Student Thesis 211

Stability and Performance in transient average constrained Economic Model Predictive Control without terminal constraints

Student Thesis

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Abstract

In this thesis, system theoretic properties of economic model predictive control (EMPC) without terminal constraints but subject to transient average constraints are investigated. The goal is to show performance and stability of the considered transient average constrained EMPC scheme without terminal constraints. Existing results either consider transient average constrained EMPC with terminal ingredients or EMPC without terminal ingredients and without transient average constraints. Using suitable dissipativity, continuity and controllability assumptions, we prove the desired properties for transient average constrained EMPC without terminal constraints.

In order to consider transient average constraints, an extended state of the system is introduced which additionally considers past values of the auxiliary output. By using dissipativity and an intersection of sets, a turnpike phenomenon is shown for consecutive time instants which can be interpreted as a turnpike property of the extended state. Combining the provided turnpike behaviour with suitable controllability properties yields local continuity of the value function. The bound on the locally continuous value function is used to prove closed-loop performance bounds in terms of value and trajectory convergence. Furthermore, it is shown that the results also hold for the rotated value function.

Moreover, we show input-to-state stability (ISS) of the auxiliary output storage. In contrast to economic MPC settings without transient average constraints, practical asymptotic stability cannot be proved by considering the rotated value function. We show stability by combining the rotated value function with the ISS Lyapunov function of the extended state and utilizing existing results on non-monotonic Lyapunov functions. The provided theoretical results are illustrated with a numerical example at the end.

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Contents

Nomenclature

The following register summarizes the symbols and notations which are introduced in the thesis.

Symbols

$f(x,u) \in \mathbb{R}^n$	System Dynamics
$h(x,u) \in \mathbb{R}^p$	Auxiliary Output
$H \in \mathbb{R}^{p\times(T-1)}$	Storage of past Auxiliary Output Values
$\mathbb{H}$	Set of feasible Storages H
$J_N(x,u)$	Finite Horizon Cost
$\tilde{J}_N(x,u)$	Rotated Cost, see equation (3.4)
$\ell(x,u)$	Stage Cost
$\tilde{\ell}(x,u)$	Rotated Stage Cost, see equation (3.4)
N	Prediction Horizon
q	Amount of considered trajectories in the intersection $\mathcal{P}'^{\epsilon}_{[k_l,k_u]}$
$s(x,u)$	Supply Rate, page 15
T	Time Period of the Transient Average Constraints
$u \in \mathbb{U} \subseteq \mathbb{R}^m$	Input Vector
$x \in \mathbb{X} \subseteq \mathbb{R}^n$	System State Vector
(x_s,u_s)	Optimal Steady-State pair
$\mathbb{Y}$	Transient Average Constraints Set

$\mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}$	Coupled Constraints
$\lambda(x)$	Storage Function, page 15
$\bar{\lambda} \in \mathbb{R}^p_{\geq 0}$	Multiplier
$\mu_N(x,H)$	MPC Feedback Law, page 14

Notation

$\upharpoonleft H \upharpoonright$	Norm-like measurement of the Storage H, see equation (2.24)
$J^*_N(x,H)$	Value Function
$\tilde{J}^*_N(x,H)$	Rotated Value Function
J^{cl}_K	Closed Loop Cost
$\mathcal{P}'^{\epsilon}_{[k_l,k_u]}$	Time Instants in $\mathcal{B}_\epsilon(x_s,us)$ for all considered trajectories in the interval $\mathbb{I}_{[k_l,k_u]}$, see equation (3.20)
$\mathcal{P}^\epsilon(u,x)$	Set of Time Instants of the trajectory $x_u(\cdot,x)$ which are in a neighborhood of the steady-state, see equation (3.9)
$\mathcal{Q}^\epsilon(u,x)$	Amount of Time Instants in the set $\mathcal{P}^\epsilon(u,x)$, page 25
$u^*_{N,x,H}$	Optimal Open Loop Input Sequence w. r. t. the Initial Conditions (x,H)
$u_N \in \mathbb{U}^N(x,H)$	Feasible Control Sequences of length N w. r. t. the Initial Conditions (x,H)
$x_{u_N}(\cdot,x)$	Resulting Trajectory of the Control Sequence u_N starting in x

1. Introduction

1.1. Motivation

Model predictive control (MPC) [Rawlings and Mayne, 2009] is a popular control method that computes the control input by repeatedly solving a finite horizon optimal control problem online in each sampling instant which yields an implicite controller. The prime advantages of MPC are that it can deal with complex nonlinear dynamics, general objective functions, multiple-input-multiple-output (MIMO) systems as well as arbitrary input and state constraints.

In stabilizing MPC, the corresponding stage cost usually expresses the distance to a desired steady-state which can be described by using a positive definite function. More generally, the main objective does not necessarily need to be stability but can rather be optimal performance with respect to an economic criterion which results in a cost function that does not have to be positive definite with respect to any setpoint. This variant of MPC is called economic MPC (EMPC) [Angeli et al., 2012], [Faulwasser et al., 2018].

Usually, an optimal steady-state (or periodic orbit) is determined for the the infinite horizon problem and then, is used as a terminal condition for the finite horizon problem. These terminal conditions are often omitted in practical applications. Reasons are, that these conditions can limit the operating region of the controller and moreover, that the absence of these additional constraints makes the finite horizon optimal control problem in each step easier to solve. Furthermore, terminal costs and regions can be complicated to design, i. e., for optimization problems without terminal conditions the amount of offline computation is reduced. The price to pay for removing the terminal conditions are a more sophisticated analysis as well as stronger assumptions and weaker properties [Grüne, 2013, Grüne and Stieler, 2014].

Additionally to pointwise in time constraints, it stands to reason to consider constraints on average values of state and input variables, e. g., this can be constraints of asymptotic average values. For instance, they need to be considered when determining the steady-state which we want to stabilize.

In addition, constraints on states and inputs averaged over some finite time period T can be interesting in different applications such as in the process industry in order to consider the average amount of inflow or in building climate control in order to deal with small average temperature deviations. Other examples are an electric motor where we want to avoid overheating by limiting values over a period of time or, limiting the frequency deviation in power grids. These transient average constraints, just like any kind of constraints, can explicitly be incorporated into the MPC controller design by taking them into account in the finite horizon optimization problem at each sampling instant. So far, results for transient average constraints in EMPC have only been shown by using terminal ingredients. Hence, the question arises whether we can consider such transient average constraints without imposing terminal constraints in order to benefit from the advantages of both properties (simple design and transient average constraints). We derive theoretical guarantees in terms of performance guarantees as well as stability for the transient average constrained EMPC scheme without terminal constraints.

1.2. Related Work

EMPC has been investigated with a terminal equality constraint [Diehl et al., 2011] as well as for a terminal cost and a terminal region [Amrit et al., 2011]. Furthermore, bounds on the average performance of EMPC are given in [Angeli et al., 2012] and non-averaged and transient performance estimates for EMPC have been investigated in [Grüne and Panin, 2015]. EMPC without terminal constraints is introduced in [Grüne, 2013] where also near optimal performance and convergence of the closed loop to a neighborhood of the optimal steady-state have been proved. These results are further developed in [Grüne and Stieler, 2014] where practical asymptotic stability and approximate transient optimality of the closed loop have been shown. In [Müller et al., 2013,Müller et al., 2014b], convergence of averagely constrained EMPC with terminal ingredients is considered. [Köhler et al., 2017] present a transient, nonaveraged performance estimate for the closed loop with asymptotic average constraints. The stricter form of transient average constraints can be found in [Müller et al., 2014a]. There, closed-loop average performance bounds and convergence results are proved for EMPC with transient average constraints by imposing a terminal region and a terminal cost. We highlight that transient average constrained EMPC without terminal ingredients has not yet been investigated.

1.3. Contribution

So far, results for transient average constrains in EMPC have been shown by imposing terminal ingredients [Müller et al., 2014a]. However, transient average constrained economic MPC without terminal constraints has not been investigated. This thesis bridges this gap by the following contributions.

We describe the transient average constrained EMPC scheme using an extended state which contains past values of the auxiliary output. As a first contribution, we extend existing turnpike arguments in order to conclude a turnpike property over multiple consecutive time steps, which implies a turnpike for this extended state. Then, we provide transient performance guarantees and show trajectory convergence, similar to the derivation of [Grüne, 2013] and [Grüne and Stieler, 2014].

For the analysis of practical asymptotic stability, we show that contrary to most economic MPC schemes, in the considered formulation the rotated value function is *not* a suitable Lyapunov function. Instead, we use a Lyapunov function consisting of the rotated value function and an input-to-state (ISS) Lyapunov function that describes the finite-memory property of added state variables. Furthermore, to ensure a monotonic decrease of this function, we consider the sum of this derived function over the time period of the transient average constraints using an approach from [Ahmadi and Parrilo, 2008]. With this novel Lyapunov function, we proof practical asymptotic stability of the closed loop. The theoretical results are illustrated with the academic example from [Müller et al., 2013,Köhler et al., 2017].

1.4. Outline

This thesis is structured as follows: In Chapter 2, we formulate the control problem and provide a more detailed description of the transient average constraints. In Chapter 3, turnpike properties of single time instants as well as consecutive time points are given. Local continuity of the value function is shown in Chapter 4. Chapter 5 contains performance guarantees in terms of value and trajectory convergence. In Chapter 6, we derive conditions for practical asymptotic stability of the closed loop. Our results are illustrated with a numerical example in Chapter 7. Chapter 8 concludes the thesis.

1.5. Notation

In this thesis, the following notation is used: The positive real numbers are denoted by $\mathbb{R}_{>0} := \{r \in \mathbb{R} | r > 0\}$, and respectively $\mathbb{R}_{\geq 0} = \mathbb{R}_{\geq 0} \cup \{0\}$. The set of integers in the interval $[a,b] \subseteq \mathbb{R}$ is denoted by $\mathbb{I}_{[a,b]}$, and the set of integers greater or equal to a is denoted by $\mathbb{I}_{\geq a}$.

The ball with radius $r > 0$ around a point y is denoted by $\mathcal{B}_r(y) := \{x \in \mathbb{R}^n | \, \|x - y\| \leq r\}$. For $c \in \mathbb{R}$, $\lceil c \rceil$ is defined as the smallest integer greater or equal to c. Equivalently, $\lfloor c \rfloor$ is defined as the largest integer smaller or equal to c.

Comparison functions

The following notation is taken from [Khalil, 2002, Grüne, 2013]. For further information on comparison functions we refer to [Kellett, 2014].

A continuous function $\alpha : [0,a) \to [0,\infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.

A continuous function $\beta : [0,a) \times [0,\infty) \to [0,\infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r,s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $s \to \infty$.

A function $\delta : \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{L}_{\mathbb{N}}$ if it is (not necessarily strictly) decreasing with $\lim_{k\to\infty} \delta(k) = 0$.

2. EMPC with Transient Average Constraints

The goal of the following chapter is to provide a brief overview of the transient average constrained EMPC setting we are dealing with in this thesis. We start by providing the problem setup and introducing the terminal average constraints as well as an extended state for the past values of the auxiliary output which will be used throughout this thesis. Furthermore, the MPC optimization problem and the corresponding optimal trajectories are introduced and the dissipativity property as well as optimal operation at steady-state are defined. We close this chapter by taking a closer look at the transient average constraints and its properties in Section 2.2.

2.1. Problem Setup

We consider discrete-time nonlinear systems of the form

$$x(k+1) = f\left(x(k), u(k)\right), \qquad x(0) = x,\, k \in \mathbb{I}_{\geq 0}, \tag{2.1}$$

with a continuous map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, state $x \in \mathbb{R}^n$ and control values $u \in \mathbb{R}^m$. We assume that the system is subject to state and input constraints and get admissible sets of states $\mathbb{X} \subseteq \mathbb{R}^n$ and control values $\mathbb{U} \subseteq \mathbb{R}^m$ which can be possibly coupled and hence, we write for any $k \in \mathbb{I}_{\geq 0}$

$$(x(k), u(k)) \in \mathbb{Z} \tag{2.2}$$

for some compact set $\mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}$.

In addition to the point-wise in time constraints (2.2), the system is subject to average constraints. These are expressed in terms of an auxiliary output variable

$$y = h(x, u) \in \mathbb{R}^p.$$

Since we want to consider transient average constraints, we require that for some given time period $T \in \mathbb{I}_{\geq 1}$ and for all $k \geq 0$

$$\sum_{j=k}^{k+T-1} \frac{h\left(x(j), u(j)\right)}{T} \in \mathbb{Y} \tag{2.3}$$

is satisfied. In case that $T = 1$ holds, we recover standard point-wise in time constraints which can be incorporated into the set (2.2). In the following, we assume without loss of generality (w. l. o. g.) for the set $\mathbb{Y}$ that

$$\mathbb{Y} := \{y \in \mathbb{R}^p \mid y \leq 0\} = \mathbb{R}^p_{\leq 0} \tag{2.4}$$

which is not restrictive since the output map $h(x,u)$ can be some general nonlinear function.

For a given control sequence with prediction horizon $N \in \mathbb{N}$

$$u_N = (u(0), \ldots, u(N-1)) \in \mathbb{U}^N$$

(or $u = (u(0), u(1), \ldots) \in \mathbb{U}^\infty$, respectively), we denote the solution of (2.1) by $x_{u_N}(k,x)$ where $x = x_{u_N}(0,x) \in \mathbb{X}$ is the initial value at which the input sequence starts. Furthermore, system (2.1) has a continuous stage cost $\ell : \mathbb{Z} \to \mathbb{R}$ which is assumed to be bounded from below on $\mathbb{Z}$ but does not need to be positive definite. The goal is to minimize this stage cost $\ell(x,u)$ over the prediction horizon N subject to the point-wise in time constraints (2.2) and the transient average constraints (2.3). To this end, we formulate the MPC optimization problem. Given an initial state x, the open-loop cost of a control sequence $u(\cdot) \in \mathbb{U}^N$ is defined as

$$J_N(x,u) := \sum_{k=0}^{N-1} \ell\left(x_u(k,x), u(k)\right). \tag{2.5}$$

Since feasibility of input sequences for transient average constrained EMPC as well as the solution of the optimal control problem depend on the initial condition x and moreover, on all $T-1$ past values of the auxiliary output y, we introduce a variable H corresponding to the previous trajectory. It can be seen as an additional state of system (2.1) and a finite memory storing $T-1$ past values of the auxiliary output. It reads

$$H(k) := \left[h(x(k-T+1), u(k-T+1)), \quad \ldots, \quad h(x(k-1), u(k-1))\right]. \tag{2.6}$$

with $H \in \mathbb{R}^{p \times (T-1)}$ and we write $H_j(k)$ for the j-th column of $H(k)$ which is equal to $h(x(k-T+j), u(k-T+j))$ where $j \in \mathbb{I}_{[1,T-1]}$. As described later more precisely (cf. Ch. 6), we can formulate similar to the dynamics for the state x in (2.1) a system for the state H which reads

$$H(k+1) = f_H(H(k), h(x(k), u(k))). \tag{2.7}$$

Analogous to the initial value of the state, we denote H as the initial condition and for the set of all feasible H

$$\mathbb{H} := \left\{ H \in \mathbb{R}^{p\times(T-1)} : \underline{h} \leq H_j \leq \bar{h}, j \in \mathbb{I}_{[1,T-1]} \right\}, \tag{2.8}$$

with $\underline{h}_i := \inf_{(x\times u)\in\mathbb{Z}} h_i(x,u)$ and $\bar{h}_i := \sup_{(x\times u)\in\mathbb{Z}} h_i(x,u)$ for $i \in \mathbb{I}_{[1,p]}$. For further information on the storage H and transient average constraints we refer to Section 2.2.

Given a state $(x,H) \in \mathbb{X} \times \mathbb{H}$, the set of admissible control sequences is denoted by $\mathbb{U}^N(x,H)$,[1] which is given by the following constraints:

$$(x_u(k,x),u(k)) \in \mathbb{Z}, \qquad k \in \mathbb{I}_{[0,N-1]} \tag{2.9a}$$

$$\sum_{i=j}^{T-1} H_i + \sum_{k=0}^{j-1} h(x_u(k,x),u(k)) \leq 0, \qquad j \in \mathbb{I}_{[1,T-1]} \tag{2.9b}$$

$$\sum_{k=i}^{i+T-1} h(x_u(k,x),u(k)) \leq 0, \qquad i \in \mathbb{I}_{[0,N-T]}. \tag{2.9c}$$

As previously mentioned, it is the overall goal of the MPC optimization problem to minimize the cost functional (2.5) with the input sequence $u(\cdot) := \{u(0), \ldots, u(N-1)\}$ (over the prediction horizon N w. r. t. the constraints) as the minimization variable. This yields the optimization problem

$$J_N^*(x,H) := \min_{u\in\mathbb{U}^N(x,H)} J_N(x,u) = \min_{u\in\mathbb{U}^N(x,H)} \sum_{k=0}^{N-1} \ell\left(x_u(k,x), u(k)\right). \tag{2.10}$$

where $J_N^*(x,H)$ denotes the corresponding value function. Furthermore, we write $u^*_{N,x,H}$ for the optimal open-loop input sequence for initial conditions (x,H), i. e., $u^*_{N,x,H}$ denotes the unique minimizer of $J_N(x,u)$. If the minimizer is not unique, we just assign a unique selection map to select one.

2.1.1. Closed Loop

The optimal control problem (2.10) is repeatedly solved in each sampling instant and hence, can be used to implicitly define a feedback law. Given a prediction horizon $N \in \mathbb{N}$, at each time k we accomplish the following steps:

1. Measure the current state $x = x(k)$ and update $H = H(k)$ using (2.7).

[1]For simplicity $\mathbb{U}^\infty(x,H) \neq \emptyset$ is assumed for all $x \in \mathbb{X}$ and all $H \in \mathbb{H}$.

2. Solve the optimization problem (2.10) and denote the resulting optimal control sequence by $u^*_{N,x,H}$.
3. Apply the first element of the optimal control sequence, i. e., $u(k) = u^*_{N,x,H}(0) =: \mu_N(x,H)$, where μ_N denotes the corresponding feedback law.

It follows a closed-loop system which reads

$$\begin{aligned} x(k+1) &= f(x(k),\mu_N(x(k),H(k))), \\ H(k+1) &= f_H\left(H(k),h(x(k),\mu_N(x(k),H(k)))\right). \end{aligned} \tag{2.11}$$

By recursively applying the MPC feedback law μ_N, we write for the resulting extended state and the feedback for an initial condition (x,H) after k steps

$$x_{\mu_N}(k,x,H), \qquad H^{\mathrm{cl}}(k,x,H), \qquad \mu_N(k,x,H). \tag{2.12}$$

Using this, we introduce an analogous closed-loop notation for the auxiliary output $h(x,u)$ which reads as follows

$$h_{\mu_N}(k,x,H) := h\left(x_{\mu_N}(k,x,H),\ \mu_N(k,x,H)\right). \tag{2.13}$$

Now, similar to the open-loop cost (2.5), we define for any $K \in \mathbb{N}$ the cost of the closed-loop solution by

$$J_K^{\mathrm{cl}}(x,H) := \sum_{k=0}^{K-1} \ell(x_{\mu_N}(k,x,H), \mu_N(k,x,H)). \tag{2.14}$$

2.1.2. Definitions

As it is standard in systems and control theory, we use comparison functions which were introduced in Section 1.5. Additionally to the class $\mathcal{K}$, $\mathcal{KL}$ and $\mathcal{L}_{\mathbb{N}}$ functions, we introduce another class in the following.

Definition 1. [Grüne, 2013, Def. 5.4] By $\mathcal{KLS}$ we denote the class of summable $\mathcal{KL}$ functions which sum up to a $\mathcal{K}$ function, i. e., the class of functions $\beta \in \mathcal{KL}$ for which $\sum_{k=0}^{\infty} \beta(r,k)$ is finite for all $r \geq 0$ and for which $\gamma_\beta : \mathbb{R}_{\geq 0} \to \mathbb{R}$ given by $\gamma_\beta(r) := \sum_{k=0}^{\infty} \beta(r,k)$ satisfies $\gamma_\beta \in \mathcal{K}$.

A fundamental property for our considerations is the property of dissipativity which was introduced in [Willems, 1972] for continuous systems and [Byrnes and Wei Lin, 1994] for discrete-time systems.

Definition 2. [Müller et al., 2014a, Def. 1] The system (2.1) is *dissipative* on $\mathbb{Z}$ with supply rate $s : \mathbb{Z} \to \mathbb{R}$ if there exists a bounded storage function $\lambda : \mathbb{X} \to \mathbb{R}$ such that the following inequality is satisfied for all $(x,u) \in \mathbb{Z}$:

$$\lambda\left(f(x,u)\right) - \lambda(x) \leq s(x,u). \tag{2.15}$$

If, in addition, there exists a function $\rho \in \mathcal{K}_\infty$ such that

$$\lambda\left(f(x,u)\right) - \lambda(x) \leq -\rho(\|(x - x_s, u - u_s)\|) + s(x,u) \tag{2.16}$$

for all $(x,u) \in \mathbb{Z}$, then the system (2.1) is *strictly dissipative* on $\mathbb{Z}$.

Remark 1. As it is discussed in [Angeli et al., 2012], the description of dissipativity in Definition 2 is equivalent to the existence of a steady-state and a function $\lambda(\cdot)$ which is bounded on $\mathbb{Z}$ such that it holds

$$\min_{(x,u)\in\mathbb{Z}} s(x,u) + \lambda(x) - \lambda(f(x,u)) \geq 0. \tag{2.17}$$

Moreover, if (with $\rho \in \mathcal{K}_\infty$) it holds $\min_{(x,u)\in\mathbb{Z}} s(x,u) + \lambda(x) - \lambda(f(x,u)) \geq \rho(\|(x - x_s, u - u_s)\|)$ for all $(x,u) \in \mathbb{Z}$, we have strict dissipativity.

We define the optimal steady-state pair (x_s,u_s) for transient average constrained EMPC as

$$\ell(x_s,u_s) = \min\{\ell(x,u) \mid (x,u) \in \mathbb{Z},\, h(x,u) \leq 0,\; x = f(x,u)\}. \tag{2.18}$$

Furthermore, optimal operation at steady-state is defined as follows which is a modification of [Angeli et al., 2012, Müller et al., 2015].

Definition 3. For the optimal steady-state (2.18), we say that system (2.1) is *optimally operated at steady-state* w. r. t. the stage cost $\ell(x,u)$ and constraints (point-wise (2.2) and transient average (2.3)), if for each initial condition $(x,H) \in \mathbb{X} \times \mathbb{H}$ and any input $u \in \mathbb{U}^\infty(x,H)$, it holds

$$\liminf_{\tau\to\infty} \frac{\sum_{k=0}^{\tau-1} \ell(x_u(k,x),u(k))}{\tau} \geq \ell(x_s,u_s). \tag{2.19}$$

Furthermore, if at least one of the conditions below holds for all $u \in \mathbb{U}^\infty(x,H)$

$$\liminf_{\tau\to\infty} \frac{\sum_{k=0}^{\tau-1} \ell(x_u(k,x),u(k))}{\tau} > \ell(x_s,u_s), \tag{2.20a}$$

$$\liminf_{k\to\infty} |x_u(k,x) - x_s| = 0, \tag{2.20b}$$

then the system is *suboptimally operated off steady-state*.

We note that dissipativity (strict dissipativity) implies optimal operation at the steady-state (2.18) (suboptimal operation off steady-state) [Angeli et al., 2012, Müller et al., 2015] (cf. Prop. 1). Furthermore, we remark that there is the stronger property of *uniformly suboptimally operated off steady-state* [Müller, 2014].

2.2. Further Investigation of Transient Average Constraints

Transient average constraints (2.3) show different interesting properties and hence, it stands to reasons to further investigate them as well as the introduced memory H (2.6). We start by remarking that the satisfaction of transient average constraints imply that asymptotic average constraints are satisfied for the same auxiliary output.

Remark 2. As shown in [Müller et al., 2014a], satisfaction of the transient average constraints (2.3) with some $T \geq 1$ and $\mathbb{Y}$ given by (2.4) implies that the asymptotic average constraints are satisfied. Due to the continuity-assumption of $h(x,u)$ and since $\mathbb{Z}$ is compact (Ass. 1), it holds

$$\limsup_{\tau\to\infty} \sum_{k=0}^{\tau} \frac{h(x(k),u(k))}{\tau+1} \leq 0, \tag{2.21}$$

i. e., the asymptotic average constraints are satisfied.

Moreover, we note that satisfaction of the transient average constraints (2.3) with some time period $T \geq 1$ implies that also transient average constraints with any time period $\tilde{T} = jT$ are satisfied for all $j \in \mathbb{I}_{\geq 1}$. In the following, we use this property in order to bound the transient average constraints over an arbitrary prediction horizon $N \in \mathbb{N}$ using the past values of the auxiliary output which are stored in H.

Bound on Auxiliary Output over the Prediction Horizon

Due to the arguments above, we can bound the transient average constraints for an arbitrary prediction horizon N as follows and as illustrated in Figure 2.1. In addition to the time steps included in the predicted horizon ($k \in \mathbb{I}_{[0,N-1]}$), we consider[2] $k_{T,N}$ previous time steps such that the amount

[2] The first argument in the index represents the time period T and the second argument N in the index represents the horizon this value is corresponding to.

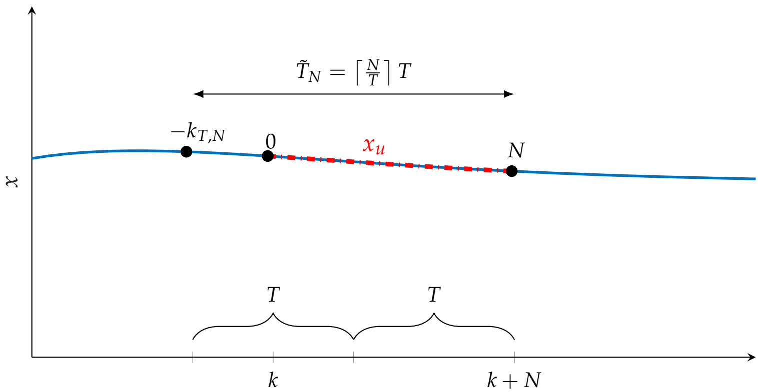


Figure 2.1.: Approach to develop a upper bound on the transient average constraints over the prediction horizon N.

of all time steps over both intervals is equal to $\tilde{T} = jT$ for some $j \in \mathbb{I}_{\geq 1}$. We can express this by

$$\tilde{T}_N := \left\lceil \frac{N}{T} \right\rceil T \quad \text{and} \quad k_{T,N} := \tilde{T}_N - N \tag{2.22}$$

which yields $k_{T,N} \in \mathbb{I}_{[0,T-1]}$. Now, using that the satisfaction of the transient average constraints implies that they are satisfied over some time interval $\tilde{T} = jT$ with $j \in \mathbb{I}_{\geq 1}$, we get (we recall that this inequality has to hold element-wise)

$$\sum_{i=1}^{k_{T,N}} H_{T-i} + \sum_{k=0}^{N-1} h(x_u(k,x),u(k)) \leq 0.$$

Reformulating this yields

$$\sum_{k=0}^{N-1} h(x_u(k,x),u(k)) \leq - \sum_{i=1}^{k_{T,N}} H_{T-i} =: Y_{T,N} \tag{2.23}$$

which represents an upper bound on the transient average constraints (2.3) over the prediction horizon. Again, we can interpret $Y_{T,N} \in \mathbb{R}^p$ as some storage of the past values of the auxiliary output y which either allows us, in case $Y_{i,T,N}(k) > 0$ (where $i \in \mathbb{I}_{[1,p]}$ represents the i-th row of the

p-dimensional vector), to spend time over the prediction horizon N in a region where $\sum_{k=0}^{N-1} h_i(x_u(k,x),u(k)) > 0$. Or in case $Y_{i,T,N}(k) < 0$, it tightens the auxiliary output constraints over the horizon N, i. e., due to the previous trajectory (which has spent time in a region where $\sum_{i=1}^{k_{T,N}} H_{T-i} > 0$) the upcoming trajectory has to fulfill even stronger transient average constraints over the horizon N. Note that this implication is *not* an if and only if relation. For instance, there are possibilities such that (2.23) holds but the transient average constraints (2.3) are not satisfied. Reverse, it is true that the satisfaction of the transient average constraints (2.3) implies (2.23).

Additionally, if we consider future values of the storage, we write for an input sequence $u \in \mathbb{U}^N$ starting in x

$$H(x,u) := \left[h(x,u(0))\,, \quad \dots \quad , h(x_u(T-2),u(T-2))\right].$$

2.2.1. Norm-like Property of the Storage H

In order to compare different past values of the transient average constraints, we introduce a norm-like property of the storage H. Usual norms are not reasonable since they are a measurement for the absolute values of the entries of a matrix, i. e., they do not take the sign of the entries into account. However, for the case of H it is vital to consider the sign of the entries; just entries $H_{i,j} > 0$ should contribute to our measurement. Therefore, it follows the key feature that our measurement is zero if all entries of H are less or equal zero.

Now, the goal is to introduce a variable which is describing the "worst" column h of H, in the sense of not satisfying $h \leq 0$. Hence, we project the entries to the positive ortant and then sum them. We write

$$\hat{H}_k := \sum_{i=1}^{p} \max\{H_{i,k},0\} \in \mathbb{R}_{\geq 0}.$$

for all $k \in \mathbb{I}_{[1,T-1]}$ and identify which of these sums is the biggest in H, i. e., which time step (column) is the "worst". We denote this by

$$\upharpoonleft H \upharpoonright := \max_{k \in \mathbb{I}_{[1,T-1]}} \{\hat{H}_k\} \in \mathbb{R}_{\geq 0}. \tag{2.24}$$

Hence, the norm-like measurement $\upharpoonleft H \upharpoonright$ provides information about the worst time step among the $T-1$ past in the sense of satisfying the transient average constraints. Therefore, it allows us to compare previous trajectories.

For instance, $\rceil H \lceil \gg 0$ can be an indicator that a feasible subsequent trajectory needs to pass a region where it holds $h(x,u) \ll 0$.

Remark 3. In case that $\rceil H \lceil = 0$ holds, any open-loop input trajectory u that satisfies the transient average constraints point-wise in time, i. e., $h(x_u(k,x),u(k)) \leq 0 \; \forall k \in \mathbb{I}_{[0,N-1]}$, is a feasible solution $u \in \mathbb{U}^N(x,H)$.

Furthermore, if it holds $\rceil H \lceil = 0$, all $T-1$ columns of H satisfy $H_k \leq 0$ and hence the corresponding trajectory can always be appended to the beginning or end of an exisiting trajectory for which the first or last $T-1$ auxiliary output values satisfy $h \leq 0$. We are aware of the fact that this is not the ultimate measure to identify which H is crucial in order to combine two trajectories, for example. We note that in Appendix A additional conditions are given for which we can compare different trajectories. However, (2.24) represents one possible measure expressing how tough it is to connect an existing trajectory described by H with another (candidate) trajectory.

We close this chapter by providing a special case consideration for the storage H and furthermore, give an overview of the different descriptions of the transient average constraints in Table 2.1.

Remark 4. We note that for the special case $T = 1$, i. e., the transient average constraints degenerate to point-wise in time constraints, the matrix H has no entries and hence, is neglectable. Thus, it follows $\rceil H \lceil = 0$.

Table 2.1.: Overview of representations of the auxiliary output $h(x,u)$.

Variable	Context	Dimension	Description
$h(x,u)$	-	$\mathbb{R}^p$	Auxiliary output variable
H	(2.6)	$\mathbb{R}^{p \times (T-1)}$	Storage of the past $T-1$ auxiliary output values
$\rceil H \lceil$	(2.24)	$\mathbb{R}$	Expresses and quantifies the "worst" entry h of H

3. The Turnpike Phenomenon

In this chapter, we deal with the so-called turnpike property of optimal control problems which was introduced in [Dorfman et al., 1987]. It describes the phenomenon that the optimal solution approaches the neighborhood of the optimal steady-state, stays within this neighborhood and then, might leaves this neighborhood again. As analyzed for discrete-time and continuous-time optimal control problems [Grüne and Müller, 2016, Faulwasser et al., 2017a], there is a relation between the turnpike property and dissipativity which again is connected to optimal operation at steady-state. We start this chapter by introducing dissipativity and continuity conditions for the transient average constrained EMPC scheme, before we move on to the investigation of the turnpike property in Section 3.2. There, we start by introducing a modified turnpike property of [Grüne, 2013] in Section 3.2.1. In Section 3.2.2, this property is extended such that we obtain a turnpike phenomenon for T consecutive time instants which can also be interpreted as a turnpike property of the extended state (x,H).

3.1. Dissipativity and Optimal Operation at Steady-State

For reasons of clarity, we start by summarizing some assumptions of the transient averaged constrained EMPC scheme of Chapter 2 in the following.

Assumption 1. The constraint set $\mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}$ is compact and the maps $f : \mathbb{Z} \to \mathbb{X}$ and $\ell : \mathbb{Z} \to \mathbb{R}$ are continuous, i. e., there exist $\alpha_f, \alpha_l \in \mathcal{K}_\infty$ such that

$$\begin{aligned} \|f(x_1,u_1) - f(x_2,u_2)\| &\leq \alpha_f(\|(x_1 - x_2, u_1 - u_2)\|), \\ |\ell(x_1,u_1) - \ell(x_2,u_2)| &\leq \alpha_l(\|(x_1 - x_2, u_1 - u_2)\|) \end{aligned}$$

hold for all $(x_1,u_1) \in \mathbb{Z}$ and $(x_2,u_2) \in \mathbb{Z}$. Furthermore, $\ell(x,u)$ is bounded from below on $\mathbb{Z}$ and the output map $h : \mathbb{Z} \to \mathbb{R}^p$ is Lipschitz continuous where we suppose w. l. o. g. that the Lipschitz constant satisfies $L_h > 0$.

The introduced Lipschitz continuity of $h(x,u)$ implies that for $u_1 \in \mathbb{U}^N(x_1,H^1)$ and $u_2 \in \mathbb{U}^N(x_2,H^2)$ it holds for all $k \in \mathbb{I}_{[0,N-1]}$ with $L_h \in \mathbb{R}_{>0}$

$$\begin{aligned} &\|h(x_{u_1}(k,x_1),u_1(k)) - h(x_{u_2}(k,x_2),u_2(k))\| \\ \leq &L_h \left\|(x_{u_1}(k,x_1) - x_{u_2}(k,x_2),\, u_1(k) - u_2(k))\right\|. \end{aligned} \tag{3.1}$$

Remark 5. We highlight that the results of this thesis can also be extended by using the weaker concept of continuity of the auxiliary output $h(x,u)$ instead of Lipschitz continuity.

As in [Köhler et al., 2017, Faulwasser et al., 2018, Faulwasser et al., 2017b], we require strict dissipativity in the state x *and* the input u in order to easily derive turnpike properties for the auxiliary output $h(x,u)$ (cf. Corollary 1, 2 or 3).

Assumption 2. There exist a bounded storage function $\lambda : \mathbb{X} \to \mathbb{R}$, a multiplier $\bar{\lambda} \in \mathbb{R}^p_{\geq 0}$ and a function $\rho \in \mathcal{K}_\infty$ such that for all $(x,u) \in \mathbb{Z}$ system (2.1) is strictly dissipative on $\mathbb{Z}$ with respect to the supply rate

$$s(x,u) := \ell(x,u) - \ell(x_s,u_s) + \bar{\lambda}^\top h(x,u). \tag{3.2}$$

Moreover, $\lambda(x)$ is continuous, i. e., there exists a function $\alpha_\lambda \in \mathcal{K}_\infty$ with $|\lambda(x_1) - \lambda(x_2)| \leq \alpha_\lambda(\|x_1 - x_2\|)$ and w. l. o. g. $\lambda(x_s) = 0$.

Recalling the definition of dissipativity (Def. 2), Assumption 4 implies that the following holds for all $(x,u) \in \mathbb{Z}$

$$\begin{aligned} \tilde{\ell}(x,u) :=& \ell(x,u) - \ell(x_s,u_s) + \lambda(x) - \lambda(f(x,u)) + \bar{\lambda}^\top h(x,u) \\ \geq& \rho(\|(x - x_s, u - u_s)\|), \end{aligned}$$

where $\tilde{\ell}$ denotes the positive definite stage cost (which we also consider later in this section). In the following proposition, which is a modification of [Angeli et al., 2012, Prop. 6.4], we show that dissipativity implies optimal operation at steady-state (Def. 3).

Proposition 1. *Assume that the system (2.1) is dissipative (strictly dissipative) w. r. t. the supply rate (3.2). Then the system is optimally operated at steady-state (suboptimally operated off steady-state).*

Proof. We get by simple manipulations for any $u \in \mathbb{U}^\infty(x,H)$

$$\begin{aligned}
0 &= \liminf_{\tau\to\infty} \frac{\lambda(x_u(\tau,x)) - \lambda(x_u(0,x))}{\tau} \\
&= \liminf_{\tau\to\infty} \frac{\sum_{k=0}^{\tau-1} \lambda(x_u(k+1,x)) - \lambda(x_u(k,x))}{\tau} \\
&\overset{\text{Def. 2}}{\leq} \liminf_{\tau\to\infty} \frac{\sum_{k=0}^{\tau-1} s(x_u(k,x),u(k))}{\tau} \\
&\overset{(3.2)}{=} \liminf_{\tau\to\infty} \frac{\sum_{k=0}^{\tau-1} \ell(x_u(k,x),u(k)) - \ell(x_s,u_s) + \bar{\lambda}^\top h(x_u(k,x),u(k))}{\tau} \\
&\leq \liminf_{\tau\to\infty} \frac{\sum_{k=0}^{\tau-1} \ell(x_u(k,x),u(k))}{\tau} - \ell(x_s,u_s) \\
&\quad + \bar{\lambda}^\top \limsup_{\tau\to\infty} \frac{\sum_{k=0}^{\tau-1} h(x_u(k,x),u(k))}{\tau} \\
&\overset{(2.21)}{\leq} \liminf_{\tau\to\infty} \frac{\sum_{k=0}^{\tau-1} \ell(x_u(k,x),u(k))}{\tau} - \ell(x_s,u_s)
\end{aligned}$$

and therefore, it holds

$$\ell(x_s,u_s) \leq \liminf_{\tau\to\infty} \frac{\sum_{k=0}^{\tau-1} \ell(x_u(k,x),u(k))}{\tau}$$

which proves the claim of optimal operation at the steady-state (2.19) under dissipativity w. r. t. the supply rate (3.2). Note that the third inequality follows from Remark 2 and non-negativity of the multiplier $\bar{\lambda}$.

In case of strict dissipativity, similar manipulations lead to

$$\begin{aligned}
&\liminf_{\tau\to\infty} \frac{\sum_{k=0}^{\tau-1} \rho(\|(x_u(k,x) - x_s, u(k) - u_s)\|)}{\tau} \\
&\leq \liminf_{\tau\to\infty} \frac{\sum_{k=0}^{\tau-1} \ell(x_u(k,x),u(k))}{\tau} - \ell(x_s,u_s)
\end{aligned} \tag{3.3}$$

for which two cases are possible due to $\rho \in \mathcal{K}_\infty$:

(i) $\liminf_{\tau\to\infty} \sum_{k=0}^{\tau-1} \frac{\ell(x_u(k,x),u(k))}{\tau} > \ell(x_s,u_s)$ which yields (2.20a) or

(ii) $\liminf_{\tau\to\infty} \sum_{k=0}^{\tau-1} \frac{\ell(x_u(k,x),u(k))}{\tau} = \ell(x_s,u_s)$ where we get from (3.3)

$$\liminf_{\tau\to\infty} \frac{\sum_{k=0}^{\tau-1} \rho(\|(x_u(k,x) - x_s, u(k) - u_s)\|)}{\tau} = 0$$

which in turn implies (2.20b) since $\rho \in \mathcal{K}_\infty$, i. e.,

$$\liminf_{k\to\infty} |x(k) - x_s| = 0.$$

This concludes the proof of suboptimal operation off steady-state in case of strict dissipativity. □

Rotated Cost and corresponding Value Function

In order to prove the main property of this chapter, the turnpike behaviour, it is fundamental to use the *rotated cost* and the corresponding rotated value function, where $\lambda(x)$ denotes the storage function and $\bar{\lambda}$ the multiplier from Ass. 2. They read

$$\tilde{\ell}(x,u) := \ell(x,u) - \ell(x_s,u_s) + \lambda(x) - \lambda(f(x,u)) + \bar{\lambda}^\top h(x,u), \tag{3.4a}$$

$$\begin{aligned} \tilde{J}_N(x,u) &:= \sum_{k=0}^{N-1} \tilde{\ell}(x_u(k,x),u(k)) \\ &= J_N(x,u) - N\ell(x_s,u_s) + \lambda(x) - \lambda(x_u(N,x)) \\ &+ \sum_{k=0}^{N-1} \bar{\lambda}^\top h(x_u(k,x),u(k)). \end{aligned} \tag{3.4b}$$

This definition immediatelly implies for the steady-state $\tilde{\ell}(x_s,u_s) = 0$. Analogously to (2.10) we denote the rotated optimization problem

$$\tilde{J}_N^*(x,H) := \inf_{u\in\mathbb{U}^N(x,H)} \sum_{k=0}^{N-1} \tilde{\ell}\,(x_u(k,x), u(k)) \tag{3.5}$$

and write for the corresponding optimal control sequence $\tilde{u}^*_{N,x,H} \in \mathbb{U}^N(x,H)$. Using the equivalent notation of dissipativity in Remark 1, strict dissipativity from Assumption 2 implies that the following holds for all $(x,u) \in \mathbb{Z}$

$$\tilde{\ell}(x,u) \geq \rho(\|(x - x_s,u - u_s)\|) \geq 0. \tag{3.6}$$

3.2. Turnpike Properties of the System

In this section, we will show that the previously mentioned turnpike property is a consequence of strict dissipativity, controllability and continuity assumptions. Moreover, in Theorem 2 we conclude that the optimal solution "passes by" the optimal steady-state until a specific time instant. We highlight that all our results are based on this turnpike behaviour.

Contrary to EMPC without transient average constraints [Grüne, 2013], we need later in the proof of closed-loop properties a stronger turnpike behaviour. In particular, we prove that consecutive points over a certain time period show the turnpike phenomenon such that we can ensure that the transient average constraints are satisfied. Equivalently, this consecutive turnpike property of x and u can be seen as a turnpike phenomenon of the extended state (x,H). Before starting with the first results, we introduce a lower and upper bound for the auxiliary output in combination with the multilpier $\bar{\lambda}$ which reads

$$\underline{\vartheta}_h := \inf_{(x \times u) \in \mathbb{Z}} \bar{\lambda}^\top h(x,u), \tag{3.7a}$$

$$\bar{\vartheta}_h := \sup_{(x \times u) \in \mathbb{Z}} \bar{\lambda}^\top h(x,u). \tag{3.7b}$$

This implies that it holds for all $u \in \mathbb{U}^N(x,H)$ with $(x,H) \in \mathbb{X} \times \mathbb{H}$, $N \geq T$

$$\mathbf{1}^{1\times(T-1)}\underline{\vartheta}_h \leq \bar{\lambda}^\top H \leq \mathbf{1}^{1\times(T-1)}\bar{\vartheta}_h, \tag{3.8}$$

where the inequalities should be interpreted element-wise and $\mathbf{1}^{x\times y}$ is a one-matrix of dimension $\mathbb{R}^{x\times y}$. Note that (3.8) is a weaker condition for H than $H \in \mathbb{H}$ since entries of the multiplier $\bar{\lambda}$ can be zero. We remark that $\underline{\vartheta}_h \leq 0$, since $\bar{\lambda} \in \mathbb{R}^p_{\geq 0}$ and $h(x_s,u_s) \leq 0$ with $(x_s,u_s) \in \mathbb{Z}$.

3.2.1. General Turnpike Propery

The following turnpike property (which is a modification of [Grüne, 2013, Thm. 5.3]) describes the phenomenon that open-loop trajectories satisfying a condition, "most of the time" stay close to the optimal steady-state [Grüne and Müller, 2016]. To this end, we denote

$$\mathcal{P}^\epsilon(u,x) := \{k \in \mathbb{I}_{[0,N-1]} \,|\, \|(x_u(k,x) - x_s,\, u(k) - u_s)\| \leq \epsilon\}. \tag{3.9}$$

as the set of time instants for which the state trajectory x_u and input trajectory u with initial condition x are in the steady-state neighborhood $\mathcal{B}_\epsilon(x_s,u_s)$ and write for the amount of time instants $Q^\epsilon(u,x) = \#\mathcal{P}^\epsilon(u,x) = \mathrm{card}(\mathcal{P}^\epsilon(u,x))$.

Theorem 1. *Let Assumptions 1 and 2 hold. Then for each $(x,H) \in \mathbb{X} \times \mathbb{H}$, each $\epsilon, \delta > 0$, each control sequence $u \in \mathbb{U}^N(x,H)$ satisfying*

$$J_N(x,u) \leq N\ell(x_s,u_s) + \delta, \tag{3.10}$$

the following inequality holds with $C := 2\sup_{x\in\mathbb{X}} |\lambda(x)|$

$$Q^\epsilon(u,x) \geq N - \frac{\delta + C - \bar{\lambda}^\top \sum_{i=1}^{k_{T,N}} H_{T-i}}{\rho(\epsilon)} \geq N - \frac{\delta + C - k_{T,N}\underline{\vartheta}_h}{\rho(\epsilon)}, \tag{3.11}$$

with $\rho \in \mathcal{K}_\infty$ from Ass. 2 and $k_{T,N}$ according to (2.22).

Proof. For each $(x,H) \in \mathbb{X} \times \mathbb{H}$ and $u \in \mathbb{U}^N(x,H)$ the following inequality holds using the definition of C and the bound (3.10):

$$\begin{aligned}\tilde{J}_N(x,u) &\overset{(3.4b)}{\leq} J_N(x,u) - N\ell(x_s,u_s) + C + \sum_{k=0}^{N-1} \bar{\lambda}^\top h(x_u(k,x),u(k)) \\ &\overset{(3.10)}{\leq} \delta + C + \sum_{k=0}^{N-1} \bar{\lambda}^\top h(x_u(k,x),u(k)) \overset{(2.23)}{\leq} \delta + C - \bar{\lambda}^\top \sum_{i=1}^{k_{T,N}} H_{T-i}.\end{aligned} \tag{3.12}$$

Now, assume for contradiction that it holds

$$Q^\epsilon(u,x) < N - \frac{\delta + C - \bar{\lambda}^\top \sum_{i=1}^{k_{T,N}} H_{T-i}}{\rho(\epsilon)}.$$

This means that there exists a set $\mathcal{N} \subseteq \mathbb{I}_{[0,N-1]}$ of

$$\#\mathcal{N} = N - Q^\epsilon(u,x) > \frac{\delta + C - \bar{\lambda}^\top \sum_{i=1}^{k_{T,N}} H_{T-i}}{\rho(\epsilon)} \tag{3.13}$$

time instants such that $\|(x_u(k,x) - x_s,\, u(k) - u_s)\| > \epsilon$ holds for all $k \in \mathcal{N}$. Using the strict dissipativity Assumption 2, we know that it holds for these k

$$\tilde{\ell}(x_u(k,x),u(k)) \geq \rho(\epsilon). \tag{3.14}$$

This implies

$$\begin{aligned}\tilde{J}_N(x,u) &= \sum_{k\in\mathcal{N}} \tilde{\ell}(x_u(k,x),u(k)) + \sum_{k\in\{0,\dots,N-1\}\setminus\mathcal{N}} \tilde{\ell}(x_u(k,x),u(k)) \\ &\geq (N - Q^\epsilon(u,x))\rho(\epsilon) \overset{(3.13)}{>} \delta + C - \bar{\lambda}^\top \sum_{i=1}^{k_{T,N}} H_{T-i}\end{aligned}$$

which contradicts (3.12) and thus proves the first inequality of the theorem. The second inequality of (3.11) immediately follows from (3.7) since each column H_k of H represents an auxiliary output h at a specific time instant.

□

Remark 6. For the special case $T = 1$ it holds from (2.22) $k_{T,N} = 0$ and hence, we get

$$Q^{\epsilon}(u,x) \geq N - \frac{\delta + C}{\rho(\epsilon)}$$

which is similar to the results of EMPC without transient average constraints [Grüne, 2013, Thm. 5.3].

Since the multiplier $\bar{\lambda} \in \mathbb{R}^p_{\geq 0}$ is a constant vector we can write

$$\sum_{k=0}^{N-1} \bar{\lambda}^\top h(x_u(k,x),u(k)) = \bar{\lambda}^\top \sum_{k=0}^{N-1} h(x_u(k,x),u(k)),$$

and recall from (2.23) that this does not necessarily has to be less or equal zero. Moreover, if the memory state H satisfies[1] $\bar{\lambda}^\top \sum_{i=1}^{k_{T,N}} H_{T-i} > 0$, the estimate for $Q^{\epsilon}(u,x)$ gets larger than in the "standard" case without transient average constraints (2.3).

Due to the assumed Lipschitz continuity of the auxiliary output $h(x,u)$ and the turnpike property of the state and input, we can derive from Theorem 1 as well a turnpike property for $h(x,u)$ if the state x and the input u show this property. However, note that in general, we can not infer bounds on x or u if the auxiliary output is bounded, i. e., it holds

$$\begin{aligned} \|(x - x_s, u - u_s)\| \leq \epsilon \quad &\Rightarrow \quad \|h(x,u) - h(x_s,u_s)\| \overset{(3.1)}{\leq} L_h\epsilon \\ \|h(x,u) - h(x_s,u_s)\| \leq L_h\epsilon \quad &\not\Rightarrow \quad \|(x - x_s, u - u_s)\| \leq \epsilon. \end{aligned}$$

For the following corollary, we consider with the Lipschitz constant $L_h > 0$ (Ass. 1) the estimates

$$\|h(x_u(k,x),u(k)) - h(x_s,u_s)\| \leq L_h\epsilon, \tag{3.15a}$$

$$\|(x_u(k,x) - x_s,\, u(k) - u_s)\| \leq \epsilon. \tag{3.15b}$$

[1]This means that there exists some $j \in \mathbb{I}_{[1,p]}$ such that $\sum_{i=1}^{k_{T,N}} H_{j,T-i} > 0$.

Corollary 1. *Let Assumptions 1 and 2 hold. Then for each $(x,H) \in \mathbb{X} \times \mathbb{H}$, each $\epsilon, \delta > 0$ and each control sequence $u \in \mathbb{U}^N(x,H)$ satisfying (3.10), the value*

$$Q_h^\epsilon(u,x) := \#\{k \in \mathbb{I}_{[0,N-1]} \,|\, \textit{s. t. } (3.15a) \textit{ and } (3.15b) \textit{ hold}\}$$

satisfies with $C := 2\sup_{x\in\mathbb{X}} |\lambda(x)|$ and $k_{T,N}$ according to (2.22)

$$Q_h^\epsilon(u,x) \geq N - \frac{\delta + C - \bar{\lambda}^\top \sum_{i=1}^{k_{T,N}} H_{T-i}}{\rho(\epsilon)} \geq N - \frac{\delta + C - k_{T,N}\underline{\vartheta}_h}{\rho(\epsilon)}.$$

Proof. Since Assumption 1 as well as 2 hold and the control sequence satisfies (3.10), we can use Theorem 1. From Lipschitz continuity of $h(x,u)$ (Ass. 1) we get that it holds for all $k \in \mathbb{I}_{[0,N-1]}$

$$\|h(x_u(k,x),u(k)) - h(x_s,u_s)\| \overset{(3.1)}{\leq} L_h \|(x_u(k,x) - x_s,\, u(k) - u_s)\|.$$

Hence, we know that for all $k \in \mathbb{I}_{[0,N-1]}$ which are in $Q^\epsilon(u,x)$ (cf. Theorem 1) it also holds

$$\|h(x_u(k,x),u(k)) - h(x_s,u_s)\| \leq L_h(\|(x_u(k,x) - x_s,\, u(k) - u_s)\|) \leq L_h\epsilon.$$

Therefore, it follows $Q_h^\epsilon(u,x) = Q^\epsilon(u,x) \geq N - \frac{\delta+C-\bar{\lambda}^\top \sum_{i=1}^{k_{T,N}} H_{T-i}}{\rho(\epsilon)}$ and we have shown the assertion by the proof of Theorem 1. □

After we have shown the turnpike property for the system with transient average constraints, we introduce an asymptotic controllability condition w. r. t. to the stage cost $\ell(x,u)$ and the rotated stage cost $\tilde{\ell}(x,u)$. In this assumption, which is a modification of [Grüne, 2013, Ass. 5.5], we use class $\mathcal{KLS}$ functions (cf. Def. 1). Contrary to [Grüne, 2013], we additionally consider transient average constraints and therefore, we consider not only the initial value x but also the past $T-1$ time steps via β_2 and a comparison of the storage H with the steady-state storage

$$H^s := H(x_s,u_s(\cdot)) = \begin{bmatrix} h(x_s,u_s), & \dots, & h(x_s,u_s) \end{bmatrix} \in \mathbb{R}_{\leq 0}^{p\times(T-1)}.$$

We note that together with the Assumptions 1 and 2, the subsequent property implies finite time controllability of the system for any initial condition $(x,H) \in \mathbb{X} \times \mathbb{H}$ into an arbitrarily small neighborhood of the steady-state which is later shown as the result of Lemma 6.

Assumption 3. There exist $\beta_1, \beta_2 \in \mathcal{KLS}$ such that for each $(x,H) \in \mathbb{X} \times \mathbb{H}$ and each $N \in \mathbb{N}$ there exists a control sequence $u \in \mathbb{U}^N(x,H)$ such that it holds for all $k \in \mathbb{I}_{[0,N-1]}$

$$\begin{aligned} \ell(x_u(k,x),u(k)) &\leq \ell(x_s,u_s) + \beta_1(\|x - x_s\|,k) + \beta_2(\rceil H - H^s \rceil,k), \\ \tilde{\ell}(x_u(k,x),u(k)) &\leq \tilde{\ell}(x_s,u_s) + \beta_1(\|x - x_s\|,k) + \beta_2(\rceil H - H^s \rceil,k). \end{aligned} \tag{3.16}$$

The additional term β_2 is necessary which we explain in the following. For instance, we can start at the steady-state but the previous trajectory is such that we have to move away from the steady-state in order to satisfy the transient average constraints. Without the term $\beta_2(\rceil H - H^s \rceil,k)$ there would not exist a sequence $u \in \mathbb{U}^N(x,H)$ satisfying (3.16) since $\beta_1(\|x - x_s\|,k) = 0$ and $\tilde{\ell}(x_s,u_s) = \min_{(x,u)\in\mathbb{Z}} \tilde{\ell}(x,u)$. Furthermore, we note from (3.4a) that $\tilde{\ell}(x,u) = \ell(x,u) - \ell(x_s,u_s) + \lambda(x) - \lambda(f(x,u)) + \bar{\lambda}^\top h(x,u)$ and, we know from the Assumptions 1 and 2 that $f(x,u)$, $h(x,u)$ and $\lambda(x)$ are continuous. Hence, it is not too restrictive that there exist the $\mathcal{KLS}$ functions β_1 and β_2 such that the second inequality holds if the first one is satisfied. The second inequality is used later (cf. Corollary 6) to show that the following turnpike results also hold for the rotated optimization problem (3.5). Moreover, since $\mathbb{Z}$ is compact, it follows that $\|x - x_s\|$ as well as $h(x,u)$ are bounded and, hence, that $\rceil H - H^s \rceil$ is bounded. Using the definition of class $\mathcal{KLS}$ functions, we define

$$\delta := \max_{x\in\mathbb{X}} \gamma_{\beta_1}(\|x - x_s\|) + \max_{H\in\mathbb{H}} \gamma_{\beta_2}(\rceil H - H^s \rceil). \tag{3.17}$$

which implies by optimality (cf. Proof of Thm. 2) that the value function satisfies $J_N^*(x,H) \leq N\ell(x_s,u_s) + \delta$ for any $N \in \mathbb{N}$ and any $(x,H) \in \mathbb{X} \times \mathbb{H}$.

The following theorem (which is a modification of [Grüne, 2013, Thm. 5.6]) shows that strict dissipativity and the asymptotic controllability property imply that the optimal open-loop solution is passing by near the steady-state. Moreover, the distance between the optimal trajectory at some time instant k_s and the steady-state is upper bounded by a class $\mathcal{L}_\mathbb{N}$ function, i. e., the distance is decreasing for an increasing prediction horizon N. Hence, there exists a time instant k_s such that

$$\lim_{N\to\infty} \left\| (x_{u^*_{N,x,H}}(k_s,x) - x_s,\, u^*_{N,x,H}(k_s) - u_s) \right\| = 0.$$

Theorem 2. *Let the Assumptions 1-3 hold. For each $N_1 \in \mathbb{N}$, each $(x,H) \in \mathbb{X} \times \mathbb{H}$ and each $N \geq N_1$, there exist $\sigma_1 \in \mathcal{L}_\mathbb{N}$ and $k_s \in \mathbb{I}_{[0,N-N_1]}$ such that the* *optimal trajectory* $u^*_{N,x,H} \in \mathbb{U}^N(x,H)$ *satisfies*

$$\left\| (x_{u^*_{N,x,H}}(k_s,x) - x_s,\, u^*_{N,x,H}(k_s) - u_s) \right\| \le \sigma_1(N - N_1). \tag{3.18}$$

Proof. Using the control sequence $u \in \mathbb{U}^N(x,H)$ from Assumption 3, we obtain

$$\begin{aligned} J_N(x,u) &= \sum_{k=0}^{N-1} l(x_u(k,x),u(k)) \\ &\overset{\text{Ass.3}}{\le} N\ell(x_s,u_s) + \gamma_{\beta_1}(\|x - x_s\|) + \gamma_{\beta_2}(|H - H^s|) \overset{(3.17)}{\le} N\ell(x_s,u_s) + \delta. \end{aligned}$$

Now, optimality yields $J^*_N(x,H) \le J_N(x,u) \le N\ell(x_s,u_s) + \delta$ and hence, we can use the turnpike property from Theorem 1. As in the proof of Thm. 1, we set $C := 2\sup_{x\in\mathbb{X}} |\lambda(x)| < \infty$ and furthermore,

$$\sigma_1(N - N_1) := \rho^{-1}\left(\frac{\delta + C - k_{T,N}\underline{\vartheta}_h}{N - N_1}\right) \tag{3.19}$$

with $k_{T,N} = \left\lceil \frac{N}{T} \right\rceil T - N$. Since the argument $N - N_1$ is in the denominator of the $\mathcal{K}_\infty$ function ρ, it follows $\sigma_1 \in \mathcal{L}_{\mathbb{N}}$. From Theorem 1, we obtain for this choice of the neighborhood $\sigma_1(N - N_1)$

$$Q^{\sigma_1(N-N_1)}(u^*_{N,x,H},x) \ge N - \frac{\delta + C - k_{T,N}\underline{\vartheta}_h}{\rho(\sigma_1(N - N_1))} = N_1.$$

Therefore, there are at least N_1 time instants k satisfying

$$\left\| (x_{u^*_{N,x,H}}(k,x) - x_s,\, u^*_{N,x,H}(k) - x_s) \right\| \le \sigma_1(N - N_1)$$

and thus, at least one of these k's must satisfy $k \in \mathbb{I}_{[0,N-N_1]}$. The assertion follows if we choose k_s as this k. □

Again, we highlight that for the special case $T = 1$ the resulting neighborhood (3.19) degenerates to $\sigma_1(N - N_1) = \rho^{-1}((\delta + C)/(N - N_1))$ which is the same result as given in [Grüne, 2013, Proof of Thm. 5.6], i. e., for the EMPC settings without transient average constraints. Analogous to Corollary 1, Lipschitz continuity of $h(x,u)$ immediately implies that the results of Theorem 2 hold in a similar fashion for the auxiliary output.

Corollary 2. *Let the Assumptions 1-3 hold. For each $N_1 \in \mathbb{N}$, each $(x,H) \in \mathbb{X} \times \mathbb{H}$ and each $N \geq N_1$, there exist $\sigma_1 \in \mathcal{L}_{\mathbb{N}}$ and $k_s \in \mathbb{I}_{[0,N-N_1]}$ such that the optimal trajectory $u^*_{N,x,H} \in \mathbb{U}^N(x,H)$ satisfies (3.18) as well as*

$$\left\| h(x_{u^*_{N,x,H}}(k_s,x), u^*_{N,x,H}(k_s)) - h(x_s,u_s) \right\| \leq L_h \sigma_1(N - N_1)$$

with the Lipschitz constant $L_h > 0$.

Proof. Since the assumptions in Theorem 2 are satisfied, we know that there exists some $k_s \in \mathbb{I}_{[0,N-N_1]}$ such that (3.18) holds. As in the proof of Corollary 1, we use Lipschitz continuity of $h(x,u)$ which yields additionally to (3.18)

$$\begin{aligned} \|h(x_u(k_s,x),u(k_s)) - h(x_s,u_s)\| &\overset{\text{Lip.}}{\leq} L_h(\|x_u(k_s,x) - x_s\| + \|u(k_s) - u_s\|) \\ &\overset{(3.18)}{\leq} L_h \sigma_1(N - N_1) \end{aligned}$$

for the beforementioned k_s. This means that the same function $\sigma_1 \in \mathcal{L}_{\mathbb{N}}$ from (3.19) implies the assertion. □

3.2.2. Turnpike Property of consecutive Time Instants

After we have shown that the optimal open-loop solution is passing by arbitrarily close to the steady-state at least at one time instant k_s, we now consider several trajectories at once as it is done in [Grüne, 2013, Sec. 7] or [Köhler et al., 2018, Lem. 2]. In the following, we consider $q \in \mathbb{N}$ trajectories and use the set (3.9) of each trajectory in order to build the intersection

$$\mathcal{P}'^{\epsilon}_{[k_l,k_u]}\left((u_1,x_1),\ldots,(u_q,x_q)\right) := \mathcal{P}^{\epsilon}(u_1,x_1) \cap \cdots \cap \mathcal{P}^{\epsilon}(u_q,x_q) \cap \{k_l,\ldots,k_u\} \tag{3.20}$$

where k_l and k_u are given parameters. This means that this intersection set contains all time instants for which the corresponding trajectories x_{u_i} and u_i with $i \in \mathbb{I}_{[1,q]}$ are in the same neighborhood $\mathcal{B}_\epsilon(x_s,u_s)$ of the steady-state. Moreover, the last set $\{k_l,\ldots,k_u\}$ tightens our intersection such that only time instants $k \in \mathbb{I}_{[k_l,k_u]}$ are admissible, with $0 \leq k_l < k_u \leq \hat{N} - 1$ and the largest prediction horizon of all considered trajectories

$$\hat{N} := \max_{i \in \mathbb{I}_{[1,q]}} N_i. \tag{3.21}$$

A general Consideration of Multiple Trajectories

We start by introducing a property which directly follows from building an intersection of sets. Since this is a general procedure, we provide some further information in Appendix B.

Proposition 2. *For any $\epsilon > 0$, for any trajectories $u_i \in \mathbb{U}^{N_i}(x_i,H^i)$, with $i \in \mathbb{I}_{[1,q]}$ and any $(x_i,H^i) \in \mathbb{X} \times \mathbb{H}$ and $Q^\epsilon(u_i,x_i) \geq M$, the intersection $\mathcal{P}'^{\epsilon}_{[k_l,k_u]}\left((u_1,x_1),\ldots,(u_q,x_q)\right)$ in (3.20) contains that least $m \geq k_u - k_l - q(\hat{N} - M)$ elements, with $\hat{N}$ according to (3.21).*

Proof. This property directly follows by combinatorially building the intersection of sets which is shown in Appendix B (cf. especially (B.1)). Using the notation of the Appendix, we have $\alpha = q$ and $a_i = M$ for all $i \in \mathbb{I}_{[1,q]}$ which yields the assertion. □

Now, we want to focus on the question how we can ensure that all the considered trajectories satisfy the conditions of Proposition 2 such that we can ensure that at least a specific amount of time instants is contained in the intersection (3.20). Therefore, we construct a neighborhood ϵ of the steady-state such that this is the case. To this end, we define the maximal difference between the largest prediction horizon $\hat{N}$ from (3.21) and the smallest one via

$$\Delta_N := \max_{i \in \mathbb{I}_{[1,q]}} \hat{N} - N_i \in \mathbb{I}_{\geq 0}. \tag{3.22}$$

Proposition 3. *Let Ass. 1 and 2 hold. For any trajectories $u_i \in \mathbb{U}^{N_i}(x_i,H^i)$, with $i \in \mathbb{I}_{[1,q]}$, $(x_i,H^i) \in \mathbb{X} \times \mathbb{H}$ and $N_i \in \mathbb{N}$ satisfying $J_{N_i}(x_i,u_i) \leq N_i \ell(x_s,u_s) + \delta$ with δ according to (3.17), as well as for any $k_l \in \mathbb{N}_0$, $k_u \in \mathbb{N}$ with $0 \leq k_l < k_u \leq \hat{N} - 1$ and any $m \in \mathbb{N}$ satisfying*

$$k_u - k_l - m - q\Delta_N > 0, \tag{3.23}$$

there exists a steady-state neighborhood $\epsilon > 0$ according to (3.26) such that the intersection $\mathcal{P}'^{\epsilon}_{[k_l,k_u]}\left((u_1,x_1),\ldots,(u_q,x_q)\right)$ in (3.20) contains at least m elements.

Proof. Since the Ass. 1 and 2 hold as well as the inequality $J_{N_i}(x_i,u_i) \leq N_i \ell(x_s,u_s) + \delta$ for each $i \in \mathbb{I}_{[1,q]}$, we can apply Theorem 1. Hence, we can estimate the amount of time instants in the set (3.9) by

$$\#\mathcal{P}^\epsilon(u_i,x_i) = Q^\epsilon(u_i,x_i) \geq N_i - \frac{\delta + C - k_{T,N_i}\underline{\vartheta}_h}{\rho(\epsilon)} = N_i - \frac{C_i'}{\rho(\epsilon)}. \tag{3.24}$$

with $C_i' := \delta + C - k_{T,N_i}\underline{\vartheta}_h$. We define

$$\hat{C}' := \delta + C - (T-1)\underline{\vartheta}_h \tag{3.25}$$

which implies $\hat{C}' \geq C_i'$ for all $i \in \mathbb{I}_{[1,q]}$. Then, by considering the neighborhood

$$\epsilon := \rho^{-1}\left(\frac{q\hat{C}'}{k_u - k_l - m - q\Delta_N}\right) \tag{3.26}$$

for all trajectories $x_{u_i}(\cdot,x_i), u_i(\cdot)$, we get that for the amount of time instants satisfying (3.9) it holds for all $i \in \mathbb{I}_{[1,q]}$

$$\begin{aligned}
Q^\epsilon =& Q^\epsilon(u_i,x_i) \overset{(3.24)}{\geq} N_i - \frac{C_i'}{\rho(\epsilon)} \\
\overset{(3.26)}{=}& N_i - \frac{C_i'}{q\hat{C}'}(k_u - k_l - m - q\Delta_N) \overset{C_i' \leq \hat{C}'}{\geq} N_i - \frac{1}{q}(k_u - k_l - m - q\Delta_N) \\
=& N_i + \Delta_N - \frac{1}{q}(k_u - k_l - m) \overset{(3.22)}{\geq} \hat{N} - \frac{1}{q}(k_u - k_l - m).
\end{aligned}$$

Setting $M := \hat{N} - \frac{1}{q}(k_u - k_l - m)$ ensure $Q^\epsilon(u_i,x_i) \geq M$ for all $i \in \mathbb{I}_{[1,q]}$. Hence, the choice of the steady-state neighborhood (3.26) guarantees due to Proposition 2 that the intersection (3.20) contains at least $m \geq k_u - k_l - q(\hat{N} - M) = m$ elements. □

Remark 7. By considering $k_u = \hat{N} - 1$, i.e. $\mathcal{P}'^{\epsilon}_{[k_l,\hat{N}-1]}\left((u_1,x_1),\ldots,(u_q,x_q)\right)$, we choose the last interval such that there are entries in $\mathbb{I}_{[k_l,\hat{N}-1]}$ allowed and we get for the neighborhood of Proposition 3

$$\epsilon = \rho^{-1}\left(\frac{q\hat{C}'}{\hat{N} - 1 - k_l - m - q\Delta_N}\right).$$

Now, if we increase all prediction horizons N_i, Δ_N is constant, but $\hat{N}$ is increasing which results in a decreasing "common" steady-state neighborhood ϵ.

Intersection of Optimal Trajectories

So far, we considered intersections of trajectories in general. Now, by using optimal trajectories we state a theorem which ensures that under some technical conditions, the optimal open-loop trajectory has at least T consecutive time instants in a neighborhood around the optimal steady-state. We highlight that this property is crucial for considering transient average constraints since the optimal input trajectory not only depends on the current state x but moreover, also on the $T-1$ past values of the state x and the input u since we need to satisfy the transient average constraints.

Theorem 3. *Let the Assumptions 1-3 hold. There exist N_T, $k_x \in \mathbb{N}$ and $\sigma_T \in \mathcal{L}_{\mathbb{N}}$ such that for any $(x,H) \in \mathbb{X} \times \mathbb{H}$ and any $k'_l \in \mathbb{N}_0$, $k'_u \in \mathbb{N}$ satisfying*

$$0 \le k'_l < k'_u - T^2 \quad \text{as well as} \quad k'_u \le N-1, \tag{3.27}$$

there exists $k_x \in \mathbb{I}_{[k'_l+T-1,k'_u]}$ such that the optimal trajectory $u^*_{N,x,H} \in \mathbb{U}^N(x,H)$ satisfies $\left\| (x_{u^*_{N,x,H}}(k,x) - x_s, u^*_{N,x,H}(k) - u_s) \right\| \le \epsilon$ for all $k \in \mathbb{I}_{[k_x-T+1,k_x]}$ with $\epsilon = \sigma_T(k'_u - N_T)$.

Proof. We divide this proof in three different parts. In the first part we will construct T trajectories using the considered trajectory $x_{u^*_{N,x,H}}$ in order to build an intersection of them. In the second part, we choose based on Proposition 3 a neighborhood ϵ to show that the intersection of all considered trajectories contains at least one time instant ($m = 1$). Finally, in the third part we conclude that this implies the assertion.

Part I: Construction of trajectories

We construct from the given optimal input sequence $u^*_{N,x,H}$ T other optimal trajectories

$$u^*_i \in \mathbb{U}^{N_i}(x_i,H^i) \tag{3.28}$$

where $N_i = N - i$ and $x_i = x_{u^*_{N,x,H}}(i,x)$ for all $i \in \mathbb{I}_{[0,T-1]}$ (H^i see later). This means that $x_{u^*_{N,x,H}} = x_{u^*_0}$. For reasons of clarity, we use in the following $x_{u^*_0}$ instead of $x_{u^*_{N,x,H}}$, x_0 instead of x and H^0 instead of H. We get that the "first" constructed trajectory $x_{u^*_1}$ starts at the first point after u^*_0 has been applied and has a prediction horizon N_1 which is one time instant shorter than N_0. The initial condition of the second trajectory (with $N_2 = N_0 - 2$) $x_{u^*_2}$ is the first point after u^*_1 has been applied which is the same as the second point

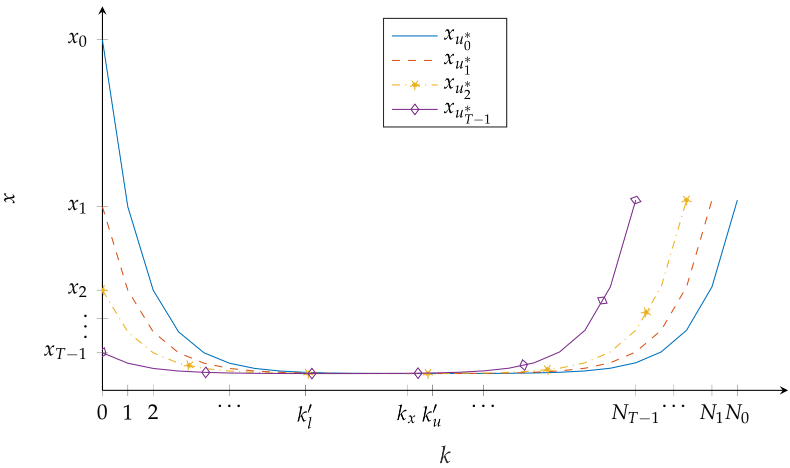


Figure 3.1.: Idea of repeatedly constructing optimal trajectories in order to prove Theorem 3.

of the original trajectory, i. e., $x_{u_0^*}(2,x_0)$. All the other trajectories are built up in the same way. Since we consider optimal trajectories, it follows that the constructed inputs (3.28) result again in optimal trajectories (which are the same as the original $x_{u_0^*}(x_0)$ just shorter and later in time) for which it holds

$$\begin{aligned} x_{u_i^*}(k,x_i) &= x_{u_0^*}(k+i,x_0) \\ u_i^*(k) &= u_0^*(k+i) \end{aligned} \tag{3.29}$$

for all $k \in \mathbb{I}_{[0,N_i-1]} = \mathbb{I}_{[0,N_0-i-1]}$ and any $i \in \mathbb{I}_{[0,T-1]}$. These trajectories and their corresponding turnpike behaviour are shown in Figure 3.1. The matrices which store the auxiliary output's past values are given by (we recall that H_j denotes the j-th column of H for $j \in \mathbb{I}_{[1,T-1]}$)

$$\begin{aligned} H^0 &:= \begin{bmatrix} H_1, & \ldots, & H_{T-1} \end{bmatrix}, \\ H^i &= \Big[H_{i+1}, \ \ldots, \ H_{T-1}, \ h(x_{u_0^*}(0,x_0),u_0^*), \ \ldots, \ h(x_{u_0^*}(i-1,x_0),u_0^*(i-1)) \Big] \end{aligned} \tag{3.30}$$

for $i \in \mathbb{I}_{[0,T-2]}$ and in case $i = T-1$, we get

$$H^{T-1} = H(x_0, u_0^*(0:T-2)).$$

Part II: Choice of the neighborhood
Now, we choose the neighborhood of the steady-state as given in Proposition 3. First, we show that all necessary conditions of this Proposition are fulfilled. It follows from Assumption 3 that it holds for each $i \in \mathbb{I}_{[0,T-1]}$ that the optimal trajectories satisfy $J^*_{N_i}(x_i, H^i) \leq N_i \ell(x_s, u_s) + \delta$. By considering the trajectories $x_{u_0^*}(\cdot, x_0), \ldots, x_{u^*_{T-1}}(\cdot, x_{T-1})$ we get $q = T$ and from (3.22) $\Delta_N = T-1$ for the values of Prop. 3. The first condition of (3.27) ensures that condition (3.23) in Prop. 3 holds with $k_l = k'_l \geq 0$ and $k_u = k'_u - (T-1)$. Furthermore, note that the condition $k'_u \leq N-1$ immediately implies that it also hold $k_u \leq N-1$. Hence, all conditions of Proposition 3 are satisfied.

As before, we choose $\hat{C}'$ from (3.25) which yields $\hat{C}' \geq C'_i = \delta + C - k_{T,N_i} \underline{\vartheta}_h$ with $i \in \mathbb{I}_{[0,T-1]}$ and $C := 2 \sup_{x \in \mathbb{X}} |\lambda(x)|$. Now, we apply Proposition 3 and choose the steady-state neighborhood according to (3.26) as

$$\epsilon = \rho^{-1} \left(\frac{T\hat{C}'}{(k'_u - (T-1)) - k'_l - 1 - T(T-1)} \right) = \rho^{-1} \left(\frac{T\hat{C}'}{k'_u - k'_l - T^2} \right) \tag{3.31}$$

which ensures that the following intersection contains at least one element k_s

$$\begin{aligned} &\mathcal{P}'^{\epsilon}_{[k'_l, k'_u - (T-1)]} \left((u_0^*, x_0), \ldots, (u^*_{T-1}, x_{T-1}) \right) \\ =&\mathcal{P}^{\epsilon}(u_0^*, x_0) \cap \ldots \mathcal{P}^{\epsilon}(u^*_{T-1}, x_{T-1}) \cap \{k'_l, \ldots, k'_u - (T-1)\}. \end{aligned} \tag{3.32}$$

Part III: Conclude consecutiveness using the intersection
Consider $k_x := k_s + T - 1$. Due to Proposition 3 it holds for all $i \in \mathbb{I}_{[0,T-1]}$

$$\left\| \left(x_{u_i^*}(k_x - T + 1, x_i) - x_s,\, u_i^*(k_x - T + 1) - u_s \right) \right\| \leq \epsilon. \tag{3.33}$$

Additionally, from (3.29) we get that it holds for all $i \in \mathbb{I}_{[0,T-1]}$

$$\begin{aligned} x_{u_i^*}(k_x - T + 1, x_i) &= x_{u_0^*}(k_x - T + 1 + i, x_0), \\ u_i^*(k_x - T + 1) &= u_0^*(k_x - T + 1 + i). \end{aligned} \tag{3.34}$$

This property is also shown in Figure 3.2. Combining Equalities (3.34)

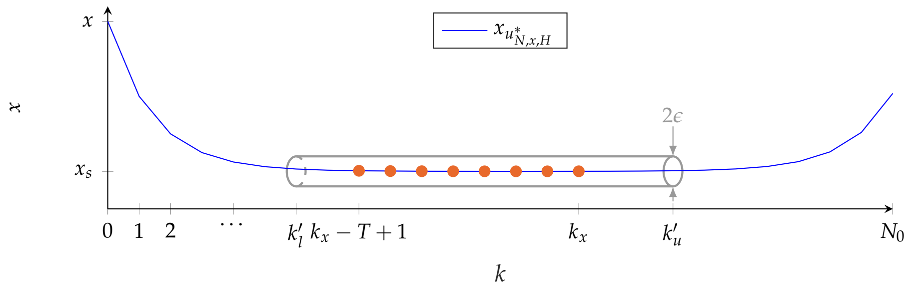


Figure 3.2.: Result of repeatedly using (3.33) and (3.34).

and (3.33), we obtain for all $k \in \mathbb{I}_{[k_x - T+1, k_x]}$

$$\left\| (x_{u_0^*}(k, x_0) - x_s,\, u_0^*(k) - u_s) \right\| \le \epsilon$$

which proves the assertion since $u_0^* = u^*_{N,x,H}$ and $x_0 = x$. Finally, it follows from (3.31) that $\epsilon = \sigma_T(k'_u - N_T)$ with $N_T = k'_l + T^2$ where $\sigma_T(r) := \rho^{-1}(\frac{T\hat{C}'}{r})$, i. e., $\sigma_T \in \mathcal{L}_\mathbb{N}$. Furthermore, note that we can choose k'_u large enough (by choosing N large enough) such that the assertion holds for any $\epsilon > 0$. □

Remark 8. Considering the special case $T = 1$, the neighborhood (3.31) of the steady-state in which the consecutive time instants lie reads

$$\epsilon = \rho^{-1}\left(\frac{\delta + C}{k'_u - k'_l - 1} \right)$$

since it holds $k_{T,N_i} = 0$ for any $N_i \in \mathbb{N}$. Furthermore, we get from Remark 6 that it holds

$$Q^\epsilon(u^*_{N,x}, x) \ge N - (k'_u - k'_l - 1) = N - (k'_u - k'_l) + 1$$

which immediately implies that there is at least one time instant $k \in \mathbb{I}_{[k'_l, k'_u]}$ in the corresponding neighborhood $\mathcal{B}_\epsilon(x_s, u_s)$. Therefore, in the special case $T = 1$, $k'_l = 0$ and $k'_u = N_0$, Theorem 3 contains the results in [Grüne, 2013, Thm. 5.6] as a special case.

Analogously to the previous results, we can use Lipschitz continuity of $h(x,u)$ in order to show that the property of Theorem 3 also holds for the same consecutive time instants for the auxiliary output.

Corollary 3. *Let the Assumptions 1-3 hold. Then, there exist N_T, $k_x \in \mathbb{N}$ and $\sigma_T \in \mathcal{L}_{\mathbb{N}}$ such that for any $(x,H) \in \mathbb{X} \times \mathbb{H}$ and any $k'_l \in \mathbb{N}_0$, $k'_u \in \mathbb{N}$ satisfying (3.27), there exists $k_x \in \mathbb{I}_{[k'_l+T-1,k'_u]}$ such that the optimal trajectory $u^*_{N,x,H} \in \mathbb{U}^N(x,H)$ satisfies*

$$\begin{aligned} \left\|(x_{u^*_{N,x,H}}(k,x) - x_s, u^*_{N,x,H}(k) - u_s)\right\| &\leq \epsilon, \\ \left\|h(x_{u^*_{N,x,H}}(k,x), u^*_{N,x,H}(k)) - h(x_s,u_s)\right\| &\leq L_h\epsilon \end{aligned} \tag{3.35}$$

for all $k \in \mathbb{I}_{[k_x-T+1,k_x]}$ with $\epsilon = \sigma_T(k'_u - N_T)$ and the Lipschitz constant $L_h > 0$.

Proof. Since all conditions of Theorem 3 are satisfied, we know that the choice $\epsilon = \sigma_T(k'_u - N_T)$ from (3.31) ensures $\left\|(x_{u^*_{N,x,H}}(k,x) - x_s, u^*_{N,x,H}(k) - u_s)\right\| \leq \epsilon$ for all $k \in \mathbb{I}_{[k_x-T+1,k_x]}$. Again, it immediately follows from Lipschitz continuity of the auxiliary output h that this implies (3.35). Hence, the analogous choices

$$\begin{aligned} \epsilon &= \rho^{-1}\left(\frac{T\hat{C}'}{k'_u - k'_l - T^2}\right) = \sigma_T(k'_u - N_T), \\ N_T &= k'_l + T^2 \end{aligned}$$

imply the assertion. □

Corollary 3 ensures by (3.35) that T consecutive values of the auxiliary output are bounded for all $k \in \mathbb{I}_{[k_x-T+1,k_x]}$ and hence, we can use this to choose a sufficient horizon N and $k'_u = N-1$, such that the neighborhood ϵ implies $\rceil H(x_{u^*_{N,x,H}}(k_x - T + 1,x), u^*_{N,x,H}(k_x - T + 1 : k_x - 1)) - H^s \rceil \leq E$ for any desired $E > 0$. This property will be used in Lemma 2 and Theorem 5 to construct trajectories which are in a sufficiently small neighborhood of the steady-state such that a local controllability condition can be applied.

Corollary 4. *Consider any $(x,H) \in \mathbb{X} \times \mathbb{H}$, any $\epsilon > 0$ and any trajectory $u \in \mathbb{U}^N(x,H)$ with $N \geq T$. Then, satisfying*

$$\|h(x_u(k,x), u(k)) - h(x_s,u_s)\| \leq L_h\epsilon$$

for all $k \in \mathbb{I}_{[0,T-1]}$ *implies*

$$\rceil H(x,u(0:T-2))\lceil \leq \rceil H(x,u(0:T-2)) - H^s \lceil \leq \sqrt{p} L_h \epsilon$$

Proof. We denote $\tilde{H} := H(x,u(0:T-2)) - H^s$ as well as $\tilde{H}_{i,j}$ for the entry in the i-th row of $\tilde{H}_j$ which is the j-th column of $\tilde{H}$. By the definition of $\rceil H \lceil$ in (2.24), it follows

$$\rceil \tilde{H} \lceil \leq \left\| \tilde{H} \right\|_1$$

where the norm represents the maximum absolute column sum norm (l_1 norm). Furthermore, using norm properties we obtain

$$\begin{aligned} \rceil \tilde{H} \lceil \leq \left\| \tilde{H} \right\|_1 \overset{\ell_1 \text{ norm}}{=} \max_{j \in \mathbb{I}_{[1,T-1]}} \sum_{i=1}^{p} |\tilde{H}_{i,j}| = \max_{j \in \mathbb{I}_{[1,T-1]}} \left\| \tilde{H}_j \right\|_1 \\ \leq \sqrt{p} \max_{j \in \mathbb{I}_{[1,T-1]}} \left\| \tilde{H}_j \right\|_2 \leq \sqrt{p} L_h \epsilon \end{aligned} \tag{3.36}$$

where the last inequality follows from the condition in the assertion. Now, using $H^s \in \mathbb{R}_{\leq 0}^{p \times (T-1)}$ yields $\rceil H(x,u(0:T-2))\lceil \leq \rceil H(x,u(0:T-2)) - H^s \lceil$ and thus, we have shown the assertion. □

Using Corollary 4, we can satisfy with k_x from Cor. 3 for any $E > 0$

$$\rceil H(x_{u^*_{N,x,H}}(k_x - T + 1), u^*_{N,x,H}(k_x - T + 1 : k_x - 1)) - H^s \lceil \leq E$$

by a steady-state neighborhood ϵ according to (3.35) of the size

$$\epsilon \leq \frac{E}{\sqrt{p} L_h}. \tag{3.37}$$

In this chapter, we have developed conditions that guarantee the turnpike property. Moreover, Theorem 3 ensures that we can choose the prediction horizon N large enough to guarantee that there are at least T consecutive time instants in an arbitrarily small neighborhood of the steady-state.

4. Local Continuity Value Function

In this chapter, we use the previous turnpike results to show that the difference between the value function with initial conditions in a specific neighborhood of the steady-state and the value function starting from the steady-state (with a previous trajectory at the steady-state) is bounded. To this end, a local controllability property, which is an extension of [Grüne, 2013, Ass. 6.1], is introduced in Assumption 4. Using this condition, we derive in Section 4.2 a turnpike property for consecutive time instants for optimal trajectories starting in a neighborhood of the steady-state. In Theorem 4, we use this property to derive local continuity bounds on the value function, which is the main contribution of this chapter.

4.1. Local Controllability Condition

In [Grüne, 2013, Ass. 6.1], it is assumed that the difference of the states and inputs of two trajectories can locally be bounded by class $\mathcal{K}_\infty$ functions which consider the difference of the two initial conditions (x_c and x_2) and the difference of the two end values ($x_{u_c}(d,x_c)$ and x_3), respectively. Due to the transient average constraints, we consider not only the states x_c, x_2 but also, the past and subsequent trajectories in terms of the auxiliary output h using H. Hence, we modify the local steady-state neighborhood for which the assumption holds as well as the arguments in the class $\mathcal{K}_\infty$ functions (cf. ζ in Ass. 4) such that we consider not only single points to be connected, but moreover also the past and subsequent $T-1$ values of the corresponding trajectories. Additionally, we sketch the idea of the local controllability assumption in Figure 4.1.

Assumption 4. There exist constants $\delta_c, E_h > 0$, $d \in \mathbb{I}_{\geq T}$ and functions $\gamma_x, \gamma_u \in \mathcal{K}_\infty$ such that for any trajectory $x_{u_c}(k,x_c)$ with $u_c \in \mathbb{U}^{d+T}(x_c,H^c)$ satisfying $\upharpoonleft H^c \upharpoonright \leq E_h$ and $x_{u_c}(k,x_c) \in \mathcal{B}_{\delta_c}(x_s)$ for all $k \in \mathbb{I}_{[0,d+T]}$ and for any trajectory $u_1 \in \mathbb{U}^{N_1}(x_1,H^1)$ with $N_1 \geq d+T$, $(x_1,H^1) \in \mathbb{X} \times \mathbb{H}$, $x_3 :=$

$x_{u_1}(d,x_1)$ and $H^3 := H(x_{u_1}(d,x_1),u_1(d:d+T-2))$ satisfying

$$\rceil H^3 - H^c(d+T-1)\lceil \leq E_h \quad \text{and} \quad x_3 \in \mathcal{B}_{\delta_c}(x_s),$$

where we denote $H^c(d+T-1) := H(x_{u_c}(d,x_c),u_c(d:d+T-2))$, such that for any $x_2 \in \mathcal{B}_{\delta_c}(x_s)$ and any $H^2 \in \mathbb{H}$ with $\rceil H^2 - H^c\lceil \leq E_h$ there exists an input $u_2 \in \mathbb{U}^d(x_2,H^2)$ with $x_{u_2}(d,x_2) = x_3$ such that

$$u_3(k) = \begin{cases} u_2(k) & k \in \mathbb{I}_{[0,d-1]} \\ u_1(k) & k \in \mathbb{I}_{[d,N_1-1]} \end{cases}$$

satisfies $u_3 \in \mathbb{U}^{N_1}(x_2,H^2)$ and it holds $k \in \mathbb{I}_{[0,d]}$

$$\|x_{u_2}(k,x_2) - x_{u_c}(k,x_c)\| \leq \gamma_x(\zeta), \tag{4.1a}$$

$$\|u_2(k) - u_c(k)\| \leq \gamma_u(\zeta), \tag{4.1b}$$

with

$$\zeta := \max\Big\{ \|x_2 - x_c\| + \rceil H^2 - H^c\lceil, \quad \|x_3 - x_{u_c}(d,x_c)\| + \rceil H^3 - H^c(d+T-1)\lceil \Big\}. \tag{4.2}$$

Remark 9. A naive, possibly more intuitive, condition would be to replace $\rceil H^3 - H^c(d+T-1)\lceil$ by $\rceil H(x_{u_2}(d-T+1),u_2(d-T+1,d-1))\lceil$ for the norm-replacement in the second argument of ζ. But, since we want to bound the trajectory x_{u_2} we cannot use values of itself as an upper bound. However, since the connecting trajectory is feasbile we remark that there is a relation between the auxiliary output values of the subsequent values of d (resulting from x_{u_1}) and the previous values of d (resulting from x_{u_2}) since the transient average constraints in the overlapping period are satisfied, too.

The estimates (4.1) hold for all controllable linear systems in $\mathbb{R}^n$ if the steady-state (x_s,u_s) is in the interior of the constraint set $\mathbb{Z}$ [Grüne, 2013] and $h(x_s,u_s) \in \text{int}(\mathbb{Y})$, i.e., $h(x_s,u_s) < 0$. Moreover, using arguments from [Limon et al., 2018], we know that continuity implies that the previous estimates also hold for nonlinear systems if the linearization of f in (x_s,u_s) is controllable, f is differentiable and the trajectories lie in the interiors of $\mathbb{Z}$ and $\mathbb{Y}$, respectively.

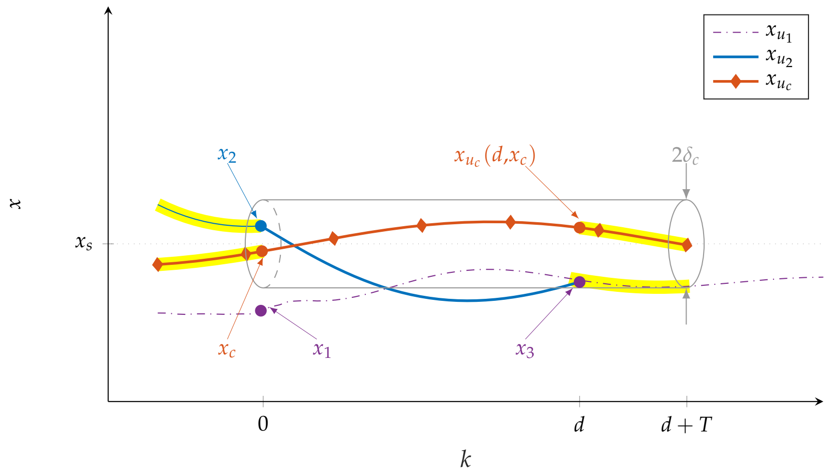


Figure 4.1.: Sketch of the local controllability condition.

Remark 10. Note that due to the definition (2.24), the norm-like value $\rceil H \lceil$ is not commutative, i. e., it holds in general $\rceil H^1 - H^2 \lceil \neq \rceil H^2 - H^1 \lceil$. However, since we will often use the steady-state trajectory as the reference trajectory u_c, i. e. $u_c(\cdot) \equiv u_s$, $x_c = x_s$ and $H^c = H^s$, it holds $\rceil H^2 - H^s \lceil \geq \rceil H^2 \lceil$ and $\rceil H^3 - H^s \lceil \geq \rceil H^3 \lceil$ since H^s only contains non-positive values. Furthermore, we note that if a storage H has only values which are "better" than the steady-state values, i. e. $H_i \leq h(x_s,u_s)$ holds for all $i \in \mathbb{I}_{[1,T-1]}$, we obtain $\rceil H - H^s \lceil = 0$.

In the following lemma we will again use continuity in order to bound $\bar{\lambda}^\top h(x,u)$ as well as the rotated cost $\tilde{\ell}(x,u)$ for all feasible x and u by two different $\mathcal{K}_\infty$-functions. This lemma is later used in Proposition 4 to bound the original and rotated stage cost as well as the auxiliary output for the trajectories of Assumption 4.

Lemma 1. *Let Assumption 1 and 2 hold. Then there exists α_h, $\alpha_u \in \mathcal{K}_\infty$ such that the following holds for all $(x,u) \in \mathbb{Z}$*

$$|\bar{\lambda}^\top h(x,u)| \leq \alpha_h\left(\|x - x_s\| + \|u - u_s\|\right), \tag{4.3a}$$

$$\tilde{\ell}(x,u) \leq \alpha_u\left(\|x - x_s\| + \|u - u_s\|\right). \tag{4.3b}$$

Proof. We start by showing that (4.3a) is satisfied under the given conditions. From Lipschitz continuity of $h(x,u)$ and since it holds $\bar{\lambda}^\top h(x_s,u_s) = 0$, we get with some arbitrary small constant $\varphi > 0$

$$\begin{aligned}\left|\bar{\lambda}^\top h(x,u)\right| &= \left|\bar{\lambda}^\top (h(x,u) - h(x_s,u_s))\right| \leq \|\bar{\lambda}\| \, \|h(x,u) - h(x_s,u_s)\| \\ &\leq \|\bar{\lambda}\| \, L_h \left(\|x - x_s\| + \|u - u_s\|\right) \\ &\leq \max\{\|\bar{\lambda}\| \, L_h, \varphi\} \cdot \|x - x_s\| + \|u - u_s\| \\ &=: \alpha_h \left(\|x - x_s\| + \|u - u_s\|\right)\end{aligned}$$

which implies $\alpha_h \in \mathcal{K}_\infty$ since $\bar{\lambda} \in \mathbb{R}^p_{\geq 0}$ and $L_h > 0$ are finite. Furthermore, from the definition of the rotated cost in (3.4a) we obtain

$$\tilde{\ell}(x,u) \leq \left|\tilde{\ell}(x,u)\right| \leq |\ell(x,u) - \ell(x_s,u_s)| + |\lambda(x)| + |\lambda(f(x,u))| + \left|\bar{\lambda}^\top h(x,u)\right|.$$

For the first term $|\ell(x,u) - \ell(x_s,u_s)|$ we can use continuity of the stage cost (Ass. 1) and for the last term we can use (4.3a) which is already proved. Moreover, Assumption 2 yields $\lambda(x_s) = 0$ and hence, it holds $|\lambda(x)| \leq \alpha_\lambda(\|x - x_s\|)$ as well as

$$\begin{aligned}|\lambda(f(x,u))| = |\lambda(f(x,u)) - \lambda(f(x_s,u_s))| &\overset{\text{Ass.2}}{\leq} \alpha_\lambda \left(\|f(x,u) - f(x_s,u_s)\|\right) \\ &\overset{\text{Ass.1}}{\leq} \alpha_\lambda \left(\alpha_f \left(\|(x - x_s, u - u_s)\|\right)\right).\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\tilde{\ell}(x,u) \leq & \alpha_l(\|x - x_s\| + \|u - u_s\|) + \alpha_h(\|x - x_s\| + \|u - u_s\|) \\ & + \alpha_\lambda(\|x - x_s\|) + \alpha_\lambda \left(\alpha_f \left(\|x - x_s\| + \|u - u_s\|\right)\right) \\ =: & \alpha_u(\|x - x_s\| + \|u - u_s\|)\end{aligned}$$

with $\alpha_u \in \mathcal{K}_\infty$ since $\alpha_f, \alpha_l, \alpha_\lambda, \alpha_h \in \mathcal{K}_\infty$. □

As already mentioned, by using the continuity assumptions we can conclude that the inequalities (4.1) imply estimates for the stage cost ℓ, the rotated stage cost $\tilde{\ell}$ and the auxiliary output h (multiplied with $\bar{\lambda}^\top$) for the setting given in Ass. 4.

Proposition 4. *Suppose Assumptions 1, 2 and 4 are satisfied. There exists $\gamma_c, \gamma_h \in \mathcal{K}_\infty$ such that the following inequalitites hold*

$$|\ell\left(x_{u_2}(k,x_2),u_2(k)\right) - \ell\left(x_{u_c}(k,x_c),u_c(k)\right)| \leq \gamma_c\left(\zeta\right), \tag{4.4a}$$

$$\left|\tilde{\ell}\left(x_{u_2}(k,x_2),u_2(k)\right) - \tilde{\ell}\left(x_{u_c}(k,x_c),u_c(k)\right)\right| \leq \gamma_c\left(\zeta\right), \tag{4.4b}$$

$$\left|\bar{\lambda}^\top\left[h(x_{u_2}(k,x_2),u_2(k)) - h(x_{u_c}(k,x_c),u_c(k))\right]\right| \leq \gamma_h(\zeta), \tag{4.4c}$$

for any x_c, x_2, u_2 and u_c from Assumption 4 with ζ according to (4.2).

Proof. Condition (4.4c) follows from (4.3a) and the estimates (4.1) with $\gamma_h(\zeta) := \alpha_h(\gamma_x(\zeta) + \gamma_u(\zeta))$, $\gamma_h \in \mathcal{K}_\infty$ since $\alpha_h, \gamma_x, \gamma_u \in \mathcal{K}_\infty$. Furthermore, (4.4a) and (4.4b) follow from (4.1) and (4.3b) with

$$\gamma_c(\zeta) := \alpha_l\left(\gamma_x(\zeta) + \gamma_u(\zeta)\right) + \alpha_\lambda(\gamma_x(\zeta)) + \alpha_\lambda(\gamma_f(\gamma_x(\zeta) + \gamma_u(\zeta))) + \gamma_h(\zeta) \tag{4.5}$$

and it holds $\gamma_c \in \mathcal{K}_\infty$ since $\alpha_l, \gamma_x, \gamma_u, \gamma_f, \alpha_\lambda \in \mathcal{K}_\infty$. □

Analogous to Ass. 3, inequality (4.4b) is used later in Corollary 6 to conclude that the results in Theorem 4 also hold for the optimal solution of the rotated cost functional (3.4b).

4.2. Local Initial Turnpike Property

The subsequent lemma is an extension of [Grüne, 2013, Lem. 6.3] and ensures that the optimal open-loop trajectory starting in a neighborhood of the steady-state (Ass. 4) stays in a neighborhood $\mathcal{B}_\eta(x_s,u_s)$ for some consecutive time instants. We prove this lemma by constructing a feasible candidate input sequence and a corresponding state trajectory with the help of the previous local controllability condition (Ass. 4). Then, we bound the rotated value function of this candidate and by using strict dissipativity, we show the assertion by contradiction.

Lemma 2. *Suppose that the Assumptions 1-4 hold. Then there exist $N_\eta > 0$, a function $\eta : \mathbb{N} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\eta(N,r) \to 0$ if $N \to \infty$ and $r \to 0$ such that for any $x \in \mathcal{B}_{\delta_c}(x_s)$, any $H \in \mathbb{H}$ with $\urcorner H - H^s \urcorner \leq E_h$ and horizon $N \geq N_\eta$, the optimal trajectory $u^*_{N,x,H} \in \mathbb{U}^N(x,H)$ satisfies for all $k \in \mathbb{I}_{[0,N/2+T-1]}$*

$$\left\|\left(x_{u^*_{N,x,H}}(k,x) - x_s,\, u^*_{N,x,H}(k) - u_s\right)\right\| \leq \eta\left(N, \|x - x_s\| + \urcorner H - H^s \urcorner\right),$$

with E_h, δ_c from Ass. 4.

Proof. We divide this proof into three different parts. In the first part we introduce a candidate input sequence and show its feasibility. In the second part, we bound the rotated value function using the candidate trajectory and in the last part, it is shown that the assertion holds using the bounds on the rotated cost. In the following, we use the same notation for the storage H as in Assumption 4, i. e.,

$$H^j(k) := H(x_{u_j}(k-T+1,x_j),\, u_j(k-T+1:k-1)) \tag{4.6}$$

for any $k \in \mathbb{I}_{[T-1,N]}$ and $j \in \{1,2,c\}$. Note that we analogously write $H^*(k)$ for matrices corresponding to the optimal input sequence $u^*_{N,x,H}$ and state trajectory $x_{u^*_{N,x,H}}$ as well as $\tilde{H}^*(k)$ for matrices corresponding to the optimal input sequence $\tilde{u}^*_{N,x,H}$ and state trajectory $x_{\tilde{u}^*_{N,x,H}}$.

Part I: Construction of a feasible candidate sequence

We start by choosing the prediction horizon large enough such that the optimal trajectory $x_{u^*_{N,x,H}}$ has at least T consecutive values in a small enough neighborhood of the steady-state which follows from Thm 3, Cor. 3 and Cor. 4. We set

$$\begin{aligned} &N_{N,T} := 2\left(T^2+1\right), && N_{d,T} := 4d+2T, \\ &N \geq \max\{N_{d,T}, N_{N,T}\}, && k'_u = N-1, \qquad k'_l = \frac{N}{2} \end{aligned} \tag{4.7}$$

which ensures satisfaction of (3.27), which is needed in Theorem 3. As shown in Corollary 4, we can choose the ϵ-neighborhood of the steady-state small enough (i. e. N large enough) such that it holds with k_x from Cor. 3

$$\lceil H^*(k_x) - H^s \rceil \leq E_h$$

which can be ensured via (3.37). Combining this with the bound to ensure that T consecutive time instants are in a ball around the steady-state from (3.31), we get with (4.7) and $\hat{C}'$ from (3.25)

$$\begin{aligned} \epsilon &\overset{(3.31)}{=} \rho^{-1}\left(\frac{T\hat{C}'}{k'_u - k'_l - T^2}\right) \\ &= \rho^{-1}\left(\frac{T\hat{C}'}{\frac{N}{2} - T^2 - 1}\right) =: \sigma_T(N/2 - T^2 - 1) \\ &\overset{!}{\leq} \frac{E_h}{\sqrt{p}L_h} \end{aligned}$$

since $k_l' = N/2$ and $k_u' = N-1$. Therefore, we choose N such that

$$N \geq 2\left(T^2 + 1 + \frac{T\hat{C}'}{\rho\left(\frac{E_h}{\sqrt{p}L_h}\right)}\right) =: N_{E_h} \tag{4.8}$$

holds and we remark $N_{E_h} > N_{N,T}$. Furthermore, we denote

$$k_y := k_x - T + 1$$

and make sure that $x_{u^*_{N,x,H}}(k,x)$ is in the neighborhood $\mathcal{B}_{\delta_c}(x_s)$ for all $k \in \mathbb{I}_{[k_y, k_y+T-1]}$. Hence, we again use (3.31) and aim for

$$\epsilon = \rho^{-1}\left(\frac{T\hat{C}'}{\frac{N}{2} - T^2 - 1}\right) =: \sigma_T(N/2 - T^2 - 1) \leq \delta_c$$

which is ensured if we choose

$$N \geq 2\left(T^2 + 1 + \frac{T\hat{C}'}{\rho\left(\delta_c\right)}\right) =: N_{\delta_c}. \tag{4.9}$$

By using the two beforementioned estimates for the neighborhood around the steady-state, we formulate the condition

$$\begin{aligned} \epsilon = \rho^{-1}\left(\frac{T\hat{C}'}{\frac{N}{2} - T^2 - 1}\right) &=: \sigma_T(N/2 - T^2 - 1) \\ \leq \min\left\{\delta_c, \ \frac{E_h}{\sqrt{p}L_h}\right\} &=: \epsilon_{E_h}. \end{aligned} \tag{4.10}$$

It follows from (4.7), (4.8) and (4.9) that the choice

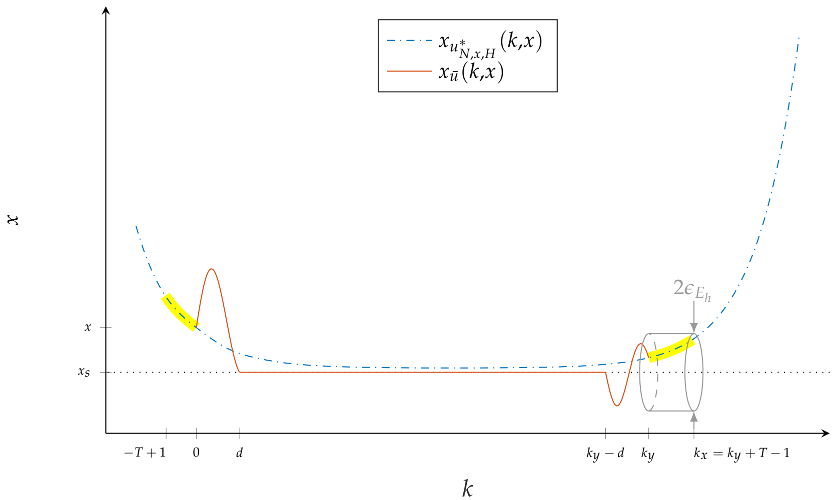
$$N \geq \max\{N_{d,T}, N_{E_h}, \ N_{\delta_c}\} =: N_\eta \tag{4.11}$$

ensures (4.10) and hence, that it holds for the optimal trajectory

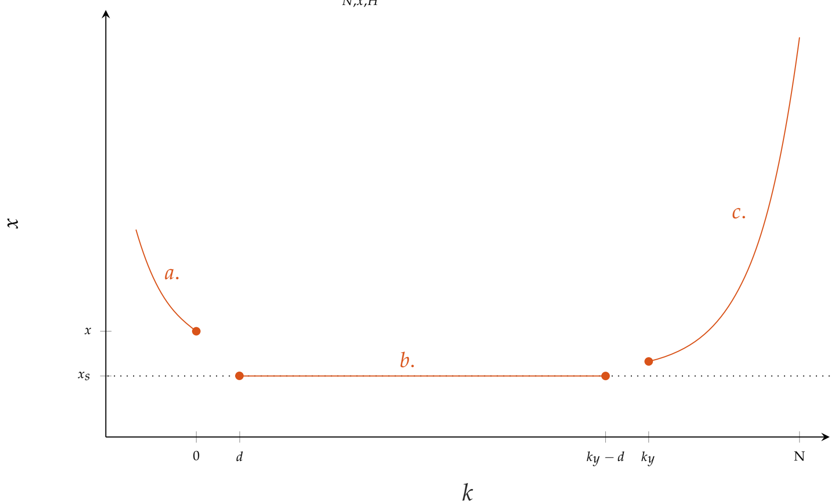
$$|H^*(k_y + T - 1) - H^s| \leq E_h \quad \text{and} \quad x_{u^*_{N,x,H}}(k_y, x_x) \in \mathcal{B}_{\delta_c}(x_s). \tag{4.12}$$

Now, after we have ensured these conditions, we construct a feasible input sequence which will later be used to show the assertion.

We construct a feasible candidate input sequence $\bar{u}$, which is composed of four different parts. In the first stage, since $x \in \mathcal{B}_{\delta_c}$ and $|H - H^s| \leq E_h$



(a) Construction of the candidate trajectory $x_{\bar{u}}$ (——), based on the steady-state x_s and the optimal trajectory $x_{u^*_{N,x,H}}$ (-·-·-).



(b) Stages of the candidate trajectory $x_{\bar{u}}$ we connect using the local controllability condition (Ass. 4).

Figure 4.2.: Candidate trajectory to prove Lemma 2.

holds, we use the local controllability condition (Ass. 4) with the reference trajectory $x_c = x_s$ and $u_c(\cdot) = u_s$, i. e. $H^c = H^s$ which implies $\rceil H^c \lceil = 0 \leq E_h$. The trajectory to be connected is given by $x_1 = x_s$ and $u_1(\cdot) = u_s$ such that $x_3 = x_s$ and $H^3 = H^s$ follow. This means, we start in $x_2 = x$ and end up at the steady-state x_s after d time instants. Once we ended up at the steady-state due to u_2, we apply u_s for another $k_y - 2d$ time instants, and therefore, stay at the steady-state for this second stage of the candidate sequence $\bar{u}$. In the third stage, we steer the system in d steps from x_s to $x_{u^*_{N,x,H}}(k_y,x)$. More precisely, we use again Assumption 4 with the steady-state reference $x_c = x_s$, $H^c = H^s$, $u_c(\cdot) = u_s$ (which implies $H^c(d + T - 1) = H^s$) and the trajectory to be connected as $x_1 = x_{u^*_{N,x,H}}(k_y - d,x)$, $u_1(\cdot) = u^*_{N,x,H}(k_y - d : N - 1)$ which implies $x_3 = x_{u^*_{N,x,H}}(k_y,x)$ and $H^3 = H^*(k_y + T - 1)$ where we recall $k_y = k_x - T + 1$. We write for the resulting connecting control sequence $\bar{u}_2$ and end the candidate control sequence $\bar{u}$ with the fourth stage by appending the last $N - k_y$ values of the original optimal sequence[1] starting at the end point $x_{u^*_{N,x,H}}(k_y,x)$ of the third stage. In conclusion, the designed candidate control sequence reads

$$\bar{u}(k) = \begin{cases} u_2(k), & k \in \mathbb{I}_{[0,d-1]} \\ u_s, & k \in \mathbb{I}_{[d,k_y-d-1]} \\ \bar{u}_2(k - k_y + d), & k \in \mathbb{I}_{[k_y-d,k_y-1]} \\ u^*_{N,x,H}(k,x), & k \in \mathbb{I}_{[k_y,N-1]} \end{cases} \tag{4.13}$$

and yields the corresponding state trajectory

$$x_{\bar{u}}(k,x) = \begin{cases} x_{u_2}(k,x), & k \in \mathbb{I}_{[0,d]} \\ x_s, & k \in \mathbb{I}_{[d,k_y-d]} \\ x_{\bar{u}_2}(k - k_y + d,x_s), & k \in \mathbb{I}_{[k_y-d,k_y]} \\ x_{u^*_{N,x,H}}(k,x), & k \in \mathbb{I}_{[k_y,N]} \end{cases} \tag{4.14}$$

which is also shown in Figure 4.2a. Furthermore, Figure 4.2b illustrates the seperate trajectories which we are connecting by the local controllability property.

Now, we will show that the candidate control sequence $\bar{u}$ is feasible. Since we have set $k_l' \geq 2d + T$, we know that the second stage (steady-state) of

[1]This sequence is optimal in the sense of starting at x for prediction horizon N with a previous trajectory H.

Table 4.1.: Overview of the connecting trajectories of the proof of Lemma 2 using the notation of Ass. 4 for given initial conditions (IC).

	IC	Subsequent trajectory	Reference trajectory
Stage 1	$x_2 = x$, $H^2 = H$	Steady-state trajectory $x_3 = x_s$, $H^3 = H^s$, $u_3(k) = u_s$ $k \in \mathbb{I}_{\geq d}$	Steady-state trajectory $x_c = x_s$, $H^c = H^s$, $u_c(\cdot) = u_s$
Stage 3	$x_2 = x_s$, $H^2 = H^s$	Optimal trajectory $x_3 = x_{u^*_{N,x,H}}(k_y,x)$, $H^3 = H^*(k_y + T - 1)$, $u_3(k+d) = u^*_{N,x,H}(k+k_y)$, $k \in \mathbb{I}_{[0,N-k_y-1]}$	Steady-state trajectory $x_c = x_s$, $H^c = H^s$, $u_c(\cdot) = u_s$

the trajectory is at least T steps long, i. e., we can consider the feasibility of both connecting sequences u_2 and $\bar{u}_2$ sperately. We start by considering the first input sequence u_2. It holds for this stage with the notation of Ass. 4 $x_1 = x_c = x_s$ $x_{u_1}(\cdot,x_1) = x_{u_c}(\cdot,x_c) = x_s$ and thus, $H^3 = H^c(d+T-1)$ as well as $x_3 = x_s$. Since $\rceil H - H^s \lceil \leq E_h$, $x \in \mathcal{B}_{\delta_c}(x_s)$ holds, Assumption 4 ensures that there exists a feasible candidate input $\bar{u} \in \mathbb{U}^{k_y-d}(x,H)$. Furthermore, due to the choice of the prediction horizon in (4.11) it holds (4.12). Again, it immediately follows from Ass. 4 that there exists an input sequence $\bar{u}_2$ which is feasibly connecting the steady-state (x_s,H^s) and $(x_{u^*_{N,x,H}}(k_y,x), H^*(k_y+T-1))$, i. e. $\bar{u}_2 \in \mathbb{U}^d(x_s,H^s)$ with $x_{\bar{u}_2}(d,x_s) = x_{u^*_{N,x,H}}(k_y,x)$ and satisfying the constraints. We additionally give an overview over the connecting trajectories (Stage 1 and 3) in Table 4.1.

Part II: Bounds on the Rotated Value Function

Before starting, we remark that we have constructed the connecting trajectories via Assumption 4 and hence, the estimates (4.1) and (4.4) (Prop. 4) hold. Furthermore, we note that it holds

$$k_x \geq k_l' + T - 1 = \frac{N}{2} + T - 1 \geq 2d + 2T - 1. \tag{4.15}$$

First, we consider the different stages of the candidate solution in order to derive a bound on the rotated cost of this candidate trajectory.

Stage 1. Considering the first stage, we recall $x_2 = x$, $H^2 = H$, $x_{u_2}(d,x_2) = x_3 = x_s = x_c$ and note that due to the given choices, it follows (see also Figure 4.3a) $\|x_3 - x_{u_c}(d,x_c)\| = 0$. As previously mentioned, it also holds

$u_c(\cdot) = u_s$ and we get $H^c = H^c(d+T-1) = H^s$, i.e., x_{u_c} remains at the steady-state before and during the regarded interval $k \in \mathbb{I}_{[0,d+T]}$. Furthermore, we choose the subsequent trajectory as the steady-state trajectory such that $H^3 = H^s$ which reduces ζ in Assumption 4 to

$$\zeta_1 := \|x - x_s\| + \rceil H - H^s \lceil \, . \tag{4.16}$$

Hence, we get from (4.4a) the estimate

$$J_d(x,u_2) \leq d\ell(x_s,u_s) + d\gamma_c(\zeta_1) \, .$$

Stage 3. For the third stage we use Assumption 4 with $x_2 = x_c = x_s$, $u_c(\cdot) = u_s$, $H^2 = H^c = H^c(d+T-1) = H^s$ and $x_3 = x_{u^*_{N,x,H}}(k_x,x)$. Since the subsequent trajectory is the optimal trajectory, i.e., $H^3 = H^*(k_y+T-1)$ we obtain

$$\zeta_3 := \left\| x_{u^*_{N,x,H}}(k_y,x) - x_s \right\| + \rceil H^*(k_y+T-1) - H^s \lceil \, . \tag{4.17}$$

Again, we can use Theorem 3, Corollary 3 and Corollary 4, respectively, to bound the right hand side of (4.17) since ζ_3 depends on the size of the ball in which the T consecutive time instants $k \in \mathbb{I}_{[k_x-T+1,k_x]} = \mathbb{I}_{[k_y,k_y+T-1]}$ are. For an increasing prediction horizon N, this ball is shrinking (cf. $\sigma_T(N/2 - T^2 - 1)$ in (4.10)) and hence, we can bound both terms in ζ_3 by

$$\begin{aligned} \left\| x_{u^*_{N,x,H}}(k_y,x) - x_s \right\| &\leq \sigma_T(N/2 - T^2 - 1), \\ \rceil H^*(k_y+T-1) - H^s \lceil \; &\leq \sqrt{p} L_h \sigma_T(N/2 - T^2 - 1) \end{aligned}$$

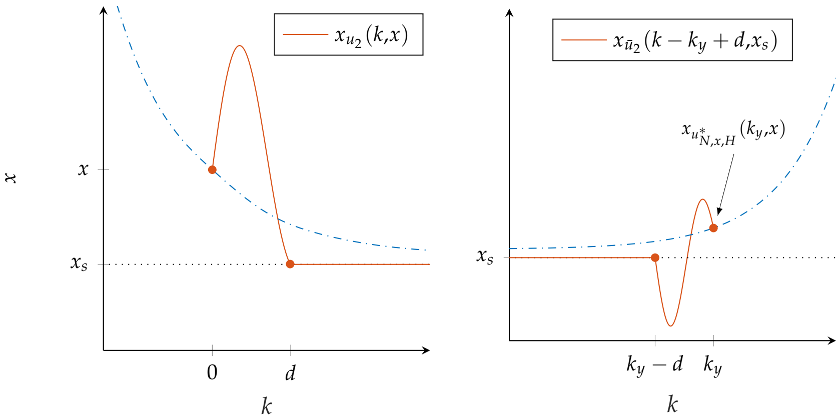
which follows from (4.10) and Theorem 3 for the first estimate as well as Corollary 3 and 4 for the second estimate. Combining these inequalities with (4.17) results in

$$\zeta_3 \leq (1 + \sqrt{p} L_h)\sigma_T(N/2 - T^2 - 1) =: \sigma_{\zeta_3}(N/2 - T^2 - 1) \tag{4.18}$$

with $\sigma_{\zeta_3} \in \mathcal{L}_{\mathbb{N}}$. In the following, we abbreviate $\sigma_{\zeta_3} := \sigma_{\zeta_3}(N/2 - T^2 - 1)$. Using (4.4a), the previous estimate ensures that the candidate sequence $\bar{u}_2$ satisfies

$$J_d(x_s,\bar{u}_2) \leq d\ell(x_s,u_s) + d\gamma_c(\zeta_3) \leq d\ell(x_s,u_s) + d\gamma_c(\sigma_{\zeta_3}(N/2 - T^2 - 1)).$$

After we have investigated the first and third part of the candidate trajectory $\bar{u}$ and $x_{\bar{u}}$, respectively, we now consider the complete trajectory.



(a) First stage x_{u_2}.

(b) Third stage $x_{\bar{u}_2}$.

Figure 4.3.: Connecting stages of the candidate trajectory $x_{\bar{u}}$ designed by Assumption 4 and connected to $x_{u^*_{N,x,H}}$ (-·-·-).

Complete trajectory. From optimality of $u^*_{N,x,H}$ and feasibility of the candidate $\bar{u}$ it follows

$$J^*_N(x,H) \le J_N(x,\bar{u}).$$

Considering the notation $J_{a\to b}(x,u) = \sum_{k=a}^{b-1} \ell(x_u(k,x),u(k))$, we can split up both functionals which yields

$$J_{k_x}(x,u^*_{N,x,H}) + J_{k_x\to N}(x,u^*_{N,x,H}) \le J_{k_x}(x,\bar{u}) + J_{k_x\to N}(x,\bar{u}).$$

Since we know from (4.14) that the last piece of the trajectory $x_{\bar{u}}$ ($k \in \mathbb{I}_{\ge k_x}$) is equal to the optimal trajectory, i. e. $J_{k_x\to N}(x,u^*_{N,x,H}) = J_{k_x\to N}(x,\bar{u})$, it follows

$$J_{k_x}(x,u^*_{N,x,H}) \le J_{k_x}(x,\bar{u}). \tag{4.19}$$

By using the definition of the rotated stage cost (3.4a) and the rotated cost (3.4b), we can write for the optimal trajectory and the candidate sequence

$$\begin{aligned}\tilde{J}_{k_x}(x,u^*_{N,x,H}) &= J_{k_x}(x,u^*_{N,x,H}) - k_x\ell(x_s,u_s) + \lambda(x)\\ &\quad -\lambda(x_{u^*_{N,x,H}}(k_x,x)) + \sum_{k=0}^{k_x-1} \bar{\lambda}^\top h(x_{u^*_{N,x,H}}(k,x),u^*_{N,x,H}(k))\end{aligned} \tag{4.20}$$

as well as

$$\begin{aligned}\tilde{J}_{k_x}(x,\bar{u}) = J_{k_x}(x,\bar{u}) - k_x\ell(x_s,u_s) + \lambda(x) \\ -\lambda(x_{\bar{u}}(k_x,x)) + \sum_{k=0}^{k_x-1} \bar{\lambda}^\top h(x_{\bar{u}}(k,x),\bar{u}(k)).\end{aligned} \tag{4.21}$$

Now, we choose the smallest $k_h \in \mathbb{I}_{[k_y,k_x]} = \mathbb{I}_{[k_y,k_y+T-1]}$ such that $k_h = jT$ holds for some $j \in \mathbb{N}$. Note that it holds $(x_{u^*_{N,x,H}}(k),u^*_{N,x,H}(k)) \in \mathcal{B}_{\epsilon_{E_h}}(x_s,u_s)$ for all $k \in \mathbb{I}_{[k_y,k_y+T-1]}$. By definition (2.22), the choice of k_h implies

$$k_{T,k_h} = \left\lceil \frac{k_h}{T} \right\rceil T - k_h = 0$$

and definition (2.23) ensures $-\sum_{i=1}^{k_{T,k_h}} H_{T-i} = 0$. Therefore, we obtain

$$\begin{aligned}&\sum_{k=0}^{k_x-1} h(x_{u^*_{N,x,H}}(k,x),u^*_{N,x,H}(k)) \\ &= \sum_{k=0}^{k_h-1} h(x_{u^*_{N,x,H}}(k,x),u^*_{N,x,H}(k)) + \sum_{k=k_h}^{k_x-1} h(x_{u^*_{N,x,H}}(k,x),u^*_{N,x,H}(k)) \\ &\overset{(2.23)}{\leq} \sum_{k=k_h}^{k_x-1} h(x_{u^*_{N,x,H}}(k,x),u^*_{N,x,H}(k)).\end{aligned} \tag{4.22}$$

Since the candidate trajectory is the same as the optimal one for $k \geq k_y$ (4.14), it follows that (see also Figure 4.2a)

$$\sum_{k=k_h}^{k_x-1} \bar{\lambda}^\top h(x_{u^*_{N,x,H}}(k,x),u^*_{N,x,H}(k)) = \sum_{k=k_h}^{k_x-1} \bar{\lambda}^\top h(x_{\bar{u}}(k,x),\bar{u}(k)).$$

Thus, subtracting (4.21) from (4.20) and using $x_{u^*_{N,x,H}}(k_x,x) = x_{\bar{u}}(k_x,x)$ we arrive at

$$\begin{aligned}&\tilde{J}_{k_x}(x,u^*_{N,x,H}) - \tilde{J}_{k_x}(x,\bar{u}) \leq \\ &J_{k_x}(x,u^*_{N,x,H}) - J_{k_x}(x,\bar{u}) - \bar{\lambda}^\top \sum_{k=0}^{k_h-1} h(x_{\bar{u}}(k,x),\bar{u}(k)).\end{aligned}$$

By using the previous result (4.19), it follows

$$\begin{aligned}\tilde{J}_{k_x}(x,u^*_{N,x,H}) &\leq \tilde{J}_{k_x}(x,\bar{u}) - \bar{\lambda}^\top \sum_{k=0}^{k_h-1} h(x_{\bar{u}}(k,x),\bar{u}(k)) \\ &\leq \tilde{J}_{k_x}(x,\bar{u}) + \sum_{k=0}^{k_h-1} \left|\bar{\lambda}^\top h(x_{\bar{u}}(k,x),\bar{u}(k))\right|.\end{aligned}$$

Furthermore, for the second stage $k \in \mathbb{I}_{[d,k_y-d-1]}$ of the candidate $x_{\bar{u}}$ we stay at the steady-state (cf. (4.14)) which implies $\bar{\lambda}^\top h(x_s,u_s) = 0$ and hence, we can rewrite the previous estimate such that

$$\begin{aligned}&\tilde{J}_{k_x}(x,u^*_{N,x,H}) \leq \\ &\tilde{J}_{k_x}(x,\bar{u}) + \sum_{k=0}^{d-1} \left|\bar{\lambda}^\top h(x_{\bar{u}}(k,x),\bar{u}(k))\right| + \sum_{k=k_y-d}^{k_y-1} \left|\bar{\lambda}^\top h(x_{\bar{u}}(k,x),\bar{u}(k))\right| \\ &+ \sum_{k=k_y}^{k_h-1} \left|\bar{\lambda}^\top h(x_{\bar{u}}(k,x),\bar{u}(k))\right|.\end{aligned} \tag{4.23}$$

Since we have constructed the first and third part of $\bar{u}$ in (4.13) via Assumption 4, the estimates for the first stage $k \in \mathbb{I}_{[0,d-1]}$ (with ζ_1 from (4.16))

$$\begin{aligned}\|\bar{u}(k) - u_s\| \leq \gamma_u(\zeta_1), \quad \|x_{\bar{u}}(k,x) - x_s\| \leq \gamma_x(\zeta_1), \\ \left|\bar{\lambda}^\top \left[h(x_{\bar{u}}(k,x),\bar{u}(k)) - h(x_s,u_s)\right]\right| = \left|\bar{\lambda}^\top h(x_{\bar{u}}(k,x),\bar{u}(k))\right| \leq \gamma_h(\zeta_1)\end{aligned} \tag{4.24}$$

hold as well as for the third stage $k \in \mathbb{I}_{[k_y-d,k_y-1]}$ (with ζ_3 from (4.17))

$$\begin{aligned}\|\bar{u}(k) - u_s\| \leq \gamma_u(\zeta_3) \leq \gamma_u(\sigma_{\zeta_3}), \quad \|x_{\bar{u}}(k,x) - x_s\| \leq \gamma_x(\zeta_3) \leq \gamma_x(\sigma_{\zeta_3}), \\ \left|\bar{\lambda}^\top \left[h(x_{\bar{u}}(k,x),\bar{u}(k)) - h(x_s,u_s)\right]\right| = \left|\bar{\lambda}^\top h(x_{\bar{u}}(k,x),\bar{u}(k))\right| \leq \gamma_h(\sigma_{\zeta_3}).\end{aligned} \tag{4.25}$$

For the second stage we stay at the steady-state and therefore, we get $x_{\bar{u}}(k,x) = x_s$ and $\bar{u}(k) = u_s$ for $k \in \mathbb{I}_{[d,k_y-d-1]}$. Furthermore, since k_h is the smallest $k \geq k_y$ which is a multiple of T and since all state and input values in this interval are due to the choice of N (cf. (4.11)) bounded by (4.10), we can use Lemma 1 in order to estimate

$$\sum_{k=k_y}^{k_h-1} \left|\bar{\lambda}^\top h(x_{\bar{u}}(k,x_1),\bar{u}(k))\right| \leq (T-1)\alpha_h(2\sigma_T(N/2 - T^2 - 1)),$$

where the argument of α_h follows from (4.10). In the following, we abbreviate $\sigma_T := \sigma_T(N/2 - T^2 - 1)$

Using these estimates and equalities, we can rewrite (4.23) as

$$\tilde{J}_{k_x}(x,u^*_{N,x,H}) \leq \tilde{J}_{k_x}(x,\bar{u}) + d\gamma_h(\zeta_1) + d\gamma_h(\sigma_{\zeta_3}) + (T-1)\alpha_h(2\sigma_T). \tag{4.26}$$

Furthermore, due to Theorem 3 we know that $\|(x_{\bar{u}}(k,x) - x_s, \bar{u}(k) - u_s)\| \leq \sigma_T$ holds for all $k \in \mathbb{I}_{[k_y,k_x]}$ and hence, Inequality (4.3b) in Lemma 1 ensures

$$\tilde{J}_{k_y \to k_x}(x,\bar{u}) \leq T\alpha_u(2\sigma_T).$$

Using this together with (4.24) and (4.25) as well as using Lemma 1, the modified functional can be rewritten as

$$\begin{aligned}\tilde{J}_{k_x}(x,\bar{u}) \leq d\alpha_u\left(\gamma_x(\zeta_1) + \gamma_u(\zeta_1)\right) + d\alpha_u\left(\gamma_x(\sigma_{\zeta_3}) + \gamma_u(\sigma_{\zeta_3})\right)& \\ + T\alpha_u(2\sigma_T).&\end{aligned} \tag{4.27}$$

Combining (4.26) and (4.27), we arrive at the following bound on the rotated cost

$$\begin{aligned}\tilde{J}_{k_x}(x,u^*_{N,x,H}) \leq & d\alpha_u\left(\gamma_x(\zeta_1) + \gamma_u(\zeta_1)\right) + d\alpha_u\left(\gamma_x(\sigma_{\zeta_3}) + \gamma_u(\sigma_{\zeta_3})\right) \\ & + T\alpha_u(2\sigma_T) + d\gamma_h(\zeta_1) + d\gamma_h(\sigma_{\zeta_3}) + (T-1)\alpha_h(2\sigma_T).\end{aligned} \tag{4.28}$$

Part III: Showing Assertion by Contradiction

Assume that $\left\|(x_{u^*_{N,x,H}}(k,x) - x_s,\, u^*_{N,x,H}(k) - u_s)\right\| \geq \Delta$ for some $k \in \mathbb{I}_{[0,k_x]}$ and some $\Delta > 0$. Then dissipativity (Ass. 2) ensures

$$\tilde{J}_{k_x}(x,u^*_{N,x,H}) \geq \rho(\|(x - x_s, u - u_s)\|) \geq \rho(\Delta). \tag{4.29}$$

In case

$$\begin{aligned}\Delta > \rho^{-1}\big(& d\alpha_u\left(\gamma_x(\zeta_1) + \gamma_u(\zeta_1)\right) + d\alpha_u\left(\gamma_x(\sigma_{\zeta_3}) + \gamma_u(\sigma_{\zeta_3})\right) \\ & + T\alpha_u(2\sigma_T) + d\gamma_h(\zeta_1) + d\gamma_h(\sigma_{\zeta_3}) + (T-1)\alpha_h(2\sigma_T)\big),\end{aligned}$$

this contradicts (4.28). Thus, we get

$$\begin{aligned}\Delta \leq \rho^{-1}\big(& d\alpha_u\left(\gamma_x(\zeta_1) + \gamma_u(\zeta_1)\right) + d\alpha_u\left(\gamma_x(\sigma_{\zeta_3}) + \gamma_u(\sigma_{\zeta_3})\right) \\ & + T\alpha_u(2\sigma_T) + d\gamma_h(\zeta_1) + d\gamma_h(\sigma_{\zeta_3}) + (T-1)\alpha_h(2\sigma_T)\big),\end{aligned}$$

for all $k \in \mathbb{I}_{[0,k_x]}$. Hence, a prediction horizon $N \geq N_\eta$ with N_η from (4.11) ensures $k_x \geq 4d + 2T$ and feasibility of the candidate $\bar{u}$. Defining

$$\begin{aligned}\eta(N,r) := \rho^{-1}\Big(& d\alpha_u\left(\gamma_x(r) + \gamma_u(r)\right) + d\alpha_u\big(\gamma_x(\sigma_{\zeta_3}(N/2 - T^2 - 1)) \\ & + \gamma_u(\sigma_{\zeta_3}(N/2 - T^2 - 1))\big) + d\gamma_h(r) \\ & + d\gamma_h(\sigma_{\zeta_3}(N/2 - T^2 - 1)) + T\alpha_u(2\sigma_T(N/2 - T^2 - 1)) \\ & + (T-1)\alpha_h(2\sigma_T(N/2 - T^2 - 1))\Big)\end{aligned}$$

finally shows the assertion with $\sigma_{\zeta_3} \in \mathcal{L}_\mathbb{N}$ from (4.18) and $\sigma_T \in \mathcal{L}_\mathbb{N}$ from (4.10). It follows that $\eta(N,r) \to 0$ if $N \to \infty$ and $r \to 0$. $\square$

For the special case $T = 1$, i. e., the transient average constraints degenerate to point-wise in time constraints, we get $P(N) \geq N/2$ and $N_\eta \geq N_{d,T} \geq 4d$. Therefore, the assertion is conceptionally the same for this special case as in [Grüne, 2013, Lem. 6.3]. We just have an additional constraint on h (i. e. another point-wise in time constraint) and we choose a more conservative N_η in order to stay at the second stage of the candidate input (steady-state) at least T consecutive time instants to avoid overlapping transient average constraints of the two connecting trajectories (x_{u_2} and $x_{\bar{u}_2}$).

4.3. Local Continuity of the Value Function

Using the previous Lemma 2, we can show analogously to [Grüne, 2013, Thm. 6.4] that under the assumptions in Lemma 2 there exists a prediction horizon such that the difference between the optimal trajectory starting at the steady-state (with a previous steady-state trajectory) and the optimal trajectory starting in a neighborhood of the steady-state can be upper bounded. In the following theorem, we use again the local controllability property (Ass. 4) in order to show local continuity of the value function. To this end, we note in the following a bound on the norm-replacement.

Remark 11. As given in Corollary 4, it holds $\upharpoonleft H \upharpoonright \leq \|H\|_1$ for any $H \in \mathbb{R}^{p\times(T-1)}$. Hence, it also holds $\upharpoonleft -H \upharpoonright \leq \|-H\|_1 = \|H\|_1$ which yields (note that $\upharpoonleft H \upharpoonright \geq 0$ by definition)

$$\max\left\{\upharpoonleft \tilde{H} \upharpoonright, \ \upharpoonleft -\tilde{H} \upharpoonright\right\} \leq \max\left\{\left\|\tilde{H}\right\|_1, \ \left\|-\tilde{H}\right\|_1\right\} = \left\|\tilde{H}\right\|_1. \qquad (4.30)$$

Thus, by setting $\tilde{H} := H^a - H^b$ we arrive at

$$\max\left\{\upharpoonleft H^a - H^b \upharpoonright,\ \upharpoonleft H^b - H^a \upharpoonright\right\} \le \left\|H^a - H^b\right\|_1 \tag{4.31}$$

for any $H^a, H^b \in \mathbb{H}$.

Theorem 4. *Let Assumptions 1-4 hold. Then there exist $N_2 \in \mathbb{N}$ and a function $\gamma_v \in \mathcal{K}_\infty$ such that for all $\delta \in (0, \delta_c]$, all $E \in (0, E_h]$ all $N \in \mathbb{N}$ with $N \ge N_2$, all $x \in \mathcal{B}_\delta(x_s)$ and all $\|H - H^s\|_1 \le E$ it holds*

$$|J_N^*(x, H) - J_N^*(x_s, H^s)| \le \gamma_v(\delta + E). \tag{4.32}$$

Proof. Since the same assumptions hold as in Lemma 2, we can consider the prediction horizon N_η and the function η from this Lemma as well as δ_c and E_h from Assumption 4. We choose $N_2 \ge N_\eta$ such that (we recall that L_h is the Lipschitz constant of the auxiliary output and p the dimension of $h(x,u)$) it holds from Lemma 2 for all $N \ge N_2$ and $r \in [0, \delta_c + E_h]$

$$\eta(N, r) \le \min\left\{\delta_c, \frac{E_h}{\sqrt{p}L_h}\right\}. \tag{4.33}$$

We highlight that Remark 11 ensures that it holds by the given conditions

$$\|H - H^s\|_1 \le E_h \quad \Rightarrow \quad \upharpoonleft \pm (H - H^s) \upharpoonright \le E_h, \ \ \upharpoonleft H \upharpoonright \le \upharpoonleft H - H^s \upharpoonright \le E_h.$$

Furthermore, by Lemma 2 and since $\eta(N, \delta_c + E_h) \le \min\left\{\delta_c, \frac{E_h}{\sqrt{p}L_h}\right\}$ and $k_x \ge 4d + 2T$ (cf. the proof of Lemma 2) we get that the optimal trajectory starting in x satisfies $x_{u^*_{N,x,H}}(k, x) \in \mathcal{B}_{\delta_c}(x_s)$ for all $k \in \mathbb{I}_{[0,d]}$ as well as $\upharpoonleft H(x_{u^*_{N,x,H}}(d), u^*_{N,x_1,H^1}(d : d + T - 2)) - H^s \upharpoonright \le E_h$ which will later be used to apply Assumption 4. The corresponding trajectories and conditions of Ass. 4 are illustrated in Table 4.2. Since we additionally consider different initial conditions (x, H) and (x_s, H^s) of the optimal trajectories, we abbreviate $H^{*,x}(k) := H(x_{u^*_{N,x,H}}(k - T + 1), u^*_{N,x,H}(k - T + 1 : k - 1))$ and $H^{*,x_s}(k) := H(x_{u^*_{N,x_s,H^s}}(k - T + 1), u^*_{N,x_s,H^s}(k - T + 1 : k - 1))$ for any $k \in \mathbb{I}_{[T-1,N]}$. In order to conclude that the absolute difference of the values functions in (4.32) is bounded, we show the inequality in two different directions and therefore, split this proof into three parts. In the first two parts we show both directions and in the third part we conclude the bound on the absolute value.

Part I: First direction of the inequality

Due to the previous arguments, there exists a input $u_2 \in \mathbb{U}^N(x_s, H_s)$ with

Table 4.2.: Overview of the connecting trajectories of the proof of Theorem 4 using the notation of Ass. 4 for two Initial Conditions (IC).

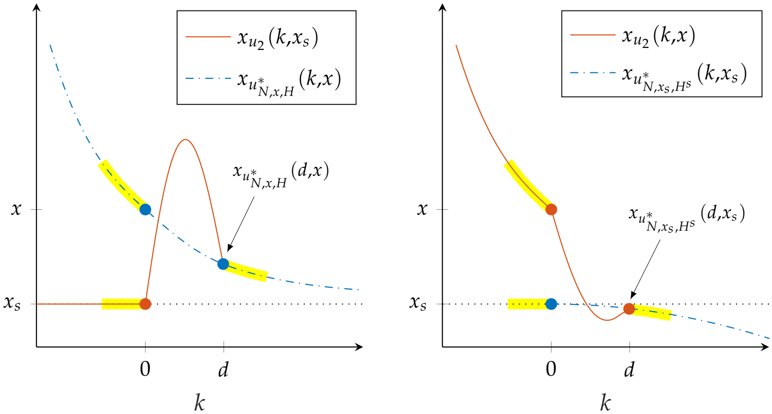
	IC	Subsequent trajectory	Reference trajectory
Part I	$x_2 = x_s$, $H^2 = H^s$	Opt. traj. with IC (x,H) $x_3 = x_{u^*_{N,x,H}}(d,x)$, $H^3 = H^{*,x}(d+T-1)$, $u_3(k+d) = u^*_{N,x,H}(k+d)$, $k \in \mathbb{I}_{[0,N-d-1]}$	Opt. traj. with IC (x,H) $x_c = x$, $H^c = H$, $u_c(\cdot) = u^*_{N,x,H}(\cdot)$
Part II	$x_2 = x$, $H^2 = H$	Opt. traj. with IC (x_s,H_s) $x_3 = x_{u^*_{N,x_s,H^s}}(d,x_s)$, $H^3 = H^{*,x_s}(d+T-1)$, $u_3(k+d) = u^*_{N,x_2,H_2}(k+d)$, $k \in \mathbb{I}_{[0,N-d-1]}$	Opt. traj. with IC (x_s,H^s) $x_c = x_s$, $H^c = H^s$, $u_c(\cdot) = u^*_{N,x_s,H_s}(\cdot)$

$x_{u_2}(d,x_s) = x_{u^*_{N,x,H}}(d,x)$, $u_2(k) = u^*_{N,x,H}(k)$ $k \in \mathbb{I}_{[d,N-1]}$. Furthermore, (4.33) yields $x_{u^*_{N,x,H}}(d,x) \in \mathcal{B}_{\delta_c}(x_s)$, $\rceil H(d+T-1) - H^s \lceil \leq E_h$ and we choose for the reference u_c from Assumption 4 $u_c = u^*_{N,x,H}$ which is possible since we have ensured $x_{u^*_{N,x,H}} \in \mathcal{B}_{\delta_c}(x_s)$ for $k \in \mathbb{I}_{[0,d+T]}$, $\rceil H \lceil \leq E_h$ and $\rceil H^s - H \lceil \leq E_h$. Note that we give an additional overview of the connections in Table 4.2. Extending u_2 via $u_2(k) := u^*_{N,x,H}(k)$ for $k \in \mathbb{I}_{[d,N-1]}$, it follows for all $k \in \mathbb{I}_{[0,d-1]}$ from (4.4a)

$$\ell(x_{u_2}(k,x_s),u_2(k)) \leq \ell(x_{u^*_{N,x,H}}(k,x), u^*_{N,x,H}(k)) + \gamma_c(\|x_s - x\| + \rceil H^s - H \lceil) \tag{4.34}$$

since $x_3 = x_{u^*_{N,x,H}}(d,x) = x_{u_c(d,x_c)}$ and $H(d+T-1) = H^c(d+T-1)$ (which simplifies the expression ζ in Assumption 4). Moreover, due to the extension of u_2 it holds for $k \in \mathbb{I}_{[d,N-1]}$ $\ell(x_{u_2}(k,x_s),u_2(k)) = \ell(x_{u^*_{N,x,H}}(k,x), u^*_{N,x,H}(k))$ which yields

$$\begin{aligned} J_N^*(x_s,H^s) \leq & J_N(x_s,u_2) \\ \overset{(4.34)}{\leq} & \sum_{k=0}^{N-1} l(x_{u^*_{N,x,H}}(k,x), u^*_{N,x,H}) + d\gamma_c(\|x_s - x\| + \rceil H^s - H \lceil) \\ = & J_N^*(x,H) + d\gamma_c(\|x_s - x\| + \rceil H^s - H \lceil) \\ \overset{(4.31)}{\leq} & J_N^*(x,H) + d\gamma_c(\|x_s - x\| + \|H^s - H\|_1). \end{aligned}$$



(a) Considering an optimal trajectory with initial condition (x,H).

(b) Considering an optimal trajectory with initial condition x_s.

Figure 4.4.: Connection of trajectories in order to prove Theorem 4.

where the first inequality follows from optimality and the second one from the construction of u_2 via Assumption 4. By setting

$$\gamma_v := d\gamma_c \in \mathcal{K}_\infty$$

we obtain

$$J_N^*(x_s,H^s) - J_N^*(x,H) \leq \gamma_v(\|x_s - x\| + \|H^s - H\|_1). \tag{4.35}$$

Part II: Second direction of the inequality
Now, we consider a trajectory which is connecting the initial point of the optimal trajectory $x_{u^*_{N,x,H}}$ with the optimal trajectory starting at the steady-state, i. e., $x_3 = x_{u^*_{N,x_s,H^s}}(d,x)$. We choose for the reference trajectory from Assumption 4 the optimal trajectory starting in the steady-state, i. e., $u_c = u^*_{N,x_s,H^s}$ which is again possible since we ensured $x_{u^*_{N,x_s,H^s}} \in \mathcal{B}_{\delta_c}(x_s)$ for all $k \in \mathbb{I}_{[0,d+T]}$, $\rceil H - H^s \lceil \leq E_h$, $\rceil H^s \lceil = 0 < E_h$. Again, all conditions of Assumption 4 are satisfied. The resulting connecting trajectory is also shown in Figure 4.4b. As before, the time period T, in which the values of the auxiliary output are relevant for the overlapping period, are highlighted

yellow in Figure 4.4. Note that the optimal trajectory starting in x_s does not necessarily have to stay at the steady-state all the time. Extending u_2 via $u_2(k) := u^*_{N,x_s,H^s}(k)$ for $k \in \mathbb{I}_{[d,N-1]}$, it follows from (4.4a) for all $k \in \mathbb{I}_{[0,d-1]}$

$$\begin{aligned} &l(x_{u_2}(k,x_1),u_2(k)) \\ &\quad \leq l(x_{u^*_{N,x_s,H^s}}(k,x_s), u^*_{N,x_s,H^s}(k)) + \gamma_c(\|x - x_s\| + \lceil H - H^s \rceil) \end{aligned} \tag{4.36}$$

because of $x_3 = x_{u^*_{N,x_s,H^s}}(d,x_s)$ and since the subsequent trajectories are the same for all $k \in \mathbb{I}_{[d,d+T-2]}$ (which reduces ζ in Assumption 4). Moreover, the extension of u_2 brings $\ell(x_{u_2}(k,x_s),u_2(k)) = \ell(x_{u^*_{N,x_s,H^s}}(k,x_s), u^*_{N,x_s,H^s}(k))$ for all $k \in \mathbb{I}_{[d,N-1]}$ which yields

$$\begin{aligned} J^*_N(x,H) \leq & J_N(x_1, u_2) \\ \overset{(4.36)}{\leq} & \sum_{k=0}^{N-1} l(x_{u^*_{N,x_s,H^s}}(k,x_s), u^*_{N,x_s,H^s}(k)) + d\gamma_c(\|x - x_s\| + \lceil H - H^s \rceil) \\ = & J^*_N(x_s,H^s) + d\gamma_c(\|x - x_s\| + \lceil H - H^s \rceil) \\ \leq & J^*_N(x_s,H^s) + d\gamma_c(\|x - x_s\| + \|H - H^s\|_1), \end{aligned}$$

where the first inequality follows from optimality and the second one from the construction of u_2 via Ass. 4. Again, using $\gamma_v(\|x - x_s\| + \|H - H^s\|_1) = d\gamma_c(\|x - x_s\| + \|H - H^s\|_1)$ from Part I of the proof, we arrive at

$$J^*_N(x,H) - J^*_N(x_s,H^s) \leq +\gamma_v(\|x - x_s\| + \|H - H^s\|_1). \tag{4.37}$$

Part III: Conclusion
Combining (4.35) and (4.37) immediately brings that it holds

$$\begin{aligned} |J^*_N(x,H) - J^*_N(x_s,H^s)| &\leq \gamma_v(\|x - x_s\| + \|H - H^s\|_1) \\ &\leq \gamma_v(\delta + E), \end{aligned}$$

where the last inequality follows from the conditions $x \in \mathcal{B}_\delta(x_s)$ and $\|H - H^s\| \leq E$ as well as $\gamma_v \in \mathcal{K}_\infty$. □

In the following remark, we consider the special case $T = 1$ which means that the transient average constraints degenerate to additional point-wise in time constraints.

Remark 12. As discussed in Remark 4, the storage H vanishes for $T = 1$ and our result from Theorem 4 simplifies to

$$|J_N^*(x) - J_N^*(x_s)| \leq \gamma_v(\|x - x_s\|) \leq \gamma_v(\delta)$$

for all $x \in \mathcal{B}_{\delta_c}$ with $\gamma_v(\|x - x_s\|) = d\gamma_c(\|x - x_s\|)$ and $\gamma_v, \gamma_c \in \mathcal{K}_\infty$. Thus, Theorem 4 contains the special case $T = 1$ which is described in [Grüne, 2013, Thm. 6.4].

In this chapter, we introduced a local controllability condition (Ass. 4) in order to construct different candidate trajectories which were used to show local continuity of the value function (Thm. 4). This result is used in the following such that we obtain convergence of the closed-loop cost.

5. Performance Guarantees

In this chapter, we use the previous bounds (Thm. 4) in order to show convergence of the closed-loop cost function. Furthermore, by showing that similar turnpike results and value function bounds (e. g. Thm. 3, 2, 3 4) also hold for the minimization of the rotated optimization problem, we prove convergence of the closed-loop rotated cost fucntional. Additionally, Section 5.2 shows trajectory convergence of the closed-loop based on the results of Section 5.1.

5.1. Value Convergence

In the following section, we investigate the closed-loop costs $J_K^{\mathrm{cl}}(x,H)$ from (2.14). The main contributions of this section are Theorem 5 and 6.

5.1.1. Convergence of the Closed-Loop Cost Function

In the subsequent proof of Theorem 5, we extend a suitable control sequence to obtain a sequence of length $N+1$ for which the difference between J_N^* and J_{N+1} can be bounded. As discussed in [Grüne, 2013, Sec. 4], this is a standard MPC technique to give an upper bound on the closed-loop cost functional. Since the open-loop optimal trajectories for a finite prediction horizon do not end up in specific regions, we do not construct an extended control sequence by appending an additional element,e. g. using a local auxiliary control law (as it is done if a terminal region/ cost is imposed [Müller et al., 2014a]). Analogously to [Grüne, 2013], we insert an additional element at an arbitrary place in the control sequence where the extended state (x,H) is in a neighborhood of the optimal steady-state and denote the corresponding time instant k_x.

We start by introducing the dynamic programming principle [Bertsekas, 1995]. The dynamic programming principle (DPP) states that end-pieces of optimal trajectories are again optimal. For the optimal control sequence $u^*_{N,x,H}$ with finite horizon N and each $K \in \mathbb{I}_{[1,N-1]}$, we can split up the original cost functional, i. e., it holds

$$J_N^*(x,H) = \sum_{k=0}^{K-1} \ell\left(x_{u_{N,x,H}^*}(k,x), u_{N,x,H}^*(k)\right) + J_{N-K}^*(x_{u_{N,x,H}^*}(K,x),H(K)) \quad (5.1)$$

with $H(K) = H(x_{u_{N,x,H}^*}(K-T+1,x), u_{N,x,H}^*(K-T+1:K-1))$, i. e., $H(K)$ stores the $T-1$ past auxiliary output values of $x_{u_{N,x,H}^*}(K,x)$. We remark that the notation for the matrix representation H in the closed cloop was introduced in Section 2.1.1.

Since we append in the proof of the following theorem an optimal control sequence to a point in a specific neighborhood of the steady-state with a previous trajectory such that $\rceil H - H^s \lceil$ is small enough, we note that using Theorem 4 implies that there exists $u \in \mathbb{U}^N(x,H)$ for any $x \in \mathcal{B}_{\delta_c}(x_s)$ and any $\rceil H - H^s \lceil \leq E_h$. Moreover, we show in the following Proposition that for the same neighborhood around the steady-state (i. e. δ_c and E_h) there is a single control step such that the state, auxiliary output and stage cost can be bounded by class $\mathcal{K}_\infty$ functions w. r. t. to the initial state and previous trajectory.

Proposition 5. *Let Ass. 1 and 4 hold. There exist γ_f, γ_l, $\gamma_y \in \mathcal{K}_\infty$ such that for all $\delta \in (0,\delta_c]$, all $x \in \mathcal{B}_\delta(x_s)$, all $E \in (0,E_h]$ and all $\rceil H - H^s \lceil \leq E$ there exists $u_x \in \mathbb{U}(x,H)$ such that $f(x,u_x) \in \mathbb{X}$ and the following inequalities hold:*

$$\|f(x,u_x) - x_s\| \leq \gamma_f(\delta + E), \quad (5.2a)$$

$$\|h(x,u_x) - h(x_s,u_s)\| \leq \gamma_y(\delta + E), \quad (5.2b)$$

$$\ell(x,u_x) - \ell(x_s,u_s) \leq \gamma_l(\delta + E). \quad (5.2c)$$

Proof. We show that the assertions hold for $u_x = u_2(0)$ with u_2 from Ass. 4. Therefore, we choose as the reference trajectory the steady-state-trajectory, i. e., $u_c(\cdot) \equiv u_s$ with $x_c = x_s$ and $H^c = H^s$. Now, for the given initial conditions we know from Ass. 4 that there exists $u_2 \in \mathbb{U}^d(x,H)$ which is connecting the initial values (x,H) with the steady-state trajectory in d steps, i. e., we also chooce $x_1 = x_s$ $x_{u_1}(\cdot,x_1) \equiv x_s$, $x_3 = x_s$ and $H^3 = H^s$. This yields

$$\zeta = \|x - x_s\| + \rceil H - H^s \lceil \leq \delta + E$$

and hence, we obtain from (4.1)

$$\|x_{u_2}(k,x) - x_{u_c}(k,x_c)\| \overset{(4.1a)}{\leq} \gamma_x(\delta + E),$$

$$\|u_2(k) - u_c(k)\| \overset{(4.1b)}{\leq} \gamma_u(\delta + E),$$

for all $k \in \mathbb{I}_{[0,d]}$. Thus, with $u_x = u_2(0)$ (5.2a) holds with $\gamma_f := \gamma_x$. Furthermore, Lipschitz continuity of $h(x,u)$ and continuity of $\ell(x,u)$ from Ass. 1 imply

$$\begin{aligned}\|h(x,u_x) - h(x_s,u_s)\| &\leq L_h(\gamma_x(\delta + E) + \gamma_u(\delta + E))\\ \ell(x,u_x) - \ell(x_s,u_s) &\leq \alpha_l(\gamma_x(\delta + E) + \gamma_u(\delta + E))\end{aligned}$$

and hence, (5.2b) holds with $\gamma_y(\delta + E)) := L_h(\gamma_x(\delta + E) + \gamma_u(\delta + E))$ and (5.2c) holds with $\gamma_l(\delta + E) := \alpha_l(\gamma_x(\delta + E) + \gamma_u(\delta + E))$. Note that $\gamma_f\, \gamma_y, \gamma_l \in \mathcal{K}_\infty$ since $\gamma_x, \gamma_u, \alpha_l \in \mathcal{K}_\infty$ and $L_h > 0$. □

The following Theorem is a generalization of [Grüne, 2013, Prop. 4.1] and [Grüne, 2013, Thm. 4.2] to EMPC schemes subject to transient average constraints of the form (2.3).

Theorem 5. *Let Assumptions 1-4 hold. There exist $N_3 \in \mathbb{I}_{\geq N_2+T^2}$ and $\sigma_3 \in \mathcal{L}_{\mathbb{N}}$ such that for all $(x,H) \in \mathbb{X} \times \mathbb{H}$, all $K \in \mathbb{N}$ and all $N \geq N_3 + 1$ it holds with N_2 from Theorem 4*

$$\begin{aligned}J_K^{\mathrm{cl}}(x,H) \leq & J_N^*(x,H) - J_N^*(x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H))\\ &+ K\left(\ell(x_s,u_s) + \sigma_3(N - N_2 - T^2 - 1)\right).\end{aligned} \tag{5.3}$$

Proof. We split the following proof into three parts. In the first part we construct a feasible candidate input sequence. As already mentioned before, we insert one control sequence at a specific time instant k_x and hence, construct a sequence of length $N+1$ which we denote by $\hat{u}_{N,x,H} \in \mathbb{U}^{N+1}(x,H)$. For the second part of the proof we introduce a modified cost functional which excludes one time instant of the trajectory $\hat{u}_{N,x,H} \in \mathbb{U}^{N+1}(x,H)$ such that it is again a sum of N time instants, i. e.,

$$J_N'(x,\hat{u}_{N,x,H}) := \sum_{\substack{k=0\\ k\neq k_x}}^{N} \ell(x_{\hat{u}_{N,x,H}}(k,x), \hat{u}_{N,x,H}(k)). \tag{5.4}$$

Then, we show that this control sequence $\hat{u}_{N,x,H} \in \mathbb{U}^{N+1}(x,H)$ satisfies with some functions $\delta_1, \delta_2 \in \mathcal{L}_{\mathbb{N}}$

$$J_N'(x,\hat{u}_{N,x,H}) \leq J_N^*(x,H) + \delta_1(N - N_2 - T^2), \tag{$\mathcal{C}_1$}$$

$$\ell(x_{\hat{u}_{N,x,H}}(k_x,x), \hat{u}_{N,x,H}(k_x)) \leq \ell(x_s,u_s) + \delta_2(N - N_2 - T^2) \tag{$\mathcal{C}_2$}$$

with $J'_N(x)$ from (5.4), some later specified $k_x \in \mathbb{I}_{[0,N-N_2]}$ and N_2 from Theorem 4. These conditions mean that the additional cost of the candidate trajectory $\hat{u}_{N,x,H}$ (i. e. $J'_N(x,\hat{u})$) is smaller than the optimal cost plus a term which is decreasing for an increasing prediction horizon N. In addition, we can also bound the cost of the additionally inserted control value at time k_x with the second condition. Finally, it is shown in the third part by using the dynamic programming principle that the conditions ($\mathcal{C}_1$)-($\mathcal{C}_2$) from above imply assertion (5.3).

Part I: Constructing a candidate input sequence
We assemble the candidate input sequence from three different parts. First, we choose the prediction horizon large enough such that we can ensure by Theorem 3 that there exists a time instant k_x close enough to the steady-state and hence, can apply Proposition 5 to append one control input at this time instant. For the third stage, we use Theorem 4 in order to ensure that we can append an optimal trajectory starting from the end of the second stage. Note that it is crucial to choose the prediction horizon large enough in the first stage such that the neighborhood of the steady-state is small enough (cf. Theorem 3) in order to ensure that we can use Theorem 4 (third stage) after we constructed the second stage via Proposition 5. We start with the first stage which is constructed by Theorem 3 and Corollary 3, respectively. We set

$$k'_l = 0 \quad \text{and} \quad k'_u = N - N_2 \geq T^2 + 1 \tag{5.6}$$

with N_2 from Theorem 4. Note that it holds

$$N_2 \overset{\text{Thm.4}}{\geq} N_1 \overset{(4.9)}{\geq} 2 \tag{5.7}$$

and hence, it holds $k'_u \leq N-1$ which is a condition of Theorem 3. This choice of the bounds k'_l and k'_u implies that there are T consecutive time instants in the interval $\mathbb{I}_{[0,N-N_2]}$ in a specific neighborhood of the steady-state for the state, input and auxiliary output. We choose this upper bound k'_u to ensure that the appended trajectory in the third stage has an prediction horizon greater or equal to N_2 such that the conditions of Theorem 4 are satisfied.

As in Theorem 3, we denote the interval of time instants for which T consecutive values are in a neighborhood of the steady-state by $\mathbb{I}_{[k_x-T+1,k_x]}$, i. e., we denote for the last time instant of the consecutive points k_x. Now, we construct the candidate input sequence and show afterwards that this input sequence is feasible for N sufficiently large. Consider the abbreviations

$$\begin{aligned}
x' &:= x_{u^*_{N,x,H}}(k_x,x),\\
H' &:= H(x_{u^*_{N,x,H}}(k_x-T+1),\, u^*_{N,x,H}(k_x-T+1:k_x-1)),\\
x'' &:= f(x',u'_{x'}),\\
H'' &:= \begin{bmatrix} H'_2, & H'_3, & \dots, & H'_{T-1}, & h(x',u'_{x'}) \end{bmatrix},
\end{aligned}$$

where $u'_{x'}$ is the control value from Prop. 5 for $x = x'$ and $H = H'$. The candidate input sequence $\hat{u}_{N,x,H} \in \mathbb{U}^{N+1}(x,H)$ reads

$$\hat{u}_{N,x,H}(k) := \begin{cases} u^*_{N,x,H}(k), & k \in \mathbb{I}_{[0,k_x-1]} \\ u'_{x'}, & k = k_x \\ u^*_{N-k_x,x'',H''}(k-k_x-1), & k \in \mathbb{I}_{[k_x+1,N]} \end{cases}. \tag{5.8}$$

Note that $u^*_{N-k_x,x'',H''}$ is the optimal control sequence for initial conditions (x'',H'') and prediction horizon $N-k_x \geq N_2$. The corresponding state trajectory of this candidate input sequence (5.8) is shown in Figure 5.1. As shown there, we apply Prop. 5 at the time instant k_x, i.e., we append a single control value and hence, end up at x'' at time k_x+1. There, we use Theorem 4 in order to set up the third stage of the candidate which consists of at least N_2 time steps.

Now, we show that the candidate sequence (5.8) is feasible and start by recalling some known results. From Theorem 3 and Corollary 3 we know that due to our choices (5.6), T consecutive values are in a specific neighborhood of the steady-state which reads for the state and the auxiliary output as follows

$$\begin{aligned}
\left\| (x_{u^*_{N,x,H}}(k,x) - x_s,\, u^*_{N,x,H}(k) - u_s) \right\| &\leq \rho^{-1}\left(\frac{T\hat{C}'}{N-N_2-T^2}\right),\\
\left\| h(x_{u^*_{N,x,H}}(k,x), u^*_{N,x,H}(k)) - h(x_s,u_s) \right\| &\leq L_h\rho^{-1}\left(\frac{T\hat{C}'}{N-N_2-T^2}\right),
\end{aligned} \tag{5.9}$$

with $\hat{C}'$ from (3.25), for all $k \in \mathbb{I}_{[k_x-T+1,k_x]}$ where it holds $k_x \in \mathbb{I}_{[T-1,N-N_2]}$. Analogously to the construction of the candidate in Lemma 2, we can choose $N-N_2$ large enough to ensure that these neighborhoods are sufficiently

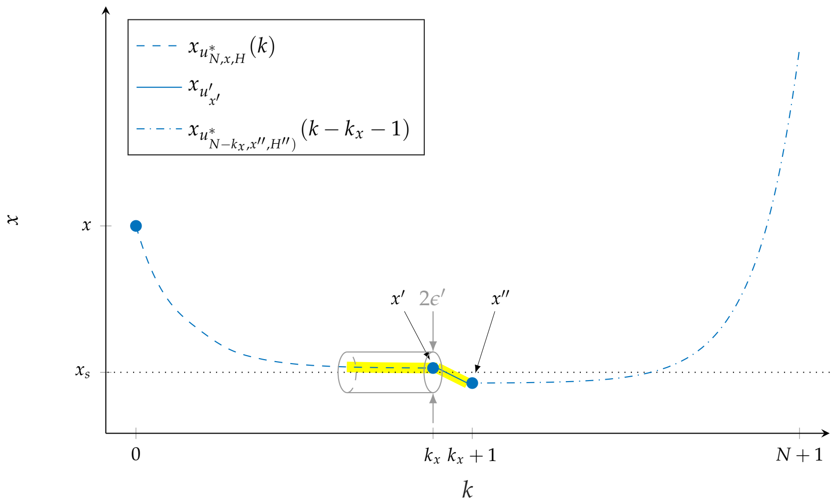


Figure 5.1.: Candidate state trajectory to prove Theorem 5.

small. In order to satisfy the conditions of Theorem 4, we have to make sure that it holds (note that $x_{u^*_{N,x,H}}(k_x,x) = x'$)

$$\begin{aligned} &\left\|(x_{\hat{u}_{N,x,H}}(k_x,x) - x_s,\ \hat{u}_{N,x,H}(k_x) - u_s)\right\| \\ =&\left\|(x_{u^*_{N,x,H}}(k_x,x) - x_s,\ u^*_{N,x,H}(k_x) - u_s)\right\| \leq \delta' \leq \delta_c \end{aligned} \tag{5.10}$$

and

$$0 \leq \left\|H' - H^s\right\|_1 \leq E' \leq E_h, \tag{5.11}$$

where δ' and E' need to be such that it also holds (note that $x_{\hat{u}_{N,x,H}}(k_x + 1,x) = x''$)

$$\begin{aligned} \left\|x_{\hat{u}_{N,x,H}}(k_x+1,x) - x_s\right\| &\leq \delta_c, \\ \left\|H'' - H^s\right\|_1 &\leq E_h. \end{aligned} \tag{5.12}$$

We show in the following that the choices

$$\delta' = \min\left\{\frac{1}{2}\gamma',\ \delta_c\right\} \quad \text{and} \quad E' = \min\left\{\frac{1}{2}\gamma',\ E_h\right\} \tag{5.13}$$

with

$$\gamma' := \min\left\{\gamma_f^{-1}(\delta_c),\, \gamma_y^{-1}(\frac{E_h}{\sqrt{p}})\right\} \tag{5.14}$$

ensure that (5.10)-(5.12) are satisfied and hence, the constructed input sequence is feasible.

It immediately follows that the choices (5.13) satisfy (5.10) and (5.11) and hence, that we can apply Prop. 5 in order to append a feasible single control input $u'_{x'}$ to the first stage. Now, we consider the next step after applying this control value. We obtain

$$\begin{aligned} \left\|x'' - x_s\right\| &\overset{(5.2a)}{\leq} \gamma_f(\delta' + E') \overset{(5.13)}{\leq} \gamma_f(\gamma') \overset{(5.14)}{\leq} \delta_c, \\ \left\|h(x',u'_{x'}) - h(x_s,u_s)\right\| &\overset{(5.2b)}{\leq} \gamma_y(\delta' + E') \overset{(5.13)}{\leq} \gamma_y(\gamma') \overset{(5.14)}{\leq} \frac{E_h}{\sqrt{p}}, \end{aligned} \tag{5.15}$$

where p is the dimension of the auxiliary output. Using monotonicity of l_p norms yields $\left\|h(x',u'_{x'}) - h(x_s,u_s)\right\| \leq \left\|h(x',u'_{x'}) - h(x_s,u_s)\right\|_1$ and we obtain by the definition of H

$$\left\|H'' - H^s\right\|_1 \leq \max\left\{\left\|H' - H^s\right\|_1,\, \sqrt{p}\left\|h(x',u'_{x'}) - h(x_s,u_s)\right\|\right\} \overset{(5.11),(5.15)}{\leq} E_h$$

as well as $\|x'' - x_s\| \leq \delta_c$. This means that we have chosen the neighborhood, in which T consecutive time instants are, small enough such that we know that we are still in a sufficiently small neighborhood after applying one control value (Prop. 4). Thus, (5.12) holds and we can use Theorem 4 at time $k_x + 1$ and formulate the condition for the size of the neighborhood around the steady-state. It follows from (5.9) (with σ_T from Theorem 3)

$$\begin{aligned} \epsilon = \rho^{-1}\left(\frac{T\hat{C}'}{N - N_2 - T^2}\right) &\overset{\text{Thm. } 3}{=} \sigma_T(N - N_2 - T^2) \\ &\overset{!}{\leq} \min\left\{\delta',\, \frac{E'}{\sqrt{p}L_h}\right\} =: \epsilon' \end{aligned} \tag{5.16}$$

and therefore, we choose the prediction horizon $N \in \mathbb{N}$ as follows

$$N \geq N_2 + \frac{T\hat{C}'}{\rho(\epsilon')} + T^2 =: N_3. \tag{5.17}$$

Note that the maximal possible neighborhood ϵ' is also shown exemplarily in Figure 5.1.

To conclude, we briefly recall why the candidate input sequence (5.8) and its corresponding trajectory is feasible. We pick $N \geq N_3$ and note that

$$\begin{aligned} \left\| x_{u^*_{N,x,H}}(k,x) - x_s,\, u^*_{N,x,H}(k) - u_s \right\| &\overset{(5.9)}{\leq} \sigma_T(N - N_2 - T^2) \overset{(5.16)}{\leq} \delta', \\ \left\| h(x_{u^*_{N,x,H}}(k,x), u^*_{N,x,H}(k)) - h(x_s,u_s) \right\| &\overset{(5.9)}{\leq} L_h \sigma_T(N - N_2 - T^2) \overset{(5.16)}{\leq} \frac{E'}{\sqrt{p}}, \end{aligned} \tag{5.18}$$

hold for all $k \in \mathbb{I}_{[k_x - T+1, k_x]}$ with $k_x \in \mathbb{I}_{[T-1, N-N_2]}$ for any $(x,H) \in \mathbb{X} \times \mathbb{H}$. Using properties of ℓ_p norms, the second inequality implies that (with $k \in \mathbb{I}_{[k_x - T+1, k_x]}$)

$$\begin{aligned} \|H' - H^s\|_1 &= \max_{j \in \mathbb{I}_{[1,T-1]}} \left\| H'_j - h(x_s,u_s) \right\|_1 \\ &\leq \sqrt{p} \left\| h(x_{u^*_{N,x,H}}(k,x), u^*_{N,x,H}) - h(x_s,u_s) \right\| \leq E' \end{aligned} \tag{5.19}$$

and hence, it holds from (5.13) $x_{u^*_{N,x,H}}(k_x,x) = x' \in \mathcal{B}_{\delta_c}(x_s)$ as well as $\|H' - H^s\|_1 \leq E_h$ such that we can apply Prop. 5. Therefore, we get that the second stage $u'_{x'}$ is feasible (also w. r. t. the transient average constraints). Furthermore, combining the choices (5.13) with (5.2a) and (5.2b) yields from (5.15)

$$\|x'' - x_s\| \leq \delta_c \quad \text{and} \quad \|H'' - H^s\|_1 \leq E_h.$$

Due to this and since $N - k_x \geq N_2$ (i. e., the third stage $u^*_{N-k_x, x'', H''}$ has at least the length N_2), we can conclude from Theorem 4 that there exists a feasible control sequence starting in x'' and H'', i. e., the third stage is feasible and hence, the candidate (5.8) is feasible.

After having shown feasibility of the candidate sequence, we give some estimates which will be used in Part II and III of this proof. First, we give an upper bound on $\|H'' - H^s\|_1$ as it is done in (5.19) for $\|H' - H^s\|_1$. We know that the auxiliary output at time k_x can be bounded via (5.2b) by

$$\begin{aligned} &\left\| h(x_{\hat{u}_{N,x,H}}(k_x,x), \hat{u}_{N,x,H}(k_x)) - h(x_s,u_s) \right\| \\ &\leq \gamma_y(\delta + E) \overset{(5.18)}{\leq} \gamma_y\left((1 + \sqrt{p} L_h) \sigma_T(N - N_2 - T^2) \right) \end{aligned} \tag{5.20}$$

which implies

$$\begin{aligned}
&\left\|h(x_{\hat{u}_{N,x,H}}(k_x,x),\hat{u}_{N,x,H}(k_x)) - h(x_s,u_s)\right\|_1\\
\leq&\sqrt{p}\left\|h(x_{\hat{u}_{N,x,H}}(k_x,x),\hat{u}_{N,x,H}(k_x)) - h(x_s,u_s)\right\|\\
\leq&\sqrt{p}\gamma_y\left((1+\sqrt{p}L_h)\sigma_T(N-N_2-T^2)\right).
\end{aligned}$$

where we use as before that it holds for all $x \in \mathbb{R}^{n_x}$ $\|x\|_1 \leq \sqrt{n_x}\,\|x\|_2 = \sqrt{n_x}\,\|x\|$. Then, by using (5.19) it follows

$$\begin{aligned}
&\left\|H'' - H^s\right\|_1\\
\leq&\max\left\{\left\|H' - H^s\right\|_1,\ \sqrt{p}\gamma_y\left((1+\sqrt{p}L_h)\sigma_T(N-N_2-T^2)\right)\right\}\\
\overset{(5.19)}{\leq}&\max\left\{\sqrt{p}L_h\sigma_T(N-N_2-T^2),\ \sqrt{p}\gamma_y\left((1+\sqrt{p}L_h)\sigma_T(N-N_2-T^2)\right)\right\}\\
=:&\sigma''_H(N-N_2-T^2) \overset{(5.19)}{\leq} E_h
\end{aligned} \tag{5.21}$$

Since both arguments of the maximum value are $\mathcal{L}_{\mathbb{N}}$ functions it follows $\sigma''_H \in \mathcal{L}_{\mathbb{N}}$, where we abbreviate in the following $\sigma''_H := \sigma''_H(N-N_2-T^2)$, $\sigma_T := \sigma_T(N-N_2-T^2)$. Furthermore, the candidate sequence (5.8) implies that $x_{\hat{u}_{N,x,H}}(k,x) = x_{u^*_{N,x,H}}(k,x)$ for $k \in \mathbb{I}_{[0,k_x]}$ as well as

$$\left\|x' - x_s\right\| \leq \sigma_T(N-N_2-T^2), \tag{5.22a}$$

$$\left\|x'' - x_s\right\| \leq \gamma_f((1+\sqrt{p}L_h)\sigma_T(N-N_2-T^2)), \tag{5.22b}$$

$$\ell(x',u'_{x'}) - \ell(x_s,u_s) \leq \gamma_l((1+\sqrt{p}L_h)\sigma_T(N-N_2-T^2)), \tag{5.22c}$$

where the second and third inequality follow from Proposition 5.

Part II: Show that the conditions ($\mathcal{C}_1$) and ($\mathcal{C}_2$) are satisfied
Since the conditions in Thm. 4 hold with $x', x'' \in \mathcal{B}_{\delta_c}(x_s)$ and $\|H' - H^s\|_1 \leq E_h$, $\|H'' - H^s\|_1 \leq E_h$ (cf. Part I) and since $K_1 := N - k_x \geq N_2$, we can apply Theorem 4 at (x',H') and (x'',H''). Hence, we can conclude

$$\begin{aligned}
J^*_{K_1}(x'',H'') - J^*_{K_1}(x_s,H^s) &\leq \left|J^*_{K_1}(x'',H'') - J^*_{K_1}(x_s,H^s)\right|\\
&\overset{\text{Thm. 4}}{\leq} \gamma_v\left(\gamma_f(\sigma_T + \sqrt{p}L_h\sigma_T) + \sigma''_H\right),
\end{aligned} \tag{5.23}$$

since it holds $\|x'' - x_s\| \leq \gamma_f(\sigma_T + \sqrt{p}L_h\sigma_T)$ from (5.22b) and from (5.21) $\|H'' - H^s\|_1 \leq \sigma''_H$. Analogously, we can apply Theorem 4 for $x = x'$ and $H = H'$ and get

$$J^*_{K_1}(x_s, H^s) - J^*_{K_1}(x', H') \overset{\text{Thm. 4}}{\leq} \gamma_v(\sigma_T + \sqrt{p}L_h\sigma_T), \tag{5.24}$$

since it holds $\|x' - x_s\| \leq \sigma_T$ from (5.22a) and $\|H' - H^s\|_1 \leq \sqrt{p}L_h\sigma_T$ from (5.19). By combining (5.23) and (5.24) it follows

$$\begin{aligned} &J^*_{K_1}(x'', H'') \\ \leq &J^*_{K_1}(x', H') + \gamma_v\left(\gamma_f(\sigma_T + \sqrt{p}L_h\sigma_T) + \sigma''_H\right) + \gamma_v(\sigma_T + \sqrt{p}L_h\sigma_T) \\ =: &J^*_{K_1}(x', H') + \delta_1(N - N_2 - T^2), \end{aligned} \tag{5.25}$$

with $\delta_1 \in \mathcal{L}_{\mathbb{N}}$ since $\gamma_f, \gamma_v \in \mathcal{K}_\infty$ and $\sigma_T, \sigma''_H \in \mathcal{L}_{\mathbb{N}}$. Note that we omitted the arguments of $\sigma_T(N - N_2 - T^2)$ and $\sigma''_H(N - N_2 - T^2)$ for reasons of clarity. Both inequalities (5.23) and (5.24) hold for any $K_1 \in \mathbb{N}$ with $K_1 = N - k_x \geq N_2$. Now, we can use (5.25) in order to obtain

$$\begin{aligned} \sum_{k=k_x+1}^{N} \ell(x_{\hat{u}_{N,x,H}}(k,x), \hat{u}_{N,x,H}(k)) &\overset{(5.8)}{=} J_{N-k_x}(x'', u^*_{N-k_x,x'',H''}) \\ &= J^*_{N-k_x}(x'', H'') \overset{(5.25)}{\leq} J^*_{N-k_x}(x', H') + \delta_1(N - N_2 - T^2). \end{aligned} \tag{5.26}$$

By definition of $J'_N(x)$ in (5.4) and the construction of our candidate (5.8) we obtain

$$\begin{aligned} &J'_N(x, \hat{u}_{N,x,H}) \\ \overset{(5.8),(5.4)}{=} &\sum_{k=0}^{k_x-1} \ell(x_{u^*_{N,x,H}}(k,x), u^*_{N,x,H}(k)) + \sum_{k=k_x+1}^{N} \ell(x_{\hat{u}_{N,x,H}}(k,x), \hat{u}_{N,x,H}(k)) \\ \overset{\text{DPP}}{=} &J^*_N(x, H) - J^*_{N-k_x}(x', H') + \sum_{k=k_x+1}^{N} \ell(x_{\hat{u}_{N,x,H}}(k,x), \hat{u}_{N,x,H}(k)) \\ \overset{(5.26)}{\leq} &J^*_N(x, H) + \delta_1(N - N_2 - T^2) \end{aligned}$$

which shows ($\mathcal{C}_1$) with (cf. (5.25))

$$\delta_1 = \gamma_v\left(\gamma_f(\sigma_T + \sqrt{p}L_h\sigma_T) + \sigma''_H\right) + \gamma_v(\sigma_T + \sqrt{p}L_h\sigma_T)$$

where $\sigma_T = \sigma_T(N - N_2 - T^2)$ and $\sigma_H'' = \sigma_H''(N - N_2 - T^2)$.

This means that we can extend our optimal control sequence by inserting a single additional control input at some time point k_x such that the difference between this modified input sequence, where the inserted time instant is excluded by definition of J_N', and the optimal one can be upper bounded. Now, we take a look at this additionally inserted time instant k_x. From (5.22c) we get

$$\ell(x_{\hat{u}_{N,x,H}}(k_x,x), \hat{u}_{N,x,H}(k_x)) \leq \ell(x_s,u_s) + \gamma_l\left((1+\sqrt{p}L_h)\sigma_T(N - N_2 - T^2)\right),$$

which shows ($\mathcal{C}_2$) with

$$\delta_2(N - N_2 - T^2) := \gamma_l\left((1+\sqrt{p}L_h)\sigma_T(N - N_2 - T^2)\right)$$

and hence, the cost of this additional input at time k_x can be upper bounded, too. Thus, we have shown that there are $N_3 \in \mathbb{N}$ and $\delta_1, \delta_2 \in \mathcal{L}_{\mathbb{N}}$ such that for each $(x,H) \in \mathbb{X} \times \mathbb{H}$ and $N \geq N_3$ the candidate input sequence (5.8) and corresponding trajectory satisfy the conditions ($\mathcal{C}_1$) and ($\mathcal{C}_2$) with $\delta_1, \delta_2 \in \mathcal{L}_{\mathbb{N}}$ as given above and $k_x \in \mathbb{I}_{[T-1,N-N_2]}$.

Part III: Showing the assertion (5.3)

Now, we consider the closed-loop and fix $(x,H) \in \mathbb{X} \times \mathbb{H}$ as well as $N \geq N_3 + 1$. From the dynamic programming principle (5.1) with $K = 1$, we get for any time instant $k \geq 0$ by using the closed-loop notation from Section 2.1.1

$$\begin{aligned}&\ell\left(x_{\mu_N}(k,x,H), \mu_N(k,x,H)\right)\\ =&J_N^*(x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H)) - J_{N-1}^*(x_{\mu_N}(k+1,x,H),H^{\mathrm{cl}}(k+1,x,H)).\end{aligned}$$

By summing up this equality for $k = 0, \ldots, K-1$, we obtain

$$\begin{aligned}J_K^{\mathrm{cl}}(x,H) =& \sum_{k=0}^{K-1} \ell(x_{\mu_N}(k,x,H),\mu_N(k,x,H))\\ =&\sum_{k=0}^{K-1}\left(J_N^*(x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H)) - J_{N-1}^*(x_{\mu_N}(k+1,x,H),H^{\mathrm{cl}}(k+1,x,H))\right)\\ =&J_N^*(x,H) - J_{N-1}^*(x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H))\\ &+\sum_{k=1}^{K-1} J_N^*(x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H)) - J_{N-1}^*(x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H)).\end{aligned} \tag{5.27}$$

Now, we investigate the terms in the sum of the right hand side of (5.27). Using ($\mathcal{C}_1$) with $N-1$ instead of N implies

$$\begin{aligned}&J^*_{N-1}(x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H))\\ \geq &J'_{N-1}(x_{\mu_N}(k,x,H),\hat{u}_{N-1,x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H)}) - \delta_1(N-N_2-T^2-1).\end{aligned} \tag{5.28}$$

We remark that $\hat{u}_{N-1,x,H} \in \mathbb{U}^N(x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H))$ and note that we can use the prediction horizon $\tilde{N} = N-1$ since we choose $N \geq N_3 + 1$ such that $\tilde{N} \geq N_3$ is still satisfied. Furthermore, due to optimality of J^*_N we get that it holds

$$\begin{aligned}&J^*_N(x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H))\\ \leq &J_N(x_{\mu_N}(k,x,H),\hat{u}_{N-1,x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H)}).\end{aligned}$$

By using this for the first term in the sum of (5.27) as well as substituting the second term there with (5.28), we can upper bound the summands with

$$\begin{aligned}&J^*_N(x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H)) - J^*_{N-1}(x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H))\\ \leq &J_N(x_{\mu_N}(k,x,H),\hat{u}_{N-1,x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H)})\\ &- J'_{N-1}(x_{\mu_N}(k,x,H),\hat{u}_{N-1,x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H)}) + \delta_1(N-N_2-T^2-1).\end{aligned} \tag{5.29}$$

Note that from the definition in (5.4) it follows

$$\begin{aligned}&J_N(x_{\mu_N}(k,x,H),\hat{u}_{N-1,x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H)})\\ &\quad - J'_{N-1}(x_{\mu_N}(k,x,H),\hat{u}_{N-1,x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H)})\\ =&\ell(x_{\hat{u}_{N-1,x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H)}}(k_x,x_{\mu_N}(k,x,H)),\ \hat{u}_{N-1,x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H)}(k_x))\\ \overset{(\mathcal{C}_2)}{\leq} &\ell(x_s,u_s) + \delta_2(N-N_2-T^2-1),\end{aligned}$$

where the inequality follows from ($\mathcal{C}_2$) with $N-1$ instead of N. The equality follows directly from the definition of J'_{N-1} in (5.4). By inserting this inequality in (5.29) we obtain

$$\begin{aligned}&J^*_N(x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H)) - J^*_{N-1}(x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H))\\ \leq &\ell(x_s,u_s) + \delta_2(N-N_2-T^2-1) + \delta_1(N-N_2-T^2-1).\end{aligned} \tag{5.30}$$

Now, combining this with (5.27) for $k \in \mathbb{I}_{[1,K-1]}$ yields

$$\begin{aligned}
&J_K^{\mathrm{cl}}(x,H) \\
&\leq \left(\sum_{k=1}^{K-1} \ell(x_s,u_s) + \delta_2(N - N_2 - T^2 - 1) + \delta_1(N - N_2 - T^2 - 1) \right) \\
&\quad + J_N^*(x,H) - J_{N-1}^*(x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H)) \\
&=(K-1)(\ell(x_s,u_s) + \delta_2(N - N_2 - T^2 - 1) + \delta_1(N - N_2 - T^2 - 1)) \\
&\quad + J_N^*(x,H) - J_{N-1}^*(x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H)).
\end{aligned}$$

Furthermore, using (5.30) for $k = K$ yields

$$\begin{aligned}
J_K^{\mathrm{cl}}(x,H) \leq J_N^*(x,H) &- J_N^*(x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H)) \\
&+ K\left(\ell(x_s,u_s) + \delta_1(N - N_2 - T^2 - 1) + \delta_2(N - N_2 - T^2 - 1)\right)
\end{aligned}$$

with $\delta_1, \delta_2 \in \mathcal{L}_{\mathbb{N}}$ from the end of Part II of this proof. This shows the assertion (5.3) for

$$\sigma_3(N - N_2 - T^2 - 1) := \delta_1(N - N_2 - T^2 - 1) + \delta_2(N - N_2 - T^2 - 1),$$

for any $N \geq N_3 + 1$ with N_3 from (5.17) and $\sigma_3 \in \mathcal{L}_{\mathbb{N}}$. □

In order to consider the infinite horizon closed-loop costs, we modify the finite time closed-loop cost functional (2.14) in order to obtain an averaged functional which reads

$$\begin{aligned}
\overline{J}_\infty^{\mathrm{cl}}(x,H) &:= \limsup_{K\to\infty} \frac{1}{K} J_K^{\mathrm{cl}}(x,H) \\
&\overset{(2.14)}{=} \limsup_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} \ell(x_{\mu_N}(k,x,H),\mu_N(x_{\mu_N}(k,x,H))).
\end{aligned} \tag{5.31}$$

After having introduced this value, we can extend Theorem 5 in the following Corollary.

Corollary 5. *Let Assumptions 1, 2, 3, and 4 hold. Then it holds for all* $(x,H) \in \mathbb{X} \times \mathbb{H}$ *and all* $N \geq N_3 + 1$

$$\overline{J}_\infty^{\mathrm{cl}}(x,H) \leq \ell(x_s,u_s) + \sigma_3(N - N_2 - T^2 - 1)$$

with $N_2 \in \mathbb{N}$ *from Thm. 4 as well as* $\sigma_3 \in \mathcal{L}_{\mathbb{N}}$ *and* $N_3 \in \mathbb{N}$ *from Thm. 5.*

Proof. It follows that (5.3) holds with $\sigma_3(N-N_2-T^2-1)$ as it is given in the proof of Theorem 5 and $N_3 = N_2 + \frac{T\hat{C}'}{\rho(\epsilon')} + T^2$ with N_2 from Theorem 4 and ϵ' from (5.16). We note that due to Assumption 1, the constraint set is compact (and therefore, bounded) and thus, the stage cost $\ell(x,u)$ is bounded. Hence, we get that for finite N the cost functionals $J_N^*(x,H)$ and $J_N^*(x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H))$ are finite by assumption. By using (5.31), we obtain

$$\begin{aligned}
&\overline{J}_\infty^{\mathrm{cl}}(x,H) = \limsup_{K\to\infty} \frac{1}{K} J_K^{\mathrm{cl}}(x,H)\\
&\overset{(5.3)}{\leq} \limsup_{K\to\infty} \frac{1}{K}\left(J_N^*(x,H) - J_N^*(x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H)) + K\left(\ell(x_s,u_s)+\sigma_3\right)\right)\\
&=\ell(x_s,u_s) + \sigma_3(N-N_2-T^2-1)
\end{aligned}$$

with $\sigma_3 = \sigma_3(N-N_2-T^2-1)$ which shows the assertion of Corollary 5. $\square$

The following remark considers the special case $T=1$ which means that the transient average constraints degenerate to additional point-wise in time constraints.

Remark 13. We recall that for the special case $T=1$, Remark 4 implies that the matrix representation H vanishes since we do not consider more than one consecutive time instant. Using this, we obtain from (5.3)

$$J_K^{\mathrm{cl}}(x) \leq J_N^*(x) - J_N^*(x_{\mu_N}(K,x)) + K\left(\ell(x_s,u_s) + \sigma_3(N-N_2-2)\right)$$

with $\sigma_3(N-N_2-2) = \delta_1(N-N_2-2) + \delta_2(N-N_2-2)$, where

$$\begin{aligned}
\delta_1(N-N_2-2) &= \gamma_v\left(\gamma_f(\sigma_T + \sqrt{p}L_h\sigma_T)\right) + \gamma_v(\sigma_T + \sqrt{p}L_h\sigma_T),\\
\delta_2(N-N_2-2) &= \gamma_l\left(\sigma_T + \sqrt{p}L_h\sigma_T\right),
\end{aligned}$$

if we choose $\sigma_H'' = 0$ which is possible since we the storage H vanishes for $T=1$. From (3.25) we get $\hat{C}' = \delta + C$ for $T=1$ and moreover, we demand that $k_x \in \mathbb{I}_{[0,N-N_2]}$ (cf. proof of Thm. 5). Furthermore, for $T=1$ we get from Theorem 3 the neighborhood of the steady-state

$$\sigma_T(N-N_2-2) = \rho^{-1}\left(\frac{(\delta+C)}{(N-N_2-2)}\right). \tag{5.32}$$

as well as the required prediction horizon $N \geq N_2 + \frac{\delta + C}{\rho(\epsilon')} + 2$. This yields that Theorem 5 includes for the special case $T = 1$ conceptionally the same result as in [Grüne, 2013, Sec. 4] with

$$\sigma(N) = \sigma_T(N - N_2 - 1) + \sqrt{p} L_h \sigma_T(N - N_2 - 1)$$

where the additional second term follows from the additional point-wise in time constraints. These additional constraints also lead to a slightly larger and more conservative prediction horizon.

5.1.2. Convergence of the Rotated Closed-Loop Cost Function

In order to prove different Lemmas in the following, we use Theorem 3 which ensures that we can choose the prediction horizon large enough such that the optimal trajectory (w. r. t. to the original as well as the rotated optimization problem) has at least T consecutive points in a neighborhood around the optimal steady-state. As in the proofs of Lemma 2 or Theorem 5 we will therefore construct these trajectories such that we can apply other theorems (for example Theorem 4). Similar to (3.9), we define the set

$$\begin{aligned} &\mathcal{T}^{\epsilon}_{[a,b]}(x,u) := \\ &\left\{ k_x \in \mathbb{I}_{[a,b]} : \|x_u(k_x - i,x) - x_s, u(k_x - i) - u_s\| \leq \epsilon,\ i \in \mathbb{I}_{[0,T-1]} \right\}. \end{aligned} \tag{5.33}$$

To ensure that the points k_x in this set satisfy the conditions in Theorem 3, we require $0 \leq a - T + 1 \leq b - T^2$ and $b \leq N - 1$ (with $k'_l = a - T + 1$ and $k'_u = b$, cf. (3.27)). This means that this set $\mathcal{T}^{\epsilon}_{[a,b]}(x,u)$ contains all time instants k_x for which the input sequence and the corresponding states are in neighborhood $\mathcal{B}_\epsilon(x_s,u_s)$ of the steady-state for this time instant and the past $T - 1$ time instants, where all these points are in the interval $\mathbb{I}_{[a-T+1,b]}$. This set contains all time instants k_x for which we can apply Theorem 4 to bound the cost function of trajectories starting at k_x.

Analogously to the intersection $\mathcal{P}'^{\epsilon}_{[k_l,k_u]}((u_1,x_1),\ldots,(u_q,x_q))$ from (3.20) we introduce the intersection set

$$\mathcal{T}'^{\epsilon}_{[a,b]}((u_1,x_1),\ldots,(u_q,x_q)) := \mathcal{T}^{\epsilon}_{[a,b]}(x_1,u_1) \cap \cdots \cap \mathcal{T}^{\epsilon}_{[a,b]}(x_q,u_q). \tag{5.34}$$

Note that compared to (3.20), the last intersection with the set $\{k_l,\ldots,k_u\}$ is not needed here since we already bound the interval of the possible time instants k_x by definition in (5.33). Analogously to (3.20), the input pair

(u_i,x_i) denotes the trajectory and its corresponding initial condition x_i. Since we want to use later in Lemma 5 that the set (5.34) is nonempty for specific bounds a and b, we show in Lemma 4 below how to choose the prediction horizon N such that this is ensured.

We continue by introducing the following lemma which is a modification of [Grüne, 2013, Lem. 7.3]. It states that the difference between the value function starting in (x,H) and the cost functional up to a time instant k_x and the value function for a prediction horizon $N-k_x$ starting in the steady-state is bounded by a class $\mathcal{K}$ function. This assertion holds for all time instants $k_x \in \mathcal{T}^{\epsilon}_{[T-1,N-N_2]}(x,u^*_{N,x,H})$ of the optimal trajectory, i. e., all time instants for which the point itself and the previous $T-1$ ones are in a specific steady-state neighborhood. Moreover, the size of the neighborhood is the argument of the previously mentioned $\mathcal{K}$ function. In order to characterize this neighborhood we introduce with δ_c and E_h from Assumption 4

$$\bar{\epsilon} := \min\left\{\delta_c, \frac{E_h}{\sqrt{p}L_h}\right\}. \tag{5.35}$$

Lemma 3. *Let Assumptions 1-4 hold. There exists $N_4 \in \mathbb{N}$ such that for all $(x,H) \in \mathbb{X}\times\mathbb{H}$, any $N \geq N_4$, the set $\mathcal{T}^{\epsilon}_{[T-1,N-N_2]}(x,u^*_{N,x,H})$ from (5.33) is nonempty with $\epsilon = \sigma_T(N-N_2-T^2) \leq \bar{\epsilon}$ and $\sigma_T \in \mathcal{L}_{\mathbb{N}}$, N_2 from Thm. 4, Thm. 3, respectively. Furthermore, there exists $R_1 \in \mathcal{K}_\infty$ such that for each $k_x \in \mathcal{T}^{\epsilon}_{[T-1,N-N_2]}(x,u^*_{N,x,H})$ it holds*

$$\left|J^*_N(x,H) - J_{k_x}(x,u^*_{N,x,H}) - J^*_{N-k_x}(x_s,H^s)\right| \leq R_1(\epsilon). \tag{5.36}$$

Proof. To show the assertion we proceed in two steps. First, we choose the prediction horizon large enough such that we can conclude that the optimal trajectory is in a sufficiently small neighborhood of the steady-state for some consecutive time instants. Then, we apply Theorem 4 and show (5.36) in the second part of the proof.

Part I: Trajectory construction

Since Assumptions 1-3 hold, we can apply Theorem 3. As in the proof of Theorem 5 we choose $k_l' = 0$ and $k_u' = N - N_2$ with N_2 from Theorem 4. Furthermore, we write as before k_x for the last time instant of the T consecutive points. Analogous to the proof of Theorem 5 we abbreviate

$$\begin{aligned} x' &:= x_{u^*_{N,x,H}}(k_x,x), \\ H' &:= H(k_x) = H(x_{u^*_{N,x,H}}(k_x-T+1,x),\, u^*_{N,x,H}(k_x-T+1:k_x-1)). \end{aligned}$$

In order to apply Theorem 4 in the second part of this proof, we need to satisfy

$$\begin{aligned} \left\|(x' - x_s, u^*_{N,x,H}(k_x) - u_s)\right\| &\leq \delta_c \\ \left\|H' - H^s\right\|_1 &\leq E_h. \end{aligned} \tag{5.37}$$

As in (5.16), we can ensure (5.37) due to

$$\begin{aligned} \epsilon = \rho^{-1}\left(\frac{T\hat{C}'}{N - N_2 - T^2}\right) &\overset{\text{Thm. 3}}{=} \sigma_T(N - N_2 - T^2) \\ &\overset{!}{\leq} \min\left\{\delta_c, \frac{E_h}{\sqrt{p}L_h}\right\} = \bar{\epsilon} \end{aligned} \tag{5.38}$$

by choosing

$$N \geq N_2 + \frac{T\hat{C}'}{\rho(\bar{\epsilon})} + T^2 =: N_4. \tag{5.39}$$

Note that $\delta' \leq \delta_c$ as well as $E' \leq E_h$ hold which implies $\epsilon' \leq \bar{\epsilon}$ (cf. (5.16)) and hence from (5.17) $N_3 \geq N_4$. Now, we know that a prediction horizon $N \geq N_4$ guarantees that there exists at least one $k_x \in \mathbb{I}_{[T-1,N-N_2]}$ such that the optimal trajectory $u^*_{N,x,H}$ satisfies

$$\left\|(x_{u^*_{N,x,H}}(k,x) - x_s, u^*_{N,x,H}(k) - u_s)\right\| \leq \sigma_T(N - N_2 - T^2) \leq \bar{\epsilon}$$

for all $k = \mathbb{I}_{[k_x-T+1,k_x]}$, i. e., the set $\mathcal{T}^{\epsilon}_{[T-1,N-N_2]}(x,u^*_{N,x,H})$ with $\epsilon = \sigma_T(N - N_2 - T^2)$ is nonempty for any $(x,H) \in \mathbb{X} \times \mathbb{H}$ and all $N \geq N_4$. Furthermore, this implies that (5.37) is satisfied for all $k_x \in \mathcal{T}^{\epsilon}_{[T-1,N-N_2]}(x,u^*_{N,x,H})$ and we can apply Theorem 4 for all these time instants.

Part II: Showing the assertion

From the dynamic programming principle (5.1) we get

$$J^*_N(x,H) = J_{k_x}(x,u^*_{N,x,H}) + J^*_{N-k_x}(x',H'). \tag{5.40}$$

with x' and H' as defined above. Since we have ensured by the choice of N and k_x that

$$\begin{aligned} \left\|(x_{u^*_{N,x,H}}(k,x) - x_s,\, u^*_{N,x,H}(k) - u_s)\right\| &\overset{(5.18)}{\leq} \sigma_T(N - N_2 - T^2) \leq \delta_c, \\ \left\|h(x_{u^*_{N,x,H}}(k,x),u^*_{N,x,H}(k)) - h(x_s,u_s)\right\| &\overset{(5.18)}{\leq} L_h\sigma_T(N - N_2 - T^2) \leq \frac{E_h}{\sqrt{p}}, \end{aligned}$$

hold for all $k \in \mathbb{I}_{[k_x-T+1,k_x]}$, it follows

$$\begin{aligned} \left\|x' - x_s\right\| &\leq \sigma_T(N - N_2 - T^2) \leq \delta_c, \\ \left\|H' - H^s\right\|_1 &\overset{\text{Cor.4}}{\leq} \sqrt{p} L_h \sigma_T(N - N_2 - T^2) \leq E_h, \end{aligned}$$

for any $k_x \in \mathcal{T}^{\sigma_T(N-N_2-T^2)}_{[T-1,N-N_2]}(x,u^*_{N,x,H})$. Now, we can apply Theorem 4 for all these k_x since it holds $N - k_x \geq N_2$. This implies

$$\left| J^*_{N-k_x}(x',H') - J^*_{N-k_x}(x_s,H^s) \right| \overset{(4.32)}{\leq} \gamma_v(\epsilon + \sqrt{p} L_h \epsilon), \tag{5.41}$$

with $\epsilon = \sigma_T(N - N_2 - T^2)$. By combining (5.40) and (5.41), we obtain

$$\begin{aligned} J^*_N(x,H) &\leq J_{k_x}(x,u^*_{N,x,H}) + J^*_{N-k_x}(x_s,H^s) + \gamma_v(\epsilon + \sqrt{p} L_h \epsilon), \\ J^*_N(x,H) &\geq J_{k_x}(x,u^*_{N,x,H}) + J^*_{N-k_x}(x_s,H^s) - \gamma_v(\epsilon + \sqrt{p} L_h \epsilon), \end{aligned}$$

which yields assertion (5.36) with $R_1(\epsilon) := \gamma_v(\epsilon + \sqrt{p} L_h \epsilon)$ and $R_1 \in \mathcal{K}_\infty$. $\square$

It follows from Theorem 3 that $u^*_{N,x,H}(0 : k_x - 1)$ lies in

$$\begin{aligned} &\overline{\mathbb{U}}^{k_x}(x,H,\epsilon) \\ &:= \left\{ u \in \mathbb{U}^{k_x}(x,H) \;\middle|\; \|(x_u(k,x) - x_s,\, u(k) - u_s)\| \leq \epsilon,\, k \in \mathbb{I}_{[k_x-T+1,k_x]} \right\} \end{aligned} \tag{5.42}$$

for all $k_x \in \mathcal{T}^{\epsilon}_{[T-1,N-N_2]}(x,u^*_{N,x,H}) \subseteq \mathcal{T}^{\bar{\epsilon}}_{[T-1,N-N_2]}(x,u^*_{N,x,H})$ since we require $\epsilon \leq \bar{\epsilon}$ in Lemma 3 which is ensured by the sufficiently large prediction horizon $N \geq N_4$. This means that the set $\overline{\mathbb{U}}^{k_x}(x,H,\epsilon)$ contains all input sequences for which the last T time instants of the state and input are in a neighborhood $\mathcal{B}_\epsilon(x_s,u_s)$.

Turnpike and Local Continuity Results for the Rotated Value Function

From now on, we will often correlate the results of the original optimization problem (2.10) and the rotated optimization problem (3.5). Therefore, we state in the following corollary under which conditions previous results also hold for the optimal open-loop trajectories of the original and the rotated problem. In order to show that some results also hold for the optimal open-loop trajectories of the rotated problem, we consider the rotated stage cost (3.4a)

$$\tilde{\ell}(x,u) \overset{(3.4a)}{=} \ell(x,u) - \ell(x_s,u_s) + \lambda(x) - \lambda(f(x,u)) + \bar{\lambda}^\top h(x,u)$$

and recall that it holds $\tilde{\ell}(x_s,u_s) = 0$.

Corollary 6. *Suppose the Assumptions 1, 2, 3 and 4 hold (for the original stage cost $\ell(x,u)$). Then, the statements in the Theorems 1, 2, 3 and 4, in the Corollaries 1 and 3 as well as in the Lemmas 2 and 3 also hold for the rotated stage cost $\tilde{\ell}(x,u)$ and the optimal open-loop trajectories of the rotated problem.*

Proof. We start the proof by showing that the corresponding assumptions also hold for the rotated stage cost $\tilde{\ell}(x,u)$ (3.4a), if they hold for the original cost $\ell(x,u)$. Therefore, we consider that the assumptions hold for the original stage cost $\ell(x,u)$ and show that then, they also hold for the rotated stage cost $\tilde{\ell}(x,u)$.

Part I: Showing the Assumptions

Continuity (Ass. 1):

Note that this Assumption always occurs together with the dissipativity Assumption 2 which ensures that the storage function $\lambda(x)$ is continuous. Furthermore, from (3.4a) it follows that $\tilde{\ell}(x,u)$ is continuous since $\ell(x,u)$, $f(x,u)$ and $h(x,u)$ are continuous (Ass. 1) as well as the storage function $\lambda(x)$ is continuous and the multiplier $\bar{\lambda}^\top$ is finite (Ass. 2).

Dissipativity (Ass. 2):

Given that the system is dissipative w. r. t. the supply rate $s(x,u) = \ell(x,u) - \ell(x_s,u_s) + \bar{\lambda}^\top h(x,u)$, we consider $\tilde{s}(x,u) = \tilde{\ell}(x,u)$, i. e, we choose the rotated multiplier $\tilde{\bar{\lambda}}^T \in \mathbb{R}^p_{\geq 0} = 0$ and note that $\tilde{\ell}(x_s,u_s) = 0$. Furthermore we set the rotated storage function to $\tilde{\lambda}(x) = 0$. Hence, we obtain

$$\begin{aligned}\tilde{s}(x,u) = \tilde{\ell}(x,u) &= \ell(x,u) - \ell(x_s,u_s) + \lambda(x) - \lambda(f(x,u)) + \bar{\lambda}^\top h(x,u)\\ &= s(x,u) + \lambda(x) - \lambda(f(x,u)) \geq \rho(\|(x - x_s, u - u_s)\|).\end{aligned}$$

The last inequality follows since we assume dissipativity of the supply rate $s(x,u)$. Hence, the system is strictly dissipative on $\mathbb{Z}$ with respect to the supply rate $\tilde{s}(x,u)$ with the storage function $\tilde{\lambda}(x) = 0$ and the multiplier $\tilde{\bar{\lambda}} = 0$. Hence, the rotation of the rotated stage cost is given by the rotated stage cost, i. e.,

$$\tilde{\tilde{\ell}}(x,u) = \tilde{\ell}(x,u). \tag{5.43}$$

Asymptotic controllability (Ass. 3):
This assumption holds by definition for $\tilde{\ell}(x,u)$ and since $\tilde{\tilde{\ell}}(x,u) = \tilde{\ell}(x,u)$ it also holds for the „rotation of the rotated cost".
Local controllability (Ass. 4):
This assumption does not contain any stage cost and hence holds for $\tilde{\ell}(x,u)$ as well.
Part II: Conclusion
Since all Assumptions are satisfied for the rotated stage $\tilde{\ell}(x,u)$ cost if they are satisfied for the original cost $\ell(x,u)$, we can conclude that the theorems, corollaries and lemmas mentioned in the assertion (which include the optimal open-loop trajectories of the original optimization problem (2.10)) also hold for the optimal open-loop trajectories of the rotated optimization problem (3.5) with different class $\mathcal{K}_\infty$ functions. □

Remark 14. As given in the proof of Corollary 6, the results of the original optimization problem given in the assertion of the corollary also hold for the rotated optimization problem, but with different class $\mathcal{K}_\infty$ functions. For simplicity, in the following we simply use the previously defined $\mathcal{K}_\infty$ functions to define the maximum of both, the function of the original and rotated optimization problem, i. e., $\sigma_1 := \max\{\sigma_1, \tilde{\sigma}_1\}$, $\sigma_T := \max\{\sigma_T, \tilde{\sigma}_T\}$, $\gamma_v := \max\{\gamma_v, \tilde{\gamma}_v\}$, $\eta := \max\{\eta, \tilde{\eta}\}$ and $R_1 := \max\{R_1, \tilde{R}_1\}$ and correspondingly N_i (with $i \in \{1, 2, 4, T, \eta\}$) defined with these new $\mathcal{K}_\infty$ functions.

Optimal Trajectories of the Open Loop Optimal Control Problem (2.10) and the Rotated Problem (3.5)

Since we use later in Lem. 5 that the set $\mathcal{T}'^{\epsilon}_{[T-1,N-N_2]}((u^*_{N,x,H},x), (\tilde{u}^*_{N,x,H},x))$ from (5.34) is nonempty, we show in the following Lemma 4 how to choose the prediction horizon such that this is ensured for a given $\epsilon > 0$.

Lemma 4. *Let the Assumptions 1, 2 and 3 hold. Then, there exist $\tilde{N}_T$, $k_x \in \mathbb{N}$ and $\tilde{\sigma}_T \in \mathcal{L}_{\mathbb{N}}$ such that for any $(x,H) \in \mathbb{X} \times \mathbb{H}$ and any k'_l, $k'_u \in \mathbb{N}$ satisfying*

$$0 \le k'_l < k'_u - T(2T-1) \quad \text{as well as} \quad \tilde{N}_T \le k'_u \le N-1, \tag{5.44}$$

there exists $k_x \in \mathbb{I}_{[k'_l+T-1,k'_u]}$ that the optimal trajectories $u^*_{N,x,H} \in \mathbb{U}^N(x,H)$ and $\tilde{u}^*_{N,x,H} \in \mathbb{U}^N(x,H)$ satisfy

$$\left\| (x_{u^*_{N,x,H}}(k,x) - x_s, u^*_{N,x,H}(k) - u_s) \right\| \le \epsilon,$$

$$\left\| (x_{\tilde{u}^*_{N,x,H}}(k,x) - x_s, \tilde{u}^*_{N,x,H}(k) - u_s) \right\| \le \epsilon,$$

for all $k \in \mathbb{I}_{[k_x-T+1,k_x]}$ *where* $\epsilon = \tilde{\sigma}_T(k'_u - \tilde{N}_T)$.

Proof. We proceed similar to the proofs of Theorem 3 and Corollary 3. In the first part we will construct $2T$ trajectories from the considered trajectories $u^*_{N,x,H} \in \mathbb{U}^N(x,H)$ and $\tilde{u}^*_{N,x,H} \in \mathbb{U}^N(x,H)$ in order to build an intersection of the corresponding sets. In the second part we choose based on Proposition 3 a neighborhood to show that the intersection of all considered trajectories contains at least one time instant k_x.

Part I: Constructed trajectories
As mentioned before, we construct the trajectories in the same way as we did it in the first part of the proof of Theorem 3 and therefore, we do not explicitly write down each step. We construct for both trajectories $u^*_{N,x,H} \in \mathbb{U}^N(x,H)$ and $\tilde{u}^*_{N,x,H} \in \mathbb{U}^N(x,H)$, T other optimal trajectories

$$u^*_i \in \mathbb{U}^{N_i}(x_i,H^i) \quad \text{and} \quad \tilde{u}^*_i \in \mathbb{U}^{N_i}(\tilde{x}_i,\tilde{H}^i)$$

with $i \in \mathbb{I}_{[0,T-1]}$, $N_i = N - i$, $x_i = x_{u^*_{N,x,H}}(i,x)$, $\tilde{x}_i = x_{\tilde{u}^*_{N,x,H}}(i,x)$, H^i as in (3.30) and $\tilde{H}^i$ defined analogously with $\tilde{u}^*_{N,x,H}$ instead of $u^*_{N,x,H}$. As in the proof of Theorem 3, we consider optimal trajectories[1] and hence it holds

$$\begin{aligned} x_{u^*_i}(k,x_i) &= x_{u^*_0}(k+i,x_0), & x_{\tilde{u}^*_i}(k,\tilde{x}_i) &= x_{\tilde{u}^*_0}(k+i,x_0), \\ u^*_i(k) &= u^*_0(k+i), & \tilde{u}^*_i(k) &= \tilde{u}^*_0(k+i), \end{aligned} \tag{5.45}$$

for all $k \in \mathbb{I}_{[0,N_i-1]} = \mathbb{I}_{[0,N_0-i-1]}$ and any $i \in \mathbb{I}_{[0,T-1]}$. For an illustration of $x_{u^*_0}$ and its shifted trajectories see Figure 3.1 and note that the optimal trajectory of the rotated problem, i. e., $x_{\tilde{u}^*}$ and its shifted trajectories share a similar turnpike property.

Part II: Choice of the neighborhood
First, we show that all necessary conditions Proposition 3 are fulfilled, namely we show that

$$J_{N_i}(x_i,u^*_i) \le N_i\ell(x_s,u_s) + \delta \quad \text{and} \quad \tilde{J}_{N_i}(\tilde{x}_i,\tilde{u}^*_i) \le \delta \tag{5.46}$$

hold for all $i \in \mathbb{I}_{[0,T-1]}$. The same arguments as in the proof of Theorem 3 (cf. Assumption 3) ensure that for all u^*_i with $i \in \mathbb{I}_{[0,T-1]}$ it holds $J^*_{N_i}(x_i,H^i) \le$

[1] They are optimal in the sense of minimizing the original cost functional (2.10) and the rotated cost functional (3.4b), respectively, for a prediction horion N_i with the corresponding initial conditions (x_i,H^i) and $(\tilde{x}_i,\tilde{H}^i)$, respectively.

$N_i\ell(x_s,u_s)+\delta$ with δ from (3.17). The same follows for the rotated cost functional from Assumption 3, i. e., we get

$$\tilde{J}_{N_i}(\tilde{x}_i,\tilde{u}_i^*) \overset{\text{Ass.3}}{\leq} N_i\tilde{\ell}(x_s,u_s)+\gamma_{\beta_1}(\|\tilde{x}_i-x_s\|)+\gamma_{\beta_2}(|\tilde{H}^i-H^s|)\leq\delta$$

with δ from (3.17) since $\tilde{\ell}(x_s,u_s)=0$ and hence, (5.46) holds for both types of control sequence, i. e., for the optimal open-loop trajectories resulting from the original and rotated problem. Together with the Assumptions 1 and 2, all conditions of Proposition 3 are satisfied such that we can apply it in the following. By considering the trajectories

$$x_{u_0^*}(\cdot,x_0),\ \ldots\ ,x_{u_{T-1}^*}(\cdot,x_{T-1}),x_{\tilde{u}_0^*}(\cdot,x_0),\ \ldots\ ,x_{\tilde{u}_{T-1}^*}(\cdot,\tilde{x}_{T-1})$$

we get $q=2T$, $\Delta_N=T-1$ from (3.22) and note that we aim for $m=1$. As in Theorem 3 and Corollary 3 we get $k_l=k_l'$ and $k_u=k_u'-(T-1)$. Now, we apply Proposition 3 and choose from (3.26) with $\hat{C}'$ from (3.25) the neighborhood

$$\begin{aligned}\epsilon&=\rho^{-1}\left(\frac{2T\hat{C}'}{(k_u'-(T-1))-k_l'-1-2T(T-1)}\right)\\&=\rho^{-1}\left(\frac{2T\hat{C}'}{k_u'-k_l'-T(2T-1)}\right)\end{aligned}\tag{5.47}$$

of the steady-state which ensures that the intersection

$$\begin{aligned}&\mathcal{P}'^{\epsilon}_{[k_l',k_u'-(T-1)]}\left((u_0^*,x),\ldots,(u_{T-1}^*,x_{T-1}),(\tilde{u}_0^*,x),\ldots,(\tilde{u}_{T-1}^*,\tilde{x}_{T-1})\right)\\:=&\mathcal{P}^{\epsilon}(u_0^*,x)\cap\cdots\cap\mathcal{P}^{\epsilon}(u_{T-1}^*,x_{T-1})\cap\mathcal{P}^{\epsilon}(\tilde{u}_0^*,x)\cap\cdots\cap\mathcal{P}^{\epsilon}(\tilde{u}_{T-1}^*,\tilde{x}_{T-1})\\&\cap\{k_l',\ldots,k_u'-(T-1)\}\end{aligned}$$

contains at least one element.

Part III: Conclude assertion
We denote the beforementioned element which lies in $\mathcal{P}'^{\epsilon}_{[k_l',k_u'-(T-1)]}$ by k_x, i. e., it holds

$$\begin{aligned}\left\|\left(x_{u_i^*}(k_x,x_i)-x_s,\ u_i^*(k_x)-u_s\right)\right\|&\leq\epsilon,\\\left\|\left(x_{\tilde{u}_i^*}(k_x,\tilde{x}_i)-x_s,\ \tilde{u}_i^*(k_x)-u_s\right)\right\|&\leq\epsilon,\end{aligned}\tag{5.48}$$

for all $i \in \mathbb{I}_{[0,T-1]}$ with ϵ from (5.47). Moreover, from (5.45) we obtain for all $i \in \mathbb{I}_{[0,T-1]}$

$$x_{u_i^*}(k_x,x_i) = x_{u_0^*}(k_x+i,x_0), \qquad u_i^*(k_x) = u_0^*(k_x+1),$$
$$x_{\tilde{u}_i^*}(k_x,\tilde{x}_i) = x_{\tilde{u}_0^*}(k_x+i,x_0), \qquad \tilde{u}_i^*(k_x) = \tilde{u}_0^*(k_x+1).$$

Combining these equalities with (5.48) yields for all $k \in \mathbb{I}_{[k_x-T+1,k_x]}$

$$\left\| \left(x_{u_0^*}(k,x_0) - x_s,\, u_0^*(k) - u_s \right) \right\| \leq \epsilon,$$
$$\left\| \left(x_{\tilde{u}_0^*}(k,x_0) - x_s,\, \tilde{u}_0^*(k) - u_s \right) \right\| \leq \epsilon,$$

which proves the assertion. Finally, it follows from (5.47) that $\epsilon =: \tilde{\sigma}_T(k_u' - \tilde{N}_T)$ with $\tilde{N}_T := k_l' + T(2T-1)$ and it holds $\tilde{\sigma}_T \in \mathcal{L}_{\mathbb{N}}$ since we have $k_u' \geq \tilde{N}_T$ due to (5.44). □

Remark 15. We highlight that Lemma 4 implies that the intersection set $\mathcal{T}'^{\epsilon}_{[k_l'+T-1,k_u']}((u^*_{N,x,H},x),\, (\tilde{u}^*_{N,x,H},x))$ from (5.34) is nonempty, given that k_l', k_u' and N satisfy the conditions of Lemma 4. Hence, we can ensure that $\mathcal{T}'^{\epsilon}_{[T-1,N-N_2]}((u^*_{N,x,H},x),\, (\tilde{u}^*_{N,x,H},x))$ is nonempty for any $\epsilon \leq \bar{\epsilon}$ by satisfying (we note again that N_2 is defined in Theorem 4)

$$\epsilon = \rho^{-1}\left(\frac{2T\hat{C}'}{N - N_2 - T(2T-1)} \right) \overset{!}{\leq} \bar{\epsilon}$$

which follows from (5.47) as well as $k_l' = 0$, $k_u' = N - N_2$. With $\epsilon = \bar{\epsilon}$ from (5.35), this is ensured for a prediction horizon

$$N \geq N_2 + \frac{2T\hat{C}'}{\rho(\bar{\epsilon})} + T(2T-1).$$

Remark 15 describes that Lemma 4 ensures the existence of k_x such that the optimal open-loop trajectories resulting from the original and the rotated problem, respectively, lie in the neighborhood $\mathcal{B}_\epsilon(x_s,u_s)$ with $\epsilon = \tilde{\sigma}_T(N - N_2 - T(2T-1))$ where $\tilde{\sigma}_T \in \mathcal{L}_{\mathbb{N}}$ for the time instant k_x and the previous $T-1$ ones. Since we will later also consider the intersection set $\mathcal{T}'^{\epsilon}_{[k_l'+T-1,k_u']}((u^*_{N,x_i,H^i},x_i),\, \ldots,\, (\tilde{u}^*_{N,\tilde{x}_i,\tilde{H}^i},\tilde{x}_i),\, \ldots)$, i.e., of more than two trajectories which may have different initial conditions but the same prediction horizon N, we generalize the results from Lemma 4 in the following corollary. There, we consider in total $\tilde{q}$ optimal trajectories which can be optimal trajectories resulting from the original or the rotated problem.

Corollary 7. *Let Assumptions 1-3 hold. For any $\tilde{q} \in \mathbb{N}$, there exist $\tilde{N}_{\tilde{q}}$, $k_x \in \mathbb{N}$ and $\tilde{\sigma}_{\tilde{q}} \in \mathcal{L}_{\mathbb{N}}$ such that for any $x_i, \tilde{x}_i \in \mathbb{X}$, any $H^i, \tilde{H}^i \in \mathbb{H}$, with $i \in \mathbb{I}_{[1,\tilde{q}]}$ and any k'_l, $k'_u \in \mathbb{N}$ satisfying*

$$0 \leq k'_l < k'_u - 1 - \tilde{q}(T^2 - 1) \quad \text{and} \quad \tilde{N}_{\tilde{q}} \leq k'_u \leq N - 1, \tag{5.49}$$

there is $k_x \in \mathbb{I}_{[k'_l+T-1,k'_u]}$ such that the optimal trajectories $u^*_{N,x_i,H^i} \in \mathbb{U}^N(x_i,H^i)$ and $\tilde{u}^*_{N,\tilde{x}_i,\tilde{H}i} \in \mathbb{U}^N(\tilde{x}_i,\tilde{H}^i)$ satisfy

$$\mathcal{T}'^{\epsilon}_{[k'_l+T-1,k'_u]}((u^*_{N,x_i,H^i},x_i),\ \ldots,\ (\tilde{u}^*_{N,\tilde{x}_i,\tilde{H}^i},\tilde{x}_i),\ \ldots) \neq \emptyset$$

where $\epsilon = \tilde{\sigma}_{\tilde{q}}(k'_u - \tilde{N}_T)$.

Proof. We use the same conceptional argumentation as in Theorem 3 and Lemma 4, i. e., we construct from each of the optimal trajectories $x_{u^*_{N,x_i,H^i}}$ and $x_{\tilde{u}^*_{N,\tilde{u}_i,\tilde{H}^i}}$, respectively, T other optimal trajectories which are shifted in time. Furthermore, the same argumentation as in the proof of Lemma 4 ensures that the conditions of Proposition 3 are satisfied and hence, we can use this Proposition with the amount of considered trajectories $q = \tilde{q}T$, $\Delta_N = T - 1$ from (3.22) and $m = 1$. As in Lemma 4 we get $k_l = k'_l$ and $k_u = k'_u - (T - 1)$. Now, we apply Proposition 3 and choose the neighborhood

$$\begin{aligned}\epsilon &= \rho^{-1}\left(\frac{\tilde{q}T\hat{C}'}{k'_u - (T-1) - k'_l - 1 - \tilde{q}T(T-1)}\right)\\ &= \rho^{-1}\left(\frac{\tilde{q}T\hat{C}'}{k'_u - k'_l - 1 - (T-1)(\tilde{q}T+1)}\right)\end{aligned}$$

which ensures that the intersection of all the optimal trajectories contains at least one element. Then, the same argumentation as in the proof of Lemma 4 and in Remark 15 ensures that there exists

$$k_x \in \mathcal{T}'^{\epsilon}_{[k'_l+T-1,k'_u]}((u^*_{N,x_i,H^i},x_i),\ \ldots,\ (\tilde{u}^*_{N,\tilde{x}_i,\tilde{H}^i},\tilde{x}_i),\ \ldots)$$

which proves the assertion. We get that $\epsilon = \tilde{\sigma}_{\tilde{q}}(k'_u - \tilde{N}_{\tilde{q}})$ with $\tilde{N}_{\tilde{q}} = k'_l + 1 + (T-1)(\tilde{q}T+1)$ and it holds $\tilde{\sigma}_{\tilde{q}} \in \mathcal{L}_{\mathbb{N}}$ since we have that $k'_u \geq \tilde{N}_{\tilde{q}}$. ☐

We note again that we can choose k_u' large enough by choosing a sufficiently large common prediction horizon N such that the assertion holds for any $\epsilon > 0$. Furthermore, for $\tilde{q} = 2$, we exactly obtain Lemma 4 and the neighborhood given in the proof above becomes the neighborhood in (5.47).

Since the rotated cost (3.4b) has an additional term which takes the auxiliary output into account, we can not proceed similarly to the case without transient average constraints [Grüne, 2013, Sec. 7]. It is possible to upper bound the auxiliary output values over the prediciton horizon N by using arguments of Sec. 2.2. However, we cannot lower bound the sum of the auxiliary output values over the prediction horizon and thus, we do not obtain absolute bounds on the rotated closed-loop cost as in Theorem 5. In order to prove value convergence of the rotated closed-loop cost, we introduce the following assumption on the auxiliary output of the optimal trajctories resulting from the rotated optimization problem (3.5). We remark that this assumption is similar to the asymptotic controllability property (Ass. 3), but this time for the auxiliary output h and the optimal trajectory of the rotated problem.

Assumption 5. For any $k_\psi \in \mathbb{I}_{\geq T-1}$, there exists a function $\psi \in \mathcal{K}_\infty$ such that the optimal trajectory resulting from the rotated optimization problem (3.5) satisfies

$$-\bar{\lambda}^\top \sum_{k=0}^{k_\psi - 1} h(x_{\tilde{u}^*_{N,x,H}}(k,x), \tilde{u}^*_{N,x,H}(k)) \leq \psi(\|x - x_s\|_1 + \|H - H^s\|_1) \quad (5.50)$$

for all $N \in \mathbb{N}$, $(x,H) \in \mathbb{X} \times \mathbb{H}$.

Since the rotated stage cost $\tilde{\ell}(x,u)$ is positive definite w. r. t. the optimal steady-state, we know that an optimal trajectory resulting from the rotated optimization problem (3.5), i. e. $x_{\tilde{u}^*_{N,x,H}}$, $\tilde{u}^*_{N,x,H}$, with the steady-state (x_s, H^s) as initial conditions, stays at the optimal steady-state which yields for all $k \in \mathbb{I}_{[0,N-1]}$ $\bar{\lambda}^\top h(x_{\tilde{u}^*_{N,x_s,H^s}}(k,x), \tilde{u}^*_{N,x_s,H^s}(k)) = 0$. Furthermore, this assumption is reasonable since the asymptotic controllability property (Ass. 3) implies in combination with the positive definite rotated stage cost $\tilde{\ell}$ that $x_{\tilde{u}^*_{N,x,H}}$ is converging to the optimal steady-state and since the constraint set $\mathbb{Z}$ is compact (Ass. 1). Moreover, if the system is exponentially stabilizable and the rotated stage cost is quadratic, then $x_{\tilde{u}^*_{N,x,H}}$ converges exponentially to x_s and thus Assumption 5 holds (using the Lipschitz bound (Ass. 1)).

The following Lemma, which is a generalization of [Grüne, 2013, Lem. 7.4, 7.5] for EMPC schemes subject to transient average constraints, shows a property of the optimal trajectories resulting from the original (2.10) and rotated optimization problem (3.5). In particular, it provides a relation between the costs of the optimal trajectories resulting from the original and rotated problem up to a time instant for which the previous $T-1$ steps are in a specific steady-state neighborhood.

Lemma 5. *Let Assumptions 1-5 hold. There exists $N_5 \in \mathbb{N}$ and $R_2 \in \mathcal{K}_\infty$ such that for all $(x,H) \in \mathbb{X} \times \mathbb{H}$, any $N \geq N_5$ and $\epsilon = \tilde{\sigma}_T(N - N_2 - T(2T-1))$, the intersection set $\mathcal{T}'^{\epsilon}_{[T-1,N-N_2]}((u^*_{N,x,H},x), (\tilde{u}^*_{N,x,H},x))$ from (5.34) is nonempty and for each $k_x \in \mathcal{T}'^{\epsilon}_{[T-1,N-N_2]}((u^*_{N,x,H},x), (\tilde{u}^*_{N,x,H},x))$ it holds*

$$
\begin{aligned}
J_{k_x}(x,u^*_{N,x,H}) \leq & \tilde{J}_{k_x}(x,\tilde{u}^*_{N,x,H}) + k_x \ell(x_s,u_s) - \lambda(x) & \text{(5.51a)}\\
& + \psi(\|x - x_s\|_1 + \|H - H^s\|_1) + R_2(\epsilon), \\
\tilde{J}_{k_x}(x,\tilde{u}^*_{N,x,H}) \leq & J_{k_x}(x,u^*_{N,x,H}) - k_x \ell(x_s,u_s) + \lambda(x) + R_2(\epsilon) & \text{(5.51b)}\\
& + (T-1) \left\|\bar{\lambda}\right\| L_h \epsilon,
\end{aligned}
$$

$\tilde{\sigma}_T \in \mathcal{L}_{\mathbb{N}}$ from Lemma 4.

Proof. Before splitting our proof in different parts, we note that we can choose from Lemma 4 the prediction horizon N large enough such that there exists $k_x \in \mathcal{T}'^{\epsilon}_{[T-1,N-N_2]}((u^*_{N,x,H},x), (\tilde{u}^*_{N,x,H},x))$, i. e., the set is nonempty for an arbitrary small $\epsilon > 0$. Namely, from (5.47) we get with $k'_u = N - N_2$ (with N_2 from Theorem 4) and $k'_l = 0$

$$
\epsilon = \rho^{-1}\left(\frac{2T\hat{C}'}{N - N_2 - T(2T-1)}\right) = \tilde{\sigma}_T\left(N - N_2 - T(2T-1)\right) \overset{!}{\leq} \bar{\epsilon}
$$

and hence, the choice

$$
N \geq N_2 + \frac{2T\hat{C}'}{\rho(\bar{\epsilon})} + T(2T-1) =: N_5, \tag{5.52}
$$

ensures that the set is nonempty and that it holds for $N \geq N_5$

$$
u^*_{N,x,H},\, \tilde{u}^*_{N,x,H} \in \overline{\mathbb{U}}^{k_x}(x,H,\tilde{\sigma}_T(N - N_2 - T(2T-1))). \tag{5.53}
$$

Now, we pick any time instant $k_x \in \mathcal{T}'^{\epsilon}_{[T-1,N-N_2]}((u^*_{N,x,H},x), (\tilde{u}^*_{N,x,H},x))$ and show the assertions in two steps. In the first step we show (5.51a) and in the second (5.51b). Note that it holds $N_5 \geq N_4$ by the definitions in (5.39) and (5.52), since the time period of the transient average constraints satisfies $T \in \mathbb{I}_{\geq 1}$. Hence, we can apply Lemma 3 for the input sequences $u^*_{N,x,H}$ and $\tilde{u}^*_{N,x,H}$ due to Corollary 6. We write in the following for any $k \geq T-1$

$$\begin{aligned}
H(k) :=& H(x_{u^*_{N,x,H}}(k-T+1,x), u^*_{N,x,H}(k-T+1:k-1)),\\
\tilde{H}(k) :=& H(x_{\tilde{u}^*_{N,x,H}}(k-T+1,x), \tilde{u}^*_{N,x,H}(k-T+1:k-1)).
\end{aligned}$$

Part I: Showing (5.51a).
We consider an input trajectory $u_1 \in \mathbb{U}^N(x,H)$

$$u_1(k) := \begin{cases} \tilde{u}^*_{N,x,H}(k) & k \in \mathbb{I}_{[0,k_x-1]} \\ u^*_{N-k_x, x_{\tilde{u}^*_{N,x,H}}(k_x,x), \tilde{H}(k_x)} & k \in \mathbb{I}_{[k_x,N-1]} \end{cases} \tag{5.54}$$

and note that $\|(x_{u_1}(k,x)-x_s,\, u_1(k)-u_s)\| \leq \epsilon$ holds for all $k \in \mathbb{I}_{[k_x-T+1,k_x]}$ with $\epsilon = \tilde{\sigma}_T(N-N_2-T(2T-1))$. Hence, Theorem 4 ensures the existence and feasibility of the input sequence constructed in (5.54) since $N-k_x \geq N_2$. Using Theorem 4 and Lemma 3 yields

$$\begin{aligned}
&J_{k_x}(x,u^*_{N,x,H}) + J^*_{N-k_x}(x_s,H^s) - R_1(\epsilon) \overset{\text{Lem.3}}{\leq} J^*_N(x,H) \leq J_N(x,u_1)\\
\overset{(5.54)}{=}& J_{k_x}(x,\tilde{u}^*_{N,x,H}) + J^*_{N-k_x}(x_{\tilde{u}_{N,x,H}}(k_x,x),\tilde{H}(k_x))\\
\overset{\text{Thm.4}}{\leq}& J_{k_x}(x,\tilde{u}^*_{N,x,H}) + J^*_{N-k_x}(x_s,H^s) + \gamma_v(\epsilon + \sqrt{p}L_h\epsilon)
\end{aligned} \tag{5.55}$$

and thus,

$$\begin{aligned}
J_{k_x}(x,\tilde{u}^*_{N,x,H}) \geq& J_{k_x}(x,u^*_{N,x,H}) - R_1(\epsilon) - \gamma_v(\epsilon+\sqrt{p}L_h\epsilon)\\
\overset{\text{Lem.3}}{=}& J_{k_x}(x,u^*_{N,x,H}) - 2\gamma_v(\epsilon+\sqrt{p}L_h\epsilon).
\end{aligned}$$

Now, using the definition of the rotated cost (3.4) yields

$$\begin{aligned}
&J_{k_x}(x,\tilde{u}^*_{N,x,H}) = \tilde{J}_{k_x}(x,\tilde{u}^*_{N,x,H}) + k_x\ell(x_s,u_s) - \lambda(x) + \lambda(x_{\tilde{u}^*_{N,x,H}}(k_x,x))\\
&- \sum_{k=0}^{k_x-1} \bar{\lambda}^\top h(x_{\tilde{u}^*_{N,x,H}}(k,x),\tilde{u}^*_{N,x,H}(k)) \geq J_{k_x}(x,u^*_{N,x,H}) - 2\gamma_v(\epsilon+\sqrt{p}L_h\epsilon).
\end{aligned}$$

By applying Assumption 5 and since it holds $\lambda(x_{\tilde{u}^*_{N,x,H}}(k_x,x)) \leq \alpha_\lambda(\epsilon)$ (cf. Ass. 2), we get

$$\begin{aligned}&J_{k_x}(x,u^*_{N,x,H})\\ \leq&\tilde{J}_{k_x}(x,\tilde{u}^*_{N,x,H}) + k_x\ell(x_s,u_s) - \lambda(x) + \psi(\|x-x_s\|_1 + \|H-H^s\|_1) + R_2(\epsilon),\end{aligned}$$

with $R_2 \in \mathcal{K}_\infty$ as follows

$$R_2(\epsilon) := 2\gamma_v(\epsilon + \sqrt{p}L_h\epsilon) + \alpha_\lambda(\epsilon), \tag{5.56}$$

which implies (5.51a).

Part II: Showing (5.51b).

We consider an input trajectory $u_2 \in \mathbb{U}^N(x,H)$

$$u_2(k) := \begin{cases} u^*_{N,x,H}(k) & k \in \mathbb{I}_{[0,k_x-1]} \\ \tilde{u}^*_{N-k_x,x_{u^*_{N,x,H}}(k_x,x),H(k_x)} & k \in \mathbb{I}_{[k_x,N-1]} \end{cases} \tag{5.57}$$

and note that it holds due to the choice of the prediction horizon N for all $k \in \mathbb{I}_{[k_x-T+1,k_x]}$

$$\|(x_{u_2}(k,x) - x_s,\, u_2(k) - u_s)\| \leq \epsilon, \tag{5.58}$$

with $\epsilon = \tilde{\sigma}_T(N - N_2 - T(2T-1))$. Hence, Theorem 4 ensures the existence and feasibility of the input sequence constructed in (5.57) since $N - k_x \geq N_2$. By using the same arguments as in Part I, it follows

$$\begin{aligned}\tilde{J}_{k_x}(x,\tilde{u}^*_{N,x,H}) + \tilde{J}^*_{N-k_x}(x_s,H^s) - R_1(\epsilon) &\overset{\text{Lem.3}}{\leq} \tilde{J}^*_N(x,H) \leq \tilde{J}_N(x,u_2)\\ &\overset{(5.57)}{=} \tilde{J}_{k_x}(x,u^*_{N,x,H}) + \tilde{J}^*_{N-k_x}(x_{u^*_{N,x,H}}(k_x,x),H(k_x))\\ &\overset{\text{Thm.4}}{\leq} \tilde{J}_{k_x}(x,u^*_{N,x,H}) + \tilde{J}^*_{N-k_x}(x_s,H^s) + \gamma_v(\epsilon + \sqrt{p}L_h\epsilon)\end{aligned} \tag{5.59}$$

and thus, we obtain

$$\tilde{J}_{k_x}(x,u^*_{N,x,H}) \geq \tilde{J}_{k_x}(x,\tilde{u}^*_{N,x,H}) - R_1(\epsilon) - \gamma_v(\epsilon + \sqrt{p}L_h\epsilon).$$

Using the definition of the rotated cost (3.4) yields

$$\begin{aligned}&\tilde{J}_{k_x}(x,\tilde{u}^*_{N,x,H}) - R_1(\epsilon) - \gamma_v(\epsilon + \sqrt{p}L_h\epsilon) \leq J_{k_x}(x,u^*_{N,x,H}) - k_x\ell(x_s,u_s) + \lambda(x)\\ &- \lambda(x_{u^*_{N,x,H}}(k_x,x)) + \sum_{k=0}^{k_x-1} \bar{\lambda}^\top h(x_{u^*_{N,x,H}}(k,x),u^*_{N,x,H}(k)).\end{aligned}$$

Now, we choose the largest $k_h \in \mathbb{I}_{[k_x-T+1,k_x]}$ which is a multiple of T, i. e., $k_h = jT$ with $j \in \mathbb{I}_{\geq 1}$ and obtain $\sum_{k=0}^{k_h-1} h(x_{u^*_{N,x,H}}(k,x),u^*_{N,x,H}(k)) \leq 0$ which yields together with (5.58) and Lipschitz continuity of h that it holds

$$\sum_{k=0}^{k_x-1} \bar{\lambda}^\top h(x_{u^*_{N,x,H}}(k,x),u^*_{N,x,H}(k)) \leq (T-1)\left\|\bar{\lambda}\right\| L_h \epsilon.$$

Using this result as well as $\lambda(x_{\tilde{u}^*_{N,x,H}}(k_x,x)) \leq \alpha(\epsilon)$ (cf. Ass. 2) implies

$$\begin{aligned}\tilde{J}_{k_x}(x,\tilde{u}^*_{N,x,H}) \leq & J_{k_x}(x,u^*_{N,x,H}) - k_x \ell(x_s,u_s) + \lambda(x) + R_2(\epsilon) \\ & + (T-1)\left\|\bar{\lambda}\right\| L_h \epsilon,\end{aligned}$$

with $R_2 \in \mathcal{K}_\infty$ from (5.56) and hence, (5.51b) is shown. □

The following theorem provides the main contribution of this chapter since it becomes fundamental later in order to prove practical asymptotic stability. More detailed, it is used in Lemma 7 in order to provide bounds on the closed-loop rotated value function. In contrast to EMPC without transient average constraints [Grüne, 2013, Proof of Thm. 7.6], the upper bound on the closed-loop rotated cost has two additional terms due to the transient average constraints. In particular, one term (ψ) follows from Ass. 5 and the second term follows from the closed-loop auxiliary output values.

Theorem 6. *Let Ass. 1-5 hold. There exists $N_6 \in \mathbb{N}$ and $\sigma_6 \in \mathcal{L}_{\mathbb{N}}$ such that*

$$\begin{aligned}&\tilde{J}^{\text{cl}}_K(x,H) \\ \leq & \tilde{J}^*_N(x,H) - \tilde{J}^*_N(x_{\mu_N}(K,x,H),H^{\text{cl}}(K,x,H)) + K\sigma_3(N-N_2-T^2-1) \\ & + \sigma_6(N-N_2-(4T+1)(T-1)-1) + \psi(\|x-x_s\|_1 + \|H-H^s\|_1) \\ & + \sum_{k=0}^{K-1} \bar{\lambda}^\top h_{\mu_N}(k,x,H)\end{aligned} \tag{5.60}$$

holds for any $(x,H) \in \mathbb{X} \times \mathbb{H}$, all $K \in \mathbb{N}$ and all $N \geq N_6 + 1$ with $\sigma_3 \in \mathcal{L}_{\mathbb{N}}$ from Theorem 5 and $\psi \in \mathcal{K}_\infty$ from Ass. 5. Furthermore, the last term satisfies $\sum_{k=0}^{K-1} \bar{\lambda}^\top h_{\mu_N}(k,x,H) \leq \min\{K,T-1\}\bar{\vartheta}_h$ for all $K \in \mathbb{N}$.

Proof. In order to show (5.60) we will apply Lemma 3 to the original and the rotated problem at initial values (x, H) and $(x_{\mu_N}(K,x,H), H^{\text{cl}}(K,x,H))$.

Hence, we need to ensure that those four trajectories (cf. Figure 5.2) have at least one time instant

$$\begin{aligned} k_x \in \mathcal{T}'^{\epsilon}_{[T-1,N-N_2]}\big((u^*_{N,x,H},x),\ &(u^*_{N,x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H)},x_{\mu_N}(K,x,H)),\\ (\tilde{u}^*_{N,x,H},x),\ &(\tilde{u}^*_{N,x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H)},x_{\mu_N}(K,x,H))\big) \end{aligned} \tag{5.61}$$

in common. We can ensure this property by choosing the prediction horizon large enough which follows from Corollary 7 with $\tilde{q} = 4$, $k'_l = 0$ and $k'_u = N - N_2$. Hence, we get from Corollary 7

$$\begin{aligned} \epsilon =& \rho^{-1}\left(\frac{4T\hat{C}'}{N - N_2 - (4T+1)(T-1) - 1}\right) \\ =&: \tilde{\sigma}_{c,1}(N - N_2 - (4T+1)(T-1) - 1), \end{aligned} \tag{5.62}$$

with $\tilde{\sigma}_{c,1} \in \mathcal{L}_{\mathbb{N}}$. Therefore, we choose

$$N \geq \max\left\{N_2 + \frac{4T\hat{C}'}{\rho(\bar{\epsilon})} + (4T+1)(T-1) + 1,\ N_3 + 1\right\} =: N_6 \tag{5.63}$$

with $\bar{\epsilon}$ from (5.35) and highlight that this ensures that the neighborhood (5.62) is small enough, i. e., $\epsilon \leq \bar{\epsilon}$ for $N \geq N_6$. Note that we additionally ensure $N \geq N_6 + 1 \geq N_3 + 1$ and hence, we can apply Theorem 5 later. This means that (5.63) ensures that the there exists at least one k_x satisfying (5.61) with $\epsilon \leq \bar{\epsilon}$ and hence, we can apply in the following Lemma 3 and Lemma 5 for these four trajectories at the same time instant k_x. Thus, we obtain from Lemma 3, Corollary 6 and Remark 14

$$\begin{aligned} J^*_N(x,H) \leq& J_{k_x}(x,u^*_{N,x,H}) + J^*_{N-k_x}(x_s,H^s) \\ &+ R_1(\epsilon), && (5.64a)\\ -J^*_N(x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H)) \leq& - J_{k_x}(x_{\mu_N}(K,x,H),u^*_{N,x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H)}) \\ &- J^*_{N-k_x}(x_s,H^s) + R_1(\epsilon), && (5.64b)\\ \tilde{J}_{k_x}(x,\tilde{u}^*_{N,x,H}) \leq& \tilde{J}^*_N(x,H) - \tilde{J}^*_{N-k_x}(x_s,H^s) + R_1(\epsilon), && (5.64c)\\ \tilde{J}^*_N(x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H)) \leq& \tilde{J}_{k_x}(x_{\mu_N}(K,x,H),\tilde{u}^*_{N,x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H)}) \\ &+ \tilde{J}^*_{N-k_x}(x_s,H^s) + R_1(\epsilon). && (5.64d) \end{aligned}$$

Furthermore, we use Lemma 5, in particular (5.51a) with initial condition

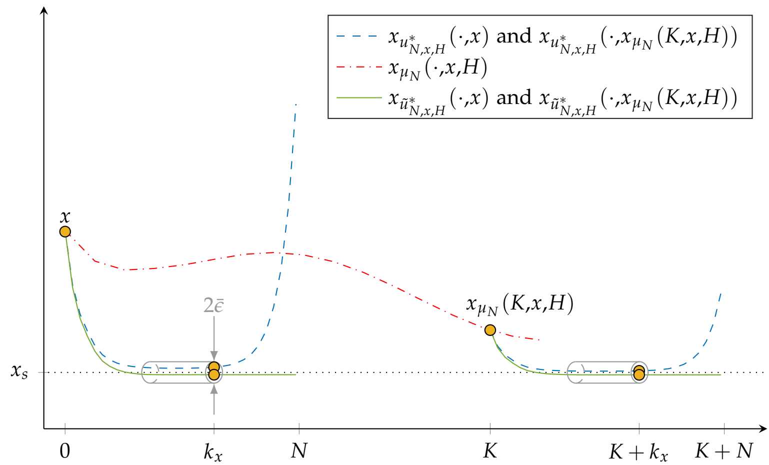


Figure 5.2.: Four trajectories which have at least one time instant k_x from (5.61) in common in order to show (5.60).

(x,H) and (5.51b) with initial condition $(x_{\mu_N}(K,x,H), H^{\mathrm{cl}}(K,x,H))$ which implies

$$\begin{aligned} & J_{k_x}(x,u^*_{N,x,H}) \\ \leq & \tilde{J}_{k_x}(x,\tilde{u}^*_{N,x,H}) + k_x\ell(x_s,u_s) - \lambda(x) + \psi(\|x-x_s\|_1 + \|H-H^s\|_1) + R_2(\epsilon), \end{aligned} \tag{5.65a}$$

$$\begin{aligned} & \tilde{J}_{k_x}(x_{\mu_N}(K,x,H),\tilde{u}^*_{N,x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H)}) \\ \leq & -k_x\ell(x_s,u_s) + \lambda(x_{\mu_N}(K,x,H)) + J_{k_x}(x_{\mu_N}(K,x,H),u^*_{N,x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H)}) \\ & + R_2(\epsilon) + (T-1)\left\|\bar{\lambda}\right\| L_h\epsilon. \end{aligned} \tag{5.65b}$$

Using the equations above, we can write

$$
\begin{aligned}
& J_N^*(x,H) - J_N^*(x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H)) \\
\overset{(5.64a),(5.64b)}{\leq}\; & J_{k_x}(x,u^*_{N,x,H}) - J_{k_x}(x_{\mu_N}(K,x,H),u^*_{N,x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H)}) \\
& + 2R_1(\epsilon) \\
\overset{(5.65a),(5.65b)}{\leq}\; & \tilde{J}_{k_x}(x,\tilde{u}^*_{N,x,H}) - \lambda(x) + 2R_2(\epsilon) + 2R_1(\epsilon) + (T-1)\left\|\bar{\lambda}\right\| L_h\epsilon \\
& - \tilde{J}_{k_x}(x_{\mu_N}(K,x,H),\tilde{u}^*_{N,x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H)}) + \lambda(x_{\mu_N}(K,x,H)) \\
& + \psi(\|x-x_s\|_1 + \|H-H^s\|_1) \\
\overset{(5.64c),(5.64d)}{\leq}\; & \tilde{J}_N^*(x,H) - \lambda(x) + 2R_2(\epsilon) + 4R_1(\epsilon) + (T-1)\left\|\bar{\lambda}\right\| L_h\epsilon \\
& - \tilde{J}_N^*(x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H)) + \lambda(x_{\mu_N}(K,x,H)) \\
& + \psi(\|x-x_s\|_1 + \|H-H^s\|_1)
\end{aligned}
$$

and it follows

$$
\begin{aligned}
& J_N^*(x,H) - J_N^*(x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H)) \\
\leq & \tilde{J}_N^*(x,H) - \tilde{J}_N^*(x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H)) - \lambda(x) + \lambda(x_{\mu_N}(K,x,H)) + R(\epsilon) \\
& + \psi(\|x-x_s\|_1 + \|H-H^s\|_1),
\end{aligned} \tag{5.66}
$$

with

$$
\begin{aligned}
R(\epsilon) =& 4R_1(\epsilon) + 2R_2(\epsilon) + (T-1)\left\|\bar{\lambda}\right\| L_h\epsilon \\
\overset{\text{Lem.3,5}}{=} & 8\gamma_v(\epsilon + \sqrt{p}L_h\epsilon) + 2\alpha_\lambda(\epsilon) + (T-1)\left\|\bar{\lambda}\right\| L_h\epsilon, \quad R \in \mathcal{K}_\infty.
\end{aligned}
$$

Note that we get from (5.62) that $\epsilon = \tilde{\sigma}_{c,1}(N - N_2 - (4T+1)(T-1) - 1)$ and hence, $R(\epsilon) = R(\tilde{\sigma}_{c,1}(N - N_2 - (4T+1)(T-1) - 1) =: \sigma_6(N - N_2 - (4T+1)(T-1) - 1)$ with $\sigma_6 \in \mathcal{L}_{\mathbb{N}}$.

Using the definition of the rotated cost from (3.4a) for the closed loop, i. e.

$$
\begin{aligned}
\tilde{J}_K^{\mathrm{cl}}(x,H) =& J_K^{\mathrm{cl}}(x,H) - K\ell(x_s,u_s) + \lambda(x) - \lambda(x_{\mu_N}(K,x,H)) \\
& + \sum_{k=0}^{K-1} \bar{\lambda}^\top h_{\mu_N}(k,x,H),
\end{aligned} \tag{5.67}
$$

and combining this with (5.3) (which we can apply since $N_6 \geq N_3 + 1$, cf. (5.63)) and (5.66) yields

$$
\begin{aligned}
&\tilde{J}_K^{\mathrm{cl}}(x,H) \\
&\overset{(5.3),(5.67)}{\leq} J_N^*(x,H) - J_N^*(x_{\mu_N}(K,x,H)),H^{\mathrm{cl}}(K,x,H)) + K\sigma_3(N - N_2 - T^2 - 1) \\
&\qquad + \lambda(x) - \lambda(x_{\mu_N}(K,x,H)) + \sum_{k=0}^{K-1} \bar{\lambda}^\top h_{\mu_N}(k,x,H) \\
&\overset{(5.66)}{\leq} \tilde{J}_N^*(x,H) - \tilde{J}_N^*(x_{\mu_N,x,H}(K),H^{\mathrm{cl}}(K,x,H)) + K\sigma_3(N - N_2 - T^2 - 1) \\
&\qquad + \sigma_6(N - N_2 - (4T+1)(T-1) - 1) + \sum_{k=0}^{K-1} \bar{\lambda}^\top h_{\mu_N}(k,x,H) \\
&\qquad + \psi(\|x - x_s\|_1 + \|H - H^s\|_1).
\end{aligned}
$$

for all $(x,H) \in \mathbb{X} \times \mathbb{H}$, all $K \in \mathbb{N}$ and $N \geq N_6 + 1$ with σ_3, $\sigma_6 \in \mathcal{L}_{\mathbb{N}}$ and hence, we have shown (5.60).

The second assertion immediately follows by using (3.7), $\bar{\lambda}^\top h(x,u) \leq \bar{\vartheta}_h$ for all $(x,u) \in \mathbb{Z}$ and since it holds $\sum_{k=0}^{jT-1} \bar{\lambda}^\top h_{\mu_N}(k,x,H) \leq 0$ for any $j \in \mathbb{I}_{\geq 0}$. □

We highlight that the assertion of Theorem 6 is especially relevant for $K = 1$ since we use this later in Chapter 6 to show stability.

5.2. Trajectory Convergence

In this chapter, we give conditions under which the convergence of the closed-loop trajectories to a neighborhood of the the optimal steady-state can be shown after we have introduced bounds for the closed-loop cost in the previous Section 5.1. We start with the following Theorem 7 which bounds the distance of the closed-loop trajectory to the optimal steady-state for a sufficiently large prediction horizon. Note that this Theorem is a generalization of [Grüne, 2013, Thm. 7.1] to EMPC schemes subject to transient average constraints.

Theorem 7. *Let Assumptions 1, 2, 3 and 4 hold.*

(i) *Assume furthermore that there exists a function $\nu \in \mathcal{L}_{\mathbb{N}}$ such that for $N \geq N_3$ with N_3 from Thm. 5, the inequality*

$$J_N^*(x,H) \geq N\ell(x_s,u_s) + N\alpha(\|x - x_s\| + \|H - H^s\|_1) \tag{5.68}$$

holds for all $x \in \mathbb{X} \setminus \{x_s\}$ and all $H \in \mathbb{H} \setminus \{H^s\}$ with $\|x - x_s\| + \|H - H^s\|_1 > \nu(N)$ and some $\alpha \in \mathcal{K}_\infty$. Then for all $N \geq N_3 + 1$ and all $k \geq N$ the inequality

$$\begin{aligned} &\left\|x_{\mu_N}(k,x,H) - x_s\right\| + \left\|H^{\mathrm{cl}}(k,x,H) - H^s\right\| \\ &\leq \max\{\nu(N),\, \alpha^{-1}(\sigma_3(N - N_2 - T^2 - 1))\} \end{aligned} \tag{5.69}$$

holds with σ_3 from Theorem 5 and N_2 from Theorem 4.

(ii) *If, moreover, for all $N \geq N_3$ the inequality*

$$J_N^*(x,H) \leq N\ell(x_s,u_s) + N\bar{\alpha}(\|x - x_s\| + \|H - H^s\|_1) \tag{5.70}$$

holds for some $\bar{\alpha} \in \mathcal{K}_\infty$, then for all $N \geq N_3 + 1$ the inequality

$$\begin{aligned} &\left\|x_{\mu_N}(k,x,H) - x_s\right\| + \left\|H^{\mathrm{cl}}(k,x,H) - H^s\right\| \leq \max\Bigg\{\nu(N), \\ &\alpha^{-1}\left(\bar{\alpha}\left(\|x - x_s\| + \|H - H^s\|_1\right) + \frac{k}{N}\sigma_3(N - N_2 - T^2 - 1)\right)\Bigg\} \end{aligned} \tag{5.71}$$

holds for all $k \in \mathbb{I}_{[1,N-1]}$, all $(x,H) \in \mathbb{X} \times \mathbb{H}$ with $J_k^*(x,H) \geq k\ell(x_s,u_s)$.

Proof. We split the proof into two different parts where we show in the first part that assertion (5.69) holds and in the second part we show (5.71).

Part I: Showing assertion (5.69)

Since all conditions of Theorem 5 hold, we get that (5.3), i. e.,

$$\begin{aligned} J_K^{\mathrm{cl}}(x,H) \leq& J_N^*(x,H) - J_N^*(x_{\mu_N}(K,x,H),H^{\mathrm{cl}}(K,x,H)) \\ &+ K\left(\ell(x_s,u_s) + \sigma_3(N - N_2 - T^2 - 1)\right) \end{aligned} \tag{5.72}$$

holds for all $(x,H) \in \mathbb{X} \times \mathbb{H}$, all $K \in \mathbb{N}$ and $N \geq N_3 + 1$ with $\sigma_3 \in \mathcal{L}_{\mathbb{N}}$ as given in the proof of Theorem 5. Hence, we can apply inequality (5.3) for all $N \geq N_3 + 1$, all $k \geq N$, all $x(k-N,x,H) \in \mathbb{X}$ and all $H(k-N,x,H) \in \mathbb{H}$ and $K = N$ which yields

$$
\begin{aligned}
&J_N^*(x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H)) \\
\leq &J_N^*(x_{\mu_N}(k-N,x,H),H^{\mathrm{cl}}(k-N,x,H)) + N\sigma_3(N-N_2-T^2-1) \\
&+ N\ell(x_s,u_s) - J_N^{\mathrm{cl}}(x_{\mu_N}(k-N,x,H),H^{\mathrm{cl}}(k-N,x,H)).
\end{aligned}
$$

Furthermore, optimality yields

$$
\begin{aligned}
&J_N^*(x_{\mu_N}(k-N,x,H),H^{\mathrm{cl}}(k-N,x,H)) \\
\overset{\text{opt.}}{\leq} &J_N^{\mathrm{cl}}(x_{\mu_N}(k-N,x,H),H^{\mathrm{cl}}(k-N,x,H))
\end{aligned}
$$

and by combining the previous two inequalities, we obtain

$$
J_N^*(x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H)) \leq N\left(\ell(x_s,u_s) + \sigma_3(N-N_2-T^2-1)\right). \tag{5.73}
$$

Now, suppose that $\|x_{\mu_N}(k,x,H) - x_s\| + \left\|H^{\mathrm{cl}}(k,x,H) - H^s\right\| > \nu(N)$ holds. We get from (5.68) and (5.73)

$$
\begin{aligned}
&N\ell(x_s,u_s) + N\alpha\left(\|x_{\mu_N}(k,x,H) - x_s\| + \left\|H^{\mathrm{cl}}(k,x,H) - H^s\right\|\right) \\
\overset{(5.68)}{\leq} &J_N^*(x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H)) \overset{(5.73)}{\leq} N\left(\ell(x_s,u_s) + \sigma_3(N-N_2-T^2-1)\right)
\end{aligned}
$$

which yields

$$
\|x_{\mu_N}(k,x,H) - x_s\| + \left\|H^{\mathrm{cl}}(k,x,H) - H^s\right\| \leq \alpha^{-1}\left(\sigma_3(N-N_2-T^2-1)\right).
$$

Now, the assertion (5.69) follows by combining this estimate with the second case $\|x_{\mu_N}(k,x,H) - x_s\| + \left\|H^{\mathrm{cl}}(k,x,H) - H^s\right\| \leq \nu(N)$.

Part II: Showing assertion (5.71)

Again, we suppose $\|x_{\mu_N}(k,x,H) - x_s\| + \left\|H^{\mathrm{cl}}(k,x,H) - H^s\right\| > \nu(N)$. From the assumptions of Thm. 7 and by optimality, we get

$$
k\ell(x_s,u_s) \overset{\text{Ass.}}{\leq} J_k^*(x,H) \overset{\text{opt.}}{\leq} J_k^{\mathrm{cl}}(x,H). \tag{5.74}
$$

Furthermore, we again use (5.3) but this time with $K = k \leq N$ and get

$$\begin{aligned} & J_N^*(x_{\mu_N}(k,x,H),H^{\text{cl}}(k,x,H)) \\ \leq & J_N^*(x,H) - J_k^{\text{cl}}(x,H) + k\left(\ell(x_s,u_s) + \sigma_3(N - N_2 - T^2 - 1)\right) \\ \overset{(5.74)}{\leq} & J_N^*(x,H) + k\sigma_3(N - N_2 - T^2 - 1). \end{aligned} \tag{5.75}$$

It follows

$$\begin{aligned} & N\alpha\left(\left\|x_{\mu_N}(k,x,H) - x_s\right\| + \left\|H^{\text{cl}}(k,x,H) - H^s\right\|\right) \\ \overset{(5.68)}{\leq} & J_N^*(x_{\mu_N}(k,x,H),H^{\text{cl}}(k,x,H)) - N\ell(x_s,u_s) \\ \overset{(5.75)}{\leq} & J_N^*(x,H) - N\ell(x_s,u_s) + k\sigma_3(N - N_2 - T^2 - 1) \\ \overset{(5.70)}{\leq} & N\bar{\alpha}\left(\|x - x_s\| + \|H - H^s\|_1\right) + k\sigma_3(N - N_2 - T^2 - 1). \end{aligned}$$

Again, combining this inequality with the second case $\left\|x_{\mu_N}(k,x,H) - x_s\right\| + \left\|H^{\text{cl}}(k,x,H) - H^s\right\| \leq \nu(N)$ yields

$$\begin{aligned} \left\|x_{\mu_N}(k,x,H) - x_s\right\| + \left\|H^{\text{cl}}(k,x,H) - H^s\right\| \leq \max\Bigg\{ & \nu(N), \\ \alpha^{-1}\left(\bar{\alpha}\left(\|x - x_s\| + \|H - H^s\|_1\right) + \frac{k}{N}\sigma_3(N - N_2 - T^2 - 1)\right) & \Bigg\} \end{aligned}$$

which shows assertion (5.71). □

As discussed in [Grüne, 2013], we note that Result (i) in Theorem 7 provides a bound for $k \geq N$ and Result (ii) yields a bound for $k \in \mathbb{I}_{[1,N-1]}$. For the case that both estimates hold, we could also construct an upper bound of the form $\beta\left(\|x - x_s\| + \|H - H^s\|_1, k\right) + \epsilon(N)$ as in more standard practical stability estimates, compare [Grüne and Stieler, 2014].

Now, with the help of the previous lemmas we can formulate convergence as in Theorem 7, but this time with the rotated cost functional.

Theorem 8. *Let Assumptions 1, 2, 3, 4 and 5 hold.*

(i) *Assume that the value function* $\tilde{J}_N^*(x,H)$ *of the rotated problem satisfies* (5.68), *i.e., there exists a function* $\nu \in \mathcal{L}_{\mathbb{N}}$ *such that for* $N \geq N_6$ *with* N_6 *from Thm. 6, the inequality*

$$\tilde{J}_N^*(x,H) \geq N\alpha(\|x - x_s\| + \|H - H^s\|_1) \tag{5.76}$$

holds for all $x \in \mathbb{X} \setminus \{x_s\}$ *and all* $H \in \mathbb{H} \setminus \{H^s\}$ *for which* $\|x - x_s\| + \|H - H^s\|_1 > \nu(N)$. *Then there exists* $\tilde{\sigma}_6 \in \mathcal{L}_{\mathbb{N}}$ *such that for all* $(x,H) \in \mathbb{X} \times \mathbb{H}$ *and* $N \geq N_6 + 1$ *it holds for all* $k \geq N$

$$\begin{aligned}\left\|x_{\mu_N}(k,x,H) - x_s\right\| &+ \left\|H^{\mathrm{cl}}(k,x,H) - H^s\right\| \\ &\leq \max\left\{\nu(N), \alpha^{-1}\left(\tilde{\sigma}_6(N - N_6)\right\}.\end{aligned} \tag{5.77}$$

(ii) *If, moreover, for all* $N \geq N_6$ *the inequality*

$$\tilde{J}_N^*(x,H) \leq N\bar{\alpha}(\|x - x_s\| + \|H - H^s\|_1) \tag{5.78}$$

holds for some $\bar{\alpha} \in \mathcal{K}_\infty$, *then for all* $N \geq N_6 + 1$ *the inequality*

$$\begin{aligned}&\left\|x_{\mu_N,x,H}(k) - x_s\right\| + \left\|H^{\mathrm{cl}}(k,x,H) - H^s\right\| \\ \leq &\max\left\{\nu(N),\ \alpha^{-1}\left(\bar{\alpha}\left(\|x - x_s\| + \|H - H^s\|_1\right) + \tilde{\sigma}_6(N - N_6)\right)\right\}\end{aligned} \tag{5.79}$$

holds for all $k \in \mathbb{I}_{[1,N-1]}$ *and all* $(x,H) \in \mathbb{X} \times \mathbb{H}$.

Proof. We split this proof into two parts. In the first part we show that assertion (5.77) follows and in the second part we show how to obtain assertion (5.79).

Part I: Show the assertion (5.77)

We can use the same arguments as in the proof of Theorem 7, i.e., since it holds $N \geq N_6 + 1$ we can use (5.60) with $k \geq N$, any $x_{\mu_N}(k - N,x,H) \in \mathbb{X}$ and any $H^{\mathrm{cl}}(k - N,x,H) \in \mathbb{H}$ with $K = N$. This yields from (5.60) and by optimality (cf. procedure in Part I of the proof of Theorem 7) with $\tau := T - 1$

$$\begin{aligned}&\tilde{J}_N^*(x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H)) \\ \leq &N\sigma_3(N - N_2 - T^2 - 1) + \sigma_6(N - N_2 - (4T + 1)(T - 1) - 1) + \tau\bar{\vartheta}_h \\ &+ \psi(\|x - x_s\|_1 + \|H - H^s\|_1).\end{aligned} \tag{5.80}$$

As in the proof of Theorem 7, we suppose that it holds $\left\|x_{\mu_N}(k,x,H) - x_s\right\| + \left\|H^{\mathrm{cl}}(k,x,H) - H^s\right\| > \nu(N)$ and get

$$\begin{aligned}
&N\alpha(\left\|x_{\mu_N}(k,x,H) - x_s\right\| + \left\|H^{\mathrm{cl}}(k,x,H) - H^s\right\|) \\
\overset{(5.76)}{\leq}\;& \tilde{J}_N^*(x_{\mu_N}(k,x,H),H^{\mathrm{cl}}(k,x,H)) \\
\overset{(5.80)}{\leq}\;& N\sigma_3(N - N_2 - T^2 - 1) + \sigma_6\left(N - N_2 - (4T+1)(T-1) - 1\right) + \tau\bar{\vartheta}_h \\
&+ \psi(\|x - x_s\|_1 + \|H - H^s\|_1),
\end{aligned}$$

which we can rearrange to

$$\begin{aligned}
&\alpha(\left\|x_{\mu_N}(k,x,H) - x_s\right\| + \left\|H^{\mathrm{cl}}(k,x,H) - H^s\right\|) \\
\leq&\sigma_3(N - N_2 - T^2 - 1) + \frac{1}{N}\sigma_6\left(N - N_2 - (4T+1)(T-1) - 1\right) + \frac{1}{N}\tau\bar{\vartheta}_h \\
&+ \frac{1}{N}\psi(\|x - x_s\|_1 + \|H - H^s\|_1).
\end{aligned} \tag{5.81}$$

Furthermore, we define

$$\psi_{\max} := \max_{(x,H)\in\mathbb{X}\times\mathbb{H}} \psi(\|x - x_s\|_1 + \|H - H^s\|_1). \tag{5.82}$$

Since $N_6 \geq N_3 + 1$ due to (5.63), we have $N_6 \geq N_2 + T^2 + 1$ using (5.17) and hence, we get from (5.63)

$$\begin{aligned}
&\sigma_3(N - N_2 - T^2 - 1) + \frac{1}{N}\sigma_6\left(N - N_2 - (4T+1)(T-1) - 1\right) + \frac{\tau\bar{\vartheta}_h}{N} \\
&+ \psi(\|x - x_s\|_1 + \|H - H^s\|_1) \\
\overset{(5.63)}{\leq}\;& \sigma_3(N - N_6) + \frac{1}{N}\sigma_6\left(N - N_6\right) + \frac{1}{N}\tau\bar{\vartheta}_h \\
&+ \frac{1}{N}\psi(\|x - x_s\|_1 + \|H - H^s\|_1) \\
\overset{N\geq N_6+1}{\leq}\;& \sigma_3(N - N_6) + \frac{1}{N - N_6}\sigma_6\left(N - N_6\right) + \frac{1}{N - N_6}\tau\bar{\vartheta}_h \\
&+ \frac{1}{N - N_6}\psi(\|x - x_s\|_1 + \|H - H^s\|_1) \qquad (5.83) \\
\overset{(5.82)}{\leq}\;& \sigma_3(N - N_6) + \frac{\sigma_6(N - N_6) + \tau\bar{\vartheta}_h + \psi_{\max}}{N - N_6} =: \tilde{\sigma}_6(N - N_6),
\end{aligned}$$

with $\tilde{\sigma}_6 \in \mathcal{L}_{\mathbb{N}}$ since $\sigma_3, \sigma_6 \in \mathcal{L}_{\mathbb{N}}$ and $\tau \leq T-1, \bar{\vartheta}_h, \psi_{\max}$ are constants. This yields together with (5.81)

$$\left\|x_{\mu_N}(k,x,H) - x_s\right\| + \left\|H^{\mathrm{cl}}(k,x,H) - H^s\right\| \leq \alpha^{-1}\left(\tilde{\sigma}_6(N-N_6)\right).$$

Combining this with the case that $\left\|x_{\mu_N}(k,x,H) - x_s\right\| + \left\|H^{\mathrm{cl}}(k,x,H) - H^s\right\| \leq \nu(N)$ yields the assertion (5.77).

Part II: Show assertion (5.79)

Again, we suppose $\left\|x_{\mu_N}(k,x,H) - x_s\right\| + \left\|H^{\mathrm{cl}}(k,x,H) - H^s\right\| > \nu(N)$. From the properties of the rotated costs and by optimality, we get

$$0 \leq \tilde{J}_k^*(x,H) \overset{\text{opt.}}{\leq} \tilde{J}_k^{\mathrm{cl}}(x,H). \tag{5.84}$$

Furthermore, we again use (5.60) but this time with $K = k \leq N$ and get

$$\begin{aligned}
&\tilde{J}_N^*(x_{\mu_N}(k,x,H), H^{\mathrm{cl}}(k,x,H)) \\
&\overset{(5.60)}{\leq} \tilde{J}_N^*(x,H) - \tilde{J}_k^{\mathrm{cl}}(x,H) + \sigma_6(N - N_2 - (4T+1)(T-1) - 1) \\
&\qquad + k\sigma_3(N - N_2 - T^2 - 1) + \tau\bar{\vartheta}_h + \psi(\|x - x_s\|_1 + \|H - H^s\|_1) \qquad (5.85) \\
&\overset{(5.84)}{\leq} \tilde{J}_N^*(x,H) + \sigma_6(N - N_2 - (4T+1)(T-1) - 1) \\
&\qquad + k\sigma_3(N - N_2 - T^2 - 1) + \tau\bar{\vartheta}_h + \psi(\|x - x_s\|_1 + \|H - H^s\|_1).
\end{aligned}$$

We obtain

$$\begin{aligned}
&N\alpha\left(\left\|x_{\mu_N}(k,x,H) - x_s\right\| + \left\|H^{\mathrm{cl}}(k,x,H) - H^s\right\|\right) \\
&\overset{(5.76)}{\leq} \tilde{J}_N^*(x_{\mu_N}(k,x,H), H^{\mathrm{cl}}(k,x,H)) \\
&\overset{(5.85)}{\leq} \tilde{J}_N^*(x,H) + \sigma_6(N - N_2 - (4T+1)(T-1) - 1) \\
&\qquad + k\sigma_3(N - N_2 - T^2 - 1) + \tau\bar{\vartheta}_h + \psi(\|x - x_s\|_1 + \|H - H^s\|_1) \\
&\overset{(5.78)}{\leq} N\bar{\alpha}\left(\|x - x_s\| + \|H - H^s\|_1\right) + \sigma_6(N - N_2 - (4T+1)(T-1) - 1) \\
&\qquad + k\sigma_3(N - N_2 - T^2 - 1) + \tau\bar{\vartheta}_h + \psi(\|x - x_s\|_1 + \|H - H^s\|_1)
\end{aligned}$$

which yields

$$\begin{aligned}
&\alpha\left(\left\|x_{\mu_N}(k,x,H)-x_s\right\|+\left\|H^{\mathrm{cl}}(k,x,H)-H^s\right\|\right)\\
\leq&\bar{\alpha}\left(\|x-x_s\|+\|H-H^s\|_1\right)+\frac{1}{N}\sigma_6(N-N_2-(4T+1)(T-1)-1)\\
&+\frac{k}{N}\sigma_3(N-N_2-T^2-1)+\frac{1}{N}\tau\bar{\vartheta}_h+\frac{1}{N}\psi_{\max}\\
\overset{k\leq N}{\leq}&\bar{\alpha}\left(\|x-x_s\|+\|H-H^s\|_1\right)+\tilde{\sigma}_6(N-N_6),
\end{aligned}$$

where the last inequality follows from (5.83) since it holds $k \leq N$ for this part of the proof. Combining this with the second case $\|x(k,x,H)-x_s\| + \left\|H^{\mathrm{cl}}(k,x,H)-H^s\right\| \leq \nu(N)$, we obtain

$$\begin{aligned}
&\left\|x_{\mu_N}(k,x,H)-x_s\right\|+\left\|H^{\mathrm{cl}}(k,x,H)-H^s\right\|\\
\leq&\max\left\{\nu(N),\ \alpha^{-1}\left(\bar{\alpha}\left(\|x-x_s\|+\|H-H^s\|_1\right)+\tilde{\sigma}_6(N-N_6)\right)\right\}
\end{aligned}$$

which shows assertion (5.79). □

Remark 16. We note that for the special case $T = 1$, it follows $\tau = 0$ and we get $\tilde{\sigma}_6(N-N_6) = \sigma_3(N-N_6) + \frac{1}{N-N_6}\sigma_6(N-N_6) + \frac{1}{N-N_6}\psi_{\max}$ from (5.83), where $\sigma_3(N-N_6)$ is conceptionally equal to $\delta(N)$ from [Grüne, 2013, Thm. 7.1] (cf. Theorem 7) and $\sigma_6(N-N_6)$ is conceptionally equivalent to $\delta_1(N)$ from [Grüne, 2013, Proof of Thm. 7.6]. Hence, the results of Theorem 8 include the special case $T = 1$ from [Grüne, 2013], where we need a larger prediction horizon N since we have additional point-wise in time constraints due to the auxiliary output. Furthermore, the same arguments hold for $K = T$ since it holds

$$\bar{\lambda}^\top \sum_{k=0}^{K-1} h\left(x_{\mu_N}(k,x,H)\mu_N(k,x,H)\right) \leq 0$$

by definition of the transient average constraints and hence, (5.67) simplifies to

$$\tilde{J}_T^{\mathrm{cl}}(x,H) \leq J_T^{\mathrm{cl}}(x,H) - T\ell(x_s,u_s) + \lambda(x) - \lambda(x_{\mu_N}(T,x,H)),$$

and the same arguments from the beginning of this Remark hold. Note that this also holds for $K = jT$ with $j \in \mathbb{I}_{\geq 1}$.

6. Stability Results

In this chapter, we will combine the previous convergence results of Chapter 5 and input-to-state stability of the memory H of the auxiliary output in oder to prove practical asymptotic stability of the closed loop. The method used in the following is based on the case without transient average constraints from [Grüne and Stieler, 2014] as well as on non-monotonic Lyapunov functions from [Ahmadi and Parrilo, 2008].

6.1. Bounds on the Rotated Value Function

Before we are able to show lower and upper bounds on the optimal rotated cost function $\tilde{J}_N^*$ as well as an upper bound on the decrease of it, we introduce a lemma which ensures finite time controllability of the system into $x \in \mathcal{B}_{\delta_c}(x_s)$ and $\|H - H^s\|_1 \leq E_h$ with $\delta_c, E_h > 0$ from Assumption 4. The idea is to use Theorem 3 which ensures that the optimal input sequence has T consecutive time instants in an arbitrarily small neighborhood of the steady-state in order to conclude that there exists a control sequence which steers the system from any initial condition (x,H) into an arbitrarily small neighborhood of (x_s,H^s).

Lemma 6. *Let Assumptions 1-3 hold. Then, there exists $K_c \in \mathbb{N}$ such that for each $(x,H) \in \mathbb{X} \times \mathbb{H}$ there exists an input sequence $u \in \mathbb{U}^{K_c}(x,H)$ and some $k_f \leq K_c$ such that it holds for δ_c and E_h from Ass. 4*

$$x_u(k_f,x) \in \mathcal{B}_{\delta_c}(x_s) \qquad \left\|H(k_f) - H^s\right\|_1 \leq E_h \tag{6.1}$$

with

$$\begin{aligned} H(k_f) &= H(x(k_f - T + 1), u(k_f - T + 1 : k_f - 1)) \\ &= \left[h(x(k_f - T + 1,x), u(k_f - T + 1)), \quad \ldots, \quad h(x_u(k_f - 1,x), u(k_f - 1))\right] \end{aligned}$$

in case [1] $k_f \geq T - 1$.

[1] W. l. o. g. we assume $k_f \geq T - 1$. In case that $k_f < T - 1$ we obtain

$$H(k_f) = \left[H_{k_f+1}, \quad \ldots, \quad H_{T-1}, \quad h(x_u(0),u(0)), \quad \ldots, \quad h(x_u(k_f - 1,x),u(k_f - 1))\right]$$

Proof. We define

$$\epsilon_c := \min\left\{\delta_c, \frac{E_h}{\sqrt{p}L_h}\right\}$$

and show in the following that the assertion holds for

$$K_c = \frac{T\hat{C}'}{\rho(\epsilon_c)} + T^2, \tag{6.2}$$

with $L_h > 0$ from Ass. 1, $\rho \in \mathcal{K}_\infty$ from Ass. 2 and $\hat{C}'$ from (3.25). We can use the assertions of Theorem 3 and Corollary 3 since all assumptions are satisfied. We set $k_l' = 0$, $k_u' = K_c$ and get that there exists $k_x \in \mathbb{I}_{[T-1,K_c]}$ such that the resulting optimal control input sequence (with prediction horizon $N \geq K_c + 1$) satisfies for all $k \in \mathbb{I}_{[k_x-T+1,k_x]}$

$$\begin{aligned}
&\left\|(x_{u^*_{N,x,H}}(k,x) - x_s, u^*_{N,x,H}(k) - u_s)\right\| \\
\leq&\sigma_T(K_c - T^2) \overset{(3.31)}{\leq} \rho^{-1}\left(\frac{T\hat{C}'}{K_c - T^2}\right) \overset{(6.2)}{\leq} \delta_c,
\end{aligned} \tag{6.3a}$$

$$\begin{aligned}
&\left\|h(x_{u^*_{N,x,H}}(k,x),u^*_{N,x,H}(k)) - h(x_s,u_s)\right\| \\
\leq&L_h\sigma_T(K_c - T^2) \overset{(3.31)}{\leq} L_h\rho^{-1}\left(\frac{T\hat{C}'}{K_c - T^2}\right) \overset{(6.2)}{\leq} \frac{E_h}{\sqrt{p}}.
\end{aligned} \tag{6.3b}$$

Now, we set $k_f = k_x \leq K_c = k_u'$ and define the input sequence $u \in \mathbb{U}^{k_f}(x,H)$

$$u(k) = u^*_{N,x,H}(k)$$

for all $k \in \mathbb{I}_{[0,k_f]}$. This means, we take the optimal control input sequence for which the turnpike property ensures that it has at least T consecutive time instants in an arbitrarily small neighborhood of the steady-state and truncate it such that the input sequence $u \in \mathbb{U}^{k_f}(x,H)$ ends at k_x. Note that k_x and its $T-1$ previous time instants are in the requested neighborhood of the steady-state. Furthermore, this input sequence satisfies (6.1) since (6.3b) implies $\left\|H(k_f) - H^s\right\|_1 \leq E_h$ (cf. Corollary 4) and hence, the assertion follows. □

.

The following Lemma introduces an upper and lower bound on the rotated cost functional as well as an upper bound on the rotated cost function in the subsequent step.

Lemma 7. *Let Assumptions 1-5 hold. Then there exist functions $\delta_7 \in \mathcal{L}_{\mathbb{N}}$ and $\alpha_7 \in \mathcal{K}_\infty$ such that the inequalities*

$$\rho(\|(x - x_s, u - u_s)\|) \leq \tilde{J}_N^*(x,H) \leq \alpha_7(\|x - x_s\| + \|H - H^s\|_1) \tag{6.4}$$

and

$$\begin{aligned} &\tilde{J}_N^*(x_{\mu_N}(1,x,H), H^{\mathrm{cl}}(1,x,H)) \\ \leq &\tilde{J}_N^*(x,H) - \rho(\|(x - x_s, \mu_N(x,H) - u_s)\|) + \psi(\|x - x_s\|_1 + \|H - H^s\|_1) \\ &+ \delta_7(N - N_6) + \bar{\lambda}^\top h(x, \mu_N(x,H)), \end{aligned} \tag{6.5}$$

hold for all $N \geq N_6 + 1$ and all $(x,H) \in \mathbb{X} \times \mathbb{H}$ with $\rho \in \mathcal{K}_\infty$ from Ass. 2, ψ from Ass. 5 and N_6 from Thm. 6.

Proof. We split the proof into two parts where we show in the first part Inequality (6.4) and in the second part Inequality (6.5).

Part I: Showing assertion (6.4)

The lower bound follows from strict dissipativity (Ass. 2), i. e.,

$$\tilde{J}_N^*(x,H) \geq \rho(\|(x - x_s, u - u_s)\|). \tag{6.6}$$

The upper bound can be derived from Ass. 4 and Lemma 6 as follows.

In case that either $x \notin \mathcal{B}_{\delta_c}(x_s)$ or $\rceil H - H^s \lceil > E_h$ with $\delta_c, E_h > 0$ from Ass. 4, it follows from Lemma 6 that there is a control sequence u that steers the system in at most K_c steps into a neighborhood of the steady-state such that we can apply Ass. 4 in order to steer x in d steps into the steady-state. Hence, we can steer the system into the steady-state in at most $K_c + d$ steps and stay there for an arbitrary number of time steps. Note that this case is shown in Figure 6.1 (——). For each $N \in \mathbb{N}$ with $N \geq N_6 \geq K_c + d$ it follows from Ass. 3 that it holds

$$\tilde{J}_N^*(x,H) \leq \max_{x \in \mathbb{X}} \gamma_{\beta_1}(\|x - x_s\|) + \max_{H \in \mathbb{H}} \gamma_{\beta_2}(\rceil H - H^s \lceil) = \delta.$$

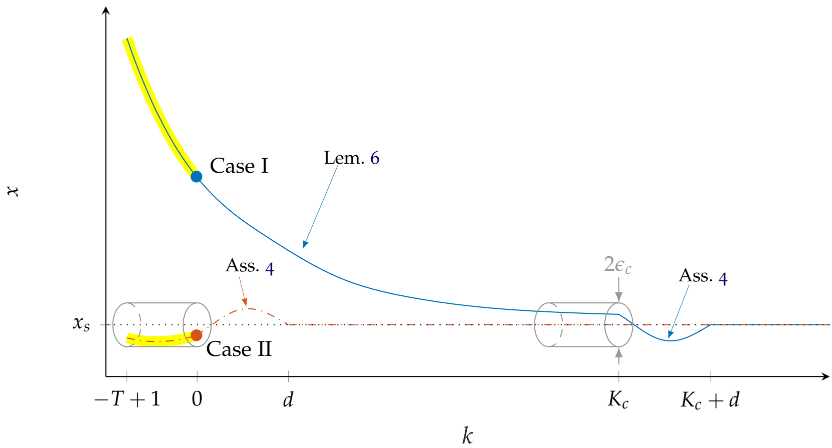


Figure 6.1.: Steering the system into the steady-state in order to prove $\tilde{J}_N^*(x,H) \leq \alpha_7(\|x - x_s\| + \|H - H^s\|_1)$.

Furthermore, since it either holds $x \notin \mathcal{B}_{\delta_c}(x_s)$ or $\lceil H - H^s \rceil > E_h$ we can estimate with $\alpha_{7,1} \in \mathcal{K}_\infty$

$$\tilde{J}_N^*(x,H) \leq \delta \leq \left(\frac{\delta}{\min\{\delta_c,\ E_h\}}\right)(\|x - x_s\| + \lceil H - H^s \rceil)$$
$$\leq \left(\frac{\delta}{\min\{\delta_c,\ E_h\}}\right)(\|x - x_s\| + \|H - H^s\|_1) =: \alpha_{7,1}(\|x - x_s\| + \|H - H^s\|_1).$$

In case $x \in \mathcal{B}_{\delta_c}(x_s)$ and $\lceil H - H^s \rceil \leq E_h$, we get from Ass. 4 with the reference trajectory $x_c = x_s$, $H^c = H^s$, $x_{u_c}(\cdot,x_c) \equiv x_s$ and $u_c(\cdot) \equiv u_s$ that there exists a control sequence $u_2 \in \mathbb{U}^d(x,H)$ satisfying $x_{u_2}(d,x) = x_s$ as well as

$$\|x_{u_2}(k,x) - x_s\| \overset{\text{Ass.4}}{\leq} \gamma_x(\|x - x_s\| + \lceil H - H^s \rceil),$$
$$\|u_2(k) - u_s\| \overset{\text{Ass.4}}{\leq} \gamma_u(\|x - x_s\| + \lceil H - H^s \rceil),$$

for all $k \in \mathbb{I}_{[0,d-1]}$ since we stay at the steady-state afterwards and hence, it holds $\zeta = \|x - x_s\| + \upharpoonleft H - H^s \upharpoonright$. Combining this with Lemma 1 yields

$$
\begin{aligned}
\tilde{J}_N^*(x,H) \leq & \tilde{J}_N(x,u_1) \leq \tilde{J}_d(x,u_1) \\
\overset{\text{Lem.1}}{\leq} & \sum_{k=0}^{d-1} \alpha_u \left(\|x_{u_2}(k,x) - x_s\| + \|u_2(k) - u_s\| \right) \\
\leq & d\alpha_u \left(\gamma_x(\|x - x_s\| + \upharpoonleft H - H^s \upharpoonright) + \gamma_u(\|x - x_s\| + \upharpoonleft H - H^s \upharpoonright) \right) \\
\leq & d\alpha_u \left(\gamma_x(\|x - x_s\| + \|H - H^s\|_1) + \gamma_u(\|x - x_s\| + \|H - H^s\|_1) \right) \\
=: & \alpha_{7,2}(\|x - x_s\| + \|H - H^s\|_1)
\end{aligned}
$$

We immediately get from α_u, γ_x, $\gamma_u \in \mathcal{K}_\infty$ that $\tilde{\alpha} \in \mathcal{K}_\infty$. This case is also illustrated in Figure 6.1 (- · - · -). With $\alpha_7 := \max\{\alpha_{7,1}, \alpha_{7,2}\} \in \mathcal{K}_\infty$ and by combining both cases, we can upper bound the rotated cost functional by

$$
\tilde{J}_N^*(x,H) \leq \alpha_7(\|x - x_s\| + \|H - H^s\|_1). \tag{6.7}
$$

Part II: Showing assertion (6.5)

Given Inequality (5.60) from Theorem 6 for $K = 1$, the following inequality holds for all $(x,H) \in \mathbb{X} \times \mathbb{H}$ and $N \geq N_6 + 1$

$$
\begin{aligned}
& \tilde{J}_N^*(x_{\mu_N}(1,x,H), H^{\text{cl}}(1,x,H)) \\
\overset{(5.60)}{\leq} & \tilde{J}_N^*(x,H) - \tilde{J}_1^{\text{cl}}(x,H) + \sigma_3(N - N_2 - T^2 - 1) \\
& + \sigma_6(N - N_2 - (4T+1)(T-1) - 1) + \psi(\|x - x_s\|_1 + \|H - H^s\|_1) \\
& + \bar{\lambda}^\top h\left(x, \mu_N(x,H)\right) \\
= & \tilde{J}_N^*(x,H) - \tilde{\ell}(x,\mu_N(x,H)) + \sigma_6(N - N_2 - (4T+1)(T-1) - 1) \\
& + \sigma_3(N - N_2 - T^2 - 1) + \psi(\|x - x_s\|_1 + \|H - H^s\|_1) \\
& + \bar{\lambda}^\top h(x,\mu_N(x,H)) \\
\overset{\text{dissip.}}{\leq} & \tilde{J}_N^*(x,H) - \rho(\|(x - x_s, \mu_N(x,H) - u_s)\|) + \sigma_3(N - N_2 - T^2 - 1) \\
& + \sigma_6(N - N_2 - (4T+1)(T-1) - 1) + \psi(\|x - x_s\|_1 + \|H - H^s\|_1) \\
& + \bar{\lambda}^\top h(x,\mu_N(x,H))
\end{aligned}
$$

which implies inequality (6.5) with

$$
\begin{aligned}
& \sigma_6(N - N_2 - (4T+1)(T-1) - 1) + \sigma_3(N - N_2 - T^2 - 1) \\
\leq & \sigma_6(N - N_6) + \sigma_3(N - N_6) =: \delta_7(N - N_6).
\end{aligned}
$$

$\square$

6.2. Investigations on the Auxiliary Output Storage H

As shown in Lemma 7, the rotated value function shows no decrease corresponding to the storage H and hence, the rotated value function is not sufficient in order to show practical asymptotic stability of the extended state (x,H). Moreover, stability of the state x can not be shown by using the rotated value function due to the additional term $\bar{\lambda}^\top$ in (6.5). Therefore, we further investigate the storage H in order to combine it with the previous results. We start by explicitly stating the dynamics of the variable H.

6.2.1. Linear Dynamics of the Storage

We consider an arbitrary time step k, where we assume w. l. o. g. $k \geq T$. We recall that $H(k)$ consists of the $T-1$ previous values of $h(x(k),u(k))$ and hence, we can consider $H(k)$ as a memory of the auxiliary output. Therefore, comparing the memory $H(k+1)$ of a subsequent time step with the current memory $H(k)$ yields that the first $T-2$ columns of $H(k+1)$ are exactly the columns $H_j(k)$ with $j \in \mathbb{I}_{[2,T-1]}$ and furthermore, the last column of $H(k+1)$, i. e., $H_{T-1}(k+1)$ is equal to the current value at the auxiliary output, $h(x(k),u(k))$. This means, by denoting $H(k+1)$ as the subsequent memory of $H(k)$, we can write

$$\begin{aligned} H(k+1) &= \begin{bmatrix} H_1(k+1), & \ldots, & H_{T-2}(k+1), & H_{T-1}(k+1) \end{bmatrix} \\ &= \begin{bmatrix} H_2(k), & \ldots, & H_{T-1}(k), & h(x(k),u(k)) \end{bmatrix} \\ &= f_H(H(k),h(x(k),u(k))). \end{aligned} \tag{6.8}$$

Moreover, we can formulate these dynamics of the memory as a discrete-time linear system which reads

$$\begin{aligned} &H(k+1) \\ =&H(k) \underbrace{\begin{bmatrix} \mathbf{0}^{1\times(T-2)} & \mathbf{0}^{1\times 1} \\ \mathbf{I}^{T-2} & \mathbf{0}^{(T-2)\times 1} \end{bmatrix}}_{=:A_H} + h(x(k),u(k)) \underbrace{\begin{bmatrix} \mathbf{0}^{1\times(T-2)} & \mathbf{1}^{1\times 1} \end{bmatrix}}_{=:B_H} \\ =:&f_H(H(k),h(x(k),u(k))) \end{aligned} \tag{6.9}$$

where we denote $\mathbf{0}^{n\times m}$ as a zero-matrix of n-rows and m-columns ($\mathbf{1}$ analogously) and $\mathbf{I}^n$ as the $n \times n$ identity matrix.

Remark 17. It follows by using submultiplicativity of matrix norms that the linear map $f_H : \mathbb{R}^{p\times(T-1)} \times \mathbb{R}^p \to \mathbb{R}^{p\times(T-1)}$ is Lipschitz continuous, namely (we write $h_i = h(x_i,u_i)$ for $i = \{1,2\}$)

$$\begin{aligned} &\left\| f_H(H^1,h_1) - f_H(H^2,h_2) \right\| = \left\| H^1 A_H + h_1 B_H - H^2 A_H - h_2 B_H \right\| \\ =&\left\| (H^1 - H^2)A_H + (h_1 - h_2)B_H \right\| \leq \|A_H\| \cdot \left\| H^1 - H^2 \right\| + \|B_H\| \cdot \|h_1 - h_2\| \\ \leq& \max\{\|A_H\|, \|B_H\|\} \left(\left\| H^1 - H^2 \right\| + \|h_1 - h_2\| \right). \end{aligned} \tag{6.10}$$

Furthermore, we can use Lipschitz continuity of the auxiliary output h such that it follows

$$\begin{aligned} &\left\| f_H(H^1,h_1) - f_H(H^2,h_2) \right\| \overset{(6.10)}{\leq} \|A_H\| \cdot \|H_1 - H_2\| + \|B_H\| \cdot \|h_1 - h_2\| \\ =& \|A_H\| \cdot \|H_1 - H_2\| + \|B_H\| \cdot \|h(x_1,u_1) - h(x_2,u_2)\| \\ \overset{\text{Lip.}}{\leq}& \|A_H\| \cdot \|H_1 - H_2\| + L_h \|B_H\| \cdot \|(x_1 - x_2, u_1 - u_2)\| \\ \leq& K_H \left(\|H_1 - H_2\| + \|(x_1 - x_2, u_1 - u_2)\| \right) \end{aligned}$$

where $K_H = \max\{\|A_H\|, \|B_H\| L_h\} > 0$.

Note that we use in the following the arguments $f_H(H,x,u)$ since the auxiliary output $h(x,u)$ immediately follows from the state x and the control u.

Remark 18. It is equally possible to write H as a vector $\mathbf{h} \in \mathbb{R}^{p(T-1)\times 1}$, i. e.

$$\begin{aligned} &\mathbf{h}(x(k),u(k:k+T-2)) \\ :=& \left[h(x(k),u(k))^\top, \quad \ldots, \quad h(x_u(k+T-2),u(k+T-2))^\top \right]^\top, \end{aligned}$$

which yields a linear system of the form $\mathbf{h}(k+1) = A\mathbf{h}(k) + Bh(x(k),u(k))$.

6.2.2. Input-To-State-Stability

As we can see from (6.8), the alteration of H only depends on the actual value of the auxiliary output and we can consider the auxiliary output $h(k) = h(x(k),u(k))$ as an input into the memory H. Thus the question arises whether the memory H is input-to-state-stable as described in [Jiang and Wang, 2001] for discrete-time systems w. r. t. its inputs x and u. We introduce for this purpose the property of input-to-state stability as given in [Jiang and Wang, 2001, Def. 3.1].

Definition 4. System (2.1) is *input-to-state stable* (ISS) w. r. t. u if there exists a $\mathcal{KL}$-function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a $\mathcal{K}$-function γ such that, for any $x \in \mathbb{X}$ and any $u \in l_\infty^m$, it holds that

$$\|x_u(k,x)\| \leq \beta\left(\|x - x_s\|, k\right) + \gamma(\|u(\cdot)\|)$$

for each $k \in \mathbb{N}_0$.

We show in the following Lemma that there exists an ISS-Lyapunov function in the sense of [Jiang and Wang, 2001, Def. 3.2] which yields input-to-state stability of system (6.9) by using [Jiang and Wang, 2001, Lem. 3.5].

Lemma 8. *For any $\kappa \in (0,\infty)$, there exists a function $\hat{V}_\kappa : \mathbb{R}^{p\times(T-1)} \to \mathbb{R}_{\geq 0}$ satisfying*

$$\|H - H^s\|_1^\kappa \leq \hat{V}_\kappa(H) \leq (T-1)^2 \|H - H^s\|_1^\kappa \tag{6.11}$$

and

$$\hat{V}_\kappa(f_H(H,x,u)) - \hat{V}_\kappa(H) \leq -\|H - H^s\|_1^\kappa + (T-1)\|h(x,u) - h(x_s,u_s)\|_1^\kappa. \tag{6.12}$$

Furthermore, system (6.9) is ISS w. r. t. $h(x,u) - h(x_s,u_s)$.

Proof. Consider the function

$$\hat{V}_\kappa(H) := \sum_{i=1}^{T-1} i \cdot \|H_i - H_i^s\|_1^\kappa = \sum_{i=1}^{T-1} i \cdot \|H_i - h(x_s,u_s)\|_1^\kappa, \tag{6.13}$$

where H_i denotes again the i-th column of H with $i \in \mathbb{I}_{[1,T-1]}$. We note that it holds

$$\|H - H^s\|_1^\kappa = \max_{j\in\mathbb{I}_{[1,T-1]}} \left\{ \left\| H_j - h(x_s,u_s) \right\|_1^\kappa \right\} \overset{(6.13)}{\leq} \hat{V}_\kappa(H).$$

as well as

$$\begin{aligned}\hat{V}_\kappa(H) &\overset{(6.13)}{=} \sum_{i=1}^{T-1} i \cdot \|H_i - h(x_s,u_s)\|_1^\kappa \leq (T-1)\sum_{i=1}^{T-1} \|H_i - h(x_s,u_s)\|_1^\kappa \\ &\leq (T-1)^2 \|H - H^s\|_1^\kappa,\end{aligned}$$

which shows assertion (6.11). By denoting $H^+ := f_H(H,x,u)$ and using $H_i^+ = H_{i+1}$ for $i \in \mathbb{I}_{[1,T-2]}$ as well as

$$\left\| H_{T-1}^+ - h(x_s,u_s) \right\|_1^\kappa = \| h(x,u) - h(x_s,u_s) \|_1^\kappa, \tag{6.14a}$$

$$\| H - H^s \|_1^\kappa \leq \sum_{i=1}^{T-1} \| H_i - h(x_s,u_s) \|_1^\kappa \tag{6.14b}$$

we obtatin

$$\begin{aligned}
&\hat{V}_\kappa(H^+) - \hat{V}_\kappa(H) \\
&= \sum_{i=1}^{T-1} i \cdot \| H_i^+ - h(x_s,u_s) \|_1^\kappa - \sum_{j=1}^{T-1} j \cdot \left\| H_j - h(x_s,u_s) \right\|_1^\kappa \\
&= \sum_{i=1}^{T-2} i \cdot \| H_{i+1} - h(x_s,u_s) \|_1^\kappa + \left\| H_{T-1}^+ - h(x_s,u_s) \right\|_1^\kappa (T-1) \\
&\quad - \sum_{j=1}^{T-1} j \cdot \left\| H_j - h(x_s,u_s) \right\|_1^\kappa \\
&\overset{(6.14a)}{=} - \sum_{j=1}^{T-1} \left\| H_j - h(x_s,u_s) \right\|_1^\kappa + (T-1) \| h(x,u) - h(x_s,u_s) \|_1^\kappa \\
&\overset{(6.14b)}{\leq} - \| H - H^s \|_1^\kappa + (T-1) \| h(x,u) - h(x_s,u_s) \|_1^\kappa .
\end{aligned}$$

Hence, we have shown assertion (6.12) and input-to-state-stability of the system (6.9) immediately follows from [Jiang and Wang, 2001, Lem. 3.5]. □

Remark 19. Furthermore, we can give an upper bound on a special constellation of future auxiliary output values. If we consider $T-2$ future time steps of a time instant k, it follows from (2.23) that we can bound them by using the initial condition H, otherwise the transient average constraints would be violated. Hence, we can write with the notation from Section 2.1.1

$$
\begin{aligned}
&\bar{\lambda}^\top \sum_{j=1}^{T-1} \sum_{k=0}^{j-1} h_{\mu_N}(k,x,H) \\
=&\bar{\lambda}^\top \left[\sum_{i=0}^{T-2} h_{\mu_N}(i,x,H) + \sum_{i=0}^{T-3} h_{\mu_N}(i,x,H) + \ldots + \sum_{i=0}^{0} h_{\mu_N}(i,x,H) \right] \\
\overset{(2.23)}{\leq}& \bar{\lambda}^\top \left[- \sum_{k=T-1}^{T-1} H_k - \sum_{k=T-2}^{T-1} H_k - \ldots - \sum_{k=2}^{T-1} H_k - \sum_{k=1}^{T-1} H_k \right]
\end{aligned}
$$

where we abbreviate $h_s := h(x_s,u_s)$. Now, using $\bar{\lambda}^\top h(x_s,u_s) = 0$ and monotonicity of l_p norms yields with the previous results

$$
\begin{aligned}
&\bar{\lambda}^\top \sum_{j=1}^{T-1} \sum_{k=0}^{j-1} h_{\mu_N}(k,x,H) \\
\overset{\bar{\lambda}^\top h_s = 0}{\leq}& \bar{\lambda}^\top \left[- \sum_{k=T-1}^{T-1} (H_k - h_s) - \sum_{k=T-2}^{T-1} (H_k - h_s) - \cdots - \sum_{k=1}^{T-1} (H_k - h_s) \right] \\
=& - \bar{\lambda}^\top \sum_{i=1}^{T-1} i \cdot (H_i - h(x_s,u_s)) \\
\leq& \sum_{i=1}^{T-1} i \cdot \|\bar{\lambda}\| \cdot \|H_i - h_s\| \leq (T-1) \|\bar{\lambda}\| \sum_{i=1}^{T-1} \|H_i - h_s\| \\
\leq& (T-1)^2 \|\bar{\lambda}\| \cdot \|H - H^s\|_1 .
\end{aligned}
$$

6.3. Practical Asymptotic Stability

Finally, we introduce the characteristic of practical asymptotic stability. We highlight that for transient averaged constrained EMPC it is not sufficient to consider stability of the steady-state x_s. For example, even if we start at the steady-state $x = x_s$, but the previous trajectory is such that $H \neq H^s$ holds, the resulting trajectory will not stay at the steady-state at the beginning but will rather steer to a region in order to ensure that the transient average constraints are satisfied or to get in a region with lower stage cost (cf. example in Chapter 7). Therefore, we consider stability of the extended state (x,H) and extend the definition of practical asymptotic stability from [Grüne and Stieler, 2014, Def. 2.2] for settings with transient average constraints, i. e, dealing with the extended state (x,H)

Definition 5. The steady-state (x_s,H^s) is called *practically asymptotically stable* w. r. t. $\epsilon \geq 0$ on a set $\mathcal{S} \subseteq \mathbb{X} \times \mathbb{H}$ with $(x_s,H^s) \in \mathcal{S}$ if there exists $\beta \in \mathcal{KL}$ such that

$$\begin{aligned} &\left\| x_{\mu_N}(k,x,H) - x_s \right\| + \left\| H^{\mathrm{cl}}(k,x,H) - H^s \right\| \\ \leq &\max \left\{ \beta(\|x - x_s\| + \|H - H^s\|, k), \epsilon \right\} \end{aligned} \tag{6.15}$$

holds for all $(x,H) \in \mathcal{S}$ and all $k \in \mathbb{N}_0$.

In the following, we conclude practical asymptotic stability using a practical Lyapunov function [Grüne and Stieler, 2014, Faulwasser et al., 2018]. As given there, it is sufficient to show the existence of a practical Lyapunov function in order to prove practical asymptotic stability. As previously mentioned and in contrast to [Grüne and Stieler, 2014], the rotated cost functional $\tilde{J}_N^*(x,H)$ is not a practical Lyapunov function since it is just decreasing in $\|x - x_s\|$ and not necessarily in $\|H - H^s\|$ (cf. Lemma 7), which is a part of our extended state. Furthermore, the lower bound in (6.4) is independent of $\|H - H^s\|$. We highlight that the following two assumptions are only needed to prove practical asymptotic stability, i. e., that the derived performance guarantees (value and trajectory convergence) from Chapter 5 hold without this two additional assumptions. Future work could consider relaxing these assumptions.

In order to construct a practical Lyapunov function, we introduce a condition on the positive definite term ρ of the dissipativity setting. Namely, we assume that this function can be lower bounded by a polynomial.

Assumption 6. There exist constants $a, \omega > 0$ such that it holds

$$\rho(r) \geq a \cdot r^{\omega} \tag{6.16}$$

for all $r \in [0, r_{\max}]$ with $r_{\max} := \max_{(x,u) \in \mathbb{Z}} \|(x - x_s, u - u_s)\|$ and $\rho \in \mathcal{K}_\infty$ from Ass. 2.

Furthermore, we assume that the function $\psi \in \mathcal{K}_\infty$ from Ass. 5 satisfies the following upper bound.

Assumption 7. The function $\psi \in \mathcal{K}_\infty$ from Ass. 5 is sufficiently small. In particular, it holds for $T \geq 2$ and for all $(x,H) \in \mathbb{X} \times \mathbb{H}$ with $(x,H) \neq (x_s,H^s)$

$$\begin{aligned}&\psi(\|x - x_s\|_1 + \|H - H^s\|_1)\\ <&\frac{1}{2}a(n+m)^{-\frac{\omega}{2}}\|x - x_s\|_1^\omega + \frac{1}{2}\frac{a(n+m)^{-\frac{\omega}{2}}}{L_h(T-1)}\|H - H^s\|_1^\omega,\end{aligned} \tag{6.17}$$

with a, ω from Ass. 6.

This assumption is difficult to verify a priori, but we again point out that this assumption is only needed in order to show stability (Thm. 9) and not performance/ convergence guarantees (Chapter 5).

In case that the optimal steady-state (2.18) does not lie in the active set of the transient average constraints, i.e. it holds $h(x_s,u_s) < 0$, the multiplier satisfies $\bar{\lambda} = 0$. Thus, Ass. 5 is satisfied with $\psi \in \mathcal{K}_\infty$ arbitrary small and hence Ass. 7 holds.

Finally, the following theorem provides a practical Lyapunov function. In addition to the rotated value function, we consider input-to-state-stability of the storage H such that we obtain a non-monotonic practical Lyapunov function. Then, using the approach from [Ahmadi and Parrilo, 2008], we construct a more complex monotonic practical Lyapunov function.

Theorem 9. *Let Assumption 1-7 hold. Then for $T \geq 2$, there exists a constant $c > 0$, functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\delta_1, \delta_2 \in \mathcal{L}_\mathbb{N}$ such that the function*

$$\begin{aligned}W(x,H) &:= \sum_{j=0}^{T-1} \hat{W}\left(x_{\mu_N}(j,x,H), H^{\mathrm{cl}}(j,x,H)\right),\\ \hat{W}(x,H) &:= \tilde{J}_N^*(x,H) + c\sum_{i=1}^{T-1} i \cdot \|H_i - h(x_s,u_s)\|_1^\omega,\end{aligned} \tag{6.18}$$

satisfies

$$\alpha_1(\chi(x,H)) \leq W(x,H) \leq \alpha_2(\chi(x,H)) + \delta_1(N - N_6), \tag{6.19}$$

$$W(x_{\mu_N}(1,x,H), H^{\mathrm{cl}}(1,x,H)) \leq W(x,H) - \alpha_3(\chi(x,H)) + \delta_2(N - N_6), \tag{6.20}$$

for all $(x,H) \in \mathbb{X} \times \mathbb{H}$, all $N \geq N_6 + 1$ with $\chi(x,H) := \|x - x_s\| + \|H - H^s\|_1$ and $N_6 \in \mathbb{N}$ from Lemma 7.

Moreover, the steady-state (x_s,H^s) is practically asymptotically stable for all $(x,H) \in \mathbb{X} \times \mathbb{H}$ w.r.t. $\epsilon \to 0$ as $N \to \infty$.

Proof. We split this proof in three parts. In the first part we show that there exist constants $\omega, c > 0$ and functions $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3 \in \mathcal{K}_\infty$ such that the function $\hat{W}(x,H)$ satisfies

$$\hat{\alpha}_1(\chi(x,H)) \leq \hat{W}(x,H) \leq \hat{\alpha}_2(\chi(x,H)) \tag{6.21}$$

and

$$\begin{aligned} &\hat{W}(x_{\mu_N}(1,x,H), H^{\mathrm{cl}}(1,x,H)) - \hat{W}(x,H) \\ \leq& -\hat{\alpha}_3(\chi(x,H)) + \delta_7(N - N_6) + \bar{\lambda}^\top h(x,\mu_N(x,H)) \end{aligned} \tag{6.22}$$

for all $N \geq N_6 + 1$, all $(x,H) \in \mathbb{X} \times \mathbb{H}$ with $\delta_7 \in \mathcal{L}_\mathbb{N}$ from Lemma 7. Then, in the second part we show that the proposed function $W(x,H)$ satisfies the inequalities (6.19) and (6.20). We close the proof by concluding practical asymptotic stability in the third part.

Part I: Showing (6.21) *and* (6.22)
We take the rotated cost functional $\tilde{J}_N^*(x,H)$ from Lemma 7 as well as the function $\hat{V}_\omega(H)$ from Lemma 8 and use

$$\hat{W}(x,H) := \tilde{J}_N^*(x,H) + c\hat{V}_\omega(H) = \tilde{J}_N^*(x,H) + c\sum_{i=1}^{T-1} i \cdot \|H_i - h(x_s,u_s)\|_1^\omega. \tag{6.23}$$

In the following, we show that Inequalities (6.21) and (6.22) hold with

$$c := \frac{1}{2}\frac{a(n+m)^{-\frac{\omega}{2}}}{L_h(T-1)} > 0. \tag{6.24}$$

Note that we just consider the case $T \geq 2$ since in case $T = 1$ we can use known results [Grüne, 2013]. Using the previous results (6.4) and (6.11) yields

$$\begin{aligned} &\rho(\|x - x_s\|) + c\|H - H^s\|_1^\omega \leq \rho(\|(x - x_s, u - u_s)\|) + c\|H - H^s\|_1^\omega \\ \leq& \hat{W}(x,H) \leq \alpha_7(\|x - x_s\| + \|H - H^s\|_1) + c(T-1)^2\|H - H^s\|_1^\omega. \end{aligned}$$

Furthermore, from the properties of comparison functions (cf. [Kellett, 2014]) we know that there exist $\hat{\alpha}_1, \hat{\alpha}_2 \in \mathcal{K}_\infty$ satisfying

$$\begin{aligned} \hat{\alpha}_1(\|x - x_s\| + \|H - H^s\|_1) \leq& \rho(\|x - x_s\|) + c\|H - H^s\|_1^\omega, \\ \hat{\alpha}_2(\|x - x_s\| + \|H - H^s\|_1) \geq& \alpha_7(\|x - x_s\| + \|H - H^s\|_1) \\ &+ c(T-1)^2\|H - H^s\|_1^\omega. \end{aligned}$$

since ρ, $\alpha_7 \in \mathcal{K}_\infty$. Thus, (6.21) is shown. From (6.5), (6.12) and (6.23) it follows with $h(x,\mu_N(x,H)) = h_{\mu_N}(0,x,H)$ (cf. Sec. 2.1.1)

$$\begin{aligned}
&\hat{W}(x_{\mu_N}(1,x,H), H^{\mathrm{cl}}(1,x,H)) - \hat{W}(x,H) \\
\overset{(6.23)}{=} &\tilde{J}_N^*(x_{\mu_N}(1,x,H), H^{\mathrm{cl}}(1,x,H)) - \tilde{J}_N^*(x,H) \\
&+ c\hat{V}_\omega(H^{\mathrm{cl}}(1,x,H)) - c\hat{V}_\omega(H) \\
\overset{(6.5),(6.12)}{\leq} &- \rho(\|(x - x_s, \mu_N(x,H) - u_s)\|) + \psi(\|x - x_s\|_1 + \|H - H^s\|_1) \\
&+ \delta_7(N - N_6) + \bar{\lambda}^\top h_{\mu_N}(0,x,H) \\
&- c\,\|H - H^s\|_1^\omega + c(T-1)\,\|h(x,\mu_N(x,H)) - h(x_s,u_s)\|_1^\omega \\
\leq &- \rho(\|(x - x_s, \mu_N(x,H) - u_s)\|) + \psi(\|x - x_s\|_1 + \|H - H^s\|_1) \\
&+ \delta_7(N - N_6) + \bar{\lambda}^\top h_{\mu_N}(0,x,H) \\
&- c\,\|H - H^s\|_1^\omega + c(T-1)L_h\,\|(x - x_s, \mu_N(x,H) - u_s)\|_1^\omega,
\end{aligned} \tag{6.25}$$

where the last inequality follows from Lipschitz continuity of the auxiliary output. Now, from Ass. 6 it follows that $\rho(r)$ can be lower bounded by a polynomial. Using the previous estimate, we obtain with c from (6.24)

$$\begin{aligned}
&\hat{W}(x_{\mu_N}(1,x,H), H^{\mathrm{cl}}(1,x,H)) - \hat{W}(x,H) \\
\overset{(6.16)}{\leq} &- a\,\|(x - x_s, \mu_N(x,H) - u_s)\|^\omega + \psi(\|x - x_s\|_1 + \|H - H^s\|_1) \\
&+ \delta_7(N - N_6) + \bar{\lambda}^\top h_{\mu_N}(0,x,H) \\
&- c\,\|H - H^s\|_1^\omega + \frac{a(n+m)^{-\frac{\omega}{2}}}{2L_h(T-1)}(T-1)L_h\,\|(x - x_s, \mu_N(x,H) - u_s)\|_1^\omega \\
\leq &- \frac{1}{2}a(n+m)^{-\frac{\omega}{2}}\,\|(x - x_s, \mu_N(x,H) - u_s)\|_1^\omega + \delta_7(N - N_6) \\
&+ \psi(\|x - x_s\|_1 + \|H - H^s\|_1) + \bar{\lambda}^\top h_{\mu_N}(0,x,H) - c\,\|H - H^s\|_1^\omega \\
\leq &- \frac{1}{2}a(n+m)^{-\frac{\omega}{2}}\,\|(x - x_s)\|_1^\omega + \delta_7(N - N_6) \\
&+ \psi(\|x - x_s\|_1 + \|H - H^s\|_1) + \bar{\lambda}^\top h_{\mu_N}(0,x,H) - c\,\|H - H^s\|_1^\omega,
\end{aligned}$$

where we recall that n is the dimension of the state and m the dimension of the input, i. e., $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Now, using Ass. 7 and the properties of comparison functions (cf. [Kellett, 2014]), we conclude that there exists $\hat{\alpha}_3 \in \mathcal{K}_\infty$ such that it holds

$$\begin{aligned}&\frac{1}{2}a(n+m)^{-\frac{\omega}{2}}\|x-x_s\|_1^{\omega}+c\|H-H^s\|_1^{\omega}-\psi(\|x-x_s\|_1+\|H-H^s\|_1)\\ \geq&\hat{\alpha}_3(\|x-x_s\|_1+\|H-H^s\|_1)\overset{(6.17)}{\geq}\hat{\alpha}_3(\|x-x_s\|+\|H-H^s\|_1),\end{aligned}$$

where the last inequality follows from monotonicity of ℓ_p norms.

Part II: Practical Lyapunov Function $W(x,H)$
We remark that $W(x,H)$ from (6.18) is constructed analogously to considerations in [Ahmadi and Parrilo, 2008] in order to get rid of the transient average constraints in (6.22) since they are less or equal zero over a time period of T steps. This yields that the corresponding function is practically monotonically decreasing in each step and moreover, has no additional term corresponding to the actual value of the auxiliary output as it is the case in (6.22).

Hence, $W(x,H)$ is constructed as follows

$$W(x,H):=\sum_{j=0}^{T-1}\hat{W}\left(x_{\mu_N}(j,x,H),H^{\mathrm{cl}}(j,x,H)\right) \tag{6.26}$$

which is equal to (6.18). Since we know from (6.21) that it holds $\hat{W}(x,H)\geq 0$ for all $(x,H)\in\mathbb{X}\times\mathbb{H}$ it follows

$$\begin{aligned}W(x,H)\geq\hat{W}(x,H)\overset{(6.21)}{\geq}&\hat{\alpha}_1(\|x-x_s\|+\|H-H^s\|_1)\\ =:&\alpha_1(\|x-x_s\|+\|H-H^s\|_1).\end{aligned} \tag{6.27}$$

Furthermore, using the the results from Part I yields

$$\begin{aligned}&\hat{W}(x_{\mu_N}(j+1,x,H),H^{\mathrm{cl}}(j+1,x,H))\\ \overset{(6.22)}{\leq}&\hat{W}(x_{\mu_N}(j,x,H),H^{\mathrm{cl}}(j,x,H))+\delta_7(N-N_6)+\bar{\lambda}^\top h_{\mu_N}(j,x,H),\end{aligned} \tag{6.28}$$

and therefore, we get

$$
\begin{aligned}
W(x,H) &\overset{(6.26)}{=} \hat{W}(x,H) + \sum_{j=1}^{T-1} \hat{W}\left(x_{\mu_N}(j,x,H), H^{\mathrm{cl}}(j,x,H)\right) \\
&\overset{(6.28)}{\leq} \hat{W}(x,H) + \sum_{j=1}^{T-1} \left(\hat{W}(x,H) + \sum_{k=0}^{j-1} \delta_7(N-N_6) + \bar{\lambda}^\top h_{\mu_N}(k,x,H) \right) \\
&= T\hat{W}(x,H) + \sum_{j=1}^{T-1}\sum_{k=0}^{j-1} \delta_7(N-N_6) + \bar{\lambda}^\top h_{\mu_N}(k,x,H) \\
&= T\hat{W}(x,H) + \frac{T(T-1)}{2}\delta_7(N-N_6) + \sum_{j=1}^{T-1}\sum_{k=0}^{j-1} \bar{\lambda}^\top h_{\mu_N}(k,x,H) \\
&\overset{\text{Rem.19}}{\leq} T\hat{W}(x,H) + \frac{T(T-1)}{2}\delta_7(N-N_6) + (T-1)^2 \left\|\bar{\lambda}\right\| \cdot \|H-H^s\|_1 \\
&\overset{(6.21)}{\leq} T\hat{\alpha}_2(\|x-x_s\| + \|H-H^s\|_1) + (T-1)^2 \left\|\bar{\lambda}\right\| \cdot \|H-H^s\|_1 \\
&\quad + \frac{T(T-1)}{2}\delta_7(N-N_6) \\
&\leq \alpha_2(\|x-x_s\| + \|H-H^s\|_1) + \delta_1(N-N_6),
\end{aligned}
$$

with $\alpha_2(r) := T\hat{\alpha}_2(r) + (T-1)^2 \left\|\bar{\lambda}\right\| \cdot r$ and $\delta_1 := \frac{1}{2}T(T-1)\delta_7$ and hence, together with (6.27) we have shown the bounds in (6.19). Now, we focus on proving assertion (6.20). From (6.22) and the definition of the transient average constraints it follows

$$
\begin{aligned}
&W(x_{\mu_N}(1,x,H), H^{\mathrm{cl}}(1,x,H)) - W(x,H) \\
&= \sum_{j=1}^{T} \hat{W}\left(x_{\mu_N}(j,x,H), H^{\mathrm{cl}}(j,x,H)\right) - \sum_{j=0}^{T-1} \hat{W}\left(x_{\mu_N}(j,x,H), H^{\mathrm{cl}}(j,x,H)\right) \\
&= \hat{W}\left(x_{\mu_N}(T,x,H), H^{\mathrm{cl}}(T,x,H)\right) - \hat{W}(x,H) \\
&\overset{(6.22)}{\leq} -\hat{\alpha}_3(\|x-x_s\| + \|H-H^s\|_1) + T\delta_7(N-N_6) + \sum_{k=0}^{T-1} \bar{\lambda}^\top h_{\mu_N}(k,x,H) \\
&\leq -\hat{\alpha}_3(\|x-x_s\| + \|H-H^s\|_1) + T\delta_7(N-N_6)
\end{aligned}
$$

since $\sum_{k=0}^{T-1} \bar{\lambda}^\top h(\cdot) \leq 0$ holds by definition of the transient average constraints. This shows assertion (6.20) with $\alpha_3(r) := \hat{\alpha}_3(r)$ and $\delta_2(N-N_6) =$

$T\delta_7(N - N_6)$ with $\delta_7(N - N_6)$ from Lemma 7.

Part III: Concluding Practical Asymptotic Stability

From [Faulwasser et al., 2018, Prop. 4.3] practical asymptotic stability immediately follows with respect to

$$\begin{aligned}\epsilon = \alpha_1^{-1}\Big(\alpha_2\left(\alpha_3^{-1}(\delta_2(N - N_6)) + \delta_2(N - N_6)\right)\\ + \delta_1(N - N_6) + \delta_2(N - N_6)\Big)\end{aligned}$$

and hence, $\epsilon \to 0$ as $N \to \infty$ since $\delta_1, \delta_2 \in \mathcal{L}_{\mathbb{N}}$. □

We conjecture that with similar tools which were used in this chapter in order to conclude practical asymptotic stability of transient average constrained EMPC without terminal constraints, stability with terminal ingredients [Müller et al., 2014a] can be studied, where previously only convergence was shown.

7. Numerical Example

So far, we focused on the theoretical results of transient average constrained economic model predictive control without terminal constraints. In this chapter, we want to illustrate these results by means of a numerical example. We consider a control system which is also investigated in [Köhler et al., 2017, Müller et al., 2014b]. It reads

$$x(k+1) = x(k)u(k) \tag{7.1}$$

with the state and input constraint set $\mathbb{Z} = \mathbb{X} \times \mathbb{U} := [-10,10]^2$. Furthermore, transient average constraints of the form (2.3) with

$$y = h(x,u) = 2x + u - 5 \tag{7.2}$$

are given and the stage cost is chosen as $\ell(x,u) := (x-3)^2 + u^2$. Hence, the optimal steady-state defined in (2.18) is given by $(x_s,u_s) = (2,1)$. Before showing some results of the simulations, we take a closer look at the assumptions we need to satisfy.

7.1. Satisfaction of Assumptions

From the system dynamics (7.1) and the chosen stage cost, it follows that $f(x,u)$ and $\ell(x,u)$ are continuous in x and u. Additionally, the transient average constraints are a linear function and hence, h is Lipschitz continuous. Therefore, Assumption 1 is satisfied. It follows from

$$\begin{aligned} \|h(x_1,u_1) - h(x_2,u_2)\| &= \|2x_1 + u_1 - 5 - (2x_2 + u_2 - 5)\| \\ \leq 2\|x_1 - x_2\| + \|u_1 - u_2\| &\leq 3\max\{\|x_1 - x_2\|, \|u_1 - u_2\|\} \end{aligned}$$

that $L_h = 3$ is a possible Lipschitz constant. Now, we want to show that the strict dissipativity Assumption 2 is satisfied with $\bar{\lambda} = 1$ and $\lambda(x) = \frac{3}{2}(x-2)$ with respect to the optimal steady-state. Using the supply rate $s(x,u)$ from

Assumption 2 and the beforementioned multilpier $\bar{\lambda}$ as well as the storage function $\lambda(x)$, we obtain

$$\ell(x,u) - \ell(x_s,u_s) + \bar{\lambda}^\top h(x,u) + \lambda(x) - \lambda(f(x,u)) =$$
$$(x-3)^2 + u^2 - 2 + 2x + u - 5 + \frac{3}{2}x - \frac{3}{2}xu$$

which can be rearranged to (note that $x_s = 2$ and $u_s = 1$)

$$\begin{aligned}
&(x-3)^2 + u^2 - 2 + 2x + u - 5 + \frac{3}{2}x - 3 - \frac{3}{2}xu + 3 \\
=&(x-2)^2 + (u-1)^2 - \frac{3}{2}(x-2)(u-1) \\
=&\begin{bmatrix} x-2 \\ u-1 \end{bmatrix}^\top \begin{bmatrix} 1 & -3/4 \\ -3/4 & 1 \end{bmatrix} \begin{bmatrix} x-2 \\ u-1 \end{bmatrix} =: z^\top A z =: V(z)
\end{aligned}$$

where we used the shifted coordinate

$$z := \begin{bmatrix} x - x_s \\ u - u_s \end{bmatrix} = \begin{bmatrix} x-2 \\ u-1 \end{bmatrix}.$$

With the positive definiteness of A we can write

$$\lambda_{\min}(A)\,\|z\|_2^2 \le z^\top A z \le \lambda_{\max}(A)\,\|z\|_2^2$$

which implies [Khalil, 2002, Lem. 4.3]

$$\begin{aligned}
V(z) \ge \lambda_{\min}(A)\,\|z\|_2^2 = \frac{1}{4}\,\|z\|_2^2 &= \frac{1}{4}\,\|(x-x_s,\, u-u_s)\|_2^2 \\
&=: \rho(\|(x-x_s,\, u-u_s)\|)
\end{aligned} \tag{7.3}$$

with $\rho \in \mathcal{K}_\infty$. Furthermore, it holds $\lambda(x_s = 2) = 0$ and we can write

$$|\lambda(x)| = \left|\frac{3}{2}(x-2)\right| = \frac{3}{2}\,\|x - x_s\| = \gamma_\lambda(\|x - x_s\|), \quad \gamma_\lambda \in \mathcal{K}_\infty.$$

This shows strict dissipativity (Assumption 2) of system (7.1) with respect to the supply rate $s(x,u) = \ell(x,u) - \ell(x_s,u_s) + \bar{\lambda}^\top h(x,u)$. Furthermore, (7.3) implies that Assumption 6 holds with $a = \frac{1}{4}$ and $\omega = 2$.

7.1.1. Controllability Assumptions

Asymptotic Controllability Condition

This controllability condition (Ass. 3) is difficult to show as stated (i. e., for all $\mathbb{X} \times \mathbb{H}$). We conjecture that the results can be modified such that asymptotic controllability on a control invariant sublevel set ($\mathcal{X} \times \mathcal{H}$) is sufficient using arguments from [Boccia et al., 2014, Köhler et al., 2018].

Local Controllability Condition

Now, consider the local controllability condition from Ass. 4. We show in the following that there exists an input sequence which is feasibly connecting local states (x,H) with the steady-state and we choose

$$E_h = \frac{1}{4(T-1)} \quad \text{and} \quad \delta_c = \frac{1}{8} \tag{7.4}$$

in case $T \geq 2$. In case $T = 1$, it holds for any $E_h \in \mathbb{R}_{\geq 0}$ since the transient average constraints degenerate to point-wise in time constraints and hence, the matrix H is vanishing. We just focus in the following on the case $T \geq 2$. With the choice (7.4), we show that there exist $u_2(0)$ and $u_2(1)$ such that with $d = \max\{3,T\}$

$$u_2 := \begin{bmatrix} u_2(0), & u_2(1), & u_2(2) & u_s, & \dots, & u_s \end{bmatrix} \in \mathbb{U}^d(x,H)$$

holds for all $x \in \mathbb{B}_{\delta_c}(x_s)$ and all $\lceil H - H^s \rceil \leq E_h$. In particular, we show feasibility by constructing u_2 where we leave a steady-state region in the first time step in order to ensure feasibility of the transient average constraints and then steer the system back to the steady-state in two steps and then, remain at the steady-state.

From $\lceil H^c \rceil \leq E_h = \frac{1}{4(T-1)}$ and $\lceil H - H^c \rceil \leq E_h = \frac{1}{4(T-1)}$, we obtain that it holds for all $j \in \mathbb{I}_{[1,T-1]}$

$$H_j \leq 2E_h = \frac{1}{2(T-1)}. \tag{7.5}$$

Furthermore, choosing $u_2(0) = \frac{1}{4} \in \mathbb{U}$ yields

$$h(x,u_2(0)) \overset{(7.2)}{=} 2x + u_2(0) - 5 \overset{x \in \mathcal{B}_{\delta_c}(x_s)}{\leq} 2(x_s + \delta_c) + u_2(0) - 5 \overset{(7.4)}{=} -\frac{1}{2}. \tag{7.6}$$

Now, we arrive at $h(x,u_2(0)) + \sum_{k=1}^{T-1} H_k \overset{(7.5)}{\leq} -\frac{1}{2} + (T-1)\frac{1}{2(T-1)} = 0$ and hence it holds for all $j \in \mathbb{I}_{[1,T-1]}$

$$h(x,u_2(0)) + \sum_{k=j}^{T-1} H_k \leq 0,$$

i. e., it follows $u_2 \in \mathbb{U}^d(x,H)$ if it holds $h(x(k),u_2(k)) \leq 0$ for all $k \in \mathbb{I}_{[1,d]}$. From the system dynamics (7.1), we obtain $x_{u_2}(1,x) = x \cdot u_2(0) = \frac{x}{4} \in \mathbb{X}$. Now, we choose $u_2(1) = \frac{x_s}{x_{u_2}(1,x)} = \frac{2x_s}{x} \in \mathbb{U}$ such that $x_{u_2}(2,x) = \frac{x_s}{2} \in \mathbb{X}$ holds. This yields $h(x_{u_2}(1,x),u_2(1)) \leq 0$ since $x \in [\frac{15}{8},\frac{17}{8}] \subset \mathbb{X}$ due to $\delta_c = \frac{1}{8}$ and $x_s = 2$. Then, applying $u_2(2) = 2$ yields $x_{u_2}(3,x) = x_s$ and $h(x_{u_2}(2,x),u_2(2)) = -1 \leq 0$. As previously described, we stay at the steady-state for the subsequent trajectory ($k \geq 3$) where it holds $h(x_s,u_s) = 0$. Hence, it holds for the choices (7.4)

$$u_2 = \left[\tfrac{1}{4}, \quad 2\tfrac{x_s}{x}, \quad 2, \quad u_s, \quad \ldots, \quad u_s\right] \in \mathbb{U}^d(x,H)$$

and we get the corresponding trajectory

$$x_{u_2}(\cdot,x) = \left[x, \quad \tfrac{x}{4}, \quad \tfrac{x_s}{2}, \quad x_s, \quad \ldots, \quad x_s\right],$$

i. e., we have shown that there exists a sequence. Note that we can use similiar argument to ensure feasibility of the transient average constraints at the end of the trajectory. However, we did not show the bounds (4.1).

Bound on the Auxiliary Output of the Optimal Rrajectory resulting from the rotated Optimization Problem

As mentioned in Chapter 6, it is difficult to verify Assumption 7 a priori. However, numerical investigations can be made. Exemplarily, we consider the following conditions. The prediction horizon is $N = 20$, the time period $T = 6$, $k_\psi = 12$ and the previous trajectory reads $H = [h(1,1), \ldots, h(1,1), h(1,x)]$, where x is the initial condition. Then, numerical investigations show that Ass. 7 is satisfied for all $x \in [1.85,3]$.

7.2. Turnpike Properties

In this section, we illustrate some results obtained in Chapter 3. Figure 7.1 illustrates the turnpike property for the state x where we consider an initial value $x = 1$, a time period $T = 3$ for the transient average constraints and two different prediction horizons $N_1 = 10$ and $N_2 = 12$. Note that the previous trajectory is such that we also stay at $x = 1$ for the last $T - 1$ time instants, i. e.,

$$H = \begin{bmatrix} h(1,1), & h(1,1) \end{bmatrix} = \begin{bmatrix} -2 & -2 \end{bmatrix} \in \mathbb{R}^{1\times(T-1)}$$

and hence, it follows $\rceil H \lceil = 0$. Then, the open-loop trajectories tend towards a neighborhood of the optimal steady-state $x_s = 2$ during the transient phase. By increasing N, the amount of time instants for which the open-loop optimal trajectory is in a neighborhood of the steady-state gets larger, as it is shown in Theorem 1 (3.11). Since all assumptions of Theorem 1 hold, we can also conclude Corollary 1 which means that the value of the auxiliary output is in a neighborhood of $h(x_s,u_s)$ during the transient phase and obviously, that more time instants are in this neighborhood if the prediction horizon is larger. Simulations have shown the turnpike property of $h(x,u)$ for this example as well; however, it is not shown in this thesis for reasons of clarity. Furthermore, Figure 7.1 shows the assertion of Theorem 2 which says that with an increasing prediction horizon N, the neighborhood of the steady-state, to which the optimal trajectory is converging to during the transient phase, is shrinking.

Effects of the Time Period T of the Transient Average Constraints

In addition to the case of economic MPC without terminal constraints [Grüne, 2013], we also consider the influence of the time period T of the transient average constraints (2.3) on the optimal open-loop trajectories of the system. Hence, we choose $N = 12$, the initial value $x = 1$ as well as the two periods of the transient average constraints $T_1 = 3$ and $T_2 = 6$ with previous trajectories $H^1 = \begin{bmatrix} h(1,1), & h(1,1) \end{bmatrix}$, $H^2 = \begin{bmatrix} h(1,1), & h(1,1), & h(1,1), & h(1,1), & h(1,1) \end{bmatrix}$, respectively. Analogously to Figure 7.1, the resulting state trajectories are illustrated in Figure 7.2. One can see that the optimal trajectory resulting from the larger time period T_2 (——) is allowed to stay longer in a "cheap" region (w. r. t. the stage cost ℓ) where it holds $h(x,u) > 0$. This implies that there are fewer time instants close to the optimal steady-state and moreover, that the neighborhood the trajectory is in, is also increasing for an increasing time period T (cf. Figure 7.2) which is in accordance with the results in Thm. 3 and Cor. 3. We note that this behaviour is typical for the consideration of transient average constraints since the primal goal is to minimize the cost functional over the prediction horizon, i. e., if possible we

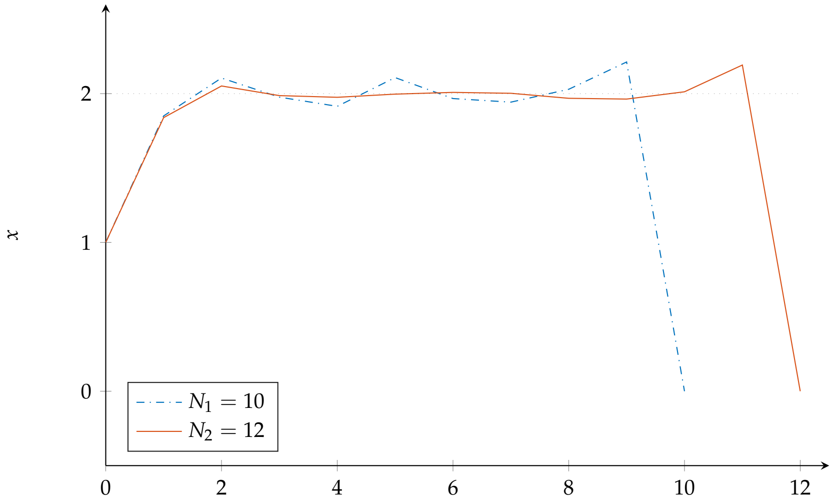


Figure 7.1.: Open loop optimal solution of (7.1) with $T = 3$, $x = 1$ and $N_1 = 10$ (-·-·-), $N_2 = 12$ (——).

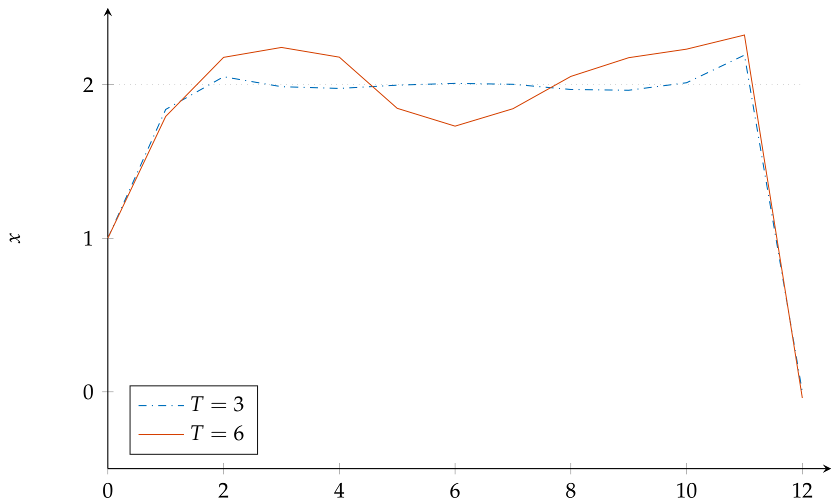


Figure 7.2.: Open loop optimal solutions of (7.1) with $N = 12$, $x = 1$ and $T = 3$ (-·-·-), $T = 6$ (——).

stay in a region where the stage cost $\ell(x,u)$ is small as long as it is possible w. r. t. the transient average constraints. If this period is longer we can also move further away to minimize $\ell(x,u)$ with an input which is not too large (to keep $\ell(x,u)$ small). In order to compensate the time instants with $h(x,u) > 0$ the system has to steer afterwards in a region where it holds $h(x,u) < 0$ such that the transient average constraints are satisfied. The result is a periodic like behaviour as we can see in the Figures 7.1 and 7.2.

7.3. Closed Loop Results

After having investigated the turnpike phenomenon for different settings of N and T, we now consider the closed loop (2.11). In the following, we present the closed-loop trajectories $x_{\mu_N}(\cdot,x,H)$ and $\mu_N(\cdot,x,H)$ (cf. Sec. 2.1.1) as well as the decrease of the Lyapunov function $W(x,H)$ and show that the rotated value function $\tilde{J}_N(x,H)$ is not a valid Lyapunov function. Furthermore, we illustrate the behaviour of the closed loop over the prediction horizon N. For this section, we use the prediction horizon $N = 12$, the time period $T = 6$ and the initial conditions

$$x = 2, \quad H = \begin{bmatrix} h(1,1), & h(1,1), & h(1,1), & h(1,1), & h(1,2) \end{bmatrix} \qquad (7.7)$$

which implies $\rceil H \lceil = 0$.

Closed Loop Trajectories

As we expect from the results of Chapter 6, the closed-loop state and input trajectories converge to a neighborhood of the optimal steady-state (x_s,u_s) (cf. Figure 7.3). Since the past trajectories are in regions where it holds $h(x,u) < 0$, the closed-loop trajectories first move to a region with lower stage cost $\ell(x,u)$ but then, are steered back to a region with higher cost in order to satisfy the transient average constraints. However, for a sufficiently large prediction horizon N and in contrast to the open-loop optimal trajectories (cf. Section 7.2) the neighborhoods of the optimal steady-state (x_s,u_s) in which the closed-loop trajectories are, are decreasing for an increasing number of MPC iterations k. Moreover, the closed loop is practically asymptotically converging to the optimal steady-state (x_s,u_s). We remark, that this implies the same behavior for the auxiliary output $h(x,u)$.

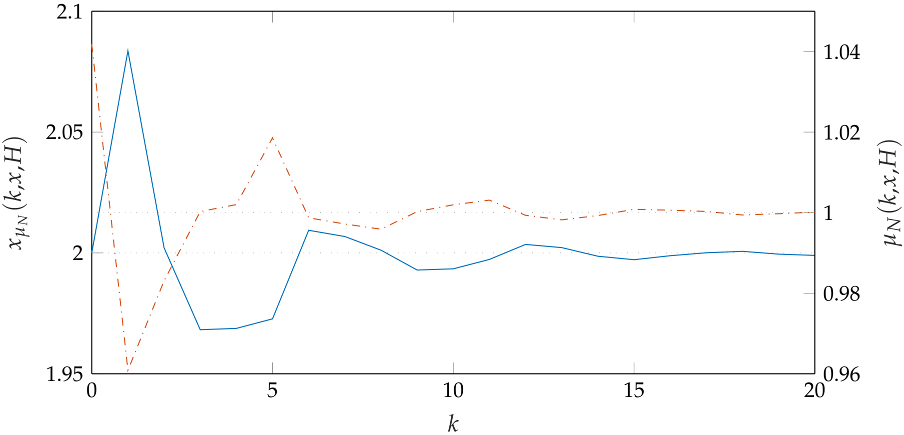


Figure 7.3.: Closed-loop state trajectory (——) and input trajectory (-·-·-) for the initial conditions (7.7).

Lyapunov Function

Now, we focus on the rotated value function $\tilde{J}_N^*$ and the novel Lyapunov function W from the previous Chapter. As illustrated in Figure 7.4, the simulation of our numerical example confirms the derived stability results. Namely, the rotated value function $\tilde{J}_N^*$ (-·-·-) which is usually used as a Lyapunov function [Grüne and Stieler, 2014] is not decreasing. Since we have chosen our initial conditions such that we start at the steady-state it holds $\tilde{J}_N^*(x_s,H) = 0$. But due to the initial condition H from (7.7), the closed loop steers in a region with lower cost such that it follows $\tilde{J}_N^* > 0$ and thus, the rotated value function can not be a valid Lyapunov function. Now, using the sum of $\tilde{J}_N^*$ over the time period T as well as the ISS property of the storage H yields $W(x,H)$ which is decreasing along the closed-loop trajectory for sufficiently large N (cf. Figure 7.4). Furthermore, simulations have shown that neglecting the ISS property $(c = 0)$ results in a function which is not necessarily decreasing, i. e., summing up $\tilde{J}_N^*$ over the time period T, does also not result in a valid Lyapunov function.

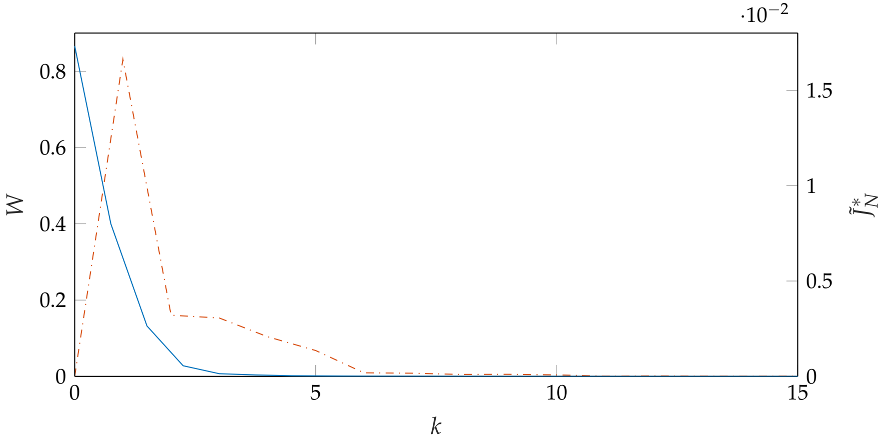


Figure 7.4.: Lyapunov function W (——) and rotated value function $\tilde{J}_N^*$ (-·-·-) along the closed loop trajectory for the initial conditions (7.7).

Closed Loop Cost over N

The question arises, how the choice of the prediction horizon N affect the closed-loop costs $J_K^{\mathrm{cl}}(x,H)$. Therefore, we consider the closed loop with the same initial conditions as in (7.7). This means that again the previous trajectory stays at $x = 1$ by applying $u = 1$ and in the last past step we set $u = 2$ such that the closed-loop state trajectory starts at the steady-state $x = x_s = 2$. We remark that we consider 100 closed-loop iterations, i. e., $K = 100$. Therefore, the (feasible) steady-state trajectory has a value of $K \cdot \ell(x_s,u_s) = 200$

Now, if we vary the prediction horizon N and fix $T = 6$, we obtain the closed-loop costs as shown in Figure (7.5). There, the closed-loop costs are not monotonically decreasing for an increasing prediction horizon. However, we highlight that our theory (cf. Chapter 6) provides a function which upper bounds the closed-loop cost and is monotonically decreasing. It is worth pointing out that the closed-loop costs show a local peak every T steps. In particular, the local peak occurs for prediction horizons $N = jT + \frac{T}{2}$ with $j \in \mathbb{I}_{\geq 1}$. Furthermore, for prediction horizons $N = iT$ with $i \in \mathbb{I}_{\geq 2}$ the closed-loop costs show a local minimum with $J_{qT}^{\mathrm{cl}}(x,H) \approx qT\ell(x_s,u_s)$ for

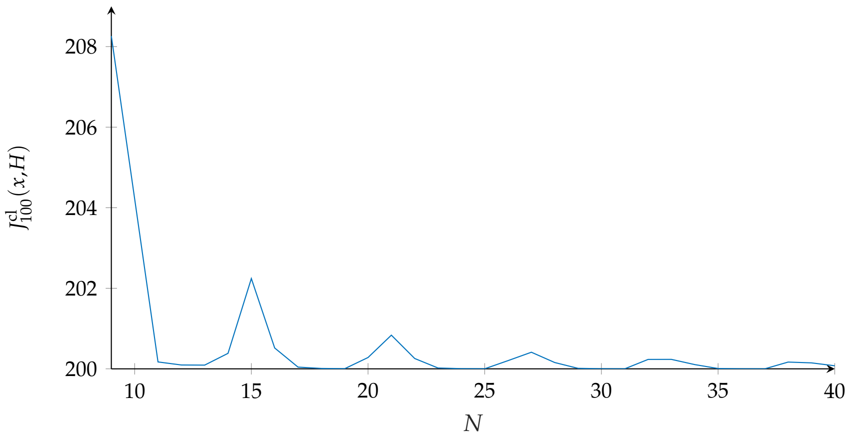


Figure 7.5.: Closed loop cost (2.14) for $K = 100$ and a time period $T = 6$ over the prediction horizon N for an initial condition (7.7).

$q \in \mathbb{I}_{\geq 3}$. A possible explanation for this phenomenon is that for a prediction horizon N which is a multiple of T, the optimal open loop can use the full “potential” of the transient average constraints and steer the system in a region with lower stage cost, while for $N = jT + \frac{T}{2}$, the controller is acting more reserved.

8. Conclusion

This chapter summarizes the main results of this thesis and provides an outlook on possible future work building on this work.

8.1. Summary

The overall goal of this thesis was to show stability of transient average constrained economic MPC without terminal conditions which can be interpreted as a combination of [Grüne, 2013, Grüne and Stieler, 2014, Müller et al., 2014a].

To this end, we first introduced an additional state storing past values of the auxiliary output and provided turnpike properties for this variable which is equal to a turnpike property for consecutive time instants of the state and input. By using a local controllability assumption, we obtained from the turnpike conditions local continuity of the value function. This is mainly an extension of [Grüne, 2013].

Then convergence of the closed-loop cost was proved and it was shown that several results considering optimal trajectories also hold for trajectories resulting from the minimization of the rotated cost function. This resulted in convergence of the rotated closed-loop cost by considering intersections of sets. In addition to that, trajectory convergence of the closed loop was shown.

It was proved that in contrast to [Grüne and Stieler, 2014], the rotated value function is not a valid Lyapunov function of the extended state. However, a combination of the rotated value function, input-to-state stability of the auxiliary output storage and an approach from [Ahmadi and Parrilo, 2008] resulted in a novel practical Lyapunov function and hence, practical asymptotic stability of EMPC subject to transient average constraints without terminal ingredients was shown. Some of the system theoretic properties were illustrated by means of a numerical example at the end of this thesis.

8.2. Future Work

Future work could consider asymptotic average constrained EMPC [Müller et al., 2013,Köhler et al., 2017] without terminal constraints; we conjecture that similar guarantees would be natural. Furthermore, the extension to robust performance guarantees [Bayer et al., 2018] in this setting is an open problem.

A. Feasibility of connecting Trajectories

In this appendix, we provide tools in order to compare different storages H of the auxiliary output. Considering any two trajectories $u_1 \in \mathbb{U}^{T-1}(x_1)$ and $u_2 \in \mathbb{U}^{T-1}(x_2)$, we write if it holds

$$\sum_{k=1}^{T-i} H_k(x_2,u_2) \leq \sum_{k=1}^{T-i} H_k(x_1,u_1),\ \forall i \in \mathbb{I}_{[1,T-1]}$$

the equivalent notation

$$H(x_2,u_2) \overrightarrow{\precsim} H(x_1,u_1). \tag{A.1}$$

Furthermore, if it holds

$$\sum_{k=T-i}^{T-1} H_k(x_2,u_2) \leq \sum_{k=T-i}^{T-1} H_k(x_1,u_1),\ \forall i \in \mathbb{I}_{[1,T-1]},$$

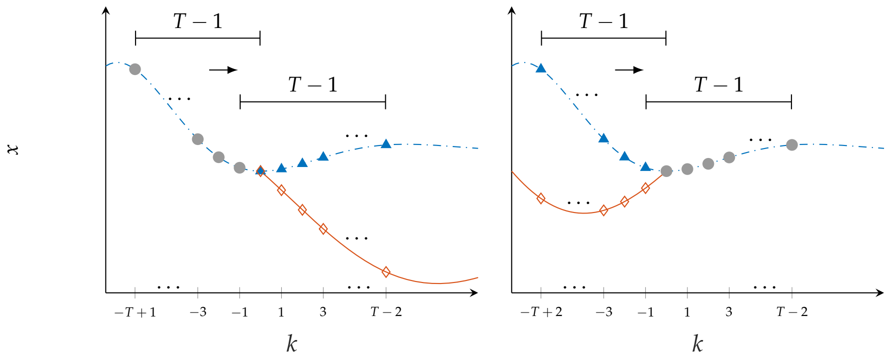
we write

$$H(x_2,u_2) \overleftarrow{\precsim} H(x_1,u_1). \tag{A.2}$$

The first property (A.1) expresses that for the same initial state, we can append x_{u_2} to the past trajectory of x_{u_1} (i. e., (x_1,H^1)) (cf. Fig. A.1a) and retain feasibility of the transient average constraints if the previous trajectory satisfied the constraints, i. e., if we consider $u_1 \in \mathbb{U}^{N_1}(x_1,H^1)$, $u_2 \in \mathbb{U}^{N_2}(x_2,H^2)$ with $N_1, N_2 \geq T$ and $x_1 = x_2$, then $H(x_2,u_2(0:T-2)) \overrightarrow{\precsim} H(x_1,u_1(0:T-2))$ implies $u_2 \in \mathbb{U}^{N_2}(x_2,H^1)$.

The second property is similar, just that we have stricter conditions for the last entries of H and weaker conditions for the first entries of H (it is the other way round for the first property). This means that this condition implies that we can set (x_1,H^2) as the previous trajectory of $x_{u_1}(k,x_1)$ for $k \in \mathbb{I}_{[0,N_1]}$ (cf. Fig A.1b).

Note that for both figures, the values of $h(x,u)$ with corresponding state values (—▲— and —◇—) appear in both storages of the auxiliary output. We can see that in the case of a possible subsequent trajectory (Figure A.1a) especially the "first" value of the auxiliary output $h(x,u)$ (which is equivalent



(a) Compare with subsequent trajectory. **(b)** Compare with previous trajectory.

Figure A.1.: Comparison of a candidate trajectory x_{u_2} (——) with an existing trajectory x_{u_1} (-·-·-) in order to conclude feasibility.

to the first column of $H(x_i, u_i(0 : T-2))$ with $i \in \{1,2\}$) is important. This is considered by (A.1). The same holds the other way round for a previous trajectory (cf. Figure A.1b). Here, especially the "last" value of the auxiliary output (equivalent to the last column of H) matters since it appears in every consideration of the overlapping period of the transient average constraints in order to conclude feasibility by (A.2).

The following proposition formalizes the feasibility results based on the notation in (A.1), (A.2), compare Figure A.2 for an illustration of the property.

Proposition 6. *With $i \in \{1,2,3\}$, let $N_i \geq T$ and $(x_i, H^i) \in \mathbb{X} \times \mathbb{H}$. Considering the inupt sequences $u_1 \in \mathbb{U}^{N_1}(x_1, H^1)$, $u_3 \in \mathbb{U}^{N_3}(x_3, H^3)$ and suppose there exists an input trajectory $u_2 \in \mathbb{U}^{N_2}(x_1)$ satisfying for all $j \in \mathbb{I}_{[0,N_2-T]}$*

$$x_{u_2}(N_2, x_1) = x_3, \qquad \sum_{k=j}^{j+T-1} h(x_{u_2}(k, x_s), u_2(k)) \leq 0. \tag{A.3}$$

Then, satisfaction of

$$\begin{aligned} H(x_2, u_2(0 : T-2)) &\overrightarrow{\lesssim} H(x_1, u_1(0 : T-2)), \\ H(x_{u_2}(N_2^-, x_2), u_2(N_2^- : N_2 - 1)) &\overleftarrow{\lesssim} H^3, \end{aligned} \tag{A.4}$$

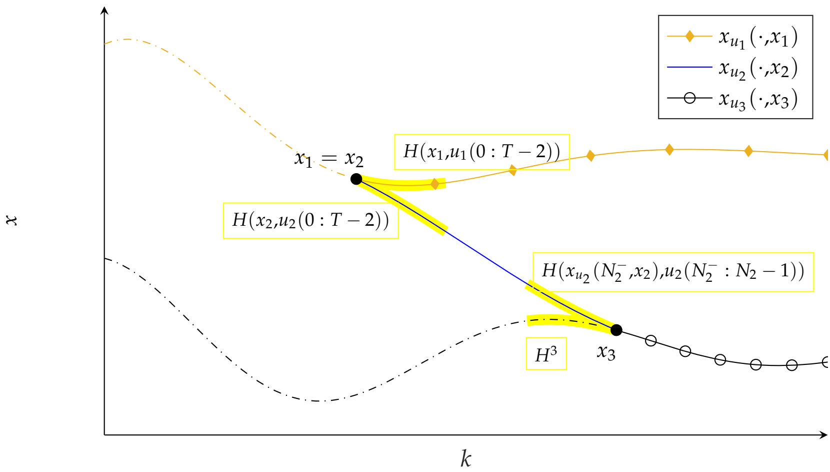


Figure A.2.: The property of connecting two existing trajectories with respect to the transient average constraints.

with $N_2^- := N_2 - T + 1$ *implies* $\hat{u} \in \mathbb{U}^{N_2+N_3}(x_1,H^1)$ *with* $\hat{u}(k) = u_2(k)$, $k \in \mathbb{I}_{[0,N_2-1]}$ *and* $\hat{u}(k) = u_3(k - N_2)$, $k \in \mathbb{I}_{[N_2,N_2+N_3-1]}$ *and we call* u_2 *trajectory-to-trajectory feasible w. r. t.* (x_1,u_1,H^1) *and* (x_3,u_3,H^3) *.*

Proof. We set the previous trajectory of x_{u_2} equal to the previous trajectory of x_{u_1}, i. e., we use (x_1,H^1) as the initial conditions of x_{u_2}, u_2. Equivalently, we set $x_{u_3}(k,x_3)$, u_3 for all $k \in \mathbb{I}_{[0,N_3-1]}$ as the subsequent trajectory of x_{u_2}, i. e., we append it to the "end" point $x_{u_2}(N_2,x_2)$. The resulting trajectory is $x_{\hat{u}}(k,x_1)$, $\hat{u}(k)$ with $k \in \mathbb{I}_{[0,N_2+N_3-1]}$ and $x_2 = x_1$ as given in the statement. The scheme is also given in Figure A.2. Now, we consider the transient average constraints of this candidate trajectory. It follows that

$$\sum_{k=j_1}^{j_1+T-1} h(x_{\hat{u}}(k,x_1),\hat{u}(k)) = \sum_{k=j_1}^{j_1+T-1} h(x_{u_2}(k,x_1),u_2(k)) \leq 0,$$

$$\sum_{k=j_2}^{j_2+T-1} h(x_{\hat{u}}(k,x_1),\hat{u}(k)) = \sum_{k=j_2}^{j_2+T-1} h(x_{u_3}(k-N_2,x_3),u_1(k-N_2)) \leq 0$$

holds for all $j_1 \in \mathbb{I}_{[0,N_2-T]}$, $j_2 \in \mathbb{I}_{[N_2,N_2+N_3-T]}$. Note that these inequalities follow from (A.3) and $u_3 \in \mathbb{U}^{N_3}(x_3,H_3)$, respectively. It remains to show that in the overlapping area of (x_1,H^1) and x_{u_2}, u_2 as well as x_{u_2}, u_2 and x_{u_3}, u_3, we retain feasibility of the transient average constraints. We know from (A.4) that it holds for all $i \in \mathbb{I}_{[1,T-1]}$ (cf. the definitions (A.1) and (A.2))

$$\sum_{k=1}^{T-i} H_k(x_2,u_2(0:T-2)) \leq \sum_{k=1}^{T-i} H_k(x_1,u_1(0:T-2)), \tag{A.5a}$$

$$\sum_{k=T-i}^{T-1} H_k(x_{u_2}(N_2^-),u_2(N_2^-:N_2-1)) \leq \sum_{k=T-i}^{T-1} H_k^3. \tag{A.5b}$$

where we recall the abbreviation $N_2^- := N_2 - T + 1$. This implies that it holds

$$\sum_{k=1+j_3}^{T-1} H_k^1 + \sum_{k=0}^{j_3} h(x_{\hat{u}}(k,x_1),\hat{u}(k)) = \sum_{k=1+j_3}^{T-1} H_k^1 + \sum_{k=0}^{j_3} h(x_{u_2}(k,x_2),u_2(k))$$

$$\overset{\text{(A.5a)}}{\leq} \sum_{k=1+j_3}^{T-1} H_k^1 + \sum_{k=0}^{j_3} h(x_{u_1}(k,x_1),u_1(k)) \leq 0,$$

with $j_3 \in \mathbb{I}_{[0,T-2]}$ where $u_1 \in \mathbb{U}^{N_1}(x_1,H_1)$ implies the last inequality. Furthermore, we get

$$\begin{aligned}
&\sum_{k=j_4}^{j_4+T-1} h(x_{\hat{u}}(k,x_1),\hat{u}(k)) \\
&= \sum_{k=j_4}^{N_2-1} h(x_{u_2}(k,x_2),u_2(k)) + \sum_{k=0}^{j_4-N_2+T-1} h(x_{u_3}(k,x_3),u_3(k)) \\
&\overset{\text{(A.5b)}}{\leq} \sum_{k=j_4-N_2+T}^{T-1} H_k^3 + \sum_{k=0}^{j_4-N_2+T-1} h(x_{u_3}(k,x_3),u_3(k)) \leq 0,
\end{aligned}$$

with $j_4 \in \mathbb{I}_{[N_2-T+1,N_2-1]}$ where the last inequality follows equivalently from $u_3 \in \mathbb{U}^{N_3}(x_3,H_3)$. Therefore, the transient average constraints are satisfied for the candidate $\hat{u}(k)$ for all $k \in \mathbb{I}_{[0,N_2+N_3-1]}$, i. e., along the complete trajectory and hence $\hat{u}$ is feasible. □

B. Intersection of Sets

In order to conclude that different optimal trajectories share the same time instants for which the turnpike property is fulfilled (cf. Proposition 2), we want to consider intersection of sets in general. Here, we consider sets with entries of values $k \in \mathbb{I}_{[0,\hat{N}-1]}$. As an easy case, we want to start with the intersection of two sets $\mathcal{A}_1$ and $\mathcal{A}_2$ as it is shown in Figure B.1. The first set $\mathcal{A}_1$ has a_1 entries and the second set has a_2 entries. Then at most $(\hat{N} - a_1) + (\hat{N} - a_2)$ entries are not in the intersection $\mathcal{A}' := \mathcal{A}_1 \cap \mathcal{A}_2$. This implies that there are at least

$$\hat{N} - (\hat{N} - a_1) - (\hat{N} - a_2) = a_1 + a_2 - \hat{N}$$

elements in the intersection set $\mathcal{A}'$. Applying this argument repeatedly, we get for the intersection of α sets $\mathcal{A}_i$ (which each consists of a_i entries)

$$\#\mathcal{A}' \geq \hat{N} - \sum_{i=1}^{\alpha} (\hat{N} - a_i) = -(\alpha - 1)\hat{N} + \sum_{i=1}^{\alpha} a_i.$$

Note that for the special case $\alpha = 1$, i. e, we build an intersection of one set, we obtain $\#\mathcal{A}' \geq a_1$ which is the amount of time entries the set has. Furthermore, if we just want to consider entries within a specific interval $k \in \mathbb{I}_{[k_l,k_u]}$, we tighten the intersection to

$$\mathcal{A}' := \mathcal{A}_1 \cap \cdots \cap \mathcal{A}_\alpha \cap \{k_l, \ldots, k_u\},$$

with $0 \leq k_l < k_u \leq \hat{N} - 1$ and get as an estimate for the amounts of entries in the intersection set

$$\#\mathcal{A}' \geq \hat{N} - (\hat{N} - (k_u - k_l)) - \sum_{i=1}^{\alpha} (\hat{N} - a_i) = -\alpha\hat{N} + (k_u - k_l) + \sum_{i=1}^{\alpha} a_i. \quad \text{(B.1)}$$

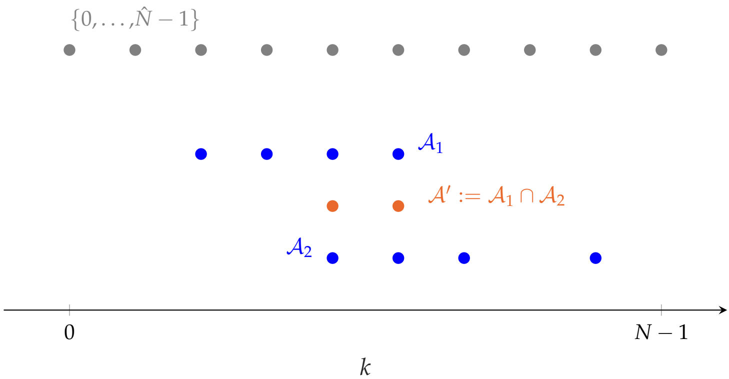


Figure B.1.: Intersection of two sets which consist of just specific time instants $k \in \mathbb{I}_{[0,\hat{N}-1]}$.

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Eigenständigkeitserklärung

Ich versichere hiermit, dass ich, Mario Rosenfelder, die vorliegende Arbeit selbstständig angefertigt, keine anderen als die angegebenen Hilfsmittel benutzt und sowohl wörtliche, als auch sinngemäß entlehnte Stellen als solche kenntlich gemacht habe. Die Arbeit hat in gleicher oder ähnlicher Form noch keiner anderen Prüfungsbehörde vorgelegen. Weiterhin bestätige ich, dass das elektronische Exemplar mit den anderen Exemplaren übereinstimmt.

Ort, Datum

Unterschrift

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