Institute of Formal Methods in Computer Science

University of Stuttgart
Universitätsstraße 38
D–70569 Stuttgart

Bachelorarbeit

Constructions and Closure Properties for Partial and Complete Automaton Structures

Janette Welker

Course of Study: Softwaretechnik

Examiner: Volker Diekert

Supervisor: Jan Philipp Wächter

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Abstract

An interesting aspect to study on the topic of semigroups are their closure properties. In particular, this work aims to expand on previous results on the closure under the semigroup free product. These results stem from work done by Tara Brough and Alan J. Cain, who proved that automaton semigroups which have an idempotent or are homogeneous will form an automaton semigroup under the free product. In this work, we show that these restrictions can be loosened to encompass a greater number of automaton semigroups. We accomplish this by proving that two automaton semigroups form another automaton semigroup under free product construction, if there exist maps between their state sets that each extend into homomorphisms between their generated semigroups.

Kurzfassung

Ein interessanter Aspekt, den man beim Thema Automatenhalbgruppen betrachten kann, sind deren Abschlusseigenschaften. Insbesondere ist das Ziel dieser Arbeit, bisherige Ergebnisse des Abschlusses unter freiem Produkt von Halbgruppen zu erweitern. Diese Ergebnisse stammen aus der Arbeit von Tara Brough und Alan J. Cain, welche bewiesen haben, dass Automatenhalbgruppen, welche mindestens ein idempotentes Element oder homogenen Charakter haben, unter dem freien Produkt wieder eine Automatenhalbgruppe bilden. In dieser Arbeit zeigen wir, dass diese Einschränkungen etwas gelockert werden können, um mehr Automatenhalbgruppen abzudecken. Wir erreichen dies, indem wir beweisen, dass zwei Automatenhalbgruppen eine weitere Automatenhalbgruppe unter freier Produktkonstruktion bilden, wenn es Abbildungen zwischen ihren Zustandsmengen gibt, die sich jeweils zu Homomorphismen zwischen ihren generierten Halbgruppen erweitern lassen.

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1 Introduction

The structure of deterministic finite state transducers — forthwith referred to as automata — allows us to represent operations on sets in a different way. We can use them to represent algebraic structures in a more intuitively accessible manner, giving rise to automaton groups and generalizations thereof.

Automaton semigroups emerged as one such generalization. Interestingly, not all semigroups can be described with these automata. For example, $(\mathbb{N}\backslash\{0\}, +)$ [Cai09, Proposition 4.3] is not an automaton semigroup. This circumstance raises the question of how to determine which semigroups are automaton semigroups and which are not. Studying under what kinds of semigroup constructions the class of automaton semigroups is closed helps us get closer to the answer to this question.

In particular, this work aims to expand previous results on the closure under the semigroup free product. These results stem from work done by Tara Brough and Alan J. Cain, who, among other things, proved that automaton semigroups meeting certain criteria, namely having at least one idempotent or being homogeneous, will form an automaton semigroup under the free product [BC17, Theorem 4]. They note the possibility of the condition given in that Theorem being a necessary criterion for this to be the case. In this work we refute this claim, by proving that adhering to a set of looser restrictions will still result in an automaton semigroup under the free product.

We begin by introducing how automata operate and describe cross diagrams to more easily visualize their processes. This method is used throughout this paper. We then show that two automaton semigroups form another automaton semigroup under free product construction, if there exist maps between their state sets that each extend into homomorphisms between their generated semigroups. Lastly, we show that there are indeed automaton semigroups that are included in this new definition, that couldn't be captured before.

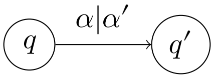
2 Preliminaries

In this chapter we go over the prerequisite definitions for this paper.

2.1 Automata

Definition 2.1.1. An *automaton* is a triple $\mathcal{A} = (Q, \Sigma, \delta)$ with Q is a finite set of *states*, Σ is a finite alphabet of *symbols*, and $\delta : Q \times \Sigma \to Q \times \Sigma$.

$\mathcal{A}$ can be viewed as a directed labeled graph. Hereby the vertices are made up of the set Q and there is an edge from q to q' with $\alpha|\alpha'$ for each $\delta(q, \alpha) = (q', \alpha')$:



Put another way, if the automaton is in a state $q \in Q$ and reads a symbol $\alpha \in \Sigma$, then it moves into a state q' while returning a symbol α'.
This interpretation can be extended in the following manner: if the automaton is in a state q_1 and reads a sequence of symbols $\alpha_1\alpha_2 \ldots \alpha_n \in \Sigma^*$ then it moves through some states in Q while returning a sequence $\alpha'_1\alpha'_2 \ldots \alpha'_n \in \Sigma^*$, if for all $1 \leq i \leq n \in \mathbb{N}$ we have $\delta(q_i, \alpha_i) = (q_{i+1}, \alpha'_i)$. We define $\hat{\delta} : Q^+ \times \Sigma^* \to Q^+ \times \Sigma^*$ to express this relationship.

Definition 2.1.2. The *action of a state* q on a sequence $\alpha \in \Sigma^*$ is defined as the output sequence α' of the automaton, when it starts in the state q and reads α, i.e. the second argument of $\hat{\delta}(q, \alpha)$. We write this as $q \circ \alpha = \alpha'$.

Definition 2.1.3. Building on 2.1.2, the *action of a word* $\mathbf{q} = q_n q_{n-1} \ldots q_2 q_1 \in Q^+$ is defined as q_1 through q_n acting on the sequence of symbols consecutively, i.e. $\mathbf{q} \circ \alpha = (q_n \circ \ldots (q_2 \circ (q_1 \circ \alpha)))$.

Definition 2.1.4. To refer to the first argument of $\hat{\delta}(q, \alpha)$, we write $q \cdot \alpha = q'$. For words in Q^+ and sequences in Σ^*, we expand the definition accordingly: $\mathbf{q} \cdot \alpha = (q_n \cdot (q_{n-1} \ldots q_1 \circ \alpha)) \ldots (q_2 \cdot (q_1 \circ \alpha))(q_1 \cdot \alpha)$.

Example 2.1.5. Let $\mathcal{A} = (Q, \Sigma, \delta)$ (see figure 2.1), with $Q = \{a, b\}$, $\Sigma = \{a, b\}$ and

$$\delta : (a, a) \mapsto (a, a), (a, b) \mapsto (b, a), (b, b) \mapsto (b, b), (b, a) \mapsto (a, b).$$

Now, consider the word $bab \in Q^+$ acting on the sequence $aaa \in \Sigma^*$. Then we have

$$\begin{aligned} bab \circ aaa &= (b \circ (a \circ (b \circ aaa))) \\ &= (b \circ (a \circ baa)) \\ &= b \circ aba \\ &= bab \end{aligned} \quad \text{and} \quad \begin{aligned} bab \cdot aaa &= (b \cdot (a \circ (b \circ aaa)))(a \cdot (b \circ aaa))(b \cdot aaa) \\ &= (b \cdot (a \circ baa))(a \cdot baa)a \\ &= (b \cdot aba)aa \\ &= aaa. \end{aligned}$$

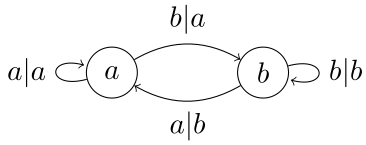


Figure 2.1: automaton $\mathcal{A}$ with $\mathcal{S}(\mathcal{A}) = \{a, b\}^+$

Cross diagrams are used to visualize these transformations. Every crossing of arrows represents the state on the left acting on the symbol above, resulting in the state on the right and the symbol below (see figure 2.2).

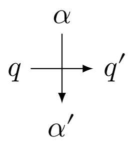


Figure 2.2: cross diagram showing $\delta(q, \alpha) = (q', \alpha')$

Naturally, we can expand this representation to words in Q^+ and sequences in Σ^*. We write the word $\mathbf{q} = q_n \dots q_1 \in Q^+$ on the left hand side and the sequence of symbols $\alpha = \alpha_1 \dots \alpha_n \in \Sigma^*$ at the top. The full grid shows every transformation that occurs. The resulting word $\mathbf{q}' = q_n^{(n)} \dots q_1^{(n)}$ can then be found in the rightmost column and the resulting sequence of symbols $\alpha' = \alpha_1^{(n)} \dots \alpha_n^{(n)}$ in the bottom row (see figure 2.3).

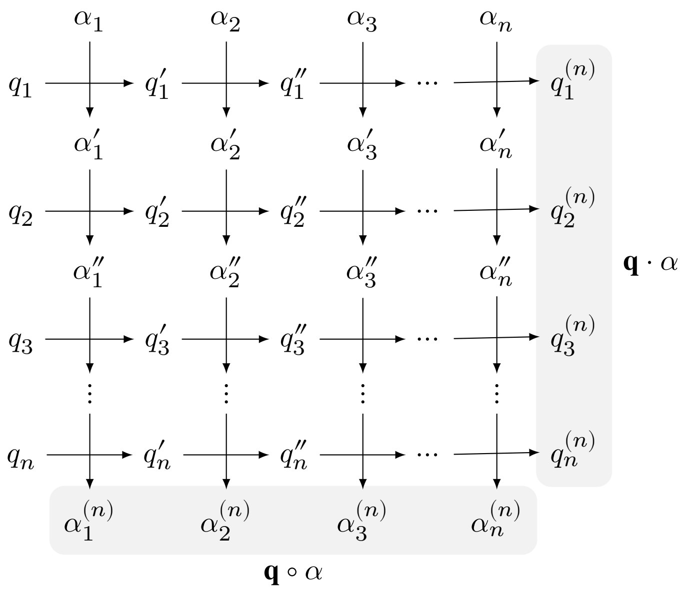
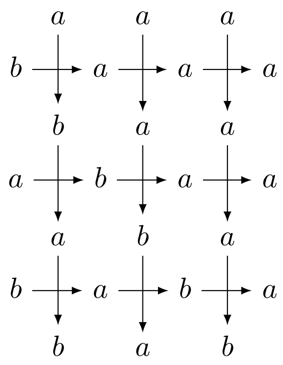


Figure 2.3: cross diagram showing $\hat{\delta}(\mathbf{q}, \alpha) = (\mathbf{q}', \alpha')$

Example 2.1.6. We can implement this representation with the example given in 2.1.5, to make it more readable:



Indeed, we find $bab \cdot aaa = aaa$ on the right and $bab \circ aaa = bab$ at the bottom.

2.2 Automaton Semigroups

Definition 2.2.1. A semigroup S is an *automaton semigroup* if there exists an automaton $\mathcal{A} = (Q, \Sigma, \delta)$ with $S \cong \mathcal{S}(\mathcal{A})$, where $\mathcal{S}(\mathcal{A}) = Q^+/_{\equiv_{\mathcal{A}}}$. We define $\equiv_{\mathcal{A}}$ via

$$\mathbf{q} \equiv_{\mathcal{A}} \mathbf{q}' \iff \forall \alpha \in \Sigma^* : \mathbf{q} \circ_{\mathcal{A}} \alpha = \mathbf{q}' \circ_{\mathcal{A}} \alpha.$$

Hereby $\mathbf{q} \circ_{\mathcal{A}} \alpha$ means that $\mathbf{q}$ acts on α within the automaton $\mathcal{A}$.

Definition 2.2.2 (semigroup free product)**.** Let $\mathcal{S}(\mathcal{A}_q) = Q^+/_{\equiv_{\mathcal{A}_q}}$ and $\mathcal{S}(\mathcal{A}_p) = P^+/_{\equiv_{\mathcal{A}_p}}$ be automaton semigroups with $\mathcal{A}_q = (Q, \Gamma, \delta)$ and $\mathcal{A}_p = (P, \Sigma, \tau)$ and ε be the empty word.

Then every word $\mathbf{r} \in (Q \cup P)^+$ can be written as $\mathbf{p}_n \mathbf{q}_n \ldots \mathbf{p}_1 \mathbf{q}_1 \mathbf{p}_0$ with $\mathbf{q}_1, \ldots, \mathbf{q}_n \in Q^+$, $\mathbf{p}_1, \ldots, \mathbf{p}_{n-1} \in P^+$ and $\mathbf{p}_0, \mathbf{p}_n \in P^*$ (We refer to these partitions as *blocks*). Analogously, we write $\mathbf{r}'$ as $\mathbf{p}'_m \mathbf{q}'_m \ldots \mathbf{p}'_1 \mathbf{q}'_1 \mathbf{p}'_0$ with $\mathbf{q}'_1, \ldots, \mathbf{q}'_m \in Q^+$, $\mathbf{p}'_1, \ldots, \mathbf{p}'_{m-1} \in P^+$ and $\mathbf{p}'_0, \mathbf{p}'_m \in P^*$ and define

$$\mathbf{r} \equiv \mathbf{r}' \iff \begin{cases} n = m & (2.1) \\ p_0 = \varepsilon \iff p'_0 = \varepsilon & (2.2) \\ p_n = \varepsilon \iff p'_m = \varepsilon & (2.3) \\ \forall 0 \leq i \leq n = m : \mathbf{q}_i \equiv_{\mathcal{A}_q} \mathbf{q}'_i & (2.4) \\ \forall 0 \leq i \leq n = m : \mathbf{p}_i \equiv_{\mathcal{A}_p} \mathbf{p}'_i & (2.5) \end{cases}$$

With this, the *free product* of two automaton semigroups $\mathcal{S}(\mathcal{A}_q)$ and $\mathcal{S}(\mathcal{A}_p)$ is defined as

$$Q^+/_{\equiv_{\mathcal{A}_q}} \star P^+/_{\equiv_{\mathcal{A}_p}} := (Q \cup P)^+/_{\equiv}.$$

3 Free Product

In this chapter we expand the proof in *Automaton semigroups: new constructions results and examples of non-automaton semigroups* [BC17, Theorem 4] by generalizing its restrictions.

The construction used here is mostly based on the construction given in the aforementioned source.

Theorem 3.0.1. *Let $\mathcal{A}_q = (Q, \Sigma, \delta)$ and $\mathcal{A}_p = (P, \Gamma, \tau)$ be automata with $\mathcal{S}(\mathcal{A}_q) = S$ and $\mathcal{S}(\mathcal{A}_p) = T$. If there are maps $\psi : Q \to P$ and $\vartheta : P \to Q$ that extend into homomorphisms $\Psi : S \to T$ and $\Theta : T \to S$, respectively, then $S \star T$ is an automaton semigroup.*

We extend ψ and ϑ into homomorphisms $\psi : Q^+ \to P^+$ and $\vartheta : P^+ \to Q^+$. Assume without loss of generality that $Q \cap P = \emptyset$.

Now we construct $\mathcal{A} = ((Q \cup P), \Lambda, \rho)$ with $\Lambda = \{x, x^S, x^T, x^\circ, y, y^\circ, \widehat{y}, \overline{y} \mid x \in X, y \in Y\}$, $X = \left\{\left(a|b\right), \boxed{b|a} \mid a \in \Sigma, b \in \Gamma\right\}$ and $Y = \{\$, \#\}$.

For $q \in Q$, $p \in P$, $a \in \Sigma$, $b \in \Gamma$ suppose that $\delta(q, a) = (q', a')$ and $\tau(p, b) = (p', b')$. With this in mind, the transitions ρ are shown in Figure 3.1.

	q	p
$\boxed{a\|b}$	$(q', \boxed{a'\|b}^S)$	$(p, \boxed{a\|b})$
$\boxed{a\|b}^S$	$(q', \boxed{a'\|b}^S)$	$(p', \boxed{a\|b'}^T)$
$\boxed{a\|b}^T$	$(q, \boxed{a\|b}^\circ)$	$(p', \boxed{a\|b'}^T)$
$\boxed{a\|b}^\circ$	$(q, \boxed{a\|b}^\circ)$	$(p, \boxed{a\|b}^\circ)$
$\overline{\$}$	$(\psi(q), \widehat{\$})$	$(p, \overline{\$})$
$\widehat{\$}$	$(\psi(q), \widehat{\$})$	$(p, \$)$
$\$$	$(q, \$^\circ)$	$(p, \$)$
$\$^\circ$	$(q, \$^\circ)$	$(p, \$^\circ)$

(a) actions of $Q \cup P$ on $\boxed{\ \|\ }$-symbols and \$-gates

	p	q
$\left(b\|a\right)$	$(p', \left(b'\|a\right)^T)$	$(q, \left(b\|a\right))$
$\left(b\|a\right)^T$	$(p', \left(b'\|a\right)^T)$	$(q', \left(b\|a'\right)^S)$
$\left(b\|a\right)^S$	$(p, \left(b\|a\right)^\circ)$	$(q', \left(b\|a'\right)^S)$
$\left(b\|a\right)^\circ$	$(p, \left(b\|a\right)^\circ)$	$(q, \left(b\|a\right)^\circ)$
$\overline{\#}$	$(\vartheta(p), \widehat{\#})$	$(q, \overline{\#})$
$\widehat{\#}$	$(\vartheta(p), \widehat{\#})$	$(q, \#)$
$\#$	$(p, \#^\circ)$	$(q, \#)$
$\#^\circ$	$(p, \#^\circ)$	$(q, \#^\circ)$

(b) actions of $Q \cup P$ on $\left(\ \|\ \right)$-symbols and \$-gates

Figure 3.1: Transitions $\rho : (Q \cup P) \times \Lambda \to (Q \cup P) \times \Lambda$ in $\mathcal{A}$

Let $\varphi : S \star T = (Q \cup P)^+/_{\equiv} \to \mathcal{S}(\mathcal{A}) = (Q \cup P)^+/_{\equiv_{\mathcal{A}}}$ with $[\mathbf{r}]_{\equiv} \mapsto [\mathbf{r}]_{\equiv_{\mathcal{A}}}$. We show that $\mathcal{S}(\mathcal{A}_q) \star \mathcal{S}(\mathcal{A}_p) \cong \mathcal{S}(\mathcal{A})$ by proving φ is a well-defined bijective homomorphism.

Remark 3.0.2 (surjective homomorphism). The properties φ is surjective and φ is a homomorphism are true by definition and will thus not be further discussed.

3.1 φ is injective

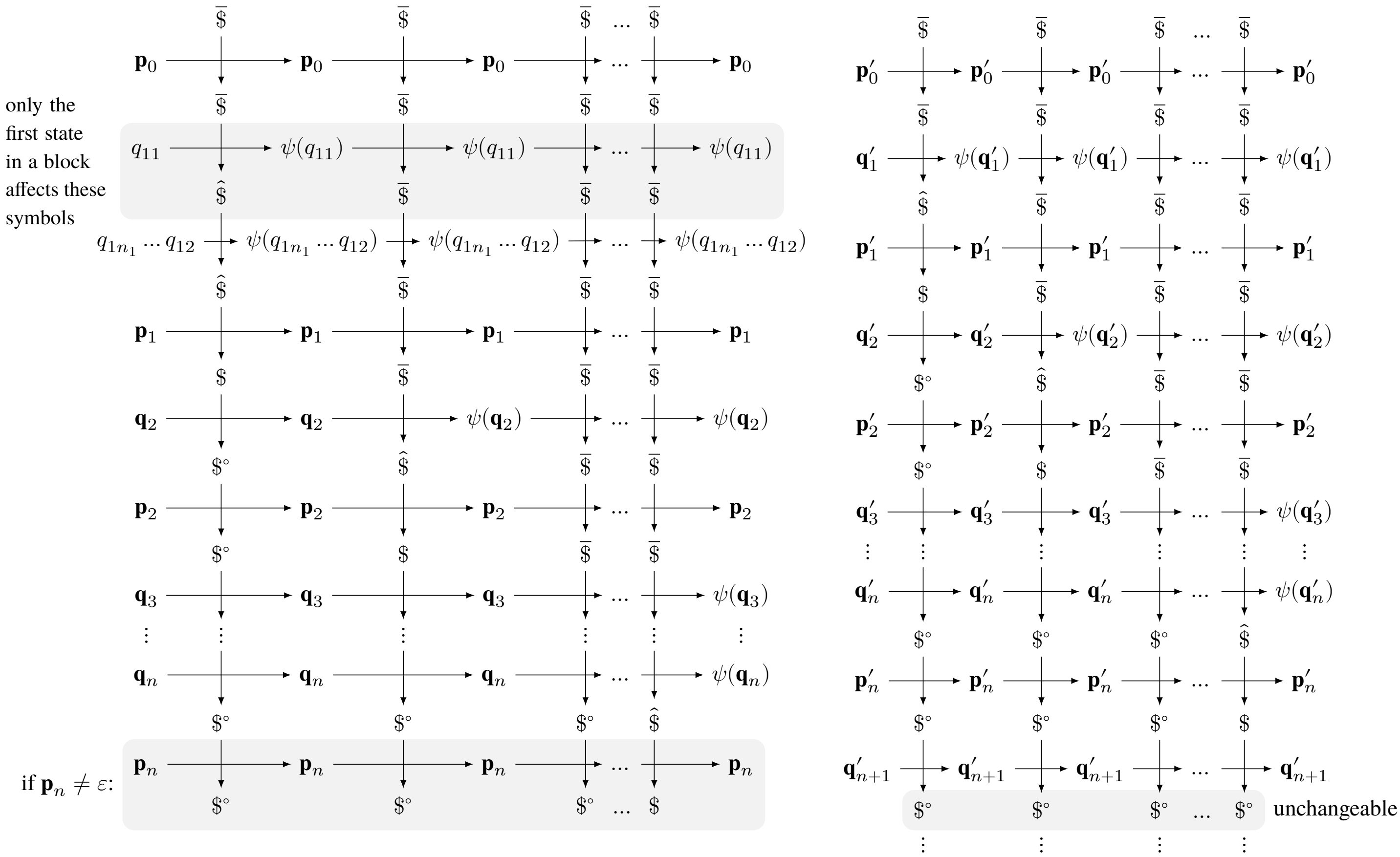
Lemma 3.1.1 (φ is injective)**.** $\forall \mathbf{r}, \mathbf{r}' \in (Q \cup P)^+ : \mathbf{r} \not\equiv \mathbf{r}' \implies \mathbf{r} \not\equiv_{\mathcal{A}} \mathbf{r}'$.

Proof. We divide the problem into separate cases, each showing a variation in how $\mathbf{r}$ and $\mathbf{r}'$ (written as has been described in Definition 2.2.2) can potentially differ in $(Q \cup P)^+$ and show that their image must then differ in $\mathcal{S}(\mathcal{A})$ as well. All these cases refer to the cases in Definition 2.2.2. As it can take several passes for the first $\overline{\$}$ or $\overline{\#}$ to get $^\circ$-marked, we assume $n \geq 2$.

Case 1 $\mathbf{r} \not\equiv \mathbf{r}'$ due to (2.1) $n = m$ not applying.
Let $n \neq m$ w.l.o.g. $n < m$. Then we have

$$\mathbf{r} \circ_{\mathcal{A}} (\overline{\$})^n = \left\{ \begin{array}{ll} (\$^\circ)^{n-1}\hat{\$} & \mathbf{p}_n = \varepsilon \\ (\$^\circ)^{n-1}\$ & \mathbf{p}_n \neq \varepsilon \end{array} \right\} \neq (\$^\circ)^n = \mathbf{r}' \circ_{\mathcal{A}} (\overline{\$})^n$$

as we have these cross diagrams:

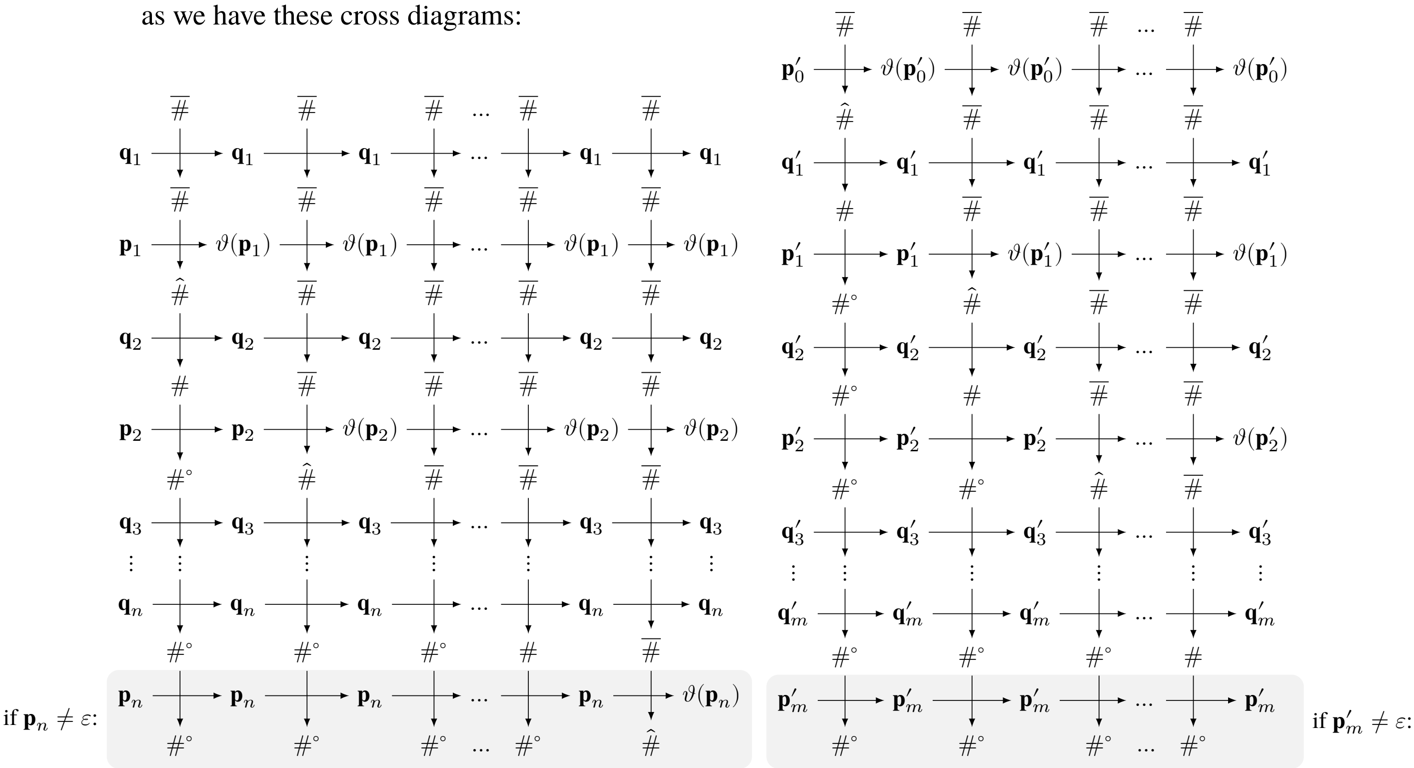


Thus we have $\mathbf{r} \not\equiv_{\mathcal{A}} \mathbf{r}' \impliedby n \neq m$.

Case 2 $\mathbf{r} \not\equiv \mathbf{r}'$ due to (2.2) $p_0 = \varepsilon \iff p'_0 = \varepsilon$ not applying.
Let $n = m$ and w.l.o.g. $\mathbf{p}_0 = \varepsilon$ and $\mathbf{p}'_0 \neq \varepsilon$. Then we have

$$\mathbf{r} \circ_{\mathcal{A}} (\overline{\#})^n = \left\{ \begin{array}{ll} (\#^\circ)^{n-2}\#\overline{\#} & \mathbf{p}_n = \varepsilon \\ (\#^\circ)^{n-1}\hat{\#} & \mathbf{p}_n \neq \varepsilon \end{array} \right\} \neq \left\{ \begin{array}{ll} (\#^\circ)^{n-1}\# & \mathbf{p}'_m = \varepsilon \\ (\#^\circ)^{n} & \mathbf{p}'_m \neq \varepsilon \end{array} \right\} = \mathbf{r}' \circ_{\mathcal{A}} (\overline{\#})^n$$

as we have these cross diagrams:

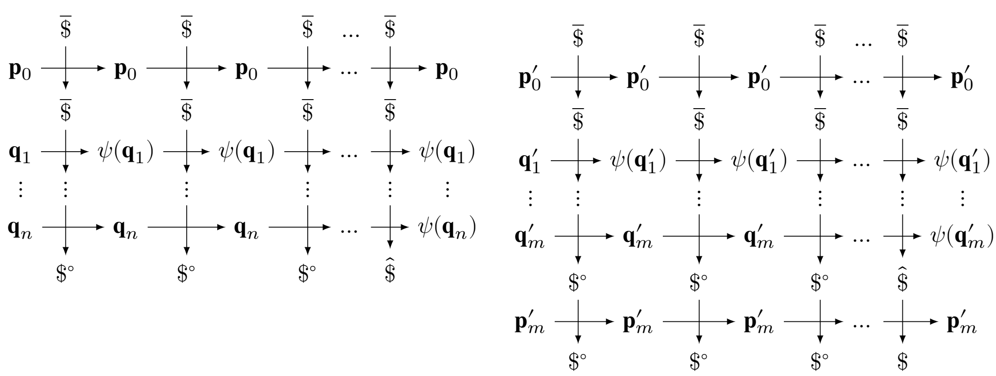


Thus we have $\mathbf{r} \not\equiv_{\mathcal{A}} \mathbf{r}'$ in this case.

Case 3 $\mathbf{r} \not\equiv \mathbf{r}'$ due to (2.3) $p_n = \varepsilon \iff p'_m = \varepsilon$ not applying.
Let $n = m$ and w.l.o.g. $\mathbf{p}_n = \varepsilon$ and $\mathbf{p}'_m \neq \varepsilon$. Then we have

$$\mathbf{r} \circ_{\mathcal{A}} (\bar{\$})^n = (\$^\circ)^{n-1}\hat{\$} \neq (\$^\circ)^{n-1}\$ = \mathbf{r}' \circ_{\mathcal{A}} (\bar{\$})^n$$

as we have these cross diagrams:

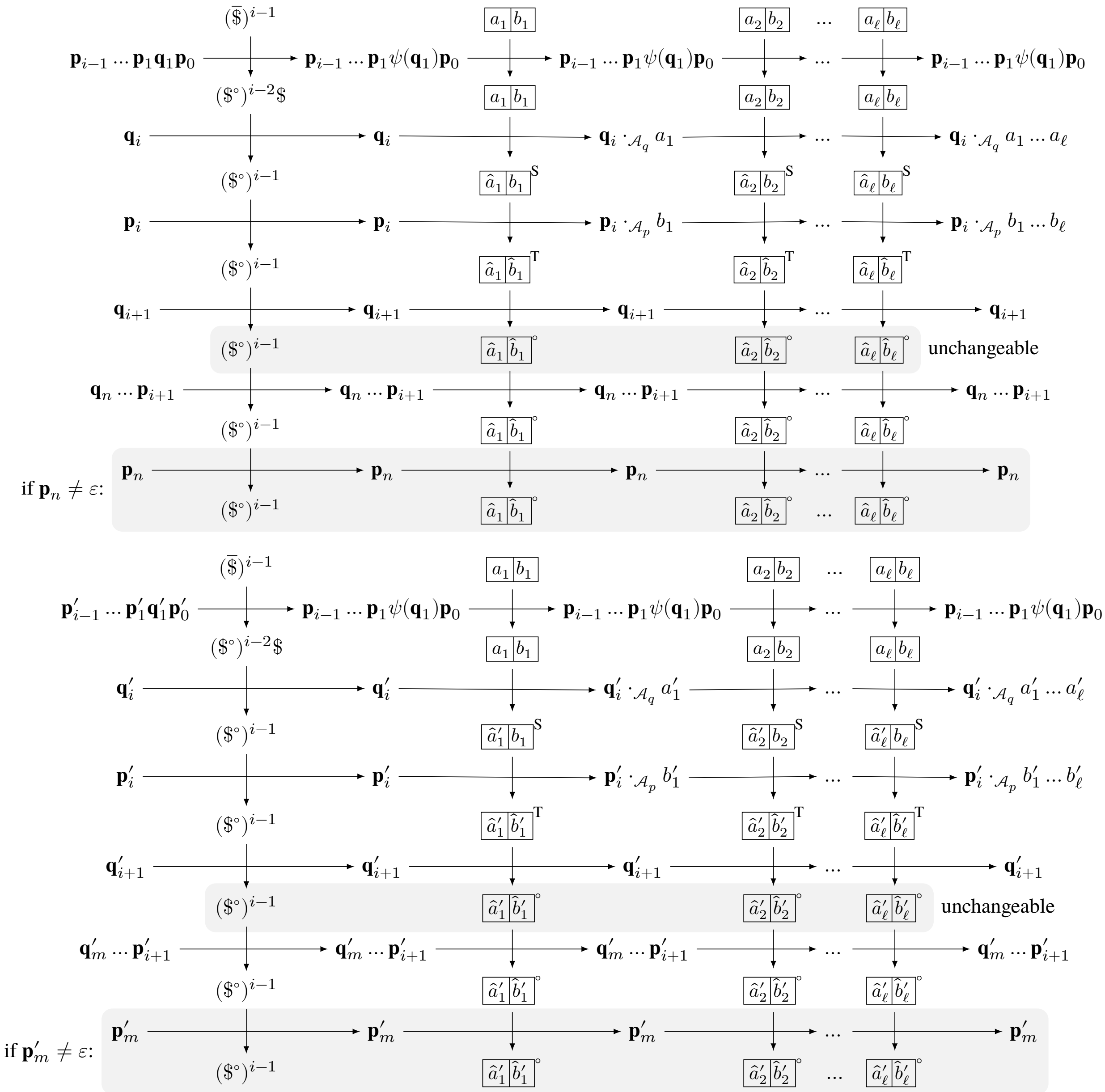


Thus we have $\mathbf{r} \not\equiv_{\mathcal{A}} \mathbf{r}'$ in this case. (for a more detailed cross diagram, see Case 1)

Case 4 $\mathbf{r} \not\equiv \mathbf{r}'$ due to 2.4 or 2.5 not applying.
Let $n = m$, $|\lambda| = i - 1 + \ell$ with $\lambda \in \Lambda^*$, $i, \ell \in \mathbb{N}$ and w.l.o.g. $\mathbf{q}_i \not\equiv_{\mathcal{A}_q} \mathbf{q}'_i \vee \mathbf{p}_i \not\equiv_{\mathcal{A}_p} \mathbf{p}'_i$ be the first differing block in $\mathbf{r}$ and $\mathbf{r}'$ with $1 < i < n = m$.
If either 2.4 or 2.5 don't hold, then

$$\exists \alpha = a_1 \dots a_\ell \in \Sigma^* : \mathbf{q}_i \circ_{\mathcal{A}_q} \alpha = \hat{a}_1 \dots \hat{a}_\ell \neq \hat{a}'_1 \dots \hat{a}'_\ell = \mathbf{q}'_i \circ_{\mathcal{A}_q} \alpha \tag{3.1}$$

$$\vee \exists \beta = b_1 \dots b_\ell \in \Gamma^* : \mathbf{p}_i \circ_{\mathcal{A}_p} \beta = \hat{b}_1 \dots \hat{b}_\ell \neq \hat{b}'_1 \dots \hat{b}'_\ell = \mathbf{p}'_i \circ_{\mathcal{A}_p} \beta. \tag{3.2}$$



When 3.1 is true, then the following inequality is true due to there existing at least one a_k with $1 \leq k \leq \ell$ where

$$\boxed{\hat{a}_k \,|\, }^{\circ} = \boxed{(q_i \cdot_{\mathcal{A}_q} a_1 \dots a_{k-1}) \circ_{\mathcal{A}_q} a_k \,|\, }^{\circ} \neq \boxed{(q'_i \cdot_{\mathcal{A}_q} a'_1 \dots a'_{k-1}) \circ_{\mathcal{A}_q} a'_k \,|\, }^{\circ} = \boxed{\hat{a}'_k \,|\, }^{\circ}.$$

When 3.2 is true, then similarly, there exists at least one b_k with $1 \le k \le \ell$ where

$$\boxed{|\hat{b}_k}^{\circ} = \boxed{|(q_i \cdot_{\mathcal{A}_p} b_1 \dots b_{k-1}) \circ_{\mathcal{A}_p} b_k}^{\circ} \neq \boxed{|(q'_i \cdot_{\mathcal{A}_p} b'_1 \dots b'_{k-1}) \circ_{\mathcal{A}_p} b'_k}^{\circ} = \boxed{|\hat{b}'_k}^{\circ}.$$

$$\begin{aligned} \mathbf{r} \circ_{\mathcal{A}} (\overline{\$})^{i-1} \boxed{a_1|b_1}\,\boxed{a_2|b_2} \dots \boxed{a_\ell|b_\ell} &= (\$^\circ)^{i-1} \boxed{\hat{a}_1|\hat{b}_1}^{\circ} \boxed{\hat{a}_2|\hat{b}_2}^{\circ} \dots \boxed{\hat{a}_\ell|\hat{b}_\ell}^{\circ} \\ \neq \mathbf{r}' \circ_{\mathcal{A}} (\overline{\$})^{i-1} \boxed{a_1|b_1}\,\boxed{a_2|b_2} \dots \boxed{a_\ell|b_\ell} &= (\$^\circ)^{i-1} \boxed{\hat{a}'_1|\hat{b}'_1}^{\circ} \boxed{\hat{a}'_2|\hat{b}'_2}^{\circ} \dots \boxed{\hat{a}'_\ell|\hat{b}'_\ell}^{\circ} \end{aligned}$$

However, we need to consider the following edge cases. While the transformations here may differ slightly, the reasons for the inequalities are the same.

$(i = n)$: We have

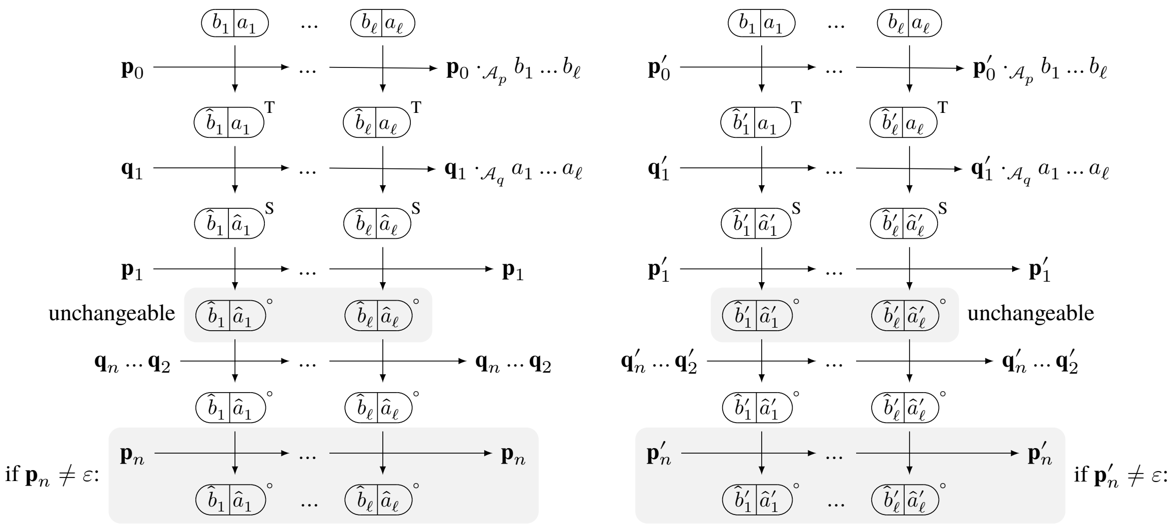
$$\begin{aligned} \mathbf{r} \circ_{\mathcal{A}} (\overline{\$})^{n-1} \boxed{a_1|b_1}\,\boxed{a_2|b_2} \dots \boxed{a_\ell|b_\ell} &= \left\{ \begin{array}{ll} (\$^\circ)^{n-1} \boxed{\hat{a}'_1|b'_1}^{\mathrm{S}} \boxed{\hat{a}'_2|b'_2}^{\mathrm{S}} \dots \boxed{\hat{a}'_\ell|b'_\ell}^{\mathrm{S}} & \mathbf{p}_n = \varepsilon \\ (\$^\circ)^{n-1} \boxed{\hat{a}'_1|\hat{b}'_1}^{\mathrm{T}} \boxed{\hat{a}'_2|\hat{b}'_2}^{\mathrm{T}} \dots \boxed{\hat{a}'_\ell|\hat{b}'_\ell}^{\mathrm{T}} & \mathbf{p}_n \neq \varepsilon \end{array} \right\} \\ \neq \mathbf{r}' \circ_{\mathcal{A}} (\overline{\$})^{n-1} \boxed{a_1|b_1}\,\boxed{a_2|b_2} \dots \boxed{a_\ell|b_\ell} &= \left\{ \begin{array}{ll} (\$^\circ)^{n-1} \boxed{\hat{a}'_1|b'_1}^{\mathrm{S}} \boxed{\hat{a}'_2|b'_2}^{\mathrm{S}} \dots \boxed{\hat{a}'_\ell|b'_\ell}^{\mathrm{S}} & \mathbf{p}'_n = \varepsilon \\ (\$^\circ)^{n-1} \boxed{\hat{a}'_1|\hat{b}'_1}^{\mathrm{T}} \boxed{\hat{a}'_2|\hat{b}'_2}^{\mathrm{T}} \dots \boxed{\hat{a}'_\ell|\hat{b}'_\ell}^{\mathrm{T}} & \mathbf{p}'_n \neq \varepsilon \end{array} \right\} \end{aligned}$$

as can be extrapolated from the prior cross diagram.

$(i = 0)$: We have

$$\mathbf{r} \circ_{\mathcal{A}} (b_1|a_1) \dots (b_\ell|a_\ell) = (\hat{b}_1|\hat{a}_1)^{\circ} \dots (\hat{b}_\ell|\hat{a}_\ell)^{\circ} \neq (\hat{b}'_1|\hat{a}'_1)^{\circ} \dots (\hat{b}'_\ell|\hat{a}'_\ell)^{\circ} = \mathbf{r}' \circ_{\mathcal{A}} (b_1|a_1) \dots (b_\ell|a_\ell)$$

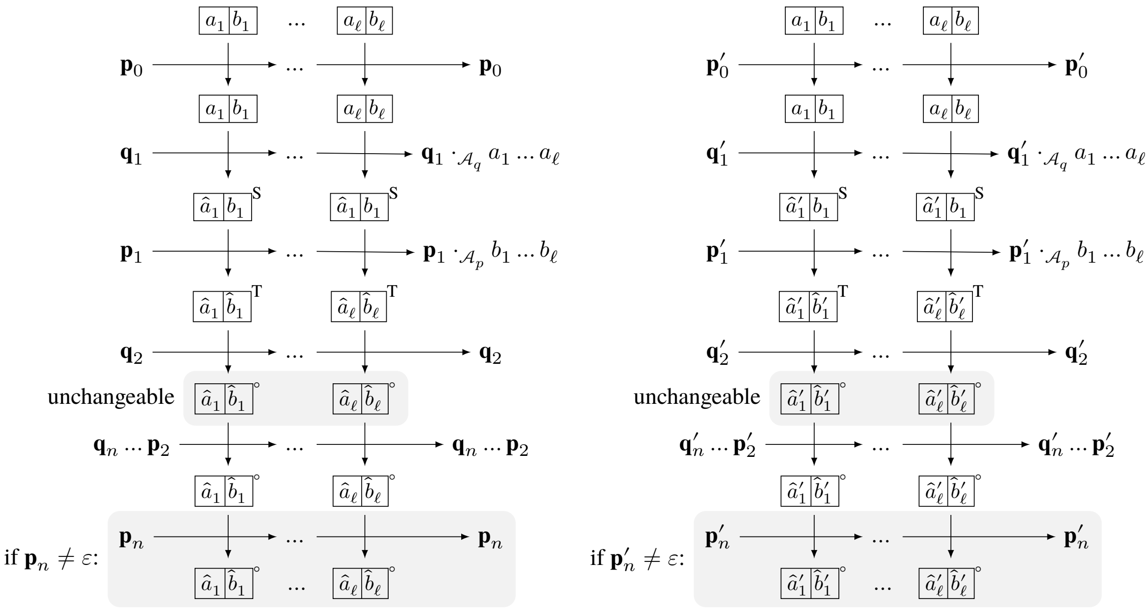
due to these cross diagrams:



$(i = 1)$: We have

$$\mathbf{r} \circ_{\mathcal{A}} \boxed{a_1 | b_1} \dots \boxed{a_\ell | b_\ell} = \boxed{\hat{a}_1 | \hat{b}_1}^{\circ} \dots \boxed{\hat{a}_\ell | \hat{b}_\ell}^{\circ} \neq \boxed{\hat{a}'_1 | \hat{b}'_1}^{\circ} \dots \boxed{\hat{a}'_\ell | \hat{b}'_\ell}^{\circ} = \mathbf{r}' \circ_{\mathcal{A}} \boxed{a_1 | b_1} \dots \boxed{a_\ell | b_\ell}$$

due to these cross diagrams:



Thus we have $\mathbf{r} \not\equiv_{\mathcal{A}} \mathbf{r}'$ in this case.

With all these cases, any manner in which $\mathbf{r}$ and $\mathbf{r}'$ can differ in $(Q \cup P)^+/_{\equiv}$ has been covered and there is no case in which $\mathbf{r} \not\equiv \mathbf{r}' \implies \mathbf{r} \equiv_{\mathcal{A}} \mathbf{r}'$, so φ must be injective. $\square$

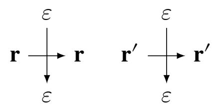
3.2 φ is well-defined

Lemma 3.2.1 (φ is well defined)**.** $\forall \mathbf{r}, \mathbf{r}' \in (Q \cup P)^+/_{\equiv} : \mathbf{r} \equiv \mathbf{r}' \implies \mathbf{r} \equiv_{\mathcal{A}} \mathbf{r}'$,
i.e. $\forall \lambda \in \Lambda^* : \mathbf{r} \circ_{\mathcal{A}} \lambda = \mathbf{r}' \circ_{\mathcal{A}} \lambda$.

Proof. Let $\mathbf{r} = \mathbf{p}_n\mathbf{q}_n \dots \mathbf{p}_1\mathbf{q}_1\mathbf{p}_0$ and $\mathbf{r}' = \mathbf{p}'_n\mathbf{q}'_n \dots \mathbf{p}'_1\mathbf{q}'_1\mathbf{p}'_0$ (as described in Definition 2.2.2), such that for each block in Q^+ we have $\mathbf{q}_i \equiv_{\mathcal{A}_q} \mathbf{q}'_i$ and for each block in P^+ we have $\mathbf{p}_i \equiv_{\mathcal{A}_p} \mathbf{p}'_i$,
i.e. $\forall \alpha \in \Sigma^* : \mathbf{q}_i \circ_{\mathcal{A}_q} \alpha = \mathbf{q}'_i \circ_{\mathcal{A}_q} \alpha$ and $\forall \beta \in \Gamma^* : \mathbf{p}_i \circ_{\mathcal{A}_p} \beta = \mathbf{p}'_i \circ_{\mathcal{A}_p} \beta$.

We propose the invariant $P(k) := \forall \lambda \in \Lambda^k, \forall \mathbf{r}, \mathbf{r}' \in (Q \cup P)^+ : \mathbf{r} \equiv \mathbf{r}' \implies \mathbf{r} \equiv_{\mathcal{A}} \mathbf{r}'$.

Base Case $k = 0$: Let $\mathbf{r}, \mathbf{r}'$ be arbitrary with $\mathbf{r} \equiv \mathbf{r}'$ and $\lambda_1 = \varepsilon$

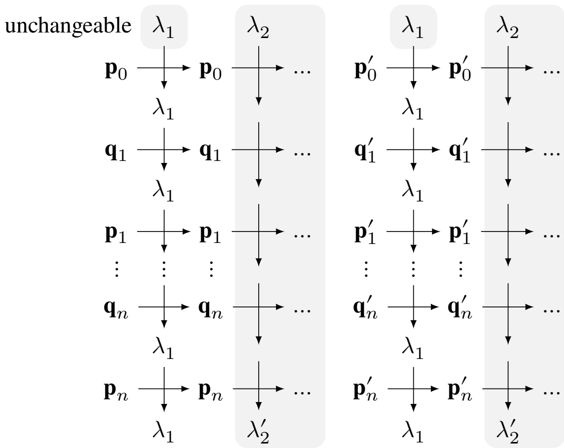


We can see that the invariant is true for the base case.

Inductive Step $k > 0$: Let $\mathbf{r}, \mathbf{r}'$ be arbitrary with $\mathbf{r} \equiv \mathbf{r}'$, $\lambda = \lambda_1\lambda_2$ with $\lambda_1 \in \Lambda$ and $\lambda_2 \in \Lambda^{k-1}$.
We assume the invariant holds for $k - 1$.

In the following cases, we analyze whether the invariant still holds when prepending an element $\lambda_1 \in \Lambda$ to λ_2.

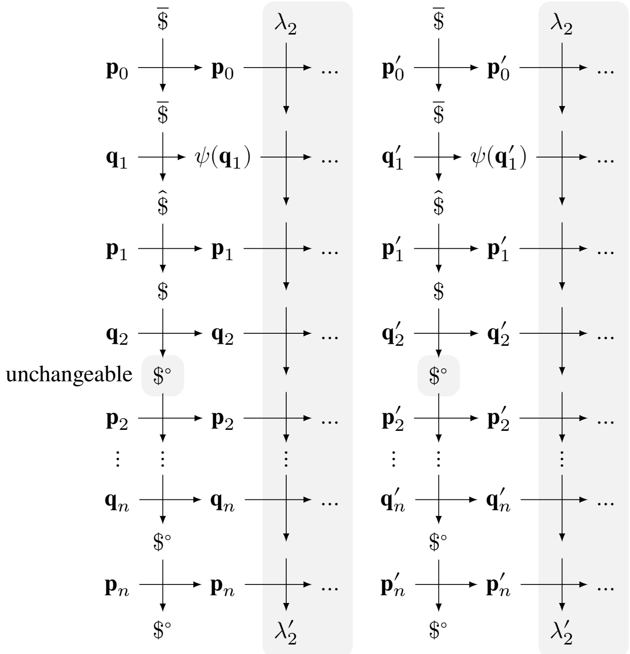
Case 1 $\lambda_1 \in \left\{\boxed{a\,|\,b}^\circ, (b\,|\,a)^\circ, \$^\circ, \#^\circ\right\}$



We can see, that $\mathbf{r} \circ_{\mathcal{A}} \lambda_1 = \lambda_1 = \mathbf{r}' \circ_{\mathcal{A}} \lambda_1$ and $\mathbf{r} \cdot_{\mathcal{A}} \lambda_1 \equiv \lambda_1 = \mathbf{r}' \cdot_{\mathcal{A}} \lambda_1$ hold.

With the induction hypothesis we have $\mathbf{r} \circ_{\mathcal{A}} \lambda_1\lambda_2 = \mathbf{r}' \circ_{\mathcal{A}} \lambda_1\lambda_2$ and $\mathbf{r} \equiv_{\mathcal{A}} \mathbf{r}'$ for $\lambda_1\lambda_2$ with $\lambda_1 \in \left\{\boxed{a\,|\,b}^\circ, (b\,|\,a)^\circ, \$^\circ, \#^\circ\right\}$ and $\lambda_2 \in \Lambda^{k-1}$.

Case 2 $\lambda_1 \in \left\{\overline{\$}, \hat{\$}, \$\right\}$

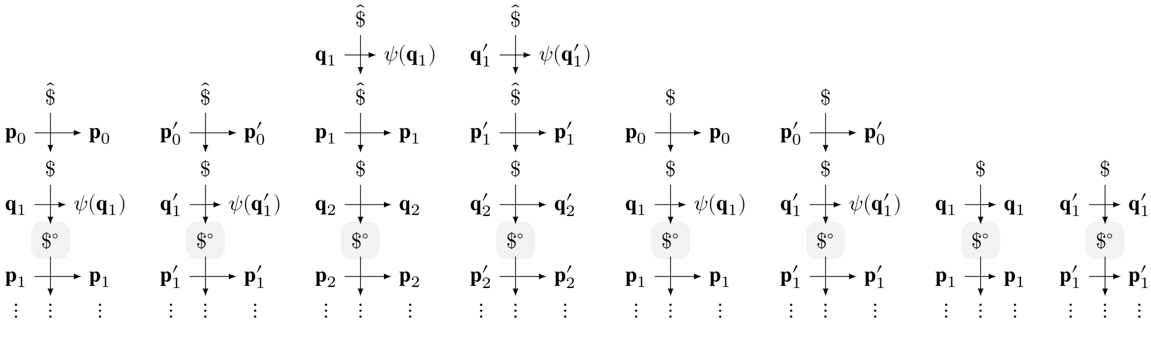


We can see, that $\mathbf{r} \circ_{\mathcal{A}} \overline{\$} = \mathbf{r}' \circ_{\mathcal{A}} \overline{\$}$ holds and

$$\begin{aligned} \mathbf{r} \ \cdot_{\mathcal{A}} \overline{\$} &= \mathbf{p}_n \mathbf{q}_n \dots \mathbf{q}_2 \mathbf{p}_1 \psi(\mathbf{q}_1) \mathbf{p}_0 \\ \equiv \mathbf{r}' \cdot_{\mathcal{A}} \overline{\$} &= \mathbf{p}'_n \mathbf{q}'_n \dots \mathbf{q}'_2 \mathbf{p}'_1 \psi(\mathbf{q}'_1) \mathbf{p}'_0 \end{aligned}$$

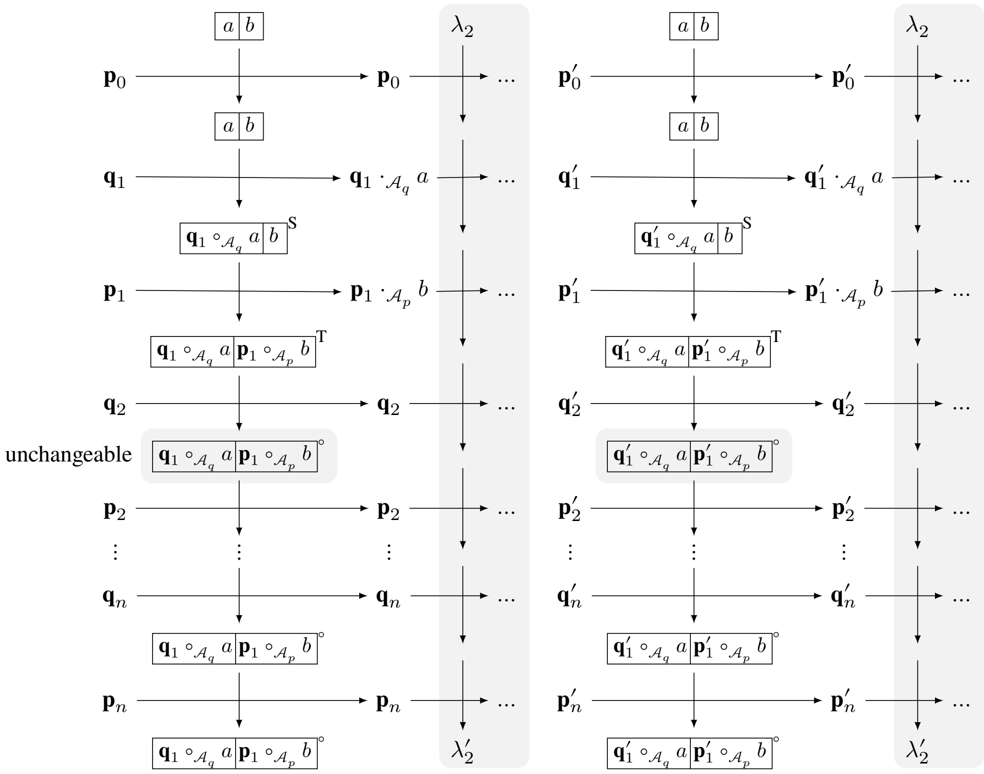
due to $\mathbf{p}_n \mathbf{q}_n \dots \mathbf{q}_2 \equiv \mathbf{p}'_n \mathbf{q}'_n \dots \mathbf{q}'_2$ as specified in the premise and $\psi(\mathbf{q}_1) \equiv_{\mathcal{A}} \psi(\mathbf{q}_1)$ as ψ is a homomorphism.

To extrapolate how $\lambda_1 \in \left\{\hat{\$}, \$\right\}$ affects the outcome, we can examine the cross diagrams below and employ similar reasoning as with $\overline{\$}$.



With the induction hypothesis we have $\mathbf{r} \circ_{\mathcal{A}} \lambda_1 \lambda_2 = \mathbf{r}' \circ_{\mathcal{A}} \lambda_1 \lambda_2$ and $\mathbf{r} \equiv_{\mathcal{A}} \mathbf{r}'$ for $\lambda_1 \lambda_2$ with $\lambda_1 \in \left\{\overline{\$}, \hat{\$}, \$\right\}$ and $\lambda_2 \in \Lambda^{k-1}$.

Case 3 $\lambda_1 \in \left\{ \boxed{a|b}, \boxed{a|b}^S, \boxed{a|b}^T \right\}$

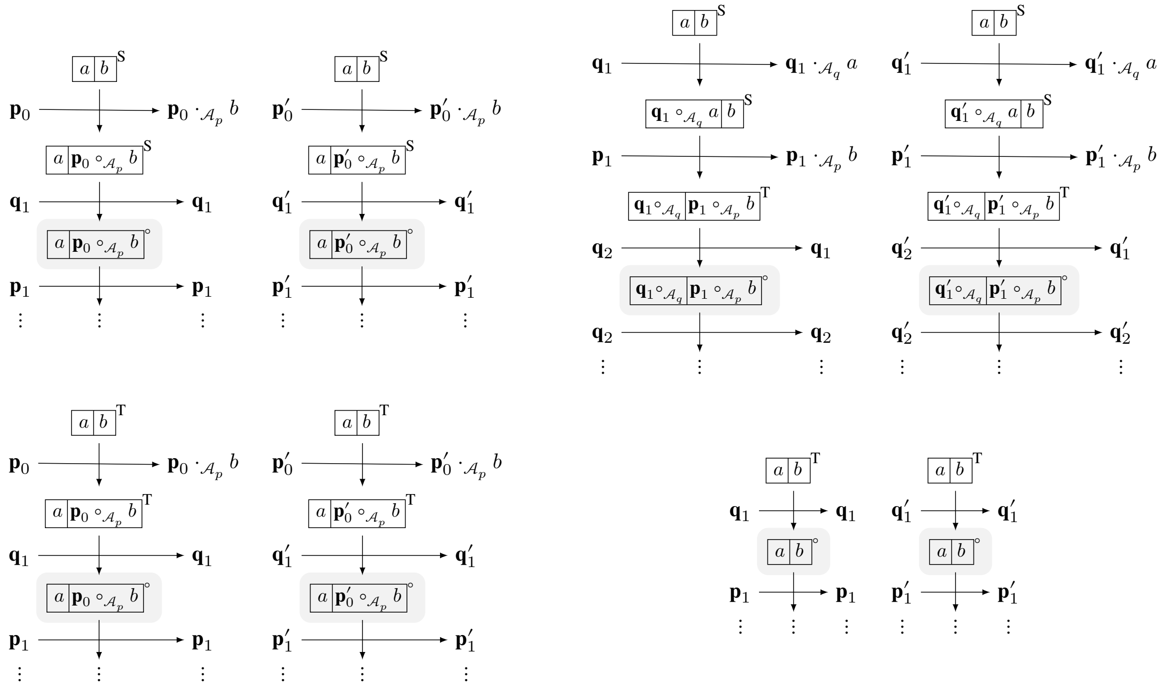


We can see, that $\mathbf{r} \circ_{\mathcal{A}} \boxed{a|b} = \mathbf{r}' \circ_{\mathcal{A}} \boxed{a|b}$ holds and

$$\begin{aligned} \mathbf{r} \;\cdot_{\mathcal{A}} \boxed{a|b} &= \mathbf{p}_n\mathbf{q}_n \dots \mathbf{p}_2\mathbf{q}_2(\mathbf{p}_1 \cdot_{\mathcal{A}_p} b)(\mathbf{q}_1 \cdot_{\mathcal{A}_q} a)\mathbf{p}_0 \\ \equiv \mathbf{r}' \cdot_{\mathcal{A}} \boxed{a|b} &= \mathbf{p}'_n\mathbf{q}'_n \dots \mathbf{p}'_2\mathbf{q}'_2(\mathbf{p}'_1 \cdot_{\mathcal{A}_p} b)(\mathbf{q}'_1 \cdot_{\mathcal{A}_q} a)\mathbf{p}'_0 \end{aligned}$$

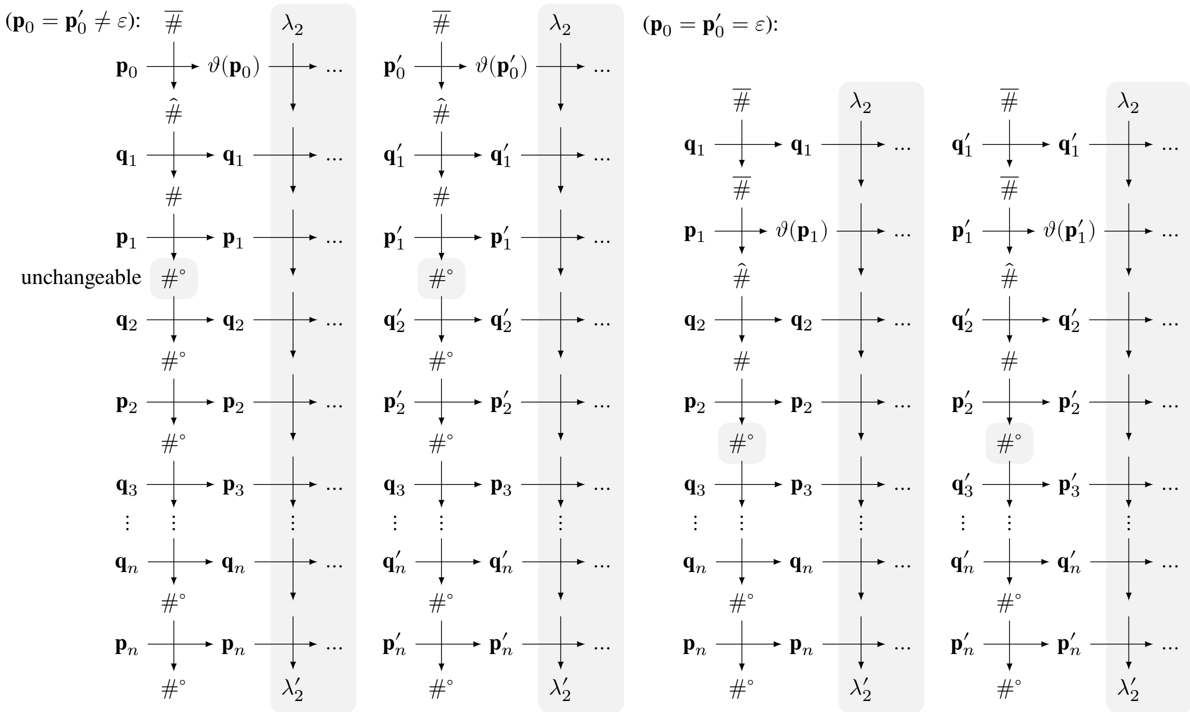
due to $\mathbf{p}_n\mathbf{q}_n \dots \mathbf{p}_2\mathbf{q}_2 \equiv \mathbf{p}'_n\mathbf{q}'_n \dots \mathbf{p}'_2\mathbf{q}'_2$ as specified in the premise and $(\mathbf{p}_1 \cdot_{\mathcal{A}_p} b)(\mathbf{q}_1 \cdot_{\mathcal{A}_q} a) \equiv_{\mathcal{A}} (\mathbf{p}'_1 \cdot_{\mathcal{A}_p} b)(\mathbf{q}'_1 \cdot_{\mathcal{A}_q} a)$ as $\mathbf{p}_1 \equiv_{\mathcal{A}_p} \mathbf{p}'_1$ and $\mathbf{q}_1 \equiv_{\mathcal{A}_q} \mathbf{q}'_1$.

To extrapolate how $\lambda_1 \in \left\{ \boxed{a \mid b}^S, \boxed{a \mid b}^T \right\}$ affects the outcome, we can examine the cross diagrams below and employ similar reasoning as with $\boxed{a \mid b}$.



With the induction hypothesis we have $\mathbf{r} \circ_{\mathcal{A}} \lambda_1\lambda_2 = \mathbf{r}' \circ_{\mathcal{A}} \lambda_1\lambda_2$ and $\mathbf{r} \equiv_{\mathcal{A}} \mathbf{r}'$ for $\lambda_1\lambda_2$ with $\lambda_1 \in \left\{ \boxed{a \mid b}, \boxed{a \mid b}^S, \boxed{a \mid b}^T \right\}$ and $\lambda_2 \in \Lambda^{k-1}$.

Case 4 $\lambda_1 \in \left\{\overline{\#}, \hat{\#}, \#\right\}$



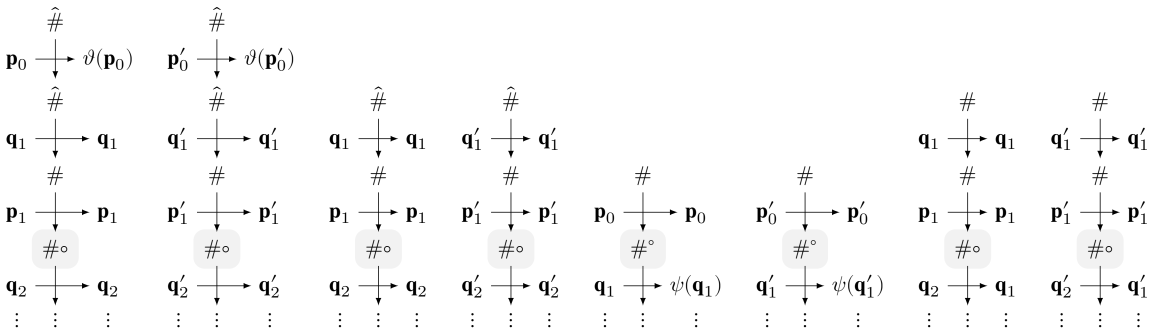
We can see, that $\mathbf{r} \circ_{\mathcal{A}} \overline{\#} = \mathbf{r}' \circ_{\mathcal{A}} \overline{\#}$ holds and

$$\mathbf{r} \cdot_{\mathcal{A}} \overline{\#} = \left\{ \begin{array}{ll} \mathbf{p}_n\mathbf{q}_n \dots \mathbf{p}_2\mathbf{q}_2\mathbf{p}_1\mathbf{q}_1\vartheta(\mathbf{p}_0) & \mathbf{p}_0 = \varepsilon \\ \mathbf{p}_n\mathbf{q}_n \dots \mathbf{p}_2\mathbf{q}_2\mathbf{q}_2\vartheta(\mathbf{p}_1)\mathbf{q}_1 & \mathbf{p}_0 \neq \varepsilon \end{array} \right\}$$

$$\equiv \mathbf{r}' \cdot_{\mathcal{A}} \overline{\#} = \left\{ \begin{array}{ll} \mathbf{p}'_n\mathbf{q}'_n \dots \mathbf{p}'_2\mathbf{q}'_2\mathbf{p}'_1\mathbf{q}'_1\vartheta(\mathbf{p}'_0) & \mathbf{p}'_0 = \varepsilon \\ \mathbf{p}'_n\mathbf{q}'_n \dots \mathbf{p}'_2\mathbf{q}'_2\mathbf{q}'_2\vartheta(\mathbf{p}'_1)\mathbf{q}'_1 & \mathbf{p}'_0 \neq \varepsilon \end{array} \right\}$$

due to $\mathbf{p}_n\mathbf{q}_n \dots \mathbf{p}_2\mathbf{q}_2\mathbf{p}_1\mathbf{q}_1 \equiv \mathbf{p}'_n\mathbf{q}'_n \dots \mathbf{p}'_2\mathbf{q}'_2\mathbf{p}'_1\mathbf{q}'_1$ or $\mathbf{p}_n\mathbf{q}_n \dots \mathbf{p}_2\mathbf{q}_2 \equiv \mathbf{p}'_n\mathbf{q}'_n \dots \mathbf{p}'_2\mathbf{q}'_2$ and $\mathbf{q}_1 \equiv_{\mathcal{A}} \mathbf{q}'_1$ as specified in the premise and $\vartheta(\mathbf{p}_0) \equiv_{\mathcal{A}} \vartheta(\mathbf{p}'_0)$ or $\vartheta(\mathbf{p}_1) \equiv_{\mathcal{A}} \vartheta(\mathbf{p}'_1)$ as ϑ is a homomorphism.

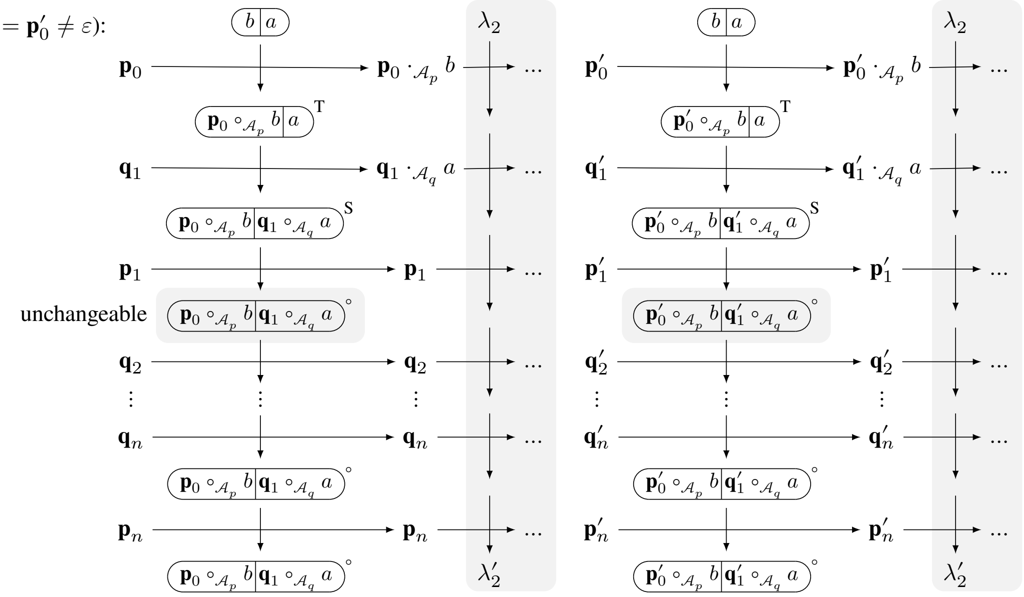
To extrapolate how $\lambda_1 \in \left\{\hat{\#}, \#\right\}$ affects the outcome, we can examine the cross diagrams below and employ similar reasoning as with $\overline{\#}$.



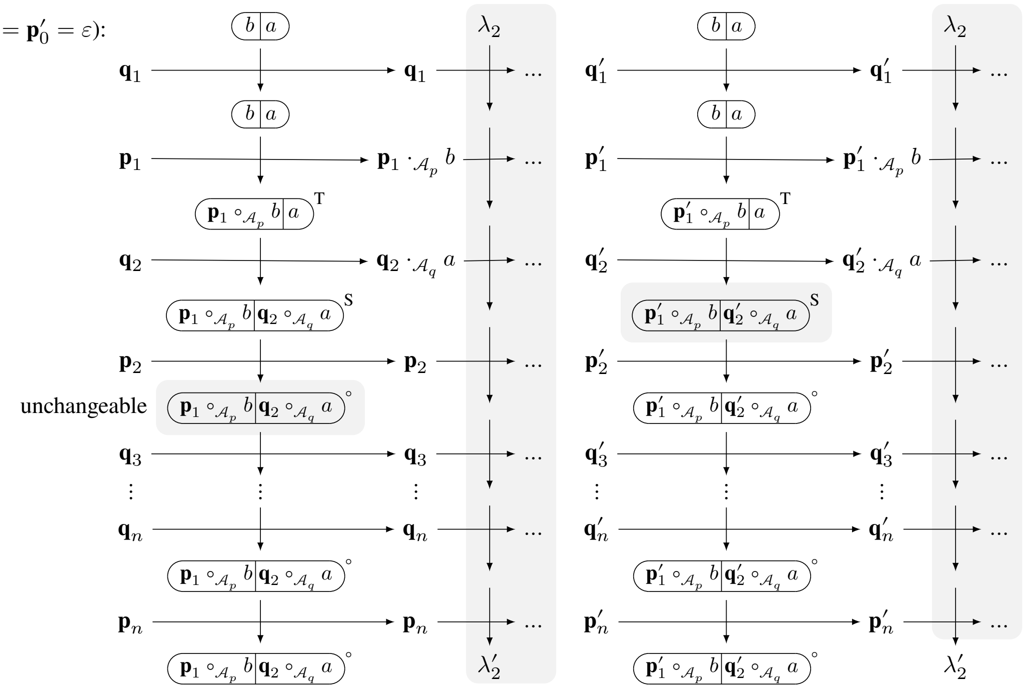
With the induction hypothesis we have $\mathbf{r} \circ_{\mathcal{A}} \lambda_1\lambda_2 = \mathbf{r}' \circ_{\mathcal{A}} \lambda_1\lambda_2$ and $\mathbf{r} \equiv_{\mathcal{A}} \mathbf{r}'$ for $\lambda_1\lambda_2$ with $\lambda_1 \in \left\{\overline{\#}, \hat{\#}, \#\right\}$ and $\lambda_2 \in \Lambda^{k-1}$.

Case 5 $\lambda_1 \in \left\{ \boxed{b \mid a}, \boxed{b \mid a}^{\mathrm{T}}, \boxed{b \mid a}^{\mathrm{S}} \right\}$

$(\mathbf{p}_0 = \mathbf{p}_0' \neq \varepsilon)$:



$(\mathbf{p}_0 = \mathbf{p}_0' = \varepsilon)$:



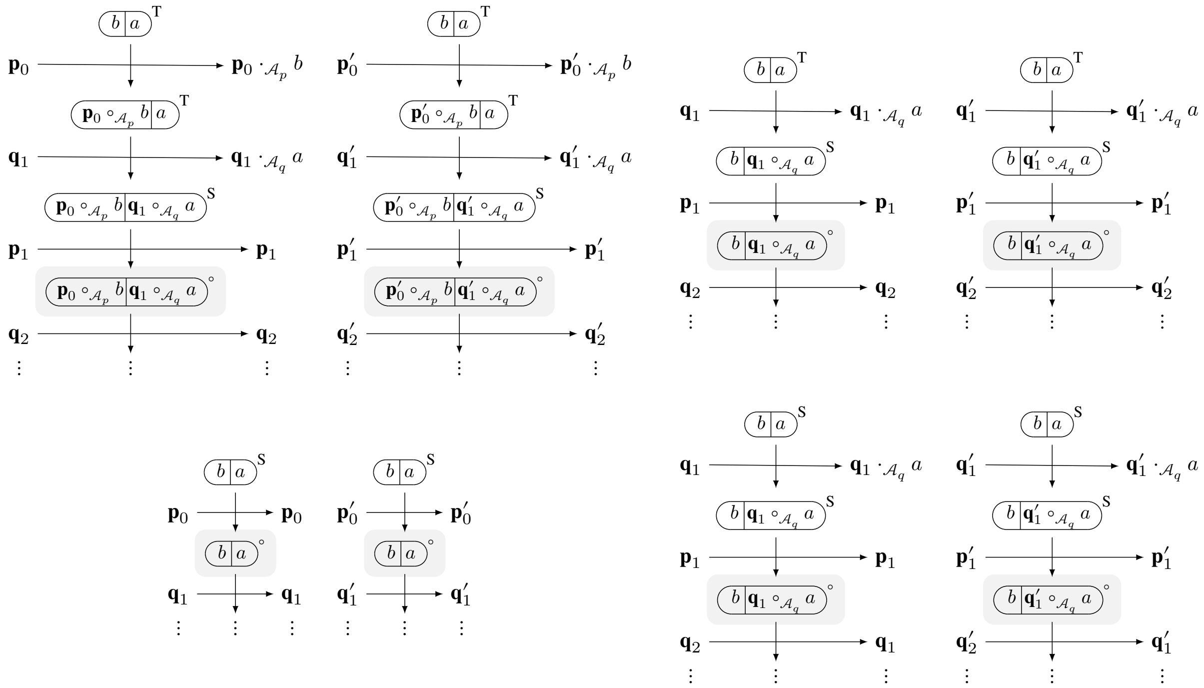
We can see, that $\mathbf{r} \circ_{\mathcal{A}} \boxed{b \mid a} = \mathbf{r}' \circ_{\mathcal{A}} \boxed{b \mid a}$ holds and

$$\mathbf{r} \cdot_{\mathcal{A}} \boxed{b \mid a} = \begin{cases} \mathbf{p}_n \mathbf{q}_n \dots \mathbf{p}_2 (\mathbf{q}_2 \cdot_{\mathcal{A}_q} a)(\mathbf{p}_1 \cdot_{\mathcal{A}_p} b) \mathbf{q}_1 & \mathbf{p}_0 = \varepsilon \\ \mathbf{p}_n \mathbf{q}_n \dots \mathbf{p}_2 \mathbf{q}_2 \mathbf{p}_1 (\mathbf{q}_1 \cdot_{\mathcal{A}_q} a)(\mathbf{p}_0 \cdot_{\mathcal{A}_q} b) & \mathbf{p}_0 \neq \varepsilon \end{cases}$$

$$\equiv \mathbf{r}' \cdot_{\mathcal{A}} \boxed{b \mid a} = \begin{cases} \mathbf{p}_n' \mathbf{q}_n \dots \mathbf{p}_2' (\mathbf{q}_2' \cdot_{\mathcal{A}_q} a)(\mathbf{p}_1' \cdot_{\mathcal{A}_p} b) \mathbf{q}_1' & \mathbf{p}_0' = \varepsilon \\ \mathbf{p}_n' \mathbf{q}_n \dots \mathbf{p}_2' \mathbf{q}_2' \mathbf{p}_1 (\mathbf{q}_1' \cdot_{\mathcal{A}_q} a)(\mathbf{p}_0' \cdot_{\mathcal{A}_q} b) & \mathbf{p}_0' \neq \varepsilon \end{cases}$$

due to $\mathbf{p}_n\mathbf{q}_n \dots \mathbf{p}_2 \equiv \mathbf{p}'_n\mathbf{q}_n \dots \mathbf{p}'_2$ and $\mathbf{q}_1 \equiv \mathbf{q}'_1$ or $\mathbf{p}_n\mathbf{q}_n \dots \mathbf{p}_2\mathbf{q}_2\mathbf{p}_1 \equiv \mathbf{p}'_n\mathbf{q}_n \dots \mathbf{p}'_2\mathbf{q}'_2\mathbf{p}_1$ as specified in the premise and $(\mathbf{q}_2 \cdot_{\mathcal{A}_q} a)(\mathbf{p}_1 \cdot_{\mathcal{A}_p} b) \equiv_{\mathcal{A}} (\mathbf{q}'_2 \cdot_{\mathcal{A}_q} a)(\mathbf{p}'_1 \cdot_{\mathcal{A}_p} b)$ or $(\mathbf{q}_1 \cdot_{\mathcal{A}_q} a)(\mathbf{p}_0 \cdot_{\mathcal{A}_q} b) \equiv_{\mathcal{A}} (\mathbf{q}'_1 \cdot_{\mathcal{A}_q} a)(\mathbf{p}'_0 \cdot_{\mathcal{A}_q} b)$ as $\mathbf{q}_2 \equiv_{\mathcal{A}_q} \mathbf{q}'_2, \mathbf{p}_1 \equiv_{\mathcal{A}_p} \mathbf{p}'_1, \mathbf{q}_1 \equiv_{\mathcal{A}_q} \mathbf{q}'_1$ and $\mathbf{p}_0 \equiv_{\mathcal{A}_p} \mathbf{p}'_0$.

To extrapolate how $\lambda_1 \in \left\{\boxed{b \mid a}^{\mathrm{T}}, \boxed{b \mid a}^{\mathrm{S}}\right\}$ affects the outcome, we can examine the cross diagrams below and employ similar reasoning as with $\boxed{b \mid a}$.



With the induction hypothesis we have $\mathbf{r} \circ_{\mathcal{A}} \lambda_1\lambda_2 = \mathbf{r}' \circ_{\mathcal{A}} \lambda_1\lambda_2$ and $\mathbf{r} \equiv_{\mathcal{A}} \mathbf{r}'$ for $\lambda_1\lambda_2$ with $\lambda_1 \in \left\{\boxed{b \mid a}, \boxed{b \mid a}^{\mathrm{T}}, \boxed{b \mid a}^{\mathrm{S}}\right\}$ and $\lambda_2 \in \Lambda^{k-1}$.

To summarize: As the invariant has been shown to be true for $P(0)$ and for $P(k)$ — on the condition that it is also true for $P(k-1)$ — then by virtue of induction, it will be true for any $k \in \mathbb{N}$. $\square$

3.3 special case

Corollary 3.3.1 (special case)**.** *S, T are automaton semigroups, $\mathcal{A}_q = (Q, \Sigma, \delta)$ and $\mathcal{A}_p = (P, \Gamma, \tau)$ are automata with $S \cong \mathcal{S}(\mathcal{A}_q)$ and $T \cong \mathcal{S}(\mathcal{A}_p)$ and suppose there exist elements $e \in S, f \in T$ such that*

$$(u =_S u' \vee u =_T u') \implies (e^{|u|} = e^{|u'|} \wedge f^{|u|} = f^{|u'|})$$

holds, with u, u' are either words over the state set Q of $\mathcal{A}_q$ or P of $\mathcal{A}_p$. [BC17, Theorem 4]
This condition can only be satisfied, if either both semigroups contain an idempotent, or both semigroups are homogeneous with respect to their generating sets, i.e. $|u| \neq |u'| \implies u \neq_S u' \vee u \neq_T u'$.

Let w.l.o.g. $e \in Q, f \in P$ and $\psi : Q \to P, q \mapsto f, \vartheta : P \to Q, p \mapsto e$. So for $Q^+ \to P^+$ we have $\mathbf{q} \mapsto f^{|\mathbf{q}|}$ and for $P^+ \to Q^+$ we have $\mathbf{p} \mapsto e^{|\mathbf{p}|}$. Then ψ extends into a homomorphism $\Psi : S \to T$ and ϑ into a homomorphism $\Theta : T \to S$. This is well defined due to the restrictions mentioned.

Lemma 3.3.2. *There exist automaton semigroups S and T that do not satisfy the condition of corollary 3.3.1 but that satisfy the condition of Theorem 3.0.1.*

Example 3.3.3. Let $\mathcal{A} = (Q, \Sigma, \delta)$ be the automaton depicted in figure 2.1 on page 10 with $\mathcal{S}(\mathcal{A}) = \{a, b\}^+$, which is a homogeneous semigroup with no idempotent.

We then construct a second automaton semigroup $T = \mathcal{S}(\mathcal{B})$ with elements of the shape $\binom{\alpha}{\beta}$ where $\alpha \in \{a, b\}^+, \beta \in \{id, tog\}$ in the following manner:
Compose automaton $\mathcal{A}$ with itself to get $\mathcal{A}^2$ and have $\mathcal{A} \cup \mathcal{A}^2$ act on the α-tape. Use identity $(0 \mapsto 0, 1 \mapsto 1)$ for the β-tape where $\mathcal{A}$ acts on the α-tape and toggle $(0 \mapsto 1, 1 \mapsto 0)$ for the β-tape where $\mathcal{A}^2$ acts on the α-tape. We then have the automaton $\mathcal{B} = (P, \Gamma, \tau)$ depicted in figure 3.2.
$\mathcal{S}(\mathcal{B})$ does not contain any idempotent as $\{a, b\}^+$ does not. It is not homogeneous, as can be seen

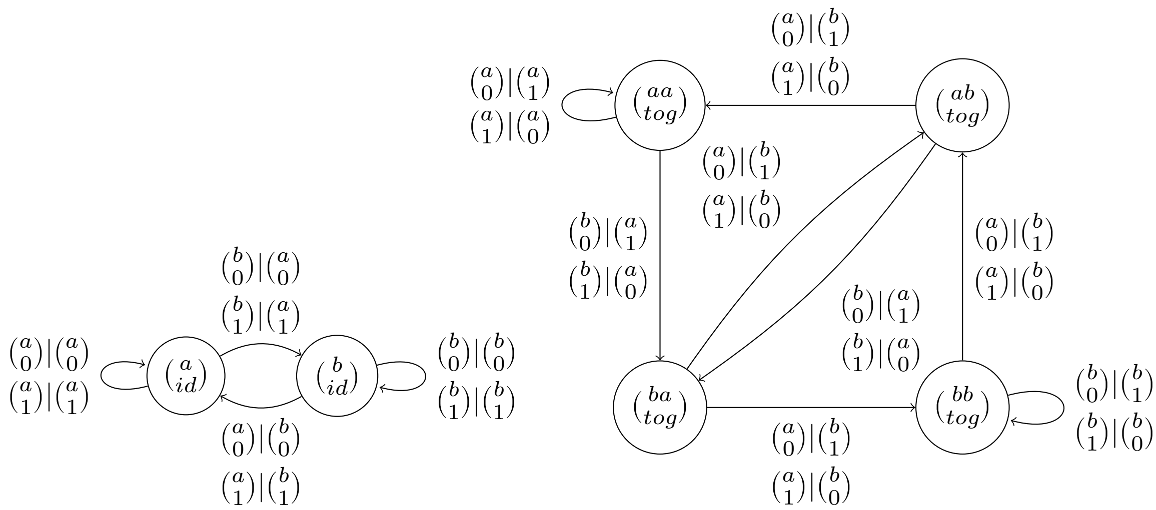


Figure 3.2: Automaton $\mathcal{B}$ such tha $\mathcal{S}(\mathcal{B})$ is neither homogenous nor has idempotents

in this example, where $\binom{ab}{tog}$ has to be a member of the generating set, as it can not be made up of a combination of other elements:

$$\begin{pmatrix} ab \\ tog \end{pmatrix}^2 = \begin{pmatrix} abab \\ id \end{pmatrix} = \begin{pmatrix} a \\ id \end{pmatrix}\begin{pmatrix} b \\ id \end{pmatrix}\begin{pmatrix} a \\ id \end{pmatrix}\begin{pmatrix} b \\ id \end{pmatrix}$$

We can see that a word made up of two elements can be rewritten to be made up of four.

Now let ψ and ϑ be defined as follows ($\alpha \in \{a, b\}^+, \beta \in \{id, tog\}$):

$$\psi : Q \to P, \alpha \mapsto \begin{pmatrix} \alpha \\ id \end{pmatrix},$$

$$\vartheta : P \to Q, \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \alpha$$

ψ extends to an homomorphism $\Psi : S \to T$ due to $\mathcal{S}(\mathcal{A})$ being isomorphic to the semigroup generated by $\mathcal{B}$ containing the elements $\{\binom{a}{id}, \binom{b}{id}\}$.

ϑ extends to an homomorphism $\Theta : T \to S$ due to this map yielding $\{a, b\}^+$.

So $\mathcal{S}(\mathcal{A}) \star \mathcal{S}(\mathcal{B})$ is covered by theorem 3.0.1.

4 Conclusion and Outlook

We have shown, that it is not necessary for automaton semigroups to fulfill the conditions given by Tara Brough and Alan J. Cain [BC17, Theorem 4] to form another automaton semigroup under free product construction.
It is sufficient for automaton semigroups S and T to have maps ψ and ϑ between the state sets of the automatons that generate them — provided ψ and ϑ can be extended into homomorphisms Ψ and Θ, such that $\Psi : S \to T$ and $\Theta : T \to S$.

Outlook

In line with the new discoveries made, it would be worthwhile to investigate, whether the criterion can be further generalized. A possible direction for this might be automaton semigroups without mutual homomorphisms.

It would also be interesting to explore whether the construction for the free product can be extended to graph products.

The term automaton semigroups, as it has been hitherto discussed, refers to semigroups generated by *complete* deterministic automata. It would be useful to expand the definition to include partial automata. This could, for example, offer the possibility to represent inverse semigroups in a natural way.

Bibliography

[BC17] T. Brough, A. J. Cain. “Automaton semigroups: new constructions results and examples of non-automaton semigroups”. In: *Theoretical Computer Science* 674 (2017), pp. 1–15 (cit. on pp. 7, 13, 26, 29).

[Cai09] A. J. Cain. “Automaton semigroups”. In: *Theoretical Computer Science* 410.47-49 (2009), pp. 5022–5038 (cit. on p. 7).

All links were last followed on October 14, 2019.

Declaration

I hereby declare that the work presented in this thesis is entirely my own and that I did not use any other sources and references than the listed ones. I have marked all direct or indirect statements from other sources contained therein as quotations. Neither this work nor significant parts of it were part of another examination procedure. I have not published this work in whole or in part before. The electronic copy is consistent with all submitted copies.

place, date, signature