# Bank Credit, Inside Money, and Debt Deflation in a Continuous-Time Macro Finance Model with Heterogeneous Agents

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# Contents

# List of Tables

# List of Figures

# List of Abbreviations

| | |
|---|---|
| BEA | Bureau of Economic Analysis |
| BGG | Bernanke, Gertler, and Gilchrist (1999) |
| BS | Brunnermeier and Sannikov |
| BVP | boundary value problem |
| CAPM | Capital Asset Pricing Model |
| CFI | circular flow of income |
| CRRA | constant relative risk aversion |
| CRS | constant returns to scale |
| CSV | Costly State Verification |
| CTMF | Continuous-Time Macro Finance |
| CVF | change-of-variables formula |
| DSGE | Dynamic Stochastic General Equilibrium |
| EFP | external finance premium |
| EIS | elasticity of intertemporal substitution |
| FOC | first-order condition |
| FRB | Board of Governors of the Federal Reserve System |
| GDP | gross domestic product |
| HARA | hyperbolic absolute risk aversion |
| IVP | initial value problem |
| KKT | Karush-Kuhn-Tucker |
| KM | Kiyotaki and Moore (1997) |
| LHS | left-hand side |
| MMR | macroprudential money supply rule |
| NK | New Keynesian |
| NKPC | New Keynesian Phillips Curve NKPC |
| ODE | ordinary differential equation |
| PDE | partial differential equation |
| QE | quantitative easing |

| | |
|---|---|
| RBC | Real Business Cycle |
| RDE | functional differential equation of the retarded type |
| RHS | right-hand side |
| RK | Runge-Kutta |
| SDC | standard debt contract |
| SDE | stochastic differential equation |
| TFP | total factor productivity |
| TFPQ | physical total factor productivity |
| TFPR | revenue total factor productivity |
| VaR | value-at-risk |
| w.r.t. | with respect to |

# List of Symbols

### Mathematical Operators and Functions

| | |
|---|---|
| Cov | covariance |
| $d$ | differential operator |
| $\mathbb{E}$ | expectation operator |
| $\mathbb{M}$ | mean |
| $O$ | order |
| Pr | probability |
| S.D. | standard deviation |
| Var | variance |

### Model Variables and Parameters

| | Description | Dimension |
|---|---|---|
| $\alpha^{e,p}$ | parameter function in prudent entrepreneurs' value function | utility |
| $\alpha^{m}$ | parameter function in managers' value function | utility |
| $\Gamma$ | external finance premium | 1/time |
| $\gamma$ | capital goods production function parameter | dimensionless |
| $\delta$ | capital depreciation rate | 1/time |
| $\eta$ | wealth share of entrepreneurs | dimensionless |
| $\eta^{\psi}$ | wealth share of entrepreneurs at beginning of optimal capital allocation region | dimensionless |
| $\eta^{s}$ | wealth share of entrepreneurs near steady state | dimensionless |
| $\Theta$ | disutility of bankers' auditing effort | utility/time |
| $\theta$ | value share of capital | dimensionless |
| $\iota$ | aggregate investment rate | output/capital/time |
| $\iota^{e,i}$ | investment rate of individual imprudent entrepreneur | output/capital/time |
| $\iota_i^{e,p}$ | investment rate of individual prudent entrepreneur | output/capital/time |

| | | |
|---|---|---|
| $\iota_i^m$ | investment rate of individual manager | output/capital/time |
| $\kappa$ | percentage drop in aggregate capital due to aggregate risk | 1/time |
| $\underline{\kappa}^{e,p}$ | percentage drop in capital of individual imprudent entrepreneur due to idiosyncratic risk | 1/time |
| $\underline{\kappa}^m$ | percentage drop in capital of individual manager due to idiosyncratic risk | 1/time |
| $\overline{\kappa}^{e,p}$ | percentage jump in capital of individual imprudent entrepreneur due to idiosyncratic risk | 1/time |
| $\overline{\kappa}^m$ | percentage jump in capital of individual manager due to idiosyncratic risk | 1/time |
| $\Lambda$ | money multiplier | dimensionless |
| $\lambda$ | intensity of Poisson process for aggregate risk | 1/time |
| $\lambda^s$ | intensity of Poisson process for entrepreneurs' sector-specific productivity shock | 1/time |
| $\lambda^{s,m}$ | intensity of Poisson process for managers' sector-specific productivity shock | 1/time |
| $\xi$ | utility weight of real balances | utility/output/time |
| $\Pi_t^b$ | banks' real aggregate profits per unit of managers' aggregate wealth | dimensionless |
| $\pi$ | inflation rate | 1/time |
| $\rho$ | time preference rate | 1/time |
| $\overline{\varrho}$ | monetary policy reaction function parameter | 1/time |
| $\varrho_\eta$ | monetary policy reaction function parameter | 1/time |
| $\tau_t$ | real transfer resulting from variations in outside money | dimensionless |
| $\Phi$ | capital goods production function | 1/time |
| $\phi$ | share of individuals exposed to idiosyncratic drop in capital | dimensionless |
| $\phi^s$ | share of entrepreneurs exposed to the sector-specific productivity shock | dimensionless |
| $\phi^{s,m}$ | share of managers exposed to the sector-specific productivity shock | dimensionless |
| $\varphi$ | share of imprudent entrepreneurs in the population of entrepreneurs | dimensionless |
| $\psi$ | entrepreneurs' share in the aggregate capital stock | dimensionless |
| $\Omega$ | managers' wealth share | dimensionless |
| $\omega$ | disutility of auditing parameter | utility/time/output |
| $A$ | aggregate TFP | output/capital/time |
| $a^e$ | entrepreneurs' productivity parameter | output/capital/time |
| $a^m$ | managers' productivity parameter | output/capital/time |

| | | |
|---|---|---|
| $B$ | auxiliary Variable | dimensionless |
| $c^b$ | bankers' instantaneous real aggregate consumption | output/time |
| $c^{e,i}$ | imprudent entrepreneurs' instantaneous real aggregate consumption | output/time |
| $c^{e,p}$ | prudent entrepreneurs' instantaneous real aggregate consumption | output/time |
| $c^m$ | managers' instantaneous real aggregate consumption | output/time |
| $c_i^{e,i}$ | instantaneous real consumption of individual imprudent entrepreneur | output/time |
| $c_i^{e,p}$ | instantaneous real consumption of individual prudent entrepreneur | output/time |
| $c_i^m$ | instantaneous real consumption of individual manager | output/time |
| $dr_i^{K,e,i}$ | instantaneous real return to capital of individual imprudent entrepreneur | dimensionless |
| $dr_i^{K,e,p}$ | instantaneous real return to capital of individual prudent entrepreneur | dimensionless |
| $dr_i^{K,m}$ | instantaneous real return to capital of individual manager | dimensionless |
| $dr_i^{L,b}$ | instantaneous real return to loans of individual bank | dimensionless |
| $dr^{L,e}$ | instantaneous real loan rate paid by entrepreneurs | dimensionless |
| $dr^M$ | instantaneous real return on money | dimensionless |
| $dr_i^{P,e,i}$ | instantaneous real portfolio return of individual imprudent entrepreneur | dimensionless |
| $dr_i^{P,e,p}$ | instantaneous real portfolio return of individual prudent entrepreneur | dimensionless |
| $dr_i^{P,m}$ | instantaneous real portfolio return of individual manager | dimensionless |
| $e^e$ | entrepreneurs' real aggregate external investment | output/time |
| $\mathbb{I}^b$ | set of bankers | individuals |
| $\mathbb{I}^{e,i}$ | set of imprudent entrepreneurs | individuals |
| $\mathbb{I}^{e,p}$ | set of prudent entrepreneurs | individuals |
| $\mathbb{I}^m$ | set of managers | individuals |
| $i_i^{e,i}$ | instantaneous real internal investment of individual imprudent entrepreneur | output/time |
| $i_i^{e,p}$ | instantaneous real internal investment of individual prudent entrepreneur | output/time |
| $i_i^m$ | instantaneous real internal investment of individual manager | output/time |
| $i^{e,i}$ | imprudent entrepreneurs' instantaneous real aggregate internal investment | output/time |

| | | |
|---|---|---|
| $i^{e,p}$ | prudent entrepreneurs' instantaneous real aggregate internal investment | output/time |
| $i^m$ | managers' instantaneous real aggregate internal investment | output/time |
| $J$ | index for set of managers, imprudent entrepreneurs, and prudent entrepreneurs | dimensionless |
| $j$ | index for set of managers and prudent entrepreneurs | dimensionless |
| $K$ | aggregate capital stock | capital |
| $k^e$ | entrepreneurs' aggregate capital stock | capital |
| $k^{e,i}$ | imprudent entrepreneurs' aggregate capital stock | capital |
| $k^{e,p}$ | prudent entrepreneurs' aggregate capital stock | capital |
| $k^m$ | managers' aggregate capital stock | capital |
| $k_i^{e,i}$ | capital stock of individual imprudent entrepreneur | capital |
| $k_i^{e,p}$ | capital stock of individual prudent entrepreneur | capital |
| $k_i^m$ | capital stock of individual manager | capital |
| $L^{e,p}$ | prudent entrepreneurs' leverage ratio | dimensionless |
| $l^b$ | aggregate nominal value of all loans extended by banks | money |
| $l^{b,i}$ | aggregate nominal value of loans extended by banks to imprudent entrepreneurs | money |
| $l^{b,p}$ | aggregate nominal value of loans extended by banks to prudent entrepreneurs | money |
| $l_i^{e,i}$ | nominal value of loans taken out by individual imprudent entrepreneur | money |
| $l_i^{e,p}$ | nominal value of loans taken out by individual prudent entrepreneur | money |
| $l_{i,t}^{e,p,r}$ | nominal value of loans taken out by the representative prudent entrepreneur | money |
| $M$ | total money supply | money |
| $M^I$ | inside money supply | money |
| $M^O$ | outside money supply | money |
| $\mathcal{M}$ | pre- relative to post-jump stock of outside money | dimensionless |
| $m^b$ | aggregate nominal money holdings in the banking sector | money |
| $m_i^{e,p}$ | nominal money holdings of individual prudent entrepreneur | money |
| $m_i^m$ | nominal money holdings of individual manager | money |
| $N$ | aggregate wealth | output |
| $\mathcal{N}$ | Poisson process for aggregate risk | dimensionless |
| $\overline{\mathcal{N}}_i$ | Poisson process for idiosyncratic risk | dimensionless |
| $\underline{\mathcal{N}}_i$ | Poisson process for idiosyncratic risk | dimensionless |

| | | |
|---|---|---|
| $\mathcal{N}_i^s$ | Poisson process for sector-specific productivity shock of individual entrepreneur | dimensionless |
| $\mathcal{N}_i^{s,m}$ | Poisson process for sector-specific productivity shock of individual manager | dimensionless |
| $n^e$ | real aggregate wealth of prudent entrepreneurs | output |
| $n_i^{e,i}$ | real wealth of individual imprudent entrepreneur | output |
| $n_i^{e,p}$ | real wealth of individual prudent entrepreneur | output |
| $n_i^m$ | real wealth of individual manager | output |
| $n_i^m$ | real aggregate wealth of managers | output |
| $P$ | price of money in final goods | output/money |
| $\mathcal{P}$ | price level | money/output |
| $p$ | price of outside money in final goods per unit of aggregate capital (model without policy) | output/money/capital |
| $p$ | value of all outside money in final goods per unit of aggregate capital (model with policy) | output/money/capital |
| $q$ | price of capital in final goods | output/capital |
| $R^{e,p}$ | prudent entrepreneurs' required risk premium on capital over money | 1/time |
| $R^m$ | managers' required risk premium on capital over money | 1/time |
| $R_a^m$ | managers' actual risk premium on capital over money | 1/time |
| $r^R$ | nominal rate on reserves | 1/time |
| $t$ | time | time |
| $t^{\mathcal{M}}$ | instant of time of unanticipated monetary policy intervention | time |
| $t_i^{\mathcal{N}}$ | instant of time individual $i$ is first exposed to a jump | time |
| $u$ | instantaneous utility function | utility/time |
| $V^{e,p}$ | prudent entrepreneurs' value function | utility |
| $V^m$ | managers' value function | utility |
| $\mathcal{V}$ | velocity of money | 1/time |
| $w^b$ | total real wage payments earned by bankers per unit of time | output/time |
| $x^{b,l}$ | banks' portfolio weight on loans in terms of bank owners' net worth | dimensionless |
| $x_{1,i}^{e,p}$ | portfolio weight on capital of individual prudent entrepreneur | dimensionless |
| $x_{2,i}^{e,p}$ | portfolio weight on money of individual prudent entrepreneur | dimensionless |
| $x_i^m$ | portfolio weight on capital of individual manager | dimensionless |
| $Y$ | instantaneous aggregate supply of final goods | output/time |

| | | |
|---|---|---|
| $Y^d$ | instantaneous aggregate demand for final goods | output/time |
| $y_i^e$ | instantaneous output of individual entrepreneur | output/time |
| $y_i^m$ | instantaneous output of individual manager | output/time |
| $z_i$ | stochastic productivity level of individual $i$ | output/capital/time |

# Abstract

We develop a continuous-time macro finance model in which banks create inside money by extending credit to a subgroup of production units termed “entrepreneurs”. These agents use the acquired funds to purchase physical capital from less productive producers, which we refer to as “managers”. The adoption of nominal debt as a means to exchange financial claims concentrates endogenous price risks on end-borrowers’ balance sheets. Entrepreneurs hold capital on the asset sides of their balance sheets, while they are financed by bank debt as well as inside equity. Thus, to the extent that changes in the values of capital and money differ, the real net worth of those individuals will be affected.

Conversely, banks’ assets and liabilities, namely loans and deposits, respectively, are both denominated in nominal terms. It follows that in tranquil episodes without loan defaults, the real equity of banks will not depend on adjustments in prices. If banks have to write off loans, they recover physical capital from bankrupt debtors. Consequently, they absorb possible movements in the value of capital. Furthermore, changes in the real value of deposits that backed the failing loans may contribute to fluctuations in banks’ real wealth. Accordingly, the portion of banks’ balance sheets that is exposed to adverse variations in asset prices is identical to the share of defaulting debtors in banks’ loan portfolios. Yet, in line with data on corporate failures, that share in our model economy is small.

The adoption of the continuous-time macro finance framework allows us to characterise equilibrium dynamics across the entire state space in a tractable way. Due to heterogeneity and incomplete markets those dynamics are governed by endogenous changes in the wealth distribution between borrowers and lenders. During tranquil episodes without exogenous shocks, entrepreneurs earn higher returns than individuals from other sectors due to their superior production technology. In the process, risk averse entrepreneurs accumulate more and more equity relative to other sectors, which induces the former to buy additional capital financed by debt. Consequently, the money multiplier, i.e. the ratio of inside to outside money, expands. Agents opt to hold money for two reasons. First, in contrast to capital, it is an asset that is not subject to idiosyncratic risk. Second, agents have a transaction motive.

Once the economy is exposed to an exogenous shock that reduces the productivity levels of a share of entrepreneurs, the aggregate demand for credit and, therefore, the stock of inside money falls. The ensuing deflationary pressure raises the real value of end-borrowers' debt, which leads to further decreases in the volume of credit and the price level. Consequently, an adverse feedback loop between price adjustments and deteriorating balance sheets in the entrepreneurial sector emerges. Entrepreneurs reduce the outstanding amount of their debt by using the proceeds from capital sales to less productive managers. As a consequence of these asset fire sales, the allocation of capital worsens and output declines. At the same time, banks adjust their lending rates, which, depending on the current state of the economy, may further contribute to the contraction on the credit market. The response of macroeconomic aggregates to exogenous shocks is much more pronounced away from the stochastic steady state. The main reason for that behaviour is the countercyclicality of firms' leverage ratios.

Conventional interest rate policies conducted by the central bank are ineffective in preventing the nominal as well as the real consequences of detrimental shocks in our flexible-price economy without nominal bonds. However, this does not hold true for policies that affect the real return on money via adjustments in the money supply. In our model, the monetary authority implements that objective by means of a helicopter drop of money à la Friedman (1969). Yet, since individuals in our model have rational expectations, that approach might backfire. Indeed, we identify a "paradox of monetary expansion": anticipated expansionary interventions in bad times exacerbate the ramifications of adverse shocks on the price level. This is due to entrepreneurs' endogenous portfolio choice: anticipating the monetary impulse and the ensuing depreciation in the value of money, they borrow more in normal times. Once adverse shocks hit the entrepreneurial sector, however, the pre-crisis debt boom is reversed: then, the reductions in the stock of inside money and the price level are even larger than without the policy. This is detrimental to agents' welfare for a number of reasons, including the intensified exposure to endogenous price risks.

As a corollary, policies that cut the supply of base money once a shock materialises counteract the adverse feedback loop in our credit model and lead to welfare improvements. We further show that these outcomes are reversed if agents do not anticipate interventions by the monetary authority. In this case, which is more similar to monetary policy analysis in linearised DSGE models, monetary expansion in the event of an adverse disturbance in the entrepreneurial sector does not lead to ex ante portfolio adjustments in the private sector and thus mitigates deflationary pressure. Yet, such policy leads to welfare reductions on the side of lenders. Our results suggest that policy makers who consider the implementation of measures that aim at expanding the money supply during disinflationary or even deflationary episodes should take into account to what extent the policy is anticipated.

# Kurzzusammenfassung

Diese Dissertationsschrift entwickelt ein zeitstetiges Makro-Finance-Modell, in dem Banken durch die Vergabe von Krediten an eine Untergruppe von Produktionseinheiten, die als „Unternehmer“ bezeichnet werden, Innengeld erschaffen. Diese Agenten verwenden die erhaltenen Finanzierungsmittel, um physisches Kapital von weniger produktiven Produzenten zu erwerben, die als „Manager“ bezeichnet werden. Die Finanzierung über Schulden, die mit Geldeinheiten beglichen werden müssen, konzentriert endogene Preisrisiken auf den Bilanzen der Endkreditnehmer. Unternehmer halten Kapital auf der Aktivseite ihrer Bilanzen, während sie sowohl durch Bankkredite als auch durch über einbehaltene Gewinne akkumuliertes Eigenkapital finanziert sind. Divergierende Entwicklungen des Kapitalpreises und des Geldwertes verändern folglich das reale Vermögen dieser Agenten.

Dagegen sind sowohl die Aktiva der Banken, in Form von Krediten, als auch deren Passiva, in Form von Sichteinlagen, in Geldeinheiten denominiert. Dies impliziert, dass in Phasen ohne Kreditausfälle das reale Eigenkapital der Banken nicht von Preisanpassungen abhängt. Wenn Banken Kredite abschreiben müssen, übernehmen und verwerten diese das physische Kapital von insolventen Schuldnern. Folglich absorbieren die Bankbilanzen in diesem Fall Anpassungen des Kapitalpreises. Darüber hinaus tragen Änderungen des Realwertes der Einlagen, die den ausfallenden Krediten gegenüberstehen, zu Schwankungen des realen Eigenkapitals der Banken bei. Dementsprechend ist der Anteil der Bankbilanzen, der Preisänderungsrisiken ausgesetzt ist, identisch mit dem Anteil der ausfallenden Schuldner in den Kreditportefeuilles der Banken. Gemäß den empirischen Daten zu Unternehmensinsolvenzen ist dieser Anteil in der modellierten Volkswirtschaft jedoch gering.

Die Methodik der zeitstetigen Makro-Finance-Literatur ermöglicht es, die Gleichgewichtsdynamik über den gesamten Zustandsraum hinweg nachvollziehbar zu charakterisieren. Aufgrund von Heterogenität und unvollständigen Märkten wird diese Dynamik durch endogene Veränderungen in der Vermögensverteilung zwischen Schuldnern und Gläubigern bestimmt. In ruhigen Phasen ohne exogene Schocks erzielen Unternehmer aufgrund ihrer überlegenen Produktionstechnologie Überschussrenditen relativ zu Agenten aus anderen Sektoren. Dadurch akkumulieren die risikoaversen Unternehmer relativ zu anderen Sektoren mehr und mehr Eigenkapital, wodurch erstere dazu bereit sind, zusätzliche Schulden aufzunehmen. Folglich vergrößert sich der Geldmengenmultiplikator, definiert als das Verhältnis von Innengeldmenge zu Außengeldmenge. Agenten halten

aus zwei Gründen Geld. Erstens handelt es sich, im Gegensatz zu physischem Kapital, um einen Vermögenswert, der keinem idiosynkratischen Risiko ausgesetzt ist. Zweitens haben die Agenten ein Transaktionsmotiv.

Sobald die Modellvolkswirtschaft einem exogenen Schock ausgesetzt ist, der die Produktivität eines Teils der Unternehmer verringert, sinkt die aggregierte Nachfrage nach Krediten und damit die Innengeldmenge. Bei unveränderter Außengeldmenge verringert sich folglich der Geldmengenmultiplikator. Der daraus resultierende deflationäre Druck erhöht den Realwert der Schulden der Endkreditnehmer, was zu einem weiteren Rückgang des Kreditvolumens und des Preisniveaus führt. Infolgedessen entsteht eine nachteilige Rückkopplung zwischen Preisanpassungen und realen Verlusten im Unternehmenssektor. Da die Unternehmer ihre Schulden durch den Verkauf von Kapital an weniger produktive Manager senken, kommt es zu einer zunehmenden Fehlallokation des Kapitals, wodurch sich die totale Faktorproduktivität und damit die aggregierte Produktionsmenge verringert. Gleichzeitig passen die Banken die Kreditzinsen an, was in Abhängigkeit von der aktuellen Wirtschaftslage zu einem weiteren Rückgang des Kreditvolumens führen kann. Die Reaktion makroökonomischer Aggregate auf exogene Schocks ist wesentlich stärker ausgeprägt wenn sich die Volkswirtschaft schon vor dem jeweiligen Schock unterhalb des stochastischen Steady States befunden hat. Der Hauptgrund für diese Eigenschaft ist die Antizyklizität der Verschuldungsquoten der Unternehmer.

Konventionelle Zinspolitik der Zentralbank verhindert weder die nominellen noch die realen Folgen negativer Schocks in der betrachteten Volkswirtschaft, die durch perfekte Preisflexibilität und die Abwesenheit nominaler Anleihen gekennzeichnet ist. Dies gilt jedoch nicht für Maßnahmen, die sich über Anpassungen der Geldmenge auf die reale Rendite des Geldes auswirken. Im betrachteten Modell setzt die Notenbank dieses Ziel durch die Verteilung von „Helikoptergeld“ à la Friedman (1969) um. Diese Maßnahme schlägt jedoch fehl wenn sie von den Wirtschaftssubjekten antizipiert wird: Erwartete expansive Interventionen nach negativen Schocks verschärfen in diesem Fall die Auswirkungen von negativen Schocks auf das Preisniveau. Dieses „Paradoxon der monetären Expansion“ lässt sich auf die endogene Portfoliowahl der Wirtschaftssubjekte zurückzuführen: In Erwartung des monetären Impulses und der damit verbundenen Verringerung des Geldwerts nehmen die Unternehmer in Perioden ohne Schocks mehr Kredite auf. Sobald der Unternehmenssektor negativen Produktivitätsschocks ausgesetzt ist, verkehrt sich der der Krise vorausgehende Schuldenboom jedoch ins Gegenteil: In diesem Fall kommt es zu stärkeren Verringerungen des Bestands an Innengeld und des Preisniveaus, als ohne die Geldmengenausweitung. Dies wirkt sich aus mehreren Gründen nachteilig auf das Wohlfahrtsniveau der Agenten aus. Dazu gehört die verstärkte Exposition gegenüber endogenen Preisrisiken.

Dagegen wirken Maßnahmen, die das Angebot an Basisgeldern reduzieren sobald ein Schock eintritt, der negativen Rückkopplungsschleife im Modell entgegen und führen zu Wohlfahrtsverbesserungen sowohl auf Seiten der Kreditgeber als auch der Kreditnehmer. Es wird ferner gezeigt, dass sich diese Ergebnisse umkehren, sofern die Akteure die Interventionen der Geldpolitik nicht antizipieren. In diesem Fall, der Parallelen zur geldpolitischen Analyse in linearisierten DSGE-Modellen aufweist, führt die Geldmengenausweitung bei einem Produktivitätsschock nicht zu Ex-ante-Portfolioanpassungen im privaten Sektor und mindert somit den deflationären Druck. Eine solche Politik führt jedoch zu Wohlfahrtseinbußen bei den Kreditgebern. Die Ergebnisse legen nahe, dass bei geldpolitischen Entscheidungen über Veränderungen der Geldmenge vor dem Hintergrund deflationärer Tendenzen berücksichtigt werden sollte, inwieweit diese Maßnahmen von den Wirtschaftssubjekten antizipiert werden.

# Chapter 1

# Introduction

In the years leading up to the financial crisis of 2007-08, mainstream academic macroeconomists paid little attention to the role of the financial sector in explaining macroeconomic fluctuations. In many of the standard business cycle models that were developed at the time, the financial sector is merely a veil: it allocates financial means to its first-best use efficiently and without frictions.[1] That modelling approach was called into question after the outbreak of the aforementioned crisis since standard models were unable to predict and explain both the depth and the persistence of the ensuing Great Recession.[2] That failure brought the integration of financial market frictions into state-of-the-art business cycle models back to the forefront of the research agenda, as demonstrated by the increasingly vast literature on the subject.[3]

Somewhat surprisingly, in that endeavour, the fundamental role of banks as creators of inside money was largely ignored, apart from a few exceptions. One might be tempted to suspect that this is due to the standard practice in modern monetary theory to abstract from monetary aggregates altogether.[4] Such practice is in stark contrast to the debate during the aftermath of the Great Depression, in which banks' ability to create purchasing power out of thin air was viewed as a major source of instability. A prominent example is Irving Fisher's (1933) classic account of the Great Depression. In his "debt-deflation theory", even minor adverse disturbances may push an economy with highly indebted sectors into deep crisis. This is due to the interaction of debt liquidation, contractions in the inside money supply, and rising real debt burdens in the face of deflationary pressure.

One of the aforementioned exceptions in the contemporary macroeconomic literature on financial

[1] For instance, the influential estimated Dynamic Stochastic General Equilibrium models by Smets and Wouters (2003) and Christiano et al. (2005) do not feature imperfections in financial markets. The situation was somewhat different in policy institutions. Coenen et al. (cf. 2012, Table 1) compare seven models employed in policy institutions prior to the crisis and document that four out of these encompass financial frictions. Remarkably, the NAWM model utilised by the European Central Bank does not fall in that category.

[2] Cf. e.g. Del Negro and Schorfheide (2013, pp. 122f.) or Lindé et al. (2016, pp. 2204f.).

[3] Cf. the surveys by Quadrini (2011) and Brunnermeier et al. (2012).

[4] Cf. e.g. Woodford's (2003, pp. 64-74) cashless economy.

frictions is the "I Theory of Money" developed in Brunnermeier and Sannikov - henceforth, BS - (2014d) and (2016a), in which the "I" stands for intermediation or inside money. In those two versions of the I Theory of Money, the authors embed the creation of deposits by financial institutions into a version of the stochastic growth model with incomplete markets due to Bewley (1980). This allows them to formalise Fisher's intuition of the debt-deflation mechanism. In their theory, the asset sides of financial intermediaries are exposed to small exogenous shocks, which may, depending on the current state of the economy, translate into large losses. The amplification of small negative shocks is due to intermediaries' desire to deleverage, which causes the inside money supply to shrink. Since intermediaries hold long-term assets that are not denominated in nominal terms, the resulting surge in the value of money leads to further reductions in their (real) equity. In effect, an adverse feedback loop between price adjustments and deteriorating balance sheets emerges. The negative effects spill over to other sectors since intermediaries cut their financing activities in the process.

The I Theory of Money deviates from the bulk of work on financial frictions in a second regard: it is part of an emerging, yet already influential[5], literature, namely the Continuous-Time Macro Finance (CTMF) literature. Characteristically, CTMF models examine the macroeconomic consequences of financial market imperfections in a continuous-time setting. As in the more traditional literature on financial frictions, incomplete markets and heterogeneity imply that the wealth distribution matters for aggregate outcomes. One of the advantages of adopting the continuous-time methodology is that it allows for deriving global model solutions in a comparatively straightforward way. Therefore, CTMF models represent an attractive framework to study the causes and consequences of financial crises, which typically exhibit pronounced nonlinearities and large deviations from "normal" macroeconomic conditions. In contrast, conventional discrete-time Dynamic Stochastic General Equilibrium (DSGE) models with financial frictions only consider linearised solutions in the vicinity of a deterministic steady state, i.e. a point of attraction in which all risk parameters are set to zero. Many authors have argued that this feature makes DSGE models inadequate in explaining the sources and effects of deep and prolonged crisis episodes.[6]

A distinct characteristic of the I Theory of Money is that financial institutions exclusively finance production units by purchasing the equity of the latter on their own accounts. This assumption implies that deposits on the liabilities sides of those institutions' balance sheets are backed by equity claims on the asset sides. Thus, inside money creation can be interpreted as a "byproduct" of investment banking. Yet, in the real world, a substantial amount of financing provided by creators of inside money comes in the form of loans. For instance, in the consolidated balance sheet of euro area monetary financial institutions, the share of loans to private sector residents in total asset is approximately equal to 40 percent as of September 2018, while equity holdings and investment fund

[5] For instance, Brunnermeier and Sannikov (2014a), which initiated the CTMF literature, is among the top ten recent research items in economics as of January 2019, according to the Research Papers in Economics network (2019).

[6] Cf. e.g. Milne (2009, p. 614) and Benes et al. (2014, pp. 12f.).

shares only account for about three percent.[7] Taking such observations into account, in this thesis, we develop a model, which we will refer to as the "credit model" in the remainder, that (inter alia) departs from the I Theory in the fact that producers are entirely financed externally by bank credit.

To that end, we undertake numerous modifications of the model economy. First and foremost, we introduce a market for bank credit. Crucially, we assume that credit is denominated in nominal terms, i.e borrowers must repay their creditors in money. To motivate the agents' use of debt instruments, we assume informational asymmetries between banks and their borrowers, which make the usage of pure equity contracts inefficient. A demand for bank credit arises since financially constrained corporate end-borrowers, which we refer to as "entrepreneurs", require external funds to finance purchases of physical capital from less productive agents. Holding capital exposes entrepreneurs to two types of fundamental risk. First, they may suffer shocks that reduce the productivity of capital. Second, their capital stocks may be subject to idiosyncratic and aggregate depreciation shocks. Moreover, we add credit risk to assign a nontrivial role to banks. That is, in times when the economy is hit by adverse shocks, a share of debtors' declare bankruptcy. Subsequently, banks write-off the corresponding loans and take over the remaining assets of those borrowers. Anticipating the ensuing losses and taking into account agency costs resulting from the asymmetries of information, banks demand a premium on the lending rate over the deposit rate in equilibrium.

As it will turn out, our adjustments have far-reaching consequences for the distribution of endogenous risk, i.e. risk that derives from endogenous changes in prices. In particular, the adoption of nominal debt concentrates endogenous risk on end-borrowers' balance sheets. The intuition is as follows. Entrepreneurs hold capital on the asset sides of their balance sheets, while they are financed by bank debt as well as inside equity. Thus, to the extent that changes in the values of capital and money differ, the real net worth of those individuals will be affected. Conversely, banks' assets and liabilities, namely loans and deposits, respectively, are both denominated in nominal terms. It follows that in tranquil episodes without loan defaults, the real equity of banks will not depend on adjustments in prices. The situation is different if banks have to write off loans. Then, they recover physical capital from bankrupt debtors. Consequently, they absorb possible movements in the value of capital. Furthermore, changes in the real value of deposits that backed the failing loans may contribute to fluctuations in banks' real wealth. As becomes clear from these considerations, the portion of banks' balance sheets that is exposed to adverse variations in asset prices is identical to the share of defaulting debtors in banks' loan portfolios. Yet, in line with data on corporate failures, that share in our model economy is small.

In order to reduce the dimension of the state space to unity, we are forced to choose whether to include a bank lending channel or a balance sheet channel in our model.[8] In the first case,

[7] Own calculations based on data provided in European Central Bank (2018a), (2018b), and (2018c).

[8] In our framework, including both channels would raise the number of state variables to two. While this would clearly enrich the model, it would also complicate the numerical solution. The literature has applied the recently proposed "iterative method" to CTMF settings with two state variables under aggregate Brownian uncertainty (cf. Di Tella, 2017, Appendix B). However, we are not aware of any attempts in the literature to solve models with

amplification of exogenous shocks is primarily due to impaired balance sheets in the intermediary sector, as in the I Theory of Money. In the second approach, which was also pursued in the more traditional financial frictions literature, distressed balance sheets in the real sector are the main source of instability.[9] Given the mentioned limitations, a natural choice in our framework is the second approach since endogenous risk is mainly concentrated on the balance sheets of production units. To implement that choice in our credit model, we abstract from financial frictions in the banking sector.

Three main research questions arise from our modifications. First, given the fact that our focus lies on the repercussions of financial frictions for macroeconomic fluctuations, we ask how our model economy with financially constrained agents responds to negative exogenous shocks. We explore this research question along two dimensions. On the one hand, does an adverse adverse feedback loop with marked swings in the quantity and value of money in the spirit of Fisher (1933) arise under our modelling assumptions and, if so, what underlying forces are at work? This relates to the interplay of balance sheets in the entrepreneurial sector, deleveraging phenomena, and deflationary pressure. On the other hand, is the amplification mechanism sufficiently potent to translate small exogenous disturbances into large fluctuations in macroeconomic aggregates under plausible parameter constellations? In addition, our quantitative analysis also allows us to gauge the importance of adjustments in the value of money relative to other sources of amplification.

Second, how can the dynamics of the credit cycle be characterised in our credit model? This mainly concerns the question of how demand- and supply-side factors on the credit market interact in the determination of the credit cycle. In particular, we ask whether these two factors either reinforce or work against each other. On the demand side, borrowers' willingness to take on additional debt varies with their relative equity position, which, in turn, fluctuates due to the possibility of shocks. On the supply side, bankruptcy and agency costs that originate on the side of banks vary with the state of the economy. Since these costs are ultimately borne by debtors, variations in banks' mark-up over the deposit rate will have consequences for the equilibrium in the credit market. In this context, we will also explore whether that mark-up is countercyclical as in other business cycles models with financial frictions.

Third, given that one possible source of amplification is due to monetary phenomena, namely variations in the inside money supply and associated adjustments in the price level, what are the options of monetary policy to prevent or mitigate the adverse repercussions of shocks? This relates to both the choice of appropriate instruments and the stance of monetary policy given a specific instrument, i.e. should policy be expansive or restrictive in a particular situation. While a specific

aggregate Poisson uncertainty, which is also featured in the credit model, via the iterative method. Moreover, our attempt to solve the credit model using that method did not prove successful. Therefore, our modelling strategy entails the reduction of the state space's dimension to unity. This allows for the application of the "shooting method", which has been used in the literature to solve models with Poisson uncertainty (cf. Brunnermeier and Sannikov, 2014d, Appendix A).

[9] Cf. Brunnermeier et al. (2012, p. 6).

policy might be effective in containing amplification resulting from variations in the value of money, such policy need not be efficient in a Pareto sense. This is even more so since such policies in general have redistributive consequences. Taking such concerns into account, we conduct a formal welfare analysis of the effects of monetary policy.

We will see that dynamics in our model economy with incomplete markets and heterogeneity are driven by changes in the cross-sector wealth distribution. During tranquil times, entrepreneurs earn higher returns than individuals from other sectors due to their superior production technology. In the process, risk averse entrepreneurs accumulate more and more equity, which induces them to buy additional capital financed by debt. Consequently, the money multiplier, i.e. the ratio of inside to outside money, expands. Agents opt to hold money for two reasons. First, it is an asset that is not subject to idiosyncratic risk. Second, agents have a transaction motive. If adverse productivity shocks materialise in the entrepreneurial sector, the affected individuals engage in fire sales of capital to less productive production units and use the proceeds to pay back debt. Subsequently, the supply of inside money contracts and the value of money surges. As the allocation of capital worsens, output declines and the price of capital goods falls. Entrepreneurs' balance sheets are impaired since the real value of their assets drops and the real value of their liabilities rises. This leads to further deleveraging and movements in prices and so on. Absent monetary policy interventions, the induced changes in allocations and the value of money are, in general, quite substantial if the economy is sufficiently far away from its stochastic steady state, a result which can be traced back to strong nonlinearities.

While entrepreneurs reduce their demand for debt in the face of negative shocks, banks adjust their lending rates, which, depending on the current state of the economy, may further contribute to the contraction on the credit market. Yet, due to the absence of a bank lending channel, the effects of those adjustments are limited. Thus, the course of events is hard to reconcile with recent events during the financial crisis of 2007-08, which was characterised by distressed balance sheets in the intermediary sector. As we will argue, the response of the economy to shocks is more reminiscent of real-world crisis episodes which were driven by voluntary corporate deleveraging and ensuing deflation, such as the Lost Decade in Japan.

Turning to monetary policy, conventional interest rate policies are ineffective in preventing the consequences of detrimental shocks in our flexible-price economy without nominal bonds. However, this does not hold true for policies that affect the real return on money via adjustments in the money supply. In particular, a helicopter drop of money à la Friedman (1969) appears to be an obvious solution to mitigate deflationary pressure induced by contractions in the inside money supply. Yet, since individuals in our model have rational expectations, that approach might backfire. Indeed, we identify a "paradox of monetary expansion": anticipated expansionary interventions in bad times exacerbate the ramifications of adverse shocks on the price level. This is due to entrepreneurs' endogenous portfolio choice: anticipating the monetary impulse and the ensuing depreciation in the value of money, they borrow more in normal times. Once adverse shocks hit the entrepreneurial sector, however, the pre-crisis debt boom "bites back"[10]: then, the reductions in the stock of inside money and the price level are even larger than without the policy. This is detrimental to agents' welfare for a number of reasons, including the intensified exposure to endogenous risk.

As a corollary, policies that cut the supply of base money once a shock materialises counteract the adverse feedback loop in our credit model and lead to welfare improvements. We further show that these outcomes are reversed if agents do not anticipate interventions by the monetary authority. In this case, which is more similar to monetary policy analysis in linearised DSGE models, monetary expansion in the event of an adverse disturbance in the entrepreneurial sector does not lead to ex ante portfolio adjustments in the private sector and thus mitigates deflationary pressure. Yet, such policy leads to welfare reductions on the side of lenders. Besides these crisis interventions, we also consider ex ante policies that vary the outside money supply in tranquil times without shocks. These policies can stabilise the economy if they reduce the base money supply in times when growth in inside money is strong.

This thesis is structured as follows. Chapter 2 discusses some relevant theoretical concepts that relate to credit frictions. This includes the normative question as to why borrowers and financiers might opt for exchanging financial claims on the basis of credit contracts rather than other forms of financing, such as the issuance of equity. We also take a positive perspective and explain how banks create inside money by extending credit to the nonbank sector. In addition, we shortly present two seminal business cycle models with credit frictions and discuss shortcomings of these works that are addressed by the new CTMF literature.

Chapter 3 introduces the reader to various aspects of the CTMF literature. It first describes methodological foundations including overviews over some basic concepts from stochastic calculus, typical assumptions on preferences and technologies, and solution procedures. Furthermore, Chapter 3 provides a review of models that are referenced repeatedly throughout the thesis. The chapter concludes with a discussion of strengths and current limitations of the CTMF literature.

Chapter 4 develops the credit model. It details our assumptions and derives the set of equilibrium equations. The focus lies on individuals' portfolio selection problems, which are solved by means of dynamic optimisation techniques. Moreover, we show how the number of state variables can be reduced to unity and explain our equilibrium concept.

In Chapter 5, we calibrate model parameters to U.S. data and present the results generated by our model in the absence of monetary policy interventions. Besides analysing the full-blown credit model with endogenous risk, we also consider three special cases, which facilitate intuition about model properties. Furthermore, we discern the effects of parameter variations on equilibrium outcomes, which, inter alia, allows us to shed light on credit market dynamics. Lastly, we compare our model and its implications to related works.

Chapter 6 is concerned with the macroeconomic effects of monetary policy in the credit model. The chapter begins with a discussion of the ramifications of interest rate policies. Afterwards, we

[10] Jordà et al. (2013, p. 6)

derive the set of equilibrium equations under different money supply rules. The effects of those policies on equilibrium outcomes and agents' welfare are presented subsequently.

Lastly, Chapter 7 summarises our findings and outlines some directions for future research.

# Chapter 2

# On the Economics of Credit

Given that our main departure from the I Theory of Money is the introduction of the credit market as a means to exchange financial claims between end-borrowers and banks, the aim of this chapter is to discuss some relevant theoretical concepts that relate to credit frictions and also play a crucial role in the model developed in this thesis. Even though many mainstream macroeconomic models have abstracted from financial markets - especially before the outbreak of the recent financial crisis -, there is a longstanding tradition of incorporating credit frictions in macroeconomics. This development was made possible by advances in the economics of information during the 1970s and 1980s.[11] Some of these approaches are summarised in this chapter. Since the ultimate objective of this thesis is to embed the credit market into a dynamic macroeconomic model with optimising agents, we also provide a short review of two seminal business cycle models that incorporate the aforementioned informational frictions, namely the Bernanke, Gertler, and Gilchrist (1999) - henceforth, BGG - model and Kiyotaki and Moore (1997) - henceforth, KM - model. These models are chosen since they share some similarities with the model developed in this thesis. The attractive feature of the BGG as well as the KM model is that they include amplification mechanisms that translate small exogenous shocks into larger swings in output. In contrast, models without such mechanisms usually require large (and unexplained) exogenous shocks to generate output fluctuations that match the data.[12] Despite this contribution, the mentioned models have two main shortcomings: first, under empirically plausible parameter constellations they generate rather low degrees of amplification. Second, they implicitly assume a loanable funds framework, i.e. a framework in which savers lend real savings to end-borrowers either directly or indirectly via financial intermediaries. This is in stark contrast to the workings of a fractional-reserve monetary economy, in which banks are able to *create* their own funding in the form of inside money by extending financing to the nonbank sector.

This chapter is structured as follows. Section 2.1 takes a normative perspective by considering the question of why credit exists in the first place. To that end, a brief review of the strand of the

[11] Cf. Bernanke et al. (1999, p. 1344).
[12] Cf. Cordoba and Ripoll (2004, p. 1011).

financial contracting literature that has dealt with this issue is presented. As we will see, the dominant explanatory approaches rely on information asymmetries in the borrower-lender relationship. In Section 2.2 we present theoretical foundations, model setups, and implications of the BGG and the KM model. Special emphasis is put on the former since it shares with our model the motivation for adopting the *standard debt contract* (SDC) as a means of organising payment streams between borrowers and lenders. We also delve into some selected criticisms directed at the traditional business cycle literature with financial frictions. In this context, we limit ourselves to issues that are specifically addressed by the models presented in Chapters 3 and 4. Finally, Section 2.3 explains how banks can create inside money by means of credit extension. This section also offers some important implications of banks' role as suppliers of inside money.

## 2.1 Explaining the Use of Credit

### 2.1.1 Financing Contracts Under Symmetric Information

Given the pervasiveness of debt contracts, it is natural to ask why this type of financial arrangement exists in the first place. To this end, let us at first analyse whether a debt contract can be optimal in a simple two-period production economy with a single physical good and symmetric information.[13] This economy is populated by a single firm and two consumers indexed by $i = 1, 2$. It is assumed that consumption is possible only in the second period and that there is no discounting. The firm is endowed with a risky investment project that requires a fixed amount $I$ of the physical good at time $t = 0$. The project generates a random amount $\tilde{y}$ of that same good at time $t = 1$. Realisation $y$ of this random variable can take on $S$ possible values $y_1, ..., y_S$, which occur with probabilities $\pi_1, ..., \pi_S$.[14] The realisation of the payoff is common knowledge. As by assumption $\mathbb{E}_0 [\tilde{y}] > I$, it is efficient to implement the project. Consumer 1 (henceforth: the borrower, indexed by $b$) owns the firm but his endowment of the commodity in $t = 0$ is zero. In contrast, consumer 2 (henceforth: the lender, indexed by $l$) owns $I$ units of the commodity at the initial date but is not endowed with any share.[15]

Next, we will look for characteristics of an efficient lending contract between the borrower and the lender. To this end, let us define a repayment rule $R(\cdot)$. This rule requires mutual agreement and determines the amount of the good the borrower has to pay to the lender when uncertainty is revealed in $t = 1$. In addition, let us impose one further assumption, namely that consumption is possible only in the second period.

[13] The analysis of this economy closely follows Freixas and Rochet (2008, section 4.1).

[14] As the realisations of output are allowed to vary across states, the economy is exposed to aggregate risk, which in turn implies that perfect risk-sharing is not possible.

[15] Cf. Freixas and Rochet (2008, pp. 128f.).

The optimal contract chosen by the borrower solves:[16]

$$\max_{R(\cdot)} \sum_{s=1}^{S} \pi_s u_b\left(y_s - R\left(y_s\right)\right) \tag{2.1.1}$$
$$s.t. \sum_{s=1}^{S} \pi_s u_l\left(R\left(y_s\right)\right) \geq U_l^0,$$

in which $U_l^0$ is the reservation utility of the lender determined by his outside option, which is not explicitly modelled. As the utility functions are assumed to be monotonous, the constraint in the above program is binding. Thus, maximising with respect to (w.r.t.) $R(y)$ gives

$$\frac{u_b'\left(y - R\left(y\right)\right)}{u_l'\left(R\left(y\right)\right)} = \lambda, \tag{2.1.2}$$

where $\lambda$ is the Lagrangian associated with the above program.[17]

Now, we want to determine how the optimal repayment varies with the project's payoff. Taking the logarithm of (2.1.2) and subsequently differentiating with respect to $y$ yields

$$\frac{u_b''\left(y - R\left(y\right)\right)}{u_b'\left(y - R\left(y\right)\right)}\left[1 - R'\left(y\right)\right] - \frac{u_l''\left(y\right)}{u_l'\left(R\left(y\right)\right)} R'\left(y\right) = 0. \tag{2.1.3}$$

Solving for $R'(y)$ gives

$$R'\left(y\right) = \frac{A_b\left(y - R\left(y\right)\right)}{A_b\left(y - R\left(y\right)\right) + A_l\left(R\left(y\right)\right)}, \tag{2.1.4}$$

in which

$$A_b\left(\cdot\right) \equiv -\frac{u_b''\left(\cdot\right)}{u_b'\left(\cdot\right)} \quad \text{and} \quad A_l\left(\cdot\right) \equiv -\frac{u_l''\left(\cdot\right)}{u_l'\left(\cdot\right)} \tag{2.1.5}$$

are the borrower's and lender's Arrow-Pratt measures of absolute risk aversion, respectively. Equation (2.1.4) characterises the optimal repayment rule: if e.g. the borrower is strictly risk averse (i.e. if $A_b(\cdot) > 0$) and the lender is as well, $0 < R'(y) < 1$ holds. This implies that the repayment to the lender is increasing in the project outcome $y$. Hence, in this case the borrower and the lender share at least a part of the project's risk for all realisations of $y$. Yet, this feature of the optimal contract is inconsistent with the characteristics of a loan arrangement as the latter typically does not allow for complete risk-sharing. Rather, a loan contract concentrates the project risk entirely

[16] The characteristics of the resulting contract are identical to those if the lender were to choose the contractual form (cf. Freixas and Rochet, 2008, p. 129).

[17] For any two realisations $y_s$ and $y_{s'}$ in the support of $\tilde{y}$ the first-order conditions imply,

$$\frac{u_b'\left(y_1 - R\left(y_1\right)\right)}{u_b'\left(y_2 - R\left(y_2\right)\right)} = \frac{u_l'\left(R\left(y_1\right)\right)}{u_l'\left(R\left(y_2\right)\right)},$$

which states that under the optimal contract agents marginal rates of substitution across states are equalised (cf. Freixas and Rochet, 2008, p. 129).

on the borrower, provided that the latter is able to repay. Conversely, if the debtor defaults, the lender seizes control over the borrower's assets and thus absorbs the payoff risk.[18]

The situation is different if the borrower is risk neutral and the lender is risk averse. In this case $A_b = 0$ and thus the repayment does not vary with $y$. The repayment is then a constant $\overline{R}$ which can be determined from the (binding) constraint in (2.1.1):

$$\overline{R} = u_l^{-1}\left(U_l^0\right).$$

Hence, the optimal reimbursement depends only on the characteristics of the lender's utility function and the reservation utility $U_l^0$. Since the repayment is constant across all states, this contract can be interpreted as riskless debt.[19,20]

Yet, as Lacker (1991) argues, the assumption that borrowers are risk neutral and lenders are risk averse is highly implausible. He provides three arguments: first, assuming risk neutrality for a large part of the population is incompatible with the widespread prevalence of insurance contracts. Second, there is no compelling theoretical reason why lenders should be systematically more risk averse than borrowers. The final argument is related to the fact that risk neutral agents behave in the same way as risk averse agents that are exposed to *idiosyncratic risk*[21] in a complete capital market. Therefore the assumption of risk neutrality can serve as a shortcut for a more elaborate setting with risk aversion. However, it is more likely for lenders to have access to a complete capital market than it is for borrowers.[22]

Before we conclude this subsection let us offer some remarks on the interpretation of the presented contracting problem. In program (2.1.1) it was implicitly assumed that agents can trade claims contingent on any *state of the world* $s$, i.e. the agents can write a contract that specifies the reimbursement in a any state of nature and that state is observed by both parties. Accordingly, it is possible to recast the contracting problem in terms of the *Arrow Debreu* framework. Before this is explained, let us briefly turn to the notion of the latter. In the basic Arrow Debreu economy there are $L$ physical commodities and $S$ states, which constitute complete descriptions of uncertainty outcomes. Agents' contractual arrangements to trade resources over states are captured by *contingent commodities*, alternatively referred to as *contingent claims*. A contingent commodity is

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[18] Cf. Freixas and Rochet (2008, p. 129 f.)

[19] Cf. Lacker (1991, pp. 5f.)

[20] Note that the constant repayment result depends crucially on the assumption that the borrower does not enjoy limited liability protection, i.e. $y - R(y)$ is not restricted to be nonnegative. Conversely, if the constraint $y - R(y) \geq 0$ is added to problem (2.1.1)) it can be shown that the optimal contract entails a constant repayment if realisation $y$ is above a certain threshold and a repayment $R(y) = y$ otherwise (cf. Freixas and Rochet, 2008, pp. 157f.). This can be interpreted as a more typical debt contract (c.f. the subsequent section for a detailed definition of a standard debt contract).

[21] The concept of idiosyncratic risk is most easily illustrated within the framework of the Arrow-Debreu model without production. In that model risk is said to be idiosyncratic if the aggregate endowment is identical in each state. The converse holds true in the case of aggregate risk (cf. Mas-Colell et al., 1995, pp. 692f.).

[22] Cf. Lacker (1991, fn. 4).

an entitlement to receive one unit of physical commodity $l = 1, ..., L$ in a state $s = 1, ..., S$.[23]

An important concept in this context is that of *market completeness*. A model economy is said to exhibit market completeness if a market for every contingent commodity exists.[24] If this is the case, every contingent commodity $ls$ naturally has a price $p_{ls}$.[25] Under complete markets and the "standard assumptions"[26] of general equilibrium theory a competitive equilibrium in the Arrow Debreu model exists and the neoclassical welfare theorems extend to the case with uncertainty.[27] This implies that the gradient of agents' utility functions is equalised to the price vector. Now, if markets are complete, any security's payoff can be replicated by a complete set of contingent claims. What is more, in that case the price of a particular security can be calculated from contingent claim prices and the payoff vector of the security by the law of one price.[28] In reverse, the prices of contingent claims can be recovered if the dimension of the market span equals the number of states of nature.[29]

To conclude, this subsection has shown that under plausible assumptions about lenders' and borrowers' preferences in a complete markets setting with perfect information the optimal financial contract entails some degree of risk-sharing across all realisations of states of nature, which is incompatible with debt contracts. In the next section we will briefly review the theory of financial contracts under asymmetric information and show that debt contracts arise in such settings as optimal financial arrangements, even under more general assumptions about preferences.

### 2.1.2 Financing Contracts Under Asymmetric Information

This subsection presents arguments for the optimality of using a SDC[30] as a means of exchanging financial claims under asymmetric information.

**Definition 1.** Standard Debt Contract[31]

A standard debt contract is a contract which requires a fixed repayment when the debtor is solvent, requires the debtor to be declared bankrupt if this fixed payment cannot be met, and allows the creditor to recoup as much of the debt as possible from the debtor's assets in case of bankruptcy.

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[23] Cf. Mas-Colell et al. (1995, p. 688).

[24] If sequential trading with spot markets for physical commodities is allowed for, a single numéraire commodity, e.g. money, can be used to transfer wealth across states. This reduces the number of required markets from $L \times S$ to $S$ (cf. Mas-Colell et al., 1995, p. 695).

[25] Cf. Mas-Colell et al. (1995, p. 691).

[26] The "standard conditions" for the existence of a competitive equilibrium include, among others, the convexity and monotonicity of consumers' and firms' decision problems Mas-Colell et al. (cf. 1995, p. 579f).

[27] Cf. Radner (1982, p. 927).

[28] Cf. Lengwiler (2004, pp. 42 f.).

[29] Cf. Lengwiler (2004, p. 54).

[30] In the remainder of this thesis the terms "credit", "loan", and "standard debt contract" will be used interchangeably.

[31] Definition 1 is adapted from Gale and Hellwig (1985, p. 648).

According to this definition a crucial distinction to equity financing is that the repayment function under the SDC is state-dependent only when the debtor declares bankruptcy. That, is the SDC entails incomplete risk sharing.[32] The arguments for the optimality of the SDC can be divided into three categories relating to the verifiability of cash flows (or profits) generated by an externally financed investment project. Cash flows can be either (i) verifiable, (ii) semi-verifiable or (iii) nonverifiable by outside investors.[33] A common theme in each of the approaches is that the optimal contractual arrangement minimises expected agency costs. The last two possibilities rely on an *ex-post information asymmetry*, while the first is characterised by an *ex ante* asymmetry in the language of Choe (1998).[34] We will focus on settings with semi- and nonverifiable income since it is ex post information asymmetries that are usually assumed in the macro finance literature.[35] When adopting possibilities (ii) or (iii), one has to deal with an incomplete markets economy. In particular, these approaches entail a form of *endogenous market incompleteness*: contingent claim markets are missing since agents endogenously choose to trade claims that are not entirely contingent on future realisations.[36] An explanation as to why profits may not be directly observable or verifiable by outside investors is that the latter may find it difficult to determine if the debtor's expenses are necessary for the project's success or simply reflect the diversion of funds for private benefit.[37]

Semi-verifiable payoffs are an integral part of the *Costly State Verification* (CSV) literature initiated by Townsend (1979) and Gale and Hellwig (1985). In the latter model (as in the preceding subsection) an entrepreneur is endowed with an investment project but lacks the funds required to realise the project. Hence, investment must be financed externally at least in part. External finance is provided by investors which are not equipped with an investment technology. However, they can perfectly diversify their investments across entrepreneurs. Thus, they are able to raise deposits by offering a risk-free rate of return. The investment project generates random cash flows and has a positive expected net present value. It follows that it is efficient to implement the project, as in the model considered in the previous section. In addition, both entrepreneurs and investors are assumed to be risk neutral. Risk neutrality in combination with the assumption that investors have unlimited access to deposits at the risk free rate allows for focussing on a bilateral contracting problem between a representative entrepreneur and a representative investor.[38] Crucially, the CSV approach due to Gale and Hellwig (1985) postulates that the entrepreneur, which plays the role of a borrower, can costlessly observe the outcome of the investment projects and the investor cannot. As a consequence, borrowers have an incentive to underreport the proceeds of the investment project as long as the unequal distribution of information persists. However, financiers are able to overcome

[32] Cf. Tirole (2010, p. 132).
[33] Cf. Tirole (2010, p. 131).
[34] Cf. Choe (1998, pp. 237f.).
[35] For a setting with ex ante private information in which the SDC emerges as an optimal contract the reader is referred to Innes (cf. 1990, especially p. 47).
[36] Cf. Quadrini (2011, p. 214).
[37] Cf. Bolton and Scharfstein (1990, p. 95).
[38] Cf. Gale and Hellwig (1985, pp. 650f.).

the information asymmetry by paying *verification* or *audit cost*s, which take the form of deadweight losses.[39]

The contracting problem consists of (i) the objective function, which is the entrepreneurs' payoff, (ii) a participation constraint that guarantees the investor an expected repayment equal to the deposit rate, (iii) an *incentive compatibility constraint* that requires truthful reporting of project outcomes,[40] (iv) a limited liability condition, and (v) additional feasibility constraints.[41] A first step in proving the optimality of the SDC is to determine which classes of contracts are incentive compatible. By simple arguments it can be proven that a contract is incentive compatible if there exists a constant $\overline{R}$ that satisfies the following repayment scheme:

$$
\begin{aligned}
(a)\, \forall y \notin \mathcal{A} \quad & R(y) = \overline{R}, \\
(b)\, \forall y \in \mathcal{A} \quad & R(y) \leq \overline{R},
\end{aligned}
$$

in which $\mathcal{A}$ is the audit region. Hence, the repayment is equal to a constant $\overline{R}$, whenever the borrower is not audited. On the contrary, when the realised project outcome falls in the audit region the entrepreneur repays at most $\overline{R}$.[42] A contract that satisfies conditions (a) and (b) can be interpreted as a debt contract.

**Definition 2.** Debt Contract[43]

A debt contract is a contract which specifies a fixed repayment $\overline{R}$ if the project payoff is above a cut-off point and a repayment that is not higher than $\overline{R}$ otherwise.

Definition 2 refers to a broad set of debt contracts, including the SDC and risk-free debt with a constant reimbursement $\overline{R}$. Debt contracts are incentive-compatible, in the sense that nontruthful reporting is not in the borrower's interest, for the following reasons: if the true state is in the audit region, a false report $\widehat{y} \in \mathcal{A}$ would be detected. Announcing $\widehat{y} \notin \mathcal{A}$, on the other hand, would not alter the reimbursement. Conversely, if $y \in \mathcal{A}$, the borrower has no incentive to report a state that falls in the no-audit region since the reimbursement would be strictly less than or equal to $\overline{R}$.[44]

Incentive compatibility is a necessary but not sufficient condition for an optimal contract in the standard CSV setting. Proving that the standard debt contract is efficient in the sense of minimised auditing costs among the class of incentive-compatible contracts is more involved than proving incentive compatibility. Thus, the reader is referred to Gale and Hellwig (1985) for details.[45]

[39] Cf. Gale and Hellwig (1985, pp. 651f.)

[40] Imposing the incentive compatibility constraint in a CSV context can be justified by referring to the *revelation principle* (cf. Townsend, 1988, pp. 416ff.). This principle states that if one is to choose an optimal contract out of the set of all possible contracts, attention can be restricted to contracts that induce truthful reporting (cf. Bolton and Dewatripont, 2005, pp. 16f.).

[41] Cf. Gale and Hellwig (1985, p. 653).

[42] Cf. Freixas and Rochet (2008, p. 131).

[43] Adapted from Krasa and Villamil (1994, p. 168) and Freixas and Rochet (2008, p. 131).

[44] Cf. Freixas and Rochet (2008, p. 131).

[45] Cf. Gale and Hellwig (1985, pp. 655f.).

Intuitively, if the financier receives the maximum reimbursement in the audit region, it follows from his participation constraint (which holds with strict equality) that the constant repayment in the no-audit region can be reduced. This, however, reduces the probability of default and thereby expected deadweight losses in the form of audit costs.[46] Further, in the context of the optimal debt contract, audits can be interpreted as bankruptcy processes, in which the creditor takes stock of the defaulting debtor's remaining assets.[47] In the presence of costly state verification complete risk sharing between banks and entrepreneurs, e.g. in the form of an equity contract, is not optimal as this would necessitate permanent auditing, resulting in an excessive waste of resources.[48]

The optimality of the SDC in the CSV framework does not necessarily carry over to settings with alternative assumptions. For instance, Mookherjee and Png (1989) show that debt contracts are not efficient if stochastic audits are allowed for.[49] Yet, what matters most with regard to the model developed in this thesis are the assumptions about agents' attitudes towards risk. If entrepreneurs are risk averse and investors remain risk neutral, the optimal contractual arrangement does not resemble standard debt. Rather, risk-sharing considerations dictate that borrowers retain a strictly positive (and constant) amount of wealth in the bankruptcy region.[50] If both debtors and creditors are risk averse, the optimality result does not hold either. Then, as shown by Winton (1995), the repayment function in the audit region has a slope between zero and unity, depending on the relative degree of risk aversion, and corresponds to a an optimal risk-sharing rule. It should be added, however, that the optimal contracts in the two aforementioned examples still belong to the class of debt contracts since incentive compatibility conditions are not affected by the introduction of risk aversion.[51]

Diamond (1984) shows that in a setting in which borrowers have private information about their payoffs a contract resembling the SDC emerges if bankrupt borrowers are subject to *nonpecuniary penalties*.[52] These penalties act as a disciplinary device to induce managers to report truthfully. They can take the form of opportunity costs incurred by the manager that are related to the time spent in bankruptcy negotiations, the manager's loss of reputation, or search costs of a discharged manager.[53] Hellwig (2001) shows that the optimal contract in a setting with nonpecuniary penalties belongs to the class of debt contracts, but does not take the form of a SDC, if the borrower is risk averse.[54]

As mentioned, a second possibility is to assume that project payoffs are entirely unobservable or unverifiable by outside financiers. Bolton and Scharfstein (1990) develop a two-period model

[46] Cf. Bolton and Dewatripont (2005, p. 193 f.).
[47] Cf. Tirole (2010, p. 132).
[48] Cf. Bolton and Dewatripont (2005, p. 191).
[49] Cf. Mookherjee and Png (1989, especially pp. 412f.).
[50] Cf. Gale and Hellwig (1985, p. 661)
[51] Cf. Winton (1995, p. 99).
[52] Cf. Diamond (1984, Section 2).
[53] Cf. Diamond (1984, p. 396).
[54] Cf. Hellwig (2001, p. 417).

with two possible project outcomes that are realised at the end of each period. The firm requires external funding at the start of each period to realise the project. The impossibility to verify outcomes requires the adoption of an alternative sanctioning mechanism which deters borrowers from diverting funds. Financiers achieve this by threatening to cut refinancing at the start of the second period if the firm reports a bad payoff at the end of the first period.[55] The model implies an inefficiency: the project funding may be terminated if the entrepreneur (truthfully) reports the bad outcome even though the expected net return is positive and independent of the first period realisation. The optimal contract in this setting is a SDC as it entails a fixed payment if reports are above a certain threshold and a transfer of all project proceeds to the investor if an outcome below that threshold is reported.[56,57]

The optimality of the SDC can also be established in the context of adverse selection problems. Myers and Majluf (1984) examine a static setting in which a firm can raise outside finance only by issuing new shares. The management of the firm acts in the interest of old shareholders and has private information about the firm value. The firm faces a trade-off when issuing equity: on the one hand, it can raise funds which can be used to finance an otherwise infeasible investment project. On the other hand, new shareholders receive a claim on the firm's assets and income.[58] It can easily be shown that firms are more likely to refrain from issuing new shares if the value of their assets are high. Potential investors rationally anticipate this adverse selection. The decision to raise new equity thus signals "bad news" about the firm. This signal affects the price of new shares and, in turn, the equity issuance trade-off.[59] As a result, a firm may pass up an investment opportunity even if the latter has a positive net present value.[60] Myers and Majluf (1984) propose the issuance of a less informationally sensitive financing contract such as debt to alleviate this inefficiency, however without proving optimality of the SDC.[61] DeMarzo and Duffie (1999) assume this task, but depart from Myers and Majluf (1984) in two regards: first, the former authors allow for the design of general financial contracts. Second, they allow the firm to raise a variable amount of funding.[62] This implies the possibility of a separating equilibrium in which better issuers can signal their type through retaining a larger part of cash flows generated by the investment than inferior types. Under these conditions, the authors show that a contract which closely resembles the SDC is optimal if, roughly speaking, the issuer's private informational advantage over the market is relatively high.[63]

[55] Cf. Bolton and Scharfstein (1990, pp. 94ff.).
[56] Cf. Bolton and Scharfstein (1990, pp. 97f.)
[57] DeMarzo and Sannikov (2006) extend the threat of termination approach to an infinite-horizon setting in continuous time and find that the optimal contract can be implemented through a capital structure consisting of a mix of a credit line, perpetual debt and equity.
[58] Cf. Myers and Majluf (1984, pp. 189ff.)
[59] Cf. Myers and Majluf (1984, p. 203)
[60] Cf. Myers and Majluf (1984, p. 200).
[61] Cf. Myers and Majluf (1984, pp. 207f.).
[62] Cf. DeMarzo and Duffie (1999, pp. 68f.).
[63] Cf. DeMarzo and Duffie (1999, pp. 88f.).

## 2.2 Credit Frictions and Macroeconomic Fluctuations: Two Seminal Models

### 2.2.1 Key Concepts: Incomplete Markets and Heterogeneity

The *representative agent* paradigm is adopted extensively in mainstream macroeconomic models. If the considered economy admits a representative agent representation, the analysis is greatly simplified: then, the equilibrium price vector of the multi-agent economy is equal to the gradient of this single agent's utility function at his endowment vector.[64] Thus, it is sufficient to solve a single maximisation problem. Several approaches can be applied to justify this practice. Gorman (1953) and Rubinstein (1974) provide early proofs of the existence of a representative individual.[65] Yet, these approaches have two main limitations: first, they require strong restrictions on individuals' utility functions and second, they abstract from agent's exposure to idiosyncratic risk.[66,67] For that reason, the use of the representative agent in the context of idiosyncratic risk is often justified by appealing to the assumption of complete markets. As stated in Section 2.1.1, under complete markets the general equilibrium is Pareto efficient. Negishi (1960) showed that any Pareto efficient equilibrium can be replicated by a social planner who maximises a *social welfare function*.[68] That function is the value of a program in which a weighted sum of individual utility functions is maximised subject to the economy's resource constraint. The weights of individual utilities are equal to the reciprocal of the respective shadow prices of wealth in order to make the equilibrium allocations affordable for each individual.[69] Constantinides (1982) utilises the Negishi result to show that under complete markets a representative agent exists, even if idiosyncratic risk is allowed for. Importantly, the utility function of the representative individual is identical to the social welfare function.[70] However, to the extent that individual shadow prices of wealth depend on individual wealth levels, the choices of the representative individual and, accordingly, equilibrium prices will depend on the wealth distribution. A possibility to obtain demand aggregation is to impose the restrictions in Rubinstein (1974).[71] Another possibility is to assume linearity in preferences and

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[64] Cf. Lengwiler (2004, p. 33).
[65] Cf. Gorman (1953, p. 55) and Rubinstein (1974, Theorem 2).
[66] Cf. Guvenen (2011, p. 262).
[67] Gorman (cf. 1953, p. 53) considers a static setting without any risk. The Gorman form of preferences is satisfied by homothetic or quasi-linear utility functions (cf. Varian, 1992, p. 225). In addition, the marginal propensities to consume out of wealth must be identical (cf. Varian, 1992, pp. 153f.). In contrast, Rubinstein (cf. 1974, p. 226) allows for aggregate uncertainty. He identifies different sets of assumptions that allow for the construction of a representative agent (cf. Rubinstein, 1974, Theorem 2). A necessary condition in each of these sets is that each individual's utility function displays linear risk tolerance $T(c) \equiv -U'(c)/U''(c) = \rho + \gamma c$ with common parameter $\gamma$ (cf. Guvenen, 2011, pp. 261f.).
[68] Cf. Negishi (1960, Theorem 1)
[69] Cf. Lengwiler (2004, pp. 30ff.).
[70] Cf. Constantinides (1982, Lemma 1).
[71] Cf. Danthine and Donaldson (2014, p. 273).

technologies.[72]

If markets are incomplete, the planner solution and the decentralised equilibrium do not coincide in general.[73] In that case, the social welfare function cannot be used to construct a representative agent and, accordingly, the computation of equilibrium prices and allocations requires the solution of multiple optimisation problems of *heterogeneous* decision-makers.[74] A crucial implication is that those equilibrium objects typically depend on the distribution of wealth among individuals, as we will discuss in detail in the remainder of this thesis. Further, it is interesting to note that the Modigliani and Miller theorem, which states that the value of a firm is independent of its capital structure[75], does not hold if the economy exhibits market incompleteness.[76] Intuitively, if there is a complete set of contingent claims, shareholders can replicate any capital structure by holding a portfolio of those claims.[77] In the opposite case, investors are not in general able to do so and the Modigliani Miller theorem fails.

As a simplification, many business cycle models with financial frictions[78], including the two models that will be discussed momentarily, feature *sector-specific representative agents*, such as a representative household or a representative entrepreneur.[79] This, of course, presupposes that the assumptions of a particular aggregation theorem hold *within* each sector. For instance, it could be assumed that individuals within a certain group are not restricted to write contracts with each other, but are so with individuals from another group.[80] Compared with full-blown heterogeneous agents models[81] the sector-specific representative agent approach has the advantage that the dimension of the state space is drastically reduced. Whereas in the former approach the number of state variables is at least as high as the number of individuals in the economy, in the latter the dimension of the state space usually equals the number of representative individuals.[82,83] The obvious drawback is that the assumptions required for the validity of the chosen aggregation procedure have to be adopted.

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72 Cf. Quadrini (2011, p. 215).

73 An exception is the case of *quasi-completeness*. An economy is said to exhibit quasi completeness, if the market span (accidentally) includes the Pareto efficient allocation, even though the set of contingent commodities is not complete (cf. Lengwiler, 2004, p. 63).

74 Cf. Lengwiler (2004, pp. 60ff.).

75 Cf. Modigliani and Miller (1958, Proposition 1).

76 Cf. Huang and Litzenberger (1988, p. 128)

77 Cf. Sannikov (2013, p. 72).

78 An economy with financial frictions can be defined as an economy in which state contingent trade is restricted in the sense that one or more markets for contingent claims are missing (cf. Quadrini, 2011, p. 213).

79 Another possibility is to consider subsector-specifc representative agents. For instance, the economy of Eggertsson and Krugman (cf. 2012, p. 1480) is populated by representative patient and impatient consumers.

80 Cf. Isohätälä et al. (2016, p. 239).

81 The most well-known of this class of models include Huggett (1993), Aiyagari (1994), and Krusell and Smith (1998). Since this strand of literature is less concerned with the role of the financial sector in the propagation of exogenous shocks in a business cycle context, which is our focus here, the reader is referred to Heathcote et al. (2009) or Guvenen (2011) for in-depth surveys of that literature.

82 Cf. Isohätälä et al. (2016, p. 239).

83 The number of state variables can be further reduced by additional simplifying assumptions. This will be discussed in Chapters 3 and 4.

Market incompleteness has direct as well as indirect welfare implications. If markets for contingent claims are missing, individuals' marginal rates of substitution will typically not be equalised across all states of nature, which entails unexploited gains from trade.[84] If this is the case, the equilibrium will not be Pareto efficient.[85] More subtle effects arise from *pecuniary externalities*, which, in distinction from technological externalities, affect the well-being of individuals indirectly via price changes.[86] Pecuniary externalities do not lead to inefficiencies in the Arrow Debreu model since agents' marginal rates of substitution across states are equalised in equilibrium.[87] While the wealth distribution is distorted by changes in prices, the associated transfers exactly net out: buyers' losses in case of a price surge equal sellers' gains.[88] Thus, in a representative agent setting these transfers can be neglected.[89]

Again, matters are different if markets are incomplete. In that case, price changes can lead to first-order welfare consequences since marginal rates of substitution usually differ across individuals. If agents are price takers, these effects are not internalised.[90] As a consequence, the decentralised solution is typically not *constrained Pareto efficient*[91] as was first shown by Greenwald and Stiglitz (1986).[92] In addition, changes in the distribution of wealth induced by price variations have non-negligible repercussions in heterogeneous agents economies. This is because the redistribution of wealth leads to further price changes as certain sectors or groups of individuals can become undercapitalised, resulting in fire sales of goods or assets. Moreover, an *adverse feedback loop* may emerge since these price changes lead to yet another redistribution of wealth.[93]

It is interesting to note that Irving Fisher's (1933) debt deflation theory can be interpreted in terms of pecuniary externalities and fire sales.[94] In Fisher's story an initial modest exogenous shock leads to debt liquidation and soaring interest rates on unsafe debt. This causes asset prices to plummet and thereby forces more debt liquidation. The consequential drop in the money supply is accompanied by deflationary pressure, which drives up the real value of debt. The ensuing reduction in borrowers' net worth precipitates bankruptcies and growing pessimism. Eventually, adverse financial conditions spill over to the real sector and induce protracted slumps and unemployment.[95]

[84] Again, an exception is the case of quasicompleteness (cf. Lengwiler, 2004, pp. 63f.).
[85] Cf. Lengwiler (2004, p. 60).
[86] Cf. Mas-Colell et al. (1995, p. 352).
[87] Cf. Brunnermeier and Oehmke (2013, p. 1245).
[88] Cf. Greenwald and Stiglitz (1986, p. 230).
[89] Cf. Eden (2016, p. 215).
[90] Cf. Brunnermeier and Oehmke (2013, p. 1245).
[91] An equilibrium is said to be constrained Pareto efficient if a planner that faces the same constraints as the market cannot achieve Pareto-improvements relative to the decentralised solution (cf. Mas-Colell et al., 1995, p. 710).
[92] Cf. Greenwald and Stiglitz (1986, pp. 257f.).
[93] Cf. Geanakoplos (cf. 1990, p. 26).
[94] Cf. Shleifer and Vishny (cf. 2011, p. 41).
[95] Cf. Fisher (1933, pp. 341 ff.).

### 2.2.2 The Bernanke, Gertler and Gilchrist (1999) Model

A key concept in the BGG model is the *"financial accelerator"* which, in general terms, states that detrimental shocks to the economy are amplified by deteriorating conditions in the credit market.[96] BGG (1999) embed such a mechanism in an otherwise standard discrete-time Dynamic Stochastic General Equilibrium (DSGE) model of the New Keynesian (NK) Type.[97] Compared to earlier and more stylised macroeconomic models that studied the financial accelerator [98], this has two main advantages: first, it allows for making qualified quantitative predictions of the effects of credit frictions on the business cycle and second, it permits to study how these frictions and the conduct of monetary policy interact in determining real equilibrium outcomes.[99]

The BGG model is inhabited by three types of agents: households, entrepreneurs, and retailers. Households and entrepreneurs assume the roles of ultimate lenders and borrowers, respectively. Households are risk averse with logarithmic utility and solve standard consumption-savings and labour-leisure trade-offs. Firms, in contrast, are risk neutral, extremely myopic in the sense that they only plan one period ahead[100], and do not consume until they exit the economy which happens with a fixed exogenous probability. If hit by the retirement shock, they consume their entire wealth. This assumption serves to preclude the possibility that entrepreneurs do not rely on external finance in the long-run.[101] Entrepreneurs combine capital, their own labour input, and labour supplied by households to produce wholesale goods under perfect competition and constant returns to scale (CRS).[102] Retailers buy and subsequently differentiate the wholesale goods. The resulting final goods are sold on monopolistically competitive markets, further characterised by sticky prices.[103] Retailers' profits are passed on lumpsum to the household sector.[104]

The financial accelerator arises from a CSV problem between firms and financial intermediaries. Firms' demand for external funds derives from their desire to finance capital purchases that exceed the value of their own net worth. Entrepreneurs' capital returns are subject both to idiosyncratic as well as aggregate risk.[105] Despite the presence of uninsurable idiosyncratic risk the model can

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[96] Cf. Bernanke et al. (1996, p. 1).

[97] The NK literature introduces two main ingredients into the infinite-horizon optimising framework of New Classical DSGE models: monopolistic competition in goods and inputs markets and nominal rigidities. Under these assumptions monetary policy in general has real effects in the short-run (cf. Galí, 2008, p. 5). For detailed expositions the reader is referred to Galí (2008) or Woodford (2003).

[98] Among these models is Bernanke and Gertler (1989), who add the financial accelerator to a Real Business Cycle (RBC) model with overlapping generations. Another crucial distinguishing feature is that the contractual relationship between borrowers and lenders in their model does not resemble the SDC. This follows from the assumption that the imposed CSV problem allows for stochastic auditing (cf. Bernanke and Gertler, 1989, fn. 10).

[99] Cf. Bernanke, Gertler and Gilchrist (1999, p. 1345).

[100] Cf. Dmitriev and Hoddenbagh (2015, p. 2).

[101] Cf. Bernanke, Gertler and Gilchrist (1999, p. 1347).

[102] Cf. Bernanke, Gertler and Gilchrist (1999, p. 1356).

[103] Shifting price inertia and imperfect competition to a later stage of production facilitates aggregation (cf. Bernanke, Gertler and Gilchrist, 1999, p. 1348).

[104] Cf. Bernanke, Gertler and Gilchrist (1999, pp. 1347f.).

[105] Cf. Bernanke, Gertler and Gilchrist (1999, pp. 1347ff.).

be solved by deriving the decision rules of a representative entrepreneur. The reason is that the CRS assumption and the linearity of preferences is sufficient to achieve aggregation in the demand for capital, given the form of the contracting problem.[106] External finance is provided by financial intermediaries in the form of short-term debt.[107] These institutions play a limited role and simply equalise the expected return on credit to the deposit rate. In the model context the optimal contractual arrangement takes on the form of the SDC by the usual arguments as shown by Carlstrom et al. (2016).[108] BGG further assume that risk neutral entrepreneurs offer ultimate lenders, namely risk averse households, a predetermined return that is independent of aggregate innovations. Specifically, the *nondefault* repayment is made contingent on the realisation of the aggregate return to capital to compensate lenders for the variation in default risk. In effect, households are perfectly insured by entrepreneurs against aggregate risk.[109] Since diversification of loan portfolios allows intermediaries to remove all idiosyncratic risk associated with lending, they can guarantee households a risk-free rate of return on deposits.[110] In equilibrium, entrepreneurs internalise expected default costs, i.e. the expected costs of auditing.[111]

The financial friction drives a wedge between the return on capital and the risk-free rate. This wedge is referred to as the "*external finance premium*" (EFP) by the authors.[112] Importantly, the EFP can be shown to depend on borrowers' aggregate wealth level. This is due to the following mechanism. Ceteris paribus, a reduction in a single debtor's equity raises his probability of default. In the aggregate, banks react to the heightened credit risk by quoting higher loan rates. Conversely, increases in borrowers' wealth relative to their capital demand cause the EFP to fall.[113] Accordingly, a crucial state variable in the model is entrepreneurs' aggregate net worth. Its evolution over time depends on three sources: entrepreneurs' profits form investment projects, labour income, and consumption by retiring individuals.[114]

Investment adjustment costs are introduced to generate an additional source of variability in entrepreneurs' aggregate wealth.[115] More specifically, the authors assume a capital goods production function which is concave in the investment rate, i.e. the ratio of investment to capital. This feature

[106] Cf. Bernanke, Gertler and Gilchrist (1999, p. 1348).
[107] Cf. Bernanke, Gertler and Gilchrist (1999, pp. 1350f.).
[108] Cf. Carlstrom et al. (2016, p. 146).
[109] Cf. Bernanke, Gertler and Gilchrist (1999, p. 1352).
[110] Carlstrom et al. (cf. 2016, pp. 119f.) criticise BGGs' assumption that lenders' return is fixed. They show that the optimal repayment scheme instead depends on the realisation of aggregate shocks, which are costlessly observable by definition in contrast to idiosyncratic shocks. The authors (cf. 2016, Section III) demonstrate that allowing for state-dependent reimbursement largely diminishes the financial accelerator.
[111] Cf. Bernanke, Gertler and Gilchrist (1999, pp. 1352f.).
[112] Cf. Bernanke, Gertler and Gilchrist (1999, p. 1354).
[113] Cf. Bernanke, Gertler and Gilchrist (1999, pp. 1353ff.).
[114] Cf. Bernanke, Gertler and Gilchrist (1999, pp. 1349f.).
[115] Hayashi (cf. 1982, Sections 2 and 3) introduced investment adjustment costs into a neoclassical investment model. This type of costs might arise due to expenses accruing from training workers to operate the new capital or from installing the new machines (cf. Romer, 2012, p. 408). Hayashi (cf. 1982, pp. 221 ff.) also provides early empirical evidence that marginal adjustment costs are increasing in the rate of investment, a feature that is satisfied by function the capital goods production function in BGG.

is referred to as "technological illiquidity" by Brunnermeier et al. (2012) as a higher degree of concavity in the capital goods production function makes investment less reversible.[116] An immediate consequence is that the price of capital goods in units of output no longer equals unity.[117,118] In equilibrium the price of capital will vary with aggregate conditions and with it the aggregate level of entrepreneurial equity.[119]

Turning to model results and implications, the authors examine the effects of unanticipated innovations in the short-term interest rate set by the monetary authority, the technology parameter, and aggregate demand as well as the ramifications of an unanticipated lumpsum redistribution of wealth form households to the entrepreneurial sector. In each case the outcomes are compared to an otherwise identical model in which the financial accelerator is "turned off". In the presence of the financial friction the effect of a cut in the policy rate on output is shown to be roughly 50 % stronger, while the effect on investment is even twice as large. In addition, the persistence of policy-induced real effect is enlarged.[120] Amplification of the initial shock stems from the following effects: the reduction in the policy rate decreases lenders' opportunity costs and thereby leads to a contraction in the EFP. This drives up investment and the demand for capital, which in turn raises the price of the productive asset. The appreciation in the value of capital is associated with a pecuniary externality since it entails improvements in borrowers' balance sheets. These improvements are substantial due to the leverage effect and cause the EFP to fall further. The sequence of events is repeated and, in effect, a positive feedback loop of mutually reinforcing asset price and net worth variations emerges. The induced rise in entrepreneurs' equity is long-lasting and therefore the real effects persist.[121] The ramifications of the technology and the demand shock in the financial accelerator model compared to the reference model are quantitatively similar to the reduction in the short-term interest rate. Further, a redistribution of wealth which initially increases entrepreneurial wealth by one percent is demonstrated to lead to significant amplification. Specifically, real gross domestic product (GDP) is boosted at an annual rate of one percent.[122] The central mechanism in each of these cases is the same as in the case of the monetary policy shock: the increase in the demand for capital leads to asset price surges that increase the financial position of debtors and thereby drive down the EFP. To sum up, the system's reaction to exogenous shocks exhibits persistence and amplification, which can be attributed to the countercyclical behaviour of the EFP.[123]

[116] Cf. Brunnermeier et al. (2012, p. 12).
[117] Cf. Bernanke, Gertler and Gilchrist (1999, p. 1356).
[118] If there were no adjustment costs present, output could be transformed to capital via an one-to-one relation. In that case, output and capital would have the same price.
[119] Similarly, variations in the price of the collateralised asset play a crucial role in determining equilibrium dynamics in the KM model, which will be presented in the subsequent subsection. Further, variability in the capital price resulting from technological illiquidity is an important source of fluctuations in the Continuous-Time Macro Finance literature, as will be discussed in the subsequent chapter.
[120] Cf. Bernanke, Gertler and Gilchrist (1999, p. 1370).
[121] Cf. Bernanke, Gertler and Gilchrist (1999, p. 1370).
[122] Cf. Bernanke, Gertler and Gilchrist (1999, p. 1372).
[123] Cf. Bernanke, Gertler and Gilchrist (1999, p. 1345).

### 2.2.3 The Kiyotaki and Moore (1997) Model

Kiyotaki and Moore (1997) develop a stylised discrete-time RBC model with financial frictions in which "farmers" and "gatherers" trade the single productive asset, namely land. Land is assumed to be in fixed supply and produces fruit. Fruit acts as the unit of account and can be either consumed, lent out, or used to purchase land. Both types of agents have linear preferences. However, gatherers are assumed to be more patient than farmers, which implies that the latter borrow from the former in equilibrium. Further, it is presumed that production technologies are heterogeneous: farmers have a linear production function, while gatherers operate under decreasing returns to scale. A labour market is abstracted from.[124]

The key friction is a collateral constraint which restricts the amount of borrowing to not exceed the value of collateral, which is precisely the value of the farmer's land. Hence, land plays a dual role: it serves as an input to the production process as well as a collateral.[125] The collateral constraint results from the borrower's inability to commit to the repayment of debt, even if he has the funds available to do so. In turn, the inability to commit to reimbursement is based on two assumptions: first, growing fruits on a farmer's land requires the specific skill of that particular farmer. Second, the farmer has the ability to withdraw his labour input at any time. This setting opens up the possibility of the borrower holding-up the lender: the farmer may threaten to withdraw from the project unless the debt repayment is renegotiated in his favour. This threat is credible since the liquidation value, which is equal to the value of the farmer's land, is strictly less than the value of the project if carried out by the farmer.[126,127] Crucially, the debtor is assumed to have the ability to negotiate the repayment down to the liquidation value. Since the borrower anticipates the debtor's behaviour, he will only extend credit if it is secured by the amount of the liquidation value.[128,129]

To explore the workings of their model, Kiyotaki and Moore (1997) consider the consequences of an unanticipated negative productivity shock, which materialises while the economy is at its steady state.[130] This shock leads to a reduction in the current-period net worth of farmers and thereby to a reduction in the demand for land. For the land market to clear, the demand by gatherers has to rise. The necessary condition for this to happen is that the price of land falls.[131] This has direct as well as indirect repercussions for the current and future demand for land. The direct effect is the usual demand effect, which leads to heightened demand, ceteris paribus. Yet, this is overcompensated by two pecuniary externalities: the price decrease (i) further tightens the borrowing constraint, and

[124] Cf. Kiyotaki and Moore (1997, pp. 215f.).
[125] Cf. Kiyotaki and Moore (1997, p. 218).
[126] Cf. Kiyotaki and Moore (1997, pp. 217f.).
[127] Technically, this is due to the assumption that land cultivated in the current period produces fruit in the next period (cf. Kiyotaki and Moore, 1997, p. 216).
[128] Cf. Kiyotaki and Moore (1997, p. 217).
[129] The financial friction considered here also implies that a debt contract secured by collateral is the only reliable type of financial arrangement between borrowers and financiers (cf. Kiyotaki and Moore, 1997, fn. 8).
[130] Cf. Kiyotaki and Moore (1997, pp. 224f).
[131] Cf. Kiyotaki and Moore (1997, p. 212).

(ii) drives down borrowers' net worth levels. A dynamic amplification effect causes the demand for the productive asset to fall even more.[132] In fact, Kiyotaki and Moore (1997) show that it is the decreases in *future* land prices, which feed in into the current land price, that are the main drivers of the amplification mechanism.[133] Further, the effect on agents' land holdings is persistent since the decision of how much land to buy depends on the net worth accumulated in previous periods.[134] Turning to the real side of the economy, it is first important to note that the economy is characterised by a misallocation of land since farmers are credit constrained. This drives a wedge between the marginal products of farmers and gatherers, with the marginal product of the former exceeding that of the latter. Accordingly, farmers' fire sales of land that are induced by the detrimental productivity shock further exacerbate the misallocation of land.[135]

### 2.2.4 Selected Criticisms

Kocherlakota (2000) builds a model similar to KM, with the main difference being that farmers' production function includes capital as a second factor.[136] He finds that the degree of amplification, which is defined as the second-period deviation of output from its steady state level relative to the size of the exogenous shock[137], is sensitive on the income share of capital. In particular, for an empirically plausible value of 0.6, the price of land does not react much to an exogenous negative disturbance. Accordingly, amplification in that case is low.[138] This is the essence of the *"Kocherlakota critique"* as termed by Brunnermeier and Sannikov (2014a).[139] Cordoba and Ripoll (2004) assess the financial accelerator mechanism in KM under more standard assumptions about technologies and preferences. Specifically, they consider preferences with constant relative risk aversion and a Cobb-Douglas production technology.[140] They demonstrate that amplification is low or even nonexistent unless the elasticity of intertemporal substitution (EIS) in consumption is small and the share of capital, which acts as the collateralised asset in their model, is higher than usually assumed.[141] In a similar vein, Dmitriev and Hoddenbagh (2015) show that the ability of the BGG model to produce significant amplification hinges on the assumptions of passive monetary policy and extremely persistent technology shocks.[142]

What lies at the root of the low degree of amplification generated by these models under more general assumptions and plausible parameter constellations? To answer that question, it is important to recognise that the effects of exogenous shocks on endogenous variables are evaluated in the

[132] Cf. Kiyotaki and Moore (1997, p. 221).
[133] Cf. Kiyotaki and Moore (1997, pp. 227f.).
[134] Cf. Kiyotaki and Moore (1997, p. 226).
[135] Cf. Kiyotaki and Moore (1997, p. 226).
[136] Cf. Kocherlakota (2000, p.4).
[137] Cf. Kocherlakota (2000, p. 5).
[138] Cf. Kocherlakota (2000, p. 7).
[139] Brunnermeier and Sannikov (2014a, p. 406).
[140] Cf. Cordoba and Ripoll (2004, p. 1014).
[141] Cf. Cordoba and Ripoll (2004, pp. 1025f.).
[142] Cf. Dmitriev and Hoddenbagh (2015, p. 21).

two models by means of the same methodology as employed in most standard DSGE models, namely by conducting a *log-linearisation* around a *deterministic* steady state. Crucially, in order to calculate a deterministic steady state, one has to set any exogenous risk equal to zero.[143] Kocherlakota (2000) points out that under these circumstances agents act as if they would never be exposed to a shock, i.e. they do not rationally anticipate stochastic exogenous disturbances. He asserts that if agents instead were to anticipate shocks, the propagation mechanisms in the discussed models could potentially work quite differently.[144] This critique is laid out in more detail by Brunnermeier and Sannikov (2014a). They argue that in the mentioned models the adopted methodology implies that the economy reverts back to the steady state with certainty after an isolated and unanticipated shock has materialised. This has repercussions for the paths of asset prices: since the economy is on a path back to the steady state and agents do not anticipate additional shocks, asset prices are known to increase back to their initial level with certainty.[145] For instance, in the KM model unconstrained gatherers regard such situations as attractive investment opportunities. Their additional demand for land prevents the land price from falling to a higher extent. Even though the BGG model does not feature a second-best use of capital, related mechanisms are at work in their model.

The procedure to calculate log-linearised solutions around the deterministic steady state has a further shortcoming in the context of models with financial frictions. In general, that procedure is equivalent to a first-order perturbation method performed in logs.[146] Applying a perturbation method to a DSGE model produces an asymptotically correct solution only in the proximity of the deterministic steady state.[147] For instance, a first-order perturbation solution of the stochastic growth model, which forms the basis of conventional DSGE models, has been shown to deteriorate quickly as one moves away from the steady state by Aruoba et al. (2006).[148] Thus, the models discussed in the previous section generate accurate results in normal times, when the economy is close to its long-run equilibrium and volatility measures are low. However, episodes of financial instability that push the economy far away from the steady state are marked by severe nonlinearities as pointed out e.g. by Milne (2009).[149] Benes et al. (2014) argue that linearised business cycle models are unable to capture such phenomena, rendering these models inappropriate to study crisis events and the ensuing roles of monetary as well as macroprudential policy.[150]

[143] Cf. Brunnermeier and Sannikov (2014a, pp. 396f.).
[144] Cf. Kocherlakota (2000, p. 4).
[145] Cf. Brunnermeier and Sannikov (2014a, p. 406).
[146] Cf. Aruoba et al. (2006, p. 2479).
[147] Cf. Aruoba et al. (2006, p. 2484).
[148] Cf. Aruoba et al. (2006, especially p. 2493).
[149] Cf. Milne (2009, p. 614).
[150] Cf. Benes et al. (2014, pp. 12f.).

## 2.3 The Link Between Bank Credit and Inside Money

Credit extension has direct repercussions for the money supply. Before this mechanism is explained, let us first define some relevant concepts. Money is an asset that is generally accepted as a means of payment and serves as a medium of exchange.[151] Two categories of the money supply are *inside* and *outside money.*

**Definition 3.** Outside Money[152]

Outside money is money that is either of a fiat nature (unbacked) or backed by some asset that is not in zero net supply within the private sector of the economy.

**Definition 4.** Inside Money[153]

Inside money is an asset representing, or backed by, any form of private credit that circulates as a medium of exchange

According to the above definitions, a crucial distinguishing feature between both types of money relates to the concept of *zero net supply.* An asset is said to be in zero net supply if for each unit of that asset there is an offsetting liability.[154] Thus, outside money, unlike inside money, expresses the contribution of government institutions within the monetary system, namely the national treasury and the central bank, to the private sector's net wealth.[155] It includes fiat, i.e. intrinsically worthless, money such as notes or private banks' reserves held at the central bank, as well as commodity money, e.g. in the form of coins.[156] The terms "outside money" and "base money" are often used interchangeably[157], with the latter being a concept that measures the sum of banks' reserves held at the central bank and currency in circulation. However, if the original definition of outside money by Gurley and Shaw (1960)[158] is applied, the latter exceeds the former by the amount of private debt held by the central bank.[159] To clarify this, Lagos (2010) considers an open market operation by the central bank. If the monetary authority purchases bonds emitted by the private sector against newly created money, the latter qualifies as inside money. This is for two reasons: first, the new amount of money is backed by private debt and second, the aggregate private sector wealth is unchanged.[160] In fact, money backed by private debt was viewed by Gurley and Shaw (1960) as the *only* type of inside money.[161] However, contemporary definitions are different since those count *any* type of private sector debt that circulates as a medium of exchange

[151] Cf. B. Moore (2006, p. 198).
[152] Definition 3 is adopted from Lagos (2010, p. 132).
[153] Definition 4 is adopted from Lagos (2010, p. 132).
[154] Cf. Lengwiler (2004, p. 50).
[155] Cf. Brunner (1989, p. 175).
[156] Cf. B. Moore (2006, pp. 198f.).
[157] Cf. e.g. Tobin (1989, p. 37).
[158] Cf. Gurley and Shaw (1960, p. 73).
[159] Cf. Brunner (1989, p. 175).
[160] Cf. Lagos (2010, pp. 133f.).
[161] Cf. Gurley and Shaw (1960, p. 73).

(a) Before Loan Extension

| A | Bank | | L |
|---|---|---|---|
| Reserves | 20 | Equity | 20 |

(b) After Loan Extension

| A | Bank | | L |
|---|---|---|---|
| Reserves | 20 | Equity | 20 |
| Loan | 100 | Deposits | 100 |

Figure 2.3.1: Inside Money Creation by Means of Bank Lending

as inside money. Definition 4 is more general than either of the two aforementioned concepts as it encompasses both.[162]

With the relevant concepts at hand, we can now focus on the nexus of inside money, outside money, and credit. Figure 2.3.1 illustrates the implications of the granting of a bank loan with face value of 100 units for the stylised balance sheet of a single bank. Panel (a) depicts the bank's balance sheet before the loan is granted. The sole assets are reserves held at the central bank of value 20, which have been acquired from bond sales to the central bank. Reserves are entirely backed by equity by assumption. We further assume that the reserve ratio, i.e. the minimum required ratio of reserves to deposits set by the monetary authority, is as high as 20 percent.[163] The new loan increases the asset as well as the liabilities side by 100 units since the bank credits the borrower's deposit account by the face value of the loan (Panel (b)). Further, the reserve requirement is satisfied: the ratio of reserves to deposits is now exactly equal to 20 percent.

This simple example suggests that a single bank can create deposits *"ex nihilo"*[164], an ability that constitutes the paradigm of the *credit creation theory of banking*, as termed by Werner (2014a).[165,166] This leads us to the question of what distinguishes bank credit from lending by other institutions. To shed light on this issue, Werner (2014b) proposes a comparative accounting approach that disaggregates the lending processes of banks and private nonbanks, the latter of which could be either nonfinancial corporations or nonbank financial institutions, such as security broker-dealers.[167]

[162] Cf. Lagos (2010, pp. 134f.).

[163] This value is chosen for ease of the graphical representation. In practice, reserve ratios are lower. For instance, as of May 2018, the reserve requirement of the Federal Reserve System amounts to 10 percent (cf. Board of Governors of the Federal Reserve System, 2018b).

[164] Benes and Kumhof (2012, fn. 5).

[165] Werner (2014a, p. 2).

[166] Interestingly, the debate over banks' ability to create money commenced around the beginning of the 20th century and subsequently involved contributions by prominent economists such as Wicksell, Keynes, or Schumpeter. For enlightening overviews of the history of the credit creation and *fractional reserve* schools of thought, the latter of which will be discussed momentarily, the reader is referred to Werner (2014a, Section 2) and Werner (2016, Section 2). These references also discuss another competing theory, namely the modern *intermediation theory of banking*, which regards banks as no different from other intermediaries.

[167] Cf. Werner (2014b, p. 72).

In general, when a loan contract is signed, the lender purchases a promissory note emitted by the borrower against an obligation to deliver the loan principal.[168] Before the principal is paid out, this obligation appears on the liabilities side of a nonbank creditor in the form of "accounts payable". Once the disbursement takes place, the lender draws down the loan's face value from his cash vault or deposit account and erases the corresponding accounts payable. In effect, an asset swap occurs that leaves the balance sheet size unchanged.[169] In case of bank lending, the analogue to accounts payable is represented by deposits. Yet, in contrast to the former case, a bank does not transfer funds from other sources to make the payment, but, somewhat paradoxically, remains indebted to the borrower in terms of the money to be delivered.[170] Consequently, (and in accord with our example in Figure 2.3.1) the bank's balance sheet expands by the face value of the new loan.[171] Since the bank's new liabilities are part of monetary aggregates M1 and higher, it has created new money. This money qualifies as inside money since it (i) represents private debt and (ii) is accepted as a medium of exchange.[172]

While new inside money comes into existence by the extension of a bank loan, that same amount of money is destroyed, once the borrower repays his debt.[173] Even though in our example the new purchasing power was produced by means of credit extension, there are other sources as well. For instance, new deposits are also created when banks purchase securities from nonbanks.[174] More generally, new inside money is created by banks, whenever they lend to or purchase assets from households or nonbank corporations.[175] Werner (2014b) argues that banks' ability to create money and credit is facilitated by deposit regulation. He demonstrates this by the example of "*Client Money Rules*" that are imposed on lenders in the UK. These rules require nonbanks to keep any client money off their balance sheets. It follows that the latter remain the legal owners of their money. Banks, on the other hand, are exempt from such regulation.[176] This implies that bank depositors receive the legal status of creditors to the bank, which is the underlying reason for the expansion of a bank's balance sheet in case of a loan extension.[177]

Yet, our example neglects the fact that borrowers do not usually take out a loan in order to keep the principal in their deposit accounts, but rather use it to purchase goods, services, or assets

[168] Cf. Werner (2014b, p. 71).
[169] Cf. Werner (2014b, pp. 73f.).
[170] To be more precise, this debt is short-term, while the bank holds long term debt emitted by the borrower. Of course, at this stage the mutual claims exactly net out.
[171] Cf. Werner (2014b, p. 73).
[172] The years leading up to the recent financial crisis witnessed a dramatic surge in "*shadow money*", i.e. short-term claims emitted by shadow banks (cf. e.g. Pozsar et al., 2012, Section 2.2). As becomes clear from the preceding discussion, this expansion was made possible by increased bank lending (cf. also Pozsar, 2014, p. 33).
[173] Cf. McLeay et al. (2014, p. 16).
[174] Cf. McLeay et al. (2014, p. 15) for a more comprehensive list of sources of inside money creation and destruction.
[175] Cf. McLeay et al. (2014, p. 25).
[176] Accordingly, banks can be defined as financial institutions that (i) issue deposits and (ii) are exempt from deposit regulation that requires other financial institutions to keep client money separated from the balance sheets of the former.
[177] Cf. Werner (2014b, p. 75).

from a third party in the economy. In our example such outflow of funds produces a deficit of 80 in terms of reserves, provided that the third party holds its deposits at a different bank. This is because reserves represent the ultimate means of settlement for banks.[178] This fact is one of the reasons which lead proponents of the *fractional reserve theory of banking* to question the ability of a single bank to create money out of thin air. Rather, they maintained that any individual bank must acquire additional reserves, either through engaging in the central bank's open market operations or attracting new depositors from other banks, *before* they can create new deposits via loan granting.[179] In the above example this reasoning implies that the bank could only extend a loan worth of 20, i.e. the amount of excess reserves the bank owns. The third party's bank would then lend out $20 - 20 \times 0.2 = 16$ units, producing a corresponding amount of deposits in the process.[180] The resulting deposit outflow would induce another bank to lend more, which would translate into yet another increase in the inside money supply, namely of value $16 - 16 \times 0.2 = 12.8$. This "chain of deposits creation"[181] would halt once there are no excess reserves left in the banking system, which exactly would be the case if the supply of inside money has risen by 100 units. Fractional reserve theorists claim that it is this course of events that enables the banking system *as a whole* to create money, while an *individual* bank does not have the ability to do so.[182] Formally, the described mechanism is reflected by the *money multiplier* concept. At its core the money multiplier is just defined as the ratio of the money stock (usually taken to be the sum of currency and deposits) to base money. The magnitude of that ratio depends on factors such as the required reserve ratio and the public's desired holdings of currency relative to deposits. According to the traditional interpretation of the multiplier concept, the central bank governs money in circulation by adjusting the supply of base money. Any increase in base money is thus "multiplied up" into an even higher increase in money in circulation.[183]

That reasoning is correct provided that (i) reserves are a binding constraint in banks' lending activities, and (ii) the central bank fixes the supply of reserves.[184] Neither of these two conditions holds in reality. This is because monetary authorities typically do not fix the supply of outside money, but rather set a short-term policy rate. This implies that the money multiplier provides information about the amount of base money the central bank *has* to supply to achieve the amount of money in circulation that is consistent with the policy rate target. Thus, the true causality runs in the opposite direction, namely from money in circulation to base money.[185,186] A further

[178] Cf. McLeay et al. (2014, p. 21).
[179] Cf. Werner (2016, p. 372).
[180] Our example abstracts from the public's desire to hold currency.
[181] Samuelson (1948, p. 328, as cited by Werner, 2014b, p. 8).
[182] Cf. Werner (2014a, p. 2).
[183] Cf. Goodhart (2009, pp. 824f.).
[184] Cf. McLeay et al. (cf. 2014, p. 15).
[185] Cf. Goodhart (2009, p. 825).
[186] The literature has also produced empirical evidence on the direction of causality. Kydland and Prescott (cf. 1990, Table 4) found that monetary aggregate M1 weakly led and base money slightly lagged the business cycle in the years from 1954 to 1989. These results are in accordance with the credit creation theory, but cannot be reconciled

shortcoming of the fractional reserve theory is that the "chain of deposit creation" is fictitious. Any bank that faces an outflow of funds and lacks the corresponding amount of reserves has the option to acquire new reserves via the interbank market or the central bank's discount window. Such operations will simply result in a liabilities swap in the balance sheet of the loan-extending bank: the money owed to the depositor will then simply be replaced by a liability to another bank or the monetary authority of equal face value. Accordingly, the size of the balance sheet will remain constant.[187] Hence, the outflow of deposits does not directly hinder the capacity of a single bank to create money.[188]

It should be fairly obvious from the discussion of accounting principles, reserve requirements, and the conduct of monetary policy that the credit creation view is valid.[189] While this view has also been expressed in various statements by central bank officials[190], it has not yet transpired to mainstream academic macroeconomists.[191] Indeed, in most orthodox macroeconomic models that feature banks, these entities do not issue additional purchasing power.[192] Instead, banks are modelled as *intermediaries of loanable funds* between savers and end-borrowers. There are three main (and related) arguments why this view is highly problematic. First, it is misleading to describe a bank's business in terms of intermediation. In reality, when a bank grants a loan, it creates its own funding.[193] Hence, at this stage no intermediation takes place whatsoever.[194,195] Second, the intermediation theory also suggests that any new lending is initiated by an abstention from consumption on the part of savers. In actual fact, when an individual saves money in the form of deposits, these deposits cannot be used in transactions with a third party, which otherwise might have chosen to hold the proceeds in the form of inside money. Accordingly, additional saving of purchasing power does not provide banks with additional funds available for their lending business.[196] Third, since loanable funds (at least implicitly) take the form of *goods* in the respective models, banks essentially

with the fractional reserve story. Jakab and Kumhof (cf. 2015, fn. 10) interpret this evidence to demonstrate that the availability of central bank reserves did not even constrain banks' money creation activities in times when the Fed explicitly targeted monetary aggregates.

[187] Cf. Werner (2016, p. 372).

[188] It should be added that borrowing from other banks or the central bank may be associated with higher funding costs. Yet, this possibility does not impede the ability to produce inside money per se.

[189] Werner (cf. 2014a, especially pp. 15f.) has also provided empirical support for the credit creation story by documenting the actual credit extension process of a small German bank.

[190] Cf. the overview in Jakab and Kumhof (2015, pp. 6f.).

[191] There are a few exceptions. For instance, Benes and Kumhof (2012) and Jakab and Kumhof (2015) construct DSGE models that feature inside money creation by commercial banks. The latter authors (cf. 2015, pp. 29-33) compare a standard loanable funds DSGE model to a DSGE model with inside money creation and find that the former model severely underestimates the effects of financial sector shocks on the real economy. Since the mentioned papers are less concerned with the consequences of changes in the money supply on the price level and the ensuing business cycle implications, which is our focus in the later part of this thesis, these models are not presented in more detail.

[192] The BGG model belongs to this class of models (cf. Jakab and Kumhof, 2015, pp. 14f.).

[193] This funding is external since it is lent to banks by depositors.

[194] Cf. Jakab and Kumhof (2015, p. 3).

[195] After the borrower has spent the money, the deposits of another agent will have increased. Yet, this additional financial saving is a result rather than a cause of the new bank loan (cf. Jakab and Kumhof, 2015, p. 12).

[196] Cf. McLeay et al. (cf. 2014, p. 15).

are to be considered as barter rather than monetary institutions. Obviously, this interpretation is starkly at odds with the real world.[197]

Does the preceding discussion imply that banks are unconstrained in their money creation activities? This is not so: as Jakab and Kumhof (2015) argue, the amount of credit and money creation depends on the interaction of the portfolio optimisation problems of banks and their customers.[198] Hence, it is factors such as the demand for credit, the public's willingness to hold deposits, banks' attitudes towards risk, their refinancing costs, or their equity constraints imposed by macroprudential policy that limit banks' capacity to create money.[199,200] Constraints of these types represent critical ingredients in some of the models that will be discussed in the remainder of this thesis.

[197] Cf. Jakab and Kumhof (2015, p. 11).

[198] Cf. Jakab and Kumhof (2015, p. 8).

[199] The last of these example requires some qualification: Werner (cf. 2014b, p. 76) reports the case of the Barclays banks, which issued new shares in 2008 to avoid a partial takeover by the UK government. Remarkably, the additional equity was supplied by Gulf sovereign investors, who in turn obtained the required funds by borrowing money from Barclays. According to Werner (cf. 2014b, ibid.), regulators tolerated this practice, even though it violated company law.

[200] For a detailed discussion of banks' constraints on money creation the reader is referred to McLeay et al. (2014, pp. 17-21).

# Chapter 3

# The New Continuous-Time Macro Finance Literature

The previous chapter presented some shortcomings of linearised discrete-time business cycle models with financial frictions. Specifically, it was argued that these models do not generate significant amplification of exogenous shocks for plausible parameter values. Chapter 3 introduces the reader to the CTMF literature initiated by BS (2014a), which the model developed in this study is part of. Characteristically, this new area of research merges methods utilised in macroeconomics and finance to build general equilibrium models set in continuous time. Importantly, CTMF models do not merely represent reformulations of existing approaches to model financial frictions in continuous time, but explicitly address the shortcomings of more traditional models mentioned at the outset. Since the solution of CTMF models requires the application of tools that are nonstandard in macroeconomics, a substantial part of this chapter is devoted to methodological foundations. This part contributes to the literature by contrasting the necessary steps to find model solutions if aggregate risk is either governed by a *Brownian motion* or a *Poisson process*, which are two important continuous-time stochastic processes. Previous methodological introductions have exclusively focused on the Brownian case[201], which is also predominantly adopted in the literature. Here, we consider Poisson uncertainty as well since it represents a crucial ingredient in the model developed in this thesis.

Further, this chapter presents a selective review of the CTMF literature that focuses on models that are related to ours.[202] Among these are two versions of the I Theory of Money, BS (2016a) and BS (2014d), which form the basis of the model developed in Chapter 4.[203] This work shows that another theme from Chapter 2, namely the creation of inside money, can be introduced into the CTMF framework in a straightforward way. Indeed, variations in the amount of inside money are

[201] Cf. Isohätälä et al. (2016, Appendix A) and Brunnermeier and Sannikov (2016b, Section 3).
[202] This part of the chapter updates and expands the previous literature review by Isohätälä et al. (2016, Section 3).
[203] Cf. Isohätälä et al. (2016, Section 3).

shown to generate substantial fluctuations in the value of money and therefore represent a powerful amplification mechanism.

This chapter proceeds as follows. Section 3.1 explains the methodological foundations and assumptions of CTMF models. This section covers a discussion of typically adopted preferences and technologies and also offers some basic principles of stochastic calculus to familiarise the nonspecialist reader with these concepts. It also goes through the steps necessary to solve models from this new line of research, including a discussion of two distinct numerical methods that can be applied to solve the arising differential equations. Section 3.2 provides a literature overview, focusing on the seminal BS (2014a) model and the I Theory of Money. Finally, Section 3.3 contrasts the relative strengths and weaknesses of CTMF and linearised DSGE models.

## 3.1 Methodological Foundations and Assumptions

### 3.1.1 Model Set-Up

#### 3.1.1.1 Continuous-Time Stochastic Processes as the Fundamental Sources of Uncertainty

Characteristically, the Continuous-time Macro Finance (CTMF) literature postulates continuous-time stochastic processes as the fundamental sources of uncertainty. A formal definition of a stochastic process is provided in the following.

**Definition 5.** Stochastic Process[204]

A stochastic process $\underline{\mathbf{X}} = \{X(t), t \in T\}$ is a collection of random variables X(t).

A stochastic process is discrete in time, denoted by $t$, if the index set $T$ is a countable set and continuous in time if $T$ is a continuum.[205] Two specific continuous-time stochastic processes are of particular importance in the CTMF literature and this thesis: (i) Brownian motions and (ii) Poisson processes.[206]

[204] Definition 5 is adopted from Ross (1996, p. 41).
[205] Cf. Ross (1996, p. 41).
[206] The application of continuous-time stochastic processes to economic problems has a long tradition. This was initiated by Merton (1969) and Merton (1971), who analysed optimal portfolio and consumption choices under the assumption that asset prices follow Brownian motions and/or Poisson processes. Black and Scholes (1973) derived the famous Black-Scholes equation by adopting Brownian motions. Merton (1976) generalised the Black-Scholes formula by considering Poisson jump processes. Other well-known examples for uses of Poisson processes include growth models with product improvements ("quality ladders") such as Grossman and Helpman (1991) and Aghion and Howitt (1992). For a more comprehensive list of economic models involving continuous-time stochastic processes the reader is referred to Sennewald and Wälde (cf. 2006, p. 1).

**Definition 6.** Brownian Motion[207]

A stochastic process $\{Z(t), t \geq 0\}$ is a Brownian motion if: (i) $Z(0) = 0$, (ii) the process has stationary independent increments, and (iii) for every $t > 0$, $Z(t)$ is normally distributed with mean 0 and variance $\sigma^2 t$.

The first condition is just a normalisation of the process. Condition (ii) requires an increment $z(t_4) - z(t_3)$ to be independent of an increment $z(t_2) - z(t_1)$ for two disjoint intervals $[t_1, t_2]$ and $[t_3, t_4]$.[208] Hence, a Brownian motion obeys the Markov property. A random Variable $Z(t)$ is said to be Markov if its probability distribution at any future point in time does only depend on the current realisation, not on past ones.[209] Condition (iii) implies that the variance of a Brownian motion increases linearly with time. It follows that a Brownian motion is not stationary.[210,211] Usually, in the CTMF literature the evolution of the capital stock is assumed to be described by a geometric Brownian motion[212] with drift. For instance, let $k_{i,t}$ measure the capital stock of an individual $i$ at time $t$.[213] Then,

$$\frac{dk_{i,t}}{k_{i,t}} = \mu_{i,t}^{k}\, dt + \sigma\, dZ_t \tag{3.1.1}$$

is the stochastic law of motion of $k_{i,t}$ in the absence of capital purchases or sales.[214] Equation (3.1.1) is a stochastic differential equation (SDE) that consists of two terms: a deterministic drift rate term $\mu_{i,t}^{k}\, dt$, which can be interpreted as the trend component, and a stochastic volatility term $\sigma\, dZ_t$, which, according to the former interpretation, is the stochastic variation around the trend. The drift rate depends on investment in and deterministic depreciation of capital and will be specified momentarily. The stochastic component consists of the volatility rate $\sigma$ and increment $dZ_t$. It can be observed from (3.1.1) that subscript $i$ is absent from that increment. This is because $dZ_t$ is an *aggregate shock* that affects all individuals in the same way. The specific formulation of the equation of motion for capital implies that negative (positive) innovations due to the stochastic term can be interpreted as depreciation (appreciation) shocks.

An alternative to Brownian motions are Poisson processes. The latter belong to the class of counting processes, which count the number of specific events that have occurred until a point in time $t$.[215]

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207 Definition 6 is adopted from Ross (1996, p. 357).
208 Cf.Wälde (2011, p .226).
209 Cf. Hull (2006, p. 264).
210 Cf. Wälde (2011, p .226).
211 In addition, it can be shown that a Brownian motion is the continuous time limit of a symmetric random walk (cf. Ross, 1996, pp. 356f.).
212 The increments of a geometric Brownian motion are proportional to the absolute value of the dependent variable. In the "standard" formulation of a Brownian motion this is not the case.
213 For the sake of clarity, time-dependent *model* variables are denoted by subscript $t$ in the remainder, rather than writing variables as functions of time.
214 Cf. e.g. Brunnermeier and Sannikov (2014a, p. 384).
215 Cf. Ross (1996, pp. 59f.).

**Definition 7.** Poisson Process[216]

A stochastic process $\{\mathcal{N}(t), t \geq 0\}$ is a Poisson process with intensity $\lambda > 0$ (also referred to as the "arrival rate") if: (i) $\mathcal{N}(0) = 0$, (ii) the process has stationary independent increments, and (iii) the number of events in any interval of length $\Delta t$ is Poisson distributed with mean $\lambda \, \Delta t$. That is, for all t, $\Delta t \geq 0$

$$\Pr\{\mathcal{N}(t + \Delta t) - \mathcal{N}(t) = n\} = e^{-\lambda \Delta t} \frac{(\lambda \, \Delta t)^n}{n!}, \quad n = 0, 1, ... \tag{3.1.2}$$

In equation (3.1.2) increment $\mathcal{N}(t + \Delta t) - \mathcal{N}(t)$ counts the number of events $n$ during interval $\Delta t$. Letting $\Delta t \to 0$, one can establish that at most one event occurs during this very small interval. The increment is then written as $d\mathcal{N}(t)$ with approximated probability distribution

$$d\mathcal{N}(t) = \begin{cases} 0 & \text{with prob. } 1 - \lambda \, dt \\ 1 & \text{with prob. } \lambda \, dt, \end{cases} \tag{3.1.3}$$

which implies an expected value $\mathbb{E}_t d\mathcal{N}(t) = \lambda \, dt$.[217] Replacing the volatility term in (3.1.1) with a Poisson term gives

$$\frac{dk_{i,t}}{k_{i,t}} = \mu^k_{i,t} \, dt - \kappa \, d\mathcal{N}_t, \tag{3.1.4}$$

with

$$\kappa \equiv \frac{k_{i,t} - \widetilde{k}_{i,t}}{k_{i,t}} \tag{3.1.5}$$

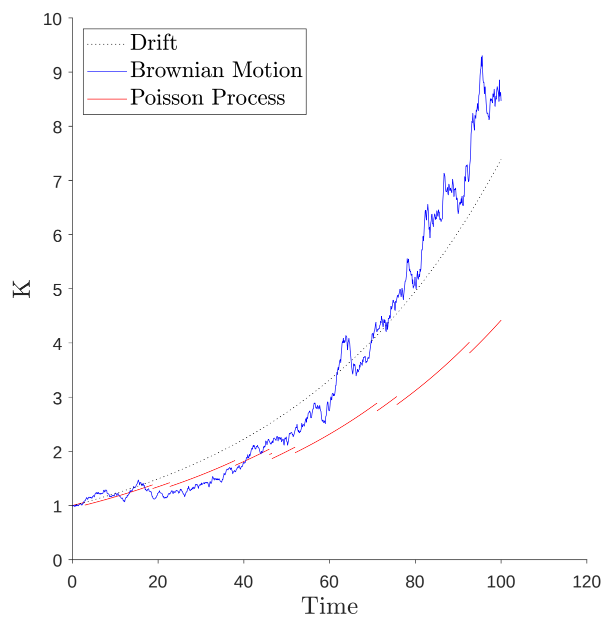
where $\kappa$ is the constant percentage depreciation of capital following a shock and $\widetilde{k}_{i,t} < k_{i,t}$ is the post-jump value of the capital stock.[218] Equation (3.1.4) can be interpreted as follows. Unless a jump is realised at time $t$, $d\mathcal{N}(t) = 0$ holds and the capital stock grows by its deterministic drift rate $\mu^k_{i,t}$. Conversely, if a jump occurs in a given period of time, i.e. $d\mathcal{N}(t) = 1$, the capital stock *immediately* decreases by $\kappa$ percent to its post-jump value $\widetilde{k}_{i,t}$. More specifically, the duration of the jump is equal to zero, i.e. $dt = 0$ during the jump. Thus, we can set terms that are proportionate to $dt$ equal to zero and arrive at[219]

$$\frac{dk_{i,t}}{k_{i,t}} = -\frac{k_{i,t} - \widetilde{k}_{i,t}}{k_{i,t}} = -\kappa, \quad \text{if } d\mathcal{N}_t = 1. \tag{3.1.6}$$

Afterwards, i.e. during the period ranging from point in time $t$ to $t + dt$, the capital stock grows at rate $\mu^k_{i,t}$ again. This behaviour is due to the *càdlàg* property of the Poisson process. That term is an abbreviation for the French expression "continue à gauche, limite à droite", which means "continuous from the right with left limits".[220]

[216] Definition 7 is slightly adapted from Ross (1996, pp. 59f.).
[217] Cf. Wälde (2011, p. 227).
[218] In the remainder, all variables with a tilde ($\tilde{}$) will denote post-jump values.
[219] Cf. Wälde (2011, pp. 229f.).
[220] Cf. Wälde (2011, p. 229).

Figure 3.1.1: Simulated Paths of a Brownian Motion and a Poisson Process




*Notes*: The figure shows simulated paths for the capital stock under specifications (3.1.1) and (3.1.4). Parameter values are set as follows: $\mu_i^k = 0.02$; $\sigma = 0.05$; $\lambda = 0.5$; $\kappa = 0.05$.

A Poisson process can be interpreted as a piecewise-deterministic process: jump timings occur randomly, but between two consecutive jumps, the respective variable evolves deterministically depending on the drift term.[221] Moreover, comparing definitions 6 and 7, it becomes clear that Poisson processes are better suited to model rare, discrete events, while a process driven by Brownian motions is to be preferred if the underlying variable is subject to continuous random movements. An advantage of employing Brownian motions is that the effect of varying jump sizes can be computed directly. In addition, the Brownian formulation in (3.1.1) allows for two-sided stochastic innovations, whereas the specification of Poisson process (3.1.4) is one-sided.[222] These distinctions are illustrated in Figure 3.1.1, which depicts simulated paths for the capital stock under specifications (3.1.1) and (3.1.4).

[221] Cf. Bayer and Wälde (2010, p. 12).

[222] In general, two-sided formulations of Poisson processes can also be constructed. Yet, in the context of CTMF models two-sided jumps that model aggregate risk are difficult to implement numerically (cf. the discussion in Subsection 3.1.2.4).

#### 3.1.1.2 Production Technologies

Typically, the literature employs a production function of the AK type:[223]

$$y_{i,t} = a_i k_{i,t}, \tag{3.1.7}$$

where $y_{i,t}$ is agent $i$'s output between dates $t$ and $t + dt$[224] and $a_i$ is his idiosyncratic capital productivity.[225] This production technology implies that productivity is deterministic, while the capital stock is stochastic. This is exactly opposite to the case in conventional DSGE models, in which productivity is stochastic and capital is deterministic. We will turn to an interpretation momentarily.

Before that, let us define the drift rate of the individual capital stock:

$$\mu^k_{i,t} \equiv \Phi(\iota_{i,t}) - \delta_i. \tag{3.1.8}$$

The right-hand side (RHS) of this expression contains the capital goods production function $\Phi(\iota_{i,t})$ with argument $\iota_{i,t}$, which is the internal gross investment rate, defined as the ratio of investment $i_{i,t}$ to capital $k_{i,t}$, and the constant, but possibly idiosyncratic, depreciation rate $\delta_i$. Hence, the drift term is equal to the net investment rate.[226] It is usually assumed that all agents that have the knowledge to produce capital goods are endowed with the same technology to do so. Output goods used for investment purposes $i_{i,t}$ are transformed to capital goods by means of technology $\Phi(\iota_{i,t})$. As in the BGG model, the capital goods production function is concave due to adjustment costs, i.e. $\Phi(\cdot)' > 0$ and $\Phi(\cdot)'' < 0$, in order to allow for a varying price of capital.[227] The degree of concavity determines the degree of technological illiquidity, as defined in Section 2.2.2. One specific function that has been used in the literature, e.g. by BS (2016a), is:

$$\Phi(\iota_{i,t}) = \frac{1}{\gamma}\log(\gamma\iota_{i,t} + 1) \tag{3.1.9}$$

in which $\gamma$ is the adjustment cost parameter. For $\gamma \to 0$ function $\Phi(\cdot)$ becomes nearly linear, i.e. no adjustment costs are present in this case.[228] An alternative is the quadratic adjustment cost function

$$\Phi(\iota_{i,t}) = \frac{1}{\gamma}\left(\sqrt{1 + 2\gamma\iota_{i,t}} - 1\right), \tag{3.1.10}$$

[223] Challenges that arise from considering a more general Cobb-Douglas production function are discussed in Section 7.2.1.

[224] As output is a stochastic flow variable, mathematical rigour would require us to refer to it by $dy_{i,t}$. However, in the literature stochastic flow variables such as consumption savings, production, or investment are usually written without operator $d$. We follow this lead. Further, increments are suppressed in supply and demand functions. Since ultimately market clearing conditions and relative rates of changes are of interest, this procedure is unproblematic.

[225] Cf. e.g. Brunnermeier and Sannikov (2014a, p. 384).

[226] Cf. e.g. Brunnermeier and Sannikov (2014a, p. 384).

[227] Cf. Brunnermeier and Sannikov (2014a, p. 384).

[228] Cf. Brunnermeier and Sannikov (2016a, p. 18).

which is utilised in BS (2014a).[229]

Another important question is yet to be answered at this point: how can the postulation of a stochastic process for capital be motivated? The answer depends on the definition of the capital stock and the adopted form of the production function. As mentioned, most of the models in this new strand of literature employ AK production functions, which traditionally assume a broad definition of capital including physical and human capital as well as intellectual property rights.[230] According to this interpretation adverse stochastic shocks to broad capital could be caused e.g. by unexpected "brain drains" or intellectual property rights becoming obsolete.

It is also possible to regard capital only in terms of physical units. Stochastic variations in this narrow capital stock are less common in the theoretical literature. One exception is Ambler and Paquet (1994), who explore the role of stochastic depreciations in an otherwise standard RBC model. They also provide some possible interpretations of the sources of random depreciations, two of which are (i) unusually strong weather conditions such as hurricanes destroying production facilities or very low temperatures deteriorating infrastructures such as roads, which in turn can harm commercial vehicles, and (ii) economic obsolescence of capital goods due to unexpected product innovations, e.g. the invention of new computer chips, which make older generations obsolete.[231] The problem with this interpretation is that the volatility of the depreciation rate is low: in the U.S. the standard deviation of the detrended component of the real annual depreciation rate was at a mere 0.17 percent during the period form 1946 to 2015.[232,233]

Brunnermeier and Sannikov (2014a) interpret the stochastic shocks to capital to reflect the fact that agents cannot instantly assess how effective the capital stock is. Rather, capital owners learn about its effectiveness over time.[234] This implies that $k_{i,t}$ stands for $i$'s capital in efficiency units, which is measured in expected future output. Di Tella (2017) formalises this notion by distinguishing between physical capital $k_{i,t}^{\text{phys.}}$ and effective capital $k_{i,t}$, which can be defined by[235]

$$k_{i,t} \equiv a_t k_{i,t}^{\text{phys.}}. \tag{3.1.11}$$

[229] Cf. Brunnermeier and Sannikov (2014a, p. 396).
[230] Cf. Aghion and Howitt (2009, p. 13).
[231] Cf. Ambler and Paquet (1994, p. 104).
[232] This figure is calculated from the fixed assets accounts assembled by the Bureau of Economic Analysis (BEA). In particular, the data in BEA (2016c) and (2016d) are used to arrive at the annual depreciation of capital in current dollars. In addition, BEA series (2016e) and (2016f) are utilised to compute a measure for the aggregate capital stock in current dollars. Afterwards these time series are deflated using the data in (2016b) and (2016a), respectively. Finally, the resulting series are detrended via the Hodrick–Prescott filter.
[233] Greenwood et al. (cf. 1997, p. 348) point to the fact that the figures typically reported in depreciation datasets (including the BEA dataset) are computed by using a straight-line depreciation method, in which the *amount* of depreciation is constant over the asset's life span, whereas most macroeconomic models (including the models presented in this and the subsequent chapters) assume a constant *rate* of depreciation. Ambler and Paquet (cf. 1994, p. 105) argue that this issue might cause the measured volatility of depreciation to be significantly underestimated.
[234] Cf. Brunnermeier and Sannikov (2014a, p. 385). They add that this interpretation of capital establishes a link to the literature on news-driven business cycles as e.g. in Jaimovich and Rebelo (2009).
[235] Cf. Di Tella (cf. 2017, fn. 7).

Inserting this into the production function (3.1.7) yields

$$y_{i,t} = a_i a_t k_{i,t}^{\text{phys.}}, \tag{3.1.12}$$

which shows that physical capital productivity is split into two multiplicative terms $a_i$ and $a_t$. The former term can be interpreted as the idiosyncratic fixed productivity component and the latter as the time-varying and stochastic aggregate component. In the remainder of this study this last interpretation of the capital stock will be employed.

#### 3.1.1.3 Derivation of Returns

Once fundamental risk is specified, a typical question in the literature is how the value of a portfolio position in capital evolves over time. That is, one is interested in the capital gains rate of a unit of capital valued at price $q_t$, which is measured in units of output. To answer that question, we must determine the evolution of product $q_t k_{i,t}$, which is the value of $i's$ capital holdings at time $t$. A complication that arises at this point is that capital as well as its price follow stochastic processes. If fundamental risk is described by a Brownian motion, a stochastic process

$$\frac{dq_t}{q_t} = \mu_t^q \, dt + \sigma_t^q \, dZ_t \tag{3.1.13}$$

can be postulated. In (3.1.13) $\mu_t^q$ is the drift rate of the capital price and $\sigma_t^q$ is its variance rate.[236] It is important to recognise that the stochastic variation in $q_t$ is proportional to $dZ_t$, which is the random increment in the equation of motion for capital given by (3.1.1). The proportionality is due to the fact that risk embodied in the capital stock represents the fundamental risk in the economy. If e.g. a strongly negative innovation is realised, agents that suffer balance sheet losses in the process might want to sell some portion of their capital stake subsequently, in order to reduce their risk exposure. Such transactions will lead to changes in the price of capital if the trades occur between agents who value capital differently.

In order to find an expression for the capital gains rate a so-called change-of-variables formula (CVF) has to be applied to product $q_t k_{i,t}$. Generally speaking, a CVF allows for deriving differentials of functions of stochastic processes.[237] The appropriate CVF for deriving differentials of functions that contain variables whose evolutions are described by Brownian motions is called *Itô's Lemma.*

**Lemma 1.** Itô's Lemma[238]

*Let there be $n$ stochastic processes $x_i$ and define the vector $\mathbf{x}(t)$ according to $\mathbf{x}(t) = (x_1(t), ..., x_n(t))^T$. Let the stochastic processes be described by $n$ SDEs, which contain $m$ Brownian motions each:*

[236] Cf. e.g. Brunnermeier and Sannikov (2014a, pp. 386f.).
[237] Cf. Wälde (2011, p. 232).
[238] Lemma 1 is adapted from Wälde (2011, pp. 234f.).

$$dx_i\left(t\right) = \alpha_i(\cdot)\,dt + \sigma_{i,1}(\cdot)\,dZ_1\left(t\right) + ... + \sigma_{i,m}(\cdot)\,dZ_m\left(t\right), \quad i = 1, ...n,$$

*in which* $(\cdot)$ *stands for* $(\mathbf{x}(t), t)$ *and* $\alpha_i$ *as well as* $\sigma_{i,1}, ..., \sigma_{i,m}$, $i, ..., n$, *are continuous and non-stochastic parameter functions. Consider further a function* $F(\cdot)$ *depending on* $\mathbf{x}(t)$ *and* $t$ *that is at least twice differentiable in* $\mathbf{x}(t)$ *and once in* $t$. *Then,*

$$dF(\cdot) = \frac{\partial F(\cdot)}{\partial t}\,dt + \sum_{i=1}^{n} \frac{\partial F(\cdot)}{\partial x_i}\,dx_i\left(t\right) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 F(\cdot)}{\partial x_i \partial x_j}\left(dx_i\left(t\right)\,dx_j\left(t\right)\right). \tag{3.1.14}$$

*The* $nm$ *possible combinations* $dx_i dx_j$ *in (3.1.14) are calculated by using*[239]

$$dt\,dt = dt\,dZ_i\left(t\right) = dZ_i\left(t\right)\,dt = 0 \text{ and } dZ_i\left(t\right)\,dZ_j\left(t\right) = \rho_{i,j}\,dt. \tag{3.1.15}$$

*where* $\rho_{i,j}$ *is the correlation coefficient between* $dZ_i$ *and* $dZ_j$.

Now, we can derive the capital gains rate of capital by applying Itô's Lemma to product $q_t k_{i,t}$:[240,241]

$$\frac{d\left(q_t k_{i,t}\right)}{q_t k_{i,t}} = \left(\mu_{i,t}^k + \mu_t^q + \sigma\sigma_t^q\right)dt + \left(\sigma + \sigma_t^q\right)\,dZ_t, \tag{3.1.16}$$

in which we have made use of the fact that the correlation coefficient between $dZ_{i,t}$ and $dZ_{j,t}$ is equal to unity for $i = j$ and thus $dZ_{i,t}\,dZ_{j,t} = dt$. Uncertainty in the capital gains rate is due to the second term on the RHS of (3.1.16), which consists of fundamental or *exogenous risk* $\sigma\,dZ_t$ and *endogenous risk* $\sigma_t^q\,dZ_t$. The former risk component is exogenous since parameter $\sigma$ is exogenously specified as a constant. In contrast, the latter term is endogenous as the volatility rate of the capital price $\sigma_t^q$ is determined endogenously in equilibrium.[242] This volatility rate will typically vary over time, just as the drift rate $\mu_t^q$, which will also be determined in general equilibrium.

Capital gains rates of different assets can be used to obtain returns, which in turn can be employed for deriving stochastic budget constraints. In addition to the capital gains rate, the

[239] The intuition for these "rules" can be described as follows. The formula in (3.1.14) is derived from a Taylor expansion of function $F(\cdot)$ w.r.t. $\mathbf{x}$ and $t$. This Taylor series contains, among others, differences such as $\Delta x_i^2$, $\Delta x_i \Delta t$ and $\Delta x_i \Delta x_j$. A discretised version of a Brownian motion is $\Delta x_i = \sum_{j=1}^{m} \sigma_{i,j}(\cdot)\Delta Z_j$, where $\Delta Z_j = \epsilon_j\sqrt{\Delta t}$. In the last equality $\epsilon_j$ is a random variable that obeys the standard normal distribution. In the limit $\Delta t \to 0$ any terms that are of a order higher than $dt$ can be neglected in the series since they approach zero at a faster rate than $dt$. From this it becomes clear that terms involving $dZ_i\,dt$ are of order $dt^{3/2}$ and thus can be dropped. On the other hand, terms proportional to $dZ_i^2$ and $dZ_i\,dZ_j$ are of order $dt$ and thus must be kept (cf. Hull, 2006, pp. 607f.).

[240] For this purpose we set $n = 2$ and $m = 1$ in equation (3.1.14) and let $F = F(q_t, k_{i,t}) = q_t k_{i,t}$.

[241] It might be surprising that term $\sigma\sigma_t^q$ appears in the drift rate of (3.1.16). This fact arises due to a "Jensen's inequality effect" (cf. Cochrane, 2005, p. 495). To gain a deeper intuition, first note that the drift term is the expected value of the increment in any Brownian motion with drift. Here, the cross partial derivative of function $F = q_t k_{i,t}$ is positive. It follows that the expected value of this function $\mathbb{E}_t\left[q_t k_{i,t}\right]$ is higher than the product of expected values of $q_t$ and $k_{i,t}$, which is given by $\mathbb{E}_t\left[q_t\right]\mathbb{E}_t\left[k_{i,t}\right]$. Hence, the effect on the mean of the increment would be neglected if term $\sigma\sigma_t^q$ was left out.

[242] Cf. Brunnermeier and Sannikov (2014a, p. 387).

return of a given asset also includes the payoff yield.[243] This component is due to the deterministic income after internal investment per unit of time generated by holding one unit of the asset relative to the asset's price. Since output usually represents the numéraire in CTMF models, the income from holding an additional unit of capital is the marginal product of capital. It follows that $i$'s return on capital $dr^K_{i,t}$ is given by[244,245]

$$dr^K_{i,t} = \underbrace{\frac{a_i - \iota_{i,t}}{q_t}\,dt}_{\text{payoff yield}} + \underbrace{\left(\Phi(\iota_{i,t}) - \delta_i + \mu^q_t + \sigma\sigma^q_t\right)dt + \left(\sigma + \sigma^q_t\right)\,dZ_t}_{\text{capital gains rate}}. \tag{3.1.18}$$

It should be added that the return on capital is independent of the agent's capital stock. This is a direct consequence of the implicit CRS assumption embodied in the AK production function.

If uncertainty is due to a Poisson process rather than a Brownian motion, the evolution of $q_t$ is to be modelled according to

$$\frac{dq_t}{q_t} = \mu^q_t\,dt + \frac{\widetilde{q}_t - q_t}{q_t}\,d\mathcal{N}_t, \tag{3.1.19}$$

in which $\widetilde{q}_t$ is the post-jump value of the capital price and, analogously to equation (3.1.13), the stochastic variation is proportional to increment $d\mathcal{N}_t$. The appropriate CVF in this case is stated in the following lemma.[246]

**Lemma 2.** *A CVF for Poisson Processes*[247]

*Let there be $n$ stochastic processes $x_i(t)$ and define the vector $\mathbf{x}(t)$ according to $\mathbf{x}(t) = (x_1(t), ..., x_n(t))^T$. Let the stochastic processes be described by $n$ SDEs, which contain $m$ jump terms each:*

$$dx_i(t) = \alpha_i(\cdot)\,dt + \beta_{i,1}(\cdot)\,d\mathcal{N}_1(t) + ... + \beta_{i,m}(\cdot)\,d\mathcal{N}_m(t), \quad i = 1, ..., n,$$

*where $(\cdot)$ stands for $(\mathbf{x}(t), t)$ and $\alpha_i$ and $\beta_{i,1}, ..., \beta_{i,m}$, $i, ..., n$, are continuous and nonstochastic parameter functions. Then, for an at least once continuously differentiable function $F(\cdot)$, the differential*

[243] In the literature, the payoff yield is referred to as the "dividend yield" (cf. e.g. Brunnermeier and Sannikov, 2014a, p. 387). The latter notion might create the impression that financing costs are subtracted when returns are calculated. As this is not the case, we use the former notion.

[244] Cf. e.g. Brunnermeier and Sannikov (2014a, p. 387).

[245] Note that variable $r^K_{i,t}$ measures the cumulative return in time interval $[0, t]$, while $dr^K_{i,t}$ denotes the *instantaneous* return between dates $t$ and $dt$. The instantaneous return is dimensionless. This can be easily demonstrated by assuming that $dr^K_{i,t}$ only contains the payoff yield:

$$\left[dr^K_{i,t}\right] = \frac{[a_i] - [\iota_{i,t}]}{[q_t]}\,[dt] \Leftrightarrow \left[dr^K_{i,t}\right] = \frac{[\text{output/capital/time}] - [\text{output/capital/time}]}{[\text{output/capital}]}\,[\text{time}] = [1], \tag{3.1.17}$$

where square brackets denote units. In contrast, the return per unit of time (e.g. one year) $dr^K_{i,t}/dt$ has the familiar dimension [1/time].

[246] Formal proofs of several CVFs for Poisson processes are provided in Sennewald (2007). As these proofs are very technical, the reader is referred to this article for details. Sennewald and Wälde (cf. 2006, p. 8) discuss why methods based on simply taking the total derivative of function $F$ (as e.g. in Dixit and Pindyck (1994, p. 85)) are inappropriate.

[247] Lemma 2 is adapted from *Wälde (2011, p. 238).*

*of this function is*

$$dF(\cdot) = \left\{ \frac{\partial F(\cdot)}{\partial t} + \sum_{i=1}^{n} \frac{\partial F(\cdot)}{\partial x_i} \alpha_i(\cdot) \right\} dt + \sum_{j=1}^{m} \left\{ F\left(\mathbf{x}(\cdot) + \boldsymbol{\beta}_j(\cdot), t\right) - F\left(\mathbf{x}(t), t\right) \right\} d\mathcal{N}_m(t), \quad (3.1.20)$$

*in which* $\boldsymbol{\beta}_j(\cdot)$ *is the n-dimensional vector function* $(\beta_{1,j}(\cdot), ..., \beta_{n,j}(\cdot))^T$.

Using (3.1.20) in combination with (3.1.4) and (3.1.19), we get for the capital gains rate:[248]

$$\frac{d(q_t k_{i,t})}{q_t k_{i,t}} = \left\{ \Phi(\iota_{i,t}) - \delta_i + \mu_t^q \right\} dt + \frac{\widetilde{q}_t (1-\kappa) - q_t}{q_t} d\mathcal{N}_t. \quad (3.1.21)$$

Here, exogenous risk is captured by parameter $\kappa$ and endogenous risk by the post-jump value $\widetilde{q}_t$ (relative to the pre-jump value $q_t$). It follows from the above equation that agent $i$'s return on capital is

$$dr_{i,t}^K = \underbrace{\frac{a_i - \iota_{i,t}}{q_t} dt}_{\text{payoff yield}} + \underbrace{\left\{ \Phi(\iota_{i,t}) - \delta_i + \mu_t^q \right\} dt + \frac{\widetilde{q}_t (1-\kappa) - q_t}{q_t} d\mathcal{N}_t}_{\text{capital gains rate}}. \quad (3.1.22)$$

#### 3.1.1.4 Optimisation Problem and Preferences

Once expressions for returns are derived, a combined rational expectations[249] consumption-savings and portfolio selection problem in continuous time in the spirit of Merton (1969) and (1971) must be solved for each agent.[250] In the simplest case the agent decides on the allocation of wealth between capital and risk-free debt, which trades at a constant real interest rate. This leads to a stochastic flow budget constraint of the form

$$\frac{dn_{i,t}}{n_{i,t}} = x_{i,t} dr_{i,t}^K + (1 - x_{i,t}) r_t \, dt + \frac{c_{i,t}}{n_{i,t}}, \quad (3.1.23)$$

where $n_{i,t}$ is $i$'s wealth, $x_{i,t}$ and $(1 - x_{i,t})$ are the portfolio weights on capital and the risk-free asset, respectively, $r_t \, dt$ is the deterministic instantaneous risk-free rate[251] and $c_{i,t}$ is $i$'s consumption.[252] The portfolio weight on capital is restricted to be nonnegative. This prevents agents form holding short positions in capital.[253] On the contrary, the portfolio weight on the risk-free asset may either be positive or negative. A negative weight $(1 - x_{i,t})$ implies that the agent borrows the risk-free

[248] Similarly to above, we set $n = 2$ and $m = 1$ in equation (3.1.20) and let $F = F(q_t, k_{i,t}) = q_t k_{i,t}$.

[249] Rational expectations in our context refer to the assumption that agents are informed about the model-consistent parameter functions and probability spaces of each stochastic process, including price processes.

[250] A crucial difference to those standard portfolio selection models is that processes for asset prices are not exogenous, but rather determined in general equilibrium (cf. Brunnermeier and Sannikov, 2016a, p. 8).

[251] As the instantaneous risk-free rate is deterministic, it is denoted by $r_t \, dt$ rather than $dr_t$.

[252] Cf. e.g. Brunnermeier and Sannikov (2014a, p. 387).

[253] Cf. Brunnermeier and Sannikov (2016b, p. 1516).

asset in order to increase his capital stock. Agent $i$ chooses portfolio weights, consumption and the investment rate in order to maximise expected lifetime utility discounted at rate $\rho_i$. When choosing portfolios and consumption streams, each agent takes prices of goods and assets as given since individuals are assumed to be atomistic.[254] Then, the optimisation problem is

$$\max_{x_{i,t},c_{i,t},\iota_{i,t}} \mathbb{E}_0\left[\int_0^\infty e^{-\rho_i t} u\left(c_{i,t}\right)\, dt\right], \tag{3.1.24}$$

subject to constraint (3.1.23) and condition $n_{i,t} \geq 0$, which is the solvency constraint. In (3.1.24) $u(\cdot)$ is the instantaneous utility function, which belongs to the constant relative risk aversion (CRRA) class. Specific types of CRRA preferences which have been employed in the literature are logarithmic, linear and power utility functions.[255] CRRA utility functions, in turn, are a subset of the hyperbolic absolute risk aversion (HARA) class.[256] A crucial advantage of using HARA functions is that they lead to consumption and asset demand functions that are linear in the agent's wealth level provided that returns are independent of wealth, as was shown by Merton (cf. 1971, p. 391).[257] Merton's (1971) linearity result applies to the setting in this section since the CRS property of production function (3.1.7) ensures that the return on capital is independent of capital and thereby wealth. Linear demand functions are desirable as they considerably simplify aggregation.

### 3.1.2 Solution Procedure

#### 3.1.2.1 Derivation of Decision Rules

The derivation of the optimal investment rate $\iota^*_{i,t}$ is straightforward as it entails an entirely static and deterministic problem: $\iota^*_{i,t}$ simply maximises the drift rate of the return on capital:[258]

$$\iota^*_{i,t} = \arg\max_{\iota_{i,t}} \frac{a_i - \iota_{i,t}}{q_t} + \mu^q_t + \Phi(\iota_{i,t}) - \delta_i. \tag{3.1.25}$$

Thus, the investment rate is chosen according to the following rule:

$$\frac{d\Phi(\iota^*_t)}{d\iota_t} = \frac{1}{q_t}, \tag{3.1.26}$$

where subscript $i$ is absent since the optimal investment rate choice does not depend on individual characteristics. As $\Phi(\iota_t)$ is a concave function of $\iota_t$, the optimal investment rate increases with the

[254] Cf. Brunnermeier and Sannikov (2014a, p. 389).
[255] An exception is Di Tella (cf. 2017, p. 2045), who assumes recursive Epstein-Zin preferences.
[256] The "HARA" notion was introduced by Merton (cf. 1971, pp. 388f.). Formally, a utility function is said to be of the HARA type if the reciprocal of the coefficient of absolute risk aversion is an affine function of wealth: $1/A(n_{i,t}) = a + bn_{i,t}$, where $a$ and $b$ are some constant coefficients (cf. Lengwiler, 2004, p. 92).
[257] Cass and Stiglitz (cf. 1970, p. 149) were the first to show that CRRA utility functions lead to linear asset demands.
[258] Maximising the drift of the return on capital is equivalent to maximising the instantaneous return on capital as the stochastic part of the latter does not depend on the investment rate.

price of capital $q_t$.[259] Condition (3.1.26) can be interpreted if it is rearranged to $q_t = d\iota_t/d\Phi(\iota_t^*)$. This shows that agents choose $\iota_t^*$ such that the market value of a unit of capital is equalised to its replacement value. This is a result in accord with Tobin's q-Theory of Investment.[260] Optimal values $\iota_t^*$ and $\Phi(\iota_t^*)$ can be explicitly solved for if a specific capital production technology is adopted. If $\Phi(\cdot)$ takes on the form implied by (3.1.9), we get

$$\iota_t^* = \frac{1}{\gamma}\left(q_t - 1\right) \quad \text{and} \quad \Phi(\iota_t^*) = \frac{1}{\gamma}\log q_t. \tag{3.1.27}$$

If $\Phi(\cdot)$ is instead determined by (3.1.10), we have

$$\iota_t^* = \frac{1}{2\gamma}\left(q_t^2 - 1\right) \quad \text{and} \quad \Phi(\iota_t^*) = \frac{1}{\gamma}\left(q_t - 1\right). \tag{3.1.28}$$

The Problem of maximising (3.1.24) subject to (3.1.23) and the solvency constraint with respect to $x_{i,t}$ and $c_{i,t}$ is more complex and requires the application of dynamic optimisation techniques. As the suitable method depends on the structure of the problem, we do not describe specific methods here[261], but rather provide general optimal portfolio rules in the case which is best suited for our expositional purpose, namely that of risk averse borrowers with logarithmic utility and risk neutral, financially unconstrained lenders.[262] The assumptions about lenders imply that the risk-free rate is constant.[263] Under Brownian uncertainty the optimal portfolio weight on capital is given by a function

$$x_{i,t}^* = x\left(dr_i^K\left(q_t, \mu_t^q, \sigma_t^q, \mathbf{z}_{i,t}\right), r\,dt\right), \tag{3.1.29}$$

where the general function $dr_{i,t}^K = dr_i^K(\cdot)$ is used, $\mathbf{z}_{i,t}$ is a vector of parameters and other variables included in the equation for the return on capital, and the time subscript of the risk-free rate is dropped. If uncertainty is due to a Poisson process, we instead have

$$x_{i,t}^* = x\left(dr_i^K\left(q_t, \mu_t^q, \widetilde{q}_t, \mathbf{z}_{i,t}\right), r\,dt\right). \tag{3.1.30}$$

#### 3.1.2.2 The State Variable and Characterisation of Equilibrium

Typically, in CTMF models the state space is reduced to dimension one. Yet, at this point, the model contains infinitely many state variables. This is because there are infinitely many agents with different characteristics and each agent's actions depend on his net worth level (cf. the discussion in Subsection 2.2.1). In addition, the aggregate capital stock enters the set of model equations,

[259] Cf. Brunnermeier and Sannikov (2014a, p. 390).
[260] Cf. Tobin (1969, p. 21) and Brunnermeier and Sannikov (2014a, p. 390).
[261] Brunnermeier and Sannikov (cf. 2016b, Section 3.1) explain optimal portfolio choice for cases in which the underlying uncertainty stems from Brownian motions. The case of Poisson uncertainty is extensively discussed in Section 4.3 of this thesis.
[262] This is a departure from the setup in Brunnermeier and Sannikov (2014a, p. 385), who assume that borrowers are risk neutral and subject to a solvency as well as nonnegative consumption constraint.
[263] The economic intuition for this result will be discussed in Section 3.2.1.

e.g. through the clearing condition in the capital market. The reduction of the state space is achieved by three simplifying assumptions. First, attention is restricted to the wealth dynamics of two distinct groups of agents, which we denote by “borrowers” and “lenders” here. That is, the sector-specific representative agents approach introduced in Subsection 2.2.1 is adopted. Borrowers have some advantage over lenders, which is related to one or more aspects of their respective technologies. For instance, it could be assumed that borrowers are more productive in generating final goods than lenders. Alternatively, it might be assumed that borrowers are able to eliminate any idiosyncratic risk associated with holding capital by diversifying their investments and lenders are not.[264] The mentioned restriction is facilitated by either assuming that all members of each group are identical or that the within-group heterogeneity does not matter for aggregate outcomes.[265] As in the models discussed in Section 2.2, this implies that it is sufficient to analyse the behaviour of sector-specific representative agents. In effect, the model now contains three state variables: the respective aggregate wealth levels of lenders and borrowers, and the aggregate capital stock. The dimension of the state space can be further reduced by adopting production technologies of the AK type and HARA preferences. As mentioned, the first assumption leads to asset returns that are independent of wealth.[266] The second assumption implies that consumption and asset demand functions are linear in wealth. Both together imply that (i) the aggregate capital stock $K_t$ can be removed from all relevant model equations and (ii) that wealth *shares* rather than levels can be considered. This is demonstrated by reformulating the clearing condition in the capital market. To this end let us first define the aggregate wealth in the economy $N_t$:[267,268]

$$N_t \equiv q_t K_t. \tag{3.1.31}$$

It should be noted that the risk-free asset is in zero net supply. For this reason the risk-free asset does not enter (3.1.31).

The capital market clearing condition can be obtained by aggregating the value of individual capital demands and equalising the result to the value of the aggregate capital stock:

$$\int_{\mathbb{I}} x^*_{i,t} n_{i,t}\, di + \int_{\mathbb{J}} x^*_{j,t} n_{j,t}\, dj = q_t K_t, \tag{3.1.32}$$

[264] Idiosyncratic risk is not included in the equations of motion for capital in equations (3.1.1) and (3.1.4), but is introduced in Section 3.2.2.
[265] Cf. Isohätälä et al. (2016, p. 239).
[266] Returns to capital are also independent of wealth under more general CRS technologies. However, if production technologies obey the CRS property, but are not of the AK type, the capital stock cannot be removed from all model equations in general. This is demonstrated in Section 7.2.1 for the case of a Cobb-Douglas production function in which labour and capital are the inputs.
[267] Cf. Brunnermeier and Sannikov (2016b, p. 1504).
[268] In the remainder all variables at the highest aggregation, i.e. economy-wide, level are denoted by capital letters and agents’ individual choice and state variables by small letters.

where borrowers are index by $i \in \mathbb{I}$ and lenders by $j \in \mathbb{J}$.[269] It is important to realise that the products within the integrals on the left-hand side (LHS) of (3.1.32) measure agents' *real* capital demands as net worth is measured in units of output. Due to the mentioned assumptions, agents' portfolio weights within each group are identical, i.e. we have $\overline{x}_t = x^*_{i,t}, \forall i$ for borrowers' common optimal portfolio weight and $\underline{x}_t = x^*_{j,t}, \forall j$ for lenders' portfolio weight $\underline{x}_t$. It follows that the portfolio weights can be pulled out of the integrals in (3.1.32). Dividing the resulting expression through aggregate wealth $q_t K_t$ then gives:

$$\overline{x}_t \frac{\int_{\mathbb{I}} n_{i,t}\, di}{q_t K_t} + \underline{x}_t \frac{\int_{\mathbb{J}} n_{j,t}\, dj}{q_t K_t} = 1. \tag{3.1.33}$$

The final step in the reduction of the state space is to define

$$\eta_t \equiv \frac{\int_{\mathbb{I}} n_{i,t}\, di}{q_t K_t} \quad \text{and} \quad 1 - \eta_t \equiv \frac{\int_{\mathbb{J}} n_{j,t}\, dj}{q_t K_t}, \tag{3.1.34}$$

where $\eta_t$ and $1 - \eta_t$ are borrowers' and lenders' shares in aggregate wealth, respectively.[270] As a result, we are left with a single state variable $\eta_t$, which measures the distribution of wealth between borrowers and lenders. Since the model economy is characterised by financial frictions, that distribution matters for aggregate outcomes and a balance sheet channel emerges.

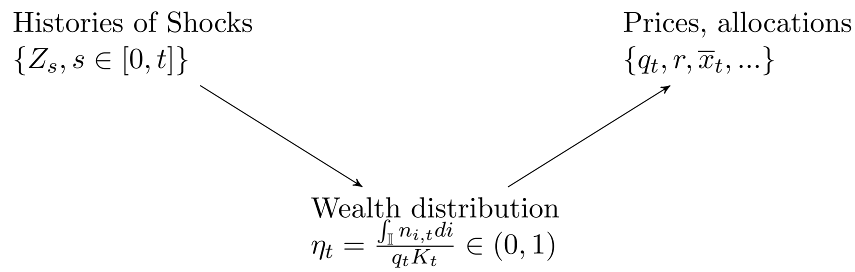
The independence of the model equations from the aggregate capital stock is referred to by Brunnermeier and Sannikov (2014a) as the *scale-invariance property*. This property implies that two or more economies that differ only with respect to their initial endowments of assets and the realisations of stochastic innovations will obey the same equilibrium processes for key endogenous variables such as the price of capital.[271] The next step is to postulate a SDE for the state variable. Under Brownian uncertainty we have[272]

$$\frac{d\eta_t}{\eta_t} = \mu^\eta_t\, dt + \sigma^\eta_t\, dZ_t. \tag{3.1.35}$$

Expressions for parameter functions $\mu^\eta_t$ and $\sigma^\eta_t$ can be obtained from the application of Itô's Lemma. Both variables are determined endogenously in equilibrium. Drift rate $\mu^\eta_t$ depends on the relative consumption and portfolio choices of borrowers and lenders in addition to the deterministic parts of asset returns. Volatility rate $\sigma^\eta_t$ depends on portfolio choices and the stochastic part of the return on capital.[273]

[269] Cf. Brunnermeier and Sannikov (2014a, p. 389).
[270] Cf. Brunnermeier and Sannikov (2014a, p. 393).
[271] Cf. Brunnermeier and Sannikov (2014a, p. 393).
[272] The remainder of this section equally applies to the case of Poisson uncertainty, with the exceptions that post-jump terms have to be considered instead of volatility terms and that the Brownian term $Z_t$ has to be replaced by term $\mathcal{N}_t$.
[273] Cf. Brunnermeier and Sannikov (2014a, pp. 393f.).

Figure 3.1.2: Equilibrium as a Map From Histories of Shocks to Prices and Allocations.




*Notes*: Slightly adapted from Brunnermeier and Sannikov (2014e).

We can now state the definition of equilibrium.

**Definition 8.** *CTMF Markov Equilibrium*[274]

*Given any initial allocation of capital among the agents, an equilibrium is a map from histories of shocks* $\{Z_s, s \in [0,t]\}$ *to price* $q_t$*, drift rate* $\mu_t^q$*, volatility rate* $\sigma_t^q$*, risk-free return* $r_t$*, portfolio weights* $\{\overline{x}_t, \underline{x}_t\}$ *, investment rate* $\iota_t$*, and consumption levels* $\{\overline{c}_t, \underline{c}_t\}$ *such that*

(i) *the markets for capital, final goods, and the risk-free asset clear,*

(ii) *all agents choose portfolios and consumption levels to maximise utility,*

(iii) *and agents satisfy their budget constraints.*

In order to derive the equilibrium, the histories of shocks are first mapped into $\eta_t$ through the equation of motion for the state variable given by (3.1.35). This yields a specific value for $\eta_t$ at a point in time $t$. Afterwards the resulting time series for the state variable is mapped into prices and allocations via the set of equilibrium equations. This step completes the equilibrium. As a result, all equilibrium processes can be expressed as functions of the current value of the state variable. The equilibrium concept is illustrated in Figure 3.1.2.

The equilibrium as defined in Definition 8 belongs to the class of *Markov equilibria.*[275] Those equilibria are characterised by the fact that optimal choices as well as equilibrium prices in the current period depend only on current values of the state variables and not on their histories.[276] The Markovian structure of the model originates from our previous assumption which states that the fundamental uncertainty in the economy is described by a Markov processes.

Crucially, the stated equilibrium concept is distinct from the steady state concept: a *stochastic steady state* is defined as an equilibrium state $\eta_t = \eta^{\text{ss}}$, in which the drift rate of the state variable

[274] Definition 8 is adapted from Brunnermeier and Sannikov (2014d, p. 11) and Brunnermeier and Sannikov (2016a, p. 13).
[275] Cf. Brunnermeier and Sannikov (2014a, p. 394).
[276] Cf. Ljungqvist and Sargent (2004, p. 198).

is zero, i.e. where $\mu_t^\eta = 0$. This point is where the system returns to, at least in the long-run, after it is hit by an exogenous shock.[277] According to Definition 8, the economy is in equilibrium at the stochastic steady state, but is so at any point away from the steady state $\eta_t \neq \eta^{\text{ss}}$ as well, provided that the requirements in the above definition are satisfied. The adjective “stochastic” refers to the fact that the steady state is *not* derived by setting the exogenous shocks to zero, as is done in the computation of deterministic steady states of traditional DSGE models.[278]

#### 3.1.2.3 Derivation of Differential Equations for the Price Function and Discretisation

The challenge arising in the solution of CTMF models is that the number of endogenous variables is larger than the number of model equations. This is due to the presence of endogenous variables $\mu_t^q$, $\sigma_t^q$, or $\widetilde{q}_t$, i.e. the parameter functions of the stochastic processes for asset prices. While optimal consumption and portfolio choices can be derived from first-order conditions (FOCs)[279] and current price levels from market clearing conditions, additional equations for the aforementioned variables are not available. However, the indeterminacy of the system can be resolved by deriving additional differential equations through either Itô’s Lemma or CVF (3.1.20) and thereby closing the model.

The first step is to postulate a general function $q(\eta_t)$, which depends on time only indirectly through the state variable. In the Brownian case the second step is to express terms $\mu_t^q$ and $\sigma_t^q$ as functions of $q(\eta_t)$ via the application of Itô’s Lemma.[280] This results in

$$\mu_t^q q_t = \frac{dq(\eta_t)}{d\eta_t}\mu_t^\eta \eta_t + \frac{d^2 q(\eta_t)}{d\eta_t^2}\frac{(\sigma_t^\eta \eta_t)^2}{2} \tag{3.1.36}$$

and

$$\sigma_t^q q(\eta_t) = \frac{dq(\eta_t)}{d\eta_t}\sigma_t^\eta \eta_t. \tag{3.1.37}$$

By imposing these differential equations, the problem of solving the model essentially boils down to a problem of finding the unknown function $q(\eta_t)$, i.e. one has to solve a *functional equation* problem.[281,282] Both of the above equations are *ordinary differential equations* (ODEs) since function $q(\cdot)$ depends only on one argument, namely $\eta_t$. Further, (3.1.36) is an ODE of *second order* as the highest derivative is of order two, while (3.1.37) is an ODE of *first order*.[283] The ODEs are then

[277] Cf. Brunnermeier and Sannikov (2014a, p. 380).
[278] Cf. Brunnermeier and Sannikov (2014a, pp. 396f.).
[279] If lenders are risk neutral and financially unconstrained, their FOCs pin down the risk-free rate as well as the expected return to capital, provided that they hold this asset (cf. Brunnermeier and Sannikov, 2014a, p. 390).
[280] Cf. Brunnermeier and Sannikov (2014a, pp. 394f.).
[281] Cf. Judd (1998, p. 335).
[282] As Brunnermeier and Sannikov (cf. 2016b, pp. 1520 f.) point out, the solution procedure is very similar to that in option pricing models. For instance, in the classic Black-Scholes model the option price is obtained by deriving a SDE for the option price via Itô’s Lemma. In that SDE the price of the underlying asset is an independent variable (cf. Black and Scholes, 1973, p. 643). Here, the pricing formula is a function of the wealth distribution rather than a function of the underlying price.
[283] If individuals from *both* groups have logarithmic preferences, it is sufficient to solve the first order ODE (3.1.37)

solved numerically via the *finite-difference method*. A crucial feature of this class of methods is the discretisation of the independent variable.[284] Accordingly, one has to define a grid for the state variable with grid points $m = 1, 2, ..., M$. Given an initial value for $\eta_m$, which we set to $\eta_0 = 0$, the values of the independent variable $\eta_m$ on the grid can be calculated from the following difference equation:

$$\eta_{m+1} = \eta_m + h_\eta, \tag{3.1.38}$$

where the distance between neighbouring grid points $h_\eta$ is some small constant. The step size $h_\eta$ is chosen such that the value of the state variable at grid point $M$ equals unity, i.e. $\eta_M = 1$. Thus, through this procedure $M$ values for entrepreneurs' wealth share in the range $[0, 1]$ are obtained.[285,286] The next question is how the derivatives in (3.1.36) and (3.1.37) should be approximated. The most basic finite-difference method is the Euler method. A scheme which converges faster to the true solution than the Euler method, in the sense that less grid points are necessary to achieve a given degree of accuracy, is the *Runge-Kutta method* (RK). Both methods are described in Appendix A.

It is important to reiterate at this stage that function $q(\eta_t)$ depends on time only indirectly through the state variable. It is thus a time-invariant function. When solved, the indeterminedness of the system of model equations is eliminated and all endogenous variables can be calculated. One can then capture the evolution of $\eta_t$ by SDE (4.5.7a) and map any value of $\eta_t$ into function $q(\eta_t)$. Thus, that procedure corresponds exactly to our definition of equilibrium (cf. Definition 8). Of course, this reasoning also applies to any other nonscalable endogenous variable: each of these variables is a function of the state variable and depends on time only via the state variable. In the numerical solution of CTMF models the calculation of equilibrium processes is achieved in two stages. At the first, the price function is solved for from the relevant differential equations. At the second stage, shocks are mapped into the process for the state variable via a given Monte Carlo method. It follows that index $t$ of each variable can be replaced by index $m$ in the numerical solution of the ODEs.

System (3.1.36)-(3.1.37) could be iterated forward easily if two initial conditions $q(\eta_1) = q_0$ and $q'(\eta_1) = q'_0$ were available. The number of required initial conditions is two since the highest order of the derivative with respect to the state variable is two. The problem of solving a differential equation or a system of differential equations subject to some initial conditions is referred to as an *initial value problem* (IVP) in the numerical literature on differential equations.[287] While the initial

(cf. Brunnermeier and Sannikov, 2014a, Section III.C).

[284] Cf. Judd (1998, p. 335).

[285] It should be noted at this point that we abstract from "zombie" agents, who continue operations with negative equity.

[286] Note that a discretisation of the state space, rather than a time discretisation, is performed here: one simply attributes $M$ values to the state variable in a range where the minimum value $\eta^{\min} = 0$ corresponds to the case in which borrowers do not own any wealth and the maximum $\eta^{\max} = 1$ to the other extreme in which these individuals own the entire aggregate wealth.

[287] Cf. Judd (1998, p. 335).

price in CTMF models can usually be calculated from so-called "autarky" solutions at grid point $m = 1$, where $\eta_m = 0$,[288] this does not hold true for the initial derivative. Before we discuss two methods that are suitable for solving the ODEs, let us turn to the Poisson uncertainty case.

If capital follows a Poisson process, the strategy is to obtain a differential equation from the drift rate of the price of capital. Applying CVF (3.1.20) to function $q(\eta_t)$ yields

$$\mu_t^q q(\eta_t) = \frac{dq(\eta_t)}{d\eta_t} \mu_t^\eta \eta_t. \tag{3.1.39}$$

A crucial difference to (3.1.37) is that differential equation (3.1.39) depends on post-jump values of $q_t$ through term $\mu_t^q$. This can be seen from equation (3.1.30), which implicitly determines $\mu_t^q$. Following an adverse shock, the net worth share of leveraged agents typically falls. Hence, the state variable drops to $\widetilde{\eta}_t < \eta_t$, or, in the discretised case, to $\widetilde{\eta}_m < \eta_m$, where $\widetilde{\eta}_m$ is the post-jump wealth share of borrowers if the jump occurs at state $\eta_m$. As $\widetilde{\eta}_m < \eta_m$, borrowers' post-jump wealth share can alternatively be written as $\eta_k$, where $k$ is a prior grid point satisfying $k < m$. Of course, $\eta_k$ is associated with a specific value of function $q(\cdot)$. This value is the post jump-level of the capital price: $\widetilde{q}_m = q(\eta_k)$.[289] A differential equation whose derivative depends on the value of the unknown function at prior grid points is said to be *a functional differential equation of the retarded type* (RDE).[290]

RDE (3.1.39) can be solved by the Euler method.[291] However, the retarded structure of the differential equation for the price of capital complicates the numerical solution of the model compared to the Brownian case. One of these difficulties arises from the fact that the solution of a first-order RDE such as (3.1.39) requires the specification of some *initial history*. Since the derivative at the grid point where the integration of the differential equation is started - denote this point by $\check{m}$ - depends on prior values of the unknown function, an initial condition at a specific value of the independent value is not sufficient.[292] If e.g. a given RDE involves a constant "delay" $m - k$, it is necessary to assign $k + 1$ values to the unknown function at grid points $m = 1, ..., k + 1$.[293] These values constitute the initial history. If the initial history is specified, the RDE can be iterated

[288] These autarky solutions can be obtained without solving differential equations at first as they entail only one group of active agents (see Section for details). This allows for setting terms such as $\mu_t^q$, $\sigma_t^q$ or $\widetilde{q}_t$ to zero. Autarky solutions are attractive because they can offer intuition for the more complex dynamic models. In some cases, it is even possible to derive tractable analytical expression for the equilibrium values of model variables, which can further enhance intuition.

[289] Cf. Brunnermeier and Sannikov (2014d, p. 35).

[290] Cf. *Posch and Trimborn (2013, p. 2606).*

[291] One drawback of using Poisson uncertainty is that the Runge-Kutta method is difficult to implement due to the retarded form of the differential equation for $q_t$. While MATLAB provides various optimised solvers for RDEs, none of these allows for accessing prior values of the sought function, which are required to find the allocation of capital, during the iteration. Thus, the most appropriate procedure is to use the Euler method with a large number of grid points.

[292] Cf. Richard (2003, p. 1669).

[293] If e.g. only $k$ values of $\theta(\eta_{m+1})$ were specified, the numerical scheme would try to access grid point $m = 0$, which is clearly not defined.

forward, starting at node $\check{m} = k + 1$.[294] In our context $m - k$ is not constant since the absolute drop in the state variable resulting from a shock to capital might differ depending on the current value of the state variable. Thus, it is not a priori clear how to choose the number of elements in the initial history vector. Yet, it is typically assumed that agents do not default. It follows that the drop in net worth can never be sufficiently strong to push agents into bankruptcy. Thus, $\eta_1 = 0$ is the fixed lower limit of the state space and the number of grid points in the predetermined range from node 1 to $\check{m}$ only matters for the precision of the approximation. Despite this difference to the Brownian case, the challenge in solving difference equation (3.1.39) is similar: a sufficient initial history is not available.

#### 3.1.2.4 Two Methods for Solving the Differential Equations

In what follows, we will discuss two methods for solving differential equation (3.1.37). We focus on the case of Brownian uncertainty as the application of the two methods in order to solve (3.1.39) is analogous in most parts. In addition, the solution of a RDE is discussed in detail in Section C.4. The first method utilises the fact that some properties of the price function can usually be deducted from economic theory.[295] These properties typically concern multiple values on the domain. Thus, the problem at hand can be characterised as a *boundary value problem* (BVP).[296] A simple finite-difference method for solving BVPs is the *shooting method*.[297] The essence of this method is the transformation of a BVP to an IVP.[298] In terms of our problem from the previous section the method requires an initial guess for the first derivative at the initial grid point: $q'(\eta_1) = q'_{\text{init.}}$. Using this guess, the ODE can be iterated forward by applying either the Euler or the RK method. Usually, the initial guess leads to a violation of at least one of the boundary conditions at some point on the grid. If this is the case, the initial guess is adjusted according to some prespecified rules and the iteration is restarted, using the adjusted guess. This procedure is repeated until all boundary conditions are satisfied.[299,300] A drawback of applying the shooting method to solve models with Poisson uncertainty is that it only allows for gauging the effects of *one-sided* jumps in the state

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[294] Cf. Brunnermeier and Sannikov (2014d, p. 35).

[295] An example in the risk neutral case of Brunnermeier and Sannikov (cf. 2014a, p. 395) is the condition that the price at any point on the grid must not exceed the first best price in the absence of financial frictions. This price can be calculated analytically.

[296] Cf. Judd (1998, p. 336).

[297] The application of the shooting method to a CTMF framework is explained in Brunnermeier and Sannikov (2016b, Section 3.4)

[298] Cf. Judd (1998, p. 350).

[299] Cf. Brunnermeier and Sannikov (2014a, p. 395) and Brunnermeier and Sannikov (2016b, Section 3.4).

[300] Another application of the shooting algorithm in macroeconomics concerns the solution of neoclassical growth models of the Ramsey type. In this approach the first steps are to guess an initial value for the consumption $C_0$ and to fix an initial value of the capital stock $K_0$. Then, consumption and capital paths are iterated forward via the consumption Euler equation and the equation of motion for the capital stock until an appropriately chosen terminal date $T$ is reached. If the value of capital obtained at this date $K_T$ differs from the steady state value of capital $K_{SS}$, the initial guess has to be adjusted. This procedure is repeated until $K_T$ converges to $K_{SS}$ (cf. Ljungqvist and Sargent, 2004, p. 332).

variable. The reason underlying this fact is that at each grid point $m$ only the values of $q(\cdot)$ at grid points $1, ..., m$ are known. However, in order to calculate the consequences of jumps that raise the state variable, one would need to know the values of the price function at grid points $m+1, ..., M$.

The second method is the *iterative method*.[301] This method introduces time as an additional independent variable into the general function for the price of capital: $q(\eta_t, t)$.[302] Again, one has to derive expressions for the drift and volatility rate of the price of capital via Itô's Lemma. This leads to

$$\mu_t^q q(\eta_t, t) = \frac{\partial q(\eta_t, t)}{\partial \eta_t} \mu_t^\eta \eta_t + \frac{\partial^2 q(\eta_t, t)}{\partial \eta_t^2} \frac{(\sigma_t^\eta \eta_t)^2}{2} + \frac{\partial q(\eta_t, t)}{\partial t}. \tag{3.1.40}$$

and

$$\sigma_t^q q(\eta_t, t) = \frac{dq(\eta_t, t)}{d\eta_t} \sigma_t^\eta \eta_t. \tag{3.1.41}$$

The structure of ODE (3.1.41) compared to that of ODE (3.1.37) remains unchanged as time is deterministic. However, the expression involving $\mu_t^q$ (3.1.40) now is a *partial differential equation* (PDE) since it contains two partial derivatives. The critical term here is the partial derivative of the price function with respect to time. It plays the role of a *"false transient"*[303] and allows for solving the PDE backwards in time, starting from a terminal condition at a terminal date $t = T$. As usually infinite horizon economies are considered, $T$ must be set sufficiently large to approximate the true solution.[304] The terminal condition has to be set along the entire $\eta_t$ dimension, i.e. one has to specify a function $q(\eta_t, T)$. The question of how to choose the terminal condition is not of particular importance. As Di Tella (2017) notes, this condition merely plays a role similar to the initial guess in the numerical solution of nonlinear equations.[305] Brunnermeier and Sannikov (2016b) report that while they have not conducted a theoretical analysis of viable terminal conditions, many reasonable guesses lead to the desired solutions for a wide range of parameter constellations.[306] Once the terminal condition is set, the iterative method consists of repeatedly following two separate steps. The first step is the *static step*. This step involves the calculation of endogenous variables $\mu_t^q$, $\sigma_t^q$, $\mu_t^\eta$ and $\sigma_t^\eta$, given values of $q(\eta_t, t)$ at time $t$. In the *dynamic step* these results are utilised to solve for the time derivative in (3.1.40).[307] In turn, the time derivative can be used to iterate the PDE backwards in time to obtain $q(\eta_{t-\Delta}, t - \Delta t)$. The backwards iteration is aborted once partial derivative $\partial q(\eta, t) / \partial t$ disappears. If this is the case, one has found the sought time-invariant

[301] The iterative method is explained in detail in Brunnermeier and Sannikov (2016b, Section 3.5) in the context of a more complex model with general CRRA preferences. The remainder of this subsection is mainly based on that source.
[302] Cf. Brunnermeier and Sannikov (2016a, p. 50).
[303] Di Tella (2017, p. 2077). The false transient method was developed by Mallinson and de Vahl Davis (1973) to solve elliptic PDEs arising in physics. Its defining characteristic is the addition of a pseudo time derivative to the differential equation (cf. Northrop et al., 2013, p. 33). Here, the time derivative is a pseudo derivative as the price function does not in fact depend on time directly.
[304] Cf. Brunnermeier and Sannikov (2016b, pp. 1525f.).
[305] Cf. Di Tella (2017, p. 2078).
[306] Cf. Brunnermeier and Sannikov (2016b, p. 1528).
[307] Cf. Brunnermeier and Sannikov (2016b, pp. 1528f.).

solution $q(\eta_t)$.[308]

Brunnermeier and Sannikov (2016a) emphasise two advantages of the iterative method relative to the shooting method. First, it can be extended straightforwardly to include several price functions. On the contrary, the shooting method is difficult to apply in such environments. Second, the iterative method avoids explosive solutions which can arise if the shooting method is used.[309] A third advantage is that the iterative method can in principle be used to analyse models with two-sided jumps in the state variable.[310]

## 3.2 Literature Overview

### 3.2.1 The Brunnermeier and Sannikov (2014a) Model

#### 3.2.1.1 Model Set-Up

**Assets.** In the baseline model there are two assets, capital and the risk free-asset. As usual, the output good is the numéraire in the model economy, which implies that the price of capital is measured in units of output. Further, agents that short the risk-free asset have to repay their creditors in final goods. Since fundamental risk in the economy is due to a Brownian process, the evolution of the capital price is described by (3.1.13).[311]

**Technologies.** There are two types of agents, *experts* and *households*. Individuals within each group are homogeneous. The former play the role of borrowers and the latter that of lenders. Both types of agents have the ability to manage projects that generate output goods. The projects managed by experts and households produce final goods via production functions

$$y_t = a\,k_t \quad \text{and} \quad \underline{y}_t = \underline{a}\,\underline{k}_t, \tag{3.2.1}$$

respectively.[312] Experts' productivity parameter $a$ is assumed to be larger than households' parameter $\underline{a}$ , i.e. the projects of the former are more productive. Capital in the hands of each agent follows (3.1.1) in conjunction with (3.1.8), i.e. capital is exposed to aggregate Brownian uncertainty and does not carry idiosyncratic risk. The capital goods production technology is given by (3.1.10).[313]

[308] Cf. Brunnermeier and Sannikov (2016a, p. 54).
[309] Cf. Brunnermeier and Sannikov (2016a, p. 50).
[310] It should be added that to the best knowledge of the author, the iterative method has not been applied to CTMF models with Poisson uncertainty, yet.
[311] Cf. Brunnermeier and Sannikov (2014a, p. 384).
[312] Individuals from each sector can either be interpreted as producers with heterogeneous productivity levels or as financial investors that allocate a portion of their wealth to projects that generate output at rates $a$ or $\underline{a}$.
[313] Cf. Brunnermeier and Sannikov (2014a, pp. 384ff.).

**Returns.** Experts' return on capital is

$$dr_t^K = \frac{a - \iota_t}{q_t}\, dt + \{\Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q\}\, dt + \{\sigma + \sigma_t^q\}\, dZ_t \tag{3.2.2}$$

and that of households is

$$d\underline{r}_t^K = \frac{\underline{a} - \iota_t}{q_t}\, dt + \{\Phi(\iota_t) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\}\, dt + \{\sigma + \sigma_t^q\}\, dZ_t. \tag{3.2.3}$$

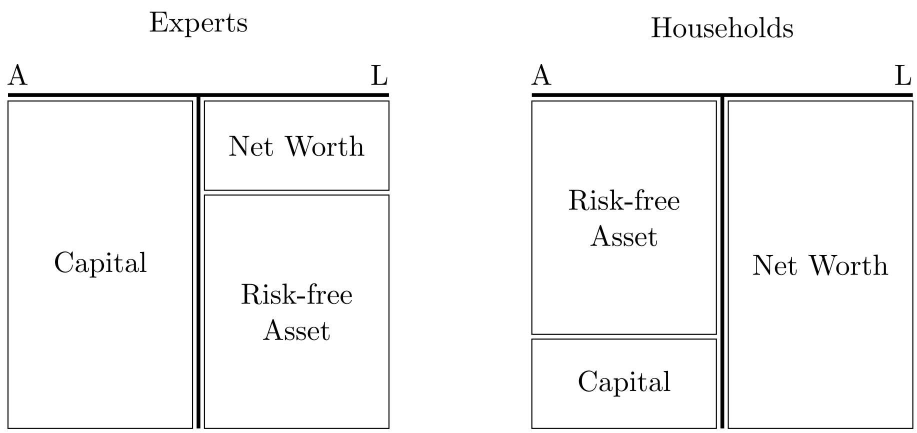
These two expressions contain the optimal investment rate $\iota_t$[314], which is identical across sectors and determined via (3.1.28). Further, comparison of (3.2.2) and (3.2.3) shows that experts' and households' deterministic depreciation rates are allowed to differ.[315]

**Preferences.** Individuals from both groups are assumed to be risk neutral in the baseline model. However, only households can consume negatively.[316] This ability allows them to generate additional funds if necessary. As the authors note, the disutility associated with negative consumption can be interpreted as the disutility from additional labour input required to produce any additional amount of output.[317] A further difference between the two sectors is that households are assumed to be more patient than experts. That is, the time preference rate of the former $r$ is lower than that of the latter $\rho$.[318] This assumption ensures for a broad set of preferences that (i) experts, who are endowed with a superior technology, do not pay off the entirety of their debt in the long run and (ii) aggregate wealth in the household sector does not approach zero in the long-run.[319,320]

**Financial Structure.** Experts can obtain external funds only by issuing risk-free debt to households. This simple capital structure may be motivated by imposing an agency problem between experts and households which leads to a "skin in the game constraint" as the authors show in the online appendix to their paper.[321] The mentioned constraint serves to align the interests of inside

[314] Here and in the remainder superscript "$*$" is removed from optimal choices to ease on notation, except for cases where the distinction might be unclear.
[315] Cf. Brunnermeier and Sannikov (2014a, p. 387).
[316] Cf. Brunnermeier and Sannikov (2014a, p. 385).
[317] Cf. Brunnermeier and Sannikov (2014a, fn. 9).
[318] Cf. Brunnermeier and Sannikov (2014a, p. 385).
[319] Similar assumptions are frequent in the financial frictions literature. As mentioned in Subsection 2.2.2, BGG assume that entrepreneurs exit the market with a fixed probability in each period. If market exit occurs, each affected entrepreneur consumes his entire wealth. These assumptions prevent entrepreneurs from growing out of the financial constraint. An early discussion of the properties of the stationary wealth distribution in economies with two different types of investors is provided in Dumas (cf. 1989, Sections 4 and 5).
[320] Cf. also Brunnermeier and Sannikov (2014d, pp. 9f.).
[321] Cf. Brunnermeier and Sannikov (2014b, Section A1). Therein the authors present a modified model in which expert's equity issuance policy is endogenous. The key friction is that experts can divert capital returns. The diversion rate is chosen endogenously by experts. However, diversion is allowed to be inefficient in the sense that experts can only recover a fraction of diverted capital returns. The authors show that the solution to the agency problem requires experts to be financed entirely by debt if diversion is not associated with deadweight losses. Thus, the assumptions imply a form of endogenously incomplete markets.

Figure 3.2.1: Agents' Balance Sheets in the Brunnermeier and Sannikov (2014a) Model




*Notes*: Adapted from Brunnermeier and Sannikov (2016a, p. 10).

and outside equity holders by requiring the former to absorb at least a portion of the project risk through their own equity.[322] Typically, the agent's incentive to behave prudently are most pronounced when they retain the entire equity stake. This case is examined in the baseline model. Further, both groups are subject to solvency constraints, i.e. their wealth is required to be non-negative at any point in time.[323] This constraint guarantees that household lending to experts is indeed risk-free. The balance sheets of the two sectors are depicted in Figure 3.2.1.

**Choices.** The assumptions on households' preferences and constraints imply that their marginal utility of consumption, which is equal to unity, is always equal to their marginal utility of wealth. This condition ensures that households are indifferent between consuming and saving at any point in time. Further, households' portfolio choices imply two conditions: first, the risk-free return is always equalised to households' time preference rate $r$. Hence, their supply of the risk-free asset is perfectly elastic at that rate. Second, households' expected return to capital is equalised to the time preference rate at any time, i.e. $\mathbb{E}_t\left[d\underline{r}_t^K\right]/dt = r$ for all $t$, provided that they choose to hold a strictly positive amount of capital.[324]

In contrast, experts act as if they were risk averse even though their instantaneous utility function is linear in consumption. This is due to the assumption that experts are neither allowed to issue outside equity to households nor to consume negatively, which in turn implies that experts require a positive amount of inside equity to avoid hitting the solvency constraint if adverse shocks to capital materialise. As a consequence, experts, in distinction from households, require risk premia for

[322] As Brunnermeier and Sannikov (cf. 2014a, p. 386) note, agency problems of this type have along tradition in corporate finance and can be traced back to the classic paper by Jensen and Meckling (1976).
[323] Cf. Brunnermeier and Sannikov (2014a, p. 388).
[324] Cf. Brunnermeier and Sannikov (2014a, p. 388).

holding capital and adjust their capital demand depending on their net worth position. Formally, these considerations are captured by introducing a stochastic process for experts' marginal utility of wealth $\zeta_t$:

$$d\zeta_t = \mu_t^\zeta \, dt + \sigma_t^\zeta \, dZ_t. \tag{3.2.4}$$

where both parameter functions $\mu_t^\zeta$ and $\sigma_t^\zeta$ are endogenously determined in equilibrium. Experts' marginal utility of wealth varies over time as it captures *stochastic investment opportunities*, i.e. future expected laws of motion of the capital price, which in turn depend on the future expected paths of experts' net worth in equilibrium.[325] Since experts cannot generate additional equity through negative consumption, they have to take into account future expected asset prices in addition to the current price of capital when deciding on their portfolios.[326] In equilibrium $\sigma_t^\zeta < 0$ holds, i.e. negative shocks to capital $dZ_t < 0$ increase the marginal utility of net worth.[327] The intuition for this result is as follows. If a negative innovation in the stochastic process for capital is realised, the *aggregate* equity position of experts, who are levered, deteriorates. If the reduction in wealth is sufficiently strong, they cut their risk exposure by selling risky capital to less productive households. However, households are only willing to buy this asset at a price discount as they are less productive. Any *individual* expert anticipates this behaviour and tries to hedge these stochastic investment opportunities. More precisely, a risk neutral expert prefers to have more net worth when the marginal utility of wealth is high, i.e. in times when the price of capital is low.[328] Put differently, experts require a risk premium for taking on risky capital as doing so might reduce their net worth when it is most valuable.[329]

Consumption policies also differ between the two sectors. Experts refrain from consumption whenever their marginal utility of wealth is above their marginal utility of consumption, which is equal to unity. Once the two marginal utilities are equalised, experts consume any additional income entirely.[330] Households, on the other hand, consume the whole aggregate output net of investment as long as experts do not consume. When experts reach the consumption threshold, households consume the residual output.

#### 3.2.1.2 Results

**Price and Allocation of Capital.** Panel (a) in Figure 3.2.2 shows that the price of capital is a monotonously increasing function of the state variable. This result can be traced back to the

[325] The marginal utility of wealth $\zeta_t$ is related to the time-varying stochastic discount factor $\xi_t$ via equation $\xi_t = e^{-\rho t}\zeta_t$ (cf. Brunnermeier and Sannikov, 2016b, p. 1512). Further, $\zeta_t$ is also equal to the sum of unity and the shadow value of the nonnegativity constraint on consumption (cf. Phelan, 2016, p. 207).

[326] Cf. Brunnermeier and Sannikov (2014a, pp. 390ff.).

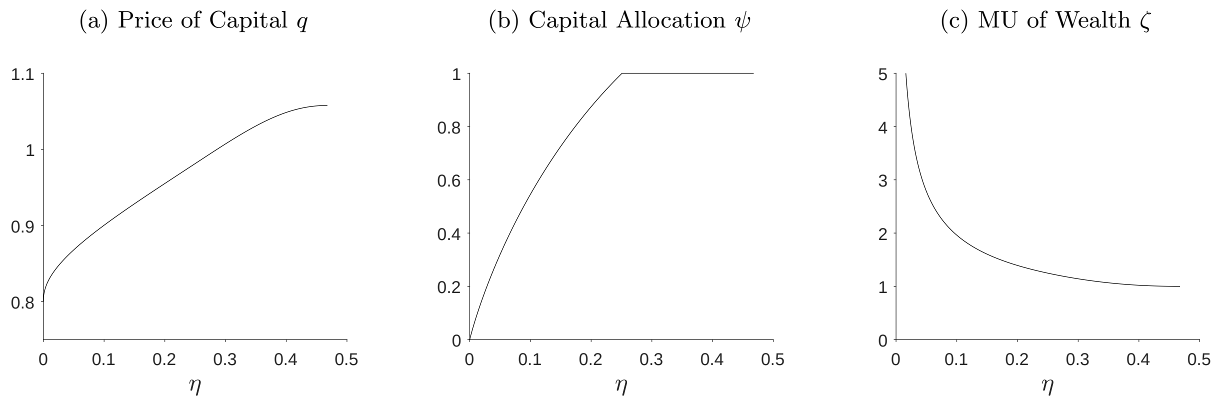
[327] Cf. Brunnermeier and Sannikov (2014a, p. 392).

[328] Cf. Di Tella (2017, p. 2053).

[329] Matters would be different if experts were allowed to consume negatively. For instance, if they were close to hitting the solvency constraint, they could simply generate additional net worth buffers via negative consumption in order to be able to absorb possible adverse shocks to capital.

[330] Cf. Brunnermeier and Sannikov (2014a, p. 394).

Figure 3.2.2: Functions $q(\eta)$, $\psi(\eta)$, and $\zeta(\eta)$ in the Brunnermeier and Sannikov (2014a) Model




*Notes*: Functions are calculated using code provided by Brunnermeier and Sannikov (2014c).

fact that experts' share in the aggregate capital stock $\psi_t$ is increasing in their aggregate net worth position, as can be observed in Panel (b). Put differently, the demand for capital by experts, who value capital more than households, rises with $\eta_t$. That result reflects experts' risk aversion with respect to their portfolio choice: they require sufficient net worth buffers as an insurance against bad shocks before they are willing to pick up additional capital financed by debt. Thus, experts' leverage is restricted by a precautionary motive.[331] The optimal capital allocation $\psi = 1$ is reached at point $\eta^{\psi}$. Domain $\left[0, \eta^{\psi}\right]$ can be regarded as the crisis region of the state space since it is characterised by a misallocation of capital.[332] Adverse shocks that push the economy into the crisis region lead to fire-sales of capital to households, who are less productive than experts. Thus, severe drops in the value of the productive asset may occur. Households are willing to increase their capital stocks in such situations and thereby act as liquidity providers[333] due to a speculative motive: they expect to sell capital back to experts at higher future prices.[334]

**Investment Opportunities.** Panel (c) visualises that the stochastic investment opportunities embedded in $\zeta_t$ fall with $\eta_t$, which is a result of the positive relationship between $q_t$ and the state variable. The stochastic steady state is reached at the point where the marginal utility of net worth $\zeta_t$ is equal to one, which is the constant value of the marginal utility of consumption. Then, experts are indifferent between consuming and saving. In the range $[0, \eta^{ss}]$ these individuals entirely abstain from consumption. Once $\eta^{ss}$ is reached, they start consuming and use any additional income for

[331] Cf. Brunnermeier and Sannikov (2014a, p. 392).
[332] Cf. Brunnermeier and Sannikov (2016b, p. 1511).
[333] According to that interpretation, market illiquidity is measured by productivity differential $a - \underline{a}$ (cf. Brunnermeier and Sannikov, 2016b, p. 1519).
[334] Cf. Brunnermeier and Sannikov (2014a, p. 389).

that purpose.[335] However, strong negative shocks to capital will occur at some point in time, which will set the economy back to a value $\eta_t < \eta^{ss}$.[336]

**Evolution of the State Variable.** The drift and volatility rate of the state variable, experts' portfolio weight on capital $x_t$, which also measures leverage (assets $q_t k_t$ to wealth $n_t$), and volatility rate $\sigma_t^q$ are presented as functions of $\eta_t$ in Figure 3.2.3. Each of these curves has a kink at point $\eta^\psi$. Drift rate $\mu_t^\eta$ is initially large, as can be seen from Panel (a). This is due to two reasons: first, households consume, while experts do not. Second, experts' portfolio return is large compared to that of households. This is because experts are highly leveraged and the return differential $dr_t^K - d\underline{r}_t^K$ is relatively large due to the low price of capital. The deterministic growth in the state variable declines as experts become less levered and the price of capital rises. However, it stays positive in the entire range $(0, \eta^{ss}]$. Hence, the system reverts back to the steady state in expectation after the occurrence of adverse shocks. When $\eta^{ss}$ is reached, experts start consuming and the drift of $\eta_t$ drops to zero.[337]

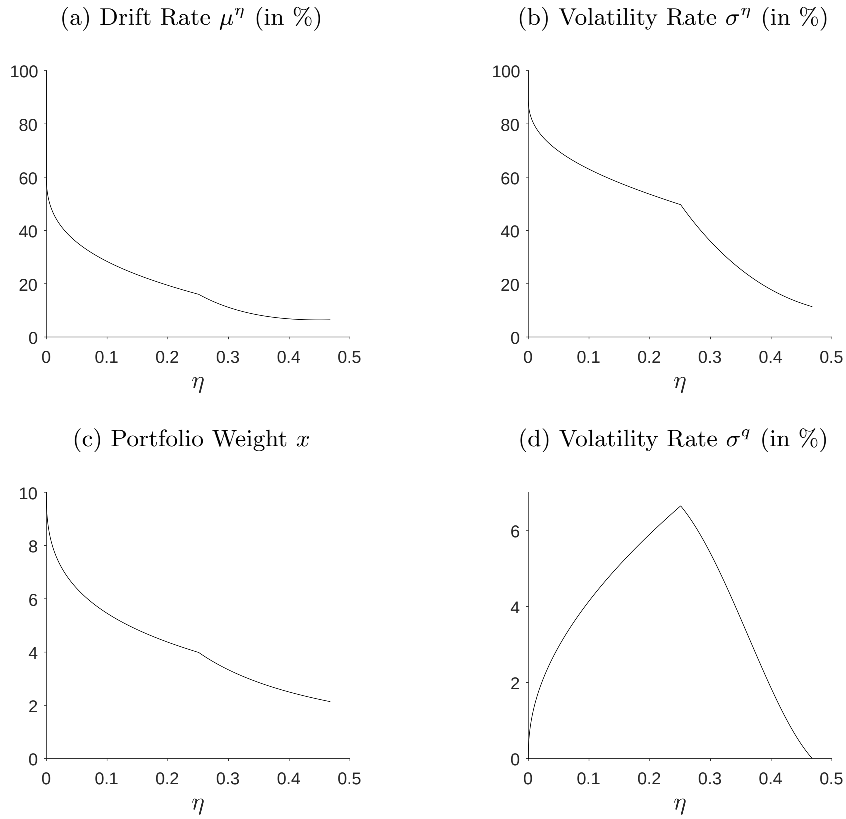
Volatility rate $\sigma^\eta$ is a monotonously decreasing function of $\eta_t$ (Panel (b)) since experts reduce their leverage with an accelerating capital price and thus become less exposed to adverse shocks. In addition, net worth losses from price changes become less severe as experts' wealth share rises. The effects of innovations in the stochastic process for capital on the state variable can formally be expressed by

$$\sigma_t^\eta = \frac{(x_t - 1)\,\sigma}{1 - \underbrace{\frac{q'(\eta_t)}{q(\eta_t)/\eta_t}(x_t - 1)}_{\text{amplification}}}. \tag{3.2.5}$$

The numerator of (3.2.5) captures the leverage effect in the absence of price movements. Term $x_t - 1$ measures the percentage drop in the state variable due to a 1 percent drop in experts' capital $k_t$ before price changes, which increases with leverage. The denominator entails an amplification term which is equal to the product of the elasticity of the capital price with respect to the state variable, which reflects the "market illiquidity" of capital, and term $x_t - 1$. This product is the percentage reduction in the price of capital due to the aforementioned shock.[338] The resulting change in $q_t$ causes a further deterioration of experts' balance sheets, which, in turn, leads to further drops in the demand for capital and thereby in $q_t$ and so on.[339,340] This adverse feedback loop is illustrated

[335] Cf. Brunnermeier and Sannikov (2014a, p. 396).
[336] If a small adverse shock occurs at the steady state, experts can avoid balance sheet losses by cutting consumption expenditures (cf. Brunnermeier and Sannikov, 2014a, p. 380).
[337] Cf. Brunnermeier and Sannikov (2014a, p. 396).
[338] Cf. Brunnermeier and Sannikov (2016b, p. 1519).
[339] Cf. Brunnermeier and Sannikov (2014a, p. 399).
[340] Note that the demand for capital *falls* with $q_t$ in that case. Thus, the demand curve for capital can be upward-sloping. This is A typical consequence of a pecuniary externality in an incomplete markets economy (cf. also the two models in Section 2.2). Under these circumstances individual actions can be regarded as strategic complements (cf. Brunnermeier and Oehmke, 2013, p. 1246).

Figure 3.2.3: Functions $\mu^{\eta}(\eta_t)$, $\sigma^{\eta}(\eta_t)$, $x(\eta_t)$, and $\sigma^{q}(\eta_t)$ in the Brunnermeier and Sannikov (2014a) Model



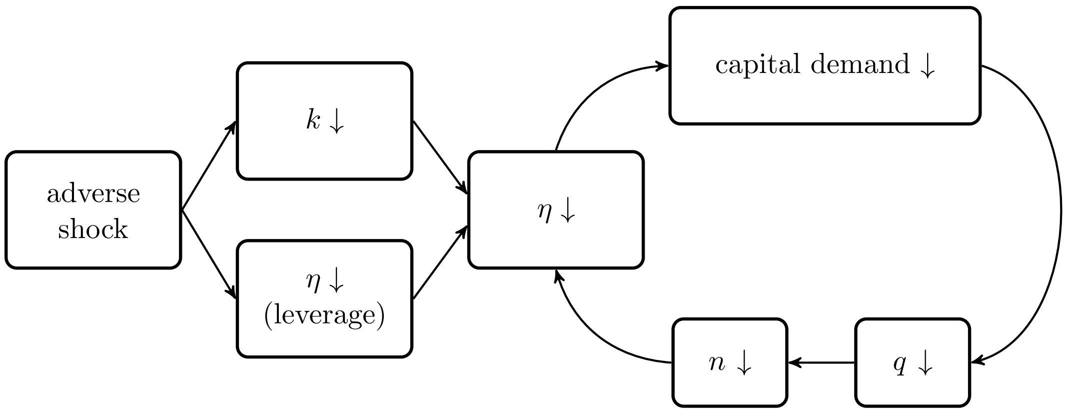

*Notes*: Functions are calculated using code provided by Brunnermeier and Sannikov (2014c).

in Figure 3.2.4. Mathematically, $\sigma_t^{\eta}$ is described by an infinite geometric series.[341] This is because the elasticity of $q_t$ is positive and experts are levered in equilibrium, i.e. $x_t > 1$ holds.

Another interesting result is the shape of function $\sigma^{q}(\eta_t)$ (lower right panel of Figure 3.2.3). At the stochastic steady state endogenous risk is zero as adverse shocks are absorbed at least in part through adjusting payouts $c_t$. Below $\eta^{ss}$ amplification through price changes and leverage can become significant. This shows that system dynamics are strongly *nonlinear*: large shocks (or an unusual series of smaller shocks) that push the economy away from the stochastic steady state are significantly more amplified than small shocks. Further, a series of negative shocks can push the economy into the lower region of the state space, which is characterised by small (absolute) changes in the state variable. Thus, low growth episodes can become protracted and there can be significant persistence of temporary shocks. Endogenous risk peaks at $\eta^{\psi}$, the point where experts first engage

[341] Cf. Brunnermeier and Sannikov (2016b, p. 1519).

Figure 3.2.4: Adverse Feedback Loop in the Brunnermeier and Sannikov (2014a) Model




*Notes*: Based on Brunnermeier and Sannikov (2014a, p. 400).

in fire sales of capital to less productive households after adverse shocks.[342]

**Real Effects of Shocks.** Shocks to capital can influence the level of output as well as its growth rate. For instance, an adverse innovation in (3.2.2) that pushes the economy into the crisis region, in which a positive amount of capital is held by less productive households, reduces the level of output. This is accompanied with a drop in the price of capital, which, in turn, reduces the investment rate according to equation (3.1.28) and hence mitigates economic growth. It should be added that these effect do not occur in the absence of financial frictions, since in that case the entire capital stock is always operated by experts and $q_t$ is permanently at its first-best level.[343]

**The Volatility Paradox.** The "*Volatility Paradox*" refers to the result that endogenous risk $\sigma_t^q$ does not disappear as exogenous risk $\sigma \to 0$. In fact, the authors show that the maximum value of endogenous risk is largely insensitive to variations in $\sigma$ and in some cases even slightly *increases* with that parameter. This implies that exogenous shocks can be subject to large amplification even in an environment of low exogenous risk. It follows that the time the system spends in a crisis regime does not approach zero. These surprising facts can be explained by the endogenous leverage response of financially constrained experts: the reduction in exogenous risk induces these agents to take on more leverage, especially in states in which the allocation of capital is inefficient.[344] Accordingly, the arrival of bad shocks leads to greater cuts in leverage and the probability of capital sales to less productive households rises.[345] The authors also demonstrate that market illiquidity

[342] Cf. Brunnermeier and Sannikov (2014a, p. 400).
[343] Cf. Brunnermeier and Sannikov (2014a, pp. 385f.).
[344] Cf. Brunnermeier and Sannikov (2014a, pp. 406f.).
[345] Adrian and Brunnermeier (cf. 2016, Section IV) provide empirical support for the Volatility Paradox hypothesis. Using weekly panel data spanning from 1971 to 2013 for all publicly traded U.S. commercial banks, they show

measured by the productivity differential has a significant impact on endogenous risk: if households are more productive, capital sales from experts to households lead to lower variations in $q_t$.[346]

**Macroprudential Policies.** The authors demonstrate that a social planner can attain the first-best outcome, i.e. the outcome without financial frictions, by creating an insurance asset, which is subsequently held by experts. The planner can alter this asset's value via open market operations, that is by issuing or repurchasing the asset, or through payouts.[347] It can be shown that there exists a specific risk profile of the insurance asset which drives down the volatility of the state variable to zero. This policy is characterised by the fact that the asset's value is negatively correlated to experts' long positions in capital. If $\sigma_t^\eta = 0$, experts no longer require a risk premium and immediately purchase the entire aggregate capital stock.[348] The drawback of the described policy is that the planner has to continuously engage in large open market operations or in the redistribution of wealth. Alternatively, the planner can offer a tail risk insurance, by recapitalising experts only when the state variable reaches a lower bound. This policy can be shown to be effective when exogenous risk is low. When exogenous risk increases, however, insurance costs, i.e. the welfare reductions associated with wealth transfers, outweigh any efficiency gains.[349]

### 3.2.2 The Brunnermeier and Sannikov (2016a) Model

#### 3.2.2.1 Model Set-Up

**Assets.** In the latest version of the I Theory of Money there are two main departures from the assumptions in BS(2014a): in the former model financial intermediation by specialised agents is explicitly considered and the risk-free asset is replaced by money. Money is distinct from the risk-free asset in BS(2014a) in that the value of the former in real terms fluctuates, while the value of the latter does not. Despite the existence of money, the unit of account is again assumed to be the final good. The total stock of money consists of outside money, which is supplied by the central bank and can be interpreted as currency, and inside money, which is generated by intermediaries.[350] Outside money is normalised to unity in the baseline model, which is tantamount to a constant stock of that asset.[351] In the following, the price of one unit of outside money in units of output is denoted by $P_t$.

From the perspective of money holders inside money is a perfect substitute for outside money. The reason is that inside money is denominated in outside money, i.e. intermediaries promise to repay depositors in outside money. Since intermediaries never default in equilibrium, intermediaries

that systemic risk builds up in times when contemporaneous market volatility is low.

[346] Cf. Brunnermeier and Sannikov (2014a, p. 408).

[347] These policies can by financed by taxes that are proportionate to individual wealth levels.

[348] Cf. Brunnermeier and Sannikov (2014a, pp. 415f.).

[349] Cf. Brunnermeier and Sannikov (2014a, pp. 416f.).

[350] Cf. Brunnermeier and Sannikov (2016a, p. 2).

[351] Cf. Brunnermeier and Sannikov (2016a, p. 8).

can in fact guarantee to service their debt. An implication is that inside money must have the same price and return profile as outside money.[352] Inside money, in contrast to outside money, is in zero net supply and for that reason does not enter (private) aggregate wealth:[353,354]

$$N_t \equiv q_t K_t + P_t. \tag{3.2.6}$$

It is important to recognise that all prices in the model, including the price of money, are completely flexible.[355]

**Technologies.** Besides intermediaries, the model is populated by households. Households are endowed with two distinct production technologies, which enable them to produce intermediate goods. In both technologies, capital is the sole factor of production. The output of the first technology is good $a$ and that of the second is good $b$. At the second stage of production, intermediate goods $a$ and $b$ are combined to the single final good via technology $A(\psi_t) K_t$, where $\psi_t$ is the share of aggregate capital utilised in the production of good $b$**.** The complimentary share $(1-\psi)$ is the share of aggregate capital allocated to the production of good $a$. Function $A(\psi_t)$ is specified to feature an internal maximum $\psi^*$.[356,357]

While households have the knowledge to produce intermediate goods via each of the two technologies $a$ and $b$, they are restricted to manage only a single project with either technology $a$ or $b$ at any point in time. However, they are allowed to costlessly switch to the respective other technology after one instant of time has passed. Costless switching implies that households must be indifferent between investing in either of the two technologies in equilibrium. Prices of the intermediate goods adjust such that this condition holds.[358] The mentioned assumption ensures that the wealth distribution between households active in technology $a$ and those in technology $b$ does not matter for aggregate outcomes. This allows for reducing the state space to dimension one and thus significantly simplifies the solution of the model.

Capital utilised in the production of good $a$ follows

$$\frac{dk_{i,t}}{k_{i,t}} = \{\Phi(\iota_t) - \delta\}\, dt + \sigma^a\, dZ_t^a + \breve{\sigma}^a\, d\breve{Z}_{i,t}^a, \tag{3.2.7}$$

[352] Cf. Brunnermeier and Sannikov (2016a, p. 5).
[353] Cf. Brunnermeier and Sannikov (2016a, p. 10).
[354] Technically, outside money is a liability of the central bank. Thus, outside money enters aggregate private wealth (cf. the discussion in Section 2.3).
[355] Cf. Brunnermeier and Sannikov (2016a, p. 5).
[356] Cf. Brunnermeier and Sannikov (2016a, p. 8).
[357] The function employed by the authors is the constant elasticity of substitution technology

$$A(\psi_t) \equiv \mathcal{A}\left[\frac{1}{2}\psi_t^{\frac{s-1}{s}} + \frac{1}{2}(1-\psi)^{\frac{s-1}{s}}\right]^{\frac{s}{s-1}},$$

in which $\mathcal{A}$ and $s$ are exogenous parameters (cf. Brunnermeier and Sannikov, 2016a, p. 8).
[358] Cf. Brunnermeier and Sannikov (2016a, p. 12).

where the specific form of function $\Phi(\cdot)$ is given by (3.1.9), $dZ_t^a$ is a sector-specific shock, or, put differently, a shock which affects all individuals in sector $a$, and $d\breve{Z}_{i,t}^a$ is an idiosyncratic shock that affects individuals in sector $a$. By definition, the idiosyncratic shock washes away at the aggregate level. The evolution of capital in sector $b$ is described by an equation analogous to (3.2.7).[359] The aggregate capital stock is the sum of $k_t$, the amount of capital held in sector $a$, and $\underline{k}_t$, the capital in sector $b$: $K_t = k_t + \underline{k}_t$. It follows from Itô's Lemma that $K_t$ evolves according to[360]

$$\frac{dK_t}{K_t} = \{\Phi(\iota_t) - \delta\}\; dt + (1-\psi_t)\,\sigma^a\, dZ_t^a + \psi_t \sigma^b\, dZ_t^b. \tag{3.2.8}$$

Further, it is conjectured that the real value of money is proportionate to the aggregate capital stock since aggregate output is as well and the scale invariance property holds.[361] This conjecture allows for cancelling out the aggregate capital stock from the set of equilibrium equations. It follows that one can replace $P_t$ by $p_t K_t$, where $p_t$ is the price of money in output goods per unit of aggregate capital, in equation (3.2.6). Prices $q_t$ and $p_t$ follow

$$\frac{dq_t}{q_t} = \mu_t^q\, dt + \left(\sigma_t^q\right)^T\, dZ_t \quad \text{and} \quad \frac{dp_t}{p_t} = \mu_t^p\, dt + \left(\sigma_t^p\right)^T\, dZ_t, \tag{3.2.9}$$

in which $\sigma_t^q \equiv \left[\sigma_t^{q,a}, \sigma_t^{q,b}\right]^T$ and $\sigma_t^p \equiv \left[\sigma_t^{p,a}, \sigma_t^{p,b}\right]^T$ are the volatility rate vectors of the prices of capital and money, the latter measured per unit of aggregate capital, respectively, and $dZ_t \equiv \left[dZ_t^a, dZ_t^b\right]^T$ is the vector of sector-specific shocks.[362]

**Preferences.** Both households and intermediaries are assumed to be risk averse with logarithmic utility.[363] Obviously, this type of utility function prevents agents from consuming negatively. Further, it can be shown that optimal attainable lifetime utility as a function of current wealth is of the logarithmic type as well.[364] Hence, the marginal utility of wealth approaches infinity as wealth approaches zero. It follows that agents with logarithmic instantaneous utility functions will never

[359] Cf. Brunnermeier and Sannikov (2016a, p. 9).

[360] Cf. Brunnermeier and Sannikov (2016a, p. 11).

[361] To gain intuition, one can consider the equation of exchange in two extreme cases: an "autarky" case in which intermediaries are absent from the model, e.g. because they do not own any wealth, and a case in which intermediaries own the entire aggregate wealth. The economy in the fully dynamic model evolves between the two cases but never quite reaches any one of them (cf. Brunnermeier and Sannikov, 2016a, pp. 2f.). In both polar cases, portfolio weights, allocations, the velocity of money $\mathcal{V}_t$, and the money supply $M_t$ are constant. The equation of exchange in combination with the aggregate production function $A(\psi_t)\, K_t$ implies

$$P_t = \frac{A(\psi_t)}{M_t \mathcal{V}_t} K_t.$$

Hence, it is immediately clear that in both polar cases the value of money is proportionate to the aggregate capital stock.

[362] Cf. Brunnermeier and Sannikov (2016a, p. 11).

[363] Cf. Brunnermeier and Sannikov (2016a, p. 9).

[364] This implication of logarithmic preferences is discussed in more detail in Subsection 4.3.1 and proven in Appendix B.1.5 in the context of the model developed in this thesis.

default under rational expectations even in the absence of a solvency constraint.[365]

Agents' time preference rates are identical across all individuals.[366] In the present model none of the two sectors can outgrow the other in the long run. The reason is that the price of intermediate goods produced in a particular sector increases due to the assumed form of the production technology if that sector becomes undercapitalised.[367] Hence, there is no need to assume heterogeneous time preference rates.

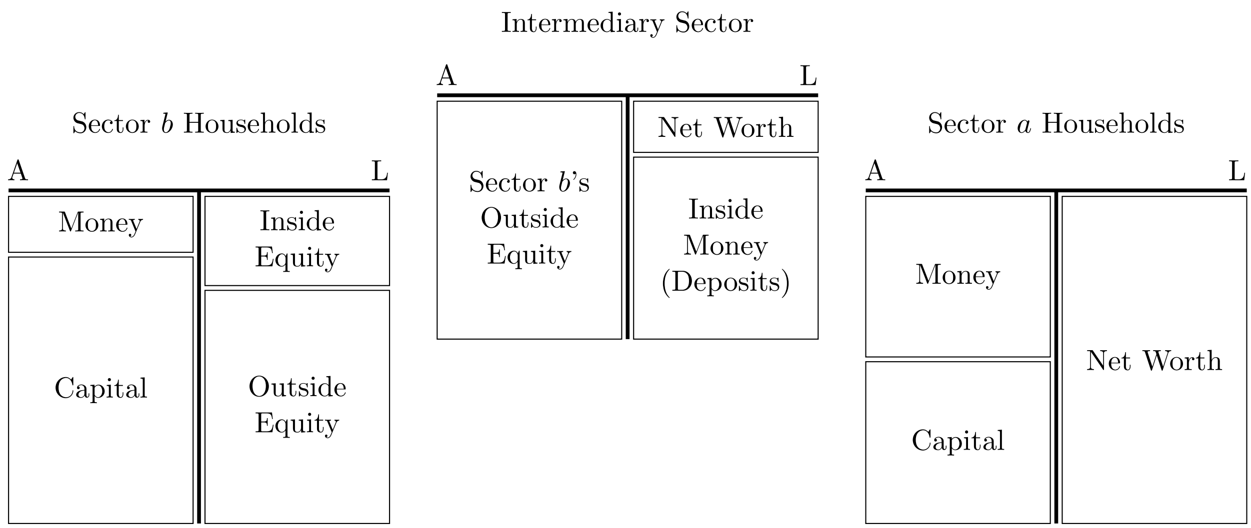
**Financial Structure.** Households can transfer some of the risk associated with holding capital by issuing outside equity to intermediaries.[368] However, household can only offload a part of their risk to intermediaries, i.e. the former have to retain some amount of inside equity. This is the first financial friction. More specifically, it is assumed that $0 \leq \overline{\chi}^a < \overline{\chi}^b \leq 1$, in which $\overline{\chi}^a$ and $\overline{\chi}^b$ are the maximum shares of equity households can sell to intermediaries in sectors $a$ and $b$, respectively.[369] The reason for this assumption is explained momentarily.

Intermediaries require outside funds in order to finance the purchases of households' equity.[370] It is assumed that the former can only obtain external funding by issuing deposits to the latter.[371] The fact that intermediaries are restricted from issuing equity is the second financial friction. Further, intermediaries have an advantage over households as they are able to diversify their portfolios by investing across a large number of projects within each sector. Diversification allows them to remove the idiosyncratic risk associated with any single project. Households, on the other hand, can only manage a single project at a given point in time and are thus exposed to idiosyncratic risk.[372] This is the third and final financial friction. It generates an incentive for households to sell their shares to intermediaries in exchange for money.

The assumption that $\overline{\chi}^a \neq \overline{\chi}^b$ is crucial for the existence of an intermediary balance sheet channel in this model: if this condition was not satisfied, intermediaries could always construct a risk-free portfolio consisting of a long position in capital and a short position in money. In such a portfolio the change in the real value of the capital stock due to a shock is always equal to the change in the real value of money.[373] Under these conditions intermediaries would not require any

[365] Cf. Brunnermeier and Sannikov (2014d, p. 12).
[366] Cf. Brunnermeier and Sannikov (2016a, p. 9).
[367] Cf. Brunnermeier and Sannikov (2016a, p. 26).
[368] Formally, equity investment by intermediaries is modelled as if these agents were directly holding capital to produce output goods.
[369] Cf. Brunnermeier and Sannikov (2016a, p. 9).
[370] Of course, this is conditional on the presumption that intermediaries do not already own the entire aggregate wealth in the economy.
[371] Cf. Brunnermeier and Sannikov (2016a, pp. 9f.).
[372] Cf. Brunnermeier and Sannikov (2016a, p. 7).
[373] For clarification, consider the case of $\overline{\chi}^a = \overline{\chi}^b = 1$, which implies that intermediaries are not constrained in their project choices. Further, assume that (i) the representative intermediary holds a long position in capital equal to the aggregate capital stock, which is entirely financed by inside money, and (ii) that this agent allocates his capital between the two technologies to maximise output, i.e. $\psi_t = \psi^*$. Then, in the absence of movements in $q_t$ and $p_t$ a large adverse shock to sector $a$ capital $dZ_t^a = -\epsilon$ reduces the real value of the intermediary's assets by $(1 - \psi^*) K_t \sigma^a \epsilon$, provided that terms of order $dt$ are set to zero. Yet, according to (3.2.8) this is also the reduction

Figure 3.2.5: Agents' Balance Sheets in the Brunnermeier and Sannikov (2016a) Model




*Notes*: Based on Brunnermeier and Sannikov (2016a, p. 10).

net worth buffers and the economy would immediately jump to the first-best solution, in which all capital is intermediated.[374] For simplicity, the authors set $\overline{\chi}^a = 0$ in the baseline model, i.e. households active in sector $a$ are not allowed to sell any equity to intermediaries. On the contrary, the value of $\overline{\chi}^b$ is set close to unity.[375]

Figure 3.2.5 depicts the balance sheets of households and intermediaries. It is interesting to note that intermediaries play the role end-borrowers in the model economy: they are the only agents that are financed by debt.[376] Before discussing optimal choices, let us offer a brief interpretation of intermediaries' activities. These are distinct from the traditional banking business: rather than lending to productive agents, intermediaries purchase shares of households on their own accounts. Hence, this behaviour can be interpreted as a form of investment banking.[377] In this sense the creation of inside money in the model is a "byproduct" of investment banking.

**Choices.** The fact that households and intermediaries have logarithmic preferences implies that optimal portfolio and consumption choices are relatively simple compared to the case in the

in the real value of money $p_t K_t$. In effect, the intermediary would not suffer any net worth losses. If this is the case, the conjecture that $q_t$ and $p_t$ do not change was indeed correct. Conversely, households cannot diversify away idiosyncratic risk and thus are not able to construct risk-free portfolios even though they are allowed to invest in both technologies.

[374] It was assumed above that $\overline{\chi}^a < \overline{\chi}^b$. Yet, by symmetry of the problem, it could also be assumed that $\overline{\chi}^a > \overline{\chi}^b$.

[375] Cf. Brunnermeier and Sannikov (2016a, p. 9).

[376] Brunnermeier and Sannikov (cf. 2016a, p. 3) refer to sector $b$ households as "end-borrowers". Yet, (at least) in a narrow sense this is not correct since sector $b$ households are not financed by debt at any point in time.

[377] As we are dealing with intermediaries that issue deposits, these agents can be characterised as commercial banks which engage in proprietary trading ("universal banks").

preceding section: agents act myopic, i.e. their choices do not depend on stochastic investment opportunities.[378] In terms of the portfolio decision this is reflected by the general function for intermediaries' portfolio weight on sector $b$ households' outside equity

$$x_t = x\left(dr^{bI}\left(q_t, \mu_t^q, \sigma_t^q, \mathbf{z}_t\right), dr^M\left(p_t, \mu_t^p, \sigma_t^p, \mu_t^K, \sigma_t^K, \right)\right), \tag{3.2.10}$$

in which the general functions for intermediaries' return on outside equity issued by sector $b$ households $dr_t^{bI} = dr^{bI}(\cdot)$ and for the return on money $dr_t^M = dr^M(\cdot)$ are used. The latter contains terms $\mu_t^K \equiv \Phi(\iota_t) - \delta$ and $\sigma_t^K \equiv \left[(1-\psi_t)\sigma^a, \psi_t\sigma^b\right]^T$ as the value of money is proportionate to the aggregate capital stock.[379] Equation (3.2.10) also shows that the optimal portfolio weight does only depend on current returns, not on future investment opportunities, which is a consequence of logarithmic utility.

Further, $x_t$ is allowed to be greater than unity, which implies that the portfolio weight on money $1-x_t$ can become negative. A negative portfolio weight on money is tantamount to a short position in this asset.[380] Indeed, $1 - x_t < 0$ will always hold true in the model equilibrium. Importantly, by increasing its short position in money, an individual intermediary raises its liabilities and therefore expands its balance sheet. Therefore, depositors (implicitly) receive the legal status of creditors to the intermediary, a fact that is also a central pillar to the credit creation theory of banking, as pointed out by Werner (2014b).[381] Since the intermediary's liabilities represent private debt and serve as a medium of exchange, it creates inside money in the process.[382] Moreover, intermediaries' optimisation problems do not include a "reserve-in-advance constraint"[383], as would be required under the fractional reserve theory of banking, but rather depend on factors such as their attitudes towards risk, the return on assets, or the publics' demand for purchasing power, which feeds back into the real return on money. As mentioned in Section 2.3, these latter constraints belong to the constraints on money creation that banks face in the real world. Finally, it should be recognised that intermediaries do not intermediate loanable funds in the form of real savings, but rather create their own funding by issuing deposits. Against this backdrop, using the term "intermediaries" is problematic.

Another simplification that results from logarithmic utility is that agents consume a constant fraction of their current wealth in each instant. This fraction is equal to the time preference rate multiplied by $dt$.[384] An additional equilibrium outcome is the fraction of households' outside equity

[378] Cf. Brunnermeier and Sannikov (2014a, p. 402).
[379] Cf. Brunnermeier and Sannikov (2016a, pp. 11f.).
[380] Cf. Brunnermeier and Sannikov (2014d, p. 7).
[381] Cf. Section 2.3.
[382] Cf. Definition 4.
[383] As will be explained in the succeeding subsection, outside money can be interpreted as reserves under certain conditions. Not surprisingly, the introduction of a minimum reserve requirement would then limit the capacity of the intermediary sector *as a whole* to create inside money in the baseline model with a fixed supply of outside money.
[384] Cf. Brunnermeier and Sannikov (2016a, p. 13).

in sector $b$ held by intermediaries $\chi_t^b$. The answer to the question whether constraint $\chi_t^b \leq \overline{\chi}^b$ binds, depends on intermediaries' required return on outside equity $dr_t^{bI}$ relative to households' required return on inside equity in sector $b$ given by $dr_t^{bH}$. If $dr_t^{bI} < dr_t^{bH}$, households sell the maximum amount of equity to intermediaries. This happens when the latter are sufficiently capitalised and are willing to accept a return discount. If, on the contrary, the constraint does not bind, the required returns are equalised and households do not earn a return premium on equity.[385]

#### 3.2.2.2 Results

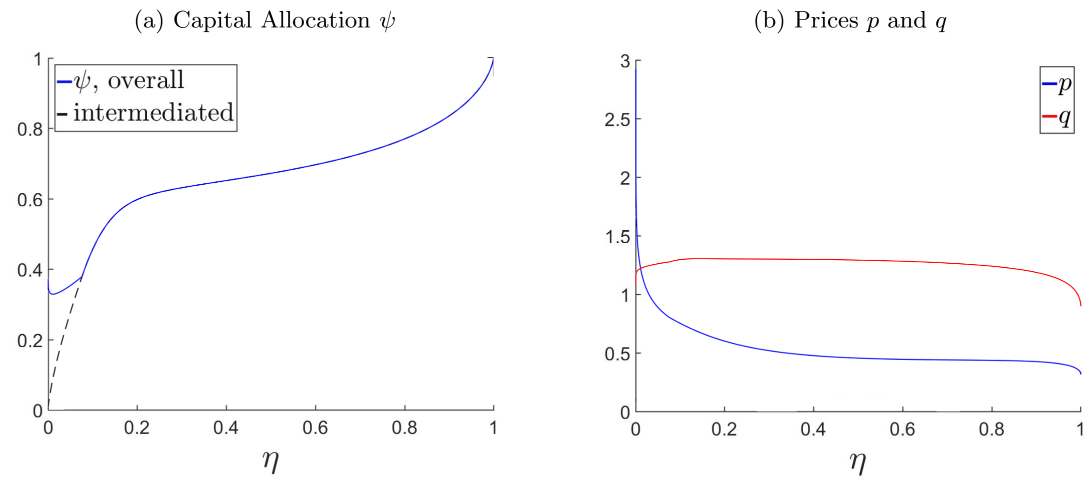
**Prices and Capital Allocation.** Figure 3.2.6 depicts some results under the baseline calibration. Panel (a) shows the share of aggregate capital allocated to sector $b$, denoted by $\psi_t$, as function of the state variable $\eta_t$, which measures intermediaries' share in aggregate wealth. The overall capital in that sector is the sum of intermediated capital, i.e. the capital financed by intermediaries' outside equity holdings, and capital financed by the inside equity of sector $b$ households. Four results are worth of notice here. First, the output maximising allocation $\psi^* = 0.5$ is not realised if intermediaries are absent from the model, that is if $\eta_t = 0$. This is because the idiosyncratic risk in sector $b$ is assumed to be greater than that in sector $a$ under the baseline calculation. Second, allocation $\psi^*$ is not reached until $\eta_t \approx 0.12$. This point will be referred to as $\eta^\psi$ in the remainder. Third, for large enough $\eta_t$ ($\eta >\approx 0.1$) capital in sector $b$ is entirely intermediated, i.e. households' active in that sector are only financed by outside equity. This is because intermediaries' required returns fall with $\eta_t$ and are lower than those of households in the range $\approx 0.1 < \eta \leq 1$. Fourth, $\psi_t$ increases with $\eta_t$. This fact reflects intermediaries' risk-aversion: the demand for risky equity stakes is an increasing function of intermediaries' aggregate net worth position.[386]

It can be observed from panel (b) that the price of capital is a nonmonotonic function of the state variable: $q_t$ rises as intermediaries become better capitalised but begins to fall at the point in the state space where the output-maximising capital allocation $\eta^\psi$ is reached. The latter fact is due to the resulting capital misallocation: since intermediaries can only buy the equity of sector $b$ households, too much capital is used in that sector relative to sector $a$. This results in depressed output due to the form of the aggregate production function and a falling price of the productive asset.

In contrast, the price of money per unit of capital is a monotonously decreasing function of $\eta_t$. Here, two main forces are at work: first, the supply of inside money increases with the state variable as intermediaries are increasingly willing to scale up their borrowing activities by issuing more and more deposits since their capital buffers rise.[387] Second, the demand for money falls with $\eta_t$. Holding money in this model is a means for individuals to reduce the exposure of their assets to idiosyncratic risk embodied in capital. Thus, the motive for holding money is the same as in Bewley

[385] Cf. Brunnermeier and Sannikov (2016a, p. 13).
[386] Cf. Brunnermeier and Sannikov (2016a, p. 22).
[387] Cf. Brunnermeier and Sannikov (2016a, p. 23).

Figure 3.2.6: Functions $\psi(\eta)$, $q(\eta)$ and $p(\eta)$ in the Brunnermeier and Sannikov (2016a) Model




*Notes*: Functions are calculated using code provided by the authors.

(1980) and Scheinkman and Weiss (1986): it serves as an insurance against uninsurable idiosyncratic risk.[388] As mentioned, households can sell off risk by issuing outside equity to intermediaries. The better intermediaries are capitalised, the more they are willing to absorb sector $b$ households' risk via outside equity purchases and the less the latter are inclined to hold money as an insurance against idiosyncratic shocks to capital.[389] In addition, there is a third effect that influences the value of money. However, the direction of this effect changes with the allocation of capital. As long as $\psi_t < \psi^*$ the *level* of output rises with the state variable due to the improving capital allocation. According to the equation of exchange this effect is conducive to the value of money. On the contrary, if $\psi_t \geq \psi^*$ any additional unit of capital allocated to sector $b$ decreases the value of money, ceteris paribus.[390] The shape of function $p(\eta_t)$ suggests, however, that this effect is small under the baseline calibration compared to the other two forces.

**Evolution of the State Variable.** As in the BS (2014a) model, a crucial question is how small exogenous shocks are amplified via price movements. The volatility of intermediaries' wealth

[388] Cf. Brunnermeier and Sannikov (2014d, p. 4) and Brunnermeier and Sannikov (2016a, p. 8).
[389] Cf. Brunnermeier and Sannikov (2016a, p. 21).
[390] The relationship between $\psi_t$, the level of output, and the value of money is not mentioned by Brunnermeier and Sannikov (2016a).

share in the baseline model can be expressed via

$$\sigma_t^\eta = \frac{x_t\left(\sigma^b \mathbf{1}^b + \sigma_t^K\right)}{1 + \underbrace{\frac{\vartheta'(\eta_t)}{\vartheta(\eta_t)/\eta_t}\left[\frac{x_t}{1-\vartheta(\eta_t)} - 1\right]}_{\text{amplification}}}, \tag{3.2.11}$$

in which $\mathbf{1}^b$ is a column vector with a 0 in position $a$ and a 1 in position $b$. Further, $\vartheta(\cdot)$ denotes the value share of money, which is defined as the value of all outside money $p_t K_t$ relative to aggregate wealth $(q_t + p_t) K_t$, as a function of $\eta_t$. In the absence of an intermediary balance sheet channel, prices do not change with the state variable, i.e. $\vartheta'(\eta_t) = 0$. Then, the volatility of intermediaries' net worth share is given by the numerator of (3.2.11), which, accordingly, can be interpreted as the fundamental risk component. We will denote this component by $\sigma_{f,t}^\eta$ in the following. It is due to the endogenous leverage governed by $x_t$ and the fact that intermediaries can only invest in $b$ projects, while the economy as a whole is exposed to shocks in both sectors.[391]

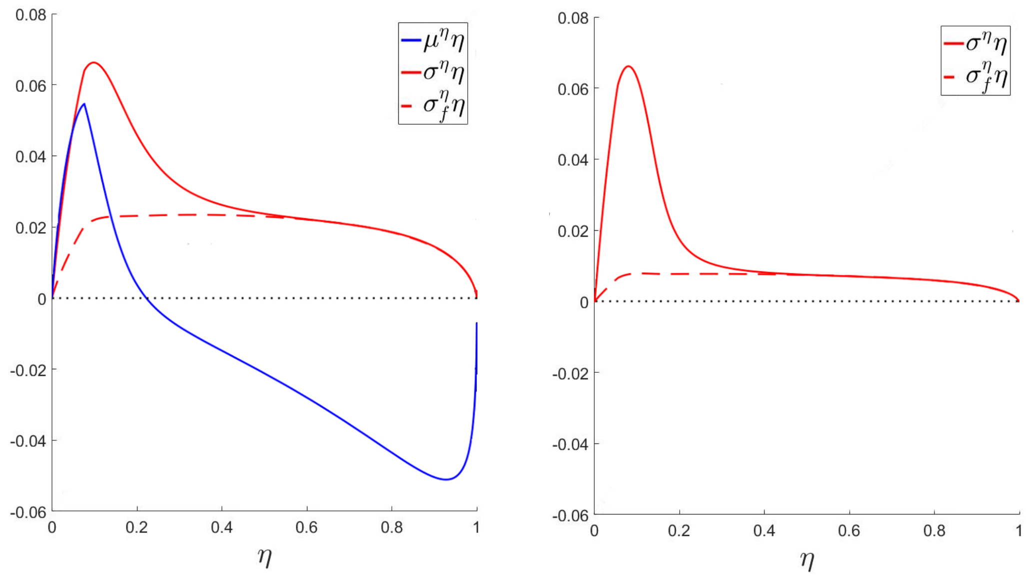
Similarly to equation (3.2.5), amplification arises from the second term in the denominator of (3.2.11). Exogenous shocks are amplified through price movements if this term is negative. This is e.g. the case if $\vartheta'(\eta_t) < 0$ and $x_t > 1 - \vartheta_t(\eta_t)$ hold true.[392] The latter condition requires intermediaries' portfolio weight on households' outside equity in sector $b$ to exceed the value share of capital. As intermediaries are leveraged at any time, i.e $x_t > 1, \forall t$, this condition is satisfied. What is more, the first condition, which requires the value of money relative to the value of capital to fall with the state variable, also holds true in equilibrium for all values of the state variable. Thus, the amplification term in (3.2.11) is negative and volatility rate $\sigma_t^\eta$ again is described by an infinite geometric series.

It can be observed from Panel (a) that volatility $\eta_t \sigma_t^\eta$ is highest around point $\eta^\psi$ in the baseline model. The main reason for this fact is that two spirals emerge if $\eta_t \in (0, \eta^\psi)$: a Fisherian *disinflationary spiral,* which boosts the value of money per unit of capital $p_t$, and, in addition, the familiar liquidity spiral due to a reduction in the price of capital. This can be seen from Panel (b): in the mentioned region $q_t$ rises and $p_t$ falls with the state variable. Thus, an exogenous adverse shock to sector $b$ which reduces intermediaries' wealth share leads to a drop in $q_t$ and a surge in $p_t$. The intuition for this is as follows. Any individual intermediary reacts to the initial losses by reducing its risk exposure. This is achieved via a reduction in leverage and sales of risky equity stakes. As all intermediaries are affected in the same way, the outside equity of $b$ households can only be sold back to the household sector. However, households are willing to accept these trades only at a discount in price $q_t$. The disinflationary spiral is mainly due to the reduction in the inside money supply, which results form the sale of equity shares to households in exchange for deposits.[393]

[391] Cf. Brunnermeier and Sannikov (2016a, pp. 23f.).
[392] Cf. Brunnermeier and Sannikov (2016a, pp. 23f.).
[393] Cf. Brunnermeier and Sannikov (2016a, p. 24).

Figure 3.2.7: Drift and Volatility of the State Variable in the Brunnermeier and Sannikov (2016a) Model

(a) Drift and Volatility of $\eta$ (baseline) (b) Drift and Volatility of $\eta$ (low $\sigma_f^\eta$)




*Notes*: Functions are calculated using code provided by the authors.

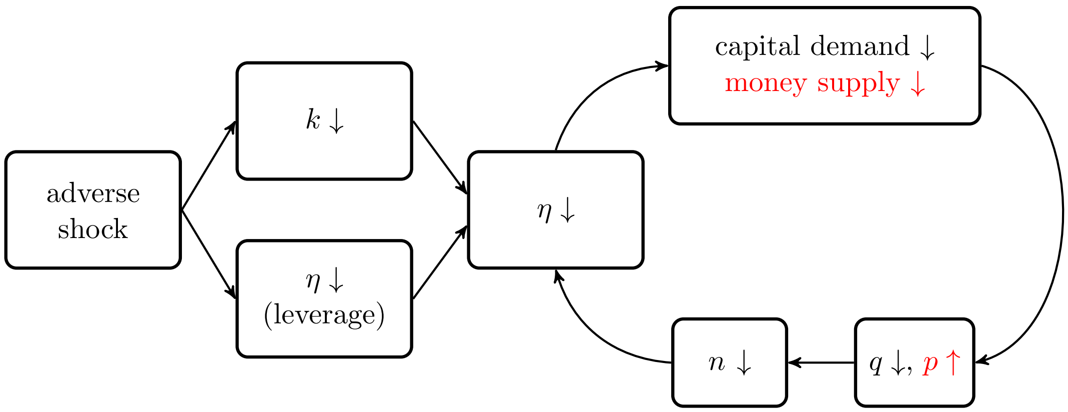
The liquidity spiral shrinks the value of assets on intermediaries' balance sheets and the disinflationary spiral increases the value of liabilities, i.e. inside money. In this way the initial reduction in aggregate intermediary net worth $n_t$ is exacerbated, resulting in further fire sales of equity shares and reductions in inside money and so on. The adverse feedback loop which emerges from the two spirals is visualised in Figure 3.2.8. The described phenomena are referred to as the "*Paradox of Prudence*" by the authors: individual intermediaries act microprudently as they reduce their leverage, however, this behaviour turns out to be macroimprudent since each individual does not take into account his contribution to the resulting changes on asset prices.[394]

Amplification is less severe for higher values of the state variable as price changes become less pronounced. As $\eta_t \to 1$, volatility $\sigma_t^\eta \eta_t$ approaches fundamental volatility $\sigma_{f,t}^\eta \eta_t$, which reflects the fact that amplification approaches zero. Drift $\mu_t^\eta \eta_t$ has a similar shape as the volatility of the state variable due to the risk-return trade-off: the more risk agents take, the higher their expected returns. Here, the stochastic steady state is the point where the drift equals zero.

**The Volatility Paradox.** The authors demonstrate that the Volatility Paradox also emerges in the present model, by comparing the model equilibrium under parameter values $\sigma^a = \sigma^b = 0.03$

[394] Cf. Brunnermeier and Sannikov (2016a, p. 24).

Figure 3.2.8: Adverse Feedback Loop in the Brunnermeier and Sannikov (2016a) Model.




*Notes*: Based on Brunnermeier and Sannikov (2014a, p. 400); the effects of the disinflationary spiral are highlighted in red.

to the results in the baseline model, in which these parameters are set according to $\sigma^a = \sigma^b = 0.1$. Panel (b) depicts the volatility of the state variable under reduced exogenous risk. Comparison to Panel (a) highlights the result that the maximum level of volatility is virtually unchanged, while the fundamental risk component is significantly reduced. The reason for this is that intermediaries operate with increased leverage ratios under lower fundamental volatility, which, in turn, leads to stronger amplification of exogenous shocks through the resulting price changes.[395]

**Inefficiencies.** Three inefficiencies arise in the model: first, the sharing of idiosyncratic risk is incomplete. Intermediaries are able to diversify individual-specific risk, but they can do so only in sector $b$. Further, the former might become undercapitalised, which hampers risk sharing within sector $b$ as well. Second, the sharing of aggregate risk is also inefficient since intermediaries are only exposed to the sector-specific risk of households active in sector $b$. Third, a productive inefficiency arises if either intermediaries or households become undercapitalised. Too much capital employed in one sector relative to the other leads to depressed output due to the assumed form of the final goods production technology. In fact, the authors show that a lumpsum redistribution of wealth from the overcapitalised to the undercapitalised sector benefits all agents in the economy at both extremes of the wealth distribution (i.e. as $\eta \to 0$ or $\eta \to 1$). This can be attributed in part to the productive inefficiency.[396]

**Monetary Policy.** In order to analyse the consequences of monetary policy, an additional class of assets, namely nominal perpetual bonds, is introduced. These bonds pay an exogenously

[395] Cf. Brunnermeier and Sannikov (2016a, p. 24).
[396] Cf. Brunnermeier and Sannikov (2016a, pp. 25f.).

fixed interest rate $i^B$ in units of outside money. The bond price $B_t$[397] follows

$$\frac{dB_t}{B_t} = \mu_t^B \, dt + \left(\sigma_t^B\right)^T \, dZ_t, \tag{3.2.12}$$

where $\sigma_t^B \equiv \left[\sigma_t^{B,a}, \sigma_t^{B,b}\right]^T$.[398] Further, the central bank sets a nominal interest rate $i_t$ on outside money. It can be assumed without loss of generality that the entire stock of outside money is in the possession of intermediaries at any time and that intermediaries issue a corresponding amount of deposits.[399] Then, outside money holdings can be interpreted as reserves those agents maintain in accounts with the central bank and interest rate $i_t$ as the rate paid by the latter on reserves. Both the interest on outside money and bonds are paid out by the central bank via the printing of new outside money.[400]

The second instrument of monetary policy are open market operations, in which the central bank trades bonds against outside money. Through these operations the central bank determines the quantity of outstanding bonds as well as the outside money supply, given interest payments on outside money and bonds. Open market operations are modelled by assuming that the central bank directly controls ratio $b_t/p_t$. In this ratio $b_t$ is the real value of *all* outstanding bonds per unit of aggregate capital $K_t$ and $p_t$ is the real value of *all* outstanding nominal assets, i.e. bonds and outside money, per unit of $K_t$.[401,402]

In the simplest case, monetary policy fixes ratio $b_t/p_t$ via open market operations and adjusts $i_t$ according to the current value of the state variable $\eta_t$. In particular, the monetary authority is assumed to lower the interest rate when intermediaries suffer losses (and vice versa) in order to achieve a negative correlation between the state variable and the bond price. Since cuts in $i_t$ lead to a surge in the bond price, any intermediary that chooses to hold a long position in bonds obtains an insurance against adverse shocks.[403] Hence, monetary policy can at least partially complete the market for aggregate risk.[404] Risk aversion implies that intermediaries indeed find it profitable to invest in bonds in equilibrium.

The effects of monetary policy on the volatility of the state variable can be understood by

[397] To be more precise, $B_t$ measures the bond price *per unit of interest* $i^B$. This distinction is, however, not material to the subsequent discussion.
[398] Cf. Brunnermeier and Sannikov (2016a, pp. 31f.).
[399] Cf. Brunnermeier and Sannikov (2016a, p. 10).
[400] Cf. Brunnermeier and Sannikov (2016a, p. 30).
[401] Cf. Brunnermeier and Sannikov (2016a, p. 31).
[402] It follows that ratio $b_t/p_t$ is the value share of bonds in the real value of all nominal assets.
[403] Cf. Brunnermeier and Sannikov (2016a, p. 33).
[404] Cf. Brunnermeier and Sannikov (2016a, pp. 35f.).

examining

$$\sigma_t^\eta = \frac{x_t\left(\sigma^b \mathbf{1}^b + \sigma_t^K\right)}{1 + \underbrace{\frac{\vartheta'(\eta_t)}{\vartheta(\eta_t)/\eta_t}\left[\frac{x_t}{1-\vartheta(\eta_t)} - 1\right]}_{\text{amplification}} - \underbrace{\frac{b_t}{p_t}\frac{B'(\eta_t)}{B(\eta_t)/\eta_t}\left[x_t + \left(\frac{1}{\eta_t} - 1\right)\vartheta(\eta_t)\right]}_{\text{mitigation}}}. \tag{3.2.13}$$

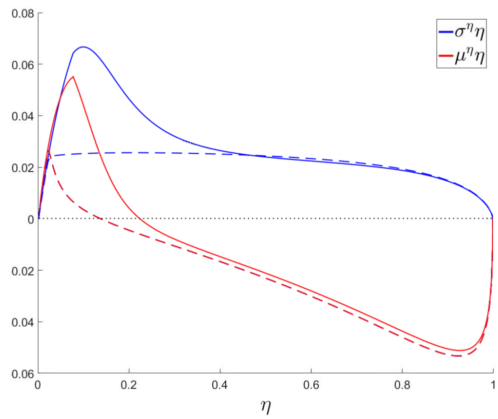
Comparison of this expression to (3.2.11) shows that a mitigation term emerges with active monetary policy. A key component of this term is the elasticity of the bond price with respect to intermediaries' wealth share $B'(\eta_t)/[B(\eta_t)/\eta_t]$. This elasticity is negative due to the interest rate policy followed by the central bank. Hence, volatility rate $\sigma_t^\eta$ is lower under a more aggressive policy that raises the elasticity of the bond price in absolute terms, ceteris paribus.[405]

Figure 3.2.9 shows the drift and volatility of the state variable under the baseline calibration and in a special case in which monetary policy completely removes endogenous risk by equalising the mitigation term in (3.2.13) to the amplification term.[406] It can be observed that volatility is significantly reduced in the crisis region, i.e. in states of the economy in which intermediaries are undercapitalised. This is due to insurance aspect of monetary policy in this model. However, as with any insurance, moral hazard arises. This is reflected in the fact that intermediaries increase their leverage ratios. This behaviour boosts the price of capital and leads to a fall in the value of money. As a consequence, intermediaries earn lower risk premia and the drift of $\eta_t$ is lower in the entire state space compared to the case without intervention by the central bank.[407] Welfare analysis shows that lifetime utility of intermediaries falls and that of households rises under a policy that removes endogenous risk compared to the baseline case. The former result arises as intermediaries earn lower risk premia. However, the authors do not provide a more general welfare analysis, which e.g. could answer questions relating to the welfare maximising monetary policy.[408]

To conclude, monetary policy has real effects, even in the absence of nominal rigidities. Policy works by affecting the dynamics of the wealth distribution and thereby influencing prices and allocations. It follows that the real consequences of central bank policy are due to wealth effects: central bank interventions in times of crisis boost the value of bonds, which improves the net worth position of intermediaries. Since no equity injections by the government are involved, the authors refer to this form of recapitalisation by the term "*stealth recapitalisation*". In contrast, in standard New Keynesian models monetary affects the real economy via substitution effects: in the presence of price inertia nominal interest rate cuts reduce the real interest rate, which induces households to prepone consumption and firms to scale up investment.[409]

[405] Cf. Brunnermeier and Sannikov (2016a, pp. 33f.).
[406] Brunnermeier and Sannikov (cf. 2016a, pp. 39f.) discuss another special case, namely a policy that sets $\sigma_t^\eta \to 0$. More general cases, such as ad-hoc policy rules, are not considered.
[407] Cf. Brunnermeier and Sannikov (2016a, pp. 35f.).
[408] Cf. Brunnermeier and Sannikov (2016a, pp. 39f.).
[409] Cf. Brunnermeier and Sannikov (2016a, pp. 3f.).

Figure 3.2.9: Effects of Monetary Policy on the Drift and Volatility of the State Variable in the Brunnermeier and Sannikov (2016a) Model




*Notes*: Functions are calculated using code provided by the authors; functions are plotted for the baseline case without policy (solid) and the case with policy (dashed).

**Macroprudential Policy.** Macroprudential policy has the potential to counteract increased risk-taking induced by countercyclical monetary policy. The authors analyse a special case in which the central bank policy allows for perfect sharing of aggregate risk and the regulator directly controls asset allocations, portfolios and asset returns. They show that welfare in the case with macroprudential and monetary policy is significantly higher for all agents relative to both the baseline model and the case with monetary policy only. The beneficial effects in the household sector arise as macroprudential regulation can prevent the former from being overexposed to idiosyncratic risk. This overexposure under monetary policy results from the fall in the value of money, which induces households to hold excessive stakes in risky capital.[410]

### 3.2.3 The Brunnermeier and Sannikov (2014d) Model

#### 3.2.3.1 Model Set-Up

**Assets.** As in the model presented in the previous subsection, in the 2014 version of the I Theory of Money agents can hold their wealth in the form of three assets: capital, inside money, and outside money. There are four other main assumptions on assets that are shared with that model. First, the value of outside money is again proportionate to the aggregate capital stock. Second, inside money does not enter aggregate wealth as its value nets out in the aggregate. These two assumptions imply that aggregate wealth is determined from

$$N_t \equiv (q_t + p_t)\, K_t, \tag{3.2.14}$$

[410] Cf. Brunnermeier and Sannikov (2016a, pp. 43f.).

as before.[411] Third, money holders regard inside money as a perfect substitute for outside money. Fourth, intermediaries can obtain external funding only by issuing deposits.[412]

A main departure from the BS (2016a) model is that fundamental risk, which will be specified momentarily, is governed by a Poisson process $\mathcal{N}_t$ with intensity $\lambda$.[413] It follows that the equation of motion for the price of capital is given by (3.1.19) and that for the price of money per unit of capital by

$$\frac{dp_t}{p_t} = \mu_t^p \, dt + \frac{\widetilde{p}_t - p_t}{p_t} \, d\mathcal{N}_t, \tag{3.2.15}$$

in which $\mu_t^p$ and $\widetilde{p}_t$ are the drift rate and the post-jump value of the price of money per unit of capital, respectively. Both asset prices are again measured in units of output.[414]

Besides intermediaries, the model features two other groups of agents: households and entrepreneurs.[415] Both households and intermediaries are allowed to own capital, which can be rent out to entrepreneurs. Entrepreneurs use capital to produce final goods.[416]

**Technologies.** As Entrepreneurs are the only agents that are endowed with a production technology, they operate the entire aggregate capital stock at any point in time. The production technology is of the standard AK type: $Y_t = aK_t$, where $a$ is entrepreneurs' productivity level.[417] The proceeds from the sales of output are always entirely passed along to the owners of capital by assumption. Formally, renting out capital is modelled as if households and intermediaries were directly holding capital, that generates payoffs in the form of final goods.

While entrepreneurs do not accumulate wealth from producing final goods, they can divert capital from capital owners, who lend the productive asset to the former.[418] To be more precise, in the event of a jump, i.e. if $d\mathcal{N}_t = 1$, a share of entrepreneurs receive the opportunity to divert the entire capital lent to them.[419] Capital diversion has different effects on households' and intermediaries' balance sheets since the latter are able to diversify their investments, while the former are not.

Let us first turn to households' capital stakes. If an innovation in Process $\mathcal{N}_t$ arrives, a share $\underline{\phi}$ of the pool of entrepreneurs that have borrowed capital from households divert the productive asset. Households are not able to diversify across projects and thus lose their entire capital with probability $\underline{\phi}$. Conversely, they do not suffer any capital losses with probability $1 - \underline{\phi}$. Any entrepreneur who

[411] Cf. Brunnermeier and Sannikov (2014d, p. 7).
[412] Cf. Brunnermeier and Sannikov (2014d, p. 7).
[413] Cf. Brunnermeier and Sannikov (2014d, p. 6).
[414] Cf. Brunnermeier and Sannikov (2014d, p. 7).
[415] In Brunnermeier and Sannikov (cf. 2014d, p. 6) entrepreneurs are referred to as "end-borrowers". Again, this notion is not correct in a narrow sense as these individuals do not issue standard debt.
[416] Cf. Brunnermeier and Sannikov (2014d, p. 6).
[417] Cf. Brunnermeier and Sannikov (2014d, p. 6).
[418] Diversion of capital corresponds to a classic setting in principal-agent theory, as discussed e.g. in DeMarzo and Fishman (2007) or Biais et al. (2007). In that literature, the agent can divert funds from projects he carries out on behalf of the principal.
[419] Cf. Brunnermeier and Sannikov (2014d, p. 6).

received the opportunity to divert capital subsequently continues his life as a regular household.[420] To formalise this form of fundamental risk, one has to introduce an additional Poisson process $\underline{\mathcal{N}}_{i,t}$ with intensity $\lambda\underline{\phi}$. Importantly, the increment of this process can only be strictly positive if $d\mathcal{N}_t = 1$.

Now, the evolution of capital a single household $i$ lends to the entrepreneurial sector is described by

$$\frac{d\underline{k}_{i,t}}{\underline{k}_{i,t}} = g\,dt - d\underline{\mathcal{N}}_{i,t}, \tag{3.2.16}$$

where $g$ is the exogenously set growth rate of capital in the absence of jumps.[421] It should be noted that the relative change in household $i$'s capital is equal to $-1$ if $d\underline{\mathcal{N}}_{i,t} = 1$. This reflects the fact that the entire capital stock is lost in that case.

There are two main differences between households' and intermediaries' equations of motion for capital. First, it is assumed that the idiosyncratic probability of capital diversion in projects funded by intermediaries $\phi$ is lower than $\underline{\phi}$. This is due to intermediaries' superior monitoring technology. Second, in case of a jump, intermediaries have to write off a *fraction* $\phi$ of their assets with certainty rather than the entire capital stock with probability $\phi$. This follows from their ability to diversify across entrepreneurs.[422] Taken together, these two facts imply that capital owned by an intermediary evolves according to

$$\frac{dk_{i,t}}{k_{i,t}} = g\,dt - \phi\,d\mathcal{N}_t. \tag{3.2.17}$$

The aggregate capital stakes of intermediaries and households change in absolute value by

$$dk_t = gk_t\,dt - \phi k_t\,d\mathcal{N}_t \quad \text{and} \quad d\underline{k}_t = g\underline{k}_t\,dt + \phi k_t\,d\mathcal{N}_t, \tag{3.2.18}$$

respectively. It is important to note that the expression for the absolute change in $\underline{k}_t$ contains term $\phi k_t\,d\mathcal{N}_t$, the absolute change in intermediaries' capital in the event of a jump. This is due to the assumption that entrepreneurs that were able to divert capital from households or intermediaries continue their lives as households. The equations in (3.2.18) also show that at the sector level capital diversion can be interpreted as *sector-specific* risk that cancels out in the aggregate: this risk is sector-specific as in case an innovation in $d\mathcal{N}_t$ occurs, the aggregate capital in the intermediary sector is reduced and that in the household sector rises. The risk agents are exposed to washes away in the aggregate since these changes in capital stakes exactly offset each other at the economy-wide level. This implies that the evolution of the aggregate capital stock $K_t \equiv k_t + \underline{k}_t$ is deterministic:[423]

$$\frac{dK_t}{K_t} = g\,dt. \tag{3.2.19}$$

[420] Cf. Brunnermeier and Sannikov (2014d, p. 6).
[421] Cf. Brunnermeier and Sannikov (2014d, p. 6).
[422] Cf. Brunnermeier and Sannikov (2014d, p. 6).
[423] Cf. Brunnermeier and Sannikov (2014d, p. 6).

The fact that aggregate capital does not change in the event of a jump prevents intermediaries from constructing risk-free portfolios, in which a long position in capital is entirely financed through deposits.[424]

**Preferences.** Intermediaries as well as households are assumed to have logarithmic preferences[425] with expected discounted lifetime utilities

$$\mathbb{E}_0 \left[ \int_0^\infty e^{-\rho t} \log c_{i,t} \, dt \right] \quad \text{and} \quad \mathbb{E}_0 \left[ \int_0^\infty e^{-r t} \log \underline{c}_{i,t} \, dt \right],$$

respectively. The time preference rates satisfy $\rho > r$, i.e. intermediaries are more impatient than households.[426] Entrepreneurs' preferences are not specified. This is unproblematic since the payoffs of the projects they manage are always entirely paid out to the owners of capital.

**Financial Structure.** On an intuitive level, the lending of capital to entrepreneurs corresponds to a financial structure in which these agents are exclusively financed by outside equity, which is held by households and intermediaries. This is because lenders of capital receive the full payoffs generated by entrepreneurs. In addition, changes in the value of capital are fully absorbed by the former. According to this interpretation, variations in the price of capital arise from the trading of shares between different groups of investors.

There are two financial frictions in the model: first, and in parallel to the latest version of the I Theory of Money, intermediaries cannot issue equity to households. That is, they can obtain outside funds only by accepting deposits from households. Second, households, in distinction from intermediaries, do not possess the ability to diversify their asset holdings. Figure 3.2.10 shows agents' balance sheets. As before, intermediaries are the end borrowers in the economy.

**Returns.** When deriving agents' returns to capital, an additional assumption that has not yet been mentioned must be taken into account: the government taxes output net of investment at a constant and proportionate rate $\tau \in [0, 1)$.[427] This fact together with the form of the production function, CVF (3.1.20), as well as equations (3.2.17) and (3.1.19) leads to the following formula for intermediaries' real return on capital:[428]

$$dr_t^K = \underbrace{\left\{ \frac{(1-\tau)\, a}{q_t} + g + \mu_t^q \right\}}_{\equiv \mu_t^{r^K}} dt + \frac{(1-\phi)\, \widetilde{q}_t - q_t}{q_t} \, d\mathcal{N}_t, \tag{3.2.20}$$

[424] If e.g. aggregate capital were to fall by factor $\phi$, construction of such portfolios would be possible. Then, the economy would immediately jump to the first-best solution (cf. the discussion in Subsection 3.2.2).

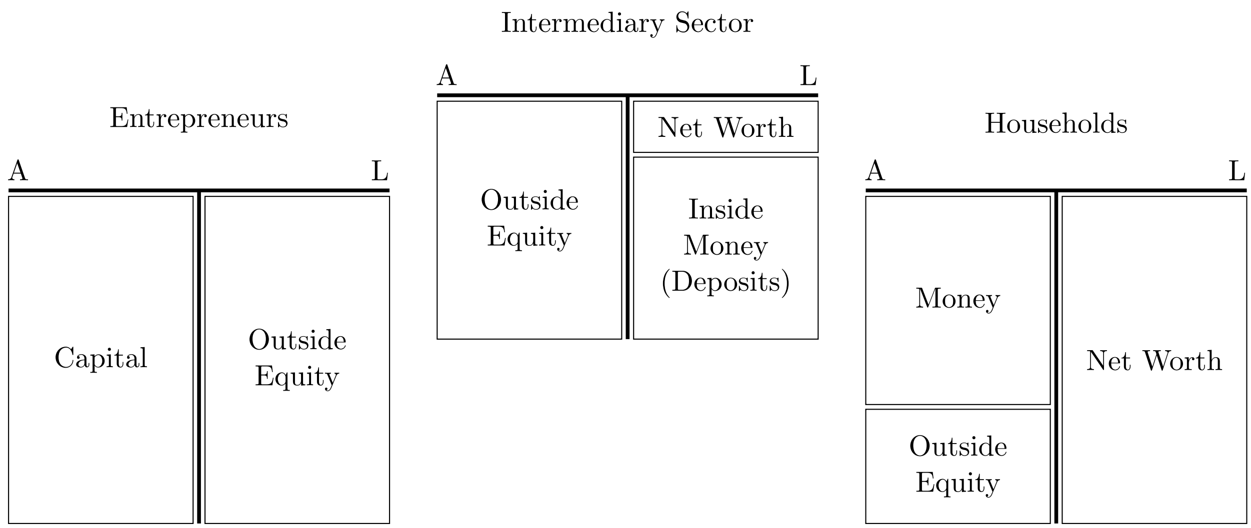
[425] Cf. Brunnermeier and Sannikov (2014d, p. 6).

[426] Cf. Brunnermeier and Sannikov (2014d, p. 6).

[427] Cf. Brunnermeier and Sannikov (2014d, p. 7).

[428] In (3.2.20) we have dropped subscript $i$ since it does not contain idiosyncratic elements.

Figure 3.2.10: Agents' Balance Sheets in the Brunnermeier and Sannikov (2014d) Model




*Notes*: Adapted from Brunnermeier and Sannikov (2016a, p. 10).

in which $\mu_t^{r^K}$ is the drift rate of the stochastic process for the return on capital. The real return on capital owned by households, on the other hand, is

$$d\underline{r}^K_{i,t} = \underbrace{\left\{\frac{(1-\tau)\,a}{q_t} + g + \mu^q_t\right\}}_{=\mu_t^{r^K}} dt - d\underline{\mathcal{N}}_{i,t} + \frac{\widetilde{q}_t - q_t}{q_t}\, d\overline{\mathcal{N}}_{i,t}, \tag{3.2.21}$$

where $\overline{\mathcal{N}}_{i,t}$ is an idiosyncratic Poisson process, whose increment takes on the value 1 if $d\underline{\mathcal{N}}_{i,t} = 0$, i.e. if the agent does not draw the capital diversion shock.

By assumption, the government redistributes the proceeds from taxes on output net of investment to holders of outside money. Introducing this type of fiscal backing of money generates an additional motive for holding money, namely a return motive. This has the advantage that equilibria in which money has no value cannot occur, even for extremely small values of $\tau$.[429,430] Taking into account the redistribution of tax proceeds, as well as equations (3.1.20), (3.2.15), and (3.2.19), we get for the real return on outside money:

$$dr^M_t = \underbrace{\left\{\frac{\tau a}{p_t} + g + \mu^p_t\right\}}_{\equiv\mu_t^{r^M}} dt + \frac{\widetilde{p}_t - p_t}{p_t}\, d\mathcal{N}_t. \tag{3.2.22}$$

[429] Cf. Brunnermeier and Sannikov (2014d, p. 7).

[430] Brunnermeier and Sannikov (cf. 2014d, fn. 3) note that in this regard the model resembles the *Fiscal Theory of the Price Level* advocated e.g. by Leeper (1991) or Woodford (1995).

In order to obtain outside financing by issuing deposits, intermediaries have to offer depositors a return at least as high as the return on outside money. The reason is that inside money and outside money are perfect substitutes from the perspective of investors. Thus, in equilibrium the real returns on inside and outside money are equalised.

**Choices.** Intermediaries' and households' portfolio choices can be described by means of two general portfolio weight functions

$$x_t = x\left(\mu_t^{r^K}, \mu_t^{r^M}, \frac{\widetilde{q}_t}{q_t}, \frac{\widetilde{p}_t}{p_t}, \lambda, \phi\right) \quad \text{and} \quad \underline{x}_t = \underline{x}\left(\mu_t^{r^K}, \mu_t^{r^M}, \frac{\widetilde{q}_t}{q_t}, \frac{\widetilde{p}_t}{p_t}, \lambda, \underline{\phi}\right), \tag{3.2.23}$$

where $x_t$ and $\underline{x}_t$ are intermediaries' and households' desired portfolio weights on capital, respectively. Since there are only two assets in the economy, $1 - x_t$ and $1 - \underline{x}_t$ are the corresponding portfolio weights on money. Function $x_t$ satisfies

$$\frac{\partial x(\cdot)}{\partial \mu_t^{r^K}} > 0, \frac{\partial x(\cdot)}{\partial \mu_t^{r^M}} < 0, \frac{\partial x(\cdot)}{\partial (\widetilde{q}_t/q_t)} > 0, \frac{\partial x(\cdot)}{\partial (\widetilde{p}_t/p_t)} < 0, \frac{\partial x(\cdot)}{\partial \lambda} < 0, \frac{\partial x(\cdot)}{\partial \phi} < 0. \tag{3.2.24}$$

Function $\underline{x}(\cdot)$ obeys analogous properties. The last two inequalities in (3.2.24) reflect the fact that increases in the frequency and severity of adverse shocks to capital tilt the agent's portfolio choice towards money, ceteris paribus.

Since intermediaries and households face the same drift rates of asset returns, the allocation of capital between the two sectors is driven by the difference in returns in case of a jump. As mentioned, intermediaries have two advantages over households in funding entrepreneurs' projects: they have a superior monitoring technology, which is reflected by assumption $\underline{\phi} > \phi$, and they can diversify across projects. The ability to diversify benefits intermediaries as risk averse agents in general prefer a small loss with high probability to a large loss with low probability, provided that losses in the two cases are equal in expectation. As a result of the two advantages, intermediaries have a higher willingness to pay for an additional unit of capital than households if both hold identical portfolios.

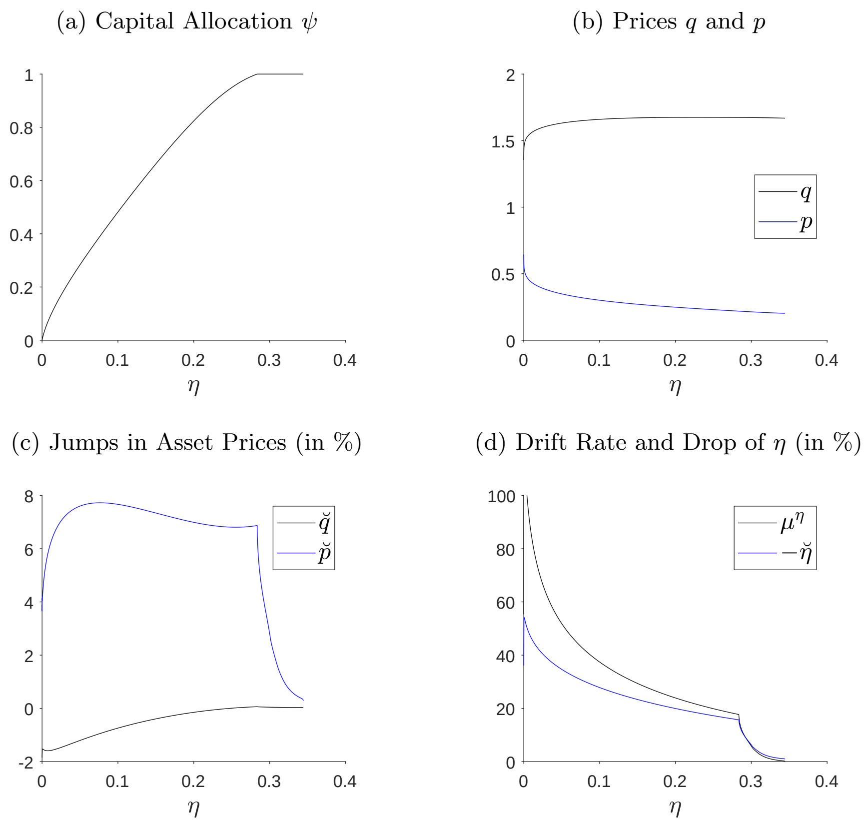
Households will always hold some strictly positive amount of money since investing their entire wealth into capital would entail a strictly positive probability of losing everything. Logarithmic preferences imply that utility would approach $-\infty$ in that case.[431] Thus, a part of households' demand for money stems from a precautionary motive, as in BS (2016a).[432]

[431] Cf. Brunnermeier and Sannikov (2014d, p. 7).
[432] Cf. Brunnermeier and Sannikov (2014d, p. 4).

#### 3.2.3.2 Results

Figure 3.2.11 depicts some important equilibrium functions under the baseline calibration, which entails parameter constellation $\lambda = 1$, $\phi = 0.002$, and $\underline{\phi} = 0.02$.[433] The equilibrium functions

Figure 3.2.11: Key Equilibrium Functions in the Brunnermeier and Sannikov (2014d) Model




*Notes*: Functions are calculated using code written on the basis of Brunnermeier and Sannikov (2014d, Appendix A).

depend on the single state variable $\eta_t$, which stands for intermediaries' wealth share.[434,435] Panel (a) shows the familiar result that intermediaries' share in the aggregate capital stock $\psi_t \equiv k_t/K_t$ increases with $\eta_t$ due to their risk averse behaviour. The optimal capital allocation in the model economy is $\psi^* = 1$ since the maximum attainable degree of risk-sharing under incomplete markets is achieved once intermediaries hold the entire capital stock.[436] Allocation $\psi^*$ is reached at $\eta^\psi \approx 0.28$. Two other main outcomes are displayed in Panel (b). First, the capital price is a monotonously increasing function of the state variable. This is due to the fact that the demand for capital by

433 Cf. Brunnermeier and Sannikov (2014d, p. 4).
434 Cf. Brunnermeier and Sannikov (2014d, pp. 13f.).
435 The state space can be reduced to dimension unity for two reasons: first, entrepreneurs' wealth is equal to zero at any point in time and second, the scale invariance property holds (cf. Brunnermeier and Sannikov, 2014d, p. 13).
436 Cf. Brunnermeier and Sannikov (2014d, p. 15).

intermediaries, who value capital more than households across the entire state space, rises with $\eta_t$.[437] Second, the price of money per unit of aggregate capital $p_t$ falls with intermediaries' wealth share. The reasons are very similar to those in the BS (2016a) model. First, as intermediaries become better capitalised, they create more inside money, which leads to a depreciation in the value of money, ceteris paribus. Second, the precautionary demand for money falls with $\eta_t$ since households offload more and more risk embodied in capital to intermediaries. This effect further erodes the value of money.

The two key properties of the price functions are reflected in the shapes of functions $\breve{q}(\eta_t)$ and $\breve{p}(\eta_t)$, which measure the percentage changes due to a jump in prices $q_t$ and $p_t$, respectively.[438] As can be seen from Panel (c), the arrival of a jump causes the price of capital to fall. Leveraged intermediaries experience a drop in their net worth positions and thus sell capital to households. The effect is most accentuated within the lower region of the state space since intermediaries' leverage is relatively high in that region. Further, the economy experiences deflationary pressure after the occurrence of a jump. This is due to a drop in the supply of inside money and a boost in the precautionary demand for money by households.[439] Function $\breve{p}(\eta_t)$ resembles an inverted U: for low values of the state variable the percentage appreciation in the value of money is increasing in $\eta$ since inside money initially is a small percentage of the total money supply. As the size of the intermediary sector increases, the mentioned share does as well and, accordingly, the drop in the supply of inside money due to a jump has stronger repercussions for the value of money. At $\eta_t \approx 0.07$ function $\breve{p}(\eta_t)$ begins to fall since the decrease in the supply of inside money is less severe as intermediaries become better capitalised. A kink occurs at point $\eta^{\psi}$, which marks the end of the fire sale region. After that point, adverse shocks do not induce intermediaries to sell capital to households. Accordingly, changes in asset prices are small.[440] To summarise, the arising adverse feedback loop features the same mechanisms as in the BS (2016a) model.

Finally, Panel (d) depicts the dynamics of the state variable. Drift rate $\mu_t^{\eta}$ as well as the percentage drop in the state variable due to a jump $-\breve{\eta}_t$ are decreasing functions of intermediaries' wealth share. In the lower domain of the state space, the intermediary sector's growth rate is large since leverage ratios within that sector and risk premia are high. As $\eta_t$ rises, endogenous price risk and leverage ratios fall, and so does $\mu_t^{\eta}$.[441] The stochastic steady state is approached around $\eta_t = 0.35$.[442] Function $-\breve{\eta}(\eta_t)$ comoves closely with $\mu^{\eta}(\eta_t)$ as a result of the risk-return trade-off, analogously to the previously discussed models. The modelling and implications of monetary policy in BS (2014d, Section 3) are very similar to those in BS (2016a) and are thus not discussed here.

[437] Cf. Brunnermeier and Sannikov (2014d, p. 17).
[438] In the remainder, all variables with a breve ( $\breve{}$ ) will denote percentage changes resulting from a jump.
[439] Cf. Brunnermeier and Sannikov (2014d, p. 16).
[440] Cf. Brunnermeier and Sannikov (2014d, p. 17).
[441] Cf. Brunnermeier and Sannikov (2014d, p. 17).
[442] The algorithm does not fully reach the stochastic steady state identified by condition $\mu_t^{\eta} = 0$. Rather, the integration of the differential equation is aborted at a point where $\mu_t^{\eta} \approx 0.1$ percent due to the violation of a boundary condition.

### 3.2.4 Other Literature

#### 3.2.4.1 The He and Krishnamurthy (2013) Model

The model is inhibited by households and specialists, both assumed to be risk averse, and features two assets: a risky asset and a short-term riskless bond. The risky asset generates a random payoff, which follows a geometric Brownian motion. The payoff of the risky asset can either be consumed or invested and thus plays the role of a final good. The parameter functions of the stochastic process for the payoff are fully exogenous. This implies that production is not endogenised as e.g. in the Lucas (1978) endowment economy.[443]

At each point in time every specialist is randomly matched with a single household to create an intermediary. After an instant of time has passed, each of these matches is dissolved. Specialists are the decision-makers within intermediaries and choose the asset allocation policies of these entities to maximise their own welfare.[444] The household sector consists of two different types. The first type are "risky asset households", who can invest in the riskless bond emitted by intermediaries as well as in intermediary equity. Individuals of the second type are "debt households", who are only allowed to hold the risk-free bond. The reason for introducing debt households is that in the absence of these individuals the demand for the risk-free asset would be zero in the unconstrained region of the state space, which will be defined momentarily. Accordingly, intermediary leverage would always be zero in that region, which is, of course, counterfactual.[445]

Intermediaries' holdings of the risky asset are financed by shorting the risk-free bond and by issuing equity to specialists and households. The profits from intermediation activities are split between the two types of agents in proportion to their initial equity contributions.[446] Households can access the market for the risky asset only indirectly by purchasing intermediaries' equity. The rationale for this assumption is that investing in the risky asset requires a degree of sophistication which is only available to specialists.[447,448]

At the heart of the model is an equity constraint which requires a household's wealth contribution to the respective intermediary to not exceed a certain threshold. That threshold is proportionate to the contribution of the matched specialist.[449] In a companion paper He and Krishnamurthy (2012) show that this "skin in the game constraint" arises as an optimal device to align the interests of inside and outside investors in the face of a specific form of moral hazard.[450] Intuitively, households demand specialists to contribute sufficient wealth to the mutual projects. The equity constraint ties

[443] Cf. He and Krishnamurthy (2013, p. 736).
[444] Cf. He and Krishnamurthy (2013, pp. 736f.).
[445] Cf. He and Krishnamurthy (2013, pp. 739f.).
[446] Cf. He and Krishnamurthy (2013, p. 738).
[447] Cf. He and Krishnamurthy (2013, pp. 735f.).
[448] As He and Krishnamurthy (cf. 2013, p. 734) remark, in that sense the model is related to the limited participation literature, such as Basak and Cuoco (1998).
[449] Cf. He and Krishnamurthy (2013, p. 738).
[450] Cf. He and Krishnamurthy (2012, Section 2).

the inside investor's wealth level to his access to external finance, similarly e.g. to the constraint in KM, and thus introduces an amplification channel.[451]

The equity constraint establishes a strong link between specialists' wealth share and the premium on the risky asset in the constrained region, i.e. the region in which the equity constraint binds. Reductions in specialists' relative net worth position increase intermediaries' leverage ceteris paribus. Besides the direct effect via specialists' inside equity contribution to intermediation projects this is due to the indirect effect arising from the equity constraint, which induces households to supply less outside equity. The risk averse behaviour of specialists then implies that a more leveraged position in the risky asset is only accepted if the risk premium rises. In the unconstrained region the link between specialists' wealth share and the risk premium is much weaker since reductions in specialists' wealth do not cause households to cut their outside equity contribution.[452]

A careful calibration of the model parameters allows the authors to make qualified quantitative predictions about the consequences of negative shocks to the payoff of the risky asset. Specifically, they choose the parameters of the stochastic process for the return on the risky asset to match precrisis moments of risk premia in the U.S. mortgage backed securities markets in the region in which the equity constraint does not bind.[453] The model results show that the risk premium is a decreasing function of the single state variable, namely specialists' aggregate wealth share, in the constrained region. In contrast, in the subset of the state space in which the equity constraint does not bind the risk premium is nearly constant. Due to the equity constraint that shape is closely mimicked by the function that maps the state variable to intermediaries' leverage ratio.[454]

Moreover, they show that the probability for the risk premium on the risky asset to exceed twice its normal level is 1.33 percent. This indicates that the probability of crisis episodes in the model is rather low.[455] Further, starting from a crisis state, in which the risk premium is as high as 12 percent, the risk premium returns halfway to the precrisis level within about 10 months. This figure is consistent with U.S. data in the 1998 crisis episode.[456]

The authors also analyse the effects of three government policies that are aimed at accelerating the speed of recovery out of a given crisis state: (i) subsidies on intermediaries' borrowing rates, (ii) purchases of the risky asset, and (iii) direct equity injections into intermediaries. Each of these policies are financed by lump sum taxes imposed on households. Importantly, the policies are unanticipated by the agents in the private sectors of the economy.[457] Out of the three examined policies, the equity injection policy is demonstrated to provide the most "bang for the buck": the recovery time can be significantly reduced by injecting only a relatively small amount of equity. The reason is that equity provision by the government tackles the source of inefficiency, namely the

[451] Cf. He and Krishnamurthy (2013, p. 738).
[452] Cf. He and Krishnamurthy (2013, pp. 743f.).
[453] Cf. He and Krishnamurthy (2013, p. 747).
[454] Cf. He and Krishnamurthy (2013, pp. 748ff.).
[455] Cf. He and Krishnamurthy (2013, pp. 754f.).
[456] Cf. He and Krishnamurthy (2013, pp. 756f.).
[457] Cf. He and Krishnamurthy (2013, pp. 758f.).

equity constraint, at its core.[458]

#### 3.2.4.2 The Adrian and Boyarchenko (2012) Model

Most CTMF models feature countercyclical leverage: leverage ratios of constrained agents are typically decreasing functions of the state variable.[459,460] High (low) levels of the state variable, in turn, are mostly associated with high (low) asset prices and overall strong (weak) economic conditions. Yet, recent empirical research suggests that leverage of financial firms is *procyclical*. For instance, Adrian and Shin (2010) present evidence that growth in marked-to-market assets of U.S. security brokers and dealers[461] was positively related to their leverage growth in the years leading up to the crisis of 2007/08.[462] As growth in marked-to-market balance sheets is usually accompanied by surging asset prices, this result hints at procyclical leverage.[463] According to the authors, a possible explanation for this result lies in the fact that financial intermediaries try to maintain a constant Value-at-Risk (VaR) to equity ratio.[464] Possible reasons for that behaviour are regulatory requirements resulting from the Basel capital accord or reasons relating to credit ratings.

In light of these results, Adrian and Boyarchenko (2012) develop a model in which intermediaries are subject to a risk-based equity constraint imposed by the regulator. This constraint sets a lower bound on intermediaries' equity. The lower bound is positively related to the risk on the asset sides of intermediaries' balance sheet.[465,466] Intermediaries, as households, can invest in the capital stock of projects endowed with an AK technology, in which the productivity level follows a geometric Brownian motion.[467] Intermediaries have a comparative advantage in these activities over households as the projects they invest in have recourse to an investment technology, while the projects managed by households have not.[468] Capital holdings by intermediaries are financed by retained earnings and bond issues to households. Bonds mature with strictly positive probability

[458] Cf. He and Krishnamurthy (2013, p. 762).
[459] This can e.g. be observed from Panel (c) in Figure 3.2.3, which depicts experts' leverage ratio as a function of the state variable in the Brunnermeier and Sannikov (2014a) model.
[460] Procyclical leverage is not only a prediction of some CTMF models, but also arises in more traditional business cycle models with financial frictions, such as the BGG and KM models (cf. Jakab and Kumhof, 2015, p. 30). Interestingly, an exception is the aforementioned DSGE model by Jakab and Kumhof (cf. 2015, pp. 30f.). They argue that the procyclicality in their model derives from banks' ability to create their own funding in the form of deposits in the act of credit extension.
[461] That category includes investment banks.
[462] Cf. Adrian and Shin (2010, Section 3).
[463] Further evidence is offered by Adrian et al. (cf. 2010, Section 4.2), who find in a dataset including U.S. financial intermediaries that security-dealer leverage growth predicts lower excess returns of equity and corporate bonds and, accordingly, higher security prices.
[464] If that ratio is fixed, the countercyclical nature of VaR directly implies procyclical leverage. For a detailed explanation cf. Adrian and Shin (2010, p. 431).
[465] Cf. Adrian and Boyarchenko (2012, p. 10).
[466] The risk-based capital constraint in the model is similar to a traditional VaR constraint. However, in contrast to a VaR constraint, the former does not imply a constant volatility of intermediaries' equity (cf. Adrian and Boyarchenko, 2012, p. 10).
[467] Cf. Adrian and Boyarchenko (2012, pp. 5f.).
[468] Cf. Adrian and Boyarchenko (2012, pp. 9f.).

at each point time.[469] Further, intermediaries are assumed to be extremely myopic in the sense that they are only concerned about the current mean growth and variance of their portfolios.[470] Myopia on the side of intermediaries in combination with the assumption that bonds randomly mature introduces a role for default of the representative intermediary and thus for systemic risk in the financial sector.[471,472]

Households are risk averse with logarithmic utility and allocate their wealth between capital, bonds and the risk-free asset, the latter of which can also be shorted. In addition, households are potentially exposed to a preference shock, which alters their effective discount rate. This type of shock can be interpreted as a liquidity preference shock. It guarantees that households always hold intermediary debt as well as capital in equilibrium.[473]

The authors assume that intermediaries' risk aversion is sufficiently low for the risk-based equity constraint to always bind.[474] Hence, leverage of the intermediary sector is always dictated by that constraint and becomes procyclical. In this environment, adverse productivity shocks are associated with subsequent decreases in asset return volatility. The intuition for this result is as follows. The negative productivity disturbance leads to losses on banks' balance sheets, which induces these agents to sell capital to households, who attribute a lower valuation to capital. The resulting drop in the price of capital increases banks' leverage, which is accompanied by lower return volatility due to the risk-based equity constraint.[475] The productivity shock is shown to have persistent effects on leverage in the financial sector as equilibrium volatility remains depressed in the long-run.[476] Amplification and the possibility of defaults in the intermediary sector give rise to a variation of the Volatility Paradox: the probability of intermediary default is demonstrated to be a *decreasing* function of return volatility. The reason is that low volatility is associated with high intermediary leverage due to the risk-based equity constraint. High leverage, in turn, makes default more likely.[477]

Welfare analysis shows that households' expected discounted lifetime utility is an inversely U-shaped function of the tightness of the equity constraint. On the one hand, an overly loose constraint increases the probability of intermediary distress. This is detrimental to households' welfare since these agents ultimately have to bear losses which push intermediaries into bankruptcy. On the other hand, an excessively tight constraint mitigates the risk-sharing capacity provided by intermediaries.[478] Optimal macroprudential policy balances these two counteracting forces.[479]

[469] Cf. Adrian and Boyarchenko (2012, p. 7).
[470] Cf. Adrian and Boyarchenko (2012, p. 12).
[471] Cf. Adrian and Boyarchenko (2012, fn. 5).
[472] Note that default risk is not present in the previously discussed CTMF models.
[473] Cf. Adrian and Boyarchenko (2012, pp. 6f.).
[474] Cf. Adrian and Boyarchenko (2012, p. 13).
[475] Cf. Adrian and Boyarchenko (2012, p. 17).
[476] Cf. Adrian and Boyarchenko (2012, pp. 19f.).
[477] Cf. Adrian and Boyarchenko (2012, pp. 25f.).
[478] Cf. Adrian and Boyarchenko (2012, p. 30).
[479] Cf. Adrian and Boyarchenko (2012, pp. 31f.).

#### 3.2.4.3 The Di Tella (2017) Model

Di Tella (2017) introduces a complete financial market which allows agents to trade experts' equity into a setting that is based on the BS (2014a) model.[480] There are four further departures from this model. First, experts are allowed to issue equity. However, they must retain an exogenous fraction of their inside equity due to a moral hazard problem. Second, in addition to aggregate total factor productivity (TFP) shocks, experts are subject to idiosyncratic risk when holding capital and to aggregate uncertainty shocks which lead to stochastic variations in the volatility of the idiosyncratic shock. Third, only experts are allowed to hold capital, which excludes the possibility of fire sales of capital. Fourth, both experts and households have general Epstein-Zin preferences.[481]

The author shows that the balance sheet channel in BS (2014a) disappears when uncertainty shocks are switched off. That is, experts' wealth share does not play any role in amplifying exogenous shocks.[482] The intuition for this result is that the complete financial market allows experts to hedge the aggregate risk associated with holding capital by an opposite position in the market portfolio.[483] Thus, experts can separately decide on how much capital to hold and on how much aggregate risk to bear. What is more, idiosyncratic shocks cancel out in the aggregate and thus do not affect the aggregate wealth share of experts. Since the only source of aggregate risk on experts' balance sheets can be removed through hedging, the economy follows a deterministic path up to the changes in the aggregate effective capital stock induced by the aggregate TFP shock.[484]

With the possibility of uncertainty shocks, experts demand a higher risk premium if this type of shock arrives.[485] This leads to a drop in the price of capital, which is tantamount to an increase in experts' investment opportunities relative to households' since the latter cannot hold capital. Risk averse experts want to stabilise their utility across different states of the world and thus prefer to have low net worth in states where investment opportunities are high.[486] Hence, experts' leverage ratios increase as these individuals suffer losses due to the induced fall in the capital price. The higher exposure to idiosyncratic risk is only accepted by experts if the price of capital falls further. In effect, a two-way feedback loop between weak balance sheets of experts and asset prices emerges, even though the model features a complete financial market. That loop gives rise to balance sheet recessions since investment is linked to the price of capital as in BS (2014a).[487]

[480] Cf. Di Tella (2017, p. 2045).
[481] Cf. Di Tella (2017, pp. 2045f.).
[482] Cf. Di Tella (2017, Proposition 2).
[483] In that sense, the balance sheet neutrality result is reminiscent of the finding in Carlstrom et al. (2016), who show that allowing for debt contracts that are contingent on the realisation of aggregate risk significantly dampens the financial accelerator in the BGG model (cf. Section 2.2.4 for details).
[484] Cf. Di Tella (2017, p. 2055).
[485] Cf. Di Tella (2017, p. 2057).
[486] Cf. Di Tella (2017, p. 2053).
[487] Cf. Di Tella (2017, pp. 2057f.).

#### 3.2.4.4 The Brunnermeier and Sannikov (2015) Model

Brunnermeier and Sannikov (2015) apply the basic CTMF methodology to an international setting. Their two-country, two-(intermediate-)good model with incomplete markets allows them to study the macroeconomic consequences of capital controls that close the international debt market. Each firm in one particular country has a comparative advantage over firms in the respective other in producing one of the two intermediate goods. Hence, specialisation in production is beneficial. The intermediate goods are combined at the final stage of production by means of a constant elasticity of substitution technology.[488] The basic financial friction in this model is that domestic firms cannot issue equity to their foreign counterparts and vice versa. This can be interpreted as an “equity home bias”, which is well documented in the empirical international finance literature, according to the authors. In the baseline model, a representative firm in one country can obtain funds via retained earnings or short-term borrowing from the foreign representative firm.[489]

After the occurrence of an adverse shock, firms in the affected country suffer balance sheet losses, which are amplified by leverage and movements in the price of capital. These effects are mitigated by a so-called ”terms of trade hedge”: as firms sell capital to reduce their risk exposures, the supply of goods produced by these firms decreases. This boosts the price of export goods and thus the terms of trade as well.[490] When balance sheets in one country are weak, further losses can lead to “sudden stops” of funding. On the contrary, if the country is spared from additional negative shocks, short-term debt allows them to recover quickly. These episodes are referred to as “Phoenix miracles”.[491]

The competitive equilibrium is not constrained efficient since agents do not internalise the price effects of their individual decisions.[492] The authors also examine whether a closed capital account can improve welfare relative to the case with unrestricted debt markets. Closing the international debt market improves stability since leverage is prevented and the terms of trade hedge becomes more powerful. The latter effect results from the fact that with the possibility to borrow production in the indebted country is higher relative to the case with capital controls. Thus, the price of export goods is in general lower and this undermines the terms of trade hedge. However, the absence of debt financing is detrimental to growth as the persistence of capital misallocation is exacerbated.[493] The welfare analysis shows that closing the capital account can indeed be beneficial. The magnitude of the welfare improvement depends on parameters such as the elasticity of substitution between the two intermediate goods and the comparative advantage between countries.[494]

[488] Cf. Brunnermeier and Sannikov (2015, p. 302).
[489] Cf. Brunnermeier and Sannikov (2015, p. 304).
[490] Cf. Brunnermeier and Sannikov (2015, p. 310).
[491] Cf. Brunnermeier and Sannikov (2015, pp. 315ff.).
[492] Cf. Brunnermeier and Sannikov (2015, p. 322).
[493] Cf. Brunnermeier and Sannikov (2015, pp. 320f.).
[494] Cf. Brunnermeier and Sannikov (2015, pp. 325ff.).

#### 3.2.4.5 The Phelan (2016) Model

Phelan (2016) employs a two-(intermediate-)good model with banks that are owned by households to investigate the effects of leverage regulation. Households and banks have a comparative advantage over the respective other sector in managing the production of one specific good.[495] The key frictions in this model are that banks' equity and dividends to households must not become negative. The latter restriction implies that banks are not allowed to issue new equity to households after adverse shocks. Banks can fund themselves only from retained earnings or by borrowing from households at the risk free rate. A strictly positive demand for banks' debt is ensured by a liquidity-in-the-utility motive.[496] Similarly to the models discussed before, these assumptions imply that banks' capitalisation is crucial in determining equilibrium dynamics.[497]

The paper compares welfare in the competitive equilibrium to welfare in equilibria under differing exogenous and endogenous leverage constraints. The results show that the competitive equilibrium is not constrained Pareto efficient.[498] There is room for macroprudential policy to improve welfare since banks do not take into account the higher volatility of the asset price and the aggregate level of banks' equity when levering up their portfolios. The trade-off the regulator faces is that limiting leverage depresses flow utility in the constrained region due to the misallocation of the productive asset, while it also reduces the volatility of banks' aggregate wealth share, which makes those states less likely.[499] The effect of leverage regulation on welfare is shown to be nonmonotonic: while intermediate leverage restrictions can improve welfare across the entire state space, overly tight leverage constraints reduce welfare relative to the competitive equilibrium.[500]

#### 3.2.4.6 The Klimenko et al. (2016) Model

Klimenko et al. (2016) also examine the repercussions of minimum equity ratios on bank policies and financial stability. There are three main differences to Phelan's (2016) setup: first, banks extend credit to productive firms rather than managing investment projects themselves.[501] Second, issuance of additional bank equity to shareholders is possible but entails a deadweight cost. The cost of issuance creates a crucial role for the accumulation of equity via retained earnings in determining equilibrium dynamics. Third, there is only one good in the economy. This good can be consumed or used for investment purposes and is the sole input in the production process.[502]

Banks obtain funds by offering deposits to households. The demand for deposits derives from a

[495] Cf. Phelan (2016, p. 202).
[496] Cf. Phelan (2016, p. 203).
[497] Cf. Phelan (2016, p. 206).
[498] Cf. Phelan (2016, p. 214).
[499] Cf. Phelan (2016, pp. 218f.).
[500] Cf. Phelan (2016, p. 217).
[501] As mentioned, in the previously discussed CTMF models with intermediaries, these agents invest in the net worth of productive agents. On the contrary, credit financing effectively separates production and intermediation activities (cf. Klimenko et al., 2016, p. 3).
[502] Cf. Klimenko et al. (2016, pp. 5ff.).

liquidity-in-the-utility motive and the assumption that households cannot lend directly to firms.[503] When extending credit to firms, banks have to take into account the possibility of borrowers' default. Default probabilities exogenously depend on the severity of aggregate Brownian shocks. In particular, the default probability is assumed to be proportionate to the Brownian increment. This has the drawback that the default rate is not restricted to interval $[0, 1]$.[504] Firms are not able to save by assumption and thus are prevented from accumulating net worth as a buffer against adverse shocks. If a single borrower defaults, his entire assets are wiped out by assumption. Hence, banks have to write-off the corresponding loan completely. Losses resulting from this exogenous credit risk are amplified by banks' leverage.[505] Since capital and money are absent in the model economy, price fluctuations, which might contribute to systemic risk, are excluded.

In optimum, banks' dividend payout and recapitalisation policies are of the "barrier" type: banks issue new equity if the aggregate net worth of the banking sector falls below a specific threshold and pay out each additional Dollar earned to shareholders once an upper critical level is reached. Both of these thresholds are determined endogenously in equilibrium. Between the barriers, banks' aggregate wealth fluctuates depending on bank policies and the severity of shocks.[506] As in other CTMF models, the competitive equilibrium is not constrained Pareto efficient due to pecuniary externalities. Here, each bank does not take into account the effect of its decisions on the behaviour of aggregate bank equity. This leads to excessive lending and financial instability.[507] Minimum equity ratios unanimously reduce lending and thus improve stability. Interestingly, capital constraints can reduce lending substantially even in unconstrained regions. This is due to a precautionary motive: banks anticipate that the constraint might bind in the future and thus decrease their current leverage. Optimal macroprudential policy balances the effects of higher lending on stability and equity accumulation against the misallocation of resources, which depresses output.[508]

#### 3.2.4.7 The Li (2017) Model

The model features three types of agents: entrepreneurs, bankers, and households, who play a limited role. All individuals are risk neutral and have identical discount rates.[509] The crucial model property is that firms hold liquidity buffers in the form of deposits that are created by the banking sector. The demand for liquidity stems from the possible exposure to liquidity shocks, which arrive with a constant Poisson intensity. Liquidity shocks destroy the entire capital stock of affected firms unless they undertake further investment in capital. Importantly, investment must be financed internally, which can be motivated by assuming that the newly created capital is not pledgeable.[510]

[503] Cf. Klimenko et al. (2016, p. 8).
[504] Cf. Klimenko et al. (2016, pp. 6f.).
[505] Cf. Klimenko et al. (2016, pp. 5ff.).
[506] Cf. Klimenko et al. (2016, pp. 14f.).
[507] Cf. Klimenko et al. (2016, p. 17).
[508] Cf. Klimenko et al. (2016, pp. 22ff.).
[509] Cf. Li (2017, p. 2).
[510] Li (cf. 2017, p. 8) notes that this mechanism is similar to that in Holmström and Tirole (cf. 1998, p. 3).

This assumption gives rise to the demand for inside money issued by banks in the form of a money-in-advance constraint.[511] In addition to the liquidity shock, productive agents' capital holdings are subject to the usual Brownian disturbances.[512] Banks serve two roles: first, they provide external finance for entrepreneurs' projects in the form of bank credit and second, they feed the demand for liquidity by productive agents. Deposit creation is risky because it requires the extension of loans. Since banks cannot enforce the repayment of loans, they impose a collateral constraint on lenders. Adverse shocks to capital destroy a fraction of collateral and lead to write-offs on banks' loan portfolios.[513] Accordingly, it is assumed that the default rate is proportional to the Brownian increment in the equation of motion for capital.[514] As in Klimenko et al. (2016), this implies that the default rate may not lie in the interval $[0, 1]$.

The model's most interesting dynamics derive from a bank balance sheet channel, which, in turn, results from an equity issuance friction on the side of banks as in Phelan (2016) and Klimenko et al. (2016). In contrast, the entrepreneurial balance sheet channel is switched off as a result of firms' ability to costlessly issue equity to outside investors.[515] Once a negative innovation in the equation of motion for capital occurs, the shadow value of bank equity surges. This causes banks to charge a higher money premium, which is defined as the difference between the discount rate and the return on deposits.[516] The ensuing drop in the supply of inside money drains firms' liquidity buffers and thus leads to lower investment. The length of the resulting slump depends on the relative net worth of banks. This mechanism is amplified by the procyclicality of banks' leverage.[517]

Stochastic variations in the price of capital do not lead to the typical balance sheet fluctuations as e.g. in BS (2014a). Rather, an intertemporal complementary between money demand and the price of capital arises: positive shocks to banks' equity improve current and expected future money market conditions. Thus, firms expect to hold more capital on average. This induces the current and expected capital prices to rise, which, in turn, leads to a higher demand for investment and money. The procyclicality of firms' money demand causes banks' liabilities to grow faster than their equity. In effect, banks are more leveraged in booms.[518] The upshot of this latter type of procyclicality is that slumps are prolonged: in the aftermath of adverse shocks that push the economy into a depressed state, banks rebuild equity slowly since their leverage ratios are low. On the contrary, when the economy is in upswing, highly levered banks make the system vulnerable to bad shocks, which is reflected by spikes in volatility measures.[519]

Outside money in the form of government bonds is introduced in an extension of the model.

[511] Cf. Li (2017, pp. 9ff.).
[512] Cf. Li (2017, p. 16).
[513] Cf. Li (2017, p. 12).
[514] Cf. Li (2017, p. 17).
[515] Cf. Li (2017, p. 16).
[516] Cf. Li (2017, p. 13).
[517] Cf. Li (2017, p. 23).
[518] Cf. Li (2017, pp. 23ff.).
[519] Cf. Li (2017, p. 27).

Outside money is assumed to be a perfect substitute to inside money and thus can potentially cure the liquidity shortage that arises when banks are undercapitalised. Indeed, the supply of government bonds decreases the money premium in every state of the world. However, the side effect is that banks' profits are depressed. Hence, the economy spends more time in regions of the state space in which banks are undercapitalised. In effect, the *average* supply of liquidity is lower than in the model without government debt and economic growth is impeded.[520]

## 3.3 Appraisal

Before turning to the strengths and weaknesses of the CTMF framework in specific, let us discuss some general considerations the model builder faces upon deciding on a particular timing structure. There are (at least) four advantages of continuous- compared to discrete-time modelling. First, and obviously, the former is more realistic - "the economy does not cease to exist in between observations", as put by the econometrician Albert R. Bergstrom.[521] An example that shows why this fact matters in the context of our discussion relates to the EIS. In discrete-time models, individuals are assumed to only consume at the end of each period. This implicitly corresponds to a linear time aggregation *within* each period. Accordingly, the EIS is finite only *across* periods. In contrast, the EIS is dictated by the (finite) curvature of the utility function at any instant of time in continuous-time models.[522] Second, continuous-time models can be more tractable compared to their discrete time counterparts in the sense that they typically allow for more analytical steps.[523] This property is facilitated by the application of Itô's Lemma, which provides flexibility in the transformation of functions.[524] Third, the computation of model equilibria can potentially be sped up considerably if the problem at hand is formulated in continuous time.[525] As an example, continuous-time setups admit closed-form expressions for agents' FOCs in some special cases, whereas such expressions are not attainable in discrete time.[526] Thus, one has to employ computationally intensive numerical methods to find optimal choices in the latter case. Fourth, using continuous-time modelling allows the macroeconomic researcher to adopt methods developed in the finance literature, which has extensively employed continuous-time stochastic processes.[527] Indeed, an important example of this fact is the CTMF literature, in which portfolio selection problems set up in continuous time interact with asset price variations that are determined in general equilibrium.

There are also some disadvantages that relate to the estimation of structural models. Conventional DSGE models, either of the RBC or NK vintage, are set in discrete-time. As mentioned in

[520] Cf. Li (2017, p. 39f.).
[521] Phillips (1988, p. 315).
[522] Cf. Brunnermeier and Sannikov (2016b, p. 1502).
[523] Cf. Brunnermeier and Sannikov (2016b, pp. 1501f.).
[524] Cf. Parra-Alvarez (2013, p. 2).
[525] Cf. Achdou et al. (2015, pp. 28f.).
[526] This is demonstrated in Appendix B.2 in the context of the model developed in Chapter 4.
[527] Cf. Bergstrom and Nowman (2007, p. 7 ).

Subsection 2.2.4, a DSGE model is usually solved by computing a loglinear approximation around its steady state. Linearised DSGE models, in turn, can be nested in vector autoregressions and therefore allow for straightforward implementations of estimation procedures.[528] When a particular continuous-time model is taken to the data, on the other hand, a major issue is that macroeconomic data comes in low frequency (annually, quarterly, or monthly).[529] Under these circumstances, standard econometric techniques might generate biased estimates if the underlying model is formulated in continuous time.[530] For that reason, continuous-time models are often translated into discrete-time approximations for estimation purposes.[531,532]

We now turn to a comparison of the CTMF literature with the more traditional discrete-time DSGE models with financial frictions, such as KM and BGG. There are some important similarities between the two frameworks, which relate to the underlying assumptions as well as model implications. First and foremost, in both literatures incomplete markets give rise to financial frictions. As a consequence, the distribution of wealth matters for equilibrium outcomes.[533] A second similarity is the adoption of sector-specific representative agents. This is achieved by either assuming that agents within a particular sector are identical or that within-sector heterogeneity does not matter for aggregate outcomes. Yet, assuming market incompleteness across sectors but not within sectors might appear as a rather ad hoc way of incorporating financial frictions.[534,535] The discussion in Subsection 2.2.1 highlighted the fact that the upshot of such assumptions is a drastic reduction in the dimension of the state space. Thirdly, both approaches typically assume that exogenous disturbances, such as productivity or preference shocks, arise in the real sector and subsequently spill over to other sectors. An alternative is to explicitly account for shocks that originate in the financial sector and directly impair households' or firms' access to credit markets.[536] While models that incorporate those types of shocks have been recently developed in the DSGE literature[537], this does not yet hold true for the CTMF literature to the best knowledge of the author.

Turning to model implications, an additional analogy is that exogenous shocks are amplified via endogenous mechanisms. Typically, exogenous disturbances would generate (mild) recessions even

[528] Cf. Del Negro and Schorfheide (2006, p. 25).
[529] Cf. Bergstrom and Nowman (2007, pp. 1f. ).
[530] Cf. Bergstrom (cf. 1996, p. 5).
[531] Cf. Cochrane (2005, p. 28).
[532] An alternative is to use advanced econometric methods that explicitly account for such bias. Early examples are provided in Phillips (1972) or Bergstrom (1985).
[533] Cf. Brunnermeier and Sannikov (2016b, pp. 1539f.).
[534] Cf. Isohätälä et al. (2016, p. 239).
[535] A promising approach to address this issue in continuous time is introduced by Achdou et al. (cf. 2015, p. 5). They show that the solution of models with a large number of heterogeneous agents can be reduced to solving two coupled partial differential equations: a Hamilton-Jacobi-Bellman equation for individuals' decision problems and a Kolmogorov forward equation for the evolution of the wealth distribution. However, their paper is not concerned with aggregate fluctuations due to adverse feedback loops involving asset price variations as encountered in the CTMF literature.
[536] Cf. Quadrini (2011, p. 240)
[537] Examples are Christiano et al. (2010), Gertler and Karadi (2011), or Kiyotaki and Moore (2012). For a more exhaustive list, the reader is referred to Quadrini (cf. 2011, Section 7).

in the absence of amplification devices.[538] As discussed in Section 2.2, a key contribution of KM and BGG was to show that financial frictions have the potential to exacerbate the ramifications of exogenous shocks. A central mechanism in these models are adverse feedback loops that are characterised by the interaction of adverse movements in asset prices, pecuniary externalities, and weakening balance sheets of financially constrained agents. As the discussion in this chapter has made clear, these interactions also form the basis of amplification mechanisms in CTMF models. A further and related similarity is that models of both types are able to generate protracted slumps that display persistence of weak overall economic conditions.

The characteristic strengths and weaknesses of the CTMF compared to the linearised DSGE approach, on the other hand, are exactly opposite to each other. The latter class of models is very general in terms of admissible assumptions. This may relate to the form of adopted preferences, technologies, or frictions. The power of DSGE models in that regard is most apparent in the large-scale DSGE models employed in central banks, which typically feature sophisticated model economies with dozens of equilibrium relations. Even a comparatively simple model such as BGG features a production technology that requires labour in addition to capital input, financial as well as nominal frictions, two endogenous state variables, namely the aggregate capital stock and entrepreneurs' wealth, and various exogenous shock processes. Yet, the employed local solution method allows the researcher only to consider isolated shocks that start at the deterministic steady state. A further implication of that procedure is that individuals do not rationally anticipate random disturbances (cf. Subsection 2.2.4).

In contrast, a characterising feature of CTMF models is that such restrictions need not be imposed. Rather, the algorithms produce global equilibrium solutions along the entire state space. Importantly, that literature demonstrates that model economies may behave quite differently at different regions of the state space.[539] In particular, there are highly *nonlinear* amplification effects: endogenous risk is typically much larger below the steady state, i.e. in periods when the balance sheets of constrained agents are weak, than at the steady state.[540] Related, the stationary distributions of state variables typically differ from those in linearised models, which take the form of normal distributions centred at the steady state.[541] Depending on model assumptions and calibration, the economy may spend most of the time at its steady state or away from it. For instance, in BS (2014a), the stationary distribution of experts' wealth share is U-shaped. That is, the economy usually is close to its steady state, where amplification is low but may occasionally get "trapped" in depressed states characterised by low growth and asset misallocation after a series of bad shocks.[542]

[538] Cf. Quadrini (2011, p. 212)
[539] The discrete-time DSGE literature has recently developed models that feature global nonlinearities. Examples are Boissay et al. (2013) and Benes et al. (2014). However, these approaches usually require complex and computationally intensive numerical techniques. In contrast, the procedure to numerically solve CTMF models is comparatively simple: it boils down to the solution of one or more differential equations.
[540] Cf. Brunnermeier and Sannikov (2014a, p. 419).
[541] Cf. Brunnermeier and Sannikov (2016b, p. 1502).
[542] Cf. Brunnermeier and Sannikov (2014a, p. 381).

A major contribution of the CTMF literature is that it resolves the Kocherlakota critique, which states that amplification in linearised DSGE models with financial frictions, such as KM or BGG, is low under reasonable parameter constellations.[543] More specifically, in the former class of models the Volatility Paradox predicates that amplification becomes arbitrarily large as exogenous risk approaches zero. As discussed, the reason is that agents' portfolio choices are endogenous: lower exogenous risk induces individuals to take on additional leverage, which causes endogenous risk to surge. In addition, agents anticipate that further adverse shocks might materialise after an initial exogenous disturbance. Put differently, the duration and severity of recessions is *stochastic*. As a consequence, drops in asset prices can be large, leading to heightened amplification and persistence.[544] Such repercussions cannot occur in models that employ linearisations around the deterministic steady state. Since individuals do not anticipate shocks, all assets must earn the same expected returns in those models. Thus, agents' portfolio choice is indeterminate.[545] It follows that a reduction in exogenous risk does not affect their desired portfolios. In addition, the solution procedure implies that the economy reverts back to the steady state with certainty after the arrival of an exogenous shock - the length and severity of slumps is *deterministic*. This feature prevents asset prices from dropping to a larger extent.[546]

Next, let us return to the property that continuous-time compared to discrete-time formulations often allow for more analytical steps and therefore aid intuition. In the context of CTMF models with Brownian uncertainty, this is exemplified by amplification terms such as (3.2.5) and (3.2.11). The derivation of these expressions is facilitated by the fact that the derivative of the price function is sufficient to characterise amplification, which, in turn, results from the application of Itô's Lemma.[547] Such characterisations are not possible when aggregate risk is driven by Poisson processes. The reason is that amplification in those cases depends on discrete changes in the value of the price function, i.e. on the difference between post- and pre-jump values. Thus, continuous-time models with aggregate Poisson uncertainty are not preferable to discrete-time models in that regard.

The discussed advantages are currently bought by the need to adopt some restrictive assumptions. In general, the modelling strategy is to keep the dimension of the state space small. In fact, most of the respective models include only a single state variable. An example is the use of AK production technologies, which facilitates the removal of the aggregate capital stock from the relevant equilibrium equations. Another typical simplifying assumption is that only two distinct groups of individuals accumulate wealth, which has the upshot that the distribution of wealth between sectors can be described by a single measure. The reason for adopting such assumptions is as follows. Adding more state variables implies that the differential equations for prices (e.g.

[543] Cf. Brunnermeier and Sannikov (2014a, p. 406).
[544] Cf. Brunnermeier and Sannikov (2014a, p. 381).
[545] Cf. Coeurdacier and Rey (2013, p. 69).
[546] Cf. Brunnermeier and Sannikov (2014a, p. 406).
[547] Cf. Brunnermeier and Sannikov (2016b, p. 1502).

equation (3.1.36)) will have multiple independent variables, i.e. these equations become PDEs. This makes the shooting method inapplicable since it can only be reasonably applied to differential equations with a single independent variable.[548] The recently developed iterative method, on the other hand, represents a promising approach to solve more complex problems with multiple state variables since it entails the solution of PDEs along the time dimension.[549] This is demonstrated by Di Tella (2017), who develops a model based on BS (2014a) that includes two state variables, namely experts' wealth share and idiosyncratic risk.[550] The iterative method also allows for the solution of models with general CRRA and Epstein-Zin preferences as shown by Brunnermeier and Sannikov (2016b) and Di Tella (2017), respectively.[551] Further, an extension to cases with multiple asset price functions is possible in principle.[552] As mentioned, it should be noted that the iterative method has not yet been applied to solve models with aggregate risk that is governed by Poisson uncertainty to the knowledge of the author.

To conclude, since the potential power of the iterative method has not yet been fully utilised, CTMF models at this stage can be considered as abstractions that yield new economic insights, rather than realistic descriptions of macroeconomic data.[553]

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[548] In principle, the shooting method could be used to solve PDEs. Yet, such an approach would not be practically feasible. To clarify this, consider the following example. Let us suppose that the model involves two state variables and that each state variable is discretised along a grid with $N$ grid points. Further, let us assume that there is one price function to be solved for. Using the shooting method in this setting would involve $N$ guesses of initial slopes. If each of these guesses were to be updated $J$ times, the PDE would have to be solved up to $M = J^2 \times (N-1)!$ times. To get an intuition for the magnitude of that problem, suppose that $N = 100$ and $J = 50$. Then, we would get an astonishing $M = 9.3 \times 10^{157}$.

[549] Cf. Brunnermeier and Sannikov (2016b, p. 1540).

[550] Cf. Di Tella (2017, pp. 2050f.).

[551] Cf. Brunnermeier and Sannikov (2016b, Section 3.5) and Di Tella (2017, Appendix B).

[552] Cf. Brunnermeier and Sannikov (2016a, p. 50).

[553] Indeed, structural estimation procedures, which could benefit the descriptive realism, have not yet been applied to CTMF models. Nevertheless, such attempts would run into the problem that the estimation of nonlinear models in general requires much larger sample sizes than the estimation of linearised models. For a more detailed discussion of this issue, the reader is referred to Benes et al. (2014, p. 48).

# Chapter 4

# The Credit Model: A CTMF Model with Bank Lending and Inside Money

The present chapter develops an integrated theory of bank lending, inside money, and debt deflation. It does so by drawing on concepts from the CTMF literature, in particular BS (2014a), (2014d), and (2016a). Similarly to the first of these references, physical capital in our credit model is traded between more productive agents ("entrepreneurs") and less productive agents ("managers") in our credit model. As the I Theory of Money, the credit model explicitly features money, which comes in the form of either outside money supplied by the monetary authority or inside money created by financial institutions that we refer to as "banks".[554] Our model departs from the I Theory of Money in that banks create money as a byproduct of loan extension rather than investment in the equity of productive agents. The demand for credit arises from entrepreneurs' desire to finance a part of their capital holdings by credit. In order to motivate a nontrivial role for banks, we also add credit risk to our framework. This is achieved by introducing a subpopulation of borrowers, who do not exhibit forward-looking behaviour and therefore do not accumulate sufficient net worth buffers to protect themselves against bankruptcy in every state of the world.

Given the fact that in CTMF models (in contrast to linearised DSGE models) portfolio choice is fully endogenous, a substantial part of this chapter is devoted to the derivation of asset demand functions from first principles. A key insight in our model is that the adopted form of financial contracting concentrates the lion's share of endogenous asset price risk on the balance sheets of entrepreneurs rather than banks. Taking into account the current limitations in the CTMF literature with regard to the dimension of the state space, we will thus make assumptions that ensure that entrepreneurs' capitalisation relative to other sectors governs equilibrium prices and allocations.

We proceed as follows. Section 4.1 discusses the model assumptions. Section 4.2 deduces agents'

[554] We use the term "banks" here, rather than "intermediaries" as in the I Theory of Money, since we regard the latter term as unfortunate, given the discussion in Section 2.3.

stochastic flow budget constraints and expressions for asset returns, which, in turn, can most transparently be derived from the former. Stochastic flow budget constraints represent crucial ingredients in agents' intertemporal optimisation problems, which are solved in Section 4.3 to obtain individual consumption and portfolio rules. Market clearing conditions necessitate the aggregation of supply and demand market sides. Accordingly, in Section 4.4 we first derive some key macroeconomic variables before turning to clearing conditions. Finally, Section 4.5 closes the model by (i) showing that equilibrium equations can be expressed as functions of a single state variable, namely the share of entrepreneurs in aggregate wealth, and (ii) deriving a differential equation for the value share of capital, which contains the state variable as the sole argument.

## 4.1 Assumptions

### 4.1.1 Assets and Prices

There are eight main assumptions on assets and asset prices that are shared with the BS (2014d) model, which is the work most closely related to ours.

(i) The asset space includes capital, outside money, and inside money.

(ii) The nominal stock of outside money $M_t^O$ is exclusively supplied by the central bank and normalised to one in the baseline model. Hence, the change in the nominal outside money supply is always equal to zero: $dM_t^O = 0, \forall t$.

(iii) Inside money is created by banks and is in zero net supply. It follows from the zero net supply assumption that the aggregate real wealth in the economy $N_t$ is the sum of the real values of outside money and capital, which, in turn, implies that $N_t$ again is determined from equation (3.2.14).

(iv) Inside money is denominated in outside money and thus is a perfect substitute for the latter from the perspective of money holders.

(v) Money has intrinsic value.

(vi) The aggregate risk in the economy is due to a Poisson process $\mathcal{N}_t$ with intensity $\lambda$. Thus, the law of motion for the price of capital is given by (3.1.19) and that for the price of outside money per unit of capital by (3.2.15).

(vii) The unit of account in the economy is the final good, which implies that all prices are measured in units of output.

(viii) Short-sales of capital are not allowed. In addition, banks are the only agents in the economy that are allowed to short-sell money.

Three major departures from BS's (2014d) assumptions on assets stand out:

(i) The credit model presented in this thesis features credit in the form of the SDC as an additional asset. Credit is the mirror image of inside money: this type of money is created solely by banks by means of credit extension. For that reason, credit shares two characteristics with inside money. First, credit is in zero net supply since it emanates from financial contracts between borrowers and lenders. Second, credit is also denominated in outside money: agents that hold a short position in the former asset have to repay their creditors in the latter. Thus, only nominal debt contracts are considered. An additional parallel is that both types of financial contracts are assumed to be short-term, i.e. they only last for an instant of time.[555]

(ii) In distinction from equation (3.2.19), the aggregate capital stock is allowed to be stochastic. Its evolution over time is described by[556]

$$\frac{dK_t}{K_t} = \underbrace{\{\Phi(\iota_t) - \delta\}}_{\equiv \mu_t^K K_t} dt - \kappa \, d\mathcal{N}_t, \tag{4.1.1}$$

where $d\mathcal{N}_t$ is the increment of Poisson process $\mathcal{N}_t$. If positive (negative), the constant proportionality factor $\kappa$ measures the percentage reduction (increase) in the aggregate capital stock due to a jump. The stochastic term in (4.1.1) contributes to the aggregate risk associated with holding capital.[557] In addition, individual capital holdings can be subject to idiosyncratic shocks which cancel out in the aggregate. Both aggregate and idiosyncratic shocks to capital are described in Section 4.1.3 in more detail.

(iii) In contrast to the BS (2014d) model, the intrinsic value of money here is not due to fiscal backing by the government. Rather, we assume that trading of goods is facilitated by using money as a medium of exchange.

### 4.1.2 Agents and Preferences

In the following, we describe the three types of private agents in the model: entrepreneurs, managers, and bankers. Each of these sectors consists of a continuum of agents with measure unity. Individuals are identified by the following indexations. We denote the set of all entrepreneurs by $\mathbb{I}^e = [0, 1]$, the

[555] There is a large literature in finance concerned with the optimal duration of debt contracts. This literature shows that short-term debt contracts allow for a mitigation of moral hazard and adverse selection problems between lenders and borrowers (cf. He and Xiong, 2012, p. 1805). Further, the assumption of short-term debt contracts is common in the financial frictions literature. For instance, in discrete-time models such as Bernanke et al. (cf. 1999, p. 1348) and Carlstrom and Fuerst (cf. 1997, p. 894) it is assumed that only one-period loan contracts are feasible. The rationale for this assumption is that there is a high degree of anonymity in lending markets. For a macro model with long-term contracts see Gertler (1992).

[556] This equation of motion will be derived in Subsection 4.4.1 from individual capital holdings.

[557] Variations in the price of capital are the second source of that form of aggregate risk.

set of all managers by $\mathbb{I}^m = (1, 2]$ and the set of all bankers by $\mathbb{I}^b = (2, 3]$.[558]

Agents are endowed with different technologies, which are specified in more detail in Subsections 4.1.3 and 4.1.4. Both managers and entrepreneurs have the knowledge to produce final goods using capital as an input to production.[559] Importantly, entrepreneurs have a comparative advantage over managers in the production of final goods due to a higher productivity level.

As entrepreneurs are more productive than managers in producing final goods, the former have a higher willingness to pay for the productive asset than the latter. Any demand for capital by entrepreneurs that exceeds available equity generates a demand for external finance. In order to introduce a role for default on external finance, we divide entrepreneurs into two subgroups: a constant fraction $0 < \varphi < 1$ of these agents are *imprudent*. Individuals from the complimentary fraction $(1 - \varphi)$ are *prudent*. The set of all prudent entrepreneurs is defined as $\mathbb{I}^{e,p} = [0, 1 - \varphi]$ and the set of their imprudent counterparts as $\mathbb{I}^{e,i} = (1 - \varphi, 1]$.[560] The shares of these groups in the total population of entrepreneurs are constant and common knowledge. In the calibration of our model, we will set $\varphi$ such that the population of imprudent debtors is small relative to that of prudent debtors.

Prudent entrepreneurs, just as managers, have infinite horizons. The transaction motive for holding money, which was mentioned in the previous section, is implemented by adopting the money-in-the-utility (MIU) approach due to Sidrauski (1967).[561,562] The expected discounted lifetime utilities of the two groups are described by

$$U_{i,0}^j \equiv \mathbb{E}_0 \int_0^\infty e^{-\rho t} u\left(c_{i,t}^j, p_t K_t m_{i,t}^j\right) dt, \quad j = m, (e, p), \tag{4.1.2}$$

in which $c_{i,t}^j$ denotes real consumption expenditures of individual $i$ from sector $j$ at time instant $t$, $m_{i,t}^j$ his nominal money holdings at $t$, and $\rho \in (0, 1)$ the common time preference rate. The assumed

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558 In the remainder, variables pertaining to entrepreneurs, managers, and banks will be denoted by superscripts $e$, $m$, and $b$ respectively.

559 This is a departure from the assumptions in the BS (2014d) model, in which households, who play a role similar to that of managers in our model, are not endowed with a production technology, but rather lend their capital stakes to entrepreneurs directly.

560 In the remainder, variables pertaining to imprudent and prudent entrepreneurs will be denoted by superscripts $e, i$ and $e, p$, respectively.

561 Cf. Sidrauski (1967, p. 535).

562 The direct utility from holding money is usually interpreted as a shortcut for a transaction motive (cf. Walsh, 2010, p. 36). In fact, Feenstra (cf. 1986, Proposition 1) finds that microfounded models with liquidity costs in the budget constraint and the MIU model are equivalent up to their functional notation under some restrictions about preferences and liquidity costs. A transaction motive by production units can stem from mechanisms developed in classic finance papers such as Baumol (cf. 1952, pp. 545-549) or Miller and Orr (cf. 1966, pp. 416-423). An alternative to motivate the adoption of the MIU paradigm in the present framework is to regard the dependence of utility on money as a reduced form expression for firms' demand for safe or money-like assets to buffer liquidity shocks as modelled in e.g. Li (cf. 2017, p. 8). Regardless of the motivation for corporate cash holdings, the empirical literature has established that nonfinancial firms have accumulated large cash stocks in recent years (cf. e.g. Bates et al., 2009, pp. 1990ff.).

form of the instantaneous utility function in both sectors is

$$u\left(c_{i,t}^{j}, p_t K_t m_{i,t}^{j}\right) = \log c_{i,t}^{j} + \xi \log \left(p_t K_t m_{i,t}^{j}\right). \tag{4.1.3}$$

That is, preferences are assumed to be logarithmic and additively separable in consumption and real balances.[563] In the above relation, parameter $\xi \geq 0$ is a utility weighting factor. Another property of this utility function is that it restricts consumption and real balances to be nonnegative.

Imprudent entrepreneurs, in contrast to their prudent counterparts, consume their entire profits in each instant of time without a jump. As noted in Subsection 3.2.2, the adoption of logarithmic utility in combination with rational expectations about the impacts of shocks ensures that prudent agents accumulate sufficient net worth to avoid bankruptcy in case these shocks materialise. In order to introduce default, which is crucial for banks to play a nontrivial role, at least a fraction of debtors must not have sufficient equity buffers to absorb losses. This precisely is the case for imprudent debtors, who refrain from saving precautionarily and thus do not accumulate any net worth in normal times without jumps.[564] More specifically, these agents declare bankruptcy if their real return on capital is lower than the real loan rate after an adverse shock, a condition which will always hold true in equilibrium. In turn, defaults of "bad risks" lead to loan write-offs on the asset side of banks' balance sheets.[565]

Let us lastly turn to the third sector, namely the banking sector. Banks perform lending and deposit-taking services. They offer external finance in the form of credit to potential borrowers, which might be utilised by the latter for asset purchases or consumption expenditures. Banks are owned by managers and operated by bankers. Bankers earn wage income as a compensation for their effort required to carry out the aforementioned tasks. Bankers are assumed to be risk neutral and extremely myopic in the sense that they care only about current consumption. These assumptions imply that they consume their entire wage income instantly. Further, neither bankers as individuals nor banks as entities have a transaction motive and thus do not hold money on the asset sides of their balance sheets.[566]

[563] This form of the instantaneous utility function is also employed in Bernanke et al. (cf. 1999, p. 1387), except for the fact that the representative household's utility function also includes leisure in their model.

[564] Similarly, Klimenko et al. (cf. 2016, p. 5) assume in their model that borrowers are not able to save as a straightforward way to allow for the possibility of default.

[565] Another virtue of assuming that imprudent entrepreneurs consume their entire income is that there is no need for adding an additional state variable for the net worth of these individuals. The inclusion of an extra state variable would complicate the solution of the model severely.

[566] Again, this is a simplifying assumption. Alternatively, and without loss of generality, the approach by Brunnermeier and Sannikov (cf. 2016a, p. 10) could be adopted, who assume that banks always hold the entire supply of outside money at any time and issue a corresponding amount of deposits. Then, outside money would take the form of reserves, which could be used by banks for transaction purposes.

### 4.1.3 Productive Agents' Technologies

As mentioned in Subsection 4.1.1, capital stocks are subject to Poisson jumps that occur with intensity $\lambda$. In the event of a jump, each individual that holds capital is subject to the aggregate shock to capital, which causes his capital stock to change by $\kappa$ percent. In addition, managers and prudent entrepreneurs are exposed to an idiosyncratic shock to capital with zero mean if a jump materialises, i.e. if $d\mathcal{N}_t = 1$. By the definition of idiosyncratic risk, this type of shock cancels out in the aggregate.[567] It follows that the evolution of the aggregate capital stock does not depend on this type of shock and is indeed described by (4.1.1).

To formalise the idiosyncratic shock, we assume that the amplitude of this type of disturbance is $\underline{\kappa}^j k^j_{i,t}$ (the "bad" state) with probability $\phi$ conditional on the realisation of a jump and $\overline{\kappa}^j k^j_{i,t}$ (the "good" state) with conditional probability $1-\phi$ . Proportionality factors $\underline{\kappa}^j$ and $\overline{\kappa}^j$ are identical across individuals within one sector but are allowed to differ between sectors. Further, we set each $\underline{\kappa}^j$ to a positive value and assume that in case the malign idiosyncratic shock is realised, agent $i$ from sector $j$ has to write off capital of amount $\overline{\kappa}^j k^j_{i,t}$. Due to the zero mean assumption restriction $\phi\underline{\kappa}^j = -(1-\phi)\overline{\kappa}^j$ has to be imposed. Hence, $\underline{\kappa}^j > 0 > \overline{\kappa}^j$ must hold. It follows that in case agent $i$ draws the benign idiosyncratic shock, his capital stock increases by amount $-\overline{\kappa}^j k^j_{i,t}$.

As Ross (2014) notes, a Poisson process with two different outcomes can be treated as two independent Poisson processes.[568] Hence, we introduce two new idiosyncratic jump terms $d\underline{\mathcal{N}}_{i,t}$ and $d\overline{\mathcal{N}}_{i,t}$, which govern the realisations of individual-specific innovations. The distributions of these increments obey

$$d\underline{\mathcal{N}}_{i,t} = \begin{cases} 1 & \text{with prob. } \lambda\phi\, dt \\ 0 & \text{with prob. } 1-\lambda\phi\, dt \end{cases}$$

and

$$d\overline{\mathcal{N}}_{i,t} = \begin{cases} 1 & \text{with prob. } \lambda\left(1-\phi\right)\, dt \\ 0 & \text{with prob. } 1-\lambda\left(1-\phi\right)\, dt \end{cases},$$

with expected values $\mathbb{E}_t\left[d\underline{\mathcal{N}}_{i,t}\right] = \lambda\phi\, dt$ and $\mathbb{E}_t\left[d\overline{\mathcal{N}}_{i,t}\right] = \lambda\left(1-\phi\right)\, dt$, respectively. Increments $d\underline{\mathcal{N}}_{i,t}$ and $d\overline{\mathcal{N}}_{i,t}$ are assumed to be i.i.d. across all holders of capital.

[567] The adoption of idiosyncratic shocks to capital is inspired by BS (2016a), who assume Brownian project-specific shocks to capital which cancel out in the aggregate (cf. Equation 3.2.7).
[568] Cf. Ross (2014, pp. 303f.).

The assumptions on the shock structure lead us to the following stochastic process for $k^j_{i,t}$:[569]

$$\frac{dk^j_{i,t}}{k^j_{i,t}} = \left\{\Phi(\iota^j_{i,t}) - \delta\right\} dt - \left\{\kappa + \underline{\kappa}^j\right\} d\underline{\mathcal{N}}_{i,t} - \left\{\kappa + \overline{\kappa}^j\right\} d\overline{\mathcal{N}}_{i,t}, \tag{4.1.4}$$

where $\delta$ is the deterministic depreciation rate that is assumed not to differ across agents and $\Phi(\iota^j_{i,t})-\delta$ is the net internal investment rate of individual $i$ from sector $j$. Again, output goods can be transformed to capital goods via technology $\Phi(\cdot)$, which has the usual properties $\Phi(\cdot)' > 0$ and $\Phi(\cdot)'' < 0$. The investment technology is identical across all agents that are allowed to hold capital. The specific form of function $\Phi(\cdot)$ employed in the baseline model is the logarithmic form contained in equation (3.1.9).

In order to keep the baseline model as tractable as possible, we make the simplifying assumption that imprudent entrepreneurs are not subject to idiosyncratic risk. Hence, their capital follows

$$\frac{dk^{e,i}_{i,t}}{k^{e,i}_{i,t}} = \left\{\Phi(\iota^{e,i}_{i,t}) - \delta\right\} dt - \kappa \, d\mathcal{N}_t. \tag{4.1.5}$$

Both managers and entrepreneurs generate final goods by means of an AK production function. The final goods production function of any agent agent endowed with a final goods production technology is described by

$$y^J_{i,t} = z_{i,t} k^J_{i,t}, \quad J = m, (e,i), (e,p). \tag{4.1.6}$$

In this equation agent $i$'s stochastic productivity level $z_{i,t} = a^e, a^m$ is a binary random variable with $a^e > a^m$. Agents with current productivity level $z_{i,t} = a^e$ are identified as "entrepreneurs" and individuals with $z_{i,t} = a^m$ as "managers".[570] The stochastic equation of motion for $z_{i,t}$ is

$$dz_{i,t} = \{a^e - a^m\} \, d\mathcal{N}^{s,m}_{i,t} - \{a^e - a^m\} \, d\mathcal{N}^s_{i,t}, \tag{4.1.7}$$

where $\mathcal{N}^{s,m}_{i,t}$ is a Poisson process that counts how often the individual's productivity level changes from $a^m$ to $a^e$ and $\mathcal{N}^s_{i,t}$ is a Poisson process that counts the reverse.[571] The two processes obey state-dependent intensities $\lambda^{s,m}(z_t)$ and $\lambda^s(z_t)$, which are listed in Table 4.1.[572]

[569] As is standard in the CTMF literature, this equation of motion measures changes in the capital stock in the absence of capital purchases or sales. Put differently, external investment $e^j_{i,t}$ is set to zero.

[570] It is convenient to assume that productivity levels can only take on two values as this implies that a share of managers become become entrepreneurs if the $d\mathcal{N}^{s,m}_{i,t}$ materialises and vice versa if $d\mathcal{N}^s_{i,t} = 1$. Assuming that productivity changes to any other level would imply that a new class of agents would emerge. To solve the model under these circumstances one would have to expand the state space by the aggregate net worth of that class.

[571] The role of shocks to productivity is discussed in Subsection 5.2.3 in detail.

[572] It is interesting to note that there is a similarity between our formulation of productivity shocks and labour market models with search and matching frictions. In the latter type of models the individual household's income is stochastic due to variations in its employment status. In the most basic case, job creation and destruction events are described by two Poisson processes with exogenous intensities, which correspond to our processes $\mathcal{N}^{s,m}_{i,t}$

Table 4.1: State-Dependent Intensities of Poisson Processes $\mathcal{N}_{i,t}^{s,m}$ and $\mathcal{N}_{i,t}^{s}$

| $z_t$ | $a^e$ | $a^m$ |
|---|---|---|
| $\lambda^{s,m}(z_t)$ | 0 | $\lambda^{s,m}\phi^{s,m}$ |
| $\lambda^{s}(z_t)$ | $\lambda^{s}\phi^{s}$ | 0 |

*Notes*: Based on Wälde (2011, p. 279).

The interpretation is as follows. Once a $d\mathcal{N}_{i,t}^{s,m}$ shock hits, the productivity level of a fraction $\phi^{s,m}$ of managers is lifted up from $a^m$ to $a^e$ - they effectively become entrepreneurs. On the contrary, if $d\mathcal{N}_{i,t}^{s} = 1$ materialises, the productivity level of a share $\phi^s$ of entrepreneurs is reduced to $a^m$. We will see later on that the single state variable in the model is entrepreneurs' wealth share, i.e. the wealth share of agents with current productivity $a^e$. Accordingly, the former type of shock increases the state variable, while the latter reduces entrepreneurs' share in aggregate wealth. Unfortunately, the CTMF literature has not yet developed numerical tools to solve models with two-sided Poisson processes.[573] To address this issue it is assumed that state $z_{i,t} = a^m$ is *absorbing*[574], i.e. once an individual draws level $a^m$, his productivity stays constant afterwards. Formally, this requires us to set $\lambda^{s,m}\phi^{s,m} = 0$. Managers' final goods production technology is thus described by equation

$$y_{i,t}^{m} = a^m k_{i,t}^{m}, \tag{4.1.8}$$

in which $y_{i,t}^{m}$ is managers' output at the firm level and the productivity level is deterministic.

Prudent and imprudent entrepreneurs' production functions are now given by

$$y_{i,t}^{e,p} = z_{i,t} k_{i,t}^{e,p} \quad \text{and} \quad y_{i,t}^{e,i} = z_{i,t} k_{i,t}^{e,i}, \tag{4.1.9}$$

respectively. Random variable $z_{i,t}$ takes on value $a^e$ at the date of birth of each agent that is endowed with the superior productivity level: $z_{i,0}^{e} = a^e$. Productivity remains at level $a^e$ until the individual is exposed to the $d\mathcal{N}_{i,t}^{s}$ shock. This type of shock is referred to as the *sector-specific productivity shock*, or, simply, the *sector-specific shock*, in the remainder. Since $z_{i,t} = a^m$ is an absorbing state, the sector-specific shock leads to a *permanent* drop in the individual's productivity parameter from

and $\mathcal{N}_{i,t}^{s}$ . A model of this type is e.g. presented in Pissarides (cf. 2000, Chapter 1). The modeling approach in the present subsection is based on the short account of the household's problem in the labour market search and matching literature by Wälde (cf. 2011, pp. 279f.).

[573] The shooting method only allows for solving models with one-sided jumps in the state variable (cf. the discussion in Subsection 3.1.2.4). The iterative method appears to be more promising in that regard. However, attempts by the author of this thesis to solve the model developed in this thesis via the iterative method have not been successful.

[574] Cf. Bayer and Wälde (2010, p. 8).

$z_{i,t} = a^e$ to level $a^m$. The evolution of $z_{i,t}$ is now described by

$$dz_{i,t} = \{a^m - a^e\}\, d\mathcal{N}^s_{i,t}, \tag{4.1.10}$$

as long as $\mathcal{N}^s_{i,t} = 0$. Process $z_{i,t}$ becomes "degenerate" if $\mathcal{N}^s_{i,t}$ reaches unity at some instant $t$: then, we have $dz_{i,s} = 0, \forall s > t$. Further, it is assumed for simplicity that the sector-specific shock is tied to Poisson process $\mathcal{N}_t$. More specifically, the shock hits a share $\phi^s$ of entrepreneurs if and only if an innovation $d\mathcal{N}_t = 1$ occurs.[575] It follows that the intensity of $\mathcal{N}^s_{i,t}$ becomes $\lambda^s(z_t) = \lambda\phi^s$. Hence, the probability distribution over the stochastic increment in (4.1.10) is given by

$$d\mathcal{N}^s_{i,t} = \begin{cases} 1 & \text{with prob. } \lambda\phi^s\, dt \\ 0 & \text{with prob. } 1 - \lambda\phi^s\, dt. \end{cases} \tag{4.1.11}$$

From the individual's perspective, $\phi^s$ is the probability of exposure to the sector-wide shock conditional on the arrival of a jump.

To keep the respective masses of entrepreneurs and managers at unity, market entry and exit is introduced. In particular, once a jump materialises, a cohort of newborn entrepreneurs with mass $\phi^s$ enter the market. In order to start operations, they require a strictly positive amount of net worth. This amount is provided by "old" prudent entrepreneurs via lump sum transfers. However, this transfer is neglected in the remainder for two reasons: first, the transfer can be made arbitrarily small and second, only aggregate entrepreneurial net worth matters for equilibrium outcomes.[576] At the same time, a mass $\phi^s$ of managers retires. These individuals transfer their wealth to nonretiring managers. Since the distribution of wealth among managers does not influence equilibrium allocations and prices, this type of transfer is ignored as well. Retirements and wealth transfers are not anticipated by managers and therefore do not affect their expected discounted lifetime utilities.

### 4.1.4 Financial Structure

Financial structure in the model economy is characterised by several financial frictions. The first financial friction is due to managers' inability to directly lend to entrepreneurs, and vice versa.[577] In contrast to the other types of private agents, banks have the capability to provide external finance. Financial transactions between banks and potential borrowers are complicated by the fact that the outcomes of borrowers' production activities, their balance sheet positions, as well as their

[575] The advantage of that simplifying assumption is that we only have to keep track of a single source of fundamental risk, namely process $\mathcal{N}_t$.

[576] This reasoning is analogous to that in Carlstrom et al. (cf. 2016, p. 125).

[577] This might be motivated by assuming that banks have a comparative advantage in lending activities over other agents resulting from their ability to diversify across assets as e.g. in Diamond and Dybvig (1983) or Diamond (1984), although such mechanisms are not explicitly modelled here.

transactions are private knowledge. However, these can be revealed with certainty by bankers via costly verification of lenders' balance sheets and activities. In particular, we adopt a variation of the standard CSV approach: auditing requires utility-reducing effort by bankers.[578,579] This assumption provides a possible rationale as to why borrowers and banks agree to exchange funds on the basis of a debt contract rather than equity issuance.[580]

If imprudent debtors declare bankruptcy, bankers exert effort to assess the value of the assets owned by the former. Afterwards, banks take over these assets, which implies that a fraction $\varphi$ of the loan portfolio is replaced by the remaining assets of bankrupt debtors. It follows that the nominal value of loans extended by banks to imprudent entrepreneurs $l_t^{b,i}$ changes stochastically. The negative of difference $\widetilde{l}_t^{b,i} - l_t^{b,i}$ is the nominal amount of credit that banks have to write off in the case of bankruptcy of bad risks, i.e. in the case of a jump. Any capital received from these borrowers is sold immediately on the market for capital as banks are not allowed to hold capital.[581] The complementary share of entrepreneurs $1 - \varphi$ are good risks since they continue operations after the occurrence of a jump due to their precautionary savings. Thus, the process for loans extended to prudent borrowers $l_t^{b,p}$ is deterministic.[582]

In addition, it is assumed that banks do not lend to managers.[583] This assumption can be motivated by demand and supply side factors. As for the supply side, it could be the case that banks specialise in lending to entrepreneurs.[584] This might e.g. be due to a specialisation in the auditing of entrepreneurs.[585] A possible demand side explanation is that managers are entirely equity financed because they are debt averse. Debt aversion may result from a reluctance to transfer

578 A similar version of the CSV problem is employed by Winton (cf. 1995, p. 95).

579 The reason for departing from the standard CSV framework, in which verification entails some deadweight losses of resources, is discussed in Subsection 5.5.2.

580 The optimality of the standard debt contract in the context of the examined model is not analysed in this thesis, but left for future research. Hence, the set of possible contracts between banks and borrowers is exogenous and only includes the standard debt contract.

581 Without this assumption banks might find it profitable to keep the capital even though they are not endowed with a production technology. This might be the case if banks want to speculate on a rising price of capital.

582 We abstract from the possibility that banks implement self-selection mechanisms for borrowers in the spirit of Bester (1985).

583 The model results are qualitatively the same in large parts relative to the baseline model if managers are allowed to borrow. Intuitively, this result can be related to the fact that managers are less productive than entrepreneurs and thus have less incentive to take out credit for financing capital purchases. We choose to prevent managers from borrowing in the baseline model as the alternative would necessitate some strong assumptions to keep the model tractable. An example is the assumption that the share of defaulting borrowers is constant, regardless of whether managers choose to demand loans or not. In addition, one has to assume that the amplitude of the idiosyncratic shock is higher for managers than for entrepreneurs or that the productivity differential $a^e - a^m$ is wider compared to the baseline calibration in order to create sufficient incentive for managers to sell capital to entrepreneurs.

584 Paravisini et al. (cf. 2017, Section 6) analyse a matched sample of Peruvian banks and exporters and find that banks have a sector-specific advantage in their lending activities, which is distinct from firm-specific advantages highlighted by the traditional literature on relationship banking.

585 If managers were allowed to obtain credit from banks, auditing of the reports by the former would be necessary since managers, just as entrepreneurs, have an incentive to underreport the outcomes of investment projects. This is because managers hold equity stakes in a large number of banks. If, on the contrary, each manager were only able to invest in the equity of a single bank, an agent of type $m$ could obtain a loan from the bank he owns without being confronted with incentive compatibility issues.

control rights to lenders.[586]

Banks' ability to perform lending services is facilitated by the assumption that they are able to perfectly diversify their loan portfolios. This implies that in case of an adverse shock which pushes imprudent borrowers into bankruptcy, banks have to write off a *share* $\varphi$, which is the fraction of imprudent entrepreneurs in the population of entrepreneurs, of their total debt claims.[587] It also follows from the perfect diversification assumption that the behaviour of the banking sector can be described by focusing on the actions of a single representative bank which holds the entire volume of credit in the economy.

As noted, a crucial feature of debt contracts in this model is that they last only for one period of length $dt$. Each instant of time prudent entrepreneurs choose their loan demand, repay old loans with either money raised from new loans, profits or a combination of both.

Let us now address the liability sides of banks' balance sheets. As mentioned, banks are allowed to hold short positions in money, which provides an additional source of liquidity to the money market. As in the I Theory of Money, a short position in money is tantamount to the issuance of deposits and leads to an expansion in a bank's balance sheet. By issuing deposits, banks create their own financing in the form of inside money. In distinction from the I Theory of Money, however, this financing is used to back credit extended to entrepreneurs rather than shares purchased from households. Accordingly, inside money creation can be regarded as a "byproduct" of banks' lending business rather than a consequence of investment banking.[588] This way of modelling inside money creation has much more empirical relevance, as mentioned in the Introduction, and also displays more parallels to the credit creation theory of banking as described by Werner (2014b).[589]

If the economy is hit by a jump, bank owners recapitalise banks to cover the resulting losses on banks' balance sheets. In normal times without jumps, all debtors repay their debt and banks pay out the profits accrued from their lending activities to bank owners right away.[590] Figure 4.1.1 depicts agents' aggregated balance sheets.

[586] Cressy (cf. 1995, p. 293) argues that lending is associated with a transfer of control rights from borrowers to lenders. The author (cf. 1995, Section 5) develops a model in which a decrease in control rights induced by an increase in debt reduces utility and shows that borrowers might find it preferable to refrain from borrowing altogether if firm owner preferences against external control are sufficiently strong. According to the author (cf. 1995, p. 293) the model offers a possible explanation for the two distinct empirical observations that a large share of borrowers express a reluctance to cede control rights and an even larger share is entirely self-financed.

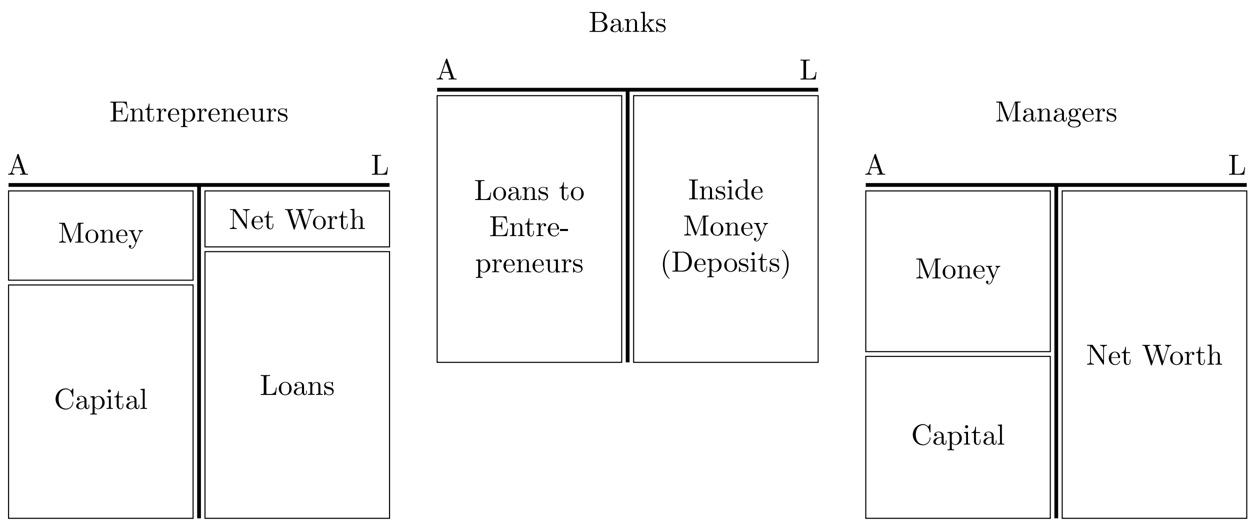
[587] In the other extreme, each bank would only be able to lend to a single borrower. This would imply that in case of an adverse shock each bank has to write-off the entire loan volume on the asset side of its balance sheet with probability $\varphi$ and nothing at all with probability $1 - \varphi$.

[588] Apart from this difference and the fact that banks act in a risk neutral way in the credit model, the modelling of deposit creation is virtually identical to that in the I Theory of Money (cf. Subection 3.2.2.1).

[589] Cf. Section 2.3.

[590] Since recapitalisation of banks does not entail any deadweight losses or costs, it is immaterial to the model outcomes whether banks accumulate equity by retaining profits or pay out any earnings to bank owners instantly. Conversely, Klimenko et al. (2016) examine a framework in which recapitalisation of banks by bank owners is associated with deadweight losses. As mentioned in Section 3.2.4, this friction implies that bank equity is a crucial state variable in their model.

Figure 4.1.1: Agents' Balance Sheets in the Baseline Credit Model




*Notes*: Based on Brunnermeier and Sannikov (2016a, p. 10).

### 4.1.5 Transactions

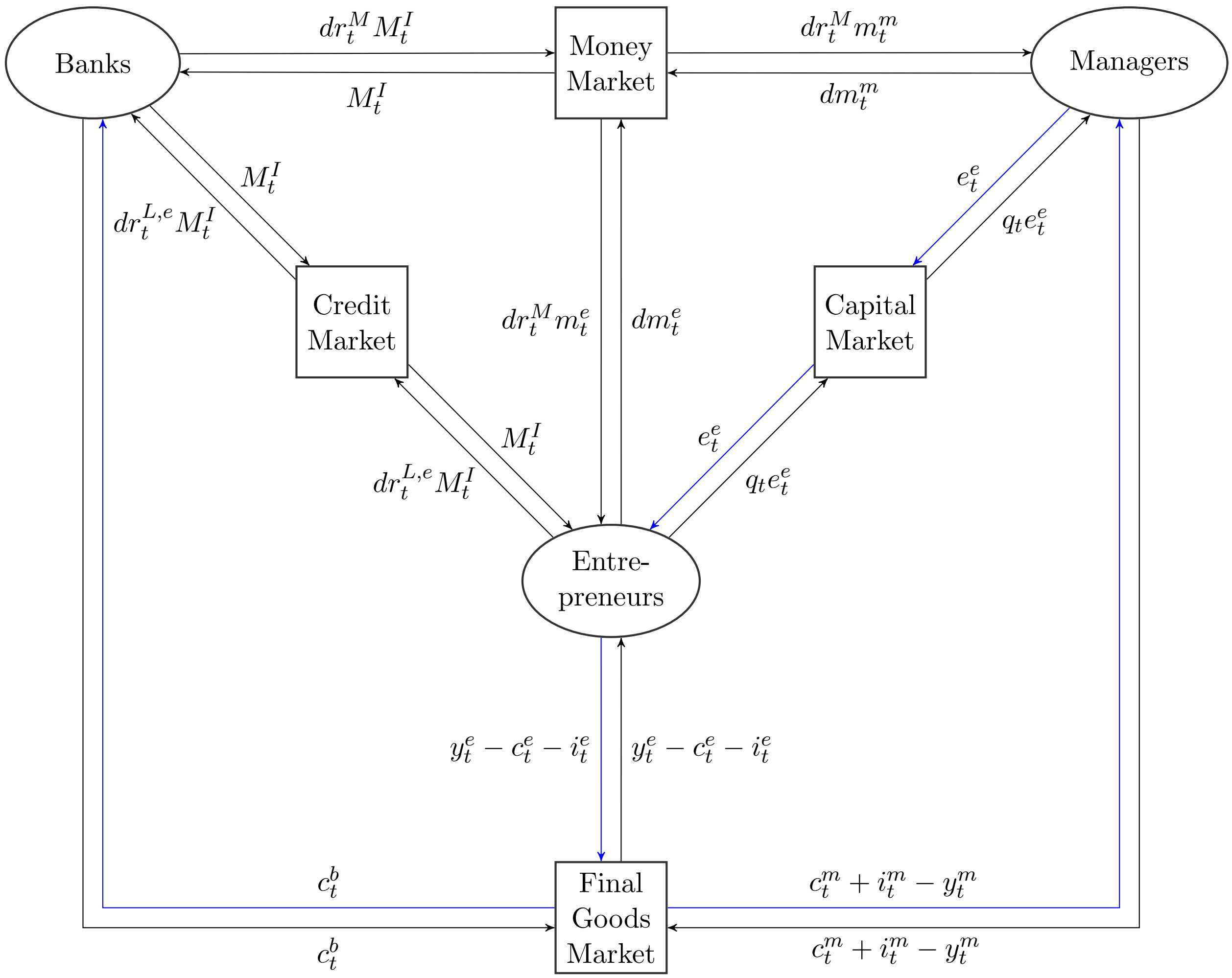
Exchange takes place on four different markets: the markets for capital, final goods, credit, and money. Market transactions can be visualised by the circular flow of income (CFI) for the presented economy. Figure 4.1.2 shows the CFI in the absence of adverse shocks. Monetary flows are depicted by black arrows and flows of goods by blue arrows.

Banks provide entrepreneurs with funds $M_t^I$, which is the stock of inside money[591], and in return receive a debt claim. The real instantaneous loan rate entrepreneurs have to pay for borrowing one unit of money is denoted by $dr_t^{L,e}$. As mentioned, banks issue deposits in order to create their external funding. They have to pay depositors the real instantaneous return on outside money $dr_t^M$ since inside money and outside money are perfect substitutes from the perspective of money holders. While the *nominal* return to money is always equal to zero, the real return fluctuates depending on the state of the economy. Banks' income from their lending activities is used in part to pay out wages to bankers. Bankers spend their entire wage income to purchase final goods for consumption purposes. Variable $c_t^b$ stands for bankers' real consumption expenditures. Banks' remaining profits are paid out to managers, who are the owners of banks (not shown). Transactions on the market for bank shares are not modeled by deriving explicit demand and supply functions from first principles for simplicity. Rather, it is assumed that only managers hold bank shares and always have perfectly diversified[592] bank equity positions that are proportionate to their wealth levels.

[591] We write $M_t^I$ rather than $dM_t^I$ as financial contracts only last for one instant of time.

[592] The assumption that managers hold ownership stakes across a large number of banks does not affect any equilibrium outcomes as banks are able to diversify away any idiosyncratic risk associated with their lending activities. However,

Figure 4.1.2: The Circular Flow of Income in the Absence of Adverse Shocks.




*Notes*: Monetary flows (measured in real units) are depicted by black arrows and flows of goods by blue arrows.

Entrepreneurs allocate available funds between additional nominal money balances $dm_t^e$ and purchases of existing capital from managers $e_t^e$ at price $q_t$. Besides this *external* investment each agent of type $e$ can also raise his capital stock from internal investment $i_t^e$. Internal investment measures the amount of final goods that is transformed to capital goods via the production technology for the productive asset. Entrepreneurs use capital to produce final goods $y_t^e$, which in equilibrium is always higher than the sum of their consumption expenditure $c_t^e$ and internal investment $i_t^e$. It follows that these agents are "net" suppliers of output on the final goods market. Managers allocate their wealth between money balances $m_t^m$ and capital $k_t^m$. The sum of managers' consumption expenditure $c_t^m$ and their internal investment $i_t^m$ in the depicted period is larger than managers' output $y_t^m$. This is the case in the model equilibrium if entrepreneurs have a sufficiently high share in aggregate wealth and accordingly hold a substantial fraction of the economy's aggregate capital stock.

If an adverse shock arrives in a given period, the CFI for this period differs from that in normal times in one respect: in those cases, entrepreneurs experience reductions in their equity levels, which are exacerbated by their leverage positions. As a consequence of their risk aversion, they reduce their risk exposure by selling capital to managers, who experience less losses or even attain positive profits since they are not levered.

## 4.2 Budget Constraints and Returns

### 4.2.1 Managers

A single manager $i$, who holds a portfolio consisting of capital and money, owns real wealth $n_{i,t}^m$ given by:

$$n_{i,t}^m \equiv q_t k_{i,t}^m + p_t K_t m_{i,t}^m, \tag{4.2.1}$$

where $k_{i,t}^m$ and $m_{i,t}^m$ are agent $i$'s stocks of capital and money, respectively, the latter of which measured in nominal terms. The change in managers equity is due to three sources: profits accruing from their production activities, dividends or capital injections resulting from their ownership stakes in banks, and capital gains or losses on assets held. Managers' income from banks' intermediation activities can be formalised by introducing process

$$\frac{d\Pi_t^b}{\Pi_t^b} = \mu_t^{\Pi^b}\, dt + \frac{\widetilde{\Pi}_t^b - \Pi_t^b}{\Pi_t^b}\, d\mathcal{N}_t, \tag{4.2.2}$$

in which $d\Pi_t^b$ measures banks' real profits per unit of managers' aggregate wealth $n_t^m \equiv \int_1^2 n_{i,t}^m \, di$. In equilibrium, the drift rate of this process will be positive and the term proportional to $d\mathcal{N}_t$ will be negative. That is, in normal times banks earn positive profits, which are immediately paid out

this assumption facilitates some interpretations of the model.

to bank owners. If a jump arrives, banks suffer losses and shareholders recapitalise banks out of their own net worth to cover these losses.

Regarding capital gains, it should be noted that both the SDE for the capital stock and the SDEs for prices will lead to fluctuations on agents' balance sheets. The stochastic jump term in the former induces exogenous risk and the jump terms in the latter induce endogenous risk as the size of price changes will depend on characteristics of the state of the economy. The following lemma takes the three sources of managers' income into account and provides an equation that describes the evolution of net worth $n_{i,t}^m$:

**Lemma 3.** *The stochastic flow budget constraint of an individual manager is given by*

$$\begin{aligned} dn_{i,t}^m = &\left\{y_{i,t}^m - i_{i,t}^m - c_{i,t}^m + q_t\left[\mu_t^q + \Phi(\iota_{i,t}^m) - \delta\right]k_{i,t}^m + p_t\left[\mu_t^p + \Phi(\iota_t) - \delta\right]K_t m_{i,t}^m\right\}dt \\ &+ \left\{\left[\widetilde{q}_t\left(1-\kappa-\underline{\kappa}^m\right) - q_t\right]k_{i,t}^m + \left[\widetilde{p}_t\left(1-\kappa\right) - p_t\right]K_t m_{i,t}^m\right\} d\underline{\mathcal{N}}_{i,t} \\ &+ \left\{\left[\widetilde{q}_t\left(1-\kappa-\overline{\kappa}^m\right) - q_t\right]k_{i,t}^m + \left[\widetilde{p}_t\left(1-\kappa\right) - p_t\right]K_t m_{i,t}^m\right\} d\overline{\mathcal{N}}_{i,t} + d\Pi_t^b n_{i,t}^m, \end{aligned} \tag{4.2.3}$$

*Proof.* See Appendix B.1.1.

It is important to recognise that constraint (4.2.3) is formulated in real terms since $n_{i,t}^m$ measures the individual's real wealth. Before interpreting the stochastic flow budget constraint further, it will prove useful to apply some manipulations. First, in order to substitute savings into (4.2.3), it can be rewritten further by introducing asset returns and portfolio weights. To this end, let

$$k_{i,t}^m = \frac{x_{i,t}^m n_{i,t}^m}{q_t} \quad \text{and} \quad m_{i,t}^m = \frac{(1-x_{i,t}^m)n_{i,t}^m}{p_t K_t}, \tag{4.2.4}$$

where $x_{i,t}^m$ is $i$'s portfolio weight on capital, i.e. the share of net worth invested into capital holdings and $1 - x_{i,t}^m$ is the portfolio weight on money. As net worth is measured in real terms, $x_{i,t}^m n_{i,t}^m$ is equal to the *real value* of capital $q_t k_{i,t}^m$. The same reasoning applies to the real value of money in the second part of (4.2.4). Substituting relations (4.2.1), (4.1.8), and (4.2.4) into (4.2.3) as well as taking into account (4.2.2) implies:

$$\begin{aligned} dn_{i,t}^m = &\underbrace{\left\{\left[x_{i,t}^m \mu_{i,t}^{r^{K,m}} + \left(1-x_{i,t}^m\right)\mu_t^{r^M} + \mu_t^{\Pi^b}\Pi_t^b\right]n_{i,t}^m - c_{i,t}^m\right\}}_{\equiv \mu_{i,t}^{n^m} n_{i,t}^m} dt \\ &+ \underbrace{\left\{\left[x_{i,t}^m\left(1-\kappa-\underline{\kappa}^m\right)\frac{\widetilde{q}_t}{q_t} + \left(1-x_{i,t}^m\right)\left(1-\kappa\right)\frac{\widetilde{p}_t}{p_t} + \widetilde{\Pi}_t^b - \Pi_t^b\right]n_{i,t}^m - n_{i,t}^m\right\}}_{\equiv \underline{\widetilde{n}}_{i,t}^m - n_{i,t}^m} d\underline{\mathcal{N}}_{i,t} \\ &+ \underbrace{\left\{\left[x_{i,t}^m\left(1-\kappa-\overline{\kappa}^m\right)\frac{\widetilde{q}_t}{q_t} + \left(1-x_{i,t}^m\right)\left(1-\kappa\right)\frac{\widetilde{p}_t}{p_t} + \widetilde{\Pi}_t^b - \Pi_t^b\right]n_{i,t}^m - n_{i,t}^m\right\}}_{\equiv \overline{\widetilde{n}}_{i,t}^m - n_{i,t}^m} d\overline{\mathcal{N}}_{i,t}, \end{aligned} \tag{4.2.5}$$

with

$$\mu_{i,t}^{r^{K,m}} \equiv \underbrace{\frac{a^m - \iota_{i,t}^m}{q_t}}_{\text{payoff yield}} + \underbrace{\mu_t^q + \Phi(\iota_{i,t}^m) - \delta}_{\text{drift of capital gains rate}} \quad \text{and} \tag{4.2.6}$$

$$\mu_t^{r^M} \equiv \underbrace{\mu_t^p + \Phi(\iota_t) - \delta}_{\text{drift of capital gains rate}} \quad , \tag{4.2.7}$$

where $\mu_{i,t}^{n^m}$ is the drift rate of $i$'s net worth, $\underline{\widetilde{n}}_{i,t}^m$ is his post-jump wealth level if he suffers the "bad" idiosyncratic shock, $\widetilde{\overline{n}}_{i,t}^m{}_{i,t}$ the post-jump wealth level in the corresponding "good" state, and $\mu_{i,t}^{r^K,m}$ and $\mu_t^{r^M}$ are the drift rates of the stochastic processes for the real returns on manager $i$'s capital and money, respectively. Equation (4.2.5) is similar to the standard intertemporal budget constraint in models without uncertainty, with the exceptions that a portfolio selection problem under Poisson uncertainty is embedded and that it contains no labour income. Drift rate $\mu_{i,t}^{r^{K,m}}$ is the deterministic part of the return on $i$'s capital, i.e. the return in the absence of shocks. This return has a payoff yield and a capital gains rate component. The former is due to the average product of capital net of investment, while the latter consists of the deterministic appreciation of the capital price and stock. On the contrary, the return on money $dr_t^M$ only consists of a capital gains rate. Term $\Phi(\iota_t) - \delta$, which is the deterministic net appreciation of the *aggregate* capital stock, is included in the return on money since $p_t$ measures the price of money per unit of capital. Thus, an increase in $K_t$ raises the price of money $P_t$, ceteris paribus. Since the price of money is measured in final goods, the drift of the return on money is the negative inflation rate in the absence of jumps. The drift of i's wealth can also be expressed as

$$\mu_{i,t}^{n^m} n_{i,t}^m = \mu_{i,t}^{r^{P,m}} n_{i,t}^m - c_{i,t}^m, \tag{4.2.8}$$

where

$$\mu_{i,t}^{r^{P,m}} \equiv x_{i,t}^m \mu_{i,t}^{r^{K,m}} + \left(1 - x_{i,t}^m\right) \mu_t^{r^M} + \mu_t^{\Pi^b} \Pi_t^b \tag{4.2.9}$$

is the drift rate of the agent's portfolio return.

The post-shock wealth level depends on post-shock values of the capital stock, given by $x_{i,t}^m$ $\left(1 - \kappa - \overline{\kappa}^m\right) \frac{\widetilde{q}_t}{q_t} n_{i,t}^m$ in the "good" state of the idiosyncratic disturbance and $x_{i,t}^m \left(1 - \kappa - \underline{\kappa}^m\right) \frac{\widetilde{q}_t}{q_t} n_{i,t}^m$ in the "bad" state, as well as the post-jump value of the money stock, which is $\left(1 - x_{i,t}^m\right)(1 - \kappa) \frac{\widetilde{p}_t}{p_t} n_{i,t}^m$. Similarly to equation (4.2.7), the depreciation rate of aggregate capital due to a shock $\kappa$ is included in the post-jump value of $i's$ money balance since $p_t$ is measured in units of the aggregate capital stock. The values of assets after a shock has occurred, in turn, are influenced by the amplitude of the exogenous shocks, the respective post-jump asset price relative to the pre-jump price, and the portfolio position. In equilibrium, $\underline{\widetilde{n}}^m_{i,t}$ as well as $\underline{\widetilde{n}}^m_{i,t}$ are decreasing in $x_{i,t}$ and $\mu^{r^{K,m}}_{i,t} > \mu^{r^M}_t$ holds. Hence, when allocating his portfolio between capital and money, the agent faces a risk-return trade-off.

Finally, we can also derive the stochastic processes for $r^{K,m}_{i,t}$ and $r^M_t$ by adding the payoff yields to the stochastic processes for the capital gains rates of capital and money. The latter are found via the application of CVF (3.1.20) to value stocks $q_t k^m_{i,t}$ and $p_t K_t m^m_{i,t}$, taking into account equations (4.1.4), (3.1.19), (3.2.15), and (4.1.1) and dropping asset purchases. The real instantaneous returns on capital and money derived this way are described by

$$\begin{aligned} dr^{K,m}_{i,t} = & \underbrace{\frac{a^m - \iota^m_{i,t}}{q_t}}_{\text{payoff yield}} dt \\ & + \underbrace{\left\{\mu^q_t + \Phi(\iota^m_{i,t}) - \delta\right\} dt + \frac{\widetilde{q}_t\,(1-\kappa-\underline{\kappa}^m) - q_t}{q_t}\, d\underline{\mathcal{N}}_{i,t} + \frac{\widetilde{q}_t\,(1-\kappa-\overline{\kappa}^m) - q_t}{q_t}\, d\overline{\mathcal{N}}_{i,t}}_{\text{capital gains rate}} \end{aligned} \tag{4.2.10}$$

and

$$dr^M_t = \underbrace{\{\mu^p_t + \Phi(\iota_t) - \delta\}\, dt + \frac{\widetilde{p}_t(1-\kappa) - p_t}{p_t}\, d\mathcal{N}_t,}_{\text{capital gains rate}} \tag{4.2.11}$$

respectively. To save on notation it is from here on assumed that the investment rate is at its optimal level, i.e. the level that maximises the return to capital. Since the investment technology is described by (3.1.9), the optimal investment rate satisfies the first expression in (3.1.27).

Taking expectations, while recognising the fact that the expected value of the idiosyncratic shock is equal to zero, results in

$$\mathbb{E}_t\left[dr^{K,m}_{i,t}\right] = \left\{\frac{a^m - \iota^m_{i,t}}{q_t} + \mu^q_t + \Phi(\iota^m_{i,t}) - \delta + \lambda\frac{\widetilde{q}_t\,(1-\kappa) - q_t}{q_t}\right\} dt \tag{4.2.12}$$

and

$$\mathbb{E}_t\left[dr^M_t\right] = \left\{\mu^p_t + \Phi(\iota_t) - \delta + \lambda\frac{\widetilde{p}_t(1-\kappa) - p_t}{p_t}\right\} dt, \tag{4.2.13}$$

which are deterministic differential equations.

### 4.2.2 Prudent Entrepreneurs

In order to arrive at prudent entrepreneurs' returns on assets and flow budget constraint, we have to take into account four main differences between these agents and managers, the first two of which have already been stated in Subsection 4.1.2. First, entrepreneurs have access to external finance in the form of credit, which is denominated in money. It follows that the balance sheet identity of an individual prudent entrepreneur is

$$n_{i,t}^{e,p} \equiv q_t k_{i,t}^{e,p} + p_t K_t \left( m_{i,t}^{e,p} + l_{i,t}^{e,p} \right), \tag{4.2.14}$$

where the *nominal* value of loans $l_{i,t}^{e,p}$ is restricted to be nonpositive. That is, entrepreneurs are allowed to hold a short position in loans, but are not able to extend loans. The change in the agent's real wealth is provided in the following lemma.

**Lemma 4.** *The stochastic flow budget constraint of an individual prudent entrepreneur is given by*

$$\begin{aligned} dn_{i,t}^{e,p} = & \left\{ y_{i,t}^{e,p} + \Gamma_t p_t K_t l_{i,t}^{e,p} - i_{i,t}^{e,p} - c_{i,t}^{e,p} \right\} dt \\ & + \left\{ q_t \left[ \mu_t^q + \Phi(\iota_{i,t}^{e,p}) - \delta \right] k_{i,t}^{e,p} + p_t \left[ \mu_t^p + \Phi(\iota_t) - \delta \right] K_t \left( m_{i,t}^{e,p} + l_{i,t}^{e,p} \right) \right\} dt \\ & + \left\{ \left[ \widetilde{q}_t \left( 1 - \kappa - \underline{\kappa}^{e,p} \right) - q_t \right] k_{i,t}^{e,p} + \left[ \widetilde{p}_t \left( 1 - \kappa \right) - p_t \right] K_t \left( m_{i,t}^{e,p} + l_{i,t}^{e,p} \right) \right\} d\underline{\mathcal{N}}_{i,t} \\ & + \left\{ \left[ \widetilde{q}_t \left( 1 - \kappa - \overline{\kappa}^{e,p} \right) - q_t \right] k_{i,t}^{e,p} + \left[ \widetilde{p}_t \left( 1 - \kappa \right) - p_t \right] K_t \left( m_{i,t}^{e,p} + l_{i,t}^{e,p} \right) \right\} d\overline{\mathcal{N}}_{i,t}. \end{aligned} \tag{4.2.15}$$

*Proof.* See Appendix B.1.2.

In (4.2.15) the negative of product $[\Gamma_t + \mu_t^p + \Phi(\iota_t) - \delta]\, p_t K_t l_{i,t}^{e,p}$ measures real interest payments which accrue if the agent decides to borrow from banks. The interpretation of variable $\Gamma_t$ will become clear momentarily. Utilising relations

$$k_{i,t}^{e,p} = \frac{x_{1,i,t}^{e,p} n_{i,t}^{e,p}}{q_t}, \quad m_{i,t}^{e,p} = \frac{x_{2,i,t}^{e,p} n_{i,t}^{e,p}}{p_t K_t} \quad \text{and} \quad l_{i,t}^{e,p} = \frac{(1 - x_{1,i,t}^{e,p} - x_{2,i,t}^{e,p}) n_{i,t}^{e,p}}{p_t K_t}, \tag{4.2.16}$$

in which $x_{1,i,t}^{e,p}$, $x_{2,i,t}^{e,p}$, and $1 - x_{1,i,t}^{e,p} - x_{2,i,t}^{e,p}$ are entrepreneur $i$'s portfolio weights on capital, money and credit, respectively, as well as (4.1.9) and (4.1.10), we get

$$\begin{aligned} dn_{i,t}^{e,p} = & \underbrace{\left\{ \left[ x_{1,i,t}^{e,p} \mu_{i,t}^{r^{K,e}} + x_{2,i,t}^{e,p} \mu_t^{r^M} + \left( 1 - x_{1,i,t}^{e,p} - x_{2,i,t}^{e,p} \right) \mu_t^{r^{L,e}} \right] n_{i,t}^{e,p} - c_{i,t}^{e,p} \right\}}_{\equiv \mu_{i,t}^{n^{e,p}} n_{i,t}^{e,p}} dt \\ & + \underbrace{\left\{ \left[ x_{1,i,t}^{e,p} \left( 1 - \kappa - \underline{\kappa}^{e,p} \right) \frac{\widetilde{q}_t}{q_t} + \left( 1 - x_{1,i,t}^{e,p} \right) (1 - \kappa) \frac{\widetilde{p}_t}{p_t} \right] n_{i,t}^{e,p} - n_{i,t}^{e,p} \right\}}_{\equiv \underline{\widetilde{n}}_{i,t}^{e,p} - n_{i,t}^{e,p}} d\underline{\mathcal{N}}_{i,t} \\ & + \underbrace{\left\{ \left[ x_{1,i,t}^{e,p} \left( 1 - \kappa - \overline{\kappa}^{e,p} \right) \frac{\widetilde{q}_t}{q_t} + \left( 1 - x_{1,i,t}^{e,p} \right) (1 - \kappa) \frac{\widetilde{p}_t}{p_t} \right] n_{i,t}^{e,p} - n_{i,t}^{e,p} \right\}}_{\equiv \overline{\widetilde{n}}_{i,t}^{e,p} - n_{i,t}^{e,p}} d\overline{\mathcal{N}}_{i,t} \\ & + \underbrace{\frac{a^m - a^e}{q_t} x_{1,i,t}^{e,p} n_{i,t}^{e,p} \, dt \, d\mathcal{N}_{i,t}^s,}_{=0} \end{aligned} \tag{4.2.17}$$

with

$$\mu_{i,t}^{r^{K,e}} \equiv \frac{a^e - \iota_{i,t}^{e,p}}{q_t} + \mu_t^q + \Phi(\iota_{i,t}^{e,p}) - \delta \quad \text{and} \tag{4.2.18}$$

$$\mu_t^{r^{L,e}} \equiv \Gamma_t + \mu_t^p + \Phi(\iota_t) - \delta. \tag{4.2.19}$$

where $\mu_{i,t}^{n^{e,p}}$ is the drift rate of prudent entrepreneurs net worth, $\mu_{i,t}^{r^{K,e}}$ is the drift rate of the return on entrepreneurs' capital, and $\mu_t^{r^{L,e}}$ is the drift rate of the stochastic process for the real loan rate individuals of type $e$ have to pay for taking out a loan. The term in the fourth line of (4.2.17) is specific to entrepreneurs and arises due to the possible exposure to the sector-specific shock $d\mathcal{N}_{i,t}^s$, which reduces $i$'s output per instant of time due to the reduction in productivity. However, since product $dt\, d\mathcal{N}_{i,t}^s$ equals zero, which is an implication of the càdlàg property of the Poisson process, the term in the fourth line of (4.2.17) is zero as well. It follows that in case of a jump, prudent entrepreneur $i$'s equity drops to either $\underline{\widetilde{n}}_{i,t}^{e,p}$ or $\overline{\widetilde{n}}_{i,t}^{e,p}$ regardless of whether the agent is subject to the sector-specific shock or not.[593,594] An alternative expression for drift $\mu_{i,t}^{n^{e,p}} n_t^{e,p}$ is

$$\mu_{i,t}^{n^{e,p}} n_t^{e,p} = \mu_{i,t}^{r^{P,e,p}} n_{i,t}^{e,p} - c_{i,t}^{e,p}, \tag{4.2.20}$$

in which the drift rate of prudent entrepreneurs' portfolio return is defined according to

$$\mu_{i,t}^{r^{P,e,p}} \equiv x_{1,i,t}^{e,p} \mu_{i,t}^{r^{K,e}} + x_{2,i,t}^{e,p} \mu_t^{r^M} + \left(1 - x_{1,i,t}^{e,p} - x_{2,i,t}^{e,p}\right) \mu_t^{r^{L,e}}. \tag{4.2.21}$$

What is more, the second and third lines of (4.2.17) show that $i$'s post-jump wealth share does not depend on the portfolio allocation between money and credit. This results from the fact that credit is denominated in money, which, in turn, implies that the real value of credit changes by factor $(1-\kappa)\,\widetilde{p}_t/p_t$ due to the arrival of a shock, just as the real value of money holdings.

The stochastic process for prudent entrepreneurs' real return on capital is characterised by

$$\begin{aligned} dr_{i,t}^{K,e,p} = & \left\{ \frac{a^e - \iota_{i,t}^{e,p}}{q_t} + \mu_t^q + \Phi(\iota_{i,t}^{e,p}) - \delta \right\} dt \\ & + \frac{\widetilde{q}_t\,(1-\kappa-\underline{\kappa}^{e,p}) - q_t}{q_t}\, d\underline{\mathcal{N}}_{i,t} + \frac{\widetilde{q}_t\,(1-\kappa-\overline{\kappa}^{e,p}) - q_t}{q_t}\, d\overline{\mathcal{N}}_{i,t} - \underbrace{\frac{a^e - a^m}{q_t}\, dt\, d\mathcal{N}_{i,t}^s}_{=0}, \end{aligned} \tag{4.2.22}$$

where the last term in the second line is due to the possible exposure to the sector-wide shock. As in the previous subsection, it will be assumed in the remainder that the investment rate is at its optimal value. This equation also reiterates that in case of a jump, i.e. in case of $d\mathcal{N}_t = 1$, each

[593] We will see in Section 4.3.2 that this is a useful fact in the derivation of entrepreneurs' optimal portfolio choice.
[594] Intuitively, output during the jump is zero since the duration of the jump is zero.

individual entrepreneur is potentially subject to three distinct types of disturbances: the aggregate, idiosyncratic and sector-specific shocks, which enter (4.2.22) via parameters $\kappa$, $\underline{\kappa}^{e,p}$, $\overline{\kappa}^{e,p}$ and $a^e - a^m$. Taking the expected value of (4.2.22) gives

$$\mathbb{E}_t\left[dr_{i,t}^{K,e,p}\right] = \left\{\frac{a^e - \iota_{i,t}^{e,p}}{q_t} + \mu_t^q + \Phi(\iota_{i,t}^{e,p}) - \delta + \lambda\frac{\widetilde{q}_t\,(1-\kappa) - q_t}{q_t}\right\} dt. \tag{4.2.23}$$

This shows that the expected return on entrepreneurs' capital differs from managers' only with respect to the payoff yield, even though the former are subject to the sector-specific shock.

The real loan rate evolves according to

$$dr_t^{L,e} = \underbrace{\left\{\Gamma_t + \underbrace{\mu_t^p + \Phi(\iota_t) - \delta}_{=\mu_t^{r^M}}\right\}}_{=\mu_t^{r^{L,e}}} dt + \frac{\widetilde{p}_t - p_t}{p_t}\, d\mathcal{N}_t. \tag{4.2.24}$$

Equation (4.2.24) shows that $\Gamma_t \equiv dr_t^{L,e} - dr_t^M$ is the mark-up over the real deposit rate banks charge to borrowers. This mark-up will be determined endogenously in equilibrium.[595] It can also be interpreted as the external finance premium since $dr_{i,t}^M$ is the opportunity cost of financing a unit of a given asset internally.[596] According to the above equation, the EFP is deterministic. This indeed is plausible since a crucial characteristic of the loan contract is that interest payments are not contingent on the states of nature in the no bankruptcy region.[597]

### 4.2.3 Imprudent Entrepreneurs

Imprudent entrepreneurs, in contrast to their prudent counterparts, do not hold money by assumption.[598] For this reason their equity is defined as

$$n_{i,t}^{e,i} \equiv q_t k_{i,t}^{e,i} + p_t K_t l_{i,t}^{e,i}. \tag{4.2.25}$$

Variables $k_{i,t}^{e,i}$ and $l_{i,t}^{e,i}$ must satisfy the same restrictions as $k_{i,t}^{e,p}$ and $l_{i,t}^{e,p}$, namely

$$k_{i,t}^{e,i} \geq 0, l_{i,t}^{e,i} \leq 0.$$

Lemma 5 states the change in wealth level $n_{i,t}^{e,i}$:

[595] Cf. Subsection 4.3.4.
[596] Cf. Bernanke et al. (1999, p. 1345).
[597] Cf. Subsection 2.1.2.
[598] This assumption leads to a more intuitive expression for the EFP and also eases the numerical solution of the model.

**Lemma 5.** *Imprudent entrepreneurs' stochastic flow budget constraint is given by*

$$\begin{aligned} dn_{i,t}^{e,i} = &\left( q_t k_{i,t}^{e,i} \mu_{i,t}^{r^{K,e}} + p_t K_t l_{i,t}^{e,i} \mu_t^{r^{L,e}} - c_{i,t}^{e,i} \right) dt \\ &+ (1-\kappa) \left( \widetilde{q}_t k_{i,t}^{e,i} + \widetilde{p}_t K_t l_{i,t}^{e,i} \right) d\mathcal{N}_t . \end{aligned} \tag{4.2.26}$$

*Proof.* Equation (4.2.26) can be derived by following similar steps as in the derivation of (4.2.17), while noting that imprudent entrepreneurs do not hold real balances and are not exposed to idiosyncratic risk. □

It should be reiterated that both types of entrepreneurs face the same productivity parameter $a^e$. For this reason $\mu_{i,t}^{r^{K,e}}$ is used rather than $\mu_{i,t}^{r^{K,e,p}}$ and $\mu_{i,t}^{r^{K,e,i}}$ in (4.2.17) and (4.2.26), respectively. Yet, the stochastic part of the process for the return on capital differs since imprudent individuals are not exposed to the project-specific shock. Hence, we have

$$dr_{i,t}^{K,e,i} = \left\{ \frac{a^e - \iota_{i,t}^{e,i}}{q_t} + \mu_t^q + \Phi(\iota_{i,t}^{e,i}) - \delta \right\} dt + \frac{\widetilde{q}_t (1-\kappa) - q_t}{q_t} d\mathcal{N}_t + \underbrace{\frac{a^e - a^m}{q_t} dt \, d\mathcal{N}_{i,t}^s}_{=0} . \tag{4.2.27}$$

Comparing capital returns (4.2.10), (4.2.22), and (4.2.27), it becomes clear that all agents that are endowed with an investment technology choose identical investment rates. Hence, we can set $\iota_{i,t}^m = \iota_{i,t}^{e,p} = \iota_{i,t}^{e,i} = \iota_t, \forall i \in \mathbb{I}^m \cup \mathbb{I}^e$ in the remainder.[599] This also implies that the deterministic components of agents' returns on capital and portfolio returns do not include idiosyncratic elements. Thus, subscript $i$ can be removed from those variables as well.

Further, agents of types $e, p$ and $e, i$ pay the same real rate on loans since banks are not able to distinguish between the two groups. It follows that imprudent entrepreneurs' real marginal cost of external finance is determined by (4.2.24).

### 4.2.4 Banks

Similarly to before, we derive the representative bank's stochastic flow budget constraint from banks' aggregate wealth, which is defined by

$$n_t^b \equiv \left( m_t^b + l_t^{b,p} + l_t^{b,i} \right) p_t K_t , \tag{4.2.28}$$

where $m_t^b$ is banks' aggregate money stock, $l_t^{b,p}$ is the aggregate nominal value of loans granted to prudent entrepreneurs, and $l_t^{b,i}$ is the aggregate nominal value of banks' loans to imprudent

[599] It might be concluded from this finding, that the economy-wide investment rate $\iota_t$ is independent of the allocation of capital between sectors . However, this conclusion does not hold true: the distribution of the production factor influences the capital price $q_t$ as we will see in Section 5.3, resulting in an indirect effect of the allocation of capital on aggregate investment.

entrepreneurs.[600] Banks hold short positions in money and long positions in loans. That is, $m_t^b$ will be negative and $l_t^{b,p}$ as well as $l_t^{b,i}$ will be positive in equilibrium. As already mentioned, a short position in money is equivalent to the creation of inside money.

**Lemma 6.** *Aggregate bank wealth follows*

$$\begin{aligned} dn_t^b = &\left\{\Gamma_t p_t K_t \left(l_t^{b,p} + l_t^{b,i}\right) + \left(m_t^b + l_t^{b,p} + l_t^{b,i}\right) p_t K_t \left[\mu_t^p + \Phi(\iota_t) - \delta\right] - w_t^b\right\} dt \\ &+ \left\{\left(m_t^b + l_t^{b,p} + \widetilde{l}_t^{b,i}\right)(1-\kappa)\,\widetilde{p}_t K_t - \left(m_t^b + l_t^{b,p} + l_t^{b,i}\right) p_t K_t\right\} d\mathcal{N}_t \\ &- \mu_t^{\Pi^b} \Pi_t^b n_t^m \, dt - \left(\widetilde{\Pi}_t^b - \Pi_t^b\right) n_t^m \, d\mathcal{N}_t. \end{aligned} \tag{4.2.29}$$

*Proof.* See Appendix B.1.3.

In (4.2.29) product $\left[\Gamma_t + \mu_t^p + \Phi(\iota_t) - \delta\right] p_t K_t \left(l_t^{b,p} + l_t^{b,i}\right)$ stands for banks' aggregate real proceeds from their lending activities. Lenders have to be compensated for potential bankruptcy costs and thus demand a premium over the deposit rate for extending loans. That is, $\Gamma_t > 0$ will hold true in equilibrium. It is important to note that banks can only choose sum $l_t^{b,p} + l_t^{b,i}$ rather than (planned) loan extensions to prudent and imprudent end-borrowers separately. The reason is that banks are not able to distinguish between the two different classes of borrowers. Variable $w_t^b$ measures the total wage payments to bankers per unit of time.

If a jump occurs, imprudent borrowers default and banks write off loans extended to the former. The assumed form of the financial contract then allows banks to seize the bankrupt debtors' remaining assets of value $(1-\kappa)\,\widetilde{q}_t k_t^{e,i}$, where $k_t^{e,i} \equiv \int_{1-\varphi}^{1} k_{i,t}^{e,i}\, di$ is imprudent agents' aggregate capital holding. Thus, term $(1-\kappa)\,\widetilde{p}_t K_t \widetilde{l}_t^{b,i}$ can be replaced in (4.2.29) and we get

$$\begin{aligned} dn_t^b = &\left\{\Gamma_t p_t K_t \left(l_t^{b,p} + l_t^{b,i}\right) + \left(m_t^b + l_t^{b,p} + l_t^{b,i}\right) p_t K_t \left[\mu_t^p + \Phi(\iota_t) - \delta\right] - w_t^b\right\} dt \\ &+ \left\{(1-\kappa)\,\widetilde{q}_t k_t^{e,i} + \left(m_t^b + l_t^{b,p}\right)(1-\kappa)\,\widetilde{p}_t K_t - \left(m_t^b + l_t^{b,p} + l_t^{b,i}\right) p_t K_t\right\} d\mathcal{N}_t \\ &- \mu_t^{\Pi^b} \Pi_t^b n_t^m \, dt - \left(\widetilde{\Pi}_t^b - \Pi_t^b\right) n_t^m \, d\mathcal{N}_t. \end{aligned} \tag{4.2.30}$$

Since banks pay out any profits to managers and as the latter recapitalise the former to cover losses, $n_t^b$ must be equal to zero at any point in time. If, in addition, banks start their businesses without any wealth, which is from here on assumed, banks' net worth is always equal to zero. These facts can be utilised to determine the parameter functions of process $\Pi_t^b$: its drift is given by

$$\mu_t^{\Pi^b} \Pi_t^b = \frac{1}{n_t^m} \left[\Gamma_t p_t K_t \left(l_t^{b,p} + l_t^{b,i}\right) - w_t^b\right] \tag{4.2.31}$$

[600] It should be noted that the variables in (4.2.28) are not denoted by subscript $i$. This is because they measure the aggregate asset positions of the banking sector.

and the change in $\Pi_t^b$ in case of a jump by

$$\widetilde{\Pi}_t^b - \Pi_t^b = \frac{1}{n_t^m}\left[(1-\kappa)\,\widetilde{q}_t k_t^{e,i} + \left(m_t^b + l_t^{b,p}\right)(1-\kappa)\,\widetilde{p}_t K_t\right]. \tag{4.2.32}$$

Banks' return on credit $dr_t^{L,b}$ is yet to be determined. This return is not identical to $dr_{i,t}^{L,e}$, the rate borrowers have to pay to banks, as the former also includes potential losses from defaults on loans extended by banks. To derive process $dr_t^{L,b}$ and its expected value, we first note that the percentage change in the real value of banks' loans to imprudent borrowers $p_t K_t l_{i,t}^{b,i}$ in case of a jump is described by the following expression:

$$\frac{d\left(p_t K_t l_t^{b,i}\right)}{p_t K_t l_t^{b,i}} = \frac{(1-\kappa)\,\widetilde{q}_t k_t^{e,i} - p_t K_t l_t^{b,i}}{p_t K_t l_t^{b,i}} \quad \text{if } d\mathcal{N}_t = 1. \tag{4.2.33}$$

This expression can be reformulated by recognising that condition $l_t^{b,i} = -l_t^{e,i}$, where $-l_t^{e,i} \equiv -\int_{1-\varphi}^{1} l_{i,t}^{e,i}\,di$ is the aggregate nominal loan demand of imprudent entrepreneurs, has to hold.[601] In addition, it was assumed in Subsection 4.1.2 that imprudent debtors do not accumulate any equity in normal times without jumps. Accordingly, aggregating equation (4.2.25) implies $q_t k_t^{e,i} = -p_t K_t l_t^{e,i}$. Taking these facts into account leads us to

$$\frac{d\left(p_t K_t l_t^{b,i}\right)}{p_t K_t l_t^{b,i}} = \frac{(1-\kappa)\,\widetilde{q}_t - q_t}{q_t} \quad \text{if } d\mathcal{N}_t = 1. \tag{4.2.34}$$

The percentage change in the value of loans to prudent entrepreneurs, on the other hand, is given by

$$\frac{d\left(p_t K_t l_t^{b,p}\right)}{p_t K_t l_t^{b,p}} = \frac{(1-\kappa)\,\widetilde{p}_t - p_t}{p_t} \quad \text{if } d\mathcal{N}_t = 1, \tag{4.2.35}$$

which is identical to the capital gains rate component of the return on money in the case of a jump. This is for two reasons: first, prudent borrowers do not default and second, loans are denominated in money. Since the deterministic return $\mu_t^{r^{L,b}}$ on both loan portfolios is identical, $l_t^{b,i} = \varphi l_t^b$, where $l_t^b$ is the total volume of credit extended by banks, holds, and $l_t^{b,p} = (1-\varphi)\, l_t^b$ is satisfied, we get

$$\begin{aligned} dr_t^{L,b} = {} & \underbrace{\left\{\Gamma_t + \mu_t^p + \Phi(\iota_t) - \delta\right\}}_{\equiv \mu_t^{r^{L,b}}} dt \\ & + \left\{\varphi \frac{(1-\kappa)\,\widetilde{q}_t - q_t}{q_t} + (1-\varphi)\,\frac{(1-\kappa)\,\widetilde{p}_t - p_t}{p_t}\right\} d\mathcal{N}_t. \end{aligned} \tag{4.2.36}$$

601 This condition simply states that the nominal value of loans granted to imprudent borrowers must equal the nominal value of these borrowers' loan volume. In order to avoid confusion at this point it should be reiterated that banks are not able to distinguish between loan applicants and thus cannot choose $l_t^{b,i}$ separately.

Taking the expected value of (4.2.36) results in

$$\mathbb{E}_t\left[dr_t^{L,b}\right] = \Big\{\Gamma_t + \mu_t^p + \Phi(\iota_t) - \delta + \lambda\varphi\frac{(1-\kappa)\,\widetilde{q}_t - q_t}{q_t} + \lambda\,(1-\varphi)\,\frac{(1-\kappa)\,\widetilde{p}_t - p_t}{p_t}\Big\}\,dt. \tag{4.2.37}$$

Equation (4.2.36) can be interpreted as follows. In normal times absent a macro shock, banks receive nominal interest rate $\Gamma_t$ from borrowers per instant of time and also have to take into account that the real value of the loan portfolio changes deterministically by the drift rate of the capital gains rate on money, i.e. the negative of the inflation rate. If a jump arrives, imprudent entrepreneurs declare bankruptcy and banks take over the assets of the former, which implies that a fraction $\varphi$ of the loan portfolio is replaced by the remaining assets of defaulting debtors on banks' balance sheets. This aspect is captured by the first fraction in the second line of (4.2.36). The complementary share of entrepreneurs $1-\varphi$ do not default. Hence, the change in value of the corresponding portion of banks' loan portfolio is only due to the change in the real value of money.

Finally, using the same relations as in the derivation of (4.2.34), equation (4.2.32) can be reformulated to

$$\widetilde{\Pi}_t^b - \Pi_t^b = \frac{1}{n_t^m}\,(1-\kappa)\left(\frac{\widetilde{q}_t - q_t}{q_t} - \frac{\widetilde{p}_t - p_t}{p_t}\right) q_t k_t^{e,i}. \tag{4.2.38}$$

The presence of percentage change $(\widetilde{p}_t - p_t)\,/p_t$ in the above equation for banks' losses is due to the fact that loans are funded entirely by inside money. It follows that real bank losses are not solely due to the change in the real value of defaulting debtors' assets $\widetilde{q}_t k_t^{e,i}$ but also to the change in the real value of the deposits that finance loans extended to those debtors. This fact is visualised in Figure 4.2.1, in which the real value of deposits on the liabilities side of banks' post-jump consolidated balance sheet is split into the portion that finances loans to prudent entrepreneurs $(1-\varphi)\,\widetilde{p}_t\widetilde{K}_t m_t^b$ and the portion that funds loans to imprudent entrepreneurs $\varphi\widetilde{p}_t\widetilde{K}_t m_t^b$. It can be observed that equity injection $\left(\widetilde{\Pi}_t^b - \Pi_t^b\right) n_t^m$ exactly offsets the losses on the loan portfolio. Further, adverse movements in asset prices only affect a small share of the loan portfolio, namely the share of loans extended to imprudent entrepreneurs $\varphi$. This is in stark contrast to the I Theory of Money, in which intermediaries' *entire* balance sheets are affected by endogenous price risk (cf. Subsections 3.2.2 and 3.2.3).

## 4.3 Decision Rules

### 4.3.1 Managers

We begin this section by considering the utility maximisation problem of managers. In order to gain additional insight, the associated program and the resulting optimality conditions are first presented in their general forms. Managers choose an *infinite sequence* of consumption and portfolio weights

Figure 4.2.1: Banks' Post-Jump Consolidated Balance Sheet

| A | L |
|---|---|
| $\widetilde{p}_t \widetilde{K}_t l_t^{b,p}$ | $(1-\varphi)\, \widetilde{p}_t \widetilde{K}_t m_t^b$ |
| $\widetilde{q}_t \widetilde{k}_t^{e,i}$ | $\varphi \widetilde{p}_t \widetilde{K}_t m_t^b$ |
| $\left(\widetilde{\Pi}_t^b - \Pi_t^b\right) n_t^m$ | |

*Notes*: In the baseline calibration in Section 5.1 the share of defaulting debtors is set to 8 percent. The model results under the baseline calibration show that banks' losses as a share of the consolidated balance sheet size are smaller than depicted.

$[c_t^m, x_t^m]_{t=0}^{\infty}$ that maximise expected lifetime utility $U_{i,0}^m$ evaluated at time $t = 0$ and discounted at rate $\rho$ subject to the stochastic flow budget constraint (4.2.5). As each individual agent is atomistic, he takes prices and asset returns as given. These facts imply the following program.

$$\max_{[c_{i,t}^m, 1 \geq x_{i,t}^m \geq 0]_{t=0}^{\infty}} U_{i,0}^m \equiv E_0 \int_0^{\infty} e^{-\rho t} u\left(c_{i,t}^m,\, (1 - x_{i,t}^m) n_{i,t}^m\right)\, dt, \tag{4.3.1a}$$

$$\text{s.t.} \quad dn_{i,t}^m = \mu_{i,t}^{n^m} n_{i,t}^m \, dt + \left\{\underline{\widetilde{n}}_{i,t}^m - n_{i,t}^m\right\} d\underline{\mathcal{N}}_t + \left\{\widetilde{\overline{n}}_{i,t}^m - n_{i,t}^m\right\} d\overline{\mathcal{N}}_t, \tag{4.3.1b}$$

$$\mu_{i,t}^{n^m} = x_{i,t}^m \mu_t^{r^{K,m}} + \left(1 - x_{i,t}^m\right) \mu_t^{r^M} - \frac{c_{i,t}^m}{n_{i,t}^m} + \mu_t^{\Pi^b} \Pi_t^b, \tag{4.3.1c}$$

$$\underline{\widetilde{n}}_{i,t}^m = \left[x_{i,t}^m \left(1 - \kappa - \underline{\kappa}\right) \frac{\widetilde{q}_t}{q_t} + \left(1 - x_{i,t}^m\right)(1 - \kappa) \frac{\widetilde{p}_t}{p_t} + \widetilde{\Pi}_t^b - \Pi_t^b\right] n_{i,t}^m, \tag{4.3.1d}$$

$$\widetilde{\overline{n}}_{i,t}^m = \left[x_{i,t}^m \left(1 - \kappa - \overline{\kappa}\right) \frac{\widetilde{q}_t}{q_t} + \left(1 - x_{i,t}^m\right)(1 - \kappa) \frac{\widetilde{p}_t}{p_t} + \widetilde{\Pi}_t^b - \Pi_t^b\right] n_{i,t}^m, \tag{4.3.1e}$$

$$n_{i,0}^m > 0, \quad \underline{\widetilde{n}}_{i,t}^m \geq 0, \quad \widetilde{\overline{n}}_{i,t}^m \geq 0, \tag{4.3.1f}$$

where $u(\cdot)$ is an additively separable instantaneous utility function in consumption $c_{i,t}^m$ and real balances $(1 - x_{i,t})n_{i,t}$, which are derived by multiplying the second equality in (4.2.4) by $p_t K_t$. A few more points are worth of notice here. First, program (4.3.1a)-(4.3.1f) contains terms $\underline{\widetilde{n}}_{i,t}^m$ and $\widetilde{\overline{n}}_{i,t}^m$ (which, in turn, depend on $\widetilde{q}$ and $\widetilde{p}_t$) through the stochastic flow budget constraint. Since those are the "true", i.e. model-consistent values after a shock has occurred, rational expectations are assumed implicitly. Second, the condition on $x_{i,t}^m$ implies that managers can neither have a short position in capital nor in money. We will see that in equilibrium managers indeed do not wish to have a short position in any asset, i.e. the mentioned condition never binds. Fourth, there is no constraint on consumption $c_{i,t}^m$. In particular, consumption is allowed to take on negative values for the moment. Fifth, the first condition in (4.3.1f) requires the individual's initial wealth to be strictly positive. Sixth, the second and third conditions in (4.3.1f) are solvency constraints. We conjecture that these constraints always hold with strict inequality in equilibrium and verify that this is indeed the case later on.

The dynamic optimisation problem (4.3.1a)-(4.3.1f) above can be solved by dynamic programming. A crucial ingredient to the technique of dynamic programming is the *value function*. Managers' value function $V^m(\cdot)$ is defined according to the following definition.

**Definition 9.** Managers' Value Function[602]

Value function $V^m(\cdot)$ expresses the optimal value of the original program (4.3.1a)-(4.3.1f) starting from an initial value of the state variable $n_{i,t}^m$ at time $s = t$ and is given by

$$V^m\left(t,\, n_{i,t}^m\right) \equiv \max_{\left[c_{i,t}^m, 1 \geq x_{i,t}^m \geq 0\right]} \mathbb{E}_t \int_t^\infty e^{-\rho(s-t)} u\left(c_{i,s}^m,\, \left(1 - x_{i,s}^m\right) n_{i,s}^m\right) ds, \tag{4.3.2}$$

subject to (4.3.1b)-(4.3.1f), where indices $t$ are replaced by indices $s$.

It follows that the value function gives the maximum attainable lifetime utility starting at time $s = t$ with wealth $n_{i,t}^m$. In Definition 9 and the above program, $c_{i,t}^m$ and $x_{i,t}^m$ are *control variables,* which can be chosen by the agent in the current period and $n_{i,t}^m$ is a *state variable*, which is taken as given in $t$. Generally, the value function expresses maximum attainable lifetime utility as a function of all state variables. In a strict sense, *all* variables that influence the agents' control variables are to be considered as state variables.[603] According to this notion, the value function should additionally include prices $q_t$ and $p_t$ as well as their respective post-jump levels and drift rates as function arguments. Since this would be notationally inconvenient, we restrict our attention to a single state variable $n_{i,t}^m$.[604] This can be theoretically justified by the fact that the shadow price of wealth, i.e. the derivative of the value function with respect to $n_{i,t}^m$, rather than the shadow prices of other state variables, is all that matters for the agent's decision problem, as we will see momentarily.[605] Using (4.3.2), the following lemma states an important intermediate step in solving the maximisation problem.

**Lemma 7.** *Solving the dynamic utility maximisation problem* (4.3.1a)-(4.3.1f) *is equivalent to solv-*

[602] Cf. also Ljungqvist and Sargent (2004, pp. 85f.).
[603] Cf. Wälde (2011, p. 56).
[604] Since the value function depends on time only through the state variables, we can also drop the time argument.
[605] Cf. also Wälde (2011, p. 56).

*ing the recursive Hamilton-Jacobi-Bellman equation given by*

$$\rho V^m(n_{i,t}^m) = \max_{c_{i,t}^m, 1 \geq x_{i,t}^m \geq 0} \Bigg\{ u\left(c_{i,t}^m, \left(1 - x_{i,t}^m\right) n_{i,t}^m\right) + \frac{dV^m(n_{i,t}^m)}{dn_{i,t}^m} \mu_{i,t}^{n^m} n_{i,t}^m \tag{4.3.3}$$
$$+ \lambda\phi \left[V^m\left(\underline{\widetilde{n}}_{i,t}^m\right) - V^m\left(n_{i,t}^m\right)\right] + \lambda\left(1-\phi\right)\left[V^m\left(\overline{\widetilde{n}}_{i,t}^m\right) - V^m\left(n_{i,t}^m\right)\right]\Bigg\},$$

*subject to* (4.3.1c)-(4.3.1f)*.*

*Proof.* See Appendix B.1.4.

Lemma 7 is an application of the *Equivalence of Values* theorem in dynamic programming, which states that both the sequence problem - program (4.3.1a)-(4.3.1f) in our context - and the recursive[606] Hamilton-Jacobi-Bellman (HJB) equation yield the same value.[607] In effect, we have transformed the infinite sequence problem to the problem of finding value function $V^m(\cdot)$ and the optimal control variables at time $t$.[608] This is the essence of dynamic programming. Intuitively, the optimal plan consists of two parts: function $u(\cdot)$, which stands for the current return, and the continuation return captured by terms involving $V^m(\cdot)$.[609] Derivative $dV^m(n_{i,t}^m)/dn_{i,t}^m$ is the shadow price of net worth, i.e. the marginal increase of the objective function (4.3.1a) due to an additional unit of $n_{i,t,}^m$.[610] In order to facilitate intuition of equation (4.3.3) further, $V^m(\cdot)$ can be thought of as being equal to the value of an asset, e.g. an asset traded in the stock market. According to this interpretation, the HJB equation is equivalent to a no-arbitrage asset pricing condition.[611] This can be seen if we rewrite (4.3.3) as

$$\rho V^m(n_{i,t}^m) = u\left(c_{i,t}^{m*}, \left(1 - x_{i,t}^{m*}\right) n_{i,t}^m\right) + \frac{dV^m(n_{i,t}^m)}{dt} \tag{4.3.4}$$
$$+ \lambda\phi \left[V^m\left(\underline{\widetilde{n}}_{i,t}^m\right) - V^m\left(n_{i,t}^m\right)\right] + \lambda\left(1-\phi\right)\left[V^m\left(\overline{\widetilde{n}}_{i,t}^m\right) - V^m\left(n_{i,t}^m\right)\right],$$

in which the elimination of the maximum operator results from the substitution of optimal choices of the control variables and $dV^m(n_{i,t}^m)/dt$ replaces the second term on the RHS of (4.3.3) via the application of the chain rule.[612] The no-arbitrage condition then demands the required return (i.e. the asset's value discounted at rate $\rho$) to be equal to the sum of the current "dividend" $u(\cdot)$, the deterministic change in the asset's value $dV^m(n_{i,t}^m)/dt$, and the expected change in value due to the occurrence of jumps, which is captured by the second line of the RHS in (4.3.4).

[606] Equation (4.3.3) is characterised as recursive since $V^m(\cdot)$ and its derivative with respect to $n_{i,t}^m$ appear on the RHS.
[607] Cf. Acemoglu (2008, p. 222).
[608] Cf. Ljungqvist and Sargent (2004, p. 86).
[609] Cf. Acemoglu (2008, p. 218).
[610] Cf. Wälde (2011, p. 269).
[611] Cf. Acemoglu (2008, pp. 280f.).
[612] Remember in this context that $\mu_{i,t}^{n^m}$ is the growth rate of $n_{i,t}^m$ in the absence of shocks.

Agents' optimal behaviour can now simply be characterised by maximising the HJB equation w.r.t the controls, while keeping in mind that (i) real balances negatively depend on $x_{i,t}^{m}$ through (4.2.4), (ii) drift rate $\mu_{i,t}^{n^m}$ positively depends on $x_{i,t}^{m}$ and negatively on $c_{i,t}^{m}$, and (iii) that post-jump wealth levels depend negatively on $x_{i,t}^{m}$. The first-order condition with respect to $c_{i,t}^{m}$ is

$$\frac{du\left(c_{i,t}^{m},\left(1-x_{i,t}^{m}\right)n_{i,t}^{m}\right)}{dc_{i,t}^{m},}=-\frac{dV^{m}(n_{i,t}^{m})}{dn_{i,t}^{m}}\frac{d\left(\mu_{i,t}^{n^m}n_{i,t}^{m}\right)}{dc_{i,t}^{m}}=\frac{dV^{m}(n_{i,t}^{m})}{dn_{i,t}^{m}}. \tag{4.3.5}$$

This condition has the well-known interpretation that in order to maximise utility, the agent has to equate the marginal utility of current consumption to the marginal lifetime utility of higher future consumption due to more saving today, as expressed through the shadow price of wealth $dV^{m}(n_{i,t}^{m})/dn_{i,t}^{m}$.[613] The first-order condition with respect to $x_{i,t}^{m}$ is

$$\begin{aligned}\frac{dV^{m}(n_{i,t}^{m})}{dn_{i,t}^{m}}\frac{d\left(\mu_{i,t}^{n^m}n_{i,t}^{m}\right)}{dx_{i,t}^{m}}\leq{}&\frac{du\left(c_{i,t}^{m},\left(1-x_{i,t}^{m}\right)n_{i,t}^{m}\right)}{d\left(\left(1-x_{i,t}^{m}\right)n_{i,t}^{m}\right)}n_{i,t}^{m}\\&-\lambda\phi\frac{dV^{m}\left(\underline{\widetilde{n}}_{i,t}^{m}\right)}{dn_{i,t}^{m}}\frac{d\underline{\widetilde{n}}_{i,t}^{m}}{dx_{i,t}^{m}}-\lambda\left(1-\phi\right)\frac{dV^{m}\left(\overline{\widetilde{n}}_{i,t}^{m}\right)}{dn_{i,t}^{m}}\frac{d\overline{\widetilde{n}}_{i,t}^{m}}{dx_{i,t}^{m}}.\end{aligned} \tag{4.3.6}$$

Importantly, the FOC in the above equation is an inequality. The reason is that less productive managers might be confronted with situations in which they prefer to hold no capital at all. In such situations, the risk premium on capital over money does not sufficiently compensate managers for the risk and opportunity costs associated with holding capital.[614] Let us now turn to the components of equation (4.3.6). The term on the LHS is the shadow price multiplied by the marginal increase in the drift of net worth due to an additional unit of $x_{i,t}^{m}$ and the terms on the RHS are the marginal utility boosts arising from an infinitesimal increase in the portfolio weight on money. The first term on the RHS is the marginal instantaneous utility resulting from one marginal unit less of $x_{i,t}^{m}$. This term arises as a consequence of the MIU assumption. The sum of the terms in the second line is the expected marginal increase in post-shock utility due to a marginal decrease in the portfolio weight on capital. Intuitively, when raising the portfolio weight on capital, the agent has to balance the utility gain due to a higher deterministic return against the instantaneous utility loss from holding less money and having a higher risk exposure from holding additional capital.

Since the HJB equation and the FOCs are functional equations[615], they cannot be used directly to determine optimal choices. Fortunately, a simple closed form solution of the value function exists if an instantaneous utility function of the logarithmic type is adopted. Rewriting (4.1.3) using the

[613] Cf. e.g. Stokey and Lucas (1989, p. 14).
[614] Cf. also Brunnermeier and Sannikov (2014d, p. 36).
[615] Functional equations are equations including unknown functions (cf. Ljungqvist and Sargent, 2004, p. 86).

second equality in (4.2.4) gives

$$u\left(c_{i,t}^m, (1-x_{i,t}^m)n_{i,t}^m\right) = \log c_{i,t}^m + \xi \log\left(\left(1-x_{i,t}^m\right)n_{i,t}^m\right). \tag{4.3.7}$$

This leads us to the following lemma.

**Lemma 8.** *Under instantaneous utility function (4.3.7) the value function corresponding to problem* (4.3.1a)-(4.3.1f) *is of the following form:*

$$V^m(n_{i,t}^m) = \alpha_t^m + \frac{1+\xi}{\rho}\log n_{i,t}^m, \tag{4.3.8}$$

*where $\alpha_t^m$ is a time-varying parameter function, which captures terms not depending on $n_{i,t}^m$.*

*Proof.* See Appendix B.1.5.

As Appendix B.1.5 shows, the result in Lemma 8 hinges upon the adoption of a CRS production technology and the usage of a utility function from the HARA class, which make the return on capital and the portfolio choice independent of wealth $n_{i,t}^m$. An important implication of Lemma 8 is that the logarithmic form of the value function in combination with rational expectations precludes the possibility of the agent's default after an adverse shock has occurred. This is because an individual with logarithmic utility prefers to avoid situations in which $V^m\left(\underline{\widetilde{n}}_{i,t}^m\right) \to -\infty$ as $\underline{\widetilde{n}}_{i,t}^m \to 0$.[616] Agents can circumvent such situations by precautionary behaviour. More specifically, they may accumulate sufficient net worth as a buffer against shocks or allocate a higher portion of their wealth to the "safe" asset money. With the value function at hand, we can now derive a specific form of the HJB equation.[617]

**Corollary 1.** *Under instantaneous utility function (4.3.7), managers solve HJB equation*

$$\begin{aligned}\rho V^m(n_{i,t}^m) = \max_{c_{i,t}^m, 1\geq x_{i,t}^m \geq 0} \Bigg\{ & \log c_{i,t}^m + \xi\log\left(\left(1-x_{i,t}^m\right)n_{i,t}^m\right) \\ & + \frac{1+\xi}{\rho n_{i,t}^m}\left[\left(x_{i,t}^m \mu_t^{r^{K,m}} + \left(1-x_{i,t}^m\right)\mu_t^{r^M} + \mu_t^{\Pi^b}\Pi_t^b\right)n_{i,t}^m - c_{i,t}^m\right] \\ & + \lambda\phi\frac{1+\xi}{\rho}\log\left(x_{i,t}^m\left(1-\kappa-\underline{\kappa}^m\right)\frac{\widetilde{q}_t}{q_t} + \left(1-x_{i,t}^m\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t} + \widetilde{\Pi}_t^b - \Pi_t^b\right) \\ & + \lambda(1-\phi)\frac{1+\xi}{\rho}\log\left(x_{i,t}^m\left(1-\kappa-\overline{\kappa}^m\right)\frac{\widetilde{q}_t}{q_t} + \left(1-x_{i,t}^m\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t} + \widetilde{\Pi}_t^b - \Pi_t^b\right)\Bigg\}.\end{aligned} \tag{4.3.9}$$

*Proof.* Substituting (4.3.1c)-(4.3.1e) and (4.3.7) into (4.3.3) and employing the closed form solution of the value function (4.3.8) yields (4.3.9). □

[616] Put differently, as $\underline{\widetilde{n}}_{i,t}^m \to 0$, marginal utility of net worth becomes arbitrarily high. Thus, that level of $\underline{\widetilde{n}}_{i,t}^m$ cannot be an optimal choice, even if the shock occurs with low probability (cf. Brunnermeier and Sannikov, 2014d, p. 7).

[617] Appendix B.2 derives a discrete-time formulation of managers' problem and also discusses some further advantages of utilising continuous- rather than discrete-time formulations..

Corollary 1 allows us to find explicit *policy functions*.[618] The first-order condition with respect to $c_t^m$ implies the policy function for consumption:

$$c_{i,t}^m = \frac{\rho}{1+\xi} n_{i,t}^m. \tag{4.3.10}$$

According to equation (4.3.10), managers consume a constant fraction $\rho/1+\xi$ of their wealth each period. Consumption increases with the agent's time preference rate and decreases with the utility weight on real balances. As is well-known, optimal current consumption is independent of returns in the logarithmic utility case, since income and substitution effects of changes in returns on current consumption exactly cancel out.[619] In that sense, logarithmic utility agents act myopic with respect to their consumption choice. However, consumption *growth* is affected by the portfolio return. This can be seen from the Keynes-Ramsey rule for optimal consumption growth, which results from combining (4.3.10), the defining equation for the drift rate of managers' portfolio return (4.2.9) and the equation of motion for $n_{i,t}^m$ implied by (4.2.3):

$$\frac{1+\xi}{\rho} dc_{i,t}^m = \left(\mu_{i,t}^{r^{P,m}} - \rho\right) n_{i,t}^m \, dt + \left(\underline{\widetilde{n}}_{i,t}^m - n_{i,t}^m\right) \, d\underline{\mathcal{N}}_t + \left(\widetilde{\overline{n}}_{i,t}^m - n_{i,t}^m\right) \, d\overline{\mathcal{N}}_t.$$

Clearly, consumption growth rises with managers' portfolio return in the absence of shocks and this is due to the substitution effect.

Next, we discern the specific form of the first-order condition with respect to $x_t^m$ under logarithmic utility. It is given by

$$\begin{aligned}\mu_t^{r^{K,m}} - \mu_t^{r^M} \leq{} & \frac{\xi\rho}{(1+\xi)\left(1-x_{i,t}^m\right)} \\ & - \lambda\phi \frac{\left(1-\kappa-\underline{\kappa}^m\right)\frac{\widetilde{q}_t}{q_t} - (1-\kappa)\frac{\widetilde{p}_t}{p_t}}{x_{i,t}^m\left(1-\kappa-\underline{\kappa}^m\right)\frac{\widetilde{q}_t}{q_t} + \left(1-x_{i,t}^m\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t} + \widetilde{\Pi}_t^b - \Pi_t^b} \\ & - \lambda(1-\phi)\frac{\left(1-\kappa-\overline{\kappa}^m\right)\frac{\widetilde{q}_t}{q_t} - (1-\kappa)\frac{\widetilde{p}_t}{p_t}}{x_{i,t}^m\left(1-\kappa-\overline{\kappa}^m\right)\frac{\widetilde{q}_t}{q_t} + \left(1-x_{i,t}^m\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t} + \widetilde{\Pi}_t^b - \Pi_t^b}.\end{aligned} \tag{4.3.11}$$

According to the above condition, the *required risk premium* on capital over money on the RHS is equalised to the *actual risk premium* $\mu_t^{r^{K,m}} - \mu_t^{r^M}$ on the LHS if the individual chooses to hold a strictly positive amount of capital. Conversely, if the required risk premium exceeds the actual, the agent abstains from holding capital altogether. It can further be observed from (4.3.11) that the optimal portfolio weight does not depend on individual characteristics. Accordingly, we will remove subscript $i$ from that variable in the remainder. In the equilibrium of the model economy,

[618] In the language of dynamic programming, policy functions represent optimal choices as functions of one or more state variables (cf. Stokey and Lucas, 1989, p. 77).
[619] Cf. e.g. Merton (1969, p. 254).

conditions $\widetilde{q}_t < q_t$ and $\widetilde{p}_t > p_t$ will always hold true. Thus, the actual risk premium will be the higher, the higher the portfolio weight on capital $x_t^m$, provided that managers hold capital. This is another manifestation of the risk-return trade-off. Since equation (4.3.11) implies a complicated cubic function in managers' optimal portfolio weight $x_t^m$, we only state a general policy function here:

$$x_t^m = x^m \left( \mu_t^{r^{K,m}} - \mu_t^{r^M}, \frac{\widetilde{q}_t}{q_t}, \frac{\widetilde{p}_t}{p_t}, \widetilde{\Pi}_t^b - \Pi_t^b, \xi, \lambda, \phi, \kappa, \underline{\kappa}^m, \overline{\kappa}^m \right), \tag{4.3.12}$$

As long as the agent chooses to hold a strictly positive amount of capital, the variables on the RHS of the above equation satisfy

$$\frac{\partial x_t^m}{\partial \left( \mu_t^{r^{K,m}} - \mu_t^{r^M} \right)} > 0, \frac{\partial x_t^m}{\partial \left( \widetilde{q}_t / q_t \right)} > 0, \frac{\partial x_t^m}{\partial \left( \widetilde{p}_t / p_t \right)} < 0, \frac{\partial x_t^m}{\partial \left( \widetilde{\Pi}_t^b - \Pi_t^b \right)} > 0. \tag{4.3.13}$$

Thus, the optimal portfolio weight on capital positively depends on the risk premium on capital, the post- relative to the pre-jump price of capital, and the negative of the equity injection into banks in case of an adverse shock, ceteris paribus. The last relation is due to the fact that the equity injection raises the post-jump marginal utility of wealth and therefore induces risk averse individuals to reduce their risk exposure. In addition, the third inequality in (4.3.13) shows that the portfolio choice is tilted towards money if the post- relative to the pre-jump value of money surges, ceteris paribus. The independence of $x_t^m$ from wealth $n_{i,t}^m$ results from the adoption of a CRS production technology and a utility function of the HARA type, as mentioned. In turn, that property of the optimal portfolio weight leads to asset demand functions that are linear in wealth. Further, it should be noted that the optimal portfolio weight depends on current pre- and post-jump asset prices, but not on the future (expected) values of these variables. Hence, logarithmic utility also implies myopia with respect to asset demands. This finding reflects a classic result in Finance theory which states that current portfolio choices do not depend on future investment opportunities under logarithmic utility.[620,621] The relatively simple structures of policy functions for consumption and portfolio weights are a major advantage of using logarithmic utility functions.

### 4.3.2 Prudent Entrepreneurs

Turning to the entrepreneurial sector, we begin by characterising optimal behaviour of imprudent entrepreneurs. Taking into account the second equality in (4.2.16), (4.1.10), and (4.2.17) their

[620] Cf. e.g. Merton (1971, p. 403).

[621] This can be related to the form of the value function under log utility. In the considered case, the marginal utility of wealth is a constant equal to $\rho / (1 + \xi)$. In contrast, the value function of a risk neutral agent that is financially constrained is $\zeta_t n_{i,t}$, where the marginal utility of wealth $\zeta_t$ captures future investment opportunities (cf. Subsection 3.2.1.1).

maximisation problem is

$$\max_{[c^{e,p}_{i,t}, x^{e,p}_{1,i,t}\geq 0, x^{e,p}_{2,i,t}\geq 0]^{\infty}_{t=0}} U^{e,p}_{i,0} \equiv E_0 \int_0^{\infty} e^{-\rho t} u\left(c^{e,p}_{i,t},\, x^{e}_{2,i,t} n^{e,p}_{i,t}\right) dt, \tag{4.3.14a}$$

$$\text{s.t.} \quad dn^{e,p}_{i,t} = \mu^{n^{e,p}}_{i,t} n^{e,p}_{i,t}\, dt + \left\{\underline{\widetilde{n}}^{e,p}_{i,t} - n^{e,p}_{i,t}\right\} d\underline{\mathcal{N}}_t + \left\{\overline{\widetilde{n}}^{e,p}_{i,t} - n^{e,p}_{i,t}\right\} d\overline{\mathcal{N}}_t \tag{4.3.14b}$$

$$\mu^{n^{e,p}}_{i,t} = x^{e,p}_{1,i,t}\mu^{r^{K,e}}_t + x^{e,p}_{2,i,t}\mu^{r^M}_t + \left(1 - x^{e,p}_{1,i,t} - x^{e,p}_{2,i,t}\right)\mu^{r^{L,e}}_t - \frac{c^{e,p}_{i,t}}{n^{e,p}_{i,t}}, \tag{4.3.14c}$$

$$\underline{\widetilde{n}}^{e,p}_{i,t} = \left[x^{e,p}_{1,i,t}\left(1-\kappa-\underline{\kappa}^{e,p}\right)\frac{\widetilde{q}_t}{q_t} + \left(1 - x^{e,p}_{1,i,t}\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t}\right] n^{e,p}_{i,t}, \tag{4.3.14d}$$

$$\overline{\widetilde{n}}^{e,p}_{i,t} = \left[x^{e,p}_{1,i,t}\left(1-\kappa-\overline{\kappa}^{e,p}\right)\frac{\widetilde{q}_t}{q_t} + \left(1 - x^{e,p}_{1,i,t}\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t}\right] n^{e,p}_{i,t}, \tag{4.3.14e}$$

$$dz_{i,t} = \{a^m - a^e\}\, d\mathcal{N}^s_{i,t}, \tag{4.3.14f}$$

$$n^{e,p}_{i,t} > 0, \quad \underline{\widetilde{n}}^{e,p}_{i,t} \geq 0, \quad \overline{\widetilde{n}}^{e,p}_{i,t} \geq 0, \quad x^{e,p}_{1,i,t} + x^{e,p}_{2,i,t} \geq 0. \tag{4.3.14g}$$

In contrast to managers' problem, this program has two state variables $n^{e,p}_{i,t}$ and $z_{i,t}$. The second state variable enters the program since entrepreneurs are potentially affected by the sector-specific disturbance that affects their stochastic productivity level $z_{i,t}$. The conditions in (4.3.14a) and the fourth condition in (4.3.14g) prevent the agent from having short positions in money and capital as well as having a long position in credit. The remaining conditions in (4.3.14g) are analogous to those in managers' problem. It will again turn out that neither of the mentioned conditions bind in equilibrium. The following lemma states prudent entrepreneurs' HJB equation:

**Lemma 9.** *The HJB equation corresponding to problem* (4.3.14a)-(4.3.14g) *is*

$$\begin{aligned}\rho V^{e,p}(n^{e,p}_{i,t}) = \max_{c^{e,p}_{i,t}, x^{e,p}_{1,i,t}\geq 0, x^{e,p}_{2,i,t}\geq 0} \Bigg\{ & u\left(c^{e,p}_{i,t}, x^{e,p}_{2,i,t} n^{e,p}_{i,t}\right) + \frac{dV^e(n^e_{i,t})}{dn^e_{i,t}}\mu^{n^e}_{i,t} n^e_{i,t} \\ & + \lambda(1-\phi^s)\left(\phi\left[V^{e,p}\left(\underline{\widetilde{n}}^{e,p}_{i,t}\right) - V^{e,p}\left(n^{e,p}_{i,t}\right)\right] + (1-\phi)\left[V^{e,p}\left(\overline{\widetilde{n}}^{e,p}_{i,t}\right) - V^{e,p}\left(n^{e,p}_{i,t}\right)\right]\right) \\ & + \lambda\phi^s\left(\phi\left[V^{m}\left(\underline{\widetilde{n}}^{e,p}_{i,t}\right) - V^{e,p}\left(n^{e,p}_{i,t}\right)\right] + (1-\phi)\left[V^{m}\left(\overline{\widetilde{n}}^{e,p}_{i,t}\right) - V^{e,p}\left(n^{e,p}_{i,t}\right)\right]\right)\Bigg\}.\end{aligned} \tag{4.3.15}$$

*Proof.* See Appendix B.1.6.

The third line of (4.3.15) shows that the change in utility due to the sector specific-shock depends on managers' value function. When the agent experiences that type of shock, his productivity parameter drops to $a^m$ and from then on he continues operations as a manager. The lifetime utility of managers is summarised by value function $V^m(\cdot)$, while the change in utility due to the sector-specific shock is given by $V^m(\cdot) - V^{e,p}(\cdot)$. This change is negative as managers' productivity parameter $a^m$ is lower than $a^e$. The assumption that $a^e$ drops to $a^m$ implies that the HJB equation can be solved using two different value functions $V^{e,p}(\cdot)$ and $V^m(\cdot)$ with only one argument $n_{i,t}^{e,p}$, rather than solving a problem with one value function $V^{e,p}(\cdot)$ and two state variables $n_{i,t}^{e,p}$ and $z_{i,t}$. This is much simpler in our context.

Rearranging utility function (4.1.3) using the second equality in (4.2.16) leads to

$$u\left(c_{i,t}^{e,p}, x_{2,i,t}^{e} n_{i,t}^{e,p}\right) = \log c_{i,t}^{e,p} + \xi \log\left(x_{2,i,t}^{e,p} n_{i,t}^{e,p}\right). \tag{4.3.16}$$

This specific form of preferences is utilised in the following lemma to solve for entrepreneurs' value function.

**Lemma 10.** *Under instantaneous utility function* (4.3.16) *the value function corresponding to problem* (4.3.14a)-(4.3.14g) *is of the following form:*

$$V^{e,p}(n_{i,t}^{e,p}) = \alpha_t^{e,p} + \frac{1+\xi}{\rho} \log n_{i,t}^{e,p}, \tag{4.3.17}$$

*where $\alpha_t^{e,p}$ is a time-varying parameter function, which captures terms not depending on $n_{i,t}^{e,p}$.*

*Proof.* See Appendix B.1.7.

**Corollary 2.** *Under instantaneous utility function* (4.3.16), *prudent entrepreneurs solve HJB equation*

$$\begin{aligned}
\rho V^{e,p}(n_{i,t}^{e,p}) = \max_{c_{i,t}^{e,p}, x_{1,i,t}^{e,p} \geq 0, x_{2,i,t}^{e,p} \geq 0} \Bigg\{ & \log c_{i,t}^{e,p} + \xi \log\left(x_{2,i,t}^{e,p} n_{i,t}^{e,p}\right) \\
& + \frac{1+\xi}{\rho n_{i,t}^{e,p}} \left(\left[x_{1,i,t}^{e,p} \mu_t^{r^{K,e}} + x_{2,i,t}^{e,p} \mu_t^{r^M} + \left(1 - x_{1,i,t}^{e,p} - x_{2,i,t}^{e,p}\right) \mu_t^{r^{L,e}}\right] n_{i,t}^{e,p} - c_{i,t}^{e,p}\right) \\
& + \lambda\phi \frac{1+\xi}{\rho} \log\left(x_{1,i,t}^{e,p} \left(1 - \kappa - \underline{\kappa}^{e,p}\right) \frac{\widetilde{q}_t}{q_t} + \left(1 - x_{1,i,t}^{e,p}\right) (1-\kappa) \frac{\widetilde{p}_t}{p_t}\right) \\
& + \lambda (1-\phi) \frac{1+\xi}{\rho} \log\left(x_{1,i,t}^{e,p} \left(1 - \kappa - \overline{\kappa}^{e,p}\right) \frac{\widetilde{q}_t}{q_t} + \left(1 - x_{1,i,t}^{e,p}\right) (1-\kappa) \frac{\widetilde{p}_t}{p_t}\right) \\
& + \lambda \phi^s \left(\alpha_t^m - \alpha_t^{e,p}\right) \Bigg\}.
\end{aligned} \tag{4.3.18}$$

*Proof.* Substituting (4.3.16), (4.3.8), (4.3.17), and (4.3.14c)-(4.3.14e) into (4.3.15) yields (4.3.18). □

Note that the term in the final line of (4.3.18) is the expected change in utility due to the sector-specific shock. Parameters $\alpha_t^m$ and $\alpha_t^{e,p}$ capture the time-varying investment opportunities of managers and prudent entrepreneurs, respectively. Since the former are less productive than the latter, $\alpha_t^m < \alpha_t^{e,p}$ holds and prudent entrepreneurs suffer a drop in utility if they are exposed to the sector-specific shock.

The consumption policy rule resulting from HJB (4.3.18) is

$$c_{i,t}^{e,p} = \frac{\rho}{1+\xi} n_{i,t}^{e,p} \tag{4.3.19}$$

and the FOC for $x_{1,i,t}^{e,p}$ is

$$\begin{aligned} \mu_t^{r^{K,e}} - \mu_t^{r^{L,e}} &= -\lambda\phi \frac{\left(1-\kappa-\underline{\kappa}^{e,p}\right)\frac{\widetilde{q}_t}{q_t} - (1-\kappa)\frac{\widetilde{p}_t}{p_t}}{x_{1,i,t}^{e,p}\left(1-\kappa-\underline{\kappa}^{e,p}\right)\frac{\widetilde{q}_t}{q_t} + \left(1-x_{1,i,t}^{e,p}\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t}} \\ &\quad - \lambda(1-\phi)\frac{\left(1-\kappa-\overline{\kappa}^{e,p}\right)\frac{\widetilde{q}_t}{q_t} - (1-\kappa)\frac{\widetilde{p}_t}{p_t}}{x_{1,i,t}^{e,p}\left(1-\kappa-\overline{\kappa}^{e,p}\right)\frac{\widetilde{q}_t}{q_t} + \left(1-x_{1,i,t}^{e,p}\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t}}, \end{aligned} \tag{4.3.20}$$

which shows that the portfolio weight on capital is identical across agents of type $e,p$ in optimum. For this reason, we remove subscript $i$ from that variable in the remainder. In addition, (4.3.20) implies a general policy function for $x_{1,t}^{e}$ of the form:

$$x_{1,t}^{e,p} = x_1^{e,p}\left(\mu_t^{r^{K,e}} - \mu_t^{r^{L,e}}, \frac{\widetilde{q}_t}{q_t}, \frac{\widetilde{p}_t}{p_t}, \lambda, \phi, \kappa, \underline{\kappa}^{e,p}, \overline{\kappa}^{e,p}\right). \tag{4.3.21}$$

It should be noted that the optimal capital choice does not directly depend on the probability of the sector-specific shock. Put differently, entrepreneurs ignore the possibility of a permanent reduction in capital productivity when choosing their capital demands. The reason is related to the logarithmic form of the utility function, which implies that agent's stochastic investment opportunities are not captured in the marginal utility of net worth. This can be observed from value functions (4.3.8) and (4.3.17). The marginal utility of net worth, however, is crucial in determining asset demands, as can be observed from (4.3.6). Since the time preference rate and the utility weight of real balances are the same for managers and prudent entrepreneurs, the marginal utilities in the two sectors are identical. Thus, the marginal post-jump utility due to an additional marginal unit of $x_{1,t}^{e,p}$ does not depend on the sector-specific shock.

To determine the agent's optimal money holdings, we now consider the FOC with respect to the portfolio weight on money $x_{2,i,t}^{e}$ resulting from maximising (4.3.15):

$$\frac{\xi}{x_{2,i,t}^{e,p}} = \frac{1+\xi}{\rho}\left(\mu_t^{r^{L,e}} - \mu_t^{r^M}\right). \tag{4.3.22}$$

Thus, in optimum the marginal instantaneous utility from allocating an additional unit of wealth to money must equal the marginal lifetime utility of more future consumption due to taking out less credit today for the purpose of financing real balances. Since condition (4.3.22) does not include idiosyncratic terms, subscript $i$ of variable $x^{e,p}_{2,i,t}$ can be neglected in the following. Solving for $x^{e,p}_{2,t}$ and using $\mu^{r^{L,e}}_t - \mu^{r^M}_t = \Gamma_t$ yields the policy function

$$x^{e,p}_{2,t} = \frac{\xi\rho}{(1+\xi)\,\Gamma_t}, \tag{4.3.23}$$

implying that the agent's optimal portfolio weight on money increases with the utility weight on money and the time preference rate and decreases with the external finance premium. Choosing $x^{e,p}_{1,t}$ and $x^{e,p}_{2,t}$ implicitly determines loan demand $l^{e,p}_{i,t}$ given by

$$l^{e,p}_{i,t} = \frac{\left(1 - x^{e,p}_{1,t} - x^{e,p}_{2,t}\right) n^{e,p}_{i,t}}{p_t K_t}, \tag{4.3.24}$$

for which only nonpositive values are allowed.[622]

### 4.3.3 Imprudent Entrepreneurs

We now focus on the behaviour of imprudent entrepreneurs. In order to keep the model as tractable as possible, we make four simplifying assumptions. First, and as already mentioned, they consume their entire profits in each instant of time without jumps. Second, they start their lives without any equity. Third, they do not hold real balances. Fourth, any individual imprudent entrepreneur demands as much credit as his representative imprudent counterpart does.[623] It follows from the first and second assumption that

$$n^{e,i}_{i,t} = q_t k^{e,i}_{i,t} + p_t K_t l^{e,i}_{i,t} = 0, \quad \forall t \in \left[t_{i,0}, t^{\mathcal{N}}_i\right), \tag{4.3.25}$$

where $t_{i,0}$ is the birth date of the imprudent individual and $t^{\mathcal{N}}_i$ is the first point in time in his life in which he experiences a jump, i.e.

$$t^{\mathcal{N}}_i = \inf\left\{t \geq t_{i,0} \mid d\mathcal{N}_t = 1\right\}.$$

Condition (4.3.25) shows that agents of type $e, i$ do not accumulate equity buffers that could protect them from insolvency in the event of a jump.

Next, we determine asset demands of imprudent entrepreneurs. The fourth assumption in the present subsection implies $l^{e,i}_{i,t} = l^{e,p,r}_{i,t}$, in which $l^{e,p,r}_{i,t}$ is the nominal loan demand of the representative prudent entrepreneur. Using (4.3.24), the definition of subset $\mathbb{I}^e$ introduced in Section 4.1.2, and

[622] Taking out credit is tantamount to a short position in loans and thus to a negative value of $l^{e,p}_{i,t}$.

[623] It is shown in Appendix B.3 that the consumption and asset demand functions resulting from these assumptions are optimal if imprudent entrepreneurs are (i) risk neutral, (ii) extremely myopic in the sense that they only care about current consumption, (iii) subject to a borrowing constraint which prevents them from borrowing more than the representative prudent entrepreneur does, and (iv) not subject to a no-Ponzi game condition in their decisions.

averaging implicitly determines $l_{i,t}^{e,p,r}$ :

$$p_t K_t l_{i,t}^{e,p,r} = \frac{1}{1-\varphi}\left(1 - x_{1,t}^{e,p} - x_{2,t}^{e,p}\right)\int_0^{1-\varphi} n_{i,t}^{e,p} di = \frac{1}{1-\varphi}\left(1 - x_{1,t}^{e,p} - x_{2,t}^{e,p}\right) n_t^e, \tag{4.3.26}$$

where entrepreneurs' aggregate wealth is defined according to $n_t^e \equiv \int_0^{1-\varphi} n_{i,t}^{e,p} di$.[624] It should be noted that $l_{i,t}^{e,p,r}$ is the loan demand by an "artificial" individual that has mass unity and that the mass of all prudent entrepreneurs is equal to $1-\varphi$. Due to that reason the loan demand of the former exceeds that of the latter.[625] Combining (4.3.25) with (4.3.26) and condition $l_{i,t}^{e,i} = l_{i,t}^{e,p,r}$ yields the value of the representative imprudent agent's capital stock:[626]

$$q_t k_{i,t}^{e,i} = \frac{1}{1-\varphi}\left(x_{1,t}^{e,p} + x_{2,t}^{e,p} - 1\right) n_t^e, \quad \forall t \in \left[t_{i,0}, t_i^{\mathcal{N}}\right). \tag{4.3.27}$$

The next step is to rewrite stochastic flow budget constraint (4.2.26) using (4.3.25):[627]

$$\begin{aligned} dn_{i,t}^{e,i} = \max\Bigg[ &\left\{\left(\mu_t^{r^{K,e}} - \mu_t^{r^{L,e}}\right) q_t k_{i,t}^{e,i} - c_{i,t}^{e,i}\right\} dt \\ &+ (1-\kappa)\, p_t K_t l_{i,t}^{e,i}\left(\frac{\widetilde{p}_t}{p_t} - \frac{\widetilde{q}_t}{q_t}\right) d\mathcal{N}_t,\ 0\Bigg], \end{aligned} \tag{4.3.28}$$

The maximum operator is due to agents' option to declare default. As mentioned, imprudent agents do not accumulate equity. Limited liability then implies that the change in individual net worth cannot be strictly negative. Further, the stochastic term in this equation shows that bankruptcy is declared in case of a shock if and only if $\widetilde{p}_t/p_t > \widetilde{q}_t/q_t$.[628,629] This condition is satisfied in equilibrium since the price of money per unit of capital will rise and the price of capital will fall relative to their respective pre-jump levels if a shock occurs. On the other hand, in normal times absent a jump, individuals of type $e,i$ never default, since $\mu_t^{r^{K,e}} > \mu_t^{r^{L,e}}$ always holds true in equilibrium. To sum up, imprudent entrepreneurs always default if $d\mathcal{N}_t = 1$ and never default in the opposite case $d\mathcal{N}_t = 0$. If a jump materialises, all agents of type $e,p$ announce bankruptcy and subsequently exit the economy with zero wealth and consumption. Then, new imprudent individuals are assumed to enter the economy so as to keep the share of bad risks in the population of entrepreneurs constant.

Consumption in the absence of shocks $c_{i,t}^{e,i}$ can be derived by combining (4.3.26), condition

[624] The aggregate wealth of prudent entrepreneurs is denoted by $n_t^e$ rather than $n_t^{e,p}$ since it is equal to the equity of the entire entrepreneurial sector.
[625] This can be observed from (4.3.26), while recognising that $0 < \varphi < 1$.
[626] Deriving the aggregate value of imprudent entrepreneurs' capital stock requires one to integrate the RHS of (4.3.27) over the range $(1-\varphi, 1]$. This takes place in Section 4.4.1.
[627] Comparing the following equation with (4.2.38) shows that banks fully absorb imprudent entrepreneurs' losses in case of a jump.
[628] Recall that $l_{i,t}^{e,i}$ is always negative.
[629] Imprudent entrepreneurs will always default if this condition is satisfied since the flow return is zero during the jump.

$l_{i,t}^{e,i} = l_{i,t}^{e,p,r}$, (4.3.27), and (4.3.28) and subsequently setting $d\mathcal{N}_t$ as well as $dn_{i,t}^n$ to zero. If a shock occurs, bad risks default and, as consequence of limited liability, their consumption equals zero.[630] Hence, we have:

$$c_{i,t}^{e,i} = \left(\mu_t^{r^{K,e}} - \mu_t^{r^{L,e}}\right) \frac{1}{1-\varphi} \left(x_{1,t}^{e,p} + x_{2,t}^{e,p} - 1\right) n_t^e \quad \forall t \in \left[t_{i,0}, t_i^{\mathcal{N}}\right). \tag{4.3.29}$$

Obviously, consumption in the no-jump states increases with entrepreneurs' risk premium $\mu_t^{r^{K,e}} - \mu_t^{r^L}$ and with the value of the agent's capital stock.

### 4.3.4 Banks

If a jump arrives, imprudent entrepreneurs declare bankruptcy and bankers verify the payoffs of the former. In order to verify, bankers must exert effort. The disutility of bankers' effort $\Theta_t$ is captured by function

$$\Theta_t = \Theta\left(\varphi x_t^{b,l} n_t^m\right),$$

where $x_t^{b,l}$ is banks' portfolio weight on loans in terms of bank owners' aggregate net worth $n_t^m$ and product $\varphi x_t^{b,l} n_t^m$ is the real value of loans extended by the banking sector to imprudent borrowers. In particular, we postulate

$$\Theta\left(\varphi x_t^{b,l} n_t^m\right) = \varphi\,(1-\kappa)\,\omega \frac{\widetilde{q}_t}{q_t} x_t^{b,l} n_t^m \tag{4.3.30}$$

and show later on that this effort function is tantamount to a disutility of effort incurred from verification in case of a jump that is a constant share $\omega$ of the real value of defaulting debtors' assets. This property makes verification costs comparable to those in other macro models with CSV, such as BGG.[631] Further, we assume that bankers are compensated for the expected disutility of verification by earning wage income

$$w_t^b\,dt = \mathbb{E}_t\left[\varphi\,(1-\kappa)\,\omega \frac{\widetilde{q}_t}{q_t} x_t^{b,l} n_t^m\,d\mathcal{N}_t\right]. \tag{4.3.31}$$

The above condition reflects bankers' risk neutrality. Bankers choose the loan volume such that expected bank profits are maximised. This behaviour does not maximise the expected utility of risk averse bank owners.[632] Yet, that assumption has two main advantages. First, it allows for the derivation of an intuitive equation for the EFP. Second, it greatly simplifies the numerical solution

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[630] Assuming limited liability implies that the borrower's consumption level in the case of bankruptcy must be non-negative (cf. Freixas and Rochet, 2008, p. 130).

[631] In Bernanke et al. (cf. 1999, p. 1350), verification costs are deadweight losses of resources that are proportionate to the defaulting entrepreneurs' assets.

[632] An explanation for that kind of behaviour might be that bankers are risk neutral and earn bonuses that are proportional to bank profits. This interpretation would introduce a moral hazard problem between bankers and bank owners. However, this mechanism is not modelled here for the sake of simplicity.

of the model.[633] It follows that the representative banker solves

$$\max_{[x_t^{b,l}\geq 0]_{t=0}^{\infty}} \left\{ \left( \mathbb{E}_t \left[ dr_t^{L,b} \right] - \mathbb{E}_t \left[ dr_t^{M} \right] \right) x_t^{b,l} n_t^m - \mathbb{E}_t \left[ \varphi \left(1-\kappa\right) \omega \frac{\widetilde{q}_t}{q_t} x_t^{b,l} n_t^m \, d\mathcal{N}_t \right] \right\}. \tag{4.3.32}$$

The fact that the expected return difference $\mathbb{E}_t \left[ dr_t^{L,b} \right] - \mathbb{E}_t \left[ dr_t^{M} \right]$ is multiplied by the loan volume is due to the assumption that the value of bank loans on the asset side of the representative bank's balance sheet always equals the value of inside money on the liability side, which was mentioned in Section 4.1.4.[634] Problem (4.3.32) implies that the expected real loan return $\mathbb{E}_t \left[ dr^{L,b} \right]$ has to be equal to the sum of the expected real deposit rate and the expected marginal disutility of verification from allocating a marginal unit of real wealth to the loan portfolio in equilibrium:

$$\mathbb{E}_t \left[ dr_t^{L,b} \right] = \mathbb{E}_t \left[ dr_t^{M} \right] + \mathbb{E}_t \left[ \varphi \left(1-\kappa\right) \omega \frac{\widetilde{q}_t}{q_t} \, d\mathcal{N}_t \right]. \tag{4.3.33}$$

As a consequence of bankers' risk neutral behaviour and the ability to costlessly issue new equity, banks do not demand a risk premium. Rather, banks' loan supply is perfectly elastic at the current loan rate. Condition (4.3.33) also shows that the entrepreneurial sector internalises the disutility of auditing by paying a mark-up over the deposit rate.[635] Equipped with the above condition, we are now in a position to provide an expression for the EFP.

**Theorem 1.** *If bankers choose banks' portfolios in a risk neutral way, the external finance premium is determined by*

$$\Gamma_t = \lambda \varphi \left(1-\kappa\right) \left( \frac{q_t - \left(1-\omega\right) \widetilde{q}_t}{q_t} + \frac{\widetilde{p}_t - p_t}{p_t} \right). \tag{4.3.34}$$

*Proof.* Equation (4.3.34) follows from substituting (4.2.37) as well as (4.2.13) into (4.3.33), using $\mathbb{E}_t \left[ d\mathcal{N}_t \right] = \lambda \, dt$, and subsequent rearranging. □

The interpretation of the first term in the parentheses of (4.3.34) is as follows. The EFP increases with the percentage loss in the value of loans extended to defaulting debtors. This loss is the higher, the higher the disutility of auditing parameter $\omega$, the higher the percentage depreciation in the capital stock and the more the capital price drops relative to its pre-jump level. The dependence of the EFP on the post-jump price of capital is due to the fact that each bank takes over the assets of

[633] Letting bankers maximise the expected utility of bank owners necessitates the repeated solution of a system of two nonlinear equations at each step of the algorithm, which drives up the computation time required to solve the model to a large extent.

[634] Further, note that managers' portfolio weights satisfy $x_t^m + x_t^b + x_t^{b,m} + \left(1 - x_t^m - x_t^b - x_t^{b,m}\right) = 1$, in which $x_t^{b,m}$ is banks' portfolio weight on inside money in terms of managers' wealth and the term in the brackets is managers' portfolio weight on money. As bank loans are entirely financed by inside money, condition $x_t^b = -x_t^{b,m}$ has to hold.

[635] This is similar to BGG (cf. 1999, p. 1353), who arrive at the result that entrepreneurs internalise the deadweight costs resulting from banks' auditing of bankrupt borrowers.

their bankrupt debtors and sells these assets on the market for capital at price $\widetilde{q}_t$. The second term is related to the fact that all loans are financed entirely by deposits. Thus, the real value of deposits that finance loans extended to imprudent entrepreneurs changes by factor $(1-\kappa)(\widetilde{p}_t - p_t)/p_t$ once a jump arrives.[636]

Interestingly, the EFP decreases with the aggregate shock if the term in the parenthesis is positive. This is due to the fact that in this case the expected return on money falls more with $\kappa$ than banks' expected return on credit. Further, note that in the absence of an aggregate shock and jump-induced price movements, the EFP simplifies to the expected audit disutility $\lambda\varphi\omega$ per unit of the real value of loans. It is also interesting to recognise that under the condition that $\omega = 1$ the EFP in *nominal* terms is equal to the expected default rate per unit of time $\lambda\varphi$.[637] Setting $\omega = 1$ implies that the difference between the expected value of assets that can be recovered from defaulting debtors and expected costs of verification $w_t^b$ is zero.[638]

As in BGG[639], rationing equilibria can in principle occur. This is the case if the real return on money is sufficiently high such that the zero expected profit condition (4.3.33) implies an actual risk premium on entrepreneurs' capital, which is given by the LHS of (4.3.20), that is lower than prudent entrepreneurs' required risk premium given by the RHS of that same equation. Then, these agents will drop out of the credit market and only imprudent entrepreneurs will remain since the latter only care about expected returns. This, in turn, implies that increases in the loan rate can reduce the expected return of banks. Hence, an adverse selection effect of higher credit rates à la Stiglitz and Weiss (1981) may emerge. For simplicity, we follow BGG[640] in restricting attention to equilibria without rationing.[641]

Let us now return to a claim made in the beginning of this subsection concerning the interpretation of the assumed form of effort function $\Theta(\cdot)$.

**Proposition 1.** *If the economy is exposed to a jump, i.e. if $d\mathcal{N}_t = 1$, bankers' disutility from auditing $\Theta\left(\varphi x_t^{b,l} n_t^m\right)$ is a constant share $0 < \omega \leq 1$ of the value of the defaulting debtors' assets.*

*Proof.* We have to show that

$$\omega \widetilde{q}_t \widetilde{k}_t^{e,i} = \varphi (1-\kappa)\, \omega \frac{\widetilde{q}_t}{q_t} x_t^{b,l} n_t^m,$$

[636] See the discussion in Subsection 4.2.4.

[637] This can be seen more directly if (4.3.34) is rearranged to

$$\Gamma_t = \lambda\varphi \left( (1-\kappa)(1-\omega)\frac{\widetilde{q}_t}{q_t} + (1-\kappa)\frac{\widetilde{p}_t}{p_t} \right).$$

If none of the defaulting debtors' assets can be recovered, the first term in the parenthesis of the above equation is zero. In addition, if the EFP in nominal terms is considered, the second term in the parenthesis has to be replaced by 1 since that term measures the relative change in the real value of money.

[638] Similarly, a classic result from the credit risk literature states that under the assumption that default risk is diversifiable the nominal mark-up of the loan rate over the deposit rate equals the expected default rate if the recovery rate of loans is zero (cf. Freixas and Rochet, 2008, p. 268).

[639] Cf. Bernanke et al. (1999, p. 1351).

[640] Cf. Bernanke et al. (1999, p. 1351).

[641] This was already implicitly assumed in Section 4.3.2 by having let FOC (4.3.20) hold with strict equality.

in which $\widetilde{q}_t \widetilde{k}_t^{e,i}$ is the post-jump value of imprudent entrepreneurs' assets and the RHS is $\Theta(\cdot)$, holds true. Substituting $\varphi x_t^{b,l} n_t^m = p_t K_t l_t^{b,i}$, where $l_t^{b,i}$ is the nominal value of loans extended by banks to imprudent borrowers, as well as $\widetilde{k}_t^{e,i} = (1-\kappa)\, k_t^{e,i}$ and subsequent rearranging leads to

$$q_t k_t^{e,i} = p_t K_t l_t^{b,i}.$$

It follows from balance sheet identity (4.3.25) that the value of imprudent agents' assets is equal to the value of their liabilities, i.e. $q_t k_t^{e,i} = -p_t K_t l_t^{e,i}$, in the absence of jumps. Thus, we end up with $l_t^{b,i} + l_t^{e,i} = 0$. Clearing on the credit market requires this condition to hold. □

Let us finally turn to bankers' consumption choice. By assumption, these agents consume their entire wage income $w_t^b\, dt$ at each instant of time. It will turn out to be useful to write the expression for $w_t^b$ in terms of entrepreneurs' aggregate net worth rather than managers'. To this end, (4.3.31) is combined with relation $\varphi x_t^{b,l} n_t^m = q_t k_t^{e,i}$, while acknowledging the fact that $\mathbb{E}_t [d\mathcal{N}_t] = \lambda\, dt$, to arrive at

$$w_t^b = \lambda (1-\kappa)\, \omega \widetilde{q}_t k_t^{e,i}. \tag{4.3.35}$$

Substituting $k_t^{e,i} \equiv \int_{1-\varphi}^{1} k_{i,t}^{e,i}\, di$ into this equation, inserting (4.3.27) into the resulting expression, and setting bankers' consumption $c_t^b = w_t^b$ yields

$$c_t^b = w_t^b = \lambda (1-\kappa)\, \omega \frac{\widetilde{q}_t}{q_t} \frac{1}{1-\varphi} \left( x_{1,t}^{e,p} + x_{2,t}^{e,p} - 1 \right) n_t^e \int_{1-\varphi}^{1} di. \tag{4.3.36}$$

## 4.4 Aggregation and Market Clearing

### 4.4.1 Derivation of Key Macroeconomic Variables

#### 4.4.1.1 The Aggregate Capital Stock

As of yet, the equation of motion for the aggregate capital stock in (4.1.1) has been treated as given. In the following, it is derived by aggregating individual capital stakes. To this end, we recall the indexations of agents introduced in Subsection 4.1.2 and write

$$K_t \equiv \underbrace{\int_0^{1-\varphi} k_{i,t}^{e,p}\, di}_{\equiv k_t^{e,p}} + \underbrace{\int_{1-\varphi}^{1} k_{i,t}^{e,i}\, di}_{\equiv k_t^{e,i}} + \underbrace{\int_1^2 k_{i,t}^{m}\, di}_{\equiv k_t^{m}}, \tag{4.4.1}$$

in which $k_t^{e,p}$ and $k_t^m$ are defined as prudent entrepreneurs' and managers' aggregate capital stocks, respectively.

The law of motion for $K_t$ can be obtained from aggregating the equations of motion of agents' individual capital stocks:

$$dK_t \equiv \left[\left\{\Phi\left(\iota_t\right) - \delta\right\}\, dt - \kappa\, d\mathcal{N}_t\right] \left( \int_0^{1-\varphi} k_{i,t}^{e,p}\, di, + \int_{1-\varphi}^{1} k_{i,t}^{e,i}\, di + \int_1^2 k_{i,t}^{m}\, di \right), \tag{4.4.2}$$

where we made use of the fact that capital transactions between agents as well as idiosyncratic shocks to capital net out at the aggregate level. Using definition (4.4.1), we get (4.1.1). This equation of motion implies that the change in aggregate capital can be expressed independently of individual capital holdings. This result is due to the assumptions that the amplitudes of shocks to capital are proportionate to individual capital stocks and that the capital goods production functions $\Phi\left(\cdot\right)$ do not differ across agents.

#### 4.4.1.2 The Aggregate Supply of Final Goods

Defining the amount of capital held in the entire entrepreneurial sector $k_t^e$ as the sum of prudent and imprudent entrepreneurs' aggregate capital, i.e. $k_t^e \equiv k_t^{e,p} + k_t^{e,i}$, aggregate output $Y_t$ can be written as

$$Y_t = a^e k_t^e + a^m k_t^m, \tag{4.4.3}$$

or, alternatively, using entrepreneurs' share in the aggregate capital stock $\psi_t \equiv k_t^e / K_t$ and managers' share $1 - \psi_t \equiv k_t^m / K_t$, as

$$Y_t = \underbrace{\left[\psi_t a^e + (1 - \psi_t) a^m\right]}_{\equiv A_t} K_t, \tag{4.4.4}$$

in which $A_t = A\left(\psi_t\right)$ is an weighted average of individual capital productivities, which we will interpret as aggregate TFP. Equation (4.4.4) shows that aggregate output can be characterised by an aggregate production function of the AK type. This fact results from the CRS property of individual production functions.[642]

It will also be of interest to observe how real GDP evolves over the time. Before a corresponding equation is derived, it is useful to provide a stochastic process for variable $\psi_t$ :

$$\frac{d\psi_t}{\psi_t} = \mu_t^{\psi}\, dt + \frac{\widetilde{\psi}_t - \psi_t}{\psi_t}\, d\mathcal{N}_t. \tag{4.4.5}$$

In order to obtain an equation of motion for aggregate output, Lemma 2 needs to be applied to the aggregate production function $Y_t = A_t K_t$, taking into account (4.4.4), (4.4.5) and (4.1.1). This

[642] The aggregate production function result is also obtained in the model by Brunnermeier and Sannikov (2014a), which features heterogeneous productivity levels and CRS technologies, as our model. Moll (cf. 2014, p. 3188) arrives at similar result in a continuous-time heterogeneous producers model, in which capital and labour serve as inputs to CRS production technologies. The relations of his model to ours are discussed in more detail in Subsection 5.5.3.

yields

$$\frac{dY_t}{Y} = \mu_t^Y \, dt + \frac{\widetilde{Y}_t - Y_t}{Y_t} \, d\mathcal{N}, \tag{4.4.6}$$

where

$$\mu_t^Y = \mu_t^A + \mu_t^K = \frac{(a^e - a^m)\mu_t^\psi \psi_t}{\psi_t a^e + (1-\psi_t)a^m} + \Xi(\iota) - \delta \tag{4.4.7}$$

and

$$\widetilde{Y}_t = \underbrace{\left(\widetilde{\psi}_t a^e + (1-\widetilde{\psi}_t)a^m\right)}_{\equiv \widetilde{A}_t} (1-\kappa) \, K_t. \tag{4.4.8}$$

The interpretation of equation (4.4.7) is straightforward: the drift rate of GDP is the higher, the wider the productivity gap, the higher the drift of the capital allocation $\mu_t^\psi \psi_t$ as well as the investment rate, and the lower the depreciation rate. The post-jump level of output can be employed to find the rate of change in GDP in case the economy is exposed to a jump:

$$\breve{Y}_t \equiv \frac{\widetilde{Y}_t - Y_t}{Y_t} = \frac{\widetilde{A}_t \, (1-\kappa) - A_t}{A_t} = \frac{\widetilde{\psi}_t a^e + (1-\widetilde{\psi}_t)a^m}{\psi_t a^e + (1-\psi_t)a^m} \, (1-\kappa) - 1. \tag{4.4.9}$$

Thus, the percentage drop in output due to a jump is the higher the greater the productivity differential, the more entrepreneurs sell capital to managers, i.e. the lower $\widetilde{\psi}_t$ is relative to $\psi_t$, and the higher the percentage amplitude of the aggregate shock $\kappa$. Similarly to BS (2014a)[643], entrepreneurs' sales of capital to managers can be interpreted as "fire sales" of the productive asset since capital is allocated to its second-best use.

#### 4.4.1.3 The Aggregate Demand for Final Goods

The aggregate demand function $Y_t^d$ is derived by adding up real consumption and internal investment expenditures by individual agents. Accordingly,

$$Y_t^d = \int_0^{1-\varphi} \left(c_{i,t}^{e,p} + i_{i,t}^{e,p}\right) \, di + \int_{1-\varphi}^{1} \left(c_{i,t}^{e,i} + i_{i,t}^{e,i}\right) \, di + \int_1^2 \left(c_{i,t}^m + i_{i,t}^m\right) \, di + c_t^b. \tag{4.4.10}$$

[643] Cf. Subsection 3.2.1.2 of this thesis.

After the substitution of optimal consumption choices (4.3.19), (4.3.29), (4.3.10), (4.3.36), and relations $i^m_{i,t}/k^m_{i,t} = i^{e,p}_{i,t}/k^{e,p}_{i,t} = i^{e,i}_{i,t}/k^{e,i}_{i,t} = \iota_t$, we get

$$\begin{aligned} Y^d_t = {} & \frac{\rho}{1+\xi}\int_0^{1-\varphi} n^{e,p}_{i,t}\,di + \left(\mu_t^{r^{K,e}} - \mu_t^{r^{L,e}}\right)\frac{1}{1-\varphi}\left(x^{e,p}_{1,t} + x^{e,p}_{2,t} - 1\right) n^e_t \int_{1-\varphi}^1 di \\ & + \frac{\rho}{1+\xi}\int_1^2 n^m_{i,t}\,di + \lambda\,(1-\kappa)\,\omega\frac{\widetilde{q}_t}{q_t}\frac{1}{1-\varphi}\left(x^{e,p}_{1,t} + x^{e,p}_{2,t} - 1\right) n^e_t \int_{1-\varphi}^1 di \\ & + \iota_t\left(\int_0^{1-\varphi} k^{e,p}_{i,t}\,di + \int_{1-\varphi}^1 k^{e,i}_{i,t}\,di + \int_1^2 k^m_{i,t}\,di\right). \end{aligned} \tag{4.4.11}$$

Using the already defined aggregate wealth levels of entrepreneurs and managers $n^e_t = \int_0^{1-\varphi} n^{e,p}_{i,t}\,di$ and $n^m_t = \int_1^2 n^m_{i,t}\,di$, respectively, recognising that the term in the parenthesis in the third line of the above equation equals the aggregate capital stock, and solving the remaining integrals leads us to

$$\begin{aligned} Y^d_t = {} & \underbrace{\frac{\rho}{1+\xi} n^e_t}_{\equiv c^{e,p}_t} + \underbrace{\left(\mu_t^{r^{K,e}} - \mu_t^{r^{L,e}}\right)\frac{\varphi}{1-\varphi}\left(x^{e,p}_{1,t} + x^{e,p}_{2,t} - 1\right) n^e_t}_{\equiv c^{e,i}_t} \\ & + \underbrace{\frac{\rho}{1+\xi} n^m_t}_{\equiv c^m_t} + \underbrace{\lambda\,(1-\kappa)\,\omega\frac{\widetilde{q}_t}{q_t}\frac{\varphi}{1-\varphi}\left(x^{e,p}_{1,t} + x^{e,p}_{2,t} - 1\right) n^e_t}_{=c^b_t} + \underbrace{\iota_t K_t}_{=i^{e,p}_t + i^{e,i}_t + i^m_t}, \end{aligned} \tag{4.4.12}$$

in which $c^{e,p}_t$, $c^{e,i}_t$ and $c^m_t$ are prudent entrepreneurs', imprudent entrepreneurs' and managers' aggregate consumption levels, respectively and $i^{e,p}_t$, $i^{e,i}_t$ and $i^m_t$ are the respective agents' aggregate internal investment levels. It is interesting to note that a lump sum redistribution of wealth from managers to prudent entrepreneurs increases aggregate demand even though both groups have identical propensities to consume out of wealth. This result arises since a higher value of $n^e_t$ not only increases the consumption of agents of type $e, p$ but also that of types $e, i$ and $b$. This, in turn, is due to two effects that occur as $n^e_t$ rises. First, due to their risk aversion, an improved net worth position induces prudent entrepreneurs to take out more loans. Since their imprudent counterparts always demand the same volume of credit as the average prudent individual does, debt levels of the former also increase. This allows them to buy more capital and earn higher profits in periods without a jump, which are consumed instantly. Second, the higher volume of bank debt raises bankers' expected auditing effort and thereby their income $w^b_t$. This effect also leads to a boost in aggregate demand since bankers, just as imprudent entrepreneurs, consume their entire income immediately.

#### 4.4.1.4 The Money Multiplier, the Velocity of Money, and Inflation

As BS (2014d)[644], we define the money multiplier $\Lambda_t$ as the ratio of inside to outside money. Since banks operate without any positive net worth, the real value of deposits is equal to the real value of loans, given by $\left(x_{1,t}^{e,p} + x_{2,t}^{e,p} - 1\right) n_t^e / (1 - \varphi)$. Hence, we have

$$\Lambda_t \equiv \frac{M_t^I}{M_t^O} = \frac{1}{1-\varphi} \left(x_{1,t}^{e,p} + x_{2,t}^{e,p} - 1\right) \frac{n_t^e}{p_t K_t}. \tag{4.4.13}$$

According to this equation, the money multiplier is equal to the nominal value of deposits. This is not surprising since the supply of outside money is normalised to unity.

The velocity of money $\mathcal{V}_t$ can be calculated by first noting that the nominal demand for money is given by

$$\frac{x_{2,t}^{e,p} n_t^e + (1 - x_t^m) n_t^m}{p_t K_t}.$$

Using the money market equilibrium condition, the nominal demand for money, aggregate production function (4.4.4), and relation $\mathcal{P}_t = (p_t K_t)^{-1}$, in which $\mathcal{P}_t$ is the price level, in the defining equation for the velocity of money yields

$$\mathcal{V}_t \equiv \frac{\mathcal{P}_t Y_t}{M_t} = \frac{A(\psi_t) K_t}{\left[x_{2,t}^{e,p} n_t^e + (1 - x_t^m) n_t^m\right]}, \tag{4.4.14}$$

where $M_t = M_t^I + M_t^O$ is the total money supply. Equation (4.4.14) states that $\mathcal{V}_t$ is equal to the ratio of output to the real money demand.

Finally, let us determine the instantaneous inflation rate $d\pi_t \equiv d\mathcal{P}_t/\mathcal{P}_t$. Applying CVF (3.1.20) to function $\mathcal{P}_t = (p_t K_t)^{-1}$, while recalling that $\mu_t^K = \Phi(\iota_t) - \delta$ and $\widetilde{K}_t = (1-\kappa) K_t$, yields

$$d\pi_t = d\mathcal{P}_t/\mathcal{P}_t = \underbrace{- \left\{\mu_t^p + \Phi(\iota_t) - \delta\right\}}_{\equiv \mu_t^{\mathcal{P}}} dt + \underbrace{\frac{\left[\widetilde{p}_t (1-\kappa)\right]^{-1} - p_t^{-1}}{p_t^{-1}}}_{\equiv \check{\mathcal{P}}_t} d\mathcal{N}_t. \tag{4.4.15}$$

It follows from this equation that the inflation rate in normal times without jumps is determined by drift rates $\mu_t^p$ and $\mu_t^K$. Further intuition on the determinants of inflation can be gained by applying CVF (3.1.20) to the equation of exchange and subsequently using (4.4.7) as well as relation $\mu_t^{\mathcal{P}} = -\mu_t^p - \mu_t^K$. This gives

$$\mu_t^{\mathcal{P}} = \underbrace{\mu_t^{\mathcal{V}} + \mu_t^M - \mu_t^A}_{= -\mu_t^p} - \mu_t^K, \tag{4.4.16}$$

where $\mu_t^{\mathcal{V}}$ and $\mu_t^M$ are the drift rates of the velocity of money and the total money supply, respectively. Thus, we can deduce that $\mu_t^p$ reflects changes in the money supply, asset allocations between

[644] Cf. Brunnermeier and Sannikov (2014d, p. 10).

heterogeneous groups of individuals, and portfolio weights. If asset allocations and portfolio weights are constant, the supply and velocity of money are constant as well.[645] Then, the growth rate of real GDP in the absence of jumps is exclusively determined by the positive (negative) growth rate of the aggregate capital stock and the economy experiences a deflationary (inflationary) episode.

Further, by comparing (4.4.15) to (4.2.11) it becomes clear that the inflation is rate is the negative of the real return on money provided that no innovations in $\mathcal{N}_t$ occur. Intuition for this result can be obtained from the Fisher equation, which implies that inflation is approximately equal to the difference between the nominal and the real interest rate. Here, the nominal return on money is zero and thus the inflation rate is equal to the real return on money.[646] On the contrary, if shocks materialise, changes in the price level and the value of money are discrete and the real return on money does not equal the negative of the inflation rate.

### 4.4.2 Market Clearing Conditions

**The Final Goods Market.** Equalising supply and demand equations (4.4.4) and (4.4.12) leads to the market clearing condition in the final goods market:

$$\begin{aligned}\left[\psi_t a^e + (1-\psi_t)a^m\right] K_t = {} & \frac{\rho}{1+\xi}\left(n_t^e + n_t^m\right) + \iota_t K_t \\ & + \frac{\varphi}{1-\varphi}\left(x_{1,t}^{e,p} + x_{2,t}^{e,p} - 1\right)\left[\mu_t^{r^{K,e}} - \mu_t^{r^{L,e}} + \lambda\,(1-\kappa)\,\omega\frac{\widetilde{q}_t}{q_t}\right] n_t^e.\end{aligned} \tag{4.4.17}$$

**The Capital Market.** The market clearing condition in the capital market is

$$\int_0^{1-\varphi} k_{i,t}^{e,p}\,di + \int_{1-\varphi}^{1} k_{i,t}^{e,i}\,di + \int_1^2 k_{i,t}^m\,di = K_t, \tag{4.4.18}$$

which states that the integral over agents' desired capital holdings must equal the aggregate capital stock. Substituting $k_{i,t}^{e,p} = x_{1,t}^{e,p} n_{i,t}^{e,p}/q_t$ as well as the first relation in (4.2.4), using equation (4.3.27), and subsequent rearranging leads to

$$x_{1,t}^{e,p}\int_0^{1-\varphi} n_{i,t}^{e,p}\,di + \frac{1}{1-\varphi}\left(x_{1,t}^{e,p} + x_{2,t}^{e,p} - 1\right)\int_{1-\varphi}^{1}\int_0^{1-\varphi} n_{i,t}^{e,p}\,di\,dj + x_t^m\int_1^2 n_{i,t}^m\,di = q_t K_t, \tag{4.4.19}$$

which equalises the integral over the *value* of agents' desired capital stocks to the *value* of the aggregate capital stock. After solving the integrals in the above equation, we get

$$x_{1,t}^{e,p} n_t^e + \frac{\varphi}{1-\varphi}\left(x_{1,t}^{e,p} + x_{2,t}^{e,p} - 1\right) n_t^e + x_t^m n_t^m = q_t K_t. \tag{4.4.20}$$

[645] This will be discussed in detail in Subsection 5.3.6.

[646] In the Brunnermeier and Sannikov (cf. 2016a, p. 30) model with monetary policy the central banks pays a nominal interest rate on outside money, which is exclusively held by intermediaries in the form of reserves. Accordingly, the Fisher equation in their model implies that inflation is approximately equal to the difference between the nominal and the real interest rate.

**The Credit and the Money Market.** The credit market clearing condition is redundant, as the loan rate is already determined via (4.2.24) and (4.3.34). The money market, on the other hand, can be dropped from the analysis in accordance with Walras' law since the markets for final goods, capital, and credit are assumed to be cleared.

Finally, note that market clearing conditions do not depend on individual wealth levels. This aggregation result implies that we do not have to keep track of the within-sector wealth distributions, but rather can focus on the behaviour of representative entrepreneurs and managers as well as aggregate wealth levels $n_t^e$ and $n_t^m$. This is due to asset demands and consumption policy functions being linear in wealth, which, in turn, results from CRS production technologies and HARA preferences.

## 4.5 Closing the Model

### 4.5.1 The State Variable

As can be observed from the market clearing conditions, dynamic behaviour of the economy at this point is governed by three state variables: the aggregate capital stock, entrepreneurs' net worth $n_t^e$ and managers' net worth $n_t^m$. The evolution of the aggregate capital stock is described by (4.1.1). Equations of motions for $n_t^e$ and $n_t^m$ can be obtained by aggregating the stochastic flow budget constraint of individuals within in each respective sector. The results of this exercise are summarised in the following lemma.

**Lemma 11.** *Aggregate wealth of entrepreneurs follows the stochastic processes*

$$\frac{dn_t^e}{n_t^e} = \mu_t^{n^e}\,dt + \frac{\widetilde{n}_t^e - n_t^e}{n_t^e}\,d\mathcal{N}_t, \quad \textit{where} \tag{4.5.1a}$$

$$\mu_t^{n^e} = \mu_t^{r^M} + \left(\mu_t^{r^{K,e}} - \mu_t^{r^M}\right) x_{1,t}^{e,p} + \left(1 - x_{1,t}^{e,p} - x_{2,t}^{e,p}\right)\Gamma_t - \frac{\rho}{1+\xi} \quad \textit{and} \tag{4.5.1b}$$

$$\widetilde{n}_t^e = (1-\phi^s)(1-\kappa)\left[x_{1,t}^{e,p}\frac{\widetilde{q}_t}{q_t} + \left(1 - x_{1,t}^{e,p}\right)\frac{\widetilde{p}_t}{p_t}\right] n_t^e. \tag{4.5.1c}$$

*The corresponding evolution of managers' aggregate net worth is described by*

$$\frac{dn_t^m}{n_t^m} = \mu_t^{n^m}\,dt + \frac{\widetilde{n}_t^m - n_t^m}{n_t^m}\,d\mathcal{N}_t, \quad \textit{where} \tag{4.5.2a}$$

$$\mu_t^{n^m} = \mu_t^{r^M} + \left(\mu_t^{r^{K,m}} - \mu_t^{r^M}\right) x_t^m + \mu_t^{\Pi^b}\Pi_t^b - \frac{\rho}{1+\xi} \quad \textit{and} \tag{4.5.2b}$$

$$\widetilde{n}_t^m = (1-\kappa)\left(\left[x_t^m\frac{\widetilde{q}_t}{q_t} + (1-x_t^m)\frac{\widetilde{p}_t}{p_t} + \widetilde{\Pi}_t^b - \Pi_t^b\right] n_t^m + \phi^s\left[x_{1,t}^{e,p}\frac{\widetilde{q}_t}{q_t} + \left(1 - x_{1,t}^{e,p}\right)\frac{\widetilde{p}_t}{p_t}\right] n_t^e\right). \tag{4.5.2c}$$

*Proof.* See Appendix B.1.8.

It should be noted that drift rates (4.5.1b) and (4.5.2b) are now expressed in terms of risk premia

$\mu_t^{r^{K,e}} - \mu_t^{r^M}$ and $\mu_t^{r^{K,m}} - \mu_t^{r^M}$. In each case, a higher portfolio weight on capital boosts deterministic growth but also results in more severe losses if a shock materialises due to the endogenous response of asset prices. In addition, the drift rate of entrepreneurial wealth depends negatively on interest expenses per unit of net worth (third term in (4.5.1b)) and the propensity to consume out of wealth (fourth term). In contrast, managers do not face interest expenses since they are not leveraged, but earn additional income on their ownership stakes in banks (third term in (4.5.2b)).

Further, the post-jump aggregate wealth levels in (4.5.1c) and (4.5.2c) do not depend on the severity of idiosyncratic shocks to capital. This is not surprising since idiosyncratic shocks cancel out in the aggregate by definition. On the contrary, the sector-specific shock reduces $\widetilde{n}_t^e$ and raises $\widetilde{n}_t^m$. This is because that type of shock reduces the productivity of a share $\phi^s$ of entrepreneurs from level $a^e$ to $a^m$.

The goal in the remainder of this section is to show how the number of state variables can be reduced to unity by (i) considering wealth *shares* instead of levels and (ii) by eliminating the aggregate capital stock $K_t$ from the equations. It will finally turn out that it is sufficient to focus on a single state variable: entrepreneurs' aggregate wealth share.

We begin our effort by describing the distribution of aggregate wealth $(p_t + q_t)K_t$:

$$n_t^m = (p_t + q_t)K_t - n_t^e, \tag{4.5.3}$$

which is a consequence of the fact that imprudent entrepreneurs' and banks' equity levels are zero at any point in time. Dividing by $(p_t + q_t)K_t$ yields

$$\Omega_t = 1 - \eta_t, \tag{4.5.4a}$$

with

$$\Omega_t \equiv \frac{n_t^m}{(p_t + q_t)K} \quad \text{and} \quad \eta_t \equiv \frac{n_t^e}{(p_t + q_t)K}, \tag{4.5.4b}$$

which are managers' and entrepreneurs' respective shares in aggregate wealth. Additionally, it will prove convenient to define the share of aggregate capital in aggregate wealth[647] $\theta_t$ and the corresponding share of money $1 - \theta_t$ according to

$$\theta_t \equiv \frac{q_t}{p_t + q_t} \quad \text{and} \quad (1 - \theta) \equiv \frac{p_t}{p_t + q_t}. \tag{4.5.5}$$

Now, the equilibrium equations can be rewritten by replacing $n_t^m$, $n_t^e$, and $K_t$, using (4.5.4b) and (4.5.5). This is summarised in the following proposition.

**Proposition 2.** *Agents' FOCs with respect to the portfolio weights on capital, entrepreneurs' share in the aggregate capital stock, and market clearing conditions can be expressed as a nonlinear system*

[647] We will alternatively refer to that variable as the "value share of capital".

*of equations in a single state variable $\eta_t$:*

$$\mu_t^q - \mu_t^p \leq \frac{\xi\rho}{(1+\xi)(1-x_t^m)} - \lambda\phi \frac{(1-\kappa-\underline{\kappa}^m)\frac{\widetilde{\theta}_t}{\theta_t} - (1-\kappa)\frac{1-\widetilde{\theta}_t}{1-\theta_t}}{x_t^m(1-\kappa-\underline{\kappa}^m)\frac{\widetilde{\theta}_t}{\theta_t} + (1-x_t^m)(1-\kappa)\frac{1-\widetilde{\theta}_t}{1-\theta_t} + B_t}$$
$$- \lambda(1-\phi)\frac{(1-\kappa-\overline{\kappa}^m)\frac{\widetilde{\theta}_t}{\theta_t} - (1-\kappa)\frac{1-\widetilde{\theta}_t}{1-\theta_t}}{x_t^m(1-\kappa-\overline{\kappa}^m)\frac{\widetilde{\theta}_t}{\theta_t} + (1-x_t^m)(1-\kappa)\frac{1-\widetilde{\theta}_t}{1-\theta_t} + B_t} - \frac{a^m-\iota_t}{q_t}, \tag{4.5.6a}$$

$$B_t \equiv (1-\kappa)\left(\frac{\widetilde{\theta}_t}{\theta_t} - \frac{1-\widetilde{\theta}_t}{1-\theta_t}\right)\frac{\varphi}{1-\varphi}\left(x_{1,t}^{e,p} + x_{2,t}^{e,p} - 1\right)\frac{\eta_t}{1-\eta_t}, \tag{4.5.6b}$$

$$\mu_t^q - \mu_t^p = -\lambda\phi\frac{(1-\kappa-\underline{\kappa}^{e,p})\frac{\widetilde{\theta}_t}{\theta_t} - (1-\kappa)\frac{1-\widetilde{\theta}_t}{1-\theta_t}}{x_{1,t}^{e,p}(1-\kappa-\underline{\kappa}^{e,p})\frac{\widetilde{\theta}_t}{\theta_t} + \left(1-x_{1,t}^{e,p}\right)(1-\kappa)\frac{1-\widetilde{\theta}_t}{1-\theta_t}}$$
$$- \lambda(1-\phi)\frac{(1-\kappa-\overline{\kappa}^{e,p})\frac{\widetilde{\theta}_t}{\theta_t} - (1-\kappa)\frac{1-\widetilde{\theta}_t}{1-\theta_t}}{x_{1,t}^{e,p}(1-\kappa-\overline{\kappa}^{e,p})\frac{\widetilde{\theta}_t}{\theta_t} + \left(1-x_{1,t}^{e,p}\right)(1-\kappa)\frac{1-\widetilde{\theta}_t}{1-\theta_t}} - \frac{a^e-\iota_t}{q_t} + \Gamma_t, \tag{4.5.6c}$$

$$\psi_t = \left[x_{1,i,t}^{e,p} + \varphi\left(x_{2,i,t}^{e,p} - 1\right)\right]\frac{\eta_t}{(1-\varphi)\theta_t}, \tag{4.5.6d}$$

$$q_t = \frac{\theta_t(\psi_t a^e + (1-\psi_t)a^m - \iota_t) - \lambda(1-\kappa)\omega\widetilde{q}_t\frac{\varphi}{1-\varphi}\left(x_{1,t}^{e,p} + x_{2,t}^{e,p} - 1\right)\eta_t}{\frac{\rho}{1+\xi} + \frac{\varphi}{1-\varphi}\left(x_{1,t}^{e,p} + x_{2,t}^{e,p} - 1\right)\left(\mu_t^{r^{K,e}} - \mu_t^{r^{L,e}}\right)\eta_t}, \tag{4.5.6e}$$

$$\theta_t = \left[x_{1,t}^{e,p} + \varphi\left(x_{2,i,t}^{e,p} - 1\right)\right]\frac{\eta_t}{1-\varphi} + x_t^m(1-\eta_t), \tag{4.5.6f}$$

*in which (4.5.6a) is managers' FOC w.r.t. $x_t^m$, (4.5.6b) is an auxiliary variable, (4.5.6c) is Entrepreneurs' FOC w.r.t. $x_{1,t}^{e,p}$ , (4.5.6d) is entrepreneurs' share in aggregate capital, (4.5.6e) is the clearing condition in the final goods market, and (4.5.6f) is the clearing condition in the capital market.*

*In addition, the external finance premium can be written as:*

$$\Gamma_t = \lambda\varphi(1-\kappa)\left[\frac{\left(1-\widetilde{\theta}_t\right)\theta_t}{\left(1-\theta_t\right)\widetilde{\theta}_t} - (1-\omega)\right]\frac{\widetilde{q}_t}{q_t}. \tag{4.5.6g}$$

*Proof.* The formulas follow from straightforward algebraic manipulations of equations (4.2.4), (4.3.27), (4.2.38), (4.3.11), (4.3.20), (4.4.17), (4.4.20), and (4.3.34), using (4.5.4a)-(4.5.5). □

A crucial implication of Proposition 2 is that the aggregate capital stock can be removed from the equilibrium equations. Accordingly, the model obeys the scale-invariance property. As mentioned in Section 3.1.2.2, this property is a result of the dual assumption of HARA utility functions and AK production technologies.[648] Considering wealth shares instead of levels and eliminating the aggregate capital stock $K_t$ thus allows us to reduce the state space to dimension unity - the only remaining state variable is $\eta_t$, which is entrepreneurs' share in aggregate wealth.

Given the importance of variable $\eta_t$ in determining equilibrium outcomes, we now take a closer look at the stochastic process governing its evolution over time.

**Proposition 3.** *State variable $\eta_t$ follows the stochastic process*

$$\frac{d\eta_t}{\eta_t} = \mu_t^\eta \, dt + \frac{(\widetilde{\eta}_t - \eta_t)}{\eta_t} \, d\mathcal{N}_t, \tag{4.5.7a}$$

*where $\mu_t^\eta$ is the drift rate of entrepreneurs' wealth share given by*

$$\mu_t^\eta = x_{1,t}^{e,p} \frac{a^e - \iota_t}{q_t} + \left(1 - x_{1,t}^{e,p} - x_{2,t}^{e,p}\right) \Gamma_t + \left(x_{1,t}^{e,p} - \theta_t\right) \left(\mu_t^q - \mu_t^p\right) - \frac{\rho}{1+\xi} \tag{4.5.7b}$$

*and $\widetilde{\eta}_t$ is entrepreneurs post-jump wealth share determined from*

$$\widetilde{\eta}_t = (1 - \phi^s) \left( x_{1,t}^{e,p} \frac{\widetilde{\theta}_t}{\theta_t} + (1 - x_{1,t}^{e,p}) \frac{1 - \widetilde{\theta}_t}{1 - \theta_t} \right) \eta_t. \tag{4.5.7c}$$

*Proof.* See Appendix B.1.9.

Regarding (4.5.7b), note that it can be reformulated to

$$\mu_t^\eta = \underbrace{\mu_t^{r^M} + \left(\mu_t^{r^{K,e}} - \mu_t^{r^M}\right) x_{1,t}^{e,p} + \left(1 - x_{1,t}^{e,p} - x_{2,t}^{e,p}\right) \Gamma_t - \frac{\rho}{1+\xi}}_{=\mu_t^{n^e}} - \underbrace{\left[\theta_t \mu_t^q + (1-\theta_t)\, \mu_t^p + \mu_t^K\right]}_{=\mu_t^N}. \tag{4.5.8}$$

Hence, the drift of the state variable is equal to the difference of $\mu_t^{n^e}$ and the drift rate of aggregate wealth $\mu_t^N$. The latter is obtained by adding the drift rate of the sum prices $q_t + p_t$ to the drift rate of the aggregate capital stock. Appendix B.4 provides additional insight by demonstrating that $\mu_t^\eta$ can be expressed in terms of (a) entrepreneurs' and managers' profit differential and (b) entrepreneurs' post-jump wealth share.

It can be observed from equation (4.5.7c) that there is an exogenous and an endogenous source of variation in the state variable in case of a jump. The exogenous source is solely due to the sector-specific productivity shock. This is in contrast to the post-jump level of entrepreneurs' aggregate net worth given by (4.5.1c), which also depends on the exogenous parameter $\kappa$. This discrepancy

[648] Section 7.2.1 discusses a model variation which allows for a CRS production function with labour input.

results form the fact that changes in *both* entrepreneurs' and managers' post-jump net worth levels are scaled by the exogenous factor $1-\kappa$, regardless of the agents' portfolio positions (cf. equations (4.5.1c) and (4.5.2c)).

Endogenous variation arises from the changes in the value shares of capital and money, captured by terms $\widetilde{\theta}_t/\theta_t$ and $\left(1-\widetilde{\theta}_t\right)/\left(1-\theta_t\right)$, respectively, and entrepreneurs' risk-taking as expressed through the portfolio weight on capital $x^{e,p}_{1,t}$. What is more, as long as $\widetilde{\theta}_t/\theta_t < 1$ and $\left(1-\widetilde{\theta}_t\right)/\left(1-\theta_t\right) > 1$ hold, i.e. if the value share of capital drops and the value share of money rises, a higher weight on the risky asset capital will decrease the post-jump wealth share of entrepreneurs relative to their pre-jump share.

### 4.5.2 Definition of Equilibrium and Derivation of a RDE for the Value Share of Capital

We are now in a position to state a definition of equilibrium.

**Definition 10.** *Markov Equilibrium*[649]

*Given any initial allocation of capital and money among the agents, an equilibrium is a map from histories of shocks* $\{\mathcal{N}_s, s\in[0,t]\}$ *to prices* $\{q_t, p_t\}$, *drift rate differential* $\mu^q_t-\mu^p_t$, *post-jump levels of prices* $\{\widetilde{q}_t, \widetilde{p}_t\}$, *external finance premium* $\Gamma_t$, *portfolio weights* $\left\{x^m_t, x^e_{1,t}, x^e_{2,t}\right\}$, *investment rate* $\iota_t$, *and consumption ratios* $\left\{c^m_t/n^m_t, c^{e,p}_t/n^e_t, c^{e,i}_t/n^e_t, c^b_t/n^e_t\right\}$ *such that*

(i) *the markets for capital, credit, money and final goods clear,*

(ii) *all agents choose portfolios and consumption rates to maximise utility,*

(iii) *and agents' satisfy their budget constraints.*

Having defined the concept of equilibrium in our model, we can now derive a RDE for the value share of capital. As mentioned in Subsection 3.1.2.3, a challenge in solving CTMF models arises from the fact that the number of endogenous variables exceeds the number of available equations. In the model presented in this chapter the set of endogenous variables consists of the level, drift rate, and post-jump value of entrepreneurs' wealth share $\{\eta_t, \mu^\eta_t, \widetilde{\eta}_t\}$, the level and post-jump value of capital's share in aggregate wealth $\left\{\theta_t, \widetilde{\theta}_t\right\}$, prices $\{q_t, p_t, \widetilde{q}_t, \widetilde{p}_t\}$, the external finance premium $\Gamma_t$, drift rate differential $\mu^q_t-\mu^p_t$[650], and allocations $\left\{x^m_t, x^{e,p}_{1,t}, x^{e,p}_{2,t}, c^m_t/n^m_t, c^{e,p}_t/n^e_t, c^{e,i}_t/n^e_t, c^b_t/n^e_t, \iota_t\right\}$. On the other hand, we have an exogenous initial value for the state variable $\eta_0$, equations for the drift rate and the post jump-level of the state variable (4.5.7b) and (4.5.7c), a defining equation for $\theta_t$ given by

[649] Definition 10 is adapted from Brunnermeier and Sannikov (2014d, p. 11) and Brunnermeier and Sannikov (2016a, p. 13).

[650] The price drift rate differential is part of the return drift rate differentials $\mu^{r^{K,m}}_t - \mu^{r^M}_t$ and $\mu^{r^{K,e}}_t - \mu^{r^{L,e}}_t$ in equations (4.5.6a) and (4.5.6c), respectively.

the first part of (4.5.5), which also implies the post-jump value $\widetilde{\theta}_t$ given $\widetilde{q}_t$ and $\widetilde{p}_t$[651], market clearing conditions (4.5.6e) and (4.5.6f), an equation for the external finance premium (4.5.6g), FOCs for portfolio weights (4.5.6a), (4.5.6c), and (4.3.23), consumption policy rules (4.3.10), (4.3.19), (4.3.29), and (4.3.36)[652], as well as an optimal investment rule (3.1.27) that is identical across all productive agents. Hence, the number of endogenous variables exceeds the number of model equations by three. Here, this results from the presence of terms $\mu_t^q - \mu_t^p$, $\widetilde{\theta}_t$, and $\widetilde{q}_t$.

As is usual in the CTMF literature, the indeterminedness of the system is resolved by imposing a BVP. The first step is to derive a RDE for the value share of capital $\theta_t$, which contains $\eta_t$ as an independent variable. The desired differential equation is provided in Proposition 4.

**Proposition 4.** *In the baseline credit model the value share of capital $\theta_t$ satisfies the following RDE:*

$$\frac{d\theta\left(\eta_t\right)}{d\eta_t} = \frac{\mu_t^q - \mu_t^p}{\mu_t^\eta \eta_t}\theta\left(\eta_t\right)\left[1 - \theta\left(\eta_t\right)\right]. \tag{4.5.9}$$

*Proof.* Let us postulate three general functions of the form $q_t = q\left(\eta_t\right)$, $p_t = p(\eta_t)$, and $\theta_t = \theta(\eta_t)$. Variables $q_t$, $p_t$, and $\theta_t$ are related by the two equations in (4.5.5) which, using the newly defined general functions, can be rearranged to

$$q\left(\eta_t\right) = \theta\left(\eta_t\right)\left[q\left(\eta_t\right) + p\left(\eta_t\right)\right] \tag{4.5.10}$$

and

$$p\left(\eta_t\right) = \left[1 - \theta\left(\eta_t\right)\right]\left[q\left(\eta_t\right) + p\left(\eta_t\right)\right]. \tag{4.5.11}$$

We can get general expressions for the drift rates of prices by applying CVF (3.1.20) to the general price functions. This leads to the following drift rate rate for the price of capital:

$$\mu_t^q = \left[\frac{\theta'_{\eta_t}\left(\eta_t\right)}{\theta\left(\eta_t\right)} + \frac{q'_{\eta_t}\left(\eta_t\right) + p'_{\eta_t}\left(\eta_t\right)}{q\left(\eta_t\right) + p\left(\eta_t\right)}\right]\mu_t^\eta \eta_t \tag{4.5.12}$$

and to the drift rate of the price of money per unit of capital:

$$\mu_t^p = \left[-\frac{\theta'_{\eta_t}\left(\eta_t\right)}{1 - \theta\left(\eta_t\right)} + \frac{q'_{\eta_t}\left(\eta_t\right) + p'_{\eta_t}\left(\eta_t\right)}{q\left(\eta_t\right) + p\left(\eta_t\right)}\right]\mu_t^\eta \eta_t. \tag{4.5.13}$$

Subtracting (4.5.13) from (4.5.12) eliminates terms involving derivatives of $q_t$ and $p_t$. Subsequent rearranging leads to (4.5.9). □

RDE (4.5.9) is identical to the RDE for $\theta_t$ in BS (2014d), except for the fact that in their model,

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[651] Put differently, $\widetilde{\theta}_t = \widetilde{q}_t / (\widetilde{q}_t + \widetilde{p}_t)$ is an additional model equation.

[652] Optimal consumption choices are already inserted into the final goods market clearing condition (4.5.6e).

$\eta_t$ measures intermediaries' rather than entrepreneurs' aggregate wealth share.[653]

[653] Cf. Brunnermeier and Sannikov (2014d, eq. (2.21)).

# Chapter 5

# Results in the Credit Model

The previous chapter developed a CTMF model in which bank lending is tantamount to the creation of inside money. Financial contracting between banks and entrepreneurs, who assume the role of end-borrowers, is hampered by the presence of informational asymmetries, which motivate the usage of debt contracts. This type of financial contract, it was argued, concentrates endogenous asset price risk on the balance sheets of end-borrowers and mitigates their ability to share risk. Thus, risk averse entrepreneurs opt to self-insure by accumulating net worth. Conversely, agency problems do not arise between banks and their financiers. This simplifying assumption implies that banks' balance sheet conditions are immaterial in determining equilibrium outcomes. Taken together with the scale-invariance property of the model, the mentioned assumptions facilitate the reduction of the state space to a single state variable, namely entrepreneurs' share in aggregate wealth. In the present chapter, we discern the results generated by the baseline model. We do so by presenting key equilibrium objects such as asset prices and allocations as functions of the state variable. To gain additional insight, we apply Monte Carlo simulations to generate moments of those variables and those that cannot be expressed as functions of a single state variable. Further, model simulations allow for calculating distributions of entrepreneurs' wealth share, which, inter alia, aid in judging whether adverse conditions display persistence.

The model results will show that the model generates significant and nonlinear amplification of adverse productivity shocks. Moreover, we will see that financial frictions precipitate a persistent misallocation of capital. Both results are in line with the recent continuous-time literature on financial frictions. In our model, detrimental productivity shocks reduce the demand for capital in the entrepreneurial sector. Since part of that demand is externally financed by bank credit, entrepreneurs opt for debt repayment and the inside money supply falls in the process. These factors set in motion an adverse feedback loop characterised by the interaction of adverse movements in balance sheets and asset prices. In particular, deflationary pressure during such episodes is shown to be a potent source of reductions in entrepreneurs' real net worth. Distressed balance sheets in the entrepreneurial sector, in turn, cause the allocation of capital to deteriorate. Meanwhile, banks adjust the terms of credit availability. The direction and extent of that adjustments depend on the current state of the economy. That is, banks' endogenous response may either exacerbate or mitigate the adverse feedback loop. However, due to the absence of a bank balance sheet channel, the demand side on the loan market is the main driver of the credit cycle.

This chapter is structured as follows. Section 3.2 calibrates the model to U.S. post World-War II data. The motivation for this choice is data availability. In Section 5.2, we consider solutions to three special cases that do not feature endogenous risk, namely the first-best and autarky solutions and the model without sector-specific productivity shocks. This is for two reasons: first, these cases facilitate intuition about some central model mechanisms and second, the solutions are helpful in solving the full model. Afterwards, Section 5.3 presents some important results generated by the full-fledged baseline model with endogenous risk, including the aforementioned adverse feedback loop. In Section 5.4, we perform parameter variations and discern the impact on equilibrium values of endogenous variables. In particular, we explore the question of whether supply or demand side factors are the main causes of fluctuations in the credit market. Finally, Section 5.5 offers a comparison to related models. Therein, special attention is devoted to relations with the CTMF literature, which was presented in Section 3.2.

## 5.1 Calibration

In this section, the model is parameterised to the objective of fitting post World-War II U.S. data. The parameters can be broadly categorised into two sets: on the one hand, parameters which can be set a priori and, on the other hand, those which are set to generate specific moments of endogenous variables in the simulated model consistent with the data. The second set of model parameters is calibrated using moments calculated by the author from the BEA fixed assets dataset for the period from the first quarter of 1959 to the third quarter of 2017. For the calibration exercise and the simulation of the model, we assume that a unit of time is equal to one year. Each year is divided into subintervals of length $dt = 1/360$, i.e. approximately one day. Table 5.1 summarises the chosen parameter values in the baseline calibration.

The time preference rate $\rho$ is set to equal 3.0 percent. This is lower than the value typically chosen in calibrated business cycle models of around 4.0 percent.[654] The reason for choosing lower values of the time preference rates is related to the fact that the model is inhabited by agents (imprudent entrepreneurs and bankers) which consume their entire income at any time. Effectively, these individuals act as if their time preference rates were equal to 100 percent. The strong demand for output goods by these agents puts upward pressure on the price level and thus downward pressure on the value of money. The value of parameters $\rho$ is chosen to partially offset this effect.

[654] It should be noted that in the mentioned literature, the value of the time preference rate usually is not obtained from microeconometric estimates but, rather, is calibrated to achieve a steady state riskless interest rate of 4.0 percent (cf. e.g. Bernanke et al., 1999, p. 1367).

Table 5.1: Baseline Calibration in the Credit Model

| Parameter | Description | Value |
|---|---|---|
| *Panel (a): Preferences* | | |
| $\rho$ | Rate of time preference | 3% |
| $\xi$ | Utility weight on real balances | 0.3 |
| *Panel (b): Technologies* | | |
| $a^e$ | Entrepreneurs' productivity level | 27.6 |
| $a^m$ | Managers' productivity level | 20 |
| $\gamma$ | Investment adjustment cost parameter | 140 |
| $\delta$ | Deterministic capital depreciation rate | 1% |
| *Panel (c): Shocks* | | |
| $\kappa$ | Severity of the aggregate capital shock | 1% |
| $\underline{\kappa}^{e,p}$ | Severity of the malign idiosyncratic capital shock $(e, p)$ | 5% |
| $\overline{\kappa}^{e,p}$ | Severity of the benign idiosyncratic capital shock $(e, p)$ | −5% |
| $\underline{\kappa}^{m}$ | Severity of the malign idiosyncratic capital shock $(m)$ | 5% |
| $\overline{\kappa}^{m}$ | Severity of the benign idiosyncratic capital shock $(m)$ | −5% |
| $\lambda$ | Intensity of fundamental risk | 0.5 |
| $\phi$ | Conditional probability of the malign idiosyncratic capital shock | 50% |
| $\phi^s$ | Share of entrepreneurs exposed to the sector-specific productivity shock | 5% |
| *Panel (d): External finance premium* | | |
| $\omega$ | Bankers' disutility of auditing parameter | 0.15 |
| $\varphi$ | Share of imprudent entrepreneurs in the population of entrepreneurs | 8% |

Several authors in the financial friction literature such as BGG or Carlstrom and Fuerst (1997) calibrate the default rate of borrowers in their models to match the quarterly bankruptcy rate of U.S. nonfinancial firms in the period from 1984 to 1994 calculated by Fisher (1999).[655] This bankruptcy rate is equal to 1.0 percent and is computed from a (discontinued) dataset assembled by the Dun and Bradstreet Corporation. The advantage of using this sample is that it takes into account considerations such as the possibility that some firms may strategically declare bankruptcy but actually continue operations.[656] Emery and Cantor (2005) analyse a more recent dataset of U.S. firms that have emitted corporate bonds as well as loans which both were still outstanding at the date of sampling. Their dataset spans from 1995 to 2004 and was assembled by rating agency Moody's. They find that the one-year cumulative loan default rate[657] of investment grade firms is 4.1 percent.[658] Drawing on these results, we set $\varphi = 0.04/\lambda$, which is tantamount to an average yearly default rate of four percent.

The calibration of parameter $\omega$, which governs bankers' disutility of auditing[659], is less straightforward. The reason for this is that in the literature, the inefficiency associated with auditing usually arises from a a deadweight loss of resources rather than the disutility incurred by the auditor, as in our model. More specifically, $\omega$ typically measures lenders' losses from auditing and asset liquidation expressed as a percentage share of the remaining value of the defaulted debtors' assets. In the literature, this parameter is interpreted to measure the costs of financial distress that accrue on the side of firms.[660] Theses costs are separated into direct costs and indirect costs of bankruptcy. Direct costs include expenses for lawyers, accountants, restructuring advisers, and other professional individuals. Indirect costs, on the other hand, measure various forms of opportunity costs, which are in general difficult to quantify. An example for this form of costs is that a distressed firm may experience drops in sales and profits due to customers refraining from dealing with such a company.[661]

Earlier studies have not lead to a consensus on direct costs: estimates range between 0.65 percent and 7.5 percent of the debtor's assets on average in the United States.[662] A recent study by Bris et al. (2006) differs from earlier work in that the authors analyse a larger sample, which also includes smaller firms, and consider Chapter 7 cases in addition to Chapter 11 cases.[663] They find that direct costs have a mean of 8.1 percent in Chapter 7 cases and a mean of 16.9 percent in Chapter 11 cases

[655] Cf. Bernanke et al. (1999, p. 1368), Carlstrom and Fuerst (1997, p. 900), and Fisher (1999, p. 198).
[656] Cf. Fisher (1999, p. 198).
[657] The first step in calculating the cumulative default rate is the formation of a cohort of debtors on the basis of the rating held on the cohort formation date. The one-year cumulative default rate then measures the probability of default from the cohort formation date up to the ending of the first year after this date (cf. Moody's, 2006, p. 5).
[658] Cf. Emery and Cantor (2005, p. 1579).
[659] Recall that in Section 4.3.4 the form of function $\Theta(\cdot)$ was chosen such that the disutility of auditing is a constant share $\omega$ of the real value of defaulting debtors' assets.
[660] Cf. e.g. Carlstrom and Fuerst (1997, p. 900).
[661] Cf. Altman and Hotchkiss (2006, pp. 93f.).
[662] Cf. the overview in Altman and Hotchkiss (2006, pp. 95f.).
[663] Filing for bankruptcy under Chapter 11 of the U.S. Bankruptcy Code allows for reorganisation of a bankrupt firm, while bankruptcy under Chapter 7 entails liquidation of the firm (cf. Altman and Hotchkiss, 2006, p. 3).

(again in percent of the firm's value), indicating much larger costs than those suggested in the bulk of the previous work.[664,665]

The debate in the literature on the magnitude of $\omega$ also concerns the question to what extent indirect bankruptcy costs in addition to direct bankruptcy costs have to be considered.[666] Some papers such as Carlstrom and Fuerst (1997) or Hackbarth et al. (2006) use rather large values in the range from 0.25 to 0.4.[667] These values are based on a very broad interpretation of indirect bankruptcy costs, which also include liquidation costs. One source for liquidation costs is Alderson and Betker (1995), who define liquidation costs as the difference between the going-concern value and the liquidation value.[668] They find that mean liquidation costs in a small sample of U.S. firms are as high as 36.5 percent of the average firm's going-concern value.[669]

It becomes clear from this short overview that the costs discussed so far only capture costs which are realised on the side of debtors and eventually are borne by banks. On the contrary, in the context of the model in this thesis, parameter $\omega$ should reflect costs that accrue on the side of banks due to the auditing of defaulted borrowers. Unfortunately, to the best knowledge of the author, there are no studies that have tried to quantify these kinds of costs. Despite the difference in interpretations of parameter $\omega$, we thus base the calibration of this parameter on the mentioned results and set $\omega$ to an intermediate value of 0.15 for the sake of comparability.

In order to calibrate the productivity differential $a^e - a^m$, we make use of results from a growing empirical literature on the firm-level distribution of TFP. In the calculation of dispersion measures, physical TFP, termed TFPQ, is typically proxied by revenue based TFP, termed TFPR, since firm-level datasets mostly do not contain information on product prices or quantities in contrast to revenues.[670,671] An exception is the seminal study by Hsieh and Klenow (2009), who estimate the distribution of TFPQ from the U.S. Census of Manufacturing from 1997. They report a very large ratio of the third quartile to the first quartile of 3.2.[672] An issue with their results is that TFPQ can be calculated only indirectly from revenue data under specific assumptions about the production and demand sides, such as Cobb-Douglas production technologies under CRS and isoelastic demand functions. Some of these assumptions are likely not to hold in the data as recent research suggests.[673] In addition, Syverson (2004), Foster et al. (2008), and Foster et al. (2016b) all calculate TFPQ directly from datasets with producer-specific prices and quantities and report much lower estimates

[664] Cf. Bris et al. (2006, p. 1279).

[665] A likely reason for the disparate results is that direct costs appear to contain a large fixed costs component. A sample with a higher prevalence of small firms will thus be characterised by a higher mean of the share of direct bankruptcy costs in the debtor's assets (cf. Altman and Hotchkiss, 2006, p. 97).

[666] Cf. e.g. the discussion in Carlstrom and Fuerst (1997, p. 900).

[667] Cf. Carlstrom and Fuerst (1997, p. 900) and Hackbarth et al. (2006, p. 532).

[668] Cf. Alderson and Betker (1995, p. 46).

[669] Cf. Alderson and Betker (1995, p. 52).

[670] Cf. Foster et al. (2016a, p. 95).

[671] The relationship between both concepts is captured by $\text{TFPR}_i = P_i \text{TFPQ}_i$, where $P_i$ is the price of firm $i$'s output (cf. Foster et al., 2016a, p. 95).

[672] Cf. Hsieh and Klenow (2009, p. 1416).

[673] Cf. e.g. Foster et al. (2016a, pp. 97f.).

of TFPQ dispersion than Hsieh and Klenow (2009).[674,675] These analyses cast further doubt on Hsieh and Klenow's results, even though the aforementioned studies are based on small and/or possibly nonrepresentative samples.

Due to the methodological difficulties involved in estimating TFPQ, we choose to utilise results by Foster et al. (2017) for the baseline calibration. They employ different methods suggested in the literature to infer the TFPR distribution from U.S. Census of Manufacturing data on the 50 largest industries covering the period from 1972 to 2010.[676] To set the productivity differential, we take the following steps. We first calculate the productivity ratios of the third to the first quartile from the interquartile ranges of logarithmised TFPR levels provided in Table 3 in Foster et al. (2017). We then take the average of these ratios across the different methods.[677] Finally, ratio $a^e/a^m$ is set to 1.38, which is equal to that average.[678]

Parameters $a^m$, $\gamma$, $\delta$, $\kappa$, $\overline{\kappa}^m$, $\overline{\kappa}^{e,p}$, and $\xi$ are set with the aim of generating model-implied moments that accord with empirical moments. The first four of those are chosen to target quarterly BEA data in the period ranging from the first quarter of 1959 to the third quarter of 2017.[679] Parameter $\gamma$, which determines the concavity of the capital goods production function $\Phi(\cdot)$, is set to achieve an gross investment-to-GDP ratio of 17.3 percent, which is consistent with the data.[680] Setting $\gamma = 140$ implies a mean gross investment-to-GDP ratio of 19.9 percent, which is deemed to be sufficiently close to the reference value.

The absolute value of parameter $a^m$ is not of particular importance since only its value relative to $a^e$ appears in the equations of Proposition 2. Consequently, varying $a^m$, while keeping fraction $a^e/a^m$ constant, does not affect any other endogenous variable except for the absolute values of prices, the investment rate, and, through the latter, the growth rates of real GDP and the price level. We choose to set managers' productivity parameter to a relatively high value of $a^m = 20$,

[674] Cf. Syverson (2004, p. 535), Foster et al. (2008, p. 404), and Foster et al. (2016b, p. 94).

[675] A further result from those papers is that the dispersion of TFPQ is slightly higher than that of TFPR. For instance, Foster et al. (cf. 2008, p. 404) report a standard deviation of 0.26 for the former and a standard deviation of 0.22 for the latter, which is much less of a divergence than in Hsieh and Klenow (cf. 2009, p. 1418).

[676] Subsection 5.4.3 presents equilibrium outcomes under a parameter constellation implied by the results in Hsieh and Klenow (2009).

[677] When computing the average, we exclude the ordinary least squares estimate of the interquartile range as it is most likely biased (cf. Foster et al., 2017, p. 10).

[678] Bartelsman and Doms (cf. 2000, p. 583) note that in firm level studies on productivity dispersion, moments are typically calculated from an *unweighted* productivity distribution at a specific point in time. In contrast, the distribution of productivities *weighted* by entrepreneurs' capital share $\psi_t$ in the model is changing over time according to equation (4.4.5). In order to reconcile the way in which the data was generated with the parameterisation, one has to make the assumption that the unweighted distribution is time-invariant. Since the productivity differential is constant, this is the case if the relative population shares of entrepreneurs and managers are held constant. Fortunately, this was already implicitly assumed in Section 4.1.2, in which the masses of entrepreneurs and managers were set to unity.

[679] The BEA samples date back to the first quarter of 1947. Yet, we restrict them to a shorter horizon since the dataset used to calibrate parameter $\xi$ starts in the first quarter of 1959. In addition, the time series for real GDP, which plays a crucial role in the calibration exercise, appears to have a structural break at the end of the 1950s. This is also recognised by e.g. Smets and Wouters (cf. 2007, fn. 7).

[680] The average gross investment-to-GDP ratio in the sample is calculated using data contained in BEA (2017b).

which translates into a slightly higher growth rate of real GDP relative to a calibration in which the productivity level is normalised to unity as e.g. in Di Tella (2017).[681,682]

The deterministic depreciation rate on capital $\delta$ does not appear in Proposition 2 and thus only affects the accumulation of capital and thereby the growth rate of real GDP and the inflation rate. When setting $\delta$ to ten percent, as is common in the literature, the model-implied average quarterly growth rate of real GDP is negative, compared to the 0.76 percent found in the data.[683] The reason for this inability of the model to fit the data lies in the fact that the model only features one-sided negative shocks to the state variable. In order to circumvent a negative growth rate of output, the depreciation rate is decreased to one percent, which implies and average growth rate of quarterly GDP of around 0.79 percent. The value chosen for $\kappa$, the amplitude of the aggregate shock to capital, is equal to one percent. This corresponds to a standard deviation of the quarterly real GDP growth rate of 0.91 percent, which is slightly higher than the quarterly volatility in the BEA sample of 0.84 percent.

In order to assign a value to the amplitude of idiosyncratic shocks, we follow Di Tella (2017) in employing monthly data by Campbell et al. (2001) on the idiosyncratic volatility component of stock returns in the Center for Research in Security Prices dataset spanning the period from July 1962 to December 1997.[684,685] In particular, we set $\underline{\kappa}^m = \underline{\kappa}^{e,p} = 5$ percent, which results in a value of 23.6 percent for the standard deviation of the idiosyncratic component of entrepreneurs' monthly annualised nominal portfolio return.[686] This is reasonably close to the corresponding component of monthly annualised stock returns of 22.4 percent found by Campbell et al. (2001).[687] The share of agents that are subject to the adverse idiosyncratic shock $\phi$ in case of a jump is set to 50 percent. The choices of $\phi$, $\underline{\kappa}^m$, and $\underline{\kappa}^{e,p}$ together with condition $\phi\underline{\kappa}^j = -(1-\phi)\overline{\kappa}^j$, $j = \{(e,p), m\}$ imply amplitudes of the benign idiosyncratic shocks of $\overline{\kappa}^m = \overline{\kappa}^{e,p} = -5$ percent.

Finally, we employ monthly data from the Board of Governors of the Federal Reserve System (FRB) on the M2 monetary aggregate and the currency component of M1 to calculate the ratio (M2-currency)/currency in the period from January 1959 to October 2017. This ratio is used to calibrate the utility weight on real balances $\xi$.[688] Parameter $\xi$ determines the demand for money and, by

[681] Cf. Di Tella (2017, p. 2057).

[682] It might be considered to set $a^m$ to target an average output to capital ratio of around 0.85 as is found in the data by means of equation $Y_t/K_t = \psi_t a^e + (1-\psi_t)\, a^m$, which follows from the aggregate production function in (4.4.4). This reasoning, however, neglects the fact that the capital stock in the model is measured in effective rather than in physical units.

[683] The first and second order moments of the quarterly real GDP growth rate are computed from the series contained in BEA (2017a).

[684] Cf. Di Tella (2017, pp. 2078f.).

[685] The method to extract the idiosyncratic volatility component of returns and its application to our context is explained in Appendix C.2.

[686] Di Tella (cf. 2017, pp. 2078f.) utilises the data by Campbell et al. (2001) to calibrate the volatility of idiosyncratic shocks to the return on capital. Given the circumstance that entrepreneurs assume the role of production units in the presented model, it appears to be more appropriate to match the volatility of portfolio returns generated by the model to the data on the volatility of stock returns.

[687] Cf. Campbell et al. (2001, Table III).

[688] The data on the currency component of M1 is from FRB (2017a) and the data on the M2 aggregate is from FRB

association, the amount of inside money created by banks. We interpret the money multiplier as defined in (4.4.13) to be the model equivalent of the mentioned ratio. Setting $\xi = 0.3$ yields a value of the so defined money multiplier of 11.9 in the model, compared to 10.8 in the data.[689]

The share of entrepreneurs that are exposed to the sector-specific shock $\phi^s$ and the intensity $\lambda$ of jump process $\mathcal{N}_t$ do not have clear-cut empirical counterparts. In addition, these parameters do not have isolated effects on only a narrow set of endogenous variables in equilibrium. Thus, they are arbitrarily set to the following values: $\phi^s = 5$ percent and $\lambda = 0.5$. The latter value implies that the economy is exposed to a jump once every two years on average.

## 5.2 Three Special Cases

### 5.2.1 The First-Best Solution

The first-best case is characterised by the absence of financial frictions and asymmetries of information. It follows that the entire capital stock must be allocated to its most efficient use, which is the production process of entrepreneurs, at any point in time.[690] These agents finance their capital holdings by issuing equity to managers.[691] Since the two types of agents can exchange claims directly and without information asymmetries, banks are superfluous and inside money creation does not occur. Similarly to BS (2014d) and (2016a), equity investment is modelled as if managers were directly buying capital, which is subsequently utilised in entrepreneurs' production activities.[692] In addition, managers are able to diversify their equity stakes by assumption. This enables them to eliminate any idiosyncratic risk that would occur if they invested only in a small amount of firms. Since neither idiosyncratic risk nor financial frictions are present, the only motive for holding money is the transaction motive, which is modeled via the MIU approach.[693]

For the sake of comparability to the model variant presented in the next subsection, we assume that the entire aggregate wealth is always in the hands of managers. This is without loss of generality since the distribution of wealth does not matter in the first-best case. The reason is twofold:

(2017b).

[689] It could be argued that a broader monetary aggregate such as M3 is a better measure for the amount of inside money created by the banking sector. However, data availability restricts us from employing the M3 monetary aggregate: its time series was discontinued in 2006 as it "does not appear to have any additional information about economic activity that is not already embodied in M2 and has not played a role in the monetary policy process for many years" according to the FRB (2006).

[690] The distinction between prudent and imprudent individuals is not of relevance here since imprudent entrepreneurs do not default in equilibrium under the adopted assumptions, as we will see momentarily.

[691] This is analogous to the first-best economy in Brunnermeier and Sannikov (cf. 2014a, pp. 385f.), in which experts issue equity to households.

[692] Cf. Subsections 3.2.2 and 3.2.3 of this thesis.

[693] As a consequence, the model in the first-best case is a version of the model in Sidrauski (1967). The two main differences in assumptions between both are as follows. First, Sidrauski (cf. 1967, p. 535) adopts a final goods production technology characterised by a decreasing marginal product of capital, whereas our model employs a production function which is linear in capital. Second, in his model (cf. p. 536), the capital goods production technology does not feature adjustment costs.

firstly, resources are always allocated to their most efficient use in a complete markets economy and secondly, both prudent entrepreneurs and managers have the same propensity to consume out of wealth. It also follows from these considerations that wealth distribution variable $\eta_t$ can be removed from the state space. An immediate consequence is that prices $q_t$ and $p_t$ must be constant.[694] Under these circumstances, we can drop the time subscripts of variables that are not proportionate to the aggregate capital stock, such as prices $q$ and $p$.[695]

For expositional purposes, we first consider the case in which investment rate $\iota$ is exogenously fixed at a constant level. Hence, managers' optimisation problem is[696]

$$\begin{aligned}\rho V_{F,I}^{m}(n_{i,t}^{m}) = \max_{c_{i,t}^{m},x_{F,I}^{m}} & \log c_{i,t}^{m} + \xi \log\left[\left(1-x_{F,I}^{m}\right)n_{i,t}^{m}\right] \\ & + \frac{1+\xi}{\rho n_{i,t}^{m}}\left(\left[x_{F,I}^{m}\frac{a^{e}-\iota}{q_{F,I}}+\Phi(\iota)-\delta\right]n_{i,t}^{m}-c_{i,t}^{m}\right)+\lambda\frac{1+\xi}{\rho}\log\left(1-\kappa\right),\end{aligned} \tag{5.2.1}$$

in which $x_{F,I}^{m}$ is managers' portfolio weight on capital employed in entrepreneurs' production processes. The FOC with respect to $x_{F,I}^{m}$ implies

$$\frac{1+\xi}{\rho}\frac{a^{e}-\iota}{q_{F,I}} = \xi\frac{1}{1-x_{F,I}^{m}}. \tag{5.2.2}$$

In contrast to the full baseline model, the optimal portfolio choice does neither depend on the frequency nor severity of shocks according to (5.2.2). The reason is that both money and capital depreciate by factor $(1-\kappa)$ following a shock due to the absence of idiosyncratic risk.

The optimal consumption policy neither depends directly on prices nor price changes and is thus still given by (4.3.10). Thus, we can write the market clearing conditions in the final goods and capital market as[697]

$$\frac{\rho}{1+\xi}\left(q_{F,I}+p_{F,I}\right) = a^{e}-\iota \quad \text{and} \quad x_{F,I}^{m}\left(q_{F,I}+p_{F,I}\right) = q_{F,I}, \tag{5.2.3}$$

respectively. The prices of capital and money per unit of aggregate capital can be solved for by

[694] Recall that in the baseline model with incomplete markets all endogenous variables, except for scaled variables such as $P_t = p_t K_t$ or $Y_t = A\left(\cdot\right) K_t$, can be expressed as functions of a single state variable $\eta_t$. Thus, variations in the former are only due to changes in the state variable. Since the distribution of wealth does not matter here, there is no force that causes prices $q$ and $p$ to change. These results also emerge e.g. in the first-best model of Brunnermeier and Sannikov (cf. 2014d, pp. 385f.).

[695] Constant prices imply that imprudent entrepreneurs do not default according to equation (4.3.28), which confirms our earlier claim in footnote 690 that equity investors are indifferent between funding prudent and imprudent agents.

[696] In the first-best case with exogenous investment, time-invariant variables that are not proportionate to the aggregate capital stock are denoted by subscript $F, I$.

[697] The two market clearing conditions again result from the aggregation of individual demand and supply decisions. In the final goods market, aggregation implies: $\frac{\rho}{1+\xi}\int_1^2 n_{i,t}^m\,di = \int_1^2\left(a^m-\iota\right)k_{i,t}^m\,di$. The corresponding condition in the capital market is $x_{F,I}^m\int_1^2 n_{i,t}^m\,di = q_{F,I}K_t$. Solving the integrals, using $n_t^m = \left(q_{F,I}+p_{F,I}\right)K_t$ and rearranging gives (5.2.3).

combining equation (5.2.2) with the two market clearing conditions in (5.2.3). This results in the first-best prices $q_{F,I}$ and $p_{F,I}$:

$$q_{F,I} = \frac{a^e - \iota}{\rho} \quad \text{and} \quad p_{F,I} = \xi \frac{a^e - \iota}{\rho}. \tag{5.2.4}$$

The first part of (5.2.4) is a version of the *Gordon growth formula*, which, in general, can be used to price any asset characterised by an infinite stream of constant payoffs (or dividends as in Gordon's original formulation) discounted at a constant rate.[698] In our case, the Gordon growth formula is the solution to the integral $(a^e - \iota) \int_t^\infty e^{-\rho t}$, which is the continuously discounted stream of payoffs generated by holding a unit of capital from time $t$ until infinity.[699]

The interpretation of the second part of (5.2.4) is straightforward as well: the value of money per unit of capital is a multiple of the capital price. The former rises proportionately with the price of capital if the utility weight on real balances $\xi$ is greater than zero, i.e. if there is a transaction motive. The reason for this relationship is that the value of money proportionately increases with output if the money supply remains constant. Again, this infinite stream of real payoffs is discounted at rate $\rho$.[700] On the other hand, if $\xi = 0$, money does not have any value.

The relationship between the value of money and output can be observed more directly if we multiply both sides of the expression for $p_{F,I}$ in (5.2.4) by the aggregate capital stock $K_t$ and set $\iota = 0$. This results in: $P_{F,I,t} = \xi Y_t / \rho$, where $P_{F,I,t}$ is the price of one unit of money in the first-best case. Since the supply of outside money is fixed at unity and the price of final goods is $\mathcal{P}_{F,I,t} = 1/P_{F,I,t}$, the previous equation can also be expressed as $M\rho/\xi = \mathcal{P}_{F,I,t} Y_t$, which includes the velocity of money $\mathcal{V}_{F,I} = \rho/\xi$ and thus represents the equation of exchange in the first-best economy without investment. Velocity $\mathcal{V}_{F,I}$ is increasing in the time preference rate since the value of money depends negatively on that parameter according to the second expression in (5.2.4). Consequently, increases in $\rho$ are associated with a higher nominal value of output. The demand for money rises with the utility weight on real balances $\xi$ and thus velocity falls.

Next, we endogenise investment by substituting optimal investment rule (3.1.27) into the two expressions in (5.2.4). This gives the first-best prices with endogenous investment denoted by subscript $F$:

[698] Cf. Gordon (1959, p. 101).

[699] In contrast to the formula provided by Brunnermeier and Sannikov (cf. 2014a, p. 386) in their model's first best case, the Gordon growth formula does not depend on the drift rate of the capital stock $\Phi(\iota_t) - \delta$ here. The reason is that the asset allocation decision in program (5.2.1) does not depend on the drift rate of the capital stock, which, in turn, results from the fact that both the value of money per unit of aggregate capital and the value of capital appreciate with that drift rate.

[700] The second part of (5.2.4) can alternatively be interpreted as follows. A rise in $a^e - \iota$ increases the supply of final goods. Together with a constant demand, this implies a drop in the price of output or, alternatively, a surge in the value of money. On the other hand, consumption demand rises with parameter $\rho$, which results in a drop of $p_{F,I}$ if supply stays constant.

$$q_F = \frac{\gamma a^e + 1}{\gamma\rho + 1} \quad \text{and} \quad p_F = \xi\frac{\gamma a^e + 1}{\gamma\rho + 1}. \tag{5.2.5}$$

As mentioned in Subsection (3.1.1.2), for $\gamma \to 0$, investment adjustment costs approach zero. The first equality in (5.2.5) tells us that the price of capital in output goods then approaches unity, which results from the fact that output can be (nearly) costlessly transformed to capital and vice versa. What is more, in that case, the price of money per unit of aggregate capital approaches parameter $\xi$. We can interpret (5.2.5) further if we substitute the expression for $q_F$ back into (3.1.27). This yields the optimal investment rate $\iota_F = (a^e - \rho) / (\gamma\rho + 1)$. This result has three implications. First, investment is nonnegative if entrepreneurs' productivity $a^e$ is not smaller than managers' discount rate $\rho$, which is in line with the parameter specifications in Section 5.1. Second, $\iota_F$ falls with investment parameter $\gamma$. A lower investment rate, in turn, is tantamount to a lower growth rate of the economy since the latter is identical to the growth rate of the aggregate capital stock. Third and lastly, $q_F$ increases with $\gamma$, which is a direct consequence of the second result: lower investment raises the price of capital since the payoff per unit of capital $a^e - \iota_F$ increases.

### 5.2.2 The Autarky Solution

Similarly to BS (2014d) and BS (2016a)[701], the autarky case in our model is characterised by extreme financial frictions: auditing costs per unit of the borrower's assets (in terms of bankers' disutility of auditing) are set to their maximum value, i.e. $\omega = 1$, which makes lending unprofitable.[702] In addition, entrepreneurs are not endowed with any assets. The upshot of these two assumptions is that entrepreneurs as well as banks are absent from the model.

As in the full baseline model, managers allocate their portfolios between money and capital used for their own production activities. This decision is now again influenced by the fact that the idiosyncratic risk embodied in each agent's individual capital stock cannot be diversified away. Since the economy is only populated by managers, trading of assets does merely occur between agents who only differ with respect to their wealth levels. Hence, the price of capital must be constant at any point time. Using this fact in (4.3.9) and dropping time indices of time-invariant variables, the HJB equation in the autarky model becomes[703]

[701] Cf. Brunnermeier and Sannikov (2014d, Section 2.1) and Brunnermeier and Sannikov (2016a, Section 3).
[702] Audits are not worthwhile from the lender's perspective, if this results in a complete depletion of the borrower's assets. Anticipating this, borrowers will always report the worst possible outcome of their projects.
[703] In the autarky model, time-invariant variables are denoted by subscript $A$.

$$\begin{aligned}\rho V_A^m(n_{i,t}^m) = & \max_{c_{i,t}^m, x_A^m} \log c_{i,t}^m + \xi \log \left[ (1 - x_A^m) \, n_{i,t}^m \right] \\ & + \frac{1+\xi}{\rho n_{i,t}^m} \left( \left[ x_A^m \frac{a^m - \iota_A}{q_A} + \Phi(\iota_A) - \delta \right] n_{i,t}^m - c_{i,t}^m \right) \\ & + \lambda \frac{1+\xi}{\rho} \left[ \phi \log \left( 1 - \kappa - \underline{\kappa} x_A^m \right) + (1 - \phi) \log \left( 1 - \kappa - \overline{\kappa} x_A^m \right) \right]\end{aligned} \tag{5.2.6}$$

and the FOC with respect to $x_A^m$

$$\frac{a^m - \iota_A}{q_A} = \frac{\xi \rho}{(1+\xi)\left(1 - x_A^m\right)} + \lambda \left[ \phi \frac{\underline{\kappa}}{1 - \kappa - \underline{\kappa} x_A^m} + (1 - \phi) \frac{\overline{\kappa}}{1 - \kappa - \overline{\kappa} x_M^m} \right]. \tag{5.2.7}$$

This FOC shows that the agent now has to take into account the higher risk exposure captured by the second and third term on the RHS when tilting his portfolio towards a higher weight on capital. To close the model, we impose market clearing conditions on the final goods and capital market. These conditions are $\rho \left( q_A + p_A \right) / \left( 1 + \xi \right) = a^m - \iota_A$ and $x_A^m = q_A / \left( q_A + p_A \right)$, respectively. Combining FOC (5.2.7) and the market clearing conditions leads to a cubic equation in $q_A$. The procedure to solve for the equilibrium is explained in C.3 of the Appendix and numerical results for variations of parameters $\lambda$, $\kappa$, and $\underline{\kappa}$ are shown in Figure C.5.1 in Appendix C.5. We focus on these parameters for two reasons. First, they represent the nonstandard parameters in the autarky model. Second, the remaining parameters qualitatively influence endogenous variables in the same way as in the first-best case.

As expected, Panel (c) in Figure C.5.1 shows a negative relationship between $\lambda$ and the portfolio weight on capital: as adverse shocks to capital become more likely, the demand for capital decreases in favour of the demand for money. However, the strength of this relationship is relatively modest: even for a very large value $\lambda = 10$, about two thirds of agents' wealth levels are still held in the form of capital. Consequently, the price of capital and the investment rate, which depends linearly on $q_A$, do not change much with an increasing value of $\lambda$. Panels (d)-(f) report the influence of the size of the aggregate shock $\kappa$ on endogenous variables. The demand for capital falls with $\kappa$ since the expected marginal loss in post-shock utility due to an additional unit of $x_A^m$, given by the sum in the square bracket of equation (5.2.7), increases. This marginal loss rises with $\kappa$ since the drop in wealth after a shock has occurred rises with that parameter and as the value function is concave. It can be observed that the influence of the magnitude of the aggregate shock on equilibrium values is small until $\kappa$ attains a value of about 0.6. As $\kappa$ approaches its maximum value $\kappa = 1 - \underline{\kappa}$, the effects of changing $\kappa$ become stronger. In contrast to the preceding comparative statics exercise, the repercussions of altering the size of the "bad" idiosyncratic shock $\underline{\kappa}$ are more pronounced for lower parameter values as Panels (g)-(i) show, even though $\overline{\kappa}$ is varied at the same time to preserve the zero expected value property of the individual shock. As before, the expected marginal loss in post-shock utility due to an additional unit of $x_A^m$ rises, ceteris paribus. Yet, the resulting drop in demand for capital is now higher since capital is the only asset affected by idiosyncratic risk, whereas in the case of an increasing value of $\kappa$, both money and capital bear a higher risk.

### 5.2.3 The Model without Sector-Specific Risk

Before analysing the results generated by the full baseline model, let us consider a further special case, in which there is no sector-wide risk, i.e. $\phi^s = 0$.

**Theorem 2.** *Without sector-specific productivity risk, equilibrium processes are deterministic up to changes in effective aggregate capital described by equation (4.1.1).*

*Proof.* If sector-specific productivity risk is switched off, i.e. if $\phi^s = 0$, entrepreneurs' post-jump wealth share is given by

$$\widetilde{\eta}_t = \left( x^e_{1,t} \frac{\widetilde{\theta}_t}{\theta_t} + (1 - x^e_{1,t}) \frac{1 - \widetilde{\theta}_t}{1 - \theta_t} \right) \eta_t. \tag{5.2.8}$$

Variables $\theta_t$ and $\widetilde{\theta}_t$ on the RHS of (5.2.8) are functions of the endogenous state variable $\eta_t$ in the baseline model with sector-specific risk. Setting $\phi^s = 0$ eliminates all exogenous variation in $\eta_t$. Without exogenous variation, however, changes in $\theta_t$ as a consequence of jumps cannot occur. Thus, we have $\widetilde{\theta}_t = \theta_t$ and, accordingly, $\widetilde{\eta}_t = \eta_t$. It follows that equilibrium paths of the endogenous variables stated in Proposition 2 are deterministic. □

In order to gain intuition for Theorem 2, consider first the reaction of the entrepreneurial sector to a jump. Initially, there are neither movements in $q_t$ nor in $p_t$. Thus, $n_t^e$ initially drops by factor $(1 - \kappa)$ according to (4.5.1c). As discussed, this result is due to the fact that in the absence of changes in $q_t$ and $p_t$, the value of any long (short) position in money, credit, or capital falls (increases) by factor $(1 - \kappa)$. This, in turn, implies that both the values of assets and liabilities fall on entrepreneurs' balance sheets by that same factor.

Since asset demands are linear in wealth due to the combined assumption of HARA utility and AK production technologies, the total demand for capital of the entrepreneurial sector also decreases by factor $(1 - \kappa)$. The effect of the idiosyncratic shock on the total demand for capital by entrepreneurs cancels out because the expected value of that shock is equal to zero: the amount of capital unlucky individuals want to sell due to the adverse idiosyncratic shock exactly equals the amount individuals exposed to the beneficial individual shock want to buy. By symmetry, the same reasoning applies to the managerial sector. Hence, capital market clearing implies

$$(1 - \kappa) \left[ x^{e,p}_{1,t} n^e_t + \frac{\varphi}{1 - \varphi} \left( x^{e,p}_{1,i,t} + x^{e,p}_{2,i,t} - 1 \right) n^e_t + x^m_t n^m_t \right] = \widetilde{q}_t \,(1 - \kappa)\, K_t. \tag{5.2.9}$$

Comparing this to (4.4.20), it becomes obvious that $\widetilde{q}_t = q_t$ has to hold. On the money market, the price of money falls by factor $(1 - \kappa)$, but the price of money per unit of capital $p_t$ stays constant.

In summary, the wealth distribution between sectors is unaffected by exogenous shocks and so are prices $q_t$ and $p_t$. Yet, these observations do not imply that entrepreneurs' relative net worth position is immaterial. Prudent entrepreneurs still require net worth buffers to absorb idiosyncratic shocks to capital. As a consequence, their demand for capital still depends on their wealth levels and so do asset prices. In the long-run, the economy will deterministically approach a steady-state in which prudent entrepreneurs own the entirety of aggregate wealth. This is related to the fact that the EFP is zero at any point in time due to the absence of endogenous asset price risk (cf. equation (4.3.34)). Thus, we have

$$\mu_t^{r^{K,e}} - \mu_t^{r^L} = \mu_t^{r^{K,e}} - \mu_t^{r^M} > \mu_t^{r^{K,m}} - \mu_t^{r^M}, \ \forall t. \tag{5.2.10}$$

In words, the premium on entrepreneurs' capital over the real loan rate is greater than managers' premium on capital over money at each $t$. In effect, entrepreneurs' real profits are always greater than managers' (cf. also equation (B.4.1) in Appendix B.4). Once the steady state is reached, the economy stays there forever. Steady-state values of endogenous variables can be calculated by using the same procedure as in the autarky case, except for the fact that productivity level $a^m$ has to be replaced by $a^e$.

Finally, it is worthwhile to compare Theorem 2 to a result in Di Tella (2017). In his complete markets model, agents proportionately share aggregate risk by trading a market index of experts' equity in the absence of uncertainty shocks. As a consequence, the wealth distribution evolves deterministically and amplification of shocks does not occur.[704] As the preceding discussion has demonstrated, an analogous result arises in our model without sector-specific shocks. However, this is not due to the presence of complete markets but, rather, to the fact that both the value of money and capital shrink by the same factor following a shock. Thus, the possibility of investing in money and taking out loans denominated in money in the absence of any sector-specific risk plays the same role as agents' ability to trade the market index in the model by Di Tella (2017): it allows for the perfect sharing of aggregate risk.[705]

## 5.3 The Full Baseline Model

### 5.3.1 The Allocation and Price of Capital

Figure 5.3.1 presents some key results in the baseline credit model.[706] Panel (a) depicts entrepreneurs' share in the aggregate capital stock $\psi_t$ as a function of entrepreneurs' wealth share in the

[704] Cf. Subsection 3.2.4.3 in this thesis.

[705] It should be noted that this result would not hold under a production technology in which output is not proportionate to capital. This is because the value of money would not be proportionate to the aggregate capital stock in that case.

[706] The procedure to solve the baseline model is explained in Appendix C.4.

domain $[0, \eta^s]$, in which $\eta^s$ is a point close to the stochastic steady state.[707,708] That panel reveals a result that is in accordance with the CTMF literature: the capital allocation improves with the wealth share of constrained agents until the optimal allocation is reached at state $\eta_t = \eta^{\psi}$. Here, entrepreneurs play the role of constrained agents and the optimal capital allocation is $\psi_t = 1$. The positive relationship between $\psi_t$ and $\eta_t$ is due to prudent entrepreneurs' risk aversion: they require sufficient net worth buffers before they are willing to pick up additional capital. Logarithmic utility guarantees that these buffers are sufficient to avoid default in any case. Obviously, function $\psi(\eta_t)$ is concave: additions to the stock of capital in the entrepreneurial sector per unit of $\eta_t$ are larger when the state variable is low. The explanation is that in the lower region of the state space, assets are essentially priced by less productive managers. Hence, the current price of capital is close to its autarky level. This offers an attractive opportunity for entrepreneurs and induces them to buy relatively large amounts of capital per unit of $\eta_t$ from managers.[709]

As entrepreneurs' demand for the productive asset rises with the state variable, the price of capital $q_t$ surges (Panel (b)) since those agents value capital more than less productive managers. Initially, the change in $q_t$ is most pronounced since the allocation of capital improves at a high rate. As $q_t$ rises, the return on capital falls and entrepreneurs are less willing to buy additional capital. As a result, the slope of function $q(\eta_t)$ decreases until point $\eta^{\psi}$. At that point, a discontinuity emerges, which causes slope $q'(\eta_t)$ to jump up, although this is hardly visible in Panel (b). Once that subset of the state space is reached, the additional demand for capital in the absence of shocks cannot be satisfied by further external investment. Accordingly, the price of capital has to rise at a faster rate to achieve capital market clearing. Effectively, external investment is replaced by additional internal investment, which is associated with a higher price of capital. As $\eta_t \to \eta^s$, the price of capital approaches its first-best level $q_{FB}$, although the latter is not fully reached. This is because financial frictions and endogenous risk do not disappear, even if the economy is at its stochastic steady state.

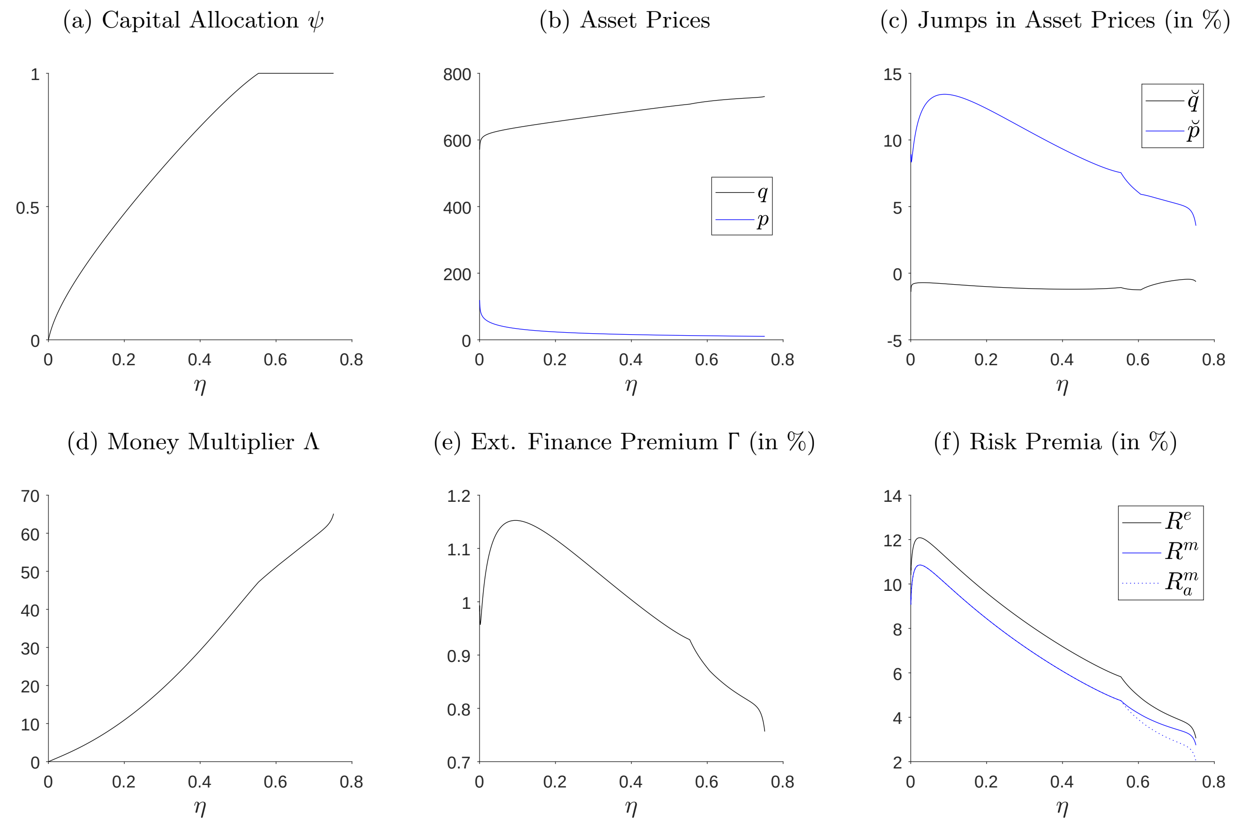
The fact that the price of capital is a monotonously increasing function of the state variable is in line with the results generated by the BS (2014a) baseline model and the BS (2014d) (cf. Figures 3.2.2 and b). Another outcome emerges in BS (2016a). In their model the price of capital falls in the upper region of the state space since term $A(\cdot)$ in the final goods production function $Y_t = A(\psi_t) K_t$ has an internal maximum (cf. Subsection 3.2.2.2). In contrast, the model of Chapter 4 does not feature an aggregate production technology with an internal maximum - the optimal capital allocation is $\psi_t = 1$. It follows that situations in which entrepreneurs demand "too much" capital do not emerge.

[707] The algorithm halts at a point close to the stochastic steady state. For details, the reader is referred to Subsection 5.3.5 and Appendix C.4.

[708] The procedure to solve the baseline model is explained in Appendix

[709] The same logic applies to the models in Brunnermeier and Sannikov (2014a) and (2014d). At the lower region of the state space, the price of capital is very close to the price that would ensue if unconstrained agents were to hold the aggregate capital stock forever. Consequently, capital purchases by constrained agents who have a higher valuation for capital are especially high per unit of the state variable.

Figure 5.3.1: Equilibrium Quantities and Prices in the Baseline Credit Model




In the event of a jump, the price of capital drops in the entire considered subset of the state space (Panel (c)). This is a direct consequence of the monotonously increasing behaviour of function $q(\cdot)$. Intuitively, in case of a jump, the wealth share of entrepreneurs declines, which induces them to cut back their demand for the productive asset. This eventually results in a reduction of $q_t$. As in other CTMF models, the drop in the price of the productive asset is more severe if the reduction in constrained agents' demand for capital triggers fire sales to less efficient agents. Here, this ensues at each point below $\eta_t \approx 0.6$, the point at which the second kink in the curves of Panel (c) occurs. Accordingly, asset prices adjustments become less pronounced above that point.

### 5.3.2 The Price of Money per Unit of Capital and the Money Multiplier

The price of money per unit of aggregate capital $p_t$ is a monotonously decreasing function of $\eta_t$ (Panel (b)). There are two main forces that drive this negative relationship and also arise in the two presented versions of the I Theory of Money. The first is the expansion in the supply of inside money. As Entrepreneurs accumulate more and more equity, they increase their stocks of money and capital. These increases are at least in part financed by additional debt. Banks, on the other hand, are willing to provide arbitrary amounts of credit at the prevailing loan rate. Since banks issue corresponding amounts of deposits to money holders, the money multiplier $\Lambda_t$ expands (Panel (d)). It follows from these considerations that the expansion in the inside money supply is driven by entrepreneurs' demand for credit. In contrast, in the I Theory of Money, inside money creation is determined by intermediaries' investment in productive agents' shares. The second force is the drop in demand for money by less productive agents, namely managers, due to improved risk-sharing in the economy: entrepreneurs hold more and more risky capital, which reduces managers' exposure to idiosyncratic risk. In parallel to BS (2016a), there is a counteracting force on the value of money due to the improvement in the capital allocation which causes the *level* of real GDP to rise with the state variable.[710] Similarly to their model, this effect is overcompensated by the two effects discussed previously, as the negative slope of function $p(\eta_t)$ suggests. The resulting downward pressure on $p_t$ is dampened in our model by an effect that arises from the presence of real balances in managers' and entrepreneurs' instantaneous utility function. Since a drop in the value of money directly impairs their utility levels, ceteris paribus, they increase their nominal demand for money to counteract the welfare loss.

It is worthwhile to compare the shape of function $\Lambda(\eta_t)$ in Panel (d) to that of the corresponding function in BS (2014d). The latter is increasing in intermediaries' wealth share, which is the state variable in their model. However, in contrast to $\Lambda(\cdot)$, it has a concave shape with a global internal maximum at $\eta^\psi$, the point at which the optimal capital allocation is reached. The concavity is related to the fact that $p_t$ is decreasing and, accordingly, intermediaries' refinancing costs are increasing in the state variable, as can be observed from equation (3.2.22). Once $\eta^\psi$ is reached, intermediaries hold the entire aggregate capital stock and they begin to pay down debt, which causes the money multiplier to fall. Here, entrepreneurs' costs of external financing are nonmonotonous, as we will see momentarily. In addition, entrepreneurs prefer to retain at least a part of their income in the form of additional real balances rather than to pay off loans after point $\eta^\psi$.[711] In BS (2014d), there is no transaction motive for holding real balances and thus intermediaries opt for accepting less deposits once they own the entire aggregate capital stock.

As discussed, in the credit model, the inside money supply is the mirror image of the volume of credit extended to entrepreneurs. Our model implies procyclicality of entrepreneurs' debt: the money multiplier is high in times when entrepreneurs' are well equipped with equity and low in the opposite case. In the former case, the price of capital is close to its first-best level and the "fundamental" component of the growth rate of real GDP $\mu_t^K = \Phi(\iota_t^*) - \delta = \frac{1}{\gamma}\log q_t - \delta$[712] is near its maximum level. Table C.1 in Appendix C.6 summarises the pairwise correlation coefficients of the growth rates of the money multiplier, the state variable, and real GDP. The correlation between

[710] This effect is not discussed by the authors.

[711] The kink in function $\Lambda(\cdot)$ at point $\eta^\psi$ and the subsequent behaviour of that function are due to two counteracting forces. First, at $\eta^\psi$ and above, entrepreneurs hold the entire aggregate capital stock. Accordingly, prudent entrepreneurs' demand for credit to finance external investments in capital drops to zero. Second, as we will see shortly, the EFP falls sharply in that region, increasing those individuals' demand for credit to finance holdings of real balances. The second effects becomes stronger relative to the first when $\eta_t$ rises.

[712] This component is the growth rate of real GDP in the absence of shocks and changes in the capital allocation.

$\hat{\Lambda}_t$[713] and $\hat{\eta}_t$ is almost perfect, as suggest by Panel (d) in Figure 5.3.1. Since $\hat{\eta}_t$ is highly positively correlated to $\hat{Y}_t$, $\hat{\Lambda}_t$ is as well. The procyclical behaviour of corporate debt is a well documented fact in the empirical macro literature.[714]

Function $\breve{p}(\eta_t) = \widetilde{p}(\eta_t)/p(\eta_t) - 1$ relates the current state to the percentage change in the price of money per unit of capital in case of a jump and therefore can be regarded as a measure for the volatility of $p_t$. Panel (c) shows that $\breve{p}(\cdot)$ is always positive in the interval $\left(0, \eta^s\right]$. The reason is that entrepreneurs decrease their leverage after the arrival of an adverse shock by lowering their demand for capital and paying back debt. The resulting drop in the inside money supply subsequently leads to deflationary pressure. At high values of $\eta_t$, the appreciation in the value of money becomes less severe since prudent entrepreneurs are well-capitalised and do not deleverage by much. Comparing functions $\breve{q}(\cdot)$ and $\breve{p}(\cdot)$, it becomes clear that the amplitude of the jump in $p_t$ is much larger in percentage terms than that of the price of capital for most of the state space. We can thus conclude that endogenous risk is primarily due to variations in the value of money.

### 5.3.3 The External Finance Premium, Returns, and Risk Premia

According to equation (4.3.34), variation in the EFP is driven by the percentage changes in the prices of capital and money resulting from a jump. Panel (e) shows that the shape of function $\Gamma(\eta_t)$ largely resembles that of $\breve{p}(\eta_t)$. This suggests that the variation in the mark-up over the real deposit rate is mainly due to adjustments in the value of money. Since the EFP is determined by banks' losses in case of a jump, that result also implies that the deflationary rather than the liquidity spiral is the most important source of amplification on banks' balance sheets. Interestingly, the EFP is not a monotonously decreasing function of borrowers' aggregate wealth as in other models with credit frictions.[715] Here, *defaulting* debtors always enter loan agreements with zero wealth. Hence, the (procyclical) *aggregate* wealth of borrowers does not directly affect the EFP. Rather, the latter depends on the state variable only indirectly through amplification terms $\breve{q}_t$ and $\breve{p}_t$. This explains why the loan rate mark-up can either be counter- or procyclical, depending on the current state of the economy. However, countercyclicality holds true in expectation: the estimated correlation coefficient between $\hat{\Gamma}_t$ and $\hat{Y}_t$ is approximately equal to $-0.58$.

Next, let us gauge the impact of the EFP on entrepreneurs' real funding costs. Table 5.2 lists the annualised mean values and standard deviations of entrepreneurs' real return on capital, the real return on money, and the EFP predicted by the model. On average, the EFP per quarter takes on a value of about 0.28 percent, while the mean of the real return on money is approximately equal to

713 In the following, variables with a hat $\hat{()}$ will denote growth rates.

714 Cf. e.g. Quadrini (2011, pp. 210f.) or Covas and Den Haan (2011, p. 893). Quadrini (2011, p. 210) argues that the procyclicality of debt hints at the presence of financial frictions since financial structure is irrelevant under complete markets and one would not expect credit flows to vary with the business cycle. Indeed, if entrepreneurs in the presented model were able to hedge idiosyncratic and aggregate risks, they would immediately buy the entire capital stock and finance any demand for the productive asset that exceeds their equity by debt (cf. the discussion of the first-best case in Subsection 5.2.1).

715 This distinction is discussed in more detail in Section 5.5.2.

Table 5.2: Model-Implied First- and Second-Order Moments of Real Asset Returns and the EFP

| $\mathbb{M}\left[dr_{i,t}^{K,e,p}\right]$ | S.D. $\left[dr_{i,t}^{K,e,p}\right]$ | $\mathbb{M}\left[dr_t^M\right]$ | S.D. $\left[dr_t^M\right]$ | $\mathbb{M}\left[\Gamma_t\right]$ | S.D. $\left[\Gamma_t\right]$ |
|---|---|---|---|---|---|
| 1.68% | 1.94% | 0.85% | 3.93% | 0.28% | 0.01% |

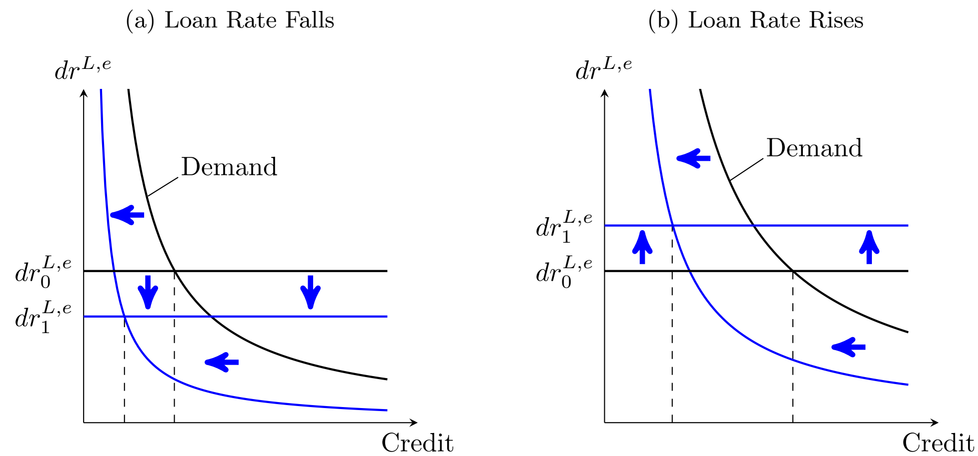
*Notes*: Quarterly returns; Model-implied returns are computed from time series generated by a Monte Carlo simulation of the baseline model. Each of the simulated series contains 9000000 realisations (360 days $\times$ 25000 years).

0.85 percent. These numbers imply that roughly one quarter of entrepreneurs' real financing costs are due to the EFP and the remaining 75 percent are explained by the real deposit rate on average. Turning to standard deviations, we observe that the volatility of $\Gamma_t$ is considerably lower than that of the two asset returns. This can be explained by recalling that $\Gamma_t$ enters the drift rate of the SDE for the real loan rate, while the two returns are described by SDEs themselves. In effect, changes in the former are always of order $dt$, while changes in the latter are of order unity. The interactions of supply and demand in the credit market induced by a jump are depicted in Figure 5.3.2. The first case is depicted in Panel (a) and corresponds to a situation in which the economy is on the increasing arc of function $\Gamma(\cdot)$. Banks, whose supply of loans is perfectly elastic at the current loan rate, react to the shock by cutting that rate. This counteracts the drop in demand caused by entrepreneurs' losses. Yet, we can deduce from the monotonically increasing shape of function $\Lambda(\cdot)$ that the improvement in the conditions of credit availability is not sufficient to prevent a reduction in the volume of credit and thereby inside money. Panel (b) visualises the case in which the drop in entrepreneurs' loan demand is reinforced by a surge in the lending rate. This case occurs if the post-jump value of entrepreneurs' net worth share is on the decreasing arc of function $\Gamma(\cdot)$.[716]

Turning to relative asset returns, note that entrepreneurs' capital earns a higher real return than money, even though the latter is more volatile than the former. According to the consumption capital asset pricing model, the covariance between a particular asset's return and the investor's consumption growth determines the return spread, rather than the variance of that asset. Indeed, the simulation of asset returns and consumption paths shows that the consumption growth of entrepreneurs, who play the role of marginal investors in capital in instants without jumps, is positively correlated to the return on capital and negatively to the return on money. Panel (f) in Figure 5.3.1 provides further information on managers' and prudent entrepreneurs' real required risk premia on capital over money, which are denoted by $R_t^m$ and $R_t^{e,p}$, respectively. Risk premium $R_t^{e,p}$ is determined by the sum of $\Gamma_t$ and the RHS of equation (4.3.20). As can be observed from this equation, variation in the risk premium is determined by the EFP, entrepreneurs' portfolio weight on capital, and the post-jump values of the capital price and the price of money per unit of

[716] Using firm-level data from the Euro Area in the years from 2009 to 2011, Holton et al. (cf. 2014, Table 3) present empirical evidence that credit demand contracted and the terms of credit supply worsened during that period.

Figure 5.3.2: Credit Market Adjustment after a Jump




*Notes*: Stylised depictions of credit market adjustments in case of a jump.

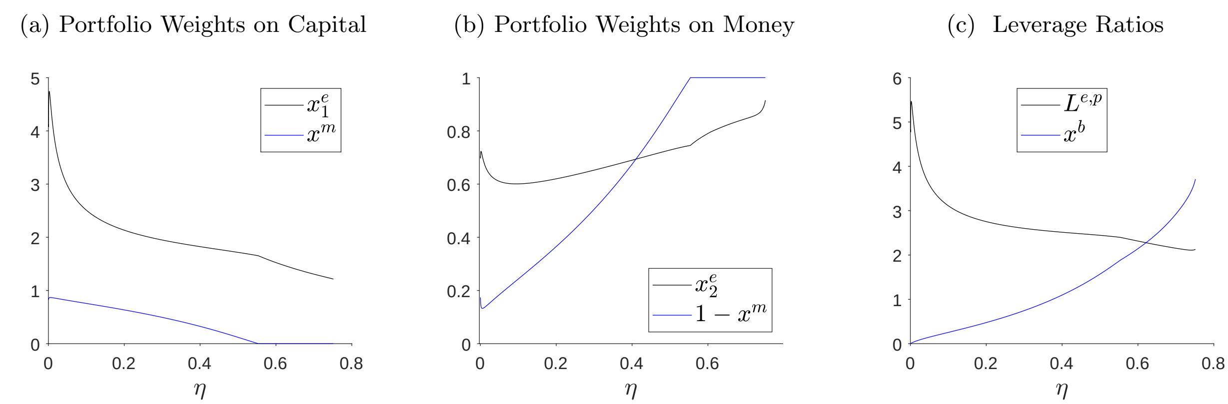
aggregate capital, both relative to their pre-jump levels. As entrepreneurs' relative equity position improves, their portfolio weight on capital declines, which drives down their required risk premium, ceteris paribus. However, this downward pressure is overcompensated by rising endogenous risk, i.e. potential movements in asset prices, on the increasing arc of function $R^{e,p}(\eta_t)$. Afterwards, price changes induced by jumps are less severe and, accordingly, the risk premium falls.

Variable $R_t^m$ can be calculated from the RHS of equation (4.3.11). Obviously, the shapes of the two curves in Panel (f) comove closely, indicating that both groups face similar sources of risk. Yet, managers required risk premium is lower than entrepreneurs' since the former are not leveraged in contrast to the latter. Within segment $\left(0, \eta^{\psi}\right)$, both types of agents hold a strictly positive amount of capital, which implies that required and actual excess returns offered by the market are equalised in each sector. From point $\eta^{\psi}$ on, managers' required excess return exceeds the actual excess return on managers' capital $R_a^m$, which is given by the LHS of (4.3.11) and depicted by the blue dotted line in Panel (f). Consequently, at point $\eta^{\psi}$ and above, managers do not hold any capital. For $\eta_t \in \left(0, \eta^{\psi}\right)$, the vertical distance between the two two solid lines is equal to the difference in the payoff yields of the two groups of agents, given by $(a^e - a^m)/q_t$. Within $\left[\eta^{\psi}, \eta^s\right]$, this difference is equal to the gap between the black solid line and the blue dotted line. Since $q_t$ depends positively on $\eta_t$, the payoff yield differential narrows as the economy approaches its stochastic steady state.

### 5.3.4 Portfolio Weights

It can be observed from Panel (a) in Figure 5.3.3 that our earlier claim that entrepreneurs' portfolio weight on capital falls with the state variable indeed holds true. This portfolio weight is high when entrepreneurs' current investment opportunities are attractive, i.e. when the price of capital is low. During the transition towards the stochastic steady state, the price of capital rises in the absence

Figure 5.3.3: Portfolio Weights in the Baseline Credit Model




of shocks. Further, endogenous risk arising from changes in asset prices becomes less pronounced, which compresses risk premia. As a consequence, entrepreneurs' reduce the share of their wealth invested in capital. The convexity of function $x_1^{e,p}(\eta_t)$ is mainly due to the concave shape of $q(\eta_t)$. In turn, the former property implies that $\psi(\eta_t)$ is concave.

The function for prudent entrepreneurs' portfolio weight on money (Panel (b)) mirrors $\Gamma(\eta_t)$: initially, the former is increasing as the latter is decreasing. This continues until $x_2^{e,p}(\eta_t)$ reaches its global minimum at the same point at witch $\Gamma(\eta_t)$ peaks. The reason for the reverse shapes of the two functions is that agents of type $e,p$ fund their real balances (at least in part) by debt. Hence, the relevant marginal cost of adding a unit of money to the portfolio is the external finance premium, as long as these agents are leveraged.[717]

Panel (c) depicts prudent entrepreneurs' leverage ratio $L_t^{e,p} \equiv x_{1,t}^{e,p} + x_{2,t}^{e,p}$, calculated as assets over equity. Obviously, the leverage ratio is decreasing with the state variable. This relationship is primarily driven by the behaviour of function $x_1^{e,p}(\eta_t)$. As discussed in Subsection 5.3.1, the price of capital is the higher, the better entrepreneurs are capitalised. Thus, prudent entrepreneurs' leverage ratio is *countercyclical* with respect to the price of the productive asset, and hence with respect to growth component $\mu_t^K = \frac{1}{\gamma}\log q_t - \delta$. This reasoning is confirmed by Table C.2, which reports various correlation coefficients calculated from a Monte Carlo simulation of the model. The correlation coefficient between the growth rate of leverage $\hat{L}_t^{e,p}$ and the growth rate of the capital price $\hat{q}_t$ is strongly negative. Since $\hat{q}_t$ and the growth rate of real GDP $\hat{Y}_t$ are positively correlated, $\hat{L}_t^{e,p}$ and $\hat{Y}_t$ are negatively related. These results are in accordance with evidence produced by the empirical corporate capital structure literature. For instance, Welch (2004) analyses a large

[717] This can be seen from combining FOCs (4.3.19) and (4.3.23) to arrive at

$$\frac{\xi\left(x_{2,t}^{e,p} n_{i,t}^{e,p}\right)^{-1}}{\left(c_{i,t}^{e,p}\right)^{-1}} = \Gamma_t, \tag{5.3.1}$$

which states that in optimum, prudent entrepreneurs equalise the ratio of the marginal utilities of real balances and consumption, which is the marginal rate of substitution between real balances and consumption, to the EFP.

sample of publicly traded U.S. nonfinancial corporations spanning from 1962 to 2000 and concludes that higher stock prices cause leverage ratios to fall.[718] Adrian and Shin (2010) present additional evidence for the countercyclicality of nonfinancial firms' leverage. They find that growth in the balance sheet size of firms in the U.S. flow of funds database is negatively associated with leverage in the period form 1963 to 2006.[719]

Turning to managers' portfolio allocation, Panel (a) reveals that their portfolio weight on capital is a decreasing function of entrepreneurs' wealth share for $\eta_t \in \left(0, \eta^{\psi}\right)$. At $\eta_t = \eta^{\psi}$ and beyond, entrepreneurs hold the entire aggregate capital stock and the share of wealth allocated by managers to the productive asset is zero. Panel (b) makes clear that capital is substituted by money in managers' portfolios until they eventually hold their wealth entirely in the form of money.

Further, Panel (c) in Figure 5.3.3 visualises that $x_t^b$, which stands for banks' leverage ratio in terms of bank owners' aggregate wealth $n_t^m$, goes up with entrepreneurs' relative equity position. Intuitively, entrepreneurs' demand for debt is large when $\eta_t$ is high. As banks' are willing to satisfy any demand for credit at the current loan rate, the numerator of banks' leverage ratio is large in these situations. At the same time, managers' equity, which is the denominator of banks' leverage ratio, is low. The third row of Table C.2 in Appendix C.6 reports the correlation coefficients of leverage growth rate $\hat{x}_t^b$ with $\hat{q}_t$ and $\hat{Y}_t$, respectively. The correlation between $\hat{x}_t^b$ and $\hat{q}_t$ is high since entrepreneurs are well-capitalised when banks' are strongly leveraged. Consequently, $\hat{x}_t^b$ is positively related to output growth. As mentioned in Subsection 3.2.4.2, the *procyclical* behaviour of financial institutions' leverage is confirmed in recent empirical work, but does neither arise in BS (2014d) nor BS (2016a).

This raises the question of what lies at the root of the procyclicality of $\hat{x}_t^b$. In a modification of the baseline model in which bankers choose lending policies such that bank owners' utility is maximised, the positive association between leverage and output growth is preserved, indicating that managers view the income accruing from their equity stakes in banks as a valuable investment opportunity in times when the risk premium on capital is depressed. This suggests that the procyclicality of banks' leverage is not due to the risk neutral behaviour of bankers, which is not in the interest of risk averse managers. Rather, that result stems from the procyclicality in the demand for money and the demand for credit. Consider first an instant of time in which a jump materialises. The appreciation in the value of money boosts agents' real balances. Since any individual seeks to maintain a (nearly) constant level of real money holdings due to the MIU motive, the nominal demand for money falls. At the same time, entrepreneurs cut back their demand for credit. As, in addition, the wealth share of bank owners improves, banks' assets-to-equity ratio in terms of owners' wealth plummets. On the contrary, in normal times, money depreciates in value, resulting in a heightened nominal demand for the liquid asset. The concurrent boost in the demand for credit from the improvement in borrowers' balance sheets reinforces the positive effect of the demand for

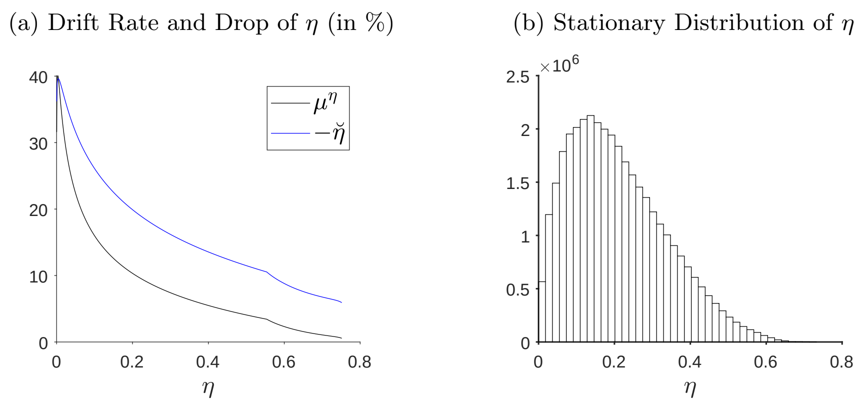
[718] Cf. Welch (2004, p. 114).
[719] Cf. Adrian and Shin (2010, p. 421).

money on banks' assets-to-equity ratios.[720]

### 5.3.5 Dynamics and Distribution of the State Variable

The drift rate of the state variable declines with $\eta_t$ (Panel (a) in Figure 5.3.4). Equation (4.5.7b) elucidates that $\mu_t^\eta$ depends on entrepreneurs' risk premium and risk-taking as expressed through portfolio weight $x_{1,t}^{e,p}$. Both variables were shown to be relatively large when the state variable is low in the preceding sections. As $\eta_t$ rises, entrepreneurs' risk premium and leverage fall and so does $\mu_t^\eta$. Again, a kink arises at point $\eta^\psi$, which can be explained by the discontinuous drops in $x_{1,t}^{e,p}$ and the risk premium. For $\eta_t \to \eta^s$ prudent entrepreneurs' real profits are increasingly offset by the propensity to consume out of wealth $\rho/\left(1+\xi\right)$ and drift rate $\mu_t^\eta$ approaches zero. However, the algorithm does not fully reach the stochastic steady state characterised by condition $\mu_t^\eta = 0$, but halts at point $\eta^s$ where $\mu_t^\eta \approx 0.5$ percent.

Figure 5.3.4: Dynamics and Distribution of the State Variable in the Baseline Credit Model



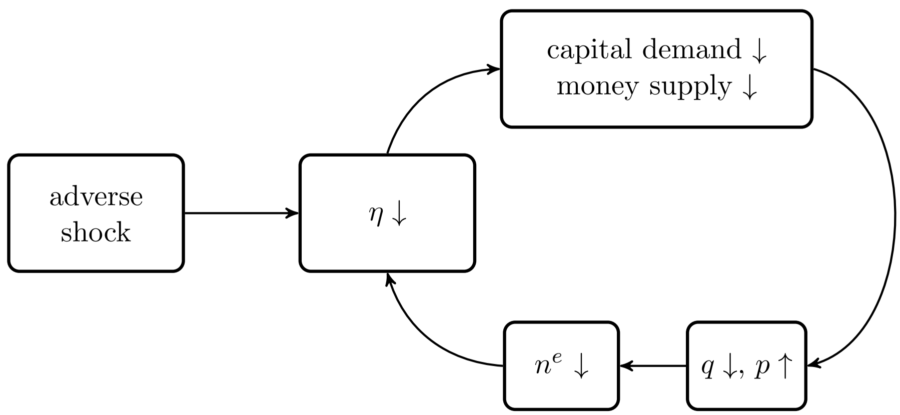

*Notes*: The stationary distribution in Panel (b) is computed from a simulated time series for the state variable. The simulated series includes 36000000 data points (360 days $\times$ 100000 years).

Panel (a) also depicts the negative of the percentage change in the state variable conditional on the arrival of a jump $-\breve{\eta}\left(\eta_t\right) = 1 - \widetilde{\eta}\left(\eta_t\right)/\eta_t$. Since $-\breve{\eta}\left(\eta_t\right)$ only takes on positive values, shocks are associated with reductions in the wealth share of entrepreneurs at any point in the state space. This outcome is due to exogenous as well as endogenous forces. Once a shock materialises, a share $\phi^s = 0.05$ of entrepreneurs experience a drop in their productivity levels, which causes their risk premia on capital to contract. To restore optimality, affected individuals reduce their required risk premia. This can only be achieved by selling capital in exchange for money. A part of the proceeds is used to pay back outstanding debt. The endogenous portfolio response sets in motion an adverse

[720] Introducing financial frictions on the side of banks might change the behaviour of banks' leverage. If e.g. deadweight costs of equity issuing as in Klimenko et al. (2016) were present, a bank balance sheet channel would emerge, i.e. the distinction between bank equity and managers' equity would matter for aggregate outcomes. This could potentially counteract the effects of expanding (contracting) demand for bank debt in normal (crisis) times.

feedback loop, which is characterised by changes in asset prices and further reductions in the state variable (cf. Figure 5.3.5). In particular, capital sales by entrepreneurs that were exposed to the sector-specific productivity shock lead to a fall in the price of capital since other individuals are only willing to buy additional capital at a price discount. At the same time, the reduction in the volume of outstanding bank loans causes the money supply to shrink and the value of money to rise.

Figure 5.3.5: Adverse Feedback Loop in the Credit Model




*Notes*: Based on Brunnermeier and Sannikov (2014a, p. 400).

The resulting price changes cause the initial shock to spill over to the entire entrepreneurial sector. This is because the drop in the price of capital shrinks the value of assets and the boost in the value of money increases the value of liabilities on their balance sheets. As a consequence of the adverse price movements, imprudent entrepreneurs are pushed into bankruptcy, which precipitates losses in the banking sector. Prudent entrepreneurs, on the other hand, stay solvent, but, by virtue of risk aversion, react to balance sheet losses by cutting their demand for capital as well as debt. Yet, this form of deleveraging is self-defeating since it leads to further adverse changes in asset prices. In the terminology of BS (2016a), the "Paradox of Prudence" emerges, albeit it does so in the final goods producing sector rather than the financial sector. Again, the amplification of the exogenous shock is considerably stronger in the lower part of the state space than in the upper: at $\eta_t \approx 0$, ratio $-\breve{\eta}\left(\eta_t\right)/\phi^s$, which can be interpreted as a measure of amplification, is at a value of about 8.0, while at $\eta_t = \eta^s$ that same ratio is approximately equal to 1.2. This is for the same underlying reason as to why drift rate $\mu_t^\eta$ decreases with the state variable: when $\eta_t$ is low, leverage and endogenous risk embodied in adverse price adjustments are relatively high. This observation also explains, why the two curves in Panel (a) comove closely.[721]

With the dynamics of the state variable at hand, its distribution can be calculated from a

[721] Cf. Appendix B.4 for an algebraic expression of the relation between the drift rate and percentage drop of the state variable.

simulated time series.[722] Panel (b) of Figure 5.3.4 shows that the stationary distribution is unimodal with a peak at about $\eta_t = 0.13$. The single-peaked shape of the estimated probability distribution function can be explained as follows. If $\eta_t$ is low, both $\mu_t^\eta$ and $\breve{\eta}$ in absolute terms are near their maxima. However, the expected (annualised) relative change in the state variable $\mu_t^\eta + \lambda\breve{\eta}_t$ is close to its maximum since the gap between the two curves in Panel (a) of Figure 5.3.4 is close to its minimum. Accordingly, if the economy is ever pushed to a depressed state near the origin, it leaves that region fast. However, $\mu_t^\eta + \lambda\breve{\eta}_t$ falls with the state variable and becomes approximately equal to zero at the mean of the simulated time series for $\eta_t$. That mean takes on a value of 0.21. Afterwards, $\eta_t$ falls in expectation. Hence, the distribution puts less weight on higher values of the state variable. Interestingly, the probability to reach point $\eta^s$ is extremely small, namely at less than 0.01 percent. This is because drift rate $\mu_t^\eta$ approaches zero while the negative of $\breve{\eta}_t$ approaches its minimum level of $\phi^s = 5$ percent. The economy spends only about a mere two percent of the time in the region where entrepreneurs own the entire aggregate capital stock, implying that misallocation of capital is the norm rather than the exception under the baseline calibration. Indeed, the mean of capital allocation variable $\psi_t$ is at 0.49, which means that on average less than half of the aggregate capital stock is allocated to its most efficient use.

### 5.3.6 Dynamics of Output and Inflation

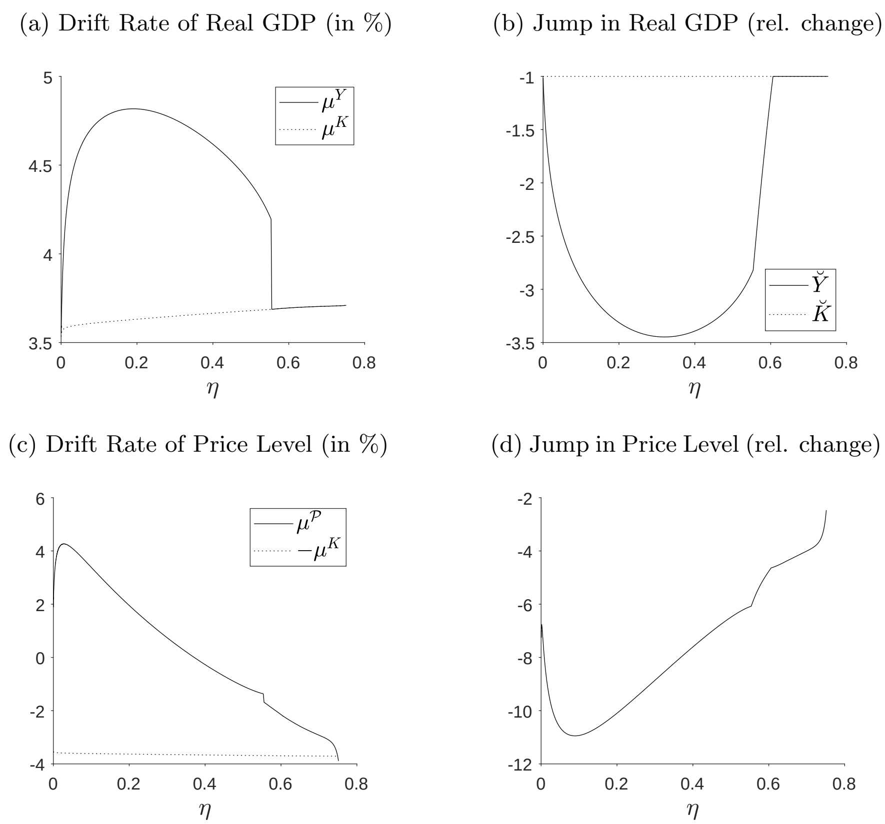
As mentioned in Subsection 4.4.1.2, equation (4.4.7) implies that the drift rate of real GDP $\mu_t^Y$ is equal to the sum of the drift rate of TFP $\mu_t^A$, which depends on the change in the allocation of capital in the absence of jumps, and the drift rate of the aggregate capital stock $\mu_t^K$. Panel (a) in Figure 5.3.6 reveals that function $\mu^Y(\eta_t)$ has an inverted-U shape. At $\eta_t \approx 0$ drift rate $\mu_t^Y$ is very close to $\mu_t^K$. This is because the change in the allocation of capital is negligible when the entrepreneurial sector is small. As $\eta_t$ increases, two forces that affect $\mu_t^A$ arise. First, entrepreneurs accumulate more and more equity. The instantaneous absolute change in the wealth share of that sector $\mu_t^\eta \eta_t$ is positively related to $\mu_t^\psi \psi_t$ and thus also to the drift rate of real GDP according to (4.4.7). The reason for those relations is that a part of the additional equity is used by prudent entrepreneurs to fund purchases of capital.

Second, entrepreneurs allocate less and less of their profits to purchases of existing capital. This is reflected in the concavity of function $\psi(\cdot)$ in Panel (a) of Figure 5.3.1 and the convexity of function $x_1^{e,p}(\eta_t)$ in Panel (a) of Figure 5.3.3. This effect leads to a downward pressure on $\mu_t^Y$. In the lower domain of the state space, the first effect dominates and $\mu_t^Y$ surges. In effect, the gap between functions $\mu^Y(\cdot)$ and $\mu^K(\cdot)$ widens until $\eta_t \approx 0.15$, which is close to the global maximum of the former. After that point, the drift rate of real GDP falls since the second effect starts to overcompensate the first and the gap between the two functions in Panel (a) narrows. At $\eta_t = \eta^\psi$, function $\mu^Y(\cdot)$ displays a kink. Subsequently, the drift rates of output and aggregate capital coincide

[722] Cf. Appendix C.4.4 for details.

since the allocation of capital and therefore aggregate TFP remain constant, provided no jumps materialise. Function $\breve{Y}(\eta_t)$, which yields the relative change in GDP due to a jump, is U-shaped (Panel (b)). The behaviour of that function is a mirror image of function $\mu^Y(\cdot)$ in large part and can be explained by inspecting equation (4.4.9). Initially, $\breve{Y}(\cdot)$ is approximately equal to $\breve{K}_t = -\kappa$, the relative change in $K_t$ in case of a jump, since the amount of capital entrepreneurs sell to managers is small. On the decreasing arc of function $\breve{Y}(\cdot)$, fire-sales to less productive managers rise in relative terms until the global minimum of that function is reached. Interestingly, the function exhibits *two* kinks on its increasing arc. The first kink occurs at $\eta_t = \eta^{\psi}$. After that point, capital sales by entrepreneurs drop markedly in relative as well as absolute terms. The second kink marks the point at which fire-sales of capital cease. Hence, from that point on, the relative changes in output and capital resulting from a jump are equal, i.e. $\breve{Y}(\cdot) = -\kappa$ holds.

Figure 5.3.6: Output, Inside Money, and Inflation Dynamics in the Baseline Credit Model




Panel (c) depicts the drift rate of the price level $\mu_t^{\mathcal{P}}$ and the negative of drift rate $\mu_t^K$ as functions of $\eta_t$. The behaviour of function $\mu^{\mathcal{P}}(\eta_t)$ is primarily governed by changes in the money supply $M_t$. At $\eta_t \approx 0$, the entrepreneurial sector grows at a high but declining rate and thus inside money does as well. Yet, since inside money $M_t^I$ initially is a small share of the total money supply, the drift rate of $M_t$ rises and with it $\mu_t^{\mathcal{P}}$. For higher values of the entrepreneurs' wealth share, the drift rate of the price level begins to fall as the share of $M_t^I$ in $M_t$ gets larger and the relative change in the money supply decreases further. For $\eta_t \to \eta^s$, drift rate $\mu_t^{\mathcal{P}}$ approaches the negative of $\mu_t^K$. This can be explained as follows. Equation (4.4.16) implies that the difference between $\mu_t^{\mathcal{P}}$ and the negative of $\mu_t^K$ is the negative of drift rate $\mu_t^p$. As discussed in Subsection 4.4.1.4, variations in $p_t$ reflect changes in asset allocations, portfolio weights, and the supply of money. However, for $\eta_t \to \eta^s$, these changes become smaller and smaller, until eventually the drift rate of the price level is exclusively determined by the drift rate of the aggregate capital stock.[723]

Finally, Panel (d) displays function $\breve{\mathcal{P}}(\eta_t)$, which yields the relative change in the price level if a jump arrives. Across the entire considered subset of the state space, $\breve{\mathcal{P}}(\cdot)$ stays negative, i.e. the economy is pushed into deflation if a shock occurs. By comparing Panel (d) in Figure 5.3.6 to Panel (c) in Figure 5.3.1, it becomes clear that the shape of $\breve{\mathcal{P}}(\cdot)$ closely resembles that of $\breve{p}(\cdot)$. Hence, the change in the price level is almost exclusively due to the change in $p_t$. In the lower region of the state space, deflation is relatively small in absolute terms even though the percentage drop in inside money is large. Again, the explanation is that inside money initially is a small share of the total money supply. As $\eta_t$ rises, this share becomes larger and the percentage change in inside money resulting from a shock smaller. These forces determine the shape of function $\breve{\mathcal{P}}(\cdot)$.

Table 5.3 reports model-implied and empirical means and standard deviations of two variables: the quarterly growth rate of real GDP $\hat{Y}_t$ and the quarterly inflation rate $\pi_t$. The first-order moments of the simulated time series for $\hat{Y}_t$ are close to the U.S. data since parameters $\delta$ and $\kappa$ were selected to match these empirical moments (cf. Section 5.1). In contrast, the mean and standard deviation of the model-implied inflation rate are quite distant from the data. Let us investigate why the simulated mean of the inflation rate is smaller than its counterpart in the U.S. data at first. Equation (4.4.15) shows that the drift rate of the price level depends linearly on the capital depreciation rate $\delta$. As discussed in Section 5.1, the equations in Proposition 2 do not depend on that parameter. Thus, the only model variables that are affected by $\delta$ are the inflation rate, returns, and the growth rates of capital and real GDP. Setting $\delta = 10$ percent, as is usual in the business cycle literature, implies $\mathbb{M}\left[\hat{Y}_t\right] \approx -1.4$ percent and $\mathbb{M}[\pi_t] \approx 1.5$ percent. Hence, the fit with respect to the inflation rate can be improved by raising the depreciation rate above its baseline level towards 10 percent. The drawback is that the mean growth rate of output falls in the process.

[723] This becomes clear by considering the elements of (4.4.16) in the neighbourhood of the stochastic steady state. First, using (4.5.4b) and (4.5.5) let us reformulate (4.4.13), (4.4.14), and $A(\psi_t) = \psi_t a^e + (1-\psi_t)a^m$ to

$$\begin{aligned}
\Lambda(\eta_t) &= \frac{1}{1-\varphi}\left(x_{1,t}^{e,p}(\eta_t) + x_{2,t}^{e,p}(\eta_t) - 1\right)\frac{\eta_t^e}{1-\theta(\eta_t)}, \\
\mathcal{V}(\eta_t) &= \frac{A(\psi_t(\eta_t))}{[q(\eta_t)+p(\eta_t)]\left[x_{2,t}^{e,p}(\eta_t)\,\eta_t + [1 - x_t^m(\eta_t)](1-\eta_t)\right]}, \quad \text{and} \\
A(\psi_t(\eta_t)) &= \psi_t(\eta_t)\,a^e + [1-\psi_t(\eta)]\,a^m.
\end{aligned}$$

It follows immediately from these expressions that neither the money supply, the velocity of money, nor the allocation of capital change deterministically when $\mu_t^\eta = 0$. Hence, we have $\mu_t^{\mathcal{P}} \to -\mu_t^K$ as $\eta_t \to \eta^s$. Then, the deflation rate is equal to the growth rate of the economy in the absence of shocks.

Table 5.3: Comparison Between Model-Implied and Empirical Moments of Output Growth and Inflation

| | $\mathbb{M}\left[\hat{Y}\right]$ | S.D. $\left[\hat{Y}\right]$ | $\mathbb{M}\left[\pi\right]$ | S.D. $\left[\pi\right]$ |
|---|---|---|---|---|
| Baseline | 0.79% | 0.91% | -0.71% | 3.48% |
| U.S. Data | 0.75% | 0.83% | 0.93% | 0.82% |

*Notes*: Moments are calculated from quarterly growth rates; model-implied moments are computed from time series generated by a Monte Carlo simulation of the baseline model. Each of the simulated series contains 9000000 realisations (360 days $\times$ 25000 years); U.S. data moments are calculated from the following time series: quarterly growth rate of real GDP: 1960Q1-2017Q3, source: BEA (2017a); quarterly growth rate of monetary aggregate M2: 1960Q1-2017Q3, source:FRB (2017b); quarterly inflation rate: 1960Q1-2017Q3, source: Organisation for Economic Co-operation and Development (2017).

In the determination of the standard deviation of inflation, parameter $\kappa$ assumes a role similar to the role $\delta$ plays in determining the mean of inflation: a higher value of the former leads to less volatility in $\pi_t$ but reduces mean growth. Another approach to reduce the volatility of inflation without directly affecting the growth rate of real GDP would be to dispense with the assumption of perfectly flexible prices. However, this approach is not pursued in this thesis.[724]

Further insight in the determinants of inflation is provided by Table C.3 , which summarises pairwise correlation coefficients between the growth rates of output, money, and the price level. As can be observed, the correlation between money growth and inflation is almost perfect. Moreover, a simple regression of $\pi_t$ on money growth $\hat{M}_t$ shows that changes in the latter explain about 99 percent of the variation in the former, indicating that money plays a significant role in the determination of inflation dynamics, as conjectured earlier. A possible explanation for the positive correlation between output growth and inflation is that changes in the money supply cause variations in real GDP rather than vice versa. Indeed, this rationale is supported by the model: a drop in the inflation rate induced by a contraction in the money supply leads to an appreciation in the real value of entrepreneurs' liabilities, which causes them to reduce their demand for capital. As explained, this reaction can deteriorate the allocation of capital and thereby output.

Let us finally discuss the relationship between expected (instantaneous) inflation $\left\{\mu_t^{\mathcal{P}} + \lambda\breve{\mathcal{P}}_t\right\} dt$ and the efficiency of the capital allocation $\psi_t - 1 \in [-1, 0]$, which might be interpreted as a measure of the output gap.[725] We know from our previous results that the so-defined output gap is large in

[724] The introduction of price inertia constitutes a promising research avenue since it would e.g. allow for analysing whether price inertia can dampen the adverse feedback loop introduced in the preceding section and thereby increase welfare.

[725] We could also adopt a more conventional measure such as the percentage deviation of current output from output

absolute terms when entrepreneurs own little equity relative to aggregate wealth. Expected inflation, on the other hand, is high when the state variable is low: the correlation coefficient between that variable and $\eta_t$ is equal to $-0.93$. Thus, inflation is high on average when the output gap is strongly negative. This is in stark contrast to the New Keynesian Phillips Curve (NKPC) without cost-push shocks, which implies a positive relation between those two variables, given expected future inflation. In the latter case, the output gap proxies for producers' real marginal costs, which are typically procyclical. Increases in real marginal costs, in turn, induce firms to raise prices.[726] The mechanism in our model is entirely different. If the output gap is highly negative, growth in debt and thus inside money is strong. Accordingly, the inflation rate is relatively large in such episodes.

### 5.3.7 Welfare Analysis

The model economy is characterised by inefficiencies that arise from the combination of financial frictions and multiple pecuniary externalities that are analogous to those in the I Theory of Money in large part. The first type of externality arises through variations in the price of capital. Any additional demand for capital raises the price of capital and therefore boosts the value of marked-to-market assets on other individuals' balance sheets. At the same time, a higher price of capital compresses risk premia, which deteriorates stochastic investment opportunities and lowers income growth on the side of capital owners ceteris paribus. On the upside, investment increases with the value of capital, resulting in accelerated growth.[727] If a shock materialises, the drop in the demand for capital leads to reverse effects and the "Paradox of Prudence" arises.

The second form of pecuniary externalities is due to changes in the value of money. In the absence of jumps, the economy grows, the money multiplier surges, and the value of money falls. This is anticipated by agents and therefore induces them to economise on real balances. However, as money can be utilised as a (partial) insurance against idiosyncratic risk associated with holding capital, the endogenous portfolio response can be harmful to welfare.[728] If a jump arrives, entrepreneurs reduce their outstanding debt but do not take into account that the appreciation of money raises the real value of liabilities within the entire entrepreneurial sector. A further inefficiency, which is not present in the I Theory of Money, results from the adoption of the MIU approach: in normal times without shocks, the expansion in the money multiplier causes real balances in agents' utility functions to shrink and thereby directly reduces welfare. In the opposite case, deflation has a positive direct impact on welfare.[729]

in the economy without frictions. Yet, we opt for the degree of capital misallocation since it does not depend on the aggregate capital stock, in contrast to the former example. Nevertheless, a traditional measure of the output gap would also be an increasing function of the state variable in our model.

[726] Cf. Walsh (2010, pp. 337f.).

[727] Cf. Brunnermeier and Sannikov (2016a, p. 27).

[728] Cf. Brunnermeier and Sannikov (2016a, p. 27).

[729] Similar externalities arise in more standard business cycle models that feature money as an argument of the utility function (cf. e.g. Farmer, 1997, p. 569).

To evaluate welfare and inefficiencies formally, welfare measures need to be derived. To this end, the welfare of a representative manager who owns the entire aggregate wealth of managers $n_t^m$ is considered at first.[730] From Lemma 8 we know that managers' value function is logarithmic in wealth. Applying that result to the case of the representative manager with net worth $n_t^m$ gives

$$V^m(n_t^m, \eta_t) = \alpha^m(\eta_t) + \frac{1+\xi}{\rho}\log n_t^m, \tag{5.3.2}$$

which yields the expected discounted lifetime utility of a manager that owns net worth $n_t^m$ at instant $t$, while the economy is at state $\eta_t$. In analogy to BS (2014d), the value function in (5.3.2) has two arguments $n_t^m$ and $\eta$.[731] The second argument is (in part) due to the fact that the stochastic investment opportunities, i.e. current and expected future asset returns and prices, captured by $\alpha^m(\cdot)$ change with state variable $\eta_t$.[732,733] Other factors that influence $\alpha^m(\cdot)$ are the flow utility derived from consumption and real balances, which also depends on $\eta_t$, and the discounting of future cash flows. Next, the aim is to solve for function $\alpha^m(\cdot)$. Proposition 5 provides a differential equation which allows us to achieve just that.[734]

**Proposition 5.** *Parameter function $\alpha^m(\cdot)$ in the value function of the representative manager who owns the entirety of managers' aggregate net worth satisfies the HJB equation*

$$\begin{aligned}\rho\alpha^m(\eta_t) = \alpha_{0,t}^m &+ \lambda\frac{1+\xi}{\rho}\left[\phi\log\left(\frac{\widetilde{\underline{n}}_t^m}{n_t^m}\right) + (1-\phi)\log\left(\frac{\widetilde{\overline{n}}_t^m}{n_t^m}\right)\right] \\ &+ \mu_t^\eta\eta_t\frac{d\alpha^m(\eta_t)}{d\eta_t} + \lambda\left[\alpha^m(\widetilde{\eta}_t) - \alpha^m(\eta_t)\right],\end{aligned} \tag{5.3.3a}$$

*where*

$$\alpha_{0,t}^m \equiv \log\frac{\rho}{1+\xi} + \xi\log(1-x_t^m) + \frac{1+\xi}{\rho}\mu_t^{r^{P,m}} - 1, \tag{5.3.3b}$$

$$\frac{\widetilde{\underline{n}}_t^m}{n_t^m} \equiv x_t^m(1-\kappa-\underline{\kappa}^m)\frac{\widetilde{q}_t}{q_t} + (1-x_t^m)(1-\kappa)\frac{\widetilde{p}_t}{p_t} + \widetilde{\Pi}_t^b - \Pi_t^b, \quad \textit{and} \tag{5.3.3c}$$

$$\frac{\widetilde{\overline{n}}_t^m}{n_t^m} \equiv x_t^m(1-\kappa-\overline{\kappa}^m)\frac{\widetilde{q}_t}{q_t} + (1-x_t^m)(1-\kappa)\frac{\widetilde{p}_t}{p_t} + \widetilde{\Pi}_t^b - \Pi_t^b. \tag{5.3.3d}$$

---

[730] It is possible to consider the so defined representative manager since individuals from that group only differ with respect to their wealth levels and as all policy functions are linear in wealth.

[731] Cf. Brunnermeier and Sannikov (2014d, p. 29).

[732] Cf. Brunnermeier and Sannikov (2015, p. 323).

[733] It should be noted that the dependence of the value function on $\eta_t$ was omitted in value function (4.3.8) for convenience. This is unproblematic since the optimal choices under logarithmic utility do not depend on $\alpha^m(\cdot)$. Conversely, when deriving welfare, the dependence of stochastic investment opportunities on the state variable cannot be neglected.

[734] The derivation of agents' HJB equations in this section is based on Brunnermeier and Sannikov (2014d, Section 4).

*Proof.* See Appendix C.1.1.

The representative prudent entrepreneur who owns the entirety of aggregate entrepreneurial wealth $n_t^e$ has value function

$$V^{e,p}(n_t^e, \eta_t) = \alpha^{e,p}(\eta_t) + \frac{1+\xi}{\rho} \log n_t^e. \tag{5.3.4}$$

**Proposition 6.** *Parameter function $\alpha^{e,p}(\cdot)$ in the value function of the representative prudent entrepreneur who owns the entirety of entrepreneurs' aggregate net worth satisfies the HJB equation*

$$\begin{aligned}\rho\alpha^{e,p}(\eta_t) = \alpha_{0,t}^{e,p} + \lambda\frac{1+\xi}{\rho}\left[\phi\log\left(\frac{\underset{\sim}{\widetilde{n}}_t^e}{n_t^e}\right) + (1-\phi)\log\left(\frac{\overset{\approx}{n}_t^e}{n_t^e}\right)\right] \\ + \mu_t^\eta\eta_t\frac{d\alpha^{e,p}(\eta_t)}{d\eta_t} + \lambda\left[(1-\phi^s)\alpha^{e,p}(\widetilde{\eta}_t) + \phi^s\alpha^m(\widetilde{\eta}_t) - \alpha^{e,p}(\eta_t)\right],\end{aligned} \tag{5.3.5a}$$

*where*

$$\alpha_{0,t}^{e,p} \equiv \log\frac{\rho}{1+\xi} + \xi\log x_{2,t}^{e,p} + \frac{1+\xi}{\rho}\mu_t^{r^{P,e,p}} - 1, \tag{5.3.5b}$$

$$\frac{\underset{\sim}{\widetilde{n}}_t^e}{n_t^e} \equiv x_{1,t}^{e,p}\left(1-\kappa-\underline{\kappa}^{e,p}\right)\frac{\widetilde{q}_t}{q_t} + \left(1-x_{1,t}^{e,p}\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t}, \quad \text{and} \tag{5.3.5c}$$

$$\frac{\overset{\approx}{n}_t^e}{n_t^e} \equiv x_{1,t}^{e,p}\left(1-\kappa-\overline{\kappa}^{e,p}\right)\frac{\widetilde{q}_t}{q_t} + \left(1-x_{1,t}^{e,p}\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t}. \tag{5.3.5d}$$

*Proof.* See Appendix C.1.2.

One notable difference of (5.3.5a) compared to (5.3.3a) is that the former includes term $(1-\phi^s)$ $\alpha^{e,p}(\widetilde{\eta}_t) + \phi^s\alpha^m(\widetilde{\eta}_t)$, rather than just $\alpha^{e,p}(\widetilde{\eta}_t)$. This results from prudent entrepreneurs' possible exposure to the sector-wide shock: if any individual suffers this type of disturbance, his productivity level is immediately reduced to $a^m$. In addition, he no longer has access to external finance from banks. Both effects reduce his lifetime utility. This effect is reflected in equilibrium by the result that $\alpha^{e,p}(\cdot) > \alpha^m(\cdot), \forall\eta_t$.

Functions $\alpha^m(\cdot)$ and $\alpha^{e,p}(\cdot)$ both are solved for via value function iteration.[735] In value function iteration, an initial guess for the Value function $V_0$ is mapped into an updated guess via the Bellman or HJB equation. The updated guess is then fed into the Bellman equation to arrive at a further updated guess. This procedure is repeated until convergence.[736] A difference to usual value function

[735] Brunnermeier and Sannikov (2014d) do not provide a numerical method to solve a HJB equation of the type considered here. Brunnermeier and Sannikov (cf. 2015, Appendix B) are able to analytically pin down a boundary condition and solve for the HJB via a finite-difference method. In the model presented here, it is not possible to derive a boundary condition analytically. Arbitrarily chosen initial or boundary conditions do not produce reliable results. For these reasons value function iteration is chosen as a means to solve for the value function.

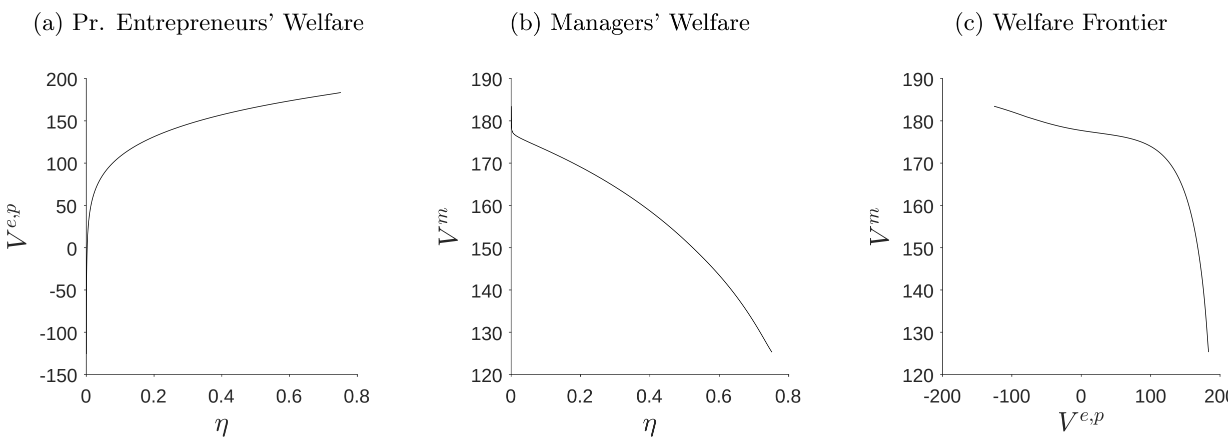
[736] Cf. e.g. Ljungqvist and Sargent (2004, p. 88).

iteration exercises is that one iterates over the HJB equations in order to find parameter functions $\alpha^m(\cdot)$ and $\alpha^{e,p}(\cdot)$ rather than value functions $V^m(\cdot)$ and $V^{e,p}(\cdot)$ themselves.[737] Once the two functions are known, state-dependent welfare levels can be computed by attributing a value to the initial capital stock. Following BS (2014d) this value is set to unity: $K_0 = 1$.[738] Then, the pair of value functions becomes

$$V^m(\eta_t) = \alpha^m(\eta_t) + \frac{1+\xi}{\rho}\log\left((1-\eta)(q_t+p_t)\right), \tag{5.3.6}$$

$$V^{e,p}(\eta_t) = \alpha^{e,p}(\eta_t) + \frac{1+\xi}{\rho}\log\left(\eta(q_t+p_t)\right). \tag{5.3.7}$$

Figure 5.3.7: Welfare in the Baseline Credit Model




*Notes*: Panel (c) plots the state-dependent value function of the representative manager against the state-dependent value function of the representative prudent entrepreneur.

Panels (a) and (b) in Figure 5.3.7 depict the two value functions for different values of $\eta_t$.[739] Welfare of the representative prudent entrepreneur is monotonously increasing in $\eta_t$, even though his stochastic investment opportunities fall with that variable. Hence, the reduction in $\alpha^{e,p}(\cdot)$ is overcompensated by the increase in the second term of (5.3.7) at any point in the considered range of the state space. The fact that the value function can take on negative values is due to the

[737] Numerical testing shows that this method leads to convergence in the state-dependent value function in all examined cases.

[738] Cf. Brunnermeier and Sannikov (2014d, p. 30).

[739] In addition to prudent entrepreneurs and managers, the economy is also populated by imprudent entrepreneurs and bankers. Bankers' expected utility is always equal to zero since the real wage they receive offsets the expected utility losses from verification. Imprudent entrepreneurs' utility, on the contrary, varies with the state variable. In particular, their utility is high when $\eta_t$ is low since the risk premium on capital is large in these situations. Nevertheless, in the following, the focus lies on the welfare of prudent entrepreneurs and managers, exclusively. This can be justified by acknowledging that imprudent individuals only form a small fraction of the entrepreneurial population. In addition, these agents represent a "nuisance" in the economy: they are pushed into bankruptcy if a jump arrives since they do not save precautionarily, a fact that increases the financing costs of all other entrepreneurs.

value function being logarithmic in wealth.[740] Likewise, managers' value function is monotonous in $\eta_t$. Yet, this function is decreasing in the entire range, which is mainly due to the declining wealth share of the representative manager. Panel (c) presents the *welfare frontier*, i.e. the function pair $(V^m(\eta_t), V^{e,p}(\eta_t))$ for any value in the interval $(0, \eta^s]$. As the state variable is increased, one moves along the welfare frontier towards the south-east direction.[741] The fact that the welfare frontier is monotonously decreasing implies that joint welfare improvements via forced lumpsum redistributions of wealth by a social planner are infeasible.

## 5.4 Parameter Variations

### 5.4.1 Conditional Probability of the Sector-Specific Productivity Shock

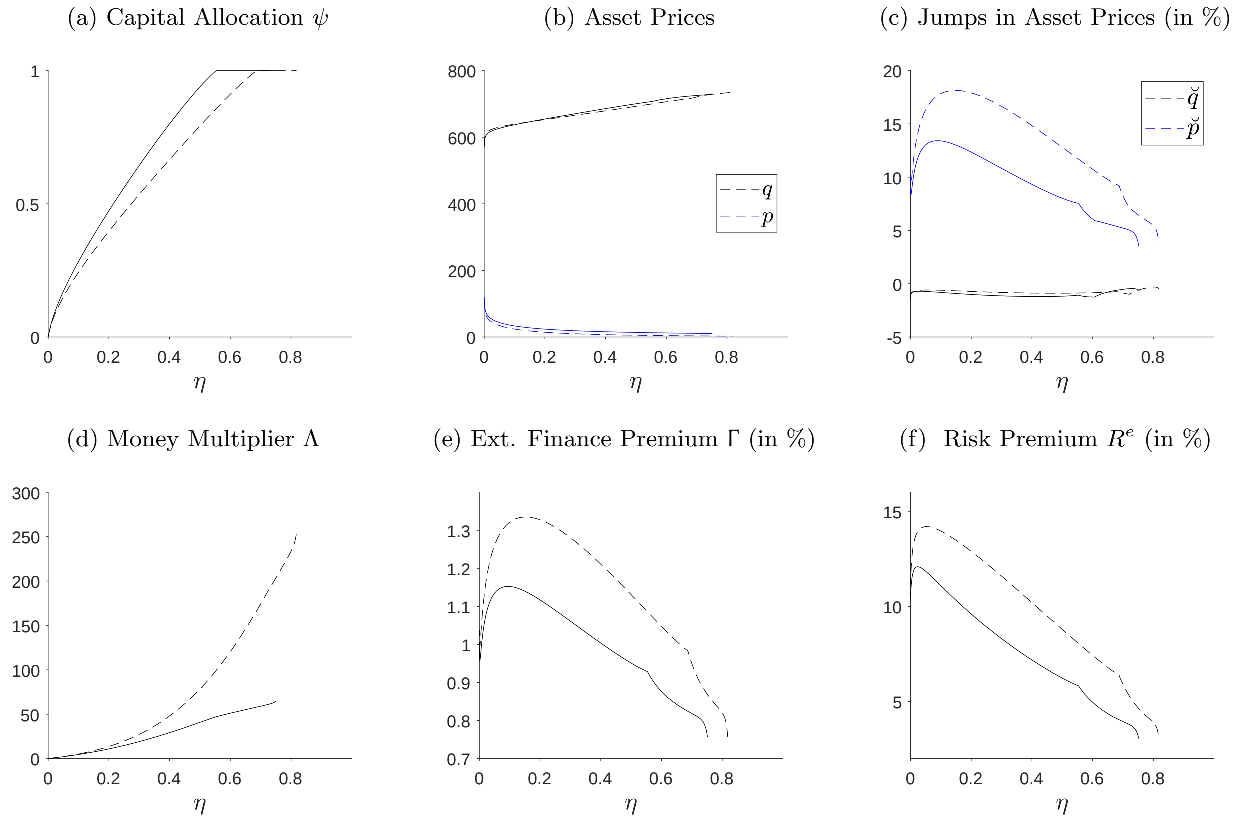
A reduction in the fraction of entrepreneurs exposed to the sector-wide productivity shock from $\phi^s = 5$ to 2.5 percent lowers the value of money per unit of capital $p_t$ throughout the entire examined subset of the state space compared to the baseline calibration (Panel (b) in Figure 5.4.1). This outcome is caused by an expansion in the money multiplier (Panel (d)). This raises the question of what is the underlying reason for the increased nominal volume of debt. Compared to the initial equilibrium with $\phi^s = 5$ percent, each individual borrower takes out more credit since the exogenous source of variation in the percentage drop of the state variable in case of a shock declines (cf. equation (4.5.7c)). More specifically, the drop in demand for capital after a jump is less accentuated since fewer entrepreneurs experience productivity shocks. It follows that the percentage drop in the value of capital is reduced (Panel (c)). Accordingly, entrepreneurs anticipate diminishing losses due to revaluations of their capital stocks, which pushes up their demand for external finance.

An upshot of that behaviour is that the drop in the money multiplier conditional on the economy being exposed to a jump, which results from entrepreneurs' desire to deleverage, is much higher. Thus, the appreciation in the value of money surges and peaks at a value close to 20 percent, compared to the maximum value of about 13 percent under the baseline calibration (Panel (c)). This is a version of the Volatility Paradox identified by BS (2014a): due to the endogenous asset choice of individuals, lower exogenous risk is associated with larger responses of the price of money to exogenous shocks. As a consequence of heightened endogenous asset price risk, the EFP widens and entrepreneurs require higher premia for taking on additional capital (Panels (e) and (f)).

The higher amplitude of endogenous risk directly tilts prudent entrepreneurs' portfolio choice towards a lower weight on capital via FOC (4.3.20) and indirectly through its impact on the external finance premium. It might seem counterintuitive that the portfolio weight on capital falls when adverse shocks to capital productivity become less likely. Yet, portfolio choice under logarithmic

[740] Cf. Brunnermeier and Sannikov (2015, p. 324).
[741] Cf. Brunnermeier and Sannikov (2015, p. 325).

Figure 5.4.1: Equilibrium Quantities and Prices for $\phi^s = 5$ Percent (solid) and $\phi^s = 2.5$ Percent (dashed)




utility implies that the optimal portfolio weight does not directly depend on the agents' future investment opportunities.[742] Under more general preferences, this result does not hold and the portfolio weight on capital might rise. The decrease in the value of money relative to the value of capital has repercussions for the allocation of capital: according to (4.5.6d) entrepreneurs' share in the aggregate capital stock $\psi_t$ depends negatively on the value share of capital $\theta_t$. The negative effect on $\psi_t$ is reinforced by the decrease in portfolio weight $x^{e,p}_{1,t}$. In effect, the curve in Panel (a) under $\phi^s = 2.5$ percent rotates downwards relative to the baseline calibration.

Turning to the dynamics of the state variable, we see from Panel (a) in Figure 5.4.2 that the drift rate of $\eta_t$ is virtually unchanged. This suggests that the effects of the increase in entrepreneurs' risk premia and the decrease in the portion of wealth these individuals allocate to capital offset each other. Further, the percentage drop of the state variable is below its original level owing to the lower exogenous component $\phi^s$. These observations explain why the stationary distribution of $\eta_t$ expands and shifts to the right (Panel (b) ). It follows that states of extreme crisis, in which entrepreneurs'

[742] This is reflected in prudent entrepreneurs' optimal rule for the portfolio weight on capital (4.3.20) by the absence of parameter $\phi^s$.

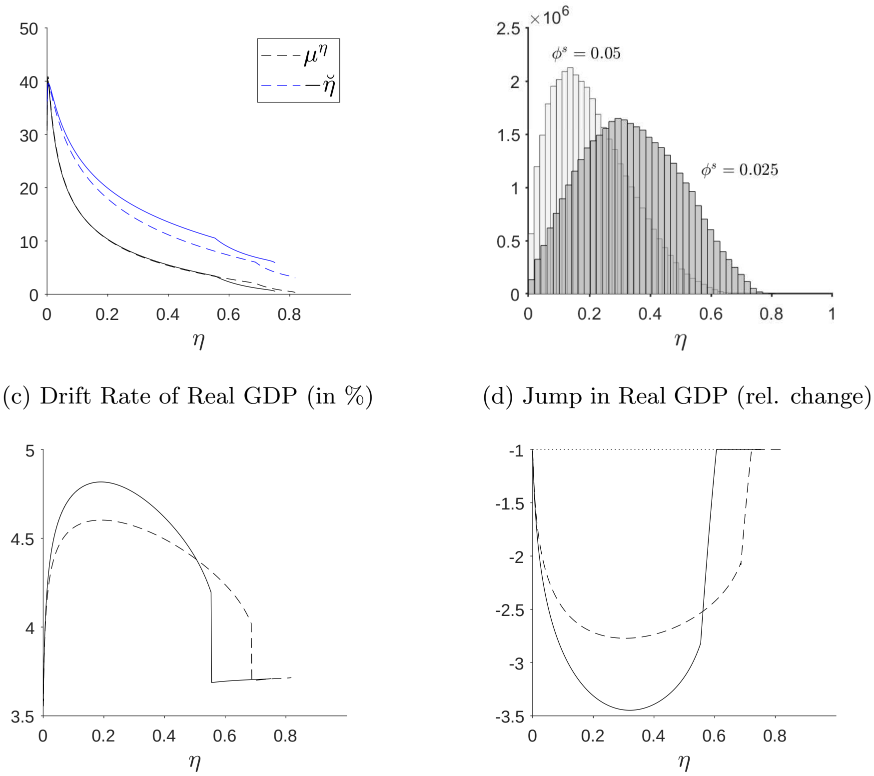
Figure 5.4.2: Dynamics of the State Variable and Output for $\phi^s = 5$ Percent (solid) and $\phi^s = 2.5$ Percent (dashed)

(a) Drift Rate and Drop of $\eta$ (in %)

(b) Stationary Distribution of $\eta$

(c) Drift Rate of Real GDP (in %)

(d) Jump in Real GDP (rel. change)



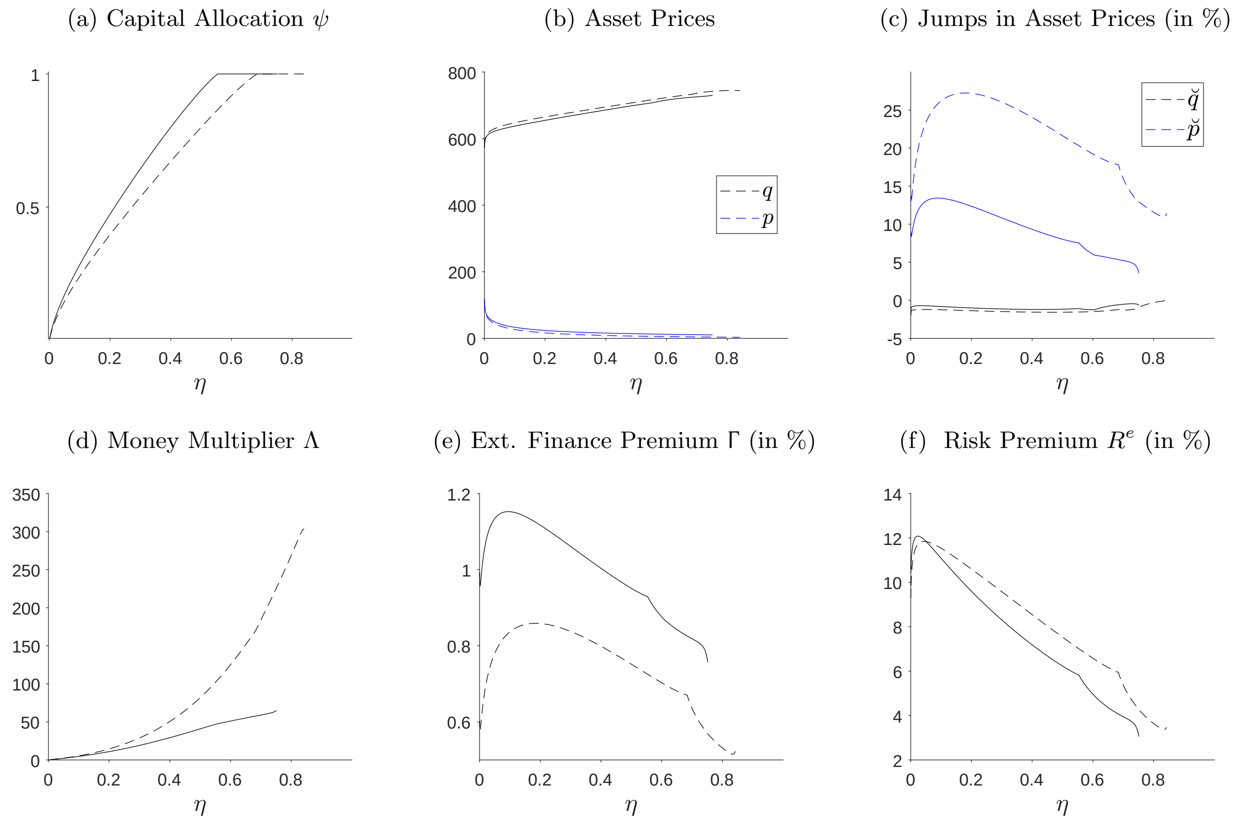

*Notes*: The stationary distribution in Panel (b) is computed from a simulated time series of $\eta$ (cf. notes to Table 5.3.4 for details).

balance sheets are highly impaired, become less likely. Further, more probability mass is assigned to states in which the most productive agents are well-capitalised. This explains why the capital allocation improves *on average* (from $\mathbb{M}[\psi_t] = 0.49$ to 0.59) despite the fact that function $\psi(\cdot)$ rotates downwards. As implied by Panels (c) and (d), the volatility of the growth rate of real GDP is below its level under the baseline parameter constellation. In periods without shocks, the drift rate of $Y_t$ declines since the speed of capital reallocation slows down. The percentage change $\breve{Y}_t$, on the other hand, is lower than before in absolute terms as fire sales of capital in case of a jump are reduced. In contrast, the average growth rate of output is virtually unchanged.

### 5.4.2 Frequency of Jumps

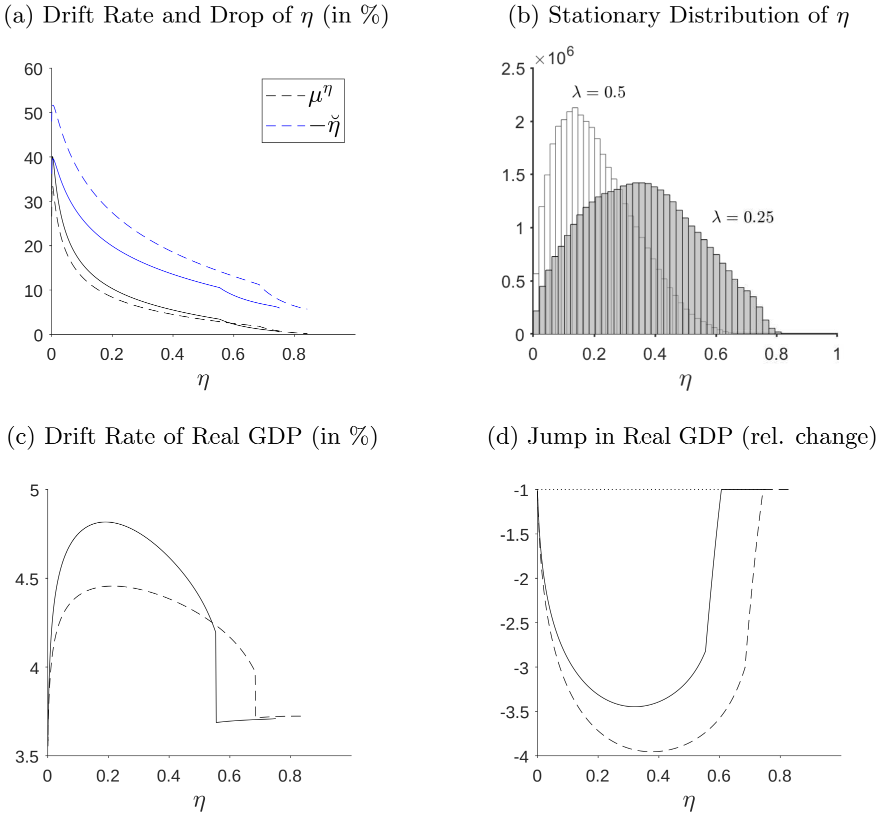
In this subsection, we investigate the effects of reductions in risk parameters further by lowering the intensity $\lambda$ of Poisson process $\mathcal{N}_t$ from 0.5 to 0.25. It follows that the economy is now exposed to a jump every four years on average. Figures 5.4.3 and 5.4.4 present the results. The reduced probability of jumps makes adverse shocks to agents' capital holdings less likely. In effect, the demand for the productive asset by entrepreneurs as well as managers rises. Therefore, the price of capital does as well. This implies that each entrepreneur has to borrow more money to finance the purchase of an additional unit of capital for a given level of the state variable. The resulting expansion in the stock of inside money mitigates the value of money and thus reinforces the aforementioned effect. The upshot of these adjustments is that for fixed $\eta_t$, entrepreneurs hold a lower share of the aggregate capital stock in the misallocation region.

Figure 5.4.3: Equilibrium Quantities and Prices for $\lambda = 0.5$ (solid) and $\lambda = 0.25$ (dashed)




While shocks arrive less frequently, their effects are more pronounced - both the liquidity and the deflationary spiral are more potent relative to the baseline calibration, as can be observed from Panel (c) in Figure 5.4.3. The reason is that the faster expansion in debt in normal times bites back in periods with jumps. Then, entrepreneurs pay off more loans to reduce their exposure to risk embodied in capital. The induced asset price variations have the familiar effects on entrepreneurs' balance sheets. The EFP declines throughout due to the fall in the (exogenous) default probability $\lambda\varphi\, dt$. Yet, the EFP is less than halved since asset prices respond more to jumps, which raises banks' losses in case of loan write-offs. The higher amplitude of asset price adjustments also causes the risk premium to rise except for states of extreme crisis.

Figure 5.4.4: Dynamics of the State Variable and Output for $\lambda = 0.5$ (solid) and $\lambda = 0.25$ (dashed)




*Notes*: The stationary distribution in Panel (b) is computed from a simulated time series of $\eta$ (cf. notes to Table 5.3.4 for details).

Moreover, the drift rate of the state variable shrinks since entrepreneurs own a lower fraction of the aggregate capital stock at each $\eta_t$. At the same time, the percentage drop in $\eta_t$ increases. Despite these facts, the distribution of entrepreneurs' wealth share widens and shifts to the right. Of course, this is a consequence of the lower probability of shocks. Since states in which entrepreneurs are well-capitalised are more likely, the average capital allocation improves (by about 20 percent) and the mean growth rate of output does as well (by 10 percent).

The bottom line of this subsection is that mild changes in shock parameters can have nontrivial effects on asset prices, returns, and allocations in our multi-sector economy. In contrast, such changes were shown to have minor repercussions in the autarky and first-best models. The reason is that in the two latter cases, endogenous risk and leverage are nonexistent. In the multi-sector economy, even small changes in shock parameters can lead to large swings in endogenous risk, which is concentrated on leveraged agents' balance sheets.

### 5.4.3 Productivity Levels

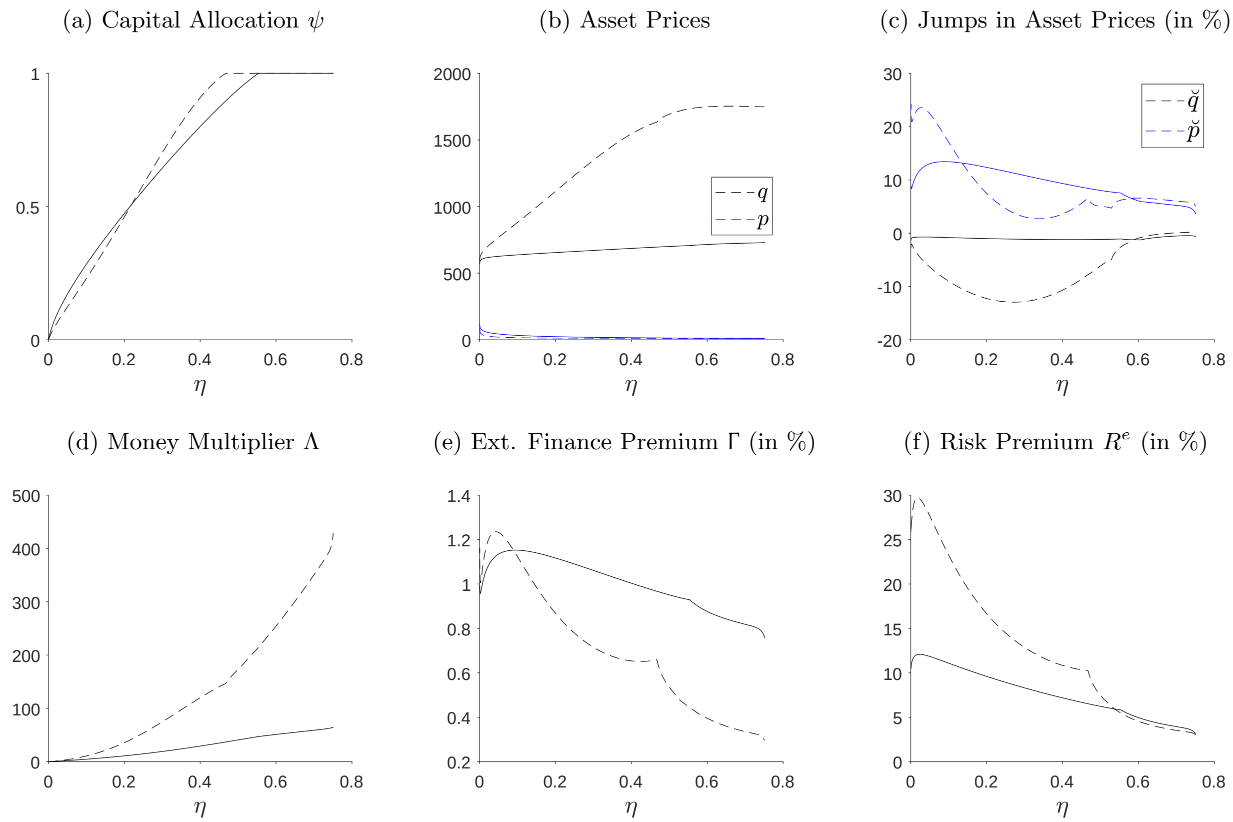
In Section 5.1, we provided a brief discussion of the disparate findings in the empirical literature on the degree of productivity dispersion across U.S. firms. We now widen the productivity gap by setting $a^e = 64$, while keeping $a^m$ fixed. We interpret the new ratio $a^e/a^m = 3.2$ to correspond to the ratio of the third quartile to the first quartile of the productivity distribution in the U.S. Census of Manufacturing from 1997 calculated by Hsieh and Klenow (2009). Figures 5.4.5 and 5.4.6 present the resulting equilibrium functions.

The productivity boost in the entrepreneurial sector strongly pushes up the price of capital. Meanwhile, the money multiplier expands since entrepreneurs take out more loans. In effect, the value of money plummets. Ironically, for low values of the state variable, the increase in the price of capital is driven by a portfolio switch on the side of managers: anticipating the accelerated decline in the value of money, they are less willing to sell capital to entrepreneurs in exchange for liquidity (Panel (a) in Figure 5.4.5). As $\eta_t$ rises, however, the depreciation in the value of money becomes less severe and thus managers are more inclined to trade capital against money. Then, the appreciation in the value of capital is mainly driven by entrepreneurs' desire to hold more capital. The intuition is a follows. Due to the widening of the productivity differential $a^e - a^m$, the reallocation of capital, which goes hand in hand with the improvement in entrepreneurial balance sheets, leads to an accelerated growth rate of real GDP in the absence of shocks. Yet, that effect is comparatively mild for low $\eta_t$. For higher values of the state variable, the *absolute* change in $\eta_t$ rises, and, with it, the drift of $\psi_t$, which affects $\mu_t^Y$ according to equation (4.4.7), until all peak at $\eta_t$ close to 0.3. Around this point, the drift rate of output is as high as approximately 15 percent. The changes in $\mu_t^Y$, in turn, counteract the inflationary pressure due to the expansion in the money multiplier.

Turning to the behaviour of the economy after the arrival of a jump, we observe significant changes in equilibrium functions relative to the baseline calibration from both figures. The liquidity spiral, which depresses the capital price, now plays a much more vital role in endogenous risk dynamics relative to the deflationary spiral, except for states in which entrepreneurs' balance sheets are highly impaired. In those states, further shocks induce individuals of type $e$ to deleverage at a high rate, leading to marked swings in the value of money. The percentage drop in the capital price is much more pronounced in the misallocation region where $\psi_t < 1$ relative to the baseline scenario. In particular, this holds true for $\eta_t \approx 0.3$, where the change in capital allocation is at its maximum. Accordingly, once a jump materialises, fire sales of the productive asset to managers are high. The changes in the capital allocation against the backdrop of an expanding productivity gap also explain the shape of function $\check{Y}(\eta_t)$ in Panel (d) of Figure 5.4.6. Again, the flip side of precipitated growth in output in times without innovations in the stochastic processes is the elevated percentage drop in real GDP after a jump, which is mainly driven by a decline in aggregate TFP.

Moreover, the sharp reduction in output after shocks causes the deflationary pressure to become

Figure 5.4.5: Equilibrium Quantities and Prices for $a^e = 27.6$ (solid) and $a^e = 64$ (dashed)

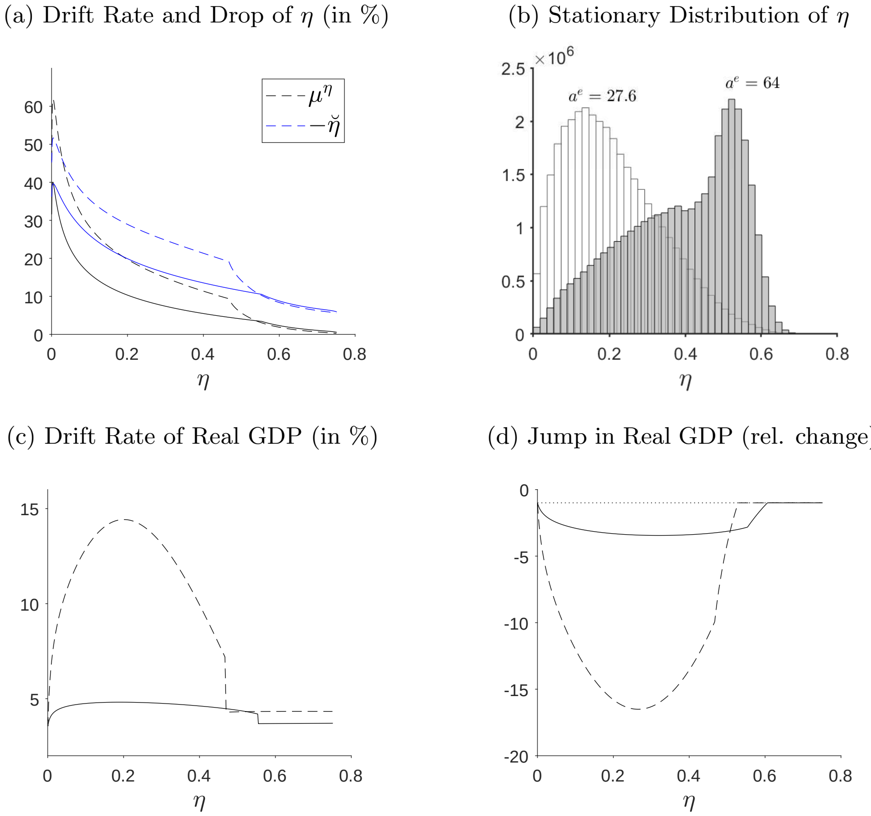



less severe in the middle region of the state space. Nevertheless, endogenous risk as a whole is much stronger throughout the misallocation region, as exemplified by the expanding risk premium on capital. Our results are in line with findings in BS (2014a), who argue that the productivity differential between more and less productive agents is a measure of the market illiquidity of capital.[743] They show that higher market illiquidity raises volatility and risk premia distinctly in misallocation states.[744]

Volatility spikes in our model also show up in other measures such as the percentage drop in the state variable, which is at a higher level for the most part. Conversely, heightened risk premia accelerate the growth of $\eta_t$ in normal times. The stationary distribution of entrepreneurs' wealth share shifts to the right. In particular, less probability mass is assigned to depressed states and the fraction of time the economy spends in the optimal capital allocation region surges from two to about 40 percent. Further, and as suggested by Figure 5.4.6, the standard deviation of the growth rate of output nearly quadruples from 0.91 to 3.43 percent per quarter, a figure which is clearly counterfactual. On the upside, mean growth rises from 0.79 to 0.99 percent per quarter.

[743] Cf. Brunnermeier and Sannikov (2014a, p. 408).
[744] Cf. Brunnermeier and Sannikov (2014a, p. 408).

Figure 5.4.6: Dynamics of the State Variable and Output for $a^e = 27.6$ (solid) and $a^e = 64$ (dashed)




*Notes*: The stationary distribution in Panel (b) is computed from a simulated time series of $\eta$ (cf. notes to Table 5.3.4 for details).

### 5.4.4 Disutility of Verification

Finally, we reduce parameter $\omega$, which governs the disutility of verification in bankers' utility functions, from value 0.15 to 0.05. According to equation (4.3.34), a reduction in that parameter decreases the EFP, ceteris paribus. Figure 5.4.7 confirms that the EFP falls indeed. The improved terms of borrowing induce entrepreneurs to take out additional credit, which is reflected in an expansion of the money multiplier. Despite the increase in the supply of inside money, the value of money is virtually unchanged. This is because the decline in financing costs causes entrepreneurs to demand more real balances. Another part of the additional loans is used to finance capital purchases. Accordingly, the value of capital rises.

While debt grows faster in normal times compared to the baseline scenario, the reduction in outstanding loans in case of a jump is more severe, which is a result of higher leverage ratios in the entrepreneurial sector. In effect, the ensuing deflationary pressure after the arrival of a jump is more pronounced. The boost in the value of entrepreneurs' liabilities leads to more fire sales of the productive asset to managers relative to the case with $\omega = 0.15$ and, accordingly, to larger drops in the value of capital. The rise in endogenous risk also shows up in entrepreneurs' risk premium, which widens in the region in which managers hold capital. Further, heightened endogenous risk in part counteracts the decrease in the EFP due to the reduction in parameter $\omega$.

Figure 5.4.7: Equilibrium Quantities and Prices for $\omega = 0.15$ (solid) and $\omega = 0.05$ (dashed)

(a) Capital Allocation $\psi$ (b) Asset Prices (c) Jumps in Asset Prices (in %)

(d) Money Multiplier $\Lambda$ (e) Ext. Finance Premium $\Gamma$ (in %) (f) Risk Premium $R^e$ (in %)

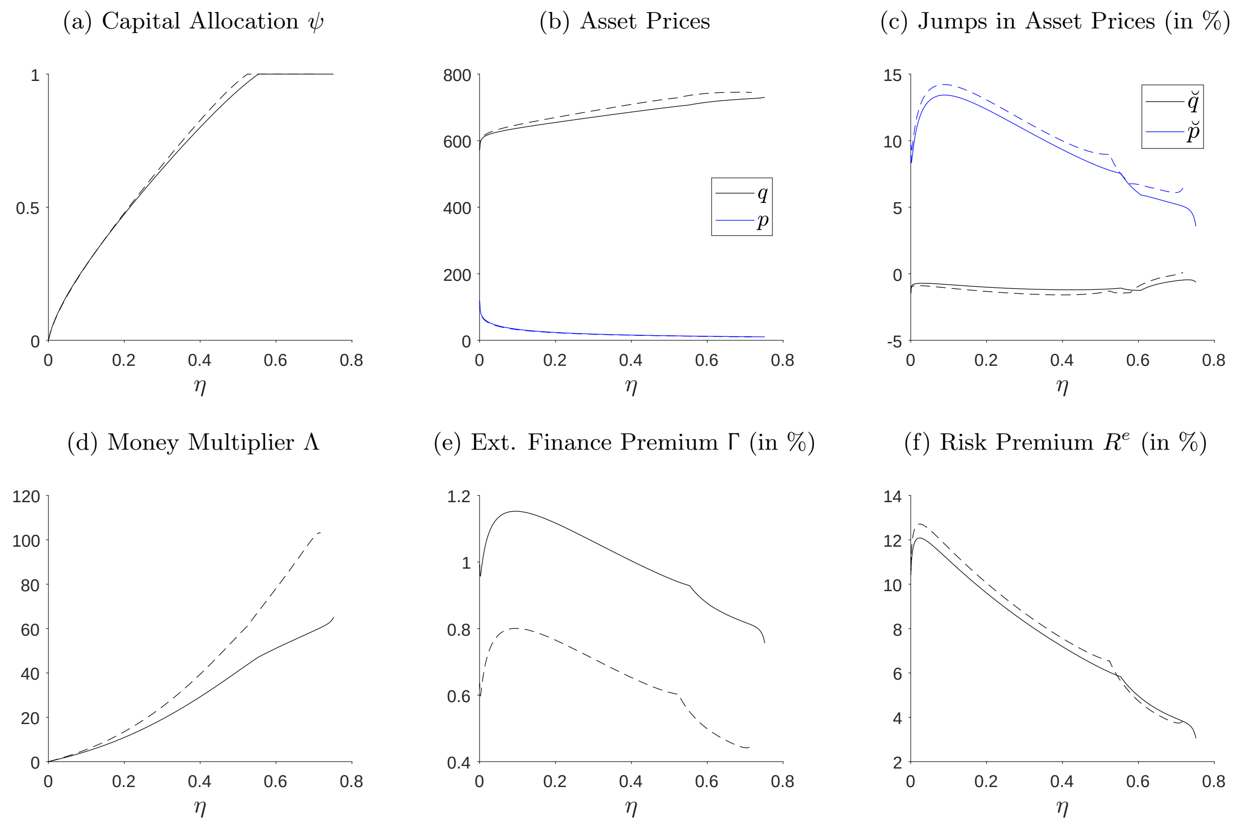
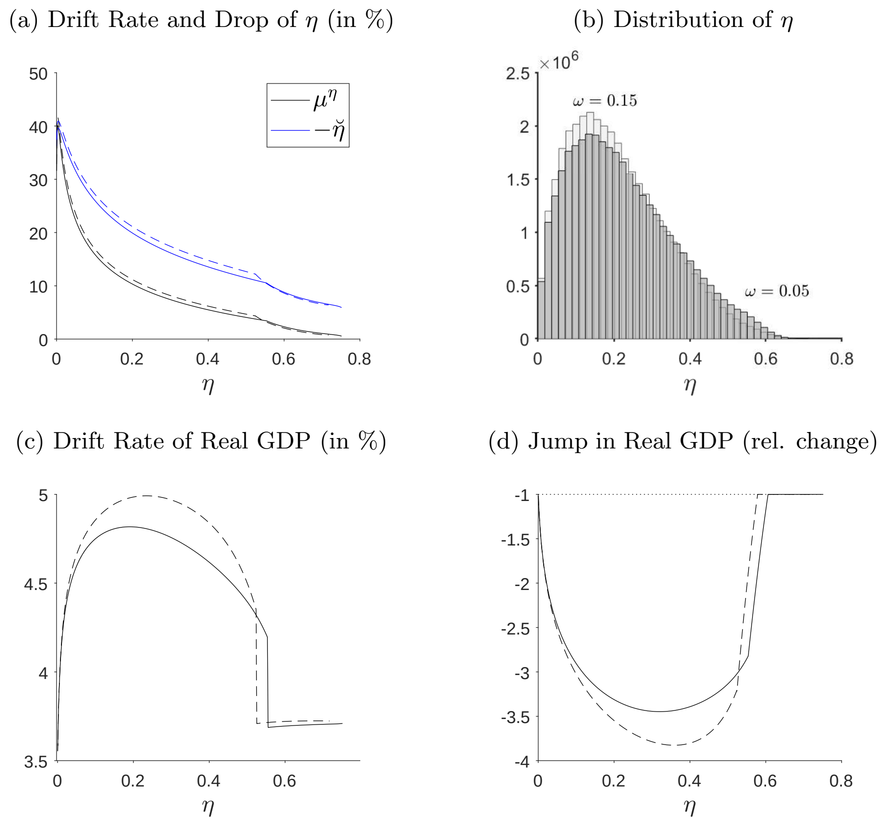



Figure 5.4.8 demonstrates that the drift rate of the state variable is raised compared to the baseline calibration for the most part of the domain. The accelerated growth in $\eta_t$ in normal times is due to larger profits of the entrepreneurial relative to the managerial sector. This results from the enlarged risk premium on entrepreneurs' capital over money and lower financing costs. However, the system becomes more unstable due to the surge in endogenous risk, as demonstrated by the increase in the percentage drop of the state variable due to a jump. Again, this outcome can be traced back to heightened endogenous risk. These observations are mirrored in the stationary distribution of $\eta_t$, which is now slightly more spread. Moreover, the mean of the entrepreneurs' wealth share increases since entrepreneurs earn higher relative profits.

We can also observe from Figure 5.4.8 the adjustments of the dynamics of real GDP. In periods without shocks, the growth rate of output is boosted since the capital allocation improves at a faster rate, which is fuelled by the growth in debt. Conversely, since entrepreneurs suffer greater losses if shocks materialise, they sell more capital to less efficient managers in such episodes. The changes in $\mu_t^Y$ and $\breve{Y}_t$ offset each other in the calculation of the average growth rate in real GDP, which is still at 0.79 percent per quarter. Thus, the decline in the disutility of auditing implies a mean-preserving spread in that rate. In light of the strong decline in the EFP induced by lower agency costs, the overall differences to the baseline scenario are fairly small. This suggests that credit market dynamics are mainly driven by demand factors such as the health of borrowers' balance sheets.

Figure 5.4.8: Dynamics of the State Variable and Output for $\omega = 0.15$ (solid) and $\omega = 0.05$ (dashed)




*Notes*: The stationary distribution in Panel (b) is computed from a simulated time series of $\eta$ (cf. notes to Table 5.3.4 for details).

With these results at hand, we can now compare our model predictions with some real-world crisis episodes. The adverse feedback loop generated by our model in periods with adverse shocks resembles many developments that occurred during the deflationary period in Japan that began in the 1990ies. Traditionally, Japan has been a bank- rather than market-oriented country in terms of corporate financing.[745] Rajan and Zingales (1995) document that a large share of Japanese firms' financing was due to the issuance of debt in the years from 1984 to 1991.[746] Koo (2009) argues that the situation in the mid 1990ies was characterised by weak corporate balance sheets

[745] Cf. Antoniou et al. (2008, p. 61).
[746] Cf. Rajan and Zingales (1995, p. 1439).

caused by plunging asset prices and high debt burdens. Nonfinancial corporation responded by paying down debt despite the fact that interest rates were plummeting.[747] According to the author, debt repayment, in turn, lead to a collapse of the money multiplier.[748] Koo (2009) also reasons that voluntary corporate deleveraging contributed to the Great Depression.[749] In light of recent empirical evidence, the credit model appears to be less suited to account for the financial crisis of 2007/08. For instance, Adrian et al. (cf. 2013, p. 194) conclude that the supply rather than the demand side was the driver of contractions in intermediated credit in the United States during that crisis episode.

## 5.5 Further Relations to the Literature

### 5.5.1 The CTMF Literature

Table 5.4 highlights the crucial differences and similarities between the credit model and the three models that are most closely related to the former. The assumptions on the production side in BS (2014a) resemble those in our model in two regards. First, both models are inhabited by two agent groups who have recourse to production functions with sector-specific productivity levels. Second, the productivity level of one group exceeds that of the other. Moreover, in both settings, productive agents are externally financed by debt.

The credit model introduces an additional layer between borrowers (entrepreneurs) and ultimate lenders (managers), namely the banking sector. Further, the usage of credit as a financing instrument is motivated by the adoption of the CSV paradigm here, whereas BS (2014a) adopt a skin in the game constraint that arises from moral hazard considerations. In an extension of their model, the authors add idiosyncratic jump risk to their framework, which may push experts into bankruptcy. Further, they introduce a CSV problem between experts and households that, in effect, drives a wedge between the lending and the risk-free rate. However, this premium on external finance is exogenously specified in contrast to our model.[750] Nevertheless, their model variant generates predictions that are similar to those discussed in Subsection 5.4.4. In particular, they find that higher costs of verification lead to more stable system dynamics, as expressed through reductions in the volatility of the state variable. The driver of this outcome is the endogenous portfolio response of levered experts, who cut their leverage ratios. In the process, endogenous risk drops and asset prices are depressed, which leads to lower investment.[751,752]Another distinction to the credit model is that BS (2014a) does not feature money. Thus, in their baseline model, experts offer households

[747] Cf. Koo (cf. 2009, pp. 11ff.)
[748] Cf. Koo (cf. 2009, pp. 30f.)
[749] Cf. Koo (cf. 2009, pp. 100f.)
[750] Cf. Brunnermeier and Sannikov (2014a, pp. 409f.).
[751] Cf. Brunnermeier and Sannikov (2014a, pp. 410f.).
[752] Related, Phelan (cf. 2016, pp. 217ff.) shows that tighter exogenous leverage constraints enhance stability.

Table 5.4: Comparison of Key Assumptions in Four CTMF Models

| | Credit model | BS (2014a) | BS (2014d) | BS (2016a) |
|---|---|---|---|---|
| Agents | Entrepreneurs (prudent/imprudent), managers, bankers | Experts, households | Entrepreneurs, households, intermediaries | Households (sectors $a/b$), intermediaries |
| Assets | Capital, money, credit | Capital, risk-free asset | Capital, money, bonds | Capital, money, bonds |
| Production technology | AK with two sector-specific productivity levels | AK with two sector-specific fixed productivity levels | AK with homogeneous productivity levels | Aggregate CES technology with two intermediate goods inputs |
| Shocks | Aggregate and idiosyncratic shocks to capital, sector-specific productivity shock | Aggregate shock to capital | Sector-specific shock to capital | Idiosyncratic and aggregate shocks to capital |
| Type of stoch. processes | Poisson | Brownian motion | Poisson | Brownian motion |
| State variable | Entrepreneurs' wealth share | Experts' wealth share | Intermediaries' wealth share | Intermediaries' wealth share |

a return on debt that is fixed in nominal as well as real terms. Here, the ultimate lenders, i.e. managers, receive a return on deposits that is predetermined only in nominal terms.

The two main differences of our framework compared to the two versions of the I Theory of Money, namely BS (2014d) and (2016a), are as follows. First, banks finance production units by means of credit rather than purchases of equity issued by the latter. Second, instead of banks'/intermediaries' wealth share, the wealth share of entrepreneurs acts as a state variable here. This property ensues from the assumptions on financial structure laid out in Subsection 4.1.4. More specifically, entrepreneurs assume the role of financially constrained agents - they can only obtain external funds by taking out credit. This type of financial contract concentrates the lion's share of endogenous risk on the balance sheets of borrowers since a drop in the price of capital decreases the real value of their assets and a surge in the value of money increases the real value of their liabilities. Hence, the arrival of a bad shock leads to large losses that are absorbed by previously accumulated equity. Banks, on the other hand, do not maintain net worth buffers since they can always raise additional outside equity from bank owners to cover losses.[753] In addition, only a small share of their balance sheets, namely the share of defaulting debtors in the population of all borrowers $\varphi$, is exposed to the reverse movement of assets and liabilities induced by asset price changes in case of a jump (cf. Figure 4.2.1). This motivates the assumptions on financial structure which imply that the relative net worth position of the entrepreneurial sector acts as a state variable.[754]

In contrast, in both versions of the I Theory of Money, intermediaries can obtain external funding only by issuing deposits.[755] The absence of risk-sharing opportunities requires these institutions to accumulate inside equity in order to absorb adverse shocks. The assumptions in those two models imply that intermediaries engage in liquidity and maturity transformation - they invest in long-term, illiquid assets, namely capital held by production units, and issue short-term liquid deposits.[756] Those two functions do not arise on the side of banks in the credit model since credit is a short-term asset. What remains are banks' capacities in the transformation of risk. This is facilitated by their abilities to diversify their loan portfolios and to audit debtors. Furthermore, in BS (2014d), entrepreneurs' net worth does not play a role in the determination of equilibrium dynamics since these agents are always entirely financed by outside equity. In BS (2016a), the distribution of wealth between households active in technology $a$ and those active in technology $b$ does not matter for aggregate outcomes. This is because households that employ a particular technology have the option to costlessly switch to the respective other technology.

It follows from our assumptions on financial structure that the reduction in the inside money supply in case of adverse shocks is primarily the consequence of a drop in the *demand* for external finance, rather than a decrease in its supply induced by distressed balance sheets of financial institutions, as in the I Theory of Money. The underlying reason is that entrepreneurs are financially constrained - losses are absorbed by entrepreneurs' previously accumulated *inside* equity. Banks, on the other hand, act in a risk neutral way and are always able to raise additional outside equity from bank owners. This ensures that banks are willing to supply arbitrary amounts of credit at the current loan rate (cf. Section 4.3.4).

Moreover, the sector-specific productivity shock, rather than shocks to agents' capital stocks, causes the adverse feedback loop. This can be explained by recognising that the model does not feature sector-specific, but only idiosyncratic and economy-wide shocks to the productive asset, which do not affect the relative equity position of entrepreneurs.[757] Even though shocks to agents' capital stocks do not directly contribute to the feedback loop, they are still important in determining equilibrium outcomes since they influence required risk-premia and, thereby, the portfolio adjustments

[753] In parallel, Klimenko et al. (2016) and Phelan (2016) assume that intermediaries are owned by households. Yet, in their models, banks face costs of equity issuance.

[754] It would likely be fruitful to study a model variation in which banks are not able to raise additional outside equity frictionlessly. In such a variation, the state space includes entrepreneurs' and banks' wealth shares. The resulting complications in the solution procedure are discussed in Subection 7.2.2.

[755] As noted in Subsection 3.2.2.1, intermediaries' equity takes on the form of inside equity exclusively.

[756] Cf. Brunnermeier and Sannikov (2016a, p. 2).

[757] That difference also becomes apparent when comparing Figures 3.2.8 and 5.3.5.

in case of a jump.

The credit model shares with Klimenko et al. (2016) and Li (2017) the property that productive agents are externally financed by bank credit rather than equity. As noted, this effectively separates the production from the intermediation side. While these models can generate rich dynamics in the demand for and the supply of inside money, they pay less attention to the modelling of default. For instance, a drawback in both approaches is that the default probability can exceed unity, implying that banks derive additional profits from positive shocks on top of the loan rate that was agreed upon an instant of time earlier. The reason for this issue is that the default probability is assumed to be proportional to a Brownian increment, i.e. there is upside as well as downside risk in loan extension. Obviously, this is not a typical feature of a standard debt contract. In the credit model, Poisson shocks always lead to reductions in borrowers' net worth. The value of imprudent entrepreneurs' assets falls below the value of liabilities which causes them to declare default. Accordingly, employing a one-sided Poisson shock leads to a simple but consistent condition for borrowers' default (cf. Subsection 4.3.3). A further limitation in the two aforementioned models is the fact that all financial contracts are denominated in real rather than nominal terms. This assumption precludes variations in the value of money, which were shown to play a substantial role in the determination of the supply of and demand for banks loans in the presented model.

A common theme in the part of the literature that explicitly incorporates an intermediary sector is the demand for debt issued by intermediaries, i.e. inside money. The models developed in that literature typically fall into two categories: first, models in which money is a bubble and second, models in which that asset has intrinsic value. Let us first discuss models from the second category. Phelan (2016) and Klimenko et al. (2016) incorporate a liquidity-in-the-utility motive, which can be considered as a shortcut for a transaction motive, as mentioned in footnote 562. In Li (2017), the demand for inside money derives from a from a precautionary motive: random liquidity shocks give rise to money-in-advance constraint on the side of firms. Adrian and Boyarchenko (2012) include a preference shock which affects households' effective discount rate. The ramifications of this type of shock for the demand for intermediary debt are similar to those of a liquidity shock.

Other models, such as He and Krishnamurthy (2013) and BS (2016a) emphasise the role of money as a store of value. The former authors assume that a fraction of savers can store their wealth only in the form of debt emitted by intermediaries. BS (2016a) take another route. In their model, all agents have the option to invest in capital and money. The sole motive for holding money is that it acts as an insurance against idiosyncratic shocks to capital along the lines of Bewley (1980) and Scheinkman and Weiss (1986). Hence, money is a bubble.[758] BS (2014d) is a special case. In their baseline specification, the government redistributes tax proceeds to holders of outside money. Yet they also derive conditions under which money has positive value even in the absence of such redistributions.[759] Similarly to BS (2016a), in that case, agents hold money in equilibrium even

[758] Cf. Brunnermeier and Sannikov (2016a, p. 2).
[759] Cf. Brunnermeier and Sannikov (2014d, p. 9).

though it does not have intrinsic value. Again, the reason is that money enables individuals to reduce the exposure to shocks to capital. In parallel to BS (2016a), a part of the motivation for holding money in our model stems from a desire to insure against idiosyncratic risk. Yet, the value of money also has a fundamental component that arises from the transaction motive, as modelled via the MIU approach.[760]

The procyclical nature of banks' leverage is in line with the results in Adrian and Boyarchenko (2012) and Li (2017). As noted in Subsection 3.2.4.2, other CTMF models imply countercyclical assets-to-equity ratios in the banking sector, which stands in contrast to empirical evidence. As explained by Li (2017), this result can be related to the static/passive demand for intermediaries' debt in these models. Against this backdrop, negative shocks to leveraged intermediary balance sheets impede equity to a higher degree than assets.[761] Adrian and Boyarchenko (2012) achieve procyclical leverage by imposing a risk-based capital constraint on myopic intermediaries, which becomes more slack when the economy is in upswing. In contrast, the money-in-advance constraint introduced by Li (2017) generates an intertemporal complementary between the demand for money and the price of capital, which leads to an expansion in entrepreneurs' demand for liquidity in booms. Similarly, in the credit model, the upward movement in the demand for money when the economy is in upswing contributes to the procyclicality of banks' leverage in terms of owners' wealth. However, the source of variations in demand for inside money is different. Here, agents desire to maintain a (nearly) constant level of real balances in the face of a declining value of money due the money-in-the-utility motive.

Our credit model departs from the bulk of the literature in employing Poisson processes as the fundamental sources of uncertainty. Working with such processes in the context of our model offers two main advantages. First, a simple condition for the default of imprudent entrepreneurs arises. In particular, this condition does not depend on the amplitude or direction of the stochastic increment, as would be the case under Brownian uncertainty. Second, banks' balance sheets would not be affected by adverse price movements if Brownian instead of Poisson uncertainty were imposed. To see this, note that in the former case, the share of defaulting debtors would be of order $dt$[762], while asset price variations would be of order $dZ$. Hence, the share of banks' losses induced by the adverse price variation (which accrue from banks' seizure of defaulting borrowers' assets) would be of order $dt\, dZ = 0$. In contrast, with Poisson shocks, this share is proportionate to a strictly positive term, namely the share of imprudent entrepreneurs.

[760] It is interesting to recognise that the motives for holding money discussed so far stand in contrast to most of the New Keynesian literature, in which money just serves as a unit of account (cf. e.g. Woodford, 2003, p. 63).
[761] Cf. Li (2017, p. 5).
[762] One could e.g. assume that at each instant a share $\varphi\, dt$ of entrepreneurs are exposed to *idiosyncratic* Poisson shocks that push these agents into bankruptcy, while the fundamental risk is governed by Brownian motions.

### 5.5.2 DSGE Models with Financial Frictions

There are two important similarities between the credit model, KM, and BGG. First, in each of the three models, the balance sheets of production units are the main drivers of the business cycle. In KM, a deterioration in the balance sheet of leveraged firms causes the exogenous borrowing constraint to tighten and thereby directly reduces lending. Net worth reductions on the side of entrepreneurs in the BGG model, on the other hand, increase the probability of lenders' default and thereby expected agency costs. As a consequence, the EFP rises and entrepreneurs' demand for capital falls. Financial institutions are either entirely absent as in KM or play a limited role due to the absence of a banking sector balance sheet channel as in BGG and our model.

Second, in all of the mentioned models, end-borrowers obtain external funds exclusively in the form of debt.[763] This is due to borrowers' inability to commit to reimbursement in KM. Conversely, BGG adopt a CSV problem to motivate that type of capital structure. Since this approach is shared with the credit model, we discuss the similarities and differences in the implementation of the CSV paradigm between both models in more detail. In the former, borrowers are risk neutral and lenders are risk averse. Under theses conditions, the SDC can be shown to constitute the optimal contract, as discussed in Subsection 2.1.2. In the latter model, borrowers *as well as* ultimate lenders are risk averse. Further, in addition to the ex post information asymmetry due to creditors' inability to observe idiosyncratic project returns, our setting features *ex ante* private information: banks, by assumption, cannot observe the types of loan applicants (prudent or imprudent) before the financing contracts are signed. It follows that financiers might find it preferable to implement self-selection procedures to reveal loan applicants' types.[764] However, there are certain conditions under which pooling equilibria are the only sustainable outcomes in settings characterised by the presence of ex ante as well as ex post private information.[765]

Neglecting the possibility of separating equilibria, one might be tempted to presuppose that the optimal contract resembles that in Winton (1995), who considers risk averse borrowers and lenders. As mentioned in Subsection 2.1.2, under these circumstances, the optimal contract belongs to the class of debt contracts, but is not identical to the SDC. Analogously to other CSV settings, incentive compatibility in our model requires the reimbursement to be constant across idiosyncratic realisations of debtors' payoffs in the no-audit region. Thus, we can conclude that the optimal

[763] This is in contrast to many recently developed DSGE models that incorporate a role for financial institutions' balance sheets in the determination of the business cycle. Examples are Christiano et al. (2010), Curdia and Woodford (2010), and Gertler and Karadi (2011).

[764] For instance, in Bester (cf. 1985, pp. 850f.) lenders achieve self-selection of heterogeneous applicants by offering them a set of contracts which allows borrowers to choose among different combinations of interest payments and collateral requirements. Intuitively, debtors with a low probability of failure are more willing to pay a lower interest rate in exchange for an increase in collateral requirements than those with a high probability of bankruptcy.

[765] Such a setting is studied by Choe (cf. 1998, p. 242), who distinguishes between two cases, one in which the lender precommits to the contract and one in which he does not. In the former case, a separating equilibrium cannot emerge since financiers would not provide finance to applicants who reported an adverse signal. Borrowers anticipate lenders' behaviour and, accordingly, do not reveal the private information on their types (cf. Choe, 1998, p. 244).

contract at least involves debt-like elements. Yet, since we do not solve for the optimal contract here, but rather leave this issue open for future research, the contracting space is to be regarded as exogenous.[766]

A difference between both models relates to the inefficiency associated with the auditing of defaulting borrowers. In Subsection 4.1.2, it was assumed that verification necessitates a utility-reducing effort by bankers. In contrast, the standard CSV approach posits that verification entails a deadweight loss of the output good. If this approach were followed here, the finals goods market clearing condition would include a term proportionate to $d\mathcal{N}_t$ since these deadweight losses would only materialise in case of a jump. Since the supply of final goods is of order $dt$ and the verification costs in output goods would be of order unity, there could never be enough supply of final goods to cover the verification costs.[767]

Turning to implications of the CSV paradigm in both models, two distinctions are worth of notice. First, the EFP is not *directly* related to borrowers' net worth in the credit model. In BGG, an increase in an individual borrower's net worth reduces his probability of default. Therefore, intermediaries respond to improvements in debtors' balance sheets by quoting lower loan rates.[768] This mechanism is at the core of the countercyclical fluctuation in the EFP. In contrast, in the credit model, there is no feedback from entrepreneurs' equity to default probabilities. Prudent entrepreneurs never default since they always save precautionarily. Imprudent entrepreneurs' constant probability of failure, on the other hand, is given by $\lambda\, dt$, the intensity of the macro shock. Since these agent never maintain any net worth buffers, they always declare bankruptcy in case of a jump. However, as noted, the EFP does depend *indirectly* on borrowers' aggregate wealth share through the effects of the state variable on asset prices. Depending on the current state of the economy, the mark-up over the deposit rate can be either pro- or countercyclical. Since adverse price changes are less severe on average when entrepreneurs' balance sheets are healthy, the EFP is countercyclical in expectation.

Second, while the two models share the implication that entrepreneurs' demands for capital and credit increase linearly in their equity, the underlying reasons for that behaviour differs: in BGG, the reduction in loan rates that comes along with higher levels of borrower equity induces risk neutral entrepreneurs to take out more loans to finance additional purchases of real capital.

[766] An issue that was touched upon in Subsection 2.2.2 is BGGs' assumption that lenders' return is fixed. As discussed, Carlstrom et al. (2016) find that (i) optimal repayment depends on the realisation of aggregate shocks due to lenders' risk aversion and (ii) the financial accelerator breaks down if state-dependent reimbursement is allowed for. The credit model is potentially subject to similar concerns since managers, who assume the role of ultimate lenders, are risk averse. In this context, the result of Di Tella (2017), who shows that the balance sheet channel disappears in BS (2014a) if there is a complete market for aggregate risk, also suggests that indexation to such risk might mitigate or even eliminate the amplification of exogenous shocks in our model.

[767] One could instead assume that verification costs entail a reduction in the capital stock of an insolvent debtor. However, this approach would complicate the numerical solution of the model considerably as in that case, the post-jump value of the aggregate capital stock would depend on the allocation of capital between defaulting and nondefaulting debtors.

[768] A negative correlation between agency costs and debtors' wealth is a common property in the literature on financial frictions (cf. e.g. BGG 1996, p. 2).

This reaction can be interpreted in terms of a substitution effect. The credit market adjustment is more complex in our model since changes in net worth cause *both* the demand and supply schedules to shift.[769] As discussed in Subsection 5.3.3, the outward shift of the demand curve results from entrepreneurs' risk aversion. The credit supply curve can either shift up- or downwards, depending on the current state of the economy. It follows that variations in the demand for credit derive from substitution as well as wealth effects.

Since bankruptcy plays a major role in the determination of the external finance premium in the credit model, let us now widen the focus to include other works that explicitly incorporate the possibility of default. In general, introducing the possibility of default is not a straightforward endeavour since rational borrowers may want to hedge against negative shocks either through accumulation of net worth or through risk-sharing. In the literature, different (sometimes arbitrary) assumptions are made to sidestep this issue.

Krishnamurthy (2003) offers an explanation as to why incomplete hedging in a collateral constraint model in the spirit of KM may occur. He assumes that suppliers of insurance must provide collateral to credibly insure the lender. In effect, the supply of insurance is limited.[770] However, that paper does not include the possibility of default. Gertler et al. (2007) simply rule out debtors' ability to insure against shocks by assumption.[771] Cúrdia and Woodford (2010) and (2016) assume that borrowers randomly receive an opportunity to divert the borrowed amount from financiers. In both models, banks are not able to seize the assets of the defaulting borrowers. Consequently, the recovery rate is always equal to zero.[772] Conversely, the recovery rate is strictly positive in our model. This has the advantage that swings in the value of defaulting borrowers' asset reinforce fluctuations in the EFP.

Much of the literature adopts BGGs' approach to model entrepreneurs as risk neutral, myopic agents, who do not care about the possibility of default.[773] Examples are Christensen and Dib (2008), Fernández-Villaverde (2010), and Christiano et al. (2014).[774] More formally, that approach entails the absence of a no-Ponzi-game condition and a nonnegativity constraint on consumption.[775] The modelling of end-borrowers in the aforementioned models is thus very much in parallel to that of imprudent entrepreneurs in Appendix B.3. In that context, the only substantial difference is that these agents always consume their entire profits, which is tantamount to their net worth being equal to zero in periods without adverse shocks.

Let us finally address models that feature debt-deflation mechanisms à la Fisher (1933). Due

[769] In both the BGG and the credit model, the net worth of intermediaries/banks does not matter for the supply of credit. Rather, entrepreneurs' wealth and their wealth share are the crucial state variables, respectively.

[770] Cf. Krishnamurthy (2003, p. 285).

[771] Cf. Gertler et al. (2007, p. 308).

[772] Cf. Curdia and Woodford (2010, p. 10) and Cúrdia and Woodford (2016, p. 38).

[773] Cf. Dmitriev and Hoddenbagh (2015, p. 4).

[774] Cf. Christensen and Dib (cf. 2008, p. 159), Fernández-Villaverde (cf. 2010, p. 36), and Christiano et al. (cf. 2014, p. 36).

[775] Aiyagari (cf. 1994, pp. 665f.) has shown that a "natural" borrowing limit, which precludes the possibility of default, emerges if these two conditions are imposed.

to its (implicit) pronunciation of pecuniary externalities and undercapitalised sectors, incorporating elements of Fisher's story in microfounded general equilibrium models requires the adoption of heterogeneity and incomplete markets (cf. Subsection 2.2.1). Thus, models such as KM or BGG in principle have the potential to generate such elements. In fact, BGG show that a forced unanticipated redistribution of wealth between borrowers and lenders can have significant effects on output and asset prices, which cause further changes in the wealth distribution. According to the authors, "this case is [...] reminiscent of (and motivated by) Fisher's (1933) 'debt-deflation' argument, that redistributions between creditors and debtors arising from unanticipated price changes can have important real effects."[776] Similar points can be made regarding KM since a redistribution of wealth in their model may affect the borrowing constraint and asset prices. Yet, the two models correspond to the debt-deflation theory only in a wider sense since debt is denominated in *real* terms in both. This assumption precludes the possibility of borrowers being exposed to a rising real burden of debt induced by increases in the value of money.

A closer replication is due to Eggertsson and Krugman (2012), who develop a NK model with heterogeneous households, nominal debt, and an exogenously imposed debt limit.[777] The household sector can be summarised by describing the behaviour of two representative households, who differ in their time preference rates. Households with the high discount rate borrow from to those with the low discount rate in equilibrium. Consequently, the former are referred to as "borrowers" and the latter as "savers".[778] Importantly, the debt limit is assumed to be always binding on the side of borrowers. This gives rise to a marginal propensity to consume out of borrowers' current income that is exactly equal to unity.[779] Moreover, the borrowing constraint is formulated in real terms, which causes the natural rate of interest to depend on the inflation rate. Intuitively, a lower current price level tightens the borrowing constraint and thus induces borrowers to deleverage. Accordingly, the natural interest rate has to fall to incentivise savers to spend more.[780]

To study the effects of an unanticipated deleveraging shock, which might be interpreted as a "Minsky moment"[781], the authors decrease the debt limit and distinguish between two distinct regimes. In the first, the shock is relatively small. Then, the drop in borrowers' consumption expenditures can be compensated for by more spending from lenders, which is induced by the central bank cutting the policy rate. If, conversely, the reduction in the debt limit is sufficiently large to push the economy against the zero lower bound and the natural rate into negative territory, a vicious cycle between falling prices and surges in the real value of debt can occur. The reason is that a negative natural rate can only be achieved in the face of a binding zero lower bound via future expected inflation. Thus, given the expected future path of the price level, the current price

[776] Bernanke et al. (1999, p. 1372)
[777] Cf. Eggertsson and Krugman (2012, p. 1479).
[778] Cf. Eggertsson and Krugman (2012, p. 1474).
[779] Cf. Eggertsson and Krugman (2012, p. 1482).
[780] Cf. Eggertsson and Krugman (2012, p. 1479 and p. 1483).
[781] Cf. Eggertsson and Krugman (2012, p. 1475).

level must fall. The lower price level, however, tightens the borrowing constraint and the subsequent deleveraging further reduces the natural rate and so on.[782]

The Eggertsson and Krugman (2012) model shares with the credit model the assumption that debt is denominated in nominal terms. In an economy characterised by financial frictions, that assumption has the potential to generate deflationary spirals, which mitigate the debt capacity in the economy. The financial friction in their framework takes the form of an exogenous borrowing limit, whereas in ours financial frictions arise from agents' inability to share different types of risk. Turning to spillovers to the real sector, note that prices are sticky in their economy. If the deleveraging shock is large and economy ends up near the zero lower bound, spillovers in their baseline model arise from a reduction in aggregate consumption expenditures. In contrast, in our flexible-price model, a worsening misallocation of capital is the main driver of recessions. This result is not conditional on the economy being in a liquidity trap. In fact, the nominal rate on money is equal to zero at any point in time. Further, our model does not require unexplained deleveraging shocks since leverage is endogenous - agents optimally reduce their debt levels after the arrival of productivity shocks that lead to impaired balance sheets.[783]

### 5.5.3 The Growth Literature

The most obvious relation to the growth literature is that the credit model, as other CTMF models, at its core is an endogenous growth model. The reason is that final goods are generated by means of AK production technologies. These models therefore stand in the tradition of works such as Frankel (1962).[784] In distinction from the early work on endogenous growth, aggregate TFP in our model is endogenous. This in parallel to more recent growth models such as Romer (1986; 1990) and Aghion and Howitt (1992) who attribute variations in TFP to factors such as learning-by-doing, expansions in product varieties, or improvements in the quality of goods. In contrast, in our model, swings in the wealth distribution between more and less productive agents lead to reallocations of capital and thereby to changes in aggregate TFP. The fundamental reason as to why the wealth distribution matters is the presence of financial frictions. Further, there is no clear trend in aggregate TFP in our model. The capital allocation improves in tranquil times provided that the optimal distribution has not already been attained. Conversely, bad shocks that push the economy (deeper) into the misallocation region lead to reductions in TFP. If the economy experiences a prolonged episode without the occurrence of jumps, the optimal capital distribution is eventually reached. Then, growth in aggregate TFP halts. Depending on parameter constellations, the economy may experience sustained growth nevertheless since the marginal product of capital is constant.

[782] Cf. Eggertsson and Krugman (2012, pp. 1484ff.).

[783] It should be added that the limitations of DSGE models discussed in Subsection 2.2.4 also apply to the Eggertsson and Krugman (2012) model. In particular, they only consider unexpected shocks that start at the deterministic steady state of a linearised system (cf. Eggertsson and Krugman, 2012, p. 1480).

[784] Cf. Isohätälä et al. (2016, p. 264).

The notion that capital misallocation induced by financial frictions matters for variations in aggregate TFP is shared with a new strand of the growth literature. Out of this literature, the work most closely related to the model developed in this thesis is Moll (2014) since his model features continuous-time stochastic processes.[785] More specifically, uncertainty in the *idiosyncratic* productivity levels of atomistic producers is captured by means of Ohrnstein-Uhlenbeck processes, which are mean-reverting Brownian motions and thus allow for autocorrelation.[786] More productive firms can borrow capital from their unproductive counterparts but must provide collateral in doing so.[787] Capital is combined with labour to produce output by means of standard Cobb-Douglas technologies with CRS.[788] As in our model, the CRS property allows for representing output as an aggregate production function in which TFP is a weighted average of individual productivity levels. In particular, the weighting factors are identical to individual wealth shares.[789,790] Inefficient allocation of capital results from the fact that not every productive firm can afford collateral. Yet, this lack of external finance can potentially be compensated for by self-financing.[791]

The substitutability of external finance with respect to internal finance is closely related to the persistence of productivity shocks. If shocks are transitory, transition to the steady state is fast, but productivity losses in the steady state are large. Conversely, if shocks are persistent, transition is prolonged, but steady state productivity losses are less severe. The reason for this result is that self-financing takes time. In turn, sufficient time to self-finance is only available if productivity shocks are persistent.[792] To summarise, in his model, financial frictions affect the allocation of capital both during transition and at the steady state, as long as TFP shocks are not perfectly persistent. In contrast, in our model, market incompleteness affects the short- as well as the long-run distribution of capital even though productivity shocks are extremely persistent.[793] This is because entrepreneurs do net end up owning all the wealth in the economy in the long-run.

Turning to empirics, it is well-known that the growth and development accounting literature[794] attributes a substantial role to TFP in explaining variations in output growth. In an influential study, Klenow and Rodriguez-Clare (1997) report that the growth of TFP explains nearly half of the unweighted average growth of income per worker in a sample of 98 countries during the period from

[785] Some notable examples that are formulated in discrete time are Restuccia and Rogerson (2008), Buera and Shin (2013), and Caselli and Gennaioli (2013).
[786] Cf. Moll (2014, p. 3202).
[787] Cf. Moll (2014, p. 3192).
[788] Cf. Moll (2014, p. 3191).
[789] Cf. Moll (2014, pp. 3188).
[790] In the credit model, the corresponding weighting factors are represented by capital rather than wealth shares, as can be observed from equation (4.4.4). The reason is that agents have the option to invest in a second asset that is in positive net supply, namely outside money.
[791] Cf. Moll (2014, p. 3187).
[792] Cf. Moll (2014, pp. 3187f.)
[793] Recall that once an entrepreneur is hit by that type of shock, his productivity level is permanently reduced.
[794] Growth and development accounting are similar but distinct concepts. The latter uses data on cross-country income differences, whereas the former utilises cross-time differences from a single country (cf. Caselli, 2005, p. 681).

1960 to 1985.[795] Applying the development accounting approach to a more recent cross-country dataset, Caselli (2005) finds that the fraction of the variance of income explained by factor levels lies between 34 and 39 percent, depending on specification.[796] Moreover, authors such as Rajan and Zingales (1998) and Levine and Zervos (1998) have presented cross-country empirical evidence that financial development facilitates economic growth. As discussed, the magnitude of financial distortion will show up in measured aggregate TFP.

There is also a growing empirical literature that investigates the contribution of distortions in the allocation of capital to TFP growth. In a seminal study, that was already mentioned in Subsection 5.1, Hsieh and Klenow (2009) compare the distribution of capital in China and India to that of the United States, which is selected as a benchmark due to its comparatively undistorted allocation. They find that adopting the U.S. efficiency benchmark from 1997 would boost current TFP by 30 to 50 percent in China and by 40 to 60 percent in India.[797,798] Foster et al. (2008) report that reallocation of capital accounts for about 25% of aggregate TFP growth in a sample of U.S. manufacturing firms during the period from 1977 to 1997, while within-firm TFP growth explains about two third of that growth.[799]

How do these findings relate to our model? Engaging in a growth accounting exercise[800] shows that the variance of the growth rate of capital only accounts for 15 percent of the variation in the growth rate of output, with the rest being due to the variance of TFP growth. Those shares are not very responsive to variations in parameters, as further testing suggests. Presumably, different outcomes could be achieved by augmenting the model with labour input and an elastic supply of labour. Moreover, note from equation (4.4.4), which includes an expression for aggregate TFP $A(\psi_t)$, and the calibration of parameter $a^e$ in Table 5.1 that the optimal capital allocation $\psi^* = 1$ is tantamount to a value $A(\psi^*) = 27.6$. Since the mean of variable $\psi_t$ is approximately equal to 0.5 under the baseline calibration, financial friction reduce aggregate TFP by about 14 percent on average.[801] Further, we cannot account for the observed variations in within-firm TFP since our model assigns no role to technological progress. To conclude, a large portion of GDP growth in the credit model is due to changes in TFP, which, in turn, are entirely driven by adjustments in the distribution of capital.

[795] Cf. Klenow and Rodriguez-Clare (1997, p. 94).
[796] Cf. Caselli (2005, p. 687).
[797] Cf. Hsieh and Klenow (2009, p. 1405).
[798] It should be recalled, however, that these calculations require some critical assumptions, which are disputed in the literature (cf. the discussion in Subsection 5.1).
[799] Cf. Foster et al. (2008, pp. 418f.).
[800] In that calculation, we follow the method described by Caselli (2005, pp. 686f.).
[801] We have $a^m = 20$ and $a^e = 27.6$ under the baseline parameter constellation. Thus,

$$1 - \frac{\mathbb{M}\left[A(\psi_t)\right]}{A(\psi^*)} \approx 1 - \frac{0.5\,(20 + 27.6)}{27.6} \approx 0.14.$$

# Chapter 6

# Monetary Policy in the Credit Model

The previous chapter has demonstrated that negative productivity shocks to the entrepreneurial sector trigger adverse feedback loops, which are characterised by mutually reinforcing movements in the inter-sectoral wealth distribution and asset prices. Under our baseline calibration, the main driver of those interactions are appreciations in the real value of money, which, in turn, result from contractions in the money multiplier caused by voluntary deleveraging in the entrepreneurial sector against the backdrop of distressed balance sheets. The occurrence of adverse feedback loops is detrimental to agents' welfare for multiple reasons, the most important being excessive fluctuations in wealth levels and thereby consumption due to the amplification of exogenous shocks and, in addition, a persistent misallocation of capital, which reduces output relative to the first-best scenario. In general, the competitive equilibrium in our economy is not constrained Pareto efficient due to nominal frictions in the form of a MIU motive for holding money as well as financial frictions due to market incompleteness. These frictions also imply that pecuniary externalities play an important role in determining aggregate outcomes. For these reasons, policies that interfere with market outcomes have the potential to improve on individuals' welfare.

In this chapter, we ask whether (a) monetary policy can be effective in counteracting deflationary pressure that ensues after the arrival of negative shocks and (b) if such policies are advisable from a welfare perspective. As BS (2016a), we focus on the redistributive effects of monetary policy effects. Since entrepreneurs in our setting do not hold nominal bonds and prices are fully flexible, the central bank cannot alter the distribution of wealth via variations in the short-term nominal interest rate. Due to that reason, we consider policies that directly target the (outside) money supply. Such policies have real effects since they affect the value of liabilities on entrepreneurs' balance sheets and, therefore, their relative equity position.

Amongst the specific measures undertaken by the monetary authority, we investigate policies that peg the growth rate in the outside money supply à la Friedman (1959). Even though such $k$-*percent rules* might appear unattractive due to their inflexibility, they actually lead to surprisingly complex adjustments in macroeconomic dynamics. Remarkably, the adoption of these rules can improve system stability as well as welfare. The underlying reason for that result is that agents adjust their behaviour ex ante, i.e. before the realisation of shocks. While k-percent rules have the potential to make agents better off, they do not directly address the collapse in the money multiplier in crisis states. This is because the drop in inside money after a jump is discrete, whereas those rules only lead to infinitesimal variations in the money supply at any given point in time. For that reason, we also examine "crisis policies" that entail discrete adjustments in the quantity of outside money in case of adverse events. Since such policies are stochastic, individuals' expectation formation with respect to the stance of monetary policy turns out to be crucial for the efficacy of the adopted measure.

This chapter is structured as follows. In Section 6.1, we discern the macroeconomic effects of conventional interest rate policies. To that end, we introduce reserves and nominal interest payments on reserves that are financed by the central bank via newly printed money. Section 6.2 derives the relevant model equations under money supply rules. Section 6.3 assumes that the monetary authority sets a constant rate of change in the money supply. Within that section, we first revisit two simplified model variants, namely the first-best and the autarky case. We do so since these variants offer additional insight into forces that are also at work in the full model with endogenous risk, which is examined subsequently. In Section 6.4, we consider two types of monetary policy reaction functions: first, rules that condition the deterministic growth rate on the inter-sectoral distribution of wealth and second, crisis policies that lead to discrete adjustments in the money supply in the event of shocks.

## 6.1 Equilibrium with Interest Rate Policies

In this section, we follow BS (2016a) in assuming that (a) banks hold the entire stock of outside money in the form of reserves at any point in time and issue a corresponding amount of deposits, (b) the monetary authority pays a nominal interest rate $r_t^R$ on reserves, and (c) the central bank funds interest payments by creating additional base money.[802] Then, the evolution of outside money is described by[803]

$$\frac{dM_t^O}{M_t^O} = \mu_t^{M^O}\, dt = r_t^R\, dt, \tag{6.1.1}$$

in which $\mu_t^{M^O}$ is the drift rate of the base money stock. Moreover, we follow BS (2016c) in redefining variable $p$ as the value of the *total* outside money supply per unit of the aggregate capital stock.[804] This implies that the price of one unit of outside money $P_t$ is equal to $p_t K_t / M_t^O$. The following proposition and its corollary summarise the effects of monetary policy under the aforementioned assumptions.

[802] In this section, we use terms "base money", "reserves", and "outside money" interchangeably.
[803] Cf. Brunnermeier and Sannikov (2016a, p. 30).
[804] Cf. Brunnermeier and Sannikov (2016c, p. 485).

**Proposition 7.** *If the central bank pays nominal interest on reserves and funds the interest expenses by issuing additional reserves, money is superneutral. That is, adjustments in the rate of change in the outside money supply induced by changes in the rate on reserves do not affect real equilibrium outcomes.*

*Proof.* The real return on outside money is now given by

$$dr_t^M = \Big\{ \underbrace{r_t^R + \underbrace{\mu_t^p - \mu_t^{M^O} + \Phi(\iota_t) - \delta}_{=-\mu_t^{\mathcal{P}}}}_{=\mu_t^{r^M}} \Big\} dt + \frac{\widetilde{p}_t(1-\kappa) - p_t}{p_t} \, d\mathcal{N}_t. \tag{6.1.2}$$

where $\mu_t^{\mathcal{P}}$ again is the inflation rate in the absence of jumps. Then, it immediately follows from (6.1.1) that the real return on money is not affected by adjustments in $M_t^O$. Since neither of the two variables $r_t^R$ and $\mu_t^{M^O}$ enters any other of the equilibrium relations in Proposition 2, money is superneutral. □

**Corollary 3.** *Adjustments in the nominal rate on reserves lead to identical adjustments in the inflation rate.*

*Proof.* Equation (6.1.2) implies

$$\mu_t^{r^M} = r_t^R - \mu_t^{\mathcal{P}}, \tag{6.1.3}$$

in words: the real return on money equals the difference between the nominal return on money and inflation, each in the absence of jumps.[805] Since we know from Proposition 7 that the real return on money remains unchanged, changes in $r_t^R$ lead to identical changes in $\mu_t^{\mathcal{P}}$. □

Proposition 7, Corollary 3, and the respective proofs are exactly analogous to the reasoning in BS (2016a).[806] The key assumptions that generate the superneutrality result in our as well as their model are the absence of price inertia and nominal bonds.

We could proceed with our exploration of the effects of interest rate policies by introducing nominal bonds, as BS (2016a). In their model, the central bank affects the price path of nominal bonds held by intermediaries by adjusting the nominal rate on outside money. In particular, they consider interest rate cuts that boost the value of bonds when adverse shocks to intermediaries' balance sheets materialise. These policies have the potential to reduce endogenous risk and, therefore, to stabilise the economy (cf. Subsection 3.2.2.2). In distinction, in the remainder of this chapter, we discern the effects of policies that adjust the outside money supply by means other than interest payments on reserves. Importantly, such policies affect the real return on money even in the absence of nominal bonds. We depart from their modelling of monetary policy for the following

[805] It is interesting to note that (6.1.3) is the famous Fisher equation, which holds with exact equality in our continuous-time setting (cf. also Brunnermeier and Sannikov, 2016a, p. 30).

[806] Cf. Brunnermeier and Sannikov (2016a, p. 30).

reason. While it is true that firms hold bonds, empirical evidence suggests that the share of wealth these entities allocate to bonds is significantly smaller than that of banks. For instance, Duchin et al. (2017) document that in a sample of S&P 500 firms from 2012, the shares of government and corporate securities in total assets are equal to about 3.0 and 1.5 percent, respectively.[807] In contrast, the corresponding shares in the consolidated balance sheet of depository institutions were at about 16.0 and 7.5 percent in that same year.[808] Now, to the extent that in our model banks hold a higher share of bonds in their portfolios than entrepreneurs, policies that boost the value of bonds in cases of negative shocks could exacerbate the consequences of those shocks. This is because such policies will affect the distribution of wealth between entrepreneurs and bank owners.[809]

## 6.2 Equilibrium Equations and Welfare with Money Supply Policies

### 6.2.1 The Outside Money Supply Process and Associated Transfers

Given the fact that the classical dichotomy holds under conventional interest rate policies, we now ask whether policies that affect the quantity of money in circulation affect real equilibrium outcomes. To this end, we continue to assume that $p_t$ stands for the value of the total outside money supply per unit of aggregate capital. Conversely, we no longer assume that outside money is held in the form of reserves at the central bank. Rather, we think of outside money as representing currency in circulation.[810] The central bank alters that quantity by following a rule of the family

$$\frac{dM_t^O}{M_t^O} = \mu_t^{M^O}\, dt + \frac{\widetilde{M}_t^O - M_t^O}{M_t^O}\, d\mathcal{N}_t = \mu_t^{M^O}\, dt + \left\{\frac{1}{\mathcal{M}_t} - 1\right\} d\mathcal{N}_t, \tag{6.2.1}$$

in which $\widetilde{M}_t$ is the post-jump level of money created by the monetary authority and $\mathcal{M}_t \equiv M_t^O/\widetilde{M}_t^O$. Accordingly, $\mathcal{M}_t < 1$ ($\mathcal{M}_t > 1$) is tantamount to an increase (decrease) in the stock of outside money in case of a jump. In the following, $\mu_t^{M^O}$ and $\mathcal{M}_t$ will act as the relevant policy variables. Further, policies that target $\mu_t^{M^O}$ are referred to as *"deterministic policies"* and measures that lead to changes in $\widetilde{M}_t^O$ as *"stochastic policies"*. In Sections (6.3) and (6.4), we will examine the effects of various variants of (6.2.1) on equilibrium outcomes.

[807] Cf. Duchin et al. (2017, Table I).

[808] These figures are calculated by the author of this thesis from data provided in FRB (2018a).

[809] Of course, in crisis episodes, the central bank could lower the value of bonds primarily held by banks by raising the nominal interest rate. In our model, this could indeed mitigate the adverse consequences of shocks. Yet, in more realistic models with price inertia, for instance, such policies could have other detrimental effects.

[810] We could, as BS (2016a), continue to assume that outside money represents banks' reserves held at the central bank and that the former issue identical amounts of deposits to the public at any time. Yet, in reality, there is no compelling reason as to why banks should always act in that way. Indeed, during the aftermath of the recent financial crisis, banks ramped up significant amounts of excess reserves (cf. e.g. Keister and McAndrews, 2009, pp. 1f.).

One possibility to specify the way in which currency in circulation is adjusted would be to incorporate quantitative easing (QE) into our model. Indeed, one aim of central banks' QE measures conducted after the crisis of 2007-08 was to increase the quantity of monetary aggregates other than reserves by means of security purchases from the nonbank private sector.[811] Unfortunately, QE measures are not straightforward to implement since such approach would necessitate the introduction of bonds and, accordingly, an additional price process.[812] For that reason, we take a pragmatic approach and follow Friedman (1969) in assuming that (a) increases in the supply of currency are achieved via "helicopter drops"[813] and (b) decreases via the imposition of taxes payable in money, the revenues of which are subsequently destroyed.[814] Assumption (b) requires us to merge the monetary and fiscal authorities. For convenience, we continue to refer to the resulting government agency as the "central bank". Further, to remain consistent with rule (6.2.1), we have to assume that taxes are payable in terms of *outside* money.[815] Finally, to prevent direct distributional ramifications, taxes as well as windfalls are assumed to be proportionate to agents' wealth levels.[816]

It follows from the above considerations that total real windfalls/taxes are given by $p_t K_t \mu_t^{M^O}$ in normal times without shocks and in the opposite case by

$$\frac{\widetilde{p}_t \left(1-\kappa\right) K_t}{\widetilde{M}_t^O} \left(\widetilde{M}_t^O - M_t^O\right) = \widetilde{p}_t \left(1-\kappa\right) K_t \left(1-\mathcal{M}_t\right),$$

in which the LHS stands for the product of the post-jump price of money and the change in the base money supply. Using these expressions, the defining equations for pre- and post-jump aggregate wealth (3.2.14) and $\widetilde{N}_t \equiv \left(\widetilde{q}_t + \widetilde{p}_t\right)\left(1-\kappa\right) K_t$, as well as the second equality in (4.5.5), we can write

[811] Cf. McLeay et al. (2014, p. 24).

[812] In effect, one would have to solve a system of two RDEs, one for the value share of capital and one for the price of the bond. The literature has not yet proposed a solution for that type of problem under Poisson uncertainty.

[813] In recent discussions, the distribution of helicopter money is usually regarded as a multi-stage process that starts with the monetary authority crediting the treasury's account at the central bank in exchange for newly issued government bonds. The treasury then brings the new money in circulation by buying goods or services from the private sector, granting tax rebates, or directly crediting households' accounts (cf. e.g. Bernanke, 2016; Cecchetti and Schoenholtz, 2016). Clearly, this course of events has a fiscal component. Yet, for instance in the Euro Area, this way of raising the money supply is prohibited by EU law since it amounts to the monetisation of government debt (cf. Buiter, 2014, p. 40). This issue would not arise if the central bank provided households with a direct payment as e.g. suggested by Muellbauer (2014). Our treatment of helicopter money corresponds to this procedure and therefore does not involve the fiscal authority.

[814] Cf. Friedman (1969, pp. 4 and 16).

[815] If we assumed that the outside money supply is always held in the form of reserves and that banks always issue a corresponding amount of inside money as in the previous section, we could allow for taxes being paid in terms of inside money. This would not alter the results presented in this chapter.

[816] Similarly, Brunnermeier and Sannikov (cf. 2016c, p. 485) assume that seigniorage revenues are passed on to households in proportion to individuals' wealth levels.

down the following process:

$$d\tau_t = \underbrace{(1-\theta_t)\,\mu^{M^O}}_{\equiv \mu_t^\tau \tau_t}\,dt + \underbrace{\left(1-\widetilde{\theta}_t\right)(1-\mathcal{M}_t)}_{\equiv \widetilde{\tau}_t - \tau_t}\,d\mathcal{N}_t. \tag{6.2.2}$$

This process measures the (instantaneous) real transfers to the private sector associated with adjustments in the outside money supply per unit of aggregate wealth.[817] Note that in the absence of shocks, these transfers are negative if $\mu^{M^O} < 0$. Then, $-\mu_t^\tau \tau_t\,dt$ is the instantaneous wealth tax rate. Analogously, if $d\mathcal{N}_t = 1$ and $\mathcal{M}_t > 1$, $\tau_t - \widetilde{\tau}_t$ is equal to that rate.

Since transfers are proportionate to post-jump wealth, an individual manager that suffers the bad idiosyncratic shock holds post-jump equity

$$\underline{\widetilde{n}}_{i,t}^m = (1+\widetilde{\tau}_t - \tau_t)\left[x_t^m\,(1-\kappa-\underline{\kappa}^m)\,\frac{\widetilde{q}_t}{q_t} + (1-x_t^m)\,(1-\kappa)\,\frac{\widetilde{p}_t}{p_t}\mathcal{M}_t + \widetilde{\Pi}_t^b - \Pi_t^b\right] n_{i,t}^m. \tag{6.2.3}$$

An analogous equation applies if the individual draws the beneficial idiosyncratic shock. For an individual prudent entrepreneur, we have

$$\underline{\widetilde{n}}_{i,t}^{e,p} = (1+\widetilde{\tau}_t - \tau_t)\left[x_{1,t}^{e,p}\,(1-\kappa-\underline{\kappa}^{e,p})\,\frac{\widetilde{q}_t}{q_t} + \left(1-x_{1,t}^{e,p}\right)(1-\kappa)\,\frac{\widetilde{p}_t}{p_t}\mathcal{M}_t\right] n_{i,t}^{e,p}. \tag{6.2.4}$$

Again, an analogous expression can be derived for $\widetilde{\widetilde{n}}_{i,t}^{e,p}$.

### 6.2.2 Returns

The return on money is now given by

$$dr_t^M = \underbrace{\left\{\mu_t^p + \Phi(\iota_t) - \delta - \mu_t^{M^O}\right\}}_{=\mu_t^{r^M}}\,dt + \left\{(1-\kappa)\frac{\widetilde{p}_t}{p_t}\mathcal{M}_t - 1\right\}\,d\mathcal{N}_t. \tag{6.2.5}$$

From this equation, it is immediately clear that variations in the quantity of outside money affect the real return on money, ceteris paribus, in contrast to the case discussed in Section 6.1.

To determine the external finance premium under monetary policy let us first consider the condition for imprudent entrepreneurs' default:

$$\begin{aligned} dn_{i,t}^{e,i} = \max\Bigg[ &\left(q_t k_{i,t}^{e,i}\mu_t^{r^{K,e}} + p_t K_t l_{i,t}^{e,i}\mu_t^{r^{L,e}} - c_{i,t}^{e,i}\right)\,dt \\ &+ (1-\kappa)\,p_t K_t l_{i,t}^{e,i}\left(\frac{\widetilde{p}_t}{p_t}\mathcal{M}_t - \frac{\widetilde{q}_t}{q_t}\right)\,d\mathcal{N}_t,\ 0\Bigg], \end{aligned} \tag{6.2.6}$$

In the model without active monetary policy, the bankruptcy condition was $\widetilde{p}_t/p_t > \widetilde{q}_t/q_t$ (cf.

[817] A similar transfer process is derived in Brunnermeier and Sannikov (cf. 2016d, p. 13).

Subsection 4.3.3). Now, by altering the outside money supply after jumps, the central bank can potentially prevent imprudent agents form defaulting. To avoid wealth accumulation on the side of these agents in that case, we assume that the surplus capital, i.e. the amount of capital that is equal in value to real equity, is transformed to final goods by means of technology $\Phi^{-1}(\cdot)$, which are subsequently consumed.[818] It follows from these considerations that if agents anticipate the policy, banks' return on loans is described by

$$dr_t^{L,b} = \begin{cases} \left\{\Gamma_t + \mu_t^{r^M}\right\} dt + (1-\kappa)\left\{\varphi \frac{\widetilde{q}_t}{q_t} + (1-\varphi)\frac{\widetilde{p}_t}{p_t}\mathcal{M}_t\right\} d\mathcal{N}_t & \text{if } \frac{\widetilde{p}_t}{p_t}\mathcal{M}_t > \frac{\widetilde{q}_t}{q_t} \\ dr_t^M & \text{otherwise,} \end{cases} \tag{6.2.7}$$

(anticipated) banks' losses per unit of managers' aggregate wealth by[819]

$$\widetilde{\Pi}_t^b - \Pi_t^b = \begin{cases} (1-\kappa)\left(\frac{\widetilde{q}_t}{q_t} - \frac{\widetilde{p}_t}{p_t}\mathcal{M}_t\right)\frac{\varphi}{1-\varphi}\left(x_{1,t}^{e,p} + x_{2,t}^{e,p} - 1\right)\frac{n_t^e}{n_t^m} & \text{if } \frac{\widetilde{p}_t}{p_t}\mathcal{M}_t > \frac{\widetilde{q}_t}{q_t} \\ 0 & \text{otherwise,} \end{cases} \tag{6.2.8}$$

and the EFP by

$$\Gamma_t = \begin{cases} \lambda\varphi(1-\kappa)\left((1-\omega)\frac{\widetilde{q}_t}{q_t} + \frac{\widetilde{p}_t}{p_t}\mathcal{M}_t\right) & \text{if } \frac{\widetilde{p}_t}{p_t}\mathcal{M}_t > \frac{\widetilde{q}_t}{q_t} \\ 0 & \text{otherwise.} \end{cases} \tag{6.2.9}$$

If the bankruptcy condition is satisfied, the EFP is an increasing function of $\mathcal{M}_t$, all else equal. This is because the post-jump value of banks' liabilities rises with that parameter. If all debtors stay solvent after a potential shock, banks experience losses with probability zero and the EFP is zero. In order to avoid an infinite value of entrepreneurs' portfolio weight on money in that case, we have to assume a saturation point for real balances (cf. FOC (4.3.22)).[820] The most straightforward possibility is to assume that additional real balances do not raise utility further if some threshold level of the portfolio weight on money is reached. In our calculations we set this threshold value to 2.0, a value which was never exceeded in our previous calculations of equilibria. Managers' and entrepreneurs' returns on capital are unchanged and thus still determined from (4.2.10), (4.2.22), and (4.2.27).

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[818] Due to numerical complications, we neglect the effect of that transformation on the aggregate capital stock in our calculation of equilibrium. The resulting error is small since (a) imprudent entrepreneurs' surplus capital is a small share of the aggregate capital stock and (b) adjustments in the aggregate capital stock have minor repercussions for the equilibrium values of endogenous variables.

[819] Of course, banks' losses will always materialise whether or not agents anticipate a particular monetary policy intervention. Yet, term $\widetilde{\Pi}_t^b - \Pi_t^b$ matters for equilibrium outcomes only to the extent that it is anticipated.

[820] Such saturation point is also often assumed in other MIU models (cf. Walsh, 2010, p. 35).

### 6.2.3 Portfolio Choices

Turning to portfolio selection, it can be shown that managers' FOC with respect to the portfolio weight on capital is

$$\begin{aligned}\mu_t^{r^{K,m}} - \mu_t^{r^M} = &\frac{\xi\rho}{(1+\xi)(1-x_t^m)} \\ &- \lambda\phi\frac{(1-\kappa-\underline{\kappa}^m)\frac{\widetilde{q}_t}{q_t} - (1-\kappa)\frac{\widetilde{p}_t}{p_t}\mathcal{M}_t}{x_t^m(1-\kappa-\underline{\kappa}^m)\frac{\widetilde{q}_t}{q_t} + (1-x_t^m)(1-\kappa)\frac{\widetilde{p}_t}{p_t}\mathcal{M}_t + \widetilde{\Pi}_t^b - \Pi_t^b} \\ &- \lambda(1-\phi)\frac{(1-\kappa-\overline{\kappa}^m)\frac{\widetilde{q}_t}{q_t} - (1-\kappa)\frac{\widetilde{p}_t}{p_t}\mathcal{M}_t}{x_t^m(1-\kappa-\overline{\kappa}^m)\frac{\widetilde{q}_t}{q_t} + (1-x_t^m)(1-\kappa)\frac{\widetilde{p}_t}{p_t}\mathcal{M}_t + \widetilde{\Pi}_t^b - \Pi_t^b}.\end{aligned} \tag{6.2.10}$$

Through its negative effect on the real return on money, a higher deterministic money growth rate $\mu_t^{MO}$ tilts managers' portfolio towards a higher weight on capital, ceteris paribus. As we will see later on, any (nondegenerate) policy of the form (6.2.1) also has indirect effects on portfolio choices in the fully articulated model with endogenous risk through the ramifications on the equilibrium paths of $q_t$ and $p_t$. Of particular importance are the effects on post- relative to pre-jump prices since these lead to discrete changes in agents' wealth levels. Accordingly, we will refer to these effects as the "wealth effects" of monetary policy. In contrast, the repercussions for the deterministic components of the real return on money will be referred to as "substitution effects".

Since $x_t^m$ always lies in the range $[0,1]$, the RHS of (6.2.10) is clearly increasing in $\mathcal{M}_t$. That is, all else equal, the required risk premium is the higher, the more the central bank reduces the base money supply in the event of a jump arrives. Interestingly, the wealth allocation decision does not depend on transfers $d\tau_t$. As can be observed from (6.2.3), the transfer proportionately alters post-jump wealth. However, logarithmic utility implies that proportionate changes in net worth lead to proportionate changes in all asset demands. Hence, optimal portfolio weights are not affected.

Prudent entrepreneurs' optimal portfolio choice satisfies

$$\begin{aligned}\mu_t^{r^{K,e}} - \left(\Gamma_t + \mu_t^{r^M}\right) = &-\lambda\phi\frac{(1-\kappa-\underline{\kappa}^{e,p})\frac{\widetilde{q}_t}{q_t} - (1-\kappa)\frac{\widetilde{p}_t}{p_t}\mathcal{M}_t}{x_{1,t}^{e,p}(1-\kappa-\underline{\kappa}^{e,p})\frac{\widetilde{q}_t}{q_t} + \left(1-x_{1,t}^{e,p}\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t}\mathcal{M}_t} \\ &- \lambda(1-\phi)\frac{(1-\kappa-\overline{\kappa}^{e,p})\frac{\widetilde{q}_t}{q_t} - (1-\kappa)\frac{\widetilde{p}_t}{p_t}}{x_{1,t}^{e,p}(1-\kappa-\overline{\kappa}^{e,p})\frac{\widetilde{q}_t}{q_t} + \left(1-x_{1,t}^{e,p}\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t}\mathcal{M}_t}.\end{aligned} \tag{6.2.11}$$

Again, accelerated outside money growth leads to a portfolio shift towards capital, ceteris paribus. However, in distinction from (6.2.10), it is not immediately clear whether the RHS of the above equation is increasing in $\mathcal{M}_t$. The reason is that entrepreneurs are allowed to borrow, which is tantamount to a negative value of portfolio weight $1-x_{1,t}^{e,p}$. Yet, differentiating the RHS of (6.2.11) yields

$$\lambda\phi \frac{\left(1-\kappa-\underline{\kappa}^{e,p}\right)\frac{\widetilde{q}_t}{q_t}\frac{\widetilde{p}_t}{p_t}}{\left[x_{1,t}^{e,p}\left(1-\kappa-\underline{\kappa}^{e,p}\right)\frac{\widetilde{q}_t}{q_t}+\left(1-x_{1,t}^{e,p}\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t}\mathcal{M}_t\right]^2} + \lambda(1-\phi)\frac{\left(1-\kappa-\overline{\kappa}^{e,p}\right)\frac{\widetilde{q}_t}{q_t}\frac{\widetilde{p}_t}{p_t}}{\left[x_{1,t}^{e,p}\left(1-\kappa-\overline{\kappa}^{e,p}\right)\frac{\widetilde{q}_t}{q_t}+\left(1-x_{1,t}^{e,p}\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t}\mathcal{M}_t\right]^2},$$

which is obviously positive. Further, when deciding on their wealth allocation, entrepreneurs have to take into account that monetary policy also affects the EFP, which enters the LHS of (6.2.11) through term $\Gamma_t$.

In sum, monetary policy may affect entrepreneurs' portfolio decision via three direct channels.[821] First, a higher (lower) deterministic outside money growth rate decreases (increases) the deterministic part of the loan rate $\mu_t^{r^{L,e}}$ and thereby raises (reduces) the actual premium on capital. Second, increases (decreases) in the outside money stock conditional on the arrival of a jump reduce (raise) the required risk premium. Third, a lower (higher) value of $\mathcal{M}_t$ expands (contracts) the actual risk premium due to the resulting drop (surge) in the EFP. In all cases, the result is a higher (lower) portfolio weight on capital $x_{1,t}^{e,p}$. In addition, in each case, indirect effects emerge through the implied adjustments in equilibrium paths of $q_t$ and $p_t$, as discussed.

### 6.2.4 The Process for the State Variable

Since we do not consider policies that depend on the aggregate capital stock, equilibrium dynamics are still driven by variations in a single state variable, namely entrepreneurs' share in aggregate wealth. Applying steps similar to those in Appendix B.1.8, we can solve for the drift rate of the state variable and its post-jump value:

$$\mu_t^{\eta} = x_{1,t}^{e,p}\frac{a^e-\iota_t}{q_t} - \left(1-x_{1,t}^{e,p}\right)\mu^{M^O} + \left(1-x_{1,t}^{e,p}-x_{2,t}^{e,p}\right)\Gamma_t + \left(x_{1,t}^{e,p}-\theta_t\right)\left(\mu_t^q-\mu_t^p\right) - \frac{\rho}{1+\xi}, \quad \text{(6.2.12a)}$$

$$\widetilde{\eta}_t = (1-\phi^s)\left(x_{1,t}^{e}\frac{\widetilde{\theta}_t}{\theta_t} + (1-x_{1,t}^{e})\frac{1-\widetilde{\theta}_t}{1-\theta_t}\mathcal{M}_t\right)\eta_t. \quad \text{(6.2.12b)}$$

Given the fact that entrepreneurs are levered, expansive policies that increase the quantity of currency, either in times with or without jumps, lead to greater changes in the state variable, all else unchanged.

[821] In each of the following cases, the ceteris paribus condition is adopted.

### 6.2.5 Computation of Welfare

At last let us turn to welfare analysis. The general forms of managers' and prudent entrepreneurs' value functions are not affected by monetary policy, i.e. (5.3.2) and (5.3.4) still apply. However, this does not hold true for functions $\alpha^m(\eta_t)$ and $\alpha^{e,p}(\eta_t)$, as Proposition 8 shows:

**Proposition 8.** *Function $\alpha^m(\cdot)$ satisfies HJB equation*

$$\begin{aligned}\rho\alpha^m(\eta_t) = \alpha_0^m + \lambda\frac{1+\xi}{\rho}\left[\phi\log\left(\frac{\underline{\widetilde{n}}_t^m}{n_t^m}\right) + (1-\phi)\log\left(\frac{\overline{\widetilde{n}}_t^m}{n_t^m}\right)\right] \\ + \mu_t^\eta\eta_t\frac{d\alpha^m(\eta_t)}{d\eta_t} + \lambda\left[\alpha^m(\widetilde{\eta}_t) - \alpha^m(\eta_t)\right],\end{aligned} \tag{6.2.13a}$$

*where*

$$\alpha_0^m \equiv \log\frac{\rho}{1+\xi} + \xi\log(1-x_t^m) + \frac{1+\xi}{\rho}\left(\mu_t^{r^{P,m}} + \mu_t^\tau\tau_t\right) - 1, \tag{6.2.13b}$$

$$\frac{\underline{\widetilde{n}}_t^m}{n_t^m} = (1+\widetilde{\tau}_t-\tau_t)\left[x_t^m(1-\kappa-\underline{\kappa}^m)\frac{\widetilde{q}_t}{q_t} + (1-x_t^m)(1-\kappa)\frac{\widetilde{p}_t}{p_t}\mathcal{M}_t\right], \quad \textit{and} \tag{6.2.13c}$$

$$\frac{\overline{\widetilde{n}}_t^m}{n_t^m} = (1+\widetilde{\tau}_t-\tau_t)\left[x_t^m(1-\kappa-\overline{\kappa}^m)\frac{\widetilde{q}_t}{q_t} + (1-x_t^m)(1-\kappa)\frac{\widetilde{p}_t}{p_t}\mathcal{M}_t\right]. \tag{6.2.13d}$$

*Function $\alpha^{e,p}(\cdot)$ satisfies*

$$\begin{aligned}\rho\alpha^{e,p}(\eta_t) = \alpha_0^{e,p} + \lambda\frac{1+\xi}{\rho}\left[\phi\log\left(\frac{\underline{\widetilde{n}}_t^e}{n_t^e}\right) + (1-\phi)\log\left(\frac{\overline{\widetilde{n}}_t^e}{n_t^e}\right)\right] \\ + \mu_t^\eta\eta_t\frac{d\alpha^{e,p}(\eta_t)}{d\eta_t} + \lambda\left[(1-\phi^s)\alpha^{e,p}(\widetilde{\eta}_t) + \phi^s\alpha^m(\widetilde{\eta}_t) - \alpha^{e,p}(\eta_t)\right],\end{aligned} \tag{6.2.14a}$$

*where*

$$\alpha_0^{e,p} \equiv \log\frac{\rho}{1+\xi} + \xi\log x_{2,t}^{e,p} + \frac{1+\xi}{\rho}\left(\mu_t^{r^{P,e,p}} + \mu_t^\tau\tau_t\right) - 1, \tag{6.2.14b}$$

$$\frac{\underline{\widetilde{n}}_t^e}{n_t^e} = (1+\widetilde{\tau}_t-\tau_t)\left[x_{1,t}^{e,p}(1-\kappa-\underline{\kappa}^{e,p})\frac{\widetilde{q}_t}{q_t} + \left(1-x_{1,t}^{e,p}\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t}\mathcal{M}_t\right], \quad \textit{and} \tag{6.2.14c}$$

$$\frac{\overline{\widetilde{n}}_t^e}{n_t^e} = (1+\widetilde{\tau}_t-\tau_t)\left[x_{1,t}^{e,p}(1-\kappa-\overline{\kappa}^{e,p})\frac{\widetilde{q}_t}{q_t} + \left(1-x_{1,t}^{e,p}\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t}\mathcal{M}_t\right]. \tag{6.2.14d}$$

*Proof.* See Appendix D.1.1.

An important channel through which the monetary authority affects agents' welfare is the asset price channel. On the one hand, asset price *levels* influence real wealth levels as well as the distribution of wealth and, therefore, the state variables in agents' value functions. On the other hand, *adjustments* in prices induced by policy interventions alter investment opportunities via their effects on risk premia and post-jump wealth levels. Furthermore, since individuals take into account the central banks' reaction function, portfolio choices are impacted, which, in turn, lead to further asset price variations. For instance, policies that reduce the value of money are associated with portfolio shifts away from money. While this might foster long-term growth, it reduces flow utility derived from holding real balances and, in addition, exposes agents to higher idiosyncratic risk embodied in capital. Nevertheless, since the baseline equilibrium without interventions in the credit model is characterised by multiple pecuniary externalities, monetary policy has the potential to create Pareto improvements. Finally, note that while transfers have no repercussion for portfolio choices, the former do have consequences for welfare.

As mentioned in the introduction to this chapter, we will start the examination of policy implications for equilibrium outcomes by focusing on the first-best and autarky cases. One of the reasons is that both cases allow for closed form expressions of agents' value functions. This allows us to solve for optimal policy in a straightforward fashion. In particular, we restrict ourselves in that exercise to measures that vary the deterministic outside money growth rate. The identification of optimal policy in these model variants is not intended to motivate policy recommendations. This is because those variants represent overly simplified economic environments. Rather, the derivation of optimal policy aims at facilitating intuition for some basic forces that are also at work in the full-fledged model with endogenous risk.

Characterising optimal monetary policy is considerably more complex in the full model than in the autarky and first-best cases. This is due to three reasons. First, the monetary authority has to decide on $\mu_t^{MO}$ as well as $\mathcal{M}_t$, which can both be state-dependent. This allows for an infinite number of combinations of these policy objectives. Second, due to heterogeneity, a welfare criterion which assigns weights to the welfare levels of all groups of agents ought to be selected.[822] It may be argued that such procedure entails some degree of arbitrariness. Third, the welfare of any agent group depends on the current state of the economy $\eta_t$ and thus cannot be expressed by a single figure in a straightforward way. For these reasons, we decide to not engage in the calculation of optimal monetary policy in the full credit model. Rather, we compare the model results under various rules of the form (6.2.1) to the results generated by the baseline model without policy intervention and discern whether entrepreneurs' and managers' welfare can be improved.

## 6.3 k-Percent Rules

### 6.3.1 First-Best Case

We now turn to the effects of policies that affect the deterministic outside money growth in the first-best case. The assumptions and solution procedure in this subsection are very similar to those in BS (2016c). In particular, we follow these authors in adopting three key assumptions. First, we

[822] Cf. also Drechsler et al. (2018, fn. 4).

consider an one-agent-group economy. An immediate corollary is that the price of capital and the value of the money supply per unit of aggregate capital are constant. Second, $\mu_t^{M^O} = \mu^{M^O}\ \forall t$, i.e. the drift rate of the base money supply is constant. Hence, we have a particularly simple variant of rule (6.2.1), namely a k-percent rule in the spirit of Friedman (1959):[823]

$$\frac{dM_t^O}{M_t^O} = \mu^{M^O},\ \mu^{M^O} = \text{const.} \tag{6.3.1}$$

Third, inside money creation is absent from the model. We depart from BS (2016c) in three main regards. First, they exclusively motivate the demand for money by agents' desire to (partially) insure against idiosyncratic risk associated with holding capital, whereas our model variant only features a transaction motive.[824] Second, in their model, agents cannot invest in the firms of other agents. Third, they consider a discrete-time framework.[825]

Since variable $p_{F,M}$ is constant, the drift rate of the return on money is given by $\mu_{F,M}^{r^M} = \Phi\left(\iota_{F,M}\right) - \delta - \mu^{M^O}$.[826] The adoption of money supply rule (6.2.1) and policy-induced transfers changes the drift rate of managers' portfolio return to

$$\mu_{F,M}^{r^{P,m}} = x_{F,M}^m \frac{a^e - \iota_{F,M}}{q_{F,M}} - \left(1 - x_{F,M}^m\right)\mu^{M^O} + \Phi(\iota_{F,M}) - \delta + \mu_{F,M}^{\tau}\tau_t \tag{6.3.2}$$

and, hence, their FOC with respect to $x_{F,M}^m$ to

$$\frac{1+\xi}{\rho}\left(\frac{a^e - \iota_{F,M}}{q_{F,M}} + \mu^{M^O}\right) = \xi\frac{1}{1 - x_{F,M}^m}. \tag{6.3.3}$$

Due to the substitution effect of monetary policy, a greater rate of change $\mu^{M^O}$ induces the agent to put a higher portfolio weight on capital, ceteris paribus. Note that policy-induced wealth effects are absent in our one-agent group economy by definition.

Due to the presence of term $\mu^{M^O}$ in FOC (6.3.3), its combination with the unchanged market clearing conditions in (5.2.3) results in a complicated quadratic equation. Instead of solving this equation now, we first derive more convenient analytical expressions for the equilibrium values of endogenous variables. To this end, we follow BS (2016c) in defining a "transformed" money growth rate $\breve{\mu}^{M^O} \equiv \left(1 - x_{F,M}^{m*}\right)\mu^{M^O}$, where $1 - x_{F,M}^{m*}$ is managers' optimal portfolio weight on money.[827] In an equilibrium with money, the relationship between $\breve{\mu}^{M^O}$ and $\mu^{M^O}$ is strictly positive by definition: in such an equilibrium money has value and, thus, there has to be a demand for it. It follows that the optimal portfolio weight on money $1 - x_{F,M}^{m*}$ must be strictly positive.[828] Now, equilibrium can

[823] Cf. Friedman (1959, pp. 92f.).
[824] The autarky case considered in the next subsection reintroduces idiosyncratic risk.
[825] Cf. Brunnermeier and Sannikov (2016c, p. 485) for their modelling assumptions.
[826] Subscript "$F, M$" denotes time-invariant variables in the first-best case with monetary policy.
[827] Cf. Brunnermeier and Sannikov (cf. 2016c, p. 487).
[828] In the remainder, we suppress superscript "$*$" to simplify notation.

be determined by the combination of the previous definition with FOC (6.3.3) and the conditions in (5.2.3). The results are summarised in the following proposition.

**Proposition 9.** *In the first-best equilibrium with monetary policy rule (6.3.1) and a strictly positive value of money, the price of capital $q_{F,M}$, the value of the outside money supply per unit of capital $p_{F,M}$, the optimal portfolio choice $x^m_{F,M}$, and the optimal investment rate $\iota_{F,M}$ are given by*

$$q_{F,M} = \frac{\gamma a^e + 1}{\gamma\left(\rho - \breve{\mu}^{MO}\right) + 1}, \tag{6.3.4a}$$

$$p_{F,M} = \frac{\xi\rho - (1+\xi)\,\breve{\mu}^{MO}}{\rho}\frac{\gamma a^e + 1}{\gamma\left(\rho - \breve{\mu}^{MO}\right) + 1}, \tag{6.3.4b}$$

$$x^m_{F,M} = \frac{\rho}{\left(\rho - \breve{\mu}^{MO}\right)(1+\xi)}, \quad \textit{and} \tag{6.3.4c}$$

$$\iota_{F,M} = \frac{a^e - \left(\rho - \breve{\mu}^{MO}\right)}{\gamma\left(\rho - \breve{\mu}^{MO}\right) + 1}. \tag{6.3.4d}$$

*A moneyless equilibrium exists if and only if $\xi = 0$ and $\breve{\mu}^{MO} \geq 0$ hold simultaneously. In this case, prices, optimal portfolio and investment decisions satisfy $q_{F,M} = q_F$, $p_{F,M} = 0$, $x^m_{F,M} = 1$ and $\iota_{F,M} = \iota_F$.*

*Proof.* See Appendix D.1.2.

An important implication of Proposition 9 is that money is *not* neutral provided that money has value. A greater value of $\breve{\mu}^{MO}$ raises the optimal investment rate $\iota_{F,M}$, as can be seen from equation (6.3.4d). The growth rate of the economy increases as well since it is identical to the growth rate of the aggregate capital stock $\Phi(\iota_{F,M}) - \delta$. The intuition for the nonneutrality result is as follows. An accelerated rate of change in the money supply makes money less attractive relative to capital, according to (6.3.4c). The heightened demand for capital raises $q_{F,M}$, as can be observed from equation (6.3.4a). The appreciation in the value of capital, in turn, stimulates investment demand and thus raises $\iota_{F,M}$. The nonneutrality result holds even in the long run. Accelerated money growth permanently raises the price of capital. Since the marginal product of capital is constant, the effect is a permanent increase in the growth rate of real GDP.[829] The nonneutrality result is in accordance with the model in BS (2016c).[830] As these authors note, the portfolio composition effect of monetary policy was already identified by Tobin (1965).[831] In contrast, in the standard MIU model with a decreasing marginal product of capital, money is superneutral at the steady state.[832]

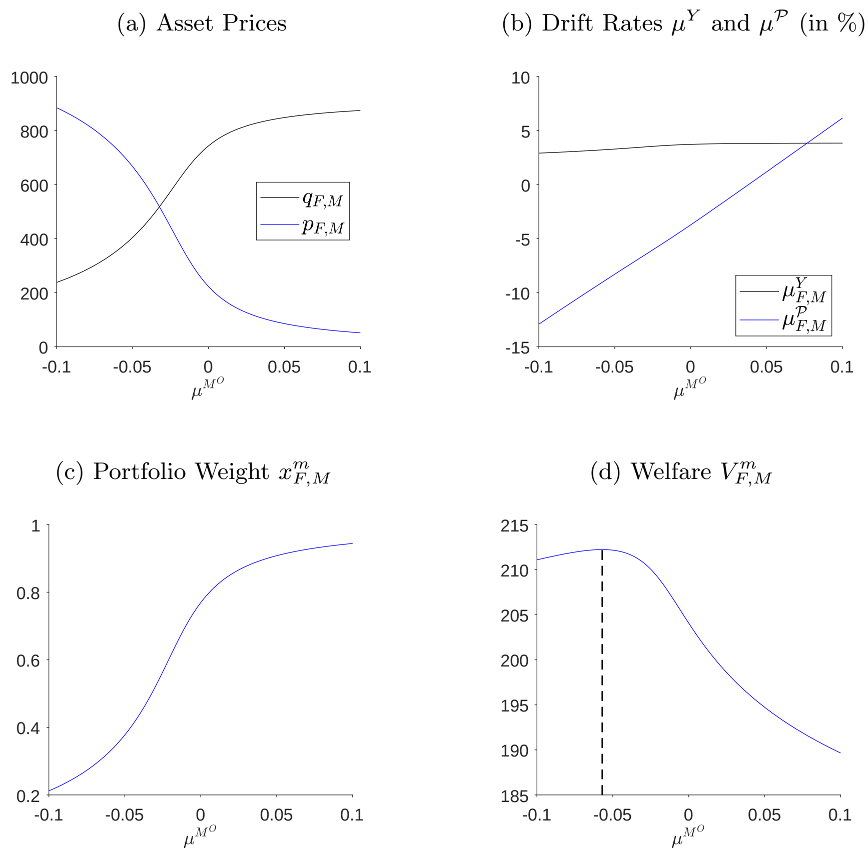
[829] If the marginal product of capital was decreasing rather than constant, capital would be accumulated until the condition $\Phi(\iota_{F,M}) = \delta$ would be met. Inserting (3.1.9) into that condition and solving for the price of capital implies $q_{F,M} = e^{\delta\gamma}$, which obviously does not depend on monetary policy.

[830] Cf. Brunnermeier and Sannikov (2016c, Corollary 1).

[831] Cf. Brunnermeier and Sannikov (2016c, p. 488) and Tobin (1965, p. 677).

[832] Sidrauski (cf. 1967, pp. 543f.)

Figure 6.3.1: Equilibrium Outcomes and Welfare in the First-Best Case under Different k-Percent Rules




However, nonneutrality can emerge during the transition towards the steady state unless preferences are log separable between consumption and real balances, as shown by Fischer (1979).[833]

Two further implications of Proposition 9 are worth of notice. First, if $\breve{\mu}^{M^O} < 0$ holds, money has value even if there is no transactions demand for it, i.e. even if $\xi = 0$. This results from the fact that in this case, the capital gains rate of money, which is equal to $\Phi\left(\iota_{F,M}\right) - \delta - \mu^{M^O}$, is greater than that of capital, given by $\Phi\left(\iota_{F,M}\right) - \delta$. Hence, compared to the case with $\xi > 0$ and $\mu^{M^O} \geq 0$, there is a return motive for holding money rather than a transaction motive. Second, the optimal portfolio choice in (6.3.4c) does not depend on the productivity level $a^e$. An intuition can be gained by substituting (6.3.4a) and (6.3.4d) into the payoff yield of capital, which leads to $\left(a^e - \iota_{F,M}\right)/q_{F,M} = \rho - \breve{\mu}^{M^O}$. This yield is independent of the productivity level since a rise in $a^e$ is exactly offset by the resulting increases in $q_{F,M}$ and $\iota_{F,M}$.

In order to assess how monetary policy affects equilibrium outcomes quantitatively, we now express the endogenous variables as functions of the growth rate of the outside money supply $\mu^{M^O}$. Since the derivation of these functions involves the solution of a complicated quadratic equation in

[833] Cf. Fischer (1979, p. 1438).

$q$, we explain the solution procedure in Appendix D.2 and report the results under the parameter calibrations listed in Section 5.1 in Figure 6.3.1. The upper left panel demonstrates that the price of capital increases and the value of the money supply per unit of capital shrinks with the money growth rate, as suggested by Proposition 9. These relationships are most accentuated in the region of negative money growth. As $\mu^{M^O}$ increases, the price of capital approaches its value in the moneyless equilibrium. At the same time, $p_{F,M}$ falls towards zero. However, the moneyless equilibrium is never fully reached as the utility weight $\xi$ is strictly greater than zero in the baseline calibration. Overall, the effects on output growth are rather mild: that rate is only about one percentage point higher at $\mu^{M^O} = 0.1$ than at $\mu^{M^O} = -0.1$. This observation can be explained by the strong curvature of the capital goods production function. As a consequence, changes in $\mu^{M^O}$ nearly translate to one-to-one changes in the inflation rate given by $d\pi = -\mu^{r^M} = \mu^{M^O} - \Phi\left(\iota_{F,M}\right) - \delta$. As suggested by FOC (6.3.3), managers' portfolios are shifted towards a higher portfolio weight on entrepreneurs' equity $x_{F,M}$ at the expense of the weight on money. This shift in *desired* asset allocation is reflected in the behaviour of $q_{F,M}$ and $p_{F,M}$ in the upper left panel.

Next, we evaluate the effects of adjustments in the k-percent rule on welfare. Proposition 10 provides a closed form solution of the value function of an individual manager, who is endowed with capital of amount $k^m_{i,0} = K_0 = 1$.

**Proposition 10.** *In the first-best equilibrium with monetary policy rule (6.3.1), the value function of a representative manager endowed at time $t = 0$ with wealth $\left(q_{F,M} + p_{F,M}\right) K_0$, where the initial capital stock $K_0$ is normalised to unity, is given by:*

$$\begin{aligned} V^m_{F,M}(n^m_{i,0}) &= V_0 + \frac{\xi}{\rho} \log\left(1 - x^m_{F,M}\right) + \frac{1+\xi}{\rho} \log\left(q_{F,M} + p_{F,M}\right) \\ &\quad + \frac{1+\xi}{\rho^2}\left[\mu^{r^{P,m}}\left(x^m_{F,M}\right) + \mu^{\tau}_{F,M}\tau_t\right], \end{aligned} \tag{6.3.5a}$$

$$\textit{where} \quad V_0 \equiv \frac{1}{\rho}\left[\log\left(\frac{\rho}{1+\xi}\right) - 1 + \lambda \frac{1+\xi}{\rho}\log\left(1-\kappa\right)\right] \quad \textit{and} \tag{6.3.5b}$$

$$\mu^{r^{P,m}}\left(x^m_{F,M}\right) = x^m_{F,M}\frac{a^e - \iota_{F,M}}{q_{F,M}} - \left(1 - x^m_{F,M}\right)\mu^{M^O} + \Phi(\iota_{F,M}) - \delta. \tag{6.3.5c}$$

*Proof.* See Appendix D.1.3.

This proposition tells us that an increase in the money growth rate affects welfare via three channels. First, as a result of the induced decline in the portfolio weight on money, the flow utility from holding real balances falls. This is captured by the second term on the RHS of equation (6.3.5a). Second, accelerated money growth alters initial wealth through its impact on prices (third term). Third, it alters the sum of the deterministic portfolio return $\mu^{r^{P,m}}\left(x^m_{F,M}\right)$ and transfers per unit of net worth (fourth term).[834]

The lower right panel of Figure 6.3.1 depicts maximum lifetime utility $V^m_{F,M}(\cdot)$ for different values of the money growth rate. Obviously, there is a unique optimal rate, which takes on a value approximately equal to $-6.0$ percent under the baseline calibration. Thus, the competitive equilibrium of the first-best economy without money growth is *not* Pareto efficient. This is mainly due to the fact that agents' holdings of real balances are too low in the equilibrium without policy[835], a result which arises from a pecuniary externality inherent in MIU models: when adjusting their demand for money, individuals do not take into account the repercussions for the price level and, thus, the value of real balances in the utility functions of other agents.[836] By increasing the real return on money, the central bank tilts the portfolio choice towards purchasing power, which induces an appreciation in its value. An intuition for that outcome was first developed by Friedman (1969).[837] The drawback of a policy that continuously lowers the money supply is that it slows down growth. Yet, that effect is relatively mild under our baseline calibration, as discussed. Accordingly, the effect on welfare is attenuated.

The fact that the inflation rate is negative at the welfare-maximising level of money growth in our model might lead one to assume that the optimal rate of inflation is negative in general. Such a result would be in accordance with the *Friedman rule.* Friedman recommended to set the nominal interest rate to zero in order to minimise the opportunity cost of holding money. Such a policy will result in a negative inflation rate, provided that the real return is positive.[838] However, the welfare-maximising inflation rate need not be negative in our setting. This can be demonstrated by raising the deterministic capital depreciation rate by ten percentage points to 11 percent. This parameter adjustment leaves optimal money growth unchanged but raises the inflation rate to a slightly positive value. This can be explained by recognising that the opportunity cost for holding money in our model is the real return on capital. A change in the capital depreciation rate, however, leads to *identical* adjustments in the real returns on money and capital, a result which can be traced back to the adoption of AK production technologies.

The presented outcomes raise the question whether the optimal money growth rate is negative under different parameter constellations as well. In particular, we lower parameters $\xi$, the utility weight on real balances, and $\gamma$, which governs the curvature of the capital goods production function.

[834] The effect of accelerated money growth on the deterministic portfolio return is not immediately clear, but rather depends on parameters. This can be seen from the following calculations. Substituting (6.3.4a), (6.3.4c), (6.3.4d) and relation $\mu^{M^O} = \breve{\mu}^{M^O} / \left(1 - x^m_{F,M}\right)$ into (6.3.5c) leads to $\mu^{r^{P,m}}\left(x^m_{F,M}\right) = \rho/(1+\xi) - \breve{\mu}^{M^O} + \Phi(\iota_{F,M}) - \delta$. Thus, monetary policy can influence the portfolio return through the payoff yield form holding money and the capital gains rate $\Phi(\iota_{F,M}) - \delta$. The contribution of the payoff on capital to the portfolio return $\rho/(1+\xi)$, on the other hand, does not depend on $\breve{\mu}^{M^O}$ since the effect of a higher portfolio weight $x^m_{F,M}$ is exactly offset by increases in $q_{F,M}$ and $\iota_{F,M}$.

[835] As the numerical solution of the first-best case shows, amongst the terms on the RHS of (6.3.5a), the second is most responsive to variations in $\mu^{M^O}$.

[836] Cf. footnote 729.

[837] Cf. Friedman (1969, pp. 14f.).

[838] Cf. Friedman (1969, p. 34).

Numerical testing shows that the welfare-maximising money growth rate increases but remains negative, even for extremely low values of those parameters. This is related to the fact that real balances enter flow utility in a logarithmic form. Thus, if agents cut their portfolio weights on money towards zero, instantaneous utility approaches $-\infty$, regardless of the utility weight on real balances.

### 6.3.2 Autarky Case

Considering the autarky case allows us to assess the role of idiosyncratic risk in determining the optimal policy in a simple one-agent-group model. Figure 6.3.2 compares the equilibrium values of endogenous variables in the autarky and the first-best cases for different k-percent rules. Both the price of capital and the value of money per unit of capital are lower in the former case (Panel (a)). The drop in $q$ causes the growth rate of real GDP to slow down, which, in turn, leads to accelerated inflation at any given value of $\mu^{M^O}$ (Panel (b)). Yet, these results are mainly due to the reduction in the productivity level rather than the introduction of idiosyncratic risk. This can be shown by solving the autarky model with managers' productivity level set equal to that of entrepreneurs. Under this calibration, the mentioned differences between the two cases virtually disappear. Turning to portfolio selection, recall from the previous subsection that the portfolio weight is not influenced by the productivity level. Thus, changes in the portfolio choice of managers relative to the first-best case are solely due the presence of idiosyncratic shocks. Since idiosyncratic risk makes holding capital less attractive for risk averse individuals, managers' portfolio weight on capital falls (Panel (c)). However, the portfolio adjustments are small.

**Proposition 11.** *In the autarky equilibrium with monetary policy rule (6.3.1), the value function of a manager endowed at time $t = 0$ with wealth $(q_{A,M} + p_{A,M})\,K_0$, in which the initial capital stock $K_0$ is normalised to unity, is given by*

$$\begin{aligned} V^m_{A,M}(n^m_{i,0}) = V_0 &+ \frac{\xi}{\rho}\log\left(1 - x^m_{A,M}\right) \\ &+ \frac{1+\xi}{\rho}\log\left(q_{A,M} + p_{A,M}\right) + \frac{1+\xi}{\rho^2}\left[\mu^{r^{P,m}}\left(x^m_{A,M}\right) + \mu^{\tau}_{A,M}\tau_t\right] \\ &+ \lambda\frac{1+\xi}{\rho^2}\left[\phi\log\left(1 - \kappa - x^m_{A,M}\underline{\kappa}\right) + (1-\phi)\log\left(1 - \kappa - x^m_{A,M}\overline{\kappa}\right)\right], \end{aligned} \tag{6.3.6a}$$

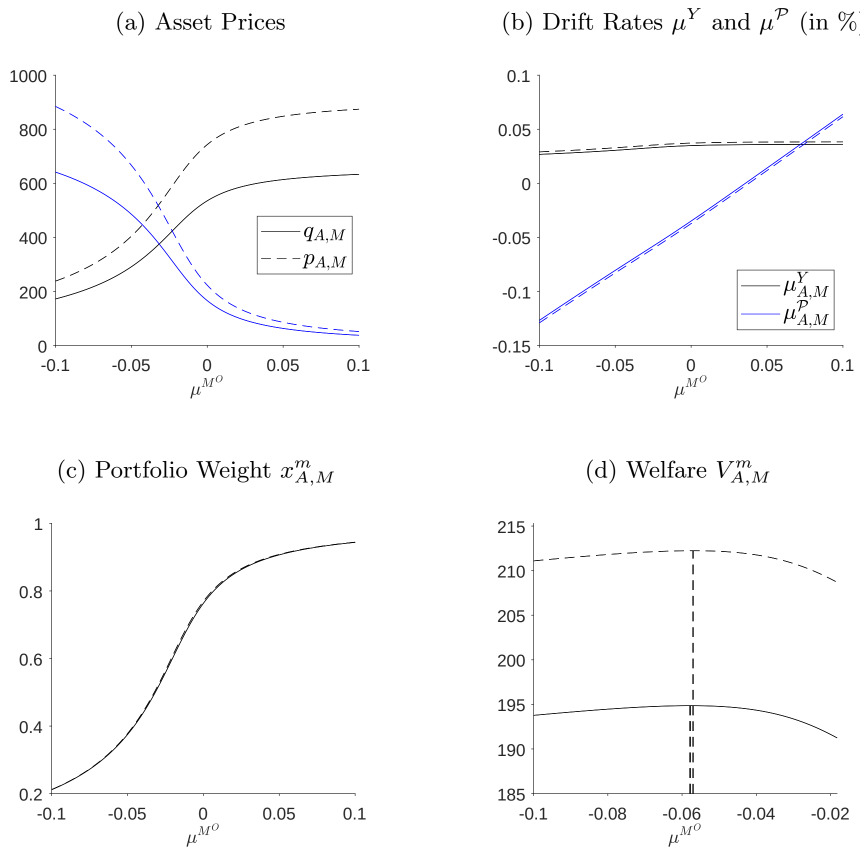
$$\textit{where}\quad V_0 \equiv \frac{1}{\rho}\left[\log\left(\frac{\rho}{1+\xi}\right) - 1\right] \quad \textit{and} \tag{6.3.6b}$$

$$\mu^{r^{P,m}}\left(x^m_{A,M}\right) = x^m_{A,M}\frac{a^m - \iota_{A,M}}{q_{A,M}} - \left(1 - x^m_{A,M}\right)\mu^{M^O} + \Phi(\iota_{A,M}) - \delta. \tag{6.3.6c}$$

*Proof.* See Appendix D.1.4.

The lower right panel of Figure 6.3.2 plots the value function against different values of $\mu^{M^O}$. This graph reveals that the optimal money growth rate is lower in the autarky case, albeit merely by a very small amount. This fact is visible only in a magnified excerpt of the $V^m_{A,M}$, $\mu^{M^O}$ plane. The reason for the decline in welfare-maximising money growth is that a policy which cuts the money supply exposes individuals to less idiosyncratic risk by raising their demand for money. The ensuing boost in the value of money, which raises agents' flow utility, is not taken into account by individuals in their portfolio decisions.

Figure 6.3.2: Equilibrium Outcomes and Welfare in the Autarky (solid) and First-Best Case (dashed) under Different k-Percent Rules




As in the previous subsection, we examine the optimal k-percent rule further by varying parameters. Increasing the amplitude of idiosyncratic shocks further reduces optimal money growth due to the aforementioned effect. However, this relationship depends on parameter $\gamma$, the curvature parameter in the capital goods production function. Lowering $\gamma$ from 140 to 20 implies that the welfare-maximising money growth rate remains negative but is now *increasing* in the amplitude of idiosyncratic shocks. This is because another pecuniary externality becomes more potent: when tilting their portfolio towards money, individuals neglect the resulting effect on the price of capital. The ensuing depreciation in the value of capital harms economic growth. This effect is the stronger, the lower parameter $\gamma$. It should be added, however, that a value of $\gamma = 20$ implies implausibly large ratios of investment to GDP. For instance, in the Autarky model with a constant outside money supply, that value is equal to about 62 percent.

The aforementioned pecuniary externality with respect to the price of capital is also present in BS (2016c), which leads them to conclude that optimal money growth is positive in their model, provided that idiosyncratic risk is sufficiently high.[839] The difference in results is due to the fact that their model does not feature a pecuniary externality arising from the presence of real balances in agents' utility function.

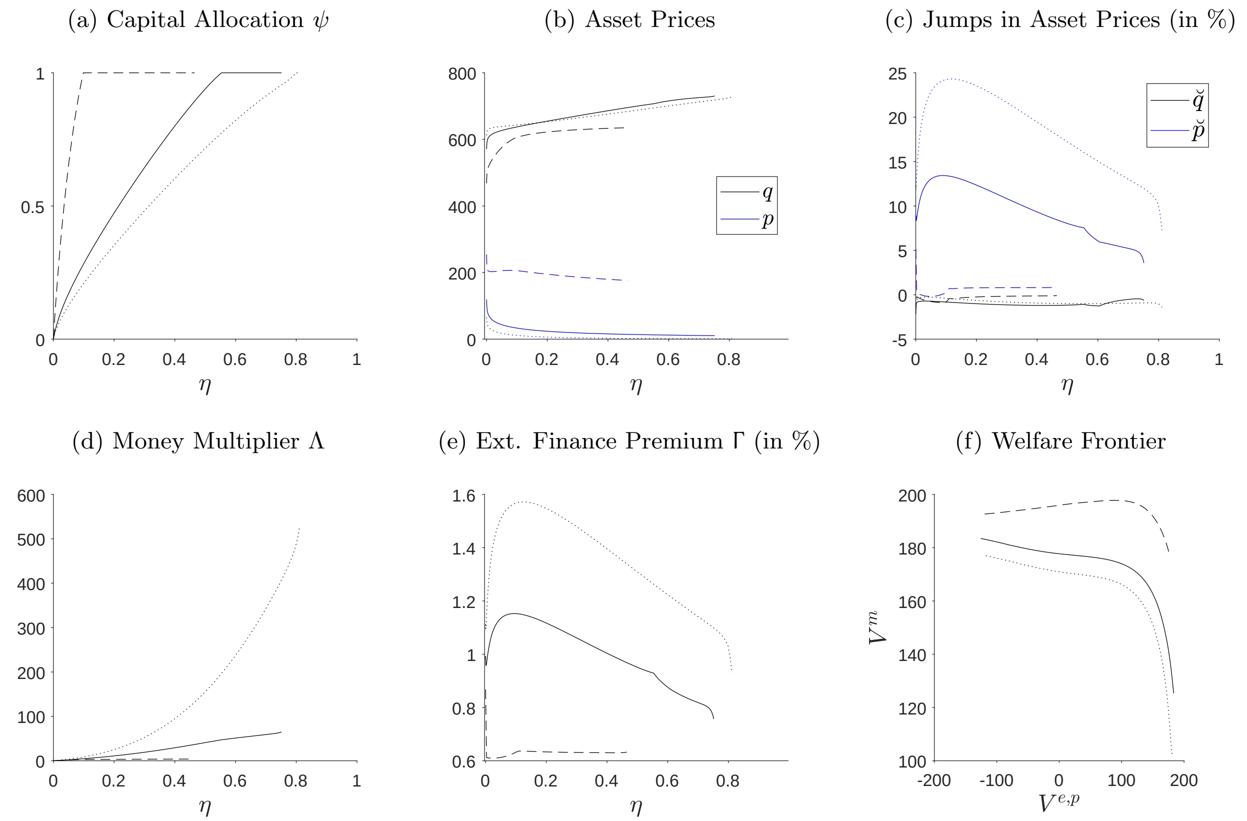
### 6.3.3 Full Baseline Model

In the remainder of this chapter, we focus on the full credit model with endogenous risk. In this subsection, we consider two cases, in which the outside money growth rate is pegged at $\mu^{M^O} = -2.5$ percent and $\mu^{M^O} = 2.5$ percent, respectively. Let us first examine the former case. Panel (b) in Figure 6.3.3 shows that the capital price is lower than in the baseline model without policy. The reasoning is identical to that in the previously explored one-agent-group economies: since holding money becomes more attractive as a result of negative base money growth, the aggregate demand for capital falls.

Further, the value of money is now much higher than in the baseline model with passive policy[840] due to the continuous reduction in the stock of outside money. This result is due to the fact that agents expect the negative outside money growth rate to counteract the surge in inside money associated with the steadily increasing volume of bank lending in periods without shocks. This outcome is also part of the reason why the price of money per unit of aggregate capital is now more stable. The second reason is that the growth in the money multiplier is reduced under the prescribed central bank rule (Panel (d)). Remarkably, this holds true even though the nominal stock of money created by the central bank, which is the denominator in the defining equation for the money multiplier (4.4.13), is lower. The explanation for this observation is that the numerator of the mentioned equation falls even more as a consequence of the reduced demand for bank loans in *nominal* terms: due to the drop in the value of capital relative to money, a given amount of capital can be bought with less money. Again, the mirror image of decelerated growth in debt in normal times is a mitigated percentage drop in the volume of loans in crisis episodes. This is reflected in the modest response of $p_t$ to shocks, as depicted in Panel (c). Simulation of the model corroborates these findings: the mean of the inflation rate falls to $-1.39$ percent, while the standard deviation is significantly reduced to about 0.1 percent (third line of Table D.1 in Appendix D.3). Since banks' losses in real terms are less severe due to a decrease in deflationary pressure in case of a jump, the EFP drops (Panel (e)).

[839] Cf. Brunnermeier and Sannikov (2016c, p. 489).
[840] With the term "passive policy", we refer to the case of a constant outside money supply.

Figure 6.3.3: Equilibrium Quantities, Prices, and Welfare for No-Policy Regime (solid), $\mu^{M^O} = -2.5$ (dashed), and $\mu^{M^O} = 2.5$ Percent (dotted)

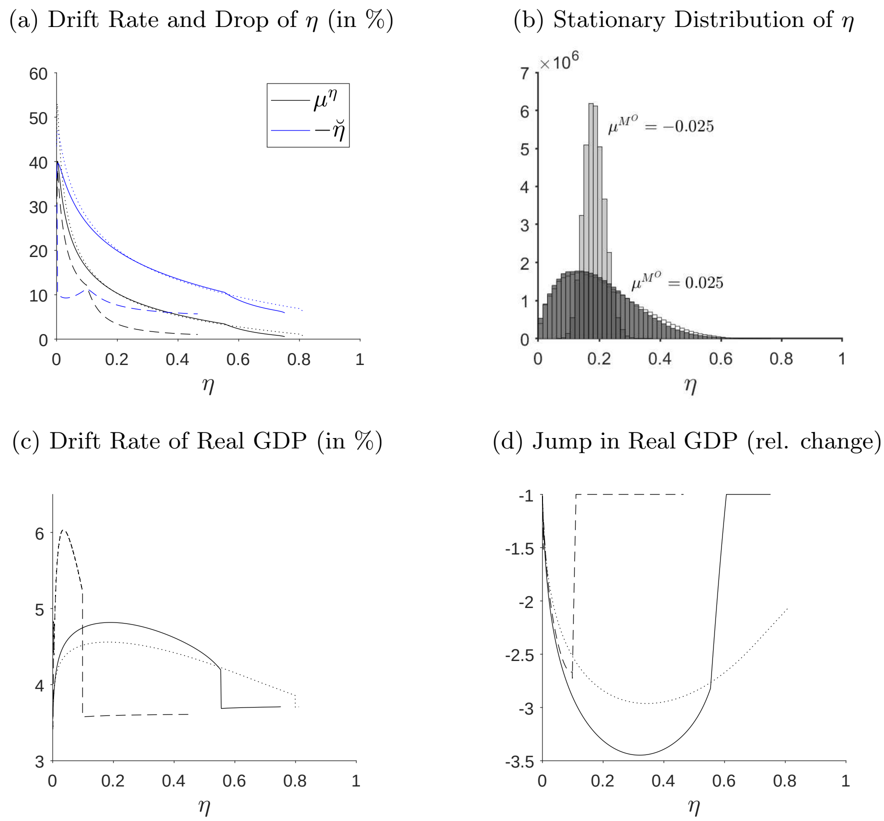



Panel (a) demonstrates that the optimal capital allocation $\psi_t$ is reached at a much lower value of the state variable. Two main factors are responsible for this result. First, given prudent entrepreneurs' portfolio weights on capital and money, a decline in the value share of capital $\theta_t$ causes entrepreneurs' share in the aggregate capital stock $\psi_t$ to rise (cf. equation (4.5.6d)). Second, for any $\eta_t \in (0, \eta^s]$, prudent entrepreneurs' portfolio weight on capital $x_{1,t}^{e,p}$ is higher relative to the case without policy. This might seem surprising, given the fact that entrepreneurs' premium on capital and, thus, the share of wealth those individuals allocate to capital are increasing in $\mu^{M^O}$, ceteris paribus, according to equation (6.2.5) and FOC (6.2.11). Yet, in contrast to the one-agent-group economies discussed in the previous subsections, two other factors come into play, namely, the reductions in the EFP and endogenous risk. As a consequence, $x_{1,t}^{e,p}$ spirals upwards. Given the outcome that entrepreneurs allocate a higher portion of their wealth to capital, it is obvious that the drop in the aggregate demand for the productive asset is due to lower demand by managers. Indeed, anticipating that the future value of money will be higher on average, managers are more willing to sell capital to entrepreneurs in exchange for purchasing power. We can gain further intuition by focusing on policy-induced substitution and wealth effects, the latter of which not being present in the previously examined one-agent-group economies. The wealth effect is more important to prudent entrepreneurs than to managers, since the former are levered: a fall in endogenous risk implies a marked reduction in losses resulting from shocks. It follows that agents of type $e, p$ are willing to take on additional capital even though the risk premium is reduced - the wealth effect dominates. Conversely, managers suffer lower losses than entrepreneurs or even experience small (real) profits once shocks hit since they are not indebted. It follows that the substitution effect overcompensates the wealth effect. It is important to recognise that these results are driven by agents' risk aversion: the concavity of the value function implies that large reductions in wealth lead to overproportionate losses in utility.

The decline in endogenous price risk also has repercussions for the evolution of the state variable (Panel (a) in Figure 6.3.4). Despite heightened leverage, the percentage drop in the former variable triggered by the arrival of a jump is lower for the most part. Due to the risk return-return trade-off, the drift rate of the state variable is also reduced, albeit only slightly as a result of higher leverage ratios. Interestingly, the percentage drop of entrepreneurs' wealth share is U-shaped in the misallocation region. This is mainly due to the behaviour of function $\breve{p}(\eta_t)$, which has a similar shape. In turn, the latter result is related to the fact that the drop in output is now more significant relative to the drop in inside money, especially at the centre of the misallocation region. The distribution of the state variable is much more concentrated under the contractionary monetary policy (Panel (b)). Specifically, the distribution is centred at about $\eta_t = 0.2$, which lies in the subset of the state space where the allocation of capital is at its efficient level. Moreover, the fraction of time the economy spends in the misallocation region plummets to less than one percent. Thus, the persistence of capital misallocation is strongly reduced. This result is due to the fact that the drop in the state variable is now much lower in percentage terms than the drift rate of that variable, whenever $\psi_t < 1$.

The drift rate of output surges relative to the case with a passive central bank in the misallocation region since the capital allocation improves at a faster rate due to increased leverage (Panel (c)). The percentage drop in output, on the other hand, is at a similar scale as in the model without policy intervention whenever $\psi_t < 1$, despite the reduction in the percentage drop of the state variable (Panel (d)). This is because function $\psi(\eta_t)$ now has a steeper slope in that case. Yet, as the economy only occasionally enters states in which managers hold capital, these outcomes have limited effect on the mean growth rate of real GDP. Consequently, average growth is mainly driven by the rate of change in the capital stock. The mean of the latter decelerates since the price of capital is at a lower level throughout the state space. These observations explain why average output growth decreases slightly to 0.77 percent (third line of Table D.1). Further, the rate of change in average real GDP now exhibits less volatility as fire-sales of capital to managers rarely occur. Concerning the absolute value of output, note that monetary policy exerts a level as well as a composition effect: on the one hand, the mitigation of capital growth reduces output, ceteris paribus. This negative effect is counteracted by the improved allocation of the productive asset. As

Figure 6.3.4: Dynamics of the State Variable and Output for No-Policy Regime (solid), $\mu^{M^O} = -2.5$ (dashed), and $\mu^{M^O} = 2.5$ Percent (dotted)




*Notes*: The stationary distribution in Panel (b) is computed from a simulated time series of $\eta$ (cf. notes to Table 5.3.4 for details); the stationary distribution under no policy in Panel (b) is displayed by white bars; in Panel (d), a linear interpolation of the dotted curve in the proximity of $\eta^{\psi}$ is applied (cf. Appendix C.4.2 for details).

time passes, the former effect becomes more important relative to the second.

Turning to welfare analysis, Panel (f) reveals that the welfare frontier is shifted outwards under rule $\mu^{M^O} = -2.5$ percent. Thus, prudent entrepreneurs' as well as managers' lifetime utility improves for any value $\eta_t \in (0, \eta^s]$.[841] As in the first-best and autarky case, this is because monetary policy counteracts different forms of pecuniary externalities. In the full model with inside money creation by banks, three additional - and novel - channels emerge. For one, recall that while deleveraging, any individual entrepreneur does not take into account his contribution to changes in the price level. As discussed, the continuous reduction in the outside money stock attenuates the resulting upward pressure in the value of money via its effect on agents' expectations of future price paths. The second and third forms of externalities result from entrepreneurs' neglect of their own contribution to changes in the value of money, when demanding additional credit in periods

[841] Simultaneous increases in the welfare of prudent entrepreneurs and managers need not be tantamount to Pareto improvements. This is because the economy is also populated by imprudent entrepreneurs and bankers.

without jumps. The upshot is a suboptimally low price of money, which induces agents to take on excessive idiosyncratic risk and drains their real balances. Accordingly, contractionary policies that raise the value of purchasing power alleviate the negative consequences of those externalities. On the downside, risk premia on capital are compressed and the growth in the capital stock slows down. Both effects shrink the profits of capital holders in normal times, all else equal. Yet, these ramifications are overcompensated by the aforementioned beneficial effects.

One might expect that different k-percent rules have symmetric effects on equilibrium outcomes. Yet, the example of $\mu^{M^O} = 2.5$ percent demonstrates that this need not be the case for all variables. In particular, equilibrium functions for the capital price, the dynamics of output and the state variable, as well as the distribution of the latter are comparable to the case of a constant stock of base money. The most accentuated differences include the value of money and the amount of inside money in circulation. Anticipating accelerated outside money growth and inflation, entrepreneurs are willing to let their debt grow at a faster rate, which, in turn, significantly raises the volatility of the inflation rate (fourth line of Table D.1). Banks react by quoting higher mark-ups over the deposit rate. As a result of these developments, welfare of managers and prudent entrepreneurs is harmed.

To conclude, in our model under the baseline calibration of nonpolicy parameters, the monetary authority may achieve output growth and inflation stabilisation at the same time by means of continuous reductions in the stock of outside money. That result is reminiscent of the "divine coincidence" arising in NK models without real rigidities, which states that the central bank does not face a trade-off between stabilising inflation and the output gap, with the latter being defined as the distance of current output to potential output, i.e. output under a frictionless benchmark.[842] A closer comparison to NK models would thus require us to compute the standard deviation of the difference between current output and output in the first-best case without financial frictions. However, such procedure would run into the problem that current and efficient output obey differing trend growth. Therefore, we remove the trend component from both series by first-differencing and subsequently calculate the difference of the two detrended series. We find that the volatility of the so-defined output gap is significantly reduced under the contractionary policy relative to the baseline case, namely by one order of magnitude. Adopting our previously introduced alternative measure of the output gap $1-\psi_t$ confirms that result. Yet, it should be recognised that in our model, stabilisation policies might be detrimental to welfare. The reason is that the stance of monetary policy affects the trend components of actual *as well as* potential output, in distinction from NK models.[843] Indeed, in our example of negative outside money growth, average economic growth is lower than in the model with passive policy.

While in both frameworks monetary policy does not face a trade-off in stabilising inflation and

[842] Cf. Blanchard and Galí (2007, p. 36).

[843] Cf. e.g. Rotemberg and Woodford (1997, p. 335) or Clarida et al. (1999, pp. 1668f.) on the optimality of stabilisation policies in NK models.

output gap measures, the mechanism in our model differs from that in NK models without real rigidities: whereas in the latter class of models the procyclicality of firms' real marginal costs drives that result[844], in the credit model it is due to the fact that a decrease in deflationary pressure after a jump induced by continuous reductions in the narrow money supply leads to less fire sales of capital to less productive managers on average.

## 6.4 Monetary Policy Reaction Functions

### 6.4.1 Deterministic Outside Money Growth

In this and the succeeding subsection, we allow for variations in money growth. Firstly, we focus on policies that affect deterministic money growth $\mu_t^{M^O}$ but do not entail discrete changes in the money supply in case of a jump. With regard to the equation of exchange, a straightforward way to incorporate the former type of policies would be to adopt a deterministic money growth rule of the from $\mu_t^{M^O} = \mu^{M^O}(dY_t/Y_t, d\pi_t)$, in words, a rule that sets $\mu_t^{M^O}$ as a function of the current growth rate of output and the current inflation rate. Yet, we do not take this route since reaction functions of that type are difficult to implement numerically. This is because the current growth rate of real GDP and the inflation rate depend on variables $\mu_t^{\psi}$ and $\mu_t^{p}$, respectively. These variables can only be calculated ex post, i.e. *after* the differential equation for the value share of capital has been solved.[845] Taking these considerations into account, we employ an alternative rule of the family

$$\mu_t^{M^O} = \overline{\varrho} + \varrho_\eta \eta_t,\ \overline{\varrho}, \varrho_\eta = \text{const.} \tag{6.4.1}$$

Rule (6.4.1) is attractive since it is simple to implement numerically. This is because the only argument is the state variable. How can that rule be interpreted? One regularity in our model is the *negative* correlation between loan growth in tranquil times without jumps and entrepreneurs' wealth share, which holds across different parameter constellations and policies. The driving force of this relationship is the countercyclicality of entrepreneurs' leverage, which, in turn, depends on the countercyclicality of risk premia. Therefore, a policy that sets $\varrho_\eta > 0$ "leans against the financial wind"[846], i.e. it lowers the outside money growth rate when deterministic debt growth is strong on average. In this sense, our rule is similar to macroprudential Taylor rules in NK models which set the short-term nominal rate in response to variations in nominal credit growth.[847] For this reason, we refer to rules of type (6.4.1) as "macroprudential money supply rules" (MMRs). It should be recalled in this context that growth in debt in our model can be "excessive" in the sense that agents

[844] Cf. Walsh (2010, pp. 337f. and 348f.).

[845] Cf. Appendix C.4.3 for details. Further, note that the set of equilibrium equations in Proposition 2 only contains drift rate differential $\mu_t^q - \mu_t^p$, rather than $\mu_t^p$ itself.

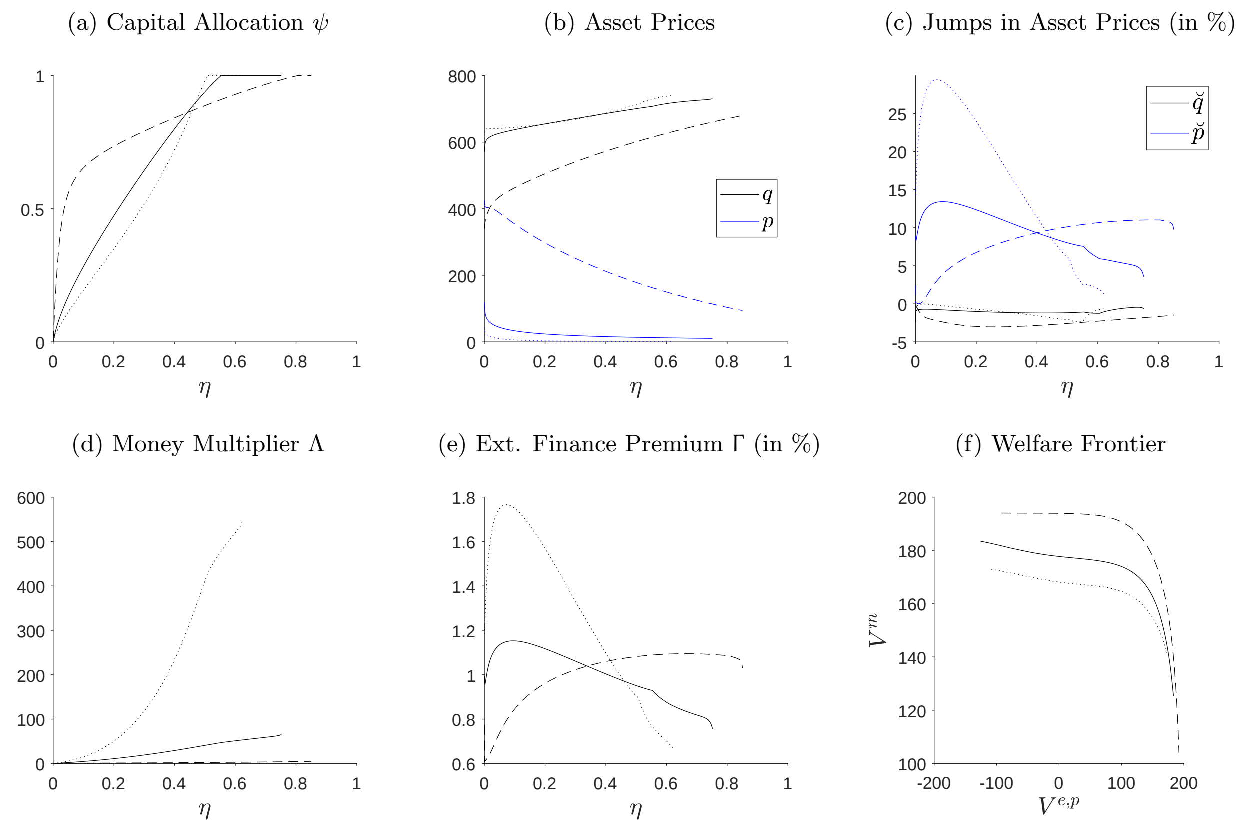
[846] Beau et al. (2012, p. 11)

[847] Some examples of NK models that incorporate a macroprudential Taylor rule reacting to credit growth are Kannan et al. (cf. 2012, p. 13), Ozkan and Unsal (cf. 2014, p. 17), and Quint and Rabanal (cf. 2014, p. 202).

do not take into account pecuniary externalities when levering up. In addition, accelerated growth in debt in normal times is associated with excessive deleveraging when the economy is hit by adverse shocks. Again, the repercussions for prices are ignored by debtors and the Paradox of Prudence emerges (cf. Subsection 5.3.5).

For illustrative purposes, we examine two policies, which we refer to as MMR1 and MMR2, that set the policy tuple $(\overline{\varrho}, \varrho_\eta)$ to $(-0.05, 0.1)$ and $(0.05, -0.1)$, respectively. Note that under both rules, the value of $\mu_t^{MO}$ at the midpoint of interval $\eta_t \in [0, 1]$ is equal to zero. Figures 6.4.1 and 6.4.2 plot equilibrium functions and welfare under the two outside money growth regimes compared to the baseline scenario absent monetary policy. Since the deterministic outside money growth rate is negative in the lower half of the state space under MMR1, many results from the previously discussed case with constant negative outside money growth carry over to that region. The increase in the supply of inside money in the absence of jumps is in part compensated for by the decrease in the outside money supply. As a result, inflation in normal times shrinks relative to the scenario without interventions by the central bank. The ensuing slowdown in the growth of debt improves system stability, which, in turn, leads to a more efficient capital allocation at any value of $\eta_t$.

Figure 6.4.1: Equilibrium Quantities, Prices, and Welfare for No-Policy Regime (solid), MMR1 (dashed), and MMR2 (dotted)



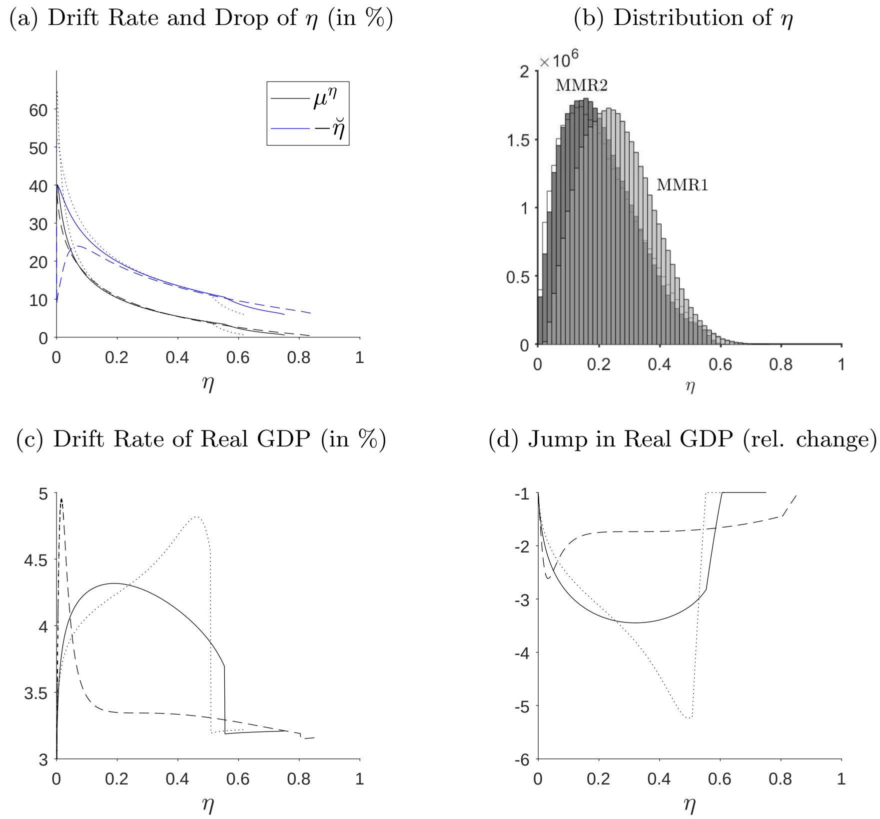
Some differences to the k-percent case $\mu^{MO} = -2.5$ percent arise because the stance of monetary policy becomes decreasingly restrictive as the state variable rises. For one, the ensuing increase in the aggregate demand for capital exerts additional upward pressure on the price of the productive asset. It follows that function $q(\cdot)$ is steeper than in the case with a passive monetary authority. This explains why the percentage drop in the value of capital is more pronounced throughout. Furthermore, deflationary pressure after a macro shock is now an increasing function of entrepreneurs' wealth share for the most part. One of the reasons is that the percentage drop in the money multiplier is markedly diminished in the lower half of the state space relative to the case without policy, a result which is the mirror image of the slower growth in debt in normal times.

Moreover, while the capital allocation improves at a fast rate initially, that speed slows down markedly after $\eta \approx 0.1$ due to heightened asset price risk. As a consequence, the drift rate of real GDP is large at that point and falters afterwards. Analogously, the deterioration in the capital allocation after the occurrence of a jump peaks at $\eta \approx 0.1$ and falls after that point, which explains the shape of function $\check{Y}(\cdot)$. The fact that the percentage drop in output is a monotonously decreasing function of the state variable for the most part further contributes to the attenuation of the deflationary spiral, especially at higher values of entrepreneurs' share in aggregate wealth. These facts also explain why both the standard deviation of output growth and inflation are reduced (fifth line of Table D.1). The slowdown in average economic growth is due to two factors, namely the decelerated improvement in the allocation of capital at higher values of the state variable and the drop in the price of capital. Another interesting outcome is that the value of the outside money supply remains well above that in the case of passive monetary policy even in interval $\eta_t \in (0.5, \eta^s]$, where the deterministic money growth rate is positive. The reason is agents' anticipation of future shocks that may push the economy into subsets of the state space where the stock of outside money shrinks.

The percentage drop of the state variable is low at extreme crisis states due to less accentuated endogenous risk. However, the former rises with entrepreneurs' wealth share initially since the percentage jump in the value of money does as well. This effect becomes weaker at higher levels of entrepreneurs' share in aggregate wealth. Then, the decline in leverage drives the reduction in the percentage drop of the state variable. Since, in addition, the drift rate of entrepreneurs' wealth share is virtually unchanged relative to the case of inactive policy, the distribution of $\eta_t$ shifts to the right. As a consequence, states of severe crisis become more unlikely. These outcomes imply that the allocation of capital improves on average. Further, the welfare frontier is shifted towards the north-east direction, which reflects the fact that entrepreneurs' and managers' lifetime utility is higher at any point in the considered subset of the state space. This is due to similar reasons as in the case of a constant negative outside money growth rate.

The results under MMR2 are similar in large part to those under k-percent rule $\mu^{MO} = 2.5$ percent. In particular, MMR2 accelerates the already fast growth in the total money supply at lower values of $\eta_t$. Via the previously described effects, the variability of inflation increases, reflecting the

Figure 6.4.2: Dynamics of the State Variable and Output for No-Policy Regime (solid), MMR1 (dashed), and MMR2 (dotted)




*Notes*: The stationary distribution in Panel (b) is computed from a simulated time series of $\eta$ (cf. notes to Table 5.3.4 for details); the stationary distribution under no policy in Panel (b) is displayed by white bars.

heightened magnitude of the deflationary spiral. Again, some distinctions result from the more accommodative stance of monetary policy at higher values of the state variable. In particular, the speed of adjustment in the capital allocation begins to improve at $\eta_t$ close to 0.3 since prudent entrepreneurs' portfolio weight does so as well until the efficient allocation is reached. In that subset of the state space, the substitution effect associated with greater outside money growth dominates the wealth effect. This is because endogenous risk is already relatively low in the upper region of the state space and thus affected by increases in money growth to a lesser extent. These developments reinforce output growth in the absence of jumps. Nevertheless, the distribution of the state variable under MMR2 is virtually unchanged relative to the baseline case without interventions by the central bank. This does not hold true for the welfare levels of managers and entrepreneurs, which are reduced at any value of the inter-sectoral wealth distribution.

At last, let us return to the relation between the expected instantaneous inflation rate and $\psi_t - 1$, which we interpreted as a measure for the output gap. In Subsection 5.3.6, we found that the correlation between those two variables is negative in the credit model without policy. Hence, inflation tends to increase when the output gap becomes more negative. As mentioned, that result is at odds with the NKPC in the absence of cost-push shocks. A striking result in the model with MMR1 is that the correlation between expected instantaneous inflation and our measure of the output gap turns *positive*. Thus, rather than being able to exploit a (short-run) trade-off between the output gap and inflation, monetary policy affects the direction of correlation between these two variables.

In summary, our welfare analysis in this subsection provides support for macroprudential monetary policy measures that "lean against the wind" in the sense that monetary tightening occurs in times of strong credit growth. Notably, under regime MMR1, the stance of the central banks is especially restrictive when the output gap measured by $\psi_t - 1$ is strongly negative. This is because the outside money growth rate is close to its minimum when entrepreneurs own little wealth relative to managers. While such policies reduce average growth, entrepreneurs and managers are better off in terms of welfare due to the reduction in volatility measures. How do our findings relate to the aforementioned literature on NK models with macroprudential Taylor Rules that condition on nominal credit growth? Quint and Rabanal (2014) show in an estimated model of the Euro Area that a policy that leans against the wind improves the lifetime utility of lenders but reduces that of borrowers.[848] In contrast, in our model, the central bank can raise the welfare of both lenders and savers. Kannan et al. (2012) do not derive separate welfare measures for borrowers and lenders, but rather consider a welfare criterion which penalises variability in inflation and the output gap.[849] In accordance to our results, they find that volatility measures are reduced if the stance of monetary policy becomes more restrictive in response to faster credit growth.[850] Finally, Ozkan and Unsal (2014) focus on the welfare of a representative household.[851] In their model, the introduction of nominal credit growth into the Taylor rule facilitates welfare. However, the improvement is rather small, namely at an equivalent of 0.03 percent of steady-state consumption.[852] A fact which might contribute to that result is that their solution strategy, as well as that of the aforementioned authors, only allows for analysing the effects of small shocks that materialise at the deterministic steady state of a linearised system. In contrast, our model has the ability to generate substantial amplification of exogenous shocks. Thus, policies that reduce amplification by leaning against the financial wind can lead to marked welfare improvements.

[848] Cf. Quint and Rabanal (2014, p. 203).
[849] Cf. Kannan et al. (2012, p. 14).
[850] Cf. Kannan et al. (2012, p. 29).
[851] Cf. Ozkan and Unsal (2014, p. 19).
[852] Cf. Ozkan and Unsal (2014, p. 20).

### 6.4.2 Crisis Policies

#### 6.4.2.1 Unanticipated Policies

While in the previous subsection we considered a policy that adjusts base money growth after a jump, that adjustment is, in general, insufficient to compensate for the drop in the inside money supply induced by corporate deleveraging. This is because that drop entails a discrete change in the money supply, while the policy induced-adjustment is of order $dt$. In this section, we first ask whether an unanticipated one-off helicopter drop that leads to a discrete increase in the quantity of outside money in case of a jump can mitigate the adverse feedback loop identified in Subsection 5.3.5. Such intervention is a form of a pure ex-post policy in the sense that (a) monetary policy only takes action in crisis times, i.e. once adverse shocks are realised, and (b) agents do not anticipate the policy ex ante. In contrast, the measures undertaken by the monetary authority in the previous subsections are examples of ex-ante policies.[853]

Compared to the setting in the next subsection, in which agents anticipate the reaction of the central bank, the setting with unanticipated interventions is more similar to that in conventional NK models.[854] This is because in NK models that are linearised around the deterministic steady state, individuals do not expect shocks and, consequently, measures undertaken by the central bank to alleviate the ensuing ramifications.[855] In distinction from standard NK models, we continue to assume that agents take the possibility of shocks into account. Yet, in general, the expected post-jump values will differ from the corresponding realised values due to the incorrect prediction of the central bank's reaction.

In specific, we adopt a policy of the form

$$\frac{dM_t^O}{M_t^O} = \left\{\frac{1}{\mathcal{M}_t} - 1\right\} d\mathcal{N}_t, \tag{6.4.2}$$
$$\text{with} \quad \mathcal{M}_t = \begin{cases} 0.95 & \text{if } t = t^{\mathcal{M}}, \\ 1 & \text{otherwise,} \end{cases}$$

where $t^{\mathcal{M}}$ is the point in time when the central bank applies the policy.[856] Then, according to rule (6.4.2), the outside money supply is raised by about 5.0 percent. If the reaction by the central bank is not anticipated and agents do not expect future interventions[857], equilibrium equations are the same as in the baseline model except for the *actual* change in the state variable, which is governed by (6.2.12b). In contrast, the change in $\eta_t$ *expected* by the agents is still determined via (4.5.7c).

[853] Cf. Brunnermeier and Sannikov (2016a, p. 29) on the distinction between ex-post and ex-ante policies.
[854] In CTMF models, policies are usually anticipated by the agents. An exception is He and Krishnamurthy (cf. 2013, p. 758).
[855] Cf. also Brunnermeier and Sannikov (2016a, p. 29).
[856] Note that we have set deterministic money growth to zero. This assumption is maintained in the remainder of this chapter. Setting $\mu_t^{M^O} \neq 0$ does not yield substantive additional insights.
[857] We will discuss this issue at the end of this subsection.

Figure 6.4.3: Jumps in Prices, Output, and the State Variable for No-Policy Regime (solid) and Unanticipated Increase in Outside Money (dashed)

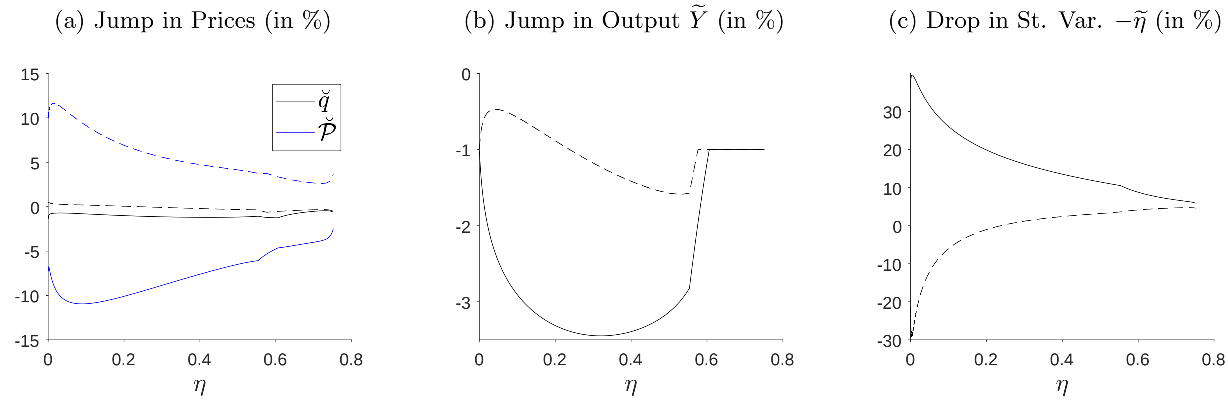



Figure 6.4.3 depicts the effects of the unanticipated, one-off helicopter drop of outside money after the arrival of a macro shock. In order to understand the effects of that policy, it is instructive to focus on Panel (c), which depicts the percentage drop of the state variable after the arrival of a macro shock $-\widetilde{\eta}_t$, at first. Two striking outcomes can be observed. First, that drop is now an increasing function of $\eta_t$. Second, within subset $\eta_t \in (0, \sim 0.25]$, the dependent variable is even negative, or, put differently, in that range, a jump causes $\eta_t$ to rise. Intuition for this result can be gained by considering equation (6.2.12b) in the absence of changes in the value share of capital, i.e. for $\widetilde{\theta}_t = \theta_t$. Then, two effects determine the value of $\widetilde{\eta}_t$. On the one hand, that variable is directly reduced by the sector specific shock, which is captured by factor $1 - \phi^s$. Conversely, the increase in the quantity of outside money, which entails a value $\mathcal{M}_{t^{\mathcal{M}}} < 1$, raises $\widetilde{\eta}_t$ due to the leveraged portfolio position of entrepreneurs. The second effect is the stronger, the greater portfolio weight $x^{e,p}_{1,t}$ is. Since this portfolio weight is decreasing with $\eta_t$, as we have seen in Panel (a) of Figure 5.3.3, the second effect dominates the first when entrepreneurs own little equity relative to aggregate wealth. Then, the inter-sectoral wealth distribution shifts in favour of entrepreneurs.

If that is the case, a *positive* feedback loop emerges. After the arrival of a jump, agents of type $e$ buy additional capital and take out more credit. This behaviour bids up the price of capital and causes the price of money to decline, as can be observed from Panel (a).[858] Both effects raise entrepreneurs' equity relative managers', which reinforces the adjustments in prices. Moreover, as a consequence of the improvement in the capital allocation in that range, output falls by less than $\kappa = 1$ percent, which is the percentage drop in the capital stock resulting from the aggregate shock to capital, as Panel (b) shows.

As entrepreneurs' leverage falls, the positive impact of shocks on the state variable becomes weaker until at point $\eta_t \approx 0.25$, the two aforementioned effects on entrepreneurs' post-shock wealth share exactly offset each other. Afterwards, some of the patterns identified in the baseline model

[858] Note that in Panel (a) we depict the percentage change in the price level due to a jump, rather than the percentage change in the value of the money supply per unit of aggregate capital $\breve{p}_t$.

without policy emerge. Most importantly, the arrival of a jump now causes the state variable to fall. This induces entrepreneurs to sell capital to managers in exchange for money. In the process, aggregate TFP falls due to the worsening capital allocation and, thus, the percentage drop in real GDP exceeds that in the aggregate capital stock. Yet, the effects are clearly dampened relative to the baseline case. Not all (qualitative) results carry over, however: while the inflationary pressure weakens, it is sustained even across the upper part of the state space. The reason is related to the boost in the relative price of capital to money, which implies that a given unit of externally financed capital is to be backed by additional nominal debt. This effect leads to an expansion in the inside money supply, which, in turn, exacerbates inflation.

It is also interesting to note that the more the economy approaches the stochastic steady state, the more the post-jump value of the state variable under the unanticipated surge in the outside money supply converges to its post-jump value in the absence of monetary intervention. The reason for this result is that in the vicinity of the stochastic steady state, the portfolio weight on capital is close to unity. Accordingly, capital is almost entirely financed by equity in that region. Then, from equation (6.2.12b) any policy that sets $\mathcal{M}_{t^{\mathcal{M}}} < 1$ can only have small effects on $\widetilde{\eta}_t$.

What are the welfare consequences of the unanticipated one-off helicopter drop? Using (5.3.2) and (5.3.4), we can gauge the effects by subtracting the post-jump value functions in the absence of the intervention from those under the policy:

$$V^m(\widetilde{\eta}_t) - V^m(\widetilde{\eta}_t^{np}) = \alpha^m(\widetilde{\eta}_t) - \alpha^m(\widetilde{\eta}_t^{np}) + \frac{1+\xi}{\rho}\log\left(\frac{(1-\widetilde{\eta}_t)(\widetilde{q}_t+\widetilde{p}_t)}{(1-\widetilde{\eta}_t^{np})(\widetilde{q}_t^{np}+\widetilde{p}_t^{np})}\right), \tag{6.4.3}$$

$$V^{e,p}(\widetilde{\eta}_t) - V^{e,p}(\widetilde{\eta}_t^{np}) = \alpha^{e,p}(\widetilde{\eta}_t) - \alpha^{e,p}(\widetilde{\eta}_t^{np}) + \frac{1+\xi}{\rho}\log\left(\frac{\widetilde{\eta}_t(\widetilde{q}_t+\widetilde{p}_t)}{\widetilde{\eta}_t^{np}(\widetilde{q}_t^{np}+\widetilde{p}_t^{np})}\right), \tag{6.4.4}$$

where superscript "$np$" stands for the regime without active monetary policy and post-jump variables without this superscript refer to the unanticipated increase in the outside money supply. Further, note that $\alpha^m(\widetilde{\eta}_t)$ and $\alpha^{e,p}(\widetilde{\eta}_t)$ are still jointly determined from the equations in Propositions 5 and 6. This is because agents do not take into account further policy interventions by assumption. Accordingly, the change in welfare levels due to the intervention corresponds to a movement towards the eastern direction along the welfare frontier in the case without an active central bank, which is displayed in Panel (c) of Figure 5.3.7, relative to the point pinned down by $\widetilde{\eta}_t^{np}$. Since $\widetilde{\eta}_t > \widetilde{\eta}_t^{np}$ holds at any point in the state space, the lifetime utility of prudent entrepreneurs is raised by the policy and that of mangers reduced. Therefore, from a welfare perspective, two of the ex-ante policies discussed in the previous subsections, namely k-percent rule $\mu^{M^O} = -2.5$ percent and MMR1, are preferable to the unanticipated helicopter drop.[859]

As with any discretionary policy, time inconsistency issues à la Kydland and Prescott (1977)

[859] This finding differs from the result in Benigno et al. (cf. 2013, p. 467), who show that in their economy with collateral constraints, ex-ante macroprudential policies are inferior to ex-post policies.

arise in our example. Prior to the intervention, agents' behaviour was based on the premise that the central bank will stick to a rule of zero base money growth forever. If they were informed of the timing and magnitude of the actual discretionary policy, they would have adjusted their plans accordingly. For instance, any manager that would have known about the inflationary effect of the helicopter drop, would have faced an incentive to reduce his demand for money ex ante. Put differently, a part of managers' demand for money arises from a speculative motive: they expect to buy back capital from entrepreneurs in exchange for money at a lower price once jumps materialise.[860] An unanticipated policy, however, increases the post-jump price of capital in terms of money relative to agents' expectations and therefore undermines managers' welfare derived from buying capital at depressed prices. Furthermore, any informed entrepreneur would have chosen to become more indebted. The ensuing collective revisions of plans, however, would have lead to entirely different equilibrium outcomes, as we will see in the next subsection. Moreover, since an unanticipated expansion in the outside money supply has the potential to alleviate adverse effects on the real economy such as the drop in output, rational individuals might "learn the trick"[861] and then take into account additional future discretionary policies as in Barro and Gordon (1983). In the above analysis, we have abstracted from such difficulties by simply assuming that agents do not revise their expectations of a zero base money growth rule.[862]

#### 6.4.2.2 Anticipated Policies

In the first part of this subsection, we continue our examination of helicopter drops in crisis states. We depart from the analysis in the previous section by assuming that (a) the central bank conducts a drop each time a jump arrives and (b) agents anticipate the interventions. We make two further assumptions: first, the drops entail constant relative changes in the outside money supply and second, the central bank can commit to the policy. Accordingly, we utilise a rule of the form

$$\frac{dM_t^O}{M_t^O} = \left\{ \frac{1}{\mathcal{M}} - 1 \right\} d\mathcal{N}_t, \; \forall t, \tag{6.4.5}$$

in which we set $\mathcal{M}$ to 0.95 as in the previous subsection. Rule (6.4.5) is explicitly taken into account by individuals when forming their expectations. It follows that the adoption of that rule will lead to adjustments in agents' choices, in particular, in asset demands, relative to the no-intervention case.

The dotted lines in Figures 6.4.4 and 6.4.5 depict the equilibrium functions. Most importantly,

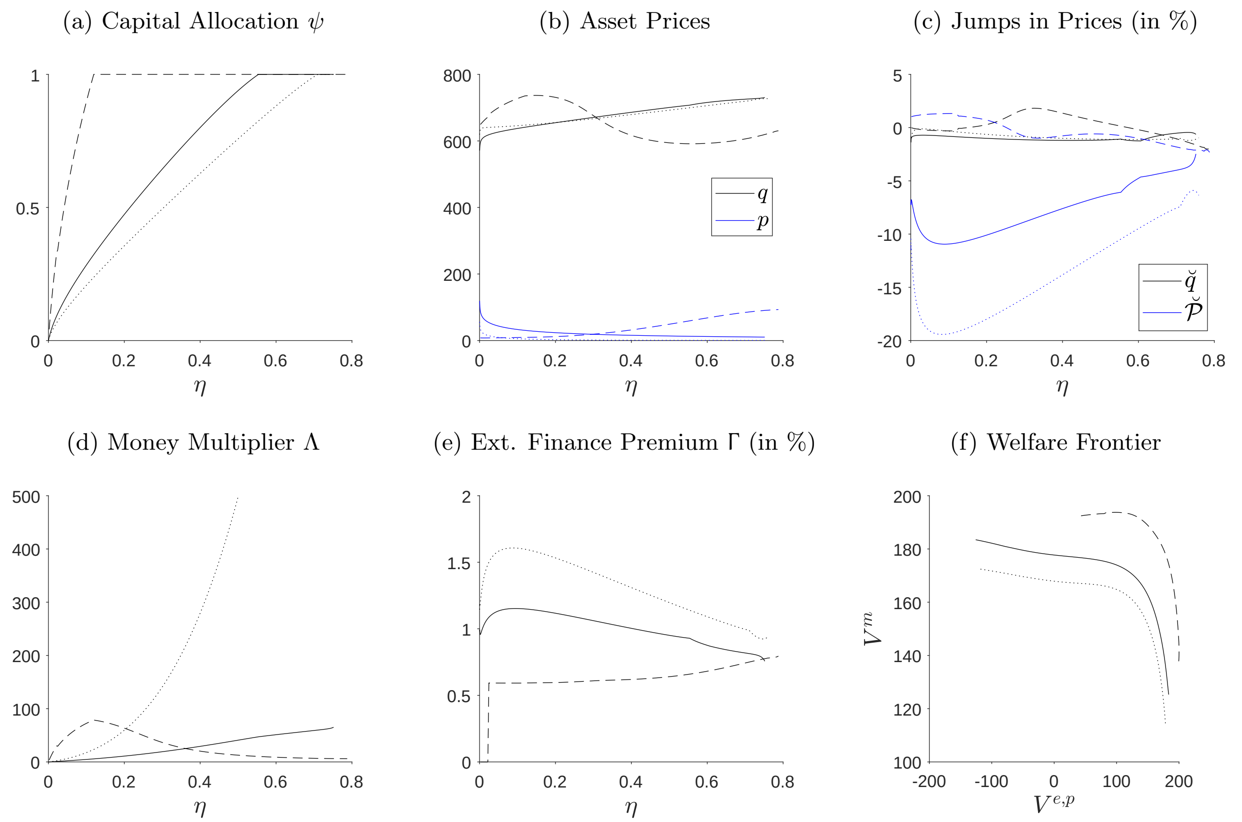
[860] Brunnermeier and Sannikov (cf. 2014a, p. 389) point out a similar speculative motive for households' investment in the risk-free asset. A speculative motive for holding money is also present in the I Theory of Money, although it is not mentioned by the authors.

[861] Calvo (1978, p. 1420)

[862] If this assumption is satisfied, monetary policy can also utilise surprise inflation induced by discrete expansions in the outside money stock to manipulate the wealth distribution in times without jumps. Such policy would always boost the wealth share of entrepreneurs since no counteracting force via term $1 - \phi^s$ would be present.

the value of money plummets since agents expect future additions to the base money stock, which reduce the real return on money, ceteris paribus. Since the value of capital remains relatively constant, the relative price of capital to money falls. Thus, in order to finance an additional unit of capital externally, each entrepreneur must borrow more in nominal terms. In the aggregate, this behaviour leads to a strong expansion in the money multiplier, which further erodes the value of purchasing power in normal times. Again, the flipside of accelerated loan growth in the absence of shocks is a collapse in the money multiplier when jumps arrive. Consequently, a "paradox of monetary expansion" emerges: the deflationary spiral is intensified in jump states, despite the provision of additional outside money.

Figure 6.4.4: Equilibrium Quantities, Prices, and Welfare for No-Policy Regime (solid), NA Regime (dashed), and $\mathcal{M} = 0.95$ (dotted)




*Notes*: In each panel, a linear interpolation is applied to the dashed curve in the proximity of $\eta_t \approx 0.025$, the point at which the EFP turns strictly positive.

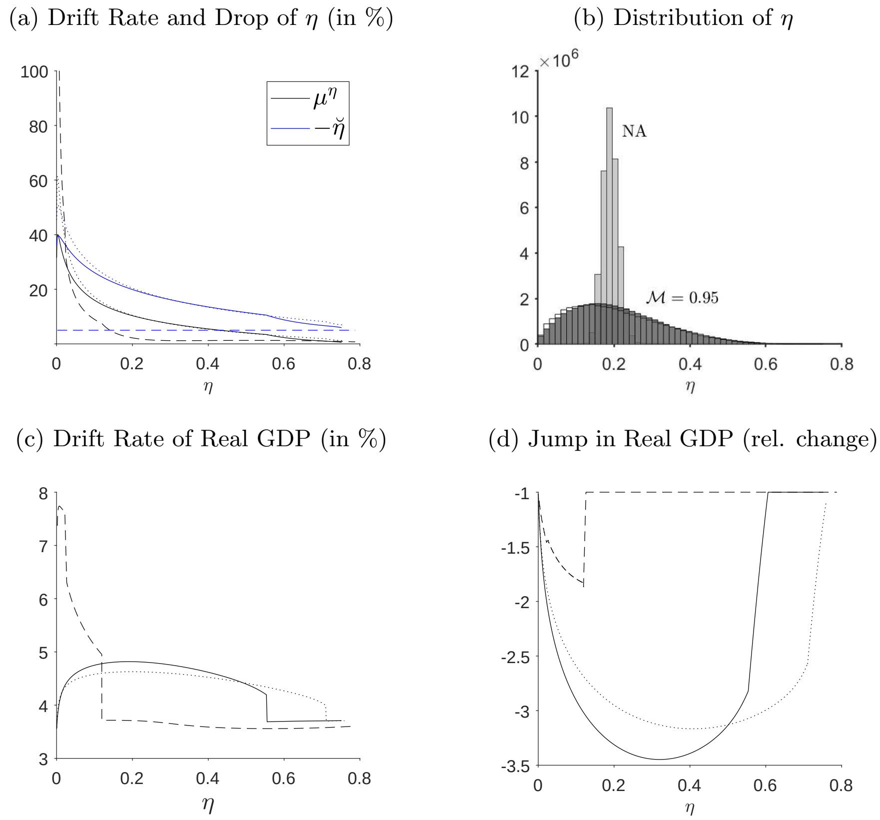
Due to the surge in endogenous risk, entrepreneurs cut their leverage ratios relative to the no-policy case, which implies that a greater share of the aggregate capital stock is in the possession of managers at any value of $\eta_t$ in the misallocation region. Since the growth in the capital allocation is muted, the drift rate of output decelerates mildly. On the upside, fire sales of capital to managers are reduced, which, taken together with the former finding, explains that the standard deviation of quarterly output growth declines (line seven of Table D.1). While the central bank might aim to mitigate the drop in output by printing new money, the mechanism in our model with anticipated interventions works quite differently than expected. Rather than compensating for the surge in the real value of debt as in the case of an unanticipated expansion in the base money supply, the central bank in fact exacerbates debt deflation via the helicopter drop. Taking this outcome into account, entrepreneurs become more cautious and hold a lower share of risky capital in their portfolios *ex ante*. Accordingly, once jumps materialise, they are not forced to sell a large portion of their capital holdings to less productive managers in order to reduce their risk exposure, an outcome that alleviates the drop in aggregate TFP. One can gain additional intuition for the different outcomes under anticipated and surprise interventions by recognising that in the latter cases, the monetary authority takes equilibrium functions as given and thus can "move" along these functions. In contrast, when agents take stochastic increases in base money into account, equilibrium functions will shift in general since individuals will adjust their behaviour. Comparing our results to those in the previous subsection, we can conclude that the effects of crisis interventions depend crucially on whether the latter are anticipated by the agents. While in the case of an unanticipated policy the expansionary measures undertaken by the central bank can attenuate or even reverse the deflationary spiral associated with corporate deleveraging, the opposite is true in the case considered here.

Before we turn to the examination of the last policy in this chapter, note that the qualitative as well as quantitative results uncovered thus far in this subsection are very similar to those under k-percent rule $\mu^{M^O} = 2.5$ percent, which implies (nearly) the same expected rate of change in the outside money supply given by $\left\{\mu^{M^O} + \lambda\left(1/\mathcal{M} - 1\right)\right\} dt$. Both policies reduce the value of money relative to capital and therefore set in motion a rapid expansion in debt in tranquil times, a development that bites back in crisis episodes. These similarities also explain why both policies have comparable effects on the distribution of the state variable and welfare. In particular, the welfare of both managers and prudent entrepreneurs is lower compared to the no-policy case at any value of $\eta_t$. One notable difference is the slightly more pronounced downward movement in the price level after jumps in the k-percent case. The explanation lies in the lack of accommodative behaviour by the central bank in those events.

Given the above results, one might expect that a welfare-improving policy *cuts* the quantity of base money in circulation if macro shocks arrive. Rather than considering interventions of the form $\mathcal{M} = \text{const} > 1$, however, we now aim to characterise a policy that removes the adverse feedback loop identified in Subsection 5.3.5. Such a policy fixes the percentage drop in entrepreneurs' wealth share $-\breve{\eta} = \phi^s$ at the exogenous source of variation $\phi^s$, which is the share of entrepreneurs exposed to the sector-specific productivity shock in case of a jump. Using (6.2.12b), we can implicitly characterise policy $\mathcal{M}_t^{\text{NA}}$, where superscript "NA" stands for "no adverse feedback loop":

$$1 = x_{1,t}^{e,p} \frac{\widetilde{\theta}_t}{\theta_t} + (1 - x_{1,t}^{e,p}) \frac{1 - \widetilde{\theta}_t}{1 - \theta_t} \mathcal{M}_t^{\text{NA}}. \tag{6.4.6}$$

Figure 6.4.5: Dynamics of the State Variable and Output for No-Policy Regime (solid), NA Regime (dashed), and $\mathcal{M} = 0.95$ (dotted)




*Notes*: The stationary distribution in Panel (b) is computed from a simulated time series of $\eta$ (cf. notes to Table 5.3.4 for details); the stationary distribution under no policy in Panel (b) is displayed by white bars; in each panel, a linear interpolation is applied to the dashed curve in in the proximity of $\eta_t \approx 0.025$, the point at which the EFP turns strictly positive.

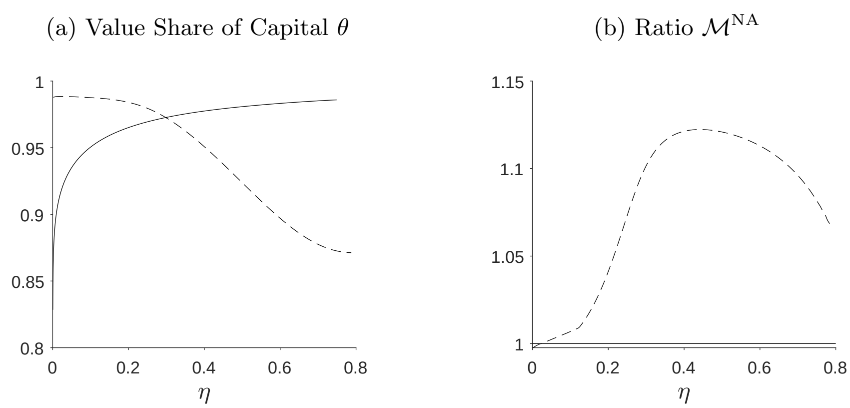
Solving for $\mathcal{M}_t^{\mathrm{NA}}$ yields

$$\mathcal{M}_t^{\mathrm{NA}} = \frac{1 - x_{1,t}^{e,p} \frac{\widetilde{\theta}_t}{\theta_t}}{(1 - x_{1,t}^{e,p}) \frac{1-\widetilde{\theta}_t}{1-\theta_t}}. \tag{6.4.7}$$

A notable property of the above function is the implied positive relationship between $\mathcal{M}_t^{\mathrm{NA}}$ and prudent entrepreneurs' portfolio weight on capital, provided that the value share of capital falls after a jump, i.e. $\widetilde{\theta}_t < \theta_t$, a condition which held true in each of our previously examined examples. A crucial difference between the NA regime and policies that keep $\mathcal{M}$ constant is that in the former case, complex interactions between agents' anticipation of the central bank's stance and the actual conduct of monetary policy emerge. By choosing a particular path of $\mathcal{M}_t^{\mathrm{NA}}$, the central bank induces agents to adjust their expectations of time paths of equilibrium values accordingly. The ensuing change in behaviour feeds back into prices and allocations, which the monetary authority has to take into account.

Since equilibrium function $\theta(\eta_t) = q(\eta_t) / [q(\eta_t) + p(\eta_t)]$ plays an important role in determining the RHS of rule (6.4.7), we plot the former in Figure 6.4.6 besides function $\mathcal{M}^{\mathrm{NA}}(\eta_t)$. Other equilibrium functions are represented by the dashed lines in Figures 6.4.4 and 6.4.5. In distinction to the no-policy case, the value share of capital is now a decreasing function of the state variable (Panel (a) in Figure 6.4.6). Panel (b) shows that function $\mathcal{M}^{\mathrm{NA}}(\cdot)$ is inverted-u-shaped with a kink at $\eta_t \approx 0.15$. This value of the state variable marks $\eta^{\psi}$, the point where the optimal capital allocation is reached (cf. Panel (a) in Figure 6.4.4). Moreover, the pre- relative to the post-jump value of the base money supply is greater than unity except for extremely low values of prudent entrepreneurs' wealth share. Thus, a policy that eliminates the adverse feedback loop cuts the quantity of outside money in circulation.

Figure 6.4.6: Value Share of Capital and Pre- to Post-Jump Ratio of Outside Money for No-Policy (solid) and NA Regime (dashed)




What accounts for these results? Firstly, note that our results presented in the first half of this subsection showed that an expansionary monetary policy actually intensifies the deflationary pressure after a jump. Raising the money supply has the opposite effect, as demonstrated by Panel (c) in Figure 6.4.4 - a “paradox of monetary tightening” is at work. This outcome is conducive to the relative net worth position of entrepreneurs. We have also argued that prudent entrepreneurs respond to reductions in endogenous risk by allocating a greater portion of their wealth to capital. As long as entrepreneurs are levered and condition

$$\frac{\widetilde{\theta}_t}{\theta_t} < \frac{1-\widetilde{\theta}_t}{1-\theta_t}\mathcal{M}_t^{\mathrm{NA}} \tag{6.4.8}$$

holds, that behaviour is detrimental to entrepreneurs' post- relative to pre-shock wealth share, according to equation (6.2.12b), making it more difficult for the central bank to achieve target rate of change $-\breve{\eta}_t = \phi^s$. A possible way out of this dilemma would arise if the value share of capital $\theta_t$ were to rise, or, equivalently, if the value share of money $1-\theta_t$ were to fall after a jump. In fact, the monetary authority can achieve just that by promising stronger cuts in the outside money supply at higher levels of the state variable. As a consequence, the value share of capital becomes a decreasing function of $\eta_t$, which immediately implies that any exogenous shock that shifts the inter-sectoral wealth distribution in favour of managers causes $\theta_t$ to rise.

Since in the extreme lower region of the state space prudent entrepreneurs' portfolio weight is relatively high, from the perspective of the central bank, it is sufficient to implement a slightly negative slope of function $\theta(\cdot)$ and a value of $\mathcal{M}_t^{\mathrm{NA}}$ slightly above unity. Interestingly, once the optimal allocation of capital is reached at point $\eta^{\psi}$, entrepreneurs start to pay down debt instead of bidding up the price of capital as in the previously analysed examples. As a consequence, $q(\eta_t)$ begins to fall and the slope of function $p(\eta_t)$ increases more strongly (Panel (b) in Figure 6.4.4). Both outcomes are reflected in the behaviour of function $\theta(\cdot)$. The reason for entrepreneurs' reaction is their expectation of stronger (percentage) reductions in the outside money supply at higher values of the state variable. Two factors call for a more aggressive response by the central bank in case of aggregate shocks after point $\eta^{\psi}$: first, the ensuing drop in entrepreneurs' portfolio weights on capital and second, the reduction in the percentage drop of output $\breve{Y}_t$, which raises deflationary pressure, ceteris paribus. However, as the state variable increases, entrepreneurs pay down more and more debt, a fact which raises the value of money in tranquil times without shocks. Accordingly, the central bank can reduce $\mathcal{M}_t^{\mathrm{NA}}$ in the upper part of the state space.

With regard to some other equilibrium outcomes we can draw a comparison to another policy that is also restrictive in the sense that expected growth in inside money is negative, namely k-percent rule $\mu^{M^O} = -2.5$ percent. Both policies reduce endogenous risk by mitigating fluctuations in the value of money. That reduction has three main positive effects. First, it induces entrepreneurs to allocate a higher share of their wealth to capital, which, in turn improves the average allocation of capital and the growth rate of the economy in the misallocation region. Second, it causes the EFP to contract. The policy that removes the adverse feedback loop has the side effect that imprudent individuals are not pushed into bankruptcy if jumps arrive in the lower region of the state space. Accordingly, the EFP drops to zero in that region. This effect boosts the deterministic growth rates of real GDP and the state variable. Third, the decline in endogenous risk improves system stability. This is reflected in the distribution of the state variable, which is much more centred in both cases than in the no-policy regime.

In light of these positive effects, it is not surprising that welfare levels of both managers and prudent entrepreneurs are higher than in the no-intervention case at any point in the state space. Interestingly, there is now potential for joint welfare improvements through lumpsum redistributions of wealth from managers to entrepreneurs enforced by a social planner. This is reflected by the increasing arc of the welfare frontier.[863] Yet, this arc corresponds to a lower subset of the state space, namely $\eta_t \in (0, \approx 0.05)$, which is only reached with extremely small probability.

[863] Similarly, in Brunnermeier and Sannikov (cf. 2015, p. 325), unanticipated wealth transfers forced by the social planner are associated with Pareto improvements in some subset of the state space.

While the NA policy drastically mitigates endogenous price risk and thereby improves overall welfare, it might be difficult to implement in practice. This is because the central bank needs to take into account agents' response to variations in the money supply, in order to be able to achieve a given target for $\breve{\eta}_t$. More specifically, the central bank needs to be aware of agents' policy functions. It may be doubted that such information is actually available. From this perspective, simple non-state-dependent rules that raise the money supply either in times with or without jumps might be preferable, provided that agents anticipate the interventions.

Even though our setting departs from BS (2016a) in many regards, the above results confirm their finding that the central bank can boost welfare and reduce volatility measures by removing the endogenous source of stochastic variation in the state variable.[864] One crucial difference of their treatment of monetary policy compared to ours is that in the latter case, the central bank adjusts the money supply to counteract the deflationary pressure after adverse events. In that sense, monetary policy works since its repercussions assist indebted agents, including banks, in repairing the *liabilities* sides of their balance sheets. Thus, our modelling of monetary policy accords to the "money view" originally put forth by Friedman and Schwartz (1963).[865] In contrast, in the I Theory of Money, the central banks' measures are directed at the asset sides of financial institutions. In specific, interest rate cuts inflate the value of nominal bonds held by those institutions. Therefore, their setting is closer in spirit to the "credit view" which was advocated by Tobin and Brainard (1963), among others.[866]

[864] Cf. Brunnermeier and Sannikov (2016a, pp. 34ff.).
[865] Cf. also Brunnermeier et al. (2012, p. 75).
[866] Cf. Brunnermeier and Sannikov (2016a, p. 5).

# Chapter 7

# Epilogue

## 7.1 Conclusion

In this thesis, we developed a CTMF model of deflationary crises induced by voluntary deleveraging in the entrepreneurial sector in the face of adverse productivity shocks. The adoption of the CTMF framework allowed us to circumvent limitations arising in more traditional discrete-time business cycle models with financial frictions. In specific, the typical approach in the latter strand of literature is the application of a linearisation around a deterministic steady state. This has two main shortcomings. First, the produced solutions are local in nature in the sense that first-order perturbations are valid only in the vicinity of the steady state. Second, the fact that exogenous risk is neglected in the computation of deterministic steady states implies that individuals in the respective model economies do not rationally anticipate shocks.

These model properties have a number of undesired repercussions. For one, after the arrival of an exogenous shock, individuals expect the economy to drift back to the steady state with certainty. That is, they do not take into account the possibility of further disturbances. Moreover, since individuals do not anticipate shocks, variations in the exogenous severity or probability of stochastic innovations do not lead to ex ante portfolio adjustments. The Kocherlakota critique has attributed the rather modest amplification of small exogenous shocks generated by DSGE models with financial frictions under plausible parameter constellations to these shortcomings. Further, authors such as Milne (2009) and Benes et al. (2014) have argued that episodes of financial turmoil may push the economy far away from the steady state and display severe nonlinearities. Linearised DSGE models are unable to capture such phenomena.

In contrast, the CTMF framework entails a comparatively straightforward numerical approach to solve for global equilibrium dynamics, namely, the solution of one or more differential equations. Usually, in these equations, the independent variables are wealth shares of financially constrained sectors and the dependent variables are equilibrium prices. Typically, CTMF models produce highly nonlinear system dynamics and significant degrees of amplification. Furthermore, steady states in these models are stochastic and individuals rationally anticipate shocks at any point in time. One implication is that adjustments in exogenous shock parameters actually induce agents to reallocate their portfolios. This property gives rise to results such as the "Volatility Paradox", which states that a reduction in exogenous risk may indeed make the system less stable in the sense that certain volatility measures may spike.

In our credit model, individuals of two types, namely risk averse managers and entrepreneurs, trade physical capital against money. Importantly, the latter class of agents is assumed to be endowed with a superior final goods production technology than the former. Holding capital exposes individuals to different forms of risk. For one, entrepreneurs may suffer shocks that permanently reduce their capital productivity. In addition, capital is subject to idiosyncratic and aggregate depreciation shocks. Each type of shock arrives with a fixed Poisson intensity. Besides investing in capital, individuals may also allocate a portion of their wealth to money. Money is an asset that is risk-free in nominal terms. Therefore, agents may reduce their risk exposure by tilting their portfolios towards money. This also boosts flow utility since real balances enter the instantaneous utility functions, but comes at the expense of lower expected portfolio returns.

The fact that entrepreneurs can only obtain external financing provided by banks in the form of standard debt constitutes the main financial friction. Since such contracts preclude risk sharing, risk averse entrepreneurs require sufficient net worth buffers to absorb negative cash flow shocks before they are willing to buy additional capital. Our assumptions give rise to a simple capital structure: entrepreneurs' assets, which consist of capital and money, are backed by inside equity and bank credit, exclusively. Since loans are denominated in nominal terms, the real value of entrepreneurs' inside equity is vulnerable to swings in the value of money. Besides providing financial means to entrepreneurs, bank credit serves a second role: since the act of extending a loan to the nonbanking sector is tantamount to the creation of inside money, additional bank credit increases the supply of liquidity available to the private sector.

The model also features credit risk on the side of banks. This type of risk derives from the fact that individuals from a subpopulation of borrowers, which we referred to as "imprudent entrepreneurs", do not accumulate sufficient equity buffers to avoid default in the event of adverse shocks. If these borrowers declare default, banks verify whether the announcements are correct and subsequently take over the remaining physical capital of the bankrupt debtors. This exposes banks to endogenous price risk for two reasons: first, the defaulting loans are backed by inside money and second, the capital price and the value of money will, in general, not move in tandem. It turned out that the discrete nature of price changes, which can be traced back to the adoption of Poisson processes as the fundamental sources of uncertainty, is critical for the exposure of banks' balance sheets to endogenous risk. In contrast, under Brownian uncertainty, such effects would be of lower order. Moreover, our model differs from other CTMF models with default in that the recovery rate, i.e. the percentage of a loan's face value that can be recovered from bankrupt debtors, is endogenous.

Banks quote a single lending rate for all borrowers as they are not able to distinguish between good and bad risks. Since banks set the lending rate in a risk neutral manner, the supply of loans is perfectly elastic at the prevailing lending rate. These two facts also imply that prudent as well as imprudent debtors eventually bear the entire costs of default. The mark-up banks set over the deposit rate varies with the adjustments in prices after jumps that push imprudent entrepreneurs into bankruptcy. In turn, these adjustments depend on the state of the economy.

Turning to model dynamics, we showed that without credit frictions, entrepreneurs would immediately buy the entire capital stock and finance the purchases by selling shares to managers. Thus, the economy would instantly jump to the first-best efficient outcome. While the economy would still be exposed to occasional shocks, amplification of these shocks and fire sales of capital to less efficient sectors would not occur.

In contrast, under financial frictions, model dynamics are driven by changes in the inter-sectoral wealth distribution between entrepreneurs and managers. In particular, we reduced the state space to dimension unity: the single state variable in our model is the wealth share of entrepreneurs. Two assumptions were pivotal to achieve that outcome. First, the adoption of AK production technologies, which allows for the removal of the aggregate capital stock from the relevant equilibrium equations. Second, the abstraction from frictions in the financing of banks, which implies that banks are free to issue outside equity to bank owners at their discretion.

Both of those assumptions served to simplify the solution procedure. In particular, they imply that the differential equation that is to be solved for obtaining equilibrium is of the retarded type. In contrast, with multiple state variables, the respective equation is a retarded partial differential equation. In principle, such equations can be solved in the context of CTMF models by applying the iterative method. However, this method has not yet been applied to models with Poisson uncertainty in the literature. Moreover, our attempt to solve the baseline credit model via the iterative method was not successful.

The necessity to reduce the state space to dimension unity required us to choose between a balance sheet channel in the entrepreneurial sector and a bank lending channel. We opted for the former since endogenous price risk is concentrated in the entrepreneurial rather than the banking sector in our model. The underlying reason is the assumption that end-borrowers and banks exchange financial claims on the basis of debt contracts which specify reimbursement in nominal terms. As a consequence, physical capital on the asset sides of entrepreneurs' balance sheets is backed by nominal debt and equity. While banks are also exposed to endogenous risk, the portion of their balance sheets that is exposed to adverse movements in prices is small. More specifically, that portion is equal to the share of imprudent entrepreneurs, who constitute a small share in the pool of all borrowers.

The adoption of Poisson shock processes effectively separates model dynamics into two parts. In tranquil times without jumps, entrepreneurs earn higher portfolio returns than managers since the former are levered and endowed with a superior production technology. This causes the intersectoral wealth distribution to shift in favour of entrepreneurs. As individuals from that former group accumulate more equity, they are willing to take on additional debt to finance purchases of capital from managers. Consequently, the money multiplier expands and the value of money deteriorates. At the same time, the value of capital appreciates since entrepreneurs value capital more than managers. The improvement in the allocation of capital boosts output. This effect is reinforced over time due to higher investment expenditures. As the capital price increases, the return premium on the productive asset over money shrinks. This induces entrepreneurs to cut their leverage. In effect, the growth of the entrepreneurial sector slows down and the economy approaches its stochastic steady state.

Yet, the convergence to the steady state may be temporarily interrupted due to the arrival of jumps, which entail productivity losses and capital depreciation. Entrepreneurs who suffer negative productivity shocks react by selling capital and using the proceeds to pay down debt. In effect, the capital price falls and, due to the contraction of the money multiplier, the value of money surges. Price adjustments cause the effects of the initial productivity shock to spill over to agents that were not directly affected. In particular, the ensuing liquidity spiral due to the drop in the capital price reduces the real value of entrepreneurs' assets, while deflationary pressure raises the value of their liabilities. The resulting drop in real wealth levels leads to further debt liquidation on the side of these agents, fire sales of capital to managers, and subsequent price changes. Moreover, real GDP drops due to the worsening allocation of the productive asset.

In effect, an adverse feedback loop between weakening balance sheets in the entrepreneurial sector and adverse price adjustments emerges. These effects are most pronounced in the lower region of the state space, where entrepreneurs own a small share of aggregate wealth and the economy is far away from its steady state. This is because leverage is large in that region. We also showed that the resulting deflationary pressure can be quite substantial, peaking at about 15 percent under the baseline calibration, while the response of the capital price is comparatively muted. Induced changes in the EFP may either reinforce or weaken the deflationary spiral. In the lower part of the state space, banks cut the mark-up over the deposit rate since they expect lower deflationary impulses from further jumps. This behaviour, in part, counteracts the reduction in debt and therefore inside money. These outcomes are reversed when entrepreneurs own a larger share of aggregate wealth.

Monte Carlo simulations of the credit model demonstrated that under the baseline calibration, the economy spends a large fraction of time in states in which entrepreneurs are undercapitalised. In fact, the probability for the system to reach the optimal capital allocation in which the entire capital stock is in the hands of entrepreneurs is only at about one percent. Therefore, misallocation of capital due to financial frictions is the norm rather than the exception. Moreover, when entrepreneurs own a small fraction of total wealth, variations in prices are most pronounced, an outcome that contributes to the excess volatility of the inflation rate. Despite this inability of the model to fit the data, our model can replicate some other empirical facts such as the procyclicality of debt and the countercyclicality of firms' leverage.

We further examined model properties by varying exogenous parameters. Reductions in both the share of entrepreneurs exposed to the sector-specific productivity shock and the frequency of jumps were shown to induce ex ante portfolio reallocations, which cause the growth in nominal debt to accelerate in tranquil times without shocks. Since these developments are reversed once jumps hit the economy, the amplitudes of the liquidity as well as the deflationary spiral increase in both cases. These outcomes can be viewed as manifestations of the Volatility Paradox. To shed additional light on the dynamics of the credit cycle, we also reduced exogenous agency costs which originate on the side of banks. Despite the ensuing drastic reduction in the EFP, changes in equilibrium quantities and prices are relatively modest. This finding suggests that demand rather than supply side factors are the main drivers of quantity variations in the credit market.

The above results were derived under the assumption that the monetary authority is entirely passive in the sense that the stock of outside money is kept constant. In our effort to discern the effects of active monetary policy on equilibrium outcomes we first considered a case in which the central bank varies the money supply by paying interest on outside money. Since our models does neither feature nominal bonds nor price inertia, such policies do not alter equilibrium values of real variables. We further showed that monetary policy has real effects if central bank operations affect the real return on money.

One option to achieve that outcome in our credit model is a helicopter drop of outside money that does not involve the fiscal authority and takes place in case the economy is hit by a jump. We found that agents' anticipation of the intervention is critical for the efficacy of the latter. If the policy is not anticipated, the economy's response is as expected: then, the injection of additional outside money counteracts the drop in inside money and therefore mitigates or even reverses the deflationary spiral. If the monetary impulse is large and entrepreneurs are highly levered, these individuals may even experience real gains in equity, which shift the inter-sectoral wealth distribution in favour of the entrepreneurial sector. However, since unanticipated inflation hurts lenders, such policies are not Pareto efficient in general.

If the helicopter drop is expected, agents adjust their behaviour ex ante: taking into account future additions to the outside money supply and the associated depreciation in the value of money, agents increase their nominal demand for credit in normal times. This causes a strong expansion in the money multiplier, which further erodes the value of money. The debt boom is reversed once jumps arrive. Despite the increase in the outside money supply, the policy in fact exacerbates the deflationary spiral due to the collapse in inside money. Therefore, a "paradox of monetary expansion" emerges. Conversely, if the monetary authority reduces the outside money supply, the results are reversed. In particular, deflationary pressure is mitigated, which is tantamount to a reduction in endogenous risk and thus raises welfare. Importantly, such outcomes cannot arise in standard linearised DSGE models, in which agents neither anticipate shocks nor the monetary policy response and banks do not have the capacity to create purchasing power. Furthermore, our results suggest that policy makers who consider the implementation of measures that aim at expanding the money supply, such as QE, should take into account to what extent the policy is anticipated.

We also showed that ex ante policies that adjust the outside money supply in periods without shocks have the potential to improve welfare. We distinguished between two families of money supply rules, namely k-percent rules, which peg the growth rate of base money, and rules that adjust deterministic money growth depending on the current value of the state variable. Among the latter type of rules, we identified policies that imply a positive relation between deterministic outside money growth and the state variable to be preferable from a welfare point of view. Such policies can be interpreted as macroprudential money supply rules since they entail a contractive stance in times when growth in nominal debt is large. The positive performance can be related to the fact that continuous reductions in outside money counteract the inflationary effect of inside money growth in normal times without shocks and thus lead to more stability in the value of money and less endogenous risk. For similar reasons, k-percent rules with negative base money growth reduce the volatility in the inflation rate and improve welfare.

On a final note, despite the important insights that have already arisen from CTMF models, the literature is still in its infancy. This is reflected in the comparatively high degree of abstraction, which mainly concerns modelling assumptions. The final section of this thesis sketches out two approaches to make our credit model more realistic.

## 7.2 Two Directions for Future Research

### 7.2.1 Labour as an Input to Production

A limitation of the CTMF literature is the neglect of labour as an input to production. In this subsection, we show how the baseline credit model can be modified to attribute a role to labour input in managers' and entrepreneurs' production technologies. Specifically, we consider a production function of the Cobb-Douglas type with constant returns to scale:

$$y_{i,t} = a_i(\ell_{i,t})^{\alpha}(k_{i,t})^{1-\alpha},\ 0 < \alpha < 1 \tag{7.2.1}$$

where $\ell_{i,t}$ is the amount of labour an agent $i \in \mathbb{I}^e \cup \mathbb{I}^m$ employs in his production process, $\alpha$ is the output elasticity of labour, and $1-\alpha$ that of capital. The drift of $i$'s net worth thus follows

$$\mu^n_{i,t} = \left(a_i(\ell_{i,t})^{\alpha}(k_{i,t})^{1-\alpha} - i_{i,t} + q_t\left(\mu^q_t + \Phi(\iota_t) - \delta\right)k_{i,t} + p_t\left(\mu^p_t + \Phi(\iota_t) - \delta\right)K_t m_{i,t} - w_t \ell_{i,t}\right)\,dt, \tag{7.2.2}$$

in which $w_t$ is the real wage rate paid to workers, which is taken as given by producers. It follows that the labour demand decision is an entirely static problem and yields

$$\ell_{i,t}^{*} = \left(\frac{\alpha a_i}{w_t}\right)^{\frac{1}{1-\alpha}} k_{i,t}. \tag{7.2.3}$$

Labour is supplied by a new class of agents, named workers. To keep matters simple, we assume that workers (i) only live for an instant of time[867], (ii) start their lives with zero net worth, (iii) do not bequest any wealth to their descendants, (iv) are replaced after each time period by a newborn generation of workers, and (v) do not have a transactions demand for money. These assumptions have two virtues: first, workers' optimal behaviour can be derived via a static optimisation problem. This is because it is not optimal for workers to be left with a positive amount of wealth at the end of their lives and, hence, there is neither a need to solve a consumption-savings problem nor a portfolio-selection problem. Second, since wealth accumulation on the side of these agents does not occur, no additional state variable for their wealth share is required.

In order to endogenise the labour supply decision, we opt for implementing the standard labour-leisure trade-off: each worker is equipped with a time endowment normalised to unity, which can be allocated between working $\ell_{i,t}^{w}$ units of time and leisure $1 - \ell_{i,t}^{w}$ .[868] Working additional hours raises income and thus allows for higher consumption, but on the other hand reduces utility as less leisure can be enjoyed. The utility function of the representative worker living in the period between instants $t$ and $t + dt$ is assumed to be of the log-separable:

$$U_{i,t}^{w} = \log c_{i,t}^{w} + \varsigma \log\left(1 - \ell_{i,t}^{w}\right), \tag{7.2.4}$$

in which $\varsigma$ is the utility weight on leisure. Maximising (7.2.4) subject to the constraint $c_{i,t}^{w} = w_t \ell_{i,t}^{w}$ and taking the wage level as given yields the optimal labour supply:

$$\ell_{i,t}^{w} = \frac{1}{1+\varsigma}, \tag{7.2.5}$$

which does not depend on the real wage $w_t$.

Next, we combine supply and demand to solve for the real wage rate that clears the labour market. Before we do this, we first define the set of workers as $\mathbb{I}^w = (3, 4]$ and index an individual worker by $i \in \mathbb{I}^w$. Then, the equilibrium condition in the labour market is:

$$\int_3^4 \ell_{i,t}^{w}\, di = \int_0^1 \ell_{i,t}^{e}\, di + \int_1^2 \ell_{i,t}^{m}\, di. \tag{7.2.6}$$

[867] The short-lifetime assumption is inspired by He and Krishnamurthy (cf. 2013, p. 739), who assume that households in their model only live for an instance of time. However, in contrast to the model in this section, they allow for bequests between generations of households.

[868] Workers' choice variables are indicated by superscript $w$.

where $\ell_{i,t}^{e}$ and $\ell_{i,t}^{e}$ are the optimal labour demand levels by an individual entrepreneur and an individual manager, respectively. Inserting optimality conditions (7.2.5) and (7.2.3), solving the integrals, and subsequently solving for $w_t$ yields

$$w_t = w\left(K_t\right) \equiv \alpha \left(\left[(a^e)^{\frac{1}{1-\alpha}}\,\psi_t + (a^m)^{\frac{1}{1-\alpha}}\left(1-\psi_t\right)\right] K_t\left[1+\varsigma\right]\right)^{1-\alpha}. \tag{7.2.7}$$

We can see from this expression that the equilibrium real wage is a nonlinear function of the aggregate capital stock. Since $a^e > a^m$, $w_t$ rises with entrepreneurs' share in the aggregate capital stock $\psi_t$. This is because in equilibrium, the real wage is equalised to a weighted average of entrepreneurs' and managers' marginal productivities of labour and $\psi_t$ is the weighting factor of that average.

We also have to take into account that producers' deterministic return on capital has to be adjusted. Substituting (7.2.3) into the drift of individual $i's$ wealth (7.2.2), applying similar steps as in the derivation of asset returns in Section 4.2, and subsequently inserting the general function for the equilibrium real wage from equation (7.2.7) leads to

$$\mu_{i,t}^{r^K} = \mu_i^{r^K}\left(K_t\right) = \frac{\left[\alpha / w\left(K_t\right)\right]^{\frac{\alpha}{1-\alpha}} a_i^{\frac{1}{1-\alpha}} - \iota_t}{q_t} + \mu_t^q + \Phi(\iota_t) - \delta. \tag{7.2.8}$$

Since that expression does neither depend on individual capital stocks nor wealth levels, entrepreneurs' and agents' value functions are still described by (4.3.8) and (4.3.17), respectively. However, the left hand sides of entrepreneurs' and managers' FOCs with respect to the portfolio weight on capital now include terms depending on the aggregate capital stock, namely, the deterministic parts of the respective returns on capital.[869] The same issue arises in the final goods market clearing condition, which is now given by

$$q_t = q\left(K_t\right) = \frac{\theta_t(\psi_t a^e + (1-\psi_t)a^m - \iota_t) - \lambda\left(1-\kappa\right)\omega \widetilde{q}_t \frac{\varphi}{1-\varphi}\left(x_{1,t}^{e,p} + x_{2,t}^{e,p} - 1\right)\eta_t}{\frac{\rho}{1+\xi} + \frac{\varphi}{1-\varphi}\left(x_{1,t}^{e,p} + x_{2,t}^{e,p} - 1\right)\left(\mu_t^{r^{K,e}} - \mu_t^{r^{L,e}}\right)\eta_t + \frac{w(K_t)\ell_t^w}{1+\varsigma}}.$$

It follows that the scale-invariance property does not hold in this setting and, as a consequence, that the number of state variables cannot be reduced to unity.

### 7.2.2 Financial Frictions in the Banking Sector

As mentioned in Subsection 5.4.4, the empirical literature has produced evidence that the supply rather than the demand side was the main driver of the contraction in intermediated credit markets in the U.S. during and after the financial crisis of 2007/08. Our model cannot account for such

[869] This can be seen by inserting (7.2.8) into (4.5.6a) and (4.5.6c).

events due to the absence of a banking sector balance sheet channel. This property of our model follows from our previous assumption that banks can issue equity to managers, who assume the role of bank owners, at any point in time and without costs. To introduce financial frictions in the banking sector, we could follow Klimenko et al. (2016) and Phelan (2016) in assuming deadweight costs of equity issuance. Adopting this assumption would make bank equity distinct from the equity held by outside investors. Yet, the most straightforward approach is to introduce a new class of agents, namely "bank owners", who own the entire equity emitted by banks and cannot issue additional shares to outside investors. Thus, as in the I Theory of Money, the only source of external finance available to banks comes in the form of deposits. Since bank owners are assumed to be risk averse, they require sufficient net worth buffers before they are willing to extend additional credit. Therefore, a role for bank equity in determining aggregate equilibrium outcomes emerges.

Under the above assumptions, the evolution of bank owners' net worth is described by[870]

$$dn_{i,t}^{bo} = \mu_t^{n^{bo}} n_{i,t}^{bo}\, dt + \left\{ \widetilde{n}_{i,t}^{bo} - n_{i,t}^{b,o} \right\} d\mathcal{N}_t, \tag{7.2.9}$$

$$\text{where} \quad \mu_t^{n^{bo}} n_{i,t}^{bo} = \left[ \Gamma_t x_t^{bo} + \mu_t^p + \Phi(\iota_t) - \delta \right] n_{i,t}^{bo} - w_t^b - c_{i,t}^{bo}, \tag{7.2.10}$$

$$\widetilde{n}_{i,t}^{bo} = (1-\kappa) \left[ \frac{\widetilde{p}_t}{p_t} + \varphi x_t^{bo} \left( \frac{\widetilde{q}_t}{q_t} - \frac{\widetilde{p}_t}{p_t} \right) \right] n_{i,t}^{bo}, \tag{7.2.11}$$

and $x_t^{bo}$ is banks' portfolio weight on loans in terms of bank owners' wealth. For simplicity, we assume that bank owners have logarithmic preferences and that real balances do not enter their utility functions. Then, their maximisation problem is as follows:

$$\max_{[c_{i,t}^b, x_t^b]_{t=0}^\infty} \mathbb{E}_0 \int_0^\infty e^{-\rho^{bo} t} \log c_{i,t}^{bo}\, dt, \tag{7.2.12}$$

s.t. (7.2.9)-(7.2.11). Applying similar steps as in the proof of Proposition 8, the value function can be shown to be of the form

$$V^{bo}\left(n_{i,t}^{bo}\right) = \alpha_t^{bo} + \frac{1}{\rho^{bo}} \log n_{i,t}^{bo}. \tag{7.2.13}$$

Thus, the HJB equation corresponding to the above program is

$$\begin{aligned} \rho^{bo} V^{bo}(n_{i,t}^{bo}) = \max_{c_{i,t}^{bo}, x_t^{bo}} \Bigg\{ & \log c_{i,t}^{bo} + \frac{1}{\rho^{bo}} \left( \left[ \Gamma_t - \lambda \varphi (1-\kappa) \omega \frac{\widetilde{q}_t}{q_t} \right] x_t^{bo} + \mu_t^p + \Phi(\iota_t) - \delta - \frac{c_{i,t}^{bo}}{n_{i,t}^{bo}} \right) \\ & + \frac{\lambda}{\rho^{bo}} \log \left( (1-\kappa) \left[ \frac{\widetilde{p}_t}{p_t} + \varphi x_t^{bo} \left( \frac{\widetilde{q}_t}{q_t} - \frac{\widetilde{p}_t}{p_t} \right) \right] \right) \Bigg\}, \end{aligned} \tag{7.2.14}$$

in which we have used (4.3.35) in conjunction with relation $q_t k_t^{e,i} = \varphi x_t^{bo} n_{i,t}^{bo}$ to substitute out wage

[870] In the remainder, variables pertaining to bank owners are denoted by superscript "$bo$".

payments $w_t^b$ to bankers. Maximisation with respect to $x_t^b$ yields an equation for the EFP:

$$\Gamma_t = \lambda\varphi\left(1-\kappa\right)\left[\omega\frac{\widetilde{q}_t}{q_t} + \Upsilon_t\left(\frac{\widetilde{p}_t}{p_t} - \frac{\widetilde{q}_t}{q_t}\right)\right], \tag{7.2.15}$$

$$\text{where}\quad \Upsilon_t \equiv \frac{1}{\left(1-\kappa\right)\left[\frac{\widetilde{p}_t}{p_t} + \varphi x_t^b\left(\frac{\widetilde{q}_t}{q_t} - \frac{\widetilde{p}_t}{p_t}\right)\right]} = \frac{n_t^{bo}}{\widetilde{n}_t^{bo}} \tag{7.2.16}$$

measures the ratio of bank owners' pre- to post-jump wealth. Comparing this to the equation for $\Gamma_t$ in the baseline model (4.3.34), it becomes clear that variable $\Upsilon_t$ governs the size of the risk premium banks require to extend a marginal unit of credit. If, on the contrary, bank owners acted in a risk neutral way and were not to face constraints on equity issuance, that term would not arise and, consequently, (7.2.15) would collapse to (4.3.34). To summarise, credit market dynamics in the model variant presented in this subsection are much richer compared to the baseline model due to two factors. First, bank owners' risk aversion introduces an additional time-varying wedge into the EFP. Second, the credit supply is proportionate to bank equity, which fluctuates due to stochastically arriving defaults on loans. Accordingly, this model variant allows for examining situations in which both credit demand and supply are depressed due to concurrent balance sheet crises in the entrepreneurial and the banking sector.

We now sketch an approach to solve for the equilibrium. Besides imposing market clearing conditions, closing the model again requires us to derive a differential equation for $\theta_t$.

**Proposition 12.** *In the credit model with financial frictions in the banking sector, the differential equation relating function $\theta\left(\eta_t, \eta_t^{bo}\right)$, in which the second argument stands for bank owners' wealth share, to its derivatives is given by*

$$\theta'_{\eta_t}\left(\eta_t, \eta_t^{bo}\right)\mu_t^{\eta}\eta_t + \theta'_{\eta_t^{bo}}\left(\eta_t, \eta_t^{bo}\right)\mu_t^{\eta^{bo}}\eta_t^{bo} = \left(\mu_t^q - \mu_t^p\right)\theta\left(\eta_t, \eta_t^{bo}\right)\left[1-\theta\left(\eta_t, \eta_t^{bo}\right)\right], \tag{7.2.17}$$

$$\textit{where}\quad \theta'_{\eta_t} \equiv \frac{\partial\theta\left(\eta_t, \eta_t^{bo}\right)}{\partial\eta_t} \quad \textit{and}\quad \theta'_{\eta_t^{bo}} \equiv \frac{\partial\theta\left(\eta_t, \eta_t^{bo}\right)}{\partial\eta_t^{bo}}.$$

*Proof.* See Appendix E.

The crucial difference to the baseline case without financial frictions in the banking sector is that the differential equation for $\theta_t$ now includes two partial derivatives. Put differently, the differential equation for $\theta_t$ is a retarded PDE. This fact arises from the introduction of a second state variable, namely bank owners' share in aggregate wealth. In principle, the iterative method is well-suited for solving PDEs arsing in the context of CTMF models (cf. Subsection 3.1.2.4). As mentioned, our attempt to solve the baseline credit model via the iterative method was not successful. This suggests that further research is required to adjust the iterative method for dealing with retarded differential equations that arise under Poisson uncertainty.

# Appendices

# Appendix A

# Appendix to Chapter 3

In this appendix we will briefly present two finite-difference methods for solving CTMF models. In general, when employing the finite-difference method, approximations of derivatives are obtained via Taylor series approximations of the unknown function.[871] Expanding function $q(\eta_m)$ with respect to $\eta_m$ gives

$$q(\eta_{m+1}) = q(\eta_m) + \frac{dq(\eta_m)}{d\eta_m}\frac{h_\eta}{1!} + \frac{d^2 q(\eta_m)}{d\eta_m^2}\frac{h_\eta^2}{2!} + \frac{d^3 q(\eta_m)}{d\eta_m^3}\frac{h_\eta^3}{3!} + \cdots \tag{A.0.1}$$

This can be rewritten to

$$\frac{dq(\eta_m)}{d\eta_m} = \frac{q(\eta_{m+1}) - q(\eta_m)}{h_\eta} + O(h_\eta), \tag{A.0.2}$$

where $q(\eta_{m+1}) - q(\eta_m) = q(\eta_m + h_\eta) - q(\eta_m)$ is called the *forward finite difference* of function $q(\eta_m)$ with step size $h_\eta$ and $O(h_\eta)$ is a function that summarises all terms of order $h_\eta$ and higher.[872]

It should be noted that (A.0.2) holds with equality and thus does not yield an approximation of the derivative. An obvious solution is to simply truncate term $O(h_\eta)$ and thereby to approximate the derivative with the ratio of the forward difference to the step size. In fact, this is the defining characteristic of the *Euler method*. The truncation error of this scheme is term $O(h_\eta)$, i.e. the error is proportionate to $h_\eta$. It follows that in order to obtain an accurate approximation, the step size has to be set sufficiently small.[873,874] When solving an ODE of second order, the problem is usually transformed to a problem of solving a system of two first-order ODEs.[875] In our context, this is achieved by defining an auxiliary variable $z(\eta_m) \equiv dq(\eta_m)/d\eta_m$, which allows for rewriting (3.1.36) to

$$\frac{dz(\eta_m)}{d\eta_m} = \frac{2\left[\mu_m^q q(\eta_m) - z(\eta_m)\,\mu_m^\eta \eta_m\right]}{\left(\sigma_m^\eta \eta_m\right)^2}. \tag{A.0.3}$$

[871] Cf. Bleecker and Csordas (2003, pp. 505f.).
[872] Cf. Bleecker and Csordas (2003, p. 506).
[873] Cf. Judd (1998, pp. 341f.).
[874] An example for this is given by Judd (cf. 1998, p. 343): suppose that the target for the approximation error is 0.0001. Then, the step size must at least be 0.0001, depending on the degree of differentiability of the function.
[875] Cf. Judd (1998, p. 336).

Using the forward finite difference approximations of $dq(\eta_m)/d\eta_m$ and $dz(\eta_m)/d\eta_m$ leads to

$$z(\eta_{m+1}) = \left( \frac{2\mu_m^q q(\eta_m)}{(\sigma_m^\eta \eta_m)^2} + z(\eta_m) \left[ 1 - \frac{2\mu_m^\eta \eta_m}{(\sigma_m^\eta \eta_m)^2} \right] \right) h_\eta \quad \text{and} \tag{A.0.4}$$

$$q(\eta_{m+1}) = q(\eta_m) + z(\eta_m) h_\eta. \tag{A.0.5}$$

In addition, one can approximate $\sigma_m^q$ via

$$\sigma_m^q = \frac{z(\eta_m)}{q(\eta_m)} \sigma_m^\eta \eta_m, \tag{A.0.6}$$

which follows from (3.1.37). System (A.0.4)-(A.0.6) can be iterated forward if two initial conditions $q(\eta_1) = q_0$ and $z(\eta_1) = z_0$ are available.

A drawback of the Euler method is its implicit assumption that the slope between two grid points is constant. In the CTMF literature, the sought price function often is concave. If the derivative is evaluated at grid point $m$, that fact implies that the Euler method overshoots the true solution, i.e. the approximated value $q(\eta_{m+1})$ will be too high. On the contrary, if at grid point $m$ the derivative at point $m+1$ is used for the forward iteration, $q(\eta_{m+1})$ will be too low. The first-order RK method takes an average of these two values and thereby converges faster to the true solution.[876] We demonstrate this principle in terms of ODE (A.0.3). The first-order RK approximation of this equation is

$$z(\eta_{m+1}) \approx z(\eta_m) + \frac{h_\eta}{2} \left[ f(\eta_m, z(\eta_m)) + f(\eta_m + h_\eta, z(\eta_m) + h_\eta f) \right],$$

in which $f = f(\eta_m, z(\eta_m))$ is defined to equal the RHS of (A.0.3). Even faster converge can be achieved by utilising higher-order RK methods. For a derivation of these schemes, the reader is referred to Miranda and Fackler (2002).[877]

[876] Cf. Judd (1998, pp. 344f.).
[877] Cf. Miranda and Fackler (2002, pp. 106f.).

# Appendix B

# Appendix to Chapter 4

## B.1 Proofs

### B.1.1 Proof of Lemma 3

To derive managers' stochastic flow budget constraint, the evolutions of asset positions, including asset purchases or sales, have to be defined at first. For an individual manager's capital stock we have

$$\frac{dk_{i,t}^m}{k_{i,t}^m} = \left\{ \frac{\chi_{i,t}^m \left( s_{i,t}^m - i_{i,t}^m \right)}{q_t k_{i,t}^m} + \Phi(\iota_{i,t}^m) - \delta \right\} dt - \left\{ \kappa + \underline{\kappa}^m \right\} d\underline{\mathcal{N}}_{i,t} - \left\{ \kappa + \overline{\kappa}^m \right\} d\overline{\mathcal{N}}_{i,t}, \tag{B.1.1}$$

where $\chi_{i,t}^m$ is the share of real savings $s_{i,t}^m$ less internal investment $i_{i,t}^m$ allocated by the manager to external investment or divestment, i.e. to purchases or sales of existing capital.[878] A positive value is tantamount to the purchasing of capital and a negative value to capital sales at price $q_t$.[879] The corresponding equation of motion for nominal money holdings is

$$\frac{dm_{i,t}^m}{m_{i,t}^m} = \frac{\left(1 - \chi_{i,t}^m\right)\left(s_{i,t}^m - i_{i,t}^m\right)}{p_t K_t m_{i,t}^m} dt. \tag{B.1.2}$$

If positive (negative), $1 - \chi_{i,t}^m$ measures the share of difference $s_{i,t}^m - i_{i,t}^m$ appropriated for increases (decreases) in the money stock. The agent's real savings $s_{i,t}^m$ are defined as the difference between

[878] The idea of using the share of savings less investment in order to determine capital purchases or sales is adopted from Wälde (cf. 2011, p. 264).

[879] More specifically, if $\chi_{i,t}^m > 0$, the agent buys $\chi_{i,t}^m \left(s_{i,t}^m - i_{i,t}^m\right)/q_t$ units of the capital good. On the contrary, if $\chi_{i,t}^m < 0$, he sells $\chi_{i,t}^m \left(s_{i,t}^m - i_{i,t}^m\right)/q_t$ units of the productive asset.

his production of final goods and his real consumption expenditures:[880]

$$s_{i,t}^m \equiv y_{i,t}^m - c_{i,t}^m. \tag{B.1.3}$$

In order to arrive at an equation of motion for net worth $n_{i,t}^m$, CVF (3.1.20) has to be applied to (4.2.1) since the variables on the RHS of that equation, with the exception of $m_{i,t}^m$, follow Poisson processes. That CVF can be applied to (4.2.1) by setting $n = 4$ as there are four processes describing the variables on the RHS of (4.2.1) and $m = 2$ since the maximum number of jump terms in these processes is two.[881] Taking into account equations (B.1.1), (B.1.2), (3.1.19), (3.2.15), and (4.1.1), this yields the stochastic flow budget constraint of agent $i$ *net of bank profits*:

$$\begin{aligned} dn_{i,t}^{m,nb} &= \left\{s_{i,t}^m - i_{i,t}^m + q_t\left[\mu_t^q + \Phi(\iota_{i,t}^m) - \delta\right]k_{i,t}^m + p_t\left[\mu_t^p + \Phi(\iota_t) - \delta\right]K_t m_{i,t}^m\right\}dt \\ &+ \left\{\left[\widetilde{q}_t\left(1-\kappa-\underline{\kappa}^m\right) - q_t\right]k_{i,t}^m + \left[\widetilde{p}_t\left(1-\kappa\right) - p_t\right]K_t m_{i,t}^m\right\}d\underline{\mathcal{N}}_{i,t} \\ &+ \left\{\left[\widetilde{q}_t\left(1-\kappa-\overline{\kappa}^m\right) - q_t\right]k_{i,t}^m + \left[\widetilde{p}_t\left(1-\kappa\right) - p_t\right]K_t m_{i,t}^m\right\}d\overline{\mathcal{N}}_{i,t}. \end{aligned} \tag{B.1.4}$$

The change in the individual's real wealth is determined by adding the income accruing from his equity stakes in banks to $dn_{i,t}^{m,nb}$:

$$dn_{i,t}^m = dn_{i,t}^{m,nb} + d\Pi_t^b n_{i,t}^m. \tag{B.1.5}$$

Inserting process (B.1.4) as well as using (B.1.3) leads to (4.2.3). □

### B.1.2 Proof of Lemma 4

In the presence of purchases or sales of the respective asset, an individual prudent entrepreneur's stocks of capital $k_{i,t}^{e,p}$, money $m_{i,t}^{e,p}$, and debt $l_{i,t}^{e,p}$ evolve according to

$$\frac{dk_{i,t}^{e,p}}{k_{i,t}^{e,p}} = \left\{\frac{\chi_{1,i,t}^{e,p}\left(s_{i,t}^{e,p} - i_{i,t}^{e,p}\right)}{q_t k_{i,t}^{e,p}} + \Phi(\iota_{i,t}^{e,p}) - \delta\right\}dt - \left\{\kappa + \underline{\kappa}^{e,p}\right\}d\underline{\mathcal{N}}_{i,t} - \left\{\kappa + \overline{\kappa}^{e,p}\right\}d\overline{\mathcal{N}}_{i,t}, \tag{B.1.6}$$

$$\frac{dm_{i,t}^{e,p}}{m_{i,t}^{e,p}} = \frac{\chi_{2,i,t}^{e,p}\left(s_{i,t}^{e,p} - i_{i,t}^{e,p}\right)}{p_t K_t m_{i,t}^{e,p}}dt, \quad \text{and} \tag{B.1.7}$$

$$\frac{dl_{i,t}^{e,p}}{l_{i,t}^{e,p}} = \frac{\left(1 - \chi_{1,i,t}^{e,p} - \chi_{2,i,t}^{e,p}\right)\left(s_{i,t}^{e,p} - i_{i,t}^{e,p}\right)}{p_t K_t m_{i,t}^{e,p}}dt, \tag{B.1.8}$$

[880] As in Wälde (cf. 2011, p. 241) savings are defined to not include capital gains.
[881] Note in this context that it is possible to write the stochastic process for $q_t$ as $\frac{dq_t}{q_t} = \mu_t^q\,dt + \frac{\widetilde{q}_t - q_t}{q_t}\,d\underline{\mathcal{N}}_t + \frac{\widetilde{q}_t - q_t}{q_t}\,d\overline{\mathcal{N}}_t$. The process for $p_t$ can be reformulated in an analogous way.

respectively. The interpretations of the variables $\chi_{1,i,t}^{e,p}$, $\chi_{2,i,t}^{e,p}$, and $\left(1-\chi_{1,i,t}^{e,p}-\chi_{2,i,t}^{e,p}\right)$ are analogous to those of variables $\chi_{i,t}^{m}$ and $1-\chi_{i,t}^{m}$ in Proof B.1.1: they measure the share of real savings less internal investment devoted to purchases or sales of the respective asset. Shares $\chi_{1,i,t}^{e,p}$ and $\chi_{2,i,t}^{e,p}$ can either be positive or negative, as long as the stock of the respective asset stays nonnegative. Conversely, share $1-\chi_{1,i,t}^{e,p}-\chi_{2,i,t}^{e,p}$ may be positive or negative, provided that $l_{i,t}^{e,p}$ remains nonpositive. This is because entrepreneurs are not allowed to hold a long position in loans.

Defining the individual's real savings according to

$$s_{i,t}^{e,p} \equiv y_{i,t}^{e,p} + \Gamma_t p_t K_t l_{i,t}^{e,p} - c_{i,t}^{e,p}, \tag{B.1.9}$$

where the second term arises from real interest payments, applying CVF (3.1.20) to (4.2.14), using (B.1.6)-(B.1.9), (3.1.19), (3.2.15), and (4.1.1) yields (4.2.15). □

### B.1.3 Proof of Lemma 6

The stochastic equations of for the nominal values of loans extended to imprudent entrepreneurs, loans extended to prudent entrepreneurs and the representative bank's money holdings in the presence of asset purchases or sales are:

$$\frac{dl_t^{b,i}}{l_t^{b,i}} = \frac{\chi_{1,t}^b s_t^b}{p_t K_t l_t^{b,i}}\, dt + \frac{\widetilde{l}_t^{b,i} - l_t^{b,i}}{l_t^{b,i}}\, d\mathcal{N}_t, \tag{B.1.10}$$

$$\frac{dl_t^{b,p}}{l_t^{b,p}} = \frac{\chi_{2,t}^b s_t^b}{p_t K_t l_t^{b,p}}\, dt, \quad \text{and} \tag{B.1.11}$$

$$\frac{dm_t^b}{m_t^b} = \frac{\left(1-\chi_{1,t}^b-\chi_{2,t}^b\right) s_t^b}{p_t K_t m_t^b}\, dt, \tag{B.1.12}$$

respectively. In (B.1.10)-(B.1.12) $\chi_{1,t}^b$ and $\chi_{2,t}^b$, and $1-\chi_{1,t}^b-\chi_{2,t}^b$ stand for the shares of the bank's real savings devoted to increasing (or decreasing if negative) the nominal stocks of loans to imprudent debtors, prudent debtors, and money respectively.

The bank's real savings are defined as

$$s_t^b \equiv \Gamma_t p_t K_t \left(l_t^{b,p} + l_t^{b,i}\right) - w_t^b, \tag{B.1.13}$$

which states that $s_t^b$ is the difference between loan proceeds in the absence of shocks and wage payments to bankers. The application of (3.1.20) to (4.2.28), using (4.1.1), (3.2.15), and (B.1.10)-(B.1.13), results in

$$\begin{aligned} dn_t^{b,nb} = {} & \left\{\Gamma_t p_t K_t \left(l_t^{b,p} + l_t^{b,i}\right) + \left(m_t^b + l_t^{b,p} + l_t^{b,i}\right) p_t K_t \left[\mu_t^p + \Phi(\iota_t) - \delta\right] - w_t^b\right\} dt \\ & + \left\{\left(m_t^b + l_t^{b,p} + \widetilde{l}_t^{b,i}\right)(1-\kappa)\, \widetilde{p}_t K_t - \left(m_t^b + l_t^{b,p} + l_t^{b,i}\right) p_t K_t\right\} d\mathcal{N}_t, \end{aligned} \tag{B.1.14}$$

which is the change in the bank's equity in the absence of payouts to and recapitalisation from bank owners. Subtracting $d\Pi_t^b n_{i,t}^m$ from (B.1.14) implies (4.2.29). $\square$

### B.1.4 Proof of Lemma 7

The proof consists of two parts. First, adopting the approach from Sennewald and Wälde (2006)[882], a general formula for the HJB equation is derived. In the second part, which follows Wälde (2011)[883], this general formulation in combination with CVF (3.1.20) is employed in order to arrive at HJB equation (4.3.3).

We start the proof by rearranging (4.3.2) to

$$0 = \max_{[c_{i,s}^m, x_{i,s}^m]_{s=t}^{\infty}} \mathbb{E}_t \int_t^{\infty} e^{-\rho(s-t)} u\left(c_{i,s}^m, \left(1 - x_{i,s}^m\right) n_{i,s}^m\right) ds - V^m\left(t, n_{i,t}^m\right). \tag{B.1.15}$$

Then, the integral is expanded by considering two subintervals $[t, h]$ and $[h, \infty]$ for some small $h > 0$. Subsequently, the *law of iterated expectations*[884] is applied, giving

$$0 = \max_{[c_{i,s}^m, x_{i,s}^m]_{s=t}^{\infty}} \left\{ \begin{array}{c} \mathbb{E}_t \int_t^{t+h} e^{-\rho(s-t)} u\left(c_{i,s}^m, \left(1 - x_{i,s}^m\right) n_{i,s}^m\right) ds \\ +\mathbb{E}_t \left[e^{-\rho^m h} \mathbb{E}_{t+h} \int_{t+h}^{\infty} e^{-\rho[s-(t+h)]} u\left(c_{i,s}^m, \left(1 - x_{i,s}^m\right) n_{i,s}^m\right) ds\right] \end{array} \right\} - V^m\left(t, n_{i,t}^m\right), \tag{B.1.16}$$

in which term $\mathbb{E}_{t+h} \int_{t+h}^{\infty} e^{-\rho[s-(t+h)]} u\left(c_{i,s}^m, \left(1 - x_{i,s}^m\right) n_{i,s}^m\right) ds$ is the expected discounted lifetime utility for an agent starting with net worth $n_{i,t+h}^m$ at time $t+h$. From equation (4.3.2) we have

$$V^m\left(t+h, n_{i,t+h}^m\right) = \max_{[c_{i,s}^m, x_{i,s}^m]_{s=t+h}^{\infty}} \mathbb{E}_{t+h} \int_{t+h}^{\infty} e^{-\rho(s-(t+h))} u\left(c_{i,s}^m, \left(1 - x_{i,s}^m\right) n_{i,s}^m\right) ds. \tag{B.1.17}$$

Substituting (B.1.17) into (B.1.16) yields

$$0 = \max_{[c_{i,s}^m, x_{i,s}^m]_{s=t}^{t+h}} \left\{ \begin{array}{c} \mathbb{E}_t \int_t^{t+h} e^{-\rho(s-t)} u\left(c_{i,s}^m, \left(1 - x_{i,s}^m\right) n_{i,s}^m\right) ds \\ +\mathbb{E}_t \left[e^{-\rho h} V^m\left(t+h, n_{i,t+h}^m\right)\right] \end{array} \right\} - V^m\left(t, n_{i,t}^m\right),$$

which shows that the original infinite sequence problem was reduced to finding the optimal value between $t$ and $t+h$. We now divide by $h$ , let $h \to 0$, and rearrange:

$$0 = \max_{c_{i,t}^m, x_{i,t}^m} \left\{ \begin{array}{c} \lim\limits_{h\to 0} \mathbb{E}_t \frac{1}{h} \int_t^{t+h} e^{-\rho(s-t)} u\left(c_{i,s}^m, \left(1 - x_{i,s}^m\right) n_{i,s}^m\right) ds \\ +\lim\limits_{h\to 0} \mathbb{E}_t \frac{1}{h} \left[e^{-\rho h} V^m\left(t+h, n_{i,t+h}^m\right) - V^m\left(t, n_{i,t}^m\right)\right] \end{array} \right\}, \tag{B.1.18}$$

[882] Cf. Sennewald and Wälde (2006, pp. 31f.).
[883] Cf. Wälde (2011, pp. 269f.).
[884] The law of iterated expectations states that for a random variable $X$ and two information sets $J$, $I$ with $J \subset I$ the equality $\mathbb{E}\left[\mathbb{E}\left(X \mid I\right) \mid J\right] = \mathbb{E}\left(X \mid J\right)$ holds (cf. Ljungqvist and Sargent, 2004, p. 33). In a time series context this implies that $\mathbb{E}_t \mathbb{E}_{t+1}\left(X_{t+1}\right) = \mathbb{E}_t\left(X_{t+1}\right)$ (cf. Ljungqvist and Sargent, 2004, p. 401).

which reduces the problem further to finding optimal choices of the controls only at time $t$. The second line in the curly bracket of the above equation is the derivative of $e^{-\rho h}V\left(t+h, n^m_{i,t+h}\right)$ with respect to $h$ at $h = 0$. This derivative, using the product rule, can also be expressed as

$$-\rho V\left(t, n^m_{i,t}\right) + \frac{d}{dt}\mathbb{E}_t V^m\left(t, n^m_{i,t}\right), \tag{B.1.19}$$

where we have used the fact that $\frac{d}{dh}\mathbb{E}_t V^m\left(t+h, n^m_{i,t+h}\right) = \frac{d}{dt}\mathbb{E}_t V^m\left(t, n^m_{i,t}\right)$ at $h = 0$. Using (B.1.19) in (B.1.18) and acknowledging that wealth at time $t$ is independent of the current controls, yields

$$\rho V^m\left(t, n^m_{i,t}\right) = \max_{c^m_{i,t}, x^m_{i,t}}\left\{\lim_{h\to 0}\ \mathbb{E}_t \tfrac{1}{h}\int_t^{t+h} e^{-\rho(s-t)}u\left(c^m_{i,s}, \left(1-x^m_{i,s}\right)n^m_{i,s}\right)\,ds\ + \frac{d}{dt}\mathbb{E}_t V^m\left(t, n^m_{i,t}\right)\right\}. \tag{B.1.20}$$

Using the fact that the limit on the RHS of the above equation can simply be replaced by $u\Big(c^m_{i,t}, \left(1-x^m_{i,t}\right)n^m_{i,t}\Big)$ and applying the *theorem of bounded convergence*, which implies that the expectation and derivative in the second term in the curly bracket of the above equation can be interchanged[885], we have

$$\rho V^m\left(t, n^m_{i,t}\right) = \max_{c^m_{i,t}, x^m_{i,t}}\left\{u\left(c^m_{i,t}, \left(1-x^m_{i,t}\right)n^m_{i,t}\right) + \frac{1}{dt}\mathbb{E}_t\, dV^m\left(t, n^m_{i,t}\right)\right\}. \tag{B.1.21}$$

The derivation of this general HJB equation completes the first part of the proof.[886]

The first step of the second part is to apply CVF (3.1.20) to $V^m(\cdot)$, taking into account the stochastic process for the representative manager's net worth (4.3.1b), in order find an expression for $dV^m(\cdot)$. This leads to

$$\begin{aligned} dV^m\left(t, n^m_{i,t}\right) = & \left\{\frac{dV^m\left(t, n^m_{i,t}\right)}{dt} + \frac{dV^m\left(t, n^m_{i,t}\right)}{dn^m_{i,t}}\mu^{n^m}_{i,t}\, n^m_{i,t}\right\} dt \\ & + \left\{V^m\left(t, \underline{\tilde{n}}^m_{i,t}\right) - V^m\left(t, n^m_{i,t}\right)\right\} d\underline{\mathcal{N}}_t + \left\{V^m\left(t, \overline{\tilde{n}}^m_{i,t}\right) - V^m\left(t, n^m_{i,t}\right)\right\} d\overline{\mathcal{N}}_t, \end{aligned} \tag{B.1.22}$$

or, after applying the expectation operator, to

$$\begin{aligned} \mathbb{E}_t dV^m\left(t, n^m_{i,t}\right) = & \left\{\frac{dV^m\left(t, n^m_{i,t}\right)}{dt} + \frac{dV^m\left(t, n^m_{i,t}\right)}{dn^m_{i,t}}\mu^{n^m}_{i,t}\, n^m_{i,t}\right\} dt \\ & + \left\{V^m\left(t, \underline{\tilde{n}}^m_{i,t}\right) - V^m\left(t, n^m_{i,t}\right)\right\}\mathbb{E}_t\, d\underline{\mathcal{N}}_t + \left\{V^m\left(t, \overline{\tilde{n}}^m_{i,t}\right) - V^m\left(t, n^m_{i,t}\right)\right\}\mathbb{E}_t\, d\overline{\mathcal{N}}_t. \end{aligned} \tag{B.1.23}$$

[885] Cf. Appendix C in Sennewald (2007) for a discussion of conditions for the theorem of bounded convergence to hold in this context.

[886] Equation (B.1.21) corresponds to Equation (13) in Sennewald and Wälde (2006).

Substituting (B.1.23) into (B.1.21), while evaluating the expectations in the former equation according to $\mathbb{E}_t \, d\underline{\mathcal{N}}_t = \lambda\phi \, dt$ and $\mathbb{E}_t \, d\overline{\mathcal{N}}_t = \lambda \left(1-\phi\right) \, dt$, implies

$$\rho V^m \left(t, n_{i,t}^m\right) = \max_{c_{i,t}^m, x_{i,t}^m} \left\{ \begin{array}{c} u\left(c_{i,t}^m, \left(1 - x_{i,t}^m\right) n_{i,t}^m\right) + \frac{dV^m\left(t, n_{i,t}^m\right)}{dt} + \frac{dV^m\left(t, n_{i,t}^m\right)}{dn_{i,t}^m} \mu_{i,t}^{n^m} n_{i,t}^m \\ +\lambda\phi\left(V^m\left(t, \underset{\sim}{\widetilde{n}}_{i,t}^m\right) - V^m\left(t, n_{i,t}^m\right)\right) \\ +\lambda\left(1-\phi\right)\left(V^m\left(t, \widetilde{\widetilde{n}}_{i,t}^m\right) - V^m\left(t, n_{i,t}^m\right)\right) \end{array} \right\}. \tag{B.1.24}$$

As mentioned in footnote 604, the value function does not depend on time directly. Hence, we can replace $V^m\left(t, n_{i,t}^m\right)$ by $V^m\left(n_{i,t}^m\right)$ and set $dV^m\left(t, n_{i,t}^m\right)/dt = 0$. This is the final step of the proof and results in (4.3.3). □

### B.1.5 Proof of Lemma 8

The goal of this appendix is to show that the value function associated with problem (4.3.1a)-(4.3.1f) is of the logarithmic type.[887] As a means to prove this, the method of undetermined coefficients is employed. The first step is to guess that the value function is of the logarithmic type. This guess is motivated by Merton's classic result which states that the value function inherits the functional form of the instantaneous utility function if the latter belongs to the HARA class.[888] Accordingly, for arbitrary $t$, the guess for the value function is

$$V^m\left(n_{i,t}^m\right) = \alpha_t^m + \beta_t^m \log n_{i,t}^m, \tag{B.1.25}$$

where $\alpha_t^m$ and $\beta_t^m$ are undetermined and potentially time-varying coefficients. Hence, the derivative of the guessed value function is

$$\frac{dV^m\left(n_{i,t}^m\right)}{dn_{i,t}^m} = \frac{\beta_t^m}{n_{i,t}^m}. \tag{B.1.26}$$

Substituting guesses (B.1.25) and (B.1.26) into the RHS of HJB equation (4.3.3) and taking into account (4.3.1c)-(4.3.1e) as well as (4.3.7) results in

[887] The proof is inspired by Moll (cf. 2014, pp. 3216f.).
[888] Cf. Merton (1971, Theorem IV.).

$$\begin{aligned}\rho V^m\left(n_{i,t}^m\right) = \max_{c_{i,t}^m, x_{i,t}^m} &\Big\{\log c_{i,t}^m + \xi \log\left(\left(1-x_{i,t}^m\right) n_{i,t}^m\right) \\ &+ \frac{\beta_t^m}{n_{i,t}^m}\left[\left(x_{i,t}^m \mu_t^{r^{K,m}} + \left(1-x_{i,t}^m\right)\mu_t^{r^M} + \mu_t^{\Pi^b}\Pi_t^b\right) n_{i,t}^m - c_{i,t}^m\right] \\ &+ \lambda\phi\beta_t^m \log\left(x_{i,t}^m\left(1-\kappa-\underline{\kappa}^m\right)\frac{\widetilde{q}_t}{q_t} + \left(1-x_{i,t}^m\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t} + \widetilde{\Pi}_t^b - \Pi_t^b\right) \\ &+ \lambda(1-\phi)\beta_t^m \log\left(x_{i,t}^m\left(1-\kappa-\overline{\kappa}^m\right)\frac{\widetilde{q}_t}{q_t} + \left(1-x_{i,t}^m\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t} + \widetilde{\Pi}_t^b - \Pi_t^b\right)\Big\}.\end{aligned} \tag{B.1.27}$$

Maximising equation (B.1.27) with respect to $c_t^m$ and rearranging yields the optimal consumption guess:

$$c_{i,t}^{m*} = \frac{n_{i,t}^m}{\beta_t^m}. \tag{B.1.28}$$

The first-order condition for the portfolio weight is

$$\begin{aligned}&-\frac{\xi}{1-x_{i,t}^m} + \beta_t^e\left(\mu_t^{r^{K,m}} + \mu_t^{r^M}\right) \\ &+ \lambda\phi\beta_t^e \frac{\left(1-\kappa+\underline{\kappa}^m\right)\frac{\widetilde{q}_t}{q_t} + (1-\kappa)\frac{\widetilde{p}_t}{p_t}}{x_{i,t}^m\left(1-\kappa+\underline{\kappa}^m\right)\frac{\widetilde{q}_t}{q_t} + \left(1-x_{i,t}^m\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t} + \widetilde{\Pi}_t^b - \Pi_t^b} \\ &+ \lambda(1-\phi)\beta_t^e \frac{\left(1-\kappa+\overline{\kappa}^m\right)\frac{\widetilde{q}_t}{q_t} + (1-\kappa)\frac{\widetilde{p}_t}{p_t}}{x_{i,t}^m\left(1-\kappa+\overline{\kappa}^m\right)\frac{\widetilde{q}_t}{q_t} + \left(1-x_{i,t}^m\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t} + \widetilde{\Pi}_t^b - \Pi_t^b} \leq 0.\end{aligned} \tag{B.1.29}$$

This equation implicitly defines the optimal portfolio weight $x_{i,t}^{m*}$, which does not depend on the agent's wealth level $n_{i,t}^m$. This results from the adoption of an instantaneous utility function that belongs to the HARA class and a CRS production technology.

In Section 4.2.1 it was assumed that the investment rate is always at its optimal level. Hence, we also have to check whether the optimal investment rate is independent of the state variable $n_{i,t}^m$. The former is found by maximising the drift rate of the return on capital:

$$\max_{\iota_t} \mu_t^{r^{K,m}} \equiv \frac{a^m - \iota_t}{q_t} + \mu_t^q + \Phi(\iota_t) - \delta. \tag{B.1.30}$$

Obviously, the optimal investment rate does not depend on $n_{i,t}^m$.

The next step is to substitute guess (B.1.25) on the LHS of (B.1.27) and FOC (B.1.28) as well as optimal portfolio weight $x_{i,t}^{m*}$ on the RHS of that equation. Subsequent rearranging yields[889]

$$\begin{aligned}\rho\left(\alpha_t^m+\beta_t^m\log n_{i,t}^m\right)&=\log n_{i,t}^m-\log\beta_t^m+\xi\left[\log\left(1-x_{i,t}^{m*}\right)+\log n_{i,t}^m\right]\\&+\beta_t^m\left[x_{i,t}^{m*}\mu_t^{r^{K,m}}+\left(1-x_{i,t}^{m*}\right)\mu_t^{r^M}+\mu_t^{\Pi^b}\Pi_t^b\right]-1\\&+\lambda\phi\beta_t^m\log\left(x_{i,t}^{m*}\left(1-\kappa+\underline{\kappa}^m\right)\frac{\widetilde{q}_t}{q_t}+\left(1-x_{i,t}^{m*}\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t}+\widetilde{\Pi}_t^b-\Pi_t^b\right)\\&+\lambda\left(1-\phi\right)\beta_t^m\log\left(x_{i,t}^{m*}\left(1-\kappa+\overline{\kappa}^m\right)\frac{\widetilde{q}_t}{q_t}+\left(1-x_{i,t}^{m*}\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t}+\widetilde{\Pi}_t^b-\Pi_t^b\right).\end{aligned}\tag{B.1.31}$$

Collecting terms involving $n_t^m$ on the left and right hand sides leads to

$$\rho\beta_t^m\log n_{i,t}^m=\log n_{i,t}^m+\xi\log n_{i,t}^m.\tag{B.1.32}$$

This can be used to determine $\beta_t^m$:

$$\beta_t^m=\frac{1+\xi}{\rho}.$$

It follows that the value function takes the following form:

$$V^m\left(n_{i,t}^m\right)=\alpha_t^m+\frac{(1+\xi)}{\rho}\log n_{i,t}^m.\tag{B.1.33}$$

To sum up, the simple form of the value function is a result of the adoption of logarithmic utility and a CRS production function. In general, HARA preferences and CRS technologies allow for closed form solutions since these two assumptions make asset demands linear in wealth, as mentioned. □

### B.1.6 Proof of Lemma 9

Compared with the derivation of managers' HJB equation in Lemma 7, solving for entrepreneurs' HJB equation is complicated by the fact that the state space associated with the maximisation problem of the latter agent type consists of two state variables $n_{i,t}^{e,p}$ and $z_{i,t}$. To tackle this issue, we commence the proof by redefining the stochastic processes for an individual entrepreneur's productivity level and equity, which are given by (4.1.10) and (4.2.17), respectively. Process (4.1.10) is rewritten to

$$dz_{i,t}=\{a^m-a^e\}\,d\underline{\mathcal{N}}_{i,t}^s+\{a^m-a^e\}\,d\overline{\mathcal{N}}_{i,t}^s,\tag{B.1.34}$$

which is obtained by dividing stochastic increment $d\mathcal{N}_{i,t}^s$ into two subincrements $d\underline{\mathcal{N}}_{i,t}^s$, which equals unity if the agent draws the adverse individual shock as well as the sector-specific shock, and $d\overline{\mathcal{N}}_{i,t}^s$, which is equal to one if he is subject to the beneficial individual shock but has to suffer the sector-

[889] Substituting optimal values consumption and the portfolio weight into the HJB equation eliminates the maximum operator.

specific shock. These increments obey distributions

$$d\underline{\mathcal{N}}^{s}_{i,t} = \begin{cases} 1 & \text{with prob. } \lambda\phi\phi^{s}\, dt \\ 0 & \text{with prob. } 1 - \lambda\phi\phi^{s}\, dt \end{cases} \tag{B.1.35a}$$

and

$$d\overline{\mathcal{N}}^{s}_{i,t}, = \begin{cases} 1 & \text{with prob. } \lambda\left(1-\phi\right)\phi^{s}\, dt \\ 0 & \text{with prob. } 1 - \lambda\left(1-\phi\right)\phi^{s}\, dt. \end{cases} \tag{B.1.35b}$$

Comparing these distributions to that in (4.1.11), it becomes clear that (B.1.34) is just a reformulation of the original process (4.1.10).

Next, we define two increments $d\underline{\mathcal{N}}^{ns}_{i,t}$, which equals unity if the agent is exposed to the adverse idiosyncratic shock to capital but not to the sector-specific shock, and $d\overline{\mathcal{N}}^{ns}_{i,t}$, which is equal to one if the agent is subject to benign idiosyncratic shock but not to the sector-specific shock. These random variables have distributions

$$d\underline{\mathcal{N}}^{ns}_{i,t} = \begin{cases} 1 & \text{with prob. } \lambda\phi\left(1-\phi^{s}\right)\, dt \\ 0 & \text{with prob. } 1 - \lambda\phi\left(1-\phi^{s}\right)\, dt \end{cases} \tag{B.1.36a}$$

and

$$d\overline{\mathcal{N}}^{ns}_{i,t}, = \begin{cases} 1 & \text{with prob. } \lambda\left(1-\phi\right)\left(1-\phi^{s}\right)\, dt \\ 0 & \text{with prob. } 1 - \lambda\left(1-\phi\right)\left(1-\phi^{s}\right)\, dt. \end{cases} \tag{B.1.36b}$$

Both increments can be used together with $d\underline{\mathcal{N}}^{s}_{i,t}$ and $d\overline{\mathcal{N}}^{s}_{i,t}$ to reformulate (4.2.17) to

$$\begin{aligned} dn^{e,p}_{i,t} = \mu^{n^{e,p}}_{i,t} n^{e,p}_{i,t}\, dt &+ \left\{\underline{\widetilde{n}}^{e,p}_{i,t} - n^{e,p}_{i,t}\right\} d\underline{\mathcal{N}}^{ns}_{i,t} + \left\{\overline{\widetilde{n}}^{e,p}_{i,t} - n^{e,p}_{i,t}\right\} d\overline{\mathcal{N}}^{ns}_{i,t} \\ &+ \left\{\underline{\widetilde{\widetilde{n}}}^{e,p}_{i,t} - n^{e,p}_{i,t}\right\} d\underline{\mathcal{N}}^{s}_{i,t} + \left\{\overline{\widetilde{\widetilde{n}}}^{e,p}_{i,t} - n^{e,p}_{i,t}\right\} d\overline{\mathcal{N}}^{s}_{i,t}. \end{aligned} \tag{B.1.37}$$

The advantage of the alternative formulations of the two stochastic processes is that they can be used to derive the differential of value function $V\left(n_{i,t}^{e,p}, z_{i,t}\right)$ via CVF (3.1.20):

$$
\begin{aligned}
dV\left(n_{i,t}^{e,p}, z_{i,t}\right) = & \frac{dV\left(n_{i,t}^{e,p}, z_{i,t}\right)}{dn_{i,t}^{m}} \mu_{i,t}^{n^{e,p}} n_{i,t}^{e,p}\, dt \\
& + \left\{V\left(\underline{\widetilde{n}}_{i,t}^{e,p}, z_{i,t}\right) - V\left(n_{i,t}^{e,p}, z_{i,t}\right)\right\} d\underline{\mathcal{N}}_{i,t}^{ns} \\
& + \left\{V\left(\overline{\widetilde{n}}_{i,t}^{e,p}, z_{i,t}\right) - V\left(n_{i,t}^{e,p}, z_{i,t}\right)\right\} d\overline{\mathcal{N}}_{i,t}^{ns} \\
& + \left\{V\left(\underline{\widetilde{n}}_{i,t}^{e,p}, z_{i,t} + \Delta_z\right) - V\left(n_{i,t}^{e,p}, z_{i,t}\right)\right\} d\underline{\mathcal{N}}_{i,t}^{s} \\
& + \left\{V\left(\overline{\widetilde{n}}_{i,t}^{e,p}, z_{i,t} + \Delta_z\right) - V\left(n_{i,t}^{e,p}, z_{i,t}\right)\right\} d\overline{\mathcal{N}}_{i,t}^{s},
\end{aligned}
\tag{B.1.38}
$$

in which $\Delta_z \equiv a^m - a^e$. This can be substituted into

$$
\rho V\left(n_{i,t}^{e,p}, z_{i,t}\right) = \max_{c_{i,t}^{e,p}, x_{1,i,t}^{e,p}, x_{2,i,t}^{e,p}} \left\{u\left(c_{i,t}^{e,p}, x_{2,i,t}^{e,p} n_{i,t}^{e,p}\right) + \frac{1}{dt}\mathbb{E}_t\left[dV\left(n_{i,t}^{e,p}, z_{i,t}\right)\right]\right\}, \tag{B.1.39}
$$

which is the analogue to (B.1.21) in prudent entrepreneur $i$'s problem, to arrive at HJB equation

$$
\begin{aligned}
\rho V\left(n_{i,t}^{e,p}, z_{i,t}\right) = & \max_{c_{i,t}^{e,p}, x_{1,i,t}^{e,p}, x_{2,i,t}^{e,p}} \Bigg\{u\left(c_{i,t}^{e,p}, x_{2,i,t}^{e,p} n_{i,t}^{e,p}\right) + \frac{dV^e\left(n_{i,t}^{e,p}, z_{i,t}\right)}{dn_{i,t}^{e}} \mu_{i,t}^{n^e} n_{i,t}^{e} \\
& + \lambda\left(1-\phi^s\right)\phi\left[V\left(\underline{\widetilde{n}}_{i,t}^{e,p}, z_{i,t}\right) - V\left(n_{i,t}^{e,p}, z_{i,t}\right)\right] \\
& + \lambda\left(1-\phi^s\right)\left(1-\phi\right)\left[V\left(\overline{\widetilde{n}}_{i,t}^{e,p}, z_{i,t}\right) - V\left(n_{i,t}^{e,p}, z_{i,t}\right)\right] \\
& + \lambda\phi^s\phi\left[V\left(\underline{\widetilde{n}}_{i,t}^{e,p}, z_{i,t} + \Delta_z\right) - V\left(n_{i,t}^{e,p}, z_{i,t}\right)\right] \\
& + \lambda\phi^s\left(1-\phi\right)\left[V\left(\overline{\widetilde{n}}_{i,t}^{e,p}, z_{i,t} + \Delta_z\right) - V\left(n_{i,t}^{e,p}, z_{i,t}\right)\right]\Bigg\},
\end{aligned}
\tag{B.1.40}
$$

where we have used

$$
\begin{aligned}
& \mathbb{E}_t\left[d\underline{\mathcal{N}}_{i,t}^{ns}\right] = \lambda\left(1-\phi^s\right)\phi\, dt, && \text{(B.1.41a)} \\
& \mathbb{E}_t\left[d\overline{\mathcal{N}}_{i,t}^{ns}\right] = \lambda\left(1-\phi^s\right)\left(1-\phi\right)\, dt, && \text{(B.1.41b)} \\
& \mathbb{E}_t\left[d\underline{\mathcal{N}}_{i,t}^{s}\right] = \lambda\phi^s\phi\, dt, && \text{(B.1.41c)} \\
\text{and}\quad & \mathbb{E}_t\left[d\overline{\mathcal{N}}_{i,t}^{s}\right] = \lambda\phi^s\left(1-\phi\right)\, dt. && \text{(B.1.41d)}
\end{aligned}
$$

The next steps are to define $V^{e,p}\left(n_{i,t}^{e,p}\right) \equiv V\left(n_{i,t}^{e,p}, z_{i,t}\right)$ and to set $V^m\left(n_{i,t}^{e,p}\right) = V\left(n_{i,t}^{e,p}, z_{i,t} + \Delta_z\right)$. The last equality follows from the fact that in case entrepreneur $i$ is exposed to the sector-specific shock, he effectively continues operations as a manager. Using these equalities in (B.1.40), yields

$$
\begin{aligned}
\rho V^{e,p}\left(n_{i,t}^{e,p}\right) = \max_{c_{i,t}^{e,p}, x_{1,i,t}^{e,p}, x_{2,i,t}^{e,p}} \Bigg\{ & u\left(c_{i,t}^{e,p}, x_{2,i,t}^{e,p} n_{i,t}^{e,p}\right) + \frac{dV^{e,p}\left(n_{i,t}^{e,p}\right)}{dn_{i,t}^{e}} \mu_{i,t}^{n^e} n_{i,t}^{e} \\
& + \lambda\left(1-\phi^s\right)\phi\left[V^{e,p}\left(\underline{\widetilde{n}}_{i,t}^{e,p}\right) - V^{e,p}\left(n_{i,t}^{e,p}\right)\right] \\
& + \lambda\left(1-\phi^s\right)\left(1-\phi\right)\left[V^{e,p}\left(\overline{\widetilde{n}}_{i,t}^{e,p}\right) - V^{e,p}\left(n_{i,t}^{e,p}, z_{i,t}\right)\right] \\
& + \lambda\phi^s\phi\left[V^{m}\left(\underline{\widetilde{n}}_{i,t}^{e,p}\right) - V^{e,p}\left(n_{i,t}^{e,p}\right)\right] \\
& + \lambda\phi^s\left(1-\phi\right)\left[V^{m}\left(\overline{\widetilde{n}}_{i,t}^{e,p}\right) - V^{e,p}\left(n_{i,t}^{e,p}\right)\right]\Bigg\}.
\end{aligned} \tag{B.1.42}
$$

Hence, the dimension of the state space reduces to unity. The crucial assumption that $z_{i,t}$ can only take on two values facilitates this simplification.[890] Equation (4.3.15) can be obtained easily by performing further algebra. □

### B.1.7 Proof of Lemma 10

In analogy to the proof in Appendix B.1.5, the guess for the value function is

$$
V^{e,p}(n_{i,t}^{e,p}) = \alpha_t^{e,p} + \beta_t^{e,p} \log n_{i,t}^{e,p}. \tag{B.1.43}
$$

Substituting this value function, its derivative, and managers' value function (4.3.8) on the RHS of (4.3.15) implies

$$
\begin{aligned}
\rho V^{e,p}(n_{i,t}^{e,p}) = \max_{c_{i,t}^{e,p}, x_{1,i,t}^{e,p}, x_{2,i,t}^{e,p}} \Bigg\{ & \log c_{i,t}^{e,p} + \xi \log\left(x_{2,i,t}^{e,p} n_{i,t}^{e,p}\right) \\
& + \frac{\beta_t^{e,p}}{n_{i,t}^{e,p}}\left(\left[x_{1,i,t}^{e,p}\mu_t^{r^{K,e}} + x_{2,i,t}^{e,p}\mu_t^{r^M} + \left(1 - x_{1,i,t}^{e,p} - x_{2,i,t}^{e,p}\right)\mu_t^{r^{L,e}}\right] n_{i,t}^{e,p} - c_{i,t}^{e,p}\right) \\
& + \lambda\left[\left(1-\phi^s\right)\beta^{e,p} + \phi^s \frac{1+\xi}{\rho}\right]\Bigg[\phi \log\Bigg(x_{1,i,t}^{e,p}\left(1-\kappa-\underline{\kappa}^{e,p}\right)\frac{\widetilde{q}_t}{q_t} \\
& + \left(1 - x_{1,i,t}^{e,p}\right)\left(1-\kappa\right)\frac{\widetilde{p}_t}{p_t}\Bigg) + \left(1-\phi\right)\log\Bigg(x_{1,i,t}^{e,p}\left(1-\kappa-\overline{\kappa}^{e,p}\right)\frac{\widetilde{q}_t}{q_t} \\
& + \left(1 - x_{1,i,t}^{e,p}\right)\left(1-\kappa\right)\frac{\widetilde{p}_t}{p_t}\Bigg)\Bigg] + \lambda\phi^s\left[\alpha_t^m - \alpha_t^{e,p} + \left(\frac{1+\xi}{\rho} - \beta_t^{e,p}\right)\log n_{i,t}^{e,p}\right]\Bigg\}.
\end{aligned} \tag{B.1.44}
$$

Optimal consumption is

$$
c_{i,t}^{e,p} = \frac{n_{i,t}^{e,p}}{\beta_t^{e,p}}. \tag{B.1.45}
$$

[890] The structure and derivation of entrepreneurs' HJB equation are very similar to those in the labour market matching literature (cf. Wälde, 2011, p. 280). In that literature workers switch between two employment statuses, namely "employed" and "unemployed". A notable difference is that the individual's wealth level is stochastic here.

The FOCs with respect to $x_{1,i,t}^{e,p}$, $x_{2,i,t}^{e,p}$, and $\iota_t$ can easily be shown to be independent of $n_{i,t}^{e,p}$. Inserting optimal policies on the RHS, value function (B.1.43) on the LHS, and rearranging leads to

$$\begin{aligned}
\log n_t^{e,p} &= \frac{1}{\rho\beta_t^{e,p}}\left[(1+\xi)\log n_{i,t}^{e,p} - \log\beta_t^{e,p} + \xi\log x_{2,i,t}^{e,p*}\right] \\
&+ \frac{1}{\rho}\left[x_{1,i,t}^{e,p*}\mu_t^{r^{K,e}} + x_{2,i,t}^{e,p}\mu_t^{r^M} + \left(1 - x_{1,i,t}^{e,p*} - x_{2,i,t}^{e,p*}\right)\mu_t^{r^{L,e}} - \frac{1}{\beta_t^{e,p}}\right] \\
&+ \frac{\lambda}{\rho}\left[1 + \left(\frac{1+\xi}{\rho\beta_t^{e,p}} - 1\right)\phi^s\right]\left[\phi\log\!\left(x_{1,i,t}^{e,p*}\left(1-\kappa-\underline{\kappa}^{e,p}\right)\frac{\widetilde{q}_t}{q_t} + \left(1 - x_{1,i,t}^{e,p*}\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t}\right)\right. \\
&\left. + (1-\phi)\log\!\left(x_{1,i,t}^{e,p*}\left(1-\kappa-\overline{\kappa}^{e,p}\right)\frac{\widetilde{q}_t}{q_t} + \left(1 - x_{1,i,t}^{e,p*}\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t}\right)\right] \\
&+ \frac{\lambda\phi^s}{\rho\beta_t^{e,p}}\left[\alpha_t^m - \alpha_t^{e,p} + \left(\frac{1+\xi}{\rho} - \beta_t^{e,p}\right)\log n_{i,t}^{e,p}\right] - \frac{\alpha_t^{e,p}}{\beta_t^{e,p}}.
\end{aligned} \tag{B.1.46}$$

Collecting terms involving $n_{i,t}^{e,p}$ gives

$$\log n_{i,t}^{e,p} = \frac{1+\xi}{\rho\beta_t^{e,p}}\log n_{i,t}^{e,p} + \frac{\lambda\phi^s}{\rho\beta_t^{e,p}}\left(\frac{1+\xi}{\rho} - \beta_t^{e,p}\right)\log n_{i,t}^{e,p}. \tag{B.1.47}$$

This can be solved for $\beta_t^{e,p}$:

$$\beta_t^{e,p} = \frac{1+\xi}{\rho}.$$

□

### B.1.8 Proof of Lemma 11

We start the proof by deriving the drift of the stochastic process for aggregate entrepreneurial wealth $n_t^e$. Using (4.3.14c) and optimal consumption choice (4.3.19), aggregation of the drifts of individual wealth levels yields

$$\int_0^{1-\varphi}\mu_{i,t}^{n^{e,p}} n_{i,t}^{e,p}\,di = \int_0^{1-\varphi}\left[x_{1,i,t}^{e,p}\mu_t^{r^{K,e}} + x_{2,i,t}^{e,p}\mu_t^{r^M} + \left(1 - x_{1,i,t}^{e,p} - x_{2,i,t}^{e,p}\right)\mu_t^{r^{L,e}} - \frac{\rho}{1+\xi}\right] n_{i,t}^{e,p}\,di, \tag{B.1.48}$$

in which we have integrated over prudent entrepreneurs only since their imprudent counterparts own zero wealth. Recalling that portfolio weights do not differ among the group of prudent entrepreneurs and hence, drift rate $\mu_{i,t}^{n^{e,p}}$ does neither, we can drop the subscript $i$ of these variables and express the above equation as

$$\mu_t^{n^e} n_t^e = \left[x_{1,t}^{e,p}\mu_t^{r^{K,e}} + x_{2,t}^{e,p}\mu_t^{r^M} + \left(1 - x_{1,t}^{e,p} - x_{2,t}^{e,p}\right)\mu_t^{r^{L,e}} - \frac{\rho}{1+\xi}\right] n_t^e. \tag{B.1.49}$$

Again, aggregation is facilitated by the independence of optimal portfolio weights on wealth and the linear relationship between optimal consumption and wealth. Some additional algebra leads to

(4.5.1b). Employing a similar procedure, the corresponding drift for managers' net worth can be derived as

$$\mu_t^{n^m} n_t^m = \left[ x_t^m \mu_t^{r^{K,m}} + (1 - x_t^m)\, \mu_t^{r^M} + \mu_t^{\Pi^b} \Pi_t^b - \frac{\rho}{1+\xi} \right] n_t^m. \tag{B.1.50}$$

We now have determined the deterministic parts of the stochastic equations of motion for $n_t^e$ and $n_t^m$. Fully describing the dynamic behaviour of theses variables also requires us to determine how they change as a consequence of a jump. To this end, let us first consider the post-jump aggregate wealth level of all individuals which belonged to the group of prudent entrepreneurs prior to the jump. This level is

$$\int_0^{1-\varphi} \left[ \widetilde{q}_t \widetilde{k}_{i,t}^{e,p} + \widetilde{p}_t \widetilde{K}_t \left( m_{i,t}^{e,p} + l_{i,t}^{e,p} \right) \right] di.$$

This can be reformulated to

$$\int_0^{1-\varphi} \left[ \left(1 - \kappa - \underline{\kappa}^{e,p}\right) \widetilde{q}_t k_{i,t}^{e,p}\, d\underline{\mathcal{N}}_{i,t} + \left(1 - \kappa - \overline{\kappa}^{e,p}\right) \widetilde{q}_t k_{i,t}^{e,p}\, d\overline{\mathcal{N}}_{i,t} + (1-\kappa)\, \widetilde{p}_t K_t \left( m_{i,t}^{e,p} + l_{i,t}^{e,p} \right) \right] di.$$

Further manipulation leads to

$$\begin{aligned} (1-\kappa) \int_0^{1-\varphi} & \left[ \widetilde{q}_t \left( k_{i,t}^{e,p}\, d\underline{\mathcal{N}}_{i,t} + k_{i,t}^{e,p}\, d\overline{\mathcal{N}}_{i,t} \right) + \widetilde{p}_t K_t \left( m_{i,t}^{e,p} + l_{i,t}^{e,p} \right) \right] di \\ & - \widetilde{q}_t \int_0^{1-\varphi} \left( \underline{\kappa}^{e,p} k_{i,t}^{e,p}\, d\underline{\mathcal{N}}_{i,t} + \overline{\kappa}^{e,p} k_{i,t}^{e,p}\, d\overline{\mathcal{N}}_{i,t} \right) di. \end{aligned}$$

Using $\int_0^{1-\varphi} k_{i,t}^{e,p}\, di = \int_0^{1-\varphi} \left( k_{i,t}^{e,p}\, d\underline{\mathcal{N}}_{i,t} + k_{i,t}^{e,p}\, d\overline{\mathcal{N}}_{i,t} \right) di$ and recognising that the second integral in the above expression is zero as the idiosyncratic shock cancels out in the aggregate by definition, yields

$$(1-\kappa) \int_0^{1-\varphi} \left[ \widetilde{q}_t k_{i,t}^{e,p} + \widetilde{p}_t K_t \left( m_{i,t}^{e,p} + l_{i,t}^{e,p} \right) \right] di.$$

Utilising the three expressions in (4.2.16) and aggregating individual wealth levels gives

$$(1-\kappa) \left[ x_{1,t}^{e,p} \frac{\widetilde{q}_t}{q_t} + \left(1 - x_{1,t}^{e,p}\right) \frac{\widetilde{p}_t}{p_t} \right] n_t^e.$$

Finally, equation (4.5.1c) follows from the additional assumption that the average wealth level in the group of prudent entrepreneurs that are exposed to the sector-wide disturbance is identical to that in the complementary group of agents who are not subject to this type of shock.[891]

In order to find the post-shock level of managers' wealth, we have to add the post-shock wealth of agents who have already been managers prior to the jump to the post-jump wealth of "new"

[891] This assumption is similar to an implicit assumption by Bernanke et al. (cf. 1999, p. 1347). They assume that a single entrepreneur does not survive until the next period with probability $1-\gamma$. If the entrepreneur exits the market, he consumes his entire wealth . Later on, they subtract the product of $1-\gamma$ and current *aggregate* entrepreneurial equity in the equation of motion for aggregate entrepreneurial net worth (cf. Bernanke et al., 1999, p. 1358).

managers, i.e. former entrepreneurs that were hit by the sector-specific shock. Applying similar steps as in the derivation of $\widetilde{n}_t^e$ implies

$$\widetilde{n}_t^m = (1-\kappa)\left(\left[x_t^m \frac{\widetilde{q}_t}{q_t} + (1-x_t^m)\frac{\widetilde{p}_t}{p_t} + \widetilde{\Pi}_t^b - \Pi_t^b\right] n_t^m + \phi^s \left[x_{1,t}^{e,p}\frac{\widetilde{q}_t}{q_t} + \left(1-x_{1,t}^{e,p}\right)\frac{\widetilde{p}_t}{p_t}\right] n_t^e\right). \quad \text{(B.1.51)}$$

□

### B.1.9 Proof of Proposition 3

The post-jump level of state variable $\eta_t$ as expressed through equation (4.5.7c) can simply be derived by multiplying both sides of (4.5.1c) by factor $(q_t + p_t) / (\widetilde{q}_t + \widetilde{p}_t)$, using the fact that $\widetilde{K}_t = (1-\kappa) K_t$, and rearranging.

Proving that (4.5.7b) follows from (B.1.49) is a bit more involved. First, we can determine the drift of the state variable by applying CVF (3.1.20) to function $\eta_t = n_t / [(q_t + p_t) K_t]$, taking into account (4.5.1a), (3.1.19), (3.2.15), and (4.1.1):

$$\eta_t \mu_t^\eta = \frac{1}{(q_t + p_t) K_t}\mu_t^{n^e} n_t^e - \frac{n_t^e}{(q_t + p_t)^2 K_t}\mu_t^q q_t - \frac{n_t^e}{(q_t + p_t)^2 K_t}\mu_t^p p_t - \frac{n_t^e}{(q_t + p_t) K_t}\mu_t^K. \quad \text{(B.1.52)}$$

Utilising the second expression in (4.5.4b) and both expressions in (4.5.5), this can be reformulated to

$$\mu_t^\eta = \mu_t^{n^e} - \theta_t \mu_t^q - (1-\theta_t)\mu_t^p - \mu_t^K. \quad \text{(B.1.53)}$$

Substituting (4.1.1) as well as the drift rate of $n_t^e$ (4.5.1b) into (B.1.53) and subsequently using (4.2.18), (4.2.7), and $\mu_t^{r^{L,e}}$ from (4.2.24) yields

$$\mu_t^\eta = x_{1,t}^{e,p}\left(\frac{a^e - \iota_t}{q_t} + \mu_t^q\right) + \left(1 - x_{1,t}^{e,p}\right)\mu_t^p + \left(1 - x_{1,t}^{e,p} - x_{2,t}^{e,p}\right)\Gamma_t - \frac{\rho}{1+\xi} - \theta_t \mu_t^q - (1-\theta_t)\mu_t^p, \quad \text{(B.1.54)}$$

where we have made use of the fact that investment rates are identical across sectors. Finally, equation (4.5.7b) follows from rearranging (B.1.54). □

## B.2 Discrete-time Version of Managers' Problem

This part of the Appendix shows how managers' decision problem can be formulated in discrete time. The purpose of this endeavour is to demonstrate a major advantage of formulating our model in continuous time. To that end, let us consider the *Bellman equation,* which is the discrete-time equivalent of the HJB equation. First, we state the flow budget constraint[892]

$$n_{t+\Delta}^m = \Delta\left(\mathcal{Z}_{t+1}^m + r_t^{P,m} n_t^m - c_t^m\right) + n_t^m, \quad \text{(B.2.1)}$$

[892] In this section we leave out subscript $i$, which identifies an individual, for convenience.

where $\Delta$ is the length of the time period, $r_t^{P,m}$ is the deterministic return on managers' portfolio between points in time $t$ and $t+\Delta$ defined by

$$r_t^{P.m} \equiv x_t^m r_t^{K,m} + \left(1 - x_t^m\right) r_t^M,$$

in which $r_t^{K,m}$ and $r_t^M$ are the deterministic returns on managers' capital and money per unit of time, respectively, and $\mathcal{Z}_{t+1}^m$ is a random variable with distribution

$$\mathcal{Z}_{t+1}^m = \begin{cases} 0 & \text{with prob. } e^{-\phi\lambda\Delta} \\ \underline{\widetilde{n}}_{t+1}^m - n_t^m & \text{with prob. } \phi\left(1 - e^{-\lambda\Delta}\right) \\ \widetilde{\widetilde{n}}_{t+1}^m - n_t^m & \text{with prob. } \left(1-\phi\right)\left(1 - e^{-\lambda\Delta}\right). \end{cases} \tag{B.2.2}$$

In (B.2.2), differences $\underline{\widetilde{n}}_{t+1}^m - n_t^m$ and $\widetilde{\widetilde{n}}_{t+1}^m - n_t^m$ are the changes in net worth if the individual is hit by an adverse and a beneficial idiosyncratic jump, respectively. Now, we are in a position to state the Bellman equation.

**Proposition 13.** *The discrete-time equivalent to HJB equation (4.3.3) is given by the following recursive Bellman equation:*

$$\begin{aligned} v^m\left(n_t^m, \mathcal{Z}_t\right) = \max_{c_t^m, x_t^m} \Big\{ & \log c_t^m \Delta + \xi \log\left(\left(1 - x_t^m\right) n_t^m\right) \Delta \\ & + e^{-\rho\Delta} \mathbb{E}_t\left[v^m\left(n_{t+\Delta}^m, \mathcal{Z}_{t+\Delta}^m\right)\right] \Big\}, \end{aligned} \tag{B.2.3}$$

*in which $v^m(\cdot)$ is managers' value function in the discrete-time case.*

*Proof.* The starting point of the proof is equation (B.2.3) with an expanded expectation term, assuming that the agent is in the "good" state absent a jump in the current period:[893]

$$\begin{aligned} v^m\left(n_t^m, 0\right) = \max_{c_t^m, x_t^m} \Big\{ & \log c_t^m \Delta + \xi \log\left(\left(1 - x_t^m\right) n_t^m\right) \Delta + e^{-\rho\Delta}\Big(e^{-\lambda\Delta} v^m\left(n_{t+\Delta}^m, 0\right) \\ & + \left(1 - e^{-\lambda\Delta}\right)\left[\phi v^m\left(n_{t+\Delta}^m, \widetilde{\widetilde{n}}_{t+\Delta}^m - n_t^m\right) + \left(1-\phi\right) v^m\left(n_{t+\Delta}^m, \underline{\widetilde{n}}_t^m - n_t^m\right)\right]\Big)\Big\}. \end{aligned} \tag{B.2.4}$$

Since we will take the continuous time limit of the Bellman equation ultimately, we can already make use of the fact that

$$e^{-\rho\Delta} \approx 1 - \rho\Delta \quad \text{and} \quad e^{-\lambda\Delta} \approx 1 - \lambda\Delta$$

[893] This proof closely follows Achdou et al. (2015, pp. 40f.).

for small $\Delta$.[894] Inserting these approximations into the Bellman equation yields

$$\begin{aligned} v^m\left(n_t^m, 0\right) = \max_{c_t^m, x_t^m} \Big\{ & \log c_t^m \Delta + \xi \log\left(\left(1 - x_t^m\right) n_t^m\right) \Delta \\ & + \left(1 - \rho\Delta\right) \Big( \left(1 - \lambda\right) \Delta v^m\left(n_{t+\Delta}^m, 0\right) \\ & + \lambda\Delta \Big[ \phi v^m\left(n_{t+\Delta}^m, \widetilde{\overline{n}}_{t+\Delta}^m - n_t^m\right) + \left(1 - \phi\right) v^m\left(n_{t+\Delta}^m, \underline{\widetilde{n}}_t^m - n_t^m\right) \Big] \Big) \Big\}. \end{aligned} \tag{B.2.5}$$

Subsequent subtracting of $\left(1 - \rho^m \Delta\right) v^m\left(n_t^m, 0\right)$ on both sides implies

$$\begin{aligned} \rho\Delta v^m\left(n_t^m, 0\right) = \max_{c_t^m, x_t^m} \Big\{ & \log c_t^m \Delta + \xi \log\left(\left(1 - x_t^m\right) n_t^m\right) \Delta \\ & + \left(1 - \lambda\Delta\right) \Big( v^m\left(n_{t+\Delta}^m, 0\right) - v^m(n_t^m, 0) \\ & + \lambda\Delta \Big[ \phi v^m\left(n_{t+\Delta}^m, \widetilde{\overline{n}}_{t+\Delta}^m - n_t^m\right) \\ & + \left(1 - \phi\right) v^m\left(n_{t+\Delta}^m, \underline{\widetilde{n}}_t^m - n_{t+\Delta}^m\right) - v^m\left(n_{t+\Delta}^m, 0\right) \Big] \Big) \Big\}. \end{aligned} \tag{B.2.6}$$

Dividing by $\Delta$, letting $\Delta \to 0$, and taking into account that[895]

$$\begin{aligned} & \lim_{\Delta \to 0} \frac{\left(1 - \lambda\Delta\right) v^m\left(n_{t+\Delta}^m, 0\right) - v^m(n_t^m, 0)}{\Delta} \\ & = \lim_{\Delta \to 0} \frac{\left(1 - \lambda\Delta\right) v^m\left(\Delta\left(r_t^{P,m} n_t^m - c_t^m\right) + n_t^m, 0\right) - v^m(n_t^m, 0)}{\Delta} \\ & = \frac{\partial}{\partial n_t^m} v^m(n_t^m, 0) \left(r_t^{P,m} n_t^m - c_t^m\right), \end{aligned}$$

in which the last equality follows from the chain rule[896], leads to (4.3.3). □

Now, substituting (B.2.1) into (B.2.3) , expanding the expectation using (B.2.2), and considering time periods of length $\Delta = 1$, program (B.2.3) can be written as

$$\begin{aligned} v^m\left(n_t^m, \mathcal{Z}_t\right) = \max_{c_t^m, x_t^m} \Big\{ & \log c_t^m + \xi \log\left(\left(1 - x_t^m\right) n_t^m\right) \\ & + e^{-\rho} \Big( e^{-\lambda} v^m\left(r_t^{P,m} n_t^m - c_t^m + n_t^m\right) \\ & + \left(1 - e^{-\lambda}\right) \Big[ \phi v^m\left(r_t^{P,m} n_t^m - c_t^m + \widetilde{\overline{n}}_{t+1}^m\right) + \left(1 - \phi\right) v^m\left(r_t^{P,m} n_t^m - c_t^m + \underline{\widetilde{n}}_{t+1}^m\right) \Big] \Big) \Big\}. \end{aligned} \tag{B.2.7}$$

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[894] Cf. Achdou et al. (2015, p. 40).

[895] Taking the limit implies that stochastic terms involving $\lambda$ do not depend on current consumption.

[896] The chain rule implies that

$$\lim_{\Delta \to 0} \frac{f\left(x + \Delta y\right) - f\left(x\right)}{\Delta} = f'\left(x\right) y.$$

Maximising with respect to consumption and rearranging gives

$$\frac{1}{c_t^m} = e^{-\rho}\Bigg(e^{-\lambda}\frac{\partial}{\partial n_{t+1}^m}v^m\left(r_t^{P,m}n_t^m - c_t^m + n_t^m\right) \\ + \left(1-e^{-\lambda}\right)\left[\phi\frac{\partial}{\partial n_{t+1}^m}v^m\left(r_t^{P,m}n_t^m - c_t^m + \widetilde{\overline{n}}_{t+1}^m\right) + (1-\phi)\frac{\partial}{\partial n_{t+1}^m}v^m\left(r_t^{P,m}n_t^m - c_t^m + \widetilde{\underline{n}}_{t+1}^m\right)\right]\Bigg). \tag{B.2.8}$$

In contrast to the optimality condition in the continuous time case given by (4.3.5), the above FOC determines optimal consumption as an implicit function of the derivatives of the value function. In addition, the derivatives need to be evaluated at different values of the function arguments.[897] Hence, it is infeasible to solve the FOC analytically by the method of undetermined coefficients. Rather, it is necessary to obtain optimal choices via value function iteration[898], which is computationally expensive. Conversely, in the continuous-time case, the stochastic terms, i.e. terms involving $\lambda$, do not depend on current consumption. This results from the fact that there is no distinction between "today" and "tomorrow" in distinction from the discrete-time case.[899] In effect, the FOC with respect to consumption can be solved by the method of undetermined coefficients.

The time intensity of the solution of the FOC for consumption in discrete time is further exacerbated if a shock with a continuous distribution is utilised. For a general continuously distributed Markov process the FOC becomes

$$\frac{1}{c_t^m} = (1+\rho)^{-1}\int\frac{\partial}{\partial n_t^m}v^m\left(n_{t+1}, \mathcal{Z}_{t+1}\right)dF\left(\mathcal{Z}_{t+1} \mid \mathcal{Z}_t\right), \tag{B.2.9}$$

in which $F(\cdot)$ is the conditional density function of the Markov process.[900] In the continuous-time case, it is not necessary to solve any such integral arising from an expectation, as can again be observed from FOC (4.3.5).

Aside from that, Achdou et al. (2015) point to a third advantage of the continuous-time formulation that is related to the treatment of occasionally binding constraints such as borrowing constraints or zero lower bounds of monetary policy. Binding constraints result in the FOCs not holding with equality, which again poses difficulties for computation. In continuous-time this issue is avoided as these constraints never bind at any interior point of the state space.[901] The reason is that the constraint leads to a so-called *state constraint boundary condition,* which can be used to solve the HJB equation numerically.[902]

[897] Cf. also Achdou et al. (2015, p. 28).
[898] Cf. e.g. Ljungqvist and Sargent (2004, Section 3.1.1.) for a short discussion of value function iteration.
[899] Cf. Achdou et al. (2015, p. 28).
[900] Cf. Achdou et al. (2015, p. 28).
[901] Cf. Achdou et al. (2015, p. 28).
[902] See Section 2.2 in Achdou et al. (2015) for a more detailed explanation.

## B.3 Imprudent Entrepreneurs' Choices Under Risk Neutrality, Extreme Myopia and an Exogenous Borrowing Limit

This section studies imprudent entrepreneurs' behaviour under the following assumptions. First, they are risk neutral and extremely myopic in the sense that they only care about *current* consumption. Second, real balances do not enter their preferences. Third, they are not subject to a no-Ponzi-game condition. Fourth, they face a borrowing limit: their loan demand must not exceed that of the representative prudent entrepreneur.

How can a borrowing constraint of the mentioned type be motivated in the presented framework? As banks are prevented from imposing self-selection mechanisms on borrowers, one could conceive of a role for costly screening of loan applicants that reveals their types in banks' loan policies: since banks are aware that imprudent individuals will default in the event of an adverse shock and prudent debtors will not, they might have an incentive to employ the screening technology to identify and sort out the bad credit risks.[903] In turn, imprudent borrowers would anticipate banks' lending policies and demand a loan volume just below the screening threshold in order to avoid revelation of their type. In this context, prudent entrepreneurs' borrowing limit can be interpreted as an (ad hoc) shortcut to a more complex problem in which banks endogenously choose the screening intensity.

The benefit of introducing a borrowing limit for imprudent entrepreneurs is as follows. Since they are risk neutral and do not face a no-Ponzi-game condition, these individuals would increase their capital stakes and finance these purchases by taking out more and more credit as long as their expected return on capital exceeds the real loan rate in the absence of a borrowing limit. In the process, risk averse prudent entrepreneurs would be driven out of the market, which in turn would deprive the model of its most interesting dynamics.

A natural interpretation of imprudent individuals' debt capacity arises in a special case of the considered model in which entrepreneurs' capital is not subject to idiosyncratic risk and prudent entrepreneurs start their lives with identical wealth levels. In this case, the equity levels are identical across individuals from that group at any point time. Since their utility functions belong to the HARA class, their loan demands are linear in net worth and thus differ neither. It follows that any loan demand by imprudent agents that deviates from that of good risks identifies them as bad risks.[904]

[903] Revelation of loan applicants' types might e.g. be achieved by observing their prior payout (i.e. consumption) policies.

[904] This reasoning is similar to that in a classical paper from the partial equilibrium credit rationing literature by Jaffee and Russell (1976). They develop a model with two types of borrowers, honest and dishonest individuals, who face an exogenous cost of default. Honest borrowers always repay their debt, even if the cost of default is less than the loan repayment. Dishonest borrowers on the contrary choose to declare bankruptcy if that condition is satisfied. Jaffee and Russell (cf. 1976, p. 653) then argue that the loan demand of dishonest individuals must equal that of honest individuals as otherwise lenders would identify the former and, as a consequence, would not lend to them.

In the remainder of this section, it is shown how imprudent entrepreneurs choose consumption and portfolios under the aforementioned assumptions and in the presence of idiosyncratic risk. To model the fact that these agents only plan one period ahead, the decision problem is solved in the discrete-time case, in which periods are of length $\Delta$ and time is indexed by $t$, $t+\Delta$, $t+2\Delta, ...$, at first. In the second step the continuous-time limit is taken, which implies that the planning horizon of imprudent individuals becomes infinitesimally small.[905]

Let us first postulate the utility of an individual of type $e, i$ that only plans for dates $t$ and $t+\Delta$:

$$U_{i,t}^{e,i} = c_{i,t}^{e,i} + \left(1 - \rho^{e,i}\Delta\right) \mathbb{E}_t \left[c_{i,t+\Delta}^{e,i}\right], \tag{B.3.1}$$

where $\rho^{e,i}$ is imprudent entrepreneurs' time preference rate.[906] We assume this time preference rate to be lower than the difference of the drift rate of the real loan rate and the intensity parameter $\lambda$ of Poisson process $\mathcal{N}_t$ at any point in time. Since in equilibrium the drift rate of the return on capital is always higher than the drift rate of the real loan rate,[907] relation

$$\mu_t^{r^{K,e}} > \mu_t^{r^{L,e}} > \lambda + \rho^{e,i}, \ \forall t \tag{B.3.2}$$

holds. By assumption, each imprudent agent starts his life without any wealth. Hence, the budget constraint at time $t$ is given by

$$q_t k_{i,t}^{e,i} + c_{i,t}^{e,i} = -p_t K_t l_{i,t}^{e,i}, \tag{B.3.3}$$

in which restrictions $k_{i,t}^{e,i} \geq 0$ and $l_{i,t}^{e,i} \leq 0$ are imposed in order to prevent agents of type $e, i$ from short-selling capital and extending credit. It follows from (B.3.3) that in the first period of life the agent has to decide on the loan volume and on the question of how to allocate the external financing to capital purchases and current consumption. The second-period budget constraint is described by

$$c_{i,t+\Delta}^{e,i} = \Pi_{i,t+\Delta}^{e,i} \left[1 - \left(\mathcal{N}_{t+\Delta}^d - \mathcal{N}_t^d\right)\right] + \max\left[0, \widetilde{\Pi}_{i,t+\Delta}^{e,i}\right] \left(\mathcal{N}_{t+\Delta}^d - \mathcal{N}_t^d\right) \tag{B.3.4a}$$

in which $\mathcal{N}_{t+\Delta}^d - \mathcal{N}_t^d$ is the increment of the discretised version of Poisson process $\mathcal{N}_t$ with intensity

[905] This procedure is inspired by He and Krishnamurthy (cf. 2013, p. 739), who assume that households in their model only live for two periods in order to keep these agents' maximisation problem simple. In contrast to the model in this section, these authors assume that agents living for two periods have a preference for bequests and thus accumulate wealth over generations.

[906] For small $\Delta$, we have $e^{-\rho^{e,i}\Delta} \approx 1 - \rho^{e,i}\Delta$ (cf. Achdou et al., 2015, p. 40).

[907] Risk neutral imprudent entrepreneurs are prevented from equalising their expected return on capital and the expected real loan rate by the constraint that their debt must not be higher than that of the representative prudent counterpart.

$\lambda$ per unit of time,

$$\Pi^{e,i}_{i,t+\Delta} \equiv \left\{ (a^e - \iota_t)\, k^{e,i}_{i,t} + \Gamma_t p_t K_t l^{e,i}_{i,t} \right\} \Delta \\ + (1 + \{\Phi(\iota_t) - \delta\} \Delta) \left[ (1 + \mu^q_t \Delta)\, q_t k^{e,i}_{i,t} + (1 + \mu^p_t \Delta)\, p_t K_t l^{e,i}_{i,t} \right] \tag{B.3.4b}$$

is the real gross portfolio income, i.e. the gross capital income less gross external financing costs, if no jump occurs at time $t + \Delta$, and

$$\widetilde{\Pi}^{e,i}_{i,t+\Delta} \equiv \left\{ (a^e - \iota_t)\, k^{e,i}_{i,t} + \Gamma_t p_t K_t l^{e,i}_{i,t} \right\} \Delta + [\widetilde{q}_t (1 - \kappa) - q_t]\, k^{e,i}_{i,t} + [\widetilde{p}_t (1 - \kappa) - p_t]\, K_t l^{e,i}_{i,t} \tag{B.3.4c}$$

is the gross portfolio income in the contrary case. The presence of the maximum operator in (B.3.4a) is due to limited liability: if $\widetilde{\Pi}^{e,i}_{i,t+\Delta} < 0$, imprudent borrowers have the option to declare default, which allows them to avoid negative consumption. In equilibrium, condition $[\widetilde{q}_t (1 - \kappa) - q_t] / q_t < [\widetilde{p}_t (1 - \kappa) - p_t] / p_t$ will be satisfied, i.e. the real loan rate will be higher than the return on capital if a jump hits the economy.[908] If the additional condition $[\widetilde{p}_t (1 - \kappa) - p_t] > 0$, which also holds true in equilibrium, is imposed, it can be shown that the sum of the second and third term in (B.3.4c) cannot be strictly positive.[909] Since this sum is of order unity and the flow return is of order $dt = 0$ in the continuous-time limit, the RHS of (B.3.4c) can never be positive in that case. Thus, the term involving the maximum operator can be dropped from (B.3.4a).

Besides (B.3.3) and (B.3.4a), each individual has to take into account two inequality constraints.[910] First, consumption at time $t$ must be nonnegative, i.e. $c^{e,i}_{i,t} \geq 0$. Using (B.3.3), this can be reformulated to

$$q_t k^{e,i}_{i,t} \leq -p_t K_t l^{e,i}_{i,t}. \tag{B.3.5}$$

Second, the imprudent individual has to take into account the borrowing limit, which states that his

[908] In states without a jump the real return on capital is higher than the real loan rate. If the former were at least as high as the latter in jump states, a long position in capital financed by a short position in credit would generate riskless income.

[909] Using (B.3.3), the sum of the second and third term can be written as

$$\left[ \frac{\widetilde{q}_t (1 - \kappa) - q_t}{q_t} - \frac{\widetilde{p}_t (1 - \kappa) - p_t}{p_t} \right] q_t k^{e,i}_{i,t} - \frac{\widetilde{p}_t (1 - \kappa) - p_t}{p_t} c^{e,i}_t .$$

As $k^{e,i}_{i,t} \geq 0$ and $c^{e,i}_{i,t} \geq 0$, this expression is nonpositive under the mentioned conditions.

[910] For the sake of brevity, we do not include conditions $c^{e,i}_{i,t+\Delta} \geq 0$, $k^{e,i}_{i,t} \geq 0$ and $l^{e,i}_{i,t} \leq 0$ in the optimisation problem. Not considering the first of these is unproblematic as no further assumptions have to be imposed for showing that the associated shadow value is zero in any case. Intuitively, if the agent decides to borrow in order to finance capital holdings, the real income in the second period is strictly positive if, in addition, no jump arrives. The budget constraint then requires that this income be consumed entirely. If, conversely, $\mathcal{N}^d_{t+\Delta} - \mathcal{N}^d_t = 1$, the agent has the option to default, which spares him from suffering the disutility of consuming negatively. In addition, we will only look for equilibria in which prudent entrepreneurs hold capital. If these agents find it profitable to do so, their imprudent counterparts do as well since the latter do not demand a risk premium for holding capital. Finally, if $k^{e,i}_{i,t} > 0$ and $c^{e,i}_{i,t+\Delta} \geq 0$ constraint (B.3.3) implies $l^{e,i}_{i,t} < 0$.

real demand for debt must be at most as high as that of the representative prudent entrepreneur:

$$-p_t K_t l_{i,t}^{e,i} \leq -p_t K_t l_{i,t}^{e,p,r}. \tag{B.3.6}$$

It follows that the maximisation problem is

$$\max_{c_{i,t}^{e,i}, c_{i,t+\Delta}^{e,i}, k_{i,t}^{e,i}, l_{i,t}^{e,i}} U_{i,t}^{e,i} = c_{i,t}^{e,i} + \left(1 - \rho^{e,i}\Delta\right) \mathbb{E}_t \left[c_{i,t+\Delta}^{e,i}\right] \tag{B.3.7}$$

subject to (B.3.3)-(B.3.6).[911] The Lagrangian emanating from this problem is

$$\begin{aligned}\mathcal{L} = &- \left(q_t k_{i,t}^{e,i} + p_t K_t l_{i,t}^{e,i}\right) + \left(1 - \rho^{e,i}\Delta\right)(1 - \lambda\Delta)\left(\left\{(a^e - \iota_t)\, k_{i,t}^{e,i} + \Gamma_t p_t K_t l_{i,t}^{e,i}\right\}\Delta\right. \\ &+ \left(1 + \{\Phi(\iota_t) - \delta\}\Delta\right)\left[\left(1 + \mu_t^q\Delta\right) q_t k_{i,t}^{e,i} + \left(1 + \mu_t^p\Delta\right) p_t K_t l_{i,t}^{e,i}\right]\Big) \\ &- \nu_1 \left(q_t k_{i,t}^{e,i} + p_t K_t l_{i,t}^{e,i}\right) + \nu_2 \left(l_{i,t}^{e,i} - l_{i,t}^{e,p,r}\right) p_t K_t,\end{aligned} \tag{B.3.8}$$

where $\nu_1$ and $\nu_2$ are the Lagrange multipliers associated with inequality constraints (B.3.5) and (B.3.6), respectively. After rearranging, the *Karush-Kuhn-Tucker* (KKT) conditions are[912]

$$\frac{\partial \mathcal{L}}{\partial k_{i,t}^{e,i}} = -1 + \left(1 - \rho^{e,i}\Delta\right)(1 - \lambda\Delta)\left[\frac{a^e - \iota_t}{q_t}\Delta + \left(1 + \{\Phi(\iota_t) - \delta\}\Delta\right)\left(1 + \mu_t^q\Delta\right)\right] - \nu_1 = 0, \tag{B.3.9}$$

$$\frac{\partial \mathcal{L}}{\partial l_{i,t}^{e,i}} = -1 + \left(1 - \rho^{e,i}\Delta\right)(1 - \lambda\Delta)\left[\Gamma_t\Delta + \left(1 + \{\Phi(\iota_t) - \delta\}\Delta\right)\left(1 + \mu_t^p\Delta\right)\right] - \nu_1 + \nu_2 = 0, \tag{B.3.10}$$

$$\nu_1 \geq 0, \quad q_t k_{i,t}^{e,i} \leq -p_t K_t l_{i,t}^{e,i}, \quad -\nu_1 \left(q_t k_{i,t}^{e,i} + p_t K_t l_{i,t}^{e,i}\right) = 0, \tag{B.3.11}$$

$$\nu_2 \geq 0, \quad -p_t K_t l_{i,t}^{e,i} \leq -p_t K_t l_{i,t}^{e,p,r}, \quad \nu_2 \left(l_{i,t}^{e,i} - l_{i,t}^{e,p,r}\right) p_t K_t = 0. \tag{B.3.12}$$

Taking the continuous-time limit $\Delta \to 0$ of (B.3.9) and (B.3.10) yields

$$\frac{\partial \mathcal{L}}{\partial k_{i,t}^{e,i}} = \underbrace{\left\{\frac{a^e - \iota_t}{q_t} + \mu_t^q + \Phi(\iota_t) - \delta\right\}}_{=\mu_t^{r^{K,e}}} dt - \left\{\lambda + \rho^{e,i}\right\} dt - \nu_1 = 0 \tag{B.3.13}$$

[911] It should be noted that a no-Ponzi-game condition has not been imposed. If one introduced that condition in addition to a nonnegativity constraint on consumption, a natural borrowing limit which precludes the possibility of default would emerge (cf. Aiyagari, 1994, pp. 665f.).

[912] The KKT conditions are explained in many texts on mathematical economics such as Simon and Blume (cf. 1994, Section 18.6).

and

$$\frac{\partial \mathcal{L}}{\partial l_{i,t}^{e,i}} = \underbrace{\left\{\Gamma_t + \mu_t^p + \Phi\left(\iota_t\right) - \delta\right\}}_{=\mu_t^{r^{L,e}}} dt - \left\{\lambda + \rho^{e,i}\right\} dt - \nu_1 + \nu_2 = 0, \tag{B.3.14}$$

which are obtained by taking into account that all terms of order $(dt)^2$ or higher can be dropped since they approach zero at a faster rate than $dt$.

First, we examine the case in which both inequality constraints bind. Then, we have $\nu_1 > 0$, $\nu_2 > 0$, $l_{i,t}^{e,i} = l_{i,t}^{e,p,r}$, and $k_{i,t}^{e,i} = -p_t K_t l_{i,t}^{e,p,r}/q_t$. Solving (B.3.13) for $\nu_1$ leads to

$$\nu_1 = \left\{\mu^{r^{K,e}} - \lambda - \rho^{e,i}\right\} dt, \tag{B.3.15}$$

Thus, multiplier $\nu_1$ is strictly positive if and only if the drift rate of the return on capital is higher than the sum of intensity parameter $\lambda$ and discount rate $\rho^{e,i}$. Since we have assumed in (B.3.2) that this holds true, constraint $q_t k_{i,t}^{e,i} \leq -p_t K_t l_{i,t}^{e,i}$ is indeed binding: the amount of loans taken out is used exclusively for purchasing capital at the first period of life and consumption in this period is zero. Next, we solve for $\nu_2$ by combining (B.3.13) and (B.3.14). This yields

$$\nu_2 = \left\{\mu_t^{r^{K,e}} - \mu_t^{r^{L,e}}\right\} dt, \tag{B.3.16}$$

which implies that $\nu_2$ is strictly positive if and only if the drift rate of the return on capital is strictly greater than the drift rate of the real loan rate. This condition will hold in equilibrium at any point in time. Hence, constraint (B.3.6) binds as well: imprudent entrepreneurs always demand the maximum volume of loans $l_{i,t}^{e,p,r}$. Since both multipliers $\nu_1$ and $\nu_2$ are strictly greater than zero, we have identified a KKT point, i.e. a solution to problem (B.3.7) subject to (B.3.3)-(B.3.6).

To check whether this is the unique solution, we have to analyse the remaining three cases. If neither inequality constraint binds, $\nu_1$ and $\nu_2$ are equal to zero. Then, we get

$$\frac{\partial \mathcal{L}}{\partial k_{i,t}^{e,i}} = \mu_t^{r^{K,e}} - \left(\lambda + \rho^{e,i}\right) = 0 \tag{B.3.17}$$

from FOC (B.3.13) and

$$\frac{\partial \mathcal{L}}{\partial l_{i,t}^{e,i}} = \mu_t^{r^{L,e}} - \left(\lambda + \rho^{e,i}\right) = 0 \tag{B.3.18}$$

from (B.3.14). Since we have assumed the expected return on capital to be higher than $\lambda + \rho^{e,i}$, the demand for capital is infinitely high. Condition (B.3.18) on the other hand implies that the optimal choice for $l_{i,t}^{e,i}$ is zero, which is the maximal value of this variable. Yet, for positive values of the price of capital this choice contradicts condition $q_t k_{i,t}^{e,i} < -p_t K_t l_{i,t}^{e,i}$. Hence, setting $\nu_1 = \nu_2 = 0$ does not lead to a KKT point.

The situation where (B.3.6) is satisfied with equality and (B.3.5) is not is similar: choosing $\nu_1 = 0$ implies an infinite demand for capital. Parameter $\nu_2$ can be obtained from (B.3.14) and is given by

$$\nu_2 = \left\{ \lambda + \rho^{e,i} - \mu_t^{r^{L,e}} \right\} dt. \tag{B.3.19}$$

As $\lambda + \rho^{e.n}\, dt < \mu_t^{r^{L,e}}$, parameter $\nu_2$ is negative. This is a contradiction to KKT condition (B.3.12).

Let us finally determine whether a KKT point results if constraint (B.3.5) binds and (B.3.6) does not. In this case, $\nu_1$ again is given by (B.3.15). Inserting this result into (B.3.14) and setting $\nu_2 = 0$ leads to

$$\frac{\partial \mathcal{L}}{\partial l_{i,t}^{e,i}} = \mu_t^{r^{L,e}} - \mu_t^{r^{K,e}} = 0. \tag{B.3.20}$$

Since $\mu_t^{r^{L,e}} < \mu_t^{r^{K,e}}$, the demand for loans is infinite. This, however, is a contradiction to (B.3.12).

In summary, the unique KKT point is characterised by portfolio choices $q_t k_t^{e,i} = -p_t K_t l_{i,t}^{e,p,r}$ and $l_{i,t}^{e,i} = l_{i,t}^{e,p,r}$. In addition, consumption is given by

$$c_{i,t}^{e,i} = \begin{cases} \mu_t^{r^{K,e}} q_t k_t^{e,i} + \mu_t^{r^{L,e}} p_t K_t l_{i,t}^{e,i} & \text{if } d\mathcal{N}_t = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{B.3.21}$$

which, using the portfolio choices and (4.3.26), can be reformulated to (4.3.29). Thus, imprudent agents' optimal consumption and asset holding choices in this finite-horizon model coincide with the assumed choices in Subsection 4.3.3.

## B.4 Two Alternative Expressions for the Drift Rate of the State Variable

In order to gain additional insight into the deterministic evolution of state variable $\eta_t$, it is useful to rewrite equation (4.5.7b). Proposition 14 states an alternative expression for the drift rate of the state variable.

**Proposition 14.** *The drift rate of entrepreneurs' wealth share can alternatively be expressed as*

$$\mu_t^\eta = (1-\eta_t) \Bigg[ \underbrace{x_{1,t}^{e,p} \frac{a^e - a^m}{q_t}}_{\textit{return difference}} + \underbrace{\left(\mu_t^{r^{K,m}} - \mu_t^{r^M}\right)\left(x_{1,t}^{e,p} - x_t^m\right)}_{\textit{relative risk-taking}} + \underbrace{\left(1 - x_{1,t}^{e,p} - x_{2,t}^{e,p}\right)\Gamma_t}_{\textit{e's interest expenses}} - \underbrace{\mu_t^{\Pi^b}\Pi_t^b}_{\textit{b's profits}} \Bigg]. \tag{B.4.1}$$

*Proof.* Applying CVF to function $\eta_t = n_t^e / \left(n_t^e + n_t^m\right)$ leads to

$$\mu_t^\eta = (1-\eta_t)\left(\mu_t^{n^e} - \mu_t^{n^m}\right) \tag{B.4.2}$$

after some algebra. Inserting (4.5.1b) as well as (4.5.2b) into (B.4.2) and further rearranging yields (B.4.1). □

Expression (B.4.1) shows that the drift rate of the state variable can be expresses in terms of managers and prudent entrepreneurs' relative real profits. The first term in the square bracket is due to the difference in the returns on capital between the two sectors, which in turn arises from entrepreneurs' superior productivity level. The return differential is amplified by portfolio weight $x_{1,t}^{e,p}$, i.e. by the share of equity individuals of type $e, p$ allocate to capital. The second term is equal to the product of managers' risk premium and the difference of the portfolio weights on capital. That difference reflects the relative risk-taking of the two types of agents. In addition, relative earnings also depend on entrepreneurs' interest expenses and banks' profits, which are entirely paid out to managers (third and fourth term).

Yet another equation for $\mu_t^\eta$ is provided by the following proposition.

**Proposition 15.** *The drift rate of entrepreneurs' wealth share can alternatively be expressed as*

$$\mu_t^\eta = \theta_t \frac{a^e - \iota_t}{q_t} + (1-\theta_t)\,\Gamma_t + \lambda\,(1-\phi^s)\,\frac{\widetilde{\theta}_t}{1-\kappa}\left(\phi\underline{\kappa}^{e,p}\frac{\eta_t}{\underline{\widetilde{\eta}}_t} + (1-\phi)\,\overline{\kappa}^{e,p}\frac{\eta_t}{\overline{\widetilde{\eta}}_t}\right) - \rho, \tag{B.4.3a}$$

*in which $\underline{\widetilde{\eta}}_t$ is the hypothetical post-jump wealth share of entrepreneurs if each of these agents were to suffer the adverse individual shock. This variable is given by*

$$\underline{\widetilde{\eta}}_t \equiv (1-\phi^s)\left[x_{1,t}^e\left(1-\frac{\underline{\kappa}^{e,p}}{1-\kappa}\right)\frac{\widetilde{\theta}_t}{\theta_t} + (1-x_{1,t}^e)\frac{1-\widetilde{\theta}_t}{1-\theta_t}\right]\eta_t. \tag{B.4.3b}$$

*The corresponding share if all entrepreneurs were to enjoy the beneficial individual shock is*

$$\overline{\widetilde{\eta}}_t \equiv (1-\phi^s)\left[x_{1,t}^e\left(1-\frac{\overline{\kappa}^{e,p}}{1-\kappa}\right)\frac{\widetilde{\theta}_t}{\theta_t} + (1-x_{1,t}^e)\frac{1-\widetilde{\theta}_t}{1-\theta_t}\right]\eta_t. \tag{B.4.3c}$$

*Proof.* Inserting FOC (4.5.6c) into the drift rate of $\eta_t$ given by (4.5.7b) results in

$$\begin{aligned}\mu_t^\eta = {} & \theta_t \frac{a^e-\iota_t}{q_t} + (1-\theta_t)\left(1 - x_{2,t}^{e,p} - \theta_t\right)\Gamma_t - \frac{\rho}{1+\xi} \\ & - \lambda\phi\frac{\left(x_1^{e,p}-\theta_t\right)\left[\left(1-\kappa-\underline{\kappa}^{e,p}\right)\frac{\widetilde{\theta}_t}{\theta_t} - (1-\kappa)\frac{1-\widetilde{\theta}_t}{1-\theta_t}\right]}{x_{1,t}^{e,p}\left(1-\kappa-\underline{\kappa}^{e,p}\right)\frac{\widetilde{\theta}_t}{\theta_t} + \left(1-x_{1,t}^{e,p}\right)(1-\kappa)\frac{1-\widetilde{\theta}_t}{1-\theta_t}} \\ & - \lambda\,(1-\phi)\,\frac{\left(x_1^{e,p}-\theta_t\right)\left[\left(1-\kappa-\overline{\kappa}^{e,p}\right)\frac{\widetilde{\theta}_t}{\theta_t} - (1-\kappa)\frac{1-\widetilde{\theta}_t}{1-\theta_t}\right]}{x_{1,t}^{e}\left(1-\kappa-\overline{\kappa}^{e,p}\right)\frac{\widetilde{\theta}_t}{\theta_t} + \left(1-x_{1,t}^{e}\right)(1-\kappa)\frac{1-\widetilde{\theta}_t}{1-\theta_t}}.\end{aligned} \tag{B.4.4}$$

This can be rewritten to

$$
\begin{aligned}
\mu_t^\eta = {} & \theta_t \frac{a^e - \iota_t}{q_t} + (1-\theta_t)\left(1 - x_{2,t}^{e,p} - \theta_t\right)\Gamma_t - \frac{\rho}{1+\xi} \\
& - \lambda\phi\left(1 - \frac{(1-\kappa)\frac{1-\widetilde{\theta}_t}{1-\theta_t} + \theta_t\left[\left(1-\kappa-\underline{\kappa}^{e,p}\right)\frac{\widetilde{\theta}_t}{\theta_t} - (1-\kappa)\frac{1-\widetilde{\theta}_t}{1-\theta_t}\right]}{x_{1,t}^{e,p}\left(1-\kappa-\underline{\kappa}^{e,p}\right)\frac{\widetilde{\theta}_t}{\theta_t} + \left(1 - x_{1,t}^{e,p}\right)(1-\kappa)\frac{1-\widetilde{\theta}_t}{1-\theta_t}}\right) \\
& - \lambda(1-\phi)\left(1 - \frac{(1-\kappa)\frac{1-\widetilde{\theta}_t}{1-\theta_t} + \theta_t\left[\left(1-\kappa-\overline{\kappa}^{e,p}\right)\frac{\widetilde{\theta}_t}{\theta_t} - (1-\kappa)\frac{1-\widetilde{\theta}_t}{1-\theta_t}\right]}{x_{1,t}^{e,p}\left(1-\kappa-\overline{\kappa}^{e,p}\right)\frac{\widetilde{\theta}_t}{\theta_t} + \left(1 - x_{1,t}^{e,p}\right)(1-\kappa)\frac{1-\widetilde{\theta}_t}{1-\theta_t}}\right).
\end{aligned} \tag{B.4.5}
$$

Employing definitions (B.4.3b) as well as (B.4.3c) and rearranging leads to

$$
\begin{aligned}
\mu_t^\eta = {} & \theta_t \frac{a^e - \iota_t}{q_t} + (1-\theta_t)\left(1 - x_{2,t}^{e,p} - \theta_t\right)\Gamma_t - \frac{\rho}{1+\xi} \\
& + \lambda\left(1-\phi^s\right)\left[\phi \frac{\underline{\kappa}^{e,p}}{1-\kappa}\widetilde{\theta}_t \frac{\eta_t}{\underline{\widetilde{\eta}}_t} + (1-\phi)\frac{\overline{\kappa}^{e,p}}{1-\kappa}\widetilde{\theta}_t \frac{\eta_t}{\overline{\widetilde{\eta}}_t}\right].
\end{aligned} \tag{B.4.6}
$$

To get from the above equation to (B.4.3a) simply requires one to insert entrepreneurs' optimal portfolio weight on money (4.3.23) and further rearranging. □

Equation (B.4.3a) implies a positive relationship between risk exposure, which is captured by the sum in the large parenthesis on the RHS, and drift rate $\mu_t^\eta$. Risk exposure is a weighted average of the inverse growth factors $\eta_t/\underline{\widetilde{\eta}}_t$ and $\eta_t/\overline{\widetilde{\eta}}_t$. Post-jump shares $\underline{\widetilde{\eta}}_t$ and $\overline{\widetilde{\eta}}_t$, in turn, are tied to the portfolio weight on the risky asset (capital) via equations (B.4.3b) and (B.4.3c). These equations also enable us to put restrictions on the behaviour of the value share of capital after a jump. To this end, we note that the terms in the square bracket of (B.4.3b) are common across all individuals. Hence, $\underline{\widetilde{\eta}}_t/\eta_t$ is equal to $\underline{\widetilde{\eta}}_{i,t}/\eta_{i,t}$, which is the ratio of the post-jump to the pre-jump wealth share of an individual that is subject to the adverse idiosyncratic shock. Since holding capital is risky, we expect the marginal unit of wealth invested into capital to decrease the post-jump wealth share in the case of a "bad" idiosyncratic shock. Hence, the partial derivative $\partial\left(\underline{\widetilde{\eta}}_{i,t}/\eta_{i,t}\right)/\partial x_{1,t}^e$ should be negative. This condition is satisfied if

$$
1 > \left(1 - \frac{\underline{\kappa}^{e,p}}{1-\kappa}\right)\frac{\frac{\widetilde{\theta}_t}{\theta_t}}{\frac{1-\widetilde{\theta}_t}{1-\theta_t}}. \tag{B.4.7}
$$

While the first factor on the RHS of this condition can be restricted to be smaller than unity through parameter choices, the second factor consists of endogenous variables and therefore can in principle be greater than unity. That is, an equilibrium in which holding capital is risky allows for the possibility that the value share of capital rises due to a shock to capital. Yet, as $\overline{\kappa}^{e,p} < \underline{\kappa}^{e,p}$ holds, condition (B.4.7) does not guarantee that more wealth allocated to capital reduces the post-jump wealth share in the "good" state, where the idiosyncratic shock is equal to$\overline{\kappa}^{e,p}$. The model results will show, however, that $\partial\left(\widetilde{\overline{\eta}}_{i,t}/\eta_{i,t}\right)/\partial x^e_{1,t} < 0$ on the entire state space. It follows that a higher portfolio weight on capital is associated with an increased risk exposure in both states and thus with a higher drift rate $\mu^\eta_t$. Hence, entrepreneurs are compensated for more risk-taking by higher deterministic growth in their wealth shares - the risk-return trade-off also materialises when the evolution of their wealth share is considered.

# Appendix C

# Appendix to Chapter 5

## C.1 Proofs

### C.1.1 Proof of Proposition 5

The equity of a representative manager, who owns the entire aggregate wealth of managers, follows

$$dn_t^m = \left\{\mu_t^{r^{P,m}} n_t^m - c_t^m\right\} dt + \left\{\underline{\widetilde{n}}_t^m - n_t^m\right\} d\underline{\mathcal{N}}_t + \left\{\overline{\widetilde{n}}_t^m - n_t^m\right\} d\overline{\mathcal{N}}_t, \tag{C.1.1a}$$

where

$$\underline{\widetilde{n}}_t^m \equiv \left[x_t^m \left(1-\kappa-\underline{\kappa}^m\right)\frac{\widetilde{q}_t}{q_t} + \left(1-x_t^m\right)\left(1-\kappa\right)\frac{\widetilde{p}_t}{p_t} + \widetilde{\Pi}_t^b - \Pi_t^b\right] n_t^m \quad \text{and} \tag{C.1.1b}$$

$$\overline{\widetilde{n}}_t^m \equiv \left[x_t^m \left(1-\kappa-\overline{\kappa}^m\right)\frac{\widetilde{q}_t}{q_t} + \left(1-x_t^m\right)\left(1-\kappa\right)\frac{\widetilde{p}_t}{p_t} + \widetilde{\Pi}_t^b - \Pi_t^b\right] n_t^m. \tag{C.1.1c}$$

The HJB equation for this individual is

$$\begin{aligned}\rho V^m(n_t^m, \eta_t) = \max_{c_t^m, x_t^m} \Bigg\{ & \log c_t^m + \xi \log\left(\left(1-x_t^m\right) n_t^m\right) \\ & + \frac{\partial V^m(n_t^m,\eta_t)}{\partial n_t^m}\left(\mu_t^{r^{P,m}} n_t^m - c_t^m\right) + \frac{\partial V^m(n_t^m,\eta_t)}{\partial \eta_t}\mu_t^\eta \eta_t \\ & + \lambda\phi\left[V^m\left(\underline{\widetilde{n}}_t^m, \widetilde{\eta}_t\right) - V^m\left(n_t^m,\eta_t\right)\right] + \lambda\left(1-\phi\right)\left[V^m\left(\overline{\widetilde{n}}_t^m, \widetilde{\eta}_t\right) - V^m\left(n_t^m,\eta_t\right)\right]\Bigg\},\end{aligned} \tag{C.1.2}$$

which was derived by employing (B.1.21) and applying CVF (3.1.20) to (5.3.2), while taking into account (C.1.1a) as well as (4.5.7a). Inserting (5.3.2), its partial derivatives, and optimal consumption choice (4.3.10) gives

$$\rho\alpha^m(\eta_t) = \log\frac{\rho}{1+\xi} + \xi\log(1-x_t^m) + \frac{1+\xi}{\rho}\mu_t^{r^{P,m}} - 1 + \frac{d\alpha^m(\eta_t)}{d\eta_t}\mu_t^\eta\eta_t \\ + \lambda\left(\frac{1+\xi}{\rho}\left[\phi\log\underline{\widetilde{n}}_t^m + (1-\phi)\log\widetilde{\overline{n}}_t^m - \log n_t^m\right] + \alpha^m(\widetilde{\eta}_t^e) - \alpha^m(\eta_t)\right) \tag{C.1.3}$$

after rearranging. Further manipulating this equation, using definitions (5.3.3b)-(5.3.3d), implies (5.3.3a). □

### C.1.2 Proof of Proposition 6

The approach in this appendix is analogous to that in the preceding section. One first has to derive the equation of motion for the net worth of a single prudent entrepreneur who owns the aggregate entrepreneurial wealth:

$$dn_t^e = \left\{\mu_t^{r^{P,e,p}} n_t^e - c_t^{e,p}\right\} dt + \left\{\underline{\widetilde{n}}_t^e - n_t^{e,}\right\} d\underline{\mathcal{N}}_t + \left\{\widetilde{\overline{n}}_t^e - n_t^e\right\} d\overline{\mathcal{N}}_t, \tag{C.1.4a}$$

in which

$$\underline{\widetilde{n}}_t^e \equiv \left[x_{1,t}^{e,p}(1-\kappa-\underline{\kappa}^{e,p})\frac{\widetilde{q}_t}{q_t} + \left(1-x_{1,t}^{e,p}\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t}\right]n_t^e \quad \text{and} \tag{C.1.4b}$$

$$\widetilde{\overline{n}}_t^m \equiv \left[x_{1,t}^{e,p}(1-\kappa-\overline{\kappa}^{e,p})\frac{\widetilde{q}_t}{q_t} + \left(1-x_{1,t}^{e,p}\right)(1-\kappa)\frac{\widetilde{p}_t}{p_t}\right]n_t^e. \tag{C.1.4c}$$

In optimum,

$$\rho V^{e,p}(n_t^e,\eta_t) = \max_{c_t^{e,p},x_{1,t}^{e,p},x_{2,t}^{e,p}}\left\{\log c_t^{e,p} + \xi\log\left(x_{2,t}^{e,p}n_t^e\right)\right. \\ + \frac{\partial V^{e,p}(n_t^e,\eta_t)}{\partial n_t^e}\left(\mu_t^{r^{P,m}}n_t^e - c_t^{e,p}\right) + \frac{\partial V^{e,p}(n_t^e,\eta_t)}{\partial\eta_t}\mu_t^\eta\eta_t \\ + \lambda(1-\phi^s)\left[\phi V^{e,p}(\underline{\widetilde{n}}_t^e,\widetilde{\eta}_t) + (1-\phi)V^{e,p}\left(\widetilde{\overline{n}}_t^e,\widetilde{\eta}_t\right) - V^{e,p}(n_t^e,\eta_t)\right] \\ \left. + \lambda\phi^s\left[\phi V^m(\underline{\widetilde{n}}_t^e,\widetilde{\eta}_t) + (1-\phi)V^m\left(\widetilde{\overline{n}}_t^e,\widetilde{\eta}_t\right) - V^{e,p}(n_t^e,\eta_t)\right]\right\} \tag{C.1.5}$$

has to hold. Substituting (5.3.4), its partial derivatives, and optimal consumption choice (4.3.19) leads to

$$\begin{aligned}\rho\alpha^{e,p}\left(\eta_t\right) = \log\frac{\rho}{1+\xi} + \xi\log + \frac{1+\xi}{\rho}\mu_t^{r^{P,e,p}} - 1 + \frac{d\alpha^{e,p}(\eta_t)}{d\eta_t}\mu_t^{\eta}\eta_t \\ + \lambda\left(\frac{1+\xi}{\rho}\left[\phi\log\underline{\widetilde{n}}_t^e + (1-\phi)\log\overline{\widetilde{n}}_t^e - \log n_t^e\right]\right) \\ + \lambda\left[\left(1-\phi^s\right)\alpha^{e,p}\left(\widetilde{\eta}_t\right) + \phi^s\alpha^m\left(\widetilde{\eta}_t\right) - \alpha^{e,p}\left(\eta_t\right)\right].\end{aligned} \tag{C.1.6}$$

From that, one arrives at (5.3.5a) by employing definitions (5.3.5b)-(5.3.5d) and some further algebra. □

## C.2 The Idiosyncratic Volatility Component of Prudent Entrepreneurs' Portfolio Return

This appendix consists of two parts. First, the idiosyncratic and sector-wide nominal portfolio return of prudent entrepreneurs are derived. This measure is the correct counterpart to empirical stock return data in our context. Afterwards, the approach by Campbell et al. (2001) to isolate the idiosyncratic component of stock returns volatility is applied to arrive at the corresponding component of prudent entrepreneurs' nominal idiosyncratic portfolio return $d\hat{r}_{i,t}^{P,e,p}$.[913]

The real individual portfolio return $dr_{i,t}^{P,e,p}$ is given by

$$dr_{i,t}^{P,e,p} \equiv x_{1,t}^{e,p} dr_{i,t}^{K,e,p} + x_{2,t}^{e,p} dr_t^M + \left(1 - x_{1,t}^{e,p} - x_{2,t}^{e,p}\right) dr_t^{L,e}, \tag{C.2.1}$$

where $dr_{i,t}^{K,e,p}$ and $dr_t^M$ are determined by (4.2.22) and (4.2.11), respectively, and $dr_t^{L,e} = dr_t^M + \Gamma_t\, dt$. Since the nominal return to money is always zero, the nominal portfolio return of individuals of type $e, p$ is

$$d\hat{r}_{i,t}^{P,e,p} \equiv x_{1,t}^{e,p} d\hat{r}_{i,t}^{K,e,p} + \left(1 - x_{1,t}^{e,p} - x_{2,t}^{e,p}\right)\Gamma_t\, dt, \tag{C.2.2}$$

in which $d\hat{r}_{i,t}^{K,e,p} = dr_{i,t}^{K,e,p} + \pi_t$ is the nominal idiosyncratic return to capital.

Idiosyncratic risk cancels out in the sector wide-return $d\hat{r}_t^{K,e,p}$ by definition. Hence,

$$d\hat{r}_t^{K,e,p} = \left\{\frac{a^e - \iota_t}{q_t} + \mu_t^q + \Phi(\iota_t) - \delta\right\} dt + \frac{\widetilde{q}_t\left(1-\kappa\right) - q_t}{q_t}\, d\mathcal{N}_t + \pi_t. \tag{C.2.3}$$

It follows that the sector-wide nominal portfolio return is

$$d\hat{r}_t^{P,e,p} \equiv x_{1,t}^{e,p} d\hat{r}_t^{K,e,p} + \left(1 - x_{1,t}^{e,p} - x_{2,t}^{e,p}\right)\Gamma_t\, dt. \tag{C.2.4}$$

[913] Cf. Campbell et al. (2001, pp. 4f.)

In the simulation of the model, one unit of time equals on year and time increment $dt$ is approximated by 1/360, i.e. about one day (cf. Appendix C.4.4). To match the amplitude of idiosyncratic risk correctly to the results by Campbell et al. (2001)[914], the daily returns in (C.2.2) and (C.2.3) first have to be compounded to monthly returns. Afterwards, these monthly returns are annualised, i.e. multiplied by factor 12. The resulting annualised monthly nominal returns are denoted by $\hat{r}_{i,\mathrm{mo}}^{P,e,p}$ and $\hat{r}_{\mathrm{mo}}^{P,e,p}$, respectively, in which the time subscript mo refers to the current month.

The next task is to identify the idiosyncratic volatility component of $\hat{r}_{i,\mathrm{mo}}^{P,e,p}$ via the procedure suggested by Campbell et al. (cf. 2001, pp. 4f.).[915] The Capital Asset Pricing Model (CAPM) motivates the following regression equation[916]

$$\hat{r}_{i,\mathrm{mo}}^{P,e,p} = b^{P,e,p}\hat{r}_{\mathrm{mo}}^{P,e,p} + u_{\mathrm{mo}}^{P,e,p}, \tag{C.2.5}$$

where $u_{\mathrm{mo}}^{P,e,p}$ is the error term and

$$b^{P,e,p} = \frac{\mathrm{Cov}\left[\hat{r}_{i,\mathrm{mo}}^{P,e,p}, \hat{r}_{\mathrm{mo}}^{P,e,p}\right]}{\mathrm{Var}\left[\hat{r}_{\mathrm{mo}}^{P,e,p}\right]}. \tag{C.2.6}$$

Coefficient $b^{P,e,p}$ can be shown to equal unity. This is for two reasons. First, the expected value of the idiosyncratic shock equals zero. Second, and related, the expected values of the individual and market portfolio returns are equal. Hence, the error term in (C.2.5) is equal to the difference of $\hat{r}_{i,\mathrm{mo}}^{P,e,p}$ and $\hat{r}_{\mathrm{mo}}^{P,e,p}$. In parallel to Campbell et al. (2001), the variance of error term $u_{\mathrm{mo}}^{P,e,p}$ is adopted as a measure of the idiosyncratic volatility component of stock returns.[917]

## C.3 Solution Procedure in the Autarky Case

We first derive an cubic equation in $q_A$ by combining the FOC with respect to $x_A^m$ with market clearing conditions. The FOC with respect to $x_A^m$ is

$$\begin{aligned}\frac{a^m - \iota_A}{q_A} = & \frac{\xi\rho}{(1+\xi)\left(1-x_A^m\right)} + \lambda\phi\frac{\underline{\kappa}}{1-\kappa-\underline{\kappa}x_A^m} \\ & + \lambda\left(1-\phi\right)\frac{\overline{\kappa}}{1-\kappa-\overline{\kappa}x_A^m}.\end{aligned} \tag{C.3.1}$$

The capital market clearing condition $x_A^m = q_A/\left(q_A + p_A\right)$ together with the corresponding condition in the final goods market $\rho\left(q_A + p_A\right)/\left(1+\xi\right) = a^m - \iota_A$ gives an expression for the optimal portfolio

[914] Cf. Campbell et al. (2001, Table III and p. 19).

[915] Cf. Campbell et al. (2001, pp. 4f.).

[916] The intercept in regression equations of this type is equal to zero under rational expectations (cf. Berk and DeMarzo, 2011, p. 148). Further, it should be recognised that $\hat{r}_{i,\mathrm{mo}}^{P,e,p}$ and $\hat{r}_{\mathrm{mo}}^{P,e,p}$ also denote the risk premia over the nominal return to money.

[917] Cf. Campbell et al. (2001, p. 5).

weight:

$$x_A^m = \frac{q_A \rho}{(a^m - \iota_A)(1+\xi)} \tag{C.3.2}$$

Inserting (C.3.2) into (C.3.1) after some manipulations yields

$$\begin{aligned} &\frac{1}{q_A} - \frac{\xi\rho}{(a^m - \iota_A)(1+\xi) - q_A\rho} \\ -&\frac{\lambda\phi\underline{\kappa}(1+\xi)}{(1-\kappa)(a^m-\iota_A)(1+\xi) - q_A\rho\underline{\kappa}} - \frac{\lambda(1-\phi)\overline{\kappa}(1+\xi)}{(1-\kappa)(a^m-\iota_A)(1+\xi)-q_A\rho\overline{\kappa}} = 0. \end{aligned} \tag{C.3.3}$$

This equation is numerically solved, while taking into account optimal investment rule (3.1.27). The number of feasible solutions can be reduced to one by imposing the restriction that price $q_A$ must be smaller than the price of capital in the first-best case.[918] Once $q_A$ is determined, equilibrium values of all other endogenous variables can be obtained from simple substitutions.

## C.4 Solution Procedure in the Baseline Model

### C.4.1 Determination of the RDE Parameters

In this Section it is shown how parameters $(\mu^q - \mu^p)_m$ and $\mu_m^\eta$, which are required to obtain a solution to RDE (4.5.9), can be calculated.[919] The first step at any grid point $m \geq \check{m}$[920] is to determine the post-jump values of entrepreneurs' wealth share and the value share of capital. As discussed in Section 3.1.2.3, after the occurrence of a shock, the system is thrown back to a lower value of the state variable $\eta_k < \eta_m$, where $k < m$ is a prior grid point. Thus, the first task is to identify $k$. The initial guess for this grid point is $k = 1$. Since the value share of capital is known at each grid point $1, ..., m$, choosing $k$ results in a tuple $(\eta_k, \theta_k)$.[921] Using this fact in the discretised version of the equation for entrepreneurs' post-jump wealth share given by (4.5.7c) allows for calculating entrepreneurs' portfolio weight corresponding to the choice of $k$:

$$x_{1,k\_}^{e,p} = \frac{\frac{1-\theta_k}{1-\theta_m} - \frac{1}{(1-\phi^s)}\frac{\eta_k}{\eta_m}}{\frac{1-\theta_k}{1-\theta_m} - \frac{\theta_k}{\theta_m}}, \tag{C.4.1}$$

[918] As mentioned in Section 5.2.2, the presence of idiosyncratic risk reduces the demand for capital, ceteris paribus. In addition, the productivity level in the autarky case is lower than that in the first-best model. Thus, $q_A < q_F$ must hold.

[919] The approach to determine the parameters of the RDE is based on Brunnermeier and Sannikov (2014d, Appendix A).

[920] As mentioned in Section 3.1.2.3, $\check{m}$ is the grid point at which the integration of the RDE commences.

[921] In the remainder, we will use $\theta_m$ and $\theta_k$ instead of $\theta(\eta_m)$ and $\theta(\eta_k)$, respectively, to simplify notation. This simplification is also applied to any other function of the state variable.

where $\theta_m$ is used as a shorthand notation for $\theta(\eta_m)$ in order to ease on notation. Variable $x^e_{1,k_-}$ can be interpreted as the portfolio weight on capital required to sink $\eta_m$ to $\eta_k$.[922,923] The next step is to calculate the price of capital which results from our choice of $k$. To this end, entrepreneurs' risk premium needs to be determined first. This can be achieved by solving the following equation:

$$\left(\mu^{r^{K,e}} - \mu^{r^{L,e}}\right)_{k_-} = -\mathcal{A}_{k_-} - \mathcal{B}_{k_-}, \tag{C.4.2a}$$

where

$$\mathcal{A}_{k_-} \equiv \lambda\phi \frac{\left(1-\kappa-\underline{\kappa}^{e,p}\right)\frac{\theta_k}{\theta_m} - (1-\kappa)\frac{1-\theta_k}{1-\theta_m}}{x^{e,p}_{1,k_-}\left(1-\kappa-\underline{\kappa}^{e,p}\right)\frac{\theta_k}{\theta_m} + \left(1-x^{e,p}_{1,k_-}\right)(1-\kappa)\frac{1-\theta_k}{1-\theta_m}} \quad \text{and} \tag{C.4.2b}$$

$$\mathcal{B}_{k_-} \equiv \lambda(1-\phi)\frac{\left(1-\kappa-\overline{\kappa}^{e,p}\right)\frac{\theta_k}{\theta_m} - (1-\kappa)\frac{1-\theta_k}{1-\theta m}}{x^{e,p}_{1,k_-}\left(1-\kappa-\overline{\kappa}^{e,p}\right)\frac{\theta_k}{\theta_m} + \left(1-x^{e,p}_{1,k_-}\right)(1-\kappa)\frac{1-\theta_k}{1-\theta_m}}, \tag{C.4.2c}$$

which is obtained from (4.5.6c). This can be utilised together with optimal investment policy $\iota_{k_-} = (q_{k_-} - 1)/\gamma$ in the discretised final goods market clearing condition

$$\begin{aligned}&\frac{\rho}{1+\xi} + \frac{\varphi}{1-\varphi}\left(x^{e,p}_{1,k_-} + x^{e,p}_{2,k_-} - 1\right)\left[\left(\mu^{r^{K,e}} - \mu^{r^L}\right)_{k_-} + \lambda(1-\kappa)\,\omega\frac{q_k}{q_{k_-}}\right]\eta_m \\ &+ \frac{\theta_m}{\gamma} - \frac{\theta_m}{q_{k_-}}\left[a^m + \psi_{k_-}(a^e - a^m) + \frac{1}{\gamma}\right] = 0,\end{aligned} \tag{C.4.3}$$

which follows from manipulating (4.5.6e). In (C.4.3), $q_k$ is the post-jump level of the capital price resulting from the guess for $k$. Entrepreneurs' share in the aggregate capital stock $\psi_{k_-}$ and their portfolio weight on money $x^{e,p}_{2,k_-}$ are yet to be determined. Thus, we first insert

$$\psi_{k_-} = \left[x^{e,p}_{1,k_-} + \varphi\left(x^{e,p}_{2,k_-} - 1\right)\right]\frac{\eta_m}{(1-\varphi)\,\theta_m} \tag{C.4.4}$$

[922] Cf. Brunnermeier and Sannikov (2014d, p. 35).

[923] Note that $x^{e,p}_{1,k_-}$ does not denote the post-jump value of entrepreneurs' portfolio weight on capital. Rather, it is the (pre-jump) portfolio weight at grid point $m$ that has to ensue if the economy jumps to point $k$. Choosing a specific value for this variable results in corresponding values for any other endogenous variable, except for entrepreneurs' wealth share and the value share of capital. For this reason, the former variables will also be denoted by subscript $k_-$ in the following.

into (C.4.3), which yields

$$
\begin{aligned}
& q_{k_-}\left(\frac{\rho}{1+\xi}+\frac{\varphi}{1-\varphi}\left(x_{1,k_-}^{e,p}-1\right)\left[\left(\mu^{r^{K,e}}-\mu^{r^L}\right)_{k_-}+\lambda\left(1-\kappa\right)\omega\frac{q_k}{q_{k_-}}\right]\eta_m+\frac{\theta_m}{\gamma}\right) \\
& +q_{k_-}\frac{\varphi}{1-\varphi}x_{2,k_-}^{e,p}\left(\left[\left(\mu^{r^{K,e}}-\mu^{r^L}\right)_{k_-}+\lambda\left(1-\kappa\right)\omega\frac{q_k}{q_{k_-}}\right]-\left(a^e-a^m\right)\right)\eta_m \\
& -\theta_m\left(a^m+\frac{1}{\gamma}\right)-\left(a^e-a^m\right)\frac{x_{1,k_-}^{e,p}-\varphi}{1-\varphi}\eta_m=0 \qquad \text{(C.4.5)}
\end{aligned}
$$

after rearranging. The next step is to provide an equation for $x_{2,k_-}^{e,p}$. From (4.3.23), we get

$$
x_{2,k_-}^{e,p}=\frac{\xi\rho}{\left(1+\xi\right)\Gamma_{k_-}}. \qquad \text{(C.4.6)}
$$

The discretised version of (4.5.6g) yields the EFP:

$$
\Gamma_{k_-}=\lambda\varphi\left(1-\kappa\right)\left[\frac{\left(1-\theta_k\right)\theta_m}{\left(1-\theta_m\right)\theta_k}-\left(1-\omega\right)\right]\frac{q_k}{q_{k_-}}, \qquad \text{(C.4.7)}
$$

Substituting (C.4.7) into (C.4.6) gives

$$
x_{2,k_-}^{e,p}=\mathcal{C}_{k_-}q_{k_-} \qquad \text{(C.4.8a)}
$$

with

$$
\mathcal{C}_{k_-}\equiv\frac{\xi\rho}{\left(1+\xi\right)\lambda\varphi\left(1-\kappa\right)\left[\frac{(1-\theta_k)\theta_m}{(1-\theta_m)\theta_k}-\left(1-\omega\right)\right]q_k}. \qquad \text{(C.4.8b)}
$$

Combining (C.4.8a) and (C.4.7) with (C.4.5) finally leads to a quadratic equation in $q_{k_-}$:

$$
\mathcal{D}_{k_-}q_{k_-}^2+\mathcal{E}_{k_-}q_{k_-}+\mathcal{F}_{k_-}=0, \qquad \text{(C.4.9a)}
$$

in which

$$
\begin{aligned}
\mathcal{D}_{k_-} &\equiv \frac{\varphi}{1-\varphi}\left(\mu^{r^{K,e}}-\mu^{r^L}\right)_{k_-}\mathcal{C}_{k_-}\eta_m, \qquad \text{(C.4.9b)} \\
\mathcal{E}_{k_-} &\equiv \frac{\varphi}{1-\varphi}\left(\left(\mu^{r^{K,e}}-\mu^{r^L}\right)_{k_-}\left(x_{1,k_-}^{e,p}-1\right)+\left[\lambda\left(1-\kappa\right)\omega q_k-\left(a^e-a^m\right)\right]\mathcal{C}_{k_-}\right)\eta_m \qquad \text{(C.4.9c)} \\
&\quad +\frac{\rho}{1+\xi}+\frac{\theta_m}{\gamma}, \\
\mathcal{F}_{k_-} &\equiv \left[\lambda\left(1-\kappa\right)\omega q_k\left(x_{1,k_-}^{e,p}-1\right)-\left(a^e-a^m\right)\left(\frac{x_{1,k_-}^{e,p}}{\varphi}-1\right)\right]\frac{\varphi}{1-\varphi}\eta_m-\theta_m\left(a^m+\frac{1}{\gamma}\right). \qquad \text{(C.4.9d)}
\end{aligned}
$$

The numerical solution of the model shows that an unique solution can be obtained for $q_{k\_}$ by restricting this variable to lie in the range $q_A \leq q_{k\_} \leq q_F$. This restriction follows from recognising that the autarky price is a lower bound for the price of capital and the first-best price is an upper bound. Given $q_{k\_}$, the EFP can be derived from (C.4.7). This result allows for calculating entrepreneurs' portfolio weight on money through equation (C.4.6) and together with the latter result for deriving entrepreneurs' share in the aggregate capital stock via equation (C.4.4). Then, one can arrive at managers' portfolio weight on capital by combining the discretised version of the capital market clearing condition (4.5.6f) with (C.4.4):

$$x^m_{k\_} = \frac{(1-\psi_{k\_})\,\theta_m}{1-\eta_m}. \tag{C.4.10}$$

After this is done, we have to check whether $x^{e,p}_{k\_}$ and $x^m_{k\_}$ satisfy their respective first-order conditions. To this end we discretise (4.5.6a) as well as (4.5.6c) and subsequently combine both to arrive at a condition which combination $\left(x^{e,p}_{1,k\_}, x^m_{k\_}\right)$ has to satisfy:

$$\mathcal{G}_{k\_} \equiv \frac{a^e - a^m}{q_{k\_}} - \Gamma_{k\_} + \frac{\xi\rho}{(1+\xi)\left(1-x^m_{k\_}\right)} + \mathcal{A}_{k\_} + \mathcal{B}_{k\_} - \mathcal{H}_{k\_} - \mathcal{I}_{k\_} \geq 0, \tag{C.4.11a}$$

where

$$\mathcal{H}_{k\_} \equiv \lambda\phi \frac{(1-\kappa-\underline{\kappa}^m)\,\frac{\theta_k}{\theta_m} - (1-\kappa)\,\frac{1-\theta_k}{1-\theta_m}}{x^m_{k\_}\,(1-\kappa-\underline{\kappa}^m)\,\frac{\theta_k}{\theta_m} + \left(1-x^m_{k\_}\right)(1-\kappa)\,\frac{1-\theta_k}{1-\theta_m} + B_{k\_}}, \tag{C.4.11b}$$

$$\mathcal{I}_{k\_} \equiv \lambda\,(1-\phi)\,\frac{(1-\kappa-\overline{\kappa}^m)\,\frac{\theta_k}{\theta_m} - (1-\kappa)\,\frac{1-\theta_k}{1-\theta_m}}{x^m_{k\_}\,(1-\kappa-\overline{\kappa}^m)\,\frac{\theta_k}{\theta_m} + \left(1-x^m_{k\_}\right)(1-\kappa)\,\frac{1-\theta_k}{1-\theta_m} + B_{k\_}}, \tag{C.4.11c}$$

and

$$B_{k\_} = (1-\kappa)\left(\frac{\theta_k}{\theta_m} - \frac{1-\theta_k}{1-\theta_m}\right)\frac{\varphi}{1-\varphi}\left(x^{e,p}_{1,k\_} + x^{e,p}_{2,k\_} - 1\right)\frac{\eta_m}{1-\eta_m}. \tag{C.4.11d}$$

Condition (C.4.11a) holds with strict equality if $0 < \psi_{k\_} < 1$ and with strict inequality if $\psi_{k\_} = 1$. Thus, as long as entrepreneurs do not own the entire aggregate capital stock, the task is to find a value of $k$ which approximately solves $\mathcal{G}_{k\_} = 0$.[924] An additional condition that has to be satisfied is that due to the logarithmic form of the value function any prudent entrepreneur must not become bankrupt after being hit by a shock. The specific condition that has to hold for this to be true is

$$\mathcal{J}_{k\_} \equiv x^{e,p}_{1,k\_}\left(1 - \frac{\underline{\kappa}^{e,p}}{1-\kappa}\right)\frac{\theta_k}{\theta_m} + (1-x^{e,p}_{1,k\_})\frac{1-\theta_k}{1-\theta_m} > 0, \tag{C.4.12}$$

which follows from the equation for the post-jump wealth share of an individual entrepreneur who is exposed to the adverse idiosyncratic shock in (B.4.3b). If this condition is satisfied, entrepreneurs

[924] As the model is solved numerically, it is infeasible to find a value of $k$ for which $\mathcal{G}_{k\_}$ exactly equals zero.

who are subject to the beneficial individual disturbance to capital will also stay solvent.

As mentioned, the initial guess for $k$ at any grid point $m$ is $k = 1$. Yet, at the first grid point we have $\eta_1 = 0$. If entrepreneurs' wealth share drops to zero when a jump arrives, condition (C.4.12) can never hold and $k = 1$ cannot be a solution to the above problem. Thus, the guess for $k$ has to be revised upwards. If the value share of capital is a monotonously increasing function of the state variable, increasing $k$ will eventually result in a slightly positive value of $\mathcal{J}_{k\_}$.[925] However, at this point, the marginal post-shock utility in the "bad state" due to an additional unit of $x^{e,p}_{1,k\_}$, which is the fraction on the RHS of (C.4.2b), approaches $-\infty$ if step size $h_{\eta^e}$ is sufficiently small and thus $\mathcal{G}_{k\_} \to -\infty$. Further increasing $k$ from this point on raises $\mathcal{G}_{k\_}$: intuitively, a higher value of $\eta_k$ is associated with less risk-taking by entrepreneurs. This reduces the required risk premia of these agents, which, in turn, leads to a higher value of $\mathcal{G}_{k\_}$.[926] It follows from the above considerations that $k$ is identified if $\mathcal{J}_{k\_} > 0$, $\mathcal{G}_{k\_} > 0$, and $\psi_{k\_} \leq 1$ for the first time. In this case, the "true" post-jump value $\widetilde{\eta}_m$ must lie between $\eta_{k-1}$ and $\eta_k$. The post-shock value is then calculated via linear interpolation formula

$$\widetilde{\eta}_m = w_m \eta_k + (1 - w_m)\eta_{k-1}, \tag{C.4.13a}$$

in which $0 \leq w_m \leq 1$ is a weighting factor. If $\mathcal{G}_{k\_}\mathcal{G}_{(k-1)\_} < 0$, it is set according to

$$w_m = \frac{\mathcal{G}_{(k-1)\_}}{\mathcal{G}_{(k-1)\_} - \mathcal{G}_{k\_}}$$

since in that case we look for a solution with $\mathcal{G}_{k\_} = 0$. If, on the other hand, $\psi_{(k-1)\_} > 1$, the weighting factor is now determined from

$$w_m = \frac{\psi_{(k-1)\_} - 1}{\psi_{(k-1)\_} - \psi_{k\_}} \tag{C.4.13b}$$

as now we look for a solution with $\psi_{k\_} = 1$. Linear interpolation is also applied to calculate the post-shock value of the value share of capital:

$$\widetilde{\theta}_m = w_m \theta_k + (1 - w_m)\theta_{k-1}. \tag{C.4.13c}$$

Having derived post-shock values, parameters $(\mu^q - \mu^p)_m$ and $\mu^\eta_m$ can be derived in a straightforward fashion. First, we can calculate $x^{e,p}_{1,m}$, $x^{e,p}_{2,m}$, $x^m_m$, $q_m$, and $\Gamma_m$, using equations (C.4.1)-(C.4.10), where we just have to use subscript $m$ instead of subscript $(k-1)_-$ and replace $\theta_k$ and $\eta_k$ by $\widetilde{\theta}_m$ and $\widetilde{\eta}_m$ in each case. The optimal investment rate can then be determined from $\iota_m = (q_m - 1)/\gamma$.

[925] If $\theta(\cdot)$ is not monotonously increasing in the state variable, nonunique solutions are in principle possible. However, testing of the algorithm offers no evidence for such behaviour of function $\theta(\cdot)$ . Hence, we will assume that $\theta(\cdot)$ is indeed increasing in $\eta$ in the remainder of this appendix.

[926] Observation of the numerical results shows that $\mathcal{G}_{k\_}$ is monotonously increasing in $k$.

Given entrepreneurs' portfolio choice, $q_m$, $\Gamma_m$, and $\iota_m$, we can determine $(\mu^q - \mu^p)_m$ via

$$(\mu^q - \mu^p)_m = -\lambda\phi \frac{\left(1-\kappa-\underline{\kappa}^{e,p}\right)\frac{\widetilde{\theta}_m}{\theta_m} - (1-\kappa)\frac{1-\widetilde{\theta}_m}{1-\theta_m}}{x^{e,p}_{1,m}\left(1-\kappa-\underline{\kappa}^{e,p}\right)\frac{\widetilde{\theta}_m}{\theta_m} + \left(1-x^{e,p}_{1,m}\right)(1-\kappa)\frac{1-\widetilde{\theta}_m}{1-\theta_m}} - \lambda(1-\phi)\frac{\left(1-\kappa-\overline{\kappa}^{e,p}\right)\frac{\widetilde{\theta}_m}{\theta_m} - (1-\kappa)\frac{1-\widetilde{\theta}_m}{1-\theta_m}}{x^{e,p}_{1,m}\left(1-\kappa-\overline{\kappa}^{e,p}\right)\frac{\widetilde{\theta}_m}{\theta_m} + \left(1-x^{e,p}_{1,m}\right)(1-\kappa)\frac{1-\widetilde{\theta}_m}{1-\theta_m}} - \frac{a^e-\iota_m}{q_m} + \Gamma_m, \tag{C.4.14}$$

which follows from (4.5.6c). Finally, the drift rate of the state variable is given by

$$\mu^\eta_m = x^{e,p}_{1,m}\frac{a^e-\iota_m}{q_m} + \left(1-x^{e,p}_{1,m}-x^{e,p}_{2,m}\right)\Gamma_m + \left(x^{e,p}_{1,m}-\theta_m\right)(\mu^q-\mu^p)_m - \frac{\rho}{1+\xi}, \tag{C.4.15}$$

which is the discretised version of (4.5.7b).

### C.4.2 Shooting Algorithm

RDE (4.5.9) is solved via the shooting method. This method is implemented by repeatedly following four steps:[927]

(i) Guess an initial history for the value share of capital. To this end, value $\theta(\eta_1 = 0)$ at the initial state $\eta_1 = 0$ is set at first. Since entrepreneurs do not operate without equity[928], this value is equal to the autarky solution derived in Section 5.2.2: $\theta(\eta_1 = 0) = \theta_A$.[929] Then, the initial guess is imposed by setting the values of $\theta_m$ for each $m = 1, ..., \check{m}$ according to

$$\theta_m = \theta_A + (m-1)\epsilon_1, \tag{C.4.16}$$

in which $\epsilon_1$ is constant, which is specified momentarily. Further, we set $\eta_{\check{m}} = \epsilon_2$, where Parameter $\epsilon_2$ is a small constant[930], for the value of entrepreneurs' wealth share at the end point of the initial history. This value can be used to pin down $\check{m}$, given the grid for the state variable. On the contrary, $\epsilon_1$ is the shooting parameter, i.e. the parameter that is adjusted if the restrictions on the model variables that are yet to be specified are violated. It

[927] The following procedure closely resembles that described in Brunnermeier and Sannikov (2014d, Appendix A), with two exceptions. First, the second step in our algorithm is not required to solve their model. Second, rather than using the linear scheme in (3.1.38) to discretise the state variable, we follow Brunnermeier and Sannikov (cf. 2016a, p. 54), who adopt an unevenly-spaced grid in $M+1$ grid points, satisfying $\eta_m = 3m^2/M^2 - 2m^3/M^3$. This scheme has the advantage that the distance between grid points is smaller at the extremes of the state space.

[928] Entrepreneurs must be equipped with a strictly positive endowment of wealth in order for them to be able to start operations as in Bernanke et al. (cf. 1999, p. 1357). The underlying reason is that asset demands are linear in wealth due to the adoption of logarithmic utility functions, which belong to the CRRA family (cf. Subsection 3.1.1.4).

[929] Recall from Section 5.2.2 that the value share of capital is equal to the portfolio weight on capital in the autarky case.

[930] In the algorithm, we set $\epsilon_2 = 3.5 \times 10^{-4}$.

is determined from

$$\epsilon_1 = \frac{\epsilon_L + \epsilon_R}{2}. \tag{C.4.17}$$

In order to calculate $\epsilon_1$, we set $\epsilon_L = 0$ and $\epsilon_R = (1 - \theta_A) / (\check{m} - 1)$ as an initial guess. The initial choices of $\epsilon_L$ and $\epsilon_R$ determine the lower and upper bounds of $\epsilon_1$, respectively. The lower bound is chosen to equal zero since $\theta_A$ is the minimum value the value share of capital can take on: in the autarky model, entrepreneurs, who are more productive than managers, are not present. Hence, the demand for credit and the stock of inside money are equal to zero. Once entrepreneurs enter the economy, the demand for capital surges since these agents value capital more than managers. This, in turn, leads to an increase in the price of capital. Moreover, at least a part of entrepreneurs' capital purchases are financed by bank credit. As a byproduct of credit extension, inside money is created. This additional money supply causes the value of money to fall. Both effects imply that the value share of capital should be at least as high as $\theta_A$ for $\eta_m > 0$. On the other hand, recalling that $\theta(\cdot)$ measures the share of capital in aggregate wealth, it becomes clear that the value of this variable must not exceed unity. The initial choice for $\epsilon_1$ guarantees that this condition is not violated in the range from grid point $m = 1$ to $m = \check{m}$.

(ii) Guess an initial history for the price of capital. This step is necessary due to the presence of equation (4.5.6g) in the set of equilibrium equations. That equation determines the EFP and contains the post-jump price of capital. The initial history of $q_m$ is approximated by applying a fixed point method. This method requires an initial guess for the history of the capital price. In order to obtain this guess, we first set $q_m$ to its autarky value $q_A$ for each $m = 1, ..., \check{m}$. Given the history of the value share of capital, $q_{\check{m}}$ can be determined via equation (4.5.6e), the market clearing condition in the final goods market. Afterwards, the initial guess is updated by setting $q_m = q_A + (q_{\check{m}} - q_A)(m - 1) / (\check{m} - 1)$ for each $m = 1, ..., \check{m}$. This procedure is repeated until $q_{\check{m}}$ is sufficiently close to its value from the previous iteration.

(iii) Solve the RDE for the value share of capital. After initial histories have been assigned to $\theta_m$ and $q_m$, the discretised RDE is iterated forward via the Euler Method. Inserting the forward finite difference of derivative $d\theta(\eta_m) / d\eta_m$ into the discretised version of (4.5.9) and subsequent rearranging leads to

$$\theta_{m+1} = \left[ \frac{(\mu^q - \mu^p)_m}{\mu_m^\eta \eta_m} \theta_m (1 - \theta_m) \right] h_\eta + \theta_m. \tag{C.4.18}$$

The solution of the difference equation commences at point $m = \check{m}$. At each grid point $m$, parameters $(\mu^q - \mu^p)_m$ and $\mu_m^\eta$ are obtained from the procedure described in the previous subsection of the Appendix.

(iv) Adjust the guesses. The integration is terminated if any of the following three conditions are met. If at any point $m+1$ on the grid $\theta_{m+1} > 1$ ensues, the algorithm has "overshot" the true solution. Thus, the initial guess for $\epsilon_1$ was too high and, as a consequence, this parameter is adjusted by setting $\epsilon_R = \epsilon_1$. If on the contrary $\theta_{m+1} < 0$, the true solution is "undershot". Hence, $\epsilon_1$ needs to be revised upwards by setting $\epsilon_L = \epsilon_1$. Finally, as the algorithm moves in the forward direction of the grid (and thereby increases entrepreneurs' wealth share), $\theta_m$ is expected to rise. This is because an improved net worth position of entrepreneurs raises their demand for credit and thus leads to more money creation, which lowers the value of money. In addition, the value of capital is expected to increase with $\eta_m$ since entrepreneurs' demand for capital is an increasing function of their wealth. Accordingly, if $\theta_{m+1} < \theta_m$ at some point $m$, the guess for $\epsilon_1$ is increased by setting $\epsilon_L = \epsilon_1$. In any of the three cases, the integration of the RDE has to be restarted at grid point $\check{m}$. Before that, the initial history must be adjusted, using the new value of $\epsilon_1$. The procedure is repeated until parameter $\epsilon_1$converges.

Let us discuss three weaknesses of the algorithm at last. First, under some parameter values a slight inaccuracy in the (very narrow) vicinity of $\eta^\psi$, the value of the state variable at which the optimal capital allocation is reached, arises. This is visible only in the graphs depicting the drift rates of output and the price level as well as the percentage drop in output. If this inaccuracy occurs, a linear interpolation in the proximity of $\eta^\psi$ is applied to the mentioned functions.

Second, under some parameter constellations entrepreneurs' share in the aggregate capital stock *falls* as the state variable approaches its steady state value. This result is highly counterintuitive since the more productive agents have no apparent incentive to sell capital to less productive agents as the relative equity position of the former improves. Rather, it is more likely that the mentioned result is due to a numerical error.[931] To address this issue, an additional restriction is added to the fourth step of the algorithm in those cases. This restriction requires the drift rate of the state variable to fall in the affected region of the state space.[932] Since the additional condition is not based on theoretical reasoning, this procedure might be problematic. Yet, testing of the algorithm has shown that the mentioned condition effectively resolves the described problem and does not lead to counterintuitive results.

Third, the algorithm does not reach the stochastic steady state, which is characterised by condition $\mu_t^\eta = 0$ . Rather, the algorithm terminates at point $\eta^s$, where $\mu_t^\eta \approx 0.5$ percent in the baseline model. This is due to the fact that one of the conditions in step four is violated, while at the same time the change in $\epsilon_1$ approaches zero.[933] Under the baseline calibration, the condition which causes the algorithm to halt is $\theta_{m+1} < \theta_m$, i.e. the value share of capital falls at that point. It should be recognised, however, that the affected region of the grid only contains values of the state variable

[931] Indeed, it can be observed from (C.4.18) that the absolute change $\theta_{m+1} - \theta_m$ can become very large as $\mu_m^\eta \to 0$. This property of the RDE can exacerbate approximation errors and might lie at the root of the problem.

[932] In contrast, Brunnermeier and Sannikov (cf. 2014d, p. 37) impose this restriction on the *entire* grid in in the solution procedure of their model.

[933] This problem also occurs in the Brunnermeier and Sannikov (2014d) model (cf. Figure 3.2.11 of this thesis ). Yet, it is not discussed by the authors.

that are very unlikely to occur, as shown by the Monte Carlo simulation of the baseline model in Section 5.3.5.

### C.4.3 Determination of Other Endogenous Variables

Some key macroeconomic variables, such as the parameters of the stochastic process for output, are yet to be determined. These variables are not necessary for solving the RDE and thus for closing the model, yet, they offer some important economic insights. As can be seen form equation (4.4.7), the drift rate of the stochastic process for GDP depends on $\mu_t^\psi$, the drift of entrepreneurs' capital share. However, no equation for the drift of capital allocation has been provided so far. Hence, we express it as a function of $\eta_t$ and apply CVF (3.1.20) to this function to arrive at

$$\mu_t^\psi = \frac{d\psi_t}{d\eta_t}\frac{\mu_t^\eta \eta_t}{\psi_t}. \tag{C.4.19}$$

Substituting (C.4.19) into (4.4.7) results in

$$\mu_t^Y = \mu_t^A + \mu_t^K = \frac{d\psi_t}{d\eta_t}\frac{(a^e - a^m)\mu_t^\eta \eta_t}{\psi_t a^e + (1-\psi_t)a^m} + \Xi(\iota) - \delta. \tag{C.4.20}$$

Next, we discretise and replace derivative $d\psi_t/d\eta_t$ by its finite difference. This yields

$$\mu_m^Y = \frac{\psi_{m+1} - \psi_m}{\eta_{m+1} - \eta_m}\frac{(a^e - a^m)\mu_m^\eta \eta_m}{\psi_m a^e + (1-\psi_m)a^m} + \Xi(\iota_m) - \delta. \tag{C.4.21}$$

The presence of term $\psi_{m+1}$ shows that $\mu_m^Y$ cannot be calculated at the current grid point $m$. Rather, it can only be determined "ex post", i.e. at grid point $m+1$. It follows that the model would not be solvable via the shooting method if any of the model equations in Proposition 2 included $\mu_t^Y$.[934] The post-jump *level* of output cannot be expressed independently of the aggregate capital stock. Conversely, this issue does not arise in the calculation of the *percentage* drop in aggregate output due to the occurrence of a shock given by

$$\breve{Y}_m = \frac{\psi_k a^e + (1-\psi_k)a^m}{\psi_m a^e + (1-\psi_m)a^m}(1-\kappa) - 1, \tag{C.4.22}$$

which was obtained from (4.4.9).

The money multiplier was defined in Subsection 4.4.1.4 as the ratio of the inside money supply to the outside money supply. A drawback in definition (4.4.13) is that the money multiplier depends on the aggregate capital stock. Using the second equality in (4.5.4b) and the first equality in (4.5.5), we can reformulate the defining equation for the money multiplier according to

[934] Indeed, one could approximate the derivative in (C.4.20) by the backward difference. Yet, doing so would not address the problem that a (potentially large) subset of the equations in Appendix C.4.1 would have to be solved simultaneously. This is due to the presence of term $\mu_m^\eta$ in (C.4.20), which is the last variable to be solved for in the algorithm.

$$\Lambda_m = \frac{(x^{e,p}_{1,m} + x^{e,p}_{2,m} - 1)\eta_m}{(1-\varphi)(1-\theta_m)}. \tag{C.4.23}$$

The term on the RHS does not depend on the aggregate capital stock and thus can be calculated without simulating paths for the latter.

It is also possible to provide simple characterisations of the drift rates of prices, using the same procedure as in the derivation of $\mu^{\psi}_m$. Applying (3.1.20) to general function $q(\eta_t)$, we get

$$\mu^q_t = \frac{dq_t}{d\eta_t}\frac{\mu^{\eta}_t \eta_t}{q_t}. \tag{C.4.24}$$

Discretisation of this equation implies

$$\mu^q_m = \frac{q_{m+1} - q_m}{\eta_{m+1} - \eta_m}\frac{\mu^{\eta}_m \eta_m}{q_m}. \tag{C.4.25}$$

Drift rate $\mu^p_m$ is determined in an analogous way. Similarly to $\mu^Y_m$, drift rates $\mu^q_m$ and $\mu^p_m$ can only determined if the respective absolute values at grid point $m+1$ are known.

### C.4.4 Monte Carlo Simulation

A central task in the Monte Carlo simulation is to simulate a path for the state variable $\eta_t$. Given such a path, all other endogenous variables can be determined accordingly. To that end, the first step is to discretise the time dimension. As Di Tella (2017), we choose a unit of time to equal one year and approximate the time step $dt$ by1/360, i.e. about one day.[935] The time horizon $T$ for the simulation is 100000 years. Next, timings for jump events need to be generated. This can be achieved by drawing random samples from an exponential distribution since this distribution measures the time passed between events generated by a Poisson distribution.[936] In particular, the samples are drawn from an exponential distribution with mean $\frac{1}{\lambda\, dt}$, in which $\lambda\, dt$ is the probability of a jump event during a period of length $dt$ and $\frac{1}{\lambda\, dt}$ is the expected waiting time between events. Sampling and subsequent rounding results in a set $\mathcal{S} \subseteq \mathbb{N}$ of discrete jump dates. Since time step $dt$ is small but does not approach zero, it is possible that multiple jumps occur per period of length $dt$.[937] Hence, stochastic increment $d\mathcal{N}_t$ can take on values greater than unity.[938]

[935] Cf. Di Tella (2017, p. 2079).
[936] Cf. Ross (2014, p. 301).
[937] Cf. the discussion of the Poisson process in Section 3.1.1.1.
[938] In contrast to considering the continuous-time limit, this implies that the state variable can become negative. The reason for this discrepancy is that in the former case, agents can reduce their exposure to risky capital instantaneously between consecutive jumps (cf. Brunnermeier and Sannikov, 2016b, p. 1502). Conversely, if the time dimension is discretised and the economy is exposed to multiple jumps during a given period, agents are implicitly prevented from deleveraging between these within-period jumps. Judging from the simulation results, a negative value of the state variable seems to be a very unlikely scenario. Yet, in order to avoid this case, the simulation algorithm is programmed to set $\eta_t$ to a slightly positive value if a negative value results from a realisation of $d\mathcal{N}_t$ greater than unity.

Afterwards, state variables $\eta_t$ and $K_t$ are iterated forward in time by applying the Euler method. State variable $\eta_t$ is iterated forward according to

$$\eta_{t+1} = (1 + \mu_t^\eta \, dt)\, \eta_t + (\widetilde{\eta}_t - \eta_t)\; d\mathcal{N}_t, \tag{C.4.26}$$

starting at some arbitrary (but strictly positive) initial value $\eta_0 \in (0, \eta^s)$.[939] Since the solution method described in the previous subsections produces values for $\mu_t^\eta$ and $\widetilde{\eta}_t$ on a discrete grid $m = 1, ..., M$, the integration procedure requires us to find the value $\eta_m$ closest to $\eta_t$ and to use this value in order to approximately match $\mu_t^\eta$ and $\widetilde{\eta}_t$. The discretised equation of motion for the aggregate capital stock is

$$K_{t+1} = (1 + [\Phi\,(\iota_t) - \delta]\; dt)\, K_t - \kappa \, d\mathcal{N}_t. \tag{C.4.27}$$

Initial value $K_0$ is chosen to be a very small, but strictly positive number.[940]

With the paths of the state variables at hand, the stationary distribution of the state variable can be computed. The term "stationary" in this context refers to the fact that neither adding additional realisations to the time series nor altering the starting value $\eta_0$ affects the distribution or measured moments.[941] In addition, one can easily obtain the paths of any other endogenous variable. To calculate flow variables such as output or consumption, the approach suggested by Di Tella (2017) is adopted.[942] This is exemplified by the calculation of quarterly real GDP. First, output per day is calculated from

$$Y_t = [\psi_t a^e + (1 - \psi_t) a^m]\, K_t \, dt, \tag{C.4.28}$$

Then, output per quarter is computed by integrating over periods of 30 days:

$$Y_Q = \int_{Q \times 0.25}^{(Q+1)\times 0.25} Y_t \, dt, \tag{C.4.29}$$

in which $Q = 1, 2, ..., T$ denotes quarters. Once quarterly flow variables are derived, the resulting time series are used to obtain quarterly growth rates.[943] To calculate the quarterly growth rates

[939] For large enough $T$, the particular value of $\eta_0$ in interval $(0, \eta^s)$ does not matter for the calculation of moments or distributions, as we will discuss below.

[940] If $K_0$ is not small enough, $K_t$ can be become too large for MATLAB to handle for high values of $t$ under some parameter constellations.

[941] Cf. He and Krishnamurthy (2013, p. 768).

[942] Cf. Di Tella (2017, p. 2079).

[943] An alternative would be to calculate the daily growth rate of GDP first and then to annualise the resulting value. The problem with this procedure is that it leads to a serious overestimation of volatility. This is due to the fact that in case of a jump, the daily growth rate is of order $d\mathcal{N}_t/dt$, while in normal times it is of order one:

$$\frac{dY_t/dt}{Y_t} = \mu_t^Y + \frac{\widetilde{Y}_t - Y_t}{Y_t}\frac{d\mathcal{N}_t}{dt}.$$

of stock variables and prices, beginning of quarter values are utilised, as in Di Tella (2017).[944] Finally, daily returns are computed from (4.2.10), (4.2.11), (4.2.22), and (4.3.34). Subsequently, daily returns are compounded to arrive at quarterly returns.[945]

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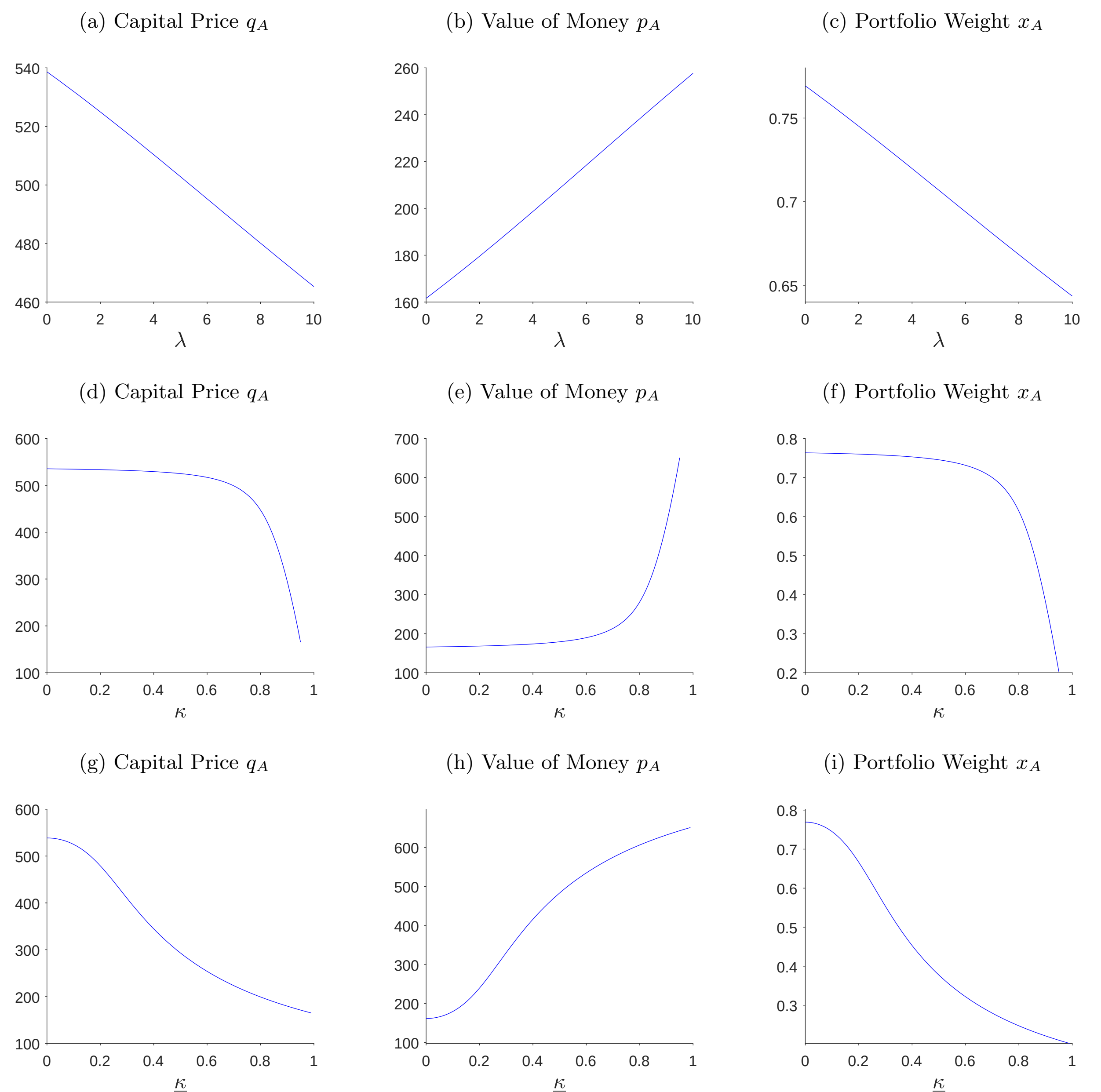
Since jumps arrive infrequently and $dt$ is low, jumps can be considered as extreme outliers. Since the standard deviation is sensitive to extreme values, we do not opt for the annualisation method.

[944] Cf. Di Tella (2017, p. 2079).

[945] In the empirical Finance literature, high frequency returns are usually compounded to obtain monthly returns (cf. e.g. Fama and French, 2007, p. 47). However, since other flow variables are reported at quarterly frequencies here, quarterly returns are selected instead.

## C.5 Supplementary Figure

Figure C.5.1: Variation of Shock Parameters in the Autarky Model



## C.6 Supplementary Tables

Table C.1: Model-Implied Correlation Matrix for $\hat{\Lambda}$, $\hat{\eta}$, and $\hat{Y}$

| | $\hat{\Lambda}$ | $\hat{\eta}$ | $\hat{Y}$ |
|---|---|---|---|
| $\hat{\Lambda}$ | 1 | 0.99 | 0.95 |
| $\hat{\eta}$ | | 1 | 0.91 |
| $\hat{Y}$ | | | 1 |

*Notes*: Model-implied correlation coefficients between quarterly growth rates; correlation coefficients are computed from time series generated by a Monte Carlo simulation of the baseline model. Each of the simulated series contains 9000000 realisations (360 days $\times$ 25000 years).

Table C.2: Model-Implied Correlation Matrix for $\hat{L}_t^{e,p}$, $\hat{x}^b$, $\hat{q}$, and $\hat{Y}$

| | $\hat{L}_t^{e,p}$ | $\hat{x}^b$ | $\hat{q}$ | $\hat{Y}$ |
|---|---|---|---|---|
| $\hat{L}_t^{e,p}$ | 1 | -0.94 | -0.82 | -0.53 |
| $\hat{x}^b$ | | 1 | 0.95 | 0.60 |
| $\hat{q}$ | | | 1 | 0.62 |
| $\hat{Y}$ | | | | 1 |

*Notes*: Model-implied correlation coefficients between quarterly growth rates; correlation coefficients are computed from time series generated by a Monte Carlo simulation of the baseline model. Each of the simulated series contains 9000000 realisations (360 days $\times$ 25000 years).

Table C.3: Model-Implied Correlation Matrix for $\hat{Y}$, $\hat{M}$, and $\pi$

| | $\hat{Y}$ | $\hat{M}$ | $\pi$ |
|---|---|---|---|
| $\hat{Y}$ | 1 | 0.62 | 0.62 |
| $\hat{M}$ | | 1 | 0.99 |
| $\pi$ | | | 1 |

*Notes*: Model-implied correlation coefficients between quarterly growth rates; correlation coefficients are computed from time series generated by a Monte Carlo simulation of the baseline model. Each of the simulated series contains 9000000 realisations (360 days $\times$ 25000 years).

# Appendix D

# Appendix to Chapter 6

## D.1 Proofs

### D.1.1 Proof of Proposition 8

The drift of agent $m$'s net worth is

$$dn_t^m = \left\{ \left( \mu_t^{r^{P,m}} + \mu_t^\tau \tau_t \right) n_t^m - c_t^m \right\} dt + \left\{ \underline{\widetilde{n}}_t^m - n_t^m \right\} d\underline{\mathcal{N}}_t + \left\{ \overline{\widetilde{n}}_t^m - n_t^m \right\} d\overline{\mathcal{N}}_t, \tag{D.1.1}$$

in which

$$\underline{\widetilde{n}}_t^m \equiv (1 + \widetilde{\tau}_t - \tau_t) \left[ x_t^m (1 - \kappa - \underline{\kappa}^m) \frac{\widetilde{q}_t}{q_t} + (1 - x_t^m)(1 - \kappa) \frac{\widetilde{p}_t}{p_t} \mathcal{M}_t + \widetilde{\Pi}_t^b - \Pi_t^b \right] n_t^m \quad \text{and}$$
$$\overline{\widetilde{n}}_t^m \equiv (1 + \widetilde{\tau}_t - \tau_t) \left[ x_t^m (1 - \kappa - \overline{\kappa}^m) \frac{\widetilde{q}_t}{q_t} + (1 - x_t^m)(1 - \kappa) \frac{\widetilde{p}_t}{p_t} \mathcal{M}_t + \widetilde{\Pi}_t^b - \Pi_t^b \right] n_t^m.$$

Optimality requires

$$\begin{aligned} \rho V^m(n_t^m, \eta_t) = \max_{c_t^m,\, 1 \geq x_t^m \geq 0} \Bigg\{ & \log c_t^m + \xi \log\left( (1 - x_t^m)\, n_t^m \right) \\ & + \frac{\partial V^m(n_t^m, \eta_t)}{\partial n_t^m} \left[ \left( \mu_t^{r^{P,m}} + \mu_t^\tau \tau_t \right) n_t^m - c_t^m \right] + \frac{\partial V^m(n_t^m, \eta_t)}{\partial \eta_t} \mu_t^\eta \eta_t \\ & + \lambda \phi \left[ V^m\left( \underline{\widetilde{n}}_t^m, \widetilde{\eta}_t^e \right) - V^m\left( n_t^m, \eta_t \right) \right] + \lambda (1 - \phi) \left[ V^m\left( \overline{\widetilde{n}}_t^m, \widetilde{\eta}_t^e \right) - V^m\left( n_t^m, \eta_t \right) \right] \Bigg\}. \end{aligned} \tag{D.1.2}$$

Using value function (5.3.2) and optimal choices leads to

$$\begin{aligned} \rho \alpha^m(\eta_t) = {} & \log \frac{\rho}{1 + \xi} + \xi \log(1 - x_t^m) + \frac{1 + \xi}{\rho} \left( \mu_t^{r^{P,m}} + \mu_t^\tau \tau_t \right) - 1 + \frac{d\alpha^m(\eta_t)}{d\eta_t} \mu_t^\eta \eta_t \\ & + \lambda \left( \frac{1 + \xi}{\rho} \left[ \phi \log \underline{\widetilde{n}}_t^m + (1 - \phi) \log \overline{\widetilde{n}}_t^m - \log n_t^m \right] + \alpha^m(\widetilde{\eta}_t^e) - \alpha^m(\eta_t) \right). \end{aligned} \tag{D.1.3}$$

Further rearranging implies (6.2.13a). A differential equation that function $\alpha^{e,p}(\cdot)$ has to satisfy can be obtained analogously. □

### D.1.2 Proof of Proposition 9

Substituting relation $\mu^{MO} = \breve{\mu}^{MO} / \left(1 - x^m_{F,M}\right)$, which is implied by the definition of the transformed money growth rate, into FOC (6.3.3) yields

$$\frac{1+\xi}{\rho}\left(\frac{a^e - \iota}{q} + \frac{\breve{\mu}^{MO}}{1 - x^m_{F,M}}\right) = \xi \frac{1}{1 - x^m_{F,M}}. \tag{D.1.4}$$

Hence,

$$\frac{1}{1 - x^m_{F,M}} = \frac{a^e - \iota}{q} \frac{1+\xi}{\xi\rho - (1+\xi)\,\breve{\mu}^{MO}}. \tag{D.1.5}$$

Next, we combine the two equations in (5.2.3) to

$$x^m_{F,M} = \frac{q\rho}{(a^e - \iota)(1+\xi)}. \tag{D.1.6}$$

We then get an expression for $q$ by substituting (D.1.6) into (D.1.5) and rearranging:

$$q = \frac{a^e - \iota}{\rho - \breve{\mu}^{MO}}. \tag{D.1.7}$$

Taking into account investment rule (3.1.27) and again solving for $q$ results in (6.3.4a).

The value of money per unit of capital $p$ can be obtained by substituting the final goods market clearing condition in (5.2.3) into (D.1.7) and successively solving for $p$:

$$p = \frac{\xi\rho - (1+\xi)\,\breve{\mu}^{MO}}{\rho} q. \tag{D.1.8}$$

Inserting (6.3.4a) leads to (6.3.4b). On the other hand, the optimum investment rate in (6.3.4d) follows from inserting (6.3.4a) into the optimality condition $\iota_{F,M} = (q_{F,M} - 1)/\gamma$ and rearranging.

Finally, we turn to the question under which conditions a moneyless equilibrium occurs. For this purpose, we first examine whether the price of capital can become nonpositive. Observation of equation (6.3.4a) suggests that a nonpositive price of capital is possible if $\breve{\mu}^{MO} > \rho + 1/\gamma$. However, this condition can never be satisfied as the comparison of (6.3.4a) with the maximum price of capital $q_{\max}$ shows. The latter price results form the notion that it is economically infeasible for the investment-output ratio to exceed unity: $I_t/Y_t = \iota/a^e \leq 1$. Price $q_{\max}$ can be solved for by letting the preceding inequality hold with strict equality and subsequently substituting (3.1.27) into the resulting equation. These steps lead us to an expression for the maximum price of capital: $q_{\max} = a^e\gamma + 1$. Next, we impose condition $q_{F,M} \leq q_{\max} = a^e\gamma + 1$, which together with (6.3.4a) reduces to $\rho \geq \breve{\mu}^{M^O}$. Hence, an equilibrium with a nonpositive price of capital is not feasible.

As the price of capital is strictly positive, (6.3.4b) implies that money does not have a positive value if condition $\breve{\mu}^{M^O} \leq \xi/(1-\xi)\rho$ holds. We can exclude cases with $\xi > 0$ since the substitution of condition $(1 - x_{F,M}) = 0$ into FOC (6.3.3) implies an infinitely high RHS of this very equation. Hence, a moneyless equilibrium can only occur if $\xi = 0$.[946] The preceding condition for a moneyless equilibrium thus reduces to $\breve{\mu}^{M^O} \leq 0$, while $\xi = 0$ must hold at the same time. □

### D.1.3 Proof of Proposition 10

The HJB equation in the first-best case with monetary policy rule (6.3.1) is given by

$$\begin{aligned}\rho V^m_{F,M}(n^m_{i,t}) = \max_{c^m_{i,t}, x^m_{F,M}} \Bigg\{ &\log c^m_{i,t} + \xi \log\left[\left(1 - x^m_{F,M}\right) n^m_{i,t}\right] \\ &+ \frac{1+\xi}{\rho n^m_{i,t}} \left[\left(x^m_{F,M} \frac{a^e - \iota_{F,M}}{q_{F,M}} - \left(1 - x^m_{F,M}\right)\mu^{M^O} + \Phi(\iota_{F,M}) - \delta + \mu^\tau_{F,M}\tau_t\right) n^m_{i,t} - c^m_{i,t}\right] \\ &+ \lambda \frac{1+\xi}{\rho} \log(1-\kappa) \Bigg\}.\end{aligned} \tag{D.1.9}$$

It can be proven by following steps analogous to those in Appendix B.1.5 that $V^m_{F,M}(\cdot)$ obeys the same general form as the value function in Proposition 8. Using this fact on the LHS of (D.1.9), inserting consumption policy (4.3.10) on the RHS, replacing the other control variables by their optimal values, and subsequently substituting (6.3.5c) leads to[947]

$$\begin{aligned}\rho\left(\alpha + \frac{1+\xi}{\rho}\log n^m_{i,t}\right) = &\log\left(\frac{\rho}{1+\xi} n^m_{i,t}\right) + \xi \log\left(\left(1 - x^m_{F,M}\right) n^m_{i,t}\right) \\ &+ \frac{1+\xi}{\rho}\left[\mu^{r^{P,m}}\left(x^m_{F,M}\right) + \lambda \log(1-\kappa)\right] - 1.\end{aligned} \tag{D.1.10}$$

Solving for $\alpha$ results in

$$\alpha = V_0 + \frac{\xi}{\rho}\log\left(1 - x^m_{F,M}\right) + \frac{1+\xi}{\rho^2}\left[\mu^{r^{P,m}}\left(x^m_{F,M}\right) + \mu^\tau_{F,M}\tau_t\right], \tag{D.1.11a}$$

$$\text{where} \quad V_0 \equiv \frac{1}{\rho}\left[\log\left(\frac{\rho}{1+\xi}\right) - 1 + \lambda\frac{1+\xi}{\rho}\log(1-\kappa)\right]. \tag{D.1.11b}$$

An agent endowed with the entire aggregate wealth owns initial wealth $n^m_{i,0} = (q_{F,M} + p_{F,M}) K_0$. Normalising $K_0$ to unity and inserting the wealth endowment as well as equation (D.1.11a) back into the general form of the value function (4.3.8) yields (6.3.5a). □

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[946] We have restricted $\xi$ to be nonnegative in (4.3.7).
[947] We have dropped the time subscript of $\alpha$ since this parameter is constant in the first-best case.

### D.1.4 Proof of Proposition 11

In the autarky case with money supply rule (6.3.1), managers' HJB equation is

$$\begin{aligned}\rho V^m_{A,M}(n^m_{i,t}) = \max_{c^m_{i,t},x^m_{A,M}} \Bigg\{ &\log c^m_{i,t} + \xi \log\left(\left(1-x^m_{A,M}\right) n^m_{i,t}\right) \\ &+ \frac{1+\xi}{\rho n^m_{i,t}} \left[\left(x^m_{A,M}\frac{a^m - \iota_{A,M}}{q_{A,M}} - \left(1-x^m_{A,M}\right)\mu^{M^O} + \Phi(\iota_{F,A}) - \delta + \mu^\tau_{A,M}\tau_t\right) n^m_{i,t} - c^m_{i,t}\right] \\ &+ \lambda\frac{1+\xi}{\rho}\left[\phi\log\left(1-\kappa - x^m_{A,M}\underline{\kappa}\right) + (1-\phi)\log\left(1-\kappa - x^m_{A,M}\overline{\kappa}\right)\right]\Bigg\}.\end{aligned} \tag{D.1.12}$$

One can then solve for value function parameter $\alpha$ by applying the same approach as in Appendix D.1.3. This implies

$$\begin{aligned}\alpha = V_0 + \frac{\xi}{\rho}\log\left(1-x^m_{A,M}\right) + \frac{1+\xi}{\rho^2}\left[\mu^{r^{P,m}}\left(x^m_{A,M}\right) + \mu^\tau_{A,M}\tau_t\right] \\ + \lambda\frac{1+\xi}{\rho^2}\left[\phi\log\left(1-\kappa - x^m_{A,M}\underline{\kappa}\right) + (1-\phi)\log\left(1-\kappa - x^m_{A,M}\overline{\kappa}\right)\right],\end{aligned} \tag{D.1.13a}$$

$$\text{where} \quad V_0 \equiv \frac{1}{\rho}\left[\log\left(\frac{\rho}{1+\xi}\right) - 1\right]. \tag{D.1.13b}$$

Thus, value function $V^m_{F,M}(n^m_{i,t})$ of an agent starting his life at time $t = 0$ with wealth $n^m_{i,0} = \left(q_{A,M} + p_{A,M}\right) K_0$ and capital $K_0 = 1$ takes on the form in (6.3.6a) □

## D.2 Solution Procedure in the First-Best Case with Monetary Policy

Combining the final goods market clearing condition $\rho\left(q_{F,M} + p_{F,M}\right)/\left(1+\xi\right) = a^e - \iota_{F,M}$, the capital market clearing condition $x^m_{F,M}\left(q_{F,M} + p_{F,M}\right) = q_{F,M}$, and FOC (6.3.3) leads to

$$\frac{\rho\mu^{M^O}}{1+\xi}\left(q_{F,M}\right)^2 + \left(\rho - \mu^{M^O}\right)\left(a^e - \iota_{F,M}\right) q_{F,M} - \left(a^e - \iota_{F,M}\right)^2. \tag{D.2.1}$$

Using optimal investment rule (3.1.27), the quadratic equation in $q_{F,M}$ can be expressed as

$$A\left(q_{F,M}\right)^2 + Bq_{F,M} + C = 0, \tag{D.2.2a}$$

$$\text{with} \quad A \equiv \frac{\rho\mu^{M^O}}{1+\xi} - \frac{\left(\rho - \mu^{M^O}\right)}{\gamma} - \frac{1}{\gamma^2}, \tag{D.2.2b}$$

$$B \equiv \left(\rho - \mu^{M^O} + \frac{2}{\gamma}\right)\left(a^e + \frac{1}{\gamma}\right), \tag{D.2.2c}$$

$$C \equiv -\left(a^e + \frac{1}{\gamma}\right)^2. \tag{D.2.2d}$$

Solving the quadratic equation, of course, entails two solutions for the price of capital. However, we can use our preceding findings as well as economic theory to restrict the range of possible values for the equilibrium price of capital. First, the price must not be lower than zero. Second, since the price of capital increases in the growth rate of money according to Proposition 9, $q_{F,M}$ must not be higher than $q_F$ if that rate is smaller than zero. Third, we exclude prices of capital, for which investment is higher than output.[948] Applying these restrictions, it fortunately turns out that there is only one feasible solution to the quadratic equation for a broad range of values for $\mu^{M^O}$. Having determined $q_{F,M}$, the remaining endogenous variables can simply be found by consecutively substituting into equations (6.3.4b)-(6.3.4d).

[948] The price at which the investment-output ratio equals unity is given by: $q_{\max} = a^e\gamma + 1$, as we have stated in Appendix D.1.2.

## D.3 Supplementary Table

Table D.1: Model-Implied Moments of Output Growth and Inflation under Different Policies

| | $\mathbb{M}\left[\hat{Y}\right]$ | S.D. $\left[\hat{Y}\right]$ | $\mathbb{M}\left[\pi\right]$ | S.D. $\left[\pi\right]$ |
|---|---|---|---|---|
| No Policy | 0.79% | 0.91% | -0.71% | 3.48% |
| $\mu^{MO} = -2.5\%$ | 0.77% | 0.30% | -1.39% | 0.09% |
| $\mu^{MO} = 2.5\%$ | 0.79% | 0.79% | 0.07% | 6.30% |
| MMR1 | 0.74% | 0.53% | -1.31% | 2.14% |
| MMR2 | 0.79% | 0.94% | 0.18% | 6.36% |
| $\mathcal{M} = 0.95$ | 0.79% | 0.83% | 0.07% | 6.17% |
| NA | 0.80% | 0.29% | -1.18% | 0.44% |
| U.S. Data | 0.75% | 0.83% | 0.93% | 0.82% |

*Notes*: Moments are calculated from quarterly growth rates; model-implied moments are computed from time series generated by a Monte Carlo simulation of the baseline model. Each of the simulated series contains 9000000 realisations (360 days $\times$ 25000 years); U.S. data moments are calculated from the following time series: quarterly growth rate of real GDP: 1960Q1-2017Q3, source: BEA (2017a); quarterly growth rate of monetary aggregate M2: 1960Q1-2017Q3, source:FRB (2017b); quarterly inflation rate: 1960Q1-2017Q3, source: Organisation for Economic Co-operation and Development (2017).

# Appendix E

# Appendix to Chapter 7

In this section of the Appendix, we provide a proof of Proposition 12. The problem of finding the relevant differential equation for $\theta_t$ can be solved analogously to the procedure employed in the proof of Proposition 4. We just have to take into account that now there are two state variables instead of one. Hence, we postulate three general functions $q_t = q\left(\eta_t, \eta_t^{bo}\right)$, $p_t = p\left(\eta_t, \eta_t^{bo}\right)$, and $\theta_t = \theta\left(\eta_t, \eta_t^{bo}\right)$ in the state variables $\eta_t$ and $\eta_t^{bo}$. Inserting these functions into the two equations in (4.5.5) and rearranging gives

$$q\left(\eta_t, \eta_t^{bo}\right) = \theta\left(\eta_t, \eta_t^{bo}\right)\left[q\left(\eta_t, \eta_t^{bo}\right) + p\left(\eta_t, \eta_t^{bo}\right)\right] \tag{E.0.1}$$

and

$$p\left(\eta_t, \eta_t^{bo}\right) = \left[1 - \theta\left(\eta_t, \eta_t^{bo}\right)\right]\left[q\left(\eta_t, \eta_t^{bo}\right) + p\left(\eta_t, \eta_t^{bo}\right)\right]. \tag{E.0.2}$$

Applying CVF (3.1.20) to these two functions yields

$$\begin{aligned}\mu_t^q = &\left[\frac{\theta'_{\eta_t^{bo}}\left(\eta_t, \eta_t^{bo}\right)}{\theta\left(\eta_t, \eta_t^{bo}\right)} + \frac{q'_{\eta_t^{bo}}\left(\eta_t, \eta_t^{bo}\right) + p'_{\eta_t^{bo}}\left(\eta_t, \eta_t^{bo}\right)}{q\left(\eta_t, \eta_t^{bo}\right) + p\left(\eta_t, \eta_t^{bo}\right)}\right]\mu_t^{\eta^{bo}}\eta_t^{bo} \\ &+ \left[\frac{\theta'_{\eta_t}\left(\eta_t, \eta_t^{bo}\right)}{\theta\left(\eta_t, \eta_t^{bo}\right)} + \frac{q'_{\eta_t}\left(\eta_t, \eta_t^{bo}\right) + p'_{\eta_t}\left(\eta_t, \eta_t^{bo}\right)}{q\left(\eta_t, \eta_t^{bo}\right) + p\left(\eta_t, \eta_t^{bo}\right)}\right]\mu_t^{\eta}\eta_t\end{aligned} \tag{E.0.3}$$

and

$$\begin{aligned}\mu_t^p = &\left[-\frac{\theta'_{\eta_t^{bo}}\left(\eta_t, \eta_t^{bo}\right)}{1 - \theta\left(\eta_t, \eta_t^{bo}\right)} + \frac{q'_{\eta_t^{bo}}\left(\eta_t, \eta_t^{bo}\right) + p'_{\eta_t^{bo}}\left(\eta_t, \eta_t^{bo}\right)}{q\left(\eta_t, \eta_t^{bo}\right) + p\left(\eta_t, \eta_t^{bo}\right)}\right]\mu_t^{\eta^{bo}}\eta_t^{bo} \\ &+ \left[-\frac{\theta'_{\eta_t}\left(\eta_t, \eta_t^{bo}\right)}{1 - \theta\left(\eta_t, \eta_t^{bo}\right)} + \frac{q'_{\eta_t^{bo}}\left(\eta_t, \eta_t^{bo}\right) + p'_{\eta_t^{bo}}\left(\eta_t, \eta_t^{bo}\right)}{q\left(\eta_t, \eta_t^{bo}\right) + p\left(\eta_t, \eta_t^{bo}\right)}\right]\mu_t^{\eta}\eta_t.\end{aligned} \tag{E.0.4}$$

Subtracting (E.0.4) from (E.0.3) leads to PDE

$$\mu_t^q - \mu_t^p = \left[\frac{\theta'_{\eta_t^{bo}}\left(\eta_t, \eta_t^{bo}\right)}{\theta\left(\eta_t, \eta_t^{bo}\right)} + \frac{\theta'_{\eta_t^{bo}}\left(\eta_t, \eta_t^{bo}\right)}{1-\theta\left(\eta_t, \eta_t^{bo}\right)}\right] \mu_t^{\eta^{bo}} \eta_t^{bo} + \left[\frac{\theta'_{\eta_t}\left(\eta_t, \eta_t^{bo}\right)}{\theta\left(\eta_t, \eta_t^{bo}\right)} + \frac{\theta'_{\eta_t}\left(\eta_t, \eta_t^{bo}\right)}{1-\theta\left(\eta_t, \eta_t^{bo}\right)}\right] \mu_t^{\eta^e} \eta_t, \tag{E.0.5}$$

which can be reformulated to (7.2.17) in a straightforward fashion. □

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Hiermit versichere ich, dass ich die Arbeit selbstständig verfasst und keine anderen als die angegebenen Hilfsmittel und Quellen benutzt habe.

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