hat{\eta}$ is a $(3n+4)\times 1$ vector. The first 4 elements of $\Delta\eta$ are the corrections of ξ and the following $3n$ elements are the corrections of y_a, which are the corrections for the initial design matrix A. The final transformation parameters are $\hat{\xi} = \Delta\xi + \xi_0$, with ξ_0 calculated from the LS solution.

2.2.2 Transformation without centering

The original transformation model can be wriiten as follow:

$$\begin{bmatrix} \boldsymbol{X}_G \\ \boldsymbol{Y}_G \\ \boldsymbol{Z}_G \end{bmatrix} = (1+d\lambda)\begin{bmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{X}_L \\ \boldsymbol{Y}_L \\ \boldsymbol{Z}_L \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$
$$\boldsymbol{X}_G = \lambda \begin{bmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{bmatrix} \boldsymbol{X}_L + \boldsymbol{T}_L \tag{2.14}$$

Where λ is scale factor, α, β, γ are rotation angles. The translation terms T_X, T_Y, T_Z are the coordinates of the origin of the 3-D network.

In Converted Total Least Squares for the n couple of coordinates with the same transformation model, the observation equations are written as:

$$\boldsymbol{E}\left\{\begin{bmatrix} X_{G1} \\ \vdots \\ X_{Gn} \\ Y_{G1} \\ \vdots \\ Y_{Gn} \\ Z_{G1} \\ \vdots \\ Z_{Gn} \end{bmatrix}\right\} =: \boldsymbol{E}\left\{\begin{bmatrix} 1 & 0 & 0 & 0 & -Z_{L1} & Y_{L1} & X_{L1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & -Z_{Ln} & Y_{Ln} & X_{Ln} \\ 0 & 1 & 0 & Z_{L1} & 0 & -X_{L1} & Y_{L1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & Z_{Ln} & 0 & -X_{Ln} & Y_{Ln} \\ 0 & 0 & 1 & -Y_{L1} & X_{L1} & 0 & Z_{L1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & -Y_{Ln} & X_{Ln} & 0 & Z_{Ln} \end{bmatrix}\right\} \begin{bmatrix} T_X \\ T_Y \\ T_Z \\ \delta\alpha \\ \delta\beta \\ \delta\gamma \\ \lambda \end{bmatrix}$$

Reform the EIV observation model from

$$\boldsymbol{y} - \boldsymbol{e_y} = (\boldsymbol{A} - \boldsymbol{E_A})\boldsymbol{\xi}$$

to

$$\boldsymbol{z} = \boldsymbol{A_\eta}\boldsymbol{\Delta\eta} + \boldsymbol{e_z} \tag{2.15}$$

Convert $\boldsymbol{E_A}\boldsymbol{\xi^0}$ to $\boldsymbol{B\Delta a}$.

Create the correspond matrixs

$$\underset{(3n\times 7)}{\boldsymbol{E_A}} = \begin{bmatrix} 1 & 0 & 0 & 0 & -\Delta z_{l1} & \Delta y_{l1} & \Delta x_{l1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & -\Delta z_{ln} & \Delta y_{ln} & \Delta x_{ln} \\ 0 & 1 & 0 & \Delta z_{l1} & 0 & -\Delta x_{l1} & \Delta y_{l1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & \Delta z_{ln} & 0 & -\Delta x_{ln} & \Delta y_{ln} \\ 0 & 0 & 1 & -\Delta y_{l1} & \Delta x_{l1} & 0 & \Delta z_{l1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & -\Delta y_{ln} & \Delta x_{ln} & 0 & \Delta z_{ln} \end{bmatrix}, \quad \underset{(7\times 1)}{\boldsymbol{\xi^0}} = \begin{bmatrix} T_X \\ T_Y \\ T_Z \\ \xi^0_{11} \\ \xi^0_{21} \\ \xi^0_{12} \\ \xi^0_{22} \end{bmatrix}, \quad \underset{(4n\times 1)}{\boldsymbol{\Delta a}} = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \Delta x_{l1} \\ \vdots \\ \Delta x_{ln} \\ \Delta y_{l1} \\ \vdots \\ \Delta y_{ln} \\ \Delta z_{l1} \\ \vdots \\ \Delta z_{ln} \end{bmatrix}$$

$$\boldsymbol{E_A \xi^0} = \boldsymbol{B \Delta a} = (\begin{bmatrix} T_X & \xi^0_{22} & \xi^0_{12} & -\xi^0_{21} \\ T_Y & -\xi^0_{12} & \xi^0_{22} & \xi^0_{11} \\ T_Z & \xi^0_{21} & -\xi^0_{11} & \xi^0_{22} \end{bmatrix} \otimes \boldsymbol{I_n}) \boldsymbol{\Delta a}$$

$$\underset{(3n\times 4n)}{\boldsymbol{B}} = \begin{bmatrix} T_X & \xi^0_{22} & \xi^0_{12} & -\xi^0_{21} \\ T_Y & -\xi^0_{12} & \xi^0_{22} & \xi^0_{11} \\ T_Z & \xi^0_{21} & -\xi^0_{11} & \xi^0_{22} \end{bmatrix} \otimes \boldsymbol{I_n}$$
$$= \begin{bmatrix} T^0_X & 0 & 0 & \xi^0_{22} & 0 & 0 & \xi^0_{12} & 0 & 0 & -\xi^0_{21} & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & 0 & T^0_X & 0 & 0 & \xi^0_{22} & 0 & 0 & \xi^0_{12} & 0 & 0 & -\xi^0_{21} \\ T^0_Y & 0 & 0 & -\xi^0_{12} & 0 & 0 & \xi^0_{22} & 0 & 0 & \xi^0_{11} & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & 0 & T^0_Y & 0 & 0 & -\xi^0_{12} & 0 & 0 & \xi^0_{22} & 0 & 0 & \xi^0_{11} \\ T^0_Z & 0 & 0 & \xi^0_{21} & 0 & 0 & -\xi^0_{11} & 0 & 0 & \xi^0_{22} & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & 0 & T^0_Z & 0 & 0 & \xi^0_{21} & 0 & 0 & -\xi^0_{11} & 0 & 0 & \xi^0_{22} \end{bmatrix}$$

The solution of CTLS is

$$\boldsymbol{\Delta \hat{\eta}} = (\boldsymbol{A^T_\eta P_z A_\eta})^{-1} \boldsymbol{A^T_\eta P_z z} \tag{2.16}$$

The same as affine transformation,equation element $\Delta \boldsymbol{a}$ contains the corrections for the variables in design matrix and also contains the elements ′1′ due to the structure of transformation model. These ′1′ are necessary to bring the 3 translation parameters into estimation progress, which violats the original idea of CTLS. Although the solutions can also be estimated but there exist corrections for the elements ′1′. Hence we do not discuss with CTLS hier. The results are presented with LS methode.

2.2.3 Presentation and comparison of the results

Table 2.3: *Comparison of 7-parameter Helmert transformation parameters LS*

Transformation Deformation	7-parameter Helmert transformation GK(DHDN)-UTM(ETRS89) $T_X(m)$	$T_Y(m)$	$T_Z(m)$	$\alpha('')$	$\beta('')$	$\gamma('')$	$d\lambda(\times 10^{-6})$
With	582.901711	112.168080	405.603061	-2.255032	-0.335003	2.068369	9.117208
Without	582.901711	112.168080	405.603061	-2.255032	-0.335003	2.068369	9.117208

Table 2.4: *Numerical deviation of 7-parameter Helmert transformation with LS*

Transformation Deformation	Collocated sites	Absolute mean of Residuals (m) $[V_N]$	$[V_E]$	Max.absolute mean of Residuals(m) $[V_N]$	$[V_E]$	RMS (m)	Standard deviation of unit weight (m)
With	B-W 131	0.1051	0.0843	0.4212	0.3112	0.1240	0.1026
Without	B-W 131	0.1051	0.0843	0.4212	0.3112	0.1240	0.1026

The statistical results above, the transfromation model with or without deformation has no affect for the transformation parameters in 7-parameter Helmert transformation model. And the CTLS is not adaptive for the no-centering model in coordinate transformation.

Chapter 3

3D Helmert-transformation in Gauss-Helmert model and Gauss-Markoff model

3.1 Introduction and Background

The 3D Helmert transformation have been calculated, where both the coordinates of the start system and the coordinates of the target system are installed as random variables with identical covariance matrices, and where the unknown transformation parameters are determined by the Least Squares method, it is known that this transformation is independent of the choice of transformation direction. Independent of the choice of the start or target system for the coordinate transformation, we obtain consistent results, that is, definitely transformation parameters. This also be theoretically exponded by Koch (2001), since Lenzmann and Lenzmann (2001b) had claimed that the dependence on the direction of transformation. In the case that coordinates of the start and target systems are installed as a random variable, Reinking (2001) pointed out that the Least Squares method can be used in the definitely coordinate transformations. Using identical results, he demonstrates his statement with the example of the line equalization chosen by Lenzmann and Lenzmann (2001b).

Nevertheless, Lenzmann and Lenzmann (2001a) insist on their statement of the dependence of the coordinate transformation on the transformation direction with the scale parameters provided, which in the case is the Helmert-transformation. Dependence on the direction of transformation is also claimed by Lenzmann (2001). The authors rely on his proof in Lenzmann and Lenzmann (2001b), which is not correct. He wanted to show that the simple transformation

$$y = mx \tag{3.1}$$

depends on the transformation direction. The vector $\boldsymbol{x}$ contains the variable coordinates of arbitrary dimensions of points in the start system and $\boldsymbol{y}$ the variable coordinates in the target system, m is the unknown scale. They determine the estimated value of m in the Gauss-Helmert model (Wolf, 1978), the general case of the equalization calculation, since in (3.1) there is a coordinate of the starting system, one of the target system and the unknown scale in each equation. For linearization it is assumed, as one can easily calculate, that the approximate value for m equals one and the approximations for the x coordinates of the starting system are identical to the coordinates themselves. Lenzmann and Lenzmann (2001b) prove the dependence on the transformation direction for this approximate solution. However, this approximate solution is not identical to the solution that results from iteration, assuming convergence. In this respect, the proof is wrong, since the dependence of the approximate solution on the direction of transformation is proved. The solution itself is independent of the transformation direction, because due to the property of the least squares method, the points determined by x and y have the shortest distances to the allowable set of points satisfying transformation (3.1). This will be discussed in more detail in the 4th section. Any examples chosen demonstrate that the transformation (3.1) is independent of the transformation direction, if $m \neq 0$, x and y are random vectors having identical covariance matrices, and the approximate values are chosen such that the iterations converge. The explicit calculation of the lot roots is not required. However, it may be useful for the calculation of the shortest distances, also called orthogonal distance regression, the changes in the lotus points are iteratively calculated as a function of the changes of the unknown parameters of the estimation problem in order to obtain efficient calculation methods (Helfrich and Zwick 1993, 1995, 1996).

For the straight line adjustment, Reinking (2001) transfers the Gauss-Helmert model to the Gauss-Markoff model by additional unknown parameters. This change of the model is also described by Koch (2001) and generally discussed by Koch (2000b). It is convenient because computational programs developed for the frequently used Gauss-Markoff model can also be applied to parameter estimates that require the Gauss-Helmert model. From this transfer into the Gauss-Markoff model, Lenzmann and Lenzmann (2001a) claimed that the original Gauss-Helmert model is abandoned, meaning that different results could be expected in the two models because, as they put it, "Identity restrictions" are introduced. This claim is incorrect. The additional unknown parameters that may be introduced into the Gauss-Helmert model merely bring the Gauss-Helmert model into the shape of the Gauss-Markoff model. By way of proof, the following example of the spatial Helmert transformation is used to show that identical transformation results are obtained in the Gauss-Helmert model and after conversion into the Gauss-Markoff model. In addition, due to the property of the Least Squares method, the independence of the Helmert transform from the transformation direction is shown. An analytical proof of independence can be found in the presented analytical solution of the seven-parameter transformation of Awange and Grafarend (2002). Correspondingly, one can also proceed with other transformations and with the line equalization, so that Reinking (2001) has obtained the results of Lenzmann and Lenzmann (2001b) in a completely correct way with the method of least squares.

3.2 3D Helmert-transformation

The spatial Helmert transformation, which represents an orthogonal transformation and includes as special cases the seven-parameter transformation and the planar Helmert transformation, is defined as follows: Does the vector x_{si} contain the three-dimensional coordinates of the start system of a point i and the vector x_{ti} the Coordinates of the target system of the same point i, we obtain the spatial Helmert transformation of the coordinates of the point i, see, for example, Schmid and Heggli (1978) or Koch (1999).

$$\boldsymbol{t} + \boldsymbol{R}\boldsymbol{M}\boldsymbol{x}_{si} = \boldsymbol{x}_{ti} \quad for \quad i \in \{1, \cdots, p\} \tag{3.2}$$

with

$$\boldsymbol{t} = \begin{vmatrix} t_x \\ t_y \\ t_z \end{vmatrix}, \quad \boldsymbol{M} = \begin{vmatrix} m_x & 0 & 0 \\ 0 & m_y & 0 \\ 0 & 0 & m_z \end{vmatrix}, \quad \boldsymbol{x}_{si} = \begin{vmatrix} x_{si} \\ y_{si} \\ z_{si} \end{vmatrix}, \quad \boldsymbol{x}_{ti} = \begin{vmatrix} x_{ti} \\ y_{ti} \\ z_{ti} \end{vmatrix}, \quad \boldsymbol{R} = \boldsymbol{R}_3(\gamma)\boldsymbol{R}_2(\beta)\boldsymbol{R}_1(\alpha) \tag{3.3}$$

Here p denotes the number of points to be transformed, $\boldsymbol{t}$ the vector of the unknown translation parameters t_x, t_y, t_z, $\boldsymbol{M}$ the matrix of the unknown dimensional parameters m_x, m_y, m_z and $\boldsymbol{R}$ the rotation matrix resulting from the rotation $\boldsymbol{R}_3(\gamma)$ in the x_1, x_2 plane by the unknown angle γ, from the rotation $\boldsymbol{R}_2(\beta)$ in the x_1, x_3 plane by the unknown angle β, and from the rotation $\boldsymbol{R}_1(\alpha)$ in the x_2, x_3 plane around the Unknown angle α results. The three-dimensional transformation (3.2) thus contains nine unknown parameters. Substituting $mx = my = mz = m$ results in the seven-parameter transformation. Both the coordinates of the starting system and the coordinates of the target system are random variables, ie results of measurements or estimates. Their covariance matrices are given by the weight matrix $\boldsymbol{P}_s$ of the coordinates of the starting system and by the weight matrix $\boldsymbol{P}_t$ of the coordinates of the target system as well as by the variance factor σ^2.

$$\boldsymbol{D}(\begin{vmatrix} x_{s1} \\ x_{s2} \\ \cdots \\ x_{sp} \end{vmatrix}) = \sigma^2 \boldsymbol{P}_s^{-1}, \quad \boldsymbol{D}(\begin{vmatrix} x_{t1} \\ x_{t2} \\ \cdots \\ x_{zp} \end{vmatrix}) = \sigma^2 \boldsymbol{P}_t^{-1} \tag{3.4}$$

The coordinates of the starting system are independent of those of the target system. If x_{si} and x_{zi} exist which satisfy the Helmert transformation (3.2) given the transformation parameters, the transformation is independent of the transformation direction

$$x_{si} = \boldsymbol{M}^{-1}\boldsymbol{R}' x_{ti} - \boldsymbol{M}^{-1}\boldsymbol{R}' t \quad for \quad i \in \{1, \cdots, p\} \tag{3.5}$$

because $\boldsymbol{M}$ and $\boldsymbol{R}$ are regular matrix, then $\boldsymbol{R}^{-1} = \boldsymbol{R}'$.

3.3 Parameter estimation with Gauss-Helmert model

As mentioned in the introduction, the new unknown parameters of the spatial Helmert transform (3.2) are first estimated in the Gauss-Helmert model. For a better overview, the parameter estimation is presented again in this model. It is characterized by that general nonlinear relations between the observations and the exist unknown parameters, see for example Wolf (1968),

$$h_i\,(y_1 + e1, \cdots, y_n + e_n, \beta_1, \cdots, \beta_n) = 0 \quad i \in \{1, \cdots, r\} \tag{3.6}$$

in which h_i denote general, differentiable functions, y_j with $j \in \{1, \cdots, n\}$ the observations, e_j their errors and β_k with $k \in \{1, \cdots, u\}$ the unknown parameters. For the linearization of (3.6) the perfect or "true" observations $\bar{y}_j$ are defined by

$$\bar{y}_j = y_i + e_i, \quad j \in \{1, \cdots, n\} \quad with \quad \bar{y}_j = \boldsymbol{E}(y_i) \quad \boldsymbol{E}\left(e_j = 0\right) \tag{3.7}$$

Approximate values $\bar{y}_{j0}$ for $\bar{y}_j$ are given, so that with the corrections $\bar{e}_j$ of the approximate values applies

$$\bar{y}_j = \bar{y}_{i0} + \bar{e}_i, \quad j \in \{1, \cdots, n\} \tag{3.8}$$

For the unknown parameters β_k, the approximate values β_{k0} are known, so that with their corrections $\Delta\beta_k$ is obtained,

$$\beta_j = \beta_{k0} + \Delta\beta_k, \quad k \in \{1, \cdots, u\} \tag{3.9}$$

The Taylor expansion for (3.6) on the approximation value is linearized. After the linearization gets

$$\begin{aligned} h_i(\bar{y}_1, \cdots, \bar{y}_n, \beta_1, \cdots, \beta_u) = h_i(\bar{y}_{10}, \cdots, \bar{y}_{n0}, \beta_{10}, \cdots, \beta_{u0}) \\ + \left.\tfrac{\partial h_i}{\partial \bar{y}_1}\right|_0 \bar{e}_1 + \cdots + \left.\tfrac{\partial h_i}{\partial \bar{y}_n}\right|_0 \bar{e}_n + \left.\tfrac{\partial h_i}{\partial \bar{\beta}_1}\right|_0 \Delta\beta_1 + \cdots + \left.\tfrac{\partial h_i}{\partial \bar{\beta}_u}\right|_0 \Delta\beta_u \end{aligned} \tag{3.10}$$

The partial derivatives in (3.10) are denoted as follows

$$\left.\tfrac{\partial h_i}{\partial \bar{y}_j}\right|_0 = z_{ij} \quad and \quad \left.\tfrac{\partial h_i}{\partial \bar{\beta}_k}\right|_0 = x_{ik} \tag{3.11}$$

and summarized in the matrices $\boldsymbol{X}$ and $\boldsymbol{Z}$.

$$\boldsymbol{X} = (x_{ik}) \quad and \quad \boldsymbol{Z} = (z_{ij}) \tag{3.12}$$

with the vector

$$\begin{aligned} \boldsymbol{\beta} = \begin{vmatrix} \Delta\beta_1 \\ \cdots \\ \Delta\beta_u \end{vmatrix}, \quad \bar{\boldsymbol{e}} = \begin{vmatrix} \bar{e}_1 \\ \cdots \\ \bar{e}_n \end{vmatrix}, \\ \bar{\boldsymbol{\omega}} = \begin{vmatrix} h_1(\bar{y}_{10}, \cdots, \bar{y}_{n0}, \beta_{10}, \cdots, \beta_{u0}) \\ \cdots \\ h_r(\bar{y}_{10}, \cdots, \bar{y}_{n0}, \beta_{10}, \cdots, \beta_{u0}) \end{vmatrix} \end{aligned} \tag{3.13}$$

the linearized model results instead of (3.6)

$$\boldsymbol{X\beta} + \boldsymbol{Z}\bar{\boldsymbol{e}} + \bar{\boldsymbol{\omega}} = \boldsymbol{0} \tag{3.14}$$

The covariance matrix of the observation vector $\boldsymbol{y}$ with $\boldsymbol{y} = (y_j)$ and $j \in \{1, \cdots, n\}$ given by

$$\boldsymbol{D}(\boldsymbol{y}) = \sigma^2 \boldsymbol{\Sigma} \tag{3.15}$$

where σ^2 denotes the variance factor and Σ is a positive definite matrix. The method of the Least Squares demands that with $\bar{\boldsymbol{y}} = (\bar{y}_j)$, $\boldsymbol{e} = (e_j)$ and with $\bar{\boldsymbol{y}} = E(y)$ from (3.7) the weighted sum of squares of the errors

$$\frac{1}{\sigma^2}\boldsymbol{e}'\boldsymbol{\Sigma}^{-1}\boldsymbol{e} \tag{3.16}$$

becomes minimal, see for example Koch (1999), where the condition is that the model (3.14) is fulfilled. In (3.14), therefore, the error vector $\bar{e}$ must be replaced by the error vector e in (3.16). With $\bar{\boldsymbol{y}}_0 = (\bar{y}_{j0})$ get we from (3.13)

$$\bar{\boldsymbol{e}} = \bar{\boldsymbol{y}} - \bar{\boldsymbol{y}}_0 = \bar{\boldsymbol{y}} - \boldsymbol{y} + \boldsymbol{y} - \bar{\boldsymbol{y}}_0 \tag{3.17}$$

and with the equation (3.7)

$$\bar{\boldsymbol{e}} = \boldsymbol{e} + \boldsymbol{y} - \bar{\boldsymbol{y}}_0 \tag{3.18}$$

and finally instead of (3.14)

$$\boldsymbol{X\beta} + \boldsymbol{Ze} + \boldsymbol{\omega} = \boldsymbol{0} \tag{3.19}$$

with

$$\boldsymbol{\omega} = \boldsymbol{Z}(\boldsymbol{y} - \bar{\boldsymbol{y}}_0) + \bar{\boldsymbol{\omega}} \tag{3.20}$$

Through a Taylor development we get hier more $\omega_i = h_1(y_1, \cdots, y_n, \beta_{10}, \cdots, \beta_{u0})$. The square shape (3.16) is to be minimized under the restriction (3.19). It is therefore the Lagrange function

$$\omega\left(\boldsymbol{\beta}, \boldsymbol{e}\right) = \frac{1}{\sigma^2}\boldsymbol{e}'\boldsymbol{\Sigma}^{-1}\boldsymbol{e} - \frac{2}{\sigma^2}\boldsymbol{k}'\left(\boldsymbol{X\beta} + \boldsymbol{Ze} + \boldsymbol{\omega}\right) \tag{3.21}$$

introduced, in the $-2\boldsymbol{k}/\sigma^2$ the Lagrange vector called multipliers. The derivatives of Lagrange function is set to zero after β and e

$$\frac{\partial\omega(\boldsymbol{\beta}, \boldsymbol{e})}{\partial\boldsymbol{\beta}} = -\frac{2}{\sigma^2}\boldsymbol{X}'\boldsymbol{k} = \boldsymbol{0} \tag{3.22}$$

$$\frac{\partial\omega(\boldsymbol{\beta}, \boldsymbol{e})}{\partial\boldsymbol{e}} = \frac{2}{\sigma^2}\boldsymbol{\Sigma}'\boldsymbol{e} - \frac{2}{\sigma^2}\boldsymbol{Z}'\boldsymbol{k} = \boldsymbol{0} \tag{3.23}$$

result from (3.23) the estimate $\hat{e}$ of e, that is, the residuals for

$$\hat{\boldsymbol{e}} = \boldsymbol{\Sigma}\boldsymbol{Z}'\boldsymbol{k} \tag{3.24}$$

This result, used in (3.19), merges with (3.22) to the normal equation system for the estimation $\hat{\boldsymbol{\beta}}$ of unknown parameter $\boldsymbol{\beta}$ and for estimation from k

$$\begin{vmatrix} \boldsymbol{Z\Sigma Z'} & \boldsymbol{X} \\ \boldsymbol{X'} & \boldsymbol{0} \end{vmatrix} \begin{vmatrix} \boldsymbol{k} \\ \hat{\boldsymbol{\beta}} \end{vmatrix} = \begin{vmatrix} -\boldsymbol{\omega} \\ \boldsymbol{0} \end{vmatrix} \tag{3.25}$$

The matrix $\boldsymbol{Z}$ has full row rank, so that $\boldsymbol{Z\Sigma Z'}$ is positive definite. In addition, $\boldsymbol{X}$ has full column rank. The elimination of $\boldsymbol{k}$ from (3.25) then gives the known Estimation $\hat{\boldsymbol{\beta}}$ of the parameter β in the Gauss-Markoff modell.

$$\hat{\boldsymbol{\beta}} = -(\boldsymbol{X'}\left(\boldsymbol{Z\Sigma Z'}\right)^{-1}\boldsymbol{X})^{-1}\boldsymbol{X'}\left(\boldsymbol{Z\Sigma Z'}\right)^{-1}\boldsymbol{\omega} \tag{3.26}$$

For the vector $\boldsymbol{k}$ we obtain from (3.25)

$$\boldsymbol{k} = \left(\boldsymbol{Z\Sigma Z'}\right)^{-1}(-\boldsymbol{\omega} - \boldsymbol{X}\hat{\boldsymbol{\beta}}) \tag{3.27}$$

so that the vector $\boldsymbol{e}$ of the residuals is known from (3.24)to

$$\hat{\boldsymbol{e}} = \boldsymbol{\Sigma Z'}\left(\boldsymbol{Z\Sigma Z'}\right)^{-1}(-\boldsymbol{\omega} - \boldsymbol{X}\hat{\boldsymbol{\beta}}) \tag{3.28}$$

The relationships between the Gauss-Markoff-Model and the mixed model are given in (Koch 2000a).
To the new unknown parameters of spatial Helmert transformation (3.2) in the Gauss-Markoff model (3.6), (3.2) is presented according to (3.6) by

$$\boldsymbol{h}_i = \boldsymbol{t} + \boldsymbol{RM}(x_{si} + \boldsymbol{e}_{si}) - (x_{ti} + \boldsymbol{e}_{ti}) \quad for \quad i \in \{1, \cdots, p\} \tag{3.29}$$

where $\boldsymbol{e}_{si}$ are the errors of the spatial coordinates of the point i in the starting system and $\boldsymbol{e}_{ti}$ the errors in the target system. According to (3.7) and (3.8) we can get

$$\bar{\boldsymbol{x}}_{si} = \boldsymbol{x}_{si} + \boldsymbol{e}_{si} = \bar{\boldsymbol{x}}_{si0} + \bar{\boldsymbol{e}}_{si} \tag{3.30}$$

$$\bar{\boldsymbol{x}}_{ti} = \boldsymbol{x}_{ti} + \boldsymbol{e}_{ti} = \bar{\boldsymbol{x}}_{ti0} + \bar{\boldsymbol{e}}_{ti} \tag{3.31}$$

where $\bar{\boldsymbol{x}}_{si0}$ and $\bar{\boldsymbol{x}}_{ti0}$ denote the approximations of the 'true' coordinates $\bar{\boldsymbol{x}}_{si}$ and $\bar{\boldsymbol{x}}_{ti}$ of the start and target system and $\bar{\boldsymbol{e}}_{si}$ and $\bar{\boldsymbol{e}}_{ti}$ their corrections. According to (3.9), approximate values are given for the unknown transformation parameters, therefore

$$\boldsymbol{t} = \boldsymbol{t}_0 + \Delta\boldsymbol{t}, \quad \boldsymbol{\alpha} = \boldsymbol{\alpha}_0 + \Delta\boldsymbol{\alpha}, \quad \boldsymbol{m} = \boldsymbol{m}_0 + \Delta\boldsymbol{m} \tag{3.32}$$

with

$$\boldsymbol{\alpha} = |\alpha, \beta, \gamma|' \quad and \quad \boldsymbol{m} = |m_x, m_y, m_z|' \tag{3.33}$$

For the linearization of (3.29) corresponding to (3.10), the differential quotients are formed,

$$\begin{aligned} &\left.\frac{\partial \boldsymbol{h}_i}{\partial \boldsymbol{t}}\right|_0 = \boldsymbol{I}, \quad \left.\frac{\partial \boldsymbol{h}_i}{\partial \boldsymbol{\alpha}}\right|_0 = \boldsymbol{B}_i, \quad \left.\frac{\partial \boldsymbol{h}_i}{\partial \boldsymbol{m}}\right|_0 = \boldsymbol{C}_i, \\ &\left.\frac{\partial \boldsymbol{h}_i}{\partial \bar{\boldsymbol{x}}_{si}}\right|_0 = \boldsymbol{R}_0\boldsymbol{M}_0, \quad \left.\frac{\partial \boldsymbol{h}_i}{\partial \bar{\boldsymbol{x}}_{ti}}\right|_0 = -\boldsymbol{I}_0 \end{aligned} \tag{3.34}$$

where $\boldsymbol{R}_0$ and $\boldsymbol{M}_0$ denote the matrices $\boldsymbol{R}$ and $\boldsymbol{M}$ calculated with the approximation values of the transformation parameters. Summarize

$$\begin{aligned} &\boldsymbol{X}_i = |\boldsymbol{I}, \boldsymbol{B}_i, \boldsymbol{C}_i| \\ &\boldsymbol{X} = \begin{vmatrix} \boldsymbol{X}_1 \\ \cdots \\ \boldsymbol{X}_p \end{vmatrix}, \quad \boldsymbol{\beta} = \begin{vmatrix} \Delta\boldsymbol{t} \\ \Delta\boldsymbol{\alpha} \\ \Delta\boldsymbol{m} \end{vmatrix} \\ &\boldsymbol{Z} = \begin{vmatrix} \boldsymbol{R}_0\boldsymbol{M_0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & -\boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R}_0\boldsymbol{M_0} & \cdots & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} & \cdots & \boldsymbol{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{R}_0\boldsymbol{M_0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & -\boldsymbol{I} \end{vmatrix} \\ &\boldsymbol{e} = \begin{vmatrix} \boldsymbol{e}_{s1} \\ \cdots \\ \boldsymbol{e}_{sp} \\ \boldsymbol{e}_{t1} \\ \cdots \\ \boldsymbol{e}_{tp} \end{vmatrix}, \quad \boldsymbol{\omega} = \begin{vmatrix} \boldsymbol{\omega}_1 \\ \boldsymbol{\omega}_2 \\ \cdots \\ \boldsymbol{\omega}_p \end{vmatrix} \\ &\boldsymbol{\omega}_i = \boldsymbol{R}_0\boldsymbol{M}_0\left(\boldsymbol{x}_{si} - \bar{\boldsymbol{x}}_{si0}\right) - \left(\boldsymbol{x}_{ti} - \bar{\boldsymbol{x}}_{ti0}\right) + \boldsymbol{t}_0 + \boldsymbol{R}_0\boldsymbol{M}_0\bar{\boldsymbol{x}}_{si0} - \bar{\boldsymbol{x}}_{ti0} \\ &\quad = \boldsymbol{t}_0 + \boldsymbol{R}_0\boldsymbol{M}_0\boldsymbol{x}_{si} - \boldsymbol{x}_{ti} = -\boldsymbol{y}_{ti} \end{aligned} \tag{3.35}$$

we obtain the linearized Gauss-Helmert model (3.19) for the Helmert transformation of the p points with $\boldsymbol{y}_t = (\boldsymbol{y}_{ti})$ to

$$\boldsymbol{X\beta} + \boldsymbol{Ze} - \boldsymbol{y}_z = \boldsymbol{0} \tag{3.36}$$

The covariance matrix $\sigma^2\boldsymbol{\Sigma}$ in (3.15) of the observations $\boldsymbol{x}_{si}$ and $\boldsymbol{x}_{ti}$ is obtained because of (3.4), with

$$\boldsymbol{\Sigma} = \begin{vmatrix} \boldsymbol{P}_s^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}_t^{-1} \end{vmatrix} \tag{3.37}$$

used in (3.36)

$$\boldsymbol{Z} = \begin{vmatrix} \boldsymbol{W} & -\boldsymbol{I} \end{vmatrix} \tag{3.38}$$

$\boldsymbol{Z\Sigma Z}'$ results from (3.20) to

$$\boldsymbol{Z\Sigma Z}' = \boldsymbol{W}\boldsymbol{P}_s^{-1}\boldsymbol{W}' + \boldsymbol{P}_t^{-1} \tag{3.39}$$

With this result and with (3.36) the unknown transformation parameter $\boldsymbol{\beta}$ follows the estimate $\hat{\boldsymbol{\beta}}$ from (3.26) to

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'(\boldsymbol{W}\boldsymbol{P}_s^{-1}\boldsymbol{W}' + \boldsymbol{P}_t^{-1})^{-1}\boldsymbol{X})^{-1} \times \boldsymbol{X}'(\boldsymbol{W}\boldsymbol{P}_s^{-1}\boldsymbol{W}' + \boldsymbol{P}_t^{-1})^{-1}\boldsymbol{y}_t \tag{3.40}$$

For comparison with the estimation result of the Gauss-Markoff-Model, which is to be deduced in the next but one section, the inverse of $\boldsymbol{W}\boldsymbol{P}_s^{-1}\boldsymbol{W}' + \boldsymbol{P}_t^{-1}$ is transformed by an identity (Koch 2000b), and we get

$$\begin{aligned}\hat{\boldsymbol{\beta}} = {} & (\boldsymbol{X}'\boldsymbol{P}_t\boldsymbol{X} - \boldsymbol{X}'\boldsymbol{P}_t\boldsymbol{W}(\boldsymbol{W}'\boldsymbol{P}_t\boldsymbol{W} + \boldsymbol{P}_s)^{-1}\boldsymbol{W}'\boldsymbol{P}_t\boldsymbol{X})^{-1} \\ & \times (\boldsymbol{X}'\boldsymbol{P}_t\boldsymbol{y}_t - \boldsymbol{X}'\boldsymbol{P}_t\boldsymbol{W}(\boldsymbol{W}'\boldsymbol{P}_t\boldsymbol{W} + \boldsymbol{P}_s)^{-1}\boldsymbol{W}'\boldsymbol{P}_t\boldsymbol{y}_t)\end{aligned} \tag{3.41}$$

With the residuals $\hat{\boldsymbol{e}}_s = \hat{\boldsymbol{e}}_{si}$ and $\hat{\boldsymbol{e}}_t = \hat{\boldsymbol{e}}_{ti}$, the estimates of the errors $\boldsymbol{e}_{si}$ and $\boldsymbol{e}_{ti}$ in (3.30) and (3.31), the vector e follows the residuals from (3.28) up (3.36) until (3.40) to

$$\hat{\boldsymbol{e}} = \begin{vmatrix}\hat{\boldsymbol{e}}_s \\ \hat{\boldsymbol{e}}_t\end{vmatrix} = \begin{vmatrix}\boldsymbol{P}_s^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}_t^{-1}\end{vmatrix} \times \begin{vmatrix}\boldsymbol{W}' \\ -\boldsymbol{I}\end{vmatrix} (\boldsymbol{W}\boldsymbol{P}_s^{-1}\boldsymbol{W}' + \boldsymbol{P}_t^{-1})^{-1}(\boldsymbol{y}_t - \boldsymbol{X}\hat{\boldsymbol{\beta}}) \tag{3.42}$$

or

$$\hat{\boldsymbol{e}}_s = \boldsymbol{P}_s^{-1}\boldsymbol{W}'(\boldsymbol{W}\boldsymbol{P}_s^{-1}\boldsymbol{W}' + \boldsymbol{P}_t^{-1})^{-1}(\boldsymbol{y}_t - \boldsymbol{X}\hat{\boldsymbol{\beta}}) \tag{3.43}$$

$$\hat{\boldsymbol{e}}_t = -\boldsymbol{P}_t^{-1}(\boldsymbol{W}\boldsymbol{P}_s^{-1}\boldsymbol{W}' + \boldsymbol{P}_t^{-1})^{-1}(\boldsymbol{y}_t - \boldsymbol{X}\hat{\boldsymbol{\beta}}) \tag{3.44}$$

With another template identity, (3.43) is transformed (Koch 2000a) and (3.44) with the identity that leads to (3.41), so that it is finally obtained

$$\hat{\boldsymbol{e}}_s = (\boldsymbol{W}'\boldsymbol{P}_t\boldsymbol{W} + \boldsymbol{P}_s)^{-1}\boldsymbol{W}'\boldsymbol{P}_t(\boldsymbol{y}_t - \boldsymbol{X}\hat{\boldsymbol{\beta}}) \tag{3.45}$$

$$\hat{\boldsymbol{e}}_t = (-\boldsymbol{I} + \boldsymbol{W}(\boldsymbol{W}'\boldsymbol{P}_t\boldsymbol{W} + \boldsymbol{P}_s)^{-1}\boldsymbol{W}'\boldsymbol{P}_t) \times (\boldsymbol{y}_t - \boldsymbol{X}\hat{\boldsymbol{\beta}}) \tag{3.46}$$

The estimated values $\hat{\boldsymbol{\beta}}$ of the unknown nine transformation parameters $\boldsymbol{\beta}$ thus follow from (3.40) or (3.41) and the residuals $\hat{\boldsymbol{e}}_s$ of the coordinates of the starting system from (3.43) or (3.45) and the residuals $\hat{\boldsymbol{e}}_t$ of the coordinates of the target system (3.44) or (3.46)

3.4 Parameter estimation with Gauss-Markoff model

The coordinates $\bar{\boldsymbol{x}}_{si}$ in (3.25) of the point i in the start system are interpreted according to (3.7) as 'true' observations. We can see them in the Gauss-Helmert model (3.29) but also as unknown parameters, then must but introduce their definition (3.35) as an additional observation equation. Now replace the Gauss -Helmert model (3.29) to the Gauss-Markoff model (Koch et al., 2000)

$$\begin{aligned} \boldsymbol{t} + \boldsymbol{R}\boldsymbol{M}\bar{\boldsymbol{x}}_{si} &= \boldsymbol{x}_{ti} + \boldsymbol{e}_{ti} \\ \bar{\boldsymbol{x}}_{si} &= \boldsymbol{x}_{si} + \boldsymbol{e}_{si} \quad for \quad i \in \{1, \cdots, p\} \end{aligned} \tag{3.47}$$

The Gauss-Helmert model (3.29) and the Gauss-Markoff model (3.47) are equivalent. In both models result identical parameter estimates and residuals. Around to show that is initially linearized. (3.32) introduces the approximate values for the new unknown transformation parameters and (3.30) with

$$\bar{\boldsymbol{x}}_{si} = \bar{\boldsymbol{x}}_{si0} + \Delta\boldsymbol{x}_{si} \tag{3.48}$$

the approximate values $\bar{\boldsymbol{x}}_{si0}$ of $\bar{\boldsymbol{x}}_{si}$ and the unknown ones Corrections $\Delta\boldsymbol{x}_{si}$. With the partial derivatives corresponding to (3.34)

$$\begin{aligned} &\left.\frac{\partial \boldsymbol{x}_{ti}}{\partial \boldsymbol{t}}\right|_0 = \boldsymbol{I}, \quad \left.\frac{\partial \boldsymbol{x}_{ti}}{\partial \boldsymbol{\alpha}}\right|_0 = \boldsymbol{B}_i, \quad \left.\frac{\partial \boldsymbol{x}_{ti}}{\partial \boldsymbol{m}}\right|_0 = \boldsymbol{C}_i, \\ &\left.\frac{\partial \boldsymbol{x}_{ti}}{\partial \bar{\boldsymbol{x}}_{si}}\right|_0 = \boldsymbol{R}_0\boldsymbol{M}_0, \quad \left.\frac{\partial \boldsymbol{x}_{si}}{\partial \boldsymbol{t}}\right|_0 = \boldsymbol{0}, \quad \left.\frac{\partial \boldsymbol{x}_{si}}{\partial \boldsymbol{\alpha}}\right|_0 = \boldsymbol{0}, \\ &\left.\frac{\partial \boldsymbol{x}_{si}}{\partial \boldsymbol{m}}\right|_0 = \boldsymbol{0}, \quad \left.\frac{\partial \boldsymbol{x}_{si}}{\partial \bar{\boldsymbol{x}}_{si}}\right|_0 = \boldsymbol{I}, \end{aligned} \tag{3.49}$$

the linearized model is given by (3.35)

$$\begin{aligned} \boldsymbol{R}_0\boldsymbol{M}_0\Delta\boldsymbol{x}_{si} + \boldsymbol{X}_i\boldsymbol{\beta} &= \boldsymbol{y}_{ti} + \boldsymbol{e}_{ti} \\ \boldsymbol{I}\Delta\boldsymbol{x}_{si} &= \boldsymbol{y}_{si} + \boldsymbol{e}_{si} \end{aligned} \tag{3.50}$$

with the observations

$$\begin{aligned} \boldsymbol{y}_{ti} &= \boldsymbol{x}_{ti} - \boldsymbol{t}_0 - \boldsymbol{R}_0\boldsymbol{M}_0\bar{\boldsymbol{x}}_{si0} \\ \boldsymbol{y}_{si} &= \boldsymbol{x}_{si} - \bar{\boldsymbol{x}}_{si0} \end{aligned} \tag{3.51}$$

With$\Delta\boldsymbol{x}_s = (\Delta\boldsymbol{x}_{si}), \boldsymbol{y}_t = (\boldsymbol{y}_{ti}), \boldsymbol{y}_s = (\boldsymbol{y}_{si}), \boldsymbol{e}_t = (\boldsymbol{e}_{ti}), \boldsymbol{e}_s = (\boldsymbol{e}_{si})$, the matrix W from (3.38) and the covariance matrix of the observations from (3.3) follow the observation equations for all p points to be transformed with

$$\begin{aligned} &\begin{vmatrix} \boldsymbol{W} & \boldsymbol{X} \\ \boldsymbol{I} & \boldsymbol{0} \end{vmatrix} \begin{vmatrix} \Delta\boldsymbol{x}_s \\ \boldsymbol{\beta} \end{vmatrix} = \begin{vmatrix} \boldsymbol{y}_t + \boldsymbol{e}_t \\ \boldsymbol{y}_s + \boldsymbol{e}_s \end{vmatrix} \\ and \quad &D(\begin{vmatrix} \boldsymbol{y}_t \\ \boldsymbol{y}_s \end{vmatrix}) = \sigma^2 \begin{vmatrix} \boldsymbol{P}_t^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}_s^{-1} \end{vmatrix} \end{aligned} \tag{3.52}$$

The estimated values $\Delta\hat{\boldsymbol{x}}_s$ and $\hat{\boldsymbol{\beta}}$ of the unknown parameters $\Delta\boldsymbol{x}_s$ and $\boldsymbol{\beta}$ are known to be obtained with

$$\begin{vmatrix} \Delta\hat{\boldsymbol{x}}_s \\ \hat{\boldsymbol{\beta}} \end{vmatrix} = \begin{vmatrix} \boldsymbol{W}'\boldsymbol{P}_t\boldsymbol{W} + \boldsymbol{P}_s & \boldsymbol{W}'\boldsymbol{P}_t\boldsymbol{W} \\ \boldsymbol{X}'\boldsymbol{P}_t\boldsymbol{W} & \boldsymbol{X}'\boldsymbol{P}_t\boldsymbol{X} \end{vmatrix}^{-1} \times \begin{vmatrix} \boldsymbol{W}'\boldsymbol{P}_t\boldsymbol{y}_t + \boldsymbol{P}_s\boldsymbol{y}_s \\ \boldsymbol{X}'\boldsymbol{P}_t\boldsymbol{y}_t \end{vmatrix} \tag{3.53}$$

The estimates are dependent on the approximations, so it must be iterated. The vector $\bar{\boldsymbol{x}}_{si}$ occurs linearly in the Helmert transform (3.47) if the transformation parameters are given. It is therefore possible to choose any approximate value $\bar{\boldsymbol{x}}_{si0}$ in (3.48), although the matrix $\boldsymbol{X}_i$ are to be calculated with the respective estimates for $\bar{\boldsymbol{x}}_{si}$. It is through

$$\bar{\boldsymbol{x}}_{si0} = \boldsymbol{x}_{si} \tag{3.54}$$

setted, so that from (3.50) and (3.51) follows

$$\boldsymbol{y}_s = \boldsymbol{0} \quad \textit{and} \quad \Delta\hat{\boldsymbol{x}}_s = \hat{\boldsymbol{e}}_s \tag{3.55}$$

With this substitution and the inverse of a block-matrix, see for example Koch (1999) the estimated values $\hat{\boldsymbol{\beta}}$ of the transformation parameters

$$\begin{aligned} \hat{\boldsymbol{\beta}} &= (\boldsymbol{X}'\boldsymbol{P}_t\boldsymbol{X} - \boldsymbol{X}'\boldsymbol{P}_t\boldsymbol{W}(\boldsymbol{W}'\boldsymbol{P}_t\boldsymbol{W} + \boldsymbol{P}_s)^{-1}\boldsymbol{W}'\boldsymbol{P}_t\boldsymbol{X})^{-1} \\ &\times (\boldsymbol{X}'\boldsymbol{P}_t\boldsymbol{y}_t - \boldsymbol{X}'\boldsymbol{P}_t\boldsymbol{W}(\boldsymbol{W}'\boldsymbol{P}_t\boldsymbol{W} + \boldsymbol{P}_s)^{-1}\boldsymbol{W}'\boldsymbol{P}_t\boldsymbol{y}_t) \end{aligned} \tag{3.56}$$

Residuals $\hat{\boldsymbol{e}}_s$ follow it the coordinates of the starting system with (3.55) by substitution of $\hat{\boldsymbol{\beta}}$ in the (3.53) corresponding normal equation system from

$$\hat{\boldsymbol{e}}_s = (\boldsymbol{W}'\boldsymbol{P}_t\boldsymbol{W} + \boldsymbol{P}_s)^{-1}\boldsymbol{W}'\boldsymbol{P}_t(\boldsymbol{y}_t - \boldsymbol{X}\hat{\boldsymbol{\beta}}) \tag{3.57}$$

The residuals $\hat{\boldsymbol{e}}_s$ give the coordinates of the target system finally from (3.52) with (3.55) and (3.57) to

$$\hat{\boldsymbol{e}}_t = (-\boldsymbol{I} + \boldsymbol{W}(\boldsymbol{W}'\boldsymbol{P}_t\boldsymbol{W} + \boldsymbol{P}_s)^{-1}\boldsymbol{W}'\boldsymbol{P}_t) \times (\boldsymbol{y}_t - \boldsymbol{X}\hat{\boldsymbol{\beta}}) \tag{3.58}$$

The estimates $\hat{\boldsymbol{\beta}}$ from (3.56) obtained here in the Gauss-Markoff model and the residuals es and ez from (3.57) and (3.58) are in agreement with those in the Gauss-Markoff model obtained Results (3.41), (3.45) and (3.46).

3.5 Applications in coordinates transformation and comparison with CTLS

In this application the coordinates transformation will also be implamented in Baden-Württemberg. The Helmert-transformation in Gauss-Helmert model read

3.5.1 Transformation with Gauss-Helmert model

$$\boldsymbol{E}\left\{\begin{bmatrix} x_{t1} \\ \vdots \\ x_{tn} \\ y_{t1} \\ \vdots \\ y_{tn} \\ z_{t1} \\ \vdots \\ z_{tn} \end{bmatrix}\right\} =: \boldsymbol{E}\left\{\begin{bmatrix} 1 & 0 & 0 & 0 & -z_{s1} & y_{s1} & x_{s1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & -z_{sn} & y_{sn} & x_{sn} \\ 0 & 1 & 0 & z_{s1} & 0 & -x_{s1} & y_{s1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & z_{ln} & 0 & -x_{sn} & y_{sn} \\ 0 & 0 & 1 & -y_{s1} & x_{s1} & 0 & z_{s1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & -y_{sn} & x_{sn} & 0 & z_{sn} \end{bmatrix}\right\} \begin{bmatrix} T_x \\ T_y \\ T_z \\ \delta\alpha \\ \delta\beta \\ \delta\gamma \\ \lambda \end{bmatrix}$$

Step 1: Replace the Gauss-Helmert model to Gauss-Markoff model and creat the observation equations and estimate the approximate value $\bar{\boldsymbol{x}}_{si0}$ of $\bar{\boldsymbol{x}}_{si}$ with LS.

$$\begin{aligned} \boldsymbol{t} + \boldsymbol{RM}\bar{\boldsymbol{x}}_{si} &= \boldsymbol{x}_{ti} + \boldsymbol{e}_{ti} \\ \bar{\boldsymbol{x}}_{si} &= \boldsymbol{x}_{si} + \boldsymbol{e}_{si} \quad for \quad i \in \{1, \cdots, n\} \end{aligned}$$

Step 2: creat the linear model and complete the elements togethere with the inside structure in observation equations.
with

$$\underset{(3n\times 7)}{\boldsymbol{X}} = \begin{bmatrix} 1 & 0 & 0 & 0 & -z_{s1} & y_{s1} & x_{s1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & -z_{sn} & y_{sn} & x_{sn} \\ 0 & 1 & 0 & z_{s1} & 0 & -x_{s1} & y_{s1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & z_{ln} & 0 & -x_{sn} & y_{sn} \\ 0 & 0 & 1 & -y_{s1} & x_{s1} & 0 & z_{s1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & -y_{sn} & x_{sn} & 0 & z_{sn} \end{bmatrix},$$

$$\begin{aligned} \underset{(3n\times 3n)}{\boldsymbol{W0}} &= \begin{bmatrix} \lambda & \delta\gamma & -\delta\beta \\ -\delta\gamma & \lambda & \delta\alpha \\ \delta\beta & -\delta\alpha & \lambda \end{bmatrix} \otimes \boldsymbol{I_n} \\ &= \begin{bmatrix} \lambda & 0 & 0 & \delta\gamma & 0 & 0 & -\delta\beta & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & 0 & \lambda & 0 & 0 & \delta\gamma & 0 & 0 & -\delta\beta \\ -\delta\gamma & 0 & 0 & \lambda & 0 & 0 & \delta\alpha & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & 0 & -\delta\gamma & 0 & 0 & \lambda & 0 & 0 & \delta\alpha \\ \delta\beta & 0 & 0 & -\delta\alpha & 0 & 0 & \lambda & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & 0 & \delta\beta & 0 & 0 & -\delta\alpha & 0 & 0 & \lambda \end{bmatrix} \end{aligned}$$

$$\boldsymbol{y_t} = \boldsymbol{y_{ti}} - \boldsymbol{t} - \boldsymbol{W0} * \boldsymbol{y_{si}}$$

Step 3: Estimate the solution $\hat{\boldsymbol{\beta}}$ for the corrections by transformation parameters and the corresponding coordinates.

$$\begin{aligned}\hat{\boldsymbol{\beta}} = &(\boldsymbol{X}'\boldsymbol{P}_t\boldsymbol{X} - \boldsymbol{X}'\boldsymbol{P}_t\boldsymbol{W}(\boldsymbol{W}'\boldsymbol{P}_t\boldsymbol{W} + \boldsymbol{P}_s)^{-1}\boldsymbol{W}'\boldsymbol{P}_t\boldsymbol{X})^{-1} \\ &\times (\boldsymbol{X}'\boldsymbol{P}_t\boldsymbol{y}_t - \boldsymbol{X}'\boldsymbol{P}_t\boldsymbol{W}(\boldsymbol{W}'\boldsymbol{P}_t\boldsymbol{W} + \boldsymbol{P}_s)^{-1}\boldsymbol{W}'\boldsymbol{P}_t\boldsymbol{y}_t)\end{aligned}$$

Step 4: calculate the estimated transformation parameters and the statistical residuals and make accuracy assessment.

Residuals $\hat{\boldsymbol{e}}_s$, the coordinates of the start system with

$$\hat{\boldsymbol{e}}_s = (\boldsymbol{W}'\boldsymbol{P}_t\boldsymbol{W} + \boldsymbol{P}_s)^{-1}\boldsymbol{W}'\boldsymbol{P}_t(\boldsymbol{y}_t - \boldsymbol{X}\hat{\boldsymbol{\beta}})$$

The residuals $\hat{\boldsymbol{e}}_t$, the coordinates of the target system with

$$\hat{\boldsymbol{e}}_t = (-\boldsymbol{I} + \boldsymbol{W}(\boldsymbol{W}'\boldsymbol{P}_t\boldsymbol{W} + \boldsymbol{P}_s)^{-1}\boldsymbol{W}'\boldsymbol{P}_t) \times (\boldsymbol{y}_t - \boldsymbol{X}\hat{\boldsymbol{\beta}})$$

3.5.2 Presentation and Comparison of the results

Statistical data by the quadratics sums of the residuals for 3 estimators. The $\hat{e}$ is the residual of observation and $\hat{E}$ is the residual of design matrix.
LS:

$$\hat{\boldsymbol{e}}_{LS}^T\hat{\boldsymbol{e}}_{LS} = 4.063234 \quad (m^2)$$

Gauss-Helmert:

$$\begin{aligned}\hat{\boldsymbol{e}}_t^T\hat{\boldsymbol{e}}_t &= 1.015790 \quad (m^2) \\ \hat{\boldsymbol{E}}_s^T\hat{\boldsymbol{E}}_s &= 1.015808 \quad (m^2) \\ \hat{\boldsymbol{e}}_t^T\hat{\boldsymbol{e}}_t + \hat{\boldsymbol{E}}_s^T\hat{\boldsymbol{E}}_s &= 2.031598 \quad (m^2)\end{aligned}$$

CTLS:

$$\begin{aligned}\hat{\boldsymbol{e}}_{CTLS}^T\hat{\boldsymbol{e}}_{CTLS} &= 1.015790 \quad (m^2) \\ \hat{\boldsymbol{E}}_{CTLS}^T\hat{\boldsymbol{E}}_{CTLS} &= 1.015808 \quad (m^2) \\ \hat{\boldsymbol{e}}_{CTLS}^T\hat{\boldsymbol{e}}_{CTLS} + \hat{\boldsymbol{E}}_{CTLS}^T\hat{\boldsymbol{E}}_{CTLS} &= 2.031598 \quad (m^2)\end{aligned}$$

Table 3.1: *Comparison of 7-parameter Helmert transformation parameters with 3 estimators*

Transformation models	7-parameter Helmert transformation GK(DHDN)-UTM(ETRS89)						
	$T_X(m)$	$T_Y(m)$	$T_Z(m)$	$\alpha('')$	$\beta('')$	$\gamma('')$	$d\lambda(\times 10^{-6})$
LS	582.901711	112.168080	405.603061	-2.255032	-0.335003	2.068369	9.117208
Gauss-Helmert	582.901711	112.168080	405.603061	-2.255032	-0.335003	2.068369	9.117208
CTLS	582.901711	112.168080	405.603061	-2.255032	-0.335003	2.068369	9.117208

Table 3.2: *Numerical deviation of 7-parameter Helmert transformation with 3 estimators*

Transformation model	Collocated sites	Absolute mean of Residuals (m)		Max.absolute mean of Residuals(m)		RMS (m)	Standard deviation of unit weight (m)
		$[V_N]$	$[V_E]$	$[V_N]$	$[V_E]$		
LS	B-W 131	0.1051	0.0843	0.4212	0.3112	0.1240	0.1026
Gauss-Helmert	B-W 131	0.0526	0.0421	0.2106	0.1556	0.0620	0.0513
CTLS	B-W 131	0.0526	0.0421	0.2106	0.1556	0.0620	0.0513

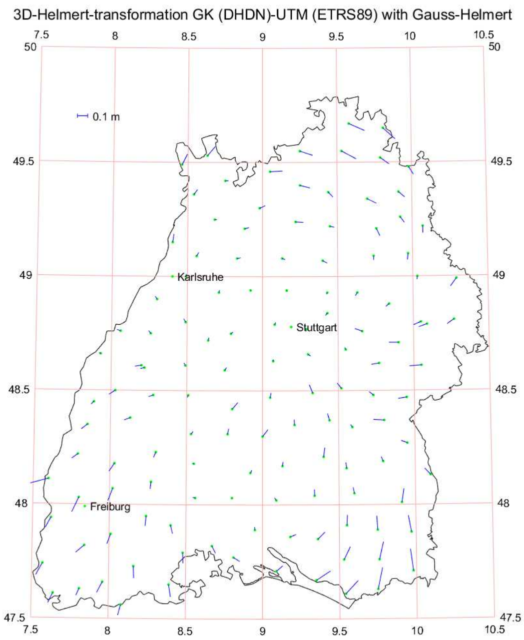

Figure 3.1: *Horizontal residuals after 7-parameter Helmert transformation in Baden-Württemberg network*

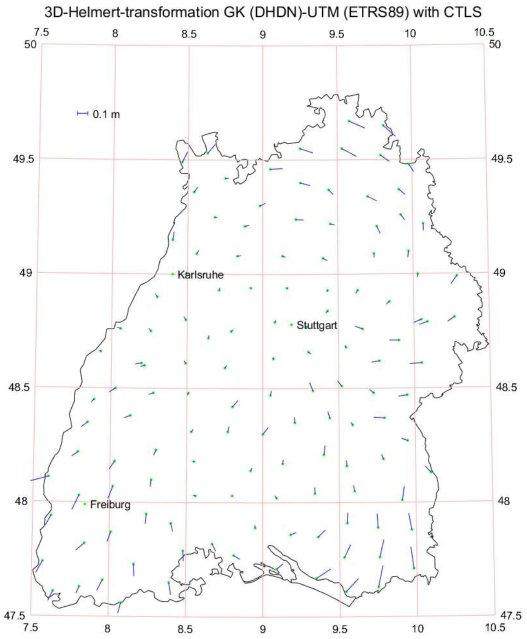

Figure 3.2: Horizontal residuals after 7-parameter Helmert transformation in Baden-Württemberg network

Conclusions: From the statistical data we can conclude that, if the least squares method estimates the unknown parameters of coordinate transformations, for example, the 3D-Helmert transformation, the results are independent of the transformation direction, as long as the coordinates of the start and target systems are treated as random variables with identical covariance matrix. The process can be done in the Gauss-Helmert model or after adding unknown parameters in the Gauss-Markoff model, the results are identical.
Compare with the LS method, the Gauss-Helmert model has a better accuracy with smaller residuals. It considers the random elements in design matrix and solves the theoretical weakness in LS method. Meanwhile, the Gauss-Helmert model has a identical consequence with CTLS. Study on the theory algorithm, we can find that, the original motivation of the development for both estimators were not identical. But these two estimators share a same processing method, which is to extract the unknow parameters and create a new observation equation. Although the later process for both estimators are not the same, they have the identical results in the case of coordiante transformation.

Chapter 4

Applications to the realizations of ITRF

4.1 Introduction

Earth observation is fundamental to addressing scientific challenges pertaining to the quantification of changes that are affecting the Earth system. Global geodesy is one of the key Earth science disciplines that not only measures changes of the Earth system in space and time but also is the only science that provides the indispensable standard against which the changes and their variability are quantified and properly referenced. In order to fundamental understanding the Earth dynamics, and also to precisely determine the orbits of the Earth-observing artificial satellites, it is critically important to ensure the continuous availability and updates of an accurate, long-term stable and truly global Terrestrial Reference Frame. The development of space geodetic techniques in the 1950s made it possible to establish an Earth reference frame. With the adoption of satellite navigation and positioning technology, the Earth reference frame is also becoming more continuous availability, more accuracy and reliable. The establishment and maintenance of a long-term stable high-precision earth reference frame can not only provide high-precision positioning and orientation benchmarks for the economy and national defense, but also be one of the foundations for conducting earth science research and many practical applications. For exsample the the International Terrestrial Reference Frame (ITRF). The recent resolution adopted on 26 February 2015 by the General Assem-bly of the United Nations on the Global Geodetic Reference Frame (GGRF) for Sustainable Development, recognizing the adoption of the ITRF by the scientific community, is a testimony of the critical importance of the reference frame for science and society.

The origin of ITRS (International Terrestrial Reference System) is defined as CM (Center of Mass of the Earth System), while that of ITRF approximates to CF (Center of surface Figure). Surface mass redistribution would induce relative motion of CF to CM, which is defined as geocenter motion here. With the improving accuracy of geodetic observations, geocenter motion has become one major error in realizing and maintaining ITRF and also monitoring sea level rising. Therefore it is important for us to deeply study geocenter motion, narrowing the inconsistency between the definition and realization of ITRF orgin.

The ITRS Center of the IERS, hosted by IGN France, is responsible for the maintenance of the ITRS/ITRF and official ITRF solutions. Two other ITRS combination centers are also generating combined solutions using ITRF input data: Deutsches Geodätisches Forschungsinstitut (DGFI) an der Technischen Universität München (TUM) [Seitz et al., 2012] and Jet Propulsion

Laboratory (JPL) [Wu et al., 2015].

4.2 Space Geodesy Solutions

4.2.1 The space geodetic techniques

The space geodetic techniques that contribute to the ITRF construction are Doppler orbitography and radiopositioning integrated by satellite (DORIS), Global Navigation Satellite Systems (GNSS), satellite laser ranging (SLR), and very long baseline interferometry (VLBI). These techniques are organized as scientific services within the International Association of Geodesy (IAG) and known by the International Earth Rotation and Reference Systems Service (IERS) as Technique Centers (TCs): the International DORIS Service (IDS) [Willis et al., 2010], the International GNSS Service, formerly the International GPS Service (IGS) [Dow et al., 2009], the International Laser Ranging Service (ILRS) [Pearlman et al., 2002], and the International VLBI Service (IVS) [Schuh and Behrend, 2012]. As none of the four space geodetic techniques is able to provide the full reference frame-defining parameters, the ITRF is demonstrated to be the most accurate reference frame available today, gathering the strengths of the four space geodesy techniques contributing to its construction and compensating for their weaknesses and systematic errors.

The IVS, ILRS, IGS, IDS provide the data for VLBI, SLR, GNSS, DORIS in SINEX file. Until 2015, the VLBI has 159 stations and started to provide data from 1980. The SLR has 142 stations and comprises 244 fortnightly solutions, started to provide data from 1983. The GNSS has 1810 stations and comprise 7714 dayly solutions, started to provide data from 1994. DORIS has 160 stations and comprises 1140 weekly solutions, started to provide data from 1993.

The effects that each technique contribute to the system can be summarized as:

DORIS:

- Improve SRP modelling to reduce draconitics
- Minimize the SAA effect

GNSS:

- Rearch near-field signal multipath and develop methods to calibrate in-situ position biases at all reference frame stations
- Investigate methods to mitigate pervasive draconitic signals
- Improve radiation force modeling, especally associated with attitude changes during eclipse
- Try harder to minimize equipment- and local-induced position offsets

SLR:

- Add estimation/handling of station Range Biases(RB)

- Use updated CoM offsets
- Add estimation/handling of Time Biases (TB)
- Include applied RB and TB in SINEX file for next contribution to ITRF with their constraint information

VLBI:

- Validate Nothnagel model for VLBI thermal effects
- Structural gravitational deformation: Update software to apply models to as many antennas as possible
- Relativity: Evaluate extra term (<1 ps) in trial basis in calc/solve, and check with formulation of Soffel et al., 2016
- Source Structure: Better/improved strategy

Table 4.1: *Summary of Submitted Solutions to ITRF (2015)*

Techniques	Data Span	Sampling	Solution Type	Constraints	EOPs
IVS	1980.0-2015.0	Daily	Normal equation	None	PM,PMr,LOD,UT1-UTC
ILRS	1983.0-1993.0	Fortnightly	Variance-covariance	Loose	PM,LOD
	1993.0-2015.0	Weekly	Variance-covariance	Loose	PM,LOD
IGS	1994.0-2015.1	Daily	Variance-covariance	Minimun	PM,PMr,LOD
IDS	1993.0-2015.0	Weekly	Variance-covariance	Minimun	PM

[a]PM: polar motion, PMr: polar motion rate, LOD: length of day.

During the ITRF2000 combination processing, 20 years of LLR, VBLI, SLR data and 10 years of GPS, DORIS data were submitted. The ITRF2005 data used GPS, SLR, DORIS weekly solution and VLBI session-wise solution (24 hours) as input data. ITRF2008 used VLBI, SLR, GPS and DORIS solution files for approximately 29 years, 26 years, 12, 5 years and 16 years, of which VLBI is a session-wise solution and the other three are weekly solutions. The four technique combined time series submitted to the ITRF2014 summarizes the data span, the sampling integration for station positions (daily for GNSS, session-wise for VLBI, weekly for DORIS, and fortnightly and weekly for SLR). For each ITRFn, the solution type (normal equations or variance-covariance), the constraints applied for the reference frame definition (free, loose, or minimum constraints), and the Earth Orientation Parameters (EOPs) provided in addition to station positions. Each per-technique time series is already a combination of the individual analysis center (AC) solutions of that technique.

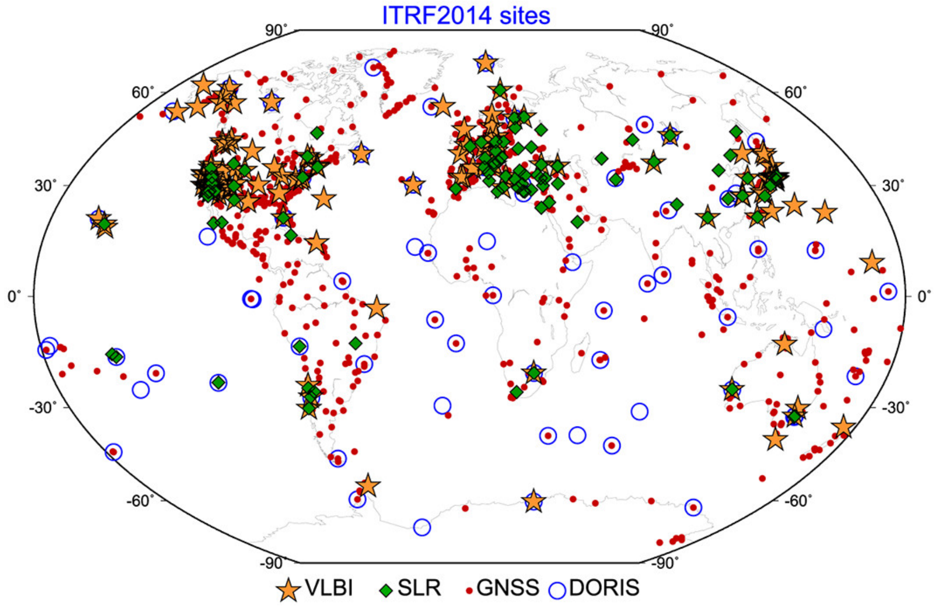

Figure 4.1: *ITRF2014 network highlighting VLBI,SLR,DORIS sites colocated with GNSS [Altamimi, 2016]*

4.2.2 Combination Model

Different geodetic techniques have different sensitivety to the Earth reference frame benchmarks and are not able to provide the unified data to establish the Earth Reference Framework. IGN is the official organization of IERS, it is responsible for ITRF's calculation and release. IGN uses the software CATREF to calculate the parameters.
Combination model is the core calculation method in CATREF. This model takes the position and velocity of each station into consideration. For each point i ,X_s^i (at epoch t_s^i) and $\dot{X}_s^i$ are position and velocity of technique solution s and X_c^i (at epoch t_0) and $\dot{X}_c^i$ are those of the combined solution c. For each individual frame k, as implicitly defined by solution s, D_k is the scale factor, T_k is the translation vector, and R_k is the rotation matrix. The dotted parameteres designate their derivatives with respect to time. The translation vector T_k is composed of three origin components, namely, T_x, T_y, T_z, and the rotation matrix of three small rotation parameters, R_x, R_y, R_z, following the three axes, respectively, X, Y and Z. t_k is a conventionally selected epoch of the seven transformation parameters.

$$
\begin{aligned}
\boldsymbol{X}_s^i &= \boldsymbol{X}_c^i + \left(t_s^i - t_0\right) \dot{\boldsymbol{X}}_c^i \\
&\quad + \boldsymbol{T}_k + \boldsymbol{D}_k \boldsymbol{X}_c^i + \boldsymbol{R}_k \boldsymbol{X}_c^i \\
&\quad + \left(t_s^i - t_0\right) \left[\dot{\boldsymbol{T}}_k + \dot{\boldsymbol{D}}_k \boldsymbol{X}_c^i + \dot{\boldsymbol{R}}_k \boldsymbol{X}_c^i\right] \\
\dot{\boldsymbol{X}}_s^i &= \dot{\boldsymbol{X}}_c^i + \dot{\boldsymbol{T}}_k + \dot{\boldsymbol{D}}_k \boldsymbol{X}_c^i + \dot{\boldsymbol{R}}_k \boldsymbol{X}_c^i
\end{aligned}
\tag{4.1}
$$

To be simplify, $\boldsymbol{X}$ stands for the position $\boldsymbol{X}_s^i$ and the velocity $\dot{\boldsymbol{X}}_s^i$, the 7 transformation parameters and corresponding rate is $\boldsymbol{T}_k$. The above equation can be written as:

$$\begin{pmatrix} \boldsymbol{A}_{1s}^T \\ \boldsymbol{A}_{2s}^T \end{pmatrix} \boldsymbol{P}_s \begin{pmatrix} \boldsymbol{A}_{1s} & \boldsymbol{A}_{2s} \end{pmatrix} \begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{T}_k \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_{1s}^T \boldsymbol{P}_s \boldsymbol{B}_s \\ \boldsymbol{A}_{2s}^T \boldsymbol{P}_s \boldsymbol{B}_s \end{pmatrix} \tag{4.2}$$

Where $\boldsymbol{A}_{1s}$ and $\boldsymbol{A}_{2s}$ are the design matrixs for each station.

$$\boldsymbol{A}_{1s}^i = \begin{bmatrix} \boldsymbol{I} & dt_s^i \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}, \quad \boldsymbol{A2}_s^i = \begin{bmatrix} \boldsymbol{I} & dt_k^i \boldsymbol{A}_s^i \\ \boldsymbol{0} & \boldsymbol{A}_s^i \end{bmatrix} \tag{4.3}$$

Here $dt_s^i = t_s^i - t_0$, $dt_k^i = t_s^i - t_k$, $\boldsymbol{P}_s$ is the weight matrix, $\boldsymbol{B}_s$ is the constant, means the difference between observed and calculated values, inside $\boldsymbol{A}_s^i$ are the approximate coordinates of the station. When it only take the coordiantes into consideration, the $\boldsymbol{A}_s^i$ reads

$$\boldsymbol{A}_s^i = \begin{bmatrix} . & . & . & . & . & . & . \\ 1 & 0 & 0 & x_0^i & 0 & z_0^i & -y_0^i \\ 0 & 1 & 0 & y_0^i & -z_0^i & 0 & x_0^i \\ 0 & 0 & 1 & z_0^i & y_0^i & -x_0^i & 0 \\ . & . & . & . & . & . & . \end{bmatrix} \tag{4.4}$$

When it considers the coordinates and velocity together, the A_s^i reads

$$\boldsymbol{A}_s^i = \begin{bmatrix} . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 1 & 0 & 0 & x_0^i & 0 & z_0^i & -y_0^i & & & & & & & \\ 0 & 1 & 0 & y_0^i & -z_0^i & 0 & x_0^i & & & & \approx & & & \\ 0 & 0 & 1 & z_0^i & y_0^i & -x_0^i & 0 & & & & & & & \\ & & & & & & & 1 & 0 & 0 & x_0^i & 0 & z_0^i & -y_0^i \\ & & & \approx & & & & 0 & 1 & 0 & y_0^i & -z_0^i & 0 & x_0^i \\ & & & & & & & 0 & 0 & 1 & z_0^i & y_0^i & -x_0^i & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \end{bmatrix} \tag{4.5}$$

When the calculation considers the EOPs parameters, it also need to add the following equation.

$$\begin{aligned} \boldsymbol{x}_s^p &= \boldsymbol{x}_c^p + \boldsymbol{R}_{2k} \\ \boldsymbol{y}_s^p &= \boldsymbol{y}_c^p + \boldsymbol{R}_{1k} \\ \boldsymbol{UT}_s &= \boldsymbol{UT}_c - \frac{\boldsymbol{1}}{\boldsymbol{f}} \boldsymbol{R}_{3k} \\ \dot{\boldsymbol{x}}_s^p &= \dot{\boldsymbol{x}}_c^p + \dot{\boldsymbol{R}}_{2k} \\ \dot{\boldsymbol{y}}_s^p &= \dot{\boldsymbol{x}}_c^p + \dot{\boldsymbol{R}}_{1k} \\ \boldsymbol{LOD}_s &= \boldsymbol{LOD}_c \end{aligned} \tag{4.6}$$

Here for each solution s the pole coordinates $\boldsymbol{x}_s^p$ and $\boldsymbol{y}_s^p$, universal time $\boldsymbol{UT}_s$ as well as their daily rates $\dot{\boldsymbol{x}}_s^p$, $\dot{\boldsymbol{y}}_s^p$ and $\boldsymbol{LOD}_s$ are used in the equation. Where $f = 1.002737909350795$ is the conversion factor from $\boldsymbol{UT}$ into sidereal time. The link between the combined frame and the $\boldsymbol{EOPs}$ is ensured via the three rotation parameters appearing in the first three lines of (4.6).
In the first step of the ITRF construction, the first two lines of (4.1) and the entire equation

(4.6) are used to estimate long-term solutions for each technique, by accumulating (rigorously stacking) the indivdual technique time series of stations and ***EOPs***. In the second step, the entire two equations are used to combine the long-term solution obtained in step 1, together with local ties in colocation sites.

This model (IGN) is used in Intra-technique combination. The main deviation steps are
1) With the speed of the coordiantes by colocated sites, transform the coodinates in the combinated reference frame under epoch t_0 to epoch t_k.

$$\boldsymbol{X}_c^i\left(t_k\right)=\boldsymbol{X}_c^i\left(t_0\right)+\left(t_k-t_0\right)\dot{\boldsymbol{X}}_c^i \tag{4.7}$$

2) Transform the coordinates $\boldsymbol{X}_c^i$ in combinated reference frame to the corresponding coordinates in reference frame TRFk with Helmert-transformation.

$$\boldsymbol{X}_s^i\left(t_k\right)=\boldsymbol{X}_c^i\left(t_k\right)+\boldsymbol{T}_k+\boldsymbol{D}_k\boldsymbol{X}_c^i\left(t_k\right)+\boldsymbol{R}_k\boldsymbol{X}_c^i\left(t_k\right) \tag{4.8}$$

3) With the speed of the coordiantes by colocated sites, transform the coodinates in the TRFk reference frame under epoch t_k to epoch t_s.

$$\boldsymbol{X}_s^i\left(t_s^i\right)=\boldsymbol{X}_s^i\left(t_k\right)+\left(t_s^i-t_k\right)\dot{\boldsymbol{X}}_s^i \tag{4.9}$$

Combinate (4.7), (4.8) and (4.9)

$$\boldsymbol{X}_s^i\left(t_s^i\right)=\left[\boldsymbol{X}_c^i\left(t_k\right)+\left(t_k^i-t_0\right)\dot{\boldsymbol{X}}_c^i\right]+\boldsymbol{T}_k+\boldsymbol{D}_k\boldsymbol{X}_c^i\left(t_k\right)+\boldsymbol{R}_k\boldsymbol{X}_c^i\left(t_k\right)+\left(t_s^i-t_k\right)\dot{\boldsymbol{X}}_s^i \tag{4.10}$$

Make $\boldsymbol{\Delta X}=\boldsymbol{T}_k+\boldsymbol{D}_k\boldsymbol{X}_c^i\left(t_k\right)+\boldsymbol{R}_k\boldsymbol{X}_c^i\left(t_k\right)$

$$\boldsymbol{X}_s^i\left(t_s^i\right)=\left[\boldsymbol{X}_c^i\left(t_k\right)+\left(t_k^i-t_0\right)\dot{\boldsymbol{X}}_c^i\right]+\boldsymbol{\Delta X}+\left(t_s^i-t_k\right)\dot{\boldsymbol{X}}_s^i \tag{4.11}$$

with

$$\begin{aligned}\boldsymbol{\Delta X}&=\boldsymbol{T}_k+\boldsymbol{D}_k\left[\boldsymbol{X}_c^i\left(t_k\right)+\left(t_k^i-t_0\right)\dot{\boldsymbol{X}}_c^i\right]+\boldsymbol{R}_k\left[\boldsymbol{X}_c^i\left(t_k\right)+\left(t_k^i-t_0\right)\dot{\boldsymbol{X}}_c^i\right]\\&\cong\boldsymbol{T}_k+\boldsymbol{D}_k\boldsymbol{X}_c^i\left(t_0\right)+\boldsymbol{R}_k\boldsymbol{X}_c^i\left(t_0\right)\end{aligned} \tag{4.12}$$

Here $\boldsymbol{\Delta X}$ is the difference, that frame under epoch t_0 with Helmert-transformation, the $\boldsymbol{D}_k\boldsymbol{X}_c^i\left(t_k\right)$ and $\boldsymbol{R}_k\boldsymbol{X}_c^i\left(t_k\right)$ can be ignored. Because $\boldsymbol{D},\boldsymbol{R}$ are with the magnitude 10^{-5}.

$$\boldsymbol{\Delta\dot{X}}=\dot{\boldsymbol{T}}_k+\dot{\boldsymbol{D}}_k\boldsymbol{X}_c^i\left(t_0\right)+\dot{\boldsymbol{R}}_k\boldsymbol{X}_c^i\left(t_0\right)=\dot{\boldsymbol{X}}_s^i-\dot{\boldsymbol{X}}_c^i \tag{4.13}$$

We can get

$$\dot{\boldsymbol{X}}_s^i=\dot{\boldsymbol{X}}_c^i+\dot{\boldsymbol{T}}_k+\dot{\boldsymbol{D}}_k\boldsymbol{X}_c^i\left(t_0\right)+\dot{\boldsymbol{R}}_k\boldsymbol{X}_c^i\left(t_0\right) \tag{4.14}$$

The equation (4.11) can be writen as

$$
\begin{aligned}
\boldsymbol{X}_s^i\left(t_s^i\right) &= \boldsymbol{X}_c^i\left(t_k\right)+\left(t_k^i-t_0\right)\dot{\boldsymbol{X}}_c^i+\boldsymbol{T}_k+\boldsymbol{D}_k\boldsymbol{X}_c^i\left(t_0\right)+\boldsymbol{R}_k\boldsymbol{X}_c^i\left(t_0\right)\\
&+\left(t_s^i-t_k\right)\left[\dot{\boldsymbol{X}}_c^i+\dot{\boldsymbol{T}}_k+\dot{\boldsymbol{D}}_k\boldsymbol{X}_c^i\left(t_0\right)+\dot{\boldsymbol{R}}_k\boldsymbol{X}_c^i\left(t_0\right)\right]\\
&= \boldsymbol{X}_c^i\left(t_k\right)+\left(t_k^i-t_0\right)\dot{\boldsymbol{X}}_c^i\left(t_0\right)\\
&+\boldsymbol{T}_k+\boldsymbol{D}_k\boldsymbol{X}_c^i\left(t_0\right)+\boldsymbol{R}_k\boldsymbol{X}_c^i\left(t_0\right)\\
&+\left(t_s^i-t_k\right)\left[\dot{\boldsymbol{T}}_k+\dot{\boldsymbol{D}}_k\boldsymbol{X}_c^i\left(t_0\right)+\dot{\boldsymbol{R}}_k\boldsymbol{X}_c^i\left(t_0\right)\right]
\end{aligned} \tag{4.15}
$$

4.2.3 Intra-technique Combination

The procedure adopted for the ITRF formation involves two steps [Altamimi et al., 2002a, 2007, 2011]: 1)stacking the individual time series to estimate a long-term solution per technique comprising station positions at a reference epoch, station velocities, and daily EOPs. 2) combining the resulting long-term solutions of the four techniques together with the local ties and colocated sites.
The establishment of the reference frame is based on the Intra-technique combination and Inter-technique combination within the space geodetic techniques. Each technique data service center uses Intra-technique combination to calculate the unified frame solutions and provide to the next service center, which will use Inter-technique combination with all the 4 techniques and calculate the ITRF solutions. The illustrate below show the processing for ITRF solutions.

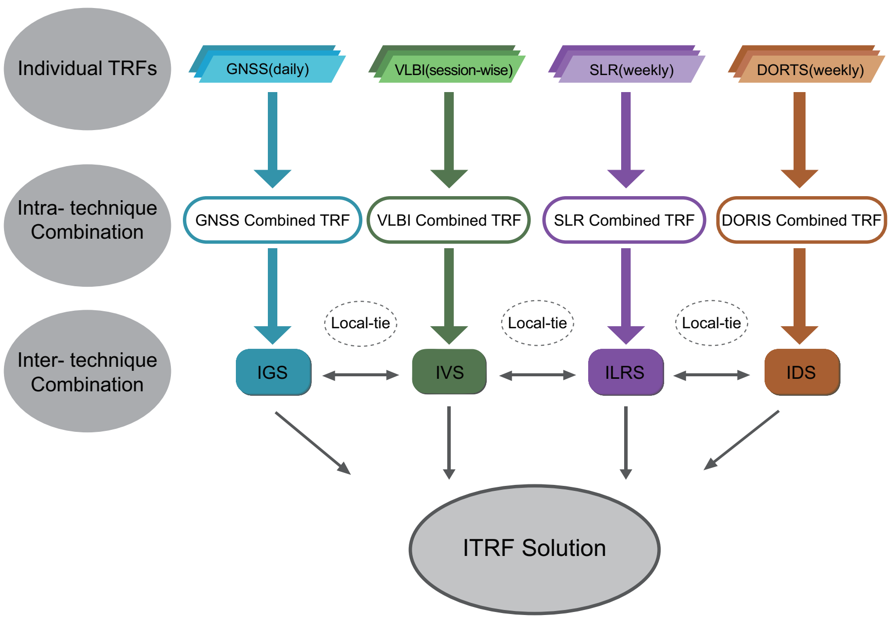

Figure 4.2: *Illustrate of the main calculation steps for ITRF*

From the illustrate we can see that, the Intra-technique combination is the basis for Inter-technique combination. In Intra-technique combination, the mathematical model is Helmert transformation. Transformation between different coordiante systems are made by three translations, three rotations and one scale. If the velocity is considered, the nummber of transformation parameter will be 14. The function model is

$$\boldsymbol{X}_2 = \boldsymbol{X}_1 + \boldsymbol{A}_s\boldsymbol{\theta} \tag{4.16}$$

With $\theta = (\boldsymbol{T_1, T_2, T_3, R_1, R_2, R_3, D, \dot{T}_1, \dot{T}_2, \dot{T}_3, \dot{R}_1, \dot{R}_2, \dot{R}_3, \dot{D}})^{\boldsymbol{T}}$,$\boldsymbol{A}_s$ is the design matrix see (4.5).

The normal equation is

$$\boldsymbol{\theta} = \left(\boldsymbol{A}_s^T\boldsymbol{P}_x\boldsymbol{A}_s\right)^{-1}\boldsymbol{A}_s^T\boldsymbol{P}_x\left(\boldsymbol{X}_2 - \boldsymbol{X}_1\right) \tag{4.17}$$

During the estimation for the transformation parameters, it's important to select the weight matrix $\boldsymbol{P}_x$. There are three main solutions. One is to select unit weight matrix, that is $\boldsymbol{P}_x = I$. Another is the inverse matrix of the matrix consisting of the values on the diagonal of the variance covariance matrix associated with $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$. The third is the inverse matrix of fully covariance matrix. Normally, these three weight matrix can have different values for the transformation parameters. Which means the 7 transformation parameters (no velocity) have a certain correlation with each other.

In the Intra-technique combination, the solutions are usually in a short time, which normally are daily solution or weekly solution. The velocity for the sites with the same epoch can be ignored. Hence the mathematical model is

$$\begin{bmatrix} x_s^i \\ y_s^i \\ z_s^i \end{bmatrix} = \begin{bmatrix} x^i \\ y^i \\ z^i \end{bmatrix} + \boldsymbol{T}_k + \boldsymbol{D}_k \begin{bmatrix} x^i \\ y^i \\ z^i \end{bmatrix} + \boldsymbol{R}_k \begin{bmatrix} x^i \\ y^i \\ z^i \end{bmatrix} \tag{4.18}$$

In matrix form is

$$\begin{bmatrix} x_s^i \\ y_s^i \\ z_s^i \end{bmatrix} = \boldsymbol{A}_1 \begin{bmatrix} x^i \\ y^i \\ z^i \end{bmatrix} + \boldsymbol{A}_2\theta \tag{4.19}$$

With $\boldsymbol{A1}$ is the unit matrix and $\boldsymbol{A2} = \begin{bmatrix} . & . & . & . & . & . & . \\ 1 & 0 & 0 & x_0^i & 0 & z_0^i & -y_0^i \\ 0 & 1 & 0 & y_0^i & -z_0^i & 0 & x_0^i \\ 0 & 0 & 1 & z_0^i & y_0^i & -x_0^i & 0 \\ . & . & . & . & . & . & . \end{bmatrix}$

For each subsolution i, the above equation can be written as

$$\boldsymbol{L}_i = \boldsymbol{A}_{1i}\boldsymbol{X} + \boldsymbol{A}_{2i}\boldsymbol{\theta}_i \tag{4.20}$$

The relationship between the k single subsolutions and the combination solutions is

$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{bmatrix} = \begin{bmatrix} A1_1 & A2_1 & 0 & 0 & 0 \\ A1_2 & 0 & A2_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A1_k & 0 & 0 & \cdots & A2_k \end{bmatrix} \begin{bmatrix} X \\ \theta_1 \\ \theta_2 \\ \vdots \\ \theta_k \end{bmatrix} \tag{4.21}$$

The observation is

$$\boldsymbol{L} = \boldsymbol{A} \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{\theta} \end{bmatrix} \tag{4.22}$$

Covariance matrix is

$$\boldsymbol{D} = \begin{bmatrix} \sigma_1^2 \varrho_1 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 \varrho_2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_k^2 \varrho_k \end{bmatrix} \tag{4.23}$$

The normal equation for the combination solution is

$$Nx = b$$

with

$$\boldsymbol{N} = \begin{bmatrix} \sum\limits_{s \in S} A_{1s}^T P_s A_{1s} & \sum\limits_{s \in S_1} A_{1s}^T P_s A_{1s} & \cdots & \cdots & \sum\limits_{s \in S_k} A_{1s}^T P_s A_{1s} \\ \sum\limits_{s \in S_1} A_{1s}^T P_s A_{1s} & \sum\limits_{s \in S_1} A_{1s}^T P_s A_{1s} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \sum\limits_{s \in S_k} A_{1s}^T P_s A_{1s} & 0 & & 0 & \sum\limits_{s \in S_k} A_{1s}^T P_s A_{1s} \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} \sum\limits_{s \in S} A_{1s}^T P_s B_s \\ \sum\limits_{s \in S_1} A_{1s}^T P_s B_s \\ \vdots \\ \vdots \\ \sum\limits_{s \in S_k} A_{1s}^T P_s B_s \end{bmatrix} \tag{4.24}$$

Here $\boldsymbol{S} = (S_1, S_2, \cdots, S_k)$.

The calculation and process of Intra-technique combination for GNSS, SLR, DORIS and VLBI are almost identical. Nur the SINEX file by VLBI provides directly no-constraint normal equation. There is no need to do the inverse calculation. The main process is
1) Read the SINEX file by 4 techniques and eliminate the outlier.

2) Calculate the inverse matrix of covariance matrix, remove a priori constraint and reform the normal matrix

3) Input Helmert parameters

4) Creat the combinated normal matrix

5) Estimation and get the solutions

By the step of estimation, there are many kinds of methods with constraint. Details will not be explained in this article.

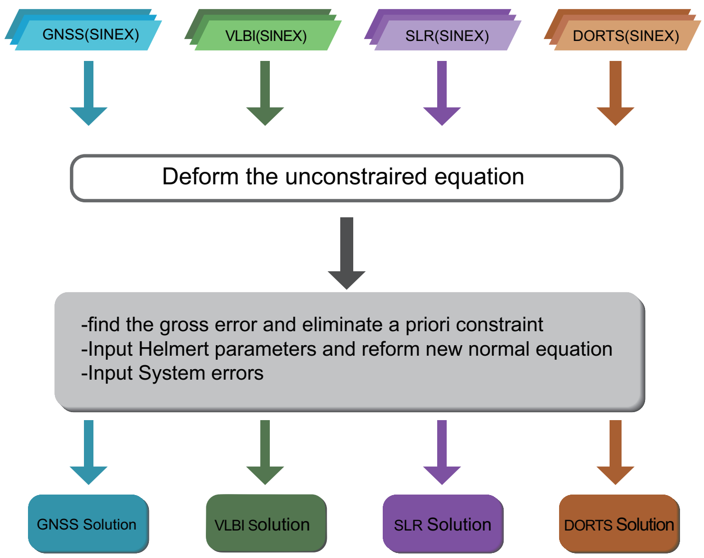

Figure 4.3: *Illustrate of the main steps for Intra-technique Combination*

4.2.4 Local Ties in Colocation Sites

Before we talk about Inter-technique combination, we need to understand the concept about the colocation sites in ITRF and the local ties between them.

When we need to combinate more than two techniques to esimate the parameters. It is nessesary to have sufficient quantity and even distribution colocation sites together with the Local-tie among them in oder to contribute to the ITRF solutions. Clocation sites are the observation stations that have two or more geodetic techniques. Otherwise, the observation stations are close to each other. The distance between those stations can be mesured by local survey or by GPS technique. These distances are named local ties. [Altamimi, 2002] The local survey usually provides the direction angles, distances and spirit leveling at the measure place towards the station. The national agencies operating ITRF colocated site provides the difference of coordiantes of each technique instruments at the reference points, which is the local ties.

The distance between two observation stations at one colocation site is normally on a hundred meters level. The biggest distance between the stations can be setted under requirment. Considerd the current colocation sites, the distance between two observation stations can upto $30km$. Since the local measurement is so important to the ITFR solution, the local ties is required to be long-term stable. Hence, the establishment of the tracking nets and repeatment of the local measurments is very important.

As far as the development of local ties, the accuracy is on 1-3mm level. Higher accuray requierment is the goal for the development by geodetic techniques. The quality of the local ties is the main factor that affects the accuracy of ITRF solution. Hence, its nessesary to test and analyse the colocation sites and local ties, if they are accuracy enough as the constraint condition to the Inter-technique combination. Before the calculation for the ITRF solution, we need to select the colocation sites.

Take ITRF2014 as an exsample. The table 4.2 shows the residuals between the 139 local ties's values by real measurement and the values by calculation with ITRF2014 solution. As we can see that, 59% of the local ties are under $5mm$, which is identical to geodetic solution. 41% of the local ties are bigger than $5mm$, and 21% of which are more than $10mm$. These residuals are caused by many reasons: the system errors in the geodetic technique, measurement errors, the movement of the measure equipments at colocation sites and so on.

Table 4.2: *Residuals of local ties on the Colocation sites with ITRF2014*

Residuals	GPS-SLR	GPS-VLBI	GPS-DORIS	VLBI-SLR	VLBI-DORIS	SLR-DORIS
under 5mm	34	52	59	11	16	10
5mm-10mm	4	10	30	1	5	10
over 10mm	16	4	29	3	8	5
Total sites	54	66	118	15	29	25

From the ITRF88, the ITRF solution had already make use of the colocated sites and local ties. At that time only VLBI and SLR were used. Since ITRF92, the GPS technique was brought into calculation. After that in ITRF94, DORIS was also included. The colocated sites are from 22 in ITRF94 to 139 in ITRF2014. In ITRF2014, in addition to the local ties udes in the ITRF2008 computation, a certain number of local ties used are new, resulting either from new colocated sites or from new survey. 36 new surveys were conducted since the release of ITRF2008.

4.2.5 Inter-technique Combination

Different geodetic techniques have their advantages. Combined processing can make full use of the advantages of various technologies to make up for some of the shortcomings. For exsample, the SLR is recognized as a technology that accurately estimates geocentric motion, but it also has serious shortcomings, like the SLR will be huge influenced by the weather. Ihe tracking stations of SLR are unevenly distributed on the earth, especially on the southern earth. When we combinate the SLR and GNSS data together, the accuracy of the time resolution and the geometry distribution of the colocated sites can be improved.

The Inter-technique combination is to combine the normal matrix of GNSS, SLR, DORIS, VLBI. The combined normal matrix is similar to the normal matrix in Intra-technique combination.

$$\boldsymbol{N}_c = \begin{bmatrix} \sum\limits_{s\in S} A_{1s}^T P_s A_{1s} & \sum\limits_{s\in S_1} A_{1s}^T P_s A_{1s} & \cdots & \cdots & \sum\limits_{s\in S_4} A_{1s}^T P_s A_{1s} \\ \sum\limits_{s\in S_1} A_{1s}^T P_s A_{1s} & \sum\limits_{s\in S_1} A_{1s}^T P_s A_{1s} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \sum\limits_{s\in S_4} A_{1s}^T P_s A_{1s} & 0 & & 0 & \sum\limits_{s\in S_4} A_{1s}^T P_s A_{1s} \end{bmatrix}, \quad \boldsymbol{b}_c = \begin{bmatrix} \sum\limits_{s\in S} A_{1s}^T P_s B_s \\ \sum\limits_{s\in S_1} A_{2s}^T P_s B_s \\ \vdots \\ \vdots \\ \sum\limits_{s\in S_4} A_{2s}^T P_s B_s \end{bmatrix} \tag{4.25}$$

Here $\boldsymbol{S} = (S_1, S_2, \cdots, S_c)$, S_1 is GNSS, S_2 is SLR, S_3 is DORIS, S_4 is VLBI.

Different geodetic techniques have different effects for the calculation with the benchmark. The realization and maintain of the earth reference frame require planty numbers of points with precise coodinates. These coordinates contain the information about the benchmark of earth reference frame, including the origin geocenter, scale, orientation and the change rate with time series.

The ideal origin Geocenter Motion for the reference frame is the CM, which is the mass center of solid earth together with atmospheric, ocean. Due to the dynamic principle, the center of satellit mission should be CM. GNSS, SLR and DORIS are satellit dynamic techniques, their motions are sensitive to CM. In fact, the system errors by GNSS and DORIS make the center not match the CM. Hence, the SLR technique can have a good accuracy for the determine of CM. When in the estimation processing, the translations by SLR will not be contrainted and the accuracy for origin can be improved. For GNSS, VLBI,DORIS under the same epoch, the translations between the combined solution and ITRFn are 0. By the scale, VLBI has a advantage over other three techniques. When in the estimation processing, the scale by VLBI will not be contrainted. Under the same epoch, the scale of the other three techniques between combined solution and ITRFn are 0. These whole four techniques are not sensitive to orientations. Under the same epoch, the orientation of all techniques between the combined solution and ITRFn are 0. These are all constaint conditions during Inter-technique combination processing.

In order to improve the reliability of the combination of these four techniques, it's important to define the weight for each technique. One solution is, combine the Helmert method variance component estimation with the empirical weighting method.
This step consists of combining the derived four long-term solutions with local ties at colocated sites, which including station positions, velocities and EOPs. A certain number of test combinations were performed, by varying the weighting of the four technique solutions as well as the local ties. It is very difficult to adequately use a mathematically or statistically prescribed method of variance component estimation as the degree of freedom or Helmert method. The main reason is that we have observations and constraints at colocation sites of different types: global space geodesy solutions, local ties, and velocity equality. In addition, there are significant tie and velocity discrepancies between technique solutions at a number of colocation sites that necessitate an iterative combination and empirical weighting process.

For the normal matrix of GNSS, SLR, DORIS and VLBI, take the prior weight factor inside

$$\sum_{i=1}^{4} \frac{1}{\sigma_{0_i}^2} N_i x_i = \sum_{i=1}^{4} \frac{1}{\sigma_{0_i}^2} b_i \tag{4.26}$$

Here i represents DNSS, SLR, DORIS and VLBI. $\frac{1}{\sigma_{0_i}^2}$ is the prior weight factor. The variance factor is

$$\sigma_i^2 = \frac{\boldsymbol{V}_i^T \boldsymbol{P}_i \boldsymbol{V}_i}{f_i} \tag{4.27}$$

Take the variance factor into (4.27). Do the iteration until the ratio of unit weight variance factors between techniques is close to 1. [Yao., 2018]

The Inter-technique combination is based on the local tie at the colocated sites. The input data is the normal matrix after the processing by Intra-technique combination. The main steps are

1) Read the SINEX file by each technique and restore the normal matix.

2) Input the Helmert parameters.

3) Add the information of colocated sites and local tie as constraint condition.

4) Establish the nenchmark of the reference frame.

5) estimate the combinated normal matrix and calculate the unknow parameters

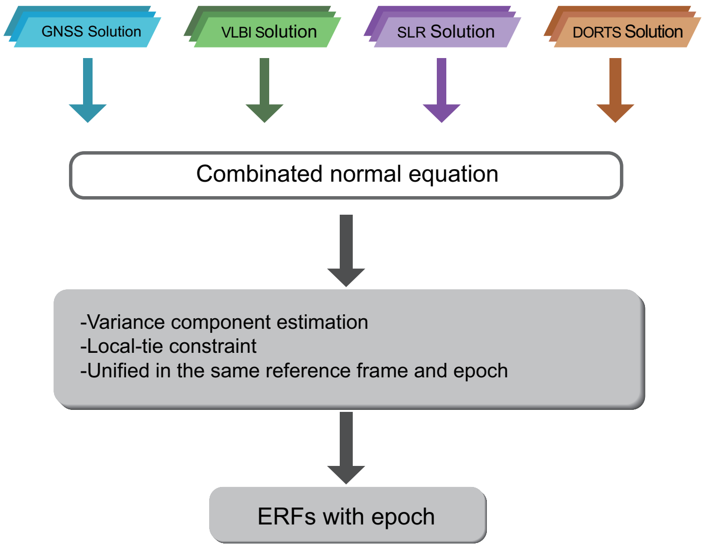

Figure 4.4: *Illustrate of the main steps for Inter-technique Combination*

4.3 Postseismic Deformation

After the release of ITRF2008 in 2010, it became more and more obvious that stations impacted by major earthquakes, and in particular the devastating ones in Sumatra (2004), Chile (2010), and Japan (2011), have nonlinear trajectories after these tragic events. Modeling the postseismic deformation (PSD) by piecewise lin-ear functions as in the past ITRF versions is no longer an appropriate approach, at least because the estimated linear velocities of the segmented station time series are imprecise and do not adequately describe the real station postseismic trajectories. [Altamimi ., 2016]

Modeling the PSD for ITRF2014 sites could of course be done using possible available models con-structed for each major earthquake individually [Freed et al., 2006; Pollitz, 1997, 2014; Trubienko et al., 2013]. However, not all earthquakes impacting the ITRF2014 sites have corresponding published models nor would it be manageable for us to evaluate and test the performance of available models against ITRF2014 input data.[Altamimi ., 2016]

For the ITRF2014 and in order to account for the PSDs of stations subject to major earthquakes, we adopted a more pragmatic approach by fitting parametric models to the ITRF2014 input time series of station positions. The four retained parametric models are (1) (Log)arithmic, (2) (Exp)onential, (3) Log+Exp, and (4) Exp+Exp. It is known that the PSDs have different structures, such as ’transient after slip creep’ behavior [Marone et al., 1991; Perfettini and Avouac, 2004; Savage et al., 2005] tending to follow a logarithmic function or of ’viscoelastic relaxation’ type [Savage and Prescott, 1978; Pollitz, 1997] that is better described by an exponential decay. Logarithmic models were used by Bevis and Brown [2014] to describe the trajectory of earthquake sites using GPS time series, while Freed et al. [2010] used a combination of a logarithmic variation and an exponential decay.[Altamimi ., 2016]

We used the IGS GNSS contributed daily time series to fit parametric models for stations where PSD was judged visually significant, including a few stations impacted by major earthquakes that occurred prior to the start of their observations. The PSD models were fitted separately in each east, north, and up component, simul-taneously with piecewise linear functions, annual, and semiannual signals. In case of a series with a unique earthquake causing PSD, 10 different models were first tried: None (0), Log (1), Exp (2), Log+Exp (3), and Exp+Exp (4), each combined with either a position-only or a position+velocity coseismic discontinuity. Among the tested models, those for which the relaxation time of at least one logarithmic or exponential function did not converge were discarded, as well as those leading to at least one insignificant estimated parameter (i.e., smaller than its formal error). Among the remaining models, we finally selected the model with the low-est Bayes Information Criterion [Schwarz, 1978], [Kass and Raftery, 1995]. For series with n > 1 earthquakes causing PSD, all possible 10n model combinations were similarly tried and the best model combination was selected based on the same criteria.[Altamimi ., 2016]

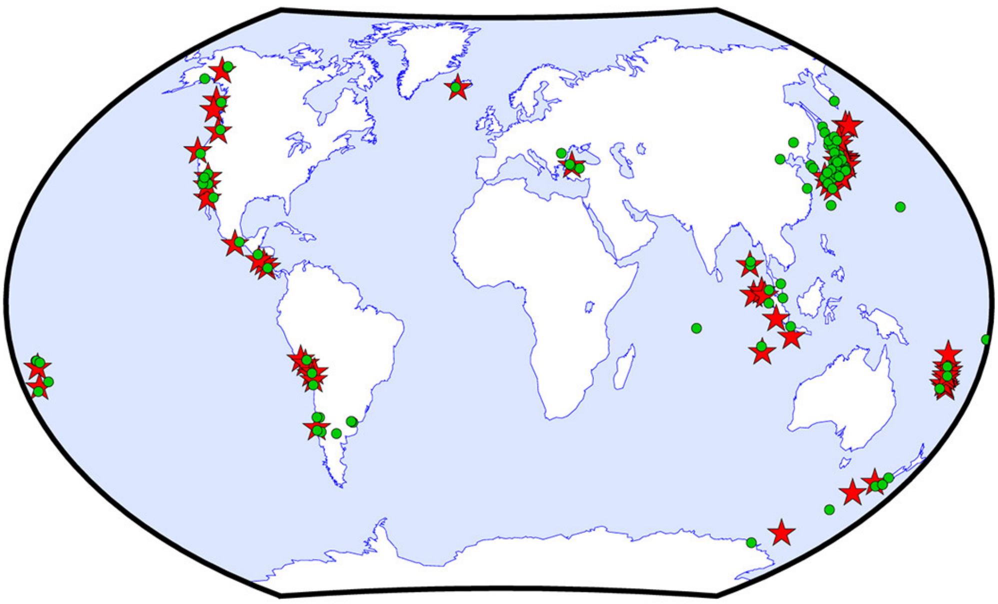

Figure 4.5: *Distribution of earthquake epicenters (red) and ITRF2014 sites (green) impacted by postseismic deformation [Altamimi, 2016]*

Figure 4.5 illustrates in red the location of 59 earthquake epicenters that caused significant PSD at ITRF2014 sites and in green the impacted 123 stations located at 117 sites. We then applied the corrections predicted by the GNSS fitted models to the nearby stations of the three other techniques at earthquake colocation sites, before stacking their respective time series. In order to illustrate the performance of the PSD parametric models, Figure 5 displays the position time series of GNSS/GPS and the colocated VLBI stations at Tsukuba (Japan): in blue the raw data, in green the piecewise linear trajectories given by the ITRF2014 coordinates, and in red the trajectories obtained when adding the parametric PSD model. In that figure, one can see the remarkable fit of the PSD model, not only to the GNSS but also to the VLBI data.[Altamimi ., 2016]

While the ITRF2014 solution provides the usual/classical estimates: station positions at epoch 2010.0, station velocities, and EOPs, the PSD models are also part of the ITRF2014 products.[Altamimi ., 2016]

4.4 Tranformation Parameters between ITRFs with LS and CTLS

4.4.1 Transformation model

In order to ensure the link between ITRFs solutions, for many geodetic applications, it is essential to provide the 14 transformation parameters with respect to the past ITRFs.

This section will provide 2 applications with transformtions between ITRF2005-ITRF2008 and ITRF2000-ITRF2005, with their cooresponding epochs. The published data are estimated with LS method. This section will also use CTLS method to estimate the 14 parameters transformation.

The transformation formula for ITRF2014-ITRF2008 is

$$\begin{cases} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{i05} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{i08} + \boldsymbol{T} + \boldsymbol{D} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{i08} + \boldsymbol{R} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{i08} \\ \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}_{i05} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}_{i08} + \dot{\boldsymbol{T}} + \dot{\boldsymbol{D}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{i08} + \dot{\boldsymbol{R}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{i08} \end{cases} \tag{4.28}$$

where $i05$ designates ITRF2005 and $i08$ is ITRF2008, $\boldsymbol{T}$ is the translation vector, $\boldsymbol{T} = [T_x, T_y, T_z]^T$, $\boldsymbol{D}$ is the scale factor, and $\boldsymbol{R}$ is the matrix containing the rotation angles, given by

$$\boldsymbol{R} = \begin{bmatrix} 0 & -R_z & R_y \\ R_z & 0 & -R_x \\ -R_y & R_x & 0 \end{bmatrix} \tag{4.29}$$

The dotted parameters designate their time derivatives. Note that the inverse transformation from ITRF2005 to ITRF2008 follows by interchanging ($i08$) with ($i05$) and changing the sign of the transformation parameters.

4.4.2 Main steps

1) Input data file by ITRF2005 and ITRF2008 including coordinates, velocities and their corresponding formal errors.

2) Select the reference points.

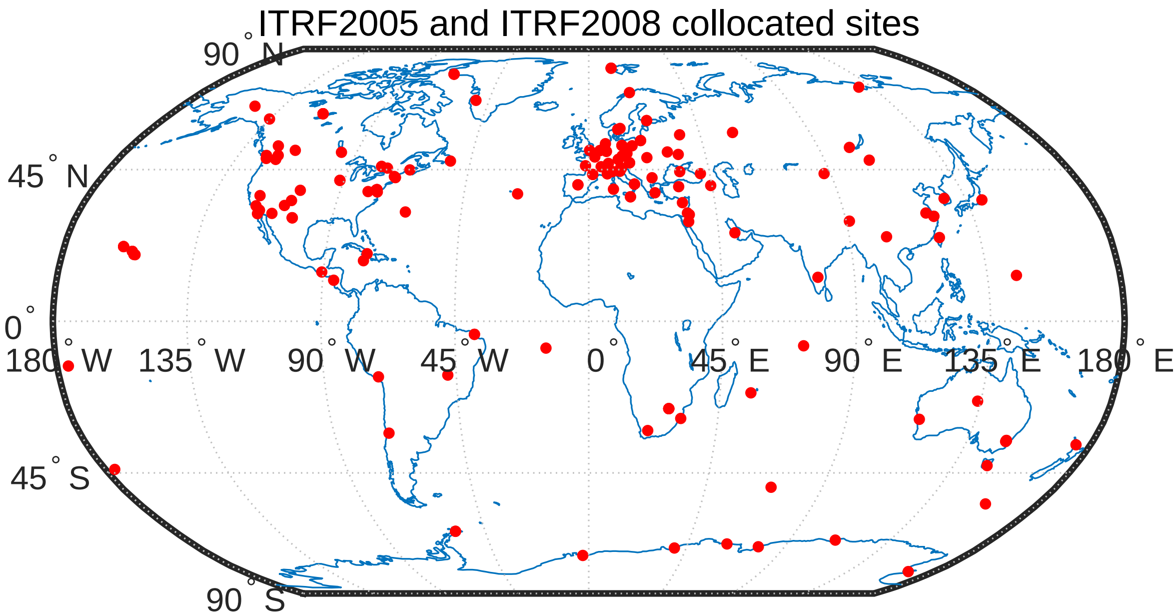

Figure 4.6: *Location of the reference frame sites used in the estimation of the 14 transformation parameters between ITRF2005 and ITRF2008*

3) Unified the two frame in the same epoch with the fomula.

$$\boldsymbol{X}(t)_{ITRF_{zz}} = \boldsymbol{X}(t_0)_{ITRF_{zz}} + (t - t_0)\boldsymbol{v}_x(t_0)_{ITRF_{zz}} \tag{4.30}$$

4) CTLS model for the case of n independent observations y_i and u independent observations y_a with different weights.

$$\begin{aligned} \boldsymbol{y} &= \boldsymbol{A}\boldsymbol{\xi} + \boldsymbol{e}_y \\ \boldsymbol{y}_a &= \boldsymbol{\xi}_a + \boldsymbol{e}_a \end{aligned} \tag{4.31}$$

Define $\boldsymbol{z} = \begin{bmatrix} \boldsymbol{y} - \boldsymbol{A}_\xi^0 \boldsymbol{\xi}^0 \\ \boldsymbol{a} - \boldsymbol{y_a} \end{bmatrix}$, $\boldsymbol{A_z} = \begin{bmatrix} \boldsymbol{A}_\xi^0 & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{E} \end{bmatrix}$, $\boldsymbol{\Delta\eta} = \begin{bmatrix} \boldsymbol{\Delta\xi} \\ \boldsymbol{\Delta a} \end{bmatrix}$, $\boldsymbol{e_z} = \begin{bmatrix} \boldsymbol{e_y} \\ \boldsymbol{e_a} \end{bmatrix}$, reduced to:

$$\boldsymbol{z} = \boldsymbol{A_\eta \Delta\eta} + \boldsymbol{e_z} \tag{4.32}$$

With the new weight matrix on the variance-covariance matrix

$$\boldsymbol{P}_z = \begin{bmatrix} \boldsymbol{P}_y & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}_a \end{bmatrix} \quad \boldsymbol{\Sigma}_z = \sigma_{z_0}^2 \begin{bmatrix} \boldsymbol{P}_y^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}_a^{-1} \end{bmatrix} \tag{4.33}$$

With detail

$$\boldsymbol{\Sigma}_z = \begin{bmatrix} \sigma_{y1}^2 & & & & & \\ & \ddots & & & \boldsymbol{0} & \\ & & \sigma_{yn}^2 & & & \\ & & & \sigma_{a1}^2 & & \\ & \boldsymbol{0} & & & \ddots & \\ & & & & & \sigma_{an}^2 \end{bmatrix} = \sigma_{z_0}^2 \begin{bmatrix} \sigma_{y1}^2/\sigma_{z_0}^2 & & & & & \\ & \ddots & & & \boldsymbol{0} & \\ & & \sigma_{yn}^2/\sigma_{z_0}^2 & & & \\ & & & \sigma_{a1}^2/\sigma_{z_0}^2 & & \\ & \boldsymbol{0} & & & \ddots & \\ & & & & & \sigma_{an}^2/\sigma_{z_0}^2 \end{bmatrix} \tag{4.34}$$

with

$$\boldsymbol{\Sigma}_z = \sigma_{z_0}^2 P^{-1}, \quad \boldsymbol{P}_i = \frac{\sigma_{z_0}^2}{\sigma_{yi}^2} \tag{4.35}$$

The estimation criterion is to get the minimum of the residual sum of the squares:

$$\boldsymbol{e_z^T P_z e_z} = \boldsymbol{e_y^T P_y e_y} + \boldsymbol{e_a^T P_a e_a} \rightarrow min \tag{4.36}$$

The estimated $\Delta\hat{\eta}$ is

$$\boldsymbol{\Delta\hat{\eta}} = (\boldsymbol{A_\eta^T \Sigma_z^{-1} A_\eta})^{-1} \boldsymbol{A_\eta^T \Sigma_z^{-1} z} = (\boldsymbol{A_\eta^T P_z A_\eta})^{-1} \boldsymbol{A_\eta^T P_z z} \tag{4.37}$$

together with the variance-covariance matrix

$$\boldsymbol{\Sigma}_{\Delta\hat{\eta}} = \sigma_{z_0}^2 (\boldsymbol{A_\eta^T P_z A_\eta})^{-1} \tag{4.38}$$

with the estimation (4.37), the corrections of observations can be derived:

$$\boldsymbol{\hat{e}_z} = \boldsymbol{A\Delta\hat{\eta}} - \boldsymbol{\eta} \tag{4.39}$$

And the unit weight variance for CTLS can be estimated

$$\hat{\sigma}^2_{z_0} = \frac{\hat{\boldsymbol{e}}_z^T \boldsymbol{P}_z \hat{\boldsymbol{e}}_z}{(n+u)-(m+u)} = \frac{\hat{\boldsymbol{e}}_z^T \boldsymbol{P}_z \hat{\boldsymbol{e}}_z}{(n-m)} \tag{4.40}$$

Here n is the number of the original observations, u is the number of the independent random elements in the design matrix A, m is the number of unknown parameters.

The estimation of the variance-covariance matrix can be derived by (different with equation(4.48))

$$\hat{\boldsymbol{\Sigma}}_{\Delta\hat{z}} = \hat{\sigma}^2_{z_0}(\boldsymbol{A}_\eta^T \boldsymbol{P}_z \boldsymbol{A}_\eta)^{-1} \tag{4.41}$$

Creat the design matrix of the 14 transformation model.

$$\underset{6n\times 14}{\boldsymbol{A}} = \begin{bmatrix}
1 & 0 & 0 & 0 & -zl_1 & yl_1 & xl_1 & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & & & & \\
1 & 0 & 0 & 0 & -zl_n & yl_n & xl_n & & & & & & & \\
0 & 1 & 0 & zl_1 & 0 & -xl_1 & yl_1 & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \mathbf{0} & & & \\
0 & 1 & 0 & zl_n & 0 & -xl_n & yl_n & & & & & & & \\
0 & 0 & 1 & -yl_1 & xl_1 & 0 & zl_1 & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & & & & \\
0 & 0 & 1 & -yl_n & xl_n & 0 & zl_n & & & & & & & \\
 & & & & & & & 1 & 0 & 0 & 0 & -zl_1 & yl_1 & xl_1 \\
 & & & & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & & & & & 1 & 0 & 0 & 0 & -zl_n & yl_n & xl_n \\
 & & & & & & & 0 & 1 & 0 & zl_1 & 0 & -xl_1 & yl_1 \\
 & & & & \mathbf{0} & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & & & & & 0 & 1 & 0 & zl_n & 0 & -xl_n & yl_n \\
 & & & & & & & 0 & 0 & 1 & -yl_1 & xl_1 & 0 & zl_1 \\
 & & & & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & & & & & 0 & 0 & 1 & -yl_n & xl_n & 0 & zl_n
\end{bmatrix} \tag{4.42}$$

By the CTLS method, we also need the new design matrix B.

$$
\underset{(6n\times 4n)}{\boldsymbol{B}} = \begin{bmatrix} T_X & \zeta_{22}^0 & \zeta_{12}^0 & -\zeta_{21}^0 \\ T_Y & -\zeta_{12}^0 & \zeta_{22}^0 & \zeta_{11}^0 \\ T_Z & \zeta_{21}^0 & -\zeta_{11}^0 & \zeta_{22}^0 \\ \dot{T}_X & \dot{\zeta}_{22}^0 & \dot{\zeta}_{12}^0 & -\dot{\zeta}_{21}^0 \\ \dot{T}_Y & -\dot{\zeta}_{12}^0 & \dot{\zeta}_{22}^0 & \dot{\zeta}_{11}^0 \\ \dot{T}_Z & \dot{\zeta}_{21}^0 & -\dot{\zeta}_{11}^0 & \dot{\zeta}_{22}^0 \end{bmatrix} \otimes \boldsymbol{I_n}
$$

$$
= \begin{bmatrix}
T_X^0 & 0 & 0 & \zeta_{22}^0 & 0 & 0 & \zeta_{12}^0 & 0 & 0 & -\zeta_{21}^0 & 0 & 0 \\
0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 \\
0 & 0 & T_X^0 & 0 & 0 & \zeta_{22}^0 & 0 & 0 & \zeta_{12}^0 & 0 & 0 & -\zeta_{21}^0 \\
T_Y^0 & 0 & 0 & -\zeta_{12}^0 & 0 & 0 & \zeta_{22}^0 & 0 & 0 & \zeta_{11}^0 & 0 & 0 \\
0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 \\
0 & 0 & T_Y^0 & 0 & 0 & -\zeta_{12}^0 & 0 & 0 & \zeta_{22}^0 & 0 & 0 & \zeta_{11}^0 \\
T_Z^0 & 0 & 0 & \zeta_{21}^0 & 0 & 0 & -\zeta_{11}^0 & 0 & 0 & \zeta_{22}^0 & 0 & 0 \\
0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 \\
0 & 0 & T_Z^0 & 0 & 0 & \zeta_{21}^0 & 0 & 0 & -\zeta_{11}^0 & 0 & 0 & \zeta_{22}^0 \\
\dot{T}_X^0 & 0 & 0 & \dot{\zeta}_{22}^0 & 0 & 0 & \dot{\zeta}_{12}^0 & 0 & 0 & -\dot{\zeta}_{21}^0 & 0 & 0 \\
0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 \\
0 & 0 & \dot{T}_X^0 & 0 & 0 & \dot{\zeta}_{22}^0 & 0 & 0 & \dot{\zeta}_{12}^0 & 0 & 0 & -\dot{\zeta}_{21}^0 \\
\dot{T}_Y^0 & 0 & 0 & -\dot{\zeta}_{12}^0 & 0 & 0 & \dot{\zeta}_{22}^0 & 0 & 0 & \dot{\zeta}_{11}^0 & 0 & 0 \\
0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 \\
0 & 0 & \dot{T}_Y^0 & 0 & 0 & -\dot{\zeta}_{12}^0 & 0 & 0 & \dot{\zeta}_{22}^0 & 0 & 0 & \dot{\zeta}_{11}^0 \\
\dot{T}_Z^0 & 0 & 0 & \dot{\zeta}_{21}^0 & 0 & 0 & -\dot{\zeta}_{11}^0 & 0 & 0 & \dot{\zeta}_{22}^0 & 0 & 0 \\
0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 \\
0 & 0 & \dot{T}_Z^0 & 0 & 0 & \dot{\zeta}_{21}^0 & 0 & 0 & -\dot{\zeta}_{11}^0 & 0 & 0 & \dot{\zeta}_{22}^0
\end{bmatrix}
$$

Here shows more detail information and exsample with weight matrix used in the application. Set the constant $\sigma_{z_0}^2 = (0.001m)^2 = (0.001)^2 m^2$. Which mean σ_{z_0} is $1mm$, and $p_i = \sigma_{z_0}^2 / \sigma_i^2$.

Exsample for collocated sites.

La Rochelle (10023M001): $p_{a1} = \frac{0.001^2}{0.0006^2} = 2.7778$
Onsala (10402S002): $p_{a1} = \frac{0.001^2}{0.0007^2} = 2.0408$
Kitab (12334S006): $p_{a1} = \frac{0.001^2}{0.0018^2} = 0.3086$
The exsample shows that larger sigma(formal error) brings smaller weight.

4.4.3 Presentation and comparison of the results

Statistical data by the quadratics sums of the residuals for 2 estimators. The $\hat{e}$ is the residual of observation and $\hat{E}$ is the residual of design matrix.
The first case will shows the transformation between ITRF2008-ITRF2005.

LS:

$$\hat{\boldsymbol{e}}_{LS}^{T}\boldsymbol{P}_{y}\hat{\boldsymbol{e}}_{LS} = 12223.8 \quad (mm^2)$$

CTLS:

$$\begin{aligned}
\hat{\boldsymbol{e}}_{CTLS}^{T}\boldsymbol{P}_{y}\hat{\boldsymbol{e}}_{CTLS} &= 8435.8 \quad (mm^2)\\
\hat{\boldsymbol{E}}_{CTLS}^{T}\boldsymbol{P}_{a}\hat{\boldsymbol{E}}_{CTLS} &= 1272.2 \quad (mm^2)\\
\hat{\boldsymbol{e}}_{CTLS}^{T}\boldsymbol{P}_{y}\hat{\boldsymbol{e}}_{CTLS} + \hat{\boldsymbol{E}}_{CTLS}^{T}\boldsymbol{P}_{a}\hat{\boldsymbol{E}}_{CTLS} &= 9708.0 \quad (mm^2)
\end{aligned}$$

Table 4.3: *Transformation parameters with errors between ITRF2008-ITRF2005 in epoch 2005 with LS*

Least Squares method	$T_X(mm)$ $\dot{T}_X(mm/a)$	$T_Y(mm)$ $\dot{T}_Y(mm/a)$	$T_Z(mm)$ $\dot{T}_Z(mm/a)$	$D(10^{-9})$ $\dot{D}(10^{-9}/a)$	$R_1(mas)$ $\dot{R}_1(mas/a)$	$R_2(mas)$ $\dot{R}_2(mas/a)$	$R_3(mas)$ $\dot{R}_3(mas/a)$
	-0.7	-0.9	-4.2	0.9	0.00	0.00	0.00
±	0.33	0.33	0.32	0.005	0.001	0.001	0.001
	0.4	0.4	0.04	0.00	0.00	0.00	0.00
±	0.33	0.33	0.32	0.005	0.001	0.001	0.001

Table 4.4: *Transformation parameters with errors between ITRF2008-ITRF2005 in epoch 2005 with CTLS*

CTLS method	$T_X(mm)$ $\dot{T}_X(mm/a)$	$T_Y(mm)$ $\dot{T}_Y(mm/a)$	$T_Z(mm)$ $\dot{T}_Z(mm/a)$	$D(10^{-9})$ $\dot{D}(10^{-9}/a)$	$R_1(mas)$ $\dot{R}_1(mas/a)$	$R_2(mas)$ $\dot{R}_2(mas/a)$	$R_3(mas)$ $\dot{R}_3(mas/a)$
	-0.8	-0.8	-4.5	0.9	0.00	0.00	0.00
±	0.26	0.25	0.26	0.004	0.001	0.001	0.001
	0.05	0.0	0.0	0.00	0.00	0.00	0.00
±	0.26	0.25	0.26	0.004	0.001	0.001	0.001

Table 4.5: *Numerical deviation of 14-parameter transformation with 2 estimators*

Transformation model	Collocated sites	Absolute mean of Residuals (mm)			Max.absolute mean of Residuals(mm)			RMS (mm)	Standard deviation of unit weight (mm)
		$[V_X]$	$[V_Y]$	$[V_Z]$	$[V_X]$	$[V_Y]$	$[V_Z]$		
LS	169	2.83	2.92	2.95	9.23	34.53	17.91	3.1566	3.4963
CTLS	169	1.65	1.54	1.66	7.29	17.92	14.54	3.0957	3.1158

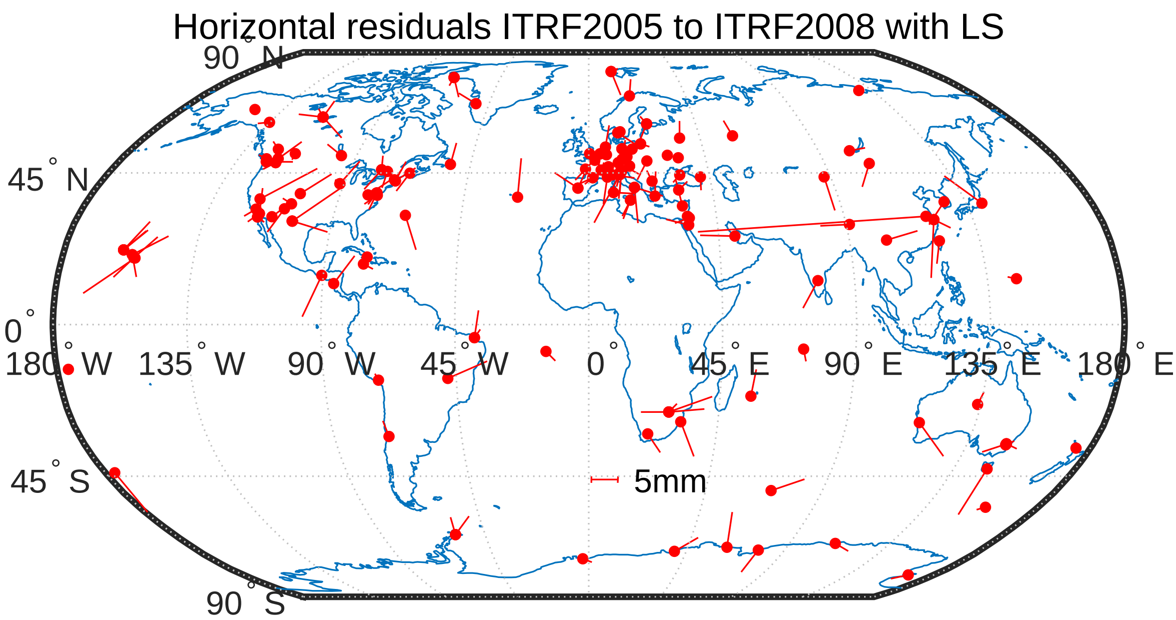

***Figure 4.7:** Sites and horizontal residuals after LS transformation between ITRF2005 and ITRF2008*

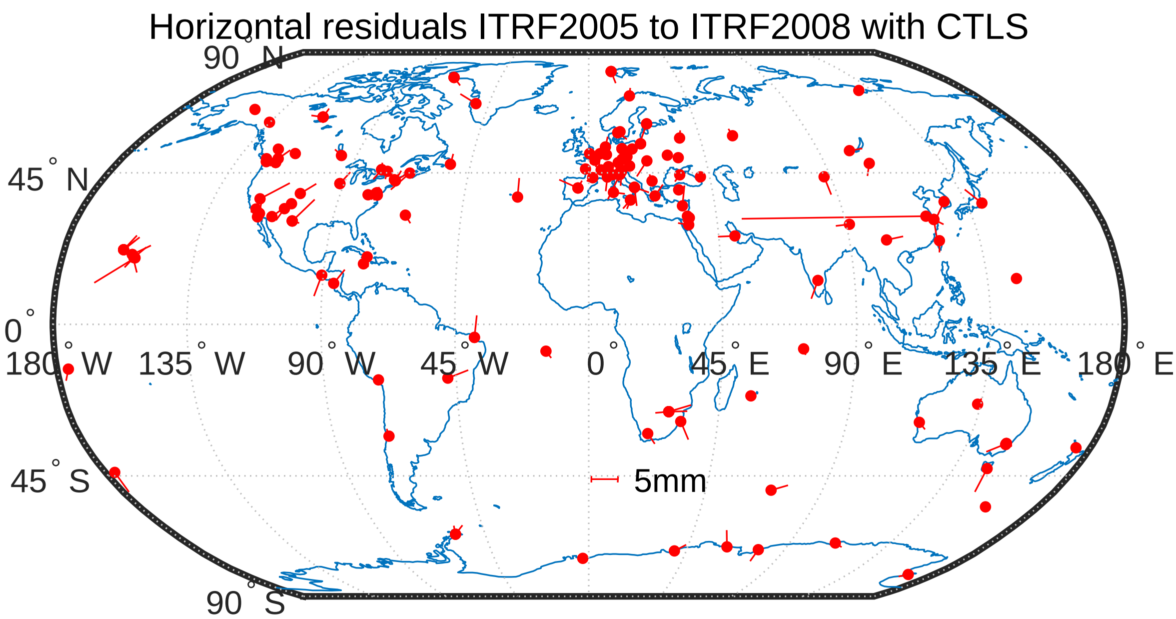

***Figure 4.8:** Sites and horizontal residuals after CTLS transformation between ITRF2005 and ITRF2008*

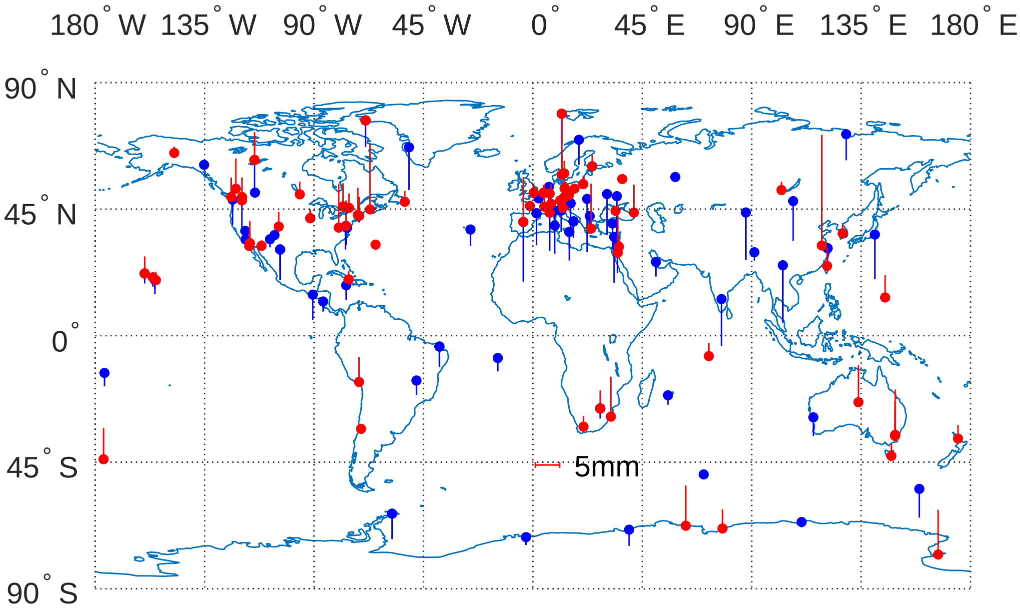

Figure 4.9: *Sites and vertical residuals after LS transformation between ITRF2005 and ITRF2008*

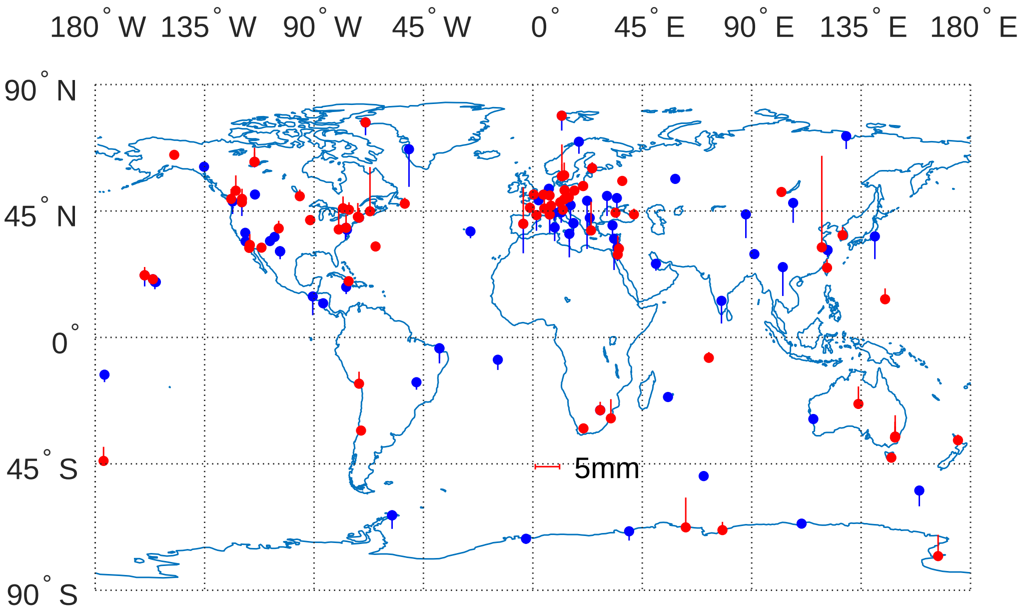

Figure 4.10: *Sites and vertical residuals after CTLS transformation between ITRF2005 and ITRF2008*

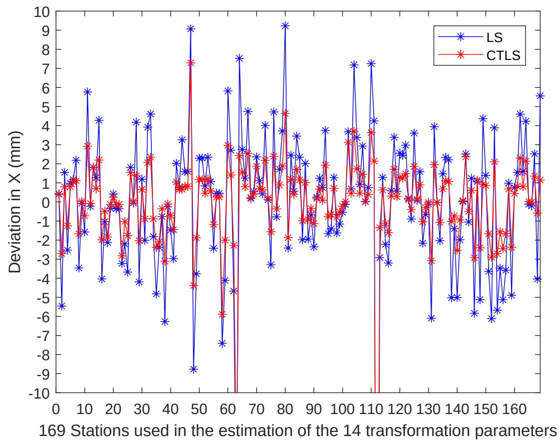

Figure 4.11: *Comparison of the deviations in X direction between LS and CTLS*

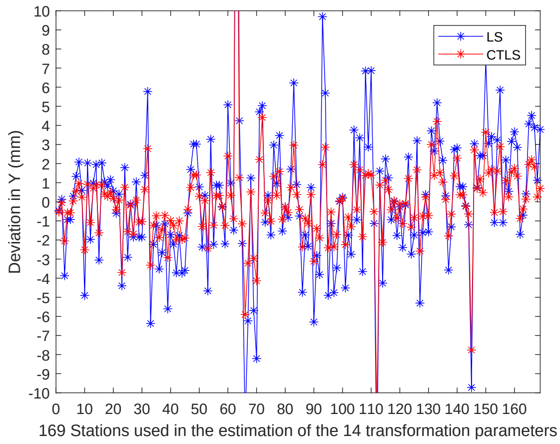

Figure 4.12: *Comparison of the deviations in Y direction between LS and CTLS*

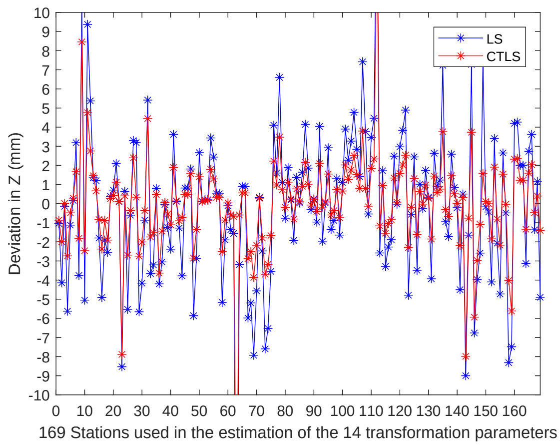

Figure 4.13: *Comparison of the deviations in Z direction between LS and CTLS*

The second case will show the transformation between ITRF2005-ITRF2000. The selected reference points used as collocated sites are presented on the map below.

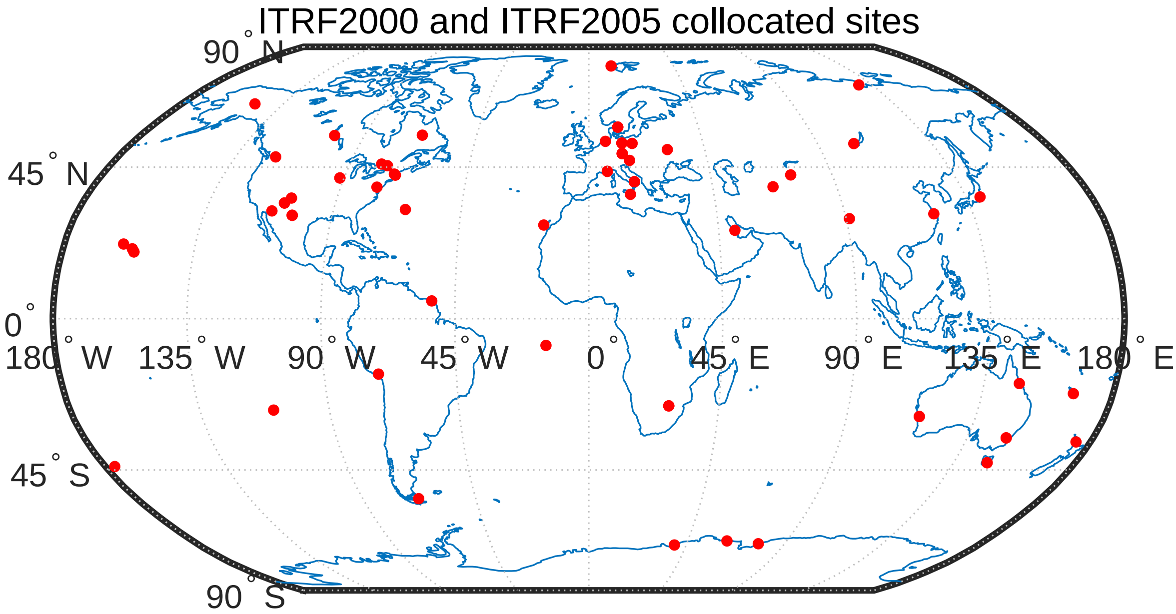

Figure 4.14: *Location of the reference frame sites used in the estimation of the 14 transformation parameters between ITRF2000 and ITRF2005*

LS:

$$\hat{e}_{LS}^{T} P_y \hat{e}_{LS} = 1497.2 \quad (mm^2)$$

CTLS:

$$\hat{e}_{CTLS}^{T} P_y \hat{e}_{CTLS} = 1174.5 \quad (mm^2)$$
$$\hat{E}_{CTLS}^{T} P_a \hat{E}_{CTLS} = 123.5 \quad (mm^2)$$
$$\hat{e}_{CTLS}^{T} P_y \hat{e}_{CTLS} + \hat{E}_{CTLS}^{T} P_a \hat{E}_{CTLS} = 1298.0 \quad (mm^2)$$

Table 4.6: *Transformation parameters with errors between ITRF2005-ITRF2000 in epoch 1997 with LS*

Least Squares method	$T_X(mm)$ $\dot{T}_X(mm/a)$	$T_Y(mm)$ $\dot{T}_Y(mm/a)$	$T_Z(mm)$ $\dot{T}_Z(mm/a)$	$D(10^{-9})$ $\dot{D}(10^{-9}/a)$	$R_1(mas)$ $\dot{R}_1(mas/a)$	$R_2(mas)$ $\dot{R}_2(mas/a)$	$R_3(mas)$ $\dot{R}_3(mas/a)$
	0.16	-1.05	-5.73	0.40	0.00	0.00	0.00
±	0.38	0.36	0.50	0.007	0.016	0.016	0.013
	-0.11	0.06	-1.77	0.02	0.00	0.00	0.00
±	0.38	0.36	0.50	0.007	0.016	0.016	0.013

Table 4.7: *Transformation parameters with errors between ITRF2005-ITRF2000 in epoch 1997 with CTLS*

CTLS method	$T_X(mm)$ $\dot{T}_X(mm/a)$	$T_Y(mm)$ $\dot{T}_Y(mm/a)$	$T_Z(mm)$ $\dot{T}_Z(mm/a)$	$D(10^{-9})$ $\dot{D}(10^{-9}/a)$	$R_1(mas)$ $\dot{R}_1(mas/a)$	$R_2(mas)$ $\dot{R}_2(mas/a)$	$R_3(mas)$ $\dot{R}_3(mas/a)$
	0.02	-0.93	-5.82	0.40	0.00	0.00	0.00
$\pm$	0.30	0.30	0.43	0.007	0.015	0.015	0.012
	-0.14	0.06	-1.74	0.002	0.00	0.00	0.00
$\pm$	0.30	0.30	0.43	0.007	0.015	0.015	0.012

Table 4.8: *Numerical deviation of 14-parameter transformation with 2 estimators*

Transformation model	Collocated sites	Absolute mean of Residuals (mm)			Max.absolute mean of Residuals(mm)			RMS (mm)	Standard deviation of unit weight (mm)
		$[V_X]$	$[V_Y]$	$[V_Z]$	$[V_X]$	$[V_Y]$	$[V_Z]$		
LS	69	2.46	2.18	2.62	10.25	6.49	8.11	1.9040	1.9347
CTLS	69	1.79	1.60	2.11	8.01	5.9	8.20	1.7728	1.8014

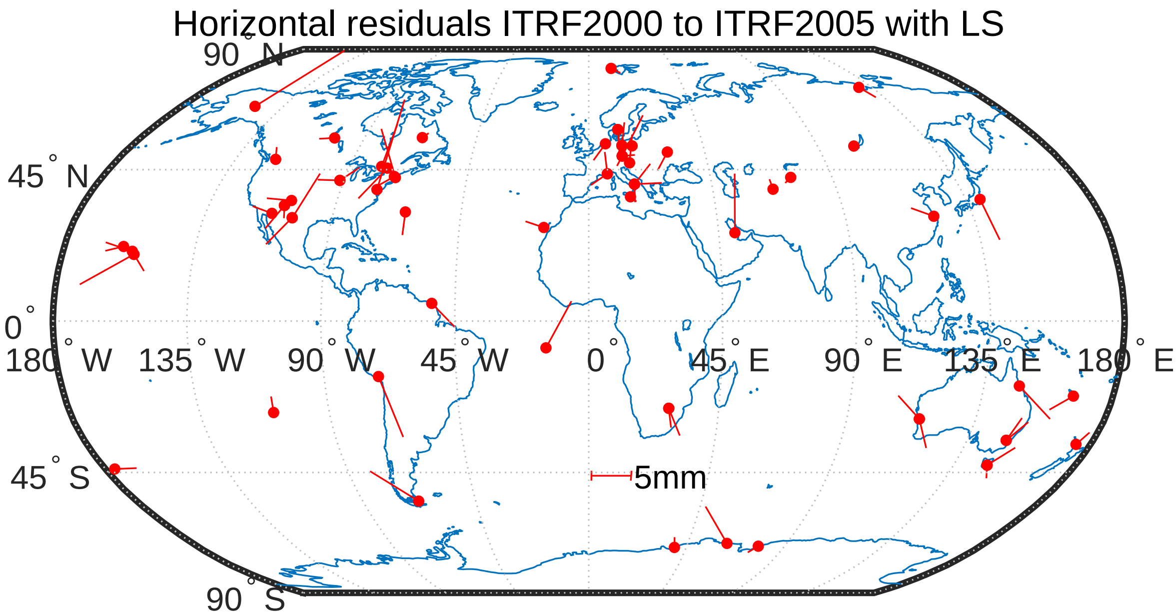

Figure 4.15: *Sites and horizontal residuals after LS transformation between ITRF2000 and ITRF2005*

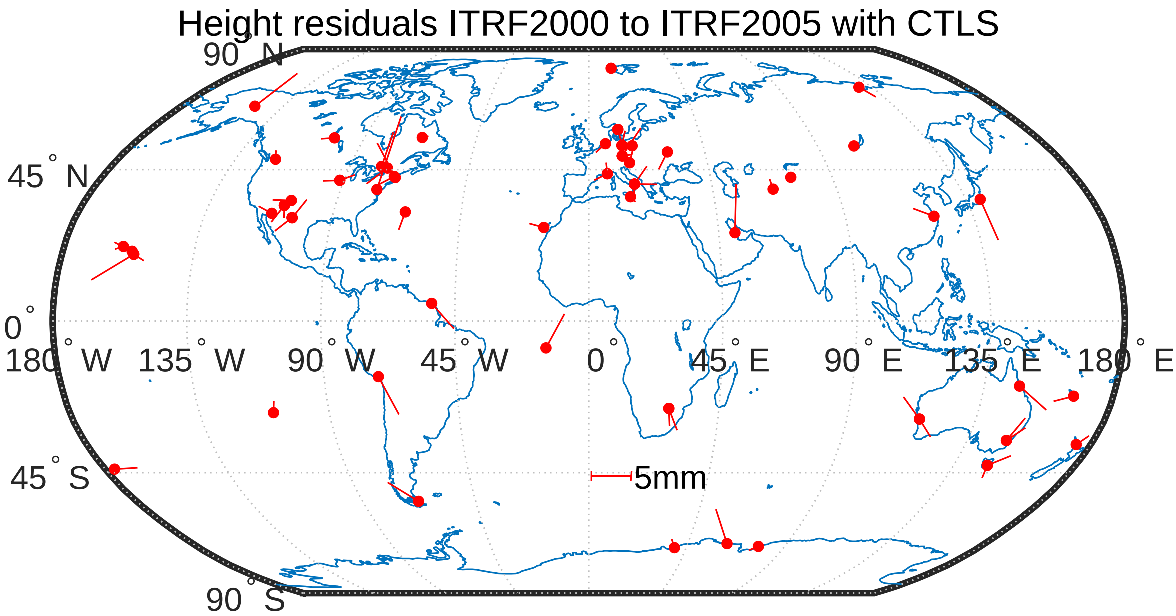

Figure 4.16: *Sites and horizontal residuals after CTLS transformation between ITRF2000 and ITRF2005*

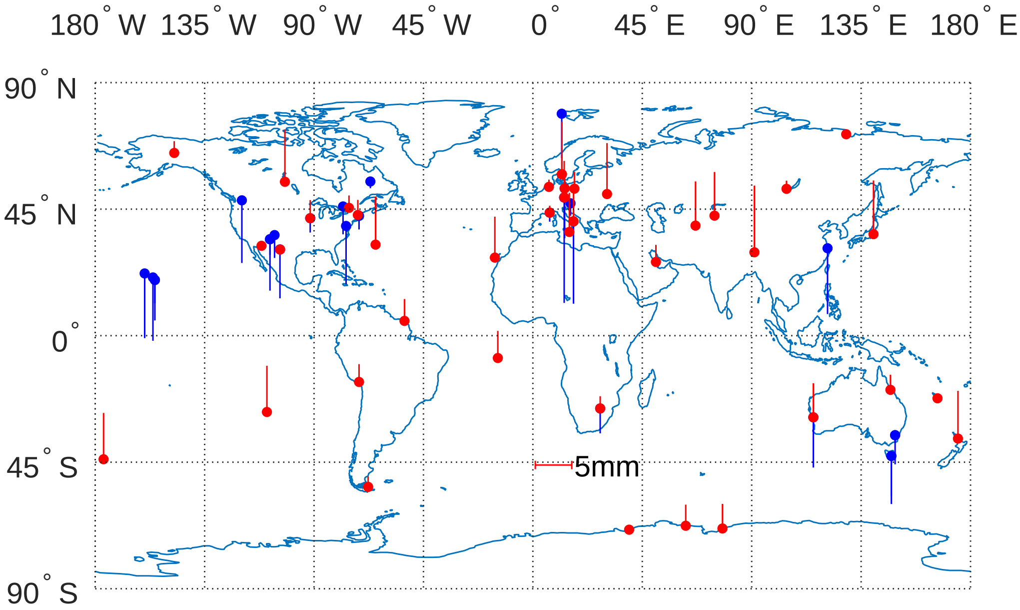

Figure 4.17: *Sites and vertical residuals after LS transformation between ITRF2000 and ITRF2005*

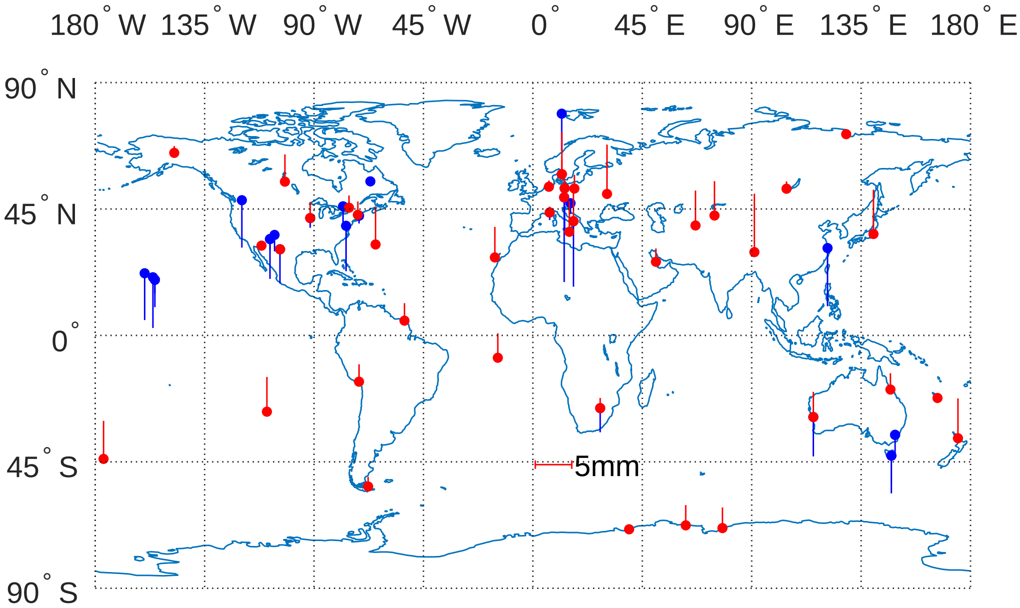

Figure 4.18: *Sites and vertical residuals after CTLS transformation between ITRF2000 and ITRF2005*

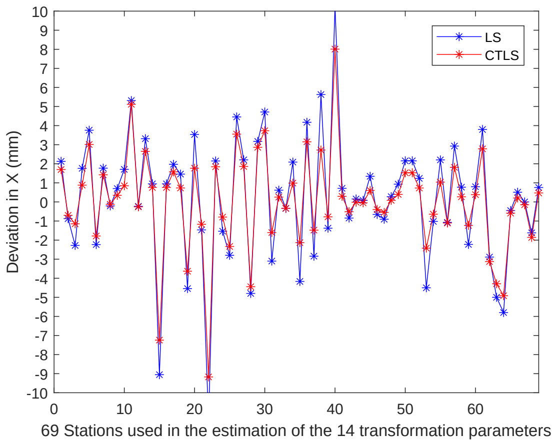

Figure 4.19: *Comparison of the deviations in X direction between LS and CTLS*

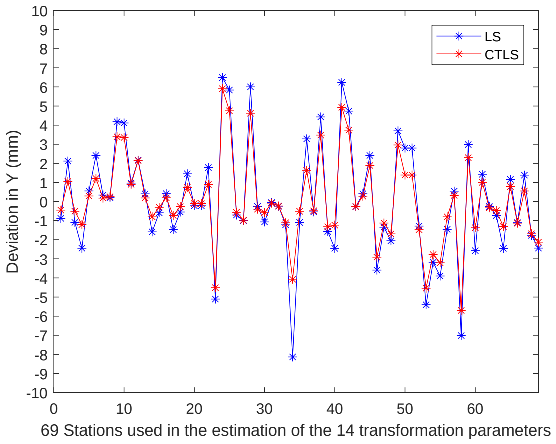

Figure 4.20: *Comparison of the deviations in Y direction between LS and CTLS*

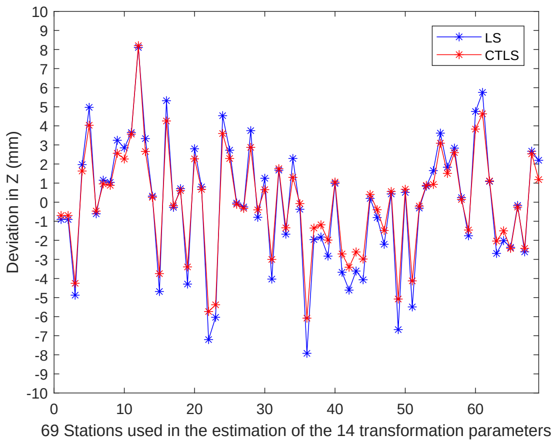

Figure 4.21: *Comparison of the deviations in Z direction between LS and CTLS*

Conclusions: After the application of coordiante transformation among ITRF realization, it is found that, the whole process of the data collection and data preparation in ITRF is very complicated and every step is so important that it can affect the final production. This thesis has a brief introduction for the way that how to collecte data together with their technology. The method that used to estimate the parameters in every step and their mathematic theory. The main part in this thesis is to compare the LS and CTLS in 14-parameter transformation. In oder to make precise accuracy assessment, this thesis introduced how to use the weight information as the weight matrix in real case estimation. And the errors for every transformation parameters are also estimated.

From the results above, we can have some conclusions.

- The weight information is quite important for ITRF realization. Any error and changes of the weights for collocated sites might result in different solutions.
- The accuracy for collocated sites on the north earth are better than the accuracy on the south earth, because the number of observation sites on north earth are more than that in south earth.
- The accuracy for collocated sites in Europe and North America are better than other sites, because the density of the collocated sites and the observation technology are higher and more stable than other place on earth.
- Different collocated sites with different observation technology have different sensitivities to the transformation parameters. Which result in that the residuals for different sites have quite different directions.
- The horizontal residuals and the height residuals estimated by CTLS are smaller than estimated by LS. CTLS has proved itself to have obvious improvement for the accuracy in ITRF realizations.

Chapter 5

Conclusion

The further study on Converted Total Least Squares method has been made with 3 main parts. The first part compares the 2D and 3D transformation model with and without centering, and their difference between LS and CTLS in application with coordinates transformation in Baden-Württemberg. After that this thesis gives a complete introduction to Gauss-Helmert model and finds out the connections with CTLS. The comparison is also made through the application with coordinates transformation in Baden-Württemberg. The third part introduces the International Terrestrial Reference Frame, with the main calculation and estimation steps together with the data processing techniques and methods for each step. The transformation among ITRF realizations with the weight information are estimated with LS and CTLS.

The results have been tested and discussed here. Based on these analyses and comparisons with the 3 main parts of this thesis, the following points can be concluded:

- The transformation model has no difference with and without centering. However, for the application by CTLS, when the observation equation contains the parameters that independent to observed values, it is not suitable to be applied. The Gauss-Helmert model is still useful in the case.
- This notable development of the CTLS reveals that CTLS estimator is identical to Gauss-Helmert model estimator in dealing with EIV models, especially in the case of coordinate transformation.
- Successful application to the estimation of the transformation parameters and related transformed residuals among ITRF realization with obvious improvement of their accuracies.

As the study goes on, there are still some further considerations for the thesis:

- ITRF is a quite systematic and complex solution for the earth reference frame. Due to the luck of the original data and the related software, the results in this article is based on the general data provided by Dr. Altamimi. Further study has better to apply the original data from collocated sites and to improve the solutions of ITRF.
- CTLS has proved itself in many kinds of cases in coordinate transformation. Further study can be performed for the other applications.

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