Resolution and Realisation Functors

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Abstract

For algebraic triangulated categories, we construct bounded realisation functors of t-structures and resolution functors of bounded w-structures. Under suitable assumptions, we also construct the corresponding unbounded versions of these functors. We show that our resolution and realisation functors yield an adjunction between the algebraic triangulated category and the derived category of the heart of the t-structure for adjacent t- and w-structures.
For silting and cosilting complexes, we obtain a connection between the derived categories of the heart of the associated t-structure and the module category that generalises the connection on the abelian level described by the silting and cosilting theorems for two-term complexes. We also deduce a Morita theorem for derived categories of suitable abelian categories which include left-complete Grothendieck categories.

Zusammenfassung

Wir konstruieren beschränkte Realisierungsfunktoren von t-Strukturen und Auflösungsfunktoren von beschränkten w-Strukturen für algebraische triangulierte Kategorien. Unter geeigneten Voraussetzungen konstruieren wir auch die zugehörigen unbeschränkten Varianten dieser Funktoren. Wir zeigen, dass unsere Auflösungs- und Realisierungsfunktoren eine Adjunktion zwischen der algebraischen triangulierten Kategorie und der derivierten Kategorie des Herzes der t-Struktur liefern, wenn die t- und w-Strukturen benachbart sind.
Für Silting- und Kosiltingkomplexe erhalten wir eine Beziehung zwischen den derivierten Kategorien des Herzes der assoziierten t-Struktur und der Modulkategorie, die die Beziehung auf der abelschen Ebene verallgemeinert, welche von den Silting- und Kosiltingtheoremen für Komplexe mit nur zwei nichttrivialen Einträgen beschrieben wird. Wir leiten zudem ein Moritatheorem für derivierte Kategorien von geeigneten abelschen Kategorien her, welche die linksvollständigen Grothendieckkategorien einschließen.

Introduction

Motivation

To compare similar categories of quiver representations, J. Bernstein, I. M. Gel′fand and V. A. Ponomarev [BGP73] introduced reflection functors and were able to prove Gabriel's theorem [Gab72] using them. For example, consider the following two quivers Q and R.

$$Q\colon \circ \longleftarrow \circ \longleftarrow \circ \qquad\qquad R\colon \circ \longleftarrow \circ \longrightarrow \circ$$

We refer to [ASS06, examples 4.10, 5.15.(a), 4.8.(a)] and [Kel07, example 2.10] for details on our examples. For simplicity, we always work over the complex numbers $\mathbf{C}$ in this section. The categories of quiver representations mod-$\mathbf{C}Q$ and mod-$\mathbf{C}R$ are not equivalent since their Auslander-Reiten quivers AR_Q and AR_R are different:

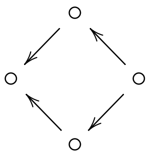
$$\mathrm{AR}_Q\colon \begin{tikzcd} & & \bullet \arrow[dr] & & \\ & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[dr] & \\ \bullet \arrow[ur] & & \bullet \arrow[ur] & & \bullet \end{tikzcd} \qquad \mathrm{AR}_R\colon \begin{tikzcd} \circledast \arrow[dr] & & \circledast \arrow[dr] & \\ & \circledast \arrow[ur] \arrow[dr] & & \circledast \\ \circledast \arrow[ur] & & \circledast \arrow[ur] & \end{tikzcd}$$

But one can compare the two categories with reflection functors. It was realised by D. Happel [Hap87] that the (bounded) derived categories $\mathrm{D}^{\mathrm{b}}(\text{mod-}\mathbf{C}Q)$ and $\mathrm{D}^{\mathrm{b}}(\text{mod-}\mathbf{C}R)$ are equivalent. We obtain graphical evidence of this fact by looking at the following Auslander-Reiten quiver of $\mathrm{D}^{\mathrm{b}}(\text{mod-}\mathbf{C}Q)$ which is the repetition quiver of AR_Q .

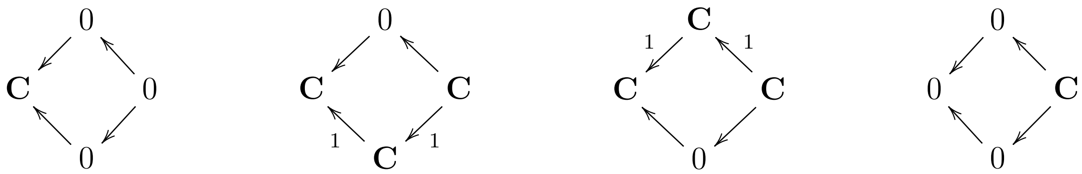
$$\begin{tikzcd}[column sep=small] & & \circ \arrow[dr] & & \circ \arrow[dr] & & \bullet \arrow[dr] & & \circ \arrow[dr] & & \circ \arrow[dr] & & \circledast \arrow[dr] & & \circledast \arrow[dr] & & \circ \arrow[dr] & & \\ \cdots & \circ \arrow[ur] \arrow[dr] & & \circ \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] \arrow[dr] & & \circ \arrow[ur] \arrow[dr] & & \circ \arrow[ur] \arrow[dr] & & \circledast \arrow[ur] \arrow[dr] & & \circledast \arrow[ur] \arrow[dr] & & \circ & \cdots \\ & & \circ \arrow[ur] & & \bullet \arrow[ur] & & \bullet \arrow[ur] & & \bullet \arrow[ur] & & \circ \arrow[ur] & & \circledast \arrow[ur] & & \circledast \arrow[ur] & & \circ \arrow[ur] & & \end{tikzcd}$$

The filled circles again form AR_Q and we also find AR_R as the subquiver formed by the circles marked with an asterisk. From our point of view, the reason for this is that the category mod-$\mathbf{C}R$ is the heart of a t-structure on the derived category $\mathrm{D}^{\mathrm{b}}(\text{mod-}\mathbf{C}Q)$. Moreover, an equivalence $\mathrm{D}^{\mathrm{b}}(\text{mod-}\mathbf{C}R) \to \mathrm{D}^{\mathrm{b}}(\text{mod-}\mathbf{C}Q)$ is given by a (bounded) realisation functor of this t-structure. In this way, such a realisation functor can be seen as a generalised version of a reflection functor.

In fact, D. Happel's result works in the more general context of tilting modules. Consider the following tame quiver S

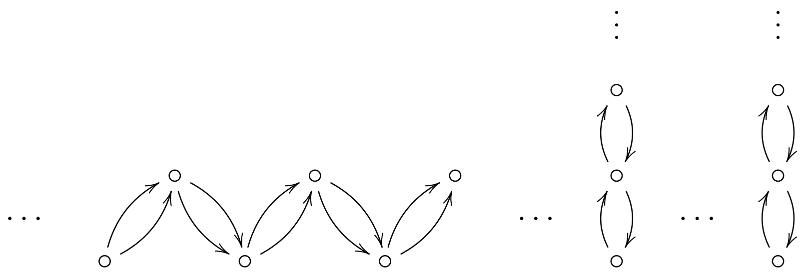
$S:$ 

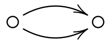
and the direct sum T of the following four indecomposable representations of S.

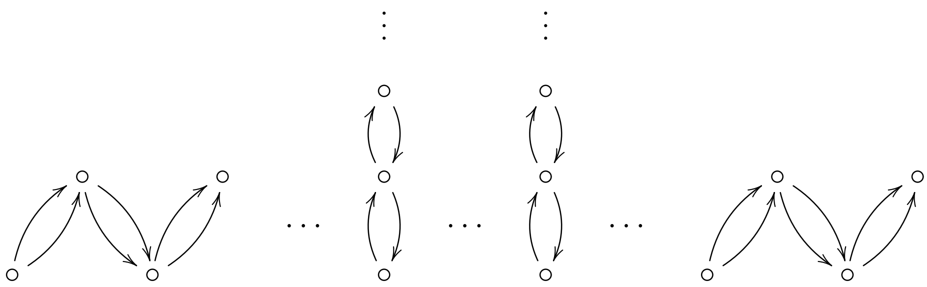


Then T is a tilting module such that its endomorphism algebra $\mathrm{End}(T)$ is representation-finite. According to D. Happel, the derived categories $\mathrm{D}^{\mathrm{b}}(\text{mod-}\mathbf{C}S)$ and $\mathrm{D}^{\mathrm{b}}(\text{mod-End}(T))$ are equivalent. Again, a realisation functor of a t-structure provides such an equivalence. So we obtain a derived equivalence between algebras of different representation type.

Realisation functors can also be used to obtain connections between algebraic geometry and representation theory. Consider the category of coherent sheaves on the projective line coh-$\mathbb{P}^1$. Its indecomposable objects and irreducible morphisms can be visualised in the following quiver.



The left part is formed by the line bundles and the right part by the skyscraper sheaves. The Auslander-Reiten quiver of the derived category $\mathrm{D}^{\mathrm{b}}(\text{coh-}\mathbb{P}^1)$ is the repetition quiver of the one above. It contains the following Auslander-Reiten quiver AR_K as a subquiver, where K is the Kronecker quiver .

$\mathrm{AR}_K:$ 

The reason is once more that the category of quiver representations mod-$\mathbf{C}K$ is the heart of a t-structure on $\mathrm{D}^{\mathrm{b}}(\text{coh-}\mathbb{P}^1)$. Again, an equivalence $\mathrm{D}^{\mathrm{b}}(\text{mod-}\mathbf{C}K) \to \mathrm{D}^{\mathrm{b}}(\text{coh-}\mathbb{P}^1)$ is given by a realisation functor. One main goal of this thesis is the construction of realisation functors not only for bounded derived categories but also for unbounded derived categories.

Overview

Triangulated categories arise in various branches of mathematics. They were introduced by J.-L. Verdier to provide an appropriate framework for derived categories and derived functors [Ver96, introduction 4]. For a suitable abelian category, its derived category is equivalent to the stable category of a Frobenius category. Since most triangulated categories occurring in algebra are of this form, they are called algebraic triangulated categories [Kel07, section 8]. In [BBD82], t-structures on triangulated categories were introduced. Each t-structure on a triangulated category $\mathscr{D}$ has a heart $\mathscr{H}$ which is an abelian subcategory. Since every abelian category is a full subcategory of its derived category $\mathrm{D}(\mathscr{H})$, it is natural to ask how the two inclusions $\mathscr{H} \hookrightarrow \mathscr{D}$ and $\mathscr{H} \hookrightarrow \mathrm{D}(\mathscr{H})$ are related. So one would like to have a realisation functor $\mathrm{D}(\mathscr{H}) \to \mathscr{D}$ that prolongs the inclusion $\mathscr{H} \hookrightarrow \mathrm{D}(\mathscr{H})$ to provide an answer to this question.
A. A. Beĭlinson, J. Bernstein and P. Deligne [BBD82] constructed a bounded realisation functor $\mathrm{D}^{\mathrm{b}}(\mathscr{H}) \to \mathscr{D}$ under the assumption that $\mathscr{D}$ is a triangulated subcategory of a left-bounded derived category of an abelian category with enough injectives. To axiomatise the properties of $\mathscr{D}$ that are needed to construct such a bounded realisation functor, A. A. Beĭlinson introduced the concept of a filtered triangulated category in [Beĭ87, Appendix]. Assuming an additional axiom, E. Cabezuelo Fernández and O. M. Schnürer [PV18, Appendix A] showed that this functor is exact. Different approaches to bounded realisation functors for algebraic triangulated categories can be found in [KV87], [Kel90] and to half-bounded versions for stable infinity-categories in [Lur11, section 1.3.3].
Bounded realisation functors are useful in many applications including Koszul duality [AR13], simple-minded collections [KY14], HRS-tilting [CHZ18], hereditary triangulated categories [CR18] and Cohen-Macaulay modules [Han19]. L. Alonso Tarrío, A. Jeremías López, M. J. Souto Salorio [ATJLSS03, remark following theorem 6.6] and C. Psaroudakis, J. Vitória [PV18, section 5] remarked that an unbounded version would be desirable to simplify their arguments. One result of this thesis is the construction of such a realisation functor $\mathrm{D}(\mathscr{H}) \to \mathscr{D}$, where $\mathscr{H}$ is the heart of a non-degenerate t-structure on a suitable algebraic triangulated category, e.g. on the derived category of a Grothendieck category.
In independent work, S. Virili [Vir18] uses derivators to extend bounded realisation functors to unbounded derived categories and obtains similar results concerning (co)tilting equivalences which we will discuss below. To achieve this, he uses homotopy (co)limits and t-structure truncations, so he has to impose some conditions on the heart of the t-structure, namely that it has enough injectives and that it is (Ab.4*)-k for some $k \in \mathbf{N}$ [Vir18, pp. 53-54 and definition

2.7]. While we also use (co)limits, our construction does not involve t-structure truncations and we only have to assume that the t-structure is non-degenerate. In fact, it relies on the decomposition of complexes with respect to the standard w-structure on $\mathrm{K}(\mathscr{H})$.

W-structures were independently introduced by M. V. Bondarko [Bon10] (who called them weight structures) and D. Pauksztello [Pau08] (who called them co-t-structures). Each w-structure on a triangulated category has a core $\mathscr{C}$ (also called heart or co-heart) which is an additive subcategory. Since every additive category is a full subcategory of its homotopy category, it is again natural to ask how the two inclusions $\mathscr{C} \hookrightarrow \mathscr{D}$ and $\mathscr{C} \hookrightarrow \mathrm{K}(\mathscr{C})$ are related. So one would like to have a functor $\mathscr{D} \to \mathrm{K}(\mathscr{C})$ that prolongs the inclusion $\mathscr{C} \hookrightarrow \mathrm{K}(\mathscr{C})$ to provide an answer to this question. M. V. Bondarko constructed a weight complex functor $\mathscr{D} \to \mathrm{K}_{\mathfrak{w}}(\mathscr{C})$, where $\mathrm{K}_{\mathfrak{w}}(\mathscr{C})$ is a factor category of $\mathrm{K}(\mathscr{C})$ which might no longer be triangulated. He showed that in some bounded cases, this functor lifts to a functor $\mathscr{D} \to \mathrm{K}(\mathscr{C})$ which he called strong weight complex functor. For filtered triangulated categories, he attributed the arguments to A. A. Beĭlinson [Bon10, section 8.4]. However, O. M. Schnürer [Sch11] had to impose the additional axiom mentioned above again to construct this lift for filtered triangulated categories.
We use the term resolution functor for the desired functor $\mathscr{D} \to \mathrm{K}(\mathscr{C})$ since for the standard w-structures on derived categories, it maps an object to its resolution.
An approach to resolution functors of bounded or compactly generated w-structures on stable infinity-categories can be found in [Sos17].
It seems that in general w-structures allow the construction of functors out of the given triangulated category, whereas t-structures can be used to map into the given triangulated category. Therefore the direction of resolution functors $\mathscr{D} \to \mathrm{K}(\mathscr{C})$ is opposite to the direction of realisation functors $\mathrm{D}(\mathscr{H}) \to \mathscr{D}$. The results of J. Lurie [Lur11, theorem 1.3.3.2] for t-structures and V. Sosnilo [Sos17, proof of corollary 3.5] for w-structures provide further evidence of this claim.
For algebraic triangulated categories, we construct resolution functors of arbitrary w-structures under the assumption that the underlying Frobenius category has countable products and coproducts of bijectives. We study the interplay between resolution and realisation functors for adjacent w- and t-structures and obtain an adjunction $\mathscr{D} \rightleftarrows \mathrm{D}(\mathscr{H})$ under suitable hypotheses.

Tilting theory is a fundamental tool in representation theory. The starting point was the tilting theorem due to S. Brenner and M. C. R. Butler [BB80] which relates the module categories of a ring R and of the endomorphism ring of a tilting module over R in terms of an equivalence of torsion pairs. E. Cline, B. Parshall, L. Scott [CPS86] and D. Happel [Hap87] showed that the involved functors induce an equivalence between the corresponding bounded derived categories. J. Rickard introduced tilting complexes as a generalisation of tilting modules and obtained a Morita theorem for derived categories of module categories in [Ric89]. As remarked in [ATJLSS03, theorem 6.6 and the following remark], bounded realisation functors can be used to recover Rickard's equivalence. C. Psaroudakis and J. Vitória [PV18] systematically used bounded realisation functors in tilting theory and obtained results on (co)tilting equivalences, standard forms of derived equivalences and on recollements.

In this thesis, we pursue two lines of research in this area. On the one hand, we obtain a Morita theorem for derived categories of suitable abelian categories which include left-complete Grothendieck categories. To this end, we introduce the notion of w-cotilting objects in triangulated categories and use our resolution and realisation functors. Moreover, we show that resolution functors associated to w-cotilting objects in derived categories are usually full and faithful, e.g. in the ones of Grothendieck categories.

On the other hand, we study silting and cosilting complexes in derived categories of module categories. They are examples of w-silting resp. w-cosilting objects by [AHMV16, theorem 4.6] and [MV18, theorem 3.13] for which one can not expect that our functors yield equivalences [PV18, proposition 5.1]. However, there are silting and cosilting theorems for two-term complexes relating the involved abelian categories in terms of equivalences of torsion pairs [BM17] [BZ16] [Pop17a].

For a cosilting complex S with associated heart $\mathscr{H}_S$ over a ring R, we show that the realisation functor $\mathrm{D}(\mathscr{H}_S) \to \mathrm{D}(\mathrm{Mod}\text{-}R)$ has a right-adjoint which is provided by the resolution functor. This adjunction yields a cosilting theorem on the abelian level for two-term complexes. A dual version for silting complexes is also true.

Results

Basic constructions

We introduce the concept of strict Frobenius categories for which the functorial Frobenius categories of [Kün07, definition A.5] are predecessors. The category of complexes $\mathrm{C}(\mathscr{A})$ with entries in an additive category $\mathscr{A}$ is the archetypical example of a strict Frobenius category. Following an idea of B. Keller [Kel94, section 4.3], we show that every algebraic triangulated category is equivalent to the stable category of a strict Frobenius category. So all of our results involving strict Frobenius categories can be applied to algebraic triangulated categories.

For a strict Frobenius category $\mathscr{F}$, we introduce the diagram category $\mathrm{FO}(\mathscr{F})$ of filtered objects, whose objects are diagrams of the following form, subject to exactness conditions.

$$\begin{tikzcd}
 & & & & X_\omega \arrow[dlll, "\shortmid" marking] \arrow[dll, "\shortmid" marking] & & & & \\
\cdots & X_{k+1|} \arrow[l, "\shortmid" marking] & X_{k|} \arrow[l, "\shortmid" marking] & \cdots \arrow[l, "\shortmid" marking] & & \cdots & X_{|k+1} \arrow[l, "\bullet" marking] \arrow[ull, twoheadrightarrow, "\bullet" marking] & X_{|k} \arrow[l, "\bullet" marking] \arrow[ulll, twoheadrightarrow, "\bullet" marking] & \cdots \arrow[l, "\bullet" marking]
\end{tikzcd}$$

The projection functor $\mathrm{P}_{\omega,\mathscr{F}} \colon \mathrm{FO}(\mathscr{F}) \to \mathscr{F}$ maps the diagram above to X_ω . We also introduce the category $\nabla(\mathscr{F})$ of ∇-diagrams, whose objects are diagrams of the following form, subject to exactness conditions.

$$
\begin{tikzcd}
\ddots & \vdots & \vdots & \vdots & \iddots \\
0 & X_{k+2/k+1} \arrow[l, "+" marking] \arrow[u, "\bullet" marking] & X_{k+2/k} \arrow[l, "+" marking] \arrow[u, "\bullet" marking] & X_{k+2/k-1} \arrow[l, "+" marking] \arrow[u, "\bullet" marking] & \cdots \arrow[l, "+" marking] \\
& 0 \arrow[u, "\bullet" marking] & X_{k+1/k} \arrow[l, "+" marking] \arrow[u, "\bullet" marking] & X_{k+1/k-1} \arrow[l, "+" marking] \arrow[u, "\bullet" marking] & \cdots \arrow[l, "+" marking] \\
& & 0 \arrow[u, "\bullet" marking] & X_{k/k-1} \arrow[l, "+" marking] \arrow[u, "\bullet" marking] & \cdots \arrow[l, "+" marking] \\
& & & 0 \arrow[u, "\bullet" marking] & \ddots
\end{tikzcd}
$$

We construct the filtered cokernel functor $\Xi_{\mathscr{F}} \colon \mathrm{FO}(\mathscr{F}) \to \nabla(\mathscr{F})$ by choosing cokernels of pure monomorphisms. For a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$, where $\underline{\mathscr{F}}$ is the stable category of $\mathscr{F}$, we have the full subcategory $\nabla_{\mathscr{S}}(\mathscr{F}) \subseteq \nabla(\mathscr{F})$ and the delta functor $\Delta_{\mathscr{S},\mathscr{F}} \colon \nabla_{\mathscr{S}}(\mathscr{F}) \to \mathrm{C}(\mathscr{S})$ which maps the diagram above to the complex

$$\cdots \longrightarrow X_{k+2/k+1}^{[-k-2]} \longrightarrow X_{k+1/k}^{[-k-1]} \longrightarrow X_{k/k-1}^{[-k]} \longrightarrow \cdots,$$

its differentials being given by connecting morphisms of triangles.

To construct our realisation and resolution functors, we study these functors $\mathrm{P}_{\omega,\mathscr{F}}$, $\Xi_{\mathscr{F}}$, $\Delta_{\mathscr{S},\mathscr{F}}$ and the ones induced by them on various factor categories and subcategories.

The bounded case

Suppose given a full triangulated subcategory $\mathscr{D} \subseteq \underline{\mathscr{F}}$.

For a bounded w-structure $\mathscr{W}$ on $\mathscr{D}$ with core $\mathscr{C}$, we obtain an equivalence $\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \colon \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}) \to \mathscr{D}$ and choose a quasi-inverse $\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \colon \mathscr{D} \to \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})$. The bounded resolution functor $\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \colon \mathscr{D} \to \mathrm{K}^{\mathrm{b}}(\mathscr{C})$ is defined as the following composite.

$$\mathscr{D} \xrightarrow{\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}} \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}) \xrightarrow{\underline{\underline{\Xi}}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}}} \underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{C}}(\mathscr{F}) \xrightarrow{\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}}} \mathrm{K}^{\mathrm{b}}(\mathscr{C})$$

Theorem A (4.3.40, 4.3.43)**.** Suppose given a strict Frobenius category $\mathscr{F}$, a full triangulated subcategory $\mathscr{D} \subseteq \underline{\mathscr{F}}$ and a bounded w-structure $\mathscr{W}$ on $\mathscr{D}$ with core $\mathscr{C}$. The bounded resolution functor $\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \colon \mathscr{D} \to \mathrm{K}^{\mathrm{b}}(\mathscr{C})$ is w-exact and the functors $\mathrm{Inc}^{\mathscr{D}}_{\mathscr{C}} \cdot \mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ and $\mathrm{I}_{\mathrm{K}^{\mathrm{b}},\mathscr{C}}$ from $\mathscr{C}$ to $\mathrm{K}^{\mathrm{b}}(\mathscr{C})$ are isomorphic. $\diamondsuit$

For a t-structure $\mathscr{T}$ on $\mathscr{D}$ with heart $\mathscr{H}$, we obtain equivalences

$$\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \colon \underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F}) \to \mathrm{K}^{\mathrm{b}}(\mathscr{H}), \qquad\qquad \underline{\Xi}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \colon \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F}) \to \underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})$$

and choose quasi-inverses

$$\underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \colon \mathrm{K}^{\mathrm{b}}(\mathscr{H}) \to \underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F}), \qquad\qquad \underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \colon \underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F}) \to \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F}).$$

We define the bounded realisation functor $\mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}} \colon \mathrm{D}^{\mathrm{b}}(\mathscr{H}) \to \mathscr{D}$ to be the unique exact functor such that the following diagram is commutative.

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{b}}(\mathscr{H}) \arrow[r, "\underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}"] \arrow[d, "\mathrm{L}^{\mathrm{b}}_{\mathscr{H}}"'] & \underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F}) \arrow[r, "\underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}"] & \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F}) \arrow[r, "\underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{H},\mathscr{F}}|^{\mathscr{D}}"] & \mathscr{D} \\
\mathrm{D}^{\mathrm{b}}(\mathscr{H}) \arrow[urrr, bend right, "\mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}}"'] & & &
\end{tikzcd}$$

Theorem B (4.4.25, 4.4.28)**.** Suppose given a strict Frobenius category $\mathscr{F}$, a full triangulated subcategory $\mathscr{D} \subseteq \underline{\mathscr{F}}$ and a t-structure $\mathscr{T}$ on $\mathscr{D}$ with heart $\mathscr{H}$. The bounded realisation functor $\mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}} \colon \mathrm{D}^{\mathrm{b}}(\mathscr{H}) \to \mathscr{D}$ is t-exact and the functors $\mathrm{I}_{\mathrm{D}^{\mathrm{b}},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}}$ and $\mathrm{Inc}^{\mathscr{D}}_{\mathscr{H}}$ from $\mathscr{H}$ to $\mathscr{D}$ are isomorphic. $\diamondsuit$

For adjacent t- and w-structures, we obtain the following adjunction.

Theorem C (4.5.4)**.** Suppose given a strict Frobenius category $\mathscr{F}$ and a full triangulated subcategory $\mathscr{D} \subseteq \underline{\mathscr{F}}$. Suppose given a bounded w-structure $\mathscr{W}$ on $\mathscr{D}$ and a t-structure $\mathscr{T}$ on $\mathscr{D}$. Suppose that that $\mathscr{T}_{[0} = \mathscr{W}_{[0}$, i.e. that $\mathscr{T}$ is left-adjacent to $\mathscr{W}$. We denote the heart of $\mathscr{T}$ by $\mathscr{H} = \mathscr{T}_{[0,0]}$, the homology functor of $\mathscr{T}$ by $\mathrm{H} = \mathrm{H}_{\mathscr{T}}$ and the core of $\mathscr{W}$ by $\mathscr{C} = \mathscr{W}_{[0,0]}$.

(a) The functor $\mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}}$ is left-adjoint to $\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}^{\mathrm{b}}_{\mathscr{H}}$.
The functors $\big(\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}^{\mathrm{b}}_{\mathscr{H}} \cdot \mathbb{H}^{\mathrm{b}}_{\mathscr{H}}\big)|_{\mathscr{H}}$ and $(\mathrm{I}_{\mathrm{D}^{\mathrm{b}},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}})|^{\mathscr{H}}$ from $\mathscr{H}$ to $\mathscr{H}$ are isomorphic to $1_{\mathscr{H}}$.

(b) Suppose that $\mathscr{C} \subseteq \mathscr{H}$.
Then the functors $\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$, $\mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}})$ and $\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}^{\mathrm{b}}_{\mathscr{H}}$ are full and faithful.

$$\begin{tikzcd}
 & \mathrm{K}^{\mathrm{b}}(\mathscr{C}) \arrow[r, "\mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}})"] & \mathrm{K}^{\mathrm{b}}(\mathscr{H}) \arrow[dr, "\mathrm{L}^{\mathrm{b}}_{\mathscr{H}}"] & \\
\mathscr{D} \arrow[ur, "\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}"] & & & \mathrm{D}^{\mathrm{b}}(\mathscr{H}) \arrow[lll, "\mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}}"'] \arrow[d, bend left, "\mathbb{H}^{\mathrm{b}}_{\mathscr{H}}"] \\
\mathscr{H} \arrow[u, "\mathrm{Inc}^{\mathscr{D}}_{\mathscr{H}}"] & & & \mathscr{H} \arrow[u, bend left, "\mathrm{I}_{\mathrm{D}^{\mathrm{b}},\mathscr{H}}"]
\end{tikzcd}$$

$\diamondsuit$

The unbounded case

Suppose that $\mathscr{F}$ and $\mathscr{F}^{\mathrm{op}}$ have countable products of bijectives. For a w-structure $\mathscr{W}$ on $\mathscr{D}$ with core $\mathscr{C}$, we obtain an equivalence $\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}} \colon \underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F}) \to \mathscr{D}$ and choose a quasi-inverse $\mathrm{W}_{\mathscr{W},\mathscr{F}} \colon \mathscr{D} \to \underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})$. The resolution functor $\mathrm{Res}_{\mathscr{W},\mathscr{F}} \colon \mathscr{D} \to \mathrm{K}(\mathscr{C})$ is defined as the following composite.

$$\mathscr{D} \xrightarrow{\mathrm{W}_{\mathscr{W},\mathscr{F}}} \underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F}) \xrightarrow{\mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{C}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})}} \underline{\underline{\mathrm{FO}}}_{\mathscr{C}}(\mathscr{F}) \xrightarrow{\Xi_{\mathscr{C},\mathscr{F}}} \underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F}) \xrightarrow{\underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}}} \mathrm{K}(\mathscr{C})$$

Theorem D (4.3.41, 4.3.44)**.** Suppose given a strict Frobenius category $\mathscr{F}$ such that $\mathscr{F}$ and $\mathscr{F}^{\mathrm{op}}$ have countable products of bijectives. Suppose given a full triangulated subcategory $\mathscr{D} \subseteq \underline{\mathscr{F}}$ and a w-structure $\mathscr{W}$ on $\mathscr{D}$ with core $\mathscr{C}$. The resolution functor $\mathrm{Res}_{\mathscr{W},\mathscr{F}} \colon \mathscr{D} \to \mathrm{K}(\mathscr{C})$ is w-exact and the functors $\mathrm{Inc}^{\mathscr{D}}_{\mathscr{C}} \cdot \mathrm{Res}_{\mathscr{W},\mathscr{F}}$ and $\mathrm{I}_{\mathrm{K},\mathscr{C}}$ from $\mathscr{C}$ to $\mathrm{K}(\mathscr{C})$ are isomorphic. $\Diamond$

Suppose that $\mathscr{F}$ has epilimits and monocolimits and that $\mathscr{D}$ is closed under countable products in $\underline{\mathscr{F}}$. Suppose given a functor $A \colon \underline{\mathscr{F}} \to \mathscr{D}$ that is left-adjoint to $\mathrm{Inc}^{\underline{\mathscr{F}}}_{\mathscr{D}}$. For a non-degenerate t-structure on $\mathscr{T}$ on $\mathscr{D}$ with heart $\mathscr{H}$, we obtain equivalences

$$\underline{\underline{\Delta}}_{\mathscr{H},\mathscr{F}} \colon \underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F}) \to \mathrm{K}(\mathscr{H}), \qquad \Xi^{\nabla}_{\mathscr{H},\mathscr{F}} \colon \underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{H}}(\mathscr{F}) \to \underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F})$$

and choose quasi-inverses

$$\underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}} \colon \mathrm{K}(\mathscr{H}) \to \underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F}), \qquad \underline{\mathrm{Lim}}_{\mathscr{H},\mathscr{F}} \colon \underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F}) \to \underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{H}}(\mathscr{F}).$$

We define the realisation functor $\mathrm{Real}_{A,\mathscr{T},\mathscr{F}} \colon \mathrm{D}(\mathscr{H}) \to \mathscr{D}$ to be the unique exact functor such that the following diagram is commutative.

$$\begin{tikzcd}
\mathrm{K}(\mathscr{H}) \arrow[r, "\underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}}"] \arrow[d, "\mathrm{L}_{\mathscr{H}}"'] & \underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F}) \arrow[r, "\underline{\mathrm{Lim}}_{\mathscr{H},\mathscr{F}}"] & \underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{H}}(\mathscr{F}) \arrow[r, "\underline{\mathrm{P}}^{\nabla}_{\omega,\mathscr{H},\mathscr{F}}"] & \underline{\mathscr{F}} \arrow[r, "A"] & \mathscr{D} \\
\mathrm{D}(\mathscr{H}) \arrow[urrrr, bend right, "\mathrm{Real}_{A,\mathscr{T},\mathscr{F}}"'] & & & &
\end{tikzcd}$$

Theorem E (4.4.26, 4.4.29)**.** Suppose given a strict Frobenius category $\mathscr{F}$ with epilimits and monocolimits. Suppose given a full triangulated subcategory $\mathscr{D} \subseteq \underline{\mathscr{F}}$ that is closed under countable products in $\underline{\mathscr{F}}$. Suppose given a functor $A \colon \underline{\mathscr{F}} \to \mathscr{D}$ that is left-adjoint to $\mathrm{Inc}^{\underline{\mathscr{F}}}_{\mathscr{D}}$. Suppose given a non-degenerate t-structure $\mathscr{T}$ on $\mathscr{D}$ with heart $\mathscr{H}$. The realisation functor $\mathrm{Real}_{A,\mathscr{T},\mathscr{F}} \colon \mathrm{D}(\mathscr{H}) \to \mathscr{D}$ is t-exact and the functors $\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}_{A,\mathscr{T},\mathscr{F}}$ and $\mathrm{Inc}^{\mathscr{D}}_{\mathscr{H}}$ from $\mathscr{H}$ to $\mathscr{D}$ are isomorphic. $\Diamond$

For adjacent t- and w-structures, we obtain the following adjunction.

Theorem F (4.5.1)**.** Suppose given a strict Frobenius category $\mathscr{F}$ with epilimits and monocolimits. Suppose given a full triangulated subcategory $\mathscr{D} \subseteq \underline{\mathscr{F}}$ that is closed under countable products in $\underline{\mathscr{F}}$. Suppose given a functor $A\colon \underline{\mathscr{F}} \to \mathscr{D}$ that is left-adjoint to $\mathrm{Inc}^{\underline{\mathscr{F}}}_{\mathscr{D}}$. Suppose given a t-structure $\mathscr{T} = (\mathscr{T}_{0]}, \mathscr{T}_{[0})$ and a w-structure $\mathscr{W} = (\mathscr{W}_{[0}, \mathscr{W}_{0]})$ on $\mathscr{D}$. Suppose that $\mathscr{T}$ is non-degenerate and that $\mathscr{T}_{[0} = \mathscr{W}_{[0}$, i.e. that $\mathscr{T}$ is left-adjacent to $\mathscr{W}$. We abbreviate $\mathscr{H} = \mathscr{T}_{[0,0]}$, $\mathrm{H} = \mathrm{H}_{\mathscr{T}}$ and $\mathscr{C} = \mathscr{W}_{[0,0]}$.

(a) The functor $\mathrm{Real}_{A,\mathscr{T},\mathscr{F}}$ is left-adjoint to $\mathrm{Res}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}}$.
The functors $\big(\mathrm{Res}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}} \cdot \mathbb{H}_{\mathscr{H}}\big)|_{\mathscr{H}}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}_{A,\mathscr{T},\mathscr{F}})|^{\mathscr{H}}$ from $\mathscr{H}$ to $\mathscr{H}$ are isomorphic to $1_{\mathscr{H}}$.

(b) Suppose that there exists an $n \in \mathbf{Z}$ such that $\mathscr{W}_{0]} \subseteq \mathscr{T}_{n]}$. Suppose that $\mathscr{C} \subseteq \mathscr{H}$.
Then the functors $\mathrm{Res}_{\mathscr{W},\mathscr{F}}$, $\mathrm{K}(\mathrm{H}|_{\mathscr{C}})$ and $\mathrm{Res}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}}$ are full and faithful.

$$\begin{tikzcd}
 & \mathrm{K}(\mathscr{C}) \arrow[r, "\mathrm{K}(\mathrm{H}|_{\mathscr{C}})"] & \mathrm{K}(\mathscr{H}) \arrow[dr, "\mathrm{L}_{\mathscr{H}}"] & \\
\mathscr{D} \arrow[ur, "\mathrm{Res}_{\mathscr{W},\mathscr{F}}"] & & & \mathrm{D}(\mathscr{H}) \arrow[lll, "\mathrm{Real}_{A,\mathscr{T},\mathscr{F}}"'] \arrow[d, bend left, "\mathbb{H}_{\mathscr{H}}"] \\
\mathscr{H} \arrow[u, "\mathrm{Inc}^{\mathscr{D}}_{\mathscr{H}}"] & & & \mathscr{H} \arrow[u, bend left, "\mathrm{I}_{\mathrm{D},\mathscr{H}}"]
\end{tikzcd}$$

$\Diamond$

Application to derived categories

Suppose given an abelian category $\mathscr{A}$ with countable products and countable coproducts. Suppose that $\mathscr{A}$ has enough K-injectives. Then we have an equivalence $\mathrm{K}^{\mathrm{inj}}(\mathscr{A}) \xrightarrow{\sim} \mathrm{D}(\mathscr{A})$, where $\mathrm{K}^{\mathrm{inj}}(\mathscr{A})$ is the full triangulated subcategory of $\mathrm{K}(\mathscr{A})$ whose objects are the K-injective complexes. Now $\mathrm{K}(\mathscr{A})$ is the stable category of the strict Frobenius category $\mathrm{C}(\mathscr{A})$ which has epilimits and monocolimits and the inclusion $\mathrm{K}^{\mathrm{inj}}(\mathscr{A}) \hookrightarrow \mathrm{K}(\mathscr{A})$ has a left adjoint. So for a w-structure $\mathscr{W}$ on $\mathrm{D}(\mathscr{A})$ with core $\mathscr{C}$, we obtain a resolution functor $\mathrm{Res}^{\mathrm{inj}}_{\mathscr{W},\mathscr{A}}\colon \mathrm{D}(\mathscr{A}) \to \mathrm{K}(\mathscr{C})$ and for a non-degenerate t-structure $\mathscr{T}$ on $\mathrm{D}(\mathscr{A})$ with heart $\mathscr{H}$, we get a realisation functor $\mathrm{Real}^{\mathrm{inj}}_{\mathscr{T},\mathscr{A}}\colon \mathrm{D}(\mathscr{H}) \to \mathrm{D}(\mathscr{A})$. We have the following version of theorem F.

Theorem G (4.6.2)**.**
Suppose given an abelian category $\mathscr{A}$ with countable products and countable coproducts. Suppose that $\mathscr{A}$ has enough K-injectives. Suppose given a t-structure $\mathscr{T} = (\mathscr{T}_{0]}, \mathscr{T}_{[0})$ and a w-structure $\mathscr{W} = (\mathscr{W}_{[0}, \mathscr{W}_{0]})$ on $\mathrm{D}(\mathscr{A})$. Suppose that $\mathscr{T}$ is non-degenerate and that $\mathscr{T}_{[0} = \mathscr{W}_{[0}$. We abbreviate $\mathscr{H} = \mathscr{T}_{[0,0]}$, $\mathrm{H} = \mathrm{H}_{\mathscr{T}}$ and $\mathscr{C} = \mathscr{W}_{[0,0]}$.

(a) The functor $\mathrm{Real}^{\mathrm{inj}}_{\mathscr{T},\mathscr{A}}$ is left-adjoint to $\mathrm{Res}^{\mathrm{inj}}_{\mathscr{W},\mathscr{A}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}}$.
The functors $\big(\mathrm{Res}^{\mathrm{inj}}_{\mathscr{W},\mathscr{A}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}} \cdot \mathbb{H}_{\mathscr{H}}\big)|_{\mathscr{H}}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{inj}}_{\mathscr{T},\mathscr{A}})|^{\mathscr{H}}$ from $\mathscr{H}$ to $\mathscr{H}$ are isomorphic to $1_{\mathscr{H}}$.

(b) Suppose that there exists an $n \in \mathbf{Z}$ such that $\mathscr{W}_{0]} \subseteq \mathscr{T}_{n]}$. Suppose that $\mathscr{C} \subseteq \mathscr{H}$.
Then the functors $\mathrm{Res}^{\mathrm{inj}}_{\mathscr{W},\mathscr{A}}$, $\mathrm{K}(\mathrm{H}|_{\mathscr{C}})$ and $\mathrm{Res}^{\mathrm{inj}}_{\mathscr{W},\mathscr{A}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}}$ are full and faithful.

$$\begin{tikzcd}
 & \mathrm{K}(\mathscr{C}) \arrow[r, "\mathrm{K}(\mathrm{H}|_{\mathscr{C}})"] & \mathrm{K}(\mathscr{H}) \arrow[dr, "\mathrm{L}_{\mathscr{H}}"] & \\
\mathrm{D}(\mathscr{A}) \arrow[ur, "\mathrm{Res}^{\mathrm{inj}}_{\mathscr{W},\mathscr{A}}"] & & & \mathrm{D}(\mathscr{H}) \arrow[lll, "\mathrm{Real}^{\mathrm{inj}}_{\mathscr{T},\mathscr{A}}"'] \arrow[d, bend left, "\mathbb{H}_{\mathscr{H}}"] \\
\mathscr{H} \arrow[u, "\mathrm{Inc}^{\mathrm{D}(\mathscr{A})}_{\mathscr{H}}"] & & & \mathscr{H} \arrow[u, bend left, "\mathrm{I}_{\mathrm{D},\mathscr{H}}"]
\end{tikzcd}$$
◇

We introduce the notion of w-silting and w-cosilting objects in triangulated categories. They are silting and cosilting objects in the sense of [PV18, definition 4.1] whose associated t-structures have adjacent w-structures. For a w-cosilting object S with heart $\mathscr{H}_S$ and core $\mathscr{C}_S$ in a derived category $\mathrm{D}(\mathscr{A})$ as above, we say that $\mathrm{Real}^{\mathrm{cos}}_{S,\mathscr{A}} = \mathrm{Real}^{\mathrm{inj}}_{\mathscr{T}^S,\mathscr{A}} \colon \mathrm{D}(\mathscr{H}_S) \to \mathrm{D}(\mathscr{A})$ is the cosilting realisation functor and that $\mathrm{Res}^{\mathrm{cos}}_{S,\mathscr{A}} = \mathrm{Res}^{\mathrm{inj}}_{\mathscr{W}^S,\mathscr{A}} \cdot \mathrm{K}(\mathrm{H}_S|_{\mathscr{C}_S}) \cdot \mathrm{L}_{\mathscr{H}_S} \colon \mathrm{D}(\mathscr{A}) \to \mathrm{D}(\mathscr{H}_S)$ is the cosilting resolution functor.

Theorem H (5.1.19)**.** Suppose given an abelian category $\mathscr{A}$ with products, coproducts and enough K-injectives. Suppose given a w-cosilting object S in $\mathrm{D}(\mathscr{A})$.

(a) The cosilting realisation functor $\mathrm{Real}^{\mathrm{cos}}_{S,\mathscr{A}} \colon \mathrm{D}(\mathscr{H}_S) \to \mathrm{D}(\mathscr{A})$ is left-adjoint to the cosilting resolution functor $\mathrm{Res}^{\mathrm{cos}}_{S,\mathscr{A}} \colon \mathrm{D}(\mathscr{A}) \to \mathrm{D}(\mathscr{H}_S)$. Both functors are exact.
The functors $(\mathrm{Res}^{\mathrm{cos}}_{S,\mathscr{A}} \cdot \mathbb{H}_{\mathscr{H}_S})|_{\mathscr{H}_S}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}_S} \cdot \mathrm{Real}^{\mathrm{cos}}_{S,\mathscr{A}})|^{\mathscr{H}_S}$ from $\mathscr{H}_S$ to $\mathscr{H}_S$ are isomorphic to $1_{\mathscr{H}_S}$.

(b) Suppose that S is a w-cotilting object in $\mathrm{D}(\mathscr{A})$. Then the cosilting resolution functor $\mathrm{Res}^{\mathrm{cos}}_{S,\mathscr{A}}$ is full and faithful. If $S\mathrm{I}_{\mathrm{D},\mathscr{H}_S}$ cogenerates $\mathrm{D}(\mathscr{H}_S)$, then $\mathrm{Res}^{\mathrm{cos}}_{S,\mathscr{A}}$ and $\mathrm{Real}^{\mathrm{cos}}_{S,\mathscr{A}}$ are mutually quasi-inverse equivalences. ◇

The second part of the previous theorem H allows us to deduce the following Morita theorem for derived categories of injectively complete abelian categories.

Theorem I (5.2.4)**.** Suppose given injectively complete abelian categories $\mathscr{A}$ and $\mathscr{B}$. The following two statements are equivalent.

(a) There is an exact equivalence $F \colon \mathrm{D}(\mathscr{B}) \to \mathrm{D}(\mathscr{A})$.

(b) There is a w-cotilting object S in $\mathrm{D}(\mathscr{A})$ and an equivalence $G \colon \mathscr{B} \to \mathscr{H}_S$. ◇

Silting and cosilting complexes in derived categories of module categories are examples of w-silting resp. w-cosilting objects by [AHMV16, theorem 4.6] and [MV18, theorem 3.13]. So theorem H and its dual yield the following two results.

Theorem J (5.3.5)**.** Suppose given a ring R and a silting complex $S \in \mathrm{Ob}(\mathrm{C}^{\mathrm{b}}(\text{Proj-}R))$. The silting realisation functor $\mathrm{Real}^{\mathrm{s}}_{S,\text{Mod-}R} \colon \mathrm{D}(\mathscr{H}_S) \to \mathrm{D}(\text{Mod-}R)$ is right-adjoint to the silting resolution functor $\mathrm{Res}^{\mathrm{s}}_{S,\text{Mod-}R} \colon \mathrm{D}(\text{Mod-}R) \to \mathrm{D}(\mathscr{H}_S)$. The functors $(\mathrm{Res}^{\mathrm{s}}_{S,\text{Mod-}R} \cdot \mathbb{H}_{\mathscr{H}_S})|_{\mathscr{H}_S}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}_S} \cdot \mathrm{Real}^{\mathrm{s}}_{S,\text{Mod-}R})|^{\mathscr{H}_S}$ from $\mathscr{H}_S$ to $\mathscr{H}_S$ are isomorphic to $1_{\mathscr{H}_S}$. ◇

Theorem K (5.3.6)**.** Suppose given a ring R and a cosilting complex $S \in \mathrm{Ob}(\mathrm{C}^{\mathrm{b}}(\text{Inj-}R))$. The cosilting realisation functor $\mathrm{Real}^{\mathrm{cos}}_{S,\text{Mod-}R}\colon \mathrm{D}(\mathscr{H}_S) \to \mathrm{D}(\text{Mod-}R)$ is left-adjoint to the cosilting resolution functor $\mathrm{Res}^{\mathrm{cos}}_{S,\text{Mod-}R}\colon \mathrm{D}(\text{Mod-}R) \to \mathrm{D}(\mathscr{H}_S)$. The functors $(\mathrm{Res}^{\mathrm{cos}}_{S,\mathscr{A}} \cdot \mathbb{H}_{\mathscr{H}_S})|_{\mathscr{H}_S}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}_S} \cdot \mathrm{Real}^{\mathrm{cos}}_{S,\mathscr{A}})|^{\mathscr{H}_S}$ from $\mathscr{H}_S$ to $\mathscr{H}_S$ are isomorphic to $1_{\mathscr{H}_S}$. $\diamondsuit$

We conclude by stating our version of a silting theorem and our version of a cosilting theorem.

Theorem L (5.3.8)**.** Suppose given a ring R and a silting complex $S \in \mathrm{Ob}(\mathrm{C}^{[1,0]}(\text{Proj-}R))$. Let $\mathscr{T}$ denote the associated t-structure in $\mathrm{D}(\text{Mod-}R)$. We abbreviate $\mathscr{H} = \mathscr{T}_{[0,0]}$. Let $\mathscr{U}$ denote the standard t-structure in $\mathrm{D}(\text{Mod-}R)$. Note that one may identify $\mathscr{U}_{[0,0]}$ with Mod-R via $\mathrm{I}_{\mathrm{D},\text{Mod-}R}$. Let $\mathcal{Y} = \mathscr{H} \cap \mathscr{U}^{[1]}_{[0,0]}$ and $\mathcal{X} = \mathscr{H} \cap \mathscr{U}_{[0,0]}$. Then $(\mathcal{Y}, \mathcal{X})$ is a torsion pair in $\mathscr{H}$ and $(\mathcal{X}, \mathcal{Y}^{[-1]})$ is a torsion pair in $\mathscr{U}_{[0,0]}$.

The functors $(\mathrm{Res}^{\mathrm{s}}_{S,\text{Mod-}R} \cdot \mathbb{H}_{\mathscr{H}})|_{\mathscr{H}}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{s}}_{S,\text{Mod-}R})|^{\mathscr{H}}$ from $\mathscr{H}$ to $\mathscr{H}$ are isomorphic to $1_{\mathscr{H}}$. In particular, we have the following.

- The functors $(\mathrm{Res}^{\mathrm{s}}_{S,\text{Mod-}R} \cdot \mathbb{H}_{\mathscr{H}})|^{\mathcal{X}}_{\mathcal{X}}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{s}}_{S,\text{Mod-}R})|^{\mathcal{X}}_{\mathcal{X}}$ are mutually quasi-inverse equivalences.

- The functors $(\Sigma_{\mathrm{D},\text{Mod-}R} \cdot \mathrm{Res}^{\mathrm{s}}_{S,\text{Mod-}R} \cdot \mathbb{H}_{\mathscr{H}})|^{\mathcal{Y}}_{\mathcal{Y}^{[-1]}}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{s}}_{S,\text{Mod-}R} \cdot \Sigma^{-1}_{\mathrm{D},\text{Mod-}R})|^{\mathcal{Y}^{[-1]}}_{\mathcal{Y}}$ are mutually quasi-inverse equivalences.

$$\begin{tikzcd}
\mathrm{D}(\text{Mod-}R) \arrow[r, shift left, "\mathrm{Res}^{\mathrm{s}}_{S,\text{Mod-}R}"] & \mathrm{D}(\mathscr{H}) \arrow[l, shift left, "\mathrm{Real}^{\mathrm{s}}_{S,\text{Mod-}R}"] \arrow[d, bend left, "\mathbb{H}_{\mathscr{H}}"] \\
\text{Mod-}R \arrow[u, "\mathrm{I}_{\mathrm{D},\text{Mod-}R}"] & \mathscr{H} \arrow[u, bend left, "\mathrm{I}_{\mathrm{D},\mathscr{H}}"]
\end{tikzcd}$$
$\diamondsuit$

Theorem M (5.3.9)**.** Suppose given a ring R and a cosilting complex $S \in \mathrm{Ob}(\mathrm{C}^{[0,-1]}(\text{Inj-}R))$. Let $\mathscr{U}$ denote the associated t-structure in $\mathrm{D}(\text{Mod-}R)$. We abbreviate $\mathscr{H} = \mathscr{U}_{[0,0]}$. Let $\mathscr{T}$ denote the standard t-structure in $\mathrm{D}(\text{Mod-}R)$. Note that one may identify $\mathscr{T}_{[0,0]}$ with Mod-R via $\mathrm{I}_{\mathrm{D},\text{Mod-}R}$. Let $\mathcal{Y} = \mathscr{T}_{[0,0]} \cap \mathscr{H}^{[1]}$ and $\mathcal{X} = \mathscr{T}_{[0,0]} \cap \mathscr{H}$. Then $(\mathcal{Y}, \mathcal{X})$ is a torsion pair in $\mathscr{T}_{[0,0]}$ and $(\mathcal{X}, \mathcal{Y}^{[-1]})$ is a torsion pair in $\mathscr{H}$.

The functors $(\mathrm{Res}^{\mathrm{cos}}_{S,\text{Mod-}R} \cdot \mathbb{H}_{\mathscr{H}})|_{\mathscr{H}}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{cos}}_{S,\text{Mod-}R})|^{\mathscr{H}}$ from $\mathscr{H}$ to $\mathscr{H}$ are isomorphic to $1_{\mathscr{H}}$. In particular, we have the following.

- The functors $(\mathrm{Res}^{\mathrm{cos}}_{S,\text{Mod-}R} \cdot \mathbb{H}_{\mathscr{H}})|^{\mathcal{X}}_{\mathcal{X}}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{cos}}_{S,\text{Mod-}R})|^{\mathcal{X}}_{\mathcal{X}}$ are mutually quasi-inverse equivalences.

- The functors $(\Sigma^{-1}_{\mathrm{D},\text{Mod-}R} \cdot \mathrm{Res}^{\mathrm{cos}}_{S,\text{Mod-}R} \cdot \mathbb{H}_{\mathscr{H}})|^{\mathcal{Y}^{[-1]}}_{\mathcal{Y}}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{cos}}_{S,\text{Mod-}R} \cdot \Sigma_{\mathrm{D},\text{Mod-}R})|^{\mathcal{Y}}_{\mathcal{Y}^{[-1]}}$ are mutually quasi-inverse equivalences.

$$\begin{tikzcd} \mathrm{D}(\text{Mod-}R) \arrow[r, shift left, "\mathrm{Res}^{\cos}_{S,\text{Mod-}R}"] & \mathrm{D}(\mathscr{H}) \arrow[l, shift left, "\mathrm{Real}^{\cos}_{S,\text{Mod-}R}"] \arrow[d, bend left, "\mathbb{H}_{\mathscr{H}}"] \\ \text{Mod-}R \arrow[u, "\mathrm{I}_{\mathrm{D},\text{Mod-}R}"] & \mathscr{H} \arrow[u, bend left, "\mathrm{I}_{\mathrm{D},\mathscr{H}}"] \end{tikzcd}$$ ◇

Comparison with others' results

A. A. Beĭlinson, J. Bernstein and P. Deligne introduced the concept of a t-structure with heart $\mathscr{H}$ on a triangulated category $\mathscr{D}$ and gave the first construction of a bounded realisation functor $\mathrm{D}^{\mathrm{b}}(\mathscr{H}) \to \mathscr{D}$ in [BBD82]. They assumed that $\mathscr{D}$ is a triangulated subcategory of $\mathrm{D}^{+}(\mathscr{A})$, where $\mathscr{A}$ is an abelian category with enough injectives. In particular, $\mathscr{D}$ is an algebraic triangulated category and thus our theorem B contains their result as a special case. Later, A. A. Beĭlinson [Beĭ87, Appendix] introduced filtered triangulated categories to axiomatise the properties that are needed to construct such a bounded realisation functor adapting the arguments from [BBD82]. However, it is not clear if an additional axiom is needed to show that the functor is exact, cf. [PV18, Appendix A].

A different approach to bounded realisation functors for algebraic triangulated categories is due to B. Keller and D. Vossieck [KV87]. They constructed the functor on the level of the homotopy category under the same assumptions as we do in definition 4.1.20. The equivalence that they used to map out of the homotopy category of the heart is studied in detail in the recent preprint [CC19]. Our theorem B recovers the result of B. Keller and D. Vossieck on the existence of bounded realisation functors.

C. Psaroudakis and J. Vitória [PV18] use A. A. Beĭlinson's approach to construct bounded realisation functors via filtered triangulated categories. Their Morita theorem [PV18, theorem A] is a predecessor of our theorem I and its dual, theorem 5.2.8. They show that a restrictable equivalence $\mathrm{D}(\mathscr{B}) \to \mathrm{D}(\mathscr{A})$, i.e. one that restricts to an equivalence $\mathrm{D}^{\mathrm{b}}(\mathscr{B}) \to \mathrm{D}^{\mathrm{b}}(\mathscr{A})$, gives rise to a bounded tilting object, where $\mathscr{A}$ and $\mathscr{B}$ are suitable abelian categories. Given such a bounded tilting object, the associated bounded realisation functor yields an equivalence $\mathrm{D}^{\mathrm{b}}(\mathscr{B}) \to \mathrm{D}^{\mathrm{b}}(\mathscr{A})$. But they can not show that in this case the bounded realisation functor can be extended to an equivalence between the unbounded derived categories and they conjecture that one can prove that using a different approach to realisation functors.

The main goal of S. Virili's paper [Vir18] is to do just that using derivators. He works with t-structures on triangulated categories at the base of a strong and stable derivator $\mathbb{D}$. Examples of such triangulated categories include derived categories of Grothendieck categories. At first, he constructs a morphism of prederivators $\mathrm{real}^{\mathrm{b}}\colon \mathbf{D}^{\mathrm{b}}_{\mathscr{H}} \to \mathbb{D}$ which is the analogon of a bounded realisation functor in the language of derivators. Our theorem B and his theorem 4.13 can be seen as essentially the same but in different settings. He then proceeds to extend $\mathrm{real}^{\mathrm{b}}$ in two steps to a morphism of prederivators $\mathrm{real}\colon \mathbf{D}_{\mathscr{H}} \to \mathbb{D}$, the analogon of a realisation functor for derivators. To map out of the prederivator $\mathbf{D}_{\mathscr{H}}$ enhancing the derived category of $\mathscr{H}$, he relies on certain model approximations for chain complexes which are only available if $\mathscr{H}$ has enough injectives and is (Ab.4*)-k for some $k \in \mathbf{N}$ or if the dual conditions are satisfied, cf. [Vir18, proposition 2.8]. These approximations involve t-structure truncations and thus it turns out that real is obtained from $\mathrm{real}^{\mathrm{b}}$ by taking homotopy limits and colimits of diagrams obtained by such truncations. While we also use limits and colimts in our construction of realisation functors in the unbounded setting, the use of t-structure truncations seems considerably different to our approach, where the decomposition of a complex with respect to the standard w-structure naturally extends the construction of the bounded version. Apart from the different assumptions on the underlying triangulated categories, our theorem E yields realisation functors for non-degenerate t-structures and S. Virili's theorem B yields morphisms of prederivators real: $\mathbf{D}_{\mathscr{H}} \to \mathbb{D}$ for t-structures whose heart has enough injectives and is (Ab.4*)-k for some $k \in \mathbf{N}$. His theorem also includes conditions for real to be full and faithful allowing him to prove that real is an equivalence for certain (co)tilting t-structures. We have chosen a different path to obtain equivalences in that we also consider w-structures and resolution functors. Let us now compare the resulting derived Morita theorems ([Vir18, theorem E] and our theorem I and its dual) for unbounded derived categories. A notable difference is that S. Virili only considers restrictable equivalences whereas we deal with arbitrary exact equivalences of derived categories. While our results for w-tilting and w-cotilting objects are dual to each other, he works with tilting objects in derived categories of Grothendieck categories and with cotilting objects in derived categories of Grothendieck categories that are (Ab.4*)-k for some $k \in \mathbf{N}$. Since the latter are left-complete Grothendieck categories by [Vir18, proposition 5.10], our theorem I can be seen as a extension of S. Virili's Morita theorem to not necessarily restrictable equivalences in the cotilting case, cf. example 5.2.3. However, in the tilting case we work with abelian categories with enough projectives and thus his result for arbitrary Grothendieck categories is more general than ours.

In [Bec18], H. Becker studies a realisation problem for algebraic triangulated categories of special origin. For a cofibrantly generated and hereditary abelian model structure $\mathcal{M}$ over a Grothendieck category $\mathscr{A}$, its homotopy category $\mathrm{ho}(\mathcal{M})$ is an algebraic triangulated category and he constructs a functor real: $\mathrm{D}(\mathscr{A}) \to \mathrm{ho}(\mathcal{M})$ using a left stabilisation functor of a butterfly of abelian model structures on $\mathrm{C}(\mathscr{A})$.

Outline

We summarise the required preliminaries in chapter 1.

In chapter 2, we introduce the concept of strict Frobenius categories. We show that every algebraic triangulated category is equivalent to the stable category of a strict Frobenius category in section 2.3.

We study the diagram categories that are involved in our constructions in chapter 3. In particular, this chapter deals with filtrations and cofiltrations, ∇-diagrams and filtered objects.

Chapter 4 contains all of our main constructions. The functors $\Delta_{\mathscr{S},\mathscr{F}}$ and $\Xi_{\mathscr{F}}$ are defined and studied in the sections 4.1 and 4.2. We construct resolution functors in section 4.3 and realisation functors in section 4.4. The theorems for adjacent t- and w-structures are located in section 4.5. The constructions are applied to derived categories in section 4.6.

The topic of chapter 5 is tilting and silting theory. We follow the treatment of [PV18, section 4]. Additionally, we introduce the notion of w-silting and w-cosilting objects in triangulated categories. Section 5.2 contains our Morita theorem for derived categories. We conclude by applying our results to silting and cosilting complexes in section 5.3.

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Conventions

1. We denote the poset of integers by $\mathbf{Z}$. We use symbols like $\mathbf{Z}_{\geq n}$, $\mathbf{Z}_{<n}$, $\mathbf{Z}_{[m,n]} = [m, n]$, where $m, n \in \mathbf{Z}$, for subsets of $\mathbf{Z}$ with the obvious associated meaning.

2. We treat categories as algebraic objects. There is no need to distinguish between small and large categories or between sets and classes in our constructions. In chapter 5, we want to talk about 'all' products/coproducts/modules, so we work inside a fixed Grothendieck universe there.

3. The opposite (or dual) category of a category $\mathscr{C}$ is denoted by $\mathscr{C}^{\mathrm{op}}$. Similarly, the opposite functor of a functor F is denoted by by F^{op} and the opposite of a transformation α is denoted by α^{op}.

4. For a category $\mathscr{C}$, we denote the set of objects in $\mathscr{C}$ by $\mathrm{Ob}(\mathscr{C})$ and the set of morphisms in $\mathscr{C}$ by $\mathrm{Mor}(\mathscr{C})$. For $X, Y \in \mathrm{Ob}(\mathscr{C})$, we write ${}_{\mathscr{C}}(X, Y)$ for the set of morphisms from X to Y in $\mathscr{C}$. For $X \in \mathrm{Ob}(\mathscr{C})$, we denote the identity morphism by $1_X \colon X \to X$ and abbreviate $1 = 1_X$ if unambiguous. If $f \in \mathrm{Mor}(\mathscr{C})$ is an isomorphism, we denote its inverse by f^{-1}.

5. The composition of morphisms is written naturally: $\big(\xrightarrow{f}\xrightarrow{g}\big) = \big(\xrightarrow{f\cdot g}\big) = \big(\xrightarrow{fg}\big)$

6. The composition of functors is written naturally: $\big(\xrightarrow{F}\xrightarrow{G}\big) = \big(\xrightarrow{F\cdot G}\big) = \big(\xrightarrow{FG}\big)$

7. Suppose given a functor $F \colon \mathscr{C} \to \mathscr{D}$. The image of $X \in \mathrm{Ob}(\mathscr{C})$ is denoted by XF. The image of $f \in \mathrm{Mor}(\mathscr{C})$ is denoted by fF.

If F is an isomorphism of categories, we denote its inverse by F^{-1}. If additionally $\mathscr{C} = \mathscr{D}$, we use the notation F^k for the corresponding composites, where $k \in \mathbf{Z}$.

8. For a category $\mathscr{A}$, we denote the identity functor of $\mathscr{A}$ by $1_{\mathscr{A}} \colon \mathscr{A} \to \mathscr{A}$.

9. For a full subcategory $\mathscr{A}$ of a category $\mathscr{C}$, we denote the inclusion functor from $\mathscr{A}$ to $\mathscr{C}$ by $\mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}} \colon \mathscr{A} \to \mathscr{C}$.

10. A strictly full subcategory is a full subcategory that is closed under isomorphisms.

11. Suppose given categories, functors and transformations as follows:

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left=50, "F"] \arrow[r, "G" description] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, shift left=2.5, "\Downarrow\lambda"] \arrow[r, phantom, shift right=2.5, "\Downarrow\mu"] & \mathscr{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "I"'] \arrow[r, phantom, "\Downarrow\nu"] & \mathscr{C} \end{tikzcd} .$$

We denote the vertical composite of λ and μ by $\lambda \cdot \mu \colon F \to H$. For $X \in \mathrm{Ob}(\mathscr{A})$, we have $X(\lambda \cdot \mu) = X\lambda \cdot X\mu$.

We denote the horizontal composite of λ and ν by $\lambda \star \nu \colon FH \to GI$. For $X \in \mathrm{Ob}(\mathscr{A})$, we have $X(\lambda \star \nu) = X\lambda H \cdot XG\nu = XF\nu \cdot X\lambda I$.

We denote the identity transformation of F by 1_F . We sometimes abbreviate $F \star \nu = 1_F \star \nu$ and similarly $\lambda \star H = \lambda \star 1_H$.

Using this notation, we have $\lambda \star \nu = (\lambda \star H) \cdot (G \star \nu) = (F \star \nu) \cdot (\lambda \star I)$.

12. We have collected some facts about adjoint functors, equivalences and isomorphisms of categories in section 1.6.

13. Suppose given a functor $F \colon \mathscr{C} \to \mathscr{D}$. Suppose given a full subcategory $\mathscr{A}$ of $\mathscr{C}$ and a full subcategory $\mathscr{B}$ of $\mathscr{D}$ such that $XF \in \mathrm{Ob}(\mathscr{B})$ for $X \in \mathrm{Ob}(\mathscr{A})$. There exists a unique functor $F|_{\mathscr{A}}^{\mathscr{B}} \colon \mathscr{A} \to \mathscr{B}$ such that $F|_{\mathscr{A}}^{\mathscr{B}} \cdot \mathrm{Inc}_{\mathscr{B}}^{\mathscr{D}} = \mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}} \cdot F$. If $\mathscr{A} = \mathscr{C}$, we also write $F|^{\mathscr{B}} = F|_{\mathscr{A}}^{\mathscr{B}}$. If $\mathscr{B} = \mathscr{D}$, we also write $F|_{\mathscr{A}} = F|_{\mathscr{A}}^{\mathscr{B}}$.

14. Suppose given functors $F, G \colon \mathscr{C} \to \mathscr{D}$. Suppose given a full subcategory $\mathscr{A}$ of $\mathscr{C}$ and a full subcategory $\mathscr{B}$ of $\mathscr{D}$ such that $XF, XG \in \mathrm{Ob}(\mathscr{B})$ for $X \in \mathrm{Ob}(\mathscr{A})$. Suppose given a transformation $\lambda \colon F \to G$. There exists a unique transformation $\lambda|_{\mathscr{A}}^{\mathscr{B}} \colon F|_{\mathscr{A}}^{\mathscr{B}} \to G|_{\mathscr{A}}^{\mathscr{B}}$ such that $\lambda|_{\mathscr{A}}^{\mathscr{B}} \star \mathrm{Inc}_{\mathscr{B}}^{\mathscr{D}} = \mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}} \star \lambda$. If $\mathscr{A} = \mathscr{C}$, we also write $\lambda|^{\mathscr{B}} = \lambda|_{\mathscr{A}}^{\mathscr{B}}$. If $\mathscr{B} = \mathscr{D}$, we also write $\lambda|_{\mathscr{A}} = \lambda|_{\mathscr{A}}^{\mathscr{B}}$.

15. We call a commutative diagram of the following form a rectangle.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ U \arrow[r] & V \end{tikzcd}$$

A square is a rectangle that is both a pushout and a pullback.

16. Suppose given categories $\mathscr{C}$ and $\mathscr{A}$. The category of functors from $\mathscr{C}$ to $\mathscr{A}$ is denoted by $\mathscr{C}(\mathscr{A})$. We have collected some facts about functor categories in section 1.4.

17. We may consider posets as categories. Suppose given a poset P. For $a, b \in P$ with $a \leq b$, we denote the unique morphism from a to b in P by $a \to b$. For a category $\mathscr{A}$, we have the functor category $P(\mathscr{A})$ and use the following notation. For $X \in \mathrm{Ob}(P(\mathscr{A}))$ and $a \in P$, we write $X_a = aX$. For $X \in \mathrm{Ob}(P(\mathscr{A}))$ and $a \to b$ in P, we write $X_{a \to b} = (a \to b)X$. For $f \in \mathrm{Mor}(P(\mathscr{A}))$ and $a \in P$, we write $f_a = af$.
    Similarly, we use the following notation in the functor category $P^{\mathrm{op}}(\mathscr{A})$.
    For $X \in \mathrm{Ob}(P^{\mathrm{op}}(\mathscr{A}))$ and $a \in P$, we write $X_a = aX$. For $X \in \mathrm{Ob}(P^{\mathrm{op}}(\mathscr{A}))$ and $a \to b$ in P, we write $X_{b \to a} = (a \to b)^{\mathrm{op}} X$. For $f \in \mathrm{Mor}(P^{\mathrm{op}}(\mathscr{A}))$ and $a \in P$, we write $f_a = af$.

18. Shift functors in triangulated categories are usually denoted by variants of the letter Σ. For example, the shift functor in the derived category $\mathrm{D}(\mathscr{A})$ of an abelian category $\mathscr{A}$ is denoted by $\Sigma_{\mathrm{D},\mathscr{A}} \colon \mathrm{D}(\mathscr{A}) \to \mathrm{D}(\mathscr{A})$. We often use abbreviations like $X^{[1]} = X\Sigma$ for such a shift functor Σ and, if Σ is an isomorphism of categories, we also write $X^{[k]} = X\Sigma^k$ for $k \in \mathbf{Z}$.

    In some diagram categories, we also use translation functors that are usually denoted by variants of the letter T. For example, the translation functor in the category of ∇-diagrams $\nabla(\mathscr{F})$ in a strict Frobenius category $\mathscr{F}$ is denoted by $\mathrm{T}_{\nabla,\mathscr{F}} \colon \nabla(\mathscr{F}) \to \nabla(\mathscr{F})$. We often use abbreviations like $X_{[k]} = X\mathrm{T}^k$ for such a translation functor and $k \in \mathbf{Z}$.

19. Suppose given an exact category $\mathscr{A}$ and an object $X \in \mathrm{Ob}(\mathscr{A})$. We say that X is bijective in $\mathscr{A}$ if X is both projective and injective in $\mathscr{A}$. Cf. definition 1.3.4.

Chapter 1

Preliminaries

In this chapter, we collect basic and well-known results and present some common definitions in our notation. We do not claim originality.

1.1 Miscellaneous

1.1.1 Lemma (pasting lemma)**.** Suppose given a category $\mathscr{C}$. Suppose given the following commutative diagram in $\mathscr{C}$.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Z \arrow[r, "c"] \arrow[d, "g"] & U \arrow[d, "h"] \\ Y \arrow[r, "b"] & W \arrow[r, "d"] & V \end{tikzcd}$$

(a) Suppose that the left rectangle (a, f, g, b) is a pushout. The right rectangle (c, g, h, d) is a pushout if and only if the outer rectangle $(a \cdot c, f, h, b \cdot d)$ is a pushout.

(b) Suppose that the right rectangle (c, g, h, d) is a pullback. The left rectangle (a, f, g, b) is a pullback if and only if the outer rectangle $(a \cdot c, f, h, b \cdot d)$ is a pullback.

$\diamondsuit$

Proof. Ad (a). Suppose that the right rectangle (c, g, h, d) is a pushout. Suppose given $Y \xrightarrow{s} T$ and $U \xrightarrow{t} T$ in $\mathscr{A}$ such that $f \cdot s = a \cdot c \cdot t$. Since the left rectangle (a, f, g, b) is a pushout, there exists a unique morphism $W \xrightarrow{u} T$ in $\mathscr{A}$ such that $b \cdot u = s$ and $g \cdot u = c \cdot t$. Since the right rectangle (c, g, h, d) is a pushout, there exists a unique morphism $V \xrightarrow{v} T$ in $\mathscr{A}$ such that $d \cdot v = u$ and $h \cdot v = t$. So $b \cdot d \cdot v = b \cdot u = s$.

Suppose given $V \xrightarrow{w} T$ such that $b \cdot d \cdot w = s$ and $h \cdot w = t$. We have $d \cdot w = u$ since $b \cdot d \cdot w = s$ and $g \cdot d \cdot w = c \cdot h \cdot w = c \cdot t$. Thus $w = v$.

Conversely, suppose that the outer rectangle $(a \cdot c, f, h, b \cdot d)$ is a pushout. Suppose given $W \xrightarrow{s} T$ and $U \xrightarrow{t} T$ in $\mathscr{A}$ such that $g \cdot s = c \cdot t$. We have $f \cdot b \cdot s = a \cdot g \cdot s = a \cdot c \cdot t$. Since the outer rectangle $(a \cdot c, f, h, b \cdot d)$ is a pushout, there exists a unique morphism $W \xrightarrow{u} T$

in $\mathscr{A}$ such that $b \cdot d \cdot u = b \cdot s$ and $h \cdot u = t$. We have $d \cdot u = s$ since $b \cdot d \cdot u = b \cdot s$, $g \cdot d \cdot u = c \cdot h \cdot u = c \cdot t = g \cdot s$ and since the left rectangle (a, f, g, b) is a pushout.

Suppose given $V \xrightarrow{v} T$ in $\mathscr{A}$ such that $d \cdot v = s$ and $h \cdot v = t$. So $b \cdot d \cdot v = b \cdot s$ and $h \cdot v = t$. Thus $v = u$.

Ad (b). This is dual to (a). □

1.1.2 Lemma. Suppose given a category $\mathscr{C}$. Suppose given the following pushouts in $\mathscr{C}$.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "b"] & W \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[r, "a'"] \arrow[d, "f'"'] & Z' \arrow[d, "g'"] \\ Y' \arrow[r, "b'"] & W' \end{tikzcd}$$

Suppose given morphisms $X \xrightarrow{x} X'$, $Y \xrightarrow{y} Y'$ and $Z \xrightarrow{z} Z'$ in $\mathscr{C}$ such that $x \cdot f' = f \cdot y$ and $x \cdot a' = a \cdot z$. Then there exists a unique morphism $W \xrightarrow{w} W'$ in $\mathscr{C}$ such that $y \cdot b' = b \cdot w$ and $z \cdot g' = g \cdot w$. ◇

Proof. Consider the morphisms $Y \xrightarrow{y \cdot b'} W'$ and $Z \xrightarrow{z \cdot g'} W'$ in $\mathscr{C}$. We have

$$f \cdot y \cdot b' = x \cdot f' \cdot b' = x \cdot a' \cdot g' = a \cdot z \cdot g'.$$

So the result follows from the pushout property. □

1.1.3 Lemma. Suppose given a category $\mathscr{C}$. Suppose given the following pullbacks in $\mathscr{C}$.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "b"] & W \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[r, "a'"] \arrow[d, "f'"'] & Z' \arrow[d, "g'"] \\ Y' \arrow[r, "b'"] & W' \end{tikzcd}$$

Suppose given morphisms $W \xrightarrow{w} W'$, $Y \xrightarrow{y} Y'$ and $Z \xrightarrow{z} Z'$ in $\mathscr{C}$ such that $y \cdot b' = b \cdot w$ and $z \cdot g' = g \cdot w$. Then there exists a unique morphism $X \xrightarrow{x} X'$ in $\mathscr{C}$ such that $x \cdot f' = f \cdot y$ and $x \cdot a' = a \cdot z$. ◇

Proof. This is dual to the previous lemma 1.1.2. □

1.1.4 Lemma (elementary properties of functors and transformations)**.**

(a) Suppose given categories and functors as follows: $\mathscr{A} \xrightarrow{F} \mathscr{B} \xrightarrow{G} \mathscr{C}$.
We have $1_F \star 1_G = 1_{FG}$.

(b) Suppose given categories, functors and transformations as follows: $\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\lambda\Downarrow"] & \mathscr{B} \end{tikzcd}$. We have $1_{\mathscr{A}} \star \lambda = \lambda = \lambda \star 1_{\mathscr{B}}$ and $1_F \cdot \lambda = \lambda = \lambda \cdot 1_G$.

(c) Suppose given categories, functors and transformations as follows: $\begin{tikzcd} \mathscr{A} \arrow[r, bend left=75, "F"{name=F}] \arrow[r, bend left=25, "G"{name=G}] \arrow[r, bend right=25, "H"'{name=H}] \arrow[r, bend right=75, "I"'{name=I}] \arrow[from=F, to=G, Rightarrow, "\lambda"] \arrow[from=G, to=H, Rightarrow, "\mu"] \arrow[from=H, to=I, Rightarrow, "\nu"] & \mathscr{B} \end{tikzcd}$.

We have $(\lambda \cdot \mu) \cdot \nu = \lambda \cdot (\mu \cdot \nu)$.

(d) Suppose given categories, functors and transformations as follows:
$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"{name=F}] \arrow[r, bend right, "G"'{name=G}] \arrow[from=F, to=G, Rightarrow, "\lambda"] & \mathscr{B} \arrow[r, bend left, "H"{name=H}] \arrow[r, bend right, "I"'{name=I}] \arrow[from=H, to=I, Rightarrow, "\mu"] & \mathscr{C} \arrow[r, bend left, "J"{name=J}] \arrow[r, bend right, "K"'{name=K}] \arrow[from=J, to=K, Rightarrow, "\nu"] & \mathscr{D} \end{tikzcd}$. We have $(\lambda \star \mu) \star \nu = \lambda \star (\mu \star \nu)$.

(e) Suppose given categories, functors and transformations as follows:

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left=50, "F"{name=F}] \arrow[r, "G"{name=G, description}] \arrow[r, bend right=50, "H"'{name=H}] \arrow[from=F, to=G, Rightarrow, "\lambda"] \arrow[from=G, to=H, Rightarrow, "\mu"] & \mathscr{B} \arrow[r, bend left=50, "I"{name=I}] \arrow[r, "J"{name=J, description}] \arrow[r, bend right=50, "K"'{name=K}] \arrow[from=I, to=J, Rightarrow, "\nu"] \arrow[from=J, to=K, Rightarrow, "\xi"] & \mathscr{C} \end{tikzcd}$$

. We have $(\lambda \star \nu) \cdot (\mu \star \xi) = (\lambda \cdot \mu) \star (\nu \cdot \xi)$.

◊

Proof. Ad (a). Suppose given $X \in \mathrm{Ob}(\mathscr{A})$. We have $X(1_F \star 1_G) = XF1_G = XFG = X1_{FG}$.
Ad (b). Suppose given $X \in \mathrm{Ob}(\mathscr{A})$. We have $X(1_{\mathscr{A}} \star \lambda) = X1_{\mathscr{A}}\lambda = X\lambda = X\lambda 1_{\mathscr{B}} = X(\lambda \star 1_{\mathscr{B}})$ and $X(1_F \cdot \lambda) = X1_F \cdot X\lambda = 1_{XF} \cdot X\lambda = X\lambda = X\lambda \cdot 1_{XG} = X\lambda \cdot X1_G = X(\lambda \cdot 1_G)$.
Ad (c). Suppose given $X \in \mathrm{Ob}(\mathscr{A})$. We have
$X((\lambda \cdot \mu) \cdot \nu) = X(\lambda \cdot \mu) \cdot X\nu = X\lambda \cdot X\mu \cdot X\nu = X\lambda \cdot X(\mu \cdot \nu) = X(\lambda \cdot (\mu \cdot \nu))$.
Ad (d). Suppose given $X \in \mathrm{Ob}(\mathscr{A})$. We have

$$\begin{aligned} X((\lambda \star \mu) \star \nu) &= X(\lambda \star \mu)J \cdot XGI\nu = X\lambda HJ \cdot XG\mu J \cdot XGI\nu = X\lambda HJ \cdot XG(\mu \star \nu) \\ &= X(\lambda \star (\mu \star \nu)). \end{aligned}$$

Ad (e). Suppose given $X \in \mathrm{Ob}(\mathscr{A})$. Note that we have $XG\nu \cdot X\mu J = X\mu I \cdot XH\nu$. We have

$$\begin{aligned} X((\lambda \star \nu) \cdot (\mu \star \xi)) &= X(\lambda \star \nu) \cdot X(\mu \star \xi) = X\lambda I \cdot XG\nu \cdot X\nu J \cdot XH\xi \\ &= X\lambda I \cdot X\mu I \cdot XH\nu \cdot XH\xi = X(\lambda \cdot \mu)I \cdot XH(\nu \cdot \xi) \\ &= X((\lambda \cdot \mu) \star (\nu \cdot \xi)). \end{aligned}$$

□

1.1.5 Lemma (composition of isomorphic functors)**.** Suppose given categories and functors as follows: $\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "F"] \arrow[r, shift right, "H"'] & \mathscr{D} \arrow[r, shift left, "G"] \arrow[r, shift right, "I"'] & \mathscr{E} \end{tikzcd}$. Suppose that F is isomorphic to H in $\mathscr{C}(\mathscr{D})$ and that G is isomorphic to I in $\mathscr{D}(\mathscr{E})$. Then FG is isomorphic to HI in $\mathscr{C}(\mathscr{E})$. ◊

Proof. Choose isotransformations $F \xrightarrow{\lambda} H$ and $G \xrightarrow{\mu} I$. Then $FG \xrightarrow{\lambda \star \mu} HF$ is an isotransformation as well. □

1.1.6 Lemma (restriction of isomorphic functors)**.** Suppose given functors $F, G\colon \mathscr{C} \to \mathscr{D}$ and a full subcategory $\mathscr{A} \subseteq \mathscr{D}$. Suppose that F and G are isomorphic in $\mathscr{C}(\mathscr{D})$. Suppose that $XF, XG \in \mathrm{Ob}(\mathscr{A})$ for $X \in \mathrm{Ob}(\mathscr{C})$. Then the functors $F|^{\mathscr{A}}$ and $G|^{\mathscr{A}}$ are isomorphic in $\mathscr{C}(\mathscr{A})$ as well. ◊

Proof. Choose an isotransformation $F \xrightarrow{\lambda} G$. Then $\lambda|^{\mathscr{A}} \colon F|^{\mathscr{A}} \to G|^{\mathscr{A}}$ is an isotransformation as well. □

1.1.7 Lemma. Suppose given functors $\mathscr{A} \xrightarrow{I} \mathscr{C}$ and $\mathscr{A} \xrightarrow{J} \mathscr{D}$. Suppose given mutually quasi-inverse equivalences $\mathscr{C} \xrightarrow{F} \mathscr{D}$ and $\mathscr{D} \xrightarrow{G} \mathscr{C}$. If $I \cdot F$ is isomorphic to J in $\mathscr{A}(\mathscr{D})$, then $J \cdot G$ is isomorphic to I in $\mathscr{A}(\mathscr{C})$.

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "F"] & \mathscr{D} \arrow[l, shift left, "G"] \\ \mathscr{A} \arrow[u, "I"] \arrow[ur, "J"'] & \end{tikzcd}$$

◇

Proof. By lemma 1.1.5, $I = I \cdot 1_{\mathscr{C}}$ and $I \cdot F \cdot G$ are isomorphic in $\mathscr{A}(\mathscr{C})$. Again by lemma 1.1.5, $I \cdot F \cdot G$ and $J \cdot G$ are isomorphic in $\mathscr{A}(\mathscr{C})$. We conclude that $J \cdot G$ is isomorphic to I in $\mathscr{A}(\mathscr{C})$. □

1.2 Additive categories

1.2.1 Definition. Suppose given an additive category $\mathscr{A}$. We define the full subcategory $\mathrm{Z}_{\mathscr{A}}$ of $\mathscr{A}$ by setting $\mathrm{Ob}(\mathrm{Z}_{\mathscr{A}}) = \{X \in \mathrm{Ob}(\mathscr{A}) \colon X \text{ is a zero object in } \mathscr{A}\}$ and call $\mathrm{Z}_{\mathscr{A}}$ the *subcategory of zero objects* in $\mathscr{A}$. We choose a zero object $0_{\mathscr{A}} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}})$.

For $X, Y \in \mathrm{Ob}(\mathscr{A})$, ${}_{\mathscr{A}}(X, Y)$ is an abelian group, written additively, and we denote the zero morphism in ${}_{\mathscr{A}}(X, Y)$ by $0_{X,Y}$. We abbreviate $0 = 0_{X,Y}$ if unambiguous.

For $X \xrightarrow{f} Y$ in $\mathscr{A}$ and $Z \in \mathrm{Ob}(\mathscr{A})$, the maps ${}_{\mathscr{A}}(Z, f) \colon {}_{\mathscr{A}}(Z, X) \to {}_{\mathscr{A}}(Z, Y) \colon g \mapsto g \cdot f$ and ${}_{\mathscr{A}}(f, Z) \colon {}_{\mathscr{A}}(Y, Z) \to {}_{\mathscr{A}}(X, Z) \colon g \mapsto f \cdot g$ are group homomorphisms.

We say that a diagram $X \underset{s}{\overset{i}{\rightleftarrows}} D \underset{p}{\overset{t}{\leftrightarrows}} Y$ in $\mathscr{A}$ is a *direct sum* of X and Y in $\mathscr{A}$ if $i \cdot s = 1$, $t \cdot p = 1$ and if $s \cdot i + p \cdot t = 1$. In this case, we also say that the sequence $X \xrightarrow{i} D \xrightarrow{p} Y$ is *split short exact* in $\mathscr{A}$, that i is a *split monomorphism*, that p is a *split epimorphism* and that X and Y are *summands* of D in $\mathscr{A}$.

For $X, Y \in \mathrm{Ob}(\mathscr{A})$, we choose a direct sum $X \underset{\binom{1}{0}}{\overset{(1\ 0)}{\rightleftarrows}} X \oplus Y \underset{\binom{0}{1}}{\overset{(0\ 1)}{\leftrightarrows}} Y$ in $\mathscr{A}$ and use the usual matrix notation for morphisms involving such direct sums. ◇

1.2.2 Definition. Suppose given an additive category $\mathscr{A}$. For $X, Y \in \mathrm{Ob}(\mathscr{A})$, we write ${}_{\mathscr{A}}(X, Y) = 0$ if ${}_{\mathscr{A}}(X, Y)$ has a single element and ${}_{\mathscr{A}}(X, Y) \neq 0$ otherwise.

Suppose given a full subcategory $\mathscr{B}$ and $Y \in \mathrm{Ob}(\mathscr{A})$. We write ${}_{\mathscr{A}}(\mathscr{B}, Y) = 0$ if ${}_{\mathscr{A}}(X, Y) = 0$ for $X \in \mathrm{Ob}(\mathscr{B})$ and ${}_{\mathscr{A}}(\mathscr{B}, Y) \neq 0$ otherwise. We write ${}_{\mathscr{A}}(Y, \mathscr{B}) = 0$ if ${}_{\mathscr{A}}(Y, X) = 0$ for $X \in \mathrm{Ob}(\mathscr{B})$ and ${}_{\mathscr{A}}(\mathscr{B}, Y) \neq 0$ otherwise.

Suppose given full subcategories $\mathscr{B}, \mathscr{C} \subseteq \mathscr{A}$. We write ${}_{\mathscr{A}}(\mathscr{B}, \mathscr{C}) = 0$ if ${}_{\mathscr{A}}(X, Y) = 0$ for $X \in \mathrm{Ob}(\mathscr{B})$ and $Y \in \mathrm{Ob}(\mathscr{C})$. Otherwise, we write ${}_{\mathscr{A}}(\mathscr{B}, \mathscr{C}) \neq 0$.

For a full subcategory $\mathscr{B} \subseteq \mathscr{A}$, we define the full subcategories $\mathscr{B}^\perp$ and ${}^\perp\mathscr{B}$ of $\mathscr{A}$ by setting

$$\mathrm{Ob}(\mathscr{B}^\perp) = \{Y \in \mathrm{Ob}(\mathscr{A})\colon {}_{\mathscr{A}}(\mathscr{B}, Y) = 0\} \text{ and } \mathrm{Ob}({}^\perp\mathscr{B}) = \{Y \in \mathrm{Ob}(\mathscr{A})\colon {}_{\mathscr{A}}(Y, \mathscr{B}) = 0\}.$$

$\diamondsuit$

1.2.3 Definition. Suppose given an additive category $\mathscr{A}$. A *full additive subcategory* $\mathscr{B}$ of $\mathscr{A}$ is a full subcategory $\mathscr{B} \subseteq \mathscr{A}$ such that there exists a zero object $Z \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}}) \cap \mathrm{Ob}(\mathscr{B})$ and such that for $X, Y \in \mathrm{Ob}(\mathscr{B})$, there exists a direct sum $X \underset{s}{\overset{i}{\rightleftarrows}} D \underset{p}{\overset{t}{\leftrightarrows}} Y$ in $\mathscr{A}$ with $D \in \mathrm{Ob}(\mathscr{B})$. $\diamondsuit$

1.2.4 Remark. Suppose given an additive category $\mathscr{A}$ and strictly full additive subcategories $\mathscr{B}, \mathscr{C} \subseteq \mathscr{A}$. Then $\mathscr{B} \cap \mathscr{C}$ is a strictly full additive subcategory of $\mathscr{A}$ as well. $\diamondsuit$

1.2.5 Remark. Suppose given additive categories $\mathscr{A}$, $\mathscr{B}$ and a functor $F\colon \mathscr{A} \to \mathscr{B}$.

(a) Suppose that F is an equivalence. Then F is additive.
More generally, adjoints of additive functors are additive since they preserve (co)limits.

(b) Suppose that F is additive. Suppose given a full additive subcategory $\mathscr{C}$ of $\mathscr{A}$ and a full additive subcategory $\mathscr{D}$ of $\mathscr{B}$ such that $XF \in \mathrm{Ob}(\mathscr{D})$ for $X \in \mathrm{Ob}(\mathscr{C})$. Then $F|_{\mathscr{C}}^{\mathscr{D}}$ is additive as well.

$\diamondsuit$

1.2.6 Lemma. Suppose given an additive category $\mathscr{A}$. Suppose given the following rectangle in $\mathscr{A}$.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "b"] & W \end{tikzcd}$$

(a) The rectangle is a pushout if and only if $\left(\begin{smallmatrix} g \\ b \end{smallmatrix}\right)$ is a cokernel of $\left(\begin{smallmatrix} a & -f \end{smallmatrix}\right)$.

(b) The rectangle is a pullback if and only if $\left(\begin{smallmatrix} a & -f \end{smallmatrix}\right)$ is a kernel of $\left(\begin{smallmatrix} g \\ b \end{smallmatrix}\right)$.

(c) The rectangle is a square if and only if $(\left(\begin{smallmatrix} a & -f \end{smallmatrix}\right), \left(\begin{smallmatrix} g \\ b \end{smallmatrix}\right))$ is a kernel-cokernel-pair. $\diamondsuit$

Proof. Ad (a). Note that we have $\left(\begin{smallmatrix} a & -f \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} g \\ b \end{smallmatrix}\right) = a \cdot g - f \cdot b = 0$.

Suppose that the rectangle is a pushout.

Suppose given $Z \oplus Y \xrightarrow{\left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)} T$ in $\mathscr{A}$ such that $\left(\begin{smallmatrix} a & -f \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right) = 0$, i.e. such that $a \cdot s = f \cdot t$. Since the rectangle is a pushout, there exists a unique morphism $W \xrightarrow{u} T$ in $\mathscr{A}$ such that $g \cdot u = s$ and $b \cdot u = t$, i.e. such that $\left(\begin{smallmatrix} g \\ b \end{smallmatrix}\right) \cdot u = \left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)$.

Conversely, suppose that $\left(\begin{smallmatrix} g \\ b \end{smallmatrix}\right)$ is a cokernel of $\left(\begin{smallmatrix} a & -f \end{smallmatrix}\right)$. Suppose given $Z \xrightarrow{s} T$ and $Y \xrightarrow{t} T$ in $\mathscr{A}$ such that $a \cdot s = f \cdot t$, i.e. such that $\left(\begin{smallmatrix} a & -f \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right) = 0$. Since $\left(\begin{smallmatrix} g \\ b \end{smallmatrix}\right)$ is a cokernel of $\left(\begin{smallmatrix} a & -f \end{smallmatrix}\right)$, there exists a unique morphism $W \xrightarrow{u} T$ in $\mathscr{A}$ such that $\left(\begin{smallmatrix} g \\ b \end{smallmatrix}\right) \cdot u = \left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)$, i.e. such that $g \cdot u = s$ and $b \cdot u = t$.

Ad (b). This is dual to (a).

Ad (c). This follows from (a) and (b). $\square$

1.2.7 Lemma. Suppose given an additive category $\mathscr{A}$. Suppose given the following pushout in $\mathscr{A}$.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "b"] & W \end{tikzcd}$$

(a) Suppose given a cokernel $W \xrightarrow{d} C$ of b. Then $g \cdot d$ is a cokernel of a.

(b) Suppose given a cokernel $Z \xrightarrow{c} C$ of a. Then there exists a unique morphism $W \xrightarrow{d} C$ in $\mathscr{A}$ such that $g \cdot d = c$ and $b \cdot d = 0$. Moreover, d is a cokernel of b.

(c) If a is an epimorphism, then b is an epimorphism. ◊

Proof. Ad (a). Consider the following commutative diagram in $\mathscr{A}$.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "b"] \arrow[d] & W \arrow[d, "d"] \\ 0 \arrow[r] & C \end{tikzcd}$$

The lower rectangle $(b, 0, d, 0)$ is a pushout since d is a cokernel of b. By the pasting lemma 1.1.1.(a), the outer rectangle $(a, 0, g \cdot d, 0)$ is a pushout as well. So $g \cdot d$ is a cokernel of a.

Ad (b). We have $a \cdot c = 0$. So there is a unique morphism $W \xrightarrow{d} C$ in $\mathscr{A}$ such that $g \cdot d = c$ and $b \cdot d = 0$ since the rectangle (a, f, g, b) is a pushout.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Z \arrow[d, "g"] \arrow[ddr, bend left, "c"] & \\ Y \arrow[r, "b"] \arrow[drr, bend right, "0"'] & W \arrow[dr, "d"] & \\ & & C \end{tikzcd}$$

Consider the following commutative diagram in $\mathscr{A}$.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "b"] \arrow[d] & W \arrow[d, "d"] \\ 0 \arrow[r] & C \end{tikzcd}$$

The outer rectangle $(a, 0, g \cdot d, 0)$ is a pushout since $g \cdot d = c$ is a cokernel of a. By the pasting lemma 1.1.1.(a), the lower rectangle $(b, 0, d, 0)$ is a pushout as well. Thus d is a cokernel of b.

Ad (c). Suppose given $W \xrightarrow{t} T$ in $\mathscr{A}$ such that $b \cdot t = 0$. We have $a \cdot g \cdot t = f \cdot b \cdot t = 0$. Since a is an epimorphism, we have $g \cdot t = 0$. Since (a, f, g, b) is a pushout, we conclude that $t = 0$. □

1.2.8 Lemma. Suppose given an additive category $\mathscr{A}$. Suppose given the following pullback in $\mathscr{A}$.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "b"] & W \end{tikzcd}$$

(a) Suppose given a kernel $C \xrightarrow{c} X$ of a. Then $c \cdot f$ is a kernel of b.

(b) Suppose given a kernel $D \xrightarrow{d} Y$ of b. Then there exists a unique morphism $D \xrightarrow{c} X$ in $\mathscr{A}$ such that $c \cdot f = d$ and $c \cdot a = 0$. Moreover, c is a kernel of a.

(c) If b is a monomorphism, then a is a monomorphism.

◇

Proof. This is dual to the previous lemma 1.2.7. □

1.2.9 Lemma. Suppose given an additive category $\mathscr{A}$. Suppose given the following square in $\mathscr{A}$.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "b"] & W \end{tikzcd}$$

(a) Suppose given $K \xrightarrow{k} X$ in $\mathscr{A}$ such that (k, f) is a kernel-cokernel-pair. Then $(k \cdot a, g)$ is a kernel-cokernel-pair as well.

(b) Suppose given $W \xrightarrow{c} C$ in $\mathscr{A}$ such that (g, c) is a kernel-cokernel-pair. Then $(f, b \cdot c)$ is a kernel-cokernel-pair as well.

◇

Proof. Consider the following commutative diagram in $\mathscr{A}$.

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d] & X \arrow[r, "a"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\ 0 \arrow[r] & Y \arrow[r, "b"] & W \end{tikzcd}$$

Since (k, f) is a kernel-cokernel-pair, the left rectangle $(k, 0, f, 0)$ is a square. By the pasting lemma 1.1.1, the outer rectangle $(k \cdot a, 0, g, 0)$ is a square as well. Thus $(k \cdot a, g)$ is a kernel-cokernel-pair.

Ad (b). This is dual to (a). □

1.2.10 Lemma. Suppose given an additive category $\mathscr{A}$. Suppose given

$$\begin{tikzcd} & X \arrow[d, "f"] & \\ K \arrow[r, "k"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

in $\mathscr{A}$ such that k is a kernel of g. Then the following diagram is a pullback in $\mathscr{A}$.

$$\begin{tikzcd} X \oplus K \arrow[r, "\left(\begin{smallmatrix} f \\ k \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f \cdot g"] & Z \end{tikzcd}$$

$\diamondsuit$

Proof. We have $\left(\begin{smallmatrix} f \\ k \end{smallmatrix}\right) \cdot g = \left(\begin{smallmatrix} f \cdot g \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \cdot f \cdot g$.

Suppose given $T \xrightarrow{s} Y$ and $T \xrightarrow{t} X$ such that $s \cdot g = t \cdot f \cdot g$. Thus $(s - t \cdot f) \cdot g = 0$. Since k is a kernel of g, there exists a unique morphism $T \xrightarrow{u} K$ in $\mathscr{A}$ such that $u \cdot k = s - t \cdot f$. We have $\left(\begin{smallmatrix} t & u \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = t$ and $\left(\begin{smallmatrix} t & u \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} f \\ k \end{smallmatrix}\right) = t \cdot f + u \cdot k = t \cdot f + s - t \cdot f = s$.

$$\begin{tikzcd} T \arrow[dr, "\left(\begin{smallmatrix} t & u \end{smallmatrix}\right)"] \arrow[drr, bend left, "s"] \arrow[ddr, bend right, "t"'] & & \\ & X \oplus K \arrow[r, "\left(\begin{smallmatrix} f \\ k \end{smallmatrix}\right)"'] \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f \cdot g"] & Z \end{tikzcd}$$

Given $T \xrightarrow{\left(\begin{smallmatrix} v & w \end{smallmatrix}\right)} X \oplus K$ with $\left(\begin{smallmatrix} v & w \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = t$ and $\left(\begin{smallmatrix} v & w \end{smallmatrix}\right) \left(\begin{smallmatrix} f \\ k \end{smallmatrix}\right) = s$, we necessarily have $v = t$ and $w \cdot k = s - v \cdot f = s - t \cdot f$. Thus $u = w$. We conclude that the rectangle $\left(\left(\begin{smallmatrix} f \\ k \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), g, f \cdot g\right)$ is a pullback. $\square$

1.2.11 Lemma (splitting of kernel-cokernel-pairs)**.** Suppose given an additive category $\mathscr{A}$ and a kernel-cokernel-pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathscr{A}$.

(a) Suppose given $Y \xrightarrow{s} X$ in $\mathscr{A}$ such that $f \cdot s = 1_X$. Then there exists a unique morphism $Z \xrightarrow{t} Y$ in $\mathscr{A}$ such that $1_Y - s \cdot f = g \cdot t$. Moreover, we have $t \cdot g = 1_Z$. Thus $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a split short exact sequence.

(b) Suppose given $Z \xrightarrow{t} Y$ in $\mathscr{A}$ such that $t \cdot g = 1_Z$. Then there exists a unique morphism $Y \xrightarrow{s} X$ in $\mathscr{A}$ such that $1_Y - g \cdot t = s \cdot f$. Moreover, we have $f \cdot s = 1_X$. Thus $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a split short exact sequence.

$\diamondsuit$

Proof. Ad (a). We have $f \cdot (1_Y - s \cdot f) = f - f = 0$. So there exists a unique morphism $Z \xrightarrow{t} Y$ in $\mathscr{A}$ such that $1_Y - s \cdot f = g \cdot t$. We also have $t \cdot g = 1_Z$ since $g \cdot t \cdot g = (1_Y - s \cdot f) \cdot g = g$ and since g is an epimorphism.

Ad (b). This is dual to (a). $\square$

Factor categories

We follow [Ste12, chapter 2].

1.2.12 Definition. Suppose given an additive category $\mathscr{A}$ and a subset $\mathfrak{J} \subseteq \mathrm{Mor}(\mathscr{A})$. We write ${}_{\mathscr{A},\mathfrak{J}}(X,Y) = {}_{\mathscr{A}}(X,Y) \cap \mathfrak{J}$ for $X, Y \in \mathrm{Ob}(\mathscr{A})$. We say that $\mathfrak{J}$ is an *ideal* in $\mathscr{A}$ if the following two conditions hold.

(I1) Suppose given $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ in $\mathscr{A}$ such that $g \in \mathfrak{J}$. Then we have $f \cdot g \cdot h \in \mathfrak{J}$ as well.

(I2) For $X, Y \in \mathrm{Ob}(\mathscr{A})$, ${}_{\mathscr{A},\mathfrak{J}}(X,Y)$ is a subgroup of the abelian group ${}_{\mathscr{A}}(X,Y)$.

$\diamondsuit$

1.2.13 Definition. Supppose given an additive category $\mathscr{A}$ and an ideal $\mathfrak{J}$ in $\mathscr{A}$. We denote the *factor category* of $\mathscr{A}$ modulo $\mathfrak{J}$ by $\mathscr{A}/\mathfrak{J}$ and the associated *residue class functor* by $\mathfrak{R}_{\mathscr{A},\mathfrak{J}} \colon \mathscr{A} \to \mathscr{A}/\mathfrak{J}$. We have $\mathrm{Ob}(\mathscr{A}/\mathfrak{J}) = \mathrm{Ob}(\mathscr{A})$ and ${}_{\mathscr{A}/\mathfrak{J}}(X,Y) = {}_{\mathscr{A}}(X,Y)/{}_{\mathscr{A},\mathfrak{J}}(X,Y)$ for $X, Y \in \mathrm{Ob}(\mathscr{A})$. $\mathscr{A}/\mathfrak{J}$ is an additive category and $\mathfrak{R}_{\mathscr{A},\mathfrak{J}}$ is an additive functor.
For $X \xrightarrow{f} Y$ in $\mathscr{A}$, we have $X\mathfrak{R}_{\mathscr{A},\mathfrak{J}} = X$ and $f\mathfrak{R}_{\mathscr{A},\mathfrak{J}} = f + {}_{\mathscr{A},\mathfrak{J}}(X,Y)$. $\diamondsuit$

1.2.14 Remark. Supppose given an additive category $\mathscr{A}$ and an ideal $\mathfrak{J}$ in $\mathscr{A}$. Suppose given a full additive subcategory $\mathscr{B} \subseteq \mathscr{A}$. Let $\underline{\mathscr{B}}$ denote the full subcategory of $\mathscr{A}/\mathfrak{J}$ defined by $\mathrm{Ob}(\underline{\mathscr{B}}) = \mathrm{Ob}(\mathscr{B})$. Then $\underline{\mathscr{B}}$ is a full additive subcategory of $\mathscr{A}/\mathfrak{J}$. Moreover, $\mathfrak{J} \cap \mathrm{Mor}(\mathscr{B})$ is an ideal in $\mathscr{B}$ and we have $\underline{\mathscr{B}} = \mathscr{B}/(\mathfrak{J} \cap \mathrm{Mor}(\mathscr{B}))$. So $\underline{\mathscr{B}}$ is a factor category itself. $\diamondsuit$

Proof. Since $\mathfrak{R}_{\mathscr{A},\mathfrak{J}}$ is additive, it preserves zero objects and direct sums. Thus $\underline{\mathscr{B}}$ is a full additive subcategory of $\mathscr{A}/\mathfrak{J}$.

Suppose given $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ in $\mathscr{B}$ such that $g \in \mathfrak{J} \cap \mathrm{Mor}(\mathscr{B})$. Then we have $f \cdot g \cdot h \in \mathfrak{J} \cap \mathrm{Mor}(\mathscr{B})$ as well.

Note that we have ${}_{\mathscr{B},\mathfrak{J}\cap\mathrm{Mor}(\mathscr{B})}(X,Y) = {}_{\mathscr{B}}(X,Y) \cap \mathfrak{J} \cap \mathrm{Mor}(\mathscr{B}) = {}_{\mathscr{A}}(X,Y) \cap \mathfrak{J} = {}_{\mathscr{A},\mathfrak{J}}(X,Y)$ for $X, Y \in \mathrm{Ob}(\mathscr{B})$ since $\mathscr{B}$ is a full subcategory of $\mathscr{A}$. Consequently, ${}_{\mathscr{B},\mathfrak{J}\cap\mathrm{Mor}(\mathscr{B})}(X,Y) = {}_{\mathscr{A},\mathfrak{J}}(X,Y)$ is a subgroup of ${}_{\mathscr{B}}(X,Y) = {}_{\mathscr{A}}(X,Y)$ for $X, Y \in \mathrm{Ob}(\mathscr{B})$.

So $\mathfrak{J} \cap \mathrm{Mor}(\mathscr{B})$ is in fact an ideal in $\mathscr{B}$. Moreover, we have $\underline{\mathscr{B}} = \mathscr{B}/(\mathfrak{J} \cap \mathrm{Mor}(\mathscr{B}))$ since $\mathrm{Ob}(\underline{\mathscr{B}}) = \mathrm{Ob}(\mathscr{B}) = \mathrm{Ob}(\mathscr{B}/(\mathfrak{J} \cap \mathrm{Mor}(\mathscr{B})))$, since

$$
{}_{\underline{\mathscr{B}}}(X,Y) = {}_{\mathscr{A}/\mathfrak{J}}(X,Y) = {}_{\mathscr{A}}(X,Y)/{}_{\mathscr{A},\mathfrak{J}}(X,Y) = {}_{\mathscr{B}}(X,Y)/{}_{\mathscr{B},\mathfrak{J}\cap\mathrm{Mor}(\mathscr{B})}(X,Y) = {}_{\mathscr{B}/(\mathfrak{J}\cap\mathrm{Mor}(\mathscr{B}))}(X,Y)
$$

for $X, Y \in \mathrm{Ob}(\mathscr{B})$ and since composites and identities are induced from the ones in $\mathscr{B}$. $\square$

1.2.15 Lemma (universal property)**.** [Ste12, theorem 21]
Supppose given an additive category $\mathscr{A}$ and an ideal $\mathfrak{J}$ in $\mathscr{A}$.

(a) Suppose given an additive category $\mathscr{B}$ and an additive functor $F \colon \mathscr{A} \to \mathscr{B}$ such that $fF = 0$ in $\mathscr{B}$ for $f \in \mathfrak{J}$. Then there exists a unique functor $\underline{F} \colon \mathscr{A}/\mathfrak{J} \to \mathscr{B}$ such that

$\mathfrak{R}_{\mathscr{A},\mathfrak{I}} \cdot \underline{F} = F$. Moreover, this functor $\underline{F}$ is additive.

$$\begin{tikzcd} \mathscr{A} \arrow[r, "F"] \arrow[d, "\mathfrak{R}_{\mathscr{A},\mathfrak{I}}"'] & \mathscr{B} \\ \mathscr{A}/\mathfrak{I} \arrow[ur, "\underline{F}"'] & \end{tikzcd}$$

(b) Suppose given an additive category $\mathscr{B}$ and additive functors $F, G \colon \mathscr{A} \to \mathscr{B}$ such that $fF = 0$ and $fG = 0$ in $\mathscr{B}$ for $f \in \mathfrak{I}$. Let $\underline{F} \colon \mathscr{A}/\mathfrak{I} \to \mathscr{B}$ denote the unique functor such that $\mathfrak{R}_{\mathscr{A},\mathfrak{I}} \cdot \underline{F} = F$ and let $\underline{G} \colon \mathscr{A}/\mathfrak{I} \to \mathscr{B}$ denote the unique functor such that $\mathfrak{R}_{\mathscr{A},\mathfrak{I}} \cdot \underline{G} = G$. Suppose given a transformation $\lambda \colon F \to G$. Then there exists a unique transformation $\underline{\lambda} \colon \underline{F} \to \underline{G}$ such that $\mathfrak{R}_{\mathscr{A},\mathfrak{I}} \star \underline{\lambda} = \lambda$.

$\diamondsuit$

1.2.16 Lemma. Suppose given an additive category $\mathscr{C}$ and an ideal $\mathfrak{I}$ in $\mathscr{C}$. Suppose given an additive category $\mathscr{D}$ and a full additive functor $F \colon \mathscr{C} \to \mathscr{D}$ such that for $f \in \mathrm{Mor}(\mathscr{C})$, we have $f \in \mathfrak{I}$ if and only if $fF = 0$ in $\mathscr{D}$. Let $\underline{F} \colon \mathscr{C}/\mathfrak{I} \to \mathscr{D}$ denote the unique functor such that $\mathfrak{R}_{\mathscr{C},\mathfrak{I}} \cdot \underline{F} = F$.

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "\mathfrak{R}_{\mathscr{C},\mathfrak{I}}"'] & \mathscr{D} \\ \mathscr{C}/\mathfrak{I} \arrow[ur, "\underline{F}"'] & \end{tikzcd}$$

For $X \in \mathrm{Ob}(\mathscr{D})$, suppose given an object $XG \in \mathrm{Ob}(\mathscr{C})$ and an isomorphism $X \xrightarrow{X\varsigma} XG\underline{F}$ in $\mathscr{D}$. For $X \xrightarrow{f} Y$ in $\mathscr{D}$, there exists a unique morphism $XG \xrightarrow{fG} YG$ in $\mathscr{C}/\mathfrak{I}$ such that $f = X\varsigma \cdot fG\underline{F} \cdot (Y\varsigma)^{-1}$. This yields a functor $G \colon \mathscr{D} \to \mathscr{C}/\mathfrak{I}$ and an isotransformation $\varsigma \colon 1_{\mathscr{D}} \to G\underline{F}$. Moreover, $\underline{F}$ and G are mutually quasi-inverse equivalences. $\diamondsuit$

Proof. Note that $\underline{F}$ is full and faithful since for $f \in \mathrm{Mor}(\mathscr{C})$, we have $f\mathfrak{R}_{\mathscr{C},\mathfrak{I}} = 0$ if and only if $f \in \mathfrak{I}$. Thus the result follows from lemma 1.6.5. $\square$

1.2.17 Lemma. Suppose given an additive category $\mathscr{C}$ and an ideal $\mathfrak{I}$ in $\mathscr{C}$. Suppose given an additive category $\mathscr{D}$ and an ideal $\mathfrak{J}$ in $\mathscr{D}$. Suppose given a full and faithful additive functor $F \colon \mathscr{C} \to \mathscr{D}$ such that for $f \in \mathrm{Mor}(\mathscr{C})$, we have $f \in \mathfrak{I}$ if and only if $fF\mathfrak{R}_{\mathscr{D},\mathfrak{J}} = 0$ in $\mathscr{D}/\mathfrak{J}$. Let $\underline{F} \colon \mathscr{C}/\mathfrak{I} \to \mathscr{D}/\mathfrak{J}$ denote the unique functor such that $\mathfrak{R}_{\mathscr{C},\mathfrak{I}} \cdot \underline{F} = F \cdot \mathfrak{R}_{\mathscr{D},\mathfrak{J}}$.

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "\mathfrak{R}_{\mathscr{C},\mathfrak{I}}"] & \mathscr{D} \arrow[d, "\mathfrak{R}_{\mathscr{D},\mathfrak{J}}"] \\ \mathscr{C}/\mathfrak{I} \arrow[r, "\underline{F}"'] & \mathscr{D}/\mathfrak{J} \end{tikzcd}$$

For $X \in \mathrm{Ob}(\mathscr{D})$, suppose given an object $XG \in \mathrm{Ob}(\mathscr{C})$ and an isomorphism $X \xrightarrow{X\varsigma} XGF$ in $\mathscr{D}$. Lemma 1.6.5 yields the functor $G \colon \mathscr{D} \to \mathscr{C}$, where for $X \xrightarrow{f} Y$ in $\mathscr{D}$, $XG \xrightarrow{fG} YG$ is the unique morphism in $\mathscr{C}$ such that $f = X\varsigma \cdot fGF \cdot (Y\varsigma)^{-1}$. The functors F and G are mutually quasi-inverse equivalences. Moreover, we obtain the isotransformation $\varsigma\colon 1_{\mathscr{D}} \to GF$.

Now for $X \in \mathrm{Ob}(\mathscr{D})$, $X \xrightarrow{X\varsigma\mathfrak{R}_{\mathscr{D},\mathfrak{J}}} XGF$ is an isomorphism in $\mathscr{D}/\mathfrak{J}$. So lemma 1.2.16 yields the functor $\underline{G}\colon \mathscr{D}/\mathfrak{J} \to \mathscr{C}/\mathfrak{I}$, where for $X \xrightarrow{f} Y$ in $\mathscr{D}$, $XG \xrightarrow{f\mathfrak{R}_{\mathscr{D},\mathfrak{J}}\underline{G}} YG$ is the unique morphism in $\mathscr{C}/\mathfrak{I}$ such that $f\mathfrak{R}_{\mathscr{D},\mathfrak{J}} = X\varsigma\mathfrak{R}_{\mathscr{D},\mathfrak{J}} \cdot f\mathfrak{R}_{\mathscr{D},\mathfrak{J}}\underline{G}\,\underline{F} \cdot (Y\varsigma)^{-1}\mathfrak{R}_{\mathscr{D},\mathfrak{J}}$. The functors $\underline{F}$ and $\underline{G}$ are mutually quasi-inverse equivalences. Moreover, we obtain the isotransformation $\underline{\varsigma}\colon 1_{\mathscr{D}/\mathfrak{J}} \to \underline{G} \cdot \underline{F}$ with $X\underline{\varsigma} = X\varsigma\mathfrak{R}_{\mathscr{D},\mathfrak{J}}$ for $X \in \mathrm{Ob}(\mathscr{D})$. We have $G \cdot \mathfrak{R}_{\mathscr{C},\mathfrak{I}} = \mathfrak{R}_{\mathscr{D},\mathfrak{J}} \cdot \underline{G}$.

$$\begin{tikzcd}
\mathscr{C} \arrow[d, "\mathfrak{R}_{\mathscr{C},\mathfrak{I}}"] & \mathscr{D} \arrow[l, "G"'] \arrow[d, "\mathfrak{R}_{\mathscr{D},\mathfrak{J}}"] \\
\mathscr{C}/\mathfrak{I} & \mathscr{D}/\mathfrak{J} \arrow[l, "\underline{G}"']
\end{tikzcd}$$

$\diamondsuit$

Proof. Suppose given $X \xrightarrow{f} Y$ in $\mathscr{D}$. We have

$$\begin{aligned}
X\varsigma\mathfrak{R}_{\mathscr{D},\mathfrak{J}} \cdot fG\mathfrak{R}_{\mathscr{C},\mathfrak{I}}\,\underline{F} \cdot (Y\varsigma)^{-1}\mathfrak{R}_{\mathscr{D},\mathfrak{J}} &= X\varsigma\mathfrak{R}_{\mathscr{D},\mathfrak{J}} \cdot fGF\mathfrak{R}_{\mathscr{D},\mathfrak{J}} \cdot (Y\varsigma)^{-1}\mathfrak{R}_{\mathscr{D},\mathfrak{J}} \\
&= (X\varsigma \cdot fGF \cdot (Y\varsigma)^{-1})\mathfrak{R}_{\mathscr{D},\mathfrak{J}} \\
&= f\mathfrak{R}_{\mathscr{D},\mathfrak{J}} .
\end{aligned}$$

Thus $fG\mathfrak{R}_{\mathscr{C},\mathfrak{I}} = f\mathfrak{R}_{\mathscr{D},\mathfrak{J}}\underline{G}$. We conclude that $G \cdot \mathfrak{R}_{\mathscr{C},\mathfrak{I}} = \mathfrak{R}_{\mathscr{D},\mathfrak{J}} \cdot \underline{G}$. $\square$

1.2.18 Lemma. Suppose given an additive category $\mathscr{C}$ and ideals $\mathfrak{I}, \mathfrak{I}'$ in $\mathscr{C}$ such that $\mathfrak{I} \subseteq \mathfrak{I}'$. Let $\mathfrak{S}\colon \mathscr{C}/\mathfrak{I} \to \mathscr{C}/\mathfrak{I}'$ denote the unique functor such that $\mathfrak{R}_{\mathscr{C},\mathfrak{I}} \cdot \mathfrak{S} = \mathfrak{R}_{\mathscr{C},\mathfrak{I}'}$. Suppose given an additive category $\mathscr{D}$ and an ideal $\mathfrak{J}$ in $\mathscr{D}$. Suppose given an additive functor $F\colon \mathscr{C} \to \mathscr{D}$ such that for $f \in \mathfrak{I}$, we have $fF = 0$ in $\mathscr{D}$ and such that for $f \in \mathfrak{I}'$, we have $fF\mathfrak{R}_{\mathscr{D},\mathfrak{J}} = 0$ in $\mathscr{D}/\mathfrak{J}$. Let $\underline{F}\colon \mathscr{C}/\mathfrak{I} \to \mathscr{D}$ denote the unique functor such that $\mathfrak{R}_{\mathscr{C},\mathfrak{I}} \cdot \underline{F} = F$. Let $\underline{\underline{F}}\colon \mathscr{C}/\mathfrak{I}' \to \mathscr{D}/\mathfrak{J}$ denote the unique functor such that $\mathfrak{R}_{\mathscr{C},\mathfrak{I}'} \cdot \underline{\underline{F}} = F \cdot \mathfrak{R}_{\mathscr{D},\mathfrak{J}}$. We have $\underline{F} \cdot \mathfrak{R}_{\mathscr{D},\mathfrak{J}} = \mathfrak{S} \cdot \underline{\underline{F}}$.

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[d, "\mathfrak{R}_{\mathscr{C},\mathfrak{I}}"] \arrow[dd, bend right=60, "\mathfrak{R}_{\mathscr{C},\mathfrak{I}'}"'] & \mathscr{D} \arrow[d, "\mathfrak{R}_{\mathscr{D},\mathfrak{J}}"] \\
\mathscr{C}/\mathfrak{I} \arrow[ur, "\underline{F}"'] \arrow[d, "\mathfrak{S}"] & \mathscr{D}/\mathfrak{J} \\
\mathscr{C}/\mathfrak{I}' \arrow[ur, "\underline{\underline{F}}"'] &
\end{tikzcd}$$

$\diamondsuit$

Proof. We have $\underline{F} \cdot \mathfrak{R}_{\mathscr{D},\mathfrak{J}} = \mathfrak{S} \cdot \underline{\underline{F}}$ since $\mathfrak{R}_{\mathscr{C},\mathfrak{I}} \cdot \underline{F} \cdot \mathfrak{R}_{\mathscr{D},\mathfrak{J}} = F \cdot \mathfrak{R}_{\mathscr{D},\mathfrak{J}} = \mathfrak{R}_{\mathscr{C},\mathfrak{I}'} \cdot \underline{\underline{F}} = \mathfrak{R}_{\mathscr{C},\mathfrak{I}} \cdot \mathfrak{S} \cdot \underline{\underline{F}}$. $\square$

1.2.19 Lemma. Suppose given an additive category $\mathscr{C}$ and ideals $\mathfrak{I}, \mathfrak{I}'$ in $\mathscr{C}$ such that $\mathfrak{I} \subseteq \mathfrak{I}'$. Let $\mathfrak{S}\colon \mathscr{C}/\mathfrak{I} \to \mathscr{C}/\mathfrak{I}'$ denote the unique functor such that $\mathfrak{R}_{\mathscr{C},\mathfrak{I}} \cdot \mathfrak{S} = \mathfrak{R}_{\mathscr{C},\mathfrak{I}'}$. Suppose given an additive category $\mathscr{D}$ and an ideal $\mathfrak{J}$ in $\mathscr{D}$. Suppose given a full additive functor $F\colon \mathscr{C} \to \mathscr{D}$ such that for $f \in \mathrm{Mor}(\mathscr{C})$, the following two statements hold.

- We have $f \in \mathfrak{I}$ if and only if $fF = 0$ in $\mathscr{D}$.

- We have $f \in \mathfrak{J}'$ if and only if $fF\mathfrak{R}_{\mathscr{D},\mathfrak{J}} = 0$ in $\mathscr{D}/\mathfrak{J}$.

Let $\underline{F}\colon \mathscr{C}/\mathfrak{J} \to \mathscr{D}$ denote the unique functor such that $\mathfrak{R}_{\mathscr{C},\mathfrak{J}} \cdot \underline{F} = F$. Let $\underline{\underline{F}}\colon \mathscr{C}/\mathfrak{J}' \to \mathscr{D}/\mathfrak{J}$ denote the unique functor such that $\mathfrak{R}_{\mathscr{C},\mathfrak{J}'} \cdot \underline{\underline{F}} = F \cdot \mathfrak{R}_{\mathscr{D},\mathfrak{J}}$. We have $\underline{F} \cdot \mathfrak{R}_{\mathscr{D},\mathfrak{J}} = \mathfrak{S} \cdot \underline{\underline{F}}$ by lemma 1.2.18.

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[d, "\mathfrak{R}_{\mathscr{C},\mathfrak{J}}"] \arrow[dd, bend right, "\mathfrak{R}_{\mathscr{C},\mathfrak{J}'}"'] & \mathscr{D} \arrow[d, "\mathfrak{R}_{\mathscr{D},\mathfrak{J}}"] \\
\mathscr{C}/\mathfrak{J} \arrow[ur, "\underline{F}"'] \arrow[d, "\mathfrak{S}"] & \mathscr{D}/\mathfrak{J} \\
\mathscr{C}/\mathfrak{J}' \arrow[ur, "\underline{\underline{F}}"'] &
\end{tikzcd}$$

For $X \in \mathrm{Ob}(\mathscr{D})$, suppose given an object $XG \in \mathrm{Ob}(\mathscr{C})$ and an isomorphism $X \xrightarrow{X\varsigma} XGF$ in $\mathscr{D}$. Lemma 1.2.16 yields the functor $G\colon \mathscr{D} \to \mathscr{C}/\mathfrak{J}$, where for $X \xrightarrow{f} Y$ in $\mathscr{D}$, $XG \xrightarrow{fG} YG$ is the unique morphism in $\mathscr{C}/\mathfrak{J}$ such that $f = X\varsigma \cdot fG\underline{F} \cdot (Y\varsigma)^{-1}$. The functors $\underline{F}$ and G are mutually quasi-inverse equivalences.

Now for $X \in \mathrm{Ob}(\mathscr{D})$, $X \xrightarrow{X\varsigma\mathfrak{R}_{\mathscr{D},\mathfrak{J}}} XGF$ is an isomorphism in $\mathscr{D}/\mathfrak{J}$. So lemma 1.2.16 yields the functor $\underline{G}\colon \mathscr{D}/\mathfrak{J} \to \mathscr{C}/\mathfrak{J}'$, where for $X \xrightarrow{f} Y$ in $\mathscr{D}$, $XG \xrightarrow{f\mathfrak{R}_{\mathscr{D},\mathfrak{J}}\underline{G}} YG$ is the unique morphism in $\mathscr{C}/\mathfrak{J}'$ such that $f\mathfrak{R}_{\mathscr{D},\mathfrak{J}} = X\varsigma\mathfrak{R}_{\mathscr{D},\mathfrak{J}} \cdot f\mathfrak{R}_{\mathscr{D},\mathfrak{J}}\underline{G}\,\underline{\underline{F}} \cdot (Y\varsigma)^{-1}\mathfrak{R}_{\mathscr{D},\mathfrak{J}}$. The functors $\underline{\underline{F}}$ and $\underline{G}$ are mutually quasi-inverse equivalences. We have $G \cdot \mathfrak{S} = \mathfrak{R}_{\mathscr{D},\mathfrak{J}} \cdot \underline{G}$.

$$\begin{tikzcd}
\mathscr{C}/\mathfrak{J} \arrow[d, "\mathfrak{S}"] & \mathscr{D} \arrow[l, "G"'] \arrow[d, "\mathfrak{R}_{\mathscr{D},\mathfrak{J}}"] \\
\mathscr{C}/\mathfrak{J}' & \mathscr{D}/\mathfrak{J} \arrow[l, "\underline{G}"']
\end{tikzcd}$$

◇

Proof. Suppose given $X \xrightarrow{f} Y$ in $\mathscr{D}$. Using lemma 1.2.18, we obtain

$$\begin{aligned}
X\varsigma\mathfrak{R}_{\mathscr{D},\mathfrak{J}} \cdot fG\mathfrak{S}\underline{\underline{F}} \cdot (Y\varsigma)^{-1}\mathfrak{R}_{\mathscr{D},\mathfrak{J}} &= X\varsigma\mathfrak{R}_{\mathscr{D},\mathfrak{J}} \cdot fG\underline{F}\mathfrak{R}_{\mathscr{D},\mathfrak{J}} \cdot (Y\varsigma)^{-1}\mathfrak{R}_{\mathscr{D},\mathfrak{J}} \\
&= (X\varsigma \cdot fG\underline{F} \cdot (Y\varsigma)^{-1})\mathfrak{R}_{\mathscr{D},\mathfrak{J}} \\
&= f\mathfrak{R}_{\mathscr{D},\mathfrak{J}} \,.
\end{aligned}$$

Thus $fG\mathfrak{S} = f\mathfrak{R}_{\mathscr{D},\mathfrak{J}}\underline{G}$. We conclude that $G \cdot \mathfrak{S} = \mathfrak{R}_{\mathscr{D},\mathfrak{J}} \cdot \underline{G}$. □

1.3 Exact categories

We use [Büh10] as basic reference for exact categories.

1.3.1 Definition. Suppose given an additive category $\mathscr{A}$. We define $\mathrm{KCP}(\mathscr{A})$ to be the full subcategory of $\mathbf{Z}_{[0,2]}(\mathscr{A})$ defined by

$$\mathrm{Ob}(\mathrm{KCP}(\mathscr{A})) = \{U \in \mathrm{Ob}(\mathbf{Z}_{[0,2]}(\mathscr{A}))\colon (U_{0\to 1}\,,\, U_{1\to 2}) \text{ is a kernel-cokernel-pair in } \mathscr{A}\}$$

and call $\mathrm{KCP}(\mathscr{A})$ the *category of kernel-cokernel-pairs* in $\mathscr{A}$. Objects $U \in \mathrm{Ob}(\mathrm{KCP}(\mathscr{A}))$ are usually denoted by $U = (f, g)$ or $U = \big(X \xrightarrow{f} Y \xrightarrow{g} Z\big)$, where $X = U_0$, $Y = U_1$, $Z = U_2$, $f = U_{0\to 1}$ and $g = U_{1\to 2}$.

Suppose given a full subcategory $\mathscr{E} \subseteq \mathrm{KCP}(\mathscr{A})$.

We write $\mathscr{E}_{\mathrm{k}} = \{f \in \mathrm{Mor}(\mathscr{A})\colon$ There exists $g \in \mathrm{Mor}(\mathscr{A})$ such that $(f, g) \in \mathrm{Ob}(\mathscr{E})\}$ for the set of kernels in kernel-cokernel-pairs in $\mathscr{E}$ and

$\mathscr{E}_{\mathrm{c}} = \{g \in \mathrm{Mor}(\mathscr{A})\colon$ There exists $f \in \mathrm{Mor}(\mathscr{A})$ such that $(f, g) \in \mathrm{Ob}(\mathscr{E})\}$ for the set of cokernels in kernel-cokernel-pairs in $\mathscr{E}$. $\diamondsuit$

1.3.2 Definition. Suppose given an additive category $\mathscr{A}$ and a strictly full subcategory $\mathscr{E} \subseteq \mathrm{KCP}(\mathscr{A})$ such that the following six conditions hold. Cf. definition 1.3.1.

(E1) For $X \in \mathrm{Ob}(\mathscr{A})$, we have $1_X \in \mathscr{E}_{\mathrm{k}}$.

(E2) For $X \in \mathrm{Ob}(\mathscr{A})$, we have $1_X \in \mathscr{E}_{\mathrm{c}}$.

(E3) Suppose given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathscr{A}$ such that $f, g \in \mathscr{E}_{\mathrm{k}}$. Then we have $f \cdot g \in \mathscr{E}_{\mathrm{k}}$.

(E4) Suppose given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathscr{A}$ such that $f, g \in \mathscr{E}_{\mathrm{c}}$. Then we have $f \cdot g \in \mathscr{E}_{\mathrm{c}}$.

(E5) Suppose given $X \xrightarrow{i} Y$ and $X \xrightarrow{f} U$ in $\mathscr{A}$ such that $i \in \mathscr{E}_{\mathrm{k}}$. Then there exists a pushout

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ U \arrow[r, "m"] & V \end{tikzcd}$$

in $\mathscr{A}$ such that $m \in \mathscr{E}_{\mathrm{k}}$.

(E6) Suppose given $Y \xrightarrow{p} Z$ and $W \xrightarrow{f} Z$ in $\mathscr{A}$ such that $p \in \mathscr{E}_{\mathrm{c}}$. Then there exists a pullback

$$\begin{tikzcd} V \arrow[r, "e"] \arrow[d, "g"'] & W \arrow[d, "f"] \\ Y \arrow[r, "p"] & Z \end{tikzcd}$$

in $\mathscr{A}$ such that $e \in \mathscr{E}_{\mathrm{c}}$.

We call the pair $(\mathscr{A}, \mathscr{E})$ an *exact category.*

The objects of $\mathscr{E}$ are called *pure short exact sequences* in $(\mathscr{A}, \mathscr{E})$. We usually write (f, g) or $X \xrightarrow{f} Y \xrightarrow{g} Z$ for such a pure short exact sequence, cf. definition 1.3.1.

A morphism $f \in \mathscr{E}_{\mathrm{k}}$ is called a *pure monomorphism* in $(\mathscr{A}, \mathscr{E})$. In diagrams, we usually mark pure monomorphisms with a dot: $X \overset{f}{\longrightarrow\!\!\!\!\!\!\!\bullet\;\;} Y$. Note that pure monomorphisms in $(\mathscr{A}, \mathscr{E})$ are in fact monomorphisms in $\mathscr{A}$.

A morphism $f \in \mathscr{E}_{\mathrm{c}}$ is called a *pure epimorphism* in $(\mathscr{A}, \mathscr{E})$. In diagrams, we usually mark pure epimorphisms with a bar: $X \overset{f}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$. Note that pure epimorphisms in $(\mathscr{A}, \mathscr{E})$ are in fact epimorphisms in $\mathscr{A}$.

We abbreviate $\mathscr{A} = (\mathscr{A}, \mathscr{E})$ if unambiguous. ◇

1.3.3 Definition. Suppose given exact categories $(\mathscr{A}, \mathscr{E})$ and $(\mathscr{B}, \mathscr{F})$. An additive functor $F \colon \mathscr{A} \to \mathscr{B}$ is called *exact* with respect to $\mathscr{E}$ and $\mathscr{F}$ if for every pure short exact sequence $X \overset{i}{\longrightarrow\!\!\!\!\!\!\!\bullet\;\;} Y \overset{p}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Z$ in $(\mathscr{A}, \mathscr{E})$, the sequence $XF \xrightarrow{iF} YF \xrightarrow{pF} ZF$ is pure short exact in $(\mathscr{B}, \mathscr{F})$. ◇

1.3.4 Definition. Suppose given an exact category $\mathscr{A}$ and $A \in \mathrm{Ob}(\mathscr{A})$.

(a) We say that A is *projective* in $\mathscr{A}$ if for every diagram

$$\begin{tikzcd} & A \arrow[d, "f"] \\ X \arrow[r, "p"] & Y \end{tikzcd}$$

in $\mathscr{A}$ such that p is a pure epimorphism, there exists $A \xrightarrow{g} X$ in $\mathscr{A}$ such that $g \cdot p = f$.

(b) We say that A is *injective* in $\mathscr{A}$ if for every diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & Y \\ A & \end{tikzcd}$$

in $\mathscr{A}$ such that i is a pure monomorphism, there exists $Y \xrightarrow{g} A$ in $\mathscr{A}$ such that $i \cdot g = f$.

(c) We say that A is *bijective* in $\mathscr{A}$ if it is both injective and projective in $\mathscr{A}$. ◇

1.3.5 Lemma. [Büh10, proposition 11.3]
Suppose given an exact category $\mathscr{A}$ and $P \in \mathrm{Ob}(\mathscr{A})$. The following statements are equivalent.

(a) P is projective in $\mathscr{A}$.

(b) For every pure epimorphism $X \overset{p}{\longrightarrow\!\!\!\!\!\!\!+\;\;} P$ in $\mathscr{A}$, there exists $P \xrightarrow{u} X$ in $\mathscr{A}$ such that $u \cdot p = 1$.

(c) For every pure short exact sequence $X \overset{i}{\longrightarrow\!\!\!\!\!\!\!\bullet\;\;} Y \overset{p}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Z$ in $\mathscr{A}$, the sequence of abelian groups ${}_{\mathscr{A}}(P, X) \xrightarrow{\mathscr{A}(P,i)} {}_{\mathscr{A}}(P, Y) \xrightarrow{\mathscr{A}(P,p)} {}_{\mathscr{A}}(P, Z)$ is short exact. ◇

1.3.6 Lemma. Suppose given an exact category $\mathscr{A}$ and $I \in \mathrm{Ob}(\mathscr{A})$. The following statements are equivalent.

(a) I is injective in $\mathscr{A}$.

(b) For every pure monomorphism $I \xrightarrow{i} X$ in $\mathscr{A}$, there exists $X \xrightarrow{u} I$ in $\mathscr{A}$ such that $i \cdot u = 1$.

(c) For every pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathscr{A}$, the sequence of abelian groups ${}_{\mathscr{A}}(Z, I) \xrightarrow{\mathscr{A}(p,I)} {}_{\mathscr{A}}(Y, I) \xrightarrow{\mathscr{A}(i,I)} {}_{\mathscr{A}}(X, I)$ is short exact.

$\diamondsuit$

Proof. This is dual to the previous lemma 1.3.5. $\square$

1.3.7 Remark. [Büh10, corollary 11.7]
Suppose given an exact category $\mathscr{A}$ and a set J.

Suppose given projective objects P_i in $\mathscr{A}$ for $i \in J$ and a coproduct $(C, (c_i)_{i \in J})$ for $(P_i)_{i \in J}$ in $\mathscr{A}$. Then C is projective in $\mathscr{A}$ as well.

Suppose given injective objects I_i in $\mathscr{A}$ for $i \in J$ and a product $(P, (p_i)_{i \in J})$ for $(I_i)_{i \in J}$ in $\mathscr{A}$. Then P is injective in $\mathscr{A}$ as well. $\diamondsuit$

1.3.8 Remark. [Büh10, corollary 11.6]
Suppose given an exact category $\mathscr{A}$. Summands of projective objects in $\mathscr{A}$ are projective in $\mathscr{A}$ as well. Summands of injective objects in $\mathscr{A}$ are injective in $\mathscr{A}$ as well. $\diamondsuit$

1.3.9 Definition. Suppose given an exact category $\mathscr{A}$. We say that $\mathscr{A}$ *has enough projectives* if for $X \in \mathrm{Ob}(\mathscr{A})$, there exists a pure epimorphism $P \xrightarrow{p} X$ in $\mathscr{A}$ such that P is projective in $\mathscr{A}$. We say that $\mathscr{A}$ *has enough injectives* if for $X \in \mathrm{Ob}(\mathscr{A})$, there exists a pure monomorphism $X \xrightarrow{i} I$ in $\mathscr{A}$ such that I is injective in $\mathscr{A}$. $\diamondsuit$

1.3.10 Lemma. Suppose given an exact category $\mathscr{A}$. Suppose given a pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ and the following commutative diagrams in $\mathscr{A}$.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & Y \arrow[r, "p"] \arrow[d, "g"] & Z \arrow[d, "h"] \\ A \arrow[r, "j"] & B \arrow[r, "q"] & C \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & Y \arrow[r, "p"] \arrow[d, "g'"] & Z \arrow[d, "h'"] \\ A \arrow[r, "j"] & B \arrow[r, "q"] & C \end{tikzcd}$$

Then there exists a unique morphism $Z \xrightarrow{u} B$ in $\mathscr{A}$ such that $g = g' + p \cdot u$. Moreover, we have $h = h' + u \cdot q$. $\diamondsuit$

Proof. We have $i \cdot (g - g') = f \cdot j - f \cdot j = 0$. Since p is a cokernel of i, there exists a unique morphism $Z \xrightarrow{u} B$ in $\mathscr{A}$ such that $g - g' = p \cdot u$, i.e. $g = g' + p \cdot u$. Moreover, we have $p \cdot h = g \cdot q = g' \cdot q + p \cdot u \cdot q = p \cdot (h' + u \cdot q)$. Since p is a pure epimorphism (In particular, p is an epimorphism, cf. definition 1.3.2.), we conclude that $h = h' + u \cdot q$. $\square$

1.3.11 Lemma. Suppose given an exact category $\mathscr{A}$ and a pure short exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathscr{A}$.

(a) Suppose that X is injective in $\mathscr{A}$. Then the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is split short exact. In particular, we may choose $Z \xrightarrow{t} Y \xrightarrow{s} X$ in $\mathscr{A}$ such that $t \cdot g = 1_Z$ and $f \cdot s = 1_X$.

(b) Suppose that Z is projective in $\mathscr{A}$. Then the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is split short exact. In particular, we may choose $Z \xrightarrow{t} Y \xrightarrow{s} X$ in $\mathscr{A}$ such that $t \cdot g = 1_Z$ and $f \cdot s = 1_X$.

◊

Proof. Ad (a). This follows from lemmata 1.3.6 and 1.2.11.

Ad (b). This follows from lemmata 1.3.5 and 1.2.11. □

1.3.12 Lemma. Suppose given an exact category $\mathscr{A}$ and a pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathscr{A}$.

(a) If X and Y are injective in $\mathscr{A}$, then Z is injective in $\mathscr{A}$ as well. If X and Z are injective in $\mathscr{A}$, then Y is injective in $\mathscr{A}$ as well.

(b) If Y and Z are projective in $\mathscr{A}$, then X is projective in $\mathscr{A}$ as well. If X and Z are projective in $\mathscr{A}$, then Y is projective in $\mathscr{A}$ as well.

◊

Proof. Ad (a). This follows from lemma 1.3.11.(a) and remarks 1.3.7, 1.3.8.

Ad (b). This follows from lemma 1.3.11.(b) and remarks 1.3.7, 1.3.8. □

1.3.13 Lemma. Suppose given an exact category $\mathscr{A}$. Suppose given pure short exact sequences $X \xrightarrow{i} Y \xrightarrow{p} C$, $Y \xrightarrow{j} Z \xrightarrow{q} D$ and $X \xrightarrow{i \cdot j} Z \xrightarrow{r} E$ in $\mathscr{A}$. There exists a unique morphism $C \xrightarrow{k} E$ in $\mathscr{A}$ such that $j \cdot r = p \cdot k$. There exists a unique morphism $E \xrightarrow{s} D$ in $\mathscr{A}$ such that $r \cdot s = q$. Moreover, $C \xrightarrow{k} E \xrightarrow{s} D$ is a pure short exact sequence in $\mathscr{A}$.

$$
\begin{tikzcd}
 & & C \arrow[ddrr, "k"] & & \\
 & Y \arrow[ur, "p"] \arrow[dr, "j"] & & & \\
X \arrow[ur, "i"] \arrow[rr, "i \cdot j"'] & & Z \arrow[rr, "r"] \arrow[dr, "q"'] & & E \arrow[dl, "s"] \\
 & & & D &
\end{tikzcd}
$$

◊

Proof. We have $i \cdot j \cdot r = 0$. Since p is a cokernel of i, there exists a unique morphism $C \xrightarrow{k} E$ in $\mathscr{A}$ such that $j \cdot r = p \cdot k$.

We have $i \cdot j \cdot q = 0$. Since r is a cokernel of $i \cdot j$, there exists a unique morphism $E \xrightarrow{s} D$ in $\mathscr{A}$ such that $r \cdot s = q$.

We have the following commutative diagram in $\mathscr{A}$.

$$
\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "1"'] & Y \arrow[r, "p"] \arrow[d, "j"] & C \arrow[d, "k"] \\
X \arrow[r, "i \cdot j"] & Z \arrow[r, "r"] & E
\end{tikzcd}
$$

By [Büh10, (the dual of) proposition 2.12], the right rectangle (p, j, k, r) is a square. Thus k is a pure monomorphism. Moreover, we have $r \cdot s = q$ and $k \cdot s = 0$ since $p \cdot k \cdot s = j \cdot r \cdot s = j \cdot q = 0$ and since p is a pure epimorphism. Thus s is a cokernel of k by lemma 1.2.7.(b). $\square$

1.3.14 Lemma. Suppose given an exact category $\mathscr{A}$. Suppose given pure short exact sequences $C \xrightarrow{j} Y \xrightarrow{q} Z$, $D \xrightarrow{i} X \xrightarrow{p} Y$ and $E \xrightarrow{h} X \xrightarrow{p \cdot q} E$ in $\mathscr{A}$. There exists a unique morphism $D \xrightarrow{k} E$ in $\mathscr{A}$ such that $k \cdot h = i$. There exists a unique morphism $E \xrightarrow{s} C$ in $\mathscr{A}$ such that $s \cdot j = h \cdot p$. Moreover, $D \xrightarrow{k} E \xrightarrow{s} C$ is a pure short exact sequence in $\mathscr{A}$.

$$
\begin{tikzcd}
 & & C \arrow[dr, "j"] & & \\
 & & & Y \arrow[dr, "q"] & \\
E \arrow[uurr, "s"] \arrow[rr, "h"] & & X \arrow[ur, "p"] \arrow[rr, "p \cdot q"'] & & Z \\
 & D \arrow[ul, "k"] \arrow[ur, "i"'] & & &
\end{tikzcd}
$$

$\diamondsuit$

Proof. This is dual to the previous lemma 1.3.13. $\square$

1.3.15 Lemma. [Büh10, corollary 3.6, exercise 3.7]
Suppose given an exact category $\mathscr{A}$ and a commutative diagram

$$
\begin{tikzcd}
X' \arrow[r, "i'"] \arrow[d, "k"'] & Y' \arrow[r, "p'"] \arrow[d, "\ell"'] & Z' \\
X \arrow[r, "i"] \arrow[d, "c"'] & Y \arrow[r, "p"] \arrow[d, "d"'] & Z \\
X'' \arrow[r, "i''"] & Y'' \arrow[r, "p''"] & Z''
\end{tikzcd}
$$

in $\mathscr{A}$ such that (i', p'), (i, p), (i'', p''), (k, c) and (ℓ, d) are pure short exact sequences.
Then there exist unique morphisms $Z' \xrightarrow{m} Z$ and $Z \xrightarrow{e} Z''$ in $\mathscr{A}$ such that $p' \cdot m = \ell \cdot p$ and $p \cdot e = d \cdot p''$. Moreover, (m, e) is a pure short exact sequence. $\diamondsuit$

1.3.16 Lemma. Suppose given an exact category $\mathscr{A}$ and a commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "i'", "\bullet" marking] & Y' \arrow[r, "p'", "+" marking] \arrow[d, "\ell"', "\bullet" marking] & Z' \arrow[d, "m", "\bullet" marking] \\
X \arrow[r, "i", "\bullet" marking] & Y \arrow[r, "p", "+" marking] \arrow[d, "d"', "+" marking] & Z \arrow[d, "e", "+" marking] \\
X'' \arrow[r, "i''", "\bullet" marking] & Y'' \arrow[r, "p''", "+" marking] & Z''
\end{tikzcd}$$

in $\mathscr{A}$ such that $(i', p'), (i, p), (i'', p''), (m, e)$ and (ℓ, d) are pure short exact sequences. Then there exist unique morphisms $X' \xrightarrow{k} X$ and $X \xrightarrow{c} X''$ in $\mathscr{A}$ such that $k \cdot i = i \cdot \ell$ and $c \cdot i'' = i \cdot d$. Moreover, (k, c) is a pure short exact sequence. $\Diamond$

Proof. This is dual to the previous lemma 1.3.15. $\square$

1.3.17 Lemma. [Büh10, corollary 3.6]
Suppose given an exact category $\mathscr{A}$ and a commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "i'", "\bullet" marking] \arrow[d, "k"', "\bullet" marking] & Y' \arrow[r, "p'", "+" marking] \arrow[d, "\ell"'] & Z' \arrow[d, "m", "\bullet" marking] \\
X \arrow[r, "i", "\bullet" marking] \arrow[d, "c"', "+" marking] & Y \arrow[r, "p", "+" marking] \arrow[d, "d"'] & Z \arrow[d, "e", "+" marking] \\
X'' \arrow[r, "i''", "\bullet" marking] & Y'' \arrow[r, "p''", "+" marking] & Z''
\end{tikzcd}$$

in $\mathscr{A}$ such that $(i', p'), (i, p), (i'', p''), (k, c)$ and (m, e) are pure short exact sequences and such that $\ell \cdot d = 0$. Then (ℓ, d) is a pure short exact sequence in $\mathscr{A}$ as well. $\Diamond$

1.3.18 Lemma. [Büh10, exercise 11.10]
Suppose given an exact category $\mathscr{A}$ with enough projectives. Suppose given a sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathscr{A}$. Then (i, p) is a pure short exact sequence in $\mathscr{A}$ if and only if for each projective object P in $\mathscr{A}$, the sequence ${}_{\mathscr{A}}(P, X) \xrightarrow{\mathscr{A}(P,i)} {}_{\mathscr{A}}(P, Y) \xrightarrow{\mathscr{A}(P,p)} {}_{\mathscr{A}}(P, Z)$ of abelian groups is short exact. $\Diamond$

Proof. Suppose that (i, p) is a pure short exact sequence in $\mathscr{A}$. By lemma 1.3.5, the sequence ${}_{\mathscr{A}}(P, X) \xrightarrow{\mathscr{A}(P,i)} {}_{\mathscr{A}}(P, Y) \xrightarrow{\mathscr{A}(P,p)} {}_{\mathscr{A}}(P, Z)$ of abelian groups is short exact for each projective object P in $\mathscr{A}$.

Conversely, suppose that the sequence ${}_{\mathscr{A}}(P, X) \xrightarrow{\mathscr{A}(P,i)} {}_{\mathscr{A}}(P, Y) \xrightarrow{\mathscr{A}(P,p)} {}_{\mathscr{A}}(P, Z)$ of abelian groups is short exact for each projective object P in $\mathscr{A}$.

We want to show that i is a monomorphism. Suppose given $T \xrightarrow{t} X$ in $\mathscr{A}$ such that $t \cdot i = 0$. Choose a pure epimorphism $P \xrightarrow{q} T$ such that P is projective in $\mathscr{A}$. We have $q \cdot t \cdot i = 0$. Since ${}_{\mathscr{A}}(P, i)$ is injective, we have $q \cdot t = 0$. Thus $t = 0$ since q is a pure epimorphism. We conclude that i is a monomorphism.

We want to show that i is a kernel of p. Suppose given $T \xrightarrow{t} Y$ in $\mathscr{A}$ such that $t \cdot p = 0$. Choose a pure short exact sequence $K \overset{j}{\rightarrowtail} P \overset{q}{\twoheadrightarrow} T$ such that P is projective in $\mathscr{A}$. We have $q \cdot t \cdot p = 0$. Since ${}_{\mathscr{A}}(P, i)$ is a kernel of ${}_{\mathscr{A}}(P, p)$, we may choose $P \xrightarrow{u} X$ in $\mathscr{A}$ such that $u \cdot i = q \cdot t$. We have $j \cdot u = 0$ since $j \cdot u \cdot i = j \cdot q \cdot t = 0$ and since i is a monomorphism. So there exists a unique morphism $T \xrightarrow{v} X$ in $\mathscr{A}$ such that $q \cdot v = u$. We have $v \cdot i = t$ since $q \cdot v \cdot i = u \cdot i = q \cdot t$ and since q is a pure epimorphism. Since we already know that i is monomorphic, we conclude that i is a kernel of p.

It remains to show that p is a pure epimorphism. Choose a pure epimorphism $P \overset{q}{\twoheadrightarrow} Z$ such that P is projective in $\mathscr{A}$. Since ${}_{\mathscr{A}}(P, p)$ is surjective, we may choose $P \xrightarrow{r} Y$ in $\mathscr{A}$ such that $r \cdot p = q$. Note that we have alredy shown that p has a kernel. By the dual of the obscure axiom [Büh10, proposition 2.16], we conclude that p is a pure epimorphism. $\square$

1.3.19 Lemma. [Büh10, corollary 3.2]
Suppose given an exact category $\mathscr{A}$.
Suppose given pure short exact sequences $X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$ and $X' \overset{i'}{\rightarrowtail} Y' \overset{p'}{\twoheadrightarrow} Z'$ in $\mathscr{A}$.
Suppose given the following commutative diagram in $\mathscr{A}$.

$$\begin{tikzcd} X \arrow[r, tail, "i"] \arrow[d, "f"'] & Y \arrow[r, two heads, "p"] \arrow[d, "g"] & Z \arrow[d, "h"] \\ X' \arrow[r, tail, "i'"] & Y' \arrow[r, two heads, "p'"] & Z' \end{tikzcd}$$

(a) If f and h are pure monomorphisms, then g is a pure monomorphism as well.

(b) If f and h are pure epimorphisms, then g is a pure epimorphism as well.

(c) If f and h are isomorphisms, then g is an isomorphism as well.

$\diamondsuit$

Exact subcategories

1.3.20 Definition. Suppose given exact categories $(\mathscr{A}, \mathscr{E})$ and $(\mathscr{B}, \mathscr{F})$ such that $\mathscr{B}$ is a full subcategory of $\mathscr{A}$. We say that $(\mathscr{B}, \mathscr{F})$ is an *exact subcategory* of $(\mathscr{A}, \mathscr{E})$ if the inclusion functor $\mathrm{Inc}_{\mathscr{B}}^{\mathscr{A}} \colon \mathscr{B} \to \mathscr{A}$ is exact with respect to $\mathscr{F}$ and $\mathscr{E}$. $\diamondsuit$

1.3.21 Lemma/Definition. Suppose given an exact category $(\mathscr{A}, \mathscr{E})$ and a full additive subcategory $\mathscr{B}$ such that the following four conditions hold.

(RE1) Suppose given pure short exact sequences $X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} C$, $Y \overset{j}{\rightarrowtail} Z \overset{q}{\twoheadrightarrow} D$ and $X \overset{i \cdot j}{\rightarrowtail} Z \overset{r}{\twoheadrightarrow} E$ in $(\mathscr{A}, \mathscr{E})$ such that $X, Y, Z, C, D \in \mathrm{Ob}(\mathscr{B})$.

Then we have $E \in \mathrm{Ob}(\mathscr{B})$ as well.

$$\begin{tikzcd}
& & C & & \\
& Y \arrow[ur, "p"] \arrow[dr, "j"] & & & \\
X \arrow[ur, "i"] \arrow[rr, "i \cdot j"'] & & Z \arrow[rr, "r"] \arrow[dr, "q"'] & & E \\
& & & D &
\end{tikzcd}$$

(RE2) Suppose given pure short exact sequences $D \xrightarrow{i} X \xrightarrow{p} Y$, $C \xrightarrow{j} Y \xrightarrow{q} Z$ and $E \xrightarrow{h} X \xrightarrow{p \cdot q} Z$ in $(\mathscr{A}, \mathscr{E})$ such that $X, Y, Z, C, D \in \mathrm{Ob}(\mathscr{B})$.
Then we have $E \in \mathrm{Ob}(\mathscr{B})$ as well.

$$\begin{tikzcd}
& & C \arrow[dr, "j"] & & \\
& & & Y \arrow[dr, "q"] & \\
E \arrow[rr, "h"] & & X \arrow[ur, "p"] \arrow[rr, "p \cdot q"'] & & Z \\
& D \arrow[ur, "i"'] & & &
\end{tikzcd}$$

(RE3) Suppose given a pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $(\mathscr{A}, \mathscr{E})$ and the following pushout in $\mathscr{A}$.

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\
U \arrow[r, "m"] & V
\end{tikzcd}$$

Suppose that $X, Y, Z, U \in \mathrm{Ob}(\mathscr{B})$. Then we have $V \in \mathrm{Ob}(\mathscr{B})$ as well.

(RE4) Suppose given a pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $(\mathscr{A}, \mathscr{E})$ and the following pullback in $\mathscr{A}$.

$$\begin{tikzcd}
V \arrow[r, "e"] \arrow[d, "g"'] & W \arrow[d, "f"] \\
Y \arrow[r, "p"] & Z
\end{tikzcd}$$

Suppose that $X, Y, Z, W \in \mathrm{Ob}(\mathscr{B})$. Then we have $V \in \mathrm{Ob}(\mathscr{B})$ as well.

We define the *restricted exact structure* of $(\mathscr{A}, \mathscr{E})$ on $\mathscr{B}$ to be the full subcategory

$\mathscr{E}|_{\mathscr{B}} \subseteq \operatorname{KCP}(\mathscr{B})$ defined by

$$\operatorname{Ob}(\mathscr{E}|_{\mathscr{B}}) = \{U|^{\mathscr{B}} : U \in \operatorname{Ob}(\mathscr{E}) \text{ such that } U_k \in \operatorname{Ob}(\mathscr{B}) \text{ for all } k \in \mathbf{Z}_{[0,2]}\}.$$

The pair $(\mathscr{B}, \mathscr{E}|_{\mathscr{B}})$ is in fact an exact subcategory of $(\mathscr{A}, \mathscr{E})$.
A sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathscr{B}$ is pure short exact in $(\mathscr{B}, \mathscr{E}|_{\mathscr{B}})$ if and only if the sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ is pure short exact in $(\mathscr{A}, \mathscr{E})$. $\diamondsuit$

Proof. We abbreviate $\mathscr{F} = \mathscr{E}|_{\mathscr{B}}$.

Note that $\mathscr{F}$ is a strictly full subcategory of $\operatorname{KCP}(\mathscr{B})$ since $U, V \in \operatorname{KCP}(\mathscr{B})$ are isomorphic if and only if $U \cdot \operatorname{Inc}^{\mathscr{A}}_{\mathscr{B}}$, $V \cdot \operatorname{Inc}^{\mathscr{A}}_{\mathscr{B}}$ are isomorphic in $\operatorname{KCP}(\mathscr{A})$.

Ad (E1). Suppose given $X \in \operatorname{Ob}(\mathscr{B})$. We have $1_X \in \mathscr{F}_{\mathrm{k}}$ since $X \xrightarrow{1_X} X \xrightarrow{0} 0_{\mathscr{B}}$ is pure short exact in $(\mathscr{A}, \mathscr{E})$.

Ad (E2). This is dual to (E1).

Ad (E3). This follows from (RE1).

Ad (E4). This follows from (RE2).

Ad (E5). This follows from (RE3).

Ad (E6). This follows from (RE4).

Note that the inclusion functor $\operatorname{Inc}^{\mathscr{B}}_{\mathscr{A}}$ is exact by construction. $\square$

1.3.22 Remark. Suppose given exact categories $(\mathscr{A}, \mathscr{E})$, $(\mathscr{B}, \mathscr{F})$ and an exact functor $F : \mathscr{A} \to \mathscr{B}$. Suppose given given full additive subcategories $\mathscr{C} \subseteq \mathscr{A}$ and $\mathscr{D} \subseteq \mathscr{B}$ that satisfy the conditions (RE1), (RE2), (RE3) and (RE4) of definition 1.3.21. Suppose that we have $XF \in \operatorname{Ob}(\mathscr{D})$ for $X \in \operatorname{Ob}(\mathscr{C})$. Then the functor $F|^{\mathscr{D}}_{\mathscr{C}}$ is exact with respect to $\mathscr{E}|_{\mathscr{C}}$ and $\mathscr{F}|_{\mathscr{D}}$. $\diamondsuit$

1.3.23 Definition. Suppose given an exact category $\mathscr{A}$ and a full subcategory $\mathscr{B}$. We say that $\mathscr{B}$ is *extension-closed* in $\mathscr{A}$ if $\mathrm{Z}_{\mathscr{A}} \subseteq \mathscr{B}$ and if for every pure short exact sequence $X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$ in $\mathscr{A}$ with $X, Z \in \operatorname{Ob}(\mathscr{B})$, we have $Y \in \operatorname{Ob}(\mathscr{B})$ as well. $\diamondsuit$

1.3.24 Remark. Suppose given an exact category $\mathscr{A}$ and extension-closed full subcategories $\mathscr{B} \subseteq \mathscr{A}$ and $\mathscr{C} \subseteq \mathscr{A}$. Then $\mathscr{B} \cap \mathscr{C}$ is an extension-closed full subcategory of $\mathscr{A}$ as well. $\diamondsuit$

1.3.25 Lemma. Suppose given an exact category $\mathscr{A}$ and an extension-closed full subcategory $\mathscr{B} \subseteq \mathscr{A}$. Then $\mathscr{B}$ is a strictly full additive subcategory of $\mathscr{A}$ and satisfies the conditions (RE1), (RE2), (RE3) and (RE4) of definition 1.3.21. Consequently, $(\mathscr{B}, \mathscr{E}|_{\mathscr{B}})$ is an exact subcategory of $(\mathscr{A}, \mathscr{E})$. $\diamondsuit$

Proof. By definition, $\mathscr{B}$ contains the zero objects of $\mathscr{A}$. Since split short exact sequences are pure short exact by [Büh10, lemma 2.7], $\mathscr{B}$ is also closed under directs sums. Consequently, $\mathscr{B}$ is a strictly full additive subcategory of $\mathscr{A}$.

Ad (RE1). This follows from lemma 1.3.13.

Ad (RE2). This is dual to (RE1).

Ad (RE3). This follows from lemma 1.2.7.(b).

Ad (RE4). This is dual to (RE3). □

Frobenius categories

1.3.26 Definition. An exact category $\mathscr{F}$ is called a *Frobenius category* if for each object $X \in \mathrm{Ob}(\mathscr{F})$, there exists a pure epimorphism $B \twoheadrightarrow X$ and a pure monomorphism $X \rightarrowtail C$ such that B and C are bijective in $\mathscr{F}$. ♢

1.3.27 Lemma/Definition. Suppose given a Frobenius category $\mathscr{F}$. Let $\mathfrak{J}_{\mathscr{F}}$ denote the set of morphisms $X \xrightarrow{f} Y$ in $\mathscr{F}$ for which there exists $X \xrightarrow{u} B \xrightarrow{v} Y$ in $\mathscr{F}$ such that $f = u \cdot v$ and such that B is bijective in $\mathscr{F}$. The set $\mathfrak{J}_{\mathscr{F}}$ is an ideal in $\mathscr{F}$, cf. definition 1.2.12. We usually denote the corresponding factor category by $\underline{\mathscr{F}} = \mathscr{F}/\mathfrak{J}_{\mathscr{F}}$ and the corresponding residue class functor by $\mathfrak{P}_{\mathscr{F}} = \mathfrak{R}_{\mathscr{F},\mathfrak{J}_{\mathscr{F}}} \colon \mathscr{F} \to \underline{\mathscr{F}}$, cf. definition 1.2.13. The category $\underline{\mathscr{F}}$ is called the *stable category of* $\mathscr{F}$ and $\mathfrak{P}_{\mathscr{F}}$ is called the *stabilisation functor* of $\mathscr{F}$. For $f \in \mathrm{Mor}(\mathscr{F})$, we usually abbreviate $\underline{f} = f\mathfrak{P}_{\mathscr{F}}$. ♢

Proof. We abbreviate $\mathfrak{J} = \mathfrak{J}_{\mathscr{F}}$. Suppose given $W \xrightarrow{a} X \xrightarrow{f} Y \xrightarrow{b} Z$ in $\mathscr{F}$ with $f \in \mathfrak{J}$. We want to show that $a \cdot f \cdot b \in \mathfrak{J}$. We may choose $X \xrightarrow{u} B \xrightarrow{v} Y$ in $\mathscr{F}$ such that $f = u \cdot v$ and such that B is bijective in $\mathscr{F}$. We have $a \cdot f \cdot b = a \cdot u \cdot v \cdot b$ and thus $a \cdot f \cdot b \in \mathfrak{J}$.

Suppose given $X, Y \in \mathrm{Ob}(\mathscr{A})$. We want to show that ${}_{\mathscr{F}}(X,Y) \cap \mathfrak{J}$ is a subgroup of ${}_{\mathscr{F}}(X,Y)$. We have $0_{X,Y} \in \mathfrak{J}$ since $0_{X,Y} = 0_{X,0_{\mathscr{F}}} \cdot 0_{0_{\mathscr{F}},Y}$ and since $0_{\mathscr{F}}$ is bijective in $\mathscr{F}$.

Suppose given $f, g \in {}_{\mathscr{F}}(X,Y) \cap \mathfrak{J}$. We may choose $X \xrightarrow{u} B \xrightarrow{v} Y$ in $\mathscr{F}$ such that $f = u \cdot v$ and such that B is bijective in $\mathscr{F}$. We may choose $X \xrightarrow{r} C \xrightarrow{s} Y$ in $\mathscr{F}$ such that $g = r \cdot s$ and such that C is bijective in $\mathscr{F}$. Note that $B \oplus C$ is bijective in $\mathscr{F}$ as well. We have $f - g = u \cdot v - r \cdot s = \left(\begin{smallmatrix} u & -r \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} v \\ s \end{smallmatrix}\right)$ and thus $f - g \in \mathfrak{J}$.

$$\begin{tikzcd} X \arrow[rr, "f-g"] \arrow[dr, "\left(\begin{smallmatrix} u & -r \end{smallmatrix}\right)"'] & & Y \\ & B \oplus C \arrow[ur, "\left(\begin{smallmatrix} v \\ s \end{smallmatrix}\right)"'] & \end{tikzcd}$$

□

1.3.28 Lemma. Suppose given a Frobenius category $\mathscr{F}$ and $X \xrightarrow{f} Y$ in $\mathscr{F}$. Suppose given a pure monomorphism $X \overset{i}{\rightarrowtail} C$ and a pure epimorphism $B \overset{p}{\twoheadrightarrow} Y$ in $\mathscr{F}$ such that B and C are bijective in $\mathscr{F}$. The following four statements are equivalent.

(a) We have $\underline{f} = 0$ in $\underline{\mathscr{F}}$.

(b) There exists $C \xrightarrow{g} Y$ in $\mathscr{F}$ such that $f = i \cdot g$.

(c) There exists $C \xrightarrow{h} B$ in $\mathscr{F}$ such that $f = i \cdot h \cdot p$.

(d) There exists $X \xrightarrow{e} B$ in $\mathscr{F}$ such that $f = e \cdot p$. $\diamondsuit$

Proof. Ad (a)→(b). We may choose $X \xrightarrow{u} D \xrightarrow{v} Y$ in $\mathscr{F}$ such that $f = u \cdot v$ and such that D is bijective in $\mathscr{F}$. Since D is injective in $\mathscr{F}$, we may choose $C \xrightarrow{a} D$ in $\mathscr{F}$ such that $i \cdot a = u$. Let $g = a \cdot v$. We have $i \cdot g = i \cdot a \cdot v = u \cdot v = f$.

Ad (b)→(c). This follows from the fact that C is projective in $\mathscr{F}$.

The remaining implications are trivial. $\square$

1.3.29 Remark. Suppose given a Frobenius category $\mathscr{F}$. An object $X \in \mathrm{Ob}(\mathscr{F})$ is projective in $\mathscr{F}$ if and only if it is injective in $\mathscr{F}$. $\diamondsuit$

Proof. Suppose given a projective object $X \in \mathrm{Ob}(\mathscr{F})$. Choose a pure epimorphism $B \overset{p}{\twoheadrightarrow} X$ in $\mathscr{F}$ such that B is bijective in $\mathscr{F}$. By lemma 1.3.11.(b), X is a summand of B and thus X is injective in $\mathscr{F}$. Dually, injective objects in $\mathscr{F}$ are also projective in $\mathscr{F}$. $\square$

The stable category of a Frobenius category is triangulated, cf. [Hap88, chapter I, section 2]. We have the following lemma.

1.3.30 Lemma. [Hap88, lemma I.2.8] [Kel96, example 8.1]
Suppose given Frobenius categories $\mathscr{F}$ and $\mathscr{G}$. Suppose given an exact functor $F\colon \mathscr{F} \to \mathscr{G}$ such that XF is bijective in $\mathscr{G}$ for all bijectives $X \in \mathrm{Ob}(\mathscr{F})$. Then there exists a unique functor $\underline{F}\colon \underline{\mathscr{F}} \to \underline{\mathscr{G}}$ such that $\mathfrak{P}_{\mathscr{F}} \cdot \underline{F} = F \cdot \mathfrak{P}_{\mathscr{G}}$. Moreover, this functor $\underline{F}$ is exact.

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "F"] \arrow[d, "\mathfrak{P}_{\mathscr{F}}"'] & \mathscr{G} \arrow[d, "\mathfrak{P}_{\mathscr{G}}"] \\
\underline{\mathscr{F}} \arrow[r, "\underline{F}"] & \underline{\mathscr{G}}
\end{tikzcd}$$

$\diamondsuit$

1.4 Functor categories

Suppose given a category $\mathscr{C}$.

1.4.1 Definition. Suppose given a category $\mathscr{A}$. We denote the category of functors from $\mathscr{C}$ to $\mathscr{A}$ by $\mathscr{C}(\mathscr{A})$. Its objects are the functors from $\mathscr{C}$ to $\mathscr{A}$ and its morphisms are the transformations between such functors. For $X \in \mathrm{Ob}(\mathscr{C}(\mathscr{A}))$, the transformation 1_X is the identity morphism in $\mathscr{C}(\mathscr{A})$. For $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathscr{C}(\mathscr{A})$, the vertical composite $f \cdot g$ of the transformations is the composite in $\mathscr{C}(\mathscr{A})$.

If $\mathscr{A}$ is an additive category, then $\mathscr{C}(\mathscr{A})$ is an additive category as well. In this case, for $X \overset{f}{\underset{h}{\rightrightarrows}} Y$ in $\mathscr{C}(\mathscr{A})$, the sum $f+h$ in $\mathscr{C}(\mathscr{A})$ is given by $A(f+h) = Af+Ah$ for $A \in \mathrm{Ob}(\mathscr{C})$. $\diamondsuit$

1.4.2 Remark. Limits and colimits in functor categories are formed pointwise, provided that the pointwise (co)limits exist, cf. [Bor94, proposition 2.15.1]. We will use this fact without comment. $\diamondsuit$

1.4.3 Lemma/Definition. Suppose given categories $\mathscr{A}$ and $\mathscr{B}$ and a functor $F\colon \mathscr{A} \to \mathscr{B}$. We define the functor $\mathscr{C}(F)\colon \mathscr{C}(\mathscr{A}) \to \mathscr{C}(\mathscr{B})$ by setting $X\mathscr{C}(F) = XF$ for $X \in \mathrm{Ob}(\mathscr{C}(\mathscr{A}))$ and $f\mathscr{C}(F) = f \star 1_F$ for $f \in \mathrm{Mor}(\mathscr{C}(\mathscr{A}))$. This in fact defines a functor.

If $\mathscr{A}$, $\mathscr{B}$ are additive categories and F is an additive functor, then $\mathscr{C}(F)$ is additive as well. $\diamondsuit$

Proof. Suppose given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathscr{C}(\mathscr{A})$. We have $1_X\mathscr{C}(F) = 1_X \star 1_F = 1_{XF} = 1_{X\mathscr{C}(F)}$ and $(f \cdot g)\mathscr{C}(F) = (f \cdot g) \star 1_F = (f \cdot g) \star (1_F \cdot 1_F) = (f \star 1_F) \cdot (g \star 1_F) = f\mathscr{C}(F) \cdot g\mathscr{C}(F)$.

Suppose that $\mathscr{A}$, $\mathscr{B}$ are additive categories and that F is an additive functor.

Suppose given $X \underset{h}{\overset{f}{\rightrightarrows}} Y$ in $\mathscr{C}(\mathscr{A})$. Then we have $(f+h)\mathscr{C}(F) = f\mathscr{C}(F) + h\mathscr{C}(F)$ since we have

$$\begin{aligned} A((f+h)\mathscr{C}(F)) &= A((f+h) \star 1_F) = A(f+h)F = (Af + Ah)F = AfF + AhF \\ &= A(f \star 1_F) + A(h \star 1_F) = A(f\mathscr{C}(F) + h\mathscr{C}(F)) \end{aligned}$$

for $A \in \mathrm{Ob}(\mathscr{C})$. $\square$

1.4.4 Lemma. Suppose given additive categories $\mathscr{A}$, $\mathscr{B}$ and an additive functor $F\colon \mathscr{A} \to \mathscr{C}$. Suppose given a category $\mathscr{C}$ and $X \in \mathrm{Ob}(\mathscr{C}(\mathscr{A}))$.

(a) Suppose given $A \xrightarrow{a} B$ in $\mathscr{C}$ such that $aX = 0$. Then we have $a(X\mathscr{C}(F)) = 0$ as well.

(b) Suppose given $A \in \mathrm{Ob}(\mathscr{C})$ such that $AX \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}})$.
Then we have $A(X\mathscr{C}(F)) \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{B}})$.

$\diamondsuit$

Proof. Ad (a). We have $a(X\mathscr{C}(F)) = aXF = 0F = 0$ since F is additive.

Ad (b). We have $A(X\mathscr{C}(F)) = AXF \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{B}})$ since F is additive. $\square$

1.4.5 Lemma/Definition. Suppose given categories $\mathscr{A}$, $\mathscr{B}$, functors $F, G\colon \mathscr{A} \to \mathscr{B}$ and a transformation $\lambda\colon F \to G$. We define the transformation $\mathscr{C}(\lambda)\colon \mathscr{C}(F) \to \mathscr{C}(G)$ by setting $X\mathscr{C}(\lambda) = 1_X \star \lambda$ for $X \in \mathrm{Ob}(\mathscr{C}(\mathscr{A}))$.

This in fact defines a transformation. $\diamondsuit$

Proof. Suppose given $X \xrightarrow{f} Y$ in $\mathscr{C}(\mathscr{A})$. We have

$$\begin{aligned} X\mathscr{C}(\lambda) \cdot f\mathscr{C}(G) &= (1_X \star \lambda) \cdot (f \star 1_G) = (1_X \cdot f) \star (\lambda \cdot 1_G) = f \star \lambda = (f \cdot 1_Y) \star (1_F \cdot \lambda) \\ &= (f \star 1_F) \cdot (1_Y \star \lambda) = f\mathscr{C}(F) \cdot Y\mathscr{C}(\lambda). \end{aligned}$$ $\square$

1.4.6 Lemma.

(a) Suppose given a category $\mathscr{A}$. We have $\mathscr{C}(1_{\mathscr{A}}) = 1_{\mathscr{C}(\mathscr{A})}$.

(b) Suppose given categories and functors as follows: $\mathscr{A} \xrightarrow{F} \mathscr{B} \xrightarrow{G} \mathscr{D}$.
We have $\mathscr{C}(F \cdot G) = \mathscr{C}(F) \cdot \mathscr{C}(G)$.

(c) Suppose given categories and functors as follows: $\mathscr{A} \xrightarrow{F} \mathscr{B}$. We have $\mathscr{C}(1_F) = 1_{\mathscr{C}(F)}$.

(d) Suppose given categories, functors and transformations as follows: $\begin{tikzcd} \mathscr{A} \arrow[r, bend left=50, "F"{name=F}] \arrow[r, "G" description, ""{name=G}] \arrow[r, bend right=50, "H"'{name=H}] \arrow[from=F, to=G, phantom, "\Downarrow\lambda"] \arrow[from=G, to=H, phantom, "\Downarrow\mu"] & \mathscr{B} \end{tikzcd}$. We have $\mathscr{C}(\lambda \cdot \mu) = \mathscr{C}(\lambda) \cdot \mathscr{C}(\mu)$.

(e) Suppose given categories, functors and transformations as follows: $\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"{name=F}] \arrow[r, bend right, "G"'{name=G}] \arrow[from=F, to=G, phantom, "\Downarrow\lambda"] & \mathscr{B} \arrow[r, bend left, "H"{name=H}] \arrow[r, bend right, "I"'{name=I}] \arrow[from=H, to=I, phantom, "\Downarrow\mu"] & \mathscr{D} \end{tikzcd}$. We have $\mathscr{C}(\lambda \star \mu) = \mathscr{C}(\lambda) \star \mathscr{C}(\mu)$.

(f) Suppose given a category $\mathscr{A}$, an additive category $\mathscr{B}$ and functors $F, G \colon \mathscr{A} \to \mathscr{B}$. Suppose given transformations $\lambda, \mu \colon F \to G$. We have $\mathscr{C}(\lambda + \mu) = \mathscr{C}(\lambda) + \mathscr{C}(\mu)$.

$\diamondsuit$

Proof. Ad (a). Suppose given $f \in \operatorname{Mor}(\mathscr{C}(\mathscr{A}))$. We have $f\mathscr{C}(1_{\mathscr{A}}) = f \star 1_{\mathscr{A}} = f = f1_{\mathscr{C}(\mathscr{A})}$.

Ad (b). Suppose given $f \in \operatorname{Mor}(\mathscr{C}(\mathscr{A}))$. We have

$$\begin{aligned} f(\mathscr{C}(F) \cdot \mathscr{C}(G)) &= f\mathscr{C}(F)\mathscr{C}(G) = (f \star 1_F)\mathscr{C}(G) = (f \star 1_F) \star 1_G = f \star (1_F \star 1_G) = f \star 1_{FG} \\ &= f\mathscr{C}(F \cdot G). \end{aligned}$$

Ad (c). Suppose given $X \in \operatorname{Ob}(\mathscr{C}(\mathscr{A}))$. We have $X\mathscr{C}(1_F) = 1_X \star 1_F = 1_{XF} = 1_{X\mathscr{C}(F)}$.

Ad (d). Suppose given $X \in \operatorname{Ob}(\mathscr{C}(\mathscr{A}))$. We have

$$\begin{aligned} X(\mathscr{C}(\lambda) \cdot \mathscr{C}(\mu)) &= X\mathscr{C}(\lambda) \cdot X\mathscr{C}(\mu) = (1_X \star \lambda) \cdot (1_X \star \mu) = (1_X \cdot 1_X) \star (\lambda \cdot \mu) = 1_X \star (\lambda \cdot \mu) \\ &= X\mathscr{C}(\lambda \cdot \mu). \end{aligned}$$

Ad (e). Suppose given $X \in \operatorname{Ob}(\mathscr{C}(\mathscr{A}))$. We have

$$\begin{aligned} X(\mathscr{C}(\lambda) \star \mathscr{C}(\mu)) &= X\mathscr{C}(\lambda)\mathscr{C}(H) \cdot X\mathscr{C}(G)\mathscr{C}(\mu) = ((1_X \star \lambda)\mathscr{C}(H)) \cdot ((X\mathscr{C}(G) \star \mu)) \\ &= ((1_X \star \lambda) \star H) \cdot (XG \star \mu) = (1_X \star \lambda) \star \mu = 1_X \star (\lambda \star \mu) = X\mathscr{C}(\lambda \star \mu). \end{aligned}$$

Ad (f). Suppose given $X \in \operatorname{Ob}(\mathscr{C}(\mathscr{A}))$. We have $X(\mathscr{C}(\lambda + \mu)) = X(\mathscr{C}(\lambda) + \mathscr{C}(\mu))$ since

$$\begin{aligned} A(X(\mathscr{C}(\lambda + \mu))) &= A(1_X \star (\lambda + \mu)) = AX(\lambda + \mu) = AX\lambda + AX\mu = A(1_X \star \lambda) + A(1_X \star \mu) \\ &= A(1_X \star \lambda + 1_X \star \mu) = A(X\mathscr{C}(\lambda) + X\mathscr{C}(\mu)) = A(X(\mathscr{C}(\lambda) + \mathscr{C}(\mu))) \end{aligned}$$

for $A \in \operatorname{Ob}(\mathscr{C})$. $\square$

Functor categories of exact categories

Suppose given a category $\mathscr{C}$ and an exact category $\mathscr{A} = (\mathscr{A}, \mathscr{E})$.

1.4.7 Definition. We consider the functor category $\mathscr{C}(\mathscr{A})$ as an exact category equipped with the pointwise exact structure, cf. [Büh10, example 13.11]. A sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathscr{C}(\mathscr{A})$ is pure short exact in $\mathscr{C}(\mathscr{A})$ if and only if the sequence $AX \xrightarrow{Ai} AY \xrightarrow{Ap} AZ$ is pure short exact in $\mathscr{A}$ for $A \in \mathrm{Ob}(\mathscr{C})$. $\diamondsuit$

1.4.8 Lemma. Suppose given a pure short exact sequence $X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$ in $\mathscr{C}(\mathscr{A})$. Suppose given $A \xrightarrow{a} B$ in $\mathscr{C}$.

(a) Suppose that aX and aZ are pure monomorphisms. Then aY is a pure monomorphism as well.

(b) Suppose that aX and aZ are pure epimorphisms. Then aY is a pure epimorphism as well.

(c) Suppose that aX and aZ are isomorphisms. Then aY is an isomorphism as well.

(d) Suppose that $AX, AZ \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}})$. Then we have $AY \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}})$ as well. $\diamondsuit$

Proof. Ad (a,b,c). This follows from lemma 1.3.19.

Ad (d).

We have the pure short exact sequence $AX \overset{Ai}{\rightarrowtail} AY \overset{Ap}{\twoheadrightarrow} AZ$ in $\mathscr{A}$ with $AX, AZ \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}})$. Thus we have $AY \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}})$ as well. $\square$

1.4.9 Lemma. Suppose given a pure short exact sequence $X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$ in $\mathscr{C}(\mathscr{A})$. Suppose given $A \xrightarrow{a} B \xrightarrow{b} C$ in $\mathscr{C}$.

(a) Suppose that (aX, bX) and (aY, bY) are pure short exact sequences in $\mathscr{A}$. Then (aZ, bZ) is a pure short exact sequence in $\mathscr{A}$ as well.

(b) Suppose that (aZ, bZ) and (aY, bY) are pure short exact sequences in $\mathscr{A}$. Then (aX, bX) is a pure short exact sequence in $\mathscr{A}$ as well. $\diamondsuit$

Proof. Ad (a). This follows from lemma 1.3.15.

Ad (b). This is dual to (a). $\square$

1.4.10 Remark. Suppose given $Z \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{C}(\mathscr{A})})$. Then we have $AZ \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}})$ for $A \in \mathrm{Ob}(\mathscr{C})$. Consequently, the following statements are also true.

(a) For $A \xrightarrow{a} B$ in $\mathscr{C}$, the morphism aZ is an isomorphism. In particular, it is both a pure monomorphism and a pure epimorphism.

(b) For $A \xrightarrow{a} B \xrightarrow{b} C$ in $\mathscr{C}$, the pair (aZ, bZ) is a pure short exact sequence in $\mathscr{A}$. $\diamondsuit$

1.4.11 Lemma. Suppose given $X, Y \in \mathrm{Ob}(\mathscr{C}(\mathscr{A}))$ and a direct sum $X \underset{s}{\overset{i}{\rightleftarrows}} D \underset{p}{\overset{t}{\leftrightarrows}} Y$ of X and Y in $\mathscr{C}(\mathscr{A})$.

(a) Suppose given $A \xrightarrow{a} B$ in $\mathscr{C}$ such that aX and aY are pure monomorphisms. Then aD is a pure monomorphism as well.

(b) Suppose given $A \xrightarrow{a} B$ in $\mathscr{C}$ such that aX and aY are pure epimorphisms. Then aD is a pure epimorphism as well.

(c) Suppose given $A \xrightarrow{a} B$ in $\mathscr{C}$ such that aX and aY are isomorphisms. Then aD is an isomorphism as well.

(d) Suppose given $A \in \mathrm{Ob}(\mathscr{C})$ such that $AX, AY \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}})$. Then we have $aD \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}})$ as well.

(e) Suppose given $A \xrightarrow{a} B \xrightarrow{b} C$ in $\mathscr{C}$ such that (aX, bX) and (aY, bY) are pure short exact sequences in $\mathscr{A}$. Then (aD, bD) is a pure short exact sequence in $\mathscr{A}$ as well.

$\diamondsuit$

Proof. Ad (a,b,c,d). This follows from lemma 1.4.8.

Ad (e). This follows from [Büh10, proposition 2.9]. $\square$

1.4.12 Lemma. Suppose given pure short exact sequences $X \xrightarrow{i} Y \xrightarrow{p} C$, $Y \xrightarrow{j} Z \xrightarrow{q} D$ and $X \xrightarrow{i \cdot j} Z \xrightarrow{r} E$ in $\mathscr{C}(\mathscr{A})$.

(a) Suppose given $A \xrightarrow{a} B$ in $\mathscr{C}$ such that aC, aD are pure monomorphisms. Then aE is a pure monomorphism as well.

(b) Suppose given $A \xrightarrow{a} B$ in $\mathscr{C}$ such that aC, aD are pure epimorphisms. Then aE is a pure epimorphism as well.

(c) Suppose given $A \xrightarrow{a} B$ in $\mathscr{C}$ such that aC, aD are isomorphisms. Then aE is an isomorphism as well.

(d) Suppose given $A \xrightarrow{a} B \xrightarrow{b} C$ in $\mathscr{C}$ such that (aX, bX) and (aZ, bZ) are pure short exact sequences in $\mathscr{A}$. Then (aE, bE) is a pure short exact sequence in $\mathscr{A}$ as well.

$$\begin{tikzcd}
 & & C & & \\
 & Y \arrow[ur, "p"] \arrow[dr, "j"] & & & \\
X \arrow[ur, "i"] \arrow[rr, "i \cdot j"'] & & Z \arrow[rr, "r"] \arrow[dr, "q"'] & & E \\
 & & & D &
\end{tikzcd}$$

$\diamondsuit$

Proof. Ad (a). This follows from lemmata 1.3.13 and 1.4.8.(a).

Ad (b). This follows from lemmata 1.3.13 and 1.4.8.(b).

Ad (c). This follows from lemmata 1.3.13 and 1.4.8.(c).

Ad (d). This follows from lemma 1.4.9.(a). ☐

1.4.13 Lemma. Suppose given pure short exact sequences $D \xrightarrow{i} X \xrightarrow{p} Y$, $C \xrightarrow{j} Y \xrightarrow{q} Z$ and $E \xrightarrow{h} X \xrightarrow{p \cdot q} Z$ in $\mathscr{C}(\mathscr{A})$.

(a) Suppose given $A \xrightarrow{a} B$ in $\mathscr{C}$ such that aC, aD are pure monomorphisms. Then aE is a pure monomorphism as well.

(b) Suppose given $A \xrightarrow{a} B$ in $\mathscr{C}$ such that aC, aD are pure epimorphisms. Then aE is a pure epimorphism as well.

(c) Suppose given $A \xrightarrow{a} B$ in $\mathscr{C}$ such that aC, aD are isomorphisms. Then aE is an isomorphism as well.

(d) Suppose given $A \xrightarrow{a} B \xrightarrow{b} C$ in $\mathscr{C}$ such that (aX, bX) and (aZ, bZ) are pure short exact sequences in $\mathscr{A}$. Then (aE, bE) is a pure short exact sequence in $\mathscr{A}$ as well.

$$\begin{tikzcd}
 & & C \arrow[dr, "j", "\bullet" marking] & & \\
 & & & Y \arrow[dr, "q", "|" marking] & \\
E \arrow[rr, "h", "\bullet" marking] & & X \arrow[ur, "p", "|" marking] \arrow[rr, "p\cdot q"', "|" marking] & & Z \\
 & D \arrow[ur, "i"', "\bullet" marking] & & &
\end{tikzcd}$$

◇

Proof. This is dual to the previous lemma 1.4.13. ☐

1.4.14 Lemma. Suppose given a pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathscr{C}(\mathscr{A})$ and the following pushout in $\mathscr{C}(\mathscr{A})$.

$$\begin{tikzcd}
X \arrow[r, "i", "\bullet" marking] \arrow[d, "f"'] & Y \arrow[d, "g"] \\
U \arrow[r, "m", "\bullet" marking] & V
\end{tikzcd}$$

(a) Suppose given $A \xrightarrow{a} B$ in $\mathscr{C}$ such that aU, aZ are pure monomorphisms. Then aV is a pure monomorphism as well.

(b) Suppose given $A \xrightarrow{a} B$ in $\mathscr{C}$ such that aU, aZ are pure epimorphisms. Then aV is a pure epimorphism as well.

(c) Suppose given $A \xrightarrow{a} B$ in $\mathscr{C}$ such that aU, aZ are isomorphisms. Then aV is an isomorphism as well.

(d) Suppose given $A \xrightarrow{a} B \xrightarrow{b} C$ in $\mathscr{C}$ such that (aX, bX), (aY, bY) and (aU, bU) are pure short exact sequences in $\mathscr{A}$. Then (aV, bV) is a pure short exact sequence in $\mathscr{A}$ as well. ◇

Proof. We may choose a cokernel $V \xrightarrow{e} Z$ of m in $\mathscr{C}(\mathscr{A})$, cf. lemma 1.2.7.(b).

Ad (a). This follows from lemma 1.4.8.(a).

Ad (b). This follows from lemma 1.4.8.(b).

Ad (c). This follows from lemma 1.4.8.(c).

Ad (d). The sequence $X \xrightarrow{(-i\ f)} Y \oplus U \xrightarrow{\left(\begin{smallmatrix} g \\ m \end{smallmatrix}\right)} V$ is pure short exact in $\mathscr{C}(\mathscr{A})$, cf. [Büh10, proposition 2.12]. So the result follows from lemmata 1.4.11.(e) and 1.4.9.(a). □

1.4.15 Lemma. Suppose given a pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathscr{C}(\mathscr{A})$ and the following pullback in $\mathscr{C}(\mathscr{A})$.

$$\begin{tikzcd}
V \arrow[r, "e"] \arrow[d, "g"'] & W \arrow[d, "f"] \\
Y \arrow[r, "p"] & Z
\end{tikzcd}$$

(a) Suppose given $A \xrightarrow{a} B$ in $\mathscr{C}$ such that aX, aW are pure monomorphisms. Then aV is a pure monomorphism as well.

(b) Suppose given $A \xrightarrow{a} B$ in $\mathscr{C}$ such that aX, aW are pure epimorphisms. Then aV is a pure epimorphism as well.

(c) Suppose given $A \xrightarrow{a} B$ in $\mathscr{C}$ such that aX, aW are isomorphisms. Then aV is an isomorphism as well.

(d) Suppose given $A \xrightarrow{a} B \xrightarrow{b} C$ in $\mathscr{C}$ such that (aW, bW), (aY, bY) and (aZ, bZ) are pure short exact sequences in $\mathscr{A}$. Then (aV, bV) is a pure short exact sequence in $\mathscr{A}$ as well. ◇

Proof. This is dual to the previous lemma 1.4.14. □

1.4.16 Lemma. Suppose given an exact functor $F \colon \mathscr{A} \to \mathscr{A}$.

(a) The functor $\mathscr{C}(F)$ is exact as well, cf. definition 1.4.3.

(b) Suppose given $X \in \mathrm{Ob}(\mathscr{C}(\mathscr{A}))$ and $A \xrightarrow{a} B \xrightarrow{b} C$ in $\mathscr{C}$ such that (aX, bX) is a pure short exact sequence in $\mathscr{A}$. Then $(a(X\mathscr{C}(F)), b(X\mathscr{C}(F)))$ is a pure short exact sequence in $\mathscr{A}$ as well.

(c) Suppose given $X \in \mathrm{Ob}(\mathscr{C}(\mathscr{A}))$ and $A \xrightarrow{a} B$ in $\mathscr{C}$ such that aX is a pure monomorphism. Then $a(X\mathscr{C}(F))$ is a pure monomorphism as well.

(d) Suppose given $X \in \mathrm{Ob}(\mathscr{C}(\mathscr{A}))$ and $A \xrightarrow{a} B$ in $\mathscr{C}$ such that aX is a pure epimorphism. Then $a(X\mathscr{C}(F))$ is a pure epimorphism as well. $\diamondsuit$

Proof. Ad (a). The functor $\mathscr{C}(F)$ is additive, cf. definition 1.4.3.
Suppose given a pure short exact sequence $X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$ in $\mathscr{C}(\mathscr{A})$. For $A \in \mathrm{Ob}(\mathscr{C})$, the sequence $\big(A(i\mathscr{C}(F)), A(p\mathscr{C}(F))\big) = \big(A(i \star 1_F), A(p \star 1_F)\big) = (AiF, ApF)$ is pure short exact in $\mathscr{A}$ since F is exact. Thus $(i\mathscr{C}(F), p\mathscr{C}(F))$ is pure short exact in $\mathscr{C}(\mathscr{A})$. We conclude that $\mathscr{C}(F)$ is exact.

Ad (b). The sequence $(a(X\mathscr{C}(F)), b(X\mathscr{C}(F))) = (aXF, bXF)$ is pure short exact in $\mathscr{A}$ since F is exact.

Ad (c). The morphism $a(X\mathscr{C}(F)) = aXF$ is a pure monomorphism since F is exact.

Ad (d). The morphism $a(X\mathscr{C}(F)) = aXF$ is a pure epimorphism since F is exact. $\square$

1.5 Triangulated categories

1.5.1 Definition. Suppose given an additive category $\mathscr{A}$ and a functor $\Sigma\colon \mathscr{A} \to \mathscr{A}$. Let $\mathrm{CT}_\Sigma(\mathscr{A})$ denote the subcategory of $\mathbf{Z}_{[0,3]}(\mathscr{A})$ defined by

$$\mathrm{Ob}(\mathrm{CT}_\Sigma(\mathscr{A})) = \{T \in \mathrm{Ob}(\mathbf{Z}_{[0,3]}(\mathscr{A}))\colon T_3 = T_0\Sigma\}$$

and

$$\mathrm{Mor}(\mathrm{CT}_\Sigma(\mathscr{A})) = \{f \in \mathrm{Mor}(\mathbf{Z}_{[0,3]}(\mathscr{A}))\colon f_3 = f_0\Sigma\}.$$

The objects of $\mathrm{CT}_\Sigma(\mathscr{A})$ are called *candidate triangles* and $\mathrm{CT}_\Sigma(\mathscr{A})$ is called the *category of candiate triangles* in $\mathscr{A}$ with respect to Σ. Objects $T \in \mathrm{Ob}(\mathrm{CT}_\Sigma(\mathscr{A}))$ are usually denoted by $T = (f, g, h)$ or $T = \big(X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X\Sigma\big)$, where $X = T_0$, $Y = T_1$, $Z = T_2$, $f = T_{0 \to 1}$, $g = T_{1 \to 2}$ and $h = T_{2 \to 3}$. $\diamondsuit$

1.5.2 Remark. We will also use candidate triangles to introduce pseudo-triangles in some exact categories, cf. definitions 1.9.16, 2.2.8, 3.3.24 and 3.4.17. Sometimes they give rise to triangles in a triangulated category, cf. definition 2.2.14. $\diamondsuit$

1.5.3 Definition. Suppose given an additive category $\mathscr{A}$, an equivalence $\Sigma\colon \mathscr{A} \to \mathscr{A}$ and a strictly full subcategory $\mathfrak{T} \subseteq \mathrm{CT}_\Sigma(\mathscr{A})$ such that the following four conditions hold. Cf. definition 1.5.1.

(TR1) For $X \in \mathrm{Ob}(\mathscr{A})$, we have $\big(X \xrightarrow{1} X \longrightarrow 0_{\mathscr{A}} \longrightarrow X\Sigma\big) \in \mathrm{Ob}(\mathfrak{T})$.
For $X \xrightarrow{f} Y$ in $\mathscr{A}$, there exists $\big(X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X\Sigma\big) \in \mathrm{Ob}(\mathfrak{T})$.

(TR2) Suppose given $\big(X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X\Sigma\big) \in \mathrm{Ob}(\mathrm{CT}_\Sigma(\mathscr{A}))$.
We have $\big(X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X\Sigma\big) \in \mathrm{Ob}(\mathfrak{T})$ if and only if
$\big(Y \xrightarrow{g} Z \xrightarrow{h} X\Sigma \xrightarrow{-f\Sigma} Y\Sigma\big) \in \mathrm{Ob}(\mathfrak{T})$.

(TR3) Suppose given $\big(X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X\Sigma\big), \big(X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X\Sigma\big) \in \mathrm{Ob}(\mathfrak{T})$.
Suppose given $X \xrightarrow{f} X'$ and $Y \xrightarrow{g} Y'$ in $\mathscr{A}$ such that $u \cdot g = f \cdot u'$.
Then there exists $Z \xrightarrow{h} Z'$ in $\mathscr{A}$ such that $v \cdot h = g \cdot v'$ and $w \cdot f\Sigma = h \cdot w'$.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & X\Sigma \arrow[d, "f\Sigma"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X\Sigma
\end{tikzcd}$$

(TR4) Suppose given $\big(X \xrightarrow{i} Y \xrightarrow{p} C \xrightarrow{u} X\Sigma\big), \big(Y \xrightarrow{j} Z \xrightarrow{q} D \xrightarrow{v} Y\Sigma\big)$,
$\big(X \xrightarrow{i \cdot j} Z \xrightarrow{r} E \xrightarrow{w} X\Sigma\big) \in \mathrm{Ob}(\mathfrak{T})$. Then there exists
$\big(C \xrightarrow{k} E \xrightarrow{s} D \xrightarrow{x} C\Sigma\big) \in \mathrm{Ob}(\mathfrak{T})$ such that $p \cdot k = j \cdot r$, $r \cdot s = q$, $k \cdot w = u$, $s \cdot v = w \cdot i\Sigma$ and such that $v \cdot p\Sigma = x$.

$$\begin{tikzcd}
 & & & X\Sigma \arrow[dddrr, "1"] & & \\
 & & C \arrow[ur, "u"] \arrow[ddrr, "k"] & & & \\
 & Y \arrow[ur, "p"] \arrow[dr, "j"] & & & & \\
X \arrow[ur, "i"] \arrow[rr, "i \cdot j"'] & & Z \arrow[rr, "r"] \arrow[dr, "q"'] & & E \arrow[r, "w"] \arrow[dl, "s"] & X\Sigma \arrow[ddl, "i\Sigma"] \\
 & & & D \arrow[dl, "x"'] \arrow[dr, "v"] & & \\
 & & C\Sigma & & Y\Sigma \arrow[ll, "p\Sigma"'] &
\end{tikzcd}$$

We call the tuple $(\mathscr{A}, \Sigma, \mathfrak{T})$ a *triangulated category*. The objects of $\mathfrak{T}$ are called *triangles* in $(\mathscr{A}, \Sigma, \mathfrak{T})$. If Σ is an isomorphism of categories, we say that $(\mathscr{A}, \Sigma, \mathfrak{T})$ is a *strict triangulated category*. We abbreviate $\mathscr{A} = (\mathscr{A}, \Sigma, \mathfrak{T})$ if unambiguous.

Suppose that $\mathscr{A} = (\mathscr{A}, \Sigma, \mathfrak{T})$ is a strict triangulated category. For $X \in \mathrm{Ob}(\mathscr{A})$ and $k \in \mathbf{Z}$, we usually write $X^{[k]} = X\Sigma^k$. For $f \in \mathrm{Mor}(\mathscr{A})$ and $k \in \mathbf{Z}$, we usually write $f^{[k]} = f\Sigma^k$.

We remark that (TR3) follows from the other axioms by [May01, lemma 2.2].

For a full subcategory $\mathscr{S} \subseteq \mathscr{A}$ and $k \in \mathbf{Z}$, we define the full subcategory $\mathscr{S}^{[k]}$ of $\mathscr{A}$ by setting $\mathrm{Ob}(\mathscr{S}^{[k]}) = \{X^{[k]} \in \mathrm{Ob}(\mathscr{A}) \colon X \in \mathrm{Ob}(\mathscr{S})\}$.

For $S \in \mathrm{Ob}(\mathscr{A})$ and $k \in \mathbf{Z}$, we define the full subcategories $S^{\perp_{<k}}$, $S^{\perp_{>k}}$, ${}^{\perp_{<k}}S$ and ${}^{\perp_{>k}}S$ of

$\mathscr{A}$ by setting

$$\mathrm{Ob}(S^{\perp_{<k}}) = \{X \in \mathrm{Ob}(\mathscr{A})\colon {}_{\mathscr{A}}(S, X^{[\ell]}) = 0 \text{ for } \ell \in \mathbf{Z}_{<k}\},$$

$$\mathrm{Ob}(S^{\perp_{>k}}) = \{X \in \mathrm{Ob}(\mathscr{A})\colon {}_{\mathscr{A}}(S, X^{[\ell]}) = 0 \text{ for } \ell \in \mathbf{Z}_{>k}\},$$

$$\mathrm{Ob}({}^{\perp_{<k}}S) = \{X \in \mathrm{Ob}(\mathscr{A})\colon {}_{\mathscr{A}}(X, S^{[\ell]}) = 0 \text{ for } \ell \in \mathbf{Z}_{<k}\}$$

and

$$\mathrm{Ob}({}^{\perp_{>k}}S) = \{X \in \mathrm{Ob}(\mathscr{A})\colon {}_{\mathscr{A}}(X, S^{[\ell]}) = 0 \text{ for } \ell \in \mathbf{Z}_{>k}\}.$$

$\diamondsuit$

1.5.4 Definition. Suppose given triangulated categories $(\mathscr{C}, \Sigma, \mathfrak{T})$, $(\mathscr{D}, \Sigma', \mathfrak{S})$ and an additive functor $F\colon \mathscr{C} \to \mathscr{D}$. We say that F is *exact* if there exists an isotransformation $\Sigma F \xrightarrow{\lambda} F\Sigma'$ such that for $\big(X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X\Sigma\big) \in \mathrm{Ob}(\mathfrak{T})$, we have $\big(XF \xrightarrow{fF} YF \xrightarrow{gF} ZF \xrightarrow{hF\cdot X\lambda} XF\Sigma'\big) \in \mathrm{Ob}(\mathfrak{S})$. $\diamondsuit$

1.5.5 Definition. Suppose given a triangulated category $(\mathscr{C}, \Sigma, \mathfrak{T})$, an abelian category $\mathscr{A}$ and an additive functor $F\colon \mathscr{C} \to \mathscr{A}$. We say that F is *homological* if for $\big(X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X\Sigma\big) \in \mathrm{Ob}(\mathfrak{T})$, the sequence $XF \xrightarrow{fF} YF \xrightarrow{gF} ZF$ is exact in $\mathscr{A}$. $\diamondsuit$

1.5.6 Definition. Suppose given a strict triangulated category $\mathscr{C}$ and $m \in \mathbf{Z}$. Suppose given full subcategories $\mathscr{S}_k \subseteq \mathscr{C}$ for $k \in [0, m]$. We recursively define the full subcategory $\mathop{\ast}\limits_{k\in[0,m]} \mathscr{S}_k$ of $\mathscr{C}$ as follows. If $m < 0$, let $\mathop{\ast}\limits_{k\in[0,m]} \mathscr{S}_k = \mathrm{Z}_{\mathscr{C}}$. If $m \geq 0$, let

$$\mathrm{Ob}\Big(\mathop{\ast}\limits_{k\in[0,m]} \mathscr{S}_k\Big) = \{Y \in \mathrm{Ob}(\mathscr{C})\colon \text{There exists a triangle } X \longrightarrow Y \longrightarrow Z \longrightarrow X^{[1]} \text{ in } \mathscr{C} \text{ such that } X \in \mathrm{Ob}\Big(\mathop{\ast}\limits_{k\in[0,m-1]} \mathscr{S}_k\Big) \text{ and } Z \in \mathrm{Ob}(\mathscr{S}_m)\}.$$

We call $\mathop{\ast}\limits_{k\in[0,m]} \mathscr{S}_k$ a *category of extensions* in $\mathscr{C}$.

In case $m = 1$, we also write $\mathscr{S}_0 * \mathscr{S}_1 = \mathop{\ast}\limits_{k\in[0,1]} \mathscr{S}_k$.

In case $m = 2$, we also write $\mathscr{S}_0 * \mathscr{S}_1 * \mathscr{S}_2 = \mathop{\ast}\limits_{k\in[0,2]} \mathscr{S}_k$. $\diamondsuit$

1.5.7 Definition. Suppose given a strict triangulated category $\mathscr{C}$ and $a, b \in \mathbf{Z}$. Suppose given full subcategories $\mathscr{S}_k \subseteq \mathscr{C}$ for $k \in [a, b]$. We write $\mathop{\ast}\limits_{k\in[a,b]} \mathscr{S}_k = \mathop{\ast}\limits_{k\in[0,b-a]} \mathscr{S}_{a+k}$. $\diamondsuit$

1.5.8 Definition. Suppose given a strict triangulated category $\mathscr{C}$ and a full additive subcategory $\mathscr{D} \subseteq \mathscr{C}$. We say that $\mathscr{D}$ is a *full triangulated subcategory* of $\mathscr{C}$ if $\mathscr{D} * \mathscr{D} = \mathscr{D}$ and if $X^{[1]}, X^{[-1]} \in \mathrm{Ob}(\mathscr{D})$ for $X \in \mathrm{Ob}(\mathscr{D})$. $\diamondsuit$

1.5.9 Lemma. Suppose given a strict triangulated category $\mathscr{C}$, $m \in \mathbf{Z}$ and full subcategories $\mathscr{S}_k \subseteq \mathscr{C}$ for $k \in [0,m]$. Suppose given $Y \in \mathrm{Ob}\left(\mathop{\ast}\limits_{k\in[0,m]} \mathscr{S}_k\right)$. Then we have $Y^{[1]} \in \mathrm{Ob}\left(\mathop{\ast}\limits_{k\in[0,m]} \mathscr{S}_k^{[1]}\right)$. ◇

Proof. We proceed by induction on m. If $m{<}0$, then we have $Y^{[1]} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{C}})$ since $Y \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{C}})$. If $m{\geq}0$, we may choose a triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X^{[1]}$ in $\mathscr{C}$ such that $X \in \mathrm{Ob}\left(\mathop{\ast}\limits_{k\in[0,m-1]} \mathscr{S}_k\right)$ and $Z \in \mathrm{Ob}(\mathscr{S}_m)$. Rotation yields a triangle $X^{[1]} \longrightarrow Y^{[1]} \longrightarrow Z^{[1]} \longrightarrow X^{[2]}$ in $\mathscr{C}$. We have $Z^{[1]} \in \mathrm{Ob}(\mathscr{S}_m^{[1]})$ and $X^{[1]} \in \mathrm{Ob}\left(\mathop{\ast}\limits_{k\in[0,m-1]} \mathscr{S}_k^{[1]}\right)$ by induction. We conclude that $Y^{[1]} \in \mathrm{Ob}\left(\mathop{\ast}\limits_{k\in[0,m]} \mathscr{S}_k^{[1]}\right)$. □

1.5.10 Lemma. Suppose given a strict triangulated category $\mathscr{C}$. Suppose given $m, n \in \mathbf{Z}$, full subcategories $\mathscr{S}_k \subseteq \mathscr{C}$ for $k \in [0,m]$ and full subcategories $\mathscr{R}_\ell \subseteq \mathscr{C}$ for $\ell \in [0,n]$ such that ${}_{\mathscr{C}}(\mathscr{S}_k\,, \mathscr{R}_\ell) = 0$ for $k \in [0,m]$ and $\ell \in [0,n]$. Then we have ${}_{\mathscr{C}}\left(\mathop{\ast}\limits_{k\in[0,m]} \mathscr{S}_k\,, \mathop{\ast}\limits_{\ell\in[0,n]} \mathscr{R}_\ell\right) = 0$. ◇

Proof. Suppose given $\ell \in [0,n]$.
Using induction on m, we want to show that ${}_{\mathscr{C}}\left(\mathop{\ast}\limits_{k\in[0,m]} \mathscr{S}_k\,, \mathscr{R}_\ell\right) = 0$. If $m < 0$, this follows from $\mathop{\ast}\limits_{k\in[0,m]} \mathscr{S}_k = \mathrm{Z}_{\mathscr{C}}$. Suppose that $m \geq 0$ and suppose given $Y \in \mathrm{Ob}\left(\mathop{\ast}\limits_{k\in[0,m]} \mathscr{S}_k\right)$. We may choose a triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X^{[1]}$ in $\mathscr{C}$ such that $X \in \mathrm{Ob}\left(\mathop{\ast}\limits_{k\in[0,m-1]} \mathscr{S}_k\right)$ and $Z \in \mathrm{Ob}(\mathscr{S}_m)$. We have ${}_{\mathscr{C}}(\mathscr{S}_m\,, \mathscr{R}_\ell) = 0$ and ${}_{\mathscr{C}}\left(\mathop{\ast}\limits_{k\in[0,m-1]} \mathscr{S}_k\,, \mathscr{R}_\ell\right) = 0$ by induction. Thus ${}_{\mathscr{C}}(Y, \mathscr{R}_\ell) = 0$. We conclude that ${}_{\mathscr{C}}\left(\mathop{\ast}\limits_{k\in[0,m]} \mathscr{S}_k\,, \mathscr{R}_\ell\right) = 0$.
Using induction on n, we want to show that ${}_{\mathscr{C}}\left(\mathop{\ast}\limits_{k\in[0,m]} \mathscr{S}_k\,, \mathop{\ast}\limits_{\ell\in[0,n]} \mathscr{R}_\ell\right) = 0$. If $n < 0$, this follows from $\mathop{\ast}\limits_{\ell\in[0,n]} \mathscr{R}_\ell = \mathrm{Z}_{\mathscr{C}}$. Suppose that $n \geq 0$ and suppose given $Y \in \mathrm{Ob}\left(\mathop{\ast}\limits_{\ell\in[0,n]} \mathscr{R}_\ell\right)$. We may choose a triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X^{[1]}$ in $\mathscr{C}$ such that $X \in \mathrm{Ob}\left(\mathop{\ast}\limits_{\ell\in[0,n-1]} \mathscr{R}_\ell\right)$ and $Z \in \mathrm{Ob}(\mathscr{R}_n)$. We have ${}_{\mathscr{C}}\left(\mathop{\ast}\limits_{k\in[0,m]} \mathscr{S}_k\,, \mathscr{R}_n\right) = 0$ and ${}_{\mathscr{C}}\left(\mathop{\ast}\limits_{k\in[0,m]} \mathscr{S}_k\,, \mathop{\ast}\limits_{\ell\in[0,n-1]} \mathscr{R}_\ell\right) = 0$ by induction. Thus ${}_{\mathscr{C}}\left(\mathop{\ast}\limits_{k\in[0,m]} \mathscr{S}_k\,, Y\right) = 0$. We conclude that ${}_{\mathscr{C}}\left(\mathop{\ast}\limits_{k\in[0,m]} \mathscr{S}_k\,, \mathop{\ast}\limits_{\ell\in[0,n]} \mathscr{R}_\ell\right) = 0$. □

1.5.11 Definition. For a strict triangulated category $\mathscr{C}$ and a full triangulated subcategory $\mathscr{S} \subseteq \mathscr{C}$, we denote the *Verdier quotient* of $\mathscr{C}$ by $\mathscr{S}$ by $\mathscr{C}/\!\!/\mathscr{S}$ and the associated *quotient functor* (or *localisation functor*) by $\mathfrak{L}_{\mathscr{C},\mathscr{S}}\colon \mathscr{C} \to \mathscr{C}/\!\!/\mathscr{S}$, cf. [Kra10, section 4.6]. We have the following universal property. ◇

1.5.12 Lemma (universal property)**.** [Kra10, proposition 4.6.2.(4)]
Suppose given a strict triangulated category $\mathscr{C}$ and a full triangulated subcategory $\mathscr{S} \subseteq \mathscr{C}$.

Suppose given a triangulated category $\mathscr{D}$ and an exact functor $F\colon \mathscr{C} \to \mathscr{D}$ such that $XF \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$ for $X \in \mathrm{Ob}(\mathscr{S})$. Then there exists a unique exact functor $\hat{F}\colon \mathscr{C}/\!\!/\mathscr{S} \to \mathscr{D}$ such that $\mathfrak{L}_{\mathscr{C},\mathscr{S}} \cdot \hat{F} = F$.

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[d, "\mathfrak{L}_{\mathscr{C},\mathscr{S}}"'] & \mathscr{D} \\
\mathscr{C}/\!\!/\mathscr{S} \arrow[ur, "\hat{F}"'] &
\end{tikzcd}$$

$\diamondsuit$

1.6 Adjoint functors

1.6.1 Definition. Suppose given categories $\mathscr{A}$, $\mathscr{B}$ and functors $F\colon \mathscr{A} \to \mathscr{B}$, $G\colon \mathscr{B} \to \mathscr{A}$. Suppose given transformations $\eta\colon 1_{\mathscr{A}} \to FG$ and $\varepsilon\colon GF \to 1_{\mathscr{B}}$. We say that the tuple $(F, G, \eta, \varepsilon)$ is an *adjunction* if we have $(\eta \star 1_F) \cdot (1_F \star \varepsilon) = 1_F$ and $(1_G \star \eta) \cdot (\varepsilon \star 1_G) = 1_G$.

In this case, we say that η is the *unit* and that ε is the *counit* of the adjunction. $\diamondsuit$

1.6.2 Definition. Suppose given categories $\mathscr{A}$, $\mathscr{B}$ and functors $F\colon \mathscr{A} \to \mathscr{B}$, $G\colon \mathscr{B} \to \mathscr{A}$. We say that F is *left-adjoint* to G if there exists an adjunction $(F, G, \eta, \varepsilon)$. In this case, we also say that G is *right-adjoint* to F and write $F \dashv G$ or $G \vdash F$.

We say that F and G are mutually quasi-inverse equivalences if there exists an adjunction $(F, G, \eta, \varepsilon)$ such that η and ε are isotransformations. In this case, we also say that F and G are equivalences.

We say that F and G are mutually inverse isomorphisms of categories if there exists an adjunction $(F, G, \eta, \varepsilon)$ such that $\eta = 1_{1_{\mathscr{A}}}$ and $\varepsilon = 1_{1_{\mathscr{B}}}$, i.e. such that $FG = 1_{\mathscr{A}}$ and such that $GF = 1_{\mathscr{B}}$. In this case, we also say that F and G are isomorphisms of categories. $\diamondsuit$

1.6.3 Remark. Suppose given categories $\mathscr{A}$, $\mathscr{B}$ and functors $F\colon \mathscr{A} \to \mathscr{B}$, $G\colon \mathscr{B} \to \mathscr{A}$.
F is left-adjoint to G if and only if there exist bijective maps $\Phi_{X,Y}\colon {}_{\mathscr{B}}(XF, Y) \to {}_{\mathscr{A}}(X, YG)$ for $X \in \mathrm{Ob}(\mathscr{A})$ and $Y \in \mathrm{Ob}(\mathscr{B})$ such that $(gF \cdot f \cdot h)\Phi_{X',Y'} = g \cdot f\Phi_{X,Y} \cdot hG$ for $X' \xrightarrow{g} X$ in $\mathscr{A}$ and $XF \xrightarrow{f} Y$, $Y \xrightarrow{h} Y'$ in $\mathscr{B}$. The maps $\Phi_{X,Y}$ are called *natural bijections.*
Cf. [Mac71, section IV.1]. $\diamondsuit$

1.6.4 Remark. Suppose given categories $\mathscr{A}$, $\mathscr{B}$ and a functor $F\colon \mathscr{A} \to \mathscr{B}$. F is an equivalence if and only if it is full, faithful and dense, cf. [Mac71, theorem IV.4.1]. A quasi-inverse of such a full, faithful and dense functor can be constructed using the following lemma 1.6.5. $\diamondsuit$

1.6.5 Lemma. [Mac71, (proof of) theorem IV.4.1]
Suppose given categories $\mathscr{C}$, $\mathscr{D}$ and a full and faithful functor $F\colon \mathscr{C} \to \mathscr{D}$. For $X \in \mathrm{Ob}(\mathscr{D})$, suppose given an object $XG \in \mathrm{Ob}(\mathscr{C})$ and an isomorphism $X \xrightarrow{X\varsigma} XGF$ in $\mathscr{D}$. For $X \xrightarrow{f} Y$ in $\mathscr{D}$, there exists a unique morphism $XG \xrightarrow{fG} YG$ in $\mathscr{C}$ such that $f = X\varsigma \cdot fGF \cdot (Y\varsigma)^{-1}$. This yields a functor $G\colon \mathscr{D} \to \mathscr{C}$ and an isotransformation $\varsigma\colon 1_{\mathscr{D}} \to GF$. Moreover, F and G are mutually quasi-inverse equivalences. $\diamondsuit$

1.6.6 Lemma. [Bor94, proposition 3.4.1]
Suppose given an adjunction $(F, G, \eta, \varepsilon)$.

(a) The functor F is full and faithful if and only if η is an isotransformation.

(b) The functor G is full and faithful if and only if ε is an isotransformation.

$\diamondsuit$

1.6.7 Lemma (composition of adjunctions)**.** Suppose given categories and functors as follows: $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B} \underset{I}{\overset{H}{\rightleftarrows}} \mathscr{C}$. If F is left-adjoint to G and H is left-adjoint to I, then $F \cdot H$ is left-adjoint to $I \cdot G$. More precisely, suppose given adjunctions $(F, G, \eta, \varepsilon)$ and (H, I, λ, μ). Then $(F \cdot H, I \cdot G, \eta \cdot (1_F \star \lambda \star 1_G), (1_I \star \varepsilon \star 1_H) \cdot \mu)$ is an adjunction as well. $\diamondsuit$

Proof. We have $(\eta \star 1_F) \cdot (1_F \star \varepsilon) = 1_F$, $(1_G \star \eta) \cdot (\varepsilon \star 1_G) = 1_G$, $(\lambda \star 1_H) \cdot (1_H \star \mu) = 1_H$ and $(1_I \star \lambda) \cdot (\mu \star 1_I) = 1_I$. Note that we have $\lambda \star \varepsilon = (1_{\mathscr{B}} \star \varepsilon) \cdot (\lambda \star 1_{\mathscr{B}}) = (\varepsilon \star 1_{\mathscr{B}}) \cdot (1_{\mathscr{B}} \star \lambda) = \varepsilon \star \lambda$. We obtain

$$\begin{aligned}
& ((\eta \cdot (1_F \star \lambda \star 1_G)) \star 1_{FH}) \cdot (1_{FH} \star ((1_I \star \varepsilon \star 1_H) \cdot \mu)) \\
={}& ((\eta \cdot (1_F \star \lambda \star 1_G)) \star (1_{FH} \cdot 1_{FH})) \cdot ((1_{FH} \cdot 1_{FH}) \star ((1_I \star \varepsilon \star 1_H) \cdot \mu)) \\
={}& (\eta \star 1_{FH}) \cdot (1_F \star \lambda \star 1_{GFH}) \cdot (1_{FHI} \star \varepsilon \star 1_H) \cdot (1_{FH} \star \mu) \\
={}& (\eta \star 1_{FH}) \cdot (1_F \star (\lambda \star 1_{GF}) \star 1_H) \cdot (1_F \star (1_{HI} \star \varepsilon) \star 1_H) \cdot (1_{FH} \star \mu) \\
={}& (\eta \star 1_{FH}) \cdot (1_F \star ((\lambda \star 1_{GF}) \cdot (1_{HI} \star \varepsilon)) \star 1_H) \cdot (1_{FH} \star \mu) \\
={}& (\eta \star 1_{FH}) \cdot (1_F \star \lambda \star \varepsilon \star 1_H) \cdot (1_{FH} \star \mu) \\
={}& (\eta \star 1_F \star 1_{\mathscr{B}} \star 1_H) \cdot (1_F \star \varepsilon \star \lambda \star 1_H) \cdot (1_{FH} \star \mu) \\
={}& (1_F \star \lambda \star 1_H) \cdot (1_F \star 1_H \star \mu) \\
={}& 1_F \star 1_H \\
={}& 1_{FH}
\end{aligned}$$

and

$$\begin{aligned}
& (1_{IG} \star (\eta \cdot (1_F \star \lambda \star 1_G))) \cdot (((1_I \star \varepsilon \star 1_H) \cdot \mu)) \star 1_{IG}) \\
={}& (1_{IG} \star \eta) \cdot (1_{IGF} \star \lambda \star 1_G) \cdot (1_I \star \varepsilon \star 1_{HIG}) \cdot (\mu \star 1_{IG}) \\
={}& (1_{IG} \star \eta) \cdot (1_I \star ((1_{GF} \star \lambda) \cdot (\varepsilon \star 1_{HI})) \star 1_G) \cdot (\mu \star 1_{IG}) \\
={}& (1_{IG} \star \eta) \cdot (1_I \star \varepsilon \star \lambda \star 1_G) \cdot (\mu \star 1_{IG}) \\
={}& (1_I \star \lambda \star 1_G) \cdot (\mu \star 1_{IG}) \\
={}& 1_I \star 1_G \\
={}& 1_{IG}
\end{aligned}$$

We conclude that $(F \cdot H, I \cdot G, \eta \cdot (1_F \star \lambda \star 1_G), (1_I \star \varepsilon \star 1_H) \cdot \mu)$ is an adjunction. $\square$

1.6.8 Lemma. Suppose given categories $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$. Suppose given functors $F\colon \mathscr{A} \to \mathscr{B}$ and $G\colon \mathscr{B} \to \mathscr{A}$ such that F is left-adjoint to G. Then $\mathscr{C}(F)$ is left-adjoint to $\mathscr{C}(G)$, cf. definition 1.4.3.

More precisely, suppose given an adjunction $(F, G, \eta, \varepsilon)$. Then $(\mathscr{C}(F), \mathscr{C}(G), \mathscr{C}(\eta), \mathscr{C}(\varepsilon))$ is an adjunction as well. $\diamondsuit$

Proof. We have $(\eta \star 1_F) \cdot (1_F \star \varepsilon) = 1_F$ and $(1_G \star \eta) \cdot (\varepsilon \star 1_G) = 1_G$. Using lemma 1.4.6, we obtain the transformations $1_{\mathscr{C}(\mathscr{A})} \xrightarrow{\mathscr{C}(\eta)} \mathscr{C}(F)\mathscr{C}(G)$ and $\mathscr{C}(G)\mathscr{C}(F) \xrightarrow{\mathscr{C}(\varepsilon)} 1_{\mathscr{C}(\mathscr{B})}$. Moreover, we have

$$\begin{aligned}
(\mathscr{C}(\eta) \star 1_{\mathscr{C}(F)}) \cdot (1_{\mathscr{C}(F)} \star \mathscr{C}(\varepsilon)) &= (\mathscr{C}(\eta) \star \mathscr{C}(1_F)) \cdot (\mathscr{C}(1_F) \star \mathscr{C}(\varepsilon)) \\
&= \mathscr{C}(\eta \star 1_F) \cdot \mathscr{C}(1_F \star \varepsilon) = \mathscr{C}((\eta \star 1_F) \cdot (1_F \star \varepsilon)) \\
&= \mathscr{C}(1_F) = 1_{\mathscr{C}(F)}
\end{aligned}$$

and

$$\begin{aligned}
(1_{\mathscr{C}(G)} \star \mathscr{C}(\eta)) \cdot (\mathscr{C}(\varepsilon) \star 1_{\mathscr{C}(G)}) &= (\mathscr{C}(1_G) \star \mathscr{C}(\eta)) \cdot (\mathscr{C}(\varepsilon) \star \mathscr{C}(1_G)) \\
&= \mathscr{C}(1_G \star \eta) \cdot \mathscr{C}(\varepsilon \star 1_G) = \mathscr{C}((1_G \star \eta) \cdot (\varepsilon \star 1_G)) \\
&= \mathscr{C}(1_G) = 1_{\mathscr{C}(G)} \ .
\end{aligned}$$

We conclude that $(\mathscr{C}(F), \mathscr{C}(G), \mathscr{C}(\eta), \mathscr{C}(\varepsilon))$ is an adjunction. $\square$

1.6.9 Lemma. Suppose given functors $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ such that F is left-adjoint to G. Suppose given a full subcategory $\mathscr{A}$ of $\mathscr{C}$ and a full subcategory $\mathscr{B}$ of $\mathscr{D}$ such that $XF \in \mathrm{Ob}(\mathscr{B})$ for $X \in \mathrm{Ob}(\mathscr{A})$ and $YG \in \mathrm{Ob}(\mathscr{A})$ for $Y \in \mathrm{Ob}(\mathscr{B})$. Then $F|_{\mathscr{A}}^{\mathscr{B}}\colon \mathscr{A} \to \mathscr{B}$ is left-adjoint to $G|_{\mathscr{B}}^{\mathscr{A}}\colon \mathscr{B} \to \mathscr{A}$ as well.
More precisely, suppose given an adjunction $(F, G, \eta, \varepsilon)$. Then $(F|_{\mathscr{A}}^{\mathscr{B}}, G|_{\mathscr{B}}^{\mathscr{A}}, \eta|_{\mathscr{A}}^{\mathscr{A}}, \varepsilon|_{\mathscr{B}}^{\mathscr{B}})$ is an adjunction as well. $\diamondsuit$

Proof. We abbreviate $\underline{F} = F|_{\mathscr{A}}^{\mathscr{B}}$, $\underline{G} = G|_{\mathscr{B}}^{\mathscr{A}}$, $\underline{\eta} = \eta|_{\mathscr{A}}^{\mathscr{A}}$ and $\underline{\varepsilon} = \varepsilon|_{\mathscr{B}}^{\mathscr{B}}$.
We have $(\eta \star 1_F) \cdot (1_F \star \varepsilon) = 1_F$ and $(1_G \star \eta) \cdot (\varepsilon \star 1_G) = 1_G$.
We have $(\underline{\eta} \star 1_{\underline{F}}) \cdot (1_{\underline{F}} \star \underline{\varepsilon}) = 1_{\underline{F}}$ since

$$\begin{aligned}
((\underline{\eta} \star 1_{\underline{F}}) \cdot (1_{\underline{F}} \star \underline{\varepsilon})) \star \mathrm{Inc}_{\mathscr{B}}^{\mathscr{D}} &= ((\underline{\eta} \star 1_{\underline{F}}) \cdot (1_{\underline{F}} \star \underline{\varepsilon})) \star (1_{\mathrm{Inc}_{\mathscr{B}}^{\mathscr{D}}} \cdot 1_{\mathrm{Inc}_{\mathscr{B}}^{\mathscr{D}}}) \\
&= (\underline{\eta} \star 1_{\underline{F}} \star 1_{\mathrm{Inc}_{\mathscr{B}}^{\mathscr{D}}}) \cdot (1_{\underline{F}} \star \underline{\varepsilon} \star 1_{\mathrm{Inc}_{\mathscr{B}}^{\mathscr{D}}}) \\
&= (\underline{\eta} \star 1_{\mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}}} \star 1_F) \cdot (1_{\underline{F}} \star 1_{\mathrm{Inc}_{\mathscr{B}}^{\mathscr{D}}} \star \varepsilon) \\
&= (1_{\mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}}} \star \eta \star 1_F) \cdot (1_{\mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}}} \star 1_F \star \varepsilon) \\
&= (1_{\mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}}} \cdot 1_{\mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}}}) \star ((\eta \star 1_F) \cdot (1_F \star \varepsilon)) \\
&= 1_{\mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}}} \star 1_F \\
&= 1_{\underline{F}} \star \mathrm{Inc}_{\mathscr{B}}^{\mathscr{D}} \ .
\end{aligned}$$

We have $(1_{\underline{G}} \star \underline{\eta}) \cdot (\underline{\varepsilon} \star 1_{\underline{G}}) = 1_{\underline{G}}$ since

$$\begin{aligned}
((1_{\underline{G}} \star \underline{\eta}) \cdot (\underline{\varepsilon} \star 1_{\underline{G}})) \star \mathrm{Inc}^{\mathscr{C}}_{\mathscr{A}} &= ((1_{\underline{G}} \star \underline{\eta} \star 1_{\mathrm{Inc}^{\mathscr{C}}_{\mathscr{A}}}) \cdot (\underline{\varepsilon} \star 1_{\underline{G}} \star 1_{\mathrm{Inc}^{\mathscr{C}}_{\mathscr{A}}})) \\
&= ((1_{\underline{G}} \star 1_{\mathrm{Inc}^{\mathscr{C}}_{\mathscr{A}}} \star \eta) \cdot (\underline{\varepsilon} \star 1_{\mathrm{Inc}^{\mathscr{D}}_{\mathscr{B}}} \star 1_G)) \\
&= ((1_{\mathrm{Inc}^{\mathscr{D}}_{\mathscr{B}}} \star 1_G \star \eta) \cdot (1_{\mathrm{Inc}^{\mathscr{D}}_{\mathscr{B}}} \star \varepsilon \star 1_G)) \\
&= 1_{\mathrm{Inc}^{\mathscr{D}}_{\mathscr{B}}} \star ((1_G \star \eta) \cdot (\varepsilon \star 1_G)) \\
&= 1_{\mathrm{Inc}^{\mathscr{D}}_{\mathscr{B}}} \star 1_G \\
&= 1_{\underline{G}} \star \mathrm{Inc}^{\mathscr{C}}_{\mathscr{A}} \ .
\end{aligned}$$

We conclude that $(\underline{F}, \underline{G}, \underline{\eta}, \underline{\varepsilon})$ is an adjunction. $\square$

1.6.10 Lemma. Suppose given categories $\mathscr{C}$, $\mathscr{D}$ and mutually quasi-inverse equivalences $F\colon \mathscr{C} \to \mathscr{D}$, $G\colon \mathscr{D} \to \mathscr{C}$. Suppose given a full subcategory $\mathscr{A}$ of $\mathscr{C}$ and a full subcategory $\mathscr{B}$ of $\mathscr{D}$ such that $XF \in \mathrm{Ob}(\mathscr{B})$ for $X \in \mathrm{Ob}(\mathscr{A})$ and $YG \in \mathrm{Ob}(\mathscr{A})$ for $Y \in \mathrm{Ob}(\mathscr{B})$. Then $F|^{\mathscr{B}}_{\mathscr{A}}\colon \mathscr{A} \to \mathscr{B}$ and $G|^{\mathscr{A}}_{\mathscr{B}}\colon \mathscr{B} \to \mathscr{A}$ are mutually quasi-inverse equivalences as well.

More precisely, suppose given an adjunction $(F, G, \eta, \varepsilon)$ such that η and ε are isotransformations. Then $(F|^{\mathscr{B}}_{\mathscr{A}}\, , G|^{\mathscr{A}}_{\mathscr{B}}\, , \eta|^{\mathscr{A}}_{\mathscr{A}}\, , \varepsilon|^{\mathscr{B}}_{\mathscr{B}})$ is an adjunction such that $\eta|^{\mathscr{A}}_{\mathscr{A}}$ and $\varepsilon|^{\mathscr{B}}_{\mathscr{B}}$ are isotransformations as well.

In particular, if F and G are mutually inverse isomorphisms of categories, then $F|^{\mathscr{B}}_{\mathscr{A}}\colon \mathscr{A} \to \mathscr{B}$ and $G|^{\mathscr{A}}_{\mathscr{B}}\colon \mathscr{B} \to \mathscr{A}$ are mutually inverse isomorphisms of categories as well. $\diamondsuit$

Proof. By lemma 1.6.9, $(F|^{\mathscr{B}}_{\mathscr{A}}\, , G|^{\mathscr{A}}_{\mathscr{B}}\, , \eta|^{\mathscr{A}}_{\mathscr{A}}\, , \varepsilon|^{\mathscr{B}}_{\mathscr{B}})$ is an adjunction as well. Now $\eta|^{\mathscr{A}}_{\mathscr{A}}$ is an isotransformation since $X\eta|^{\mathscr{A}}_{\mathscr{A}} = X\eta$ is an isomorphism in $\mathscr{A}$ for $X \in \mathrm{Ob}(\mathscr{A})$. Similarly, $\varepsilon|^{\mathscr{B}}_{\mathscr{B}}$ is an isotransformation since $Y\varepsilon|^{\mathscr{B}}_{\mathscr{B}} = Y\varepsilon$ is an isomorphism in $\mathscr{B}$ for $Y \in \mathrm{Ob}(\mathscr{B})$. $\square$

1.6.11 Lemma. Suppose given additive categories $\mathscr{A}$ and $\mathscr{B}$. Suppose given an ideal $\mathfrak{I}$ in $\mathscr{A}$ and an ideal $\mathfrak{J}$ in $\mathscr{B}$. Suppose given an additive functor $F\colon \mathscr{A} \to \mathscr{B}$ such that $fF \in \mathfrak{J}$ for $f \in \mathfrak{I}$. Suppose given an additive functor $G\colon \mathscr{B} \to \mathscr{A}$ such that $fF \in \mathfrak{I}$ for $f \in \mathfrak{J}$.
Let $\underline{F}\colon \mathscr{A}/\mathfrak{I} \to \mathscr{B}/\mathfrak{J}$ denote the unique additive functor such that $\mathfrak{R}_{\mathscr{A},\mathfrak{I}} \cdot \underline{F} = F \cdot \mathfrak{R}_{\mathscr{B},\mathfrak{J}}$. Let $\underline{G}\colon \mathscr{B}/\mathfrak{J} \to \mathscr{A}/\mathfrak{I}$ denote the unique additive functor such that $\mathfrak{R}_{\mathscr{B},\mathfrak{J}} \cdot \underline{G} = G \cdot \mathfrak{R}_{\mathscr{A},\mathfrak{I}}$. Suppose that F is left-adjoint to G. Then $\underline{F}$ is left-adjoint to $\underline{G}$.

More precisely, suppose given an adjunction $(F, G, \eta, \varepsilon)$. Let $\underline{\eta}\colon 1_{\mathscr{A}/\mathfrak{I}} \to \underline{F}\,\underline{G}$ denote the unique transformation such that $\mathfrak{R}_{\mathscr{A},\mathfrak{I}} \star \underline{\eta} = \eta \star \mathfrak{R}_{\mathscr{A},\mathfrak{I}}$. Let $\underline{\varepsilon}\colon \underline{G}\,\underline{F} \to 1_{\mathscr{B}/\mathfrak{J}}$ denote the unique transformation such that $\mathfrak{R}_{\mathscr{B},\mathfrak{J}} \star \underline{\varepsilon} = \varepsilon \star \mathfrak{R}_{\mathscr{B},\mathfrak{J}}$. Then $(\underline{F}, \underline{G}, \underline{\eta}, \underline{\varepsilon})$ is an adjunction as well. $\diamondsuit$

Proof. We have $(\eta \star 1_F) \cdot (1_F \star \varepsilon) = 1_F$ and $(1_G \star \eta) \cdot (\varepsilon \star 1_G) = 1_G$.

We have $(\underline{\eta} \star 1_{\underline{F}}) \cdot (1_{\underline{F}} \star \underline{\varepsilon}) = 1_{\underline{F}}$ since

$$
\begin{aligned}
\mathfrak{R}_{\mathscr{A},\mathfrak{J}} \star ((\underline{\eta} \star 1_{\underline{F}}) \cdot (1_{\underline{F}} \star \underline{\varepsilon})) &= (1_{\mathfrak{R}_{\mathscr{A},\mathfrak{J}}} \cdot 1_{\mathfrak{R}_{\mathscr{A},\mathfrak{J}}}) \star ((\underline{\eta} \star 1_{\underline{F}}) \cdot (1_{\underline{F}} \star \underline{\varepsilon})) \\
&= (1_{\mathfrak{R}_{\mathscr{A},\mathfrak{J}}} \star \underline{\eta} \star 1_{\underline{F}}) \cdot (1_{\mathfrak{R}_{\mathscr{A},\mathfrak{J}}} \star 1_{\underline{F}} \star \underline{\varepsilon}) \\
&= (\eta \star 1_{\mathfrak{R}_{\mathscr{A},\mathfrak{J}}} \star 1_{\underline{F}}) \cdot (1_F \star 1_{\mathfrak{R}_{\mathscr{B},\mathfrak{J}}} \star \underline{\varepsilon}) \\
&= (\eta \star 1_F \star 1_{\mathfrak{R}_{\mathscr{B},\mathfrak{J}}}) \cdot (1_F \star \varepsilon \star 1_{\mathfrak{R}_{\mathscr{B},\mathfrak{J}}}) \\
&= ((\eta \star 1_F) \cdot (1_F \star \varepsilon)) \star (1_{\mathfrak{R}_{\mathscr{B},\mathfrak{J}}} \cdot 1_{\mathfrak{R}_{\mathscr{B},\mathfrak{J}}}) \\
&= 1_F \star 1_{\mathfrak{R}_{\mathscr{B},\mathfrak{J}}} \\
&= \mathfrak{R}_{\mathscr{A},\mathfrak{J}} \star 1_{\underline{F}} \; .
\end{aligned}
$$

We have $(1_{\underline{G}} \star \underline{\eta}) \cdot (\underline{\varepsilon} \star 1_{\underline{G}}) = 1_{\underline{G}}$ since

$$
\begin{aligned}
\mathfrak{R}_{\mathscr{B},\mathfrak{J}} \star ((1_{\underline{G}} \star \underline{\eta}) \cdot (\underline{\varepsilon} \star 1_{\underline{G}})) &= ((1_{\mathfrak{R}_{\mathscr{B},\mathfrak{J}}} \star 1_{\underline{G}} \star \underline{\eta}) \cdot (1_{\mathfrak{R}_{\mathscr{B},\mathfrak{J}}} \star \underline{\varepsilon} \star 1_{\underline{G}})) \\
&= ((1_G \star 1_{\mathfrak{R}_{\mathscr{A},\mathfrak{J}}} \star \underline{\eta}) \cdot (\varepsilon \star 1_{\mathfrak{R}_{\mathscr{B},\mathfrak{J}}} \star 1_{\underline{G}})) \\
&= ((1_G \star \eta \star 1_{\mathfrak{R}_{\mathscr{A},\mathfrak{J}}}) \cdot (\varepsilon \star 1_G \star 1_{\mathfrak{R}_{\mathscr{A},\mathfrak{J}}})) \\
&= ((1_G \star \eta) \cdot (\varepsilon \star 1_G)) \star 1_{\mathfrak{R}_{\mathscr{A},\mathfrak{J}}} \\
&= 1_G \star 1_{\mathfrak{R}_{\mathscr{A},\mathfrak{J}}} \\
&= \mathfrak{R}_{\mathscr{B},\mathfrak{J}} \star 1_{\underline{G}} \; .
\end{aligned}
$$

We conclude that $(\underline{F}, \underline{G}, \underline{\eta}, \underline{\varepsilon})$ is an adjunction. □

1.6.12 Lemma. Suppose given additive categories $\mathscr{A}$ and $\mathscr{B}$. Suppose given an ideal $\mathfrak{J}$ in $\mathscr{A}$ and an ideal $\mathfrak{J}$ in $\mathscr{B}$. Suppose given an additive functor $F\colon \mathscr{A} \to \mathscr{B}$ such that $fF \in \mathfrak{J}$ for $f \in \mathfrak{J}$. Suppose given an additive functor $G\colon \mathscr{B} \to \mathscr{A}$ such that $fF \in \mathfrak{J}$ for $f \in \mathfrak{J}$.
Let $\underline{F}\colon \mathscr{A}/\mathfrak{J} \to \mathscr{B}/\mathfrak{J}$ denote the unique additive functor such that $\mathfrak{R}_{\mathscr{A},\mathfrak{J}} \cdot \underline{F} = F \cdot \mathfrak{R}_{\mathscr{B},\mathfrak{J}}$.
Let $\underline{G}\colon \mathscr{B}/\mathfrak{J} \to \mathscr{A}/\mathfrak{J}$ denote the unique additive functor such that $\mathfrak{R}_{\mathscr{B},\mathfrak{J}} \cdot \underline{G} = G \cdot \mathfrak{R}_{\mathscr{A},\mathfrak{J}}$.
Suppose that F and G are mutually quasi-inverse equivalences. Then $\underline{F}$ and $\underline{G}$ are mutually quasi-inverse equivalences as well.

More precisely, suppose given an adjunction $(F, G, \eta, \varepsilon)$ such that η and ε are isotransformations. Let $\underline{\eta}\colon 1_{\mathscr{A}/\mathfrak{J}} \to \underline{F}\,\underline{G}$ denote the unique transformation such that $\mathfrak{R}_{\mathscr{A},\mathfrak{J}} \star \underline{\eta} = \eta \star \mathfrak{R}_{\mathscr{A},\mathfrak{J}}$.
Let $\underline{\varepsilon}\colon \underline{G}\,\underline{F} \to 1_{\mathscr{B}/\mathfrak{J}}$ denote the unique transformation such that $\mathfrak{R}_{\mathscr{B},\mathfrak{J}} \star \underline{\varepsilon} = \varepsilon \star \mathfrak{R}_{\mathscr{B},\mathfrak{J}}$.
Then $(\underline{F}, \underline{G}, \underline{\eta}, \underline{\varepsilon})$ is an adjunction such that $\underline{\eta}$ and $\underline{\varepsilon}$ are isotransformations as well.

In particular, if F and G are mutually inverse isomorphisms of categories, then $\underline{F}$ and $\underline{G}$ are mutually inverse isomorphisms of categories as well. ♢

Proof. By lemma 1.6.11, $(\underline{F}, \underline{G}, \underline{\eta}, \underline{\varepsilon})$ is an adjunction as well. Now $\underline{\eta}$ is an isotransformation since $X\underline{\eta} = X\eta\mathfrak{R}_{\mathscr{A},\mathfrak{J}}$ is an isomorphism in $\mathscr{A}/\mathfrak{J}$ for $X \in \mathrm{Ob}(\mathscr{A})$. Similarly, $\underline{\varepsilon}$ is an isotransformation since $Y\underline{\varepsilon} = Y\varepsilon\mathfrak{R}_{\mathscr{B},\mathfrak{J}}$ is an isomorphism in $\mathscr{B}/\mathfrak{J}$ for $Y \in \mathrm{Ob}(\mathscr{B})$. □

1.6.13 Lemma. [Nee01, lemma 5.3.6]
Suppose given triangulated categories $\mathscr{C}$, $\mathscr{D}$ and functors $F\colon \mathscr{C} \to \mathscr{D}$, $G\colon \mathscr{D} \to \mathscr{C}$ such that F is left-adjoint to G. Then F is exact if and only if G is exact. $\diamondsuit$

1.6.14 Lemma. Suppose given a triangulated category $\mathscr{C}$ and a full triangulated subcategory $\mathscr{S} \subseteq \mathscr{C}$. Suppose given a triangulated category $\mathscr{D}$ and an exact functor $F\colon \mathscr{C} \to \mathscr{D}$ such that $XF \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$ for $X \in \mathrm{Ob}(\mathscr{S})$. Let $\hat{F}\colon \mathscr{C}/\!\!/\mathscr{S} \to \mathscr{D}$ denote the unique exact functor such that $\mathfrak{L}_{\mathscr{C},\mathscr{S}} \cdot \hat{F} = F$, cf. lemma 1.5.12. Suppose given a functor $G\colon \mathscr{D} \to \mathscr{C}$ such that F is left-adjoint to G.

$$\begin{tikzcd}
\mathscr{C} \arrow[r, shift left, "F"] \arrow[d, "\mathfrak{L}_{\mathscr{C},\mathscr{S}}"'] & \mathscr{D} \arrow[l, shift left, "G"] \\
\mathscr{C}/\!\!/\mathscr{S} \arrow[ur, "\hat{F}"'] &
\end{tikzcd}$$

(a) For $Y \in \mathrm{Ob}(\mathscr{D})$, we have $YG \in \mathrm{Ob}(\mathscr{S}^{\perp})$.

(b) If G is full and faithful, then $G \cdot \mathfrak{L}_{\mathscr{C},\mathscr{S}}$ is full and faithful as well.

(c) The functor $\hat{F}$ is left-adjoint to $G \cdot \mathfrak{L}_{\mathscr{C},\mathscr{S}}$.

$\diamondsuit$

Proof. Since $F \dashv G$, we may choose natural bijections $\Phi_{X,Y}\colon {}_{\mathscr{D}}(XF, Y) \to {}_{\mathscr{C}}(X, YG)$ for $X \in \mathrm{Ob}(\mathscr{C})$ and $Y \in \mathrm{Ob}(\mathscr{D})$.

Ad (a). Suppose given $Y \in \mathrm{Ob}(\mathscr{D})$ and $X \in \mathrm{Ob}(\mathscr{S})$.
We have ${}_{\mathscr{D}}(XF, Y) = 0$ since $XF \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$ by assumption. Since $\Phi_{X,Y}$ is a bijection, we also have ${}_{\mathscr{C}}(X, YG) = 0$. Thus $YG \in \mathrm{Ob}(\mathscr{S}^{\perp})$.

Ad (b). This follows from (a) and [Kra10, lemma 4.8.1].

Ad (c). For $X \in \mathrm{Ob}(\mathscr{C})$ and $Y \in \mathrm{Ob}(\mathscr{D})$, let $\Psi_{X,Y}\colon {}_{\mathscr{C}}(X, YG) \to {}_{\mathscr{C}/\!\!/\mathscr{S}}(X, YG)\colon f \mapsto f\mathfrak{L}_{\mathscr{C},\mathscr{S}}$. This is a bijection by (a) and [Kra10, lemma 4.8.1]. Consider the bijections $\Phi_{X,Y} \cdot \Psi_{X,Y}$ for $X \in \mathrm{Ob}(\mathscr{C})$ and $Y \in \mathrm{Ob}(\mathscr{D})$. It remains to show that they are natural.

We use a calculus of fraction as explained in [Kra10, sections 3 and 4.6]. Suppose given a left fraction $X' \xrightarrow{g} X'' \xleftarrow{s} X$ in $\mathscr{C}/\!\!/\mathscr{S}$, where $X' \xrightarrow{g} X''$ and $X \xrightarrow{s} X''$ are morphisms in $\mathscr{C}$. We write g/s for this left fraction. Suppose given $Y \xrightarrow{h} Y'$ in $\mathscr{D}$. We have

$$\begin{aligned}
(g/s\hat{F} \cdot f \cdot h)\Phi_{X',Y'}\Psi_{X',Y'} &= (gF \cdot (sF)^{-1} \cdot f \cdot h)\Phi_{X',Y'}\mathfrak{L}_{\mathscr{C},\mathscr{S}} \\
&= (g \cdot ((sF)^{-1} \cdot f)\Phi_{X'',Y} \cdot hG)\mathfrak{L}_{\mathscr{C},\mathscr{S}} \\
&= g\mathfrak{L}_{\mathscr{C},\mathscr{S}} \cdot ((sF)^{-1} \cdot f)\Phi_{X'',Y}\mathfrak{L}_{\mathscr{C},\mathscr{S}} \cdot hG\mathfrak{L}_{\mathscr{C},\mathscr{S}} \\
&= g/s \cdot s\mathfrak{L}_{\mathscr{C},\mathscr{S}} \cdot ((sF)^{-1} \cdot f)\Phi_{X'',Y}\mathfrak{L}_{\mathscr{C},\mathscr{S}} \cdot hG\mathfrak{L}_{\mathscr{C},\mathscr{S}} \\
&= g/s \cdot (s \cdot ((sF)^{-1} \cdot f)\Phi_{X'',Y})\mathfrak{L}_{\mathscr{C},\mathscr{S}} \cdot hG\mathfrak{L}_{\mathscr{C},\mathscr{S}} \\
&= g/s \cdot (sF \cdot (sF)^{-1} \cdot f)\Phi_{X,Y}\mathfrak{L}_{\mathscr{C},\mathscr{S}} \cdot hG\mathfrak{L}_{\mathscr{C},\mathscr{S}} \\
&= g/s \cdot f\Phi_{X,Y}\Psi_{X,Y} \cdot hG\mathfrak{L}_{\mathscr{C},\mathscr{S}} \,.
\end{aligned}$$

$\square$

1.6.15 Lemma. Suppose given a category $\mathscr{C}$ and a full subcategory $\mathscr{A}$. Suppose given a functor $F\colon \mathscr{C} \to \mathscr{D}$ such that the map $\Psi_{X,Y}\colon {}_{\mathscr{C}}(X,Y) \to {}_{\mathscr{D}}(XF, YF)\colon f \mapsto fF$ is bijective for $X \in \mathrm{Ob}(\mathscr{A})$ and $Y \in \mathrm{Ob}(\mathscr{C})$. Suppose given a functor $G\colon \mathscr{D} \to \mathscr{A}$ such that $\mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}} \cdot F$ is left-adjoint to G. Then $\mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}}$ is left-adjoint to $F \cdot G$.

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] & \mathscr{D} \arrow[dl, "G"] \\
\mathscr{A} \arrow[u, "\mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}}"] &
\end{tikzcd}$$

$\diamondsuit$

Proof. Since $\mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}} \cdot F \dashv G$, we may choose natural bijections $\Phi_{X,Z}\colon {}_{\mathscr{D}}(XF, Z) \to {}_{\mathscr{A}}(X, ZG)$ for $X \in \mathrm{Ob}(\mathscr{A})$ and $Z \in \mathrm{Ob}(\mathscr{D})$. Consider the bijections $\Psi_{X,Y} \cdot \Phi_{X,YF}\colon {}_{\mathscr{C}}(X,Y) \to {}_{\mathscr{A}}(X, YFG)$ for $X \in \mathrm{Ob}(\mathscr{A})$ and $Y \in \mathrm{Ob}(\mathscr{C})$. It remains to show that they are natural.

Suppose given $X' \xrightarrow{g} X$ in $\mathscr{A}$ and $X \xrightarrow{f} Y$, $Y \xrightarrow{h} Y'$ in $\mathscr{C}$. We have

$$\begin{aligned}
(g \cdot f \cdot h)\Psi_{X',Y'}\Phi_{X',Y'F} &= (g \cdot f \cdot h)F\Phi_{X',Y'F} = (gF \cdot fF \cdot hF)\Phi_{X',Y'F} \\
&= g \cdot fF\Phi_{X,YF} \cdot hFG = g \cdot f\Psi_{X,Y}\Phi_{X,YF} \cdot hFG.
\end{aligned}$$

$\square$

1.6.16 Lemma. Suppose given a category $\mathscr{C}$ and a full subcategory $\mathscr{A}$. Suppose given mutually quasi-inverse equivalences $E\colon \mathscr{A} \to \mathscr{B}$ and $D\colon \mathscr{B} \to \mathscr{A}$. Suppose given functors $I\colon \mathscr{B} \to \mathscr{D}$ and $H\colon \mathscr{D} \to \mathscr{B}$ such that I is left-adjoint to H. Suppose given a functor $F\colon \mathscr{C} \to \mathscr{D}$ such that the map ${}_{\mathscr{C}}(X,Y) \to {}_{\mathscr{D}}(XF, YF)\colon f \mapsto fF$ is bijective for $X \in \mathrm{Ob}(\mathscr{A})$ and $Y \in \mathrm{Ob}(\mathscr{C})$. Suppose that $\mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}} \cdot F = E \cdot I$. Then $D \cdot \mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}}$ is left-adjoint to $F \cdot H$.

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] & \mathscr{D} \arrow[d, bend left, "H"] \\
\mathscr{A} \arrow[u, "\mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}}"] \arrow[r, bend left, "E"] & \mathscr{B} \arrow[u, bend left, "I"] \arrow[l, bend left, "D"]
\end{tikzcd}$$

$\diamondsuit$

Proof. We have $E \dashv D$ and $I \dashv H$ and thus $E \cdot I \dashv H \cdot D$. So $\mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}} \cdot F \dashv H \cdot D$ since $E \cdot I = \mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}} \cdot F$. By lemma 1.6.15, we have $\mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}} \dashv F \cdot H \cdot D$. Since $D \dashv E$, we obtain $D \cdot \mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}} \dashv F \cdot H \cdot D \cdot E$. Now D and E are mutually quasi-inverse equivalences and thus $D \cdot E$ is isomorphic to $1_{\mathscr{B}}$ in $\mathscr{B}(\mathscr{B})$. We conclude that $D \cdot \mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}} \dashv F \cdot H$. $\square$

1.6.17 Lemma. Suppose given a category $\mathscr{C}$ and a full subcategory $\mathscr{A}$. Suppose given mutually quasi-inverse equivalences $E\colon \mathscr{A} \to \mathscr{B}$ and $D\colon \mathscr{B} \to \mathscr{A}$. Suppose given functors $I\colon \mathscr{B} \to \mathscr{D}$ and $H\colon \mathscr{D} \to \mathscr{B}$ such that H is left-adjoint to I. Suppose given a functor $F\colon \mathscr{C} \to \mathscr{D}$ such that the map ${}_{\mathscr{C}}(X,Y) \to {}_{\mathscr{D}}(XF, YF)\colon f \mapsto fF$ is bijective for $X \in \mathrm{Ob}(\mathscr{C})$

and $Y \in \mathrm{Ob}(\mathscr{A})$. Suppose that $\mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}} \cdot F = E \cdot I$. Then $F \cdot H$ is left-adjoint to $D \cdot \mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}}$.

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] & \mathscr{D} \arrow[d, bend left, "H"] \\
\mathscr{A} \arrow[u, "\mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}}"] \arrow[r, bend left, "E"] & \mathscr{B} \arrow[u, bend left, "I"] \arrow[l, bend left, "D"]
\end{tikzcd}$$

◇

Proof. This is dual to the previous lemma 1.6.16. □

1.7 W-structures

We refer to [Bon10, section 1] for details and proofs.

1.7.1 Definition. Suppose given a triangulated category $\mathscr{C}$.
A pair $\mathscr{W} = (\mathscr{U}, \mathscr{V})$ of summand-closed full subcategories $\mathscr{U}, \mathscr{V} \subseteq \mathscr{C}$ is called a *w-structure* (or *weight structure*) on $\mathscr{C}$ if the following three conditions hold.

(W1) ${}_{\mathscr{C}}(\mathscr{U}, \mathscr{V}^{[1]}) = 0$

(W2) $\mathscr{V}^{[1]} \subseteq \mathscr{V}$

(W3) $\mathscr{C} = \mathscr{U} * \mathscr{V}^{[1]}$

◇

1.7.2 Lemma/Definition. Suppose given a strict triangulated category $\mathscr{C}$ and a w-structure $\mathscr{W} = (\mathscr{U}, \mathscr{V})$ on $\mathscr{C}$. For $k \in \mathbf{Z}$, we write $\mathscr{W}_{[k} = \mathscr{U}^{[k]}$ and $\mathscr{W}_{k]} = \mathscr{V}^{[k]}$. For $k, \ell \in \mathbf{Z}$, we write $\mathscr{W}_{[k,\ell]} = \mathscr{W}_{[k} \cap \mathscr{W}_{\ell]}$. We also write $\mathscr{W}^{\mathrm{b}} = \bigcup\limits_{k,\ell \in \mathbf{Z}} \mathscr{W}_{[k,\ell]}$.
We say that $\mathscr{W}$ is *left-non-degenerate* if $\bigcap\limits_{k \in \mathbf{Z}} \mathscr{W}_{[k} = \mathrm{Z}_{\mathscr{C}}$. We say that $\mathscr{W}$ is *right-non-degenerate* if $\bigcap\limits_{k \in \mathbf{Z}} \mathscr{W}_{k]} = \mathrm{Z}_{\mathscr{C}}$. We say that $\mathscr{W}$ is *non-degenerate* if it is left- and right-non-degenerate. We say that $\mathscr{W}$ is *bounded* if $\mathscr{C} = \mathscr{W}^{\mathrm{b}}$. The full additive subcategory $\mathscr{W}_{[0,0]}$ of $\mathscr{C}$ is called the *core* of the w-structure $\mathscr{W}$.
Note that $\mathscr{W}|_{\mathscr{W}^{\mathrm{b}}} = (\mathscr{W}_{[0} \cap \mathscr{W}^{\mathrm{b}}, \mathscr{W}_{0]} \cap \mathscr{W}^{\mathrm{b}})$ is a w-structure on the full triangulated subcategory $\mathscr{W}^{\mathrm{b}}$ of $\mathscr{C}$ by [Sch11, lemma 4.5]. ◇

1.7.3 Lemma. Suppose given a strict triangulated category $\mathscr{C}$ and a w-structure $\mathscr{W} = (\mathscr{W}_{[0}, \mathscr{W}_{0]})$ on $\mathscr{C}$.

(a) For $k, \ell \in \mathbf{Z}$, we have $\mathscr{W}_{[k}^{[\ell]} = \mathscr{W}_{[k+\ell}$ and $\mathscr{W}_{k]}^{[\ell]} = \mathscr{W}_{k+\ell]}$.

(b) Suppose given $k < \ell$ in $\mathbf{Z}$. We have ${}_{\mathscr{C}}(\mathscr{W}_{[k}, \mathscr{W}_{\ell]}) = 0$.

(c) Suppose given $k \in \mathbf{Z}$. We have $\mathscr{W}_{[k} = {}^{\perp}(\mathscr{W}_{k+1]})$ and $(\mathscr{W}_{[k})^{\perp} = \mathscr{W}_{k+1]}$.

(d) Suppose given $k \in \mathbf{Z}$. The subcategory $\mathscr{W}_{[k}$ is closed under coproducts in $\mathscr{C}$ and the subcategory $\mathscr{W}_{k]}$ is closed under products in $\mathscr{C}$.

(e) Suppose given $k \le \ell$ in $\mathbf{Z}$. We have $\mathscr{W}_{[k} \subseteq \mathscr{W}_{[\ell}$ and $\mathscr{W}_{\ell]} \subseteq \mathscr{W}_{k]}$.

(f) For $k \in \mathbf{Z}$, we have $\mathscr{W}_{[k} * \mathscr{W}_{[k} = \mathscr{W}_{[k}$ and $\mathscr{W}_{k]} * \mathscr{W}_{k]} = \mathscr{W}_{k]}$.

(g) For $k \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathscr{C})$, we may choose a triangle $X_{[k} \longrightarrow X \longrightarrow X_{k+1]} \longrightarrow X_{[k}^{[1]}$ in $\mathscr{C}$ such that $X_{[k} \in \mathrm{Ob}(\mathscr{W}_{[k})$ and $X_{k+1]} \in \mathrm{Ob}(\mathscr{W}_{k+1]})$.

(h) Suppose given $k \in \mathbf{Z}$. Suppose given triangles $X_{[k-1} \xrightarrow{j_{k-1}} X \xrightarrow{q_k} X_{k]} \longrightarrow X_{[k-1}^{[1]}$ and $X_{[k} \xrightarrow{j_k} X \xrightarrow{q_{k+1}} X_{k+1]} \longrightarrow X_{[k}^{[1]}$ in $\mathscr{C}$ such that $X_{[k-1} \in \mathrm{Ob}(\mathscr{W}_{[k-1})$, $X_{[k} \in \mathrm{Ob}(\mathscr{W}_{[k})$, $X_{k]} \in \mathrm{Ob}(\mathscr{W}_{k]})$ and $X_{k+1]} \in \mathrm{Ob}(\mathscr{W}_{k+1]})$. Then there exist unique morphisms $X_{[k-1} \xrightarrow{a} X_{[k}$ and $X_{k]} \xrightarrow{b} X_{k+1]}$ in $\mathscr{C}$ such that $j_{k-1} = a \cdot j_k$ and $q_{k+1} = q_k \cdot b$.

$\diamondsuit$

1.7.4 Definition. Suppose given triangulated categories $\mathscr{C}, \mathscr{D}$, a functor $F\colon \mathscr{C} \to \mathscr{D}$ and w-structures $\mathscr{W} = (\mathscr{W}_{[0}\,, \mathscr{W}_{0]})$ on $\mathscr{C}$ and $\mathscr{V} = (\mathscr{V}_{[0}\,, \mathscr{V}_{0]})$ on $\mathscr{D}$. We say that F is *w-exact* with respect to $\mathscr{W}$ and $\mathscr{V}$ if F is exact, $XF \in \mathrm{Ob}(\mathscr{V}_{0]})$ for $X \in \mathrm{Ob}(\mathscr{W}_{0]})$ and if $XF \in \mathrm{Ob}(\mathscr{V}_{[0})$ for $X \in \mathrm{Ob}(\mathscr{W}_{[0})$. $\diamondsuit$

1.7.5 Lemma/Definition. Suppose given triangulated categories $\mathscr{C}$, $\mathscr{D}$ and a w-structure $\mathscr{W} = (\mathscr{U}, \mathscr{V})$ on $\mathscr{C}$. Suppose given an exact equivalence $F\colon \mathscr{C} \to \mathscr{D}$. We obtain a w-structure $\mathscr{W}F = (\mathscr{U}', \mathscr{V}')$ on $\mathscr{D}$ by setting

$$\mathrm{Ob}(\mathscr{U}') = \{Y \in \mathrm{Ob}(\mathscr{D})\colon \text{There exists } X \in \mathrm{Ob}(\mathscr{U}) \text{ such that } XF \text{ is isomorphic to } Y \text{ in } \mathscr{D}\}$$

and

$$\mathrm{Ob}(\mathscr{V}') = \{Y \in \mathrm{Ob}(\mathscr{D})\colon \text{There exists } X \in \mathrm{Ob}(\mathscr{V}) \text{ such that } XF \text{ is isomorphic to } Y \text{ in } \mathscr{D}\}.$$

$\diamondsuit$

1.8 T-structures

We refer to [BBD82, section 1.3] [ATJLSS03, section 1] for details and proofs.

1.8.1 Definition. Suppose given a triangulated category $\mathscr{C}$. A pair $\mathscr{T} = (\mathscr{U}, \mathscr{V})$ of full subcategories $\mathscr{U}, \mathscr{V} \subseteq \mathscr{C}$ is called a *t-structure* (or *truncation structure*) on $\mathscr{C}$ if the following three conditions hold.

(T1) ${}_{\mathscr{C}}(\mathscr{U}^{[1]}, \mathscr{V}) = 0$

(T2) $\mathscr{U}^{[1]} \subseteq \mathscr{U}$

(T3) $\mathscr{C} = \mathscr{U}^{[1]} * \mathscr{V}$

$\diamondsuit$

1.8.2 Lemma/Definition. Suppose given a strict triangulated category $\mathscr{C}$ and a t-structure $\mathscr{T} = (\mathscr{U}, \mathscr{V})$ on $\mathscr{C}$. For $k \in \mathbf{Z}$, we write $\mathscr{T}_{k\rfloor} = \mathscr{U}^{[k]}$ and $\mathscr{T}_{\lfloor k} = \mathscr{V}^{[k]}$. For $k, \ell \in \mathbf{Z}$, we write $\mathscr{T}_{\lfloor k,\ell\rfloor} = \mathscr{T}_{\lfloor k} \cap \mathscr{T}_{\ell\rfloor}$. We also write $\mathscr{T}^{\mathrm{b}} = \bigcup_{k,\ell\in\mathbf{Z}} \mathscr{T}_{\lfloor k,\ell\rfloor}$.

We say that $\mathscr{T}$ is *left-non-degenerate* if $\bigcap_{k\in\mathbf{Z}} \mathscr{T}_{k\rfloor} = \mathrm{Z}_{\mathscr{C}}$. We say that $\mathscr{T}$ is *right-non-degenerate* if $\bigcap_{k\in\mathbf{Z}} \mathscr{T}_{\lfloor k} = \mathrm{Z}_{\mathscr{C}}$. We say that $\mathscr{T}$ is *non-degenerate* if it is left- and right-non-degenerate. We say that $\mathscr{T}$ is *bounded* if $\mathscr{C} = \mathscr{T}^{\mathrm{b}}$.

We refer to [BBD82, proposition 1.3.3 and théorème 1.3.6] for the following notions and results.

The subcategory $\mathscr{T}_{\lfloor 0,0\rfloor}$ is called the *heart* of the t-structure $\mathscr{T}$. The heart $\mathscr{T}_{\lfloor 0,0\rfloor}$ is an abelian category. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathscr{T}_{\lfloor 0,0\rfloor}$ is short exact if and only if it can be completed to a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow X^{[1]}$ in $\mathscr{C}$.

Suppose given $k \in \mathbf{Z}$.

We denote the truncation functors by $\mathrm{T}_{k\rfloor} = \mathrm{T}^{\mathscr{T}}_{k\rfloor} \colon \mathscr{C} \to \mathscr{T}_{k\rfloor}$ and $\mathrm{T}_{\lfloor k} = \mathrm{T}^{\mathscr{T}}_{\lfloor k} \colon \mathscr{C} \to \mathscr{T}_{\lfloor k}$. The functor $\mathrm{T}_{k\rfloor}$ is right-adjoint to $\mathrm{Inc}^{\mathscr{C}}_{\mathscr{T}_{k\rfloor}}$ and the functor $\mathrm{T}_{\lfloor k}$ is left-adjoint to $\mathrm{Inc}^{\mathscr{C}}_{\mathscr{T}_{\lfloor k}}$.

We denote the homology functor by $\mathrm{H}_{\mathscr{T}} \colon \mathscr{C} \to \mathscr{T}_{\lfloor 0,0\rfloor}$. The homology functor is a homological functor. The restriction $\mathrm{H}_{\mathscr{T}}|_{\mathscr{T}_{\lfloor 0}} \colon \mathscr{T}_{\lfloor 0} \to \mathscr{T}_{\lfloor 0,0\rfloor}$ is isomorphic to $\mathrm{T}_{0\rfloor}|^{\mathscr{T}_{\lfloor 0,0\rfloor}}_{\mathscr{T}_{\lfloor 0}}$ in $\mathscr{T}_{\lfloor 0}(\mathscr{T}_{\lfloor 0,0\rfloor})$. Consequently, $\mathrm{H}_{\mathscr{T}}|_{\mathscr{T}_{\lfloor 0}} \colon \mathscr{T}_{\lfloor 0} \to \mathscr{T}_{\lfloor 0,0\rfloor}$ right-adjoint to $\mathrm{Inc}^{\mathscr{T}_{\lfloor 0}}_{\mathscr{T}_{\lfloor 0,0\rfloor}}$, cf. lemma 1.6.9. $\diamondsuit$

1.8.3 Lemma. Suppose given a strict triangulated category $\mathscr{C}$ and a t-structure $\mathscr{T} = (\mathscr{T}_{0\rfloor}, \mathscr{T}_{\lfloor 0})$ on $\mathscr{C}$.

(a) For $k, \ell \in \mathbf{Z}$, we have $\mathscr{T}^{[\ell]}_{\lfloor k} = \mathscr{T}_{\lfloor k+\ell}$ and $\mathscr{T}^{[\ell]}_{k\rfloor} = \mathscr{T}_{k+\ell\rfloor}$.

(b) Suppose given $k > \ell$ in $\mathbf{Z}$. We have ${}_{\mathscr{C}}(\mathscr{T}_{k\rfloor}, \mathscr{T}_{\lfloor \ell}) = 0$.

(c) Suppose given $k \in \mathbf{Z}$. We have $\mathscr{T}_{k+1\rfloor} = {}^{\perp}(\mathscr{T}_{\lfloor k})$ and $(\mathscr{T}_{k+1\rfloor})^{\perp} = \mathscr{T}_{\lfloor k}$.

(d) Suppose given $k \in \mathbf{Z}$. The subcategory $\mathscr{T}_{k\rfloor}$ is closed under coproducts in $\mathscr{C}$ and the subcategory $\mathscr{T}_{\lfloor k}$ is closed under products in $\mathscr{C}$.

(e) We have ${}_{\mathscr{C}}(\mathscr{T}^{[k]}_{\lfloor 0,0\rfloor}, \mathscr{T}_{\lfloor 0}) = 0$ and ${}_{\mathscr{C}}(\mathscr{T}^{[k]}_{\lfloor 0,0\rfloor}, \mathscr{T}_{\lfloor 0,0\rfloor}) = 0$ for $k \in \mathbf{Z}_{\geq 1}$.

(f) Suppose given $k \geq \ell$ in $\mathbf{Z}$. We have $\mathscr{T}_{k\rfloor} \subseteq \mathscr{T}_{\ell\rfloor}$ and $\mathscr{T}_{\lfloor \ell} \subseteq \mathscr{T}_{\lfloor k}$.

(g) For $k \in \mathbf{Z}$, we have $\mathscr{T}_{\lfloor k} * \mathscr{T}_{\lfloor k} = \mathscr{T}_{\lfloor k}$ and $\mathscr{T}_{k\rfloor} * \mathscr{T}_{k\rfloor} = \mathscr{T}_{k\rfloor}$.

(h) For $k \in \mathbf{Z}$, we may choose a triangle $X_{k+1\rfloor} \longrightarrow X \longrightarrow X_{\lfloor k} \longrightarrow X^{[1]}_{k+1\rfloor}$ in $\mathscr{C}$ such that $X_{k+1\rfloor} \in \mathrm{Ob}(\mathscr{T}_{k+1\rfloor})$ and $X_{\lfloor k} \in \mathrm{Ob}(\mathscr{T}_{\lfloor k})$.

(i) Suppose that $\mathscr{T}$ is non-degenerate. Suppose given $X \in \mathrm{Ob}(\mathscr{C})$. We have $X \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{C}})$ if and only if $X^{[-k]}\mathrm{H}_{\mathscr{T}} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{T}_{\lfloor 0,0\rfloor}})$ for $k \in \mathbf{Z}$.

$\diamondsuit$

1.8.4 Definition. Suppose given triangulated categories $\mathscr{C}, \mathscr{D}$, a functor $F\colon \mathscr{C} \to \mathscr{D}$ and t-structures $\mathscr{T} = (\mathscr{T}_{0]}\,, \mathscr{T}_{[0})$ on $\mathscr{C}$ and $\mathscr{U} = (\mathscr{U}_{0]}\,, \mathscr{U}_{[0})$ on $\mathscr{D}$. We say that F is *t-exact* with respect to $\mathscr{T}$ and $\mathscr{U}$ if F is exact, $XF \in \mathrm{Ob}(\mathscr{U}_{0]})$ for $X \in \mathrm{Ob}(\mathscr{T}_{0]})$ and if $XF \in \mathrm{Ob}(\mathscr{U}_{[0})$ for $X \in \mathrm{Ob}(\mathscr{T}_{[0})$. $\diamondsuit$

1.8.5 Lemma/Definition. Suppose given triangulated categories $\mathscr{C}$, $\mathscr{D}$ and a t-structure $\mathscr{T} = (\mathscr{U}, \mathscr{V})$ on $\mathscr{C}$. Suppose given an exact equivalence $F\colon \mathscr{C} \to \mathscr{D}$. We obtain a t-structure $\mathscr{T}F = (\mathscr{U}', \mathscr{V}')$ on $\mathscr{D}$ by setting

$$\mathrm{Ob}(\mathscr{U}') = \{Y \in \mathrm{Ob}(\mathscr{D})\colon \text{There exists } X \in \mathrm{Ob}(\mathscr{U}) \text{ such that } XF \text{ is isomorphic to } Y \text{ in } \mathscr{D}\}$$

and

$$\mathrm{Ob}(\mathscr{V}') = \{Y \in \mathrm{Ob}(\mathscr{D})\colon \text{There exists } X \in \mathrm{Ob}(\mathscr{V}) \text{ such that } XF \text{ is isomorphic to } Y \text{ in } \mathscr{D}\}.$$

$\diamondsuit$

1.8.6 Definition. Suppose given a triangulated category $\mathscr{C}$, a t-structure $\mathscr{T} = (\mathscr{T}_{0]}, \mathscr{T}_{[0})$ on $\mathscr{C}$ and a w-structure $\mathscr{W} = (\mathscr{W}_{[0}, \mathscr{W}_{0]})$ on $\mathscr{C}$.

(a) We say that $\mathscr{T}$ is *left-adjacent* to $\mathscr{W}$ if $\mathscr{T}_{[0} = \mathscr{W}_{[0}$. In this case, we also say that $\mathscr{W}$ is *right-adjacent* to $\mathscr{T}$.

(b) We say that $\mathscr{W}$ is *left-adjacent* to $\mathscr{T}$ if $\mathscr{W}_{0]} = \mathscr{T}_{0]}$. In this case, we also say that $\mathscr{T}$ is *right-adjacent* to $\mathscr{W}$.

$\diamondsuit$

1.9 Complexes

The additive case

1.9.1 Definition. Suppose given an additive category $\mathscr{A}$. For $X \in \mathrm{Ob}(\mathbf{Z}^{\mathrm{op}}(\mathscr{A}))$ and $k \in \mathbf{Z}$, we write $x_k = X_{k \to k-1}$, cf. convention 17. $\diamondsuit$

1.9.2 Definition. Suppose given an additive category $\mathscr{A}$. The *category of complexes* with entries in $\mathscr{A}$ is the full subcategory $\mathrm{C}(\mathscr{A})$ of $\mathbf{Z}^{\mathrm{op}}(\mathscr{A})$ defined by

$$\mathrm{Ob}(\mathrm{C}(\mathscr{A})) = \{X \in \mathrm{Ob}(\mathbf{Z}^{\mathrm{op}}(\mathscr{A}))\colon x_k \cdot x_{k-1} = 0 \text{ for } k \in \mathbf{Z}\}.$$

$$\cdots \longrightarrow X_{k+1} \xrightarrow{x_{k+1}} X_k \xrightarrow{x_k} X_{k-1} \xrightarrow{x_{k-1}} X_{k-2} \longrightarrow \cdots$$

We always equip $\mathrm{C}(\mathscr{A})$ with the pointwise split exact structure: A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathrm{C}(\mathscr{A})$ is pure short exact in $\mathrm{C}(\mathscr{A})$ if and only if $X_k \xrightarrow{f_k} Y_k \xrightarrow{g_k} Z_k$ is split short exact in $\mathscr{A}$ for $k \in \mathbf{Z}$. Cf. [Büh10, lemma 9.1]. $\diamondsuit$

1.9.3 Definition. Suppose given an exact category $\mathscr{A}$ and $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$. We say that X is *acyclic* in $\mathrm{C}(\mathscr{A})$ if there exist pure short exact sequences $S_{k+1} \xrightarrow{i_k} X_k \xrightarrow{p_k} S_k$ in $\mathscr{A}$ such that $x_k = p_k \cdot i_{k-1}$ for $k \in \mathbf{Z}$. $\diamondsuit$

1.9.4 Definition. Suppose given an additive category $\mathscr{A}$ and $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$. We say that X is *split acyclic* in $\mathrm{C}(\mathscr{A})$ if there exist direct sums $S_{k+1} \underset{s_k}{\overset{i_k}{\rightleftarrows}} X_k \underset{p_k}{\overset{t_k}{\leftrightarrows}} S_k$ in $\mathscr{A}$ such that $x_k = p_k \cdot i_{k-1}$ for $k \in \mathbf{Z}$. $\diamondsuit$

1.9.5 Lemma. Suppose given an additive category $\mathscr{A}$ and a split acyclic complex $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$. Then X is a bijective object in $\mathrm{C}(\mathscr{A})$. $\diamondsuit$

Proof. Choose direct sums $S_{k+1} \underset{s_k}{\overset{i_k}{\rightleftarrows}} X_k \underset{p_k}{\overset{t_k}{\leftrightarrows}} S_k$ in $\mathscr{A}$ such that $x_k = p_k \cdot i_{k-1}$ for $k \in \mathbf{Z}$.

We want to show that X is injective in $\mathscr{A}$. Suppose given a pure monomorphism $X \xrightarrow{f} Y$ in $\mathrm{C}(\mathscr{A})$. For $k \in \mathbf{Z}$, we may choose $Y_k \xrightarrow{h_k} X_k$ in $\mathscr{A}$ such that $f_k \cdot h_k = 1$.
For $k \in \mathbf{Z}$, let $g_k = h_k \cdot s_k \cdot i_k + y_k \cdot h_{k-1} \cdot s_{k-1} \cdot t_k$.
Suppose given $k \in \mathbf{Z}$. We have

$$\begin{aligned} g_k \cdot x_k &= h_k \cdot s_k \cdot i_k \cdot x_k + y_k \cdot h_{k-1} \cdot s_{k-1} \cdot t_k \cdot x_k \\ &= h_k \cdot s_k \cdot i_k \cdot p_k \cdot i_{k-1} + y_k \cdot h_{k-1} \cdot s_{k-1} \cdot t_k \cdot p_k \cdot i_{k-1} \\ &= y_k \cdot h_{k-1} \cdot s_{k-1} \cdot i_{k-1} \\ &= y_k \cdot g_{k-1} \end{aligned}$$

and

$$\begin{aligned} f_k \cdot g_k &= f_k \cdot h_k \cdot s_k \cdot i_k + f_k \cdot y_k \cdot h_{k-1} \cdot s_{k-1} \cdot t_k = s_k \cdot i_k + x_k \cdot f_{k-1} \cdot h_{k-1} \cdot s_{k-1} \cdot t_k \\ &= s_k \cdot i_k + p_k \cdot i_{k-1} \cdot s_{k-1} \cdot t_k = s_k \cdot i_k + p_k \cdot t_k = 1. \end{aligned}$$

Thus we have a morphism $Y \xrightarrow{g} X$ in $\mathrm{C}(\mathscr{A})$ such that $f \cdot g = 1$. We conclude that X is injective in $\mathrm{C}(\mathscr{A})$. Dually, X is also projective in $\mathrm{C}(\mathscr{A})$. $\square$

1.9.6 Lemma. Suppose given an additive category $\mathscr{A}$. Suppose given $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ and direct sums $X_k \underset{s_k}{\overset{i_k}{\rightleftarrows}} B_k \underset{p_k}{\overset{t_k}{\leftrightarrows}} X_{k-1}$ in $\mathscr{A}$ for $k \in \mathbf{Z}$. Let $b_k = s_k \cdot x_k \cdot i_{k-1} + p_k \cdot i_{k-1} - p_k \cdot x_{k-1} \cdot t_{k-1}$ for $k \in \mathbf{Z}$. This yields a bijective object $B \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$. $\diamondsuit$

Proof. For $k \in \mathbf{Z}$, we have the direct sums $X_k \underset{s_k}{\overset{i_k - x_k \cdot t_k}{\rightleftarrows}} B_k \underset{p_k + s_k \cdot x_k}{\overset{t_k}{\leftrightarrows}} X_{k-1}$ in $\mathscr{A}$. We have $(p_k + s_k \cdot x_k) \cdot (i_{k-1} - x_{k-1} \cdot t_{k-1}) = s_k \cdot x_k \cdot i_{k-1} + p_k \cdot i_{k-1} - p_k \cdot x_{k-1} \cdot t_{k-1}$ for $k \in \mathbf{Z}$.
Thus X is split acyclic. The result now follows from lemma 1.9.5. $\square$

1.9.7 Lemma/Definition. Suppose given an additive category $\mathscr{A}$. We define the functor $\mathrm{B}_{\mathrm{C},\mathscr{A}} \colon \mathrm{C}(\mathscr{A}) \to \mathrm{C}(\mathscr{A})$ as follows. For $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ and $k \in \mathbf{Z}$, let $(X\mathrm{B}_{\mathrm{C},\mathscr{A}})_k = X_k \oplus X_{k-1}$ and $(X\mathrm{B}_{\mathrm{C},\mathscr{A}})_{k \to k-1} = \left(\begin{smallmatrix} x_k & 0 \\ 1 & -x_{k-1} \end{smallmatrix}\right)$. This object $X\mathrm{B}_{\mathrm{C},\mathscr{A}}$ is often called the mapping cone of the identity 1_X . For $f \in \mathrm{Mor}(\mathrm{C}(\mathscr{A}))$, let $(f\mathrm{B}_{\mathrm{C},\mathscr{A}})_k = \left(\begin{smallmatrix} f_k & 0 \\ 0 & f_{k-1} \end{smallmatrix}\right)$. This in fact defines an additive functor. Note that $X\mathrm{B}_{\mathrm{C},\mathscr{A}}$ is bijective in $\mathrm{C}(\mathscr{A})$ for $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ by lemma 1.9.6. We abbreviate $\mathrm{B} = \mathrm{B}_{\mathrm{C},\mathscr{A}}$ if unambiguous. $\diamondsuit$

Proof. Suppose given $X \overset{f}{\underset{h}{\rightrightarrows}} Y \xrightarrow{g} Z$ in $\mathrm{C}(\mathscr{A})$.
Note that for $k \in \mathbf{Z}$, we have

$$\begin{aligned}
(X\mathrm{B})_{k \to k-1} \cdot (f\mathrm{B})_{k-1} &= \left(\begin{smallmatrix} x_k & 0 \\ 1 & -x_{k-1} \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} f_{k-1} & 0 \\ 0 & f_{k-2} \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_k \cdot f_{k-1} & 0 \\ f_{k-1} & -x_{k-1} \cdot f_{k-2} \end{smallmatrix}\right) \\
&= \left(\begin{smallmatrix} f_k \cdot y_k & 0 \\ f_{k-1} & -f_{k-1} \cdot y_{k-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} f_k & 0 \\ 0 & f_{k-1} \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} y_k & 0 \\ 1 & -y_{k-1} \end{smallmatrix}\right) \\
&= (f\mathrm{B})_k \cdot (Y\mathrm{B})_{k \to k-1} \,.
\end{aligned}$$

We have $1_X\mathrm{B} = 1_{X\mathrm{B}}$ since $(1_X\mathrm{B})_k = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = 1_{X_k \oplus X_{k-1}} = (1_{X\mathrm{B}})_k$ for $k \in \mathbf{Z}$.
We have $(f \cdot g)\mathrm{B} = f\mathrm{B} \cdot g\mathrm{B}$ since
$((f \cdot g)\mathrm{B})_k = \left(\begin{smallmatrix} f_k \cdot g_k & 0 \\ 0 & f_{k-1} \cdot g_{k-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} f_k & 0 \\ 0 & f_{k-1} \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} g_k & 0 \\ 0 & g_{k-1} \end{smallmatrix}\right) = (f\mathrm{B} \cdot g\mathrm{B})_k$ for $k \in \mathbf{Z}$.
We have $(f + h)\mathrm{B} = f\mathrm{B} + h\mathrm{B}$ since
$((f + h)\mathrm{B})_k = \left(\begin{smallmatrix} f_k + h_k & 0 \\ 0 & f_{k-1} + h_{k-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} f_k & 0 \\ 0 & f_{k-1} \end{smallmatrix}\right) + \left(\begin{smallmatrix} h_k & 0 \\ 0 & h_{k-1} \end{smallmatrix}\right) = (f\mathrm{B} + h\mathrm{B})_k$ for $k \in \mathbf{Z}$. $\square$

1.9.8 Lemma/Definition. Suppose given an additive category $\mathscr{A}$. We define the shift functors $\Sigma_{\mathrm{C},\mathscr{A}}, \Sigma^-_{\mathrm{C},\mathscr{A}} \colon \mathrm{C}(\mathscr{A}) \to \mathrm{C}(\mathscr{A})$ as follows. For $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ and $k \in \mathbf{Z}$, let $(X\Sigma_{\mathrm{C},\mathscr{A}})_k = X_{k-1}$, $(X\Sigma^-_{\mathrm{C},\mathscr{A}})_k = X_{k+1}$, $(X\Sigma_{\mathrm{C},\mathscr{A}})_{k \to k-1} = -x_{k-1}$ and $(X\Sigma^-_{\mathrm{C},\mathscr{A}})_{k \to k-1} = -x_{k+1}$.
For $f \in \mathrm{Mor}(\mathrm{C}(\mathscr{A}))$, let $(f\Sigma_{\mathrm{C},\mathscr{A}})_k = f_{k-1}$ and $(f\Sigma^-_{\mathrm{C},\mathscr{A}})_k = f_{k+1}$.
The functors $\Sigma_{\mathrm{C},\mathscr{A}}$ and $\Sigma^-_{\mathrm{C},\mathscr{A}}$ are mutually inverse isomorphisms of categories. For $k \in \mathbf{Z}$ and $X \xrightarrow{f} Y$ in $\mathrm{C}(\mathscr{A})$, we often write $X^{[k]} = X\Sigma^k_{\mathrm{C},\mathscr{A}}$ and $f^{[k]} = f\Sigma^k_{\mathrm{C},\mathscr{A}}$. $\diamondsuit$

Proof. We abbreviate $\Sigma = \Sigma_{\mathrm{C},\mathscr{A}}$ and $\Sigma^- = \Sigma^-_{\mathrm{C},\mathscr{A}}$. Suppose given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathrm{C}(\mathscr{A})$.
We have $1_X\Sigma = 1_{X\Sigma}$ since $(1_X\Sigma)_k = 1_{X_{k-1}} = (1_{X\Sigma})_k$ for $k \in \mathbf{Z}$.
We have $(f \cdot g)\Sigma = f\Sigma \cdot g\Sigma$ since $((f \cdot g)\Sigma)_k = f_{k-1} \cdot g_{k-1} = (f\Sigma \cdot g\Sigma)_k$ for $k \in \mathbf{Z}$.
We have $1_X\Sigma^- = 1_{X\Sigma^-}$ since $(1_X\Sigma^-)_k = 1_{X_{k+1}} = (1_{X\Sigma^-})_k$ for $k \in \mathbf{Z}$.
We have $(f \cdot g)\Sigma^- = f\Sigma^- \cdot g\Sigma^-$ since $((f \cdot g)\Sigma^-)_k = f_{k+1} \cdot g_{k+1} = (f\Sigma^- \cdot g\Sigma^-)_k$ for $k \in \mathbf{Z}$.
We have $f\Sigma\Sigma^- = f$ since $(f\Sigma\Sigma^-)_k = f_k$ for $k \in \mathbf{Z}$.
We have $f\Sigma^-\Sigma = f$ since $(f\Sigma^-\Sigma)_k = f_k$ for $k \in \mathbf{Z}$. $\square$

1.9.9 Lemma/Definition. Suppose given an additive category $\mathscr{A}$. We define the transformation $\iota_{\mathrm{C},\mathscr{A}} \colon 1_{\mathrm{C}(\mathscr{A})} \to \mathrm{B}_{\mathrm{C},\mathscr{A}}$ by setting $(X\iota_{\mathrm{C},\mathscr{A}})_k = \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) : X_k \to X_k \oplus X_{k-1}$ for $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ and $k \in \mathbf{Z}$. We abbreviate $\iota = \iota_{\mathrm{C},\mathscr{A}}$ if unambiguous.

We define the transformation $\pi_{\mathrm{C},\mathscr{A}} \colon \mathrm{B}_{\mathrm{C},\mathscr{A}} \to \Sigma_{\mathrm{C},\mathscr{A}}$ by setting
$(X\pi_{\mathrm{C},\mathscr{A}})_k = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) : X_k \oplus X_{k-1} \to X_{k-1}$ for $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ and $k \in \mathbf{Z}$. We abbreviate $\pi = \pi_{\mathrm{C},\mathscr{A}}$ if unambiguous.

Note that for $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$, the sequence $(X\iota_{\mathrm{C},\mathscr{A}}, X\pi_{\mathrm{C},\mathscr{A}})$ is pure short exact in $\mathrm{C}(\mathscr{A})$. $\diamondsuit$

Proof. Suppose given $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ and $k \in \mathbf{Z}$. We have

$$\begin{aligned}(X\iota_{\mathrm{C},\mathscr{A}})_k \cdot (X\mathrm{B}_{\mathrm{C},\mathscr{A}})_{k \to k-1} &= \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x_k & 0 \\ 1 & -x_{k-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_k & 0 \end{smallmatrix}\right) = x_k \cdot \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) \\ &= (X1_{\mathrm{C}(\mathscr{A})})_{k \to k-1} \cdot (X\iota_{\mathrm{C},\mathscr{A}})_{k-1}\end{aligned}$$

and

$$\begin{aligned}(X\mathrm{B}_{\mathrm{C},\mathscr{A}})_{k \to k-1} \cdot (X\pi_{\mathrm{C},\mathscr{A}})_{k-1} &= \left(\begin{smallmatrix} x_k & 0 \\ 1 & -x_{k-1} \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ -x_{k-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \cdot (-x_{k-1}) \\ &= (X\pi_{\mathrm{C},\mathscr{A}})_k \cdot (X\Sigma_{\mathrm{C},\mathscr{A}})_{k \to k-1} .\end{aligned}$$

Suppose given $X \xrightarrow{f} Y$ in $\mathrm{C}(\mathscr{A})$ and $k \in \mathbf{Z}$. We have

$$\begin{aligned}(X\iota_{\mathrm{C},\mathscr{A}})_k \cdot (f\mathrm{B}_{\mathrm{C},\mathscr{A}})_k &= \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) \cdot \begin{pmatrix} f_k & 0 \\ 0 & f_{k-1} \end{pmatrix} = \left(\begin{smallmatrix} f_k & 0 \end{smallmatrix}\right) = f_k \cdot \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) \\ &= (f1_{\mathrm{C}(\mathscr{A})})_k \cdot (Y\iota_{\mathrm{C},\mathscr{A}})_k\end{aligned}$$

and

$$\begin{aligned}(f\mathrm{B}_{\mathrm{C},\mathscr{A}})_k \cdot (Y\pi_{\mathrm{C},\mathscr{A}})_k &= \begin{pmatrix} f_k & 0 \\ 0 & f_{k-1} \end{pmatrix} \cdot \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ f_{k-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \cdot f_{k-1} \\ &= (X\pi_{\mathrm{C},\mathscr{A}})_k \cdot (f\Sigma_{\mathrm{C},\mathscr{A}})_k .\end{aligned}$$ □

1.9.10 Lemma/Definition. Suppose given an additive category $\mathscr{A}$. We define the isotransformation $\alpha_{\mathrm{C},\mathscr{A}} \colon \Sigma_{\mathrm{C},\mathscr{A}}\mathrm{B}_{\mathrm{C},\mathscr{A}} \to \mathrm{B}_{\mathrm{C},\mathscr{A}}\Sigma_{\mathrm{C},\mathscr{A}}$ by setting
$(X\alpha_{\mathrm{C},\mathscr{A}})_k = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \colon X_{k-1} \oplus X_{k-2} \to X_{k-1} \oplus X_{k-2}$ for $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ and $k \in \mathbf{Z}$. We abbreviate $\alpha = \alpha_{\mathrm{C},\mathscr{A}}$ if unambiguous. ◊

Proof. Suppose given $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ and $k \in \mathbf{Z}$. We have

$$\begin{aligned}(X\alpha_{\mathrm{C},\mathscr{A}})_k \cdot (X\mathrm{B}_{\mathrm{C},\mathscr{A}}\Sigma_{\mathrm{C},\mathscr{A}})_{k \to k-1} &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \cdot \begin{pmatrix} -x_{k-1} & 0 \\ -1 & x_{k-2} \end{pmatrix} = \begin{pmatrix} -x_{k-1} & 0 \\ 1 & -x_{k-2} \end{pmatrix} \\ &= \begin{pmatrix} -x_{k-1} & 0 \\ 1 & x_{k-2} \end{pmatrix} \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = (X\Sigma_{\mathrm{C},\mathscr{A}}\mathrm{B}_{\mathrm{C},\mathscr{A}})_{k \to k-1} \cdot (X\alpha_{\mathrm{C},\mathscr{A}})_{k-1} .\end{aligned}$$

Suppose given $X \xrightarrow{f} Y$ in $\mathrm{C}(\mathscr{A})$ and $k \in \mathbf{Z}$. We have

$$\begin{aligned}(f\Sigma_{\mathrm{C},\mathscr{A}}\mathrm{B}_{\mathrm{C},\mathscr{A}})_k \cdot (Y\alpha_{\mathrm{C},\mathscr{A}})_k &= \begin{pmatrix} f_{k-1} & 0 \\ 0 & f_{k-2} \end{pmatrix} \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = \begin{pmatrix} f_{k-1} & 0 \\ 0 & -f_{k-2} \end{pmatrix} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \cdot \begin{pmatrix} f_{k-1} & 0 \\ 0 & f_{k-2} \end{pmatrix} \\ &= (X\alpha_{\mathrm{C},\mathscr{A}})_k \cdot (f\mathrm{B}_{\mathrm{C},\mathscr{A}}\Sigma_{\mathrm{C},\mathscr{A}})_k .\end{aligned}$$ □

1.9.11 Lemma. Suppose given an additive category $\mathscr{A}$.
We have $(\Sigma_{\mathrm{C},\mathscr{A}} \star \iota_{\mathrm{C},\mathscr{A}}) \cdot \alpha_{\mathrm{C},\mathscr{A}} = \iota_{\mathrm{C},\mathscr{A}} \star \Sigma_{\mathrm{C},\mathscr{A}}$ and $\Sigma_{\mathrm{C},\mathscr{A}} \star \pi_{\mathrm{C},\mathscr{A}} = -\alpha_{\mathrm{C},\mathscr{A}} \cdot (\pi_{\mathrm{C},\mathscr{A}} \star \Sigma_{\mathrm{C},\mathscr{A}})$. ◊

Proof. Suppose given $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ and $k \in \mathbf{Z}$. We have

$$\begin{aligned}(X((\Sigma_{\mathrm{C},\mathscr{A}} \star \iota_{\mathrm{C},\mathscr{A}}) \cdot \alpha_{\mathrm{C},\mathscr{A}}))_k &= (X^{[1]}\iota \cdot X\alpha)_k = \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) = ((X\iota)^{[1]})_k \\ &= (X(\iota_{\mathrm{C},\mathscr{A}} \star \Sigma_{\mathrm{C},\mathscr{A}}))_k\end{aligned}$$

and

$$\begin{aligned}(X(-\alpha_{\mathrm{C},\mathscr{A}} \cdot (\pi_{\mathrm{C},\mathscr{A}} \star \Sigma_{\mathrm{C},\mathscr{A}})))_k &= (-X\alpha \cdot (X\pi)^{[1]})_k = -\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = (X^{[1]}\pi)_k \\ &= (X(\Sigma_{\mathrm{C},\mathscr{A}} \star \pi_{\mathrm{C},\mathscr{A}}))_k \,.\end{aligned}$$ □

1.9.12 Lemma/Definition. Suppose given an additive category $\mathscr{A}$. The exact category $\mathrm{C}(\mathscr{A})$ is a Frobenius category. We denote its stable category by $\mathrm{K}(\mathscr{A}) = \underline{\mathrm{C}(\mathscr{A})}$ and call it the *homotopy category* of $\mathscr{A}$, cf. definition 1.3.27. ◇

Proof. Suppose given $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$. We have the pure epimorphism $X^{[-1]}\mathrm{B} \xrightarrow{X^{[-1]}\pi} X$ and the pure monomorphism $X \xrightarrow{X\iota} X\mathrm{B}$ with $X^{[-1]}\mathrm{B}$, $X\mathrm{B}$ bijective in $\mathscr{F}$, cf. definitions 1.9.7 and 1.9.9. □

1.9.13 Lemma. Suppose given an additive category $\mathscr{A}$ and $X \xrightarrow{f} Y$ in $\mathrm{C}(\mathscr{A})$. The following three statements are equivalent.

(a) We have $\underline{f} = 0$ in $\mathrm{K}(\mathscr{A})$.

(b) There exists $X\mathrm{B} \xrightarrow{g} Y$ in $\mathrm{C}(\mathscr{A})$ such that $X\iota \cdot g = f$.

(c) There exist morphisms $X_k \xrightarrow{h_k} Y_{k+1}$ in $\mathscr{A}$ such that $h_k \cdot y_{k+1} + x_k \cdot h_{k-1} = f_k$ for $k \in \mathbf{Z}$. ◇

Proof. Ad (a)↔(b). This follows from lemma 1.3.28.(a,b).
Ad (b)→(c). For $k \in \mathbf{Z}$, write $g_k = \left(\begin{smallmatrix} u_k \\ h_{k-1} \end{smallmatrix}\right) : X_k \oplus X_{k-1} \to Y_k$.
Suppose given $k \in \mathbf{Z}$. We have $u_k = (X\iota)_k \cdot \left(\begin{smallmatrix} u_k \\ h_{k-1} \end{smallmatrix}\right) = (X\iota \cdot g)_k = f_k$ and

$$\left(\begin{smallmatrix} u_{k+1} \\ h_k \end{smallmatrix}\right) \cdot y_{k+1} = g_{k+1} \cdot y_{k+1} = (X\mathrm{B})_{k+1 \to k} \cdot g_k = \left(\begin{smallmatrix} x_{k+1} & 0 \\ 1 & -x_k \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} u_k \\ h_{k-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_{k+1} \cdot u_k \\ u_k - x_k \cdot h_{k-1} \end{smallmatrix}\right) .$$

Thus $h_k \cdot y_{k+1} + x_k \cdot h_{k-1} = f_k$.
Ad (c)→(b). For $k \in \mathbf{Z}$, let $g_k = \begin{pmatrix} f_k \\ h_{k-1} \end{pmatrix} : X_k \oplus X_{k-1} \to Y_k$.
Suppose given $k \in \mathbf{Z}$. We have

$$g_k \cdot y_k = \begin{pmatrix} f_k \\ h_{k-1} \end{pmatrix} \cdot y_k = \begin{pmatrix} x_k \cdot f_{k-1} \\ f_{k-1} - x_{k-1} \cdot h_{k-2} \end{pmatrix} = \left(\begin{smallmatrix} x_k & 0 \\ 1 & -x_{k-1} \end{smallmatrix}\right) \cdot \begin{pmatrix} f_{k-1} \\ h_{k-2} \end{pmatrix} = (X\mathrm{B})_{k \to k-1} \cdot g_{k-1}$$

and $(X\iota)_k \cdot g_k = \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) \cdot \begin{pmatrix} f_k \\ h_{k-1} \end{pmatrix} = f_k$. □

1.9.14 Definition. Suppose given an additive category $\mathscr{A}$. For $n \in \mathbf{Z}$, we define the full subcategories $\mathrm{C}^{[n}(\mathscr{A})$ and $\mathrm{C}^{n]}(\mathscr{A})$ of $\mathrm{C}(\mathscr{A})$ by setting

$$\mathrm{Ob}(\mathrm{C}^{[n}(\mathscr{A})) = \{X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))\colon X_k \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}}) \text{ for } k \in \mathbf{Z}_{>n}\}$$

and

$$\mathrm{Ob}(\mathrm{C}^{n]}(\mathscr{A})) = \{X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))\colon X_k \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}}) \text{ for } k \in \mathbf{Z}_{<n}\}.$$

For $m, n \in \mathbf{Z}$, let $\mathrm{C}^{[m,n]}(\mathscr{A}) = \mathrm{C}^{[m}(\mathscr{A}) \cap \mathrm{C}^{n]}(\mathscr{A})$. ◊

1.9.15 Definition. Suppose given an additive category $\mathscr{A}$. Let $\mathrm{C}^{\mathrm{b}}(\mathscr{A}) = \bigcup\limits_{m,n\in\mathbf{Z}} \mathrm{C}^{[m,n]}(\mathscr{A})$. Note that $\mathrm{C}^{\mathrm{b}}(\mathscr{A})$ is an extension-closed subcategory of $\mathrm{C}(\mathscr{A})$, cf. definition 1.3.23.

We define the full subcategory $\mathrm{K}^{\mathrm{b}}(\mathscr{A})$ of $\mathrm{K}(\mathscr{A})$ by setting $\mathrm{Ob}(\mathrm{K}^{\mathrm{b}}(\mathscr{A})) = \mathrm{Ob}(\mathrm{C}^{\mathrm{b}}(\mathscr{A}))$. Note that $\mathrm{K}^{\mathrm{b}}(\mathscr{A})$ is a full additive subcategory of $\mathrm{K}(\mathscr{A})$, cf. remark 1.2.14. ◊

1.9.16 Definition. Suppose given an additive category $\mathscr{A}$. A candidate triangle $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}$ in $\mathrm{C}(\mathscr{A})$ with respect to $\Sigma_{\mathrm{C},\mathscr{A}}$ is called a *pseudo-triangle* in $\mathrm{C}(\mathscr{A})$ if (i, p) is a pure short exact sequence in $\mathrm{C}(\mathscr{A})$ and if there exists $X\mathrm{B} \xrightarrow{g} Z$ in $\mathrm{C}(\mathscr{A})$ such that the following diagram is commutative in $\mathrm{C}(\mathscr{A})$.

$$\begin{tikzcd} X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[r, "X\pi"] \arrow[d, "g"] & X^{[1]} \arrow[d, "1"] \\ Y \arrow[r, "i"] & Z \arrow[r, "p"] & X^{[1]} \end{tikzcd}$$

We use the term pseudo-triangle since it will be a special case of definition 2.2.8 below. We will also explain the reasons for using this term there.

Cf. also [Wei94, definition 10.1.3] and [Büh10, definition 9.2], where the term 'strict triangle' is used to describe similar notions. ◊

1.9.17 Lemma. Suppose given an additive category $\mathscr{A}$. Suppose given a candidate triangle $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}$ in $\mathrm{C}(\mathscr{A})$. It is a pseudo-triangle if and only if we may choose $Z_k \xrightarrow{s_k} Y_k$ and $X_{k-1} \xrightarrow{t_k} Z_k$ for $k \in \mathbf{Z}$ such that $Y_k \overset{i_k}{\underset{s_k}{\rightleftarrows}} Z_k \overset{t_k}{\underset{p_k}{\leftrightarrows}} X_{k-1}$ is a direct sum in $\mathscr{A}$ and such that

$$z_k = s_k \cdot y_k \cdot i_{k-1} + p_k \cdot f_{k-1} \cdot i_{k-1} - p_k \cdot x_{k-1} \cdot t_{k-1}$$

for $k \in \mathbf{Z}$. Cf. definition 1.9.16. ◊

Proof. Suppose given $Z_k \xrightarrow{s_k} Y_k$ and $X_{k-1} \xrightarrow{t_k} Z_k$ for $k \in \mathbf{Z}$ such that $Y_k \overset{i_k}{\underset{s_k}{\rightleftarrows}} Z_k \overset{t_k}{\underset{p_k}{\leftrightarrows}} X_{k-1}$ is a direct sum in $\mathscr{A}$ and such that

$$z_k = s_k \cdot y_k \cdot i_{k-1} + p_k \cdot f_{k-1} \cdot i_{k-1} - p_k \cdot x_{k-1} \cdot t_{k-1}$$

for $k \in \mathbf{Z}$.

Note that $(i_k\,,\, p_k)$ is a split short exact sequence for $k \in \mathbf{Z}$. Thus (i,p) is a pure short exact sequence in $\mathrm{C}(\mathscr{A})$.

We define $X\mathrm{B} \xrightarrow{g} Z$ by setting $g_k = \left(\begin{smallmatrix} f_k \cdot i_k \\ t_k \end{smallmatrix}\right) : X_k \oplus X_{k-1} \to Z_k$ for $k \in \mathbf{Z}$.

For $k \in \mathbf{Z}$, we have

$$\begin{aligned} g_k \cdot z_k &= \left(\begin{smallmatrix} f_k \cdot i_k \\ t_k \end{smallmatrix}\right) \cdot z_k = \left(\begin{smallmatrix} f_k \cdot y_k \cdot i_{k-1} \\ f_{k-1} \cdot i_{k-1} - x_{k-1} \cdot t_{k-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_k \cdot f_{k-1} \cdot i_{k-1} \\ f_{k-1} \cdot i_{k-1} - x_{k-1} \cdot t_{k-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_k & 0 \\ 1 & -x_{k-1} \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} f_{k-1} \cdot i_{k-1} \\ t_{k-1} \end{smallmatrix}\right) \\ &= (X\mathrm{B})_{k \to k-1} \cdot g_{k-1}\,. \end{aligned}$$

It remains to show that the following diagram is commutative.

$$\begin{tikzcd} X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[r, "X\pi"] \arrow[d, "g"] & X^{[1]} \arrow[d, "1"] \\ Y \arrow[r, "i"] & Z \arrow[r, "p"] & X^{[1]} \end{tikzcd}$$

Indeed, we have $\left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) \cdot g_k = \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} f_k \cdot i_k \\ t_k \end{smallmatrix}\right) = f_k \cdot i_k$ and $g_k \cdot p_k = \left(\begin{smallmatrix} f_k \cdot i_k \\ t_k \end{smallmatrix}\right) \cdot p_k = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ for $k \in \mathbf{Z}$.

Conversely, suppose given $X\mathrm{B} \xrightarrow{g} Z$ in $\mathrm{C}(\mathscr{A})$ such that the following diagram is commutative.

$$\begin{tikzcd} X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[r, "X\pi"] \arrow[d, "g"] & X^{[1]} \arrow[d, "1"] \\ Y \arrow[r, "i"] & Z \arrow[r, "p"] & X^{[1]} \end{tikzcd}$$

Write $g_k = \left(\begin{smallmatrix} u_k \\ t_k \end{smallmatrix}\right) : X_k \oplus X_{k-1} \to Z_k$ for $k \in \mathbf{Z}$.

Suppose given $k \in \mathbf{Z}$. We have $f_k \cdot i_k = \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) \cdot g_k = \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} u_k \\ t_k \end{smallmatrix}\right) = u_k$ and $\left(\begin{smallmatrix} u_k \\ t_k \end{smallmatrix}\right) \cdot p_k = g_k \cdot p_k = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$.

So $t_k \cdot p_k = 1$. By lemma 1.2.11, we may choose $Z_k \xrightarrow{s_k} Y_k$ in $\mathscr{A}$ such that $Y_k \underset{s_k}{\overset{i_k}{\rightleftarrows}} Z_k \underset{p_k}{\overset{t_k}{\leftrightarrows}} X_{k-1}$ is a direct sum in $\mathscr{A}$.

Suppose given $k \in \mathbf{Z}$. We have the following (pointwise) pushout in $\mathscr{A}$.

$$\begin{tikzcd} X_k \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)"] \arrow[d, "f_k"'] & X_k \oplus X_{k-1} \arrow[d, "g_k = \left(\begin{smallmatrix} u_k \\ t_k \end{smallmatrix}\right) = \left(\begin{smallmatrix} f_k \cdot i_k \\ t_k \end{smallmatrix}\right)"] \\ Y_k \arrow[r, "i_k"] & Z_k \end{tikzcd}$$

We have $z_k = s_k \cdot y_k \cdot i_{k-1} + p_k \cdot f_{k-1} \cdot i_{k-1} - p_k \cdot x_{k-1} \cdot t_{k-1}$ since

$$\begin{aligned}
\left(\begin{smallmatrix} i_k \\ g_k \end{smallmatrix}\right) \cdot z_k &= \left(\begin{smallmatrix} y_k \cdot i_{k-1} \\ \left(\begin{smallmatrix} x_k & 0 \\ 1 & -x_{k-1} \end{smallmatrix}\right) \cdot g_{k-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} i_k \cdot s_k \cdot y_k \cdot i_{k-1} \\ \left(\begin{smallmatrix} x_k & 0 \\ 1 & -x_{k-1} \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} f_{k-1} \cdot i_{k-1} \\ t_{k-1} \end{smallmatrix}\right) \end{smallmatrix}\right) = \left(\begin{smallmatrix} i_k \cdot s_k \cdot y_k \cdot i_{k-1} \\ \left(\begin{smallmatrix} x_k \cdot f_{k-1} \cdot i_{k-1} \\ f_{k-1} \cdot i_{k-1} - x_{k-1} \cdot t_{k-1} \end{smallmatrix}\right) \end{smallmatrix}\right) \\
&= \left(\begin{smallmatrix} i_k \cdot s_k \cdot y_k \cdot i_{k-1} \\ \left(\begin{smallmatrix} f_k \cdot y_k \cdot i_{k-1} \\ f_{k-1} \cdot i_{k-1} - x_{k-1} \cdot t_{k-1} \end{smallmatrix}\right) \end{smallmatrix}\right) = \left(\begin{smallmatrix} i_k \cdot s_k \cdot y_k \cdot i_{k-1} \\ \left(\begin{smallmatrix} f_k \cdot i_k \cdot s_k \cdot y_k \cdot i_{k-1} \\ t_k \cdot p_k \cdot f_{k-1} \cdot i_{k-1} - t_k \cdot p_k \cdot x_{k-1} \cdot t_{k-1} \end{smallmatrix}\right) \end{smallmatrix}\right) \\
&= \left(\begin{smallmatrix} i_k \cdot (s_k \cdot y_k \cdot i_{k-1} + p_k \cdot f_{k-1} \cdot i_{k-1} - p_k \cdot x_{k-1} \cdot t_{k-1}) \\ \left(\begin{smallmatrix} f_k \cdot i_k \\ t_k \end{smallmatrix}\right) \cdot (s_k \cdot y_k \cdot i_{k-1} + p_k \cdot f_{k-1} \cdot i_{k-1} - p_k \cdot x_{k-1} \cdot t_{k-1}) \end{smallmatrix}\right) \\
&= \left(\begin{smallmatrix} i_k \\ g_k \end{smallmatrix}\right) \cdot (s_k \cdot y_k \cdot i_{k-1} + p_k \cdot f_{k-1} \cdot i_{k-1} - p_k \cdot x_{k-1} \cdot t_{k-1})
\end{aligned}$$

and since $\left(\begin{smallmatrix} i_k \\ g_k \end{smallmatrix}\right)$ is a split epimorphism. $\square$

1.9.18 Lemma. Suppose given an additive category $\mathscr{A}$. Suppose given a pseudo-triangle $X \xrightarrow{1} X \xrightarrow{i} B \xrightarrow{p} X^{[1]}$ in $\mathrm{C}(\mathscr{A})$.

(a) The object B is bijective in $\mathrm{C}(\mathscr{A})$.

(b) Suppose given $X \xrightarrow{f} Y$ in $\mathrm{C}(\mathscr{A})$. Then $\underline{f} = 0$ in $\mathrm{K}(\mathscr{A})$ if and only if there exists $B \xrightarrow{g} Y$ in $\mathrm{C}(\mathscr{A})$ such that $f = i \cdot g$.

$\diamondsuit$

Proof. Ad (a). This follows from lemmata 1.9.17 and 1.9.6.

Ad (b). This follows from (a) and from the fact that i is a pure monomorphism. $\square$

1.9.19 Lemma/Definition. Suppose given an additive category $\mathscr{A}$. We have $f\Sigma_{\mathrm{C},\mathscr{A}}\mathfrak{P}_{\mathrm{C}(\mathscr{A})} = 0$ and $f\Sigma^{-}_{\mathrm{C},\mathscr{A}}\mathfrak{P}_{\mathrm{C}(\mathscr{A})} = 0$ in $\mathrm{K}(\mathscr{A})$ for $f \in \mathfrak{J}_{\mathrm{C}(\mathscr{A})}$. Let $\Sigma_{\mathrm{K},\mathscr{A}} \colon \mathrm{K}(\mathscr{A}) \to \mathrm{K}(\mathscr{A})$ denote the unique additive functor such that $\mathfrak{P}_{\mathrm{C}(\mathscr{A})} \cdot \Sigma_{\mathrm{K},\mathscr{A}} = \Sigma_{\mathrm{C},\mathscr{A}} \cdot \mathfrak{P}_{\mathrm{C}(\mathscr{A})}$, cf. lemma 1.2.15. Note that $\Sigma_{\mathrm{K},\mathscr{A}}$ is an isomorphism of categories, cf. lemma 1.6.12.

Let $\Sigma_{\mathrm{K^b},\mathscr{A}} = \Sigma_{\mathrm{K},\mathscr{A}}|^{\mathrm{K^b}(\mathscr{A})}_{\mathrm{K^b}(\mathscr{A})} \colon \mathrm{K^b}(\mathscr{A}) \to \mathrm{K^b}(\mathscr{A})$. Note that $\Sigma_{\mathrm{K^b},\mathscr{A}}$ is an isomorphism of categories as well, cf. lemma 1.6.10. $\diamondsuit$

Proof. Suppose given $X \xrightarrow{f} Y$ in $\mathrm{C}(\mathscr{A})$ such that $f \in \mathfrak{J}_{\mathrm{C}(\mathscr{A})}$.

We want to show that $f\Sigma_{\mathrm{C},\mathscr{A}}\mathfrak{P}_{\mathrm{C}(\mathscr{A})} = 0$ and that $f\Sigma^{-}_{\mathrm{C},\mathscr{A}}\mathfrak{P}_{\mathrm{C}(\mathscr{A})} = 0$ in $\mathrm{K}(\mathscr{A})$.

We may choose $X_k \xrightarrow{h_k} Y_{k+1}$ in $\mathscr{A}$ such that $h_k \cdot y_{k+1} + x_k \cdot h_{k-1} = f_k$ for $k \in \mathbf{Z}$. So for $k \in \mathbf{Z}$, we have $-h_k \cdot (-y_{k+1}) + (-x_k) \cdot (-h_{k-1}) = f_k$. Thus $f\Sigma_{\mathrm{C},\mathscr{A}}\mathfrak{P}_{\mathrm{C}(\mathscr{A})} = 0$ and $f\Sigma^{-}_{\mathrm{C},\mathscr{A}}\mathfrak{P}_{\mathrm{C}(\mathscr{A})} = 0$ in $\mathrm{K}(\mathscr{A})$. $\square$

1.9.20 Definition. Suppose given an additive category $\mathscr{A}$.

Suppose given $T \in \mathrm{Ob}(\mathrm{CT}_{\Sigma_{\mathrm{K},\mathscr{A}}}(\mathrm{K}(\mathscr{A})))$. We say that T is a *triangle* in $\mathrm{K}(\mathscr{A})$ if there exists a pseudo-triangle $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}$ in $\mathrm{C}(\mathscr{A})$ such that T is isomorphic to $X \xrightarrow{\underline{f}} Y \xrightarrow{\underline{i}} Z \xrightarrow{\underline{p}} X^{[1]}$ in $\mathrm{CT}_{\Sigma_{\mathrm{K},\mathscr{A}}}(\mathrm{K}(\mathscr{A}))$.

We define the full subcategory $\mathfrak{T}_{\mathrm{K},\mathscr{A}}$ of $\mathrm{CT}_{\Sigma_{\mathrm{K},\mathscr{A}}}(\mathrm{K}(\mathscr{A}))$ by setting

$$\mathrm{Ob}(\mathfrak{T}_{\mathrm{K},\mathscr{A}}) = \{T \in \mathrm{Ob}(\mathrm{CT}_{\Sigma_{\mathrm{K},\mathscr{A}}}(\mathrm{K}(\mathscr{A}))) \colon T \text{ is a triangle in } \mathrm{K}(\mathscr{A})\}.$$

The tuple $(\mathrm{K}(\mathscr{A}), \Sigma_{\mathrm{K},\mathscr{A}}, \mathfrak{T}_{\mathrm{K},\mathscr{A}})$ is a strict triangulated category.

Similarly, we define the full subcategory $\mathfrak{T}^{\mathrm{b}}_{\mathrm{K},\mathscr{A}}$ of $\mathrm{CT}_{\Sigma_{\mathrm{K^b},\mathscr{A}}}(\mathrm{K^b}(\mathscr{A}))$ by setting

$$\mathrm{Ob}(\mathfrak{T}^{\mathrm{b}}_{\mathrm{K},\mathscr{A}}) = \{T \in \mathrm{Ob}(\mathrm{CT}_{\Sigma_{\mathrm{K^b},\mathscr{A}}}(\mathrm{K^b}(\mathscr{A}))) \colon T \cdot \mathrm{Inc}^{\mathrm{K}(\mathscr{A})}_{\mathrm{K^b}(\mathscr{A})} \text{ is a triangle in } \mathrm{K}(\mathscr{A})\}.$$

The tuple $(\mathrm{K^b}(\mathscr{A}), \Sigma_{\mathrm{K^b},\mathscr{A}}, \mathfrak{T}^{\mathrm{b}}_{\mathrm{K},\mathscr{A}})$ is a strict triangulated category as well.

We will give proofs in the more general context of strict Frobenius categories below, cf. examples 2.1.4 and 2.1.33 and lemma 2.2.14. $\diamondsuit$

1.9.21 Lemma/Definition. Suppose given additive categories $\mathscr{A}$, $\mathscr{B}$ and an additive functor $F \colon \mathscr{A} \to \mathscr{B}$. Let $\mathrm{C}(F) = \mathbf{Z}^{\mathrm{op}}(F)|^{\mathrm{C}(\mathscr{B})}_{\mathrm{C}(\mathscr{A})}$, cf. definition 1.4.3 and lemma 1.4.4.(a). Note that $\mathrm{C}(F)$ is an additive functor as well, cf. remark 1.2.5.(b).

We have $f\mathrm{C}(F)\mathfrak{P}_{\mathrm{C}(\mathscr{B})} = 0$ in $\mathrm{K}(\mathscr{B})$ for $f \in \mathfrak{J}_{\mathrm{C}(\mathscr{A})}$. Let $\mathrm{K}(F) \colon \mathrm{K}(\mathscr{A}) \to \mathrm{K}(\mathscr{B})$ denote the unique additive functor such that $\mathfrak{P}_{\mathrm{C}(\mathscr{A})} \cdot \mathrm{K}(F) = \mathrm{C}(F) \cdot \mathfrak{P}_{\mathrm{C}(\mathscr{B})}$, cf. lemma 1.2.15.

$$\begin{tikzcd}
\mathrm{C}(\mathscr{A}) \arrow[r, "\mathrm{C}(F)"] \arrow[d, "\mathfrak{P}_{\mathrm{C}(\mathscr{A})}"'] & \mathrm{C}(\mathscr{B}) \arrow[d, "\mathfrak{P}_{\mathrm{C}(\mathscr{B})}"] \\
\mathrm{K}(\mathscr{A}) \arrow[r, "\mathrm{K}(F)"] & \mathrm{K}(\mathscr{B})
\end{tikzcd}$$

Let $\mathrm{K^b}(F) = \mathrm{K}(F)|^{\mathrm{K^b}(\mathscr{B})}_{\mathrm{K^b}(\mathscr{A})} \colon \mathrm{K^b}(\mathscr{A}) \to \mathrm{K^b}(\mathscr{B})$, cf. lemma 1.4.4.(b). Note that $\mathrm{K^b}(F)$ is an additive functor as well, cf. remark 1.2.5.(b). $\diamondsuit$

Proof. Suppose given $X \xrightarrow{f} Y$ in $\mathrm{C}(\mathscr{A})$ such that $f \in \mathfrak{J}_{\mathrm{C}(\mathscr{A})}$.
We want to show that $f\mathrm{C}(F)\mathfrak{P}_{\mathrm{C}(\mathscr{B})} = 0$ in $\mathrm{K}(\mathscr{B})$.
We may choose $X_k \xrightarrow{h_k} Y_{k+1}$ in $\mathscr{A}$ such that $h_k \cdot y_{k+1} + x_k \cdot h_{k-1} = f_k$ for $k \in \mathbf{Z}$.
We have $h_kF \cdot y_{k+1}F + x_kF \cdot h_{k-1}F = (h_k \cdot y_{k+1} + x_k \cdot h_{k-1})F = f_kF = (f\mathrm{C}(F))_k$ for $k \in \mathbf{Z}$.
Thus $f\mathrm{C}(F)\mathfrak{P}_{\mathrm{C}(\mathscr{B})} = 0$ in $\mathrm{K}(\mathscr{B})$. $\square$

1.9.22 Lemma. Suppose given additive categories $\mathscr{C}$ and $\mathscr{D}$. Suppose given a full and faithful additive functor $F \colon \mathscr{C} \to \mathscr{D}$. Then $\mathrm{K}(F) \colon \mathrm{K}(\mathscr{C}) \to \mathrm{K}(\mathscr{D})$ is full and faithful as well. $\diamondsuit$

Proof. Suppose given $X, Y \in \mathrm{Ob}(\mathrm{K}(\mathscr{C}))$ and $X\mathrm{K}(F) \xrightarrow{f} Y\mathrm{K}(F)$ in $\mathrm{C}(\mathscr{D})$. Since F is full, we may choose $X_k \xrightarrow{g_k} Y_k$ in $\mathscr{C}$ such that $g_kF = f_k$ for $k \in \mathbf{Z}$. For $k \in \mathbf{Z}$, we have $x_k \cdot g_{k-1} = g_k \cdot y_k$ since $(x_k \cdot g_{k-1})F = x_kF \cdot f_{k-1} = f_k \cdot y_kF = (g_k \cdot y_k)F$ and since F is faithful. So we obtain a morphism $X \xrightarrow{g} Y$ in $\mathrm{C}(\mathscr{C})$ with $\underline{g}\mathrm{K}(F) = \underline{f}$ since $(g\mathrm{C}(F))_k = g_kF = f_k$ for $k \in \mathbf{Z}$. We conclude that $\mathrm{K}(F)$ is full.

Suppose given $X \xrightarrow{f} Y$ in $\mathrm{C}(\mathscr{C})$ such that $\underline{f}\mathrm{K}(F) = 0$. We may choose $X_kF \xrightarrow{h_k} Y_{k+1}F$ in $\mathscr{D}$ such that $h_k \cdot y_{k+1}F + x_kF \cdot h_{k-1} = f_kF$ for $k \in \mathbf{Z}$. Since F is full, we may choose $X_k \xrightarrow{g_k} Y_{k+1}$ in $\mathscr{C}$ such that $g_kF = h_k$ for $k \in \mathbf{Z}$. We have $g_k \cdot y_{k+1} + x_k \cdot g_{k-1} = f_k$ since $(g_k \cdot y_{k+1} + x_k \cdot g_{k-1})F = h_k \cdot y_{k+1}F + x_kF \cdot h_{k-1} = f_kF$ and since F is faithful. We conclude that $\mathrm{K}(F)$ is faithful. $\square$

1.9.23 Lemma. Suppose given additive categories $\mathscr{C}$ and $\mathscr{D}$. Suppose given a full and faithful additive functor $F\colon \mathscr{C} \to \mathscr{D}$. Then $\mathrm{K}^{\mathrm{b}}(F)\colon \mathrm{K}^{\mathrm{b}}(\mathscr{C}) \to \mathrm{K}^{\mathrm{b}}(\mathscr{D})$ is full and faithful as well. $\diamondsuit$

Proof. This follows from lemma 1.9.22 since $\mathrm{K}^{\mathrm{b}}(F) = \mathrm{K}(F)|_{\mathrm{K}^{\mathrm{b}}(\mathscr{C})}^{\mathrm{K}^{\mathrm{b}}(\mathscr{D})}$. $\square$

1.9.24 Lemma. Suppose given additive categories $\mathscr{C}$ and $\mathscr{D}$.
Suppose given functors $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ such that F is left-adjoint to G. Note that F and G are necessarily additive, cf. remark 1.2.5.(a). Then $\mathrm{K}(F)\colon \mathrm{K}(\mathscr{C}) \to \mathrm{K}(\mathscr{D})$ is left-adjoint to $\mathrm{K}(G)\colon \mathrm{K}(\mathscr{D}) \to \mathrm{K}(\mathscr{C})$. $\diamondsuit$

Proof. This follows from lemmata 1.6.8, 1.6.9 and 1.6.11. $\square$

1.9.25 Lemma/Definition. Suppose given an additive category $\mathscr{A}$. We define the functor $\mathrm{I}_{\mathrm{C},\mathscr{A}}\colon \mathscr{A} \to \mathrm{C}(\mathscr{A})$ as follows. For $X \in \mathrm{Ob}(\mathscr{A})$, let $(X\mathrm{I}_{\mathrm{C},\mathscr{A}})_0 = X$ and $(X\mathrm{I}_{\mathrm{C},\mathscr{A}})_k = 0_{\mathscr{A}}$ for $k \in \mathbf{Z} \setminus \{0\}$. For $f \in \mathrm{Mor}(\mathscr{A})$, let $(X\mathrm{I}_{\mathrm{C},\mathscr{A}})_0 = f$.

This in fact defines an additive functor.

Let $\mathrm{I}_{\mathrm{K},\mathscr{A}} = \mathrm{I}_{\mathrm{C},\mathscr{A}} \cdot \mathfrak{P}_{\mathrm{C}(\mathscr{A})}\colon \mathscr{A} \to \mathrm{K}(\mathscr{A})$, $\mathrm{I}_{\mathrm{C}^{\mathrm{b}},\mathscr{A}} = \mathrm{I}_{\mathrm{C},\mathscr{A}}|^{\mathrm{C}^{\mathrm{b}}(\mathscr{A})}\colon \mathscr{A} \to \mathrm{C}^{\mathrm{b}}(\mathscr{A})$ and
$\mathrm{I}_{\mathrm{K}^{\mathrm{b}},\mathscr{A}} = \mathrm{I}_{\mathrm{K},\mathscr{A}}|^{\mathrm{K}^{\mathrm{b}}(\mathscr{A})}\colon \mathscr{A} \to \mathrm{K}^{\mathrm{b}}(\mathscr{A})$. $\diamondsuit$

Proof. We abbreviate $I = \mathrm{I}_{\mathrm{C},\mathscr{A}}$. Suppose given $X \underset{h}{\overset{f}{\rightrightarrows}} Y \xrightarrow{g} Z$ in $\mathscr{A}$. We have $1_X I = 1_{XI}$ since $(1_X I)_0 = 1_X = (1_{XI})_0$. We have $(f \cdot g)I = fI \cdot gI$ since $((f \cdot g)I)_0 = f \cdot g = (fI \cdot gI)_0$. We have $(f + h)I = fI + hI$ since $((f + h)I)_0 = f + h = (fI + hI)_0$. $\square$

1.9.26 Definition. Suppose given an additive category $\mathscr{A}$, $n \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$. We define the pure short exact sequence $X\mathbb{S}^{\mathscr{A}}_{\lceil n} \xrightarrow{X\mathbbm{s}^{\mathscr{A}}_{\lceil n}} X \xrightarrow{X\mathbbm{s}^{\mathscr{A}}_{n+1\rceil}} X\mathbb{S}^{\mathscr{A}}_{n+1\rceil}$ in $\mathrm{C}(\mathscr{A})$ as follows.

For $k \in \mathbf{Z}_{>n}$, let $(X\mathbb{S}^{\mathscr{A}}_{\lceil n})_k = 0_{\mathscr{F}}$, $(X\mathbb{S}^{\mathscr{A}}_{n+1\rceil})_k = X_k$, $(X\mathbb{S}^{\mathscr{A}}_{n+1\rceil})_{k+1 \to k} = x_{k+1}$, $(X\mathbbm{s}^{\mathscr{A}}_{n+1\rceil})_k = 1_{X_k}$.

For $k \in \mathbf{Z}_{\leq n}$, let $(X\mathbb{S}^{\mathscr{A}}_{\lceil n})_k = X_k$, $(X\mathbb{S}^{\mathscr{A}}_{\lceil n})_{k \to k-1} = x_k$, $(X\mathbb{S}^{\mathscr{A}}_{n+1\rceil})_k = 0_{\mathscr{A}}$, $(X\mathbbm{s}^{\mathscr{A}}_{\lceil n})_k = 1_{X_k}$.

The objects $X\mathbb{S}^{\mathscr{A}}_{\lceil n}$ and $X\mathbb{S}^{\mathscr{A}}_{n+1\rceil}$ are sometimes called simple/hard/brutal/naive/stupid truncations of X.

$$
\begin{tikzcd}
X\mathbb{S}^{\mathscr{A}}_{\lceil n} \arrow[d, "X\mathbbm{s}^{\mathscr{A}}_{\lceil n}"'] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X_n \arrow[r, "x_n"] \arrow[d, "1"] & X_{n-1} \arrow[r, "x_{n-1}"] \arrow[d, "1"] & X_{n-2} \arrow[r] \arrow[d, "1"] & \cdots \\
X \arrow[d, "X\mathbbm{s}^{\mathscr{A}}_{n+1\rceil}"'] & \cdots \arrow[r] & X_{n+2} \arrow[r, "x_{n+2}"] \arrow[d, "1"] & X_{n+1} \arrow[r, "x_{n+1}"] \arrow[d, "1"] & X_n \arrow[r, "x_n"] \arrow[d] & X_{n-1} \arrow[r, "x_{n-1}"] \arrow[d] & X_{n-2} \arrow[r] \arrow[d] & \cdots \\
X\mathbb{S}^{\mathscr{A}}_{n+1\rceil} & \cdots \arrow[r] & X_{n+2} \arrow[r, "x_{n+2}"] & X_{n+1} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$
$\diamondsuit$

1.9.27 Definition. An additive category $\mathscr{A}$ is called *idempotent complete* if for every morphism $X \xrightarrow{e} X$ in $\mathscr{A}$ with $e \cdot e = e$, there exists a direct sum $A \underset{s}{\overset{i}{\rightleftarrows}} X \underset{p}{\overset{t}{\leftrightarrows}} B$ in $\mathscr{A}$ such that $e = s \cdot i$. Cf. [Büh10, definition 6.1]. $\diamondsuit$

1.9.28 Example. An abelian category is idempotent complete by [Büh10, remark 6.2]. $\diamondsuit$

1.9.29 Lemma. Suppose given an additive category $\mathscr{A}$, $n \in \mathbf{Z}$ and $Y \in \mathrm{Ob}(\mathrm{C}^{n]}(\mathscr{A}))$. Suppose given $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ such that X is a summand of Y in $\mathrm{K}(\mathscr{A})$. Then there exists a complex $C \in \mathrm{Ob}(\mathrm{C}^{n]}(\mathscr{A}))$ such that X is isomorphic to C in $\mathrm{K}(\mathscr{A})$. If $\mathscr{A}$ is idempotent complete, then $X \xrightarrow{X\mathrm{s}^{\mathscr{A}}_{n]}} X\mathbb{S}^{\mathscr{A}}_{n]}$ is an isomorphism in $\mathrm{K}(\mathscr{A})$. ◊

Proof. Since X is a summand of Y in $\mathrm{K}(\mathscr{A})$, we may choose $X \underset{v}{\overset{u}{\rightleftarrows}} Y$ in $\mathrm{C}(\mathscr{A})$ such that $\underline{u} \cdot \underline{v} = 1$ in $\mathrm{K}(\mathscr{A})$. We have $\underline{X\mathrm{s}^{\mathscr{A}}_{[n-1}} = \underline{X\mathrm{s}^{\mathscr{A}}_{[n-1}} \cdot \underline{u} \cdot \underline{v} = 0$ in $\mathrm{K}(\mathscr{A})$ since ${}_{\mathrm{C}(\mathscr{A})}(X\mathbb{S}^{\mathscr{A}}_{[n-1}, Y) = 0$. So we may choose $X_k \xrightarrow{h_k} X_{k+1}$ such that $h_k \cdot x_{k+1} + x_k \cdot h_{k-1} = 1$ for $k \in \mathbf{Z}_{<n}$, cf. lemma 1.9.13.

Let $e = x_n \cdot h_{n-1} \colon X_n \to X_n$.
We have $x_{n+1} \cdot e = 0$, $e \cdot x_n = x_n \cdot (1 - x_{n-1} \cdot h_{n-2}) = x_n$, $e \cdot e = e \cdot x_n \cdot h_{n-1} = x_n \cdot h_{n-1} = e$, $(1-e) \cdot x_n = x_n - x_n = 0$ and $(1-e) \cdot e = e - e = 0$.

We define $M \in \mathrm{Ob}(\mathrm{C}^{n]}(\mathscr{A}))$ by setting $M_k = 0_{\mathscr{A}}$ for $k \in \mathbf{Z}_{<n}$, $M_k = X_n$ for $k \in \mathbf{Z}_{\geq n}$, $m_{n+2k} = e$ for $k \in \mathbf{Z}_{\geq 1}$ and $m_{n+2k-1} = 1 - e$ for $k \in \mathbf{Z}_{\geq 1}$.

$$\cdots \longrightarrow X_n \xrightarrow{1-e} X_n \xrightarrow{e} X_n \xrightarrow{1-e} X_n \longrightarrow 0 \longrightarrow \cdots$$

Note that if $\mathscr{A}$ is idempotent complete, then M is split acyclic and, consequently, we have $M \in \mathrm{Ob}(\mathrm{Z}_{\mathrm{K}(\mathscr{A})})$ in this case, cf. definitions 1.9.27, 1.9.4 and lemma 1.9.5.

We define $C \in \mathrm{Ob}(\mathrm{C}^{n]}(\mathscr{A}))$ by setting $C_k = 0_{\mathscr{A}}$ for $k \in \mathbf{Z}_{<n}$, $C_n = X_n$, $C_k = X_k \oplus X_n$ for $k \in \mathbf{Z}_{>n}$, $m_{n+1} = \left(\begin{smallmatrix} x_{n+1} \\ e \end{smallmatrix}\right)$, $m_{n+2k+1} = \left(\begin{smallmatrix} x_{n+2k+1} & 0 \\ 0 & -e \end{smallmatrix}\right)$ for $k \in \mathbf{Z}_{\geq 1}$ and $m_{n+2k} = \left(\begin{smallmatrix} x_{n+2k} & 0 \\ 0 & e-1 \end{smallmatrix}\right)$ for $k \in \mathbf{Z}_{\geq 1}$.

$$\cdots \longrightarrow X_{n+3} \oplus X_n \xrightarrow{\left(\begin{smallmatrix} x_{n+3} & 0 \\ 0 & -e \end{smallmatrix}\right)} X_{n+2} \oplus X_n \xrightarrow{\left(\begin{smallmatrix} x_{n+2} & 0 \\ 0 & e-1 \end{smallmatrix}\right)} X_{n+1} \oplus X_n \xrightarrow{\left(\begin{smallmatrix} x_{n+1} \\ e \end{smallmatrix}\right)} X_n \longrightarrow 0 \longrightarrow \cdots$$

We obtain a pseudo-triangle $M \xrightarrow{f} X\mathbb{S}^{\mathscr{A}}_{n]} \xrightarrow{i} C \xrightarrow{p} M^{[1]}$ in $\mathrm{C}(\mathscr{A})$ by setting $f_n = e$, $f_k = 0$ for $k \in \mathbf{Z}_{>n}$, $i_n = 1$, $i_k = \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)$ for $k \in \mathbf{Z}_{>n}$ and $p_k = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ for $k \in \mathbf{Z}_{>n}$, cf. lemma 1.9.17.

$$\begin{tikzcd}
M \arrow[d, "f"'] & \cdots \arrow[r] & X_n \arrow[r, "1-e"] \arrow[d, "0"] & X_n \arrow[r, "e"] \arrow[d, "0"] & X_n \arrow[r, "1-e"] \arrow[d, "0"] & X_n \arrow[r] \arrow[d, "e"] & 0 \arrow[r] \arrow[d] & \cdots \\
X\mathbb{S}^{\mathscr{A}}_{n]} \arrow[d, "i"'] & \cdots \arrow[r] & X_{n+3} \arrow[r, "x_{n+3}"] \arrow[d, "(1\ 0)"'] & X_{n+2} \arrow[r, "x_{n+2}"] \arrow[d, "(1\ 0)"] & X_{n+1} \arrow[r, "x_{n+1}"] \arrow[d, "(1\ 0)"'] & X_n \arrow[r] \arrow[d, "1"] & 0 \arrow[r] \arrow[d] & \cdots \\
C \arrow[d, "p"'] & \cdots \arrow[r] & X_{n+3} \oplus X_n \arrow[r, "\left(\begin{smallmatrix} x_{n+3} & 0 \\ 0 & -e \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"'] & X_{n+2} \oplus X_n \arrow[r, "\left(\begin{smallmatrix} x_{n+2} & 0 \\ 0 & e-1 \end{smallmatrix}\right)"'] \arrow[d, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"'] & X_{n+1} \oplus X_n \arrow[r, "\left(\begin{smallmatrix} x_{n+1} \\ e \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] & X_n \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
M^{[1]} & \cdots \arrow[r] & X_n \arrow[r, "-e"] & X_n \arrow[r, "e-1"] & X_n \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Note that $M \xrightarrow{\underline{f}} X\mathbb{S}^{\mathscr{A}}_{n]} \xrightarrow{\underline{i}} C \xrightarrow{\underline{p}} M^{[1]}$ is a triangle in $\mathrm{K}(\mathscr{A})$. If $\mathscr{A}$ is idempotent complete, then $\underline{i}$ is an isomorphism in $\mathrm{K}(\mathscr{A})$ since $M \in \mathrm{Ob}(\mathrm{Z}_{\mathrm{K}(\mathscr{A})})$ in this case.

So it suffices to show that $\underline{X\mathfrak{s}^{\mathscr{A}}_{n]} \cdot i}\colon X \to C$ is an isomorphism in $\mathrm{K}(\mathscr{A})$.

We define $C \xrightarrow{g} X$ in $\mathrm{C}(\mathscr{A})$ by setting $g_n = 1 - e$ and $g_k = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ for $k \in \mathbf{Z}_{>n}$.

$$\begin{tikzcd}
C \arrow[d, "g"'] & \cdots \arrow[r] & X_{n+3} \oplus X_n \arrow[r, "\left(\begin{smallmatrix} x_{n+3} & 0 \\ 0 & -e \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] & X_{n+2} \oplus X_n \arrow[r, "\left(\begin{smallmatrix} x_{n+2} & 0 \\ 0 & e-1 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] & X_{n+1} \oplus X_n \arrow[r, "\left(\begin{smallmatrix} x_{n+1} \\ e \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] & X_n \arrow[r] \arrow[d, "1-e"] & 0 \arrow[r] \arrow[d] & \cdots \\
X & \cdots \arrow[r] & X_{n+3} \arrow[r, "x_{n+3}"] & X_{n+2} \arrow[r, "x_{n+2}"] & X_{n+1} \arrow[r, "x_{n+1}"] & X_n \arrow[r, "x_n"] & X_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

We want to show that $\underline{1 - X\mathfrak{s}^{\mathscr{A}}_{n]} \cdot i \cdot g} = 0$ in $\mathrm{K}(\mathscr{A})$. Let $a_k = 0\colon X_k \to X_{k+1}$ for $k \in \mathbf{Z}_{\geq n}$ and $a_k = h_k\colon X_k \to X_{k+1}$ for $k \in \mathbf{Z}_{<n}$.

For $k \in \mathbf{Z}_{>n}$, we have $a_k \cdot x_{k+1} + x_k \cdot a_{k-1} = 0 = 1 - (X\mathfrak{s}^{\mathscr{A}}_{n]})_k \cdot i_k \cdot g_k$.
We have $a_n \cdot x_{n+1} + x_n \cdot a_{n-1} = x_n \cdot h_{n-1} = e = 1 - (X\mathfrak{s}^{\mathscr{A}}_{n]})_k \cdot i_k \cdot g_k$.
For $k \in \mathbf{Z}_{<n}$, we have $a_k \cdot x_{k+1} + x_k \cdot a_{k-1} = h_k \cdot x_{k+1} + x_k \cdot h_{k-1} = 1 = 1 - (X\mathfrak{s}^{\mathscr{A}}_{n]})_k \cdot i_k \cdot g_k$.

We want to show that $\underline{1 - g \cdot X\mathfrak{s}^{\mathscr{A}}_{n]} \cdot i} = 0$ in $\mathrm{K}(\mathscr{A})$. Let $b_k = \left(\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right) : C_k \to C_{k+1}$ for $k \in \mathbf{Z}_{>n+1}$, $b_{n+1} = \left(\begin{smallmatrix} 0 & x_{n+1} \\ 0 & -1 \end{smallmatrix}\right) : C_{n+1} \to C_{n+2}$, $b_n = \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right) : C_n \to C_{n+1}$ and $b_k = 0\colon C_k \to C_{k+1}$ for $k \in \mathbf{Z}_{<n}$. For $k \in \mathbf{Z}_{\geq 1}$, we have

$$\begin{aligned} b_{n+2k+1} \cdot c_{n+2k+2} + c_{n+2k+1} \cdot b_{n+2k} &= \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1-e \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & e \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \\ &= 1 - g_{n+2k+1} \cdot (X\mathfrak{s}^{\mathscr{A}}_{n]})_{n+2k+1} \cdot i_{n+2k+1} . \end{aligned}$$

For $k \in \mathbf{Z}_{>1}$, we have

$$\begin{aligned} b_{n+2k} \cdot c_{n+2k+1} + c_{n+2k} \cdot b_{n+2k-1} &= \left(\begin{smallmatrix} 0 & 0 \\ 0 & e \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1-e \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \\ &= 1 - g_{n+2k} \cdot (X\mathfrak{s}^{\mathscr{A}}_{n]})_{n+2k} \cdot i_{n+2k} . \end{aligned}$$

We have $b_{n+2} \cdot c_{n+3} + c_{n+2} \cdot b_{n+1} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & e \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1-e \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = 1 - g_{n+2} \cdot (X\mathfrak{s}^{\mathscr{A}}_{n]})_{n+2} \cdot i_{n+2}$, $b_{n+1} \cdot c_{n+2} + c_{n+1} \cdot b_n = \left(\begin{smallmatrix} 0 & -x_{n+1} \\ 0 & 1-e \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & x_{n+1} \\ 0 & e \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = 1 - g_{n+1} \cdot (X\mathfrak{s}^{\mathscr{A}}_{n]})_{n+1} \cdot i_{n+1}$ and $b_n \cdot c_{n+1} + c_n \cdot b_{n-1} = e = 1 - g_n \cdot (X\mathfrak{s}^{\mathscr{A}}_{n]})_n \cdot i_n$.
For $k \in \mathbf{Z}_{<n}$, we have $b_k \cdot c_{k+1} + c_k \cdot b_{k-1} = 0 = 1 - g_k \cdot (X\mathfrak{s}^{\mathscr{A}}_{n]})_k \cdot i_k$.

We conclude that $\underline{X\mathfrak{s}^{\mathscr{A}}_{n]} \cdot i}$ and $\underline{g}$ are mutually inverse isomorphisms in $\mathrm{K}(\mathscr{A})$. $\square$

1.9.30 Lemma. Suppose given an additive category $\mathscr{A}$, $n \in \mathbf{Z}$ and $Y \in \mathrm{Ob}(\mathrm{C}^{[n}(\mathscr{A}))$. Suppose given $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ such that X is a summand of Y in $\mathrm{K}(\mathscr{A})$. Then there exists a complex $C \in \mathrm{Ob}(\mathrm{C}^{[n}(\mathscr{A}))$ such that X is isomorphic to C in $\mathrm{K}(\mathscr{A})$. If $\mathscr{A}$ is idempotent complete, then $X\mathfrak{S}^{\mathscr{A}}_{[n} \xrightarrow{\underline{X\mathfrak{s}^{\mathscr{A}}_{[n}}} X$ is an isomorphism in $\mathrm{K}(\mathscr{A})$. $\diamondsuit$

Proof. This is dual to the previous lemma 1.9.29. $\square$

1.9.31 Lemma/Definition. Suppose given an additive category $\mathscr{A}$. We obtain a w-structure

$\mathcal{W}^{\mathscr{A}}$ on $\mathrm{K}(\mathscr{A})$ as follows. For $k \in \mathbf{Z}$, let $\mathcal{W}^{\mathscr{A}}_{k]}$ be the full subcategory of $\mathrm{K}(\mathscr{A})$ defined by

$$\mathrm{Ob}(\mathcal{W}^{\mathscr{A}}_{k]}) = \{X \in \mathrm{Ob}(\mathrm{K}(\mathscr{A}))\colon \text{There exists } C \in \mathrm{Ob}(\mathrm{C}^{k]}(\mathscr{A})) \text{ and an isomorphism } X \longrightarrow C \text{ in } \mathrm{K}(\mathscr{A})\}$$

and let $\mathcal{W}^{\mathscr{A}}_{[k}$ be the full subcategory of $\mathrm{K}(\mathscr{A})$ defined by

$$\mathrm{Ob}(\mathcal{W}^{\mathscr{A}}_{[k}) = \{X \in \mathrm{Ob}(\mathrm{K}(\mathscr{A}))\colon \text{There exists } C \in \mathrm{Ob}(\mathrm{C}^{[k}(\mathscr{A})) \text{ and an isomorphism } X \longrightarrow C \text{ in } \mathrm{K}(\mathscr{A})\}.$$

We call $\mathcal{W}^{\mathscr{A}}$ the *standard w-structure* on $\mathrm{K}(\mathscr{A})$.

We obtain a w-structure $\mathcal{W}^{\mathscr{A},\mathrm{b}}$ on $\mathrm{K}^{\mathrm{b}}(\mathscr{A})$ as follows. For $k \in \mathbf{Z}$, let $\mathcal{W}^{\mathscr{A},\mathrm{b}}_{k]}$ be the full subcategory of $\mathrm{K}^{\mathrm{b}}(\mathscr{A})$ defined by

$$\mathrm{Ob}(\mathcal{W}^{\mathscr{A},\mathrm{b}}_{k]}) = \{X \in \mathrm{Ob}(\mathrm{K}^{\mathrm{b}}(\mathscr{A}))\colon \text{There exists } C \in \mathrm{Ob}(\mathrm{C}^{k]}(\mathscr{A})) \text{ and an isomorphism } X \longrightarrow C \text{ in } \mathrm{K}(\mathscr{A})\}$$

and let $\mathcal{W}^{\mathscr{A},\mathrm{b}}_{[k}$ be the full subcategory of $\mathrm{K}^{\mathrm{b}}(\mathscr{A})$ defined by

$$\mathrm{Ob}(\mathcal{W}^{\mathscr{A},\mathrm{b}}_{[k}) = \{X \in \mathrm{Ob}(\mathrm{K}^{\mathrm{b}}(\mathscr{A}))\colon \text{There exists } C \in \mathrm{Ob}(\mathrm{C}^{[k}(\mathscr{A})) \text{ and an isomorphism } X \longrightarrow C \text{ in } \mathrm{K}(\mathscr{A})\}.$$

We call $\mathcal{W}^{\mathscr{A},\mathrm{b}}$ the *standard w-structure* on $\mathrm{K}^{\mathrm{b}}(\mathscr{A})$. Cf. [Sch11, proposition 4.6]. $\diamondsuit$

The abelian case

We assume some familiarity with abelian categories and derived categories. We refer to [Wei94] for an introduction to these notions and to [Ste75] for Grothendieck categories.

1.9.32 Definition. Suppose given an abelian category $\mathscr{A}$. Let $\mathrm{Inj}(\mathscr{A}) \subseteq \mathscr{A}$ denote the full subcategory of injective objects in $\mathscr{A}$. Let $\mathrm{Proj}(\mathscr{A}) \subseteq \mathscr{A}$ denote the full subcategory of projective objects in $\mathscr{A}$. $\diamondsuit$

1.9.33 Definition. Suppose given an abelian category $\mathscr{A}$. Let $\mathrm{Ac}(\mathscr{A}) \subseteq \mathrm{K}(\mathscr{A})$ denote the full subcategory of acyclic complexes in $\mathrm{K}(\mathscr{A})$. Let $\mathrm{Ac}^{\mathrm{b}}(\mathscr{A}) = \mathrm{Ac}(\mathscr{A}) \cap \mathrm{K}^{\mathrm{b}}(\mathscr{A})$.

We say that a morphism $X \xrightarrow{f} Y$ in $\mathrm{K}(\mathscr{A})$ is a *quasi-isomorphism* if there is a triangle $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X^{[1]}$ in $\mathrm{K}(\mathscr{A})$ such that $Z \in \mathrm{Ob}(\mathrm{Ac}(\mathscr{A}))$.

We say that a morphism $X \xrightarrow{f} Y$ in $\mathrm{K}^{\mathrm{b}}(\mathscr{A})$ is a *quasi-isomorphism* if there is a triangle $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X^{[1]}$ in $\mathrm{K}^{\mathrm{b}}(\mathscr{A})$ such that $Z \in \mathrm{Ob}(\mathrm{Ac}^{\mathrm{b}}(\mathscr{A}))$.

The *derived category* of $\mathscr{A}$ is the Verdier quotient $\mathrm{D}(\mathscr{A}) = \mathrm{K}(\mathscr{A}) /\!\!/ \mathrm{Ac}(\mathscr{A})$. We denote the associated quotient functor by $\mathrm{L}_{\mathscr{A}}\colon \mathrm{K}(\mathscr{A}) \to \mathrm{D}(\mathscr{A})$. Let $\mathrm{I}_{\mathrm{D},\mathscr{A}} = \mathrm{I}_{\mathrm{K},\mathscr{A}} \cdot \mathrm{L}_{\mathscr{A}}\colon \mathscr{A} \to \mathrm{D}(\mathscr{A})$. Let

$\mathbb{H}_{\mathscr{A}} \colon \mathrm{D}(\mathscr{A}) \to \mathscr{A}$ denote the *homology functor* that maps an object to its zeroth homology.

The *bounded derived category* of $\mathscr{A}$ is the Verdier quotient $\mathrm{K}^{\mathrm{b}}(\mathscr{A}) /\!\!/ \mathrm{Ac}^{\mathrm{b}}(\mathscr{A})$. We denote the corresponding quotient functor by $\mathrm{L}^{\mathrm{b}}_{\mathscr{A}} \colon \mathrm{K}^{\mathrm{b}}(\mathscr{A}) \to \mathrm{D}^{\mathrm{b}}(\mathscr{A})$.

Let $\mathrm{I}_{\mathrm{D}^{\mathrm{b}},\mathscr{A}} = \mathrm{I}_{\mathrm{K}^{\mathrm{b}},\mathscr{A}} \cdot \mathrm{L}^{\mathrm{b}}_{\mathscr{A}} \colon \mathscr{A} \to \mathrm{D}^{\mathrm{b}}(\mathscr{A})$. Let $\mathbb{H}^{\mathrm{b}}_{\mathscr{A}} \colon \mathrm{D}^{\mathrm{b}}(\mathscr{A}) \to \mathscr{A}$ denote the *bounded homology functor* that maps an object to its zeroth homology. $\diamondsuit$

1.9.34 Definition. Suppose given an abelian category $\mathscr{A}$, $n \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$. We define the sequence $X\mathbb{T}^{\mathscr{A}}_{n+1\rfloor} \xrightarrow{X\mathbb{t}^{\mathscr{A}}_{n+1\rfloor}} X \xrightarrow{X\mathbb{t}^{\mathscr{A}}_{\lfloor n}} X\mathbb{T}^{\mathscr{A}}_{\lfloor n}$ in $\mathrm{C}(\mathscr{A})$ as follows.

Choose a cokernel $X_{n+1} \xrightarrow{c_{n+1}} C_n$ of x_{n+2} and let $C_n \xrightarrow{d_n} X_n$ denote the unique morphism in $\mathscr{A}$ such that $c_{n+1} \cdot d_n = x_{n+1}$. Choose a kernel $E_{n+1} \xrightarrow{f_{n+1}} X_n$ of x_n and let $X_{n+1} \xrightarrow{e_{n+1}} E_{n+1}$ denote the unique morphism in $\mathscr{A}$ such that $e_{n+1} \cdot f_{n+1} = x_{n+1}$. For $k \in \mathbf{Z}_{>n+1}$,
let $(X\mathbb{T}^{\mathscr{A}}_{n+1\rfloor})_{k-1} = X_{k-1}$, $(X\mathbb{T}^{\mathscr{A}}_{n+1\rfloor})_{k \to k-1} = x_k$, $(X\mathbb{T}^{\mathscr{A}}_{\lfloor n})_k = 0_{\mathscr{A}}$, $(X\mathbb{t}^{\mathscr{A}}_{n+1\rfloor})_{k-1} = 1_{X_{k-1}}$.
Let $(X\mathbb{T}^{\mathscr{A}}_{n+1\rfloor})_n = C_n$, $(X\mathbb{T}^{\mathscr{A}}_{n+1\rfloor})_{n+1 \to n} = c_{n+1}$ and $(X\mathbb{t}^{\mathscr{A}}_{n+1\rfloor})_n = d_n$.
Let $(X\mathbb{T}^{\mathscr{A}}_{\lfloor n})_{n+1} = E_{n+1}$, $(X\mathbb{T}^{\mathscr{A}}_{\lfloor n})_{n+1 \to n} = f_{n+1}$ and $(X\mathbb{t}^{\mathscr{A}}_{\lfloor n})_{n+1} = e_{n+1}$.
For $k \in \mathbf{Z}_{<n}$,
let $(X\mathbb{T}^{\mathscr{A}}_{n+1\rfloor})_k = 0_{\mathscr{A}}$, $(X\mathbb{T}^{\mathscr{A}}_{\lfloor n})_{k+1} = X_{k+1}$, $(X\mathbb{T}^{\mathscr{A}}_{\lfloor n})_{k+1 \to k} = x_{k+1}$, $(X\mathbb{t}^{\mathscr{A}}_{\lfloor n})_{k+1} = 1_{X_{k+1}}$.
The objects $X\mathbb{T}^{\mathscr{A}}_{n+1\rfloor}$ and $X\mathbb{T}^{\mathscr{A}}_{\lfloor n}$ are sometimes called canonical/soft/smart/intelligent truncations of X.

$$\begin{tikzcd}
X\mathbb{T}^{\mathscr{A}}_{n+1\rfloor} \arrow[d, "X\mathbb{t}^{\mathscr{A}}_{n+1\rfloor}"'] & \cdots \arrow[r] & X_{n+2} \arrow[r, "x_{n+2}"] \arrow[d, "1"] & X_{n+1} \arrow[r, "c_{n+1}"] \arrow[d, "1"] & C_n \arrow[r] \arrow[d, "d_n"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
X \arrow[d, "X\mathbb{t}^{\mathscr{A}}_{\lfloor n}"'] & \cdots \arrow[r] & X_{n+2} \arrow[r, "x_{n+2}"] \arrow[d] & X_{n+1} \arrow[r, "x_{n+1}"] \arrow[d, "e_{n+1}"] & X_n \arrow[r, "x_n"] \arrow[d, "1"] & X_{n-1} \arrow[r, "x_{n-1}"] \arrow[d, "1"] & X_{n-2} \arrow[r] \arrow[d, "1"] & \cdots \\
X\mathbb{T}^{\mathscr{A}}_{\lfloor n} & \cdots \arrow[r] & 0 \arrow[r] & E_{n+1} \arrow[r, "f_{n+1}"] & X_n \arrow[r, "x_n"] & X_{n-1} \arrow[r, "x_{n-1}"] & X_{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

$\diamondsuit$

1.9.35 Lemma/Definition. Suppose given an abelian category $\mathscr{A}$. We obtain a t-structure $\mathcal{T}^{\mathscr{A}}$ on $\mathrm{D}(\mathscr{A})$ as follows. For $k \in \mathbf{Z}$, let $\mathcal{T}^{\mathscr{A}}_{k\rfloor}$ be the full subcategory of $\mathrm{D}(\mathscr{A})$ defined by

$$\mathrm{Ob}(\mathcal{T}^{\mathscr{A}}_{k\rfloor}) = \{X \in \mathrm{Ob}(\mathrm{D}(\mathscr{A})) \colon \underline{X\mathbb{t}_{k\rfloor}} \text{ is a quasi-isomorphism in } \mathrm{K}(\mathscr{A})\}$$

and let $\mathcal{T}^{\mathscr{A}}_{\lfloor k}$ be the full subcategory of $\mathrm{D}(\mathscr{A})$ defined by

$$\mathrm{Ob}(\mathcal{T}^{\mathscr{A}}_{\lfloor k}) = \{X \in \mathrm{Ob}(\mathrm{D}(\mathscr{A})) \colon \underline{X\mathbb{t}_{\lfloor k}} \text{ is a quasi-isomorphism in } \mathrm{K}(\mathscr{A})\}.$$

We call $\mathcal{T}^{\mathscr{A}}$ the *standard t-structure* on $\mathrm{D}(\mathscr{A})$.

For $k, \ell \in \mathbf{Z}$, $\mathcal{T}^{\mathscr{A}}_{[k,\ell]}$ is the full subcategory of $\mathrm{D}(\mathscr{A})$ with

$$\mathrm{Ob}(\mathcal{T}^{\mathscr{A}}_{[k,\ell]}) = \{X \in \mathrm{Ob}(\mathrm{D}(\mathscr{A})) \colon X \text{ is isomorphic to a complex } Y \in \mathrm{Ob}(\mathrm{C}^{[k,\ell]}(\mathscr{A})) \text{ in } \mathrm{D}(\mathscr{A})\}.$$

We obtain a t-structure $\mathcal{T}^{\mathscr{A},\mathrm{b}}$ on $\mathrm{D}^{\mathrm{b}}(\mathscr{A})$ as follows. For $k \in \mathbf{Z}$, let $\mathcal{T}^{\mathscr{A},\mathrm{b}}_{k\rfloor}$ be the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathscr{A})$ defined by

$$\mathrm{Ob}(\mathcal{T}^{\mathscr{A},\mathrm{b}}_{k\rfloor}) = \{X \in \mathrm{Ob}(\mathrm{D}^{\mathrm{b}}(\mathscr{A}))\colon \underline{X\mathfrak{t}_{k\rfloor}} \text{ is a quasi-isomorphism in } \mathrm{K}^{\mathrm{b}}(\mathscr{A})\}$$

and let $\mathcal{T}^{\mathscr{A},\mathrm{b}}_{\lfloor k}$ be the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathscr{A})$ defined by

$$\mathrm{Ob}(\mathcal{T}^{\mathscr{A},\mathrm{b}}_{\lfloor k}) = \{X \in \mathrm{Ob}(\mathrm{D}^{\mathrm{b}}(\mathscr{A}))\colon \underline{X\mathfrak{t}_{\lfloor k}} \text{ is a quasi-isomorphism in } \mathrm{K}^{\mathrm{b}}(\mathscr{A})\}.$$

We call $\mathcal{T}^{\mathscr{A},\mathrm{b}}$ the *standard t-structure* on $\mathrm{D}^{\mathrm{b}}(\mathscr{A})$.
For $k, \ell \in \mathbf{Z}$, $\mathcal{T}^{\mathscr{A},\mathrm{b}}_{\lfloor k,\ell\rfloor}$ is the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathscr{A})$ with

$$\mathrm{Ob}(\mathcal{T}^{\mathscr{A},\mathrm{b}}_{\lfloor k,\ell\rfloor}) = \{X \in \mathrm{Ob}(\mathrm{D}^{\mathrm{b}}(\mathscr{A}))\colon X \text{ is isomorphic to a complex } Y{\in}\mathrm{Ob}(\mathrm{C}^{\lceil k,\ell\rceil}(\mathscr{A})) \text{ in } \mathrm{D}^{\mathrm{b}}(\mathscr{A})\}.$$

Cf. [BBD82, example 1.3.2.(i)]. $\diamondsuit$

1.9.36 Definition. Suppose given an abelian category $\mathscr{A}$. A complex $I \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ is called *K-injective* if ${}_{\mathrm{K}(\mathscr{A})}(X, I) = 0$ for every acyclic complex $X \in \mathrm{Ob}(\mathrm{Ac}(\mathscr{A}))$.
Let $\mathrm{K}^{\mathrm{inj}}(\mathscr{A}) \subseteq \mathrm{K}(\mathscr{A})$ denote the full subcategory of K-injective complexes in $\mathrm{K}(\mathscr{A})$. Note that $\mathrm{K}^{\mathrm{inj}}(\mathscr{A})$ is closed under products in $\mathrm{K}(\mathscr{A})$.
Cf. [Spa88, definition 1.1] [BN93, remark 2.13]. $\diamondsuit$

1.9.37 Definition. Suppose given an abelian category $\mathscr{A}$. A complex $P \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ is called *K-projective* if ${}_{\mathrm{K}(\mathscr{A})}(P, X) = 0$ for every acyclic complex $X \in \mathrm{Ob}(\mathrm{Ac}(\mathscr{A}))$.
Let $\mathrm{K}^{\mathrm{proj}}(\mathscr{A}) \subseteq \mathrm{K}(\mathscr{A})$ denote the full subcategory of K-projective complexes in $\mathrm{K}(\mathscr{A})$. Note that $\mathrm{K}^{\mathrm{proj}}(\mathscr{A})$ is closed under coproducts in $\mathrm{K}(\mathscr{A})$. $\diamondsuit$

1.9.38 Lemma. [Sta, lemma 070I] [Mur06, proposition 51]
Suppose given an abelian category $\mathscr{A}$ and a complex $I \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$. The following three statements are equivalent.

(a) The complex I is K-injective.

(b) For every complex $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$, the map ${}_{\mathrm{K}(\mathscr{A})}(X, I) \to {}_{\mathrm{D}(\mathscr{A})}(X, I)\colon f \mapsto f\mathrm{L}_{\mathscr{A}}$ is an isomorphism of abelian groups.

(c) For every quasi-isomorphism $X \xrightarrow{f} Y$ in $\mathrm{K}(\mathscr{A})$, the map ${}_{\mathrm{K}(\mathscr{A})}(f, I)\colon {}_{\mathrm{K}(\mathscr{A})}(Y, I) \to {}_{\mathrm{K}(\mathscr{A})}(X, I)\colon g \mapsto f \cdot g$ is an isomorphism of abelian groups.

$\diamondsuit$

1.9.39 Lemma. Suppose given an abelian category $\mathscr{A}$ and a complex $P \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$. The following three statements are equivalent.

(a) The complex P is K-projective.

(b) For every complex $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$, the map ${}_{\mathrm{K}(\mathscr{A})}(P, X) \to {}_{\mathrm{D}(\mathscr{A})}(P, X)\colon f \mapsto f\mathrm{L}_{\mathscr{A}}$ is an isomorphism of abelian groups.

(c) For every quasi-isomorphism $X \xrightarrow{f} Y$ in $\mathrm{K}(\mathscr{A})$, the map ${}_{\mathrm{K}(\mathscr{A})}(P, f)\colon {}_{\mathrm{K}(\mathscr{A})}(P, X) \to {}_{\mathrm{K}(\mathscr{A})}(P, Y)\colon g \mapsto g \cdot f$ is an isomorphism of abelian groups. $\diamondsuit$

1.9.40 Lemma. [Sta, lemma 090X]
Suppose given an abelian category $\mathscr{A}$ and a triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X^{[1]}$ in $\mathrm{K}(\mathscr{A})$. If two out of the three objects X, Y , Z are K-injective, then the remaining one is K-injective as well. $\diamondsuit$

1.9.41 Lemma. [Sta, lemma 070J] [Mur06, proposition 47]
Suppose given an abelian category $\mathscr{A}$, $n \in \mathbf{Z}$ and a complex $X \in \mathrm{Ob}(\mathrm{C}^{\lceil n}(\mathscr{A}))$ such that $X_k \in \mathrm{Ob}(\mathrm{Inj}(\mathscr{A}))$ for $k \in \mathbf{Z}$. Then X is K-injective. $\diamondsuit$

1.9.42 Definition. We say that an abelian category $\mathscr{A}$ *has enough K-injectives* if for every $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$, there exists a K-injective complex $I \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ and a quasi-isomorphism $X \longrightarrow I$ in $\mathrm{K}(\mathscr{A})$. $\diamondsuit$

1.9.43 Definition. We say that an abelian category $\mathscr{A}$ *has enough K-projectives* if for every $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$, there exists a K-projective complex $P \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ and a quasi-isomorphism $P \longrightarrow X$ in $\mathrm{K}(\mathscr{A})$. $\diamondsuit$

1.9.44 Definition. Suppose given an abelian category $\mathscr{A}$. A complex $I \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ is called *DG-injective* if it is K-injective and if $I_k \in \mathrm{Ob}(\mathrm{Inj}(\mathscr{A}))$ for $k \in \mathbf{Z}$. $\diamondsuit$

1.9.45 Definition. Suppose given an abelian category $\mathscr{A}$. A complex $P \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ is called *DG-projective* if it is K-projective and if $P_k \in \mathrm{Ob}(\mathrm{Proj}(\mathscr{A}))$ for $k \in \mathbf{Z}$. $\diamondsuit$

1.9.46 Remark. Suppose given an abelian category $\mathscr{A}$ and a complex $X \in \mathrm{Ob}(\mathscr{A})$. By definition, X is DG-projective if and only if it is K-projective and has projective entries. Not every K-projective complex has projective entries since any zero object in $\mathrm{K}(\mathscr{A})$ is K-projective and so split acyclic complexes of the form $\cdots \longrightarrow 0 \longrightarrow Y \xrightarrow{1} Y \longrightarrow 0 \longrightarrow \cdots$ are always K-projective. Also not every complex with projective entries is K-projective. For example, let $\mathscr{A} = \text{Mod-}\mathbf{Z}/4\mathbf{Z}$ and consider the complex $X = \big(\cdots \xrightarrow{2} \mathbf{Z}/4\mathbf{Z} \xrightarrow{2} \mathbf{Z}/4\mathbf{Z} \xrightarrow{2} \cdots\big)$ with projective (free) entries. It is acyclic and thus ${}_{\mathrm{K}(\mathscr{A})}(X, X) = 0$ if it were K-projective. But X is not a zero object in $\mathrm{K}(\mathscr{A})$ which can be seen by tensoring with $\mathbf{Z}/2\mathbf{Z}$. Cf. [Spa88, p. 124] and [Dol60, section 3.4].

Dually, X is DG-injective if and only if it is K-injective and has injective entries. Not every K-injective complex has injective entries and not every complex with injective entries is K-injective. $\diamondsuit$

1.9.47 Definition. We say that an abelian category $\mathscr{A}$ *has enough DG-injectives* if for every $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$, there exists a DG-injective complex $I \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ and a quasi-isomorphism $X \longrightarrow I$ in $\mathrm{K}(\mathscr{A})$. $\diamondsuit$

1.9.48 Example. A Grothendieck category has enough DG-injectives by [Ser03, theorem 3.13, lemma 3.7] [Sta, theorem 079P]. $\diamondsuit$

1.9.49 Definition. We say that an abelian category $\mathscr{A}$ *has enough DG-projectives* if for every $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$, there exists a DG-projective complex $P \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ and a quasi-isomorphism $P \longrightarrow X$ in $\mathrm{K}(\mathscr{A})$. $\diamondsuit$

1.9.50 Example. The module category Mod-R, where R is a ring, has enough DG-injectives and enough DG-projectives by [AF91, 1.6]. $\diamondsuit$

1.9.51 Lemma. Suppose given an abelian category $\mathscr{A}$, $n \in \mathbf{Z}$ and a DG-injective complex $I \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$. Then $I\mathbb{S}^{\mathscr{A}}_{\lceil n} \in \mathrm{Ob}(\mathrm{C}^{\lceil n}(\mathscr{A}))$ and $I\mathbb{S}^{\mathscr{A}}_{n+1\rceil} \in \mathrm{Ob}(\mathrm{C}^{n+1\rceil}(\mathscr{A}))$ are DG-injective as well. $\diamondsuit$

Proof. It suffices to show that $I\mathbb{S}^{\mathscr{A}}_{\lceil n}$ and $I\mathbb{S}^{\mathscr{A}}_{n+1\rceil}$ are K-injective. Now $I\mathbb{S}_{\lceil n}$ is K-injective by lemma 1.9.41. Since we have a triangle $I\mathbb{S}^{\mathscr{A}}_{\lceil n} \xrightarrow{I\mathbb{s}^{\mathscr{A}}_{\lceil n}} I \xrightarrow{I\mathbb{s}^{\mathscr{A}}_{n+1\rceil}} I\mathbb{S}^{\mathscr{A}}_{n+1\rceil} \longrightarrow (I\mathbb{S}^{\mathscr{A}}_{\lceil n})^{[1]}$ in $\mathrm{K}(\mathscr{A})$, $I\mathbb{S}^{\mathscr{A}}_{n+1\rceil}$ is K-injective as well by lemma 1.9.40. $\square$

1.9.52 Lemma. Suppose given an abelian category $\mathscr{A}$. Suppose given a direct sum $X \underset{s}{\overset{i}{\rightleftarrows}} D \underset{p}{\overset{t}{\leftrightarrows}} Y$ in $\mathrm{D}(\mathscr{A})$. Suppose given quasi-isomorphisms $X \xrightarrow{f} I$ $D \xrightarrow{g} J$ and $Y \xrightarrow{h} K$ in $\mathrm{K}(\mathscr{A})$ such that I, J, K are K-injective. Then there exists a direct sum $I \underset{s'}{\overset{i'}{\rightleftarrows}} J \underset{p'}{\overset{t'}{\leftrightarrows}} K$ in $\mathrm{K}(\mathscr{A})$. $\diamondsuit$

Proof. We will repeatedly use the properties 1.9.38.(b,c) of K-injective complexes. We abbreviate $\mathrm{L} = \mathrm{L}_{\mathscr{A}} \colon \mathrm{K}(\mathscr{A}) \to \mathrm{D}(\mathscr{A})$. Note that we have $i \cdot s = 1$, $t \cdot p = 1$ and $s \cdot i + p \cdot t = 1$ in $\mathrm{D}(\mathscr{A})$. By lemma 1.9.38, there exists

- a unique morphism $I \xrightarrow{i'} J$ in $\mathrm{K}(\mathscr{A})$ such that $(f \cdot i')\mathrm{L} = i \cdot g\mathrm{L}$.
- a unique morphism $J \xrightarrow{s'} I$ in $\mathrm{K}(\mathscr{A})$ such that $(g \cdot s')\mathrm{L} = s \cdot f\mathrm{L}$.
- a unique morphism $J \xrightarrow{p'} K$ in $\mathrm{K}(\mathscr{A})$ such that $(g \cdot p')\mathrm{L} = p \cdot h\mathrm{L}$.
- a unique morphism $K \xrightarrow{t'} J$ in $\mathrm{K}(\mathscr{A})$ such that $(h \cdot t')\mathrm{L} = t \cdot g\mathrm{L}$.

It remains to show that $i' \cdot s' = 1$, $t' \cdot p' = 1$ and $s' \cdot i' + p' \cdot t' = 1$ in $\mathrm{K}(\mathscr{A})$.

Using lemma 1.9.38, we have $i' \cdot s' = 1$ since

$$(f \cdot i' \cdot s')\mathrm{L} = i \cdot g\mathrm{L} \cdot s'\mathrm{L} = i \cdot s \cdot f\mathrm{L} = f\mathrm{L} = (f \cdot 1)\mathrm{L}.$$

Using lemma 1.9.38, we have $t' \cdot p' = 1$ since

$$(h \cdot t' \cdot p')\mathrm{L} = t \cdot g\mathrm{L} \cdot p'\mathrm{L} = t \cdot p \cdot h\mathrm{L} = h\mathrm{L} = (h \cdot 1)\mathrm{L}.$$

Using lemma 1.9.38, we have $t' \cdot p' = 1$ since

$$(h \cdot t' \cdot p')\mathrm{L} = t \cdot g\mathrm{L} \cdot p'\mathrm{L} = t \cdot p \cdot h\mathrm{L} = h\mathrm{L} = (h \cdot 1)\mathrm{L}.$$

Using lemma 1.9.38, we have $s' \cdot i' + p' \cdot t' = 1$ since

$$(g \cdot (s' \cdot i' + p' \cdot t'))\mathrm{L} = s \cdot f\mathrm{L} \cdot i'\mathrm{L} + p \cdot h\mathrm{L} \cdot t'\mathrm{L} = s \cdot i \cdot g\mathrm{L} + p \cdot t \cdot g\mathrm{L} = g\mathrm{L} = (g \cdot 1)\mathrm{L}.$$

$$\begin{tikzcd} X \arrow[r, shift left, "i"] \arrow[d, "f"'] & D \arrow[l, shift left, "s"] \arrow[r, shift right, "p"'] \arrow[d, "g"] & Y \arrow[l, shift right, "t"'] \arrow[d, "h"] \\ I \arrow[r, shift left, "i'"] & J \arrow[l, shift left, "s'"] \arrow[r, shift right, "p'"'] & K \arrow[l, shift right, "t'"'] \end{tikzcd}$$

$\square$

1.9.53 Lemma/Definition.
Suppose given an abelian category $\mathscr{A}$ with enough DG-injectives. We define the *standard injective w-structure* $\mathcal{W}^{\mathscr{A},\mathrm{inj}} = (\mathcal{W}^{\mathscr{A},\mathrm{inj}}_{\lceil 0}, \mathcal{W}^{\mathscr{A},\mathrm{inj}}_{0\rceil})$ on $\mathrm{D}(\mathscr{A})$ as follows. For $k \in \mathbf{Z}$, $\mathcal{W}^{\mathscr{A},\mathrm{inj}}_{k\rceil}$ is the full subcategory of $\mathrm{D}(\mathscr{A})$ with

$$\mathrm{Ob}(\mathcal{W}^{\mathscr{A},\mathrm{inj}}_{k\rceil}) = \{X \in \mathrm{Ob}(\mathrm{D}(\mathscr{A}))\colon \text{There exists a DG-injective complex } I \in \mathrm{Ob}(\mathrm{C}^{k\rceil}(\mathscr{A})) \text{ and a quasi-isomorphism } X \longrightarrow I \text{ in } \mathrm{K}(\mathscr{A})\}$$

and $\mathcal{W}^{\mathscr{A},\mathrm{inj}}_{\lceil k}$ is the full subcategory of $\mathrm{D}(\mathscr{A})$ with

$$\mathrm{Ob}(\mathcal{W}^{\mathscr{A},\mathrm{inj}}_{\lceil k}) = \{X \in \mathrm{Ob}(\mathrm{D}(\mathscr{A}))\colon \text{There exists a DG-injective complex } I \in \mathrm{Ob}(\mathrm{C}^{\lceil k}(\mathscr{A})) \text{ and a quasi-isomorphism } X \longrightarrow I \text{ in } \mathrm{K}(\mathscr{A})\}.$$

(a) For $k, \ell \in \mathbf{Z}$, $\mathcal{W}^{\mathscr{A},\mathrm{inj}}_{\lceil k,\ell\rceil}$ is the full subcategory of $\mathrm{D}(\mathscr{A})$ with

$$\mathrm{Ob}(\mathcal{W}^{\mathscr{A},\mathrm{inj}}_{\lceil k,\ell\rceil}) = \{X \in \mathrm{Ob}(\mathrm{D}(\mathscr{A}))\colon \text{There exists a DG-injective complex } I \in \mathrm{Ob}(\mathrm{C}^{\lceil k,\ell\rceil}(\mathscr{A})) \text{ and a quasi-isomorphism } X \longrightarrow I \text{ in } \mathrm{K}(\mathscr{A})\}.$$

(b) We have $\mathcal{W}^{\mathscr{A},\mathrm{inj}}_{0\rceil} \subseteq \mathcal{T}^{\mathscr{A}}_{0]}$.

(c) We have $\mathcal{W}^{\mathscr{A},\mathrm{inj}}_{\lceil 0} = \mathcal{T}^{\mathscr{A}}_{[0}$, i.e. $\mathcal{T}^{\mathscr{A}}$ is left-adjacent to $\mathcal{W}^{\mathscr{A},\mathrm{inj}}$. Cf. definition 1.8.6.(a).

$\diamondsuit$

Proof. We abbreviate $\mathcal{W} = \mathcal{W}^{\mathscr{A},\mathrm{inj}}$.

We want to show that $\mathcal{W}$ is a w-structure on $\mathrm{D}(\mathscr{A})$.

Suppose given $Y \in \mathrm{Ob}(\mathcal{W}_{\lceil 0})$ and $X \in \mathrm{Ob}(\mathrm{D}(\mathscr{A}))$ such that X is a summand of Y in $\mathrm{D}(\mathscr{A})$. Choose quasi-isomorphisms $X \xrightarrow{f} I$ and $Y \xrightarrow{g} J$ in $\mathrm{K}(\mathscr{A})$ such that $I \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ and $J \in \mathrm{Ob}(\mathrm{C}^{\lceil 0}(\mathscr{A}))$ are DG-injective. Thus I is a summand of J in $\mathrm{K}(\mathscr{A})$ by lemma 1.9.52. By lemma 1.9.30, I is isomorphic to $I\mathbb{S}^{\mathscr{A}}_{\lceil 0} \in \mathrm{Ob}(\mathrm{C}^{\lceil 0}(\mathscr{A}))$ in $\mathrm{K}(\mathscr{A})$.
We conclude that $X \in \mathrm{Ob}(\mathcal{W}_{\lceil 0})$.

Suppose given $Y \in \mathrm{Ob}(\mathcal{W}_{0\rceil})$ and $X \in \mathrm{Ob}(\mathrm{D}(\mathscr{A}))$ such that X is a summand of Y in $\mathrm{D}(\mathscr{A})$. Choose quasi-isomorphisms $X \xrightarrow{f} I$ and $Y \xrightarrow{g} J$ in $\mathrm{K}(\mathscr{A})$ such that $I \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ and

$J \in \mathrm{Ob}(\mathrm{C}^{0]}(\mathscr{A}))$ are DG-injective. Thus I is a summand of J in $\mathrm{K}(\mathscr{A})$ by lemma 1.9.52. By lemma 1.9.29, I is isomorphic to $I\mathbb{S}^{\mathscr{A}}_{0]} \in \mathrm{Ob}(\mathrm{C}^{0]}(\mathscr{A}))$ in $\mathrm{K}(\mathscr{A})$.
We conclude that $X \in \mathrm{Ob}(\mathcal{W}_{0]})$.

Ad (W1). Suppose given $X \in \mathrm{Ob}(\mathcal{W}_{[0})$ and $Y \in \mathrm{Ob}(\mathcal{W}_{1]})$. Choose quasi-isomorphisms $X \xrightarrow{f} I$ and $Y \xrightarrow{g} J$ in $\mathrm{K}(\mathscr{A})$ such that $I \in \mathrm{Ob}(\mathrm{C}^{[0}(\mathscr{A}))$ and $J \in \mathrm{Ob}(\mathrm{C}^{1]}(\mathscr{A}))$ are DG-injective. We have ${}_{\mathrm{D}(\mathscr{A})}(X,Y) = 0$ since ${}_{\mathrm{K}(\mathscr{A})}(I,J) = 0$.

Ad (W2). Suppose given $X \in \mathrm{Ob}(\mathcal{W}_{0]})$. Choose a quasi-isomorphism $X \xrightarrow{f} I$ in $\mathrm{K}(\mathscr{A})$ such that $I \in \mathrm{Ob}(\mathrm{C}^{0]}(\mathscr{A}))$ is DG-injective. Then $X^{[1]} \xrightarrow{f^{[1]}} I^{[1]}$ is a quasi-isomorphism in $\mathrm{K}(\mathscr{A})$ such that $I^{[1]} \in \mathrm{Ob}(\mathrm{C}^{1]}(\mathscr{A})) \subseteq \mathrm{Ob}(\mathrm{C}^{0]}(\mathscr{A}))$.

Ad (W3). Suppose given $X \in \mathrm{Ob}(\mathrm{D}(\mathscr{A}))$. Choose a quasi-isomorphism $X \xrightarrow{f} I$ in $\mathrm{K}(\mathscr{A})$ such that $I \in \mathrm{Ob}(\mathrm{C}(\mathscr{A}))$ is DG-injective. We have a triangle $I\mathbb{S}^{\mathscr{A}}_{[0} \xrightarrow{\mathrm{s}^{\mathscr{A}}_{[0}} I \xrightarrow{\mathrm{s}^{\mathscr{A}}_{1]}} I\mathbb{S}^{\mathscr{A}}_{1]} \longrightarrow (I\mathbb{S}^{\mathscr{A}}_{[0})^{[1]}$ in $\mathrm{K}(\mathscr{A})$ with $I\mathbb{S}^{\mathscr{A}}_{[0}$, $I\mathbb{S}^{\mathscr{A}}_{1]}$ DG-injective by lemma 1.9.51. This yields a triangle
$I\mathbb{S}^{\mathscr{A}}_{[0} \longrightarrow X \longrightarrow I\mathbb{S}^{\mathscr{A}}_{1]} \longrightarrow (I\mathbb{S}^{\mathscr{A}}_{[0})^{[1]}$ in $\mathrm{D}(\mathscr{A})$.
We conclude that $\mathcal{W}$ is a w-structure on $\mathrm{D}(\mathscr{A})$.

Ad (a). Suppose given $X \in \mathrm{Ob}(\mathcal{W}_{[k,\ell]})$. Choose quasi-isomorphisms $X \xrightarrow{f} I$ and $X \xrightarrow{g} J$ in $\mathrm{K}(\mathscr{A})$ such that $I \in \mathrm{Ob}(\mathrm{C}^{\ell]}(\mathscr{A}))$ and $J \in \mathrm{Ob}(\mathrm{C}^{[k}(\mathscr{A}))$ are DG-injective. Since X is a summand of X in $\mathrm{D}(\mathscr{A})$, J is a summand of I in $\mathrm{K}(\mathscr{A})$ by lemma 1.9.52. By lemma 1.9.29, J is isomorphic to $J\mathbb{S}^{\mathscr{A}}_{\ell]} \in \mathrm{Ob}(\mathrm{C}^{[k,\ell]}(\mathscr{A}))$ in $\mathrm{K}(\mathscr{A})$.

Ad (b). Suppose given $X \in \mathrm{Ob}(\mathcal{W}_{0]})$. Choose a quasi-isomorphism $X \xrightarrow{f} I$ in $\mathrm{K}(\mathscr{A})$ such that $I \in \mathrm{Ob}(\mathrm{C}^{0]}(\mathscr{A}))$ is DG-injective. We have $I \in \mathrm{Ob}(\mathcal{T}^{\mathscr{A}}_{0]})$ and thus $X \in \mathrm{Ob}(\mathcal{T}^{\mathscr{A}}_{0]})$.

Ad (c). Suppose given $X \in \mathrm{Ob}(\mathcal{W}_{[0})$. Choose a quasi-isomorphism $X \xrightarrow{f} I$ in $\mathrm{K}(\mathscr{A})$ such that $I \in \mathrm{Ob}(\mathrm{C}^{[0}(\mathscr{A}))$ is DG-injective. We have $I \in \mathrm{Ob}(\mathcal{T}^{\mathscr{A}}_{[0})$ and thus $X \in \mathrm{Ob}(\mathcal{T}^{\mathscr{A}}_{[0})$.
Conversely, suppose given $X \in \mathrm{Ob}(\mathcal{T}^{\mathscr{A}}_{[0})$. It suffices to show that $X \in \mathrm{Ob}({}^{\perp}\mathcal{W}_{1]})$. Suppose given $Y \in \mathrm{Ob}(\mathcal{W}_{1]})$. Choose a quasi-isomorphism $Y \xrightarrow{f} I$ in $\mathrm{K}(\mathscr{A})$ such that $I \in \mathrm{Ob}(\mathrm{C}^{1]}(\mathscr{A}))$ is DG-injective. We have ${}_{\mathrm{D}(\mathscr{A})}(X,Y) = 0$ since ${}_{\mathrm{K}(\mathscr{A})}(X\mathbb{T}^{\mathscr{A}}_{[0}, I) = 0$. $\square$

1.9.54 Lemma/Definition. Suppose given an abelian category $\mathscr{A}$ with enough DG-projectives. We define the *standard projective w-structure* $\mathcal{W}^{\mathscr{A},\mathrm{proj}} = (\mathcal{W}^{\mathscr{A},\mathrm{proj}}_{[0}, \mathcal{W}^{\mathscr{A},\mathrm{proj}}_{0]})$ on $\mathrm{D}(\mathscr{A})$ as follows. For $k \in \mathbf{Z}$, $\mathcal{W}^{\mathscr{A},\mathrm{proj}}_{k]}$ is the full subcategory of $\mathrm{D}(\mathscr{A})$ with

$$\mathrm{Ob}(\mathcal{W}^{\mathscr{A},\mathrm{proj}}_{k]}) = \{X \in \mathrm{Ob}(\mathrm{D}(\mathscr{A}))\colon \text{There exists a DG-projective complex } P \in \mathrm{Ob}(\mathrm{C}^{k]}(\mathscr{A})) \text{ and a quasi-isomorphism } P \longrightarrow X \text{ in } \mathrm{K}(\mathscr{A})\}$$

and $\mathcal{W}^{\mathscr{A},\mathrm{proj}}_{[k}$ is the full subcategory of $\mathrm{D}(\mathscr{A})$ with

$$\mathrm{Ob}(\mathcal{W}^{\mathscr{A},\mathrm{proj}}_{[k}) = \{X \in \mathrm{Ob}(\mathrm{D}(\mathscr{A}))\colon \text{There exists a DG-projective complex } P \in \mathrm{Ob}(\mathrm{C}^{[k}(\mathscr{A})) \text{ and a quasi-isomorphism } P \longrightarrow X \text{ in } \mathrm{K}(\mathscr{A})\}.$$

(a) For $k, \ell \in \mathbf{Z}$, $\mathcal{W}^{\mathscr{A},\mathrm{proj}}_{[k,\ell]}$ is the full subcategory of $\mathrm{D}(\mathscr{A})$ with

$$\mathrm{Ob}(\mathcal{W}^{\mathscr{A},\mathrm{proj}}_{[k,\ell]}) = \{X \in \mathrm{Ob}(\mathrm{D}(\mathscr{A}))\colon \text{There exists a DG-projective complex } P \in \mathrm{Ob}(\mathrm{C}^{[k,\ell]}(\mathscr{A})) \text{ and a quasi-isomorphism } P \longrightarrow X \text{ in } \mathrm{K}(\mathscr{A})\}.$$

(b) We have $\mathcal{W}^{\mathscr{A},\mathrm{proj}}_{[0} \subseteq \mathcal{T}^{\mathscr{A}}_{[0}$.

(c) We have $\mathcal{W}^{\mathscr{A},\mathrm{proj}}_{0]} = \mathcal{T}^{\mathscr{A}}_{0]}$, i.e. $\mathcal{T}^{\mathscr{A}}$ is right-adjacent to $\mathcal{W}^{\mathscr{A},\mathrm{proj}}$. Cf. definition 1.8.6.(b). ♢

Proof. This is dual to the previous lemma 1.9.53 □

1.9.55 Lemma. Suppose given an abelian category $\mathscr{A}$, an acyclic complex $X \in \mathrm{Ob}(\mathrm{Ac}(\mathscr{A}))$ and $m \in \mathbf{Z}$. There exists a triangle $T \longrightarrow X \longrightarrow U \longrightarrow T^{[1]}$ in $\mathrm{K}(\mathscr{A})$ such that $T, U \in \mathrm{Ob}(\mathrm{Ac}(\mathscr{A}))$, $T \in \mathrm{Ob}(\mathrm{C}^{m]}(\mathscr{A}))$, $U \in \mathrm{Ob}(\mathrm{C}^{[m+1}(\mathscr{A}))$ and such that the following two statements hold.

- If $X \in \mathrm{Ob}(\mathrm{C}^{[n}(\mathscr{A}))$, where $n \in \mathbf{Z}$, then $T \in \mathrm{Ob}(\mathrm{C}^{[n}(\mathscr{A}))$ as well.
- If $X \in \mathrm{Ob}(\mathrm{C}^{n]}(\mathscr{A}))$, where $n \in \mathbf{Z}$, then $U \in \mathrm{Ob}(\mathrm{C}^{n]}(\mathscr{A}))$ as well.

♢

Proof. Choose an image $X_{m+1} \xrightarrow{p} M \xrightarrow{i} X_m$ of x_{m+1} in $\mathscr{A}$.
Consider the pseudo-triangle $T \xrightarrow{h} X \xrightarrow{j} C \xrightarrow{q} T^{[1]}$ in $\mathrm{C}(\mathscr{A})$ with

- $T_k = X_k$, $t_{k+1} = x_{k+1}$ for $k \in \mathbf{Z}_{>m}$, $T_m = M$, $t_{m+1} = p$, $T_k = 0_{\mathscr{A}}$ for $k \in \mathbf{Z}_{<m}$,
- $h_k = 1$ for $k \in \mathbf{Z}_{>m}$, $h_m = i$,
- $C_k = X_k \oplus X_{k-1}$, $c_{k+1} = \left(\begin{smallmatrix} x_{k+1} & 0 \\ 1 & -x_k \end{smallmatrix}\right)$ for $k \in \mathbf{Z}_{>m+1}$, $C_{m+1} = X_{m+1} \oplus M$, $c_{m+2} = \left(\begin{smallmatrix} x_{m+2} & 0 \\ 1 & -p \end{smallmatrix}\right)$, $c_{m+1} = \left(\begin{smallmatrix} x_{m+1} \\ i \end{smallmatrix}\right)$, $C_k = X_k$, $c_k = x_k$ for $k \in \mathbf{Z}_{\leq m}$,
- $q_k = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ for $k \in \mathbf{Z}_{>m}$.

$$\begin{tikzcd}
T \arrow[d, "h"'] & \cdots \arrow[r] & X_{m+3} \arrow[r, "x_{m+3}"] \arrow[d, "1"'] & X_{m+2} \arrow[r, "x_{m+2}"] \arrow[d, "1"] & X_{m+1} \arrow[r, "p"] \arrow[d, "1"] & M \arrow[r] \arrow[d, "i"] & 0 \arrow[r] \arrow[d] & \cdots \\
X \arrow[d, "j"'] & \cdots \arrow[r] & X_{m+3} \arrow[r, "x_{m+3}"] \arrow[d, "(1\ 0)"'] & X_{m+2} \arrow[r, "x_{m+2}"] \arrow[d, "(1\ 0)"] & X_{m+1} \arrow[r, "x_{m+1}"] \arrow[d, "(1\ 0)"'] & X_m \arrow[r, "x_m"] \arrow[d, "1"] & X_{m-1} \arrow[r] \arrow[d, "1"] & \cdots \\
C \arrow[d, "q"'] & \cdots \arrow[r] & X_{m+3} \oplus X_{m+2} \arrow[r, "\left(\begin{smallmatrix} x_{m+3} & 0 \\ 1 & -x_{m+2} \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"'] & X_{m+2} \oplus X_{m+1} \arrow[r, "\left(\begin{smallmatrix} x_{m+2} & 0 \\ 1 & -p \end{smallmatrix}\right)"'] \arrow[d, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"'] & X_{m+1} \oplus M \arrow[r, "\left(\begin{smallmatrix} x_{m+1} \\ i \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] & X_m \arrow[r, "x_m"] \arrow[d] & X_{m-1} \arrow[r] \arrow[d] & \cdots \\
T^{[1]} & \cdots \arrow[r] & X_{m+2} \arrow[r, "-x_{m+2}"] & X_{m+1} \arrow[r, "-p"] & M \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Note that $T \in \mathrm{Ob}(\mathrm{Ac}(\mathscr{A}))$ and $T \in \mathrm{Ob}(\mathrm{C}^{m\rceil}(\mathscr{A}))$ by construction. If $X \in \mathrm{Ob}(\mathrm{C}^{\lceil n}(\mathscr{A}))$, where $n \in \mathbf{Z}$, then $T \in \mathrm{Ob}(\mathrm{C}^{\lceil n}(\mathscr{A}))$ by construction as well.

We define $U \in \mathrm{C}(\mathscr{A})$ by setting $U_k = 0_{\mathscr{A}}$ for $k \in \mathbf{Z}_{>m+1}$, $U_{m+1} = M$, $u_{m+1} = i$, $U_k = X_k$, $u_k = x_k$ for $k \in \mathbf{Z}_{\leq m}$.

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{i} X_m \xrightarrow{x_m} X_{m-1} \xrightarrow{x_{m-1}} X_{m-2} \longrightarrow \cdots$$

Note that $U \in \mathrm{Ob}(\mathrm{Ac}(\mathscr{A}))$ and $U \in \mathrm{Ob}(\mathrm{C}^{\lceil m+1}(\mathscr{A}))$ by construction. If $X \in \mathrm{Ob}(\mathrm{C}^{n\rceil}(\mathscr{A}))$, where $n \in \mathbf{Z}$, then $U \in \mathrm{Ob}(\mathrm{C}^{n\rceil}(\mathscr{A}))$ by construction as well.

It remains to show that C and U are isomorphic in $\mathrm{K}(\mathscr{A})$.

We define $C \xrightarrow{f} U$ in $\mathrm{C}(\mathscr{A})$ by setting $f_{m+1} = \left(\begin{smallmatrix} p \\ 1 \end{smallmatrix}\right)$ and $f_k = 1$ for $k \in \mathbf{Z}_{\leq m}$.

$$\begin{tikzcd}
C \arrow[d, "f"'] & \cdots \arrow[r] & X_{m+3}\oplus X_{m+2} \arrow[r, "\left(\begin{smallmatrix} x_{m+3} & 0 \\ 1 & -x_{m+2} \end{smallmatrix}\right)"] \arrow[d] & X_{m+2}\oplus X_{m+1} \arrow[r, "\left(\begin{smallmatrix} x_{m+2} & 0 \\ 1 & -p \end{smallmatrix}\right)"] \arrow[d] & X_{m+1}\oplus M \arrow[r, "\left(\begin{smallmatrix} x_{m+1} \\ i \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} p \\ 1 \end{smallmatrix}\right)"] & X_m \arrow[r, "x_m"] \arrow[d, "1"] & X_{m-1} \arrow[r] \arrow[d, "1"] & \cdots \\
U & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r, "i"] & X_m \arrow[r, "x_m"] & X_{m-1} \arrow[r] & \cdots
\end{tikzcd}$$

We define $U \xrightarrow{g} C$ in $\mathrm{C}(\mathscr{A})$ by setting $g_{m+1} = \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right)$ and $g_k = 1$ for $k \in \mathbf{Z}_{\leq m}$.

$$\begin{tikzcd}
U \arrow[d, "g"'] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & M \arrow[r, "i"] \arrow[d, "\left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right)"] & X_m \arrow[r, "x_m"] \arrow[d, "1"] & X_{m-1} \arrow[r] \arrow[d, "1"] & \cdots \\
C & \cdots \arrow[r] & X_{m+3}\oplus X_{m+2} \arrow[r, "\left(\begin{smallmatrix} x_{m+3} & 0 \\ 1 & -x_{m+2} \end{smallmatrix}\right)"'] & X_{m+2}\oplus X_{m+1} \arrow[r, "\left(\begin{smallmatrix} x_{m+2} & 0 \\ 1 & -p \end{smallmatrix}\right)"'] & X_{m+1}\oplus M \arrow[r, "\left(\begin{smallmatrix} x_{m+1} \\ i \end{smallmatrix}\right)"'] & X_m \arrow[r, "x_m"'] & X_{m-1} \arrow[r] & \cdots
\end{tikzcd}$$

Note that we have $g \cdot f = 1_U$ by construction. Thus it suffices to show that $\underline{1 - f \cdot g} = 0$ in $\mathrm{K}(\mathscr{A})$. Let $h_k = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) : C_k \to C_{k+1}$ for $k \in \mathbf{Z}_{>m}$ and $h_k = 0 \colon C_k \to C_{k+1}$ for $k \in \mathbf{Z}_{\leq m}$. For $k \in \mathbf{Z}_{>m+2}$, we have

$$(1 - f \cdot g)_k = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x_{k+1} & 0 \\ 1 & -x_k \end{smallmatrix}\right) + \left(\begin{smallmatrix} x_k & 0 \\ 1 & -x_{k-1} \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = h_k \cdot c_{k+1} + c_k \cdot h_{k-1} \,.$$

We have

$$(1 - f \cdot g)_{m+2} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x_{m+3} & 0 \\ 1 & -x_{m+2} \end{smallmatrix}\right) + \left(\begin{smallmatrix} x_{m+2} & 0 \\ 1 & -p \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = h_{m+2} \cdot c_{m+3} + c_{m+2} \cdot h_{m+1}$$

and

$$(1 - f \cdot g)_{m+1} = \left(\begin{smallmatrix} 1 & -p \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x_{m+2} & 0 \\ 1 & -p \end{smallmatrix}\right) + \left(\begin{smallmatrix} x_{m+1} \\ i \end{smallmatrix}\right) \cdot 0 = h_{m+1} \cdot c_{m+2} + c_{m+1} \cdot h_m \,.$$

For $k \in \mathbf{Z}_{\leq m}$, we have $(1 - f \cdot g)_k = 0 = h_k \cdot c_{k+1} + c_k \cdot h_{k-1}$. □

1.9.56 Definition. Suppose given an abelian category $\mathscr{A}$. An acyclic complex $X \in \mathrm{Ob}(\mathrm{Ac}(\mathscr{A}))$ is called 2-*acyclic* if there exists $k \in \mathbf{Z}$ such that $X \in \mathrm{Ob}(\mathrm{C}^{\lceil k+1,k-1\rceil}(\mathscr{A}))$.

We define the full subcategory $\mathrm{Ac}^2(\mathscr{A})$ of $\mathrm{K}(\mathscr{A})$ by setting

$$\mathrm{Ob}(\mathrm{Ac}^2(\mathscr{A})) = \{X \in \mathrm{Ob}(\mathrm{Ac}(\mathscr{A}))\colon X \text{ is 2-acyclic}\}.$$

Note that $\mathrm{Ac}^2(\mathscr{A}) \subseteq \mathrm{Ac}^{\mathrm{b}}(\mathscr{A}) \subseteq \mathrm{K}^{\mathrm{b}}(\mathscr{A})$. ◇

1.9.57 Lemma. Suppose given an abelian category $\mathscr{A}$. Suppose given a triangulated category $\mathscr{D}$ and an exact functor $F\colon \mathrm{K}^{\mathrm{b}}(\mathscr{A}) \to \mathscr{D}$ such that $XF \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$ for $X \in \mathrm{Ob}(\mathrm{Ac}^2(\mathscr{A}))$. Then we have $XF \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$ for $X \in \mathrm{Ob}(\mathrm{Ac}^{\mathrm{b}}(\mathscr{A}))$. ◇

Proof. Suppose given $X \in \mathrm{Ob}(\mathrm{Ac}^{\mathrm{b}}(\mathscr{A}))$.
We may choose $m, n \in \mathbf{Z}$ such that $X \in \mathrm{Ob}(\mathrm{C}^{[m,n]}(\mathscr{A}))$. We use induction on $m - n \in \mathbf{Z}$.

If $m - n \in \mathbf{Z}_{\leq 1}$, we have $X \in \mathrm{Ob}(\mathrm{Z}_{\mathrm{K}^{\mathrm{b}}(\mathscr{A})})$. Thus $XF \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$ since F is additive.

If $m - n = 2$, we have $XF \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$ by assumption.

If $m - n \in \mathbf{Z}_{>2}$, we may choose a triangle $T \longrightarrow X \longrightarrow U \longrightarrow T^{[1]}$ in $\mathrm{K}(\mathscr{A})$ such that $T, U \in \mathrm{Ob}(\mathrm{Ac}(\mathscr{A}))$, $T \in \mathrm{Ob}(\mathrm{C}^{[m,m-2]}(\mathscr{A}))$ and $U \in \mathrm{Ob}(\mathrm{C}^{[m-1,n]}(\mathscr{A}))$ by lemma 1.9.55. We have $TF \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$ since T is 2-acyclic and $UF \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$ by induction. We conclude that $XF \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$ since F is exact. □

1.9.58 Lemma. Suppose given an abelian category $\mathscr{A}$. Suppose given a triangulated category $\mathscr{D}$ and a non-degenerate t-structure $\mathscr{T}$ on $\mathscr{D}$. Suppose given an exact functor $F\colon \mathrm{K}(\mathscr{A}) \to \mathscr{D}$ such that

(a) $XF \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$ for $X \in \mathrm{Ob}(\mathrm{Ac}^2(\mathscr{A}))$,

(b) $XF \in \mathrm{Ob}(\mathscr{T}_{[m})$ for $m \in \mathbf{Z}$, $X \in \mathrm{Ob}(\mathrm{C}^{[m}(\mathscr{A}))$ and such that

(c) $XF \in \mathrm{Ob}(\mathscr{T}_{m]})$ for $m \in \mathbf{Z}$, $X \in \mathrm{Ob}(\mathrm{C}^{m]}(\mathscr{A}))$.

Then we have $XF \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$ for all $X \in \mathrm{Ob}(\mathrm{Ac}(\mathscr{A}))$. ◇

Proof. We want prove the following four statements.

(i) For $X \in \mathrm{Ob}(\mathrm{Ac}^{\mathrm{b}}(\mathscr{A}))$, we have $XF \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$.

(ii) For $m \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{Ac}(\mathscr{A})) \cap \mathrm{Ob}(\mathrm{C}^{[m}(\mathscr{A}))$, we have $XF \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$.

(iii) For $m \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{Ac}(\mathscr{A})) \cap \mathrm{Ob}(\mathrm{C}^{m]}(\mathscr{A}))$, we have $XF \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$.

(iv) For $X \in \mathrm{Ob}(\mathrm{Ac}(\mathscr{A}))$, we have $XF \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$.

Ad (i). This follows from lemma 1.9.57 applied to the exact functor $\mathrm{Inc}_{\mathrm{K}^{\mathrm{b}}(\mathscr{A})}^{\mathrm{K}(\mathscr{A})} \cdot F\colon \mathrm{K}^{\mathrm{b}}(\mathscr{A}) \to \mathscr{D}$.

Ad (ii). We abbreviate $\mathscr{H} = \mathscr{T}_{[0,0]}$. Suppose given $m \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{Ac}(\mathscr{A})) \cap \mathrm{Ob}(\mathrm{C}^{[m}(\mathscr{A}))$. It suffices to show that $XF^{[-k]}\mathrm{H}_{\mathscr{T}} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{H}})$ for $k \in \mathbf{Z}$ since $\mathscr{T}$ is non-degenerate.

Suppose given $k \in \mathbf{Z}$. By lemma 1.9.55, we may choose a triangle $T \longrightarrow X \longrightarrow U \longrightarrow T^{[1]}$ in $\mathrm{K}(\mathscr{A})$ such that $T, U \in \mathrm{Ob}(\mathrm{Ac}(\mathscr{A}))$, $T \in \mathrm{Ob}(\mathrm{C}^{[m,k-2]}(\mathscr{A}))$ and $U \in \mathrm{Ob}(\mathrm{C}^{[k-1}(\mathscr{A}))$. We have

$TF^{[-k]}\mathrm{H}_{\mathscr{T}} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{H}})$ by (i) and $UF^{[-k]}\mathrm{H}_{\mathscr{T}} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{H}})$ since $UF \in \mathrm{Ob}(\mathscr{T}_{\lfloor k-1})$ by assumption. We conclude that $XF^{[-k]}\mathrm{H}_{\mathscr{T}} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{H}})$ since $F \cdot \Sigma^{-k} \cdot \mathrm{H}_{\mathscr{T}}$ is homological.

Ad (iii). This is dual to (ii).

Ad (iv). Suppose given $X \in \mathrm{Ob}(\mathrm{Ac}(\mathscr{A}))$. We may choose a triangle $T \longrightarrow X \longrightarrow U \longrightarrow T^{[1]}$ in $\mathrm{K}(\mathscr{A})$ such that $T, U \in \mathrm{Ob}(\mathrm{Ac}(\mathscr{A}))$, $T \in \mathrm{Ob}(\mathrm{C}^{m]}(\mathscr{A}))$ and $U \in \mathrm{Ob}(\mathrm{C}^{[m+1}(\mathscr{A}))$ by lemma 1.9.55. We have $TF \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$ by (iii) and $UF \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$ by (ii).
We conclude that $XF \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$ since F is exact. $\square$

Chapter 2

Strict Frobenius categories

Some aspects of the general theory of triangulated categories are not satisfactory. Examples include the non-functoriality of cones, homotopy limits and homotopy colimits. One way to overcome these problems is to work with enhancements such as Frobenius categories [Hap88], dg-categories [CS17], A_∞-categories [Kel06], derivators [Gro13] or infinity-categories [Lur11]. We choose to work with Frobenius categories. Triangulated categories that have this kind of enhancement are called algebraic. When working with diagrams as in chapter 3, it will be advantageous to have functorial short exact sequences $X \xrightarrow{X\iota} X\mathrm{B} \xrightarrow{X\pi} X\Sigma$ on the level of the Frobenius category. To this end, we develop the theory of strict Frobenius categories for which the functorial Frobenius categories of [Kün07, definition A.5] are predecessors. Strict Frobenius categories will provide the language for all of the constructions in the chapters 3 and 4.

Section 2.1 contains the basic definitions and we deduce elementary properties of strict Frobenius categories: They are in fact Frobenius categories (proposition 2.1.27) and the functors B and Σ are exact (lemmata 2.1.31, 2.1.30).

We define and study connecting morphisms ('deltas') in section 2.2. These are used to show that the stable category of a strict Frobenius category is a strict triangulated category in lemma 2.2.14.

In section 2.3, we show that every algebraic triangulated category is equivalent to the stable category of a strict Frobenius category. Since every algebraic triangulated category is equivalent to the stable category of a Frobenius category by definition, it suffices to show that the stable category of a Frobenius category is equivalent to the stable category of a strict Frobenius category. The idea behind the construction is due to B. Keller [Kel94, section 4.3] and was also described in [Kün07, lemma A.8], [Kra07, proof of theorem 7.5] and [CS17, proposition 3.1]. We briefly discuss that the construction can be extended to functors and transformations in remark 2.3.10. However, the construction is not needed in our main example: The category of complexes with entries in an additive category is already a strict Frobenius category. Here the construction yields a category of double complexes.

2.1 Definitions and elementary properties

2.1.1 Definition. Suppose given an additive category $\mathscr{A}$.

A morphism $X \xrightarrow{i} Y$ in $\mathscr{A}$ is called a *pairable monomorphism* if there exists $Y \xrightarrow{p} Z$ in $\mathscr{A}$ such that (i, p) is a kernel-cokernel-pair.

A morphism $Y \xrightarrow{p} Z$ in $\mathscr{A}$ is called a *pairable epimorphism* if there exists $X \xrightarrow{i} Y$ in $\mathscr{A}$ such that (i, p) is a kernel-cokernel-pair. ◇

2.1.2 Definition. Suppose given an additive category $\mathscr{F}$, an additive functor $\mathrm{B}\colon \mathscr{F} \to \mathscr{F}$, an isomorphism of categories $\Sigma\colon \mathscr{F} \to \mathscr{F}$, transformations $\iota\colon 1_{\mathscr{F}} \to \mathrm{B}$, $\pi\colon \mathrm{B} \to \Sigma$ and an isotransformation $\alpha\colon \Sigma\mathrm{B} \to \mathrm{B}\Sigma$ such that the following four conditions hold.

(SF1) The pair $(X\iota, X\pi)$ is a kernel-cokernel-pair in $\mathscr{F}$ for $X \in \mathrm{Ob}(\mathscr{F})$.

(SF2) Suppose given $X \xrightarrow{f} Y$ in $\mathscr{F}$.

(a) There exists a pushout

$$\begin{tikzcd} X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[d, "f'"] \\ Y \arrow[r, "i"] & Z \end{tikzcd}$$

in $\mathscr{F}$ such that i is a pairable monomorphism.

(b) There exists a pullback

$$\begin{tikzcd} W \arrow[r, "p"] \arrow[d, "f''"'] & X \arrow[d, "f"] \\ Y\Sigma^{-1}\mathrm{B} \arrow[r, "(Y\Sigma^{-1})\pi"] & Y \end{tikzcd}$$

in $\mathscr{F}$ such that p is a pairable epimorphism.

(SF3) Suppose given $X, Y, Z \in \mathrm{Ob}(\mathscr{F})$ and $Y\mathrm{B} \xrightarrow{f} X\Sigma$, $X \xrightarrow{g} Z\mathrm{B}$ in $\mathscr{F}$. Then there exist $Y\mathrm{B} \xrightarrow{f'} X\mathrm{B}$ and $X\mathrm{B} \xrightarrow{g'} Z\mathrm{B}$ in $\mathscr{F}$ such that $f' \cdot X\pi = f$ and $X\iota \cdot g' = g$.

$$\begin{tikzcd} & & Y\mathrm{B} \arrow[dl, "f'"'] \arrow[d, "f"] \\ X \arrow[r, "X\iota"] \arrow[d, "g"'] & X\mathrm{B} \arrow[r, "X\pi"'] \arrow[dl, "g'"] & X\Sigma \\ Z\mathrm{B} & & \end{tikzcd}$$

(SF4) We have $(\Sigma \star \iota) \cdot \alpha = \iota \star \Sigma$ and $\Sigma \star \pi = -\alpha \cdot (\pi \star \Sigma)$.

We call such a tuple $(\mathscr{F}, \mathrm{B}, \Sigma, \iota, \pi, \alpha)$ a *strict Frobenius category.*
We abbreviate $\mathscr{F} = (\mathscr{F}, \mathrm{B}, \Sigma, \iota, \pi, \alpha)$ if unambiguous. ◇

2.1.3 Remark. Every strict Frobenius category is a Frobenius category when equipped with a suitable exact structure, cf. proposition 2.1.27 below. Moreover, the stable category of a strict Frobenius category is a strict triangulated category, cf. lemma 2.2.14 below. $\diamondsuit$

2.1.4 Example. Suppose given an additive category $\mathscr{A}$. $(\mathrm{C}(\mathscr{A}), \mathrm{B}_{\mathrm{C},\mathscr{A}}, \Sigma_{\mathrm{C},\mathscr{A}}, \iota_{\mathrm{C},\mathscr{A}}, \pi_{\mathrm{C},\mathscr{A}}, \alpha_{\mathrm{C},\mathscr{A}})$ is a strict Frobenius category, cf. definitions 1.9.7, 1.9.8, 1.9.9, 1.9.10 and lemma 1.9.11. $\diamondsuit$

Suppose given a strict Frobenius category $\mathscr{F} = (\mathscr{F}, \mathrm{B}, \Sigma, \iota, \pi, \alpha)$ for the remainder of this section.

2.1.5 Remark. Note that $(\mathscr{F}^{\mathrm{op}}, (\Sigma^{-1}\mathrm{B})^{\mathrm{op}}, (\Sigma^{-1})^{\mathrm{op}}, (\Sigma^{-1}\star\pi)^{\mathrm{op}}, (\Sigma^{-1}\star\iota)^{\mathrm{op}}, (\Sigma^{-2}\star-\alpha\star\Sigma^{-1})^{\mathrm{op}})$ is a strict Frobenius category as well. This allows for reasoning by dualisation. $\diamondsuit$

2.1.6 Remark. For $X \in \mathrm{Ob}(\mathscr{F})$, we have $(X\Sigma)\iota \cdot X\alpha = (X\iota)\Sigma$ and $-(X\Sigma)\pi = X\alpha \cdot (X\pi)\Sigma$ by (SF4).

$$\begin{tikzcd}
X\Sigma \arrow[r, "(X\Sigma)\iota"] \arrow[d, "1"'] & X\Sigma\mathrm{B} \arrow[r, "(X\Sigma)\pi"] \arrow[d, "X\alpha"] & X\Sigma^2 \arrow[d, "-1"] \\
X\Sigma \arrow[r, "(X\iota)\Sigma"] & X\mathrm{B}\Sigma \arrow[r, "(X\pi)\Sigma"] & X\Sigma^2
\end{tikzcd}$$

$\diamondsuit$

2.1.7 Lemma. Suppose given a pushout in $\mathscr{F}$ as follows.

$$\begin{tikzcd}
X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[d, "f'"] \\
Y \arrow[r, "i"] & Z
\end{tikzcd}$$

Then i is a pairable monomorphism in $\mathscr{F}$. $\diamondsuit$

Proof. Since pushouts are unique up to isomorphism, i is the composite of a pairable monomorphism and an isomorphism, cf. (SF2). $\square$

2.1.8 Lemma. Suppose given a pullback in $\mathscr{F}$ as follows.

$$\begin{tikzcd}
W \arrow[r, "p"] \arrow[d, "f''"'] & X \arrow[d, "f"] \\
Y\Sigma^{-1}\mathrm{B} \arrow[r, "Y\Sigma^{-1}\pi"] & Y
\end{tikzcd}$$

Then p is a pairable epimorphism in $\mathscr{F}$. $\diamondsuit$

Proof. This is dual to the previous lemma 2.1.7. $\square$

2.1.9 Lemma. Suppose given a commutative diagram in $\mathscr{F}$ as follows.

$$\begin{tikzcd}
X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[r, "X\pi"] \arrow[d, "f'"] & X\Sigma \arrow[d, "1"] \\
Y \arrow[r, "i"] & Z \arrow[r, "p"] & X\Sigma
\end{tikzcd}$$

Suppose that i is a kernel of p. Then the left rectangle $(X\iota, f, f', i)$ is a square and (i, p) is a kernel-cokernel-pair in $\mathscr{F}$. $\diamondsuit$

Proof. Consider the following commutative diagram in $\mathscr{F}$.

$$\begin{tikzcd} X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[d, "f'"] \\ Y \arrow[r, "i"] \arrow[d] & Z \arrow[d, "p"] \\ 0 \arrow[r] & X\Sigma \end{tikzcd}$$

The lower rectangle $(i, 0, p, 0)$ is a pullback since i is a kernel of p. The outer rectangle $(X\iota, 0, f' \cdot p, 0)$ is a pullback since $X\iota$ is a kernel of $f' \cdot p = X\pi$. By the pasting lemma 1.1.1.(b), the upper rectangle $(X\iota, f, f', i)$ is a pullback as well. Thus $\begin{pmatrix} X\iota & -f \end{pmatrix}$ is a kernel of $\begin{pmatrix} f' \\ i \end{pmatrix}$ by lemma 1.2.6.(b). By lemma 1.2.10, the following rectangle is a pullback in $\mathscr{F}$.

$$\begin{tikzcd} X\mathrm{B} \oplus Y \arrow[r, "\left(\begin{smallmatrix} f' \\ i \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] & Z \arrow[d, "p"] \\ X\mathrm{B} \arrow[r, "X\pi"] & X\Sigma \end{tikzcd}$$

Thus $\begin{pmatrix} f' \\ i \end{pmatrix}$ is a pairable epimorphism by lemma 2.1.8. So $\left(\begin{pmatrix} X\iota & -f \end{pmatrix}, \begin{pmatrix} f' \\ i \end{pmatrix}\right)$ is a kernel-cokernel-pair. We conclude that $(X\iota, f, f', i)$ is a square, cf. lemma 1.2.6.(c). Now p is a cokernel of i by lemma 1.2.7.(b). Thus (i, p) is a kernel-cokernel-pair. $\square$

2.1.10 Lemma. Suppose given a commutative diagram in $\mathscr{F}$ as follows.

$$\begin{tikzcd} Y\Sigma^{-1} \arrow[r, "i"] \arrow[d, "1"'] & W \arrow[r, "p"] \arrow[d, "f''"] & X \arrow[d, "f"] \\ Y\Sigma^{-1} \arrow[r, "Y\Sigma^{-1}\iota"] & Y\Sigma^{-1}\mathrm{B} \arrow[r, "Y\Sigma^{-1}\pi"] & Y \end{tikzcd}$$

Suppose that p is a cokernel of i. Then the right rectangle $(p, f'', f, Y\Sigma^{-1}\pi)$ is a square and (i, p) is a kernel-cokernel-pair. $\diamondsuit$

Proof. This is dual to the previous lemma 2.1.9. $\square$

2.1.11 Lemma. Suppose given a pushout in $\mathscr{F}$ as follows.

$$\begin{tikzcd} X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[d, "f'"] \\ Y \arrow[r, "i"] & Z \end{tikzcd}$$

(a) This pushout is a square.

(b) We may choose $Z \xrightarrow{p} X\Sigma$ in $\mathscr{F}$ such that $f' \cdot p = X\pi$ and such that (i, p) is a kernel-cokernel-pair.

(c) Suppose given $Z \xrightarrow{p} X\Sigma$ in $\mathscr{F}$ such that $f' \cdot p = X\pi$ and such that (i, p) is a kernel-cokernel-pair. We may choose $Z \xrightarrow{h} Y\mathrm{B}$ in $\mathscr{F}$ such that $i \cdot h = Y\iota$, $f' \cdot h = f\mathrm{B}$ and such that the rectangle

$$\begin{tikzcd} Z \arrow[r, "p"] \arrow[d, "h"'] & X\Sigma \arrow[d, "f\Sigma"] \\ Y\mathrm{B} \arrow[r, "Y\pi"] & Y\Sigma \end{tikzcd}$$

is a square in $\mathscr{F}$.

$$\begin{tikzcd} X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[r, "X\pi"] \arrow[d, "f'"] & X\Sigma \arrow[d, "1"] \\ Y \arrow[r, "i"] \arrow[d, "1"'] & Z \arrow[r, "p"] \arrow[d, "h"] & X\Sigma \arrow[d, "f\Sigma"] \\ Y \arrow[r, "Y\iota"] & Y\mathrm{B} \arrow[r, "Y\pi"] & Y\Sigma \end{tikzcd}$$

$\diamondsuit$

Proof. Ad (b). The morphism $X\pi$ is a cokernel of $X\iota$ by (SF1). By lemma 1.2.7.(b), we get a cokernel $Z \xrightarrow{p} X\Sigma$ of i such that $f' \cdot p = X\pi$. Since i is a pairable monomorphism by lemma 2.1.7, the pair (i, p) is a kernel-cokernel-pair.

Ad (a). Using (b), lemma 2.1.9 yields that the pushout $(X\iota, f, f', i)$ is a square.

Ad (c). We have $X\iota \cdot f\mathrm{B} = f \cdot Y\iota$ since ι is a transformation. By the pushout property, we get $Z \xrightarrow{h} Y\mathrm{B}$ such that $i \cdot h = Y\iota$ and $f' \cdot h = f\mathrm{B}$.

$$\begin{tikzcd} X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[d, "f'"] \arrow[ddr, bend left, "f\mathrm{B}"] & \\ Y \arrow[r, "i"] \arrow[drr, bend right, "Y\iota"'] & Z \arrow[dr, "h"] & \\ & & Y\mathrm{B} \end{tikzcd}$$

We have $\left(\begin{smallmatrix} f' \\ i \end{smallmatrix}\right) \cdot p \cdot f\Sigma = \left(\begin{smallmatrix} X\pi \\ 0 \end{smallmatrix}\right) \cdot f\Sigma = \left(\begin{smallmatrix} f\mathrm{B} \cdot Y\pi \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} f\mathrm{B} \\ Y\iota \end{smallmatrix}\right) \cdot Y\pi = \left(\begin{smallmatrix} f' \\ i \end{smallmatrix}\right) \cdot h \cdot Y\pi$. Since $\left(\begin{smallmatrix} f' \\ i \end{smallmatrix}\right)$ is an epimorphism by lemma 1.2.6.(a), we have $p \cdot f\Sigma = h \cdot Y\pi$.

Consider the following commutative diagram in $\mathscr{F}$.

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "1"'] & Z \arrow[r, "p"] \arrow[d, "h"'] & X\Sigma \arrow[d, "f\Sigma"] \\ Y \arrow[r, "Y\iota"] & Y\mathrm{B} \arrow[r, "Y\pi"] & Y\Sigma \end{tikzcd}$$

Since p is a cokernel of i, the right rectangle $(p, h, f\Sigma, Y\pi)$ is a square by lemma 2.1.10. □

2.1.12 Lemma. Suppose given a pullback in $\mathscr{F}$ as follows.

$$\begin{tikzcd} W \arrow[r, "p"] \arrow[d, "\tilde{f}"'] & X \arrow[d, "f"] \\ Y\Sigma^{-1}\mathrm{B} \arrow[r, "Y\Sigma^{-1}\pi"] & Y \end{tikzcd}$$

(a) This pullback is a square.

(b) We may choose $Y\Sigma^{-1} \xrightarrow{i} W$ in $\mathscr{F}$ such that $i \cdot \tilde{f} = Y\Sigma^{-1}\iota$ and such that (i, p) is a kernel-cokernel-pair.

(c) Suppose given $Y\Sigma^{-1} \xrightarrow{i} W$ in $\mathscr{F}$ such that $i \cdot \tilde{f} = Y\Sigma^{-1}\iota$ and such that (i, p) is a kernel-cokernel-pair. We may choose $X\Sigma^{-1}\mathrm{B} \xrightarrow{h} W$ in $\mathscr{F}$ such that $h \cdot p = X\Sigma^{-1}\pi$, $h \cdot \tilde{f} = f\Sigma^{-1}\mathrm{B}$ and such that the rectangle

$$\begin{tikzcd} X\Sigma^{-1} \arrow[r, "X\Sigma^{-1}\iota"] \arrow[d, "f\Sigma^{-1}"'] & X\Sigma^{-1}\mathrm{B} \arrow[d, "h"] \\ Y\Sigma^{-1} \arrow[r, "i"] & W \end{tikzcd}$$

is a square in $\mathscr{F}$.

$$\begin{tikzcd} X\Sigma^{-1} \arrow[r, "X\Sigma^{-1}\iota"] \arrow[d, "f\Sigma^{-1}"'] & X\Sigma^{-1}\mathrm{B} \arrow[r, "X\Sigma^{-1}\pi"] \arrow[d, "h"] & X \arrow[d, "1"] \\ Y\Sigma^{-1} \arrow[r, "i"] \arrow[d, "1"'] & W \arrow[r, "p"] \arrow[d, "\tilde{f}"] & X \arrow[d, "f"] \\ Y\Sigma^{-1} \arrow[r, "Y\Sigma^{-1}\iota"] & Y\Sigma^{-1}\mathrm{B} \arrow[r, "Y\Sigma^{-1}\pi"] & Y \end{tikzcd}$$ ◇

Proof. This is dual to the previous lemma 2.1.11. □

2.1.13 Definition. A morphism $Y \xrightarrow{i} Z$ in $\mathscr{F}$ is called a *pure monomorphism* in $\mathscr{F}$ if there exists a pushout in $\mathscr{F}$ as follows.

$$\begin{tikzcd} X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[d, "f'"] \\ Y \arrow[r, "i"] & Z \end{tikzcd}$$

A morphism $W \xrightarrow{q} X$ in $\mathscr{F}$ is called a *pure epimorphism* in $\mathscr{F}$ if there exists a pullback in

$\mathscr{F}$ as follows.

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[d, "\tilde{f}"'] & X \arrow[d, "f"] \\ Y\Sigma^{-1}\mathrm{B} \arrow[r, "Y\Sigma^{-1}\pi"] & Y \end{tikzcd}$$

◇

2.1.14 Remark. Suppose given $X \in \mathrm{Ob}(\mathscr{F})$. The morphism $X\iota$ is a pure monomorphism and the morphism $X\pi$ is a pure epimorphism in $\mathscr{F}$. ◇

Proof. The following diagram is a pushout in $\mathscr{F}$.

$$\begin{tikzcd} X \arrow[r, "X\iota"] \arrow[d, "1"'] & X\mathrm{B} \arrow[d, "1"] \\ X \arrow[r, "X\iota"] & X\mathrm{B} \end{tikzcd}$$

Thus $X\iota$ is a pure monomorphism in $\mathscr{F}$.

The following diagram is a pullback in $\mathscr{F}$.

$$\begin{tikzcd} X\mathrm{B} \arrow[r, "X\pi"] \arrow[d, "1"'] & X\Sigma \arrow[d, "1"] \\ X\mathrm{B} \arrow[r, "X\pi"] & X\Sigma \end{tikzcd}$$

Thus $X\pi$ is a pure epimorphism in $\mathscr{F}$. □

2.1.15 Lemma.

(a) Pure monomorphisms are closed under isomorphisms.
More precisely, given a pure monomorphism $Y \xrightarrow{i} Z$ in $\mathscr{F}$ and a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "g"'] & Z \arrow[d, "h"] \\ W \arrow[r, "j"] & Q \end{tikzcd}$$

in $\mathscr{F}$ with g, h isomorphisms, then j is a pure monomorphism as well.

(b) Pure epimorphisms are closed under isomorphisms.

(c) Every pure monomorphism has a cokernel that is a pure epimorphism.

(d) Every pure epimorphism has a kernel that is a pure monomorphism.

(e) Suppose given a kernel-cokernel-pair (i, p) in $\mathscr{F}$. If i is a pure monomorphism, then p is a pure epimorphism. If p is a pure epimorphism, then i is a pure monomorphism.

◇

Proof. Ad (a). Since i is a pure monomorphism, there is a pushout

$$\begin{tikzcd} X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[d, "f'"] \\ Y \arrow[r, "i"] & Z \end{tikzcd}$$

in $\mathscr{F}$. By the pasting lemma 1.1.1.(a), the diagram

$$\begin{tikzcd} X \arrow[r, "X\iota"] \arrow[d, "f\cdot g"'] & X\mathrm{B} \arrow[d, "f'\cdot h"] \\ W \arrow[r, "j"] & Y \end{tikzcd}$$

is a pushout as well. Thus j is a pure monomorphism.

Ad (b). This is dual to (a).

Ad (c). This follows from lemma 2.1.11.(b,c).

Ad (d). This is dual to (c).

Ad (e). If i is a pure monomorphism, then it has a cokernel that is a pure epimorphism by (c). By (a), p is also a pure epimorphism.

If p is a pure epimorphism, then it has a kernel that is a pure monomorphism by (d). By (b), i is also a pure monomorphism. □

2.1.16 Definition. A kernel-cokernel-pair (i, p) in $\mathscr{F}$ is called a *pure short exact sequence* in $\mathscr{F}$ if i is a pure monomorphism and if p is a pure epimorphism, cf. definition 2.1.13. Let $\mathscr{E}^{\mathrm{p}}_{\mathscr{F}}$ denote the full subcategory of $\mathrm{KCP}(\mathscr{F})$ defined by

$$\mathrm{Ob}(\mathscr{E}^{\mathrm{p}}_{\mathscr{F}}) = \{(i,p) \in \mathrm{Ob}(\mathrm{KCP}(\mathscr{F}))\colon (i,p) \text{ is a pure short exact sequence in } \mathscr{F}\}.$$

◇

2.1.17 Remark. Suppose given $X \in \mathrm{Ob}(\mathscr{F})$. The kernel-cokernel-pair $(X\iota, X\pi)$ is a pure short exact sequence in $\mathscr{F}$. ◇

Proof. This follows from remark 2.1.14 and lemma 2.1.15.(e). □

2.1.18 Remark. $\mathscr{E}^{\mathrm{p}}_{\mathscr{F}}$ is a strictly full subcategory of $\mathrm{KCP}(\mathscr{F})$. ◇

Proof. This follows from lemma 2.1.15.(a,e). □

2.1.19 Lemma. Suppose given $X \in \mathrm{Ob}(\mathscr{F})$. The identity morphism 1_X is a pure monomorphism in $\mathscr{F}$. ◇

Proof. The following rectangle is a pushout since B is additive.

$$\begin{tikzcd} 0_{\mathscr{F}} \arrow[r, "0_{\mathscr{F}}\iota"] \arrow[d, "0"'] & 0_{\mathscr{F}}\mathrm{B} \arrow[d, "0"] \\ X \arrow[r, "1_X"] & X \end{tikzcd}$$

Thus 1_X is a pure monomorphism in $\mathscr{F}$. $\square$

2.1.20 Lemma. Suppose given $X \in \mathrm{Ob}(\mathscr{F})$. The identity morphism 1_X is a pure epimorphism in $\mathscr{F}$. $\diamondsuit$

Proof. This is dual to the previous lemma 2.1.19. $\square$

2.1.21 Lemma. Suppose given a morphism $Y \xrightarrow{g} W$ and pure monomorphism $Y \xrightarrow{i} Z$ in $\mathscr{F}$. There exists a pushout in $\mathscr{F}$ as follows such that j is a pure monomorphism in $\mathscr{F}$.

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "g"'] & Z \arrow[d, "g'"] \\ W \arrow[r, "j"] & Q \end{tikzcd}$$

Moreover, this pushout is a square. $\diamondsuit$

Proof. Since i is a pure monomorphism, we may choose a pushout in $\mathscr{F}$ as follows.

$$\begin{tikzcd} X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[d, "f'"] \\ Y \arrow[r, "i"] & Z \end{tikzcd}$$

By (SF2), we may choose a pushout in $\mathscr{F}$ as follows such that j is a pure monomorphism.

$$\begin{tikzcd} X \arrow[r, "X\iota"] \arrow[d, "f\cdot g"'] & X\mathrm{B} \arrow[d, "u"] \\ W \arrow[r, "j"] & Q \end{tikzcd}$$

We have $X\iota \cdot u = f \cdot g \cdot j$. By the pushout property, we get $Z \xrightarrow{g'} Q$ in $\mathscr{F}$ such that $i \cdot g' = g \cdot j$ and $f' \cdot g' = u$.

$$\begin{tikzcd} X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[d, "f'"] \arrow[ddr, bend left, "u"] & \\ Y \arrow[r, "i"] \arrow[drr, bend right, "g\cdot j"'] & Z \arrow[dr, "g'"] & \\ & & Q \end{tikzcd}$$

Consider the following commutative diagram in $\mathscr{F}$.

$$\begin{tikzcd}
X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[d, "f'"] \\
Y \arrow[r, "i"] \arrow[d, "g"'] & Z \arrow[d, "g'"] \\
W \arrow[r, "j"] & Q
\end{tikzcd}$$

The lower rectangle (i, g, g', j) is a pushout by the pasting lemma 1.1.1.(a). Since j is a pure monomorphism, we may choose $Q \xrightarrow{q} R$ in $\mathscr{F}$ such that (j, q) is a kernel-cokernel-pair.

Consider the following commutative diagram in $\mathscr{F}$.

$$\begin{tikzcd}
Y \arrow[r, "i"] \arrow[d, "g"'] & Z \arrow[d, "g'"] \\
W \arrow[r, "j"] \arrow[d] & Q \arrow[d, "q"] \\
0 \arrow[r] & R
\end{tikzcd}$$

The lower rectangle $(j, 0, q, 0)$ is a pullback since (j, q) is a kernel-cokernel-pair. By lemma 1.2.7.(a), $g' \cdot q$ is a cokernel of i. So $(i, g' \cdot q)$ is a kernel-cokernel-pair since i is a pure monomorphism. Thus the outer rectangle $(i, 0, g' \cdot q, 0)$ is a pullback. By the pasting lemma 1.1.1.(b), the upper rectangle (i, g, g', j) is a pullback as well. $\square$

2.1.22 Lemma. Suppose given a morphism $W \xrightarrow{g} Z$ and pure epimorphism $Y \xrightarrow{p} Z$ in $\mathscr{F}$. There exists a pullback in $\mathscr{F}$ as follows such that q is a pure epimorphism in $\mathscr{F}$.

$$\begin{tikzcd}
P \arrow[r, "q"] \arrow[d, "\tilde{g}"'] & W \arrow[d, "g"] \\
Y \arrow[r, "p"] & Z
\end{tikzcd}$$

Moreover, this pullback is a square. $\diamondsuit$

Proof. This is dual to the previous lemma 2.1.21. $\square$

2.1.23 Lemma. Suppose given $W \in \mathrm{Ob}(\mathscr{F})$, a morphism $Y \xrightarrow{g} W\mathrm{B}$ and pure monomorphism $Y \xrightarrow{i} Z$ in $\mathscr{F}$. There exists $Z \xrightarrow{g'} W\mathrm{B}$ in $\mathscr{F}$ such that $i \cdot g' = g$.

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "g"'] & Z \arrow[dl, "g'"] \\ W\mathrm{B} \end{tikzcd}$$
$\diamondsuit$

Proof. Since i is a pure monomorphism, we may choose a pushout in $\mathscr{F}$ as follows.

$$\begin{tikzcd} X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[d, "f'"] \\ Y \arrow[r, "i"] & Z \end{tikzcd}$$

We get $X\mathrm{B} \xrightarrow{u} W\mathrm{B}$ in $\mathscr{F}$ such that $X\iota \cdot u = f \cdot g$ by (SF3).

$$\begin{tikzcd} X \arrow[r, "X\iota"] \arrow[d, "f \cdot g"'] & X\mathrm{B} \arrow[dl, "u"] \\ W\mathrm{B} \end{tikzcd}$$

By the pushout property, we get $Z \xrightarrow{g'} W\mathrm{B}$ such that $i \cdot g' = g$ and $f' \cdot g' = u$.

$$\begin{tikzcd} X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[d, "f'"] \arrow[ddr, bend left, "u"] \\ Y \arrow[r, "i"] \arrow[drr, bend right, "g"'] & Z \arrow[dr, "g'"] \\ & & W\mathrm{B} \end{tikzcd}$$

$\square$

2.1.24 Lemma. Suppose given $W \in \mathrm{Ob}(\mathscr{F})$, a morphism $W\mathrm{B} \xrightarrow{g} Z$ and pure epimorphism $Y \xrightarrow{p} Z$ in $\mathscr{F}$. There exists $W\mathrm{B} \xrightarrow{g'} Y$ in $\mathscr{F}$ such that $g' \cdot p = g$.

$$\begin{tikzcd} & W\mathrm{B} \arrow[dl, "g'"'] \arrow[d, "g"] \\ Y \arrow[r, "p"'] & Z \end{tikzcd}$$
$\diamondsuit$

Proof. This is dual to the previous lemma 2.1.23. $\square$

2.1.25 Lemma. The composite of two pure monomorphisms is a pure monomorphism. More precisely, suppose given pure monomorphisms $Y \xrightarrow{i} Z$ and $Z \xrightarrow{j} W$ in $\mathscr{F}$. Then the composite $i \cdot j$ is a pure monomorphism in $\mathscr{F}$ as well. $\diamondsuit$

Proof. Since i is a pure monomorphism, we may choose a pushout in $\mathscr{F}$ as follows.

$$\begin{tikzcd} X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[d, "f'"] \\ Y \arrow[r, "i"] & Z \end{tikzcd}$$

By lemma 2.1.11.(b,c), we may choose a pullback

$$\begin{tikzcd} Z \arrow[r, "p"] \arrow[d, "h"'] & X\Sigma \arrow[d, "f\Sigma"] \\ Y\mathrm{B} \arrow[r, "Y\pi"] & Y\Sigma \end{tikzcd}$$

in $\mathscr{F}$ such that (i, p) a kernel-cokernel-pair and such that $i \cdot h = Y\iota$.

By lemma 2.1.23, we may choose $W \xrightarrow{c} Y\mathrm{B}$ in $\mathscr{F}$ such that $j \cdot c = h$.

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[d, "h"'] & W \arrow[dl, "c"] \\ Y\mathrm{B} & \end{tikzcd}$$

By lemma 2.1.21, we may choose a square in $\mathscr{F}$ as follows.

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[d, "p"'] & W \arrow[d, "w"] \\ X\Sigma \arrow[r, "s"] & Q \end{tikzcd}$$

By lemma 1.2.9.(a), $(i \cdot j, w)$ is a kernel-cokernel-pair in $\mathscr{F}$. We have $j \cdot c \cdot Y\pi = h \cdot Y\pi = p \cdot f\Sigma$. By the pushout property, we get $Q \xrightarrow{k} Y\Sigma$ in $\mathscr{F}$ such that $s \cdot k = f\Sigma$ and $w \cdot k = c \cdot Y\pi$.

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[d, "p"'] & W \arrow[d, "w"] \arrow[ddr, bend left, "c\cdot Y\pi"] & \\ X\Sigma \arrow[r, "s"] \arrow[drr, bend right, "f\Sigma"'] & Q \arrow[dr, "k"] & \\ & & Y\Sigma \end{tikzcd}$$

We have the following commutative diagram in $\mathscr{F}$.

$$\begin{tikzcd} Y \arrow[r, "i\cdot j"] \arrow[d, "1"'] & W \arrow[r, "w"] \arrow[d, "c"] & Q \arrow[d, "k"] \\ Y \arrow[r, "Y\iota"] & Y\mathrm{B} \arrow[r, "Y\pi"] & Y\Sigma \end{tikzcd}$$

Moreover, w is a cokernel of $i \cdot j$. Thus the right rectangle $(w, c, k, Y\pi)$ is a pullback by lemma 2.1.10. We conclude that w is a pure epimorphism and, consequently, $i \cdot j$ is a pure monomorphism in $\mathscr{F}$ by lemma 2.1.15.(e). $\square$

2.1.26 Lemma. The composite of two pure epimorphisms is a pure epimorphism.
More precisely, suppose given pure epimorphisms $Y \xrightarrow{p} Z$ and $Z \xrightarrow{q} W$ in $\mathscr{F}$. Then the composite $p \cdot q$ is a pure epimorphism in $\mathscr{F}$ as well. $\diamondsuit$

Proof. This is dual to the previous lemma 2.1.25. $\square$

2.1.27 Proposition. The pair $(\mathscr{F}, \mathscr{E}^{\mathrm{p}}_{\mathscr{F}})$ is a Frobenius category, cf. definitions 1.3.2, 1.3.26, 2.1.2 and 2.1.16. $\diamondsuit$

Proof. The pair $(\mathscr{F}, \mathscr{E}^{\mathrm{p}}_{\mathscr{F}})$ is an exact category by remark 2.1.18 and lemmata 2.1.19, 2.1.20, 2.1.25, 2.1.26, 2.1.21, 2.1.22.

For $X \in \mathrm{Ob}(\mathscr{F})$, we have the pure epimorphism $X\Sigma^{-1}\mathrm{B} \xrightarrow{(X\Sigma^{-1})\pi} X$ and the pure monomorphism $X \xrightarrow{X\iota} X\mathrm{B}$ with $X\Sigma^{-1}\mathrm{B}$, $X\mathrm{B}$ bijective in $\mathscr{F}$ by remark 2.1.14 and lemmata 2.1.23, 2.1.24. $\square$

2.1.28 Remark. Since $\mathscr{F}$ is a Frobenius category by proposition 2.1.27, we have the ideal $\mathfrak{J}_{\mathscr{F}} \subseteq \mathrm{Mor}(\mathscr{F})$, the stable category $\underline{\mathscr{F}} = \mathscr{F}/\mathfrak{P}_{\mathscr{F}}$ and the residue class functor $\mathfrak{P}_{\mathscr{F}} \colon \mathscr{F} \to \underline{\mathscr{F}}$. For $f \in \mathrm{Mor}(\mathscr{F})$, we usually abbreviate $\underline{f} = f\mathfrak{P}_{\mathscr{F}}$. Cf. definition 1.3.27 and lemma 1.3.28. $\diamondsuit$

2.1.29 Lemma. Suppose given an additive category $\mathscr{A}$ and an additive functor $F \colon \mathscr{F} \to \mathscr{A}$ such that $X\mathrm{B}F \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}})$ for $X \in \mathrm{Ob}(\mathscr{F})$. Then we have $fF = 0$ for $f \in \mathfrak{J}_{\mathscr{F}}$. Consequently, there exists a unique additive functor $\underline{F} \colon \underline{\mathscr{F}} \to \mathscr{A}$ such that $\mathfrak{P}_{\mathscr{F}} \cdot \underline{F} = F$, cf. lemma 1.2.15. $\diamondsuit$

Proof. Suppose given $X \xrightarrow{f} Y$ in $\mathscr{F}$ such that $f \in \mathfrak{J}_{\mathscr{F}}$. By lemma 1.3.28, we may choose $X\mathrm{B} \xrightarrow{g} Y$ in $\mathscr{F}$ such that $f = X\iota \cdot g$. We have $fF = X\iota F \cdot gF = 0$ since $X\mathrm{B}F \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}})$. $\square$

2.1.30 Lemma/Definition. The functor $\Sigma \colon \mathscr{F} \to \mathscr{F}$ is exact.
For $X \in \mathrm{Ob}(\mathscr{F})$, we have $X\mathrm{B}\Sigma\mathfrak{P}_{\mathscr{F}} , X\mathrm{B}\Sigma^{-1}\mathfrak{P}_{\mathscr{F}} \in \mathrm{Ob}(\mathrm{Z}_{\underline{\mathscr{F}}})$.
Let $\underline{\Sigma} \colon \underline{\mathscr{F}} \to \underline{\mathscr{F}}$ denote the unique additive functor such that $\mathfrak{P}_{\mathscr{F}} \cdot \underline{\Sigma} = \Sigma \cdot \mathfrak{P}_{\mathscr{F}}$, cf. lemma 2.1.29. Note that $\underline{\Sigma}$ is an isomorphism of categories, cf. lemma 1.6.12.
For $X \in \mathrm{Ob}(\mathscr{F})$, we often write $X^{[k]} = X\Sigma^k = X\underline{\Sigma}^k$ for $k \in \mathbf{Z}$. For $f \in \mathrm{Mor}(\mathscr{F})$, we often write $f^{[k]} = f\Sigma^k$ and $\underline{f}^{[k]} = \underline{f^{[k]}} = \underline{f}\,\underline{\Sigma}^k$ for $k \in \mathbf{Z}$. $\diamondsuit$

Proof. Suppose given a pure short exact sequence $Y \xrightarrow{i} Z \xrightarrow{p} W$ in $\mathscr{F}$. Then $(i\Sigma, p\Sigma)$ is a kernel-cokernel-pair in $\mathscr{F}$ since Σ is an isomorphism of categories. We may choose a pushout in $\mathscr{F}$ as follows since i is a pure monomorphism in $\mathscr{F}$.

$$\begin{tikzcd} X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[d, "f'"] \\ Y \arrow[r, "i"] & Z \end{tikzcd}$$

We have the the following commutative diagram in $\mathscr{F}$.

$$\begin{tikzcd} X\Sigma \arrow[r, "(X\Sigma)\iota"] \arrow[d, "1"'] & X\Sigma\mathrm{B} \arrow[d, "X\alpha"] \\ X\Sigma \arrow[r, "(X\iota)\Sigma"] \arrow[d, "f\Sigma"'] & X\mathrm{B}\Sigma \arrow[d, "f'\Sigma"] \\ Y\Sigma \arrow[r, "i\Sigma"] & Z\Sigma \end{tikzcd}$$

By the pasting lemma 1.1.1.(a), the outer rectangle

$$\begin{tikzcd} X\Sigma \arrow[r, "(X\Sigma)\iota"] \arrow[d, "f\Sigma"'] & X\Sigma\mathrm{B} \arrow[d, "X\alpha\cdot f'\Sigma"] \\ Y\Sigma \arrow[r, "i\Sigma"] & Z\Sigma \end{tikzcd}$$

is a pushout. Thus $i\Sigma$ is a pure monomorphism and, consequently, $(i\Sigma, p\Sigma)$ is a pure short exact sequence in $\mathscr{F}$.

Suppose given $X \in \mathrm{Ob}(\mathscr{F})$.

We have the isomorphisms $X\Sigma\mathrm{B} \xrightarrow{X\alpha} X\mathrm{B}\Sigma$ and $X\mathrm{B}\Sigma^{-1} \xrightarrow{X\Sigma^{-1}\alpha\Sigma^{-1}} X\Sigma^{-1}\mathrm{B}$ in $\mathscr{F}$.

Thus $X\mathrm{B}\Sigma\mathfrak{P}_{\mathscr{F}}, X\mathrm{B}\Sigma^{-1}\mathfrak{P}_{\mathscr{F}} \in \mathrm{Ob}(\mathrm{Z}_{\underline{\mathscr{F}}})$. □

2.1.31 Lemma. The functor $\mathrm{B}\colon \mathscr{F} \to \mathscr{F}$ is exact. ◇

Proof. This follows from lemmata 2.1.30 and 1.3.17. □

2.1.32 Lemma/Definition. Suppose given a full subcategory $\mathscr{G} \subseteq \mathscr{F}$. We say that $\mathscr{G}$ is a *strict Frobenius subcategory* of $\mathscr{F}$ if it extension-closed in $\mathscr{F}$ and if $X^{[1]}, X^{[-1]} \in \mathrm{Ob}(\mathscr{G})$ for $X \in \mathrm{Ob}(\mathscr{G})$. In this case, $(\mathscr{G}, \mathrm{B}|_{\mathscr{G}}^{\mathscr{G}}, \Sigma|_{\mathscr{G}}^{\mathscr{G}}, \iota|_{\mathscr{G}}^{\mathscr{G}}, \pi|_{\mathscr{G}}^{\mathscr{G}}, \alpha|_{\mathscr{G}}^{\mathscr{G}})$ is a strict Frobenius category as well and $\underline{\mathscr{G}} \subseteq \underline{\mathscr{F}}$ is a full additive subcategory, cf. definition 1.3.23 and remark 1.2.14. ◇

Proof. Note that $\mathscr{G}$ is a strictly full additive subcategory of $\mathscr{G}$, $\mathrm{B}|_{\mathscr{G}}^{\mathscr{G}}$ is an additive functor, $\Sigma|_{\mathscr{G}}^{\mathscr{G}}$ is an isomorphism of categories and $\alpha|_{\mathscr{G}}^{\mathscr{G}}$ is an isotransformation, cf. lemmata 1.3.25, 1.6.10 and remark 1.2.5.(b).

Ad (SF1),(SF3). This follows directly from (SF1),(SF3) in $\mathscr{F}$.

Ad (SF2). This follows from (RE3) and (RE4) of definition 1.3.21, cf. lemma 1.3.25.

Ad (SF4). We have $(\Sigma|_{\mathscr{G}}^{\mathscr{G}} \star \iota|_{\mathscr{G}}^{\mathscr{G}}) \cdot \alpha|_{\mathscr{G}}^{\mathscr{G}} = \iota|_{\mathscr{G}}^{\mathscr{G}} \star \Sigma|_{\mathscr{G}}^{\mathscr{G}}$ since

$$
\begin{aligned}
((\Sigma|_{\mathscr{G}}^{\mathscr{G}} \star \iota|_{\mathscr{G}}^{\mathscr{G}}) \cdot \alpha|_{\mathscr{G}}^{\mathscr{G}}) \star \mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}} &= ((\Sigma|_{\mathscr{G}}^{\mathscr{G}} \star \iota|_{\mathscr{G}}^{\mathscr{G}}) \cdot \alpha|_{\mathscr{G}}^{\mathscr{G}}) \star (1_{\mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}}} \cdot 1_{\mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}}}) \\
&= ((\Sigma|_{\mathscr{G}}^{\mathscr{G}} \star \iota|_{\mathscr{G}}^{\mathscr{G}} \star 1_{\mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}}}) \cdot (\alpha|_{\mathscr{G}}^{\mathscr{G}} \star 1_{\mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}}}) \\
&= (1_{\mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}}} \star \Sigma \star \iota) \cdot (1_{\mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}}} \star \alpha) \\
&= (1_{\mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}}} \cdot 1_{\mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}}}) \star ((\Sigma \star \iota) \cdot \alpha) \\
&= 1_{\mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}}} \star (\iota \star \Sigma) \\
&= 1_{\mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}}} \star \iota \star \Sigma \\
&= \iota|_{\mathscr{G}}^{\mathscr{G}} \star \Sigma|_{\mathscr{G}}^{\mathscr{G}} \star 1_{\mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}}} \\
&= (\iota|_{\mathscr{G}}^{\mathscr{G}} \star \Sigma|_{\mathscr{G}}^{\mathscr{G}}) \star \mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}} \,.
\end{aligned}
$$

We have $\Sigma|_{\mathscr{G}}^{\mathscr{G}} \star \pi|_{\mathscr{G}}^{\mathscr{G}} = -\alpha|_{\mathscr{G}}^{\mathscr{G}} \cdot (\pi|_{\mathscr{G}}^{\mathscr{G}} \star \Sigma|_{\mathscr{G}}^{\mathscr{G}})$ since

$$
\begin{aligned}
(-\alpha|_{\mathscr{G}}^{\mathscr{G}} \cdot (\pi|_{\mathscr{G}}^{\mathscr{G}} \star \Sigma|_{\mathscr{G}}^{\mathscr{G}})) \star \mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}} &= (-\alpha|_{\mathscr{G}}^{\mathscr{G}} \cdot (\pi|_{\mathscr{G}}^{\mathscr{G}} \star \Sigma|_{\mathscr{G}}^{\mathscr{G}})) \star (1_{\mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}}} \cdot 1_{\mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}}}) \\
&= ((-\alpha|_{\mathscr{G}}^{\mathscr{G}} \star 1_{\mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}}}) \cdot (\pi|_{\mathscr{G}}^{\mathscr{G}} \star \Sigma|_{\mathscr{G}}^{\mathscr{G}} \star 1_{\mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}}}) \\
&= ((1_{\mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}}} \star -\alpha) \cdot (1_{\mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}}} \star \pi \star \Sigma) \\
&= 1_{\mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}}} \star (-\alpha \cdot (\pi \star \Sigma)) \\
&= 1_{\mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}}} \star \Sigma \star \pi \\
&= (\Sigma|_{\mathscr{G}}^{\mathscr{G}} \star \pi|_{\mathscr{G}}^{\mathscr{G}}) \star \mathrm{Inc}_{\mathscr{G}}^{\mathscr{F}} \,.
\end{aligned}
$$

□

2.1.33 Example. Suppose given an additive category $\mathscr{A}$.
The full subcategory $\mathrm{C}^{\mathrm{b}}(\mathscr{A}) \subseteq \mathrm{C}(\mathscr{A})$ is a strict Frobenius subcategory of $\mathrm{C}(\mathscr{A})$, cf. example 2.1.4 and definition 1.9.15. Note that we have $\underline{\mathrm{C}^{\mathrm{b}}(\mathscr{A})} = \mathrm{K}^{\mathrm{b}}(\mathscr{A}) \subseteq \mathrm{K}(\mathscr{A})$. ◇

2.1.34 Lemma. Suppose given a strictly full subcategory $\mathscr{E} \subseteq \mathrm{KCP}(\mathscr{F})$ such that $(\mathscr{F}, \mathscr{E})$ is an exact category. Suppose that $(X\iota, X\pi) \in \mathrm{Ob}(\mathscr{E})$ and that $X\mathrm{B}$ is projective in $(\mathscr{F}, \mathscr{E})$ for $X \in \mathrm{Ob}(\mathscr{F})$. Then we have $\mathscr{E} = \mathscr{E}_{\mathscr{F}}^{\mathrm{p}}$. ◇

Proof. Suppose given $\big(Y \xrightarrow{i} Z \xrightarrow{p} W\big) \in \mathrm{Ob}(\mathscr{E}_{\mathscr{F}}^{\mathrm{p}})$. We may choose a pushout in $\mathscr{F}$ as follows, cf. definition 2.1.13.

$$
\begin{tikzcd}
X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[d, "f'"] \\
Y \arrow[r, "i"] & Z
\end{tikzcd}
$$

We conclude that $(i, p) \in \mathrm{Ob}(\mathscr{E})$ since $(X\iota, X\pi) \in \mathrm{Ob}(\mathscr{E})$.

Conversely, suppose given $\big(Y \xrightarrow{i} Z \xrightarrow{p} W\big) \in \mathrm{Ob}(\mathscr{E})$. Since $W^{[-1]}\mathrm{B}$ is projective in $(\mathscr{F}, \mathscr{E})$, we may choose $W^{[-1]}\mathrm{B} \xrightarrow{g} Z$ in $\mathscr{F}$ such that $g \cdot p = W^{[-1]}\pi$. Note that we have $W^{[-1]}\iota \cdot g \cdot p = W^{[-1]}\iota \cdot W^{[-1]}\pi = 0$. Since i is a kernel of p, we may choose $W^{[-1]} \xrightarrow{f} Y$ in $\mathscr{F}$

such that $f \cdot i = W^{[-1]}\iota \cdot g$.

$$\begin{tikzcd}
W^{[-1]} \arrow[r, "W^{[-1]}\iota"] \arrow[d, "f"'] & W^{[-1]}\mathrm{B} \arrow[r, "W^{[-1]}\pi"] \arrow[d, "g"] & W \arrow[d, "1"] \\
Y \arrow[r, "i"] & Z \arrow[r, "p"] & W
\end{tikzcd}$$

By [Büh10, proposition 2.12], the left rectangle $(W^{[-1]}\iota, f, g, i)$ is a pushout. We conclude that $(i, p) \in \mathrm{Ob}(\mathscr{E}^{\mathrm{p}}_{\mathscr{F}})$. $\square$

2.1.35 Lemma. Suppose given pure short exact sequences $X \xrightarrow{i} Y \xrightarrow{p} Z$, $X' \xrightarrow{i'} Y' \xrightarrow{p'} Z'$ and the following commutative diagram in $\mathscr{F}$.

$$\begin{tikzcd}
X \arrow[r, "i", "\bullet" marking] \arrow[d, "f"'] & Y \arrow[r, "p", "\shortmid" marking] \arrow[d, "g"] & Z \arrow[d, "h"] \\
X' \arrow[r, "i'", "\bullet" marking] & Y' \arrow[r, "p'", "\shortmid" marking] & Z'
\end{tikzcd}$$

Suppose given $X\mathrm{B} \xrightarrow{\tilde{f}} X'^{[-1]}\mathrm{B}$ and $Z\mathrm{B} \xrightarrow{\tilde{h}} Z'^{[-1]}\mathrm{B}$ in $\mathscr{F}$ such that $f = X\iota \cdot \tilde{f} \cdot X'^{[-1]}\pi$ and $h = Z\iota \cdot \tilde{h} \cdot Z'^{[-1]}\pi$. Suppose that $\underline{\mathscr{F}}(Z, X') = 0$. Then there exists $Y\mathrm{B} \xrightarrow{\tilde{g}} Y'^{[-1]}\mathrm{B}$ in $\mathscr{F}$ such that $g = Y\iota \cdot \tilde{g} \cdot Y'^{[-1]}\pi$, $\tilde{f} \cdot i'^{[-1]}\mathrm{B} = i\mathrm{B} \cdot \tilde{g}$ and $\tilde{g} \cdot p'^{[-1]}\mathrm{B} = p\mathrm{B} \cdot \tilde{h}$.

$$\begin{tikzcd}
X \arrow[r, "i", "\bullet" marking] \arrow[d, "X\iota"', "\bullet" marking] & Y \arrow[r, "p", "\shortmid" marking] \arrow[d, "Y\iota", "\bullet" marking] & Z \arrow[d, "Z\iota", "\bullet" marking] \\
X\mathrm{B} \arrow[r, "i\mathrm{B}", "\bullet" marking] \arrow[d, "\tilde{f}"'] & Y\mathrm{B} \arrow[r, "p\mathrm{B}", "\shortmid" marking] \arrow[d, "\tilde{g}"] & Z\mathrm{B} \arrow[d, "\tilde{h}"] \\
X'^{[-1]}\mathrm{B} \arrow[r, "i'^{[-1]}\mathrm{B}", "\bullet" marking] \arrow[d, "X'^{[-1]}\pi"', "\shortmid" marking] & Y'^{[-1]}\mathrm{B} \arrow[r, "p'^{[-1]}\mathrm{B}", "\shortmid" marking] \arrow[d, "Y'^{[-1]}\pi", "\shortmid" marking] & Z'^{[-1]}\mathrm{B} \arrow[d, "Z'^{[-1]}\pi", "\shortmid" marking] \\
X' \arrow[r, "i'", "\bullet" marking] & Y' \arrow[r, "p'", "\shortmid" marking] & Z'
\end{tikzcd}$$

$\diamondsuit$

Proof. We may choose $Y\mathrm{B} \xrightarrow{s} X\mathrm{B}$ such that $i\mathrm{B} \cdot s = 1$ and $Z'^{[-1]}\mathrm{B} \xrightarrow{v} Y'^{[-1]}\mathrm{B}$ such that $v \cdot p'^{[-1]}\mathrm{B} = 1$ by lemma 1.3.11.

Let $a = Y\iota \cdot s \cdot \tilde{f} \cdot i'^{[-1]}\mathrm{B} \cdot Y'^{[-1]}\pi \colon Y \to Y'$ and $b = Y\iota \cdot p\mathrm{B} \cdot \tilde{h} \cdot v \cdot Y'^{[-1]}\pi \colon Y \to Y'$.

We have

$$\begin{aligned}
i \cdot (g - a - b) &= f \cdot i' - X\iota \cdot i\mathrm{B} \cdot s \cdot \tilde{f} \cdot i'^{[-1]}\mathrm{B} \cdot Y'^{[-1]}\pi - i \cdot p \cdot Z\iota \cdot \tilde{h} \cdot v \cdot Y'^{[-1]}\pi \\
&= f \cdot i' - X\iota \cdot \tilde{f} \cdot X'^{[-1]}\pi \cdot i' = 0.
\end{aligned}$$

So there exists a unique morphism $Z \xrightarrow{q} Y'$ in $\mathscr{F}$ such that $p \cdot q = g - a - b$.

We have $q \cdot p' = 0$ since

$$\begin{aligned} p \cdot q \cdot p' &= g \cdot p' - a \cdot p' - b \cdot p' \\ &= p \cdot h - Y\iota \cdot s \cdot \tilde{f} \cdot X'^{[-1]}\pi \cdot i' \cdot p' - Y\iota \cdot p\mathrm{B} \cdot \tilde{h} \cdot v \cdot p'^{[-1]}\mathrm{B} \cdot Z'^{[-1]}\pi \\ &= p \cdot h - p \cdot Z\iota \cdot \tilde{h} \cdot Z'^{[-1]}\pi = 0 \end{aligned}$$

and since p is a pure epimorphism (In particular, p is an epimorphism, cf. definition 1.3.2.). So there exists a unique morphism $Z \xrightarrow{r} X'$ in $\mathscr{F}$ such that $q = r \cdot i'$. Since ${}_{\underline{\mathscr{F}}}(Z, X') = 0$, we may choose $Z\mathrm{B} \xrightarrow{d} X'^{[-1]}\mathrm{B}$ in $\mathscr{F}$ such that $r = Z\iota \cdot d \cdot X'^{[-1]}\pi$.
Let $\tilde{g} = s \cdot \tilde{f} \cdot i'^{[-1]}\mathrm{B} + p\mathrm{B} \cdot \tilde{h} \cdot v + p\mathrm{B} \cdot d \cdot i'^{[-1]}\mathrm{B}$.
We have $i\mathrm{B} \cdot \tilde{g} = \tilde{f} \cdot i'^{[-1]}\mathrm{B}$, $\tilde{g} \cdot p'^{[-1]}\mathrm{B} = p\mathrm{B} \cdot \tilde{h}$ and

$$\begin{aligned} Y\iota \cdot \tilde{g} \cdot Y'^{[-1]}\pi &= a + b + Y\iota \cdot p\mathrm{B} \cdot d \cdot i^{[-1]}\mathrm{B} \cdot Y'^{[-1]}\pi \\ &= a + b + p \cdot Z\iota \cdot d \cdot X'^{[-1]}\pi \cdot i' \\ &= a + b + p \cdot r \cdot i' = a + b + p \cdot q \\ &= a + b + g - a - b = g. \end{aligned}$$

□

2.1.36 Corollary. Suppose given pure short exact sequences $X \xrightarrow{i} Y \xrightarrow{p} Z$, $X' \xrightarrow{i'} Y' \xrightarrow{p'} Z'$ and the following commutative diagram in $\mathscr{F}$.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[r, "p"] \arrow[d, "g"] & Z \arrow[d, "h"] \\ X' \arrow[r, "i'"] & Y' \arrow[r, "p'"] & Z' \end{tikzcd}$$

Suppose given $X\mathrm{B} \xrightarrow{\hat{f}} X'$ and $Z\mathrm{B} \xrightarrow{\hat{h}} Z'$ in $\mathscr{F}$ such that $f = X\iota \cdot \hat{f}$ and $h = Z\iota \cdot \hat{h}$. Suppose that ${}_{\underline{\mathscr{F}}}(Z, X') = 0$. Then there exists $Y\mathrm{B} \xrightarrow{\hat{g}} Y'$ in $\mathscr{F}$ such that $g = Y\iota \cdot \hat{g}$, $\hat{f} \cdot i' = i\mathrm{B} \cdot \hat{g}$ and $\hat{g} \cdot p' = p\mathrm{B} \cdot \hat{h}$.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "X\iota"'] & Y \arrow[r, "p"] \arrow[d, "Y\iota"] & Z \arrow[d, "Z\iota"] \\ X\mathrm{B} \arrow[r, "i\mathrm{B}"] \arrow[d, "\hat{f}"'] & Y\mathrm{B} \arrow[r, "p\mathrm{B}"] \arrow[d, "\hat{g}"] & Z\mathrm{B} \arrow[d, "\hat{h}"] \\ X' \arrow[r, "i'"] & Y' \arrow[r, "p'"] & Z' \end{tikzcd}$$

◇

Proof. We may choose $X\mathrm{B} \xrightarrow{\tilde{f}} X'^{[-1]}\mathrm{B}$ and $Z\mathrm{B} \xrightarrow{\tilde{h}} Z'^{[-1]}\mathrm{B}$ in $\mathscr{F}$ such that $\hat{f} = \tilde{f} \cdot X'^{[-1]}\pi$ and $\hat{h} = \tilde{h} \cdot Z'^{[-1]}\pi$. By the previous lemma 2.1.35, we may choose $Y\mathrm{B} \xrightarrow{\tilde{g}} Y'^{[-1]}\mathrm{B}$ in $\mathscr{F}$ such that $g = Y\iota \cdot \tilde{g} \cdot Y'^{[-1]}\pi$, $\tilde{f} \cdot i'^{[-1]}\mathrm{B} = i\mathrm{B} \cdot \tilde{g}$ and $\tilde{g} \cdot p'^{[-1]}\mathrm{B} = p\mathrm{B} \cdot \tilde{h}$.

Let $\hat{g} = \tilde{g} \cdot Y'^{[-1]}\pi$. We have $Y\iota \cdot \hat{g} = Y\iota \cdot \tilde{g} \cdot Y'^{[-1]}\pi = g$,

$$\hat{f} \cdot i' = \tilde{f} \cdot X'^{[-1]}\pi \cdot i' = \tilde{f} \cdot i'^{[-1]}\mathrm{B} \cdot Y'^{[-1]}\pi = i\mathrm{B} \cdot \tilde{g} \cdot Y'^{[-1]}\pi = i\mathrm{B} \cdot \hat{g} \quad \text{and}$$
$$\hat{g} \cdot p' = \tilde{g} \cdot Y'^{[-1]}\pi \cdot p' = \tilde{g} \cdot p'^{[-1]}\mathrm{B} \cdot Z'^{[-1]}\pi = p\mathrm{B} \cdot \tilde{h} \cdot Z'^{[-1]}\pi = p\mathrm{B} \cdot \hat{h}. \qquad \square$$

2.1.37 Lemma. Suppose given a pure short exact sequence $X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$ and a pure monomorphism $V \overset{j}{\rightarrowtail} Z$ in $\mathscr{F}$. Then we may choose a pullback

$$\begin{tikzcd}
U \arrow[r, "q"] \arrow[d, "k"'] & V \arrow[d, "j", "\bullet" marking] \\
Y \arrow[r, "p", "\shortmid" marking] & Z
\end{tikzcd}$$

and a morphism $X \xrightarrow{\ell} U$ in $\mathscr{F}$ such that (ℓ, q) is a pure short exact sequence, $\ell \cdot k = i$ and such that k is a pure monomorphism in $\mathscr{F}$. Moreover, the pullback is a square.

Suppose given a cokernel $Z \overset{r}{\twoheadrightarrow} C$ of j in $\mathscr{F}$. Then $(k, p \cdot r)$ is a pure short exact sequence in $\mathscr{F}$. $\diamondsuit$

Proof. We may choose a pullback as follows since p is a pure epimorphism.

$$\begin{tikzcd}
U \arrow[r, "q"] \arrow[d, "k"'] & V \arrow[d, "j", "\bullet" marking] \\
Y \arrow[r, "p", "\shortmid" marking] & Z
\end{tikzcd}$$

Moreover, this pullback is a square and q is a pure epimorphism since p is a pure epimorphism. By lemma 1.2.8.(b), we may choose $X \xrightarrow{\ell} U$ in $\mathscr{F}$ such that $\ell \cdot k = i$ and such that ℓ is a kernel of q. Thus (ℓ, q) is a pure short exact sequence.

Suppose given a cokernel $Z \overset{r}{\twoheadrightarrow} C$ of j. So r is a pure epimorphism and, consequently, $p \cdot r$ is a pure epimorphism as well. By lemma 1.2.9.(b), $(k, p \cdot r)$ is a pure short exact sequence. In particular, k is a pure monomorphism in $\mathscr{F}$. $\square$

2.1.38 Lemma. Suppose given a pure short exact sequence $X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$ and pure epimorphism $X \overset{q}{\twoheadrightarrow} U$ in $\mathscr{F}$. Then we may choose a pushout

$$\begin{tikzcd}
X \arrow[r, "i", "\bullet" marking] \arrow[d, "q"', "\shortmid" marking] & Y \arrow[d, "k"] \\
U \arrow[r, "j"] & V
\end{tikzcd}$$

and a morphism $V \xrightarrow{r} W$ in $\mathscr{F}$ such that (j, r) is a pure short exact sequence, $k \cdot r = p$ and such that k is a pure epimorphism in $\mathscr{F}$. Moreover, the pushout is a square.

Suppose given a kernel $T \overset{h}{\rightarrowtail} X$ of q in $\mathscr{F}$. Then $(h \cdot i, k)$ is a pure short exact sequence in $\mathscr{F}$. $\diamondsuit$

Proof. This is dual to the previous lemma 2.1.37. $\square$

2.1.39 Lemma. Suppose given a pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathscr{F}$. If two out of the three objects X, Y, Z are bijective in $\mathscr{F}$, then the remaining third one is bijective in $\mathscr{F}$ as well. $\diamondsuit$

Proof. This follows from lemma 1.3.12 and remark 1.3.29. $\square$

2.1.40 Lemma. Suppose given a set I and $X_k \in \mathrm{Ob}(\mathscr{F})$ for $k \in I$. Suppose given a coproduct $(C, (c_k)_{k\in I})$ for $(X_k)_{k\in I}$ in $\mathscr{F}$ and a coproduct $(D, (d_k)_{k\in I})$ for $(X_k\mathrm{B})_{k\in I}$ in $\mathscr{F}$. Then $(C, (\underline{c_k})_{k\in I})$ is a coproduct for $(X_k)_{k\in I}$ in $\underline{\mathscr{F}}$. $\diamondsuit$

Proof. Since $(C, (c_k)_{k\in I})$ is a coproduct for $(X_k)_{k\in I}$ in $\mathscr{F}$, we may choose $C \xrightarrow{i} D$ in $\mathscr{F}$ such that $c_k \cdot i = X_k\iota \cdot d_k$ for $k \in I$.

Suppose given morphisms $X_k \xrightarrow{f_k} T$ in $\mathscr{F}$ for $k \in I$. Since $(C, (c_k)_{k\in I})$ is a coproduct for $(X_k)_{k\in I}$ in $\mathscr{F}$, we may choose $C \xrightarrow{f} T$ in $\mathscr{F}$ such that $c_k \cdot f = f_k$ in $\mathscr{F}$ for $k \in I$. In particular, we have $\underline{c_k} \cdot \underline{f} = \underline{c_k \cdot f} = \underline{f_k}$ in $\underline{\mathscr{F}}$ for $k \in I$. Suppose given $C \xrightarrow{g} T$ in $\mathscr{F}$ such that $\underline{c_k} \cdot \underline{g} = \underline{f_k}$ for $k \in I$. So $\underline{c_k \cdot (f-g)} = \underline{c_k} \cdot \underline{f} - \underline{c_k} \cdot \underline{g} = \underline{f_k} - \underline{f_k} = 0$ in $\underline{\mathscr{F}}$. Thus we may choose $X_k\mathrm{B} \xrightarrow{r_k} T$ in $\mathscr{F}$ such that $X_k\iota \cdot r_k = c_k \cdot (f-g)$ for $k \in I$. Since $(D, (d_k)_{k\in I})$ is a coproduct for $(X_k\mathrm{B})_{k\in I}$ in $\mathscr{F}$, we may may choose $D \xrightarrow{r} T$ in $\mathscr{F}$ such that $d_k \cdot r = r_k$ for $k \in I$. We have $f - g = i \cdot r$ since $c_k \cdot i \cdot r = X_k\iota \cdot d_k \cdot r = X_k\iota \cdot r_k = c_k \cdot (f-g)$ for $k \in I$. We conclude that $\underline{f} = \underline{g}$ in $\underline{\mathscr{F}}$, cf. remark 1.3.7. $\square$

2.1.41 Lemma. Suppose given a set I and $X_k \in \mathrm{Ob}(\mathscr{F})$ for $k \in I$. Suppose given a product $(P, (p_k)_{k\in I})$ for $(X_k)_{k\in I}$ in $\mathscr{F}$ and a product $(Q, (q_k)_{k\in I})$ for $(X_k^{[-1]}\mathrm{B})_{k\in I}$ in $\mathscr{F}$. Then $(P, (\underline{p_k})_{k\in I})$ is a product for $(X_k)_{k\in I}$ in $\underline{\mathscr{F}}$. $\diamondsuit$

Proof. This is dual to the previous lemma 2.1.41. $\square$

2.2 Triangulated structure

Suppose given a strict Frobenius category $\mathscr{F} = (\mathscr{F}, \mathrm{B}, \Sigma, \iota, \pi, \alpha)$.

2.2.1 Definition. For each pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathscr{F}$, we choose morphisms $Y \xrightarrow{\gamma_{(i,p)}} X\mathrm{B}$ and $Z \xrightarrow{\delta_{(i,p)}} X^{[1]}$ in $\mathscr{F}$ as follows.

Since $X\mathrm{B}$ is injective and i is a pure monomorphism in $\mathscr{F}$, we may choose $Y \xrightarrow{\gamma_{(i,p)}} X\mathrm{B}$ such that $i \cdot \gamma_{(i,p)} = X\iota$. We have $i \cdot \gamma_{(i,p)} \cdot X\pi = X\iota \cdot X\pi = 0$. Since p is a cokernel of i, there exists a unique morphism $Z \xrightarrow{\delta_{(i,p)}} X^{[1]}$ in $\mathscr{F}$ such that $p \cdot \delta_{(i,p)} = \gamma_{(i,p)} \cdot X\pi$.

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "1"'] & Y \arrow[r, "p"] \arrow[d, "\gamma_{(i,p)}"] & Z \arrow[d, "\delta_{(i,p)}"] \\
X \arrow[r, "X\iota"] & X\mathrm{B} \arrow[r, "X\pi"] & X^{[1]}
\end{tikzcd}$$

Note that the right rectangle is a square by lemma 2.1.10. Also note that $\underline{p \cdot \delta_{(i,p)}} = 0$ in $\underline{\mathscr{F}}$. $\diamondsuit$

2.2.2 Remark. Suppose given a pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathscr{F}$ and the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "i", "\bullet" marking] \arrow[d, "1"'] & Y \arrow[r, "p", "|" marking] \arrow[d, "c"] & Z \arrow[d, "d"] \\
X \arrow[r, "X\iota", "\bullet" marking] & X\mathrm{B} \arrow[r, "X\pi", "|" marking] & X^{[1]}
\end{tikzcd}$$

Then we have $\underline{d} = \underline{\delta_{(i,p)}}$ in $\underline{\mathscr{F}}$. $\diamondsuit$

Proof. Lemma 1.3.10 yields a morphism $Z \xrightarrow{u} X\mathrm{B}$ in $\mathscr{F}$ such that $d = \delta_{(i,p)} + u \cdot X\pi$. So $\underline{d} = \underline{\delta_{(i,p)}}$ in $\underline{\mathscr{F}}$. $\square$

2.2.3 Lemma. Suppose given $X \in \mathrm{Ob}(\mathscr{F})$. We have $\underline{\delta_{(X\iota, X\pi)}} = \underline{1_{X^{[1]}}}$ in $\underline{\mathscr{F}}$. $\diamondsuit$

Proof. We have the following commutative diagram in $\mathscr{F}$.

$$\begin{tikzcd}
X \arrow[r, "X\iota", "\bullet" marking] \arrow[d, "1"'] & X\mathrm{B} \arrow[r, "X\pi", "|" marking] \arrow[d, "1"] & X^{[1]} \arrow[d, "1"] \\
X \arrow[r, "X\iota", "\bullet" marking] & X\mathrm{B} \arrow[r, "X\pi", "|" marking] & X^{[1]}
\end{tikzcd}$$

So the result follows from remark 2.2.2. $\square$

2.2.4 Lemma. Suppose given pure short exact sequences $X \xrightarrow{i} Y \xrightarrow{p} Z$, $X' \xrightarrow{i'} Y' \xrightarrow{p'} Z'$ and the following commutative diagram in $\mathscr{F}$.

$$\begin{tikzcd}
X \arrow[r, "i", "\bullet" marking] \arrow[d, "f"'] & Y \arrow[r, "p", "|" marking] \arrow[d, "g"] & Z \arrow[d, "h"] \\
X' \arrow[r, "i'", "\bullet" marking] & Y' \arrow[r, "p'", "|" marking] & Z'
\end{tikzcd}$$

Then we have $\underline{h \cdot \delta_{(i',p')}} = \underline{\delta_{(i,p)} \cdot f^{[1]}}$ in $\underline{\mathscr{F}}$.

$$\begin{tikzcd}
X \arrow[r, "i", "\bullet" marking] \arrow[d, "f"'] & Y \arrow[r, "p", "|" marking] \arrow[d, "g"] & Z \arrow[r, "\delta_{(i,p)}"] \arrow[d, "h"] & X^{[1]} \arrow[d, "f^{[1]}"] \\
X' \arrow[r, "i'", "\bullet" marking] & Y' \arrow[r, "p'", "|" marking] & Z' \arrow[r, "\delta_{(i',p')}"] & X'^{[1]}
\end{tikzcd}$$

$\diamondsuit$

Proof. We want to show that the following diagrams are commutative.

$$\begin{tikzcd}
X \arrow[r, "i", "\bullet" marking] \arrow[d, "f"'] & Y \arrow[r, "p", "|" marking] \arrow[d, "g \cdot \gamma_{(i',p')}"] & Z \arrow[d, "h \cdot \delta_{(i',p')}"] \\
X' \arrow[r, "X'\iota", "\bullet" marking] & X'\mathrm{B} \arrow[r, "X'\pi", "|" marking] & X'^{[1]}
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "i", "\bullet" marking] \arrow[d, "f"'] & Y \arrow[r, "p", "|" marking] \arrow[d, "\gamma_{(i,p)} \cdot f\mathrm{B}"] & Z \arrow[d, "\delta_{(i,p)} \cdot f^{[1]}"] \\
X' \arrow[r, "X'\iota", "\bullet" marking] & X'\mathrm{B} \arrow[r, "X'\pi", "|" marking] & X'^{[1]}
\end{tikzcd}$$

We have $i \cdot g \cdot \gamma_{(i',p')} = f \cdot i' \cdot \gamma_{(i',p')} = f \cdot X'\iota$ and $p \cdot h \cdot \delta_{(i',p')} = g \cdot p' \cdot \delta_{(i',p')} = g \cdot \gamma_{(i',p')} \cdot X'\pi$.
We have $i \cdot \gamma_{(i,p)} \cdot f\mathrm{B} = X\iota \cdot f\mathrm{B} = f \cdot X'\iota$ and $p \cdot \delta_{(i,p)} \cdot f^{[1]} = \gamma_{(i,p)} \cdot X\pi \cdot f^{[1]} = \gamma_{(i,p)} \cdot f\mathrm{B} \cdot X'\pi$.
Now lemma 1.3.10 yields a morphism $Z \xrightarrow{u} X'\mathrm{B}$ such that $h \cdot \delta_{(i',p')} = \delta_{(i,p)} \cdot f^{[1]} + u \cdot X'\pi$.
Thus $\underline{h \cdot \delta_{(i',p')}} = \underline{\delta_{(i,p)} \cdot f^{[1]}}$ in $\underline{\mathscr{F}}$. □

2.2.5 Lemma. Suppose given a pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathscr{F}$. Then we have $\underline{\delta_{(i^{[1]},p^{[1]})}} = \underline{-\delta^{[1]}_{(i,p)}}$ in $\underline{\mathscr{F}}$. ◊

Proof. We want to show that the following diagram is commutative.

$$\begin{tikzcd}
X^{[1]} \arrow[r, "i^{[1]}"] \arrow[d, "1"'] & Y^{[1]} \arrow[r, "p^{[1]}"] \arrow[d, "\gamma^{[1]}_{(i,p)} \cdot X\alpha^{-1}"] & Z^{[1]} \arrow[d, "-\delta^{[1]}_{(i,p)}"] \\
X^{[1]} \arrow[r, "X^{[1]}\iota"] & X^{[1]}\mathrm{B} \arrow[r, "X^{[1]}\pi"] & X^{[2]}
\end{tikzcd}$$

We have $i^{[1]} \cdot \gamma^{[1]}_{(i,p)} \cdot X\alpha^{-1} = (i \cdot \gamma_{(i,p)})^{[1]} \cdot X\alpha^{-1} = (X\iota)^{[1]} \cdot X\alpha^{-1} = X^{[1]}\iota$, cf. remark 2.1.6.
We have $-p^{[1]} \cdot \delta^{[1]}_{(i,p)} = -(p \cdot \delta_{(i,p)})^{[1]} = -(\gamma_{(i,p)} \cdot X\pi)^{[1]} = \gamma^{[1]}_{(i,p)} \cdot (-X\pi)^{[1]} = \gamma^{[1]}_{(i,p)} \cdot X\alpha^{-1} \cdot X^{[1]}\pi$.
By remark 2.2.2, we have $\underline{\delta_{(i^{[1]},p^{[1]})}} = \underline{-\delta^{[1]}_{(i,p)}}$ in $\underline{\mathscr{F}}$. □

2.2.6 Lemma. Suppose given pure short exact sequences $X \xrightarrow{i} Y \xrightarrow{p} Z$ and $X' \xrightarrow{i'} Y' \xrightarrow{p'} Z'$ in $\mathscr{F}$. Suppose given morphisms $Z \xrightarrow{h} Z'$ and $X \xrightarrow{f} X'$ in $\mathscr{F}$ such that $\underline{h \cdot \delta_{(i',p')}} = \underline{\delta_{(i,p)} \cdot f^{[1]}}$ in $\underline{\mathscr{F}}$. Then there exists $Y \xrightarrow{g} Y'$ in $\mathscr{F}$ such that $f \cdot i' = i \cdot g$ and $g \cdot p' = p \cdot h$.

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[r, "p"] \arrow[d, "g"] & Z \arrow[r, "\delta_{(i,p)}"] \arrow[d, "h"] & X^{[1]} \arrow[d, "f^{[1]}"] \\
X' \arrow[r, "i'"] & Y' \arrow[r, "p'"] & Z' \arrow[r, "\delta_{(i',p')}"] & X'^{[1]}
\end{tikzcd}$$
◊

Proof. We may choose $Z \xrightarrow{a} X'\mathrm{B}$ in $\mathscr{F}$ such that $h \cdot \delta_{(i',p')} - \delta_{(i,p)} \cdot f^{[1]} = a \cdot X'\pi$ since we have $\underline{h \cdot \delta_{(i',p')}} = \underline{\delta_{(i,p)} \cdot f^{[1]}}$ in $\underline{\mathscr{F}}$.
We use the fact that the following rectangle is a pullback, cf. definition 2.2.1.

$$\begin{tikzcd}
Y' \arrow[r, "p'"] \arrow[d, "\gamma_{(i',p')}"'] & Z' \arrow[d, "\delta_{(i',p')}"] \\
X'\mathrm{B} \arrow[r, "X'\pi"] & X'^{[1]}
\end{tikzcd}$$

We have

$$\begin{aligned}
(p \cdot a + \gamma_{(i,p)} \cdot f\mathrm{B}) \cdot X'\pi &= p \cdot h \cdot \delta_{(i',p')} - p \cdot \delta_{(i,p)} \cdot f^{[1]} + \gamma_{(i,p)} \cdot X\pi \cdot f^{[1]} \\
&= p \cdot h \cdot \delta_{(i',p')} - p \cdot \delta_{(i,p)} \cdot f^{[1]} + p \cdot \delta_{(i,p)} \cdot f^{[1]} \\
&= p \cdot h \cdot \delta_{(i',p')} .
\end{aligned}$$

So we may choose $Y \xrightarrow{g} Y'$ in $\mathscr{F}$ such that $g \cdot p' = p \cdot h$ and $g \cdot \gamma_{(i',p')} = p \cdot a + \gamma_{(i,p)} \cdot f\mathrm{B}$.
We have $f \cdot i' = i \cdot g$ since

$$f \cdot i' \cdot \begin{pmatrix} p' \\ \gamma_{(i',p')} \end{pmatrix} = \begin{pmatrix} 0 \\ f \cdot X'\iota \end{pmatrix} = \begin{pmatrix} 0 \\ X\iota \cdot f\mathrm{B} \end{pmatrix} = \begin{pmatrix} 0 \\ i \cdot \gamma_{(i,p)} \cdot f\mathrm{B} \end{pmatrix} = \begin{pmatrix} i \cdot p \cdot h \\ i \cdot (p \cdot a + \gamma_{(i,p)} \cdot f\mathrm{B}) \end{pmatrix} = i \cdot g \cdot \begin{pmatrix} p' \\ \gamma_{(i',p')} \end{pmatrix}$$

and since $\begin{pmatrix} p' \\ \gamma_{(i',p')} \end{pmatrix}$ is a monomorphism. □

2.2.7 Lemma. Suppose given $Z \xrightarrow{d} X^{[1]}$ in $\mathscr{F}$. Then there exists a pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathscr{F}$ such that $\underline{\delta_{(i,p)}} = \underline{d}$. ♢

Proof. Choose a pullback in $\mathscr{F}$ as follows.

$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[d, "c"'] & Z \arrow[d, "d"] \\ X\mathrm{B} \arrow[r, "X\pi"] & X^{[1]} \end{tikzcd}$$

Note that p is a pure epimorphism. By lemma 1.2.8.(b), we may choose $X \xrightarrow{i} Y$ in $\mathscr{F}$ such that $i \cdot c = X\iota$ and such that i is a kernel of p. Thus (i,p) is a pure short exact sequence and the following diagram is commutative.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "1"'] & Y \arrow[r, "p"] \arrow[d, "c"'] & Z \arrow[d, "d"] \\ X \arrow[r, "X\iota"] & X\mathrm{B} \arrow[r, "X\pi"] & X^{[1]} \end{tikzcd}$$

So $\underline{\delta_{(i,p)}} = \underline{d}$ by remark 2.2.2. □

2.2.8 Definition. A candidate triangle $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}$ in $\mathscr{F}$ with respect to Σ is called a *pseudo-triangle* in $\mathscr{F}$ if (i,p) is a pure short exact sequence in $\mathscr{F}$ and if there exists $X\mathrm{B} \xrightarrow{g} Z$ in $\mathscr{F}$ such that the following diagram is commutative in $\mathscr{F}$.

$$\begin{tikzcd} X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[r, "X\pi"] \arrow[d, "g"] & X^{[1]} \arrow[d, "1"] \\ Y \arrow[r, "i"] & Z \arrow[r, "p"] & X^{[1]} \end{tikzcd}$$

Note that in this case the left rectangle is a square, cf. lemma 2.1.9.

On the level of exact categories, these pseudo-triangles do not have the properties that triangles in triangulated categories have. For example, rotation of a pseudo-triangle may not yield a pseudo-triangle. Moreover, the term will also be used in similar contexts such as in definition 3.4.17. There the underlying exact category is not necessarily a Frobenius category, so the pseudo-triangles do not give rise to triangles in a stable category. ♢

2.2.9 Lemma. Suppose given a pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathscr{F}$. Then $Z^{[-1]} \xrightarrow{\delta^{[-1]}_{(i,p)}} X \xrightarrow{i} Y \xrightarrow{p} Z$ is a pseudo-triangle in $\mathscr{F}$. ♢

Proof. We have the following commutative diagram in $\mathscr{F}$.

$$\begin{tikzcd}
X \arrow[r, "i", "\bullet" marking] \arrow[d, "1"'] & Y \arrow[r, "p", "|" marking] \arrow[d, "\gamma_{(i,p)}"] & Z \arrow[d, "\delta_{(i,p)}"] \\
X \arrow[r, "X\iota", "\bullet" marking] & X\mathrm{B} \arrow[r, "X\pi", "|" marking] & X^{[1]}
\end{tikzcd}$$

Lemma 2.1.12.(c) yields $Z^{[-1]}\mathrm{B} \xrightarrow{g} Y$ in $\mathscr{F}$ such that the following diagram is commutative in $\mathscr{F}$.

$$\begin{tikzcd}
Z^{[-1]} \arrow[r, "Z^{[-1]}\iota", "\bullet" marking] \arrow[d, "\delta^{[-1]}_{(i,p)}"'] & Z^{[-1]}\mathrm{B} \arrow[r, "Z^{[-1]}\pi", "|" marking] \arrow[d, "g"] & Z \arrow[d, "1"] \\
X \arrow[r, "i", "\bullet" marking] & Y \arrow[r, "p", "|" marking] & Z
\end{tikzcd}$$

□

2.2.10 Lemma. Suppose given a pseudo-triangle $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}$ in $\mathscr{F}$. We have $\underline{\delta_{(i,p)}} = \underline{f^{[1]}}$ and $\underline{\delta^{[-1]}_{(i,p)}} = \underline{f}$. ◇

Proof. We may choose $X\mathrm{B} \xrightarrow{g} Z$ in $\mathscr{F}$ such that $X\iota \cdot g = f \cdot i$, $g \cdot p = X\pi$ and such that

$$\begin{tikzcd}
X \arrow[r, "X\iota", "\bullet" marking] \arrow[d, "f"'] & X\mathrm{B} \arrow[d, "g"] \\
Y \arrow[r, "i", "\bullet" marking] & Z
\end{tikzcd}$$

is a square in $\mathscr{F}$, cf. definition 2.2.8. By lemma 2.1.11.(c), we may choose $Z \xrightarrow{h} Y\mathrm{B}$ in $\mathscr{F}$ such that $g \cdot h = f\mathrm{B}$, $i \cdot h = Y\iota$ and such that $h \cdot Y\pi = p \cdot f^{[1]}$.

$$\begin{tikzcd}
Y \arrow[r, "i", "\bullet" marking] \arrow[d, "1"'] & Z \arrow[r, "p", "|" marking] \arrow[d, "h"] & X^{[1]} \arrow[d, "f^{[1]}"] \\
Y \arrow[r, "Y\iota", "\bullet" marking] & Y\mathrm{B} \arrow[r, "Y\pi", "|" marking] & Y^{[1]}
\end{tikzcd}$$

So the result follows from remark 2.2.2. □

2.2.11 Lemma. Suppose given $X \xrightarrow{f} Y$ in $\mathscr{F}$. Then there exists a pseudo-triangle $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}$ in $\mathscr{F}$. ◇

Proof. Choose a pushout in $\mathscr{F}$ as follows.

$$\begin{tikzcd}
X \arrow[r, "X\iota", "\bullet" marking] \arrow[d, "f"'] & X\mathrm{B} \arrow[d, "g"] \\
Y \arrow[r, "i", "\bullet" marking] & Z
\end{tikzcd}$$

By lemma 2.1.11.(b), we may choose $Z \xrightarrow{p} X^{[1]}$ in $\mathscr{F}$ such that $g \cdot p = X\pi$ and such that (i, p) is a pure short exact sequence. Thus $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}$ is a pseudo-triangle in $\mathscr{F}$. □

2.2.12 Lemma.

(a) Suppose given a pseudo-triangle $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}$ in $\mathscr{F}$. Then there exists a pure short exact sequence $X \xrightarrow{j} Y' \xrightarrow{q} Z$ in $\mathscr{F}$ such that the candidate triangles $X \xrightarrow{\underline{f}} Y \xrightarrow{\underline{i}} Z \xrightarrow{\underline{p}} X^{[1]}$ and $X \xrightarrow{\underline{j}} Y' \xrightarrow{\underline{q}} Z \xrightarrow{\underline{-\delta_{(j,q)}}} X^{[1]}$ are isomorphic in $\mathrm{CT}_{\underline{\Sigma}}(\underline{\mathscr{F}})$. Note that we have $\underline{f} = \underline{\delta_{(i,p)}^{[-1]}}$ by lemma 2.2.7.

(b) Suppose given a pure short exact sequence $X \xrightarrow{j} Y \xrightarrow{q} Z$ in $\mathscr{F}$. Then there exists a pseudo-triangle $X \xrightarrow{f} Y \xrightarrow{i} Z' \xrightarrow{p} X^{[1]}$ in $\mathscr{F}$ such that the candidate triangles $X \xrightarrow{\underline{f}} Y \xrightarrow{\underline{i}} Z' \xrightarrow{\underline{p}} X^{[1]}$ and $X \xrightarrow{\underline{j}} Y \xrightarrow{\underline{q}} Z \xrightarrow{\underline{-\delta_{(j,q)}}} X^{[1]}$ are isomorphic in $\mathrm{CT}_{\underline{\Sigma}}(\underline{\mathscr{F}})$. Again, note that we have $\underline{f} = \underline{\delta_{(i,p)}^{[-1]}}$ by lemma 2.2.7.

$\diamond$

Proof. Ad (a). We may choose $X\mathrm{B} \xrightarrow{g} Z$ such that $X\iota \cdot g = f \cdot i$, $g \cdot p = X\pi$ and such that

$$\begin{tikzcd}
X \arrow[r, "X\iota"] \arrow[d, "f"'] & X\mathrm{B} \arrow[d, "g"] \\
Y \arrow[r, "i"] & Z
\end{tikzcd}$$

is a square in $\mathscr{F}$, cf. definition 2.2.8.

We have the pure short exact sequence $X \xrightarrow{(\,X\iota\ f\,)} X\mathrm{B} \oplus Y \xrightarrow{\left(\begin{smallmatrix} -g \\ i \end{smallmatrix}\right)} Z$ in $\mathscr{F}$, cf. [Büh10, proposition 2.12]. Let $Y' = X\mathrm{B} \oplus Y$, $j = (\,X\iota\ f\,)$ and $q = \left(\begin{smallmatrix} -g \\ i \end{smallmatrix}\right)$. We have $\underline{-\delta_{(j,q)}} = \underline{p}$ in $\underline{\mathscr{F}}$ since the following diagram is commutative in $\mathscr{F}$, cf. remark 2.2.2.

$$\begin{tikzcd}
X \arrow[r, "(\,X\iota\ f\,)"] \arrow[d, "1"'] & X\mathrm{B} \oplus Y \arrow[r, "\left(\begin{smallmatrix} -g \\ i \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] & Z \arrow[d, "-p"] \\
X \arrow[r, "X\iota"] & X\mathrm{B} \arrow[r, "X\pi"] & X^{[1]}
\end{tikzcd}$$

The result now follows from the following commutative diagram in $\underline{\mathscr{F}}$ whose columns are isomorphisms in $\underline{\mathscr{F}}$.

$$\begin{tikzcd}
X \arrow[r, "\underline{(\,X\iota\ f\,)}"] \arrow[d, "1"'] & X\mathrm{B} \oplus Y \arrow[r, "\underline{\left(\begin{smallmatrix} -g \\ i \end{smallmatrix}\right)}"] \arrow[d, "\underline{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)}"] & Z \arrow[r, "\underline{-\delta_{(j,q)}}"] \arrow[d, "1"] & X^{[1]} \arrow[d, "1"] \\
X \arrow[r, "\underline{f}"] & Y \arrow[r, "\underline{i}"] & Z \arrow[r, "\underline{p}"] & X^{[1]}
\end{tikzcd}$$

Ad (b). We have the pure short exact sequence $Y \xrightarrow{(\,\gamma_{(j,q)}\ q\,)} X\mathrm{B} \oplus Z \xrightarrow{\left(\begin{smallmatrix} X\pi \\ -\delta_{(j,q)} \end{smallmatrix}\right)} X^{[1]}$ in $\mathscr{F}$, cf. definition 2.2.1 and [Büh10, proposition 2.12].

Moreover, $X \xrightarrow{j} Y \xrightarrow{(\,\gamma_{(j,q)}\ q\,)} X\mathrm{B} \oplus Z \xrightarrow{\left(\begin{smallmatrix} X\pi \\ -\delta_{(j,q)} \end{smallmatrix}\right)} X^{[1]}$ is a pseudo-triangle in $\mathscr{F}$ since the following diagram is commutative in $\mathscr{F}$.

$$\begin{tikzcd}
X \arrow[r, "X\iota"] \arrow[d, "j"'] & X\mathrm{B} \arrow[r, "X\pi"] \arrow[d, "(1\ 0)"] & X^{[1]} \arrow[d, "1"] \\
Y \arrow[r, "(\gamma_{(j,q)}\ q)"] & X\mathrm{B}\oplus Z \arrow[r, "\left(\begin{smallmatrix} X\pi \\ -\delta_{(j,q)} \end{smallmatrix}\right)"] & X^{[1]}
\end{tikzcd}$$

The result now follows from the following commutative diagram in $\underline{\mathscr{F}}$ whose columns are isomorphisms in $\underline{\mathscr{F}}$.

$$\begin{tikzcd}
X \arrow[r, "\underline{j}"] \arrow[d, "1"'] & Y \arrow[r, "\underline{q}"] \arrow[d, "1"] & Z \arrow[r, "\underline{-\delta_{(j,q)}}"] \arrow[d, "\underline{(0\ 1)}"] & X^{[1]} \arrow[d, "1"] \\
X \arrow[r, "\underline{j}"] & Y \arrow[r, "\underline{(\gamma_{(j,q)}\ q)}"] & X\mathrm{B}\oplus Z \arrow[r, "\underline{\left(\begin{smallmatrix} X\pi \\ -\delta_{(j,q)} \end{smallmatrix}\right)}"] & X^{[1]}
\end{tikzcd}$$

□

2.2.13 Lemma. Suppose given pure short exact sequences $X \xrightarrow{i} Y \xrightarrow{p} C$, $Y \xrightarrow{j} Z \xrightarrow{q} D$ and $X \xrightarrow{i\cdot j} Z \xrightarrow{r} E$ in $\mathscr{F}$. By lemma 1.3.13, There exists a unique morphism $C \xrightarrow{k} E$ in $\mathscr{F}$ such that $j \cdot r = p \cdot k$ and a unique morphism $E \xrightarrow{s} D$ in $\mathscr{F}$ such that $r \cdot s = q$. Moreover, $C \xrightarrow{k} E \xrightarrow{s} D$ is a pure short exact sequence in $\mathscr{F}$.
We also have $\underline{k\cdot\delta_{(i\cdot j,r)}} = \underline{\delta_{(i,p)}}$, $\underline{\delta_{(i\cdot j,r)}\cdot i^{[1]}} = \underline{s\cdot\delta_{(j,q)}}$ and $\underline{\delta_{(j,q)}\cdot p^{[1]}} = \underline{\delta_{(k,s)}}$ in $\underline{\mathscr{F}}$.

$$\begin{tikzcd}
 & & & X^{[1]} \arrow[dddrr, "1"] & & \\
 & & C \arrow[ur, "\delta_{(i,p)}"] \arrow[ddrr, "k"] & & & \\
 & Y \arrow[ur, "p"] \arrow[dr, "j"] & & & & \\
X \arrow[ur, "i"] \arrow[rr, "i\cdot j"'] & & Z \arrow[rr, "r"] \arrow[dr, "q"'] & & E \arrow[r, "\delta_{(i\cdot j,r)}"] \arrow[dl, "s"] & X^{[1]} \arrow[ddl, "i^{[1]}"] \\
 & & & D \arrow[dl, "\delta_{(k,s)}"'] \arrow[dr, "\delta_{(j,q)}"] & & \\
 & & C^{[1]} & & Y^{[1]} \arrow[ll, "p^{[1]}"'] &
\end{tikzcd}$$

◊

Proof. We have the following commutative diagrams in $\mathscr{F}$.

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "1"'] & Y \arrow[r, "p"] \arrow[d, "j"] & C \arrow[d, "k"] \\
X \arrow[r, "i\cdot j"] & Z \arrow[r, "r"] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "i\cdot j"] \arrow[d, "i"'] & Z \arrow[r, "r"] \arrow[d, "1"] & E \arrow[d, "s"] \\
Y \arrow[r, "j"] & Z \arrow[r, "q"] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "j"] \arrow[d, "p"'] & Z \arrow[r, "q"] \arrow[d, "r"] & D \arrow[d, "1"] \\
C \arrow[r, "k"] & E \arrow[r, "s"] & D
\end{tikzcd}$$

The result now follows from lemma 2.2.4. □

2.2.14 Lemma/Definition. Suppose given $T \in \mathrm{Ob}(\mathrm{CT}_{\underline{\Sigma}}(\underline{\mathscr{F}}))$. We say that T is a *triangle* in $\underline{\mathscr{F}}$ if there exists a pseudo-triangle $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}$ in $\mathscr{F}$ such that T is isomorphic to $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}$ in $\mathrm{CT}_{\underline{\Sigma}}(\underline{\mathscr{F}})$. We define the full subcategory $\mathfrak{T}_{\mathscr{F}}$ of $\mathrm{CT}_{\underline{\Sigma}}(\underline{\mathscr{F}})$ by setting

$$\mathrm{Ob}(\mathfrak{T}_{\mathscr{F}}) = \{T \in \mathrm{Ob}(\mathrm{CT}_{\underline{\Sigma}}(\underline{\mathscr{F}}))\colon T \text{ is a triangle in } \underline{\mathscr{F}}\}.$$

The tuple $(\underline{\mathscr{F}}, \underline{\Sigma}, \mathfrak{T}_{\mathscr{F}})$ is in fact a strict triangulated category. $\diamondsuit$

Proof. Ad (TR1). This follows from lemmata 2.2.9, 2.2.3 and 2.2.11.

Ad (TR2). This follows from lemmata 2.2.12 and 2.2.9.

Ad (TR3). This follows from lemmata 2.2.6, 2.2.10

Ad (TR4). This follows from lemmata 2.2.12 and 2.2.13. $\square$

2.3 Algebraic triangulated categories

Suppose given a Frobenius category $\mathscr{F}$. Let $\mathscr{B}$ denote the strictly full additive subcategory of $\mathscr{F}$ whose objects are bijective in $\mathscr{F}$. We will define a strict Frobenius catgory $\mathscr{B}^{\mathrm{ac}}$ and an exact functor $I\colon \mathscr{B}^{\mathrm{ac}} \to \mathscr{F}$ that induces an equivalence $\underline{I}\colon \underline{\mathscr{B}^{\mathrm{ac}}} \to \underline{\mathscr{F}}$. We briefly discuss that the construction can be extended to functors and transformations in remark 2.3.10.

2.3.1 Definition. We define the full subcategory $\mathscr{B}^{\mathrm{ac}}$ of $\mathrm{C}(\mathscr{B})$ by setting

$$\mathrm{Ob}(\mathscr{B}^{\mathrm{ac}}) = \{X \in \mathrm{Ob}(\mathrm{C}(\mathscr{B}))\colon X\mathrm{C}(\mathrm{Inc}_{\mathscr{B}}^{\mathscr{F}}) \text{ is acyclic in } \mathrm{C}(\mathscr{F})\},$$

cf. definitions 1.9.3 and 1.9.21. $\diamondsuit$

2.3.2 Lemma. $\mathscr{B}^{\mathrm{ac}}$ is a strict Frobenius subcategory of $\mathrm{C}(\mathscr{B})$, cf. definition 2.1.32. In particular, it is a strict Frobenius category itself. $\diamondsuit$

Proof. Note that we have $\mathrm{Z}_{\mathrm{C}(\mathscr{B})} \subseteq \mathscr{B}^{\mathrm{ac}}$ and that $X^{[1]}, X^{[-1]} \in \mathrm{Ob}(\mathscr{B}^{\mathrm{ac}})$ for $X \in \mathrm{Ob}(\mathscr{B}^{\mathrm{ac}})$. Suppose given a pure short exact sequence $X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$ in $\mathrm{C}(\mathscr{B})$ such that $X, Z \in \mathrm{Ob}(\mathscr{B}^{\mathrm{ac}})$. Then Y is a mapping cone (see [Büh10, definition 9.2]) of $\delta_{(i,p)}^{[-1]}$ by lemmata 2.2.9 and 1.9.17. So $Y \in \mathrm{Ob}(\mathscr{B}^{\mathrm{ac}})$ by [Nee90, lemma 1.1] [Büh10, lemma 10.3]. $\square$

2.3.3 Lemma. Suppose given $X \xrightarrow{f} Y$ in $\mathscr{B}^{\mathrm{ac}}$. We have $\underline{f} = 0$ in $\underline{\mathscr{B}^{\mathrm{ac}}}$ if and only if there exist morphisms $X_k \xrightarrow{h_k} Y_{k+1}$ in $\mathscr{F}$ such that $h_k \cdot y_{k+1} + x_k \cdot h_{k-1} = f_k$ for $k \in \mathbf{Z}$. $\diamondsuit$

Proof. This follows from lemma 1.9.13. $\square$

2.3.4 Definition. Suppose given $X \in \mathrm{Ob}(\mathscr{B}^{\mathrm{ac}})$. For $k \in \mathbf{Z}$, we choose pure short exact sequences $X_{\overline{k+1}} \xrightarrow{x_{\overset{\bullet}{k}}} X_k \xrightarrow{x_{\overline{k}}} X_{\overline{k}}$ in $\mathscr{F}$ such that $x_{\overline{k}} \cdot x_{\overset{\bullet}{k-1}} = x_k$.

$$\begin{tikzcd}
\cdots \arrow[r] & X_{k+1} \arrow[rr, "x_{k+1}"] \arrow[dr, "x_{\overline{k+1}}"'] & & X_k \arrow[rr, "x_k"] \arrow[dr, "x_{\overline{k}}"'] & & X_{k-1} \arrow[r] & \cdots \\
\cdots & & X_{\overline{k+1}} \arrow[ur, "x_{\overset{\bullet}{k}}"'] & & X_{\overline{k}} \arrow[ur, "x_{\overset{\bullet}{k-1}}"'] & & \cdots
\end{tikzcd}$$

Suppose given $X \xrightarrow{f} Y$ in $\mathscr{B}^{\mathrm{ac}}$. For $k \in \mathbf{Z}$, let $X_{\overline{k}} \xrightarrow{f_{\overline{k}}} Y_{\overline{k}}$ denote the unique morphism in $\mathscr{F}$ such that $x_{\overline{k}} \cdot f_{\overline{k}} = f_k \cdot y_{\overline{k}}$. Note that we also have $f_{\overline{k}} \cdot y_{\overset{\bullet}{k-1}} = x_{\overset{\bullet}{k-1}} \cdot f_{k-1}$ for $k \in \mathbf{Z}$. ◊

2.3.5 Lemma/Definition. We define the functor $I\colon \mathscr{B}^{\mathrm{ac}} \to \mathscr{F}$ by setting $XI = X_{\overline{0}}$ for $X \in \mathrm{Ob}(\mathscr{B}^{\mathrm{ac}})$ and $fI = f_{\overline{0}}$ for $f \in \mathrm{Mor}(\mathscr{B}^{\mathrm{ac}})$. This in fact defines an exact functor. The letter I stands for *image* and in fact the object $XI = X_{\overline{0}}$ belongs to the image $X_0 \xrightarrow{x_{\overline{0}}} X_{\overline{0}} \xrightarrow{x_{\overset{\bullet}{-1}}} X_{-1}$ of x_0 for $X \in \mathrm{Ob}(\mathscr{B}^{\mathrm{ac}})$. ◊

Proof. Suppose given $X \underset{h}{\overset{f}{\rightrightarrows}} Y \xrightarrow{g} Z$ in $\mathscr{B}^{\mathrm{ac}}$. We have $1_X I = 1_{XI}$ since $x_{\overline{0}} \cdot 1_{XI} = 1_{X_0} \cdot x_{\overline{0}}$. We have $(f \cdot g)I = fI \cdot gI$ since $x_{\overline{0}} \cdot fI \cdot gI = f_0 \cdot y_{\overline{0}} \cdot gI = f_0 \cdot g_0 \cdot z_{\overline{0}} = (f \cdot g)_0 \cdot z_{\overline{0}}$. We have $(f+h)I = fI+hI$ since $x_{\overline{0}}\cdot(fI+hI) = x_{\overline{0}}\cdot fI+x_{\overline{0}}\cdot hI = f_0\cdot y_{\overline{0}}+h_0\cdot y_{\overline{0}} = (f_0+h_0)\cdot y_{\overline{0}} = (f+h)_0\cdot y_{\overline{0}}$. Suppose that (f, g) is a pure short exact sequence. Then Y is a mapping cone (see [Büh10, definition 9.2]) of $\delta^{[-1]}_{(f,g)}$ by lemmata 2.2.9 and 1.9.17. So $(fI, gI) = (f_{\overline{0}}, g_{\overline{0}})$ is pure short exact in $\mathscr{F}$ by [Büh10, remark 10.4]. □

2.3.6 Lemma.

(a) The functor $I\colon \mathscr{B}^{\mathrm{ac}} \to \mathscr{F}$ is dense.

(b) The functor $I\colon \mathscr{B}^{\mathrm{ac}} \to \mathscr{F}$ is full.

(c) Suppose given a bijective object $X \in \mathrm{Ob}(\mathscr{B}^{\mathrm{ac}})$. The object $XI \in \mathrm{Ob}(\mathscr{F})$ is bijective as well.

◊

Proof. Ad (a). Suppose given $I_0 \in \mathrm{Ob}(\mathscr{F})$.

For $k \in \mathbf{Z}_{\leq 0}$, we may recursively choose pure short exact sequences $I_k \xrightarrow{i_{k-1}} X_{k-1} \xrightarrow{p_{k-1}} I_{k-1}$ such that X_{k-1} is bijective in $\mathscr{F}$.

For $k \in \mathbf{Z}_{\geq 1}$, we may recursively choose pure short exact sequences $I_k \xrightarrow{i_{k-1}} X_{k-1} \xrightarrow{p_{k-1}} I_{k-1}$ such that X_{k-1} is bijective in $\mathscr{F}$.

We obtain $X \in \mathrm{Ob}(\mathscr{B}^{\mathrm{ac}})$ by setting $x_k = p_k \cdot i_{k-1}$ for $k \in \mathbf{Z}$.

By [Büh10, lemma 8.4], the objects I_0 and $XI = X_{\overline{0}}$ are isomorphic in $\mathscr{F}$.

Ad (b). Suppose given $X, Y \in \mathrm{Ob}(\mathscr{B}^{\mathrm{ac}})$ and $X_{\overline{0}} \xrightarrow{f_0} Y_{\overline{0}}$ in $\mathscr{F}$.

For $k \in \mathbf{Z}_{<0}$, we want to recursively construct $X_k \xrightarrow{g_k} Y_k$ and $X_{\overline{k}} \xrightarrow{f_k} Y_{\overline{k}}$ in $\mathscr{F}$ such that $x_{\overset{\bullet}{k}} \cdot g_k = f_{k+1} \cdot y_{\overset{\bullet}{k}}$ and $x_{\overline{k}} \cdot f_k = g_k \cdot y_{\overline{k}}$.

Suppose given $k \in \mathbf{Z}_{<0}$. Suppose we have already constructed g_ℓ , f_ℓ for $\ell \in [k+1, -1]$. Since Y_k is injective in $\mathscr{F}$, we may choose $X_k \xrightarrow{g_k} Y_k$ in $\mathscr{F}$ such that $x_k^{\bullet} \cdot g_k = f_{k+1} \cdot y_k^{\bullet}$. We have $x_k^{\bullet} \cdot g_k \cdot y_{\overline{k}} = f_{k+1} \cdot y_k^{\bullet} \cdot y_{\overline{k}} = 0$. So we may choose $X_{\overline{k}} \xrightarrow{f_k} Y_{\overline{k}}$ in $\mathscr{F}$ such that $x_{\overline{k}} \cdot f_k = g_k \cdot y_{\overline{k}}$.

For $k \in \mathbf{Z}_{\geq 0}$, we want to recursively construct $X_k \xrightarrow{g_k} Y_k$ and $X_{\overline{k+1}} \xrightarrow{f_{k+1}} Y_{\overline{k+1}}$ in $\mathscr{F}$ such that $g_k \cdot y_{\overline{k}} = x_{\overline{k}} \cdot f_k$ and $f_{k+1} \cdot y_k^{\bullet} = x_k^{\bullet} \cdot g_k$.

Suppose given $k \in \mathbf{Z}_{\geq 0}$. Suppose we have already constructed g_ℓ , $f_{\ell+1}$ for $\ell \in [0, k-1]$. Since X_k is projective in $\mathscr{F}$, we may choose $X_k \xrightarrow{g_k} Y_k$ in $\mathscr{F}$ such that $g_k \cdot y_{\overline{k}} = x_{\overline{k}} \cdot f_k$. We have $x_k^{\bullet} \cdot g_k \cdot y_{\overline{k}} = x_k^{\bullet} \cdot x_{\overline{k}} \cdot f_k = 0$. So we may choose $X_{\overline{k+1}} \xrightarrow{f_{k+1}} Y_{\overline{k+1}}$ in $\mathscr{F}$ such that $f_{k+1} \cdot y_k^{\bullet} = x_k^{\bullet} \cdot g_k$.

We obtain a morphism $X \xrightarrow{g} Y$ in $\mathscr{B}^{\mathrm{ac}}$ since we have

$g_k \cdot y_k = g_k \cdot y_{\overline{k}} \cdot y_{k-1}^{\bullet} = x_{\overline{k}} \cdot f_k \cdot y_{k-1}^{\bullet} = x_{\overline{k}} \cdot x_{k-1}^{\bullet} \cdot g_{k-1} = x_k \cdot g_{k-1}$ for $k \in \mathbf{Z}$.

Moreover, we have $gI = f_0$ since $x_{\overline{0}} \cdot f_0 = g_0 \cdot y_{\overline{0}}$. We conclude that I is full.

Ad (c). We have $\underline{1_X} = 0$ in $\underline{\mathscr{B}^{\mathrm{ac}}}$. So we may choose $X_k \xrightarrow{h_k} X_{k+1}$ in $\mathscr{F}$ such that $h_k \cdot y_{k+1} + x_k \cdot h_{k-1} = 1$ for $k \in \mathbf{Z}$ by lemma 2.3.3.

Consider the morphism $t = x_{-1}^{\bullet} \cdot h_{-1} \colon X_{\overline{0}} \to X_0$ in $\mathscr{F}$. We have $t \cdot x_{\overline{0}} = 1$ since

$$t \cdot x_{\overline{0}} \cdot x_{-1}^{\bullet} = x_{-1}^{\bullet} \cdot h_{-1} \cdot x_0 = x_{-1}^{\bullet} \cdot (1 - x_{-1} \cdot h_{-2}) = x_{-1}^{\bullet}$$

and since $x_{-1}^{\bullet}$ is a pure monomorphism. So $X_{\overline{0}}$ is a summand of X_0 , cf. lemma 1.2.11.(b). We conclude that $XI = X_{\overline{0}}$ is bijective in $\mathscr{F}$. $\square$

2.3.7 Lemma. Suppose given $X \xrightarrow{f} Y$ in $\mathscr{B}^{\mathrm{ac}}$. Suppose given $X_{-1} \xrightarrow{h_{-1}} Y_0$ in $\mathscr{F}$ such that $x_0 \cdot h_{-1} \cdot y_0 = f_0 \cdot y_0$. Then we have $\underline{f} = 0$ in $\underline{\mathscr{B}^{\mathrm{ac}}}$. $\diamondsuit$

Proof. For $k \in \mathbf{Z}_{\geq 0}$, we want to recursively construct $X_k \xrightarrow{h_k} Y_{k+1}$ in $\mathscr{F}$ such that $h_k \cdot y_{k+1} + x_k \cdot h_{k-1} = f_k$.

We have $(f_0 - x_0 \cdot h_{-1}) \cdot y_0 = 0$. So we may choose $X_0 \xrightarrow{g_0} Y_{\overline{1}}$ in $\mathscr{F}$ such that $g_0 \cdot y_0^{\bullet} = f_0 - x_0 \cdot h_{-1}$. Since X_0 is projective in $\mathscr{F}$, we may choose $X_0 \xrightarrow{h_0} Y_1$ in $\mathscr{F}$ such that $h_0 \cdot y_{\overline{1}} = g_0$. We have $h_0 \cdot y_1 + x_0 \cdot h_{-1} = h_0 \cdot y_{\overline{1}} \cdot y_0^{\bullet} + x_0 \cdot h_{-1} = g_0 \cdot y_0^{\bullet} + x_0 \cdot h_{-1} = f_0$.

Suppose given $k \in \mathbf{Z}_{>0}$. Suppose we have already constructed h_ℓ for $\ell \in [k-1, 0]$. We have $(f_k - x_k \cdot h_{k-1}) \cdot y_k = x_k \cdot f_{k-1} - x_k \cdot (f_{k-1} - x_{k-1} \cdot h_{k-2}) = 0$. So we may choose $X_k \xrightarrow{g_k} Y_{\overline{k+1}}$ in $\mathscr{F}$ such that $g_k \cdot y_k^{\bullet} = f_k - x_k \cdot h_{k-1}$. Since X_k is projective in $\mathscr{F}$, we may choose $X_k \xrightarrow{h_k} Y_{k+1}$ in $\mathscr{F}$ such that $h_k \cdot y_{\overline{k+1}} = g_k$. We have

$h_k \cdot y_{k+1} + x_k \cdot h_{k-1} = h_k \cdot y_{\overline{k+1}} \cdot y_k^{\bullet} + x_k \cdot h_{k-1} = g_k \cdot y_k^{\bullet} + x_k \cdot h_{k-1} = f_k$.

For $k \in \mathbf{Z}_{<-1}$, we want to recursively construct $X_k \xrightarrow{h_k} Y_{k+1}$ in $\mathscr{F}$ such that $h_{k+1} \cdot y_{k+2} + x_{k+1} \cdot h_k = f_{k+1}$.

We have $x_0 \cdot (f_{-1} - h_{-1} \cdot y_0) = f_0 \cdot y_0 - f_0 \cdot y_0 = 0$. So we may choose $X_{\overline{-1}} \xrightarrow{g_{-2}} Y_{-1}$ in $\mathscr{F}$ such that $x_{\overline{-1}} \cdot g_{-2} = f_{-1} - h_{-1} \cdot y_0$. Since Y_{-1} is injective in $\mathscr{F}$, we may choose $X_{-2} \xrightarrow{h_{-2}} Y_{-1}$ in $\mathscr{F}$ such that $x_{-2}^{\bullet} \cdot h_{-2} = g_{-2}$. We have

$h_{-1} \cdot y_0 + x_{-1} \cdot h_{-2} = h_{-1} \cdot y_0 + x_{\overline{-1}} \cdot x^{\bullet}_{\underline{-2}} \cdot h_{-2} = h_{-1} \cdot y_0 + x_{\overline{-1}} \cdot g_{-2} = f_{-1}$.

Suppose given $k \in \mathbf{Z}_{<-2}$. Suppose we have already constructed h_ℓ for $\ell \in [-1, k+1]$. We have $x_{k+2} \cdot (f_{k+1} - h_{k+1} \cdot y_{k+2}) = f_{k+2} \cdot y_{k+2} - (f_{k+2} - h_{k+2} \cdot y_{k+3}) \cdot y_{k+2} = 0$. So we may choose $X_{\overline{k+1}} \xrightarrow{g_k} Y_{k+1}$ in $\mathscr{F}$ such that $x_{\overline{k+1}} \cdot g_k = f_{k+1} - h_{k+1} \cdot y_{k+2}$. Since Y_{k+1} is injective in $\mathscr{F}$, we may choose $X_k \xrightarrow{h_k} Y_{k+1}$ in $\mathscr{F}$ such that $x^{\bullet}_{\underline{k}} \cdot h_k = g_k$. We have $h_{k+1} \cdot y_{k+2} + x_{k+1} \cdot h_k = h_{k+1} \cdot y_{k+2} + x_{\overline{k+1}} \cdot x^{\bullet}_{\underline{k}} \cdot h_k = h_{k+1} \cdot y_{k+2} + x_{\overline{k+1}} \cdot g_k = f_{k+1}$.

By lemma 2.3.3, we have $\underline{f} = 0$ in $\underline{\mathscr{B}^{\mathrm{ac}}}$. □

2.3.8 Definition. Let $\underline{I} \colon \underline{\mathscr{B}^{\mathrm{ac}}} \to \underline{\mathscr{F}}$ denote the unique functor such that $\mathfrak{P}_{\mathscr{B}^{\mathrm{ac}}} \cdot \underline{I} = I \cdot \mathfrak{P}_{\mathscr{F}}$, cf. lemmata 2.3.5, 2.3.6.(c) and 1.3.30. Note that this functor $\underline{I}$ is exact.

$$\begin{tikzcd}
\mathscr{B}^{\mathrm{ac}} \arrow[r, "I"] \arrow[d, "\mathfrak{P}_{\mathscr{B}^{\mathrm{ac}}}"'] & \mathscr{F} \arrow[d, "\mathfrak{P}_{\mathscr{F}}"] \\
\underline{\mathscr{B}^{\mathrm{ac}}} \arrow[r, "\underline{I}"] & \underline{\mathscr{F}}
\end{tikzcd}$$

◊

2.3.9 Proposition. The functor $\underline{I} \colon \underline{\mathscr{B}^{\mathrm{ac}}} \to \underline{\mathscr{F}}$ is an exact equivalence. ◊

Proof. The functor $\underline{I}$ is exact, dense and full, cf. definition 2.3.8 and lemma 2.3.6.(a,b). So it remains to show that $\underline{I}$ is faithful.

Suppose given $X \xrightarrow{f} Y$ in $\mathscr{B}^{\mathrm{ac}}$ such that $\underline{f}\,\underline{I} = 0$. Thus we have $\underline{f_{\bar{0}}} = \underline{fI} = \underline{f}\,\underline{I} = 0$ in $\underline{\mathscr{F}}$. So we may choose bijective objects $I, P \in \mathrm{Ob}(\mathscr{F})$, a morphism $I \xrightarrow{g} P$, a pure monomorphism $X_{\bar{0}} \xrightarrow{i} I$ and a pure epimorphism $P \xrightarrow{p} Y_{\bar{0}}$ in $\mathscr{F}$ such that $f_{\bar{0}} = i \cdot g \cdot p$.

Since I is injective in $\mathscr{F}$, we may choose $X_{-1} \xrightarrow{a} I$ in $\mathscr{F}$ such that $x^{\bullet}_{\underline{-1}} \cdot a = i$. Since P is projective in $\mathscr{F}$, we may choose $P \xrightarrow{b} Y_0$ in $\mathscr{F}$ such that $b \cdot y_{\bar{0}} = p$.

$$\begin{tikzcd}
X_0 \arrow[r, "x_{\bar{0}}"] & X_{\bar{0}} \arrow[r, "x^{\bullet}_{\underline{-1}}"] \arrow[d, "i"] & X_{-1} \arrow[dl, "a"] \\
& I \arrow[d, "g"] & \\
& P \arrow[dl, "b"'] \arrow[d, "p"] & \\
Y_0 \arrow[r, "y_{\bar{0}}"] & Y_{\bar{0}} \arrow[r, "y^{\bullet}_{\underline{-1}}"] & Y_{-1}
\end{tikzcd}$$

Let $h = a \cdot g \cdot b$. We have

$$\begin{aligned}
x_0 \cdot h \cdot y_0 &= x_{\bar{0}} \cdot x^{\bullet}_{\underline{-1}} \cdot h \cdot y_{\bar{0}} \cdot y^{\bullet}_{\underline{-1}} = x_{\bar{0}} \cdot x^{\bullet}_{\underline{-1}} \cdot a \cdot g \cdot b \cdot y_{\bar{0}} \cdot y^{\bullet}_{\underline{-1}} = x_{\bar{0}} \cdot i \cdot g \cdot p \cdot y^{\bullet}_{\underline{-1}} \\
&= x_{\bar{0}} \cdot f_{\bar{0}} \cdot y^{\bullet}_{\underline{-1}} = f_0 \cdot y_{\bar{0}} \cdot y^{\bullet}_{\underline{-1}} = f_0 \cdot y_0 \,.
\end{aligned}$$

Now lemma 2.3.7 yields that $\underline{f} = 0$ in $\underline{\mathscr{B}^{\mathrm{ac}}}$. We conclude that $\underline{I}$ is faithful. □

2.3.10 Remark (construction for functors and transformations)**.** Suppose given another Frobenius category $\mathscr{F}'$ and an exact functor $G\colon \mathscr{F} \to \mathscr{F}'$ that sends bijective objects in $\mathscr{F}$ to bijective objects in $\mathscr{F}'$. Let $\mathscr{B}'$ denote the full subcategory of $\mathscr{F}'$ whosae objects are bijective in $\mathscr{F}'$. We obtain a functor $G^{\mathrm{ac}}\colon \mathscr{B}^{\mathrm{ac}} \to \mathscr{B}'^{\mathrm{ac}}$ by setting $G^{\mathrm{ac}} = \mathbf{Z}^{\mathrm{op}}(G|_{\mathscr{B}}^{\mathscr{B}'})|_{\mathscr{B}^{\mathrm{ac}}}^{\mathscr{B}'^{\mathrm{ac}}}$, cf. definition 1.9.21.

Suppose given another exact functor $H\colon \mathscr{F} \to \mathscr{F}'$ that sends bijective objects in $\mathscr{F}$ to bijective objects in $\mathscr{F}'$ and a transformation $\varphi\colon G \to H$. We obtain a transformation $\varphi^{\mathrm{ac}}\colon G^{\mathrm{ac}} \to H^{\mathrm{ac}}$ by setting $\varphi^{\mathrm{ac}} = \mathbf{Z}^{\mathrm{op}}(\varphi|_{\mathscr{B}}^{\mathscr{B}'})|_{\mathscr{B}^{\mathrm{ac}}}^{\mathscr{B}'^{\mathrm{ac}}}$. $\Diamond$

Chapter 3

Diagram categories

The previous constructions of (bounded) realisation functors in [BBD82] and [Beĭ87] used filtered derived categories and the more general filtered triangulated categories. We will also use filtrations in our constructions but in order to carry them out in the unbounded case, we have to introduce four different kinds of categories that will be equivalent on bounded levels: the category of filtrations $\mathrm{F}(\mathscr{F})$, the category of cofiltrations $\mathrm{CF}(\mathscr{F})$, the category of ∇-diagrams $\nabla(\mathscr{F})$ and the category of filtered objects $\mathrm{FO}(\mathscr{F})$ with entries in strict Frobenius category $\mathscr{F}$. This chapter deals with the definition of these categories, the study of their properties and the functors between them and between various factor categories and subcategories. The construction of resolution and realisation functors in the next chapter 4 will heavily rely on the results presented in this chapter 3.

In the brief first section 3.1, we describe how to obtain an ideal in an additive category with translation functor. This will be applied in the sections 3.3 and 3.4 to obtain the factor categories $\underline{\underline{\nabla}}(\mathscr{F})$ and $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$.

Section 3.2 defines and studies the category of filtrations $\mathrm{F}(\mathscr{A})$ and the category of cofiltrations $\mathrm{CF}(\mathscr{A})$ with entries in an exact category $\mathscr{A}$. The properties of limits, colimits and of morphisms induced between them will be important for the sequel, as well as the new concept of projective and injective families which are weak forms of (co)limits.

The first major result in section 3.3 is theorem 3.3.8, where we prove that the category of ∇-diagrams $\nabla(\mathscr{F})$ with entries in a strict Frobenius category $\mathscr{F}$ is a strict Frobenius category itself. So we may form its stable category $\underline{\nabla}(\mathscr{F})$ and have the stabilisation functor $\mathfrak{P}_{\nabla(\mathscr{F})}\colon \nabla(\mathscr{F}) \to \underline{\nabla}(\mathscr{F})$. We proceed to introduce another factor category $\underline{\underline{\nabla}}(\mathscr{F})$ with residue class functor $\mathfrak{Q}_{\nabla,\mathscr{F}}\colon \nabla(\mathscr{F}) \to \underline{\underline{\nabla}}(\mathscr{F})$ by factoring out more morphisms, so there exists a natural functor $\mathfrak{S}\colon \underline{\nabla}(\mathscr{F}) \to \underline{\underline{\nabla}}(\mathscr{F})$. We discuss projection functors $\Psi_{\ell,\mathrm{CF},\mathscr{F}}\colon \nabla(\mathscr{F}) \to \mathrm{CF}(\mathscr{F})$, $\Psi_{k,\mathrm{F},\mathscr{F}}\colon \nabla(\mathscr{F}) \to \mathrm{CF}(\mathscr{F})$ and also how (co)limits can be used to obtain cofiltrations $X\Psi_{\infty,\mathrm{CF},\mathscr{F}}$ and filtrations $X\Psi_{-\infty,\mathrm{F},\mathscr{F}}$ from a ∇-diagram X.

In section 3.4, we introduce the category of filtered objects $\mathrm{FO}(\mathscr{F})$ with entries in a strict Frobenius category $\mathscr{F}$. It is related to the categories in the previous sections via the projection functors $\mathrm{P}_{\omega,\mathscr{F}}\colon \mathrm{FO}(\mathscr{F}) \to \mathscr{F}$, $\mathrm{P}_{\mathrm{CF},\mathscr{F}}\colon \mathrm{FO}(\mathscr{F}) \to \mathrm{CF}(\mathscr{F})$ and $\mathrm{P}_{\mathrm{F},\mathscr{F}}\colon \mathrm{FO}(\mathscr{F}) \to \mathrm{F}(\mathscr{F})$. We define the factor category $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$ and also various subcategories whose objects consist of (co)limits, projective families or injective families such as $\mathrm{FO}^{\nabla}(\mathscr{F})$, $\underline{\underline{\mathrm{FO}}}^{\nabla}(\mathscr{F})$, $\mathrm{FO}^{\mathrm{proj}}(\mathscr{F})$ or $\mathrm{FO}^{\mathrm{inj}}(\mathscr{F})$. Finally, we construct the embedding $\mathrm{E}_{\mathscr{F}} \colon \mathscr{F} \to \mathrm{FO}(\mathscr{F})$ of the strict Frobenius category $\mathscr{F}$ into the category of filtered objects.

3.1 Ideals in additive categories with translation

3.1.1 Lemma/Definition. Suppose given an additive category $\mathscr{A}$, a functor $T \colon \mathscr{A} \to \mathscr{A}$, a transformation $\rho \colon 1_{\mathscr{A}} \to T$ and a full additive subcategory $\mathscr{B} \subseteq \mathscr{A}$. Let $\mathfrak{J}_{\mathscr{A},T,\rho,\mathscr{B}}$ denote the set of morphisms $X \xrightarrow{f} Y$ in $\mathscr{A}$ for which there exists $X \xrightarrow{u} B \xrightarrow{v} YT$ in $\mathscr{A}$ such that $B \in \mathrm{Ob}(\mathscr{B})$ and $f \cdot Y\rho = u \cdot v$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "Y\rho"] \\ B \arrow[r, "v"] & YT \end{tikzcd}$$

The set $\mathfrak{J}_{\mathscr{A},T,\rho,\mathscr{B}}$ is an ideal in $\mathscr{A}$, cf. definition 1.2.12. $\diamondsuit$

Proof. We abbreviate $\mathfrak{J} = \mathfrak{J}_{\mathscr{A},T,\rho,\mathscr{B}}$. Suppose given $W \xrightarrow{a} X \xrightarrow{f} Y \xrightarrow{b} Z$ in $\mathscr{A}$ with $f \in \mathfrak{J}$. We want to show that $a \cdot f \cdot b \in \mathfrak{J}$. We may choose $X \xrightarrow{u} B \xrightarrow{v} YT$ in $\mathscr{A}$ such that $B \in \mathrm{Ob}(\mathscr{B})$ and $f \cdot Y\rho = u \cdot v$. We have $a \cdot f \cdot b \cdot Z\rho = a \cdot f \cdot Y\rho \cdot bT = a \cdot u \cdot v \cdot bT$ and thus $a \cdot f \cdot b \in \mathfrak{J}$.

$$\begin{tikzcd} W \arrow[r, "a"] \arrow[d, "a\cdot u"'] & X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "b"] \arrow[d, "Y\rho"] & Z \arrow[d, "Z\rho"] \\ B \arrow[r, "1"] & B \arrow[r, "v"] & YT \arrow[r, "bT"] & ZT \end{tikzcd}$$

Suppose given $X, Y \in \mathrm{Ob}(\mathscr{A})$. We want to show that ${}_{\mathscr{A}}(X,Y) \cap \mathfrak{J}$ is a subgroup of ${}_{\mathscr{A}}(X,Y)$. Choose a zero object $Z \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}}) \cap \mathrm{Ob}(\mathscr{B})$. We have $0_{X,Y} \in \mathfrak{J}$ since $0_{X,Y} \cdot Y\rho = 0_{X,Z} \cdot 0_{Z,YT}$.

$$\begin{tikzcd} X \arrow[r, "0"] \arrow[d, "0"'] & Y \arrow[d, "Y\rho"] \\ Z \arrow[r, "0"] & YT \end{tikzcd}$$

Suppose given $f, g \in {}_{\mathscr{A}}(X,Y) \cap \mathfrak{J}$. We may choose $X \xrightarrow{u} B \xrightarrow{v} YT$ in $\mathscr{A}$ such that $B \in \mathrm{Ob}(\mathscr{B})$ and $f \cdot Y\rho = u \cdot v$. We may choose $X \xrightarrow{r} C \xrightarrow{s} YT$ in $\mathscr{A}$ such that $C \in \mathrm{Ob}(\mathscr{B})$ and $g \cdot Y\rho = r \cdot s$. We may choose a direct sum $B \oplus C \in \mathrm{Ob}(\mathscr{B})$ of B and C in $\mathscr{A}$. We have

$(f-g) \cdot Y\rho = f \cdot Y\rho - g \cdot Y\rho = u \cdot v - r \cdot s = \begin{pmatrix} u & -r \end{pmatrix} \cdot \begin{pmatrix} v \\ s \end{pmatrix}$ and thus $f - g \in \mathfrak{J}$.

$$\begin{tikzcd} X \arrow[r, "f-g"] \arrow[d, "\begin{pmatrix} u & -r \end{pmatrix}"'] & Y \arrow[d, "Y\rho"] \\ B \oplus C \arrow[r, "\begin{pmatrix} v \\ s \end{pmatrix}"] & YT \end{tikzcd}$$

□

3.1.2 Lemma. Suppose given an additive category $\mathscr{A}$, an isomorphism $T\colon \mathscr{A} \to \mathscr{A}$ with inverse $T^{-1}\colon \mathscr{A} \to \mathscr{A}$, a transformation $\rho\colon 1_{\mathscr{A}} \to T$ and a full additive subcategory $\mathscr{B} \subseteq \mathscr{A}$. Suppose that we have $\rho \star T = T \star \rho$ and $BT, BT^{-1} \in \mathrm{Ob}(\mathscr{B})$ for $B \in \mathrm{Ob}(\mathscr{B})$. Suppose given $X \xrightarrow{f} Y$ in $\mathscr{A}$. We have $f \in \mathfrak{J}_{\mathscr{A},T,\rho,\mathscr{B}}$ if and only if there exists $XT^{-1} \xrightarrow{r} B \xrightarrow{s} Y$ in $\mathscr{A}$ such that $B \in \mathrm{Ob}(\mathscr{B})$ and $XT^{-1}\rho \cdot f = r \cdot s$.

$$\begin{tikzcd} XT^{-1} \arrow[r, "r"] \arrow[d, "XT^{-1}\rho"'] & B \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

◇

Proof. Note that we have $\rho \star T^{-1} = T^{-1} \star T \star \rho \star T^{-1} = T^{-1} \star \rho \star T \star T^{-1} = T^{-1} \star \rho$.

We abbreviate $\mathfrak{J} = \mathfrak{J}_{\mathscr{A},T,\rho,\mathscr{B}}$.

Suppose that $f \in \mathfrak{J}$. We may choose $X \xrightarrow{u} B \xrightarrow{v} YT$ in $\mathscr{A}$ such that $B \in \mathrm{Ob}(\mathscr{B})$ and $f \cdot Y\rho = u \cdot v$. We have $XT^{-1}\rho \cdot f = fT^{-1} \cdot YT^{-1}\rho = (f \cdot Y\rho)T^{-1} = (u \cdot v)T^{-1} = uT^{-1} \cdot vT^{-1}$.

$$\begin{tikzcd} XT^{-1} \arrow[r, "uT^{-1}"] \arrow[d, "XT^{-1}\rho"'] & BT^{-1} \arrow[d, "vT^{-1}"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Conversely, suppose given $XT^{-1} \xrightarrow{r} B \xrightarrow{s} Y$ in $\mathscr{A}$ such that $B \in \mathrm{Ob}(\mathscr{B})$ and $XT^{-1}\rho \cdot f = r \cdot s$. We have

$$f \cdot Y\rho = X\rho \cdot fT = (X\rho T^{-1} \cdot fTT^{-1})T = (XT^{-1}\rho \cdot f)T = (r \cdot s)T = rT \cdot sT.$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "rT"'] & Y \arrow[d, "Y\rho"] \\ BT \arrow[r, "sT"] & YT \end{tikzcd}$$

□

3.2 Filtrations and cofiltrations

For exact categories

Suppose given an exact category $\mathscr{A}$.

3.2.1 Definition. We consider the functor category $\mathbf{Z}(\mathscr{A})$ as an exact category equipped with the pointwise exact structure, cf. convention 17 and definition 1.4.7.
For $X \in \mathrm{Ob}(\mathbf{Z}(\mathscr{A}))$ and $k \in \mathbf{Z}$, we write $x_k = X_{k \to k+1}$.
We define the full subcategories $\mathrm{F}(\mathscr{A})$ and $\mathrm{CF}(\mathscr{A})$ of $\mathbf{Z}(\mathscr{A})$ by setting

$$\mathrm{Ob}(\mathrm{F}(\mathscr{A})) = \{X \in \mathrm{Ob}(\mathbf{Z}(\mathscr{A}))\colon x_k \text{ is a pure monomorphism in } \mathscr{A} \text{ for } k \in \mathbf{Z}\}$$

and

$$\mathrm{Ob}(\mathrm{CF}(\mathscr{A})) = \{X \in \mathrm{Ob}(\mathbf{Z}(\mathscr{A}))\colon x_k \text{ is a pure epimorphism in } \mathscr{A} \text{ for } k \in \mathbf{Z}\}.$$

We may think of objects $X \in \mathrm{Ob}(\mathrm{F}(\mathscr{A}))$ as diagrams of the following form.

$$\cdots \longleftarrow X_{k+2} \xleftarrow{x_{k+1}} X_{k+1} \xleftarrow{x_k} X_k \xleftarrow{x_{k-1}} X_{k-1} \longleftarrow \cdots$$

We call such an object $X \in \mathrm{Ob}(\mathrm{F}(\mathscr{A}))$ a *filtration* in $\mathscr{A}$ and $\mathrm{F}(\mathscr{A})$ the *category of filtrations* in $\mathscr{A}$. We may think of objects $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$ as diagrams of the following form.

$$\cdots \longleftarrow X_{k+2} \xleftarrow{x_{k+1}} X_{k+1} \xleftarrow{x_k} X_k \xleftarrow{x_{k-1}} X_{k-1} \longleftarrow \cdots$$

We call such an object $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$ a *cofiltration* in $\mathscr{A}$ and $\mathrm{CF}(\mathscr{A})$ the *category of cofiltrations* in $\mathscr{A}$. The full subcategories $\mathrm{F}(\mathscr{A})$ and $\mathrm{CF}(\mathscr{A})$ are extension-closed in $\mathbf{Z}(\mathscr{A})$, cf. definition 1.3.23, lemma 1.4.8 and remark 1.4.10.

We equip $\mathrm{F}(\mathscr{A})$ and $\mathrm{CF}(\mathscr{A})$ with the restricted exact structures of the pointwise exact structure on $\mathbf{Z}(\mathscr{A})$, cf. definition 1.3.21 and 1.3.25.

A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathrm{F}(\mathscr{A})$ is a pure short exact sequence if and only if $X_k \xrightarrow{f_k} Y_k \xrightarrow{g_k} Z_k$ is a pure short exact sequence in $\mathscr{A}$ for $k \in \mathbf{Z}$.

A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathrm{CF}(\mathscr{A})$ is a pure short exact sequence if and only if $X_k \xrightarrow{f_k} Y_k \xrightarrow{g_k} Z_k$ is a pure short exact sequence in $\mathscr{A}$ for $k \in \mathbf{Z}$. $\Diamond$

3.2.2 Definition. Suppose given $m \in \mathbf{Z}$. We define the full subcategories $\mathrm{CF}^{\lceil m}(\mathscr{A})$ and $\mathrm{CF}^{m\rceil}(\mathscr{A})$ of $\mathrm{CF}(\mathscr{A})$ by setting

$$\mathrm{Ob}(\mathrm{CF}^{\lceil m}(\mathscr{A})) = \{X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))\colon X_k \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}}) \text{ for } k \in \mathbf{Z}_{>m}\}$$

and

$$\mathrm{Ob}(\mathrm{CF}^{m\rceil}(\mathscr{A})) = \{X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))\colon x_k \text{ is an isomorphism in } \mathscr{A} \text{ for } k \in \mathbf{Z}_{<m}\}.$$

The full subcategories $\mathrm{CF}^{\lceil m}(\mathscr{A})$ and $\mathrm{CF}^{m\rceil}(\mathscr{A})$ are extension-closed in $\mathrm{CF}(\mathscr{A})$, cf. definition 1.3.23, lemma 1.4.8 and remark 1.4.10.

We equip $\mathrm{CF}^{\lceil m}(\mathscr{A})$ and $\mathrm{CF}^{m\rceil}(\mathscr{A})$ with the restricted exact structures of the pointwise exact structure on $\mathrm{CF}(\mathscr{A})$, cf. definition 1.3.21 and 1.3.25. $\diamondsuit$

3.2.3 Definition. Suppose given $m \in \mathbf{Z}$. We define the full subcategories $\mathrm{F}^{m\rceil}(\mathscr{A})$ and $\mathrm{F}^{\lceil m}(\mathscr{A})$ of $\mathrm{F}(\mathscr{A})$ by setting

$$\mathrm{Ob}(\mathrm{F}^{m\rceil}(\mathscr{A})) = \{X \in \mathrm{Ob}(\mathrm{F}(\mathscr{A}))\colon X_k \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}}) \text{ for } k \in \mathbf{Z}_{<m}\}$$

and

$$\mathrm{Ob}(\mathrm{F}^{\lceil m}(\mathscr{A})) = \{X \in \mathrm{Ob}(\mathrm{F}(\mathscr{A}))\colon x_k \text{ is an isomorphism in } \mathscr{A} \text{ for } k \in \mathbf{Z}_{\geq m}\}.$$

The full subcategories $\mathrm{F}^{\lceil m}(\mathscr{A})$ and $\mathrm{F}^{m\rceil}(\mathscr{A})$ are extension-closed in $\mathrm{F}(\mathscr{A})$, cf. definition 1.3.23, lemma 1.4.8 and remark 1.4.10.

We equip $\mathrm{F}^{\lceil m}(\mathscr{A})$ and $\mathrm{F}^{m\rceil}(\mathscr{A})$ with the restricted exact structures of the pointwise exact structure on $\mathrm{F}(\mathscr{A})$, cf. definition 1.3.21 and 1.3.25. $\diamondsuit$

3.2.4 Definition. Suppose given $m, n \in \mathbf{Z}$. Let $\mathrm{CF}^{\lceil m,n\rceil}(\mathscr{A}) = \mathrm{CF}^{\lceil m}(\mathscr{A}) \cap \mathrm{CF}^{n\rceil}(\mathscr{A})$ and let $\mathrm{F}^{\lceil m,n\rceil}(\mathscr{A}) = \mathrm{F}^{\lceil m}(\mathscr{A}) \cap \mathrm{F}^{n\rceil}(\mathscr{A})$. $\diamondsuit$

3.2.5 Definition. Let $\mathrm{CF}^{\mathrm{b}}(\mathscr{A}) = \bigcup\limits_{m,n\in\mathbf{Z}} \mathrm{CF}^{\lceil m,n\rceil}(\mathscr{A})$ and let $\mathrm{F}^{\mathrm{b}}(\mathscr{A}) = \bigcup\limits_{m,n\in\mathbf{Z}} \mathrm{F}^{\lceil m,n\rceil}(\mathscr{A})$. $\diamondsuit$

3.2.6 Definition. For $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$ and $k \in \mathbf{Z}$, we choose kernels $X_{\dot{k}} \xrightarrow{x_{\dot{k}}} X_k$ of x_k .

For $X \xrightarrow{f} Y$ in $\mathrm{CF}(\mathscr{A})$ and $k \in \mathbf{Z}$, let $X_{\dot{k}} \xrightarrow{f_{\dot{k}}} Y_{\dot{k}}$ denote the unique morphism in $\mathscr{A}$ such that $f_{\dot{k}} \cdot y_{\dot{k}} = x_{\dot{k}} \cdot f_k$. Note that for a pure short exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathrm{CF}(\mathscr{A})$, the sequence $X_{\dot{k}} \xrightarrow{f_{\dot{k}}} Y_{\dot{k}} \xrightarrow{g_{\dot{k}}} Z_{\dot{k}}$ is pure short exact in $\mathscr{A}$ for $k \in \mathbf{Z}$ by lemma 1.3.16. $\diamondsuit$

3.2.7 Definition. For $X \in \mathrm{Ob}(\mathrm{F}(\mathscr{A}))$ and $k \in \mathbf{Z}$, we choose cokernels $X_k \xrightarrow{x_{\overline{k}}} X_{\overline{k}}$ of x_{k-1} . For $X \xrightarrow{f} Y$ in $\mathrm{F}(\mathscr{A})$ and $k \in \mathbf{Z}$, let $X_{\overline{k}} \xrightarrow{f_{\overline{k}}} Y_{\overline{k}}$ denote the unique morphism in $\mathscr{A}$ such that $x_{\overline{k}} \cdot f_{\overline{k}} = f_k \cdot y_{\overline{k}}$. Note that for a pure short exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathrm{F}(\mathscr{A})$, the sequence $X_{\overline{k}} \xrightarrow{f_{\overline{k}}} Y_{\overline{k}} \xrightarrow{g_{\overline{k}}} Z_{\overline{k}}$ is pure short exact in $\mathscr{A}$ for $k \in \mathbf{Z}$ by lemma 1.3.15. $\diamondsuit$

3.2.8 Definition. Suppose given $m \in \mathbf{Z}$. We define the functor $\Gamma_{\lceil m}\colon \mathrm{CF}(\mathscr{A}) \to \mathrm{CF}^{\lceil m}(\mathscr{A})$ as follows.

Suppose given $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$. For $k \in \mathbf{Z}_{\leq m}$, let $(X\Gamma_{\lceil m})_k = X_k$ and $(X\Gamma_{\lceil m})_{k-1 \to k} = x_{k-1}$.

For $k \in \mathbf{Z}_{>m}$, let $(X\Gamma_{\lceil m})_k = 0_{\mathscr{A}}$.
Suppose given $f \in \mathrm{Mor}(\mathrm{CF}(\mathscr{A}))$. For $k \in \mathbf{Z}_{\leq m}$, let $(f\Gamma_{\lceil m})_k = f_k$.
This in fact defines an exact functor. $\diamond$

Proof. We abbreviate $\Gamma = \Gamma_{\lceil m}$. Suppose given $X \overset{f}{\underset{h}{\rightrightarrows}} Y \xrightarrow{g} Z$ in $\mathrm{CF}(\mathscr{A})$.
We have $1_X\Gamma = 1_{X\Gamma}$ since $(1_X\Gamma)_k = 1_{X_k} = (1_{X\Gamma})_k$ for $k \in \mathbf{Z}_{\leq m}$.
We have $(f \cdot g)\Gamma = f\Gamma \cdot g\Gamma$ since since $((f \cdot g)\Gamma)_k = f_k \cdot g_k = (f\Gamma \cdot g\Gamma)_k$ for $k \in \mathbf{Z}_{\leq m}$.
We have $(f + h)\Gamma = f\Gamma + h\Gamma$ since since $((f + h)\Gamma)_k = f_k + h_k = (f\Gamma + h\Gamma)_k$ for $k \in \mathbf{Z}_{\leq m}$.
If (f, g) is a pure short exact sequence in $\mathrm{CF}(\mathscr{A})$, then (f_k , g_k) is a pure short exact sequence in $\mathscr{A}$ for $k \in \mathbf{Z}$. Consequently, $(f\Gamma, g\Gamma)$ is a pure short exact sequence in $\mathrm{CF}^{\lceil m}(\mathscr{A})$. $\square$

3.2.9 Definition. Suppose given $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$. We define the cofiltration $X_{[1]} \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$ by setting $(X_{[1]})_k = X_{k+1}$ and $(X_{[1]})_{k \to k+1} = x_{k+1}$ for $k \in \mathbf{Z}$. $\diamond$

3.2.10 Definition. Suppose given $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$. Suppose given morphisms $A \xrightarrow{a_k} X_k$ in $\mathscr{A}$ for $k \in \mathbf{Z}$. We say that the pair $(A, (a_k)_{k\in\mathbf{Z}})$ is a *compatible family* for X if we have $a_{k-1} \cdot x_{k-1} = a_k$ for $k \in \mathbf{Z}$. $\diamond$

3.2.11 Definition. Suppose given $X \in \mathrm{Ob}(\mathrm{F}(\mathscr{A}))$. Suppose given morphisms $X_k \xrightarrow{a_k} A$ in $\mathscr{A}$ for $k \in \mathbf{Z}$. We say that the pair $(A, (a_k)_{k\in\mathbf{Z}})$ is a *compatible family* for X if we have $x_{k-1} \cdot a_k = a_{k-1}$ for $k \in \mathbf{Z}$. $\diamond$

3.2.12 Definition. Suppose given $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$. A compatible family $(A, (a_k)_{k\in\mathbf{Z}})$ for X is called a *limit* for X if for every compatible family $(B, (f_k)_{k\in\mathbf{Z}})$ for X, there exists a unique morphism $B \xrightarrow{u} A$ in $\mathscr{A}$ such that $u \cdot a_k = f_k$ for $k \in \mathbf{Z}$. $\diamond$

3.2.13 Definition. Suppose given $X \in \mathrm{Ob}(\mathrm{F}(\mathscr{A}))$. A compatible family $(A, (a_k)_{k\in\mathbf{Z}})$ for X is called an *colimit* for X if for every compatible family $(B, (f_k)_{k\in\mathbf{Z}})$ for X, there exists a unique morphism $A \xrightarrow{u} B$ in $\mathscr{A}$ such that $a_k \cdot u = f_k$ for $k \in \mathbf{Z}$. $\diamond$

3.2.14 Lemma (cofinality of limits)**.** Suppose given $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$.

(a) Suppose given a limit $(A, (a_k)_{k\in\mathbf{Z}})$ for X. Suppose given $m \in \mathbf{Z}$ and $T \xrightarrow{t_k} X_k$ in $\mathscr{A}$ such that $t_{k-1} \cdot x_{k-1} = t_k$ for $k \in \mathbf{Z}_{\leq m}$. Then there exists a unique morphism $T \xrightarrow{t} A$ in $\mathscr{A}$ such that $t \cdot a_k = t_k$ for $k \in \mathbf{Z}_{\leq m}$.

(b) Suppose given $m \in \mathbf{Z}$ and $A \xrightarrow{a_k} X_k$ such that $a_{k-1} \cdot x_{k-1} = a_k$ for $k \in \mathbf{Z}_{\leq m}$.
Suppose that for $T \xrightarrow{t_k} X_k$ in $\mathscr{A}$ with $t_{k-1} \cdot x_{k-1} = t_k$ for $k \in \mathbf{Z}_{\leq m}$, there exists a unique morphism $T \xrightarrow{t} A$ in $\mathscr{A}$ such that $t \cdot a_k = t_k$ for $k \in \mathbf{Z}_{\leq m}$.
Let $b_k = a_k$ for $k \in \mathbf{Z}_{\leq m}$ and $b_k = a_m \cdot X_{m \to k}$ for $k \in \mathbf{Z}_{>m}$. Then $(A, (b_k)_{k\in\mathbf{Z}})$ is a limit for X.

$\diamond$

Proof. Ad (a). Let $u_k = t_k$ for $k \in \mathbf{Z}_{\leq m}$ and let $u_k = t_m \cdot X_{m \to k}$ for $k \in \mathbf{Z}_{>m}$.

For $k \in \mathbf{Z}_{\leq m}$, we have $u_{k-1} \cdot x_{k-1} = t_{k-1} \cdot x_{k-1} = t_k = u_k$.

For $k \in \mathbf{Z}_{>m}$, we have $u_{k-1} \cdot x_{k-1} = t_m \cdot X_{m \to k-1} \cdot X_{k-1 \to k} = t_m \cdot X_{m \to k} = u_k$.

Since $(A, (a_k)_{k \in \mathbf{Z}})$ is a limit for X, there exists a unique morphism $T \xrightarrow{t} A$ in $\mathscr{A}$ such that $t \cdot a_k = u_k$ for $k \in \mathbf{Z}$.

In particular, we have $t \cdot a_k = u_k = t_k$ for $k \in \mathbf{Z}_{\leq m}$.

Suppose given $T \xrightarrow{s} A$ in $\mathscr{A}$ such that $s \cdot a_k = t_k = u_k$ for $k \in \mathbf{Z}_{\leq m}$. Then we also have $s \cdot a_k = s \cdot a_m \cdot X_{m \to k} = t_m \cdot X_{m \to k} = u_k$ for $k \in \mathbf{Z}_{>m}$. Thus $s = t$.

Ad (b). For $k \in \mathbf{Z}_{\leq m}$, we have $b_{k-1} \cdot x_{k-1} = a_{k-1} \cdot x_{k-1} = a_k = b_k$.

For $k \in \mathbf{Z}_{>m}$, we have $b_{k-1} \cdot x_{k-1} = a_m \cdot X_{m \to k-1} \cdot X_{k-1 \to k} = a_m \cdot X_{m \to k} = b_k$.

Suppose given $T \xrightarrow{t_k} X_k$ in $\mathscr{A}$ such that $t_{k-1} \cdot x_{k-1} = t_k$ for $k \in \mathbf{Z}$. By assumption, there exists a unique morphism $T \xrightarrow{t} A$ in $\mathscr{A}$ such that $t \cdot a_k = t_k$ for $k \in \mathbf{Z}_{\leq m}$. The result now follows since we also have $t \cdot b_k = t \cdot a_m \cdot X_{m \to k} = t_m \cdot X_{m \to k} = t_k$ for $k \in \mathbf{Z}_{>m}$. □

3.2.15 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$ and $m \in \mathbf{Z}$.

(a) Suppose given a limit $(A, (a_k)_{k \in \mathbf{Z}})$ for X. Let $\tilde{a}_k = a_k \colon A \to X_k$ for $k \in \mathbf{Z}_{\leq m}$ and $\tilde{a}_k = 0 \colon A \to 0_{\mathscr{A}}$ for $k \in \mathbf{Z}_{>m}$. Then $(A, (\tilde{a}_k)_{k \in \mathbf{Z}})$ is a limit for $X\Gamma_{\lceil m}$.

(b) Suppose given a limit $(A, (\tilde{a}_k)_{k \in \mathbf{Z}})$ for $X\Gamma_{\lceil m}$. Let $a_k = \tilde{a}_k \colon A \to X_k$ for $k \in \mathbf{Z}_{\leq m}$ and $a_k = \tilde{a}_m \cdot X_{m \to k}$ for $k \in \mathbf{Z}_{>m}$. Then $(A, (a_k)_{k \in \mathbf{Z}})$ is a limit for X.

◇

Proof. This follows from lemma 3.2.14. □

3.2.16 Remark. Suppose given an isomorphism $X \xrightarrow{f} Y$ in $\mathrm{CF}(\mathscr{A})$ and a limit $(A, (a_k)_{k \in \mathbf{Z}})$ for X. Then $(A, (a_k \cdot f_k)_{k \in \mathbf{Z}})$ is a limit for Y. ◇

Proof. We write $g = f^{-1} \colon Y \to X$.

Suppose given $T \xrightarrow{t_k} Y_k$ in $\mathscr{A}$ such that $t_k \cdot y_k = t_{k+1}$ for $k \in \mathbf{Z}$.

For $k \in \mathbf{Z}$, we have $t_k \cdot g_k \cdot x_k = t_{k+1} \cdot g_{k+1}$ since

$$t_k \cdot g_k \cdot x_k \cdot f_{k+1} = t_k \cdot g_k \cdot f_k \cdot y_k = t_k \cdot y_k = t_{k+1} = t_{k+1} \cdot g_{k+1} \cdot f_{k+1}$$

and since f_{k+1} is an isomorphism. Since $(A, (a_k)_{k \in \mathbf{Z}})$ is a limit for X, there exists a unique morphism $T \xrightarrow{t} A$ in $\mathscr{A}$ such that $t \cdot a_k = t_k \cdot g_k$ for $k \in \mathbf{Z}$.

For $k \in \mathbf{Z}$, we have $t \cdot a_k \cdot f_k = t_k \cdot g_k \cdot f_k = t_k$.

Suppose given $T \xrightarrow{u} A$ in $\mathscr{A}$ such that $u \cdot a_k \cdot f_k = t_k$ for $k \in \mathbf{Z}$.
We have $u \cdot a_k = u \cdot a_k \cdot f_k \cdot g_k = t_k \cdot g_k$ for $k \in \mathbf{Z}$. Thus $u = t$. □

3.2.17 Lemma. Suppose given $m \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$. Suppose given a compatible family $(A, (a_k)_{k \in \mathbf{Z}})$ for X such that a_k is an isomorphism in $\mathscr{A}$ for $k \in \mathbf{Z}_{\leq m}$. Then $(A, (a_k)_{k \in \mathbf{Z}})$ is a limit for X. ◇

Proof. Note we have the isomorphisms $X_{k \to m} = a_k^{-1} \cdot a_m$ for $k \in \mathbf{Z}_{\leq m}$ since

$$a_k \cdot X_{k \to m} \cdot a_m^{-1} = a_m \cdot a_m^{-1} = 1 = a_k \cdot a_k^{-1} \cdot a_m \cdot a_m^{-1}$$

and since a_k and a_m are isomorphisms.

We use lemma 3.2.14.(b). Suppose given $T \xrightarrow{t_k} X_k$ in $\mathscr{A}$ with $t_{k-1} \cdot x_{k-1} = t_k$ for $k \in \mathbf{Z}_{\leq m}$. Let $t = t_m \cdot a_m^{-1}$. For $k \in \mathbf{Z}_{\leq m}$, we have $t \cdot a_k = t_k$ since $t \cdot a_k \cdot X_{k \to m} = t_m \cdot a_m^{-1} \cdot a_m = t_m = t_k \cdot X_{k \to m}$ and since $X_{k \to m}$ is an isomorphism.

Suppose given $T \xrightarrow{s} A$ in $\mathscr{A}$ such that $s \cdot a_k = t_k$ for $k \in \mathbf{Z}_{\leq m}$.
Then $s = t$ since $s \cdot a_m = t_m = t_m \cdot a_m^{-1} \cdot a_m = t \cdot a_m$ and since a_m is an isomorphism. □

3.2.18 Lemma. Suppose given $m \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{CF}^{m\rceil}(\mathscr{A}))$. Let $a_k = X_{m \to k}$ for $k \in \mathbf{Z}_{\geq m}$ and $a_k = \big(X_{k \to m}\big)^{-1}$ for $k \in \mathbf{Z}_{<m}$. Then $(X_m\,,\,(a_k)_{k \in \mathbf{Z}})$ is a limit for X. ◇

Proof. For $k \in \mathbf{Z}_{\geq m}$, we have $a_k \cdot x_k = X_{m \to k} \cdot X_{k \to k+1} = X_{m \to k+1} = a_{k+1}$.
For $k \in \mathbf{Z}_{<m}$, we have $a_k \cdot x_k = a_{k+1}$ since
$a_k \cdot x_k \cdot X_{k+1 \to m} = (X_{k \to m})^{-1} \cdot X_{k \to k+1} \cdot X_{k+1 \to m} = 1 = (X_{k+1 \to m})^{-1} \cdot X_{k+1 \to m} = a_{k+1} \cdot X_{k+1 \to m}$ and since $X_{k+1 \to m}$ is an isomorphism. The result now follows from lemma 3.2.17. □

3.2.19 Lemma. Suppose given $m \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{F}^{\lceil m}(\mathscr{A}))$. Let $a_k = X_{k \to m}$ for $k \in \mathbf{Z}_{\leq m}$ and $a_k = \big(X_{m \to k}\big)^{-1}$ for $k \in \mathbf{Z}_{>m}$. Then $(X_m\,,\,(a_k)_{k \in \mathbf{Z}})$ is a colimit for X. ◇

Proof. This is dual to the previous lemma 3.2.18. □

3.2.20 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$.

(a) Suppose given a compatible family $(A, (a_k)_{k \in \mathbf{Z}})$ for X.
Then $(A, (a_{k+1})_{k \in \mathbf{Z}})$ is a compatible family for $X_{[1]}$.

(b) Suppose given a compatible family $(A, (a_{k+1})_{k \in \mathbf{Z}})$ for $X_{[1]}$.
Then $(A, (a_k)_{k \in \mathbf{Z}})$ is a compatible family for X.

◇

Proof. Suppose given morphisms $A \xrightarrow{a_k} X_k$ in $\mathscr{A}$ for $k \in \mathbf{Z}$. For $k \in \mathbf{Z}$, we have $a_k \cdot x_k = a_{k+1}$ if and only $a_k \cdot (X_{[1]})_{k-1 \to k} = a_{k+1}$ for $k \in \mathbf{Z}$. □

3.2.21 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$.

(a) Suppose given a limit $(A, (a_k)_{k \in \mathbf{Z}})$ for X. Then $(A, (a_{k+1})_{k \in \mathbf{Z}})$ is a limit for $X_{[1]}$.

(b) Suppose given a limit $(A, (a_{k+1})_{k \in \mathbf{Z}})$ for $X_{[1]}$. Then $(A, (a_k)_{k \in \mathbf{Z}})$ is a limit for X.

◇

Proof. This follows from lemma 3.2.20. □

3.2.22 Lemma/Definition. Suppose given $X \xrightarrow{f} Y$ in $\mathrm{CF}(\mathscr{A})$. Suppose given a compatible family $(A, (a_k)_{k\in\mathbf{Z}})$ for X and a limit $(B, (b_k)_{k\in\mathbf{Z}})$ for Y. There exists a unique morphism $f\rceil^{(B,(b_k)_{k\in\mathbf{Z}})}_{(A,(a_k)_{k\in\mathbf{Z}})} \colon A \to B$ in $\mathscr{A}$ such that $a_k \cdot f_k = f\rceil^{(B,(b_k)_{k\in\mathbf{Z}})}_{(A,(a_k)_{k\in\mathbf{Z}})} \cdot b_k$ for $k \in \mathbf{Z}$.
We abbreviate $f\rceil = f\rceil^{(B,(b_k)_{k\in\mathbf{Z}})}_{(A,(a_k)_{k\in\mathbf{Z}})} \colon A \to B$ if unambiguous. $\diamondsuit$

Proof. For $k \in \mathbf{Z}$, we have $a_k \cdot f_k \cdot y_k = a_k \cdot x_k \cdot f_{k+1} = a_{k+1} \cdot f_{k+1}$. Since $(B, (b_k)_{k\in\mathbf{Z}})$ is a limit for Y, there exists a unique morphism $f\rceil^{(B,(b_k)_{k\in\mathbf{Z}})}_{(A,(a_k)_{k\in\mathbf{Z}})} \colon A \to B$ in $\mathscr{A}$ such that $a_k \cdot f_k = f\rceil^{(B,(b_k)_{k\in\mathbf{Z}})}_{(A,(a_k)_{k\in\mathbf{Z}})} \cdot b_k$ for $k \in \mathbf{Z}$. $\square$

3.2.23 Lemma/Definition. Suppose given $X \xrightarrow{f} Y$ in $\mathrm{F}(\mathscr{A})$. Suppose given a colimit $(A, (a_k)_{k\in\mathbf{Z}})$ for X and a compatible family $(B, (b_k)_{k\in\mathbf{Z}})$ for Y. There exists a unique morphism $f\rceil^{(B,(b_k)_{k\in\mathbf{Z}})}_{(A,(a_k)_{k\in\mathbf{Z}})} \colon A \to B$ in $\mathscr{A}$ such that $a_k \cdot f\rceil^{(B,(b_k)_{k\in\mathbf{Z}})}_{(A,(a_k)_{k\in\mathbf{Z}})} = f_k \cdot b_k$ for $k \in \mathbf{Z}$.
We abbreviate $f\rceil = f\rceil^{(B,(b_k)_{k\in\mathbf{Z}})}_{(A,(a_k)_{k\in\mathbf{Z}})} \colon A \to B$ if unambiguous. $\diamondsuit$

Proof. This is dual to the previous lemma 3.2.22. $\square$

3.2.24 Lemma. Suppose given $X \underset{h}{\overset{f}{\rightrightarrows}} Y \xrightarrow{g} Z$ in $\mathrm{CF}(\mathscr{A})$.

(a) Suppose given a limit $(A, (a_k)_{k\in\mathbf{Z}})$ for X. We have $1_X\rceil = 1_A$.

(b) Suppose given a compatible family $(A, (a_k)_{k\in\mathbf{Z}})$ for X. Suppose given a limit $(B, (b_k)_{k\in\mathbf{Z}})$ for Y and a limit $(C, (c_k)_{k\in\mathbf{Z}})$ for Z. We have $(f \cdot g)\rceil = f\rceil \cdot g\rceil$.

(c) Suppose given a compatible family $(A, (a_k)_{k\in\mathbf{Z}})$ for X and a limit $(B, (b_k)_{k\in\mathbf{Z}})$ for Y. We have $(f + h)\rceil = f\rceil + h\rceil$.

(d) Suppose given a compatible family $(A, (a_k)_{k\in\mathbf{Z}})$ for X and a limit $(B, (b_k)_{k\in\mathbf{Z}})$ for Y. If $f = 0$ in $\mathrm{CF}(\mathscr{A})$, then $f\rceil = 0$ in $\mathscr{A}$.

$\diamondsuit$

Proof. Ad (a). For $k \in \mathbf{Z}$, we have $a_k \cdot 1_A = a_k = (1_X)_k \cdot a_k$.

Ad (b). For $k \in \mathbf{Z}$, we have $a_k \cdot f\rceil \cdot g\rceil = f_k \cdot b_k \cdot g\rceil = f_k \cdot g_k \cdot c_k = (f \cdot g)_k \cdot c_k$.

Ad (c). For $k \in \mathbf{Z}$, we have
$a_k \cdot (f\rceil + h\rceil) = a_k \cdot f\rceil + a_k \cdot h\rceil = f_k \cdot b_k + h_k \cdot b_k = (f_k + h_k) \cdot b_k = (f + h)_k \cdot b_k$.

Ad (d). For $k \in \mathbf{Z}$, we have $a_k \cdot 0 = 0 = f_k \cdot b_k$. $\square$

3.2.25 Remark. Suppose given $X \xrightarrow{f} Y$ in $\mathrm{CF}(\mathscr{A})$. Suppose given a limit $(A, (a_k)_{k\in\mathbf{Z}})$ for X and a limit $(B, (b_k)_{k\in\mathbf{Z}})$ for Y. If f is an isomorphism in $\mathrm{CF}(\mathscr{A})$, then $f\rceil$ is an isomorphism in $\mathscr{A}$. $\diamondsuit$

Proof. This follows from lemma 3.2.24.(a,b). $\square$

3.2.26 Remark. Suppose given $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$ and a limit $(A, (a_k)_{k\in\mathbf{Z}})$ for X.
If $X \in \mathrm{Ob}\big(\mathrm{Z}_{\mathrm{CF}(\mathscr{A})}\big)$, then $A \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}})$. $\diamondsuit$

Proof. This follows from lemma 3.2.18 and remark 3.2.25. $\square$

3.2.27 Lemma. Suppose given $X \xrightarrow{f} Y$ in $\mathrm{CF}(\mathscr{A})$ and $n \in \mathbf{Z}$. Suppose given a limit $(A, (a_k)_{k\in\mathbf{Z}})$ for X. Let $\tilde{a}_k = a_k \colon A \to X_k$ for $k \in \mathbf{Z}_{\leq n}$ and $\tilde{a}_k = 0 \colon A \to 0_{\mathscr{A}}$ for $k \in \mathbf{Z}_{>n}$. Then $(A, (\tilde{a}_k)_{k\in\mathbf{Z}})$ is a limit for $X\Gamma_{\lceil n}$ by lemma 3.2.15. Suppose given a limit $(B, (b_k)_{k\in\mathbf{Z}})$ for Y. Let $\tilde{b}_k = b_k \colon B \to Y_k$ for $k \in \mathbf{Z}_{\leq n}$ and $\tilde{b}_k = 0 \colon B \to 0_{\mathscr{A}}$ for $k \in \mathbf{Z}_{>n}$. Then $(B, (\tilde{b}_k)_{k\in\mathbf{Z}})$ is a limit for $Y\Gamma_{\lceil n}$ by lemma 3.2.15. We have $f\rceil = f\Gamma_{\lceil n}\rceil$. $\diamondsuit$

Proof. For $k \in \mathbf{Z}_{>n}$, we have $\tilde{a}_k \cdot f_k = 0 = f\rceil \cdot \tilde{b}_k$.
For $k \in \mathbf{Z}_{\leq n}$, we have $\tilde{a}_k \cdot f_k = a_k \cdot f_k = f\rceil \cdot b_k = f\rceil \cdot \tilde{b}_k$. Thus $f\rceil = f\Gamma_{\lceil n}\rceil$. $\square$

3.2.28 Lemma/Definition. Suppose given an object $P \in \mathrm{Ob}(\mathscr{A})$.
We define the cofiltration $\overline{P} \in \mathrm{Ob}(\mathrm{CF}^{\lceil 0}(\mathscr{A}))$ by setting $\overline{P}_k = 0_{\mathscr{A}}$ for $k \in \mathbf{Z}_{>0}$ and $\overline{P}_k = P$, $\overline{p}_{k-1} = 1$ for $k \in \mathbf{Z}_{\leq 0}$.
We obtain a limit $(P, (p_k)_{k\in\mathbf{Z}})$ for $\overline{P}$ by setting $p_k = 1$ for $k \in \mathbf{Z}_{\leq 0}$, cf. lemma 3.2.18.
For a cofiltration $X \in \mathrm{Ob}(\mathrm{CF}^{\lceil 0}(\mathscr{A}))$ and a limit $(A, (a_k)_{k\in\mathbf{Z}})$ for X, we have the following isomorphism of abelian groups.

$$\Lambda_{P,X}^{(A,(a_k)_{k\in\mathbf{Z}})} \colon {}_{\mathrm{CF}(\mathscr{A})}(\overline{P}, X) \to {}_{\mathscr{A}}(P, A) \colon g \mapsto g\rceil_{(P,(p_k)_{k\in\mathbf{Z}})}^{(A,(a_k)_{k\in\mathbf{Z}})} ,$$

cf. definition 3.2.22.
For a morphism $X \xrightarrow{f} Y$ in $\mathrm{CF}(\mathscr{A})$, a limit $(A, (a_k)_{k\in\mathbf{Z}})$ for X and a limit $(B, (b_k)_{k\in\mathbf{Z}})$ for Y, we have $\Lambda_{P,X}^{(A,(a_k)_{k\in\mathbf{Z}})} \cdot {}_{\mathscr{A}}(P, f\rceil) = {}_{\mathrm{CF}(\mathscr{A})}(\overline{P}, f) \cdot \Lambda_{P,Y}^{(B,(b_k)_{k\in\mathbf{Z}})}$.

$$\begin{tikzcd}
{}_{\mathrm{CF}(\mathscr{A})}(\overline{P}, X) \arrow[r, "\Lambda_{P,X}^{(A,(a_k)_{k\in\mathbf{Z}})}"] \arrow[d, "{}_{\mathrm{CF}(\mathscr{A})}(\overline{P},f)"'] & {}_{\mathscr{A}}(P, A) \arrow[d, "{}_{\mathscr{A}}(P,f\rceil)"] \\
{}_{\mathrm{CF}(\mathscr{A})}(\overline{P}, Y) \arrow[r, "\Lambda_{P,Y}^{(B,(b_k)_{k\in\mathbf{Z}})}"] & {}_{\mathscr{A}}(P, B)
\end{tikzcd}$$
$\diamondsuit$

Proof. Suppose given $X \in \mathrm{Ob}(\mathrm{CF}^{\lceil 0}(\mathscr{A}))$ and a limit $(A, (a_k)_{k\in\mathbf{Z}})$ for X.
For $g, h \in {}_{\mathrm{CF}(\mathscr{A})}(\overline{P}, X)$, we have
$(g + h)\Lambda_{P,X}^{(A,(a_k)_{k\in\mathbf{Z}})} = (g + h)\rceil = g\rceil + h\rceil = g\Lambda_{P,X}^{(A,(a_k)_{k\in\mathbf{Z}})} + h\Lambda_{P,X}^{(A,(a_k)_{k\in\mathbf{Z}})}$ by lemma 3.2.24.(c).
Suppose given $g, h \in {}_{\mathrm{CF}(\mathscr{A})}(\overline{P}, X)$ such that $g\Lambda_{P,X}^{(A,(a_k)_{k\in\mathbf{Z}})} = h\Lambda_{P,X}^{(A,(a_k)_{k\in\mathbf{Z}})}$, i.e. $g\rceil = h\rceil$.
For $k \in \mathbf{Z}_{\leq 0}$, we have $h_k = 1 \cdot h_k = h\rceil \cdot a_k = g\rceil \cdot a_k = 1 \cdot g_k = g_k$. Thus $g = h$.
Suppose given $h \in {}_{\mathscr{A}}(P, A)$. Let $g_k = h \cdot a_k$ for $k \in \mathbf{Z}_{\leq 0}$.
We have $g_{k-1} \cdot x_{k-1} = h \cdot a_{k-1} \cdot x_{k-1} = h \cdot a_k = g_k = 1 \cdot g_k$ for $k \in \mathbf{Z}_{\leq 0}$. So we obtain a morphism $g \in {}_{\mathrm{CF}(\mathscr{A})}(\overline{P}, X)$ with $h = g\Lambda_{P,X}^{(A,(a_k)_{k\in\mathbf{Z}})} = g\rceil$ since $h \cdot a_k = 1 \cdot g_k$ for $k \in \mathbf{Z}_{\leq 0}$.
We conclude that $\Lambda_{P,X}^{(A,(a_k)_{k\in\mathbf{Z}})}$ is an isomorphism of abelian groups.

Suppose given a morphism $X \xrightarrow{f} Y$ in $\mathrm{CF}(\mathscr{A})$, a limit $(A, (a_k)_{k\in\mathbf{Z}})$ for X and a limit $(B, (b_k)_{k\in\mathbf{Z}})$ for Y.

Suppose given $g \in {}_{\mathrm{CF}(\mathscr{A})}(\overline{P}, X)$. We have

$$g\Lambda_{P,X}^{(A,(a_k)_{k\in\mathbf{Z}})}{}_{\mathscr{A}}(P, f\rceil) = g\rceil \cdot f\rceil = (gf)\rceil = g_{\mathrm{CF}(\mathscr{A})}(\overline{P}, f)\Lambda_{P,Y}^{(B,(b_k)_{k\in\mathbf{Z}})}$$

by lemma 3.2.24.(b). □

3.2.29 Lemma. Suppose given a projective object $P \in \mathrm{Ob}(\mathscr{A})$. Then $\overline{P}$ is projective in $\mathrm{CF}^{\lceil 0}(\mathscr{A})$. ♢

Proof. Suppose given a pure epimorphism $X \xrightarrow{f} \overline{P}$ in $\mathrm{CF}^{\lceil 0}(\mathscr{A})$. Since f_0 is a pure epimorphism and P is projective in $\mathscr{A}$, we may choose $P \xrightarrow{g_0} X_0$ in $\mathscr{A}$ such that $g_0 \cdot f_0 = 1$. For $k \in \mathbf{Z}_{<0}$, we may recursively choose $P \xrightarrow{g_k} X_k$ in $\mathscr{A}$ such that $g_k \cdot x_k = g_{k+1}$. So we obtain a morphism $\overline{P} \xrightarrow{g} X$ in $\mathrm{CF}^{\lceil 0}(\mathscr{A})$. We have $g \cdot f = 1$ since $g_0 \cdot f_0 = 1$ and since we have $g_k \cdot f_k = g_k \cdot f_k \cdot 1 = g_k \cdot x_k \cdot f_{k+1} = g_{k+1} \cdot f_{k+1} = 1$ for $k \in \mathbf{Z}_{<0}$ by induction. We conclude that $\overline{P}$ is projective in $\mathrm{CF}^{\lceil 0}(\mathscr{A})$. □

3.2.30 Lemma (exactness of limits)**.** Suppose that $\mathscr{A}$ has enough projectives. Suppose given a pure short exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathrm{CF}(\mathscr{A})$. Suppose given a limit $(A, (a_k)_{k\in\mathbf{Z}})$ for X, a limit $(B, (b_k)_{k\in\mathbf{Z}})$ for Y and a limit $(C, (c_k)_{k\in\mathbf{Z}})$ for Z. Then the sequence $A \xrightarrow{f\rceil} B \xrightarrow{g\rceil} C$ is pure short exact in $\mathscr{A}$, cf. definition 3.2.22.

Cf. [Kel90, section 5]. ♢

Proof. Let $\tilde{a}_k = a_k \colon A \to X_k$ for $k \in \mathbf{Z}_{\leq 0}$ and $\tilde{a}_k = 0 \colon A \to 0_{\mathscr{A}}$ for $k \in \mathbf{Z}_{>0}$. Then $(A, (\tilde{a}_k)_{k\in\mathbf{Z}})$ is a limit for $X\Gamma_{\lceil 0}$ by lemma 3.2.15.

Let $\tilde{b}_k = b_k \colon B \to Y_k$ for $k \in \mathbf{Z}_{\leq 0}$ and $\tilde{b}_k = 0 \colon B \to 0_{\mathscr{A}}$ for $k \in \mathbf{Z}_{>0}$. Then $(B, (\tilde{b}_k)_{k\in\mathbf{Z}})$ is a limit for $Y\Gamma_{\lceil 0}$ by lemma 3.2.15.

Let $\tilde{c}_k = c_k \colon C \to Z_k$ for $k \in \mathbf{Z}_{\leq 0}$ and $\tilde{c}_k = 0 \colon C \to 0_{\mathscr{A}}$ for $k \in \mathbf{Z}_{>0}$. Then $(C, (\tilde{c}_k)_{k\in\mathbf{Z}})$ is a limit for $Z\Gamma_{\lceil 0}$ by lemma 3.2.15.

Note that we have $f\rceil = f\Gamma_{\lceil 0}\rceil$ and $g\rceil = g\Gamma_{\lceil 0}\rceil$ by lemma 3.2.27.

Suppose given a projective object $P \in \mathrm{Ob}(\mathscr{A})$.

Since the sequence $X\Gamma_{\lceil 0} \xrightarrow{f\Gamma_{\lceil 0}} Y\Gamma_{\lceil 0} \xrightarrow{g\Gamma_{\lceil 0}} Z\Gamma_{\lceil 0}$ is pure short exact in $\mathrm{CF}(\mathscr{A})$, lemma 3.2.29 yields the short exact sequence ${}_{\mathrm{CF}(\mathscr{A})}(\overline{P}, X\Gamma_{\lceil 0}) \xrightarrow{{}_{\mathrm{CF}(\mathscr{A})}(\overline{P}, f\Gamma_{\lceil 0})} {}_{\mathrm{CF}(\mathscr{A})}(\overline{P}, Y\Gamma_{\lceil 0}) \xrightarrow{{}_{\mathrm{CF}(\mathscr{A})}(\overline{P}, g\Gamma_{\lceil 0})} {}_{\mathrm{CF}(\mathscr{A})}(\overline{P}, Z\Gamma_{\lceil 0})$ of abelian groups. Lemma 3.2.28 yields the following commutative diagram of abelian groups whose columns are isomorphisms.

$$\begin{tikzcd}
{}_{\mathrm{CF}(\mathscr{A})}(\overline{P}, X\Gamma_{\lceil 0}) \arrow[r, "{}_{\mathrm{CF}(\mathscr{A})}(\overline{P}, f\Gamma_{\lceil 0})"] \arrow[d, "\Lambda_{P,X\Gamma_{\lceil 0}}^{(A,(\tilde{a}_k)_{k\in\mathbf{Z}})}"'] & {}_{\mathrm{CF}(\mathscr{A})}(\overline{P}, Y\Gamma_{\lceil 0}) \arrow[r, "{}_{\mathrm{CF}(\mathscr{A})}(\overline{P}, g\Gamma_{\lceil 0})"] \arrow[d, "\Lambda_{P,Y\Gamma_{\lceil 0}}^{(B,(\tilde{b}_k)_{k\in\mathbf{Z}})}"'] & {}_{\mathrm{CF}(\mathscr{A})}(\overline{P}, Z\Gamma_{\lceil 0}) \arrow[d, "\Lambda_{P,Z\Gamma_{\lceil 0}}^{(C,(\tilde{c}_k)_{k\in\mathbf{Z}})}"] \\
{}_{\mathscr{A}}(P, A) \arrow[r, "{}_{\mathscr{A}}(P, f\rceil)"] & {}_{\mathscr{A}}(P, B) \arrow[r, "{}_{\mathscr{A}}(P, g\rceil)"] & {}_{\mathscr{A}}(P, C)
\end{tikzcd}$$

Thus ${}_{\mathscr{A}}(P,A) \xrightarrow{{}_{\mathscr{A}}(P,f\rceil)} {}_{\mathscr{A}}(P,B) \xrightarrow{{}_{\mathscr{A}}(P,g\rceil)} {}_{\mathscr{A}}(P,C)$ is a short exact sequence of abelian groups as well. So the sequence $A \xrightarrow{f\rceil} B \xrightarrow{g\rceil} C$ is pure short exact in $\mathscr{A}$ by lemma 1.3.18. $\square$

3.2.31 Lemma/Definition. Suppose given $n \in \mathbf{Z}$ and $X_k \in \mathrm{Ob}(\mathscr{A})$ for $k \in \mathbf{Z}_{\leq n}$. We obtain a cofiltration $\mathrm{D} = \mathrm{D}_{(X_k)_{k \in \mathbf{Z}_{\leq n}}} \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$ recursively as follows. Let $\mathrm{D}_k = 0_{\mathscr{A}}$ for $k \in \mathbf{Z}_{>n}$. Let $\mathrm{D}_n = X_n$. For $k \in \mathbf{Z}_{<n}$, let $\mathrm{D}_k = \mathrm{D}_{k+1} \oplus X_k$ and $\mathrm{d}_k = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) : \mathrm{D}_{k+1} \oplus X_k \to \mathrm{D}_{k+1}$.

(a) Suppose given a product $(P, (p_k)_{k \in \mathbf{Z}_{\leq n}})$ for $(X_k)_{k \in \mathbf{Z}_{\leq n}}$ in $\mathscr{A}$.
We obtain a limit $(P, (a_k)_{k \in \mathbf{Z}})$ for D recursively as follows.
Let $a_n = p_n$. For $k \in \mathbf{Z}_{<n}$, let $a_k = \left(\begin{smallmatrix} a_{k+1} & p_k \end{smallmatrix}\right) : P \to \mathrm{D}_{k+1} \oplus X_k$.

(b) Suppose given a limit $(A, (a_k)_{k \in \mathbf{Z}})$ for D. We obtain a product $(A, (p_k)_{k \in \mathbf{Z}_{\leq n}})$ for $(X_k)_{k \in \mathbf{Z}_{\leq n}}$ in $\mathscr{A}$ by setting $p_n = a_n$ and $p_k = a_k \cdot \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ for $k \in \mathbf{Z}_{<n}$.

$$A \xrightarrow{a_k} \mathrm{D}_{k+1} \oplus X_k \xrightarrow{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)} X_k$$

$\diamondsuit$

Proof. We write $t_n = 1 \colon \mathrm{D}_n \to \mathrm{D}_n$ and $t_k = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) : \mathrm{D}_{k+1} \oplus X_k \to X_k$ for $k \in \mathbf{Z}_{<n}$.

Ad (a). Note that we have $a_k \cdot t_k = p_k$ for $k \in \mathbf{Z}_{\leq n}$. Also note that we have $a_k \cdot \mathrm{d}_k = a_{k+1}$ for $k \in \mathbf{Z}$ by construction.

Suppose given $T \xrightarrow{f_k} \mathrm{D}_k$ such that $f_{k-1} \cdot \mathrm{d}_{k-1} = f_k$ for $k \in \mathbf{Z}_{\leq n}$.

Let $g_k = f_k \cdot t_k \colon T \to X_k$ for $k \in \mathbf{Z}_{\leq n}$. So we have $f_k = \left(\begin{smallmatrix} f_{k+1} & g_k \end{smallmatrix}\right) : T \to \mathrm{D}_{k+1} \oplus X_k$ for $k \in \mathbf{Z}_{<n}$. Since $(P, (p_k)_{k \in \mathbf{Z}_{\leq n}})$ is a product for $(X_k)_{k \in \mathbf{Z}_{\leq n}}$ in $\mathscr{A}$, there exists a unique morphism $T \xrightarrow{g} P$ in $\mathscr{A}$ such that $g \cdot p_k = g_k$ for $k \in \mathbf{Z}_{\leq n}$.

We have $g \cdot a_n = g \cdot p_n = g_n = f_n$ and, inductively, $g \cdot a_k = g \cdot \left(\begin{smallmatrix} a_{k+1} & p_k \end{smallmatrix}\right) = \left(\begin{smallmatrix} f_{k+1} & g_k \end{smallmatrix}\right) = f_k$ for $k \in \mathbf{Z}_{<n}$.

Suppose given $T \xrightarrow{h} P$ in $\mathscr{A}$ such that $h \cdot a_k = f_k$ for $k \in \mathbf{Z}_{\leq n}$.
We have $h \cdot p_k = h \cdot a_k \cdot t_k = f_k \cdot t_k = g_k$ for $k \in \mathbf{Z}_{\leq n}$. Thus $h = g$.

Ad (b). Note that we have $p_k = a_k \cdot t_k$ for $k \in \mathbf{Z}_{\leq n}$.

Suppose given $T \xrightarrow{g_k} X_k$ for $k \in \mathbf{Z}_{\leq n}$. Let $f_n = g_n$ and $f_k = \left(\begin{smallmatrix} f_{k+1} & g_k \end{smallmatrix}\right) : T \to \mathrm{D}_{k+1} \oplus X_k$ for $k \in \mathbf{Z}_{<n}$. Note that $f_k \cdot t_k = g_k$ for $k \in \mathbf{Z}_{\leq n}$ by construction. We have $f_k \cdot \mathrm{d}_k = f_{k+1}$ for $k \in \mathbf{Z}_{<n}$ and thus there exists a unique morphism $T \xrightarrow{f} A$ in $\mathscr{A}$ such that $f \cdot a_k = f_k$ for $k \in \mathbf{Z}_{\leq n}$. We have $f \cdot p_k = f \cdot a_k \cdot t_k = f_k \cdot t_k = g_k$ for $k \in \mathbf{Z}_{\leq n}$.

Supppose given $T \xrightarrow{h} P$ in $\mathscr{A}$ such that $h \cdot p_k = g_k$ for $k \in \mathbf{Z}_{\leq n}$.
We have $h \cdot a_n = h \cdot p_n = g_n = f_n$. Suppose given $k \in \mathbf{Z}_{<n}$. We have $h \cdot a_k \cdot t_k = h \cdot p_k = g_k$ and, inductively, $h \cdot a_k \cdot \mathrm{d}_k = h \cdot a_{k+1} = f_{k+1}$. Thus $h \cdot a_k = f_k$.

We conclude that $h = f$. $\square$

3.2.32 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$ such that x_k is a split epimorphism for $k \in \mathbf{Z}$. Suppose given $n \in \mathbf{Z}$ and kernels $R_k \xrightarrow{r_k} X_k$ of x_k for $k \in \mathbf{Z}_{<n}$. We write $R_n = X_n$.

(a) In $\mathrm{CF}(\mathscr{A})$, the cofiltration $X\Gamma_{\lceil n}$ is isomorphic to the cofiltration $\mathrm{D} = \mathrm{D}_{(R_k)_{k\in\mathbf{Z}_{\leq n}}}$ constructed in definition 3.2.31.

(b) Suppose given a product $(P, (p_k)_{k\in\mathbf{Z}_{\leq n}})$ for $(R_k)_{k\in\mathbf{Z}_{\leq n}}$ in $\mathscr{A}$. Then there exist morphisms $P \xrightarrow{a_k} X_k$ for $k \in \mathbf{Z}$ such that $(P, (a_k)_{k\in\mathbf{Z}})$ is a limit for X.

$\diamondsuit$

Proof. Ad (a). For $k \in \mathbf{Z}_{\leq n}$, we want to recursively construct isomorphisms $\mathrm{D}_k \xrightarrow{f_k} X_k$ such that $\mathrm{d}_{k-1} \cdot f_k = f_{k-1} \cdot x_{k-1}$.

Let $f_n = 1$. Suppose given $k \in \mathbf{Z}_{<n}$. Suppose that we have already constructed f_ℓ for $\ell \in [k+1, n]$.

Since x_k is a split epimorphism, we may choose $X_{k+1} \xrightarrow{s_k} X_k$ in $\mathscr{A}$ such that $s_k \cdot x_k = 1$, cf. lemma 1.2.11.

Let $f_k = \begin{pmatrix} f_{k+1}\cdot s_k \\ r_k \end{pmatrix} : \mathrm{D}_{k+1} \oplus R_k \to X_k$.
We have $f_k \cdot x_k = \begin{pmatrix} f_{k+1} \\ 0 \end{pmatrix} = \mathrm{d}_k \cdot f_{k+1}$ and $\begin{pmatrix} 0 & 1 \end{pmatrix} \cdot f_k = 1 \cdot r_k \colon R_k \to X_k$.

$$\begin{tikzcd}
R_k \arrow[r, "\begin{pmatrix} 0 & 1 \end{pmatrix}"] \arrow[d, "1"'] & \mathrm{D}_{k+1} \oplus R_k \arrow[r, "\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathrm{d}_k"] \arrow[d, "f_k"] & \mathrm{D}_{k+1} \arrow[d, "f_{k+1}"] \\
R_k \arrow[r, "r_k"] & X_k \arrow[r, "x_k"] & X_{k+1}
\end{tikzcd}$$

By lemma 1.3.19.(c), f_k is an isomorphism.

So we obtain an isomorphism $\mathrm{D} \xrightarrow{f} X\Gamma_{\lceil n}$ in $\mathrm{CF}(\mathscr{A})$.

Ad (b). This follows from (a), lemmata 3.2.31.(a), 3.2.15.(b) and remark 3.2.16. $\square$

3.2.33 Lemma. Suppose that $\mathscr{A}$ has countable products of projectives. Suppose given a cofiltration $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$ such that X_k is projective in $\mathscr{A}$ for $k \in \mathbf{Z}$. Then there exists a limit $(P, (a_k)_{k\in\mathbf{Z}})$ for X such that P is a countable product of projectives in $\mathscr{A}$. $\diamondsuit$

Proof. This follows from lemma 3.2.32. $\square$

3.2.34 Lemma. Suppose that $\mathscr{A}$ has countable coproducts of injectives. Suppose given a filtration $X \in \mathrm{Ob}(\mathrm{F}(\mathscr{A}))$ such that X_k is injective in $\mathscr{A}$ for $k \in \mathbf{Z}$. Then there exists a colimit $(I, (a_k)_{k\in\mathbf{Z}})$ for X such that I is a countable coproduct of injectives in $\mathscr{A}$. $\diamondsuit$

Proof. This is dual to the previous lemma 3.2.33. $\square$

3.2.35 Lemma. Suppose that $\mathscr{A}$ has enough projectives. Suppose given $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$ and a limit $(A, (a_k)_{k\in\mathbf{Z}})$ for X. Suppose given $n \in \mathbf{Z}$ and a product $(P, (p_k)_{k\in\mathbf{Z}_{\leq n}})$ for $(X_k)_{k\in\mathbf{Z}_{\leq n}}$. Then there exists a pure short exact sequence $A \xrightarrow{i} P \xrightarrow{q} P$ in $\mathscr{A}$, where q is the unique morphism in $\mathscr{A}$ such that $q \cdot p_k = p_k - p_{k-1} \cdot x_{k-1}$ for $k \in \mathbf{Z}_{\leq n}$. $\diamondsuit$

Proof. We use the cofiltration $\mathrm{D} = \mathrm{D}_{(X_k)_{k\in\mathbf{Z}_{\leq n}}} \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$ constructed in definition 3.2.31. We write $e_n = 1\colon \mathrm{D}_n \to \mathrm{D}_n$ and $e_k = \begin{pmatrix} 0 & 1 \end{pmatrix} : X_k \to \mathrm{D}_{k+1} \oplus X_k$ for $k \in \mathbf{Z}_{<n}$. We write

$t_n = 1\colon \mathrm{D}_n \to \mathrm{D}_n$ and $t_k = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : \mathrm{D}_{k+1} \oplus X_k \to X_k$ for $k \in \mathbf{Z}_{<n}$. Note that e_k is a kernel of d_k and that we have $e_k \cdot t_k = 1$ for $k \in \mathbf{Z}_{\leq n}$. We obtain a limit $(P, (b_k)_{k\in\mathbf{Z}})$ for D recursively as in definition 3.2.31.(a) by setting $b_n = p_n$ and $b_k = \begin{pmatrix} b_{k+1} & p_k \end{pmatrix}$ for $k \in \mathbf{Z}_{<n}$. Note that we have $b_k \cdot t_k = p_k$ for $k \in \mathbf{Z}_{\leq n}$.

For $k \in \mathbf{Z}_{\leq n}$, we want to recursively construct morphisms $X_k \xrightarrow{f_k} \mathrm{D}_k$, $\mathrm{D}_k \xrightarrow{h_k} \mathrm{D}_k$ and $\mathrm{D}_k \xrightarrow{g_k} \mathrm{D}_{k+1}$ in $\mathscr{A}$ such that $x_{k-1} \cdot f_k = f_{k-1} \cdot \mathrm{d}_{k-1}$, $\mathrm{d}_{k-1} \cdot g_k = g_{k-1} \cdot \mathrm{d}_k$, $f_k \cdot h_k = e_k$, $h_k \cdot \mathrm{d}_k = g_k$, $h_k \cdot t_k = t_k$, $b_{k-1} \cdot g_{k-1} \cdot t_k = p_k - p_{k-1} \cdot x_{k-1}$ and such that (f_k, g_k) is a pure short exact sequence in $\mathscr{A}$.

Let $f_n = 1$, $h_n = 1$ and $g_n = 0$. Note that we have $f_n \cdot h_n = 1 = e_n$, $h_n \cdot \mathrm{d}_n = 0 = g_n$, $h_n \cdot t_n = 1 = e_n$ and that (f_n, g_n) is a pure short exact sequence in $\mathscr{A}$.

Suppose given $k \in \mathbf{Z}_{<n}$. Suppose we have already constructed f_ℓ , h_ℓ and g_ℓ for $\ell \in [k+1, n]$.
Let $f_k = \begin{pmatrix} x_k \cdot f_{k+1} & 1 \end{pmatrix} : X_k \to \mathrm{D}_{k+1} \oplus X_k$, $g_k = \begin{pmatrix} h_{k+1} \\ -x_k \cdot e_{k+1} \end{pmatrix} : \mathrm{D}_{k+1} \oplus X_k \to \mathrm{D}_{k+1}$ and
$h_k = \begin{pmatrix} h_{k+1} & 0 \\ -x_k \cdot e_{k+1} & 1 \end{pmatrix} : \mathrm{D}_{k+1} \oplus X_k \to \mathrm{D}_{k+1} \oplus X_k$.

We have $f_k \cdot \mathrm{d}_k = x_k \cdot f_{k+1}$, $g_k \cdot \mathrm{d}_{k+1} = \begin{pmatrix} h_{k+1} \cdot \mathrm{d}_{k+1} \\ -x_k \cdot e_{k+1} \cdot \mathrm{d}_{k+1} \end{pmatrix} = \begin{pmatrix} g_{k+1} \\ 0 \end{pmatrix} = \mathrm{d}_k \cdot g_{k+1}$, $f_k \cdot h_k = \begin{pmatrix} 0 & 1 \end{pmatrix} = e_k$, $h_k \cdot \mathrm{d}_k = \begin{pmatrix} h_{k+1} \\ -x_k \cdot e_{k+1} \end{pmatrix} = g_k$, $h_k \cdot t_k = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = t_k$ and $b_k \cdot g_k \cdot t_{k+1} = \begin{pmatrix} b_{k+1} & p_k \end{pmatrix} \cdot \begin{pmatrix} t_{k+1} \\ -x_k \end{pmatrix} = p_{k+1} - p_k \cdot x_k$.
Moreover, we have $x_{\overset{\bullet}{k}} \cdot f_k = \begin{pmatrix} 0 & x_{\overset{\bullet}{k}} \end{pmatrix} = x_{\overset{\bullet}{k}} \cdot e_k$ and $e_k \cdot g_k = -x_k \cdot e_{k+1}$. So we obtain the following commutative diagram in $\mathscr{A}$.

$$\begin{tikzcd}
X_{k+1} \arrow[d, "f_{k+1}"'] & X_k \arrow[l, "x_k"'] \arrow[d, "f_k"'] & X_{\overset{\bullet}{k}} \arrow[l, "x_{\overset{\bullet}{k}}"'] \arrow[d, "x_{\overset{\bullet}{k}}"] \\
\mathrm{D}_{k+1} \arrow[d, "g_{k+1}"'] & \mathrm{D}_k \arrow[l, "\mathrm{d}_k"'] \arrow[d, "g_k"'] & X_k \arrow[l, "e_k"'] \arrow[d, "-x_k"] \\
\mathrm{D}_{k+2} & \mathrm{D}_{k+1} \arrow[l, "\mathrm{d}_{k+1}"'] & X_{k+1} \arrow[l, "e_{k+1}"']
\end{tikzcd}$$

Since $f_k \cdot g_k = x_k \cdot e_{k+1} - x_k \cdot e_{k+1} = 0$, the sequence (f_k, g_k) is pure short exact in $\mathscr{A}$ by lemma 1.3.17.

So we obtain a pure short exact sequence $X\Gamma_{\lceil n} \xrightarrow{f} \mathrm{D} \xrightarrow{g} \mathrm{D}_{[1]}$ in $\mathrm{CF}(\mathscr{A})$.
Let $\tilde{a}_k = a_k\colon A \to X_k$ for $k \in \mathbf{Z}_{\leq n}$ and $\tilde{a}_k = 0\colon A \to 0_{\mathscr{A}}$ for $k \in \mathbf{Z}_{>n}$. Then $(A, (\tilde{a}_k)_{k\in\mathbf{Z}})$ is a limit for $X\Gamma_{\lceil n}$ by lemma 3.2.15. Moreover, $(P, (b_{k+1})_{k\in\mathbf{Z}})$ is a limit for $\mathrm{D}_{[1]}$ by lemma 3.2.21.

Let $i = f{\rceil}\colon A \to P$ and $q = g{\rceil}\colon P \to P$. The sequence (i, q) is pure short exact in $\mathscr{A}$ by lemma 3.2.30.
Moreover, we have $q \cdot p_k = q \cdot b_k \cdot t_k = b_{k-1} \cdot g_{k-1} \cdot t_k = p_k - p_{k-1} \cdot x_{k-1}$ for $k \in \mathbf{Z}_{\leq n}$. □

3.2.36 Lemma. Suppose that $\mathscr{A}$ has enough injectives. Suppose given $X \in \mathrm{Ob}(\mathrm{F}(\mathscr{A}))$ and a colimit $(A, (a_k)_{k\in\mathbf{Z}})$ for X. Suppose given $n \in \mathbf{Z}$ and a coproduct $(C, (c_k)_{k\in\mathbf{Z}_{\geq n}})$ for $(X_k)_{k\in\mathbf{Z}_{\geq n}}$. Then there exists a pure short exact sequence $C \xrightarrow{i} C \xrightarrow{q} A$ in $\mathscr{A}$, where i is the unique morphism in $\mathscr{A}$ such that $c_k \cdot i = c_k - x_k \cdot c_{k+1}$ for $k \in \mathbf{Z}_{\geq n}$. ◊

Proof. This is dual to the previous lemma 3.2.35. □

3.2.37 Definition. Suppose given $m \in \mathbf{Z}$. A morphism $X \xrightarrow{i} Y$ in $\mathrm{CF}(\mathscr{A})$ is called an m-pure monomorphism if $X_k \xrightarrow{i_k} Y_k \xrightarrow{Y_{k \to m+1}} Y_{m+1}$ is a pure short exact sequence in $\mathscr{A}$ for $k \in \mathbf{Z}_{\leq m+1}$. ♢

3.2.38 Definition. Suppose given $m \in \mathbf{Z}$. A morphism $Y \xrightarrow{p} Z$ in $\mathrm{F}(\mathscr{A})$ is called an m-pure epimorphism if $Y_{m-1} \xrightarrow{Y_{m-1 \to k}} Y_k \xrightarrow{p_k} Z_k$ is a pure short exact sequence in $\mathscr{A}$ for $k \in \mathbf{Z}_{\geq m-1}$. ♢

3.2.39 Lemma. Suppose given $m \in \mathbf{Z}$ and an m-pure monomorphism $X \xrightarrow{i} Y$ in $\mathrm{CF}(\mathscr{A})$.

(a) We have $X \in \mathrm{Ob}(\mathrm{CF}^{\lceil m}(\mathscr{A}))$.

(b) The morphism i is a pure monomorphism in $\mathrm{CF}(\mathscr{A})$.

(c) Suppose given a limit $(B, (b_k)_{k \in \mathbf{Z}})$ for Y and a pure short exact sequence $A \xrightarrow{u} B \xrightarrow{b_{m+1}} Y_{m+1}$ in $\mathscr{A}$. For $k \in \mathbf{Z}$, there exists a unique morphism $A \xrightarrow{a_k} X_k$ in $\mathscr{A}$ such that $a_k \cdot i_k = u \cdot b_k$. Moreover, $(A, (a_k)_{k \in \mathbf{Z}})$ is a limit for X.

(d) Suppose that $\mathscr{A}$ has enough projectives. Suppose given a limit $(A, (a_k)_{k \in \mathbf{Z}})$ for X and a limit $(B, (b_k)_{k \in \mathbf{Z}})$ for Y. Then $A \xrightarrow{i\upharpoonright} B \xrightarrow{b_{m+1}} Y_{m+1}$ is a pure short exact sequence in $\mathscr{A}$, cf. definition 3.2.22. ♢

Proof. Ad (a). We have $X_{m+1} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}})$ since $X_{m+1} \xrightarrow{i_{m+1}} Y_m \xrightarrow{1} Y_m$ is pure short exact in $\mathscr{A}$.

Ad (b). We define $Z \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$ and $Y \xrightarrow{p} Z$ in $\mathrm{CF}(\mathscr{A})$ as follows.

For $k \in \mathbf{Z}_{\leq m+1}$, let $Z_k = Y_{m+1}$, $z_{k-1} = 1$ and $p_k = Y_{k \to m+1}$. For $k \in \mathbf{Z}_{>m+1}$, let $Z_k = Y_k$, $z_{k-1} = y_{k-1}$ and $p_k = 1$.

$$\begin{tikzcd}
X \arrow[d, "i"] & \cdots & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & X_m \arrow[l] \arrow[d, "i_m"] & X_{m-1} \arrow[l, "x_{m-1}"'] \arrow[d, "i_{m-1}"] & X_{m-2} \arrow[l, "x_{m-2}"'] \arrow[d, "i_{m-1}"] & \cdots \arrow[l] \\
Y \arrow[d, "p"] & \cdots & Y_{m+3} \arrow[l] \arrow[d, "1"] & Y_{m+2} \arrow[l, "y_{m+2}"'] \arrow[d, "1"] & Y_{m+1} \arrow[l, "y_{m+1}"'] \arrow[d, "1"] & Y_m \arrow[l, "y_m"'] \arrow[d, "y_m"] & Y_{m-1} \arrow[l, "y_{m-1}"'] \arrow[d, "Y_{m-1 \to m+1}"] & Y_{m-2} \arrow[l, "y_{m-2}"'] \arrow[d, "Y_{m-2 \to m+1}"] & \cdots \arrow[l] \\
Z & \cdots & Y_{m+3} \arrow[l] & Y_{m+2} \arrow[l, "y_{m+2}"] & Y_{m+1} \arrow[l, "y_{m+1}"] & Y_{m+1} \arrow[l, "1"] & Y_{m+1} \arrow[l, "1"] & Y_{m+1} \arrow[l, "1"] & \cdots \arrow[l]
\end{tikzcd}$$

Note that (i, p) is a pure short exact sequence in $\mathrm{CF}(\mathscr{A})$. In particular, i is a pure monomorphism in $\mathrm{CF}(\mathscr{A})$.

Ad (c). For $k \in \mathbf{Z}_{>m}$, we have $X_k \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}})$ by (a) and $u \cdot b_k = u \cdot b_{m+1} \cdot Y_{m+1 \to k} = 0$. Thus $a_k = 0 \colon A \to X_k$ is the unique morphism such that $a_k \cdot i_k = u \cdot b_k$.

For $k \in \mathbf{Z}_{\leq m}$, we have $u \cdot b_k \cdot Y_{k \to m+1} = u \cdot b_{m+1} = 0$. Since i_k is a kernel of $Y_{k \to m+1}$, there exists a unique morphism $A \xrightarrow{a_k} X_k$ in $\mathscr{A}$ such that $a_k \cdot i_k = u \cdot b_k$.

Suppose given morphisms $T \xrightarrow{f_k} X_k$ in $\mathscr{A}$ such that $f_k \cdot x_k = f_{k+1}$ for $k \in \mathbf{Z}$. Consider the morphisms $T \xrightarrow{f_k \cdot i_k} Y_k$ for $k \in \mathbf{Z}$. We have $f_k \cdot i_k \cdot y_k = f_k \cdot x_k \cdot i_{k+1} = f_{k+1} \cdot i_{k+1}$ for $k \in \mathbf{Z}$.

Since $(B, (b_k)_{k\in\mathbf{Z}})$ is a limit for Y, there exists a unique morphism $T \xrightarrow{g} B$ in $\mathscr{A}$ such that $g \cdot b_k = f_k \cdot i_k$ for $k \in \mathbf{Z}$. In particular, we have $g \cdot b_{m+1} = f_{m+1} \cdot i_{m+1} = 0$. Since u is a kernel of b_{m+1} , we get $T \xrightarrow{h} A$ in $\mathscr{A}$ such that $h \cdot u = g$. So for $k \in \mathbf{Z}$, we have $h \cdot a_k = f_k$ since $h \cdot a_k \cdot i_k = h \cdot u \cdot b_k = g \cdot b_k = f_k \cdot i_k$ and since i_k is a pure monomorphism (In particular, i_k is a monomorphism, cf. definition 1.3.2.).

Suppose given $T \xrightarrow{e} A$ in $\mathscr{A}$ such that $e \cdot a_k = f_k$ for $k \in \mathbf{Z}$. We have $e \cdot u \cdot b_k = e \cdot a_k \cdot i_k = f_k \cdot i_k$ for $k \in \mathbf{Z}$. Thus $e \cdot u = g = h \cdot u$. We conclude that $e = h$ since u is a pure monomorphism.

Ad (d). We consider the pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathrm{CF}(\mathscr{A})$ from (b). Note that we get a limit $(Y_{m+1}\,, (c_k)_{k\in\mathbf{Z}})$ for Z with $c_k = 1$ for $k \in \mathbf{Z}_{\leq m+1}$, cf. lemma 3.2.18. Moreover, we have $p\rceil = b_{m+1}$ since $b_{m+1} \cdot c_k = b_{m+1} = b_k \cdot Y_{k \to m+1} = b_k \cdot p_k$ for $k \in \mathbf{Z}_{\leq m+1}$. The result now follows from lemma 3.2.30. □

3.2.40 Lemma. Suppose given $m \in \mathbf{Z}$ and an m-pure epimorphism $Y \xrightarrow{p} Z$ in $\mathrm{F}(\mathscr{A})$.

(a) The morphism p is a pure epimorphism in $\mathrm{F}(\mathscr{A})$.

(b) We have $Z \in \mathrm{Ob}(\mathrm{F}^{m\rceil}(\mathscr{A}))$.

(c) Suppose given a colimit $(B, (b_k)_{k\in\mathbf{Z}})$ for Y and a pure short exact sequence $Y_{m-1} \xrightarrow{b_{m-1}} B \xrightarrow{v} C$ in $\mathscr{A}$. For $k \in \mathbf{Z}$, there exists a unique morphism $Z_k \xrightarrow{c_k} C$ in $\mathscr{A}$ such that $p_k \cdot c_k = b_k \cdot v$. Moreover, $(C, (c_k)_{k\in\mathbf{Z}})$ is a colimit for Z.

(d) Suppose that $\mathscr{A}$ has enough injectives. Suppose given a colimit $(C, (c_k)_{k\in\mathbf{Z}})$ for Z and a colimit $(B, (b_k)_{k\in\mathbf{Z}})$ for Y. Then $Y_{m-1} \xrightarrow{b_{m-1}} B \xrightarrow{p\rceil} C$ is a pure short exact sequence in $\mathscr{A}$, cf. definition 3.2.23. ◇

Proof. This is dual to the previous lemma 3.2.39. □

3.2.41 Lemma. Suppose that $\mathscr{A}$ has enough projectives. Suppose given a pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathrm{CF}(\mathscr{A})$. Suppose given a limit $(A, (a_k)_{k\in\mathbf{Z}})$ for X, a limit $(B, (b_k)_{k\in\mathbf{Z}})$ for Y and a limit $(C, (c_k)_{k\in\mathbf{Z}})$ for Z. Suppose given a pure short exact sequence $A \xrightarrow{j} D \xrightarrow{q} C$ in $\mathscr{A}$. Suppose given a compatible family $(D, (d_k)_{k\in\mathbf{Z}})$ for Y such that $j \cdot d_k = a_k \cdot i_k$ and $q \cdot c_k = d_k \cdot p_k$ for $k \in \mathbf{Z}$. Then $(D, (d_k)_{k\in\mathbf{Z}})$ is a limit for Y. ◇

Proof. Let $D \xrightarrow{u} B$ be the unique morphism in $\mathscr{A}$ such that $u \cdot b_k = d_k$ for $k \in \mathbf{Z}$. We want to show that the following diagram is commutative in $\mathscr{A}$.

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[d, "1"'] & D \arrow[r, "q"] \arrow[d, "u"] & C \arrow[d, "1"] \\
A \arrow[r, "i\rceil"] & B \arrow[r, "p\rceil"] & C
\end{tikzcd}$$

For $k \in \mathbf{Z}$, we have $j \cdot u \cdot b_k = j \cdot d_k = a_k \cdot i_k = i\rceil \cdot b_k$. Since $(B, (b_k)_{k\in\mathbf{Z}})$ is a limit for Y, we obtain $j \cdot u = i\rceil$. For $k \in \mathbf{Z}$, we have $u \cdot p\rceil \cdot c_k = u \cdot b_k \cdot p_k = d_k \cdot p_k = q \cdot c_k$. Since $(C, (c_k)_{k\in\mathbf{Z}})$ is a limit for Z, we obtain $u \cdot p\rceil = q$.

Now u is an isomorphism by lemma 1.3.19.(c). We conclude that $(D, (d_k)_{k\in\mathbf{Z}})$ is a limit for Y. $\square$

We introduce a weak form of limits for cofiltrations and a weak form of colimits for filtrations.

3.2.42 Definition. Suppose given $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$. A compatible family $(A, (a_k)_{k\in\mathbf{Z}})$ for X is called a *projective family* for X if for every compatible family $(P, (f_k)_{k\in\mathbf{Z}})$ for X such that P is projective in $\mathscr{A}$, there exists $P \xrightarrow{u} A$ in $\mathscr{A}$ such that $u \cdot a_k = f_k$ for $k \in \mathbf{Z}$. $\diamondsuit$

3.2.43 Definition. Suppose given $X \in \mathrm{Ob}(\mathrm{F}(\mathscr{A}))$. A compatible family $(A, (a_k)_{k\in\mathbf{Z}})$ for X is called an *injective family* for X if for every compatible family $(I, (f_k)_{k\in\mathbf{Z}})$ for X such that I is injective in $\mathscr{A}$, there exists $A \xrightarrow{u} I$ in $\mathscr{A}$ such that $a_k \cdot u = f_k$ for $k \in \mathbf{Z}$. $\diamondsuit$

3.2.44 Remark. Suppose given $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$. If $(A, (a_k)_{k\in\mathbf{Z}})$ is a limit for X, then it is a projective family for X. $\diamondsuit$

3.2.45 Remark. Suppose given $X \in \mathrm{Ob}(\mathrm{F}(\mathscr{A}))$. If $(A, (a_k)_{k\in\mathbf{Z}})$ is a colimit for X, then it is an injective family for X. $\diamondsuit$

3.2.46 Lemma (cofinality of projective families)**.** Suppose given $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$.

(a) Suppose given a projective family $(A, (a_k)_{k\in\mathbf{Z}})$ for X.
Suppose given $m \in \mathbf{Z}$ and $P \xrightarrow{t_k} X_k$ in $\mathscr{A}$ such that P is projective in $\mathscr{A}$ and such that $t_{k-1} \cdot x_{k-1} = t_k$ for $k \in \mathbf{Z}_{\le m}$. Then there exists $P \xrightarrow{t} A$ in $\mathscr{A}$ such that $t \cdot a_k = t_k$ for $k \in \mathbf{Z}_{\le m}$.

(b) Suppose given $m \in \mathbf{Z}$ and $A \xrightarrow{a_k} X_k$ such that $a_{k-1} \cdot x_{k-1} = a_k$ for $k \in \mathbf{Z}_{\le m}$. Suppose that for $P \xrightarrow{t_k} X_k$ in $\mathscr{A}$ with P projective in $\mathscr{A}$ and $t_{k-1} \cdot x_{k-1} = t_k$ for $k \in \mathbf{Z}_{\le m}$, there exists $T \xrightarrow{t} A$ in $\mathscr{A}$ such that $t \cdot a_k = t_k$ for $k \in \mathbf{Z}_{\le m}$. Let $b_k = a_k$ for $k \in \mathbf{Z}_{\le m}$ and $b_k = a_m \cdot X_{m \to k}$ for $k \in \mathbf{Z}_{>m}$. Then $(A, (b_k)_{k\in\mathbf{Z}})$ is a projective family for X.

Cf. lemma 3.2.14. $\diamondsuit$

Proof. Ad (a). Let $u_k = t_k$ for $k \in \mathbf{Z}_{\le m}$ and let $u_k = t_m \cdot X_{m\to k}$ for $k \in \mathbf{Z}_{>m}$.

For $k \in \mathbf{Z}_{\le m}$, we have $u_{k-1} \cdot x_{k-1} = t_{k-1} \cdot x_{k-1} = t_k = u_k$.

For $k \in \mathbf{Z}_{>m}$, we have $u_{k-1} \cdot x_{k-1} = t_m \cdot X_{m\to k-1} \cdot X_{k-1\to k} = t_m \cdot X_{m\to k} = u_k$.

Since $(A, (a_k)_{k\in\mathbf{Z}})$ is a projective family for X, there exists $P \xrightarrow{t} A$ in $\mathscr{A}$ such that $t \cdot a_k = u_k$ for $k \in \mathbf{Z}$. In particular, we have $t \cdot a_k = u_k = t_k$ for $k \in \mathbf{Z}_{\le m}$.

Ad (b). For $k \in \mathbf{Z}_{\le m}$, we have $b_{k-1} \cdot x_{k-1} = a_{k-1} \cdot x_{k-1} = a_k = b_k$.

For $k \in \mathbf{Z}_{>m}$, we have $b_{k-1} \cdot x_{k-1} = a_m \cdot X_{m\to k-1} \cdot X_{k-1\to k} = a_m \cdot X_{m\to k} = b_k$.

Suppose given $P \xrightarrow{t_k} X_k$ in $\mathscr{A}$ such that P is projective in $\mathscr{A}$ and such that $t_{k-1} \cdot x_{k-1} = t_k$ for $k \in \mathbf{Z}$. By assumption, there exists a morphism $P \xrightarrow{t} A$ in $\mathscr{A}$ such that $t \cdot a_k = t_k$ for $k \in \mathbf{Z}_{\le m}$. The result now follows since we also have $t \cdot b_k = t \cdot a_m \cdot X_{m\to k} = t_m \cdot X_{m\to k} = t_k$ for $k \in \mathbf{Z}_{>m}$. $\square$

3.2.47 Lemma. Suppose given a pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathrm{CF}(\mathscr{A})$. Suppose given a projective family $(A, (a_k)_{k \in \mathbf{Z}})$ for X, a compatible family $(B, (b_k)_{k \in \mathbf{Z}})$ for Y and a projective family $(C, (c_k)_{k \in \mathbf{Z}})$ for Z. Suppose given a pure short exact sequence $A \xrightarrow{j} B \xrightarrow{q} C$ in $\mathscr{A}$ such that $a_k \cdot i_k = j \cdot b_k$ and $b_k \cdot p_k = q \cdot c_k$ for $k \in \mathbf{Z}$. Then $(B, (b_k)_{k \in \mathbf{Z}})$ is a projective family for Y as well. $\diamondsuit$

Proof. Suppose given $P \xrightarrow{f_k} Y_k$ in $\mathscr{A}$ such that P is projective and such that $f_{k-1} \cdot y_{k-1} = f_k$ for $k \in \mathbf{Z}_{\leq 0}$.

Consider the morphisms $P \xrightarrow{f_k \cdot p_k} Z_k$ for $k \in \mathbf{Z}_{\leq 0}$.

We have $f_{k-1} \cdot p_{k-1} \cdot z_{k-1} = f_{k-1} \cdot y_{k-1} \cdot p_k = f_k \cdot p_k$ for $k \in \mathbf{Z}_{\leq 0}$. Since $(C, (c_k)_{k \in \mathbf{Z}})$ is a projective family for Z, we may choose $P \xrightarrow{u} C$ in $\mathscr{A}$ such that $f_k \cdot p_k = u \cdot c_k$ for $k \in \mathbf{Z}_{\leq 0}$. Since P is projective in $\mathscr{A}$, we may choose $P \xrightarrow{v} B$ in $\mathscr{A}$ such that $v \cdot q = u$.

For $k \in \mathbf{Z}_{\leq 0}$, we want to construct $P \xrightarrow{g_k} X_k$ in $\mathscr{A}$ recursively such that $g_{k-1} \cdot x_{k-1} = g_k$ and such that $v \cdot b_k + g_k \cdot i_k = f_k$.

We have $(f_0 - v \cdot b_0) \cdot p_0 = f_0 \cdot p_0 - v \cdot q \cdot c_0 = f_0 \cdot p_0 - u \cdot c_0 = 0$. So there exists a unique morphism $P \xrightarrow{g_0} X_0$ in $\mathscr{A}$ such that $f_0 - v \cdot b_0 = g_0 \cdot i_0$. Thus $v \cdot b_0 + g_0 \cdot i_0 = f_0$.

Suppose given $k \in \mathbf{Z}_{<0}$. Suppose we have already constructed g_ℓ for $\ell \in [k+1, 0]$.

Since P is projective in $\mathscr{A}$, we may choose $P \xrightarrow{h_k} X_k$ in $\mathscr{A}$ such that $h_k \cdot x_k = g_{k+1}$. We have

$$(f_k - v \cdot b_k - h_k \cdot i_k) \cdot y_k = f_{k+1} - v \cdot b_{k+1} - h_k \cdot x_k \cdot i_{k+1} = g_{k+1} \cdot i_{k+1} - g_{k+1} \cdot i_{k+1} = 0.$$

So there exists a unique morphism $P \xrightarrow{r_k} Y_{\overset{\bullet}{k}}$ in $\mathscr{A}$ such that $r_k \cdot y_{\overset{\bullet}{k}} = f_k - v \cdot b_k - h_k \cdot i_k$. We have $r_k \cdot p_{\overset{\bullet}{k}} = 0$ since

$$r_k \cdot p_{\overset{\bullet}{k}} \cdot z_{\overset{\bullet}{k}} = r_k \cdot y_{\overset{\bullet}{k}} \cdot p_k = (f_k - v \cdot b_k - h_k \cdot i_k) \cdot p_k = u \cdot c_k - v \cdot q \cdot c_k = 0$$

and since $z_{\overset{\bullet}{k}}$ is a pure monomorphism.

So there exists a unique morphism $P \xrightarrow{s_k} X_{\overset{\bullet}{k}}$ in $\mathscr{A}$ such that $s_k \cdot i_{\overset{\bullet}{k}} = r_k$.

$$\begin{tikzcd}
P \arrow[r, "f_k - v\cdot b_k - h_k\cdot i_k"] \arrow[d, "s_k"'] \arrow[dr, "r_k"] & Y_k \arrow[r, "y_k"] & Y_{k+1} \\
X_{\overset{\bullet}{k}} \arrow[r, "i_{\overset{\bullet}{k}}"'] & Y_{\overset{\bullet}{k}} \arrow[u, "y_{\overset{\bullet}{k}}"'] &
\end{tikzcd}$$

Let $g_k = h_k + s_k \cdot x_{\overset{\bullet}{k}}$. We have $g_k \cdot x_k = g_{k+1}$ and

$$v \cdot b_k + g_k \cdot i_k = v \cdot b_k + h_k \cdot i_k + s_k \cdot i_{\overset{\bullet}{k}} \cdot y_{\overset{\bullet}{k}} = v \cdot b_k + h_k \cdot i_k + r_k \cdot y_{\overset{\bullet}{k}} = f_k \,.$$

Since $(A, (a_k)_{k \in \mathbf{Z}})$ is a projective family for X, we may choose $P \xrightarrow{w} A$ in $\mathscr{A}$ such that $w \cdot a_k = g_k$ for $k \in \mathbf{Z}_{\leq 0}$. Consider the morphism $P \xrightarrow{v + w \cdot j} B$ in $\mathscr{A}$. For $k \in \mathbf{Z}_{\leq 0}$, we have $(v + w \cdot j) \cdot b_k = v \cdot b_k + w \cdot a_k \cdot i_k = v \cdot b_k + g_k \cdot i_k = f_k$. We conclude that $(B, (b_k)_{k \in \mathbf{Z}})$ is a projective family for Y. $\square$

3.2.48 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$.

(a) Suppose given a projective family $(A, (a_k)_{k\in\mathbf{Z}})$ for X. Then $(A, (a_{k+1})_{k\in\mathbf{Z}})$ is a projective family for $X_{[1]}$.

(b) Suppose given a projective family $(A, (a_{k+1})_{k\in\mathbf{Z}})$ for $X_{[1]}$. Then $(A, (a_k)_{k\in\mathbf{Z}})$ is a projective family for X.

$\diamondsuit$

Proof. This follows from lemma 3.2.20. $\square$

3.2.49 Definition. We say that $\mathscr{A}$ *has epilimits* if there exists a limit for every cofiltration $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{A}))$. We say that $\mathscr{A}$ *has monocolimits* if there exists a colimit for every filtration $X \in \mathrm{Ob}(\mathrm{F}(\mathscr{A}))$. $\diamondsuit$

3.2.50 Lemma. Suppose given an additive category $\mathscr{B}$ with countable products. Then $\mathrm{C}(\mathscr{B})$ has epilimits. $\diamondsuit$

Proof. The pointwise limits exist by lemma 3.2.32. They yield a complex by lemma 3.2.24.(d). $\square$

3.2.51 Lemma. Suppose given an additive category $\mathscr{B}$ with countable coproducts. Then $\mathrm{C}(\mathscr{B})$ has monocolimits. $\diamondsuit$

Proof. This is dual to the previous lemma 3.2.50. $\square$

For strict Frobenius categories

Suppose given a strict Frobenius category $\mathscr{F} = (\mathscr{F}, \mathrm{B}, \Sigma, \iota, \pi, \alpha)$.

3.2.52 Definition. Suppose given an exact functor $G\colon \mathscr{F} \to \mathscr{F}$.
Let $\mathrm{F}(G) = \mathbf{Z}(G)|_{\mathrm{F}(\mathscr{F})}^{\mathrm{F}(\mathscr{F})} \colon \mathrm{F}(\mathscr{F}) \to \mathrm{F}(\mathscr{F})$, cf. definition 1.4.3 and lemma 1.4.16.(c). The functor $\mathrm{F}(G)$ is exact by lemma 1.4.16.(a) and remark 1.3.22.

We often abbreviate $\mathbf{B} = \mathrm{F}(\mathrm{B})$ and $\mathbf{\Sigma} = \mathrm{F}(\Sigma)$. $\diamondsuit$

3.2.53 Definition. Suppose given an exact functor $G\colon \mathscr{F} \to \mathscr{F}$.
Let $\mathrm{CF}(G) = \mathbf{Z}(G)|_{\mathrm{CF}(\mathscr{F})}^{\mathrm{CF}(\mathscr{F})} \colon \mathrm{CF}(\mathscr{F}) \to \mathrm{CF}(\mathscr{F})$, cf. definition 1.4.3 and lemma 1.4.16.(d). The functor $\mathrm{CF}(G)$ is exact by lemma 1.4.16.(a) and remark 1.3.22.

We often abbreviate $\mathbf{B} = \mathrm{CF}(\mathrm{B})$ and $\mathbf{\Sigma} = \mathrm{CF}(\Sigma)$. $\diamondsuit$

3.2.54 Definition. Suppose given a full subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$. We say that $\mathscr{S}$ is *closed under epilimits* if for each $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{F}))$ with $X_k \in \mathrm{Ob}(\mathscr{S})$ for $k \in \mathbf{Z}$ and each limit $(A, (a_k)_{k\in\mathbf{Z}})$ for X, we have $A \in \mathrm{Ob}(\mathscr{S})$. $\diamondsuit$

3.2.55 Lemma. Suppose given $\ell \in \mathbf{Z}$, $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{F}))$ and a limit $(A, (a_k)_{k\in\mathbf{Z}})$ for X. Then $(A^{[\ell]}, (a_k^{[\ell]})_{k\in\mathbf{Z}})$ is a limit for $X^{[\ell]}$. $\diamondsuit$

Proof. This follows from the fact that $\Sigma\colon \mathscr{F} \to \mathscr{F}$ is an isomorphism of categories. $\square$

3.2.56 Lemma. Suppose given $\ell \in \mathbf{Z}$, $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{F}))$ and a projective family $(A, (a_k)_{k\in\mathbf{Z}})$ for X. Then $(A^{[\ell]}, (a_k^{[\ell]})_{k\in\mathbf{Z}})$ is a projective family for $X^{[\ell]}$. $\diamondsuit$

Proof. This follows from the fact that $\Sigma, \Sigma^{-1}\colon \mathscr{F} \to \mathscr{F}$ are mutually inverse exact isomorphisms of categories. $\square$

3.3 ∇-diagrams

Suppose given a strict Frobenius category $\mathscr{F} = (\mathscr{F}, \mathrm{B}, \Sigma, \iota, \pi, \alpha)$.

3.3.1 Definition. Let $\mathrm{V} := \{(k, \ell) \in \mathbf{Z} \times \mathbf{Z} \colon k \leq \ell\}$. We write $\ell/k := (k, \ell)$ for $(k, \ell) \in \mathrm{V}$. We define a partial order on V by setting $j/i \leq \ell/k$ if $i \leq k$ and $j \leq \ell$ for $j/i, \ell/k \in \mathrm{V}$.

We consider the functor category $\mathrm{V}(\mathscr{F})$ as an exact category equipped with the pointwise exact structure, cf. convention 17 and definition 1.4.7.

For $X \in \mathrm{Ob}(\mathrm{V}(\mathscr{F}))$ and $j \leq k \leq \ell$ in $\mathbf{Z}$, we write $x_{k,\ell/j} = X_{k/j \to \ell/j}$ and $x_{\ell/j,k} = X_{\ell/j \to \ell/k}$. $\diamondsuit$

3.3.2 Definition. An object $X \in \mathrm{Ob}(\mathrm{V}(\mathscr{F}))$ is called a ∇-*diagram* (or *nabla-diagram*) if $(x_{k,\ell/j}\, ,\, x_{\ell/j,k})$ is a pure short exact sequence in $\mathscr{F}$ for $j \leq k \leq \ell$ in $\mathbf{Z}$. We may think of such an object as a diagram of the following form.

$$\begin{tikzcd}
\ddots & \vdots & \vdots & \vdots & \iddots \\
0 & X_{k+2/k+1} \arrow[l] \arrow[u] & X_{k+2/k} \arrow[l, "x_{k+2/k,k+1}"'] \arrow[u] & X_{k+2/k-1} \arrow[l, "x_{k+2/k-1,k}"'] \arrow[u] & \cdots \arrow[l] \\
 & 0 \arrow[u] & X_{k+1/k} \arrow[l] \arrow[u, "x_{k+1,k+2/k}"] & X_{k+1/k-1} \arrow[l, "x_{k+1/k-1,k}"'] \arrow[u, "x_{k+1,k+2/k-1}"] & \cdots \arrow[l] \\
 & & 0 \arrow[u] & X_{k/k-1} \arrow[l] \arrow[u, "x_{k,k+1/k-1}"] & \cdots \arrow[l] \\
 & & & 0 \arrow[u] & \ddots
\end{tikzcd}$$

We use the letter ∇ because its shape is reminiscent of the shape of the diagram. Let $\nabla(\mathscr{F})$ denote the full subcategory of $\mathrm{V}(\mathscr{F})$ defined by

$$\mathrm{Ob}(\nabla(\mathscr{F})) = \{X \in \mathrm{Ob}(\mathrm{V}(\mathscr{F}))\colon X \text{ is a } \nabla\text{-diagram}\}.$$

We call $\nabla(\mathscr{F})$ the *category of* ∇*-diagrams* in $\mathscr{F}$. $\nabla(\mathscr{F})$ is a full additive subcategory of $\mathrm{V}(\mathscr{F})$ that satifies the conditions (RE1), (RE2) (RE3) and (RE4) of definition 1.3.21 by remark 1.4.10 and lemmata 1.4.11, 1.4.12, 1.4.13, 1.4.14, 1.4.15. We equip $\nabla(\mathscr{F})$ with the restricted exact structure $\mathscr{E}_{\nabla,\mathscr{F}}$ of the pointwise exact structure on $\mathrm{V}(\mathscr{F})$. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\nabla(\mathscr{F})$ is a pure short exact sequence if and only if $X_{\ell/k} \xrightarrow{f_{\ell/k}} Y_{\ell/k} \xrightarrow{g_{\ell/k}} Z_{\ell/k}$ is a pure short exact sequence in $\mathscr{F}$ for $\ell/k \in \mathrm{V}$. $\diamondsuit$

3.3.3 Definition. Suppose given an exact functor $F\colon \mathscr{F} \to \mathscr{F}$.
Let $\nabla(F) = \mathrm{V}(F)|_{\nabla(\mathscr{F})}^{\nabla(\mathscr{F})}\colon \nabla(\mathscr{F}) \to \nabla(\mathscr{F})$, cf. definition 1.4.3 and lemma 1.4.16.(b). The functor $\nabla(F)$ is exact by lemma 1.4.16.(a) and remark 1.3.22.

Suppose given exact functors $F, G\colon \mathscr{F} \to \mathscr{F}$ and a transformation $\lambda\colon F \to G$.
Let $\nabla(\lambda) = \mathrm{V}(\lambda)|_{\nabla(\mathscr{F})}^{\nabla(\mathscr{F})}\colon \nabla(F) \to \nabla(G)$, cf. definition 1.4.5.

We often abbreviate $\mathbf{B} = \nabla(\mathrm{B})$, $\mathbf{\Sigma} = \nabla(\Sigma)$, $\boldsymbol{\iota} = \nabla(\iota)$, $\boldsymbol{\pi} = \nabla(\pi)$ and $\boldsymbol{\alpha} = \nabla(\alpha)$. We also write $\Sigma_{\nabla,\mathscr{F}} = \nabla(\Sigma)$. $\diamondsuit$

3.3.4 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{V}(\mathscr{F}))$. Then we have $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$ if and only if the following four conditions hold.

(N1) We have $X_{k/k} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$, i.e. $X_{k/k}$ is a zero object for $k \in \mathbf{Z}$.

(N2) The rectangle

$$\begin{tikzcd}
X_{\ell+1/k+1} & X_{\ell+1/k} \arrow[l, "x_{\ell+1/k,k+1}"'] \\
X_{\ell/k+1} \arrow[u, "x_{\ell,\ell+1/k+1}"] & X_{\ell/k} \arrow[l, "x_{\ell/k,k+1}"'] \arrow[u, "x_{\ell,\ell+1/k}"']
\end{tikzcd}$$

is a square for $k < \ell$ in $\mathbf{Z}$.

(N3) The morphism $x_{\ell,\ell+1/k}$ is a pure monomorphism for $k < \ell$ in $\mathbf{Z}$.

(N4) The morphism $x_{\ell+1/k,k+1}$ is a pure epimorphism for $k < \ell$ in $\mathbf{Z}$.

$\diamondsuit$

Proof. Suppose that $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$. Suppose given $k \in \mathbf{Z}$. We have the pure short exact sequence $(x_{k,k/k}\,,\, x_{k/k,k})$ in $\mathscr{F}$. Thus $1_{X_{k/k}} = 0$. We conclude that $X_{k/k} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$.
Note that the following rectangle is a square since $(x_{k,k+1/k-1}\,,\, x_{k+1/k-1,k})$ is a pure short exact sequence in $\mathscr{F}$.

$$\begin{tikzcd}
X_{k+1/k} & X_{k+1/k-1} \arrow[l, "x_{k+1/k-1,k}"'] \\
X_{k/k} \arrow[u] & X_{k/k-1} \arrow[l] \arrow[u, "x_{k,k+1/k-1}"']
\end{tikzcd}$$

Suppose given $k < \ell$ in $\mathbf{Z}$. We have the pure short exact sequences $X_{\ell/k} \xrightarrow{x_{\ell,\ell+1/k}} X_{\ell+1/k} \xrightarrow{x_{\ell+1/k,\ell}} X_{\ell+1/\ell}$ and $X_{k+1/k} \xrightarrow{x_{k+1,\ell+1/k}} X_{\ell+1/k} \xrightarrow{x_{\ell+1/k,k+1}} X_{\ell+1/k+1}$ in $\mathscr{F}$. It remains to show that the following rectangle is a square.

$$\begin{tikzcd}
X_{\ell+1/k+1} & X_{\ell+1/k} \arrow[l, "x_{\ell+1/k,k+1}"'] \\
X_{\ell/k+1} \arrow[u, "x_{\ell,\ell+1/k+1}"] & X_{\ell/k} \arrow[l, "x_{\ell/k,k+1}"] \arrow[u, "x_{\ell,\ell+1/k}"']
\end{tikzcd}$$

If $\ell = k+1$, then this follows from the discussion above. Suppose that $\ell > k+1$. We have the pure short exact sequences $X_{\ell/k+1} \xrightarrow{x_{\ell,\ell+1/k+1}} X_{\ell+1/k+1} \xrightarrow{x_{\ell+1/k+1,\ell}} X_{\ell+1/\ell}$ and $X_{k+1/k} \xrightarrow{x_{k+1,\ell/k}} X_{\ell/k} \xrightarrow{x_{\ell/k,k+1}} X_{\ell/k+1}$ in $\mathscr{F}$. The result now follows by applying the pasting lemma 1.1.1.(a,b) to the following diagram.

$$\begin{tikzcd}
X_{\ell+1/\ell} & X_{\ell+1/k+1} \arrow[l, "x_{\ell+1/k+1,\ell}"'] & X_{\ell+1/k} \arrow[l, "x_{\ell+1/k,k+1}"'] \\
X_{\ell/\ell} \arrow[u] & X_{\ell/k+1} \arrow[l] \arrow[u, "x_{\ell,\ell+1/k+1}"] & X_{\ell/k} \arrow[l, "x_{\ell/k,k+1}"'] \arrow[u, "x_{\ell,\ell+1/k}"'] \\
 & X_{k+1/k+1} \arrow[u] & X_{k+1/k} \arrow[l] \arrow[u, "x_{k+1,\ell/k}"']
\end{tikzcd}$$

Conversely, suppose that the conditions (N1), (N2), (N3) and (N4) hold.
By induction, $x_{k,\ell/j}$ is a pure monomorphism and $x_{\ell/j,k}$ is a pure epimorphism for $j \leq k \leq \ell$ in $\mathbf{Z}$. Also by induction, the pasting lemma 1.1.1.(a,b) yields that all rectangles of the following form, where $j/i \leq \ell/k$ in V, are squares.

$$\begin{tikzcd}
X_{\ell/k} & X_{\ell/i} \arrow[l, "x_{\ell/i,k}"'] \\
X_{j/k} \arrow[u, "x_{j,\ell/k}"] & X_{j/i} \arrow[l, "x_{j/i,k}"] \arrow[u, "x_{j,\ell/i}"']
\end{tikzcd}$$

In particular, we have the following square for $j \leq k \leq \ell$ in $\mathbf{Z}$ with $X_{k/k} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$.

$$\begin{tikzcd}
X_{\ell/k} & X_{\ell/j} \arrow[l, "x_{\ell/j,k}"'] \\
X_{k/k} \arrow[u] & X_{k/j} \arrow[l] \arrow[u, "x_{k,\ell/j}"']
\end{tikzcd}$$

We conclude that $(x_{k,\ell/j}\, , x_{\ell/j,k})$ is a pure short exact sequence in $\mathscr{F}$. $\square$

3.3.5 Lemma. Suppose given the following squares in $\mathscr{F}$ for $j+1 < \ell$ in $\mathbf{Z}$, where the vertical morphisms are pure monomorphisms and the horizontal morphisms are pure epimorphisms, as indicated.

$$\begin{tikzcd}
X_{\ell+1/j+1} & X_{\ell+1/j} \arrow[l, "x_{\ell+1/j,j+1}"'] \\
X_{\ell/j+1} \arrow[u, "x_{\ell,\ell+1/j+1}"] & X_{\ell/j} \arrow[l, "x_{\ell/j,j+1}"] \arrow[u, "x_{\ell,\ell+1/j}"']
\end{tikzcd}$$

Suppose that $X_{k/k-1} \xrightarrow{x_{k,k+1/k-1}} X_{k+1/k-1} \xrightarrow{x_{k+1/k-1,k}} X_{k+1/k}$ is a pure short exact sequence in $\mathscr{F}$ for $k \in \mathbf{Z}$. Then there exists an object $Y \in \mathrm{Ob}(\nabla(\mathscr{F}))$ such that

$$y_{\ell/j,j+1} = x_{\ell/j,j+1} \text{ and } y_{\ell,\ell+1/j+1} = x_{\ell,\ell+1/j+1}$$

for $j+1 < \ell$ in $\mathbf{Z}$. In particular, we have $Y_{\ell/j} = X_{\ell/j}$ for $j < \ell$ in $\mathbf{Z}$. $\diamondsuit$

Proof. This is an application of the previous lemma 3.3.4. $\square$

3.3.6 Definition. An object $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$ is called *pointwise bijective* if $X_{\ell/k}$ is a bijective object in $\mathscr{F}$ for $\ell/k \in \mathrm{V}$. Let $\mathfrak{B}_{\nabla,\mathscr{F}}$ denote the full subcategory of $\nabla(\mathscr{F})$ defined by $\mathrm{Ob}(\mathfrak{B}_{\nabla,\mathscr{F}}) = \{X \in \mathrm{Ob}(\nabla(\mathscr{F}))\colon X \text{ is pointwise bijective}\}$. $\diamondsuit$

3.3.7 Lemma. Suppose given a pointwise bijective object $X \in \mathrm{Ob}(\mathfrak{B}_{\nabla,\mathscr{F}})$. Then X is bijective in $\nabla(\mathscr{F})$. $\diamondsuit$

Proof. We want to show that X is injective in $\nabla(\mathscr{F})$. Suppose given $Y \xrightarrow{g} X$ and a pure monomorphism $Y \xrightarrow{m} Z$ in $\nabla(\mathscr{F})$. We want to construct $Z \xrightarrow{h} X$ in $\nabla(\mathscr{F})$ recursively.

$$\begin{tikzcd}
Y \arrow[r, "m"] \arrow[d, "g"'] & Z \arrow[dl, "h"] \\
X &
\end{tikzcd}$$

For $k \in \mathbf{Z}$, we may choose $h_{k/k} = 0\colon Z_{k/k} \to X_{k/k}$.

For $k \in \mathbf{Z}$, we may choose $Z_{k/k-1} \xrightarrow{h_{k/k-1}} X_{k/k-1}$ such that $m_{k/k-1} \cdot h_{k/k-1} = g_{k/k-1}$ since $X_{k/k-1}$ is injective in $\mathscr{F}$.

$$\begin{tikzcd}
Y_{k/k-1} \arrow[r, "m_{k/k-1}"] \arrow[d, "g_{k/k-1}"'] & Z_{k/k-1} \arrow[dl, "h_{k/k-1}"] \\
X_{k/k-1} &
\end{tikzcd}$$

So we have $m_{\ell/k} \cdot h_{\ell/k} = g_{\ell/k}$ for $\ell, k \in \mathbf{Z}$ with $0 \leq \ell - k \leq 1$.

Moreover, we have $z_{\ell/k,k+1} \cdot h_{\ell/k+1} = 0 = h_{\ell/k} \cdot x_{\ell/k,k+1}$ and $z_{\ell-1,\ell/k} \cdot h_{\ell/k} = 0 = h_{\ell-1/k} \cdot x_{\ell-1,\ell/k}$ for $\ell, k \in \mathbf{Z}$ with $\ell - k = 1$.

Suppose given $k, \ell \in \mathbf{Z}$ with $\ell - k > 1$. Suppose we have constructed $Z_{j/i} \xrightarrow{h_{j/i}} X_{j/i}$ for $i, j \in \mathbf{Z}$ with $0 \leq j - i < \ell - k$ such that $m_{j/i} \cdot h_{j/i} = g_{j/i}$, such that $z_{j/i,i+1} \cdot h_{j/i+1} = h_{j/i} \cdot x_{j/i,i+1}$ and such that $z_{j-1,j/i} \cdot h_{j/i} = h_{j-1/i} \cdot x_{j-1,j/i}$ for $0 < j - i < \ell - k$.

Since $X_{\ell/k+1}$ is projective in $\mathscr{F}$, we may choose $X_{\ell/k+1} \xrightarrow{t} X_{\ell/k}$ such that $t \cdot x_{\ell/k,k+1} = 1$, cf. lemma 1.3.11. Let $a = z_{\ell/k,k+1} \cdot h_{\ell/k+1} \cdot t \colon Z_{\ell/k} \to X_{\ell/k}$. We have $a \cdot x_{\ell/k,k+1} = z_{\ell/k,k+1} \cdot h_{\ell/k+1}$.

$$\begin{tikzcd}
Z_{\ell/k} \arrow[d, "z_{\ell/k,k+1}"'] \arrow[ddr, "a"] & \\
Z_{\ell/k+1} \arrow[d, "h_{\ell/k+1}"'] & \\
X_{\ell/k+1} & X_{\ell/k} \arrow[l, "x_{\ell/k,k+1}"']
\end{tikzcd}$$

We have

$$\begin{aligned}
& (h_{\ell-1/k} \cdot x_{\ell-1,\ell/k} - z_{\ell-1,\ell/k} \cdot a) \cdot x_{\ell/k,k+1} \\
&= h_{\ell-1/k} \cdot x_{\ell-1/k,k+1} \cdot x_{\ell-1,\ell/k+1} - z_{\ell-1,\ell/k} \cdot \ z_{\ell/k,k+1} \cdot h_{\ell/k+1} \\
&= z_{\ell-1/k,k+1} \cdot h_{\ell-1/k+1} \cdot x_{\ell-1,\ell/k+1} - z_{\ell-1/k,k+1} \cdot \ z_{\ell-1,\ell/k+1} \cdot h_{\ell/k+1} \\
&= z_{\ell-1/k,k+1} \cdot z_{\ell-1,\ell/k+1} \cdot h_{\ell/k+1} - z_{\ell-1/k,k+1} \cdot z_{\ell-1,\ell/k+1} \cdot h_{\ell/k+1} \\
&= 0.
\end{aligned}$$

Since $x_{k+1,\ell/k}$ is a kernel of $x_{\ell/k,k+1}$, there exists a unique morphism $Z_{\ell-1/k} \xrightarrow{b} X_{\ell+1/k}$ in $\mathscr{F}$ such that $b \cdot x_{k+1,\ell/k} = h_{\ell-1/k} \cdot x_{\ell-1,\ell/k} - z_{\ell-1,\ell/k} \cdot a$.
We may choose $Z_{\ell/k} \xrightarrow{c} X_{k+1/k}$ with $z_{\ell-1,\ell/k} \cdot c = b$ since $X_{k+1/k}$ is injective in $\mathscr{F}$.

$$\begin{tikzcd}
 & Z_{\ell-1/k} \arrow[r, "z_{\ell-1,\ell/k}"] \arrow[d, "h_{\ell-1/k} \cdot x_{\ell-1,\ell/k} - z_{\ell-1,\ell/k} \cdot a"'] \arrow[dr, "b"] & Z_{\ell/k} \arrow[d, "c"] \\
X_{\ell/k+1} & X_{\ell/k} \arrow[l, "x_{\ell/k,k+1}"] & X_{k+1/k} \arrow[l, "x_{k+1,\ell/k}"]
\end{tikzcd}$$

We have

$$\begin{aligned}
& y_{\ell-1,\ell/k} \cdot (g_{\ell/k} - m_{\ell/k} \cdot a - m_{\ell/k} \cdot c \cdot x_{k+1,\ell/k}) \\
&= g_{\ell-1/k} \cdot x_{\ell-1,\ell/k} - m_{\ell-1/k} \cdot z_{\ell-1,\ell/k} \cdot a - m_{\ell-1/k} \cdot z_{\ell-1,\ell/k} \cdot c \cdot x_{k+1,\ell/k} \\
&= g_{\ell-1/k} \cdot x_{\ell-1,\ell/k} - m_{\ell-1/k} \cdot z_{\ell-1,\ell/k} \cdot a - m_{\ell-1/k} \cdot b \cdot x_{k+1,\ell/k} \\
&= g_{\ell-1/k} \cdot x_{\ell-1,\ell/k} - m_{\ell-1/k} \cdot z_{\ell-1,\ell/k} \cdot a - m_{\ell-1/k} \cdot (h_{\ell-1/k} \cdot x_{\ell-1,\ell/k} - z_{\ell-1,\ell/k} \cdot a) \\
&= g_{\ell-1/k} \cdot x_{\ell-1,\ell/k} - m_{\ell-1/k} \cdot h_{\ell-1/k} \cdot x_{\ell-1,\ell/k} \\
&= (g_{\ell-1/k} - m_{\ell-1/k} \cdot h_{\ell-1/k}) \cdot x_{\ell-1,\ell/k} \\
&= 0.
\end{aligned}$$

Since $y_{\ell/k,\ell-1}$ is a cokernel of $y_{\ell-1,\ell/k}$, there exists a unique morphism $Y_{\ell/\ell-1} \xrightarrow{d} X_{\ell/k}$ in $\mathscr{F}$ such that $y_{\ell/k,\ell-1} \cdot d = g_{\ell/k} - m_{\ell/k} \cdot a - m_{\ell/k} \cdot c \cdot x_{k+1,\ell/k}$.

We have

$$
\begin{aligned}
& y_{\ell/k,\ell-1} \cdot d \cdot x_{\ell/k,k+1} \\
&= (g_{\ell/k} - m_{\ell/k} \cdot a - m_{\ell/k} \cdot c \cdot x_{k+1,\ell/k}) \cdot x_{\ell/k,k+1} \\
&= y_{\ell/k,k+1} \cdot g_{\ell/k+1} - m_{\ell/k} \cdot z_{\ell/k,k+1} \cdot h_{\ell/k+1} - m_{\ell/k} \cdot c \cdot 0 \\
&= y_{\ell/k,k+1} \cdot g_{\ell/k+1} - y_{\ell/k,k+1} \cdot m_{\ell/k+1} \cdot h_{\ell/k+1} \\
&= y_{\ell/k,k+1} \cdot (g_{\ell/k+1} - m_{\ell/k+1} \cdot h_{\ell/k+1}) \\
&= 0.
\end{aligned}
$$

Since $y_{\ell/k,\ell-1}$ is a pure epimorphism, we conclude that $d \cdot x_{\ell/k,k+1} = 0$. Since $x_{k+1,\ell/k}$ is a kernel of $x_{\ell/k,k+1}$, there exists a unique morphism $Y_{\ell/\ell-1} \xrightarrow{e} X_{k+1/k}$ in $\mathscr{F}$ such that $e \cdot x_{k+1,\ell/k} = d$. We may choose $Z_{\ell/\ell-1} \xrightarrow{f} X_{k+1/k}$ with $m_{\ell/\ell-1} \cdot f = e$ since $X_{k+1/k}$ is injective in $\mathscr{F}$.

$$
\begin{tikzcd}
Y_{\ell-1/k} \arrow[r, "y_{\ell-1,\ell/k}"] & Y_{\ell/k} \arrow[r, "y_{\ell/k,\ell-1}"] \arrow[d, "g_{\ell/k}-m_{\ell/k}\cdot a-m_{\ell/k}\cdot c\cdot x_{k+1,\ell/k}"'] & Y_{\ell/\ell-1} \arrow[r, "m_{\ell/\ell-1}"] \arrow[d, "e"] \arrow[dl, "d"'] & Z_{\ell/\ell-1} \arrow[dl, "f"] \\
X_{\ell/k+1} & X_{\ell/k} \arrow[l, "x_{\ell/k,k+1}"] & X_{k+1/k} \arrow[l, "x_{k+1,\ell/k}"] &
\end{tikzcd}
$$

Let $h_{\ell/k} = a + c \cdot x_{k+1,\ell/k} + z_{\ell/k,\ell-1} \cdot f \cdot x_{k+1,\ell/k}$.
We have

$$
\begin{aligned}
m_{\ell/k} \cdot h_{\ell/k} &= m_{\ell/k} \cdot a + m_{\ell/k} \cdot c \cdot x_{k+1,\ell/k} + m_{\ell/k} \cdot z_{\ell/k,\ell-1} \cdot f \cdot x_{k+1,\ell/k} \\
&= m_{\ell/k} \cdot a + m_{\ell/k} \cdot c \cdot x_{k+1,\ell/k} + y_{\ell/k,\ell-1} \cdot m_{\ell/\ell-1} \cdot f \cdot x_{k+1,\ell/k} \\
&= m_{\ell/k} \cdot a + m_{\ell/k} \cdot c \cdot x_{k+1,\ell/k} + y_{\ell/k,\ell-1} \cdot e \cdot x_{k+1,\ell/k} \\
&= m_{\ell/k} \cdot a + m_{\ell/k} \cdot c \cdot x_{k+1,\ell/k} + y_{\ell/k,\ell-1} \cdot d \\
&= m_{\ell/k} \cdot a + m_{\ell/k} \cdot c \cdot x_{k+1,\ell/k} + g_{\ell/k} - m_{\ell/k} \cdot a - m_{\ell/k} \cdot c \cdot x_{k+1,\ell/k} \\
&= g_{\ell/k} ,
\end{aligned}
$$

$$
\begin{aligned}
h_{\ell/k} \cdot x_{\ell/k,k+1} &= a \cdot x_{\ell/k,k+1} + c \cdot x_{k+1,\ell/k} \cdot x_{\ell/k,k+1} + z_{\ell/k,\ell-1} \cdot f \cdot x_{k+1,\ell/k} \cdot x_{\ell/k,k+1} \\
&= z_{\ell/k,k+1} \cdot h_{\ell/k+1}
\end{aligned}
$$

and

$$
\begin{aligned}
z_{\ell-1,\ell/k} \cdot h_{\ell/k} &= z_{\ell-1,\ell/k} \cdot a + z_{\ell-1,\ell/k} \cdot c \cdot x_{k+1,\ell/k} + z_{\ell-1,\ell/k} \cdot z_{\ell/k,\ell-1} \cdot f \cdot x_{k+1,\ell/k} \\
&= z_{\ell-1,\ell/k} \cdot a + b \cdot x_{k+1,\ell/k} \\
&= z_{\ell-1,\ell/k} \cdot a + h_{\ell-1/k} \cdot x_{\ell-1,\ell/k} - z_{\ell-1,\ell/k} \cdot a \\
&= h_{\ell-1/k} \cdot x_{\ell-1,\ell/k} .
\end{aligned}
$$

$$
\begin{tikzcd}
Y_{\ell/k+1} \arrow[drrrr, "\bullet" marking, "m_{\ell/k+1}"' near end] \arrow[ddddr, "g_{\ell/k+1}"'] & & Y_{\ell/k} \arrow[ll, "\shortmid" marking] \arrow[drrr, "\bullet" marking, "m_{\ell/k}"] \arrow[ddddr, "g_{\ell/k}"] & & & \\
& & & & Z_{\ell/k+1} \arrow[dddlll, bend left, "h_{\ell/k+1}"] & Z_{\ell/k} \arrow[l, "\shortmid" marking] \arrow[dddll, bend left, "h_{\ell/k}"] \\
& & Y_{\ell-1/k} \arrow[uu, "\bullet" marking] \arrow[drrr, "\bullet" marking, "m_{\ell-1/k}"] \arrow[dddr, "g_{\ell-1/k}"'] & & & \\
& & & & & Z_{\ell-1/k} \arrow[uu, "\bullet" marking] \arrow[ddll, bend left, "h_{\ell-1/k}"] \\
& X_{\ell/k+1} & & X_{\ell/k} \arrow[ll, "\shortmid" marking] & & \\
& & & X_{\ell-1/k} \arrow[u, "\bullet" marking] & &
\end{tikzcd}
$$

We conclude that X is injective in $\nabla(\mathscr{F})$. Dually, it is also projective in $\nabla(\mathscr{F})$. $\square$

3.3.8 Theorem. The tuple $(\nabla(\mathscr{F}), \mathbf{B}, \boldsymbol{\Sigma}, \boldsymbol{\iota}, \boldsymbol{\pi}, \boldsymbol{\alpha})$ is a strict Frobenius category. Moreover, we have $\mathscr{E}^{\mathrm{p}}_{\nabla(\mathscr{F})} = \mathscr{E}_{\nabla, \mathscr{F}}$, cf. definitions 3.3.2 and 2.1.16. $\diamondsuit$

Proof. The functor $\mathbf{B} = \nabla(\mathrm{B})$ is additive, cf. definition 1.4.3 and remark 1.2.5.(b). The functor $\boldsymbol{\Sigma} = \nabla(\Sigma)$ is an isomorphism of categories with inverse $\nabla(\Sigma^{-1})$ since we have

$$
\begin{aligned}
(\nabla(\Sigma) \cdot \nabla(\Sigma^{-1})) \star \mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})} &= (\nabla(\Sigma) \cdot \nabla(\Sigma^{-1})) \star (1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}} \cdot 1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}}) \\
&= (\nabla(\Sigma) \star 1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}}) \cdot (\nabla(\Sigma^{-1}) \star 1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}}) \\
&= (1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}} \star \mathrm{V}(\Sigma)) \cdot (1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}} \star \mathrm{V}(\Sigma^{-1})) \\
&= (1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}} \cdot 1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}}) \star (\mathrm{V}(\Sigma) \cdot \mathrm{V}(\Sigma^{-1})) \\
&= 1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}} \star (\mathrm{V}(\Sigma \cdot \Sigma^{-1})) \\
&= \mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})} \star \mathrm{V}(1_{\mathscr{F}}) \\
&= \mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})} \star 1_{\mathrm{V}(\mathscr{F})} \\
&= 1_{\nabla(\mathscr{F})} \star \mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}
\end{aligned}
$$

and

$$
\begin{aligned}
(\nabla(\Sigma^{-1}) \cdot \nabla(\Sigma)) \star \mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})} &= (\nabla(\Sigma^{-1}) \cdot \nabla(\Sigma)) \star (1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}} \cdot 1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}}) \\
&= (\nabla(\Sigma^{-1}) \star 1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}}) \cdot (\nabla(\Sigma) \star 1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}}) \\
&= (1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}} \star \mathrm{V}(\Sigma^{-1})) \cdot (1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}} \star \mathrm{V}(\Sigma)) \\
&= (1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}} \cdot 1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}}) \star (\mathrm{V}(\Sigma^{-1}) \cdot \mathrm{V}(\Sigma)) \\
&= 1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}} \star (\mathrm{V}(\Sigma^{-1} \cdot \Sigma)) \\
&= \mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})} \star \mathrm{V}(1_{\mathscr{F}}) \\
&= \mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})} \star 1_{\mathrm{V}(\mathscr{F})} \\
&= 1_{\nabla(\mathscr{F})} \star \mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})} ,
\end{aligned}
$$

cf. lemma 1.4.6.(a,b).

The transformation $\boldsymbol{\alpha} = \nabla(\alpha)$ is an isotransformation since $X\nabla(\alpha) = X\mathrm{V}(\alpha)$ is an isomorphism in $\mathrm{V}(\mathscr{F})$ for $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$, cf. lemma 1.4.6.(c,d).

Ad (SF4). We have $(\boldsymbol{\Sigma} \star \boldsymbol{\iota}) \cdot \boldsymbol{\alpha} = \boldsymbol{\iota} \star \boldsymbol{\Sigma}$ since

$$
\begin{aligned}
((\boldsymbol{\Sigma} \star \boldsymbol{\iota}) \cdot \boldsymbol{\alpha}) \star \mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})} &= ((\nabla(\Sigma) \star \nabla(\iota)) \cdot \nabla(\alpha)) \star (1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}} \cdot 1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}}) \\
&= ((\nabla(\Sigma) \star \nabla(\iota) \star 1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}}) \cdot (\nabla(\alpha) \star 1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}}) \\
&= (1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}} \star \mathrm{V}(\Sigma) \star \mathrm{V}(\iota)) \cdot (1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}} \star \mathrm{V}(\alpha)) \\
&= (1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}} \cdot 1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}}) \star ((\mathrm{V}(\Sigma) \star \mathrm{V}(\iota)) \cdot \mathrm{V}(\alpha)) \\
&= 1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}} \star \mathrm{V}((\Sigma \star \iota) \cdot \alpha)) \\
&= 1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}} \star \mathrm{V}(\iota \star \Sigma) \\
&= 1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}} \star \mathrm{V}(\iota) \star \mathrm{V}(\Sigma), \\
&= \nabla(\iota) \star \nabla(\Sigma) \star 1_{\mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})}} \\
&= (\boldsymbol{\iota} \star \boldsymbol{\Sigma}) \star \mathrm{Inc}^{\mathrm{V}(\mathscr{F})}_{\nabla(\mathscr{F})} ,
\end{aligned}
$$

cf. lemma 1.4.6.(d,e). We have $\boldsymbol{\Sigma} \star \boldsymbol{\pi} = -\boldsymbol{\alpha} \cdot (\boldsymbol{\pi} \star \boldsymbol{\Sigma})$ since

$$\begin{aligned}
(-\boldsymbol{\alpha} \cdot (\boldsymbol{\pi} \star \boldsymbol{\Sigma})) \star \mathrm{Inc}_{\nabla(\mathscr{F})}^{\mathrm{V}(\mathscr{F})} &= (-\nabla(\alpha) \cdot (\nabla(\pi) \star \nabla(\Sigma))) \star (1_{\mathrm{Inc}_{\nabla(\mathscr{F})}^{\mathrm{V}(\mathscr{F})}} \cdot 1_{\mathrm{Inc}_{\nabla(\mathscr{F})}^{\mathrm{V}(\mathscr{F})}}) \\
&= (-\nabla(\alpha) \star 1_{\mathrm{Inc}_{\nabla(\mathscr{F})}^{\mathrm{V}(\mathscr{F})}}) \cdot ((\nabla(\pi) \star \nabla(\Sigma) \star 1_{\mathrm{Inc}_{\nabla(\mathscr{F})}^{\mathrm{V}(\mathscr{F})}}) \\
&= (1_{\mathrm{Inc}_{\nabla(\mathscr{F})}^{\mathrm{V}(\mathscr{F})}} \star \mathrm{V}(-\alpha)) \cdot (1_{\mathrm{Inc}_{\nabla(\mathscr{F})}^{\mathrm{V}(\mathscr{F})}} \star \mathrm{V}(\pi) \star \mathrm{V}(\Sigma)) \\
&= (1_{\mathrm{Inc}_{\nabla(\mathscr{F})}^{\mathrm{V}(\mathscr{F})}} \cdot 1_{\mathrm{Inc}_{\nabla(\mathscr{F})}^{\mathrm{V}(\mathscr{F})}}) \star (\mathrm{V}(-\alpha) \cdot (\mathrm{V}(\pi) \star \mathrm{V}(\Sigma))) \\
&= 1_{\mathrm{Inc}_{\nabla(\mathscr{F})}^{\mathrm{V}(\mathscr{F})}} \star \mathrm{V}(-\alpha \cdot (\pi \star \Sigma))) \\
&= 1_{\mathrm{Inc}_{\nabla(\mathscr{F})}^{\mathrm{V}(\mathscr{F})}} \star \mathrm{V}(\Sigma \star \pi) \\
&= 1_{\mathrm{Inc}_{\nabla(\mathscr{F})}^{\mathrm{V}(\mathscr{F})}} \star \mathrm{V}(\Sigma) \star \mathrm{V}(\pi), \\
&= \nabla(\Sigma) \star \nabla(\pi) \star 1_{\mathrm{Inc}_{\nabla(\mathscr{F})}^{\mathrm{V}(\mathscr{F})}} \\
&= (\boldsymbol{\Sigma} \star \boldsymbol{\pi}) \star \mathrm{Inc}_{\nabla(\mathscr{F})}^{\mathrm{V}(\mathscr{F})},
\end{aligned}$$

cf. lemma 1.4.6.(d,e,f).

Ad (SF1),(SF2),(SF3).

Note that for $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$, $(X\boldsymbol{\iota}, X\boldsymbol{\pi})$ is a pure short exact sequence in $(\nabla(\mathscr{F}), \mathscr{E}_{\nabla,\mathscr{F}})$ since $((X\boldsymbol{\iota})_{\ell/k}\,,\, (X\boldsymbol{\pi})_{\ell/k}) = (X_{\ell/k}\iota\,,\, X_{\ell/k}\pi)$ is a pure short exact sequence in $\mathscr{F}$ for $\ell/k \in \mathrm{V}$. Also note that for $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$, the object $X\mathrm{B}$ is bijective in $(\nabla(\mathscr{F}), \mathscr{E}_{\nabla,\mathscr{F}})$ by lemma 3.3.7. This yields (SF1),(SF2) and (SF3). Moreover, we have $\mathscr{E}^{\mathrm{p}}_{\nabla(\mathscr{F})} = \mathscr{E}_{\nabla,\mathscr{F}}$ by lemma 2.1.34. □

3.3.9 Definition. By theorem 3.3.8, $\nabla(\mathscr{F}) = (\nabla(\mathscr{F}), \mathbf{B}, \boldsymbol{\Sigma}, \boldsymbol{\iota}, \boldsymbol{\pi}, \boldsymbol{\alpha})$ is a strict Frobenius category. We denote its stable category by $\underline{\nabla}(\mathscr{F}) = \underline{\nabla(\mathscr{F})}$, cf. definition 1.3.27. Note that $\mathfrak{P}_{\nabla(\mathscr{F})} \colon \nabla(\mathscr{F}) \to \underline{\nabla}(\mathscr{F})$ denotes the stabilisation functor of $\nabla(\mathscr{F})$. ◊

3.3.10 Remark. Suppose given $X \xrightarrow{f} Y$ in $\nabla(\mathscr{F})$. Then we have $\underline{f} = 0$ in $\underline{\nabla}(\mathscr{F})$ if and only if there exists $X\mathbf{B} \xrightarrow{g} Y$ in $\nabla(\mathscr{F})$ such that $X\boldsymbol{\iota} \cdot g = f$, cf. lemma 1.3.28. ◊

3.3.11 Lemma/Definition. We define the translation functors $\mathrm{T}_{\nabla,\mathscr{F}}\,,\, \mathrm{T}^{-}_{\nabla,\mathscr{F}} \colon \nabla(\mathscr{F}) \to \nabla(\mathscr{F})$ as follows. For $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$ and $j/i \leq \ell/k$ in V, let

$$(X\mathrm{T}_{\nabla,\mathscr{F}})_{\ell/k} = X_{\ell+1/k+1} \qquad , \qquad (X\mathrm{T}^{-}_{\nabla,\mathscr{F}})_{\ell/k} = X_{\ell-1/k-1}\,,$$

$$(X\mathrm{T}_{\nabla,\mathscr{F}})_{j/i \rightarrowtail \ell/k} = X_{j+1/i+1 \rightarrowtail \ell+1/k+1} \quad \text{and} \quad (X\mathrm{T}^{-}_{\nabla,\mathscr{F}})_{j/i \rightarrowtail \ell/k} = X_{j-1/i-1 \rightarrowtail \ell-1/k-1}\,.$$

For $f \in \mathrm{Mor}(\nabla(\mathscr{F}))$ and $\ell/k \in \mathrm{V}$, let $(f\mathrm{T}_{\nabla,\mathscr{F}})_{\ell/k} = f_{\ell+1/k+1}$ and $(f\mathrm{T}^{-}_{\nabla,\mathscr{F}})_{\ell/k} = f_{\ell-1/k-1}$.

The functors $\mathrm{T}_{\nabla,\mathscr{F}}$ and $\mathrm{T}^{-}_{\nabla,\mathscr{F}}$ are mutually inverse isomorphisms of categories.

For $k \in \mathbf{Z}$ and $X \xrightarrow{f} Y$ in $\nabla(\mathscr{F})$, we often write $X_{[k]} = X\mathrm{T}^{k}_{\nabla,\mathscr{F}}$ and $f_{[k]} = f\mathrm{T}^{k}_{\nabla,\mathscr{F}}$. ◊

Proof. We abbreviate $\mathrm{T} = \mathrm{T}_{\nabla,\mathscr{F}}$ and $\mathrm{T}^{-} = \mathrm{T}^{-}_{\nabla,\mathscr{F}}$. Suppose given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\nabla(\mathscr{F})$.

We have $1_X\mathrm{T} = 1_{X\mathrm{T}}$ since $(1_X\mathrm{T})_{\ell/k} = 1_{X_{\ell+1/k+1}} = (1_{X\mathrm{T}})_{\ell/k}$ for $\ell/k \in \mathrm{V}$.
We have $(fg)\mathrm{T} = f\mathrm{T} \cdot g\mathrm{T}$ since $((fg)\mathrm{T})_{\ell/k} = f_{\ell+1/k+1} \cdot g_{\ell+1/k+1} = (f\mathrm{T} \cdot g\mathrm{T})_{\ell/k}$ for $\ell/k \in \mathrm{V}$.
We have $1_X\mathrm{T}^- = 1_{X\mathrm{T}^-}$ since $(1_X\mathrm{T}^-)_{\ell/k} = 1_{X_{\ell-1/k-1}} = (1_{X\mathrm{T}^-})_{\ell/k}$ for $\ell/k \in \mathrm{V}$.
We have $(fg)\mathrm{T}^- = f\mathrm{T}^- \cdot g\mathrm{T}^-$ since $((fg)\mathrm{T}^-)_{\ell/k} = f_{\ell-1/k-1} \cdot g_{\ell-1/k-1} = (f\mathrm{T}^- \cdot g\mathrm{T}^-)_{\ell/k}$ for $\ell/k \in \mathrm{V}$.
We have $f\mathrm{T}\mathrm{T}^- = f$ since $(f\mathrm{T}\mathrm{T}^-)_{\ell/k} = f_{\ell/k}$ for $\ell/k \in \mathrm{V}$.
We have $f\mathrm{T}^-\mathrm{T} = f$ since $(f\mathrm{T}^-\mathrm{T})_{\ell/k} = f_{\ell/k}$ for $\ell/k \in \mathrm{V}$. □

3.3.12 Remark. Suppose given $k, \ell \in \mathbf{Z}$. We have $\Sigma^k_{\nabla,\mathscr{F}} \cdot \mathrm{T}^\ell_{\nabla,\mathscr{F}} = \mathrm{T}^\ell_{\nabla,\mathscr{F}} \cdot \Sigma^k_{\nabla,\mathscr{F}}$. ◊

Proof. Suppose given $X \xrightarrow{f} Y$ in $\nabla(\mathscr{F})$. We have $f\Sigma^k_{\nabla,\mathscr{F}}\mathrm{T}^\ell_{\nabla,\mathscr{F}} = f\mathrm{T}^\ell_{\nabla,\mathscr{F}}\Sigma^k_{\nabla,\mathscr{F}}$ since

$$(f\Sigma^k_{\nabla,\mathscr{F}}\mathrm{T}^\ell_{\nabla,\mathscr{F}})_{j/i} = f^{[k]}_{j+\ell/i+\ell} = (f\mathrm{T}^\ell_{\nabla,\mathscr{F}}\Sigma^k_{\nabla,\mathscr{F}})_{j/i}$$

for $j/i \in \mathrm{V}$. □

3.3.13 Lemma/Definition. We define the transformation $\rho_{\nabla,\mathscr{F}} \colon 1_{\nabla(\mathscr{F})} \to \mathrm{T}_{\nabla,\mathscr{F}}$ by setting $(X\rho_{\nabla,\mathscr{F}})_{\ell/k} = X_{\ell/k \rightarrow \ell+1/k+1}$ for $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$ and $\ell/k \in \mathrm{V}$. This in fact defines a transformation. We sometimes abbreviate $\rho = \rho_{\nabla,\mathscr{F}}$. ◊

Proof. We abbreviate $\rho = \rho_{\nabla,\mathscr{F}}$ and $\mathrm{T} = \mathrm{T}_{\nabla,\mathscr{F}}$. Suppose given $X \xrightarrow{f} Y$ in $\nabla(\mathscr{F})$.
We have $X\rho \cdot f\mathrm{T} = f \cdot Y\rho$ since

$$(X\rho \cdot f\mathrm{T})_{\ell/k} = X_{\ell/k \rightarrow \ell+1/k+1} \cdot f_{\ell+1/k+1} = f_{\ell/k} \cdot Y_{\ell/k \rightarrow \ell+1/k+1} = (f \cdot Y\rho)_{\ell/k}$$

for $\ell/k \in \mathrm{V}$. □

3.3.14 Lemma. We have $\mathrm{T}_{\nabla,\mathscr{F}} \star \rho_{\nabla,\mathscr{F}} = \rho_{\nabla,\mathscr{F}} \star \mathrm{T}_{\nabla,\mathscr{F}}$. ◊

Proof. We abbreviate $\mathrm{T} = \mathrm{T}_{\nabla,\mathscr{F}}$ and $\rho = \rho_{\nabla,\mathscr{F}}$. We have $\mathrm{T} \star \rho = \rho \star \mathrm{T}$ since
$(X(\mathrm{T} \star \rho))_{\ell/k} = X_{\ell+1/k+1 \rightarrow \ell+2/k+2} = (X(\rho \star \mathrm{T}))_{\ell/k}$ for $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$ and $\ell/k \in \mathrm{V}$. □

3.3.15 Definition. The set $\mathfrak{J}_{\nabla(\mathscr{F}),\mathrm{T}_{\nabla,\mathscr{F}},\rho_{\nabla,\mathscr{F}},\mathfrak{B}_{\nabla,\mathscr{F}}}$ is an ideal in $\nabla(\mathscr{F})$, cf. definition 3.1.1. We abbreviate $\mathfrak{J}_{\nabla,\mathscr{F}} = \mathfrak{J}_{\nabla(\mathscr{F}),\mathrm{T}_{\nabla,\mathscr{F}},\rho_{\nabla,\mathscr{F}},\mathfrak{B}_{\nabla,\mathscr{F}}}$. Let $\underline{\underline{\nabla}}(\mathscr{F}) = \nabla(\mathscr{F})/\mathfrak{J}_{\nabla,\mathscr{F}}$ denote the corresponding factor category and let $\mathfrak{Q}_{\nabla,\mathscr{F}} \colon \nabla(\mathscr{F}) \to \underline{\underline{\nabla}}(\mathscr{F})$ denote the correspoding residue class functor, cf. definition 1.2.13. For $X \xrightarrow{f} Y$ in $\nabla(\mathscr{F})$, we write $\underline{\underline{f}} = f + {}_{\nabla(\mathscr{F}),\mathfrak{Q}_{\nabla,\mathscr{F}}}(X,Y)$. ◊

3.3.16 Remark. Suppose given $X \xrightarrow{f} Y$ in $\nabla(\mathscr{F})$. The following statements are equivalent, cf. lemmata 3.1.2, 3.3.14, definition 1.3.27 and remark 3.3.10.

(a) We have $\underline{\underline{f}} = 0$ in $\underline{\underline{\nabla}}(\mathscr{F})$.

(b) We have $\underline{X_{[-1]}\rho_{\nabla,\mathscr{F}} \cdot f} = 0$ in $\underline{\nabla}(\mathscr{F})$.

(c) There exists $X_{[-1]}\mathbf{B} \xrightarrow{g} Y$ in $\nabla(\mathscr{F})$ such that $X_{[-1]}\rho_{\nabla,\mathscr{F}} \cdot f = X_{[-1]}\boldsymbol{\iota} \cdot g$. ◊

3.3.17 Definition. Suppose given a full subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$. We define the full subcategory $\nabla_{\mathscr{S}}(\mathscr{F})$ of $\nabla(\mathscr{F})$ by setting

$$\mathrm{Ob}(\nabla_{\mathscr{S}}(\mathscr{F})) = \{X \in \mathrm{Ob}(\nabla(\mathscr{F}))\colon X^{[-k]}_{k/k-1} \in \mathrm{Ob}(\mathscr{S}) \text{ for } k \in \mathbf{Z}\}.$$

We define the full subcategory $\underline{\nabla}_{\mathscr{S}}(\mathscr{F})$ of $\underline{\nabla}(\mathscr{F})$ by setting $\mathrm{Ob}(\underline{\nabla}_{\mathscr{S}}(\mathscr{F})) = \mathrm{Ob}(\nabla_{\mathscr{S}}(\mathscr{F}))$. We define the full subcategory $\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})$ of $\underline{\underline{\nabla}}(\mathscr{F})$ by setting $\mathrm{Ob}(\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})) = \mathrm{Ob}(\nabla_{\mathscr{S}}(\mathscr{F}))$. $\diamondsuit$

3.3.18 Definition. Suppose given a full subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$. Let

$$\mathfrak{P}_{\nabla(\mathscr{F}),\mathscr{S}} = \mathfrak{P}_{\nabla(\mathscr{F})}|^{\underline{\nabla}_{\mathscr{S}}(\mathscr{F})}_{\nabla_{\mathscr{S}}(\mathscr{F})}\colon \nabla_{\mathscr{S}}(\mathscr{F}) \to \underline{\nabla}_{\mathscr{S}}(\mathscr{F})$$

and let

$$\mathfrak{Q}_{\nabla,\mathscr{F},\mathscr{S}} = \mathfrak{Q}_{\nabla,\mathscr{F}}|^{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})}_{\nabla_{\mathscr{S}}(\mathscr{F})}\colon \nabla_{\mathscr{S}}(\mathscr{F}) \to \underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F}).$$

Cf. definitions 3.3.9 and 3.3.15. $\diamondsuit$

3.3.19 Lemma. Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$. Then $\nabla_{\mathscr{S}}(\mathscr{F})$ is a strictly full additive subcategory of $\nabla(\mathscr{F})$ as well. $\diamondsuit$

Proof. Note that zero objects, direct sums and isomorphisms in $\nabla(\mathscr{F})$ are formed pointwise, cf. remark 1.4.2. The result now follows from the fact that the residue class functor $\mathfrak{P}_{\mathscr{F}}\colon \mathscr{F} \to \underline{\mathscr{F}}$ is additive. $\square$

3.3.20 Remark. Suppose given a full subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$ and $X \in \mathrm{Ob}(\nabla_{\mathscr{S}}(\mathscr{F}))$. For $\ell \in \mathbf{Z}$, we have $X_{\ell/\ell-1} \in \mathrm{Ob}(\mathscr{S}^{[\ell]})$. For $k, \ell \in \mathbf{Z}$ with $k \leq \ell$, we have $X_{\ell/k} \in \mathrm{Ob}\left(\mathop{\ast}\limits_{j\in[k+1,\ell]} \mathscr{S}^{[j]}\right)$. $\diamondsuit$

Proof. For $\ell \in \mathbf{Z}$, we have $X_{\ell/\ell-1} \in \mathrm{Ob}(\mathscr{S}^{[\ell]})$ since $X^{[-\ell]}_{\ell/\ell-1} \in \mathrm{Ob}(\mathscr{S})$.
Suppose given $k, \ell \in \mathbf{Z}$ with $k \leq \ell$.
For $\ell = k$, we have $X_{\ell/k} = X_{\ell/\ell} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}}) \subseteq \mathrm{Ob}(\mathrm{Z}_{\underline{\mathscr{F}}}) = \mathrm{Ob}\left(\mathop{\ast}\limits_{j\in[k+1,\ell]} \mathscr{S}^{[j]}\right)$.
Now suppose that $k < \ell$. We proceed by induction on $\ell - k \in \mathbf{Z}_{\geq 1}$.
For $\ell - k = 1$, we have $X_{\ell/k} = X_{\ell/\ell-1} \in \mathrm{Ob}(\mathscr{S}^{[\ell]}) \subseteq \mathrm{Ob}\left(\mathop{\ast}\limits_{j\in[k+1,\ell]} \mathscr{S}^{[j]}\right)$.
Suppose that $\ell - k > 1$. We obtain a triangle

$$X_{\ell-1/k} \xrightarrow{\underline{x_{\ell-1,\ell/k}}} X_{\ell/k} \xrightarrow{\underline{x_{\ell/k,\ell-1}}} X_{\ell/\ell-1} \longrightarrow X^{[1]}_{\ell-1/k}$$

in $\underline{\mathscr{F}}$, cf. lemma 2.2.9 and definition 2.2.14. Since by induction $X_{\ell-1/k} \in \mathrm{Ob}\left(\mathop{\ast}\limits_{j\in[k+1,\ell-1]} \mathscr{S}^{[j]}\right)$ and $X_{\ell/\ell-1} \in \mathrm{Ob}(\mathscr{S}^{[\ell]})$, we have $X_{\ell/k} \in \mathrm{Ob}\left(\mathop{\ast}\limits_{j\in[k+1,\ell]} \mathscr{S}^{[j]}\right)$. $\square$

3.3.21 Lemma. Suppose given full subcategories $\mathscr{Q},\mathscr{R} \subseteq \underline{\mathscr{F}}$ such that ${}_{\underline{\mathscr{F}}}(\mathscr{Q}^{[k]},\mathscr{R}) = 0$ for $k \in \mathbf{Z}_{>0}$. Suppose given $X \in \mathrm{Ob}(\nabla_{\mathscr{Q}}(\mathscr{F}))$ and $Y \in \mathrm{Ob}(\nabla_{\mathscr{R}}(\mathscr{F}))$. For $i,j,k,\ell \in \mathbf{Z}$ with $i \leq j \leq k \leq \ell$, we have ${}_{\underline{\mathscr{F}}}(X_{\ell/k}\, ,\, Y_{j/i}) = 0$. $\diamondsuit$

Proof. Suppose given $i,j,k,\ell \in \mathbf{Z}$ with $i \leq j \leq k \leq \ell$.
By remark 3.3.20, we have $X_{\ell/k} \in \mathrm{Ob}\left(\mathop{\ast}\limits_{m\in[k+1,\ell]} \mathscr{Q}^{[m]}\right)$ and $Y_{j/i} \in \mathrm{Ob}\left(\mathop{\ast}\limits_{n\in[i+1,j]} \mathscr{R}^{[n]}\right)$. So the result follows from lemma 1.5.10. $\square$

3.3.22 Lemma. Suppose given full subcategories $\mathscr{Q},\mathscr{R} \subseteq \underline{\mathscr{F}}$ such that ${}_{\underline{\mathscr{F}}}(\mathscr{Q}^{[k]},\mathscr{R}) = 0$ for $k \in \mathbf{Z}_{>0}$. Suppose given $X \in \mathrm{Ob}(\nabla_{\mathscr{Q}}(\mathscr{F}))$ and $Y \in \mathrm{Ob}(\nabla_{\mathscr{R}}(\mathscr{F}))$. For $i,j,k,\ell \in \mathbf{Z}$ with $i \leq j < k \leq \ell$, we have ${}_{\underline{\mathscr{F}}}(X_{\ell/k}\, ,\, Y^{[1]}_{j/i}) = 0$. $\diamondsuit$

Proof. Suppose given $i,j,k,\ell \in \mathbf{Z}$ with $i \leq j < k \leq \ell$.
By remark 3.3.20, we have $X_{\ell/k} \in \mathrm{Ob}\left(\mathop{\ast}\limits_{m\in[k+1,\ell]} \mathscr{Q}^{[m]}\right)$ and $Y_{j/i} \in \mathrm{Ob}\left(\mathop{\ast}\limits_{n\in[i+1,j]} \mathscr{R}^{[n]}\right)$. So $Y^{[1]}_{j/i} \in \mathrm{Ob}\left(\mathop{\ast}\limits_{n\in[i+1,j]} \mathscr{R}^{[n+1]}\right) = \mathrm{Ob}\left(\mathop{\ast}\limits_{n\in[i+2,j+1]} \mathscr{R}^{[n]}\right)$ by lemma 1.5.9. Thus the result follows from lemma 1.5.10. $\square$

3.3.23 Remark. Suppose given a full subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$ and $X \in \mathrm{Ob}(\nabla_{\mathscr{S}}(\mathscr{F}))$. Then we have $X^{[1]}_{[-1]} \in \mathrm{Ob}(\nabla_{\mathscr{S}}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. For $k \in \mathbf{Z}$, we have $(X^{[1]}_{[-1]})^{[-k]}_{k/k-1} = (X^{[1]}_{k-1/k-2})^{[-k]} = X^{[-k+1]}_{k-1/k-2} \in \mathrm{Ob}(\mathscr{S})$. $\square$

3.3.24 Definition. A candidate triangle $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}_{[-1]}$ in $\nabla(\mathscr{F})$ with respect to $\Sigma_{\nabla,\mathscr{F}} \cdot \mathrm{T}^{-1}_{\nabla,\mathscr{F}}$ is called a *pseudo-triangle* in $\nabla(\mathscr{F})$ if (i,p) is a pure short exact sequence in $\nabla(\mathscr{F})$ and if there exists $X_{[-1]}\mathbf{B} \xrightarrow{g} Z$ in $\nabla(\mathscr{F})$ such that the following diagram is commutative in $\nabla(\mathscr{F})$.

$$\begin{tikzcd}
X_{[-1]} \arrow[r, "X_{[-1]}\boldsymbol{\iota}"] \arrow[d, "X_{[-1]}\rho\cdot f"'] & X_{[-1]}\mathbf{B} \arrow[r, "X_{[-1]}\boldsymbol{\pi}"] \arrow[d, "g"] & X^{[1]}_{[-1]} \arrow[d, "1"] \\
Y \arrow[r, "i"] & Z \arrow[r, "p"] & X^{[1]}_{[-1]}
\end{tikzcd}$$

Note that in this case the left rectangle is a square, cf. lemma 2.1.9. $\diamondsuit$

3.3.25 Lemma. Suppose given $X \xrightarrow{f} Y$ in $\nabla(\mathscr{F})$. Then there exists a pseudo-triangle $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}_{[-1]}$ in $\nabla(\mathscr{F})$. $\diamondsuit$

Proof. Choose a pushout in $\nabla(\mathscr{F})$ as follows.

$$\begin{tikzcd}
X_{[-1]} \arrow[r, "X_{[-1]}\boldsymbol{\iota}"] \arrow[d, "X_{[-1]}\rho\cdot f"'] & X_{[-1]}\mathbf{B} \arrow[d, "g"] \\
Y \arrow[r, "i"] & Z
\end{tikzcd}$$

By lemma 2.1.11.(b), we may choose $Z \xrightarrow{p} X^{[1]}_{[-1]}$ in $\mathscr{F}$ such that $g \cdot p = X_{[-1]}\boldsymbol{\pi}$ and such that (i, p) is a pure short exact sequence. Thus $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}_{[-1]}$ is a pseudo-triangle in $\nabla(\mathscr{F})$. $\square$

3.3.26 Lemma. Suppose given a pseudo-triangle $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}_{[-1]}$ in $\nabla(\mathscr{F})$ and a morphism $X_{[-1]}\mathbf{B} \xrightarrow{g} Z$ in $\nabla(\mathscr{F})$ such that the following diagram is commutative.

$$\begin{tikzcd}
X_{[-1]} \arrow[r, "X_{[-1]}\boldsymbol{\iota}"] \arrow[d, "X_{[-1]}\mathsf{\rho} \cdot f"'] & X_{[-1]}\mathbf{B} \arrow[r, "X_{[-1]}\boldsymbol{\pi}"] \arrow[d, "g"] & X^{[1]}_{[-1]} \arrow[d, "1"] \\
Y \arrow[r, "i"] & Z \arrow[r, "p"] & X^{[1]}_{[-1]}
\end{tikzcd}$$

For $k \in \mathbf{Z}$, we may choose $Z_{k/k-1} \xrightarrow{s_k} Y_{k/k-1}$ and $X^{[1]}_{k-1/k-2} \xrightarrow{t_k} Y_{k/k-1}$ in $\mathscr{F}$ such that $X_{k-1/k-2}\pi \cdot t_k = g_{k/k-1}$ and such that $Y_{k/k-1} \underset{s_k}{\overset{i_{k/k-1}}{\rightleftarrows}} Z_{k/k-1} \underset{p_{k/k-1}}{\overset{t_k}{\leftrightarrows}} X^{[1]}_{k-1/k-2}$ is a direct sum in $\mathscr{F}$. $\diamondsuit$

Proof. Suppose given $k \in \mathbf{Z}$.

We have $(X_{[-1]}\mathsf{\rho})_{k/k-1} = 0$ and thus $X_{k-1/k-2}\iota \cdot g_{k/k-1} = (X_{[-1]}\mathsf{\rho} \cdot f)_{k/k-1} \cdot i_{k/k-1} = 0$.

So we may choose $X^{[1]}_{k-1/k-2} \xrightarrow{t_k} Z_{k/k-1}$ in $\mathscr{F}$ such that $X_{k-1/k-2}\pi \cdot t_k = g_{k/k-1}$.

We have $t_k \cdot p_{k/k-1} = 1$ since $X_{k-1/k-2}\pi \cdot t_k \cdot p_{k/k-1} = g_{k/k-1} \cdot p_{k/k-1} = X_{k-1/k-2}\pi$ and since $X_{k-1/k-2}\pi$ is a pure epimorphism.

The result now follows from lemma 1.2.11. $\square$

3.3.27 Corollary. Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$. Suppose given a pseudo-triangle $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}_{[-1]}$ in $\nabla(\mathscr{F})$ with $X, Y \in \mathrm{Ob}(\nabla_{\mathscr{S}}(\mathscr{F}))$. Then we have $Z, X^{[1]}_{[-1]} \in \mathrm{Ob}(\nabla_{\mathscr{S}}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. We have $X^{[1]}_{[-1]} \in \mathrm{Ob}(\nabla_{\mathscr{S}}(\mathscr{F}))$, cf. remark 3.3.23.

For $k \in \mathbf{Z}$, lemma 3.3.26 yields a direct sum $Y_{k/k-1} \underset{s_k}{\overset{i_{k/k-1}}{\rightleftarrows}} Z_{k/k-1} \underset{p_{k/k-1}}{\overset{t_k}{\leftrightarrows}} X^{[1]}_{k-1/k-2}$ in $\mathscr{F}$.

So $Y^{[-k]}_{k/k-1} \underset{\underline{s^{[-k]}_k}}{\overset{\underline{i^{[-k]}_{k/k-1}}}{\rightleftarrows}} Z^{[-k]}_{k/k-1} \underset{\underline{p^{[-k]}_{k/k-1}}}{\overset{\underline{t^{[-k]}_k}}{\leftrightarrows}} X^{[-k+1]}_{k-1/k-2}$ is a direct sum in $\underline{\mathscr{F}}$.

Thus $Z^{[-k]}_{k/k-1} \in \mathrm{Ob}(\mathscr{S})$. We conclude that $Z \in \mathrm{Ob}(\nabla_{\mathscr{S}}(\mathscr{F}))$. $\square$

3.3.28 Lemma. Suppose given a pseudo-triangle $X \xrightarrow{1} X \xrightarrow{i} B \xrightarrow{p} X^{[1]}_{[-1]}$ in $\nabla(\mathscr{F})$. Suppose given $X \xrightarrow{f} Y$ in $\nabla(\mathscr{F})$. The following three statements are equivalent.

(a) We have $\underline{\underline{f}} = 0$.

(b) There exists $B \xrightarrow{h} Y$ in $\nabla(\mathscr{F})$ such that $f = i \cdot h$.

(c) There exists $B \xrightarrow{h} Y$ in $\nabla(\mathscr{F})$ such that $\underline{f} = \underline{i \cdot h}$.

$\diamond$

Proof. We may choose $X_{[-1]}\mathbf{B} \xrightarrow{g} Z$ in $\nabla(\mathscr{F})$ such that the following diagram is a square, cf. definition 3.3.24.

$$\begin{tikzcd} X_{[-1]} \arrow[r, "X_{[-1]}\boldsymbol{\iota}", "\bullet" marking] \arrow[d, "X_{[-1]}\rho"'] & X_{[-1]}\mathbf{B} \arrow[d, "g"] \\ X \arrow[r, "i", "\bullet" marking] & B \end{tikzcd}$$

Ad (a) $\to$ (b). Suppose that $\underline{\underline{f}} = 0$. So we may choose $X_{[-1]}\mathbf{B} \xrightarrow{e} Y$ in $\nabla(\mathscr{F})$ such that $X_{[-1]}\rho \cdot f = X_{[-1]}\boldsymbol{\iota} \cdot e$ by remark 3.3.16. Since the diagram above is a pushout, we may choose $B \xrightarrow{h} Y$ such that $i \cdot h = f$.

Ad (b) $\to$ (c). This is trivial.

Ad (c) $\to$ (a). Suppose given $B \xrightarrow{h} Y$ in $\nabla(\mathscr{F})$ such that $\underline{f} = \underline{i \cdot h}$.
Then $\underline{X_{[-1]}\rho \cdot f} = \underline{X_{[-1]}\rho \cdot i \cdot h} = \underline{X_{[-1]}\boldsymbol{\iota} \cdot g \cdot h}$. So $\underline{X_{[-1]}\rho \cdot f} = 0$. We conclude that $\underline{\underline{f}} = 0$ by remark 3.3.16. $\square$

3.3.29 Definition. Suppose given $k \in \mathbf{Z}$. We define the full subcategories $\nabla^{[k}(\mathscr{F})$ and $\nabla^{k]}(\mathscr{F})$ of $\nabla(\mathscr{F})$ by setting

$$\mathrm{Ob}(\nabla^{[k}(\mathscr{F})) = \{X \in \mathrm{Ob}(\nabla(\mathscr{F})) \colon X_{\ell/\ell-1} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}}) \text{ for } \ell \in \mathbf{Z}_{>k}\}$$

and

$$\mathrm{Ob}(\nabla^{k]}(\mathscr{F})) = \{X \in \mathrm{Ob}(\nabla(\mathscr{F})) \colon X_{\ell/\ell-1} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}}) \text{ for } \ell \in \mathbf{Z}_{<k}\}.$$

We define the full subcategories $\underline{\nabla}^{[k}(\mathscr{F})$ and $\underline{\nabla}^{k]}(\mathscr{F})$ of $\underline{\nabla}(\mathscr{F})$ by setting
$\mathrm{Ob}(\underline{\nabla}^{[k}(\mathscr{F})) = \mathrm{Ob}(\nabla^{[k}(\mathscr{F}))$ and $\mathrm{Ob}(\underline{\nabla}^{k]}(\mathscr{F})) = \mathrm{Ob}(\nabla^{k]}(\mathscr{F}))$.

We define the full subcategories $\underline{\underline{\nabla}}^{[k}(\mathscr{F})$ and $\underline{\underline{\nabla}}^{k]}(\mathscr{F})$ of $\underline{\underline{\nabla}}(\mathscr{F})$ by setting
$\mathrm{Ob}(\underline{\underline{\nabla}}^{[k}(\mathscr{F})) = \mathrm{Ob}(\nabla^{[k}(\mathscr{F}))$ and $\mathrm{Ob}(\underline{\underline{\nabla}}^{k]}(\mathscr{F})) = \mathrm{Ob}(\nabla^{k]}(\mathscr{F}))$.

Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \mathscr{F}$.
Let $\nabla^{[k}_{\mathscr{S}}(\mathscr{F}) = \nabla^{[k}(\mathscr{F}) \cap \nabla_{\mathscr{S}}(\mathscr{F})$ and $\nabla^{k]}_{\mathscr{S}}(\mathscr{F}) = \nabla^{k]}(\mathscr{F}) \cap \nabla_{\mathscr{S}}(\mathscr{F})$.
Let $\underline{\nabla}^{[k}_{\mathscr{S}}(\mathscr{F}) = \underline{\nabla}^{[k}(\mathscr{F}) \cap \underline{\nabla}_{\mathscr{S}}(\mathscr{F})$ and $\underline{\nabla}^{k]}_{\mathscr{S}}(\mathscr{F}) = \underline{\nabla}^{k]}(\mathscr{F}) \cap \underline{\nabla}_{\mathscr{S}}(\mathscr{F})$.
Let $\underline{\underline{\nabla}}^{[k}_{\mathscr{S}}(\mathscr{F}) = \underline{\underline{\nabla}}^{[k}(\mathscr{F}) \cap \underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})$ and $\underline{\underline{\nabla}}^{k]}_{\mathscr{S}}(\mathscr{F}) = \underline{\underline{\nabla}}^{k]}(\mathscr{F}) \cap \underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})$. $\diamond$

3.3.30 Definition. Suppose given $k, \ell \in \mathbf{Z}$. Let $\nabla^{[k,\ell]}(\mathscr{F}) = \nabla^{[k}(\mathscr{F}) \cap \nabla^{\ell]}(\mathscr{F})$,
$\underline{\nabla}^{[k,\ell]}(\mathscr{F}) = \underline{\nabla}^{[k}(\mathscr{F}) \cap \underline{\nabla}^{\ell]}(\mathscr{F})$ and $\underline{\underline{\nabla}}^{[k,\ell]}(\mathscr{F}) = \underline{\underline{\nabla}}^{[k}(\mathscr{F}) \cap \underline{\underline{\nabla}}^{\ell]}(\mathscr{F})$.

Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \mathscr{F}$.
Let $\nabla^{[k,\ell]}_{\mathscr{S}}(\mathscr{F}) = \nabla^{[k,\ell]}(\mathscr{F}) \cap \nabla_{\mathscr{S}}(\mathscr{F})$, $\underline{\nabla}^{[k,\ell]}_{\mathscr{S}}(\mathscr{F}) = \underline{\nabla}^{[k,\ell]}(\mathscr{F}) \cap \underline{\nabla}_{\mathscr{S}}(\mathscr{F})$ and
$\underline{\underline{\nabla}}^{[k,\ell]}_{\mathscr{S}}(\mathscr{F}) = \underline{\underline{\nabla}}^{[k,\ell]}(\mathscr{F}) \cap \underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})$. $\diamond$

3.3.31 Definition. Let $\nabla^{\mathrm{b}}(\mathscr{F}) = \bigcup\limits_{k,\ell\in\mathbf{Z}} \nabla^{\lceil k,\ell\rceil}(\mathscr{F})$, $\underline{\nabla}^{\mathrm{b}}(\mathscr{F}) = \bigcup\limits_{k,\ell\in\mathbf{Z}} \underline{\nabla}^{\lceil k,\ell\rceil}(\mathscr{F})$ and $\underline{\underline{\nabla}}^{\mathrm{b}}(\mathscr{F}) = \bigcup\limits_{k,\ell\in\mathbf{Z}} \underline{\underline{\nabla}}^{\lceil k,\ell\rceil}(\mathscr{F})$.

Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$.
Let $\nabla^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F}) = \nabla^{\mathrm{b}}(\mathscr{F}) \cap \nabla_{\mathscr{S}}(\mathscr{F})$, $\underline{\nabla}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F}) = \underline{\nabla}^{\mathrm{b}}(\mathscr{F}) \cap \underline{\nabla}_{\mathscr{S}}(\mathscr{F})$ and
$\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F}) = \underline{\underline{\nabla}}^{\mathrm{b}}(\mathscr{F}) \cap \underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})$ ◇

3.3.32 Definition. Let $\mathfrak{Q}_{\nabla^{\mathrm{b}},\mathscr{F}} = \mathfrak{Q}_{\nabla,\mathscr{F}}|^{\underline{\underline{\nabla}}^{\mathrm{b}}(\mathscr{F})}_{\nabla^{\mathrm{b}}(\mathscr{F})} \colon \nabla^{\mathrm{b}}(\mathscr{F}) \to \underline{\underline{\nabla}}^{\mathrm{b}}(\mathscr{F})$. ◇

3.3.33 Remark. Suppose given $n \in \mathbf{Z}$.

(a) Suppose given $X \in \mathrm{Ob}(\nabla^{n\rceil}(\mathscr{F}))$. Then $X_{\ell/k} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$ for $\ell/k \in \mathrm{V}$ with $\ell < n$.

(b) Suppose given $X \in \mathrm{Ob}(\nabla^{\lceil n}(\mathscr{F}))$. Then $X_{\ell/k} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$ for $\ell/k \in \mathrm{V}$ with $k > n-1$.

◇

Proof. Ad (a). We use induction on $\ell - k \in \mathbf{Z}_{\geq 0}$ with $\ell < n$.
If $\ell - k = 0$, then $X_{\ell/k} = X_{\ell/\ell} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$.
If $\ell - k = 1$ with $\ell < n$, then $X_{\ell/k} = X_{\ell/\ell-1} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$.
Suppose that $\ell - k > 1$ with $\ell < n$.
Consider the pure short exact sequence $X_{\ell-1/k} \xrightarrow{x_{\ell-1,\ell/k}} X_{\ell/k} \xrightarrow{x_{\ell/k,\ell-1}} X_{\ell/\ell-1}$ in $\mathscr{F}$. We have $X_{\ell/\ell-1} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$ and, by induction, $X_{\ell-1/k} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$. We conclude that $X_{\ell/k} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$.
Ad (b). We use induction on $\ell - k \in \mathbf{Z}_{\geq 0}$ with $k > n-1$.
If $\ell - k = 0$, then $X_{\ell/k} = X_{k/k} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$.
If $\ell - k = 1$ with $k > n-1$, then $X_{\ell/k} = X_{k+1/k} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$.
Suppose that $\ell - k > 1$ with $k > n-1$.
Consider the pure short exact sequence $X_{k+1/k} \xrightarrow{x_{k+1,\ell/k}} X_{\ell/k} \xrightarrow{x_{\ell/k,k+1}} X_{\ell/k+1}$ in $\mathscr{F}$. We have $X_{k+1/k} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$ and, by induction, $X_{\ell/k+1} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$. We conclude that $X_{\ell/k} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$. □

3.3.34 Remark. Suppose given $n \in \mathbf{Z}$.

(a) Suppose given $X \in \mathrm{Ob}(\nabla^{\lceil n}(\mathscr{F}))$. Then $x_{\ell,\ell+1/k}$ is an isomorphism in $\mathscr{F}$ for $\ell/k \in \mathrm{V}$ with $\ell \geq n$.

(b) Suppose given $X \in \mathrm{Ob}(\nabla^{n\rceil}(\mathscr{F}))$. Then $x_{\ell/k-1,k}$ is an isomorphism in $\mathscr{F}$ for $\ell/k \in \mathrm{V}$ with $k < n$.

◇

Proof. Ad (a). Suppose given $\ell/k \in \mathrm{V}$ with $\ell \geq n$. The sequence $(x_{\ell,\ell+1/k}\,,\, x_{\ell+1/k,\ell})$ is pure short exact in $\mathscr{F}$ with $x_{\ell+1/k,\ell} = 0$ since $X_{\ell+1/\ell} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$. We conclude that $x_{\ell,\ell+1/k}$ is an isomorphism in $\mathscr{F}$.

Ad (b). Suppose given $\ell/k \in \mathrm{V}$ with $k < n$. The sequence $(x_{k,\ell/k-1}\,,\, x_{\ell/k-1,k})$ is pure short exact in $\mathscr{F}$ with $x_{k,\ell/k-1} = 0$ since $X_{k/k-1} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$. We conclude that $x_{\ell/k-1,k}$ is an isomorphism in $\mathscr{F}$. □

3.3.35 Lemma. Suppose given $m \in \mathbf{Z}$ and a pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\nabla(\mathscr{F})$.

(a) If $X, Z \in \mathrm{Ob}(\nabla^{\lceil m}(\mathscr{F}))$, then $Y \in \mathrm{Ob}(\nabla^{\lceil m}(\mathscr{F}))$ as well.

(b) If $X, Z \in \mathrm{Ob}(\nabla^{m\rceil}(\mathscr{F}))$, then $Y \in \mathrm{Ob}(\nabla^{m\rceil}(\mathscr{F}))$ as well.

$\Diamond$

Proof. This follows from lemma 1.4.8.(d). $\square$

3.3.36 Lemma. Suppose given a pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\nabla(\mathscr{F})$ such that $X, Z \in \mathrm{Ob}(\nabla^{\mathrm{b}}(\mathscr{F}))$. Then we have $Y \in \mathrm{Ob}(\nabla^{\mathrm{b}}(\mathscr{F}))$ as well. $\Diamond$

Proof. This follows from lemma 3.3.35. $\square$

3.3.37 Corollary. $\nabla^{\mathrm{b}}(\mathscr{F})$ is an extension-closed subcategory of $\nabla(\mathscr{F})$, cf. definition 1.3.23, remark 1.4.10 and lemma 3.3.36. In particular, it is a strictly full additive subcategory of $\nabla(\mathscr{F})$. So $\underline{\nabla}^{\mathrm{b}}(\mathscr{F})$ is a full additive subcategory of $\underline{\nabla}(\mathscr{F})$ and $\underline{\underline{\nabla}}^{\mathrm{b}}(\mathscr{F})$ is a full additive subcategory of $\underline{\underline{\nabla}}(\mathscr{F})$, cf. remark 1.2.14.

Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$. Then $\nabla_{\mathscr{S}}(\mathscr{F})$ is a strictly full additive subcategory of $\nabla(\mathscr{F})$ as well by lemma 3.3.19. Consequently, $\nabla^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})$ is a strictly full additive subcategory of $\nabla(\mathscr{F})$, cf. remark 1.2.4. Thus $\underline{\nabla}_{\mathscr{S}}(\mathscr{F})$, $\underline{\nabla}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})$ are full additive subcategories of $\underline{\nabla}(\mathscr{F})$ and $\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})$, $\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})$ are full additive subcategories of $\underline{\underline{\nabla}}(\mathscr{F})$, cf. remark 1.2.14. $\Diamond$

3.3.38 Definition. Suppose given $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$. We write

$$\gamma_{X,j,k,\ell} = \gamma_{(x_{k,\ell/j},\, x_{\ell/j,k})} \colon X_{\ell/j} \to X_{k/j}\mathrm{B} \quad \text{and} \quad \delta_{X,j,k,\ell} = \delta_{(x_{k,\ell/j},\, x_{\ell/j,k})} \colon X_{\ell/k} \to X^{[1]}_{k/j}$$

for $j \leq k \leq \ell$ in $\mathbf{Z}$. Cf. definition 2.2.1.

$$\begin{tikzcd}
X_{k/j} \arrow[r, "x_{k,\ell/j}"] \arrow[d, "1"'] & X_{\ell/j} \arrow[r, "x_{\ell/j,k}"] \arrow[d, "\gamma_{X,j,k,\ell}"] & X_{\ell/k} \arrow[d, "\delta_{X,j,k,\ell}"] \\
X_{k/j} \arrow[r, "X_{k/j}\iota"] & X_{k/j}\mathrm{B} \arrow[r, "X_{k/j}\pi"] & X^{[1]}_{k/j}
\end{tikzcd}$$

Note that $\underline{x_{\ell/j,k} \cdot \delta_{X,j,k,\ell}} = 0$ in $\underline{\mathscr{F}}$. $\Diamond$

3.3.39 Remark. Suppose given $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$ and $j, k, \ell \in \mathbf{Z}$ with $j \leq k \leq \ell$. We have $\delta_{X_{[-1]},j,k,\ell} = \delta_{(x_{k-1,\ell-1/j-1},\, x_{\ell-1/j-1,k-1})} = \delta_{X,j-1,k-1,\ell-1}$. $\Diamond$

3.3.40 Lemma. Suppose given $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$ and $j, k, \ell \in \mathbf{Z}$ with $j \leq k \leq \ell$. We have $\underline{\delta_{X^{[1]},j,k,\ell}} = \underline{-\delta^{[1]}_{X,j,k,\ell}}$ in $\underline{\mathscr{F}}$. $\Diamond$

Proof. By lemma 2.2.5, we have $\underline{\delta_{X^{[1]},j,k,\ell}} = \underline{\delta_{(x^{[1]}_{k,\ell/j},\, x^{[1]}_{\ell/j,k})}} = \underline{-\delta^{[1]}_{(x_{k,\ell/j},\, x_{\ell/j,k})}} = \underline{-\delta^{[1]}_{X,j,k,\ell}}$. $\square$

3.3.41 Corollary. Suppose given $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$ and $j, k, \ell \in \mathbf{Z}$ with $j \leq k \leq \ell$. We have $\underline{\delta_{X^{[1]}_{[-1]},j,k,\ell}} = \underline{-\delta^{[1]}_{X,j-1,k-1,\ell-1}}$ in $\underline{\mathscr{F}}$. ♢

Proof. Using remark 3.3.39 and lemma 3.3.40, we obtain
$\underline{\delta_{X^{[1]}_{[-1]},j,k,\ell}} = \underline{\delta_{X^{[1]},j-1,k-1,\ell-1}} = \underline{-\delta^{[1]}_{X,j-1,k-1,\ell-1}}$ in $\underline{\mathscr{F}}$. □

3.3.42 Lemma. Suppose given $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$. Suppose given $k, \ell \in \mathbf{Z}$ such that $k \leq \ell$. Then we have $\underline{x_{k,\ell/k-1} \cdot \delta_{X,k-2,k-1,\ell}} = \underline{\delta_{X,k-2,k-1,k}}$. ♢

Proof. Consider the pure short exact sequences $X_{k-1/k-2} \xrightarrow{x_{k-1,k/k-2}} X_{k/k-2} \xrightarrow{x_{k/k-2,k-1}} X_{k/k-1}$ and $X_{k-1/k-2} \xrightarrow{x_{k-1,\ell/k-2}} X_{\ell/k-2} \xrightarrow{x_{\ell/k-2,k-1}} X_{\ell/k-1}$ and the following commutative diagram in $\mathscr{F}$.

$$\begin{tikzcd}
X_{k-1/k-2} \arrow[r, "x_{k-1,k/k-2}"] \arrow[d, "1"'] & X_{k/k-2} \arrow[r, "x_{k/k-2,k-1}"] \arrow[d, "x_{k,\ell/k-2}"] & X_{k/k-1} \arrow[d, "x_{k,\ell/k-1}"] \\
X_{k-1/k-2} \arrow[r, "x_{k-1,\ell/k-2}"] & X_{\ell/k-2} \arrow[r, "x_{\ell/k-2,k-1}"] & X_{\ell/k-1}
\end{tikzcd}$$

By lemma 2.2.4, we have $\underline{x_{k,\ell/k-1} \cdot \delta_{X,k-2,k-1,\ell}} = \underline{\delta_{X,k-2,k-1,k}}$.

$$\begin{tikzcd}
X_{k-1/k-2} \arrow[r, "x_{k-1,k/k-2}"] \arrow[d, "1"'] & X_{k/k-2} \arrow[r, "x_{k/k-2,k-1}"] \arrow[d, "x_{k,\ell/k-2}"] & X_{k/k-1} \arrow[r, "\delta_{X,k-2,k-1,k}"] \arrow[d, "x_{k,\ell/k-1}"] & X^{[1]}_{k-1/k-2} \arrow[d, "1"] \\
X_{k-1/k-2} \arrow[r, "x_{k-1,\ell/k-2}"] & X_{\ell/k-2} \arrow[r, "x_{\ell/k-2,k-1}"] & X_{\ell/k-1} \arrow[r, "\delta_{X,k-2,k-1,\ell}"] & X^{[1]}_{k-1/k-2}
\end{tikzcd}$$
□

3.3.43 Lemma. Suppose given $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$. Suppose given $j, k \in \mathbf{Z}$ such that $j \leq k-2$. Then we have $\underline{\delta_{X,j,k-1,k} \cdot x^{[1]}_{k-1/j,k-2}} = \underline{\delta_{X,k-2,k-1,k}}$. ♢

Proof. Consider the pure short exact sequences $X_{k-1/k-2} \xrightarrow{x_{k-1,k/k-2}} X_{k/k-2} \xrightarrow{x_{k/k-2,k-1}} X_{k/k-1}$ and $X_{k-1/k-2} \xrightarrow{x_{k-1,\ell/k-2}} X_{\ell/k-2} \xrightarrow{x_{\ell/k-2,k-1}} X_{\ell/k-1}$ and the following commutative diagram in $\mathscr{F}$.

$$\begin{tikzcd}
X_{k-1/j} \arrow[r, "x_{k-1,k/j}"] \arrow[d, "x_{k-1/j,k-2}"'] & X_{k/j} \arrow[r, "x_{k/j,k-1}"] \arrow[d, "x_{k/j,k-2}"] & X_{k/k-1} \arrow[d, "1"] \\
X_{k-1/k-2} \arrow[r, "x_{k-1,k/k-2}"] & X_{k/k-2} \arrow[r, "x_{k/k-2,k-1}"] & X_{k/k-1}
\end{tikzcd}$$

By lemma 2.2.4, we have $\underline{\delta_{X,k-2,k-1,k}} = \underline{\delta_{X,j,k-1,k} \cdot x^{[1]}_{k-1/j,k-2}}$.

$$\begin{tikzcd}
X_{k-1/j} \arrow[r, "x_{k-1,k/j}"] \arrow[d, "x_{k-1/j,k-2}"'] & X_{k/j} \arrow[r, "x_{k/j,k-1}"] \arrow[d, "x_{k/j,k-2}"] & X_{k/k-1} \arrow[r, "\delta_{X,j,k-1,k}"] \arrow[d, "1"] & X^{[1]}_{k-1/j} \arrow[d, "x^{[1]}_{k-1/j,k-2}"] \\
X_{k-1/k-2} \arrow[r, "x_{k-1,k/k-2}"] & X_{k/k-2} \arrow[r, "x_{k/k-2,k-1}"] & X_{k/k-1} \arrow[r, "\delta_{X,k-2,k-1,k}"] & X^{[1]}_{k-1/k-2}
\end{tikzcd}$$
□

3.3.44 Lemma. Suppose given $\ell/j \in \mathrm{V}$ with $j < \ell$.

(a) Suppose that $\ell+j<0$ and write $k=-j$. Then $\ell/j=\ell/-k$ with $k>0$ and $-k<\ell<k$.

(b) Suppose that $\ell+j=0$ and write $k=\ell$. Then $\ell/j=k/-k$ with $k>0$.

(c) Suppose that $\ell+j=1$ and write $k=\ell-1$. Then $\ell/j=k+1/-k$ with $k\geq 0$.

(d) Suppose that $\ell+j>1$ and write $k=\ell$. Then $\ell/j=k/j$ with $k>1$ and $-k+1<j<k$.

$\diamondsuit$

Proof. Ad (a). We have $2\cdot j<\ell+j<0$ and thus $k=-j>0$.
Moreover, we have $-k=j<\ell<-j=k$.

Ad (b). We have $2\cdot\ell>\ell+j=0$ and thus $k=\ell>0$.

Ad (c). We have $2\cdot\ell>\ell+j=1$ and thus $k=\ell-1\geq 0$.

Ad (d). We have $2\cdot\ell>\ell+j>1$ and thus $k=\ell>1$.
Moreover, we have $-k+1=-\ell+1<j<\ell=k$. $\square$

3.3.45 Lemma. Suppose given $X,Y\in\mathrm{Ob}(\nabla(\mathscr{F}))$. Suppose given $X_{k/-k}\xrightarrow{f_{k/-k}}Y_{k/-k}$ and $X_{k+1/-k}\xrightarrow{f_{k+1/-k}}Y_{k+1/-k}$ in $\mathscr{F}$ such that

$$f_{k/-k}\cdot y_{k,k+1/-k}=x_{k,k+1/-k}\cdot f_{k+1/-k} \text{ and } f_{k+1/-k-1}\cdot y_{k+1/-k-1,-k}=x_{k+1/-k-1,-k}\cdot f_{k+1/-k}$$

for $k\in\mathbf{Z}_{\geq 0}$.
Let $f_{k/k}=0\colon X_{k/k}\to Y_{k/k}$ denote the unique morphism from $X_{k/k}$ to $Y_{k/k}$ in $\mathscr{F}$ for $k\in\mathbf{Z}$.

(a) For $k\in\mathbf{Z}_{\geq 0}$, there exist unique morphisms $X_{k/j}\xrightarrow{f_{k/j}}Y_{k/j}$, where $j\in\mathbf{Z}$ with $-k+1<j<k$, and $X_{\ell/-k}\xrightarrow{f_{\ell/-k}}Y_{\ell/-k}$, where $\ell\in\mathbf{Z}$ with $-k<\ell<k$, that satisfy the following equations.

$$\begin{aligned}
f_{k/-k+1}\cdot y_{k/-k+1,j}&=x_{k/-k+1,j}\cdot f_{k/j}\\
f_{k-1/j}\cdot y_{k-1,k/j}&=x_{k-1,k/j}\cdot f_{k/j}\\
f_{\ell/-k}\cdot y_{\ell,k/-k}&=x_{\ell,k/-k}\cdot f_{k/-k}\\
f_{\ell/-k}\cdot y_{\ell/-k,-k+1}&=x_{\ell/-k,-k+1}\cdot f_{\ell/-k+1}
\end{aligned}$$

(b) Note that, using (a), we have a morphism $X_{\ell/j}\xrightarrow{f_{\ell/j}}Y_{\ell/j}$ in $\mathscr{F}$ for every $\ell/j\in\mathrm{V}$, cf. lemma 3.3.44. So we obtain a morphism $X\xrightarrow{f}Y$ in $\nabla(\mathscr{F})$.

$\diamondsuit$

Proof. Ad (a). For $k=0$, there is nothing to show. Suppose given $k\in\mathbf{Z}_{>0}$.

Suppose given $j \in \mathbf{Z}$ with $-k+1 < j < k$. We have the following pushouts in $\mathscr{F}$.

$$\begin{tikzcd}
X_{k/j} & X_{k/-k+1} \arrow[l, "x_{k/-k+1,j}"'] \\
X_{k-1/j} \arrow[u, "x_{k-1,k/j}"] & X_{k-1/-k+1} \arrow[l, "x_{k-1/-k+1,j}"'] \arrow[u, "x_{k-1,k/-k+1}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Y_{k/j} & Y_{k/-k+1} \arrow[l, "y_{k/-k+1,j}"'] \\
Y_{k-1/j} \arrow[u, "y_{k-1,k/j}"] & Y_{k-1/-k+1} \arrow[l, "y_{k-1/-k+1,j}"'] \arrow[u, "y_{k-1,k/-k+1}"']
\end{tikzcd}$$

We want to apply lemma 1.1.2.

We have $f_{k-1/-k+1} \cdot y_{k-1,k/-k+1} = x_{k-1,k/-k+1} \cdot f_{k/-k+1}$.

If $j = k-1$, then $f_{k-1/-k+1} \cdot y_{k-1/-k+1,j} = 0 = x_{k-1/-k+1,j} \cdot f_{k-1/j}$.

If $j = -k+2$, then $f_{k-1/-k+1} \cdot y_{k-1/-k+1,j} = x_{k-1/-k+1,j} \cdot f_{k-1/j}$.

If $-k+2 < j < k-1$, then we have $f_{k-1/-k+2} \cdot y_{k-1/-k+2,j} = x_{k-1/-k+2,j} \cdot f_{k-1/j}$ by induction and thus

$$\begin{aligned}
f_{k-1/-k+1} \cdot y_{k-1/-k+1,j} &= f_{k-1/-k+1} \cdot y_{k-1/-k+1,-k+2} \cdot y_{k-1/-k+2,j} \\
&= x_{k-1/-k+1,-k+2} \cdot f_{k-1/-k+2} \cdot y_{k-1/-k+2,j} \\
&= x_{k-1/-k+1,-k+2} \cdot x_{k-1/-k+2,j} \cdot f_{k-1/j} \\
&= x_{k-1/-k+1,j} \cdot f_{k-1/j} \, .
\end{aligned}$$

By lemma 1.1.2, there exists a unique morphism $X_{k/j} \xrightarrow{f_{k/j}} Y_{k/j}$ such that $f_{k/-k+1} \cdot y_{k/-k+1,j} = x_{k/-k+1,j} \cdot f_{k/j}$ and $f_{k-1/j} \cdot y_{k-1,k/j} = x_{k-1,k/j} \cdot f_{k/j}$.

Suppose given $\ell \in \mathbf{Z}$ with $-k < \ell < k$. We have the following pullbacks in $\mathscr{F}$.

$$\begin{tikzcd}
X_{k/-k+1} & X_{k/-k} \arrow[l, "x_{k/-k,-k+1}"'] \\
X_{\ell/-k+1} \arrow[u, "x_{\ell,k/-k+1}"] & X_{\ell/-k} \arrow[l, "x_{\ell/-k,-k+1}"'] \arrow[u, "x_{\ell,k/-k}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Y_{k/-k+1} & Y_{k/-k} \arrow[l, "y_{k/-k,-k+1}"'] \\
Y_{\ell/-k+1} \arrow[u, "y_{\ell,k/-k+1}"] & Y_{\ell/-k} \arrow[l, "y_{\ell/-k,-k+1}"'] \arrow[u, "y_{\ell,k/-k}"']
\end{tikzcd}$$

We want to apply lemma 1.1.3.

We have $f_{k/-k} \cdot y_{k/-k,-k+1} = x_{k/-k,-k+1} \cdot f_{k/-k+1}$.

If $\ell = -k+1$, then $f_{\ell/-k+1} \cdot y_{\ell,k/-k+1} = 0 = x_{\ell,k/-k+1} \cdot f_{k/-k+1}$.

If $\ell = k-1$, then $f_{\ell/-k+1} \cdot y_{\ell,k/-k+1} = x_{\ell,k/-k+1} \cdot f_{k/-k+1}$.

If $-k+1 < \ell < k-1$, then we have $f_{\ell/-k+1} \cdot y_{\ell,k-1/-k+1} = x_{\ell,k-1/-k+1} \cdot f_{k-1/-k+1}$ by induction and thus

$$\begin{aligned}
f_{\ell/-k+1} \cdot y_{\ell,k/-k+1} &= f_{\ell/-k+1} \cdot y_{\ell,k-1/-k+1} \cdot y_{k-1,k/-k+1} \\
&= x_{\ell,k-1/-k+1} \cdot f_{k-1/-k+1} \cdot y_{k-1,k/-k+1} \\
&= x_{\ell,k-1/-k+1} \cdot x_{k-1,k/-k+1} \cdot f_{k/-k+1} \\
&= x_{\ell,k/-k+1} \cdot f_{k/-k+1} \, .
\end{aligned}$$

By lemma 1.1.3, there exists a unique morphism $X_{\ell/-k} \xrightarrow{f_{\ell/-k}} Y_{\ell/-k}$ such that $f_{\ell/-k} \cdot y_{\ell,k/-k} = x_{\ell,k/-k} \cdot f_{k/-k}$ and $f_{\ell/-k} \cdot y_{\ell/-k,-k+1} = x_{\ell/-k,-k+1} \cdot f_{\ell/-k+1}$.

Ad (b). It suffices to show that $f_{\ell/j} \cdot y_{\ell/j,j+1} = x_{\ell/j,j+1} \cdot f_{\ell/j+1}$ and $f_{\ell-1/j} \cdot y_{\ell-1,\ell/j} = x_{\ell-1,\ell/j} \cdot f_{\ell/j}$ for $\ell/j \in \mathrm{V}$ with $\ell > j$.

We make use of lemma 3.3.44.

- Suppose given $\ell/-k \in \mathrm{V}$ with $k > 0$ and $-k < \ell < k$.

  We have $f_{\ell/-k} \cdot y_{\ell/-k,-k+1} = x_{\ell/-k,-k+1} \cdot f_{\ell/-k+1}$.

  We have $f_{\ell-1/-k} \cdot y_{\ell-1,\ell/-k} = x_{\ell-1,\ell/-k} \cdot f_{\ell/-k}$ since

$$\begin{aligned} f_{\ell-1/-k} \cdot y_{\ell-1,\ell/-k} \cdot y_{\ell,k/-k} &= f_{\ell-1/-k} \cdot y_{\ell-1,k/-k} \\ &= x_{\ell-1,k/-k} \cdot f_{k/-k} \\ &= x_{\ell-1,\ell/-k} \cdot x_{\ell,k/-k} \cdot f_{k/-k} \\ &= x_{\ell-1,\ell/-k} \cdot f_{\ell/-k} \cdot y_{\ell,k/-k} \,, \end{aligned}$$

  where either $\ell - 1 = -k$ or $-k < \ell - 1 < k$, and since $y_{\ell,k/-k}$ is a pure monomorphism.

- Suppose given $k/-k \in \mathrm{V}$ with $k > 0$.

  We have $f_{k/-k} \cdot y_{k/-k,-k+1} = x_{k/-k,-k+1} \cdot f_{k/-k+1}$.

  We have $f_{k-1/-k} \cdot y_{k-1,k/-k} = x_{k-1,k/-k} \cdot f_{k/-k}$, where $-k < k - 1 < k$.

- Suppose given $k + 1/-k \in \mathrm{V}$ with $k \geq 0$.

  We have $f_{k+1/-k} \cdot y_{k+1/-k,-k+1} = x_{k+1/-k,-k+1} \cdot f_{k+1/-k+1}$, where either $k = 0$ or $-k < -k + 1 < k + 1$ with $k + 1 > 0$.

  We have $f_{k/-k} \cdot y_{k,k+1/-k} = x_{k,k+1/-k} \cdot f_{k+1/-k}$.

- Suppose given $k/j \in \mathrm{V}$ with $k > 1$ and $-k + 1 < j < k$.

  We have $f_{k/j} \cdot y_{k/j,j+1} = x_{k/j,j+1} \cdot f_{k/j+1}$ since

$$\begin{aligned} x_{k/-k+1,j} \cdot f_{k/j} \cdot y_{k/j,j+1} &= f_{k/-k+1} \cdot y_{k/-k+1,j} \cdot y_{k/j,j+1} \\ &= f_{k/-k+1} \cdot y_{k/-k+1,j+1} \\ &= x_{k/-k+1,j+1} \cdot f_{k/j+1} \\ &= x_{k/-k+1,j} \cdot x_{k/j,j+1} \cdot f_{k/j+1} \,, \end{aligned}$$

  where either $j + 1 = k$ or $-k + 1 < j + 1 < k$, and since $x_{k/-k+1,j}$ is a pure epimorphism.

  We have $f_{k-1/j} \cdot y_{k-1,k/j} = x_{k-1,k/j} \cdot f_{k/j}$.

$\square$

3.3.46 Corollary. Suppose given $X, Y \in \mathrm{Ob}(\nabla(\mathscr{F}))$. Suppose given $X_{k/-k} \xrightarrow{f_{k/-k}} Y_{k/-k}$, $X_{k+1/-k} \xrightarrow{f_{k+1/-k}} Y_{k+1/-k}$, $X_{k+2/k+1} \xrightarrow{f_{k+2/k+1}} Y_{k+2/k+1}$ and $X_{-k/-k-1} \xrightarrow{f_{-k/-k-1}} Y_{-k/-k-1}$ in

$\mathscr{F}$ for $k \in \mathbf{Z}_{\geq 0}$ such that the following diagrams are commutative.

$$\begin{tikzcd}
X_{-k/-k-1} \arrow[r, "x_{-k,k+1/-k-1}", "\bullet" marking] \arrow[d, "f_{-k/-k-1}"'] & X_{k+1/-k-1} \arrow[r, "x_{k+1/-k-1,-k}", "\shortmid" marking] \arrow[d, "f_{k+1/-k-1}"] & X_{k+1/-k} \arrow[d, "f_{k+1/-k}"] \\
Y_{-k/-k-1} \arrow[r, "y_{-k,k+1/-k-1}", "\bullet" marking] & Y_{k+1/-k-1} \arrow[r, "y_{k+1/-k-1,-k}", "\shortmid" marking] & Y_{k+1/-k}
\end{tikzcd}$$

$$\begin{tikzcd}
X_{k/-k} \arrow[r, "x_{k,k+1/-k}", "\bullet" marking] \arrow[d, "f_{k/-k}"'] & X_{k+1/-k} \arrow[r, "x_{k+1/-k,k}", "\shortmid" marking] \arrow[d, "f_{k+1/-k}"] & X_{k+1/k} \arrow[d, "f_{k+1/k}"] \\
Y_{k/-k} \arrow[r, "y_{k,k+1/-k}", "\bullet" marking] & Y_{k+1/-k} \arrow[r, "y_{k+1/-k,k}", "\shortmid" marking] & Y_{k+1/k}
\end{tikzcd}$$

Then there exists a unique morphism $X \xrightarrow{g} Y$ in $\nabla(\mathscr{F})$ such that

$$g_{k/-k} = f_{k/-k}\ ,\ g_{k+1/-k} = f_{k+1/-k}\ ,\ g_{k+2/k+1} = f_{k+2/k+1} \text{ and } g_{-k/-k-1} = f_{-k/-k-1}$$

for $k \in \mathbf{Z}_{\geq 0}$. $\diamondsuit$

Proof. For $k \in \mathbf{Z}$, let $f_{k/k} = 0\colon X_{k/k} \to Y_{k/k}$ denote the unique morphism from $X_{k/k}$ to $Y_{k/k}$ in $\mathscr{F}$. Suppose given $k \in \mathbf{Z}_{\geq 0}$.

Note that $f_{-k/-k-1}$ is the unique morphism in $\mathscr{F}$ such that

$$f_{-k/-k-1} \cdot y_{-k/-k-1,-k} = 0 = x_{-k/-k-1,-k} \cdot f_{-k/-k}$$

and $f_{-k/-k-1} \cdot y_{-k,k+1/-k-1} = x_{-k,k+1/-k-1} \cdot f_{k+1/-k-1}$.

Also note that $f_{k+2/k+1}$ is the unique morphism in $\mathscr{F}$ such that

$$f_{k+1/k+1} \cdot y_{k+1,k+2/k+1} = 0 = x_{k+1,k+2/k+1} \cdot f_{k+2/k+1}$$

and $f_{k+2/-k-1} \cdot y_{k+2/-k-1,k+1} = x_{k+2/-k-1,k+1} \cdot f_{k+2/k+1}$.

So the result follows from lemma 3.3.45. $\square$

3.3.47 Lemma. Suppose given pure short exact sequences $X_{-k+1/-k} \xrightarrow{x_{-k+1,k/-k}} X_{k/-k} \xrightarrow{x_{k/-k,-k+1}} X_{k/-k+1}$ and $X_{k/-k} \xrightarrow{x_{k,k+1/-k}} X_{k+1/-k} \xrightarrow{x_{k+1/-k,k}} X_{k+1/k}$ for $k \in \mathbf{Z}_{\geq 1}$. Then there exists an object $Y \in \mathrm{Ob}(\nabla(\mathscr{F}))$ such that

$$y_{-k+1,k/-k} = x_{-k+1,k/-k}\ ,\ y_{k/-k,-k+1} = x_{k/-k,-k+1}\ ,\ y_{k,k+1/-k} = x_{k,k+1/-k}\ ,\ y_{k+1/-k,k} = x_{k+1/-k,k}$$

for $k \in \mathbf{Z}_{\geq 1}$. In particular, we have $Y_{-k+1/-k} = X_{-k+1/-k}$, $Y_{k/-k} = X_{k/-k}$, $Y_{k/-k+1} = X_{k/-k+1}$ and $Y_{k+1/k} = X_{k+1/k}$ for $k \in \mathbf{Z}_{\geq 1}$. $\diamondsuit$

Proof. Using recursion on $k \in \mathbf{Z}_{\geq 1}$ and (descending) recursion on $\ell \in [-k+2, k-1]$, we construct objects $X_{\ell/-k} \in \mathrm{Ob}(\mathscr{F})$ and morphisms $X_{-k+1/-k} \xrightarrow{x_{-k+1,\ell/-k}} X_{\ell/-k}$,

$X_{\ell/-k} \xrightarrow{x_{\ell,\ell+1/-k}} X_{\ell+1/-k}$, $X_{\ell/-k} \xrightarrow{x_{\ell/-k,-k+1}} X_{\ell/-k+1}$ in $\mathscr{F}$ such that

- $X_{-k+1/-k} \xrightarrow{x_{-k+1,\ell/-k}} X_{\ell/-k} \xrightarrow{x_{\ell/-k,-k+1}} X_{\ell/-k+1}$ is a pure short exact sequence in $\mathscr{F}$,
- $x_{\ell,\ell+1/-k}$ is a pure monomorphism,
- $x_{-k+1,\ell/-k} \cdot x_{\ell,\ell+1/-k} = x_{-k+1,\ell+1/-k}$ and
- such that the following diagram is a square in $\mathscr{F}$.

$$\begin{tikzcd}
X_{\ell+1/-k+1} & X_{\ell+1/-k} \arrow[l, "x_{\ell+1/-k,-k+1}"'] \\
X_{\ell/-k+1} \arrow[u, "x_{\ell,\ell+1/-k+1}"] & X_{\ell/-k} \arrow[l, "x_{\ell/-k,-k+1}"'] \arrow[u, "x_{\ell,\ell+1/-k}"']
\end{tikzcd}$$

Suppose given $k \in \mathbf{Z}_{\geq 1}$ and $\ell \in [-k+2, k-1]$. Note that $(x_{-k+1,\ell+1/-k}\,, x_{\ell+1/-k,-k+1})$ is a pure short exact sequence and that $x_{\ell,\ell+1/-k+1}$ is a pure monomorphism, either by assumption or by recursion.

By lemma 2.1.37, we may choose a pullback

$$\begin{tikzcd}
X_{\ell+1/-k+1} & X_{\ell+1/-k} \arrow[l, "x_{\ell+1/-k,-k+1}"'] \\
X_{\ell/-k+1} \arrow[u, "x_{\ell,\ell+1/-k+1}"] & X_{\ell/-k} \arrow[l, "x_{\ell/-k,-k+1}"'] \arrow[u, "x_{\ell,\ell+1/-k}"']
\end{tikzcd}$$

and a morphism $X_{-k+1/-k} \xrightarrow{x_{-k+1,\ell/-k}} X_{\ell/-k}$ in $\mathscr{F}$ such that
$X_{-k+1/-k} \xrightarrow{x_{-k+1,\ell/-k}} X_{\ell/-k} \xrightarrow{x_{\ell/-k,-k+1}} X_{\ell/-k+1}$ is a pure short exact sequence,
$x_{-k+1,\ell/-k} \cdot x_{\ell,\ell+1/-k} = x_{-k+1,\ell+1/-k}$ and such that $x_{\ell,\ell+1/-k}$ is a pure monomorphism. Moreover, the pushout is a square.

Using recursion on $k \in \mathbf{Z}_{\geq 1}$ and (ascending) recursion on $j \in [-k+1, k-1]$, we construct objects $X_{k+1/j} \in \mathrm{Ob}(\mathscr{F})$ and morphisms $X_{k/j} \xrightarrow{x_{k,k+1/j}} X_{k+1/j}$, $X_{k+1/j-1} \xrightarrow{x_{k+1/j-1,j}} X_{k+1/j}$, $X_{k+1/j} \xrightarrow{x_{k+1/j,k}} X_{k+1/k}$ in $\mathscr{F}$ such that

- $X_{k/j} \xrightarrow{x_{k,k+1/j}} X_{k+1/j} \xrightarrow{x_{k+1/j,k}} X_{k+1/k}$ is a pure short exact sequence in $\mathscr{F}$,
- $x_{k+1/j-1,j}$ is a pure epimorphism,
- $x_{k+1/j-1,j} \cdot x_{k+1/j,k} = x_{k+1/j-1,k}$ and
- such that the following diagram is a square in $\mathscr{F}$.

$$\begin{tikzcd}
X_{k+1/j} & X_{k+1/j-1} \arrow[l, "x_{k+1/j-1,j}"'] \\
X_{k/j} \arrow[u, "x_{k,k+1/j}"] & X_{k/j-1} \arrow[l, "x_{k/j-1,j}"'] \arrow[u, "x_{k,k+1/j-1}"']
\end{tikzcd}$$

Suppose given $k \in \mathbf{Z}_{\geq 1}$ and $j \in [-k+1, k-1]$. Note that $(x_{k,k+1/j-1}\,,\, x_{k+1/j-1,k})$ is a pure short exact sequence and that $x_{k/j-1,j}$ is a pure epimorphism, either by assumption or by recursion.

By lemma 2.1.38, we may choose a pushout

$$\begin{tikzcd}
X_{k+1/j} & X_{k+1/j-1} \arrow[l, "x_{k+1/j-1,j}"'] \\
X_{k/j} \arrow[u, "x_{k,k+1/j}"] & X_{k/j-1} \arrow[l, "x_{k/j-1,j}", "\shortmid" marking] \arrow[u, "x_{k,k+1/j-1}"', "\bullet" marking]
\end{tikzcd}$$

and a morphism $X_{k+1/j} \xrightarrow{x_{k+1/j,k}} X_{k+1/k}$ in $\mathscr{F}$ such that
$X_{k/j} \xrightarrow{x_{k,k+1/j}} X_{k+1/j} \xrightarrow{x_{k+1/j,k}} X_{k+1/k}$ is a pure short exact sequence,
$x_{k+1/j-1,j} \cdot x_{k+1/j,k} = x_{k+1/j-1,k}$ and such that $x_{k+1/j-1,j}$ is a pure epimorphism. Moreover, the pushout is a square.

Suppose given $i, \ell \in \mathbf{Z}$ with $i + 1 < \ell$. We show that we now have the following square in $\mathscr{F}$.

$$\begin{tikzcd}
X_{\ell+1/i+1} & X_{\ell+1/i} \arrow[l, "x_{\ell+1/i,i+1}"', "\shortmid" marking] \\
X_{\ell/i+1} \arrow[u, "x_{\ell,\ell+1/i+1}", "\bullet" marking] & X_{\ell/i} \arrow[l, "x_{\ell/i,i+1}"', "\shortmid" marking] \arrow[u, "x_{\ell,\ell+1/i}"', "\bullet" marking]
\end{tikzcd}$$

If $\ell + i < 0$, write $k = -i$. So $\ell < -i = k$. We have $k \in \mathbf{Z}_{\geq 1}$ since $2 \cdot i < \ell + i < 0$. We have $-k + 1 < \ell$ since $i + 1 < \ell$. Thus $\ell/i = \ell/{-k}$ with $k \in \mathbf{Z}_{\geq 1}$ and $\ell \in [-k+2, k-1]$. So the square was constructed in the first recursion.

If $\ell + i \geq 0$, write $k = \ell$ and $j = i + 1$. So $j = i + 1 < \ell = k$ and $j > i \geq -\ell = -k$. We have $k \in \mathbf{Z}_{\geq 1}$ since $2 \cdot k = \ell + \ell > \ell + i + 1 \geq 1$. Thus $\ell/i = k/j-1$ with $k \in \mathbf{Z}_{\geq 1}$ and $j \in [-k+1, k-1]$. So the square was constructed in the second recursion.

We show that we now have a pure short exact sequence
$X_{i/i-1} \xrightarrow{x_{i,i+1/i-1}} X_{i+1/i-1} \xrightarrow{x_{i+1/i-1,i}} X_{i+1/i}$ for $i \in \mathbf{Z}$.

For $i = 0$, we have the given pure short exact sequence $X_{0/-1} \xrightarrow{x_{0,1/-1}} X_{1/-1} \xrightarrow{x_{1/-1,0}} X_{1/0}$.

For $i < 0$, write $k = 1 - i \in \mathbf{Z}_{>1}$ and $\ell = -k + 2 \in [-k+2, k-1]$. We have the pure short exact sequence $X_{-k+1/-k} \xrightarrow{x_{-k+1,\ell/-k}} X_{\ell/-k} \xrightarrow{x_{\ell/-k,-k+1}} X_{\ell/-k+1}$ from the first recursion.

For $i > 0$, write $k = i \in \mathbf{Z}_{\geq 1}$ and $j = k - 1 \in [-k+1, k-1]$. We have the pure short exact sequence $X_{k/j} \xrightarrow{x_{k,k+1/j}} X_{k+1/j} \xrightarrow{x_{k+1/j,k}} X_{k+1/k}$ from the second recursion.

Thus lemma 3.3.5 yields an object $Y \in \mathrm{Ob}(\nabla(\mathscr{F}))$ such that

$$y_{\ell/i,i+1} = x_{\ell/i,i+1} \text{ and } y_{\ell,\ell+1/i+1} = x_{\ell,\ell+1/i+1}$$

for $i+1 < \ell$ in $\mathbf{Z}$. In particular, we have $Y_{\ell/i} = X_{\ell/i}$ for $i < \ell$ in $\mathbf{Z}$ and $y_{k/-k,-k+1} = x_{k/-k,-k+1}$, $y_{k,k+1/-k} = x_{k,k+1/-k}$ for $k \in \mathbf{Z}_{\geq 1}$.

It remains to show that

$$y_{-k+1,k/-k} = x_{-k+1,k/-k} \quad \text{and} \quad y_{k+1/-k,k} = x_{k+1/-k,k}$$

for $k \in \mathbf{Z}_{\geq 1}$.

For $k = 1$, we have $y_{0,1/-1} = x_{0,1/-1}$. Suppose given $k \in \mathbf{Z}_{>1}$. We use induction to show that $y_{-k+1,\ell/-k} = x_{-k+1,\ell/-k}$ for $\ell \in [-k+2, k]$:

For $\ell = -k+2$, we have $y_{-k+1,-k+2/-k} = x_{-k+1,-k+2/-k}$. For $\ell \in [-k+2, k-1]$, we have $y_{-k+1,\ell+1/-k} = y_{-k+1,\ell/-k} \cdot y_{\ell,\ell+1/-k} = x_{-k+1,\ell/-k} \cdot x_{\ell,\ell+1/-k} = x_{-k+1,\ell+1/-k}$.

We conclude that $y_{-k+1,k/-k} = x_{-k+1,k/-k}$ for $k \in \mathbf{Z}_{\geq 1}$.

Suppose given $k \in \mathbf{Z}_{\geq 1}$. We use induction to show that $y_{k+1/j,k} = x_{k+1/j,k}$ for $j \in [-k, k-1]$.

For $j = k-1$, we have $y_{k+1/k-1,k} = x_{k+1/k-1,k}$.

For $j \in [-k+1, k-1]$, we have $y_{k+1/j-1,k} = y_{k+1/j-1,j} \cdot y_{k+1/j,k} = x_{k+1/j-1,j} \cdot x_{k+1/j,k} = x_{k+1/j-1,k}$.

We conclude that $y_{k+1/-k,k} = x_{k+1/-k,k}$ for $k \in \mathbf{Z}_{\geq 1}$. □

3.3.48 Definition. Suppose given $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$. For $\ell < k$ in $\mathbf{Z}$, we write $X_{\ell/k} = 0_{\mathscr{F}}$. Moreover, for $\ell < k$, $j \leq k$ and $\ell \leq m$ in $\mathbf{Z}$, we write $x_{\ell,m/k} = 0\colon X_{\ell/k} \to X_{m/k}$ and $x_{\ell/j,k} = 0\colon X_{\ell/j} \to X_{\ell/k}$.

Suppose given $X \xrightarrow{f} Y$ in $\nabla(\mathscr{F})$. We write $f_{\ell/k} = 0\colon X_{\ell/k} \to Y_{\ell/k}$ for $\ell < k$ in $\mathbf{Z}$. ♢

3.3.49 Lemma/Definition. Suppose given $\ell \in \mathbf{Z}$.

We define the projection functor $\Psi_{\ell,\mathrm{CF},\mathscr{F}}\colon \nabla(\mathscr{F}) \to \mathrm{CF}(\mathscr{F})$ as follows. For $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$ and $k \in \mathbf{Z}$, let $(X\Psi_{\ell,\mathrm{CF},\mathscr{F}})_k = X_{\ell/k-1}$ and $(X\Psi_{\ell,\mathrm{CF},\mathscr{F}})_{k\to k+1} = x_{\ell/k-1,k}$.

For $X \xrightarrow{f} Y$ in $\nabla(\mathscr{F})$ and $k \in \mathbf{Z}$, let $(f\Psi_{\ell,\mathrm{CF},\mathscr{F}})_k = f_{\ell/k-1}$.

This in fact defines an exact functor. ♢

Proof. Note that for $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$ and $k \in \mathbf{Z}$, $x_{\ell/k-1,k}$ is a pure epimorphism in $\mathscr{F}$. Thus $X\Psi_{\ell,\mathrm{CF},\mathscr{F}} \in \mathrm{Ob}(\mathrm{CF}(\mathscr{F}))$ for $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$.

For $X \xrightarrow{f} Y$ in $\nabla(\mathscr{F})$ and $k \in \mathbf{Z}$, we have

$$\begin{aligned}(f\Psi_{\ell,\mathrm{CF},\mathscr{F}})_k \cdot (Y\Psi_{\ell,\mathrm{CF},\mathscr{F}})_{k\to k+1} &= f_{\ell/k-1} \cdot x_{\ell/k-1,k} = y_{\ell/k-1,k} \cdot f_{\ell/k} \\ &= (X\Psi_{\ell,\mathrm{CF},\mathscr{F}})_{k\to k+1} \cdot (f\Psi_{\ell,\mathrm{CF},\mathscr{F}})_{k+1}\,.\end{aligned}$$

Suppose given $X \underset{h}{\overset{f}{\rightrightarrows}} Y \xrightarrow{g} Z$ in $\nabla(\mathscr{F})$. For $k \in \mathbf{Z}$, we have

$$(1_X\Psi_{\ell,\mathrm{CF},\mathscr{F}})_k = (1_X)_{\ell/k-1} = 1_{X_{\ell/k-1}} = (1_{X\Psi_{\ell,\mathrm{CF},\mathscr{F}}})_k\,,$$

$$\begin{aligned}((f \cdot g)\Psi_{\ell,\mathrm{CF},\mathscr{F}})_k &= (f \cdot g)_{\ell/k-1} = f_{\ell/k-1} \cdot g_{\ell/k-1} = (f\Psi_{\ell,\mathrm{CF},\mathscr{F}})_k \cdot (g\Psi_{\ell,\mathrm{CF},\mathscr{F}})_k \\ &= (f\Psi_{\ell,\mathrm{CF},\mathscr{F}} \cdot g\Psi_{\ell,\mathrm{CF},\mathscr{F}})_k\end{aligned}$$

and

$$
\begin{aligned}
((f+h)\Psi_{\ell,\mathrm{CF},\mathscr{F}})_k = (f+h)_{\ell/k-1} = f_{\ell/k-1} + h_{\ell/k-1} &= (f\Psi_{\ell,\mathrm{CF},\mathscr{F}})_k + (h\Psi_{\ell,\mathrm{CF},\mathscr{F}})_k \\
&= (f\Psi_{\ell,\mathrm{CF},\mathscr{F}} + h\Psi_{\ell,\mathrm{CF},\mathscr{F}})_k \,.
\end{aligned}
$$

Suppose given a pure short exact sequence (i, p) in $\nabla(\mathscr{F})$. So for $k \in \mathbf{Z}$, $(i_{\ell/k-1}\, ,\, p_{\ell/k-1})$ is a pure short exact sequence in $\mathscr{F}$. We conclude that $(i\Psi_{\ell,\mathrm{CF},\mathscr{F}}\, ,\, p\Psi_{\ell,\mathrm{CF},\mathscr{F}})$ is a pure short exact sequence in $\mathrm{CF}(\mathscr{F})$. □

3.3.50 Lemma/Definition. Suppose given $k \in \mathbf{Z}$.
We define the projection functor $\Psi_{k,\mathrm{F},\mathscr{F}} \colon \nabla(\mathscr{F}) \to \mathrm{F}(\mathscr{F})$ as follows. For $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$ and $\ell \in \mathbf{Z}$, let $(X\Psi_{k,\mathrm{F},\mathscr{F}})_\ell = X_{\ell/k-1}$ and $(X\Psi_{k,\mathrm{F},\mathscr{F}})_{\ell \to \ell+1} = x_{\ell,\ell+1/k-1}$.

For $X \xrightarrow{f} Y$ in $\nabla(\mathscr{F})$ and $\ell \in \mathbf{Z}$, let $(f\Psi_{k,\mathrm{F},\mathscr{F}})_\ell = f_{\ell/k-1}$.

This in fact defines an exact functor. ◇

Proof. This is dual to the previous lemma 3.3.49. □

3.3.51 Lemma. Suppose given $\ell \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$.

(a) We have $X\Psi_{\ell,\mathrm{CF},\mathscr{F}} \in \mathrm{Ob}(\mathrm{CF}^{[\ell}(\mathscr{F}))$.

(b) Suppose given $m \in \mathbf{Z}_{\leq \ell}$. If $X \in \mathrm{Ob}(\nabla^{m]}(\mathscr{F}))$, then $X\Psi_{\ell,\mathrm{CF},\mathscr{F}} \in \mathrm{Ob}(\mathrm{CF}^{m]}(\mathscr{F}))$. ◇

Proof. Ad (a). We have $(X\Psi_{\ell,\mathrm{CF},\mathscr{F}})_k = X_{\ell/k-1} \in \mathrm{Ob}(\mathrm{Z}_\mathscr{F})$ for $k \in \mathbf{Z}_{>\ell}$, cf. definition 3.3.48.
Ad (b). For $k \in \mathbf{Z}_{<m}$, we have $(X\Psi_{\ell,\mathrm{CF},\mathscr{F}})_{k \to k+1} = x_{\ell/k-1,k}$, which is an isomorphism in $\mathscr{F}$ by remark 3.3.34.(b). □

3.3.52 Corollary. Suppose given $X \in \mathrm{Ob}(\nabla^{\mathrm{b}}(\mathscr{F}))$.

(a) For $\ell \in \mathbf{Z}$, we have $X\Psi_{\ell,\mathrm{CF},\mathscr{F}} \in \mathrm{Ob}(\mathrm{CF}^{\mathrm{b}}(\mathscr{F}))$.

(b) For $k \in \mathbf{Z}$, we have $X\Psi_{k,\mathrm{F},\mathscr{F}} \in \mathrm{Ob}(\mathrm{F}^{\mathrm{b}}(\mathscr{F}))$. ◇

Proof. Choose $a, b \in \mathbf{Z}$ such that $X \in \mathrm{Ob}(\nabla^{[a,b]}(\mathscr{F}))$.
Ad (a). Suppose given $\ell \in \mathbf{Z}$. We have $X\Psi_{\ell,\mathrm{CF},\mathscr{F}} \in \mathrm{Ob}(\mathrm{CF}^{[\ell}(\mathscr{F}))$ by lemma 3.3.51.(a).
If $b \leq \ell$, then $X\Psi_{\ell,\mathrm{CF},\mathscr{F}} \in \mathrm{Ob}(\mathrm{CF}^{b]}(\mathscr{F}))$ by lemma 3.3.51.(b).
If $b > \ell$, then $X\Psi_{\ell,\mathrm{CF},\mathscr{F}} \in \mathrm{Ob}(\mathrm{CF}^{\ell]}(\mathscr{F}))$ since for $k \in \mathbf{Z}_{<\ell}$, we have $(X\Psi_{\ell,\mathrm{CF},\mathscr{F}})_{k \to k+1} = x_{\ell/k-1,k}$, which is an isomorphism in $\mathscr{F}$ by remark 3.3.34.(b).
Ad (b). This is dual to (a). □

3.3.53 Lemma/Definition. Suppose given $\ell \leq m$ in $\mathbf{Z}$. We define the transformation $\psi_{\ell,m,\mathrm{CF},\mathscr{F}} \colon \Psi_{\ell,\mathrm{CF},\mathscr{F}} \to \Psi_{m,\mathrm{CF},\mathscr{F}}$ by setting $(X\psi_{\ell,m,\mathrm{CF},\mathscr{F}})_k = x_{\ell,m/k-1}$ for $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$ and $k \in \mathbf{Z}$. Moreover, $X\psi_{\ell,m,\mathrm{CF},\mathscr{F}}$ is an ℓ-pure monomorphism in $\mathrm{CF}(\mathscr{F})$ for $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$. ◇

Proof. For $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$ and $k \in \mathbf{Z}$, we have

$$\begin{aligned}(X\psi_{\ell,m,\mathrm{CF},\mathscr{F}})_k \cdot (X\Psi_{m,\mathrm{CF},\mathscr{F}})_{k \to k+1} &= x_{\ell,m/k-1} \cdot x_{m/k-1,k} = x_{\ell/k-1,k} \cdot x_{\ell,m/k} \\ &= (X\Psi_{\ell,\mathrm{CF},\mathscr{F}})_{k \to k+1} \cdot (X\psi_{\ell,m,\mathrm{CF},\mathscr{F}})_{k+1} \,.\end{aligned}$$

For $X \xrightarrow{f} Y$ in $\nabla(\mathscr{F})$ and $k \in \mathbf{Z}$, we have

$$\begin{aligned}(X\psi_{\ell,m,\mathrm{CF},\mathscr{F}})_k \cdot (f\Psi_{m,\mathrm{CF},\mathscr{F}})_k &= x_{\ell,m/k-1} \cdot f_{m/k-1} = f_{\ell/k-1} \cdot x_{\ell,m/k-1} \\ &= (f\Psi_{\ell,\mathrm{CF},\mathscr{F}})_k \cdot (Y\psi_{\ell,m,\mathrm{CF},\mathscr{F}})_k \,.\end{aligned}$$

Moreover, for $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$, $X\psi_{\ell,m,\mathrm{CF},\mathscr{F}}$ is an ℓ-pure monomorphism in $\mathrm{CF}(\mathscr{F})$ since for $k \in \mathbf{Z}_{\leq \ell+1}$, the sequence $\big((X\psi_{\ell,m,\mathrm{CF},\mathscr{F}})_k \,,\, (X\Psi_{m,\mathrm{CF},\mathscr{F}})_{k \to \ell+1}\big) = (x_{\ell,m/k-1} \,,\, x_{m/k-1,\ell})$ is pure short exact in $\mathscr{F}$. □

3.3.54 Lemma/Definition. Suppose given $j \leq k$ in $\mathbf{Z}$. We define the transformation $\psi_{j,k,\mathrm{F},\mathscr{F}} \colon \Psi_{j,\mathrm{F},\mathscr{F}} \to \Psi_{k,\mathrm{F},\mathscr{F}}$ by setting $(X\psi_{j,k,\mathrm{F},\mathscr{F}})_\ell = x_{\ell/j-1,k-1}$ for $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$ and $\ell \in \mathbf{Z}$. Moreover, $X\psi_{j,k,\mathrm{F},\mathscr{F}}$ is a k-pure epimorphism in $\mathrm{F}(\mathscr{F})$ for $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$. ♢

Proof. This is dual to the previous lemma 3.3.53. □

3.3.55 Lemma. Suppose given $j \leq k \leq \ell$ in $\mathbf{Z}$.

(a) We have $\psi_{\ell,\ell,\mathrm{CF},\mathscr{F}} = 1_{\Psi_{\ell,\mathrm{CF},\mathscr{F}}}$.

(b) We have $\psi_{j,\ell,\mathrm{CF},\mathscr{F}} = \psi_{j,k,\mathrm{CF},\mathscr{F}} \cdot \psi_{k,\ell,\mathrm{CF},\mathscr{F}}$.

♢

Proof. Ad (a). Suppose given $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$ and $i \in \mathbf{Z}$.
We have $(\psi_{\ell,\ell,\mathrm{CF},\mathscr{F}})_i = x_{\ell,\ell/i-1} = 1_{X_{\ell/i-1}}$.
Ad (b). Suppose given $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$ and $i \in \mathbf{Z}$.
We have $(X\psi_{j,k,\mathrm{CF},\mathscr{F}})_i \cdot (X\psi_{k,\ell,\mathrm{CF},\mathscr{F}})_i = x_{j,k/i-1} \cdot x_{k,\ell/i-1} = x_{j,\ell/i-1} = (X\psi_{j,\ell,\mathrm{CF},\mathscr{F}})_i$. □

3.3.56 Remark. Suppose given $n \in \mathbf{Z}$.

(a) Suppose given $X \in \mathrm{Ob}(\nabla^{\lceil n}(\mathscr{F}))$. Then $X\psi_{\ell,\ell+1,\mathrm{CF},\mathscr{F}}$ is an isomorphism in $\mathrm{CF}(\mathscr{F})$ for $\ell \in \mathbf{Z}_{\geq n}$.

(b) Suppose given $X \in \mathrm{Ob}(\nabla^{n\rceil}(\mathscr{F}))$. Then $X\psi_{k,k+1,\mathrm{F},\mathscr{F}}$ is an isomorphism in $\mathrm{F}(\mathscr{F})$ for $k \in \mathbf{Z}_{<n}$.

♢

Proof. Ad (a). Suppose given $\ell \in \mathbf{Z}_{\geq n}$.
For $k \in \mathbf{Z}$, $(X\psi_{\ell,\ell+1,\mathrm{CF},\mathscr{F}})_k = x_{\ell,\ell+1/k-1}$ is an isomorphism, cf. remark 3.3.34.(a).
Ad (b). This is dual to (a). □

3.3.57 Lemma. Suppose given $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$.
For $\ell \in \mathbf{Z}$, suppose given a limit $(X_{\ell/-\infty}\,,\, (x_{\ell/-\infty,k-1})_{k\in\mathbf{Z}})$ for $X\Psi_{\ell,\mathrm{CF},\mathscr{F}}$, cf. definition 3.3.49.

(a) We obtain a filtration $X\Psi_{-\infty,\mathrm{F},\mathscr{F}} \in \mathrm{Ob}(\mathrm{F}(\mathscr{F}))$ as follows.
For $\ell \in \mathbf{Z}$, let $(X\Psi_{-\infty,\mathrm{F},\mathscr{F}})_\ell = X_{\ell/-\infty}$.
For $k \leq \ell$ in $\mathbf{Z}$, let $(X\Psi_{-\infty,\mathrm{F},\mathscr{F}})_{k\to\ell} = X\psi_{k,\ell,\mathrm{CF},\mathscr{F}}\rceil \colon X_{k/-\infty} \to X_{\ell/-\infty}$, cf. definitions 3.3.53, 3.2.22 and lemma 3.2.30.

(b) For $k \leq \ell$ in $\mathbf{Z}$, the morphism $X_{\ell/-\infty} \xrightarrow{x_{\ell/-\infty,k}} X_{\ell/k}$ is a cokernel of $(X\Psi_{-\infty,\mathrm{F},\mathscr{F}})_{k\to\ell}$.
For $j/i \leq \ell/k$ in V, we have $x_{j/-\infty,i} \cdot X_{j/i\to\ell/k} = (X\Psi_{-\infty,\mathrm{F},\mathscr{F}})_{j\to\ell} \cdot x_{\ell/-\infty,k}$.

(c) Suppose given $k \in \mathbf{Z}$. We obtain a morphism $X\Psi_{-\infty,\mathrm{F},\mathscr{F}} \xrightarrow{X\psi_{-\infty,k,\mathrm{F},\mathscr{F}}} X\Psi_{k,\mathrm{F},\mathscr{F}}$ in $\mathrm{F}(\mathscr{F})$ by setting $(X\psi_{-\infty,k,\mathrm{F},\mathscr{F}})_\ell = x_{\ell/-\infty,k-1}$ for $\ell \in \mathbf{Z}$.
Moreover, $X\psi_{-\infty,k,\mathrm{F},\mathscr{F}}$ is a k-pure epimorphism in $\mathrm{F}(\mathscr{F})$.

(d) Suppose given $n \in \mathbf{Z}$. If $X \in \mathrm{Ob}(\nabla^{\lceil n}(\mathscr{F}))$, then $X\Psi_{-\infty,\mathrm{F},\mathscr{F}} \in \mathrm{Ob}(\mathrm{F}^{\lceil n}(\mathscr{F}))$.

(e) Suppose given $n \in \mathbf{Z}$. If $X \in \mathrm{Ob}(\nabla^{n\rceil}(\mathscr{F}))$, then $X\Psi_{-\infty,\mathrm{F},\mathscr{F}} \in \mathrm{Ob}(\mathrm{F}^{n\rceil}(\mathscr{F}))$.

(f) If $X \in \mathrm{Ob}(\nabla^{\mathrm{b}}(\mathscr{F}))$, then $X\Psi_{-\infty,\mathrm{F},\mathscr{F}} \in \mathrm{Ob}(\mathrm{F}^{\mathrm{b}}(\mathscr{F}))$.

$\diamondsuit$

Proof. Ad (a). For $\ell \in \mathbf{Z}$, we have $(X\Psi_{-\infty,\mathrm{F},\mathscr{F}})_{\ell\to\ell} = X\psi_{\ell,\ell,\mathrm{CF},\mathscr{F}}\rceil = 1_{X\Psi_{\ell,\mathrm{CF},\mathscr{F}}}\rceil = 1_{X_{\ell/-\infty}}$ by lemmata 3.3.55.(a) and 3.2.24.(a). For $j \leq k \leq \ell$ in $\mathbf{Z}$, we have

$$(X\Psi_{-\infty,\mathrm{F},\mathscr{F}})_{j\to k} \cdot (X\Psi_{-\infty,\mathrm{F},\mathscr{F}})_{k\to\ell} = X\psi_{j,k,\mathrm{CF},\mathscr{F}}\rceil \cdot X\psi_{k,\ell,\mathrm{CF},\mathscr{F}}\rceil = X\psi_{j,\ell,\mathrm{CF},\mathscr{F}}\rceil = (X\Psi_{-\infty,\mathrm{F},\mathscr{F}})_{j\to\ell}$$

by lemmata 3.3.55.(b) and 3.2.24.(b).
Ad (b). For $k \leq \ell$ in $\mathbf{Z}$, the morphism $X_{\ell/-\infty} \xrightarrow{x_{\ell/-\infty,k}} X_{\ell/k}$ is a cokernel of $(X\Psi_{-\infty,\mathrm{F},\mathscr{F}})_{k\to\ell}$ by lemma 3.2.39.(d).
Suppose given $j/i \leq \ell/k$ in V. We have

$$\begin{aligned} x_{j/-\infty,i} \cdot X_{j/i\to\ell/k} &= x_{j/-\infty,i} \cdot x_{j/i,k} \cdot x_{j,\ell/k} = x_{j/-\infty,k} \cdot x_{j,\ell/k} = x_{j/-\infty,k} \cdot (X\psi_{j,\ell,\mathrm{CF},\mathscr{F}})_{k+1} \\ &= X\psi_{j,\ell,\mathrm{CF},\mathscr{F}}\rceil \cdot x_{\ell/-\infty,k} = (X\Psi_{-\infty,\mathrm{F},\mathscr{F}})_{j\to\ell} \cdot x_{\ell/-\infty,k} \,. \end{aligned}$$

Ad (c). For $\ell \in \mathbf{Z}$, we have

$$\begin{aligned} (X\psi_{-\infty,k,\mathrm{F},\mathscr{F}})_\ell \cdot (X\Psi_{k,\mathrm{F},\mathscr{F}})_{\ell\to\ell+1} &= x_{\ell/-\infty,k-1} \cdot x_{\ell,\ell+1/k-1} = x_{\ell/-\infty,k-1} \cdot (X\psi_{\ell,\ell+1,\mathrm{CF},\mathscr{F}})_k \\ &= X\psi_{\ell,\ell+1,\mathrm{CF},\mathscr{F}}\rceil \cdot x_{\ell+1/-\infty,k-1} \\ &= (X\Psi_{-\infty,\mathrm{F},\mathscr{F}})_{\ell\to\ell+1} \cdot (X\psi_{-\infty,k,\mathrm{F},\mathscr{F}})_{\ell+1} \,. \end{aligned}$$

Moreover, $X\psi_{-\infty,k,\mathrm{F},\mathscr{F}}$ is a k-pure epimorphism in $\mathrm{F}(\mathscr{F})$ since for $\ell \in \mathbf{Z}_{\geq k-1}$, the sequence $\big((X\Psi_{-\infty,\mathrm{F},\mathscr{F}})_{k-1\to\ell}\,,\, (X\psi_{-\infty,k,\mathrm{F},\mathscr{F}})_\ell\big)$ is pure short exact by (b).

Ad (d). Suppose that $X \in \mathrm{Ob}(\nabla^{\lceil n}(\mathscr{F}))$. For $\ell \in \mathbf{Z}_{\geq n}$, $(X\Psi_{-\infty,\mathrm{F},\mathscr{F}})_{\ell \to \ell+1} = X\psi_{\ell,\ell+1,\mathrm{CF},\mathscr{F}}\rceil$ is an isomorphism by remarks 3.3.56.(a) and 3.2.25.

Ad (e). Suppose that $X \in \mathrm{Ob}(\nabla^{n\rceil}(\mathscr{F}))$. Suppose given $\ell \in \mathbf{Z}_{<n}$. For $k \in \mathbf{Z}$, we have $(X\Psi_{\ell,\mathrm{CF},\mathscr{F}})_k = X_{\ell/k-1} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$, cf. remark 3.3.33.(a). Thus $X_{\ell/-\infty} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$ by remark 3.2.26.

Ad (f). This follows from (d) and (e). □

3.3.58 Lemma. Suppose given $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$.
For $k \in \mathbf{Z}$, suppose given a colimit $(X_{\infty/k-1}\, , (x_{\ell,\infty/k-1})_{\ell\in\mathbf{Z}})$ for $X\Psi_{k,\mathrm{F},\mathscr{F}}$, cf. definition 3.3.50.

(a) We obtain a cofiltration $X\Psi_{\infty,\mathrm{CF},\mathscr{F}} \in \mathrm{Ob}(\mathrm{CF}(\mathscr{F}))$ as follows.
For $k \in \mathbf{Z}$, let $(X\Psi_{\infty,\mathrm{CF},\mathscr{F}})_k = X_{\infty/k-1}$.
For $k \leq \ell$ in $\mathbf{Z}$, let $(X\Psi_{\infty,\mathrm{CF},\mathscr{F}})_{k \to \ell} = X\psi_{k,\ell,\mathrm{F},\mathscr{F}}\rceil \colon X_{\infty/k-1} \to X_{\infty/\ell-1}$, cf. definitions 3.3.54 and 3.2.23.

(b) Suppose given $\ell \in \mathbf{Z}$. We obtain a morphism $X\Psi_{\ell,\mathrm{CF},\mathscr{F}} \xrightarrow{X\psi_{\ell,\infty,\mathrm{CF},\mathscr{F}}} X\Psi_{\infty,\mathrm{CF},\mathscr{F}}$ in $\mathrm{CF}(\mathscr{F})$ by setting $(X\psi_{\ell,\infty,\mathrm{CF},\mathscr{F}})_k = x_{\ell,\infty/k-1}$ for $k \in \mathbf{Z}$.
Moreover, $X\psi_{\ell,\infty,\mathrm{CF},\mathscr{F}}$ is an ℓ-pure monomorphism in $\mathrm{F}(\mathscr{F})$.

(c) Suppose given $n \in \mathbf{Z}$. If $X \in \mathrm{Ob}(\nabla^{\lceil n}(\mathscr{F}))$, then $X\Psi_{\infty,\mathrm{CF},\mathscr{F}} \in \mathrm{Ob}(\mathrm{CF}^{\lceil n}(\mathscr{F}))$.

(d) Suppose given $n \in \mathbf{Z}$. If $X \in \mathrm{Ob}(\nabla^{n\rceil}(\mathscr{F}))$, then $X\Psi_{\infty,\mathrm{CF},\mathscr{F}} \in \mathrm{Ob}(\mathrm{CF}^{n\rceil}(\mathscr{F}))$.

(e) If $X \in \mathrm{Ob}(\nabla^{\mathrm{b}}(\mathscr{F}))$, then $X\Psi_{\infty,\mathrm{CF},\mathscr{F}} \in \mathrm{Ob}(\mathrm{CF}^{\mathrm{b}}(\mathscr{F}))$. ◇

Proof. This is dual to the previous lemma 3.3.57.(a,c,d,e,f). □

3.3.59 Lemma. Suppose given $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$.
For $\ell \in \mathbf{Z}$, suppose given a limit $(X_{\ell/-\infty}\, , (x_{\ell/-\infty,k-1})_{k\in\mathbf{Z}})$ for $X\Psi_{\ell,\mathrm{CF},\mathscr{F}}$.
Lemma 3.3.57 yields the filtration $X\Psi_{-\infty,\mathrm{F},\mathscr{F}} \in \mathrm{Ob}(\mathrm{F}(\mathscr{F}))$ and k-pure epimorphisms $X\Psi_{-\infty,\mathrm{F},\mathscr{F}} \xrightarrow{X\psi_{-\infty,k,\mathrm{F},\mathscr{F}}} X\Psi_{k,\mathrm{F},\mathscr{F}}$ in $\mathrm{F}(\mathscr{F})$ for $k \in \mathbf{Z}$.
For $k \in \mathbf{Z}$, suppose given a colimit $(X_{\infty/k-1}\, , (x_{\ell,\infty/k-1})_{\ell\in\mathbf{Z}})$ for $X\Psi_{k,\mathrm{F},\mathscr{F}}$.
Lemma 3.3.58 yields the cofiltration $X\Psi_{\infty,\mathrm{CF},\mathscr{F}} \in \mathrm{Ob}(\mathrm{CF}(\mathscr{F}))$ and ℓ-pure monomorphisms $X\Psi_{\ell,\mathrm{CF},\mathscr{F}} \xrightarrow{X\psi_{\ell,\infty,\mathrm{CF},\mathscr{F}}} X\Psi_{\infty,\mathrm{CF},\mathscr{F}}$ in $\mathrm{CF}(\mathscr{F})$ for $\ell \in \mathbf{Z}$.

(a) Suppose given a colimit $(C, (c_k)_{k\in\mathbf{Z}})$ for $X\Psi_{-\infty,\mathrm{F},\mathscr{F}}$. For $k \in \mathbf{Z}$, the sequence $X_{k/-\infty} \xrightarrow{c_k} C \xrightarrow{X\psi_{-\infty,k+1,\mathrm{F},\mathscr{F}}\rceil} X_{\infty/k}$ is pure short exact in $\mathscr{F}$, cf. definition 3.2.23.

(b) Suppose given a limit $(A, (a_k)_{k\in\mathbf{Z}})$ for $X\Psi_{\infty,\mathrm{CF},\mathscr{F}}$. For $k \in \mathbf{Z}$, the sequence $X_{k/-\infty} \xrightarrow{X\psi_{k,\infty,\mathrm{CF},\mathscr{F}}\rceil} A \xrightarrow{a_{k+1}} X_{\infty/k}$ is pure short exact in $\mathscr{F}$, cf. definition 3.2.22.

(c) Suppose given a colimit $(C, (c_k)_{k\in\mathbf{Z}})$ for $X\Psi_{-\infty,\mathrm{F},\mathscr{F}}$ and a limit $(A, (a_k)_{k\in\mathbf{Z}})$ for $X\Psi_{\infty,\mathrm{CF},\mathscr{F}}$. There exists a unique morphism $C \xrightarrow{u} A$ in $\mathscr{F}$ such that $c_\ell \cdot u \cdot a_{k+1} = x_{\ell/-\infty,k} \cdot x_{\ell,\infty/k}$ for $k, \ell \in \mathbf{Z}$. Moreover, u is an isomorphism, $(C, (X\psi_{-\infty,k,\mathrm{F},\mathscr{F}}\rceil)_{k\in\mathbf{Z}})$ is a limit for $X\Psi_{\infty,\mathrm{CF},\mathscr{F}}$ and $(A, (X\psi_{k,\infty,\mathrm{CF},\mathscr{F}}\rceil)_{k\in\mathbf{Z}})$ is a colimit for $X\Psi_{-\infty,\mathrm{F},\mathscr{F}}$. ◇

Proof. Ad (a).

This follows from lemma 3.2.40.(d) since $X\psi_{-\infty,k+1,\mathrm{F},\mathscr{F}}$ is a $(k+1)$-pure epimorphism for $k \in \mathbf{Z}$.

Ad (b).

This follows from lemma 3.2.39.(d) since $X\psi_{k,\infty,\mathrm{F},\mathscr{F}}$ is a k-pure monomorphism for $k \in \mathbf{Z}$.

Ad (c).

Suppose given $\ell \in \mathbf{Z}$.

Consider the morphisms $x_{\ell/-\infty,k} \cdot x_{\ell,\infty/k} \colon X_{\ell/-\infty} \to X_{\infty/k}$ for $k \in \mathbf{Z}$. We have $x_{\ell/-\infty,k} \cdot x_{\ell,\infty/k} \cdot (X\Psi_{\infty,\mathrm{CF},\mathscr{F}})_{k+1\to k+2} = x_{\ell/-\infty,k} \cdot x_{\ell/k,k+1} \cdot x_{\ell,\infty/k+1} = x_{\ell/-\infty,k+1} \cdot x_{\ell,\infty/k+1}$.

Since $(A, (a_k)_{k\in\mathbf{Z}})$ is a limit for $X\Psi_{\infty,\mathrm{CF},\mathscr{F}}$, there exists a unique morphism $X_{\ell/-\infty} \xrightarrow{t_\ell} A$ in $\mathscr{F}$ such that $t_\ell \cdot a_{k+1} = x_{\ell/-\infty,k} \cdot x_{\ell,\infty/k}$ for $k \in \mathbf{Z}$.

Now consider the morphisms $X_{\ell/-\infty} \xrightarrow{t_\ell} A$ for $\ell \in \mathbf{Z}$. We have $(X\Psi_{-\infty,\mathrm{F},\mathscr{F}})_{\ell-1\to\ell} \cdot t_\ell = t_{\ell-1}$ since

$$\begin{aligned}
(X\Psi_{-\infty,\mathrm{F},\mathscr{F}})_{\ell-1\to\ell} \cdot t_\ell \cdot a_{k+1} &= (X\Psi_{-\infty,\mathrm{F},\mathscr{F}})_{\ell-1\to\ell} \cdot x_{\ell/-\infty,k} \cdot x_{\ell,\infty/k} \\
&= x_{\ell-1/-\infty,k} \cdot x_{\ell-1,\ell/k} \cdot x_{\ell,\infty/k} = x_{\ell-1/-\infty,k} \cdot x_{\ell-1,\infty/k} \\
&= t_{\ell-1} \cdot a_{k+1}
\end{aligned}$$

for $k \in \mathbf{Z}$ and since $(A, (a_k)_{k\in\mathbf{Z}})$ is a limit for $X\Psi_{\infty,\mathrm{CF},\mathscr{F}}$.

Since $(C, (c_\ell)_{\ell\in\mathbf{Z}})$ is a colimit for $X\Psi_{-\infty,\mathrm{F},\mathscr{F}}$, there exists a unique morphism $C \xrightarrow{u} A$ in $\mathscr{F}$ such that $c_\ell \cdot u = t_\ell$ for $\ell \in \mathbf{Z}$. So $c_\ell \cdot u \cdot a_{k+1} = t_\ell \cdot a_{k+1} = x_{\ell/-\infty,k} \cdot x_{\ell,\infty/k}$ for $k, \ell \in \mathbf{Z}$.

Suppose given $C \xrightarrow{v} A$ in $\mathscr{F}$ such that $c_\ell \cdot v \cdot a_{k+1} = x_{\ell/-\infty,k} \cdot x_{\ell,\infty/k}$ for $k, \ell \in \mathbf{Z}$.

For $\ell \in \mathbf{Z}$, we have the morphisms $X_{\ell/-\infty} \xrightarrow{t_\ell} A$ and $X_{\ell/-\infty} \xrightarrow{c_\ell \cdot v} A$ in $\mathscr{F}$ with $t_\ell \cdot a_{k+1} = x_{\ell/-\infty,k} \cdot x_{\ell,\infty/k} = c_\ell \cdot v \cdot a_{k+1}$ for $k \in \mathbf{Z}$. Since $(A, (a_k)_{k\in\mathbf{Z}})$ is a limit for $X\Psi_{\infty,\mathrm{CF},\mathscr{F}}$, we have $t_\ell = c_\ell \cdot v$ for $\ell \in \mathbf{Z}$.

Now $c_\ell \cdot v = t_\ell = c_\ell \cdot u$ for $\ell \in \mathbf{Z}$ and thus $v = u$.

We want to show that the following diagram is commutative in $\mathscr{F}$.

$$\begin{tikzcd}
X_{0/-\infty} \arrow[r, "c_0"] \arrow[d, "1"'] & C \arrow[r, "X\psi_{-\infty,1,\mathrm{F},\mathscr{F}}\rceil"] \arrow[d, "u"] & X_{\infty/0} \arrow[d, "1"] \\
X_{0/-\infty} \arrow[r, "X\psi_{0,\infty,\mathrm{CF},\mathscr{F}}\rceil"] & A \arrow[r, "a_1"] & X_{\infty/0}
\end{tikzcd}$$

We have $c_0 \cdot u = X\psi_{0,\infty,\mathrm{CF},\mathscr{F}}\rceil$ since $c_0 \cdot u \cdot a_{k+1} = x_{0/-\infty,k} \cdot x_{0,\infty/k} = X\psi_{0,\infty,\mathrm{CF},\mathscr{F}}\rceil \cdot a_{k+1}$ for $k \in \mathbf{Z}$ and since $(A, (a_k)_{k\in\mathbf{Z}})$ is a limit for $X\Psi_{\infty,\mathrm{CF},\mathscr{F}}$.

We have $u \cdot a_1 = X\psi_{-\infty,1,\mathrm{F},\mathscr{F}}\rceil$ since $c_\ell \cdot u \cdot a_1 = x_{\ell/-\infty,0} \cdot x_{\ell,\infty/0} = c_\ell \cdot X\psi_{-\infty,1,\mathrm{F},\mathscr{F}}\rceil$ for $\ell \in \mathbf{Z}$ and since $(C, (c_\ell)_{\ell\in\mathbf{Z}})$ is a colimit for $X\Psi_{-\infty,\mathrm{F},\mathscr{F}}$.

Now u is an isomorphism by lemma 1.3.19.(c).

So $(C, (u \cdot a_k)_{k\in\mathbf{Z}})$ is a limit for $X\Psi_{\infty,\mathrm{CF},\mathscr{F}}$ and $(A, (c_\ell \cdot u)_{\ell\in\mathbf{Z}})$ is a colimit for $X\Psi_{-\infty,\mathrm{F},\mathscr{F}}$.

Suppose given $k \in \mathbf{Z}$. We have $u \cdot a_k = X\psi_{-\infty,k,\mathrm{F},\mathscr{F}}1$ since

$$c_\ell \cdot u \cdot a_k = x_{\ell/-\infty,k-1} \cdot x_{\ell,\infty/k-1} = c_\ell \cdot X\psi_{-\infty,k,\mathrm{F},\mathscr{F}}1$$

for $\ell \in \mathbf{Z}$ and since $(C, (c_\ell)_{\ell\in\mathbf{Z}})$ is a colimit for $X\Psi_{-\infty,\mathrm{F},\mathscr{F}}$.

Suppose given $\ell \in \mathbf{Z}$. We have $c_\ell \cdot u = X\psi_{\ell,\infty,\mathrm{CF},\mathscr{F}}1$ since

$$c_\ell \cdot u \cdot a_{k+1} = x_{\ell/-\infty,k} \cdot x_{\ell,\infty/k} = X\psi_{\ell,\infty,\mathrm{CF},\mathscr{F}}1 \cdot a_{k+1}$$

for $k \in \mathbf{Z}$ and since $(A, (a_k)_{k\in\mathbf{Z}})$ is a limit for $X\Psi_{\infty,\mathrm{CF},\mathscr{F}}$. □

3.4 Filtered objects

Suppose given a strict Frobenius category $\mathscr{F} = (\mathscr{F}, \mathrm{B}, \Sigma, \iota, \pi, \alpha)$.

3.4.1 Definition. Let $\Omega = \mathbf{Z} \sqcup \mathbf{Z}_{[0,0]} \sqcup \mathbf{Z} = (\mathbf{Z} \times \{1\}) \cup (\mathbf{Z}_{[0,0]} \times \{2\}) \cup (\mathbf{Z} \times \{3\})$. We write $|k = (k, 1) \in \Omega$ and $k| = (k, 3) \in \Omega$ for $k \in \mathbf{Z}$. We also write $\omega = (0, 2) \in \Omega$.

We define a partial order on Ω as follows. Suppose given $k, \ell \in \mathbf{Z}$.

- We have $|k \leq |\ell$ if and only if $k \leq \ell$ in $\mathbf{Z}$.
- We have $k| \leq \ell|$ if and only if $k \leq \ell$ in $\mathbf{Z}$.
- We have $|k \leq \omega$.
- We have $\omega \leq k|$.
- We have $|k \leq \ell|$.

We consider the functor category $\Omega(\mathscr{F})$ as an exact category equipped with the pointwise exact structure, cf. convention 17 and definition 1.4.7.

For $X \in \mathrm{Ob}(\Omega(\mathscr{F}))$ and $k \in \mathbf{Z}$, we write $x_{|k} = X_{|k \to |k+1}$, $x_{k|} = X_{k| \to k+1|}$, $x^\omega_{|k} = X_{|k \to \omega}$ and $x^\omega_{k|} = X_{\omega \to k|}$. ♢

3.4.2 Definition. An object $X \in \mathrm{Ob}(\Omega(\mathscr{F}))$ is called a *filtered object* if the following three conditions hold.

(FO1) The pair $(x^\omega_{|k}\, , x^\omega_{k+1|})$ is a pure short exact sequence for $k \in \mathbf{Z}$.

(FO2) The morphism $x_{|k}$ is a pure monomorphism for $k \in \mathbf{Z}$.

(FO3) The morphism $x_{k|}$ is a pure epimorphism for $k \in \mathbf{Z}$.

Let $\mathrm{FO}(\mathscr{F})$ denote the full subcategory of $\Omega(\mathscr{F})$ defined by

$$\mathrm{Ob}(\mathrm{FO}(\mathscr{F})) = \{X \in \mathrm{Ob}(\Omega(\mathscr{F}))\colon X \text{ is a filtered object}\}.$$

We call $\mathrm{FO}(\mathscr{F})$ the *category of filtered objects* in $\mathscr{F}$.

$$\begin{tikzcd}
 & & & & X_\omega \arrow[dlll, "x^\omega_{k+1|}"'] \arrow[dll, "x^\omega_{k|}"] & & & & \\
\cdots & X_{k+1|} \arrow[l] & X_{k|} \arrow[l, "x_{k|}"] & \cdots \arrow[l] & & \cdots & X_{|k+1} \arrow[l] \arrow[ull, "x^\omega_{|k+1}"] & X_{|k} \arrow[l, "x_{|k}"] \arrow[ulll, "x^\omega_{|k}"'] & \cdots \arrow[l]
\end{tikzcd}$$

$\mathrm{FO}(\mathscr{F})$ is a full additive subcategory of $\Omega(\mathscr{F})$ that satifies the conditions (RE1), (RE2) (RE3) and (RE4) of definition 1.3.21 by remark 1.4.10 and lemmata 1.4.11, 1.4.12, 1.4.13, 1.4.14, 1.4.15.

We equip $\mathrm{FO}(\mathscr{F})$ with the restricted exact structure of the pointwise exact structure on $\Omega(\mathscr{F})$. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathrm{FO}(\mathscr{F})$ is a pure short exact sequence if and only if $X_a \xrightarrow{f_a} Y_a \xrightarrow{g_a} Z_a$ is a pure short exact sequence in $\mathscr{F}$ for $a \in \Omega$. $\diamondsuit$

3.4.3 Lemma. Suppose given $X, Y \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$.

(a) Suppose given a morphism $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$. The morphism $X_\omega \xrightarrow{f_\omega} Y_\omega$ satisfies $x^\omega_{|k} \cdot f_\omega \cdot y^\omega_{k+1|} = 0$ for $k \in \mathbf{Z}$.

(b) Suppose given a morphism $X_\omega \xrightarrow{g} Y_\omega$ in $\mathscr{F}$ such that $x^\omega_{|k} \cdot g \cdot y^\omega_{k+1|} = 0$ for $k \in \mathbf{Z}$. Then there exists a unique morphism $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$ such that $f_\omega = g$.
For $k \in \mathbf{Z}$, $X_{|k} \xrightarrow{f_{|k}} Y_{|k}$ is the unique morphism in $\mathscr{F}$ such that $f_{|k} \cdot y^\omega_{|k} = x^\omega_{|k} \cdot g$ and $X_{k|} \xrightarrow{f_{k|}} Y_{k|}$ is the unique morphism in $\mathscr{F}$ such that $x^\omega_{k|} \cdot f_{k|} = g \cdot y^\omega_{k|}$.

(c) Suppose given $f, g\colon X \to Y$ in $\mathrm{FO}(\mathscr{F})$ such that $f_\omega = g_\omega$ in $\mathscr{F}$. Then $f = g$ in $\mathrm{FO}(\mathscr{F})$. $\diamondsuit$

Proof. Ad (a). For $k \in \mathbf{Z}$, we have $x^\omega_{|k} \cdot f_\omega \cdot y^\omega_{k+1|} = x^\omega_{|k} \cdot x^\omega_{k+1|} \cdot f_{k+1|} = 0$.

Ad (b). Let $f_\omega = g$. Suppose given $k \in \mathbf{Z}$. Since $y^\omega_{|k}$ is a kernel of $y^\omega_{k+1|}$, there exists a unique morphism $X_{|k} \xrightarrow{f_{|k}} Y_{|k}$ in $\mathscr{F}$ such that $f_{|k} \cdot y^\omega_{|k} = x^\omega_{|k} \cdot g$. We also have $f_{|k} \cdot y_{|k} = x_{|k} \cdot f_{|k+1}$ since $f_{|k} \cdot y_{|k} \cdot y^\omega_{|k+1} = f_{|k} \cdot y^\omega_{|k} = x^\omega_{|k} \cdot g = x_{|k} \cdot x^\omega_{|k+1} \cdot g = x_{|k} \cdot f_{|k+1} \cdot y^\omega_{|k+1}$ and since $y^\omega_{|k+1}$ is a pure monomorphism.

Since $x^\omega_{k|}$ is a cokernel of $x^\omega_{|k-1}$, there exists a unique morphism $X_{k|} \xrightarrow{f_{k|}} Y_{k|}$ in $\mathscr{F}$ such that $x^\omega_{k|} \cdot f_{k|} = g \cdot y^\omega_{k|}$. We also have $f_{k|} \cdot y_{k|} = x_{k|} \cdot f_{k+1|}$ since
$x^\omega_{k|} \cdot f_{k|} \cdot y_{k|} = g \cdot y^\omega_{k|} \cdot y_{k|} = g \cdot y^\omega_{k+1|} = x^\omega_{k+1|} \cdot f_{k+1|} = x^\omega_{k|} \cdot x_{k|} \cdot f_{k+1|}$ and since $x^\omega_{k|}$ is a pure epimorphism.

Ad (c). This follows from (a) and (b). $\square$

3.4.4 Lemma/Definition. We define the projection functor $\mathrm{P}_{\omega,\mathscr{F}}\colon \mathrm{FO}(\mathscr{F}) \to \mathscr{F}$ by setting $X\mathrm{P}_{\omega,\mathscr{F}} = X_\omega$ for $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ and $f\mathrm{P}_{\omega,\mathscr{F}} = f_\omega$ for $f \in \mathrm{Mor}(\mathrm{FO}(\mathscr{F}))$. This in fact defines an exact functor. $\diamondsuit$

Proof. We abbreviate $\mathrm{P} = \mathrm{P}_{\omega,\mathscr{F}}$. Suppose given $X \overset{f}{\underset{h}{\rightrightarrows}} Y \xrightarrow{g} Z$ in $\mathrm{FO}(\mathscr{F})$.
We have $1_X\mathrm{P} = (1_X)_\omega = 1_{X_\omega} = 1_{X\mathrm{P}}$, $(f \cdot g)\mathrm{P} = (f \cdot g)_\omega = f_\omega \cdot g_\omega = f\mathrm{P} \cdot g\mathrm{P}$ and $(f + h)\mathrm{P} = (f + h)_\omega = f_\omega + h_\omega = f\mathrm{P} + h\mathrm{P}$.

If (f, g) is a pure short exact sequence in $\mathrm{FO}(\mathscr{F})$, then $(f\mathrm{P}, g\mathrm{P}) = (f_\omega\, , \, g_\omega)$ is a pure short exact sequence in $\mathscr{F}$. □

3.4.5 Lemma/Definition. We define the projection functor $\mathrm{P}_{\mathrm{CF},\mathscr{F}} \colon \mathrm{FO}(\mathscr{F}) \to \mathrm{CF}(\mathscr{F})$ as follows. For $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ and $k \in \mathbf{Z}$, let $(X\mathrm{P}_{\mathrm{CF},\mathscr{F}})_k = X_{k|}$ and $(X\mathrm{P}_{\mathrm{CF},\mathscr{F}})_{k\to k+1} = x_{k|}$. For $f \in \mathrm{Mor}(\mathrm{FO}(\mathscr{F}))$ and $k \in \mathbf{Z}$, let $(f\mathrm{P}_{\mathrm{CF},\mathscr{F}})_k = f_{k|}$. This in fact defines an exact functor. ♢

Proof. We abbreviate $\mathrm{P} = \mathrm{P}_{\mathrm{CF},\mathscr{F}}$. Suppose given $X \overset{f}{\underset{h}{\rightrightarrows}} Y \xrightarrow{g} Z$ in $\mathrm{FO}(\mathscr{F})$.
We have $1_X\mathrm{P} = (1_X)_\omega = 1_{X_\omega} = 1_{X\mathrm{P}}$ since $(1_X\mathrm{P})_k = (1_X)_{k|} = 1_{X_{k|}} = (1_{X\mathrm{P}})_{k|}$ for $k \in \mathbf{Z}$.
We have $(f \cdot g)\mathrm{P} = f\mathrm{P} \cdot g\mathrm{P}$ since

$$((f \cdot g)\mathrm{P})_k = (f \cdot g)_{k|} = f_{k|} \cdot g_{k|} = (f\mathrm{P})_k \cdot (g\mathrm{P})_k = (f\mathrm{P} \cdot g\mathrm{P})_k$$

for $k \in \mathbf{Z}$.
We have $(f + h)\mathrm{P} = f\mathrm{P} + h\mathrm{P}$ since

$$((f + h)\mathrm{P})_k = (f + h)_{k|} = f_{k|} + h_{k|} = (f\mathrm{P})_k + (h\mathrm{P})_k = (f\mathrm{P} + h\mathrm{P})_k$$

for $k \in \mathbf{Z}$.
Suppose that (f, g) is a pure short exact sequence in $\mathrm{FO}(\mathscr{F})$. Then $((f\mathrm{P})_k\, , \, (g\mathrm{P})_k) = (f_{k|}\, , \, g_{k|})$ is a pure short exact sequence in $\mathscr{F}$ for $k \in \mathbf{Z}$. We conclude that $(f\mathrm{P}, g\mathrm{P})$ is a pure short exact sequence in $\mathrm{CF}(\mathscr{F})$. □

3.4.6 Lemma/Definition. We define the projection functor $\mathrm{P}_{\mathrm{F},\mathscr{F}} \colon \mathrm{FO}(\mathscr{F}) \to \mathrm{F}(\mathscr{F})$ as follows. For $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ and $k \in \mathbf{Z}$, let $(X\mathrm{P}_{\mathrm{F},\mathscr{F}})_k = X_{|k}$ and $(X\mathrm{P}_{\mathrm{F},\mathscr{F}})_{k\to k+1} = x_{|k}$. For $f \in \mathrm{Mor}(\mathrm{FO}(\mathscr{F}))$ and $k \in \mathbf{Z}$, let $(f\mathrm{P}_{\mathrm{F},\mathscr{F}})_k = f_{|k}$. This in fact defines an exact functor. ♢

Proof. This is dual to the previous lemma 3.4.5 □

3.4.7 Lemma/Definition. We define the translation functors
$\mathrm{T}_{\mathrm{FO},\mathscr{F}}\, , \mathrm{T}^{-}_{\mathrm{FO},\mathscr{F}} \colon \mathrm{FO}(\mathscr{F}) \to \mathrm{FO}(\mathscr{F})$ as follows. For $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ and $k \in \mathbf{Z}$, let

- $(X\mathrm{T}_{\mathrm{FO},\mathscr{F}})_\omega = X_\omega$, $(X\mathrm{T}_{\mathrm{FO},\mathscr{F}})_{|k} = X_{|k+1}$, $(X\mathrm{T}_{\mathrm{FO},\mathscr{F}})_{k|} = X_{k+1|}$,
- $(X\mathrm{T}_{\mathrm{FO},\mathscr{F}})_{|k\to\omega} = X_{|k+1\to\omega}$, $(X\mathrm{T}_{\mathrm{FO},\mathscr{F}})_{\omega\to k|} = X_{\omega\to k+1|}$,
- $(X\mathrm{T}_{\mathrm{FO},\mathscr{F}})_{|k\to|k+1} = X_{|k+1\to|k+2}$, $(X\mathrm{T}_{\mathrm{FO},\mathscr{F}})_{k|\to k+1|} = X_{k+1|\to k+2|}$,
- $(X\mathrm{T}^{-}_{\mathrm{FO},\mathscr{F}})_\omega = X_\omega$, $(X\mathrm{T}^{-}_{\mathrm{FO},\mathscr{F}})_{|k} = X_{|k-1}$, $(X\mathrm{T}^{-}_{\mathrm{FO},\mathscr{F}})_{k|} = X_{k-1|}$,
- $(X\mathrm{T}^{-}_{\mathrm{FO},\mathscr{F}})_{|k\to\omega} = X_{|k-1\to\omega}$, $(X\mathrm{T}^{-}_{\mathrm{FO},\mathscr{F}})_{\omega\to k|} = X_{\omega\to k-1|}$,

- $(X\mathrm{T}^{-}_{\mathrm{FO},\mathscr{F}})_{|k\to|k+1} = X_{|k-1\to|k}$, $(X\mathrm{T}^{-}_{\mathrm{FO},\mathscr{F}})_{k|\to k+1|} = X_{k-1|\to k|}$.

For $f \in \mathrm{Mor}(\mathrm{FO}(\mathscr{F}))$ and $k \in \mathbf{Z}$, let

- $(f\mathrm{T}_{\mathrm{FO},\mathscr{F}})_\omega = f_\omega$, $(f\mathrm{T}_{\mathrm{FO},\mathscr{F}})_{|k} = f_{|k+1}$, $(f\mathrm{T}_{\mathrm{FO},\mathscr{F}})_{k|} = f_{k+1|}$,
- $(f\mathrm{T}^{-}_{\mathrm{FO},\mathscr{F}})_\omega = f_\omega$, $(f\mathrm{T}^{-}_{\mathrm{FO},\mathscr{F}})_{|k} = f_{|k-1}$, $(f\mathrm{T}^{-}_{\mathrm{FO},\mathscr{F}})_{k|} = f_{k-1|}$.

The functors $\mathrm{T}_{\mathrm{FO},\mathscr{F}}$ and $\mathrm{T}^{-}_{\mathrm{FO},\mathscr{F}}$ are mutually inverse isomorphisms of categories.
For $k \in \mathbf{Z}$ and $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$, we often write $X_{[k]} = X\mathrm{T}^{k}_{\mathrm{FO},\mathscr{F}}$ and $f_{[k]} = f\mathrm{T}^{k}_{\mathrm{FO},\mathscr{F}}$. ♢

Proof. We abbreviate $\mathrm{T} = \mathrm{T}_{\mathrm{FO},\mathscr{F}}$ and $\mathrm{T}^- = \mathrm{T}^{-}_{\mathrm{FO},\mathscr{F}}$. We will use lemma 3.4.3.(c).
Suppose given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathrm{FO}(\mathscr{F})$.
We have $1_X\mathrm{T} = 1_{X\mathrm{T}}$ since $(1_X\mathrm{T})_\omega = 1_{X_\omega} = (1_{X\mathrm{T}})_\omega$.
We have $(fg)\mathrm{T} = f\mathrm{T}\cdot g\mathrm{T}$ since $((fg)\mathrm{T})_\omega = f_\omega \cdot g_\omega = (f\mathrm{T}\cdot g\mathrm{T})_\omega$.
We have $1_X\mathrm{T}^- = 1_{X\mathrm{T}^-}$ since $(1_X\mathrm{T}^-)_\omega = 1_{X_\omega} = (1_{X\mathrm{T}^-})_\omega$.
We have $(fg)\mathrm{T}^- = f\mathrm{T}^-\cdot g\mathrm{T}^-$ since $((fg)\mathrm{T}^-)_\omega = f_\omega \cdot g_\omega = (f\mathrm{T}^-\cdot g\mathrm{T}^-)_\omega$.
We have $f\mathrm{T}\mathrm{T}^- = f$ since $(f\mathrm{T}\mathrm{T}^-)_\omega = f_\omega$. We have $f\mathrm{T}^-\mathrm{T} = f$ since $(f\mathrm{T}^-\mathrm{T})_\omega = f_\omega$. □

3.4.8 Lemma/Definition. We define the transformation $\rho_{\mathrm{FO},\mathscr{F}} \colon 1_{\mathrm{FO}(\mathscr{F})} \to \mathrm{T}_{\mathrm{FO},\mathscr{F}}$ by setting $(X\rho_{\mathrm{FO},\mathscr{F}})_\omega = 1_{X_\omega}$, $(X\rho_{\mathrm{FO},\mathscr{F}})_{|k} = x_{|k}$ and $(X\rho_{\mathrm{FO},\mathscr{F}})_{k|} = x_{k|}$ for $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ and $k \in \mathbf{Z}$. This in fact defines a transformation. We sometimes abbreviate $\rho = \rho_{\mathrm{FO},\mathscr{F}}$. ♢

Proof. We abbreviate $\rho = \rho_{\mathrm{FO},\mathscr{F}}$ and $\mathrm{T} = \mathrm{T}_{\mathrm{FO},\mathscr{F}}$. We will use lemma 3.4.3.(c).
Suppose given $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$. We have $X\rho\cdot f\mathrm{T} = f\cdot Y\rho$ since
$(X\rho\cdot f\mathrm{T})_\omega = 1_{X_\omega}\cdot f_\omega = f_\omega = f_\omega \cdot 1_{Y_\omega} = (f\cdot Y\rho)_\omega$. □

3.4.9 Definition. Suppose given an exact functor $F \colon \mathscr{F} \to \mathscr{F}$.
Let $\mathrm{FO}(F) = \Omega(F)|^{\mathrm{FO}(\mathscr{F})}_{\mathrm{FO}(\mathscr{F})} \colon \mathrm{FO}(\mathscr{F}) \to \mathrm{FO}(\mathscr{F})$, cf. definition 1.4.3 and lemma 1.4.16.(b,c,d). The functor $\mathrm{FO}(F)$ is exact by lemma 1.4.16.(a) and remark 1.3.22.

Suppose given exact functors $F, G \colon \mathscr{F} \to \mathscr{F}$ and a transformation $\lambda \colon F \to G$.
Let $\mathrm{FO}(\lambda) = \Omega(\lambda)|^{\mathrm{FO}(\mathscr{F})}_{\mathrm{FO}(\mathscr{F})} \colon \mathrm{FO}(F) \to \mathrm{FO}(G)$, cf. definition 1.4.5.

We often abbreviate $\mathbf{B} = \mathrm{FO}(\mathrm{B})$, $\boldsymbol{\Sigma} = \mathrm{FO}(\Sigma)$, $\boldsymbol{\iota} = \mathrm{FO}(\iota)$, $\boldsymbol{\pi} = \mathrm{FO}(\pi)$ and $\boldsymbol{\alpha} = \mathrm{FO}(\alpha)$. We also write $\Sigma_{\mathrm{FO},\mathscr{F}} = \mathrm{FO}(\Sigma)$. ♢

3.4.10 Lemma. Suppose given an exact functor $F \colon \mathscr{F} \to \mathscr{F}$.
We have $\mathrm{FO}(F)\cdot \mathrm{T}_{\mathrm{FO},\mathscr{F}} = \mathrm{T}_{\mathrm{FO},\mathscr{F}}\cdot \mathrm{FO}(F)$ and $\mathrm{FO}(F)\cdot \mathrm{T}^{-1}_{\mathrm{FO},\mathscr{F}} = \mathrm{T}^{-1}_{\mathrm{FO},\mathscr{F}}\cdot \mathrm{FO}(F)$. ♢

Proof. Suppose given $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$. We have

$$(f\mathrm{FO}(F)\mathrm{T}_{\mathrm{FO},\mathscr{F}})_\omega = (f\mathrm{FO}(F))_\omega = f_\omega F = (f\mathrm{T}_{\mathrm{FO},\mathscr{F}})_\omega F = (f\mathrm{T}_{\mathrm{FO},\mathscr{F}}\mathrm{FO}(F))_\omega\,.$$

Thus $f\mathrm{FO}(F)\mathrm{T}_{\mathrm{FO},\mathscr{F}} = f\mathrm{T}_{\mathrm{FO},\mathscr{F}}\mathrm{FO}(F)$ by lemma 3.4.3. So $\mathrm{FO}(F) \cdot \mathrm{T}_{\mathrm{FO},\mathscr{F}} = \mathrm{T}_{\mathrm{FO},\mathscr{F}} \cdot \mathrm{FO}(F)$. We also have

$$\begin{aligned} \mathrm{FO}(F) \cdot \mathrm{T}^{-1}_{\mathrm{FO},\mathscr{F}} &= \mathrm{T}^{-1}_{\mathrm{FO},\mathscr{F}} \cdot \mathrm{T}_{\mathrm{FO},\mathscr{F}} \cdot \mathrm{FO}(F) \cdot \mathrm{T}^{-1}_{\mathrm{FO},\mathscr{F}} = \mathrm{T}^{-1}_{\mathrm{FO},\mathscr{F}} \cdot \mathrm{FO}(F) \cdot \mathrm{T}_{\mathrm{FO},\mathscr{F}} \cdot \mathrm{T}^{-1}_{\mathrm{FO},\mathscr{F}} \\ &= \mathrm{T}^{-1}_{\mathrm{FO},\mathscr{F}} \cdot \mathrm{FO}(F). \end{aligned}$$ □

3.4.11 Lemma. We have $\mathrm{T}_{\mathrm{FO},\mathscr{F}} \star \rho_{\mathrm{FO},\mathscr{F}} = \rho_{\mathrm{FO},\mathscr{F}} \star \mathrm{T}_{\mathrm{FO},\mathscr{F}}$. ◊

Proof. We abbreviate $\mathrm{T} = \mathrm{T}_{\mathrm{FO},\mathscr{F}}$ and $\rho = \rho_{\mathrm{FO},\mathscr{F}}$. We will use lemma 3.4.3.(c). We have $\mathrm{T} \star \rho = \rho \star \mathrm{T}$ since $(X(\mathrm{T} \star \rho))_\omega = 1_{X_\omega} = (X(\rho \star \mathrm{T}))_\omega$ for $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$. □

3.4.12 Definition. An object $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ is called *pointwise bijective* if X_ω , $X_{|k}$ and $X_{k|}$ are bijective objects in $\mathscr{F}$ for $k \in \mathbf{Z}$. Let $\mathfrak{B}_{\mathrm{FO},\mathscr{F}}$ denote the full subcategory of $\mathrm{FO}(\mathscr{F})$ defined by $\mathrm{Ob}(\mathfrak{B}_{\mathrm{FO},\mathscr{F}}) = \{X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))\colon X \text{ is pointwise bijective}\}$. ◊

3.4.13 Definition. The set $\mathfrak{J}_{\mathrm{FO}(\mathscr{F}),\mathrm{T}_{\mathrm{FO},\mathscr{F}},\rho_{\mathrm{FO},\mathscr{F}},\mathfrak{B}_{\mathrm{FO},\mathscr{F}}}$ is an ideal in $\mathrm{FO}(\mathscr{F})$, cf. definition 3.1.1. We abbreviate $\mathfrak{J}_{\mathrm{FO},\mathscr{F}} = \mathfrak{J}_{\mathrm{FO}(\mathscr{F}),\mathrm{T}_{\mathrm{FO},\mathscr{F}},\rho_{\mathrm{FO},\mathscr{F}},\mathfrak{B}_{\mathrm{FO},\mathscr{F}}}$. Let $\underline{\underline{\mathrm{FO}}}(\mathscr{F}) = \mathrm{FO}(\mathscr{F})/\mathfrak{J}_{\mathrm{FO},\mathscr{F}}$ denote the corresponding factor category and let $\mathfrak{Q}_{\mathrm{FO},\mathscr{F}}\colon \mathrm{FO}(\mathscr{F}) \to \underline{\underline{\mathrm{FO}}}(\mathscr{F})$ denote the correspoding residue class functor, cf. definition 1.2.13.
For $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$, we write $\underline{\underline{f}} = f + {}_{\mathrm{FO}(\mathscr{F}),\mathfrak{Q}_{\mathrm{FO},\mathscr{F}}}(X,Y)$. ◊

3.4.14 Remark. Suppose given $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$.
We have $\underline{\underline{f}} = 0$ if and only if there exists $X_{[-1]} \xrightarrow{u} B \xrightarrow{g} Y$ in $\mathrm{FO}(\mathscr{F})$ such that $X_{[-1]}\rho_{\mathrm{FO},\mathscr{F}} \cdot f = u \cdot g$ and such that $B \in \mathrm{Ob}(\mathfrak{B}_{\mathrm{FO},\mathscr{F}})$, cf. lemmata 3.1.2 and 3.4.11. ◊

3.4.15 Lemma. Suppose given $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$ such that $\underline{\underline{f}} = 0$ in $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$. Then we have $\underline{f_\omega} = 0$ in $\underline{\mathscr{F}}$. ◊

Proof. We may choose $X_{[-1]} \xrightarrow{u} B \xrightarrow{g} Y$ in $\mathrm{FO}(\mathscr{F})$ such that $X_{[-1]}\rho_{\mathrm{FO},\mathscr{F}} \cdot f = u \cdot g$ and such that $B \in \mathrm{Ob}(\mathfrak{B}_{\mathrm{FO},\mathscr{F}})$, cf. remark 3.4.14. Thus $f_\omega = u_\omega \cdot g_\omega$. We conclude that $\underline{f_\omega} = 0$ in $\underline{\mathscr{F}}$, cf. lemma 1.3.28. □

3.4.16 Definition. Let $\underline{\mathrm{P}}_{\omega,\mathscr{F}}\colon \underline{\underline{\mathrm{FO}}}(\mathscr{F}) \to \underline{\mathscr{F}}$ denote the unique functor such that $\mathfrak{Q}_{\mathrm{FO},\mathscr{F}} \cdot \underline{\mathrm{P}}_{\omega,\mathscr{F}} = \mathrm{P}_{\omega,\mathscr{F}} \cdot \mathfrak{P}_{\mathscr{F}}$, cf. lemma 3.4.15.

$$\begin{tikzcd} \mathrm{FO}(\mathscr{F}) \arrow[r, "\mathrm{P}_{\omega,\mathscr{F}}"] \arrow[d, "\mathfrak{Q}_{\mathrm{FO},\mathscr{F}}"'] & \mathscr{F} \arrow[d, "\mathfrak{P}_{\mathscr{F}}"] \\ \underline{\underline{\mathrm{FO}}}(\mathscr{F}) \arrow[r, "\underline{\mathrm{P}}_{\omega,\mathscr{F}}"] & \underline{\mathscr{F}} \end{tikzcd}$$ ◊

3.4.17 Definition. A candidate triangle $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}_{[-1]}$ in $\mathrm{FO}(\mathscr{F})$ with respect to $\Sigma_{\mathrm{FO},\mathscr{F}} \cdot \mathrm{T}^{-1}_{\mathrm{FO},\mathscr{F}}$ is called a *pseudo-triangle* in $\mathrm{FO}(\mathscr{F})$ if (i,p) is a pure short exact sequence in $\mathrm{FO}(\mathscr{F})$ and if there exists $X_{[-1]}\mathbf{B} \xrightarrow{g} Z$ in $\mathrm{FO}(\mathscr{F})$ such that the following diagram is commutative in $\mathrm{FO}(\mathscr{F})$.

$$\begin{tikzcd}
X_{[-1]} \arrow[r, "X_{[-1]}\boldsymbol{\iota}"] \arrow[d, "X_{[-1]}\rho_{\mathrm{FO},\mathscr{F}} \cdot f"'] & X_{[-1]}\mathbf{B} \arrow[r, "X_{[-1]}\boldsymbol{\pi}"] \arrow[d, "g"] & X^{[1]}_{[-1]} \arrow[d, "1"] \\
Y \arrow[r, "i"] & Z \arrow[r, "p"] & X^{[1]}_{[-1]}
\end{tikzcd}$$

$\diamondsuit$

3.4.18 Lemma. Suppose given $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$. Then there exists a pseudo-triangle $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}_{[-1]}$ in $\mathrm{FO}(\mathscr{F})$. $\diamondsuit$

Proof. Choose a pushout in $\mathrm{FO}(\mathscr{F})$ as follows.

$$\begin{tikzcd}
X_{[-1]} \arrow[r, "X_{[-1]}\boldsymbol{\iota}"] \arrow[d, "X_{[-1]}\rho_{\mathrm{FO},\mathscr{F}} \cdot f"'] & X_{[-1]}\mathbf{B} \arrow[d, "g"] \\
Y \arrow[r, "i"] & Z
\end{tikzcd}$$

By lemma 1.2.7.(b), we may choose $Z \xrightarrow{p} X^{[1]}_{[-1]}$ in $\mathscr{F}$ such that $g \cdot p = X_{[-1]}\boldsymbol{\pi}$ and such that (i, p) is a pure short exact sequence. Thus $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}_{[-1]}$ is a pseudo-triangle in $\mathrm{FO}(\mathscr{F})$. $\square$

3.4.19 Lemma.

(a) We have $\Sigma_{\mathrm{FO},\mathscr{F}} \cdot \mathrm{T}^{-1}_{\mathrm{FO},\mathscr{F}} \cdot \mathrm{P}_{\omega,\mathscr{F}} = \mathrm{P}_{\omega,\mathscr{F}} \cdot \Sigma$.

(b) Suppose given a pseudo-triangle $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}_{[-1]}$ in $\mathrm{FO}(\mathscr{F})$. Then $X\mathrm{P}_{\omega,\mathscr{F}} \xrightarrow{f\mathrm{P}_{\omega,\mathscr{F}}} Y\mathrm{P}_{\omega,\mathscr{F}} \xrightarrow{i\mathrm{P}_{\omega,\mathscr{F}}} Z\mathrm{P}_{\omega,\mathscr{F}} \xrightarrow{p\mathrm{P}_{\omega,\mathscr{F}}} X^{[1]}_{[-1]}\mathrm{P}_{\omega,\mathscr{F}}$ is a pseudo-triangle in $\mathscr{F}$, cf. definitions 3.4.17 and 2.2.8.

$\diamondsuit$

Proof. Ad (a). For $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$, we have $f^{[1]}_{[-1]}\mathrm{P}_{\omega,\mathscr{F}} = \big(f^{[1]}_{[-1]}\big)_\omega = f^{[1]}_\omega = \big(f\mathrm{P}_{\omega,\mathscr{F}}\big)^{[1]}$.
Ad (b). Note that we have $X\mathrm{P}_{\omega,\mathscr{F}} = X_\omega$, $Y\mathrm{P}_{\omega,\mathscr{F}} = Y_\omega$, $Z\mathrm{P}_{\omega,\mathscr{F}} = Z_\omega$, $X^{[1]}_{[-1]}\mathrm{P}_{\omega,\mathscr{F}} = X^{[1]}_\omega$, $f\mathrm{P}_{\omega,\mathscr{F}} = f_\omega$, $i\mathrm{P}_{\omega,\mathscr{F}} = i_\omega$ and $p\mathrm{P}_{\omega,\mathscr{F}} = p_\omega$.
We may choose $X_{[-1]}\mathbf{B} \xrightarrow{g} Z$ in $\mathrm{FO}(\mathscr{F})$ such that the following diagram is commutative in $\mathrm{FO}(\mathscr{F})$.

$$\begin{tikzcd}
X_{[-1]} \arrow[r, "X_{[-1]}\boldsymbol{\iota}"] \arrow[d, "X_{[-1]}\rho_{\mathrm{FO},\mathscr{F}} \cdot f"'] & X_{[-1]}\mathbf{B} \arrow[r, "X_{[-1]}\boldsymbol{\pi}"] \arrow[d, "g"] & X^{[1]}_{[-1]} \arrow[d, "1"] \\
Y \arrow[r, "i"] & Z \arrow[r, "p"] & X^{[1]}_{[-1]}
\end{tikzcd}$$

So we get the following commutative diagram in $\mathscr{F}$.

$$\begin{tikzcd}
X_\omega \arrow[r, "X_\omega \iota"] \arrow[d, "f_\omega"'] & X_\omega \mathbf{B} \arrow[r, "X_\omega \pi"] \arrow[d, "g_\omega"] & X_\omega^{[1]} \arrow[d, "1"] \\
Y_\omega \arrow[r, "i_\omega"] & Z_\omega \arrow[r, "p_\omega"] & X_\omega^{[1]}
\end{tikzcd}$$

We conclude that $X_\omega \xrightarrow{f_\omega} Y_\omega \xrightarrow{i_\omega} Z_\omega \xrightarrow{p_\omega} X_\omega^{[1]}$ is a pseudo-triangle in $\mathscr{F}$. $\square$

3.4.20 Definition. Suppose given $k \in \mathbf{Z}$. We define the full subcategories $\mathrm{FO}^{\lceil k}(\mathscr{F})$ and $\mathrm{FO}^{k\rceil}(\mathscr{F})$ of $\mathrm{FO}(\mathscr{F})$ by setting

$$\mathrm{Ob}(\mathrm{FO}^{\lceil k}(\mathscr{F})) = \{X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))\colon X_{\ell|} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}}) \text{ for } \ell \in \mathbf{Z}_{>k}\}$$

and

$$\mathrm{Ob}(\mathrm{FO}^{k\rceil}(\mathscr{F})) = \{X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))\colon X_{|\ell} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}}) \text{ for } \ell \in \mathbf{Z}_{<k}\}.$$

We define the full subcategories $\underline{\underline{\mathrm{FO}}}^{\lceil k}(\mathscr{F})$ and $\underline{\underline{\mathrm{FO}}}^{k\rceil}(\mathscr{F})$ of $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$ by setting
$\mathrm{Ob}(\underline{\underline{\mathrm{FO}}}^{\lceil k}(\mathscr{F})) = \mathrm{Ob}(\mathrm{FO}^{\lceil k}(\mathscr{F}))$ and $\mathrm{Ob}(\underline{\underline{\mathrm{FO}}}^{k\rceil}(\mathscr{F})) = \mathrm{Ob}(\mathrm{FO}^{k\rceil}(\mathscr{F}))$. $\diamondsuit$

3.4.21 Definition. Suppose given $k, \ell \in \mathbf{Z}$. Let $\mathrm{FO}^{\lceil k,\ell\rceil}(\mathscr{F}) = \mathrm{FO}^{\lceil k}(\mathscr{F}) \cap \mathrm{FO}^{\ell\rceil}(\mathscr{F})$ and $\underline{\underline{\mathrm{FO}}}^{\lceil k,\ell\rceil}(\mathscr{F}) = \underline{\underline{\mathrm{FO}}}^{\lceil k}(\mathscr{F}) \cap \underline{\underline{\mathrm{FO}}}^{\ell\rceil}(\mathscr{F})$. $\diamondsuit$

3.4.22 Definition. Let $\mathrm{FO}^{\mathrm{b}}(\mathscr{F}) = \bigcup\limits_{k,\ell \in \mathbf{Z}} \mathrm{FO}^{\lceil k,\ell\rceil}(\mathscr{F})$ and $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}(\mathscr{F}) = \bigcup\limits_{k,\ell \in \mathbf{Z}} \underline{\underline{\mathrm{FO}}}^{\lceil k,\ell\rceil}(\mathscr{F})$. $\diamondsuit$

3.4.23 Definition. Let $\mathfrak{Q}_{\mathrm{FO}^{\mathrm{b}},\mathscr{F}} = \mathfrak{Q}_{\mathrm{FO},\mathscr{F}}|_{\mathrm{FO}^{\mathrm{b}}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}(\mathscr{F})}\colon \mathrm{FO}^{\mathrm{b}}(\mathscr{F}) \to \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}(\mathscr{F})$. $\diamondsuit$

3.4.24 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ and $m \in \mathbf{Z}$. The following three statements are equivalent.

(a) We have $X \in \mathrm{Ob}(\mathrm{FO}^{\lceil m}(\mathscr{F}))$.

(b) We have $X\mathrm{P}_{\mathrm{F},\mathscr{F}} \in \mathrm{Ob}(\mathrm{F}^{\lceil m}(\mathscr{F}))$.

(c) We have $X\mathrm{P}_{\mathrm{CF},\mathscr{F}} \in \mathrm{Ob}(\mathrm{CF}^{\lceil m}(\mathscr{F}))$.

$\diamondsuit$

Proof. Suppose given $\ell \in \mathbf{Z}_{>m}$. Consider the pure short exact sequence $X_{|\ell-1} \xrightarrow{x^\omega_{|\ell-1}} X_\omega \xrightarrow{x^\omega_{\ell|}} X_{\ell|}$ in $\mathscr{F}$. We have $X_{\ell|} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$ is and only if $x^\omega_{|\ell-1} = x_{|\ell-1} \cdot x^\omega_{|\ell}$ is an isomorphism in $\mathscr{F}$. So the statements (a), (b) and (c) are equivalent. $\square$

3.4.25 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ and $m \in \mathbf{Z}$. The following three statements are equivalent.

(a) We have $X \in \mathrm{Ob}(\mathrm{FO}^{m\rceil}(\mathscr{F}))$.

(b) We have $X\mathrm{P}_{\mathrm{F},\mathscr{F}} \in \mathrm{Ob}(\mathrm{F}^{m\rceil}(\mathscr{F}))$.

(c) We have $X\mathrm{P}_{\mathrm{CF},\mathscr{F}} \in \mathrm{Ob}(\mathrm{CF}^{m\rceil}(\mathscr{F}))$.

$\diamondsuit$

Proof. This is dual to lemma 3.4.24. $\square$

3.4.26 Corollary. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$. The following three statements are equivalent.

(a) We have $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}(\mathscr{F}))$.

(b) We have $X\mathrm{P}_{\mathrm{F},\mathscr{F}} \in \mathrm{Ob}(\mathrm{F}^{\mathrm{b}}(\mathscr{F}))$.

(c) We have $X\mathrm{P}_{\mathrm{CF},\mathscr{F}} \in \mathrm{Ob}(\mathrm{CF}^{\mathrm{b}}(\mathscr{F}))$.

$\diamondsuit$

Proof. This follows from the previous lemmata 3.4.24 and 3.4.25. $\square$

3.4.27 Remark. Suppose given $m, n \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$.

(a) If $X \in \mathrm{Ob}(\mathrm{FO}^{\lceil m}(\mathscr{F}))$, then $X_{[n]} \in \mathrm{Ob}(\mathrm{FO}^{\lceil m-n}(\mathscr{F}))$.

(b) If $X \in \mathrm{Ob}(\mathrm{FO}^{m\rceil}(\mathscr{F}))$, then $X_{[n]} \in \mathrm{Ob}(\mathrm{FO}^{m-n\rceil}(\mathscr{F}))$.

$\diamondsuit$

Proof. Ad (a). Suppose given $\ell \in \mathbf{Z}_{>m-n}$.
We have $(X_{[n]})_{\ell|} = X_{n+\ell|} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$ since $X \in \mathrm{Ob}(\mathrm{FO}^{\lceil m}(\mathscr{F}))$.
Ad (b). This is dual to (a). $\square$

3.4.28 Lemma. Suppose given $m \in \mathbf{Z}$ and a pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathrm{FO}(\mathscr{F})$.

(a) If $X, Z \in \mathrm{Ob}(\mathrm{FO}^{\lceil m}(\mathscr{F}))$, then $Y \in \mathrm{Ob}(\mathrm{FO}^{\lceil m}(\mathscr{F}))$ as well.

(b) If $X, Z \in \mathrm{Ob}(\mathrm{FO}^{m\rceil}(\mathscr{F}))$, then $Y \in \mathrm{Ob}(\mathrm{FO}^{m\rceil}(\mathscr{F}))$ as well.

$\diamondsuit$

Proof. This follows from lemma 1.4.8.(d). $\square$

3.4.29 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}(\mathscr{F}))$. We have $X_{[1]}, X_{[-1]} \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. This follows from remark 3.4.27. $\square$

3.4.30 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}(\mathscr{F}))$. We have $X^{[1]}, X^{[-1]} \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. This follows from lemma 1.4.4.(b). $\square$

3.4.31 Lemma. Suppose given a pure short exact sequence $X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$ in $\mathrm{FO}(\mathscr{F})$ such that $X, Z \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}(\mathscr{F}))$. Then we have $Y \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. This follows from lemma 3.4.28. $\square$

3.4.32 Corollary. $\mathrm{FO}^{\mathrm{b}}(\mathscr{F})$ is an extension-closed subcategory of $\mathrm{FO}(\mathscr{F})$, cf. definition 1.3.23, remark 1.4.10 and lemma 3.4.31. In particular, it is a strictly full additive subcategory of $\mathrm{FO}(\mathscr{F})$. So $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}(\mathscr{F})$ is a full additive subcategory of $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$, cf. remark 1.2.14. $\diamondsuit$

3.4.33 Lemma. Suppose given a pseudo-triangle $X \xrightarrow{f} Y \overset{i}{\rightarrowtail} Z \overset{p}{\twoheadrightarrow} X^{[1]}_{[-1]}$ in $\mathrm{FO}(\mathscr{F})$ such that $X, Y \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}(\mathscr{F}))$. Then we have $Z, X^{[1]}_{[-1]} \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. This follows from lemmata 3.4.29, 3.4.30 and 3.4.31. $\square$

3.4.34 Definition. We define the full subcategories $\mathrm{FO}^{\lim}(\mathscr{F})$ and $\mathrm{FO}^{\mathrm{colim}}(\mathscr{F})$ of $\mathrm{FO}(\mathscr{F})$ by setting

$$\mathrm{Ob}(\mathrm{FO}^{\lim}(\mathscr{F})) = \{X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))\colon (X_\omega\,,\,(x^{\omega}_{k|})_{k\in\mathbf{Z}}) \text{ is a limit for } X\mathrm{P}_{\mathrm{CF},\mathscr{F}}\}$$

and

$$\mathrm{Ob}(\mathrm{FO}^{\mathrm{colim}}(\mathscr{F})) = \{X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))\colon (X_\omega\,,\,(x^{\omega}_{|k})_{k\in\mathbf{Z}}) \text{ is a colimit for } X\mathrm{P}_{\mathrm{F},\mathscr{F}}\}.$$

Let $\mathrm{FO}^{\nabla}(\mathscr{F}) = \mathrm{FO}^{\lim}(\mathscr{F}) \cap \mathrm{FO}^{\mathrm{colim}}(\mathscr{F})$.
We define the full subcategories $\underline{\underline{\mathrm{FO}}}^{\lim}(\mathscr{F})$, $\underline{\underline{\mathrm{FO}}}^{\mathrm{colim}}(\mathscr{F})$ and $\underline{\underline{\mathrm{FO}}}^{\nabla}(\mathscr{F})$ of $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$ by setting $\mathrm{Ob}(\underline{\underline{\mathrm{FO}}}^{\lim}(\mathscr{F})) = \mathrm{Ob}(\mathrm{FO}^{\lim}(\mathscr{F}))$, $\mathrm{Ob}(\underline{\underline{\mathrm{FO}}}^{\mathrm{colim}}(\mathscr{F})) = \mathrm{Ob}(\mathrm{FO}^{\mathrm{colim}}(\mathscr{F}))$ and
$\mathrm{Ob}(\underline{\underline{\mathrm{FO}}}^{\nabla}(\mathscr{F})) = \mathrm{Ob}(\mathrm{FO}^{\nabla}(\mathscr{F}))$. $\diamondsuit$

3.4.35 Definition. Let $\mathfrak{Q}_{\mathrm{FO}^{\nabla},\mathscr{F}} = \mathfrak{Q}_{\mathrm{FO},\mathscr{F}}|^{\underline{\underline{\mathrm{FO}}}^{\nabla}(\mathscr{F})}_{\mathrm{FO}^{\nabla}(\mathscr{F})}\colon \mathrm{FO}^{\nabla}(\mathscr{F}) \to \underline{\underline{\mathrm{FO}}}^{\nabla}(\mathscr{F})$. $\diamondsuit$

3.4.36 Remark. Suppose given $m \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{FO}^{m]}(\mathscr{F}))$. Then $X \in \mathrm{Ob}(\mathrm{FO}^{\lim}(\mathscr{F}))$. $\diamondsuit$

Proof. This follows from lemma 3.2.17. $\square$

3.4.37 Remark. Suppose given $m \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{FO}^{[m}(\mathscr{F}))$. Then $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{colim}}(\mathscr{F}))$. $\diamondsuit$

Proof. This is dual to the previous remark 3.4.36. $\square$

3.4.38 Remark. We have $\mathrm{FO}^{\mathrm{b}}(\mathscr{F}) \subseteq \mathrm{FO}^{\lim}(\mathscr{F})$ and $\mathrm{FO}^{\mathrm{b}}(\mathscr{F}) \subseteq \mathrm{FO}^{\mathrm{colim}}(\mathscr{F})$, cf. remarks 3.4.36 and 3.4.37. $\diamondsuit$

3.4.39 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\lim}(\mathscr{F}))$. We have $X_{[1]}, X_{[-1]} \in \mathrm{Ob}(\mathrm{FO}^{\lim}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. This follows from lemma 3.2.21. $\square$

3.4.40 Lemma. Suppose given $X \in \operatorname{Ob}(\mathrm{FO}^{\mathrm{colim}}(\mathscr{F}))$. We have $X_{[1]}, X_{[-1]} \in \operatorname{Ob}(\mathrm{FO}^{\mathrm{colim}}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. This is dual to the previous lemma 3.4.39. $\square$

3.4.41 Lemma. Suppose given $X \in \operatorname{Ob}(\mathrm{FO}^{\nabla}(\mathscr{F}))$. We have $X_{[1]}, X_{[-1]} \in \operatorname{Ob}(\mathrm{FO}^{\nabla}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. This follows from lemmata 3.4.39 and 3.4.40. $\square$

3.4.42 Lemma. Suppose given $X \in \operatorname{Ob}(\mathrm{FO}^{\mathrm{lim}}(\mathscr{F}))$. We have $X^{[1]}, X^{[-1]} \in \operatorname{Ob}(\mathrm{FO}^{\mathrm{lim}}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. This follows from lemma 3.2.55. $\square$

3.4.43 Lemma. Suppose given $X \in \operatorname{Ob}(\mathrm{FO}^{\mathrm{colim}}(\mathscr{F}))$. We have $X^{[1]}, X^{[-1]} \in \operatorname{Ob}(\mathrm{FO}^{\mathrm{colim}}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. This is dual to the previous lemma 3.4.42. $\square$

3.4.44 Lemma. Suppose given $X \in \operatorname{Ob}(\mathrm{FO}^{\nabla}(\mathscr{F}))$. We have $X^{[1]}, X^{[-1]} \in \operatorname{Ob}(\mathrm{FO}^{\nabla}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. This follows from lemmata 3.4.42 and 3.4.43. $\square$

3.4.45 Lemma. Suppose given a pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathrm{FO}(\mathscr{F})$ such that $X, Z \in \operatorname{Ob}(\mathrm{FO}^{\mathrm{lim}}(\mathscr{F}))$.
If there exists a limit for $Y\mathrm{P}_{\mathrm{CF},\mathscr{F}}$, then we have $Y \in \operatorname{Ob}(\mathrm{FO}^{\mathrm{lim}}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. This follows from lemma 3.2.41. $\square$

3.4.46 Lemma. Suppose given a pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathrm{FO}(\mathscr{F})$ such that $X, Z \in \operatorname{Ob}(\mathrm{FO}^{\mathrm{colim}}(\mathscr{F}))$.
If there exists a colimit for $Y\mathrm{P}_{\mathrm{F},\mathscr{F}}$, then we have $Y \in \operatorname{Ob}(\mathrm{FO}^{\mathrm{colim}}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. This is dual to the previous lemma 3.4.45. $\square$

3.4.47 Lemma. Suppose given a pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathrm{FO}(\mathscr{F})$ such that $X, Z \in \operatorname{Ob}(\mathrm{FO}^{\nabla}(\mathscr{F}))$. If there exists a limit for $Y\mathrm{P}_{\mathrm{CF},\mathscr{F}}$ and a colimit for $Y\mathrm{P}_{\mathrm{F},\mathscr{F}}$, then we have $Y \in \operatorname{Ob}(\mathrm{FO}^{\nabla}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. This follows from lemmata 3.4.45 and 3.4.46. $\square$

3.4.48 Corollary. Suppose that $\mathscr{F}$ has epilimits and monocolimits. $\mathrm{FO}^{\nabla}(\mathscr{F})$ is an extension-closed subcategory of $\mathrm{FO}(\mathscr{F})$, cf. definition 1.3.23, remark 1.4.10 and lemma 3.4.47. In particular, it is a full additive subcategory of $\mathrm{FO}(\mathscr{F})$. So $\underline{\underline{\mathrm{FO}}}^{\nabla}(\mathscr{F})$ is a full additive subcategory of $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$, cf. remark 1.2.14. $\diamondsuit$

3.4.49 Lemma. Suppose that $\mathscr{F}$ has epilimits and monocolimits. Suppose given a pseudo-triangle $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}_{[-1]}$ in $\mathrm{FO}(\mathscr{F})$ such that $X, Y \in \mathrm{Ob}(\mathrm{FO}^{\nabla}(\mathscr{F}))$. Then we have $Z, X^{[1]}_{[-1]} \in \mathrm{Ob}(\mathrm{FO}^{\nabla}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. This follows from lemmata 3.4.41, 3.4.44 and 3.4.47. $\square$

3.4.50 Lemma. Suppose that $\mathscr{F}$ has countable products of bijectives.
Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\lim}(\mathscr{F}))$. Then we have $X\mathbf{B} \in \mathrm{Ob}(\mathrm{FO}^{\lim}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. Consider the pure short exact sequence $X \xrightarrow{X\iota} X\mathbf{B} \xrightarrow{X\pi} X^{[1]}$ in $\mathrm{FO}(\mathscr{F})$. We have $X^{[1]} \in \mathrm{Ob}(\mathrm{FO}^{\lim}(\mathscr{F}))$ by lemma 3.4.42. Moreover, there exists a limit for $X\mathbf{B}\mathrm{P}_{\mathrm{CF},\mathscr{F}}$ by lemma 3.2.33. Thus $X\mathbf{B} \in \mathrm{Ob}(\mathrm{FO}^{\lim}(\mathscr{F}))$ by lemma 3.4.45. $\square$

3.4.51 Lemma. Suppose that $\mathscr{F}$ has countable coproducts of bijectives. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{colim}}(\mathscr{F}))$. Then we have $X\mathbf{B} \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{colim}}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. This is dual to the previous lemma 3.4.50. $\square$

3.4.52 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{colim}}(\mathscr{F}))$ and $Y \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$. Suppose given a morphism $X\mathrm{P}_{\mathrm{F},\mathscr{F}} \xrightarrow{f} Y\mathrm{P}_{\mathrm{F},\mathscr{F}}$ in $\mathrm{F}(\mathscr{F})$. Then there exists a unique morphism $X \xrightarrow{g} Y$ in $\mathrm{FO}(\mathscr{F})$ such that $g\mathrm{P}_{\mathrm{F},\mathscr{F}} = f$. Moreover, we have $g_\omega = f\upharpoonleft^{(Y_\omega,(y^\omega_{|k})_{k\in\mathbf{Z}})}_{(X_\omega,(x^\omega_{|k})_{k\in\mathbf{Z}})}$, cf. definition 3.2.23. $\diamondsuit$

Proof. Consider the morphism $f\upharpoonleft^{(Y_\omega,(y^\omega_{|k})_{k\in\mathbf{Z}})}_{(X_\omega,(x^\omega_{|k})_{k\in\mathbf{Z}})} \colon X_\omega \to Y_\omega$. For $k \in \mathbf{Z}$, we have

$$x^\omega_{|k} \cdot f\upharpoonleft^{(Y_\omega,(y^\omega_{|k})_{k\in\mathbf{Z}})}_{(X_\omega,(x^\omega_{|k})_{k\in\mathbf{Z}})} \cdot y^\omega_{k+1|} = f_k \cdot y^\omega_{|k} \cdot y^\omega_{k+1|} = 0.$$

Now lemma 3.4.3.(b) yields the morphism $X \xrightarrow{g} Y$ such that $g_\omega = f\upharpoonleft^{(Y_\omega,(y^\omega_{|k})_{k\in\mathbf{Z}})}_{(X_\omega,(x^\omega_{|k})_{k\in\mathbf{Z}})}$. Moreover, we have $g\mathrm{P}_{\mathrm{F},\mathscr{F}} = f$ since for $k \in \mathbf{Z}$, we have $f_k \cdot y^\omega_{|k} = x^\omega_{|k} \cdot f\upharpoonleft^{(Y_\omega,(y^\omega_{|k})_{k\in\mathbf{Z}})}_{(X_\omega,(x^\omega_{|k})_{k\in\mathbf{Z}})}$ and, consequently, $f_k = g_{|k}$.
Suppose given $X \xrightarrow{e} Y$ in $\mathrm{FO}(\mathscr{F})$ such that $e\mathrm{P}_{\mathrm{F},\mathscr{F}} = f$. For $k \in \mathbf{Z}$, we have $x^\omega_{|k} \cdot e_\omega = e_{|k} \cdot y^\omega_{|k} = f_k \cdot y^\omega_{|k} = x^\omega_{|k} \cdot f\upharpoonleft^{(Y_\omega,(y^\omega_{|k})_{k\in\mathbf{Z}})}_{(X_\omega,(x^\omega_{|k})_{k\in\mathbf{Z}})}$. Thus $e_\omega = f\upharpoonleft^{(Y_\omega,(y^\omega_{|k})_{k\in\mathbf{Z}})}_{(X_\omega,(x^\omega_{|k})_{k\in\mathbf{Z}})}$. We conclude that $e = g$ by lemma 3.4.3.(c). $\square$

3.4.53 Definition. We define the full subcategories $\mathrm{FO}^{\mathrm{proj}}(\mathscr{F})$ and $\mathrm{FO}^{\mathrm{inj}}(\mathscr{F})$ of $\mathrm{FO}(\mathscr{F})$ by setting

$$\mathrm{Ob}(\mathrm{FO}^{\mathrm{proj}}(\mathscr{F})) = \{X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F})) \colon (X_\omega\,,\,(x^\omega_{k|})_{k\in\mathbf{Z}}) \text{ is a projective family for } X\mathrm{P}_{\mathrm{CF},\mathscr{F}}\}$$

and

$$\mathrm{Ob}(\mathrm{FO}^{\mathrm{inj}}(\mathscr{F})) = \{X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F})) \colon (X_\omega\,,\,(x^\omega_{|k})_{k\in\mathbf{Z}}) \text{ is an injective family for } X\mathrm{P}_{\mathrm{F},\mathscr{F}}\}.$$

Cf. definitions 3.2.42 and 3.2.43.

We define the full subcategories $\underline{\underline{\mathrm{FO}}}^{\mathrm{proj}}(\mathscr{F})$ and $\underline{\underline{\mathrm{FO}}}^{\mathrm{inj}}(\mathscr{F})$ of $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$ by setting $\mathrm{Ob}(\underline{\underline{\mathrm{FO}}}^{\mathrm{proj}}(\mathscr{F})) = \mathrm{Ob}(\mathrm{FO}^{\mathrm{proj}}(\mathscr{F}))$ and $\mathrm{Ob}(\underline{\underline{\mathrm{FO}}}^{\mathrm{inj}}(\mathscr{F})) = \mathrm{Ob}(\mathrm{FO}^{\mathrm{inj}}(\mathscr{F}))$. $\diamondsuit$

3.4.54 Remark. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\lim}(\mathscr{F}))$. Then $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{proj}}(\mathscr{F}))$, cf. remark 3.2.44. $\diamondsuit$

3.4.55 Remark. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{colim}}(\mathscr{F}))$. Then $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{inj}}(\mathscr{F}))$, cf. remark 3.2.45. $\diamondsuit$

3.4.56 Remark. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}(\mathscr{F}))$. Then we have $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{inj}}(\mathscr{F}))$ and $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{proj}}(\mathscr{F}))$, cf. remarks 3.4.38, 3.4.54 and 3.4.55. $\diamondsuit$

3.4.57 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{proj}}(\mathscr{F}))$.
Then we have $X^{[1]}, X^{[-1]} \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{proj}}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. This follows from lemma 3.2.56. $\square$

3.4.58 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{inj}}(\mathscr{F}))$.
Then we have $X^{[1]}, X^{[-1]} \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{inj}}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. This is dual to the previous lemma 3.4.54. $\square$

3.4.59 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{proj}}(\mathscr{F}))$.
Then we have $X_{[1]}, X_{[-1]} \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{proj}}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. This follows from lemma 3.2.48. $\square$

3.4.60 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{inj}}(\mathscr{F}))$.
Then we have $X_{[1]}, X_{[-1]} \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{inj}}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. This is dual to the previous lemma 3.4.59. $\square$

3.4.61 Lemma. Suppose given a pure short exact sequence $X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$ in $\mathrm{FO}(\mathscr{F})$ such that $X, Z \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{proj}}(\mathscr{F}))$. Then we have $Y \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{proj}}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. This follows from lemma 3.2.47. $\square$

3.4.62 Corollary. $\mathrm{FO}^{\mathrm{proj}}(\mathscr{F})$ is an extension-closed subcategory of $\mathrm{FO}(\mathscr{F})$, cf. definition 1.3.23, remark 1.4.10 and lemma 3.4.61. In particular, it is a strictly full additive subcategory of $\mathrm{FO}(\mathscr{F})$. So $\underline{\underline{\mathrm{FO}}}^{\mathrm{proj}}(\mathscr{F})$ is a full additive subcategory of $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$, cf. remark 1.2.14. $\diamondsuit$

3.4.63 Lemma. Suppose given a pure short exact sequence $X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$ in $\mathrm{FO}(\mathscr{F})$ such that $X, Z \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{inj}}(\mathscr{F}))$. Then we have $Y \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{inj}}(\mathscr{F}))$ as well. $\diamondsuit$

Proof. This is dual to lemma 3.4.61. $\square$

3.4.64 Corollary. $\mathrm{FO}^{\mathrm{inj}}(\mathscr{F})$ is an extension-closed subcategory of $\mathrm{FO}(\mathscr{F})$, cf. definition 1.3.23, remark 1.4.10 and lemma 3.4.63. In particular, it is a strictly full additive subcategory of $\mathrm{FO}(\mathscr{F})$. So $\underline{\underline{\mathrm{FO}}}^{\mathrm{inj}}(\mathscr{F})$ is a full additive subcategory of $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$, cf. remark 1.2.14. $\diamondsuit$

3.4.65 Lemma/Definition. We define the functor $\mathrm{E}_{\mathscr{F}} \colon \mathscr{F} \to \mathrm{FO}(\mathscr{F})$ as follows.

For $X \in \mathrm{Ob}(\mathscr{F})$, we define $X\mathrm{E}_{\mathscr{F}} \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ by setting

- $(X\mathrm{E}_{\mathscr{F}})_\omega = X$,
- $(X\mathrm{E}_{\mathscr{F}})_{|k} = (X\mathrm{E}_{\mathscr{F}})_{\text{-}k|} = X$ for $k \in \mathbf{Z}_{\geq 0}$,
- $(X\mathrm{E}_{\mathscr{F}})_{|k} = (X\mathrm{E}_{\mathscr{F}})_{\text{-}k|} = 0_{\mathscr{F}}$ for $k \in \mathbf{Z}_{<0}$,
- $(X\mathrm{E}_{\mathscr{F}})_{|k \to \omega} = (X\mathrm{E}_{\mathscr{F}})_{|k \to |k+1} = (X\mathrm{E}_{\mathscr{F}})_{\omega \to \text{-}k|} = (X\mathrm{E}_{\mathscr{F}})_{\text{-}k\text{-}1| \to \text{-}k|} = 1$ for $k \in \mathbf{Z}_{\geq 0}$.

For $f \in \mathrm{Mor}(\mathscr{F})$, we define $f\mathrm{E}_{\mathscr{F}} \in \mathrm{Mor}(\mathrm{FO}(\mathscr{F}))$ by setting

- $(f\mathrm{E}_{\mathscr{F}})_\omega = f$,
- $(f\mathrm{E}_{\mathscr{F}})_{|k} = (f\mathrm{E}_{\mathscr{F}})_{\text{-}k|} = f$ for $k \in \mathbf{Z}_{\geq 0}$.

This in fact defines an additive functor. We call $\mathrm{E}_{\mathscr{F}}$ the *embedding functor* of $\mathscr{F}$. Note that $X\mathrm{E}_{\mathscr{F}} \in \mathrm{Ob}(\mathrm{FO}^{[0,0]}(\mathscr{F}))$ for $X \in \mathrm{Ob}(\mathscr{F})$ by construction. $\diamondsuit$

Proof. We abbreviate $\mathrm{E} = \mathrm{E}_{\mathscr{F}}$. We will use lemma 3.4.3.(c). Suppose given $X \overset{f}{\underset{h}{\rightrightarrows}} Y \xrightarrow{g} Z$ in $\mathscr{F}$. We have $1_X\mathrm{E} = 1_{X\mathrm{E}}$ since $(1_X\mathrm{E})_\omega = 1_X = (1_{X\mathrm{E}})_\omega$.
We have $(f \cdot g)\mathrm{E} = f\mathrm{E} \cdot g\mathrm{E}$ since $((f \cdot g)\mathrm{E})_\omega = f \cdot g = (f\mathrm{E} \cdot g\mathrm{E})_\omega$.
We have $(f + h)\mathrm{E} = f\mathrm{E} + h\mathrm{E}$ since $((f + h)\mathrm{E})_\omega = f + h = (f\mathrm{E} + h\mathrm{E})_\omega$. $\square$

3.4.66 Lemma. Suppose given $X \xrightarrow{f} Y$ in $\mathscr{F}$ such that $\underline{f} = 0$ in $\underline{\mathscr{F}}$. Then we have $\underline{\underline{f\mathrm{E}_{\mathscr{F}}}} = 0$ in $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$. $\diamondsuit$

Proof. Choose $X\mathrm{B} \xrightarrow{g} Y$ in $\mathscr{F}$ such that $X\iota \cdot g = f$. Note that $((X\mathrm{E}_{\mathscr{F}})_{[-1]}\mathbf{B})_\omega = X\mathrm{B}$ and that $(Y\mathrm{E}_{\mathscr{F}})_\omega = Y$. For $k \in \mathbf{Z}$, we have
$((X\mathrm{E}_{\mathscr{F}})_{[-1]}\mathbf{B})_{|k \to \omega} \cdot g \cdot (Y\mathrm{E}_{\mathscr{F}})_{\omega \to k+1|} = (X\mathrm{E}_{\mathscr{F}})_{|k-1 \to \omega}\mathrm{B} \cdot g \cdot (Y\mathrm{E}_{\mathscr{F}})_{\omega \to k+1|} = 0$. So lemma 3.4.3.(b) yields a morphism $(X\mathrm{E}_{\mathscr{F}})_{[-1]}\mathbf{B} \xrightarrow{h} Y\mathrm{E}_{\mathscr{F}}$ in $\mathrm{FO}(\mathscr{F})$ with $h_\omega = g$. We have
$((X\mathrm{E}_{\mathscr{F}})_{[-1]}\rho \cdot f\mathrm{E}_{\mathscr{F}})_\omega = f = X\iota \cdot g = ((X\mathrm{E}_{\mathscr{F}})_{[-1]}\boldsymbol{\iota} \cdot h)_\omega$. So $(X\mathrm{E}_{\mathscr{F}})_{[-1]}\rho \cdot f\mathrm{E}_{\mathscr{F}} = (X\mathrm{E}_{\mathscr{F}})_{[-1]}\boldsymbol{\iota} \cdot h$ by lemma 3.4.3.(c). We conclude that $\underline{\underline{f\mathrm{E}_{\mathscr{F}}}} = 0$ in $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$. $\square$

3.4.67 Definition. Let $\underline{\mathrm{E}}_{\mathscr{F}} \colon \underline{\mathscr{F}} \to \underline{\underline{\mathrm{FO}}}(\mathscr{F})$ denote the unique functor such that $\mathfrak{P}_{\mathscr{F}} \cdot \underline{\mathrm{E}}_{\mathscr{F}} = \mathrm{E}_{\mathscr{F}} \cdot \mathfrak{Q}_{\mathrm{FO},\mathscr{F}}$, cf. lemma 3.4.66.

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\mathrm{E}_{\mathscr{F}}"] \arrow[d, "\mathfrak{P}_{\mathscr{F}}"'] & \mathrm{FO}(\mathscr{F}) \arrow[d, "\mathfrak{Q}_{\mathrm{FO},\mathscr{F}}"] \\
\underline{\mathscr{F}} \arrow[r, "\underline{\mathrm{E}}_{\mathscr{F}}"] & \underline{\underline{\mathrm{FO}}}(\mathscr{F})
\end{tikzcd}$$

$\diamondsuit$

3.4.68 Lemma. We have $\underline{\mathrm{E}}_{\mathscr{F}} \cdot \underline{\mathrm{P}}_{\omega,\mathscr{F}} = 1_{\underline{\mathscr{F}}}$. $\diamondsuit$

Proof. Suppose given $f \in \mathrm{Mor}(\mathscr{F})$. We have $\underline{f}\,\underline{\mathrm{E}}_{\mathscr{F}}\underline{\mathrm{P}}_{\omega,\mathscr{F}} = \underline{\underline{f\mathrm{E}_{\mathscr{F}}}}\,\underline{\mathrm{P}}_{\omega,\mathscr{F}} = \underline{(f\mathrm{E}_{\mathscr{F}})_\omega} = \underline{f}$. $\square$

Chapter 4

Resolution and realisation functors

We refer to the introduction for generalities on resolution and realisation functors. The main ingredients for our construction of resolution and realisation are the projection functor $\mathrm{P}_{\omega,\mathscr{F}}\colon \mathrm{FO}(\mathscr{F}) \to \mathscr{F}$ from the previous chapter 3, the delta functor $\Delta_{\mathscr{S},\mathscr{F}}\colon \nabla_{\mathscr{S}}(\mathscr{F}) \to \mathrm{C}(\mathscr{S})$ and the filtered cokernel functor $\Xi_{\mathscr{F}}\colon \mathrm{FO}(\mathscr{F}) \to \nabla(\mathscr{F})$.

In the first section 4.1, we define the delta functor and show that it induces functors $\underline{\Delta}_{\mathscr{S},\mathscr{F}}\colon \underline{\nabla}_{\mathscr{S}}(\mathscr{F}) \to \mathrm{C}(\mathscr{S})$ and $\underline{\underline{\Delta}}_{\mathscr{S},\mathscr{F}}\colon \underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F}) \to \mathrm{K}(\mathscr{S})$ on factor categories. For a heart of a t-structure $\mathscr{H} \subseteq \underline{\mathscr{F}}$, proposition 4.1.11 yields that $\underline{\Delta}_{\mathscr{H},\mathscr{F}}$ and $\underline{\underline{\Delta}}_{\mathscr{H},\mathscr{F}}$ are equivalences. We proceed to define quasi-inverses $\mathrm{R}_{\mathscr{H},\mathscr{F}}$ and $\underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}}$ of these equivalences. Proposition 4.1.24 contains the first of three adjunctions that will be composed to obtain our adjunction for adjacent t- and w-structures.

The filtered cokernel functor $\Xi_{\mathscr{F}}$ is defined in section 4.2 by choosing cokernels of pure monomorphisms. To study its properties, we compare different choices of such cokernels. We obtain several functors induced by the filtered cokernel functor on factor categories and subcategories. Under suitable assumptions, we obtain equivalences $\Xi^{\nabla}_{\mathscr{F}}\colon \mathrm{FO}^{\nabla}(\mathscr{F}) \to \nabla(\mathscr{F})$ and $\underline{\underline{\Xi}}^{\nabla}_{\mathscr{F}}\colon \underline{\underline{\mathrm{FO}}}^{\nabla}(\mathscr{F}) \to \underline{\underline{\nabla}}(\mathscr{F})$. We define the limit functors $\mathrm{Lim}_{\mathscr{F}}$ and $\underline{\underline{\mathrm{Lim}}}_{\mathscr{F}}$ as quasi-inverses of these equivalences. Proposition 4.2.60 contains the second of the three adjunctions mentioned above.

We construct resolution functors in section 4.3. To this end, we show that there is an equivalence $\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}}\colon \underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F}) \to \mathscr{D}$ (proposition 4.3.32) and choose the weight equivalence $\mathrm{W}_{\mathscr{W},\mathscr{F}}$ to be a quasi-inverse of it. This functor will be the first factor in the definition of the resolution functor as a composite. After studying the properties of resolution functors, we define a modified version of the weight equivalence under special assumptions and finally obtain the third of the adjunctions mentioned above in proposition 4.3.52.

In section 4.4, we define a preliminary version of the realisation functor on the level of the homotopy category: $\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}}\colon \mathrm{K}(\mathscr{H}) \to \underline{\mathscr{F}}$. We study its properties and finally define the realisation functor to be the one induced on the derived category.

Section 4.5 contains our main theorems 4.5.1 and 4.5.3 for adjacent t- and w-structures which show how our resolution and realisation functors yield adjunctions.

We conclude this chapter by applying the results to derived categories in section 4.6.

4.1 From ∇-diagrams to complexes: the functor $\Delta_{\mathscr{S},\mathscr{F}} \colon \nabla_{\mathscr{S}}(\mathscr{F}) \to \mathrm{C}(\mathscr{S})$

Suppose given a strict Frobenius category $\mathscr{F} = (\mathscr{F}, \mathrm{B}, \Sigma, \iota, \pi, \alpha)$.

4.1.1 Lemma/Definition. Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$. We define the functor $\Delta_{\mathscr{S},\mathscr{F}} \colon \nabla_{\mathscr{S}}(\mathscr{F}) \to \mathrm{C}(\mathscr{S})$ as follows. For $X \in \mathrm{Ob}(\nabla_{\mathscr{S}}(\mathscr{F}))$ and $k \in \mathbf{Z}$, let

$$(X\Delta_{\mathscr{S},\mathscr{F}})_k = X^{[-k]}_{k/k-1} \quad \text{and} \quad (X\Delta_{\mathscr{S},\mathscr{F}})_{k \to k-1} = \underline{\delta^{[-k]}_{X,k-2,k-1,k}}\ ,$$

cf. definition 3.3.38. For $f \in \mathrm{Mor}(\nabla_{\mathscr{S}}(\mathscr{F}))$ and $k \in \mathbf{Z}$, let $(f\Delta_{\mathscr{S},\mathscr{F}})_k = \underline{f^{[-k]}_{k/k-1}}$.
This in fact defines an additive functor, cf. lemma 3.3.19. We call $\Delta_{\mathscr{S},\mathscr{F}}$ the *delta functor* of $\mathscr{S}$ with respect to $\mathscr{F}$. Note that it is obtained by choosing connecting morphisms of triangles ('deltas'). $\diamondsuit$

Proof. We abbreviate $\Delta = \Delta_{\mathscr{S},\mathscr{F}}$.
Suppose given $X \in \mathrm{Ob}(\nabla_{\mathscr{S}}(\mathscr{F}))$ and $k \in \mathbf{Z}$. We have $(X\Delta)_k = X^{[-k]}_{k/k-1} \in \mathrm{Ob}(\mathscr{S})$, cf. definition 3.3.17. By lemma 3.3.43, we have

$$\begin{aligned}
(X\Delta)_{k+1 \to k} \cdot (X\Delta)_{k \to k-1} &= \underline{\delta^{[-k-1]}_{X,k-1,k,k+1} \cdot \delta^{[-k]}_{X,k-2,k-1,k}} \\
&= \underline{\delta^{[-k-1]}_{k-2,k,k+1} \cdot x^{[-k]}_{k/k-2,k-1} \cdot \delta^{[-k]}_{X,k-2,k-1,k}} \\
&= \underline{\delta^{[-k-1]}_{k-2,k,k+1}} \cdot \big(\underline{x_{k/k-2,k-1} \cdot \delta_{X,k-2,k-1,k}}\big)^{[-k]} \\
&= 0.
\end{aligned}$$

Cf. definition 3.3.38. Thus $X\Delta \in \mathrm{Ob}(\mathrm{C}(\mathscr{S}))$.

Suppose given $X \xrightarrow{f} Y$ in $\nabla_{\mathscr{S}}(\mathscr{F})$ and $k \in \mathbf{Z}$.

Consider the pure short exact sequences $X_{k-1/k-2} \xrightarrow{x_{k-1,k/k-2}} X_{k/k-2} \xrightarrow{x_{k/k-2,k-1}} X_{k/k-1}$ and $Y_{k-1/k-2} \xrightarrow{y_{k-1,k/k-2}} Y_{k/k-2} \xrightarrow{y_{k/k-2,k-1}} Y_{k/k-1}$ and the following commutative diagram in $\mathscr{F}$.

$$\begin{tikzcd}
X_{k-1/k-2} \arrow[r, "x_{k-1,k/k-2}"] \arrow[d, "f_{k-1/k-2}"'] & X_{k/k-2} \arrow[r, "x_{k/k-2,k-1}"] \arrow[d, "f_{k/k-2}"] & X_{k/k-1} \arrow[d, "f_{k/k-1}"] \\
Y_{k-1/k-2} \arrow[r, "y_{k-1,k/k-2}"] & Y_{k/k-2} \arrow[r, "y_{k/k-2,k-1}"] & Y_{k/k-1}
\end{tikzcd}$$

By lemma 2.2.4, we have

$$\begin{aligned}(X\Delta)_{k\to k-1}\cdot(f\Delta)_{k-1} &= \underline{\delta^{[-k]}_{X,k-2,k-1,k}\cdot f^{[-k+1]}_{k-1/k-2}} = \underline{(\delta_{X,k-2,k-1,k}\cdot f^{[1]}_{k-1/k-2})}^{[-k]}\\ &= (\underline{f_{k/k-1}\cdot\delta_{Y,k-2,k-1,k}})^{[-k]} = \underline{f^{[-k]}_{k/k-1}\cdot\delta^{[-k]}_{Y,k-2,k-1,k}}\\ &= (f\Delta)_k\cdot(Y\Delta)_{k\to k-1}\,.\end{aligned}$$

$$\begin{tikzcd} X_{k-1/k-2} \arrow[r, "x_{k-1,k/k-2}"] \arrow[d, "f_{k-1/k-2}"'] & X_{k/k-2} \arrow[r, "x_{k/k-2,k-1}"] \arrow[d, "f_{k/k-2}"] & X_{k/k-1} \arrow[r, "\delta_{X,k-2,k-1,k}"] \arrow[d, "f_{k/k-1}"] & X^{[1]}_{k-1/k-2} \arrow[d, "f^{[1]}_{k-1/k-2}"] \\ Y_{k-1/k-2} \arrow[r, "y_{k-1,k/k-2}"] & Y_{k/k-2} \arrow[r, "y_{k/k-2,k-1}"] & Y_{k/k-1} \arrow[r, "\delta_{Y,k-2,k-1,k}"] & X^{[1]}_{k/k-1} \end{tikzcd}$$

So $f\Delta \in {}_{\mathrm{C}(\mathscr{S})}(X\Delta, Y\Delta)$.

Suppose given $X \overset{f}{\underset{h}{\rightrightarrows}} Y \xrightarrow{g} Z$ in $\nabla_{\mathscr{S}}(\mathscr{F})$.

For $k\in\mathbf{Z}$, we have $(1_X\Delta)_k = \underline{(1_X)^{[-k]}_{k/k-1}} = \underline{1^{[-k]}_{X_{k/k-1}}} = \underline{1_{X^{[-k]}_{k/k-1}}} = (1_{X\Delta})_k$. So $1_X\Delta = 1_{X\Delta}$.

For $k\in\mathbf{Z}$, we have $((f\cdot g)\Delta)_k = \underline{(f\cdot g)^{[-k]}_{k/k-1}} = \underline{f^{[-k]}_{k/k-1}\cdot g^{[-k]}_{k/k-1}} = (f\Delta)_k\cdot(g\Delta)_k$.
So $(f\cdot g)\Delta = f\Delta\cdot g\Delta$.

For $k\in\mathbf{Z}$, we have $((f+h)\Delta)_k = \underline{(f+h)^{[-k]}_{k/k-1}} = \underline{f^{[-k]}_{k/k-1}+h^{[-k]}_{k/k-1}} = (f\Delta)_k+(h\Delta)_k$. So $(f+h)\Delta = f\Delta+h\Delta$.

We conclude that $\Delta = \Delta_{\mathscr{S},\mathscr{F}}\colon \nabla_{\mathscr{S}}(\mathscr{F}) \to \mathrm{C}(\mathscr{S})$ is an additive functor. □

4.1.2 Remark. Suppose given strictly full additive subcategories $\mathscr{R},\mathscr{S}\subseteq\underline{\mathscr{F}}$ with $\mathscr{R}\subseteq\mathscr{S}$. We have $\Delta_{\mathscr{R},\mathscr{F}}\cdot\mathrm{C}(\mathrm{Inc}^{\mathscr{S}}_{\mathscr{R}}) = \mathrm{Inc}^{\nabla_{\mathscr{S}}(\mathscr{F})}_{\nabla_{\mathscr{R}}(\mathscr{F})}\cdot\Delta_{\mathscr{S},\mathscr{F}}$.

$$\begin{tikzcd} \nabla_{\mathscr{S}}(\mathscr{F}) \arrow[r, "\Delta_{\mathscr{S},\mathscr{F}}"] & \mathrm{C}(\mathscr{S}) \\ \nabla_{\mathscr{R}}(\mathscr{F}) \arrow[r, "\Delta_{\mathscr{R},\mathscr{F}}"] \arrow[u, "\mathrm{Inc}^{\nabla_{\mathscr{S}}(\mathscr{F})}_{\nabla_{\mathscr{R}}(\mathscr{F})}"] & \mathrm{C}(\mathscr{R}) \arrow[u, "\mathrm{C}(\mathrm{Inc}^{\mathscr{S}}_{\mathscr{R}})"'] \end{tikzcd}$$

◇

4.1.3 Proposition. Suppose given a strictly full additive subcategory $\mathscr{S}\subseteq\underline{\mathscr{F}}$.

(a) We have $\left(\Sigma_{\nabla,\mathscr{F}}\cdot\mathrm{T}^{-1}_{\nabla,\mathscr{F}}\right)\big|^{\nabla_{\mathscr{S}}(\mathscr{F})}_{\nabla_{\mathscr{S}}(\mathscr{F})}\cdot\Delta_{\mathscr{S},\mathscr{F}} = \Delta_{\mathscr{S},\mathscr{F}}\cdot\Sigma_{\mathrm{C},\mathscr{S}}$.

(b) Suppose given a pseudo-triangle $X\xrightarrow{f}Y\xrightarrow{i}Z\xrightarrow{p}X^{[1]}_{[-1]}$ in $\nabla(\mathscr{F})$ with $X,Y\in\mathrm{Ob}(\nabla_{\mathscr{S}}(\mathscr{F}))$. Then

$$X\Delta_{\mathscr{S},\mathscr{F}} \xrightarrow{f\Delta_{\mathscr{S},\mathscr{F}}} Y\Delta_{\mathscr{S},\mathscr{F}} \xrightarrow{i\Delta_{\mathscr{S},\mathscr{F}}} Z\Delta_{\mathscr{S},\mathscr{F}} \xrightarrow{p\Delta_{\mathscr{S},\mathscr{F}}} X^{[1]}_{[-1]}\Delta_{\mathscr{S},\mathscr{F}}$$

is a pseudo-triangle in $\mathrm{C}(\mathscr{S})$.

◇

Proof. We abbreviate $\Delta = \Delta_{\mathscr{S},\mathscr{F}}$.

Ad (a). Suppose given $f \in \mathrm{Mor}(\nabla_{\mathscr{S}}(\mathscr{F}))$. We want to show that $f^{[1]}_{[-1]}\Delta = (f\Delta)^{[1]}$.

For $k \in \mathbf{Z}$, we have $(f^{[1]}_{[-1]}\Delta)_k = \underline{(f^{[1]}_{k-1/k-2})^{[-k]}} = \underline{f^{[-k+1]}_{k-1/k-2}} = \underline{(f\Delta)_{k-1}} = ((f\Delta)^{[1]})_k$.

We conclude that $\big(\Sigma_{\nabla,\mathscr{F}}\mathrm{T}^{-1}_{\nabla,\mathscr{F}}\big)\big|^{\nabla_{\mathscr{S}}(\mathscr{F})}_{\nabla_{\mathscr{S}}(\mathscr{F})} \cdot \Delta_{\mathscr{S},\mathscr{F}} = \Delta_{\mathscr{S},\mathscr{F}} \cdot \Sigma_{\mathrm{C},\mathscr{S}}$.

Ad (b). We abbreviate $X' = X^{[1]}_{[-1]}$.

Choose a morphism $X_{[-1]}\mathbf{B} \xrightarrow{g} Z$ in $\nabla(\mathscr{F})$ such that the following diagram is commutative.

$$\begin{tikzcd}
X_{[-1]} \arrow[r, "X_{[-1]}\boldsymbol{\iota}"] \arrow[d, "X_{[-1]}\rho\cdot f"'] & X_{[-1]}\mathbf{B} \arrow[r, "X_{[-1]}\boldsymbol{\pi}"] \arrow[d, "g"] & X' \arrow[d, "1"] \\
Y \arrow[r, "i"] & Z \arrow[r, "p"] & X'
\end{tikzcd}$$

For $k \in \mathbf{Z}$, we may choose $Z_{k/k-1} \xrightarrow{s_k} Y_{k/k-1}$ and $X^{[1]}_{k-1/k-2} \xrightarrow{t_k} Y_{k/k-1}$ in $\mathscr{F}$ such that $X_{k-1/k-2}\pi \cdot t_k = g_{k/k-1}$ and such that $Y_{k/k-1} \underset{s_k}{\overset{i_{k/k-1}}{\rightleftarrows}} Z_{k/k-1} \underset{p_{k/k-1}}{\overset{t_k}{\leftrightarrows}} X^{[1]}_{k-1/k-2}$ is a direct sum in $\mathscr{F}$ by lemma 3.3.26.

So $Y^{[-k]}_{k/k-1} \underset{\underline{s^{[-k]}_k}}{\overset{\underline{i^{[-k]}_{k/k-1}}}{\rightleftarrows}} Z^{[-k]}_{k/k-1} \underset{\underline{p^{[-k]}_{k/k-1}}}{\overset{\underline{t^{[-k]}_k}}{\leftrightarrows}} X^{[-k+1]}_{k-1/k-2}$ is a direct sum in $\underline{\mathscr{S}}$ for $k \in \mathbf{Z}$, cf. corollary 3.3.27. Note that, for $k \in \mathbf{Z}$, we have $(X'\Delta)_k = (X\Delta)_{k-1}$ and
$(X'\Delta)_{k \to k-1} = \underline{\delta^{[-k]}_{X',k-2,k-1,k}} = \underline{-\delta^{[-k+1]}_{X,k-3,k-2,k-1}} = -(X\Delta)_{k-1 \to k-2}$, cf. corollary 3.3.41.

Also note that, for $k \in \mathbf{Z}$, we have $(i\Delta)_k = \underline{i^{[-k]}_{k/k-1}}$ and $(p\Delta)_k = \underline{p^{[-k]}_{k/k-1}}$.

We want to apply lemma 1.9.17.

Suppose given $k \in \mathbf{Z}$.

Let $d_k = s_k \cdot \delta_{Y,k-2,k-1,k} \cdot i^{[1]}_{k-1/k-2} + p_{k/k-1} \cdot f^{[1]}_{k-1/k-2} \cdot i^{[1]}_{k-1/k-2} + p_{k/k-1} \cdot \delta_{X',k-2,k-1,k} \cdot t^{[1]}_{k-1}$.

So $d^{[-k]}_k = s^{[-k]}_k \cdot \delta^{[-k]}_{Y,k-2,k-1,k} \cdot i^{[-k+1]}_{k-1/k-2} + p^{[-k]}_{k/k-1} \cdot f^{[-k+1]}_{k-1/k-2} \cdot i^{[-k+1]}_{k-1/k-2} + p^{[-k]}_{k/k-1} \cdot \delta^{[-k]}_{X',k-2,k-1,k} \cdot t^{[-k+1]}_{k-1}$.

It remains to show that $\underline{\delta^{[-k]}_{Z,k-2,k-1,k}} = \underline{d^{[-k]}_k}$, i.e. that $\underline{\delta_{Z,k-2,k-1,k}} = \underline{d_k}$.

We have the following commutative diagrams in $\mathscr{F}$, cf. definition 3.3.38.

$$\begin{tikzcd}
Y_{k-1/k-2} \arrow[r, "y_{k-1,k/k-2}"] \arrow[d, "1"'] & Y_{k/k-2} \arrow[r, "y_{k/k-2,k-1}"] \arrow[d, "\gamma_{Y,k-2,k-1,k}"] & Y_{k/k-1} \arrow[d, "\delta_{Y,k-2,k-1,k}"] \\
Y_{k-1/k-2} \arrow[r, "Y_{k-1/k-2}\iota"] & Y_{k-1/k-2}\mathrm{B} \arrow[r, "Y_{k-1/k-2}\pi"] & Y^{[1]}_{k-1/k-2}
\end{tikzcd}$$

$$\begin{tikzcd}
X^{[1]}_{k-2/k-3} \arrow[r, "x^{[1]}_{k-2,k-1/k-3}"] \arrow[d, "1"'] & X^{[1]}_{k-1/k-3} \arrow[r, "x^{[1]}_{k-1/k-3,k-2}"] \arrow[d, "\gamma_{X',k-2,k-1,k}"] & X^{[1]}_{k-1/k-2} \arrow[d, "\delta_{X',k-2,k-1,k}"] \\
X^{[1]}_{k-2/k-3} \arrow[r, "X^{[1]}_{k-2/k-3}\iota"] & X^{[1]}_{k-2/k-3}\mathrm{B} \arrow[r, "X^{[1]}_{k-2/k-3}\pi"] & X^{[2]}_{k-2/k-3}
\end{tikzcd}$$

We have the following (pointwise) pushouts in $\mathscr{F}$.

$$
\begin{tikzcd}
X_{k-2/k-3} \arrow[r, "X_{k-2/k-3}\iota"] \arrow[d, "0"'] & X_{k-2/k-3}\mathrm{B} \arrow[d, "g_{k-1/k-2}"] \\
Y_{k-1/k-2} \arrow[r, "i_{k-1/k-2}"] & Z_{k-1/k-2}
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{k-1/k-3} \arrow[r, "X_{k-1/k-3}\iota"] \arrow[d, "X_{k-1/k-3\to k/k-2}\cdot f_{k/k-2}"'] & X_{k-1/k-3}\mathrm{B} \arrow[d, "g_{k/k-2}"] \\
Y_{k/k-2} \arrow[r, "i_{k/k-2}"] & Z_{k/k-2}
\end{tikzcd}
$$

We have

$$
\begin{aligned}
& X_{k-1/k-3\to k/k-2} \cdot f_{k/k-2} \cdot \gamma_{Y,k-2,k-1,k} \cdot i_{k-1/k-2}\mathrm{B} \\
={} & f_{k-1/k-3} \cdot Y_{k-1/k-3\to k/k-2} \cdot \gamma_{Y,k-2,k-1,k} \cdot i_{k-1/k-2}\mathrm{B} \\
={} & f_{k-1/k-3} \cdot y_{k-1/k-3,k-2} \cdot y_{k-1,k/k-2} \cdot \gamma_{Y,k-2,k-1,k} \cdot i_{k-1/k-2}\mathrm{B} \\
={} & f_{k-1/k-3} \cdot y_{k-1/k-3,k-2} \cdot Y_{k-1/k-2}\iota \cdot i_{k-1/k-2}\mathrm{B} \\
={} & f_{k-1/k-3} \cdot Y_{k-1/k-3}\iota \cdot y_{k-1/k-3,k-2}\mathrm{B} \cdot i_{k-1/k-2}\mathrm{B} \\
={} & X_{k-1/k-3}\iota \cdot f_{k-1/k-3}\mathrm{B} \cdot y_{k-1/k-3,k-2}\mathrm{B} \cdot i_{k-1/k-2}\mathrm{B} \\
={} & X_{k-1/k-3}\iota \cdot \big(f_{k-1/k-3}\mathrm{B} \cdot y_{k-1/k-3,k-2}\mathrm{B} \cdot i_{k-1/k-2}\mathrm{B} + X_{k-1/k-3}\pi \cdot \gamma_{X',k-2,k-1,k} \cdot t_{k-1}\mathrm{B}\big).
\end{aligned}
$$

By the universal property of the pushout, there exists a unique morphism $Z_{k/k-2} \xrightarrow{c_k} Z_{k-1/k-2}\mathrm{B}$ in $\mathscr{F}$ such that $i_{k/k-2} \cdot c_k = \gamma_{Y,k-2,k-1,k} \cdot i_{k-1/k-2}\mathrm{B}$ and

$$
g_{k/k-2} \cdot c_k = f_{k-1/k-3}\mathrm{B} \cdot y_{k-1/k-3,k-2}\mathrm{B} \cdot i_{k-1/k-2}\mathrm{B} + X_{k-1/k-3}\pi \cdot \gamma_{X',k-2,k-1,k} \cdot t_{k-1}\mathrm{B}.
$$

$$
\begin{tikzcd}
X_{k-1/k-3} \arrow[r, "X_{k-1/k-3}\iota"] \arrow[d, "X_{k-1/k-3\to k/k-2}\cdot f_{k/k-2}"'] & X_{k-1/k-3}\mathrm{B} \arrow[d, "g_{k/k-2}"] \arrow[ddr, bend left, "{f_{k-1/k-3}\mathrm{B}\cdot y_{k-1/k-3,k-2}\mathrm{B}\cdot i_{k-1/k-2}\mathrm{B} + X_{k-1/k-3}\pi\cdot\gamma_{X',k-2,k-1,k}\cdot t_{k-1}\mathrm{B}}"] & \\
Y_{k/k-2} \arrow[r, "i_{k/k-2}"] \arrow[drr, bend right, "\gamma_{Y,k-2,k-1,k}\cdot i_{k-1/k-2}\mathrm{B}"'] & Z_{k/k-2} \arrow[dr, "c_k"] & \\
& & Z_{k-1/k-2}\mathrm{B}
\end{tikzcd}
$$

By remark 2.2.2, it suffices to show that the following diagram is commutative.

$$
\begin{tikzcd}
Z_{k-1/k-2} \arrow[r, "z_{k-1,k/k-2}"] \arrow[d, "1"'] & Z_{k/k-2} \arrow[r, "z_{k/k-2,k-1}"] \arrow[d, "c_k"] & Z_{k/k-1} \arrow[d, "d_k"] \\
Z_{k-1/k-2} \arrow[r, "Z_{k-1/k-2}\iota"] & Z_{k-1/k-2}\mathrm{B} \arrow[r, "Z_{k-1/k-2}\pi"] & Z^{[1]}_{k-1/k-2}
\end{tikzcd}
$$

We have $z_{k/k-2,k-1} \cdot d_k = c_k \cdot Z_{k-1/k-2}\pi$ since

$$
\begin{aligned}
&\begin{pmatrix} i_{k/k-2} \\ g_{k/k-2} \end{pmatrix} \cdot z_{k/k-2,k-1} \cdot d_k = \begin{pmatrix} y_{k/k-2,k-1} \cdot i_{k/k-1} \cdot d_k \\ x_{k-1/k-3,k-2}\mathrm{B} \cdot g_{k/k-1} \cdot d_k \end{pmatrix} \\
&= \begin{pmatrix} y_{k/k-2,k-1} \cdot \delta_{Y,k-2,k-1,k} \cdot i^{[1]}_{k-1/k-2} \\ x_{k-1/k-3,k-2}\mathrm{B} \cdot X_{k-1/k-2}\pi \cdot t_k \cdot d_k \end{pmatrix} = \begin{pmatrix} \gamma_{Y,k-2,k-1,k} \cdot Y_{k-1/k-2}\pi \cdot i^{[1]}_{k-1/k-2} \\ X_{k-1/k-3}\pi \cdot x^{[1]}_{k-1/k-3,k-2} \cdot t_k \cdot d_k \end{pmatrix} \\
&= \begin{pmatrix} \gamma_{Y,k-2,k-1,k} \cdot i_{k-1/k-2}\mathrm{B} \cdot Z_{k-1/k-2}\pi \\ X_{k-1/k-3}\pi \cdot x^{[1]}_{k-1/k-3,k-2} \cdot f^{[1]}_{k-1/k-2} \cdot i^{[1]}_{k-1/k-2} + X_{k-1/k-3}\pi \cdot x^{[1]}_{k-1/k-3,k-2} \cdot \delta_{X',k-2,k-1,k} \cdot t^{[1]}_{k-1} \end{pmatrix} \\
&= \begin{pmatrix} \gamma_{Y,k-2,k-1,k} \cdot i_{k-1/k-2}\mathrm{B} \cdot Z_{k-1/k-2}\pi \\ x_{k-1/k-3,k-2}\mathrm{B} \cdot f_{k-1/k-2}\mathrm{B} \cdot i_{k-1/k-2}\mathrm{B} \cdot Z_{k-1/k-2}\pi + X_{k-1/k-3}\pi \cdot \gamma_{X',k-2,k-1,k} \cdot X^{[1]}_{k-2/k-3}\pi \cdot t^{[1]}_{k-1} \end{pmatrix} \\
&= \begin{pmatrix} \gamma_{Y,k-2,k-1,k} \cdot i_{k-1/k-2}\mathrm{B} \cdot Z_{k-1/k-2}\pi \\ f_{k-1/k-3}\mathrm{B} \cdot y_{k-1/k-3,k-2}\mathrm{B} \cdot i_{k-1/k-2}\mathrm{B} \cdot Z_{k-1/k-2}\pi + X_{k-1/k-3}\pi \cdot \gamma_{X',k-2,k-1,k} \cdot t_{k-1}\mathrm{B} \cdot Z_{k-1/k-2}\pi \end{pmatrix} \\
&= \begin{pmatrix} i_{k/k-2} \\ g_{k/k-2} \end{pmatrix} \cdot c_k \cdot Z_{k-1/k-2}\pi
\end{aligned}
$$

and since $\begin{pmatrix} i_{k/k-2} \\ g_{k/k-2} \end{pmatrix}$ is a pure epimorphism (In particular, $\begin{pmatrix} i_{k/k-2} \\ g_{k/k-2} \end{pmatrix}$ is an epimorphism, cf. definition 1.3.2.).

We will use that $x_{k-2,k-1/k-3} \cdot x_{k-1/k-3,k-2} = 0$ and thus $x_{k-2,k-1/k-3}\mathrm{B} \cdot x_{k-1/k-3,k-2}\mathrm{B} = 0$.

We have $z_{k-1,k/k-2} \cdot c_k = Z_{k-1/k-2}\iota$ since

$$
\begin{aligned}
&\begin{pmatrix} i_{k-1/k-2} \\ g_{k-1/k-2} \end{pmatrix} \cdot z_{k-1,k/k-2} \cdot c_k = \begin{pmatrix} y_{k-1,k/k-2} \cdot i_{k/k-2} \cdot c_k \\ x_{k-2,k-1/k-3}\mathrm{B} \cdot g_{k/k-2} \cdot c_k \end{pmatrix} \\
&= \begin{pmatrix} y_{k-1,k/k-2} \cdot \gamma_{Y,k-2,k-1,k} \cdot i_{k-1/k-2}\mathrm{B} \\ x_{k-2,k-1/k-3}\mathrm{B} \cdot f_{k-1/k-3}\mathrm{B} \cdot y_{k-1/k-3,k-2}\mathrm{B} \cdot i_{k-1/k-2}\mathrm{B} + x_{k-2,k-1/k-3}\mathrm{B} \cdot X_{k-1/k-3}\pi \cdot \gamma_{X',k-2,k-1,k} \cdot t_{k-1}\mathrm{B} \end{pmatrix} \\
&= \begin{pmatrix} Y_{k-1/k-2}\iota \cdot i_{k-1/k-2}\mathrm{B} \\ x_{k-2,k-1/k-3}\mathrm{B} \cdot x_{k-1/k-3,k-2}\mathrm{B} \cdot f_{k-1/k-2}\mathrm{B} \cdot i_{k-1/k-2}\mathrm{B} + X_{k-2/k-3}\pi \cdot x^{[1]}_{k-2,k-1/k-3} \cdot \gamma_{X',k-2,k-1,k} \cdot t_{k-1}\mathrm{B} \end{pmatrix} \\
&= \begin{pmatrix} i_{k-1/k-2} \cdot Z_{k-1/k-2}\iota \\ X_{k-2/k-3}\pi \cdot X^{[1]}_{k-2/k-3}\iota \cdot t_{k-1}\mathrm{B} \end{pmatrix} = \begin{pmatrix} i_{k-1/k-2} \cdot Z_{k-1/k-2}\iota \\ X_{k-2/k-3}\pi \cdot t_{k-1} \cdot Z_{k-1/k-2}\iota \end{pmatrix} \\
&= \begin{pmatrix} i_{k-1/k-2} \cdot Z_{k-1/k-2}\iota \\ g_{k-1/k-2} \cdot Z_{k-1/k-2}\iota \end{pmatrix} = \begin{pmatrix} i_{k-1/k-2} \\ g_{k-1/k-2} \end{pmatrix} \cdot Z_{k-1/k-2}\iota
\end{aligned}
$$

and since $\begin{pmatrix} i_{k-1/k-2} \\ g_{k-1/k-2} \end{pmatrix}$ is a pure epimorphism. $\square$

4.1.4 Lemma. Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$ and $X, Y \in \mathrm{Ob}(\nabla_{\mathscr{S}}(\mathscr{F}))$ such that $_{\underline{\mathscr{F}}}(X_{\ell/k}\,,\, Y^{[1]}_{j/i}) = 0$ for $i, j, k, \ell \in \mathbf{Z}$ with $i \leq j < k \leq \ell$.
Suppose given $X\Delta_{\mathscr{S},\mathscr{F}} \xrightarrow{g} Y\Delta_{\mathscr{S},\mathscr{F}}$ in $\mathrm{C}(\mathscr{S})$. Then there exists $X \xrightarrow{h} Y$ in $\nabla_{\mathscr{S}}(\mathscr{F})$ such that $h\Delta_{\mathscr{S},\mathscr{F}} = g$. $\diamondsuit$

Proof. We abbreviate $\Delta = \Delta_{\mathscr{S},\mathscr{F}}$. Let $f_{k/k} = 0 \colon X_{k/k} \to Y_{k/k}$ for $k \in \mathbf{Z}$.

For $k \in \mathbf{Z}$, choose $X_{k/k-1} \xrightarrow{f_{k/k-1}} Y_{k/k-1}$ in $\mathscr{F}$ such that $g_k = \underline{f^{[-k]}_{k/k-1}}$. Note that we have

$$
\underline{f^{[-k]}_{k/k-1} \cdot \delta^{[-k]}_{Y,k-2,k-1,k}} = g_k \cdot (Y\Delta)_{k \to k-1} = (X\Delta)_{k \to k-1} \cdot g_{k-1} = \underline{\delta^{[-k]}_{X,k-2,k-1,k} \cdot f^{[-k+1]}_{k-1/k-2}}
$$

and thus $\underline{f_{k/k-1} \cdot \delta_{Y,k-2,k-1,k}} = \underline{\delta_{X,k-2,k-1,k} \cdot f^{[1]}_{k-1/k-2}}$ for $k \in \mathbf{Z}$.

Using recursion on $k \in \mathbf{Z}_{\geq 1}$, we construct morphisms $X_{k/-k} \xrightarrow{f_{k/-k}} Y_{k/-k}$ and

$X_{k+1/-k} \xrightarrow{f_{k+1/-k}} Y_{k+1/-k}$ in $\mathscr{F}$ such that

$$\underline{f_{k+1/-k} \cdot \delta_{Y,-k-1,-k,k+1}} = \underline{\delta_{X,-k-1,-k,k+1} \cdot f^{[1]}_{-k/-k-1}}$$

and such that the following diagrams are commutative.

$$\begin{tikzcd}
X_{-k+1/-k} \arrow[r, "x_{-k+1,k/-k}", "\bullet" marking] \arrow[d, "f_{-k+1/-k}"'] & X_{k/-k} \arrow[r, "x_{k/-k,-k+1}", "|" marking] \arrow[d, "f_{k/-k}"] & X_{k/-k+1} \arrow[d, "f_{k/-k+1}"] \\
Y_{-k+1/-k} \arrow[r, "y_{-k+1,k/-k}", "\bullet" marking] & Y_{k/-k} \arrow[r, "y_{k/-k,-k+1}", "|" marking] & Y_{k/-k+1}
\end{tikzcd}$$

$$\begin{tikzcd}
X_{k/-k} \arrow[r, "x_{k,k+1/-k}", "\bullet" marking] \arrow[d, "f_{k/-k}"'] & X_{k+1/-k} \arrow[r, "x_{k+1/-k,k}", "|" marking] \arrow[d, "f_{k+1/-k}"] & X_{k+1/k} \arrow[d, "f_{k+1/k}"] \\
Y_{k/-k} \arrow[r, "y_{k,k+1/-k}", "\bullet" marking] & Y_{k+1/-k} \arrow[r, "y_{k+1/-k,k}", "|" marking] & Y_{k+1/k}
\end{tikzcd}$$

Suppose given $k \in \mathbf{Z}_{\geq 1}$. We have $\underline{f_{k/-k+1} \cdot \delta_{Y,-k,-k+1,k}} = \underline{\delta_{X,-k,-k+1,k} \cdot f^{[1]}_{-k+1/-k}}$ for $k = 1$ since $\underline{f_{1/0} \cdot \delta_{Y,-1,0,1}} = \underline{\delta_{X,-1,0,1} \cdot f^{[1]}_{0/-1}}$ and for $k > 1$ by recursion.

By lemma 2.2.6, we may choose $X_{k/-k} \xrightarrow{f_{k/-k}} Y_{k/-k}$ in $\mathscr{F}$ such that the following diagram is commutative.

$$\begin{tikzcd}
X_{-k+1/-k} \arrow[r, "x_{-k+1,k/-k}", "\bullet" marking] \arrow[d, "f_{-k+1/-k}"'] & X_{k/-k} \arrow[r, "x_{k/-k,-k+1}", "|" marking] \arrow[d, "f_{k/-k}"] & X_{k/-k+1} \arrow[d, "f_{k/-k+1}"] \\
Y_{-k+1/-k} \arrow[r, "y_{-k+1,k/-k}", "\bullet" marking] & Y_{k/-k} \arrow[r, "y_{k/-k,-k+1}", "|" marking] & Y_{k/-k+1}
\end{tikzcd}$$

Now $\underline{f_{k+1/k} \cdot \delta_{Y,-k,k,k+1}} = \underline{\delta_{X,-k,k,k+1} \cdot f^{[1]}_{k/-k}}$ since we have, using lemma 3.3.43 and recursion for $k > 1$,

$$\begin{aligned}
\underline{f_{k+1/k} \cdot \delta_{Y,-k,k,k+1} \cdot y^{[1]}_{k/-k,k-1}} &= \underline{f_{k+1/k} \cdot \delta_{Y,k-1,k,k+1}} = \underline{\delta_{X,k-1,k,k+1} \cdot f^{[1]}_{k/k-1}} \\
&= \underline{\delta_{X,-k,k,k+1} \cdot x^{[1]}_{k/-k,k-1} \cdot f^{[1]}_{k/k-1}} \\
&= \underline{\delta_{X,-k,k,k+1} \cdot x^{[1]}_{k/-k,-k+1} \cdot x^{[1]}_{k/-k+1,k-1} \cdot f^{[1]}_{k/k-1}} \\
&= \underline{\delta_{X,-k,k,k+1} \cdot x^{[1]}_{k/-k,-k+1} \cdot f^{[1]}_{k/-k+1} \cdot y^{[1]}_{k/-k+1,k-1}} \\
&= \underline{\delta_{X,-k,k,k+1} \cdot f^{[1]}_{k/-k} \cdot y^{[1]}_{k/-k,-k+1} \cdot y^{[1]}_{k/-k+1,k-1}} \\
&= \underline{\delta_{X,-k,k,k+1} \cdot f^{[1]}_{k/-k} \cdot y^{[1]}_{k/-k,k-1}}
\end{aligned}$$

and since $\underline{\mathscr{F}}(X_{k+1/k}\, ,\, Y^{[1]}_{k-1/-k}) = 0$.

By lemma 2.2.6, we may choose $X_{k+1/-k} \xrightarrow{f_{k+1/-k}} Y_{k+1/-k}$ in $\mathscr{F}$ such that the following diagram is commutative.

$$\begin{tikzcd}
X_{k/-k} \arrow[r, "x_{k,k+1/-k}"] \arrow[d, "f_{k/-k}"'] & X_{k+1/-k} \arrow[r, "x_{k+1/-k,k}"] \arrow[d, "f_{k+1/-k}"] & X_{k+1/k} \arrow[d, "f_{k+1/k}"] \\
Y_{k/-k} \arrow[r, "y_{k,k+1/-k}"] & Y_{k+1/-k} \arrow[r, "y_{k+1/-k,k}"] & Y_{k+1/k}
\end{tikzcd}$$

Now $\underline{f_{k+1/-k} \cdot \delta_{Y,-k-1,-k,k+1}} = \underline{\delta_{X,-k-1,-k,k+1} \cdot f^{[1]}_{-k/-k-1}}$ since we have, using lemma 3.3.42,

$$\begin{aligned}
\underline{x_{-k+1,k+1/-k} \cdot f_{k+1/-k} \cdot \delta_{Y,-k-1,-k,k+1}} &= \underline{x_{-k+1,k/-k} \cdot x_{k,k+1/-k} \cdot f_{k+1/-k} \cdot \delta_{Y,-k-1,-k,k+1}} \\
&= \underline{x_{-k+1,k/-k} \cdot f_{k/-k} \cdot y_{k,k+1/-k} \cdot \delta_{Y,-k-1,-k,k+1}} \\
&= \underline{f_{-k+1/-k} \cdot y_{-k+1,k/-k} \cdot y_{k,k+1/-k} \cdot \delta_{Y,-k-1,-k,k+1}} \\
&= \underline{f_{-k+1/-k} \cdot y_{-k+1,k+1/-k} \cdot \delta_{Y,-k-1,-k,k+1}} \\
&= \underline{f_{-k+1/-k} \cdot \delta_{Y,-k-1,-k,-k+1}} \\
&= \underline{\delta_{X,-k-1,-k,-k+1} \cdot f^{[1]}_{-k/-k-1}} \\
&= \underline{x_{-k+1,k+1/-k} \cdot \delta_{X,-k-1,-k,k+1} \cdot f^{[1]}_{-k/-k-1}}
\end{aligned}$$

and since ${}_{\underline{\mathscr{F}}}(X_{k+1/-k+1}\, , Y^{[1]}_{-k/-k-1}) = 0$.

Note that we also have $f_{0/0} \cdot y_{0,1/0} = 0 = x_{0,1/0} \cdot f_{1/0}$ and $f_{1/0} \cdot y_{1/0,0} = f_{1/0} = x_{1/0,0} \cdot f_{1/0}$.

So corollary 3.3.46 yields a morphism $X \xrightarrow{h} Y$ in $\nabla_{\mathscr{S}}(\mathscr{F})$ with $h_{k/k-1} = f_{k/k-1}$ for $k \in \mathbf{Z}$. Thus $(h\Delta)_k = \underline{h^{[-k]}_{k/k-1}} = \underline{f^{[-k]}_{k/k-1}} = g_k$. We conclude that $h\Delta = g$. $\square$

4.1.5 Lemma. Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$ and full subcategories $\mathscr{Q}, \mathscr{R} \subseteq \mathscr{S}$ such that ${}_{\underline{\mathscr{F}}}(\mathscr{Q}^{[k]}, \mathscr{R}) = 0$ for $k \in \mathbf{Z}_{>0}$. Suppose given $X \in \mathrm{Ob}(\nabla_{\mathscr{Q}}(\mathscr{F}))$ and $Y \in \mathrm{Ob}(\nabla_{\mathscr{R}}(\mathscr{F}))$. For each $X\Delta_{\mathscr{S},\mathscr{F}} \xrightarrow{g} Y\Delta_{\mathscr{S},\mathscr{F}}$ in $\mathrm{C}(\mathscr{S})$, there exists $X \xrightarrow{f} Y$ in $\nabla_{\mathscr{S}}(\mathscr{F})$ such that $f\Delta_{\mathscr{S},\mathscr{F}} = g$. $\diamondsuit$

Proof. Suppose given $X\Delta_{\mathscr{S},\mathscr{F}} \xrightarrow{g} Y\Delta_{\mathscr{S},\mathscr{F}}$ in $\mathrm{C}(\mathscr{S})$.

For $i, j, k, \ell \in \mathbf{Z}$ with $i \le j < k \le \ell$, we have ${}_{\underline{\mathscr{F}}}(X_{\ell/k}\, , Y^{[1]}_{j/i}) = 0$ by lemma 3.3.22. So the result follows from the previous lemma 4.1.4. $\square$

4.1.6 Lemma. Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$ and $X \xrightarrow{f} Y$ in $\nabla_{\mathscr{S}}(\mathscr{F})$ such that ${}_{\underline{\mathscr{F}}}(X_{\ell/k}\, , Y_{j/i}) = 0$ for $i, j, k, \ell \in \mathbf{Z}$ with $i \le j \le k \le \ell$. Suppose that $f\Delta_{\mathscr{S},\mathscr{F}} = 0$ in $\mathrm{C}(\mathscr{S})$. Then $\underline{f} = 0$ in $\underline{\nabla}_{\mathscr{S}}(\mathscr{F})$. $\diamondsuit$

Proof. We abbreviate $\Delta = \Delta_{\mathscr{S},\mathscr{F}}$. Let $g_{k/k} = 0\colon X_{k/k}\mathrm{B} \to Y_{k/k}$ for $k \in \mathbf{Z}$.

For $k \in \mathbf{Z}$, we have $\underline{f^{[-k]}_{k/k-1}} = (f\Delta)_k = 0$ and thus $\underline{f_{k/k-1}} = 0$ in $\underline{\mathscr{F}}$. So we may choose $X_{k/k-1}\mathrm{B} \xrightarrow{g_{k/k-1}} Y_{k/k-1}$ in $\mathscr{F}$ such that $X_{k/k-1}\iota \cdot g_{k/k-1} = f_{k/k-1}$ for $k \in \mathbf{Z}$.

Using recursion on $k \in \mathbf{Z}_{\ge 1}$, we construct morphisms $X_{k/-k}\mathrm{B} \xrightarrow{g_{k/-k}} Y_{k/-k}$ and

$X_{k+1/-k}\mathrm{B} \xrightarrow{g_{k+1/-k}} Y_{k+1/-k}$ in $\mathscr{F}$ such that

$$X_{k/-k}\iota \cdot g_{k/-k} = f_{k/-k}\,, \qquad X_{k+1/-k}\iota \cdot g_{k+1/-k} = f_{k+1/-k}$$

and such that the following diagrams are commutative.

$$\begin{tikzcd}
X_{-k+1/-k}\mathrm{B} \arrow[r, "x_{-k+1,k/-k}\mathrm{B}"] \arrow[d, "g_{-k+1/-k}"'] & X_{k/-k}\mathrm{B} \arrow[r, "x_{k/-k,-k+1}\mathrm{B}"] \arrow[d, "g_{k/-k}"] & X_{k/-k+1}\mathrm{B} \arrow[d, "g_{k/-k+1}"] \\
Y_{-k+1/-k} \arrow[r, "y_{-k+1,k/-k}"] & Y_{k/-k} \arrow[r, "y_{k/-k,-k+1}"] & Y_{k/-k+1}
\end{tikzcd}$$

$$\begin{tikzcd}
X_{k/-k}\mathrm{B} \arrow[r, "x_{k,k+1/-k}\mathrm{B}"] \arrow[d, "g_{k/-k}"'] & X_{k+1/-k}\mathrm{B} \arrow[r, "x_{k+1/-k,k}\mathrm{B}"] \arrow[d, "g_{k+1/-k}"] & X_{k+1/k}\mathrm{B} \arrow[d, "g_{k+1/k}"] \\
Y_{k/-k} \arrow[r, "y_{k,k+1/-k}"] & Y_{k+1/-k} \arrow[r, "y_{k+1/-k,k}"] & Y_{k+1/k}
\end{tikzcd}$$

Suppose given $k \in \mathbf{Z}_{\geq 1}$. We have the following commutative diagram in $\mathscr{F}$.

$$\begin{tikzcd}
X_{-k+1/-k} \arrow[r, "x_{-k+1,k/-k}"] \arrow[d, "f_{-k+1/-k}"'] & X_{k/-k} \arrow[r, "x_{k/-k,-k+1}"] \arrow[d, "f_{k/-k}"] & X_{k/-k+1} \arrow[d, "f_{k/-k+1}"] \\
Y_{-k+1/-k} \arrow[r, "y_{-k+1,k/-k}"] & Y_{k/-k} \arrow[r, "y_{k/-k,-k+1}"] & Y_{k/-k+1}
\end{tikzcd}$$

We have $f_{-k+1/-k} = X_{-k+1/-k}\iota \cdot g_{-k+1/-k}$ and, using recursion for $k > 1$, $f_{k/-k+1} = X_{k/-k+1}\iota \cdot g_{k/-k+1}$. Moreover, we have $\underline{\mathscr{F}}(X_{k/-k+1}\,,\, Y_{-k+1/-k}) = 0$.

By corollary 2.1.36, we may choose $X_{k/-k}\mathrm{B} \xrightarrow{g_{k/-k}} Y_{k/-k}$ in $\mathscr{F}$ such that $X_{k/-k}\iota \cdot g_{k/-k} = f_{k/-k}$ and such that the following diagram is commutative.

$$\begin{tikzcd}
X_{-k+1/-k}\mathrm{B} \arrow[r, "x_{-k+1,k/-k}\mathrm{B}"] \arrow[d, "g_{-k+1/-k}"'] & X_{k/-k}\mathrm{B} \arrow[r, "x_{k/-k,-k+1}\mathrm{B}"] \arrow[d, "g_{k/-k}"] & X_{k/-k+1}\mathrm{B} \arrow[d, "g_{k/-k+1}"] \\
Y_{-k+1/-k} \arrow[r, "y_{-k+1,k/-k}"] & Y_{k/-k} \arrow[r, "y_{k/-k,-k+1}"] & Y_{k/-k+1}
\end{tikzcd}$$

We have the following commutative diagram in $\mathscr{F}$.

$$\begin{tikzcd}
X_{k/-k} \arrow[r, "x_{k,k+1/-k}"] \arrow[d, "f_{k/-k}"'] & X_{k+1/-k} \arrow[r, "x_{k+1/-k,k}"] \arrow[d, "f_{k+1/-k}"] & X_{k+1/k} \arrow[d, "f_{k+1/k}"] \\
Y_{k/-k} \arrow[r, "y_{k,k+1/-k}"] & Y_{k+1/-k} \arrow[r, "y_{k+1/-k,k}"] & Y_{k+1/k}
\end{tikzcd}$$

We have $f_{k/-k} = X_{k/-k}\iota \cdot g_{k/-k}$ and $f_{k+1/k} = X_{k+1/k}\iota \cdot g_{k+1/k}$.
Moreover, we have $\underline{\mathscr{F}}(X_{k+1/k}\,,\, Y_{k/-k}) = 0$.

By corollary 2.1.36, we may choose $X_{k+1/-k}\mathrm{B} \xrightarrow{g_{k+1/-k}} Y_{k+1/-k}$ in $\mathscr{F}$ such that

$X_{k+1/-k}\iota \cdot g_{k+1/-k} = f_{k+1/-k}$ and such that the following diagram is commutative.

$$\begin{tikzcd} X_{k/-k}\mathrm{B} \arrow[r, "x_{k,k+1/-k}\mathrm{B}"] \arrow[d, "g_{k/-k}"'] & X_{k+1/-k}\mathrm{B} \arrow[r, "x_{k+1/-k,k}\mathrm{B}"] \arrow[d, "g_{k+1/-k}"] & X_{k+1/k}\mathrm{B} \arrow[d, "g_{k+1/k}"] \\ Y_{k/-k} \arrow[r, "y_{k,k+1/-k}"] & Y_{k+1/-k} \arrow[r, "y_{k+1/-k,k}"] & Y_{k+1/k} \end{tikzcd}$$

Note that we also have $g_{0/0} \cdot y_{0,1/0} = 0 = x_{0,1/0}\mathrm{B} \cdot g_{1/0}$ and $g_{1/0} \cdot y_{1/0,0} = g_{1/0} = x_{1/0,0}\mathrm{B} \cdot g_{1/0}$.

So corollary 3.3.46 yields a morphism $X\mathbf{B} \xrightarrow{h} Y$ in $\nabla_{\mathscr{S}}(\mathscr{F})$ with $h_{k/k-1} = f_{k/k-1}$ for $k \in \mathbf{Z}$, $h_{k/-k} = g_{k/-k}$ and $h_{k+1/-k} = g_{k+1/-k}$ for $k \in \mathbf{Z}_{\geq 1}$.

Again by corollary 3.3.46, we have $X\boldsymbol{\iota} \cdot h = f$ since $X_{k/k-1}\iota \cdot h_{k/k-1} = X_{k/k-1}\iota \cdot g_{k/k-1} = f_{k/k-1}$ for $k \in \mathbf{Z}$, $X_{k/-k}\iota \cdot h_{k/-k} = X_{k/-k}\iota \cdot g_{k/-k} = f_{k/-k}$ and $X_{k+1/-k}\iota \cdot h_{k+1/-k} = X_{k+1/-k}\iota \cdot g_{k+1/-k} = f_{k+1/-k}$ for $k \in \mathbf{Z}_{\geq 1}$. We conclude that $\underline{f} = 0$ in $\underline{\nabla}_{\mathscr{S}}(\mathscr{F})$. □

4.1.7 Lemma. Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$ and full subcategories $\mathscr{Q}, \mathscr{R} \subseteq \mathscr{S}$ such that ${}_{\underline{\mathscr{F}}}(\mathscr{Q}^{[k]}, \mathscr{R}) = 0$ for $k \in \mathbf{Z}_{>0}$. Suppose given $X \xrightarrow{f} Y$ in $\nabla_{\mathscr{S}}(\mathscr{F})$ with $X \in \mathrm{Ob}(\nabla_{\mathscr{Q}}(\mathscr{F}))$ and $Y \in \mathrm{Ob}(\nabla_{\mathscr{R}}(\mathscr{F}))$. Suppose that $f\Delta_{\mathscr{S},\mathscr{F}} = 0$ in $\mathrm{C}(\mathscr{S})$. Then $\underline{f} = 0$ in $\underline{\nabla}_{\mathscr{S}}(\mathscr{F})$. ♢

Proof. For $i, j, k, \ell \in \mathbf{Z}$ with $i \leq j \leq k \leq \ell$, we have ${}_{\underline{\mathscr{F}}}(X_{\ell/k}\, , Y_{j/i}) = 0$ by lemma 3.3.21. So the result follows from the previous lemma 4.1.6. □

4.1.8 Lemma. Suppose given a strictly full additive subcategory $\mathscr{H} \subseteq \underline{\mathscr{F}}$ such that ${}_{\underline{\mathscr{F}}}(\mathscr{H}^{[k]}, \mathscr{H}) = 0$ for $k \in \mathbf{Z}_{>0}$. Then the functor $\Delta_{\mathscr{H},\mathscr{F}} \colon \nabla_{\mathscr{H}}(\mathscr{F}) \to \mathrm{C}(\mathscr{H})$ is strictly dense. ♢

Proof. We abbreviate $\Delta = \Delta_{\mathscr{H},\mathscr{F}}$.

Note that ${}_{\underline{\mathscr{F}}}\Big(\mathop{\ast}\limits_{m\in[k,\ell]} \mathscr{H}^{[m]}, \mathop{\ast}\limits_{n\in[i,j]} \mathscr{H}^{[n]}\Big) = 0$ for $i \leq j < k \leq \ell$ in $\mathbf{Z}$, cf. lemma 1.5.10.

Suppose given $Z \in \mathrm{Ob}(\mathrm{C}(\mathscr{H}))$. We have to construct $Y \in \mathrm{Ob}(\nabla_{\mathscr{H}}(\mathscr{F}))$ such that $Y\Delta = Z$.

Let $X_{k/k-1} = Z_k^{[k]} \in \mathrm{Ob}(\mathscr{H}^{[k]})$ for $k \in \mathbf{Z}$.

Using recursion on $k \in \mathbf{Z}_{\geq 1}$, we construct pure short exact sequences $X_{-k+1/-k} \xrightarrow{x_{-k+1,k/-k}} X_{k/-k} \xrightarrow{x_{k/-k,-k+1}} X_{k/-k+1}$, $X_{k/-k} \xrightarrow{x_{k,k+1/-k}} X_{k+1/-k} \xrightarrow{x_{k+1/-k,k}} X_{k+1/k}$ and abbreviate $\delta_{-k,-k+1,k} = \delta_{(x_{-k+1,k/-k}\, , x_{k/-k,-k+1})}$, $\delta_{-k,k,k+1} = \delta_{(x_{k,k+1/-k}\, , x_{k+1/-k,k})}$ such that

$$X_{k/-k} \in \mathrm{Ob}\Big(\mathop{\ast}\limits_{j\in[-k+1,k]} \mathscr{H}^{[j]}\Big), \quad X_{k+1/-k} \in \mathrm{Ob}\Big(\mathop{\ast}\limits_{j\in[-k+1,k+1]} \mathscr{H}^{[j]}\Big),$$

$$\underline{\delta_{-k,-k+1,k}^{[-1]}} \cdot z_{-k+1}^{[-k+1]} = 0, \quad z_{k+2}^{[k+2]} \cdot \underline{\delta_{-k,k,k+1}^{[1]}} = 0,$$

$\underline{\delta_{-1,0,1}} = z_1^{[1]}$, $\underline{\delta_{-1,1,2} \cdot x_{1/-1,0}^{[1]}} = z_2^{[2]}$ and such that, for $k \in \mathbf{Z}_{>1}$, we have

$$\underline{x_{-k+2,k-1/-k+1} \cdot x_{k-1,k/-k+1} \cdot \delta_{-k,-k+1,k}} = z_{-k+2}^{[-k+2]}, \quad \underline{\delta_{-k,k,k+1} \cdot x_{k/-k,-k+1}^{[1]} \cdot x_{k/-k+1,k-1}^{[1]}} = z_{k+1}^{[k+1]}.$$

We may choose a pure short exact sequence $X_{0/-1} \xrightarrow{x_{0,1/-1}} X_{1/-1} \xrightarrow{x_{1/-1,0}} X_{1/0}$ in $\mathscr{F}$ such that $\underline{\delta_{-1,0,1}} = z_1^{[1]}$ by lemma 2.2.7. Note that $X_{1/-1} \in \operatorname{Ob}(\mathscr{H} * \mathscr{H}^{[1]})$ since $X_{0/-1} \in \operatorname{Ob}(\mathscr{H})$ and $X_{1/0} \in \operatorname{Ob}(\mathscr{H}^{[1]})$. We have $\underline{\delta_{-1,0,1}^{[-1]}} \cdot z_0 = z_1 \cdot z_0 = 0$.

Since $z_2^{[2]} \cdot \underline{\delta_{-1,0,1}^{[1]}} = z_2^{[2]} \cdot z_1^{[2]} = 0$, we may choose $X_{2/1} \xrightarrow{\beta} X_{1/-1}^{[1]}$ in $\mathscr{F}$ such that $\underline{\beta \cdot x_{1/-1,0}^{[1]}} = z_2^{[2]}$. Note that $z_3^{[3]} \cdot \underline{\beta^{[1]}} = 0$ since $z_3^{[3]} \cdot \underline{\beta^{[1]} \cdot x_{1/-1,0}^{[2]}} = z_3^{[3]} \cdot z_2^{[3]} = 0$ and since $\underline{\mathscr{F}}(X_{3/2}\,,\, X_{0/-1}^{[2]}) = 0$.

By lemma 2.2.7, we may choose a pure short exact sequence $X_{1/-1} \xrightarrow{x_{1,2/-1}} X_{2/-1} \xrightarrow{x_{2/-1,1}} X_{2/1}$ in $\mathscr{F}$ such that $\underline{\delta_{-1,1,2}} = \underline{\beta}$. Note that $X_{2/1} \in \operatorname{Ob}(\mathscr{H} * \mathscr{H}^{[1]} * \mathscr{H}^{[2]})$ since $X_{1/-1} \in \operatorname{Ob}(\mathscr{H} * \mathscr{H}^{[1]})$ and $X_{2/1} \in \operatorname{Ob}(\mathscr{H}^{[2]})$. We have $\underline{\delta_{-1,1,2} \cdot x_{1/-1,0}^{[1]}} = \underline{\beta \cdot x_{1/-1,0}^{[1]}} = z_2^{[2]}$ and $z_3^{[3]} \cdot \underline{\delta_{-1,1,2}^{[1]}} = z_3^{[3]} \cdot \underline{\beta^{[1]}} = 0$.

Suppose given $k \in \mathbf{Z}_{\geq 1}$ and the recursively constructed pure short exact sequences $X_{-k+1/-k} \xrightarrow{x_{-k+1,k/-k}} X_{k/-k} \xrightarrow{x_{k/-k,-k+1}} X_{k/-k+1}$, $X_{k/-k} \xrightarrow{x_{k,k+1/-k}} X_{k+1/-k} \xrightarrow{x_{k+1/-k,k}} X_{k+1/k}$ such that

$$X_{k/-k} \in \operatorname{Ob}\left(\mathop{\ast}_{j \in [-k+1,k]} \mathscr{H}^{[j]}\right), \quad X_{k+1/-k} \in \operatorname{Ob}\left(\mathop{\ast}_{j \in [-k+1,k+1]} \mathscr{H}^{[j]}\right) \text{ and}$$

$$\underline{\delta_{-k,-k+1,k}^{[-1]}} \cdot z_{-k+1}^{[-k+1]} = 0, \quad z_{k+2}^{[k+2]} \cdot \underline{\delta_{-k,k,k+1}^{[1]}} = 0.$$

Since $\underline{\delta_{-k,-k+1,k}^{[-1]}} \cdot z_{-k+1}^{[-k+1]} = 0$, we may choose $X_{k/-k} \xrightarrow{\varepsilon} X_{-k/-k-1}^{[1]}$ in $\mathscr{F}$ such that $\underline{x_{-k+1,k/-k} \cdot \varepsilon} = z_{-k+1}^{[-k+1]}$. Note that $\underline{\varepsilon^{[-1]}} \cdot z_{-k}^{[-k]} = 0$ since $\underline{x_{-k+1,k/-k}^{[-1]} \cdot \varepsilon^{[-1]}} \cdot z_{-k}^{[-k]} = z_{-k+1}^{[-k]} \cdot z_{-k}^{[-k]} = 0$ and since $\underline{\mathscr{F}}(X_{k/-k+1}^{[-1]}\,,\, X_{-k-1/-k-2}^{[1]}) = 0$.

Since $\underline{\mathscr{F}}(X_{k+1/k}^{[-1]}\,,\, X_{-k/-k-1}^{[1]}) = 0$, we may choose $X_{k+1/-k} \xrightarrow{\tilde{\varepsilon}} X_{-k/-k-1}^{[1]}$ in $\mathscr{F}$ such that $\underline{x_{k,k+1/-k} \cdot \tilde{\varepsilon}} = \underline{\varepsilon}$. Note that $\underline{\tilde{\varepsilon}^{[-1]}} \cdot z_{-k}^{[-k]} = 0$ since $\underline{x_{k,k+1/-k}^{[-1]} \cdot \tilde{\varepsilon}^{[-1]}} \cdot z_{-k}^{[-k]} = \underline{\varepsilon^{[-1]}} \cdot z_{-k}^{[-k]} = 0$ and since $\underline{\mathscr{F}}(X_{k+1/k}^{[-1]}\,, X_{-k-1/-k-2}^{[1]}) = 0$.

By lemma 2.2.7, we may choose a pure short exact sequence $X_{-k/-k-1} \xrightarrow{x_{-k,k+1/-k-1}} X_{k+1/-k-1} \xrightarrow{x_{k+1/-k-1,-k}} X_{k+1/-k}$ such that $\underline{\delta_{-k-1,-k,k+1}} = \underline{\tilde{\varepsilon}}$.

Note that $X_{k+1/-k-1} \in \operatorname{Ob}\left(\mathop{\ast}_{j \in [-k,k+1]} \mathscr{H}^{[j]}\right)$ since $X_{-k/-k-1} \in \operatorname{Ob}(\mathscr{H}^{[-k]})$ and $X_{k+1/-k} \in \operatorname{Ob}\left(\mathop{\ast}_{j \in [-k+1,k+1]} \mathscr{H}^{[j]}\right)$. We have $\underline{\delta_{-k-1,-k,k+1}^{[-1]}} \cdot z_{-k}^{[-k]} = \underline{\tilde{\varepsilon}^{[-1]}} \cdot z_{-k}^{[-k]} = 0$ and

$$\underline{x_{-k+1,k/-k} \cdot x_{k,k+1/-k} \cdot \delta_{-k-1,-k,k+1}} = \underline{x_{-k+1,k/-k} \cdot x_{k,k+1/-k} \cdot \tilde{\varepsilon}} = \underline{x_{-k+1,k/-k} \cdot \varepsilon} = z_{-k+1}^{[-k+1]}.$$

Since $z_{k+2}^{[k+2]} \cdot \underline{\delta_{-k,k,k+1}^{[1]}} = 0$, we may choose $X_{k+2/k+1} \xrightarrow{\zeta} X_{k+1/-k}^{[1]}$ such that $\underline{\zeta \cdot x_{k+1/-k,k}^{[1]}} = z_{k+2}^{[k+2]}$. Note that $z_{k+3}^{[k+3]} \cdot \underline{\zeta^{[1]}} = 0$ since $z_{k+3}^{[k+3]} \cdot \underline{\zeta^{[1]} \cdot x_{k+1/-k,k}^{[2]}} = z_{k+3}^{[k+3]} \cdot z_{k+2}^{[k+3]} = 0$ and since ${}_{\underline{\mathscr{F}}}(X_{k+3/k+2}\,,\, X_{k/-k}^{[2]}) = 0$.

Since ${}_{\underline{\mathscr{F}}}(X_{k+2/k+1}\,,\, X_{-k/-k-1}^{[2]}) = 0$, we may choose $X_{k+2/k+1} \xrightarrow{\tilde{\zeta}} X_{k+1/-k-1}^{[1]}$ in $\mathscr{F}$ such that $\underline{\tilde{\zeta} \cdot x_{k+1/-k-1,-k}^{[1]}} = \underline{\zeta}$. Note that $z_{k+3}^{[k+3]} \cdot \underline{\tilde{\zeta}^{[1]}} = 0$ since $z_{k+3}^{[k+3]} \cdot \underline{\tilde{\zeta}^{[1]} \cdot x_{k+1/-k-1,-k}^{[2]}} = z_{k+3}^{[k+3]} \cdot \underline{\zeta^{[1]}} = 0$ and since ${}_{\underline{\mathscr{F}}}(X_{k+3/k+2}\,,\, X_{-k/-k-1}^{[2]}) = 0$.

By lemma 2.2.7, we may choose a pure short exact sequence $X_{k+1/-k-1} \xrightarrow{x_{k+1,k+2/-k-1}} X_{k+2/-k-1} \xrightarrow{x_{k+2/-k-1,k+1}} X_{k+2/k+1}$ in $\mathscr{F}$ such that $\underline{\delta_{-k-1,k+1,k+2}} = \underline{\tilde{\zeta}}$.

Note that $X_{k+2/-k-1} \in \mathrm{Ob}\left(\mathop{\ast}\limits_{j\in[-k,k+2]} \mathscr{H}^{[j]}\right)$ since $X_{k+1/-k-1} \in \mathrm{Ob}\left(\mathop{\ast}\limits_{j\in[-k,k+1]} \mathscr{H}^{[j]}\right)$ and since $X_{k+2/k+1} \in \mathrm{Ob}(\mathscr{H}^{[k+2]})$. We have $z_{k+3}^{[k+3]} \cdot \underline{\delta_{-k-1,k+1,k+2}^{[1]}} = z_{k+3}^{[k+3]} \cdot \underline{\tilde{\zeta}^{[1]}} = 0$ and

$$\underline{\delta_{-k-1,k+1,k+2} \cdot x_{k+1/-k-1,-k}^{[1]} \cdot x_{k+1/-k,k}^{[1]}} = \underline{\tilde{\zeta} \cdot x_{k+1/-k-1,-k}^{[1]} \cdot x_{k+1/-k,k}^{[1]}} = \underline{\zeta \cdot x_{k+1/-k,k}^{[1]}} = z_{k+2}^{[k+2]}.$$

Now lemma 3.3.47 yields an object $Y \in \mathrm{Ob}(\nabla(\mathscr{F}))$ such that

$$y_{-k+1,k/-k} = x_{-k+1,k/-k}\,,\; y_{k/-k,-k+1} = x_{k/-k,-k+1}\,,\; y_{k,k+1/-k} = x_{k,k+1/-k}\,,\; y_{k+1/-k,k} = x_{k+1/-k,k}$$

for $k \in \mathbf{Z}_{\geq 1}$. In particular, we have $Y_{-k+1/-k} = X_{-k+1/-k}$, $Y_{k/-k} = X_{k/-k}$, $Y_{k/-k+1} = X_{k/-k+1}$ and $Y_{k+1/k} = X_{k+1/k}$ for $k \in \mathbf{Z}_{\geq 1}$.

So $Y_{k/k-1}^{[-k]} = X_{k/k-1}^{[-k]} = Z_k \in \mathrm{Ob}(\mathscr{H})$ for $k \in \mathbf{Z}$. We conclude that $Y \in \mathrm{Ob}(\nabla_{\mathscr{H}}(\mathscr{F}))$ and that $(Y\Delta)_k = Z_k$ for $k \in \mathbf{Z}$.

It remains to show that $\underline{\delta_{Y,\ell-2,\ell-1,\ell}} = z_\ell^{[\ell]}$ for $\ell \in \mathbf{Z}$.

For $\ell = 1$, we have $\underline{\delta_{Y,-1,0,1}} = \underline{\delta_{-1,0,1}} = z_1^{[1]}$.

For $\ell = 2$, we have, using lemma 3.3.43, $\underline{\delta_{Y,0,1,2}} = \underline{\delta_{Y,-1,1,2} \cdot y_{1/-1,0}^{[1]}} = \underline{\delta_{-1,1,2} \cdot x_{1/-1,0}^{[1]}} = z_2^{[2]}$.

For $\ell > 2$, let $k = \ell - 1 > 1$. We have, using lemma 3.3.43,

$$\begin{aligned}\underline{\delta_{Y,\ell-2,\ell-1,\ell}} = \underline{\delta_{Y,k-1,k,k+1}} &= \underline{\delta_{Y,-k,k,k+1} \cdot y_{k/-k,-k+1}^{[1]} \cdot y_{k/-k+1,k-1}^{[1]}} \\ &= \underline{\delta_{-k,k,k+1} \cdot x_{k/-k,-k+1}^{[1]} \cdot x_{k/-k+1,k-1}^{[1]}} = z_{k+1}^{[k+1]} = z_\ell^{[\ell]}.\end{aligned}$$

For $\ell < 1$, let $k = -\ell + 2 > 1$. We have, using lemma 3.3.42,

$$\begin{aligned}\underline{\delta_{Y,\ell-2,\ell-1,\ell}} = \underline{\delta_{Y,-k,-k+1,-k+2}} &= \underline{y_{-k+2,k-1/-k+1} \cdot y_{k-1,k/-k+1} \cdot \delta_{Y,-k,-k+1,k}} \\ &= \underline{x_{-k+2,k-1/-k+1} \cdot x_{k-1,k/-k+1} \cdot \delta_{-k,-k+1,k}} = z_{-k+2}^{[-k+2]} = z_\ell^{[\ell]}.\end{aligned}$$

□

4.1.9 Lemma. Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$. Suppose given $X \xrightarrow{f} Y$ in $\nabla_{\mathscr{S}}(\mathscr{F})$ such that $\underline{f} = 0$ in $\underline{\nabla}(\mathscr{F})$. Then $f\Delta_{\mathscr{S},\mathscr{F}} = 0$ in $\mathrm{C}(\mathscr{S})$. ◇

Proof. We abbreviate $\Delta = \Delta_{\mathscr{S},\mathscr{F}}$. We may choose $X\mathbf{B} \xrightarrow{g} Y$ in $\nabla(\mathscr{F})$ such that $f = X\boldsymbol{\iota} \cdot g$, cf. remark 3.3.10. For $k \in \mathbf{Z}$, we have $f_{k/k-1} = X_{k/k-1}\iota \cdot g_{k/k-1}$. Thus $(f\Delta)_k = \underline{f^{[-k]}_{k/k-1}} = 0$ in $\underline{\mathscr{F}}$. We conclude that $f\Delta = 0$ in $\mathrm{C}(\mathscr{S})$. □

4.1.10 Definition. Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$. Let $\underline{\Delta}_{\mathscr{S},\mathscr{F}} \colon \underline{\nabla}_{\mathscr{S}}(\mathscr{F}) \to \mathrm{C}(\mathscr{S})$ denote the unique additive functor such that $\mathfrak{P}_{\nabla(\mathscr{F}),\mathscr{S}} \cdot \underline{\Delta}_{\mathscr{S},\mathscr{F}} = \Delta_{\mathscr{S},\mathscr{F}}$, cf. lemma 4.1.9.

$$\begin{tikzcd} \nabla_{\mathscr{S}}(\mathscr{F}) \arrow[r, "\Delta_{\mathscr{S},\mathscr{F}}"] \arrow[d, "\mathfrak{P}_{\nabla(\mathscr{F}),\mathscr{S}}"'] & \mathrm{C}(\mathscr{S}) \\ \underline{\nabla}_{\mathscr{S}}(\mathscr{F}) \arrow[ur, "\underline{\Delta}_{\mathscr{S},\mathscr{F}}"'] & \end{tikzcd}$$

◇

4.1.11 Proposition. Suppose given a strictly full additive subcategory $\mathscr{H} \subseteq \underline{\mathscr{F}}$ such that $_{\underline{\mathscr{F}}}(\mathscr{H}^{[k]}, \mathscr{H}) = 0$ for $k \in \mathbf{Z}_{>0}$. Then the functor $\underline{\Delta}_{\mathscr{H},\mathscr{F}} \colon \underline{\nabla}_{\mathscr{H}}(\mathscr{F}) \to \mathrm{C}(\mathscr{H})$ is an equivalence. Moreover, it is strictly dense. ◇

Proof. The functor $\underline{\Delta}_{\mathscr{H},\mathscr{F}}$ is full, faithful and strictly dense by lemmata 4.1.5, 4.1.7, 4.1.8 and thus it is an equivalence. □

4.1.12 Remark. Suppose given strictly full additive subcategories $\mathscr{R}, \mathscr{S} \subseteq \underline{\mathscr{F}}$ with $\mathscr{R} \subseteq \mathscr{S}$. We have $\underline{\Delta}_{\mathscr{R},\mathscr{F}} \cdot \mathrm{C}(\mathrm{Inc}^{\mathscr{S}}_{\mathscr{R}}) = \mathrm{Inc}^{\underline{\nabla}_{\mathscr{S}}(\mathscr{F})}_{\underline{\nabla}_{\mathscr{R}}(\mathscr{F})} \cdot \underline{\Delta}_{\mathscr{S},\mathscr{F}}$.

$$\begin{tikzcd} \underline{\nabla}_{\mathscr{S}}(\mathscr{F}) \arrow[r, "\underline{\Delta}_{\mathscr{S},\mathscr{F}}"] & \mathrm{C}(\mathscr{S}) \\ \underline{\nabla}_{\mathscr{R}}(\mathscr{F}) \arrow[r, "\underline{\Delta}_{\mathscr{R},\mathscr{F}}"] \arrow[u, "\mathrm{Inc}^{\underline{\nabla}_{\mathscr{S}}(\mathscr{F})}_{\underline{\nabla}_{\mathscr{R}}(\mathscr{F})}"] & \mathrm{C}(\mathscr{R}) \arrow[u, "\mathrm{C}(\mathrm{Inc}^{\mathscr{S}}_{\mathscr{R}})"'] \end{tikzcd}$$

Cf. remark 4.1.2. ◇

4.1.13 Lemma. Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$. Suppose given $X \xrightarrow{f} Y$ in $\nabla_{\mathscr{S}}(\mathscr{F})$ such that $\underline{\underline{f}} = 0$ in $\underline{\underline{\nabla}}(\mathscr{F})$. Then $\underline{f\Delta_{\mathscr{S},\mathscr{F}}} = 0$ in $\mathrm{K}(\mathscr{S})$. ◇

Proof. We abbreviate $\Delta = \Delta_{\mathscr{S},\mathscr{F}}$. Choose a pseudo-triangle $X \xrightarrow{1} X \xrightarrow{i} B \xrightarrow{p} X^{[1]}_{[-1]}$ in $\nabla(\mathscr{F})$, cf. lemma 3.3.25. Note that $B \in \mathrm{Ob}(\nabla_{\mathscr{S}}(\mathscr{F}))$, cf. corollary 3.3.27. By lemma 3.3.28, we may choose $B \xrightarrow{h} Y$ in $\nabla_{\mathscr{S}}(\mathscr{F})$ such that $f = i \cdot h$.
By proposition 4.1.3.(b), $X\Delta \xrightarrow{1} X\Delta \xrightarrow{i\Delta} B\Delta \xrightarrow{p\Delta} X^{[1]}_{[-1]}\Delta$ is a pseudo-triangle in $\mathrm{C}(\mathscr{S})$. We have $f\Delta = (i \cdot h)\Delta = i\Delta \cdot h\Delta$. Thus $\underline{f\Delta} = 0$ in $\mathrm{K}(\mathscr{S})$ by lemma 1.9.18.(b). □

4.1.14 Lemma. Suppose given strictly full additive subcategories $\mathscr{S} \subseteq \underline{\mathscr{F}}$ and $\mathscr{Q} \subseteq \underline{\mathscr{F}}$ such that $\mathscr{Q} \subseteq \mathscr{S}$. Suppose given a full subcategory $\mathscr{R} \subseteq \mathscr{S}$ such that $_{\underline{\mathscr{F}}}(\mathscr{Q}^{[k]}, \mathscr{R}) = 0$ for $k \in \mathbf{Z}_{>0}$. Suppose given $X \xrightarrow{f} Y$ in $\nabla_{\mathscr{S}}(\mathscr{F})$ with $X \in \mathrm{Ob}(\nabla_{\mathscr{Q}}(\mathscr{F}))$ and $Y \in \mathrm{Ob}(\nabla_{\mathscr{R}}(\mathscr{F}))$. Suppose that $\underline{f\Delta_{\mathscr{S},\mathscr{F}}} = 0$ in $\mathrm{K}(\mathscr{S})$. Then $\underline{\underline{f}} = 0$ in $\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})$. ◇

Proof. We abbreviate $\Delta = \Delta_{\mathscr{S},\mathscr{F}}$.

Choose a pseudo-triangle $X \xrightarrow{1} X \xrightarrow{i} B \xrightarrow{p} X^{[1]}_{[-1]}$ in $\nabla(\mathscr{F})$, cf. lemma 3.3.25. Note that $B \in \operatorname{Ob}(\nabla_{\mathscr{Q}}(\mathscr{F}))$, cf. corollary 3.3.27. By proposition 4.1.3.(b) and remark 4.1.2, $X\Delta \xrightarrow{1} X\Delta \xrightarrow{i\Delta} B\Delta \xrightarrow{p\Delta} X^{[1]}_{[-1]}\Delta$ is a pseudo-triangle in $\mathrm{C}(\mathscr{S})$. By lemma 1.9.18.(b), we may choose $B\Delta \xrightarrow{g} Y\Delta$ in $\mathrm{C}(\mathscr{S})$ such that $f\Delta = i\Delta \cdot g$. By lemma 4.1.5, we may choose $B \xrightarrow{h} Y$ in $\nabla_{\mathscr{S}}(\mathscr{F})$ such that $h\Delta = g$. We have $(f - i\cdot h)\Delta = f\Delta - i\Delta\cdot g = 0$. Thus $\underline{f} = \underline{i\cdot h}$ by lemma 4.1.7. We conclude that $\underline{\underline{f}} = 0$ in $\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})$ by lemma 3.3.28. $\square$

4.1.15 Definition. Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$.
Let $\underline{\underline{\Delta}}_{\mathscr{S},\mathscr{F}} \colon \underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F}) \to \mathrm{K}(\mathscr{S})$ denote the unique additive functor such that $\mathfrak{Q}_{\nabla,\mathscr{F},\mathscr{S}} \cdot \underline{\underline{\Delta}}_{\mathscr{S},\mathscr{F}} = \Delta_{\mathscr{S},\mathscr{F}} \cdot \mathfrak{P}_{\mathrm{C}(\mathscr{S})}$, cf. lemma 4.1.13. Let $\mathfrak{S} \colon \underline{\nabla}_{\mathscr{S}}(\mathscr{F}) \to \underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})$ denote the unique functor such that $\mathfrak{P}_{\nabla(\mathscr{F}),\mathscr{S}} \cdot \mathfrak{S} = \mathfrak{Q}_{\nabla,\mathscr{F},\mathscr{S}}$. We have $\mathfrak{S} \cdot \underline{\underline{\Delta}}_{\mathscr{S},\mathscr{F}} = \underline{\Delta}_{\mathscr{S},\mathscr{F}} \cdot \mathfrak{P}_{\mathrm{C}(\mathscr{S})}$, cf. lemma 1.2.18.

$$\begin{tikzcd}
\nabla_{\mathscr{S}}(\mathscr{F}) \arrow[rr, "\Delta_{\mathscr{S},\mathscr{F}}"] \arrow[d, "\mathfrak{P}_{\nabla(\mathscr{F}),\mathscr{S}}"] \arrow[dd, bend right, "\mathfrak{Q}_{\nabla,\mathscr{F},\mathscr{S}}"'] & & \mathrm{C}(\mathscr{S}) \arrow[d, "\mathfrak{P}_{\mathrm{C}(\mathscr{S})}"] \\
\underline{\nabla}_{\mathscr{S}}(\mathscr{F}) \arrow[urr, "\underline{\Delta}_{\mathscr{S},\mathscr{F}}"'] \arrow[d, "\mathfrak{S}"] & & \mathrm{K}(\mathscr{S}) \\
\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F}) \arrow[urr, "\underline{\underline{\Delta}}_{\mathscr{S},\mathscr{F}}"'] & &
\end{tikzcd}$$

$\diamond$

4.1.16 Remark. Suppose given strictly full additive subcategories $\mathscr{R}, \mathscr{S} \subseteq \underline{\mathscr{F}}$ with $\mathscr{R} \subseteq \mathscr{S}$. We have $\underline{\underline{\Delta}}_{\mathscr{R},\mathscr{F}} \cdot \mathrm{K}(\mathrm{Inc}^{\mathscr{S}}_{\mathscr{R}}) = \mathrm{Inc}^{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}_{\mathscr{R}}(\mathscr{F})} \cdot \underline{\underline{\Delta}}_{\mathscr{S},\mathscr{F}}$.

$$\begin{tikzcd}
\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F}) \arrow[r, "\underline{\underline{\Delta}}_{\mathscr{S},\mathscr{F}}"] & \mathrm{K}(\mathscr{S}) \\
\underline{\underline{\nabla}}_{\mathscr{R}}(\mathscr{F}) \arrow[r, "\underline{\underline{\Delta}}_{\mathscr{R},\mathscr{F}}"] \arrow[u, "\mathrm{Inc}^{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}_{\mathscr{R}}(\mathscr{F})}"] & \mathrm{K}(\mathscr{R}) \arrow[u, "\mathrm{K}(\mathrm{Inc}^{\mathscr{S}}_{\mathscr{R}})"']
\end{tikzcd}$$

Cf. remarks 4.1.2 and 4.1.12. $\diamond$

4.1.17 Lemma. Suppose given $k \in \mathbf{Z}$ and an strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$.

(a) For $X \in \operatorname{Ob}(\nabla^{\lceil k}_{\mathscr{S}}(\mathscr{F}))$, we have $X\Delta_{\mathscr{S},\mathscr{F}} \in \operatorname{Ob}(\mathrm{C}^{\lceil k}(\mathscr{S}))$.

(b) For $X \in \operatorname{Ob}(\nabla^{k\rceil}_{\mathscr{S}}(\mathscr{F}))$, we have $X\Delta_{\mathscr{S},\mathscr{F}} \in \operatorname{Ob}(\mathrm{C}^{k\rceil}(\mathscr{S}))$.

$\diamond$

Proof. Ad (a). Suppose given $X \in \operatorname{Ob}(\nabla^{\lceil k}_{\mathscr{S}}(\mathscr{F}))$.
For $\ell \in \mathbf{Z}_{>k}$, we have $(X\Delta_{\mathscr{S},\mathscr{F}})_\ell = X^{[-\ell]}_{\ell/\ell-1} \in \operatorname{Ob}(\mathrm{Z}_{\mathscr{F}}) \subseteq \operatorname{Ob}(\mathrm{Z}_{\mathscr{H}})$. We conclude that $X\Delta_{\mathscr{S},\mathscr{F}} \in \operatorname{Ob}(\mathrm{C}^{\lceil k}(\mathscr{S}))$.

Ad (b). Suppose given $X \in \mathrm{Ob}(\nabla_{\mathscr{S}}^{k]}(\mathscr{F}))$.

For $\ell \in \mathbf{Z}_{<k}$, we have $(X\Delta_{\mathscr{S},\mathscr{F}})_\ell = X_{\ell/\ell-1}^{[-\ell]} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}}) \subseteq \mathrm{Ob}(\mathrm{Z}_{\mathscr{H}})$. We conclude that $X\Delta_{\mathscr{S},\mathscr{F}} \in \mathrm{Ob}(\mathrm{C}^{k]}(\mathscr{S}))$. $\square$

4.1.18 Definition. Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$. Note that for $X \in \mathrm{Ob}(\nabla_{\mathscr{S}}^{\mathrm{b}}(\mathscr{F}))$, we have $X\Delta_{\mathscr{S},\mathscr{F}} \in \mathrm{Ob}(\mathrm{C}^{\mathrm{b}}(\mathscr{S}))$ by lemma 4.1.17. We use the following notation for the corresponding restricted functors. Let

$$\Delta^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} = \Delta_{\mathscr{S},\mathscr{F}}|_{\nabla^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}^{\mathrm{C}^{\mathrm{b}}(\mathscr{S})} \colon \nabla^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F}) \to \mathrm{C}^{\mathrm{b}}(\mathscr{S}), \qquad \underline{\Delta}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} = \underline{\Delta}_{\mathscr{S},\mathscr{F}}|_{\underline{\nabla}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}^{\mathrm{C}^{\mathrm{b}}(\mathscr{S})} \colon \underline{\nabla}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F}) \to \mathrm{C}^{\mathrm{b}}(\mathscr{S})$$

and

$$\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} = \underline{\underline{\Delta}}_{\mathscr{S},\mathscr{F}}|_{\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}^{\mathrm{K}^{\mathrm{b}}(\mathscr{S})} \colon \underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F}) \to \mathrm{K}^{\mathrm{b}}(\mathscr{S}).$$

Note that all three functors are additive, cf. remark 1.2.5.(b) and corollary 3.3.37. $\diamondsuit$

The following definition will make it easier to handle bounded or half-bounded complexes.

4.1.19 Definition. Suppose given an strictly full additive subcategory $\mathscr{H} \subseteq \underline{\mathscr{F}}$. Suppose given $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{H}))$.

We define an object $X^\circ \in \mathrm{Ob}(\mathrm{C}(\mathscr{H}))$ and an isomorphism $X \xrightarrow{X\varsigma} X^\circ$ in $\mathrm{C}(\mathscr{H})$ as follows.

Let $\Upsilon_X = \{k \in \mathbf{Z} \colon X_k \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{H}})\}$.

Let $X_k^\circ = X_k$ for $k \in \mathbf{Z} \setminus \Upsilon_X$ and $X_k^\circ = 0_{\mathscr{F}}$ for $k \in \Upsilon_X$. Let $x_k^\circ = x_k$ for $k, k-1 \in \mathbf{Z} \setminus \Upsilon_X$. Let $x_k^\circ = 0$ for $k \in \Upsilon_X$ or $k - 1 \in \Upsilon_X$. Let $(X\varsigma)_k = 1$ for $k \in \mathbf{Z} \setminus \Upsilon_X$ and $(X\varsigma)_k = 0$ for $k \in \Upsilon_X$. $\diamondsuit$

4.1.20 Definition. Suppose given a strictly full additive subcategory $\mathscr{H} \subseteq \underline{\mathscr{F}}$ such that $_{\underline{\mathscr{F}}}(\mathscr{H}^{[k]}, \mathscr{H}) = 0$ for $k \in \mathbf{Z}_{>0}$. We want to construct quasi-inverses $\mathrm{R}_{\mathscr{H},\mathscr{F}}$ and $\underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}}$ of the functors $\underline{\Delta}_{\mathscr{H},\mathscr{F}} \colon \underline{\nabla}_{\mathscr{H}}(\mathscr{F}) \to \mathrm{C}(\mathscr{H})$ and $\underline{\underline{\Delta}}_{\mathscr{H},\mathscr{F}} \colon \underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F}) \to \mathrm{K}(\mathscr{H})$ following the steps of lemma 1.2.19. These quasi-inverses will play a key role in our construction of realisation functors. The letter R is the first in the word Realisation and such a functor will be the first factor in the definition of a realisation functor, cf. definition 4.4.2. Let $\mathfrak{S} \colon \underline{\nabla}_{\mathscr{H}}(\mathscr{F}) \to \underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F})$ denote the unique functor such that $\mathfrak{P}_{\nabla(\mathscr{F}),\mathscr{H}} \cdot \mathfrak{S} = \mathfrak{Q}_{\nabla,\mathscr{F},\mathscr{H}}$.

$$\begin{tikzcd}
\nabla_{\mathscr{H}}(\mathscr{F}) \arrow[rr, "\Delta_{\mathscr{H},\mathscr{F}}"] \arrow[d, "\mathfrak{P}_{\nabla(\mathscr{F}),\mathscr{H}}"] \arrow[dd, bend right=60, "\mathfrak{Q}_{\nabla,\mathscr{F},\mathscr{H}}"'] & & \mathrm{C}(\mathscr{H}) \arrow[d, "\mathfrak{P}_{\mathrm{C}(\mathscr{H})}"] \\
\underline{\nabla}_{\mathscr{H}}(\mathscr{F}) \arrow[urr, "\underline{\Delta}_{\mathscr{H},\mathscr{F}}"'] \arrow[d, "\mathfrak{S}"] & & \mathrm{K}(\mathscr{H}) \\
\underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F}) \arrow[urr, "\underline{\underline{\Delta}}_{\mathscr{H},\mathscr{F}}"'] & &
\end{tikzcd}$$

The functor $\Delta_{\mathscr{H},\mathscr{F}}$ is full by lemma 4.1.5. For $f \in \mathrm{Mor}(\nabla_{\mathscr{H}}(\mathscr{F}))$, the following two statements hold.

- We have $\underline{f} = 0$ if and only if $f\Delta_{\mathscr{H},\mathscr{F}} = 0$ by lemmata 4.1.9 and 4.1.7.
- We have $\underline{\underline{f}} = 0$ if and only if $\underline{f\Delta_{\mathscr{H},\mathscr{F}}} = 0$ by lemmata 4.1.13 and 4.1.14.

We will use definition 4.1.19. The functor $\Delta_{\mathscr{H},\mathscr{F}}$ is strictly dense by lemma 4.1.8. So for $X \in \mathrm{Ob}(\mathrm{C}(\mathscr{H}))$, we may choose $X\mathrm{R}_{\mathscr{H},\mathscr{F}} \in \mathrm{Ob}(\nabla_{\mathscr{H}}(\mathscr{F}))$ such that $X\mathrm{R}_{\mathscr{H},\mathscr{F}}\Delta_{\mathscr{H},\mathscr{F}} = X^\circ$. Moreover, we have the isomorphism $X \xrightarrow{X\varsigma} X^\circ$.

Lemma 1.2.16 yields the functor $\mathrm{R}_{\mathscr{H},\mathscr{F}} \colon \mathrm{C}(\mathscr{H}) \to \underline{\nabla}_{\mathscr{H}}(\mathscr{F})$, where for $X \xrightarrow{f} Y$ in $\mathrm{C}(\mathscr{H})$, $X\mathrm{R}_{\mathscr{H},\mathscr{F}} \xrightarrow{f\mathrm{R}_{\mathscr{H},\mathscr{F}}} Y\mathrm{R}_{\mathscr{H},\mathscr{F}}$ is the unique morphism in $\underline{\nabla}_{\mathscr{H}}(\mathscr{F})$ such that $f = X\varsigma \cdot f\mathrm{R}_{\mathscr{H},\mathscr{F}}\underline{\Delta}_{\mathscr{H},\mathscr{F}} \cdot (Y\varsigma)^{-1}$. The functors $\underline{\Delta}_{\mathscr{H},\mathscr{F}}$ and $\mathrm{R}_{\mathscr{H},\mathscr{F}}$ are mutually quasi-inverse equivalences.

Lemma 1.2.16 yields the functor $\underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}} \colon \mathrm{K}(\mathscr{H}) \to \underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F})$, where for $X \xrightarrow{f} Y$ in $\mathrm{C}(\mathscr{H})$, $X\mathrm{R}_{\mathscr{H},\mathscr{F}} \xrightarrow{\underline{f}\,\underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}}} Y\mathrm{R}_{\mathscr{H},\mathscr{F}}$ is the unique morphism in $\underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F})$ such that $\underline{f} = \underline{X\varsigma} \cdot \underline{f}\,\underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}}\underline{\underline{\Delta}}_{\mathscr{H},\mathscr{F}} \cdot \underline{(Y\varsigma)^{-1}}$. The functors $\underline{\underline{\Delta}}_{\mathscr{H},\mathscr{F}}$ and $\underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}}$ are mutually quasi-inverse equivalences. We have $\mathrm{R}_{\mathscr{H},\mathscr{F}} \cdot \mathfrak{S} = \mathfrak{P}_{\mathrm{C}(\mathscr{H})} \cdot \underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}}$.

$$\begin{tikzcd}
\underline{\nabla}_{\mathscr{H}}(\mathscr{F}) \arrow[d, "\mathfrak{S}"] & \mathrm{C}(\mathscr{H}) \arrow[l, "\mathrm{R}_{\mathscr{H},\mathscr{F}}"'] \arrow[d, "\mathfrak{P}_{\mathrm{C}(\mathscr{H})}"] \\
\underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F}) & \mathrm{K}(\mathscr{H}) \arrow[l, "\underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}}"']
\end{tikzcd}$$

Note that the functors $\mathrm{R}_{\mathscr{H},\mathscr{F}}$ and $\underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}}$ are additive, cf. remark 1.2.5.(a) and corollary 3.3.37.

$\diamondsuit$

4.1.21 Lemma. Suppose given $k \in \mathbf{Z}$ and a strictly full additive subcategory $\mathscr{H} \subseteq \underline{\mathscr{F}}$ such that ${}_{\underline{\mathscr{F}}}(\mathscr{H}^{[\ell]}, \mathscr{H}) = 0$ for $\ell \in \mathbf{Z}_{>0}$.

(a) For $X \in \mathrm{Ob}(\mathrm{C}^{[k}(\mathscr{H}))$, we have $X\mathrm{R}_{\mathscr{H},\mathscr{F}} \in \mathrm{Ob}(\nabla_{\mathscr{H}}^{[k}(\mathscr{F}))$.

(b) For $X \in \mathrm{Ob}(\mathrm{C}^{k]}(\mathscr{H}))$, we have $X\mathrm{R}_{\mathscr{H},\mathscr{F}} \in \mathrm{Ob}(\nabla_{\mathscr{H}}^{k]}(\mathscr{F}))$.

$\diamondsuit$

Proof. Ad (a). Suppose given $X \in \mathrm{Ob}(\mathrm{C}^{[k}(\mathscr{H}))$. For $\ell \in \mathbf{Z}_{>k}$, we have

$$(X\mathrm{R}_{\mathscr{H},\mathscr{F}})_{\ell/\ell-1}^{[-\ell]} = (X\mathrm{R}_{\mathscr{H},\mathscr{F}}\Delta_{\mathscr{H},\mathscr{F}})_\ell = (X^\circ)_\ell = 0_{\mathscr{F}}$$

and thus $(X\mathrm{R}_{\mathscr{H},\mathscr{F}})_{\ell/\ell-1} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$. We conclude that $X\mathrm{R}_{\mathscr{H},\mathscr{F}} \in \mathrm{Ob}(\nabla_{\mathscr{H}}^{[k}(\mathscr{F}))$.

Ad (b). Suppose given $X \in \mathrm{Ob}(\mathrm{C}^{k]}(\mathscr{H}))$. For $\ell \in \mathbf{Z}_{<k}$, we have

$$(X\mathrm{R}_{\mathscr{H},\mathscr{F}})_{\ell/\ell-1}^{[-\ell]} = (X\mathrm{R}_{\mathscr{H},\mathscr{F}}\Delta_{\mathscr{H},\mathscr{F}})_\ell = (X^\circ)_\ell = 0_{\mathscr{F}}$$

and thus $(X\mathrm{R}_{\mathscr{H},\mathscr{F}})_{\ell/\ell-1} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$. We conclude that $X\mathrm{R}_{\mathscr{H},\mathscr{F}} \in \mathrm{Ob}(\nabla_{\mathscr{H}}^{k]}(\mathscr{F}))$. $\square$

4.1.22 Definition. Suppose given a strictly full additive subcategory $\mathscr{H} \subseteq \underline{\mathscr{F}}$ such that ${}_{\underline{\mathscr{F}}}(\mathscr{H}^{[k]}, \mathscr{H}) = 0$ for $k \in \mathbf{Z}_{>0}$. Note that for $X \in \mathrm{Ob}(\mathrm{C}^{\mathrm{b}}(\mathscr{H}))$, we have
$X\mathrm{R}_{\mathscr{H},\mathscr{F}} \in \mathrm{Ob}(\nabla^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F}))$ by lemma 4.1.21. We use the following notation for the corresponding restricted functors. Let

$$\mathrm{R}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} = \mathrm{R}_{\mathscr{H},\mathscr{F}}|^{\underline{\nabla}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})}_{\mathrm{C}^{\mathrm{b}}(\mathscr{H})}\colon \mathrm{C}^{\mathrm{b}}(\mathscr{H}) \to \underline{\nabla}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F}) \text{ and } \underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} = \underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}}|^{\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})}_{\mathrm{K}^{\mathrm{b}}(\mathscr{H})}\colon \mathrm{K}^{\mathrm{b}}(\mathscr{H}) \to \underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F}).$$

Note that both functors are additive, cf. remark 1.2.5.(b) and corollary 3.3.37. $\diamondsuit$

4.1.23 Remark. Suppose given an strictly full additive subcategory $\mathscr{H} \subseteq \underline{\mathscr{F}}$ such that ${}_{\underline{\mathscr{F}}}(\mathscr{H}^{[k]}, \mathscr{H}) = 0$ for $k \in \mathbf{Z}_{>0}$.
The functors $\mathrm{R}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}$ and $\underline{\Delta}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}$ are mutually quasi-inverse equivalences by lemma 1.6.10.
The functors $\underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}$ and $\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}$ are mutually quasi-inverse equivalences by lemma 1.6.10. $\diamondsuit$

4.1.24 Proposition. Suppose given strictly full additive subcategories $\mathscr{H}, \mathscr{S} \subseteq \underline{\mathscr{F}}$ such that $\mathscr{H} \subseteq \mathscr{S}$ and such that ${}_{\underline{\mathscr{F}}}(\mathscr{H}^{[k]}, \mathscr{S}) = 0$ for $k \in \mathbf{Z}_{>0}$.
Suppose given a functor $H\colon \mathscr{S} \to \mathscr{H}$ that is right-adjoint to the inclusion functor $\mathrm{Inc}^{\mathscr{S}}_{\mathscr{H}}$.
Then $\underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F})}$ is left-adjoint to $\underline{\underline{\Delta}}_{\mathscr{S},\mathscr{F}} \cdot \mathrm{K}(H)$. $\diamondsuit$

Proof. We have the mutually quasi-inverse equivalences $\underline{\underline{\Delta}}_{\mathscr{H},\mathscr{F}}\colon \underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F}) \to \mathrm{K}(\mathscr{H})$ and $\underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}}\colon \mathrm{K}(\mathscr{H}) \to \underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F})$, cf. definition 4.1.20. Since $\mathrm{Inc}^{\mathscr{S}}_{\mathscr{H}} \dashv H$, we have $\mathrm{K}(\mathrm{Inc}^{\mathscr{S}}_{\mathscr{H}}) \dashv \mathrm{K}(H)$ by lemma 1.9.24. The map ${}_{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})}(X, Y) \to {}_{\mathrm{K}(\mathscr{H})}(X\Delta_{\mathscr{S},\mathscr{F}}\, , \, Y\Delta_{\mathscr{S},\mathscr{F}})\colon f \mapsto f\underline{\underline{\Delta}}_{\mathscr{S},\mathscr{F}}$ is bijective for $X \in \mathrm{Ob}(\nabla_{\mathscr{H}}(\mathscr{F}))$ and $Y \in \mathrm{Ob}(\nabla_{\mathscr{S}}(\mathscr{F}))$ by lemmata 4.1.5 and 4.1.14.
We have $\underline{\underline{\Delta}}_{\mathscr{H},\mathscr{F}} \cdot \mathrm{K}(\mathrm{Inc}^{\mathscr{S}}_{\mathscr{H}}) = \mathrm{Inc}^{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F})} \cdot \underline{\underline{\Delta}}_{\mathscr{S},\mathscr{F}}$, cf. remark 4.1.16. So the result follows from lemma 1.6.16.

$$\begin{tikzcd}
\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F}) \arrow[rr, "\underline{\underline{\Delta}}_{\mathscr{S},\mathscr{F}}"] & & \mathrm{K}(\mathscr{S}) \arrow[d, bend left, "\mathrm{K}(H)"] \\
\underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F}) \arrow[u, "\mathrm{Inc}^{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F})}"] \arrow[rr, bend left=10, "\underline{\underline{\Delta}}_{\mathscr{H},\mathscr{F}}"] & & \mathrm{K}(\mathscr{H}) \arrow[u, bend left, "\mathrm{K}(\mathrm{Inc}^{\mathscr{S}}_{\mathscr{H}})"] \arrow[ll, bend left=10, "\underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}}"]
\end{tikzcd}$$

$\square$

4.1.25 Proposition. Suppose given strictly full additive subcategories $\mathscr{H}, \mathscr{S} \subseteq \underline{\mathscr{F}}$ such that $\mathscr{H} \subseteq \mathscr{S}$ and such that ${}_{\underline{\mathscr{F}}}(\mathscr{H}^{[k]}, \mathscr{S}) = 0$ for $k \in \mathbf{Z}_{>0}$.
Suppose given a functor $H\colon \mathscr{S} \to \mathscr{H}$ that is right-adjoint to the inclusion functor $\mathrm{Inc}^{\mathscr{S}}_{\mathscr{H}}$.
Then $\underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})}$ is left-adjoint to $\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(H)$. $\diamondsuit$

Proof. This follows from proposition 4.1.24 and lemma 1.6.9. Cf. definitions 4.1.18 and 4.1.22.

$\square$

4.2 From filtered objects to ∇-diagrams: the functor $\Xi_{\mathscr{F}} \colon \mathrm{FO}(\mathscr{F}) \to \nabla(\mathscr{F})$

Suppose given a strict Frobenius category $\mathscr{F} = (\mathscr{F}, \mathrm{B}, \Sigma, \iota, \pi, \alpha)$.

4.2.1 Definition. For each pure monomorphism $X \xrightarrow{m} Y$ in $\mathscr{F}$, we choose a cokernel $Y \xrightarrow{\mathrm{c}_m} \mathrm{C}_m$ of m in $\mathscr{F}$. $\diamondsuit$

4.2.2 Definition. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ and $\ell/k \in \mathrm{V}$. We write $X_{\ell/k} = \mathrm{C}_{X_{|k\to|\ell}}$ and $X_{|\ell/k} = \mathrm{c}_{X_{|k\to|\ell}}$. So $X_{|\ell} \xrightarrow{X_{|\ell/k}} X_{\ell/k}$ is a cokernel of $X_{|k\to|\ell}$. By the circumference lemma 1.3.13, there exists a unique morphism $X_{\ell/k} \xrightarrow{X_{\ell/k|}} X_{k+1|}$ in $\mathscr{F}$ such that $X_{|\ell/k} \cdot X_{\ell/k|} = x^{\omega}_{|\ell} \cdot x^{\omega}_{k+1|}$. Moreover, it is a kernel of $X_{k+1|\to\ell+1|}$.

$$\begin{tikzcd}
 & & X_{\ell/k} \arrow[ddrr, "X_{\ell/k|}"] & & \\
 & X_{|\ell} \arrow[ur, "X_{|\ell/k}"] \arrow[dr, "x^{\omega}_{|\ell}"] & & & \\
X_{|k} \arrow[ur, "X_{|k\to|\ell}"] \arrow[rr, "x^{\omega}_{|k}"] & & X_{\omega} \arrow[rr, "x^{\omega}_{k+1|}"] \arrow[dr, "x^{\omega}_{\ell+1|}"'] & & X_{k+1|} \arrow[dl, "X_{k+1|\to\ell+1|}"] \\
 & & & X_{\ell+1|} &
\end{tikzcd}$$

Suppose given $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$ and $\ell/k \in \mathrm{V}$. Let $f_{\ell/k}$ denote the unique morphism in $\mathscr{F}$ such that $f_{|\ell} \cdot Y_{|\ell/k} = X_{|\ell/k} \cdot f_{\ell/k}$.

$$\begin{tikzcd}
X_{|k} \arrow[r, "X_{|k\to|\ell}"] \arrow[d, "f_{|k}"'] & X_{|\ell} \arrow[r, "X_{|\ell/k}"] \arrow[d, "f_{|\ell}"] & X_{\ell/k} \arrow[d, "f_{\ell/k}"] \\
Y_{|k} \arrow[r, "Y_{|k\to|\ell}"] & Y_{|\ell} \arrow[r, "Y_{|\ell/k}"] & Y_{\ell/k}
\end{tikzcd}$$

$\diamondsuit$

We want to construct a functor $\Xi_{\mathscr{F}} \colon \mathrm{FO}(\mathscr{F}) \to \nabla(\mathscr{F})$ such that, for $\ell/k \in \mathrm{V}$, we have $(X\Xi_{\mathscr{F}})_{\ell/k} = X_{\ell/k}$ for $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ and $(f\Xi_{\mathscr{F}})_{\ell/k} = f_{\ell/k}$ for $f \in \mathrm{Mor}(\mathrm{FO}(\mathscr{F}))$. To study its properties, it is useful to compare different choices of cokernels first.

4.2.3 Lemma/Definition. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ and cokernels $X_{|\ell} \xrightarrow{c_{\ell/k}} C_{\ell/k}$ of $X_{|k\to|\ell}$ for $\ell/k \in \mathrm{V}$ in $\mathscr{F}$. We obtain an object $X\Theta_{(c_{n/m})_{n/m\in\mathrm{V}}} \in \mathrm{Ob}(\nabla(\mathscr{F}))$ as follows.

Let $(X\Theta_{(c_{n/m})_{n/m\in\mathrm{V}}})_{\ell/k} = C_{\ell/k}$ for $\ell/k \in \mathrm{V}$. For $j/i \leq \ell/k$ in V, let $\big(X\Theta_{(c_{n/m})_{n/m\in\mathrm{V}}}\big)_{j/i\to\ell/k}$ be the unique morphism in $\mathscr{F}$ such that $c_{j/i} \cdot \big(X\Theta_{(c_{n/m})_{n/m\in \mathrm{V}}}\big)_{j/i\to\ell/k} = X_{|j\to|\ell} \cdot c_{\ell/k}$.

$$\begin{tikzcd}
X_{|i} \arrow[r, "X_{|i\to|j}"] \arrow[d, "X_{|i\to|k}"'] & X_{|j} \arrow[r, "c_{j/i}"] \arrow[d, "X_{|j\to|\ell}"] & C_{j/i} \arrow[d, "\big(X\Theta_{(c_{n/m})_{n/m\in \mathrm{V}}}\big)_{j/i\to\ell/k}"] \\
X_{|k} \arrow[r, "X_{|k\to|\ell}"] & X_{|\ell} \arrow[r, "c_{\ell/k}"] & C_{\ell/k}
\end{tikzcd}$$

$\diamondsuit$

Proof. We abbreviate $X\Theta = X\Theta_{(c_{n/m})_{n/m\in \mathrm{V}}}$. For $\ell/k \in \mathrm{V}$, we have $(X\Theta)_{j/i\to j/i} = 1_{C_{j/i}}$ since $c_{j/i} \cdot 1_{C_{j/i}} = 1_{X_{|\ell}} \cdot c_{j/i} = X_{|\ell\to|\ell} \cdot c_{j/i}$ and since $c_{j/i}$ is a pure epimorphism.
For $h/g \le j/i \le \ell/k$ in V, we have $(X\Theta)_{h/g\to j/i} \cdot (X\Theta)_{j/i\to\ell/k} = (X\Theta)_{h/g\to\ell/k}$ since

$$\begin{aligned} c_{h/g} \cdot (X\Theta)_{h/g\to j/i} \cdot (X\Theta)_{j/i\to\ell/k} &= X_{|h\to|j} \cdot c_{j/i} \cdot (X\Theta)_{j/i\to\ell/k} = X_{|h\to|j} \cdot X_{|j\to|\ell} \cdot c_{\ell/k} = X_{|h\to|\ell} \cdot c_{\ell/k} \\ &= c_{h/g} \cdot (X\Theta)_{h/g\to\ell/k} \end{aligned}$$

and since $c_{h/g}$ is a pure epimorphism. So $X\Theta \in \mathrm{Ob}(\mathrm{V}(\mathscr{F}))$.

Suppose given $j \le k \le \ell$ in $\mathbf{Z}$. The sequence $C_{k/j} \xrightarrow{(X\Theta)_{k/j\to\ell/j}} C_{\ell/j} \xrightarrow{(X\Theta)_{\ell/j\to\ell/k}} C_{\ell/k}$ is pure short exact by the circumference lemma 1.3.13.

$$\begin{tikzcd}
 & & C_{k/j} \arrow[drr, "(X\Theta)_{k/j\to\ell/j}"] & & \\
 & X_{|k} \arrow[ur, "c_{k/j}"] \arrow[dr, "X_{|k\to|\ell}"] & & & \\
X_{|j} \arrow[ur, "X_{|j\to|k}"] \arrow[rr, "X_{|j\to|\ell}"] & & X_{|\ell} \arrow[rr, "c_{\ell/j}"] \arrow[dr, "c_{\ell/k}"'] & & C_{\ell/j} \arrow[dl, "(X\Theta)_{\ell/j\to\ell/k}"] \\
 & & & C_{\ell/k} &
\end{tikzcd}$$

We conclude that $X\Theta \in \mathrm{Ob}(\nabla(\mathscr{F}))$. $\square$

4.2.4 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ and cokernels $X_{|\ell} \xrightarrow{c_{\ell/k}} C_{\ell/k}$ of $X_{|k\to|\ell}$ for $\ell/k \in \mathrm{V}$ in $\mathscr{F}$. Note that $X_{|\ell+1} \xrightarrow{c_{\ell+1/k+1}} C_{\ell+1/k+1}$ are cokernels of $(X\mathrm{T}_{\mathrm{FO},\mathscr{F}})_{|k\to|\ell} = X_{|k+1\to|\ell+1}$ for $\ell/k \in \mathrm{V}$ in $\mathscr{F}$. We have $X\mathrm{T}_{\mathrm{FO},\mathscr{F}}\Theta_{(c_{n+1/m+1})_{n/m\in\mathrm{V}}} = X\Theta_{(c_{n/m})_{n/m\in\mathrm{V}}}\mathrm{T}_{\nabla,\mathscr{F}}$. $\diamondsuit$

Proof. For $\ell/k \in \mathrm{V}$, we have

$$(X\mathrm{T}_{\mathrm{FO},\mathscr{F}}\Theta_{(c_{n+1/m+1})_{n/m\in\mathrm{V}}})_{\ell/k} = C_{\ell+1/k+1} = (X\Theta_{(c_{n/m})_{n/m\in\mathrm{V}}})_{\ell+1/k+1} = (X\Theta_{(c_{n/m})_{n/m\in\mathrm{V}}}\mathrm{T}_{\nabla,\mathscr{F}})_{\ell/k}\,.$$

For $j/i \le \ell/k$ in V, we have $(X\mathrm{T}_{\mathrm{FO},\mathscr{F}}\Theta_{(c_{n+1/m+1})_{n/m\in\mathrm{V}}})_{j/i\to\ell/k} = (X\Theta_{(c_{n/m})_{n/m\in\mathrm{V}}}\mathrm{T}_{\nabla,\mathscr{F}})_{j/i\to\ell/k}$

since

$$\begin{aligned} c_{j+1/i+1} \cdot (X \mathrm{T}_{\mathrm{FO},\mathscr{F}} \Theta_{(c_{n+1/m+1})_{n/m \in \mathrm{V}}})_{j/i \to \ell/k} &= X_{|j+1 \to |\ell+1} \cdot c_{\ell+1/k+1} \\ &= c_{j+1/i+1} \cdot (X \Theta_{(c_{n/m})_{n/m \in \mathrm{V}}})_{j+1/i+1 \to \ell+1/k+1} \\ &= c_{j+1/i+1} \cdot (X \Theta_{(c_{n/m})_{n/m \in \mathrm{V}}} \mathrm{T}_{\nabla,\mathscr{F}})_{j/i \to \ell/k} \end{aligned}$$

and since $c_{j+1/i+1}$ is a pure epimorphism. $\square$

4.2.5 Lemma. Suppose given a pointwise bijective object $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ and cokernels $X_{|\ell} \xrightarrow{c_{\ell/k}} C_{\ell/k}$ of $X_{|k \to |\ell}$ for $\ell/k \in \mathrm{V}$ in $\mathscr{F}$. Then $X\Theta_{(c_{n/m})_{n/m \in \mathrm{V}}}$ is bijective in $\nabla(\mathscr{F})$. $\diamond$

Proof. The objects $(X\Theta_{(c_{n/m})_{n/m \in \mathrm{V}}})_{\ell/k} = C_{\ell/k}$ are bijective in $\mathscr{F}$ for $\ell/k \in \mathrm{V}$ by lemma 2.1.39. So $X\Theta_{(c_{n/m})_{n/m \in \mathrm{V}}}$ is bijective in $\nabla(\mathscr{F})$ by lemma 3.3.7. $\square$

4.2.6 Lemma/Definition. Suppose given $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$. Suppose given cokernels $X_{|\ell} \xrightarrow{c_{\ell/k}} C_{\ell/k}$ of $X_{|k \to |\ell}$ for $\ell/k \in \mathrm{V}$ in $\mathscr{F}$ and cokernels $Y_{|\ell} \xrightarrow{d_{\ell/k}} D_{\ell/k}$ of $Y_{|k \to |\ell}$ for $\ell/k \in \mathrm{V}$ in $\mathscr{F}$.

We obtain a morphism $X\Theta_{(c_{n/m})_{n/m \in \mathrm{V}}} \xrightarrow{f\theta^{(d_{n/m})_{n/m \in \mathrm{V}}}_{(c_{n/m})_{n/m \in \mathrm{V}}}} Y\Theta_{(d_{n/m})_{n/m \in \mathrm{V}}}$ as follows. For $\ell/k \in \mathrm{V}$, let $\left(f\theta^{(d_{n/m})_{n/m \in \mathrm{V}}}_{(c_{n/m})_{n/m \in \mathrm{V}}}\right)_{\ell/k}$ be the unique morphism in $\mathscr{F}$ such that
$c_{\ell/k} \cdot \left(f\theta^{(d_{n/m})_{n/m \in \mathrm{V}}}_{(c_{n/m})_{n/m \in \mathrm{V}}}\right)_{\ell/k} = f_{|\ell} \cdot d_{\ell/k}$.

$$\begin{tikzcd} X_{|k} \arrow[r, "X_{|k \to |\ell}"] \arrow[d, "f_{|k}"'] & X_{|\ell} \arrow[r, "c_{\ell/k}"] \arrow[d, "f_{|\ell}"] & C_{\ell/k} \arrow[d, "\left(f\theta^{(d_{n/m})_{n/m \in \mathrm{V}}}_{(c_{n/m})_{n/m \in \mathrm{V}}}\right)_{\ell/k}"] \\ Y_{|k} \arrow[r, "Y_{|k \to |\ell}"] & Y_{|\ell} \arrow[r, "d_{\ell/k}"] & D_{\ell/k} \end{tikzcd}$$

$\diamond$

Proof. We abbreviate $X\Theta = X\Theta_{(c_{n/m})_{n/m \in \mathrm{V}}}$, $Y\Theta = Y\Theta_{(d_{n/m})_{n/m \in \mathrm{V}}}$ and $f\theta = f\theta^{(d_{n/m})_{n/m \in \mathrm{V}}}_{(c_{n/m})_{n/m \in \mathrm{V}}}$. Suppose given $j/i \leq \ell/k$ in V. We have $(X\Theta)_{j/i \to \ell/k} \cdot (f\theta)_{\ell/k} = (f\theta)_{j/i} \cdot (Y\Theta)_{j/i \to \ell/k}$ since

$$\begin{aligned} c_{j/i} \cdot (X\Theta)_{j/i \to \ell/k} \cdot (f\theta)_{\ell/k} &= X_{|j \to |\ell} \cdot c_{\ell/k} \cdot (f\theta)_{\ell/k} = X_{|j \to |\ell} \cdot f_{|\ell} \cdot d_{\ell/k} = f_{|j} \cdot Y_{|j \to |\ell} \cdot d_{\ell/k} \\ &= f_{|j} \cdot d_{j/i} \cdot (Y\Theta)_{j/i \to \ell/k} = c_{j/i} \cdot (f\theta)_{j/i} \cdot (Y\Theta)_{j/i \to \ell/k} \end{aligned}$$

and since $c_{j/i}$ is a pure epimorphism. $\square$

4.2.7 Lemma. Suppose given $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$. Suppose given cokernels $X_{|\ell} \xrightarrow{c_{\ell/k}} C_{\ell/k}$ of $X_{|k \to |\ell}$ for $\ell/k \in \mathrm{V}$ in $\mathscr{F}$ and cokernels $Y_{|\ell} \xrightarrow{d_{\ell/k}} D_{\ell/k}$ of $Y_{|k \to |\ell}$ for $\ell/k \in \mathrm{V}$ in $\mathscr{F}$.
We have $f\mathrm{T}_{\mathrm{FO},\mathscr{F}}\theta^{(d_{n+1/m+1})_{n/m \in \mathrm{V}}}_{(c_{n+1/m+1})_{n/m \in \mathrm{V}}} = f\theta^{(d_{n/m})_{n/m \in \mathrm{V}}}_{(c_{n/m})_{n/m \in \mathrm{V}}}\mathrm{T}_{\nabla,\mathscr{F}}$, cf. lemma 4.2.4. $\diamond$

Proof. For $\ell/k \in \mathrm{V}$, we have $\left(f\mathrm{T}_{\mathrm{FO},\mathscr{F}}\theta^{(d_{n+1/m+1})_{n/m\in\mathrm{V}}}_{(c_{n+1/m+1})_{n/m\in\mathrm{V}}}\right)_{\ell/k} = \left(f\theta^{(d_{n/m})_{n/m\in\mathrm{V}}}_{(c_{n/m})_{n/m\in\mathrm{V}}}\mathrm{T}_{\nabla,\mathscr{F}}\right)_{\ell/k}$ since

$$\begin{aligned} c_{\ell+1/k+1} \cdot \left(f\mathrm{T}_{\mathrm{FO},\mathscr{F}}\theta^{(d_{n+1/m+1})_{n/m\in\mathrm{V}}}_{(c_{n+1/m+1})_{n/m\in\mathrm{V}}}\right)_{\ell/k} &= f_{|\ell+1} \cdot d_{\ell+1/k+1} \\ &= c_{\ell+1/k+1} \cdot \left(f\theta^{(d_{n/m})_{n/m\in\mathrm{V}}}_{(c_{n/m})_{n/m\in\mathrm{V}}}\right)_{\ell+1/k+1} \\ &= c_{\ell+1/k+1} \cdot \left(f\theta^{(d_{n/m})_{n/m\in\mathrm{V}}}_{(c_{n/m})_{n/m\in\mathrm{V}}}\mathrm{T}_{\nabla,\mathscr{F}}\right)_{\ell/k} \end{aligned}$$

and since $c_{\ell+1/k+1}$ is a pure epimorphism. □

4.2.8 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$.
We have $X\rho_{\mathrm{FO},\mathscr{F}}\theta^{(c_{n+1/m+1})_{n/m\in\mathrm{V}}}_{(c_{n/m})_{n/m\in\mathrm{V}}} = X\Theta_{(c_{n/m})_{n/m\in\mathrm{V}}}\rho_{\nabla,\mathscr{F}}$. ◊

Proof. For $\ell/k \in \mathrm{V}$, we have $\left(X\rho_{\mathrm{FO},\mathscr{F}}\theta^{(c_{n+1/m+1})_{n/m\in\mathrm{V}}}_{(c_{n/m})_{n/m\in\mathrm{V}}}\right)_{\ell/k} = \left(X\Theta_{(c_{n/m})_{n/m\in\mathrm{V}}}\rho_{\nabla,\mathscr{F}}\right)_{\ell/k}$ since

$$\begin{aligned} c_{\ell/k} \cdot \left(X\rho_{\mathrm{FO},\mathscr{F}}\theta^{(c_{n+1/m+1})_{n/m\in\mathrm{V}}}_{(c_{n/m})_{n/m\in\mathrm{V}}}\right)_{\ell/k} &= x_{|\ell} \cdot c_{\ell+1/k+1} \\ &= c_{\ell/k} \cdot \left(X\Theta_{(c_{n/m})_{n/m\in\mathrm{V}}}\right)_{\ell/k \to \ell+1/k+1} \\ &= c_{\ell/k} \cdot \left(X\Theta_{(c_{n/m})_{n/m\in\mathrm{V}}}\rho_{\nabla,\mathscr{F}}\right)_{\ell/k} \end{aligned}$$

and since $c_{\ell/k}$ is a pure epimorphism. □

4.2.9 Lemma. Suppose given $X \overset{f}{\underset{h}{\rightrightarrows}} Y \xrightarrow{g} Z$ in $\mathrm{FO}(\mathscr{F})$.

Suppose given cokernels $X_{|\ell} \xrightarrow{c_{\ell/k}} C_{\ell/k}$ of $X_{|k \to |\ell}$ for $\ell/k \in \mathrm{V}$ in $\mathscr{F}$, cokernels $Y_{|\ell} \xrightarrow{d_{\ell/k}} D_{\ell/k}$ of $Y_{|k \to |\ell}$ for $\ell/k \in \mathrm{V}$ in $\mathscr{F}$ and cokernels $Z_{|\ell} \xrightarrow{e_{\ell/k}} E_{\ell/k}$ of $Z_{|k \to |\ell}$ for $\ell/k \in \mathrm{V}$ in $\mathscr{F}$.

(a) We have $1_X\theta^{(c_{n/m})_{n/m\in\mathrm{V}}}_{(c_{n/m})_{n/m\in\mathrm{V}}} = 1_{X\Theta_{(c_{n/m})_{n/m\in\mathrm{V}}}}$,

(b) $(fg)\theta^{(e_{n/m})_{n/m\in\mathrm{V}}}_{(c_{n/m})_{n/m\in\mathrm{V}}} = f\theta^{(d_{n/m})_{n/m\in\mathrm{V}}}_{(c_{n/m})_{n/m\in\mathrm{V}}} \cdot g\theta^{(e_{n/m})_{n/m\in\mathrm{V}}}_{(d_{n/m})_{n/m\in\mathrm{V}}}$ and

(c) $(f+h)\theta^{(d_{n/m})_{n/m\in\mathrm{V}}}_{(c_{n/m})_{n/m\in\mathrm{V}}} = f\theta^{(d_{n/m})_{n/m\in\mathrm{V}}}_{(c_{n/m})_{n/m\in\mathrm{V}}} + h\theta^{(d_{n/m})_{n/m\in\mathrm{V}}}_{(c_{n/m})_{n/m\in\mathrm{V}}}$.

(d) If (f, g) is a pure short exact sequence in $\mathrm{FO}(\mathscr{F})$, then $\left(f\theta^{(d_{n/m})_{n/m\in\mathrm{V}}}_{(c_{n/m})_{n/m\in\mathrm{V}}}, g\theta^{(e_{n/m})_{n/m\in\mathrm{V}}}_{(d_{n/m})_{n/m\in\mathrm{V}}}\right)$ is a pure short exact sequence in $\nabla(\mathscr{F})$.

◊

Proof. We abbreviate $c = (c_{n/m})_{n/m\in\mathrm{V}}$, $d = (d_{n/m})_{n/m\in\mathrm{V}}$ and $e = (e_{n/m})_{n/m\in\mathrm{V}}$.
Ad (a). Suppose given $\ell/k \in \mathrm{V}$. We have $(1_X\theta^c_c)_{\ell/k} = (1_{X\Theta_c})_{\ell/k}$ since

$$c_{\ell/k} \cdot (1_X\theta^c_c)_{\ell/k} = 1_{X_{|\ell}} \cdot c_{\ell/k} = c_{\ell/k} = c_{\ell/k} \cdot 1_{C_{\ell/k}} = c_{\ell/k} \cdot (1_{X\Theta_c})_{\ell/k}$$

and since $c_{\ell/k}$ is a pure epimorphism.

Ad (b). Suppose given $\ell/k \in \mathrm{V}$. We have $((fg)\theta_c^e)_{\ell/k} = (f\theta_c^d \cdot g\theta_d^e)_{\ell/k}$ since

$$\begin{aligned} c_{\ell/k} \cdot ((fg)\theta_c^e)_{\ell/k} &= (fg)_{|\ell} \cdot e_{\ell/k} = f_{|\ell} \cdot g_{|\ell} \cdot e_{\ell/k} = f_{|\ell} \cdot d_{\ell/k} \cdot (g\theta_d^e)_{\ell/k} = c_{\ell/k} \cdot (f\theta_c^d)_{\ell/k} \cdot (g\theta_d^e)_{\ell/k} \\ &= c_{\ell/k} \cdot (f\theta_c^d \cdot g\theta_d^e)_{\ell/k} \end{aligned}$$

and since $c_{\ell/k}$ is a pure epimorphism.

Ad (c). Suppose given $\ell/k \in \mathrm{V}$. We have $((f+h)\theta_c^d)_{\ell/k} = (f\theta_c^d + h\theta_c^d)_{\ell/k}$ since

$$\begin{aligned} c_{\ell/k} \cdot ((f+h)\theta_c^d)_{\ell/k} &= (f+h)_{|\ell} \cdot d_{\ell/k} = f_{|\ell} \cdot d_{\ell/k} + h_{|\ell} \cdot d_{\ell/k} = c_{\ell/k} \cdot (f\theta_c^d)_{\ell/k} + c_{\ell/k} \cdot (h\theta_c^d)_{\ell/k} \\ &= c_{\ell/k} \cdot (f\theta_c^d + h\theta_c^d)_{\ell/k} \end{aligned}$$

and since $c_{\ell/k}$ is a pure epimorphism.

Ad (d). Suppose given $\ell/k \in \mathrm{V}$. By assumption, $(f_{|k}\,,\, g_{|k})$ and $(f_{|\ell}\,,\, g_{|\ell})$ are pure short exact sequences in $\mathscr{F}$. By lemma 1.3.15, $((f\theta_c^d)_{\ell/k}\,,\, (g\theta_d^e)_{\ell/k})$ is a pure short exact sequence in $\mathscr{F}$ as well.

$$\begin{tikzcd} X_{|k} \arrow[r, "X_{|k \to |\ell}"] \arrow[d, "f_{|k}"'] & X_{|\ell} \arrow[r, "c_{\ell/k}"] \arrow[d, "f_{|\ell}"] & C_{\ell/k} \arrow[d, "(f\theta_c^d)_{\ell/k}"] \\ Y_{|k} \arrow[r, "Y_{|k \to |\ell}"] \arrow[d, "g_{|k}"'] & Y_{|\ell} \arrow[r, "d_{\ell/k}"] \arrow[d, "g_{|\ell}"] & D_{\ell/k} \arrow[d, "(g\theta_d^e)_{\ell/k}"] \\ Z_{|k} \arrow[r, "Z_{|k \to |\ell}"] & Z_{|\ell} \arrow[r, "e_{\ell/k}"] & E_{\ell/k} \end{tikzcd}$$

We conclude that $(f\theta_c^d\,,\, g\theta_d^e)$ is a pure short exact sequence in $\nabla(\mathscr{F})$. $\square$

4.2.10 Definition. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$. For $\ell/k \in \mathrm{V}$, we have the chosen cokernels $X_{|\ell} \xrightarrow{X_{|\ell/k}} X_{\ell/k}$ of $X_{|k \to |\ell}$, where $X_{\ell/k} = \mathrm{C}_{X_{|k\to|\ell}}$ and $X_{|\ell/k} = \mathrm{c}_{X_{|k\to|\ell}}$, cf. definition 4.2.2. We write $X\mathrm{C} = (X_{|\ell/k})_{\ell/k\in\mathrm{V}}$. Note that $X\mathrm{T}_{\mathrm{FO},\mathscr{F}}\mathrm{C} = (X_{|\ell+1/k+1})_{\ell/k\in\mathrm{V}}$. $\diamondsuit$

4.2.11 Definition. We define the functor $\Xi_{\mathscr{F}} \colon \mathrm{FO}(\mathscr{F}) \to \nabla(\mathscr{F})$ as follows.

For $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$, let $X\Xi_{\mathscr{F}} = X\Theta_{X\mathrm{C}}$. For $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$, let $f\Xi_{\mathscr{F}} = f\theta_{X\mathrm{C}}^{Y\mathrm{C}}$.

This in fact defines a functor by lemma 4.2.9.(a,b). Moreover, it is exact by lemma 4.2.9.(c,d).

Note that $(X\Xi_{\mathscr{F}})_{\ell/k} = X_{\ell/k}$ for $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ and $\ell/k \in \mathrm{V}$.

Also note that $(f\Xi_{\mathscr{F}})_{\ell/k} = f_{\ell/k}$ for $f \in \mathrm{Mor}(\mathrm{FO}(\mathscr{F}))$ and $\ell/k \in \mathrm{V}$.

For $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ and $j/i \leq \ell/k$ in V, we have $X_{|j/i} \cdot \big(X\Xi_{\mathscr{F}}\big)_{j/i \to \ell/k} = X_{|j \to |\ell} \cdot X_{|\ell/k}$.

Cf. definitions 4.2.3, 4.2.6 and 4.2.10. We also call $\Xi_{\mathscr{F}}$ the *filtered cokernel functor* of $\mathscr{F}$. The letter Ξ was chosen to be reminiscent of the rows of cofiltrations one gets by taking cokernels to obtain a diagram as in definition 3.3.2. $\diamondsuit$

4.2.12 Definition. Suppose given a full subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$. We define the full subcategory

$\mathrm{FO}_{\mathscr{S}}(\mathscr{F})$ of $\mathrm{FO}(\mathscr{F})$ by setting

$$\mathrm{Ob}(\mathrm{FO}_{\mathscr{S}}(\mathscr{F})) = \{X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))\colon X^{[-k]}_{k/k-1} \in \mathrm{Ob}(\mathscr{S}) \text{ for } k \in \mathbf{Z}\}.$$

We define the full subcategory $\underline{\underline{\mathrm{FO}}}_{\mathscr{S}}(\mathscr{F})$ of $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$ by setting $\mathrm{Ob}(\underline{\underline{\mathrm{FO}}}_{\mathscr{S}}(\mathscr{F})) = \mathrm{Ob}(\mathrm{FO}_{\mathscr{S}}(\mathscr{F}))$. $\diamondsuit$

4.2.13 Remark. Suppose given a full subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$ and $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$.
We have $(X\Xi_{\mathscr{F}})_{k/k-1} = X_{k/k-1}$ for $k \in \mathbf{Z}$. Thus we have $X \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{S}}(\mathscr{F}))$ if and only if $X\Xi_{\mathscr{F}} \in \mathrm{Ob}(\nabla_{\mathscr{S}}(\mathscr{F}))$. Cf. definitions 3.3.17 and 4.2.12. $\diamondsuit$

4.2.14 Remark. Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$. Then $\mathrm{FO}_{\mathscr{S}}(\mathscr{F})$ is a strictly full additive subcategory of $\mathrm{FO}(\mathscr{F})$ as well, cf. remark 4.2.13 and lemma 3.3.19. Thus $\underline{\underline{\mathrm{FO}}}_{\mathscr{S}}(\mathscr{F})$ is a full additive subcategory of $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$, cf. remark 1.2.14. $\diamondsuit$

4.2.15 Definition. Suppose given a full subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$.
Let $\Xi_{\mathscr{S},\mathscr{F}} = \Xi_{\mathscr{F}}|^{\nabla_{\mathscr{S}}(\mathscr{F})}_{\mathrm{FO}_{\mathscr{S}}(\mathscr{F})}\colon \mathrm{FO}_{\mathscr{S}}(\mathscr{F}) \to \nabla_{\mathscr{S}}(\mathscr{F})$, cf. remark 4.2.13. $\diamondsuit$

4.2.16 Definition. Suppose given a full subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$.
Let $\mathrm{FO}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F}) = \mathrm{FO}_{\mathscr{S}}(\mathscr{F}) \cap \mathrm{FO}^{\mathrm{b}}(\mathscr{F})$, $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F}) = \underline{\underline{\mathrm{FO}}}_{\mathscr{S}}(\mathscr{F}) \cap \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}(\mathscr{F})$,
$\mathrm{FO}^{\nabla}_{\mathscr{S}}(\mathscr{F}) = \mathrm{FO}_{\mathscr{S}}(\mathscr{F}) \cap \mathrm{FO}^{\nabla}(\mathscr{F})$, $\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{S}}(\mathscr{F}) = \underline{\underline{\mathrm{FO}}}_{\mathscr{S}}(\mathscr{F}) \cap \underline{\underline{\mathrm{FO}}}^{\nabla}(\mathscr{F})$,
$\mathrm{FO}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F}) = \mathrm{FO}_{\mathscr{S}}(\mathscr{F}) \cap \mathrm{FO}^{\mathrm{lim}}(\mathscr{F}) \cap \mathrm{FO}^{\mathrm{inj}}(\mathscr{F})$ and
$\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F}) = \underline{\underline{\mathrm{FO}}}_{\mathscr{S}}(\mathscr{F}) \cap \underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}(\mathscr{F}) \cap \underline{\underline{\mathrm{FO}}}^{\mathrm{inj}}(\mathscr{F})$. $\diamondsuit$

4.2.17 Remark. Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$. Then $\mathrm{FO}^{\mathrm{b}}(\mathscr{F})$ and $\mathrm{FO}_{\mathscr{S}}(\mathscr{F})$ are strictly full additive subcategories of $\mathrm{FO}(\mathscr{F})$, cf. remark 4.2.14 and corollary 3.4.32. Consequently, $\mathrm{FO}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})$ is a strictly full additive subcategory of $\mathrm{FO}(\mathscr{F})$ as well, cf. remark 1.2.4. Thus $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})$ is a full additive subcategory of $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$, cf. remark 1.2.14. $\diamondsuit$

4.2.18 Remark. Suppose that $\mathscr{F}$ has epilimits and monocolimits. Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$. Then $\mathrm{FO}_{\mathscr{S}}(\mathscr{F})$ and $\mathrm{FO}^{\nabla}(\mathscr{F})$ are strictly full additive subcategories of $\mathrm{FO}(\mathscr{F})$, cf. remark 4.2.14 and lemma 3.4.48. Consequently, $\mathrm{FO}^{\nabla}_{\mathscr{S}}(\mathscr{F})$ is a strictly full additive subcategory of $\mathrm{FO}(\mathscr{F})$ as well, cf. remark 1.2.4. Thus $\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{S}}(\mathscr{F})$ is a full additive subcategory of $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$, cf. remark 1.2.14. $\diamondsuit$

4.2.19 Definition. Suppose given a full subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$. Let $\mathrm{P}_{\omega,\mathscr{S},\mathscr{F}} = \mathrm{P}_{\omega,\mathscr{F}}|_{\mathrm{FO}_{\mathscr{S}}(\mathscr{F})}$, $\underline{\mathrm{P}}_{\omega,\mathscr{S},\mathscr{F}} = \underline{\mathrm{P}}_{\omega,\mathscr{F}}|_{\underline{\underline{\mathrm{FO}}}_{\mathscr{S}}(\mathscr{F})}$, $\mathrm{P}^{\mathrm{b}}_{\omega,\mathscr{S},\mathscr{F}} = \mathrm{P}_{\omega,\mathscr{F}}|_{\mathrm{FO}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}$, $\underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{S},\mathscr{F}} = \underline{\mathrm{P}}_{\omega,\mathscr{F}}|_{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}$,
$\mathrm{P}^{\nabla}_{\omega,\mathscr{S},\mathscr{F}} = \mathrm{P}_{\omega,\mathscr{F}}|_{\mathrm{FO}^{\nabla}_{\mathscr{S}}(\mathscr{F})}$, $\underline{\mathrm{P}}^{\nabla}_{\omega,\mathscr{S},\mathscr{F}} = \underline{\mathrm{P}}_{\omega,\mathscr{F}}|_{\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{S}}(\mathscr{F})}$, $\mathrm{P}^{\mathrm{lim,inj}}_{\omega,\mathscr{S},\mathscr{F}} = \mathrm{P}_{\omega,\mathscr{F}}|_{\mathrm{FO}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F})}$ and
$\underline{\mathrm{P}}^{\mathrm{lim,inj}}_{\omega,\mathscr{S},\mathscr{F}} = \underline{\mathrm{P}}_{\omega,\mathscr{F}}|_{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F})}$. $\diamondsuit$

4.2.20 Remark. Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$. Then $\underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{S},\mathscr{F}}$ is an additive functor, cf. remarks 1.2.5.(b) and 4.2.17.

Suppose that $\mathscr{F}$ has epilimits and monocolimits. Then $\underline{\mathrm{P}}^{\nabla}_{\omega,\mathscr{S},\mathscr{F}}$ is an additive functor, cf. remarks 1.2.5.(b) and 4.2.18. $\diamondsuit$

4.2.21 Lemma. We have $\Xi_{\mathscr{F}} \cdot \mathrm{T}_{\nabla,\mathscr{F}} = \mathrm{T}_{\mathrm{FO},\mathscr{F}} \cdot \Xi_{\mathscr{F}}$, $\Xi_{\mathscr{F}} \cdot \mathrm{T}^{-1}_{\nabla,\mathscr{F}} = \mathrm{T}^{-1}_{\mathrm{FO},\mathscr{F}} \cdot \Xi_{\mathscr{F}}$ and $1_{\Xi_{\mathscr{F}}} \star \rho_{\nabla,\mathscr{F}} = \rho_{\mathrm{FO},\mathscr{F}} \star 1_{\Xi_{\mathscr{F}}}$. $\diamondsuit$

Proof. Suppose given $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$. We have

$$\begin{aligned} f\Xi_{\mathscr{F}}\mathrm{T}_{\nabla,\mathscr{F}} &= f\theta^{Y\mathrm{C}}_{X\mathrm{C}}\mathrm{T}_{\nabla,\mathscr{F}} = f\theta^{(Y_{|\ell/k})_{\ell/k\in\mathrm{V}}}_{(X_{|\ell/k})_{\ell/k\in\mathrm{V}}}\mathrm{T}_{\nabla,\mathscr{F}} = f\mathrm{T}_{\mathrm{FO},\mathscr{F}}\theta^{(Y_{|\ell+1/k+1})_{\ell/k\in\mathrm{V}}}_{(X_{|\ell+1/k+1})_{\ell/k\in\mathrm{V}}} = f\mathrm{T}_{\mathrm{FO},\mathscr{F}}\theta^{Y\mathrm{T}_{\mathrm{FO},\mathscr{F}}\mathrm{C}}_{X\mathrm{T}_{\mathrm{FO},\mathscr{F}}\mathrm{C}} \\ &= f\mathrm{T}_{\mathrm{FO},\mathscr{F}}\Xi_{\mathscr{F}} \end{aligned}$$

by lemma 4.2.7. We conclude that $\Xi_{\mathscr{F}} \cdot \mathrm{T}_{\nabla,\mathscr{F}} = \mathrm{T}_{\mathrm{FO},\mathscr{F}} \cdot \Xi_{\mathscr{F}}$.

We have $\Xi_{\mathscr{F}} \cdot \mathrm{T}^{-1}_{\nabla,\mathscr{F}} = \mathrm{T}^{-1}_{\mathrm{FO},\mathscr{F}} \cdot \mathrm{T}_{\mathrm{FO},\mathscr{F}} \cdot \Xi_{\mathscr{F}} \cdot \mathrm{T}^{-1}_{\nabla,\mathscr{F}} = \mathrm{T}^{-1}_{\mathrm{FO},\mathscr{F}} \cdot \Xi_{\mathscr{F}} \cdot \mathrm{T}_{\nabla,\mathscr{F}} \cdot \mathrm{T}^{-1}_{\nabla,\mathscr{F}} = \mathrm{T}^{-1}_{\mathrm{FO},\mathscr{F}} \cdot \Xi_{\mathscr{F}}$.

Suppose given $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$. We have

$$\begin{aligned} X(1_{\Xi_{\mathscr{F}}} \star \rho_{\nabla,\mathscr{F}}) &= X\Xi_{\mathscr{F}}\rho_{\nabla,\mathscr{F}} = X\Theta_{X\mathrm{C}}\rho_{\nabla,\mathscr{F}} = X\Theta_{(X_{|\ell/k})_{\ell/k\in\mathrm{V}}}\rho_{\nabla,\mathscr{F}} = X\rho_{\mathrm{FO},\mathscr{F}}\theta^{(X_{|\ell+1/k+1})_{\ell/k\in\mathrm{V}}}_{(X_{|\ell/k})_{\ell/k\in\mathrm{V}}} \\ &= X\rho_{\mathrm{FO},\mathscr{F}}\theta^{X\mathrm{T}_{\mathrm{FO},\mathscr{F}}\mathrm{C}}_{X\mathrm{C}} = X\rho_{\mathrm{FO},\mathscr{F}}\Xi_{\mathscr{F}} = X(\rho_{\mathrm{FO},\mathscr{F}} \star 1_{\Xi_{\mathscr{F}}}) \end{aligned}$$

by lemma 4.2.8. $\square$

4.2.22 Lemma. Suppose given an exact functor $F\colon \mathscr{F} \to \mathscr{F}$.

We have the functors $\mathrm{FO}(F)\colon \mathrm{FO}(\mathscr{F}) \to \mathrm{FO}(\mathscr{F})$ and $\nabla(F)\colon \nabla(\mathscr{F}) \to \nabla(\mathscr{F})$ and abbreviate $\boldsymbol{F} = \mathrm{FO}(F)$, cf. definitions 3.4.9 and 3.3.3. Note that for $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ and $\ell/k \in \mathrm{V}$, we have the cokernel $X_{|\ell}F \xrightarrow{X_{|\ell/k}F} X_{\ell/k}F$ of $(X\boldsymbol{F})_{|k \to |\ell} = X_{|k\to|\ell}F$.

For $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$, we have $X\Xi_{\mathscr{F}}\nabla(F) = X\boldsymbol{F}\Theta_{(X_{|n/m}F)_{n/m\in\mathrm{V}}}$.

For $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$, we have $f\Xi_{\mathscr{F}}\nabla(F) = f\boldsymbol{F}\theta^{(Y_{|n/m}F)_{n/m\in\mathrm{V}}}_{(X_{|n/m}F)_{n/m\in\mathrm{V}}}$. $\diamondsuit$

Proof. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$. For $\ell/k \in \mathrm{V}$, we have

$$(X\Xi_{\mathscr{F}}\nabla(F))_{\ell/k} = (X\Xi_{\mathscr{F}})_{\ell/k}F = X_{\ell/k}F = \big(X\boldsymbol{F}\Theta_{(X_{|n/m}F)_{n/m\in\mathrm{V}}}\big)_{\ell/k} .$$

For $j/i \le \ell/k$ in V, we have $(X\Xi_{\mathscr{F}}\nabla(F))_{j/i\to\ell/k} = \big(X\boldsymbol{F}\Theta_{(X_{|n/m}F)_{n/m\in\mathrm{V}}}\big)_{j/i\to\ell/k}$ since

$$\begin{aligned} X_{|j/i}F \cdot (X\Xi_{\mathscr{F}}\nabla(F))_{j/i\to\ell/k} &= X_{|j/i}F \cdot (X\Xi_{\mathscr{F}})_{j/i\to\ell/k}F = \big(X_{|j/i} \cdot (X\Xi_{\mathscr{F}})_{j/i\to\ell/k}\big)F \\ &= (X_{|j\to|\ell} \cdot X_{|\ell/k})F = X_{|j\to|\ell}F \cdot X_{|\ell/k}F \\ &= X_{|j/i}F \cdot \big(X\boldsymbol{F}\Theta_{(X_{|n/m}F)_{n/m\in\mathrm{V}}}\big)_{j/i\to\ell/k} \end{aligned}$$

and since $X_{|j/i}F$ is a pure epimorphism. We conclude that $X\Xi_{\mathscr{F}}\nabla(F) = X\boldsymbol{F}\Theta_{(X_{|n/m}F)_{n/m\in\mathrm{V}}}$.

Suppose given $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$. For $\ell/k \in \mathrm{V}$, we have

$(f\Xi_{\mathscr{F}}\nabla(F))_{\ell/k} = \big(f\boldsymbol{F}\theta^{(Y_{|n/m}F)_{n/m\in \mathrm{V}}}_{(X_{|n/m}F)_{n/m\in \mathrm{V}}}\big)_{\ell/k}$ since

$$\begin{aligned} X_{|\ell/k}F\cdot(f\Xi_{\mathscr{F}}\nabla(F))_{\ell/k} &= X_{|\ell/k}F\cdot(f\Xi_{\mathscr{F}})_{\ell/k}F = (X_{|\ell/k}\cdot(f\Xi_{\mathscr{F}})_{\ell/k})F = (f_{|\ell}\cdot Y_{|\ell/k})F \\ &= f_{|\ell}F\cdot Y_{|\ell/k}F = (f\boldsymbol{F})_{|\ell}\cdot Y_{|\ell/k}F \\ &= X_{|\ell/k}F\cdot\big(f\boldsymbol{F}\theta^{(Y_{|n/m}F)_{n/m\in \mathrm{V}}}_{(X_{|n/m}F)_{n/m\in \mathrm{V}}}\big)_{\ell/k} \end{aligned}$$

and since $X_{|\ell/k}F$ is a pure epimorphism.
We conclude that $f\Xi_{\mathscr{F}}\nabla(F) = f\boldsymbol{F}\theta^{(Y_{|n/m}F)_{n/m\in \mathrm{V}}}_{(X_{|n/m}F)_{n/m\in \mathrm{V}}}$. □

4.2.23 Lemma/Definition. Suppose given an exact functor $F\colon \mathscr{F}\to\mathscr{F}$.
We obtain an isotransformation $\xi_{F,\mathscr{F}}\colon \mathrm{FO}(F)\cdot\Xi_{\mathscr{F}}\to\Xi_{\mathscr{F}}\cdot\nabla(F)$ as follows.
For $X\in\mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$, let $X\xi_{F,\mathscr{F}} = 1_{X\mathrm{FO}(F)}\theta^{(X_{|n/m}F)_{n/m\in \mathrm{V}}}_{X\mathrm{FO}(F)\mathrm{C}}\colon X\mathrm{FO}(F)\Xi_{\mathscr{F}}\to X\Xi_{\mathscr{F}}\nabla(F)$.
Cf. lemma 4.2.22. ◊

Proof. We sometimes abbreviate $\boldsymbol{F} = \mathrm{FO}(F)$. We will use lemmata 4.2.22 and 4.2.9.(b).
Suppose given $X\xrightarrow{f}Y$ in $\mathrm{FO}(\mathscr{F})$. We have

$$\begin{aligned} X\xi_{F,\mathscr{F}}\cdot f\Xi_{\mathscr{F}}\nabla(F) &= 1_{X\boldsymbol{F}}\theta^{(X_{|n/m}F)_{n/m\in \mathrm{V}}}_{X\boldsymbol{F}\mathrm{C}}\cdot f\boldsymbol{F}\theta^{(Y_{|n/m}F)_{n/m\in \mathrm{V}}}_{(X_{|n/m}F)_{n/m\in \mathrm{V}}} = f\boldsymbol{F}\theta^{(Y_{|n/m}F)_{n/m\in \mathrm{V}}}_{X\boldsymbol{F}\mathrm{C}} \\ &= f\boldsymbol{F}\theta^{Y\boldsymbol{F}\mathrm{C}}_{X\boldsymbol{F}\mathrm{C}}\cdot 1_{Y\boldsymbol{F}}\theta^{(Y_{|n/m}F)_{n/m\in \mathrm{V}}}_{Y\boldsymbol{F}\mathrm{C}} = f\mathrm{FO}(F)\Xi_{\mathscr{F}}\cdot Y\xi_{F,\mathscr{F}} \end{aligned}$$

$$\begin{tikzcd} X\mathrm{FO}(F)\Xi_{\mathscr{F}} \arrow[r, "X\xi_{F,\mathscr{F}}"] \arrow[d, "f\mathrm{FO}(F)\Xi_{\mathscr{F}}"'] & X\Xi_{\mathscr{F}}\nabla(F) \arrow[d, "f\Xi_{\mathscr{F}}\nabla(F)"] \\ Y\mathrm{FO}(F)\Xi_{\mathscr{F}} \arrow[r, "Y\xi_{F,\mathscr{F}}"] & Y\Xi_{\mathscr{F}}\nabla(F) \end{tikzcd}$$

Note that $X\xi_{F,\mathscr{F}} = 1_{X\mathrm{FO}(F)}\theta^{(X_{|n/m}F)_{n/m\in \mathrm{V}}}_{X\mathrm{FO}(F)\mathrm{C}}$ and $1_{X\mathrm{FO}(F)}\theta^{X\mathrm{FO}(F)\mathrm{C}}_{(X_{|n/m}F)_{n/m\in \mathrm{V}}}$ are mutually inverse isomorphisms by lemma 4.2.9.(a,b). □

4.2.24 Remark. Note that $\xi_{1_{\mathscr{F}},\mathscr{F}} = 1_{\Xi_{\mathscr{F}}}\colon \Xi_{\mathscr{F}}\to\Xi_{\mathscr{F}}$ since $X\xi_{1_{\mathscr{F}},\mathscr{F}} = 1_X\theta^{X\mathrm{FO}(F)\mathrm{C}}_{X\mathrm{FO}(F)\mathrm{C}} = 1_{X\Xi_{\mathscr{F}}}$ for $X\in\mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ by lemma 4.2.9.(a). ◊

4.2.25 Lemma. Suppose given an exact functor $F\colon \mathscr{F}\to\mathscr{F}$. We have

$$1_{\mathrm{T}_{\mathrm{FO},\mathscr{F}}}\star\xi_{F,\mathscr{F}} = \xi_{F,\mathscr{F}}\star 1_{\mathrm{T}_{\nabla,\mathscr{F}}}\colon \mathrm{FO}(F)\cdot\mathrm{T}_{\mathrm{FO},\mathscr{F}}\cdot\Xi_{\mathscr{F}}\to\Xi_{\mathscr{F}}\cdot\nabla(F)\cdot\mathrm{T}_{\nabla,\mathscr{F}}$$

and

$$1_{\mathrm{T}^{-1}_{\mathrm{FO},\mathscr{F}}}\star\xi_{F,\mathscr{F}} = \xi_{F,\mathscr{F}}\star 1_{\mathrm{T}^{-1}_{\nabla,\mathscr{F}}}\colon \mathrm{FO}(F)\cdot\mathrm{T}^{-1}_{\mathrm{FO},\mathscr{F}}\cdot\Xi_{\mathscr{F}}\to\Xi_{\mathscr{F}}\cdot\nabla(F)\cdot\mathrm{T}^{-1}_{\nabla,\mathscr{F}}\ ,$$

cf. lemmata 3.4.10 and 4.2.21. ◊

Proof. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$. We will use lemmata 3.4.10 and 4.2.7. We have

$$\begin{aligned} X(1_{\mathrm{T}_{\mathrm{FO},\mathscr{F}}} \star \xi_{F,\mathscr{F}}) &= X\mathrm{T}_{\mathrm{FO},\mathscr{F}}\xi_{F,\mathscr{F}} = 1_{X\mathrm{T}_{\mathrm{FO},\mathscr{F}}\mathrm{FO}(F)}\theta^{(X_{|n+1/m+1}F)_{n/m\in\mathrm{V}}}_{X\mathrm{T}_{\mathrm{FO},\mathscr{F}}\mathrm{FO}(F)\mathrm{C}} \\ &= 1_{X\mathrm{FO}(F)\mathrm{T}_{\mathrm{FO},\mathscr{F}}}\theta^{(X_{|n+1/m+1}F)_{n/m\in\mathrm{V}}}_{X\mathrm{FO}(F)\mathrm{T}_{\mathrm{FO},\mathscr{F}}\mathrm{C}} = 1_{X\mathrm{FO}(F)}\mathrm{T}_{\mathrm{FO},\mathscr{F}}\theta^{(X_{|n+1/m+1}F)_{n/m\in\mathrm{V}}}_{(X\mathrm{FO}(F)_{|n+1/m+1})_{n/m\in\mathrm{V}}} \\ &= 1_{X\mathrm{FO}(F)}\theta^{(X_{|n/m}F)_{n/m\in\mathrm{V}}}_{(X\mathrm{FO}(F)_{|n/m})_{n/m\in\mathrm{V}}}\mathrm{T}_{\nabla,\mathscr{F}} = 1_{X\mathrm{FO}(F)}\theta^{(X_{|n/m}F)_{n/m\in\mathrm{V}}}_{X\mathrm{FO}(F)\mathrm{C}}\mathrm{T}_{\nabla,\mathscr{F}} \\ &= X\xi_{F,\mathscr{F}}\mathrm{T}_{\nabla,\mathscr{F}} = X(\xi_{F,\mathscr{F}} \star 1_{\mathrm{T}_{\nabla,\mathscr{F}}}). \end{aligned}$$

We conclude that $1_{\mathrm{T}_{\mathrm{FO},\mathscr{F}}} \star \xi_{F,\mathscr{F}} = \xi_{F,\mathscr{F}} \star 1_{\mathrm{T}_{\nabla,\mathscr{F}}}$.

Suppose given $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$. We have

$$\begin{aligned} X\Big(1_{\mathrm{T}^{-1}_{\mathrm{FO},\mathscr{F}}} \star \xi_{F,\mathscr{F}}\Big) &= X\mathrm{T}^{-1}_{\mathrm{FO},\mathscr{F}}\xi_{F,\mathscr{F}} = X\mathrm{T}^{-1}_{\mathrm{FO},\mathscr{F}}\xi_{F,\mathscr{F}}\mathrm{T}_{\nabla,\mathscr{F}}\mathrm{T}^{-1}_{\nabla,\mathscr{F}} = X\mathrm{T}^{-1}_{\mathrm{FO},\mathscr{F}}\mathrm{T}_{\mathrm{FO},\mathscr{F}}\xi_{F,\mathscr{F}}\mathrm{T}^{-1}_{\nabla,\mathscr{F}} \\ &= X\xi_{F,\mathscr{F}}\mathrm{T}^{-1}_{\nabla,\mathscr{F}} = X\Big(\xi_{F,\mathscr{F}} \star 1_{\mathrm{T}^{-1}_{\nabla,\mathscr{F}}}\Big). \end{aligned}$$

We conclude that $1_{\mathrm{T}^{-1}_{\mathrm{FO},\mathscr{F}}} \star \xi_{F,\mathscr{F}} = \xi_{F,\mathscr{F}} \star 1_{\mathrm{T}^{-1}_{\nabla,\mathscr{F}}}$. □

4.2.26 Lemma.

Suppose given exact functors $F, G\colon \mathscr{F} \to \mathscr{F}$ and a transformation $\varphi\colon F \to G$. We have the transformations $\mathrm{FO}(\varphi)\colon \mathrm{FO}(F) \to \mathrm{FO}(G)$ and $\nabla(\varphi)\colon \nabla(F) \to \nabla(G)$, cf. definitions 3.4.9 and 3.3.3. For $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$, we have $X\Xi_{\mathscr{F}}\nabla(\varphi) = X\mathrm{FO}(\varphi)\theta^{(X_{|n/m}G)_{n/m\in\mathrm{V}}}_{(X_{|n/m}F)_{n/m\in\mathrm{V}}}$. ♢

Proof. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$.

For $\ell/k \in \mathrm{V}$, we have $(X\Xi_{\mathscr{F}}\nabla(\varphi))_{\ell/k} = \big(X\mathrm{FO}(\varphi)\theta^{(X_{|n/m}G)_{n/m\in\mathrm{V}}}_{(X_{|n/m}F)_{n/m\in\mathrm{V}}}\big)_{\ell/k}$ since

$$\begin{aligned} X_{|\ell/k}F \cdot (X\Xi_{\mathscr{F}}\nabla(\varphi))_{\ell/k} &= X_{|\ell/k}F \cdot (X\Xi_{\mathscr{F}})_{\ell/k}\varphi = X_{|\ell/k}F \cdot X_{\ell/k}\varphi = X_{|\ell}\varphi \cdot X_{|\ell/k}G \\ &= (X\mathrm{FO}(\varphi))_{|\ell} \cdot X_{|\ell/k}G = X_{|\ell/k}F \cdot \big(X\mathrm{FO}(\varphi)\theta^{(X_{|n/m}G)_{n/m\in\mathrm{V}}}_{(X_{|n/m}F)_{n/m\in\mathrm{V}}}\big)_{\ell/k} \end{aligned}$$

and since $X_{|\ell/k}F$ is a pure epimorphism.

We conclude that $X\Xi_{\mathscr{F}}\nabla(\varphi) = X\mathrm{FO}(\varphi)\theta^{(X_{|n/m}G)_{n/m\in\mathrm{V}}}_{(X_{|n/m}F)_{n/m\in\mathrm{V}}}$. □

4.2.27 Lemma.

Suppose given exact functors $F, G\colon \mathscr{F} \to \mathscr{F}$ and a transformation $\varphi\colon F \to G$. We have $\xi_{F,\mathscr{F}} \cdot (1_{\Xi_{\mathscr{F}}} \star \nabla(\varphi)) = (\mathrm{FO}(\varphi) \star 1_{\Xi_{\mathscr{F}}}) \cdot \xi_{G,\mathscr{F}}$.

$$\begin{tikzcd} \mathrm{FO}(F)\cdot\Xi_{\mathscr{F}} \arrow[r, "\xi_{F,\mathscr{F}}"] \arrow[d, "\mathrm{FO}(\varphi)\star 1_{\Xi_{\mathscr{F}}}"'] & \Xi_{\mathscr{F}}\cdot\nabla(F) \arrow[d, "1_{\Xi_{\mathscr{F}}}\star\nabla(\varphi)"] \\ \mathrm{FO}(G)\cdot\Xi_{\mathscr{F}} \arrow[r, "\xi_{G,\mathscr{F}}"] & \Xi_{\mathscr{F}}\cdot\nabla(G) \end{tikzcd}$$

We also have $\xi^{-1}_{F,\mathscr{F}} \cdot (\mathrm{FO}(\varphi) \star 1_{\Xi_{\mathscr{F}}}) = (1_{\Xi_{\mathscr{F}}} \star \nabla(\varphi)) \cdot \xi^{-1}_{G,\mathscr{F}}$. ♢

Proof. We will use lemmata 4.2.26 and 4.2.9.(b). For $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$, we have

$$\begin{aligned} X(\xi_{F,\mathscr{F}} \cdot (1_{\Xi_{\mathscr{F}}} \star \nabla(\varphi))) &= X\xi_{F,\mathscr{F}} \cdot X\Xi_{\mathscr{F}}\nabla(\varphi) = 1_{X\mathrm{FO}(F)}\theta_{X\mathrm{FO}(F)\mathrm{C}}^{(X_{|n/m}F)_{n/m\in\mathrm{V}}} \cdot X\mathrm{FO}(\varphi)\theta_{(X_{|n/m}F)_{n/m\in\mathrm{V}}}^{(X_{|n/m}G)_{n/m\in\mathrm{V}}} \\ &= X\mathrm{FO}(\varphi)\theta_{X\mathrm{FO}(F)\mathrm{C}}^{(X_{|n/m}G)_{n/m\in\mathrm{V}}} = X\mathrm{FO}(\varphi)\theta_{X\mathrm{FO}(F)\mathrm{C}}^{X\mathrm{FO}(G)\mathrm{C}} \cdot 1_{X\mathrm{FO}(G)}\theta_{X\mathrm{FO}(G)\mathrm{C}}^{(X_{|n/m}G)_{n/m\in\mathrm{V}}} \\ &= X\mathrm{FO}(\varphi)\Xi_{\mathscr{F}} \cdot X\xi_{G,\mathscr{F}} = X((\mathrm{FO}(\varphi) \star 1_{\Xi_{\mathscr{F}}}) \cdot \xi_{G,\mathscr{F}}). \end{aligned}$$

We conclude that $\xi_{F,\mathscr{F}} \cdot (1_{\Xi_{\mathscr{F}}} \star \nabla(\varphi)) = (\mathrm{FO}(\varphi) \star 1_{\Xi_{\mathscr{F}}}) \cdot \xi_{G,\mathscr{F}}$.

We also have

$$\begin{aligned} \xi_{F,\mathscr{F}}^{-1} \cdot (\mathrm{FO}(\varphi) \star 1_{\Xi_{\mathscr{F}}}) &= \xi_{F,\mathscr{F}}^{-1} \cdot (\mathrm{FO}(\varphi) \star 1_{\Xi_{\mathscr{F}}}) \cdot \xi_{G,\mathscr{F}} \cdot \xi_{G,\mathscr{F}}^{-1} \\ &= \xi_{F,\mathscr{F}}^{-1} \cdot \xi_{F,\mathscr{F}} \cdot (1_{\Xi_{\mathscr{F}}} \star \nabla(\varphi)) \cdot \xi_{G,\mathscr{F}}^{-1} \\ &= (1_{\Xi_{\mathscr{F}}} \star \nabla(\varphi)) \cdot \xi_{G,\mathscr{F}}^{-1} . \end{aligned}$$

□

4.2.28 Proposition.

(a) We have the isotransformation
$1_{\mathrm{T}_{\mathrm{FO},\mathscr{F}}^{-1}} \star \xi_{\Sigma,\mathscr{F}} = \xi_{\Sigma,\mathscr{F}} \star 1_{\mathrm{T}_{\nabla,\mathscr{F}}^{-1}} : \Sigma_{\mathrm{FO},\mathscr{F}} \cdot \mathrm{T}_{\mathrm{FO},\mathscr{F}}^{-1} \cdot \Xi_{\mathscr{F}} \to \Xi_{\mathscr{F}} \cdot \Sigma_{\nabla,\mathscr{F}} \cdot \mathrm{T}_{\nabla,\mathscr{F}}^{-1}$, cf. lemma 4.2.25.

(b) Suppose given a pseudo-triangle $X \xrightarrow{f} Y \xrightarrow{i} Y \xrightarrow{p} X_{[-1]}^{[1]}$ in $\mathrm{FO}(\mathscr{F})$. Then

$$X\Xi_{\mathscr{F}} \xrightarrow{f\Xi_{\mathscr{F}}} Y\Xi_{\mathscr{F}} \xrightarrow{i\Xi_{\mathscr{F}}} Z\Xi_{\mathscr{F}} \xrightarrow{p\Xi_{\mathscr{F}} \cdot X_{[-1]}\xi_{\Sigma,\mathscr{F}}} (X\Xi_{\mathscr{F}})_{[-1]}^{[1]}$$

is a pseudo-triangle in $\nabla(\mathscr{F})$, cf. definitions 3.3.24 and 3.4.17.

◊

Proof. Ad (b). We use abbreviations as in the sections 3.3 and 3.4. For example, we write $\mathbf{B} = \nabla(\mathrm{B})\colon \nabla(\mathscr{F}) \to \nabla(\mathscr{F})$ and $\mathbf{B} = \mathrm{FO}(\mathrm{B})\colon \mathrm{FO}(\mathscr{F}) \to \mathrm{FO}(\mathscr{F})$.

Note that $(i\Xi_{\mathscr{F}}\, ,\, p\Xi_{\mathscr{F}} \cdot X_{[-1]}\xi_{\Sigma,\mathscr{F}})$ is a pure short exact sequence since $(i\Xi_{\mathscr{F}}\, ,\, p\Xi_{\mathscr{F}})$ is a pure short exact sequence and since $X_{[-1]}\xi_{\Sigma,\mathscr{F}}$ is an isomorphism, cf. definitions 4.2.11 and 4.2.23.

We may choose $X_{[-1]}\mathbf{B} \xrightarrow{g} Z$ in $\mathrm{FO}(\mathscr{F})$ such that the following diagram is commutative.

$$\begin{tikzcd} X_{[-1]} \arrow[r, "X_{[-1]}\boldsymbol{\iota}"] \arrow[d, "X_{[-1]}\rho \cdot f"'] & X_{[-1]}\mathbf{B} \arrow[r, "X_{[-1]}\boldsymbol{\pi}"] \arrow[d, "g"] & X_{[-1]}^{[1]} \arrow[d, "1"] \\ Y \arrow[r, "i"] & Z \arrow[r, "p"] & X_{[-1]}^{[1]} \end{tikzcd}$$

We will use lemmata 4.2.21, 4.2.27 and remark 4.2.24.

Consider the morphism $X_{[-1]}\xi_{\mathrm{B},\mathscr{F}}^{-1} \cdot g\Xi_{\mathscr{F}}\colon (X\Xi_{\mathscr{F}})_{[-1]}\mathbf{B} = X_{[-1]}\Xi_{\mathscr{F}}\mathbf{B} \to Z\Xi_{\mathscr{F}}$ in $\nabla(\mathscr{F})$. We have

$$\begin{aligned}(X\Xi_{\mathscr{F}})_{[-1]}\rho \cdot f\Xi_{\mathscr{F}} \cdot i\Xi_{\mathscr{F}} &= X_{[-1]}\Xi_{\mathscr{F}}\rho \cdot f\Xi_{\mathscr{F}} \cdot i\Xi_{\mathscr{F}} = X_{[-1]}\rho\Xi_{\mathscr{F}} \cdot (f \cdot i)\Xi_{\mathscr{F}} = (X_{[-1]}\rho \cdot f \cdot i)\Xi_{\mathscr{F}} \\ &= (X_{[-1]}\boldsymbol{\iota} \cdot g)\Xi_{\mathscr{F}} = X_{[-1]}\boldsymbol{\iota}\Xi_{\mathscr{F}} \cdot g\Xi_{\mathscr{F}} = X_{[-1]}\xi^{-1}_{1_{\mathscr{F}},\mathscr{F}} \cdot X_{[-1]}\boldsymbol{\iota}\Xi_{\mathscr{F}} \cdot g\Xi_{\mathscr{F}} \\ &= X_{[-1]}\Xi_{\mathscr{F}}\boldsymbol{\iota} \cdot X_{[-1]}\xi^{-1}_{\mathrm{B},\mathscr{F}} \cdot g\Xi_{\mathscr{F}} = (X\Xi_{\mathscr{F}})_{[-1]}\boldsymbol{\iota} \cdot X_{[-1]}\xi^{-1}_{\mathrm{B},\mathscr{F}} \cdot g\Xi_{\mathscr{F}}\end{aligned}$$

and

$$\begin{aligned}X_{[-1]}\xi^{-1}_{\mathrm{B},\mathscr{F}} \cdot g\Xi_{\mathscr{F}} \cdot p\Xi_{\mathscr{F}} \cdot X_{[-1]}\xi_{\Sigma,\mathscr{F}} &= X_{[-1]}\xi^{-1}_{\mathrm{B},\mathscr{F}} \cdot (g \cdot p)\Xi_{\mathscr{F}} \cdot X_{[-1]}\xi_{\Sigma,\mathscr{F}} \\ &= X_{[-1]}\xi^{-1}_{\mathrm{B},\mathscr{F}} \cdot X_{[-1]}\boldsymbol{\pi}\Xi_{\mathscr{F}} \cdot X_{[-1]}\xi_{\Sigma,\mathscr{F}} \\ &= X_{[-1]}\Xi_{\mathscr{F}}\boldsymbol{\pi} \cdot X_{[-1]}\xi^{-1}_{\Sigma,\mathscr{F}} \cdot X_{[-1]}\xi_{\Sigma,\mathscr{F}} \\ &= (X\Xi_{\mathscr{F}})_{[-1]}\boldsymbol{\pi}.\end{aligned}$$

$$\begin{tikzcd}
(X\Xi_{\mathscr{F}})_{[-1]} \arrow[r, "(X\Xi_{\mathscr{F}})_{[-1]}\boldsymbol{\iota}"] \arrow[d, "(X\Xi_{\mathscr{F}})_{[-1]}\rho \cdot f\Xi_{\mathscr{F}}"'] & (X\Xi_{\mathscr{F}})_{[-1]}\mathbf{B} \arrow[r, "(X\Xi_{\mathscr{F}})_{[-1]}\boldsymbol{\pi}"] \arrow[d, "X_{[-1]}\xi^{-1}_{\mathrm{B},\mathscr{F}} \cdot g\Xi_{\mathscr{F}}"] & (X\Xi_{\mathscr{F}})^{[1]}_{[-1]} \arrow[d, "1"] \\
Y\Xi_{\mathscr{F}} \arrow[r, "i\Xi_{\mathscr{F}}"] & Z\Xi_{\mathscr{F}} \arrow[r, "p\Xi_{\mathscr{F}} \cdot X_{[-1]}\xi_{\Sigma,\mathscr{F}}"] & (X\Xi_{\mathscr{F}})^{[1]}_{[-1]}
\end{tikzcd}$$

We conclude that $X\Xi_{\mathscr{F}} \xrightarrow{f\Xi_{\mathscr{F}}} Y\Xi_{\mathscr{F}} \xrightarrow{i\Xi_{\mathscr{F}}} Z\Xi_{\mathscr{F}} \xrightarrow{p\Xi_{\mathscr{F}} \cdot X_{[-1]}\xi_{\Sigma,\mathscr{F}}} (X\Xi_{\mathscr{F}})^{[1]}_{[-1]}$ is a pseudo-triangle in $\nabla(\mathscr{F})$. □

4.2.29 Lemma. Suppose given an strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$. Suppose given a pseudo-triangle $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}_{[-1]}$ in $\mathrm{FO}(\mathscr{F})$ such that $X, Y \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{S}}(\mathscr{F}))$. Then we have $Z, X^{[1]}_{[-1]} \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{S}}(\mathscr{F}))$ as well. ◇

Proof. This follows from remark 4.2.13, proposition 4.2.28.(a,b), lemma 3.3.19 and corollary 3.3.27. □

4.2.30 Lemma. Suppose given $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$ such that $\underline{\underline{f}} = 0$ in $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$. Then we have $\underline{\underline{f\Xi_{\mathscr{F}}}} = 0$ in $\underline{\underline{\nabla}}(\mathscr{F})$ as well. ◇

Proof. We may choose a pointwise bijective object $B \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ and morphisms $X_{[-1]} \xrightarrow{r} B \xrightarrow{s} Y$ in $\mathrm{FO}(\mathscr{F})$ such that $X_{[-1]}\rho \cdot f = r \cdot s$.
Using lemma 4.2.21, we obtain

$$\begin{aligned}(X\Xi_{\mathscr{F}})_{[-1]}\rho \cdot f\Xi_{\mathscr{F}} &= X_{[-1]}\Xi_{\mathscr{F}}\rho \cdot f\Xi_{\mathscr{F}} = X_{[-1]}\rho\Xi_{\mathscr{F}} \cdot f\Xi_{\mathscr{F}} = (X_{[-1]}\rho \cdot f)\Xi_{\mathscr{F}} = (r \cdot s)\Xi_{\mathscr{F}} \\ &= r\Xi_{\mathscr{F}} \cdot s\Xi_{\mathscr{F}}\,.\end{aligned}$$

Since moreover $B\Xi_{\mathscr{F}}$ is bijective in $\nabla(\mathscr{F})$ by lemma 4.2.5, we conclude that $\underline{\underline{f\Xi_{\mathscr{F}}}} = 0$ in $\underline{\underline{\nabla}}(\mathscr{F})$. □

4.2.31 Definition. Let $\underline{\underline{\Xi}}_{\mathscr{F}} \colon \underline{\underline{\mathrm{FO}}}(\mathscr{F}) \to \underline{\underline{\nabla}}(\mathscr{F})$ denote the unique additive functor such that $\mathfrak{Q}_{\mathrm{FO},\mathscr{F}} \cdot \underline{\underline{\Xi}}_{\mathscr{F}} = \Xi_{\mathscr{F}} \cdot \mathfrak{Q}_{\nabla,\mathscr{F}}$, cf. lemma 4.2.30.

$$\begin{tikzcd}
\mathrm{FO}(\mathscr{F}) \arrow[r, "\Xi_{\mathscr{F}}"] \arrow[d, "\mathfrak{Q}_{\mathrm{FO},\mathscr{F}}"'] & \nabla(\mathscr{F}) \arrow[d, "\mathfrak{Q}_{\nabla,\mathscr{F}}"] \\
\underline{\underline{\mathrm{FO}}}(\mathscr{F}) \arrow[r, "\underline{\underline{\Xi}}_{\mathscr{F}}"] & \underline{\underline{\nabla}}(\mathscr{F})
\end{tikzcd}$$

$\Diamond$

4.2.32 Definition. Suppose given a full subcategory $\mathscr{S} \subseteq \mathscr{F}$.
Let $\underline{\underline{\Xi}}_{\mathscr{S},\mathscr{F}} = \underline{\underline{\Xi}}_{\mathscr{F}}|^{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{S}}(\mathscr{F})} \colon \underline{\underline{\mathrm{FO}}}_{\mathscr{S}}(\mathscr{F}) \to \underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})$, cf. remark 4.2.13 and definition 4.2.15.
If $\mathscr{S}$ is a strictly full additive subcategory of $\mathscr{F}$, then $\underline{\underline{\Xi}}_{\mathscr{S},\mathscr{F}}$ is additive, cf. remarks 1.2.5.(a), 4.2.14 and corollary 3.3.37. $\Diamond$

4.2.33 Lemma.

(a) Suppose given $m \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{FO}^{\lceil m}(\mathscr{F}))$. Then we have $X\Xi_{\mathscr{F}} \in \mathrm{Ob}(\nabla^{\lceil m}(\mathscr{F}))$.

(b) Suppose given $m \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{FO}^{m\rceil}(\mathscr{F}))$. Then we have $X\Xi_{\mathscr{F}} \in \mathrm{Ob}(\nabla^{m\rceil}(\mathscr{F}))$.

(c) Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}(\mathscr{F}))$. Then we have $X\Xi_{\mathscr{F}} \in \mathrm{Ob}(\nabla^{\mathrm{b}}(\mathscr{F}))$.

Cf. definitions 3.3.29 and 3.4.20. $\Diamond$

Proof. Ad (a). Suppose given $\ell \in \mathbf{Z}_{>m}$.
We have the pure short exact sequence $X_{\ell/\ell-1} \xrightarrow{X_{\ell/\ell-1|}} X_{\ell|} \xrightarrow{x_{\ell|}} X_{\ell+1|}$ in $\mathscr{F}$.
We have $X_{\ell|}\, , X_{\ell+1|} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$ since $X \in \mathrm{Ob}(\mathrm{FO}^{\lceil m}(\mathscr{F}))$. So $(X\Xi_{\mathscr{F}})_{\ell/\ell-1} = X_{\ell/\ell-1} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$.
We conclude that $X\Xi_{\mathscr{F}} \in \mathrm{Ob}(\nabla^{\lceil m}(\mathscr{F}))$.

Ad (b). Suppose given $\ell \in \mathbf{Z}_{<m}$.
We have the pure short exact sequence $X_{|\ell-1} \xrightarrow{x_{|\ell-1}} X_{|\ell} \xrightarrow{X_{|\ell/\ell-1}} X_{\ell/\ell-1}$ in $\mathscr{F}$.
We have $X_{|\ell}\, , X_{|\ell-1} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$ since $X \in \mathrm{Ob}(\mathrm{FO}^{m\rceil}(\mathscr{F}))$. So $(X\Xi_{\mathscr{F}})_{\ell/\ell-1} = X_{\ell/\ell-1} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$.
We conclude that $X\Xi_{\mathscr{F}} \in \mathrm{Ob}(\nabla^{m\rceil}(\mathscr{F}))$.

Ad (c). This follows from (a) and (b). $\square$

4.2.34 Definition. Let $\Xi^{\mathrm{b}}_{\mathscr{F}} = \Xi_{\mathscr{F}}|^{\nabla^{\mathrm{b}}(\mathscr{F})}_{\mathrm{FO}^{\mathrm{b}}(\mathscr{F})} \colon \mathrm{FO}^{\mathrm{b}}(\mathscr{F}) \to \nabla^{\mathrm{b}}(\mathscr{F})$ and let
$\underline{\underline{\Xi}}^{\mathrm{b}}_{\mathscr{F}} = \underline{\underline{\Xi}}_{\mathscr{F}}|^{\underline{\underline{\nabla}}^{\mathrm{b}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}(\mathscr{F})} \colon \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}(\mathscr{F}) \to \underline{\underline{\nabla}}^{\mathrm{b}}(\mathscr{F})$. Cf. lemma 4.2.33.(c). $\Diamond$

4.2.35 Definition. Let $\Xi^{\nabla}_{\mathscr{F}} = \Xi_{\mathscr{F}}|_{\mathrm{FO}^{\nabla}(\mathscr{F})} \colon \mathrm{FO}^{\nabla}(\mathscr{F}) \to \nabla(\mathscr{F})$ and let
$\underline{\underline{\Xi}}^{\nabla}_{\mathscr{F}} = \underline{\underline{\Xi}}_{\mathscr{F}}|_{\underline{\underline{\mathrm{FO}}}^{\nabla}(\mathscr{F})} \colon \underline{\underline{\mathrm{FO}}}^{\nabla}(\mathscr{F}) \to \underline{\underline{\nabla}}(\mathscr{F})$. Cf. definition 3.4.34. $\Diamond$

4.2.36 Definition. Let $\Xi^{\lim}_{\mathscr{F}} = \Xi_{\mathscr{F}}|_{\mathrm{FO}^{\lim}(\mathscr{F})} \colon \mathrm{FO}^{\lim}(\mathscr{F}) \to \nabla(\mathscr{F})$ and let
$\underline{\underline{\Xi}}^{\lim}_{\mathscr{F}} = \underline{\underline{\Xi}}_{\mathscr{F}}|_{\underline{\underline{\mathrm{FO}}}^{\lim}(\mathscr{F})} \colon \underline{\underline{\mathrm{FO}}}^{\lim}(\mathscr{F}) \to \underline{\underline{\nabla}}(\mathscr{F})$. Cf. definition 3.4.34. $\Diamond$

4.2.37 Definition. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$.
We write $X_{\ell/k} = 0_{\mathscr{F}}$, $X_{|\ell/k} = 0 \colon X_{|\ell} \to X_{\ell/k}$ and $X_{\ell/k|} = 0 \colon X_{\ell/k} \to X_{k+1|}$ for $\ell < k$ in $\mathbf{Z}$.
Note that $X_{\ell/k} = (X\Xi_{\mathscr{F}})_{\ell/k}$ for $\ell < k$ in $\mathbf{Z}$. Cf. definition 3.3.48.

Also note that, for $\ell < k$, we have $x^\omega_{|\ell} \cdot x^\omega_{k+1|} = X_{|\ell\to|k} \cdot x^\omega_{|k} \cdot x^\omega_{k+1|} = 0 = X_{|\ell/k} \cdot X_{\ell/k|}$.

Suppose given $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$. We write $f_{\ell/k} = 0\colon X_{\ell/k} \to Y_{\ell/k}$ for $\ell < k$ in $\mathbf{Z}$. Note that $f_{\ell/k} = (f\Xi_{\mathscr{F}})_{\ell/k}$ for $\ell < k$ in $\mathbf{Z}$. $\diamondsuit$

4.2.38 Lemma/Definition. Suppose given $\ell \in \mathbf{Z}$. We define the transformation $\chi_{\ell,\mathrm{CF},\mathscr{F}}\colon \Xi_{\mathscr{F}} \cdot \Psi_{\ell,\mathrm{CF},\mathscr{F}} \to \mathrm{P}_{\mathrm{CF},\mathscr{F}}$ as follows, cf. definitions 3.3.49 and 3.4.5. For $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ and $k \in \mathbf{Z}$, let $(X\chi_{\ell,\mathrm{CF},\mathscr{F}})_k = X_{\ell/k-1|}\colon X_{\ell/k-1} \to X_{k|}$, cf. definitions 4.2.2 and 4.2.37. Moreover, $X\chi_{\ell,\mathrm{CF},\mathscr{F}}$ is an ℓ-pure monomorphism in $\mathrm{CF}(\mathscr{F})$ for $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$. $\diamondsuit$

Proof. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ and $k \in \mathbf{Z}$.
We have $(X\chi_{\ell,\mathrm{CF},\mathscr{F}})_k \cdot (X\mathrm{P}_{\mathrm{CF},\mathscr{F}})_{k\to k+1} = (X\Xi_{\mathscr{F}}\Psi_{\ell,\mathrm{CF},\mathscr{F}})_{k\to k+1} \cdot (X\chi_{\ell,\mathrm{CF},\mathscr{F}})_{k+1}$ since, for $k \leq \ell$, we have

$$\begin{aligned} X_{|\ell/k-1} \cdot (X\chi_{\ell,\mathrm{CF},\mathscr{F}})_k \cdot (X\mathrm{P}_{\mathrm{CF},\mathscr{F}})_{k\to k+1} &= X_{|\ell/k-1} \cdot X_{\ell/k-1|} \cdot x_{k|} = x^\omega_{|\ell} \cdot x^\omega_{k|} \cdot x_{k|} = x^\omega_{|\ell} \cdot x^\omega_{k+1|} \\ &= X_{|\ell/k} \cdot X_{\ell/k|} = X_{|\ell/k-1} \cdot (X\Xi_{\mathscr{F}})_{\ell/k-1\to\ell/k} \cdot X_{\ell/k|} \\ &= X_{|\ell/k-1} \cdot (X\Xi_{\mathscr{F}}\Psi_{\ell,\mathrm{CF},\mathscr{F}})_{k\to k+1} \cdot (X\chi_{\ell,\mathrm{CF},\mathscr{F}})_{k+1} \end{aligned}$$

and since $X_{|\ell/k-1}$ is a pure epimorphism in $\mathscr{F}$.

Suppose given $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$. We have $X\chi_{\ell,\mathrm{CF},\mathscr{F}} \cdot f\mathrm{P}_{\mathrm{CF},\mathscr{F}} = f\Xi_{\mathscr{F}}\Psi_{\ell,\mathrm{CF},\mathscr{F}} \cdot Y\chi_{\ell,\mathrm{CF},\mathscr{F}}$ since, for $k \in \mathbf{Z}$, we have

$$\begin{aligned} X_{|\ell/k-1} \cdot (X\chi_{\ell,\mathrm{CF},\mathscr{F}})_k \cdot (f\mathrm{P}_{\mathrm{CF},\mathscr{F}})_k &= X_{|\ell/k-1} \cdot X_{\ell/k-1|} \cdot f_{k|} = x^\omega_{|\ell} \cdot x^\omega_{k|} \cdot f_{k|} = x^\omega_{|\ell} \cdot f_\omega \cdot y^\omega_{k|} \\ &= f_{|\ell} \cdot y^\omega_{|\ell} \cdot y^\omega_{k|} = f_{|\ell} \cdot Y_{|\ell/k-1} \cdot Y_{\ell/k-1|} = X_{|\ell/k-1} \cdot f_{\ell/k-1} \cdot Y_{\ell/k-1|} \\ &= X_{|\ell/k-1} \cdot (f\Xi_{\mathscr{F}}\Psi_{\ell,\mathrm{CF},\mathscr{F}})_k \cdot (Y\chi_{\ell,\mathrm{CF},\mathscr{F}})_k \end{aligned}$$

and since $X_{|\ell/k-1}$ is a pure epimorphism in $\mathscr{F}$.

Suppose given $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$. The morphism $X\chi_{\ell,\mathrm{CF},\mathscr{F}}$ is an ℓ-pure monomorphism in $\mathrm{CF}(\mathscr{F})$ since $\big((X\chi_{\ell,\mathrm{CF},\mathscr{F}})_k\,,\,(X\mathrm{P}_{\mathrm{CF},\mathscr{F}})_{k\to\ell+1}\big) = \big(X_{\ell/k-1|}\,,\,X_{k|\to\ell+1|}\big)$ is a pure short exact sequence for $k \in \mathbf{Z}_{\leq \ell+1}$. $\square$

4.2.39 Lemma. Suppose given $\ell \in \mathbf{Z}$. We have $(1_{\Xi_{\mathscr{F}}} \star \psi_{\ell,\ell+1,\mathrm{CF},\mathscr{F}}) \cdot \chi_{\ell+1,\mathrm{CF},\mathscr{F}} = \chi_{\ell,\mathrm{CF},\mathscr{F}}$, cf. definition 3.3.53. $\diamondsuit$

Proof. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$. We have $X\Xi_{\mathscr{F}}\psi_{\ell,\ell+1,\mathrm{CF},\mathscr{F}} \cdot X\chi_{\ell+1,\mathrm{CF},\mathscr{F}} = X\chi_{\ell,\mathrm{CF},\mathscr{F}}$ since, for $k \in \mathbf{Z}_{\leq \ell+1}$,

$$\begin{aligned} X_{|\ell/k-1} \cdot (X\Xi_{\mathscr{F}}\psi_{\ell,\ell+1,\mathrm{CF},\mathscr{F}})_k \cdot (X\chi_{\ell+1,\mathrm{CF},\mathscr{F}})_k &= X_{|\ell/k-1} \cdot (X\Xi_{\mathscr{F}})_{\ell/k-1\to\ell+1/k-1} \cdot X_{\ell+1/k-1|} \\ &= x_{|\ell} \cdot X_{|\ell+1/k-1} \cdot X_{\ell+1/k-1|} \\ &= x_{|\ell} \cdot x^\omega_{|\ell+1} \cdot x^\omega_{k|} = x^\omega_{|\ell} \cdot x^\omega_{k|} \\ &= X_{|\ell/k-1} \cdot X_{\ell/k-1|} = X_{|\ell/k-1} \cdot (X\chi_{\ell,\mathrm{CF},\mathscr{F}})_k \end{aligned}$$

and since $X_{|\ell/k-1}$ is a pure epimorphism. $\square$

4.2.40 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\lim}(\mathscr{F}))$ and $\ell \in \mathbf{Z}$. Then $(X_{|\ell}\,,\,(X_{|\ell/k-1})_{k\in\mathbf{Z}})$ is a limit for $X\Xi_{\mathscr{F}}\Psi_{\ell,\mathrm{CF},\mathscr{F}}$. $\diamondsuit$

Proof. We want to apply lemma 3.2.39.(c). The morphism $X\chi_{\ell,\mathrm{CF},\mathscr{F}} : X\Xi_{\mathscr{F}}\Psi_{\ell,\mathrm{CF},\mathscr{F}} \to X\mathrm{P}_{\mathrm{CF},\mathscr{F}}$ is an ℓ-pure monomorphism in $\mathrm{CF}(\mathscr{F})$, cf. definition 4.2.38. We have the limit $(X_\omega\,,\,(x^\omega_{k|})_{k\in\mathbf{Z}})$ for $X\mathrm{P}_{\mathrm{CF},\mathscr{F}}$, cf. definition 3.4.34. Note that $(x^\omega_{|\ell}\,,\,x^\omega_{\ell+1|})$ is a pure short exact sequence. For $k \in \mathbf{Z}$, we have $X_{|\ell/k-1}\cdot(X\chi_{\ell,\mathrm{CF},\mathscr{F}})_k = X_{|\ell/k-1}\cdot X_{\ell/k-1|} = x^\omega_{|\ell}\cdot x^\omega_{k|}$.
We conclude that $(X_{|\ell}\,,\,(X_{|\ell/k-1})_{k\in\mathbf{Z}})$ is a limit for $X\Xi_{\mathscr{F}}\Psi_{\ell,\mathrm{CF},\mathscr{F}}$. $\square$

4.2.41 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{colim}}(\mathscr{F}))$ and $k \in \mathbf{Z}$. Then $(X_{k|}\,,\,(X_{\ell/k-1|})_{\ell\in\mathbf{Z}})$ is a colimit for $X\Xi_{\mathscr{F}}\Psi_{k,\mathrm{F},\mathscr{F}}$. $\diamondsuit$

Proof. This is dual to the previous lemma 4.2.40. $\square$

4.2.42 Lemma. Suppose given $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$ with $Y \in \mathrm{Ob}(\mathrm{FO}^{\lim}(\mathscr{F}))$. For $\ell \in \mathbf{Z}$, we have $f_{|\ell} = f\Xi_{\mathscr{F}}\Psi_{\ell,\mathrm{CF},\mathscr{F}}\rceil^{(Y_{|\ell},(Y_{|\ell/k-1})_{k\in\mathbf{Z}})}_{(X_{|\ell},(X_{|\ell/k-1})_{k\in\mathbf{Z}})}$, cf. lemma 4.2.40 and definition 3.2.22. $\diamondsuit$

Proof. Suppose given $k, \ell \in \mathbf{Z}$. We have $f_{|\ell}\cdot Y_{|\ell/k-1} = X_{|\ell/k-1}\cdot f_{\ell/k-1} = X_{|\ell/k-1}\cdot(f\Xi_{\mathscr{F}}\Psi_{\ell,\mathrm{CF},\mathscr{F}})_k$. $\square$

4.2.43 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{colim}}(\mathscr{F}))$, $Y \in \mathrm{Ob}(\mathrm{FO}^{\lim}(\mathscr{F}))$ and $X\Xi_{\mathscr{F}} \xrightarrow{f} Y\Xi_{\mathscr{F}}$ in $\nabla(\mathscr{F})$. There exists a unique morphism $X \xrightarrow{g} Y$ in $\mathrm{FO}(\mathscr{F})$ such that $g\Xi_{\mathscr{F}} = f$. For $\ell \in \mathbf{Z}$, we have $g_{|\ell} = f\Psi_{\ell,\mathrm{CF},\mathscr{F}}\rceil^{(Y_{|\ell},(Y_{|\ell/k-1})_{k\in\mathbf{Z}})}_{(X_{|\ell},(X_{|\ell/k-1})_{k\in\mathbf{Z}})}$.
Moreover, we have $g_\omega = g\mathrm{P}_{\mathrm{F},\mathscr{F}}\rceil^{(Y_\omega,(y^\omega_{|k})_{k\in\mathbf{Z}})}_{(X_\omega,(x^\omega_{|k})_{k\in\mathbf{Z}})}$. $\diamondsuit$

Proof. We verify that we obtain a morphism $X\mathrm{P}_{\mathrm{F},\mathscr{F}} \xrightarrow{h} Y\mathrm{P}_{\mathrm{F},\mathscr{F}}$ in $\mathrm{F}(\mathscr{F})$ by setting $h_\ell = f\Psi_{\ell,\mathrm{CF},\mathscr{F}}\rceil^{(Y_{|\ell},(Y_{|\ell/k-1})_{k\in\mathbf{Z}})}_{(X_{|\ell},(X_{|\ell/k-1})_{k\in\mathbf{Z}})}$ for $\ell \in \mathbf{Z}$.
For $\ell \in \mathbf{Z}$, we have $h_\ell \cdot y_{|\ell} = x_{|\ell}\cdot h_{\ell+1}$ since

$$
\begin{aligned}
h_\ell \cdot y_{|\ell}\cdot Y_{|\ell+1/j} &= f\Psi_{\ell,\mathrm{CF},\mathscr{F}}\rceil^{(Y_{|\ell},(Y_{|\ell/k-1})_{k\in\mathbf{Z}})}_{(X_{|\ell},(X_{|\ell/k-1})_{k\in\mathbf{Z}})}\cdot Y_{|\ell/j}\cdot(Y\Xi_{\mathscr{F}})_{\ell/j\to\ell+1/j} \\
&= X_{|\ell/j}\cdot f_{\ell/j}\cdot(Y\Xi_{\mathscr{F}})_{\ell/j\to\ell+1/j} = X_{|\ell/j}\cdot(X\Xi_{\mathscr{F}})_{\ell/j\to\ell+1/j}\cdot f_{\ell+1/j} \\
&= x_{|\ell}\cdot X_{|\ell+1/j}\cdot f_{\ell+1/j} = x_{|\ell}\cdot f\Psi_{\ell+1,\mathrm{CF},\mathscr{F}}\rceil^{(Y_{|\ell+1},(Y_{|\ell+1/k-1})_{k\in\mathbf{Z}})}_{(X_{|\ell+1},(X_{|\ell+1/k-1})_{k\in\mathbf{Z}})}\cdot Y_{|\ell+1/j} \\
&= x_{|\ell}\cdot h_{\ell+1}\cdot Y_{|\ell+1/j}
\end{aligned}
$$

for $j \in \mathbf{Z}_{\leq\ell}$ and since $(Y_{|\ell+1}\,,\,(Y_{|\ell+1/k-1})_{k\in\mathbf{Z}})$ is a limit for $X\Xi_{\mathscr{F}}\Psi_{\ell+1,\mathrm{CF},\mathscr{F}}$.
Lemma 3.4.52 yields a unique morphism $X \xrightarrow{g} Y$ in $\mathrm{FO}(\mathscr{F})$ such that $g\mathrm{P}_{\mathrm{F},\mathscr{F}} = h$. Moreover, we have $g_\omega = g\mathrm{P}_{\mathrm{F},\mathscr{F}}\rceil^{(Y_\omega,(y^\omega_{|k})_{k\in\mathbf{Z}})}_{(X_\omega,(x^\omega_{|k})_{k\in\mathbf{Z}})}$.
For $\ell/j \in \mathrm{V}$, we have $(g\Xi_{\mathscr{F}})_{\ell/j} = f_{\ell/j}$ since

$$
X_{|\ell/j}\cdot(g\Xi_{\mathscr{F}})_{\ell/j} = X_{|\ell/j}\cdot g_{\ell/j} = g_{|\ell}\cdot Y_{|\ell/j} = f\Psi_{\ell,\mathrm{CF},\mathscr{F}}\rceil^{(Y_{|\ell},(Y_{|\ell/k-1})_{k\in\mathbf{Z}})}_{(X_{|\ell},(X_{|\ell/k-1})_{k\in\mathbf{Z}})}\cdot Y_{|\ell/j} = X_{|\ell/j}\cdot f_{\ell/j}
$$

and since $X_{|\ell/j}$ is a pure epimorphism. Thus $g\Xi_{\mathscr{F}} = f$.

Suppose given $X \xrightarrow{e} Y$ in $\mathrm{FO}(\mathscr{F})$ such that $e\Xi_{\mathscr{F}} = f$.
By lemma 4.2.42, we have $e_{|\ell} = f\Psi_{\ell,\mathrm{CF},\mathscr{F}}\rceil^{(Y_{|\ell},(Y_{|\ell/k-1})_{k\in\mathbf{Z}})}_{(X_{|\ell},(X_{|\ell/k-1})_{k\in\mathbf{Z}})} = h_\ell$ for $\ell \in \mathbf{Z}$. So $e\mathrm{P}_{\mathrm{F},\mathscr{F}} = h$. We conclude that $e = g$. $\square$

4.2.44 Corollary. The restrictions $\Xi^{\mathrm{b}}_{\mathscr{F}} = \Xi_{\mathscr{F}}|^{\nabla^{\mathrm{b}}(\mathscr{F})}_{\mathrm{FO}^{\mathrm{b}}(\mathscr{F})} \colon \mathrm{FO}^{\mathrm{b}}(\mathscr{F}) \to \nabla^{\mathrm{b}}(\mathscr{F})$ and $\Xi^{\nabla}_{\mathscr{F}} = \Xi_{\mathscr{F}}|_{\mathrm{FO}^{\nabla}(\mathscr{F})} \colon \mathrm{FO}^{\nabla}(\mathscr{F}) \to \nabla(\mathscr{F})$ are full and faithful, cf. remark 3.4.38. $\diamondsuit$

4.2.45 Lemma. Suppose given $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$ such that $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{colim}}(\mathscr{F}))$, $X_{[-1]}\mathbf{B} \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{colim}}(\mathscr{F}))$ and such that $Y \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{lim}}(\mathscr{F}))$. If $\underline{\underline{f\Xi_{\mathscr{F}}}} = 0$ in $\underline{\underline{\nabla}}(\mathscr{F})$, then $\underline{\underline{f}} = 0$ in $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$. $\diamondsuit$

Proof. Suppose that $\underline{\underline{f\Xi_{\mathscr{F}}}} = 0$ in $\underline{\underline{\nabla}}(\mathscr{F})$. We may choose $(X\Xi_{\mathscr{F}})_{[-1]}\mathbf{B} \xrightarrow{g} Y\Xi_{\mathscr{F}}$ in $\nabla(\mathscr{F})$ such that $(X\Xi_{\mathscr{F}})_{[-1]}\rho \cdot f\Xi_{\mathscr{F}} = (X\Xi_{\mathscr{F}})_{[-1]}\boldsymbol{\iota} \cdot g$, cf. remarks 3.3.16 and 3.3.10.
Note that $(X\Xi_{\mathscr{F}})_{[-1]}\mathbf{B} = X_{[-1]}\Xi_{\mathscr{F}}\mathbf{B}$ by lemma 4.2.21.

Consider the morphism $X_{[-1]}\mathbf{B}\Xi_{\mathscr{F}} \xrightarrow{X_{[-1]}\xi_{\mathrm{B},\mathscr{F}}\cdot g} Y\Xi_{\mathscr{F}}$ in $\nabla(\mathscr{F})$, cf. definition 4.2.23. By lemma 4.2.43, we may choose $X_{[-1]}\mathbf{B} \xrightarrow{h} Y$ in $\mathrm{FO}(\mathscr{F})$ such that $h\Xi_{\mathscr{F}} = X_{[-1]}\xi_{\mathrm{B},\mathscr{F}} \cdot g$.

We show that $X_{[-1]}\rho \cdot f = X_{[-1]}\boldsymbol{\iota} \cdot h$ using lemmata 4.2.21, 4.2.27, 4.2.43 and remark 4.2.24:

$$\begin{aligned}(X_{[-1]}\rho \cdot f)\Xi_{\mathscr{F}} &= (X\Xi_{\mathscr{F}})_{[-1]}\rho \cdot f\Xi_{\mathscr{F}} = X_{[-1]}\xi_{1_{\mathscr{F}},\mathscr{F}} \cdot (X\Xi_{\mathscr{F}})_{[-1]}\boldsymbol{\iota} \cdot g = X_{[-1]}\boldsymbol{\iota}\Xi_{\mathscr{F}} \cdot X_{[-1]}\xi_{\mathrm{B},\mathscr{F}} \cdot g \\ &= X_{[-1]}\boldsymbol{\iota}\Xi_{\mathscr{F}} \cdot h\Xi_{\mathscr{F}} = (X_{[-1]}\boldsymbol{\iota} \cdot h)\Xi_{\mathscr{F}}\,.\end{aligned}$$

We conclude that $\underline{\underline{f}} = 0$ in $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$, cf. remark 3.4.14. $\square$

4.2.46 Corollary. Suppose given $X \xrightarrow{f} Y$ in $\mathrm{FO}^{\mathrm{b}}(\mathscr{F})$. If $\underline{\underline{f\Xi_{\mathscr{F}}}} = 0$ in $\underline{\underline{\nabla}}(\mathscr{F})$, then $\underline{\underline{f}} = 0$ in $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$, cf. lemma 3.4.29 and remark 3.4.38. $\diamondsuit$

4.2.47 Corollary. Suppose that $\mathscr{F}$ has countable coproducts of bijectives. Suppose given $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$ such that $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{colim}}(\mathscr{F}))$ and such that $Y \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{lim}}(\mathscr{F}))$. If $\underline{\underline{f\Xi_{\mathscr{F}}}} = 0$ in $\underline{\underline{\nabla}}(\mathscr{F})$, then $\underline{\underline{f}} = 0$ in $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$. $\diamondsuit$

Proof. We have $X_{[-1]}\mathbf{B} \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{colim}}(\mathscr{F}))$ by lemmata 3.4.40 and 3.4.51. So the result follows from lemma 4.2.45. $\square$

4.2.48 Definition. Suppose given $X \in \mathrm{Ob}(\nabla^{\mathrm{b}}(\mathscr{F}))$. We obtain $X\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}} \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}(\mathscr{F}))$ as follows. We will use lemmata 3.2.18 and 3.2.19 repeatedly.

For $\ell \in \mathbf{Z}$, we choose a limit $(X_{\ell/-\infty}\,, (x_{\ell/-\infty,k-1})_{k\in\mathbf{Z}})$ for $X\Psi_{\ell,\mathrm{CF},\mathscr{F}}$, cf. corollary 3.3.52.(a).
Lemma 3.3.57 yields the filtration $X\Psi_{-\infty,\mathrm{F},\mathscr{F}} \in \mathrm{Ob}(\mathrm{F}^{\mathrm{b}}(\mathscr{F}))$ and morphisms $X\Psi_{-\infty,\mathrm{F},\mathscr{F}} \xrightarrow{X\psi_{-\infty,k,\mathrm{F},\mathscr{F}}} X\Psi_{k,\mathrm{F},\mathscr{F}}$ in $\mathrm{F}(\mathscr{F})$ for $k \in \mathbf{Z}$, cf. lemma 3.3.57.(f).

For $k \in \mathbf{Z}$, we choose a colimit $(X_{\infty/k-1}\,, (x_{\ell,\infty/k-1})_{\ell\in\mathbf{Z}})$ for $X\Psi_{k,\mathrm{F},\mathscr{F}}$, cf. corollary 3.3.52.(b).
Lemma 3.3.58 yields the cofiltration $X\Psi_{\infty,\mathrm{CF},\mathscr{F}} \in \mathrm{Ob}(\mathrm{CF}^{\mathrm{b}}(\mathscr{F}))$, cf. lemma 3.3.58.(e).

Choose a colimit $(X_{\infty/-\infty}\,,\,(x_{\ell,\infty/-\infty})_{\ell\in\mathbf{Z}})$ for $X\Psi_{-\infty,\mathrm{F},\mathscr{F}} \in \mathrm{Ob}(\mathrm{F}^{\mathrm{b}}(\mathscr{F}))$.
We define $X\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}} \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}(\mathscr{F}))$ by setting

- $X\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}}\mathrm{P}_{\mathrm{F},\mathscr{F}} = X\Psi_{-\infty,\mathrm{F},\mathscr{F}}$,
- $X\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}}\mathrm{P}_{\mathrm{CF},\mathscr{F}} = X\Psi_{\infty,\mathrm{CF},\mathscr{F}}$,
- $(X\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}})_{\omega} = X_{\infty/-\infty}$,
- $(X\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}})_{|k\to\omega} = x_{k,\infty/-\infty}$ for $k \in \mathbf{Z}$ and
- $(X\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}})_{\omega\to k|} = X\psi_{-\infty,k,\mathrm{F},\mathscr{F}}\rceil$ for $k \in \mathbf{Z}$.

Note that, for $k \in \mathbf{Z}$, we have $x_{k,\infty/-\infty} = (X\Psi_{-\infty,\mathrm{F},\mathscr{F}})_{k\to k+1} \cdot x_{k+1,\infty/-\infty}$ and
$X\psi_{-\infty,k,\mathrm{F},\mathscr{F}}\rceil \cdot (X\Psi_{\infty,\mathrm{CF},\mathscr{F}})_{k\to k+1} = X\psi_{-\infty,k+1,\mathrm{F},\mathscr{F}}\rceil$ by lemma 3.3.59.(c). Moreover, for $k \in \mathbf{Z}$, the sequence $X_{k/-\infty} \xrightarrow{x_{k,\infty/-\infty}} X_{\infty/-\infty} \xrightarrow{X\psi_{-\infty,k+1,\mathrm{F},\mathscr{F}}\rceil} X_{\infty/k}$ is pure short exact in $\mathscr{F}$ by lemma 3.3.59.(a).
Additionally, we obtain the isomorphism $X\sigma^{\mathrm{b}}_{\mathscr{F}} = 1_{X\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}}}\theta^{X\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}}\mathrm{C}}_{(x_{\ell/-\infty,k})_{\ell/k\in\mathrm{V}}} : X \to X\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}}\Xi^{\mathrm{b}}_{\mathscr{F}}$ in $\nabla(\mathscr{F})$, cf. lemmata 3.3.57.(b) and 4.2.9.(a,b). $\diamondsuit$

4.2.49 Definition. Suppose that $\mathscr{F}$ has epilimits and monocolimits.
Suppose given $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$. We obtain $X\mathrm{Lim}_{\mathscr{F}} \in \mathrm{Ob}(\mathrm{FO}^{\nabla}(\mathscr{F}))$ as follows.
For $X \in \mathrm{Ob}(\nabla^{\mathrm{b}}(\mathscr{F}))$, we choose $X\mathrm{Lim}_{\mathscr{F}} = X\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}} \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}(\mathscr{F}))$, cf. definition 4.2.48.
Now suppose that $X \notin \mathrm{Ob}(\nabla^{\mathrm{b}}(\mathscr{F}))$.
For $\ell \in \mathbf{Z}$, we choose a limit $(X_{\ell/-\infty}\,,\,(x_{\ell/-\infty,k-1})_{k\in\mathbf{Z}})$ for $X\Psi_{\ell,\mathrm{CF},\mathscr{F}}$.
Lemma 3.3.57 yields the filtration $X\Psi_{-\infty,\mathrm{F},\mathscr{F}} \in \mathrm{Ob}(\mathrm{F}(\mathscr{F}))$ and morphisms
$X\Psi_{-\infty,\mathrm{F},\mathscr{F}} \xrightarrow{X\psi_{-\infty,k,\mathrm{F},\mathscr{F}}} X\Psi_{k,\mathrm{F},\mathscr{F}}$ in $\mathrm{F}(\mathscr{F})$ for $k \in \mathbf{Z}$.
For $k \in \mathbf{Z}$, we choose a colimit $(X_{\infty/k-1}\,,\,(x_{\ell,\infty/k-1})_{\ell\in\mathbf{Z}})$ for $X\Psi_{k,\mathrm{F},\mathscr{F}}$.
Lemma 3.3.58 yields the cofiltration $X\Psi_{\infty,\mathrm{CF},\mathscr{F}} \in \mathrm{Ob}(\mathrm{CF}(\mathscr{F}))$.
Choose a colimit $(X_{\infty/-\infty}\,,\,(x_{\ell,\infty/-\infty})_{\ell\in\mathbf{Z}})$ for $X\Psi_{-\infty,\mathrm{F},\mathscr{F}} \in \mathrm{Ob}(\mathrm{F}(\mathscr{F}))$.
We define $X\mathrm{Lim}_{\mathscr{F}} \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ by setting

- $X\mathrm{Lim}_{\mathscr{F}}\mathrm{P}_{\mathrm{F},\mathscr{F}} = X\Psi_{-\infty,\mathrm{F},\mathscr{F}}$,
- $X\mathrm{Lim}_{\mathscr{F}}\mathrm{P}_{\mathrm{CF},\mathscr{F}} = X\Psi_{\infty,\mathrm{CF},\mathscr{F}}$,
- $(X\mathrm{Lim}_{\mathscr{F}})_{\omega} = X_{\infty/-\infty}$,
- $(X\mathrm{Lim}_{\mathscr{F}})_{|k\to\omega} = x_{k,\infty/-\infty}$ for $k \in \mathbf{Z}$ and
- $(X\mathrm{Lim}_{\mathscr{F}})_{\omega\to k|} = X\psi_{-\infty,k,\mathrm{F},\mathscr{F}}\rceil$ for $k \in \mathbf{Z}$.

Note that, for $k \in \mathbf{Z}$, we have $x_{k,\infty/-\infty} = (X\Psi_{-\infty,\mathrm{F},\mathscr{F}})_{k\to k+1} \cdot x_{k+1,\infty/-\infty}$ and
$X\psi_{-\infty,k,\mathrm{F},\mathscr{F}}\rceil \cdot (X\Psi_{\infty,\mathrm{CF},\mathscr{F}})_{k\to k+1} = X\psi_{-\infty,k+1,\mathrm{F},\mathscr{F}}\rceil$ by lemma 3.3.59.(c). Moreover, for $k \in \mathbf{Z}$,

the sequence $X_{k/-\infty} \xrightarrow{x_{k,\infty/-\infty}} X_{\infty/-\infty} \xrightarrow{X\psi_{-\infty,k+1,\mathrm{F},\mathscr{F}}\rceil} X_{\infty/k}$ is pure short exact in $\mathscr{F}$ by lemma 3.3.59.(a).

Also note that we have $X\mathrm{Lim}_{\mathscr{F}} \in \mathrm{Ob}(\mathrm{FO}^{\nabla}(\mathscr{F}))$ by lemma 3.3.59.(c).

Additionally, we obtain the isomorphism $X\sigma_{\mathscr{F}} = 1_{X\mathrm{Lim}_{\mathscr{F}}}\theta^{X\mathrm{Lim}_{\mathscr{F}}\mathrm{C}}_{(x_{\ell/-\infty,k})_{\ell/k\in\mathrm{V}}} : X \to X\mathrm{Lim}_{\mathscr{F}}\Xi_{\mathscr{F}}$ in $\nabla(\mathscr{F})$, cf. lemmata 3.3.57.(b) and 4.2.9.(a,b). $\diamondsuit$

4.2.50 Definition. We want to construct quasi-inverses $\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}}$ and $\underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{F}}$ of the functors $\Xi^{\mathrm{b}}_{\mathscr{F}} = \Xi_{\mathscr{F}}|^{\nabla^{\mathrm{b}}(\mathscr{F})}_{\mathrm{FO}^{\mathrm{b}}(\mathscr{F})} : \mathrm{FO}^{\mathrm{b}}(\mathscr{F}) \to \nabla^{\mathrm{b}}(\mathscr{F})$ and $\underline{\underline{\Xi}}^{\mathrm{b}}_{\mathscr{F}} = \underline{\underline{\Xi}}_{\mathscr{F}}|^{\underline{\underline{\nabla}}^{\mathrm{b}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}(\mathscr{F})} : \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}(\mathscr{F}) \to \underline{\underline{\nabla}}^{\mathrm{b}}(\mathscr{F})$ following the steps of lemma 1.2.17. Cf. definition 4.2.34. We call these quasi-inverses *bounded limit functors* since the are obtained by taking limits and colimits.

$$\begin{tikzcd}
\mathrm{FO}^{\mathrm{b}}(\mathscr{F}) \arrow[r, "\Xi^{\mathrm{b}}_{\mathscr{F}}"] \arrow[d, "\mathfrak{Q}_{\mathrm{FO}^{\mathrm{b}},\mathscr{F}}"'] & \nabla^{\mathrm{b}}(\mathscr{F}) \arrow[d, "\mathfrak{Q}_{\nabla^{\mathrm{b}},\mathscr{F}}"] \\
\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}(\mathscr{F}) \arrow[r, "\underline{\underline{\Xi}}^{\mathrm{b}}_{\mathscr{F}}"] & \underline{\underline{\nabla}}^{\mathrm{b}}(\mathscr{F})
\end{tikzcd}$$

The functor $\Xi^{\mathrm{b}}_{\mathscr{F}}$ is full and faithful by corollary 4.2.44. For $f \in \mathrm{Mor}(\mathrm{FO}^{\mathrm{b}}(\mathscr{F}))$, we have $\underline{\underline{f}} = 0$ if and only if $\underline{\underline{f\Xi_{\mathscr{F}}}} = 0$ by lemma 4.2.30 and corollary 4.2.46.

For $X \in \mathrm{Ob}(\nabla^{\mathrm{b}}(\mathscr{F}))$, definition 4.2.48 yields the object $X\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}} \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}(\mathscr{F}))$ and the isomorphism $X\sigma^{\mathrm{b}}_{\mathscr{F}} : X \to X\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}}\Xi^{\mathrm{b}}_{\mathscr{F}}$.

Lemma 1.6.5 yields the functor $\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}} : \nabla^{\mathrm{b}}(\mathscr{F}) \to \mathrm{FO}^{\mathrm{b}}(\mathscr{F})$, where for $X \xrightarrow{f} Y$ in $\nabla^{\mathrm{b}}(\mathscr{F})$, $X\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}} \xrightarrow{f\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}}} Y\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}}$ is the unique morphism in $\mathrm{FO}^{\mathrm{b}}(\mathscr{F})$ such that
$f = X\sigma^{\mathrm{b}}_{\mathscr{F}} \cdot f\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}}\Xi^{\mathrm{b}}_{\mathscr{F}} \cdot (Y\sigma^{\mathrm{b}}_{\mathscr{F}})^{-1}$. The functors $\Xi^{\mathrm{b}}_{\mathscr{F}}$ and $\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}}$ are mutually quasi-inverse equivalences. Moreover, we obtain the isotransformation $\sigma^{\mathrm{b}}_{\mathscr{F}} : 1_{\nabla^{\mathrm{b}}(\mathscr{F})} \to \mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}}\Xi^{\mathrm{b}}_{\mathscr{F}}$.

Lemma 1.2.16 yields the functor $\underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{F}} : \underline{\underline{\nabla}}^{\mathrm{b}}(\mathscr{F}) \to \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}(\mathscr{F})$, where for $X \xrightarrow{f} Y$ in $\nabla^{\mathrm{b}}(\mathscr{F})$, $X\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}} \xrightarrow{\underline{\underline{f}}\,\underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{F}}} Y\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}}$ is the unique morphism in $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}(\mathscr{F})$ such that
$\underline{\underline{f}} = \underline{\underline{X\sigma^{\mathrm{b}}_{\mathscr{F}}}} \cdot \underline{\underline{f}}\,\underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{F}}\underline{\underline{\Xi}}^{\mathrm{b}}_{\mathscr{F}} \cdot \underline{\underline{(Y\sigma^{\mathrm{b}}_{\mathscr{F}})^{-1}}}$.
The functors $\underline{\underline{\Xi}}^{\mathrm{b}}_{\mathscr{F}}$ and $\underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{F}}$ are mutually quasi-inverse equivalences. Moreover, we obtain the isotransformation $\underline{\sigma}^{\mathrm{b}}_{\mathscr{F}} : 1_{\underline{\underline{\nabla}}^{\mathrm{b}}(\mathscr{F})} \to \underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{F}}\underline{\underline{\Xi}}^{\mathrm{b}}_{\mathscr{F}}$ with $X\underline{\sigma}^{\mathrm{b}}_{\mathscr{F}} = \underline{\underline{X\sigma^{\mathrm{b}}_{\mathscr{F}}}}$ for $X \in \mathrm{Ob}(\nabla^{\mathrm{b}}(\mathscr{F}))$. We have $\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}} \cdot \mathfrak{Q}_{\mathrm{FO}^{\mathrm{b}},\mathscr{F}} = \mathfrak{Q}_{\nabla^{\mathrm{b}},\mathscr{F}} \cdot \underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{F}}$.

$$\begin{tikzcd}
\mathrm{FO}^{\mathrm{b}}(\mathscr{F}) \arrow[d, "\mathfrak{Q}_{\mathrm{FO}^{\mathrm{b}},\mathscr{F}}"'] & \nabla^{\mathrm{b}}(\mathscr{F}) \arrow[l, "\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}}"'] \arrow[d, "\mathfrak{Q}_{\nabla^{\mathrm{b}},\mathscr{F}}"] \\
\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}(\mathscr{F}) & \underline{\underline{\nabla}}^{\mathrm{b}}(\mathscr{F}) \arrow[l, "\underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{F}}"']
\end{tikzcd}$$

$\diamondsuit$

4.2.51 Definition. Suppose that $\mathscr{F}$ has epilimits and monocolimits. We want to construct quasi-inverses $\mathrm{Lim}_{\mathscr{F}}$ and $\underline{\mathrm{Lim}}_{\mathscr{F}}$ of the functors $\Xi^{\nabla}_{\mathscr{F}} = \Xi_{\mathscr{F}}|_{\mathrm{FO}^{\nabla}(\mathscr{F})} : \mathrm{FO}^{\nabla}(\mathscr{F}) \to \nabla(\mathscr{F})$ and $\underline{\underline{\Xi}}^{\nabla}_{\mathscr{F}} = \underline{\underline{\Xi}}_{\mathscr{F}}|_{\underline{\underline{\mathrm{FO}}}^{\nabla}(\mathscr{F})} : \underline{\underline{\mathrm{FO}}}^{\nabla}(\mathscr{F}) \to \underline{\underline{\nabla}}(\mathscr{F})$ following the steps of lemma 1.2.17. Cf. definition 4.2.35. We call these quasi-inverses *limit functors* since the are obtained by taking limits and colimits.

$$\begin{tikzcd}
\mathrm{FO}^{\nabla}(\mathscr{F}) \arrow[r, "\Xi^{\nabla}_{\mathscr{F}}"] \arrow[d, "\mathfrak{Q}_{\mathrm{FO}^{\nabla},\mathscr{F}}"'] & \nabla(\mathscr{F}) \arrow[d, "\mathfrak{Q}_{\nabla,\mathscr{F}}"] \\
\underline{\underline{\mathrm{FO}}}^{\nabla}(\mathscr{F}) \arrow[r, "\underline{\Xi}^{\nabla}_{\mathscr{F}}"] & \underline{\underline{\nabla}}(\mathscr{F})
\end{tikzcd}$$

The functor $\Xi^{\nabla}_{\mathscr{F}}$ is full and faithful by corollary 4.2.44. For $f \in \mathrm{Mor}(\mathrm{FO}^{\nabla}(\mathscr{F}))$, we have $\underline{\underline{f}} = 0$ if and only if $\underline{\underline{f\Xi_{\mathscr{F}}}} = 0$ by lemma 4.2.30 and corollary 4.2.47.

For $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$, definition 4.2.49 yields the object $X\mathrm{Lim}_{\mathscr{F}} \in \mathrm{Ob}(\mathrm{FO}^{\nabla}(\mathscr{F}))$ and the isomorphism $X\sigma_{\mathscr{F}} \colon X \to X\mathrm{Lim}_{\mathscr{F}}\Xi_{\mathscr{F}}$.

Lemma 1.6.5 yields the functor $\mathrm{Lim}_{\mathscr{F}} \colon \nabla(\mathscr{F}) \to \mathrm{FO}^{\nabla}(\mathscr{F})$, where for $X \xrightarrow{f} Y$ in $\nabla(\mathscr{F})$, $X\mathrm{Lim}_{\mathscr{F}} \xrightarrow{f\mathrm{Lim}_{\mathscr{F}}} Y\mathrm{Lim}_{\mathscr{F}}$ is the unique morphism in $\mathrm{FO}^{\nabla}(\mathscr{F})$ such that
$f = X\sigma_{\mathscr{F}} \cdot f\mathrm{Lim}_{\mathscr{F}}\Xi^{\nabla}_{\mathscr{F}} \cdot (Y\sigma_{\mathscr{F}})^{-1}$. The functors $\Xi^{\nabla}_{\mathscr{F}}$ and $\mathrm{Lim}_{\mathscr{F}}$ are mutually quasi-inverse equivalences. Moreover, we obtain the isotransformation $\sigma_{\mathscr{F}} \colon 1_{\nabla(\mathscr{F})} \to \mathrm{Lim}_{\mathscr{F}}\Xi^{\nabla}_{\mathscr{F}}$.

Lemma 1.2.16 yields the functor $\underline{\mathrm{Lim}}_{\mathscr{F}} \colon \underline{\underline{\nabla}}(\mathscr{F}) \to \underline{\underline{\mathrm{FO}}}^{\nabla}(\mathscr{F})$, where for $X \xrightarrow{f} Y$ in $\nabla(\mathscr{F})$, $X\mathrm{Lim}_{\mathscr{F}} \xrightarrow{\underline{\underline{f}}\,\underline{\mathrm{Lim}}_{\mathscr{F}}} Y\mathrm{Lim}_{\mathscr{F}}$ is the unique morphism in $\underline{\underline{\mathrm{FO}}}^{\nabla}(\mathscr{F})$ such that
$\underline{\underline{f}} = \underline{\underline{X\sigma_{\mathscr{F}}}} \cdot \underline{\underline{f}}\,\underline{\mathrm{Lim}}_{\mathscr{F}}\underline{\Xi}^{\nabla}_{\mathscr{F}} \cdot \underline{\underline{(Y\sigma_{\mathscr{F}})^{-1}}}$.
The functors $\underline{\Xi}^{\nabla}_{\mathscr{F}}$ and $\underline{\mathrm{Lim}}_{\mathscr{F}}$ are mutually quasi-inverse equivalences. Moreover, we obtain the isotransformation $\underline{\sigma}_{\mathscr{F}} \colon 1_{\underline{\underline{\nabla}}(\mathscr{F})} \to \underline{\mathrm{Lim}}_{\mathscr{F}}\underline{\Xi}^{\nabla}_{\mathscr{F}}$ with $X\underline{\sigma}_{\mathscr{F}} = \underline{\underline{X\sigma_{\mathscr{F}}}}$ for $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$. We have $\mathrm{Lim}_{\mathscr{F}} \cdot \mathfrak{Q}_{\mathrm{FO}^{\nabla},\mathscr{F}} = \mathfrak{Q}_{\nabla,\mathscr{F}} \cdot \underline{\mathrm{Lim}}_{\mathscr{F}}$.

$$\begin{tikzcd}
\mathrm{FO}^{\nabla}(\mathscr{F}) \arrow[d, "\mathfrak{Q}_{\mathrm{FO}^{\nabla},\mathscr{F}}"'] & \nabla(\mathscr{F}) \arrow[l, "\mathrm{Lim}_{\mathscr{F}}"'] \arrow[d, "\mathfrak{Q}_{\nabla,\mathscr{F}}"] \\
\underline{\underline{\mathrm{FO}}}^{\nabla}(\mathscr{F}) & \underline{\underline{\nabla}}(\mathscr{F}) \arrow[l, "\underline{\mathrm{Lim}}_{\mathscr{F}}"']
\end{tikzcd}$$

Note that we have $\mathrm{Lim}_{\mathscr{F}}|_{\nabla^{\mathrm{b}}(\mathscr{F})}^{\mathrm{FO}^{\mathrm{b}}(\mathscr{F})} = \mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}}$ and $\underline{\mathrm{Lim}}_{\mathscr{F}}|_{\underline{\underline{\nabla}}^{\mathrm{b}}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}(\mathscr{F})} = \underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{F}}$ by construction. $\Diamond$

4.2.52 Remark. Suppose given $X \in \mathrm{Ob}(\nabla^{\mathrm{b}}(\mathscr{F}))$.

(a) Suppose given $n \in \mathbf{Z}$. If $X \in \mathrm{Ob}(\nabla^{[n}(\mathscr{F}))$, then $X\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}} \in \mathrm{Ob}(\mathrm{FO}^{[n}(\mathscr{F}))$, cf. definition 4.2.48, lemma 3.3.57.(d) and lemma 3.4.24.

(b) Suppose given $n \in \mathbf{Z}$. If $X \in \mathrm{Ob}(\nabla^{n]}(\mathscr{F}))$, then $X\mathrm{Lim}^{\mathrm{b}}_{\mathscr{F}} \in \mathrm{Ob}(\mathrm{FO}^{n]}(\mathscr{F}))$, cf. definition 4.2.48, lemma 3.3.57.(e) and lemma 3.4.25.

$\Diamond$

4.2.53 Remark. Suppose that $\mathscr{F}$ has epilimits and monocolimits.
Suppose given $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$.

(a) Suppose given $n \in \mathbf{Z}$. If $X \in \mathrm{Ob}(\nabla^{[n}(\mathscr{F}))$, then $X\mathrm{Lim}_{\mathscr{F}} \in \mathrm{Ob}(\mathrm{FO}^{[n}(\mathscr{F}))$, cf. definition 4.2.49, lemma 3.3.57.(d) and lemma 3.4.24.

(b) Suppose given $n \in \mathbf{Z}$. If $X \in \operatorname{Ob}(\nabla^{n]}(\mathscr{F}))$, then $X\operatorname{Lim}_{\mathscr{F}} \in \operatorname{Ob}(\operatorname{FO}^{n]}(\mathscr{F}))$, cf. definition 4.2.49, lemma 3.3.57.(e) and lemma 3.4.25.

◊

4.2.54 Lemma. Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$. Suppose given $X \in \operatorname{Ob}(\nabla^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F}))$. Then $X\operatorname{Lim}^{\mathrm{b}}_{\mathscr{F}} \in \operatorname{Ob}(\operatorname{FO}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F}))$. ◊

Proof. We have $X\operatorname{Lim}^{\mathrm{b}}_{\mathscr{F}}\Xi^{\mathrm{b}}_{\mathscr{F}} \in \operatorname{Ob}(\nabla^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F}))$ by lemma 3.3.19. Thus $X\operatorname{Lim}^{\mathrm{b}}_{\mathscr{F}} \in \operatorname{Ob}(\operatorname{FO}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F}))$ by remark 4.2.13. □

4.2.55 Lemma. Suppose that $\mathscr{F}$ has epilimits and monocolimits.
Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$. Suppose given $X \in \operatorname{Ob}(\nabla_{\mathscr{S}}(\mathscr{F}))$. Then $X\operatorname{Lim}_{\mathscr{F}} \in \operatorname{Ob}(\operatorname{FO}_{\mathscr{S}}(\mathscr{F}))$. ◊

Proof. We have $X\operatorname{Lim}_{\mathscr{F}}\Xi^{\nabla}_{\mathscr{F}} \in \operatorname{Ob}(\nabla_{\mathscr{S}}(\mathscr{F}))$ by lemma 3.3.19. Thus $X\operatorname{Lim}_{\mathscr{F}} \in \operatorname{Ob}(\operatorname{FO}_{\mathscr{S}}(\mathscr{F}))$ by remark 4.2.13. □

4.2.56 Definition. Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$.
Let $\Xi^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} = \Xi^{\mathrm{b}}_{\mathscr{F}}|^{\nabla^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\operatorname{FO}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})} \colon \operatorname{FO}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F}) \to \nabla^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})$, $\underline{\Xi}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} = \underline{\Xi}^{\mathrm{b}}_{\mathscr{F}}|^{\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\operatorname{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})} \colon \underline{\underline{\operatorname{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F}) \to \underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})$ and $\operatorname{Lim}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} = \operatorname{Lim}^{\mathrm{b}}_{\mathscr{F}}|^{\operatorname{FO}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\nabla^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})} \colon \nabla^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F}) \to \operatorname{FO}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})$, $\underline{\operatorname{Lim}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} = \underline{\operatorname{Lim}}^{\mathrm{b}}_{\mathscr{F}}|^{\underline{\underline{\operatorname{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})} \colon \underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F}) \to \underline{\underline{\operatorname{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})$, cf. remark 4.2.13 and lemma 4.2.54.

The functors $\Xi^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}$ and $\operatorname{Lim}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}$ are mutually quasi-inverse equivalences, cf. lemma 1.6.10.
The functors $\underline{\Xi}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}$ and $\underline{\operatorname{Lim}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}$ are mutually quasi-inverse equivalences, cf. lemma 1.6.10.
Note that the functors $\Xi^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}$, $\operatorname{Lim}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}$, $\underline{\Xi}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}$ and $\underline{\operatorname{Lim}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}$ are additive, cf. remarks 1.2.5.(a), 4.2.17 and corollary 3.3.37. ◊

4.2.57 Definition. Suppose that $\mathscr{F}$ has epilimits and monocolimits.
Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$.
Let $\Xi^{\nabla}_{\mathscr{S},\mathscr{F}} = \Xi_{\mathscr{F}}|^{\nabla_{\mathscr{S}}(\mathscr{F})}_{\operatorname{FO}^{\nabla}_{\mathscr{S}}(\mathscr{F})} \colon \operatorname{FO}^{\nabla}_{\mathscr{S}}(\mathscr{F}) \to \nabla_{\mathscr{S}}(\mathscr{F})$, $\underline{\Xi}^{\nabla}_{\mathscr{S},\mathscr{F}} = \underline{\Xi}_{\mathscr{F}}|^{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\operatorname{FO}}}^{\nabla}_{\mathscr{S}}(\mathscr{F})} \colon \underline{\underline{\operatorname{FO}}}^{\nabla}_{\mathscr{S}}(\mathscr{F}) \to \underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})$ and $\operatorname{Lim}_{\mathscr{S},\mathscr{F}} = \operatorname{Lim}_{\mathscr{F}}|^{\operatorname{FO}^{\nabla}_{\mathscr{S}}(\mathscr{F})}_{\nabla_{\mathscr{S}}(\mathscr{F})} \colon \nabla_{\mathscr{S}}(\mathscr{F}) \to \operatorname{FO}^{\nabla}_{\mathscr{S}}(\mathscr{F})$, $\underline{\operatorname{Lim}}_{\mathscr{S},\mathscr{F}} = \underline{\operatorname{Lim}}_{\mathscr{F}}|^{\underline{\underline{\operatorname{FO}}}^{\nabla}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})} \colon \underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F}) \to \underline{\underline{\operatorname{FO}}}^{\nabla}_{\mathscr{S}}(\mathscr{F})$, cf. remark 4.2.13 and lemma 4.2.55.

The functors $\Xi^{\nabla}_{\mathscr{S},\mathscr{F}}$ and $\operatorname{Lim}_{\mathscr{S},\mathscr{F}}$ are mutually quasi-inverse equivalences, cf. lemma 1.6.10.
The functors $\underline{\Xi}^{\nabla}_{\mathscr{S},\mathscr{F}}$ and $\underline{\operatorname{Lim}}_{\mathscr{S},\mathscr{F}}$ are mutually quasi-inverse equivalences, cf. lemma 1.6.10.
Moreover, $\underline{\sigma}_{\mathscr{S},\mathscr{F}} = \underline{\sigma}_{\mathscr{F}}|^{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})} \colon 1_{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})} \to \underline{\operatorname{Lim}}_{\mathscr{S},\mathscr{F}}\underline{\Xi}^{\nabla}_{\mathscr{S},\mathscr{F}}$ is an isotransformation. Note that we have $\operatorname{Lim}_{\mathscr{S},\mathscr{F}}|^{\operatorname{FO}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\nabla^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})} = \operatorname{Lim}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}$ and $\underline{\operatorname{Lim}}_{\mathscr{S},\mathscr{F}}|^{\underline{\underline{\operatorname{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})} = \underline{\operatorname{Lim}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}$, cf. definition 4.2.51.
Also note that the functors $\Xi^{\nabla}_{\mathscr{S},\mathscr{F}}$, $\operatorname{Lim}^{\nabla}_{\mathscr{S},\mathscr{F}}$, $\underline{\Xi}^{\nabla}_{\mathscr{S},\mathscr{F}}$ and $\underline{\operatorname{Lim}}^{\nabla}_{\mathscr{S},\mathscr{F}}$ are additive, cf. remarks 1.2.5.(a), 4.2.18 and corollary 3.3.37. ◊

4.2.58 Lemma/Definition. Suppose that $\mathscr{F}$ has epilimits and monocolimits. Consider the functors $\operatorname{Lim}_{\mathscr{F}} \cdot \operatorname{Inc}^{\operatorname{FO}^{\lim}(\mathscr{F})}_{\operatorname{FO}^{\nabla}(\mathscr{F})} \colon \nabla(\mathscr{F}) \to \operatorname{FO}^{\lim}(\mathscr{F})$ and $\Xi^{\lim}_{\mathscr{F}} \colon \operatorname{FO}^{\lim}(\mathscr{F}) \to \nabla(\mathscr{F})$, cf. definition

4.2.36. We have the isotranformation $\sigma_{\mathscr{F}} \colon 1_{\nabla(\mathscr{F})} \to \mathrm{Lim}_{\mathscr{F}} \cdot \Xi^{\nabla}_{\mathscr{F}} = \mathrm{Lim}_{\mathscr{F}} \cdot \mathrm{Inc}^{\mathrm{FO}^{\lim}(\mathscr{F})}_{\mathrm{FO}^{\nabla}(\mathscr{F})} \cdot \Xi^{\lim}_{\mathscr{F}}$.

$$\begin{tikzcd}
\mathrm{FO}^{\lim}(\mathscr{F}) \arrow[r, "\Xi^{\lim}_{\mathscr{F}}"] & \nabla(\mathscr{F}) \arrow[dl, "\mathrm{Lim}_{\mathscr{F}}"] \\
\mathrm{FO}^{\nabla}(\mathscr{F}) \arrow[u, "\mathrm{Inc}^{\mathrm{FO}^{\lim}(\mathscr{F})}_{\mathrm{FO}^{\nabla}(\mathscr{F})}"] &
\end{tikzcd}$$

We obtain a tranformation $\tau_{\mathscr{F}} \colon \Xi^{\lim}_{\mathscr{F}} \cdot \mathrm{Lim}_{\mathscr{F}} \cdot \mathrm{Inc}^{\mathrm{FO}^{\lim}(\mathscr{F})}_{\mathrm{FO}^{\nabla}(\mathscr{F})} \to 1_{\mathrm{FO}^{\lim}(\mathscr{F})}$ as follows.
For $X \in \mathrm{Ob}(\mathrm{FO}^{\lim}(\mathscr{F}))$, let $X\tau_{\mathscr{F}} \colon X\Xi_{\mathscr{F}}\mathrm{Lim}_{\mathscr{F}} \to X$ be the unique morphism in $\mathrm{FO}(\mathscr{F})$ such that $X\tau_{\mathscr{F}}\Xi_{\mathscr{F}} = X\Xi_{\mathscr{F}}\sigma^{-1}_{\mathscr{F}} \colon X\Xi_{\mathscr{F}}\mathrm{Lim}_{\mathscr{F}}\Xi_{\mathscr{F}} \to X\Xi_{\mathscr{F}}$, cf. lemma 4.2.43.
Moreover, $\big(\mathrm{Lim}_{\mathscr{F}} \cdot \mathrm{Inc}^{\mathrm{FO}^{\lim}(\mathscr{F})}_{\mathrm{FO}^{\nabla}(\mathscr{F})}\, ,\, \Xi^{\lim}_{\mathscr{F}}\, ,\, \sigma_{\mathscr{F}}\, ,\, \tau_{\mathscr{F}}\big)$ is an adjunction, i.e. $\mathrm{Lim}_{\mathscr{F}} \cdot \mathrm{Inc}^{\mathrm{FO}^{\lim}(\mathscr{F})}_{\mathrm{FO}^{\nabla}(\mathscr{F})}$ is left-adjoint to $\Xi^{\lim}_{\mathscr{F}}$ with unit $\sigma_{\mathscr{F}}$ and counit $\tau_{\mathscr{F}}$.
For $X \in \mathrm{Ob}(\mathrm{FO}^{\lim}(\mathscr{F}))$ and $\ell \in \mathbf{Z}$, the morphism $(X\tau_{\mathscr{F}})_{|\ell}$ is an isomorphism in $\mathscr{F}$. $\Diamond$

Proof. Suppose given $X \xrightarrow{f} Y$ in $\mathrm{FO}^{\lim}(\mathscr{F})$. We have

$$\begin{aligned}
(X\tau_{\mathscr{F}} \cdot f)\Xi_{\mathscr{F}} = X\tau_{\mathscr{F}}\Xi_{\mathscr{F}} \cdot f\Xi_{\mathscr{F}} = X\Xi_{\mathscr{F}}\sigma^{-1}_{\mathscr{F}} \cdot f\Xi_{\mathscr{F}} &= f\Xi_{\mathscr{F}}\mathrm{Lim}_{\mathscr{F}}\Xi_{\mathscr{F}} \cdot Y\Xi_{\mathscr{F}}\sigma^{-1}_{\mathscr{F}} \\
&= f\Xi_{\mathscr{F}}\mathrm{Lim}_{\mathscr{F}}\Xi_{\mathscr{F}} \cdot Y\tau_{\mathscr{F}}\Xi_{\mathscr{F}} = (f\Xi_{\mathscr{F}}\mathrm{Lim}_{\mathscr{F}} \cdot Y\tau_{\mathscr{F}})\Xi_{\mathscr{F}} \,.
\end{aligned}$$

Thus $X\tau_{\mathscr{F}} \cdot f = f\Xi_{\mathscr{F}}\mathrm{Lim}_{\mathscr{F}} \cdot Y\tau_{\mathscr{F}}$ by lemma 4.2.43.
Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\lim}(\mathscr{F}))$. We have $X\Xi_{\mathscr{F}}\sigma_{\mathscr{F}} \cdot X\tau_{\mathscr{F}}\Xi_{\mathscr{F}} = X\Xi_{\mathscr{F}}\sigma_{\mathscr{F}} \cdot X\Xi_{\mathscr{F}}\sigma^{-1}_{\mathscr{F}} = 1_{X\Xi_{\mathscr{F}}}$.
Suppose given $X \in \mathrm{Ob}(\nabla(\mathscr{F}))$. We have $(X\sigma_{\mathscr{F}}\mathrm{Lim}_{\mathscr{F}} \cdot X\mathrm{Lim}_{\mathscr{F}}\tau_{\mathscr{F}})\Xi_{\mathscr{F}} = 1_{X\mathrm{Lim}_{\mathscr{F}}}\Xi_{\mathscr{F}}$ since

$$\begin{aligned}
X\sigma_{\mathscr{F}} \cdot (X\sigma_{\mathscr{F}}\mathrm{Lim}_{\mathscr{F}} \cdot X\mathrm{Lim}_{\mathscr{F}}\tau_{\mathscr{F}})\Xi_{\mathscr{F}} &= X\sigma_{\mathscr{F}} \cdot X\sigma_{\mathscr{F}}\mathrm{Lim}_{\mathscr{F}}\Xi_{\mathscr{F}} \cdot X\mathrm{Lim}_{\mathscr{F}}\tau_{\mathscr{F}}\Xi_{\mathscr{F}} \\
&= X\sigma_{\mathscr{F}} \cdot X\mathrm{Lim}_{\mathscr{F}}\Xi_{\mathscr{F}}\sigma_{\mathscr{F}} \cdot X\mathrm{Lim}_{\mathscr{F}}\Xi_{\mathscr{F}}\sigma^{-1}_{\mathscr{F}} \\
&= X\sigma_{\mathscr{F}} \cdot 1_{X\mathrm{Lim}_{\mathscr{F}}\Xi_{\mathscr{F}}} = X\sigma_{\mathscr{F}} \cdot 1_{X\mathrm{Lim}_{\mathscr{F}}}\Xi_{\mathscr{F}}
\end{aligned}$$

and since $X\sigma_{\mathscr{F}}$ is an isomorphism. Thus $X\sigma_{\mathscr{F}}\mathrm{Lim}_{\mathscr{F}} \cdot X\mathrm{Lim}_{\mathscr{F}}\tau_{\mathscr{F}} = 1_{X\mathrm{Lim}_{\mathscr{F}}}$ by lemma 4.2.43.
We conclude that $\big(\mathrm{Lim}_{\mathscr{F}} \cdot \mathrm{Inc}^{\mathrm{FO}^{\lim}(\mathscr{F})}_{\mathrm{FO}^{\nabla}(\mathscr{F})}\, ,\, \Xi^{\lim}_{\mathscr{F}}\, ,\, \sigma_{\mathscr{F}}\, ,\, \tau_{\mathscr{F}}\big)$ is an adjunction.
Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\lim}(\mathscr{F}))$ and $\ell \in \mathbf{Z}$. By lemma 4.2.43, we have
$(X\tau_{\mathscr{F}})_{|\ell} = X\Xi_{\mathscr{F}}\sigma^{-1}_{\mathscr{F}}\Psi_{\ell,\mathrm{CF},\mathscr{F}}\rceil^{(X_{|\ell},(X_{|\ell/k-1})_{k\in\mathbf{Z}})}_{((X\Xi_{\mathscr{F}}\mathrm{Lim}_{\mathscr{F}})_{|\ell},((X\Xi_{\mathscr{F}}\mathrm{Lim}_{\mathscr{F}})_{|\ell/k-1})_{k\in\mathbf{Z}})}$.
Note that $X\Xi_{\mathscr{F}}\sigma^{-1}_{\mathscr{F}}\Psi_{\ell,\mathrm{CF},\mathscr{F}}$ is an isomorphism in $\mathrm{CF}(\mathscr{F})$,
$((X\Xi_{\mathscr{F}}\mathrm{Lim}_{\mathscr{F}})_{|\ell}\, ,\, ((X\Xi_{\mathscr{F}}\mathrm{Lim}_{\mathscr{F}})_{|\ell/k-1})_{k\in\mathbf{Z}})$ is a limit for $X\Xi_{\mathscr{F}}\mathrm{Lim}_{\mathscr{F}}\Xi_{\mathscr{F}}\Psi_{\ell,\mathrm{CF},\mathscr{F}}$ and that $(X_{|\ell}\, ,\, (X_{|\ell/k-1})_{k\in\mathbf{Z}})$ is a limit for $X\Xi_{\mathscr{F}}\Psi_{\ell,\mathrm{CF},\mathscr{F}}$, cf. lemma 4.2.40.
Thus $(X\tau_{\mathscr{F}})_{|\ell}$ is an isomorphism in $\mathscr{F}$, cf. lemma 3.2.24.(a,b). $\square$

4.2.59 Definition. Suppose given a full subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$.
Let $\underline{\underline{\Xi}}^{\lim,\mathrm{inj}}_{\mathscr{S},\mathscr{F}} = \underline{\underline{\Xi}}_{\mathscr{F}}|^{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\lim,\mathrm{inj}}_{\mathscr{S}}(\mathscr{F})} \colon \underline{\underline{\mathrm{FO}}}^{\lim,\mathrm{inj}}_{\mathscr{S}}(\mathscr{F}) \to \underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})$, cf. definition 4.2.32. $\Diamond$

4.2.60 Proposition. Suppose that $\mathscr{F}$ has epilimits and monocolimits. Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$. Consider the functors

$\underline{\mathrm{Lim}}_{\mathscr{S},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\nabla}(\mathscr{F})} \colon \underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F}) \to \underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F})$ and $\underline{\underline{\Xi}}^{\mathrm{lim,inj}}_{\mathscr{S},\mathscr{F}} \colon \underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F}) \to \underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})$. By lemmata 4.2.58, 1.6.9 and 1.6.11, we obtain the adjunction

$$\Big(\underline{\mathrm{Lim}}_{\mathscr{S},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{S}}(\mathscr{F})} \;,\; \underline{\underline{\Xi}}^{\mathrm{lim,inj}}_{\mathscr{S},\mathscr{F}} \;,\; \underline{\sigma}_{\mathscr{S},\mathscr{F}} \;,\; \underline{\tau}_{\mathscr{S},\mathscr{F}}\Big)$$

with $X\underline{\tau}_{\mathscr{S},\mathscr{F}} = \underline{\underline{\underline{X\tau_{\mathscr{F}}}}}$ for $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F}))$ and $X\underline{\sigma}_{\mathscr{S},\mathscr{F}} = \underline{\underline{\underline{X\sigma_{\mathscr{F}}}}}$ for $X \in \mathrm{Ob}(\nabla_{\mathscr{S}}(\mathscr{F}))$.

$$\begin{tikzcd}
\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F}) \arrow[r, "\underline{\underline{\Xi}}^{\mathrm{lim,inj}}_{\mathscr{S},\mathscr{F}}"] & \underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F}) \arrow[dl, "\underline{\mathrm{Lim}}_{\mathscr{S},\mathscr{F}}"] \\
\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{S}}(\mathscr{F}) \arrow[u, "\mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{S}}(\mathscr{F})}"] &
\end{tikzcd}$$

$\diamondsuit$

4.2.61 Lemma. Suppose given a strictly full additive subcategory $\mathscr{S} \subseteq \underline{\mathscr{F}}$.

(a) We have $\underline{\mathrm{E}}_{\mathscr{F}}|^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\mathscr{S}} \cdot \underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{S},\mathscr{F}} = \mathrm{Inc}^{\underline{\mathscr{F}}}_{\mathscr{S}}$, cf. definitions 3.4.65 and 3.4.67.

(b) The functors $\underline{\mathrm{E}}_{\mathscr{F}}|^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\mathscr{S}} \cdot \underline{\underline{\Xi}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \cdot \underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}$ and $\mathrm{I}_{\mathrm{K^b},\mathscr{S}}$ are isomorphic in $\mathscr{S}(\mathrm{K^b}(\mathscr{S}))$.

(c) Suppose that ${}_{\underline{\mathscr{F}}}(\mathscr{S}^{[k]},\mathscr{S}) = 0$ for $k \in \mathbf{Z}_{\geq 1}$.
The functors $\underline{\mathrm{E}}_{\mathscr{F}}|^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\mathscr{S}}$ and $\mathrm{I}_{\mathrm{K^b},\mathscr{S}} \cdot \underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \cdot \underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}$ are isomorphic in $\mathscr{S}(\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F}))$.

(d) Suppose that ${}_{\underline{\mathscr{F}}}(\mathscr{S}^{[k]},\mathscr{S}) = 0$ for $k \in \mathbf{Z}_{\geq 1}$.
The functors $\mathrm{I}_{\mathrm{K^b},\mathscr{S}} \cdot \underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \cdot \underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \cdot \underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{S},\mathscr{F}}$ and $\mathrm{Inc}^{\underline{\mathscr{F}}}_{\mathscr{S}}$ are isomorphic in $\mathscr{S}(\underline{\mathscr{F}})$.

$\diamondsuit$

Proof. Ad (a). By lemma 3.4.68, we have

$$\underline{\mathrm{E}}_{\mathscr{F}}|^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\mathscr{S}} \cdot \underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{S},\mathscr{F}} = \underline{\mathrm{E}}_{\mathscr{F}}|^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\mathscr{S}} \cdot \underline{\mathrm{P}}_{\omega,\mathscr{F}}|_{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})} = \big(\underline{\mathrm{E}}_{\mathscr{F}} \cdot \underline{\mathrm{P}}_{\omega,\mathscr{F}}\big)|_{\mathscr{S}} = 1_{\underline{\mathscr{F}}}|_{\mathscr{S}} = \mathrm{Inc}^{\underline{\mathscr{F}}}_{\mathscr{S}} .$$

Ad (b). We want to construct an isotransformation $\mathrm{I}_{\mathrm{K^b},\mathscr{S}} \xrightarrow{\lambda} \underline{\mathrm{E}}_{\mathscr{F}}|^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\mathscr{S}} \cdot \underline{\underline{\Xi}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \cdot \underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}$.
Suppose given $X \in \mathrm{Ob}(\mathscr{S})$.
We have $X\underline{\mathrm{E}}_{\mathscr{F}}\underline{\underline{\Xi}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} = X\mathrm{E}_{\mathscr{F}}\Xi_{\mathscr{F}}\Delta_{\mathscr{S},\mathscr{F}} \in \mathrm{Ob}(\mathrm{C}^{[0,0]}(\mathscr{S}))$ by lemmata 4.2.33 and 4.1.17. Note that $(X\mathrm{E}_{\mathscr{F}}\Xi_{\mathscr{F}}\Delta_{\mathscr{S},\mathscr{F}})_0 = (X\mathrm{E}_{\mathscr{F}})_{0/-1}$. We have the pure short exact sequence $0_{\mathscr{F}} \xrightarrow{(X\mathrm{E}_{\mathscr{F}})_{|-1\rightarrow|0}} X \xrightarrow{(X\mathrm{E}_{\mathscr{F}})_{|0/-1}} (X\mathrm{E}_{\mathscr{F}})_{0/-1}$ in $\mathscr{F}$. Thus $(X\mathrm{E}_{\mathscr{F}})_{|0/-1} \colon X \to (X\mathrm{E}_{\mathscr{F}})_{0/-1}$ is an isomorphism in $\mathscr{F}$.
We obtain the isomorphism $X\mathrm{I}_{\mathrm{K^b},\mathscr{S}} \xrightarrow{X\mu} X\mathrm{E}_{\mathscr{F}}\Xi_{\mathscr{F}}\Delta_{\mathscr{S},\mathscr{F}}$ in $\mathrm{C^b}(\mathscr{S})$ by setting $(X\mu)_0 = \underline{(X\mathrm{E}_{\mathscr{F}})_{|0/-1}}$. Let $X\lambda = \underline{X\mu}$ in $\mathrm{K^b}(\mathscr{S})$.

Suppose given $X \xrightarrow{f} Y$ in $\mathscr{S}$. We have $X\lambda \cdot \underline{f}\, \underline{\mathrm{E}}_{\mathscr{F}}\underline{\underline{\Xi}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} = \underline{f}\,\mathrm{I}_{\mathrm{K^b},\mathscr{S}} \cdot Y\lambda$ since $X\lambda = \underline{X\mu}$,

$Y\lambda = \underline{Y\mu}$, $\underline{f}\,\underline{\mathrm{E}}_{\mathscr{F}}\underline{\Xi}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} = \underline{f\mathrm{E}_{\mathscr{F}}\Xi_{\mathscr{F}}\Delta_{\mathscr{S},\mathscr{F}}}$, $\underline{f}\,\mathrm{I}_{\mathrm{K^b},\mathscr{S}} = \underline{\underline{f}\,\mathrm{I}_{\mathrm{C^b},\mathscr{S}}}$ and since

$$\begin{aligned}(X\mu \cdot f\mathrm{E}_{\mathscr{F}}\Xi_{\mathscr{F}}\Delta_{\mathscr{S},\mathscr{F}})_0 &= \underline{(X\mathrm{E}_{\mathscr{F}})_{|0/-1}} \cdot \underline{(f\mathrm{E}_{\mathscr{F}})_{0/-1}} = \underline{(f\mathrm{E}_{\mathscr{F}})_{|0}} \cdot \underline{(Y\mathrm{E}_{\mathscr{F}})_{|0/-1}} = \underline{f} \cdot (Y\mu)_0 \\ &= (\underline{f}\,\mathrm{I}_{\mathrm{C^b},\mathscr{S}} \cdot Y\mu)_0\,.\end{aligned}$$

$$\begin{tikzcd} X\mathrm{I}_{\mathrm{K^b},\mathscr{S}} \arrow[r, "X\lambda"] \arrow[d, "\underline{f}\,\mathrm{I}_{\mathrm{K^b},\mathscr{S}}"'] & X\underline{\mathrm{E}}_{\mathscr{F}}\underline{\Xi}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \arrow[d, "\underline{f}\,\underline{\mathrm{E}}_{\mathscr{F}}\underline{\Xi}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}"] \\ Y\mathrm{I}_{\mathrm{K^b},\mathscr{S}} \arrow[r, "Y\lambda"] & Y\underline{\mathrm{E}}_{\mathscr{F}}\underline{\Xi}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \end{tikzcd}$$

We conclude that $\mathrm{I}_{\mathrm{K^b},\mathscr{S}} \xrightarrow{\lambda} \underline{\mathrm{E}}_{\mathscr{F}}|^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\mathscr{S}} \cdot \underline{\Xi}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \cdot \underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}$ is an isotransformation.

Ad (c). The functors $\underline{\Xi}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \cdot \underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}$ and $\underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \cdot \underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}$ are mutually quasi-inverse equivalences. So the result follows from (b) and lemma 1.1.7.

$$\begin{tikzcd} \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F}) \arrow[r, shift left, "\underline{\Xi}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \cdot \underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}"] & \mathrm{K^b}(\mathscr{S}) \arrow[l, shift left, "\underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \cdot \underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}"] \\ \mathscr{S} \arrow[u, "\underline{\mathrm{E}}_{\mathscr{F}}|^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\mathscr{S}}"] \arrow[ur, bend right, "\mathrm{I}_{\mathrm{K^b},\mathscr{S}}"'] & \end{tikzcd}$$

Ad (d). This follows from (c), (a) and lemma 1.1.5. □

4.3 Resolution functors

We have collected some facts about w-structures in section 1.7 which we will use now.

Suppose given a strict Frobenius category $\mathscr{F} = (\mathscr{F}, \mathrm{B}, \Sigma, \iota, \pi, \alpha)$. Suppose given a full triangulated subcategory $\mathscr{D} \subseteq \underline{\mathscr{F}}$. Suppose given a w-structure $\mathscr{W} = (\mathscr{W}_{[0}, \mathscr{W}_{0]})$ on $\mathscr{D}$.

4.3.1 Remark. Suppose given $X \in \mathrm{Ob}(\mathscr{D})$ and $k \in \mathbf{Z}$.
There is a triangle $Y \longrightarrow Z \longrightarrow X \longrightarrow Y^{[1]}$ in $\underline{\mathscr{F}}$ such that $Y \in \mathrm{Ob}(\mathscr{W}_{k]})$ and $Z \in \mathrm{Ob}(\mathscr{W}_{[k})$. ◊

Proof. We may choose a triangle $Z \longrightarrow X \longrightarrow Y^{[1]} \longrightarrow Z^{[1]}$ in $\underline{\mathscr{F}}$ such that $Z \in \mathrm{Ob}(\mathscr{W}_{[k})$ and $Y^{[1]} \in \mathrm{Ob}(\mathscr{W}_{k+1]})$. Rotation of the triangle yields the result. □

4.3.2 Definition. We define the full subcategory $\mathrm{FO}_{\mathscr{W}|}(\mathscr{F})$ of $\mathrm{FO}^{\mathrm{proj}}(\mathscr{F})$ by setting

$$\mathrm{Ob}(\mathrm{FO}_{\mathscr{W}|}(\mathscr{F})) = \{X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{proj}}(\mathscr{F}))\colon X_{k|} \in \mathrm{Ob}(\mathscr{W}_{k]}) \text{ for } k \in \mathbf{Z}\}.$$

We define the full subcategory $\mathrm{FO}_{|\mathscr{W}}(\mathscr{F})$ of $\mathrm{FO}^{\mathrm{inj}}(\mathscr{F})$ by setting

$$\mathrm{Ob}(\mathrm{FO}_{|\mathscr{W}}(\mathscr{F})) = \{X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{inj}}(\mathscr{F}))\colon X_{|k} \in \mathrm{Ob}(\mathscr{W}_{[k}) \text{ for } k \in \mathbf{Z}\}.$$

Cf. definition 3.4.53. Let $\mathrm{FO}_{\mathscr{W}}(\mathscr{F}) = \mathrm{FO}_{\mathscr{W}|}(\mathscr{F}) \cap \mathrm{FO}_{|\mathscr{W}}(\mathscr{F})$.

We define the full subcategories $\underline{\underline{\mathrm{FO}}}_{\mathscr{W}|}(\mathscr{F})$, $\underline{\underline{\mathrm{FO}}}_{|\mathscr{W}}(\mathscr{F})$ and $\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})$ of $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$ by setting $\mathrm{Ob}(\underline{\underline{\mathrm{FO}}}_{\mathscr{W}|}(\mathscr{F})) = \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}|}(\mathscr{F}))$, $\mathrm{Ob}(\underline{\underline{\mathrm{FO}}}_{|\mathscr{W}}(\mathscr{F})) = \mathrm{Ob}(\mathrm{FO}_{|\mathscr{W}}(\mathscr{F}))$ and $\mathrm{Ob}(\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})) = \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}}(\mathscr{F}))$.

Let $\mathrm{FO}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}) = \mathrm{FO}^{\mathrm{b}}(\mathscr{F}) \cap \mathrm{FO}_{\mathscr{W}}(\mathscr{F})$, $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}) = \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}(\mathscr{F}) \cap \underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})$, $\mathrm{FO}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F}) = \mathrm{FO}^{\mathrm{lim}}(\mathscr{F}) \cap \mathrm{FO}_{\mathscr{W}}(\mathscr{F})$ and $\underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F}) = \underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}(\mathscr{F}) \cap \underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})$. $\diamond$

4.3.3 Definition. Suppose given $m, n \in \mathbf{Z}$.
Let $\mathrm{FO}^{m]}_{|\mathscr{W}}(\mathscr{F}) = \mathrm{FO}^{m]}(\mathscr{F}) \cap \mathrm{FO}_{|\mathscr{W}}(\mathscr{F})$, $\mathrm{FO}^{[m}_{\mathscr{W}|}(\mathscr{F}) = \mathrm{FO}^{[m}(\mathscr{F}) \cap \mathrm{FO}_{\mathscr{W}|}(\mathscr{F})$, $\mathrm{FO}^{[m}_{\mathscr{W}}(\mathscr{F}) = \mathrm{FO}^{[m}(\mathscr{F}) \cap \mathrm{FO}_{\mathscr{W}}(\mathscr{F})$, $\mathrm{FO}^{m]}_{\mathscr{W}}(\mathscr{F}) = \mathrm{FO}^{m]}(\mathscr{F}) \cap \mathrm{FO}_{\mathscr{W}}(\mathscr{F})$ and $\mathrm{FO}^{[m,n]}_{\mathscr{W}}(\mathscr{F}) = \mathrm{FO}^{[m,n]}(\mathscr{F}) \cap \mathrm{FO}_{\mathscr{W}}(\mathscr{F})$. $\diamond$

4.3.4 Remark. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}}(\mathscr{F}))$. Then we have $X_\omega \in \mathrm{Ob}(\mathscr{D})$. $\diamond$

Proof. The pure short exact sequence $X_{|0} \xrightarrow{x^\omega_{|0}} X_\omega \xrightarrow{x^\omega_{1|}} X_{1|}$ in $\mathscr{F}$ yields a triangle $X_{|0} \longrightarrow X_\omega \longrightarrow X_{1|} \longrightarrow X^{[1]}_{|0}$ in $\underline{\mathscr{F}}$. Since $X_{|0}, X_{1|} \in \mathrm{Ob}(\mathscr{D})$, we have $X_\omega \in \mathrm{Ob}(\mathscr{D})$ as well. $\square$

4.3.5 Remark. We will use definition 4.2.2. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}(\mathscr{F}))$ and $k \in \mathbf{Z}$.

(a) If $X \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}|}(\mathscr{F}))$, then $X_{k/k-1} \in \mathrm{Ob}(\mathscr{W}_{k]})$.

(b) If $X \in \mathrm{Ob}(\mathrm{FO}_{|\mathscr{W}}(\mathscr{F}))$, then $X_{k/k-1} \in \mathrm{Ob}(\mathscr{W}_{[k})$.

(c) If $X \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}}(\mathscr{F}))$, then $X_{k/k-1} \in \mathrm{Ob}(\mathscr{W}_{[k,k]})$.

$\diamond$

Proof. The pure short exact sequences $X_{|k-1} \xrightarrow{x_{|k-1}} X_{|k} \xrightarrow{X_{|k/k-1}} X_{k/k-1}$ and $X_{k/k-1} \xrightarrow{X_{k/k-1|}} X_{k|} \xrightarrow{x_{k|}} X_{k+1|}$ in $\mathscr{F}$ yield triangles $X_{|k-1} \longrightarrow X_{|k} \longrightarrow X_{k/k-1} \longrightarrow X^{[1]}_{|k-1}$ and $X_{k/k-1} \longrightarrow X_{k|} \longrightarrow X_{k+1|} \longrightarrow X^{[1]}_{k/k-1}$ in $\underline{\mathscr{F}}$.

Ad (a). If $X \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}|}(\mathscr{F}))$, then $X_{k|}, X^{[-1]}_{k+1|} \in \mathrm{Ob}(\mathscr{W}_{k]})$ and thus $X_{k/k-1} \in \mathrm{Ob}(\mathscr{W}_{k]})$.

Ad (b). If $X \in \mathrm{Ob}(\mathrm{FO}_{|\mathscr{W}}(\mathscr{F}))$, then $X_{|k}, X^{[1]}_{|k-1} \in \mathrm{Ob}(\mathscr{W}_{[k})$ and thus $X_{k/k-1} \in \mathrm{Ob}(\mathscr{W}_{[k})$.

Ad (c). This follows from (a) and (b). $\square$

4.3.6 Corollary. We have $\mathrm{FO}_{\mathscr{W}|}(\mathscr{F}) \subseteq \mathrm{FO}_{\mathscr{W}_{0]}}(\mathscr{F})$, $\mathrm{FO}_{|\mathscr{W}}(\mathscr{F}) \subseteq \mathrm{FO}_{\mathscr{W}_{[0}}(\mathscr{F})$ and $\mathrm{FO}_{\mathscr{W}}(\mathscr{F}) \subseteq \mathrm{FO}_{\mathscr{W}_{[0,0]}}(\mathscr{F})$, cf. definition 4.2.12. $\diamond$

4.3.7 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}_{[0}}(\mathscr{F}))$. For $\ell/k \in \mathrm{V}$, we have $X_{\ell/k} \in \mathrm{Ob}(\mathscr{W}_{[\ell})$. $\diamond$

Proof. We have $X_{\ell/k} \in \mathrm{Ob}\left(\mathop{\ast}\limits_{j\in[k+1,\ell]} \mathscr{W}^{[j]}_{[0}\right) = \mathrm{Ob}\left(\mathop{\ast}\limits_{j\in[k+1,\ell]} \mathscr{W}_{[j}\right)$ by remark 3.3.20.
Thus $X_{\ell/k} \in \mathrm{Ob}(\mathscr{W}_{[\ell})$. $\square$

4.3.8 Lemma. Suppose given $m \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{FO}^{m]}_{\mathscr{W}_{[0}}(\mathscr{F}))$. Then we have $X_{|\ell} \in \mathrm{Ob}(\mathscr{W}_{[\ell})$ for $\ell \in \mathbf{Z}$. $\diamond$

Proof. For $\ell \in \mathbf{Z}_{<m}$, we have $X_{|\ell} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$ and thus $X_{|\ell} \in \mathrm{Ob}(\mathscr{W}_{[\ell})$.
Suppose given $\ell \in \mathbf{Z}_{\geq m}$. We have $X_{\ell/m-1} \in \mathrm{Ob}(\mathscr{W}_{[\ell})$ by lemma 4.3.7.
Consider the pure short exact sequence $X_{|m-1} \xrightarrow{X_{|m-1 \to |\ell}} X_{|\ell} \xrightarrow{X_{|\ell/m-1}} X_{\ell/m-1}$ in $\mathscr{F}$. The morphism $X_{|\ell/m-1}$ is an isomorphism in $\mathscr{F}$ since $X_{|m-1} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$. We conclude that $X_{|\ell} \in \mathrm{Ob}(\mathscr{W}_{[\ell})$ since $X_{\ell/m-1} \in \mathrm{Ob}(\mathscr{W}_{[\ell})$. □

4.3.9 Lemma. Suppose given $m \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{FO}^{[m}_{\mathscr{W}_{0]}}(\mathscr{F}))$. Then we have $X_{\ell|} \in \mathrm{Ob}(\mathscr{W}_{\ell]})$ for $\ell \in \mathbf{Z}$. ♢

Proof. This is dual to the previous lemma 4.3.8. □

4.3.10 Lemma. We have $\mathrm{FO}^{\mathrm{b}}_{\mathscr{W}_{[0,0]}}(\mathscr{F}) = \mathrm{FO}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})$ and $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}_{[0,0]}}(\mathscr{F}) = \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})$. ♢

Proof. Note that we have $\mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})) = \mathrm{Ob}(\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}))$.
Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}))$.
Thus $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}(\mathscr{F}))$ and $X \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}}(\mathscr{F})) \subseteq \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}_{[0,0]}}(\mathscr{F}))$ by corollary 4.3.6. We conclude that $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}_{\mathscr{W}_{[0,0]}}(\mathscr{F}))$.
Conversely, suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}_{\mathscr{W}_{[0,0]}}(\mathscr{F}))$.
We may choose $m, n \in \mathbf{Z}$ such that $X \in \mathrm{Ob}(\mathrm{FO}^{[m,n]}_{\mathscr{W}_{[0,0]}}(\mathscr{F}))$. By lemmata 4.3.8, 4.3.9 and remark 3.4.56, we have $X \in \mathrm{Ob}(\mathrm{FO}^{[m,n]}_{\mathscr{W}}(\mathscr{F}))$. We conclude that $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}))$. □

4.3.11 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}}(\mathscr{F}))$. Then $X^{[1]}_{[-1]} \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}}(\mathscr{F}))$ as well. ♢

Proof. We have $X^{[1]}_{[-1]} \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{proj}}(\mathscr{F}))$ by lemmata 3.4.57 and 3.4.59.
We have $X^{[1]}_{[-1]} \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{inj}}(\mathscr{F}))$ by lemmata 3.4.58 and 3.4.60.
For $k \in \mathbf{Z}$, we have $(X^{[1]}_{[-1]})_{k|} = X^{[1]}_{k-1|} \in \mathrm{Ob}(\mathscr{W}_{k]})$ and $(X^{[1]}_{[-1]})_{|k} = X^{[1]}_{|k-1} \in \mathrm{Ob}(\mathscr{W}_{[k})$. □

4.3.12 Lemma. Suppose given a pure short exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in $\mathrm{FO}(\mathscr{F})$ such that $X, Z \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}}(\mathscr{F}))$. Then we have $Y \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}}(\mathscr{F}))$ as well. ♢

Proof. We have $Y \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{proj}}(\mathscr{F}))$ by lemma 3.4.61. We have $Y \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{inj}}(\mathscr{F}))$ by lemma 3.4.63. Suppose given $k \in \mathbf{Z}$. The pure short exact sequences $X_{|k} \xrightarrow{i_{|k}} Y_{|k} \xrightarrow{p_{|k}} Z_{|k}$ and $X_{k|} \xrightarrow{i_{k|}} Y_{k|} \xrightarrow{p_{k|}} Z_{k|}$ in $\mathscr{F}$ yield triangles $X_{|k} \longrightarrow Y_{|k} \longrightarrow Z_{|k} \longrightarrow X^{[1]}_{|k}$ and $X_{k|} \longrightarrow Y_{k|} \longrightarrow Z_{k|} \longrightarrow X^{[1]}_{k|}$ in $\underline{\mathscr{F}}$. We have $Y_{|k} \in \mathrm{Ob}(\mathscr{W}_{[k})$ since $X_{|k}, Z_{|k} \in \mathrm{Ob}(\mathscr{W}_{[k})$. We have $Y_{k|} \in \mathrm{Ob}(\mathscr{W}_{k|})$ since $X_{k|}, Z_{k|} \in \mathrm{Ob}(\mathscr{W}_{k]})$. □

4.3.13 Corollary. $\mathrm{FO}_{\mathscr{W}}(\mathscr{F})$ is an extension-closed subcategory of $\mathrm{FO}(\mathscr{F})$, cf. definition 1.3.23, remark 1.4.10 and lemma 4.3.12. In particular, it is a strictly full additive subcategory of $\mathrm{FO}(\mathscr{F})$. Moreover, $\mathrm{FO}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})$ is an extension-closed subcategory of $\mathrm{FO}(\mathscr{F})$ as well, cf. remark 1.3.24. Thus $\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})$ and $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})$ are full additive subcategories of $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$, cf. remark 1.2.14. ♢

4.3.14 Lemma. Suppose given a pseudo-triangle $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}_{[-1]}$ in $\mathrm{FO}(\mathscr{F})$ such that $X, Y \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}}(\mathscr{F}))$. Then we have $Z, X^{[1]}_{[-1]} \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}}(\mathscr{F}))$ as well. $\Diamond$

Proof. This follows from lemmata 4.3.11 and 4.3.12. $\square$

4.3.15 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}_{|\mathscr{W}}(\mathscr{F}))$, $Y \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}|}(\mathscr{F}))$ and $X_\omega \xrightarrow{f} Y_\omega$ in $\mathscr{F}$. Then there exists $X \xrightarrow{g} Y$ in $\mathrm{FO}(\mathscr{F})$ such that $\underline{g_\omega} = \underline{f}$ in $\underline{\mathscr{F}}$. $\Diamond$

Proof. In two steps, we want to construct $X_\omega \xrightarrow{\mu} Y_\omega^{[-1]}\mathrm{B}$ in $\mathscr{F}$ such that $x^\omega_{|k} \cdot (f - \mu \cdot Y_\omega^{[-1]}\pi) \cdot y^\omega_{k+1|} = 0$, i.e. such that $x^\omega_{|k} \cdot f \cdot y^\omega_{k+1|} = x^\omega_{|k} \cdot \mu \cdot Y_\omega^{[-1]}\pi \cdot y^\omega_{k+1|}$ for $k \in \mathbf{Z}$.

First step.

For $k \in \mathbf{Z}_{\leq 1}$, we want to construct $X_{|0}\mathrm{B} \xrightarrow{\lambda_k} Y_{k|}$ in $\mathscr{F}$ recursively such that $\lambda_{k-1} \cdot y_{k-1|} = \lambda_k$ and such that $x^\omega_{|k-1} \cdot f \cdot y^\omega_{k|} = X_{|k-1\rightarrow|0} \cdot X_{|0}\iota \cdot \lambda_k$.

Since ${}_{\underline{\mathscr{F}}}(X_{|0}\,, Y_{1|}) = 0$, we may choose $X_{|0}\mathrm{B} \xrightarrow{\lambda_1} Y_{1|}$ in $\mathscr{F}$ such that $x^\omega_{|0} \cdot f \cdot y^\omega_{1|} = X_{|0}\iota \cdot \lambda_1$.

Suppose given $k \in \mathbf{Z}_{<1}$. Suppose that we have already constructed λ_ℓ for $\ell \in [k+1, 1]$.

Since $X_{|0}\mathrm{B}$ is projective in $\mathscr{F}$, we may choose $X_{|0}\mathrm{B} \xrightarrow{\zeta_k} Y_{k|}$ in $\mathscr{F}$ such that $\zeta_k \cdot y_{k|} = \lambda_{k+1}$. We have

$$\begin{aligned}(x^\omega_{|k-1} \cdot f \cdot y^\omega_{k|} - X_{|k-1\rightarrow|0} \cdot X_{|0}\iota \cdot \zeta_k) \cdot y_{k|} &= x_{|k-1} \cdot x^\omega_{|k} \cdot f \cdot y^\omega_{k+1|} - x_{|k-1} \cdot X_{|k\rightarrow|0} \cdot X_{|0}\iota \cdot \lambda_{k+1} \\ &= x_{|k-1} \cdot (x^\omega_{|k} \cdot f \cdot y^\omega_{k+1|} - X_{|k\rightarrow|0} \cdot X_{|0}\iota \cdot \lambda_{k+1}) = 0.\end{aligned}$$

Since moreover ${}_{\underline{\mathscr{F}}}(X_{|k-1}, Y_{k/k-1}) = 0$ by remark 4.3.5.(a), we may choose $X_{|0}\mathrm{B} \xrightarrow{\eta_k} Y_{k/k-1}$ in $\mathscr{F}$ such that

$$X_{|k-1\rightarrow|0} \cdot X_{|0}\iota \cdot \eta_k \cdot Y_{k/k-1|} = x^\omega_{|k-1} \cdot f \cdot y^\omega_{k|} - X_{|k-1\rightarrow|0} \cdot X_{|0}\iota \cdot \zeta_k \,.$$

$$\begin{tikzcd}
X_{|k-1} \arrow[r, "x^\omega_{|k-1}\cdot f\cdot y^\omega_{k|} - X_{|k-1\rightarrow|0}\cdot X_{|0}\iota\cdot\zeta_k"] \arrow[d, "X_{|k-1\rightarrow|0}\cdot X_{|0}\iota"'] & Y_{k|} \arrow[r, "y_{k|}"] & Y_{k+1|} \\
X_{|0}\mathrm{B} \arrow[r, "\eta_k"] & Y_{k/k-1} \arrow[u, "Y_{k/k-1|}"'] &
\end{tikzcd}$$

Let $\lambda_k = \zeta_k + \eta_k \cdot Y_{k/k-1|}$. We have $\lambda_k \cdot y_{k|} = \lambda_{k+1}$ and

$$\begin{aligned}X_{|k-1\rightarrow|0} \cdot X_{|0}\iota \cdot \lambda_k &= X_{|k-1\rightarrow|0} \cdot X_{|0}\iota \cdot \zeta_k + X_{|k-1\rightarrow|0} \cdot X_{|0}\iota \cdot \eta_k \cdot Y_{k/k-1} \\ &= X_{|k-1\rightarrow|0} \cdot X_{|0}\iota \cdot \zeta_k + x^\omega_{|k-1} \cdot f \cdot y^\omega_{k|} - X_{|k-1\rightarrow|0} \cdot X_{|0}\iota \cdot \zeta_k \\ &= x^\omega_{|k-1} \cdot f \cdot y^\omega_{k|} \,.\end{aligned}$$

Since $Y \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{proj}}(\mathscr{F}))$ and since $X_{|0}\mathrm{B}$ is projective in $\mathscr{F}$, we may choose $X_{|0}\mathrm{B} \xrightarrow{\lambda} Y_\omega$ in $\mathscr{F}$ such that $\lambda \cdot y^\omega_{k|} = \lambda_k$ for $k \in \mathbf{Z}_{\leq 1}$.

Since $X_{|0}\mathrm{B}$ is projective in $\mathscr{F}$, we may choose $X_{|0}\mathrm{B} \xrightarrow{\vartheta} Y_\omega^{[-1]}\mathrm{B}$ in $\mathscr{F}$ such that $\vartheta \cdot Y_\omega^{[-1]}\pi = \lambda$.

Second step.

For $k \in \mathbf{Z}_{\geq 0}$, we want to construct $X_{|k} \xrightarrow{\mu_k} Y_\omega^{[-1]}\mathrm{B}$ in $\mathscr{F}$ recursively such that $x_{|k} \cdot \mu_{k+1} = \mu_k$, such that $x^\omega_{|k} \cdot f \cdot y^\omega_{k+1|} = \mu_k \cdot Y_\omega^{[-1]}\pi \cdot y^\omega_{k+1|}$ and such that $\mu_0 = X_{|0}\iota \cdot \vartheta$.

We have $\mu_0 \cdot Y_\omega^{[-1]}\pi \cdot y^\omega_{1|} = X_{|0}\iota \cdot \vartheta \cdot Y_\omega^{[-1]}\pi \cdot y^\omega_{1|} = X_{|0}\iota \cdot \lambda \cdot y^\omega_{1|} = X_{|0}\iota \cdot \lambda_1 = x^\omega_{|0} \cdot f \cdot y^\omega_{1|}$.

Suppose given $k \in \mathbf{Z}_{>0}$. Suppose that we have already constructed μ_ℓ for $\ell \in [0, k-1]$.

Since $Y_\omega^{[-1]}\mathrm{B}$ is injective in $\mathscr{F}$, we may choose $X_{|k} \xrightarrow{\beta_k} Y_\omega^{[-1]}\mathrm{B}$ in $\mathscr{F}$ such that $x_{|k-1} \cdot \beta_k = \mu_{k-1}$. We have

$$\begin{aligned} x_{|k-1} \cdot (x^\omega_{|k} \cdot f \cdot y^\omega_{k+1|} - \beta_k \cdot Y_\omega^{[-1]}\pi \cdot y^\omega_{k+1|}) &= x^\omega_{|k-1} \cdot f \cdot y^\omega_{k|} \cdot y_{k|} - \mu_{k-1} \cdot Y_\omega^{[-1]}\pi \cdot y^\omega_{k|} \cdot y_{k|} \\ &= (x^\omega_{|k-1} \cdot f \cdot y^\omega_{k|} - \mu_{k-1} \cdot Y_\omega^{[-1]}\pi \cdot y^\omega_{k|}) \cdot y_{k|} = 0. \end{aligned}$$

Since moreover ${}_{\underline{\mathscr{F}}}(X_{k/k-1} , Y_{k+1|}) = 0$ by remark 4.3.5.(b), we may choose $X_{k/k-1} \xrightarrow{\nu_k} Y_\omega^{[-1]}\mathrm{B}$ in $\mathscr{F}$ such that

$$X_{|k/k-1} \cdot \nu_k \cdot Y_\omega^{[-1]}\pi \cdot y^\omega_{k+1|} = x^\omega_{|k} \cdot f \cdot y^\omega_{k+1|} - \beta_k \cdot Y_\omega^{[-1]}\pi \cdot y^\omega_{k+1|} .$$

$$\begin{tikzcd} X_{|k-1} \arrow[r, "x_{|k-1}"] & X_{|k} \arrow[rr, "x^\omega_{|k} \cdot f \cdot y^\omega_{k+1|} - \beta_k \cdot Y_\omega^{[-1]}\pi \cdot y^\omega_{k+1|}"] \arrow[d, "X_{|k/k-1}"'] & & Y_{k+1|} \\ & X_{k/k-1} \arrow[rr, "\nu_k"] & & Y_\omega^{[-1]}\mathrm{B} \arrow[u, "Y_\omega^{[-1]}\pi \cdot y^\omega_{k+1|}"'] \end{tikzcd}$$

Let $\mu_k = \beta_k + X_{|k/k-1} \cdot \nu_k$. We have $x_{|k-1} \cdot \mu_k = \mu_{k-1}$ and

$$\begin{aligned} \mu_k \cdot Y_\omega^{[-1]}\pi \cdot y^\omega_{k+1|} &= \beta_k \cdot Y_\omega^{[-1]}\pi \cdot y^\omega_{k+1|} + X_{|k/k-1} \cdot \nu_k \cdot Y_\omega^{[-1]}\pi \cdot y^\omega_{k+1|} \\ &= \beta_k \cdot Y_\omega^{[-1]}\pi \cdot y^\omega_{k+1|} + x^\omega_{|k} \cdot f \cdot y^\omega_{k+1|} - \beta_k \cdot Y_\omega^{[-1]}\pi \cdot y^\omega_{k+1|} \\ &= x^\omega_{|k} \cdot f \cdot y^\omega_{k+1|} . \end{aligned}$$

Since $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{inj}}(\mathscr{F}))$ and since $Y_\omega^{[-1]}\mathrm{B}$ is injective in $\mathscr{F}$, we may choose $X_\omega \xrightarrow{\mu} Y_\omega^{[-1]}\mathrm{B}$ in $\mathscr{F}$ such that $x^\omega_{|k} \cdot \mu = \mu_k$ for $k \in \mathbf{Z}_{\geq 0}$.

For $k \in \mathbf{Z}_{\geq 0}$, we have $x^\omega_{|k} \cdot \mu \cdot Y_\omega^{[-1]}\pi \cdot y^\omega_{k+1|} = \mu_k \cdot Y_\omega^{[-1]}\pi \cdot y^\omega_{k+1|} = x^\omega_{|k} \cdot f \cdot y^\omega_{k+1|}$.

For $k \in \mathbf{Z}_{<0}$, we have

$$\begin{aligned} x^\omega_{|k} \cdot \mu \cdot Y_\omega^{[-1]}\pi \cdot y^\omega_{k+1|} &= X_{|k \rightarrow |0} \cdot x^\omega_{|0} \cdot \mu \cdot Y_\omega^{[-1]}\pi \cdot y^\omega_{k+1|} = X_{|k \rightarrow |0} \cdot \mu_0 \cdot Y_\omega^{[-1]}\pi \cdot y^\omega_{k+1|} \\ &= X_{|k \rightarrow |0} \cdot X_{|0}\iota \cdot \vartheta \cdot Y_\omega^{[-1]}\pi \cdot y^\omega_{k+1|} = X_{|k \rightarrow |0} \cdot X_{|0}\iota \cdot \lambda \cdot y^\omega_{k+1|} \\ &= X_{|k \rightarrow |0} \cdot X_{|0}\iota \cdot \lambda_{k+1} = x^\omega_{|k} \cdot f \cdot y^\omega_{k+1|} . \end{aligned}$$

By lemma 3.4.3.(b), we may choose $X \xrightarrow{g} Y$ in $\mathrm{FO}(\mathscr{F})$ with $g_\omega = f - \mu \cdot Y_\omega^{[-1]}\pi$. We conclude that $\underline{g_\omega} = \underline{f}$ in $\underline{\mathscr{F}}$. $\square$

4.3.16 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}_{|\mathscr{W}}(\mathscr{F}))$, $Y \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}|}(\mathscr{F}))$ and $X_\omega\mathrm{B} \xrightarrow{f} Y_\omega$ in $\mathscr{F}$ such that $x^\omega_{|k} \cdot X_\omega\iota \cdot f \cdot y^\omega_{k+1|} = 0$ for $k \in \mathbf{Z}$. Then there exists $X^{[1]}_\omega \xrightarrow{\xi} Y_\omega$ in $\mathscr{F}$ such that $x^\omega_{|k-1}\mathrm{B} \cdot (f - X_\omega\pi \cdot \xi) \cdot y^\omega_{k+1|} = 0$ for $k \in \mathbf{Z}$. $\diamondsuit$

Proof. For $k \in \mathbf{Z}$, we have $X_{|k}\iota \cdot x^\omega_{|k}\mathrm{B} \cdot f \cdot y^\omega_{k+1|} = x^\omega_{|k} \cdot X_\omega\iota \cdot f \cdot y^\omega_{k+1|} = 0$, which we use frequently.

First step.

For $k \in \mathbf{Z}_{\leq 2}$, we want to construct $X^{[1]}_{|0}\mathrm{B} \xrightarrow{\varepsilon_k} Y_{k|}$ in $\mathscr{F}$ recursively such that $\varepsilon_{k-1} \cdot y_{k-1} = \varepsilon_k$ and such that $x^\omega_{|k-2}\mathrm{B} \cdot f \cdot y^\omega_{k|} = X_{|k-2}\pi \cdot X^{[1]}_{|k-2}\iota \cdot X^{[1]}_{|k-2\to|0}\mathrm{B} \cdot \varepsilon_k$.

We have $X_{|0}\iota \cdot x^\omega_{|0}\mathrm{B} \cdot f \cdot y^\omega_{2|} = X_{|0}\iota \cdot x^\omega_{|0}\mathrm{B} \cdot f \cdot y^\omega_{1|} \cdot y_{1|} = 0$. Since $\underline{\mathscr{F}}(X^{[1]}_{|0}, Y_{2|}) = 0$, we may choose $X^{[1]}_{|0}\mathrm{B} \xrightarrow{\varepsilon_2} Y_{2|}$ in $\mathscr{F}$ such that $x^\omega_{|0}\mathrm{B} \cdot f \cdot y^\omega_{2|} = X_{|0}\pi \cdot X^{[1]}_{|0}\iota \cdot \varepsilon_2$.

$$
\begin{tikzcd}
X_{|0} \arrow[r, "X_{|0}\iota"] & X_{|0}\mathrm{B} \arrow[rr, "x^\omega_{|0}\mathrm{B}\cdot f\cdot y^\omega_{2|}"] \arrow[d, "X_{|0}\pi"'] & & Y_{2|} \\
& X^{[1]}_{|0} \arrow[r, "X^{[1]}_{|0}\iota"] & X^{[1]}_{|0}\mathrm{B} \arrow[ur, "\varepsilon_2"'] &
\end{tikzcd}
$$

Suppose given $k \in \mathbf{Z}_{<2}$. Suppose that we have constructed ε_ℓ for $\ell \in [k+1, 2]$.

Since $X^{[1]}_{|0}\mathrm{B}$ is projective in $\mathscr{F}$, we may choose $X^{[1]}_{|0}\mathrm{B} \xrightarrow{\beta_k} Y_{k|}$ in $\mathscr{F}$ such that $\beta_k \cdot y_{k|} = \varepsilon_{k+1}$. We have

$$
\begin{aligned}
X_{|k-2}\iota \cdot (x^\omega_{|k-2}\mathrm{B} \cdot f \cdot y^\omega_{k|} - X_{|k-2}\pi \cdot X^{[1]}_{|k-2}\iota \cdot X^{[1]}_{|k-2\to|0}\mathrm{B} \cdot \beta_k) &= X_{|k-2}\iota \cdot x^\omega_{|k-2}\mathrm{B} \cdot f \cdot y^\omega_{k-1|} \cdot y_{k-1|} - 0 \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
& (x^\omega_{|k-2}\mathrm{B} \cdot f \cdot y^\omega_{k|} - X_{|k-2}\pi \cdot X^{[1]}_{|k-2}\iota \cdot X^{[1]}_{|k-2\to|0}\mathrm{B} \cdot \beta_k) \cdot y_{k|} \\
&= x_{|k-2}\mathrm{B} \cdot x^\omega_{|k-1}\mathrm{B} \cdot f \cdot y^\omega_{k+1|} - X_{|k-2}\pi \cdot X^{[1]}_{|k-2}\iota \cdot x^{[1]}_{|k-2}\mathrm{B} \cdot X^{[1]}_{|k-1\to|0}\mathrm{B} \cdot \varepsilon_{k+1} \\
&= x_{|k-2}\mathrm{B} \cdot x^\omega_{|k-1}\mathrm{B} \cdot f \cdot y^\omega_{k+1|} - x_{|k-2}\mathrm{B} \cdot X_{|k-1}\pi \cdot X^{[1]}_{|k-1}\iota \cdot X^{[1]}_{|k-1\to|0}\mathrm{B} \cdot \varepsilon_{k+1} \\
&= x_{|k-2}\mathrm{B} \cdot (x^\omega_{|k-1}\mathrm{B} \cdot f \cdot y^\omega_{k+1|} - X_{|k-1}\pi \cdot X^{[1]}_{|k-1}\iota \cdot X^{[1]}_{|k-1\to|0}\mathrm{B} \cdot \varepsilon_{k+1}) = 0.
\end{aligned}
$$

Since moreover $\underline{\mathscr{F}}(X^{[1]}_{|k-2}, Y_{k/k-1}) = 0$ by remark 4.3.5.(a), we may choose $X^{[1]}_{|0}\mathrm{B} \xrightarrow{\lambda_k} Y_{k/k-1}$ in $\mathscr{F}$ such that

$x^\omega_{|k-2}\mathrm{B} \cdot f \cdot y^\omega_{k|} - X_{|k-2}\pi \cdot X^{[1]}_{|k-2}\iota \cdot X^{[1]}_{|k-2\to|0}\mathrm{B} \cdot \beta_k = X_{|k-2}\pi \cdot X^{[1]}_{|k-2}\iota \cdot X^{[1]}_{|k-2\to|0}\mathrm{B} \cdot \lambda_k \cdot Y_{k/k-1|}$.

$$
\begin{tikzcd}
X_{|k-2} \arrow[r, "X_{|k-2}\iota"] & X_{|k-2}\mathrm{B} \arrow[rr, "x^\omega_{|k-2}\mathrm{B}\cdot f\cdot y^\omega_{k|} - X_{|k-2}\pi\cdot X^{[1]}_{|k-2}\iota\cdot X^{[1]}_{|k-2\to|0}\mathrm{B}\cdot\beta_k"] \arrow[d, "X_{|k-2}\pi"'] & & Y_{k|} \arrow[r, "y_{k|}"] & Y_{k+1|} \\
& X^{[1]}_{|k-2} \arrow[r, "X^{[1]}_{|k-2}\iota\cdot X^{[1]}_{|k-2\to|0}\mathrm{B}"] & X^{[1]}_{|0}\mathrm{B} \arrow[r, "\lambda_k"] & Y_{k/k-1} \arrow[u, "Y_{k/k-1|}"'] &
\end{tikzcd}
$$

Let $\varepsilon_k = \beta_k + \lambda_k \cdot Y_{k/k-1|}$. We have $\varepsilon_k \cdot y_{k|} = \varepsilon_{k+1}$ and

$$\begin{aligned} & X_{|k-2}\pi \cdot X^{[1]}_{|k-2}\iota \cdot X^{[1]}_{|k-2\to|0}\mathrm{B} \cdot \varepsilon_k \\ = & X_{|k-2}\pi \cdot X^{[1]}_{|k-2}\iota \cdot X^{[1]}_{|k-2\to|0}\mathrm{B} \cdot \beta_k + X_{|k-2}\pi \cdot X^{[1]}_{|k-2}\iota \cdot X^{[1]}_{|k-2\to|0}\mathrm{B} \cdot \lambda_k \cdot Y_{k/k-1|} \\ = & x^{\omega}_{|k-2}\mathrm{B} \cdot f \cdot y^{\omega}_{k|} \,. \end{aligned}$$

Since $Y \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{proj}}(\mathscr{F}))$ and since $X^{[1]}_{|0}\mathrm{B}$ is projective in $\mathscr{F}$, we may choose $X^{[1]}_{|0}\mathrm{B} \xrightarrow{\varepsilon} Y_\omega$ in $\mathscr{F}$ such that $\varepsilon \cdot y^{\omega}_{k|} = \varepsilon_k$ for $k \in \mathbf{Z}_{\leq 2}$.

Since $X^{[1]}_{|0}\mathrm{B}$ is projective in $\mathscr{F}$, we may choose $X^{[1]}_{|0}\mathrm{B} \xrightarrow{\mu} Y^{[-1]}_{\omega}\mathrm{B}$ in $\mathscr{F}$ such that $\mu \cdot Y^{[-1]}_{\omega}\pi = \varepsilon$.

Second step.

For $k \in \mathbf{Z}_{\geq 0}$, we want to construct $X^{[1]}_{|k} \xrightarrow{\vartheta_k} Y^{[-1]}_{\omega}\mathrm{B}$ in $\mathscr{F}$ recursively such that $\vartheta_k = x^{[1]}_{|k} \cdot \vartheta_{k+1}$, such that $x^{\omega}_{|k}\mathrm{B} \cdot f \cdot y^{\omega}_{k+2|} = X_{|k}\pi \cdot \vartheta_k \cdot Y^{[-1]}_{\omega}\pi \cdot y^{\omega}_{k+2|}$ and such that $\vartheta_0 = X^{[1]}_{|0}\iota \cdot \mu$.

We have

$$\begin{aligned} X_{|0}\pi \cdot \vartheta_0 \cdot Y^{[-1]}_{\omega}\pi \cdot y^{\omega}_{2|} &= X_{|0}\pi \cdot X^{[1]}_{|0}\iota \cdot \mu \cdot Y^{[-1]}_{\omega}\pi \cdot y^{\omega}_{2|} = X_{|0}\pi \cdot X^{[1]}_{|0}\iota \cdot \varepsilon \cdot y^{\omega}_{2|} = X_{|0}\pi \cdot X^{[1]}_{|0}\iota \cdot \varepsilon_2 \\ &= x^{\omega}_{|0} \cdot f \cdot y^{\omega}_{2|} \,. \end{aligned}$$

Suppose given $k \in \mathbf{Z}_{>0}$. Suppose that we have constructed ϑ_ℓ for $\ell \in [0, k-1]$.

Since $X^{[-1]}_{\omega}\mathrm{B}$ is injective in $\mathscr{F}$, we may choose $X^{[1]}_{|k} \xrightarrow{\varphi_k} Y^{[-1]}_{\omega}\mathrm{B}$ in $\mathscr{F}$ such that $x^{[1]}_{|k-1} \cdot \varphi_k = \vartheta_{k-1}$.

We have

$$\begin{aligned} & x_{|k-1}\mathrm{B} \cdot (x^{\omega}_{|k}\mathrm{B} \cdot f \cdot y^{\omega}_{k+2|} - X_{|k}\pi \cdot \varphi_k \cdot Y^{[-1]}_{\omega}\pi \cdot y^{\omega}_{k+2|}) \\ = & x^{\omega}_{|k-1}\mathrm{B} \cdot f \cdot y^{\omega}_{k+1|} \cdot y_{k+1|} - X_{|k-1}\pi \cdot x^{[1]}_{|k-1} \cdot \varphi_k \cdot Y^{[-1]}_{\omega}\pi \cdot y^{\omega}_{k+1|} \cdot y_{k+1|} \\ = & (x^{\omega}_{|k-1}\mathrm{B} \cdot f \cdot y^{\omega}_{k+1|} - X_{|k-1}\pi \cdot \vartheta_{k-1} \cdot Y^{[-1]}_{\omega}\pi \cdot y^{\omega}_{k+1|}) \cdot y_{k+1|} = 0. \end{aligned}$$

So we may choose $X_{k/k-1}\mathrm{B} \xrightarrow{\eta_k} Y_{k+2|}$ in $\mathscr{F}$ such that

$$X_{|k/k-1}\mathrm{B} \cdot \eta_k = x^{\omega}_{|k}\mathrm{B} \cdot f \cdot y^{\omega}_{k+2|} - X_{|k}\pi \cdot \varphi_k \cdot Y^{[-1]}_{\omega}\pi \cdot y^{\omega}_{k+2|} \,.$$

We have $X_{k/k-1}\iota \cdot \eta_k = 0$ since

$$\begin{aligned} X_{|k/k-1} \cdot X_{k/k-1}\iota \cdot \eta_k &= X_{|k}\iota \cdot X_{|k/k-1}\mathrm{B} \cdot \eta_k = X_{|k}\iota \cdot (x^{\omega}_{|k}\mathrm{B} \cdot f \cdot y^{\omega}_{k+2|} - X_{|k}\pi \cdot \varphi_k \cdot Y^{[-1]}_{\omega}\pi \cdot y^{\omega}_{k+2|}) \\ &= X_{|k}\iota \cdot x^{\omega}_{|k}\mathrm{B} \cdot f \cdot y^{\omega}_{k+1|} \cdot y_{k+1|} - 0 = 0. \end{aligned}$$

and since $X_{|k/k-1}$ is a pure epimorphism. Since moreover $\underline{\mathscr{F}}(X^{[1]}_{k/k-1}, Y_{k+2|}) = 0$ by remark

4.3.5.(b), we may choose $X^{[1]}_{k/k-1} \xrightarrow{\zeta_k} Y^{[-1]}_\omega \mathrm{B}$ in $\mathscr{F}$ such that $X_{k/k-1}\pi \cdot \zeta_k \cdot Y^{[-1]}_\omega \pi \cdot y^\omega_{k+2|} = \eta_k$.

$$\begin{tikzcd}
X_{|k-1}\mathrm{B} \arrow[r, "x_{|k-1}\mathrm{B}"] & X_{|k}\mathrm{B} \arrow[rr, "x^\omega_{|k}\mathrm{B}\cdot f\cdot y^\omega_{k+2|} - X_{|k}\pi\cdot\varphi_k\cdot Y^{[-1]}_\omega\pi\cdot y^\omega_{k+2|}"] \arrow[d, "X_{|k/k-1}\mathrm{B}"'] & & Y_{k+2|} \\
X_{k/k-1} \arrow[r, "X_{k/k-1}\iota"] & X_{k/k-1}\mathrm{B} \arrow[urr, "\eta_k"'] \arrow[d, "X_{k/k-1}\pi"'] & & \\
& X^{[1]}_{k/k-1} \arrow[rr, "\zeta_k"] & & Y^{[-1]}_\omega\mathrm{B} \arrow[uu, "Y^{[-1]}_\omega\pi\cdot y^\omega_{k+2|}"']
\end{tikzcd}$$

Let $\vartheta_k = \varphi_k + X^{[1]}_{|k/k-1} \cdot \zeta_k$. We have $x^{[1]}_{|k-1} \cdot \vartheta_k = \vartheta_{k-1}$ and

$$\begin{aligned}
X_{|k}\pi \cdot \vartheta_k \cdot Y^{[-1]}_\omega \pi \cdot y^\omega_{k+2|} &= X_{|k}\pi \cdot \varphi_k \cdot Y^{[-1]}_\omega \pi \cdot y^\omega_{k+2|} + X_{|k/k-1}\mathrm{B} \cdot X_{k/k-1}\pi \cdot \zeta_k \cdot Y^{[-1]}_\omega \pi \cdot y^\omega_{k+2|} \\
&= X_{|k}\pi \cdot \varphi_k \cdot Y^{[-1]}_\omega \pi \cdot y^\omega_{k+2|} + X_{|k/k-1}\mathrm{B} \cdot \eta_k \\
&= X_{|k}\pi \cdot \varphi_k \cdot Y^{[-1]}_\omega \pi \cdot y^\omega_{k+2|} + x^\omega_{|k}\mathrm{B} \cdot f \cdot y^\omega_{k+2|} - X_{|k}\pi \cdot \varphi_k \cdot Y^{[-1]}_\omega \pi \cdot y^\omega_{k+2|} \\
&= x^\omega_{|k}\mathrm{B} \cdot f \cdot y^\omega_{k+2|} \, .
\end{aligned}$$

Since $X^{[1]} \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{inj}}(\mathscr{F}))$ by lemma 3.4.58 and since $Y^{[-1]}_\omega\mathrm{B}$ is injective in $\mathscr{F}$, we may choose $X^{[1]}_\omega \xrightarrow{\vartheta} Y^{[-1]}_\omega\mathrm{B}$ in $\mathscr{F}$ such that $(x^\omega_{|k})^{[1]} \cdot \vartheta = \vartheta_k$ for $k \in \mathbf{Z}_{\geq 0}$. Let $\xi = \vartheta \cdot Y^{[-1]}_\omega \pi$.
For $k \in \mathbf{Z}_{\geq 1}$, we have

$$\begin{aligned}
X_{|k-1}\pi \cdot (x^\omega_{|k-1})^{[1]} \cdot \xi \cdot y^\omega_{k+1|} &= X_{|k-1}\pi \cdot (x^\omega_{|k-1})^{[1]} \cdot \vartheta \cdot Y^{[-1]}_\omega \pi \cdot y^\omega_{k+1|} \\
&= X_{|k-1}\pi \cdot \vartheta_{k-1} \cdot Y^{[-1]}_\omega \pi \cdot y^\omega_{k+1|} \\
&= x^\omega_{|k-1}\mathrm{B} \cdot f \cdot y^\omega_{k+1|} \, .
\end{aligned}$$

For $k \in \mathbf{Z}_{<1}$, we have

$$\begin{aligned}
X_{|k-1}\pi \cdot (x^\omega_{|k-1})^{[1]} \cdot \xi \cdot y^\omega_{k+1|} &= X_{|k-1}\pi \cdot X^{[1]}_{|k-1 \rightarrow |0} \cdot (x^\omega_{|0})^{[1]} \cdot \vartheta \cdot Y^{[-1]}_\omega \pi \cdot y^\omega_{k+1|} \\
&= X_{|k-1}\pi \cdot X^{[1]}_{|k-1 \rightarrow |0} \cdot \vartheta_0 \cdot Y^{[-1]}_\omega \pi \cdot y^\omega_{k+1|} \\
&= X_{|k-1}\pi \cdot X^{[1]}_{|k-1 \rightarrow |0} \cdot X^{[1]}_{|0}\iota \cdot \mu \cdot Y^{[-1]}_\omega \pi \cdot y^\omega_{k+1|} \\
&= X_{|k-1}\pi \cdot X^{[1]}_{|k-1}\iota \cdot X^{[1]}_{|k-1 \rightarrow |0}\mathrm{B} \cdot \varepsilon \cdot y^\omega_{k+1|} \\
&= X_{|k-1}\pi \cdot X^{[1]}_{|k-1}\iota \cdot X^{[1]}_{|k-1 \rightarrow |0}\mathrm{B} \cdot \varepsilon_{k+1} \\
&= x^\omega_{|k-1}\mathrm{B} \cdot f \cdot y^\omega_{k+1|} \, .
\end{aligned}$$

We conclude that

$$x^\omega_{|k-1}\mathrm{B} \cdot (f - X_\omega \pi \cdot \xi) \cdot y^\omega_{k+1|} = x^\omega_{|k-1}\mathrm{B} \cdot f \cdot y^\omega_{k+1|} - X_{|k-1}\pi \cdot (x^\omega_{|k-1})^{[1]} \cdot \xi \cdot y^\omega_{k+1|} = 0$$

for $k \in \mathbf{Z}$. $\square$

4.3.17 Corollary. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}_{|\mathscr{W}}(\mathscr{F}))$, $Y \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}|}(\mathscr{F}))$ and $X \xrightarrow{g} Y$ in $\mathrm{FO}(\mathscr{F})$. If $\underline{g_\omega} = 0$ in $\underline{\mathscr{F}}$, then $\underline{\underline{g}} = 0$ in $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$. $\diamondsuit$

Proof. Suppose that $\underline{g_\omega} = 0$ in $\underline{\mathscr{F}}$. We may choose $X_\omega \mathrm{B} \xrightarrow{f} Y_\omega$ in $\mathscr{F}$ such that $X_\omega \iota \cdot f = g_\omega$. By lemma 3.4.3.(a), we have $x^\omega_{|k} \cdot X_\omega \iota \cdot f \cdot y^\omega_{k+1|} = x^\omega_{|k} \cdot g_\omega \cdot y^\omega_{k+1|} = 0$ for $k \in \mathbf{Z}$. The previous lemma 4.3.16 yields $X^{[1]}_\omega \xrightarrow{\xi} Y_\omega$ in $\mathscr{F}$ such that $x^\omega_{|k-1}\mathrm{B} \cdot (f - X_\omega \pi \cdot \xi) \cdot y^\omega_{k+1|} = 0$ for $k \in \mathbf{Z}$. Thus we obtain a morphism $X_{[-1]}\mathbf{B} \xrightarrow{h} Y$ in $\mathrm{FO}(\mathscr{F})$ such that $h_\omega = f - X_\omega \pi \cdot \xi$, cf. lemma 3.4.3.(b). So $X_\omega \iota \cdot h_\omega = X_\omega \iota \cdot f = g_\omega$ in $\mathscr{F}$ and thus $X_{[-1]}\boldsymbol{\iota} \cdot h = X_{[-1]}\boldsymbol{\rho} \cdot g$ by lemma 3.4.3.(c). We conclude that $\underline{\underline{g}} = 0$ in $\underline{\underline{\mathrm{FO}}}(\mathscr{F})$, cf. remark 3.4.14. $\square$

4.3.18 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}}(\mathscr{F}))$ and $m \in \mathbf{Z}$.

(a) If $X \in \mathrm{Ob}(\mathrm{FO}^{\lceil m}(\mathscr{F}))$, then we have $X_\omega \in \mathrm{Ob}(\mathscr{W}_{\lceil m})$.

(b) If $X \in \mathrm{Ob}(\mathrm{FO}^{m\rceil}(\mathscr{F}))$, then we have $X_\omega \in \mathrm{Ob}(\mathscr{W}_{m\rceil})$.

$\diamondsuit$

Proof. Ad (a). Suppose that $X \in \mathrm{Ob}(\mathrm{FO}^{\lceil m}(\mathscr{F}))$. We have the pure short exact sequence $X_{|m} \xrightarrow{x^\omega_{|m}} X_\omega \xrightarrow{x^\omega_{m+1|}} X_{m+1|}$ with $X_{m+1|} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$. So $X_\omega \in \mathrm{Ob}(\mathscr{W}_{\lceil m})$ since $X_{|m} \in \mathrm{Ob}(\mathscr{W}_{\lceil m})$.
Ad (b). This is dual to (a). $\square$

4.3.19 Corollary. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}))$. Then we have $X_\omega \in \mathrm{Ob}(\mathscr{W}^{\mathrm{b}})$. $\diamondsuit$

4.3.20 Remark. Suppose given $m \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathscr{W}_{\lceil m,m\rceil})$.
We have $(X\mathrm{E}_{\mathscr{F}})_{[-m]} \in \mathrm{Ob}\big(\mathrm{FO}^{\lceil m,m\rceil}_{\mathscr{W}}(\mathscr{F})\big)$ and $\big((X\mathrm{E}_{\mathscr{F}})_{[-m]}\big)_\omega = X$, cf. definition 3.4.65 and remark 3.4.27. $\diamondsuit$

4.3.21 Lemma. Suppose given $m, n \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathscr{W}_{\lceil m,n\rceil})$.
There exists $W \in \mathrm{Ob}(\mathrm{FO}^{\lceil m,n\rceil}_{\mathscr{W}}(\mathscr{F}))$ such that W_ω is isomorphic to X in $\mathscr{W}^{\mathrm{b}}$. $\diamondsuit$

Proof. If $m < n$, then $X \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$ and we may choose $W = 0_{\mathrm{FO}(\mathscr{F})}$.
Suppose that $m \geq n$. We will use induction on $m - n \in \mathbf{Z}_{\geq 0}$.

If $m = n$, then we may choose $W = (X\mathrm{E}_{\mathscr{F}})_{[-m]}$ by remark 4.3.20.

If $m > n$, we may choose a triangle $Y \xrightarrow{f} Z \longrightarrow X \longrightarrow Y^{[1]}$ in $\underline{\mathscr{F}}$ such that $Y \in \mathrm{Ob}(\mathscr{W}_{m-1\rceil})$ and $Z \in \mathrm{Ob}(\mathscr{W}_{\lceil m-1})$, cf. remark 4.3.1.

Since $X^{[-1]}, Z \in \mathrm{Ob}(\mathscr{W}_{\lceil m-1})$, we have $Y \in \mathrm{Ob}(\mathscr{W}_{\lceil m-1,m-1\rceil})$. So we may choose $U \in \mathrm{Ob}(\mathrm{FO}^{\lceil m-1,m-1\rceil}_{\mathscr{W}}(\mathscr{F}))$ and $U_\omega \xrightarrow{a} Y$ in $\mathscr{F}$ such that $\underline{a}$ is an isomorphism in $\underline{\mathscr{F}}$.

Since $Y, X \in \mathrm{Ob}(\mathscr{W}_{n\rceil})$, we have $Z \in \mathrm{Ob}(\mathscr{W}_{\lceil m-1,n\rceil})$. By induction, we may choose $V \in \mathrm{Ob}(\mathrm{FO}^{\lceil m-1,n\rceil}_{\mathscr{W}}(\mathscr{F}))$ and $V_\omega \xrightarrow{b} Z$ in $\mathscr{F}$ such that $\underline{b}$ is an isomorphism in $\underline{\mathscr{F}}$.

Consider the morphism $U_\omega \xrightarrow{\underline{a} \cdot \underline{f} \cdot (\underline{b})^{-1}} V_\omega$ in $\underline{\mathscr{F}}$. By lemma 4.3.15, we may choose $U \xrightarrow{g} V$ in $\mathrm{FO}(\mathscr{F})$ such that $\underline{g_\omega} = \underline{a} \cdot \underline{f} \cdot (\underline{b})^{-1}$.

Choose a pseudo-triangle $U \xrightarrow{g} V \xrightarrow{i} W \xrightarrow{p} U^{[1]}_{[-1]}$ in $\mathrm{FO}(\mathscr{F})$ such that $W, U^{[1]}_{[-1]} \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}}(\mathscr{F}))$, cf. lemmata 3.4.18 and 4.3.14.

Note that $W \in \mathrm{Ob}(\mathrm{FO}^{[m,n]}_{\mathscr{W}}(\mathscr{F}))$, cf. remark 3.4.27 and lemma 3.4.28.

By lemma 3.4.19.(b), $U_\omega \xrightarrow{g_\omega} V_\omega \xrightarrow{i_\omega} W_\omega \xrightarrow{p_\omega} U^{[1]}_\omega$ is a pseudo-triangle in $\mathscr{F}$.

So $U_\omega \xrightarrow{g_\omega} V_\omega \xrightarrow{i_\omega} W_\omega \xrightarrow{p_\omega} U^{[1]}_\omega$ is a triangle in $\underline{\mathscr{F}}$, cf. definition 2.2.14. We have $\underline{g_\omega} \cdot \underline{b} = \underline{a} \cdot \underline{f}$.

$$\begin{tikzcd}
U_\omega \arrow[r, "\underline{g_\omega}"] \arrow[d, "\underline{a}"'] & V_\omega \arrow[r, "\underline{i_\omega}"] \arrow[d, "\underline{b}"] & W_\omega \arrow[r, "\underline{p_\omega}"] & U^{[1]}_\omega \arrow[d, "\underline{a^{[1]}}"] \\
Y \arrow[r, "\underline{f}"] & Z \arrow[r] & X \arrow[r] & Y^{[1]}
\end{tikzcd}$$

Since $\underline{a}$ and $\underline{b}$ are isomorphisms in $\underline{\mathscr{F}}$, we conclude that W_ω is isomorphic to X in $\underline{\mathscr{F}}$. $\square$

4.3.22 Corollary. Suppose given $X \in \mathrm{Ob}(\mathscr{W}^{\mathrm{b}})$. There exists $W \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}))$ such that W_ω is isomorphic to X in $\mathscr{W}^{\mathrm{b}}$. $\diamondsuit$

Proof. We may choose $m, n \in \mathbf{Z}$ such that $X \in \mathrm{Ob}(\mathscr{W}_{[m,n]})$. So the result follows from the previous lemma 4.3.21. $\square$

4.3.23 Remark. Suppose that $\mathscr{F}$ has countable coproducts of bijectives. Suppose given $X \in \mathrm{Ob}(\mathrm{F}(\mathscr{F}))$.

(a) There exists a colimit $(A, (a_k)_{k \in \mathbf{Z}})$ for $X\mathbf{B}$ such that A is bijective in $\mathscr{F}$.

(b) Suppose given a colimit $(C, (c_k)_{k \in \mathbf{Z}})$ for $X\mathbf{B}$. Then C is bijective in $\mathscr{F}$ and c_k is a pure monomorphism for $k \in \mathbf{Z}$.

$\diamondsuit$

Proof. Ad (a). This follows from lemma 3.2.34 and from the fact that coproducts of bijectives are bijective as well.

Ad (b). By (a), there exists a colimit $(A, (a_k)_{k \in \mathbf{Z}})$ for $X\mathbf{B}$ such that A is bijective in $\mathscr{F}$. So C is bijective in $\mathscr{F}$ as well, since it is isomorphic to A in $\mathscr{F}$.

Suppose given $m \in \mathbf{Z}$. We want to show that c_{m-1} is a pure monomorphism in $\mathscr{F}$.

We obtain a filtration $Z \in \mathrm{Ob}(\mathrm{F}(\mathscr{F}))$ and a morphism $X \xrightarrow{p} Z$ in $\mathrm{F}(\mathscr{F})$ as follows. Let $Z_k = 0_{\mathscr{F}}$ and $p_k = 0$ for $k \in \mathbf{Z}_{<m-1}$. For $k \in \mathbf{Z}_{\geq m-1}$, choose cokernels $X_k \xrightarrow{p_k} Z_k$ of $X_{m-1 \to k}$. So for $k \in \mathbf{Z}_{\geq m-1}$, the circumference lemma 1.3.13 yields pure monomorphisms $Z_k \xrightarrow{z_k} Z_{k+1}$ such that $p_k \cdot z_k = x_k \cdot p_{k+1}$.

$$\begin{tikzcd}
 & & Z_k \arrow[ddrr, "z_k"] & & \\
 & X_k \arrow[ur, "p_k"] \arrow[dr, "x_k"] & & & \\
X_{m-1} \arrow[ur, "X_{m-1 \to k}"] \arrow[rr, "X_{m-1 \to k+1}"'] & & X_{k+1} \arrow[rr, "p_{k+1}"] & & Z_{k+1}
\end{tikzcd}$$

Note that p is an m-pure epimorphism. Consequently, $X\mathbf{B} \xrightarrow{p\mathbf{B}} Z\mathbf{B}$ is an m-pure monomorphism as well. By (a), we may choose a colimit $(D, (d_k)_{k\in\mathbf{Z}})$ for $Z\mathbf{B}$. By lemma 3.2.40.(d), $X_{m-1} \xrightarrow{c_{m-1}} C \xrightarrow{p\mathbf{B}\upharpoonright} D$ is a pure short exact sequence in $\mathscr{F}$. $\square$

4.3.24 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{F}(\mathscr{F}))$, a compatible family $(A, (a_k)_{k\in\mathbf{Z}})$ for X and a colimit $(C, (c_k)_{k\in\mathbf{Z}})$ for $X\mathbf{B}$.

(a) $\Big(A \oplus C, \big(\begin{pmatrix} a_k & X_k\iota\cdot c_k \end{pmatrix}\big)_{k\in\mathbf{Z}}\Big)$ is an injective family for X.

(b) Suppose that $\mathscr{F}$ has countable coproducts of bijectives.
So c_k is a pure monomorphism for $k \in \mathbf{Z}$, cf. remark 4.3.23.(b). Consequently, $X_k\iota \cdot c_k$ is also a pure monomorphism for $k \in \mathbf{Z}$. We obtain $Y \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{inj}}(\mathscr{F}))$ as follows.
Let $Y\mathrm{P}_{\mathrm{F},\mathscr{F}} = X$, $Y_\omega = A \oplus C$ and $y^\omega_{|k} = \begin{pmatrix} a_k & X_k\iota\cdot c_k \end{pmatrix}$ for $k \in \mathbf{Z}$. Choose pushouts in $\mathscr{F}$ for $k \in \mathbf{Z}$ as follows.

$$\begin{tikzcd}
X_k \arrow[r, "a_k"] \arrow[d, "X_k\iota\cdot c_k"', "\bullet" marking] & A \arrow[d, "h_{k+1}"] \\
C \arrow[r, "m_{k+1}"] & Y_{k+1|}
\end{tikzcd}$$

Let $y^\omega_{k|} = \begin{pmatrix} -h_k \\ m_k \end{pmatrix}$ for $k \in \mathbf{Z}$. For $k \in \mathbf{Z}$, lemma 1.3.13 yields the pure epimorphism $Y_{k|} \xrightarrow{y_{k|}} Y_{k+1|}$ in $\mathscr{F}$ such that $y^\omega_{k|} \cdot y_{k|} = y^\omega_{k+1|}$.

$$\begin{tikzcd}
 & X_k \arrow[dr, "\begin{pmatrix} a_k & X_k\iota\cdot c_k \end{pmatrix}", "\bullet" marking] & & & \\
X_{k-1} \arrow[ur, "x_{k-1}", "\bullet" marking] \arrow[rr, "\begin{pmatrix} a_{k-1} & X_{k-1}\iota\cdot c_{k-1} \end{pmatrix}"', "\bullet" marking] & & A \oplus C \arrow[rr, "\begin{pmatrix} -h_k \\ m_k \end{pmatrix}", "|" marking] \arrow[dr, "\begin{pmatrix} -h_{k+1} \\ m_{k+1} \end{pmatrix}"', "|" marking] & & Y_{k|} \arrow[dl, "y_{k|}", "|" marking] \\
 & & & Y_{k+1|} &
\end{tikzcd}$$

$\diamondsuit$

Proof. Ad (a). For $k \in \mathbf{Z}$, we have $x_k \cdot \begin{pmatrix} a_{k+1} & X_{k+1}\iota\cdot c_{k+1} \end{pmatrix} = \begin{pmatrix} a_k & X_k\iota\cdot x_k\mathrm{B}\cdot c_{k+1} \end{pmatrix} = \begin{pmatrix} a_k & X_k\iota\cdot c_k \end{pmatrix}$.

Suppose given $X_k \xrightarrow{f_k} I$ in $\mathscr{F}$ such that I is injective and such that $x_k \cdot f_{k+1} = f_k$ for $k \in \mathbf{Z}_{\geq 0}$.

For $k \in \mathbf{Z}_{\geq 0}$, we want to construct $X_k\mathrm{B} \xrightarrow{g_k} I$ in $\mathscr{F}$ recursively such that $x_k\mathrm{B} \cdot g_{k+1} = g_k$ and such that $X_k\iota \cdot g_k = f_k$.

Since I is injective in $\mathscr{F}$, we may choose $X_0\mathrm{B} \xrightarrow{g_0} I$ in $\mathscr{F}$ such that $X_0\iota \cdot g_0 = f_0$.

Suppose given $k \in \mathbf{Z}_{>0}$. Suppose we have already constructed g_ℓ for $\ell \in [0, k-1]$.

We may choose $X_k\mathrm{B} \xrightarrow{s_k} X_{k-1}\mathrm{B}$ in $\mathscr{F}$ such that $x_{k-1}\mathrm{B} \cdot s_k = 1$, cf. lemma 1.3.11.

We have $x_{k-1} \cdot (f_k - X_k\iota \cdot s_k \cdot g_{k-1}) = f_{k-1} - X_{k-1}\iota \cdot x_{k-1}\mathrm{B} \cdot s_k \cdot g_{k-1} = f_{k-1} - X_{k-1}\iota \cdot g_{k-1} = 0$. Since

I is injective in $\mathscr{F}$, we may choose $X_{\overline{k}}\mathrm{B} \xrightarrow{h_k} I$ in $\mathscr{F}$ such that $x_{\overline{k}} \cdot X_{\overline{k}}\iota \cdot h_k = f_k - X_k\iota \cdot s_k \cdot g_{k-1}$.

$$\begin{tikzcd}
X_{k-1} \arrow[r, "x_{k-1}", "\bullet" marking] & X_k \arrow[r, "f_k - X_k\iota\cdot s_k\cdot g_{k-1}"] \arrow[d, "x_{\overline{k}}"', "|" marking] & I \\
& X_{\overline{k}} \arrow[r, "X_{\overline{k}}\iota"] & X_{\overline{k}}\mathrm{B} \arrow[u, "h_k"']
\end{tikzcd}$$

Let $g_k = s_k \cdot g_{k-1} + x_{\overline{k}}\mathrm{B} \cdot h_k$. We have $x_{k-1}\mathrm{B} \cdot g_k = g_{k-1}$ and

$$X_k\iota \cdot g_k = X_k\iota \cdot s_k \cdot g_{k-1} + x_{\overline{k}} \cdot X_{\overline{k}}\iota \cdot h_k = X_k\iota \cdot s_k \cdot g_{k-1} + f_k - X_k\iota \cdot s_k \cdot g_{k-1} = f_k\,.$$

Since $(C, (c_k)_{k\in\mathbf{Z}})$ is a colimit for $X\mathbf{B}$, we may choose $C \xrightarrow{u} I$ in $\mathscr{F}$ such that $c_k \cdot u = g_k$ for $k \in \mathbf{Z}_{\geq 0}$.

Consider the morphism $A \oplus C \xrightarrow{\left(\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right)} I$ in $\mathscr{F}$.
For $k \in \mathbf{Z}_{\geq 0}$, we have $\left(\begin{smallmatrix} a_k & X_k\iota\cdot c_k \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right) = X_k\iota \cdot c_k \cdot u = X_k\iota \cdot g_k = f_k$.
We conclude that $\Big(A \oplus C, \big(\left(\begin{smallmatrix} a_k & X_k\iota\cdot c_k \end{smallmatrix}\right)\big)_{k\in\mathbf{Z}}\Big)$ is an injective family for X.
Ad (b). We have $Y \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{inj}}(\mathscr{F}))$ by (a). $\square$

4.3.25 Lemma. Suppose that $\mathscr{F}$ has countable coproducts of bijectives. Suppose given $m \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathscr{W}_{m\rceil})$. There exists $Y \in \mathrm{Ob}(\mathrm{FO}^{m\rceil}_{\mathscr{W}}(\mathscr{F}))$ such that Y_ω is isomorphic to X in $\mathscr{D}$. $\diamondsuit$

Proof. For $k \in \mathbf{Z}_{\geq m}$, choose triangles $X_{\lceil k} \xrightarrow{\underline{j_k}} X \xrightarrow{\underline{q_{k+1}}} X'_{k+1\rceil} \longrightarrow X^{[1]}_{\lceil k}$ in $\underline{\mathscr{F}}$ such that $X_{\lceil k} \in \mathrm{Ob}(\mathscr{W}_{\lceil k})$ and $X_{k+1\rceil} \in \mathrm{Ob}(\mathscr{W}_{k+1\rceil})$. Moreover, for $k \in \mathbf{Z}_{\geq m}$, choose $X_{\lceil k} \xrightarrow{x_{\lceil k}} X_{\lceil k+1}$ in $\mathscr{F}$ such that $\underline{x_{\lceil k} \cdot j_{k+1}} = \underline{j_k}$ in $\underline{\mathscr{F}}$, cf. lemma 1.7.3.(f).

$$\begin{tikzcd}
X_{\lceil k} \arrow[r, "\underline{j_k}"] \arrow[d, "\underline{x_{\lceil k}}"'] & X \arrow[r, "\underline{q_{k+1}}"] \arrow[d, "\underline{1}"] & X_{k+1\rceil} \\
X_{\lceil k+1} \arrow[r, "\underline{j_{k+1}}"] & X \arrow[r, "\underline{q_{k+2}}"] & X_{k+2\rceil}
\end{tikzcd}$$

For $k \in \mathbf{Z}_{\geq m}$, we want to construct $X'_{\lceil k} \xrightarrow{x'_{\lceil k}} X'_{\lceil k+1}$, $X'_{\lceil k} \xrightarrow{\beta_k} X_{\lceil k}$ and $X'_{\lceil k} \xrightarrow{\tilde{j}_k} X$ in $\mathscr{F}$ recursively such that $X'_{\lceil k+1} \in \mathrm{Ob}(\mathscr{W}_{\lceil k+1})$, $\tilde{j}_k = x'_{\lceil k} \cdot \tilde{j}_{k+1}$, $\underline{\beta_k \cdot j_k} = \underline{\tilde{j}_k}$ and such that $\underline{\beta_k}$ is an isomorphism in $\underline{\mathscr{F}}$.
Let $X'_{\lceil m} = X_{\lceil m} \in \mathrm{Ob}(\mathscr{W}_{\lceil m})$, $\beta_m = 1$ and $\tilde{j}_m = j_m$. We have $\beta_m \cdot j_m = \tilde{j}_m$.
Suppose given $k \in \mathbf{Z}_{\geq m}$. Suppose that we have constructed $X'_{\lceil \ell}$, $\tilde{j}_\ell$, β_ℓ for $\ell \in [m, k]$ and $x'_{\lceil \ell}$ for $\ell \in [m, k-1]$.
Let $X'_{\lceil k+1} = X_{\lceil k+1} \oplus X'_{\lceil k}\mathrm{B} \in \mathrm{Ob}(\mathscr{W}_{\lceil k+1})$, $\beta_{k+1} = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) : X_{\lceil k+1} \oplus X'_{\lceil k}\mathrm{B} \to X_{\lceil k+1}$ and $x'_{\lceil k} = \left(\begin{smallmatrix} \beta_k\cdot x_{\lceil k} & X'_{\lceil k}\iota \end{smallmatrix}\right) : X'_{\lceil k} \to X_{\lceil k+1} \oplus X'_{\lceil k}\mathrm{B}$. Note that $\underline{\beta_{k+1}}$ is an isomorphism in $\underline{\mathscr{F}}$.
Since $\underline{\tilde{j}_k - \beta_k \cdot x_{\lceil k} \cdot j_{k+1}} = \underline{\tilde{j}_k - \beta_k \cdot j_k} = 0$, we may choose $X'_{\lceil k}\mathrm{B} \xrightarrow{u_{k+1}} X$ in $\mathscr{F}$ such that

$\tilde{j}_k - \beta_k \cdot x_{\lceil k} \cdot j_{k+1} = X'_{\lceil k}\iota \cdot u_{k+1}$.

$$\begin{tikzcd}
X'_{\lceil k} \arrow[r, "\tilde{j}_k - \beta_k \cdot x_{\lceil k} \cdot j_{k+1}"] \arrow[d, "X'_{\lceil k}\iota"'] & X \\
X'_{\lceil k}\mathrm{B} \arrow[ur, "u_{k+1}"'] &
\end{tikzcd}$$

Let $\tilde{j}_{k+1} = \left(\begin{smallmatrix} j_{k+1} \\ u_{k+1} \end{smallmatrix}\right) : X_{\lceil k+1} \oplus X'_{\lceil k}\mathrm{B} \to X$. We have $x'_{\lceil k} \cdot \tilde{j}_{k+1} = \beta_k \cdot x_{\lceil k} \cdot j_{k+1} + X'_{\lceil k}\iota \cdot u_{k+1} = \tilde{j}_k$ and $\underline{\beta_{k+1} \cdot j_{k+1}} = \underline{\left(\begin{smallmatrix} j_{k+1} \\ 0 \end{smallmatrix}\right)} = \underline{\left(\begin{smallmatrix} j_{k+1} \\ u_{k+1} \end{smallmatrix}\right)} = \underline{\tilde{j}_{k+1}}$.

Let $Z \in \mathrm{Ob}(\mathrm{F}(\mathscr{F}))$ denote the filtration with $Z_k = 0_{\mathscr{F}}$ for $k \in \mathbf{Z}_{<m}$ and $Z_k = X'_{\lceil k}$, $z_k = x'_{\lceil k}$ for $k \in \mathbf{Z}_{\geq m}$. Choose a colimit $(C, (c_k)_{k \in \mathbf{Z}})$ for $Z\mathbf{B} \in \mathrm{Ob}(\mathrm{F}(\mathscr{F}))$. Note that C is bijective in $\mathscr{F}$. Cf. remark 4.3.23.

Using lemma 4.3.24.(b), we obtain $Y \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{inj}}(\mathscr{F}))$ as follows. Let $Y\mathrm{P}_{\mathrm{F},\mathscr{F}} = Z$, $Y_\omega = X \oplus C$ and $y^\omega_{|k} = \left(\begin{smallmatrix} \tilde{j}_k & X'_{\lceil k}\iota \cdot c_k \end{smallmatrix}\right)$ for $k \in \mathbf{Z}$. Choose pushouts in $\mathscr{F}$ for $k \in \mathbf{Z}$ as follows.

$$\begin{tikzcd}
X'_{\lceil k} \arrow[r, "\tilde{j}_k"] \arrow[d, "X'_{\lceil k}\iota \cdot c_k"'] & X \arrow[d, "h_{k+1}"] \\
C \arrow[r, "m_{k+1}"] & Y_{k+1|}
\end{tikzcd}$$

Let $y^\omega_{k|} = \left(\begin{smallmatrix} -h_k \\ m_k \end{smallmatrix}\right)$ for $k \in \mathbf{Z}$. For $k \in \mathbf{Z}$, lemma 1.3.13 yields the pure epimorphism $Y_{k|} \xrightarrow{y_{k|}} Y_{k+1|}$ in $\mathscr{F}$ such that $y^\omega_{k|} \cdot y_{k|} = y^\omega_{k+1|}$. We have $Y \in \mathrm{Ob}(\mathrm{FO}^{m]}(\mathscr{F}))$ since $Y\mathrm{P}_{\mathrm{F},\mathscr{F}} = Z \in \mathrm{Ob}(\mathrm{F}^{m]}(\mathscr{F}))$ by construction, cf. lemma 3.4.25. Thus $Y \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{lim}}(\mathscr{F})) \subseteq \mathrm{Ob}(\mathrm{FO}^{\mathrm{proj}}(\mathscr{F}))$, cf. remark 3.4.36 and lemma 3.4.54. Note that $Y_\omega = X \oplus C$ is isomorphic to X in $\underline{\mathscr{F}}$ since C is bijective in $\mathscr{F}$. Also note that $Y_{|k} \in \mathrm{Ob}(\mathscr{W}_{\lceil k})$ for $k \in \mathbf{Z}$ by construction. It remains to show that $Y_{k+1|} \in \mathrm{Ob}(\mathscr{W}_{k+1\rceil})$ for $k \in \mathbf{Z}$.

Suppose given $k \in \mathbf{Z}_{<m}$. Consider the pure short exact sequence $Y_{|k} \xrightarrow{y^\omega_{|k}} Y_\omega \xrightarrow{y^\omega_{k+1|}} Y_{k+1|}$ in $\mathscr{F}$. We have $Y_{|k} = Z_k = 0_{\mathscr{F}}$. Thus $Y_{k+1\rceil} \in \mathrm{Ob}(\mathscr{W}_{k+1\rceil})$ since $Y_\omega = X \oplus C \in \mathrm{Ob}(\mathscr{W}_{m\rceil}) \subseteq \mathrm{Ob}(\mathscr{W}_{k+1\rceil})$.

Suppose given $k \in \mathbf{Z}_{\geq m}$. Note that the pure short exact sequence $Y_{|k} \xrightarrow{y^\omega_{|k}} Y_\omega \xrightarrow{y^\omega_{k+1|}} Y_{k+1|}$ in $\mathscr{F}$ yields a triangle $Y_{|k} \xrightarrow{\underline{y^\omega_{|k}}} Y_\omega \xrightarrow{\underline{y^\omega_{k+1|}}} Y_{k+1|} \longrightarrow Y^{[1]}_{|k}$ in $\underline{\mathscr{F}}$, cf. lemma 2.2.9 and definition 2.2.14. We have $\underline{y^\omega_{|k} \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} = \underline{\left(\begin{smallmatrix} \tilde{j}_k & X'_{\lceil k}\iota \cdot c_k \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} = \underline{\tilde{j}_k} = \underline{\beta_k \cdot j_k}$.

$$\begin{tikzcd}
X'_{\lceil k} \arrow[r, "\underline{\left(\begin{smallmatrix} \tilde{j}_k & X'_{\lceil k}\iota \cdot c_k \end{smallmatrix}\right)}"] \arrow[d, "\underline{\beta_k}"'] & X \oplus C \arrow[r, "\underline{y^\omega_{k+1|}}"] \arrow[d, "\underline{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}"] & Y_{k+1|} \arrow[r] & X'^{[1]}_{\lceil k} \arrow[d, "\underline{\beta^{[1]}_k}"] \\
X_{\lceil k} \arrow[r, "\underline{j_k}"] & X \arrow[r, "\underline{q_{k+1}}"] & X_{k+1\rceil} \arrow[r] & X^{[1]}_{\lceil k}
\end{tikzcd}$$

Since both $\underline{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}$ and $\underline{\beta_k}$ are isomorphisms in $\underline{\mathscr{F}}$, we conclude that $Y_{k+1|}$ is isomorphic to $X_{k+1\rceil}$ in $\underline{\mathscr{F}}$. Thus $Y_{k+1|} \in \mathrm{Ob}(\mathscr{W}_{k+1\rceil})$. $\square$

4.3.26 Lemma. Suppose that $\mathscr{F}$ has countable products of bijectives. Suppose given $m \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathscr{W}_{\lceil m})$. There exists $Y \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}}^{\lceil m}(\mathscr{F}))$ such that Y_ω is isomorphic to X in $\mathscr{D}$. $\diamondsuit$

Proof. This is dual to the previous lemma 4.3.25. $\square$

4.3.27 Lemma. Suppose that $\mathscr{F}$ and $\mathscr{F}^{\mathrm{op}}$ have countable products of bijectives.
Suppose given $X \in \mathrm{Ob}(\mathscr{D})$. There exists $W \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}}(\mathscr{F}))$ such that W_ω is isomorphic to X in $\mathscr{D}$. $\diamondsuit$

Proof. We may choose a triangle $Y \xrightarrow{f} Z \longrightarrow X \longrightarrow Y^{[1]}$ in $\underline{\mathscr{F}}$ such that $Y \in \mathrm{Ob}(\mathscr{W}_{0\rceil})$ and $Z \in \mathrm{Ob}(\mathscr{W}_{\lceil 0})$, cf. remark 4.3.1.

By lemma 4.3.25, we may choose $U \in \mathrm{FO}_{\mathscr{W}}(\mathscr{F})$ and $U_\omega \xrightarrow{a} Y$ in $\mathscr{F}$ such that $\underline{a}$ is an isomorphism in $\underline{\mathscr{F}}$. By lemma 4.3.26, we may choose $V \in \mathrm{FO}_{\mathscr{W}}(\mathscr{F})$ and $V_\omega \xrightarrow{b} Z$ in $\mathscr{F}$ such that $\underline{b}$ is an isomorphism in $\underline{\mathscr{F}}$.

Consider the morphism $U_\omega \xrightarrow{\underline{a}\cdot \underline{f}\cdot(\underline{b})^{-1}} V_\omega$ in $\underline{\mathscr{F}}$. By lemma 4.3.15, we may choose $U \xrightarrow{g} V$ in $\mathrm{FO}(\mathscr{F})$ such that $\underline{g_\omega} = \underline{a} \cdot \underline{f} \cdot (\underline{b})^{-1}$.

Choose a pseudo-triangle $U \xrightarrow{g} V \xrightarrow{i} W \xrightarrow{p} U^{[1]}_{[-1]}$ in $\mathrm{FO}(\mathscr{F})$ such that $W, U^{[1]}_{[-1]} \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}}(\mathscr{F}))$, cf. lemmata 3.4.18 and 4.3.14.

By lemma 3.4.19.(b), $U_\omega \xrightarrow{g_\omega} V_\omega \xrightarrow{i_\omega} W_\omega \xrightarrow{p_\omega} U_\omega^{[1]}$ is a pseudo-triangle in $\mathscr{F}$.

So $U_\omega \xrightarrow{\underline{g_\omega}} V_\omega \xrightarrow{\underline{i_\omega}} W_\omega \xrightarrow{\underline{p_\omega}} U_\omega^{[1]}$ is a triangle in $\underline{\mathscr{F}}$, cf. definition 2.2.14. We have $\underline{g_\omega} \cdot \underline{b} = \underline{a} \cdot \underline{f}$.

$$\begin{tikzcd}
U_\omega \arrow[r, "\underline{g_\omega}"] \arrow[d, "\underline{a}"'] & V_\omega \arrow[r, "\underline{i_\omega}"] \arrow[d, "\underline{b}"] & W_\omega \arrow[r, "\underline{p_\omega}"] & U_\omega^{[1]} \arrow[d, "\underline{a}^{[1]}"] \\
Y \arrow[r, "\underline{f}"] & Z \arrow[r] & X \arrow[r] & Y^{[1]}
\end{tikzcd}$$

Since $\underline{a}$ and $\underline{b}$ are isomorphisms in $\underline{\mathscr{F}}$, we conclude that W_ω is isomorphic to X in $\underline{\mathscr{F}}$. $\square$

4.3.28 Definition. Let $\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} = \underline{\mathrm{P}}_{\omega,\mathscr{F}}|^{\mathscr{W}^{\mathrm{b}}}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})} \colon \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}) \to \mathscr{W}^{\mathrm{b}}$, cf. corollary 4.3.19. $\diamondsuit$

4.3.29 Remark. Note that $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}) = \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}_{\lceil 0,0\rceil}}(\mathscr{F})$ by lemma 4.3.10.
We have $\underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{W}_{\lceil 0,0\rceil},\mathscr{F}}|^{\mathscr{W}^{\mathrm{b}}} = \underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$, cf. definition 4.2.19. $\diamondsuit$

4.3.30 Definition. Let $\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}} = \underline{\mathrm{P}}_{\omega,\mathscr{F}}|^{\mathscr{D}}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})} \colon \underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F}) \to \mathscr{D}$, cf. remark 4.3.4. $\diamondsuit$

4.3.31 Proposition. The functor $\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \colon \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}) \to \mathscr{W}^{\mathrm{b}}$ is an equivalence. $\diamondsuit$

Proof. The functor is full, faithful and dense by lemma 4.3.15 and corollaries 4.3.17, 4.3.22. $\square$

4.3.32 Proposition. Suppose that $\mathscr{F}$ and $\mathscr{F}^{\mathrm{op}}$ have countable products of bijectives. The functor $\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}} \colon \underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F}) \to \mathscr{D}$ is an equivalence. $\diamondsuit$

Proof.
The functor is full, faithful and dense by lemma 4.3.15, corollary 4.3.17 and lemma 4.3.27. $\square$

4.3.33 Definition. We want to construct a quasi-inverse $\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ of $\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \colon \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}) \to \mathscr{W}^{\mathrm{b}}$ using lemma 1.6.5. We call $\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ the *bounded weight equivalence* of $\mathscr{W}$ with respect to $\mathscr{F}$. The functor $\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ is full and faithful, cf. proposition 4.3.31.

For $X \in \mathrm{Ob}(\mathscr{W}^{\mathrm{b}})$, we may choose an object $X\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \in \mathrm{Ob}(\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}))$ and an isomorphism $X\upsilon^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \colon X \to X\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ in $\underline{\mathscr{F}}$ by corollary 4.3.22.

Lemma 1.6.5 yields the functor $\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \colon \mathscr{W}^{\mathrm{b}} \to \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})$, where for $X \xrightarrow{f} Y$ in $\mathscr{W}^{\mathrm{b}}$,
$X\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \xrightarrow{f\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}} Y\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ is the unique morphism in $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})$ such that
$f = X\upsilon^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot f\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot (Y\upsilon^{\mathrm{b}}_{\mathscr{W},\mathscr{F}})^{-1}$. The functors $\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ and $\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ are mutually quasi-inverse equivalences. Moreover, we obtain the isotransformation $\upsilon^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \colon 1_{\mathscr{W}^{\mathrm{b}}} \to \mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$.
Note that the functors $\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ and $\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ are additive, cf. remark 1.2.5.(a) and corollary 4.3.13. ♢

4.3.34 Definition. Suppose that $\mathscr{F}$ and $\mathscr{F}^{\mathrm{op}}$ have countable products of bijectives. We want to construct a quasi-inverse $\mathrm{W}_{\mathscr{W},\mathscr{F}}$ of the functor $\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}} \colon \underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F}) \to \mathscr{D}$ using lemma 1.6.5. We call $\mathrm{W}_{\mathscr{W},\mathscr{F}}$ the *weight equivalence* of $\mathscr{W}$ with respect to $\mathscr{F}$. The functor $\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}}$ is full and faithful, cf. proposition 4.3.32.

Suppose given $X \in \mathrm{Ob}(\mathscr{D})$.
If $X \in \mathrm{Ob}(\mathscr{W}^{\mathrm{b}})$, we may choose $X\mathrm{W}_{\mathscr{W},\mathscr{F}} = X\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \in \mathrm{Ob}(\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F}))$ and the isomorphism $X\upsilon_{\mathscr{W},\mathscr{F}} = X\upsilon^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \colon X \to X\mathrm{W}_{\mathscr{W},\mathscr{F}}\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}}$ in $\underline{\mathscr{F}}$, cf. definition 4.3.33.

If $X \notin \mathrm{Ob}(\mathscr{W}^{\mathrm{b}})$, we may choose an object $X\mathrm{W}_{\mathscr{W},\mathscr{F}} \in \mathrm{Ob}(\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F}))$ and an isomorphism $X\upsilon_{\mathscr{W},\mathscr{F}} \colon X \to X\mathrm{W}_{\mathscr{W},\mathscr{F}}\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}}$ in $\underline{\mathscr{F}}$ by lemma 4.3.27.

Lemma 1.6.5 yields the functor $\mathrm{W}_{\mathscr{W},\mathscr{F}} \colon \mathscr{D} \to \underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})$, where for $X \xrightarrow{f} Y$ in $\mathscr{D}$,
$X\mathrm{W}_{\mathscr{W},\mathscr{F}} \xrightarrow{f\mathrm{W}_{\mathscr{W},\mathscr{F}}} Y\mathrm{W}_{\mathscr{W},\mathscr{F}}$ is the unique morphism in $\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})$ such that
$f = X\upsilon_{\mathscr{W},\mathscr{F}} \cdot f\mathrm{W}_{\mathscr{W},\mathscr{F}}\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}} \cdot (Y\upsilon_{\mathscr{W},\mathscr{F}})^{-1}$. The functors $\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}}$ and $\mathrm{W}_{\mathscr{W},\mathscr{F}}$ are mutually quasi-inverse equivalences. Moreover, we obtain the isotransformation $\upsilon_{\mathscr{W},\mathscr{F}} \colon 1_{\mathscr{D}} \to \mathrm{W}_{\mathscr{W},\mathscr{F}}\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}}$.
Note that we have $\mathrm{Inc}^{\mathscr{D}}_{\mathscr{W}^{\mathrm{b}}} \cdot \mathrm{W}_{\mathscr{W},\mathscr{F}} = \mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})}$ by construction. Also note that the functors $\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}}$ and $\mathrm{W}_{\mathscr{W},\mathscr{F}}$ are additive, cf. remark 1.2.5.(a) and corollary 4.3.13. ♢

4.3.35 Definition. We abbreviate $\mathscr{C} = \mathscr{W}_{[0,0]}$. Note that $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}) = \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{C}}(\mathscr{F})$ by lemma 4.3.10. The composite

$$\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} = \mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \Xi^{\mathrm{b}}_{\mathscr{C},\mathscr{F}} \cdot \underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}} \colon \mathscr{W}^{\mathrm{b}} \to \mathrm{K}^{\mathrm{b}}(\mathscr{C})$$

is called the *bounded resolution functor* of $\mathscr{W}$ with respect to $\mathscr{F}$. Cf. definitions 4.3.33, 4.2.56 and 4.1.18.

$$\mathscr{W}^{\mathrm{b}} \xrightarrow{\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}} \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}) \xrightarrow{\Xi^{\mathrm{b}}_{\mathscr{C},\mathscr{F}}} \underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{C}}(\mathscr{F}) \xrightarrow{\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}}} \mathrm{K}^{\mathrm{b}}(\mathscr{C})$$

♢

4.3.36 Definition. Suppose that $\mathscr{F}$ and $\mathscr{F}^{\mathrm{op}}$ have countable products of bijectives. We abbreviate $\mathscr{C} = \mathscr{W}_{[0,0]}$. The composite

$$\mathrm{Res}_{\mathscr{W},\mathscr{F}} = \mathrm{W}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{C}}(\mathscr{F})} \cdot \Xi_{\mathscr{C},\mathscr{F}} \cdot \underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}} \colon \mathscr{D} \to \mathrm{K}(\mathscr{C})$$

is called the *resolution functor* of $\mathscr{W}$ with respect to $\mathscr{F}$. Cf. definitions 4.3.34, 4.2.32, 4.1.15 and corollary 4.3.6.

$$\mathscr{D} \xrightarrow{\mathrm{W}_{\mathscr{W},\mathscr{F}}} \underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F}) \xrightarrow{\mathrm{Inc}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{C}}(\mathscr{F})}} \underline{\underline{\mathrm{FO}}}_{\mathscr{C}}(\mathscr{F}) \xrightarrow{\Xi_{\mathscr{C},\mathscr{F}}} \underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F}) \xrightarrow{\underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}}} \mathrm{K}(\mathscr{C})$$

$\diamondsuit$

4.3.37 Remark. Suppose that $\mathscr{F}$ and $\mathscr{F}^{\mathrm{op}}$ have countable products of bijectives. We abbreviate $\mathscr{C} = \mathscr{W}_{[0,0]}$. We have $\mathrm{Inc}_{\mathscr{W}^{\mathrm{b}}}^{\mathscr{D}} \cdot \mathrm{Res}_{\mathscr{W},\mathscr{F}} = \mathrm{Res}_{\mathscr{W},\mathscr{F}}^{\mathrm{b}} \cdot \mathrm{Inc}_{\mathrm{K}^{\mathrm{b}}(\mathscr{C})}^{\mathrm{K}(\mathscr{C})}$. $\diamondsuit$

Proof. We have

$$\begin{aligned}
\mathrm{Inc}_{\mathscr{W}^{\mathrm{b}}}^{\mathscr{D}} \cdot \mathrm{Res}_{\mathscr{W},\mathscr{F}} &= \mathrm{Inc}_{\mathscr{W}^{\mathrm{b}}}^{\mathscr{D}} \cdot \mathrm{W}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{C}}(\mathscr{F})} \cdot \Xi_{\mathscr{C},\mathscr{F}} \cdot \underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}} \\
&= \mathrm{W}_{\mathscr{W},\mathscr{F}}^{\mathrm{b}} \cdot \mathrm{Inc}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}^{\mathrm{b}}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})} \cdot \mathrm{Inc}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{C}}(\mathscr{F})} \cdot \Xi_{\mathscr{C},\mathscr{F}} \cdot \underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}} \\
&= \mathrm{W}_{\mathscr{W},\mathscr{F}}^{\mathrm{b}} \cdot \mathrm{Inc}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}^{\mathrm{b}}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{C}}(\mathscr{F})} \cdot \Xi_{\mathscr{C},\mathscr{F}} \cdot \underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}} \\
&= \mathrm{W}_{\mathscr{W},\mathscr{F}}^{\mathrm{b}} \cdot \underline{\underline{\Xi}}_{\mathscr{C},\mathscr{F}}^{\mathrm{b}} \cdot \mathrm{Inc}_{\underline{\underline{\nabla}}_{\mathscr{C}}^{\mathrm{b}}(\mathscr{F})}^{\underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F})} \cdot \underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}} \\
&= \mathrm{W}_{\mathscr{W},\mathscr{F}}^{\mathrm{b}} \cdot \underline{\underline{\Xi}}_{\mathscr{C},\mathscr{F}}^{\mathrm{b}} \cdot \underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}}^{\mathrm{b}} \cdot \mathrm{Inc}_{\mathrm{K}^{\mathrm{b}}(\mathscr{C})}^{\mathrm{K}(\mathscr{C})} \\
&= \mathrm{Res}_{\mathscr{W},\mathscr{F}}^{\mathrm{b}} \cdot \mathrm{Inc}_{\mathrm{K}^{\mathrm{b}}(\mathscr{C})}^{\mathrm{K}(\mathscr{C})} .
\end{aligned}$$

Cf. definitions 4.3.34, 4.2.56 and 4.1.18. $\square$

4.3.38 Lemma. We abbreviate $\mathscr{C} = \mathscr{W}_{[0,0]}$.
The bounded resolution functor $\mathrm{Res}_{\mathscr{W},\mathscr{F}}^{\mathrm{b}} \colon \mathscr{W}^{\mathrm{b}} \to \mathrm{K}^{\mathrm{b}}(\mathscr{C})$ is exact. $\diamondsuit$

Proof. We will use lemma 3.4.19 and propositions 4.1.3, 4.2.28. We abbreviate $\mathrm{W} = \mathrm{W}_{\mathscr{W},\mathscr{F}}^{\mathrm{b}}$, $\Xi = \Xi_{\mathscr{C},\mathscr{F}}^{\mathrm{b}}$, $\underline{\underline{\Xi}} = \underline{\underline{\Xi}}_{\mathscr{C},\mathscr{F}}^{\mathrm{b}}$, $\Delta = \Delta_{\mathscr{C},\mathscr{F}}^{\mathrm{b}}$, $\underline{\underline{\Delta}} = \underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}}^{\mathrm{b}}$, $\underline{\mathrm{P}} = \underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}}^{\mathrm{b}}$, $\upsilon = \upsilon_{\mathscr{W},\mathscr{F}}^{\mathrm{b}}$ and $\xi = \xi_{\Sigma,\mathscr{F}}$.
We will use that $\underline{\mathrm{P}}$ is full and faithful, cf. proposition 4.3.31.

Note that $\mathrm{Res}_{\mathscr{W},\mathscr{F}}^{\mathrm{b}}$ is additive since it is a composite of additive functors, cf. definitions 4.3.33, 4.2.56 and 4.1.18.

First step.
We want to construct an isotransformation $\underline{\Sigma}|_{\mathscr{W}^{\mathrm{b}}}^{\mathscr{W}^{\mathrm{b}}} \cdot \mathrm{Res}_{\mathscr{W},\mathscr{F}}^{\mathrm{b}} \xrightarrow{\lambda} \mathrm{Res}_{\mathscr{W},\mathscr{F}}^{\mathrm{b}} \cdot \Sigma_{\mathrm{K}^{\mathrm{b}},\mathscr{C}}$.

Suppose given $X \in \mathrm{Ob}(\mathscr{W}^{\mathrm{b}})$.

Note that we have $X\underline{\Sigma}|_{\mathscr{W}^{\mathrm{b}}}^{\mathscr{W}^{\mathrm{b}}}\mathrm{Res}_{\mathscr{W},\mathscr{F}}^{\mathrm{b}} = X^{[1]}\mathrm{W}\underline{\underline{\Xi}}\underline{\underline{\Delta}}$ and $X\mathrm{Res}_{\mathscr{W},\mathscr{F}}^{\mathrm{b}}\Sigma_{\mathrm{K}^{\mathrm{b}},\mathscr{C}} = (X\mathrm{W}\underline{\underline{\Xi}}\underline{\underline{\Delta}})^{[1]}$. Also note that $(X\mathrm{W})_{[-1]}^{[1]}\underline{\mathrm{P}} = (X\mathrm{W}\underline{\mathrm{P}})^{[1]}$, cf. lemma 3.4.19.(a). Since $\underline{\mathrm{P}}$ is full, we may choose $X^{[1]}\mathrm{W} \xrightarrow{X\mu} (X\mathrm{W})_{[-1]}^{[1]}$ in $\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}^{\mathrm{b}}(\mathscr{F})$ such that $X\mu\underline{\mathrm{P}} = X^{[1]}\upsilon^{-1} \cdot (X\upsilon)^{[1]}$. Note that $X\mu$ is an isomorphism in $\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}^{\mathrm{b}}(\mathscr{F})$ since $X^{[1]}\upsilon^{-1}$, $(X\upsilon)^{[1]}$ are isomorphisms in $\mathscr{W}^{\mathrm{b}}$ and since $\underline{\mathrm{P}}$ is full and faithful.

Let $X\lambda = X\mu\underline{\Xi}\underline{\underline{\Delta}} \cdot \underline{\underline{\underline{X\mathrm{W}_{[-1]}\xi}}}\,\underline{\underline{\Delta}} \colon X^{[1]}\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \to (X\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}})^{[1]}$. Note that $X\lambda$ is an isomorphism in $\mathrm{K}^{\mathrm{b}}(\mathscr{C})$ since $X\mu$ is an isomorphism in $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})$ and since $\underline{\underline{\underline{X\mathrm{W}_{[-1]}\xi}}}$ is an isomorphism in $\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{C}}(\mathscr{F})$.

Suppose given $X \xrightarrow{f} Y$ in $\mathscr{W}^{\mathrm{b}}$.

Choose $X\mathrm{W} \xrightarrow{g} Y\mathrm{W}$ in $\mathrm{FO}(\mathscr{F})$ such that $\underline{\underline{g}} = f\mathrm{W}$. We have $X\mu \cdot \underline{\underline{\underline{g^{[1]}_{[-1]}}}} = f^{[1]}\mathrm{W} \cdot Y\mu$ since

$$\begin{aligned}(X\mu \cdot \underline{\underline{\underline{g^{[1]}_{[-1]}}}})\underline{\mathrm{P}} &= X^{[1]}\upsilon^{-1} \cdot (X\upsilon)^{[1]} \cdot (\underline{\underline{g}}\underline{\mathrm{P}})^{[1]} = X^{[1]}\upsilon^{-1} \cdot (X\upsilon \cdot f\mathrm{W}\underline{\mathrm{P}})^{[1]} = X^{[1]}\upsilon^{-1} \cdot (f \cdot Y\upsilon)^{[1]} \\ &= X^{[1]}\upsilon^{-1} \cdot f^{[1]} \cdot (Y\upsilon)^{[1]} = f^{[1]}\mathrm{W}\underline{\mathrm{P}} \cdot Y^{[1]}\upsilon^{-1} \cdot (Y\upsilon)^{[1]} = (f^{[1]}\mathrm{W} \cdot Y\mu)\underline{\mathrm{P}}\end{aligned}$$

and since $\underline{\mathrm{P}}$ is faithful.

$$\begin{tikzcd} X^{[1]}\mathrm{W} \arrow[r, "X\mu"] \arrow[d, "f^{[1]}\mathrm{W}"'] & (X\mathrm{W})^{[1]}_{[-1]} \arrow[d, "\underline{\underline{\underline{g^{[1]}_{[-1]}}}}"] \\ Y^{[1]}\mathrm{W} \arrow[r, "Y\mu"] & (Y\mathrm{W})^{[1]}_{[-1]} \end{tikzcd}$$

We have

$$\begin{aligned} X\lambda \cdot (f\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}})^{[1]} &= X\mu\underline{\Xi}\underline{\underline{\Delta}} \cdot \underline{\underline{\underline{X\mathrm{W}_{[-1]}\xi}}}\,\underline{\underline{\Delta}} \cdot (\underline{\underline{g}}\underline{\Xi}\underline{\underline{\Delta}})^{[1]} = X\mu\underline{\Xi}\underline{\underline{\Delta}} \cdot \underline{\underline{\underline{X\mathrm{W}_{[-1]}\xi}}}\,\underline{\underline{\Delta}} \cdot \underline{\underline{\underline{(g\Xi)^{[1]}_{[-1]}}}}\underline{\underline{\Delta}} \\ &= \big(X\mu\underline{\Xi} \cdot \underline{\underline{\underline{X\mathrm{W}_{[-1]}\xi \cdot (g\Xi)^{[1]}_{[-1]}}}}\big)\underline{\underline{\Delta}} = \big(X\mu\underline{\Xi} \cdot \underline{\underline{\underline{g^{[1]}_{[-1]}\Xi \cdot Y\mathrm{W}_{[-1]}\xi}}}\big)\underline{\underline{\Delta}} \\ &= \big(X\mu\underline{\Xi} \cdot \underline{\underline{\underline{g^{[1]}_{[-1]}}}}\underline{\Xi}\big)\underline{\underline{\Delta}} \cdot \underline{\underline{\underline{Y\mathrm{W}_{[-1]}\xi}}}\,\underline{\underline{\Delta}} = \big(X\mu \cdot \underline{\underline{\underline{g^{[1]}_{[-1]}}}}\big)\underline{\Xi}\underline{\underline{\Delta}} \cdot \underline{\underline{\underline{Y\mathrm{W}_{[-1]}\xi}}}\,\underline{\underline{\Delta}} \\ &= \big(f^{[1]}\mathrm{W} \cdot Y\mu\big)\underline{\Xi}\underline{\underline{\Delta}} \cdot \underline{\underline{\underline{Y\mathrm{W}_{[-1]}\xi}}}\,\underline{\underline{\Delta}} = f^{[1]}\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \cdot Y\mu\underline{\Xi}\underline{\underline{\Delta}} \cdot \underline{\underline{\underline{Y\mathrm{W}_{[-1]}\xi}}}\,\underline{\underline{\Delta}} \\ &= f^{[1]}\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \cdot Y\lambda. \end{aligned}$$

$$\begin{tikzcd} X^{[1]}\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \arrow[r, "X\lambda"] \arrow[d, "f^{[1]}\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}}"'] & (X\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}})^{[1]} \arrow[d, "(f\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}})^{[1]}"] \\ Y^{[1]}\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \arrow[r, "Y\lambda"] & (Y\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}})^{[1]} \end{tikzcd}$$

We conclude that $\underline{\Sigma}|^{\mathscr{W}^{\mathrm{b}}}_{\mathscr{W}^{\mathrm{b}}} \cdot \mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \xrightarrow{\lambda} \mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \Sigma_{\mathrm{K}^{\mathrm{b}},\mathscr{C}}$ is an isotransformation.

Second step.

Suppose given a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X^{[1]}$ in $\mathscr{W}^{\mathrm{b}}$. We want to show that

$$\begin{aligned} &\Big(X\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \xrightarrow{u\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}} Y\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \xrightarrow{v\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}} Z\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \xrightarrow{w\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot X\lambda} (X\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}})^{[1]}\Big) \\ = &\Big(X\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \xrightarrow{u\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}}} Y\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \xrightarrow{v\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}}} Z\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \xrightarrow{w\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \cdot X\lambda} (X\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}})^{[1]}\Big) \end{aligned}$$

is a triangle in $\mathrm{K}^{\mathrm{b}}(\mathscr{C})$.

Choose $X\mathrm{W} \xrightarrow{f} Y\mathrm{W}$ in $\mathrm{FO}(\mathscr{F})$ such that $\underline{\underline{f}} = u\mathrm{W}$.

Choose a pseudo-triangle $X\mathrm{W} \xrightarrow{f} Y\mathrm{W} \xrightarrow{i} A \xrightarrow{p} X\mathrm{W}^{[1]}_{[-1]}$ in $\mathrm{FO}(\mathscr{F})$ such that $A, X\mathrm{W}^{[1]}_{[-1]} \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}))$, cf. lemmata 3.4.18, 3.4.33 and 4.3.14.

By lemma 3.4.19.(b), $X\mathrm{W}\mathrm{P}_{\omega,\mathscr{F}} \xrightarrow{f\mathrm{P}_{\omega,\mathscr{F}}} Y\mathrm{W}\mathrm{P}_{\omega,\mathscr{F}} \xrightarrow{i\mathrm{P}_{\omega,\mathscr{F}}} A\mathrm{P}_{\omega,\mathscr{F}} \xrightarrow{p\mathrm{P}_{\omega,\mathscr{F}}} X\mathrm{W}^{[1]}_{[-1]}\mathrm{P}_{\omega,\mathscr{F}}$ is a pseudo-triangle in $\mathscr{F}$. Consequently, $X\mathrm{W}\underline{\mathrm{P}} \xrightarrow{\underline{\underline{f}}\,\underline{\mathrm{P}}} Y\mathrm{W}\underline{\mathrm{P}} \xrightarrow{\underline{\underline{i}}\,\underline{\mathrm{P}}} A\underline{\mathrm{P}} \xrightarrow{\underline{\underline{p}}\,\underline{\mathrm{P}}} X\mathrm{W}^{[1]}_{[-1]}\underline{\mathrm{P}}$ is a triangle in $\mathscr{W}^{\mathrm{b}}$.

We have $X\upsilon \cdot \underline{\underline{f}}\,\underline{\mathrm{P}} = X\upsilon \cdot u\mathrm{W}\underline{\mathrm{P}} = u \cdot Y\upsilon$. Since $X\upsilon$ and $Y\upsilon$ are isomorphisms in $\mathscr{W}^{\mathrm{b}}$, we may choose an isomorphism $Z \xrightarrow{a} A\underline{\mathrm{P}}$ in $\mathscr{W}^{\mathrm{b}}$ such that $v \cdot a = Y\upsilon \cdot \underline{\underline{i}}\,\underline{\mathrm{P}}$ and $a \cdot \underline{\underline{p}}\,\underline{\mathrm{P}} = w \cdot (X\upsilon)^{[1]}$.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "X\upsilon"'] & Y \arrow[r, "v"] \arrow[d, "Y\upsilon"] & Z \arrow[r, "w"] \arrow[d, "a"] & X^{[1]} \arrow[d, "(X\upsilon)^{[1]}"] \\
X\mathrm{W}\underline{\mathrm{P}} \arrow[r, "\underline{\underline{f}}\,\underline{\mathrm{P}}"] & Y\mathrm{W}\underline{\mathrm{P}} \arrow[r, "\underline{\underline{i}}\,\underline{\mathrm{P}}"] & A\underline{\mathrm{P}} \arrow[r, "\underline{\underline{p}}\,\underline{\mathrm{P}}"] & X\mathrm{W}^{[1]}_{[-1]}\underline{\mathrm{P}}
\end{tikzcd}$$

Since $\underline{\mathrm{P}}$ is full, we may choose $Z\mathrm{W} \xrightarrow{b} A$ in $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})$ such that $b\underline{\mathrm{P}} = Z\upsilon^{-1} \cdot a$. Note that b is an isomorphism in $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})$ since $Z\upsilon^{-1}$, a are isomorphisms in $\mathscr{W}^{\mathrm{b}}$ and since $\underline{\mathrm{P}}$ is full and faithful. We have $v\mathrm{W} \cdot b = \underline{\underline{i}}$ since $(v\mathrm{W} \cdot b)\underline{\mathrm{P}} = v\mathrm{W}\underline{\mathrm{P}} \cdot Z\upsilon^{-1} \cdot a = Y\upsilon^{-1} \cdot v \cdot a = Y\upsilon^{-1} \cdot Y\upsilon \cdot \underline{\underline{i}}\,\underline{\mathrm{P}} = \underline{\underline{i}}\,\underline{\mathrm{P}}$ and since $\underline{\mathrm{P}}$ is faithful. We have $b \cdot \underline{\underline{p}} = w\mathrm{W} \cdot X\mu$ since $(b \cdot \underline{\underline{p}})\underline{\mathrm{P}} = Z\upsilon^{-1} \cdot a \cdot \underline{\underline{p}}\,\underline{\mathrm{P}} = Z\upsilon^{-1} \cdot w \cdot (X\upsilon)^{[1]} = w\mathrm{W}\underline{\mathrm{P}} \cdot X^{[1]}\upsilon^{-1} \cdot (X\upsilon)^{[1]} = (w\mathrm{W} \cdot X\mu)\underline{\mathrm{P}}$ and since $\underline{\mathrm{P}}$ is faithful.

$$\begin{tikzcd}
X\mathrm{W} \arrow[r, "u\mathrm{W}"] \arrow[d, "1"'] & Y\mathrm{W} \arrow[r, "v\mathrm{W}"] \arrow[d, "1"] & Z\mathrm{W} \arrow[r, "w\mathrm{W}\cdot X\mu"] \arrow[d, "b"] & X\mathrm{W}^{[1]}_{[-1]} \arrow[d, "1"] \\
X\mathrm{W} \arrow[r, "\underline{\underline{f}}"] & Y\mathrm{W} \arrow[r, "\underline{\underline{i}}"] & A \arrow[r, "\underline{\underline{p}}"] & X\mathrm{W}^{[1]}_{[-1]}
\end{tikzcd}$$

By propositions 4.2.28.(b) and 4.1.3.(b),

$X\mathrm{W}\Xi\Delta \xrightarrow{f\Xi\Delta} Y\mathrm{W}\Xi\Delta \xrightarrow{i\Xi\Delta} A\Xi\Delta \xrightarrow{(p\Xi \cdot X\mathrm{W}_{[-1]}\xi)\Delta} X\mathrm{W}\Xi^{[1]}_{[-1]}\Delta$ is a pseudo-triangle in $\mathrm{C}(\mathscr{C})$. Consequently, $X\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \xrightarrow{\underline{\underline{f}}\,\underline{\Xi}\underline{\underline{\Delta}}} Y\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \xrightarrow{\underline{\underline{i}}\,\underline{\Xi}\underline{\underline{\Delta}}} A\underline{\Xi}\underline{\underline{\Delta}} \xrightarrow{\underline{\underline{p}}\,\underline{\Xi}\underline{\underline{\Delta}} \cdot \underline{\underline{X\mathrm{W}_{[-1]}\xi}}\,\underline{\underline{\Delta}}} (X\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}})^{[1]}$ is a triangle in $\mathrm{K}^{\mathrm{b}}(\mathscr{C})$. Note that $Z\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \xrightarrow{b\underline{\Xi}\underline{\underline{\Delta}}} A\underline{\Xi}\underline{\underline{\Delta}}$ is an isomorphism in $\mathrm{K}^{\mathrm{b}}(\mathscr{C})$ since b is an isomorphism in $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})$. We have $u\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} = \underline{\underline{f}}\,\underline{\Xi}\underline{\underline{\Delta}}$, $v\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \cdot b\underline{\Xi}\underline{\underline{\Delta}} = (v\mathrm{W} \cdot b)\underline{\Xi}\underline{\underline{\Delta}} = \underline{\underline{i}}\,\underline{\Xi}\underline{\underline{\Delta}}$ and

$$\begin{aligned}
w\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \cdot X\lambda &= w\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \cdot X\mu\underline{\Xi}\underline{\underline{\Delta}} \cdot \underline{\underline{X\mathrm{W}_{[-1]}\xi}}\,\underline{\underline{\Delta}} = (w\mathrm{W} \cdot X\mu)\underline{\Xi}\underline{\underline{\Delta}} \cdot \underline{\underline{X\mathrm{W}_{[-1]}\xi}}\,\underline{\underline{\Delta}} \\
&= (b \cdot \underline{\underline{p}})\underline{\Xi}\underline{\underline{\Delta}} \cdot \underline{\underline{X\mathrm{W}_{[-1]}\xi}}\,\underline{\underline{\Delta}} = b\underline{\Xi}\underline{\underline{\Delta}} \cdot \underline{\underline{p}}\,\underline{\Xi}\underline{\underline{\Delta}} \cdot \underline{\underline{X\mathrm{W}_{[-1]}\xi}}\,\underline{\underline{\Delta}}.
\end{aligned}$$

$$\begin{tikzcd}
X\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \arrow[r, "u\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}}"] \arrow[d, "1"'] & Y\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \arrow[r, "v\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}}"] \arrow[d, "1"] & Z\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \arrow[r, "w\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}}\cdot X\lambda"] \arrow[d, "b\underline{\Xi}\underline{\underline{\Delta}}"'] & (X\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}})^{[1]} \arrow[d, "1"] \\
X\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \arrow[r, "\underline{\underline{f}}\,\underline{\Xi}\underline{\underline{\Delta}}"] & Y\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \arrow[r, "\underline{\underline{i}}\,\underline{\Xi}\underline{\underline{\Delta}}"] & A\underline{\Xi}\underline{\underline{\Delta}} \arrow[r, "\underline{\underline{p}}\,\underline{\Xi}\underline{\underline{\Delta}}\cdot\underline{\underline{\underline{X\mathrm{W}_{[-1]}\xi}}}\,\underline{\underline{\Delta}}"] & (X\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}})^{[1]}
\end{tikzcd}$$

We conclude that

$$\begin{aligned}
&\Big(X\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \xrightarrow{u\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}} Y\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \xrightarrow{v\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}} Z\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \xrightarrow{w\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}\cdot X\lambda} (X\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}})^{[1]}\Big) \\
= &\Big(X\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \xrightarrow{u\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}}} Y\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \xrightarrow{v\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}}} Z\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \xrightarrow{w\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}}\cdot X\lambda} (X\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}})^{[1]}\Big)
\end{aligned}$$

is a triangle in $\mathrm{K}^{\mathrm{b}}(\mathscr{C})$. □

4.3.39 Lemma. Suppose that $\mathscr{F}$ and $\mathscr{F}^{\mathrm{op}}$ have countable products of bijectives. We abbreviate $\mathscr{C} = \mathscr{W}_{[0,0]}$. The resolution functor $\mathrm{Res}_{\mathscr{W},\mathscr{F}}\colon \mathscr{D} \to \mathrm{K}(\mathscr{C})$ is exact. ◇

Proof. We will use lemma 3.4.19 and propositions 4.1.3, 4.2.28. We abbreviate $\mathrm{W} = \mathrm{W}_{\mathscr{W},\mathscr{F}}$, $\Xi = \Xi_{\mathscr{C},\mathscr{F}}$, $\underline{\Xi} = \underline{\Xi}_{\mathscr{C},\mathscr{F}}$, $\Delta = \Delta_{\mathscr{C},\mathscr{F}}$, $\underline{\underline{\Delta}} = \underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}}$, $\underline{\mathrm{P}} = \underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}}$, $\upsilon = \upsilon_{\mathscr{W},\mathscr{F}}$ and $\xi = \xi_{\Sigma,\mathscr{F}}$.

We will use that $\underline{\mathrm{P}}$ is full and faithful, cf. proposition 4.3.32.

Note that $\mathrm{Res}_{\mathscr{W},\mathscr{F}}$ is additive since it is a composite of additive functors, cf. definition 4.3.34, corollary 4.3.13, remark 4.2.14 and definitions 4.2.32, 4.1.15.

First step.

We want to construct an isotransformation $\underline{\Sigma}|^{\mathscr{D}}_{\mathscr{D}} \cdot \mathrm{Res}_{\mathscr{W},\mathscr{F}} \xrightarrow{\lambda} \mathrm{Res}_{\mathscr{W},\mathscr{F}} \cdot \Sigma_{\mathrm{K},\mathscr{C}}$.

Suppose given $X \in \mathrm{Ob}(\mathscr{D})$.

Note that we have $X\underline{\Sigma}|^{\mathscr{D}}_{\mathscr{D}}\mathrm{Res}_{\mathscr{W},\mathscr{F}} = X^{[1]}\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}}$ and $X\mathrm{Res}_{\mathscr{W},\mathscr{F}}\Sigma_{\mathrm{K},\mathscr{C}} = (X\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}})^{[1]}$. Also note that $(X\mathrm{W})^{[1]}_{[-1]}\underline{\mathrm{P}} = (X\mathrm{W}\underline{\mathrm{P}})^{[1]}$, cf. lemma 3.4.19.(a). Since $\underline{\mathrm{P}}$ is full, we may choose $X^{[1]}\mathrm{W} \xrightarrow{X\mu} (X\mathrm{W})^{[1]}_{[-1]}$ in $\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})$ such that $X\mu\underline{\mathrm{P}} = X^{[1]}\upsilon^{-1} \cdot (X\upsilon)^{[1]}$. Note that $X\mu$ is an isomorphism in $\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})$ since $X^{[1]}\upsilon^{-1}$, $(X\upsilon)^{[1]}$ are isomorphisms in $\mathscr{D}$ and since $\underline{\mathrm{P}}$ is full and faithful.

Let $X\lambda = X\mu\underline{\Xi}\underline{\underline{\Delta}} \cdot \underline{\underline{\underline{X\mathrm{W}_{[-1]}\xi}}}\,\underline{\underline{\Delta}}\colon X^{[1]}\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \to (X\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}})^{[1]}$. Note that $X\lambda$ is an isomorphism in $\mathrm{K}(\mathscr{C})$ since $X\mu$ is an isomorphism in $\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})$ and since $\underline{\underline{\underline{X\mathrm{W}_{[-1]}\xi}}}$ is an isomorphism in $\underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F})$.

Suppose given $X \xrightarrow{f} Y$ in $\mathscr{D}$.

Choose $X\mathrm{W} \xrightarrow{g} Y\mathrm{W}$ in $\mathrm{FO}(\mathscr{F})$ such that $\underline{\underline{g}} = f\mathrm{W}$. We have $X\mu \cdot \underline{\underline{\underline{g^{[1]}_{[-1]}}}} = f^{[1]}\mathrm{W} \cdot Y\mu$ since

$$\begin{aligned}
(X\mu \cdot \underline{\underline{\underline{g^{[1]}_{[-1]}}}})\underline{\mathrm{P}} &= X^{[1]}\upsilon^{-1} \cdot (X\upsilon)^{[1]} \cdot (\underline{\underline{g}}\underline{\mathrm{P}})^{[1]} = X^{[1]}\upsilon^{-1} \cdot (X\upsilon \cdot f\mathrm{W}\underline{\mathrm{P}})^{[1]} = X^{[1]}\upsilon^{-1} \cdot (f \cdot Y\upsilon)^{[1]} \\
&= X^{[1]}\upsilon^{-1} \cdot f^{[1]} \cdot (Y\upsilon)^{[1]} = f^{[1]}\mathrm{W}\underline{\mathrm{P}} \cdot Y^{[1]}\upsilon^{-1} \cdot (Y\upsilon)^{[1]} = (f^{[1]}\mathrm{W} \cdot Y\mu)\underline{\mathrm{P}}
\end{aligned}$$

and since $\underline{\mathrm{P}}$ is faithful.

$$\begin{tikzcd}
X^{[1]}\mathrm{W} \arrow[r, "X\mu"] \arrow[d, "f^{[1]}\mathrm{W}"'] & (X\mathrm{W})^{[1]}_{[-1]} \arrow[d, "\underline{\underline{g^{[1]}_{[-1]}}}"] \\
Y^{[1]}\mathrm{W} \arrow[r, "Y\mu"] & (Y\mathrm{W})^{[1]}_{[-1]}
\end{tikzcd}$$

We have

$$\begin{aligned}
X\lambda \cdot (f\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}})^{[1]} &= X\mu\underline{\Xi}\underline{\underline{\Delta}} \cdot \underline{\underline{X\mathrm{W}_{[-1]}\xi}}\,\underline{\underline{\Delta}} \cdot (\underline{g}\underline{\Xi}\underline{\underline{\Delta}})^{[1]} = X\mu\underline{\Xi}\underline{\underline{\Delta}} \cdot \underline{\underline{X\mathrm{W}_{[-1]}\xi}}\,\underline{\underline{\Delta}} \cdot \underline{\underline{(g\Xi)^{[1]}_{[-1]}}}\underline{\underline{\Delta}} \\
&= \big(X\mu\underline{\Xi} \cdot \underline{\underline{X\mathrm{W}_{[-1]}\xi \cdot (g\Xi)^{[1]}_{[-1]}}}\big)\underline{\underline{\Delta}} = \big(X\mu\underline{\Xi} \cdot \underline{\underline{g^{[1]}_{[-1]}\Xi \cdot Y\mathrm{W}_{[-1]}\xi}}\big)\underline{\underline{\Delta}} \\
&= \big(X\mu\underline{\Xi} \cdot \underline{\underline{g^{[1]}_{[-1]}}}\underline{\Xi}\big)\underline{\underline{\Delta}} \cdot \underline{\underline{Y\mathrm{W}_{[-1]}\xi}}\,\underline{\underline{\Delta}} = \big(X\mu \cdot \underline{\underline{g^{[1]}_{[-1]}}}\big)\underline{\Xi}\underline{\underline{\Delta}} \cdot \underline{\underline{Y\mathrm{W}_{[-1]}\xi}}\,\underline{\underline{\Delta}} \\
&= \big(f^{[1]}\mathrm{W} \cdot Y\mu\big)\underline{\Xi}\underline{\underline{\Delta}} \cdot \underline{\underline{Y\mathrm{W}_{[-1]}\xi}}\,\underline{\underline{\Delta}} = f^{[1]}\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \cdot Y\mu\underline{\Xi}\underline{\underline{\Delta}} \cdot \underline{\underline{Y\mathrm{W}_{[-1]}\xi}}\,\underline{\underline{\Delta}} \\
&= f^{[1]}\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \cdot Y\lambda.
\end{aligned}$$

$$\begin{tikzcd}
X^{[1]}\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \arrow[r, "X\lambda"] \arrow[d, "f^{[1]}\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}}"'] & (X\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}})^{[1]} \arrow[d, "(f\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}})^{[1]}"] \\
Y^{[1]}\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \arrow[r, "Y\lambda"] & (Y\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}})^{[1]}
\end{tikzcd}$$

We conclude that $\underline{\Sigma}|^{\mathscr{D}}_{\mathscr{D}} \cdot \mathrm{Res}_{\mathscr{W},\mathscr{F}} \xrightarrow{\lambda} \mathrm{Res}_{\mathscr{W},\mathscr{F}} \cdot \Sigma_{\mathrm{K},\mathscr{C}}$ is an isotransformation.

Second step.

Suppose given a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X^{[1]}$ in $\mathscr{D}$. We want to show that

$$\begin{aligned}
&\Big(X\mathrm{Res}_{\mathscr{W},\mathscr{F}} \xrightarrow{u\mathrm{Res}_{\mathscr{W},\mathscr{F}}} Y\mathrm{Res}_{\mathscr{W},\mathscr{F}} \xrightarrow{v\mathrm{Res}_{\mathscr{W},\mathscr{F}}} Z\mathrm{Res}_{\mathscr{W},\mathscr{F}} \xrightarrow{w\mathrm{Res}_{\mathscr{W},\mathscr{F}} \cdot X\lambda} (X\mathrm{Res}_{\mathscr{W},\mathscr{F}})^{[1]}\Big) \\
= &\Big(X\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \xrightarrow{u\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}}} Y\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \xrightarrow{v\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}}} Z\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \xrightarrow{w\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}} \cdot X\lambda} (X\mathrm{W}\underline{\Xi}\underline{\underline{\Delta}})^{[1]}\Big)
\end{aligned}$$

is a triangle in $\mathrm{K}(\mathscr{C})$.

Choose $X\mathrm{W} \xrightarrow{f} Y\mathrm{W}$ in $\mathrm{FO}(\mathscr{F})$ such that $\underline{\underline{f}} = u\mathrm{W}$.

Choose a pseudo-triangle $X\mathrm{W} \xrightarrow{f} Y\mathrm{W} \xrightarrow{i} A \xrightarrow{p} X\mathrm{W}^{[1]}_{[-1]}$ in $\mathrm{FO}(\mathscr{F})$ such that $A, X\mathrm{W}^{[1]}_{[-1]} \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}}(\mathscr{F}))$, cf. lemmata 3.4.18 and 4.3.14.

By lemma 3.4.19.(b), $X\mathrm{W}\mathrm{P}_{\omega,\mathscr{F}} \xrightarrow{f\mathrm{P}_{\omega,\mathscr{F}}} Y\mathrm{W}\mathrm{P}_{\omega,\mathscr{F}} \xrightarrow{i\mathrm{P}_{\omega,\mathscr{F}}} A\mathrm{P}_{\omega,\mathscr{F}} \xrightarrow{p\mathrm{P}_{\omega,\mathscr{F}}} X\mathrm{W}^{[1]}_{[-1]}\mathrm{P}_{\omega,\mathscr{F}}$ is a pseudo-triangle in $\mathscr{F}$. Consequently, $X\mathrm{W}\underline{\mathrm{P}} \xrightarrow{\underline{\underline{f}}\,\underline{\mathrm{P}}} Y\mathrm{W}\underline{\mathrm{P}} \xrightarrow{\underline{\underline{i}}\,\underline{\mathrm{P}}} A\underline{\mathrm{P}} \xrightarrow{\underline{\underline{p}}\,\underline{\mathrm{P}}} X\mathrm{W}^{[1]}_{[-1]}\underline{\mathrm{P}}$ is a triangle in $\mathscr{D}$.

We have $X\upsilon \cdot \underline{\underline{f}}\,\underline{\mathrm{P}} = X\upsilon \cdot u\mathrm{W}\underline{\mathrm{P}} = u \cdot Y\upsilon$. Since $X\upsilon$ and $Y\upsilon$ are isomorphisms in $\mathscr{D}$, we may choose an isomorphism $Z \xrightarrow{a} A\underline{\mathrm{P}}$ in $\mathscr{D}$ such that $v \cdot a = Y\upsilon \cdot \underline{\underline{i}}\,\underline{\mathrm{P}}$ and $a \cdot \underline{\underline{p}}\,\underline{\mathrm{P}} = w \cdot (X\upsilon)^{[1]}$.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "X\upsilon"'] & Y \arrow[r, "v"] \arrow[d, "Y\upsilon"] & Z \arrow[r, "w"] \arrow[d, "a"] & X^{[1]} \arrow[d, "(X\upsilon)^{[1]}"] \\
X\mathrm{W}\underline{\mathrm{P}} \arrow[r, "\underline{\underline{f}}\,\underline{\mathrm{P}}"] & Y\mathrm{W}\underline{\mathrm{P}} \arrow[r, "\underline{\underline{i}}\,\underline{\mathrm{P}}"] & A\underline{\mathrm{P}} \arrow[r, "\underline{\underline{p}}\,\underline{\mathrm{P}}"] & X\mathrm{W}^{[1]}_{[-1]}\underline{\mathrm{P}}
\end{tikzcd}$$

Since $\underline{\mathrm{P}}$ is full, we may choose $Z\mathrm{W} \xrightarrow{b} A$ in $\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})$ such that $b\underline{\mathrm{P}} = Z\upsilon^{-1} \cdot a$. Note that b is an isomorphism in $\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})$ since $Z\upsilon^{-1}$, a are isomorphisms in $\mathscr{D}$ and since $\underline{\mathrm{P}}$ is full and faithful. We have $v\mathrm{W} \cdot b = \underline{\underline{i}}$ since $(v\mathrm{W} \cdot b)\underline{\mathrm{P}} = v\mathrm{W}\underline{\mathrm{P}} \cdot Z\upsilon^{-1} \cdot a = Y\upsilon^{-1} \cdot v \cdot a = Y\upsilon^{-1} \cdot Y\upsilon \cdot \underline{\underline{i}}\,\underline{\mathrm{P}} = \underline{\underline{i}}\,\underline{\mathrm{P}}$ and since $\underline{\mathrm{P}}$ is faithful. We have $b \cdot \underline{\underline{p}} = w\mathrm{W} \cdot X\mu$ since
$(b \cdot \underline{\underline{p}})\underline{\mathrm{P}} = Z\upsilon^{-1} \cdot a \cdot \underline{\underline{p}}\,\underline{\mathrm{P}} = Z\upsilon^{-1} \cdot w \cdot (X\upsilon)^{[1]} = w\mathrm{W}\underline{\mathrm{P}} \cdot X^{[1]}\upsilon^{-1} \cdot (X\upsilon)^{[1]} = (w\mathrm{W} \cdot X\mu)\underline{\mathrm{P}}$ and since $\underline{\mathrm{P}}$ is faithful.

$$\begin{tikzcd}
X\mathrm{W} \arrow[r, "u\mathrm{W}"] \arrow[d, "1"'] & Y\mathrm{W} \arrow[r, "v\mathrm{W}"] \arrow[d, "1"] & Z\mathrm{W} \arrow[r, "w\mathrm{W} \cdot X\mu"] \arrow[d, "b"] & X\mathrm{W}^{[1]}_{[-1]} \arrow[d, "1"] \\
X\mathrm{W} \arrow[r, "\underline{\underline{f}}"] & Y\mathrm{W} \arrow[r, "\underline{\underline{i}}"] & A \arrow[r, "\underline{\underline{p}}"] & X\mathrm{W}^{[1]}_{[-1]}
\end{tikzcd}$$

By propositions 4.2.28.(b) and 4.1.3.(b),
$X\mathrm{W}\Xi\Delta \xrightarrow{f\Xi\Delta} Y\mathrm{W}\Xi\Delta \xrightarrow{i\Xi\Delta} A\Xi\Delta \xrightarrow{(p\Xi \cdot X\mathrm{W}_{[-1]}\xi)\Delta} X\mathrm{W}\Xi^{[1]}_{[-1]}\Delta$ is a pseudo-triangle in $\mathrm{C}(\mathscr{C})$. Consequently, $X\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \xrightarrow{\underline{\underline{f}}\,\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}}} Y\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \xrightarrow{\underline{\underline{i}}\,\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}}} A\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \xrightarrow{\underline{\underline{p}}\,\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \cdot \underline{\underline{X\mathrm{W}_{[-1]}\xi}}\,\underline{\underline{\underline{\Delta}}}} (X\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}})^{[1]}$ is a triangle in $\mathrm{K}(\mathscr{C})$. Note that $Z\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \xrightarrow{b\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}}} A\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}}$ is an isomorphism in $\mathrm{K}(\mathscr{C})$ since b is an isomorphism in $\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})$. We have $u\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} = \underline{\underline{f}}\,\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}}$, $v\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \cdot b\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} = (v\mathrm{W} \cdot b)\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} = \underline{\underline{i}}\,\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}}$ and

$$\begin{aligned}
w\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \cdot X\lambda &= w\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \cdot X\mu\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \cdot \underline{\underline{X\mathrm{W}_{[-1]}\xi}}\,\underline{\underline{\underline{\Delta}}} = (w\mathrm{W} \cdot X\mu)\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \cdot \underline{\underline{X\mathrm{W}_{[-1]}\xi}}\,\underline{\underline{\underline{\Delta}}} \\
&= (b \cdot \underline{\underline{p}})\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \cdot \underline{\underline{X\mathrm{W}_{[-1]}\xi}}\,\underline{\underline{\underline{\Delta}}} = b\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \cdot \underline{\underline{p}}\,\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \cdot \underline{\underline{X\mathrm{W}_{[-1]}\xi}}\,\underline{\underline{\underline{\Delta}}}.
\end{aligned}$$

$$\begin{tikzcd}
X\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \arrow[r, "u\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}}"] \arrow[d, "1"'] & Y\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \arrow[r, "v\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}}"] \arrow[d, "1"] & Z\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \arrow[r, "w\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \cdot X\lambda"] \arrow[d, "b\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}}"'] & (X\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}})^{[1]} \arrow[d, "1"] \\
X\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \arrow[r, "\underline{\underline{f}}\,\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}}"] & Y\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \arrow[r, "\underline{\underline{i}}\,\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}}"] & A\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \arrow[r, "\underline{\underline{p}}\,\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \cdot \underline{\underline{X\mathrm{W}_{[-1]}\xi}}\,\underline{\underline{\underline{\Delta}}}"] & (X\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}})^{[1]}
\end{tikzcd}$$

We conclude that

$$\begin{aligned}
&\Big(X\mathrm{Res}_{\mathscr{W},\mathscr{F}} \xrightarrow{u\mathrm{Res}_{\mathscr{W},\mathscr{F}}} Y\mathrm{Res}_{\mathscr{W},\mathscr{F}} \xrightarrow{v\mathrm{Res}_{\mathscr{W},\mathscr{F}}} Z\mathrm{Res}_{\mathscr{W},\mathscr{F}} \xrightarrow{w\mathrm{Res}_{\mathscr{W},\mathscr{F}} \cdot X\lambda} (X\mathrm{Res}_{\mathscr{W},\mathscr{F}})^{[1]}\Big) \\
= &\Big(X\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \xrightarrow{u\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}}} Y\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \xrightarrow{v\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}}} Z\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \xrightarrow{w\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}} \cdot X\lambda} (X\mathrm{W}\underline{\underline{\Xi}}\,\underline{\underline{\underline{\Delta}}})^{[1]}\Big)
\end{aligned}$$

is a triangle in $\mathrm{K}(\mathscr{C})$. $\square$

4.3.40 Lemma. We abbreviate $\mathscr{C} = \mathscr{W}_{\lceil 0,0\rceil}$. The bounded resolution functor $\mathrm{Res}_{\mathscr{W},\mathscr{F}} \colon \mathscr{W}^{\mathrm{b}} \to \mathrm{K}^{\mathrm{b}}(\mathscr{C})$ is w-exact with respect to $\mathscr{W}|_{\mathscr{W}^{\mathrm{b}}}$ and $\mathcal{W}^{\mathscr{C},\mathrm{b}}$. $\diamondsuit$

Proof. The bounded resolution functor $\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ is exact by lemma 4.3.38.

Suppose given $X \in \mathrm{Ob}(\mathscr{W}_{\lceil 0} \cap \mathscr{W}^{\mathrm{b}})$. So we may choose $m \in \mathbf{Z}$ such that $X \in \mathrm{Ob}(\mathscr{W}_{\lceil 0,m\rceil})$.

By lemma 4.3.21, we may choose $Y \in \mathrm{Ob}(\mathrm{FO}^{\lceil 0,m\rceil}_{\mathscr{W}}(\mathscr{F}))$ such that $Y\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ is isomorphic to X in $\mathscr{W}^{\mathrm{b}}$. So $X\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ and $Y\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ are isomorphic in $\mathscr{W}^{\mathrm{b}}$.
Since $\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ is full and faithful, $X\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ and Y are isomorphic in $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})$.
Consequently, $X\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} = X\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}\Xi^{\mathrm{b}}_{\mathscr{C},\mathscr{F}}\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}}$ and $Y\Xi^{\mathrm{b}}_{\mathscr{C},\mathscr{F}}\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}}$ are isomorphic in $\mathrm{K}^{\mathrm{b}}(\mathscr{C})$.
By lemma 4.2.33.(a), we have $Y\Xi^{\mathrm{b}}_{\mathscr{C},\mathscr{F}} = Y\Xi_{\mathscr{F}} \in \mathrm{Ob}(\nabla^{\lceil 0}_{\mathscr{C}}(\mathscr{F}))$.
By lemma 4.1.17.(a), we have $Y\Xi^{\mathrm{b}}_{\mathscr{C},\mathscr{F}}\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}} = Y\Xi_{\mathscr{F}}\Delta_{\mathscr{C},\mathscr{F}} \in \mathrm{Ob}(\mathrm{C}^{\lceil 0}(\mathscr{C}))$.
We conclude that $X\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} = X\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}\Xi^{\mathrm{b}}_{\mathscr{C},\mathscr{F}}\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}} \in \mathrm{Ob}(\mathcal{W}^{\mathscr{C},\mathrm{b}}_{\lceil 0})$.

Suppose given $X \in \mathrm{Ob}(\mathscr{W}_{0\rceil} \cap \mathscr{W}^{\mathrm{b}})$. So we may choose $m \in \mathbf{Z}$ such that $X \in \mathrm{Ob}(\mathscr{W}_{\lceil m,0\rceil})$.

By lemma 4.3.21, we may choose $Y \in \mathrm{Ob}(\mathrm{FO}^{\lceil m,0\rceil}_{\mathscr{W}}(\mathscr{F}))$ such that $Y\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ is isomorphic to X in $\mathscr{W}^{\mathrm{b}}$. So $X\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ and $Y\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ are isomorphic in $\mathscr{W}^{\mathrm{b}}$.
Since $\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ is full and faithful, $X\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ and Y are isomorphic in $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})$.
Consequently, $X\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} = X\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}\Xi^{\mathrm{b}}_{\mathscr{C},\mathscr{F}}\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}}$ and $Y\Xi^{\mathrm{b}}_{\mathscr{C},\mathscr{F}}\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}}$ are isomorphic in $\mathrm{K}^{\mathrm{b}}(\mathscr{C})$.
By lemma 4.2.33.(a), we have $Y\Xi^{\mathrm{b}}_{\mathscr{C},\mathscr{F}} = Y\Xi_{\mathscr{F}} \in \mathrm{Ob}(\nabla^{0\rceil}_{\mathscr{C}}(\mathscr{F}))$.
By lemma 4.1.17.(a), we have $Y\Xi^{\mathrm{b}}_{\mathscr{C},\mathscr{F}}\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}} = Y\Xi_{\mathscr{F}}\Delta_{\mathscr{C},\mathscr{F}} \in \mathrm{Ob}(\mathrm{C}^{0\rceil}(\mathscr{C}))$.
We conclude that $X\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} = X\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}\Xi^{\mathrm{b}}_{\mathscr{C},\mathscr{F}}\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}} \in \mathrm{Ob}(\mathcal{W}^{\mathscr{C},\mathrm{b}}_{0\rceil})$. $\square$

4.3.41 Lemma. Suppose that $\mathscr{F}$ and $\mathscr{F}^{\mathrm{op}}$ have countable products of bijectives. We abbreviate $\mathscr{C} = \mathscr{W}_{\lceil 0,0\rceil}$. The resolution functor $\mathrm{Res}_{\mathscr{W},\mathscr{F}} \colon \mathscr{D} \to \mathrm{K}(\mathscr{C})$ is w-exact with respect to $\mathscr{W}$ and $\mathcal{W}^{\mathscr{C}}$. $\diamondsuit$

Proof. The resolution functor $\mathrm{Res}_{\mathscr{W},\mathscr{F}}$ is exact by lemma 4.3.39.

Suppose given $X \in \mathrm{Ob}(\mathscr{W}_{\lceil 0})$. By lemma 4.3.26, we may choose $Y \in \mathrm{Ob}(\mathrm{FO}^{\lceil 0}_{\mathscr{W}}(\mathscr{F}))$ such that $Y\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}}$ is isomorphic to X in $\mathscr{D}$. So $X\mathrm{W}_{\mathscr{W},\mathscr{F}}\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}}$ and $Y\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}}$ are isomorphic in $\mathscr{D}$.
Since $\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}}$ is full and faithful, $X\mathrm{W}_{\mathscr{W},\mathscr{F}}$ and Y are isomorphic in $\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})$.
Consequently, $X\mathrm{Res}_{\mathscr{W},\mathscr{F}} = X\mathrm{W}_{\mathscr{W},\mathscr{F}}\Xi_{\mathscr{C},\mathscr{F}}\underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}}$ and $Y\Xi_{\mathscr{C},\mathscr{F}}\underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}}$ are isomorphic in $\mathrm{K}(\mathscr{C})$.
By lemma 4.2.33.(a), we have $Y\Xi_{\mathscr{C},\mathscr{F}} = Y\Xi_{\mathscr{F}} \in \mathrm{Ob}(\nabla^{\lceil 0}_{\mathscr{C}}(\mathscr{F}))$.
By lemma 4.1.17.(a), we have $Y\Xi_{\mathscr{C},\mathscr{F}}\underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}} = Y\Xi_{\mathscr{F}}\Delta_{\mathscr{C},\mathscr{F}} \in \mathrm{Ob}(\mathrm{C}^{\lceil 0}(\mathscr{C}))$.
We conclude that $X\mathrm{Res}_{\mathscr{W},\mathscr{F}} = X\mathrm{W}_{\mathscr{W},\mathscr{F}}\Xi_{\mathscr{C},\mathscr{F}}\underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}} \in \mathrm{Ob}(\mathcal{W}^{\mathscr{C}}_{\lceil 0})$.

Suppose given $X \in \mathrm{Ob}(\mathscr{W}_{0\rceil})$. By lemma 4.3.25, we may choose $Y \in \mathrm{Ob}(\mathrm{FO}^{0\rceil}_{\mathscr{W}}(\mathscr{F}))$ such that $Y\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}}$ is isomorphic to X in $\mathscr{D}$. So $X\mathrm{W}_{\mathscr{W},\mathscr{F}}\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}}$ and $Y\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}}$ are isomorphic in $\mathscr{D}$.
Since $\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}}$ is full and faithful, $X\mathrm{W}_{\mathscr{W},\mathscr{F}}$ and Y are isomorphic in $\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})$.
Consequently, $X\mathrm{Res}_{\mathscr{W},\mathscr{F}} = X\mathrm{W}_{\mathscr{W},\mathscr{F}}\Xi_{\mathscr{C},\mathscr{F}}\underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}}$ and $Y\Xi_{\mathscr{C},\mathscr{F}}\underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}}$ are isomorphic in $\mathrm{K}(\mathscr{C})$.
By lemma 4.2.33.(b), we have $Y\Xi_{\mathscr{C},\mathscr{F}} = Y\Xi_{\mathscr{F}} \in \mathrm{Ob}(\nabla^{0\rceil}_{\mathscr{C}}(\mathscr{F}))$.
By lemma 4.1.17.(b), we have $Y\Xi_{\mathscr{C},\mathscr{F}}\underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}} = Y\Xi_{\mathscr{F}}\Delta_{\mathscr{C},\mathscr{F}} \in \mathrm{Ob}(\mathrm{C}^{0\rceil}(\mathscr{C}))$.
We conclude that $X\mathrm{Res}_{\mathscr{W},\mathscr{F}} = X\mathrm{W}_{\mathscr{W},\mathscr{F}}\Xi_{\mathscr{C},\mathscr{F}}\underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}} \in \mathrm{Ob}(\mathcal{W}^{\mathscr{C}}_{0\rceil})$. $\square$

4.3.42 Lemma. We abbreviate $\mathscr{C} = \mathscr{W}_{[0,0]}$. The functors $\mathrm{Inc}_{\mathscr{C}}^{\mathscr{W}^{\mathrm{b}}} \cdot \mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ and $\underline{\mathrm{E}}_{\mathscr{F}}|_{\mathscr{C}}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})}$ are isomorphic in $\mathscr{C}(\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}))$. $\diamondsuit$

Proof. Note that $\underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{C},\mathscr{F}}|^{\mathscr{W}^{\mathrm{b}}} = \underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$, cf. remark 4.3.29.

We have $\underline{\mathrm{E}}_{\mathscr{F}}|_{\mathscr{C}}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})} \cdot \underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} = \underline{\mathrm{E}}_{\mathscr{F}}|_{\mathscr{C}}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{C}}(\mathscr{F})} \cdot \underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{C},\mathscr{F}}|^{\mathscr{W}^{\mathrm{b}}} = \mathrm{Inc}_{\mathscr{C}}^{\mathscr{F}}|^{\mathscr{W}^{\mathrm{b}}} = \mathrm{Inc}_{\mathscr{C}}^{\mathscr{W}^{\mathrm{b}}}$ by lemma 4.2.61.(a). The result now follows from lemma 1.1.7 since $\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ and $\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ are mutually quasi-inverse equivalences.

$$\begin{tikzcd}
\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}) \arrow[r, shift left, "\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}"] & \mathscr{W}^{\mathrm{b}} \arrow[l, shift left, "\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}"] \\
\mathscr{C} \arrow[u, "\underline{\mathrm{E}}_{\mathscr{F}}|_{\mathscr{C}}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})}"] \arrow[ur, bend right, "\mathrm{Inc}_{\mathscr{C}}^{\mathscr{W}^{\mathrm{b}}}"']
\end{tikzcd}$$
$\square$

4.3.43 Lemma. We abbreviate $\mathscr{C} = \mathscr{W}_{[0,0]}$. The functors $\mathrm{Inc}_{\mathscr{C}}^{\mathscr{W}^{\mathrm{b}}} \cdot \mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ and $\mathrm{I}_{\mathrm{K}^{\mathrm{b}},\mathscr{C}}$ are isomorphic in $\mathscr{C}(\mathrm{K}^{\mathrm{b}}(\mathscr{C}))$. $\diamondsuit$

Proof. We have $\mathrm{Inc}_{\mathscr{C}}^{\mathscr{W}^{\mathrm{b}}} \cdot \mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} = \mathrm{Inc}_{\mathscr{C}}^{\mathscr{W}^{\mathrm{b}}} \cdot \mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \underline{\Xi}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}} \cdot \underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}}$.

Since $\mathrm{Inc}_{\mathscr{C}}^{\mathscr{W}^{\mathrm{b}}} \cdot \mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ and $\underline{\mathrm{E}}_{\mathscr{F}}|_{\mathscr{C}}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})}$ are isomorphic in $\mathscr{C}(\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}))$ by lemma 4.3.42, the functors $\mathrm{Inc}_{\mathscr{C}}^{\mathscr{W}^{\mathrm{b}}} \cdot \mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ and $\underline{\mathrm{E}}_{\mathscr{F}}|_{\mathscr{C}}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})} \cdot \underline{\Xi}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}} \cdot \underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}}$ are isomorphic in $\mathscr{C}(\mathrm{K}^{\mathrm{b}}(\mathscr{C}))$ by lemma 1.1.5. The result now follows from lemma 4.2.61.(b). $\square$

4.3.44 Lemma. Suppose that $\mathscr{F}$ and $\mathscr{F}^{\mathrm{op}}$ have countable products of bijectives. We abbreviate $\mathscr{C} = \mathscr{W}_{[0,0]}$. The functors $\mathrm{Inc}_{\mathscr{C}}^{\mathscr{D}} \cdot \mathrm{Res}_{\mathscr{W},\mathscr{F}}$ and $\mathrm{I}_{\mathrm{K},\mathscr{C}}$ are isomorphic in $\mathscr{C}(\mathrm{K}(\mathscr{C}))$. $\diamondsuit$

Proof. We have $\mathrm{Inc}_{\mathscr{C}}^{\mathscr{D}} \cdot \mathrm{Res}_{\mathscr{W},\mathscr{F}} = \mathrm{Inc}_{\mathscr{C}}^{\mathscr{W}^{\mathrm{b}}} \cdot \mathrm{Inc}_{\mathscr{W}^{\mathrm{b}}}^{\mathscr{D}} \cdot \mathrm{Res}_{\mathscr{W},\mathscr{F}} = \mathrm{Inc}_{\mathscr{C}}^{\mathscr{W}^{\mathrm{b}}} \cdot \mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}_{\mathrm{K}^{\mathrm{b}}(\mathscr{C})}^{\mathrm{K}(\mathscr{C})}$, cf. remark 4.3.37. So the functors $\mathrm{Inc}_{\mathscr{C}}^{\mathscr{D}} \cdot \mathrm{Res}_{\mathscr{W},\mathscr{F}}$ and $\mathrm{I}_{\mathrm{K}^{\mathrm{b}},\mathscr{C}} \cdot \mathrm{Inc}_{\mathrm{K}^{\mathrm{b}}(\mathscr{C})}^{\mathrm{K}(\mathscr{C})} = \mathrm{I}_{\mathrm{K},\mathscr{C}}$ are isomorphic by lemmata 4.3.43 and 1.1.5. $\square$

4.3.45 Lemma. Suppose that $\mathscr{F}$ has countable products. Suppose that $\mathscr{W}_{[0}$ is closed under countable products in $\underline{\mathscr{F}}$. Then $\mathscr{W}_{[\ell}$ is closed under epilimits for $\ell \in \mathbf{Z}$, cf. definition 3.2.54. $\diamondsuit$

Proof. Suppose given $\ell \in \mathbf{Z}$, $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{F}))$ with $X_k \in \mathrm{Ob}(\mathscr{W}_{[\ell})$ for $k \in \mathbf{Z}$ and a limit $(A, (a_k)_{k \in \mathbf{Z}})$ for X. Note that $(A^{[-\ell]}, (a_k^{[-\ell]})_{k \in \mathbf{Z}})$ is a limit for $X^{[-\ell]}$ by lemma 3.2.55. Choose a product $(P, (p_k)_{k \in \mathbf{Z}_{\leq 0}})$ for $(X_k^{[-\ell]})_{k \in \mathbf{Z}_{\leq 0}}$ in $\mathscr{F}$. We have $P \in \mathrm{Ob}(\mathscr{W}_{[0})$ since $\mathscr{W}_{[0}$ is closed under countable products in $\underline{\mathscr{F}}$, cf. lemma 2.1.41. Lemma 3.2.35 yields a pure short exact sequence $A^{[-\ell]} \xrightarrow{i} P \xrightarrow{q} P$ in $\mathscr{F}$, which in turn yields a triangle $P^{[-1]} \longrightarrow A^{[-\ell]} \longrightarrow P \longrightarrow P$ in $\underline{\mathscr{F}}$. So $A^{[-\ell]} \in \mathrm{Ob}(\mathscr{W}_{[0})$ since $P^{[-1]}, P \in \mathrm{Ob}(\mathscr{W}_{[0})$. We conclude that $A \in \mathrm{Ob}(\mathscr{W}_{[\ell})$. $\square$

4.3.46 Lemma. Suppose that $\mathscr{F}$ has epilimits and countable coproducts of bijectives. Suppose that $\mathscr{W}$ is left-non-degenerate and that $\mathscr{W}_{[0}$ is closed under countable products in $\underline{\mathscr{F}}$. Suppose given $V \in \mathrm{Ob}(\mathscr{D})$. There exists $Y \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F}))$ such that Y_ω is isomorphic to V in $\mathscr{D}$. $\diamondsuit$

Proof. By lemma 4.3.27, we may choose $X \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{W}}(\mathscr{F}))$ such that X_ω is isomorphic to V in $\mathscr{D}$. Choose a limit $(A, (a_k)_{k \in \mathbf{Z}})$ for $X\mathrm{P}_{\mathrm{CF},\mathscr{F}}$. For $\ell \in \mathbf{Z}$, choose a limit $(X_{\ell/-\infty}\,, (x_{\ell/-\infty,k-1})_{k\in\mathbf{Z}})$ for $X\Xi_{\mathscr{F}}\Psi_{\ell,\mathrm{CF},\mathscr{F}}$. Note that $\mathscr{W}_{\lceil \ell}$ is closed under epilimits for $\ell \in \mathbf{Z}$ by lemma 4.3.45. So $X_{\ell/-\infty} \in \mathrm{Ob}(\mathscr{W}_{\lceil \ell})$ for $\ell \in \mathbf{Z}$ by lemma 4.3.7.

Using lemma 3.3.57.(a), we obtain a filtration $F \in \mathrm{Ob}(\mathrm{F}(\mathscr{F}))$ as follows. For $\ell \in \mathbf{Z}$, let $F_\ell = X_{\ell/-\infty}$. For $k \leq \ell$ in $\mathbf{Z}$, let $F_{k \twoheadrightarrow \ell} = X\Xi_{\mathscr{F}}\psi_{k,\ell,\mathrm{CF},\mathscr{F}}\rceil \colon X_{k/-\infty} \to X_{\ell/-\infty}$.

Suppose given $\ell \in \mathbf{Z}$. We abbreviate $\chi_\ell = X\chi_{\ell,\mathrm{CF},\mathscr{F}}$. Note that χ_ℓ is an ℓ-pure monomorphism in $\mathrm{CF}(\mathscr{F})$, cf. definition 4.2.38. So lemma 3.2.39.(d) yields a pure short exact sequence $F_\ell \xrightarrow{\chi_\ell\rceil} A \xrightarrow{a_{\ell+1}} X_{\ell+1|}$.

Note that $(A, (\chi_\ell\rceil)_{\ell\in\mathbf{Z}})$ is a compatible family for F by lemmata 4.2.39 and 3.2.24.(b).

Let $u = 1_{X\mathrm{P}_{\mathrm{CF},\mathscr{F}}}\rceil^{(A,(a_k)_{k\in\mathbf{Z}})}_{(X_\omega,(x^\omega_{k|})_{k\in\mathbf{Z}})}$. Choose a triangle $Z \longrightarrow X_\omega \xrightarrow{\underline{u}} A \longrightarrow Z^{[1]}$ in $\underline{\mathscr{F}}$.

Suppose given $k \in \mathbf{Z}$. Note that $u \cdot a_k = x^\omega_{k|}$ for $k \in \mathbf{Z}$. The pure short exact sequences $X_{k/-\infty} \xrightarrow{\chi_k\rceil} A \xrightarrow{a_{k+1}} X_{k+1|}$ and $X_{|k} \xrightarrow{x^\omega_{|k}} X_\omega \xrightarrow{x^\omega_{k+1|}} X_{k+1|}$ in $\mathscr{F}$ yield triangles $X_{k/-\infty} \longrightarrow A \xrightarrow{\underline{a_{k+1}}} X_{k+1|} \longrightarrow X^{[1]}_{k/-\infty}$ and $X_{|k} \longrightarrow X_\omega \xrightarrow{\underline{x^\omega_{k+1|}}} X_{k+1|} \longrightarrow X^{[1]}_{|k}$ in $\underline{\mathscr{F}}$.

$$
\begin{tikzcd}
 & & X_{k/-\infty} \arrow[dr] & & \\
 & & & A \arrow[dr, "\underline{a_{k+1}}"] & \\
X_{|k} \arrow[rr] & & X_\omega \arrow[ur, "\underline{u}"] \arrow[rr, "\underline{x^\omega_{k+1|}}"'] & & X_{k+1|} \\
 & Z \arrow[ur] & & &
\end{tikzcd}
$$

The dual of (TR4) yields a triangle $X^{[-1]}_{k/-\infty} \longrightarrow Z \longrightarrow X_{|k} \longrightarrow X_{k/-\infty}$ in $\underline{\mathscr{F}}$.

Thus $Z \in \mathrm{Ob}(\mathscr{W}_{\lceil k})$ since $X^{[-1]}_{k/-\infty}\,, X_{|k} \in \mathrm{Ob}(\mathscr{W}_{\lceil k})$.

So we obtain $Z \in \mathrm{Ob}(\mathscr{W}_{\lceil k})$ for all $k \in \mathbf{Z}$. Since $\mathscr{W}$ is left-non-degenerate, we have $Z \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$. We conclude that A is isomorphic to X_ω and thus to V in $\mathscr{D}$.

Choose a colimit $(C, (c_k)_{k\in\mathbf{Z}})$ for $F\mathbf{B}$.

Using lemma 4.3.24, we obtain $Y \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{inj}}(\mathscr{F}))$ as follows.

Let $Y\mathrm{P}_{\mathrm{F},\mathscr{F}} = F$, $Y_\omega = A \oplus C$ and $y^\omega_{|k} = (\,\chi_k\rceil \;\; F_k\iota\cdot c_k\,)$ for $k \in \mathbf{Z}$. Since C is bijective in $\mathscr{F}$, Y_ω is isomorphic to A and thus to V in $\mathscr{D}$. Choose pushouts in $\mathscr{F}$ for $k \in \mathbf{Z}$ as follows.

$$
\begin{tikzcd}
F_k \arrow[r, "\chi_k\rceil"] \arrow[d, "F_k\iota\cdot c_k"'] & A \arrow[d, "h_{k+1}"] \\
C \arrow[r, "i_{k+1}"] & Y_{k+1|}
\end{tikzcd}
$$

We may choose cokernels $Y_{k+1|} \xrightarrow{p_{k+1}} X_{k+1|}$ of i_{k+1} such that $h_{k+1} \cdot p_{k+1} = a_{k+1}$ for $k \in \mathbf{Z}$ by lemma 1.2.7.(b). Note that since C is bijective in $\mathscr{F}$, the objects $X_{k|}$ and $Y_{k|}$ are isomorphic in $\underline{\mathscr{F}}$ for $k \in \mathbf{Z}$. Thus $Y_{k|} \in \mathrm{Ob}(\mathscr{W}_{k]})$ for $k \in \mathbf{Z}$.

Let $y^{\omega}_{k|} = \left(\begin{smallmatrix} -h_k \\ i_k \end{smallmatrix}\right)$ for $k \in \mathbf{Z}$. For $k \in \mathbf{Z}$, lemma 1.3.13 yields the pure epimorphism $Y_{k|} \xrightarrow{y_{k|}} Y_{k+1|}$ in $\mathscr{F}$ such that $y^{\omega}_{k|} \cdot y_{k|} = y^{\omega}_{k+1|}$.

Let $N \in \mathrm{Ob}(\mathrm{CF}(\mathscr{F}))$ denote the cofiltration with $N_k = C$ and $n_k = 1_C$ for $k \in \mathbf{Z}$.

Suppose given $k \in \mathbf{Z}$.

We have $\left(\begin{smallmatrix} -h_k \\ i_k \end{smallmatrix}\right) \cdot y_{k|} = y^{\omega}_{k|} \cdot y_{k|} = y^{\omega}_{k+1|} = \left(\begin{smallmatrix} -h_{k+1} \\ i_{k+1} \end{smallmatrix}\right)$, i.e. $i_k \cdot y_{k|} = i_{k+1}$ and $h_k \cdot y_{k|} = h_{k+1}$.

We have $p_k \cdot x_{k|} = y_{k|} \cdot p_{k+1}$ since

$$\left(\begin{smallmatrix} -h_k \\ i_k \end{smallmatrix}\right) \cdot p_k \cdot x_{k|} = \left(\begin{smallmatrix} -a_k \cdot x_{k|} \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -a_{k+1} \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -h_{k+1} \cdot p_{k+1} \\ i_{k+1} \cdot p_{k+1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} -h_k \\ i_k \end{smallmatrix}\right) \cdot y_{k|} \cdot p_{k+1}$$

and since $\left(\begin{smallmatrix} -h_k \\ i_k \end{smallmatrix}\right)$ is a pure epimorphism.

Thus the pure short exact sequences $C \xrightarrow{i_k} Y_{k|} \xrightarrow{p_k} X_{k|}$, where $k \in \mathbf{Z}$, yield the pure short exact sequence $N \xrightarrow{i} Y\mathrm{P}_{\mathrm{CF},\mathscr{F}} \xrightarrow{p} X\mathrm{P}_{\mathrm{CF},\mathscr{F}}$ in $\mathrm{CF}(\mathscr{F})$.

Consider the pure short exact sequence $C \xrightarrow{(0\ 1)} A \oplus C \xrightarrow{\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)} A$ in $\mathscr{F}$. For $k \in \mathbf{Z}$, we have $(\,0\ 1\,) \cdot y^{\omega}_{k|} = i_k = 1_C \cdot i_k$ and $y^{\omega}_{k|} \cdot p_k = \left(\begin{smallmatrix} -h_k \cdot p_k \\ i_k \cdot p_k \end{smallmatrix}\right) = \left(\begin{smallmatrix} -a_k \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right) \cdot a_k$. Thus $Y \in \mathrm{Ob}(\mathrm{FO}^{\lim}(\mathscr{F}))$ by lemma 3.2.41.

We conclude that $Y \in \mathrm{Ob}(\mathrm{FO}^{\lim}_{\mathscr{W}}(\mathscr{F}))$. $\square$

4.3.47 Definition. Let $\underline{\mathrm{P}}^{\lim}_{\mathscr{W},\mathscr{F}} = \underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}}|_{\underline{\underline{\mathrm{FO}}}^{\lim}_{\mathscr{W}}(\mathscr{F})}$. $\Diamond$

4.3.48 Proposition. Suppose that $\mathscr{F}$ has epilimits and countable coproducts of bijectives. Suppose that $\mathscr{W}$ is left-non-degenerate and that $\mathscr{W}_{[0}$ is closed under countable products in $\underline{\mathscr{F}}$. The functor $\underline{\mathrm{P}}^{\lim}_{\mathscr{W},\mathscr{F}} \colon \underline{\underline{\mathrm{FO}}}^{\lim}_{\mathscr{W}}(\mathscr{F}) \to \mathscr{D}$ is an equivalence. $\Diamond$

Proof.
The functor is full, faithful and dense by lemma 4.3.15, corollary 4.3.17 and lemma 4.3.46. $\square$

4.3.49 Definition. Suppose that $\mathscr{F}$ has epilimits and countable coproducts of bijectives. Suppose that $\mathscr{W}$ is left-non-degenerate and that $\mathscr{W}_{[0}$ is closed under countable products in $\underline{\mathscr{F}}$. We want to construct a quasi-inverse of the functor $\underline{\mathrm{P}}^{\lim}_{\mathscr{W},\mathscr{F}} \colon \underline{\underline{\mathrm{FO}}}^{\lim}_{\mathscr{W}}(\mathscr{F}) \to \mathscr{D}$ using lemma 1.6.5. The functor $\underline{\mathrm{P}}^{\lim}_{\mathscr{W},\mathscr{F}}$ is full and faithful, cf. proposition 4.3.48.

For $X \in \mathrm{Ob}(\mathscr{D})$, we may choose an object $X\mathrm{W}^{\lim}_{\mathscr{W},\mathscr{F}} \in \mathrm{Ob}(\underline{\underline{\mathrm{FO}}}^{\lim}_{\mathscr{W}}(\mathscr{F}))$ and an isomorphism $X\upsilon^{\lim}_{\mathscr{W},\mathscr{F}} \colon X \to X\mathrm{W}^{\lim}_{\mathscr{W},\mathscr{F}}\underline{\mathrm{P}}^{\lim}_{\mathscr{W},\mathscr{F}}$ in $\underline{\mathscr{F}}$ by lemma 4.3.46.

Lemma 1.6.5 yields the functor $\mathrm{W}^{\lim}_{\mathscr{W},\mathscr{F}} \colon \mathscr{D} \to \underline{\underline{\mathrm{FO}}}^{\lim}_{\mathscr{W}}(\mathscr{F})$, where for $X \xrightarrow{f} Y$ in $\mathscr{D}$, $X\mathrm{W}^{\lim}_{\mathscr{W},\mathscr{F}} \xrightarrow{f\mathrm{W}^{\lim}_{\mathscr{W},\mathscr{F}}} Y\mathrm{W}^{\lim}_{\mathscr{W},\mathscr{F}}$ is the unique morphism in $\underline{\underline{\mathrm{FO}}}^{\lim}_{\mathscr{W}}(\mathscr{F})$ such that $f = X\upsilon^{\lim}_{\mathscr{W},\mathscr{F}} \cdot f\mathrm{W}^{\lim}_{\mathscr{W},\mathscr{F}}\underline{\mathrm{P}}^{\lim}_{\mathscr{W},\mathscr{F}} \cdot (Y\upsilon^{\lim}_{\mathscr{W},\mathscr{F}})^{-1}$. The functors $\underline{\mathrm{P}}^{\lim}_{\mathscr{W},\mathscr{F}}$ and $\mathrm{W}^{\lim}_{\mathscr{W},\mathscr{F}}$ are mutually quasi-inverse equivalences. Moreover, we obtain the isotransformation $\upsilon^{\lim}_{\mathscr{W},\mathscr{F}} \colon 1_{\mathscr{D}} \to \mathrm{W}^{\lim}_{\mathscr{W},\mathscr{F}}\underline{\mathrm{P}}^{\lim}_{\mathscr{W},\mathscr{F}}$. $\Diamond$

4.3.50 Lemma. Suppose that $\mathscr{F}$ has epilimits and countable coproducts of bijectives. Suppose that $\mathscr{W}$ is left-non-degenerate and that $\mathscr{W}_{\lceil 0}$ is closed under countable products in $\underline{\mathscr{F}}$. The functors $\mathrm{W}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F})}$ and $\mathrm{W}_{\mathscr{W},\mathscr{F}}$ are isomorphic in $\mathscr{D}(\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F}))$. $\Diamond$

Proof. Note that $\mathrm{W}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}}$ and $\underline{\mathrm{P}}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}}$ are mutually quasi-inverse equivalences. Also note that $\mathrm{W}_{\mathscr{W},\mathscr{F}}$ and $\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}}$ are mutually quasi-inverse equivalences.
We have $\underline{\mathrm{P}}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}} = \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F})} \cdot \underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}}$, cf. definition 4.3.47. So the functors $\underline{\mathrm{P}}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{W}_{\mathscr{W},\mathscr{F}}$ and $\mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F})}$ are isomorphic in $\underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F})(\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F}))$ by lemma 1.1.7.

$$\begin{tikzcd}
\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F}) \arrow[rr, shift left, "\underline{\mathrm{P}}_{\mathscr{W},\mathscr{F}}"] & & \mathscr{D} \arrow[ll, shift left, "\mathrm{W}_{\mathscr{W},\mathscr{F}}"] \\
\underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F}) \arrow[u, "\mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F})}"] \arrow[urr, bend right, "\underline{\mathrm{P}}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}}"'] & &
\end{tikzcd}$$

By lemma 1.1.5, the functors $\mathrm{W}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}} \cdot \underline{\mathrm{P}}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{W}_{\mathscr{W},\mathscr{F}}$ and $\mathrm{W}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F})}$ are isomorphic in $\mathscr{D}(\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F}))$. Since the functors $1_{\mathscr{D}}$ and $\mathrm{W}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}} \cdot \underline{\mathrm{P}}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}}$ are isomorphic in $\mathscr{D}(\mathscr{D})$, the functors $\mathrm{W}_{\mathscr{W},\mathscr{F}} = 1_{\mathscr{D}} \cdot \mathrm{W}_{\mathscr{W},\mathscr{F}}$ and $\mathrm{W}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}} \cdot \underline{\mathrm{P}}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{W}_{\mathscr{W},\mathscr{F}}$ are isomorphic in $\mathscr{D}(\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F}))$ by lemma 1.1.5. We conclude that the functors $\mathrm{W}_{\mathscr{W},\mathscr{F}}$ and $\mathrm{W}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F})}$ are isomorphic in $\mathscr{D}(\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F}))$. $\square$

4.3.51 Lemma. Suppose that $\mathscr{F}$ has countable products. Suppose that $\mathscr{W}_{\lceil 0}$ is closed under countable products in $\underline{\mathscr{F}}$. Then we have $\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{W}_{\lceil 0}}(\mathscr{F}) \subseteq \underline{\underline{\mathrm{FO}}}_{|\mathscr{W}}(\mathscr{F})$. $\Diamond$

Proof. Suppose given $X \in \mathrm{Ob}(\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{W}_{\lceil 0}}(\mathscr{F}))$ and $\ell \in \mathbf{Z}$. We have to show that $X_{|\ell} \in \mathrm{Ob}(\mathscr{W}_{\lceil \ell})$. By lemma 4.3.7, we have $X_{\ell/k} \in \mathrm{Ob}(\mathscr{W}_{\lceil \ell})$ for $k \in \mathbf{Z}_{\leq \ell}$. Note that $X_{\ell/k} = 0_{\mathscr{F}} \in \mathrm{Ob}(\mathscr{W}_{\lceil \ell})$ for $k \in \mathbf{Z}_{>\ell}$. By lemma 4.2.40, $(X_{|\ell}\, ,\, (X_{|\ell/k-1})_{k\in\mathbf{Z}})$ is a limit for $X\Xi_{\mathscr{F}}\Psi_{\ell,\mathrm{CF},\mathscr{F}}$. Thus $X_{|\ell} \in \mathrm{Ob}(\mathscr{W}_{\lceil \ell})$ by lemma 4.3.45. $\square$

4.3.52 Proposition. Suppose that $\mathscr{F}$ has epilimits and countable coproducts of bijectives. Suppose that $\mathscr{W}$ is left-non-degenerate and that $\mathscr{W}_{\lceil 0}$ is closed under countable products in $\underline{\mathscr{F}}$. Suppose given a functor $A\colon \underline{\mathscr{F}} \to \mathscr{D}$ such that A is left-adjoint to $\mathrm{Inc}^{\mathscr{F}}_{\mathscr{D}}$.
Then $\underline{\mathrm{P}}^{\mathrm{lim,inj}}_{\omega,\mathscr{W}_{\lceil 0},\mathscr{F}} \cdot A$ is left-adjoint to $\mathrm{W}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{W}_{\lceil 0}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F})}$. $\Diamond$

Proof. Note that the functors $\mathrm{W}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}}$ and $\underline{\mathrm{P}}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}}$ are mutually quasi-inverse equivalences. Also note that we have $\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{W}_{\lceil 0}}(\mathscr{F}) \subseteq \underline{\underline{\mathrm{FO}}}_{|\mathscr{W}}(\mathscr{F})$ by lemma 4.3.51.
The map ${}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{W}_{\lceil 0}}(\mathscr{F})}(X, Y) \to {}_{\underline{\mathscr{F}}}(X\underline{\mathrm{P}}^{\mathrm{lim,inj}}_{\omega,\mathscr{W}_{\lceil 0},\mathscr{F}}, Y\underline{\mathrm{P}}^{\mathrm{lim,inj}}_{\omega,\mathscr{W}_{\lceil 0},\mathscr{F}})\colon f \mapsto f\underline{\mathrm{P}}^{\mathrm{lim,inj}}_{\omega,\mathscr{W}_{\lceil 0},\mathscr{F}}$ is bijective for $X \in \mathrm{Ob}(\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{W}_{\lceil 0}}(\mathscr{F}))$ and $Y \in \mathrm{Ob}(\underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F}))$ by lemma 4.3.15 and corollary 4.3.17.
We have $\mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{W}_{\lceil 0}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F})} \cdot \underline{\mathrm{P}}^{\mathrm{lim,inj}}_{\omega,\mathscr{W}_{\lceil 0},\mathscr{F}} = \underline{\mathrm{P}}_{\omega,\mathscr{F}}|_{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F})} = \underline{\mathrm{P}}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\mathscr{F}}_{\mathscr{D}}$.

So the result follows from lemma 1.6.17.

$$
\begin{tikzcd}
\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{W}_{\lceil 0}}(\mathscr{F}) \arrow[r, "\underline{\mathrm{P}}^{\mathrm{lim,inj}}_{\omega,\mathscr{W}_{\lceil 0},\mathscr{F}}"] & \mathscr{F} \arrow[d, bend left, "A"] \\
\underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F}) \arrow[u, "\mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{W}_{\lceil 0}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F})}"] \arrow[r, bend left=10, "\underline{\mathrm{P}}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}}"] & \mathscr{D} \arrow[u, bend left, "\mathrm{Inc}^{\mathscr{F}}_{\mathscr{D}}"] \arrow[l, bend left=10, "\mathrm{W}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}}"]
\end{tikzcd}
$$

□

4.3.53 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}_{\mathscr{W}_{\lceil 0}}(\mathscr{F}))$. Then we have $X_\omega \in \mathrm{Ob}(\mathscr{D})$. ◇

Proof. Choose $\ell, m \in \mathbf{Z}$ such that $X \in \mathrm{Ob}(\mathrm{FO}^{\lceil \ell, m \rceil}(\mathscr{F}))$.
By lemma 4.3.8, we have $X_{|\ell} \in \mathrm{Ob}(\mathscr{W}_{\lceil \ell}) \subseteq \mathrm{Ob}(\mathscr{D})$. Consider the pure short exact sequence $X_{|\ell} \xrightarrow{x^\omega_{|\ell}} X_\omega \xrightarrow{x^\omega_{\ell+1|}} X_{\ell+1|}$ in $\mathscr{F}$. We have $X_{\ell+1|} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$ since $X \in \mathrm{Ob}(\mathrm{FO}^{\lceil \ell}(\mathscr{F}))$ and thus $X_{|\ell}$ is isomorphic to X_ω in $\underline{\mathscr{F}}$. We conclude that $X_\omega \in \mathrm{Ob}(\mathscr{D})$. □

4.3.54 Proposition. Suppose that $\mathscr{W}$ is bounded, i.e. that $\mathscr{W}^{\mathrm{b}} = \mathscr{D}$.
Then $\underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{W}_{\lceil 0},\mathscr{F}}|^{\mathscr{D}}$ is left-adjoint to $\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}_{\lceil 0}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})}$. ◇

Proof. Note that the functors $\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ and $\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ are mutually quasi-inverse equivalences. Also note that we have $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}_{\lceil 0}}(\mathscr{F}) \subseteq \underline{\underline{\mathrm{FO}}}_{|\mathscr{W}}(\mathscr{F})$ by lemma 4.3.8.
Consider the functor $\underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{W}_{\lceil 0},\mathscr{F}}|^{\mathscr{D}} \colon \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}_{\lceil 0}}(\mathscr{F}) \to \mathscr{D}$, cf. lemma 4.3.53.
The map ${}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}_{\lceil 0}}(\mathscr{F})}(X,Y) \to {}_{\mathscr{D}}(X\underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{W}_{\lceil 0},\mathscr{F}}, Y\underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{W}_{\lceil 0},\mathscr{F}}) \colon f \mapsto f\underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{W}_{\lceil 0},\mathscr{F}}$ is bijective for $X \in \mathrm{Ob}(\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}_{\lceil 0}}(\mathscr{F}))$ and $Y \in \mathrm{Ob}(\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}))$ by lemma 4.3.15 and corollary 4.3.17.
We have $\mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}_{\lceil 0}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})} \cdot \underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{W}_{\lceil 0},\mathscr{F}}|^{\mathscr{D}} = \underline{\mathrm{P}}_{\omega,\mathscr{F}}|^{\mathscr{D}}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})} = \underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$.
So the result follows from lemma 1.6.17.

$$
\begin{tikzcd}
\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}_{\lceil 0}}(\mathscr{F}) \arrow[r, "\underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{W}_{\lceil 0},\mathscr{F}}|^{\mathscr{D}}"] & \mathscr{D} \arrow[d, bend left, "1_{\mathscr{D}}"] \\
\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F}) \arrow[u, "\mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}_{\lceil 0}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})}"] \arrow[r, bend left=10, "\underline{\mathrm{P}}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}"] & \mathscr{D} \arrow[u, bend left, "1_{\mathscr{D}}"] \arrow[l, bend left=10, "\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}"]
\end{tikzcd}
$$

□

4.4 Realisation functors

We have collected some facts about t-structures in section 1.8 which we will use now.
Suppose given a strict Frobenius category $\mathscr{F} = (\mathscr{F}, \mathrm{B}, \Sigma, \iota, \pi, \alpha)$. Suppose given a full triangulated subcategory $\mathscr{D} \subseteq \underline{\mathscr{F}}$. Suppose given a t-structure $\mathscr{T} = (\mathscr{T}_{0\rfloor}, \mathscr{T}_{\lceil 0})$ on $\mathscr{D}$. We abbreviate $\mathscr{H} = \mathscr{T}_{\lceil 0,0\rfloor}$.

4.4.1 Definition. Let $\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} = \underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \cdot \underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \cdot \underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{H},\mathscr{F}} \colon \mathrm{K}^{\mathrm{b}}(\mathscr{H}) \to \underline{\mathscr{F}}$, cf. definitions 4.1.22, 4.2.56 and 4.2.19.

$$\mathrm{K}^{\mathrm{b}}(\mathscr{H}) \xrightarrow{\underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}} \underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F}) \xrightarrow{\underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}} \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F}) \xrightarrow{\underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{H},\mathscr{F}}} \underline{\mathscr{F}}$$

$\diamondsuit$

4.4.2 Definition. Suppose that $\mathscr{F}$ has epilimits and monocolimits.
Let $\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} = \underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}} \cdot \underline{\mathrm{Lim}}_{\mathscr{H},\mathscr{F}} \cdot \underline{\mathrm{P}}^{\nabla}_{\omega,\mathscr{H},\mathscr{F}} \colon \mathrm{K}(\mathscr{H}) \to \underline{\mathscr{F}}$, cf. definitions 4.1.20, 4.2.57 and 4.2.19.

$$\mathrm{K}(\mathscr{H}) \xrightarrow{\underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}}} \underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F}) \xrightarrow{\underline{\mathrm{Lim}}_{\mathscr{H},\mathscr{F}}} \underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{H}}(\mathscr{F}) \xrightarrow{\underline{\mathrm{P}}^{\nabla}_{\omega,\mathscr{H},\mathscr{F}}} \underline{\mathscr{F}}$$

$\diamondsuit$

4.4.3 Remark. Suppose that $\mathscr{F}$ has epilimits and monocolimits.
We have $\mathrm{Inc}^{\mathrm{K}(\mathscr{H})}_{\mathrm{K}^{\mathrm{b}}(\mathscr{H})} \cdot \mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} = \mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}}$. $\diamondsuit$

Proof. We have

$$\begin{aligned}
\mathrm{Inc}^{\mathrm{K}(\mathscr{H})}_{\mathrm{K}^{\mathrm{b}}(\mathscr{H})} \cdot \mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} &= \mathrm{Inc}^{\mathrm{K}(\mathscr{H})}_{\mathrm{K}^{\mathrm{b}}(\mathscr{H})} \cdot \underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}} \cdot \underline{\mathrm{Lim}}_{\mathscr{H},\mathscr{F}} \cdot \underline{\mathrm{P}}^{\nabla}_{\omega,\mathscr{H},\mathscr{F}} \\
&= \underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F})}_{\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})} \cdot \underline{\mathrm{Lim}}_{\mathscr{H},\mathscr{F}} \cdot \underline{\mathrm{P}}^{\nabla}_{\omega,\mathscr{H},\mathscr{F}} \\
&= \underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \cdot \underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{H}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})} \cdot \underline{\mathrm{P}}_{\omega,\mathscr{F}}|_{\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{H}}(\mathscr{F})} \\
&= \underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \cdot \underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \cdot \underline{\mathrm{P}}_{\omega,\mathscr{F}}|_{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})} \\
&= \underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \cdot \underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \cdot \underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{H},\mathscr{F}} \\
&= \mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} .
\end{aligned}$$

Cf. definitions 4.1.22, 4.2.57 and 4.2.19. $\square$

4.4.4 Lemma. The functors $\mathrm{I}_{\mathrm{K}^{\mathrm{b}},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}}$ and $\mathrm{Inc}^{\mathscr{F}}_{\mathscr{H}}$ are isomorphic in $\mathscr{H}(\underline{\mathscr{F}})$. $\diamondsuit$

Proof. This follows from lemma 4.2.61.(d). $\square$

4.4.5 Lemma. Suppose that $\mathscr{F}$ has epilimits and monocolimits. The functors $\mathrm{I}_{\mathrm{K},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}}$ and $\mathrm{Inc}^{\mathscr{F}}_{\mathscr{H}}$ are isomorphic in $\mathscr{H}(\underline{\mathscr{F}})$. $\diamondsuit$

Proof. Note that $\mathrm{I}_{\mathrm{K}^{\mathrm{b}},\mathscr{H}} \cdot \mathrm{Inc}^{\mathrm{K}(\mathscr{H})}_{\mathrm{K}^{\mathrm{b}}(\mathscr{H})} = \mathrm{I}_{\mathrm{K},\mathscr{H}}$ and $\mathrm{Inc}^{\mathrm{K}(\mathscr{H})}_{\mathrm{K}^{\mathrm{b}}(\mathscr{H})} \cdot \mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} = \mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}}$, cf. remark 4.4.3. So $\mathrm{I}_{\mathrm{K},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} = \mathrm{I}_{\mathrm{K}^{\mathrm{b}},\mathscr{H}} \cdot \mathrm{Inc}^{\mathrm{K}(\mathscr{H})}_{\mathrm{K}^{\mathrm{b}}(\mathscr{H})} \cdot \mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} = \mathrm{I}_{\mathrm{K}^{\mathrm{b}},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}}$. Thus the result follows from lemma 4.4.4. $\square$

4.4.6 Lemma. The functor $\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} \colon \mathrm{K}^{\mathrm{b}}(\mathscr{H}) \to \underline{\mathscr{F}}$ is exact. $\diamondsuit$

Proof. We will use lemma 3.4.19 and propositions 4.1.3, 4.2.28.
We abbreviate $\underline{\mathrm{R}} = \underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}$, $\underline{\mathrm{Lim}} = \underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}$, $\Xi = \Xi^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}$, $\underline{\Xi} = \underline{\Xi}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}$, $\Delta = \Delta^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}$, $\underline{\underline{\Delta}} = \underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}$, $\mathrm{P} = \mathrm{P}^{\mathrm{b}}_{\omega,\mathscr{H},\mathscr{F}}$, $\underline{\mathrm{P}} = \underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{H},\mathscr{F}}$, $\underline{\sigma} = \underline{\sigma}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}$ and $\xi = \xi_{\Sigma,\mathscr{F}}$. We will use that $\underline{\Xi} \cdot \underline{\underline{\Delta}} \colon \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F}) \to \mathrm{K}^{\mathrm{b}}(\mathscr{H})$ is full and faithful, cf. definitions 4.1.22 and 4.2.56.

Note that $\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}}$ is additive since it is a composite of additive functors, cf. definitions 4.1.22, 4.2.56 and remark 4.2.20.

First step.

We want to construct an isotransformation $\Sigma_{\mathrm{K}^{\mathrm{b}},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} \xrightarrow{\lambda} \mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} \cdot \underline{\Sigma}$.

Suppose given $X \in \mathrm{Ob}(\mathrm{K}^{\mathrm{b}}(\mathscr{H}))$.

Note that we have $X\Sigma_{\mathrm{K}^{\mathrm{b}},\mathscr{H}}\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} = X^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}}$ and
$X\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}}\underline{\Sigma} = (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]} = (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}})^{[1]}_{[-1]}\underline{\mathrm{P}}$, cf. lemma 3.4.19.(a).

Since $\underline{\Xi} \cdot \underline{\underline{\Delta}}$ is full, we may choose $X^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \xrightarrow{X\mu} (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}})^{[1]}_{[-1]}$ in $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})$ such that
$X\mu\underline{\Xi}\underline{\underline{\Delta}} = X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot \underline{X\varsigma}^{[1]} \cdot (X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}}$.

Note that $X\mu$ is an isomorphism in $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})$ since $X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}}$, $\underline{X^{[1]}\varsigma}^{-1}$, $\underline{X\varsigma}^{[1]}$, $(X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]}$ and $\underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}}$ are isomorphisms in $\mathrm{K}^{\mathrm{b}}(\mathscr{H})$ and since $\underline{\Xi} \cdot \underline{\underline{\Delta}}$ is full and faithful.

Let $X\lambda = X\mu\underline{\mathrm{P}}\colon X^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \to (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]}$. Note that $X\lambda$ is an isomorphism in $\underline{\mathscr{F}}$ since $X\mu$ is an isomorphism in $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})$.

Suppose given $X \xrightarrow{f} Y$ in $\mathrm{K}^{\mathrm{b}}(\mathscr{H})$.
Choose $X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \xrightarrow{g} Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}$ in $\mathrm{FO}(\mathscr{F})$ such that $\underline{\underline{g}} = f\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}$.

We have $X\mu \cdot \underline{\underline{g^{[1]}_{[-1]}}} = f^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \cdot Y\mu$ since

$$
\begin{aligned}
&(X\mu \cdot \underline{\underline{g^{[1]}_{[-1]}}})\underline{\Xi}\underline{\underline{\Delta}} \\
&= X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot \underline{X\varsigma}^{[1]} \cdot (X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}} \cdot \underline{\underline{g^{[1]}_{[-1]}}}\,\underline{\Xi}\,\underline{\underline{\Delta}} \\
&= X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot \underline{X\varsigma}^{[1]} \cdot (X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1} \cdot g^{[1]}_{[-1]}}}\,\underline{\Xi}\,\underline{\underline{\Delta}} \\
&= X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot \underline{X\varsigma}^{[1]} \cdot (X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{g\Xi^{[1]}_{[-1]} \cdot Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}} \\
&= X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot \underline{X\varsigma}^{[1]} \cdot (X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{g\Xi\Delta^{[1]}}} \cdot \underline{\underline{Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}} \\
&= X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot \underline{X\varsigma}^{[1]} \cdot (X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot (f\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\Xi}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}} \\
&= X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot \underline{X\varsigma}^{[1]} \cdot (f\underline{\mathrm{R}}\,\underline{\underline{\Delta}})^{[1]} \cdot (Y\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}} \\
&= X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot f^{[1]} \cdot \underline{Y\varsigma}^{[1]} \cdot (Y\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}} \\
&= X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot f^{[1]}\underline{\mathrm{R}}\,\underline{\underline{\Delta}} \cdot \underline{Y^{[1]}\varsigma}^{-1} \cdot \underline{Y\varsigma}^{[1]} \cdot (Y\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}} \\
&= f^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\Xi}\,\underline{\underline{\Delta}} \cdot Y^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot \underline{Y^{[1]}\varsigma}^{-1} \cdot \underline{Y\varsigma}^{[1]} \cdot (Y\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}} \\
&= (f^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \cdot Y\mu)\underline{\Xi}\underline{\underline{\Delta}}
\end{aligned}
$$

and since $\underline{\Xi} \cdot \underline{\underline{\Delta}}$ is full and faithful.

$$\begin{tikzcd}
X^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \arrow[r, "X\mu"] \arrow[d, "f^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}"'] & (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}})^{[1]}_{[-1]} \arrow[d, "\underline{\underline{g}}^{[1]}_{[-1]}"] \\
Y^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \arrow[r, "Y\mu"] & (Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}})^{[1]}_{[-1]}
\end{tikzcd}$$

We have

$$\begin{aligned}
X\lambda \cdot (f\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]} &= X\mu\underline{\mathrm{P}} \cdot (\underline{\underline{g}}\,\underline{\mathrm{P}})^{[1]} = X\mu\underline{\mathrm{P}} \cdot \underline{\underline{g^{[1]}_{[-1]}}}\,\underline{\mathrm{P}} = (X\mu \cdot \underline{\underline{g^{[1]}_{[-1]}}})\underline{\mathrm{P}} = (f^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \cdot Y\mu)\underline{\mathrm{P}} \\
&= f^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \cdot Y\lambda .
\end{aligned}$$

$$\begin{tikzcd}
X^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \arrow[r, "X\lambda"] \arrow[d, "f^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}}"'] & (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]} \arrow[d, "(f\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]}"] \\
Y^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \arrow[r, "Y\lambda"] & (Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]}
\end{tikzcd}$$

We conclude that $\Sigma_{\mathrm{K^b},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} \xrightarrow{\lambda} \mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} \cdot \underline{\Sigma}$ is an isotransformation.

Second step.

Suppose given a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X^{[1]}$ in $\mathrm{K^b}(\mathscr{H})$. We want to show that

$$\begin{aligned}
&\Big(X\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} \xrightarrow{u\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}}} Y\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} \xrightarrow{v\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}}} Z\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} \xrightarrow{w\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} \cdot X\lambda} (X\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}})^{[1]} \Big) \\
= &\Big(X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \xrightarrow{u\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}}} Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \xrightarrow{v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}}} Z\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \xrightarrow{w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \cdot X\lambda} (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]} \Big)
\end{aligned}$$

is a triangle in $\underline{\mathscr{F}}$.

Choose $X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \xrightarrow{f} Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}$ in $\mathrm{FO}(\mathscr{F})$ such that $\underline{\underline{f}} = u\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}$.

Choose a pseudo-triangle $X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \xrightarrow{f} Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \xrightarrow{i} A \xrightarrow{p} X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}^{[1]}_{[-1]}$ in $\mathrm{FO}(\mathscr{F})$ such that $A, X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}^{[1]}_{[-1]} \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F}))$, cf. lemmata 3.4.33 and 4.2.29. By propositions 4.2.28.(b) and 4.1.3.(b), $X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\Xi\Delta \xrightarrow{f\Xi\Delta} Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\Xi\Delta \xrightarrow{i\Xi\Delta} A\Xi\Delta \xrightarrow{(p\Xi\cdot X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi)\Delta} X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\Xi^{[1]}_{[-1]}\Delta$ is a pseudo-triangle in $\mathrm{C}(\mathscr{H})$. Consequently, $X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\Xi}\,\underline{\Delta} \xrightarrow{\underline{\underline{f}}\,\underline{\Xi}\,\underline{\underline{\Delta}}} Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\Xi}\,\underline{\underline{\Delta}} \xrightarrow{\underline{\underline{i}}\,\underline{\Xi}\,\underline{\underline{\Delta}}} A\underline{\Xi}\,\underline{\underline{\Delta}} \xrightarrow{\underline{\underline{p}}\,\underline{\Xi}\,\underline{\underline{\Delta}}\cdot \underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi}}\,\underline{\underline{\Delta}}} (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\Xi}\,\underline{\underline{\Delta}})^{[1]}$ is a triangle in $\mathrm{K^b}(\mathscr{H})$. We have

$$\begin{aligned}
\underline{X\varsigma} \cdot X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}} \cdot \underline{\underline{f}}\,\underline{\Xi}\,\underline{\underline{\Delta}} &= \underline{X\varsigma} \cdot X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}} \cdot u\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\Xi}\,\underline{\underline{\Delta}} = \underline{X\varsigma} \cdot (X\underline{\mathrm{R}}\,\underline{\sigma} \cdot u\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\Xi})\underline{\underline{\Delta}} \\
&= \underline{X\varsigma} \cdot (u\underline{\mathrm{R}} \cdot Y\underline{\mathrm{R}}\,\underline{\sigma})\underline{\underline{\Delta}} = \underline{X\varsigma} \cdot u\underline{\mathrm{R}}\,\underline{\underline{\Delta}} \cdot Y\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}} = u \cdot \underline{Y\varsigma} \cdot Y\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}} .
\end{aligned}$$

Since $\underline{X\varsigma}\cdot X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}}$ and $\underline{Y\varsigma}\cdot Y\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}}$ are isomorphisms in $\mathrm{K^b}(\mathscr{H})$, we may choose an isomorphism $Z \xrightarrow{a} A\underline{\Xi}\,\underline{\underline{\Delta}}$ in $\mathrm{K^b}(\mathscr{H})$ such that $v \cdot a = \underline{Y\varsigma} \cdot Y\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}} \cdot \underline{\underline{i}}\,\underline{\Xi}\,\underline{\underline{\Delta}}$ and

$a \cdot \underline{\underline{p}}\,\Xi\,\underline{\underline{\Delta}} \cdot \underline{\underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi}}}\,\underline{\underline{\Delta}} = w \cdot (\underline{X\varsigma} \cdot X\underline{\mathrm{R}}\,\sigma\,\underline{\underline{\Delta}})^{[1]}.$

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "\underline{X\varsigma}\cdot X\underline{\mathrm{R}}\,\sigma\,\underline{\underline{\Delta}}"] & Y \arrow[r, "v"] \arrow[d, "\underline{Y\varsigma}\cdot Y\underline{\mathrm{R}}\,\sigma\,\underline{\underline{\Delta}}"] & Z \arrow[r, "w"] \arrow[d, "a"] & X^{[1]} \arrow[d, "(\underline{X\varsigma}\cdot X\underline{\mathrm{R}}\,\sigma\,\underline{\underline{\Delta}})^{[1]}"'] \\
X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\Xi\,\underline{\underline{\Delta}} \arrow[r, "\underline{\underline{f}}\,\Xi\,\underline{\underline{\Delta}}"'] & Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\Xi\,\underline{\underline{\Delta}} \arrow[r, "\underline{\underline{i}}\,\Xi\,\underline{\underline{\Delta}}"'] & A\Xi\,\underline{\underline{\Delta}} \arrow[r, "\underline{\underline{p}}\,\Xi\,\underline{\underline{\Delta}}\cdot\underline{\underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi}}}\,\underline{\underline{\Delta}}"'] & (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\Xi\,\underline{\underline{\Delta}})^{[1]}
\end{tikzcd}$$

Since $\Xi \cdot \underline{\underline{\Delta}}$ is full, we may choose $Z\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \xrightarrow{b} A$ in $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})$ such that $b\,\Xi\,\underline{\underline{\Delta}} = Z\underline{\mathrm{R}}\,\sigma^{-1}\underline{\underline{\Delta}} \cdot \underline{Z\varsigma}^{-1} \cdot a$. Note that b is an isomorphism in $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})$ since $Z\underline{\mathrm{R}}\,\sigma^{-1}\underline{\underline{\Delta}}$, $\underline{Z\varsigma}^{-1}$, a are isomorphisms in $\mathrm{K}^{\mathrm{b}}(\mathscr{H})$ and since $\Xi \cdot \underline{\underline{\Delta}}$ is full and faithful.

We have $v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \cdot b = \underline{\underline{i}}$ since

$$\begin{aligned}
(v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \cdot b)\Xi\,\underline{\underline{\Delta}} &= v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\Xi\,\underline{\underline{\Delta}} \cdot b\Xi\,\underline{\underline{\Delta}} = v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\Xi\,\underline{\underline{\Delta}} \cdot Z\underline{\mathrm{R}}\,\sigma^{-1}\underline{\underline{\Delta}} \cdot \underline{Z\varsigma}^{-1} \cdot a \\
&= (v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\Xi \cdot Z\underline{\mathrm{R}}\,\sigma^{-1})\underline{\underline{\Delta}} \cdot \underline{Z\varsigma}^{-1} \cdot a = (Y\underline{\mathrm{R}}\,\sigma^{-1} \cdot v\underline{\mathrm{R}})\underline{\underline{\Delta}} \cdot \underline{Z\varsigma}^{-1} \cdot a \\
&= Y\underline{\mathrm{R}}\,\sigma^{-1}\underline{\underline{\Delta}} \cdot \underline{Y\varsigma}^{-1} \cdot v \cdot a = Y\underline{\mathrm{R}}\,\sigma^{-1}\underline{\underline{\Delta}} \cdot \underline{Y\varsigma}^{-1} \cdot \underline{Y\varsigma} \cdot Y\underline{\mathrm{R}}\,\sigma\,\underline{\underline{\Delta}} \cdot \underline{\underline{i}}\,\Xi\,\underline{\underline{\Delta}} \\
&= Y\underline{\mathrm{R}}\,\sigma^{-1}\underline{\underline{\Delta}} \cdot Y\underline{\mathrm{R}}\,\sigma\,\underline{\underline{\Delta}} \cdot \underline{\underline{i}}\,\Xi\,\underline{\underline{\Delta}} = \underline{\underline{i}}\,\Xi\,\underline{\underline{\Delta}}
\end{aligned}$$

and since $\Xi \cdot \underline{\underline{\Delta}}$ is faithful. We have $b \cdot \underline{\underline{p}} = w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \cdot X\mu$ since

$$\begin{aligned}
(b \cdot \underline{\underline{p}})\Xi\,\underline{\underline{\Delta}} &= Z\underline{\mathrm{R}}\,\sigma^{-1}\underline{\underline{\Delta}} \cdot \underline{Z\varsigma}^{-1} \cdot a \cdot \underline{\underline{p}}\,\Xi\,\underline{\underline{\Delta}} \\
&= Z\underline{\mathrm{R}}\,\sigma^{-1}\underline{\underline{\Delta}} \cdot \underline{Z\varsigma}^{-1} \cdot w \cdot (\underline{X\varsigma} \cdot X\underline{\mathrm{R}}\,\sigma\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}}\,\underline{\underline{\Delta}} \\
&= Z\underline{\mathrm{R}}\,\sigma^{-1}\underline{\underline{\Delta}} \cdot w\underline{\mathrm{R}}\,\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot \underline{X\varsigma}^{[1]} \cdot (X\underline{\mathrm{R}}\,\sigma\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}}\,\underline{\underline{\Delta}} \\
&= (Z\underline{\mathrm{R}}\,\sigma^{-1} \cdot w\underline{\mathrm{R}})\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot \underline{X\varsigma}^{[1]} \cdot (X\underline{\mathrm{R}}\,\sigma\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}}\,\underline{\underline{\Delta}} \\
&= (w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\Xi \cdot X^{[1]}\underline{\mathrm{R}}\,\sigma^{-1})\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot \underline{X\varsigma}^{[1]} \cdot (X\underline{\mathrm{R}}\,\sigma\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}}\,\underline{\underline{\Delta}} \\
&= w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\Xi\,\underline{\underline{\Delta}} \cdot X^{[1]}\underline{\mathrm{R}}\,\sigma^{-1}\,\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot \underline{X\varsigma}^{[1]} \cdot (X\underline{\mathrm{R}}\,\sigma\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}}\,\underline{\underline{\Delta}} \\
&= w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\Xi\,\underline{\underline{\Delta}} \cdot X\mu\Xi\,\underline{\underline{\Delta}} \\
&= (w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \cdot X\mu)\Xi\,\underline{\underline{\Delta}}
\end{aligned}$$

and since $\Xi \cdot \underline{\underline{\Delta}}$ is faithful.

$$\begin{tikzcd}
X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \arrow[r, "u\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}"] \arrow[d, "1"'] & Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \arrow[r, "v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}"] \arrow[d, "1"] & Z\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \arrow[r, "w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\cdot X\mu"] \arrow[d, "b"] & X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}^{[1]}_{[-1]} \arrow[d, "1"] \\
X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \arrow[r, "\underline{\underline{f}}"] & Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \arrow[r, "\underline{\underline{i}}"] & A \arrow[r, "\underline{\underline{p}}"] & X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}^{[1]}_{[-1]}
\end{tikzcd}$$

By lemma 3.4.19.(b), $X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\mathrm{P} \xrightarrow{f\mathrm{P}} Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\mathrm{P} \xrightarrow{i\mathrm{P}} A\mathrm{P} \xrightarrow{p\mathrm{P}} X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}^{[1]}_{[-1]}\mathrm{P}$ is a pseudo-triangle in $\mathscr{F}$. Consequently, $X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \xrightarrow{\underline{\underline{f}}\,\underline{\mathrm{P}}} Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \xrightarrow{\underline{\underline{i}}\,\underline{\mathrm{P}}} A\underline{\mathrm{P}} \xrightarrow{\underline{\underline{p}}\,\underline{\mathrm{P}}} (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]}$ is a triangle in $\underline{\mathscr{F}}$.

Note that $Z\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \xrightarrow{b\underline{\mathrm{P}}} A\underline{\mathrm{P}}$ is an isomorphism in $\underline{\mathscr{F}}$ since b is an isomorphism in $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})$.
We have $u\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} = \underline{\underline{f}}\,\underline{\mathrm{P}}$, $v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \cdot b\underline{\mathrm{P}} = (v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \cdot b)\underline{\mathrm{P}} = \underline{\underline{i}}\,\underline{\mathrm{P}}$ and

$$w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \cdot X\lambda = w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \cdot X\mu\underline{\mathrm{P}} = (w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \cdot X\mu)\underline{\mathrm{P}} = (b \cdot \underline{\underline{p}})\underline{\mathrm{P}} = b\underline{\mathrm{P}} \cdot \underline{\underline{p}}\,\underline{\mathrm{P}}\,.$$

$$\begin{tikzcd}
X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \arrow[r, "u\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}}"] \arrow[d, "1"'] & Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \arrow[r, "v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}}"] \arrow[d, "1"] & Z\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \arrow[r, "w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}}\cdot X\lambda"] \arrow[d, "b\underline{\mathrm{P}}"] & (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]} \arrow[d, "1"] \\
X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \arrow[r, "\underline{\underline{f}}\,\underline{\mathrm{P}}"] & Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \arrow[r, "\underline{\underline{i}}\,\underline{\mathrm{P}}"] & A \arrow[r, "\underline{\underline{p}}\,\underline{\mathrm{P}}"] & (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]}
\end{tikzcd}$$

We conclude that

$$\begin{aligned}
&\Big(X\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} \xrightarrow{u\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}}} Y\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} \xrightarrow{v\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}}} Z\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} \xrightarrow{w\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}}\cdot X\lambda} (X\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}})^{[1]}\Big) \\
= &\Big(X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \xrightarrow{u\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}}} Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \xrightarrow{v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}}} Z\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \xrightarrow{w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}}\cdot X\lambda} (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]}\Big)
\end{aligned}$$

is a triangle in $\underline{\mathscr{F}}$. $\square$

4.4.7 Lemma. Suppose that $\mathscr{F}$ has epilimits and monocolimits.
The functor $\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} \colon \mathrm{K}(\mathscr{H}) \to \underline{\mathscr{F}}$ is exact. $\diamondsuit$

Proof. We will use lemma 3.4.19 and propositions 4.1.3, 4.2.28. We abbreviate $\underline{\mathrm{R}} = \underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}}$, $\underline{\mathrm{Lim}} = \underline{\mathrm{Lim}}_{\mathscr{H},\mathscr{F}}$, $\Xi = \Xi^{\nabla}_{\mathscr{H},\mathscr{F}}$, $\underline{\Xi} = \underline{\Xi}^{\nabla}_{\mathscr{H},\mathscr{F}}$, $\Delta = \Delta_{\mathscr{H},\mathscr{F}}$, $\underline{\underline{\Delta}} = \underline{\underline{\Delta}}_{\mathscr{H},\mathscr{F}}$, $\mathrm{P} = \mathrm{P}^{\nabla}_{\omega,\mathscr{H},\mathscr{F}}$, $\underline{\mathrm{P}} = \underline{\mathrm{P}}^{\nabla}_{\omega,\mathscr{H},\mathscr{F}}$, $\underline{\sigma} = \underline{\sigma}_{\mathscr{H},\mathscr{F}}$ and $\xi = \xi_{\Sigma,\mathscr{F}}$.
We will use that $\Xi\cdot\underline{\underline{\Delta}} \colon \underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{H}}(\mathscr{F}) \to \mathrm{K}(\mathscr{H})$ is full and faithful, cf. definitions 4.1.20 and 4.2.57.
Note that $\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}}$ is additive since it is a composite of additive functors, cf. definitions 4.1.20, 4.2.57 and remark 4.2.20.

First step.
We want to construct an isotransformation $\Sigma_{\mathrm{K},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} \xrightarrow{\lambda} \mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} \cdot \underline{\Sigma}$.
Suppose given $X \in \mathrm{Ob}(\mathrm{K}(\mathscr{H}))$.
Note that we have $X\Sigma_{\mathrm{K},\mathscr{H}}\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} = X^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}}$ and
$X\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}}\underline{\Sigma} = (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]} = (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}})^{[1]}_{[-1]}\underline{\mathrm{P}}$, cf. lemma 3.4.19.(a).

Since $\Xi\cdot\underline{\underline{\Delta}}$ is full, we may choose $X^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \xrightarrow{X\mu} (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}})^{[1]}_{[-1]}$ in $\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{H}}(\mathscr{F})$ such that
$X\mu\Xi\underline{\underline{\Delta}} = X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma^{-1}} \cdot \underline{X\varsigma^{[1]}} \cdot (X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}}$.

Note that $X\mu$ is an isomorphism in $\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{H}}(\mathscr{F})$ since $X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}}$, $\underline{X^{[1]}\varsigma^{-1}}$, $\underline{X\varsigma^{[1]}}$, $(X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]}$ and $\underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}}$ are isomorphisms in $\mathrm{K}(\mathscr{H})$ and since $\Xi\cdot\underline{\underline{\Delta}}$ is full and faithful.
Let $X\lambda = X\mu\underline{\mathrm{P}} \colon X^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \to (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]}$. Note that $X\lambda$ is an isomorphism in $\underline{\mathscr{F}}$ since $X\mu$ is an isomorphism in $\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{H}}(\mathscr{F})$.

Suppose given $X \xrightarrow{f} Y$ in $\mathrm{K}(\mathscr{H})$.

Choose $X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \xrightarrow{g} Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}$ in $\mathrm{FO}(\mathscr{F})$ such that $\underline{\underline{g}} = f\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}$.

We have $X\mu \cdot \underline{\underline{g^{[1]}_{[-1]}}} = f^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \cdot Y\mu$ since

$$
\begin{aligned}
&(X\mu \cdot \underline{\underline{g^{[1]}_{[-1]}}})\underline{\Xi}\underline{\underline{\Delta}} \\
&= X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot \underline{X\varsigma}^{[1]} \cdot (X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}} \cdot \underline{\underline{g^{[1]}_{[-1]}\Xi}}\,\underline{\underline{\Delta}} \\
&= X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot \underline{X\varsigma}^{[1]} \cdot (X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1} \cdot g^{[1]}_{[-1]}\Xi}}\,\underline{\underline{\Delta}} \\
&= X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot \underline{X\varsigma}^{[1]} \cdot (X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{g\Xi^{[1]}_{[-1]} \cdot Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}} \\
&= X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot \underline{X\varsigma}^{[1]} \cdot (X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{g\Xi\Delta^{[1]}}} \cdot \underline{\underline{Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}} \\
&= X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot \underline{X\varsigma}^{[1]} \cdot (X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot (f\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\Xi}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}} \\
&= X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot \underline{X\varsigma}^{[1]} \cdot (f\underline{\mathrm{R}}\,\underline{\underline{\Delta}})^{[1]} \cdot (Y\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}} \\
&= X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot f^{[1]} \cdot \underline{Y\varsigma}^{[1]} \cdot (Y\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}} \\
&= X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot f^{[1]}\underline{\mathrm{R}}\,\underline{\underline{\Delta}} \cdot \underline{Y^{[1]}\varsigma}^{-1} \cdot \underline{Y\varsigma}^{[1]} \cdot (Y\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}} \\
&= f^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\Xi}\,\underline{\underline{\Delta}} \cdot Y^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot \underline{Y^{[1]}\varsigma}^{-1} \cdot \underline{Y\varsigma}^{[1]} \cdot (Y\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}} \\
&= (f^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \cdot Y\mu)\underline{\Xi}\underline{\underline{\Delta}}
\end{aligned}
$$

and since $\underline{\Xi} \cdot \underline{\underline{\Delta}}$ is full and faithful.

$$
\begin{tikzcd}
X^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \arrow[r, "X\mu"] \arrow[d, "f^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}"'] & (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}})^{[1]}_{[-1]} \arrow[d, "\underline{\underline{g^{[1]}_{[-1]}}}"] \\
Y^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \arrow[r, "Y\mu"] & (Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}})^{[1]}_{[-1]}
\end{tikzcd}
$$

We have

$$
\begin{aligned}
X\lambda \cdot (f\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]} &= X\mu\underline{\mathrm{P}} \cdot (\underline{\underline{g}}\,\underline{\mathrm{P}})^{[1]} = X\mu\underline{\mathrm{P}} \cdot \underline{\underline{g^{[1]}_{[-1]}}}\,\underline{\mathrm{P}} = (X\mu \cdot \underline{\underline{g^{[1]}_{[-1]}}})\underline{\mathrm{P}} = (f^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \cdot Y\mu)\underline{\mathrm{P}} \\
&= f^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \cdot Y\lambda.
\end{aligned}
$$

$$
\begin{tikzcd}
X^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \arrow[r, "X\lambda"] \arrow[d, "f^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}}"'] & (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]} \arrow[d, "(f\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]}"] \\
Y^{[1]}\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \arrow[r, "Y\lambda"] & (Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]}
\end{tikzcd}
$$

We conclude that $\Sigma_{\mathrm{K},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} \xrightarrow{\lambda} \mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} \cdot \underline{\Sigma}$ is an isotransformation.

Second step.

Suppose given a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X^{[1]}$ in $\mathrm{K}(\mathscr{H})$. We want to show that

$$\begin{aligned}
&\Big(X\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} \xrightarrow{u\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}}} Y\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} \xrightarrow{v\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}}} Z\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} \xrightarrow{w\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}}\cdot X\lambda} (X\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}})^{[1]} \Big) \\
= &\Big(X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \xrightarrow{u\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}}} Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \xrightarrow{v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}}} Z\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \xrightarrow{w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}}\cdot X\lambda} (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]} \Big)
\end{aligned}$$

is a triangle in $\underline{\mathscr{F}}$.

Choose $X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \xrightarrow{f} Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}$ in $\mathrm{FO}(\mathscr{F})$ such that $\underline{\underline{f}} = u\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}$.

Choose a pseudo-triangle $X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \xrightarrow{f} Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \xrightarrow{i} A \xrightarrow{p} X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}^{[1]}_{[-1]}$ in $\mathrm{FO}(\mathscr{F})$ such that $A, X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}^{[1]}_{[-1]} \in \mathrm{Ob}(\mathrm{FO}^{\nabla}_{\mathscr{H}}(\mathscr{F}))$, cf. lemmata 3.4.49 and 4.2.29. By propositions 4.2.28.(b) and 4.1.3.(b), $X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\Xi\Delta \xrightarrow{f\Xi\Delta} Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\Xi\Delta \xrightarrow{i\Xi\Delta} A\Xi\Delta \xrightarrow{(p\Xi\cdot X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi)\Delta} X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\Xi^{[1]}_{[-1]}\Delta$ is a pseudo-triangle in $\mathrm{C}(\mathscr{H})$. Consequently,
$X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\Xi}\,\underline{\underline{\Delta}} \xrightarrow{\underline{\underline{f}}\,\underline{\Xi}\,\underline{\underline{\Delta}}} Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\Xi}\,\underline{\underline{\Delta}} \xrightarrow{\underline{\underline{i}}\,\underline{\Xi}\,\underline{\underline{\Delta}}} A\underline{\Xi}\,\underline{\underline{\Delta}} \xrightarrow{\underline{\underline{p}}\,\underline{\Xi}\,\underline{\underline{\Delta}}\cdot\underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi}}\,\underline{\underline{\Delta}}} (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\Xi}\,\underline{\underline{\Delta}})^{[1]}$ is a triangle in $\mathrm{K}(\mathscr{H})$. We have

$$\begin{aligned}
\underline{X\varsigma}\cdot X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}}\cdot\underline{\underline{f}}\,\underline{\Xi}\,\underline{\underline{\Delta}} &= \underline{X\varsigma}\cdot X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}}\cdot u\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\Xi}\,\underline{\underline{\Delta}} = \underline{X\varsigma}\cdot(X\underline{\mathrm{R}}\,\underline{\sigma}\cdot u\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\Xi})\underline{\underline{\Delta}} \\
&= \underline{X\varsigma}\cdot(u\underline{\mathrm{R}}\cdot Y\underline{\mathrm{R}}\,\underline{\sigma})\underline{\underline{\Delta}} = \underline{X\varsigma}\cdot u\underline{\mathrm{R}}\,\underline{\underline{\Delta}}\cdot Y\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}} = u\cdot\underline{Y\varsigma}\cdot Y\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}}\,.
\end{aligned}$$

Since $\underline{X\varsigma}\cdot X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}}$ and $\underline{Y\varsigma}\cdot Y\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}}$ are isomorphisms in $\mathrm{K}(\mathscr{H})$, we may choose an isomorphism $Z \xrightarrow{a} A\underline{\Xi}\,\underline{\underline{\Delta}}$ in $\mathrm{K}(\mathscr{H})$ such that $v\cdot a = \underline{Y\varsigma}\cdot Y\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}}\cdot\underline{\underline{i}}\,\underline{\Xi}\,\underline{\underline{\Delta}}$ and
$a\cdot\underline{\underline{p}}\,\underline{\Xi}\,\underline{\underline{\Delta}}\cdot\underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}}}\xi\,\underline{\underline{\Delta}} = w\cdot(\underline{X\varsigma}\cdot X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]}$.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "\underline{X\varsigma}\cdot X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}}"] & Y \arrow[r, "v"] \arrow[d, "\underline{Y\varsigma}\cdot Y\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}}"] & Z \arrow[r, "w"] \arrow[d, "a"] & X^{[1]} \arrow[d, "(\underline{X\varsigma}\cdot X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]}"'] \\
X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\Xi}\,\underline{\underline{\Delta}} \arrow[r, "\underline{\underline{f}}\,\underline{\Xi}\,\underline{\underline{\Delta}}"'] & Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\Xi}\,\underline{\underline{\Delta}} \arrow[r, "\underline{\underline{i}}\,\underline{\Xi}\,\underline{\underline{\Delta}}"'] & A\underline{\Xi}\,\underline{\underline{\Delta}} \arrow[r, "\underline{\underline{p}}\,\underline{\Xi}\,\underline{\underline{\Delta}}\cdot\underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}}}\xi\,\underline{\underline{\Delta}}"'] & (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\Xi}\,\underline{\underline{\Delta}})^{[1]}
\end{tikzcd}$$

Since $\underline{\Xi}\cdot\underline{\underline{\Delta}}$ is full, we may choose $Z\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \xrightarrow{b} A$ in $\underline{\mathrm{FO}}^{\nabla}_{\mathscr{H}}(\mathscr{F})$ such that
$b\,\underline{\Xi}\,\underline{\underline{\Delta}} = Z\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}}\cdot\underline{Z\varsigma}^{-1}\cdot a$. Note that b is an isomorphism in $\underline{\mathrm{FO}}^{\nabla}_{\mathscr{H}}(\mathscr{F})$ since $Z\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}}$, $\underline{Z\varsigma}^{-1}$, a are isomorphisms in $\mathrm{K}(\mathscr{H})$ and since $\underline{\Xi}\cdot\underline{\underline{\Delta}}$ is full and faithful.

We have $v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\cdot b = \underline{\underline{i}}$ since

$$\begin{aligned}
(v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\cdot b)\underline{\Xi}\,\underline{\underline{\Delta}} &= v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\Xi}\,\underline{\underline{\Delta}}\cdot b\,\underline{\Xi}\,\underline{\underline{\Delta}} = v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\Xi}\,\underline{\underline{\Delta}}\cdot Z\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}}\cdot\underline{Z\varsigma}^{-1}\cdot a \\
&= (v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\Xi}\cdot Z\underline{\mathrm{R}}\,\underline{\sigma}^{-1})\underline{\underline{\Delta}}\cdot\underline{Z\varsigma}^{-1}\cdot a = (Y\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\cdot v\underline{\mathrm{R}})\underline{\underline{\Delta}}\cdot\underline{Z\varsigma}^{-1}\cdot a \\
&= Y\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}}\cdot\underline{Y\varsigma}^{-1}\cdot v\cdot a = Y\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}}\cdot\underline{Y\varsigma}^{-1}\cdot\underline{Y\varsigma}\cdot Y\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}}\cdot\underline{\underline{i}}\,\underline{\Xi}\,\underline{\underline{\Delta}} \\
&= Y\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}}\cdot Y\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}}\cdot\underline{\underline{i}}\,\underline{\Xi}\,\underline{\underline{\Delta}} = \underline{\underline{i}}\,\underline{\Xi}\,\underline{\underline{\Delta}}
\end{aligned}$$

and since $\Xi \cdot \underline{\underline{\Delta}}$ is faithful. We have $b \cdot \underline{\underline{p}} = w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \cdot X\mu$ since

$$\begin{aligned}
(b \cdot \underline{\underline{p}})\Xi\,\underline{\underline{\Delta}} &= Z\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot \underline{Z\varsigma}^{-1} \cdot a \cdot \underline{\underline{p}}\,\Xi\,\underline{\underline{\Delta}} \\
&= Z\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot \underline{Z\varsigma}^{-1} \cdot w \cdot (\underline{X\varsigma} \cdot X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}} \\
&= Z\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\underline{\underline{\Delta}} \cdot w\underline{\mathrm{R}}\,\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot \underline{X\varsigma}^{[1]} \cdot (X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}} \\
&= (Z\underline{\mathrm{R}}\,\underline{\sigma}^{-1} \cdot w\underline{\mathrm{R}})\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot \underline{X\varsigma}^{[1]} \cdot (X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}} \\
&= (w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\Xi \cdot X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1})\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot \underline{X\varsigma}^{[1]} \cdot (X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}} \\
&= w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\Xi\,\underline{\underline{\Delta}} \cdot X^{[1]}\underline{\mathrm{R}}\,\underline{\sigma}^{-1}\,\underline{\underline{\Delta}} \cdot \underline{X^{[1]}\varsigma}^{-1} \cdot \underline{X\varsigma}^{[1]} \cdot (X\underline{\mathrm{R}}\,\underline{\sigma}\,\underline{\underline{\Delta}})^{[1]} \cdot \underline{\underline{X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}_{[-1]}\xi^{-1}}}\,\underline{\underline{\Delta}} \\
&= w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\Xi\,\underline{\underline{\Delta}} \cdot X\mu\Xi\,\underline{\underline{\Delta}} \\
&= (w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \cdot X\mu)\Xi\,\underline{\underline{\Delta}}
\end{aligned}$$

and since $\Xi \cdot \underline{\underline{\Delta}}$ is faithful.

$$\begin{tikzcd}
X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \arrow[r, "u\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}"] \arrow[d, "1"'] & Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \arrow[r, "v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}"] \arrow[d, "1"] & Z\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \arrow[r, "w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\cdot X\mu"] \arrow[d, "b"] & X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}^{[1]}_{[-1]} \arrow[d, "1"] \\
X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \arrow[r, "\underline{\underline{f}}"] & Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \arrow[r, "\underline{\underline{i}}"] & A \arrow[r, "\underline{\underline{p}}"] & X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}^{[1]}_{[-1]}
\end{tikzcd}$$

By lemma 3.4.19.(b), $X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\mathrm{P} \xrightarrow{f\mathrm{P}} Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\mathrm{P} \xrightarrow{i\mathrm{P}} A\mathrm{P} \xrightarrow{p\mathrm{P}} X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}^{[1]}_{[-1]}\mathrm{P}$ is a pseudo-triangle in $\mathscr{F}$. Consequently, $X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \xrightarrow{\underline{\underline{f}}\,\underline{\mathrm{P}}} Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \xrightarrow{\underline{\underline{i}}\,\underline{\mathrm{P}}} A\underline{\mathrm{P}} \xrightarrow{\underline{\underline{p}}\,\underline{\mathrm{P}}} (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]}$ is a triangle in $\underline{\mathscr{F}}$. Note that $Z\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \xrightarrow{b\underline{\mathrm{P}}} A\underline{\mathrm{P}}$ is an isomorphism in $\underline{\mathscr{F}}$ since b is an isomorphism in $\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{H}}(\mathscr{F})$. We have $u\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} = \underline{\underline{f}}\,\underline{\mathrm{P}}$, $v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \cdot b\underline{\mathrm{P}} = (v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \cdot b)\underline{\mathrm{P}} = \underline{\underline{i}}\,\underline{\mathrm{P}}$ and

$$w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \cdot X\lambda = w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \cdot X\mu\underline{\mathrm{P}} = (w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}} \cdot X\mu)\underline{\mathrm{P}} = (b \cdot \underline{\underline{p}})\underline{\mathrm{P}} = b\underline{\mathrm{P}} \cdot \underline{\underline{p}}\,\underline{\mathrm{P}}\,.$$

$$\begin{tikzcd}
X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \arrow[r, "u\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}}"] \arrow[d, "1"'] & Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \arrow[r, "v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}}"] \arrow[d, "1"] & Z\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \arrow[r, "w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}}\cdot X\lambda"] \arrow[d, "b\underline{\mathrm{P}}"] & (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]} \arrow[d, "1"] \\
X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \arrow[r, "\underline{\underline{f}}\,\underline{\mathrm{P}}"] & Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \arrow[r, "\underline{\underline{i}}\,\underline{\mathrm{P}}"] & A \arrow[r, "\underline{\underline{p}}\,\underline{\mathrm{P}}"] & (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]}
\end{tikzcd}$$

We conclude that

$$\begin{aligned}
&\Big(X\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} \xrightarrow{u\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}}} Y\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} \xrightarrow{v\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}}} Z\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} \xrightarrow{w\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}}\cdot X\lambda} (X\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}})^{[1]}\Big) \\
=\ &\Big(X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \xrightarrow{u\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}}} Y\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \xrightarrow{v\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}}} Z\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}} \xrightarrow{w\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}}\cdot X\lambda} (X\underline{\mathrm{R}}\,\underline{\mathrm{Lim}}\,\underline{\mathrm{P}})^{[1]}\Big)
\end{aligned}$$

is a triangle in $\underline{\mathscr{F}}$. $\square$

4.4.8 Lemma. Suppose given $n \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{T}_{n]}}(\mathscr{F}))$.
For $\ell/k \in \mathrm{V}$, we have $X_{\ell/k} \in \mathrm{Ob}(\mathscr{T}_{n+k+1]})$. $\diamondsuit$

Proof. We have $X_{\ell/k} \in \mathrm{Ob}\left(\underset{j\in[k+1,\ell]}{\ast} \mathscr{T}_{n]}^{[j]}\right) = \mathrm{Ob}\left(\underset{j\in[k+1,\ell]}{\ast} \mathscr{T}_{n+j]}\right)$ by remark 3.3.20.
Thus $X_{\ell/k} \in \mathrm{Ob}(\mathscr{T}_{n+k+1]})$. $\square$

4.4.9 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{H}}(\mathscr{F}))$. For $\ell/k \in \mathrm{V}$, we have $X_{\ell/k} \in \mathrm{Ob}(\mathscr{T}_{[\ell,k+1]})$. $\diamondsuit$

Proof. We have $X_{\ell/k} \in \mathrm{Ob}\left(\underset{j\in[k+1,\ell]}{\ast} \mathscr{H}^{[j]}\right) = \mathrm{Ob}\left(\underset{j\in[k+1,\ell]}{\ast} \mathscr{T}_{[j,j]}\right)$ by remark 3.3.20.
Thus $X_{\ell/k} \in \mathrm{Ob}(\mathscr{T}_{[\ell,k+1]})$. $\square$

4.4.10 Lemma. Suppose given $n \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{H}}^{n]}(\mathscr{F}))$. Then we have $X_{|\ell} \in \mathrm{Ob}(\mathscr{T}_{[\ell,n]})$ for $\ell \in \mathbf{Z}$. $\diamondsuit$

Proof. For $\ell \in \mathbf{Z}_{<n}$, we have $X_{|\ell} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$ and thus $X_{|\ell} \in \mathrm{Ob}(\mathscr{T}_{[\ell,n]})$.
Suppose given $\ell \in \mathbf{Z}_{\geq n}$. We have $X_{\ell/n-1} \in \mathrm{Ob}(\mathscr{T}_{[\ell,n]})$ by lemma 4.4.9.
Consider the pure short exact sequence $X_{|n-1} \xrightarrow{X_{|n-1\to|\ell}} X_{|\ell} \xrightarrow{X_{|\ell/n-1}} X_{\ell/n-1}$ in $\mathscr{F}$. The morphism $X_{|\ell/n-1}$ is an isomorphism in $\mathscr{F}$ since $X_{|n-1} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$. We conclude that
$X_{|\ell} \in \mathrm{Ob}(\mathscr{T}_{[\ell,n]})$ since $X_{\ell/n-1} \in \mathrm{Ob}(\mathscr{T}_{[\ell,n]})$. $\square$

4.4.11 Lemma. Suppose given $m, n \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{FO}_{\mathscr{H}}^{[m,n]}(\mathscr{F}))$.
Then we have $X_\omega \in \mathrm{Ob}(\mathscr{T}_{[m,n]})$. $\diamondsuit$

Proof. Consider the pure short exact sequence $X_{|m} \xrightarrow{x^\omega_{|m}} X_\omega \xrightarrow{x^\omega_{m+1|}} X_{m+1|}$ in $\mathscr{F}$. The morphism $x^\omega_{|m}$ is an isomorphism in $\mathscr{F}$ since $X_{m+1|} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$. We conclude that $X_\omega \in \mathrm{Ob}(\mathscr{T}_{[m,n]})$ since $X_{|m} \in \mathrm{Ob}(\mathscr{T}_{[m,n]})$ by lemma 4.4.10. $\square$

4.4.12 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{C}^{\mathrm{b}}(\mathscr{H}))$.

(a) Suppose given $m \in \mathbf{Z}$ and suppose that $X \in \mathrm{Ob}(\mathrm{C}^{m]}(\mathscr{H}))$.
We have $X\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} \in \mathrm{Ob}(\mathscr{T}_{m]})$.

(b) Suppose given $m \in \mathbf{Z}$ and suppose that $X \in \mathrm{Ob}(\mathrm{C}^{[m}(\mathscr{H}))$.
We have $X\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} \in \mathrm{Ob}(\mathscr{T}_{[m})$.

(c) We have $X\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} \in \mathrm{Ob}(\mathscr{T}^{\mathrm{b}})$.

$\diamondsuit$

Proof. Ad (a). We may choose $\ell \in \mathbf{Z}$ such that $X \in \mathrm{Ob}(\mathrm{C}^{[\ell,m]}(\mathscr{H}))$. By lemma 4.1.21 and remark 4.2.52, we have $X\underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}}\underline{\mathrm{Lim}}_{\mathscr{H},\mathscr{F}} \in \mathrm{Ob}(\mathrm{FO}^{[\ell,m]}(\mathscr{F}))$.
Thus $X\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} = X\underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}\underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}\underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{H},\mathscr{F}} \in \mathrm{Ob}(\mathscr{T}_{[\ell,m]}) \subseteq \mathrm{Ob}(\mathscr{T}_{m]})$ by lemma 4.4.11.

Ad (b). This is dual to (a).

Ad (c). This follows from (a) and (b). $\square$

4.4.13 Lemma. Suppose given $U \in \mathrm{Ob}(\mathrm{Ac}^2(\mathscr{H}))$. We have $U\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$.
Cf. definition 1.9.56. $\diamondsuit$

Proof. We may choose $n \in \mathbf{Z}$ such that $U \in \mathrm{Ob}(\mathrm{C}^{[n+1,n-1]}(\mathscr{H}))$. Write $X = U^{[n]}_{n+1}$, $Y = U^{[n]}_{n}$ and choose $X \xrightarrow{f} Y$ in $\mathscr{F}$ such that $\underline{f} = u^{[n]}_{n+1}$.
Choose a pseudo-triangle $X \xrightarrow{f} Y \xrightarrow{i} Z \xrightarrow{p} X^{[1]}$ in $\mathscr{F}$, cf. lemma 2.2.11.
Note that $X \xrightarrow{\underline{f}} Y \xrightarrow{\underline{i}} Z \xrightarrow{\underline{p}} X^{[1]}$ is a triangle in $\underline{\mathscr{F}}$. Since $U_{n+1} \xrightarrow{u_{n+1}} U_n \xrightarrow{u_n} U_{n-1}$ is a short exact sequence in $\mathscr{H}$, it can be completed to a triangle $U_{n+1} \xrightarrow{u_{n+1}} U_n \xrightarrow{u_n} U_{n-1} \longrightarrow U^{[1]}_{n+1}$ in $\underline{\mathscr{F}}$. Since we have $u_{n+1} = \underline{f}^{[-n]}$, we may choose an isomorphism $Z^{[-n]} \xrightarrow{a} U_{n+1}$ in $\underline{\mathscr{F}}$ such that $\underline{i}^{[-n]} \cdot a = u_n$.

$$\begin{tikzcd}
X^{[-n]} \arrow[r, "\underline{f}^{[-n]}"] \arrow[d, "1"'] & Y^{[-n]} \arrow[r, "\underline{i}^{[-n]}"] \arrow[d, "1"] & Z^{[-n]} \arrow[r] \arrow[d, "a"] & X^{[-n+1]} \arrow[d, "1"] \\
U_{n+1} \arrow[r, "u_{n+1}"] & U_n \arrow[r, "u_n"] & U_{n-1} \arrow[r] & U^{[1]}_{n+1}
\end{tikzcd}$$

By lemma 2.1.37, we may choose a pullback in $\mathscr{F}$ as follows.

$$\begin{tikzcd}
Q \arrow[r, "q"] \arrow[d, "j"'] & Y \arrow[d, "i"] \\
Z^{[-1]}\mathrm{B} \arrow[r, "Z^{[-1]}\pi"] & Z
\end{tikzcd}$$

Moreover, we may choose a kernel $Z^{[-1]} \xrightarrow{h} Q$ of q in $\mathscr{F}$ such that $h \cdot j = Z^{[-1]}\iota$ and $Z^{[-1]}\mathrm{B} \xrightarrow{Z^{[-1]}\pi \cdot p} X^{[1]}$ is a cokernel of j.
We define $V \in \mathrm{Ob}(\mathrm{FO}^{[n+1,n-1]}(\mathscr{F}))$ by setting

- $V_\omega = Z^{[-1]}\mathrm{B}$,
- $V_{|k} = Z^{[-1]}\mathrm{B}$, $v_{|k} = 1$, $v^{\omega}_{|k} = 1$ for $k \in \mathbf{Z}_{\geq n+1}$,
- $V_{|n} = Q$, $v_{|n} = j$, $v^{\omega}_{|n} = j$, $V_{|n-1} = Z^{[-1]}$, $v_{|n-1} = h$, $v^{\omega}_{|n-1} = Z^{[-1]}\iota$,
- $V_{|k} = 0_{\mathscr{F}}$ for $k \in \mathbf{Z}_{<n-1}$,
- $V_{k|} = Z^{[-1]}\mathrm{B}$, $v_{k-1|} = 1$, $v^{\omega}_{k|} = 1$ for $k \in \mathbf{Z}_{<n}$,
- $V_{n|} = Z$, $v_{n-1|} = Z^{[-1]}\pi$, $v^{\omega}_{n|} = Z^{[-1]}\pi$, $V_{n+1|} = X^{[1]}$, $v_{n|} = p$, $v^{\omega}_{n+1|} = Z^{[-1]}\pi \cdot p$ and
- $V_{k|} = 0_{\mathscr{F}}$ for $k \in \mathbf{Z}_{\geq n+2}$.

Note that $V\mathrm{P}_{\omega,\mathscr{F}} = V_\omega = Z^{[-1]}\mathrm{B} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$.
We choose cokernels $V_{|\ell} \xrightarrow{c_{\ell/k}} C_{\ell/k}$ of $V_{|k \to |\ell}$ for $\ell/k \in \mathrm{V}$ in $\mathscr{F}$ as follows. Let $c_{n-1/n-2} = 1_{Z^{[-1]}}$, $c_{n+1/n-1} = Z^{[-1]}\pi$, $c_{n/n-1} = q$, $c_{n/n-2} = 1_Q$, $c_{n+1/n} = Z^{[-1]}\pi \cdot p$ and $c_{n+1/n-2} = 1_{Z^{[-1]}\mathrm{B}}$.

We abbreviate $C = (c_{\ell/k})_{\ell/k \in \bar{\Delta}_{\mathrm{V}}}$. The object $V\Theta_C$ is isomorphic to $V\Xi_{\mathscr{F}}$ in $\nabla(\mathscr{F})$, cf. definitions 4.2.3, 4.2.11 and lemma 4.2.9.(a,b). Thus $V\Theta_C\Delta_{\mathscr{H},\mathscr{F}}$ is isomorphic to $V\Xi_{\mathscr{F}}\Delta_{\mathscr{H},\mathscr{F}}$ in $\mathrm{C}(\mathscr{H})$. By lemmata 4.2.33 and 4.1.17, we have $V\Xi_{\mathscr{F}}\Delta_{\mathscr{H},\mathscr{F}} \in \mathrm{Ob}(\mathrm{C}^{[n+1,n-1]}(\mathscr{H}))$ and thus $V\Theta_C\Delta_{\mathscr{H},\mathscr{F}} \in \mathrm{Ob}(\mathrm{C}^{[n+1,n-1]}(\mathscr{H}))$.

We have

$$\begin{aligned}(V\Theta_C)_{n-1/n-2 \to n+1/n-2} &= c_{n-1/n-2} \cdot (V\Theta_C)_{n-1/n-2 \to n+1/n-2} = V_{|n-1 \to |n+1} \cdot c_{n+1/n-2} = h \cdot j \\ &= Z^{[-1]}\iota\end{aligned}$$

and $(V\Theta_C)_{n+1/n-2 \to n+1/n-1} = c_{n+1/n-2} \cdot (V\Theta_C)_{n+1/n-2 \to n+1/n-1} = c_{n+1/n-1} = Z^{[-1]}\pi$.

We have $(V\Theta_C)_{n/n-1 \to n+1/n-1} = i$ since

$$q \cdot (V\Theta_C)_{n/n-1 \to n+1/n-1} = c_{n/n-1} \cdot (V\Theta_C)_{n/n-1 \to n+1/n-1} = v_{|n} \cdot c_{n+1/n-1} = j \cdot Z^{[-1]}\pi = q \cdot i$$

and since q is a pure epimorphism.

We have $(V\Theta_C)_{n+1/n-1 \to n+1/n} = p$ since

$$Z^{[-1]}\pi \cdot (V\Theta_C)_{n+1/n-1 \to n+1/n} = c_{n+1/n-1} \cdot (V\Theta_C)_{n+1/n-1 \to n+1/n} = c_{n+1/n} = Z^{[-1]}\pi \cdot p$$

and since $Z^{[-1]}\pi$ is a pure epimorphism.

Using lemmata 2.2.10, 2.2.3 and 3.3.42, we obtain

$$(V\Theta_C\Delta_{\mathscr{H},\mathscr{F}})_{n+1 \to n} = \underline{\delta^{[-n-1]}_{V\Theta_C,n-1,n,n+1}} = \underline{\delta^{[-n-1]}_{(i,p)}} = \underline{f^{[-n]}}$$

and

$$\begin{aligned}(V\Theta_C\Delta_{\mathscr{H},\mathscr{F}})_{n \to n-1} &= \underline{\delta^{[-n]}_{V\Theta_C,n-2,n-1,n}} = \underline{(V\Theta_C)_{n/n-1 \to n+1/n-1}}^{[-n]} \cdot \underline{\delta^{[-n]}_{V\Theta_C,n-2,n-1,n+1}} \\ &= \underline{i}^{[-n]} \cdot \underline{\delta^{[-n]}_{(Z^{[-1]}\iota,Z^{[-1]}\pi)}} = \underline{i}^{[-n]}.\end{aligned}$$

Thus we get an isomorphism $V\Theta_C\Delta_{\mathscr{H},\mathscr{F}} \xrightarrow{b} U$ in $\mathrm{C}^{[n+1,n-1]}(\mathscr{H})$ by setting $b_{n+1} = 1$, $b_n = 1$ and $b_{n-1} = a$.

Since $\Xi^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}$ and $\underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}$ as well as $\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}$ and $\underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}$ are mutually quasi-inverse equivalences, we get that V and $U\underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}\underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}$ are isomorphic in $\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})$. We conclude that $U\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} = U\underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}\underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}\underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{H},\mathscr{F}} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$ since $V\underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{H},\mathscr{F}} = V_\omega \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$. □

4.4.14 Lemma. Suppose given $X \in \mathrm{Ob}(\mathrm{Ac}^{\mathrm{b}}(\mathscr{H}))$. We have $X\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$. ◇

Proof. This follows from lemmata 4.4.6, 4.4.13 and 1.9.57. □

4.4.15 Lemma. Suppose that $\mathscr{F}$ has countable products and that $\mathscr{D}$ is closed under countable products in $\underline{\mathscr{F}}$. Then $\mathscr{T}_{[\ell}$ is closed under epilimits for $\ell \in \mathbf{Z}$, cf. definition 3.2.54. ◇

Proof. Suppose given $\ell \in \mathbf{Z}$, $X \in \mathrm{Ob}(\mathrm{CF}(\mathscr{F}))$ with $X_k \in \mathrm{Ob}(\mathscr{T}_{[\ell})$ for $k \in \mathbf{Z}$ and a limit $(A, (a_k)_{k\in\mathbf{Z}})$ for X. Choose a product $(P, (p_k)_{k\in\mathbf{Z}_{\leq 0}})$ for $(X_k)_{k\in\mathbf{Z}_{\leq 0}}$ in $\mathscr{F}$. We have $P \in \mathrm{Ob}(\mathscr{T}_{[\ell})$ since $P \in \mathrm{Ob}(\mathscr{D})$ by assumption and since $\mathscr{T}_{[\ell}$ is closed under products in $\mathscr{D}$, cf. lemma 2.1.41. Lemma 3.2.35 yields a pure short exact sequence $A \overset{i}{\rightarrowtail} P \overset{q}{\twoheadrightarrow} P$ in $\mathscr{F}$, which in turn yields a triangle $P^{[-1]} \longrightarrow A \longrightarrow P \longrightarrow P$ in $\underline{\mathscr{F}}$. So $A \in \mathrm{Ob}(\mathscr{T}_{[\ell})$ since $P^{[-1]}, P \in \mathrm{Ob}(\mathscr{T}_{[\ell})$. □

4.4.16 Lemma. Suppose that $\mathscr{F}$ has countable products and that $\mathscr{D}$ is closed under countable products in $\underline{\mathscr{F}}$. Suppose given $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{lim}}_{\mathscr{H}}(\mathscr{F}))$. For $\ell \in \mathbf{Z}$, we have $X_{|\ell} \in \mathrm{Ob}(\mathscr{T}_{[\ell})$. ◊

Proof. This follows from lemmata 4.2.40, 4.4.9 and 4.4.15. □

4.4.17 Remark. Suppose given a functor $A\colon \underline{\mathscr{F}} \to \mathscr{D}$.

(a) If A is left-adjoint to $\mathrm{Inc}_{\mathscr{D}}^{\underline{\mathscr{F}}}$, then the functors $\mathrm{Inc}_{\mathscr{D}}^{\underline{\mathscr{F}}} \cdot A$ and $1_{\mathscr{D}}$ are isomorphic in $\mathscr{D}(\mathscr{D})$.

(b) If A is right-adjoint to $\mathrm{Inc}_{\mathscr{D}}^{\underline{\mathscr{F}}}$, then the functors $\mathrm{Inc}_{\mathscr{D}}^{\underline{\mathscr{F}}} \cdot A$ and $1_{\mathscr{D}}$ are isomorphic in $\mathscr{D}(\mathscr{D})$. ◊

Proof. This follows from the fact that the inclusion functor $\mathrm{Inc}_{\mathscr{D}}^{\underline{\mathscr{F}}}$ is full and faithful, cf. lemma 1.6.6. □

4.4.18 Lemma. Suppose that $\mathscr{F}$ has countable coproducts. Suppose given a functor $A\colon \underline{\mathscr{F}} \to \mathscr{D}$ that is left-adjoint to $\mathrm{Inc}_{\mathscr{D}}^{\underline{\mathscr{F}}}$, $n \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{colim}}_{\mathscr{T}_{n]}}(\mathscr{F}))$.
For $k \in \mathbf{Z}$, we have $X_{k|}A \in \mathrm{Ob}(\mathscr{T}_{n+k]})$. ◊

Proof. Suppose given $k \in \mathbf{Z}$. By lemma 4.2.41, $(X_{k|}, (X_{\ell/k-1|})_{\ell\in\mathbf{Z}})$ is a colimit for $X\Xi_{\mathscr{F}}\Psi_{k,\mathrm{F},\mathscr{F}}$. Moreover, we have $(X\Xi_{\mathscr{F}}\Psi_{k,\mathrm{F},\mathscr{F}})_\ell = X_{\ell/k-1} \in \mathrm{Ob}(\mathscr{T}_{n+k]})$, cf. lemma 4.4.8. Choose a coproduct $(C, (c_k)_{k\in\mathbf{Z}_{\geq 0}})$ for $(X_{\ell/k-1})_{\ell\in\mathbf{Z}_{\geq 0}}$ in $\mathscr{F}$. Lemma 3.2.36 yields a pure short exact sequence $C \overset{i}{\rightarrowtail} C \overset{q}{\twoheadrightarrow} X_{k|}$ in $\mathscr{F}$, which in turn yields a triangle $C \longrightarrow C \longrightarrow X_{k|} \longrightarrow C^{[1]}$ in $\underline{\mathscr{F}}$. Since A is left-adjoint to $\mathrm{Inc}_{\mathscr{D}}^{\underline{\mathscr{F}}}$, we obtain a triangle $CA \longrightarrow CA \longrightarrow X_{k|}A \longrightarrow CA^{[1]}$ in $\underline{\mathscr{F}}$ and $(CA, (\underline{c_k}A)_{k\in\mathbf{Z}_{\geq 0}})$ is a coproduct for $(X_{\ell/k-1}A)_{\ell\in\mathbf{Z}_{\geq 0}}$ in $\mathscr{D}$, cf. lemma 2.1.40. Note that, for $\ell \in \mathbf{Z}$, we have $X_{\ell/k-1}A \in \mathrm{Ob}(\mathscr{T}_{n+k]})$ since $X_{\ell/k-1} \in \mathrm{Ob}(\mathscr{T}_{n+k]})$, cf. remark 4.4.17. Thus $CA \in \mathrm{Ob}(\mathscr{T}_{n+k]})$ since $\mathscr{T}_{n+k]}$ is closed under coproducts in $\mathscr{D}$.
We conclude that $X_{k|}A \in \mathrm{Ob}(\mathscr{T}_{n+k]})$ since $CA, CA^{[1]} \in \mathrm{Ob}(\mathscr{T}_{n+k]})$. □

4.4.19 Lemma. Suppose that $\mathscr{F}$ has epilimits and monocolimits. Suppose that $\mathscr{D}$ is closed under countable products in $\underline{\mathscr{F}}$. Suppose given a functor $A\colon \underline{\mathscr{F}} \to \mathscr{D}$ that is left-adjoint to $\mathrm{Inc}_{\mathscr{D}}^{\underline{\mathscr{F}}}$.

(a) Suppose given $m \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{C}^{m]}(\mathscr{H}))$. We have $X\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}}A \in \mathrm{Ob}(\mathscr{T}_{m]})$.

(b) Suppose given $m \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{C}^{[m}(\mathscr{H}))$. We have $X\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}}A \in \mathrm{Ob}(\mathscr{T}_{[m})$. ◊

Proof. Ad (a). We abbreviate $Y = X\underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}}\underline{\mathrm{Lim}}_{\mathscr{H},\mathscr{F}}$. By lemma 4.1.21.(b) and remark 4.2.53.(b), we have $Y \in \mathrm{Ob}(\mathrm{FO}^{m\rceil}(\mathscr{F}))$. Consider the pure short exact sequence $Y_{|m-1} \xrightarrow{y^{\omega}_{|m-1}} Y_{\omega} \xrightarrow{y^{\omega}_{m|}} Y_{m|}$ in $\mathscr{F}$. We have $Y_{|m-1} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$ since $Y \in \mathrm{Ob}(\mathrm{FO}^{m\rceil}(\mathscr{F}))$. Thus $y^{\omega}_{m|}$ is an isomorphism in $\mathscr{F}$. By lemma 4.4.18, we have $X_{m|}A \in \mathrm{Ob}(\mathscr{T}_{m\rfloor})$. We conclude that $X\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}}A = Y_{\omega}A \in \mathrm{Ob}(\mathscr{T}_{m\rfloor})$.

Ad (b). We abbreviate $Y = X\underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}}\underline{\mathrm{Lim}}_{\mathscr{H},\mathscr{F}}$. By lemma 4.1.21.(a) and remark 4.2.53.(a), we have $Y \in \mathrm{Ob}(\mathrm{FO}^{\lceil m}(\mathscr{F}))$. Consider the pure short exact sequence $Y_{|m} \xrightarrow{y^{\omega}_{|m}} Y_{\omega} \xrightarrow{y^{\omega}_{m+1|}} Y_{m+1|}$ in $\mathscr{F}$. We have $Y_{m+1|} \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{F}})$ since $Y \in \mathrm{Ob}(\mathrm{FO}^{\lceil m}(\mathscr{F}))$. Thus $y^{\omega}_{|m}$ is an isomorphism in $\mathscr{F}$. By lemma 4.4.16, we have $Y_{|m} \in \mathrm{Ob}(\mathscr{T}_{\lfloor m})$. So $Y_{\omega} \in \mathrm{Ob}(\mathscr{T}_{\lfloor m})$. We conclude that $X\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}}A = Y_{\omega}A \in \mathrm{Ob}(\mathscr{T}_{\lfloor m})$, cf. remark 4.4.17. □

4.4.20 Lemma. Suppose that $\mathscr{F}$ has epilimits and monocolimits. Suppose that $\mathscr{D}$ is closed under countable coproducts in $\underline{\mathscr{F}}$. Suppose given a functor $A\colon \underline{\mathscr{F}} \to \mathscr{D}$ that is right-adjoint to $\mathrm{Inc}^{\underline{\mathscr{F}}}_{\mathscr{D}}$.

(a) Suppose given $m \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{C}^{m\rceil}(\mathscr{H}))$. We have $X\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}}A \in \mathrm{Ob}(\mathscr{T}_{m\rfloor})$.

(b) Suppose given $m \in \mathbf{Z}$ and $X \in \mathrm{Ob}(\mathrm{C}^{\lceil m}(\mathscr{H}))$. We have $X\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}}A \in \mathrm{Ob}(\mathscr{T}_{\lfloor m})$. ◊

Proof. This is dual to the previous lemma 4.4.19. □

4.4.21 Lemma. Suppose that $\mathscr{F}$ has epilimits and monocolimits. Suppose that $\mathscr{D}$ is closed under countable products in $\underline{\mathscr{F}}$ and that $\mathscr{T}$ is non-degenerate. Suppose given a functor $A\colon \underline{\mathscr{F}} \to \mathscr{D}$ that is left-adjoint to $\mathrm{Inc}^{\underline{\mathscr{F}}}_{\mathscr{D}}$. Suppose given $X \in \mathrm{Ob}(\mathrm{Ac}(\mathscr{H}))$. We have $X\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}}A \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$. ◊

Proof. This follows from lemmata 4.4.7, 4.4.13, 4.4.19 and 1.9.58. Cf. remark 4.4.3. Also note that A is exact since it is left-adjoint to the exact functor $\mathrm{Inc}^{\underline{\mathscr{F}}}_{\mathscr{D}}$. □

4.4.22 Lemma. Suppose that $\mathscr{F}$ has epilimits and monocolimits. Suppose that $\mathscr{D}$ is closed under countable coproducts in $\underline{\mathscr{F}}$. Suppose given a functor $A\colon \underline{\mathscr{F}} \to \mathscr{D}$ that is right-adjoint to $\mathrm{Inc}^{\underline{\mathscr{F}}}_{\mathscr{D}}$. Suppose given $X \in \mathrm{Ob}(\mathrm{Ac}(\mathscr{H}))$. We have $X\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}}A \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$. ◊

Proof. This is dual to the previous lemma 4.4.21. □

4.4.23 Definition. We define $\mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}}\colon \mathrm{D}^{\mathrm{b}}(\mathscr{H}) \to \mathscr{D}$ to be the unique exact functor such that $\mathrm{L}^{\mathrm{b}}_{\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}} = \mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}}|^{\mathscr{D}}$, cf. lemmata 4.4.6, 4.4.11, 4.4.12.(c), 4.4.14 and 1.5.12. We call $\mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}}$ the *bounded realisation functor* of $\mathscr{T}$ with respect to $\mathscr{F}$.

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{b}}(\mathscr{H}) \arrow[r, "\underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}"] \arrow[d, "\mathrm{L}^{\mathrm{b}}_{\mathscr{H}}"'] & \underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F}) \arrow[r, "\underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}}"] & \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F}) \arrow[r, "\underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{H},\mathscr{F}}|^{\mathscr{D}}"] & \mathscr{D} \\
\mathrm{D}^{\mathrm{b}}(\mathscr{H}) \arrow[rrru, bend right=10, "\mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}}"'] & & &
\end{tikzcd}$$

◊

4.4.24 Definition. Suppose that $\mathscr{F}$ has epilimits and monocolimits. Suppose given a functor $A\colon \underline{\mathscr{F}} \to \mathscr{D}$. Suppose that one of the following two statements is true.

(a) $\mathscr{D}$ is closed under countable products in $\underline{\mathscr{F}}$ and A is left-adjoint to $\mathrm{Inc}_{\mathscr{D}}^{\underline{\mathscr{F}}}$.

(b) $\mathscr{D}$ is closed under countable coproducts in $\underline{\mathscr{F}}$ and A is right-adjoint to $\mathrm{Inc}_{\mathscr{D}}^{\underline{\mathscr{F}}}$.

We define $\mathrm{Real}_{A,\mathscr{T},\mathscr{F}}\colon \mathrm{D}(\mathscr{H}) \to \mathscr{D}$ to be the unique exact functor such that $\mathrm{L}_{\mathscr{H}} \cdot \mathrm{Real}_{A,\mathscr{T},\mathscr{F}} = \mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} \cdot A$, cf. lemmata 4.4.7, 4.4.21, 4.4.22 and 1.5.12. We call $\mathrm{Real}_{A,\mathscr{T},\mathscr{F}}$ the *realisation functor* of $\mathscr{T}$ with respect to A and $\mathscr{F}$.

$$\begin{tikzcd}
\mathrm{K}(\mathscr{H}) \arrow[r, "\mathrm{R}_{\mathscr{H},\mathscr{F}}"] \arrow[d, "\mathrm{L}_{\mathscr{H}}"'] & \underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F}) \arrow[r, "\underline{\mathrm{Lim}}_{\mathscr{H},\mathscr{F}}"] & \underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{H}}(\mathscr{F}) \arrow[r, "\underline{\mathrm{P}}^{\nabla}_{\omega,\mathscr{H},\mathscr{F}}"] & \underline{\mathscr{F}} \arrow[r, "A"] & \mathscr{D} \\
\mathrm{D}(\mathscr{H}) \arrow[rrrru, bend right=10, "\mathrm{Real}_{A,\mathscr{T},\mathscr{F}}"'] & & & &
\end{tikzcd}$$
$\diamondsuit$

4.4.25 Proposition. The bounded realisation functor $\mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}}\colon \mathrm{D}^{\mathrm{b}}(\mathscr{H}) \to \mathscr{D}$ is t-exact with respect to $\mathcal{T}^{\mathscr{A},\mathrm{b}}$ and $\mathscr{T}$. $\diamondsuit$

Proof. Note that $\mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}}$ is exact by construction.
Now the result follows from lemma 4.4.12.(a,b). $\square$

4.4.26 Proposition. Suppose that $\mathscr{F}$ has epilimits and monocolimits. Suppose that $\mathscr{D}$ is closed under countable products in $\underline{\mathscr{F}}$. Suppose given a functor $A\colon \underline{\mathscr{F}} \to \mathscr{D}$ that is left-adjoint to $\mathrm{Inc}_{\mathscr{D}}^{\underline{\mathscr{F}}}$.
The realisation functor $\mathrm{Real}_{A,\mathscr{T},\mathscr{F}}\colon \mathrm{D}(\mathscr{H}) \to \mathscr{D}$ is t-exact with respect to $\mathcal{T}^{\mathscr{A}}$ and $\mathscr{T}$. $\diamondsuit$

Proof. Note that $\mathrm{Real}_{A,\mathscr{T},\mathscr{F}}$ is exact by construction.
Now the result follows from lemma 4.4.19. $\square$

4.4.27 Proposition. Suppose that $\mathscr{F}$ has epilimits and monocolimits. Suppose that $\mathscr{D}$ is closed under countable coproducts in $\underline{\mathscr{F}}$. Suppose given a functor $A\colon \underline{\mathscr{F}} \to \mathscr{D}$ that is right-adjoint to $\mathrm{Inc}_{\mathscr{D}}^{\underline{\mathscr{F}}}$.
The realisation functor $\mathrm{Real}_{A,\mathscr{T},\mathscr{F}}\colon \mathrm{D}(\mathscr{H}) \to \mathscr{D}$ is t-exact with respect to $\mathcal{T}^{\mathscr{A}}$ and $\mathscr{T}$. $\diamondsuit$

Proof. This is dual to the previous proposition 4.4.26. $\square$

4.4.28 Proposition. The functors $\mathrm{I}_{\mathrm{D}^{\mathrm{b}},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}}$ and $\mathrm{Inc}^{\mathscr{D}}_{\mathscr{H}}$ are isomorphic in $\mathscr{H}(\mathscr{D})$. $\diamondsuit$

Proof. We have $\mathrm{I}_{\mathrm{D}^{\mathrm{b}},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}} = \mathrm{I}_{\mathrm{K}^{\mathrm{b}},\mathscr{H}} \cdot \mathrm{L}^{\mathrm{b}}_{\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}} = \mathrm{I}_{\mathrm{K}^{\mathrm{b}},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}}|^{\mathscr{D}}$. So the result follows from lemmata 4.4.4 and 1.1.6. $\square$

4.4.29 Proposition. Suppose that $\mathscr{F}$ has epilimits and monocolimits. Suppose that $\mathscr{D}$ is closed under countable products in $\underline{\mathscr{F}}$. Suppose given a functor $A\colon \underline{\mathscr{F}} \to \mathscr{D}$ that is left-adjoint to $\mathrm{Inc}_{\mathscr{D}}^{\underline{\mathscr{F}}}$. The functors $\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}_{A,\mathscr{T},\mathscr{F}}$ and $\mathrm{Inc}^{\mathscr{D}}_{\mathscr{H}}$ are isomorphic in $\mathscr{H}(\mathscr{D})$. $\diamondsuit$

Proof. We have $\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}_{A,\mathscr{T},\mathscr{F}} = \mathrm{I}_{\mathrm{K},\mathscr{H}} \cdot \mathrm{L}_{\mathscr{H}} \cdot \mathrm{Real}_{A,\mathscr{T},\mathscr{F}} = \mathrm{I}_{\mathrm{K},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} \cdot A$.
The functors $\mathrm{Inc}^{\mathscr{D}}_{\mathscr{H}}$ and $\mathrm{Inc}^{\mathscr{D}}_{\mathscr{H}} \cdot \mathrm{Inc}^{\mathscr{F}}_{\mathscr{D}} \cdot A = \mathrm{Inc}^{\mathscr{F}}_{\mathscr{H}} \cdot A$ are isomorphic in $\mathscr{H}(\mathscr{D})$ by remark 4.4.17 and lemma 1.1.5. Now the result follows from lemmata 4.4.5 and 1.1.5. $\square$

4.4.30 Proposition. Suppose that $\mathscr{F}$ has epilimits and monocolimits. Suppose that $\mathscr{D}$ is closed under countable coproducts in $\underline{\mathscr{F}}$. Suppose given a functor $A\colon \underline{\mathscr{F}} \to \mathscr{D}$ that is right-adjoint to $\mathrm{Inc}^{\underline{\mathscr{F}}}_{\mathscr{D}}$. The functors $\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}_{A,\mathscr{T},\mathscr{F}}$ and $\mathrm{Inc}^{\mathscr{D}}_{\mathscr{H}}$ are isomorphic in $\mathscr{H}(\mathscr{D})$. $\diamondsuit$

Proof. This is dual to the previous proposition 4.4.29. $\square$

4.4.31 Remark. It is not known if realisation functors are unique or universal in some way, cf. [Ric89, section 7] and [Ric91, corollary 3.5]. The problem has its roots in the non-uniqueness of the induced morphism in the axiom (TR3), cf. [Nee91, section 5]. $\diamondsuit$

4.5 Adjacent w- and t-structures

We have collected some facts about w- and t-structures in sections 1.7 and 1.8 which we will use now.
Suppose given a strict Frobenius category $\mathscr{F} = (\mathscr{F}, \mathrm{B}, \Sigma, \iota, \pi, \alpha)$. Suppose given a full triangulated subcategory $\mathscr{D} \subseteq \underline{\mathscr{F}}$. Suppose given a t-structure $\mathscr{T} = (\mathscr{T}_{0]}, \mathscr{T}_{[0})$ on $\mathscr{D}$. Suppose given a w-structure $\mathscr{W} = (\mathscr{W}_{[0}, \mathscr{W}_{0]})$ on $\mathscr{D}$. We abbreviate $\mathscr{H} = \mathscr{T}_{[0,0]}$, $\mathrm{H} = \mathrm{H}_{\mathscr{T}}$ and $\mathscr{C} = \mathscr{W}_{[0,0]}$.

4.5.1 Theorem (adjunction for adjacent structures)**.** Suppose that $\mathscr{F}$ has epilimits and monocolimits. Suppose that $\mathscr{D}$ is closed under countable products in $\underline{\mathscr{F}}$. Suppose given a functor $A\colon \underline{\mathscr{F}} \to \mathscr{D}$ that is left-adjoint to $\mathrm{Inc}^{\underline{\mathscr{F}}}_{\mathscr{D}}$. Suppose that $\mathscr{T}$ is non-degenerate and that $\mathscr{T}_{[0} = \mathscr{W}_{[0}$, i.e. that $\mathscr{T}$ is left-adjacent to $\mathscr{W}$.

(a) The functor $\mathrm{Real}_{A,\mathscr{T},\mathscr{F}}$ is left-adjoint to $\mathrm{Res}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}}$.
The functors $\big(\mathrm{Res}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}} \cdot \mathbb{H}_{\mathscr{H}}\big)|_{\mathscr{H}}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}_{A,\mathscr{T},\mathscr{F}})|^{\mathscr{H}}$ are isomorphic to $1_{\mathscr{H}}$ in $\mathscr{H}(\mathscr{H})$.

(b) Suppose given $n \in \mathbf{Z}$ such that $\mathscr{W}_{0]} \subseteq \mathscr{T}_{n]}$.
Then the functor $\mathrm{W}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{C}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})} \cdot \Xi_{\mathscr{C},\mathscr{F}}\colon \mathscr{D} \to \underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F})$ is full and faithful. Moreover, it is right-adjoint to $\underline{\mathrm{Lim}}_{\mathscr{C},\mathscr{F}} \cdot \underline{\mathrm{P}}^{\nabla}_{\omega,\mathscr{C},\mathscr{F}} \cdot A$.

(c) Suppose given $n \in \mathbf{Z}$ such that $\mathscr{W}_{0]} \subseteq \mathscr{T}_{n]}$. Suppose that $\mathscr{C} \subseteq \mathscr{H}$.
Then the functors $\mathrm{Res}_{\mathscr{W},\mathscr{F}}$, $\mathrm{K}(\mathrm{H}|_{\mathscr{C}})$ and $\mathrm{Res}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}}$ are full and faithful.

$$\begin{tikzcd}
 & \mathrm{K}(\mathscr{C}) \arrow[r, "\mathrm{K}(\mathrm{H}|_{\mathscr{C}})"] & \mathrm{K}(\mathscr{H}) \arrow[dr, "\mathrm{L}_{\mathscr{H}}"] & \\
\mathscr{D} \arrow[ur, "\mathrm{Res}_{\mathscr{W},\mathscr{F}}"] & & & \mathrm{D}(\mathscr{H}) \arrow[lll, "\mathrm{Real}_{A,\mathscr{T},\mathscr{F}}"'] \arrow[d, bend left, "\mathbb{H}_{\mathscr{H}}"] \\
\mathscr{H} \arrow[u, "\mathrm{Inc}^{\mathscr{D}}_{\mathscr{H}}"] & & & \mathscr{H} \arrow[u, bend left, "\mathrm{I}_{\mathrm{D},\mathscr{H}}"]
\end{tikzcd}$$

$\diamondsuit$

Proof. We abbreviate $\mathscr{S} = \mathscr{T}_{\lceil 0} = \mathscr{W}_{\lceil 0}$. Note that $\mathscr{S}$ is closed under countable products in $\underline{\mathscr{F}}$ since $\mathscr{D}$ is closed under countable products in $\underline{\mathscr{F}}$ and since $\mathscr{S} = \mathscr{T}_{\lceil 0}$. Note that $\mathscr{W}$ is left-non-degenerate since $\mathscr{T}$ is non-degenerate. Also note that $\mathrm{H}|_{\mathscr{S}}$ is right-adjoint to $\mathrm{Inc}_{\mathscr{H}}^{\mathscr{S}}$. By proposition 4.1.24, $\underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\nabla}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F})}$ is left-adjoint to $\underline{\underline{\Delta}}_{\mathscr{S},\mathscr{F}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{S}})$.

Proposition 4.2.60 yields the adjunction $\big(\underline{\mathrm{Lim}}_{\mathscr{S},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{S}}(\mathscr{F})}\, , \underline{\Xi}^{\mathrm{lim,inj}}_{\mathscr{S},\mathscr{F}}\, , \underline{\sigma}_{\mathscr{S},\mathscr{F}}\, , \underline{\tau}_{\mathscr{S},\mathscr{F}}\big)$ with $X\underline{\tau}_{\mathscr{S},\mathscr{F}} = \underline{\underline{X\tau_{\mathscr{F}}}}$ for $X \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F}))$. We abbreviate $\tau = \tau_{\mathscr{F}}$ and $\underline{\tau} = \underline{\tau}_{\mathscr{S},\mathscr{F}}$.

By proposition 4.3.52, $\underline{\mathrm{P}}^{\mathrm{lim,inj}}_{\omega,\mathscr{S},\mathscr{F}} \cdot A$ is left-adjoint to $\mathrm{W}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F})}$.

$$\begin{tikzcd}[column sep=huge]
\mathscr{D} \arrow[r, shift right, "\underline{\mathrm{P}}^{\mathrm{lim,inj}}_{\omega,\mathscr{S},\mathscr{F}}\cdot A"'] & \underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F}) \arrow[l, shift right, "\mathrm{W}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}}\cdot\mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F})}"'] \arrow[r, shift right, "\underline{\Xi}^{\mathrm{lim,inj}}_{\mathscr{S},\mathscr{F}}"'] & \underline{\nabla}_{\mathscr{S}}(\mathscr{F}) \arrow[l, shift right, "\underline{\mathrm{Lim}}_{\mathscr{S},\mathscr{F}}\cdot\mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{S}}(\mathscr{F})}"'] \arrow[r, shift right, "\underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}}\cdot\mathrm{Inc}^{\underline{\nabla}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F})}"' , leftarrow] & \mathrm{K}(\mathscr{H}) \arrow[l, shift right, "\underline{\underline{\Delta}}_{\mathscr{S},\mathscr{F}}\cdot\mathrm{K}(\mathrm{H}|_{\mathscr{S}})"' , leftarrow]
\end{tikzcd}$$

We want to prove the following six statements.

(i) We have $\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} \cdot A = \underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\nabla}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F})} \cdot \underline{\mathrm{Lim}}_{\mathscr{S},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{S}}(\mathscr{F})} \cdot \underline{\mathrm{P}}^{\mathrm{lim,inj}}_{\omega,\mathscr{S},\mathscr{F}} \cdot A.$

(ii) We have $\underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) = \mathrm{Inc}^{\underline{\nabla}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F})} \cdot \underline{\underline{\Delta}}_{\mathscr{S},\mathscr{F}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{S}}).$

(iii) The functors $\mathrm{W}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{C}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})} \cdot \underline{\Xi}_{\mathscr{C},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\nabla}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F})}$ and $\mathrm{W}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F})} \cdot \underline{\Xi}^{\mathrm{lim,inj}}_{\mathscr{S},\mathscr{F}}$ are isomorphic in $\mathscr{D}(\underline{\nabla}_{\mathscr{S}}(\mathscr{F}))$.

(iv) The functors $\mathrm{W}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{C}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})} \cdot \underline{\Xi}_{\mathscr{C},\mathscr{F}}$ and $\big(\mathrm{W}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F})} \cdot \underline{\Xi}^{\mathrm{lim,inj}}_{\mathscr{S},\mathscr{F}}\big)\big|^{\underline{\nabla}_{\mathscr{C}}(\mathscr{F})}$ are isomorphic in $\mathscr{D}(\underline{\nabla}_{\mathscr{C}}(\mathscr{F}))$.

(v) The functors $\mathrm{Res}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}})$ and $\mathrm{W}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F})} \cdot \underline{\Xi}^{\mathrm{lim,inj}}_{\mathscr{S},\mathscr{F}} \cdot \underline{\underline{\Delta}}_{\mathscr{S},\mathscr{F}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{S}})$ are isomorphic in $\mathscr{D}(\mathrm{K}(\mathscr{H}))$.

(vi) We have $\big(\underline{\mathrm{Lim}}_{\mathscr{S},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{S}}(\mathscr{F})} \cdot \underline{\mathrm{P}}^{\mathrm{lim,inj}}_{\omega,\mathscr{S},\mathscr{F}} \cdot A\big)\big|_{\underline{\nabla}_{\mathscr{C}}(\mathscr{F})} = \underline{\mathrm{Lim}}_{\mathscr{C},\mathscr{F}} \cdot \underline{\mathrm{P}}^{\nabla}_{\omega,\mathscr{C},\mathscr{F}} \cdot A.$

Ad (i). We have

$$\begin{aligned}
\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}} \cdot A &= \underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}} \cdot \underline{\mathrm{Lim}}_{\mathscr{H},\mathscr{F}} \cdot \underline{\mathrm{P}}^{\nabla}_{\omega,\mathscr{H},\mathscr{F}} \cdot A = \underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}} \cdot \underline{\mathrm{Lim}}_{\mathscr{F}}\big|^{\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{H}}(\mathscr{F})}_{\underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F})} \cdot \underline{\mathrm{P}}_{\omega,\mathscr{F}}\big|_{\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{H}}(\mathscr{F})} \cdot A \\
&= \underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}} \cdot \underline{\mathrm{Lim}}_{\mathscr{F}}\big|^{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F})} \cdot \underline{\mathrm{P}}_{\omega,\mathscr{F}}\big|_{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F})} \cdot A \\
&= \underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}} \cdot \underline{\mathrm{Lim}}_{\mathscr{F}}\big|^{\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F})} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{S}}(\mathscr{F})} \cdot \underline{\mathrm{P}}^{\mathrm{lim,inj}}_{\omega,\mathscr{S},\mathscr{F}} \cdot A \\
&= \underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\nabla}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F})} \cdot \underline{\mathrm{Lim}}_{\mathscr{F}}\big|^{\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{S}}(\mathscr{F})}_{\underline{\nabla}_{\mathscr{S}}(\mathscr{F})} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{S}}(\mathscr{F})} \cdot \underline{\mathrm{P}}^{\mathrm{lim,inj}}_{\omega,\mathscr{S},\mathscr{F}} \cdot A \\
&= \underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\nabla}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F})} \cdot \underline{\mathrm{Lim}}_{\mathscr{S},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{S}}(\mathscr{F})} \cdot \underline{\mathrm{P}}^{\mathrm{lim,inj}}_{\omega,\mathscr{S},\mathscr{F}} \cdot A.
\end{aligned}$$

Ad (ii). We have

$$\begin{aligned}\underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}}\cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) &= \underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}}\cdot \mathrm{K}(\mathrm{Inc}_{\mathscr{C}}^{\mathscr{S}}\cdot \mathrm{H}|_{\mathscr{S}}) = \underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}}\cdot \mathrm{K}(\mathrm{Inc}_{\mathscr{C}}^{\mathscr{S}})\cdot \mathrm{K}(\mathrm{H}|_{\mathscr{S}})\\ &= \mathrm{Inc}_{\underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F})}^{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})}\cdot \underline{\underline{\Delta}}_{\mathscr{S},\mathscr{F}}\cdot \mathrm{K}(\mathrm{H}|_{\mathscr{S}}),\end{aligned}$$

cf. remark 4.1.16.

Ad (iii).

The functors $\mathrm{W}_{\mathscr{W},\mathscr{F}}^{\mathrm{lim}}\cdot \mathrm{Inc}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}^{\mathrm{lim}}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})}$ and $\mathrm{W}_{\mathscr{W},\mathscr{F}}$ are isomorphic in $\mathscr{D}(\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F}))$ by lemma 4.3.50. Thus the functors $\mathrm{W}_{\mathscr{W},\mathscr{F}}\cdot \mathrm{Inc}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{C}}(\mathscr{F})}\cdot \underline{\underline{\Xi}}_{\mathscr{C},\mathscr{F}}\cdot \mathrm{Inc}_{\underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F})}^{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})}$ and $\mathrm{W}_{\mathscr{W},\mathscr{F}}^{\mathrm{lim}}\cdot \mathrm{Inc}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}^{\mathrm{lim}}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})}\cdot \mathrm{Inc}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{C}}(\mathscr{F})}\cdot \underline{\underline{\Xi}}_{\mathscr{C},\mathscr{F}}\cdot \mathrm{Inc}_{\underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F})}^{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})}$ are isomorphic by lemma 1.1.5. We have

$$\begin{aligned}\mathrm{W}_{\mathscr{W},\mathscr{F}}^{\mathrm{lim}}\cdot \mathrm{Inc}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}^{\mathrm{lim}}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})}\cdot \mathrm{Inc}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{C}}(\mathscr{F})}\cdot \underline{\underline{\Xi}}_{\mathscr{C},\mathscr{F}}\cdot \mathrm{Inc}_{\underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F})}^{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})} &= \mathrm{W}_{\mathscr{W},\mathscr{F}}^{\mathrm{lim}}\cdot \underline{\underline{\Xi}}_{\mathscr{F}}|_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}^{\mathrm{lim}}(\mathscr{F})}^{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})}\\ &= \mathrm{W}_{\mathscr{W},\mathscr{F}}^{\mathrm{lim}}\cdot \mathrm{Inc}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}^{\mathrm{lim}}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{S}}^{\mathrm{lim,inj}}(\mathscr{F})}\cdot \underline{\underline{\Xi}}_{\mathscr{F}}|_{\underline{\underline{\mathrm{FO}}}_{\mathscr{S}}^{\mathrm{lim,inj}}(\mathscr{F})}^{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})}\\ &= \mathrm{W}_{\mathscr{W},\mathscr{F}}^{\mathrm{lim}}\cdot \mathrm{Inc}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}^{\mathrm{lim}}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{S}}^{\mathrm{lim,inj}}(\mathscr{F})}\cdot \underline{\underline{\Xi}}_{\mathscr{S},\mathscr{F}}^{\mathrm{lim,inj}}\,.\end{aligned}$$

Ad (iv). This follows from (iii) and lemmata 3.3.19, 1.1.6.

Ad (v). By (ii), we have

$$\begin{aligned}\mathrm{Res}_{\mathscr{W},\mathscr{F}}\cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) &= \mathrm{W}_{\mathscr{W},\mathscr{F}}\cdot \mathrm{Inc}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{C}}(\mathscr{F})}\cdot \underline{\underline{\Xi}}_{\mathscr{C},\mathscr{F}}\cdot \underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}}\cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}})\\ &= \mathrm{W}_{\mathscr{W},\mathscr{F}}\cdot \mathrm{Inc}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{C}}(\mathscr{F})}\cdot \underline{\underline{\Xi}}_{\mathscr{C},\mathscr{F}}\cdot \mathrm{Inc}_{\underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F})}^{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})}\cdot \underline{\underline{\Delta}}_{\mathscr{S},\mathscr{F}}\cdot \mathrm{K}(\mathrm{H}|_{\mathscr{S}}).\end{aligned}$$

So the result follows from (iii) and lemma 1.1.5.

Ad (vi). We have

$$\begin{aligned}\big(\underline{\mathrm{Lim}}_{\mathscr{S},\mathscr{F}}\cdot \mathrm{Inc}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{S}}^{\nabla}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{S}}^{\mathrm{lim,inj}}(\mathscr{F})}\cdot \underline{\mathrm{P}}_{\omega,\mathscr{S},\mathscr{F}}^{\mathrm{lim,inj}}\cdot A\big)\big|_{\underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F})} &= \mathrm{Inc}_{\underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F})}^{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})}\cdot \underline{\mathrm{Lim}}_{\mathscr{S},\mathscr{F}}\cdot \mathrm{Inc}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{S}}^{\nabla}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{S}}^{\mathrm{lim,inj}}(\mathscr{F})}\cdot \underline{\mathrm{P}}_{\omega,\mathscr{S},\mathscr{F}}^{\mathrm{lim,inj}}\cdot A\\ &= \underline{\mathrm{Lim}}_{\mathscr{F}}|_{\underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{S}}^{\nabla}(\mathscr{F})}\cdot \underline{\mathrm{P}}_{\omega,\mathscr{F}}|_{\underline{\underline{\mathrm{FO}}}_{\mathscr{S}}^{\nabla}(\mathscr{F})}\cdot A\\ &= \underline{\mathrm{Lim}}_{\mathscr{F}}|_{\underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{C}}^{\nabla}(\mathscr{F})}\cdot \underline{\mathrm{P}}_{\omega,\mathscr{F}}|_{\underline{\underline{\mathrm{FO}}}_{\mathscr{C}}^{\nabla}(\mathscr{F})}\cdot A\\ &= \underline{\mathrm{Lim}}_{\mathscr{C},\mathscr{F}}\cdot \underline{\mathrm{P}}_{\omega,\mathscr{C},\mathscr{F}}^{\nabla}\cdot A.\end{aligned}$$

Ad (a). Composing the three adjunctions from above, we get that $\underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}}\cdot \mathrm{Inc}_{\underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F})}^{\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F})}\cdot \underline{\mathrm{Lim}}_{\mathscr{S},\mathscr{F}}\cdot \mathrm{Inc}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{S}}^{\nabla}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{S}}^{\mathrm{lim,inj}}(\mathscr{F})}\cdot \underline{\mathrm{P}}_{\omega,\mathscr{S},\mathscr{F}}^{\mathrm{lim,inj}}\cdot A$ is left-adjoint to $\mathrm{W}_{\mathscr{W},\mathscr{F}}^{\mathrm{lim}}\cdot \mathrm{Inc}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}^{\mathrm{lim}}(\mathscr{F})}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{S}}^{\mathrm{lim,inj}}(\mathscr{F})}\cdot \underline{\underline{\Xi}}_{\mathscr{S},\mathscr{F}}^{\mathrm{lim,inj}}\cdot \underline{\underline{\Delta}}_{\mathscr{S},\mathscr{F}}\cdot \mathrm{K}(\mathrm{H}|_{\mathscr{S}})$, cf. lemma 1.6.7. Using (i) and (v), we get that $\mathrm{Real}_{\mathscr{T},\mathscr{F}}^{\mathrm{K}}\cdot A$ is left-adjoint $\mathrm{Res}_{\mathscr{W},\mathscr{F}}\cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}})$. So $\mathrm{Real}_{A,\mathscr{T},\mathscr{F}}$ is left-adjoint to $\mathrm{Res}_{\mathscr{W},\mathscr{F}}\cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}})\cdot \mathrm{L}_{\mathscr{H}}$ by lemma 1.6.14.(c).

Suppose given $X \in \mathrm{Ob}(\mathscr{T}_{[0})$. For $Y \in \mathrm{Ob}(\mathcal{T}_{1]}^{\mathscr{H}})$, we have a bijection between

${}_{\mathscr{D}}(Y\mathrm{Real}_{A,\mathscr{T},\mathscr{F}}\,, X)$ and ${}_{\mathrm{D}(\mathscr{H})}(Y, X\mathrm{Res}_{\mathscr{W},\mathscr{F}}\mathrm{K}(\mathrm{H}|_{\mathscr{C}})\mathrm{L}_{\mathscr{H}})$. Since $\mathrm{Real}_{A,\mathscr{T},\mathscr{F}}$ is t-exact by proposition 4.4.26, we have ${}_{\mathscr{D}}(Y\mathrm{Real}_{A,\mathscr{T},\mathscr{F}}\,, X) = 0$ and thus ${}_{\mathrm{D}(\mathscr{H})}(Y, X\mathrm{Res}_{\mathscr{W},\mathscr{F}}\mathrm{K}(\mathrm{H}|_{\mathscr{C}})\mathrm{L}_{\mathscr{H}}) = 0$.
We conclude that $X\mathrm{Res}_{\mathscr{W},\mathscr{F}}\mathrm{K}(\mathrm{H}|_{\mathscr{C}})\mathrm{L}_{\mathscr{H}} \in \mathrm{Ob}(\mathcal{T}^{\mathscr{H}}_{[0})$.

So $\mathrm{Real}_{A,\mathscr{T},\mathscr{F}}|^{\mathscr{T}_{[0}}_{\mathcal{T}^{\mathscr{H}}_{[0}}$ is left-adjoint to $(\mathrm{Res}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}})|^{\mathcal{T}^{\mathscr{H}}_{[0}}_{\mathscr{T}_{[0}}$, cf. lemma 1.6.9.

Note that $\mathrm{I}_{\mathrm{D},\mathscr{H}}|^{\mathcal{T}^{\mathscr{H}}_{[0}}$ is left-adjoint to $\mathbb{H}_{\mathscr{H}}|_{\mathcal{T}^{\mathscr{H}}_{[0}}$.

So $\big(\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}_{A,\mathscr{T},\mathscr{F}}\big)\big|^{\mathscr{T}_{[0}}$ is left-adjoint to $\big(\mathrm{Res}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}} \cdot \mathbb{H}_{\mathscr{H}}\big)\big|_{\mathscr{T}_{[0}}$, cf. lemma 1.6.7.

The functors $\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}_{A,\mathscr{T},\mathscr{F}}$ and $\mathrm{Inc}^{\mathscr{D}}_{\mathscr{H}}$ are isomorphic in $\mathscr{H}(\mathscr{D})$ by proposition 4.4.29. In particular, the functors $(\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}_{A,\mathscr{T},\mathscr{F}})|^{\mathscr{H}}$ and $1_{\mathscr{H}}$ are isomorphic in $\mathscr{H}(\mathscr{H})$.

Since $(\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}_{A,\mathscr{T},\mathscr{F}})|^{\mathscr{H}}$ is left-adjoint to $\big(\mathrm{Res}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}} \cdot \mathbb{H}_{\mathscr{H}}\big)|_{\mathscr{H}}$ by lemma 1.6.9, we also obtain that the functors $\big(\mathrm{Res}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}} \cdot \mathbb{H}_{\mathscr{H}}\big)|_{\mathscr{H}}$ and $1_{\mathscr{H}}$ are isomorphic in $\mathscr{H}(\mathscr{H})$.

Ad (b). We abbreviate $F = \underline{\mathrm{P}}^{\mathrm{lim,inj}}_{\omega,\mathscr{S},\mathscr{F}} \cdot A$, $G = \mathrm{W}^{\mathrm{lim}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F})}$, $U = \underline{\mathrm{Lim}}_{\mathscr{S},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{lim,inj}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\nabla}_{\mathscr{S}}(\mathscr{F})}$ and $V = \underline{\underline{\Xi}}^{\mathrm{lim,inj}}_{\mathscr{S},\mathscr{F}}$.

Since F is left-adjoint to G , we may choose a counit $GF \xrightarrow{\lambda} 1_{\mathscr{D}}$ of the adjunction. Note that λ is an isotransformation since G is full and faithful, cf. lemma 1.6.6.

Consider the composite of the first two adjunctions from above: $U \cdot F$ is left-adjoint to $G \cdot V$ with counit $(1_G \star \underline{\tau} \star 1_F) \cdot \lambda$, cf. lemma 1.6.7.

For $X \in \mathrm{Ob}(\mathscr{D})$, we have $XGV \in \mathrm{Ob}(\underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F}))$, cf. (iv).
So lemma 1.6.9 yields that $(F{\cdot}U)|_{\underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F})}$ is left-adjoint to $(V{\cdot}G)|^{\underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F})}$ with counit $(1_G{\star}\underline{\tau}{\star}1_F){\cdot}\lambda$.
Note that $(U \cdot F)|_{\underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F})} = \underline{\mathrm{Lim}}_{\mathscr{C},\mathscr{F}} \cdot \underline{\mathrm{P}}^{\nabla}_{\omega,\mathscr{C},\mathscr{F}} \cdot A$ by (vi) and that the functors $(G \cdot V)|^{\underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F})}$ and $\mathrm{W}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{C}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})} \cdot \Xi_{\mathscr{C},\mathscr{F}}$ are isomorphic by (iv).
So it remains to show that $(1_G \star \underline{\tau} \star 1_F) \cdot \lambda$ is an isotransformation, cf. lemma 1.6.6. Since λ is an isotransformation, it suffices to show that $1_G \star \underline{\tau} \star 1_F$ is an isotransformation.

Suppose given $D \in \mathrm{Ob}(\mathscr{D})$. We abbreviate $Y = DG$, $X = Y\Xi_{\mathscr{F}}\mathrm{Lim}_{\mathscr{F}}$ and $f = Y\tau$. So we have $X \xrightarrow{f} Y$ in $\mathrm{FO}(\mathscr{F})$ with $D(1_G \star \underline{\tau} \star 1_F) = DG\underline{\tau}F = \underline{Y\tau}\,\underline{\mathrm{P}}^{\mathrm{lim,inj}}_{\omega,\mathscr{S},\mathscr{F}} \cdot A = \underline{f_\omega}A$.
Note that $Y \in \mathrm{Ob}(\mathrm{FO}^{\mathrm{lim}}_{\mathscr{W}}(\mathscr{F}))$ and that $X \in \mathrm{Ob}(\mathrm{FO}^{\nabla}_{\mathscr{C}}(\mathscr{F}))$, cf. lemma 4.2.55. Since $\mathscr{W}_{0]} \subseteq \mathscr{T}_{n]}$ by assumption, we also have $\mathscr{C} \subseteq \mathscr{T}_{n]}$ and thus $X \in \mathrm{Ob}(\mathrm{FO}^{\nabla}_{\mathscr{T}_{n]}}(\mathscr{F}))$.

Choose a triangle $X_\omega A \xrightarrow{f_\omega A} Y_\omega A \longrightarrow Z \longrightarrow X_\omega A^{[1]}$ in $\mathscr{D}$.

Suppose given $k \in \mathbf{Z}$.

The morphism $f_{|k}$ is an isomorphism in $\mathscr{F}$, cf. definition 4.2.58. Note that $f_{|k} \cdot y^{\omega}_{|k} = x^{\omega}_{|k} \cdot f_\omega$ in $\mathscr{F}$. The pure short exact sequences $X_{|k} \xrightarrow{f_{|k} \cdot y^{\omega}_{|k}} Y_\omega \xrightarrow{y^{\omega}_{k+1|}} Y_{k+1|}$ and $X_{|k} \xrightarrow{x^{\omega}_{|k}} X_\omega \xrightarrow{x^{\omega}_{k+1|}} X_{k+1|}$ in $\mathscr{F}$ yield triangles $X_{|k} \xrightarrow{\underline{x^{\omega}_{|k} \cdot f_\omega}} Y_\omega \longrightarrow Y_{k+1|} \longrightarrow X^{[1]}_{|k}$ and $X_{|k} \xrightarrow{\underline{x^{\omega}_{|k}}} X_\omega \longrightarrow X_{k+1|} \longrightarrow X^{[1]}_{|k}$ in $\underline{\mathscr{F}}$ which in turn yield triangles $X_{|k}A \xrightarrow{\underline{x^{\omega}_{|k}}A \cdot \underline{f_\omega}A} Y_\omega A \longrightarrow Y_{k+1|}A \longrightarrow X_{|k}A^{[1]}$ and
$X_{|k}A \xrightarrow{\underline{x^{\omega}_{|k}}A} X_\omega A \longrightarrow X_{k+1|}A \longrightarrow X_{|k}A^{[1]}$ in $\mathscr{D}$ since A is exact.

Now (TR4) yields a triangle $X_{k+1|}A \longrightarrow Y_{k+1|}A \longrightarrow Z \longrightarrow X_{k+1|}A^{[1]}$ in $\mathscr{D}$.

$$\begin{tikzcd}
 & & X_{k+1|}A \arrow[ddrr] & & \\
 & X_\omega A \arrow[ur] \arrow[dr, "\underline{f_\omega A}"] & & & \\
X_{|k}A \arrow[ur, "\underline{x^\omega_{|k}A}"] \arrow[rr, "\underline{x^\omega_{|k}A}\cdot\underline{f_\omega A}"'] & & Y_\omega A \arrow[rr] \arrow[dr] & & Y_{k+1|}A \arrow[dl] \\
 & & & Z &
\end{tikzcd}$$

Note that we have $Y_{k+1|}A \in \mathrm{Ob}(\mathscr{W}_{k+1\rceil}) \subseteq \mathrm{Ob}(\mathscr{T}_{n+k+1\rfloor})$, cf. remark 4.4.17.
We have $X_{k+1|}A\,, X_{k+1|}A^{[1]} \in \mathrm{Ob}(\mathscr{T}_{n+k+1\rfloor})$ by lemma 4.4.18. Thus $Z \in \mathrm{Ob}(\mathscr{T}_{n+k+1\rfloor})$ as well. So we have $Z \in \mathrm{Ob}(\mathscr{T}_{m\rfloor})$ for all $m \in \mathbf{Z}$. Since $\mathscr{T}$ is non-degenerate, we have $Z \in \mathrm{Ob}(\mathrm{Z}_{\mathscr{D}})$. Thus $\underline{f_\omega}A$ is an isomorphism in $\mathscr{D}$. We conclude that $1_G \star \underline{\tau} \star 1_F$ is an isotransformation.

Ad (c). The functor $\mathrm{W}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{C}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})} \cdot \Xi_{\mathscr{C},\mathscr{F}}$ is full and faithful by (b) and the functor $\underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}}$ is full and faithful since $\mathscr{C} \subseteq \mathscr{H}$, cf. definition 4.1.20. So $\mathrm{Res}_{\mathscr{W},\mathscr{F}} = \mathrm{W}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{C}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})} \cdot \Xi_{\mathscr{C},\mathscr{F}} \cdot \underline{\underline{\Delta}}_{\mathscr{C},\mathscr{F}}$ is full and faithful. Note that $\mathrm{H}|_{\mathscr{C}}$ is isomorphic to $\mathrm{Inc}^{\mathscr{H}}_{\mathscr{C}}$ in $\mathscr{C}(\mathscr{H})$ since $\mathscr{C} \subseteq \mathscr{H}$. Thus $\mathrm{K}(\mathrm{H}|_{\mathscr{C}})$ is full and faithful, cf. lemma 1.9.22. Now lemma 1.6.14.(b) yields that $\mathrm{Res}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}}$ is full and faithful as well. $\square$

4.5.2 Remark. One may call w-structures with the properties of theorem 4.5.1.(b) *complete.* $\diamondsuit$

4.5.3 Theorem (adjunction for adjacent structures)**.** Suppose that $\mathscr{F}$ has epilimits and monocolimits. Suppose that $\mathscr{D}$ is closed under countable coproducts in $\underline{\mathscr{F}}$. Suppose given a functor $A\colon \underline{\mathscr{F}} \to \mathscr{D}$ that is right-adjoint to $\mathrm{Inc}^{\underline{\mathscr{F}}}_{\mathscr{D}}$. Suppose that $\mathscr{T}$ is non-degenerate and that $\mathscr{T}_{0\rfloor} = \mathscr{W}_{0\rceil}$, i.e. that $\mathscr{T}$ is right-adjacent to $\mathscr{W}$.

(a) The functor $\mathrm{Real}_{A,\mathscr{T},\mathscr{F}}$ is right-adjoint to $\mathrm{Res}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}}$.
The functors $\big(\mathrm{Res}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}} \cdot \mathbb{H}_{\mathscr{H}}\big)|_{\mathscr{H}}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}_{A,\mathscr{T},\mathscr{F}})|^{\mathscr{H}}$ are isomorphic to $1_{\mathscr{H}}$ in $\mathscr{H}(\mathscr{H})$.

(b) Suppose given $n \in \mathbf{Z}$ such that $\mathscr{W}_{\lceil 0} \subseteq \mathscr{T}_{\lceil n}$.
Then the functor $\mathrm{W}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}_{\mathscr{C}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})} \cdot \Xi_{\mathscr{C},\mathscr{F}}\colon \mathscr{D} \to \underline{\underline{\nabla}}_{\mathscr{C}}(\mathscr{F})$ is full and faithful. Moreover, it is left-adjoint to $\underline{\mathrm{Lim}}_{\mathscr{C},\mathscr{F}} \cdot \underline{\mathrm{P}}^{\nabla}_{\omega,\mathscr{C},\mathscr{F}} \cdot A$.

(c) Suppose given $n \in \mathbf{Z}$ such that $\mathscr{W}_{\lceil 0} \subseteq \mathscr{T}_{\lceil n}$. Suppose that $\mathscr{C} \subseteq \mathscr{H}$.
Then the functors $\mathrm{Res}_{\mathscr{W},\mathscr{F}}$, $\mathrm{K}(\mathrm{H}|_{\mathscr{C}})$ and $\mathrm{Res}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}}$ are full and faithful.

$$\begin{tikzcd}
 & \mathrm{K}(\mathscr{C}) \arrow[r, "\mathrm{K}(\mathrm{H}|_{\mathscr{C}})"] & \mathrm{K}(\mathscr{H}) \arrow[dr, "\mathrm{L}_{\mathscr{H}}"] & \\
\mathscr{D} \arrow[ur, "\mathrm{Res}_{\mathscr{W},\mathscr{F}}"] & & & \mathrm{D}(\mathscr{H}) \arrow[lll, "\mathrm{Real}_{A,\mathscr{T},\mathscr{F}}"'] \arrow[d, bend left, "\mathbb{H}_{\mathscr{H}}"] \\
\mathscr{H} \arrow[u, "\mathrm{Inc}^{\mathscr{D}}_{\mathscr{H}}"] & & & \mathscr{H} \arrow[u, bend left, "\mathrm{I}_{\mathrm{D},\mathscr{H}}"]
\end{tikzcd}$$

$\diamondsuit$

Proof. This is dual to theorem 4.5.1. $\square$

4.5.4 Theorem (bounded adjunction for adjacent structures)**.** Suppose that $\mathscr{W}$ is bounded and that $\mathscr{T}_{[0} = \mathscr{W}_{[0}$, i.e. that $\mathscr{T}$ is left-adjacent to $\mathscr{W}$.

(a) The functor $\mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}}$ is left-adjoint to $\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}^{\mathrm{b}}_{\mathscr{H}}$.
The functors $\big(\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}^{\mathrm{b}}_{\mathscr{H}} \cdot \mathbb{H}^{\mathrm{b}}_{\mathscr{H}}\big)|_{\mathscr{H}}$ and $(\mathrm{I}_{\mathrm{D}^{\mathrm{b}},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}})|^{\mathscr{H}}$ are isomorphic to $1_{\mathscr{H}}$ in $\mathscr{H}(\mathscr{H})$.

(b) Suppose that $\mathscr{C} \subseteq \mathscr{H}$.
Then the functors $\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$, $\mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}})$ and $\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}^{\mathrm{b}}_{\mathscr{H}}$ are full and faithful.

$$\begin{tikzcd}
 & \mathrm{K}^{\mathrm{b}}(\mathscr{C}) \arrow[r, "\mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}})"] & \mathrm{K}^{\mathrm{b}}(\mathscr{H}) \arrow[dr, "\mathrm{L}^{\mathrm{b}}_{\mathscr{H}}"] & \\
\mathscr{D} \arrow[ur, "\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}"] & & & \mathrm{D}^{\mathrm{b}}(\mathscr{H}) \arrow[lll, "\mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}}"'] \arrow[d, bend left, "\mathbb{H}^{\mathrm{b}}_{\mathscr{H}}"] \\
\mathscr{H} \arrow[u, "\mathrm{Inc}^{\mathscr{D}}_{\mathscr{H}}"] & & & \mathscr{H} \arrow[u, bend left, "\mathrm{I}_{\mathrm{D}^{\mathrm{b}},\mathscr{H}}"]
\end{tikzcd}$$

$\diamondsuit$

Proof. We abbreviate $\mathscr{S} = \mathscr{T}_{[0} = \mathscr{W}_{[0}$. Note that $\mathrm{H}|_{\mathscr{S}}$ is right-adjoint to $\mathrm{Inc}^{\mathscr{S}}_{\mathscr{H}}$.
By proposition 4.1.25, $\underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})}$ is left-adjoint to $\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{S}})$.
The functors $\underline{\Xi}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}$ and $\underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}$ are mutually quasi-inverse equivalences, cf. definition 4.2.56.
By proposition 4.3.54, $\underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{S},\mathscr{F}}|^{\mathscr{D}}$ is left-adjoint to $\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})}$ since $\mathscr{W}$ is bounded.

$$\begin{tikzcd}[column sep=huge]
\mathscr{D} \arrow[r, shift left, "\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})}"] & \underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F}) \arrow[l, shift left, "\underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{S},\mathscr{F}}|^{\mathscr{D}}"] \arrow[r, shift left, "\underline{\Xi}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}"] & \underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F}) \arrow[l, shift left, "\underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}}"] \arrow[r, shift left, "\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{S}})"] & \mathrm{K}^{\mathrm{b}}(\mathscr{H}) \arrow[l, shift left, "\underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})}"]
\end{tikzcd}$$

Ad (a). Composing the three adjunctions from above, we get that
$\underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})} \cdot \underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \cdot \underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{S},\mathscr{F}}|^{\mathscr{D}}$ is left-adjoint to $\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})} \cdot \underline{\Xi}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \cdot \underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{S}})$,

cf. lemma 1.6.7. We have

$$\begin{aligned}
\underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})} \cdot \underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \cdot \underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{S},\mathscr{F}}|^{\mathscr{D}} &= \underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \cdot \underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{F}}|^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})} \cdot \underline{\mathrm{P}}_{\omega,\mathscr{F}}|^{\mathscr{D}}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})} \\
&= \underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \cdot \underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{F}}|^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})}_{\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})} \cdot \underline{\mathrm{P}}_{\omega,\mathscr{F}}|^{\mathscr{D}}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{H}}(\mathscr{F})} \\
&= \big(\underline{\mathrm{R}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \cdot \underline{\mathrm{Lim}}^{\mathrm{b}}_{\mathscr{H},\mathscr{F}} \cdot \underline{\mathrm{P}}^{\mathrm{b}}_{\omega,\mathscr{H},\mathscr{F}}\big)|^{\mathscr{D}} \\
&= \mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}}|^{\mathscr{D}}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\mathrm{FO}}}^{\mathrm{b}}_{\mathscr{W}}(\mathscr{F})} \cdot \underline{\Xi}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \cdot \underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{S}}) &= \mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \underline{\Xi}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}} \cdot \mathrm{Inc}^{\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{S}}(\mathscr{F})}_{\underline{\underline{\nabla}}^{\mathrm{b}}_{\mathscr{C}}(\mathscr{F})} \cdot \underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{S},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{S}}) \\
&= \mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \underline{\Xi}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}} \cdot \underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{Inc}^{\mathscr{S}}_{\mathscr{C}}) \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{S}}) \\
&= \mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{Inc}^{\mathscr{S}}_{\mathscr{C}} \cdot \mathrm{H}|_{\mathscr{S}}) \\
&= \mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}}),
\end{aligned}$$

cf. lemma 4.3.10, remark 4.1.16 and definition 4.1.18.

So $\mathrm{Real}^{\mathrm{K,b}}_{\mathscr{T},\mathscr{F}}|^{\mathscr{D}}$ is left-adjoint $\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}})$.

By lemma 1.6.14.(c), $\mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}}$ is left-adjoint to $\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}^{\mathrm{b}}_{\mathscr{H}}$.

Suppose given $X \in \mathrm{Ob}(\mathscr{T}_{[0})$. For $Y \in \mathrm{Ob}(\mathcal{T}^{\mathscr{H},\mathrm{b}}_{1]})$, we have a bijection between ${}_{\mathscr{D}}(Y\mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}}, X)$ and ${}_{\mathrm{D}^{\mathrm{b}}(\mathscr{H})}(Y, X\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}\mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}})\mathrm{L}^{\mathrm{b}}_{\mathscr{H}})$. Since $\mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}}$ is t-exact by proposition 4.4.25, we have ${}_{\mathscr{D}}(Y\mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}}, X) = 0$ and thus ${}_{\mathrm{D}^{\mathrm{b}}(\mathscr{H})}(Y, X\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}\mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}})\mathrm{L}^{\mathrm{b}}_{\mathscr{H}}) = 0$.
We conclude that $X\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}\mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}})\mathrm{L}^{\mathrm{b}}_{\mathscr{H}} \in \mathrm{Ob}(\mathcal{T}^{\mathscr{H},\mathrm{b}}_{[0})$.

So $\mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}}|^{\mathscr{T}_{[0}}_{\mathcal{T}^{\mathscr{H},\mathrm{b}}_{[0}}$ is left-adjoint to $(\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}^{\mathrm{b}}_{\mathscr{H}})|^{\mathcal{T}^{\mathscr{H},\mathrm{b}}_{[0}}_{\mathscr{T}_{[0}}$, cf. lemma 1.6.9.

Note that $\mathrm{I}_{\mathrm{D}^{\mathrm{b}},\mathscr{H}}|^{\mathcal{T}^{\mathscr{H},\mathrm{b}}_{[0}}$ is left-adjoint to $\mathbb{H}^{\mathrm{b}}_{\mathscr{H}}|_{\mathcal{T}^{\mathscr{H},\mathrm{b}}_{[0}}$.

So $\big(\mathrm{I}_{\mathrm{D}^{\mathrm{b}},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}}\big)\big|^{\mathscr{T}_{[0}}$ is left-adjoint to $\big(\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}^{\mathrm{b}}_{\mathscr{H}} \cdot \mathbb{H}^{\mathrm{b}}_{\mathscr{H}}\big)\big|_{\mathscr{T}_{[0}}$, cf. lemma 1.6.7.

The functors $\mathrm{I}_{\mathrm{D}^{\mathrm{b}},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}}$ and $\mathrm{Inc}^{\mathscr{D}}_{\mathscr{H}}$ are isomorphic in $\mathscr{H}(\mathscr{D})$ by proposition 4.4.28. In particular, the functors $(\mathrm{I}_{\mathrm{D}^{\mathrm{b}},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}})|^{\mathscr{H}}$ and $1_{\mathscr{H}}$ are isomorphic in $\mathscr{H}(\mathscr{H})$.

Since $(\mathrm{I}_{\mathrm{D}^{\mathrm{b}},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}})|^{\mathscr{H}}$ is left-adjoint to $\big(\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}^{\mathrm{b}}_{\mathscr{H}} \cdot \mathbb{H}^{\mathrm{b}}_{\mathscr{H}}\big)|_{\mathscr{H}}$ by lemma 1.6.9, we also obtain that the functors $\big(\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}^{\mathrm{b}}_{\mathscr{H}} \cdot \mathbb{H}^{\mathrm{b}}_{\mathscr{H}}\big)|_{\mathscr{H}}$ and $1_{\mathscr{H}}$ are isomorphic in $\mathscr{H}(\mathscr{H})$.

Ad (b).
The functor $\mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$ is full and faithful, cf. definition 4.3.33. The functor $\underline{\Xi}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}}$ is full and faithful, cf. definition 4.2.56. The functor $\underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}}$ is full and faithful since $\mathscr{C} \subseteq \mathscr{H}$, cf. remark 4.1.23. So $\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} = \mathrm{W}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \underline{\Xi}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}} \cdot \underline{\underline{\Delta}}^{\mathrm{b}}_{\mathscr{C},\mathscr{F}}$ is full and faithful. Note that $\mathrm{H}|_{\mathscr{C}}$ is isomorphic to $\mathrm{Inc}^{\mathscr{H}}_{\mathscr{C}}$ in $\mathscr{C}(\mathscr{H})$ since $\mathscr{C} \subseteq \mathscr{H}$. Thus $\mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}})$ is full and faithful, cf. lemma 1.9.23. Now lemma 1.6.14.(b) yields that $\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}^{\mathrm{b}}_{\mathscr{H}}$ is full and faithful as well. □

4.5.5 Theorem (bounded adjunction for adjacent structures)**.** Suppose that $\mathscr{W}$ is bounded and that $\mathscr{T}_{0]} = \mathscr{W}_{0]}$, i.e. that $\mathscr{T}$ is right-adjacent to $\mathscr{W}$.

(a) The functor $\mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}}$ is right-adjoint to $\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}^{\mathrm{b}}_{\mathscr{H}}$.
The functors $\big(\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}^{\mathrm{b}}_{\mathscr{H}} \cdot \mathbb{H}^{\mathrm{b}}_{\mathscr{H}}\big)|_{\mathscr{H}}$ and $(\mathrm{I}_{\mathrm{D}^{\mathrm{b}},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}})|^{\mathscr{H}}$ are isomorphic to $1_{\mathscr{H}}$ in $\mathscr{H}(\mathscr{H})$.

(b) Suppose that $\mathscr{C} \subseteq \mathscr{H}$.
Then the functors $\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}$, $\mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}})$ and $\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}} \cdot \mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}^{\mathrm{b}}_{\mathscr{H}}$ are full and faithful.

$$\begin{tikzcd}
 & \mathrm{K}^{\mathrm{b}}(\mathscr{C}) \arrow[r, "\mathrm{K}^{\mathrm{b}}(\mathrm{H}|_{\mathscr{C}})"] & \mathrm{K}^{\mathrm{b}}(\mathscr{H}) \arrow[dr, "\mathrm{L}^{\mathrm{b}}_{\mathscr{H}}"] & \\
\mathscr{D} \arrow[ur, "\mathrm{Res}^{\mathrm{b}}_{\mathscr{W},\mathscr{F}}"] & & & \mathrm{D}^{\mathrm{b}}(\mathscr{H}) \arrow[lll, "\mathrm{Real}^{\mathrm{b}}_{\mathscr{T},\mathscr{F}}"'] \arrow[d, bend left, "\mathbb{H}^{\mathrm{b}}_{\mathscr{H}}"] \\
\mathscr{H} \arrow[u, "\mathrm{Inc}^{\mathscr{D}}_{\mathscr{H}}"] & & & \mathscr{H} \arrow[u, bend left, "\mathrm{I}_{\mathrm{D}^{\mathrm{b}},\mathscr{H}}"]
\end{tikzcd}$$
$\Diamond$

Proof. This is dual to the previous theorem 4.5.4. $\square$

4.6 Application to derived categories

4.6.1 Definition. Suppose given an abelian category $\mathscr{A}$ with countable products and countable coproducts. Suppose that $\mathscr{A}$ has enough K-injectives.
We abbreviate $\mathbb{L}^{\mathrm{inj}}_{\mathscr{A}} = \mathrm{Inc}^{\mathrm{K}(\mathscr{A})}_{\mathrm{K}^{\mathrm{inj}}(\mathscr{A})} \cdot \mathrm{L}_{\mathscr{A}} \colon \mathrm{K}^{\mathrm{inj}}(\mathscr{A}) \to \mathrm{D}(\mathscr{A})$. By [Kra10, proposition 4.9.1], $\mathbb{L}^{\mathrm{inj}}_{\mathscr{A}}$ is an exact equivalence and we choose a quasi-inverse $\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}} \colon \mathrm{D}(\mathscr{A}) \to \mathrm{K}^{\mathrm{inj}}(\mathscr{A})$. Moreover, we choose a functor $\mathbb{A}^{\mathrm{inj}}_{\mathscr{A}} \colon \mathrm{K}(\mathscr{A}) \to \mathrm{K}^{\mathrm{inj}}(\mathscr{A})$ that is left-adjoint to $\mathrm{Inc}^{\mathrm{K}(\mathscr{A})}_{\mathrm{K}^{\mathrm{inj}}(\mathscr{A})}$.

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{inj}}(\mathscr{A}) \arrow[r, shift left, "\mathrm{Inc}^{\mathrm{K}(\mathscr{A})}_{\mathrm{K}^{\mathrm{inj}}(\mathscr{A})}"] \arrow[rr, bend left, "\mathbb{L}^{\mathrm{inj}}_{\mathscr{A}}"] & \mathrm{K}(\mathscr{A}) \arrow[l, shift left, "\mathbb{A}^{\mathrm{inj}}_{\mathscr{A}}"] \arrow[r, "\mathrm{L}_{\mathscr{A}}"] & \mathrm{D}(\mathscr{A}) \arrow[ll, bend left, "\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}}"]
\end{tikzcd}$$

Note that $\mathrm{C}(\mathscr{A})$ is a strict Frobenius category with epilimits and monocolimits, cf. lemmata 3.2.50 and 3.2.51. Also note that $\mathrm{K}^{\mathrm{inj}}(\mathscr{A})$ is closed under countable products in $\mathrm{K}(\mathscr{A})$.

(a) Suppose given a w-structure $\mathscr{W}$ on $\mathrm{D}(\mathscr{A})$. Since $\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}}$ is an exact equivalence, we obtain a w-structure $\mathscr{W}\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}}$ on $\mathrm{K}^{\mathrm{inj}}(\mathscr{A})$. We abbreviate $\mathscr{C} = \mathscr{W}_{[0,0]}$ and $\mathscr{C}' = (\mathscr{W}\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}})_{[0,0]}$. Note that the restricted functors $\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}}|^{\mathscr{C}'}_{\mathscr{C}}$ and $\mathbb{L}^{\mathrm{inj}}_{\mathscr{A}}|^{\mathscr{C}}_{\mathscr{C}'}$ are mutually quasi-inverse equivalences. Consequently, $\mathrm{K}\big(\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}}|^{\mathscr{C}'}_{\mathscr{C}}\big)$ and $\mathrm{K}\big(\mathbb{L}^{\mathrm{inj}}_{\mathscr{A}}|^{\mathscr{C}}_{\mathscr{C}'}\big)$ are mutually quasi-inverse equivalences as well.

We define the *injective resolution functor* of $\mathscr{W}$ over $\mathscr{A}$ to be the composite
$\mathrm{Res}^{\mathrm{inj}}_{\mathscr{W},\mathscr{A}} = \mathbb{R}^{\mathrm{inj}}_{\mathscr{A}} \cdot \mathrm{Res}_{\mathscr{W}\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}},\mathrm{C}(\mathscr{A})} \cdot \mathrm{K}\big(\mathbb{L}^{\mathrm{inj}}_{\mathscr{A}}|^{\mathscr{C}}_{\mathscr{C}'}\big) \colon \mathrm{D}(\mathscr{A}) \to \mathrm{K}(\mathscr{C})$.

$$\mathrm{D}(\mathscr{A}) \xrightarrow{\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}}} \mathrm{K}^{\mathrm{inj}}(\mathscr{A}) \xrightarrow{\mathrm{Res}_{\mathscr{W}\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}},\mathrm{C}(\mathscr{A})}} \mathrm{K}(\mathscr{C}') \xrightarrow{\mathrm{K}\left(\mathbb{L}^{\mathrm{inj}}_{\mathscr{A}}|^{\mathscr{C}}_{\mathscr{C}'}\right)} \mathrm{K}(\mathscr{C})$$

(b) Suppose given a non-degenerate t-structure $\mathscr{T}$ on $\mathrm{D}(\mathscr{A})$. Since $\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}}$ is an exact equivalence, we obtain a non-degenerate t-structure $\mathscr{T}\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}}$ on $\mathrm{K}^{\mathrm{inj}}(\mathscr{A})$.
We abbreviate $\mathscr{H} = \mathscr{T}_{[0,0]}$ and $\mathscr{H}' = (\mathscr{T}\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}})_{[0,0]}$. Note that the restricted functors $\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}}|^{\mathscr{H}'}_{\mathscr{H}}$ and $\mathbb{L}^{\mathrm{inj}}_{\mathscr{A}}|^{\mathscr{H}}_{\mathscr{H}'}$ are mutually quasi-inverse equivalences. Consequently, $\mathrm{D}\big(\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}}|^{\mathscr{H}'}_{\mathscr{H}}\big)$ and $\mathrm{D}\big(\mathbb{L}^{\mathrm{inj}}_{\mathscr{A}}|^{\mathscr{H}}_{\mathscr{H}'}\big)$ are mutually quasi-inverse equivalences as well.
We define the *injective realisation functor* of $\mathscr{T}$ over $\mathscr{A}$ to be the composite
$\mathrm{Real}^{\mathrm{inj}}_{\mathscr{T},\mathscr{A}} = \mathrm{D}\big(\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}}|^{\mathscr{H}'}_{\mathscr{H}}\big) \cdot \mathrm{Real}_{\mathbb{A}^{\mathrm{inj}}_{\mathscr{A}},\mathscr{T}\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}},\mathrm{C}(\mathscr{A})} \cdot \mathbb{L}^{\mathrm{inj}}_{\mathscr{A}} \colon \mathrm{D}(\mathscr{H}) \to \mathrm{D}(\mathscr{A})$.

$$\mathrm{D}(\mathscr{H}) \xrightarrow{\mathrm{D}\left(\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}}|^{\mathscr{H}'}_{\mathscr{H}}\right)} \mathrm{D}(\mathscr{H}') \xrightarrow{\mathrm{Real}_{\mathbb{A}^{\mathrm{inj}}_{\mathscr{A}},\mathscr{T}\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}},\mathrm{C}(\mathscr{A})}} \mathrm{K}^{\mathrm{inj}}(\mathscr{A}) \xrightarrow{\mathbb{L}^{\mathrm{inj}}_{\mathscr{A}}} \mathrm{D}(\mathscr{A})$$

$\diamondsuit$

Using the equivalences $\mathrm{D}(\mathscr{A}) \underset{\mathbb{L}^{\mathrm{inj}}_{\mathscr{A}}}{\overset{\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}}}{\rightleftarrows}} \mathrm{K}^{\mathrm{inj}}(\mathscr{A})$, $\mathrm{K}(\mathscr{C}) \underset{\mathrm{K}\left(\mathbb{L}^{\mathrm{inj}}_{\mathscr{A}}|^{\mathscr{C}}_{\mathscr{C}'}\right)}{\overset{\mathrm{K}\left(\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}}|^{\mathscr{C}'}_{\mathscr{C}}\right)}{\rightleftarrows}} \mathrm{K}(\mathscr{C}')$ and

$\mathrm{D}(\mathscr{H}) \underset{\mathrm{D}\left(\mathbb{L}^{\mathrm{inj}}_{\mathscr{A}}|^{\mathscr{H}}_{\mathscr{H}'}\right)}{\overset{\mathrm{D}\left(\mathbb{R}^{\mathrm{inj}}_{\mathscr{A}}|^{\mathscr{H}'}_{\mathscr{H}}\right)}{\rightleftarrows}} \mathrm{D}(\mathscr{H}')$ from above, we obtain the following version of theorem 4.5.1.

4.6.2 Theorem (adjunction for adjacent structures on derived categories)**.** Suppose given an abelian category $\mathscr{A}$ with countable products and countable coproducts. Suppose that $\mathscr{A}$ has enough K-injectives. Suppose given a t-structure $\mathscr{T} = (\mathscr{T}_{0]}, \mathscr{T}_{[0})$ and a w-structure $\mathscr{W} = (\mathscr{W}_{[0}, \mathscr{W}_{0]})$ on $\mathrm{D}(\mathscr{A})$. We abbreviate $\mathscr{H} = \mathscr{T}_{[0,0]}$, $\mathrm{H} = \mathrm{H}_{\mathscr{T}}$ and $\mathscr{C} = \mathscr{W}_{[0,0]}$. Suppose that $\mathscr{T}$ is non-degenerate and that $\mathscr{T}_{[0} = \mathscr{W}_{[0}$, i.e. that $\mathscr{T}$ is left-adjacent to $\mathscr{W}$.

(a) The functor $\mathrm{Real}^{\mathrm{inj}}_{\mathscr{T},\mathscr{A}}$ is left-adjoint to $\mathrm{Res}^{\mathrm{inj}}_{\mathscr{W},\mathscr{A}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}}$.
The functors $\big(\mathrm{Res}^{\mathrm{inj}}_{\mathscr{W},\mathscr{A}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}} \cdot \mathbb{H}_{\mathscr{H}}\big)|_{\mathscr{H}}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{inj}}_{\mathscr{T},\mathscr{A}})|^{\mathscr{H}}$ are isomorphic to $1_{\mathscr{H}}$ in $\mathscr{H}(\mathscr{H})$.

(b) Suppose given $n \in \mathbf{Z}$ such that $\mathscr{W}_{0]} \subseteq \mathscr{T}_{n]}$. Suppose that $\mathscr{C} \subseteq \mathscr{H}$.
Then the functors $\mathrm{Res}^{\mathrm{inj}}_{\mathscr{W},\mathscr{A}}$, $\mathrm{K}(\mathrm{H}|_{\mathscr{C}})$ and $\mathrm{Res}^{\mathrm{inj}}_{\mathscr{W},\mathscr{A}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}}$ are full and faithful.

$$\begin{tikzcd}
 & \mathrm{K}(\mathscr{C}) \arrow[r, "\mathrm{K}(\mathrm{H}|_{\mathscr{C}})"] & \mathrm{K}(\mathscr{H}) \arrow[dr, "\mathrm{L}_{\mathscr{H}}"] & \\
\mathrm{D}(\mathscr{A}) \arrow[ur, "\mathrm{Res}^{\mathrm{inj}}_{\mathscr{W},\mathscr{A}}"] & & & \mathrm{D}(\mathscr{H}) \arrow[lll, "\mathrm{Real}^{\mathrm{inj}}_{\mathscr{T},\mathscr{A}}"'] \arrow[d, bend left, "\mathbb{H}_{\mathscr{H}}"] \\
\mathscr{H} \arrow[u, "\mathrm{Inc}^{\mathrm{D}(\mathscr{A})}_{\mathscr{H}}"] & & & \mathscr{H} \arrow[u, bend left, "\mathrm{I}_{\mathrm{D},\mathscr{H}}"]
\end{tikzcd}$$

$\diamondsuit$

As usual, we also have the following dual versions of definition 4.6.1 and theorem 4.6.2.

4.6.3 Definition. Suppose given an abelian category $\mathscr{A}$ with countable products and countable coproducts. Suppose that $\mathscr{A}$ has enough K-projectives.
We abbreviate $\mathbb{L}^{\mathrm{proj}}_{\mathscr{A}} = \mathrm{Inc}^{\mathrm{K}(\mathscr{A})}_{\mathrm{K}^{\mathrm{proj}}(\mathscr{A})} \cdot \mathrm{L}_{\mathscr{A}} \colon \mathrm{K}^{\mathrm{proj}}(\mathscr{A}) \to \mathrm{D}(\mathscr{A})$. The functor $\mathbb{L}^{\mathrm{proj}}_{\mathscr{A}}$ is an exact equivalence and we choose a quasi-inverse $\mathbb{R}^{\mathrm{proj}}_{\mathscr{A}} \colon \mathrm{D}(\mathscr{A}) \to \mathrm{K}^{\mathrm{proj}}(\mathscr{A})$. Moreover, we choose a functor $\mathbb{A}^{\mathrm{proj}}_{\mathscr{A}} \colon \mathrm{K}(\mathscr{A}) \to \mathrm{K}^{\mathrm{proj}}(\mathscr{A})$ that is right-adjoint to $\mathrm{Inc}^{\mathrm{K}(\mathscr{A})}_{\mathrm{K}^{\mathrm{proj}}(\mathscr{A})}$.

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{proj}}(\mathscr{A}) \arrow[r, shift left, "\mathrm{Inc}^{\mathrm{K}(\mathscr{A})}_{\mathrm{K}^{\mathrm{proj}}(\mathscr{A})}"] \arrow[rr, bend left, "\mathbb{L}^{\mathrm{proj}}_{\mathscr{A}}"] & \mathrm{K}(\mathscr{A}) \arrow[l, shift left, "\mathbb{A}^{\mathrm{proj}}_{\mathscr{A}}"] \arrow[r, "\mathrm{L}_{\mathscr{A}}"] & \mathrm{D}(\mathscr{A}) \arrow[ll, bend left, "\mathbb{R}^{\mathrm{proj}}_{\mathscr{A}}"]
\end{tikzcd}$$

Note that $\mathrm{C}(\mathscr{A})$ is a strict Frobenius category with epilimits and monocolimits, cf. lemmata 3.2.50 and 3.2.51. Also note that $\mathrm{K}^{\mathrm{proj}}(\mathscr{A})$ is closed under countable coproducts in $\mathrm{K}(\mathscr{A})$.

(a) Suppose given a w-structure $\mathscr{W}$ on $\mathrm{D}(\mathscr{A})$. Since $\mathbb{R}^{\mathrm{proj}}_{\mathscr{A}}$ is an exact equivalence, we obtain a w-structure $\mathscr{W}\mathbb{R}^{\mathrm{proj}}_{\mathscr{A}}$ on $\mathrm{K}^{\mathrm{proj}}(\mathscr{A})$. We abbreviate $\mathscr{C} = \mathscr{W}_{[0,0]}$ and $\mathscr{C}' = (\mathscr{W}\mathbb{R}^{\mathrm{proj}}_{\mathscr{A}})_{[0,0]}$. Note that the restricted functors $\mathbb{R}^{\mathrm{proj}}_{\mathscr{A}}|^{\mathscr{C}'}_{\mathscr{C}}$ and $\mathbb{L}^{\mathrm{proj}}_{\mathscr{A}}|^{\mathscr{C}}_{\mathscr{C}'}$ are mutually quasi-inverse equivalences. Consequently, $\mathrm{K}\big(\mathbb{R}^{\mathrm{proj}}_{\mathscr{A}}|^{\mathscr{C}'}_{\mathscr{C}}\big)$ and $\mathrm{K}\big(\mathbb{L}^{\mathrm{proj}}_{\mathscr{A}}|^{\mathscr{C}}_{\mathscr{C}'}\big)$ are mutually quasi-inverse equivalences as well. We define the *projective resolution functor* of $\mathscr{W}$ over $\mathscr{A}$ to be the composite $\mathrm{Res}^{\mathrm{proj}}_{\mathscr{W},\mathscr{A}} = \mathbb{R}^{\mathrm{proj}}_{\mathscr{A}} \cdot \mathrm{Res}_{\mathscr{W}\mathbb{R}^{\mathrm{proj}}_{\mathscr{A}},\mathrm{C}(\mathscr{A})} \cdot \mathrm{K}\big(\mathbb{L}^{\mathrm{proj}}_{\mathscr{A}}|^{\mathscr{C}}_{\mathscr{C}'}\big) \colon \mathrm{D}(\mathscr{A}) \to \mathrm{K}(\mathscr{C})$.

$$\mathrm{D}(\mathscr{A}) \xrightarrow{\mathbb{R}^{\mathrm{proj}}_{\mathscr{A}}} \mathrm{K}^{\mathrm{proj}}(\mathscr{A}) \xrightarrow{\mathrm{Res}_{\mathscr{W}\mathbb{R}^{\mathrm{proj}}_{\mathscr{A}},\mathrm{C}(\mathscr{A})}} \mathrm{K}(\mathscr{C}') \xrightarrow{\mathrm{K}\left(\mathbb{L}^{\mathrm{proj}}_{\mathscr{A}}|^{\mathscr{C}}_{\mathscr{C}'}\right)} \mathrm{K}(\mathscr{C})$$

(b) Suppose given a non-degenerate t-structure $\mathscr{T}$ on $\mathrm{D}(\mathscr{A})$. Since $\mathbb{R}^{\mathrm{proj}}_{\mathscr{A}}$ is an exact equivalence, we obtain a non-degenerate t-structure $\mathscr{T}\mathbb{R}^{\mathrm{proj}}_{\mathscr{A}}$ on $\mathrm{K}^{\mathrm{proj}}(\mathscr{A})$.
We abbreviate $\mathscr{H} = \mathscr{T}_{[0,0]}$ and $\mathscr{H}' = (\mathscr{T}\mathbb{R}^{\mathrm{proj}}_{\mathscr{A}})_{[0,0]}$. Note that the restricted functors $\mathbb{R}^{\mathrm{proj}}_{\mathscr{A}}|^{\mathscr{H}'}_{\mathscr{H}}$ and $\mathbb{L}^{\mathrm{proj}}_{\mathscr{A}}|^{\mathscr{H}}_{\mathscr{H}'}$ are mutually quasi-inverse equivalences. Consequently, $\mathrm{D}\big(\mathbb{R}^{\mathrm{proj}}_{\mathscr{A}}|^{\mathscr{H}'}_{\mathscr{H}}\big)$ and $\mathrm{D}\big(\mathbb{L}^{\mathrm{proj}}_{\mathscr{A}}|^{\mathscr{H}}_{\mathscr{H}'}\big)$ are mutually quasi-inverse equivalences as well.
We define the *projective realisation functor* of $\mathscr{T}$ over $\mathscr{A}$ to be the composite

$$\mathrm{Real}^{\mathrm{proj}}_{\mathscr{T},\mathscr{A}} = \mathrm{D}\big(\mathbb{R}^{\mathrm{proj}}_{\mathscr{A}}|^{\mathscr{H}'}_{\mathscr{H}}\big) \cdot \mathrm{Real}_{\mathbb{A}^{\mathrm{proj}}_{\mathscr{A}},\mathscr{T}\mathbb{R}^{\mathrm{proj}}_{\mathscr{A}},\mathrm{C}(\mathscr{A})} \cdot \mathbb{L}^{\mathrm{proj}}_{\mathscr{A}} \colon \mathrm{D}(\mathscr{H}) \to \mathrm{D}(\mathscr{A}).$$

$$\mathrm{D}(\mathscr{H}) \xrightarrow{\mathrm{D}\big(\mathbb{R}^{\mathrm{proj}}_{\mathscr{A}}|^{\mathscr{H}'}_{\mathscr{H}}\big)} \mathrm{D}(\mathscr{H}') \xrightarrow{\mathrm{Real}_{\mathbb{A}^{\mathrm{proj}}_{\mathscr{A}},\mathscr{T}\mathbb{R}^{\mathrm{proj}}_{\mathscr{A}},\mathrm{C}(\mathscr{A})}} \mathrm{K}^{\mathrm{proj}}(\mathscr{A}) \xrightarrow{\mathbb{L}^{\mathrm{proj}}_{\mathscr{A}}} \mathrm{D}(\mathscr{A})$$

◇

4.6.4 Theorem (adjunction for adjacent structures on derived categories)**.** Suppose given an abelian category $\mathscr{A}$ with countable products and countable coproducts. Suppose that $\mathscr{A}$ has enough K-projectives. Suppose given a t-structure $\mathscr{T} = (\mathscr{T}_{0]}, \mathscr{T}_{[0})$ on $\mathrm{D}(\mathscr{A})$ and a w-structure $\mathscr{W} = (\mathscr{W}_{[0}, \mathscr{W}_{0]})$ on $\mathrm{D}(\mathscr{A})$. We abbreviate $\mathscr{H} = \mathscr{T}_{[0,0]}$, $\mathrm{H} = \mathrm{H}_{\mathscr{T}}$ and $\mathscr{C} = \mathscr{W}_{[0,0]}$. Suppose that $\mathscr{T}$ is non-degenerate and that $\mathscr{T}_{0]} = \mathscr{W}_{0]}$, i.e. that $\mathscr{T}$ is right-adjacent to $\mathscr{W}$.

(a) The functor $\mathrm{Real}^{\mathrm{proj}}_{\mathscr{T},\mathscr{A}}$ is right-adjoint to $\mathrm{Res}^{\mathrm{proj}}_{\mathscr{W},\mathscr{A}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}}$.
The functors $\big(\mathrm{Res}^{\mathrm{proj}}_{\mathscr{W},\mathscr{A}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}} \cdot \mathbb{H}_{\mathscr{H}}\big)|_{\mathscr{H}}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{proj}}_{\mathscr{T},\mathscr{A}})|^{\mathscr{H}}$ are isomorphic to $1_{\mathscr{H}}$ in $\mathscr{H}(\mathscr{H})$.

(b) Suppose given $n \in \mathbf{Z}$ such that $\mathscr{W}_{[0} \subseteq \mathscr{T}_{[n}$. Suppose that $\mathscr{C} \subseteq \mathscr{H}$.
Then the functors $\mathrm{Res}^{\mathrm{proj}}_{\mathscr{W},\mathscr{A}}$, $\mathrm{K}(\mathrm{H}|_{\mathscr{C}})$ and $\mathrm{Res}^{\mathrm{proj}}_{\mathscr{W},\mathscr{A}} \cdot \mathrm{K}(\mathrm{H}|_{\mathscr{C}}) \cdot \mathrm{L}_{\mathscr{H}}$ are full and faithful.

$$\begin{tikzcd}
 & \mathrm{K}(\mathscr{C}) \arrow[r, "\mathrm{K}(\mathrm{H}|_{\mathscr{C}})"] & \mathrm{K}(\mathscr{H}) \arrow[dr, "\mathrm{L}_{\mathscr{H}}"] & \\
\mathrm{D}(\mathscr{A}) \arrow[ur, "\mathrm{Res}^{\mathrm{proj}}_{\mathscr{W},\mathscr{A}}"] & & & \mathrm{D}(\mathscr{H}) \arrow[lll, "\mathrm{Real}^{\mathrm{proj}}_{\mathscr{T},\mathscr{A}}"'] \arrow[d, bend left, "\mathbb{H}_{\mathscr{H}}"] \\
\mathscr{H} \arrow[u, "\mathrm{Inc}^{\mathrm{D}(\mathscr{A})}_{\mathscr{H}}"] & & & \mathscr{H} \arrow[u, bend left, "\mathrm{I}_{\mathrm{D},\mathscr{H}}"]
\end{tikzcd}$$

◇

Chapter 5

Tilting and silting theory

We follow the treatment of [PV18, section 4]. In particular, we define silting and cosilting objects in triangulated categories using t-structures. The t-structure associated to a (co)silting object S will be denoted by $\mathscr{T}^S$, its heart by $\mathscr{H}_S$ and the corresponding homology functor by H_S . Additionally, we introduce the concept of w-(co)silting objects for (co)silting objects whose t-structures have adjacent w-structures. The w-structure associated to such a w-(co)silting object will be denoted by $\mathscr{W}^S$ and its core by $\mathscr{C}_S$.

We define (co)silting resolution and realisation functors associated to w-(co)silting objects in suitable derived categories (definitions 5.1.17, 5.1.20) and use the results from the previous chapter 4 to show that they are adjoint to each other (theorems 5.1.19, 5.1.22).

Section 5.2 contains our derived Morita theorems 5.2.4 and 5.2.8 for injectively (resp. projectively) complete abelian categories.

We apply our adjunctions to silting and cosilting complexes in derived categories of module categories in the last section 5.3 and deduce a silting theorem as well as a cosilting theorem (theorems 5.3.8, 5.3.9).

In this chapter, we work inside a fixed Grothendieck universe $\mathfrak{U}$ that contains the integers, cf. [SGA72, exposé I] [Sch72, sections 3.2, 3.3]. All categories are assumed to be $\mathfrak{U}$-categories. All products and coproducts are assumed to be small with respect to $\mathfrak{U}$. Note that the derived category of an abelian $\mathfrak{U}$-category $\mathscr{A}$ with enough K-injectives or enough K-projectives is again a $\mathfrak{U}$-category since it is equivalent to a subcategory of $\mathrm{K}(\mathscr{A})$.

5.1 W-silting and w-cosilting objects

5.1.1 Definition. Suppose given an additive category $\mathscr{A}$ and an object $X \in \mathrm{Ob}(\mathscr{A})$.

(a) Let $\mathrm{Coprod}_{\mathscr{A}}(X)$ denote the full subcategory of $\mathscr{A}$ whose objects are summands of a coproduct of copies of X.

(b) Let $\mathrm{Prod}_{\mathscr{A}}(X)$ denote the full subcategory of $\mathscr{A}$ whose objects are summands of a product of copies of X. $\diamondsuit$

5.1.2 Definition. [PV18, definition 4.1]
Suppose given a strict triangulated category $\mathscr{D}$ and an object $S \in \mathrm{Ob}(\mathscr{D})$.

(a) We say that S is a *silting object* in $\mathscr{D}$ if $S \in \mathrm{Ob}(S^{\perp_{>0}})$ and if $(S^{\perp_{>0}}, S^{\perp_{<0}})$ is a t-structure on $\mathscr{D}$. In this case, we write $\mathscr{T}^S = (S^{\perp_{>0}}, S^{\perp_{<0}})$ and say that $\mathscr{T}^S$ is the t-structure *associated to* S. We also write $\mathrm{H}_S = \mathrm{H}_{\mathscr{T}^S}$ and $\mathscr{H}_S = \mathscr{T}^S_{[0,0]}$.

(b) We say that S is a *cosilting object* in $\mathscr{D}$ if $S \in \mathrm{Ob}({}^{\perp_{>0}}S)$ and if $({}^{\perp_{<0}}S, {}^{\perp_{>0}}S)$ is a t-structure on $\mathscr{D}$. In this case, we write $\mathscr{T}^S = ({}^{\perp_{<0}}S, {}^{\perp_{>0}}S)$ and say that $\mathscr{T}^S$ is the t-structure *associated to* S. We also write $\mathrm{H}_S = \mathrm{H}_{\mathscr{T}^S}$ and $\mathscr{H}_S = \mathscr{T}^S_{[0,0]}$.

(c) We say that S is a *tilting object* in $\mathscr{D}$ if it is a silting object in $\mathscr{D}$ such that $\mathrm{Coprod}_{\mathscr{D}}(S) \subseteq \mathscr{H}_S$.

(d) We say that S is a *cotilting object* in $\mathscr{D}$ if it is a cosilting object in $\mathscr{D}$ such that $\mathrm{Prod}_{\mathscr{D}}(S) \subseteq \mathscr{H}_S$. $\diamondsuit$

This notion of silting and tilting objects is designed to be used in unbounded derived categories. We refer to [PV18, example 4.2] for a comparison with existing notions in the literature and to [AI12], where the authors work in the bounded setting in the first part of the paper and then turn to the unbounded setting in section 4. The following proposition shows that silting and tilting objects in derived categories of Grothendieck categories admit a perhaps more familiar description.

5.1.3 Proposition. [PV18, proposition 4.13] Suppose given a Grothendieck category $\mathscr{A}$ and an object $S \in \mathrm{Ob}(\mathrm{D}(\mathscr{A}))$. Then S is a silting object in $\mathrm{D}(\mathscr{A})$ if and only if the following three conditions hold.

(a) We have ${}_{\mathrm{D}(\mathscr{A})}(S, S^{[k]}) = 0$ for $k \in \mathbf{Z}_{>0}$.

(b) Given $X \in \mathrm{Ob}(\mathrm{D}(\mathscr{A}))$ with ${}_{\mathrm{D}(\mathscr{A})}(S, X^{[k]}) = 0$ for $k \in \mathbf{Z}$, then we have $X \in \mathrm{Ob}(\mathrm{Z}_{\mathrm{D}(\mathscr{A})})$.

(c) $S^{\perp_{>0}}$ is closed under coproducts in $\mathrm{D}(\mathscr{A})$.

Note that, by definition, such a silting object $S \in \mathrm{Ob}(\mathrm{D}(\mathscr{A}))$ is a tilting object in $\mathrm{D}(\mathscr{A})$ if and only if ${}_{\mathrm{D}(\mathscr{A})}(S, \mathrm{Coprod}_{\mathrm{D}(\mathscr{A})}(S)^{[k]}) = 0$ for $k \in \mathbf{Z}_{<0}$. $\diamondsuit$

5.1.4 Definition. Suppose given an abelian category $\mathscr{A}$ and an object $G \in \mathrm{Ob}(\mathscr{A})$.

(a) We say that G is a *projective generator* in $\mathscr{A}$ if G is projective in $\mathscr{A}$ and if we have ${}_{\mathscr{A}}(G, Y) \neq 0$ for all $Y \in \mathrm{Ob}(\mathscr{A})$ with $Y \notin \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}})$.

(b) We say that G is an *injective cogenerator* in $\mathscr{A}$ if G is injective in $\mathscr{A}$ and if we have ${}_{\mathscr{A}}(Y, G) \neq 0$ for all $Y \in \mathrm{Ob}(\mathscr{A})$ with $Y \notin \mathrm{Ob}(\mathrm{Z}_{\mathscr{A}})$. $\diamondsuit$

5.1.5 Example. [PV18, lemma 4.10 and remark 4.11]
Suppose given an abelian category $\mathscr{A}$.

(a) Suppose given a projective generator P in $\mathscr{A}$. Then $P\mathrm{I}_{\mathrm{D},\mathscr{A}}$ is a tilting object in $\mathrm{D}(\mathscr{A})$ and the associated t-structure is the standard one.

(b) Suppose given an injective cogenerator Q in $\mathscr{A}$. Then $Q\mathrm{I}_{\mathrm{D},\mathscr{A}}$ is a cotilting object in $\mathrm{D}(\mathscr{A})$ and the associated t-structure is the standard one.

$\diamondsuit$

5.1.6 Proposition. [PV18, proposition 4.3]
Suppose given a strict triangulated category $\mathscr{D}$ and a silting object S in $\mathscr{D}$. Then $\mathscr{T}^S$ is a non-degenerate t-structure on $\mathscr{D}$ and $S\mathrm{H}_S$ is a projective generator in $\mathscr{H}_S$. $\diamondsuit$

5.1.7 Proposition. [PV18, proposition 4.3]
Suppose given a strict triangulated category $\mathscr{D}$ and a cosilting object S in $\mathscr{D}$. Then $\mathscr{T}^S$ is a non-degenerate t-structure on $\mathscr{D}$ and $S\mathrm{H}_S$ is an injective cogenerator in $\mathscr{H}_S$. $\diamondsuit$

5.1.8 Definition. Suppose given a strict triangulated category $\mathscr{D}$ and an object $S \in \mathrm{Ob}(\mathscr{D})$.

(a) We say that S is a *w-silting object* in $\mathscr{D}$ if it is a silting object in $\mathscr{D}$ such that $({}^\perp(\mathscr{T}^S_{1]}), \mathscr{T}^S_{0]})$ is a w-structure on $\mathscr{D}$. In this case, we write $\mathscr{W}^S = ({}^\perp(\mathscr{T}^S_{1]}), \mathscr{T}^S_{0]})$ and say that $\mathscr{W}^S$ is the w-structure *associated to* S. We also write $\mathscr{C}_S = \mathscr{W}^S_{[0,0]}$.

(b) We say that S is a *w-cosilting object* in $\mathscr{D}$ if it is a cosilting object in $\mathscr{D}$ such that $(\mathscr{T}^S_{[0}, (\mathscr{T}^S_{[-1})^\perp)$ is a w-structure on $\mathscr{D}$. In this case, we write $\mathscr{W}^S = (\mathscr{T}^S_{[0}, (\mathscr{T}^S_{[-1})^\perp)$ and say that $\mathscr{W}^S$ is the w-structure *associated to* S. We also write $\mathscr{C}_S = \mathscr{W}^S_{[0,0]}$.

(c) Suppose given $n \in \mathbf{Z}$. We say that S is an *n-w-silting object* in $\mathscr{D}$ if it is a w-silting object in $\mathscr{D}$ such that $\mathscr{W}^S_{[0} \subseteq \mathscr{T}^S_{[n}$.

(d) Suppose given $n \in \mathbf{Z}$. We say that S is an *n-w-cosilting object* in $\mathscr{D}$ if it is a w-cosilting object in $\mathscr{D}$ such that $\mathscr{W}^S_{0]} \subseteq \mathscr{T}^S_{n]}$.

(e) We say that S is a *w-tilting object* in $\mathscr{D}$ if there exists $n \in \mathbf{Z}$ such that S is an n-w-silting object in $\mathscr{D}$ and if $\mathscr{C}_S \subseteq \mathscr{H}_S$.

(f) We say that S is a *w-cotilting object* in $\mathscr{D}$ if there exists $n \in \mathbf{Z}$ such that S is an n-w-cosilting object in $\mathscr{D}$ and if $\mathscr{C}_S \subseteq \mathscr{H}_S$.

$\diamondsuit$

5.1.9 Lemma. Suppose given a strict triangulated category $\mathscr{D}$ and a w-silting object S in $\mathscr{D}$. Suppose that $\mathscr{D}$ has coproducts. Then we have $\mathscr{C}_S = \mathrm{Coprod}_{\mathscr{D}}(S)$.
So if S is a w-tilting object in $\mathscr{D}$ then it is a tilting object in $\mathscr{D}$. $\diamondsuit$

Proof. Note that we have $\mathscr{C}_S = \mathscr{W}^S_{0]} \cap \mathscr{W}^S_{[0} = \mathscr{T}^S_{0]} \cap {}^\perp(\mathscr{T}^S_{1]}) = S^{\perp_{>0}} \cap {}^\perp((S^{\perp_{>0}})^{[1]})$. So the result follows from [PV18, lemma 4.5.(ii).(a)]. $\square$

5.1.10 Lemma. Suppose given a strict triangulated category $\mathscr{D}$ and a w-cosilting object S in $\mathscr{D}$. Suppose that $\mathscr{D}$ has products. Then we have $\mathscr{C}_S = \mathrm{Prod}_{\mathscr{D}}(S)$.
So if S is a w-cotilting object in $\mathscr{D}$ then it is a cotilting object in $\mathscr{D}$. $\diamondsuit$

Proof. This is dual to lemma 5.1.9. $\square$

5.1.11 Definition. Suppose given an abelian category $\mathscr{A}$ and an object $S \in \mathrm{Ob}(\mathrm{D}(\mathscr{A}))$.

(a) Suppose that S is a silting object in $\mathrm{D}(\mathscr{A})$. We say that S is *bounded* if there exist $n, m \in \mathbf{Z}$ such that $\mathcal{T}^{\mathscr{A}}_{m\rfloor} \subseteq \mathscr{T}^S_{0\rfloor} \subseteq \mathcal{T}^{\mathscr{A}}_{n\rfloor}$.

(b) Suppose that S is a cosilting object in $\mathrm{D}(\mathscr{A})$. We say that S is *bounded* if there exist $n, m \in \mathbf{Z}$ such that $\mathcal{T}^{\mathscr{A}}_{\lfloor m} \subseteq \mathscr{T}^S_{\lfloor 0} \subseteq \mathcal{T}^{\mathscr{A}}_{\lfloor n}$.

Cf. [PV18, lemma 4.14 and definition 4.15]. $\diamondsuit$

5.1.12 Lemma. Suppose given an abelian category $\mathscr{A}$ with enough DG-projectives and a bounded w-silting object S in $\mathrm{D}(\mathscr{A})$. Then there exists $n \in \mathbf{Z}$ such that S is a n-w-silting object in $\mathrm{D}(\mathscr{A})$. $\diamondsuit$

Proof. Note that we have $\mathcal{W}^{\mathscr{A},\mathrm{proj}}_{\lceil k} \subseteq \mathcal{T}^{\mathscr{A}}_{\lfloor k}$ for $k \in \mathbf{Z}$, cf. definition 1.9.54.
Since S is bounded, we may choose $m, n \in \mathbf{Z}$ such that $\mathcal{T}^{\mathscr{A}}_{m\rfloor} \subseteq \mathscr{T}^S_{0\rfloor} \subseteq \mathcal{T}^{\mathscr{A}}_{n\rfloor}$. So we obtain
$\mathscr{W}^S_{\lceil 0} = {}^{\perp}(\mathscr{T}^S_{1\rfloor}) \subseteq {}^{\perp}(\mathcal{T}^{\mathscr{A}}_{m+1\rfloor}) = \mathcal{W}^{\mathscr{A},\mathrm{proj}}_{\lceil m} \subseteq \mathcal{T}^{\mathscr{A}}_{\lfloor m} = (\mathcal{T}^{\mathscr{A}}_{m+1\rfloor})^{\perp} \subseteq (\mathcal{T}^S_{m-n+1\rfloor})^{\perp} = \mathscr{T}^S_{\lfloor m-n}$. $\square$

5.1.13 Corollary. Suppose given an abelian category $\mathscr{A}$ with coproducts and enough DG-projectives. Suppose given a bounded w-silting object S in $\mathrm{D}(\mathscr{A})$. Then S is a w-tilting object in $\mathrm{D}(\mathscr{A})$ if and only if it is a tilting object in $\mathrm{D}(\mathscr{A})$. Cf. lemmata 5.1.9 and 5.1.12. $\diamondsuit$

5.1.14 Lemma. Suppose given an abelian category $\mathscr{A}$ with enough DG-injectives and a bounded w-cosilting object S in $\mathrm{D}(\mathscr{A})$. Then there exists $n \in \mathbf{Z}$ such that S is a n-w-cosilting object in $\mathrm{D}(\mathscr{A})$. $\diamondsuit$

Proof. This is dual to lemma 5.1.12. $\square$

5.1.15 Corollary. Suppose given an abelian category $\mathscr{A}$ with products and enough DG-injectives. Suppose given a bounded w-cosilting object S in $\mathrm{D}(\mathscr{A})$. Then S is a w-cotilting object in $\mathrm{D}(\mathscr{A})$ if and only if it is a cotilting object in $\mathrm{D}(\mathscr{A})$. Cf. lemmata 5.1.10 and 5.1.14. $\diamondsuit$

5.1.16 Example. Suppose given an abelian category $\mathscr{A}$.

(a) Suppose that $\mathscr{A}$ has enough DG-projectives. Suppose given a projective generator P in $\mathscr{A}$. Then $P\mathrm{I}_{\mathrm{D},\mathscr{A}}$ is a bounded w-tilting object in $\mathrm{D}(\mathscr{A})$ and the associated t-structure is the standard one.

(b) Suppose that $\mathscr{A}$ has enough DG-injectives. Suppose given an injective cogenerator Q in $\mathscr{A}$. Then $Q\mathrm{I}_{\mathrm{D},\mathscr{A}}$ is a bounded w-cotilting object in $\mathrm{D}(\mathscr{A})$ and the associated t-structure is the standard one.

Cf. example 5.1.5 and definitions 1.9.53, 1.9.54. $\diamondsuit$

5.1.17 Definition. Suppose given an abelian category $\mathscr{A}$ with products, coproducts and enough K-injectives. Suppose given a w-cosilting object S in $\mathrm{D}(\mathscr{A})$. We write

$$\mathrm{Real}^{\mathrm{cos}}_{S,\mathscr{A}} = \mathrm{Real}^{\mathrm{inj}}_{\mathscr{T}^S,\mathscr{A}} \colon \mathrm{D}(\mathscr{H}_S) \to \mathrm{D}(\mathscr{A})$$

and say that $\mathrm{Real}^{\mathrm{cos}}_{S,\mathscr{A}}$ is the *cosilting realisation functor* associated to S. We also write

$$\mathrm{Res}^{\mathrm{cos}}_{S,\mathscr{A}} = \mathrm{Res}^{\mathrm{inj}}_{\mathscr{W}^S,\mathscr{A}} \cdot \mathrm{K}(\mathrm{H}_S|_{\mathscr{C}_S}) \cdot \mathrm{L}_{\mathscr{H}_S} \colon \mathrm{D}(\mathscr{A}) \to \mathrm{D}(\mathscr{H}_S)$$

and say that $\mathrm{Res}^{\mathrm{cos}}_{S,\mathscr{A}}$ is the *cosilting resolution functor* associated to S.
Cf. definition 4.6.1 and proposition 5.1.7. $\diamondsuit$

5.1.18 Definition. Suppose given a triangulated category $\mathscr{D}$. A full triangulated subcategory $\mathscr{S} \subseteq \mathscr{D}$ is called a *colocalising subcategory* of $\mathscr{D}$ if it is closed under products in $\mathscr{D}$. We say that an object $X \in \mathrm{Ob}(\mathscr{D})$ *cogenerates* $\mathscr{D}$ if $\mathscr{D}$ is the smallest colocalising subcategory of $\mathscr{D}$ that contains X.

When $\mathscr{D}$ has products, then this definition coincides with the one in [Nee11, 0. Introduction]. $\diamondsuit$

5.1.19 Theorem (adjunction for w-cosilting objects)**.** Suppose given an abelian category $\mathscr{A}$ with products, coproducts and enough K-injectives. Suppose given a w-cosilting object S in $\mathrm{D}(\mathscr{A})$.

(a) The cosilting realisation functor $\mathrm{Real}^{\mathrm{cos}}_{S,\mathscr{A}}$ is left-adjoint to the cosilting resolution functor $\mathrm{Res}^{\mathrm{cos}}_{S,\mathscr{A}}$. The functors $(\mathrm{Res}^{\mathrm{cos}}_{S,\mathscr{A}} \cdot \mathbb{H}_{\mathscr{H}_S})|_{\mathscr{H}_S}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}_S} \cdot \mathrm{Real}^{\mathrm{cos}}_{S,\mathscr{A}})|^{\mathscr{H}_S}$ are isomorphic to $1_{\mathscr{H}_S}$ in $\mathscr{H}_S(\mathscr{H}_S)$.

(b) Suppose that S is a w-cotilting object in $\mathrm{D}(\mathscr{A})$. Then the cosilting resolution functor $\mathrm{Res}^{\mathrm{cos}}_{S,\mathscr{A}}$ is full and faithful. If $S\mathrm{I}_{\mathrm{D},\mathscr{H}_S}$ cogenerates $\mathrm{D}(\mathscr{H}_S)$, then $\mathrm{Res}^{\mathrm{cos}}_{S,\mathscr{A}}$ and $\mathrm{Real}^{\mathrm{cos}}_{S,\mathscr{A}}$ are mutually quasi-inverse equivalences. $\diamondsuit$

Proof. Ad (a). This follows from theorem 4.6.2.(a).

Ad (b). The functor $\mathrm{Res}^{\mathrm{cos}}_{S,\mathscr{A}}$ is full and faithful by theorem 4.6.2.(b).

Suppose that $S\mathrm{I}_{\mathrm{D},\mathscr{H}_S}$ cogenerates $\mathrm{D}(\mathscr{H}_S)$. The essential image of $\mathrm{Res}^{\mathrm{cos}}_{S,\mathscr{A}}$ is a triangulated subcategory of $\mathrm{D}(\mathscr{H}_S)$ that is closed under products since $\mathrm{Res}^{\mathrm{cos}}_{S,\mathscr{A}}$ has a left-adjoint. Moreover, $S\mathrm{Res}^{\mathrm{cos}}_{S,\mathscr{A}} = S\mathrm{Res}^{\mathrm{inj}}_{\mathscr{W}^S,\mathscr{A}}\mathrm{K}(\mathrm{H}_S|_{\mathscr{C}_S})\mathrm{L}_{\mathscr{H}_S}$ is isomorphic to $S\mathrm{I}_{\mathrm{D},\mathscr{H}_S}$ in $\mathrm{D}(\mathscr{H}_S)$ by lemma 4.3.44 and since $S \in \mathrm{Ob}(\mathscr{C}_S) \subseteq \mathrm{Ob}(\mathscr{H}_S)$. Thus $\mathrm{D}(\mathscr{H}_S)$ is the essential image of $\mathrm{Res}^{\mathrm{cos}}_{S,\mathscr{A}}$. $\square$

5.1.20 Definition. Suppose given an abelian category $\mathscr{A}$ with products, coproducts and enough K-projectives. Suppose given a w-silting object S in $\mathrm{D}(\mathscr{A})$. We write

$$\mathrm{Real}^{\mathrm{s}}_{S,\mathscr{A}} = \mathrm{Real}^{\mathrm{proj}}_{\mathscr{T}^S,\mathscr{A}} \colon \mathrm{D}(\mathscr{H}_S) \to \mathrm{D}(\mathscr{A})$$

and say that $\mathrm{Real}^{\mathrm{s}}_{S,\mathscr{A}}$ is the *silting realisation functor* associated to S. We also write

$$\mathrm{Res}^{\mathrm{s}}_{S,\mathscr{A}} = \mathrm{Res}^{\mathrm{proj}}_{\mathscr{W}^S,\mathscr{A}} \cdot \mathrm{K}(\mathrm{H}_S|_{\mathscr{C}_S}) \cdot \mathrm{L}_{\mathscr{H}_S} \colon \mathrm{D}(\mathscr{A}) \to \mathrm{D}(\mathscr{H}_S)$$

and say that $\mathrm{Res}^{\mathrm{s}}_{S,\mathscr{A}}$ is the *silting resolution functor* associated to S.
Cf. definition 4.6.3 and proposition 5.1.6. $\diamondsuit$

5.1.21 Definition. Suppose given a triangulated category $\mathscr{D}$. A full triangulated subcategory $\mathscr{S} \subseteq \mathscr{D}$ is called a *localising subcategory* of $\mathscr{D}$ if it is closed under coproducts in $\mathscr{D}$. We say that an object $X \in \mathrm{Ob}(\mathscr{D})$ *generates* $\mathscr{D}$ if $\mathscr{D}$ is the smallest localising subcategory of $\mathscr{D}$ that contains X. $\diamondsuit$

5.1.22 Theorem (adjunction for w-silting objects)**.** Suppose given an abelian category $\mathscr{A}$ with products, coproducts and enough K-projectives. Suppose given a w-silting object S in $\mathrm{D}(\mathscr{A})$.

(a) The silting realisation functor $\mathrm{Real}^{\mathrm{s}}_{S,\mathscr{A}}$ is right-adjoint to the silting resolution functor $\mathrm{Res}^{\mathrm{s}}_{S,\mathscr{A}}$. The functors $(\mathrm{Res}^{\mathrm{s}}_{S,\mathscr{A}} \cdot \mathbb{H}_{\mathscr{H}_S})|_{\mathscr{H}_S}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}_S} \cdot \mathrm{Real}^{\mathrm{s}}_{S,\mathscr{A}})|^{\mathscr{H}_S}$ are isomorphic to $1_{\mathscr{H}_S}$ in $\mathscr{H}_S(\mathscr{H}_S)$.

(b) Suppose that S is a w-tilting object in $\mathrm{D}(\mathscr{A})$. Then the silting resolution functor $\mathrm{Res}^{\mathrm{s}}_{S,\mathscr{A}}$ is full and faithful. If $S\mathrm{I}_{\mathrm{D},\mathscr{H}_S}$ generates $\mathrm{D}(\mathscr{H}_S)$, then $\mathrm{Res}^{\mathrm{s}}_{S,\mathscr{A}}$ and $\mathrm{Real}^{\mathrm{s}}_{S,\mathscr{A}}$ are mutually quasi-inverse equivalences.

$\diamondsuit$

Proof. This is dual to theorem 5.1.19. $\square$

5.2 Morita theory for derived categories

Classical Morita theory [Mor58] tells us when two categories of modules Mod-R and Mod-S are equivalent. A necessary and sufficient condition is that there exists a progenerator in Mod-R such that its endomorphism ring is isomorphic to S. For derived categories of module categories, the corresponding theory is due to [Ric89] and [Kel94]. Here the existence of a tilting complex theorem is necessary and sufficient for two derived categories of module categories to be equivalent. We are now able to generalise this to derived categories of projectively complete abelian categories via w-tilting objects in theorem 5.2.8 and, dually, to derived categories of injectively complete abelian categories via w-cotilting objects in theorem 5.2.4.

5.2.1 Definition. We say that an abelian category $\mathscr{A}$ is *injectively complete* if

(a) it has products and coproducts,

(b) it has enough DG-injectives and if

(c) there exists an injective cogenerator Q in $\mathscr{A}$ such that $Q\mathrm{I}_{\mathrm{D},\mathscr{A}}$ cogenerates $\mathrm{D}(\mathscr{A})$. $\diamondsuit$

5.2.2 Remark. Suppose given an injectively complete abelian category $\mathscr{A}$ and an injective cogenerator R in $\mathscr{A}$. Then $R\mathrm{I}_{\mathrm{D},\mathscr{A}}$ cogenerates $\mathrm{D}(\mathscr{A})$. $\diamondsuit$

Proof. We may choose an injective cogenerator Q in $\mathscr{A}$ such that $Q\mathrm{I}_{\mathrm{D},\mathscr{A}}$ cogenerates $\mathrm{D}(\mathscr{A})$. So we have $\mathrm{Prod}_{\mathscr{A}}(R) = \mathrm{Prod}_{\mathscr{A}}(Q)$. The result now follows from the fact that $\mathrm{I}_{\mathrm{D},\mathscr{A}}$ preserves products of injectives. $\square$

5.2.3 Example. We say that a Grothendieck category $\mathscr{A}$ is *left-complete* if $\mathrm{D}(\mathscr{A})$ is left-complete in the sense of [PV18, definition 6.2]. Left-complete Grothendieck categories are injectively complete. See example 1.9.48 for (b) and [PV18, (proof of) lemma 6.6.(ii)] for (c). $\diamondsuit$

5.2.4 Theorem (derived Morita theorem for injectively complete abelian categories)**.** Suppose given injectively complete abelian categories $\mathscr{A}$ and $\mathscr{B}$. The following two statements are equivalent.

(a) There is an exact equivalence $F\colon \mathrm{D}(\mathscr{B}) \to \mathrm{D}(\mathscr{A})$.

(b) There is a w-cotilting object S in $\mathrm{D}(\mathscr{A})$ and an equivalence $G\colon \mathscr{B} \to \mathscr{H}_S$. $\diamondsuit$

Proof. Ad (a)→(b). Choose an injective cogenerator Q in $\mathscr{B}$. Then $Q\mathrm{I}_{\mathrm{D},\mathscr{B}}$ is a w-cotilting object in $\mathrm{D}(\mathscr{B})$ and the associated t-structure is the standard one, cf. example 5.1.16. Consequently, $S = Q\mathrm{I}_{\mathrm{D},\mathscr{B}}F$ is a w-cotilting object in $\mathrm{D}(\mathscr{A})$ and there is an equivalence $G\colon \mathscr{B} \to \mathscr{H}_S$.
Ad (b)→(a). Note that $S\mathrm{I}_{\mathrm{D},\mathscr{H}_S}$ cogenerates $\mathrm{D}(\mathscr{H}_S)$, cf. proposition 5.1.7 and remark 5.2.2. By theorem 5.1.19.(b), $\mathrm{Real}^{\mathrm{cos}}_{S,\mathscr{A}}\colon \mathrm{D}(\mathscr{H}_S) \to \mathrm{D}(\mathscr{A})$ is an exact equivalence.
We conclude that $\mathrm{D}(G) \cdot \mathrm{Real}^{\mathrm{cos}}_{S,\mathscr{A}}\colon \mathrm{D}(\mathscr{B}) \to \mathrm{D}(\mathscr{A})$ is an exact equivalence. $\square$

5.2.5 Remark. If the w-cotilting object S in theorem 5.2.4.(b) is bounded, then the equivalence is restrictable (see [PV18, definition 3.15]) since the functor $\mathrm{Real}^{\mathrm{cos}}_{S,\mathscr{A}}$ induces an equivalence between $(\mathcal{T}^{\mathscr{H}_S})^{\mathrm{b}}$ and $(\mathscr{T}^S)^{\mathrm{b}}$ and the latter can be identified with $\mathrm{D}^{\mathrm{b}}(\mathscr{A})$ since S is bounded. $\diamondsuit$

5.2.6 Definition. We say that an abelian category $\mathscr{A}$ is *projectively complete* if

(a) it has products and coproducts,

(b) it has enough DG-projectives and if

(c) there exists a projective generator P in $\mathscr{A}$ such that $P\mathrm{I}_{\mathrm{D},\mathscr{A}}$ generates $\mathrm{D}(\mathscr{A})$. $\diamondsuit$

5.2.7 Example. The module category Mod-R, where R is a ring, is a projectively complete abelian category. See example 1.9.50 for (b) and [Fra01, proposition 3.3] for (c). Moreover, it is also injectively complete since it is left-complete Grothendieck. $\diamondsuit$

5.2.8 Theorem (derived Morita theorem for projectively complete abelian categories)**.** Suppose given projectively complete abelian categories $\mathscr{A}$ and $\mathscr{B}$. The following two statements are equivalent.

(a) There is an exact equivalence $F\colon \mathrm{D}(\mathscr{B}) \to \mathrm{D}(\mathscr{A})$.

(b) There is a w-tilting object S in $\mathrm{D}(\mathscr{A})$ and an equivalence $G\colon \mathscr{B} \to \mathscr{H}_S$.

$\diamondsuit$

Proof. This is dual to theorem 5.2.4. $\square$

5.3 Silting and cosilting complexes

An overview of recent developments in silting theory can be found in [Ang18].

Suppose given a ring R. Let Proj-R denote the full subcategory of Mod-R whose objects are the projective R-modules. Let Inj-R denote the full subcategory of Mod-R whose objects are the injective R-modules.

We consider $\mathrm{C}(\text{Proj-}R)$ and $\mathrm{C}(\text{Inj-}R)$ as full subcategories of $\mathrm{C}(\text{Mod-}R)$ via $\mathrm{C}(\mathrm{Inc}_{\text{Proj-}R}^{\text{Mod-}R})\colon \mathrm{C}(\text{Proj-}R) \to \mathrm{C}(\text{Mod-}R)$ and $\mathrm{C}(\mathrm{Inc}_{\text{Inj-}R}^{\text{Mod-}R})\colon \mathrm{C}(\text{Inj-}R) \to \mathrm{C}(\text{Mod-}R)$.

5.3.1 Definition. Suppose given $X \in \mathrm{Ob}(\mathrm{C}^{\mathrm{b}}(\text{Proj-}R))$. We say that that X is a *silting complex* if it is a silting object in $\mathrm{D}(\text{Mod-}R)$. Cf. [AHMV16, proposition 4.2]. $\diamondsuit$

5.3.2 Definition. Suppose given $X \in \mathrm{Ob}(\mathrm{C}^{\mathrm{b}}(\text{Inj-}R))$. We say that that X is a *cosilting complex* if it is a cosilting object in $\mathrm{D}(\text{Mod-}R)$. Cf. [Ang18, proposition 6.8.(2)]. $\diamondsuit$

The following two theorems allow us to apply our results to silting and cosilting complexes.

5.3.3 Theorem. [AHMV16, theorem 4.6]
Suppose given a silting complex $X \in \mathrm{Ob}(\mathrm{C}^{\mathrm{b}}(\text{Proj-}R))$. Then X is a bounded w-silting object in $\mathrm{D}(\text{Mod-}R)$. $\diamondsuit$

5.3.4 Theorem. [MV18, theorem 3.13]
Suppose given a cosilting complex $X \in \mathrm{Ob}(\mathrm{C}^{\mathrm{b}}(\text{Inj-}R))$. Then X is a bounded w-cosilting object in $\mathrm{D}(\text{Mod-}R)$. $\diamondsuit$

5.3.5 Theorem (adjunction for silting complexes)**.** Suppose given a silting complex $S \in \mathrm{Ob}(\mathrm{C}^{\mathrm{b}}(\text{Proj-}R))$. The silting realisation functor $\mathrm{Real}^{\mathrm{s}}_{S,\text{Mod-}R}\colon \mathrm{D}(\mathscr{H}_S) \to \mathrm{D}(\text{Mod-}R)$ is right-adjoint to the silting resolution functor $\mathrm{Res}^{\mathrm{s}}_{S,\text{Mod-}R}\colon \mathrm{D}(\text{Mod-}R) \to \mathrm{D}(\mathscr{H}_S)$. The functors $(\mathrm{Res}^{\mathrm{s}}_{S,\text{Mod-}R} \cdot \mathbb{H}_{\mathscr{H}_S})|_{\mathscr{H}_S}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}_S} \cdot \mathrm{Real}^{\mathrm{s}}_{S,\text{Mod-}R})|^{\mathscr{H}_S}$ are isomorphic to $1_{\mathscr{H}_S}$ in $\mathscr{H}_S(\mathscr{H}_S)$. $\diamondsuit$

Proof. This is an application of theorem 5.1.22.(a). $\square$

5.3.6 Theorem (adjunction for cosilting complexes)**.** Suppose given a cosilting complex $S \in \mathrm{Ob}(\mathrm{C}^{\mathrm{b}}(\text{Inj-}R))$. The cosilting realisation functor $\mathrm{Real}^{\mathrm{cos}}_{S,\text{Mod-}R} \colon \mathrm{D}(\mathscr{H}_S) \to \mathrm{D}(\text{Mod-}R)$ is left-adjoint to the cosilting resolution functor $\mathrm{Res}^{\mathrm{cos}}_{S,\text{Mod-}R} \colon \mathrm{D}(\text{Mod-}R) \to \mathrm{D}(\mathscr{H}_S)$. The functors $(\mathrm{Res}^{\mathrm{cos}}_{S,\text{Mod-}R} \cdot \mathbb{H}_{\mathscr{H}_S})|_{\mathscr{H}_S}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}_S} \cdot \mathrm{Real}^{\mathrm{cos}}_{S,\text{Mod-}R})|^{\mathscr{H}_S}$ are isomorphic to $1_{\mathscr{H}_S}$ in $\mathscr{H}_S(\mathscr{H}_S)$. $\diamondsuit$

Proof. This is an application of theorem 5.1.19.(a). $\square$

There is a correspondence between t-structures and torsion pairs in abelian categories known as HRS-tilting, see e. g. [HRS96, chapter I, section 2] and [PSZ18, proposition 1]. We have the following lemma.

5.3.7 Lemma. [Pol07, lemma 1.1.2]
Suppose given a strict triangulated category $\mathscr{D}$. Suppose given t-structures $\mathscr{T}$ and $\mathscr{U}$ on $\mathscr{D}$ such that $\mathscr{T}_{0]} \subseteq \mathscr{U}_{0]} \subseteq \mathscr{T}_{-1]}$. Then $(\mathscr{T}_{[0,0]} \cap \mathscr{U}^{[1]}_{[0,0]}\,,\ \mathscr{T}_{[0,0]} \cap \mathscr{U}_{[0,0]})$ is a torsion pair in $\mathscr{T}_{[0,0]}$ and $(\mathscr{T}_{[0,0]} \cap \mathscr{U}_{[0,0]}\,,\ \mathscr{T}^{[-1]}_{[0,0]} \cap \mathscr{U}_{[0,0]})$ is a torsion pair in $\mathscr{U}_{[0,0]}$. $\diamondsuit$

5.3.8 Theorem (silting theorem)**.** Suppose given a silting complex $S \in \mathrm{Ob}(\mathrm{C}^{[1,0]}(\text{Proj-}R))$. Let $\mathscr{T}$ denote the associated t-structure in $\mathrm{D}(\text{Mod-}R)$. We abbreviate $\mathscr{H} = \mathscr{T}_{[0,0]}$. Let $\mathscr{U}$ denote the standard t-structure in $\mathrm{D}(\text{Mod-}R)$. Note that one may identify $\mathscr{U}_{[0,0]}$ with Mod-R via $\mathrm{I}_{\mathrm{D},\text{Mod-}R}$. We have $\mathscr{T}_{0]} \subseteq \mathscr{U}_{0]} \subseteq \mathscr{T}_{-1]}$, cf. [AHMV16, lemma 4.5].
Let $\mathcal{Y} = \mathscr{H} \cap \mathscr{U}^{[1]}_{[0,0]}$ and $\mathcal{X} = \mathscr{H} \cap \mathscr{U}_{[0,0]}$. By lemma 5.3.7, $(\mathcal{Y}, \mathcal{X})$ is a torsion pair in $\mathscr{H}$ and $(\mathcal{X}, \mathcal{Y}^{[-1]})$ is a torsion pair in $\mathscr{U}_{[0,0]}$.

The functors $(\mathrm{Res}^{\mathrm{s}}_{S,\text{Mod-}R} \cdot \mathbb{H}_{\mathscr{H}})|_{\mathscr{H}}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{s}}_{S,\text{Mod-}R})|^{\mathscr{H}}$ are isomorphic to $1_{\mathscr{H}}$ in $\mathscr{H}(\mathscr{H})$ by theorem 5.3.5. In particular, we have the following.

- The functors $(\mathrm{Res}^{\mathrm{s}}_{S,\text{Mod-}R} \cdot \mathbb{H}_{\mathscr{H}})|^{\mathcal{X}}_{\mathcal{X}}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{s}}_{S,\text{Mod-}R})|^{\mathcal{X}}_{\mathcal{X}}$ are mutually quasi-inverse equivalences.

- The functors $(\Sigma_{\mathrm{D},\text{Mod-}R} \cdot \mathrm{Res}^{\mathrm{s}}_{S,\text{Mod-}R} \cdot \mathbb{H}_{\mathscr{H}})|^{\mathcal{Y}}_{\mathcal{Y}^{[-1]}}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}^{\mathrm{s}}_{S,\text{Mod-}R} \cdot \Sigma^{-1}_{\mathrm{D},\text{Mod-}R})|^{\mathcal{Y}^{[-1]}}_{\mathcal{Y}}$ are mutually quasi-inverse equivalences.

$$\begin{tikzcd}
\mathrm{D}(\text{Mod-}R) \arrow[r, shift left, "\mathrm{Res}^{\mathrm{s}}_{S,\text{Mod-}R}"] & \mathrm{D}(\mathscr{H}) \arrow[l, shift left, "\mathrm{Real}^{\mathrm{s}}_{S,\text{Mod-}R}"] \arrow[d, bend left, "\mathbb{H}_{\mathscr{H}}"] \\
\text{Mod-}R \arrow[u, "\mathrm{I}_{\mathrm{D},\text{Mod-}R}"] & \mathscr{H} \arrow[u, bend left, "\mathrm{I}_{\mathrm{D},\mathscr{H}}"]
\end{tikzcd}$$
$\diamondsuit$

5.3.9 Theorem (cosilting theorem)**.** Suppose given a cosilting complex $S \in \mathrm{Ob}(\mathrm{C}^{[0,-1]}(\text{Inj-}R))$. Let $\mathscr{U}$ denote the associated t-structure in $\mathrm{D}(\text{Mod-}R)$. We abbreviate $\mathscr{H} = \mathscr{U}_{[0,0]}$. Let $\mathscr{T}$ denote the standard t-structure in $\mathrm{D}(\text{Mod-}R)$. Note that one may identify $\mathscr{T}_{[0,0]}$ with Mod-R via $\mathrm{I}_{\mathrm{D},\text{Mod-}R}$. We have $\mathscr{T}_{0]} \subseteq \mathscr{U}_{0]} \subseteq \mathscr{T}_{-1]}$, cf. [Pop17b, theorem 2.12 and proposition 2.16].

Let $\mathcal{Y} = \mathscr{T}_{[0,0]} \cap \mathscr{H}^{[1]}$ and $\mathcal{X} = \mathscr{T}_{[0,0]} \cap \mathscr{H}$. By lemma 5.3.7, $(\mathcal{Y}, \mathcal{X})$ is a torsion pair in $\mathscr{T}_{[0,0]}$ and $(\mathcal{X}, \mathcal{Y}^{[-1]})$ is a torsion pair in $\mathscr{H}$.

The functors $(\mathrm{Res}^{\cos}_{S,\mathrm{Mod}\text{-}R} \cdot \mathbb{H}_{\mathscr{H}})|_{\mathscr{H}}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}^{\cos}_{S,\mathrm{Mod}\text{-}R})|^{\mathscr{H}}$ are isomorphic to $1_{\mathscr{H}}$ in $\mathscr{H}(\mathscr{H})$ by theorem 5.3.6. In particular, we have the following.

- The functors $(\mathrm{Res}^{\cos}_{S,\mathrm{Mod}\text{-}R} \cdot \mathbb{H}_{\mathscr{H}})|^{\mathcal{X}}_{\mathcal{X}}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}^{\cos}_{S,\mathrm{Mod}\text{-}R})|^{\mathcal{X}}_{\mathcal{X}}$ are mutually quasi-inverse equivalences.
- The functors $(\Sigma^{-1}_{\mathrm{D},\mathrm{Mod}\text{-}R} \cdot \mathrm{Res}^{\cos}_{S,\mathrm{Mod}\text{-}R} \cdot \mathbb{H}_{\mathscr{H}})|^{\mathcal{Y}^{[-1]}}_{\mathcal{Y}}$ and $(\mathrm{I}_{\mathrm{D},\mathscr{H}} \cdot \mathrm{Real}^{\cos}_{S,\mathrm{Mod}\text{-}R} \cdot \Sigma_{\mathrm{D},\mathrm{Mod}\text{-}R})|^{\mathcal{Y}}_{\mathcal{Y}^{[-1]}}$ are mutually quasi-inverse equivalences.

$$\begin{tikzcd}
\mathrm{D}(\mathrm{Mod}\text{-}R) \arrow[r, shift left, "\mathrm{Res}^{\cos}_{S,\mathrm{Mod}\text{-}R}"] & \mathrm{D}(\mathscr{H}) \arrow[l, shift left, "\mathrm{Real}^{\cos}_{S,\mathrm{Mod}\text{-}R}"] \arrow[d, bend left, "\mathbb{H}_{\mathscr{H}}"] \\
\mathrm{Mod}\text{-}R \arrow[u, "\mathrm{I}_{\mathrm{D},\mathrm{Mod}\text{-}R}"] & \mathscr{H} \arrow[u, bend left, "\mathrm{I}_{\mathrm{D},\mathscr{H}}"]
\end{tikzcd}$$

$\diamondsuit$

List of symbols

Symbol	Description	Definition	Page
$\mathbf{Z}$	The poset of integers.	Conv. 1	20
$\mathscr{C}^{\mathrm{op}}$	The opposite (or dual) of a category $\mathscr{C}$. Analogously for functors and transformations.	Conv. 3	20
$\mathrm{Ob}(\mathscr{C})$	The set of objects in a category $\mathscr{C}$.	Conv. 4	20
$\mathrm{Mor}(\mathscr{C})$	The set of morphisms in a category $\mathscr{C}$.	Conv. 4	20
${}_{\mathscr{C}}(X, Y)$	The set of morphisms between objects X and Y in a category $\mathscr{C}$.	Conv. 4	20
$\mathrm{Inc}_{\mathscr{A}}^{\mathscr{C}} \colon \mathscr{A} \to \mathscr{C}$	The inclusion functor from a full subcategory $\mathscr{A}$ to a category $\mathscr{C}$.	Conv. 9	21
$F\vert_{\mathscr{A}}^{\mathscr{B}} \colon \mathscr{A} \to \mathscr{B}$	The restriction of a functor F to subcategories $\mathscr{A}$ and $\mathscr{B}$. Analogously for transformations.	Conv. 13	21
$\mathrm{Z}_{\mathscr{A}}$	The subcategory of zero objects in an additive category $\mathscr{A}$.	1.2.1	26
$\mathscr{A}/\mathfrak{I}$	The factor category of an additive category $\mathscr{A}$ modulo an ideal $\mathfrak{I}$.	1.2.13	31
$\mathfrak{R}_{\mathscr{A},\mathfrak{I}} \colon \mathscr{A} \to \mathscr{A}/\mathfrak{I}$	The residue class functor associated to a factor category $\mathscr{A}/\mathfrak{I}$.	1.2.13	31
$\mathrm{KCP}(\mathscr{A})$	The category of kernel-cokernel-pairs in an additive category $\mathscr{A}$.	1.3.1	34
$\mathscr{E}_{\mathrm{k}}$, $\mathscr{E}_{\mathrm{c}}$	The sets of kernels (resp. cokernels) in kernel-cokernel-pairs in $\mathscr{E} \subseteq \mathrm{KCP}(\mathscr{A})$.	1.3.1	34
$\overset{\bullet}{\rightarrowtail}$, $\overset{+}{\twoheadrightarrow}$	A pure monomorphism (resp. pure epimorphism) in an exact category.	1.3.2	35
$\mathscr{E}\vert_{\mathscr{B}}$	The restricted exact structure of an exact category $(\mathscr{A}, \mathscr{E})$ on a full additive subcategory $\mathscr{B}$.	1.3.21	41
$\underline{\mathscr{F}}$	The stable category of a Frobenius category $\mathscr{F}$.	1.3.27	44
$\mathfrak{P}_{\mathscr{F}} \colon \mathscr{F} \to \underline{\mathscr{F}}$	The stabilisation functor of a Frobenius category $\mathscr{F}$.	1.3.27	44
$\mathscr{C}(\mathscr{A})$	The category of functors from a category $\mathscr{C}$ to a category $\mathscr{A}$.	1.4.1	45

$\mathscr{C}(F)\colon \mathscr{C}(\mathscr{A}) \to \mathscr{C}(\mathscr{B})$	The pointwise application of a functor $F\colon \mathscr{A} \to \mathscr{B}$ to the functor categories with source category $\mathscr{C}$. Analogously for transformations.	1.4.3	46
$\mathrm{CT}_{\Sigma}(\mathscr{A})$	The category of candidate triangles in an additive category $\mathscr{A}$ with respect to a functor $\Sigma\colon \mathscr{A} \to \mathscr{A}$.	1.5.1	52
$\mathop{\ast}\limits_{k\in[a,b]} \mathscr{S}_k$	The category of extensions of full subcategories $\mathscr{S}_k$ in a strict triangulated category.	1.5.6, 1.5.7	54
$\mathscr{C}/\!\!/\mathscr{S}$	The Verdier quotient of a strict triangulated category $\mathscr{C}$ by a full triangulated subcategroy $\mathscr{S}$.	1.5.11	55
$\mathfrak{L}_{\mathscr{C},\mathscr{S}}\colon \mathscr{C} \to \mathscr{C}/\!\!/\mathscr{S}$	The quotient functor (or localisation functor) associated to the Verdier quotient $\mathscr{C}/\!\!/\mathscr{S}$.	1.5.11	55
$\mathrm{T}^{\mathscr{T}}_{k]}\colon \mathscr{C} \to \mathscr{T}_{k]}$, $\mathrm{T}^{\mathscr{T}}_{[k}\colon \mathscr{C} \to \mathscr{T}_{[k}$	The truncation functors of a t-structure $\mathscr{T}$ on a strict triangulated category $\mathscr{C}$.	1.8.2	65
$\mathrm{H}_{\mathscr{T}}\colon \mathscr{C} \to \mathscr{T}_{[0,0]}$	The homology functor of a t-structure $\mathscr{T}$ on a strict triangulated category $\mathscr{C}$.	1.8.2	65
$\mathrm{C}(\mathscr{A})$	The category of complexes with entries in an additive category $\mathscr{A}$.	1.9.2	66
$\mathrm{B}_{\mathrm{C},\mathscr{A}}\colon \mathrm{C}(\mathscr{A}) \to \mathrm{C}(\mathscr{A})$	The functor that acts on objects as the mapping cone of the identity morphism.	1.9.7	67
$\iota_{\mathrm{C},\mathscr{A}}\colon 1_{\mathrm{C}(\mathscr{A})} \to \mathrm{B}_{\mathrm{C},\mathscr{A}}$, $\pi_{\mathrm{C},\mathscr{A}}\colon \mathrm{B}_{\mathrm{C},\mathscr{A}} \to \Sigma_{\mathrm{C},\mathscr{A}}$	The transformations that yield the fundamental pure short exact sequences $(X\iota_{\mathrm{C},\mathscr{A}}, X\pi_{\mathrm{C},\mathscr{A}})$ in the strict Frobenius category $\mathrm{C}(\mathscr{A})$.	1.9.9	68
$\mathrm{K}(\mathscr{A})$	The homotopy category of an additive category $\mathscr{A}$.	1.9.12	70
$\mathrm{I}_{\mathrm{C},\mathscr{A}}\colon \mathscr{A} \to \mathrm{C}(\mathscr{A})$, $\mathrm{I}_{\mathrm{K},\mathscr{A}}\colon \mathscr{A} \to \mathrm{K}(\mathscr{A})$	The canonical inclusion functors of an additive category $\mathscr{A}$.	1.9.25	75
$X\mathbb{S}^{\mathscr{A}}_{[n}$, $X\mathbb{S}^{\mathscr{A}}_{n+1]}$	The simple/hard/brutal/naive/stupid truncations of a complex X.	1.9.26	75
$\mathcal{W}^{\mathscr{A}}$	The standard w-structure on the homotopy category $\mathrm{K}(\mathscr{A})$ of an additive category $\mathscr{A}$.	1.9.31	77
$\mathrm{Ac}(\mathscr{A})$	The full subcategory of acyclic complexes in the homotopy category $\mathrm{K}(\mathscr{A})$ of an abelian category $\mathscr{A}$.	1.9.33	78
$\mathrm{D}(\mathscr{A})$	The derived category of an abelian category $\mathscr{A}$.	1.9.33	78
$\mathrm{L}_{\mathscr{A}}\colon \mathrm{K}(\mathscr{A}) \to \mathrm{D}(\mathscr{A})$	The quotient functor associated to the Verdier quotient $\mathrm{D}(\mathscr{A}) = \mathrm{K}(\mathscr{A})/\!\!/\mathrm{Ac}(\mathscr{A})$.	1.9.33	78
$\mathrm{I}_{\mathrm{D},\mathscr{A}}\colon \mathscr{A} \to \mathrm{D}(\mathscr{A})$	The canonical inclusion functor of an abelian category $\mathscr{A}$.	1.9.33	78

$\mathbb{H}_{\mathscr{A}} \colon \mathrm{D}(\mathscr{A}) \to \mathscr{A}$	The zeroth homology functor of an abelian category $\mathscr{A}$.	1.9.33	78
$X\mathbb{T}^{\mathscr{A}}_{n+1\rfloor}$, $X\mathbb{T}^{\mathscr{A}}_{\lfloor n}$	The canonical/soft/smart/intelligent truncations of a complex X.	1.9.34	79
$\mathcal{T}^{\mathscr{A}}$	The standard t-structure on the derived category $\mathrm{D}(\mathscr{A})$ of an abelian category $\mathscr{A}$.	1.9.35	79
$\mathcal{W}^{\mathscr{A},\mathrm{inj}}$	The standard injective w-structure on the derived category $\mathrm{D}(\mathscr{A})$ of an abelian category $\mathscr{A}$ with enough DG-injectives.	1.9.53	83
$\mathcal{W}^{\mathscr{A},\mathrm{proj}}$	The standard projective w-structure on the derived category $\mathrm{D}(\mathscr{A})$ of an abelian category $\mathscr{A}$ with enough DG-projectives.	1.9.54	84
$\mathscr{B}^{\mathrm{ac}}$	The category of acyclic complexes with entries in the subcategory of bijective objects of a Frobenius category $\mathscr{F}$.	2.3.1	114
$I \colon \mathscr{B}^{\mathrm{ac}} \to \mathscr{F}$	The image functor of a Frobenius category $\mathscr{F}$.	2.3.5	115
$\mathrm{F}(\mathscr{A})$	The category of filtrations in an exact category $\mathscr{A}$.	3.2.1	122
$\mathrm{CF}(\mathscr{A})$	The category of cofiltrations in an exact category $\mathscr{A}$.	3.2.1	122
$f\!\upharpoonright \colon A \to B$	The induced morphism between a compatible family A and a limit B of a morphism f in a category of cofiltrations.	3.2.22	127
$f\!\upharpoonright \colon A \to B$	The induced morphism between a colimit A and a compatible family B of a morphism f in a category of filtrations.	3.2.23	127
V	The underlying poset of ∇-diagrams.	3.3.1	138
$\nabla(\mathscr{F})$	The category of ∇-diagrams in a strict Frobenius category $\mathscr{F}$.	3.3.2	138
$\Psi_{\ell,\mathrm{CF},\mathscr{F}} \colon \nabla(\mathscr{F}) \to \mathrm{CF}(\mathscr{F})$	The projection to the ℓ-th cofiltration of a ∇-diagram in a strict Frobenius category $\mathscr{F}$.	3.3.49	161
$\Psi_{k,\mathrm{F},\mathscr{F}} \colon \nabla(\mathscr{F}) \to \mathrm{F}(\mathscr{F})$	The projection to the k-th filtration of a ∇-diagram in a strict Frobenius category $\mathscr{F}$.	3.3.50	162
Ω	The underlying poset of filtered objects.	3.4.1	167
$\mathrm{FO}(\mathscr{F})$	The category of filtered objects in a strict Frobenius category $\mathscr{F}$.	3.4.2	167
$\mathrm{P}_{\omega,\mathscr{F}} \colon \mathrm{FO}(\mathscr{F}) \to \mathscr{F}$	The projection to the object with index ω of a filtered object in a strict Frobenius category $\mathscr{F}$.	3.4.4	168

$\mathrm{P}_{\mathrm{CF},\mathscr{F}}\colon$ $\mathrm{FO}(\mathscr{F}) \to \mathrm{CF}(\mathscr{F})$	The projection to the cofiltration of a filtered object in a strict Frobenius category $\mathscr{F}$.	3.4.5	169
$\mathrm{P}_{\mathrm{F},\mathscr{F}}\colon$ $\mathrm{FO}(\mathscr{F}) \to \mathrm{F}(\mathscr{F})$	The projection to the filtration of a filtered object in a strict Frobenius category $\mathscr{F}$.	3.4.6	169
$\underline{\mathrm{P}}_{\omega,\mathscr{F}}\colon$ $\underline{\underline{\mathrm{FO}}}(\mathscr{F}) \to \mathscr{F}$	The functor induced by $\mathrm{P}_{\omega,\mathscr{F}}$ on factor categories.	3.4.16	171
$\mathrm{E}_{\mathscr{F}}\colon$ $\mathscr{F} \to \mathrm{FO}(\mathscr{F})$	The embedding functor of a strict Frobenius category $\mathscr{F}$.	3.4.65	179
$\underline{\mathrm{E}}_{\mathscr{F}}\colon$ $\underline{\mathscr{F}} \to \underline{\underline{\mathrm{FO}}}(\mathscr{F})$	The functor induced by the embedding functor on factor categories.	3.4.67	179
$\Delta_{\mathscr{S},\mathscr{F}}\colon$ $\nabla_{\mathscr{S}}(\mathscr{F}) \to \mathrm{C}(\mathscr{S})$	The delta functor of a stricly full additive subcategory $\mathscr{S}$ with respect to a strict Frobenius category $\mathscr{F}$.	4.1.1	182
$\underline{\Delta}_{\mathscr{S},\mathscr{F}}\colon$ $\underline{\nabla}_{\mathscr{S}}(\mathscr{F}) \to \mathrm{C}(\mathscr{S})$	The functor induced by the delta functor on a factor category.	4.1.10	193
$\underline{\underline{\Delta}}_{\mathscr{S},\mathscr{F}}\colon$ $\underline{\underline{\nabla}}_{\mathscr{S}}(\mathscr{F}) \to \mathrm{K}(\mathscr{S})$	The functor induced by the delta functor on factor categories.	4.1.15	194
$\mathrm{R}_{\mathscr{H},\mathscr{F}}\colon$ $\mathrm{C}(\mathscr{H}) \to \underline{\nabla}_{\mathscr{H}}(\mathscr{F})$	A quasi-inverse of the functor $\underline{\Delta}_{\mathscr{H},\mathscr{F}}$.	4.1.20	195
$\underline{\mathrm{R}}_{\mathscr{H},\mathscr{F}}\colon$ $\mathrm{K}(\mathscr{H}) \to \underline{\underline{\nabla}}_{\mathscr{H}}(\mathscr{F})$	A quasi-inverse of the functor $\underline{\underline{\Delta}}_{\mathscr{H},\mathscr{F}}$.	4.1.20	195
$\Xi_{\mathscr{F}}\colon$ $\mathrm{FO}(\mathscr{F}) \to \nabla(\mathscr{F})$	The filtered cokernel functor of a strict Frobenius category $\mathscr{F}$.	4.2.11	202
$\underline{\Xi}_{\mathscr{F}}\colon$ $\underline{\underline{\mathrm{FO}}}(\mathscr{F}) \to \underline{\underline{\nabla}}(\mathscr{F})$	The functor induced by the filtered cokernel functor on factor categories.	4.2.31	209
$\mathrm{Lim}_{\mathscr{F}}\colon$ $\nabla(\mathscr{F}) \to \mathrm{FO}^{\nabla}(\mathscr{F})$	The limit functor obtained as a quasi-inverse of $\Xi^{\nabla}_{\mathscr{F}}$.	4.2.51	214
$\underline{\mathrm{Lim}}_{\mathscr{F}}\colon$ $\underline{\underline{\nabla}}(\mathscr{F}) \to \underline{\underline{\mathrm{FO}}}^{\nabla}(\mathscr{F})$	The limit functor obtained as a quasi-inverse of $\underline{\Xi}^{\nabla}_{\mathscr{F}}$.	4.2.51	214
$\mathrm{W}_{\mathscr{W},\mathscr{F}}\colon$ $\mathscr{D} \to \underline{\underline{\mathrm{FO}}}_{\mathscr{W}}(\mathscr{F})$	The weight equivalence of a weight structure $\mathscr{W}$ with respect to a strict Frobenius category $\mathscr{F}$.	4.3.34	233
$\mathrm{Res}_{\mathscr{W},\mathscr{F}}\colon$ $\mathscr{D} \to \mathrm{K}(\mathscr{C})$	The resolution functor of a w-structure $\mathscr{W}$ with respect to a strict Frobenius category $\mathscr{F}$.	4.3.36	233
$\mathrm{Real}^{\mathrm{K}}_{\mathscr{T},\mathscr{F}}\colon$ $\mathrm{K}(\mathscr{H}) \to \underline{\mathscr{F}}$	The preliminary version of the realisation functor on the level of the homotopy category.	4.4.2	246
$\mathrm{Real}_{A,\mathscr{T},\mathscr{F}}\colon$ $\mathrm{D}(\mathscr{H}) \to \mathscr{D}$	The realisation functor of a t-structure $\mathscr{T}$ with respect to a functor A and a strict Frobenius category $\mathscr{F}$.	4.4.24	259

$\mathrm{Res}^{\mathrm{inj}}_{\mathscr{W},\mathscr{A}}\colon \mathrm{D}(\mathscr{A}) \to \mathrm{K}(\mathscr{C})$	The injective resolution functor of a w-structure $\mathscr{W}$ with core $\mathscr{C}$ over an abelian category $\mathscr{A}$.	4.6.1	267
$\mathrm{Real}^{\mathrm{inj}}_{\mathscr{T},\mathscr{A}}\colon \mathrm{D}(\mathscr{H}) \to \mathrm{D}(\mathscr{A})$	The injective realisation functor of a non-degenerate t-structure $\mathscr{T}$ with heart $\mathscr{H}$ over an abelian category $\mathscr{A}$.	4.6.1	267
$\mathrm{Res}^{\mathrm{proj}}_{\mathscr{W},\mathscr{A}}\colon \mathrm{D}(\mathscr{A}) \to \mathrm{K}(\mathscr{C})$	The projective resolution functor of a w-structure $\mathscr{W}$ with core $\mathscr{C}$ over an abelian category $\mathscr{A}$.	4.6.3	269
$\mathrm{Real}^{\mathrm{proj}}_{\mathscr{T},\mathscr{A}}\colon \mathrm{D}(\mathscr{H}) \to \mathrm{D}(\mathscr{A})$	The projective realisation functor of a non-degenerate t-structure $\mathscr{T}$ with heart $\mathscr{H}$ over an abelian category $\mathscr{A}$.	4.6.3	269
$\mathrm{Coprod}_{\mathscr{A}}(X)$	The full subcategory of an additive category $\mathscr{A}$ whose objects are summands of a coproduct of copies of an object $X \in \mathrm{Ob}(\mathscr{A})$.	5.1.1	271
$\mathrm{Prod}_{\mathscr{A}}(X)$	The full subcategory of an additive category $\mathscr{A}$ whose objects are summands of a product of copies of an object $X \in \mathrm{Ob}(\mathscr{A})$.	5.1.1	271
$\mathscr{T}^S$	The t-structure associated to a (co)silting object S.	5.1.2	272
$\mathscr{H}_S$	The heart of the t-structure associated to a (co)silting object S.	5.1.2	272
H_S	The homology functor of the t-structure associated to a (co)silting object S.	5.1.2	272
$\mathscr{W}^S$	The w-structure associated to a w-(co)silting object S.	5.1.8	273
$\mathscr{C}_S$	The core of the w-structure associated to a w-(co)silting object S.	5.1.8	273
$\mathrm{Real}^{\mathrm{cos}}_{S,\mathscr{A}}\colon \mathrm{D}(\mathscr{H}_S) \to \mathrm{D}(\mathscr{A})$	The cosilting realisation functor associated to a w-cosilting object S in a derived category $\mathrm{D}(\mathscr{A})$.	5.1.17	275
$\mathrm{Res}^{\mathrm{cos}}_{S,\mathscr{A}}\colon \mathrm{D}(\mathscr{A}) \to \mathrm{D}(\mathscr{H}_S)$	The cosilting resolution functor associated to a w-cosilting object S in a derived category $\mathrm{D}(\mathscr{A})$.	5.1.17	275
$\mathrm{Real}^{\mathrm{s}}_{S,\mathscr{A}}\colon \mathrm{D}(\mathscr{H}_S) \to \mathrm{D}(\mathscr{A})$	The silting realisation functor associated to a w-silting object S in a derived category $\mathrm{D}(\mathscr{A})$.	5.1.20	275
$\mathrm{Res}^{\mathrm{s}}_{S,\mathscr{A}}\colon \mathrm{D}(\mathscr{A}) \to \mathrm{D}(\mathscr{H}_S)$	The silting resolution functor associated to a w-silting object S in a derived category $\mathrm{D}(\mathscr{A})$.	5.1.20	275

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