Amplitude Equations for Boussinesq and Ginzburg-Landau-like Models

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Ich versichere hiermit, dass die vorliegende Dissertation bis auf explizit gekennzeichnete Stellen von mir selbständig und nur unter Zuhilfenahme der angegebenen Hilfsmittel verfasst wurde. Die Arbeit wurde bisher nicht zur Erlangung einer Qualifizierung oder eines Titels bei einer anderen akademischen Institution eingereicht.

I hereby declare that this thesis, with the exception of the explicitly marked parts, has been written by myself and only with the help of the specified aids. It has not previously been submitted to any other academic institution to obtain a degree or a qualification.

Stuttgart, 2019

List of Symbols

$B_r(x)$	The (open) ball of radius $r > 0$ around x in any metric space. 29
C^n	Class of n times continuously differentiable functions. 31
C_c^∞	Class of infinitely many times continuously differentiable functions with compact support. 31
$C(I, \mathrm{Hol}_\sigma)$	Class of continuous functions on $I = I(\sigma')$, $\sigma' < \sigma$ with values in Hol_σ. 33
$\mathscr{F}$	Fourier transform. 29
Hol_σ^2	Holomorphic functions $u : \mathcal{S}_\sigma \to \mathbb{C}^n$ with bounded $L^2(\mathcal{S}_\tau)$ for all $0 < \tau < \sigma$. 30
Hol_σ	Holomorphic functions $u : \mathcal{S}_\sigma \to \mathbb{C}^n$ with bounded supremum of the $L^2(\mathbb{R}^m)$ norm for every translation $\mathrm{i}y$ parallel to the real axis with $\lvert y\rvert < \sigma$. 30
H^s	Sobolev space $W^{s,2}$ for $s \in \mathbb{R}$. 30
$\mathrm{L}(X, Y)$	Space of linear continuous operators from X to Y. 30
L_s^p	Function space $L^p(\mathbb{R}, (1 + \lvert x\rvert^2)^s \,\mathrm{d}x)$ for $s \in \mathbb{R}$. 30, 31
$L_s^{p,b}$	Function space $L^p(\mathbb{R}, (1 + \lvert x\rvert^{ps}) \,\mathrm{d}x)$ for $s \in \mathbb{R}$. 30, 31
$L_s^{p,h}$	Function space $L^p(\mathbb{R}, \lvert x\rvert^{ps} \,\mathrm{d}x)$ for $s \in \mathbb{R}$. 30, 31
$L_{s,\sigma}^p$	Function space $L^p(\mathbb{R}, (1 + \lvert x\rvert^2)^{ps/2} e^{p\sigma\lvert x\rvert} \,\mathrm{d}x)$ for $s \in \mathbb{R}$. 30
M_f	Multiplication operator that acts by multiplication with function f. 30
$\mathcal{O}$	$\mathcal{O}$ symbol in asymtotic notation. 34
o	o symbol in asymtotic notation. 34
Ω	Ω symbol in asymtotic notation (as used in complexity theory). 34

$\mathfrak{E}_\psi$	Residual of vector ψ in some function space. 19
Θ	Θ symbol in asymtotic notation. 34
$W^{k,p}$	Sobolev space $W^{k,p}$ for $k \in \mathbb{N}_0$ and $p \in [1, \infty]$. 30
$X_{\sigma,s,b}$	Gevrey class Fourier restriction spaces for the KdV equation. 31
$\lesssim$	Inequality holds up to some constant $C > 0$. 34
$\pi(\alpha)$	Multiset that contains for $\alpha \in \mathbb{N}^n$ the value i α_i times for $i \in \{1, \ldots, n\}$. 116
$\lvert\cdot\rvert_p$	The p norm in $\mathbb{R}^n$ and $\mathbb{C}^n$. 29
$\sqsubset$	For a multiset Q we note $P \sqsubset Q$ if P is a partition of Q. 116
u^{*^n}	The convolution of u with itself for n times. 227
$\langle \cdot, \cdot \rangle$	Scalar product in inner product spaces. 34

Zusammenfassung

Amplitudengleichungen können zur Beschreibung approximativer Lösungen partieller Differentialgleichungen genutzt werden. Sie können auftauchen im Zusammenhang von Modulationen exakter (möglicherweise distributioneller) Lösungen partieller Differentialgleichungen, als bestimmende Gleichungen eines Ansatzes oder als (formale) bestimmende Gleichungen einer asymptotischen Entwicklung. Deshalb erscheinen Amplitudengleichungen in einer Vielzahl von Zusammenhängen in den Ingenieurswissenschaften, der Physik oder auch der Mathematik nichtlinearer partieller Differentialgleichungen, da sie in der Regel einfacher zu verstehen sind als nichtlineare partielle Differentialgleichungen. Die zentrale Frage ist immer, welcher Zusammenhang zwischen einer Lösung der Amplitudengleichung und einer Lösung des ursprünglichen Problems besteht. Oder anders formuliert, ob das Verhalten der Lösung der Amplitudengleichung im ursprünglichen Problem wiedergefunden werden kann.

Die stärkste Formulierung dieser Frage lautet: „Ist es richtig, dass die Lösung der Amplitudengleichung eine Lösung des ursprünglichen Problems für lange Zeit approximiert und dass die Approximation stabil bezüglich einer Störung der Anfangsdaten ist?" In den letzten Jahrzehnten wurde diese Frage für viele sehr unterschiedliche Gleichungen und Gleichungssysteme untersucht und in etlichen Fällen bejaht. Jedoch gab es auch Beispiele, bei denen eine formal korrekt hergeleitete Amplitudengleichung das Verhalten einer Lösung einer nichtlinearen partiellen Differentialgleichung nicht für lange Zeit korrekt beschreiben konnte. Einschränkend muss man hierzu anmerken, dass Gegenbeispiele im Wesentlichen für den Fall periodischer Randbedingungen bewiesen wurden.

In dieser Arbeit wird diese Frage im Falle vierer verschiedener semilinearer Originalgleichungen nachgegangen. Die ersten beiden Gleichungen im Kapitel 2 werden ähnlich zu Boussinesq-Gleichungen aus der Theorie der Wasserwellen sein,

die anderen beiden sind die Ginsburg-Landau-Gleichung in Kapitel 3 und eine nichtlinear gekoppelte Ginsburg-Landau-Gleichung in Kapitel 4. Die Amplitudengleichungen werden eine „Dreiwelleninteraktionsgleichung“, eine „Vierwelleninteraktionsgleichung“, die Korteweg-de Vries-Gleichung und eine Gleichung in Form einer Erhaltungsgleichung sein, die als „Whithams Gleichung“ bezeichnet werden wird.
Für diese Systeme und Amplitudengleichungen wird die vorige Fragestellung betrachtet werden. In Kapitel 2 wird der Fokus auf einem Gegenbeispiel ohne periodische Randbedingungen liegen, während in den Kapiteln 3 und 4 der Fokus auf dem Beweis der Korrektheit der Approximation liegen wird.
Da alle betrachteten Problemstellungen semilinearer Natur sind und das Hauptinteresse an Lösungen in Unterräumen des Hilbertraums L^2 liegen wird, werden Energieabschätzungen eine zentrale Rolle für die Abschätzung der Fehler zwischen approximativer Lösung und echter Lösung sowie für die Frage der Stabilität der Lösung spielen.

Summary

Amplitude equations can be used to describe approximate solutions to partial differential equations (PDEs). They can be the outcome of a (formal) asymptotic expansion, appear as modulation equations of exact solutions to PDEs, possibly in the sense of distributions, or they can be derived as governing equations of an ansatz. Amplitude equations can be found in many different contexts in physics, are common tools of engineers, and used in mathematics to study nonlinear PDEs, since they can often be easier understood than the nonlinear PDE. An essential question is the relation between the solution to the amplitude equation and a solution to the original problem. We can rephrase this statement and ask whether the behaviour of the solution to the amplitude equation can be recovered in solutions to the original problem.

The strongest form of this question is as follows. Is it true that the solution to the amplitude equation is an approximate solution to the original problem for large times and is this still true under perturbations of the initial data? This issue has been investigated for many different equations and systems of equations in the last decades. In many cases it turned out that the foregoing statement is true, but in some cases it turned out wrong. There are examples where a formally correctly derived amplitude equation has solutions that do not closely follow solutions to the original nonlinear PDE for large times. However, these examples usually require periodic boundary conditions, which is clearly not the most general or most natural setting.

In this thesis we will investigate this issue for four different semilinear problems. The first two equations, which will be treated in Chapter 2, are inspired by Boussinesq's equations for the water waves problem in shallow water. The other two are the Ginzburg-Landau equation that will be considered in Chapter 3 and in Chapter 4 we will consider a nonlinearly coupled Ginzburg-Landau equation.

We will study amplitude equations of different kinds. In Chapter 2 we will use the four-wave interaction and three-wave interaction equations as amplitude equations whereas in Chapter 3 the KdV equation will be the amplitude equation and in Chapter 4 we will see a sort of conservation law, called 'Whitham's equation', playing the role of the amplitude equation.
The focus of the investigation will be on the construction of a counterexample without periodic boundary conditions in Chapter 2, i.e. we will investigate arguments why the strongest formulation of the question might in general be wrong in a situation without periodic boundary conditions. On the other hand we will prove in Chapters 3 and 4 that solutions to the amplitude equations considered in those chapters really approximate true solutions to the original problem if the initial data are close.
Since all problems are of semilinear nature and the focus will be on solutions to the original problems in subspaces of the Hilbert space L^2, we will exploit energy estimates to control the error between an approximate solution and a true solution and for some stability arguments.

Chapter 1

Introductory Comments

We will investigate the question of approximation of solutions to evolution problems which are formulated in the language of PDEs in the following chapters. We will explain what is meant by this statement in Section 1.1 and Section 1.2 in a rather abstract way and give some concrete examples. The main question will always be whether a formal approximate solution is a good approximation to a true solution – or whether it fails. We will shed some light on this issue for three different situations in the Chapters 2, 3 and 4, which are not really connected to each other, except for the models considered in Chapters 3 and 4.

1.1 On Approximate Solutions and PDEs

In this work we will consider PDEs that are semilinear. We will in general write these PDEs in the form

$$0 = \mathcal{L}u + f(u)$$

for a linear operator $\mathcal{L} : D(\mathcal{L}) \subset X \to X$ and a nonlinear map $f : O \subset X \to X$. Since we want to study evolution problems and strict solutions, X will in general be of the form $C(I, Y_0) \cap C^1(I, Y_1)$ for a time interval $I \subset \mathbb{R}$ with non-empty interior and Banach spaces Y_0, Y_1. We will assume $0 \in I$ and that initial data are given on some manifold.

In most cases that will follow it is convenient to distinguish the time dependence and write the PDE in the form

$$\partial_t u(t) = \mathcal{L}u(t) + f(t, u(t)), \qquad\qquad u(0) = u_0, \tag{1.1}$$

with $\mathcal{L} : D(\mathcal{L}) \subset Y \to Y$ and $f : O \subset I \times Y \to Y$. Since most of the nonlinear parts that we will consider do not explicitly depend on time, we usually drop the time dependence of f.

Definition 1.1.1 (Strict solution)**.** *Let $I \subset \mathbb{R}$ be an interval with non-empty interior and* $\min I = 0$. *We call a function $u \in C\big(I, D(\mathcal{L})\big) \cap C^1(I, Y)$ satisfying* (1.1) *and $u(t) \in O$ for all $t \in I$ with $u(0) = u_0$ a strict solution to equation* (1.1)*.*

This definition is used for instance by Lunardi [62, cf. Definition 7.0.1(i)]. Whilst her book treats the parabolic case in essence, this definition is useful in general. We will not give a strict definition as to what we mean by an approximate solution. This will be different in every case considered and will be clear in the applications. But for an impression of the meaning of this expression, we state it for the abstract problem above as follows: an approximate solution ψ to equation (1.1) is a function in $C\big(I, D(\mathcal{L})\big) \cap C^1(I, Y)$ and $\psi(t) \in O$ for all $t \in I$ for which the error $R \coloneqq u - \psi$ is 'small' for some time interval $0 \ni \tilde{I} \subset I$. By small we mean (relatively) smaller than the approximation itself in the norm $\sup_{t\in\tilde{I}}\|R(t)\|_Y$ or $\sup_{t\in\tilde{I}}\|R(t)\|_{\tilde{Y}}$ for some subspace $\tilde{Y} \subset Y$, or – if $R(t)$ takes values in $\mathbb{R}^n$ or $\mathbb{C}^n$ respectively – at least pointwise, i.e. $\sup_{t\in\tilde{I}}\|R(t)\|_{L^\infty(\mathbb{R})}$. Since we only consider solutions that are continuous in time with respect to these norms, we will also use the notation $L^\infty(\tilde{I}, Y)$ etc. for these norms as they are equivalent in this case.
If $u \in C\big(I, D(\mathcal{L})\big) \cap C^1\big(I, Y\big)$ is a strict solution and $\psi \in C\big(I, D(\mathcal{L})\big) \cap C^1\big(I, Y\big)$, the error R has to satisfy the equation

$$\begin{aligned} \partial_t R &= \mathcal{L}R + f(R + \psi) + \mathcal{L}\psi - \partial_t\psi \\ &= \mathcal{L}R + f(R + \psi) - f(\psi) + \big(- \partial_t\psi + \mathcal{L}\psi + f(\psi)\big). \end{aligned} \tag{1.2}$$

We note that the last part, surrounded by brackets, does not depend on R and acts as an inhomogenous part in equation (1.2). This part could make the error grow rapidly and thus is of some importance. Therefore we define the residual as follows.

Definition 1.1.2 (Residual)**.** *Let $f : O \to X$ for a subset $O \subset X$ of a Banach space X and $\mathcal{L} : D(\mathcal{L}) \subset X \to X$ a linear operator. Assume $\psi \in D(\mathcal{L}) \cap O$. Then* *we define the residual of ψ for the equation*

$$0 = \mathcal{L}v + f(v), \qquad v \in D(\mathcal{L}) \cap O$$

by

$$\mathfrak{E}_\psi := \mathcal{L}\psi + f(\psi),$$

i.e. $\mathfrak{E}_\psi$ is the remainder that is left if we insert ψ into the equation.

If $\mathfrak{E}_\psi = 0$ for equation (1.1) – meaning ψ solves the PDE (1.1) but not necessarily with initial data $\psi(0) = u_0$ – then the error equation reduces to

$$\partial_t R = \mathcal{L}R + f(R + \psi) - f(\psi)$$

and would reduce further if f were a linear map. In the linear case the error will grow with time if and only if $R(0) \neq 0$.
Let us assume that $f \in C^2(O, Y)$ and for simplicity O open and convex. Then Taylor's formula holds and with the above assumptions we have

$$\begin{aligned} f(R + \psi) - f(\psi) &= Df(\psi)[R] + \frac{1}{2}D^2 f(\psi)[R, R] + \tilde{G}(R, \psi) \\ &= Df(\psi)[R] + G(R, \psi), \end{aligned}$$

where $Df(\psi)[R]$ is the Fréchet derivative of f in ψ applied to R, $D^2 f(\psi)[R, R]$ the second order Fréchet derivative, which is a bilinear form, applied to R in the sense of a quadratic form and $\tilde{G}$ is a second order remainder. Hence $\|G(R, \psi)\|_Y = \mathrm{o}(\|R\|_Y^2)$. Under these assumptions we can write (1.2) as

$$\partial_t R = \mathcal{L}R + Df(\psi)[R] + G(R, \psi) + \mathfrak{E}_\psi. \tag{1.3}$$

This means – from a formal point of view – that the error R is primarily driven by the linear part of equation (1.3) initially if $R(0)$ is small and $\mathfrak{E}_\psi$ vanishes or is sufficiently small. These observations lead to the following program for finding formal approximate solutions to equation (1.1) and the proof that they are approximate solutions.

1. Find $\psi \in C\big(I, D(\mathcal{L})\big) \cap C^1\big(I, Y\big)$ and $\psi(t) \in O$ for all $t \in I$ such that $\mathfrak{E}_\psi$ is small in the Y-norm (or whatever norm we want to consider).

2. Prove for this ψ that the growth of the error coming from the linear part of equation (1.3) is not too fast for small $R(0)$ (or $R(0) = 0$).

We have to stress here that results about approximate solutions are of no interest if $\tilde{I}$ – the interval in which the error remains smaller than ψ – is not large in some sense since strict solutions are continuous in time and hence any function $v \in C\big(I, D(\mathcal{L})\big) \cap C^1\big(I, Y\big)$ with $v(0) - u_0$ small and $v(t) \in O$ for $t \in I$ could be considered as an approximate solution for $t \to 0$.
On the other hand these considerations indicate why some formal approximation might not be a good approximate solution. Whilst $\mathfrak{E}_\psi$ should not be too large by construction, otherwise it wouldn't be a good formal approximation, the error could grow too quickly because of the linear part of (1.3). We will investigate some ideas about the success and failure of formal approximations in the following.
In the subsequent chapters, we will usually be in the fortunate situation that $D(\mathcal{L}) \subset O$ where $D(\mathcal{L})$ is a linear space. Hence, the additional condition $\psi(t) \in O$ and O convex is trivially satisfied if we choose $O = D(\mathcal{L})$. In this situation we simply drop any notion of O.
An example for equation (1.1) that one could have in mind is the Nonlinear Schrödinger Equation (NLS) in the Sobolev space $Y = H^s$, $s \geq 0$, and $D(\mathcal{L}) = H^{s+2}$

$$i\partial_t u = -\partial_x^2 u \pm |u|^2 u.$$

Another example, see Section 3.1, could be the complex Ginzburg-Landau equation in the same space

$$\partial_t u = (1 + \mathrm{i}\alpha)\partial_x^2 u + u - (1 + \mathrm{i}\beta) u|u|^2,$$

with real α, β. However, this does not fit into the above formulation in the following sense. If we use $f(u) = \gamma|u|^2 u$, $\gamma \in \mathbb{C}$, then f is not $C^1(H^{s+2}, H^s)$ (the point is just that $z \mapsto |z|^2 z$ is not holomorphic). But we can split real and imaginary parts and consider the equation as a real-valued system. Then we can calculate the Fréchet derivative. We will use such a splitting in Section 3.5.2 for example. But let us assume for simplicity that $\tilde{f} : \mathbb{C} \to \mathbb{C}$ is entire and $\tilde{f}(0) = 0$. Then the Fréchet derivative $Df(\psi)$ of the induced map f is given by the multiplication operator which acts by multiplication with $\partial_z \tilde{f}(\psi)$. Or more concrete for $f(u) = u^p$, $p \in \mathbb{N}$, we have $Df(\psi)[u] = p\psi^{p-1}u$. Obviously, such a construction is not limited to entire functions $\tilde{f}$ but works for functions that are holomorphic in a large enough region of the complex plane or a polydisk in $\mathbb{C}^n$.

***Remark* 1.1.3.**

- We stress that the designations of $\mathcal{L}$ and f are somewhat arbitrary in the sense that we can shift linear operators of lower order than the principal part between these. For example we can use $f(u) = -(1 + \mathrm{i}\beta)u|u|^2$ or $f(u) = u - (1 + \mathrm{i}\beta)u|u|^2$ in the Ginzburg-Landau equation above.

- This ambiguity allows us to redefine $\tilde{\mathcal{L}} = \mathcal{L} + Df(0)$ and $\tilde{f} = f - Df(0)$ if $f \in C^1$. Sometimes it is useful to have a nonlinear part with $Df(0) = 0$. We exploit this splitting in Section 3.5.2 or Section 4.4.2 for instance.

- One way to find formal approximate solutions ψ is by multiple-scale analysis. We give some hints and references concerning this idea in the next section.

- Note that we could use the mapping $u \mapsto \gamma(\psi^2\overline{u} + 2|\psi|^2u)$ to replace the linear operator $Df(\psi)$ which does not exist if $f(u) = \gamma|u|^2u$. It would serve our purpose as well as $Df(\psi)$ but it is neither linear nor antilinear.

1.2 On Formal Approximations: A Simple Example

There are many PDE systems and evolution problems in physics where larger and smaller scales are involved. For example in the water waves problem, cf. [58], the extent of the ocean or even some lakes is large compared to typical heights of waves. In the case of interactions of light waves and some bulk material the wavelength is much smaller than the grain size of the material in many situations. Or there are slow motions and fast motions involved. For instance the speed of light in a lightning strike is much faster than the sonic speed that limits when the thunder can be heard. Another example in the context of reaction-diffusion systems is chemical reactions where some reactions are much faster than others. Such examples can be found in the book of Kuehn, cf. [57, Chapters 18 and 20]. It can be useful to take the multiple scales into account and separate effects on different scales. In a simple case there is only one parameter $\epsilon \in \mathbb{R}^+$ and different powers of it involved. In such a situation it might be useful to make an ansatz that explicitly respects these different scales and results in some (simpler) equations for different parts of the ansatz. Then one can hope to construct formal approximate solutions with the ansatz. This procedure is called multiple-scale analysis.

Multiple-scale analysis is widely used for phenomena that involve waves, see e.g. [9, 19, 46, 68] or in a more abstract setting for hyperbolic problems [50]. The strategy is to have $0 < \epsilon \ll 1$ and then use an ansatz in form of an expansion in higher powers of ϵ and neglect the highest powers as small error, see [21, 22, 58] for some examples in the case of the water waves problem or [66, 76] and the references therein in the case of nonlinear optics. As a simple example we consider the following four-wave interaction ansatz (FWI for short)

$$\psi = \epsilon^a \sum_{i=1}^{4} A_i(\epsilon^b \cdot, \epsilon^c \cdot) e^{\mathrm{i}(\xi_i \boldsymbol{x} + \omega_i \boldsymbol{t})} + \epsilon^a \sum_{i=1}^{4} \overline{A_i(\epsilon^b \cdot, \epsilon^c \cdot) e^{\mathrm{i}(\xi_i \boldsymbol{x} + \omega_i \boldsymbol{t})}}. \tag{1.4}$$

One interpretation of this ansatz could be that it models four waves with (angular) frequencies $\omega_i \in \mathbb{R}$, wave numbers $\xi_i \in \mathbb{R}$ and amplitudes – actually they are rather shape factors – A_i. Here the size of the shape factors is in some relation to the extent of the profile and the speed of their propagation in powers of ϵ. It is convenient to define $A_{-i} = \overline{A_i}$ and $\omega_{-i} = -\omega_i$ as well as $\xi_{-i} = -\xi_i$ for $i \in \{1, \dots, 4\}$ since we are usually interested in real-valued solutions. We can imagine this ansatz to model flat waves on a large lake or we could imagine them to be four carrier modes with some amplitudes in an optical fibre, cf. [23] in the case of water waves on a bounded domain.

We assume that we have $(\xi_i, \omega_i) \neq (\xi_j, \omega_j)$ for all $i, j \in \{1, \dots, 4\}$ and $i \neq j$. Otherwise we would reduce the ansatz and remove the redundant amplitudes. Since ψ shall be an approximate solution, some regularity assumptions have to be satisfied. Hence, these assumptions have to be satisfied by A_i in general. We have to respect these restrictions when we try to seek equations to determine the A_is. The equations that determine the A_i are called *amplitude equations*. Further this leads to the denotation *'natural time scale'* for the class $[0, \Theta(\epsilon^{-b})]$, $\epsilon \to 0$, since, on the one hand, one cannot expect that the A_i exist globally in time and on the other hand – recall that they have to be continuous in time – for shorter time scales the results become trivial for small ϵ.

If we can solve the amplitude equations that determine the A_i, then the question at hand is whether the constructed formal approximate solution is a valid approximate solution in the sense that there is a useful error bound. In the given example with the FWI ansatz we can rephrase the question as follows. ‘If we have initial data consisting of (carrier) waves with some amplitudes for the original problem, is the solution to the original problem then close to the formal approximate solution constructed above on the natural time scale of the FWI ansatz?’

In some cases the answer to this question is rather trivial. For instance if $a \geq b$ and the formal approximate solutions has values in the Sobolev space $H^s \cap W^{s,\infty}$ for some large enough $s \in \mathbb{N}$ and is continuous in time on its natural time scale with respect to this space, then we can expect an approximation result under some conditions on the natural time scale if $\mathcal{L}$ in (1.3) is dissipative and the nonlinear part is induced by an entire function $\tilde{f} : \mathbb{C} \to \mathbb{C}$ with $\tilde{f}(0) = \partial_z \tilde{f}(0) = 0$. See Theorem 2.1.34 in Chapter 2 for such a result, where we will use energy estimates. If no energy estimates are available, one could still hope for such a result simply by a mild formulation and the construction of an evolution family in the sense of [37, Definition VI.9.2]. The construction of such an evolution family might be possible with a perturbation result, see [37, Theorem VI.9.19]. In the case $a < b$ this question is non-trivial, which will become more clear in Chapter 2.

As a concrete example let us assume $a = 1, b = 2, c = 2$ and let us consider the PDE

$$0 = \mathcal{L}(\partial_t, \partial_x)u + |u|^2 u \quad \text{in } \mathbb{R}_0^+ \times \mathbb{R}, \qquad u(0) = u_0, \qquad u_t(0) = u_{t,0} \text{ in } \mathbb{R},$$

where the linear operator is given by $\mathcal{L}(\partial_t, \partial_x) \coloneqq \partial_x^2 - \partial_x^4 - 1 - \partial_t^2$ and Y shall be a subspace of $L^2(\mathbb{R})$. As announced in the previous Section 1.1 the task at hand is to make $\mathfrak{E}_\psi$ as small as possible. Hence we consider

$$\begin{aligned}
0 &= \epsilon \sum_{i=1}^{4} e^{\mathrm{i}(\xi_i \boldsymbol{x} + \omega_i \boldsymbol{t})} \mathcal{L}(\epsilon^2 \partial_t + \mathrm{i}\omega_i, \epsilon^2 \partial_x + \mathrm{i}\xi_i) A_i \\
&\quad + \epsilon^3 \sum_{j,j',j'' \in I^*} e^{\mathrm{i}(\xi_j - \xi_{j'} + \xi_{j''})\boldsymbol{x} + \mathrm{i}(\omega_j - \omega_{j'} + \omega_{j''})\boldsymbol{t}} A_j A_{-j'} A_{j''} \\
&= \epsilon \sum_{i=1}^{4} e^{\mathrm{i}(\xi_i \boldsymbol{x} + \omega_i \boldsymbol{t})} \mathcal{L}(\epsilon^2 \partial_t + \mathrm{i}\omega_i, \epsilon^2 \partial_x + \mathrm{i}\xi_i) A_i \\
&\quad + \epsilon^3 \sum_{j,j',j'' \in I^*} e^{\mathrm{i}(\xi_j + \xi_{j'} + \xi_{j''})\boldsymbol{x} + \mathrm{i}(\omega_j + \omega_{j'} + \omega_{j''})\boldsymbol{t}} A_j A_{j'} A_{j''},
\end{aligned}$$

where $I^* = \{1, \ldots, 4\} \cup \{-1, \ldots, -4\}$. We can now use a Taylor expansion of $\mathcal{L}(\boldsymbol{\omega}, \boldsymbol{\xi})$ around $(\mathrm{i}\omega_i, \mathrm{i}\xi_i)$ for $i \in \{1, \ldots, 4\}$ to obtain a formal expansion in powers of ϵ. We define an (almost everywhere locally analytic) map $\omega : \mathbb{R} \to \mathbb{R}$ by the principal branch of the algebraic equation $\mathcal{L}(\mathrm{i}\omega, \mathrm{i}\xi) = 0$. We call the map ω the dispersion relation. It is obvious that the lowest order in our formal expansion in powers of ϵ vanishes if we take $\omega_i = \omega(\xi_i)$ for $i \in \{1, \ldots, 4\}$. In order ϵ^3 we obtain PDEs for A_i as coefficients to the exponentials $e^{\mathrm{i}(\xi_i \boldsymbol{x} + \omega_i \boldsymbol{t})}$:

$$
\begin{aligned}
0 &= \partial_{\mathrm{i}\omega}\mathcal{L}(\mathrm{i}\omega_i, \mathrm{i}\xi_i)\partial_t A_i + \partial_{\mathrm{i}\xi}\mathcal{L}(\mathrm{i}\omega_i, \mathrm{i}\xi_i)\partial_x A_i + 3\sum_{j=1}^{4} |A_j|^2 A_i + \sum_{\substack{j,j',j'' \in I^* \setminus \{i\} \\ \xi_j + \xi_{j'} + \xi_{j''} = \xi_i \\ \omega_j + \omega_{j'} + \omega_{j''} = \omega_i}} A_j A_{j'} A_{j''} \\
&= -2\mathrm{i}\omega_i \partial_t A_i + 2\mathrm{i}\xi_i(1 + 2\xi_i^2)\partial_x A_i + 3\sum_{j=1}^{4} |A_j|^2 A_i + \sum_{\substack{j,j',j'' \in I^* \setminus \{i\} \\ \xi_j + \xi_{j'} + \xi_{j''} = \xi_i \\ \omega_j + \omega_{j'} + \omega_{j''} = \omega_i}} A_j A_{j'} A_{j''}.
\end{aligned}
$$

This system is the amplitude equations that determine the A_i. If $\xi_j + \xi_{j'} + \xi_{j''} = \xi_i$ and $\omega_j + \omega_{j'} + \omega_{j''} = \omega_i$ for some $i, j, j', j'' \in \{1, \ldots, 4\}$ we call the system resonant, otherwise non-resonant.

This way we have almost $\|\mathfrak{E}_\psi\|_Y \leq \epsilon^{6-\frac{1}{2}}$. Except for all nonlinear terms that do not appear in the sum above. In some cases one can get rid of them by adding some higher order *'correctors'* to the ansatz, i.e. use

$$
\psi = \epsilon^a \sum_{i=1}^{4} A_i(\epsilon^b \cdot, \epsilon^c \cdot) e^{\mathrm{i}(\xi_i \boldsymbol{x} + \omega_i \boldsymbol{t})} + \epsilon^a \sum_{i=1}^{4} \overline{A_i(\epsilon^b \cdot, \epsilon^c \cdot) e^{\mathrm{i}(\xi_i \boldsymbol{x} + \omega_i \boldsymbol{t})}} + \mathcal{O}(\epsilon^{2a}).
$$

Then, under some conditions, one really obtains $\|\mathfrak{E}_\psi\|_Y \leq \epsilon^{6-\frac{1}{2}}$ and possibly one can iterate this procedure to obtain even higher order powers in ϵ. This will become more clear in Section 2.2.1 where we use a more elaborate notation for the ansatz that we will introduce in Section 2.1.2. This notation is precisely designed for that purpose. Unfortunately, this method has one drawback: it requires more regularity of the ansatz than expected initially. Hence, more regular initial data are necessary in general.

Obviously, we could use more (or fewer) addends in the sums. This way it is possible to define for $N \in \mathbb{N}$ the N-wave interaction ansatz (abbreviation NWI ansatz or NWI approximation).

In Chapters 3 and 4 we will consider a related ansatz. We will have a look at modulations of special complex-valued periodic travelling wave solutions in Chapter 3 and in Chapter 4 of modulations of a system of a periodic travelling wave and a constant. For convenience we will use an exponential formulation in these chapters, which means, in the scaling used in Chapter 4,

$$A_0 e^{\mathrm{i}(\xi_0 \boldsymbol{x}+\omega_0 \boldsymbol{t})} e^{r(\epsilon \boldsymbol{t},\epsilon \boldsymbol{x})+\mathrm{i}\phi(\epsilon \boldsymbol{t},\epsilon \boldsymbol{x})}.$$

These are related to the ansatz in equation (1.4) in the sense that we could consider it as a complex version – meaning we drop the second sum – where $A_i \equiv 0$ for $i > 1$. However, there are some differences in the process of the derivation of the formal approximate solution to the FWI case discussed above since $a = 0$ here. We give the details about that in the mentioned chapters.

1.3 Failure and Success of Amplitude Equations

We already indicated in Section 1.1 that there is no guarantee that a formal approximate solution is an approximate solution in the sense mentioned in that section. More generally, we are interested in *'approximation properties'* as follows.

***Approximation Property* 1.3.1.** Let ψ be a formal approximate solution with properties in accordance with those mentioned in Section 1.1 and u a strict solution to a PDE of the form (1.1) with initial data u_0.
Then there is $\epsilon_0 > 0$ such that for all $\epsilon \in]0, \epsilon_0[$ the condition

$$\|u_0 - \psi(0)\|_Y \lesssim \epsilon^p.$$

is preserved within the natural time scale, i.e.

$$\|u - \psi\|_{L^\infty(\tilde{I},Y)} \lesssim \epsilon^p$$

where $0 \in \tilde{I}$ covers the natural time scale that was introduced in Section 1.2.

***Remark* 1.3.2.**

- The condition on the initial data can be considered as a stability statement. This is much stronger than just the statement that ψ is an approximate solution for some initial data, see the discussions in Sections 2.1.5, 2.2.3

and 2.3. We should emphasise that the condition on the initial data must not be understood in the sense of holding uniformly in ϵ. In Chapters 3 and 4 in particular, the only function that would satisfy the condition uniformly is the zero function.

- As stated in the foregoing Section 1.1 we are not necessarily interested in the norm of Y but the norm of any space of interest where the solution and approximate solution make sense. This could be the Sobolev norm H^s or $W^{s,\infty}$, $s \in \mathbb{N}_0$, in the examples given in Section 1.1.

There is a vast amount of literature covering such results. We only mention a few examples. In the case of geometric optics and nonlinear optics for example, there are many results due to J.-L. Joly, G. Métivier, J. Rauch and others, cf. [66, 76] and the references therein. Some examples in the case of the water waves problem were already mentioned, for instance [21, 22] and in particular the monograph of Lannes [58] with all its references in Section 1.2. Further note that there are more abstract settings available, cf. [50, 81]. All these results cover equations and systems that are hyperbolic or dispersive in some sense. But such results exist for dissipative equations, too. For example reaction-diffusion systems and the complex Ginzburg-Landau equation are mentioned in the book of Schneider and Uecker, cf. [86].

Despite all these results showing the successful application of amplitude equations for the construction of approximate solutions, there is no guarantee that this method works. There are examples in literature where a formal approximate solution that is constructed via multiple-scale analysis does not satisfy an approximation property in the sense of Approximation Property 1.3.1 or it only holds under certain restrictions. In [80] Schneider proved an approximation property for a formal approximate solution constructed by the Newell-Whitehead equation, see [80, Theorem 1]. But in his result the solution to the Newell-Whitehead equation has to be in a certain Banach space that requires that the solution is holomorphic in the Cartesian product of two strips around the real axis in the complex plane. Schneider explains in Section 4 of his article that this constraint cannot be removed.

Other examples in a similar direction are [83–85]. Schneider introduced in [83] an idea how one might prove that an approximation property does not hold for periodic boundary conditions. In essence the construction relies on the fact that ‘resonant modes’ in the three-wave interaction (TWI) system can grow quickly, see Section 2.1.4 below, too. Then he argues that solutions to the original system for specific initial data remain close to the solution constructed by the TWI system and that this is not true for the formal approximate solution constructed with the aid of the NLS equation, which was introduced in Section 1.1. But the arguments in that article remained on a formal level. Later this idea was used in [84, 85] to prove that formal approximate solutions constructed with the aid of the NLS equation or a four-wave interaction (FWI) system cannot approximate solutions to a water waves model in general. We should point out that all these counterexample were discussed and proved in the context of periodic boundary conditions and some of the techniques used exploit the fact that a representation of the solutions in Fourier series is possible.

The message of this section is that one has to be careful. One can hope but should not expect that formal approximate solutions constructed by multiple-scale analysis and amplitude equations result in approximation properties in the spirit of Approximation Property 1.3.1. We will come back to this question in Chapter 2.

1.4 Outline and Results

Recall that we will study three different problems in the next three chapters. These problems have in common that we will consider formal approximate solutions constructed with the aid of amplitude equations and that we will investigate the question whether an approximation property for these formal approximate solutions is true.

In Chapter 2 we will follow the path lead by Schneider et al. in [84]. We will consider two different Boussinesq-like equations and for each of them we will derive amplitude equations for two formal approximate solutions (the FWI and TWI system). Since we will seek approximate solutions in the Sobolev space $H^s(\mathbb{R})$, $s \in \mathbb{N}$, we have to discuss properties of these solutions to the amplitude equations. In particular, we will discuss the regularity of the solutions, which is necessary because of the choice of the definition of ‘approximate solution’ that we made before. We will prove global existence and uniqueness of solutions to the FWI and TWI system in some Sobolev spaces under certain conditions, see Theorems 2.1.6 and 2.1.16. Then we will consider some completely explicit examples of the formal

approximate solutions. We will see that they differ after some time has passed and at most one of them can remain close to a true solution to the original system. This is a first step in the direction of a counterexample without periodic boundary conditions.
Then one needs some error estimates for the completion of the proof of the counterexample. Under certain assumptions, we will prove such error estimates between approximate solutions and true solutions for a class of nonlinear PDEs that contains the two aforementioned Boussinesq-like models, see Theorems 2.1.28 and 2.2.5. We will demonstrate in Section 2.2.3 how these theorems can be used to prove that an approximation property cannot be true. Unfortunately, it is not clear at the moment whether the assumptions of Theorems 2.1.28 or 2.2.5 can be satisfied by the formal approximate solution constructed with the aid of the TWI system. This point remains open and should be subject to future research.
Finally, we will see an example where an approximation property is trivially obtained if certain restrictions are met, see Theorem 2.1.34. The sole purpose of this example is to illustrate the arguments of Section 1.2.
In Chapter 3 we will consider the Korteweg-de Vries (KdV) equation as an amplitude equation to modulations of travelling wave solutions. Van Harten formally derived in [91] that modulations of travelling wave solutions to the complex Ginzburg-Landau equation are governed by the KdV equation. Since he remained on a formal level and gave no error estimate we will do so at least for initial data that are holomorphic on a (pretty large) strip of the complex plane. More precisely, we prove in Theorems 3.5.2 and 3.5.3 that after a 'change of variables' in the Ginzburg-Landau equation an approximation property in the spirit of Approximation Property 1.3.1 holds for the formal approximate solution constructed with the aid of the KdV equation on the natural time scale of the formal approximate solution, if we assume initial data which are holomorphic on a strip in the complex plane containing the real axis. However, the 'change of variables' leads to some difficulties for the derivation of an error bound in the original variables. We will take a quick look at this problem at the end of Section 3.6 where we will give two kinds of error estimate in the original variables. Further we will point out how one could possibly obtain another counterexample in the foregoing sense.
The last chapter is somewhat similar to Chapter 3. We will consider the Ginzburg-Landau equation in Chapter 4 again but this time nonlinearly coupled to another equation. The program for this original system is similar to the one in Chap-

ter 3. Once more, we will derive a system of amplitude equations. This system of PDEs, called 'Whitham's system', has the form of a conservation law. We will show that it possesses solutions being holomorphic in space for holomorphic initial data, see Theorem 4.2.2. Then we will prove in Theorems 4.4.2 and 4.4.3 that – again after a 'change of variables' – an approximations property in the spirit of Approximation Property 1.3.1 holds for the formal approximate solution constructed with the aid of Whitham's system on the natural time scale of the formal approximate solution if we assume holomorphic initial data. The problem and results concerning the reconstruction in the original variables are the same as in the case of the Ginzburg-Landau equation before.

1.5 Notation

We will use a more or less common notation for sets, function spaces and other symbols. Nonetheless we collect most of the notation here and introduce some conventions that we will tacitly use within the following chapters.

The sets $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ are as usual the natural numbers (without 0), the integers, the real and complex numbers. $\mathbb{R}^+$ are the positive real numbers. A subscript 0 indicates the union of the set itself and the set $\{0\}$, e.g. $\mathbb{R}_0^+$ are the non-negative real numbers. $\mathbb{R}^\times, \mathbb{C}^\times$ denote the set of units in $\mathbb{R}, \mathbb{C}$, i.e. $\mathbb{R}\backslash\{0\}$ and $\mathbb{C}\backslash\{0\}$. The set $\mathbb{C}^-$ and $\mathbb{C}^+$ is the set of complex numbers with real part in $\mathbb{R}^-$ and $\mathbb{R}^+$ respectively. Therefore $\mathbb{R}^- \subset \mathbb{C}^-$ and $\mathbb{R}^+ \subset \mathbb{C}^+$.

$|\cdot|_p$, $1 \leq p \leq \infty$, denotes the p norm in $\mathbb{R}^n$ and $\mathbb{C}^n$. $B_r(x)$ denotes in any metric space the (open) ball of radius $r > 0$ around x.

As a convention we will always consider the first argument of any function that depends on time as the time variable. This means that the (partial) time derivative ∂_t denotes the derivative with respect to first argument of the function. Sometimes we will write ∂_1 to avoid confusion.

Bold mathematical symbols will denote mappings with respect to the symbol. For instance $\boldsymbol{x}$ will denote the map $x \mapsto x$ and $e^{\mathrm{i}(\boldsymbol{x}+\boldsymbol{t})}$ is the map $(t,x) \mapsto e^{\mathrm{i}(x+t)}$. If the domain of the mappings is clear from context we will not state it explicitly.

The operator $\mathscr{F} : L^2 \to L^2$, the Fourier transform, denotes, as usual, the unique continuous extension of the operator defined for every Schwartz class function u

by

$$u \mapsto \mathscr{F} u = \hat{u} := \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{\mathbb{R}^m} u(x) e^{-\mathrm{i}x\cdot} \,\mathrm{d}x.$$

Therefore it is an isometric isomorphism and the inverse operator is denoted by $\mathscr{F}^{-1}$. Further we denote the set of linear continuous operators from X to Y by $\mathrm{L}(X,Y)$ for any two Banach spaces X, Y and the operator norm accordingly. If $X = Y$ then we write $\mathrm{L}(X)$ only. For a function f we denote by $M_f : D(M_f) \subset X \to Y$ the multiplication operator that acts by multiplication with f between Banach spaces X, Y, if such an operation makes sense.
We will use some function spaces and several relations between them.

Definition 1.5.1. *Let $\Omega \subset \mathbb{R}$ and $\mathcal{S}_\sigma = \{z \in \mathbb{C} \mid |\mathrm{Im} z| < \sigma\}$. If $\Omega = \mathbb{R}^m$, $m \in \mathbb{N}$, we omit the set.*

- $L^p_s(\Omega) := L^p(\Omega, (1+|x|^2)^{ps/2} \,\mathrm{d}x)$ *for $s \in \mathbb{R}$ and $p \in [1, \infty[$.*
- $L^{p,b}_s(\Omega) := L^p(\Omega, (1+|x|^{ps}) \,\mathrm{d}x)$ *for $s \in \mathbb{R}$ and $p \in [1, \infty[$.*
- $L^{p,h}_s(\Omega) := L^p(\Omega, |x|^{ps} \,\mathrm{d}x)$ *for $s \in \mathbb{R}$ and $p \in [1, \infty[$.*
- $H^s(\Omega) := \big\{u \in L^2(\Omega) \,\big|\, \|u\|_{H^s(\Omega)} < \infty\big\}$ *and* $\|u\|^2_{H^s(\Omega)} = \sum_{l=0}^{s} \|\partial^l_x u\|^2_{L^2(\Omega)}$.
- $W^{k,p}(\Omega)$ *is the usual Sobolev space of k times weakly differentiable functions whose derivatives are in L^p, $p \in [1, \infty[$.*
- $W^{k,\infty}(\Omega)$ *is the usual Sobolev space of k times weakly differentiable functions u for which the norm $\|u\|_{W^{k,\infty}(\Omega)} = \max_{l \in \{0,\dots,k\}} \|\partial^l_x u\|_{L^\infty(\Omega)}$ is bounded.*
- $L^p_{s,\sigma}(\Omega) := L^p(\Omega, (1+|x|^2)^{ps/2} e^{p\sigma|x|} \,\mathrm{d}x)$ *for $s \in \mathbb{R}$ and $p \in [1, \infty[$.*

Further we define

$$\mathrm{Hol}_\sigma := \Big\{u : \mathcal{S}_\sigma \to \mathbb{C}^n \text{ holomorphic} \Big| \; n \in \mathbb{N}, \text{ and for all } \tau < \sigma \text{ holds } \sup_{|y| \le \tau} \|u(\cdot + \mathrm{i}y)\|_{L^2} < \infty \Big\},$$

$$\mathrm{Hol}^2_\sigma := \Big\{u : \mathcal{S}_\sigma \to \mathbb{C}^n \text{ holomorphic} \Big| \; n \in \mathbb{N}, \text{ and for all } 0 < \tau < \sigma \text{ holds } \|u\|_{L^2(\mathcal{S}_\tau)} < \infty \Big\}.$$

In general we will not distinguish between $\mathbb{R}^n$ or $\mathbb{C}^n$ valued functions for any $n \in \mathbb{N}$ in the notation except for the case where it would lead to confusion. We define $L^p_\infty(\Omega) = \bigcap_{s\geq 0} L^p_s(\Omega)$ and $H^\infty(\Omega) = \bigcap_{s\geq 0} H^s(\Omega)$. Further we denote by $C^n(\Omega, \mathbb{C}^k)$ the class of n times continuously differentiable functions that map Ω into $\mathbb{C}^k$ and we denote the class of infinitely many times continuously differentiable functions with compact support with $C^\infty_c(\Omega, \mathbb{C}^k)$ (and for real-valued functions accordingly).

Further we will use the rather particular space $X_{\sigma,s,b}$ in Chapter 3. It is defined as the closure of the Schwartz class under the norm

$$\|u\|^2_{X_{\sigma,s,b}} = \int_{\mathbb{R}^2} (1 + |\tau - \xi^3|)^{2b} \Lambda^{2s} e^{2\sigma\Lambda} |\hat{u}(\tau, \xi)|^2 \mathrm{d}(\xi, \tau)$$

where $\Lambda = (1 + |\xi|)$ and $\hat{u}$ is the space-time Fourier transform of u. We define this space for $(\xi, \tau) \in \mathbb{R}^2$ only and usually $b \in [-1, 1]$.
It is appropriate to give a few comments concerning these spaces.

***Remark* 1.5.2.**

- For $p = 2$ the spaces $L^{p,b}_s, L^p_s$ are Hilbert, for other p they are Banach. The scalar product is obvious. The space $L^{p,h}_s$ is a little bit special but we do not need many properties of it and will not discuss it further.

- For all $s \in \mathbb{R}$ we have the continuous embedding $L^{p,b}_s \hookrightarrow L^{p,h}_s$ and for all $0 \leq s \leq \tilde{s} < \infty$ the (continuous) embeddings in Figure 1.1(a) hold. Note that the continuous embeddings in Figure 1.1(a) remain valid if we take $p \neq 2$ except for the relation to H^s.

- We equip the space $X_1 \cap X_2$ with the norm $\|\cdot\|_{X_1 \cap X_2} = \|\cdot\|_{X_1} + \|\cdot\|_{X_2}$ for two Banach spaces X_1, X_2. Therefore we have $L^{p,h}_s \cap L^{p,h}_0 \xrightarrow{\cong} L^{p,b}_s$.

- There are some obvious embeddings between the spaces $X_{\sigma,s,b}$ and $L^p_{s,\sigma}$ as well, see Figures 1.1(b) and 1.1(c). These make clear why the index s is of minor concern if we can switch from σ to σ' where $\sigma' < \sigma$. The space $X_{0,s,b}$ was used by Bourgain in [11] but also earlier, cf. the discussion in Tao's book [89, Section 2.6]. For $\sigma > 0$ these spaces were used by Grujić and Kalisch in [40] to investigate 'Gevrey-class' regularity for solutions to the KdV equation.

- Let $G_{\sigma,s} := \mathscr{F}^{-1} L^2_{s,\sigma}$. Then we have for $b > \frac{1}{2}$ the embedding $X_{\sigma,s,b} \hookrightarrow C(\mathbb{R}, G_{\sigma,s})$. This follows from Sobolev's Embedding Theorem.

- Note that the spaces $(L^2_{s,\sigma}, +, *)$ are Banach algebras with respect to convolution for $s > \frac{d}{2}$ and all $\sigma \geq 0$ whereas the spaces $X_{\sigma,s,b}$ are no multiplication algebras in general.

$$\begin{tikzcd}
L^{2,b}_{\tilde{s}} \arrow[r, "\cong"] \arrow[d, hook] & L^2_{\tilde{s}} \arrow[r, hook] \arrow[d, hook] & L^{2,h}_{\tilde{s}} \arrow[d, hook] \\
L^{2,b}_{s} \arrow[r, "\cong"] & L^2_{s} \arrow[r, hook] \arrow[d, "\cong"] & L^{2,h}_{s} \\
 & H^s &
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{\sigma,s,b} \arrow[r, hook] \arrow[d, hook] & X_{\sigma',s',b'} \\
X_{\sigma'',s'',b'} &
\end{tikzcd}
\qquad
\begin{tikzcd}
L^p_{s,\sigma} \arrow[r, hook] \arrow[d, hook] & L^p_{s',\sigma'} \\
L^p_{s'',\sigma''} &
\end{tikzcd}$$

(a) Embeddings of the spaces $L^{p,h}_s, L^{p,b}_s, L^p_s$ for $0 \leq s \leq \tilde{s} < \infty$.

(b) Embeddings of the spaces $X_{\sigma,s,b}$ for all $\sigma' \leq \sigma$, $s' \leq s$, $b' \leq b$ and $\sigma'' < \sigma$ and $s'' \in \mathbb{R}$.

(c) Embeddings of the spaces $L^p_{s,\sigma}$ for all $\sigma' \leq \sigma$, $s' \leq s$ and $\sigma'' < \sigma$ and $s'' \in \mathbb{R}$.

Figure 1.1: Embeddings between the function spaces.

Note that we can estimate the constant of the embedding $L^2_{s',\sigma'} \hookrightarrow L^2_{s,\sigma}$ for $\sigma' > \sigma \geq 0$ as follows.

***Remark* 1.5.3.** Note that for each $\sigma' > \sigma \geq 0$ and all $s, s' \in \mathbb{R}$ we have the continuous embedding $L^2_{s',\sigma'} \hookrightarrow L^2_{s,\sigma}$. In case $s < s'$ the constant is at most 1. We can estimate for $\lceil s - s' \rceil = \delta \geq 0$ as follows.

$$\int |\xi|^{2s} |\hat{u}(\xi)|^2 e^{2\sigma|\xi|} \, \mathrm{d}\xi \leq \frac{(\delta!)^2}{(\sigma' - \sigma)^{2\delta}} \int |\hat{u}(\xi)|^2 |\xi|^{2s'} e^{2\sigma'|\xi|} \, \mathrm{d}\xi$$

and

$$\int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 e^{2\sigma|\xi|} \, \mathrm{d}\xi \leq \frac{2^\delta (2\delta)!}{(2(\sigma' - \sigma))^{2\delta}} \int (1 + |\xi|^2)^{s'} |\hat{u}(\xi)|^2 e^{2\sigma'|\xi|} \, \mathrm{d}\xi.$$

This means that we can estimate the constant of the embedding by a factor proportional to $(\sigma' - \sigma)^{-\delta}$. We will need this estimate in Chapter 4.

There are some equivalent characterisations of the spaces $\mathrm{Hol}_\sigma, \mathrm{Hol}^2_\sigma$. The space

Hol_σ can be characterised equivalently by

$$\begin{aligned}\mathrm{Hol}_\sigma &= \left\{u \in L^2 \,\middle|\, \hat{u} \in L^2_{0,\tau} \text{ for all } \tau < \sigma\right\}\\ &= \left\{u \in L^2 \,\middle|\, \hat{u} \in L^2_{s,\tau} \text{ for all } \tau < \sigma \text{ and any } s \in \mathbb{R}\right\}.\end{aligned}$$

More precisely, if u is contained in the set above then u has an holomorphic continuation into $\mathcal{S}_\sigma$. This is essentially a Paley-Wiener theorem, cf. [77, Theorem IX.13]. Another useful characterisation for the space Hol^2_σ in one spatial dimension can be given by introduction of the norms

$$\|u\|^2_\tau := \sum_{j=0}^{\infty} \frac{e^{2\tau j}}{(j!)^2} \|\partial_x^j u\|^2_{L^2}$$

in H^∞. Then it holds

$$\mathrm{Hol}^2_\sigma = \left\{u \in H^\infty \,\middle|\, \|u\|^2_\tau < \infty \text{ for all } e^\tau < \sigma\right\},$$

again in the sense that there is an holomorphic extension, cf. [56, Lemma 2.2]. Hence, $\mathrm{Hol}_\sigma \subset \mathrm{Hol}^2_\sigma$ in one dimension but in general these sets are not equal since we have for $u \in \mathrm{Hol}_\sigma$ and for all $\sigma' \in]0, \sigma[$

$$\|\hat{u}\|^2_{L^2_{0,\sigma'}} = \int_{\mathbb{R}} \sum_{j=0}^{\infty} \frac{(2\sigma')^j}{j!} |\xi^j u(\xi)|^2 \,\mathrm{d}\xi = \sum_{j=0}^{\infty} (2\sigma')^j a_j < \infty$$

where $a_j = \frac{\|\partial_x^j u\|^2_{L^2}}{j!}$. And on the other hand side we obtain for $u \in \mathrm{Hol}^2_\sigma$, following the comment above,

$$\sum_{j=0}^{\infty} \frac{e^{2\tau j}}{j!} a_j < \infty$$

for all $e^\tau < \sigma$. Since $\frac{z^j}{j!} \to 0$ for $j \to \infty$ for all $z \in \mathbb{C}$, we see $\frac{e^{2j\tau}}{(2\sigma')^j} \to 0$ for $j \to \infty$ and all admissible τ, σ'. This will be a minor issue in Section 3.4 of Chapter 3.

For convenience we will write – in rough abuse of notation – $u \in C(I, \mathrm{Hol}_\sigma)$ for a simply connected set $I \subset \mathbb{R}$ if

$$\hat{u} \in C(I(\sigma'), L^2_{0,\sigma'}) \quad \text{for all } \sigma' < \sigma.$$

Equivalently we could choose any $L^2_{s,\sigma'}$, $s \in \mathbb{R}$, as well. *We will use this notation even when $I(\sigma')$ depends on σ' and it might be the case that the Lebesgue measure of I shrinks to zero for $\sigma \to \sigma'$.* Sometimes we will stress the fact that $I(\sigma')$ depends on σ' but not always and in some cases it will not depend on σ' at all. For as simpler notation we introduce the symbol $\gtrsim$ as follows.

Definition 1.5.4. *We write $a \lesssim b$ and $a \gtrsim b$ if the inequality holds up to some positive constant.*

We denote a scalar product in inner product spaces with $\langle \cdot, \cdot \rangle$ to avoid having too much round brackets. We will use Landau symbols for a precise definition of asymptotics. For $a \in \mathbb{R} \cup \{\pm\infty\}$ or $a \in \mathbb{C} \cup \{\infty\}$ and functions $f, g : U(a) \to \mathbb{C}^n$ for an environment $U(a)$ of a we define

$$
\begin{aligned}
f \in \mathrm{o}(g) &:= \lim_{z \to a} \frac{|f(z)|}{|g(z)|} = 0,\\
f \in \mathcal{O}(g) &:= \limsup_{z \to a} \frac{|f(z)|}{|g(z)|} < \infty,\\
f \in \Omega(g) &:= \liminf_{z \to a} \frac{|f(z)|}{|g(z)|} > 0,\\
f \in \Theta(g) &:= 0 < \liminf_{z \to a} \frac{|f(z)|}{|g(z)|} \leq \limsup_{z \to a} \frac{|f(z)|}{|g(z)|} < \infty.
\end{aligned}
$$

If we do not specify a when we use such a symbol, we mean $a = 0$ usually.

Chapter 2

Problems with an Amplitude Equation

Amplitude equations of the form discussed in Section 1.2 and 1.3 and multiple-scale analysis are used for the investigation of water waves problems. There are several formulations and extensions of the main problem of the water waves problem consisting in the description of the surface of a liquid-air interface, cf. the monograph of Lannes [58], in particular Chapter 1 for the formulation of the problems. Euler derived his famous equations that can be used to formulate this problem in the 1750s, cf. [38, 39]. In articles treating this topic, the assumptions that the fluid is irrotational, incompressible, homogeneous and inviscid are made widely, apart from several other ones, cf. [58, Section 1.1.1]. These assumptions simplify Euler's equations and make it possible to introduce a velocity potential, cf. [58, Section 1.1.3]. Several 'simpler' or asymptotic models for Euler's equations have been suggested over the last two centuries, e.g., in the case of shallow water or deep water, cf. [58, Chapters 5-8]. Boussinesq suggested such models in the 1870s that are well-known, cf. [12–14] and again the monograph of Lannes.

If one wants to apply amplitude equations techniques for the water waves problem or for one of Boussinesq's equations, then the natural question arises whether the formal approximate solutions constructed by these amplitude equations satisfy some approximation property in the spirit of Approximation Property 1.3.1. As pointed out in Section 1.3, this question is non-trivial and of some importance since in the case of periodic boundary conditions there are the counterexamples due to Schneider et al., cf. [84, 85]. However, the question is still open in the case of non-periodic boundary conditions. We will investigate the approximation problem in this chapter in the case of two Boussinesq-like models on the real line without periodic boundary conditions for the four-wave interaction (FWI) and three-wave interaction (TWI) amplitude equations, see Section 2.1.3. The first model, considered in Section 2.1, is intended to cover a surface tension effect. We will study the second model in Section 2.2 for its simpler structure primarily. Although the two models investigated in this chapter are inspired by Boussinesq's models, they are no genuine 'Boussinesq models' but rather toy problems which shall model some aspects and difficulties of the original water waves problems. We use these toy models in the hope that the methods developed in this Chapter apply to the full water waves problem as well and that similar examples and counterexamples can be proved for it, eventually.

2.1 A 6th Order Boussinesq-like Model

2.1.1 Introduction

We will discuss the approximation properties of two different formal approximate solutions for a Boussinesq-like equation within the subsections of Section 2.1. The first formal approximate solution is constructed with the aid of the so called FWI system, the second one with the aid of the so called TWI system, see Section 2.1.3. The original Boussinesq-like problem that we will consider is as follows. We are looking for strict solutions $u : I \times \mathbb{R} \to \mathbb{R}$ in certain Sobolev spaces of the equation

$$0 = \mathcal{L}(\partial_t, \partial_x)u + \partial_x^2 u^2 \quad \text{in } I \times \mathbb{R}, \qquad u(0) = v, \qquad u_t(0) = w \quad \text{in } \mathbb{R}, \tag{2.1}$$

where $I = [0, T[$ for a $T > 0$, v, w sufficiently regular functions, see Section 2.1.6, and the linear operator is $\mathcal{L} = \mathcal{L}_\mu(\partial_t, \partial_x) \coloneqq \partial_x^2 + \partial_t^2\partial_x^2 + \mu\partial_x^6 - \partial_t^2$ for $\mu > 0$. See Theorem 2.1.28 for the exact definition of a strict solution in this case. We will omit the subscript μ in general. This equation with $\mu = 0$ was considered in [74, 75] with regard to linear stability of solitary waves and a similar equation was considered in [51, Proposition 2.6] where some error estimate between solutions to the Boussinesq equation and solutions to Euler's equation were shown for analytic functions, if we take into account the identification after the remark following Proposition 2.6 of that article. Kano also discusses in [51] the relations

to Boussinesq's equations in [12].
Equation (2.1) with $\mu > 0$ was investigated in [33]. In that article an approximation property was proved for a formal approximate solution constructed with the nonlinear Schrödinger equation (NLS) as amplitude equation. The intention of the parameter μ is to cover a surface tension effect but we have to emphasis that our model is just a toy model at this point. The idea is that for $\mu > 0$ there are some interactions between different functions A_i of a multiple-scale ansatz similar to the one in Section 1.2, cf. Section 2.1.3 below. The fact that amplitudes of different exponentials interact is sometimes called a resonance and we will require such a behaviour later in Section 2.1.3. The connection to the water waves problem with surface tension is that such resonances can appear there, too. The modifications to the water waves problem that are necessary due to the additional surface tension are explained in [58, Chapter 9].
We are interested in the question whether an approximation property similar to Approximation Property 1.3.1, or more concrete Approximation Property 2.1.20 below, is true for either of the two formal approximate solutions. Actually, we want to show that this is not the case for the one constructed with the aid of the FWI system. The motivation for this counterexample comes from the articles [84, 85] of Schneider at al. They prove such a result in the case of a water waves model with periodic boundary conditions. But in their articles they heavily exploit the fact that they can reduce the problem to an ODE system for the coefficients of a Fourier series representation of the problem and that only a finite number of Fourier coefficients are important. In the setting of the whole real line with solutions in H^s, which we will consider, this applies no more and we have to work with different tools. However, the strategy will be the same as in [84], i.e. we will

1. derive two formal approximations of three (TWI system) or four (FWI sytem) interacting waves and

2. give examples where the two formal approximate solutions have such a big difference at a certain point in time that no useful error estimate is possible.

Further we will prove an error estimate under certain conditions. Unfortunately, the results in this chapter are not strong enough to prove a counterexample. The reason is that none of the formal approximate solutions that we use meets the assumptions of the error estimate that we will prove in Section 2.1.6.

In details, the strategy is as follows. We will derive in Section 2.1.3 amplitude equations that we call the FWI and TWI system. A simplified version of the FWI system will be called sFWI equation. In Section 2.1.4 the Cauchy problem for these amplitude equations will be investigated. Further we will show some regularity results for the solutions to the FWI and TWI systems under certain assumptions. We will construct some explicit formal approximate solutions in Section 2.1.5 and show that these differ so strongly that no approximation property can be true for both. This requires us to consider the TWI formal approximate solution beyond its natural time scale and leads to some difficulties as we will see in Sections 2.1.4 and 2.1.5. In Section 2.1.6 we will prove an error estimate in Corollary 2.1.30 under certain assumptions. This estimate together with the examples in Section 2.1.5 is strong enough to rule out an approximation property of the kind of Approximation Property 2.1.20 for one of the formal approximate solutions if we meet its assumptions. Hence, Corollary 2.1.30 would be sufficient to prove a counterexample.
Last but not least note that local existence and uniqueness for the Cauchy problem with equation (2.1) is trivially established by standard methods for sufficiently regular initial data in Sobolev spaces being Hilbert. We refer to Appendix 2.B for such a result. Therefore we will not talk much about this.

2.1.2 Notation

The derivation of the amplitude equations in the next Section 2.1.3 is done similarly to the derivation in the example in Section 1.2. However, the notation used in Section 1.2 is in general not suitable to keep track of all the possible wave numbers and frequencies generated by the nonlinear part. Therefore we will introduce some notation that allows us to do so. The benefit of this notation will become more clear in the process of the derivation of the amplitude equations.
We start with the definition $I_n := \{z \in \mathbb{Z}\backslash\{0\} \,|\, |z| \leq n\}$ for any $n \in \mathbb{N}$ and $I_n^{\leq l} = \bigcup_{k \leq l} I_n^k \subset \mathbb{Z}^l$ for any $(n, l) \in \mathbb{N}^2$ where we abuse the notation in the sense that we write

$$I_n^k := \underbrace{I_n \times \ldots \times I_n}_{k} \times \underbrace{\{0\} \times \ldots \times \{0\}}_{l-k}$$

for $k \leq l$. In this sense we have $I_n = I_n^{\leq 1}$. Further we define $|J| = l$ for $J \in I_n^{\leq l}$

and for sets $\{\omega_i \in \mathbb{R} \,|\, i \in I_n\}$ and $\{\xi_i \in \mathbb{R} \,|\, i \in I_n\}$ we define $\Omega_l(J) := \sum_{i=1}^{l} \omega_{J_i}$ as well as $\Xi_l(J) := \sum_{i=1}^{l} \xi_{J_i}$, where we set $\omega_{J_i}, \xi_{J_i} = 0$ if $J_i = 0$ to compensate for the aforementioned abuse of notation. In general we will drop the index l and write $\Omega(J), \Xi(J)$ only.

For a shorter notation we abbreviate $\Omega(J, J') := \Omega(J) + \Omega(J')$ and $\Xi(J, J') := \Xi(J) + \Xi(J')$ for all $J \in I_n^{\leq l}$ and $J' \in I_n^{\leq l'}$. This is natural in the sense that we can consider (J, J') as element of $I_n^{\leq |J|+|J'|}$ by simply concatenating J and J' and applying some permutations afterwards. For this element we would obtain the right hand sides mentioned above if we evaluate Ω and Ξ for it. We define a equivalence relation on $I_n^{\leq l}$ by

$$J \sim J' \quad :\Leftrightarrow \quad \Omega(J) = \Omega(J') \wedge \Xi(J) = \Xi(J') \text{ for } J, J' \in I_n^{\leq l}.$$

We denote the quotient set ${I_n^{\leq l}}\big/{\sim}$ by $\mathcal{I}_n^{\leq l}$. The quotient set $\mathcal{I}_n^{\leq l}$ contains the special set

$$\mathring{\mathcal{I}}_n^{\leq l} =: \left\{ J \in \mathcal{I}_n^{\leq l} \,\middle|\, \mathcal{L}(\mathrm{i}\Omega(J), \mathrm{i}\Xi(J)) = 0 \right\},$$

which we will call ‘kernel set’. Further we define

$$\mathcal{M}_{2l}(\mathcal{J}, \mathcal{J}') := \# \left\{ \mathcal{J}'' \in \mathcal{I}_n^{\leq 2l} \,\middle|\, \Xi(J'') = \Xi(J, J'), \Omega(J'') = \Omega(J, J') \right\}$$

and we will shorten this to $\mathcal{M}(\mathcal{J}, \mathcal{J}')$ if no confusion is possible.

***Remark* 2.1.1.** Note that in particular if $-\xi_i = \xi_{-i}$ and $-\omega_i = \omega_{-i}$ for all $i \in I_n$, then we have $\Omega(-J) = -\Omega(J)$ and $\Xi(-J) = -\Xi(J)$. We note that this notation is somewhat similar to the notation of Kalyakin in [50, §3]. But we do not restrict it to integers and we use, in contrast to Kalyakin, finite sets only.

2.1.3 Formal Derivation of the FWI and TWI Systems

We will make an ansatz similar to the one in Section 1.2 and use it to derive the amplitude equations mentioned in Section 2.1.1. We do so by making a multiple-scale analysis for this ansatz. The slow scale stems from the slow moving amplitudes and the fast scale from the rapidly varying phase factors. The slow and fast scales are defined by a small parameter $\epsilon \in]0, 1[$ or more precisely different powers of it. We will work with five different scales in the following:

$$x, \qquad X := \epsilon^2 x, \qquad t, \qquad T := \epsilon t, \qquad \mathcal{T} := \epsilon^2 t,$$

where t, x are our 'original' variables of the system.

N.B.: we will use another time scale for the derivation of the FWI system in the following subsection than for the derivation of the TWI system in the subsection following the derivation of the FWI system. This is an essential point, which has to be kept in mind.

We have to choose wave numbers and frequencies for the phase factors. We choose them in such a way that the phase factors are, in the sense of distributions, in the kernel of the linear operator $\mathcal{L}$ of equation (2.1). With this in mind we define the set of characteristic frequencies

$$Char\mathcal{L} := \left\{(\omega, \xi) \in \mathbb{R}^2 \backslash \{(0,0)\} \,\middle|\, \mathcal{L}(\mathrm{i}\omega, \mathrm{i}\xi) = 0\right\},$$

which is not exactly the usual characteristic variety since we use the whole symbol, but useful in the following. Trivially we can give a local description of this manifold through

$$\omega_\pm(\xi) = \pm\sqrt{\frac{\xi^2 + \mu\xi^6}{1+\xi^2}}.$$

Thus we define the function $\omega : \mathbb{R} \to \mathbb{R}$, which is called 'dispersion relation', by

$$\xi \mapsto \mathrm{sign}(\xi)\omega_+(\xi) \tag{2.2}$$

and for $\xi_i \in \mathbb{R}$ we define $\omega_i := \omega(\xi_i)$ for every index $i \in \mathbb{Z}$. Note that

$$\begin{aligned} &\left\{(\omega, \xi) \in \mathbb{R}^2 \,\middle|\, \xi = 0\right\} \cap Char\mathcal{L} = \emptyset, \\ &\left\{(\omega, \xi) \in \mathbb{R}^2 \,\middle|\, \omega = 0\right\} \cap Char\mathcal{L} = \emptyset. \end{aligned} \tag{2.3}$$

We will always assume in the following that all (ξ_i, ω_i) differ from each other.

The Non-resonant FWI System

We choose four different wave numbers and frequencies

$$(\xi_1, \omega_1), \ldots, (\xi_4, \omega_4) \in Char\boldsymbol{\mathcal{L}}$$

satisfying certain constraints that we will specify and need later. Since we are looking for real valued solutions, we set $\xi_{-i} = -\xi_i$ for $i \in \{1, \ldots, 4\}$ and according ω_i. Now we let

$$\epsilon\psi_{\mathrm{FWI}} = \sum_{i=1}^{2} \sum_{J \in \mathcal{I}_4^{\leq i}} \epsilon^i A_J(\epsilon^2 \boldsymbol{t}, \epsilon^2 \boldsymbol{x}) e^{\mathrm{i}\Omega(J)\boldsymbol{t} + \mathrm{i}\xi(J)\boldsymbol{x}}$$

where $A_{-J} = \overline{A_J}$. By inserting this ansatz into equation (2.1) we obtain

$$\begin{aligned}\mathfrak{E}_{\epsilon\psi_{\mathrm{FWI}}} = &\sum_{i=1}^{2} \sum_{J \in \mathcal{I}_4^{\leq i}} \epsilon^i e^{\mathrm{i}\Omega(J)\boldsymbol{t} + \mathrm{i}\Xi(J)\boldsymbol{x}} \left(\boldsymbol{\mathcal{L}}(\epsilon^2 \partial_t + \mathrm{i}\Omega(J), \epsilon^2 \partial_x + \mathrm{i}\Xi(J)) A_J\right)(\epsilon^2 \boldsymbol{t}, \epsilon^2 \boldsymbol{x}) \\ &+ \sum_{i,j=1}^{2} \sum_{(J',J'') \in \mathcal{I}_4^{\leq i} \times \mathcal{I}_4^{\leq j}} \epsilon^{i+j} e^{\mathrm{i}\Omega(J',J'')\boldsymbol{t} + \mathrm{i}\Xi(J',J'')\boldsymbol{x}} \times \\ &\qquad\qquad \times \left(\left(\epsilon^2 \partial_x + \mathrm{i}\Xi(J', J'')\right)^2 A_{J'} A_{J''}\right)(\epsilon^2 \boldsymbol{t}, \epsilon^2 \boldsymbol{x}).\end{aligned}$$

Note that there are actually no powers of order ϵ since $(\xi_i, \omega_i) \in Char\boldsymbol{\mathcal{L}}$. In order ϵ^2 it remains

$$\begin{aligned}&\sum_{J \in \mathcal{I}_4^{\leq 2}} e^{\mathrm{i}\Omega(J)\boldsymbol{t} + \mathrm{i}\Xi(J)\boldsymbol{x}} \boldsymbol{\mathcal{L}}(\mathrm{i}\Omega(J), \mathrm{i}\Xi(J)) A_J(\epsilon^2 \boldsymbol{t}, \epsilon^2 \boldsymbol{x}) \\ &- \sum_{(J',J'') \in \mathcal{I}_4^{\leq 1} \times \mathcal{I}_4^{\leq 1}} e^{\mathrm{i}\Omega(J',J'')\boldsymbol{t} + \mathrm{i}\Xi(J',J'')\boldsymbol{x}} \Xi(J', J'')^2 (A_{J'} A_{J''})(\epsilon^2 \boldsymbol{t}, \epsilon^2 \boldsymbol{x}).\end{aligned}$$

If we assume that $\Omega(J', J'')$ and $\Xi(J', J'')$ are not elements of $\mathrm{Char}\boldsymbol{\mathcal{L}}$ for all $J', J'' \in \mathcal{I}_4^{\leq 1}$ and $J' \neq -J''$, i.e.

$$\begin{aligned}&\boldsymbol{\mathcal{L}}(\mathrm{i}\Omega(J', J''), \mathrm{i}\Xi(J', J'')) \neq 0 \\ \Leftrightarrow &(\omega_i + \omega_j)^2 - \omega(\xi_i + \xi_j)^2 \neq 0 \text{ for all } i \neq -j \in I_4,\end{aligned} \tag{2.4}$$

then we can make the whole sum vanish by defining A_J accordingly. Therefore we consider condition (2.4) as algebraic constraint. We call this constraint a non-resonance condition. Note that in the case $J' = -J''$ all addends of the last sum vanish. Hence, we define $A_J \equiv 0$ for all remaining $J \in \mathcal{I}_4^{\leq 2}$. This is the constraint mentioned at the beginning of the section.

***Remark* 2.1.2** (Non-resonance condition)**.** We note that in the case

$$(\omega_i + \omega_j)^2 - \omega(\xi_i + \xi_j)^2 = \omega_i^2 + \omega_j^2 + 2\omega_i\omega_j - \omega(\xi_i + \xi_j)^2 = 0$$

we also have

$$\big(\omega_i^2 + \omega_j^2 - \omega(\xi_i + \xi_j)^2\big)^2 = 4\omega_i^2\omega_j^2.$$

This can equivalently be written as a polynomial in ξ_i, ξ_j of degree at most 24. Hence, for any given ξ_i there are at most finitely many ξ_j that we have to avoid. We will fix ξ_3 by another condition later, see Section 2.1.5, and therefore we will have to respect this restriction but there remain uncountably many choices for ξ_1, ξ_2 and ξ_4.

Let us write $\mathcal{L} = \mathcal{L}(\boldsymbol{\omega}, \boldsymbol{\xi})$. Then we can use a Taylor expansion of $\mathcal{L}$ and we see that it remains

$$\begin{aligned}&\sum_{J \in \mathcal{I}_4^{\leq 1}} e^{\mathrm{i}\Omega(J)\boldsymbol{t} + \mathrm{i}\Xi(J)\boldsymbol{x}} \Big(\big(\partial_\omega \mathcal{L}\big(\mathrm{i}\Omega(J), \mathrm{i}\Xi(J)\big)\partial_t + \partial_\xi \mathcal{L}\big(\mathrm{i}\Omega(J), \mathrm{i}\Xi(J)\big)\partial_x\big) A_J\Big)(\epsilon^2\boldsymbol{t}, \epsilon^2\boldsymbol{x}) \\ &- 2 \sum_{(J',J'') \in \mathcal{I}_4^{\leq 1} \times \mathcal{I}_4^{\leq 2}} e^{\mathrm{i}\Omega(J',J'')\boldsymbol{t} + \mathrm{i}\Xi(J',J'')\boldsymbol{x}} \Xi(J', J'')^2 (A_{J'} A_{J''})(\epsilon^2\boldsymbol{t}, \epsilon^2\boldsymbol{x})\end{aligned}$$

in order ϵ^3. If we require the coefficients of $e^{\mathrm{i}\Omega(J)\boldsymbol{t} + \mathrm{i}\Xi(J)\boldsymbol{x}}$ to vanish, we obtain for all $J \in \mathcal{I}_4$ the FWI system

$$\partial_t A_J = c_J \partial_x A_J + \mathrm{i} d_J \sum_{\substack{(J',J'') \in \mathcal{I}_4^{\leq 1} \times \mathcal{I}_4^{\leq 2} \\ \Xi(J) = \Xi(J',J'') \\ \Omega(J) = \Omega(J',J'')}} A_{J'} A_{J''},$$

where $c_J := -\frac{\Omega(J)^2\Xi(J) - \Xi(J) - 3\mu\Xi(J)^5}{\Omega(J)(1+\Xi(J)^2)}$ and $d_J := \frac{\Xi(J)^2}{\Omega(J)(1+\Xi(J)^2)}$. Since we determined $A_{J''}$ for $J'' \in \mathcal{I}_4^{\leq 2}$ by an algebraic expression above, we can rewrite the equation as

$$\partial_t A_J = c_J \partial_x A_J + \mathrm{i}d_J \sum_{\substack{(J',J'',J''')\in\left(\mathcal{I}_4^{\leq 1}\right)^3 \\ \Xi(J)=\Xi(J',J'',J''') \\ \Omega(J)=\Omega(J',J'',J''') \\ \mathcal{L}(\mathrm{i}\Omega(J'',J'''),\mathrm{i}\Xi(J'',J'''))\neq 0}} \frac{A_{J'}A_{J''}A_{J'''}}{\mathcal{M}(J'',J''')\mathcal{L}(\mathrm{i}\Omega(J'',J'''),\mathrm{i}\Xi(J'',J'''))}$$

$$\begin{aligned} &= c_J\partial_x A_J \\ &\quad + 2\mathrm{i}d_J A_J \sum_{\substack{J'\in\mathcal{I}_4^{\leq 1} \\ \mathcal{L}(\mathrm{i}\Omega(-J',J),\mathrm{i}\Xi(-J',J))\neq 0}} \frac{|A_{J'}|^2}{\mathcal{M}(-J',J)\mathcal{L}(\mathrm{i}\Omega(-J',J),\mathrm{i}\Xi(-J',J))} \\ &\quad + \mathrm{i}d_J \sum_{\substack{(J',J'',J''')\in\left(\mathcal{I}_4^{\leq 1}\right)^3 \\ J'\neq -J'',-J''' \\ \Xi(J)=\Xi(J',J'',J''') \\ \Omega(J)=\Omega(J',J'',J''') \\ \mathcal{L}(\mathrm{i}\Omega(J'',J'''),\mathrm{i}\Xi(J'',J'''))\neq 0}} \frac{A_{J'}A_{J''}A_{J'''}}{\mathcal{M}(J'',J''')\mathcal{L}(\mathrm{i}\Omega(J'',J'''),\mathrm{i}\Xi(J'',J'''))}, \end{aligned} \tag{2.5}$$

If, additionally, there is no resonance of the wave numbers and frequencies, then the last sum vanishes and we obtain the non-resonant FWI system

$$\partial_t A_J = c_J\partial_x A_J + \mathrm{i}A_J \sum_{\substack{J'\in\mathcal{I}_4^{\leq 1} \\ \mathcal{L}(\mathrm{i}\Omega(-J',J),\mathrm{i}\Xi(-J',J))\neq 0}} d_{J,J'}|A_{J'}|^2, \tag{2.6}$$

where $d_{J,J'} = \frac{2d_J}{\mathcal{M}(-J',J)\mathcal{L}(\mathrm{i}\Omega(-J',J),\mathrm{i}\Xi(-J',J))}$.

***Remark* 2.1.3.** Because of our assumption that all (ω_i, ξ_i) are different from each other, we can conclude $(\omega_i, \xi_i) = (\omega_j, \xi_j) \Rightarrow i = j$. Thus the reduction above follows.

Since we are interested in giving a counterexample, we do another simplification and assume that $A_i(0) \equiv 0$ for $i \in \{1, 2, 4\}$. Then the non-resonant system only consists of

$$\partial_t A_3 = c_3\partial_x A_3 + \mathrm{i}d_{(3,-3)}A_3|A_3|^2 \tag{2.7}$$

and $A_i(t) \equiv 0$ for $i \in 1, 2, 4$ and $t \geq 0$ where $c_3 = -\frac{\omega_3^2\xi_3 - \xi_3 - 3\mu\xi_3^5}{\omega_3(1+\xi_3^2)}$ and $d_{(3,-3)} = \frac{2}{\mathcal{M}(3,3)\mathcal{L}(\mathrm{i}2\omega_3,\mathrm{i}2\xi_3)}\frac{\xi_3^2}{\omega_3(1+\xi_3^2)}$. We call this amplitude equation sFWI equation and we will study the formal approximate solution constructed with its aid in the next sections.

***Remark* 2.1.4** (Reduction to the sFWI equation for the general FWI system)**.** If we assume $\xi_i \neq \pm 3\xi_3$ or $\omega_i \neq 3\omega_3$ for all $i \in \mathcal{I}_4$, then the same equation also arises for these initial data in the case of the FWI system (2.5). This is obvious since $A_i(t) \equiv 0$ for $i \in 1, 2, 4$ and $t \geq 0$ is a solution – and in a suitable Hilbert space like L^2 or H^s the unique solution – such that the 'resonant part' in equation (2.5) vanishes. Note that the condition $\xi_i = \pm 3\xi_3$ and $\omega_i = 3\omega_3$ has a solution for a finite set of values of ξ_3 only.

The Resonant TWI System

We choose three different wave numbers and frequencies

$$(\xi_1, \omega_1), \ldots, (\xi_3, \omega_3) \in Char\boldsymbol{\mathcal{L}}$$

satisfying the resonance condition $\sum_{i=1}^{3} \xi_i = \sum_{i=1}^{3} \omega_i = 0$ for the next ansatz. Since we seek again real-valued solutions, we choose $\xi_{-i} = -\xi_i$ for $i \in \{1, \ldots, 3\}$ and according ω_i. Now we let

$$\epsilon\psi_{\mathrm{TWI}} = \sum_{J \in \mathcal{I}_3^{\leq 1}} \epsilon A_J(\epsilon \boldsymbol{t}, \epsilon^2 \boldsymbol{x}) e^{\mathrm{i}\Omega(J)\boldsymbol{t} + \mathrm{i}\Xi(J)\boldsymbol{x}}$$

and again $A_{-J} = \overline{A_J}$. Now we obtain

$$\begin{aligned} \mathfrak{E}_{\epsilon\psi_{\mathrm{TWI}}} = &\sum_{J \in \mathcal{I}_3^{\leq 1}} \epsilon e^{\mathrm{i}\Omega(J)\boldsymbol{t} + \mathrm{i}\Xi(J)\boldsymbol{x}} \left(\mathcal{L}(\epsilon\partial_t + \mathrm{i}\Omega(J), \epsilon^2\partial_x + \mathrm{i}\Xi(J)) A_J \right)(\epsilon \boldsymbol{t}, \epsilon^2 \boldsymbol{x}) \\ &+ \sum_{(J', J'') \in \mathcal{I}_3^{\leq 1} \times \mathcal{I}_3^{\leq 1}} \epsilon^2 e^{\mathrm{i}\Omega(J', J'')\boldsymbol{t} + \mathrm{i}\Xi(J', J'')\boldsymbol{x}} \times \\ &\qquad \times \left(\left(\epsilon^2\partial_x + \mathrm{i}\Xi(J', J'')\right)^2 A_{J'} A_{J''} \right)(\epsilon \boldsymbol{t}, \epsilon^2 \boldsymbol{x}). \end{aligned}$$

after inserting the ansatz into equation (2.1). Note again that there are no powers of order ϵ since $(\xi_i, \omega_i) \in Char\boldsymbol{\mathcal{L}}$. In order ϵ^2 it remains

$$\begin{aligned} &\sum_{J \in \mathcal{I}_3^{\leq 1}} e^{\mathrm{i}\Omega(J)\boldsymbol{t} + \mathrm{i}\Xi(J)\boldsymbol{x}} \left(\partial_\omega \mathcal{L}(\mathrm{i}\Omega(J), \mathrm{i}\Xi(J)) \partial_t A_J \right)(\epsilon \boldsymbol{t}, \epsilon^2 \boldsymbol{x}) \\ - &\sum_{(J', J'') \in \mathcal{I}_3^{\leq 1} \times \mathcal{I}_3^{\leq 1}} e^{\mathrm{i}\Omega(J', J'')\boldsymbol{t} + \mathrm{i}\Xi(J', J'')\boldsymbol{x}} \Xi(J', J'')^2 (A_{J'} A_{J''})(\epsilon \boldsymbol{t}, \epsilon^2 \boldsymbol{x}). \end{aligned}$$

We require that the coefficients of the exponentials in the first sum vanish and thus obtain

$$\partial_\omega \mathcal{L}(\mathrm{i}\Omega(J), \mathrm{i}\Xi(J))\partial_t A_J = \sum_{\substack{(J',J'')\in\mathcal{I}_3^{\leq 1}\times\mathcal{I}_3^{\leq 1}\\ \Omega(J)=\Omega(J',J'')\\ \Xi(J)=\Xi(J',J'')}} \Xi(J',J'')^2 A_{J'}A_{J''}.$$

If $\xi_i \neq 2\xi_j$ or $\omega_i \neq 2\omega_j$ for every $i,j \in \mathcal{I}_3$, this reduces to

$$\partial_t \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \mathrm{i}\Gamma \begin{pmatrix} A_{-2}A_{-3} \\ A_{-1}A_{-3} \\ A_{-1}A_{-2} \end{pmatrix} = \mathrm{i}\Gamma \overline{\begin{pmatrix} A_2A_3 \\ A_1A_3 \\ A_1A_2 \end{pmatrix}}, \tag{2.8}$$

where $\Gamma \in \mathrm{GL}(3,\mathbb{R})$ with $(\Gamma)_{i,j} = \frac{-\mathrm{i}\xi_i^2\delta_{i,j}}{\partial_\omega\mathcal{L}(\mathrm{i}\omega_i,\mathrm{i}\xi_i)}$. This follows from (2.3) clearly.

***Remark* 2.1.5** (Existence of resonant wave numbers)**.**

1. Note that there is for all $\mu > 0$ an uncountable set for choices of the wave numbers ξ_1, ξ_2, ξ_3 that satisfy the resonance condition $\sum_{i=1}^3 \xi_i = \sum_{i=1}^3 \omega_i = 0$. This is true since we obtain $\xi_3 = -(1+\alpha)\xi_2 = -(1+\alpha)\eta \in \mathbb{R}^-$ for $0 < \xi_1 = \alpha\xi_2$ and $\alpha > 0$. Then we have to consider the equation

   $$0 = \omega(\eta) + \omega(\alpha\eta) - \omega((1+\alpha)\eta).$$

   We investigate the map $\omega(\boldsymbol{\eta}) + \omega(\alpha\boldsymbol{\eta}) - \omega((1+\alpha)\boldsymbol{\eta}) \in C(\mathbb{R}^+,\mathbb{R})$ for this purpose. A short calculation shows

   $$\begin{aligned} \omega(\eta) + \omega(\alpha\eta) - \omega((1+\alpha)\eta) &= -2\sqrt{\mu}\alpha\eta^2 + \mathcal{O}(\eta), & \eta &\to \infty \\ \omega(\eta) + \omega(\alpha\eta) - \omega((1+\alpha)\eta) &= 3(\alpha^2+\alpha)\eta^3 + \mathcal{O}(\eta^4), & \eta &\to 0. \end{aligned}$$

   Hence there is a zero for a $\eta = \xi_2 > 0$ by the intermediate value theorem.

2. For the question whether $\omega_i = 2\omega_j$ and $\xi_i = 2\xi_j$ for any $i,j \in \mathcal{I}_3$, we note first that $j \in \{\pm i\}$ is impossible. But if the dispersion relation is (locally) linear, this would be true. However, with the dispersion relation given by equation (2.2) there is only a finite set of ξ_1 satisfying this relation. We assume w.l.o.g $i = 1, j = \pm 2$ to see this. Then we obtain from the resonance

condition that $j = -2$ is impossible. In the other case we find

$$\omega(2\xi_2) = \omega(\xi_1) = \omega_1 = 2\omega_2 = 2\omega(\xi_2)$$
$$\omega(3\xi_2) = -\omega(\xi_3) = -\omega_3 = 3\omega_2 = 3\omega(\xi_2)$$

such that

$$\omega(2\xi_2)^2 - \omega(3\xi_2)^2 = 4\omega(\xi_2)^2 - 9\omega(\xi_2)^2 = -5\omega(\xi_2)^2.$$

We can write this condition as a polynomial in ξ_2 of degree 12 so that there can be at most 12 real roots. Hence there can be an at most finite set of values for ξ_i that we have to avoid.

2.1.4 Dynamics of the FWI and TWI Systems

After the derivation of the formal approximate solutions, we have to discuss their behaviours. We proceed as follows for our goal of finding two formal approximate solutions that are different to a significant amount. We must find (strict) solutions to (2.7) and (2.8) respectively as well as a time $t_0 > 0$ such that $\|\epsilon\psi_{\mathrm{TWI}} - \epsilon\psi_{\mathrm{FWI}}\|_{L^\infty} \gtrsim \epsilon$. Then at least one formal 'approximate solution' cannot be correct. Further we will give some regularity estimates which are necessary for the error estimates.
We remark in advance that we will have to estimate the spatial derivatives of the solution to equation (2.8) pretty carefully. In the proof we will calculate the derivatives more or less explicitly.

Dynamics of the FWI Systems

We start the execution of the outlined program by proving existence and uniqueness of solutions to the non-resonant FWI system (2.6). The subsequent Theorem 2.1.6 gives an explicit solution for the dynamics of the non-resonant FWI system (2.6) including the special case of the simplified FWI 'system' (2.7). It also gives explicit regularity estimates. This settles the question of the behaviour of the dynamics of the non-resonant FWI system in the spaces we are interested in completely.

Theorem 2.1.6. *For $i, j \in \{1,2,3,4\}$ let $c_i \in \mathbb{R}$, $d_{i,j} \in \mathbb{R}$ and $t_0 > 0$. Then the unique solution in $A \in C^0([0,t_0], H^1) \cap C^1([0,t_0], L^2)$ to*

$$\partial_t A_i = c_i \partial_x A_i + \mathrm{i} \left(\sum_{j=1}^{4} d_{i,j} |A_j|^2 \right) A_i$$

with initial data $A(0) = a \in H^1 \cap C^1$ is given by

$$(t,x) \mapsto A(t,x) = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} (t,x) = \begin{pmatrix} a_1(x + c_1 t) e^{\mathrm{i} g_1(t,x)} \\ a_2(x + c_2 t) e^{\mathrm{i} g_2(t,x)} \\ a_3(x + c_3 t) e^{\mathrm{i} g_3(t,x)} \\ a_4(x + c_4 t) e^{\mathrm{i} g_4(t,x)} \end{pmatrix},$$

where

$$\begin{aligned} g_i(t,x) &= \sum_{j=1}^{4} \int_0^t d_{i,j} |a_j(x + c_i t + (c_j - c_i)s)|^2 \, \mathrm{d}s \\ &= \sum_{\substack{j=1 \\ j \neq i}}^{4} \int_0^t d_{i,j} |a_j(x + c_i t + (c_j - c_i)s)|^2 \, \mathrm{d}s + d_{i,i} t |A_i(t,x)|^2. \end{aligned}$$

Further we have the following regularity statements.

1. *If additionally $A(0) = a \in H^n \cap C^n$, then $\partial_t^k A \in L^\infty([0,t_0], H^l)$ for $k, l \in \mathbb{N}_0$ and $k + l \leq n$ and*

   $$\|\partial_t^k \partial_x^l A_i(t)\|_{L^2} \leq P_n(t)$$

   for a polynomial P_n whose degree is not greater than n.

2. *If additionally $A(0) = a \in W^{n,\infty} \cap C^n$, then $\partial_t^k A_3 \in L^\infty([0,t_0], W^{l,\infty})$ for $k, l \in \mathbb{N}_0$ and $k + l \leq n$ and*

   $$\|\partial_t^k \partial_x^l A_i(t)\|_{L^\infty} \leq P_n(t)$$

   for a polynomial P_n whose degree is not greater than n.

The proof of Theorem 2.1.6 can be given by a straightforward but rather tedious calculation for the explicitly given solution formula. Since no particular difficulties arise in this calculation we skip it here. For completeness we show the calculations in Appendix 2.C.

***Remark* 2.1.7.**

- Note that system (2.6) or (2.7) is actually just a non-autonomous linear system since for all sufficiently regular solutions A

  $$\partial_t |A_i|^2 = c_i \partial_x |A_i|^2$$

  and thus the equation can be written as

  $$\partial_t A_i(t) = c_i \partial_x A_i(t) + \mathrm{i}\Big(\sum_{j=1}^{4} d_{i,j} |a_j(\cdot + c_i t)|^2\Big) A_i(t),$$

  where the non-autonomous part depends on the initial data $a \in H^1$.

- Since all estimates in the proof solely involve the Sobolev norms, one could drop the assumption of classical differentiability for the initial data by density arguments.

Dynamics of the TWI Systems

We will continue the program outlined at the beginning of Section 2.1 and discuss local existence and uniqueness of solutions to system (2.8). At present, we can do this under certain restrictions only, see Hypothesis 2.1.9 and 2.1.15 below. Nonetheless, we will see in Section 2.1.5 that this is not problematic for our purpose. Our main results in this section are Lemma 2.1.10 stating existence and uniqueness of the solutions to system (2.9) and the regularity result of Theorem 2.1.16.

We start with the following observation. We assume that we have a system as follows

$$\begin{aligned}
\partial_t A_1 &= \mathrm{i}\Gamma_1 \overline{A_2 A_3},\\
\partial_t A_2 &= \mathrm{i}\Gamma_2 \overline{A_1 A_3},\\
\partial_t A_3 &= \mathrm{i}\Gamma_3 \overline{A_1 A_2},
\end{aligned}$$

where $\Gamma_i \in \mathbb{R}^\times$. If we define $A_i =: -\mathrm{i}(|\Gamma_1\Gamma_2\Gamma_3|)^{-1/2}\sqrt{|\Gamma_i|}\tilde{B}_i$, we can eliminate the factors Γ_i

$$\begin{aligned}\partial_t \tilde{B}_1 &= \operatorname{sign}(\Gamma_1)\overline{\tilde{B}_2\tilde{B}_3},\\ \partial_t \tilde{B}_2 &= \operatorname{sign}(\Gamma_2)\overline{\tilde{B}_1\tilde{B}_3},\\ \partial_t \tilde{B}_3 &= \operatorname{sign}(\Gamma_3)\overline{\tilde{B}_1\tilde{B}_2}.\end{aligned}$$

If $\operatorname{sign}(\Gamma_1) = \operatorname{sign}(\Gamma_2) = \operatorname{sign}(\Gamma_3)$, then we can make them vanish by defining $\operatorname{sign}(\Gamma_1)\tilde{B}_1 =: B_1, \tilde{B}_2 =: B_2, \tilde{B}_3 =: B_3$. If only two of the $\operatorname{sign}(\Gamma_i)$ are the same, then we can always choose B_i in such a way that only $\overline{B_1B_2}$ has a negative sign. Thus, the system can always be written in the form

$$\begin{aligned}\partial_t B_1 &= \overline{B_2B_3},\\ \partial_t B_2 &= \overline{B_1B_3},\\ \partial_t B_3 &= \pm\overline{B_1B_2}.\end{aligned} \tag{2.9}$$

We will investigate the Cauchy problem for system (2.9) for pretty smooth and specific initial data in the following. This is equivalent to the Cauchy problem for system (2.8) with pretty smooth and specific initial data by the calculations above. But it will be sufficient to give an idea for a counterexample in Section 2.1.5. Further we remark that we are only interested in the case where one sign is different from the other ones actually, since the other case cannot occur because of the resonance condition for the TWI system in Section 2.1.3. However, we will consider both cases for the moment. The following observation will be helpful for this purpose.

***Remark* 2.1.8.** There are some conserved quantities for system (2.9). Let us assume that there are classical solutions to (2.9) and let $\alpha \in \mathbb{R}^3$. We find for the function $g = \sum_{i=1}^{3} \alpha_i |B_i|^2$ in each point $x \in \mathbb{R}$ and as long as the solution exists

$$\partial_t g(\alpha, t, x) = 2(\alpha_1 + \alpha_2 \pm \alpha_3)\operatorname{Re}(B_1B_2B_3)(t, x).$$

Thus, we have conserved quantities, e.g. for $\alpha = (1, 0, \mp 1)$. In the case of the minus sign in system (2.9), this limits the absolute value of the functions B_i pointwise (otherwise it just states that the absolute values are not independent of each other).

In the following we will make some quite restrictive assumptions on the initial data. Since our goal is to construct solutions that have appropriate properties to give an counterexample, we are free to choose these.

Hypothesis 2.1.9. *We assume*

1. $B_1(\boldsymbol{x},0), B_2(\boldsymbol{x},0), B_3(\boldsymbol{x},0)$ *are real-valued bounded functions and*

2. $B_1(\boldsymbol{x},0) = B_2(\boldsymbol{x},0)$.

Under these assumptions we can prove existence and uniqueness.

Lemma 2.1.10. *Assume Hypothesis 2.1.9 is true and initial conditions* $B_1(0,\cdot) = B_2(0,\cdot) = b_1$ *and* $B_3(0,\cdot) = b_3$ *with* $b_1, b_3 \in C^0(\mathbb{R})$.
Then the unique classical solution to system (2.9) *is given by* $B_2 \equiv B_1$ *and*

1. *(in the case of the minus sign)*

$$B_1(t,x) = \begin{cases} b_1(x)\frac{\sqrt{c(x)}}{\sqrt{c(x)}\cosh\left(\sqrt{c(x)}t\right) - B_3(x,0)\sinh\left(\sqrt{c(x)}t\right)} & \text{for } x \in M_1 \\ 0 & \text{for } x \in \mathbb{R}\backslash M_1 \end{cases}$$

$$B_3(t,x) = \begin{cases} \sqrt{c(x)}\frac{b_3(x)\cosh\left(\sqrt{c(x)}t\right) - \sqrt{c(x)}\sinh\left(\sqrt{c(x)}t\right)}{\sqrt{c(x)}\cosh\left(\sqrt{c(x)}t\right) - b_3(x)\sinh\left(\sqrt{c(x)}t\right)} & \text{for } x \in M_1 \cup M_3 \\ 0 & \text{for } x \in \mathbb{R}\backslash(M_1 \cup M_3). \end{cases}$$

where $M_1 := \mathrm{supp}(b_1), M_3 := \mathrm{supp}(b_3)$ *and* $c := b_1^2 + b_3^2 = B_1(t,\cdot)^2 + B_3(t,\cdot)^2$ *for all* $t \in \mathbb{R}$ *as long as the solution exists.*

2. *(in the case of the plus sign)*

$$B_1(t,x) = \begin{cases} b_1(x)\frac{\sqrt{c(x)}}{\sqrt{c(x)}\cos(\sqrt{c(x)}t) - b_3(x)\sin(\sqrt{c(x)}t)} & b_1(x)^2 > b_3(x)^2 \\ \frac{b_1(x)}{1 - b_3(x)t} & b_1(x)^2 = b_3(x)^2 \\ b_1(x)\frac{\sqrt{d(x)}}{\sqrt{d(x)}\cosh(\sqrt{d(x)}t) - b_3(x)\sinh(\sqrt{d(x)}t)} & b_1(x,0)^2 < b_3(x)^2 \end{cases}$$

$$B_3(t,x) = \begin{cases} \sqrt{c(x)}\frac{b_3(x)\cos\left(\sqrt{c(x)}t\right) + \sqrt{c(x)}\sin\left(\sqrt{c(x)}t\right)}{\sqrt{c(x)}\cos\left(\sqrt{c(x)}t\right) - b_3(x)\sin\left(\sqrt{c(x)}t\right)} & b_1(x)^2 > b_3(x)^2 \\ \frac{b_3(x)}{1 - b_3(x)t} & b_1(x)^2 = b_3(x)^2 \\ \sqrt{d(x)}\frac{b_3(x)\cosh\left(\sqrt{d(x)}t\right) - \sqrt{d(x)}\sinh\left(\sqrt{d(x)}t\right)}{\sqrt{d(x)}\cosh\left(\sqrt{d(x)}t\right) - b_3(x)\sinh\left(\sqrt{d(x)}t\right)} & b_1(x)^2 < b_3(x)^2 \end{cases}$$

where $c := b_1^2 - b_3^2 = B_1(t,\cdot)^2 - B_3(t,\cdot)^2 = -d$ *for all* $t \in \mathbb{R}$ *as long as the solution exists.*

Proof. It is obvious that the initial conditions are satisfied and uniqueness of the solution is clear as well since system (2.9) is just an ODE satisfying a Lipschitz condition pointwise. Further the solution formulae above are defined for a time interval with non-empty interior since b_1, b_3 are bounded by assumption. Note that for $x \notin \operatorname{supp}(b_1)$, $B_1(t,x) = 0$ and $B_3(t,x) = b_3(x)$ for all t as long as the solution exists. We will not explicitly note the dependence on x in the following for a shorter notation.

We calculate as follows for the conserved quantity (we consider the case $x \in \operatorname{supp}(b_1)$ only as the other case is trivial):

1. We use $c \sinh^2\left(\sqrt{c}t\right) = \left(b_1^2 + b_3^2\right)\sinh^2\left(\sqrt{c}t\right)$ and $1 = \cosh^2 - \sinh^2$ and we obtain

$$\begin{aligned} B_1(t,\cdot)^2 + B_3(t,\cdot)^2 &= c\frac{b_1^2 + \left(b_3\cosh\left(\sqrt{c}t\right) - \sqrt{c}\sinh\left(\sqrt{c}t\right)\right)^2}{\left(\sqrt{c}\cosh\left(\sqrt{c}t\right) - b_3\sinh\left(\sqrt{c}t\right)\right)^2} \\ &= c\frac{\left(\sqrt{c}\cosh\left(\sqrt{c}t\right) - b_3\sinh\left(\sqrt{c}t\right)\right)^2}{\left(\sqrt{c}\cosh\left(\sqrt{c}t\right) - b_3\sinh\left(\sqrt{c}t\right)\right)^2} = c \end{aligned}$$

2. We do a similar calculation as before in the case of the minus sign and obtain

$$B_1(t,\cdot)^2 - B_3(t,\cdot)^2 = \begin{cases} c\frac{b_1^2 - \left(b_3\cos\left(\sqrt{c}t\right) + \sqrt{c}\sin\left(\sqrt{c}t\right)\right)^2}{\left(\sqrt{c}\cos(\sqrt{c}t) - b_3\sin(\sqrt{c}t)\right)^2} & b_1^2 > b_3^2 \\ \frac{b_1^2 - b_3^2}{(1-b_3t)^2} & b_1^2 = b_3^2 \\ d\frac{b_1^2 - \left(b_3\cosh\left(\sqrt{d}t\right) - \sqrt{d}\sinh\left(\sqrt{d}t\right)\right)^2}{\left(\sqrt{d}\cosh(\sqrt{d}t) - b_3\sinh(\sqrt{d}t)\right)^2} & b_1(x,0)^2 < b_3^2 \end{cases}$$

$$
= \begin{cases} c\dfrac{\left(\sqrt{c}\cos\left(\sqrt{c}t\right)-b_3\sin\left(\sqrt{c}t\right)\right)^2}{\left(\sqrt{c}\cos(\sqrt{c}t)-b_3\sin(\sqrt{c}t)\right)^2} & b_1^2 > b_3^2 \\ 0 & b_1^2 = b_3^2 \\ -d\dfrac{\left(\sqrt{d}\cosh\left(\sqrt{d}t\right)-b_3\sinh\left(\sqrt{d}t\right)\right)^2}{\left(\sqrt{d}\cosh(\sqrt{d}t)-b_3\sinh(\sqrt{d}t)\right)^2} & b_1(x,0)^2 < b_3^2 \end{cases}
$$

$$
= c.
$$

Now one easily calculates:

1. (in the case of the minus sign) The case $x \notin M_1$ is trivial. So consider $x \in \mathring{M}_1$:

$$
\begin{aligned}
&\partial_t B_1(t,\cdot) \\
=&b_1\frac{\sqrt{c}}{\sqrt{c}\cosh\left(\sqrt{c}t\right)-b_3\sinh\left(\sqrt{c}t\right)}\frac{\sqrt{c}\Big(b_3\cosh\left(\sqrt{c}t\right)-\sqrt{c}\sinh\left(\sqrt{c}t\right)\Big)}{\sqrt{c}\cosh\left(\sqrt{c}t\right)-b_3\sinh\left(\sqrt{c}t\right)} \\
=&B_1(t,\cdot)B_3(t,\cdot) = B_2(t,\cdot)B_3(t,\cdot).
\end{aligned}
$$

Then we note that we can write

$$
B_3(t,\cdot) = \frac{B_1(t,\cdot)}{b_1}\Big(b_3\cosh\left(\sqrt{c}t\right)-\sqrt{c}\sinh\left(\sqrt{c}t\right)\Big)
$$

for $x \in M_1$. Hence we obtain by the product rule

$$
\begin{aligned}
\partial_t B_3(t,\cdot) &= \frac{B_1(t,\cdot)B_3(t,\cdot)}{b_1}\Big(b_3\cosh\left(\sqrt{c}t\right)-\sqrt{c}\sinh\left(\sqrt{c}t\right)\Big) \\
&\quad + \sqrt{c}\frac{B_1(t,\cdot)}{b_1}\Big(b_3\sinh\left(\sqrt{c}t\right)-\sqrt{c}\cosh\left(\sqrt{c}t\right)\Big) \\
&= B_3(t,\cdot)^2 - c = -B_1(t,\cdot)^2 = -B_1(t,\cdot)B_2(t,\cdot).
\end{aligned}
$$

2. (in the case of the plus sign) Again the case where $x \in \mathbb{R}\backslash M_1$ is trivial and

we consider the case where $|b_1| > 0$ only:

$$\partial_t B_1(t,\cdot) = \begin{cases} \frac{b_1\sqrt{c}}{\sqrt{c}\cos(\sqrt{c}t)-b_3\sin(\sqrt{c}t)} \frac{\sqrt{c}\left(\sqrt{c}\sin(\sqrt{c}t)+b_3\cos(\sqrt{c}t)\right)}{\sqrt{c}\cos(\sqrt{c}t)-b_3\sin(\sqrt{c}t)} & b_1^2 > b_3^2 \\ \frac{b_1}{1-b_3t}\frac{b_3}{1-b_3t} & b_1^2 = b_3^2 \\ \frac{b_1\sqrt{d}}{\sqrt{d}\cosh(\sqrt{d}t)-b_3\sinh(\sqrt{d}t)} \frac{\sqrt{d}\left(b_3\cosh(\sqrt{d}t)-\sqrt{d}\sinh(\sqrt{d}t)\right)}{\sqrt{d}\cosh(\sqrt{d}t)-b_3\sinh(\sqrt{d}t)} & b_1^2 < b_3^2. \end{cases}$$

So, it holds $\partial_t B_1 = B_1 B_3 = B_2 B_3$ obviously again. And we find similarly

$$\partial_t B_3 = B_3^2 + c = B_1^2 = B_1 B_2.$$

□

***Remark* 2.1.11.** If we use complex roots, then we could also have written in the case of the plus sign in system (2.9)

$$B_1(t,x) = \begin{cases} \sqrt{c(x)}\frac{b_1(x)}{\sqrt{c(x)}\cos(\sqrt{c(x)}t)-b_3(x)\sin(\sqrt{c(x)}t)} & b_1(x)^2 \neq b_3(x)^2 \\ \frac{b_1(x)}{1-b_3(x)t} & b_1(x)^2 = b_3(x)^2 \end{cases}$$
$$B_3(t,x) = \begin{cases} \sqrt{c(x)}\frac{b_3(x)\cos(\sqrt{c(x)}t)+\sqrt{c(x)}\sin(\sqrt{c(x)}t)}{\sqrt{c(x)}\cos(\sqrt{c(x)}t)-b_3(x)\sin(\sqrt{c(x)}t)} & b_1(x)^2 \neq b_3(x)^2 \\ \frac{b_3(x)}{1-b_3(x)t} & b_1(x)^2 = b_3(x)^2 \end{cases}$$

where $c := b_1^2 - b_3^2 = B_1(t,\cdot)^2 - B_3(t,\cdot)^2$ for all $t \in \mathbb{R}$ as long as the solution exists. It does not matter which of the two branches of the root we choose. The natural choice would be the principal branch.

It is obvious that in some cases the solution exists globally in time and in other cases not. We give a more precise statement of this fact in the following comment.

***Remark* 2.1.12.**

1. It is clear that the solution exists as long the denominator does not vanish. This is impossible in the case of the minus sign in system (2.9) because of

$$\begin{aligned} &2\sqrt{c(x)}\cosh\left(\sqrt{c(x)}t\right) - 2b_3(x)\sinh\left(\sqrt{c(x)}t\right) \\ =&(\sqrt{c(x)} - b_3(x))e^{\sqrt{c(x)}t} + (\sqrt{c(x)} + b_3(x))e^{-\sqrt{c(x)}t}, \end{aligned}$$

which is non-zero except for x such that $b_1(x) = b_3(x) = 0$. But in this case the complete solution vanishes identically.

2. Obviously, system (2.9) is invariant under the the transform $t \to -t$ and the change of the signs of an odd number of the B_i, $i \in \{1, 2, 3\}$. Since we restricted ourselves to the case $B_1 = B_2$, the only remaining possibility is changing $t \to -t$ and $B_3 \to -B_3$ which corresponds to solving the equation backwards in time and changing the sign of the initial data of B_3.

3. In the case of the plus sign in system (2.9) there might be zeros of the denominator. If we assume $b_3(x) \geq 0$ then the solution exists for $t \in I(x)$ where $\tilde{c} = \frac{\sqrt{c(x)}}{b_3(x)}$ and

$$I(x) := \begin{cases} \left]\frac{1}{\sqrt{c(x)}} \arctan \tilde{c} - \frac{\pi}{\sqrt{c(x)}}, \frac{1}{\sqrt{c(x)}} \arctan \tilde{c}\right[& b_1(x)^2 > b_3(x)^2 > 0 \\ \left]-\frac{\pi}{2|b_1(x)|}, \frac{\pi}{2|b_1(x)|}\right[& b_1(x)^2 > b_3(x)^2 = 0 \\ \left]-\infty, b_3(x)^{-1}\right[& b_1(x)^2 = b_3(x)^2 \\ \left]-\infty, \frac{1}{\sqrt{d(x)}} \operatorname{artanh}\left(\frac{\sqrt{d(x)}}{b_3(x)}\right)\right[& b_1(x)^2 < b_3(x)^2. \end{cases}$$

We define '$\frac{1}{0} = \infty$' here. Values for $b_3(x) < 0$ can be found by the substitution $t \to -t$ according to the preceding comment. At the endpoints of these intervals, $B_1(t, x)$ becomes unbounded and, because of the conserved quantity, $B_3(t, x)$, too. Hence, in general the solution will exist for $t \in I = \cap_{x \in \mathbb{R}} I(x)$.

We need to know when $|B_1|$ becomes large or maximal for the construction of a counterexample in Section 2.1.5. The answer to this question is as follows.

***Remark* 2.1.13.** In the case of the minus sign one can ask when $|B_1(t, x)|$ becomes maximal. It is obvious that $B_1(t, x) \to 0$ for $t \to \infty$ and all $x \in \mathbb{R}$. Hence, there has to be some finite time such that $|B_1(t, x)|$ becomes maximal. Indeed, this must be the case when $B_3(t, x)$ vanishes because of the conserved quantity. Therefore $|B_1(t, x)|$ becomes maximal for $b_1(x) \neq 0$ at

$$t(x) = \frac{\operatorname{artanh}\left(\frac{b_3(x)}{\sqrt{c(x)}}\right)}{\sqrt{c(x)}} = \frac{\ln\left(\sqrt{c(x)} + b_3(x)\right) - \ln|b_1(x)|}{\sqrt{c(x)}}.$$

We conclude:

1. if $b_1(x) = 0$, $B_1(t, x) = 0$ for all $t \in \mathbb{R}$,

2. if $b_3(x) < 0$, then the maximum of $B_1(t, x)$ is in the past ($t < 0$). Or rephrased: $B_1(t, x)$ is monotonically decreasing with t.

3. We note in view of Hypothesis 2.1.15 that if $b_3(x) = \gamma > 0$ on a compact, connected component $N \subset \operatorname{supp}(b_1)$, then $\min_{x \in N} t(x) = t(x_0)$ where $x_0 \in \arg\max_{x \in N} |b_1(x)|$. This is a consequence of the fact that

$$\mathbb{R}^+ \ni y \mapsto \frac{\ln\left(\sqrt{y^2 + \gamma^2} + \gamma\right) - \ln|y|}{\sqrt{y^2 + \gamma^2}}$$

is strictly monotonic decreasing since its derivative is

$$\begin{aligned}\mathbb{R}^+ \ni y \mapsto & \frac{y}{\left(\sqrt{y^2+\gamma^2}+\gamma\right)(y^2+\gamma^2)} - \frac{1}{y\sqrt{y^2+\gamma^2}} \\ & + \frac{y\left(\ln y - \ln\left(\sqrt{y^2+\gamma^2}+\gamma\right)\right)}{\sqrt[3/2]{y^2+\gamma^2}} < 0.\end{aligned}$$

Note that this means $\sup_{x \in N} |B_1(x, t)|$ is maximal for all $t \in \mathbb{R}$, too.

In the case of the plus sign, this question does not make sense since the solution might not be bounded at all.

After the discussion of existence, uniqueness, and some other properties of the solutions to system (2.9), the remaining task is to prove a regularity result. At this point the problem becomes much more involved and we will use further restrictive assumptions in addition to Hypothesis 2.1.9. A first step in this direction is the following lemma concerning classical differentiability.

Lemma 2.1.14. *Assume Hypothesis 2.1.9 is true and additionally that* $\exists \delta > 0$ $\forall x \in \operatorname{supp}(b_1) : B_\delta(x) \subset \operatorname{supp}(b_3)$. *Assume that* $b_1, b_3 \in C^n(\mathbb{R})$ *for a* $n \in \mathbb{N}$. *Then we have* $\partial^\alpha B_1, \partial^\alpha B_3 \in C^{n-|\alpha|}(I \times \mathbb{R}, \mathbb{R})$ *for all multiindices* $\alpha \in \mathbb{N}_0^2$ *with* $|\alpha| \leq n$ *and* I *the maximal interval where the solution exists.*

Proof. We observe that by assumption Lemma 2.1.10 holds and that we have to ensure that $\sqrt{c}, \sqrt{d}$ are differentiable whenever needed, which means for $x \in \operatorname{supp}(b_1)$. Then the rest follows by product and chain rule. But this point is trivial

under the assumptions since whenever $x \notin \operatorname{supp}(b_1)$, $B_1 \equiv 0$ and $B_3(t,x) = b_3(x)$ and we need no knowledge of $\sqrt{c}, \sqrt{d}$.
On the other hand side, whenever $x \in \operatorname{supp}(b_1)$, then $c, d : B_{\frac{\delta}{2}}(x) \to \mathbb{R}^+$ and $\sqrt{\cdot} : \mathbb{R}^+ \to \mathbb{R}^+$ is differentiable. Thus the result is clear by the chain rule. $\square$

In the following we will discuss Sobolev norms of the solutions to system (2.9) given by Lemma 2.1.10. We will restrict ourselves to the case of the minus sign in system (2.9) because it has to be done in a pretty specific way. We summarise the assumptions that we will use as follows.

Hypothesis 2.1.15.

1. *The initial data for the Cauchy problem with equation* (2.9) *are real-valued and satisfy* $b_1 = b_2, b_3 \in C^n(\mathbb{R}) \cap H^n \cap W^{n,\infty}$ *for an* $n \in \mathbb{N}_0$.

2. *Let* $\operatorname{supp}(b_1)$ *be compact and* $\exists \delta > 0\ \forall x \in \operatorname{supp}(b_1) : B_\delta(x) \subset \operatorname{supp}(b_3)$.

3. *Let* N *be a connected component of* $\operatorname{supp}(b_1)$. *Then* $|b_3(x)| = \gamma > 0$ *for all* $x \in B_\delta(N) := \bigcup_{y \in N} B_\delta(y)$.

4. $|b_1(x)| \leq C_0 \varkappa, \ldots, |\partial_x^n b_1(x)| \leq C_n \varkappa$ *for a* $\varkappa > 0$ *and some constants* $C_0, \ldots, C_n \in \mathbb{R}_0^+$.

A picture of what we have in mind is shown in Figure 2.1. The intention behind these assumptions is as follows. The first one is necessary since the solution is non-smoothing. Further we ensure that Hypothesis 2.1.9 is satisfied. The second one allows us to work with pointwise estimates on a compact set for all Sobolev norms and helps, together with the third and fourth one, to obtain a regularity result in the following theorem. Actually, the second point is used to apply Lemma 2.1.14.
With these assumptions we are able to prove the regularity result.

Theorem 2.1.16. *Let us assume that Hypothesis 2.1.15 is satisfied for initial data* $B_i(0,\cdot) = b_i$, $i \in \{1,2,3\}$. *Let* $I = [0, (-\ln \varkappa + t_0)/\sqrt{\tilde{c}}]$ *for a* $t_0 > 0$, $\varkappa$ *as in Hypothesis 2.1.15 and* $\tilde{c} = \max_{x \in \operatorname{supp} b_1} c(x) = \max_{x \in \operatorname{supp} b_1} b_1(x)^2 + b_3(x)^2$.
Then the following statements are true for the Cauchy problem for system (2.9).

1. *There is a unique classical solution* $(B_1, B_2, B_3) \in C^n(\mathbb{R}^2; \mathbb{R})$.

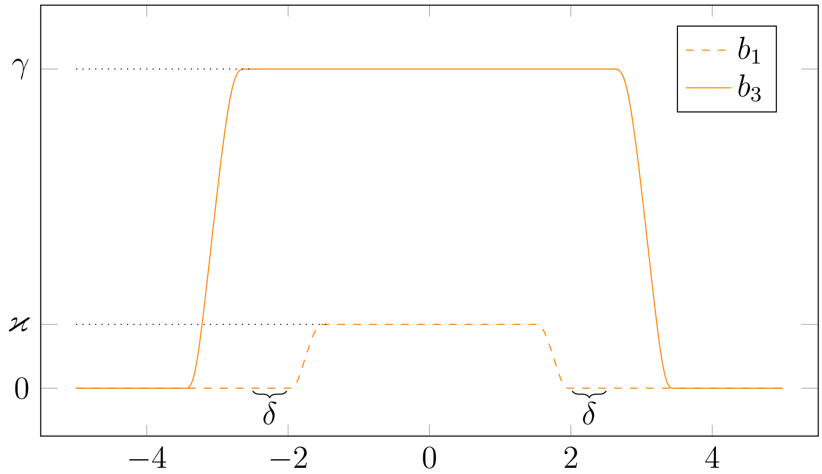


Figure 2.1: Sketch of what the assumptions in Hypothesis 2.1.15 mean, here with C_c^∞ initial data.

2. *We have* $\partial_t^l B_i \in L^\infty(I, H^k)$ *for integers* $0 \le l + k \le n$ *and*

$$\|\partial_t^l B_i\|_{L^\infty(I,H^k)} \lesssim 1.$$

3. *We have* $\partial_t^l B_i \in L^\infty(I, W^{k,\infty})$ *for integers* $0 \le l + k \le n$ *and*

$$\|\partial_t^l B_i\|_{L^\infty(I,W^{k,\infty})} \lesssim 1.$$

The hidden constants are independent of $\varkappa$*.*

The proof of Theorem 2.1.16 is a – although tedious – straightforward calculation that brings no new insights into the problem. We skip it here and refer to Appendix 2.C for the details.

***Remark* 2.1.17.** Theorem 2.1.16 states that we can bound the H^n and $W^{n,\infty}$ norms of $B_1 \equiv B_2$ and B_3 uniformly with respect to b_1 on the interval I provided b_1 satisfies the assumptions of Hypothesis 2.1.15. In particular, we can make $\varkappa$ as small as we want, as long as it does not vanish, and obtain the same bounds. Especially in the case $b_3 \ge 0$ this theorem is of interest. Together with Remark 2.1.13 it states that the uniform estimates hold at least until B_1 reaches its global maximum.

***Remark* 2.1.18.** The question whether we can get rid of the second and third assumption in Hypothesis 2.1.15 is open. We used them for the estimate

$$|\partial_x^n \sqrt{c}| \lesssim \varkappa^2 \text{ for } n \geq 1,$$

which is in general not true except for the case $|\partial_x^n(b_1(x)^2 + b_3(x)^2)| \lesssim \varkappa^2$ for all $n \geq 1$. Since we used this estimate to obtain estimates that are uniform in $\varkappa$, it is not clear whether one can obtain similar estimates without these assumptions. We see in Figures 2.2(a) and 2.3(a) why this could be necessary. There is a quite steep rise (or decline) of B_1 (or B_3 respectively) in the region where $\partial_x b_1 \neq 0$. The value depends on $\varkappa$ and if additionally $\partial_x b_3 \neq 0$ in this region, it is somewhat doubtful whether the estimate would still hold in general. But we will not investigate this question further since Theorem 2.1.16 is sufficient for our purpose.

2.1.5 Differences Between FWI and TWI Approximation in a Concrete Case

After the derivation of the FWI and TWI system in Section 2.1.3 and the discussion of their solutions and behaviour in the previous Section 2.1.4, we will compare these solutions now and show that they strongly differ at a certain time t_0. We will study a concrete example therefore, which is an easy task since we have the explicit solution formulae of Section 2.1.4. We will use $C_c^\infty(\mathbb{R})$ initial data in the form of scaled versions of the bump function $\mathrm{bmp} : \mathbb{R} \to \mathbb{R}$

$$\mathrm{bmp}(x) = \frac{f(2-x^2)}{f(x^2-1) + f(2-x^2)}, \qquad f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

Recall that the formal approximate solutions that we constructed in Section 2.1.3 with the aid of the FWI or TWI ansatz respectively are

$$\epsilon\psi_{\mathrm{FWI}} = \sum_{i=1}^{2} \sum_{J \in \tilde{\mathcal{I}}_4^{\leq i}} \epsilon^i \mathcal{A}_J(\epsilon^2 \boldsymbol{t}, \epsilon^2 \boldsymbol{x}) e^{\mathrm{i}\Omega(J)\boldsymbol{t} + \mathrm{i}\Xi(J)\boldsymbol{x}},$$

$$\epsilon\psi_{\mathrm{TWI}} = \sum_{J \in \mathcal{I}_3^{\leq 1}} \epsilon A_J(\epsilon \boldsymbol{t}, \epsilon^2 \boldsymbol{x}) e^{\mathrm{i}\Omega(J)\boldsymbol{t} + \mathrm{i}\Xi(J)\boldsymbol{x}},$$

where A_i satisfy system (2.8) and $\mathcal{A}_i$ system (2.7). We choose wave numbers and frequencies in accordance with the restrictions in Section 2.1.3. We choose $\xi_3 < 0$ to be the same for the FWI and TWI ansatz. This choice ensures $\mathrm{sign}(\Gamma_{33}) = \mathrm{sign}(\xi_3) < 0$. In the TWI ansatz we choose $\xi_1, \xi_2 > 0$ so that only $\mathrm{sign}(\Gamma_{33}) < 0$ and the other Γ_{ii}, $i \in \{1, 2\}$, are positive.

In the first example let $b_1 = b_2 = \epsilon\mathrm{bmp}$ and $b_3 = \mathrm{bmp}(\boldsymbol{x}/5)$ for the initial amplitudes of the TWI problem (2.9) in Section 2.1.4. Thus

$$\begin{aligned} A_1(0) &= -\mathrm{i}(|\Gamma_{22}\Gamma_{33}|)^{-1/2} b_1, \\ A_2(0) &= -\mathrm{i}(|\Gamma_{11}\Gamma_{33}|)^{-1/2} b_2, \\ A_3(0) &= -\mathrm{i}(|\Gamma_{11}\Gamma_{22}|)^{-1/2} b_3 \end{aligned}$$

for the TWI system (2.8) and in the non-resonant FWI system we will just use $\mathcal{A}_3(0) = A_3(0)$ and set the initial data of all other $\mathcal{A}_i$, $i \in \{1, 2, 4\}$, of the FWI problem to zero. This ensures that the simplified equation (2.7) gives the complete dynamics of the FWI system.

Obviously these initial data meet all assumptions of Hypothesis 2.1.15 and Theorem 2.1.6, too. Hence the solution formula of Section 2.1.4 apply and we can simply study the dynamics of these. In Figures 2.2(a) and 2.2(b) the absolute value of the initial data is plotted for $\epsilon = 10^{-4}$. These pictures are almost identical except for the small perturbation in form of the small functions $A_1(0), A_2(0)$ in Figure 2.2(a). These are only perceptible as thin lines at the bottom.

Now, while time evolves, the situation changes completely. Remark 2.1.13 tells us that the time when the perturbations of A_3 are maximal is round about $t_0 = \frac{-\ln\epsilon + \ln(1+\sqrt{\tilde{c}})}{\sqrt{\tilde{c}}\epsilon}$, $\tilde{c} = 1 + \epsilon^2$. If we look at the absolute values of the solutions to the amplitude equations (2.7) and (2.8) at that time, there is apparently a big difference between them, cf. Figures 2.3(a) and 2.3(b). Note that the difference between the values of $A_1(\epsilon t_0)$ and $A_2(\epsilon t_0)$ comes from the different wave numbers and frequencies resulting in a difference in Γ_{22} and Γ_{33}. The regularity results of Theorem 2.1.16 remain applicable at that time. This is obvious since we only use $\varkappa = \epsilon$ in Theorem 2.1.16 and Theorem 2.1.6 is a (almost) global result that we only need for $\epsilon^2 t_0 = \frac{-\epsilon\ln\epsilon + \epsilon\ln(1+\sqrt{\tilde{c}})}{\sqrt{\tilde{c}}} < \infty$.

To make things more precise, we do a little calculation. Because of the assumptions we can give a more explicit description of the formal approximate solutions of Sectionn 2.1.3 for these initial data. We know that the solutions to the amplitude equations are given by the formulae on Section 2.1.4 and therefore

$$\begin{aligned}\epsilon\psi_{\mathrm{FWI}}(t,x) &= 2\epsilon\frac{b_3(\epsilon^2 x + c_3\epsilon^2 t)}{\sqrt{|\Gamma_{11}\Gamma_{22}|}}\sin\big(\omega_3 t + \xi_3 x + d_{3,3}\epsilon^2 t b_3^2(\epsilon^2 x + c_3\epsilon^2 t)\big) \qquad (2.10)\\ &\quad - 2\epsilon^2\frac{4\xi_3^2 b_3^2(\epsilon^2 x + c_3\epsilon^2 t)}{|\Gamma_{11}\Gamma_{22}|\mathcal{L}(2\mathrm{i}\omega_3, 2\mathrm{i}\xi_3)}\cos 2\big(\omega_3 t + \xi_3 x + d_{3,3}\epsilon^2 t b_3^2(\epsilon^2 x + c_3\epsilon^2 t)\big),\\ \epsilon\psi_{\mathrm{TWI}}(t,x) &= 2\epsilon\sum_{i=1}^{3}\frac{\sqrt{|\Gamma_{ii}|}}{\sqrt{|\Gamma_{11}\Gamma_{22}\Gamma_{33}|}}B_i(\epsilon t, \epsilon^2 x)\sin(\omega_i t + \xi_i x). \qquad (2.11)\end{aligned}$$

Let $\delta \in \mathbb{R}$ be fixed. At time $t_0 + \delta$ and $\epsilon^2 x \in [-1,1]$ we know that

$$\begin{aligned}&B_3(\epsilon(t_0+\delta), \epsilon^2 x) = \mathcal{O}(\epsilon),\\ &B_1(\epsilon(t_0+\delta), \epsilon^2 x) = B_2(\epsilon(t_0+\delta), \epsilon^2 x) = \sqrt{1+\epsilon^2} + \mathcal{O}(\epsilon)\end{aligned}$$

for sufficiently small ϵ, where the functions B_i are given by the solution formulae of Lemma 2.1.10. This is a consequence of Remark 2.1.13 and Taylor's Theorem since we have $B_1, B_2, B_3 \in C^2(\mathbb{R}^2;\mathbb{R})$ and uniformly L^∞-bounded for $t \le t_0 + \delta$. Hence we have for sufficiently small $\epsilon > 0$

$$\begin{aligned}\epsilon\psi_{\mathrm{FWI}}(t_0+\delta, x) &= 2\epsilon(|\Gamma_{11}\Gamma_{22}|)^{-1/2}\sin\big(\omega_3 t_0 + \xi_3 x + d_{3,3}\epsilon^2 t_0 + \delta(\omega_3 + \epsilon^2 d_{3,3})\big)\\ &\quad + \mathcal{O}(\epsilon^2),\\ \epsilon\psi_{\mathrm{TWI}}(t_0+\delta, x) &= 2\epsilon\sum_{i=1}^{2}\frac{\sqrt{|\Gamma_{ii}|}}{\sqrt{|\Gamma_{11}\Gamma_{22}\Gamma_{33}|}}\sqrt{1+\epsilon^2}\sin(\omega_i t_0 + \xi_i x + \delta\omega_i) + \mathcal{O}(\epsilon^2).\end{aligned}$$

This is the case since $\epsilon^2 t_0 \to 0$ for $\epsilon \to 0$ and $b_3(x) = 1$ for $\epsilon^2 x \in [-2,2]$. By the same argument and Taylor's Theorem we obtain for sufficiently small $\epsilon > 0$

$$\begin{aligned}\epsilon\psi_{\mathrm{FWI}}(t_0+\delta, x) &= 2\epsilon(|\Gamma_{11}\Gamma_{22}|)^{-1/2}\sin(\omega_3 t_0 + \xi_3 x + \delta\omega_3) + \mathcal{O}(\epsilon^2),\\ \epsilon\psi_{\mathrm{TWI}}(t_0+\delta, x) &= 2\epsilon\sum_{i=1}^{2}\frac{\sqrt{|\Gamma_{ii}|}}{\sqrt{|\Gamma_{11}\Gamma_{22}\Gamma_{33}|}}\sin(\omega_i t_0 + \xi_i x + \delta\omega_i) + \mathcal{O}(\epsilon^2).\end{aligned}$$

Note that $\mathcal{O}(\epsilon^2)$ is in some sense dependent on δ. But this is not really problematic since we will choose δ in a compact set. Now let $\omega_3 t_0 + \xi_3 x + \delta\omega_3 = z\pi \in \mathbb{Z}\pi$. Such values exist for all $t_0 > 0$ and $\epsilon > 0$ sufficiently small since $x \in [-\epsilon^{-2}, \epsilon^{-2}]$.

Then the first expression vanishes up to $\mathcal{O}(\epsilon^2)$. The second one becomes

$$\epsilon\psi_{\mathrm{TWI}}(t_0+\delta,x) = 2\epsilon\sum_{i=1}^{2}\frac{\sqrt{|\Gamma_{ii}|}}{\sqrt{|\Gamma_{11}\Gamma_{22}\Gamma_{33}|}}\sin\left((\omega_i-\frac{\xi_i}{\xi_3}\omega_3)(t_0+\delta)+\frac{\xi_i}{\xi_3}z\pi\right)+\mathcal{O}(\epsilon^2). \tag{2.12}$$

Let us assume that $\frac{\omega_i}{\omega_3}-\frac{\xi_i}{\xi_3}\neq 0$ for at least one $i\in\{1,2\}$. We will justify this assumption later in Remark 2.1.19. Then we obtain the estimate

$$\epsilon \lesssim \|\epsilon\psi_{\mathrm{TWI}}-\epsilon\psi_{\mathrm{FWI}}\|_{L^\infty([0,t_0+\delta],L^\infty)} \tag{2.13}$$

for every sufficiently small $\epsilon_0>0$ and all $\epsilon\in]0,\epsilon_0[$ where the constant is independent of ϵ and positive. That this estimate is true can be seen as follows. Without loss of generality we assume $\frac{\omega_1}{\omega_3}-\frac{\xi_1}{\xi_3}\neq 0$ and let $\tilde{\omega}_1 := (\omega_1-\frac{\xi_1}{\xi_3}\omega_3)$, $\tilde{\omega}_2 := (\omega_2-\frac{\xi_1}{\xi_3}\omega_3)$. We distinguish some cases.

Case $\frac{\omega_2}{\omega_3}-\frac{\xi_2}{\xi_3}=0$. Here the result is obvious since the addends in the sum of equation (2.12) become

$$\sqrt{|\Gamma_{11}|}\sin\left(\tilde{\omega}_1(t_0+\delta)+\frac{\xi_1}{\xi_3}z\pi\right)+\sqrt{|\Gamma_{22}|}\sin\left(\frac{\xi_2}{\xi_3}z\pi\right).$$

If the last addend is larger or equal to $\sqrt{|\Gamma_{11}|}/2$, then we can choose $\tilde{\omega}_1\delta\in[-2\pi,0]$ such that the first addend vanishes. In the other case we can choose $\tilde{\omega}_1\delta\in[-2\pi,0]$ such that the sine in the first addend is maximal.

Case $\frac{\omega_2}{\omega_3}-\frac{\xi_2}{\xi_3}\neq 0$. If $|\Gamma_{11}|>|\Gamma_{22}|$ we choose $\tilde{\omega}_1\delta\in[-2\pi,0]$ such that the sine in the first addend is maximal and we obtain in equation (2.12)

$$\epsilon\psi_{\mathrm{TWI}}(t_0+\delta,x)\geq 2\epsilon\frac{\sqrt{|\Gamma_{11}|}-\sqrt{|\Gamma_{22}|}}{\sqrt{|\Gamma_{11}\Gamma_{22}\Gamma_{33}|}}+\mathcal{O}(\epsilon^2).$$

If $|\Gamma_{11}|<|\Gamma_{22}|$ we change the roles and choose $\tilde{\omega}_2\delta\in[-2\pi,0]$ such that the second addend is maximal. If $|\Gamma_{11}| = |\Gamma_{22}|$ the sum in equation (2.12) is

$$\begin{aligned}
&2\sqrt{|\Gamma_{11}|}\sin\left(\frac{\tilde{\omega}_1+\tilde{\omega}_2}{2}(t_0+\delta)+\frac{\xi_1+\xi_2}{2\xi_3}z\pi\right)\times \\
&\times\cos\left(\frac{\tilde{\omega}_1-\tilde{\omega}_2}{2}(t_0+\delta)+\frac{\xi_1-\xi_2}{2\xi_3}z\pi\right) \\
=&-2\sqrt{|\Gamma_{11}|}\sin\left(\frac{z\pi}{2}\right)\cos\left(\frac{\tilde{\omega}_1-\tilde{\omega}_2}{2}(t_0+\delta)+\frac{\xi_1-\xi_2}{2\xi_3}z\pi\right).
\end{aligned}$$

Since $z \in \mathbb{Z}$ we can always choose $|\tilde{\omega}_1 - \tilde{\omega}_2|\delta \in [-4\pi, 0]$ such that the cosine is maximal.

In the last case we used that the resonance condition $\sum_{i=1}^3 \xi_i = \sum_{i=1}^3 \omega_i = 0$ of the TWI system yields

$$\begin{aligned}
&\tilde{\omega}_1+\tilde{\omega}_2=\omega_1+\omega_2-\frac{\xi_1+\xi_2}{\xi_3}\omega_3=\omega_1+\omega_2+\omega_3=0, \\
&\frac{\xi_1+\xi_2}{2\xi_3}=-\frac{1}{2}.
\end{aligned}$$

***Remark* 2.1.19.** Note that the case $\frac{\omega_1}{\omega_3} - \frac{\xi_1}{\xi_3} = \frac{\omega_2}{\omega_3} - \frac{\xi_2}{\xi_3} = 0$ is impossible for the ω defined in equation (2.2) under the resonance condition $\sum_{i=1}^3 \xi_i = \sum_{i=1}^3 \omega_i = 0$ and ξ_i different from each other. This becomes obvious from

$$\frac{\omega_i}{\omega_j}=\frac{\xi_i}{\xi_j} \quad\Leftrightarrow\quad \frac{\omega_i^2}{\xi_i^2}=\frac{\omega_j^2}{\xi_j^2}=\frac{1+\mu\xi_j^4}{1+\xi_j^2} \quad\Leftrightarrow\quad (1+\mu\xi_i^4)(1+\xi_j^2)=(1+\mu\xi_j^4)(1+\xi_i^2)$$

and

$$0=(1+\mu\xi_i^4)(1+\xi_j^2)-(1+\mu\xi_j^4)(1+\xi_i^2)=(\xi_j^2-\xi_i^2)\left(1-\mu(\xi_i^2+\xi_j^2+\xi_i^2\xi_j^2)\right).$$

We know that $(\xi_j^2 - \xi_i^2) \neq 0$ for $i \neq j$ because of the resonance condition and $0 \neq \xi_i$ for all $i \in \{1, 2, 3\}$. Hence $1 = \mu(\xi_i^2 + \xi_j^2 + \xi_i^2\xi_j^2)$ would have to be true. But this is impossible for the same reason.

We summarise what we have seen from this example.

1. It is not possible that an approximation property for $t \in [0, t_0]$ holds for both formal approximate solutions in the sense of the abstract Approximation Property 1.3.1 or more concrete

***Approximation Property* 2.1.20.** Let t_0 be as before. Suppose there is a solution $\epsilon\psi \in C^l([0,t_0] \times \mathbb{R}, \mathbb{R})$ for the FWI or TWI system respectively for a $6 \leq l \in \mathbb{N}$ as constructed in Section 2.1.3. Further let $p > 1$ and suppose $\epsilon(\psi, \partial_t\psi) \in C([0,t_0], H^l \times H^{l-2})$. Finally, let (u,v) a strict solution to equation (2.1) for initial data $(u_0,v_0) \in H^l \times H^{l-2}$ in the sense of Theorem 2.B.1.

Then there is $\epsilon_0 > 0$ such that for all $\epsilon \in]0, \epsilon_0[$ the property

$$\|u_0 - \epsilon\psi(0)\|_{H^l} + \|v_0 - \epsilon\partial_t\psi(0)\|_{H^{l-2}} \lesssim \epsilon^p.$$

is preserved for $t \in \Theta(\frac{-\ln\epsilon}{\epsilon})$, i.e. it holds for a $t_0 \in \Theta(\frac{-\ln\epsilon}{\epsilon})$

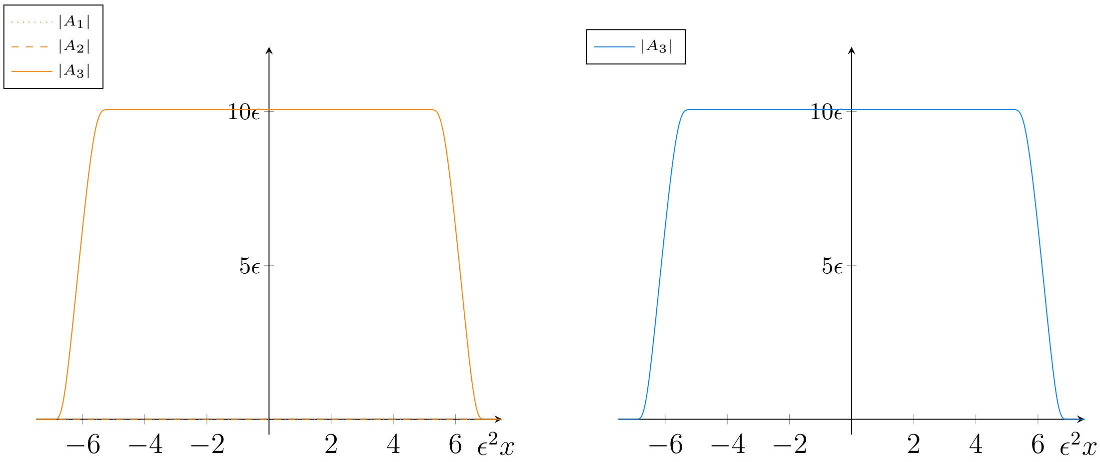
$$\|u - \epsilon\psi\|_{L^\infty([0,t_0],H^l)} \lesssim \epsilon^p.$$

Thus, at least one formal approximate solution fails. This is a simple consequence of the triangle inequality and inequality (2.13).

2. This an example that the time interval in which an approximation property holds is not easily expandable. If ψ_{FWI} fails the approximation property, then the formal approximation does not hold even on the natural time scale $\Theta(\epsilon^{-2})$. Actually, it would mean that an approximation property can hold at most for $t \in [0, \mathcal{O}(\epsilon^{-q})]$ for $q \leq 1$ but not higher.

   If ψ_{TWI} fails the approximation property, then same argument shows that it is not possible to have an extension of the interval where the approximation property holds in powers of ϵ.

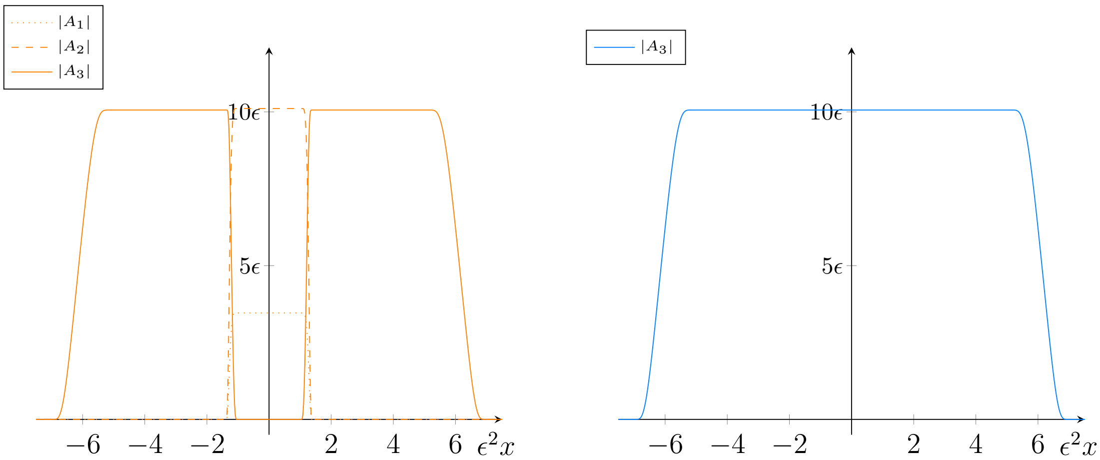
We should give one comment concerning the regularity assumptions. We require these solely because of the search for strict solutions that 'live' in the domain of the operator and continuous differentiability is assumed for the applicability of Theorem 2.1.6 and Theorem 2.1.16 of Section 2.1.4. In view of these theorems, these assumptions are restriction of the regularity of the initial data only.
In the second example we change the sign of $A_3(0)$ for the FWI and TWI system, see Figures 2.4(a) and 2.4(b) respectively. The explicit representation of the solutions are given again by the formulae (2.10) and (2.11) above with the exception of the opposite sign in front of b_3 and B_3. In this case we know that $\partial_t B_3 \geq 0$ and



(a) Absolute values of the envelopes of the TWI system.

(b) Absolute values of the envelope of the sFWI system.

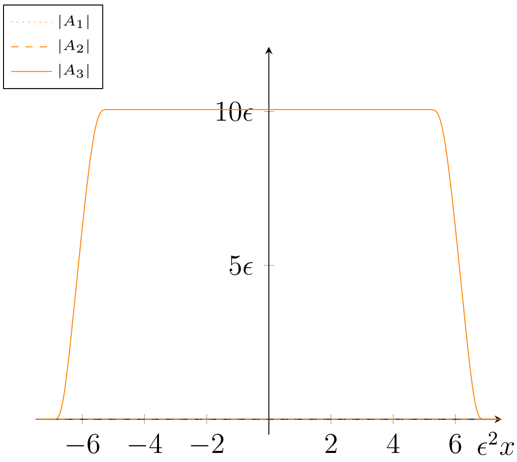
Figure 2.2: The envelopes for smooth initial data at $t = 0$ in the case of a 'unstable' perturbation of the TWI system ($\epsilon = 10^{-4}$).



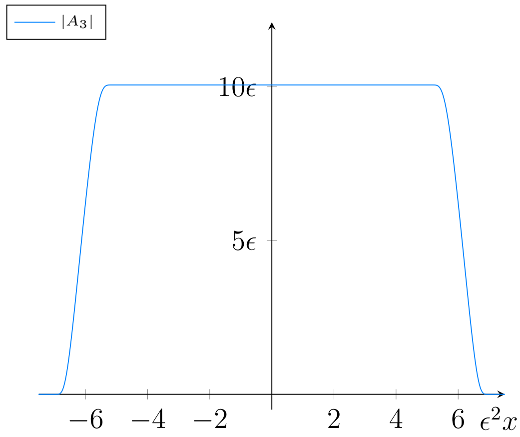
(a) Absolute values of the envelopes of the TWI system.

(b) Absolute values of the envelope of the sFWI system.

Figure 2.3: The envelopes for smooth initial data at $t \approx t_0$ in the case of a 'unstable' perturbation of the TWI system ($\epsilon = 10^{-4}$).

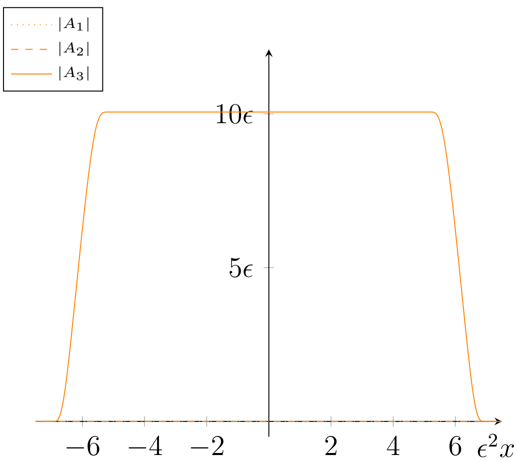


(a) Absolute values of the envelopes of the TWI system.

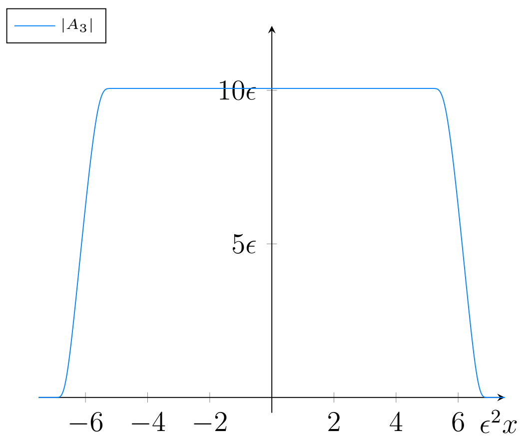


(b) Absolute values of the envelope of the sFWI system.

Figure 2.4: The envelopes for smooth initial data at $t = 0$ in the case of a 'stable' perturbation of the TWI system ($\epsilon = 10^{-4}$).



(a) Absolute values of the envelopes of the TWI system.



(b) Absolute values of the envelope of the sFWI system.

Figure 2.5: The envelopes for smooth initial data at $t \approx t_0$ in the case of a 'stable' perturbation of the TWI system ($\epsilon = 10^{-4}$).

$\partial_t B_1 \leq 0$ pointwise for all $t \geq 0$ according to Lemma 2.1.10. But the conserved quantity of the TWI system in Lemma 2.1.10 only permits

$$b_1^2 + b_3^2 - B_1(t)^2 \leq B_3(t)^2 \leq b_1^2 + b_3^2.$$

Hence we obtain

$$\begin{aligned}
&\|B_3(t) - B_3(0)\|_{L^\infty} \leq \|\sqrt{b_1^2 - B_1^2(t)}\|_{L^\infty} \leq \|b_1\|_{L^\infty} = \epsilon \qquad &&\text{for all } t \geq 0,\\
&\|B_1(t) - B_1(0)\|_{L^\infty} \leq \|b_1\|_{L^\infty} = \epsilon &&\text{for all } t \geq 0.
\end{aligned}$$

As a consequence we obtain upper bounds between the formal FWI and formal TWI approximate solution by

$$\begin{aligned}
\|\epsilon\psi_{\mathrm{TWI}}(t) - \epsilon\psi_{\mathrm{FWI}}(t)\|_{L^\infty} \leq{}& \frac{2\epsilon}{\sqrt{|\Gamma_{11}\Gamma_{22}|}}\|b_3(\epsilon^2\cdot)\sin(\omega_3 t + \xi_3\cdot)\\
&- b_3(\epsilon^2\cdot)\sin\big(\omega_3 t + \xi_3\cdot + d_{3,3}\epsilon^2 t b_3^2(\epsilon^2(\cdot + c_3 t))\big)\|_{L^\infty}\\
&+ \frac{2\epsilon}{\sqrt{|\Gamma_{11}\Gamma_{22}|}}\|b_3(\epsilon^2\cdot) - b_3(\epsilon^2(\cdot + c_3 t))\big)\|_{L^\infty} + \mathcal{O}(\epsilon^2)\\
\leq{}& 2\frac{\|b_3\|_{L^\infty}^3 |d_{3,3}| + \|\partial_x b_3\|_{L^\infty}|c_3|}{\sqrt{|\Gamma_{11}\Gamma_{22}|}}\epsilon^3 t + \mathcal{O}(\epsilon^2) + \mathcal{O}(\epsilon^3 t)
\end{aligned}$$

Figures 2.5(a) and 2.5(b) show the absolute value of the amplitudes for the chosen initial data at time ϵt_0 given above. They show almost no difference as expected by the previous estimates. Eventually, the estimates let us conclude that for these initial data the formal approximate solutions constructed with the aid of the FWI and TWI system remain close for $t \in [0, \Theta(\epsilon^{-p})]$, $p \in [0, 2[$.
There are some consequences of this calculation.

1. If one the the formal approximate solutions constructed with the aid of the FWI or TWI system is close to a true solution, so is the other for $t \in [0, \Theta(\epsilon^{-p})]$, $p \in [0, 2[$, with an error at most $\mathcal{O}(\epsilon^{3-p})$.

2. In this case we would obtain that the time scale where the TWI system delivers an approximate solution can be extended beyond its natural time scale.

3. But again at least one approximate solution fails at a certain point. If $|c_3| > 0$ we have for $t = \frac{5(\sqrt{2}-1)}{|c_3|\epsilon^2}$ in the FWI approximate solution (w.l.o.g

$c_3 > 0$, otherwise use $\epsilon^2 x_0 = -5$ instead of $\epsilon^2 x_0 = 5$)

$$b_3(5\sqrt{2}) \sin\left(\omega_3 \frac{\sqrt{2}}{|c_3|\epsilon^2} + \xi_3 x_0 \epsilon^{-2} + d_{3,3} \frac{\sqrt{2}}{|c_3|} b_3^2(5\sqrt{2})\right) = 0$$

and in the TWI approximate solution we find for all $t \geq 0$ and $\epsilon^2 |x_0| = 5$, since $\pm 5 \notin \operatorname{supp}(b_1)$ and therefore the B_i are constant in time,

$$\begin{aligned} B_3(\epsilon t, \pm 5) \sin(\omega_3 t + \xi_3 x_0 \epsilon^{-2}) &= B_3(0, \pm 5) \sin(\omega_3 t + \xi_3 x_0 \epsilon^{-2}) \\ &= \sin(\omega_3 t + \xi_3 x_0 \epsilon^{-2}). \end{aligned}$$

Hence, we obtain a difference of $\Omega(1)$ for $\epsilon \to 0$ by shifting x_0 by $\delta \in [0, 2\pi\epsilon^2]$ since the addends $i = 1$ and $i = 2$ can be neglect in the TWI ansatz for these initial data, cf. the estimate above. The reason is simply that the transport term in the FWI equation (2.11) shifts the support of the functions in the FWI system but the support of the TWI solution is invariant in time.

If $c_3 = 0$, then we can again choose $\epsilon^2 |x_0| = 5$ and obtain for the FWI approximate solution

$$b_3(5) \sin(\omega_3 t + \xi_3 x_0 \epsilon^{-2} + d_{3,3} t b_3^2(5)) = \sin(\omega_3 t + \xi_3 x_0 \epsilon^{-2} + d_{3,3} \epsilon^2 t)$$

Hence, there is a $t \in [0, 2\pi\epsilon^{-1}]$ such that we find a difference of order $\Omega(1)$ between these solutions for $\epsilon \to 0$.

***Remark* 2.1.21.** Note that if we take $b_1 = b_2 \equiv 0$ as initial data, then the behaviour of ψ_{TWI} and ψ_{FWI} are more or less the same up to t_0 and ϵ small enough. In fact, the TWI system is just static with $B_1(t) = B_2(t) \equiv 0, B_3(t) = b_3$ for all $t \geq 0$ and the simplified FWI system differs not much from its initial value since it is continuous and $\epsilon^2 t_0 \to 0$ for $\epsilon \to 0$. This will happen independently of the sign of b_3 and the results will be similar to the case of the negative sign of b_3 discussed in the previous paragraphs. This is clear by comparison with the calculations above.
The conclusion of this observation is as follows. *Approximation Property 2.1.20 can hold for the TWI system for $t \in [0, \Theta(\epsilon^{-p})]$ and $p \in [0, 1]$ but for none $p > 1$.* This is a simple consequence of the fact that the calculations above show that changing the initial data in our first example by $\Theta(\epsilon^{\nu})$, $\nu \in \mathbb{N}$ – for example

change $b_1 = b_2 \equiv 0$ as suggested – leads to an error $\Omega(\epsilon)$ for $t \in \Theta(-\epsilon^{-1} \ln \epsilon)$. Hence, we violate the stability part of the approximation property.

2.1.6 Error Estimate

We need an error estimate to be able to make a decision whether any of the constructed formal approximate solutions is a true approximate solution. We will give an error estimate in the following Corollary 2.1.30. The corollary is an immediate consequence of the more abstract Theorem 2.1.28. We will always consider the time interval $I = \left[0, \frac{-\ln \epsilon + q t_0}{q\epsilon\sqrt{\tilde{c}}}\right]$ where $t_0 > 0$ and $\tilde{c} = 1 + \epsilon^{2/q}$, $q \in [1, \infty[$, in the rest of this section. The interval I is longer than the natural time scale of the TWI system, being $\Theta(\epsilon^{-1})$, but much shorter than the natural time scale of the FWI system which is $\Theta(\epsilon^{-2})$. We choose this time interval in view of the concrete examples given in the previous Section 2.1.5. We saw in that section that the formal approximate solutions constructed using the FWI and TWI system exhibit a difference of order $\Omega(\epsilon)$ in the time interval I for $q = 1$ and $\epsilon \to 0$. For $q > 1$ the same examples and similar differences as in Section 2.1.5 would appear for $t \in I$ if we replace the initial data $b_1 = b_2 = \epsilon\mathrm{bmp}$ of Section 2.1.5 by $b_1 = b_2 = \sqrt[q]{\epsilon}\mathrm{bmp}$. This becomes clear when we recall that Remark 2.1.13 states that for such initial data B_3 vanishes for t near the end of the interval I in the case of the first example in Section 2.1.5, which means that $|B_1|$ and $|B_2|$ or equivalently $|A_1|$ and $|A_2|$ are maximal. Then the calculations of Section 2.1.5 remain essentially unchanged.

A consequence of the choice of I is that the reasoning in the rest of this section is insufficient for proving an approximation property of the FWI system on its natural time scale. But if the conditions of Corollary 2.1.30 are met by the formal approximate solution constructed with the aid of the TWI system, then we could rule out that such an approximation property can be true for the formal approximate solution constructed with the aid of the FWI system at all. A by-product of the following arguments is that we can give an error estimate under certain conditions that is sufficient to show an approximation property on the natural time scale of the TWI ansatz, see Theorem 2.1.34 and Corollary 2.1.35 below. This is one of the cases mentioned in Section 1.2 where such a result is rather trivial.

Prelude

We start implementing the program sketched in Section 1.1 for the proof of the error bound. Therefore we write for a strict solution u to the Boussinesq problem (2.1) and a formal approximate solution $\epsilon\psi$ that $u = \epsilon\psi + \epsilon^p R$ for a $p > 1$. Consequently, R has to satisfy the PDE

$$\begin{aligned} & 0 = \mathcal{L}(\partial_t, \partial_x)(\epsilon\psi + \epsilon^p R) + \partial_x^2(\epsilon\psi + \epsilon^p R)^2 \\ \Leftrightarrow \quad & 0 = \mathcal{L}(\partial_t, \partial_x)R + \partial_x^2(\epsilon\psi R + \epsilon^p R^2) + \epsilon^{-p}\mathfrak{E}_{\epsilon\psi} \end{aligned} \tag{2.14}$$

We recall that the residual $\mathfrak{E}_{\epsilon\psi}$ is defined by $\mathfrak{E}_{\epsilon\psi} = \mathcal{L}(\partial_t, \partial_x)\epsilon\psi + \partial_x^2(\epsilon\psi)^2$ for sufficiently regular functions ψ, see Definition 1.1.2 in Section 1.1. Since $p > 1$ we aim for a proof of $\|R\|_{L^\infty(I, L^\infty)} \lesssim 1$.
We need some regularity assumptions for the approximate solution and the initial data for the proof of the error estimate. Some other assumptions concerning the structure of the linear operator and the nonlinear part are required as well.
We will work in a little more abstract setting in the following. Then we are able to use the same theorem for the Boussinesq probelm (2.1) and other problems as well, see Section 2.2. We state the assumptions on the regularity of the approximate solution $\epsilon\psi$ and the structural assumptions in the following hypothesis and we will always assume these regularity assumptions subsequently.

Hypothesis 2.1.22. *Let*

$$0 = \mathcal{L}(\partial_t, \partial_x)u + f(u) \tag{2.15}$$

be the considered PDE. We assume for $(\omega, \xi) \in \mathbb{C}^2$

$$\mathcal{L}(\omega, \xi) = -\omega^2 q_t(\xi^2) - q_x(\xi^2)$$

for polynomials $0 \neq q_t, q_x$ *of degrees* $D_t \in \mathbb{N}_0$ *and* $D_x \in \mathbb{N}$. *We assume* $D_x \geq D_t$ *and require* $\lim_{z\to 0} q_t(z) = 1$ *as well as* $q_t, q_x : \mathbb{R}^- \to \mathbb{R}^+$ *and let* $d_x := \arg\max_{d\in\mathbb{R}_0^+} z^{-d} q_x(z) \in \mathcal{O}(1)$ *for* $z \to 0$.
Finally, we assume that there is $m_0 \in \mathbb{R}_0^+$ *such that*

- $M_{q_t^{-1}(-\boldsymbol{\xi}^2)} \mathscr{F} \circ f \circ \mathscr{F}^{-1} \in C^1(L^2_{D_x - D_t + m}, L^{2,b}_m \cap L^{2,b}_{-d_x})$ *for all* $m \geq m_0$.
- *There is a continuous and monotonically increasing function* $h_1 : \mathbb{R}_0^+ \to \mathbb{R}_0^+$

with $h_1 \in \mathcal{O}(|\boldsymbol{x}|)$ *for* $x \to 0$ *such that the estimates*

$$\|M_{q_t^{-1}(-\boldsymbol{\xi}^2)}\,\mathscr{F}\,Df(v)\,\mathscr{F}^{-1}\|_{\mathrm{L}(L^2_{D_x-D_t+m},L^{2,b}_m)} \leq h_1(\|v\|_{W^{D_x-D_t+m-1,\infty}})$$
$$\|M_{q_t^{-1}(-\boldsymbol{\xi}^2)}\,\mathscr{F}\,Df(v)\,\mathscr{F}^{-1}\|_{\mathrm{L}(L^2_{D_x-D_t+m},L^{2,b}_{-d_x})} \leq h_1(\|v\|_{W^{D_x-D_t+m-1,\infty}})$$

are true for all $v \in L^2_{D_x-D_t+m}$ *and* $m \geq m_0$.

- $\mathscr{F} \circ f \circ \mathscr{F}^{-1} : L^2_{D_x-D_t+m} \to L^2_{m-2D_t}$ *is Lipschitz on bounded subsets for all* $m \geq m_0$.

For the Boussinesq problem (2.1) we have $f : u \mapsto \partial_x^2 u^2$ and $q_t(z) = (1-z)$, $q_x(z) = -(z + \mu z^3)$ as well as $D_t = 1, D_x = 3, d_x = 1$. In general Proposition A.1.4 can help to check the assumptions on the nonlinear part as long as it consists of combinations of analytic functions and spatial derivatives.

We should explain some points of these assumptions.

***Remark* 2.1.23.**

1. Note that $z^{-1}(q_t(z) - 1) \in \mathcal{O}(1)$ for $z \to 0$ since q_t is a polynomial. It seems possible to easily allow for more general symbols q_t, q_x than polynomials provided that they satisfy these assumptions. Also one could think of nonlinear parts that are only defined on an open and convex sets instead of the complete space $L^2_{D_x-D_t+m}$.

2. The last assumption is used to prove existence and uniqueness to the Cauchy problem for equation (2.15), see Appendix 2.B. We require this condition albeit it is somewhat redundant in view of the second condition on the mapping f. But we keep it nonetheless since it is necessary for the way we prove existence and uniqueness of solutions and one could be tempted to modify the second condition in view of its application later.

3. The use of the structure of the linear operator $\mathcal{L}$ in Hypothesis 2.1.22 is in essence the possibility to construct a 'linear' energy which dominates some Sobolev (semi-) norms. Assume for the moment $f \equiv 0$ and let $\tilde{q}_t, \tilde{q}_x : \mathbb{R} \to \mathbb{R}^+$, $\xi \mapsto q_t(-\xi^2)$ and $\xi \mapsto q_x(-\xi^2)$ respectively. Then we can

define an energy for sufficiently regular functions R

$$\begin{aligned} E_{\hat{R}}(t) &= \frac{1}{2}\int_{\mathbb{R}} \tilde{q}_t(\xi)|\partial_t \hat{R}(\xi)|^2 + \tilde{q}_x(\xi)|\hat{R}(\xi)|^2 \,\mathrm{d}\xi \\ &= \frac{1}{2}\big(\langle \tilde{q}_t \partial_t \hat{R}, \partial_t \hat{R}\rangle_{L^2} + \langle \tilde{q}_x \hat{R}, \hat{R}\rangle_{L^2}\big) \end{aligned} \tag{2.16}$$

and it follows

$$\begin{aligned} \partial_t E_{\hat{R}}(t) &= \mathrm{Re}\big(\langle \tilde{q}_t \partial_t^2 \hat{R}, \partial_t \hat{R}\rangle_{L^2} + \langle \partial_t \hat{R}, \tilde{q}_x \hat{R}\rangle_{L^2}\big) \\ &= \mathrm{Re}\big(- \langle \tilde{q}_x \hat{R}, \partial_t \hat{R}\rangle_{L^2} + \langle \partial_t \hat{R}, \tilde{q}_x \hat{R}\rangle_{L^2}\big) \\ &= \mathrm{Re}\big(- \overline{\langle \partial_t \hat{R}, \tilde{q}_x \hat{R}\rangle_{L^2}} + \langle \partial_t \hat{R}, \tilde{q}_x \hat{R}\rangle_{L^2}\big) = 0. \end{aligned}$$

If $f \not\equiv 0$, we obtain

$$\partial_t E_{\hat{R}}(t) = \mathrm{Re}\langle \mathscr{F} f(R), \partial_t \hat{R}\rangle_{L^2}.$$

4. If $d_x = 0$ we can bound Sobolev norms H^s, $s \in [0, D_x]$ by (2.16). For Sobolev norms up to order $D_x + l$, $l \geq 0$, we can simply choose

$$\tilde{E}_{\hat{R}}(t) := E_{\hat{R}}(t) + E_{|\xi|^l \hat{R}}(t)$$

for sufficiently regular solutions R. To circumvent the problem $d_x > 0$ we could use the linear combination

$$\tilde{E}_{\hat{R}}(t) := E_{\hat{R}}(t) + E_{|\boldsymbol{\xi}|^{-d_x} \hat{R}}(t)$$

provided that e.g. $(\hat{R}, \partial_t \hat{R}) \in C(I, L^{2,b}_{D_x} \cap L^2_{D_x - d_x} \times L^2_{D_t} \cap L^2_{D_t - d_x} \cap L^{2,b}_{-d_x})$ (this requirement is slightly more strict than necessary). Since $D_x \geq d_x \geq 0$ it is equivalent

$$\hat{R}(t) \in L^{2,b}_{D_x} \cap L^2_{D_x - d_x} \qquad \Leftrightarrow \qquad \hat{R}(t) \in L^2_{D_x} \cap L^2_{D_x - d_x} \subset L^2_{D_x}.$$

The property $\partial_t \hat{R} \in C(I, L^{2,b}_{-d_x})$ is problematic in general. But if we suppose that $\partial_t \hat{R}(0) \in L^{2,b}_{-d_x}$ and R is a strict solution to (2.15) in the sense of Theorem 2.B.1, then it will hold as long as the solution exists – simply because the equation respects this property since

$$M_{q_t^{-1}(-\boldsymbol{\xi}^2)}\, \mathscr{F}^{-1} \circ f \circ \mathscr{F} : L^2_{D_x - D_t + m} \to L^{2,b}_{-d_x}$$

according to the first assumption on f.

Obviously, higher order Sobolev norms can be controlled in the same way as in the case $d_x = 0$.

In essence we will use such an energy for the proof of Theorem 2.1.28. However, going back to our original problem governed by equation (2.14), we note that here

$$\mathscr{F}^{-1} \circ f(t) \circ \mathscr{F} : L^2_2 \to L^{2,b}_{-1}, \qquad \hat{u} \mapsto \mathscr{F}\, \partial_x^2(\epsilon\psi(t)u + \epsilon^p u^2) + \mathscr{F}\, \epsilon^{-p}\mathfrak{E}_{\epsilon\psi}(t)$$

will only be true in the case of $\mathscr{F}\, \epsilon^{-p}\mathfrak{E}_{\epsilon\psi} \in C(I, L^{2,b}_{-1})$, which is not granted for free. In particular, we note that the equations (2.7) and (2.9) are somewhat problematic since $(L^{2,b}_{-1} \cap L^2_s, +, *)$ is no convolution algebra for any $s \in \mathbb{R}$.
For functions $v \in L^2$ with compact support, there is an simple characterisation of this property as follows:

$$\hat{v} \in L^{2,b}_{-1} \cap L^2_s \quad \Leftrightarrow \quad v \in H^s \wedge \int_{\mathbb{R}} v(x)\, \mathrm{d}x = 0 \quad \Leftrightarrow \quad \hat{v} \in L^2_s \wedge \hat{v}(0) = 0,$$

since $\hat{v}$ can be extended to an entire function in that case. The set of such functions is not empty since for every compactly supported function $v \in H^s$ trivially $v(\cdot) - v(\cdot + x_0) \in H^s$ with compact support and mean zero for all $x_0 \in \mathbb{R}$. For general functions in L^2 this is not enough or does not even make sense. The last statement above is clearly not sufficient since any function that looks like $\sqrt{\boldsymbol{\xi}}$ in an environment of $\xi = 0$ and that is in L^2_s satisfies the last statement but is not in $L^{2,b}_{-1}$. The middle statement might not make sense as a Lebesgue integral at all.
To cope with this problem, we combine the two observations above which had been that it is sufficient that the residual is in this space and that the initial data have to be in a suitable space as well. We achieve this by modifying the function ψ by $\tilde{\psi} := \psi + \phi$ and determining the function ϕ in such a way that $\mathscr{F}\, \epsilon^{-p}\mathfrak{E}_{\epsilon\tilde{\psi}} \in C(I, L^2_l \cap L^{2,b}_{-1})$ for a sufficiently large $l \in \mathbb{N}$. The price we have to pay is a worsening of the powers of ϵ in the estimates, cf. Remarks 2.1.33 and 2.2.6. This is the statement of Corollary 2.1.27 which is formulated in the

abstract setting for $d_x \leq 2$. For its proof we use Lemma 2.1.26 as an auxiliary result. In this lemma we estimate how strong the difference between the functions ψ and $\tilde{\psi}$ (and the residual) becomes in L^2_s and $W^{s,\infty}$ norms for $t \in I$. Note that we also use this lemma to cope with none vanishing initial data – particularly when they are not in the space $L^{2,b}_{-1}$. This is useful in the case $d_x = 0$, too, although we do not have to care much about the residual in this situation. With these tools we can deduce the error estimate in Theorem 2.1.28.
A similar idea was used by Alterman and Rauch in [3]. In essence, they used a cut-off function vanishing in an environment of 0 that was applied to the Fourier transform and looked for solution in a suitable space. Finally, they had to estimate the error between this function and the true solution.

Preparations

Some regularity and 'smallness' suppositions are necessary for all these results. We list sufficient suppositions in the next hypothesis.

Hypothesis 2.1.24. *Let $m_0 \leq l \in \mathbb{N}$,*

$$s = \max\{2(D_x - D_t) + l, l + 2\}, \qquad r = D_x - D_t + l.$$

We assume there is a function $\psi \in C^{s+\max\{2D_x, 2D_t+2\}}(I \times \mathbb{R}, \mathbb{C})$ and

$$\epsilon\psi \in L^\infty(I, W^{r,\infty}), \qquad \mathfrak{E}_{\epsilon\psi} \in L^\infty(I, H^s), \qquad \|\mathfrak{E}_{\epsilon\psi}\|_{L^\infty(I,H^s)} \leq C_0\epsilon^N$$

with bounds of these norms uniform in $\epsilon \in]0, \epsilon_0]$ for $t \in I$, an $\epsilon_0 \in]0, 1]$ and any arbitrary fixed $N \in \mathbb{N}$, which we will determine later. For the sake of simplicity, we assume $f : C^{s+\max\{2D_x, 2D_t+2\}}(I \times \mathbb{R}, \mathbb{C}) \to C^s(I \times \mathbb{R}, \mathbb{C})$.

***Remark* 2.1.25.** The last assumption,

$$f : C^{s+\max\{2D_x, 2D_t+2\}}(I \times \mathbb{R}, \mathbb{C}) \to C^s(I \times \mathbb{R}, \mathbb{C}),$$

should not be necessary. We suppose it and $\psi \in C^{s+\max\{2D_x, 2D_t+2\}}(I \times \mathbb{R}, \mathbb{C})$ since it allows us to calculate in the 'classical' way of first year calculus. The Boussinesq equations that we will consider and the examples in Section 2.1.5 satisfy these restrictions. Therefore we do not care much about it.

If $\psi(t) \in H^{s+2D_x}$, $\partial_t^2\psi(t) \in H^{s+2D_t}$ and $f : H^{s+2D_x} \to H^s$, then the regularity assumption for the residual holds. The smallness assumption, however, has to be proved nonetheless.

The next lemma states some properties of the solutions to the one dimensional inhomogeneous wave equation. This is a technical auxiliary result for the proof of Corollary 2.1.27.

Lemma 2.1.26. *Let $\epsilon \in]0, \epsilon_0]$, assume that the suppositions of Hypothesis 2.1.24 hold, and use the same notation. Further assume that there are functions $\phi_0 \in C^{s+1}(\mathbb{R}, \mathbb{C}) \cap H^{s+1}$ and $\psi_0 \in C^s(\mathbb{R}, \mathbb{C}) \cap H^s$ such that $\|\epsilon\phi_0\|_{H^{s+1}} \le C_\phi \epsilon^{N-2}$, $\|\epsilon\psi_0\|_{H^s} \le C_\psi \epsilon^N$. Let $s \ge l, k+2 \in \mathbb{N}$ and $\delta \in]0, 1]$.*
Then we obtain for the solution to the problem

$$\begin{aligned} (\partial_t^2 - \partial_x^2)\phi &= -\epsilon^{-1}\mathfrak{E}_{\epsilon\psi} && \text{in } I \times \mathbb{R} \\ (\phi, \partial_t\phi)(0) &= (\phi_0, \psi_0) && \text{in } \mathbb{R} \end{aligned}$$

the estimates

$$\begin{aligned} \|\epsilon\phi\|_{L^\infty(I,L^2)} &\lesssim \epsilon^{N-2-2\delta}, \\ \|\partial_x^l \epsilon\phi\|_{L^\infty(I,L^2)} &\lesssim \epsilon^{N-2}, \\ \|\partial_x^k\partial_t^2\epsilon\phi\|_{L^\infty(I,L^2)} = \|\partial_t^2\partial_x^k\epsilon\phi\|_{L^\infty(I,L^2)} &\lesssim \epsilon^{N-2}, \end{aligned}$$

as well as

$$\begin{aligned} \|\phi\|_{L^\infty(I,L^\infty)} &\lesssim \epsilon^{N-3-2\delta}, \\ \|\partial_x^l\phi\|_{L^\infty(I,L^\infty)} &\lesssim \epsilon^{N-3}. \end{aligned}$$

The constants depend on $C_\phi, C_\psi, C_0, q, \delta, \tilde{c}, t_0$ but not on ϵ.

Proof. The assumptions and the definition of the residual immediately show that $\mathfrak{E}_{\epsilon\psi} \in C^s(I\times\mathbb{R}, \mathbb{C})$. Then Lemma 2.A.1 applies and we obtain $\phi \in C^{s+1}(I\times\mathbb{R}, \mathbb{C})$ as well as for any $0 < \delta \le 1$ and $s \ge l, k+2 \in \mathbb{N}$

$$\begin{aligned} \|\epsilon\phi\|_{L^\infty(I,L^2)} \le \|\epsilon\phi_0\|_{L^2} &+ \|\epsilon\psi_0\|_{L^2}\frac{-\ln\epsilon + qt_0}{\epsilon q\sqrt{\tilde{c}}} \\ &+ \frac{1}{2q^2\epsilon^2\tilde{c}}\|\mathfrak{E}_{\epsilon\psi}\|_{L^\infty(I,L^2)}(-\ln\epsilon + qt_0)^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \epsilon^{N-2}C_\phi + C_\psi \epsilon^{N-1-\delta}\Big(\frac{1}{q\delta e\sqrt{\tilde{c}}} + \frac{\epsilon^{1-\delta}t_0}{\sqrt{\tilde{c}}}\Big) + \frac{\epsilon^{N-2}}{q^2\tilde{c}}C_0 q^2 t_0^2 \\
&\quad + \frac{\epsilon^{N-2-2\delta}}{(\delta e q)^2\tilde{c}}C_0, \\
\|\partial_x^l \epsilon\phi\|_{L^\infty(I,L^2)} &\leq \|\partial_x^l\phi_0\|_{L^2} + \|\partial_x^{l-1}\psi_0\|_{L^2} + \|\partial_x^{l-1}\mathfrak{E}_{\epsilon\psi}\|_{L^\infty(I,L^2)}\frac{(-\ln\epsilon + qt_0)}{\epsilon\sqrt{q\tilde{c}}} \\
&\leq C_\phi\epsilon^{N-2} + C_\psi\epsilon^N + C_0\epsilon^N\frac{(-\ln\epsilon + qt_0)}{\epsilon q\sqrt{\tilde{c}}} \\
&\leq \Big(C_\phi + C_\psi + \frac{C_0}{\delta e\sqrt{\tilde{c}}}(1+\delta e\epsilon^\delta t_0)\Big)\epsilon^{N-2}, \\
\|\partial_x^k\partial_t^2\epsilon\phi\|_{L^\infty(I,L^2)} &\leq \|\partial_x^{k+2}\phi_0\|_{L^2} + \|\partial_x^{k+1}\psi_0\|_{L^2} \\
&\quad + \|\partial_x^k\mathfrak{E}_{\epsilon\psi}\|_{L^\infty(I,L^2)} + \|\partial_x^{k+1}\mathfrak{E}_{\epsilon\psi}\|_{L^\infty(I,L^2)}\frac{(-\ln\epsilon + qt_0)}{\epsilon q\sqrt{\tilde{c}}} \\
&\leq C_\phi\epsilon^{N-2} + C_\psi\epsilon^N + \|\mathfrak{E}_{\epsilon\psi}\|_{L^\infty(I,H^{s-1})} \\
&\quad + \|\mathfrak{E}_{\epsilon\psi}\|_{L^\infty(I,H^{s-2})}\frac{(-\ln\epsilon + qt_0)}{\epsilon q\sqrt{\tilde{c}}} \\
&\leq \Big(C_\phi + C_\psi + C_0 + \frac{C_0}{\delta e q\sqrt{\tilde{c}}}(1+\delta e\epsilon^\delta qt_0)\Big)\epsilon^{N-2},
\end{aligned}
$$

because of $-\epsilon^\delta\ln\epsilon \leq \frac{1}{\delta e}$ and the assumptions. We obtain further for $1 \leq l \leq s-1$

$$
\begin{aligned}
\|\phi\|_{L^\infty(I,L^\infty)} &\leq \|\phi_0\|_{L^\infty} + \|\psi_0\|_{L^\infty}\frac{-\ln\epsilon + qt_0}{\epsilon q\sqrt{\tilde{c}}} + \frac{(-\ln\epsilon + qt_0)^2}{2q^2\epsilon^3\tilde{c}}\|\mathfrak{E}_{\epsilon\psi}\|_{L^\infty(I,L^\infty)} \\
&\leq C_\infty\Big(\|\phi_0\|_{H^1} + \|\psi_0\|_{H^1}\frac{-\ln\epsilon + qt_0}{\epsilon q\sqrt{\tilde{c}}}\Big) \\
&\quad + C_\infty\frac{(-\ln\epsilon + qt_0)^2}{2q^2\epsilon^3\tilde{c}}\|\mathfrak{E}_{\epsilon\psi}\|_{L^\infty(I,H^1)} \\
&\leq C_\infty\Big(C_\phi\epsilon^{2\delta} + C_\psi\epsilon^{1+\delta}\frac{1+qet_0}{q\delta e\sqrt{\tilde{c}}} + C_0\frac{\delta^2e^2\epsilon^{2\delta}q^2t_0^2 + 1}{q^2\sqrt{2}\delta^2e^2\tilde{c}}\Big)\epsilon^{N-3-2\delta}, \\
\|\partial_x^l\phi\|_{L^\infty(I,L^\infty)} &\leq \|\partial_x^l\phi_0\|_{L^\infty} + \|\partial_x^{l-1}\psi_0\|_{L^\infty} + \frac{1}{\sqrt{\tilde{c}}\epsilon^2}\|\partial_x^{l-1}\mathfrak{E}_{\epsilon\psi}\|_{L^\infty(I,L^\infty)}(-\ln\epsilon + t_0) \\
&\leq C_\infty\left(\|\partial_x^l\phi_0\|_{H^1} + \|\partial_x^{l-1}\psi_0\|_{H^1}\right) \\
&\quad + \frac{C_\infty}{\sqrt{\tilde{c}}\epsilon^2}\|\partial_x^{l-1}\mathfrak{E}_{\epsilon\psi}\|_{L^\infty(I,H^1)}(-\ln\epsilon + t_0) \\
&\leq C_\infty\Big(C_\phi + C_\psi\epsilon^2 + C_0\frac{(1+\epsilon^\delta e\delta qt_0)}{q\sqrt{\tilde{c}}e\delta}\epsilon^{1-\delta}\Big)\epsilon^{N-3},
\end{aligned}
$$

where C_∞ accounts for the continuous embedding $H^1 \hookrightarrow L^\infty$. $\square$

With the aid of Lemma 2.1.26 we can prove the following corollary.

Corollary 2.1.27. *Let $d_x \leq 2$ and $m_0 \leq l \in \mathbb{N}$. We use the notation of Lemma 2.1.26 and assume that its suppositions are satisfied. Additionally, we assume $N - 2 - 2\delta \geq 0$ and that the assumptions of Hypothesis 2.1.22 are met. Then it holds for $\tilde{\psi} \coloneqq \psi + \phi$ and $i \in \{-d_x, l\}$*

$$\begin{aligned}
\|\tilde{\psi}\|_{L^\infty(I,W^{D_x-D_t+l,\infty})} &\leq \|\psi\|_{L^\infty(I,W^{D_x-D_t+l,\infty})} + \mathcal{O}(\epsilon^{N-3-2\delta}),\\
\|\widehat{\mathfrak{E}_{\epsilon\tilde{\psi}}}\|_{L^\infty(I,L_i^{2,h})} &\leq C_2\epsilon^{N-2-2\delta},\\
\|\tilde{\psi} - \psi\|_{L^\infty(I,H^l)} &\lesssim C_3\epsilon^{N-3-2\delta},
\end{aligned}$$

where the constants depend again on $C_\phi, C_\psi, C_0, q, \delta, \tilde{c}, t_0$ as well as l and the coefficients of q_t, q_x.

Proof. Trivially, it holds by Lemma 2.1.26

$$\left| \|\tilde{\psi}\|_{L^\infty(I,W^{D_x-D_t+l,\infty})} - \|\psi\|_{L^\infty(I,W^{D_x+D_t+l,\infty})} \right| \leq \|\phi\|_{L^\infty(I,W^{s,\infty})} \lesssim \epsilon^{N-3-2\delta}.$$

Next we consider the residual. By definition and Lemma 2.1.26 we obtain

$$\begin{aligned}
\mathfrak{E}_{\epsilon\tilde{\psi}} &= \mathfrak{E}_{\epsilon\psi+\epsilon\phi} = \mathcal{L}\epsilon\psi + \mathcal{L}\epsilon\phi + f\big(\epsilon(\psi+\phi)\big)\\
&= \mathcal{L}\epsilon\psi + f(\epsilon\psi) + \mathcal{L}\epsilon\phi + f\big(\epsilon(\psi+\phi)\big) - f(\epsilon\psi)\\
&= \mathfrak{E}_{\epsilon\psi} + \mathcal{L}\epsilon\phi + f\big(\epsilon(\psi+\phi)\big) - f(\epsilon\psi)\\
&= \mathfrak{E}_{\epsilon\psi} + (\mathcal{L} + \partial_t^2 - \partial_x^2)\epsilon\phi + (-\partial_t^2 + \partial_x^2)\epsilon\phi + f\big(\epsilon(\psi+\phi)\big) - f(\epsilon\psi)\\
&= (\mathcal{L} + \partial_t^2 - \partial_x^2)\epsilon\phi + f\big(\epsilon(\psi+\phi)\big) - f(\epsilon\psi).
\end{aligned}$$

If we consider the Fourier transform, we can estimate for $i \in \{-d_x, l\}$

$$\begin{aligned}
&\|\tilde{q}_t^{-1}\boldsymbol{\xi}^i\widehat{\mathfrak{E}_{\epsilon\tilde{\psi}}}\|_{L^\infty(I,L^2)}\\
=&\left\|\tilde{q}_t^{-1}\boldsymbol{\xi}^i(\mathcal{L}(\partial_t, \mathrm{i}\boldsymbol{\xi}) + \partial_t^2 + \xi^2)\epsilon\hat{\phi} + \tilde{q}_t^{-1}\boldsymbol{\xi}^i\,\mathscr{F}\,(f(\epsilon(\psi+\phi)) - f(\epsilon\psi))\right\|_{L^\infty(I,L^2)}\\
\leq&\|\tilde{q}_t^{-1}\boldsymbol{\xi}^i(\tilde{q}_t - 1)\epsilon\partial_t^2\hat{\phi}\|_{L^\infty(I,L^2)} + \|\tilde{q}_t^{-1}(\boldsymbol{\xi}^i\tilde{q}_x + \boldsymbol{\xi}^{2+i})\epsilon\hat{\phi}\|_{L^\infty(I,L^2)}\\
&+ \|\tilde{q}_t^{-1}\boldsymbol{\xi}^i\,\mathscr{F}\,\big(G(\epsilon\psi, \epsilon\phi)\big)\|_{L^\infty(I,L^2)} + \|\tilde{q}_t^{-1}\boldsymbol{\xi}^i\,\mathscr{F}\,Df(\epsilon\psi)[\epsilon\phi]\|_{L^\infty(I,L^2)}
\end{aligned}$$

$$
\begin{aligned}
\lesssim & \|\epsilon\partial_t^2\hat{\phi}\|_{L^\infty(I,L^2_l)} + \|\epsilon\hat{\phi}\|_{L^\infty(I,L^2_{2(D_x-D_t)+l})} \\
& + \operatorname*{ess\,sup}_{t\in I} \max_{\delta\in[0,1]} \left\|\tilde{q}_t^{-1}\mathscr{F}\circ Df(\epsilon\psi(t)+\delta\phi(t)) - \tilde{q}_t^{-1}\mathscr{F}\circ Df(\epsilon\psi(t))\right\|_{\mathrm{L}(L^2_{D_x-D_t+l},L^{2,b}_i)} \times \\
& \times \|\epsilon\hat{\phi}\|_{L^\infty(I,L^2_{D_x-D_t+l})} \\
& + \operatorname*{ess\,sup}_{t\in I} \left\|\tilde{q}_t^{-1}\mathscr{F}\circ Df(\epsilon\psi(t))\right\|_{\mathrm{L}(L^2_{D_x-D_t+l},L^{2,b}_i)} \|\epsilon\hat{\phi}\|_{L^\infty(I,L^2_{D_x-D_t+l})} \\
\lesssim & \|\epsilon\partial_t^2\hat{\phi}\|_{L^\infty(I,L^2_l)} + \|\epsilon\hat{\phi}\|_{L^\infty(I,L^2_{2(D_x-D_t)+l})} \lesssim \epsilon^{N-2} + \epsilon^{N-2-\delta} \lesssim \epsilon^{N-2-\delta},
\end{aligned}
$$

because of the assumptions of Hypothesis 2.1.22 and

$$
\begin{aligned}
\|\hat{\phi}\|_{L^\infty(I,L^2_{2(D_x-D_t)+l})} &\lesssim \operatorname*{ess\,sup}_{t\in I}\|\phi(t)\|_{L^2} + \|\partial_x^{2(D_x-D_t)+l}\phi(t)\|_{L^2} \\
&\lesssim \epsilon^{N-3-2\delta} + \epsilon^{N-3} \le \epsilon^{N-3-2\delta}, \\
\|\partial_t^2\hat{\phi}\|_{L^\infty(I,L^2_l)} &\lesssim \operatorname*{ess\,sup}_{t\in I}\|\partial_t^2\phi(t)\|_{L^2} + \|\partial_x^l\partial_t^2\phi(t)\|_{L^2} \lesssim \epsilon^{N-3}.
\end{aligned}
$$

We note that the hidden constant is uniform in $\epsilon \in]0,\epsilon_0]$. This is a consequence of the second assumption on f in Hypothesis 2.1.22 and the uniform norm estimates in Hypothesis 2.1.24 and Lemma 2.1.26.
Finally, we already know

$$
\|\tilde{\psi} - \psi\|_{L^\infty(I,H^l)} = \|\phi\|_{L^\infty(I,H^l)} \lesssim \epsilon^{N-3-2\delta}.
$$

The constants are dependent on the constants of Lemma 2.1.26 and l as well as the coefficients of q_t, q_x. □

We remark that we could omit the restriction $d_x \le 2$ if we had a suitable substitute for Lemma 2.1.26 to correct the residual in the previous proof and $f(u) \in L^{2,b}_{-d_x}$.

Error Estimate

We can now state and prove an error estimate as follows. We will use an energy estimate and have to presume local existence and uniqueness of solutions to the Cauchy problem with equation (2.15). But this is an easy exercise, see Theorem 2.B.1 and Remark 2.B.3 in Appendix 2.B.

Theorem 2.1.28. *Let $d_x \le 2$, $m_0 \le l \in \mathbb{N}$, and $N \in \mathbb{N}$ sufficiently large, see the end of the proof. We assume that Hypothesis 2.1.22 and 2.1.24 are satisfied. Let* *u be a strict solution to the Cauchy problem with equation* (2.15) *in the sense of Theorem 2.B.1 for initial data*

$$(u, \partial_t u)(0) = (v, w) \in H^{s+1} \cap C^{s+1}(\mathbb{R}, \mathbb{C}) \times H^s \cap C^s(\mathbb{R}, \mathbb{C})$$

for the s defined in Hypothesis 2.1.24.
Then there exists $\epsilon_0 > 0$ *such that for all* $\epsilon \in]0, \epsilon_0[$*, arbitrary* $p > 1$*, and initial data*

$$\|v - \epsilon\psi(0)\|_{H^{s+1}} \lesssim \epsilon^{N-2}, \qquad \|w - \epsilon\partial_t\psi(0)\|_{H^s} \lesssim \epsilon^N,$$

we have the estimate

$$\|u - \epsilon\psi\|_{L^\infty(I,H^l)} \lesssim \epsilon^p.$$

Proof. We define $\tilde{\psi} = \psi + \phi$ and ϕ as in Lemma 2.1.26 where we use $\epsilon\phi_0 = v - \epsilon\psi(0)$, $\epsilon\psi_0 = w - \epsilon\partial_t\psi(0)$. Thus $\epsilon\tilde{\psi}(0) - v = 0$ and $\partial_t\tilde{\psi}(0) - w = 0$. Then we use Corollary 2.1.27 and require $N \geq p + 2 + 2\delta$. Hence, we can estimate by the triangle inequality

$$\|u - \epsilon\psi\|_{L^\infty(I,H^l)} \leq \|u - \epsilon\tilde{\psi}\|_{L^\infty(I,H^l)} + \|\epsilon\tilde{\psi} - \epsilon\psi\|_{L^\infty(I,H^l)}$$

and because of $N \geq p + 2 + 2\delta$ we find

$$\|\epsilon\tilde{\psi} - \epsilon\psi\|_{L^\infty(I,H^l)} \lesssim \epsilon^{N-2-2\delta} \lesssim \epsilon^p.$$

If we write $u = \epsilon\tilde{\psi} + \epsilon^p R$, we obtain

$$\|u - \epsilon\tilde{\psi}\|_{L^\infty(I,H^l)} = \epsilon^p\|R\|_{L^\infty(I,H^l)}.$$

It remains to show that the norm $\|R\|_{L^\infty(I,H^l)} \leq C < \infty$ is bounded. Recall that we have for $\mathcal{L} = \mathcal{L}(\partial_t, \partial_x)$ the equation

$$\begin{aligned} 0 &= \mathcal{L}R + \epsilon^{-p}\big(\mathcal{L}\epsilon\tilde{\psi} + f(\epsilon\tilde{\psi})\big) + \epsilon^{-p}\big(f(\epsilon^p R + \epsilon\tilde{\psi}) - f(\epsilon\tilde{\psi})\big) \\ &= \mathcal{L}R + \epsilon^{-p}\big(f(\epsilon^p R + \epsilon\tilde{\psi}) - f(\epsilon\tilde{\psi})\big) + \epsilon^{-p}\mathfrak{E}_{\epsilon\tilde{\psi}}. \end{aligned}$$

Let $S = \{-d_x, l\}$. In view of Remark 2.1.23 and Theorem 2.B.1, we define the energy

$$E_{\hat{R}} = \frac{1}{2}\sum_{i\in S}\langle \partial_t \hat{R}, \partial_t \hat{R}\rangle_{L^{2,h}_i} + \langle \tilde{q}\hat{R}, \hat{R}\rangle_{L^{2,h}_i},$$

which is well-defined for $\tilde{q} = \frac{\tilde{q}_x}{\tilde{q}_t}$ since $(\partial_t R)(0) = \epsilon^{-p}\big(\partial_t u(0) - \epsilon\partial_t \tilde{\psi}(0)\big) \equiv 0 \in L^{2,b}_{-d_x}$ and this property is respected by the PDE above – recall the discussion after Remark 2.1.23 in the case of the Boussinesq problem. Theorem 2.B.1 guarantees that there is a strict solution as long as this energy is finite since it gives a bound to the $L^2_{D_x-D_t+l} \times L^2_l$ norm. Further we know that the energy is continuously differentiable with respect to time as u and $\epsilon\psi$ are continuously differentiable with respect to time.

Let $t_* := \sup\{s \in I \,|\, E(s) < M\}$ for a $M > 0$ which we will define later. If $t_* = \sup I$ there is nothing to do. Hence we have to show that $t_* < \sup I$ leads to a contradiction. We will derive a differential inequality for this purpose. Let $t \le t_*$. We start with the observation that for any $\kappa > 0$

$$\begin{aligned}
&\partial_t E_{\hat{R}} \\
&= \sum_{i\in S} \operatorname{Re}\Big(\langle \tilde{q}_t^{-1}\,\mathscr{F}\,\epsilon^{-p}\mathfrak{E}_{\epsilon\tilde{\psi}}, \partial_t\hat{R}\rangle_{L^{2,h}_i} + \epsilon^{-p}\langle \tilde{q}_t^{-1}\,\mathscr{F}\,\big(f(\epsilon^p R + \epsilon\tilde{\psi}) - f(\epsilon\tilde{\psi})\big), \partial_t\hat{R}\rangle_{L^{2,h}_i}\Big) \\
&\le \sum_{i\in S} \|\tilde{q}_t^{-1}\,\mathscr{F}\,\epsilon^{-p}\mathfrak{E}_{\epsilon\tilde{\psi}}\|_{L^{2,h}_i}\|\partial_t\hat{R}\|_{L^{2,h}_i} \\
&\qquad + \epsilon^{-p}\|\tilde{q}_t^{-1}\,\mathscr{F}\,\big(f(\epsilon^p R + \epsilon\tilde{\psi}) - f(\epsilon\tilde{\psi})\big)\|_{L^{2,b}_i}\|\partial_t\hat{R}\|_{L^{2,h}_i} \\
&\le \sum_{i\in S} \frac{\epsilon^{-2p+1}}{2\kappa}\|\tilde{q}_t^{-1}\,\mathscr{F}\,\mathfrak{E}_{\epsilon\tilde{\psi}}\|^2_{L^{2,h}_i} + \frac{\kappa}{2}\epsilon\|\partial_t\hat{R}\|^2_{L^{2,h}_i} + \|\tilde{q}_t^{-1}\,\mathscr{F}\circ Df(\epsilon\tilde{\psi})[R]\|_{L^{2,b}_i}\|\partial_t\hat{R}\|_{L^{2,h}_i} \\
&\qquad + \epsilon^p\|\tilde{q}_t^{-1}G(\epsilon\tilde{\psi}, R, \epsilon^p R)\|_{L^{2,b}_i}\|\partial_t\hat{R}\|_{L^{2,h}_i} \\
&\le \sum_{i\in S} \frac{\epsilon^{-2p+1}}{2\kappa}\|\tilde{q}_t^{-1}\,\mathscr{F}\,\mathfrak{E}_{\epsilon\tilde{\psi}}\|^2_{L^{2,h}_i} + \frac{\kappa}{2}\epsilon\|\partial_t\hat{R}\|^2_{L^{2,h}_i} \\
&\qquad + 2h_1\big(\epsilon\|\psi\|_{L^\infty(I,W^{D_x-D_t+l,\infty})}\big)\|\hat{R}\|_{L^2_{D_x-D_t+l}}\|\partial_t\hat{R}\|_{L^{2,h}_i} \\
&\qquad + \max_{\delta\in[0,1]} h_1\big(\|\epsilon\psi + \delta\epsilon^p R\|_{L^\infty(I,W^{D_x-D_t+l,\infty})}\big)\|\hat{R}\|_{L^2_{D_x-D_t+l}}\|\partial_t\hat{R}\|_{L^{2,h}_i} \\
&\le \sum_{i\in S} \frac{\epsilon^{-2p+1}}{2\kappa}\|\tilde{q}_t^{-1}\,\mathscr{F}\,\mathfrak{E}_{\epsilon\tilde{\psi}}\|^2_{L^{2,h}_i} + \epsilon(C_0+\kappa)E
\end{aligned}$$

for a constant $C_0 \ge 0$ and ϵ_0 small enough. We used a first order remainder G in the Taylor expansion for this calculation which means that we applied the mean value theorem for its estimate. Further we exploited the second assumption on f in Hypothesis 2.1.22 and made use of $\|\hat{R}\|^2_{L^2_{D_x-D_t+l}} \lesssim E_{\hat{R}}$. Therefore the constant C_0 depends on $\epsilon_0^{p-1}M$, $\|\tilde{\psi}\|_{L^\infty(I,W^{D_x-D_t+l,\infty})}$ and ϵ_0 but can be chosen uniformly for $\epsilon \in]0,\epsilon_0[$. Note that $C_0(\boldsymbol{\epsilon_0^{p-1}M})$ is monotonically increasing in $\epsilon_0^{p-1}M$ and $C_0(\boldsymbol{\epsilon_0^{p-1}M}) \in \Omega(1)$ for $\epsilon_0^{p-1}M \to 0$. This is a consequence of the second assumption on f in Hypothesis 2.1.22. Corollary 2.1.27 allows us to write

$$\partial_t E_{\hat{R}} \leq \epsilon D_1 E_{\hat{R}} + \epsilon^{2N-4-4\delta-2p-1} D_2$$

where

$$D_1 := C_0 + \kappa, \qquad\qquad D_2 := \frac{2C_2^2}{\kappa} > 0.$$

Therefore D_1 depends on $\epsilon_0^{p-1}M$ in the same way C_0 does. We have

$$R(0) = \epsilon^{-p}\big(u(0) - \epsilon\tilde{\psi}(0)\big) \equiv 0 \in L^2_\infty$$

and also recall

$$(\partial_t R)(0) \equiv 0 \in L^2_\infty \cap L^{2,b}_{-d_x}$$

and consequently $E_{\hat{R}}(0) = 0$. By Gronwall's inequality we obtain for $t \leq t_*$

$$E_{\hat{R}}(t) \leq \frac{\epsilon^{2N-4-4\delta-2p-2} D_2}{D_1}(e^{D_1\epsilon t} - 1) \leq \frac{\epsilon^{2N-4-4\delta-2p-2} D_2}{D_1} e^{D_1 \epsilon t_*}.$$

This shows

$$\begin{aligned} &M \leq \frac{\epsilon^{2N-4-4\delta-2p-2} D_2}{D_1} e^{D_1\epsilon t_*} \\ \Leftrightarrow &\frac{-(2N-4-4\delta-2p-2)\ln\epsilon + \ln(MD_1) - \ln D_2}{D_1 \epsilon} \leq t_*. \end{aligned}$$

Recall that $\sup I = \frac{-\ln\epsilon + qt_0}{q\epsilon\sqrt{\tilde{c}}}$. We obtain the contradiction $\sup I \leq t_*$ for

$$N \geq 2 + 2\delta + p + 1 + \frac{D_1}{2q\sqrt{\tilde{c}}} \qquad \text{and} \qquad \frac{\ln(MD_1(\epsilon_0^{p-1}M)) - \ln D_2}{D_1(\epsilon_0^{p-1}M)} \geq \frac{t_0}{\sqrt{\tilde{c}}}.$$

Hence the proof is complete since the first inequality is satisfied for sufficiently large N and the second inequality is true for sufficiently small ϵ_0 and large enough M. $\square$

***Remark* 2.1.29.** In more concrete cases the proof of Theorem 2.1.28 becomes simpler and possibly we can reduce some regularity assumptions. For example if $d_x = 0$, then there is no necessity to apply Corollary 2.1.27. This allows for lower regularity assumptions on the residual – and as a consequence on $\epsilon\psi$. Additionally, we used Corollary 2.1.27 for the control of the initial values of $(R, \partial_t R)$. This resulted in a coupling of the powers of ϵ that we can allow for the initial data and the residual. Without this coupling one might be able to lower the assumptions on the initial data.
Last but not least we emphasis that some assumptions of Theorem 2.1.28 and Hypothesis 2.1.24 are not optimal but the benefits of these are simpler proofs and shorter notation. For example, if additional $W^{r,\infty}$ assumptions are introduced, one could easily lower the assumption $\|v - \epsilon\psi(0)\|_{H^{s+1}}$ to $\|v - \epsilon\psi(0)\|_{H^s}$.

Since the abstract Theorem 2.1.28 applies to the Boussinesq problem (2.1) we obtain a corollary.

Corollary 2.1.30. *Let $l \in \mathbb{N}$ and assume the suppositions of Hypothesis 2.1.24 hold for $D_x = 3, d_x = 1, D_t = 1$. Let u be a strict solution to the Cauchy problem with equation* (2.1) *in the sense of Theorem 2.B.1 for initial data*

$$(u, \partial_t u)(0) = (v, w) \in H^{s+1} \cap C^{s+1}(\mathbb{R}, \mathbb{C}) \times H^s \cap C^s(\mathbb{R}, \mathbb{C}).$$

Then there exists $\epsilon_0 > 0$ such that for all $\epsilon \in]0, \epsilon_0[$, arbitrary $p > 1$, and initial data

$$\|v - \epsilon\psi(0)\|_{H^{s+1}} \lesssim \epsilon^{N-2}, \qquad \|w - \epsilon\partial_t\psi(0)\|_{H^s} \lesssim \epsilon^N,$$

we have the estimate

$$\|u - \epsilon\psi\|_{L^\infty(I,H^l)} \lesssim \epsilon^p.$$

Proof. We meet the assumptions of Hypothesis 2.1.22 with $D_x = 3, d_x = 1, D_t = 1$ for any $m_0 \in \mathbb{N}$ since $f : u \mapsto \partial_x^2 u^2$ and $Df(v)[u] = 2\partial_x^2(uv)$ in suitable spaces. Since L^2_{2+m} is an convolution algebra and f essentially a polynomial, the map $\mathscr{F} \circ f \circ \mathscr{F}^{-1} : L^2_{2+m} \to L^2_{m-2}$ is Lipschitz on bounded subsets. Differentiability and continuity are obvious for the concrete representations above and it is obvious that

$$\|M_{q_t^{-1}(-\boldsymbol{\xi}^2)} \mathscr{F} Df(v) \mathscr{F}^{-1}\|_{\mathrm{L}(L^2_{2+m}, L^{2,b}_m)} \lesssim \|v\|_{W^{m,\infty}}$$
$$\|M_{q_t^{-1}(-\boldsymbol{\xi}^2)} \mathscr{F} Df(v) \mathscr{F}^{-1}\|_{\mathrm{L}(L^2_{2+m}, L^{2,b}_{-1})} \lesssim \|v\|_{W^{m,\infty}}$$

where the constant solely depends on some continuous embeddings. Thus Theorem 2.1.28 applies for all $l \in \mathbb{N}$. □

At this point we should give some comments.

1. The proof is not optimal in some sense. Conditions being sufficient for Corollary 2.1.30 in this situation are given in Hypothesis 2.1.31 below. Especially, we note that the previous comment applies and we could try to reduce the regularity assumptions by 1.

2. Theorem 2.1.28 does not guarantee an approximation property in the sense of Approximation Property 2.1.20 since we need in general much smaller initial data than the error bound.

Hypothesis 2.1.31. *We assume there is a function $\psi \in C^{l+10}(I \times \mathbb{R}, \mathbb{C})$ and*

$$\epsilon\psi \in L^\infty(I, H^{l+10}), \qquad \psi \in L^\infty(I, W^{l,\infty}), \qquad \epsilon\partial_t^2\psi \in L^\infty(I, H^{l+6})$$

for an $l \in \mathbb{N}$ with bounds of these norms uniform in $\epsilon \in]0, \epsilon_0[$ in the interval I. We require $\|\mathfrak{E}_{\epsilon\psi}\|_{L^\infty(I,H^{l+4})} \leq C_0\epsilon^N$ for any arbitrary chosen $N \in \mathbb{N}$ in addition to these.

***Remark* 2.1.32.** Theorem 2.1.28 shows that the satisfaction of the assumptions of Hypothesis 2.1.24 and extremely small deviations in the initial data are sufficient to obtain a useful error estimate. If we assume initial conditions $\epsilon\psi \in C_c^\infty(\mathbb{R}, \mathbb{C})$ and $\epsilon\partial_t^2\psi \in L^\infty(I, H^\infty)$, then regularity considerations are not essential if the amplitude equations preserve the regularity of the initial data or at least do not worsen them much. Thus, it seems to be necessary and sufficient to have a small residual and small deviations in the initial data for the error estimate.
We already pointed out that Theorem 2.1.28 and Corollary 2.1.30 are not appropriate to prove an approximation result in the sense of Approximation Prop-

erty 1.2. *However, if the smallness condition holds the corollary or theorem can be sufficient to prove that an approximation property cannot hold.*

Now the question arises whether we can use the formal approximate solutions of Section 2.1.3, constructed with the aid of the FWI or TWI system respectively, as function $\epsilon\psi$ in Corollary 2.1.30. As pointed out before and seen in Section 2.1.4 regularity issues are of no concern for $t \in I$. Actually, the sole problem is the question whether the residual $\mathfrak{E}_{\psi_{\text{FWI}}}$ or $\mathfrak{E}_{\psi_{\text{TWI}}}$ is small enough. We give a partial answer to this question.

***Remark* 2.1.33.** The question how large N actually has to be for the Boussinesq problem (2.1) is of some importance. We observe that we required at the beginning of the proof of Theorem 2.1.28 that $N \geq p+2+2\delta$ and $\delta \in]0,1]$. Later we introduced a condition in addition that results for $p = 1+\frac{1}{q}$, $\delta = \frac{1}{2q}$ and $q = \max\{2, \frac{D_1}{2\sqrt{\tilde{c}}}\}$ in

$$N \geq 6.$$

We point out that this is not obvious and requires a very careful examination of all constants involved in all proofs and uses some properties specific to the Boussinesq problem (2.1). Unfortunately, we made q quite large this way since, even in the case $l = 1$ and $N = 6$, a careful evaluation of the constants shows that

$$\frac{D_1}{2\sqrt{\tilde{c}}} > \frac{4\sqrt{2}\|\psi\|_{L^\infty(I,W^{1,\infty})}}{\sqrt{\tilde{c}}}.$$

This is in general larger than 2 since we recall that $\tilde{c} = \sup_{x\in\operatorname{supp} b_1} b_1(x)^2 + b_3(x)^2$ is similar to $\|\psi\|^2_{L^\infty(I,W^{1,\infty})}$. Then

$$q = \max\left\{2, \frac{D_1}{2\sqrt{\tilde{c}}}\right\} > \frac{4\sqrt{2}\|\psi\|_{L^\infty(I,W^{1,\infty})}}{\sqrt{\tilde{c}}} > 2.$$

The last remark shows that neither the ‘trivial’ residual estimate resulting from Section 2.1.3 for the FWI system nor the one for the TWI system are good enough to apply Corollary 2.1.30. At the moment it is not clear whether an error estimate holds for either of them for $t \in I$. One could resolve this issue by using a formal approximate solution that allows the required residual estimate and differs from the FWI or TWI ansatz only by ϵ^p in the considered norm (e.g. H^1 or L^∞). In [84] the same problem appears in the setting of a NLS formal approximate solution instead of the FWI approximate solution and is solved similarly by an 'improved ansatz'. However, the authors do not explicitly use the NLS or a simple TWI formal approximate solution in that article but show that the improved ansatz possesses properties similar to those of the TWI formal approximate solution.

Finally, we point out that there is no challenge in proving an approximation property for functions satisfying Hypotheses 2.1.22 and 2.1.24 for $t \in [0, \Omega(\epsilon^{-1})]$ if we require certain additional constraints for the initial data, see Theorem 2.1.34.

Theorem 2.1.34. *Let $d_x \leq 2$, $m_0 \leq l \in \mathbb{N}$, $p > 1$, and $3 + p < N \in \mathbb{N}$. We assume that Hypothesis 2.1.22 and 2.1.24 are satisfied. Let u be a strict solution to the Cauchy problem with equation* (2.15) *in the sense of Theorem 2.B.1 for initial data $(\hat{u}, \partial_t \hat{u})(0) = (\hat{v}, \hat{w}) \in L^2_s \times L^2_r \cap L^{2,b}_{-d_x}$.*
Then there exists $\epsilon_0 > 0$ such that for all $\epsilon \in]0, \epsilon_0[$ and initial data

$$\|v - \epsilon\psi(0)\|_{H^s} \lesssim \epsilon^p, \qquad \|\hat{w} - \epsilon\partial_t\hat{\psi}(0)\|_{L^{2,b}_{-d_x} \cap L^{2,b}_r} \lesssim \epsilon^p,$$

we have the estimate

$$\|u - \epsilon\psi\|_{L^\infty([0,t_0], H^l)} \lesssim \epsilon^p$$

for a $t_0 \in \Theta(\epsilon^{-1})$.

Proof. The proof consists in a repetition of the arguments of the proof of Theorem 2.1.28. We define $\tilde{\psi} = \psi + \phi$ and ϕ as in Lemma 2.1.26 where we use $\epsilon\phi_0 = 0, \epsilon\psi_0 = 0$. Thus $\epsilon\tilde{\psi}(0) = \epsilon\psi(0)$ and $\partial_t\tilde{\psi}(0) = \partial_t\psi(0)$. Then we use Corollary 2.1.27 and require $N \geq p + 2 + 2\delta$. Hence, we can estimate by the triangle inequality

$$\|u - \epsilon\psi\|_{L^\infty(I,H^l)} \leq \|u - \epsilon\tilde{\psi}\|_{L^\infty(I,H^l)} + \|\epsilon\tilde{\psi} - \epsilon\psi\|_{L^\infty(I,H^l)}$$

and because of $N \geq p + 2 + 2\delta$ we find

$$\|\epsilon\tilde{\psi} - \epsilon\psi\|_{L^\infty(I,H^l)} \lesssim \epsilon^{N-2-2\delta} \lesssim \epsilon^p.$$

If we write $u = \epsilon\tilde{\psi} + \epsilon^p R$, we obtain

$$\|u - \epsilon\tilde{\psi}\|_{L^\infty(I,H^l)} = \epsilon^p \|R\|_{L^\infty(I,H^l)}.$$

It remains to show that the norm $\|R\|_{L^\infty([0,t_0],H^l)} \leq C < \infty$ is bounded for a $t_0 \in \Theta(\epsilon^{-1})$. Let $I = [0, t_0]$ and $S = \{-d_x, l\}$. We use the same energy as before, i.e.

$$E_{\hat{R}} = \frac{1}{2} \sum_{i \in S} \langle \partial_t \hat{R}, \partial_t \hat{R} \rangle_{L_i^{2,h}} + \langle \tilde{q}\hat{R}, \hat{R} \rangle_{L_i^{2,h}},$$

which is well-defined for $\tilde{q} = \frac{\tilde{q}_x}{\tilde{q}_t}$ since $(\partial_t R)(0) = \epsilon^{-p}\big(\partial_t u(0) - \epsilon \partial_t \tilde{\psi}(0)\big) \in L^{2,b}_{-d_x}$ and this property is respected by the PDE above – recall the discussion after Remark 2.1.23 in the case of the Boussinesq problem. Theorem 2.B.1 guarantees that there is a strict solution as long as this energy is finite since it gives a bound to the $L^2_{D_x - D_t + l} \times L^2_l$ norm. Further we know that the energy is continuously differentiable with respect to time as u and $\epsilon\psi$ are continuously differentiable with respect to time.

Let $t_* \coloneqq \sup\{s \in I \,|\, E(s) < M\}$ for a $M > 0$ which we will define later. If $t_* = \sup I$ there is nothing to do. We have to show again that $t_* < \sup I$ leads to a contradiction. We can derive the same differential inequality and, with the aid of Corollary 2.1.27, write it as

$$\partial_t E_{\hat{R}} \leq \epsilon D_1 E_{\hat{R}} + \epsilon^{2N-4-4\delta-2p-1} D_2,$$

where

$$D_1 \coloneqq C_0 + \kappa, \qquad D_2 \coloneqq \frac{2C_2^2}{\kappa} > 0.$$

Recall that D_1 depends on $\epsilon^{p-1}M$ since C_0 does. We have

$$\hat{R}(0) = \epsilon^{-p}\big(\hat{u}(0) - \tilde{\hat{\epsilon}}\psi(0)\big) \in L^2_s$$

and also recall

$$(\partial_t R)(0) = \epsilon^{-p}\big(\hat{u}(0) - \tilde{\hat{\epsilon}}\psi(0)\big) \in L^{2,b}_r \cap L^{2,b}_{-d_x}$$

and consequently $E_{\hat{R}}(0) \leq C_1$. We obtain by Gronwall's inequality for $t \leq t_*$ that

$$E_{\hat{R}}(t) \leq C_1 e^{D_1 \epsilon t} + \frac{\epsilon^{2N-4-4\delta-2p-2} D_2}{D_1}(e^{D_1 \epsilon t} - 1) \leq \frac{C_1 D_1 + \epsilon^{2N-4-4\delta-2p-2} D_2}{D_1} e^{D_1 \epsilon t_*}.$$

This shows

$$\begin{aligned} M &\leq \frac{C_1 D_1 + \epsilon^{2N-4-4\delta-2p-2} D_2}{D_1} e^{D_1 \epsilon t_*} \\ \Leftrightarrow & \frac{\ln(M D_1) - \ln\left(\epsilon^{2N-4-4\delta-2p-2} D_2 + C_1 D_1\right)}{D_1 \epsilon} \leq t_*. \end{aligned}$$

Recall that $\sup I = t_0 \in \Theta(\epsilon^{-1})$. We obtain the contradiction $\sup I \leq t_*$ for

$$N \geq 2 + 2\delta + p + 1 \quad \text{and} \quad \frac{\ln(M D_1(\epsilon^{p-1} M)) - \ln\left(D_2 + C_1 D_1(\epsilon^{p-1} M)\right)}{D_1(\epsilon^{p-1} M)} \gtrsim \epsilon t_0.$$

Hence the proof is complete since the first inequality is satisfied for sufficiently large N and the second inequality is true for sufficiently small ϵ_0 and large enough M. □

Theorem 2.1.34 leads to the following corollary for the Boussinesq equation (2.1).

Corollary 2.1.35. *Let $l \in \mathbb{N}$, $p > 1$, $3 + p < N \in \mathbb{N}$ and assume that the suppositions of Hypothesis 2.1.24 hold for $D_x = 3, d_x = 1, D_t = 1$. Let u be a strict solution to the Cauchy problem with equation* (2.1) *in the sense of Theorem 2.B.1 for initial data $(\hat{u}, \partial_t \hat{u})(0) = (\hat{v}, \hat{w}) \in L^2_{l+4} \times L^2_{l+2} \cap L^{2,b}_{-1}$.*
Then there exists $\epsilon_0 > 0$ such that for all $\epsilon \in]0, \epsilon_0[$ and initial data

$$\|v - \epsilon\psi(0)\|_{H^{l+4}} \lesssim \epsilon^p, \qquad \|\hat{w} - \epsilon\partial_t \hat{\psi}(0)\|_{L^{2,b}_{-1} \cap L^{2,b}_{l+2}} \lesssim \epsilon^p,$$

we have the estimate

$$\|u - \epsilon\psi\|_{L^\infty(I, H^l)} \lesssim \epsilon^p.$$

We already laid out all necessary arguments for the proof of Corollary 2.1.35 in the proof of Corollary 2.1.30. Therefore, we refrain from stating the details again.

2.2 A 4th Order Boussinesq-like Model

We will now take a look at another Boussinesq-like model for which we can say more about the existence of an approximation property. Essentially, we proceed in complete analogy to the foregoing Section 2.1 and we will reuse most of the results. We will only point out some alterations and modifications necessary due to the different model which will lead to more interesting results.

The model we want to look at now is the Cauchy problem for $u : I \times \mathbb{R} \to \mathbb{R}$ satisfying

$$0 = \mathcal{L}(\partial_t, \partial_x)u + u^2 \quad \text{in } I \times \mathbb{R}, \qquad u(0) = v, \qquad u_t(0) = w \quad \text{in } \mathbb{R}, \quad (2.17)$$

where $I = [0, T[$ for a $T > 0$, v, w sufficiently regular functions and the linear operator $\mathcal{L}(\partial_t, \partial_x) \coloneqq \partial_x^2 - \partial_x^4 - 1 - \partial_t^2$. Local existence and uniqueness for the Cauchy problem for equation (2.17) with sufficiently regular initial data is again trivially established by standard methods. Once more, we refer the interested reader to Appendix 2.B.

The model bears some resemblance to the so called ‘good’ Boussinesq equation

$$0 = \mathcal{L}(\partial_t, \partial_x)u + \partial_x^2 u^2 \quad \text{in } I \times \mathbb{R}, \qquad u(0) = v, \qquad u_t(0) = w \quad \text{in } \mathbb{R},$$

where the linear operator is $\mathcal{L}(\partial_t, \partial_x) \coloneqq \partial_x^2 - \partial_x^4 - \partial_t^2$ and v, w are sufficiently regular functions. This equation is also known as ‘the equation of the nonlinear string’ and was discussed in literature for some time, cf. [10, 59–61, 63]. It is well-known that the problem is locally well-posed in Sobolev spaces, cf. [10] or [60] for lower regularity, that it is has global small solutions, cf. [60], and that there are blow-up phenomena in finite time, cf. [49] for results on bounded domains and [61] on $\mathbb{R}$. In the theory of water waves a similar equation appears, which is called the ‘bad’ Boussinesq equation. The difference is that the sign of ∂_x^4 is inverted. This equation was derived by Boussinesq, cf. [12, (26)] and has attracted some interest since it admits soliton solutions, is completely integrable – which is true for the good Boussinesq equation as well, cf. the discussion in [60] – but it is ill-posed, cf. [21, 60]. Approximation results for the water waves problem can be found e.g. in [21, 51].

We note that there are two differences in comparison to equation (2.17). The first is that we have a purely polynomial nonlinear part. This is of minor importance since most of the subsequent constructions work as long as the problem remains semilinear. But the second difference, the fact that $\mathcal{L}(0,0) = 1 > 0$, will be essential in Section 2.2.2.

Obviously, it is possible to derive the same two formal approximate solutions as in Subsection 2.1.3, except for some details. The issue is now the same as in the previous Section 2.1, see Subsection 2.1.1, i.e. we want to prove a counterexample and investigate whether an approximation property for the formal approximate solution constructed with aid of the FWI system can be true. There is not much that has to be changed to obtain results similar to those of Section 2.1. Since the derivation of the two formal approximate solutions is the same as in Section 2.1.3, except for some details, we will only state the necessary adaptions in Section 2.2.1. Towards the end of Section 2.2.1, we will define a 'corrector' to the ansatz for the FWI system which will lead to better residual estimates on the natural time scale of the FWI ansatz. All questions concerning the dynamics of the solutions to the derived amplitude equations were discussed in Sections 2.1.4 and 2.1.5 before. Thus, we skip these and we will only prove some error estimates comparable to those of Section 2.1.6 in Section 2.2.2. Since there are some improvements possible in the case of the Boussinesq-like equation (2.17), we will obtain slightly better results than those of Section 2.1.6. We will use them to show a non-approximation result in Section 2.2.3. Unfortunately, this is no new result and still not what we were aiming for.

2.2.1 Derivation of the FWI and TWI Systems

We define the set of characteristic frequencies $\mathrm{Char}\mathcal{L}$ and $\omega_\pm$ for the symbol $\mathcal{L}$ of the linear operator of equation (2.17) in complete analogy to the definitions in Section 2.1.3. Then we choose the mapping $\omega : \mathbb{R} \to \mathbb{R}$ in the same manner as before and write again $\omega_i \coloneqq \omega(\xi_i)$ for $\xi_i \in \mathbb{R}$ and $i \in \mathbb{Z}$. The derivation of the FWI and TWI systems is done the same way as before in Section 2.1.3. All necessary changes and checks are as follows. At the end of this section we will discuss some correctors for the approximate solutions constructed with the FWI system. This will allow us to prove the mentioned non-approximation result in Section 2.2.3.

Derivation of the FWI system

We only have to check the non-resonance condition of Section 2.1.3 for the derivation of the FWI system, i.e.

$$\Omega(J)^2 - \Xi(J)^2(1 + \Xi(J)^2) - 1 = \Omega(J)^2 - \omega(\Xi(J))^2 \neq 0$$

for all $J', J'' \in I_4^{\leq 1}$ such that $\Omega(J) = \Omega(J') + \Omega(J''), \Xi(J) = \Xi(J') + \Xi(J')$ for a $J \in I_4^{\leq 2}$.

The situation is similar to the one in Remark 2.1.2 and we observe that in this case

$$(\omega_i + \omega_j)^2 - \omega(\xi_i + \xi_j)^2 = \omega_i^2 + \omega_j^2 + 2\omega_i\omega_j - \omega(\xi_i + \xi_j)^2 = 0$$

so that

$$\left(\omega_i^2 + \omega_j^2 - \omega(\xi_i + \xi_j)^2\right)^2 = 4\omega_i^2\omega_j^2.$$

This can be written as a polynomial in ξ_i, ξ_j whose degree is at most 8. Therefore we have the same consequence as noticed in Remark 2.1.2, namely that for every chosen ξ_i there are only finitely many ξ_j that we have to avoid.

Thus we can always satisfy this constraint except for finitely many tuples (ω_i, ξ_i) and the derivation of the (s)FWI system is possible as before in Section 2.1.3.

***Remark* 2.2.1.** Note that $(2\omega_i)^2 - \omega(2\xi_i)^2 = 3 - 12\xi_i^4 \neq 0$ for $|\xi_i| \neq \frac{1}{\sqrt{2}}$. Hence, the algebraic constraint for the sFWI ansatz is satisfied whenever $|\xi_3| \neq \frac{1}{\sqrt{2}}$.

Higher Order Residual for the sFWI System

Let us assume that we derived the sFWI system of the previous section. Note that, while deriving the sFWI system, we have left the addends with (effectively) $J = (3, 3, 3), J = (-3, -3, -3)$ untouched in the residual in order ϵ^3. Suppose that $\mathcal{L}(\Omega(J), \Xi(J)) \neq 0$ for these. Then we can define

$$\psi_{\text{FWIe}} \coloneqq \psi_{\text{FWI}} + \epsilon^2 \sum_{J \in \{(-3,-3,-3),(3,3,3)\}} \frac{\Xi(J)^2}{\mathcal{L}(\Omega(J), \Xi(J))} A_J(\epsilon^2 \boldsymbol{t}, \epsilon^2 \boldsymbol{x}) e^{\mathrm{i}\Omega(J)\epsilon^2 \boldsymbol{t} + \mathrm{i}\Xi(J)\epsilon^2 \boldsymbol{x}}$$

where

$$A_{(-3,-3,-3)} = \mathcal{M}((3),(3,3))A_{(-3)}A_{(-3,-3)} \quad A_{(3,3,3)} = \mathcal{M}((3),(3,3))A_{(3)}A_{(3,3)}.$$

Consequently, we obtain for some time interval I

$$\|\mathscr{F}\,\mathfrak{E}_{\epsilon\psi_{\mathrm{FWIe}}}\|_{L^\infty(I,L^2_l)} \lesssim \epsilon^3 \tag{2.18}$$

and

$$\begin{aligned}\|\psi_{\mathrm{FWIe}} - \psi_{\mathrm{FWI}}\|_{L^\infty(I,W^{k,\infty})} &\lesssim \epsilon^2 \sum_{J\in\{(-3,-3,-3),(3,3,3)\}} \frac{\Xi(J)^2}{|\mathcal{L}(\Omega(J),\Xi(J))|}\|A_J\|_{L^\infty(I,W^{k,\infty})} \\ &\lesssim \epsilon^2 \end{aligned} \tag{2.19}$$

as long as $\|A_{(3)}\|_{L^\infty(I,W^{k,\infty})}$, $\|A_{(3)}\|_{L^\infty(I,H^{l+4})}$, $\|\partial_t A_{(3)}\|_{L^\infty(I,H^l)}$ and $\|\partial_t^2 A_{(3)}\|_{L^\infty(I,H^l)}$ are bounded. It is sufficient for the satisfaction of this condition, in view of Theorem 2.1.6, that the initial datum $A_3(0)$ is in $H^{l+4}\cap W^{k,\infty}\cap C^{\max\{k,l+4\}}(\mathbb{R},\mathbb{C})$. Note that the first estimate holds essentially by construction of the corrector above, which removes all addends up to order ϵ^3. Everything left is pointwise of order ϵ^4. Due to the fact that the L^2 norm is not invariant under the scaling $x \to \epsilon^2 x$, we loose the factor ϵ if we take the L^2_l norm of the Fourier transform of the residual $\mathfrak{E}_{\epsilon\psi_{\mathrm{FWIe}}}$. Therefore the residual estimate is valid by Theorem 2.1.6.

Remark **2.2.2.** Note that $(3\omega_3)^2 - \omega(3\xi_3)^2 = 8 - 72\xi_3^4 \neq 0$ for $|\xi_3| \neq \frac{1}{\sqrt{3}}$. Therefore we can define ψ_{FWIe} the way we did above whenever ψ_{FWI} is constructed with the aid of sFWI system, has sufficiently regular initial data, and $|\xi_3| \notin \left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}\right\}$.

Derivation of the TWI System

The only difficulty for the derivation of the TWI system consists in the choice of the resonant wave numbers. But it is easy to verify that one can choose such waver numbers. We proceed in complete analogy to Remark 2.1.5 and define for $0 < \alpha$ $0 < \xi_1 := \alpha\xi_2$. Thus we obtain $\xi_3 = -(1+\alpha)\xi_2 = -(1+\alpha)\eta \in \mathbb{R}^-$. Then we have to consider the equation

$$0 = \omega(\eta) + \omega(\alpha\eta) - \omega((1+\alpha)\eta).$$

Again we investigate the map $\omega(\boldsymbol{\eta}) + \omega(\alpha\boldsymbol{\eta}) - \omega((1+\alpha)\boldsymbol{\eta}) \in C(\mathbb{R}^+, \mathbb{R})$ for this purpose. A short calculation shows

$$\begin{aligned}
\omega(\eta) + \omega(\alpha\eta) - \omega((1+\alpha)\eta) &= -2\alpha\eta^2 + \mathcal{O}(\eta), &\qquad& \eta \to \infty,\\
\omega(\eta) + \omega(\alpha\eta) - \omega((1+\alpha)\eta) &= 1 + \mathcal{O}(\eta^2), && \eta \to 0.
\end{aligned}$$

Hence, there is a zero for a $\eta = \xi_2 > 0$ by the intermediate value theorem and we can again choose wave numbers ξ_i out of an uncountable set. Note that the rest of Remark 2.1.5 remains true as well, only the degree of the polynomial changes.

2.2.2 Error Estimate

The issue at hand in this section is the same as it was in Section 2.1.6 with the single difference that we are considering the Boussinesq-like problem (2.17). We are again seeking an error estimate to be able to make a decision whether any of the constructed formal approximate solutions is a true approximate solution. We will always consider the time interval $I = \left[0, \frac{-\ln\epsilon + qt_0}{q\epsilon\sqrt{\tilde{c}}}\right]$ where $t_0 > 0$ and $\tilde{c} = 1 + \epsilon^{2/q}$, $q \in [1, \infty[$, in rest of this section. The reason is that we know that the formal approximate solutions constructed by the FWI and TWI system respectively exhibit large differences within this time interval as explained at the beginning of Section 2.1.6. Recall that the consequence is the same as in Section 2.1.6: the arguments in the rest of this section are inadequate to a proof of an approximation property of the FWI system on its natural time scale.

We will give an error estimate in the following Corollary 2.2.3. The corollary is again an immediate consequence of Theorem 2.1.28 of the previous Section 2.1.6. Additionally, we prove another abstract Theorem 2.2.5 which exploits the properties of the Boussinesq problem (2.17) better. Then Corollary 2.2.7 gives an improved error estimate for the Boussinesq problem (2.17).

Albeit these theorems and corollaries are not sufficient to prove an approximation property in the spirit of the Approximation Property 1.3.1 for the FWI formal approximate solution on its natural time scale, as mentioned before, they are nonetheless useful to prove that an approximation property cannot hold. We will discuss this in some detail in the subsequent Section 2.2.3.

Simple Error Estimate

We note that the error estimate in Section 2.1.5 is sufficiently general to cope with this problem as well. Thus we obtain the following corollary of Theorem 2.1.28.

Corollary 2.2.3. *Let $l, N \in \mathbb{N}$, N sufficiently large, see the end of the proof of Theorem 2.1.28, and assume that the suppositions of Hypothesis 2.1.24 hold for $D_x = 2, d_x = D_t = 0$. Let u be a strict solution to the Cauchy problem with equation* (2.1) *in the sense of Theorem 2.B.1 for initial data*

$$(u, \partial_t u)(0) = (v, w) \in H^{s+1} \cap C^{s+1}(\mathbb{R}, \mathbb{C}) \times H^s \cap C^s(\mathbb{R}, \mathbb{C}).$$

Then there exists $\epsilon_0 > 0$ such that for all $\epsilon \in]0, \epsilon_0[$, arbitrary $p > 1$, and initial data

$$\|v - \epsilon\psi(0)\|_{H^{s+1}} \lesssim \epsilon^{N-2}, \qquad \|w - \epsilon\partial_t\psi(0)\|_{H^s} \lesssim \epsilon^N,$$

we have the estimate

$$\|u - \epsilon\psi\|_{L^\infty(I, H^l)} \lesssim \epsilon^p.$$

Proof. We meet the assumptions of Hypothesis 2.1.22 with $D_x = 2, d_x = D_t = 0$ for any $m_0 \in \mathbb{N}$ since $f : u \mapsto u^2$ and $Df(v)[u] = 2(uv)$ in suitable spaces. Since L^2_{2+m} is an convolution algebra and f essentially a polynomial, $\mathscr{F} \circ f \circ \mathscr{F}^{-1} : L^2_{2+m} \to L^2_m$ is Lipschitz on bounded subsets. Differentiability and continuity are obvious for the concrete representations above and it is obvious that

$$\|M_{q_t^{-1}(-\boldsymbol{\xi}^2)} \mathscr{F}\, Df(v)\, \mathscr{F}^{-1}\|_{\mathrm{L}(L^2_{2+m}, L^{2,b}_m)} \lesssim \|v\|_{W^{m,\infty}},$$
$$\|M_{q_t^{-1}(-\boldsymbol{\xi}^2)} \mathscr{F}\, Df(v)\, \mathscr{F}^{-1}\|_{\mathrm{L}(L^2_{2+m}, L^{2,b}_0)} \lesssim \|v\|_{W^{m,\infty}}$$

where the constant solely depends on some continuous embeddings. Thus Theorem 2.1.28 applies for all $l \in \mathbb{N}$. □

Improved Error Estimate

Recall that there are some improvements on Theorem 2.1.28 possible in special situations, see Remark 2.1.29. Actually, we do not need the statements of

Corollary 2.1.27 if $d_x = 0$. Thus we can easily prove that we can substitute Theorem 2.1.28 by the subsequent Theorem 2.2.5 for the abstract Cauchy problem with equation (2.15) and $d_x = 0$. The regularity assumptions of Hypothesis 2.1.24 can be lessened in this situation and we can substitute them by Hypothesis 2.2.4.

Hypothesis 2.2.4. *Let $m_0 \leq l \in \mathbb{N}$,*

$$\begin{aligned} s &= \max\{2(D_x - D_t) + l, l + 2\}, \\ r &= D_x - D_t + l. \end{aligned}$$

We assume there is a function $\psi \in C^s(I \times \mathbb{R}, \mathbb{C})$ and

$$\epsilon\psi \in L^\infty(I, W^{r,\infty}), \qquad \mathfrak{E}_{\epsilon\psi} \in L^\infty(I, H^l), \qquad \|\mathfrak{E}_{\epsilon\psi}\|_{L^\infty(I,H^l)} \leq C_2 \epsilon^N$$

with bounds of these norms uniform in $\epsilon \in]0, \epsilon_0]$ for $t \in I$, an $\epsilon_0 \in]0, 1]$ and any arbitrary fixed $N \in \mathbb{N}$, which we will determine later.

Theorem 2.2.5. *Let $d_x = 0$, $m_0 \leq l \in \mathbb{N}$, and $N \in \mathbb{N}$ sufficiently large, see the end of the proof. We assume that Hypothesis 2.1.22 and 2.2.4 are satisfied. Let u be a strict solution to the Cauchy problem with equation* (2.15) *in the sense of Theorem 2.B.1 for initial data $(u, \partial_t u)(0) = (v, w) \in H^s \times H^r$.*

Then there exists $\epsilon_0, D_1(1) > 0$ such that for all $\epsilon \in]0, \epsilon_0[$, arbitrary $p > 1$, and initial data

$$\|v - \epsilon\psi(0)\|_{H^r} + \|w - \epsilon\partial_t\psi(0)\|_{H^l} \lesssim \epsilon^{p + \frac{D_1(1)}{2q\sqrt{\tilde{c}}}},$$

we have the estimate

$$\|u - \epsilon\psi\|_{L^\infty(I,H^l)} \lesssim \epsilon^p$$

Proof. We write $u = \epsilon\psi + \epsilon^p R$. Thus we obtain

$$\|u - \epsilon\psi\|_{L^\infty(I,H^l)} = \epsilon^p \|R\|_{L^\infty(I,H^l)}.$$

and it remains to show that the norm $\|R\|_{L^\infty(I,H^l)} \leq C < \infty$ is bounded. Recall that we have for $\mathcal{L} = \mathcal{L}(\partial_t, \partial_x)$ the equation

$$0 = \mathcal{L}R + \epsilon^{-p}\big(f(\epsilon^p R + \epsilon\psi) - f(\epsilon\psi)\big) + \epsilon^{-p}\mathfrak{E}_{\epsilon\psi}.$$

In view of Remark 2.1.23 and Theorem 2.B.1 we define the energy

$$E_{\hat{R}} = \frac{1}{2}\left(\langle \partial_t \hat{R}, \partial_t \hat{R}\rangle_{L^{2,b}_l} + \langle \tilde{q}\hat{R}, \hat{R}\rangle_{L^{2,b}_l}\right).$$

Theorem 2.B.1 guarantees that there is a strict solution as long as this energy is finite since it gives a bound to the $L^2_r \times L^2_l$ norm. Further we know that the energy is continuously differentiable with respect to time as u and $\epsilon\psi$ are continuously differentiable with respect to time.

Let $t_* := \sup\{s \in I \,|\, E(s) < M\}$ for a $M > 0$, which we will define later. If $t_* = \sup I$ there is nothing to do. Hence we have to show that $t_* < \sup I$ leads to a contradiction. We will derive a differential inequality for this purpose. Let $t \leq t_*$. We start with the observation that for any $\kappa > 0$ we find as in the proof of Theorem 2.1.28

$$\begin{aligned}
&\partial_t E_{\hat{R}} \\
=&\mathrm{Re}\left(\langle \tilde{q}_t^{-1}\mathscr{F}\,\epsilon^{-p}\mathfrak{E}_{\epsilon\psi}, \partial_t\hat{R}\rangle_{L^{2,b}_l} + \epsilon^{-p}\langle \tilde{q}_t^{-1}\mathscr{F}\left(f(\epsilon^p R + \epsilon\psi) - f(\epsilon\psi)\right), \partial_t\hat{R}\rangle_{L^{2,b}_l}\right) \\
\leq&\|\tilde{q}_t^{-1}\mathscr{F}\,\epsilon^{-p}\mathfrak{E}_{\epsilon\psi}\|_{L^{2,b}_l}\|\partial_t\hat{R}\|_{L^{2,b}_l} + \epsilon^{-p}\|\tilde{q}_t^{-1}\mathscr{F}\left(f(\epsilon^p R + \epsilon\psi) - f(\epsilon\psi)\right)\|_{L^{2,b}_i}\|\partial_t\hat{R}\|_{L^{2,b}_l} \\
\leq&\frac{\epsilon^{-2p+1}}{2\kappa}\|\tilde{q}_t^{-1}\mathscr{F}\,\mathfrak{E}_{\epsilon\psi}\|^2_{L^{2,b}_l} + \frac{\kappa}{2}\epsilon\|\partial_t\hat{R}\|^2_{L^{2,b}_l} + \|\tilde{q}_t^{-1}\mathscr{F}\circ Df(\epsilon\psi)[R]\|_{L^{2,b}_i}\|\partial_t\hat{R}\|_{L^{2,b}_l} \\
&\quad + \epsilon^p\|\tilde{q}_t^{-1}G(\epsilon\psi, R, \epsilon^p R)\|_{L^{2,b}_i}\|\partial_t\hat{R}\|_{L^{2,b}_l} \\
\leq&\frac{\epsilon^{-2p+1}}{2\kappa}\|\tilde{q}_t^{-1}\mathscr{F}\,\mathfrak{E}_{\epsilon\psi}\|^2_{L^{2,b}_l} + \frac{\kappa}{2}\epsilon\|\partial_t\hat{R}\|^2_{L^{2,b}_l} + 2h_1\big(\epsilon\|\psi\|_{L^\infty(I,W^{r,\infty})}\big)\|\hat{R}\|_{L^2_r}\|\partial_t\hat{R}\|_{L^{2,b}_l} \\
&\quad + \max_{\delta\in[0,1]} h_1\big(\|\epsilon\psi + \delta\epsilon^p R\|_{L^\infty(I,W^{r,\infty})}\big)\|\hat{R}\|_{L^2_r}\|\partial_t\hat{R}\|_{L^{2,b}_l} \\
\leq&\frac{\epsilon^{-2p+1}}{2\kappa}\|\mathscr{F}\,\mathfrak{E}_{\epsilon\psi}\|^2_{L^2_{\max\{l-2D_t,0\}}} + \epsilon(C_0 + \kappa)E
\end{aligned}$$

for a constant $C_0 \geq 0$ and ϵ_0 small enough. We used a first order remainder G in the Taylor expansion for this calculation, which means that we applied the mean value theorem for the estimate of it. Further we exploited the second assumption on f in Hypothesis 2.1.22 and made use of $\|\hat{R}\|^2_{L^2_r} \lesssim E_{\hat{R}}$. Therefore the constant C_0 depends on $\epsilon_0^{p-1}M$, $\|\psi\|_{L^\infty(I,W^{r,\infty})}$ and ϵ_0 but can be chosen uniformly for $\epsilon \in]0, \epsilon_0[$.

Note that $C_0(\boldsymbol{\epsilon_0^{p-1} M})$ is monotonically increasing in $\epsilon_0^{p-1}M$ and $C_0(\boldsymbol{\epsilon_0^{p-1} M}) \in \Omega(1)$ for $\epsilon_0^{p-1}M \to 0$. This is a consequence of the second assumption on f in Hypothesis 2.1.22. Hypothesis 2.2.4 allows us to write

$$\partial_t E_{\hat{R}} \leq \epsilon D_1 E_{\hat{R}} + \epsilon^{2N-2p-1} D_2,$$

where

$$D_1 := C_0 + \kappa, \qquad\qquad D_2 := \frac{2C_2^2}{\kappa} > 0.$$

Therefore D_1 depends on $\epsilon_0^{p-1}M$ in the same way C_0 does. We have

$$\|(\hat{R}, \partial_t \hat{R})(0)\|_{L_r^2 \times L_l^2} \lesssim \epsilon^{-p}\big(\|v - \epsilon\hat{\psi}(0)\|_{L_r^2} + \|w - \epsilon\partial_t\hat{\psi}(0)\|_{L_l^2}\big)$$

and consequently $E_{\hat{R}}(0) \leq C_1 \epsilon^{\frac{D_1(1)}{q\sqrt{\tilde{c}}}}$ for a constant $C_1 > 0$. We obtain by Gronwall's inequality for $t \leq t_*$

$$E_{\hat{R}}(t) \leq E(0)e^{D_1\epsilon t} + \frac{\epsilon^{2N-2p-2}D_2}{D_1}(e^{D_1\epsilon t} - 1) \leq \frac{D_1 C_1 \epsilon^{\frac{D_1}{q\sqrt{\tilde{c}}}} + \epsilon^{2N-2p-2}D_2}{D_1} e^{D_1 \epsilon t_*}.$$

This shows

$$\begin{aligned}
&M \leq \frac{D_1 C_1 \epsilon^{\frac{D_1}{q\sqrt{\tilde{c}}}} + \epsilon^{2N-2p-2}D_2}{D_1} e^{D_1 \epsilon t_*} \\
\Leftrightarrow &\frac{\ln(D_1 M) - \ln(D_1 C_1 \epsilon^{\frac{D_1}{q\sqrt{\tilde{c}}}} + \epsilon^{2N-2p-2}D_2)}{D_1\epsilon} \leq t_*.
\end{aligned}$$

Hence we obtain for $N \geq p + 1 + \frac{D_1}{2q\sqrt{\tilde{c}}}$

$$\begin{aligned}
\frac{\ln(MD_1) - \ln(D_1C_1 + D_2) - \frac{D_1}{q\sqrt{\tilde{c}}}\ln\epsilon}{D_1\epsilon} &\leq \frac{\ln(D_1 M) - \ln(D_1 M \epsilon^{\frac{D_1}{q\sqrt{\tilde{c}}}} + \epsilon^{2N-2p-2}D_2)}{D_1\epsilon} \\
&\leq t_*.
\end{aligned}$$

Recall that $\sup I = \frac{-\ln\epsilon + qt_0}{q\epsilon\sqrt{\tilde{c}}}$. We obtain the contradiction $\sup I \leq t_*$ for

$$\frac{\ln(MD_1(\epsilon_0^{p-1}M)) - \ln(D_1(\epsilon_0^{p-1}M)C_1 + D_2)}{D_1(\epsilon_0^{p-1}M)} \geq \frac{t_0}{\sqrt{\tilde{c}}}.$$

Hence the proof is complete since this inequality is true for $\epsilon_0^{p-1}M \leq 1$ and sufficiently large M. $\square$

***Remark* 2.2.6.**

1. The advantage in this case is two fold. First, we need less regularity for the initial data. In Theorem 2.2.5 we need initial data in $H^s \times H^r$ whereas initial data in $H^{s+1} \times H^s$ had been required in Theorem 2.1.28. Secondly, there is no loss in powers of ϵ due to Corollary 2.1.27 which we do not need in this situation. This alleviates the strong condition 'N sufficiently large' to some extent.

2. Again the question how large N actually has to be arises. Sufficient is

$$N > p + 1 + \frac{C_0(0)}{2q\sqrt{\tilde{c}}}$$

since κ, ϵ_0, and $\epsilon_0^{p-1}M$ can be chosen arbitrarily small in the proof of Theorem 2.2.5. Since p will be fixed in application one can think about methods to make the third addend small. The most obvious way is to make q large. Such a choice means that it is sufficient to choose N as the smallest integer larger than $p+1$ if we accept the price of making q pretty large.

Clearly, Theorem 2.2.5 applies to the Cauchy problem for the Boussinesq equation (2.17) and we obtain a corollary.

Corollary 2.2.7. *Let $l \in \mathbb{N}$, $p > 1$ and $N \in \mathbb{N}$ sufficiently large, see the end of the proof of Theorem 2.2.5. We assume that Hypothesis 2.1.22 is satisfied for $D_x = 2, D_t = 0$. Let u be a strict solution to the Cauchy problem for the Boussinesq equation* (2.17) *in the sense of Thereom 2.B.1 for initial data* $(u, \partial_t u)(0) = (v, w) \in H^{4+l} \times H^{l+2}$.
Then there exists $\epsilon_0, D_1(1) > 0$ such that for all $\epsilon \in]0, \epsilon_0[$ and initial data

$$\|v - \epsilon\psi(0)\|_{H^{l+2}} + \|w - \epsilon\partial_t\psi(0)\|_{H^l} \lesssim \epsilon^{p+\frac{D_1(1)}{2q\sqrt{\tilde{c}}}},$$

we have the estimate

$$\|u - \epsilon\psi\|_{L^\infty(I,H^l)} \lesssim \epsilon^p.$$

Proof. We meet the assumptions of Hypothesis 2.1.22 with $D_x = 2, d_x = D_t = 0$ for any $m_0 \in \mathbb{N}$ since $f : u \mapsto u^2$ and $Df(v)[u] = 2(uv)$ in suitable spaces. Since L^2_{2+m} is an convolution algebra and f essentially a polynomial, $\mathscr{F} \circ f \circ \mathscr{F}^{-1} : L^2_{2+m} \to L^2_m$ is Lipschitz on bounded subsets. Differentiability and continuity are obvious for the concrete representations above and it is obvious that

$$\|M_{q_t^{-1}(-\boldsymbol{\xi}^2)} \mathscr{F}\, Df(v)\, \mathscr{F}^{-1}\|_{\mathrm{L}(L^2_{2+m}, L^{2,b}_m)} \lesssim \|v\|_{W^{m,\infty}}$$
$$\|M_{q_t^{-1}(-\boldsymbol{\xi}^2)} \mathscr{F}\, Df(v)\, \mathscr{F}^{-1}\|_{\mathrm{L}(L^2_{2+m}, L^{2,b}_0)} \lesssim \|v\|_{W^{m,\infty}}$$

where the constant solely depends on some continuous embeddings. Thus Theorem 2.2.5 applies for all $l \in \mathbb{N}$. □

In this concrete case we can reduce the size of N further. If we use a second order Taylor expansion in the proof of Theorem 2.2.5 for the Boussinesq problem (2.17), we can prove that for $l \leq 2$ the bound

$$N \geq p + 1 + \frac{\|\psi\|_{L^\infty}}{q\sqrt{\tilde{c}}} + \frac{\kappa}{2q\sqrt{\tilde{c}}} + \frac{\rho}{q\sqrt{\tilde{c}}}\big(\|\psi\|_{L^\infty} + \kappa\big)$$

is sufficient for any $\rho, \kappa > 0$. Or in other terms, if we accept that ϵ_0 becomes pretty small and M quite large in the proof of Theorem 2.2.5, then we can choose κ, ρ as small as we want to and obtain

$$N > p + 1 + \frac{\|\psi\|_{L^\infty(I, L^\infty)}}{q\sqrt{\tilde{c}}}. \tag{2.20}$$

This is consistent with the bound given in Remark 2.2.6 since for $l \leq 2$ we have $C_0(0) \leq 6\|\psi\|_{L^\infty(I,L^\infty)}$. The additional factor 3 is a tribute to the error of the first order Taylor expansion in the proof of Theorem 2.2.5.

2.2.3 Example of a Non-Approximation Result

We claimed in Sections 2.1.6 and 2.2.2 that the error estimates of Theorem 2.1.28 and Theorem 2.2.5 are useful to prove that in some cases no approximation property can be true. We will illustrate this for the FWI and TWI formal approximate solutions constructed in Section 2.2.1. We will demonstrate that no approximation property can hold for the TWI formal approximate solution in the case of the Boussinesq-like problem (2.17) and $t \in I = \big[0, \frac{-\ln \epsilon + q t_0}{q\epsilon\sqrt{\tilde{c}}}\big]$. However, this is a somewhat useless thing to do since I is much longer than the natural time scale of the TWI approximate solution and we already know that it is wrong, recall the discussion in Section 2.1.5. The really interesting statement would be the other way round where we would use the theorems to proof that no approximation property for the FWI formal approximate solution is true on I since I is much shorter than its natural time scale. Unfortunately, we are unable to show this at the moment since we do not have an appropriate estimate for the residual of the TWI formal approximate solution on I.

For the demonstration of the usefulness of the theorems we go back to the examples considered in Section 2.1.5. We compared the formal approximate solution constructed with the aid of the FWI and TWI system respectively in that section and stated that we can find initial data such that $\epsilon \lesssim \|\epsilon\psi_{\mathrm{TWI}} - \epsilon\psi_{\mathrm{FWI}}\|_{L^\infty(I,L^\infty)}$ on $I = \left[0, \frac{-\ln\epsilon + qt_0}{q\epsilon\sqrt{\tilde{c}}}\right]$ where

$$\tilde{c} = \sup_{x\in\mathrm{supp}(b_1)} b_1(x)^2 + b_3(x)^2 \geq \sup_{x\in\mathrm{supp}(b_1)} b_3(x)^2 = \|b_3\|^2_{L^\infty}.$$

Actually, we only considered the case $q = 1$ in that section but the results essentially remain unchanged provided that we choose $\|b_1\|_{L^\infty} = \epsilon^{1/q}$, i.e. $b_1 = \epsilon^{1/q}\mathrm{bmp}$. This leads to $2b_3 = -2\mathrm{i}\sqrt{|\gamma_1\gamma_2|}A_3(0) = -|\omega_1\omega_2|^{-1/2}a_3$ for the initial data of the FWI system so that

$$\|2b_3\|_{L^\infty} = |\omega_1\omega_2|^{-1/2}\|a_3\|_{L^\infty}.$$

Further we have $\|b_3\|_{L^\infty} = \|b_3\|_{L^\infty(\mathrm{supp}(b_1))}$. Therefore we can estimate

$$\|\psi_{\mathrm{FWI}}\|_{L^\infty} \leq \|\sum_{J\in\mathcal{I}_4^{\leq i}} A_J\|_{L^\infty} + \sum_{J\in\mathcal{I}_4^{\leq 2}} \epsilon\|A_J\|_{L^\infty} \leq \|a_3\|_{L^\infty} + \epsilon\sum_{J\in\mathcal{I}_4^{\leq 2}} \|A_J\|_{L^\infty},$$

hence

$$\frac{\|\psi_{\mathrm{FWIe}}\|_{L^\infty}}{\sqrt{\tilde{c}}} \leq 2\sqrt{|\omega_1\omega_2|} + 2\epsilon\sqrt{|\omega_1\omega_2|}\frac{\sum_{J\in\mathcal{I}_4^{\leq 2}}\|A_J\|_{L^\infty}}{\|a_3\|_{L^\infty}} + \mathcal{O}(\epsilon^2).$$

If we use estimate (2.20), the estimate above and the residual estimate for ψ_{FWIe} in (2.18), we obtain that the error estimate of Corollary 2.2.7 holds for $p \in]1,2[$

and ϵ_0 small enough provided

$$2 > p + \frac{2\sqrt{|\omega_1\omega_2|}}{q} \qquad \Leftrightarrow \qquad q > \frac{2\sqrt{|\omega_1\omega_2|}}{2-p}. \tag{2.21}$$

Thus, we know that ψ_{FWIe} is an approximate solution to the Cauchy problem of the Boussinesq-like equation (2.17) for $t \in I$ with error of order ϵ^p. On the other hand, we know

$$\epsilon \lesssim \|\epsilon\psi_{\mathrm{TWI}} - \epsilon\psi_{\mathrm{FWIe}}\|_{L^\infty(I,L^\infty)} \tag{2.22}$$

where the constant is larger than zero and uniformly bounded from below. This is a consequence of the discussion in Section 2.1.6 and equation (2.19).

Therefore we showed that no approximation property similar to the subsequent Approximation Property 2.2.8 can hold for the formal approximate solution constructed with the aid of the TWI system.

***Approximation Property* 2.2.8.** Let $4 \leq l \in \mathbb{N}$. Let $\epsilon\psi \in C^l(I \times \mathbb{R}, \mathbb{R})$ be the formal approximate solution constructed with the aid of the TWI system in Section 2.2.1. Further suppose $\epsilon(\psi, \partial_t\psi) \in C(I, H^l \times H^{l-2})$ and $p \in \left]1, 2\frac{1+\sqrt{|\omega_1\omega_2|}}{1+2\sqrt{|\omega_1\omega_2|}}\right[$. Let u be a strict solution to the Cauchy problem for the Boussinesq-like equation (2.17) in the sense of Theorem 2.B.1 with initial data $(u_0, v_0) \in H^l \times H^{l-2}$. Then there is $\epsilon_0 > 0$ such that for all $\epsilon \in]0, \epsilon_0[$ and initial data satisfying

$$\|u_0 - \epsilon\psi(0)\|_{H^l} + \|v_0 - \epsilon\partial_t\psi(0)\|_{H^{l-2}} \lesssim \epsilon^p,$$

this property is preserved for $t \in \Theta(\frac{-\ln\epsilon}{\epsilon})$, i.e. it holds for $(u, \partial_t u)(0) = (u_0, v_0)$ and

$$\|u - \epsilon\psi\|_{L^\infty(I,H^{l-2})} \lesssim \epsilon^p.$$

The result is clear if we set $p = 1 + \frac{1}{q}$ and use the resulting bound on q in inequality (2.21) together with the lower bound between the FWIe and TWI formal approximate solution in inequality (2.22) and Corollary 2.2.3 where we use $(u, \partial_t u)(0) = \epsilon(\psi_{\mathrm{FWIe}}, \partial_t\psi_{\mathrm{FWIe}})(0)$. Note that $|\omega_i| > 1$ since $\xi_i \neq 0$. Thus it is not possible to get $q < 3$ this way meaning $1 < p < 1 + \frac{1}{3}$ above. If one wishes to rule out the case $p = \frac{3}{2}$ in Approximation Property 2.2.8, which would be the analogue to the statement of Schneider et al. in [84], one needs a better residual estimate than estimate (2.18). A residual estimate for ψ_{FWIe} of order ϵ^N allows to rule out $p \in \left]1, \frac{N-1+2\sqrt{|\omega_1\omega_2|}}{1+2\sqrt{|\omega_1\omega_2|}}\right[$. A consequence is that a finite $N \in \mathbb{N}$, that cannot be chosen arbitrarily large, allows for an approximation property with large enough p. This is a certain drawback of this method that stems from the choice $p = 1 + \frac{1}{q}$ above.

In order to eliminate this drawback, one could come up with the idea to choose the initial data of the non-resonant FWI system (2.6) the same way as for the TWI system above and the wave numbers and frequencies accordingly. This would decouple the p in the approximation property from q for the initial data as follows.

- We choose $p \in]1, N-1[$ of the size that we wish for the error bound $\|u - \epsilon\psi_{\mathrm{TWI}}\|_{L^\infty(I,L^\infty)} \lesssim \epsilon^p$ and q large enough to satisfy the substitute of inequality (2.21) for $N \in \mathbb{N}$ ($N = 3$ is precisely inequality (2.21)).

- This would provide two formal approximate solutions to the same initial data $u(0) = \epsilon\psi_{\mathrm{TWI}}(0) = \epsilon\psi_{\mathrm{FWIe}}(0)$ and an error of order ϵ for the chosen initial data near the end point of I and ϵ_0 small enough. The latter is a simple consequence of the fact that $\epsilon^2 \sup I \to 0$ for $\epsilon \to 0$ and that estimates which are analogous to the estimates in Section 2.1.5 for the sFWI solution hold for the non-resonant FWI system whose solution is given in Section 2.1.4.

- By Corollary 2.2.7 the FWI solution ψ_{FWIe} has the required error bound. Hence, the approximation property cannot hold for ψ_{TWI}.

Unfortunately, this procedure is not possible since the constraints for the derivation of the FWI system conflict with the resonance condition of the TWI system. Another way to overcome the problem that q might be quite large for a fixed $N \in \mathbb{N}$ could be to reconsider Section 2.1.5 and try to find smaller intervals $\tilde{I} \subset I$ on which the FWI and TWI solutions vary to a sufficient degree. For instance we could require that $|B_1(t)|$ of the TWI solution is only sufficiently large in a certain area in $\mathrm{supp}(b_1)$ instead of requiring that $|B_1(t)|$ is maximal in that area, which means $B_3(t) = A_3(t) = 0$ in that area. This leads to the following observation supplementing Remark 2.1.13 of Section 2.1.4.

***Remark* 2.2.9.** Let $b_1, b_3 \geq 0$ and assume Hypothesis 2.1.9 is satisfied. We study the question when the component with the minus sign in the TWI system (2.9), B_3, reaches the value $B_3(t) = \alpha\sqrt{c}$ for some $\alpha \in]-1,1[$ and $x \in M_1$. In this case we obtain the equation

$$\frac{b_3(x) - \alpha\sqrt{c(x)}}{\sqrt{c(x)} - \alpha b_3(x)} = \tanh\left(\sqrt{c(x)}t\right).$$

Hence,

$$t_3(x) = \frac{\frac{1}{2}\ln\frac{1-\alpha}{1+\alpha} + \ln\left(\sqrt{c(x)} + b_3(x)\right) - \ln b_1(x)}{\sqrt{c(x)}}.$$

On the other hand, since $B_1(t)^2 = c - B_3(t)^2$, we obtain for $B_3(t) = \alpha\sqrt{c}$ that $B_1(t) = \sqrt{1-\alpha^2}\sqrt{c} = \beta\sqrt{c}$. Thus for given $\beta \in]0,1]$, the same question for B_1 results in $\alpha = \sqrt{1-\beta^2}$ and consequently

$$t_1(x) = \frac{\ln\frac{1-\sqrt{1-\beta^2}}{\beta} + \ln\left(\sqrt{c(x)} + b_3(x)\right) - \ln b_1(x)}{\sqrt{c(x)}}.$$

This shows that, under these assumptions, it is not really interesting to proceed this way since for all α independent of ϵ and $\|b_1\|_{L^\infty} \leq \epsilon^{1/q}$ we always end up with an addend $-\frac{\ln\epsilon}{q}$.

In view of the motivation mentioned at the very beginning of this Chapter and in view of the articles of Schneider et al. [84, 85], the question whether an approximation property is true for the formal approximate solution constructed with the aid of the FWI system on its natural time scale is of much more interest. In the cited articles they give a counterexample in the case of L^2 functions with periodic boundary conditions. They are able to construct a corrector for the TWI system, or rather an extended TWI system, such that the residual estimate for the formal approximate solution constructed with the aid of the extended TWI system is of order $\mathcal{O}(\epsilon^N)$ for an arbitrary $N \in \mathbb{N}$ on a sufficiently large interval I. Then they proceed as outlined above and obtain that no approximation property can hold for the formal approximate solution constructed with the aid of the NLS or the FWI equations.

Since we cannot construct a corrector to the formal approximate solution in the

TWI case, we are unable to prove such a result. Whilst the construction of such a corrector should be rather simple for $t \in \Theta(\epsilon^{-1})$, i.e. on the natural time scale, for I as above it is not clear at the moment. The reason is that by adding higher order correctors one can expect to obtain some algebraic equations and some linearised differential equations of (2.8), cf. Section 3.A, where such a corrector is constructed for another problem. One can try to solve the linearised Cauchy problem with vanishing initial data with the aid of evolution families, cf. [37, Section VI.9; 53; 79]. However, the problem with the long time interval I is that the evolution family might grow with an exponential rate, which is much too quickly for the long time interval but completely fine for all (uniformly in ϵ) bounded time intervals.

2.3 Conclusions

Section 2.2.3 showed that the Theorems 2.1.28 and 2.2.5 can be used to prove that no approximation property holds for some formal approximate solutions. This was demonstrated in the case of the TWI approximation in that section but the time scale was beyond the natural time scale of the TWI formal approximate solution. Because of this fact and because of the explicit examples of Section 2.1.5, this comes as no surprise. We already knew that there cannot hold an approximation property for the TWI formal approximate solution on a time scale containing $\Theta(\epsilon^{-p})$ for any $p > 1$, see Section 2.1.5.

Though, if we were able to prove a residual estimate of arbitrary order in ϵ for some higher order corrector to the TWI formal approximate solution on a time scale including $\Theta(-\epsilon^{-1} \ln \epsilon)$ – similar to ψ_{FWI} and ψ_{FWIe} in Section 2.2.1, see equations (2.18) and (2.19), too – then we could use Theorems 2.1.28 and 2.2.5 to prove that no approximation property in the spirit of Approximation Property 1.3.1 can hold for the FWI formal approximate solution.

We emphasise that the procedure to derive the non-approximation result is quite general. We solely depend on the following points.

- We must be able to derive the resonant TWI and sFWI amplitude equations for an ansatz of the kind used in Section 2.1.3.
- There must be a residual estimate for the TWI formal approximate solution or for the TWI formal approximate solution with some higher order correc-

tors for any arbitrary order of powers of ϵ on an time interval including $\Theta(-\epsilon^{-1}\ln\epsilon)$.

- We need an error estimates of the kind given in Theorem 2.1.28 or Theorem 2.2.5.

The first step depends on some (non-)resonance conditions only. Therefore it should be possible for every suitable linear operator $\mathcal{L}$. We solved the last step in Sections 2.1.6 and 2.2.2 for a certain class of problems that included the Boussinesq-like problems considered in this chapter. Note that almost all steps relied on certain asymptotic estimates of the symbol in 0 or $\pm\infty$ and some bounds on the symbol. Thus, there is some hope that these techniques are applicable in cases in which the linear operator $\mathcal{L}$ has a symbol that is not of polynomial type, maybe even the water waves problem in Eulerian or Lagrangian coordinates, which is the ultimate aim as stated in the introduction to Section 2.1.
The second point seems to be rather difficult, however. At the moment we have no solution to prove such a result for the two considered toy problems. While it seems rather simple to derive correctors for the formal approximate solution constructed with the aid of the TWI system so that the residual estimate of Hypothesis 2.1.24 satisfied on the natural time scale, see the comment at the end of Section 2.2.3, the question whether these can be extended to a time scale involving $\Theta(\epsilon^{-1}\ln\epsilon)$ seems to be more involved. Even the question whether the solutions to the TWI system for arbitrary initial data satisfying Hypothesis 2.1.9 remain in H^s, $s\in\mathbb{N}$, on such a long time scale and uniform for $\epsilon\in]0,\epsilon_0]$ for some $\epsilon_0>0$, seems to be arcane. While the solutions to equation (2.9) given in Lemma 2.1.10 obviously remain in L^∞ for initial data in L^∞ – hence the solution remains in L^2 for compactly supported initial data, too – we had serious problems for the regularity estimates of these solutions. Similar problems might appear for correctors as well. Therefore it would be of some interest whether – and how far – it is possible to weaken the assumptions of Hypothesis 2.1.15. And one might speculate whether similar – or maybe completely different – assumptions would be necessary to define correctors that exist in Sobolev spaces for the long time scale. Or whether there are any such assumptions, eventually.
Finally, note two points. First, note that we needed some rather hand-tailored initial data to show a difference between the formal approximate solutions that we obtain using the two ansatzes. It is not clear whether a large enough difference

would appear between these on the natural time scale of the FWI ansatz for arbitrary initial data. This was not the case in two of the explicit examples given in Section 2.1.5. It seems that much more care is needed in this case than in the case of periodic boundary conditions, which was considered in [84, 85] where they chose some initial data of an unstable manifold without a large number of restrictions, see [84, Section 7]. One could even ponder on the question whether such initial data are common or whether these are a 'small' – in what sense ever – set only. Secondly, we point out that the ansatz for the TWI system is one of the cases that we called trivial in the introduction where the outer power in ϵ is equal to the inner power of ϵ so that an approximation property is obtained in a more or less trivial way in H^s on the natural time scale if we have H^s and $W^{s,\infty}$ estimate on the formal approximate solution, see Theorem 2.1.34. And clearly, the result of Theorem 2.1.34 could be improved if we strengthen its assumptions in the way we did in Section 2.2 for the other results of Section 2.1.

Appendix

2.A Properties of the One-Dimensional Inhomogeneous Wave Equation

We used some properties of the one dimensional inhomogeneous wave equation to correct the residual in Section 2.1.6. These properties are summarised in the following lemma.

Lemma 2.A.1. *Let* $0 < t_0 < \infty$*,* $I \coloneqq (0, t_0)$*,* $f \in C^1(I \times \mathbb{R}, \mathbb{C})$*. Consider the problem*

$$\begin{aligned} (\partial_t^2 - c^2 \partial_x^2) u &= f && \text{in } I \times \mathbb{R} \\ (u, \partial_t u)(0, \cdot) &= (\phi, \psi) && \text{in } \mathbb{R} \end{aligned}$$

for $c \in \mathbb{R}^\times$*,* $\phi \in C^2(\mathbb{R}, \mathbb{C}), \psi \in C^1(\mathbb{R}, \mathbb{C})$*. Then there is a unique classical solution* $u \in C^2(I \times \mathbb{R}, \mathbb{C})$ *given by d'Alembert's formula*

$$\begin{aligned} &u(t, x) \\ =&\frac{\phi(x + ct) + \phi(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) \, \mathrm{d}y + \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s, y) \, \mathrm{d}y \, \mathrm{d}s \\ =&\frac{\phi(x + ct) + \phi(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) \, \mathrm{d}y + \frac{1}{2c} \int_0^t \int_{x-cs}^{x+cs} f(t - s, y) \, \mathrm{d}y \, \mathrm{d}s \end{aligned}$$

Further, the following properties hold.

- *If* $\overline{\bigcup_{t\in I}\operatorname{supp}(f(t,\cdot))}, \operatorname{supp}(\phi), \operatorname{supp}(\psi)$ *compact then*

$$\begin{aligned}\bigcup_{t\in I}\operatorname{supp}(u(t,\cdot)) &\subset \overline{\bigcup_{t\in I}\operatorname{supp}(f(t,\cdot))} + ct_0\\ &= \left\{ x\in\mathbb{R} \,\middle|\, \inf_{y\in\overline{\bigcup_{t\in I}\operatorname{supp}(f(t,\cdot))}} \operatorname{dist}(y,x) \le ct_0 \right\}.\end{aligned}$$

- *If* $\mathbb{N}\ni m\ge 1$, $f\in C^m(I\times\mathbb{R},\mathbb{C})$, $\phi\in C^{m+1}(\mathbb{R},\mathbb{C})$, $\psi\in C^m(\mathbb{R},\mathbb{C})$ *and for all* $\mathbb{N}\ni k\le k_0\le m_0\le m$ $\partial_t^k f\in L^\infty(I,H^{m_0-k})$, $\phi\in H^{m_0+1}$, $\psi\in H^{m_0}$, *then* $u\in C^{m+1}(I\times\mathbb{R},\mathbb{C})$ *and for all* $k_0+2\ge k$ *holds* $\partial_t^k u\in L^\infty(I,H^{m_0-k+1})$ *and for* $1\le l+k\le m_0+1$ *we find the estimates*

$$\begin{aligned}\|u\|_{L^\infty(I,L^2)} &\le \|\phi\|_{L^2} + \|\psi\|_{L^2}t_0 + \|f\|_{L^\infty(I,L^2)}\frac{t_0^2}{2}\\ \|\partial_t^k\partial_x^l u\|_{L^\infty(I,L^2)} &\le |c|^k\|\partial_x^{k+l}\phi\|_{L^2} + |c|^{k-1}\|\partial_x^{k+l-1}\psi\|_{L^2}\\ &\quad+\sum_{n=0}^{k-2}|c|^{k-n-1}\|\partial_x^{l-2-n+k}\partial_t^n f\|_{L^\infty(I,L^2)}\\ &\quad+|c|^{k-1}\|\partial_x^{l-1+k}f\|_{L^\infty(I,L^2)}t_0\end{aligned}$$

- *If* $\mathbb{N}\ni m\ge 1$, $f\in C^m(I\times\mathbb{R},\mathbb{C})$, $\phi\in C^{m+1}(\mathbb{R},\mathbb{C})$, $\psi\in C^m(\mathbb{R},\mathbb{C})$ *and for all* $\mathbb{N}\ni k\le k_0\le m_0\le m$ $\partial_t^k f\in L^\infty(I,W^{m_0-k,\infty})$, $\phi\in W^{m_0+1,\infty}$, $\psi\in W^{m_0,\infty}$, *then* $u\in C^{m+1}(I\times\mathbb{R},\mathbb{C})$ *and for all* $k_0+2\ge k$ *holds* $\partial_t^k u\in L^\infty(I,W^{m_0-k+1,\infty})$ *and for* $1\le l+k\le m_0+1$ *we find the estimates*

$$\begin{aligned}\|u\|_{L^\infty(I,L^\infty)} &\le \|\phi\|_{L^\infty} + \|\psi\|_{L^\infty}t_0 + \frac{t_0^2}{2}\|f\|_{L^\infty(I,L^\infty)}\\ \|\partial_t^k\partial_x^l u\|_{L^\infty(I,L^\infty)} &\le |c|^k\|\partial_x^{k+l}\phi\|_{L^\infty} + |c|^{k-1}\|\partial_x^{k+l-1}\psi\|_{L^\infty}\\ &\quad+\sum_{n=0}^{k-2}|c|^{k-n-1}\|\partial_x^{l-2-n+k}\partial_t^n f\|_{L^\infty(I,L^\infty)}\\ &\quad+|c|^{k-1}\|\partial_x^{l-1+k}f\|_{L^\infty(I,L^\infty)}t_0.\end{aligned}$$

Proof. **Existence and uniqueness:** Is established in literature, e.g. [67, Theorem 4.12].

Solution formulae: We define

$$
\begin{aligned}
v(t,x) &\coloneqq \frac{\phi(x+ct)+\phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,\mathrm{d}y,\\
w(t,x) &\coloneqq \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\,\mathrm{d}y\,\mathrm{d}s
\end{aligned}
$$

and clearly

$$
\begin{aligned}
\partial_x v(t,x) &= \partial_x \frac{\phi(x+ct)+\phi(x-ct)}{2} + \frac{\psi(x+ct)-\psi(x-ct)}{2c}\\
\partial_x^2 v(t,x) &= \partial_x^2 \frac{\phi(x+ct)+\phi(x-ct)}{2} + \partial_x \frac{\psi(x+ct)-\psi(x-ct)}{2c}\\
\partial_t v(t,x) &= c\partial_x \frac{\phi(x+ct)-\phi(x-ct)}{2} + \frac{\psi(x-ct)+\psi(x+ct)}{2}\\
\partial_t^2 v(t,x) &= c^2\partial_x^2 \frac{\phi(x+ct)+\phi(x-ct)}{2} + c\frac{\partial_x\psi(x+ct)-\partial_x\psi(x-ct)}{2} = c^2\partial_x^2 v(t,x)
\end{aligned}
$$

as well as (since $f \in C^1(I\times\mathbb{R},\mathbb{C})$ by dominated convergence and linearity)

$$
\begin{aligned}
\partial_x w(t,x) &= \frac{1}{2c}\int_0^t f(s,x+c(t-s)) - f(s,x-c(t-s))\,\mathrm{d}s\\
&= \frac{1}{2c}\int_0^t f(t-s,x+cs) - f(t-s,x-cs)\,\mathrm{d}s\\
\partial_x^2 w(t,x) &= \frac{1}{2c}\int_0^t \partial_x f(s,x+c(t-s)) - \partial_x f(s,x-c(t-s))\,\mathrm{d}s\\
\partial_t w(t,x) &= \frac{1}{2}\int_0^t f(s,x+c(t-s)) + f(s,x-c(t-s))\,\mathrm{d}s\\
&= \frac{1}{2}\int_0^t f(t-s,x+cs) + f(t-s,x-cs)\,\mathrm{d}s\\
\partial_t^2 w(t,x) &= f(t,x) + \frac{c}{2}\int_0^t \partial_x f(s,x+c(t-s)) - \partial_x f(s,x-c(t-s))\,\mathrm{d}s.
\end{aligned}
$$

Trivially it follows $u = v + w \in C^2(I\times\mathbb{R},\mathbb{C})$ and

$$
\begin{aligned}
(\partial_t^2 - c^2\partial_x^2)u &= f && \text{in } I\times\mathbb{R}\\
(u(0,\cdot),\partial_t u(0,\cdot)) &= (\phi,\psi) && \text{in } \mathbb{R}.
\end{aligned}
$$

Support of u: Define $\overline{\bigcup_{t\in I} \operatorname{supp}(f(\cdot,t))} = S$. Then obviously

$$\int_0^t \int_{x-cs}^{x+cs} f(t-s,y)\,\mathrm{d}y\,\mathrm{d}s = 0$$

for all $x \in \mathbb{R}\backslash(S+|ct|)$ (where $S+|ct| := \{x \in \mathbb{R}|\inf_{y\in S}\operatorname{dist}(y,x) \leq |ct|\}$) and clearly $\operatorname{supp}(\phi(\cdot+ct)+\phi(\cdot-ct)) \subset \operatorname{supp}(\phi)+|ct|$ as well as above

$$\int_{x-cs}^{x+cs} \psi(y)\,\mathrm{d}y = 0$$

for all $x \in \mathbb{R}\backslash(\operatorname{supp}(\psi)+|ct|)$. So

$$\operatorname{supp}(u(t)) \subset |\operatorname{supp}(\phi)+ct| + |\operatorname{supp}(\psi)+ct| + |S+ct|.$$

Regularity: We split the regularity considerations for the parts $u = v + w$. In the following we need to calculate L^2 norms of $v(t,\cdot), w(t,\cdot)$ etc. We will use the following transforms of the area of integration

$$\mathbb{R}^3 \ni x \mapsto x' = Ax = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} x \in \mathbb{R}^3.$$

Hence, we obtain, if any of the integrals exists,

$$\begin{aligned} &\int_{\mathbb{R}} \int_{x_1-cs_1}^{x_1+cs_1} \int_{x_1-cs_2}^{x_1+cs_2} |g(s_1,x_2)g(s_2,x_3)|\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1 \\ = &\int_{\mathbb{R}} \int_{-cs_1}^{cs_1} \int_{-cs_2}^{cs_2} |g(s_1,x_1'-x_2')g(s_2,x_1'-x_3')|\,\mathrm{d}x_3'\,\mathrm{d}x_2'\,\mathrm{d}x_1', \end{aligned}$$

since the mapping above is bijective from $\mathbb{R}^3 \to \mathbb{R}^3$ and $\det(A) = 1$ and maps the area of integration as follows: $x_1' = x_1$ and

$$x_1 - cs_1 \leq x_2 \leq x_1 + cs_1 \;\Leftrightarrow\; x_1' - cs_1 \leq x_1' - x_2' \leq x_1' + cs_1 \;\Leftrightarrow\; -cs_1 \leq x_2' \leq cs_1$$

and similarly for x_3, x_3'. Thus, we have by application of the theorems of Fubini and Hölder that

$$
\begin{aligned}
&\int_{\mathbb{R}} \int_{x-cs_1}^{x+cs_1} \int_{x-cs_2}^{x+cs_2} |g(s_1,y)g(s_2,y)| \,\mathrm{d}z \,\mathrm{d}y \,\mathrm{d}x \\
&= \int_{-cs_1}^{cs_1} \int_{-cs_2}^{cs_2} \left| \int_{\mathbb{R}} |g(s_1,x-y)g(s_2,x-z)| \,\mathrm{d}x \right| \,\mathrm{d}z \,\mathrm{d}y \\
&\leq \int_{-cs_1}^{cs_1} \int_{-cs_2}^{cs_2} \left(\int_{\mathbb{R}} |g(s_1,x-y)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |g(s_2,x-z)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \,\mathrm{d}z \,\mathrm{d}y \\
&\leq \int_{-cs_1}^{cs_1} \int_{-cs_2}^{cs_2} \|g\|_{L^\infty(I,L^2)}^2 \,\mathrm{d}z \,\mathrm{d}y = 4c^2 \|g\|_{L^\infty(I,L^2)}^2 s_1 s_2.
\end{aligned}
$$

It is obvious that $v \in C^{m+1}(I \times \mathbb{R}, \mathbb{C})$ from the explicit solution formulae above and the assumptions. Therefore we find for $0 \leq k \leq k_0 + 2$ and $1 \leq l + k \leq m_0 + 1 \leq m + 1$

$$
\begin{aligned}
\|v\|_{L^\infty(I,L^2)} &\leq \|\phi\|_{L^2} + \|\psi\|_{L^2} t_0, \\
\|\partial_x^k \partial_t^l v\|_{L^\infty(I,L^2)} &\leq |c|^k \|\partial_x^{k+l} \phi\|_{L^2} + |c|^{k-1} \|\partial_x^{k+l-1} \psi\|_{L^2},
\end{aligned}
$$

and

$$
\begin{aligned}
\|v\|_{L^\infty(I,L^\infty)} &\leq \|\phi\|_{L^\infty} + \|\psi\|_{L^\infty} t_0, \\
\|\partial_x^l \partial_t^k v\|_{L^\infty(I,L^\infty)} &\leq |c|^k \|\partial_x^{k+l} \phi\|_{L^\infty} + |c|^{k-1} \|\partial_x^{k+l-1} \psi\|_{L^\infty}.
\end{aligned}
$$

We used for the derivatives in the latter formulae that for $l + k \geq 1$ we have

$$
\begin{aligned}
\partial_t^k \partial_x^l v(t,x) = \frac{c^{k-1}}{2} \big(&\partial_x^{k+l} c\phi(x+ct) + (-1)^k c\phi(x-ct) \\
&+ \partial_x^{k+l-1} \psi(x+ct) - (-1)^k \psi(x-ct) \big)
\end{aligned}
$$

$$
\left\| \frac{1}{c} \int_{x-ct}^{x+ct} \psi(y) \,\mathrm{d}y \right\|_{L^2}^2 \leq \frac{1}{c^2} \int_{\mathbb{R}} \int_{x-ct}^{x+ct} \int_{x-ct}^{x+ct} |\psi(y)\psi(z)| \,\mathrm{d}z \,\mathrm{d}y \,\mathrm{d}x \leq 4\|\psi\|_{L^2}^2 t^2.
$$

Further, if $f \in C^m(I \times \mathbb{R}, \mathbb{C})$, then $(t,x) \mapsto f(s, x \pm c(t-s)) \in C^m(I \times \mathbb{R}, \mathbb{C})$ for all $s \in I$ and it follows

$$
\begin{aligned}
\partial_t w(t,x) &= \frac{1}{2} \int_0^t f(s, x + c(t-s)) + f(s, x - c(t-s)) \,\mathrm{d}s \in C^m(I \times \mathbb{R}, \mathbb{C}), \\
\partial_x w(t,x) &= \frac{1}{2c} \int_0^t f(s, x + c(t-s)) + f(s, x - c(t-s)) \,\mathrm{d}s \in C^m(I \times \mathbb{R}, \mathbb{C}),
\end{aligned}
$$

which means $w \in C^{m+1}(I \times \mathbb{R}, \mathbb{C})$. Further it holds for $m+1 \geq l \in \mathbb{N}$

$$\begin{aligned}\partial_t^l w(t,x) = &\sum_{n=0}^{l-2} \frac{c^{l-n-1}}{2} \partial_t^n \partial_x^{l-2-n} \big(f(t,x) + (-1)^{l-n} f(t,x)\big) \\ &+ \frac{c^{l-1}}{2} \int_0^t \partial_x^{l-1} f(s, x+c(t-s)) + (-1)^{l-1} \partial_x^{l-1} f(s, x-c(t-s)) \,\mathrm{d}s.\end{aligned}$$

We prove this by (mathematical) induction:

Base case The result is clear for $l = 1$ as a simple calculation shows

$$\partial_t w(t,x) = \frac{1}{2} \int_0^t f(s, x+c(t-s)) + f(s, x-c(t-s)) \,\mathrm{d}s.$$

Inductive step Let us assume that the statement is true up to the $(l-1)$th derivative. Then it follows

$$\begin{aligned}&\partial_t^l w(t,x) = \partial_t \partial_t^{l-1} w(t,x) \\ &= \sum_{n=0}^{l-3} \frac{c^{l-n-2}}{2} \partial_t^{n+1} \partial_x^{l-3-n} \big(f(t,x) + (-1)^{l-n-1} f(t,x)\big) \\ &\quad + \frac{c^{l-2}}{2} \partial_t \int_0^t \partial_x^{l-2} f(s, x+c(t-s)) + (-1)^{l-2} \partial_x^{l-2} f(s, x-c(t-s)) \,\mathrm{d}s \\ &= \sum_{n=0}^{l-3} \frac{c^{l-(n+1)-1}}{2} \partial_t^{n+1} \partial_x^{l-2-(n+1)} \big(f(t,x) + (-1)^{l-(n+1)} f(t,x)\big) \\ &\quad + \frac{c^{l-2}}{2} \partial_x^{l-2} f(t,x) + (-1)^{l-2} \partial_x^{l-2} f(t,x) \\ &\quad + \frac{c^{l-1}}{2} \int_0^t \partial_x^{l-1} f(s, x+c(t-s)) + (-1)^{l-1} \partial_x^{l-1} f(s, x-c(t-s)) \,\mathrm{d}s \\ &= \sum_{n=0}^{l-2} \frac{c^{l-n-1}}{2} \partial_t^n \partial_x^{l-2-n} \big(f(t,x) + (-1)^{l-n} f(t,x)\big) \\ &\quad + \frac{c^{l-1}}{2} \int_0^t \partial_x^{l-1} f(s, x+c(t-s)) + (-1)^{l-1} \partial_x^{l-1} f(s, x-c(t-s)) \,\mathrm{d}s,\end{aligned}$$

which completes the inductive step. $\Diamond$

We can now conclude

$$
\begin{aligned}
&\|w(t,\cdot)\|_{L^2}^2 = \int_{\mathbb{R}} |w(x,t)|^2 \,\mathrm{d}x = \int_{\mathbb{R}} \left| \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\,\mathrm{d}y\,\mathrm{d}s \right|^2 \mathrm{d}x \\
\leq & \frac{1}{4c^2} \int_{\mathbb{R}} \left(\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} |f(s,y)|\;\mathrm{d}y\,\mathrm{d}s \right)^2 \mathrm{d}x \\
= & \frac{1}{4c^2} \int_{\mathbb{R}} \left(\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} |f(s_1,y)|\;\mathrm{d}y\,\mathrm{d}s_1 \right) \left(\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} |f(s_2,z)|\;\mathrm{d}z\,\mathrm{d}s_2 \right) \mathrm{d}x \\
= & \frac{1}{4c^2} \int_{\mathbb{R}} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} |f(s_2,z)|\,|f(s_1,y)|\;\mathrm{d}y\,\mathrm{d}s_1\,\mathrm{d}z\,\mathrm{d}s_2\,\mathrm{d}x \\
= & \frac{1}{4c^2} \int_0^t \int_0^t \int_{\mathbb{R}} \int_{x-c(t-s)}^{x+c(t-s)} \int_{x-c(t-s)}^{x+c(t-s)} |f(z,s_2)|\,|f(y,s_1)|\;\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}s_1\,\mathrm{d}s_2 \\
\leq & \|f\|_{L^\infty(I,L^2)}^2 \int_0^t \int_0^t s_1 s_2\,\mathrm{d}s_1\,\mathrm{d}s_2 = \|f\|_{L^\infty(I,L^2)}^2 \frac{t^4}{4}
\end{aligned}
$$

Therefore we have $w \in L^\infty(I, L^2)$ and we obtain for $t \in I$

$$
\|w(t,\cdot)\|_{L^2} \leq \|f\|_{L^\infty(I,L^2)} \frac{t_0^2}{2}.
$$

We will proceed for the derivatives similarly. Let $0 \leq k \leq k_0 + 2$ and $1 \leq l + k \leq m_0 + 1, l \neq 0$ and by differentiating the formula above we find

$$
\begin{aligned}
&\|\partial_x^l \partial_t^k w(t,\cdot)\|_{L^2} = \|\partial_t^k \partial_x^l w(t,\cdot)\|_{L^2} \\
\leq & \sum_{n=0}^{k-2} \frac{c^{k-n-1}}{2} (1 + (-1)^{k-n}) \|\partial_t^n \partial_x^{l-2-n+k} f(t,\cdot)\|_{L^2} \\
& \quad + \frac{c^{k-1}}{2} \left\| \int_0^t \partial_x^{l-1+k} f(s, \boldsymbol{x} + c(t-s)) + (-1)^{k-1} \partial_x^{l-1+k} f(s, \boldsymbol{x} - c(t-s))\,\mathrm{d}s \right\|_{L^2}.
\end{aligned}
$$

The last term can be estimated by

$$
\begin{aligned}
& \frac{1}{4} \int_{\mathbb{R}} \left| \int_0^t \partial_x^{l-1+k} f(s, x + c(t-s)) + (-1)^{k-1} \partial_x^{l-1+k} f(s, x - c(t-s))\,\mathrm{d}s \right|^2 \mathrm{d}x \\
\leq & \frac{1}{4} \int_{\mathbb{R}} \Big(\int_0^t |\partial_x^{l-1+k} f(s, x + c(t-s))| + |\partial_x^{l-1+k} f(s, x - c(t-s))|\,\mathrm{d}s \Big)^2 \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{2}\int_{\mathbb{R}} \Big(\int_0^t |\partial_x^{l-1+k} f(s, x + c(t-s))| \,\mathrm{d}s \Big)^2 \,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_{\mathbb{R}} \Big(\int_0^t |\partial_x^{l-1+k} f(s, x - c(t-s))| \,\mathrm{d}s \Big)^2 \,\mathrm{d}x \\
&\leq \|\partial_x^{l-1+k} f\|^2_{L^\infty(I,L^2)} t^2 \leq \|\partial_x^{l-1+k} f\|^2_{L^\infty(I,L^2)} t_0^2
\end{aligned}
$$

where we used

$$
\begin{aligned}
&\int_{\mathbb{R}} \Big(\int_0^t |\partial_x^{l-1+k} f(s, x \pm c(t-s))| \,\mathrm{d}s \Big)^2 \,\mathrm{d}x \\
=& \int_{\mathbb{R}} \Big(\int_0^t |\partial_x^{l-1+k} f(s_1, x \pm c(t-s_1))| \,\mathrm{d}s_1 \Big) \Big(\int_0^t |\partial_x^{l-1+k} f(s_2, x \pm c(t-s_2))| \,\mathrm{d}s_2 \Big) \,\mathrm{d}x \\
=& \int_0^t \int_0^t \int_{\mathbb{R}} |\partial_x^{l-1+k} f(s_1, x \pm c(t-s_1), s_1)| |\partial_x^{l-1+k} f(s_2, x \pm c(t-s_2))| \,\mathrm{d}x \,\mathrm{d}s_2 \,\mathrm{d}s_1 \\
\leq& \int_0^t \int_0^t \|\partial_x^{l-1+k} f(s_1, \cdot)\|_{L^2} \|\partial_x^{l-1+k} f(s_2, \cdot)\|_{L^2} \,\mathrm{d}s_2 \,\mathrm{d}s_1 \\
\leq& \|\partial_x^{l-1+k} f\|^2_{L^\infty(I,L^2)} t^2.
\end{aligned}
$$

Hence, we have for $k = 0$ the estimate

$$
\begin{aligned}
\|\partial_x^l w(t, \cdot)\|^2_{L^2} &= \frac{1}{4c^2} \int_{\mathbb{R}} \Big| \int_0^t \partial_x^{l-1} f(s, x + c(t-s)) + \partial_x^{l-1} f(s, x - c(t-s)) \,\mathrm{d}s \Big|^2 \,\mathrm{d}x \\
&\leq c^{-2} \|\partial_x^{l-1} f\|^2_{L^\infty(I,L^2)} t^2 \leq c^{-2} \|\partial_x^{l-1} f\|^2_{L^\infty(I,L^2)} t_0^2.
\end{aligned}
$$

Therefore, we conclude that for $0 \leq k \leq k_0 + 2$ and $1 \leq l + k \leq m_0 + 1$

$$
\begin{aligned}
\|w\|_{L^\infty(I,L^2)} &\leq \frac{1}{2} \|f\|_{L^\infty(I,L^2)} t_0^2 \\
\|\partial_x^k \partial_t^l w\|_{L^\infty(I,L^2)} &\leq \Big(\sum_{n=0}^{l-2} \frac{|c|^{l-n-1}}{2} (1 + (-1)^{l-n}) \|\partial_x^{l-2-n+k} \partial_t^n f\|_{L^\infty(I,L^2)} \Big) \\
&\quad + |c|^{l-1} \|\partial_x^{l-1+k} f\|_{L^\infty(I,L^2)} t_0 \\
&\leq \Big(\sum_{n=0}^{l-2} |c|^{l-n-1} \|\partial_x^{l-2-n+k} \partial_t^n f\|_{L^\infty(I,L^2)} \Big) \\
&\quad + |c|^{l-1} \|\partial_x^{l-1+k} f\|_{L^\infty(I,L^2)} t_0,
\end{aligned}
$$

which proves the claim of the $L^\infty(I, H^{m_0-k+1})$ regularity. We continue with the

$L^\infty(I, W^{m_0-k+1,\infty})$ estimates. Apparently, it holds

$$\begin{aligned}
|w(t,x)| &\leq \frac{1}{2|c|}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} |f(s,y)| \,\mathrm{d}y \,\mathrm{d}s \\
&\leq \frac{1}{2|c|}\|f\|_{L^\infty(I,L^\infty)} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} 1 \,\mathrm{d}y \,\mathrm{d}s \\
&= \frac{1}{2|c|}\|f\|_{L^\infty(I,L^\infty)} \int_0^t 2|c|(t-s) \,\mathrm{d}s = \frac{1}{2}\|f\|_{L^\infty(I,L^\infty)} t^2 \leq \frac{t_0^2}{2}\|f\|_{L^\infty(I,L^\infty)}.
\end{aligned}$$

We estimate for $0 \leq k \leq k_0+2$ and $1 \leq l+k \leq m_0+1$, $l \in \mathbb{N}$ similar to the estimates above

$$\begin{aligned}
&\|\partial_x^l \partial_t^k w(t,\cdot)\|_{L^\infty} = \|\partial_t^k \partial_x^l w(t,\cdot)\|_{L^\infty} \\
&\leq \sum_{n=0}^{k-2} \frac{|c|^{l-n-1}}{2}(1+(-1)^{k-n})\|\partial_t^n \partial_x^{l-2-n+k} f(t,\cdot)\|_{L^\infty} \\
&\quad + \frac{|c|^{k-1}}{2} \operatorname*{ess\,sup}_{x\in\mathbb{R}} \left| \int_0^t \partial_x^{l-1+k} f(s, x+c(t-s)) + (-1)^{k-1}\partial_x^{l-1+k} f(s,x-c(t-s)) \,\mathrm{d}s \right| \\
&\leq \sum_{n=0}^{k-2} \frac{|c|^{k-n-1}}{2}(1+(-1)^{k-n})\|\partial_t^n \partial_x^{l-2-n+k} f(t,\cdot)\|_{L^\infty} + |c|^{k-1}\int_0^t \|\partial_x^{l-1+k} f\|_{L^\infty(I,L^\infty)} \,\mathrm{d}s \\
&\leq \sum_{n=0}^{k-2} \frac{|c|^{k-n-1}}{2}(1+(-1)^{k-n})\|\partial_t^n \partial_x^{l-2-n+k} f\|_{L^\infty(I,L^\infty)} + |c|^{k-1}\|\partial_x^{l-1+k} f\|_{L^\infty(I,L^\infty)} t_0.
\end{aligned}$$

Finally, we obtain for $k=0$

$$\begin{aligned}
\|\partial_x^k w(t,\cdot)\|_{L^2} &= \frac{1}{2|c|} \operatorname*{ess\,sup}_{x\in\mathbb{R}} \left| \int_0^t \partial_x^{k-1} f(s,x+c(t-s)) + \partial_x^{k-1} f(s,x-c(t-s)) \,\mathrm{d}s \right| \\
&\leq |c|^{-1} t \|\partial_x^{k-1} f\|_{L^\infty(I,L^\infty)} \leq |c|^{-1}\|\partial_x^{k-1} f\|_{L^\infty(I,L^\infty)} t_0.
\end{aligned}$$

So, we conclude for $0 \leq k \leq k_0+2$ and $1 \leq l+k \leq m_0+1$

$$\begin{aligned}
\|w\|_{L^\infty(I,L^\infty)} &\leq \|f\|_{L^\infty(I,L^\infty)} \frac{t_0^2}{2} \\
\|\partial_x^l \partial_t^k w\|_{L^\infty(I,L^\infty)} &\leq \sum_{n=0}^{k-2} \frac{|c|^{k-n-1}}{2}(1+(-1)^{k-n})\|\partial_x^{l-2-n+k}\partial_t^n f\|_{L^\infty(I,L^\infty)} \\
&\quad + |c|^{k-1}\|\partial_x^{l-1+k} f\|_{L^\infty(I,L^\infty)} t_0
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{n=0}^{k-2} |c|^{k-n-1} \|\partial_x^{l-2-n+k} \partial_t^n f\|_{L^\infty(I,L^\infty)} \\
&\quad + |c|^{k-1} \|\partial_x^{l-1+k} f\|_{L^\infty(I,L^\infty)} t_0,
\end{aligned}
$$

which proves the last claim. □

2.B Existence and Uniqueness of Solutions to the Boussinesq-like Models

We want to prove local existence and uniqueness for the Boussinesq-like models considered in Chapter 2. This can easily be accomplished in the setting of the abstract assumptions introduced in Section 2.1.6. Therefore we suppose that we consider an equation of the structure mentioned in Hypothesis 2.1.22. Recall the definition of $\tilde{q}_t, \tilde{q}_x$ in Remark 2.1.23 of Section 2.1.6. Using these, we can write the equation for the Fourier transform $\hat{u}$ of equation (2.15) as

$$
\tilde{q}_t \partial_t^2 \hat{u} = -\tilde{q}_x \hat{u} + \mathscr{F} \circ f \circ \mathscr{F}^{-1}(\hat{u}) \tag{2.23}
$$

in L^2_s for some $s \in \mathbb{R}$. Since the multiplication operator induced by $\tilde{q}_t$ has a bounded inverse in L^2_s, this is equivalent to

$$
\partial_t^2 \hat{u} = -\tilde{q}\hat{u} + \tilde{q}_t^{-1} \mathscr{F} \circ f \circ \mathscr{F}^{-1}(\hat{u}), \tag{2.24}
$$

where $\tilde{q} := \frac{\tilde{q}_x}{\tilde{q}_t}$. We introduce $\hat{v} = \partial_t \hat{u}$ and consequently we can write the equation as system

$$
\partial_t U = \begin{pmatrix} -1 & 1 \\ -\tilde{q} & -1 \end{pmatrix} U + \begin{pmatrix} U_1 \\ U_2 + \tilde{q}_t^{-1} \mathscr{F} \circ f \circ \mathscr{F}^{-1}(U_1) \end{pmatrix} = AU + \tilde{F}(U) \tag{2.25}
$$

where $U = (\hat{u}, \hat{v})^t$. Let $\mathcal{X} = L^2_{D_x - D_t + l} \times L^2_l$ with the scalar product

$$
\langle U, V \rangle_{\mathcal{X}} := \langle U_1, (\tilde{q}+1)V_1 \rangle_{L^2_l} + \langle U_2, V_2 \rangle_{L^2_l},
$$

which induces an equivalent norm on $L^2_{D_x-D_t+l} \times L^2_l$, $l \in \mathbb{N}_0$. Hence, A is dissipative in $\mathcal{X}$ since for all $U \in D(A)$

$$\begin{aligned}\mathrm{Re}\langle U, AU\rangle_{\mathcal{X}} &= -\langle U_1, (\tilde{q}+1)U_1\rangle_{L^2_l} - \langle U_2, U_2\rangle_{L^2_l} \\ &\quad + \mathrm{Re}\Big(\langle U_1, (\tilde{q}+1)U_2\rangle_{L^2_l} - \langle U_2, \tilde{q}U_1\rangle_{L^2_l}\Big) \\ &= -\langle U_1, (\tilde{q}+1)U_1\rangle_{L^2_l} - \langle U_2, U_2\rangle_{L^2_l} + \mathrm{Re}\Big(\langle U_1, U_2\rangle_{L^2_l}\Big) \le 0.\end{aligned}$$

Further $\det(1-\lambda A) = (1+\lambda)^2 + \lambda^2\tilde{q} > 0$ for all $\lambda > 0$ guarantees that A is maximal dissipative (short m-dissipative) in $\mathcal{X}$. We assume that $\mathscr{F} \circ f \circ \mathscr{F}^{-1} : L^2_{D_x-D_t+l} \to L^2_{l-2D_t}$ is Lipschitz on bounded subsets. Thus, $\tilde{F}$ is Lipschitz on bounded subsets of $\mathcal{X}$ meaning there is an $L(M) \ge 0$ such that

$$\|\tilde{F}(U) - \tilde{F}(V)\|_{\mathcal{X}} \le L(M)\|U-V\|_{\mathcal{X}} \qquad \text{for all } U, V \in B_M(0) \subset \mathcal{X}.$$

With this preparatory work we state the existence and uniqueness theorem.

Theorem 2.B.1. *Let $x \in \mathcal{X} = L^2_{D_x-D_t+l} \times L^2_l$, $l \in \mathbb{N}_0$, and assume that Hypothesis 2.1.22 is satisfied.*
Then there is a maximal $T = T(x)$ and a unique mild solution $U \in C([0,T[, \mathcal{X})$ of the Cauchy problem with equation (2.25)*. Further, if*

$$x \in D(A) = L^2_{2(D_x-D_t)+l} \times L^2_{D_x-D_t+l} \subset \mathcal{X},$$

then the solution is a strict solution meaning $U \in C\big([0,T[, D(A)\big) \cap C^1\big([0,T[, \mathcal{X}\big)$ and

$$\partial_t U(t) = AU(t) + \tilde{F}(U(t)), \qquad \textit{for all } t \in [0,T[.$$

If $T < \infty$, then $\lim_{t\uparrow T}\|U(t)\|_{\mathcal{X}} \to \infty$.

Proof. This is just a common result in literature since A is m-dissipative in $\mathcal{X}$ and $\tilde{F} : \mathcal{X} \to \mathcal{X}$ Lipschitz on bounded subsets of $\mathcal{X}$. Existence and uniqueness of the mild solution is clear, cf. [18, Proposition 4.3.3]. We note that $\mathcal{X}$ is a Hilbert space, thus reflexive. Consequently, the mild solution is a strict solution, cf. [18, Proposition 4.3.9], since $U(0) = x \in D(A)$. The statement on the blow-up in finite time is clear, cf. [18, Theorme 4.3.4]. $\square$

***Remark* 2.B.2.** Actually, it is even possible to write down an explicit expression for the semigroup. We have

$$\begin{aligned} e^{At} &= \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = e^{-t}\cos(\sqrt{\tilde{q}_t}t)\,1 + Ae^{-t}\frac{\sin(\sqrt{\tilde{q}}t)}{\sqrt{\tilde{q}_t}} \\ &= e^{-t}\begin{pmatrix} \cos(\sqrt{\tilde{q}}t) & \frac{\sin(\sqrt{\tilde{q}}t)}{\sqrt{\tilde{q}}} \\ \sqrt{\tilde{q}}\sin(\sqrt{\tilde{q}}t) & \cos(\sqrt{\tilde{q}}t) \end{pmatrix}. \end{aligned}$$

We denote the multiplication operator by the same symbol e^{At} and in this sense we have $e^{At} \in C_0(\mathbb{R}, \mathrm{L}(\mathcal{X}))$ with generator $A : D(A) \subset \mathcal{X} \to \mathcal{X}$. We see that it was actually not necessary to add -1 to the linear operator in equation (2.25) and compensate for it later by modifying the 'nonlinear' part.

***Remark* 2.B.3.** Note that the strict solution constructed in Theorem 2.B.1 solves equation (2.24) in the sense of L^2_l. Hence, equation (2.23) is satisfied in the sense of $L^2_{l-2D_t}$. An obvious consequence is that equation (2.23) might not in general be satisfied pointwise for $D = D_x - D_t \geq 1$ and initial data $x \in L^2_{2D}$, i.e. initial data that admit continuous differentiable representations. Sufficient for pointwise evaluation of equation (2.23) is to use initial data in $D(A)$ for $l > 2D_t + \frac{1}{2}$, i.e. $x \in L^2_{2D_x+m} \times L^2_{D_x+D_t+m}$ for $m > \frac{1}{2}$.

2.C Proofs of Regularity Estimates

We skipped some proofs for all their technicalities and their lack of insights in Chapter 2. For completeness we give them here. We will use the following notation for multisets.

Definition 2.C.1. *For any multiindex $\alpha \in \mathbb{N}^n$ we define the multiset*

$$\pi(\alpha) := \{\underbrace{1, \dots, 1}_{\alpha_1}, \dots, \underbrace{n, \dots, n}_{\alpha_n}\}$$

and we will write for a multiset Q, that $P \sqsubset Q$ if P is a partition of Q. Elements in a partition $B \in P$ are called blocks of P. By $|P| := \#\{B \,|\, B \in P\}$ we denote the number of blocks of the partition and for a block B clearly $|B| := \#\{x \,|\, x \in B\}$ is the number of elements of the multiset B (each block of a partition of a multiset Q is a (mulit-) set), where we count with multiplicity.

We will use this notation for a generalised Faà di Bruno rule, which will be used to calculate the partial derivatives of the solution formulae given in Section 2.1.4.

2.C.1 Existence, Regularity and Uniqueness for the Non-Resonant FWI System

The proof of Theorem 2.1.6 was skipped in Section 2.1.4 since it is quite straightforward. The necessary arguments are as follows.

Proof. (Theorem 2.1.6). Let $t_0 \in \mathbb{R}^+$. By assumption the a_i are continuously differentiable. Thus $a_i(\boldsymbol{x} + c\boldsymbol{t}) \in C^1$, too. Therefore $A_i \in C^1$ by chain and product rule, since $|a_i(x + ct)|^2 = \overline{a_i(x + ct)} a_i(x + ct)$. Further,

$$\begin{aligned} c_i \partial_x g_i(t,x) &= \sum_{j=1}^{4} \int_0^t d_{i,j} c_i \partial_x |a_j(x + c_i t + (c_j - c_i)s)|^2 \, \mathrm{d}s \\ \partial_t g_i(t,x) &= \sum_{j=1}^{4} \int_0^t d_{i,j} c_i \partial_x |a_j(x + c_i t + (c_j - c_i)s)|^2 \, \mathrm{d}s + \sum_{j=1}^{4} d_{i,j} |a_j(x + c_j t)|^2 \\ &= c_i \partial_x g_i(t,x) + \sum_{j=1}^{4} d_{i,j} |A_j(t,x)|^2 \end{aligned}$$

and thus A_i satisfies the PDE since

$$\begin{aligned} \partial_t A_i(t,x) &= c \partial_x a_i(x + c_i t) e^{\mathrm{i} g_i(t,x)} + a_i(x + c_i t) e^{\mathrm{i} g_i(t,x)} \partial_t g_i(t,x), \\ c_i \partial_x A_i(t,x) &= c \partial_x a_i(x + c_i t) e^{\mathrm{i} g_i(t,x)} + a_i(x + c_i t) e^{\mathrm{i} g_i(t,x)} c_i \partial_x g_i(t,x). \end{aligned}$$

Now it is clear that $A_i \in C^0([0,t_0), H^1) \cap C^1([0,t_0), L^2)$ by the explicit expression above, the continuous embedding $H^1 \hookrightarrow L^\infty$, the fact that H^1 is a Banach algebra and the translation invariance of the Lebesgue measure.
Uniqueness is trivially clear by general results for mild solutions to semilinear problems with m-dissipative linear parts, cf. [18, Lemma 4.3.2]. For this we consider the function A above as a mild solution in the Banach space H^1 and write for $A(t) = (A_1, \ldots, A_4)(t) \in H^1$ and $t > 0$

$$A(t) = S(t)a + \int_0^t S(t-s)(F \circ A)(s) \, \mathrm{d}s,$$

where $S \in C_0(\mathbb{R}, \mathrm{L}(H^1))$ is the translation group, given by

$$t \mapsto S(t)u = \begin{pmatrix} u_1(\cdot + c_1 t) \\ u_2(\cdot + c_2 t) \\ u_3(\cdot + c_3 t) \\ u_4(\cdot + c_4 t), \end{pmatrix}$$

and $F : H^1 \to H^1$, $A \mapsto F_i(A) = \mathrm{i} \sum_{j=1}^{4} d_{i,j} |A_j|^2 A_i$. A short calculation shows that F is Lipschitz on bounded subsets of H^1.

$$\|F_i(u) - F_i(v)\|_{H^1} \lesssim |d| \|u\|_{H^1}^2 \|u_i - v_i\|_{H^1} + 2|d| \|v_i\|_{H^1} (\|u\|_{H^1} + \|v\|_{H^1}) \|u - v\|_{H^1}$$

$$\begin{aligned} \|F(u) - F(v)\|_{H^1} &\lesssim \sum_{i=1}^{4} \|F_i(u) - F_i(v)\|_{H^1} \\ &\lesssim \sum_{i=1}^{4} |d| \|u\|_{H^1}^2 \|u_i - v_i\|_{H^1} \\ &\qquad + 2|d| \|v_i\|_{H^1} (\|u\|_{H^1} + \|v\|_{H^1}) \|u - v\|_{H^1} \\ &\lesssim \left(|d| \|u\|_{H^1}^2 + 2|d| \|v\|_{H^1} (\|u\|_{H^1} + \|v\|_{H^1}) \right) \|u - v\|_{H^1} \end{aligned}$$

The hidden constant depends on some embeddings but on nothing else, in particular it does not depend on u, v. Hence, we have a unique (local) solution in $C([0, \tilde{t}_0], H^1)$, for a sufficiently small $\tilde{t}_0$, and by the above calculations also in $C([0, \tilde{t}_0], H^1) \cap C^1([0, \tilde{t}_0], L^2)$. Since our solution exists on $[0, t_0[$, we can apply this result for every $t \in [0, t_0[$ and have a unique solution, eventually.

We still have to show the regularity estimates. We note that for $a \in H^n \cap C^n(\mathbb{R}, \mathbb{C})$ the same argument as before shows $A \in C^n(\mathbb{R}^2, \mathbb{C})$. Further we note that for $k \in \mathbb{N}_0$

$$\partial_t^k g_i(t, x) = (c_i \partial_x)^k g_i(t, x) + \sum_{p=0}^{k-1} \sum_{j=1}^{4} (c_i \partial_x)^p (c_j \partial_x)^{k-p-1} d_{i,j} |a_j(x + c_j t)|^2.$$

This can be proved by mathematical induction as follows.

Base case The case $k = 0$ is trivial.

Inductive step Assume the statement holds for $k-1$. Then

$$\begin{aligned}
\partial_t \partial_t^{k-1} g_i(t,x) &= (c_i\partial_x)^{k-1}\partial_t g_i(t,x) \\
&\quad + \sum_{p=0}^{k-2}\sum_{j=1}^{4}(c_i\partial_x)^p(c_j\partial_x)^{k-p-2}\partial_t d_{i,j}|a_j(x+c_jt)|^2 \\
&= (c_i\partial_x)^k g_i(t,x) + (c_i\partial_x)^{k-1}\sum_{j=1}^{4} d_{i,j}|a_j(x+c_jt)|^2 \\
&\quad + \sum_{p=0}^{k-2}\sum_{j=1}^{4}(c_i\partial_x)^p(c_j\partial_x)^{k-p-1} d_{i,j}|a_j(x+c_jt)|^2 \\
&= (c_i\partial_x)^k g_i(t,x) + \sum_{p=0}^{k-1}\sum_{j=1}^{4}(c_i\partial_x)^p(c_j\partial_x)^{k-p-1} d_{i,j}|a_j(x+c_jt)|^2.
\end{aligned}$$

$\Diamond$

Now, we can calculate the derivatives for $1 \leq k+l \leq n$ using a more general version of the Faà di Bruno-formula, cf. [43].

$$\begin{aligned}
\partial_t^k\partial_x^l e^{\mathrm{i}g_i(t,x)} &= e^{\mathrm{i}g_i(t,x)} \sum_{P\sqsubset\pi(k,l)}\prod_{\alpha\in P}\partial^\alpha\big(\mathrm{i}g_i(t,x)\big) \\
\partial_t^k\partial_x^l g_i(t,x) &= c_i^k\partial_x^{k+l} g_i(t,x) + \sum_{p=0}^{k-1}\sum_{j=1}^{4} c_i^p c_j^{k-p-1}\partial_x^{k+l-1} d_{i,j}|a_j(x+c_jt)|^2,
\end{aligned}$$

where α has to be read as multiindex of a block of this partition and where the first variable is the time variable t, which is denoted by 1 in the multiset, and the second variable is x. By Leibniz's rule we obtain

$$\begin{aligned}
|\partial_t^k\partial_x^l A_i(t,x)| &\leq \sum_{\substack{0\leq n_k\leq k\\ 0\leq n_l\leq l}} \binom{k}{n_k}\binom{l}{n_l} c_i^{k-n_k}|\partial_x^{k+l-n_k-n_l} a_i(x+c_it)||\partial_t^{n_k}\partial_x^{n_l} e^{\mathrm{i}g_i(t,x)}| \\
&\leq \sum_{\substack{0\leq n_k\leq k\\ 0\leq n_l\leq l}} \binom{k}{n_k}\binom{l}{n_l} c_i^{k-n_k}|\partial_x^{k+l-n_k-n_l} a_i(x+c_it)|\times \\
&\qquad \times \sum_{P\sqsubset\pi(n_k,n_l)}\prod_{\alpha\in P}|\partial^\alpha g_i(t,x)|.
\end{aligned}$$

Now the result follows easily, since we can calculate for $1 \leq m \leq n$ and $|P| > 1$, which means $\alpha_1 + \alpha_2 < n$ for all $\alpha \in P$,

$$\begin{aligned}
|\partial_x^m g_i(t,x)| &\leq \sum_{j=1}^4 \int_0^t |d_{i,j}| \left|\partial_x^m |a_j(x + c_i t + (c_j - c_i)s)|^2\right| \,\mathrm{d}s \leq D_i \sum_{j=1}^4 \|a_j\|_{W^{m,\infty}}^2 t, \\
\|\partial_x^m g_i(t)\|_{L^2} &\leq \sum_{j=1}^4 \int_0^t |d_{i,j}| \left\|\partial_x^m |a_j(\boldsymbol{x} + c_i t + (c_j - c_i)s)|^2|\right\|_{L^2} \,\mathrm{d}s \lesssim D_i \|a\|_{H^m}^2 t, \\
\|g_i(t)\|_{L^2} &\leq \sum_{j=1}^4 |d_{i,j}| \|a_j\|_{L^\infty} \int_0^t \|a_j\|_{L^2} \,\mathrm{d}s \lesssim D_i \|a\|_{H^1}^2 t, \\
\|g_i(t)\|_{H^m} &\lesssim D_i \|a\|_{H^m}^2 t,
\end{aligned}$$

where $D_i = \max_{j=1,\ldots,4}\{|d_{i,j}|\}$ and the hidden constant depends on some embeddings and m only. Thus there is a polynomial p of decree at most $n-1$ in t such that

$$\begin{aligned}
&\left\|\prod_{\alpha\in P} \partial^\alpha g_i(t)\right\|_{L^2} \lesssim \prod_{\alpha\in P} \|\partial^\alpha g_i(t)\|_{H^1} \\
\lesssim &\prod_{\alpha\in P} \left(|c_i|^{\alpha_1} \|\partial_x^{|\alpha|} g_i(t)\|_{H^1} + \sum_{p=0}^{\alpha_1-1} \sum_{j=1}^4 |c_i|^p |c_j|^{\alpha_1-p-1} D_i \|\partial_x^{|\alpha|-1} |a_j|^2\|_{H^1} \right) \\
\lesssim &\prod_{\alpha\in P} D_i \left(|c_i|^{\alpha_1} K \|a\|_{H^{|\alpha|+1}}^2 t + \sum_{p=0}^{\alpha_1-1} \sum_{j=1}^4 |c_i|^p |c_j|^{\alpha_1-p-1} \|a_j\|_{H^{|\alpha|}}^2 \right) \\
\lesssim &\, p(P, \|a\|_{H^n}, t, D_i, c_1, \ldots, c_4).
\end{aligned}$$

Further, we obtain for any $P \sqsubset \pi(n_k, n_l)$

$$\begin{aligned}
&\left\|\prod_{\alpha\in P} \partial^\alpha g_i(t)\right\|_{L^\infty} \leq \prod_{\alpha\in P} \|\partial^\alpha g_i(t)\|_{L^\infty} \\
\leq &\prod_{\alpha\in P} \left(|c_i|^{\alpha_1} \|\partial_x^{|\alpha|} g_i(t)\|_{L^\infty} + \sum_{p=0}^{\alpha_1-1} \sum_{j=1}^4 |c_i|^p |c_j|^{\alpha_1-p-1} D_i \|\partial_x^{|\alpha|-1} |a_j|^2\|_{L^\infty} \right) \\
\leq &\prod_{\alpha\in P} D_i \Big(|c_i|^{\alpha_1} \sum_{j=1}^4 \|a_j\|_{W^{|\alpha|,\infty}}^2 t + \sum_{p=0}^{\alpha_1-1} \sum_{j=1}^4 |c_i|^p |c_j|^{\alpha_1-p-1} \|a_j\|_{W^{|\alpha|-1,\infty}}^2 \Big) \\
\leq &\, q(P, \|a\|_{W^{n,\infty}}, t, D_i, c_1, \ldots, c_4),
\end{aligned}$$

where q is a polynomial in t of degree at most n and it follows

$$
\begin{aligned}
&\|\partial_t^k \partial_x^l A_i(t)\|_{L^2} \\
\lesssim &|c_i|^k \|a_i\|_{H^{l+k}} + \sum_{\substack{0 \le n_k \le k \\ 0 \le n_l \le l \\ (n_k, n_l) \neq (0,0)}} \binom{k}{n_k} \binom{l}{n_l} |c_i|^{k-n_k} \|a_i\|_{H^{l+k}} \sum_{P \sqsubset \pi(n_k, n_l)} \left\| \prod_{\alpha \in P} \partial^\alpha g_i(t) \right\|_{L^2} \\
\lesssim &|c_i|^k \|a_i\|_{H^{l+k}} \\
&+ \sum_{\substack{0 \le n_k \le k \\ 0 \le n_l \le l \\ (n_k, n_l) \neq (0,0)}} \binom{k}{n_k} \binom{l}{n_l} |c_i|^{k-n_k} \|a_i\|_{H^{l+k}} \sum_{\substack{P \sqsubset \pi(n_k, n_l) \\ |P| > 1}} \left\| \prod_{\alpha \in P} \partial^\alpha g_i(t) \right\|_{L^2} \\
&+ \sum_{\substack{0 \le n_k \le k \\ 0 \le n_l \le l \\ (n_k, n_l) \neq (0,0)}} \binom{k}{n_k} \binom{l}{n_l} |c_i|^{k} \|a_i\|_{H^{l+k}} \sum_{\substack{P \sqsubset \pi(n_k, n_l) \\ |P| = 1}} \|g_i(t)\|_{H^{n_k+n_l}} + D_i \|a\|^2_{H^{n_k+n_l}} \\
\lesssim &|c|^k \|a_i\|_{H^{l+k}} \\
&+ \sum_{\substack{0 \le n_k \le k \\ 0 \le n_l \le l \\ (n_k, n_l) \neq (0,0)}} \binom{k}{n_k} \binom{l}{n_l} |c_i|^{k-n_k} \|a_i\|_{H^{l+k}} \sum_{\substack{P \sqsubset \pi(n_k, n_l) \\ |P| > 1}} p(P, \|a\|_{H^n}, t, D_i, c_1, \dots, c_4) \\
&+ \sum_{\substack{0 \le n_k \le k \\ 0 \le n_l \le l \\ (n_k, n_l) \neq (0,0)}} \binom{k}{n_k} \binom{l}{n_l} |c_i|^{k} \|a_i\|_{H^{l+k}} D_i \sum_{\substack{P \sqsubset \pi(n_k, n_l) \\ |P| = 1}} \|a\|^2_{H^{n_k+n_l}} t + \|a\|^2_{H^{n_k+n_l}}
\end{aligned}
$$

where the constant depends on the same embeddings and $n_k, n_l, c_1, \dots, c_4$. Finally, we have

$$
\begin{aligned}
&\|\partial_t^k \partial_x^l A_i(t)\|_{L^\infty} \\
\le &\sum_{\substack{0 \le n_k \le k \\ 0 \le n_l \le l}} \binom{k}{n_k} \binom{l}{n_l} c^{k-n_k} \|\partial_x^{k+l-n_k-n_l} a_i(x+ct)\|_{L^\infty} \sum_{P \sqsubset \pi(n_k, n_l)} \prod_{\alpha \in P} \|\partial^\alpha g_i(t)\|_{L^\infty} \\
\le &\sum_{\substack{0 \le n_k \le k \\ 0 \le n_l \le l}} \binom{k}{n_k} \binom{l}{n_l} c^{k-n_k} \|a_i\|_{W^{n,\infty}} \sum_{P \sqsubset \pi(n_k, n_l)} q(P, \|a\|_{W^{n,\infty}}, t, D_i, c_1, \dots, c_4).
\end{aligned}
$$

□

2.C.2 Regularity of the TWI System

The proof of Theorem 2.1.16 was left out in Section 2.1.4. We give it here.

Proof. (Theorem 2.1.16). We know that there is a classical solution in $C^n(\mathbb{R}^2;\mathbb{R})$ due to the fact that we satisfy the assumptions of Lemmata 2.1.10 and 2.1.14. Further, as mentioned before, $B_1(t,x) = 0$ and $B_3(t,x) = b_3(x)$ for $x \notin \operatorname{supp}(b_1)$. Therefore, we split the area of integration and the sets on which we consider the $\operatorname{ess\,sup}$ for the H^k and $W^{k,\infty}$ norm, respectively. In abuse of notation we denote the parts of this splitting by $H^k(\mathbb{R}\setminus \operatorname{supp}(b_1))$ and $W^{k,\infty}(\mathbb{R}\setminus \operatorname{supp}(b_1))$ and the 'complements'. The argument above shows that the $H^k(\mathbb{R}\setminus \operatorname{supp}(b_1))$ and $W^{k,\infty}(\mathbb{R}\setminus \operatorname{supp}(b_1))$ norms of B_1, B_3 can be estimated by the corresponding norm estimates of the initial data.

That is why we must only consider these norms in the compact set $\operatorname{supp}(b_1)$. For a shorter notation we do not mark the dependence on x in all following estimates. Moreover we can interchange the order of the derivatives and we know that, if we take all the time derivatives first, we only have to apply the Sobolev norms $H^k(\operatorname{supp}(b_1))$, $W^{k,\infty}(\operatorname{supp}(b_1))$ to the product of $B_1^{\nu_1} B_3^{\nu_2}$ for some $\nu_1, \nu_2 \in \mathbb{N}$. Due to the fact that these spaces are Banach algebras, it is sufficient to consider $\|B_1(t,\cdot)\|_{H^k(\operatorname{supp}(b_1))}$, $\|B_1(t,\cdot)\|_{W^{k,\infty}(\operatorname{supp}(b_1))}$ and $\|B_3(t,\cdot)\|_{H^k(\operatorname{supp}(b_1))}$, $\|B_3(t,\cdot)\|_{W^{k,\infty}(\operatorname{supp}(b_1))}$ only.

For this purpose we apply the Leibniz rule to the solutions given in Lemma 2.1.10. Hence, we have to consider the sum

$$\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}_0\\ n_1+n_2+n_3=n}} \binom{n}{n_1,n_2,n_3} \left(\partial_x^{n_1}\sqrt{c}\right)\partial_x^{n_2}\left(b_3\cosh\left(\sqrt{c}t\right) - \sqrt{c}\sinh\left(\sqrt{c}t\right)\right) \times$$

$$\times\, \partial_x^{n_3}\left(\sqrt{c}\cosh\left(\sqrt{c}t\right) - b_3\sinh\left(\sqrt{c}t\right)\right)^{-1}$$

for $x \in \operatorname{supp}(b_1)$ and $n \in \mathbb{N}$ as well as the sum

$$\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}_0\\ n_1+n_2+n_3=n}} \binom{n}{n_1,n_2,n_3} \partial_x^{n_1}\sqrt{c}\,\partial_x^{n_2} b_1 \partial_x^{n_3}\left(\sqrt{c}\cosh\left(\sqrt{c}t\right) - b_3\sinh\left(\sqrt{c}t\right)\right)^{-1}.$$

We will now bound all factors of these products for each index in the sum. We use the notation of the generalised Faà di Bruno rule again – actually, it is just the classical version of the rule here – and obtain

$$
\begin{aligned}
&\partial_x^n(\sqrt{c}\cosh\left(\sqrt{c}t\right) - b_3\sinh\left(\sqrt{c}t\right))^{-1}\\
&= \sum_{P\sqsubset\pi(0,n)} (-1)^{|P|}|P|!\big(\sqrt{c}\cosh\left(\sqrt{c}t\right) - b_3\sinh\left(\sqrt{c}t\right)\big)^{-|P|-1}\times\\
&\qquad\times\prod_{\alpha\in P}\partial^\alpha(\sqrt{c}\cosh\left(\sqrt{c}t\right) - b_3\sinh\left(\sqrt{c}t\right))\\
&= \sum_{P\sqsubset\pi(0,n)} (-1)^{|P|}|P|!\big(\sqrt{c}\cosh\left(\sqrt{c}t\right) - b_3\sinh\left(\sqrt{c}t\right)\big)^{-1}\times\\
&\qquad\times\prod_{\alpha\in P}\frac{\partial^\alpha(\sqrt{c}\cosh\left(\sqrt{c}t\right) - b_3\sinh\left(\sqrt{c}t\right))}{\sqrt{c}\cosh\left(\sqrt{c}t\right) - b_3\sinh\left(\sqrt{c}t\right)}.
\end{aligned}
$$

We estimate rather roughly and pointwise for fixed $t\in I, x\in\operatorname{supp}(b_1)$

$$
\begin{aligned}
\frac{1}{|\sqrt{c}\cosh\left(\sqrt{c}t\right) - b_3\sinh\left(\sqrt{c}t\right)|} &= \frac{1}{\left|(\sqrt{c}-b_3)e^{\sqrt{c}t} + (\sqrt{c}+b_3)e^{-\sqrt{c}t}\right|} \le \frac{e^{\sqrt{c}t}}{(\sqrt{c}+b_3)}\\
&\le \frac{e^{\sqrt{\frac{c}{\tilde c}}t_0}}{\sqrt{c}+b_3}\varkappa^{-\sqrt{\frac{c}{\tilde c}}} \le \frac{e^{t_0}}{\sqrt{c}+b_3}\varkappa^{-1},\\
\frac{\cosh\left(\sqrt{c}t\right)}{|\sqrt{c}\cosh\left(\sqrt{c}t\right) - b_3\sinh\left(\sqrt{c}t\right)|} &\le \frac{2e^{2\sqrt{c}t}}{\sqrt{c}+b_3} \le \frac{2e^{2\sqrt{\frac{c}{\tilde c}}t_0}}{\sqrt{c}+b_3}\varkappa^{-2\sqrt{\frac{c}{\tilde c}}} \le \frac{2e^{2t_0}}{\sqrt{c}+b_3}\varkappa^{-2},\\
\frac{\sinh\left(\sqrt{c}t\right)}{|\sqrt{c}\cosh\left(\sqrt{c}t\right) - b_3\sinh\left(\sqrt{c}t\right)|} &\le \frac{2e^{2\sqrt{c}t}}{\sqrt{c}+b_3} \le \frac{2e^{2\sqrt{\frac{c}{\tilde c}}t_0}}{\sqrt{c}+b_3}\varkappa^{-2\sqrt{\frac{c}{\tilde c}}} \le \frac{2e^{2t_0}}{\sqrt{c}+b_3}\varkappa^{-2},\\
\frac{|\sqrt{c}\cosh\left(\sqrt{c}t\right) - b_3\sinh\left(\sqrt{c}t\right)|}{|\sqrt{c}\cosh\left(\sqrt{c}t\right) - b_3\sinh\left(\sqrt{c}t\right)|} &= 1,\\
\frac{|\sqrt{c}\sinh\left(\sqrt{c}t\right) - b_3\cosh\left(\sqrt{c}t\right)|}{|\sqrt{c}\cosh\left(\sqrt{c}t\right) - b_3\sinh\left(\sqrt{c}t\right)|} &\le \frac{\left|(\sqrt{c}-b_3)e^{\sqrt{c}t}\right|}{\left|(\sqrt{c}-b_3)e^{\sqrt{c}t} + (\sqrt{c}+b_3)e^{-\sqrt{c}t}\right|}\\
&\quad + \frac{\left|(\sqrt{c}+b_3)e^{-\sqrt{c}t}\right|}{\left|(\sqrt{c}-b_3)e^{\sqrt{c}t} + (\sqrt{c}+b_3)e^{-\sqrt{c}t}\right|}\\
&\le \frac{\left|(\sqrt{c}-b_3)e^{\sqrt{c}t}\right|}{\left|(\sqrt{c}-b_3)e^{\sqrt{c}t}\right|} + \frac{\left|(\sqrt{c}+b_3)e^{-\sqrt{c}t}\right|}{\left|(\sqrt{c}+b_3)e^{-\sqrt{c}t}\right|} = 2.
\end{aligned}
$$

And finally we obtain, again by use of the generalised Faà di Bruno rule and the

Leibniz rule, the estimates

$$|\partial_x^n \sqrt{c}| \leq \sum_{P \sqsubset \pi(0,n)} (2|P|-3)!!\sqrt{c} \prod_{\alpha \in P} \frac{|\partial^\alpha (b_1^2+b_3^2)|}{c} \leq C(n)\varkappa^2 \text{ für } n \geq 1$$

as well as

$$\begin{aligned}
&\left|\partial_x^n\Big((\sqrt{c}\cosh\big(\sqrt{c}t\big) - b_3\sinh\big(\sqrt{c}t\big)\Big)\right| \\
\leq & \sum_{k\leq n}\binom{n}{k}|\big(\partial_x^k\sqrt{c}\big)\partial_x^{n-k}\cosh\big(\sqrt{c}t\big) - \big(\partial_x^k b_3\big)\partial_x^{n-k}\sinh\big(\sqrt{c}t\big)| \\
\leq & |\sqrt{c}\partial_x^n\cosh\big(\sqrt{c}t\big) - b_3\partial_x^n\sinh\big(\sqrt{c}t\big)| + \sum_{0<k\leq n}\binom{n}{k}C(k)\varkappa^2|\partial_x^{n-k}\cosh\big(\sqrt{c}t\big)|.
\end{aligned}$$

In the last step we exploited that we only consider $x \in \operatorname{supp}(b_1)$. With the aid of these inequalities we obtain estimates that are uniform for $x \in \operatorname{supp}(b_1), t \in I$:

$$\begin{aligned}
|\partial_x^n(\sqrt{c}t)| &\leq C(n) \text{ for } n \geq 1, \\
|\partial_x^n\cosh\big(\sqrt{c}t\big)| &\leq \sum_{P\sqsubset\pi(0,n)}\begin{cases}\cosh\big(\sqrt{c}t\big)\prod_{\alpha\in P}C(\alpha), & |P| \text{ even },\\ \sinh\big(\sqrt{c}t\big)\prod_{\alpha\in P}C(\alpha), & |P| \text{ odd },\end{cases} \\
\frac{\varkappa^2|\partial_x^n\cosh\big(\sqrt{c}t\big)|}{|\sqrt{c}\cosh\big(\sqrt{c}t\big) - b_3\sinh\big(\sqrt{c}t\big)|} &\leq \sum_{P\sqsubset\pi(0,n)}\frac{2e^{2t_0}}{\sqrt{c}+b_3}\prod_{\alpha\in P}C(\alpha),
\end{aligned}$$

and

$$\begin{aligned}
&|\sqrt{c}\partial_x^n\cosh\big(\sqrt{c}t\big) - b_3\partial_x^n\sinh\big(\sqrt{c}t\big)| \\
\leq &\sum_{P\sqsubset\pi(0,n)}\begin{cases}|\sqrt{c}\cosh\big(\sqrt{c}t\big) - b_3\sinh\big(\sqrt{c}t\big)|\prod_{\alpha\in P}C(\alpha), & |P| \text{ even },\\ |\sqrt{c}\sinh\big(\sqrt{c}t\big) - b_3\cosh\big(\sqrt{c}t\big)|\prod_{\alpha\in P}C(\alpha), & |P| \text{ odd },\end{cases}
\end{aligned}$$

as well as

$$\begin{aligned}
&\frac{|\sqrt{c}\partial_x^n\cosh\big(\sqrt{c}t\big) - b_3\partial_x^n\sinh\big(\sqrt{c}t\big)|}{|\sqrt{c}\cosh\big(\sqrt{c}t\big) - b_3\sinh\big(\sqrt{c}t\big)|} \leq \sum_{P\sqsubset\pi(0,n)}\begin{cases}\prod_{\alpha\in P}C(\alpha), & |P| \text{ even}\\ 2\prod_{\alpha\in P}C(\alpha), & |P| \text{ odd}\end{cases} \\
\leq &\sum_{P\sqsubset\pi(0,n)} 2\prod_{\alpha\in P}C(\alpha),
\end{aligned}$$

and

$$\frac{\left|\partial_x^n\Big(\sqrt{c}\cosh\left(\sqrt{c}t\right)-b_3\partial_x^n\sinh\left(\sqrt{c}t\right)\Big)\right|}{|\sqrt{c}\cosh\left(\sqrt{c}t\right)-b_3\sinh\left(\sqrt{c}t\right)|}$$
$$\leq\sum_{P\sqsubset\pi(0,n)}2\prod_{\alpha\in P}C(\alpha)+\sum_{0<k\leq n}\binom{n}{k}C(k)\sum_{P\sqsubset\pi(0,n-k)}\frac{2e^{2t_0}}{\sqrt{c}+b_3}\prod_{\alpha\in P}C(\alpha)|.$$

These estimates are sufficient to bound all addends of the sums above, since we have

$$\Bigg|\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}_0\\ n_1+n_2+n_3=n}}\binom{n}{n_1,n_2,n_3}(\partial_x^{n_1}\sqrt{c})\partial_x^{n_2}\left(b_3\cosh\left(\sqrt{c}t\right)-\sqrt{c}\sinh\left(\sqrt{c}t\right)\right)\times$$
$$\times\,\partial_x^{n_3}\left(\sqrt{c}\cosh\left(\sqrt{c}t\right)-b_3\sinh\left(\sqrt{c}t\right)\right)^{-1}\Bigg|$$
$$\leq\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}_0\\ n_1+n_2+n_3=n}}\sum_{P\sqsubset\pi(0,n_3)}\binom{n}{n_1,n_2,n_3}|\partial_x^{n_1}\sqrt{c}|\frac{\left|\partial_x^{n_2}\left(b_3\cosh\left(\sqrt{c}t\right)-\sqrt{c}\sinh\left(\sqrt{c}t\right)\right)\right|}{|\sqrt{c}\cosh\left(\sqrt{c}t\right)-b_3\sinh\left(\sqrt{c}t\right)|}\times$$
$$\times\,|P|!\prod_{\alpha\in P}\frac{|\partial^\alpha(\sqrt{c}\cosh\left(\sqrt{c}t\right)-b_3\sinh\left(\sqrt{c}t\right))|}{|\sqrt{c}\cosh\left(\sqrt{c}t\right)-b_3\sinh\left(\sqrt{c}t\right)|}$$

as well as

$$\left|\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}_0\\ n_1+n_2+n_3=n}}\binom{n}{n_1,n_2,n_3}\partial_x^{n_1}\sqrt{c}\partial_x^{n_2}b_1\partial_x^{n_3}\left(\sqrt{c}\cosh\left(\sqrt{c}t\right)-b_3\sinh\left(\sqrt{c}t\right)\right)^{-1}\right|$$
$$\leq\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}_0\\ n_1+n_2+n_3=n}}\sum_{P\sqsubset\pi(0,n_3)}\binom{n}{n_1,n_2,n_3}|\partial_x^{n_1}\sqrt{c}|\frac{|\partial_x^{n_2}b_1|}{|\sqrt{c}\cosh\left(\sqrt{c}t\right)-b_3\sinh\left(\sqrt{c}t\right)|}\times$$
$$\times\,|P|!\prod_{\alpha\in P}\frac{|\partial^\alpha(\sqrt{c}\cosh\left(\sqrt{c}t\right)-b_3\sinh\left(\sqrt{c}t\right))|}{|\sqrt{c}\cosh\left(\sqrt{c}t\right)-b_3\sinh\left(\sqrt{c}t\right)|}.$$

Thus, we proved the claim since we bounded all derivatives uniformly in $\varkappa$ on the compact set $\operatorname{supp}(b_1)$. $\square$

Chapter 3

Modulation Equations Near the Eckhaus Boundary – The KdV Equation

The complex Ginzburg-Landau equation admits periodic travelling wave solutions of a very simple form, see the subsequent Section 3.1. We will consider modulations in space and time of the periodic travelling wave solutions and discuss, to some extent, under what conditions these are governed by the Korteweg-de Vries (KdV) equation.
The content of this section is published as preprint in [24]. However, we will use a slightly different notation and give partially different proofs. This allows us to use the structure outlined in the introduction in Section 1.1 and focus on some other aspects. The main results remain essentially unchanged with the different proofs, only some intermediate lemmata and theorems will be changed slightly. There are some advantages (and also disadvantages) of this approach, which we will outline in some comments.

3.1 Introduction

The complex Ginzburg-Landau equation

$$\partial_t \Psi = (1 + \mathrm{i}\alpha)\partial_x^2 \Psi + \Psi - (1 + \mathrm{i}\beta)\Psi|\Psi|^2, \quad \text{in } I \times \mathbb{R}, \quad \Psi(0) = \Psi_0 \text{ in } \mathbb{R}, \quad (3.1)$$

where Ψ is complex-valued, $I = [0, t_0[$ for a $t_0 \in \mathbb{R}^+$, and $(\alpha, \beta) \in \mathbb{R}^2$, is a 'universal amplitude equation'. It can be derived by multiple-scale analysis as amplitude equation for the description of pattern forming systems, such as reaction-diffusion systems or the Couette-Taylor problem, near the threshold of the first instability, cf. [69]. An overview of its applications in condensed-matter physics was given in [4]. A recent mathematical survey of pattern forming systems for which it can be derived was given by Schneider and Uecker, cf. [86, Chapter 10].
There exists a one-parameter family of periodic travelling wave solutions to this equation. These solutions are

$$\Psi_{\text{per}}(t, x) = A_0 e^{\mathrm{i}(\xi_0 \boldsymbol{x} + \omega_0 \boldsymbol{t})} \quad (3.2)$$

with $\xi_0 \in]-1, 1[$, $|A_0|^2 + \xi_0^2 = 1$, and $\omega_0 + \alpha\xi_0^2 + \beta|A_0|^2 = 0$. Note that there is a two-parameter family of solutions actually, if we take into account that for a solution Ψ_{per} and any $\phi \in \mathbb{R}$ another solution $\tilde{\Psi}_{\text{per}} = e^{\mathrm{i}\phi}\Psi_{\text{per}}$ exists, too. Therefore, without loss of generality, we restrict all considerations to the case $A_0 > 0$.
Spectral stability of these solutions has been discussed for some time. A first result was obtained by Eckhaus in [36]. Since then it is well known that for $(\alpha, \beta) = 0$ the solutions with $\xi_0^2 \leq 1/3$, the 'Eckhaus Boundary', are spectrally stable and spectrally unstable otherwise.
In fact, there is a wave number $\xi_{\text{EB}} = \xi_{\text{EB}}(\alpha, \beta)$ for all $(\alpha, \beta) \in \mathbb{R}^2$, which we call the 'Eckhaus Boundary', too, such that for $|\xi_0| \leq \xi_{\text{EB}}$ spectral stability is guaranteed and otherwise spectral instability is observed, cf. [91]. The discussion of the spectral stability by van Harten in [91] is included in Appendix 3.D with some more details for completeness. It is worth noticing that the Ginzburg-Landau equation and the Eckhaus Boundary seem to be of some importance for the qualitative behaviour of pattern formation in the Couette-Taylor flow, cf. [1]. In particular, the selection of the wave number of the pattern seems to be connected to the Eckhaus Boundary.
The nonlinear stability of the solutions (3.2) to the complex Ginzburg-Landau

equation (3.1) for $|\xi_0| < \xi_{\text{EB}}$ with respect to small spatially localised perturbations has been established in the case $(\alpha, \beta) = 0$ in [15, 20, 52]. Since [52] essentially relies on estimates for analytic semigroups, it probably remains true in the complex case $(\alpha, \beta) \neq 0$ as well as long as the linear part is still generator of an analytic semigroup and spectrally stable. A result concerning the nonlinear stability at the Eckhaus Boundary is known in the real case $(\alpha, \beta) = 0$ at least, cf. [41].

Van Harten derived in [91] various amplitude equations for the description of slow modulations in time and space of the solution Ψ_{per} for ξ_0 in a small neighbourhood of $\pm\xi_{\text{EB}}$. He distinguishes three different (nonintersecting) areas in the (α, β)-plane: the diagonal set $\mathcal{D} = \{(\alpha, \beta) \in \mathbb{R}^2 \,|\, \alpha = \beta\}$ and the sets $\mathcal{A}_s \backslash \mathcal{D}, \mathcal{A}_h$, see Figure 3.1(b). In the set $\mathcal{A}_h$ the periodic travelling wave solution destabilises by a Hopf-Turing instability if $|\xi_0|$ increases above ξ_{EB} whilst destabilisation through a sideband instability occurs in $\mathcal{A}_s$. He concludes that the modulations are in essence determined by a generalized Ginzburg-Landau-like system in $\mathcal{A}_h$, the subsequent KdV equation in $\mathcal{A}_s \backslash \mathcal{D}$, and a Cahn-Hilliard equation in $\mathcal{D}$. The special role of $\mathcal{D}$ is obvious from the coefficient of the right hand side of the KdV equation below. However, he only gives a *formal* derivation and no strict error estimates. The main result of this chapter will be the proof that the KdV equation

$$\partial_t A = (\beta - \alpha) \left(\frac{1 + \alpha\beta}{1 + \beta^2} \partial_x^3 A + \partial_x (A^2) \right)$$

governs the dynamics of the modulations of the solutions (3.2) to the Ginzburg-Landau equation (3.1) in a certain parameter regime on the natural time scale – see Theorems 3.5.2 and 3.5.3 below and the remarks in Section 3.6.1.

Similar results already exist if ξ_0 is not in a close neighbourhood of $\pm\xi_{\text{EB}}$. In the case of $(\alpha, \beta) = 0$ a 'phase-diffusion equation' was justified in [65] and the validity of a conservation law was shown for $(\alpha, \beta) \neq 0$ in [64]. Further, the Burgers equation was discussed as a modulation equation in [27] in this regime. Besides the Burgers equation, the validity of the KdV equation was discussed in the same work as well, cf. [27, Section 7.3]. However, it seems that the treatment in the abstract framework of [27] is incomplete and there are some non-trivial steps that remained open. We refrain from the use of this framework and try to give a complete proof of all relevant statements.

Another interesting result is that a waiting time phenomenon occurs at the Eckhaus Boundary in the case $(\alpha, \beta) = 0$, cf. [35].

We will now proceed in four steps to prove our result. First, we will discuss spectral properties of the modulations in Section 3.2. Then we will derive the KdV equation in a certain regime in Section 3.3 and introduce another (auxiliary) formal approximate solution in Section 3.4. In the last step we will present and, with the aid of the auxiliary formal approximate solution, prove the mentioned result in Section 3.5.

3.2 Modulations and Spectral Properties

3.2.1 Modulation Equations

We are interested in (small) modulations of Ψ_{per} and therefore assume $\Psi = \Psi_{\mathrm{per}} \cdot e^{r(\boldsymbol{t},\boldsymbol{x}-\boldsymbol{c}\boldsymbol{t})+\mathrm{i}\phi(\boldsymbol{t},\boldsymbol{x}-\boldsymbol{c}\boldsymbol{t})}$ for functions $r, \phi : I \times \mathbb{R} \to \mathbb{R}$ and $I = [0, t_0[$, $t_0 > 0$. The use of the co-moving frame will facilitate the formal derivation of the KdV approximation in Section 3.3. We deduce

$$\begin{aligned}\partial_t r - c\partial_x r + \mathrm{i}(\omega_0 + \partial_t\phi - c\partial_x\phi) = (1+\alpha\mathrm{i})\left(\partial_x^2 r + \mathrm{i}\partial_x^2\phi + (\partial_x r + \mathrm{i}\xi_0 + \mathrm{i}\partial_x\phi)^2\right) + 1 \\ - (1+\beta\mathrm{i})A_0^2 e^{2r}.\end{aligned}$$

We can consider the equation as as system in $\mathbb{R}^2$ and obtain

$$\begin{aligned}\partial_t r &= \partial_x^2 r + (\partial_x r)^2 + 1 - \xi_0^2 - (\partial_x\phi)^2 - 2\xi_0\partial_x\phi - \alpha\left(\partial_x^2\phi + 2\partial_x r(\xi_0 + \partial_x\phi)\right) \\ &\quad + c\partial_x r - A_0^2 e^{2r}, \\ \partial_t\phi &= \partial_x^2\phi + 2(\partial_x r)(\xi_0 + \partial_x\phi) + \alpha\left(\partial_x^2 r + (\partial_x r)^2 - \xi_0^2 - (\partial_x\phi)^2 - 2\xi_0(\partial_x\phi)\right) \\ &\quad + c\partial_x\phi - \omega_0 - \beta A_0^2 e^{2r}.\end{aligned}$$

By exploitation of the conditions for ξ_0 and ω_0 we can write this system for $U = (r, \phi)$ as follows.

$$\begin{aligned}\partial_t U &= \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}\partial_x^2 U + \begin{pmatrix} c - 2\alpha\xi_0 & -2\xi_0 \\ 2\xi_0 & c - 2\alpha\xi_0 \end{pmatrix}\partial_x U - 2A_0^2\begin{pmatrix} 1 & 0 \\ \beta & 0 \end{pmatrix} U + N(U) \\ &= \mathcal{L}(\partial_x)U + N(U), \end{aligned} \tag{3.3}$$

where $h(r) := e^{2r} - 1 - 2r$ and

$$N(r,\phi) = \begin{pmatrix} (\partial_x r)^2 - (\partial_x \phi)^2 - 2\alpha(\partial_x r)(\partial_x \phi) - A_0^2 h(r) \\ 2(\partial_x r)(\partial_x \phi) + \alpha\left((\partial_x r)^2 - (\partial_x \phi)^2\right) - \beta A_0^2 h(r) \end{pmatrix}.$$

***Remark* 3.2.1.**

- If one wants to write the perturbation in an additive way, i.e. $\Psi = \Psi_{\text{per}} + \tilde{\Psi}$, then

  $$\tilde{\Psi}(t,x) = \Psi_{\text{per}}(t,x)(e^{\tilde{r}(t,x-ct)+\mathrm{i}\tilde{\phi}(t,x-ct)} - 1).$$

  This means that the perturbation is given to leading order by

  $$\Psi_{\text{per}}(t,x)\big(\tilde{r}(t,x-ct) + \mathrm{i}\tilde{\phi}(t,x-ct)\big)$$

  for pointwise small perturbations.

- The scaling that van Harten uses in [91] is different. He scales x with $\xi_0 \in \mathbb{R}^\times$ and t with ξ_0^2. If we used this scaling, then every ξ_0 would vanish in the equations above and A_0^2 would have to be replaced by $\frac{A_0^2}{\xi_0^2}$, which is defined as σ in his article.

3.2.2 Spectral Properties

We note that the linear operator $\mathcal{L}(\partial_x)$ in equation (3.3) turns into a multiplication operator if we consider the Fourier transform of the linear problem,

$$\partial_t \hat{U} = \mathcal{L}(\mathrm{i}\xi)\hat{U}.$$

Therefore its spectrum is given by the set $\sigma(\mathcal{L}) = \{\lambda \in \mathbb{C} \mid \det(\mathcal{L} - \lambda) = 0\}$. $\det(\mathcal{L} - \lambda)$ is a quadratic polynomial in λ and the roots are obviously given by two curves $\mathbb{R} \ni \xi \mapsto \lambda_\pm(\xi) \in \mathbb{C}$ where

$$\begin{aligned} \lambda_\pm(\xi) &= \frac{\operatorname{tr}(\mathcal{L}(\mathrm{i}\xi))}{2} \pm \sqrt{\frac{(\mathcal{L}_{11}(\mathrm{i}\xi) - \mathcal{L}_{22}(\mathrm{i}\xi))^2}{4} + \mathcal{L}_{12}(\mathrm{i}\xi)\mathcal{L}_{21}(\mathrm{i}\xi)} \\ &= -A_0^2 - \xi^2 - (2\alpha\xi_0 - c)\mathrm{i}\xi \pm A_0^2\sqrt{1 - 2\beta\gamma(\xi) - \gamma(\xi)^2} \end{aligned} \tag{3.4}$$

for $\gamma(\xi) := A_0^{-2}(-\mathrm{i}\xi 2\xi_0 + \alpha\xi^2)$ and – as usual – we take the principal branch of the root.

We note further that $\lambda_-(\mathbb{R}) \subset \mathbb{C}^-$ and $\operatorname{Re}\lambda_-(\mathbb{R}) \leq -A_0^2$, actually. This is not the case for λ_+ where $\lambda_+(0) = 0$. We make the following expansion of λ_+ in a neighbourhood of 0:

$$\lambda_+(\xi) = c_1\mathrm{i}\xi - c_2\xi^2 + c_3(\mathrm{i}\xi)^3 - c_4\xi^4 + \mathcal{O}(|\xi|^5), \tag{3.5}$$

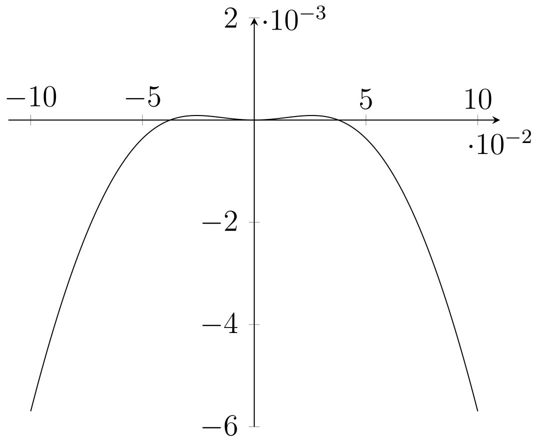
where $\eta := \frac{\xi_0^2}{A_0^2} \geq 0$ and

$$\begin{aligned}
c_1 &= c - 2(\alpha - \beta)\xi_0, \\
c_2 &= 1 + \alpha\beta - 2\eta(1 + \beta^2), \\
c_3 &= (\alpha - 2\eta\beta)(1 + \beta^2)2\xi_0 A_0^{-2}, \\
c_4 &= \left(\frac{\alpha^2}{2} - 6\alpha\beta\eta + 2\eta^2(1 + 5\beta^2)\right)(1 + \beta^2)A_0^{-2}.
\end{aligned}$$

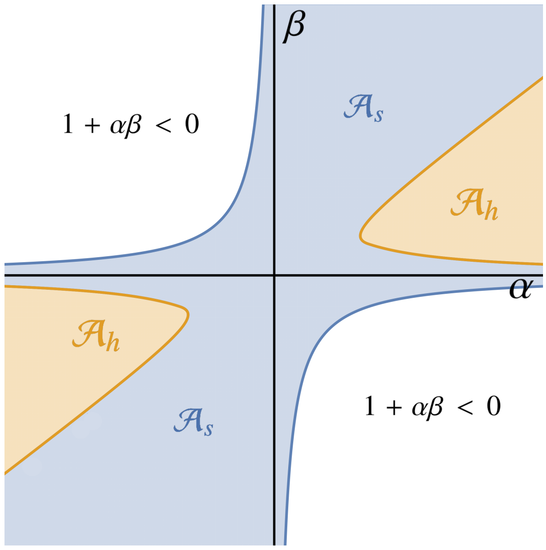
Obviously, the spectrum is unstable meaning that it contains numbers with positive real parts if $1 + \alpha\beta < 0$. This is well-known and sometimes called Benjamin-Feirs instability. Such an instability mechanism for periodic travelling wave solutions was discussed in the setting of water waves in [7, 8] and strict results were proven in [16]. We say that there is a *sideband instability in* 0 if there is $\delta_0 > 0$ such that $\operatorname{Re} z > 0$ for all $z \in \sigma(\mathcal{L}) \cap U(0)$ and all deleted neighbourhoods $U(0) \subset B_\delta(0)$ of 0 with $\delta < \delta_0$. Clearly, a sideband instability occurs for the spectrum given by the curves (3.4) if and only if

$$\frac{1 + \alpha\beta}{2(1 + \beta^2)} =: \eta_{\mathrm{EB}} < \eta \qquad \Leftrightarrow \qquad \frac{(1 + \alpha\beta)}{2(1 + \beta^2) + (1 + \alpha\beta)} =: \xi_{\mathrm{EB}}^2 < \xi_0^2,$$

where $\xi_{\mathrm{EB}} \geq 0$ is called the 'Eckhaus Boundary'. Hence if ξ_0 is increased from 0 through the threshold ξ_{EB} the periodic travelling wave solution is (spectrally) destabilised by the sideband instability. Whilst the sideband instability exists for all $(\alpha, \beta) \in \mathbb{R}^2$ and $|\xi_0| > \xi_{\mathrm{EB}}$, the periodic travelling wave is not necessarily destabilised by it. In fact, van Harten investigated the situation further and (formally) showed in [91] that destabilisation by the sideband instability occurs if and only if $(\alpha, \beta) \in \mathcal{A}_s \subset \mathbb{R}^2$. He introduced the function $r\colon]0, 1[\to \mathbb{R} \cup \{\infty\}$



(a) The spectral curve λ_+ in the parameter regime (3.7) where a sideband instability occurred. $\alpha = 1, \beta = 100, \epsilon = 10^{-2}$



(b) Regions in the (α, β)-plane marked in correspondence with the primary destabilisation mechanism. Destabilisation by a sideband instability occurs in $\mathcal{A}_s$. Courtesy of B. de Rijk.

Figure 3.1

by

$$r(z) = \begin{cases} z^{-1/2} & \text{for } z \in]0, 1/3] \\ r \in \mathbb{R}^+ : r^4 z^2(4z-3) + r^2(5z^2 - 4z + 1) + 1 = 0 & \text{for } z \in]1/3, 3/4[\\ \infty & \text{for } z \in [3/4, 1[\end{cases}$$

Then one obtains

$$\mathcal{A}_s = \left\{(\alpha, \beta) \in \mathbb{R}^2 \,\middle|\, \left(-1 < \alpha\beta < \beta^2\right) \vee (\beta = 0 \wedge \alpha \neq 0)\right\}$$
$$\cup \left\{(\alpha, \beta) \in \mathbb{R}^2 \,\middle|\, \left(0 < |\beta| \leq |\alpha| < r\left(\tfrac{\beta}{\alpha}\right)\right)\right\},$$

see Figure 3.1(b) for an idea how this region looks like.

At the Eckhaus Boundary the expressions for c_3 and c_4 become

$$\begin{aligned} c_{3,\mathrm{EB}} &= \pm 2(\alpha - \beta)\xi_{\mathrm{EB}} A_{\mathrm{EB}}^{-2}, \\ c_{4,\mathrm{EB}} &= \left(\frac{1 + (\alpha - 2\beta)^2 + \beta^2(1 + 4\alpha\beta - 3\alpha^2)}{2(1+\beta^2)}\right) A_{\mathrm{EB}}^{-2}, \end{aligned} \tag{3.6}$$

where $A_{\text{EB}}^2 = 1 - \xi_{\text{EB}}^2$ and the sign of $c_{3,\text{EB}}$ depends on the sign of ξ_0. Thus, if $(\alpha, \beta) \in \mathcal{A}_s \backslash \mathcal{D}$ near the Eckhaus Boundary, we expect that the leading order behaviour of $\partial_t - \mathcal{L}(\partial_x)$ is dispersive in the appropriate co-moving frame, i.e. $c = 2(\alpha - \beta)\xi_0$, for a long wave modulation. This will lead to a KdV equation as an amplitude equation for modulations of the solution Ψ_{per} to equation (3.3) – cf. [91] and Section 3.3. On the other hand, in the special case $\alpha = \beta$, $c_{3,\text{EB}}$ vanishes and a Cahn-Hilliard equation occurs instead – cf. [91].
However, we should emphasise that the complete analysis depends on a small parameter $0 < \epsilon \ll 1$ and $|\xi_0 - \xi_{\text{EB}}|$ has to be small in terms of ϵ. This applies – more strongly still – to our analysis. To make this more precise we require

$$(\alpha, \beta) \in \mathcal{A}_s \backslash \mathcal{D}, \qquad\qquad |\xi_0 - \xi_{\text{EB}}| \lesssim \epsilon^2. \tag{3.7}$$

Recall that this means $|\eta - \eta_{\text{EB}}| \lesssim \epsilon^2$ also. The conditions (3.6) and (3.7) imply for sufficiently small ϵ

$$c_3 = c_{3,\text{EB}} + \mathcal{O}(\epsilon^2) \neq 0,$$

yielding the desired dispersive dynamics on the linear level.
For the proof of the validity of the KdV equation as a modulation equation shortly after the periodic travelling wave has been destabilised through a sideband instability we need the following spectral estimate.

Lemma 3.2.2. *Assume condition* (3.7) *is satisfied. Then there is* $\epsilon_0 > 0$ *such that the spectral curves* $\lambda_\pm(\boldsymbol{\xi})$ *given by equation* (3.4) *enjoy for all* $\delta \in [0, 1[$ *and* $\epsilon \in]1, \epsilon_0[$ *the following bounds*

$$\operatorname{Re}(\lambda_-(\xi)) \leq -\delta A_{\text{EB}}^2 - \xi^2, \qquad\qquad \operatorname{Re}(\lambda_+(\xi)) \lesssim \epsilon^3 |\xi|, \qquad \xi \in \mathbb{R},$$

where the hidden constant is independent of ϵ.

Proof. Since we use the principal branch of the root, it has positive real part. Hence by equation (3.4) and inequality (3.7)

$$\operatorname{Re}(\lambda_-(\xi)) \leq -A_0^2 - \xi^2 \leq -A_{\text{EB}}^2 + 2\epsilon^2 C A_{\text{EB}} - \xi^2 \leq -\delta A_{\text{EB}}^2 - \xi^2,$$

where we used $|A_{\text{EB}} - A_0| \leq C\epsilon^2$ and $\epsilon^2 \leq \frac{(1-\delta)A_{\text{EB}}}{2C}$.

For the other estimate we note that the function $f : U_{\tilde\delta}(0)\times]-1,1[\to \mathbb{R}$, $(\xi,\xi_0)\mapsto \mathrm{Re}\big(\lambda_+(\xi;\xi_0)\big)$ depends smoothly on (ξ,ξ_0) for $\tilde\delta$ sufficiently small. A sufficient condition is $\tilde\delta^2 < \frac{2|\beta|-\sqrt{4\beta^2+1}}{2|\alpha|}$ (and $\tilde\delta<\infty$ for $\alpha=0$). By the Taylor expansion of λ_+ around $\xi=0$ given in (3.5) it holds

$$\begin{aligned}
&\partial_\xi^2 f(0,\xi_{\mathrm{EB}}) = 0, && \partial_\xi^j f(0,\xi_0) = 0, \qquad j=0,1,3,\\
&\partial_\xi^4 f(0,\xi_{\mathrm{EB}}) = -c_{4,\mathrm{EB}} < 0,
\end{aligned}$$

for all $\xi_0 \in]-1,1[$. The last inequality holds since $c_{4,\mathrm{EB}} = \frac{C_{4,\mathrm{EB}}}{2A_{\mathrm{EB}}^6} > 0$ in $\mathcal{A}_s$ by definition, where $C_{4,\mathrm{EB}}$ is the fourth order coefficient in van Harten's polynomial evaluated at $\xi_0=\xi_{\mathrm{EB}}$, cf. Appendix 3.D. The smoothness guarantees that there is $C>0$ such that

$$\left|\partial_\xi^2 f(0,\xi_0)\right|, \left|\partial_\xi^4 f(0,\xi_0)+c_{4,s}\right| \le C\epsilon^2$$

for $|\xi_0-\xi_{\mathrm{EB}}| \lesssim \epsilon^2$ and ϵ in a admissible compact subset of $[0,1]$ – meaning $\xi_0,\xi_{\mathrm{EB}} \in]-1,1[$. Further we find by Taylor's Theorem a ball $B_\rho(0)\subset \overline{B_{\tilde\delta}(0)}$ such that

$$\left| f(\xi,\xi_0) - \frac{1}{2}\partial_\xi^2 f(0,\xi_0)\xi^2 - \frac{1}{4!}\partial_\xi^4 f(0,\xi_0)\xi^4 \right| \le \frac{c_{4,\mathrm{EB}}\xi^4}{2\cdot 4!}$$

for all $\xi \in B_\rho(0)\times[\xi_{\mathrm{EB}}-\epsilon^2,\xi_{\mathrm{EB}}+\epsilon^2]$ and all admissible $\epsilon>0$. Combining the latter two inequalities yields

$$f(\xi,\xi_0) \le \frac{1}{2}C\epsilon^2\xi^2 - \frac{c_{4,\mathrm{EB}}\xi^4}{4\cdot 4!} \le \frac{C^{3/2}}{72\sqrt{c_{4,\mathrm{EB}}}}\epsilon^3|\xi| \tag{3.8}$$

for all $\xi \in B_\rho(0)\times[\xi_{\mathrm{EB}}-\epsilon^2,\xi_{\mathrm{EB}}+\epsilon^2]$ and every $\epsilon^2 \le \frac{c_{4,\mathrm{EB}}}{4C}$ satisfying any previous ϵ-bound. We used in the last estimate that for any $a,b>0$, the function $2b\sqrt{b}/(3\sqrt{3a})|\boldsymbol{\xi}|$ is an upper bound to the quartic $-a\boldsymbol{\xi}^4+b\boldsymbol{\xi}^2$ for all $\xi\in\mathbb{R}$.

On the other hand, since we have $(\alpha,\beta)\in\mathcal{A}_s$, the periodic travelling wave solutions undergoes a sideband instability at $\eta=\eta_{\mathrm{EB}}$, i.e., $f(\xi;\xi_{\mathrm{EB}})$ is strictly negative for all $\xi\in\mathbb{R}^\times$ and touches the origin at $\xi=0$ in a quartic tangency. Thus, since $B_\rho(0)$ is an ϵ-independent ball around of the origin, we have $f(\xi,\xi_0)<0$ for $\xi\in\mathbb{R}\backslash B_\rho(0)$ provided $\epsilon>0$ is sufficiently small (cf. Appendix 3.D for more information). Combining the latter with the expansion (3.8) yields the result. $\square$

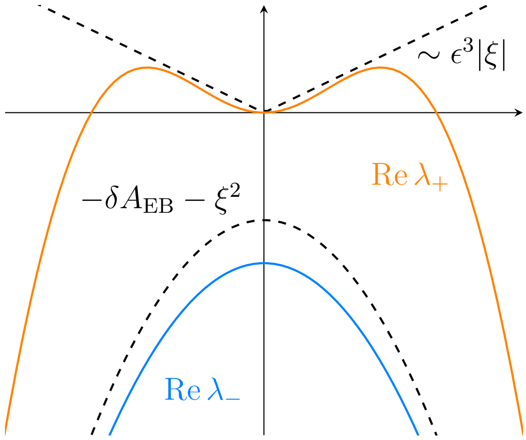


Figure 3.2: Sketch of the spectral estimates established in Lemma 3.2.2.

3.3 Derivation of the KdV Equation

Let us assume $\xi_0 \neq 0$ and change the scaling to the one used by van Harten, i.e. change every ξ_0 to 1 and every A_0^2 to η^{-1}. Especially for $\xi_{\mathrm{EB}} > 0$ this is always possible. In fact, it is somewhat doubtful whether the KdV equation is a suitable candidate for an amplitude equation of a formal approximate solution in the case $\xi_0 = 0$ since $c_{3,\mathrm{EB}} = 0$ in this case.
We notice that only spatial and temporal derivatives of ϕ appear in equation (3.3). Thus, we introduce $\psi = \partial_x \phi$ and obtain the system (in the scaling of van Harten)

$$\partial_t U = \tilde{\mathcal{L}}(\partial_x) U + \tilde{N}(U), \tag{3.9}$$

where $U = (r, \psi)$ and

$$\tilde{\mathcal{L}}(\partial_x) := \begin{pmatrix} \partial_x^2 + (c - 2\alpha)\partial_x - 2\eta^{-1} & -\alpha\partial_x - 2 \\ \alpha\partial_x^3 + 2\partial_x^2 - 2\eta^{-1}\beta\partial_x & \partial_x^2 + (c - 2\alpha)\partial_x \end{pmatrix},$$

$$\tilde{N}(r, \psi) = \begin{pmatrix} (\partial_x r)^2 - (\psi)^2 - 2\alpha(\partial_x r)(\psi) - \eta^{-1}h(r) \\ \partial_x\big(2(\partial_x r)(\psi) + \alpha\left((\partial_x r)^2 - (\psi)^2\right) - \beta\eta^{-1}h(r)\big) \end{pmatrix}.$$

We will now formally derive that the KdV equation governs the behaviour of the modulation of the periodic travelling wave solution (3.2) in leading-order in a certain parameter regime. To be more precise, we will derive that in the regime of small initial data a long wave approximation for the modulated periodic travelling wave solution is governed by the KdV equation if $|\eta - \eta_{\mathrm{EB}}|$ is small in terms of ϵ and certain additional constraints on the parameter regime are satisfied. For this purpose we make the ansatz that a small solution $U_* = (r_*, \psi_*)$ to the modulation equation (3.9) has the form

$$r_* = \epsilon^2 B(\epsilon^3 \boldsymbol{t}, \epsilon \boldsymbol{x}), \qquad \psi_* = \epsilon^2 A(\epsilon^3 \boldsymbol{t}, \epsilon \boldsymbol{x}), \tag{3.10}$$

and B can be expanded in terms of A in the following form

$$B = \nu_0 A + \nu_1 \epsilon \partial_x A + \nu_2 (\epsilon \partial_x)^2 A + \nu_3 (\epsilon A)^2,$$

where $0 < \epsilon \ll 1$ is a small parameter and the coefficients $\nu_0, \dots, \nu_3 \in \mathbb{R}$ will be determined later. This approach requires more regularity of ψ_* than needed for an ansatz where r_* is independent of ψ_* but we will ignore this point for the moment.

Inserting the ansatz (3.10) into (3.9) gives (pointwise) the system

$$\begin{aligned}\epsilon^3 \partial_t B = \epsilon^2 \partial_x^2 B + \epsilon \partial_x((c - 2\alpha)B - \alpha A) - 2\eta^{-1} B - 2A - \epsilon^2 A^2 \\ - 2\eta^{-1} \epsilon^2 B^2 + \mathcal{O}(\epsilon^4),\end{aligned} \tag{3.11a}$$

$$\begin{aligned}\epsilon^3 \partial_t A = \epsilon^3 \alpha \partial_x^3 B + \epsilon^2 \partial_x^2 (A + 2B) + \epsilon \partial_x ((c - 2\alpha) A - 2\beta \eta^{-1} B) \\ - \epsilon^3 \alpha \partial_x (A)^2 - 2\beta \eta^{-1} \epsilon^3 \partial_x B^2 + \mathcal{O}(\epsilon^5).\end{aligned} \tag{3.11b}$$

After using the expansion for B and equating terms of order 1, ϵ and ϵ^2 in equation (3.11a) we obtain the system

$$\begin{aligned} -2\eta^{-1}\nu_0 - 2 &= 0, \\ -2\eta^{-1}\nu_1 + (c - 2\alpha)\nu_0 - \alpha &= 0, \\ -2\eta^{-1}\nu_2 + \nu_0 + (c - 2\alpha)\nu_1 &= 0, \\ -2\eta^{-1}\nu_3 - 2\eta^{-1}\nu_0^2 - 1 &= 0. \end{aligned} \tag{3.12}$$

Similarly, the comparison of orders ϵ, ϵ^2 and ϵ^3 in equation in (3.11b) leads to

$$c - \beta 2\eta^{-1}\nu_0 - 2\alpha = 0, \tag{3.13}$$

$$1 + 2\nu_0 - 2\beta\eta^{-1}\nu_1 = 0, \tag{3.14}$$

as well as the KdV equation

$$\partial_T A = \gamma_{lin} \partial_X^3 A + \gamma_{non} \partial_X (A^2), \tag{3.15}$$

with coefficients

$$\begin{aligned} \gamma_{\text{lin}} &:= \alpha \nu_0 - 2\beta \eta^{-1} \nu_2 + 2\nu_1, \\ \gamma_{\text{non}} &:= -(2\beta \eta^{-1} \nu_0^2 + \alpha + 2\beta \eta^{-1} \nu_3). \end{aligned}$$

Obviously, the system (3.12) together with the velocity equation $c = 2(\beta\eta^{-1}\nu_0 + \alpha)$ has a unique solution. Thus

$$\begin{aligned} c &= 2(\alpha - \beta), \\ \gamma_{\text{lin}} &= (\alpha - \beta)\nu_0 + 2(1 + \beta^2)\nu_1 = 2(\beta - \alpha)\eta_{\text{EB}} + \mathcal{O}(|\eta - \eta_{\text{EB}}|) \\ \gamma_{\text{non}} &= -(\alpha - \beta). \end{aligned}$$

Note that $\gamma_{\text{lin}} = \pm c_{3,\text{EB}} + \mathcal{O}(|\eta - \eta_{\text{EB}}|)$. The reason why the sign of γ_{lin} is independent of ξ_0 is that we reflect the space at the origin for $\xi_0 < 0$ by introduction of the scaling. We note however that the 'KdV condition' (3.14) is not necessarily satisfied. But we observe that

$$1 + 2\nu_0 - 2\beta\eta^{-1}\nu_1 = 1 + \alpha\beta - 2(1 + \beta^2)\eta = \mathcal{O}(|\eta - \eta_{\text{EB}}|).$$

Since we are interested in the case $|\eta - \eta_{\text{EB}}| \lesssim \epsilon^2$, the above equation holds up to order ϵ^2. The conclusion is that for sufficiently regular solutions to the KdV equation

$$\partial_t A = (\beta - \alpha) \left(\frac{1 + \alpha\beta}{1 + \beta^2} \partial_x^3 A + \partial_x (A^2) \right)$$

we obtain pointwise

$$|\mathfrak{E}_{U_*}|_1 \lesssim \epsilon^5 + \epsilon^6.$$

The local (and global) existence of regular solutions to the KdV equation is clear for sufficiently regular initial data, cf. [54] for solutions in H^s.

3.4 Estimates for the Residual

We will use the space Hol_σ as an intermediate space for the proof of the main results. The reason for this choice will become clear in Section 3.5. This means that we need solutions to the KdV equation (3.15) with values in Hol_σ. Fortunately, this is a almost well known result in literature. Kato and Masuda proved in 1986 that for initial data in Hol^2_σ there is a global (mild) solution u to the KdV equation, which is analytic in $\mathcal{S}_{\sigma'}$ and $u \in C(I, \mathrm{Hol}^2_{\sigma'})$ for some $\sigma' \in]0, \sigma]$, cf. [56, Theorem 2]. Another result is due to Grujić and Kalisch, cf. [40, Theorem 1]. They proved that there is a local mild solution $\hat{u} \in C([0, t_0], L^2_{s,\sigma})$ to the KdV equation if the Fourier transform of the initial data is in $L^2_{s,\sigma}$ for some $s \geq 0, \sigma > 0$. This is sufficient for local results with values in Hol_σ.

Hence, the last comments of Section 3.3 indicate that $\|\mathscr{F}\,\mathfrak{E}_{U_*}\|_{L^2_{s,\sigma'}} \lesssim \epsilon^{9/2}$ for initial data in Hol_σ and $\sigma' < \sigma$. But this will not be sufficient for our error analysis in Section 3.5.2. For an improvement of the residual estimate we introduce

$$r_{**} = \epsilon^2 \sum_{i=0}^{N} \epsilon^i B_i(\epsilon^3 \boldsymbol{t}, \epsilon \boldsymbol{x}), \qquad \psi_{**} = \epsilon^2 \sum_{i=0}^{N} \epsilon^i A_i(\epsilon^3 \boldsymbol{t}, \epsilon \boldsymbol{x})$$

for an $N \in \mathbb{N}$. We will show in the following Lemma 3.4.2 that this ansatz is a pointwise correction of order $\mathcal{O}(\epsilon^3)$ of U_* and gives a residual estimate of order $\mathcal{O}(\epsilon^{N+\frac{9}{2}})$ (but not necessarily global in time). More precisely, we claim that Lemma 3.4.2 holds if the 'relaxed KdV condition' stated below is satisfied and if the initial datum of the function A_0 is in the space Hol_σ for a $\sigma > 0$.

Hypothesis 3.4.1 (Relaxed KdV condition)**.** *We assume $c = 2(\alpha - \beta)$ and that the 'relaxed KdV condition'*

$$\epsilon^{-2}(1 - 2\eta(1 + \beta^2) + \alpha\beta) = K(\epsilon, \alpha, \beta, \eta) \leq C(\alpha, \beta, \eta) \tag{3.16}$$

is satisfied uniformly for $\epsilon \in]0, \epsilon_0]$ and some $\epsilon_0 \leq 1$.

Lemma 3.4.2. *Suppose the assumptions of Hypothesis 3.4.1 are satisfied. Let $A \in C([0, t_0], \mathrm{Hol}_\sigma)$ be a solution to the KdV equation* (3.15) *for a $\sigma > 0$ and $U_* = (r_*, \psi_*)$ as in Section 3.3.*

Then there is an $\epsilon_0 > 0$ *and a function*

$$U_{**} \in C([0, t_1/\epsilon^3], \mathrm{Hol}_{\sigma/\epsilon}) \cap C^1([0, t_1/\epsilon^3], \mathrm{Hol}_{\sigma/\epsilon}),$$

where $t_1 = t_1(\sigma) \in]0, t_0]$ *might depend on the choice of the* σ*-norm, such that we have the estimate*

$$\|\mathscr{F}\, \mathfrak{E}_{U_{**}}\|_{L^\infty([0,t_1/\epsilon^3], L^2_{0,\sigma'/\epsilon})} \lesssim \epsilon^{8-\frac{1}{2}}$$

for all $\epsilon \in]0, \epsilon_0]$ *and* $\sigma' \in [0, \sigma[$ *as well as*

$$\|\hat{U}_{**} - \hat{U}_*\|_{L^\infty([0,t_1/\epsilon^3], L^2_{0,\sigma'/\epsilon})} \lesssim \epsilon^{5/2}$$
$$\|U_{**} - U_*\|_{L^\infty([0,t_1/\epsilon^3], L^\infty)} \lesssim \epsilon^3.$$

***Remark* 3.4.3.** We emphasise three points about the previous lemma.

- We note that the constant in the inequalities depends on σ' and is unbounded for $\sigma' \to \sigma$.
- In general t_1 will depend on the value $\sigma' < \sigma$ chosen for the norm. If one had a bound being uniform in $\sigma' < \sigma$ for $\|g\|_{X_{\sigma,s,b}}, \|f\|_{X_{\sigma,s,b'}}$ in Lemma 3.4.4 below, then one could choose t_1 independent of σ'.
- We could just as well use $L^2_{s,\sigma'/\epsilon}$, $s \in \mathbb{R}$, as norm instead of $L^2_{0,\sigma'/\epsilon}$. The norm estimates would remain unchanged except for the constants in the inequalities.

We will use the following result about solutions to a linearised KdV equation for the proof of Lemma 3.4.2. This is just a small modification of the results of Grujić and Kalisch in [40].

Lemma 3.4.4. *Let* $b, \tilde{b} \in]\frac{1}{2}, \frac{3}{4}[$, $\tilde{b} > b$ *and* $b' \in]b-1, \min\{-\frac{1}{4}, \tilde{b}-1\}[$. *Further let* $s \geq 3b$, $g \in X_{\sigma,s,b}$, $f \in X_{\sigma,s,b'}$ *and* $\hat{u}_0 \in L^2_{s,\sigma}$. *Then there is a* $T > 0$ *and a unique mild solution* u *to*

$$\begin{aligned} \partial_t u &= \partial_x^3 u + \partial_x(gu) + f && \text{in } [-T, T] \cap [-1, 1] \times \mathbb{R}, \\ u(0) &= u_0 && \text{in } \mathbb{R}, \end{aligned}$$

with $\hat{u} \in C([-T, T] \cap [-1, 1], L^2_{s,\sigma})$, $u \in X_{\sigma,s,b}$ *and* $\partial_t u \in X_{\sigma,s-3,b-1}$.

The proof of Lemma 3.4.4 is straightforward and we refer the interested reader to Appendix 3.E. Another tool that we will use is the following lemma concerning the regularity of the time derivatives, which we will need for the residual estimate. The lemma resolves the problem that the Fourier transform of a function of the space $X_{\sigma,s-3,b-1}$ is not necessarily in $C([-T,T]\cap[-1,1], L^2_{s,\sigma})$ for $b \leq \frac{3}{2}$.

Lemma 3.4.5. *Let X be a Banach space, $A : D(A) \subset X \to X$ a closed linear operator and $(Y, \|\cdot\|_Y) \overset{c}{\hookrightarrow} (D(A), \|\cdot\|_{D(A)}) \overset{c}{\hookrightarrow} (X, \|\cdot\|_X)$ with continuous embeddings for a closed subspace Y. Let $I = [0, t_0]$ for a $t_0 \in \mathbb{R}^+$ or $I = \mathbb{R}_0^+$. We assume we have a mild solution $u \in C(I, Y)$, such that*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)(f \circ u)(s)\mathrm{d}s$$

where $(T)_{t\geq 0}$ is the strongly continuous semigroup in X with generator A. Further we assume $f \in C(I \times Y, D(A))$ and $u_0 \in Y$. Then $u \in C(I, Y) \cap C^1(I, X)$ and

$$\partial_t u = Au + f \circ u.$$

Proof. This is trivial since in that case $f \circ u \in C\big(I, D(A)\big)$ so that the mappings

$$t \mapsto T(t)u_0 \qquad \text{and} \qquad t \mapsto T(t-s)(f \circ u)(s)$$

are differentiable as mappings from I to X and the statement follows easily. $\square$

Now we can start with the proof of Lemma 3.4.2.

Proof. (Lemma 3.4.2). We write $U_{**} = (r_{**}, \psi_{**})$ for functions

$$r_{**} = \epsilon^2 \sum_{i=0}^{N} \epsilon^i B_i(\epsilon^3 \boldsymbol{t}, \epsilon \boldsymbol{x}), \qquad \psi_{**} = \epsilon^2 \sum_{i=0}^{N} \epsilon^i A_i(\epsilon^3 \boldsymbol{t}, \epsilon \boldsymbol{x}).$$

Then we formally obtain the system

$$
\begin{aligned}
\sum_{i=0}^{N} \epsilon^{i+5}\partial_t B_i = \sum_{i=0}^{N} \epsilon^{i+2}\big(&\epsilon^2\partial_x^2 B_i - 2A_i - \epsilon\partial_x\left(\alpha A_i + 2\beta B_i\right)\big) \\
&- \eta^{-1}\left(\prod_{i=0}^{N} e^{2\epsilon^{2+i}B_i} - 1\right) + \sum_{i,j=0}^{N} \epsilon^{6+i+j}(\partial_x B_i)(\partial_x B_j) \\
&- \sum_{i,j=0}^{N} \epsilon^{4+i+j} A_i A_j - 2\alpha \sum_{i,j=0}^{N} \epsilon^{5+i+j}(\partial_x B_i) A_j
\end{aligned} \tag{3.17a}
$$

and

$$
\begin{aligned}
\sum_{i=0}^{N} \epsilon^{i+5}\partial_t A_i = \sum_{i=0}^{N} \epsilon^{i+2}\big(&\epsilon^2\partial_x^2 A_i + 2\epsilon^2\partial_x^2 B_i + \alpha\epsilon^3\partial_x^3 B_i - 2\beta\epsilon\partial_x A_i\big) \\
&- \partial_x\Bigg(\beta\eta^{-1}\left(\prod_{i=0}^{N} e^{2\epsilon^{2+i}B_i} - 1\right) - 2\sum_{i,j=0}^{N} \epsilon^{6+i+j}(\partial_x B_i) A_j. \\
&- \alpha \sum_{i,j=0}^{N} \epsilon^{i+j+5}\left(\epsilon^2(\partial_x B_i)(\partial_x B_j) - A_i A_j\right)\Bigg)
\end{aligned} \tag{3.17b}
$$

We will now define an iterative scheme for the reduction of the residual to the required order. By comparing orders of ϵ in equation (3.17a), we define B_i for $i \geq 0$ by

$$2B_i = -2\eta A_i - \eta\partial_x\left(\alpha A_{i-1} + 2\beta B_{i-1}\right) + \eta F_i(A_0, \ldots, A_{i-2}, B_0 \ldots, B_{i-2}),$$

where

$$F_i = \partial_x^2 B_{i-2} + \tilde{F}_i(A_0, \ldots, A_{i-2}, B_0 \ldots, B_{i-2})$$

and $A_i, B_i \equiv 0$ for $i < 0$. We stress that $\tilde{F}_i$ is completely linear in A_{i-2}, B_{i-2} and does not depend on derivatives of A_{i-2}, B_{i-2}. More details concerning the derivation of this scheme can be found in Appendix 3.A.

Using this definition of B_i, we see that $c = 2(\alpha - \beta)$ eliminates the corresponding terms of order ϵ^{3+i} of equation (3.17b), just the same way as in Section 3.3. Hence, we have in order ϵ^{i+5} in equation (3.17b)

$$
\begin{aligned}
\partial_t A_i = {} & (1+\alpha\beta-2\eta(1+\beta^2))\partial_x^2 A_{i+1} \\
& + \partial_x^3(-\eta\alpha A_i - 2\eta\beta B_i + \alpha B_i - \beta^2\eta\,(\alpha A_i + 2\beta B_i) - \beta B_i) \\
& + \partial_x^2 \beta^2 \eta F_{i+1} - \beta\partial_x \tilde{F}_{i+2} + \eta\partial_x^2 F_{i+1} \\
& - \partial_x \Big(\beta\eta^{-1} \sum_{\substack{(k_0,\dots,k_N)\in\mathbb{N}\times\mathbb{N}_0^N \\ k_N \le k_{N-1} \le \dots \le k_1 \le k_0 \\ k_0+\sum_{l=0}^N k_l = i+4, k_{i+2}=0}} 2^{k_0} \frac{B_N^{k_N}}{k_N!} \frac{B_{N-1}^{(k_{N-1}-k_N)}}{(k_{N-1}-k_N)!} \cdots \frac{B_0^{(k_0-k_1)}}{(k_0-k_1)!} \\
& \quad - 2 \sum_{\substack{j',j''\in\{0,\dots,N\}^2 \\ 1+j'+j''=i}} (\partial_x B_{j'}) A_{j''} + \alpha \sum_{\substack{j',j''\in\{0,\dots,N\}^2 \\ j'+j''=i}} A_{j'} A_{j''} \\
& \quad - \alpha \sum_{\substack{j',j''\in\{0,\dots,N\}^2 \\ 2+j'+j''=i}} (\partial_x B_{j'})(\partial_x B_{j''}) \Big).
\end{aligned}
$$

Because of the 'KdV condition' (3.16) we can shift the addend

$$\partial_x^2 (1-2\eta(1+\beta^2)+\alpha\beta) A_{i+1}$$

of order ϵ^{4+i} in this equation to order ϵ^{6+i}. Now, we need solutions to the following Cauchy problem with zero initial data for $i = 1, \dots, N-3$ in the mentioned space

$$
\begin{aligned}
\partial_t A_i = {} & K(\epsilon,\alpha,\beta,\sigma)\partial_x^2 A_{i-1} \\
& + \partial_x^3(-\eta\alpha A_i - 2\eta\beta B_i + \alpha B_i - \beta^2\eta\,(\alpha A_i + 2\beta B_i) - \beta B_i) \\
& + \partial_x^2 \beta^2 \eta F_{i+1} - \beta\partial_x \tilde{F}_{i+2} + \eta\partial_x^2 F_{i+1} \\
& - \partial_x \Big(\beta\eta^{-1} \sum_{\substack{(k_0,\dots,k_N)\in\mathbb{N}\times\mathbb{N}_0^N \\ k_N \le k_{N-1} \le \dots \le k_1 \le k_0 \\ k_0+\sum_{l=0}^N k_l = i+4, k_{i+2}=0}} 2^{k_0} \frac{B_N^{k_N}}{k_N!} \frac{B_{N-1}^{(k_{N-1}-k_N)}}{(k_{N-1}-k_N)!} \cdots \frac{B_0^{(k_0-k_1)}}{(k_0-k_1)!} \\
& \quad - 2 \sum_{\substack{j',j''\in\{0,\dots,N\}^2 \\ 1+j'+j''=i}} (\partial_x B_{j'}) A_{j''} + \alpha \sum_{\substack{j',j''\in\{0,\dots,N\}^2 \\ j'+j''=i}} A_{j'} A_{j''} \\
& \quad - \alpha \sum_{\substack{j',j''\in\{0,\dots,N\}^2 \\ 2+j'+j''=i}} (\partial_x B_{j'})(\partial_x B_{j''}) \Big).
\end{aligned}
$$

For $i = 0$ we recover the KdV equation (3.15). For $i > 0$ this is a linearised inhomogeneous KdV equation

$$\partial_t A_i = \gamma_i \partial_x^3 A_i + \kappa_i \partial_x (A_0 A_i) + f_i, \qquad A_i(0) \equiv 0,$$

where $\gamma_i, \kappa_i \in \mathbb{R}$ and $f_i = f_i(A_0, \ldots, A_{i-1})$. Since we only need pointwise estimates of order ϵ^8 for the residual, we define $A_3, \ldots, A_5 \equiv 0$. Further we choose decreasingly $b_0, b_1, b_2 \in]\frac{1}{2}, \frac{3}{4}[$. Then there is a function $A_0 \in X_{\sigma',0,b_0}$ for arbitrary $\sigma' < \sigma$ that is a mild solution of the KdV equation (3.15) ([40, Theorem 1]) and $\partial_t A_0 \in X_{\sigma',0,b_0-1}$.

The next step is to realise that f_i consists of products of (spatial derivatives of) A_j and maybe has a single addend consisting of a (spatial derivative of) $\partial_t A_j$ with $j < i$ for $i = 1, 2$ but not involving any products of these functions. Thus $f_i \in X_{\sigma',s,b_{i-1}-1} \subset X_{\sigma'',0,b_i-1}$ for $i = 1, 2$, some $s > 3b_i$ and $\sigma'' < \sigma' < \sigma$ by [40, Theorem 3] and Lemma 3.4.4 (recall the embeddings in Remark 1.5.2). Hence we can construct mild solutions to all the linearised KdV equations above with the aid of Lemma 3.4.4 and we obtain strict solutions in $C^1([0, t_1], G_{\sigma'''',s'})$ for some $\sigma'''' < \sigma'''$, $\frac{1}{2} < s'$, and $0 < t_1$ by Lemma 3.4.5. Therefore B_i is in the same space as A_i, which is a Banach algebra. Since we can eliminate all addends of the residual up to (pointwise) order ϵ^7 (included), we can estimate the residual by

$$\|\mathscr{F}\, \mathfrak{E}_{U_{**}}\|_{L^\infty([0,t_1/\epsilon^3],L^2_{0,\tilde{\sigma}/\epsilon})} \lesssim \epsilon^{8-\frac{1}{2}}$$

as claimed in the lemma thanks to the Banach algebra property. Obviously, it is always possible to choose $\tilde{\sigma}$ arbitrarily close to σ, thus the residual estimate follows. Finally,

$$\begin{aligned}
&\|U_{**} - U_*\|_{L^\infty([0,t_1/\epsilon^3],L^\infty)} \\
&\leq \sum_{i=1}^{N} \epsilon^{i+2} \|(A_i, B_i)\|_{L^\infty([0,t_1],L^\infty)} + \epsilon^2 \|\nu_1 \epsilon \partial_x A +^2 \nu_2 (\epsilon\partial_x)^2 A + \nu_3 (\epsilon A)^2\|_{L^\infty([0,t_1],L^\infty)} \\
&\lesssim \epsilon^3
\end{aligned}$$

follows because of $C([0, t_1], G_{\tilde{\sigma},s'}) \hookrightarrow C([0, t_1], L^\infty)$ for any $\tilde{\sigma} > 0$ and $A_0 = A$ as well as $B_0 = -\eta A_0$. The remaining estimate follows in the same manner. $\square$

***Remark* 3.4.6.** The advantage of Lemma 3.4.2 in comparison to [24, Lemma 7.10] is that we do not have to shrink σ with time with a constant rate. However, there is a drawback using this approach. We have a severe problem with the time derivatives and cannot easily have residual estimates of higher orders than $\epsilon^{8-\frac{1}{2}}$, since in this case there might be products of time derivatives and other functions in the inhomogeneous part f_i and $X_{\sigma,s,b}$ is in general no Banach algebra. Further, it is not clear how to possibly obtain estimates which are global in time if we use these spaces since we used a cut-off function in the proof (and this step was crucial).

The approach of Kato and Masuda in [56] has no problem with the time derivatives provided that they stay regular enough. This is easily guaranteed for sufficiently regular initial data. Further we observe that as long as the solution exists in H^∞ it is in $\mathrm{Hol}^2_{\sigma'}$ for some $\sigma' \leq \sigma$ and initial data in Hol_σ. But σ' is in general smaller than σ. This is the price to pay for the higher order residual estimates and, in that sense, similar to the procedure in [24]. However, no constant rate has to be introduced. Since it is well known that solutions to the KdV equation exist for sufficiently regular initial data globally in time, one could have t_1 as large as one wants to if the same holds for the linearised inhomogeneous KdV equation. The necessary modifications to the approach of Kato and Masuda for obtaining a result comparable to [24, Lemma 7.10] are outlined in Appendix 3.B. If one proved additionally that the inhomogeneous linearised KdV equation has strict solutions in H^∞ being global in time for sufficiently regular global inhomogeneous parts, then one could have t_1 as large as one wishes. However, we have to point out that we need solutions with values in $\mathrm{Hol}_{\sigma'}$ and not $\mathrm{Hol}^2_{\sigma'}$, which is in general not the same set, see Section 1.5.

3.5 The Error Estimates

We will prove the following theorem in this section.

Theorem 3.5.1. *Let U_{**} be the approximate solution to equation* (3.9) *of Section 3.4 and σ, t_1 as in Lemma 3.4.2. Let $\sigma' \in]0,\sigma[$, $s \in [7/4,\infty[$, $q \in [3,15/2]$ and $(\alpha,\beta) \in \mathcal{A}_s \backslash \mathcal{D}$. Finally, let U be a strict solution to system* (3.9) *in $\mathscr{F}^{-1} L^2_{s-3/4,\sigma'/\epsilon} \times L^2_{s-7/4,\sigma'/\epsilon}$ with initial data $\hat{U}(0) \in L^2_{s+5/4,\sigma'/\epsilon} \times L^2_{s+1/4,\sigma'/\epsilon}$.*

Then there exists an $\epsilon_0 > 0$ *such that for all* $\epsilon \in]0, \epsilon_0[$ *and initial data*

$$\|\hat{U}(0) - \hat{U}_{**}(0)\|_{L^2_{s+1,\sigma/\epsilon} \times L^2_{s,\sigma'/\epsilon}} \lesssim \epsilon^q \tag{3.18}$$

property (3.18) *is preserved in* H^s *for* $t \in [0, \Theta(\epsilon^{-3})]$*, i.e. there is* $t_2 \in]0, t_1]$ *such that*

$$\sup_{t \in [0, t_2/\epsilon^3]} \|U(t) - U_{**}(t)\|_{H^s} \lesssim \epsilon^q.$$

An immediate consequence of Lemma 3.4.2 is

Theorem 3.5.2. *Let* U_* *be the approximate solution constructed in Section 3.3 with aid of the KdV equation and* σ, t_1 *as in Lemma 3.4.2. Let* $\sigma' \in]0, \sigma[, s \in [7/4, \infty[$ *and* $(\alpha, \beta) \in \mathcal{A}_s \backslash \mathcal{D}$*. Finally, let* U *be a strict solution to system* (3.9) *in* $\mathscr{F}^{-1} L^2_{s-3/4,\sigma'/\epsilon} \times L^2_{s-7/4,\sigma'/\epsilon}$ *with initial data* $\hat{U}(0) \in L^2_{s+5/4,\sigma'/\epsilon} \times L^2_{s+1/4,\sigma'/\epsilon}$*. Then there exists an* $\epsilon_0 > 0$ *such that for all* $\epsilon \in]0, \epsilon_0[$ *and initial data*

$$\|\hat{U}(0) - \hat{U}_*(0)\|_{L^2_{s+1,\sigma'/\epsilon} \times L^2_{s,\sigma'/\epsilon}} \lesssim \epsilon^3 \tag{3.19}$$

the property (3.19) *is (almost) preserved in* H^s *for* $t \in [0, \Theta(\epsilon^{-3})]$*, i.e. there is* $t_2 \in]0, t_1]$ *such that*

$$\sup_{t \in [0, t_2/\epsilon^3]} \|U(t) - U_*(t)\|_{H^s} \lesssim \epsilon^{5/2}.$$

Proof. (Theorem 3.5.2) For each U_* we can construct U_{**} according to Lemma 3.4.2. Hence

$$\|\hat{U}(0) - \hat{U}_*(0)\|_{L^2_{s+1,\sigma'/\epsilon} \times L^2_{s,\sigma'/\epsilon}} = \|\hat{U}(0) - \hat{U}_{**}(0)\|_{L^2_{s+1,\sigma'/\epsilon} \times L^2_{s,\sigma'/\epsilon}} \lesssim \epsilon^3$$

and

$$\|\hat{U}_* - \hat{U}_{**}\|_{L^\infty([s,t_2/\epsilon^3], L^2_{0,\sigma'/\epsilon})} \lesssim \epsilon^{5/2}.$$

Then the rest follows from Theorem 3.5.1 and the triangle inequality and some embeddings. □

If we are satisfied with estimates in L^∞, we can avoid the loss in powers of ϵ:

Theorem 3.5.3. *Let U_* be the approximate solution constructed in Section 3.3 with aid of the KdV equation and σ, t_1 as in Lemma 3.4.2. Let $\sigma' \in]0, \sigma[, s \in [7/4, \infty[$ and $(\alpha, \beta) \in \mathcal{A}_s \backslash \mathcal{D}$. Finally, let U be a strict solution to system* (3.9) *in $\mathscr{F}^{-1} L^2_{s-3/4,\sigma'/\epsilon} \times L^2_{s-7/4,\sigma'/\epsilon}$ with initial data $\hat{U}(0) \in L^2_{s+5/4,\sigma'/\epsilon} \times L^2_{s+1/4,\sigma'/\epsilon}$. Then there exists an $\epsilon_0 > 0$ such that for all $\epsilon \in]0, \epsilon_0[$ and initial data*

$$\|\hat{U}(0) - \hat{U}_*(0)\|_{L^2_{s+1,\sigma'/\epsilon} \times L^2_{s,\sigma'/\epsilon}} \lesssim \epsilon^3 \tag{3.20}$$

the property (3.20) *is preserved in L^∞ for $t \in [0, \Theta(\epsilon^{-3})]$. More precisely, there is $t_2 \in]0, t_1]$ such that*

$$\sup_{t \in [0, t_2/\epsilon^3]} \|U(t) - U_*(t)\|_{L^\infty} \lesssim \epsilon^3.$$

Proof. The proof is the same as before for Theorem 3.5.2, we just notice that by Lemma 3.4.2 we obtain

$$\|U_* - U_{**}\|_{L^\infty([0,t_2/\epsilon^3], L^\infty)} \lesssim \epsilon^3.$$

□

***Remark* 3.5.4.** The condition $\|\hat{U}(0) - \hat{U}_{**}(0)\|_{L^2_{s+1,\sigma/\epsilon} \times L^2_{s,\sigma/\epsilon}}$ is somewhat annoying. If we want to remove the dependence on ϵ in the norm, then we need $\bigcap_{\sigma > 0} \|\hat{U}(0) - \hat{U}_{**}(0)\|_{L^2_{s+1,\sigma} \times L^2_{s,\sigma}} \lesssim \epsilon^q$. But this means that the support of the difference $\hat{U}(0) - \hat{U}_{**}(0)$ has to have measure 0. Hence $U(0) = U_{**}(0)$ (a.e.) is the only possibility.

3.5.1 Preparations

We consider equation (3.9) as a system in $H^1 \times L^2$. This is a consequence of the introduction of $\psi = \partial_x \phi$ since we decided to consider the Ginzburg-Landau system in $L^2 \times L^2$. We emphasise that the curves of eigenvalues of $\tilde{\mathcal{L}}(\mathrm{i}\xi)$, which is the Fourier transform of the linear part of the operator, remain the same as in the case of equation (3.3) and are in the scaling of van Harten given by

$$\lambda_\pm(\xi) = -\eta^{-1} - \xi^2 - (2\alpha - c)\mathrm{i}\xi \pm \eta^{-1}\sqrt{\nu(\xi)},$$

where $\gamma(\xi) = \eta(\alpha\xi^2 - \mathrm{i}\xi 2)$ as before and $\nu(\xi) = 1 - 2\beta\gamma(\xi) - \gamma(\xi)^2$. We note that the symbol $\tilde{\mathcal{L}}(\mathrm{i}\xi)$ is diagonalisable for all values of ξ since otherwise we would obtain $\alpha \neq 0$, $\eta = \frac{\alpha(1+\beta^2)}{4\beta} > 0$ and $\eta\xi^2 = -\frac{\beta}{\alpha} \geq 0$. Hence $\nu : \mathbb{R} \to \mathbb{C}^\times$ and there are two distinct eigenvectors of the symbol for all $\xi \in \mathbb{R}$ given by

$$\frac{1}{\sqrt{\nu(\xi)}} \begin{pmatrix} \tilde{\gamma}(\xi) \\ 1 \pm \sqrt{\nu(\xi)} \end{pmatrix},$$

where $\tilde{\gamma}(\xi) = \frac{\gamma(\xi)}{\mathrm{i}\xi} = -\eta(2+\mathrm{i}\alpha\xi)$. These give rise to the isomorphism $S : L^p_1 \times L^p \to L^p \times L^p$, $1 \leq p \leq \infty$, whose matrix representation is given by (in abuse of notation)

$$S^{-1} = \frac{1}{\sqrt{\nu}} \begin{pmatrix} \tilde{\gamma} & \tilde{\gamma} \\ 1 - \sqrt{\nu} & 1 + \sqrt{\nu} \end{pmatrix}$$

and

$$S = \frac{1}{2} \begin{pmatrix} -\frac{(1+\sqrt{\nu})}{\tilde{\gamma}} & 1 \\ \frac{(1-\sqrt{\nu})}{\tilde{\gamma}} & -1 \end{pmatrix} =: \begin{pmatrix} \tilde{s}_{11} & \tilde{s}_{12} \\ \mathrm{i}\xi\tilde{s}_{21} & \tilde{s}_{22} \end{pmatrix}.$$

Note that for $|\xi| \to 0$

$$1 - \sqrt{\nu(\xi)} = 1 - \sqrt{1 - 2\beta\gamma(\xi) - \gamma(\xi)^2} = -\mathrm{i}\xi\beta\tilde{\gamma}(\xi) + \mathrm{o}(|\xi|)$$

so that the operator defined by multiplication with $\tilde{s}_{21}$ is in $\mathrm{L}(L^p)$.

This isomorphism as a map $S : L^2_1 \times L^2 \to L^2 \times L^2$ diagonalises the Fourier transform of the linear part. The corresponding diagonal multiplication operator has the symbol $\Lambda = S\mathcal{L}(\mathrm{i}\xi)S^{-1} = \mathrm{diag}(\lambda_-, \lambda_+)$. Recall that we have the estimates, cf. Lemma 3.2.2,

$$\mathrm{Re}(\lambda_-(\xi)) \leq -\frac{1}{2\eta_{\mathrm{EB}}} - \xi^2, \qquad \mathrm{Re}(\lambda_+(\xi)) \lesssim \epsilon^3|\xi|$$

for the symbol. The linearisation of the nonlinear part in equation (3.9) around U_{**} will give another linear operator for the dynamics of the error. Despite being of lower order than $\mathcal{L}(\mathrm{i}\xi)$, recall that we consider a semilinear problem after all, this operator might make the error grow too quickly on the considered time scale. We will work in a time dependant scale of Banach spaces to circumvent this problem. This will cause some additional artificial decay. Therefore, we introduce for $s, s' \geq 0$ the transform $R_\kappa^{-1} : I_\kappa \times C^1(I, L^p_s \times L^p_{s'}) \to C^1(I_\kappa, L^p_{s,\sigma_t} \times L^p_{s',\sigma_t})$, $\sigma_t := (\sigma - \epsilon^3\kappa t)/\epsilon$, $1 \leq p \leq \infty$,

$$(t, u) \mapsto e^{-\sigma_t|\cdot|}u(t)$$

being obviously an isomorphism, too. Here $I_\kappa := [0, \frac{\sigma}{\epsilon^3\kappa}] \cap I$.
We abbreviate $Q_\kappa := SR_\kappa$. We transform $\hat{U} = (Q_\kappa)^{-1}V$ with the aid of these isomorphisms and obtain in $L^2 \times L^2$ the system

$$\partial_t V = (\Lambda - \epsilon^2\kappa|\boldsymbol{\xi}|)V + \mathcal{N}(V) \tag{3.21}$$

with $\Lambda = S\mathcal{L}(\mathrm{i}\xi)S^{-1} = \mathrm{diag}(\lambda_-, \lambda_+)$ and $([S, R_\kappa] = 0)$

$$\begin{aligned}\mathcal{N}(V) &= Q_\kappa \mathscr{F} \tilde{N}((Q_\kappa \mathscr{F})^{-1}V) \\ &= R_\kappa \begin{pmatrix} \tilde{s}_{11} \mathscr{F} \tilde{N}_1((Q_\kappa \mathscr{F})^{-1}V) + \tilde{s}_{12} \mathscr{F} \tilde{N}_2((Q_\kappa \mathscr{F})^{-1}V) \\ \mathrm{i}\xi\big(\tilde{s}_{21} \mathscr{F} \tilde{N}_1((Q_\kappa \mathscr{F})^{-1}V) + \tilde{s}_{22}(\mathrm{i}\xi)^{-1} \mathscr{F} \tilde{N}_2((Q_\kappa \mathscr{F})^{-1}V)\big)\end{pmatrix}.\end{aligned}$$

Here $\mathcal{N}_j$, $j = 1, 2$, is a composition of the Fourier transforms of entire functions and derivatives of them. The essential point is that the second component of $\mathcal{N}$ has an $\mathrm{i}\xi$ in front and actually no $(\mathrm{i}\xi)^{-1}$ exists in the whole expression since $\tilde{N}$ has a differential operator in front of its second component.

3.5.2 Estimates

Recall that we can satisfy the following two assumptions, which will be sufficient to prove the theorem. The first one concerns properties of the formal approximate solution and the residual. The second one is just a spectral estimate.

Hypothesis 3.5.5. *Let $I = [0, \frac{t_1}{\epsilon^3}]$, $\epsilon \in (0, 1], s' \geq \frac{7}{4}, \sigma > 0$ and $p \geq 3$. Suppose there is a function*

$$U_{**} = \epsilon^2 \begin{pmatrix} r_{**}(\epsilon^3\boldsymbol{t}, \epsilon\boldsymbol{x}) \\ \psi_{**}(\epsilon^3\boldsymbol{t}, \epsilon\boldsymbol{x}) \end{pmatrix}$$

for functions $r_{**}, \psi_{**} \in C([0, t_1], \mathrm{Hol}_\sigma)$ and $\mathfrak{E}_{U_{**}} \in C(I, \mathrm{Hol}_{\sigma/\epsilon})$,

$$\|\mathscr{F}\, \mathfrak{E}_{U_{**}}\|_{L^\infty\left(I, L^2_{s'+1,\sigma'/\epsilon}\right)} \lesssim \epsilon^{p+3},$$

for all $\sigma' < \sigma$ where

$$\mathfrak{E}_U := \big(\tilde{\mathcal{L}}(\partial_x) - \partial_t\big)U + \tilde{N}(U)$$

is the residual of a sufficiently regular function U for equation (3.9)*.*

This is in essence a summary of Lemma 3.4.2 where $p \in [3, 9/2]$.

***Remark* 3.5.6.** Obviously,

$$(\Lambda - \epsilon^2\kappa|\boldsymbol{\xi}| - \partial_t)V + \mathcal{N}(V) = Q_\kappa \mathscr{F}\left(\big(\tilde{\mathcal{L}}(\partial_x) - \partial_t\big)U + \tilde{N}(U)\right)$$

so that

$$\begin{aligned}\|(\Lambda - \epsilon^2\kappa|\boldsymbol{\xi}| - \partial_t)V(t) + \mathcal{N}(V)(t)\|_{L^2_s} &= \|Q_\kappa \mathscr{F}\,\mathfrak{E}_U(t)\|_{L^2_s} \lesssim \|R_\kappa \mathscr{F}\,\mathfrak{E}_U(t)\|_{L^2_{s+1}} \\ &\lesssim \|\mathscr{F}\,\mathfrak{E}_U(t)\|_{L^2_{s+1,\sigma/\epsilon}}\end{aligned}$$

for sufficiently regular U and $t \in I_\kappa$ defined in the previous subsection.

The following spectral assumptions are a consequence of the spectral estimate in Lemma 3.2.2 if we choose ϵ_0 sufficiently small and κ sufficiently large.

Hypothesis 3.5.7. *We assume that the symbol Λ is a diagonal matrix and that the real parts of the eigenvalues of Λ can be bounded by curves*

$$\begin{aligned}\mathrm{Re}(\lambda_- - \epsilon^2\kappa|\boldsymbol{\xi}|) &\leq -\frac{1}{2\eta_{\mathrm{EB}}} - \epsilon^2 C_1|\boldsymbol{\xi}| - C_2|\boldsymbol{\xi}|^2, \\ \mathrm{Re}(\lambda_+ - \epsilon^2\kappa|\boldsymbol{\xi}|) &\leq -\epsilon^2 C_1|\boldsymbol{\xi}|,\end{aligned}$$

$\eta_{\mathrm{EB}}, C_1, C_2 > 0$ *and independent of* $\epsilon \in (0, 1]$*.*

We start with the proof of Theorem 3.5.1 now. Obviously, it is sufficient to prove an error bound for system (3.21) in L^2_s for some $s > \frac{1}{2}$ (recall the embeddings $L^2_{s,\sigma} \hookrightarrow L^2_s$). Let σ be smaller than σ in Section 3.4 or in the estimate in Theorem 3.5.1. Then, by the embeddings, we can assume that we are working in $\bigcap_{s\geq 0} L^2_{s,\sigma}$. The trade-off might be that $t_2 \in]0, t_1[$ and not in the half closed interval.

As usual we define the error by $\epsilon^q R := V - V_{**}$ and $V_{**} = Q_\kappa \hat{U}_{**}$. Since we consider 'small' approximate solutions, we expect that the growth of the error will mainly come from the linear part of the operator and the linearisation of $\mathcal{N}$ about V_{**}. We can now easily prove the subsequent theorem. Theorem 3.5.1 is a trivial consequence of this theorem and of the observations made above that we can easily satisfy the assumptions of Hypothesis 3.5.5 and 3.5.7 for a small ϵ_0.

Theorem 3.5.8. *Suppose the assumptions of Hypotheses 3.5.5 and 3.5.7 are satisfied. Let* $V_{**} = Q_\kappa \hat{U}_{**}$, $s \in [\frac{7}{4}, s']$ *and* $q \in [3, p]$. *Let* V *be a strict solution to system* (3.21) *in* $L^2_{s-7/4}$ *for initial data* $V(0) \in L^2_{s+1/4}$.
Then there is an $\epsilon_0 > 0$ *such that for all* $\epsilon \in]0, \epsilon_0[$ *and initial data*

$$\|V(0) - V_{**}(0)\|_{L^2_s} \lesssim \epsilon^q \tag{3.22}$$

the property (3.22) *is preserved in time for* $t \in [0, \Theta(\epsilon^{-3})]$, *i.e. there is* $t_2 \in]0, t_1]$ *such that*

$$\sup_{t\in[0,t_2/\epsilon^3]} \|V(t) - V_{**}(t)\|_{L^2_s} \lesssim \epsilon^q.$$

Proof. As previously announced we write a solution to equation (3.21) as $V = V_{**} + \epsilon^q R$. Then the error function R satisfies

$$\begin{aligned}\partial_t R &= (\Lambda - \epsilon^2\kappa|\boldsymbol{\xi}|)R + \epsilon^{-q}\mathcal{N}(V_{**} + \epsilon^q R) + \epsilon^{-q}(\Lambda - \epsilon^2\kappa|\boldsymbol{\xi}| - \partial_t)V_{**} \\ &= (\Lambda - \epsilon^2\kappa|\boldsymbol{\xi}|)R + \epsilon^{-q}\mathcal{N}(V_{**} + \epsilon^q R) - \epsilon^{-q}\mathcal{N}(V_{**}) + \epsilon^{-q}\left(Q_\kappa \mathscr{F}\, \mathfrak{E}_V\right),\end{aligned}$$

see Remark 3.5.6, too.
We have to prove that the error R is in $L^\infty([0, t_2/\epsilon^3], L^2_s)$ for a $t_2 \in]0, t_1]$. For this purpose we use the energy

$$E_s = \frac{1}{2}\|R\|^2_{L^{2,b}_s}.$$

We know that $R \in C^1([0, \delta[, L^2_{s-7/4})$ for a (maximal) $\delta \in \mathbb{R}^+ \cup \{\infty\}$ by local existence and uniqueness results for the approximate solution and equation (3.9) (or the transformed equation (3.21) respectively). The first is clear from Lemma 3.4.2, the latter is clear by standard methods for parabolic semilinear problems, cf. Appendix 3.C. Further we note that $R \in C^1([0, \delta[, L^2_s)$, cf. [44, Theorem 3.5.2; 73, Chapter 6, Corollary 3.2]. Since we can use Proposition A.1.4 in combination with Lemma A.2.1 with $r = \frac{7}{4}$ and $s_0 \geq 0$ for V, we have $R \in C(]0, \delta[, L^2_{s+2})$, too.

If $\delta < \max I_\kappa$, it is clear that $\|V(t)\|_{L^2_s} \to \infty$ for $t \to \delta < \infty$ since V satisfies a mild solution formula in $C([0,\delta[, L^2_s)$. Consequently $E_s(t) \to \infty$ for $t \to \delta < \max I_\kappa$. Thus there is an ϵ-independent constant $M > 1$ satisfying $E_s(0) < M$ and

$$0 < T_2 := \sup\left\{r \in I_\kappa \,|\, E_s(r) < 3M\right\}.$$

We will show that T_2 must be of order $\Theta(\epsilon^{-3})$. This proves the claim.

Let $\tilde{\delta} \in]0, T_2/2[$ so small that $E_s(\tilde{\delta}) \le M$. The argument above shows that $E_s \in C^1([\tilde{\delta}, T_2])$ and we will see that a differential inequality of the form

$$\frac{d}{dt}E_s \lesssim \epsilon^3(1+E_s), \qquad t \in \left\{r \in I_\kappa \,\middle|\, r > \tilde{\delta} \text{ and } E_s(r) < 3M\right\}, \tag{3.23}$$

is satisfied for $\epsilon > 0$ sufficiently small (the constant will in general depend on M). Then we deduce via Gronwall's inequality

$$E_s(t) \le \left(M + \epsilon^3 Ct\right)e^{C\epsilon^3 t}, \qquad t \in \left\{r \in I_\kappa \,|\, E_s(r) < 3M\right\}. \tag{3.24}$$

Suppose $T_2 < \max I_\kappa$. Then inequality (3.24) implies

$$3M \le (M + \epsilon^3 CT_2)e^{C\epsilon^3 T_2},$$

which yields $T_2 \gtrsim \epsilon^{-3}$ for an ϵ-independent non-vanishing constant. Hence, a differential inequality of the form (3.23) yields the desired result.

The derivative of E_s in $\left\{r \in I_\kappa \,\middle|\, r > \tilde{\delta} \text{ and } E_s(r) < 3M\right\}$ is

$$\frac{d}{dt}E_s = \mathrm{Re}\langle R, (\Lambda - \epsilon^2\kappa|\boldsymbol{\xi}|)R + \epsilon^{-q}\left(\mathcal{N}(V_{**} + \epsilon^q R) - \mathcal{N}(V_{**}) + (Q_\kappa \mathscr{F}\,\mathfrak{E}_U)\right)\rangle_{L^{2,b}_s}.$$

Since Hypothesis 3.5.7 is satisfied the linear part $(\Lambda - \epsilon^2\kappa|\boldsymbol{\xi}|)R$ gives some damping terms if we choose $C_1 > 0$ (meaning κ large enough), which are

$$-(2\eta)^{-1}\|R_1\|^2_{L^{2,b}_s}, \qquad -\epsilon^2 C_1\||\xi|^{1/2}R\|^2_{L^{2,b}_s}, \qquad -C_2\||\xi|R_1\|^2_{L^{2,b}_s}.$$

We use these damping terms to derive the mentioned differential inequality

$$\frac{d}{dt}E_s \lesssim \epsilon^3(1+E_s).$$

We stress that there is no possibility to control the the L^2 norm of the second component R_2 with the damping terms. We will exploit the structure of the nonlinear part to compensate for this fact. Let us start to estimate the right hand side of $\frac{d}{dt}E_s$.

The most simple addend is the residual. By assumption, see Hypothesis 3.5.5 and Remark 3.5.6, we have

$$\mathrm{Re}\langle R, \epsilon^{-q}\left(Q_\kappa \mathscr{F}\,\mathfrak{E}_U\right)\rangle_{L^{2,b}_s} \lesssim \epsilon^{-q}\|R\|_{L^2_s}\|\mathscr{F}\,\mathfrak{E}_U\|_{L^2_{s+1,\sigma}} \lesssim \epsilon^{3+p-q}\|R\|_{L^2_s} \lesssim \epsilon^3(1+E_s),$$

where we applied Young's inequality and used the equivalence of the norms of $L^{2,b}_s$ and L^2_s for $s \geq 0$ in the last step.

Now the estimate of the difference of the nonlinear parts remains. We have to be more careful for these. As noted before, every component of $\tilde{N}$ is in essence a linear combination of entire functions and derivatives. Thus we can apply Proposition A.1.4 to every addend of $\mathcal{N}$, the transformed version of $\tilde{N}$. We obtain

$$\begin{aligned}\epsilon^{-q}\left(\mathcal{N}(V_{**}+\epsilon^q R)-\mathcal{N}(V_{**})\right) = {} & Q_\kappa D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R] \\ & + \epsilon^{-q}Q_\kappa G(Q_\kappa^{-1}V_{**}, \epsilon^q Q_\kappa^{-1}R),\end{aligned} \tag{3.25}$$

where G is the second order remainder of the Taylor expansion. By Proposition A.1.4 and application of the Bunyakovsky-Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
&\langle R, \epsilon^{-q}Q_\kappa G(Q_\kappa^{-1}V_{**}, \epsilon^q Q_\kappa^{-1}R)\rangle_{L^{2,b}_s} \\
=&\langle R, \epsilon^{-q}Q_\kappa G(Q_\kappa^{-1}V_{**}, \epsilon^q Q_\kappa^{-1}R)\rangle_{L^2} + \langle \xi^{s+1/2}R, \epsilon^{-q}\xi^{s-1/2}Q_\kappa G(Q_\kappa^{-1}V_{**}, \epsilon^q Q_\kappa^{-1}R)\rangle_{L^2} \\
\lesssim&\epsilon^{-q}\|R\|_{L^{2,b}_{s+1/2}}\|G(Q_\kappa^{-1}V_{**}, \epsilon^q Q_\kappa^{-1}R)\|_{L^2_{1,\sigma_t}\times L^2_{0,\sigma_t}} \\
&+ \epsilon^{-q}\|R\|_{L^{2,b}_{s+1/2}}\|G(Q_\kappa^{-1}V_{**}, \epsilon^q Q_\kappa^{-1}R)\|_{L^2_{s+1/2,\sigma_t}\times L^2_{s-1/2,\sigma_t}} \\
\lesssim&\epsilon^q\|R\|_{L^{2,b}_{s+1/2}}\|Q_\kappa^{-1}V_{**}\|_{L^1_{2,\sigma_t}\times L^1_{1,\sigma_t}}\|Q_\kappa^{-1}R\|_{L^2_{2,\sigma_t}\times L^2_{1,\sigma_t}}\|Q_\kappa^{-1}R\|_{L^1_{2,\sigma_t}\times L^1_{1,\sigma_t}} \\
&+ \epsilon^q\|R\|_{L^{2,b}_{s+1/2}}\|Q_\kappa^{-1}V_{**}\|_{L^1_{s+3/2,\sigma_t}\times L^1_{s+1/2,\sigma_t}}\|Q_\kappa^{-1}R\|_{L^2_{s+3/2,\sigma_t}\times L^2_{s+1/2,\sigma_t}}\|Q_\kappa^{-1}R\|_{L^1_{2,\sigma_t}\times L^1_{1,\sigma_t}} \\
\lesssim&\epsilon^q\|R\|_{L^{2,b}_{s+1/2}}\|V_{**}\|_{L^1_1}\|R\|^2_{L^2_s} + \epsilon^q\|R\|_{L^{2,b}_{s+1/2}}\|V_{**}\|_{L^1_{s+1/2}}\|R\|_{L^2_{s+1/2}}\|R\|_{L^2_s} \\
\lesssim&\epsilon^q\|R\|^2_{L^{2,b}_{s+1/2}}
\end{aligned}$$

In the antepenultimate step we exploited the fact that $\tilde{N}$ has the structure

$$\tilde{N}(r,\psi) = \begin{pmatrix} \tilde{n}_1(r,\partial_x r,\psi) \\ \partial_x \tilde{n}_2(r,\partial_x r,\psi) \end{pmatrix},$$

where $\tilde{n}_1, \tilde{n}_2$ are entire functions, and the embeddings of the $L^p_{s,\sigma}$ spaces. The hidden constant in this step depends on the entire functions given in Proposition A.1.4, which depend on

$$\|Q_\kappa^{-1} V_{**}\|_{L^1_{2,\sigma_t} \times L^1_{1,\sigma_t}} \lesssim \|V_{**}\|_{L^2_s}$$

and

$$\|Q_\kappa^{-1} R)\|_{L^1_{2,\sigma_t} \times L^1_{1,\sigma_t}} \lesssim \|R\|_{L^2_s}.$$

Since we consider $t \le T_2$ and $s \ge 1$, this expression and $\|R\|_{L^2_s}$ are bounded (with a bound depending on M). This is why the constant in the differential inequality will in general depend on M.

Finally, we have to estimate the linearisation. It is

$$\begin{aligned}\langle R, Q_\kappa D\tilde{N}(Q_\kappa^{-1} V_{**})[Q_\kappa^{-1} R]\rangle_{L^{2,b}_s} = {} & \langle R, Q_\kappa D\tilde{N}(Q_\kappa^{-1} V_{**})[Q_\kappa^{-1} R]\rangle_{L^2} \\ & + \langle R, Q_\kappa D\tilde{N}(Q_\kappa^{-1} V_{**})[Q_\kappa^{-1} R]\rangle_{L^{2,h}_s}.\end{aligned}$$

Let $Y : L^2_1 \times L^2 \to L^2 \times L^2$ be the multiplication operator that multiplies with $\begin{pmatrix} \mathrm{i}\boldsymbol{\xi} & 0 \\ 0 & 1 \end{pmatrix}$. We split the components in the scalar product. Then we find

$$\begin{aligned}
&\langle R_1, (Q_\kappa D\tilde{N}(Q_\kappa^{-1} V_{**})[Q_\kappa^{-1} R])_1\rangle_{L^{2,h}_s} \\
\le & \|R_1\|_{L^{2,h}_{s+1/2}} \|\big(Q_\kappa D\tilde{N}(Q_\kappa^{-1} V_{**})[Q_\kappa^{-1} R]\big)_1\|_{L^{2,h}_{s-1/2}} \\
\lesssim & \|R_1\|_{L^{2,h}_{s+1/2}} \|\xi^{s-1/2} D\tilde{N}(Q_\kappa^{-1} V_{**})[Q_\kappa^{-1} R]\|_{L^2_{1,\sigma_t} \times L^2_{0,\sigma_t}} \\
\lesssim & \|R_1\|_{L^{2,h}_{s+1/2}} \|\xi^{s-1/2} Q_\kappa^{-1} V_{**}\|_{L^1_{2,\sigma_t} \times L^1_{1,\sigma_t}} \|Q_\kappa^{-1} R\|_{L^2_{2,\sigma_t} \times L^2_{1,\sigma_t}} \\
& + \|R_1\|_{L^{2,h}_{s+1/2}} \|Q_\kappa^{-1} V_{**}\|_{L^1_{2,\sigma_t} \times L^1_{1,\sigma_t}} \|\xi^{s-1/2} Q_\kappa^{-1} R\|_{L^2_{2,\sigma_t} \times L^2_{1,\sigma_t}} \\
\lesssim & \|R_1\|_{L^{2,h}_{s+1/2}} \|\xi^{s-1/2} V_{**}\|_{L^1_1} \|R\|_{L^2_1} + \|R_1\|_{L^{2,h}_{s+1/2}} \|V_{**}\|_{L^1_1} \|\xi^{s-1/2} R\|_{L^2_1} \\
\lesssim & \epsilon^{s+3/2} \|R_1\|_{L^{2,h}_{s+1/2}} \|R\|_{L^2_1} + \epsilon^2 \|R_1\|_{L^{2,h}_{s+1/2}} \|R\|_{L^2_{s+1/2}}
\end{aligned}$$

$$\lesssim (\epsilon^{2s} + \epsilon)\|R_1\|^2_{L^{2,h}_{s+1/2}} + \epsilon^3 \|R\|^2_{L^2_{s+1/2}}.$$

Here we used the scaling invariance of the L^1 norm with respect to $u \mapsto \frac{1}{\epsilon}u(\frac{\cdot}{\epsilon})$, the assumptions on U_{**} and crucially that $n_1, n_2 \in \mathcal{O}(|z|^2)$ for $|z| \to 0$. We proceed similarly.

$$\begin{aligned}
\langle R_1, (Q_\kappa D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R])_1\rangle_{L^2} &\leq \|R_1\|_{L^2}\|\big(Q_\kappa D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R]\big)_1\|_{L^2} \\
&\lesssim \|R_1\|_{L^2}\|D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R]\|_{L^2_{1,\sigma_t}\times L^2_{0,\sigma_t}} \\
&\lesssim \|R_1\|_{L^2}\|Q_\kappa^{-1}V_{**}\|_{L^1_{2,\sigma_t}\times L^1_{1,\sigma_t}}\|Q_\kappa^{-1}R\|_{L^2_{2,\sigma_t}\times L^2_{1,\sigma_t}} \\
&\lesssim \|R_1\|_{L^2}\|V_{**}\|_{L^1_1}\|R\|_{L^2_1} \\
&\lesssim \epsilon^2\|R_1\|_{L^2}\|R\|_{L^2_1} \lesssim \epsilon\|R_1\|^2_{L^2} + \epsilon^3\|R\|^2_{L^2_1}.
\end{aligned}$$

Finally, we can estimate

$$\begin{aligned}
&\langle R_2, (Q_\kappa D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R])_2\rangle_{L^{2,h}_s} \\
\leq&\|R_2\|_{L^{2,h}_{s+1/2}}\|\big(Q_\kappa D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R]\big)_2\|_{L^{2,h}_{s-1/2}} \\
\lesssim&\|R_2\|_{L^{2,h}_{s+1/2}}\|\xi^{s-1/2}YD\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R]\|_{L^2_{0,\sigma_t}\times L^2_{0,\sigma_t}} \\
\lesssim&\|R_2\|_{L^{2,h}_{s+1/2}}\|\xi^{s+1/2}Q_\kappa^{-1}V_{**}\|_{L^1_{1,\sigma_t}\times L^1_{0,\sigma_t}}\|Q_\kappa^{-1}R\|_{L^2_{1,\sigma_t}\times L^2_{0,\sigma_t}} \\
&\quad + \|R_2\|_{L^{2,h}_{s+1/2}}\|Q_\kappa^{-1}V_{**}\|_{L^1_{1,\sigma_t}\times L^1_{0,\sigma_t}}\|\xi^{s+1/2}Q_\kappa^{-1}R\|_{L^2_{1,\sigma_t}\times L^2_{0,\sigma_t}} \\
\lesssim&\|R_2\|_{L^{2,h}_{s+1/2}}\|\xi^{s+1/2}V_{**}\|_{L^1}\|R\|_{L^2} + \|R_2\|_{L^{2,h}_{s+1/2}}\|V_{**}\|_{L^1}\|R\|_{L^{2,h}_{s+1/2}} \\
\lesssim&\epsilon^{s+5/2}\|R_2\|_{L^{2,h}_{s+1/2}}\|R\|_{L^2} + \epsilon^2\|R\|^2_{L^{2,h}_{s+1/2}} \\
\lesssim& \left(\epsilon^{2(s+1)} + \epsilon^2\right)\|R\|^2_{L^{2,h}_{s+1/2}} + \epsilon^3\|R\|^2_{L^2}
\end{aligned}$$

and

$$\begin{aligned}
&\langle R_2, (Q_\kappa D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R])_2\rangle_{L^2} \\
\leq&\|R_2\|_{L^{2,h}_{1/2}}\|(Q_\kappa D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R])_2\|_{L^{2,h}_{-1/2}} \\
\lesssim&\|R_2\|_{L^{2,h}_{1/2}}\|\xi^{-1/2}YD\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R]\|_{L^2_{0,\sigma_t}\times L^2_{0,\sigma_t}} \\
\lesssim&\|R_2\|_{L^{2,h}_{1/2}}\|\xi^{1/2}Q_\kappa^{-1}V_{**}\|_{L^1_{1,\sigma_t}\times L^1_{0,\sigma_t}}\|Q_\kappa^{-1}R\|_{L^2_{1,\sigma_t}\times L^2_{0,\sigma_t}} \\
&\quad + \|R_2\|_{L^{2,h}_{1/2}}\|Q_\kappa^{-1}V_{**}\|_{L^1_{1,\sigma_t}\times L^1_{0,\sigma_t}}\|\xi^{1/2}Q_\kappa^{-1}R\|_{L^2_{1,\sigma_t}\times L^2_{0,\sigma_t}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|R_2\|_{L^{2,h}_{1/2}} \|\xi^{1/2} V_{**}\|_{L^1} \|R\|_{L^2} + \|R_2\|_{L^2} \|V_{**}\|_{L^1} \|R\|_{L^{2,h}_{1/2}} \\
&\lesssim \epsilon^{5/2} \|R_2\|_{L^{2,h}_{1/2}} \|R\|_{L^2} + \epsilon^2 \|R\|^2_{L^{2,h}_{1/2}} \\
&\lesssim \epsilon^2 \|R\|^2_{L^{2,h}_{1/2}} + \epsilon^3 \|R\|^2_{L^2}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\operatorname{Re}\langle R, \epsilon^{-q}\mathcal{N}(V_{**} + \epsilon^q R) - \epsilon^{-q}\mathcal{N}(V_{**})\rangle_{L^{2,b}_s} \\
\le & C_{\mathrm{L}}\big(\epsilon \|R_1\|^2_{L^{2,b}_{s+1/2}} + \epsilon^2 \|\xi^{1/2} R\|^2_{L^{2,b}_s} + \epsilon^3 \|R\|^2_{L^{2,b}_{s+1/2}}\big) + C_{\mathrm{NL}} \epsilon^3 \|R\|^2_{L^{2,b}_{s+1/2}}.
\end{aligned}
$$

Since $\|\cdot\|^2_{L^{2,b}_{s+1/2}} \le \|\cdot\|^2_{L^{2,b}_s} + \|\xi^{1/2}\cdot\|^2_{L^{2,b}_s}$ and $\|\cdot\|^2_{L^{2,b}_{s+1/2}} \le \|\cdot\|^2_{L^{2,b}_s} + \|\xi\cdot\|^2_{L^{2,b}_s}$ we obtain the desired inequality

$$
\frac{d}{dt} E_s \lesssim \epsilon^3 (1 + E_s), \qquad t \in \left\{ r \in I_\kappa \,\middle|\, r > \tilde{\delta} \text{ and } E_s(r) < 3M \right\},
$$

if we require $\max\{C_{\mathrm{L}}, C_{\mathrm{NL}}\}\epsilon_0 = \min\{(2\eta)^{-1}, C_2, 1\}$ and $C_1 \ge C_L + 2$. □

3.6 Conclusions

In view of Theorem 3.5.1, 3.5.2 and 3.5.3 we have to notice that there are some unsatisfactory points. One of these is the question about the phase ϕ. We have a statement about $\partial_x \phi$ in the mentioned theorem only. We will shed some light on this point in Section 3.6.1.

Another issue is the strong regularity assumptions that we required in the theorems. This is at least annoying in applications and not satisfactory in the sense of a stability result. A result in the Sobolev space H^s, for some large enough s, would be more desirable since strict solutions to the KdV equation exist in these spaces, cf. [54], and existence and uniqueness theory for semilinear parabolic problems with constant coefficient differential operators is no challenge in Sobolev spaces. In fact, we only exploited the analyticity to obtain the artificial smoothing in equation (3.21). We used this smoothing to eliminate the effects of the spectral instability of the periodic travelling wave solutions and the (possible) growth of the error coming from the linearisation of the nonlinear part around the approximate solution in (3.25) in the proof of Theorem 3.5.8. But this was more convenience than need for the spectral instability since we estimated the linear growth stemming from the spectral instability in Lemma 3.2.2 by $\mathcal{O}(\epsilon^3)|\partial_x|$. This is not problematic on the natural time scale $\Theta(\epsilon^{-3})$ if we take into account that the symbol $\tilde{\mathcal{L}}(\mathrm{i}\xi)$ is asymptotically quadratically decaying for $|\xi| \to \infty$ (see Figure 3.2). More precisely, we can easily calculate that $\lambda_+(\xi) > 0$ only occurs for wave numbers ξ of the interval $[-\mathcal{O}(\epsilon), \mathcal{O}(\epsilon)]$. The real problem is the addend

$$\epsilon^2 \| |\boldsymbol{\xi}|^{1/2} R \|^2_{L^{2,b}_s}.$$

We note that we could split this term into

$$\epsilon^2 \| |\boldsymbol{\xi}|^{1/2} R \|^2_{L^{2,b}_s} \leq \epsilon^3 \| R \|^2_{L^{2,b}_s} + \epsilon \| |\boldsymbol{\xi}| R \|^2_{L^{2,b}_s}.$$

If we had suitable spectral estimates, e.g.

$$\mathrm{Re}(\lambda_+) \leq \epsilon^3 C_0 - \epsilon C_2 |\boldsymbol{\xi}|^2,$$

then we could easily repeat all calculations in the Sobolev space H^s and adapt the proofs for the residual estimate in Lemma 3.4.2 accordingly, see Appendix 3.B.1 and [54] for existence and uniqueness results in H^s for the (linearised) KdV equation. Such a spectral estimate seems to be unlikely, however, since for sufficiently small ϵ

$$\mathrm{Re}(\lambda_+)(\xi) = \mathcal{O}(\epsilon^4) + \mathcal{O}(\epsilon^3)(\xi - \epsilon) + \mathcal{O}(\epsilon^2)(\xi - \epsilon)^2 + \mathcal{O}(\epsilon^1)(\xi - \epsilon)^3 + \dots$$

whereas

$$\epsilon^2 \xi = \epsilon^3 + \epsilon^2(\xi - \epsilon)$$

in the norm above. Whilst the first addend is not problematic for $t \in [0, \Theta(\epsilon^{-3})]$ the second addend would have to be controlled by the damping terms coming from $\mathrm{Re}(\lambda_+)$. Therefore one might conjecture that such a result is not true in Sobolev spaces. A similar problem was observed in literature before, cf. [80].

Last but not least, we will quickly discuss the question what is happening in other areas of the (α, β)-plane. We already mentioned in the introduction that on the diagonal $\mathcal{D}$ the KdV equation cannot be derived. The reason is that in this case the right hand side of the formal KdV equation vanishes identically. Instead of the ansatz (3.10) one could use the ansatz

$$r_* = \epsilon^2 B(\epsilon^4 \boldsymbol{t}, \epsilon \boldsymbol{x}), \qquad \psi_* = \epsilon^2 A(\epsilon^4 \boldsymbol{t}, \epsilon \boldsymbol{x})$$

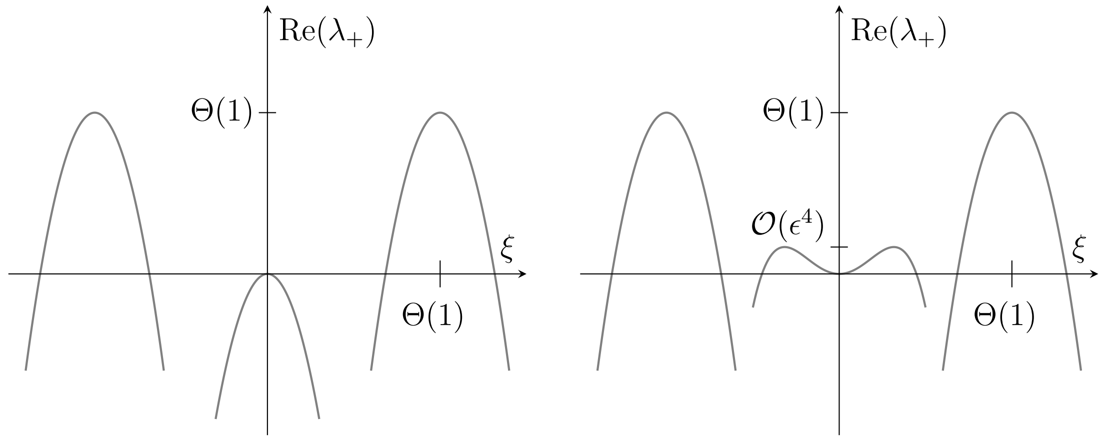
and derive a Cahn-Hilliard equation

$$\partial_t A = \nu_2 \partial_x^2 A + \nu_4 \partial_x^4 A + \nu_{non} \partial_x (A^2)$$

with real-valued coefficients ν_2, ν_{non} and $\nu_4 < 0$. An approximation result for this equation in the case $\alpha = \beta = 0$ has been established in [30]. The case $(\alpha, \beta) \in \mathcal{D} \backslash \{0\}$ seems to be open still and will be subject to future research. A possible starting point for this could be [29, 82].

Further we remark that for $(\alpha, \beta) \in \mathcal{A}_h$, where a instability of Hopf-Turing type occurs at the Eckhaus Boundary, a Ginzburg-Landau equation coupled to a Burgers equation is derived in [91]. Since the spectral situation and general setting is similar to the situation considered by Häcker, Schneider and Zimmermann in [42], the method used in their work might be adapted to proof an approximation result in this situation.

We close this section with one final remark. Assume $(\alpha, \beta) \in \mathcal{A}_h$ and $|\xi_0 - \xi_{\mathrm{BD}}| \lesssim \epsilon^2$, where $\xi_{\mathrm{BD}} := \frac{(1+\alpha\beta)}{2(1+\beta^2)+(1+\alpha\beta)}$. In general we can expect a spectral situation comparable to the one depicted in Figure 3.3(a) for $|\xi_0| < \xi_{\mathrm{BD}}$ and Figure 3.3(b) for $|\xi_0| > \xi_{\mathrm{BD}}$. We note that the derivation of the KdV equation is completely unaware of this fact. Hence, from a formal point of view, it is possible to derive the KdV equation in the same manner as in Section 3.3. The same is true for the ansatz and the residual in Section 3.4. Of course, one cannot expect that Lemma 3.2.2 is true in this situation. Thus, we think that no approximation result comparable to Theorem 3.5.8 or any of the other theorems in Section 3.5 can be proved in this situation. It would be interesting to find a counterexample and show that the operator $\tilde{\mathcal{L}}(\partial_x)$ leads to quick growth of the error in this setting. Since the complex Ginzburg-Landau equation is in essence a parabolic equation, methods that show growth for nonlinear heat equations could be applicable.



(a) Sketch of the spectral situation in $\mathcal{A}_h$ before a sideband instability occurs.

(b) Sketch of the spectral situation in $\mathcal{A}_h$ after a sideband instability occurred.

Figure 3.3: Unstable eigenvalue curve in the case of a Hopf-Turing unstable periodic travelling wave.

3.6.1 The Approximation Result in the Original Variables

We used the spatial derivative of the phase ϕ in all theorems. We can reconstruct the phase itself by integration in space

$$\phi(t,x) = \phi(t,x_0) + \int_{x_0}^{x} \psi(t,y)\ \mathrm{d}y$$

since all solutions are at least C^1 in time and space. Since we do not have L^1 estimates for ψ and have no good control of $\phi(t,x_0)$ we cannot expect global error estimates for ϕ on $\mathbb{R}$. This has been noted before, cf. [34, Corollary 2.2; 35, Section 4; 64, Section 4]. We can try a reconstruction of ϕ at least locally in a ball $B_R(x_0) \subset \mathbb{R}$, $R \in \Theta(\epsilon^{-\rho})$, for any fixed $\rho > 0$, up to the function $\phi(t,x_0)$ by

$$\begin{aligned}\Psi_*(t,x) &= \Psi_{\mathrm{per}}(t,x) e^{r_*(t,x-ct)+\mathrm{i}\int_{x_0}^{x-ct} \psi_*(t,y)\ \mathrm{d}y},\\ \Psi_{**}(t,x) &= \Psi_{\mathrm{per}}(t,x) e^{r_{**}(t,x-ct)+\mathrm{i}\int_{x_0}^{x-ct} \psi_{**}(t,y)\ \mathrm{d}y}.\end{aligned}$$

For a strict solution to the Ginzburg-Landau equation (3.1) we have

$$\Psi(t,x) = \Psi_{\mathrm{per}}(t,x) e^{r(t,x-ct)+\mathrm{i}\int_{x_0}^{x-ct} \psi(t,y)\ \mathrm{d}y + \mathrm{i}\phi(t,x_0)}.$$

We can estimate the difference, modulo the function $\phi(t, x_0)$, in $B_R(x_0)$ and $t \in \Theta(\epsilon^{-3})$, so that all solutions exist according to Theorems 3.5.1 and 3.5.2, by

$$\begin{aligned}
&|\Psi(t, x + ct)e^{-\mathrm{i}\phi(t,x_0)} - \Psi_*(t, x + ct)|_1 \\
\lesssim &|r(t, x) - r_*(t, x)| + \int_{x_0}^{x} |\psi(t, y) - \psi_*(t, y)| \, \mathrm{d}y \\
\lesssim &\epsilon^3(1 + |x - x_0|)
\end{aligned}$$

and

$$\begin{aligned}
&|\Psi(t, x + ct)e^{-\mathrm{i}\phi(t,x_0)} - \Psi_{**}(t, x + ct)|_1 \\
\lesssim &|r(t, x) - r_{**}(t, x)| + \int_{x_0}^{x} |\psi(t, y) - \psi_{**}(t, y)| \, \mathrm{d}y \\
\lesssim &\epsilon^q(1 + |x - x_0|).
\end{aligned}$$

This follows from the pointwise estimate in Theorem 3.5.3 and Theorem 3.5.1. The hidden constant does not depend on x, x_0 or t but on the size $|x - x_0|$. Since $\Psi, \Psi_*, \Psi_{**} \in \Theta(1)$ for $\epsilon \to 0$ the approximation results are at least true (modulo $\phi(t, x_0)$) in balls containing balls of size of the natural spatial scale $\Theta(\epsilon^{-1})$ of the KdV approximation. If we had L^1 estimates for $\psi - \psi_*$ or $\psi - \psi_{**}$, we would obtain global estimates. Therefore, it might make sense to consider function spaces where these can be controlled. But the use of this is somewhat limited since $\phi(t, x_0)$ is not really controlled.

Another point of view is as follows. Let (r, ϕ) a strict solution to (3.3) with initial data (r_0, ϕ_0) such that the conditions of Theorem 3.5.1 are met for some approximate solution (r_{**}, ψ_{**}) constructed according to Section 3.4. Then we can give a reconstruction

$$\Psi_{**}(t, x) = \Psi_{\mathrm{per}}(t, x)e^{r_{**}(t,x-ct)+\mathrm{i}\phi_0(x-ct)+\mathrm{i}\int_0^t \partial_s\phi_{**}(s,x-ct) \, \mathrm{d}s}$$

where we construct $\partial_s\phi_{**}$ from (r_{**}, ψ_{**}) according to (3.3). We can write for a strict solution Ψ to the Ginzburg-Landau equation (3.3)

$$\Psi(t, x) = \Psi_{\mathrm{per}}(t, x)e^{r(t,x-ct)+\mathrm{i}\phi_0(x-ct)+\mathrm{i}\int_0^t \partial_s\phi(s,x-ct) \, \mathrm{d}s}.$$

Thus we can estimate for sufficiently regular solutions, which is not a issue since our solutions are analytic in space and continuous in time,

$$\begin{aligned}\|\Psi(t) - \Psi_{**}(t)\|_{L^\infty} &\lesssim \|r(t) - r_{**}(t)\|_{L^\infty} + \int_0^t \|\partial_s \phi(s) - \partial_s \phi_{**}(s)\|_{L^\infty} \,\mathrm{d}s \\ &\lesssim \epsilon^q (1+t).\end{aligned}$$

This is true since the right hand side of equation (3.3) is a locally Lipschitz continuous map from $H^{s+2} \to H^s$ for $s \geq 0$ and Theorem 3.5.1 is applicable. However, note that the hidden constant depends on $\epsilon^q t$. But for $t \in \Theta(\epsilon^{-3})$, the natural time scale of (r_{**}, ψ_{**}), and $q > 3$ this might be a useful error bound.
We should point out that the assumptions on ϕ_0 are rather strong. We assume that ϕ_0 is defined on the strip $\mathcal{S}_{\sigma/\epsilon}$ and its derivative has to be in $\mathrm{Hol}_{\sigma/\epsilon}$ and sufficiently close to ψ_{**} in the sense of Theorem 3.5.1. A sufficient condition is $\hat{\psi}(0) \in L^2_{s,\sigma/\epsilon} \cap L^{2,b}_{-1}$ for sufficiently large σ, s. In this case we can define $\hat{\phi}_0 \coloneqq \frac{\psi(0)}{\mathrm{i}\boldsymbol{\xi}} \in L^2_{s+1,\sigma/\epsilon} \cap L^2$ and clearly $\partial_x \phi_0 \in \mathrm{Hol}_{\sigma/\epsilon}$.

Appendix

3.A Derivation of the Residual Equations

In the proof of Lemma 3.4.2 the details about the construction of the iterative scheme were missing. We will show them here. Let

$$r = \epsilon^2 \sum_{i=0}^{N} \epsilon^i B_i(\epsilon^3 \boldsymbol{t}, \epsilon \boldsymbol{x}), \qquad \psi = \epsilon^2 \sum_{i=0}^{N} \epsilon^i A_i(\epsilon^3 \boldsymbol{t}, \epsilon \boldsymbol{x})$$

be the ansatz and recall system (3.9)

$$\begin{aligned}
\partial_t r &= \partial_x^2 r - 2\psi - \alpha\left(\partial_x \psi + 2\partial_x r\right) + c\partial_x r - \eta^{-1}(e^{2r} - 1) + (\partial_x r)^2 - \psi^2 \\
&\quad - 2\alpha(\partial_x r)\psi \\
\partial_t \psi &= \partial_x \Big(\partial_x \psi + 2(\partial_x r) + \alpha\left(\partial_x^2 r - 2\psi\right) + c\psi - \beta\eta^{-1}(e^{2r} - 1) + 2(\partial_x r)\psi \\
&\quad + \alpha\left((\partial_x r)^2 - \psi^2\right)\Big).
\end{aligned}$$

By inserting the ansatz in the equations above, we obtain a system in powers of ϵ as follows:

$$\begin{aligned}
\sum_{i=0}^{N} \epsilon^{i+5} \partial_t B_i = \sum_{i=0}^{N} \epsilon^{i+2} \big(&\epsilon^2 \partial_x^2 B_i - 2A_i - \alpha\left(\epsilon \partial_x A_i + 2\epsilon \partial_x B_i\right) + c\epsilon \partial_x B_i \big) \\
&- \eta^{-1}\Big(\prod_{i=0}^{N} e^{2\epsilon^{2+i} B_i} - 1\Big) + \sum_{i,j=0}^{N} \epsilon^{6+i+j} (\partial_x B_i)(\partial_x B_j) \\
&- \sum_{i,j=0}^{N} \epsilon^{4+i+j} A_i A_j - 2\alpha \sum_{i,j=0}^{N} \epsilon^{5+i+j} (\partial_x B_i) A_j
\end{aligned} \tag{3.26}$$

$$
\begin{aligned}
\sum_{i=0}^{N}\epsilon^{i+5}\partial_t A_i = \sum_{i=0}^{N}\epsilon^{i+2}\big(&\epsilon^2\partial_x^2 A_i + 2\epsilon^2\partial_x^2 B_i + \alpha\epsilon^3\partial_x^3 B_i - (2\alpha - c)\epsilon\partial_x A_i\big)\\
&- \partial_x\Big(\epsilon\beta\eta^{-1}(\prod_{i=0}^{N} e^{2\epsilon^{2+i}B_i} - 1) - 2\sum_{i,j=0}^{N}\epsilon^{6+i+j}(\partial_x B_i)A_j \\
&- \alpha\sum_{i,j=0}^{N}\epsilon^{i+j+5}\left(\epsilon^2(\partial_x B_i)(\partial_x B_j) - A_i A_j\right)\Big).
\end{aligned}
\tag{3.27}
$$

Because of the identity, the Cauchy product formula,

$$
\sum_{k_0=0}^{\infty}\sum_{k_1=0}^{k_0}\cdots\sum_{k_n=0}^{k_{N-1}} a_{0,k_N}a_{1,k_{N-1}-k_n}\cdots a_{N,k_0-k_1} = \prod_{j=0}^{N}\left(\sum_{k_j=0}^{\infty} a_{j,k_j}\right)
$$

for pointwise bounded function $B_0, \dots, B_N$, which is no restriction since we seek solutions in $C([0,t_1], \mathrm{Hol}_\sigma)$ for $t_1, \sigma > 0$, we can expand the exponentials

$$
\begin{aligned}
&\prod_{i=0}^{N} e^{2\epsilon^{2+i}B_i} - 1\\
&= \sum_{k_0=0}^{\infty}\sum_{k_1=0}^{k_0}\cdots\sum_{k_N=0}^{k_{N-1}} \frac{\epsilon^{(N+2)k_N}(2B_N)^{k_N}}{k_N!}\frac{\epsilon^{(N+1)(k_{N-1}-k_N)}(2B_{N-1})^{(k_{N-1}-k_N)}}{(k_{N-1}-k_N)!}\cdots\\
&\qquad\qquad\cdots\frac{\epsilon^{2(k_0-k_1)}(2B_0)^{(k_0-k_1)}}{(k_0-k_1)!} - 1\\
&= \sum_{k_0=1}^{\infty} 2^{k_0}\sum_{k_1=0}^{k_0}\cdots\sum_{k_N=0}^{k_{N-1}} \epsilon^{k_0+\sum_{l=0}^{N}k_l}\frac{B_N^{k_N}}{k_N!}\frac{B_{N-1}^{(k_{N-1}-k_N)}}{(k_{N-1}-k_N)!}\cdots\frac{B_0^{(k_0-k_1)}}{(k_0-k_1)!}.
\end{aligned}
$$

We notice that we can split this expression in order ϵ^{i+2} into

$$
2B_i + \sum_{\substack{(k_0,\dots,k_N)\in\mathbb{N}\times\mathbb{N}_0^N\\ k_n\le k_{N-1}\le\dots\le k_1\le k_0\\ k_0+\sum_{l=0}^{N}k_l=i+2,k_i=0}} 2^{k_0}\frac{B_N^{k_N}}{k_N!}\frac{B_{N-1}^{(k_{N-1}-k_N)}}{(k_{N-1}-k_N)!}\cdots\frac{B_0^{(k_0-k_1)}}{(k_0-k_1)!},
$$

where the sum involves only addends with index lower than i.

Thus, we simply define B_i as solution to the problem in order ϵ^{i+2} of equation (3.26):

$$2B_i = -2\eta A_i - \eta\partial_x\left(\alpha A_{i-1} + 2\beta B_{i-1}\right) + \eta F_i(A_0, \dots, A_{i-2}, B_0 \dots, B_{i-2}),$$

where we use $c = 2(\alpha - \beta)$ and

$$\begin{aligned}
F_i &= \partial_x^2 B_{i-2} + \tilde{F}_i(A_0, \dots, A_{i-2}, B_0 \dots, B_{i-2}) \\
&= \partial_x^2 B_{i-2} - \partial_t B_{i-3} - \eta^{-1} \sum_{\substack{(k_0,\dots,k_N)\in\mathbb{N}\times\mathbb{N}_n^N \\ k_N\le k_{N-1}\le\dots\le k_1\le k_0 \\ k_0+\sum_{l=0}^N k_l=i+2, k_i=0}} 2^{k_0} \frac{B_N^{k_N}}{k_N!} \frac{B_{N-1}^{(k_{N-1}-k_N)}}{(k_{N-1}-k_N)!} \cdots \frac{B_0^{(k_0-k_1)}}{(k_0-k_1)!} \\
&\quad + \sum_{\substack{j',j''\in\{0,\dots,N\}^2 \\ 4+j'+j''=i}} (\partial_x B_{j'})(\partial_x B_{j''}) - \sum_{\substack{j',j''\in\{0,\dots,N\}^2 \\ 2+j'+j''=i}} A_{j'}A_{j''} \\
&\quad - 2\alpha \sum_{\substack{j',j''\in\{0,\dots,N\}^2 \\ 3+j'+j''=i}} (\partial_x B_{j'})A_{j''}.
\end{aligned}$$

Proceeding the same way for $\partial_t A_i$ in equation (3.27), we obtain

$$\begin{aligned}
\partial_t A_i = \partial_x^2(A_{i+1} + 2B_{i+1}) &+ \alpha\partial_x^3 B_i - 2\beta\partial_x A_{i+2} - 2\partial_x\beta\eta^{-1}B_{i+2} \\
&- \partial_x\Big(\beta\eta^{-1} \sum_{\substack{(k_0,\dots,k_N)\in\mathbb{N}\times\mathbb{N}_0^N \\ k_N\le k_{N-1}\le\dots\le k_1\le k_0 \\ k_0+\sum_{l=0}^N k_l=i+4, k_{i+2}=0}} 2^{k_0} \frac{B_N^{k_N}}{k_N!} \frac{B_{N-1}^{(k_{N-1}-k_N)}}{(k_{N-1}-k_N)!} \cdots \frac{B_0^{(k_0-k_1)}}{(k_0-k_1)!} \\
&- 2 \sum_{\substack{j',j''\in\{0,\dots,N\}^2 \\ 1+j'+j''=i}} (\partial_x B_{j'})A_{j''} + \alpha \sum_{\substack{j',j''\in\{0,\dots,N\}^2 \\ j'+j''=i}} A_{j'}A_{j''} \\
&- \alpha \sum_{\substack{j',j''\in\{0,\dots,N\}^2 \\ 2+j'+j''=i}} (\partial_x B_{j'})(\partial_x B_{j''})\Big).
\end{aligned}$$

by exploitation of $c = 2(\alpha - \beta)$ and by separation of the addend with the highest index in the sum of the exponential. With the definition of B_i above we obtain

$$\begin{aligned}\partial_t A_i = {} & (1+\alpha\beta - 2\eta(1+\beta^2))\partial_x^2 A_{i+1} + \partial_x^2 \beta^2 \eta F_{i+1} - \beta\partial_x \tilde{F}_{i+2} + \eta \partial_x^2 F_{i+1} \\ & + \partial_x^3(-\eta\alpha A_i - 2\eta\beta B_i + \alpha B_i - \beta^2\eta\left(\alpha A_i + 2\beta B_i\right) - \beta B_i) \\ & - \partial_x \Bigg(\beta\eta^{-1} \sum_{\substack{(k_0,\ldots,k_N)\in\mathbb{N}\times\mathbb{N}_0^N \\ k_N\le k_{N-1}\le\ldots\le k_1\le k_0 \\ k_0+\sum_{l=0}^N k_l = i+4, k_{i+2}=0}} 2^{k_0} \frac{B_N^{k_N}}{k_N!} \frac{B_{N-1}^{(k_{N-1}-k_N)}}{(k_{N-1}-k_N)!} \cdots \frac{B_0^{(k_0-k_1)}}{(k_0-k_1)!} \\ & \quad - 2 \sum_{\substack{j',j''\in\{0,\ldots,N\}^2 \\ 1+j'+j''=i}} (\partial_x B_{j'}) A_{j''} + \alpha \sum_{\substack{j',j''\in\{0,\ldots,N\}^2 \\ j'+j''=i}} A_{j'} A_{j''} \\ & \quad - \alpha \sum_{\substack{j',j''\in\{0,\ldots,N\}^2 \\ 2+j'+j''=i}} (\partial_x B_{j'})(\partial_x B_{j''}) \Bigg).\end{aligned}$$

Since the relaxed KdV condition is satisfied by assumption, we shift the addend with coefficient $(1+\alpha\beta-2\eta(1+\beta^2)) \in \mathcal{O}(\epsilon^2)$ by two orders and obtain the expression claimed in the proof of Lemma 3.4.2.

3.B Residual Estimates with the Method of Kato and Masuda

A similar result to Lemma 3.4.2 can be proved using the regularisation method of Kato and Masuda in [56]. In essence we proceed the same way as in the proof of Lemma 3.4.2 and obtain the same iterative scheme. In lowest order in this scheme, we obtain the well-know KdV equation of Section 3.3. [56, Theorem 2] states that there is a $\tilde{\sigma}' > 0$ for initial data in Hol_σ such that the solution to the KdV equation is in $C(I, \mathrm{Hol}^2_{\tilde{\sigma}'})$. The same is true for the time derivative of this solution if we choose $\sigma' < \tilde{\sigma}'$.

To be able to close the iterative scheme of the proof of Lemma 3.4.2, we only need to know that similar statements are true for the solutions to the linearised inhomogeneous KdV equations

$$\partial_t u + \rho \partial_x^3 u + \partial_x(a(t)u) = f(t)$$

for $a, f \in C(I, \mathrm{Hol}^2_{\sigma'})$ and $\rho \in \mathbb{R}^\times$. We follow the approach of Kato and Masuda for the proof of such statements, i.e. we will prove a existence and regularity result in H^s for the linearised KdV equation, see Lemma 3.B.1, and a regularity result for initial data and coefficients in Hol^2_σ, see Lemma 3.B.2.

For the existence result we introduce the abbreviations $X = \{u \in L^2(\mathbb{R}) : u = \overline{u}\}$, $Y = H^s \cap X$, $s \geq 3$, since we will consider real-valued solutions only.

Lemma 3.B.1. *Let $u_0 \in Y$ and $b, f \in C([0,\tilde{T}], Y)$, $a \in C([0,\tilde{T}], H^{s+1})$ real-valued for a $\tilde{T} > 0$.*

Then there is a unique solution $u \in C([0,T[, Y) \cap C^1([0,T[, H^{s-3} \cap X)$ to

$$\partial_t u + \rho \partial_x^3 u + \partial_x(a(t)u) + b(t)u = f(t) \tag{3.28}$$

for a $T \in]0,\tilde{T}]$.

Proof. We mimic the proof of Kato in [54]. Let $\tilde{I} = [0,\tilde{T}]$ for short. We define a unitary transform

$$P(t) = e^{-t\rho\partial_x^3} : X \to X$$

for every $t \in \mathbb{R}$. Now let $u(t) = P(t)v(t)$, which means that we use the interaction picture in the language of physicists. Then clearly

$$\partial_t v + A(t)v + B(t)v = F(t), \qquad\qquad v(0) = u(0) = u_0,$$

where $A(t) = P(-t)(\partial_x a(t) + a(t)\partial_x)P(t)$, $B(t) = P(-t)b(t)P(t)$ and the inhomogeneous part is $F(t) = P(-t)f(t)$.

Note that $P(t)\big|_Y : Y \to Y$ so that Y is an invariant subspace. We want to show that $A(t) + B(t) \in G(X, 1, \nu)$ for $t \in \tilde{I}$, i.e. that the operator is the generator of a strongly continuous quasi-contraction semigroup for every fixed $t \in \mathbb{R}$. Due to the fact that $P(t)$ is unitary, we just have to consider $\partial_x a(t) + a(t)\partial_x + b(t)$. We note that for $u \in Y$ and fixed $t \geq 0$ (we write $a = a(t)$ and $b = b(t)$)

$$\begin{aligned}\langle u, \partial_x(au) + bu\rangle_X &= -\langle a\partial_x u, u\rangle_X + \langle u, bu\rangle_X \\ &= \langle \big((\partial_x a) + b\big)u, u\rangle_X - \langle \partial_x(au), u\rangle_X.\end{aligned}$$

Since a, b and u are real-valued, we see

$$2\langle u, \partial_x(au) + bu\rangle_X = \langle \big(\big(\partial_x a) + 2b\big)u, u\rangle_X = -2\langle (a\partial_x - b)u, u\rangle_X.$$

Thus $2\nu \leq \|\partial_x a\|_{C(\tilde{I}, L^\infty(\mathbb{R}))} + \|b\|_{C(\tilde{I}, L^\infty(\mathbb{R}))}$. Clearly the operator is closable.
Note that $Y \hookrightarrow X$ densly and continuously. We define the map $S : Y \to X$, $u \mapsto \Lambda^s u$, $\Lambda = (1 - \partial_x^2)^{1/2}$. This is an isomorphism from Y onto X and clearly

$$[S, P(-t)(\partial_x a(t) + b(t))P(t)]S^{-1} = P(-t)\left([S, \partial_x a(t)] + [S, b(t)]\right) S^{-1} P(t),$$

where $[\cdot, \cdot]$ denotes the usual commutator of two operators, is a bounded operator in X since

$$\begin{aligned}\|\partial_x [S, a(t)] S^{-1}\| &= \|\partial_x \Lambda^{-1} \Lambda [S, a(t)] S^{-1}\| \leq c \|\partial_x a(t)\|_{H^s(\mathbb{R})}, \\ \|[S, b(t)] S^{-1}\| &\leq c \|\partial_x b(t)\|_{H^{s-1}(\mathbb{R})}\end{aligned}$$

according to [54, Lemma 2.6] and it is strongly continuous. Since $Y \subset D(A(t))$ for all $t \in \tilde{I}$ there is a unique solution to problem (3.28) by [53, Theorem 2] with the stated properties. □

Now we can give the analyticity result.

Lemma 3.B.2. *Assume there is a real-valued solution $u \in C(I, H^\infty)$ of*

$$\partial_t u = Au + \partial_x(a(t)u) + b(t)u + f(t)$$

where $A : D(A) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that $\langle v, Av\rangle_{H^s} \leq 0$ for every $s \geq 0$ and $v \in H^\infty \subset D(A)$, $a, b, f \in C(I, \mathrm{Hol}^2_\sigma)$ for a $\sigma > 0$.
If $u(0) = u_0 \in \mathrm{Hol}^2_\sigma$, then there is a $\sigma' \in]0, \sigma[$ such that $u \in C(I, \mathrm{Hol}^2_{\sigma'})$.

Proof. We follow section two of Kato and Masuda in [56]. In fact we just have to find a suitable substitute for [56, Lemma 2.3]. We use the same Lyapunov function

$$\Phi_{\tau,m}(v) = \frac{1}{2} \sum_{j=0}^{m} \frac{e^{2\tau j}}{(j!)^2} \|\partial_x^j v\|_{H^2}^2$$

as in the original proof. Thus, we have to estimate for $v \in H^{m+5}$ and the functions

$F(v) := Av + (a\partial_x v) + \tilde{b}v + f$, $\tilde{b} := \partial_x a + b$,

$$\langle F(v), D\Phi_{\tau,m}(v)\rangle = \sum_{j=0}^{m} \frac{e^{2\tau j}}{(j!)^2} \langle \partial_x^j v, \partial_x^j F(v)\rangle_{H^2}$$
$$\leq \sum_{j=0}^{m} \frac{e^{2\tau j}}{(j!)^2} \Big(\langle \partial_x^j v, \partial_x^j (a\partial_x v)\rangle_{H^2} + \langle \partial_x^j v, \partial_x^j (\tilde{b}v)\rangle_{H^2} + \langle \partial_x^j v, \partial_x^j f\rangle_{H^2} \Big).$$

Here $\langle \cdot, \cdot \rangle$ denotes the dual pairing and $D\Phi_{\tau,m}(v)$ is the Gâteaux derivative of $\Phi_{\tau,m}$ in v. This follows as in the proof of Lemma 2.3 in [56].

First we note that we have for real-valued, regular functions a, v (C denotes the constant of the embedding $H^1 \hookrightarrow L^\infty$)

$$\begin{aligned} 2\langle v, a\partial_x v\rangle_{H^2} &\leq \|\partial_x a\|_{L^\infty} \|v\|_{L^2}^2 + \|\partial_x a\|_{L^\infty} \|\partial_x v\|_{L^2}^2 + \|\partial_x^2 a\|_{L^2} \|\partial_x v\|_{L^\infty}^2 \\ &\quad + \big(\|\partial_x^2 a\|_{L^2} + 3\|\partial_x a\|_{L^\infty} \big) \|\partial_x^2 v\|_{L^2}^2 \\ &\leq 5C \|a\|_{H^2} \|v\|_{H^2}^2. \end{aligned}$$

Further we note that for $j = 0$

$$\langle v, a\partial_x v\rangle_{H^2} + \langle v, \tilde{b}v\rangle_{H^2} \leq \Big(\frac{5C}{2} \|a\|_{H^2} + \|\tilde{b}\|_{H^2} \Big) \|v\|_{H^2}^2.$$

Hence, we only have to consider $j \in \{1, \ldots, m\}$. We start with

$$\begin{aligned} &\langle \partial_x^j v, \partial_x^j (a\partial_x v)\rangle_{H^2} \\ =& \sum_{k=0}^{j} \binom{j}{k} \langle \partial_x^j v, \partial_x^k a \partial_x^{j+1-k} v\rangle_{H^2} \\ =& \sum_{k=1}^{j} \binom{j}{k} \langle \partial_x^j v, \partial_x^k a \partial_x^{j+1-k} v\rangle_{H^2} + \langle \partial_x^j v, a\partial_x^{j+1} v\rangle_{H^2} \\ \leq& \|\partial_x^j v\|_{H^2} C \sum_{k=1}^{j} \binom{j}{k} \|\partial_x^k a \partial_x^{j+1-k} v\|_{H^2} + \frac{5C}{2} \|a\|_{H^2} \|\partial_x^j v\|_{H^2}^2. \end{aligned}$$

We estimate the sum by

$$
\begin{aligned}
&\sum_{j=1}^{m}\sum_{k=1}^{j}\frac{e^{2j\tau}}{(j!)^2}\|\partial_x^j v\|_{H^2}\binom{j}{k}\|\partial_x^k a\partial_x^{j+1-k}v\|_{H^2}\\
\leq &C\sum_{j=1}^{m}\sum_{k=1}^{j}\frac{e^{j\tau}}{j!}\|\partial_x^j v\|_{H^2}e^{j\tau}\left(\frac{\|\partial_x^k a\|_{H^1}\|\partial_x^{j+1-k}v\|_{H^2}}{k!(j-k)!}+\frac{\|\partial_x^k a\|_{H^2}\|\partial_x^{j+1-k}v\|_{H^1}}{k!(j-k)!}\right)\\
\leq &C\sum_{j=1}^{m}\sum_{k=1}^{j}\frac{e^{j\tau}}{j!}\|\partial_x^j v\|_{H^2}e^{j\tau}\left(\frac{\|\partial_x^{k-1} a\|_{H^2}\|\partial_x^{j+1-k}v\|_{H^2}}{k!(j-k)!}+\frac{\|\partial_x^k a\|_{H^2}\|\partial_x^{j-k}v\|_{H^2}}{k!(j-k)!}\right)\\
&\leq 8C\left(\Phi_{\tau,m}(a)+\frac{\Phi_{\tilde\tau,m}(a)}{\tilde\tau-\tau}\right)\partial_\tau\Phi_{\tau,m}(v).
\end{aligned}
$$

The latter is a consequence of the following estimates for $a_k=\frac{e^{\tau k}}{k!}\|\partial_x^k a\|_{H^2}$ and $c_k=\frac{e^{\tau k}}{k!}\|\partial_x^k b\|_{H^2}$.

$$
\begin{aligned}
&\sum_{j=1}^{m}\sum_{k=1}^{j}c_j\left(\frac{a_{k-1}(j-k+1)c_{j+1-k}}{k}+a_kc_{j-k}\right)\\
\leq &\sum_{k=1}^{m}\sum_{j=k}^{m}c_j\left(\frac{a_{k-1}(j-k+1)c_{j+1-k}}{k}+\sqrt{\frac{j}{k}}a_kc_{j-k}\right)\\
\leq &\sum_{k=1}^{m}\Big(\frac{a_{k-1}\sum_{j=1}^{m-k+1}\sqrt{j+k-1}c_{j+k-1}\sqrt{j}c_j}{k}+\frac{\sqrt{k}}{k}a_kC\tilde C\Big)\\
\leq &\sum_{k=1}^{m}\Big(\frac{a_{k-1}\tilde C^2}{k}+\frac{\sqrt{k}}{k}a_kC\tilde C\Big)\leq 2A\tilde C^2+2\tilde AC\tilde C\leq 2(A+\tilde A)\tilde C^2,
\end{aligned}
$$

where $A^2=\sum_{k=1}^m a_k^2, C^2=\sum_{k=1}^m c_k^2, \tilde A^2=\sum_{k=1}^m ka_k^2, \tilde C^2=\sum_{k=1}^m kc_k^2$. Similarly we proceed for the other estimate:

$$
\begin{aligned}
&\sum_{j=1}^{m}\sum_{k=0}^{j}\frac{e^{2j\tau}}{(j!)^2}\|\partial_x^j v\|_{H^2}\binom{j}{k}\|\partial_x^k\tilde b\partial_x^{j-k}v\|_{H^2}\\
\leq &C\sum_{j=1}^{m}\sum_{k=1}^{j}\frac{e^{j\tau}}{j!}\|\partial_x^j v\|_{H^2}e^{j\tau}2\frac{\|\partial_x^k\tilde b\|_{H^2}\|\partial_x^{j-k}v\|_{H^2}}{k!(j-k)!}+2C\|\tilde b\|_{H^2}\sum_{j=1}^{m}e^{2j\tau}\frac{\|\partial_x^j v\|_{H^2}^2}{(j!)^2}\\
\leq &8C\frac{\Phi_{\tilde\tau,m}(\tilde b)}{\tilde\tau-\tau}\partial_\tau\Phi_{\tau,m}(v)+4C\Phi_{\tau,m}(\tilde b)\Phi_{\tau,m}(v),
\end{aligned}
$$

where we used the estimate

$$\sum_{j=1}^{m}\sum_{k=1}^{j} c_j b_k c_{j-k} \le \sum_{k=1}^{m}\sum_{j=k}^{m} c_j \sqrt{\frac{j}{k}} b_k c_{j-k} \le \sum_{k=1}^{m} \sqrt{\frac{1}{k}} b_k \tilde{C} C \le 2\tilde{B}\tilde{C}C \le 2\tilde{B}\tilde{C}^2$$

for $b_k = \frac{e^{\tau k}}{k!}\|\partial_x^k \tilde{b}\|_{H^2}$, c_k as above and $\tilde{B}^2 = \sum_{k=1}^m k b_k^2$.

Finally, the inhomogeneous part is easily estimate by

$$\sum_{j=0}^{m} \frac{e^{2\tau j}}{(j!)^2} \left|\langle \partial_x^j v, \partial_x^j f\rangle_2\right| \le \sum_{j=0}^{m} \frac{e^{2\tau j}}{(j!)^2} \|\partial_x^j v\|_{H^2} \|\partial_x^j f\|_{H^2} \le 2\sqrt{\Phi_{\tau,m}(v)\Phi_{\tau,m}(f)}.$$

Hence we have estimated

$$\begin{aligned}
\langle F(v), D\Phi_{\tau,m}(v)\rangle &\le \sum_{j=0}^{m} \frac{e^{2\tau j}}{(j!)^2} \|\partial_x^j v\|_{H^2} C \sum_{k=1}^{j} \binom{j}{k} \|\partial_x^k a \partial_x^{j+1-k} v\|_{H^2} \\
&\quad + \sum_{j=0}^{m} \frac{e^{2\tau j}}{(j!)^2} \frac{5C}{2} \|a\|_{H^2} \|\partial_x^j v\|_{H^2}^2 \\
&\quad + \sum_{j=0}^{m} \frac{e^{2\tau j}}{(j!)^2} \langle \partial_x^j v, \partial_x^j (\tilde{b} v)\rangle_2 + \sum_{j=0}^{m} \frac{e^{2\tau j}}{(j!)^2} \langle \partial_x^j v, \partial_x^j f\rangle_2 \\
&\le 8C(\Phi_{\tau,m}(a) + \frac{\Phi_{\tilde{\tau},m}(a)}{\tilde{\tau}-\tau})\partial_\tau \Phi_{\tau,m}(v) + 5C\|a\|_{H^2}\Phi_{\tau,m}(v) \\
&\quad + \big(5C\|a\|_{H^2} + 2C\|\tilde{b}\|_{H^2}\big)\Phi_{\tau,m}(v) + 8C\frac{\Phi_{\tilde{\tau},m}(\tilde{b})}{\tilde{\tau}-\tau}\partial_\tau \Phi_{\tau,m}(v) \\
&\quad + 4C\Phi_{\tau,m}(\tilde{b})\Phi_{\tau,m}(v) + 2\sqrt{\Phi_{\tau,m}(v)\Phi_{\tau,m}(f)} \\
&= C_1 \Phi_{\tau,m}(v) + C_2\sqrt{|\Phi_{\tau,m}(v)|} + C_3 \partial_\tau \Phi_{\tau,m}(v),
\end{aligned}$$

where C_1, C_2, C_3 are uniformly bounded in τ for $e^\tau \le e^{\overline{\tau}} < \tilde{\sigma}$. Thus $\alpha(r) = C_3$, $\beta(r) = C_1 r + C_2\sqrt{|r|}$ in the sense of the [56, Theorem 1] or the definition of the Lyapunov family for the problem. $\overline{r}$ can be chosen arbitrarily large, for instance in the same way as in the article of Kato and Masuda, cf. [56, equation (2.6)]. Then the result follows. □

3.C Existence and Uniqueness of Solutions to System (3.9)

We will establish that $\tilde{\mathcal{L}}(\mathrm{i}\xi)$, the symbol of the linear operator in equation (3.9), is a sectorial operator on $L^2_{1,\sigma} \times L^2_{0,\sigma}$, which yields local existence and uniqueness in $L^2_{1,\sigma} \times L^2_{0,\sigma}$ for the semilinear parabolic problem (3.9). This will be an immediate consequence of the following proposition.

Proposition 3.C.1. *Let $\mathcal{M} : D(\mathcal{M}) \subset L^2_{1,\sigma} \times L^2_{0,\sigma} \to L^2_{1,\sigma} \times L^2_{0,\sigma}$ with $D(\mathcal{M}) = L^2_{3,\sigma} \times L^2_{2,\sigma}$, where $\mathcal{M}$ is the multiplication operator acting by multiplication with the matrix-valued function*

$$\mathbb{R} \ni \xi \mapsto -\begin{pmatrix} \xi^2 & \mathrm{i}\alpha\xi \\ \mathrm{i}\alpha\xi^3 & \xi^2 \end{pmatrix} - 1 \in \mathrm{GL}(2, \mathbb{C}).$$

Then it holds

1. *$\mathcal{M}$ is a sectorial operator.*

2. *The part of $\mathcal{M}$ in $X_{7/8} := L^2_{11/4,\sigma} \times L^2_{7/4,\sigma} = D\left(M^{7/8}_{1+|\boldsymbol{\xi}|^2}\right)$ is sectorial in $X_{7/8}$.*

3. *$X_{7/8}$ is an intermediate space in the sense $D_{\mathcal{M}}(7/8, 1) \subset X_{7/8} \subset D_{\mathcal{M}}(7/8, \infty)$ (definition in Lunardi* [62, Section 2.2.1]*).*

Proof. First, we note that the graph norms of $\mathcal{M}$ and of the multiplication operator $M_{1+|\boldsymbol{\xi}|^2}$ acting by multiplication with $(1 + |\boldsymbol{\xi}|^2)$ in $L^2_{1,\sigma} \times L^2_{0,\sigma}$ are equivalent. This becomes obvious if we write $w = \mathcal{M}u$, note that $\mathcal{M}$ is an invertible operator, and that

$$-\left(1 + \frac{\alpha^2 \boldsymbol{\xi}^4}{(1+\boldsymbol{\xi}^2)^2}\right)(1 + |\boldsymbol{\xi}|^2)u = \begin{pmatrix} w_1 - \mathrm{i}\alpha\frac{\boldsymbol{\xi}}{(1+\boldsymbol{\xi}^2)} w_2 \\ w_2 - \mathrm{i}\alpha\frac{\boldsymbol{\xi}^3}{(1+\boldsymbol{\xi}^2)} w_1 \end{pmatrix}.$$

Hence, the domain of $\mathcal{M}$ is clear.
We can easily calculate the resolvent. Thus, the operator is sectorial since

$$\|\lambda R(\mathcal{M}, \lambda)\|_{\mathrm{L}(L^2_{1,\sigma} \times L^2_{0,\sigma})} \lesssim \left\|\lambda(\boldsymbol{\xi}^2 + 1 + \lambda)^{-1}\right\|_{L^\infty} < \infty$$

for all $\lambda \in \mathbb{C}^+ \cup \mathrm{i}\mathbb{R}$. Then sectoriality follows, cf. [62, Proposition 2.1.2]. The same argument holds for the second claim since the expression for the resolvent is the same and again

$$\|\lambda R(\mathcal{M},\lambda)\|_{\mathrm{L}(X_{7/8})} \lesssim \left\|\lambda(\boldsymbol{\xi}^2+1+\lambda)^{-1}\right\|_{L^\infty} < \infty.$$

For the last claim we note that the intermediate spaces, as defined by Lunardi, only depend on the domain and graph norms essentially, cf. [62, Corollary 2.2.3], which means

$$D_{\mathcal{M}}(7/8,1) = D_{M_{1+|\boldsymbol{\xi}|^2}}(7/8,1), \qquad D_{\mathcal{M}}(7/8,\infty) = D_{M_{1+|\boldsymbol{\xi}|^2}}(7/8,\infty).$$

Thus, we need a space such that

$$D_{M_{1+|\boldsymbol{\xi}|^2}}(7/8,1) \subset X_{7/8} \subset D_{M_{1+|\boldsymbol{\xi}|^2}}(7/8,\infty).$$

We know that

$$D_{M_{1+|\boldsymbol{\xi}|^2}}(7/8,1) \subset D(M_{1+|\boldsymbol{\xi}|^2}^{7/8}) \subset D_{M_{1+|\boldsymbol{\xi}|^2}}(7/8,\infty)$$

for the operator $M_{1+|\boldsymbol{\xi}|^2}$ in $L^2_{1,\sigma} \times L^2_{0,\sigma}$, cf. [62, Proposition 2.2.15], which proves the claim. $\square$

This immediately yields the corollary

Corollary 3.C.2. *The operator* $\tilde{\mathcal{L}}(\mathrm{i}\boldsymbol{\xi})\colon D\big(\tilde{\mathcal{L}}(\mathrm{i}\boldsymbol{\xi})\big) \subset L^2_{1,\sigma} \times L^2_{0,\sigma} \to L^2_{1,\sigma} \times L^2_{0,\sigma}$ *is sectorial with domain* $D\big(\tilde{\mathcal{L}}(\mathrm{i}\boldsymbol{\xi})\big) = D(\mathcal{M})$.

Proof. $\tilde{\mathcal{L}}(\mathrm{i}\boldsymbol{\xi}) - \mathcal{M}$ is (infintesimally) relatively $\mathcal{M}$-bounded, i.e.

$$\|(\tilde{\mathcal{L}}(\mathrm{i}\boldsymbol{\xi}) - \mathcal{M})u\|_{L^2_{1,\sigma}\times L^2_{0,\sigma}} \leq \delta\|\mathcal{M}u\|_{L^2_{1,\sigma}\times L^2_{0,\sigma}} + C\|u\|_{L^2_{1,\sigma}\times L^2_{0,\sigma}}$$

for all $u \in D(\mathcal{M})$, all $\delta > 0$ and a constant $C > 0$ depending on δ^{-1} ($\mathcal{M}$ is just constructed this way). Thus, $D(\mathcal{M}) = D(\tilde{\mathcal{L}}(\mathrm{i}\boldsymbol{\xi}))$ and $\tilde{\mathcal{L}}(\mathrm{i}\boldsymbol{\xi})$ is the generator of an analytic semigroup in $L^2_{1,\sigma} \times L^2_{0,\sigma}$, hence sectorial, cf. [37, Theorem III.2.10]. $\square$

Existence, uniqueness, and regularity of solutions to the Fourier transform of equation (3.9) are now given by the following proposition.

Proposition 3.C.3. *Let $\hat{U}_0 \in L^2_{3,\sigma} \times L^2_{2,\sigma}$. There exists a maximal $T_0 \in \mathbb{R}^+ \cup \{\infty\}$ such that the Fourier transform of equation* (3.9) *has a strict solution*

$$\hat{U} \in C([0, T_0[, L^2_{3,\sigma} \times L^2_{2,\sigma}) \cap C^1([0, T_0[, L^2_{1,\sigma} \times L^2_{0,\sigma})$$

with initial condition $\hat{U}(0) = \hat{U}_0$. If $T_0 < \infty$, then $\|\hat{U}(t)\|_{L^2_{11/4,\sigma} \times L^2_{7/4,\sigma}}$ must blow up as $t \to T_0$.

Proof. Let $\hat{U}$ be the Fourier transform of U and

$$F := \hat{U} \mapsto (\tilde{\mathcal{L}}(\mathrm{i}\xi) - \mathcal{M})\hat{U} + \mathscr{F}\,\tilde{N}(\mathscr{F}^{-1}\,\hat{U}).$$

We know that $F : X_{7/8} \to L^2_{1,\sigma} \times L^2_{0,\sigma}$ by Proposition A.1.4 and F is Lipschitz on bounded subsets of $X_{7/8}$.
Further $\hat{U}_0 \in D(\mathcal{M}) = D(\tilde{\mathcal{L}}(\mathrm{i}\boldsymbol{\xi}))$, as stated in the previous corollary, and $\mathcal{M}\hat{U}_0 + F(\hat{U}_0) \in \overline{D(\tilde{\mathcal{L}}(\mathrm{i}\boldsymbol{\xi}))} = L^2_{1,\sigma} \times L^2_{0,\sigma}$. Then existence, uniqueness and regularity of the solution are granted by [62, Theorem 7.1.2] in combination with [62, Proposition 7.1.10(iii)].
If $T_0 < \infty$, then $\limsup_{t \to T_0} \|U(t)\|_{X_{7/8}} = \infty$ according to [62, Proposition 7.1.8] and the remark following this proposition since F maps bounded subsets of $X_{7/8}$ into bounded subsets of $L^2_{1,\sigma} \times L^2_{0,\sigma}$. □

***Remark* 3.C.4.** Obviously the statements above remain valid for all $s \geq 0$ if $L^2_{1,\sigma} \times L^2_{0,\sigma}$ is substituted by $L^2_{s+1,\sigma} \times L^2_{s,\sigma}$ and the domains adapted.

Corollary 3.C.5. *Let $U_0 \in \mathrm{Hol}_\sigma$. There exists a maximal $T_0 = T_0(\sigma) \in \mathbb{R}^+ \cup \{\infty\}$ such that* (3.9) *has a solution $U \in C([0, T_0[, \mathrm{Hol}_\sigma) \cap C^1([0, T_0[, \mathrm{Hol}_\sigma)$ with initial condition $U(0) = \hat{U}_0$. If $T_0 < \infty$, then $\|\hat{U}(t)\|_{L^2_{11/4,\sigma'} \times L^2_{7/4,\sigma'}}$ must blow up for at least one $\sigma' \in [0, \sigma[$ as $t \to T_0$.*

Note that in general T_0 depends on σ' and $T_0(\sigma') \to 0$ for $\sigma' \to \sigma$ may be expected. This ambiguity is an unfortunate consequence of our 'definition' of $C(I, \mathrm{Hol}_\sigma)$.

Proof. For all $\sigma' \in [0, \sigma[$ the arguments of Proposition 3.C.3 apply. □

3.D Spectral Stability

Spectral stability is an essential feature of linear or linearised nonlinear problems. Spectral stability of the solutions Ψ_{per} to the complex Ginzburg-Landau equation (3.1) was completely treated by van Harten in [91]. Nonetheless, we will discuss it here again with some more details for completeness. Again we consider the curves of Section 3.2.2

$$\lambda_{\pm}(\xi) = -A_0^2 - \xi^2 - (2\alpha\xi_0 - c)\mathrm{i}\xi \pm A_0^2\sqrt{1 - 2\beta\gamma(\xi) - \gamma(\xi)^2}$$

and are interest in the question of spectral stability by which we mean $\operatorname{Re}\lambda_{\pm} \leq 0$. Recall that only λ_+ is of interest as noted in Section 3.2.2. Thus, we consider

$$\begin{aligned}
& & -A_0^2 - \xi^2 + A_0^2\operatorname{Re}\sqrt{\nu(\xi)} &\leq 0 \\
&\Leftrightarrow & 0 \leq A_0^2\operatorname{Re}\sqrt{\nu(\xi)} &\leq A_0^2 + \xi^2 \\
&\Leftrightarrow & 0 \leq A_0^4\left(\operatorname{Re}\sqrt{\nu(\xi)}\right)^2 &\leq \left(A_0^2 + \xi^2\right)^2 \\
&\Leftrightarrow & A_0^4(|\nu(\xi)| + \operatorname{Re}\nu(\xi)) &\leq 2\left(A_0^2 + \xi^2\right)^2,
\end{aligned}$$

where we introduced $\nu \coloneqq 1 - 2\beta\gamma - \gamma^2$ for short. More explicitly, this means

$$\begin{aligned}
& & A_0^8|\nu(\xi)|^2 &\leq \left(2\left(A_0^2 + \xi^2\right)^2 - A_0^4\operatorname{Re}\nu(\xi)\right)^2 \\
&\Leftrightarrow & A_0^8(\operatorname{Im}\nu(\xi))^2 &\leq 4\left(A_0^2 + \xi^2\right)^4 - 4A_0^4\left(A_0^2 + \xi^2\right)^2\operatorname{Re}\nu(\xi) \\
&\Leftrightarrow & 4A_0^4\xi_0^2(\beta\xi + \alpha\xi^3)^2 &\leq \left(A_0^2 + \xi^2\right)^4 - \left(A_0^2 + \xi^2\right)^2\left(A_0^4 - 2\beta A_0^2\alpha\xi^2 - \alpha^2\xi^4 + \xi^2 4\xi_0^2\right) \\
&\Leftrightarrow & 0 \leq P(\xi) &\coloneqq \sum_{k=1}^{4} C_{2k}\xi^{2k},
\end{aligned}$$

where P is the same polynomial as derived by van Harten in [91] and the coefficients are as follows (recall that the scaling is different from the one of van Harten)

$$\begin{aligned}
C_2 &= 2A_0^6\left(1 + \alpha\beta - 2\eta(1 + \beta^2)\right) \\
C_4 &= A_0^4\left(5 + \alpha^2 + 4\alpha\beta - 8\eta(1 + \alpha\beta)\right) \\
C_6 &= 2A_0^2\left(2 + \alpha^2 + \alpha\beta - 2\eta(1 + \alpha^2)\right) \\
C_8 &= 1 + \alpha^2.
\end{aligned}$$

We observe that there is no spectrally stable solution Ψ_{per} for $1 + \alpha\beta < 0$. There is only one spectrally stable solution with $\xi_0 = 0$ for $1 + \alpha\beta = 0$. More spectrally stable solutions may exist for $1 + \alpha\beta > 0$. This criterion is known as 'Benjamin-

Feir' stability. Note that all coefficients are strictly monotonically decreasing in η for $1+\alpha\beta > 0$. Hence, there is an upper bound to $|\xi_0|$ such that up to this bound spectral stability is obtained and no spectral stability exists beyond this bound.

Next, we consider what happens at the Eckhaus Boundary if we have a pure sideband instability. We proceed in the same way van Harten did, i.e. we evaluate $C_2, \ldots, C_8$ for $\eta = \eta_{\text{EB}}$:

$$\begin{aligned}
C_2 &= 0 \\
C_4 &= \frac{A_0^4}{1+\beta^2}\left(1 + (\alpha - 2\beta)^2 + \beta^2(1 + 4\alpha\beta - 3\alpha^2)\right) \\
C_6 &= \frac{2A_0^2}{1+\beta^2}\left(1 + 2\beta^2 + \alpha^2\beta^2 + \alpha\beta(\beta^2 - \alpha^2)\right) \\
C_8 &= 1 + \alpha^2.
\end{aligned}$$

Since the coefficients are invariant under the map $(\alpha, \beta) \mapsto -(\alpha, \beta)$, we restrict the next considerations to the half-plane $\{(\alpha, \beta) \in \mathbb{R}^2 \,|\, \beta \geq 0\}$. In view of the previous calculations, it is necessary and sufficient that $C_4 \geq 0$ and either $C_6 \geq 0$ or $C_6^2 - 4C_4C_8 \leq 0$ for $P \geq 0$. Because of the condition $1+\alpha\beta > 0$, which is necessary for spectral stability, we notice that $C_4, C_6, C_8 > 0$ in $\{(\alpha, \beta) \in \mathbb{R}^2 \,|\, \beta \geq 0 \geq \alpha \wedge 1 + \alpha\beta > 0\}$ since

$$\begin{aligned}
1 + (\alpha - 2\beta)^2 + \beta^2(1 + 4\alpha\beta - 3\alpha^2) &> 1 + (\alpha - 2\beta)^2 - 3\beta^2 + 3\alpha\beta \\
&= 1 + (\alpha - \beta/2)^2 + \frac{3}{4}\beta^2, \\
1 + 2\beta^2 + \alpha^2\beta^2 + \alpha\beta(\beta^2 - \alpha^2) &> 1 + \beta^2 + \alpha^2\beta^2 - \alpha\beta\alpha^2 > 1 + \beta^2 + \alpha^2\beta^2.
\end{aligned}$$

The situation is more complicated in the region

$$\left\{(\alpha, \beta) \in \mathbb{R}^2 \,\middle|\, \beta, \alpha > 0 \wedge 1 + \alpha\beta > 0\right\}.$$

As a first step, we note that $C_4, C_6, C_8 > 0$ in $\{(\alpha, \beta) \in \mathbb{R}^2 \,|\, \beta \geq \alpha > 0\}$. The case $0 < \beta < \alpha$ is more difficult. Apparently, $C_4, C_6 > 0$ for α, β small enough but in general there will be an upper bound $B(\beta/\alpha) > \alpha$, the function r in the article of van Harten, such that the necessary conditions are only satisfied below that bound. This becomes more clear when we consider the coefficients at the boundary on the 'rays' $\beta = \kappa\alpha$, $\kappa \in]0,1[$. We see that

$$\frac{1+\kappa^2\alpha^2}{A_0^4}C_4 = 1 + (1-2\kappa)^2\alpha^2 + \big(1 + (4\kappa-3)\alpha^2\big)\kappa^2\alpha^2,$$
$$\frac{1+\kappa^2\alpha^2}{2A_0^2}C_6 = 1 + 2\kappa^2\alpha^2 + \Big((\kappa+\frac{1}{2})^2 - \frac{5}{4}\Big)\alpha^4\kappa.$$

We notice the following points.

- For all $\alpha \in \mathbb{R}^+$ and $\kappa \in [\frac{3}{4}, 1[$, the coefficient C_4 is non-negative. The coefficient C_6 is non-negative for all $\alpha \in \mathbb{R}^+$ and $\kappa \in [\frac{\sqrt{5}-1}{2}, 1[$.
- For $\kappa \in [\frac{1}{3}, 1[$ we have the inequalities

  $$(1-2\kappa)^2 \le \kappa^2, \qquad\qquad (3\kappa-4)\kappa \le -1.$$

  Hence it follows $0 \le C_4 \le C_6$ for these values of κ by comparison of orders of α if we presume the necessary condition $C_4 \ge 0$. Thus, it is necessary and sufficient that $C_4 \ge 0$ in this region.
- Note that the above two conditions are equivalent. Therefore it holds

  $$1 + 2\kappa^2\alpha^2 + \Big((\kappa+\frac{1}{2})^2 - \frac{5}{4}\Big)\alpha^4\kappa \le 1 + (1-2\kappa)^2\alpha^2 + \big(1 + (4\kappa-3)\alpha^2\big)\kappa^2\alpha^2$$

  for $\kappa \in]0, \frac{1}{3}[$. Thus $C_6 \ge 0 \Rightarrow C_4 \ge 0$. Unfortunately, $C_6 \le 0$ is possible for sufficiently large α^2. This can occur for $\kappa > 0$ if

  $$\alpha^2 \ge \frac{\sqrt{\kappa^3 + 1 - \kappa - \kappa^2} + \sqrt{\kappa}\kappa}{(1-\kappa-\kappa^2)\sqrt{\kappa}}.$$

We note that the aforementioned bound B given by van Harten is

$$B(\kappa) = \begin{cases} \kappa^{-1/2} & \kappa \in]0, \frac{1}{3}] \\ \sqrt{\frac{(2\kappa-1)^2+\kappa^2+\sqrt{(2\kappa-1)^4+2(2\kappa-1)^2\kappa^2+\kappa^4-4\kappa^2(4\kappa-3)}}{-2\kappa^2(4\kappa-3)}} & \kappa \in]\frac{1}{3}, \frac{3}{4}[\\ \infty & \kappa \in [\frac{3}{4}, 1[\end{cases}.$$

The middle bound is just the (positive) root of

$$1 + (1-2\kappa)^2\alpha^2 + \big(1 + (4\kappa-3)\alpha^2\big)\kappa^2\alpha^2,$$

i.e. it determines when C_4 changes its sign. The region where the bound is infinity results from the fact that in this area $C_4, C_6, C_8 > 0$ for all (α, β). The last region is more difficult to understand and we refrain from further explanations.
The conclusion of this discussion is as follows. If η is in a small neighbourhood of η_{EB} and below that limit, we have spectral stability. If we cross that limit C_2 changes its sign but C_4, C_6, C_8 do not change their sign as long as η is not too large yet. Hence, we have a pure sideband instability in that case.

3.E Proof of Lemma 3.4.4

The proof of Lemma 3.4.4 can be given by a standard contraction mapping argument in a closed ball of the space $X_{\sigma,s,b}$. We follow the proof in [40].

Proof. (Lemma 3.4.4). We mimic the proof of [40, Lemma 5]. Let $\varphi \in C_c^\infty(\mathbb{R}, \mathbb{R})$ with

$$\varphi(x) = 1 \text{ for } |x| \leq 1, \qquad\qquad \varphi(x) = 0 \text{ for } |x| \geq 2$$

and $\varphi_T(t) = \varphi(t/T)$.
Let $G_{\sigma,s} = \{u \in L^2 \mid \hat{u} \in L^2_{s,\sigma}\}$. Let W be the group generated by ∂_x^3 in $G_{\sigma,s}$. Then we have the estimates

$$\begin{aligned} \|\varphi W u_0\|_{X_{\sigma,s,b}} &\leq c\|u_0\|_{G_{\sigma,s}}, \\ \|\varphi_T(t) \int_0^t W(t-\tau) v(\tau)\, \mathrm{d}\tau\|_{X_{\sigma,s,b}} &\leq c T^{1-b-b'} \|v\|_{X_{\sigma,s,b'}}, \end{aligned}$$

for $v \in X_{\sigma,s,b'}$, some $c \geq 0$ and any $s, \sigma \geq 0$, see [40, Lemma 1]. Let $R > \|u_0\|_{G_{\sigma,s}}$. The mapping $\Gamma : B_{2cR}(0) \subset X_{\sigma,s,b} \to X_{\sigma,s,b}$,

$$\Gamma[u](t) = \varphi(t) W(t) u_0 + \varphi_T(t) \int_0^t W(t-\tau)\partial_x(gu)(\tau) + W(t-\tau) f(\tau)\, \mathrm{d}\tau,$$

is a contraction for T sufficiently small since

$$\begin{aligned} \|\Gamma[u]\|_{X_{\sigma,s,b}} \leq{}& \|\varphi(t) W(t) u_0\|_{X_{\sigma,s,b}} \\ &+ \|\varphi_T(t) \int_0^t W(t-\tau)\partial_x(gu)(\tau) + W(t-\tau) f(\tau)\, \mathrm{d}\tau\|_{X_{\sigma,s,b}} \end{aligned}$$

$$\begin{aligned}&\leq c\|u_0\|_{G_{\sigma,s}} + cT^{1-b-b'}(\|\partial_x(gu)\|_{X_{\sigma,s,b'}} + \|f\|_{X_{\sigma,s,b'}})\\&\leq cR + cT^{1-b-b'}(c\|g\|_{X_{\sigma,s,b}}R + \|f\|_{X_{\sigma,s,b'}}) \leq 2cR\end{aligned}$$

for $T^{1-b-b'}(c\|g\|_{X_{\sigma,s,b}}R + \|f\|_{X_{\sigma,s,b'}}) \leq R$ by [40, Theorem 2] and similarly

$$\begin{aligned}\|\Gamma[u] - \Gamma[v]\|_{X_{\sigma,s,b}} &\leq \|\varphi_T(t)\int_0^t W(t-\tau)\partial_x(g(u-v))(\tau)\,\mathrm{d}\tau\|_{X_{\sigma,s,b}}\\&\leq cT^{1-b-b'}\|\partial_x(g(u-v))\|_{X_{\sigma,s,b'}}\\&\leq cT^{1-b-b'}c\|g\|_{X_{\sigma,s,b}}\|u-v\|_{X_{\sigma,s,b}}\end{aligned}$$

for $T^{1-b-b'}\|g\|_{X_{\sigma,s,b}} < c^{-2}$. Because of $b > \frac{1}{2}$, the fixed point $u \in X_{\sigma,s,b}$ of this map is in $C(\mathbb{R}, G_{\sigma,s})$ and satisfies for $t \in [-T,T] \cap [-1,1]$

$$u(t) = W(t)u_0 + \int_0^t W(t-\tau)\partial_x(gu)(\tau) + W(t-\tau)f(\tau)\,\mathrm{d}\tau.$$

□

Chapter 4

Phase Dynamics in a Generalised Ginzburg-Landau System: Validity of the Whitham System

The subsequent complex Ginzburg-Landau-like system (4.1) admits periodic solutions of a very simple form, see the subsequent Sections 4.1 and 4.2. In some sense, which we will discuss later, this system is a generalisation of the complex Ginzburg-Landau equation considered in Chapter 3. Hence, many of the comments concerning the complex Ginzburg-Landau equation made in Chapter 3 apply in this sense as well. But there will be some new challenges because of the structure of the system and the different parameter regime that we will consider. In general, the program that we will follow is very similar to the one of Chapter 3. We will consider modulations in space and time of the periodic solutions and, to some degree, discuss under what conditions these are governed by a conservation law-like system, called Whitham's system.

4.1 Introduction

There are pattern forming systems with a steady state, sometimes called ground state or trivial solution, possessing marginally stable long modes, i.e. the symbol of the linearisation around the steady state vanishes for $\xi = 0$ and the eigenvalues of the symbol are in the negative complex half-plane $\mathbb{C}^-$ in a deleted environment of $\xi = 0$. Examples of such systems are the Bénard-Marangoni problem [88], the flow down an inclined plane [19, Chapter 3], or the Faraday Wave Experiment [2]. In particular, we are interested in such systems if the linearisation around the steady state becomes linearly unstable by a short wave instability and exhibits a Hopf-Turing bifurcation. It is possible to derive a generalised Ginzburg-Landau-like system as universal amplitude equation by multiple-scale analysis for such systems close to the first instability. This Ginzburg-Landau-like system describes slow modulations in space and time of the most unstable linear modes and can be normalised to the following system:

$$\begin{aligned} \partial_t A &= (1+\alpha i)\partial_x^2 A + A - (1+\beta i)A|A|^2 + \gamma AB, \\ \partial_t B &= a\partial_x^2 B + c\partial_x B + d\partial_x(|A|^2), \end{aligned} \qquad (t,x) \in I \times \mathbb{R}, \tag{4.1}$$

with coefficients $\alpha, \beta, a, c, d \in \mathbb{R}$ and $\gamma \in \mathbb{C}$ and a time interval $I = [0, t_0[$, $t_0 \in \mathbb{R}^+$. We will restrict us to the case $a > 0$ so that (4.1) is a semlinear parabolic problem in L^2. However, most of the analysis should be easily transferable to the case $a, b, c \in \mathbb{C}$ as long as the principal part of the linear operator in system (4.1) is still sectorial in L^2, i.e. $\operatorname{Re} a > 0$.

In physical literature the derivation of (4.1) is usually given on a formal level or it stops after the investigation of the spectral or linear stability. But no approximation results in the spirit of Approximation Property 1.3.1 are proved in general. In mathematical literature some results in this direction exist. It has been shown for a problem related to the Bénard-Marangoni problem that system (4.1) makes correct predictions about the dynamics of such a pattern forming system close to the first instability, cf. [87]. In [32] it has been shown that solutions to the toy problem of [87] with initial data whose Fourier transform is small in $L^{1,h}_{-1} \cap L^1$ develop in such a way that after a certain time they are described by system (4.1).

Equation (4.1) possesses some trivial periodic solutions of the form

$$A_{\text{per}} = \begin{pmatrix} A_0 e^{\mathrm{i}(\xi_0 \boldsymbol{x} + \omega_0 \boldsymbol{t})} \\ B_0 \end{pmatrix},$$

where $A_0, B_0 \in \mathbb{C}$ and $\xi_0, \omega_0 \in \mathbb{R}$. The most trivial case is the one-parameter family of stationary solutions where $(A_0, B_0) \in \{0\} \times \mathbb{C}$. Obviously, these solutions are spectrally stable in L^2 if and only if $1 + \operatorname{Re}\gamma B_0 \leq 0$.

For $A_0 \neq 0$, there is a set of admissible parameters, which we will determine in Section 4.2, such that A_{per} is a solution to system (4.1). The question of spectral stability is much more involved in this case, c.f. Section 4.4.1. We will consider slow modulations in space and time of these solutions and derive a formal approximate solution that is governed by the subsequent conservation law-like system, which we call Whitham's system,

$$\begin{aligned}\partial_t \psi_* &= \partial_x \left(2(\beta-\alpha)\xi_0\psi_* + \mathrm{Im}\gamma b_* + (\beta-\alpha)\psi_*^2 - \beta\mathrm{Re}\gamma b_*\right),\\ \partial_t b_* &= \partial_x \left(cb_* - d\left(2\xi_0\psi_* - \mathrm{Re}\gamma b_* + \psi_*^2\right)\right).\end{aligned}$$

The main result of this chapter will be the proof that, under certain conditions, this formal approximate solution describes the dynamics of the modulations of the trivial solutions A_{per} to system (4.1) for $A_0 \in \mathbb{C}^\times$ on its natural time scale – see Theorems 4.4.2 and 4.4.3 below and the comments in Section 4.5.

Note that system (4.1) includes the 'classical' complex Ginzburg-Landau equation, too, if we have $d = \gamma = 0$. In that sense all comments and remarks concerning the complex Ginzburg-Landau equation in Chapter 3 apply in this case as well.

Whitham's system has originally been derived in the context of dispersive systems involving wave-like phenomena, cf. the book of Whitham [92]. But the method to derive such systems is widely used in physics and not restricted to these kind of systems. In a number of papers, cf. [48] and the references therein, it has been shown that approximate solutions constructed with the aid of a certain Whitham's system dominate the long-time behaviour of modulations of periodic travelling wave solutions to certain 'viscous conservation laws'. In the last decade some approximation properties have been proven for approximate solutions that are governed by systems of this type in the dispersive case, cf. [6, 31, 34], but not only in the dispersive setting, cf. [47].

We emphasise, however, that we do not assume strict spectral stability for our result, see Hypothesis 4.4.10 below, in contrast to e.g. [48]*. The price to pay for this is that we need an approximate solution and a strict solution to equation* (4.1) *that are analytic in some (large) strip $S_{\sigma'/\epsilon} \subset \mathbb{C}$ and in in the space L^2, which means that we need initial data in $\mathrm{Hol}_{\sigma/\epsilon}$ for pretty small $\epsilon > 0$ and $\sigma' < \sigma$.*

We will apply a method similar to the one used in Chapter 3 for the proof of the mentioned result. Again, we will proceed in four steps to prove our result. First, we will introduce a formal approximate solution for slow modulations of the periodic solutions with $A_0 > 0$ in Section 4.2. In that Section, we will also prove that there is a solution to Whithams's system that is analytic in $S_{\sigma'/\epsilon}$ for initial data in $\mathrm{Hol}_{\sigma/\epsilon}$. Then we will define another (auxiliary) formal approximate solution in Section 4.3 and prove a similar existence result for the equation system by which it is governed. In Section 4.4.1, we will introduce isomorphisms between certain spaces, that we will need for the proof of the main result, and discuss some properties of the isomorphisms as a preparation for the final step. Further, we will state some restrictions on the linear part of (4.1), which will be crucial for the final step. As final step, we will prove our result with the aid of the auxiliary formal approximate solution in Section 4.4.

4.2 Derivation of Whitham's System

Equation (4.1) possesses a two-parameter family of space and time periodic solutions

$$A_{\mathrm{per}} = \begin{pmatrix} A_0 e^{\mathrm{i}(\xi_0 \boldsymbol{x} + \omega_0 \boldsymbol{t})} \\ B_0 \end{pmatrix},$$

provided that $1 - \xi_0^2 + \mathrm{Re}\gamma B_0 \geq 0$. The case $A_0 = 0$ is somewhat special, as noted above, and therefore we exclude it, i.e. the set of admissible parameters is $\{(\xi_0, B_0) \in \mathbb{R}^2 \,|\, 1 - \xi_0^2 + \mathrm{Re}\gamma B_0 > 0\}$. For the frequency ω_0 and amplitude $A_0 \in \mathbb{R}^\times$ we find the relations

$$\begin{aligned} \omega_0 &= (\beta - \alpha)\xi_0^2 - \beta(1 + \mathrm{Re}\gamma B_0) + \mathrm{Im}\gamma B_0, \\ A_0 &= \pm\sqrt{1 - \xi_0^2 + \mathrm{Re}\gamma B_0}. \end{aligned}$$

Some comments about these solutions are advisable.

***Remark* 4.2.1.**

- It is possible to consider complex-valued A_0. Then the condition for A_0 has to be read as $|A_0| = \sqrt{1 - \xi_0^2 + \mathrm{Re}\gamma B_0}$ and the phase is arbitrary. In this case the solutions can be considered as a three-parameter family. However, since system (4.1) is invariant under the map $(A, B) \mapsto (e^{\mathrm{i}\varphi} A, B)$ for $\varphi \in \mathbb{R}$, we will only consider $A_0 > 0$.

- As already indicated in Section 4.1, it is not necessary to consider real-valued B_0 only. We will not do so in the following and the set of admissible parameters is now

$$\left\{(\xi_0, B_0) \in \mathbb{R} \times \mathbb{C} \,\middle|\, 1 - \xi_0^2 + \mathrm{Re}\gamma B_0 > 0\right\}.$$

- In the special case $\gamma = d = 0$ the system decouples and the first component is just the complex Ginzburg-Landau equation of Chapter 3. Hence, the space and time periodic solutions are the same in this situation. System (4.1) is a generalisation of the complex Ginzburg-Landau equation (3.1) in this spirit.

- For $1 + \mathrm{Re}\gamma B_0 > 0$, we can assume, without loss of generality, $B_0 = 0$ in the following sense. We can define $\tilde{A} : [0, c_t^{-1} t_0[\times \mathbb{R} \to \mathbb{C}$,

$$(t, x) \mapsto \tilde{A}(t, x) = c_A A(c_t t, c_x x) e^{-\mathrm{i} c_t \mathrm{Im}\gamma B_0 t},$$

  by assumption, where $c_t^{-1} = 1 + \mathrm{Re}\gamma B_0, c_x^{-1} = \sqrt{1 + \mathrm{Re}\gamma B_0} = c_A^{-1} \neq 0$ for any solution (A, B_0) of system (4.1). Then $(\tilde{A}, 0)$ is also a solution to system (4.1) but with different coefficients

$$\begin{aligned} a &\to a, & d &\to (1 + \mathrm{Re}\gamma B_0)^{-1/2} d, \\ c &\to (1 + \mathrm{Re}\gamma B_0)^{-1/2} c, & \gamma &\to (1 + \mathrm{Re}\gamma B_0)^{-1} \gamma. \end{aligned}$$

  The latter is actually only important for the modulations of A_{per} that we will consider subsequently. Since we will write the modulations in a multiplicative way we cannot allow $A_0 = 0$. Hence $1 + \mathrm{Re}\gamma B_0 \geq 1 - \xi_0^2 + \mathrm{Re}\gamma B_0 > 0$ is always true. We have to emphasis that this procedure changes the spectrum of the linearisation around A_{per} in a not so obvious way and reflects the fact that we eliminate the linear operator mapping $U \mapsto (\gamma B_0 U_1, 0)$ in the linearisation around A_{per} by this transformation.

As announced in the introduction we are again interested in modulations of these solutions. Therefore we consider $U : I \times \mathbb{R} \to \mathbb{C}^2$,

$$U = \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} e^{r_0 + r + \mathrm{i}(\xi_0 \boldsymbol{x} + \omega_0 \boldsymbol{t}) + \mathrm{i}\phi} \\ B_0 + b \end{pmatrix},$$

with $r_0 = \ln A_0$ and $r, \phi : I \times \mathbb{R} \to \mathbb{R}$, $b : I \times \mathbb{R} \to \mathbb{C}$. We obtain for r, ϕ, b the system

$$
\begin{aligned}
\partial_t r &= \partial_x^2 r + (\partial_x r)^2 - (\partial_x \phi)^2 - 2\xi_0 \partial_x \phi - \alpha \left(2\partial_x r \partial_x \phi + \partial_x^2 \phi + 2\xi_0 \partial_x r\right) \\
&\quad - |A_0|^2 (e^{2r} - 1) + \mathrm{Re}\gamma b, \\
\partial_t \phi &= \partial_x^2 \phi + 2\partial_x r \partial_x \phi + 2\xi_0 \partial_x r + \alpha \left((\partial_x r)^2 - (\partial_x \phi)^2 + \partial_x^2 r - 2\xi_0 \partial_x \phi\right) \\
&\quad - \beta |A_0|^2 (e^{2r} - 1) + \mathrm{Im}\gamma b, \\
\partial_t b &= a\partial_x^2 b + c\partial_x b + d|A_0|^2 \partial_x e^{2r},
\end{aligned} \tag{4.2}
$$

Again we replace this system by the system with the spatial derivative $\psi = \partial_x \phi$, which is sometimes referred to as 'local wave number', cf. Section 3.3 of Chapter 3, and thus we will study the system

$$
\partial_t U = \tilde{\mathcal{L}}(\partial_x) U + \tilde{N}(U) \tag{4.3}
$$

where $U = (r, \psi, \mathrm{Re}\, b, \mathrm{Im}\, b)$,

$$
\tilde{\mathcal{L}}(\partial_x) = \begin{pmatrix}
\partial_x^2 - 2\alpha\xi_0 \partial_x - 2|A_0|^2 & -\alpha\partial_x - 2\xi_0 & \mathrm{Re}\gamma & -\mathrm{Im}\gamma \\
\alpha\partial_x^3 + 2\xi_0 \partial_x^2 - 2\beta|A_0|^2 \partial_x & \partial_x^2 - 2\alpha\xi_0 \partial_x & \mathrm{Im}\gamma \partial_x & \mathrm{Re}\gamma \partial_x \\
2d|A_0|^2 \partial_x & 0 & a\partial_x^2 + c\partial_x & 0 \\
0 & 0 & 0 & a\partial_x^2 + c\partial_x
\end{pmatrix}
$$

and, for short $h(r) = e^{2r} - 1 - 2r$,

$$
\tilde{N}(r, \psi, \mathrm{Re}\, b, \mathrm{Im}\, b) = \begin{pmatrix}
(\partial_x r)^2 - \psi^2 - 2\alpha(\partial_x r)\psi - |A_0|^2 h(r) \\
2\partial_x((\partial_x r)\psi) + \alpha\partial_x((\partial_x r)^2 - \psi^2) - \beta|A_0|^2 \partial_x h(r) \\
d|A_0|^2 \partial_x h(r) \\
0
\end{pmatrix}.
$$

Since Whitham's system should describe the long wavelength regime, we use the ansatz $U_* : [0, t_0[\times \mathbb{R} \to \mathbb{R}^4$,

$$
(t, x) \mapsto U_*(t, x) = \begin{pmatrix}
r_*(\epsilon t, \epsilon x) \\
\psi_*(\epsilon t, \epsilon x) \\
\mathrm{Re}\, b_*(\epsilon t, \epsilon x) \\
\mathrm{Im}\, b_*(\epsilon t, \epsilon x)
\end{pmatrix}, \tag{4.4}
$$

for the derivation of a formal approximate solution, where $0 < \epsilon \ll 1$. If we insert ansatz (4.4) into system (4.3), we obtain in lowest order in ϵ the system

$$\begin{aligned} 0 &= -\xi_0\psi_* + \mathrm{Re}\gamma b_* - \psi_*^2 - |A_0|^2(h(r_*) + 2r_*) && (4.5a)\\ \partial_t\psi_* &= \partial_x\left(-2\alpha\xi_0\psi_* + \mathrm{Im}\gamma b_* - \alpha\psi_*^2 - \beta|A_0|^2(h(r_*) + 2r_*)\right) && (4.5b)\\ \partial_t b_* &= \partial_x\left(cb_* + d|A_0|^2(h(r_*) + 2r_*)\right). && (4.5c)\end{aligned}$$

A necessary condition for the existence of a solution to system (4.5) is

$$2\xi_0\psi_* - \mathrm{Re}\gamma b_* + \psi_*^2 = -|A_0|^2(h(r_*) + 2r_*) = |A_0|^2(1 - e^{2r_*}) \leq |A_0|^2.$$

Hence, a solution to (4.5) can at least exist for small values of ψ_*, b_*. In this situation, the solution is determined by the system, called Whitham's system,

$$\begin{aligned}\partial_t\psi_* &= \partial_x\left(2(\beta-\alpha)\xi_0\psi_* + \mathrm{Im}\gamma b_* + (\beta-\alpha)\psi_*^2 - \beta\mathrm{Re}\gamma b_*\right),\\ \partial_t b_* &= \partial_x\left(cb_* - d\left(2\xi_0\psi_* - \mathrm{Re}\gamma b_* + \psi_*^2\right)\right).\end{aligned} \tag{4.6}$$

This system has the structure of a conservation law. An obvious observation is that this system is a nonlinear hyperbolic system for some parameters but not for others. In general it seems not clear how to prove existence of solutions for all permissible parameter values in a common frame work *if one does not require analyticity of the solutions*. For (real) analytic initial data, system (4.6) possesses (locally in time and space) an analytic solution. This is a simple consequence of the Cauchy-Kowalevski Theorem, cf. [67, Theorem 4.1]. But we need better knowledge of the decay of the solution and other properties. We will work with a solution whose Fourier transform is $C^1([0,t_0[, L^2_{s,\sigma})$ for some $s, \sigma, t_0 > 0$.

Theorem 4.2.2 (Local existence)**.** *Let* $(\psi_{*,0}, b_{*,0}) \in \mathrm{Hol}_\sigma$ *for a* $\sigma > 0$, $\psi_{*,0}$ *real-valued, and*

$$2\|(\hat{\psi}_{*,0}, \hat{b}_{*,0})\|_{L^2_{1,\sigma'}} < \sqrt{(2|\xi_0| + |\gamma|)^2 + |A_0|^2} - 2|\xi_0| - |\gamma| \tag{4.7}$$

for all $\sigma' < \sigma$.
Then there is $t_0 = t_0(\sigma') > 0$ *such that system* (4.6) *has a solution*

$$(\psi_*, b_*) \in C([0,t_0[, \mathrm{Hol}_{\sigma'}) \cap C^1([0,t_0[, \mathrm{Hol}_{\sigma'})$$

with $(\psi_*, b_*)(0) = (\psi_{*,0}, b_{*,0})$ *for all* $0 \leq \sigma' < \sigma$ *and* ψ_* *real-valued. More precisely,* *system* (4.6) *has a solution*

$$(\hat{\psi}_*, \hat{b}_*) \in C([0, t(\sigma')[, L^2_{1,\sigma'}) \cap C^1([0, t(\sigma')[, L^2_{0,\sigma'})$$

for all $0 \le \sigma' < \sigma$*, where* $t(\sigma') \le \frac{\sigma - \sigma'}{\lambda}$ *and* $\lambda > 0$*. Further we have*

$$\|(\hat{\psi}_*, \hat{b}_*)(t)\|_{L^2_{1,\sigma'}} < \sqrt{(2|\xi_0| + |\gamma|)^2 + |A_0|^2} - 2|\xi_0| - |\gamma|$$

for all $t \in [0, t(\sigma')[$.

***Remark* 4.2.3.** There are many results about existence (and uniqueness) of solutions to nonlinear PDEs or systems of PDEs with values in Banach spaces. More or less a straightforward extension of the classical Cauchy-Kowalevski result can be found in the work of Trèves [90] where analyticity in time and space is assumed. Further results in scales of Banach spaces are e.g. [5, 17, 71, 72, 78] and others. These usually only require continuity in time (but extend to holomorphic functions in time if the PDE system is holomorphic, too). The prove of Asanov looks nice and short but we will use the statement of Safonov [78] since it is more handy for our purpose. The techniques are more or less similar and can already be found in [70].

Proof. (Theorem 4.2.2). We will use Safonov's theorem of 1995, cf. [78, Theorem 1.1]. Let

$$\left\{ X_\rho = L^2_{1,\rho} \,\middle|\, 0 \le \rho \le \rho_0 = \sigma, u(\boldsymbol{\xi}) = \overline{u(-\boldsymbol{\xi})} \right\}$$

the scale of Banach spaces and $2r = \sqrt{(2|\xi_0| + |\gamma|)^2 + |A_0|^2} - 2|\xi_0| - |\gamma|$. Then the embedding $X_\rho \hookrightarrow X_{\rho'}$ is trivial an the inclusion map has norm 1 for all $0 \le \rho' < \rho \le \rho_0$. We write

$$\begin{aligned} \tilde{F}(t,u) = &\begin{pmatrix} 2(\beta - \alpha)\xi_0 & \mathrm{Im}\gamma - \beta\mathrm{Re}\gamma & \mathrm{Re}\gamma + \beta\mathrm{Im}\gamma \\ -2d\xi_0 & c + d\mathrm{Re}\gamma & -d\mathrm{Im}\gamma \\ 0 & 0 & c \end{pmatrix} M_{\mathrm{i}\boldsymbol{\xi}} u \\ &+ \sqrt{\frac{2}{\pi}} \begin{pmatrix} (\beta - \alpha) & 0 & 0 \\ -d & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (u)_1 * M_{\mathrm{i}\boldsymbol{\xi}} u, \end{aligned}$$

where $(u)_1 * M_{\mathrm{i}\boldsymbol{\xi}}u$ denotes the convolution of each component of $M_{\mathrm{i}\boldsymbol{\xi}}u$ with the first component of u. Let $F(t,v) := \tilde{F}(t, v+u_0)$ for $u_0 \in \bigcap_{0\le\rho<\rho_0} X_\rho$. We consider the problem

$$\partial_t v = F(t,v), \qquad v(0) = 0. \tag{4.8}$$

Then we obtain a solution to (4.6) if we add $u_0 = (\hat{\psi}_{*,0}, \widehat{\operatorname{Re} b}_{*,0}, \widehat{\operatorname{Im} b}_{*,0}) \in X_{\rho_0}$ to the solution to system (4.8). We have to check Assumption (1.1) of [78].

(a) The mapping $(t,u) \mapsto (t, u+u_0)$ is in $C^\infty(\mathbb{R} \times X_\rho)$ for all $0 \le \rho < \rho_0$ and $\tilde{F} \in C^\infty(\mathbb{R} \times X_\rho, X_{\rho'})$ for all $0 \le \rho' < \rho < \rho_0$. Hence, $F \in C^\infty(\mathbb{R} \times X_\rho, X_{\rho'})$ for all $0 \le \rho' < \rho < \rho_0$ and thus the assumption is satisfied for all $r, \lambda > 0$.

(b) Let $0 < \rho' < \rho < \rho_0$. We have for $u, v \in B_r(0) \subset X_\rho$

$$\begin{aligned}\|\tilde{F}(t,u) - \tilde{F}(t,v)\|_{X_{\rho'}} &\lesssim \|M_{\mathrm{i}\boldsymbol{\xi}}(u-v)\|_{X_{\rho'}} + \|(u)_1 * M_{\mathrm{i}\boldsymbol{\xi}}u - (v)_1 * M_{\mathrm{i}\boldsymbol{\xi}}v\|_{X_{\rho'}} \\ &\lesssim \frac{1}{\rho-\rho'}\Big(\|u-v\|_{X_\rho} + \|u\|_{X_{\rho'}}\|u-v\|_{X_\rho}\Big) \\ &\quad + \frac{1}{\rho-\rho'}\|u-v\|_{X_{\rho'}}\|v\|_{X_\rho} \\ &\lesssim \frac{1+2r}{\rho-\rho'}\|u-v\|_{X_\rho}.\end{aligned}$$

The hidden constant does not depend on t, u, v, ρ or ρ' for all $t > 0$. Then it follows

$$\|F(t,u) - F(t,v)\|_{X_{\rho'}} \lesssim \frac{1+2r}{\rho-\rho'}\|u-v\|_{X_\rho}$$

and again hidden constant does not depend on t, u, v, ρ or ρ' for all $t > 0$ (but will depend on u_0).

(c) $F(t,0) = \tilde{F}(t,u_0)$. Hence $F(\cdot,0) \in C^\infty(\mathbb{R}, X_{\rho'})$ and $\|F(t,0)\|_{X_{\rho'}} \lesssim \|u_0\|_{X_{\rho_0}} + \|u_0\|^2_{X_{\rho_0}}$ for all $0 \le \rho' < \rho < \rho_0$.

Application of [78, Theorem 1.1] gives a solution $v \in C([0,t(\rho)[, X_\rho)$ to the integral version of (4.8) for any $0 < \rho < \rho_0$ and $t(\rho) = \frac{\rho_0-\rho}{\lambda}$ for some $\lambda > 0$. It holds $\|v\|_{L^2_{s,\rho}} < r$ for all $t < t(\rho)$.

Further, the maps $v(\boldsymbol{t}), M_{\mathrm{i}\boldsymbol{\xi}}v(\boldsymbol{t}), ((v)_1 * M_{\mathrm{i}\boldsymbol{\xi}}v)(\boldsymbol{t})$ are continuous with values in $L^2_{0,\rho}$

for $t \in [0, t(\rho)[$. Thus $v(\boldsymbol{t})$ is continuously differentiable with values in $L^2_{0,\rho}$ and satisfies (4.8).
We obtain a solution $v \in C([0, t(\rho)[, X_\rho) \cap C^1([0, t(\rho)[, L^2_{0,\rho})$ to system (4.6) with $u(0) = u_0$ by $u = v + u_0$.
Since $\|v\|_{L^2_{s,\rho}}, \|u_0\|_{L^2_{s,\rho}} < r$ we have

$$\|u\|_{L^2_{s,\rho}} < 2r = \sqrt{(2|\xi_0| + |\gamma|)^2 + |A_0|^2} - 2|\xi_0| - |\gamma|.$$

□

***Remark* 4.2.4.**

- Assumption (4.7) is satisfied if

$$2\|(\hat{\psi}_{*,0}, \hat{b}_{*,0})\|_{L^2_{1,\sigma}} < \sqrt{(2|\xi_0| + |\gamma|)^2 + |A_0|^2} - 2|\xi_0| - |\gamma|.$$

- In some special cases system (4.6) is of nonlinear hyperbolic character in the sense of [45, Chapter IV]. This is the case for $\xi_0 = 0$ and $c, d\mathrm{Re}\gamma > 0$ for example. In these cases one can expect to find solutions in usual Sobolev spaces. However, in general it seems impossible to do so and one might conjecture that system (4.6) is ill-posed in Sobolev spaces.

  Consider the case $\mathrm{Re}\gamma = c = 0$ and $\alpha = \beta$ for example. Then the matrix of the linear part for $\tilde{F}$ has eigenvalues $0, \pm\mathrm{i}\sqrt{d\xi_0\mathrm{Im}\gamma}$ where we used again the principal branch of the root. This means that the spectrum of the linear part of $\tilde{F}$ in Sobolev spaces $W^{m,p}$, $0 \leq p \leq \infty$, contains $\mathbb{R}$ for $d, \xi_0, \mathrm{Im}\gamma \neq 0$. Therefore the pure linear problem would be ill-posed, cf. [37, Corollary II.6.9; 67, Proposition 4.2]. Thus, weakly nonlinear theory seems hard to implement and one would have to hope that the complete quasi-linear problem has solutions.

- The condition $\|u\|_{L^2_{s,\rho}} < \sqrt{(2|\xi_0| + |\gamma|)^2 + |A_0|^2} - 2|\xi_0| - |\gamma|$ guarantees that the algebraic equation (4.5a) is resolvable.

4.3 Higher Order Residual

So far, the pointwise residual is of order ϵ and its Fourier transform in $L^2_{s,\sigma}$ norm of order $\epsilon^{\frac{1}{2}}$. We need a better residual estimate for the error estimates in Section 4.4. We can achieve such a result in the usual way, i.e. by adding higher order correctors to the ansatz. Thus, we obtain the following lemma.

Lemma 4.3.1. *Assume* $(r_*, \psi_*, b_*) \in C([0, t_0], \mathrm{Hol}_\sigma)$ *is a solution to system* (4.5) *for a* $\sigma > 0$ *as in Theorem 4.2.2. Let* U_* *as in Section 4.2 and* $N \in \mathbb{N}$.
Then there is $\epsilon_0 > 0$ *and a function*

$$U_{**} \in C([0, t_1/\epsilon], \mathrm{Hol}_{\sigma/\epsilon}) \cap C^1([0, t_1/\epsilon], \mathrm{Hol}_{\sigma/\epsilon}),$$

where $t_1 = t_1(\sigma') \in]0, t_0]$, *such that*

$$\|\mathscr{F}\,\mathfrak{E}_{U_{**}}\|_{L^\infty\left([0,t_1(\sigma')/\epsilon],L^2_{0,\sigma'/\epsilon}\right)} \lesssim \epsilon^{N+\frac{1}{2}}$$

and

$$\|\hat{U}_{**} - \hat{U}_*\|_{L^\infty\left([0,t_1(\sigma')/\epsilon],L^2_{0,\sigma'/\epsilon}\right)} \lesssim \epsilon^{1/2}$$
$$\|U_{**} - U_*\|_{L^\infty([0,t_1(\sigma')/\epsilon],L^\infty)} \lesssim \epsilon.$$

for all $\epsilon \in]0, \epsilon_0]$, $\sigma' \in [0, \sigma[$

Proof. We write

$$r = \sum_{i=0}^{N} \epsilon^i r_i, \qquad \psi = \sum_{i=0}^{N} \epsilon^i \psi_i, \qquad b = \sum_{i=0}^{N} \epsilon^i b_i,$$

for functions $(r_j, \psi_j, b_j) : [0, t_1[\times\mathbb{R} \to \mathbb{R}^2 \times \mathbb{C}$, $0 \le j \le N$. Let

$$U_{**} : [0, t_1/\epsilon[\times\mathbb{R} \to \mathbb{R}^2 \times \mathbb{C}, \quad (t, x) \mapsto (r, \psi, b)(\epsilon t, \epsilon x).$$

We will search for solutions that can be embedded into continuous functions. Hence, pointwise calculations are well-defined.

We obtain a system with addends of different powers in ϵ by inserting the ansatz U_{**} into system (4.3). As usual, we can reduce the residual by equating the coefficients of ϵ^i, $i \in \{1, \dots, N\}$, see Appendix 4.A for the details. This leads to system (4.5) in lowest order, with the initial data given in Lemma 4.3.1, and to the system

$$2|A_0|^2e^{2r_0}r_i = \mathrm{Re}\gamma b_i - 2(\xi_0+\psi_0)\psi_i + F_{r,i}, \tag{4.9a}$$

$$\partial_t\psi_i = \partial_x\big(2(\beta-\alpha)(\xi_0+\psi_0)\psi_i + \mathrm{Im}\gamma b_i - \beta\mathrm{Re}\gamma b_i + F_{\psi,i}\big), \tag{4.9b}$$

$$\partial_t b_i = \partial_x\big(cb_i - 2d(\xi_0+\psi_0)\psi_i + d\mathrm{Re}\gamma b_i + F_{b,i}\big), \tag{4.9c}$$

with vanishing initial data in order ϵ^{i+1}, or ϵ^i for the algebraic equation respectively, $1 \le i \le N$. Note that each of the functions $F_{r,i}, F_{\psi,i}, F_{b,i}$ depends on $r_0,\dots,r_{i-1},\psi_0,\dots,\psi_{i-1},b_0,\dots,b_{i-1}$. Actually,

$$F_{r,i}, F_{\psi,i}, F_{b,i} : C\big([0,\tilde{t}[,\mathscr{F}L^2_{s+3,\sigma'}\big)\cap C^1\big([0,\tilde{t}[,\mathscr{F}L^2_{s,\sigma'}\big) \to C\big([0,\tilde{t}[,\mathscr{F}L^2_{s,\sigma'}\big) \tag{4.10}$$

are holomorphic in their arguments for all $(s,\sigma')\in[0,\infty[^2$ and $\tilde{t}>0$.

Let $0<\sigma_N<\ldots<\sigma_1<\sigma_0<\sigma$ any strictly decreasing sequence of real numbers. Now we proceed by induction in i and prove that there is a decreasing sequence $0<t_N\le\ldots\le t_0$, $t_i=t_i(\sigma_i)$, and that there are solutions

$$\mathscr{F}(r_i,\psi_i,b_i)\in C([0,t_i[,L^2_{1,\sigma_i})\cap C^1([0,t_i[,L^2_{0,\sigma_i})$$

to system (4.5) with initial data as in Theorem 4.2.2 in the case $i=0$ or to system (4.9) with vanishing initial data in the case $i\in\{1,\dots,N\}$, respectively.

Base case We know that system (4.5) has a solution in

$$\mathscr{F}(r_0,\psi_0,b_0)\in C([0,t_0[,L^2_{1,\sigma_0})\cap C^1([0,t_0[,L^2_{0,\sigma_0})$$

for the required initial data by Theorem 4.2.2. Hence the result is true for $i=0$ since we can simply use $(r_0,\psi_0,b_0)=(r_*,\psi_*,b_*)$.

Inductive step We assume we have for $0<t_{i-1}\le\ldots\le t_0$, $0\le j\le i-1$, such solutions

$$\mathscr{F}(r_j,\psi_j,b_j)\in C([0,t_j[,L^2_{1,\sigma_j})\cap C^1([0,t_j[,L^2_{0,\sigma_j}).$$

We will again use Safonov's Theorem [78, Theorem 1.1] to show that we can construct the required function

$$\mathscr{F}(r_i,\psi_i,b_i)\in C([0,t_i[,L^2_{1,\sigma_i})\cap C^1([0,t_i[,L^2_{0,\sigma_i}),$$

$t_i\le t_{i-1}$. Therefore, we have to check the Assumptions (1.1) of that theorem. Let $r>0, \rho_0=\frac{\sigma_i+\sigma_{i-1}}{2}$ and $\lambda>\frac{\rho_0}{t_{i-1}}$. We define

$$\left\{X_\rho=L^2_{3,\rho}\,\middle|\,0\le\rho\le\rho_0\right\}$$

similarly as in the proof of Theorem 4.2.2.

Because of the embeddings of the spaces $L^2_{s,\sigma}$ we have

$$\mathscr{F}(r_0,\psi_0,b_0)\in C\big([0,\rho_0/\lambda],\mathscr{F}\,L^2_{6,\rho_0}\big)\cap C^1\big([0,\rho_0/\lambda],\mathscr{F}\,L^2_{3,\rho_0}\big).$$

Note that $F_{r,i},F_{\psi,i},F_{b,i}\in C\big([0,\rho_0/\lambda],\mathscr{F}^{-1}\,L^2_{3,\rho_0}\big)$ by (4.10) since $\frac{\rho_0}{\lambda}<t_{i-1}$.

We consider the problem $\partial_t\hat{u}=F(t,\hat{u})$ where the mapping

$$F:[0,\rho_0/\lambda[\times B_r(0)\subset\mathbb{R}^+_0\times X_\rho\to X_{\rho'},$$

$0\le\rho<\rho'<\rho_0$, is defined by

$$(t,u)\mapsto\begin{pmatrix}M_{\mathrm{i}\boldsymbol{\xi}}\big(2(\beta-\alpha)(\xi_0+\mathscr{F}\,M_{\psi_0(t)}\,\mathscr{F}^{-1})(u)_1+\mathrm{Im}\gamma(u)_2-\beta\mathrm{Re}\gamma(u)_2\big)\\ M_{\mathrm{i}\boldsymbol{\xi}}\big(c(u)_2-2d(\xi_0+\mathscr{F}\,M_{\psi_0(t)}\,\mathscr{F}^{-1})(u_1)+d\mathrm{Re}\gamma(u)_2\big)\end{pmatrix}$$
$$+\begin{pmatrix}\mathscr{F}\,F_{\psi,i}(t)\\ \mathscr{F}\,F_{b,i}(t)\end{pmatrix}.$$

The check of the Assumptions (1.1) is now a trivial task.

(a) Since the first addend of F is just a linear bounded operator from $X_{\rho'}\to X_\rho$, it is clearly continuous because $\hat{\psi}_0\in C\big([0,\rho_0/\lambda],L^2_{4,\rho_0}\big)$. The last addend is continuous because of the reasoning above.

(b) Let $0 < \rho' < \rho < \rho_0$, $0 \leq t < \frac{\rho_0}{\lambda}$ and $u, v \in B_r(0) \subset X_\rho$. Then

$$\begin{aligned}\|F(t,u) - F(t,v)\|_{X'_\rho} &\lesssim \|M_{\mathrm{i}\boldsymbol{\xi}}(u-v)\|_{X'_\rho} + \|\hat{\psi}_0(t) * M_{\mathrm{i}\boldsymbol{\xi}}(u-v)\|_{X'_\rho} \\ &\quad + \|(u-v) * M_{\mathrm{i}\boldsymbol{\xi}}\hat{\psi}_0(t)\|_{X'_\rho} \\ &\lesssim \frac{\|u-v\|_{X_\rho}}{\rho - \rho'}\end{aligned}$$

where the constant depends on

$$\|M_{\mathrm{i}\boldsymbol{\xi}}\hat{\psi}_0(t)\|_{L^\infty([0,\rho_0/\lambda],X'_\rho)} \leq \|\hat{\psi}_0(t)\|_{L^\infty([0,\rho_0/\lambda],X_{\sigma_0})} < \infty$$

since $\frac{\rho_0}{\lambda} < t_{i-1} \leq t_0$.

(c) Since $F(t,0) = (\mathscr{F} F_{\psi,i}(t), \mathscr{F} F_{b,i}(t))$, the restriction is trivially continuous and bounded with values in every X_ρ for all $0 \leq \rho < \rho_0$ (and the bound is independent of ρ) since $\frac{\rho_0}{\lambda} < t_{i-1}$.

Hence, [78, Theorem 1.1] is applicable and we have solutions

$$\mathscr{F}(r_i, \psi_i, b_i) \in C([0, t_i[, L^2_{3,\sigma_i}) \cap C^1([0, t_i[, L^2_{2,\sigma_i})$$

to system (4.9) for a $0 < t_i \leq t_{i-1}$ and vanishing initial data. $\Diamond$

Thus, we constructed $U_{**} \in C\big([0, t_1/\epsilon], \mathrm{Hol}_{\sigma/\epsilon}\big) \cap C^1\big([0, t_1/\epsilon], \mathrm{Hol}_{\sigma/\epsilon}\big)$ where t_1 depends on the exact choice of σ' and vanishes for $\sigma' \to \sigma$.
Further we have

$$U_{**} - U_* = \sum_{i=1}^N \epsilon^i (r_i, \psi_i, b_i)(\epsilon \boldsymbol{t}, \epsilon \boldsymbol{x}).$$

This proves for $\sigma' = \sigma_N$ and all $t_1(\sigma') \in]0, t_N[$

$$\begin{aligned}\|\hat{U}_{**} - \hat{U}_*\|_{L^\infty\left([0,t_1(\sigma')/\epsilon], L^2_{0,\sigma'/\epsilon}\right)} &\lesssim \epsilon^{1/2} \\ \|U_{**} - U_*\|_{L^\infty([0,t_1(\sigma')/\epsilon], L^\infty)} &\lesssim \epsilon.\end{aligned}$$

For the residual estimate we notice that

$$\mathscr{F}(r_i, \psi_i, b_i) \in C([0, t_N[, L^2_{3,\sigma_N}) \cap C^1([0, t_N[, L^2_{2,\sigma_N})$$

for all $i \in \{0, \ldots, N\}$. Further we observe that the linear part of system (4.3) maps $C([0, t_N[, \mathscr{F} L^2_{3,\sigma_N}) \cap C^1([0, t_N[, \mathscr{F} L^2_{2,\sigma_N}) \to C([0, t_N[, \mathscr{F} L^2_{0,\sigma_N})$. Also, we note that the nonlinear part is entire in all its the variables. Hence the nonlinear part maps $C([0, t_N[, \mathscr{F} L^2_{3,\sigma_N}) \cap C^1([0, t_N[, \mathscr{F} L^2_{2,\sigma_N}) \to C([0, t_N[, \mathscr{F} L^2_{1,\sigma_N})$ since L^2_{1,σ_N} is a Banach algebra, see Lemma A.1.2 also.
Consequently, the Fourier transform of the residual is in $C([0, t_N[, L^2_{0,\sigma_N})$ and of order $\epsilon^{N+1/2}$ by construction of the function U_{**}. For the quantitative estimate one has to estimate all remaining addends of system (4.21) in the appendix, which are all at least of order ϵ^{N+1} pointwise, and are otherwise bounded thanks to the Banach algebra property and high regularity. Therefore, we proved

$$\|\mathscr{F}\,\mathfrak{E}_{U_{**}}\|_{L^\infty\left([0,t_1(\sigma')/\epsilon],L^2_{0,\sigma'/\epsilon}\right)} \lesssim \epsilon^{N+\frac{1}{2}}.$$

□

4.4 The Error Estimates

We will discuss error estimates between an approximate solution and a strict solution to the Fourier transform of system (4.3) in the following, i.e. we consider the system

$$\partial_t \mathscr{F} U = \tilde{\mathcal{L}}(\mathrm{i}\boldsymbol{\xi})\, \mathscr{F} U + \mathscr{F}\, \tilde{N}(U).$$

We collect the error estimates in the following Theorems 4.4.1, 4.4.2 and 4.4.3. These will be a simple consequence of Theorem 4.4.11, which will be proved in Section 4.4.2. Let us use the abbreviation $X_{s,\sigma} = L^2_{s+1,\sigma} \times L^2_{s,\sigma} \times L^2_{s+1,\sigma} \times L^2_{s+1,\sigma}$ for a shorter notation.

Theorem 4.4.1. *Let U_{**} be the approximate solution to equation* (4.3) *constructed in Section 4.3 and σ, t_1 as in Lemma 4.3.1. Further let $\sigma' \in]0, \sigma[$, $s \in [\frac{7}{4}, \infty[$, $q \in [1, \infty[$ and assume Hypothesis 4.4.5 below is satisfied. Finally, let U be a strict solution to system* (4.3) *in $\mathscr{F}^{-1} X_{s-7/4,\sigma'/\epsilon}$ with initial data $\hat{U}(0) \in X_{s+1/4,\sigma'/\epsilon}$. Then there exist $C_{U_{**}}, \epsilon_0 > 0$ such that for all $\epsilon \in]0, \epsilon_0[$ and $\|U_{**}\|_{L^\infty\left([0,t_1(\sigma'),L^2_{s,\sigma/\epsilon}\right)} \leq C_{U_{**}}$ and initial data*

$$\|\hat{U}(0) - \hat{U}_{**}(0)\|_{X_{s,\sigma/\epsilon}} \lesssim \epsilon^q \tag{4.11}$$

property (4.11) *is preserved in* H^s *by the dynamics for* $t \in [0, \Theta(\epsilon^{-1})]$*, i.e. there is a* $t_2 \in]0, t_1[$ *such that*

$$\sup_{t\in[0,t_2/\epsilon]} \|U(t) - U_{**}(t)\|_{H^s} \lesssim \epsilon^q.$$

Proof. Lemma 4.3.1 guarantees that Hypothesis 4.4.8 is satisfied for all $p \geq 1$. The assumptions of Hypothesis 4.4.10 are always satisfied by Hypothesis 4.4.5 and $A_0 > 0$, see Section 4.4.1.

Theorem 4.4.11 proves the existence of $U := \mathscr{F}^{-1} Q_\kappa^{-1} V$ with the stated properties where V is the strict solution to equation (4.14). Clearly, U is a strict solution to equation (4.3), see Section 4.4.1. □

The next theorem is an immediate consequence of Theorem 4.4.1, when combined with Lemma 4.3.1.

Theorem 4.4.2. *Let* U_* *be the approximate solution to equation* (4.3) *constructed in Section 4.2 and* σ, t_1 *as in Lemma 4.3.1. Further let* $\sigma' \in]0, \sigma[$, $s \in [\frac{7}{4}, \infty[$ *and assume Hypothesis 4.4.5 below is satisfied. Finally, let* U *be a strict solution to system* (4.3) *in* $\mathscr{F}^{-1} X_{s-7/4,\sigma'/\epsilon}$ *with initial data* $\hat{U}(0) \in X_{s+1/4,\sigma'/\epsilon}$.

Then there exist $C_{U_*}, \epsilon_0 > 0$ *such that for all* $\epsilon \in]0, \epsilon_0[$ *and* $\|U_*\|_{L^\infty\left([0,t_1(\sigma'),L^2_{s,\sigma/\epsilon}\right)} \leq C_{U_*}$ *and initial data*

$$\|\hat{U}(0) - \hat{U}_*(0)\|_{X_{s,\sigma/\epsilon}} \lesssim \epsilon \tag{4.12}$$

property (4.12) *is (almost) preserved in* H^s *by the dynamics for* $t \in [0, \Theta(\epsilon^{-1})]$*, i.e. there is a* $t_2 \in]0, t_1]$ *such that*

$$\sup_{t\in[0,t_2/\epsilon]} \|U(t) - U_*(t)\|_{H^s} \lesssim \epsilon^{1/2}.$$

Proof. (Theorem 4.4.2). With the aid of Lemma 4.3.1 we can construct U_{**} for each U_*. Hence

$$\|\hat{U}(0) - \hat{U}_*(0)\|_{X_{s,\sigma/\epsilon}} = \|\hat{U}(0) - \hat{U}_{**}(0)\|_{X_{s,\sigma/\epsilon}} \lesssim \epsilon$$

and

$$\|\hat{U}_* - \hat{U}_{**}\|_{L^\infty\left([0,t_2/\epsilon],L^2_{0,\sigma'/\epsilon}\right)} \lesssim \epsilon^{1/2}.$$

Then the rest follows from Theorem 4.4.1 and the triangle inequality and some embeddings. □

If we are satisfied with estimates in L^∞, we can avoid the loss in ϵ.

Theorem 4.4.3. *Let U_* be the approximate solution to equation* (4.3) *constructed in Section 4.2 and σ, t_1 as in Lemma 4.3.1. Further let $\sigma' \in]0,\sigma[$, $s \in [\frac{7}{4}, \infty[$ and assume Hypothesis 4.4.5 below is satisfied. Finally, let U be a strict solution to system* (4.3) *in $\mathscr{F}^{-1} X_{s-7/4,\sigma'/\epsilon}$ with initial data $\hat{U}(0) \in X_{s+1/4,\sigma'/\epsilon}$.*
Then there exist $C_{U_*}, \epsilon_0 > 0$ such that for all $\epsilon \in]0, \epsilon_0[$ and $\|U_*\|_{L^\infty\left([0,t_1(\sigma'),L^2_{s,\sigma/\epsilon}\right)} \leq C_{U_*}$ and initial data

$$\|\hat{U}(0) - \hat{U}_*(0)\|_{X_{s,\sigma/\epsilon}} \lesssim \epsilon \tag{4.13}$$

property (4.13) *is preserved in L^∞ by the dynamics for $t \in [0, \Theta(\epsilon^{-1})]$, i.e. there is a $t_2 \in]0, t_1]$ such that*

$$\sup_{t\in[0,t_2/\epsilon]} \|U(t) - U_*(t)\|_{L^\infty} \lesssim \epsilon.$$

Proof. The proof is the same as for Theorem 4.4.2 before, we recall only that we also have

$$\|\hat{U}_* - \hat{U}_{**}\|_{L^\infty([0,t_2/\epsilon],L^\infty)} \lesssim \epsilon$$

by Lemma 4.3.1. □

***Remark* 4.4.4.** The condition

$$\|\hat{U}(0) - \hat{U}_{**}(0)\|_{X_{s+1/4,\sigma/\epsilon}} \lesssim \epsilon^q$$

is somewhat annoying. First, we note that

$$\|\hat{U}(0) - \hat{U}_{**}(0)\|_{L^2_{s+5/4,\sigma/\epsilon}} \lesssim \epsilon^q$$

is trivially sufficient. Secondly, we have the same problem as noted before in Chapter 3, Section 3.5. If we require an estimate uniform for $\epsilon \in]0,\epsilon_0]$, then this leads to the requirement

$$\bigcap_{\sigma>0} \|\hat{U}(0) - \hat{U}_{**}(0)\|_{X_{s,\sigma/\epsilon}} \lesssim \epsilon^q.$$

This means that the support of $\hat{U}(0) - \hat{U}_{**}(0)$ has to have measure 0. Hence $U(0) = U_{**}(0)$ (a.e.) is the only possibility. A similar statement is true for the condition $\|U_*\|_{L^\infty\left([0,t_1(\sigma'),L^2_{s,\sigma/\epsilon}\right)} \leq C_{U_*}$.

We will apply some transformations to system 4.3 and its Fourier transform for the proof of Theorem 4.4.11 below. Therefore we will do some preparatory work first.

4.4.1 Preparations

We consider the Fourier transform of system (4.3) in $L^2_1 \times L^2 \times L^2_1 \times L^2_1$ and additionally assume that for every $\xi \in \mathbb{R}$ the symbol $\tilde{\mathcal{L}}(\mathrm{i}\xi) \in \mathbb{C}^{4\times 4}$ has eigenvalues that depend continuously on ξ and that there is a basis of eigenvectors, which depends continuously on ξ, too. In the language of Kato's book on perturbation theory [55] this means that for all $\xi \in \mathbb{R}$ the eigenvalues of the symbol are semisimple and continuously depending on ξ as well as a basis of eigenvectors. We should point out that this assumption is important and does not hold in general. It is clear to hold true in the trivial case $d = \gamma = 0$ and $\xi_0 \neq 0$ since system (4.1) decouples in this case and the Ginzburg-Landau equation part is the same as in Chapter 3. A counterexample where $\tilde{\mathcal{L}}(\mathrm{i}\xi)$ is not diagonable is $\xi_0 = d = \beta = \gamma = 0$ and $\xi = \alpha = 1$. We summarise these assumptions as follows.

Hypothesis 4.4.5. *We assume that for every $\xi \in \mathbb{R}$ the symbol $\tilde{\mathcal{L}}(\mathrm{i}\xi) \in \mathbb{C}^{4\times 4}$ has eigenvalues that depend continuously on ξ and that there is a basis of eigenvectors also depending continuously on ξ for every $\xi \in \mathbb{R}$.*

We will now study the linear operator of system (4.3) more closely. The principal symbol of the linear operator in system (4.3) is in this space

$$\begin{pmatrix} -\xi^2 & -\alpha \mathrm{i}\xi & 0 & 0 \\ -\mathrm{i}\alpha\xi^3 & -\xi^2 & 0 & 0 \\ 0 & 0 & -a\xi^2 & 0 \\ 0 & 0 & 0 & -a\xi^2 \end{pmatrix}.$$

The transform $\tilde{S} : L_1^2 \times L^2 \times L_1^2 \times L_1^2 \to L^2 \times L^2 \times L^2 \times L^2$

$$u \mapsto M_{A(\boldsymbol{\xi})} u$$

transforms the principal symbol in a diagonal form where $A(\boldsymbol{\xi}) : \mathbb{R}^{\times} \to \mathbb{R}^{4\times 4}$ is the mapping

$$\xi \mapsto \frac{1}{2} \begin{pmatrix} \mathrm{sign}(\alpha)\xi & -1 & 0 & 0 \\ \xi & \mathrm{sign}(\alpha) & 0 & 0 \\ 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & \xi \end{pmatrix}.$$

Hence, there is an isomorphism $S : L_1^2 \times L^2 \times L_1^2 \times L_1^2 \to L^2 \times L^2 \times L^2 \times L^2$ such that

$$S(\xi)\tilde{\mathcal{L}}(\mathrm{i}\xi)S^{-1}(\xi) = \mathrm{diag}\left(\lambda_1(\xi), \lambda_2(\xi), \lambda_3(\xi), \lambda_4(\xi)\right) \in \mathbb{C}^{4\times 4}$$

is a diagonal matrix for all $\xi \in \mathbb{R}$ since we presumed that $\tilde{\mathcal{L}}(\mathrm{i}\xi)$ is diagonable for every $\xi \in \mathbb{R}$. Note that $\mathrm{Re}\lambda_{1/2} = -\xi^2 + \mathcal{O}(|\xi|)$ and $\mathrm{Re}\lambda_{3/4} = -a\xi^2 + \mathcal{O}(|\xi|)$ for $|\xi| \to \infty$. This becomes obvious if we look at the principal symbol.

***Remark* 4.4.6.** We should emphasise that the choice of the space – whilst probably the most natural one – is not unique. There are other possible choices where the principal symbol would be elliptic in the sense of Douglis and Nirenberg [28]. See also [26] for a more recent representation of this idea. One possible other choice is e.g. $L_1^2 \times L^2 \times L^2 \times L^2$. The choice is natural in the sense that we obtain a symbol that is similar to the symbol of the original system (4.2) in L^2.

We are only interested in choices that guarantee ellipticity in the sense of Douglis and Nirenberg for the principal symbol because problem (4.3) is of nonlinear parabolic type in this case. Thus, local existence and uniqueness of strict solutions is rather straightforward, cf. [25, Chapter II] also. We give such a result for the Fourier transform of (4.3) in the corresponding $L^2_{s,\sigma}$ spaces in Appendix 4.C .

Besides the asymptotic of the symbol for $|\xi| \to \infty$ we are also interested in the asymptotic for $\xi \to 0$ and observe that

$$\tilde{\mathcal{L}}(0) = \begin{pmatrix} -2|A_0|^2 & -2\xi_0 & \mathrm{Re}\gamma & -\mathrm{Im}\gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

obviously has eigenvalues $-2|A_0|^2$ with multiplicity 1 and 0 with algebraic (and geometric) multiplicity 3. The projection on the subspace belonging to 0 for $\xi = 0$ is given in a vicinity of $\xi = 0$ by, see Section 4.B,

$$P_0(\mathrm{i}\xi) = \begin{pmatrix} 0 & -\frac{\xi_0}{|A_0|^2} & \frac{\mathrm{Re}\gamma}{2|A_0|^2} & -\frac{\mathrm{Im}\gamma}{2|A_0|^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(|\xi|)$$

and the projection on the subspace belonging to $-2|A_0|^2$ for $\xi = 0$ by

$$P_{-2|A_0|^2}(\mathrm{i}\xi) = \begin{pmatrix} 1 & \frac{\xi_0}{|A_0|^2} & -\frac{\mathrm{Re}\gamma}{2|A_0|^2} & \frac{\mathrm{Im}\gamma}{2|A_0|^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(|\xi|).$$

Since all eigenvalues are semisimple for real ξ the mappings $\mathbb{R} \ni \xi \mapsto \lambda_i(\xi) \in \mathbb{C}$, $i \in \{1, \dots, 4\}$, are continuously differentiable, cf. [55, Theorem II.2.3], and we obtain the estimates

$$\mathrm{Re}\lambda(\xi) + \min\{1, a\}\xi^2 \lesssim |\xi|$$

for three eigenvalue curves and for the fourth

$$\mathrm{Re}\lambda(\xi) + 2|A_0|^2 + \min\{1, a\}\xi^2 \lesssim |\xi|$$

by a close look at the principal symbol (or better the transformed version of it) and the values for $\xi = 0$. We choose the isomorphism S in the following in such a way that the eigenvalue curve obeying the last estimate belongs to the first component of the transformed system.

Once again – and for the same reasons as in Section 3.5.1 – a time dependant scale of Banach spaces is used. We introduce for $s, s' \geq 0$ the transform $R_\kappa^{-1} : I_\kappa \times C^1(I, L^p_s \times L^p_{s'}) \to C^1(I_\kappa, L^p_{s,\sigma_t} \times L^p_{s',\sigma_t})$, $\sigma_t := (\sigma - \epsilon\kappa t)/\epsilon$, $1 \leq p \leq \infty$,

$$(t, u) \mapsto e^{-\sigma_t |\cdot|} u(t),$$

which is obviously an isomorphism. Here $I_\kappa := [0, \frac{\sigma}{\epsilon\kappa}] \cap I$.

We abbreviate $Q_\kappa := SR_\kappa$. Using these isomorphisms we transform the system $\hat{U} = (Q_\kappa)^{-1} V$ and obtain in $L^2 \times L^2 \times L^2 \times L^2$

$$\partial_t V = (\Lambda - \kappa|\boldsymbol{\xi}|)V + \mathcal{N}(V) \tag{4.14}$$

with $\Lambda = S\tilde{\mathcal{L}}(\mathrm{i}\xi)S^{-1}$ and

$$\mathcal{N}(V) = Q_\kappa \mathscr{F} \tilde{N}((Q_\kappa \mathscr{F})^{-1} V).$$

Here $\mathcal{N}_j$, $j \in \{1, \ldots, 4\}$, is a composition of Fourier transforms of entire functions and derivatives of them. We observe that we have

$$\partial_t P_0(\mathrm{i}\xi)\hat{U}(\xi) = \sum_{i=1}^{s} \lambda_i(\mathrm{i}\xi) P_{0,i}(\mathrm{i}\xi)\hat{U}(\xi) + P_0(\mathrm{i}\xi) \mathscr{F} \tilde{N}(U)(\xi)$$

in a deleted environment $U(0)$ of $\xi = 0$, where $P_{0,i}$ are the projections belonging to the different eigenvalues λ_i, $i = 2, \ldots, s+1$, and $P_0(\mathrm{i}\xi) = \sum_{i=1}^{s} P_{0,i}(\mathrm{i}\xi)$ is the total projection of the λ-group in the notation of Kato. s is the number of eigenvalues that split out of 0 for $\xi \in U(0)$ and might be any of $1, 2, 3$. For example, for $\alpha = \beta = \xi_0 = c = d = 0$ and $a = 1$, we obtain $s = 1$ whereas for $\alpha \neq 0$ and $d = 0$ or $\gamma = 0$ we have $s = 2$. The case $s = 3$ is not so clear but in this case necessarily $d \neq 0$ or $\gamma \neq 0$. A closer look at the expansion of the eigenvalues around $\xi = 0$ reveals that this is happening for e.g. $\xi_0 = 0$, $c \neq 0$ and $d\mathrm{Re}\gamma \in \mathbb{R}\backslash\{0, -c\}$, see Appendix 4.B.

Because of the structure of $\tilde{N} = (n_1, \partial_x n_2, \partial_x n_3, 0)$ we have

$$\begin{aligned} P_0(\mathrm{i}\xi)\,\mathscr{F}\,\tilde{N}(U)(\xi) &= \mathrm{i}\xi P_0^{(0)}(0)\,\mathscr{F}\begin{pmatrix} 0 \\ n_2(U) \\ n_3(U) \\ 0 \end{pmatrix}(\xi) \\ &\quad + \mathrm{i}\xi P_0^{(1)}(0)\,\mathscr{F}\begin{pmatrix} n_1(U) \\ \partial_x n_2(U) \\ \partial_x n_3(U) \\ 0 \end{pmatrix}(\xi) + \mathcal{O}(|\xi|^2) \end{aligned} \tag{4.15}$$

where $P_0^{(j)}(0)$ are the coefficients of the power series expansion of the projector $P_0(\xi) = \sum_{j=0}^{\infty} P_0^{(j)}\xi^j$ in this environment, meaning $P_0^{(0)}(0) = P_0(0)$ given above. If we use one of the eigenvectors for U, say $\hat{U}(\xi) = P_{0,i}(\mathrm{i}\xi)\hat{U}(\xi)$, this results in

$$\begin{aligned} \partial_t \hat{U}(\xi) &= \lambda_i(\mathrm{i}\xi)\hat{U}(\xi) + P_{0,i}(\mathrm{i}\xi)P_0(\mathrm{i}\xi)\,\mathscr{F}\,\tilde{N}(U)(\xi) \\ &= \lambda_i(\mathrm{i}\xi)\hat{U}(\xi) + P_{0,i}(\mathrm{i}\xi)\,\mathscr{F}\,\tilde{N}(U)(\xi). \end{aligned}$$

This calculation together with (4.15) shows that the nonlinear part in the diagonal system in L^2 – after application of the transformation Q_κ – is of order $\mathcal{O}(\xi)$ for $\xi \to 0$. Thus we see that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mathrm{i}\xi^{-1} & 0 & 0 \\ 0 & 0 & -\mathrm{i}\xi^{-1} & 0 \\ 0 & 0 & 0 & -\mathrm{i}\xi^{-1} \end{pmatrix} \mathcal{N}(V)(\xi)$$

is bounded for $\xi \to 0$. Recalling the structure of $\tilde{N}$ in system (4.3), we conclude that $PSG^{-1} : L_1^2 \times L^2 \times L_1^2 \times L_1^2 \to L^2$ has to be a bounded map where P is the projection to the subspace spanned by the components 2 to 4, i.e.

$$L^2 \ni (u_1, u_2, u_3, u_4) \mapsto Pu = (u_2, u_3, u_4) \in L^2,$$

and $G : L^p_{s_1,\sigma_1} \times L^p_{s_2,\sigma_2} \times L^p_{s_3,\sigma_3} \times L^p_{s_4,\sigma_4} \to L^p_{s_1,\sigma_1} \cap L^{p,h}_{-1} \times L^p_{s_2,\sigma_2} \times L^p_{s_3,\sigma_3} \times L^p_{s_4,\sigma_4}$,

$s_i, \sigma_i \in [0, \infty[$, is defined as multiplication operator acting by multiplication with

$$\begin{pmatrix} \frac{\mathrm{i}\boldsymbol{\xi}}{\sqrt{1+|\boldsymbol{\xi}|^2}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

***Remark* 4.4.7.**

- We proved by analytic perturbation theory that at least in an environment of $\xi = 0$ the symbol is always diagonable. This is a trivial consequence of the fact that $\operatorname{rank} P_0(0) = 3$ and that the set of exceptional points is discrete. Therefore, Jordan blocks cannot appear in a (small) neighbourhood of $\xi = 0$.
- In general it is not clear whether the projections $P_0^{(i)}$ depend holomorphically on ξ in an environment of $\xi = 0$. For a certain set of parameters the dependence is holomorphic, see Appendix 4.B.
- Obviously, the operator G^{-1} is in general not applicable to functions of $L_1^2 \times L^2 \times L_1^2 \times L_1^2$. But PSG^{-1} can be defined in the above way nonetheless and is bounded. This is only true because of the special structure of S and because we only consider the last three components of the vector. This fact will be used in the proof of Theorem 4.4.11.

4.4.2 Estimates

The proof of the main theorem is similar to the proof of Theorem 3.5.8 in Section 3.5. Recall that we can satisfy the following two assumptions, which will again be sufficient to prove the theorem. The first one once again concerns properties of the approximate solution and the residual. The second one is in essence the spectral estimate from above.

Hypothesis 4.4.8. *Let $I = [0, \frac{t_1}{\epsilon}]$, $\epsilon \in (0, 1], s' \geq \frac{7}{4}, \sigma > 0$ and $p \geq 1$. Suppose there is a function*

$$U_{**} = \begin{pmatrix} r_{**}(\epsilon \boldsymbol{t}, \epsilon \boldsymbol{x}) \\ \psi_{**}(\epsilon \boldsymbol{t}, \epsilon \boldsymbol{x}) \\ b_{**}(\epsilon \boldsymbol{t}, \epsilon \boldsymbol{x}) \end{pmatrix}$$

for functions $r_{**}, \psi_{**}, b_{**} \in C\big([0,t_1], \mathrm{Hol}_\sigma\big)$ *and* $\mathfrak{E}_{U_{**}} \in C(I, \mathrm{Hol}_{\sigma/\epsilon})$,

$$\|\mathscr{F}\,\mathfrak{E}_{U_{**}}\|_{L^\infty\left(I, L^2_{s'+1,\sigma'/\epsilon}\right)} \lesssim \epsilon^{p+1},$$

for $\sigma' < \sigma$ *where*

$$\mathfrak{E}_U \coloneqq \big(\tilde{\mathcal{L}}(\partial_x) - \partial_t\big)U + \tilde{N}(U)$$

is the residual of a sufficiently regular function U *for equation* (4.3).

This is in essence a summary of Lemma 4.3.1 for $p \in \mathbb{N} + \frac{1}{2}$.

***Remark* 4.4.9.** Obviously,

$$(\Lambda - \kappa|\boldsymbol{\xi}| - \partial_t)V + \mathcal{N}(V) = Q_\kappa \mathscr{F}\left(\big(\tilde{\mathcal{L}}(\partial_x) - \partial_t\big)U + \tilde{N}(U)\right)$$

so that

$$\begin{aligned}\|(\Lambda - \kappa|\boldsymbol{\xi}| - \partial_t)V(t) + \mathcal{N}(V)(t)\|_{L^2_s} = \|Q_\kappa \mathscr{F}\,\mathfrak{E}_U(t)\|_{L^2_s} &\lesssim \|R_\kappa \mathscr{F}\,\mathfrak{E}_U(t)\|_{L^2_{s+1}} \\ &\lesssim \|\mathscr{F}\,\mathfrak{E}_U(t)\|_{L^2_{s+1,\sigma/\epsilon}}\end{aligned}$$

for sufficiently regular U and $t \in I_\kappa$, which was defined in Section 4.4.1.

The following spectral assumption is necessary for the proof of the main theorem and trivially covered by the estimates in the previous Section 4.4.1.

Hypothesis 4.4.10. *We assume that the symbol* Λ *is a diagonal matrix and the real parts of the eigenvalues of* $\Lambda = diag(\lambda_1, \ldots, \lambda_4)$ *can be bounded by curves*

$$\begin{aligned}&\mathrm{Re}(\lambda_1 - \kappa|\boldsymbol{\xi}|) \le -|A_0|^2 - C_1|\boldsymbol{\xi}| - C_2|\boldsymbol{\xi}|^2, && \\ &\mathrm{Re}(\lambda_j - \kappa|\boldsymbol{\xi}|) \le -C_1|\boldsymbol{\xi}|, && j \in \{2,3,4\},\end{aligned}$$

for $C_1, C_2 > 0$ *and independent of* $\epsilon \in (0,1]$.

Large parts of the proof of Theorem 4.4.1 will be analogue to the proof of Theorem 3.5.1. Recall that it is sufficient to prove an error bound for system (4.14) in L^2_s for some $s > \frac{1}{2}$ because of the embeddings $L^2_{s,\sigma} \hookrightarrow L^2_s$. Let σ be smaller than any of Section 4.3 or of the assumptions of Theorem 4.4.1. Then, by the embeddings, we can assume that we have an approximate solution in $\bigcap_{s \ge 0} L^2_{s,\sigma}$.

Note that the consequence of this assumption might be that $t_2 \in]0, t_1[$ and not in the half closed interval.

As usual we define the error $\epsilon^q R := V - V_{**}$ for a strict solution V to equation (4.14) and $V_{**} = Q_\kappa \hat{U}_{**}$. Since we only consider 'small' errors – at least initially – we expect that the growth of the error will mainly come from the linear part of the operator and the linearisation of $\mathcal{N}$ around V_{**}. Since the linear operator is essentially diffusive, we will focus on the linearisation. We will estimate the error in a certain time interval with an energy argument and prove the following theorem.

Theorem 4.4.11. *Suppose the assumptions of Hypotheses 4.4.5, 4.4.8 and 4.4.10 are satisfied. Let* $V_{**} = Q_\kappa \hat{U}_{**}$, $s \in [\frac{7}{4}, s']$ *and* $q \in [1, p]$. *Let* V *be a strict solution to system* (4.14) *in* $L^2_{s-7/4}$ *for initial data* $V(0) \in L^2_{s+1/4}$
Then there exist $C_V, \epsilon_0 > 0$ *such that for all* $\epsilon \in]0, \epsilon_0[$ *and* $\|V_{**}\|_{L^\infty(I_\kappa, L^2_s)} \leq C_V$ *and initial data*

$$\|V(0) - V_{**}(0)\|_{L^2_s} \lesssim \epsilon^q \tag{4.16}$$

property (4.16) *is preserved by the dynamics for* $t \in [0, \Theta(\epsilon^{-1})]$, *i.e. there is a* $t_2 \in]0, t_1]$ *such that*

$$\sup_{t \in [0, t_2/\epsilon]} \|V(t) - V_{**}(t)\|_{L^2_s} \lesssim \epsilon^q.$$

Proof. As previously announced we write a solution to equation (4.14) as $V = V_{**} + \epsilon^q R$. Then the error function R satisfies, as usual, see Section 1.1,

$$\partial_t R = (\Lambda - \kappa|\boldsymbol{\xi}|)R + \epsilon^{-q}\mathcal{N}(V_{**} + \epsilon^q R) - \epsilon^{-q}\mathcal{N}(V_{**}) + \epsilon^{-q}\left(Q_\kappa \mathscr{F}\,\mathfrak{E}_U\right).$$

Instead of the residual for V we used the transformed residual for U, which is equivalent, see Remark 4.4.9. We have to prove that R is in $L^\infty([0, t_2/\epsilon], L^2_s)$ for a $t_2 \in]0, t_1]$. Then the result follows immediately since R is continuous in time. For this purpose we use the energies

$$E_s = \frac{1}{2}\|R\|^2_{L^{2,b}_s}, \qquad F_s = \frac{1}{2}\|R_1\|^2_{L^{2,h}_{1/2} \cap L^{2,h}_s} + \frac{1}{2}\sum_{j=2}^{4}\|R_j\|^2_{L^{2,b}_s}.$$

We know that $R \in C^1([0,\delta[, L^2_{s-7/4})$ for a (maximal) $\delta \in \mathbb{R}^+ \cup \{\infty\}$ by local existence and uniqueness results for the approximate solution and equation (4.3) (or the transformed equation (4.14) respectively). The first is clear from Lemma 4.3.1, the latter is clear by standard methods for parabolic semilinear problems, cf. Section 4.C. Further we note that $R \in C^1([0,\delta[, L^2_s)$, cf. [44, Theorem 3.5.2; 73, Chapter 6, Corollary 3.2]. Since we can use Proposition A.1.4 in combination with Lemma A.2.1 with $r = \frac{7}{4}$ and $s_0 \geq 0$ for V, we have $R \in C(]0,\delta[, L^2_{s+2})$, too.

If $\delta < \max I_\kappa$, it is clear that $\|V(t)\|_{L^2_s} \to \infty$ for $t \to \delta < \infty$ since V satisfies a mild solution formula in $C([0,\delta[, L^2_s)$. Consequently $E_s(t) \to \infty$ for $t \to \delta < \max I_\kappa$. Thus there is an ϵ-independent constant $M > 1$ satisfying $E_s(0) < M$ and we define

$$0 < T_2 := \sup\left\{t \in I_\kappa \,|\, E_s(t) < C_M M\right\},$$

where $C_M > C_T C_{T^{-1}}(1+C_F)$ is a constant that has to be chosen large enough and is independent of $\epsilon \in [0,\epsilon_0]$ – see the comments in Remark 4.4.12 for an estimate on its size. The constants $C_T, C_{T^{-1}}$ and C_F will be introduced while giving the proof.

The proof that $T_2 \in \Omega(\epsilon^{-1})$ is given as follows. We will introduce a function $B \in C^0(\mathbb{R}^+_0, \mathbb{R}^+)$ such that $E(t) \leq B(\epsilon t)$ for all $t \in \{r \in I_\kappa \,|\, E_s(r) < C_M M\}$. Then we will see that there is a $0 < x_0$ such that $B(x) < C_M M$ for all $x \in [0, x_0]$, which proves the claim with $t_2 = \epsilon T_2 = x_0$.

Let $\tilde{\delta} \in]0, T_2/2[$ be so small that $E(\tilde{\delta}) < M$. In a first step in the proof, we notice that $E_s, F_s \in C^1([\tilde{\delta}, T_2], L^2_s)$, see Lemma A.2.1, and $F_s \leq C_F E_s$. Therefore we can easily calculate the time derivatives for these energies:

$$\begin{aligned}
\frac{d}{dt}E_s &= \mathrm{Re}\langle R, (\Lambda - \kappa|\boldsymbol{\xi}|)R + \epsilon^{-q}\mathcal{N}(V_{**} + \epsilon^q R) - \epsilon^{-q}\mathcal{N}(V_{**}) + \epsilon^{-q}\left(Q_\kappa \mathscr{F}\, \mathfrak{E}_U\right)\rangle_{L^{2,b}_s} \\
\frac{d}{dt}F_s &= \mathrm{Re}\langle R_1, (\lambda_1 - \kappa|\boldsymbol{\xi}|)R_1 + \epsilon^{-q}\left[\mathcal{N}(V_{**} + \epsilon^q R) - \mathcal{N}(V_{**})\right]_1\rangle_{L^{2,h}_{1/2} \cap L^{2,h}_s} \\
&\quad + \mathrm{Re}\langle R_1, \epsilon^{-q}\left[(Q_\kappa \mathscr{F}\, \mathfrak{E}_U)\right]_1\rangle_{L^{2,h}_{1/2} \cap L^{2,h}_s} \\
&\quad + \sum_{j=2}^{4} \mathrm{Re}\langle R_j, (\lambda_j - \kappa|\boldsymbol{\xi}|)R_j + \epsilon^{-q}\left[\mathcal{N}(V_{**} + \epsilon^q R) - \mathcal{N}(V_{**})\right]_j\rangle_{L^{2,b}_s} \\
&\qquad + \mathrm{Re}\langle R_j, \epsilon^{-q}\left[(Q_\kappa \mathscr{F}\, \mathfrak{E}_U)\right]_j\rangle_{L^{2,b}_s}.
\end{aligned}$$

Since Hypothesis 4.4.10 is satisfied the linear part $(\Lambda - \epsilon^2\kappa|\boldsymbol{\xi}|)R$ gives some damping terms if we choose $C_1 > 0$ (meaning κ large enough), which are for E_s

$$-2|A_0|^2\|R_1\|^2_{L^{2,b}_s}, \qquad -C_1\||\boldsymbol{\xi}|^{1/2}R\|^2_{L^{2,b}_s}, \qquad -C_2\||\boldsymbol{\xi}|R_1\|^2_{L^{2,b}_s}.$$

For F_s we obtain similarly

$$\begin{aligned} -2|A_0|^2\|R_1\|^2_{L^{2,h}_{1/2}\cap L^{2,h}_s}, &\qquad -C_1\|R_1\|^2_{L^{2,h}_1\cap L^{2,h}_{s+1/2}}, \\ -C_2\|R_1\|^2_{L^{2,h}_{3/2}\cap L^{2,h}_{s+1}}, &\qquad -C_1\||\boldsymbol{\xi}|^{1/2}R_j\|^2_{L^{2,b}_s}, \end{aligned}$$

for $j \in \{2, 3, 4\}$. We will use these damping terms to derive a system of differential inequalities for the energies, see inequality (4.19) below. So, we have to estimate all nonlinear parts of the time derivatives of the energies for the derivation of this system of inequalities. We proceed again as in the proof of Theorem 3.5.8 by giving an estimate of (a) the residual parts, (b) the (in R) nonlinear parts and (c) the linearised parts (linear in R).

The most simple addend is the residual. Because of Hypothesis 4.4.8 and Remark 4.4.9, we have for $j \in \{1, \dots, 4\}$

$$\begin{aligned} \operatorname{Re}\langle R, \epsilon^{-q}\left(Q_\kappa \mathscr{F}\, \mathfrak{E}_U\right)\rangle_{L^{2,b}_s} &\lesssim \epsilon^{-q}\|R\|_{L^2_s}\|\mathscr{F}\,\mathfrak{E}_U\|_{L^2_{s+1,\sigma/\epsilon}} \\ &\lesssim \epsilon^{1+p-q}\|R\|_{L^2_s} \lesssim \epsilon(1+E_s), \\ \operatorname{Re}\langle R_j, \left[\epsilon^{-q}\left(Q_\kappa \mathscr{F}\, \mathfrak{E}_U\right)\right]_j\rangle_{L^{2,b}_s} &\lesssim \epsilon^{-q}\|R_j\|_{L^{2,b}_s}\|\mathscr{F}\,\mathfrak{E}_U\|_{L^2_{s+1,\sigma/\epsilon}} \lesssim \epsilon^{1+p-q}\|R_j\|_{L^{2,b}_s} \\ &\lesssim \epsilon(1+E_s), \\ \operatorname{Re}\langle R_1, \left[\epsilon^{-q}\left(Q_\kappa \mathscr{F}\, \mathfrak{E}_U\right)\right]_1\rangle_{L^{2,h}_{1/2}\cap L^{2,h}_s} &\lesssim \epsilon^{-q}\|R_1\|_{L^{2,h}_{1/2}\cap L^{2,h}_s}\|\mathscr{F}\,\mathfrak{E}_U\|_{L^2_{s+1,\sigma/\epsilon}} \\ &\lesssim \epsilon^{1+p-q}\|R_1\|_{L^{2,h}_{1/2}\cap L^{2,h}_s} \lesssim \epsilon(1+E_s). \end{aligned}$$

We applied Young's inequality and used the equivalence of the norms of $L^{2,b}_s$ and L^2_s for $s \geq 0$ in the last step. We denote the largest constant in these estimates with $C_{\mathrm{RES}}/2$.

Now the estimate of the difference of the nonlinear parts remains. We have to be more careful for these. We notice that every component of the nonlinear part $\tilde{N}$ is in essence an entire function (in the variables $r, \psi, \operatorname{Re} b, \operatorname{Im} b$ and their spatial derivatives). Hence, these are well suited for the spaces $L^2_{s,\sigma}$ and we can apply Proposition A.1.4 to every component of $\mathcal{N}$, the transformed version of $\tilde{N}$. Since the mapping is C^2 we can split it in a linearised part and the actual nonlinear part. We obtain

$$\begin{aligned}\epsilon^{-q}\left(\mathcal{N}(V_{**}+\epsilon^q R)-\mathcal{N}(V_{**})\right)=Q_\kappa D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R]\\ +\epsilon^{-q}Q_\kappa K(Q_\kappa^{-1}V_{**},\epsilon^q Q_\kappa^{-1}R),\end{aligned} \tag{4.17}$$

where K is the second order corrector of the Taylor expansion. We use the abbreviations $X_{s,\sigma}=L^2_{s+1,\sigma}\times L^2_{s,\sigma}\times L^2_{s+1,\sigma}\times L^2_{s+1,\sigma}$ and $Y_{s,\sigma}=L^1_{s+1,\sigma}\times L^1_{s,\sigma}\times L^1_{s+1,\sigma}\times L^1_{s+1,\sigma}$ for a shorter notation. By Proposition A.1.4 and the Bunyakovsky-Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\mathrm{Re}\langle R,\epsilon^{-q}Q_\kappa K(Q_\kappa^{-1}V_{**},\epsilon^q Q_\kappa^{-1}R)\rangle_{L^{2,b}_s}\\
=&\mathrm{Re}\langle R,\epsilon^{-q}Q_\kappa K(Q_\kappa^{-1}V_{**},\epsilon^q Q_\kappa^{-1}R)\rangle_{L^2}\\
&+\mathrm{Re}\langle \xi^{s+1/2}R,\epsilon^{-q}\xi^{s-1/2}Q_\kappa K(Q_\kappa^{-1}V_{**},\epsilon^q Q_\kappa^{-1}R)\rangle_{L^2}\\
\lesssim&\epsilon^{-q}\|R\|_{L^{2,b}_{s+1/2}}\left(\|K(Q_\kappa^{-1}V_{**},\epsilon^q Q_\kappa^{-1}R)\|_{X_{0,\sigma_t}}+\|K(Q_\kappa^{-1}V_{**},\epsilon^q Q_\kappa^{-1}R)\|_{X_{s-1/2,\sigma_t}}\right)\\
\lesssim&\epsilon^{q}\|R\|_{L^{2,b}_{s+1/2}}\|Q_\kappa^{-1}V_{**}\|_{Y_{1,\sigma_t}}\|Q_\kappa^{-1}R\|_{X_{1,\sigma_t}}\|Q_\kappa^{-1}R\|_{Y_{1,\sigma_t}}\\
&\quad+\epsilon^{q}\|R\|_{L^{2,b}_{s+1/2}}\|Q_\kappa^{-1}V_{**}\|_{Y_{s+1/2,\sigma_t}}\|Q_\kappa^{-1}R\|_{X_{s+1/2,\sigma_t}}\|Q_\kappa^{-1}R\|_{Y_{1,\sigma_t}}\\
\lesssim&\epsilon^{q}\left(\|R\|_{L^{2,b}_{s+1/2}}\|V_{**}\|_{L^1_1}\|R\|_{L^2_1}\|R\|_{L^2_s}+\|R\|_{L^{2,b}_{s+1/2}}\|V_{**}\|_{L^1_{s+1/2}}\|R\|_{L^2_{s+1/2}}\|R\|_{L^2_s}\right)\\
\lesssim&\epsilon^{q}\|R\|^2_{L^{2,b}_{s+1/2}}
\end{aligned}$$

In the antepenultimate last step we exploited the fact that $\tilde{N}$ has the structure

$$\tilde{N}(r,\psi,\mathrm{Re}\,b,\mathrm{Im}\,b)=\begin{pmatrix}\tilde{n}_1(r,\partial_x r,\psi)\\ \tilde{n}_2(r,\partial_x r,\partial_x^2 r,\psi,\partial_x\psi)\\ \tilde{n}_3(r,\partial_x r)\\ 0\end{pmatrix}=\begin{pmatrix}\tilde{m}_1(r,\partial_x r,\psi)\\ \partial_x\tilde{m}_2(r,\partial_x r,\psi)\\ \partial_x\tilde{m}_3(r)\\ 0\end{pmatrix}, \tag{4.18}$$

where $\tilde{n}_1,\tilde{n}_2,\tilde{n}_3,\tilde{m}_1,\tilde{m}_2,\tilde{m}_3$ are entire functions, which are at least quadratic for $|z|\to 0$, and the embeddings of the $L^p_{s,\sigma}$ spaces. The hidden constant in the estimate stems from the entire functions of Proposition A.1.4 depending on $\|Q_\kappa^{-1}V_{**}\|_{Y_{1,\sigma_t}}\lesssim\|V_{**}\|_{L^1_1}\lesssim\|V_{**}\|_{L^2_s}$ and $\|Q_\kappa^{-1}R\|_{Y_{1,\sigma_t}}\lesssim\|R\|_{L^1_1}\lesssim\|R\|_{L^2_s}$. Since we consider $t\le T_2$ and $\frac{7}{4}\le s$, the entire functions are bounded with a bound depending on $C_M M$. This is why the constant in the differential inequality will in general depend on M.

Note that the calculations also show

$$\begin{aligned}
&\mathrm{Re}\langle R_1, \epsilon^{-q}\left[Q_\kappa K(Q_\kappa^{-1}V_{**}, \epsilon^q Q_\kappa^{-1}R)\right]_1\rangle_{L^{2,h}_{1/2}\cap L^{2,h}_s} \\
&+\sum_{j=2}^{4}\mathrm{Re}\langle R_j, \epsilon^{-q}\left[Q_\kappa K(Q_\kappa^{-1}V_{**}, \epsilon^q Q_\kappa^{-1}\right]_j R)\rangle_{L^{2,b}_s} \\
\lesssim&\epsilon^q\|R\|^2_{L^{2,b}_{s+1/2}}.
\end{aligned}$$

Finally, we have to estimate the linearisation. It is

$$\begin{aligned}
\langle R, Q_\kappa D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R]\rangle_{L^{2,b}_s} =& \langle R, Q_\kappa D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R]\rangle_{L^2} \\
&+\langle R, Q_\kappa D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R]\rangle_{L^{2,h}_s}.
\end{aligned}$$

We split the components in the scalar product. Then we find

$$\begin{aligned}
&\langle R_1, (Q_\kappa D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R])_1\rangle_{L^{2,h}_s} \\
\leq&\|R_1\|_{L^{2,h}_{s+1/2}}\|(Q_\kappa D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R]_1\|_{L^{2,h}_{s-1/2}} \\
\lesssim&\|R_1\|_{L^{2,h}_{s+1/2}}\|\xi^{s-1/2}D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R]\|_{X_{0,\sigma_t}} \\
\lesssim&\|R_1\|_{L^{2,h}_{s+1/2}}\|\xi^{s-1/2}Q_\kappa^{-1}V_{**}\|_{Y_{1,\sigma_t}}\|Q_\kappa^{-1}R\|_{X_{1,\sigma_t}} \\
&+\|R_1\|_{L^{2,h}_{s+1/2}}\|Q_\kappa^{-1}V_{**}\|_{Y_{1,\sigma_t}}\|\xi^{s-1/2}Q_\kappa^{-1}R\|_{X_{1,\sigma_t}} \\
\lesssim&\|R_1\|_{L^{2,h}_{s+1/2}}\|\xi^{s-1/2}V_{**}\|_{L^1_1}\|R\|_{L^2_1}+\|R_1\|_{L^{2,h}_{s+1/2}}\|V_{**}\|_{L^1_1}\|\xi^{s-1/2}R\|_{L^2_1} \\
\lesssim&\epsilon^{s-1/2}\|R_1\|_{L^{2,h}_{s+1/2}}\|R\|_{L^2_1}+\|R_1\|_{L^{2,h}_{s+1/2}}\|\xi^{s-1/2}R\|_{L^2_1} \\
\lesssim&\epsilon\|R\|^2_{L^2_1}+\|\xi^{1/2}R\|^2_{L^{2,b}_s}\lesssim\epsilon\|R\|^2_{L^2}+\|\xi^{1/2}R\|^2_{L^{2,b}_s}.
\end{aligned}$$

We used the scaling invariance of the L^1 norm with respect to $u\mapsto\frac{1}{\epsilon}u(\frac{\cdot}{\epsilon})$, the assumptions on U_{**} and crucially that $\tilde{n}_1,\tilde{n}_2,\tilde{n}_3=\mathcal{O}(|z|^2)$ for $|z|\to 0$ for these estimates. Once again the hidden constant depends on M with the same argument as in the case of the quadratic remainder. In addition we need the subsequent estimate for the estimate of the time derivative of F_s.

$$\begin{aligned}
&\langle R_1, (Q_\kappa D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R])_1\rangle_{L^{2,h}_{1/2}} \\
\leq&\|R_1\|_{L^{2,h}_{1/2}}\|(Q_\kappa D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R]_1\|_{L^{2,h}_{1/2}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|R_1\|_{L^{2,h}_{1/2}} \|\xi^{1/2} D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R]\|_{X_{0,\sigma_t}} \\
&\lesssim \|R_1\|_{L^{2,h}_{1/2}} \|\xi^{1/2} Q_\kappa^{-1}V_{**}\|_{Y_{1,\sigma_t}} \|Q_\kappa^{-1}R\|_{X_{1,\sigma_t}} \\
&\quad + \|R_1\|_{L^{2,h}_{1/2}} \|Q_\kappa^{-1}V_{**}\|_{Y_{1,\sigma_t}} \|\xi^{1/2} Q_\kappa^{-1}R\|_{X_{1,\sigma_t}} \\
&\lesssim \|R_1\|_{L^{2,h}_{1/2}} \|\xi^{1/2} V_{**}\|_{L^1_1} \|R\|_{L^2_1} + \|R_1\|_{L^{2,h}_{1/2}} \|V_{**}\|_{L^1_1} \|\xi^{1/2} R\|_{L^2_1} \\
&\lesssim \epsilon^{1/2} \|R_1\|_{L^{2,h}_{1/2}} \|R\|_{L^2_1} + \|R_1\|_{L^{2,h}_{1/2}} \|\xi^{1/2} R\|_{L^2_1} \\
&\lesssim \epsilon \|R\|_{L^2_1}^2 + \|\xi^{1/2} R\|_{L^{2,b}_s}^2 \lesssim \epsilon \|R\|_{L^2}^2 + \|\xi^{1/2} R\|_{L^{2,b}_s}^2.
\end{aligned}
$$

We proceed similarly for the L^2 estimate of the first component.

$$
\begin{aligned}
\langle R_1, (Q_\kappa D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R])_1\rangle_{L^2} &\leq \|R_1\|_{L^2} \|(Q_\kappa D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R]_1\|_{L^2} \\
&\lesssim \|R_1\|_{L^2} \|D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R]\|_{X_{0,\sigma_t}} \\
&\lesssim \|R_1\|_{L^2} \|Q_\kappa^{-1}V_{**}\|_{Y_{1,\sigma_t}} \|Q_\kappa^{-1}R\|_{X_{1,\sigma_t}} \\
&\lesssim \|R_1\|_{L^2} \|V_{**}\|_{L^1_1} \|R\|_{L^2_1} \\
&\lesssim \|R_1\|_{L^2} \|R\|_{L^2_1} \lesssim \|R_1\|_{L^2}^2 + \|R\|_{L^{2,b}_s}^2.
\end{aligned}
$$

Finally, for the j'th component of the scalar product, $j \in \{2, 3, 4\}$, we estimate

$$
\begin{aligned}
&\langle R_j, (Q_\kappa D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R])_j\rangle_{L^{2,h}_s} \\
\leq& \|R_j\|_{L^{2,h}_{s+1/2}} \|(Q_\kappa D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R])_j\|_{L^{2,h}_{s-1/2}} \\
\lesssim& \|R_j\|_{L^{2,h}_{s+1/2}} \|\xi^{s-1/2} D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R]\|_{X_{0,\sigma_t}} \\
\lesssim& \|R_j\|_{L^{2,h}_{s+1/2}} \|\xi^{s-1/2} Q_\kappa^{-1}V_{**}\|_{Y_{1,\sigma_t}} \|Q_\kappa^{-1}R\|_{X_{1,\sigma_t}} \\
&+ \|R_j\|_{L^{2,h}_{s+1/2}} \|Q_\kappa^{-1}V_{**}\|_{Y_{1,\sigma_t}} \|\xi^{s-1/2} Q_\kappa^{-1}R\|_{X_{1,\sigma_t}} \\
\lesssim& \|R_j\|_{L^{2,h}_{s+1/2}} \|\xi^{s-1/2} V_{**}\|_{L^1_1} \|R\|_{L^2_1} \\
&+ \|R_j\|_{L^{2,h}_{s+1/2}} \|V_{**}\|_{L^1_1} \||\xi|^{s-1/2} R\|_{L^2_1} \\
\lesssim& \epsilon^{s-1/2} \|R_j\|_{L^{2,h}_{s+1/2}} \|R\|_{L^2_1} + \||\boldsymbol{\xi}|^{1/2} R\|_{L^{2,b}_s}^2 \\
\lesssim& \||\boldsymbol{\xi}|^{1/2} R\|_{L^{2,b}_s}^2 + \epsilon \|R\|_{L^2}^2
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle R_j, (Q_\kappa D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R])_j\rangle_{L^2} \\
\leq&\|R_j\|_{L^{2,h}_{1/2}}\|(Q_\kappa D\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R])_j\|_{L^{2,h}_{-1/2}} \\
\lesssim&\|R_j\|_{L^{2,h}_{1/2}}\|\xi^{-1/2}GD\tilde{N}(Q_\kappa^{-1}V_{**})[Q_\kappa^{-1}R]\|_{X_{0,\sigma_t}} \\
\lesssim&\|R_j\|_{L^{2,h}_{1/2}}\|\xi^{1/2}Q_\kappa^{-1}V_{**}\|_{Y_{1,\sigma_t}}\|Q_\kappa^{-1}R\|_{X_{0,\sigma_t}} \\
&\quad+\|R_j\|_{L^{2,h}_{1/2}}\|Q_\kappa^{-1}V_{**}\|_{Y_{0,\sigma_t}}\|\xi^{1/2}Q_\kappa^{-1}R\|_{X_{1,\sigma_t}} \\
\lesssim&\|R_j\|_{L^{2,h}_{1/2}}\|\xi^{1/2}V_{**}\|_{L^1}\|R\|_{L^2}+\|R_j\|_{L^{2,h}_{1/2}}\|V_{**}\|_{L^1}\||\boldsymbol{\xi}|^{1/2}R\|_{L^{2,b}_1} \\
\lesssim&\epsilon^{1/2}\|R_j\|_{L^{2,h}_{1/2}}\|R\|_{L^2}+\||\boldsymbol{\xi}|^{1/2}R\|^2_{L^{2,b}_1} \\
\lesssim&\||\boldsymbol{\xi}|^{1/2}R\|^2_{L^{2,b}_s}+\epsilon\|R\|^2_{L^2}.
\end{aligned}
$$

These estimates easily follow from Proposition A.1.4 and the nonlinear part written using the $\tilde{m}_i$ in (4.18). We have to emphasise that all hidden constants in these estimates also depend on $\|V_{**}\|_{L^2_s}$ and are at least of class $\mathcal{O}(\|V_{**}\|_{L^2_s})$ for $\|V_{**}\|_{L^2_s}\to 0$. This follows from the fact that the nonlinear part is at least quadratic and Proposition A.1.4.

Hence, there are constants $C_{\mathrm{L}}, C_{\mathrm{NL}}\geq 0$ such that

$$
\begin{aligned}
&\mathrm{Re}\langle R, \epsilon^{-q}\left(\mathcal{N}\big(V_{**}+\epsilon^q R\big)-\mathcal{N}\big(V_{**}\big)\right)\rangle_{L^{2,b}_s} \\
\leq&C_{\mathrm{L}}\big(\|R_1\|^2_{L^2}+\sum_{j=2}^{4}\|R_j\|^2_{L^2}+\||\boldsymbol{\xi}|^{1/2}R\|^2_{L^{2,b}_s}\big)+C_{\mathrm{NL}}\epsilon\|R\|^2_{L^{2,b}_{s+1/2}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{Re}\langle R_1, \epsilon^{-q}\left[\mathcal{N}(V_{**}+\epsilon^q R)-\mathcal{N}(V_{**})\right]_1\rangle_{L^{2,h}_{1/2}\cap L^{2,h}_s} \\
&+\sum_{j=2}^{4}\mathrm{Re}\langle R_j, \epsilon^{-q}\left[\mathcal{N}(V_{**}+\epsilon^q R)-\mathcal{N}(V_{**})\right]_j\rangle_{L^{2,b}_s} \\
\leq&C_{\mathrm{L}}\big(\epsilon\|R\|^2_{L^2}+\||\boldsymbol{\xi}|^{1/2}R\|^2_{L^{2,b}_s}\big)+C_{\mathrm{NL}}\epsilon\|R\|^2_{L^{2,b}_{s+1/2}},
\end{aligned}
$$

where $C_{\mathrm{L}}\in\mathcal{O}(\|V_{**}\|_{L^2_s})$ for $\|V_{**}\|_{L^2_s}\to 0$. Let us choose ϵ_0 and $\|V_{**}\|_{L^2_s}$ so small that

$$C_{\mathrm{L}} + C_{\mathrm{NL}}\epsilon_0 < 2|A_0|^2, \qquad C_L + C_{\mathrm{NL}}\epsilon_0 \le C_1, \qquad (2C_{\mathrm{NL}} + C_{\mathrm{RES}})\epsilon_0 < \eta,$$

and define $0 < \eta = 2|A_0|^2 - C_{\mathrm{L}}$, which is independent of $\epsilon \in [0, \epsilon_0]$.

Since $\|\cdot\|^2_{L^{2,b}_{s+1/2}} \le \|\cdot\|^2_{L^{2,b}_s} + \|\boldsymbol{\xi}^{1/2}\cdot\|^2_{L^{2,b}_s}$, we obtain for $t \in [\tilde{\delta}, T_2]$ the inequalities

$$\begin{aligned}
\frac{d}{dt}E_s &\le -\eta\|R_1\|^2_{L^{2,b}_s} - \eta\sum_{j=2}^{4}\|R_j\|^2_{L^{2,b}_s} + 2(C_{\mathrm{L}} + \eta)F_s + (2C_{\mathrm{NL}} + C_{\mathrm{RES}})\epsilon E_s + \epsilon C_{\mathrm{RES}}, \\
&\le -A_{11}E_s + A_{12}F_s + \epsilon C_{\mathrm{RES}}, && (4.19\mathrm{a}) \\
\frac{d}{dt}F_s &\le \epsilon A_{21}E_s + \epsilon C_{\mathrm{RES}}, && (4.19\mathrm{b})
\end{aligned}$$

where we introduced the constants $A_{11} = 2\eta - (2C_{\mathrm{NL}} + C_{\mathrm{RES}})\epsilon_0 > \eta$, $A_{12} = 2(C_{\mathrm{L}} + \eta)$ and $A_{21} = 2(C_{\mathrm{L}} + C_{\mathrm{NL}}) + C_{\mathrm{RES}}$.

We write the system of inequalities (4.19) in the form

$$\frac{d}{dt}\mathcal{E} \le A\mathcal{E} + \epsilon\mathcal{G}$$

where $\mathcal{E} = (E_s, F_s) \in C^1([\tilde{\delta}, T_2], \mathbb{R}^2)$ and $\mathcal{G} = C_{\mathrm{RES}}(1, 1) \in C^\infty(\mathbb{R}, \mathbb{R}^2)$. Further, $\|\mathcal{G}\|_{L^\infty} = C_{\mathrm{RES}} < \infty$ is obvious. We observe that

$$A = \begin{pmatrix} -A_{11} & A_{12} \\ \epsilon A_{21} & 0 \end{pmatrix}$$

has non-negative off-diagonal entries so that the matrix e^{tA} preserves the componentwise order on $\mathbb{R}^2$ for all $t \ge 0$, i.e. if $0 \le x \in \mathbb{R}^2$ for any component then $0 \le e^{tA}x$ for any component.

Hence, we obtain

$$\begin{aligned}
\mathcal{E}(t) - e^{A(t-\tilde{\delta})}\mathcal{E}(\tilde{\delta}) &= \int_{\tilde{\delta}}^{t} -Ae^{(t-s)A}\mathcal{E}(s) + e^{(t-s)A}\frac{d}{ds}\mathcal{E}(s)\,\mathrm{d}s \le \epsilon\int_{\tilde{\delta}}^{t} e^{(t-s)A}\mathcal{G}(s)\,\mathrm{d}s \\
&\le C_{\mathrm{RES}}\epsilon t \sup_{s\in[0,t]}\|e^{sA}\|\begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\end{aligned}$$

Now let $C_A = \frac{2A_{12}A_{21}}{A_{11}}$ and

$$B = C_T C_{T^{-1}}(1 + C_F) M e^{C_A \boldsymbol{x}} + 2 C_T C_{T^{-1}} C_{\mathrm{RES}} \boldsymbol{x} e^{C_A \boldsymbol{x}}.$$

Assume we proved $\|e^{sA}\| \leq C_T C_{T^{-1}} e^{\epsilon C_A s}$ for $s \in \mathbb{R}_0^+$. Then $E(t) \leq |\mathcal{E}(t)|_1 \leq B(\epsilon t)$ for all $t \in [\tilde{\delta}, T_2[$ and in particular $E(t) \leq B(\epsilon t)$ for all $t \in \{r \in I_\kappa \,|\, E_s(r) < C_M M\}$. Since

$$B(0) = C_T C_{T^{-1}}(1 + C_F) M < C_M M$$

and B is continuous and monotonically increasing, it is clear that there is $0 < x_0$ independent of ϵ such that $B(x) \leq C_M M$ for all $x \in [0, x_0]$. Thus, the proof of $\sup \{r \in I_\kappa \,|\, E_s(r) < C_M M\} \in \Omega(\epsilon^{-1})$ is complete if we prove $\|e^{sA}\| \leq C_T C_{T^{-1}} e^{\epsilon C_A s}$ for $s \in \mathbb{R}_0^+$.

The proof of this assumption is pretty simple. We note that there are two distinct eigenvalues $\mu_\pm$ of A,

$$\mu_\pm := \frac{-A_{11} \pm \sqrt{A_{11}^2 + 4\epsilon A_{12} A_{21}}}{2},$$

for all ϵ in a neighbourhood of $\mathbb{R}^+ \subset \mathbb{C}$. Since A is holomorphic in ϵ for all $\epsilon \in \mathbb{C}$ there are projections on the eigenspaces and transformation matrices T, T^{-1} that make A diagonal. These are also holomorphic in ϵ in a neighbourhood of $\mathbb{R}^+ \subset \mathbb{C}$ by standard results in analytic perturbation theory.

Thus, $\|T\|, \|T^{-1}\|$ are bounded for ϵ in any compact subset of $\mathbb{R}_0^+$. Further we note that $\mu_- < 0$ for ϵ in any compact subset of $\mathbb{R}_0^+$ and we can estimate $\mu_+ \leq \frac{2A_{12}A_{21}}{A_{11}}\epsilon = C_A \epsilon$ in this case. Hence,

$$\|e^{sA}\| = \|T^{-1} \operatorname{diag}(e^{\mu_+ s}, e^{\mu_- s}) T\| \leq C_T C_{T^{-1}} e^{\epsilon C_A s} \qquad \text{for } s \in \mathbb{R}_0^+.$$

□

***Remark* 4.4.12.** While it is not so obvious in the proof above, the choice of C_M is not completely trivial. The problem is that the constant C_{NL} actually depends on C_M and therefore the constants A_{11} and A_{21} do as well. Thus $C_T, C_{T^{-1}}$ also depend on C_M. However, these are used to define C_M in the proof. This looks a bit like a circle but the important point is that there is an ϵ_0 in A_{11} and there is

ϵA_{21} in the matrix A to cope with this problem. Therefore it is possible to choose C_M uniform for sufficiently small ϵ_0. We proceed as follows for this purpose. For all $C_M > 0$ we can choose ϵ_0 so small that $A_{11} \in [\eta/2, \eta]$. A_{12} does not depend on C_M. We note that $\|T\| = C_T, \|T^{-1}\| = C_{T^{-1}}$ do not depend on C_M for $\epsilon = 0$ under these conditions. Since T, T^{-1} are holomorphic in ϵ (or ϵA_{21}), we can use any C_M with $C_T, C_{T^{-1}}$ larger than in the case $\epsilon = 0$ if we choose ϵ_0 small enough.

4.5 Conclusions

On the one hand, since the techniques applied in Chapter 4 are similar to those of Chapter 3, there are similar shortcomings and similar points that need commenting as in Chapter 3. On the other hand there are some tremendous differences. We see that there are in essence the same unsatisfactory points in the Theorems 4.4.1, 4.4.2 and 4.4.3 as in the corresponding theorems of Chapter 3. The first issue is the reconstruction of the phase ϕ from ψ. Since this problem is the same as in Chapter 3 we only refer to the discussion in Section 3.6.1.

The second question concerns the strong regularity that we required in the theorems. This is annoying for applications and not satisfying in the sense of a stability result as before. But there is a big difference to the situation considered in Chapter 3. First, we noticed in Section 4.2 that existence and uniqueness in Sobolev spaces is somewhat unclear for system (4.6). Secondly, we exploited the analyticity to obtain the artificial smoothing in equation (4.14). We used this smoothing to eliminate the effects of the spectral instability induced by the periodic travelling wave solutions and to control the linearisation of the error around the approximate solution in equation (4.17) in the proof of Theorem 4.4.11. Since we did not make any assumptions on the size of the spectral instability, only on the asymptotics $|\xi| \to \infty$ and $\xi \to 0$, it is not clear whether the effects of the spectral instability are as small as necessary to be controlled for $t \in \Theta(\epsilon^{-1})$. Further it is not clear how to control the effects of the linearisation without the artificial smoothing.

Thirdly, one could complain that the limit of the size of the approximate solutions is annoying. But in general one cannot expect to remove it. We needed it to make sense of the algebraic condition (4.5a) and if we removed it the problem with the effects of the linearisation on the error would be even more severe.

Last but not least, the question whether the assumptions of Hypothesis 4.4.5 are necessary arises, i.e. whether it is necessary that the linear operator in equation (4.14) is diagonable and that the matrices of the transform depend on the parameter continuously. We would remove them gladly but our method is not applicable in the case of Jordan blocks. Since we cannot exclude them, as noticed in Section 4.4.1, one has to work around them. Maybe this is possible by introducing mode filters, cf. [27, Section 3.6] for this idea. For the question of the continuous dependence on ξ of the eigenvalues and eigenvectors (or projections, respectively), we have to refer to literature on perturbation theory. Since we have no explicit expression for the eigenvalues, in contrast to the situation in Section 3.5.1 of Chapter 3, this question is not so easily answered. It is well known that eigenvalues do not need to depend holomorphically on the perturbation parameter in branching points and the projections on the subspaces are even worse, see [55, Example II.2.10]. Therefore, one can not expect more than a C^1 or continuous dependence in general.

Appendix

4.A Derivation of the Residual Equations

The details of the construction of the iterative scheme had been omitted in the proof of Lemma 4.3.1. We show them here. Let

$$r = \sum_{i=0}^{N} \epsilon^i r_i, \qquad \psi = \sum_{i=0}^{N} \epsilon^i \psi_i, \qquad b = \sum_{i=0}^{N} \epsilon^i b_i,$$

for functions $(r_j, \psi_j, b_j) : [0, t_1] \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{C}$, $0 \le j \le N$. We make the ansatz $U_{**} : [0, t_1] \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{C}$, $(t, x) \mapsto (r, \psi, b)(\epsilon t, \epsilon x)$. Recall that system (4.3) was

$$\partial_t U = \tilde{\mathcal{L}}(\partial_x) U + N(U), \tag{4.20}$$

where $U = (r, \psi, \operatorname{Re} b, \operatorname{Im} b) = (r, \psi, b)$,

$$\tilde{\mathcal{L}}(\partial_x) = \begin{pmatrix} \partial_x^2 - \alpha 2\xi_0 \partial_x - 2|A_0|^2 & -\alpha\partial_x - 2\xi_0 & \mathrm{Re}\gamma & -\mathrm{Im}\gamma \\ \alpha\partial_x^3 + 2\xi_0 \partial_x^2 - 2\beta|A_0|^2\partial_x & \partial_x^2 - \alpha 2\xi_0 \partial_x & \mathrm{Im}\gamma\partial_x & \mathrm{Re}\gamma\partial_x \\ 2d|A_0|^2\partial_x & 0 & a\partial_x^2 + c\partial_x & 0 \\ 0 & 0 & 0 & a\partial_x^2 + c\partial_x \end{pmatrix}$$

and, with the abbreviation $h(r) = e^{2r} - 1 - 2r$,

$$\tilde{N}(r, \psi, \operatorname{Re} b, \operatorname{Im} b) = \begin{pmatrix} (\partial_x r)^2 - \psi^2 - 2\alpha(\partial_x r)\psi - |A_0|^2 h(r) \\ 2\partial_x((\partial_x r)\psi) + \alpha\partial_x((\partial_x r)^2 - \psi^2) - \beta|A_0|^2\partial_x h(r) \\ d|A_0|^2 \partial_x h(r) \\ 0 \end{pmatrix}.$$

By inserting the ansatz into equation (4.20) above we obtain

$$
\begin{aligned}
\sum_{i=0}^{N} \epsilon^{i+1} \partial_t r_i = &\sum_{i=0}^{N} \epsilon^i \Big((\epsilon^2 \partial_x^2 - 2\alpha\xi_0 \epsilon \partial_x) r_i - (\alpha\epsilon\partial_x + 2\xi_0)\psi_i + \mathrm{Re}\gamma b_i \Big) \\
&+ \sum_{i,j=0}^{N} \epsilon^{i+j+2} (\partial_x r_i)(\partial_x r_j) - \sum_{i,j=0}^{N} \epsilon^{i+j} \psi_i \psi_j \qquad (4.21a)\\
&- 2\alpha \sum_{i,j=0}^{N} \epsilon^{i+j+1} (\partial_x r_i)\psi_j - |A_0|^2 \Big(\prod_{i=0}^{N} e^{2\epsilon^i r_i} - 1 \Big)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=0}^{N} \epsilon^{i+1} \partial_t \psi_i = &\sum_{i=0}^{N} \epsilon^i \Big((\alpha\epsilon^3 \partial_x^3 + 2\xi_0 \epsilon^2 \partial_x^2) r_i + (\epsilon^2 \partial_x^2 - 2\alpha\xi_0\epsilon\partial_x)\psi_i + \epsilon \mathrm{Im} \partial_x \gamma b_i \Big) \\
&+ \sum_{i,j=0}^{N} \epsilon^{i+j+2} 2\partial_x ((\partial_x r_i)\psi_j) + \alpha \sum_{i,j=0}^{N} \epsilon^{i+j+3} \partial_x \big((\partial_x r_i)(\partial_x r_j) \big) \\
&- \alpha \sum_{i,j=0}^{N} \epsilon^{i+j+1} \partial_x (\psi_i \psi_j) \\
&- \beta |A_0|^2 \epsilon \partial_x \Big(\prod_{i=0}^{N} e^{2\epsilon^i r_i} - 1 \Big) \qquad (4.21b)
\end{aligned}
$$

$$
\sum_{i=0}^{N} \epsilon^{i+1} \partial_t b_i = \sum_{i=0}^{N} \epsilon^i (a\epsilon^2 \partial_x^2 + c\epsilon\partial_x) b_i + d|A_0|^2 \epsilon \partial_x \Big(\prod_{i=0}^{N} e^{2\epsilon^i r_i} - 1 \Big). \qquad (4.21c)
$$

Again because of the identity, cf. Appendix 3.A,

$$
\sum_{k_0=0}^{\infty} \sum_{k_1=0}^{k_0} \cdots \sum_{k_n=0}^{k_{N-1}} a_{0,k_N} a_{1,k_{N-1}-k_n} \cdots a_{N,k_0-k_1} = \prod_{j=0}^{N} \left(\sum_{k_j=0}^{\infty} a_{j,k_j} \right),
$$

we can expand the exponentials

$$
\begin{aligned}
&\prod_{i=0}^{N} e^{2\epsilon^i r_i} - 1 \\
&= \sum_{k_0=0}^{\infty} \sum_{k_1=0}^{k_0} \cdots \sum_{k_N=0}^{k_{N-1}} \frac{\epsilon^{Nk_N} (2r_N)^{k_N}}{k_N!} \frac{\epsilon^{(N-1)(k_{N-1}-k_N)} (2r_{N-1})^{(k_{N-1}-k_N)}}{(k_{N-1}-k_N)!} \cdots \\
&\qquad \cdots \frac{\epsilon^{0(k_0-k_1)} (2r_0)^{(k_0-k_1)}}{(k_0-k_1)!} - 1
\end{aligned}
$$

$$= \sum_{k_0=1}^{\infty} 2^{k_0} \sum_{k_1=0}^{k_0} \cdots \sum_{k_N=0}^{k_{N-1}} \epsilon^{\sum_{l=1}^N k_l} \frac{r_N^{k_N}}{k_N!} \frac{r_{N-1}^{(k_{N-1}-k_N)}}{(k_{N-1}-k_N)!} \cdots \frac{r_0^{(k_0-k_1)}}{(k_0-k_1)!}.$$

We notice that we can split this expression in order ϵ^i into

$$E_i := \begin{cases} e^{2r_0} - 1, & i = 0, \\ 2e^{2r_0} r_i + \sum_{\substack{(k_0,\dots,k_N)\in\mathbb{N}\times\mathbb{N}_0^N \\ k_N \le k_{N-1} \le \dots \le k_1 \le k_0 \\ \sum_{l=1}^N k_l = i, k_i = 0}} 2^{k_0} \frac{r_N^{k_N}}{k_N!} \frac{r_{N-1}^{(k_{N-1}-k_N)}}{(k_{N-1}-k_N)!} \cdots \frac{r_0^{(k_0-k_1)}}{(k_0-k_1)!}, & i \ge 1, \end{cases} \tag{4.22}$$

where the sum involves only addends with index lower than i. Thus, we simply define r_i as solution to the (algebraic) problem (4.21a) in order ϵ^i (functions with negative index are to be considered as zero function):

$$\begin{aligned} \mathrm{Re}\gamma b_0 - 2\xi_0\psi_0 - \psi_0^2 - |A_0|^2(e^{2r_0} - 1) = 0, \quad i = 0, \\ -2|A_0|^2 e^{2r_0} r_i - 2\xi_0\psi_i + \mathrm{Re}\gamma b_i - 2\psi_0\psi_i + F_{r,i} = 0, \quad i \ge 1, \end{aligned} \tag{4.23}$$

where $F_{r,i} = F_{r,i}(r_0, \dots, r_{i-1}, \psi_0, \dots, \psi_{i-1}, b_0, \dots, b_{i-1})$ is defined by

$$\begin{aligned} F_{r,i} = {} & \partial_x^2 r_{i-2} - \alpha 2\xi_0 \partial_x r_{i-1} - \alpha\partial_x\psi_{i-1} + \sum_{\substack{j,j'\in\{0,\dots,N\}^2 \\ j+j'+2=i}} (\partial_x r_j)(\partial_x r_{j'}) \\ & - 2\alpha \sum_{\substack{j,j'\in\{0,\dots,N\}^2 \\ j+j'+1=i}} (\partial_x r_j)\psi_{j'} - \sum_{\substack{j,j'\in\{1,\dots,N\}^2 \\ j+j'=i}} \psi_j\psi_{j'} \\ & - |A_0|^2 \sum_{\substack{(k_0,\dots,k_N)\in\mathbb{N}\times\mathbb{N}_0^N \\ k_N \le k_{N-1} \le \dots \le k_1 \le k_0 \\ \sum_{l=1}^N k_l = i, k_i = 0}} 2^{k_0} \frac{r_N^{k_N}}{k_N!} \frac{r_{N-1}^{(k_{N-1}-k_N)}}{(k_{N-1}-k_N)!} \cdots \frac{r_0^{(k_0-k_1)}}{(k_0-k_1)!} - \partial_t r_{i-1}. \end{aligned}$$

Proceeding the same way for (4.21b) and (4.21c) we obtain in ϵ^{i+1}

$$\begin{aligned} \partial_t\psi_i = {} & \alpha\partial_x^3 r_{i-2} + 2\xi_0\partial_x^2 r_{i-1} + \partial_x^2\psi_{i-1} - \alpha 2\xi_0\epsilon\partial_x\psi_i + \mathrm{Im}\partial_x\gamma b_i - 2\alpha\partial_x(\psi_0\psi_i) \\ & - \beta|A_0|^2\partial_x E_i + \sum_{\substack{j,j'\in\{0,\dots,N\}^2 \\ j+j'+1=i}} 2\partial_x((\partial_x r_j)\psi_{j'}) + \alpha \sum_{\substack{j,j'\in\{0,\dots,N\}^2 \\ j+j'+2=i}} \partial_x\big((\partial_x r_j)(\partial_x r_{j'})\big) \\ & - \alpha \sum_{\substack{j,j'\in\{1,\dots,N\}^2 \\ j+j'=i}} \partial_x(\psi_j\psi_{j'}), \end{aligned}$$

$$\partial_t b_i = a\partial_x^2 b_{i-1} + c\partial_x b_i + d|A_0|^2\partial_x E_i.$$

With the definition of r_i in (4.23) and E_i in (4.22) we obtain for $i = 0$ system (4.5) of Section 4.2 and for $i \geq 1$ the system

$$\begin{aligned}
\partial_t \psi_i &= \partial_x \left(-\alpha 2\xi_0 \psi_i + \mathrm{Im}\gamma b_i - 2\alpha\psi_0\psi_i - \beta\left(-2\xi_0\psi_i + \mathrm{Re}\gamma b_i - 2\psi_0\psi_i\right) + F_{\psi,i}\right), \\
\partial_t b_i &= \partial_x \left(c b_i + d\left(-2\xi_0\psi_i + \mathrm{Re}\gamma b_i - 2\psi_0\psi_i\right) + F_{b,i}\right),
\end{aligned}$$

where $F_{\psi,i}, F_{b,i}$ depend on $r_0, \ldots, r_{i-1}, \psi_0, \ldots, \psi_{i-1}, b_0, \ldots, b_{i-1}$ and are defined as follows

$$\begin{aligned}
F_{\psi,i} = {} & \alpha\partial_x^2 r_{i-2} + 2\xi_0\partial_x r_{i-1} + \partial_x\psi_{i-1} + \sum_{\substack{j,j' \in \{0,\ldots,N\}^2 \\ j+j'+1=i}} 2(\partial_x r_j)\psi_{j'} \\
& + \alpha \sum_{\substack{j,j' \in \{0,\ldots,N\}^2 \\ j+j'+2=i}} (\partial_x r_j)(\partial_x r_{j'}) - \beta F_{r,i} - \alpha \sum_{\substack{j,j' \in \{1,\ldots,N\}^2 \\ j+j'=i}} \psi_j \psi_{j'} \\
& - \beta|A_0|^2 \sum_{\substack{(k_0,\ldots,k_N) \in \mathbb{N}\times\mathbb{N}_0^N \\ k_N \leq k_{N-1} \leq \ldots \leq k_1 \leq k_0 \\ \sum_{l=1}^N k_l = i, k_i = 0}} 2^{k_0} \frac{r_N^{k_N}}{k_N!} \frac{r_{N-1}^{(k_{N-1}-k_N)}}{(k_{N-1}-k_N)!} \cdots \frac{r_0^{(k_0-k_1)}}{(k_0-k_1)!}, \\
F_{b,i} = {} & a\partial_x b_{i-1} + dF_{r,i} + d|A_0|^2 \sum_{\substack{(k_0,\ldots,k_N) \in \mathbb{N}\times\mathbb{N}_0^N \\ k_N \leq k_{N-1} \leq \ldots \leq k_1 \leq k_0 \\ \sum_{l=1}^N k_l = i, k_i = 0}} 2^{k_0} \frac{r_N^{k_N}}{k_N!} \frac{r_{N-1}^{(k_{N-1}-k_N)}}{(k_{N-1}-k_N)!} \cdots \frac{r_0^{(k_0-k_1)}}{(k_0-k_1)!}.
\end{aligned}$$

This is what was claimed in the proof of Lemma 4.3.1.

4.B Perturbation Theory

In Section 4.4.1 we made some statements and claimed that these simply follow from application of analytic perturbation theory for linear operators. Details concerning the perturbation arguments are as follows. We have the following expansion of the symbol $\tilde{\mathcal{L}}(\mathrm{i}\xi)$

$$
\begin{aligned}
\tilde{\mathcal{L}}(\mathrm{i}\xi) &= \sum_{k=0}^{3} \mathcal{L}^{(k)}(\mathrm{i}\xi)^k \\
&= \begin{pmatrix} -2|A_0|^2 & -2\xi_0 & \mathrm{Re}\gamma & -\mathrm{Im}\gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \mathrm{i}\xi \begin{pmatrix} -\alpha 2\xi_0 & -\alpha & 0 & 0 \\ -2\beta|A_0|^2 & -\alpha 2\xi_0 & \mathrm{Im}\gamma & \mathrm{Re}\gamma \\ 2d|A_0|^2 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \\
&\quad - \xi^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2\xi_0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} - \mathrm{i}\xi^3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

Obviously the mapping $\mathbb{C} \ni \xi \mapsto \tilde{\mathcal{L}}(\xi) \in \mathbb{C}^{4\times 4}$ is entire.

Let $\lambda \in \rho(\tilde{\mathcal{L}}(0))$. Then we can easily calculate the resolvent for $\tilde{\mathcal{L}}(0)$ and see

$$
\left(\lambda - \tilde{\mathcal{L}}(0)\right)^{-1} = \begin{pmatrix} \frac{1}{2|A_0|^2+\lambda} & \frac{-2\xi_0}{(2|A_0|^2+\lambda)\lambda} & \frac{\mathrm{Re}\gamma}{(2|A_0|^2+\lambda)\lambda} & \frac{-\mathrm{Im}\gamma}{(2|A_0|^2+\lambda)\lambda} \\ 0 & \frac{1}{\lambda} & 0 & 0 \\ 0 & 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda} \end{pmatrix}.
$$

By analytic perturbation theory the projector on the subspace belonging to the eigenvalue 0 for $\xi = 0$ is in a neighbourhood of $\xi = 0$ given by

$$
\begin{aligned}
P_0(\mathrm{i}\xi) &= \begin{pmatrix} 0 & -\frac{\xi_0}{|A_0|^2} & \frac{\mathrm{Re}\gamma}{2|A_0|^2} & -\frac{\mathrm{Im}\gamma}{2|A_0|^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&\quad + \mathrm{i}\xi \begin{pmatrix} 0 & -\frac{\xi_0}{|A_0|^2} & \frac{\mathrm{Re}\gamma}{2|A_0|^2} & -\frac{\mathrm{Im}\gamma}{2|A_0|^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2\alpha\xi_0 & -\alpha & 0 & 0 \\ -2\beta|A_0|^2 & -2\alpha\xi_0 & \mathrm{Im}\gamma & \mathrm{Re}\gamma \\ 2d|A_0|^2 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \\
&\quad + \mathcal{O}(|\xi|^2)
\end{aligned}
$$

$$
= \begin{pmatrix} 0 & -\frac{\xi_0}{|A_0|^2} & \frac{\mathrm{Re}\gamma}{2|A_0|^2} & -\frac{\mathrm{Im}\gamma}{2|A_0|^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
+ \mathrm{i}\xi \begin{pmatrix} 2\beta\xi_0 + d\mathrm{Re}\gamma & \frac{2\alpha\xi_0^2}{|A_0|^2} & \frac{c\mathrm{Re}\gamma - 2\mathrm{Im}\gamma\xi_0}{2|A_0|^2} & -\frac{2\mathrm{Re}\gamma\xi_0 + c\mathrm{Im}\gamma}{2|A_0|^2} \\ -2\beta|A_0|^2 & -2\alpha\xi_0 & \mathrm{Im}\gamma & \mathrm{Re}\gamma \\ 2d|A_0|^2 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix} + \mathcal{O}(|\xi|^2).
$$

Similarly, the projector on the subspace belonging to the eigenvalue $-2|A_0|^2$ for $\xi = 0$ is in a neighbourhood of $\xi = 0$ given by

$$
\begin{aligned}
P_{-2|A_0|^2}(\mathrm{i}\xi) = & \begin{pmatrix} 1 & \frac{\xi_0}{|A_0|^2} & -\frac{\mathrm{Re}\gamma}{2|A_0|^2} & \frac{\mathrm{Im}\gamma}{2|A_0|^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
& + \mathrm{i}\xi \begin{pmatrix} 1 & \frac{\xi_0}{|A_0|^2} & -\frac{\mathrm{Re}\gamma}{2|A_0|^2} & \frac{\mathrm{Im}\gamma}{2|A_0|^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -2\alpha\xi_0 & -\alpha & 0 & 0 \\ -2\beta|A_0|^2 & -2\alpha\xi_0 & \mathrm{Im}\gamma & \mathrm{Re}\gamma \\ 2d|A_0|^2 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \\
& + \mathcal{O}(|\xi|^2) \\
= & \begin{pmatrix} 1 & \frac{\xi_0}{|A_0|^2} & -\frac{\mathrm{Re}\gamma}{2|A_0|^2} & \frac{\mathrm{Im}\gamma}{2|A_0|^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
& - \mathrm{i}\xi \begin{pmatrix} 2(\alpha+\beta)\xi_0 + d\mathrm{Re}\gamma & \alpha\frac{|A_0|^2 + 2\xi_0^2}{|A_0|^2} & \frac{c\mathrm{Re}\gamma + 2\xi_0\mathrm{Im}\gamma}{2|A_0|^2} & -\frac{2\xi_0\mathrm{Re}\gamma + c\mathrm{Im}\gamma}{2|A_0|^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
& + \mathcal{O}(|\xi|^2).
\end{aligned}
$$

Since $\tilde{\mathcal{L}}(0)$ has semisimple eigenvalues we can use the 'reduction process' of [55, Section II, §2.3]. We calculate the first order of the, in the language of Kato, 'reduced expansion' $\widetilde{\tilde{\mathcal{L}}(0)} = P_0(0)\tilde{\mathcal{L}}^{(1)}P_0(0)$ on the subspace spanned by $P_0(0)$ as follows

$$
\begin{aligned}
&\widetilde{\tilde{\mathcal{L}}(0)} \\
&= \begin{pmatrix} 0 & -\frac{\xi_0}{|A_0|^2} & \frac{\mathrm{Re}\gamma}{2|A_0|^2} & -\frac{\mathrm{Im}\gamma}{2|A_0|^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\alpha 2\xi_0 & -\alpha & 0 & 0 \\ -2\beta|A_0|^2 & -\alpha 2\xi_0 & \mathrm{Im}\gamma & \mathrm{Re}\gamma \\ 2d|A_0|^2 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \times \\
&\quad \times \begin{pmatrix} 0 & -\frac{\xi_0}{|A_0|^2} & \frac{\mathrm{Re}\gamma}{2|A_0|^2} & -\frac{\mathrm{Im}\gamma}{2|A_0|^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\frac{\xi_0}{|A_0|^2} & \frac{\mathrm{Re}\gamma}{2|A_0|^2} & -\frac{\mathrm{Im}\gamma}{2|A_0|^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \alpha 2\frac{\xi_0^2}{|A_0|^2} - \alpha & -\alpha\xi_0\frac{\mathrm{Re}\gamma}{|A_0|^2} & \alpha\xi_0\frac{\mathrm{Im}\gamma}{|A_0|^2} \\ 0 & 2\xi_0(\beta-\alpha) & \mathrm{Im}\gamma - \beta\mathrm{Re}\gamma & \beta\mathrm{Im}\gamma + \mathrm{Re}\gamma \\ 0 & -2d\xi_0 & d\mathrm{Re}\gamma + c & -d\mathrm{Im}\gamma \\ 0 & 0 & 0 & c \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\xi_0\frac{2\xi_0^2(\beta-\alpha)^2+\mathrm{Re}\gamma d}{|A_0|^2} & \frac{2\xi_0(\beta\mathrm{Re}\gamma-\mathrm{Im}\gamma)+\mathrm{Re}\gamma(d\mathrm{Re}\gamma+c)}{2|A_0|^2} & -\frac{2\xi_0(\beta\mathrm{Im}\gamma+\mathrm{Re}\gamma)-\mathrm{Re}\gamma d\mathrm{Im}\gamma-\mathrm{Im}\gamma c}{2|A_0|^2} \\ 0 & 2\xi_0(\beta-\alpha) & \mathrm{Im}\gamma - \beta\mathrm{Re}\gamma & \beta\mathrm{Im}\gamma + \mathrm{Re}\gamma \\ 0 & -2d\xi_0 & d\mathrm{Re}\gamma + c & -d\mathrm{Im}\gamma \\ 0 & 0 & 0 & c \end{pmatrix}
\end{aligned}
$$

By [55, Theorem II.2.3] the first order of the eigenvalue expansion around $\xi = 0$ of the eigenvalues converging to 0 for $\xi \to 0$ are given by the non-zero eigenvalues of the matrix above. The eigenvalues are obviously $0, c$ and

$$\frac{2\xi_0(\beta-\alpha) + d\mathrm{Re}\gamma + c}{2} \pm \sqrt{\frac{(2\xi_0(\beta-\alpha) - d\mathrm{Re}\gamma - c)^2}{4} - 2d\xi_0(\mathrm{Im}\gamma - \beta\mathrm{Re}\gamma)}.$$

The zero value is a consequence of the fact that $P_0(0)$ is a projection on a three dimensional subspace and has to be neglected. For $\xi_0 = 0$ the expression simplifies to $0, c, d\mathrm{Re}\gamma + c, 0$ where the 'last' zero eigenvalue is actually an eigenvalue. We note that in general the matrix above has non-semisimple eigenvalues. For example for $c = \xi_0 = \mathrm{Re}\gamma = 0$, i.e.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Im}\gamma & \beta\mathrm{Im}\gamma \\ 0 & 0 & 0 & -d\mathrm{Im}\gamma \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

we have a matrix of at least rank one in general. But we would need rank zero for semisimple eigenvalues. Hence, we cannot iteratively apply the theorem.
Now, the following two points are a consequence of [55, Theorem II.2.3]. First, if all eigenvalues of the 'reduced expansion' are different from each other, then the eigenvalue 0 for $\xi = 0$ splits into three different eigenvalues for $\xi \neq 0$ in a small environment of 0. Secondly, if all the eigenvalues are different from each other then the projection on the subspaces are holomorphic in ξ in a small environment of $\xi = 0$. As a consequence the same is true for the projections $P_{0,i}$.

4.C Existence and Uniqueness of Solutions to System (4.3)

We establish that $\tilde{\mathcal{L}}(\mathrm{i}\boldsymbol{\xi})$, the symbol of the linear operator in equation (4.3), is a sectorial operator on $X = L^2_{1,\sigma} \times L^2_{0,\sigma} \times L^2_{1,\sigma} \times L^2_{1,\sigma}$, which yields local existence and uniqueness in X for the semilinear parabolic problem (4.3). The necessary arguments are completely analogous to those of Appendix 3.C.

Proposition 4.C.1. *Let $\mathcal{M} : D(\mathcal{M}) \subset X \to X$ the multiplication operator with domain $D(\mathcal{M}) = L^2_{3,\sigma} \times L^2_{2,\sigma} \times L^2_{3,\sigma} \times L^2_{3,\sigma}$ which acts by multiplication with the matrix-valued function*

$$\mathbb{R} \ni \xi \mapsto - \begin{pmatrix} \xi^2 & \mathrm{i}\alpha\xi & 0 & 0 \\ \mathrm{i}\alpha\xi^3 & \xi^2 & 0 & 0 \\ 0 & 0 & a\xi^2 & 0 \\ 0 & 0 & 0 & a\xi^2 \end{pmatrix} - 1 \in \mathrm{GL}(4, \mathbb{C}).$$

Then it holds true

1. *$\mathcal{M}$ is a sectorial operator.*

2. *The part of $\mathcal{M}$ in $X_{7/8} := L^2_{11/4,\sigma} \times L^2_{7/4,\sigma} \times L^2_{11/4,\sigma} \times L^2_{11/4,\sigma} = D\left(M^{7/8}_{1+\boldsymbol{\xi}^2}\right)$ is sectorial in $X_{7/8}$.*

3. *$X_{7/8}$ is an intermediate space in the sense $D_{\mathcal{M}}(7/8,1) \subset X_{7/8} \subset D_{\mathcal{M}}(7/8,\infty)$ (definition in Lunardi* [62, Section 2.2.1]*).*

Proof. First, we note that $\mathcal{M}$ is reducible to two (actually three) subproblems. The upper left 2×2 block matrix corresponds to the one considered in Appendix 3.C. Hence, the graph norms of $\mathcal{M}$ and of the multiplication operator $M_{1+|\boldsymbol{\xi}|^2}$ acting by multiplication with $(1 + |\boldsymbol{\xi}|^2)$ in X are equivalent, cf. Appendix 3.C.
The rest of the proof is the same as in Appendix 3.C since all assertions for each subproblem are clear (for the upper block matrix see Appendix 3.C and the claim is trivial for the lower diagonal part). □

This immediately yields a corollary.

Corollary 4.C.2. *The operator $\tilde{\mathcal{L}}(\mathrm{i}\boldsymbol{\xi})\colon D\big(\tilde{\mathcal{L}}(\mathrm{i}\boldsymbol{\xi})\big) \subset X \to X$ is sectorial with domain $D\big(\tilde{\mathcal{L}}(\mathrm{i}\boldsymbol{\xi})\big) = D(\mathcal{M})$.*

Proof. $\tilde{\mathcal{L}}(\mathrm{i}\xi) - \mathcal{M}$ is (infintesimally) relatively $\mathcal{M}$-bounded, i.e.

$$\|(\tilde{\mathcal{L}}(\mathrm{i}\xi) - \mathcal{M})u\|_X \leq \delta\|\mathcal{M}u\|_X + C\|u\|_X$$

for all $u \in D(\mathcal{M})$, all $\delta > 0$ and a constant $C > 0$ depending on δ^{-1} ($\mathcal{M}$ is just the principal symbol of $\tilde{\mathcal{L}}(\mathrm{i}\boldsymbol{\xi})$). The rest is a perturbation argument for generators of strongly continuous analytic semigroups, cf. Appendix 3.C. □

Existence, uniqueness and regularity of solutions to the Fourier transform of equation (4.3) are now given by the following proposition.

Proposition 4.C.3. *Let $\hat{U}_0 \in D(\mathcal{M})$. There exists a maximal $T_0 \in \mathbb{R}^+ \cup \{\infty\}$ such that the Fourier transform of equation* (4.3) *has a strict solution*

$$\hat{U} \in C([0,T_0[, D(\mathcal{M})) \cap C^1([0,T_0[, X)$$

with initial condition $\hat{U}(0) = \hat{U}_0$. If $T_0 < \infty$, then $\|\hat{U}(t)\|_{X_{7/8}}$ must blow up as $t \to T_0$.

Proof. Let $\hat{U}$ be the Fourier transform of U and

$$F := \hat{U} \mapsto (\tilde{\mathcal{L}}(\mathrm{i}\boldsymbol{\xi}) - \mathcal{M})\hat{U} + \mathscr{F}\,\tilde{N}(\mathscr{F}^{-1}\,\hat{U}).$$

We know that $F : X_{7/8} \to X$ by Proposition A.1.4 and F is Lipschitz on bounded subsets of $X_{7/8}$.
Further $\hat{U}_0 \in D(\mathcal{M}) = D(\tilde{\mathcal{L}}(\mathrm{i}\boldsymbol{\xi}))$, as stated in the previous corollary, and $\mathcal{M}\hat{U}_0 + F(\hat{U}_0) \in \overline{D(\tilde{\mathcal{L}}(\mathrm{i}\boldsymbol{\xi}))} = X$. Then existence, uniqueness and regularity of the solution are granted by [62, Theorem 7.1.2] in combination with [62, Proposition 7.1.10(iii)].
If $T_0 < \infty$, then $\limsup_{t \to T_0} \|U(t)\|_{X_{7/8}} = \infty$ according to [62, Proposition 7.1.8] and the remark following this proposition since F maps bounded subsets of $X_{7/8}$ into bounded subsets of X. $\square$

***Remark* 4.C.4.** Obviously the above statements remain valid for all $s \geq 0$ if X is substituted by $L^2_{s+1,\sigma} \times L^2_{s,\sigma} \times L^2_{s+1,\sigma} \times L^2_{s+1,\sigma}$ and the domains adapted.

Corollary 4.C.5. *Let* $U_0 \in \mathrm{Hol}_\sigma$. *There exists a maximal* $T_0 = T_0(\sigma) \in \mathbb{R}^+ \cup \{\infty\}$ *such that system* (4.3) *has a solution* $U \in C([0, T_0[, \mathrm{Hol}_\sigma) \cap C^1([0, T_0[, \mathrm{Hol}_\sigma)$ *with initial condition* $U(0) = \hat{U}_0$. *If* $T_0(\sigma) < \infty$, *then* $\|\hat{U}(t)\|_{L^2_{11/4,\sigma'} \times L^2_{7/4,\sigma'} \times L^2_{11/4,\sigma'} \times L^2_{11/4,\sigma'}}$ *must blow up as* $t \to T_0$ *for at least one* $\sigma' \in [0, \sigma[$.

Note that in general T_0 depends on σ' and $T_0(\sigma') \to 0$ for $\sigma' \to \sigma$ has to be expected. This ambiguity is, as before in Chapter 3, an unfortunate consequence of our 'definition' of $C(I, \mathrm{Hol}_\sigma)$.

Proof. For all $\sigma' \in [0, \sigma[$ the arguments of Proposition 4.C.3 apply. $\square$

Appendix A

A.1 Interplay Between Entire Functions and the Spaces $L^2_{s,\sigma}$

We had to consider entire functions applied to functions of $L^2_{s,\sigma}$ for the nonlinear parts in Chapters 2 to 4. Under certain conditions there is a nice interplay between these, which could be expected since the composition of an entire and a holomorphic function is clearly holomorphic again. We prove this fact and some norm estimates in the following lemma. For non-entire but locally holomorphic functions one can expect similar but more complicated statements.

Definition A.1.1. *Let $u \in L^1$ and $n \in \mathbb{Z}$. We define*

$$u^{*^n} = \begin{cases} \underbrace{u * u * \cdots * u}_{n} & n \in \mathbb{N}, \\ 1 & n = 0, \\ 0 & n < 0. \end{cases}$$

Further we use the notation $u * z = zu$ for $z \in \mathbb{C}$. For multiindices $\alpha \in \mathbb{N}_0^n$ we use this notation accordingly.

Lemma A.1.2. *Let $f, g : \mathbb{C}^n \to \mathbb{C}$ entire, $n \in \mathbb{N}$ and $f(0) = 0$. Let $u, w \in H^s$, $s > \frac{1}{2}$. Then*

$$f \circ w, \quad (g \circ w)v \in H^s.$$

Further let $f(z) = \sum_{\alpha \in \mathbb{N}_0^n} f_\alpha z^\alpha$, $g(z) = \sum_{\alpha \in \mathbb{N}_0^n} g_\alpha z^\alpha$, $f_\alpha, g_\alpha \in \mathbb{C}$. Then

- $\mathscr{F}(f \circ w), \mathscr{F}\big((g \circ w)v\big) \in L^2_s$ *and*

$$\mathscr{F}(f \circ w) = \sum_{\alpha \in \mathbb{N}_0^n} f_\alpha \hat{w}^{*^\alpha}, \quad \mathscr{F}\big((g \circ w)v\big) = \hat{v} * \sum_{\alpha \in \mathbb{N}_0^n} g_\alpha \hat{w}^{*^\alpha}$$

and there are entire functions $h_1, h_2, h_3, h_{4,1}, h_{4,2}$, whose restriction to $\mathbb{R}_0^+$ is monotonically increasing, such that for $s' \le s$, $s'' + \frac{1}{2} < s$

$$\begin{aligned}
\|\mathscr{F}(f \circ w)\|_{L^2} &\le \|\hat{w}\|_{L^2} h_1(\|\hat{w}\|_{L^1}),\\
\|\xi^{s'} \mathscr{F}(f \circ w)\|_{L^2} &\le \|\xi^{s'} \hat{w}\|_{L^2} h_2(\|\hat{w}\|_{L^1})\\
\|\mathscr{F}\big((g \circ w)v\big)\|_{L^1} &\le \|\hat{v}\|_{L^1} h_3(\|\hat{w}\|_{L^1}),\\
\|\xi^{s'} \mathscr{F}\big((g \circ w)v\big)\|_{L^2} &\le \|\hat{v}\|_{L^1} \|\xi^{s'} \hat{w}\|_{L^2} h_{4,1}(\|\hat{w}\|_{L^1}) + \|\xi^{s'} \hat{v}\|_{L^2} h_{4,2}(\|\hat{w}\|_{L^1})\\
\|\xi^{s''} \mathscr{F}\big((g \circ w)v\big)\|_{L^2} &\le \|\hat{v}\|_{L^2} \|\xi^{s''} \hat{w}\|_{L^1} h_{4,1}(\|\hat{w}\|_{L^1}) + \|\xi^{s''} \hat{v}\|_{L^2} h_{4,2}(\|\hat{w}\|_{L^1})
\end{aligned}$$

- *If $\hat{w}, \hat{v} \in L^2_{s,\sigma}$ for $s > \frac{1}{2}$, $\sigma \in [0, \infty[$, then the above statement is true in $L^2_{s,\sigma}$ and*

$$\begin{aligned}
\|\mathscr{F}(f \circ w)\|_{L^2_{0,\sigma}} &\le \|\hat{w}\|_{L^2_{0,\sigma}} h_1(\|\hat{w}\|_{L^1_{0,\sigma}}),\\
\|\xi^{s'} \mathscr{F}(f \circ w)\|_{L^2_{0,\sigma}} &\le \|\xi^{s'} \hat{w}\|_{L^2_{0,\sigma}} h_2(\|\hat{w}\|_{L^1_{0,\sigma}})\\
\|\mathscr{F}\big((g \circ w)v\big)\|_{L^1_{0,\sigma}} &\le \|\hat{v}\|_{L^1_{0,\sigma}} h_3(\|\hat{w}\|_{L^1_{0,\sigma}}),\\
\|\xi^{s'} \mathscr{F}\big((g \circ w)v\big)\|_{L^2_{0,\sigma}} &\le \|\hat{v}\|_{L^2_{0,\sigma}} \|\xi^{s'} \hat{w}\|_{L^1_{0,\sigma}} h_{4,1}(\|\hat{w}\|_{L^1_{0,\sigma}})\\
&\quad + \|\xi^{s'} \hat{v}\|_{L^2_{0,\sigma}} h_{4,2}(\|\hat{w}\|_{L^1_{0,\sigma}})\\
\|\xi^{s''} \mathscr{F}\big((g \circ w)v\big)\|_{L^2_{0,\sigma}} &\le \|\hat{v}\|_{L^1_{0,\sigma}} \|\xi^{s''} \hat{w}\|_{L^2_{0,\sigma}} h_{4,1}(\|\hat{w}\|_{L^1_{0,\sigma}})\\
&\quad + \|\xi^{s''} \hat{v}\|_{L^2_{0,\sigma}} h_{4,2}(\|\hat{w}\|_{L^1_{0,\sigma}})
\end{aligned}$$

If $f, g = \mathcal{O}(|z|^k)$, $k \in \mathbb{N}$, for $z \to 0$ then $h_1, h_2, h_3, h_{4,1}, h_{4,2} = \mathcal{O}(|z|^{k-1})$ for $z \to 0$.

Proof. Since H^s is a Banach algebra for $s > \frac{1}{2}$ the first claim is obvious. Then $\mathscr{F}(f \circ w), \mathscr{F}\big(g \circ w)v\big) \in L^2_s$ is clear as well since $L^2_s \cong H^s$ and the formulae for $\mathscr{F}(f \circ w)$ and $\mathscr{F}\big((g \circ w)v\big)$ are a simple consequence of this isomorphism because the pointwise multiplication is transformed into a convolution by the Fourier transform.

The norm estimates follow easily:

$$
\begin{aligned}
\|\mathscr{F}(f\circ w)\|_{L^2} &\le \sum_{\alpha\in\mathbb{N}_0^n} |f_\alpha| \|\hat{w}^{*^\alpha}\|_{L^2} \le \|\hat{w}\|_{L^2} \sum_{\alpha\in\mathbb{N}_0^n} |f_\alpha| \left(\sqrt{2\pi}\|\hat{w}\|_{L^1}+1\right)^{|\alpha|_1}, \\
\|\xi^s\,\mathscr{F}(f\circ w)\|_{L^2} &\le \sum_{\alpha\in\mathbb{N}_0^n} |f_\alpha| \|\xi^s \hat{w}^{*^\alpha}\|_{L^2} \\
&\le \sum_{\alpha\in\mathbb{N}_0^n}\sum_{i=1}^{n} |f_\alpha||\alpha_i| \|\xi^s\hat{w}_i\|_{L^2} \left(C_s\sqrt{2\pi}\|\hat{w}\|_{L^1}\right)^{|\alpha|_1-1} \\
&\le n\|\xi^s\hat{w}\|_{L^2} \sum_{\alpha\in\mathbb{N}_0^n} |f_\alpha||\alpha|_1 \left(C_s\sqrt{2\pi}\|\hat{w}\|_{L^1}\right)^{|\alpha|_1-1}
\end{aligned}
$$

where $C_s = \max\{1, s^{s-1}\}$. The latter holds because of

$$
|\xi|^s|u_1 * \cdots * u_k| \le C_s^{k-1}\sum_{i=1}^{k} |u_1| * \cdots * |u_{i-1}| * |\xi|^s|u_i| * |u_{i+1}| * \cdots * |u_k|
$$

for all $k\in\mathbb{N}$ and $u_i \in L^2_s$. This is reduced to

$$
\begin{aligned}
&|\xi|^s|u_1^{*^{\alpha_1}} * \cdots * u_n^{*^{\alpha_n}}| \\
\le& C_s^{|\alpha|_1-1}\sum_{i=1}^{n} \alpha_i|\xi|^s|u_i| * |u_1^{*^{\alpha_1}}| * \cdots * |u_{i-1}^{*^{\alpha_{i-1}}}| * |u_i^{*^{\alpha_i-1}}| * |u_{i+1}^{*^{\alpha_{i+1}}}| * \cdots * |u_n^{*^{\alpha_n}}|.
\end{aligned}
$$

for the foregoing estimates. Estimates for $\mathscr{F}\left((g\circ w)v\right)$ and $\xi^s\,\mathscr{F}\left((g\circ w)v\right)$ follow by splitting $g\circ w = (g\circ w - g_0) + g_0 w$, where $g_0 = g(0)\in\mathbb{C}$, and the arguments above. The estimates of the L^1 norm are trivial since L^1 is a convolution algebra. The statement about $\hat{w},\hat{v}\in L^2_{s,\sigma}$ for $s > \frac{1}{2}$, $\sigma\in[0,\infty[$ is trivial now since, obviously, the previous arguments and estimates hold and we know

$$
\|\hat{w} * \hat{v}\|_{L^2_{0,\sigma}} \le \|\hat{w}\|_{L^2_{0,\sigma}}\|\hat{v}\|_{L^1_{0,\sigma}}.
$$

Since $L^1_{0,\sigma} \hookrightarrow L^2_{s,\sigma}$ and $\|\cdot\|_{L^2_{s,\sigma}}$ is equivalent to the norm $\|\cdot\|_{L^2_{0,\sigma}} + \||\boldsymbol{\xi}|^s\cdot\|_{L^2_{0,\sigma}}$, we obtain the estimate by replacing $L^2 \to L^2_{0,\sigma}$ and $L^1 \to L^1_{0,\sigma}$ in the above estimates. Finally, decay estimates follow from the explicit representations of $\mathscr{F}(f\circ w)$ and of $\mathscr{F}\left((g\circ w)v\right)$ respectively and the estimates above. $\square$

This lemma is certainly not the most general statement possible.

***Remark* A.1.3.**

- It is not necessary to consider functions in $u, v \in H^s$ for $s > \frac{1}{2}$. Actually, we only needed Young's convolution inequality and the Fourier transform and therefore $u, v \in L^p \cap L^1$, $p \in [1, \infty]$, would have been sufficient.
- The same statements are true if $L^p_{s,\sigma}$ is replaced by $L^p\left(\mathbb{R}, (1+|\boldsymbol{\xi}|^2)^{ps/2}\omega \, \mathrm{d}\xi\right)$ for any continuous function $\omega : \mathbb{R} \to \mathbb{R}^+$ that is subexponential, by which we mean $\omega(\xi + \eta) \leq C\omega(\xi)\omega(\eta)$ for all $\xi, \eta \in \mathbb{R}$ and a $C > 0$.
- If all the coefficients in the expansions of f and g are non-negative, then the functions $h_1, \ldots, h_{4,2}$ are derivatives of first or second order of f and g (up to some constants), for example

$$\|\mathscr{F}(f \circ u)\|\|_{L^2} \lesssim \|\widehat{u}\|_{L^2} f'(C\|\widehat{u}\|_{L^1}).$$

- Last but not least, it was nowhere necessary that f, g are entire. We only used that f, g are holomorphic on a set that contains the range of f, g. But many additional complications arise if f, g are not entire.

The next proposition shows that on the one hand entire functions behave nicely on the set Hol_σ or the space $L^2_{s,\sigma}$ and on the other hand they give some norm estimates on the linearisation and the quadratic remainder of Taylor's expansion.

Proposition A.1.4. *Let $f : \mathbb{C}^n \to \mathbb{C}$ entire, $n \in \mathbb{N}$ and $f(0) = 0$.*
Then $\tilde{f} := \mathscr{F} \circ f \circ \mathscr{F}^{-1} \in C^2(L^2_{s,\sigma}, L^2_{s,\sigma})$ for all $s \in]\frac{1}{2}, \infty[, \sigma \in [0, \infty[$ and for all $u, h \in L^2_{s,\sigma}$ we have

$$\tilde{f}(u+h) = \tilde{f}(u) + D\tilde{f}(u)[h] + G(u,h)$$

where $G : L^2_{s,\sigma} \times L^2_{s,\sigma} \to L^2_{s,\sigma}$.
If $u \in L^1_{s,\sigma}$ then there are entire functions h_1, h_2, h_3, h_4, whose restrictions to $\mathbb{R}^+_0$ are strictly monotonically increasing, such that

$$\begin{aligned}
\|\xi^{s''} D\tilde{f}(u)[h]\|_{L^2_{s',\sigma'}} &\leq \|h\|_{L^2_{0,\sigma'}} \|\xi^{s''} u\|_{L^1_{s',\sigma'}} h_1(\|u\|_{L^1_{0,\sigma'}}) + \|\xi^{s''} h\|_{L^2_{s',\sigma'}} h_2(\|u\|_{L^1_{0,\sigma'}}), \\
\|G(u,h)\|_{L^2_{s',\sigma'}} &\leq \|h\|_{L^1_{0,\sigma'}} \|h\|_{L^2_{0,\sigma'}} \|u\|_{L^1_{s',\sigma'}} h_3(\|(u,h)\|_{L^1_{0,\sigma'}}) \\
&\quad + \|h\|_{L^1_{0,\sigma'}} \|h\|_{L^2_{s',\sigma'}} h_4(\|(u,h)\|_{L^1_{0,\sigma'}}),
\end{aligned}$$

for all $s', s'' \geq 0$ such that $s'+s'' \in [0,s]$ and $\sigma' \in [0,\sigma]$. Further, if $f(z) = \mathcal{O}(|z|^2)$ for $|z| \to 0$, then $h_2(z) = \mathcal{O}(|z|)$ for $|z| \to 0$.

Proof. By Lemma A.1.2 we know that $\tilde{f} : L^2_{s,\sigma} \to L^2_{s,\sigma}$ for all $s \in]\frac{1}{2}, \infty[, \sigma \in [0,\infty[$. Further we know that there is an expansion

$$f(z+h) = f(z) + Df(z) \cdot h + \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha|_1 = 2}} G_\alpha(z,h) h^\alpha$$

for all $z, h \in \mathbb{C}^n$ and $Df(z) = (\partial_{z_1} f(z), \ldots, \partial_{z_n} f(z))$ and G_α are entire functions. This is essentially Taylor's theorem for entire functions. We see $\tilde{f} \in C(L^2_{s,\sigma})$ immediately by using Lemma A.1.2. We can iterate this argument for $D\tilde{f}$, $D^2\tilde{f}$ etc. and thus $\tilde{f} \in C^2(L^2_{s,\sigma}, L^2_{s,\sigma})$.
Hence, there is an expansion

$$\tilde{f}(u+h) = \tilde{f}(u) + D\tilde{f}(u)[h] + G(u,h),$$

for $u, h \in L^2_{s,\sigma}$. Eventually, we know how $D\tilde{f}(u)$ and G look like. These are given by the convolution operators induced by the Taylor expansion of f above and the formulae in Lemma A.1.2. I.e.

$$\begin{aligned} D\tilde{f}(u)[h] &= \sum_{i=1}^n \mathscr{F}(\partial_{z_i} f \circ \mathscr{F}^{-1} u) * h_i \\ G(u,h) &= \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha|_1 = 2}} \mathscr{F}\, G_\alpha(\mathscr{F}^{-1}(u,h)) * h^{*^\alpha}. \end{aligned}$$

Now, the claims follow because of $L^2_{s,\sigma} \hookrightarrow L^2_{s',\sigma'}$ for all $s', s'' \geq 0$ such that $s' + s'' \in [0,s]$ and $\sigma' \in [0,\sigma]$, the estimates in Lemma A.1.2 and the fact that $L^2_{s,\sigma}$ is a Banach algebra.
The statement about convergence to 0 is a consequence of the fact that $\partial^\alpha f(z) = \mathcal{O}(|z|)$, $|\alpha|_1 = 1$, if $f(z) = \mathcal{O}(|z|^2)$ for $|z| \to 0$ and Lemma A.1.2. $\square$

A.2 Regularity for Mild Solutions

Mild solutions are in general only continuous with values in a Banach space or strict solutions if certain additional conditions are met. However, in the case of analytic semigroups and certain additional assumptions one can hope for better regularity than $C(I, D(A)) \cap C^1(I, X)$ where X a Banach space, $D(A)$ is the domain of the generator $A : D(A) \subset X \to X$ of the semigroup with the graph norm and I a time interval. We state the following lemma which considers the case of an analytic semigroup whose generator is the multiplication operator $M_{-|\boldsymbol{\xi}|^2}$ in L^2_s essentially.

Lemma A.2.1. *Let $s_0, r \in \mathbb{R}^+_0$ and $\{T(t)\}_{t\in\mathbb{R}^+_0}$ an analytic C_0-semigroup in $L^p_{s_0}$ with generator $A : L^p_{s_0+2} \to L^p_{s_0}$. Let $u \in C([0, t_0[, L^p_{s_0+r})$ be a mild solution for a $t_0 > 0$ satisfying*

$$u(t) = T(t)u(0) + \int_0^t T(t-t')f \circ u(t')\,\mathrm{d}t'$$

for a (nonlinear) mapping $f : L^p_{s_0+r} \to L^p_{s_0}$. Assume $T(t) \in \mathrm{L}(L^p_s, L^p_{s+r'})$ for $t > 0$ and $t \mapsto \|T(t)\|_{\mathrm{L}(L^p_s,L^p_{s+r'})}$ is bounded by a continuous function for all $t > 0$ as well as

$$\|T(t)\|_{\mathrm{L}(L^p_s,L^p_{s+r'})} \in \mathcal{O}(t^{-\frac{r'}{2}}) \text{ for } t \to 0$$

for all $s \geq s_0 \in \mathbb{R}^+_0$ and $r' \in \mathbb{R}^+_0$. Further we assume that (a)

$$f : C(]0, t_0[, L^p_{s+r}) \to C(]0, t_0[, L^p_s) \text{ for all } s \geq s_0 \in \mathbb{R}^+_0$$

and (b) $\{T(t)\}_{t\in\mathbb{R}^+_0}$ is an analytic C_0-semigroup in L^p_s, $s \geq s_0$, with generator $A : L^p_{s+2} \to L^p_s$ and there is a mild solution for the equation above in $C([0, \delta[, L^p_{s+r})$ for a $\delta = \delta(s) > 0$ if $u(0) \in L^p_{s+r}$. Then $u \in C(]0, t_0[, L^p_\infty)$ provided that $r < 2$.

Proof. Obviously $T(t)u(0) \in L^p_\infty$ for all $t \in]0, t_0[$. Further $T(\boldsymbol{t})u(0) \in C(]0, t[, L^p_\infty)$ because of the strong continuity of the semigroup and $T(t) \in \mathrm{L}(L^p_s, L^p_{s+r'})$ for all $t > 0$, $s \geq s_0$ and $r' \in \mathbb{R}^+_0$ and the continuous bound to the mapping $t \mapsto \|T(t)\|_{\mathrm{L}\left(L^p_s,L^p_{s+r'}\right)}$ for $t > 0$.

Now let $s = s_0$. Note that $t' \mapsto T(t-t')f \circ u(t') \in L^1([0, t], L^p_{s+r+r'})$ for all $r + r' < 2$ and $0 < t < t_0$ since for a sufficiently small $\epsilon \in]0, t[$ we can estimate

$$\|T(t-t')f \circ u(t')\|_{L^p_{s+r+r'}} \leq \|T(t-t')\|_{\mathrm{L}(L^p_s,L^p_{s+r+r'})}\|f \circ u(t')\|_{L^\infty([\epsilon,t],L^2_s)}$$

for $\epsilon < t' < t$ and

$$\|T(t-t')f \circ u(t')\|_{L^p_{s+r+r'}} \leq \|T(t-t')\|_{\mathrm{L}(L^p_{s_0}, L^p_{s+r+r'})} \|f \circ u(t')\|_{L^\infty([0,\epsilon], L^2_{s_0})}$$

for $0 < t' < \epsilon$, and by assumption $t' \mapsto \|T(t-t')\|_{\mathrm{L}(L^p_s, L^p_{s+r+r'})}$ is continuous and bounded for all $t' \in [0, t[$ and $\|T(t-t')\|_{\mathrm{L}(L^p_s, L^p_{s+r+r'})} \in \mathcal{O}((t-t')^{\frac{r+r'}{2}})$ for $t' \to t$. Thus

$$\int_0^t T(t-t')f \circ u(t') \,\mathrm{d}t' \in L^p_{s+r+r'}$$

for all $0 < t < t_0$ and the map $t \mapsto \int_0^t T(t-t')f \circ u(t') \,\mathrm{d}t'$ is continuous for $t \in]0, t_0[$ since for $0 < t_1 < t < t_0$

$$\begin{aligned}
&\int_0^t T(t-t')f \circ u(t') \,\mathrm{d}t' - \int_0^{t_1} T(t_1-t')f \circ u(t') \,\mathrm{d}t' \\
&= \int_{t_1}^t T(t-t')f \circ u(t') \,\mathrm{d}t' + (T(t-t_1) - 1) \int_0^{t_1} T(t_1-t')f \circ u(t') \,\mathrm{d}t' \\
&= \int_{t_1}^t T(t-t')f \circ u(t') \,\mathrm{d}t' + \int_0^{t-t_1} T(t'')A \int_0^{t_1} T(t_1-t')f \circ u(t') \,\mathrm{d}t' \,\mathrm{d}t''
\end{aligned}$$

and $A \int_0^{t_1} T(t_1-t')f \circ u(t') \,\mathrm{d}t' \in L^p_{s+r'}$ because of assumption (b). The case $0 < t < t_1 < t_0$ is similar.
Hence, $u \in C(]0, t_0[, L^p_{s_0+r+r'})$ for $0 < r + r' < 2$. We can iterate this scheme if we define $s = s_0 + r'$ and obtain after n steps $u \in C(]0, t_0[, L^p_{s_0+r+nr'})$. □

***Remark* A.2.2.** Note that we can replace L^p_s by $L^p_{s,\sigma}$ in Lemma A.2.1. Further note that if there is an upper bound such that the assumptions on f are only satisfied for $s_0 \leq s \leq s_1$, then one could use the proof and obtain that $u \in C(]0, t_0[, L^p_{s'})$ for a $s' \geq s_0$.

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