Stochastic Partial Differential Equations on Cantor-like Sets

Von der Fakultät Mathematik und Physik der Universität Stuttgart
zur Erlangung der Würde eines Doktors der Naturwissenschaften
(Dr. rer. nat.) genehmigte Abhandlung

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Tag der mündlichen Prüfung: 20.05.2020

Institut für Stochastik und Anwendungen der Universität Stuttgart
2020

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Acknowledgements

First and foremost, I would like to thank Prof. Uta Freiberg, who has been a fantastic doctoral advisor. Her exceptional way to teach students aroused my enthusiasm for the stochastics. It was a great opportunity to work with her and I am very thankful for all the trust she gave me during the last three years.

Further, it was a pleasure to work with my external supervisor Prof. Ben Hambly. I am thankful for the inspiring discussions which offered me a new perspective on my work. Working with him at the University of Oxford was one of the most impressive experiences in my life.

A major part of my work, my research stays at the University of Oxford and also my conference participations were financed by the German Science Foundation within the Research Training Group 1838, Spectral Theory and Dynamics of Quantum Systems. This program offered me optimal financial and subject-specific conditions. In particular, I want to thank the spokesman Prof. Marcel Griesemer and my supervisor Prof. Guido Schneider.

It was a pleasure to be part of the working group consisting of Elias Hauser, Stefan Kohl, Lenon Minorics and Klemens Taglieber. I am grateful for several discussions on mathematics and a lot of topics far from that. Furthermore, on the joint journeys, we collected memories that will last a lifetime.

Beyond that, I want to thank all my other colleagues at the University of Stuttgart. The everyday life would not have been the same without them.

Finally, I would like to express my gratitude to my family and friends for the great support not only but also during my doctoral studies. In particular, I want to thank my parents, my twin brother Kevin and my girlfriend Celina. Without them, this way would not have been possible.

Abstract

We study PDEs and SPDEs defined by a measure theoretic Laplacian Δ_μ with Neumann or Dirichlet boundary conditions, where μ is a Borel measure on $[0,1]$. We do not assume that μ possesses a Lebesgue density, which includes singular measures and especially self-similar measures on Cantor-like sets.

In the first part, we address the question of how to interpret a heat equation defined by Δ_μ if the support of μ is not the whole interval. We show that weak measure convergence implies convergence of the solutions to the corresponding heat equations. This provides an interpretation for the mathematical model of heat diffusion in a rod with gaps in that the heat in this model diffuses approximately like the heat in a rod possessing a strictly positive mass distribution which is small on the gaps of the former rod.

In the second part, we investigate stochastic heat and wave equations, where μ is a self-similar measure on a Cantor-like set. We prove existence and uniqueness of the mild solution under some Lipschitz and linear growth conditions. Further, we establish Hölder continuity in space and time and determine Hölder exponents. The obtained results generalize the well-known Hölder continuity properties of stochastic heat and wave equations defined by the standard Laplacian.

Zusammenfassung

Wir untersuchen PDGs und SPDGs, die durch einen maßtheoretischen Laplace-Operator Δ_μ mit Neumann- oder Dirichlet-Randbedingungen definiert sind, wobei μ ein Borelmaß auf $[0,1]$ ist. Wir stellen nicht die Annahme der Existenz einer Lebesgue-Dichte, was singuläre Maße und insbesondere auch selbstähnliche Maße auf Cantor-ähnlichen Mengen einschließt.

Im ersten Teil befassen wir uns mit der Frage, wie eine durch Δ_μ definierte Wärmeleitungsgleichung interpretiert werden kann, wenn der Träger von μ nicht das gesamte Intervall umfasst. Wir zeigen, dass schwache Maßkonvergenz Konvergenz der Lösungen der zugehörigen Wärmeleitungsgleichungen impliziert. Dies liefert eine Interpretation für das mathematische Model von Wärmeleitung in einem Stab mit Lücken: Die Wärme in diesem Modell diffundiert annähernd wie Wärme in einem lückenlosen Stab, der aber an den Lücken des zuvor betrachteten Stabs hinreichend wenig Masse besitzt.

Im zweiten Teil untersuchen wir stochastische Wärmeleitungs- und Wellengleichungen, wobei μ ein selbstähnliches Maß auf einer Cantor-ähnlichen Menge ist. Wir beweisen Existenz und Eindeutigkeit der milden Lösung unter der Annahme geeigneter Lipschitz- und linearer Wachstumsbedingungen. Weiterhin weisen wir Hölderstetigkeit in Raum und Zeit nach und bestimmen Hölderexponenten. Die erhaltenen Resultate verallgemeinern die bekannten Hölderstetigkeitseigenschaften von stochastischen Wärmeleitungs- und Wellengleichungen, die durch den Standard-Laplace-Operator definiert sind.

1 Introduction

1.1 Statement of the problem

The heat equation, first introduced by Joseph Fourier [25] around 200 years ago, constitutes the prototype of a parabolic partial differential equation and is of fundamental importance in various scientific fields. The connection to Brownian motion, to the flow of electricity, to the diffusion of solutes in liquids (compare e.g. [60]) and to the Black-Scholes partial differential equation (compare [7]) are just a few examples of countless applications. Joseph Fourier established a connection to physics: The equation

$$\rho(x)\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2}, \quad (t,x) \in [0,\infty) \times [0,1]$$

describes heat flow on some one-dimensional, for example metallic, rod with mass density $\rho : [0,1] \to (0,\infty)$. This equation has been investigated in numerous works (compare e.g. [40], [8]), where the existence of a strictly positive mass density ρ is usually assumed. But what if the rod does not possess such a mass density?

The treatment of this problem involves the generalization of the spatial derivative with respect to a measure μ. To this end, let μ be a non-atomic Borel probability measure on $[0,1]$ such that $0, 1 \in \operatorname{supp}(\mu)$, $\mathcal{L}^2([0,1],\mu)$ be the space of measurable functions f such that $\int_0^1 f^2 d\mu < \infty$ and $L^2([0,1],\mu)$ be the corresponding Hilbert space of equivalence classes. A function $g \in L^2([0,1],\mu)$ is called the μ-derivative of $f : [0,1] \to \mathbb{R}$ if

$$f(x) = \int_0^x g(y)d\mu(y), \ x \in [0,1].$$

Composing the μ-derivative with the classical first derivative yields a measure theoretic generalization of the classical one-dimensional Laplacian Δ. We define

$$\mathcal{D}_\mu^2 := \Big\{ f \in C^1([0,1]) : \text{there exists } (f')^\mu \in L^2([0,1],\mu) : \\ f'(x) = f'(0) + \int_0^x (f')^\mu (y) d\mu(y), \quad x \in [0,1] \Big\}.$$

The measure theoretic Laplacian with respect to μ is defined by

$$\Delta_\mu : \mathcal{D}_\mu^2 \subseteq L^2([0,1],\mu) \to L^2([0,1],\mu), \quad f \to (f')^\mu .$$

Consequently, $(f')^\mu$ is the $L^2([0,1],\mu)$-equivalence class of the μ-derivative of f'.

This operator has been widely studied, for example with an emphasis on addressing questions of the spectral asymptotics and further analytical properties, such as properties of the resolvent operator and Green's function [6, 11, 23, 26–33, 35–37, 57, 58, 61, 62]. Further, heat kernels and their connection to the associated Markov process, known as Quasi or gap diffusion [47, 53, 54], wave equations [10] and higher-dimensional generalizations [34, 59].

Let $u_t(x) := u(t,x)$, $(t,x) \in [0,\infty) \times [0,1]$. We are interested in the heat equation

$$\frac{\partial u}{\partial t}(t,x) = \Delta_\mu u_t(x), \quad (t,x) \in [0,\infty) \times [0,1] \tag{1}$$

with appropriate initial value and boundary conditions, especially in its physical meaning. How can we interpret a solution to this equation if the support of the mass distribution μ is not the whole interval?

The inhomogeneous problem

$$\frac{\partial u}{\partial t}(t,x) = \Delta_\mu u_t(x) + f(t,x), \quad (t,x) \in [0,\infty) \times [0,1] \tag{2}$$

allows for an external heat source. We study the case where f is a stochastic force, more precisely a stochastic process that involves a multiplicative space-time white noise, and suitable initial and boundary conditions are given. We are interested in the regularity of the solution according to an appropriate solution concept. Hambly and Yang [41] considered generalized Laplacians on some connected sets with spectral dimension $d_S \in [1,2)$, which includes the case of Hausdorff dimension $d_H \in [1,2)$, and proved that the regularity decreases as d_S increases. We will extend this result by examining the case of Hausdorff dimension less than or equal one.

In addition to that, we analyse the second-order problem

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta_\mu u_t(x) + f(t,x), \quad (t,x) \in [0,\infty) \times [0,1], \tag{3}$$

where f involves a multiplicative space-time white noise and the Hausdorff dimension of the support of μ is less than or equal one. This equation generalizes the well-known stochastic wave equation defined by Δ, which describes the motion of a vibrating string that is struck by a succession of random particles. The canonical application is a stringed instrument left out during a sandstorm (compare [69]).

1.2 Physical motivation

We give a physical motivation for heat equations defined by Δ_μ, where we follow [40, Section 1.2] and consider a metallic rod of constant cross-sectional area oriented in the x-direction occupying a region from $x = 0$ to $x = 1$ such that all thermal quantities are constant across a section. The rod can thus be considered as one-dimensional. We investigate the conduction of thermal energy on a segment from $x = a$ to $x = b$. Let the temperature at the point $x \in [a, b]$ and time $t \in [0, \infty)$ be denoted by $u(t, x)$ and the total thermal energy in the considered segment at time t by $e_{a,b}(t)$. Let $t \in [0, \infty)$ be fixed. It is well-known that

$$e_{a,b}(t) = \int_a^b u(t, x)\rho(x)dx,$$

assuming that the rod possesses a mass density $\rho : [0, 1] \to (0, \infty)$. However, if we denote the mass distribution of the rod by μ, we can write

$$e_{a,b}(t) = \int_a^b u(t, x)d\mu(x).$$

This equation involves no density. Hence, we can compute the total thermal energy even if μ has no density. The total thermal energy changes only if thermal energy flows through the boundaries $x = a$ and $x = b$. We deduce for the rate of change of thermal energy

$$\frac{d}{dt}e_{a,b}(t) = \phi(t, a) - \phi(t, b), \tag{4}$$

where $\phi(t, x)$ denotes the heat flux density at (t, x), which gives the rate of thermal energy flowing through x at time t to the right. Assuming sufficient regularity, we can rewrite (4) as

$$\int_a^b \frac{\partial}{\partial t}u(t, x)d\mu(x) = -\int_a^b \frac{d\phi_t}{d\mu}(x)d\mu(x),$$

where $\phi_t(x) \coloneqq \phi(t, x)$. With $u_t(x) \coloneqq u(t, x)$, Fourier's law of heat conduction $\phi = -u_t'$ gives

$$\int_a^b \frac{\partial}{\partial t}u(t, x)d\mu(x) = \int_a^b \frac{d}{d\mu}\frac{d}{dx}u_t(x)d\mu(x).$$

Figure 1: First steps of the iterative construction of the Cantor set

Since this is valid for all $a, b \in [0,1]$, $a < b$, it follows for μ-almost all $x \in [0,1]$

$$\frac{\partial}{\partial t} u(t,x) = \frac{d}{d\mu}\frac{d}{dx} u_t(x).$$

Applying the definition of Δ_μ yields the heat equation (1) with Dirichlet boundary conditions $u(t,0) = u(t,1) = 0$ for all $t \geq 0$ if we assume that the temperature vanishes at the boundaries or with Neumann boundary conditions $\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 0$ for all $t \geq 0$ if the boundaries are assumed to be perfectly insulated.

This provides a physical motivation for a mass distribution having full support even if it possesses no Lebesgue density. However, it is still not clear how to interpret the equation if the support of the mass distribution is not the whole interval, in particular for singular measures, such as measures on the Cantor set.

1.3 Cantor set, Cantor-like sets and Cantor measures

The classical Cantor set, also known as the Cantor ternary set or simply the Cantor set, first described by Cantor [9] and Smith [66], is a subset of the real line that enjoys a lot of remarkable properties. It can be introduced in different ways. Cantor [9] introduced it as the set of real numbers that can be written as

$$\sum_{n=1}^{\infty} \frac{c_n}{3^n}, \quad c_n \in \{0,2\}.$$

An iterative construction, which is maybe more instructive than the above definition, is visualized in Figure 1 and can be described as follows: First, remove the open middle third of the interval $[0,1]$. Then, remove the open middle third of the resulting two intervals and continue this procedure ad infinitum.

A further construction relies on the theory of iterated function systems, which goes back to Hutchinson [46]. An iterated function system (IFS) is defined as a finite set of contraction mappings $\{S_1, ..., S_N\}$ on a complete metric space, where we additionally assume that S_i is a similarity mapping for $i \in \{1, ..., N\}$. There exists a unique non-empty compact set K such that

$$\bigcup_{i=1}^{N} S_i(K) = K. \tag{5}$$

K is called self-similar with respect to $\{S_1, ...S_N\}$.

Let $S_1, S_2 : [0,1] \to \mathbb{R},\ S_1(x) := \frac{x}{3},\ S_2(x) := \frac{2}{3} + \frac{x}{3},\ x \in [0,1]$. The Cantor set is the unique non-empty compact set that is self-similar with respect to $\{S_1, S_2\}$. The theory of iterated function systems allows various generalizations of the Cantor set. We are interested in the following: Let $\{S_1, ..., S_N\}$ be an IFS consisting of affine contractions on $[0,1]$ with contraction factors $0 < r_i < 1$ such that

$$0 = S_1(0) < S_1(1) \leq S_2(0) < S_2(1) \leq ... < S_N(1) = 1.$$

We call the unique non-empty compact set satisfying (5) a Cantor-like set.

One of the most important properties when studying these sets is their Hausdorff dimension, named after Felix Hausdorff, who introduced this concept in [44]. The Hausdorff dimension of a Cantor-like set K can be calculated easily: By Hutchinson [46], the Hausdorff dimension of K, denoted by $d_H(K)$, is the unique solution d of

$$\sum_{i=1}^{N} r_i^d = 1. \tag{6}$$

It is notable that the Hausdorff dimension of Cantor-like sets for different contraction factors can be understood as an interpolation of the dimension of a single point and that of an open interval: By variation of the contraction factors, the Hausdorff dimension of Cantor-like sets can take any number $d \in (0,1)$. For example, let K be the Cantor-like set given by $S_1(x) = rx,\ S_2(x) = 1 - r + rx,\ x \in [0,1]$, where $r = 2^{-\frac{1}{d}}$. Evidently, it holds $0 < r \leq \frac{1}{2}$. Formula (6) gives $d_H(K) = d$.

We can define several measures on a Cantor-like set K. For example, Hutchinson [46] introduced the class of self-similar measures on Cantor-like sets, also called Cantor measures. Let $\mu_1, ...\mu_N \in (0,1)$ be probability weights, that is $\sum_{k=1}^{N} \mu_k = 1$.

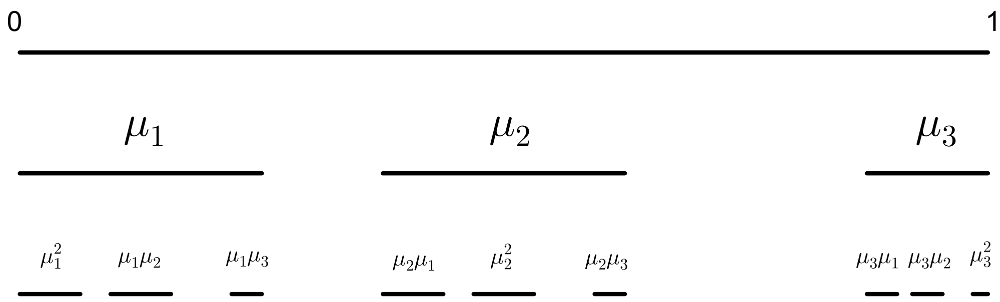

Figure 2: Self-similar measure on a Cantor-like set

By Hutchinson [46], there exists a unique Borel probability measure μ such that

$$\mu(A) = \sum_{k=1}^{N} \mu_k \mu(S_k^{-1}(A))$$

for any Borel set $A \subseteq [0,1]$ and it holds $\mathrm{supp}(\mu) = K$. The measure μ is called self-similar with respect to $(S_1, ..., S_N)$ and $(\mu_1, ...\mu_N)$. In particular, for $n \in \mathbb{N}$ and $w_1, ..., w_n \in \{1, ..., N\}$, it holds $\mu\left(S_{w_1} \circ ... \circ S_{w_n}([0,1])\right) = \mu_{w_1} \cdots \mu_{w_n}$. Figure 2 illustrates that for $n = 2$. The natural choice of weights is $\mu_i = r_i^{d_H(K)}$, $i \in \{1, ..., N\}$. The resulting measure is the normalized $d_H(K)$-dimensional Hausdorff measure, often called natural measure. If the sum of all contraction factors is less than one, the one-dimensional Lebesgue measure of K, denoted by $\lambda^1(K)$, vanishes and any measure on K is thus singular with respect to λ^1. This is the class of singular measures we are especially interested in.

1.4 White noise and the Brownian sheet

Let μ be the Lebesgue measure on $[0,1]$. Recall the inhomogeneous heat equation

$$\frac{\partial u}{\partial t}(t,x) = \Delta_\mu u_t(x) + f(t,x), \quad (t,x) \in [0,\infty) \times [0,1] \tag{7}$$

with Dirichlet boundary conditions $u(0,t) = u(1,t) = 0$ for all $t \geq 0$ and an appropriate initial condition. For a random exogenous forcing density f, we seek to find a stochastic process that solves this equation. In order to define the notion of random force we will include, let $N \geq 1$ and $(\xi_{i,j} : 1 \leq i,j \leq N)$ be a sequence of i.i.d. random variables such that $\mathbb{P}\left(\xi_{i,j} = 1\right) = \mathbb{P}\left(\xi_{i,j} = -1\right) = \frac{1}{2}$. The random variable $\xi_{i,j}$ can be interpreted as a short-term heat impulse at time $t = \frac{i}{N}$ around $x = \frac{j}{N}$. We consider the two-parameter random walk

$$S_{k,m} = \sum_{1 \le i \le k} \sum_{1 \le j \le m} \xi_{i,j}, \quad k, m \le N.$$

According to Khoshnevisan [51, Theorem 4.1.1],

$$\left\{ N^{-1} S_{\lfloor Nt \rfloor, \lfloor Nx \rfloor} : t, x \in [0,1] \right\} \to \{ B(t,x) : t, x \in [0,1] \}, \; N \to \infty$$

weakly in a suitable space, where B is a two-parameter real-valued centred Gaussian process with covariance

$$\mathbb{E}\left[B(t,x) B(t',x') \right] = \min\{t,t'\} \min\{x,x'\}, \quad t, t', x, x' \in [0,1].$$

This process is called a Brownian sheet on $[0,1]^2$. Replacing the time interval $[0,1]$ by $[m, m+1]$ for $m \ge 1$ and glueing the obtained processes B_m, $m \ge 0$ together, more precisely, by

$$B(t,x) = B_{\lfloor t \rfloor}(t - \lfloor t \rfloor, x) + \sum_{i=0}^{\lfloor t \rfloor - 1} B_i(1,x), \quad t \ge 0, \; x \in [0,1],$$

we obtain a process called Brownian sheet on $[0,\infty) \times [0,1]$. It can be understood as a two-parameter generalization of Brownian motion. Following Walsh [69], we define the random set function

$$\xi([0,t] \times [0,x]) \coloneqq B(t,x), \quad (t,x) \in [0,\infty) \times [0,1].$$

Let $\mathcal{B}([0,\infty) \times [0,1])$ be the Borel-σ-algebra on $[0,\infty) \times [0,1]$. By extending the definition of ξ to all elements of $\mathcal{B}([0,\infty) \times [0,1])$, we obtain a centred Gaussian process with covariance given by

$$\mathbb{E}\left[\xi(A_1) \xi(A_2) \right] = \lambda^2(A_1 \cap A_2), \quad A_1, A_2 \in \mathcal{B}([0,\infty) \times [0,1]),$$

where λ^2 is the two-dimensional Lebesgue measure. ξ is called space-time white noise and allows for the definition of a stochastic integral $\int_{[0,t] \times [0,1]} g(s,x) d\xi(s,x)$ for a suitable integrator g in the sense of Walsh [69].

We want to apply this concept of random noise to the inhomogeneous heat equation (7). For sufficient smooth exogenous forcing density f and zero initial condition, Duhamel's principle (see e.g. [21, Problem 7.1]) provides a solution to (7) in $L^2([0,1),\mu)$, given by

$$u(t,x)=\int_0^t\int_0^1 p^D_{t-s}(x,y)f(s,y)dyds,\quad (t,x)\in[0,\infty)\times[0,1],$$

where p^D is the heat kernel of the Dirichlet Laplacian on $L^2([0,1),\mu)$. If we assume that f is the Radon-Nikodym derivative of a Borel measure ν on $\mathcal{B}([0,\infty)\times[0,1])$, we can rewrite the previous identity as

$$u(t,x)=\int_0^t\int_0^1 p^D_{t-s}(x,y)d\nu(s,y),\quad (t,x)\in[0,\infty)\times[0,1].$$

Accordingly, we define

$$u(t,x)=\int_0^t\int_0^1 p^D_{t-s}(x,y)d\xi(s,y)$$

to be the mild solution to

$$\frac{\partial u}{\partial t}(t,x)=\Delta_\mu u_t(x)+\xi(t,x) \tag{8}$$

for $(t,x)\in[0,\infty)\times[0,1]$ with Dirichlet boundary conditions and zero initial value. The term $\xi(t,x)$ in equation (8) needs to be understood formally. In contrast to the case of ν, where f is the weak derivative of the distribution function of ν, the (pathwise) derivative of the distribution function of ξ, the Brownian sheet, only exists in a distributional sense. Changing over to heat equations in the distributional sense, equation (8) can be made more rigorous. We refer to [49] for more details.

If μ is an arbitrary Borel probability measure on $[0,1]$, we define a space-time white noise based on μ as a centred Gaussian process with

$$\mathbb{E}\left[\xi(A_1)\xi(A_2)\right]=\left(\lambda^1\otimes\mu\right)(A_1\cap A_2),\quad A_1,A_2\in\mathcal{B}([0,\infty)\times[0,1]).$$

If $A_1\subseteq[0,\infty)\times([0,1]\setminus\operatorname{supp}(\mu))$, then $\mathbb{E}\left[\xi(A_1)^2\right]=0$ and thus $\xi(A_1)=0$ almost surely. In the context of a metallic rod: There is no noise on massless parts of the rod.

1.5 Outline of the thesis

Each of the chapters 3-5 is dedicated to one of the equations (1)-(3). Before turning towards those problems, we summarize basic properties of measure theoretic Laplacians and give a brief introduction into the theory of SPDEs driven by white noise in Chapter 2.

The goal of Chapter 3 is to give an interpretation of a solution to the heat equation (1) in the case where μ is not of full support: We approximate the solution by a sequence of solutions to heat equations defined by μ_n for $n \in \mathbb{N}$ such that μ_n is of full support and converges weakly to μ for $n \to \infty$.

To this end, let $b \in \{N, D\}$ represent the boundary condition, where N denotes homogeneous Neumann and D homogeneous Dirichlet boundary conditions. Further, we assume that $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of non-atomic Borel probability measures on $[0,1]$ such that $0, 1 \in \operatorname{supp}(\mu_n)$ and $\mu_n \rightharpoonup \mu, n \to \infty$, where $\rightharpoonup$ denotes weak measure convergence.

It is well-known that Δ_μ^b is the generator of a strongly continuous semigroup $\left(T_t^b\right)_{t\geq 0}$ on $L^2([0,1],\mu)$. If $u_0 \in L^2([0,1],\mu)$, the unique solution to the initial value problem

$$\begin{aligned} \frac{du}{dt}(t) &= \Delta_\mu^b u(t), \quad t \in [0,\infty), \\ u(0) &= u_0 \end{aligned} \tag{9}$$

in $L^2([0,1],\mu)$ is given by $u(t) = T_t^b u_0$ for $t \geq 0$ according to a generalized solution concept. This motivates the investigation of strong semigroup convergence. However, for different measures, the corresponding semigroups are defined on different spaces. For the special case $\operatorname{supp}(\mu) = \operatorname{supp}(\mu_n) = [0,1]$ for all $n \in \mathbb{N}$, the results by Croydon [14] can be applied to obtain strong semigroup convergence on the space of continuous functions on $[0,1]$. To formulate a strong semigroup convergence result without this assumption, we restrict the semigroup $\left(T_t^N\right)_{t\geq 0}$ associated to Δ_μ^N on $L^2([0,1],\mu)$ to the subspace of continuous functions, denoted by $(C[0,1])_\mu^N$, which is a Banach space with the uniform norm. The semigroup $\left(T_t^D\right)_{t\geq 0}$ is restricted to the Banach space of continuous functions satisfying Dirichlet boundary conditions, denoted by $(C[0,1])_\mu^D$. We show that the restricted semigroup, denoted by $\left(\bar{T}_t^b\right)_{t\geq 0}$, is again a strongly continuous contraction semigroup and its infinitesimal generator is given by the restriction of Δ_μ^b to $\left\{f \in \mathcal{D}\left(\Delta_\mu^b\right) : \Delta_\mu^b f \in (C[0,1])_\mu^b\right\}$. We denote this operator by $\bar{\Delta}_{\mu}^{b}$. Moreover, if we assume that $\operatorname{supp}(\mu) \subseteq \operatorname{supp}(\mu_n)$ for all $n \in \mathbb{N}$, the space $(C[0,1])_{\mu}^{b}$ can be continuously embedded in $(C[0,1])_{\mu_n}^{b}$, where we denote the embedding by π_n. We will see that in this case, strong resolvent convergence implies strong semigroup convergence and strong resolvent convergence is what we will establish.

More precisely, let $f \in (C[0,1])_{\mu}^{b}$, $\lambda > 0$ and $n \in \mathbb{N}$. We define $\bar{R}_{\lambda}^{b} := \left(\lambda - \bar{\Delta}_{\mu}^{b}\right)^{-1}$ and $\bar{R}_{\lambda,n}^{b} := \left(\lambda - \bar{\Delta}_{\mu_n}^{b}\right)^{-1}$ and prove

$$\left\|\pi_n \bar{R}_{\lambda}^{b} f - \bar{R}_{\lambda,n}^{b} \pi_n f\right\|_{\infty} \to 0, \quad n \to \infty. \tag{10}$$

The proof of (10) involves the construction of measure theoretic hyperbolic functions in order to generalize the hyperbolic functions

$$\sinh(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \quad \cosh(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \ x \in [0,1]$$

by replacing $\frac{x^k}{k!}$ by generalized monomials defined by μ. This extends the theory of measure theoretic functions, developed for trigonometric functions by Arzt [2]. Then, we show that weak measure convergence implies convergence of the corresponding hyperbolic functions and that the resolvent density of the operator Δ_{μ}^{b} is a product of generalized hyperbolic functions. This leads to the desired strong resolvent convergence. We obtain for $f \in (C[0,1])_{\mu}^{b}$ and $t \geq 0$

$$\lim_{n\to\infty} \left\|\pi_n \bar{T}_t^{b} f - \bar{T}_{t,n}^{b} \pi_n f\right\|_{\infty} = 0, \tag{11}$$

uniformly on bounded time intervals. Afterwards, we will see that for $f \in (C[0,1])_{\mu}^{b}$,

$$u : [0,\infty) \to (C[0,1])_{\mu}^{b}, \ t \mapsto \bar{T}_t^{b} f$$

is the unique solution to the initial value problem

$$\begin{aligned} \frac{du}{dt}(t) &= \bar{\Delta}_{\mu}^{b} u(t), \\ u(0) &= f \end{aligned} \tag{12}$$

for $t \in [0,\infty)$ in the sense that $t \mapsto u(t)$ satisfies (12) for all $t > 0$ and is continuous with respect to $(C[0,1])_{\mu}^{b}$ for all $t \geq 0$. The same holds true if μ is replaced by μ_n

for $n \in \mathbb{N}$. Finally, combining these results and (11) yields

$$\lim_{n\to\infty} \|\pi_n u(t) - u_n(t)\|_\infty = 0,$$

uniformly on bounded time intervals.

We obtain a meaningful interpretation for the diffusion of heat in the case of a mass distribution with gaps in that the heat in a rod with mass distribution μ diffuses approximately like the heat on a rod with mass distribution μ_n for sufficiently large $n \in \mathbb{N}$.

In Chapter 4, we study the SPDE

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= \Delta_\mu^b u_t(x) + f(t,u(t,x)) + g(t,u(t,x))\xi(t,x), \\ u(0,x) &= u_0(x) \end{aligned} \tag{13}$$

for $(t,x) \in [0,T]\times[0,1]$, where $T > 0$, μ is a self-similar measure on a Cantor-like set K, ξ is a space-time white noise based on μ, f and g are predictable processes satisfying some Lipschitz and linear growth conditions and u_0 satisfies some regularity conditions.

We establish the existence of a unique mild solution to (13) as well as various regularity properties. A mild solution is defined to be a predictable $[0,T]\times[0,1]$-indexed process such that for every $(t,x) \in [0,T]\times[0,1]$ it holds almost surely

$$\begin{aligned} u(t,x) = \int_0^1 p_t^b(x,y)u_0(y)d\mu(y) &+ \int_0^t \int_0^1 p_{t-s}^b(x,y)f(s,u(s,y))d\mu(y)ds \\ &+ \int_0^t \int_0^1 p_{t-s}^b(x,y)g(s,u(s,y))\xi(s,y)d\mu(y)ds, \end{aligned} \tag{14}$$

where the last term is a stochastic integral in the sense of Walsh [69]. We review the theory of this integral in Section 2.3.

If $\mu = \lambda^1$ and u_0, f and g are uniformly bounded, it is known (see e.g. [69]) that the stochastic heat equation has a unique mild solution, which is essentially $\frac{1}{2}$-Hölder continuous in space and $\frac{1}{4}$-Hölder continuous in time. Essentially α-Hölder continuous means Hölder continuous for every exponent strictly less than α. However, in two space dimensions, the mild solution is a distribution, not a function. Hambly and Yang [41] studied these properties in the setting of a p.c.f. self-similar set (in

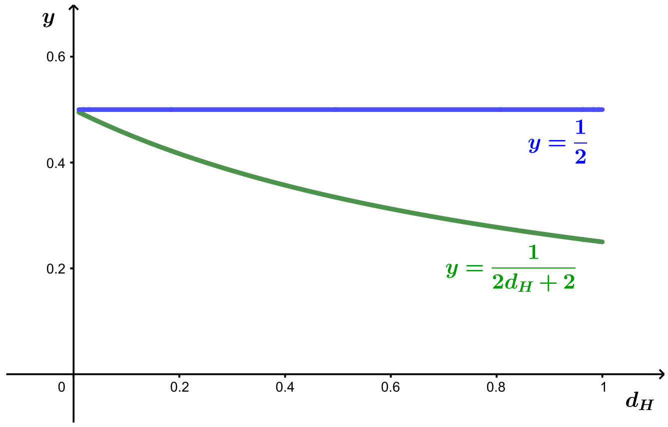

Figure 3: Hölder exponent graphs for the stochastic heat equation

the sense of [48]) with spectral dimension between one and two. It turned out that the temporal Hölder exponent decreases as the space dimension increases.

We consider the case where the Hausdorff dimension of K is less than or equal one. It will turn out that a mild solution to (13) exists and is unique. Moreover, assuming some additional regularity conditions, there exists a version that is essentially $\frac{1}{2}$-Hölder continuous in space and essentially $\frac{1}{2} - \frac{\gamma\delta}{2}$-Hölder continuous in time. Here, γ is the spectral exponent of Δ_μ^b (see Section 2.2) and $\delta := \max_{1 \le i \le N} \frac{\log \mu_i}{\log((\mu_i r_i)^\gamma)}$ can be understood as a measure for the "skewness" of μ. If μ is the normalized $d_H(K)$-dimensional Hausdorff measure, we obtain the essential temporal Hölder exponent $\frac{1}{2}(d_H(K)+1)^{-1}$. Therefore, the temporal Hölder exponent we obtained increases as $d_H(K)$ decreases. Figure 3 visualizes that.

Preliminary for proving these results, we focus on the heat kernel of Δ_μ^b. First, we establish an improved estimate on the uniform norm of the eigenfunction φ_k^b for $k \in \mathbb{N}$ (see Section 2.1 for a detailed definition of φ_k^b). In fact, we prove that there exists a constant $C > 0$ such that for all $k \in \mathbb{N}$

$$\left\|\varphi_k^b\right\|_\infty \le C k^{\frac{\delta}{2}}. \tag{15}$$

A comparable result was proven by Kigami [48, Theorem 4.5.4] for eigenfunctions of Laplacians on p.c.f. self-similar sets. This estimate along with the well-known estimates on spectral exponents (see [37]) allows us to prove several continuity properties of the heat kernel of Δ_μ^b. This will be essential in the observation of temporal Hölder continuity of the mild solution.

The key tool in order to establish spatial Hölder continuity is the approximation of $(t,y) \mapsto p_t^b(x,y)$ for fixed $x \in K$ by $(t,y) \mapsto \langle p_t^b(\cdot,y), f_n^x \rangle_\mu$ in the space $L^2([0,T] \times [0,1], \lambda^1 \otimes \mu)$ for $n \geq 1$ sufficiently large, where the sequence $(f_n^x)_{n\in\mathbb{N}}$ approximates the Delta functional δ_x. We prove that the approximating mild solutions, which are defined by replacing the heat kernel by the approximated heat kernel, have the desired spatial continuity and that the regularity is preserved upon taking the limit. The technique of approximating point evaluations of heat kernels is usually applied to investigate SPDEs in the sense of da Prato-Zabczyk (compare e.g. [24, 41–43]). We provide a way to apply this idea to Walsh SPDEs.

Besides these continuity properties, we investigate a property called intermittency. Roughly speaking, an intermittent process develops increasingly high peaks on small space-intervals when the time parameter increases. This is a phenomenon of mild solutions to stochastic heat equations, which has found much attention in the last years (compare, among many others, [3,45,49,50]). According to [49, Definition 7.5], we call a mild solution u weakly intermittent on $[0,1]$ if the lower and the upper moment Lyapunov exponents, which are respectively the functions γ and $\bar{\gamma}$ defined for $p \in (0,\infty),\ x \in [0,1]$ by

$$\gamma(p,x) := \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[|u(t,x)|^p\right], \quad \bar{\gamma}(p,x) := \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[|u(t,x)|^p\right]$$

satisfy

$$\gamma(2,x) > 0, \quad \bar{\gamma}(p,x) < \infty, \quad p \in [2,\infty),\ x \in [0,1].$$

We prove this in the Neumann case for $f = 0$ assuming some conditions on g.

In Chapter 5, we are concerned with the stochastic wave equation

$$\begin{aligned} \frac{\partial^2}{\partial t^2} u(t,x) &= \Delta_\mu^b u_t(x) + f(t,u(t,x))\xi(t,x), \\ u(0,x) = u_0(x),\ & \frac{\partial}{\partial t} u(0,x) = u_1(x) \end{aligned} \tag{16}$$

for $(t,x) \in [0,T] \times [0,1]$, where μ is a self-similar measure on a Cantor-like set K, f is a predictable process satisfying some Lipschitz and linear growth conditions and u_0 and u_1 satisfy some regularity conditions.

It is known (see [69]) that if $\mu = \lambda^1$ and f is uniformly bounded, the stochastic wave equation has a unique mild solution, which is essentially $\frac{1}{2}$-Hölder continuous

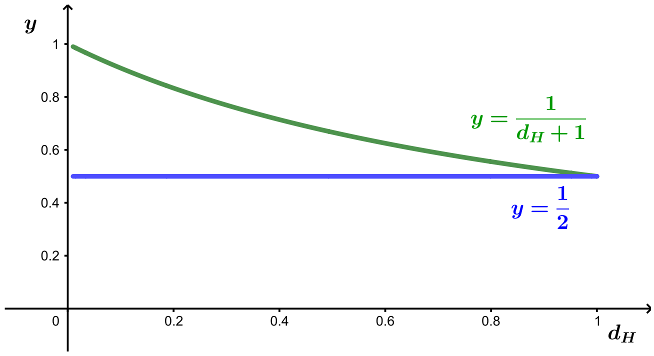

Figure 4: Hölder exponent graphs for the wave equation

in space and in time. Again, in two space dimensions, the mild solution is a distribution, not a function. Hambly and Yang [41] addressed the questions regarding these properties for stochastic wave equations in the sense of da Prato-Zabczyk on p.c.f. self-similar sets with Hausdorff dimension between one and two. According to the knowledge of the author, there are no results about second-order Walsh SPDEs defined by a fractal Laplacian.

We show that there is a unique mild solution, which has a version that is essentially $\frac{1}{2}$-Hölder continuous in space and essentially $\left(d_H(K) + 1 + \frac{\log(\min_{1\le i\le N}\nu_i)}{\log(\max_{1\le i\le N} r_i)}\right)^{-1}$-Hölder continuous in time, where $\nu_i := \frac{\mu_i}{r_i^{d_H(K)}}$, $i = 1, \dots, N$. In particular, if μ is the natural measure on K, we obtain the essential temporal Hölder exponent $\frac{1}{d_H(K)+1}$. Figure 4 indicates this result.

Essential for the proof is the examination of the wave propagator of Δ_μ^b, defined by

$$P_t^N(x,y) = t + \sum_{k\ge 2} \frac{\sin\left(\sqrt{\lambda_k^N} t\right)}{\sqrt{\lambda_k^N}} \varphi_k^N(x)\varphi_k^N(y),\ (t,x,y) \in [0,\infty) \times [0,1]^2, \tag{17}$$

where $\varphi_k^N,\ k \ge 1$ are the $L^2([0,1],\mu)$-normed eigenfunctions and $\lambda_k^N,\ k \ge 1$ the related eigenvalues of $-\Delta_\mu^N$, which will be introduced precisely in Section 2.1, and analogously for Dirichlet boundary conditions. It is the wave equation counterpart to the heat kernel. In contrast to the heat kernel, there is nothing known about the regularity of the wave propagator. Further, an upper estimate of (17) using the eigenfunction estimate (15) does not even give convergence. We approximate $y \to P_t^b(x,y)$ for fixed $(t,x) \in [0,\infty) \times K$ by $y \to \left\langle P_t^b(y,\cdot), f_n^x\right\rangle_\mu$ in $L^2([0,1],\mu)$ for n sufficiently large and conclude that $y \to P_t^b(x,y)$ is an $L^2([0,1],\mu)$-function. Then, we show that the approximated mild solutions, defined by replacing the wave propagator by these approximating functions, have the desired Hölder continuity properties and that the regularity is preserved upon taking the limit. Finally, we observe weak intermittency.

This thesis is based on the following papers (compare the references [17], [18], [19]):

- T. Ehnes, B. Hambly, *An Approximation of Solutions to Heat Equations defined by Generalized Measure Theoretic Laplacians*, Preprint, 2020.
- T. Ehnes, *Stochastic Heat Equations defined by Fractal Laplacians on Cantor-like Sets*, Preprint, 2019.
- T. Ehnes, *Stochastic Wave Equations defined by Fractal Laplacians on Cantor-like Sets*, Preprint, 2019.

2 Preliminaries

2.1 Basic properties of measure theoretic Laplacians

First, we introduce basic concepts that will be essential in this work. For a more detailed account, we refer to [38, 48].

Let μ be a Borel measure on $[0,1]$ and let $L^2([0,1],\mu)$ be the Hilbert space with inner product $\langle f,g\rangle_\mu := \int_0^1 f(x)g(x)d\mu(x)$. Further, let $C[0,1]$ be the Banach space of continuous functions with the uniform norm $\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|$.

Definition 2.1: *Let $\mathcal{D}$ be a dense subset of $L^2([0,1],\mu)$. A Dirichlet form on $L^2([0,1],\mu)$ is defined to be a symmetric bilinear function $\mathcal{E} : \mathcal{D}\times\mathcal{D} \to \mathbb{R}$ such that*

(i) $\mathcal{E}(u,u) \geq 0$ for all $u \in \mathcal{D}$.

(ii) $\mathcal{D}$ equipped with the bilinear function $\mathcal{E}_1(u,v) := \langle u,v\rangle_\mu + \mathcal{E}(u,v)$, $u,v \in \mathcal{D}$ is a Hilbert space.

(iii) If $u \in \mathcal{D}$, then $\bar{u} := (0 \vee u) \wedge 1 \in \mathcal{D}$ and $\mathcal{E}(\bar{u},\bar{u}) \leq \mathcal{E}(u,u)$.

Property (iii) is known as the Markov property. Furthermore, a Dirichlet form is regular if $\mathcal{D}\cap C[0,1]$ is dense in $\mathcal{D}$ with respect to the norm $\sqrt{\mathcal{E}_1(u,u)}$ and dense in $C[0,1]$ with respect to the uniform norm.

Definition 2.2: *Let $(A,\mathcal{D}(A))$ be a densely defined linear operator on $L^2([0,1],\mu)$. An element $y \in L^2([0,1],\mu)$ is said to belong to the domain $\mathcal{D}(A^*)$ of the adjoint operator A^* if there exists $h \in L^2([0,1],\mu)$ such that*

$$\langle Af, g\rangle = \langle f, h\rangle .$$

In this case, $A^*f := h$. Further, A is said to be self-adjoint if $\mathcal{D}(A) = \mathcal{D}(A^*)$ and $A = A^*$.

If A is a non-negative self-adjoint operator on $L^2([0,1],\mu)$, there exists a unique self-adjoint operator B on $L^2([0,1],\mu)$ such that $BBf = Af$ for all $f \in \mathcal{D}(A)$ (see e.g. [48, Proposition B.1.2]). This operator is called the square root of A and we write $B = A^{\frac{1}{2}}$.

Let $(X, \|\cdot\|)$ be a real Banach space and id_X be the identity on X.

Definition 2.3: *A strongly continuous semigroup is defined to be a family* $(T_t)_{t\geq 0}$ *of linear operators on* H *satisfying*

(i) $T_0 = \mathrm{id}_X$,

(ii) $T_{s+t} = T_s T_t$ *for* $s, t \geq 0$,

(iii) $\lim_{t\searrow 0} \|T_t f - f\| = 0$ *for* $f \in X$.

Furthermore, we define

$$\mathcal{D}(A) = \left\{ f \in X : \text{ there exists } g \in X \text{ such that } \left\| g - \lim_{t\searrow 0} \frac{T_t f - f}{t} \right\| = 0 \right\}$$

and $A := \lim_{t\searrow 0} \frac{T_t f - f}{t}$ *for* $f \in \mathcal{D}(A)$. A *is the infinitesimal generator of the strongly continuous semigroup* $(T_t)_{t\geq 0}$.

Definition 2.4: *Let* A *be an operator on* X. *The resolvent set* $\rho(A)$ *is defined as the set* $\rho(A) := \{\lambda \in \mathbb{R} : A - \lambda \mathrm{id}_X \text{ maps } \mathcal{D}(A) \text{ bijectively onto } X\}$. *For* $\lambda \in \rho(A)$, *we define* $R(\lambda, A) := (A - \lambda \mathrm{id}_X)^{-1}$. *This operator is called the resolvent operator of* A.

We now define measure theoretic Laplacians. To this end, let $b \in \{N, D\}$ and let μ be a non-atomic Borel probability measure on $[0,1]$ such that $0, 1 \in \mathrm{supp}(\mu)$. If $[0,1] \setminus \mathrm{supp}(\mu) \neq \emptyset$, $[0,1] \setminus \mathrm{supp}(\mu)$ is open in $\mathbb{R}$ and can be written as

$$[0,1] \setminus \mathrm{supp}(\mu) = \bigcup_{i\geq 1} (a_i, b_i) \tag{18}$$

with $0 < a_i < b_i < 1$, $a_i, b_i \in \mathrm{supp}(\mu)$ for $i \geq 1$. Recall from Section 1.1 that

$$\mathcal{D}_\mu^2 = \Big\{ f \in C^1([0,1]) : \text{there exists } (f')^\mu \in L^2([0,1],\mu) : \\ f'(x) = f'(0) + \int_0^x (f')^\mu (y) d\mu(y), \quad x \in [0,1] \Big\}.$$

We define

$$\mathcal{D}\left(\Delta_\mu^N\right) := \left\{ f \in \mathcal{D}_\mu^2 : f'(0) = f'(1) = 0 \right\} \subseteq L^2([0,1],\mu)$$

and Δ_μ^N as the restriction of Δ_μ to $\mathcal{D}\left(\Delta_\mu^N\right)$, that is

$$\Delta_\mu^N : \mathcal{D}\left(\Delta_\mu^N\right) \to L^2([0,1],\mu), \quad f \to (f')^\mu .$$

For $\lambda > 0$, the resolvent operator $R_\lambda^N := R\left(\lambda, \Delta_\mu^N\right) = \left(\lambda - \Delta_\mu^N\right)^{-1}$ is a continuous self-adjoint operator on $L^2([0,1],\mu)$ (see [27, Theorem 6.1]). Consequently, $\lambda - \Delta_\mu^N$ is the inverse of a self-adjoint injective operator and thus self-adjoint (see e.g. [67, Proposition A.8.2]. We conclude that Δ_μ^N is self-adjoint.

A more abstract way to introduce measure theoretic Laplacians relies on the theory of Dirichlet forms. Let

$$\mathcal{D}^1 := \Big\{ f : [0,1] \to \mathbb{R} : \text{ there exists } f' \in L^2\left([0,1], \lambda^1\right) : \\ f(x) = f(0) + \int_0^x f'(y)dy, \; x \in [0,1] \Big\}$$

and let $\mathcal{F}$ be the space of all $L^2([0,1],\mu)$-equivalence classes that possess a $\mathcal{D}^1$-representative. Note that we write λ^1 for the one-dimensional Lebesgue measure restricted to an interval $I \subseteq (-\infty, \infty)$ if the restriction is clear from the context.

We define the non-negative symmetric bilinear form $(\mathcal{E}, \mathcal{F})$ on $L^2([0,1],\mu)$ by

$$\mathcal{E}(u,v) = \int_0^1 u'(x)v'(x)dx, \quad u, v \in \mathcal{F}.$$

From now on, for each argument, which is an element of $L^2([0,1],\mu)$, we choose the $\mathcal{D}^1$-representative that is for $i \geq 1$ linear on $[a_i, b_i]$ (see Lemma A.1 for the proof of existence of this representative). By Freiberg [28, Theorem 4.1], $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form on $L^2([0,1],\mu)$. As a consequence of the basic theory of Dirichlet forms (see [38, Theorem 1.3.1]), there exists a unique self-adjoint operator $\left(A^N, \mathcal{D}\left(A^N\right)\right)$ on $L^2([0,1],\mu)$ such that

$$\left\langle -A^N u, v \right\rangle_\mu = \mathcal{E}(u,v), \quad u \in \mathcal{D}\left(A^N\right), \; v \in \mathcal{F}. \tag{19}$$

Freiberg [27, Proposition 3.1] proved that Δ_μ^N satisfies the Gauss-Green formula (19) as well. We obtain $\mathcal{D}\left(A^N\right) = \mathcal{D}\left(\Delta_\mu^N\right)$ and $A^N = \Delta_\mu^N$.

We treat the case of Dirichlet boundary conditions in the same way. Let

$$\mathcal{D}\left(\Delta_\mu^D\right) := \left\{ f \in \mathcal{D}_\mu^2 : f(0) = f(1) = 0 \right\} \subseteq L^2([0,1],\mu)$$

and let Δ_μ^D be the restriction of Δ_μ to $\mathcal{D}\left(\Delta_\mu^D\right)$. Further, let $\mathcal{F}_0$ be the space of all $L^2([0,1],\mu)$-equivalence classes having a $\mathcal{D}^1$-representative f such that $f(0) = f(1) = 0$. Then, $(\mathcal{E}, \mathcal{F}_0)$ is a Dirichlet form (see [28, Theorem 4.1]). Again, there exists an associated self-adjoint operator $\left(A^D, \mathcal{D}\left(A^D\right)\right)$ on $L^2([0,1],\mu)$ such that

$$\left\langle -A^D u, v\right\rangle_\mu = \mathcal{E}(u,v), \quad u \in \mathcal{D}\left(A^D\right),\ v \in \mathcal{F}_0. \tag{20}$$

We deduce $\mathcal{D}\left(A^D\right) = \mathcal{D}\left(\Delta_\mu^D\right)$ as well as $A^D = \Delta_\mu^D$ with a similar argumentation to the Neumann case.

By Freiberg [27, Proposition 6.3, Lemma 6.7, Corollary 6.9], there exists an orthonormal basis $\{\varphi_k^b : k \in \mathbb{N}\}$ of $L^2([0,1],\mu)$ consisting of eigenfunctions of $-\Delta_\mu^b$ and for the related ascending ordered eigenvalues $\{\lambda_k^b : k \in \mathbb{N}\}$ it holds that $0 \leq \lambda_1^b \leq \lambda_2^b \leq \ldots$, where $\lambda_1^D > 0$. Since $\{\varphi_k^b : k \geq 1\}$ is an orthonormal basis of $L^2([0,1],\mu)$, each $f \in L^2([0,1],\mu)$ can be written as $f = \sum_{k\geq 1} f_k^b \varphi_k^b$, where $f_k^b := \left\langle f, \varphi_k^b\right\rangle_\mu,\ k \geq 1$. Along with the self-adjointness, we obtain the following formula, called the spectral representation of Δ_μ^b (see e.g. [38, Section 1.3]):

$$\begin{aligned} -\Delta_\mu^b f &= \sum_{k\geq 1} \lambda_k^b f_k^b \varphi_k^b, \\ \mathcal{D}\left(\Delta_\mu^b\right) &= \left\{ f \in L^2([0,1],\mu) : \sum_{k\geq 1} \left(\lambda_k^b f_k^b\right)^2 < \infty \right\}. \end{aligned}$$

The spectral representation provides a direct way to introduce the associated semigroup. Define for $f \in L^2([0,1],\mu)$

$$T_t^b f := \sum_{k\geq 1} e^{-\lambda_k^b t} f_k^b \varphi_k^b, \quad t \geq 0. \tag{21}$$

Then, $\left(T_t^b\right)_{t\geq 0}$ is a strongly continuous semigroup on $L^2([0,1],\mu)$ and its infinitesimal generator is Δ_μ^b (see [38, Lemma 1.3.2]). Further, the Markov property of the associated Dirichlet form implies the Markov property of the associated semigroup, which means that for all $t \geq 0$ we have that $T_t^b f \in [0,1]$ μ-a.e. whenever $f \in L^2([0,1],\mu)$ and $f \in [0,1]$ μ-a.e. (see [38, Theorem 1.4.1]). Moreover, $\left(T_t^b\right)_{t\geq 0}$ is the transition semigroup of a strong Markov process (see [38, Theorem 7.2.1]), which is known as Quasi or gap diffusion (see for example [47, 53–55]). This process is not the object of this work.

2.2 Spectral asymptotics for self-similar measures

Throughout this section, let μ be self-similar with respect to $(S_1, ..., S_N)$ and $(\mu_1, ..., \mu_N)$, where $S_1, ..., S_N$ are affine contractions on $[0,1]$ with contraction factors $0 < r_i < 1$ such that

$$0 = S_1(0) \leq S_1(1) \leq S_2(0) \leq S_2(1) \leq ... \leq S_N(1) = 1$$

and $\mu_1, ..., \mu_N \in (0,1)$ are probability weights (compare Section 1.3). We further assume that $0, 1 \in \operatorname{supp}(\mu)$ and set $K \coloneqq \operatorname{supp}(\mu)$.

Let $b \in \{N, D\}$. We have already mentioned that there exists an orthonormal basis $\{\varphi_k^b : k \in \mathbb{N}\}$ of $L^2([0,1], \mu)$ consisting of eigenfunctions of $-\Delta_\mu^b$ and that for the related ascending ordered eigenvalues $\{\lambda_k^b : k \in \mathbb{N}\}$ we have that $0 \leq \lambda_1^b \leq \lambda_2^b \leq ...$, where $\lambda_1^D > 0$. Let r_i be the contraction factor of S_i and let γ be the unique solution of

$$\sum_{i=1}^{N} (\mu_i r_i)^\gamma = 1. \tag{22}$$

By Fujita [37], there exist constants $c_0, c_1 > 0$ such that for $k \geq 2$

$$c_0 k^{\frac{1}{\gamma}} \leq \lambda_k^b \leq c_1 k^{\frac{1}{\gamma}}. \tag{23}$$

γ is called the spectral exponent of Δ_μ^b.

The goal of this section is to develop an asymptotic upper estimate on the uniform norm of φ_k^b for $k \to \infty$. The only known estimate, established in [29, Section 2] and [2, Lemma 3.6], is elementary to derive and grows exponentially in k, which will turn out to be far too weak for our purposes. In the following, we establish an improved estimate, where we do not use the explicit representation of the eigenfunctions as Arzt [2]. Instead, we follow Kigami [48, Theorem 4.5.4], who has established a similar estimate for Laplacians on p.c.f. self-similar sets.

Theorem 2.5: *Let $\delta \coloneqq \max_{1 \leq i \leq N} \frac{\log \mu_i}{\log((\mu_i r_i)^\gamma)}$. Then, there exists a constant $\bar{c_2} > 0$ such that for all $k \in \mathbb{N}$*

$$\left\| \varphi_k^b \right\|_\infty \leq \bar{c_2} \left(\lambda_k^b \right)^{\frac{\gamma\delta}{2}}.$$

Since φ_k^b is an element of $L^2([0,1], \mu)$, it is not determined on $[0,1] \setminus \operatorname{supp}(\mu)$.

To overcome this ambiguity in the definition of $\left\|\varphi_k^b\right\|_\infty$, we henceforth evaluate the $\mathcal{D}_\mu^2$-representative of φ_k^b, which is for $i \geq 1$ linear on $[a_i, b_i]$.

Theorem 2.5 implies with $c_2 := c_1^{\frac{\gamma\delta}{2}} \bar{c}_2$ for all $k \in \mathbb{N}$

$$\left\|\varphi_k^b\right\|_\infty \leq c_2 k^{\frac{\delta}{2}}. \tag{24}$$

We now prepare the proof of Theorem 2.5. First, we introduce some notation.

A concept to describe Cantor-like sets is given by the so-called word or code space. Let $I := \{1, ..., N\}$, $\mathbb{W}_n := I^n$ be the set of all sequences ω of length $|\omega| = n$, $\mathbb{W}^* := \cup_{n\in\mathbb{N}} I^n$ be the set of all finite sequences and $\mathbb{W} := I^\infty$ be the set of all infinite sequences $\omega = \omega_1\omega_2\omega_3...$ with $\omega_i \in I$ for $i \in \mathbb{N}$. Then, I is called the alphabet and $\mathbb{W}$, $\mathbb{W}^*$, $\mathbb{W}^n$, $n \in \mathbb{N}$ are called word spaces. We define an ordering on $\mathbb{W}$ by denoting two words ω and σ as equal if $\omega_i = \sigma_i$ for all $i \in \mathbb{N}$ and otherwise, we write $\omega < \sigma :\Leftrightarrow \sigma_\kappa < \omega_k$ or $\omega > \sigma :\Leftrightarrow \sigma_\kappa > \omega_\kappa$, where $\kappa := \inf\{n \in \mathbb{N} : \sigma_n \neq \omega_n\}$. In addition to an ordering, we define a metric on the word space by the map $d : \mathbb{W} \times \mathbb{W} \to \mathbb{R}$, $d(\omega, \sigma) := N^{-\kappa}$ with κ defined as before. For every $x \in [0, 1]$, the map

$$\pi_x : \mathbb{W} \to K, \quad \sigma \mapsto \lim_{n\to\infty} S_{\sigma_1} \circ S_{\sigma_2} \circ ... \circ S_{\sigma_n}(x)$$

is well-defined, continuous, surjective and independent of $x \in [0, 1]$, which means that for $x, y \in [0, 1]$, $\sigma \in \mathbb{W}$ we have that $\pi_x(\sigma) = \pi_y(\sigma)$ (see [5, Theorem 2.1]). Hence, for every $x \in [0, 1]$ and every $y \in K$ there exists at least one element of $\mathbb{W}$, which is by π_x associated to y.

We need a couple of lemmas to prove Theorem 2.5.

Lemma 2.6: *If $u \in \mathcal{F}_0$, then*

$$\|u\|_\mu^2 \leq \mathcal{E}(u),$$

where $\mathcal{E}(u) := \mathcal{E}(u, u)$ and $\|u\|_\mu^2 := \langle u, u\rangle_\mu$.

Proof. Using $\lambda_1^D \geq 1$ (see e.g. [58, Lemma 4.9]), the assertion follows by the representation of the smallest eigenvalue of a Dirichlet form (see [16, Theorem 1.3]). □

Lemma 2.7: *There is a constant $c_3 > 0$ such that for all $u \in \mathcal{F}$*

$$\|u\|_\mu^2 \leq c_3 \left(\mathcal{E}(u) + \|u\|_1^2\right),$$

where $\|u\|_1 := \int_0^1 |u(x)| d\mu(x)$.

Proof. Let $u \in \mathcal{F}$ and let u_0 be the unique harmonic function with $u_0(0) = u(0)$ and $u_0(1) = u(1)$, that is $u_0(x) := u(0)(1-x) + u(1)x$, $x \in [0,1]$. We have $(u-u_0)(0) = (u-u_0)(1) = 0$ and thus $u - u_0 \in \mathcal{F}_0$. Since the space of harmonic functions on $[0,1]$ with two boundary conditions is two-dimensional, there exists $c_3' \geq 1$ such that for all harmonic functions u_0

$$\|u_0\|_\mu \leq c_3' \|u_0\|_1 .$$

Since μ is a probability measure we have for all $u \in \mathcal{F}$

$$\|u\|_1 \leq \|u\|_\mu .$$

Furthermore, for $u \in \mathcal{F}$ and the corresponding harmonic function u_0

$$\begin{aligned}
\mathcal{E}(u-u_0) &= \mathcal{E}(u) - 2\mathcal{E}(u,u_0) + \mathcal{E}(u_0) \\
&= \mathcal{E}(u) - 2\int_0^1 u'(x)(u(1)-u(0))dx + (u(1)-u(0))^2 \\
&= \mathcal{E}(u) - 2(u(1)-u(0))^2 + (u(1)-u(0))^2 \\
&= \mathcal{E}(u) - (u(1)-u(0))^2
\end{aligned}$$

and thus

$$\mathcal{E}(u-u_0) \leq \mathcal{E}(u).$$

By Lemma 2.6 and the above calculations,

$$\begin{aligned}
\|u\|_\mu &\leq \|u_0\|_\mu + \|u-u_0\|_\mu \\
&\leq c_3' \|u_0\|_1 + \sqrt{\mathcal{E}(u-u_0)} \\
&\leq c_3'(\|u\|_1 + \|u-u_0\|_1) + \sqrt{\mathcal{E}(u-u_0)} \\
&\leq c_3'(\|u\|_1 + \|u-u_0\|_\mu) + \sqrt{\mathcal{E}(u-u_0)} \\
&\leq c_3' \|u\|_1 + c_3'\sqrt{\mathcal{E}(u-u_0)} + \sqrt{\mathcal{E}(u-u_0)} \\
&\leq 2c_3' \left(\|u\|_1 + \sqrt{\mathcal{E}(u)}\right) .
\end{aligned}$$

The assertion follows from the fact that for positive numbers a, b, c with $a \leq b + c$ we have that $a^2 \leq 2(b^2 + c^2)$. $\square$

Moreover, we will need scaling properties for μ and $\mathcal{E}$. Firstly, we introduce the notion of a partition, following Kigami [48, Definition 1.3.9].

Definition 2.8: *Let $\omega \in \mathbb{W}^*$ and*

$$\Sigma_\omega := \{\sigma = \sigma_1\sigma_2 ... \in \mathbb{W} : \sigma_i = \omega_i \text{ for all } 1 \leq i \leq |\omega|\}.$$

A finite subset $\Lambda \subset \mathbb{W}^*$ is called partition of $\mathbb{W}$ if it holds $\Sigma_\omega \cap \Sigma_\sigma = \emptyset$ for $\omega \neq \sigma \in \Lambda$ and $\mathbb{W} = \bigcup_{\omega \in \Lambda} \Sigma_\omega$.

Let $w \in \mathbb{W}^*$. For a function f, we define $f_\omega := f_{\omega_1} \circ f_{\omega_2} \circ ... \circ f_{\omega_{|\omega|}}$. Further, let $r_\omega := r_{\omega_1} r_{\omega_2} \cdots r_{\omega_{|\omega|}}$ and $\mu_\omega := \mu_{\omega_1}\mu_{\omega_2}\cdots\mu_{\omega_{|\omega|}}$.

Lemma 2.9: *Let Λ be a partition. We have*

(i) $\mu = \sum_{\omega \in \Lambda} \mu_\omega (\mu \circ S_\omega^{-1})$.

(ii) $\sum_{\omega \in \Lambda} r_\omega^{-1} \mathcal{E}(u \circ S_\omega) \leq \mathcal{E}(u)$ *for all* $u \in \mathcal{F}$.

This can be verified using [1, Section 3.2.1] by induction. Since this is a standard argument, we skip the proof.

Proof of Theorem 2.5. Let $u \in \mathcal{F}$ be fixed and let Λ be a partition. Then,

$$\begin{aligned}
\|u\|_\mu^2 &= \int_0^1 u^2(x) d\mu(x) \\
&= \sum_{\omega \in \Lambda} \mu_w \int_0^1 u^2(x) d\mu \circ S_\omega^{-1}(x) && (25) \\
&= \sum_{\omega \in \Lambda} \mu_w \int_0^1 u(S_\omega(x))^2 d\mu(x) \\
&\leq c_3 \sum_{\omega \in \Lambda} \mu_w \left(\mathcal{E}(u \circ S_\omega) + \|u \circ S_\omega\|_1^2\right) && (26) \\
&\leq c_3 \left(\max_{\omega \in \Lambda}\{\mu_\omega r_\omega\} \sum_{\omega \in \Lambda} r_\omega^{-1} \mathcal{E}(u \circ S_\omega) + \sum_{\omega \in \Lambda} \mu_w^{-1} \left(\mu_\omega \int_0^1 |u \circ S_\omega(x)| d\mu(x) \right)^2 \right) \\
&\leq c_3 \left(\max_{\omega \in \Lambda}\{\mu_\omega r_\omega\} \mathcal{E}(u) + \min_{\omega \in \Lambda}\{\mu_\omega^{-1}\} \|u\|_1^2 \right). && (27)
\end{aligned}$$

In the above, equation (25) follows from Lemma 2.9(i), inequality (26) from Lemma 2.7 and inequality (27) from Lemma 2.9(ii). Now, let $\nu_i := (\mu_i r_i)^\gamma$, $i = 1, ..., N$. By (22) we have $\sum_{i=1}^N \nu_i = 1$. Let $\lambda \in (0, 1]$ and the partition Λ_λ be defined by

$$\Lambda_\lambda = \{\omega \in \mathbb{W}^* : \nu_{\omega_1} \cdots \nu_{\omega_{|\omega|-1}} > \lambda \geq \nu_\omega\}.$$

By definition of Λ_λ, for $\omega \in \Lambda_\lambda$ we have $\nu_\omega^{\frac{1}{\gamma}} = \mu_\omega r_\omega \leq \lambda^{\frac{1}{\gamma}}$ and thus $\max_{\omega\in\Lambda_\lambda}(\mu_\omega r_\omega) \leq \lambda^{\frac{1}{\gamma}}$. Furthermore, it is known from [48, Proposition 4.5.2] that there exists $c_2' > 0$ such that $\min_{\omega\in\Lambda_\lambda} \mu_\omega \geq c_2'\lambda^\delta$, from which it follows $(\min_{\omega\in\Lambda_\lambda} \mu_\omega)^{-1} \leq \frac{1}{c_2'}\lambda^{-\delta}$. This and (27) yield the existence of a constant $c_2'' > 0$ such that for all $\lambda \in (0,1]$, $u \in \mathcal{F}$

$$\|u\|_\mu^2 \leq c_2'' \left(\lambda^{\frac{1}{\gamma}}\mathcal{E}(u) + \lambda^{-\delta}\|u\|_1^2\right).$$

Let $\theta := 2\gamma\delta$. Lemma 2.6 implies

$$\|u\|_1^2 \leq \|u\|_\mu^2 \leq \mathcal{E}(u).$$

We can thus choose $\lambda \in (0,1]$ such that $\lambda^{\frac{1}{\gamma}+\delta} = \frac{\|u\|_1^2}{\mathcal{E}(u)}$. It follows

$$\|u\|_\mu^2 \leq 2c_2''\lambda^{-\delta}\|u\|_1^2 \tag{28}$$

and with $c_2''' := (2c_2'')^{1+\frac{2}{\theta}}$

$$\begin{aligned}\|u\|_\mu^{2+\frac{4}{\theta}} &\leq (2c_2'')^{1+\frac{2}{\theta}}\lambda^{-\delta-\frac{1}{\gamma}}\|u\|_1^{2+\frac{4}{\theta}} \\ &= (2c_2'')^{1+\frac{2}{\theta}}\|u\|_1^{\frac{4}{\theta}}\mathcal{E}(u) \\ &= c_2'''\|u\|_1^{\frac{4}{\theta}}\mathcal{E}(u).\end{aligned}$$

Let $\psi : L^2([0,1],\mu) \to L^2(K,\mu)$, $f \mapsto f|_K$ and $\widetilde{\Delta}_\mu^N : \psi\left(\mathcal{D}\left(\Delta_\mu^N\right)\right) \to L^2(K,\mu)$, $u \mapsto \psi\Delta_\mu^N\psi^{-1}u$. Then, $\widetilde{\Delta}_\mu^N$ is self-adjoint, has eigenvalues $-\lambda_k^N$ with eigenfunctions $\psi\varphi_k^N$ for $k \in \mathbb{N}$ and the Dirichlet form $\widetilde{\mathcal{E}}(\widetilde{u},\widetilde{v}) := \mathcal{E}(\psi^{-1}\widetilde{u},\psi^{-1}\widetilde{v})$, $\widetilde{u},\widetilde{v} \in \widetilde{\mathcal{F}} := \psi(\mathcal{F})$ is associated (see Lemma A.2). It follows that for all $\widetilde{u} \in \widetilde{\mathcal{F}}$ the Nash-type inequality

$$\|\widetilde{u}\|_\mu^{2+\frac{4}{\theta}} \leq c_2'''\widetilde{\mathcal{E}}(\widetilde{u})\|\widetilde{u}\|_1^{\frac{4}{\theta}} \tag{29}$$

is satisfied. Applying [48, Proposition B.3.7] yields the existence of $c_2'''' > 0$ such that for all $k \in \mathbb{N}$ and all $t > 0$

$$\left\|\widetilde{T}_t^N\widetilde{\varphi}_k^N\right\|_\infty \leq c_2''''t^{-\frac{\theta}{4}}, \tag{30}$$

where $\left(\widetilde{T}_t^N\right)_{t\geq 0}$ is the strongly continuous semigroup associated to $\widetilde{\Delta}_\mu^N$. Using that $\widetilde{T}_t^N\widetilde{\varphi}_k^N = e^{-\lambda_k^N t}\widetilde{\varphi}_k^N$ for $t \geq 0$ (see [48, Corollary B.2.7]) and setting $t := \frac{1}{\lambda_k^N}$ and

$\bar{c_2} := c_2''''e$ for $k \geq 2$, we obtain for all $k \in \mathbb{N}$

$$\left\|\widetilde{\varphi}_k^N\right\|_\infty \leq \bar{c_2}\left(\lambda_k^N\right)^{\frac{\gamma\delta}{2}}.$$

The assertion follows for $b = N$ since we evaluate the $\mathcal{D}_2^\mu$-representative of φ_k^b that is for $i \geq 1$ linear on $[a_i, b_i]$ (see (18) for the definition of a_i, b_i). In case of $b = D$ the proof works analogously since $\mathcal{F}_0 \subseteq \mathcal{F}$. □

2.3 Stochastic integration

In this section, we review the integration theory with respect to space-time white noise in the sense of Walsh [69]. Let μ be a finite Borel measure on $[0,1]$, let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space and let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfy the usual conditions. The white noise integration theory can be extended in various directions, such as for measures on Lusin spaces, but we won't need these generalizations and therefore refer to Walsh [69, Chapter 1].

First, we recap the definition of white noise.

Definition 2.10: *A space-time white noise based on μ w.r.t. $\mathbb{F}$ is a centred Gaussian process $\xi = (\xi(A) : A \in \mathcal{B}([0,\infty) \times [0,1]))$ such that for all $A_1, A_2 \in \mathcal{B}([0,\infty) \times [0,1])$, $A_3 \in \mathcal{B}([0,1])$*

(i) $\mathbb{E}\left[\xi(A_1)\xi(A_2)\right] = \left(\lambda^1 \otimes \mu\right)(A_1 \cap A_2)$,

(ii) $t \mapsto \xi([0,t] \times A_3)$ *is an $\mathbb{F}$-martingale.*

For the proof of the existence of such a process, we refer to Walsh [69, Chapter 1]. We will write space-time white noise, or white noise, if the measure μ and the filtration $\mathbb{F}$ are clear from the context. White noise is an $L^2(\Omega)$-valued countably-additive measure on $\mathcal{B}([0,\infty) \times [0,1])$ (see [49, Lemma 2.3]). This motivates the idea to integrate appropriate functions against it. Walsh [69] developed an integration theory for a wider class of integrators, the so-called martingale measures. White noise is a well-behaved example of a martingale measure.

First, we define the stochastic integral for a class of simple processes, as one does for the well-known Itô integral.

Definition 2.11: *A simple process $\phi : \Omega \times [0,\infty) \times [0,1] \to \mathbb{R}$ is defined as a finite sum of functions $h : \Omega \times [0,\infty) \times [0,1] \to \mathbb{R}$ of the form*

$$h(\omega, t, x) = X(\omega)\mathbb{1}_{(a,b]}(t)\mathbb{1}_B(x), \quad (\omega, t, x) \in \Omega \times [0,\infty) \times [0,1]$$

with X bounded and $\mathcal{F}_a$-measurable, $a, b \geq 0$, $a < b$ and $B \in \mathcal{B}([0,1])$.

Let $T > 0$. We define the stochastic integral for $h(\omega, t, x) = X(\omega)\mathbb{1}_{(a,b]}(t)\mathbb{1}_B(x)$ by

$$\int_0^T \int_0^1 h(\omega, t, x)\xi(t, x)d\mu(x)dt = X(w)\left(\xi([0, t \wedge b] \times B) - \xi([0, t \wedge a] \times B)\right).$$

The integral for a simple process ϕ, denoted by $\int_0^T \int_0^1 \phi(t,x)\xi(t,x)d\mu(x)dt$, is defined by linearity. As usual, we suppress the dependence on ω.

For each simple process $\phi : \Omega \times [0,\infty) \times [0,1] \to \mathbb{R}$ it can be easily shown that

$$\mathbb{E}\left[\left|\int_0^T \int_0^1 \phi(t,x)\xi(t,x)d\mu(x)dt\right|^2\right] = \int_0^T \int_0^1 \mathbb{E}\left[\phi(t,x)\right]^2 d\mu(x)dt \tag{31}$$

(compare [49, Section 4.2]). Identity (31) is known as Walsh's isometry. In stochastic calculus, the Itô isometry is essential to extend the stochastic integral to a class of predictable processes, where the predictable σ-algebra is generated by the class of simple processes on $\Omega \times [0, T]$. The extension for Walsh integrals has a similar character.

Let $\mathcal{P}_{[0,T],[0,1]}$ be the σ-algebra that is generated by the class of simple processes on $\Omega \times [0, T] \times [0,1]$. We call a process on $\Omega \times [0, T] \times [0,1]$ predictable if it is measurable from $\mathcal{P}_{[0,T],[0,1]}$ into $\mathcal{B}(\mathbb{R})$.

Definition 2.12: *Let $\mathcal{P}_{2,T}$ be the space of all processes $\phi \in L^2(\Omega \times [0,T] \times [0,1])$ such that ϕ is predictable.*

$\mathcal{P}_{2,T}$ is a Banach space, which can be checked by a standard argument, so we skip the proof here. The subspace of all simple processes, where we identify each element with all of its modifications, is dense in $\mathcal{P}_{2,T}$ (see [69, Proposition 2.3]). Now, let $\phi \in \mathcal{P}_{2,T}$ and let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence of simple processes such that $\phi_n \to \phi$ in $\mathcal{P}_{2,T}$

as $n \to \infty$. Then,

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^T \int_0^1 \phi_n(t,x)\xi(t,x)d\mu(x)dt - \int_0^T \int_0^1 \phi_m(t,x)\xi(t,x)d\mu(x)dt\right|^2\right] \\
&= \int_0^T \int_0^1 \mathbb{E}\left[\phi_n(t,x) - \phi_m(t,x)\right]^2 d\mu(x)dt \\
&= ||\phi_n - \phi_m||^2_{L^2(\Omega\times[0,T]\times[0,1])}.
\end{aligned}$$

Since $L^2(\Omega)$ is a Banach space, the stochastic integral of ϕ_n converges in $L^2(\Omega)$ and we define

$$\int_0^T \int_0^1 \phi(t,x)\xi(t,x)d\mu(x)dt \coloneqq \lim_{n\to\infty} \int_0^T \int_0^1 \phi_n(t,x)\xi(t,x)d\mu(x)dt \ \text{ in } L^2(\Omega),$$

where the limit is independent of the choice of the convergent sequence. Obviously, this integral again satisfies Walsh's isometry. Further, $(M_T)_{T\geq 0}$ defined by

$$M_T \coloneqq \int_0^T \int_0^1 \phi(t,x)\xi(t,x)d\mu(x)dt$$

is a continuous $\mathbb{F}$-martingale (see [49, Proposition 4.3]).

There are other notions of stochastic integrals. The theory of da Prato–Zabczyk [15] is another very common way to define SPDEs. However, we do not investigate such SPDEs in the present work. It should be noted that by using the results in the present work, the investigations of heat and damped stochastic wave equations on Cantor-like sets in the sense of da Prato–Zabczyk work very similar to those in [41] and [43] for p.c.f. fractals.

3 An Approximation of Solutions to Measure Theoretic Heat Equations

In this chapter, we consider heat equations defined by a generalized measure theoretic Laplacian on $[0,1]$. These equations describe heat diffusion on a rod such that the mass distribution of the rod is given by a non-atomic Borel probability measure μ. First, we develop a theory of measure theoretic hyperbolic functions in Section 3.1. Then, we establish properties of measure theoretic Laplacians on spaces of continuous functions in Section 3.2. Based on that, we show that weak measure convergence implies convergence of the corresponding measure theoretic Laplacians in the strong resolvent sense. We prove that strong semigroup convergence with respect to the uniform norm follows, which implies uniform convergence of solutions to the corresponding heat equations. This is all done in Section 3.3.

3.1 Generalized hyperbolic functions and the resolvent density

Let $b \in \{N, D\}$ and let μ be a non-atomic Borel probability measure on $[0,1]$ such that $0, 1 \in \operatorname{supp}(\mu)$. In this section, we develop a useful representation for the resolvent density of Δ_μ^b.

Let $\lambda > 0$. We consider the initial value problem

$$\begin{aligned} &\Delta_\mu g = \lambda g, \\ &g(0) = 1, \quad g'(0) = 0 \end{aligned} \tag{32}$$

on $L^2([0,1],\mu)$. (32) possesses a unique solution (see [27, Lemma 5.1]), which we denote by $g_{1,N}^\lambda$. Further, under the initial conditions

$$g(1) = 1, \quad g'(1) = 0, \tag{33}$$

$$g(0) = 0, \quad g'(0) = 1, \tag{34}$$

and

$$g(1) = 0, \quad g'(1) = 1, \tag{35}$$

respectively, the above eigenvalue problem also possesses a unique solution (see [27, Remark 5.2]), and we denote it by $g^{\lambda}_{2,N}$, $g^{\lambda}_{1,D}$ and $g^{\lambda}_{2,D}$, respectively. The resolvent density is then given as follows.

Lemma 3.1: *Let $\lambda > 0$. The resolvent operator $R^b_\lambda := (\lambda - \Delta^b_\mu)^{-1}$ is well-defined and for all $f \in L^2([0,1],\mu)$ we have*

$$R^b_\lambda f(x) = \int_0^1 \rho^b_\lambda(x,y) f(y) d\mu(y), \quad x \in [0,1],$$

where the resolvent densities are given by

$$\rho^N_\lambda(x,y) = \rho^N_\lambda(y,x) := \frac{g^{\lambda}_{1,N}(x) g^{\lambda}_{2,N}(y)}{\left(g^{\lambda}_{1,N}\right)'(1)}, \quad x,y \in [0,1],\ x \le y,$$
$$\rho^D_\lambda(x,y) = \rho^D_\lambda(y,x) := -\frac{g^{\lambda}_{1,D}(x) g^{\lambda}_{2,D}(y)}{g^{\lambda}_{1,D}(1)}, \quad x,y \in [0,1],\ x \le y.$$

Proof. See [27, Theorem 6.1]. □

It is well-known that if $\mu = \lambda^1$, the solutions to (32) and (34) are given by

$$g^{\lambda}_{1.N}(x) = \cosh\left(\sqrt{\lambda} x\right) \quad \text{and} \quad g^{\lambda}_{1.D}(x) = \frac{1}{\sqrt{\lambda}} \sinh\left(\sqrt{\lambda} x\right), \ x \in [0,1],$$

respectively. We generalize the notion of hyperbolic functions by solving (32) and (34) for an arbitrary measure μ according to the given conditions. To this end, we introduce generalized monomials as in [2].

Definition 3.2: *For $x \in [0,1]$ we set $p_0(x) = q_0(x) = 1$ and for $k \in \mathbb{N}$*

$$p_k(x) := \begin{cases} \int_0^x p_{k-1}(t) d\mu(t), & \text{if } k \text{ is odd,} \\ \int_0^x p_{k-1}(t) dt, & \text{if } k \text{ is even,} \end{cases}$$
$$q_k(x) := \begin{cases} \int_0^x q_{k-1}(t) dt, & \text{if } k \text{ is odd,} \\ \int_0^x q_{k-1}(t) d\mu(t), & \text{if } k \text{ is even.} \end{cases}$$

We note that for $x \in [0,1]$ and $k \ge 0$

$$p_{2k+1}(x) \le p_{2k}(x) \le \frac{x^k}{k!}, \quad q_{2k+1}(x) \le p_{2k}(x) \le \frac{x^k}{k!} \tag{36}$$

(see [29, Lemma 2.3]).

Definition 3.3: *We define for $x \in [0,1]$, $z \in \mathbb{R}$*

$$\sinh_z(x) := \sum_{k=0}^{\infty} z^{2k+1} q_{2k+1}(x), \quad \cosh_z(x) := \sum_{k=0}^{\infty} z^{2k} p_{2k}(x).$$

By (36) for all $z \in \mathbb{R}$

$$\|\sinh_z\|_\infty \le z e^{z^2}, \quad \|\cosh_z\|_\infty \le e^{z^2}. \tag{37}$$

Example 3.4: If $\mu = \lambda^1$, we have $q_k(x) = \frac{x^k}{k!}$, $k \ge 0$. It follows that in this case

$$\sinh_z(x) = \sum_{k=0}^{\infty} z^{2k+1} \frac{x^{2k+1}}{(2k+1)!} = \sinh(zx)$$

and analogously $\cosh_z(x) = \cosh(zx)$.

Proposition 3.5: *Let $\lambda > 0$. Then, for $x \in [0,1]$, we have*

$$\begin{aligned} g^\lambda_{1,N}(x) &= \cosh_{\sqrt{\lambda}}(x), & g^\lambda_{1,D}(x) &= \frac{1}{\sqrt{\lambda}} \sinh_{\sqrt{\lambda}}(x), \\ g^\lambda_{2,N}(x) &= \cosh_{\sqrt{\lambda}}(1-x), & g^\lambda_{2,D}(x) &= -\frac{1}{\sqrt{\lambda}} \sinh_{\sqrt{\lambda}}(1-x). \end{aligned}$$

Proof. The assertion for $g^\lambda_{1,D}$ was proven in [29, Lemma 2.3]. The proof for $g^\lambda_{1,N}$ works analogously. We verify the assertion for $g^\lambda_{2,N}$. Let $x \in [0,1]$. Then,

$$\begin{aligned} \cosh_{\sqrt{\lambda}}(1-x) &= \sum_{n=0}^{\infty} \lambda^n p_{2n}(1-x) \\ &= 1 + \sum_{n=1}^{\infty} \lambda^n \int_0^{1-x} \int_0^y p_{2n-2}(t) d\mu(t) dy \\ &= 1 + \sum_{n=1}^{\infty} \lambda^n \int_0^{1-x} \int_{1-y}^1 p_{2n-2}(1-t) d\mu(t) dy \\ &= 1 - \sum_{n=1}^{\infty} \lambda^n \int_x^1 \int_0^y p_{2n-2}(1-t) d\mu(t) dy \\ &= 1 - \sum_{n=0}^{\infty} \lambda^{n+1} \int_x^1 \int_0^y p_{2n}(1-t) d\mu(t) dy. \end{aligned}$$

Due to estimate (36) we can use the dominated convergence theorem and obtain

$$\begin{aligned}\cosh_{\sqrt{\lambda}}(1-x) &= 1-\lambda\int_x^1\int_0^y\sum_{n=0}^{\infty}\lambda^n p_{2n}(1-t)d\mu(t)dy\\ &= 1-\lambda\int_x^1\int_0^y \cosh_{\sqrt{\lambda}}(1-t)d\mu(t)dy.\end{aligned}$$

We set $f(x) := \cosh_{\sqrt{\lambda}}(1-x),\ x \in [0,1]$ and get

$$f(x) = 1-\lambda\int_x^1\int_0^y f(t)d\mu(t)dy,\ x\in[0,1]$$

and in particular

$$f(0) = 1-\lambda\int_0^1\int_0^y f(t)d\mu(t)dy.$$

Hence, for $x \in [0,1]$,

$$f(x)-f(0) = \lambda\int_0^x\int_0^y f(t)d\mu(t)dy.$$

The latter equation can be written as $\Delta_\mu f = \lambda f$. It remains to check the initial conditions. Obviously, $f(1) = \cosh_{\sqrt{\lambda}}(0) = 1$. Using (36) again, we have

$$f'(1) = -\sum_{n=1}^{\infty}\lambda^n p_{2n-1}(0) = 0.$$

The proof for $g_{2,D}^{\lambda}$ follows using the same ideas. □

This leads to the following representation for the resolvent density:

Corollary 3.6: *Let $\lambda > 0$. It holds for $x, y \in [0,1]$, $x \le y$,*

$$\begin{aligned}\rho_\lambda^N(x,y) = \rho_\lambda^N(y,x) &= \left(\cosh'_{\sqrt{\lambda}}(1)\right)^{-1}\cosh_{\sqrt{\lambda}}(x)\cosh_{\sqrt{\lambda}}(1-y),\\ \rho_\lambda^D(x,y) = \rho_\lambda^D(y,x) &= \frac{1}{\sqrt{\lambda}}\left(\sinh_{\sqrt{\lambda}}(1)\right)^{-1}\sinh_{\sqrt{\lambda}}(x)\sinh_{\sqrt{\lambda}}(1-y).\end{aligned}$$

3.2 Restricted semigroups

Let $b \in \{N, D\}$ and let μ be defined as before. Recall that Δ_μ^b is the generator of a strongly continuous Markovian semigroup $\left(T_t^b\right)_{t\geq 0}$ of contractions on $L^2([0,1],\mu)$.

Definition 3.7: *For $(t,x,y) \in (0,\infty) \times [0,1] \times [0,1]$, we define*

$$p_t^b(x,y) := \sum_{k=1}^{\infty} e^{-\lambda_k^b t} \varphi_k^b(x) \varphi_k^b(y). \tag{38}$$

This is called the heat kernel of Δ_μ^b. Note that for fixed $(t,y) \in (0,\infty) \times [0,1]$, $p_t(\cdot, y)$ is affine on the intervals outside of $\mathrm{supp}(\mu)$*, as the eigenfunctions are affine. Hence, this pointwise representation coincides with the $\mathcal{D}_\mu^2$-representative of the $L^2([0,1],\mu)$-equivalence class given by (38).*

The heat kernel is the integral kernel of T_t^b for $t > 0$. That is, for $t > 0$ and $f \in L^2([0,1],\mu)$, we can write

$$T_t^b f(x) = \int_0^1 p_t^b(x,y) f(y) d\mu(y), \quad x \in [0,1].$$

In this section, we restrict these semigroups to appropriate spaces of equivalence classes of continuous functions.

Definition 3.8: *(i) We define $(C[0,1])_\mu^N$ as the set of all $L^2([0,1],\mu)$-equivalence classes possessing a continuous representative, formally*

$$(C[0,1])_\mu^N := \left\{ f \in L^2([0,1],\mu) : f \text{ possesses a continuous representative} \right\}.$$

(ii) We further define $(C[0,1])_\mu^D$ as the set of all $L^2([0,1],\mu)$-equivalence classes possessing a continuous representative that satisfies Dirichlet boundary conditions, formally

$$(C[0,1])_\mu^D := \Big\{ f \in L^2([0,1],\mu) : f \text{ possesses a continuous representative } \bar{f} \\ \text{such that } \bar{f}(0) = \bar{f}(1) = 0 \Big\}.$$

The space $(C[0,1])_\mu^b$ is a Banach space with the norm $\|f\|_{(C[0,1])_\mu^b} := \left\| f|_{\mathrm{supp}(\mu)} \right\|_\infty$. Note that

$$\|f\|_{(C[0,1])_\mu^b} = \left\| \widetilde{f} \right\|_\infty,$$

where $\widetilde{f}$ is the continuous representative of f that is for $i \geq 1$ linear on $[a_i, b_i]$ (see (18) for the definition of a_i, b_i). This is the representative we use when evaluating $f(x)$ for $x \in [0,1] \setminus \operatorname{supp}(\mu)$. To simplify the notation, we henceforth write $\|f\|_\infty$ for $\|f\|_{(C[0,1])^b_\mu}$.

Let $u = \sum_{k\geq 1} u_k^b \varphi_k^b \in L^2([0,1],\mu)$ and let $t > 0$. We have

$$\Delta_\mu^b T_t^b u = \sum_{k\geq 1} \lambda_k^b e^{-\lambda_k^b t} u_k^b \varphi_k^b \in L^2([0,1],\mu) \tag{39}$$

and thus $T_t^b u \in \mathcal{D}\left(\Delta_\mu^b\right)$. Hence, the following inclusion holds:

$$T_t^b \left((C[0,1])^b_\mu\right) \subseteq (C[0,1])^b_\mu.$$

This motivates the definition of the restricted semigroup $\left(\bar{T}_t^b\right)_{t\geq 0}$, which is for $t \geq 0$ defined by

$$\bar{T}_t^b : (C[0,1])^b_\mu \to (C[0,1])^b_\mu,\ \bar{T}_t^b f := T_t^b f.$$

The goal of this section is to show that $\left(\bar{T}_t^b\right)_{t\geq 0}$ again defines a strongly continuous contraction semigroup. It is obvious that the semigroup property holds. Note that by the Markov property of $\left(T_t^b\right)_{t\geq 0}$ it follows with $f \equiv 1$ for $(t,x) \in (0,\infty) \times [0,1]$

$$0 \leq T_t^b f(x) = \int_0^1 p_t^b(x,y) d\mu(y) \leq 1$$

and consequently for $g \in (C[0,1])^b_\mu$

$$\left|T_t^b g(x)\right| = \left|\int_0^1 p_t^b(x,y) g(y) d\mu(y)\right| \leq \|g\|_\infty \left|\int_0^1 p_t^b(x,y) d\mu(y)\right| \leq \|g\|_\infty,\ x \in [0,1].$$

Hence, $\left(\bar{T}_t^b\right)_{t\geq 0}$ is a semigroup of contractions. It remains to prove the strong continuity. To this end, we need some preparations. We write $\mathcal{E}(f,f) := \mathcal{E}(f)$.

Lemma 3.9: *If $f \in \mathcal{F}$, then*

$$\|f\|_\infty \leq \mathcal{E}(f)^{\frac{1}{2}} + \|f\|_\mu.$$

Proof. Let $f \in \mathcal{F}$. Then, by the Cauchy-Schwarz inequality for all $x, y \in [0,1]$

$$|f(x) - f(y)| = \left| \int_x^y f'(z)dz \right| \leq \left(\int_x^y (f')^2 (z)dz \right)^{\frac{1}{2}} |x-y|^{\frac{1}{2}} = \mathcal{E}(f)^{\frac{1}{2}} |x-y|^{\frac{1}{2}} .$$

It follows by the reversed triangle inequality and by $|x-y| \leq 1$

$$|f(x)| \leq |f(y)| + \mathcal{E}(f)^{\frac{1}{2}}.$$

Further, by integrating of y w.r.t. μ,

$$|f(x)| \leq \int_0^1 |f(y)| d\mu(y) + \mathcal{E}(f)^{\frac{1}{2}}$$

and finally by the Cauchy-Schwarz inequality

$$|f(x)| \leq \|f\|_\mu + \mathcal{E}(f)^{\frac{1}{2}}.$$

$\square$

Lemma 3.10: *Let* $f \in (C[0,1])_\mu^b$. *Then,* $\lim_{t \to 0} \left\| T_t^b f - f \right\|_\infty = 0$.

Proof. We follow the proof of [48, Proposition 5.2.6]. Let $f \in \mathcal{F}$. By Lemma 3.9 and [48, Lemma B.2.4],

$$\begin{aligned}
\lim_{t \to 0} \left\| T_t^b f - f \right\|_\infty &\leq \lim_{t \to 0} \mathcal{E}\left(T_t^b f - f\right)^{\frac{1}{2}} + \left\| T_t^b f - f \right\|_\mu \\
&\leq \lim_{t \to 0} 2^{\frac{1}{2}} \left(\mathcal{E}\left(T_t^b f - f\right) + \left\| T_t^b f - f \right\|_\mu^2 \right)^{\frac{1}{2}} \\
&= 0.
\end{aligned}$$

By the fact that $\mathcal{F}$ is dense in $(C[0,1])_\mu^N$ and that, for $t \geq 0$, T_t^N is continuous on $(C[0,1])_\mu^N$, we obtain the assertion for $b = N$. To verify the case $b = D$, we prove that $\mathcal{F}_0$ is dense in $(C[0,1])_\mu^D$. Let $f \in (C[0,1])_\mu^D$. Then, by the density of $\mathcal{F}$ in $(C[0,1])_\mu^N$, there exists a sequence $(f_n)_{n \in \mathbb{N}}$ with $f_n \in \mathcal{F}$ for each $n \in \mathbb{N}$ such that $\|f - f_n\|_\infty \to 0, \ n \to \infty$. We define for $n \in \mathbb{N}$

$$f_{n,0}(x) := f_n(x) - f_n(0) - x(f_n(1) - f_n(0)), \ x \in [0,1],$$

which is an element of $\mathcal{F}_0$. Further, we have that

$$f_0(x) := f(x) - f(0) - x(f(1) - f(0)) = f(x),\ x \in [0,1]$$

since f satisfies Dirichlet boundary conditions. This implies

$$\begin{aligned}
&\lim_{n\to\infty} \|f_{n,0} - f\|_\infty \\
&= \lim_{n\to\infty} \|f_{n,0} - f_0\|_\infty \\
&\le \lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| + |f_n(0) - f(0)| + |x\,(f_n(1) - f_n(0) - (f(1) - f(0)))| \\
&= 0.
\end{aligned}$$

□

The main result of this section now follows immediately.

Corollary 3.11: $\left(\bar{T}_t^b\right)_{t\ge 0}$ *is a strongly continuous contraction semigroup on* $(C[0,1])_\mu^b$.

3.3 Convergence results

3.3.1 Strong resolvent convergence

Let μ be defined as before and let F be the distribution function of μ. We give our basic assumption.

Assumption 3.12: *Let* $(\mu_n)_{n\in\mathbb{N}}$ *be a sequence of non-atomic Borel probability measures on* $[0,1]$ *such that* $0,1 \in \operatorname{supp}(\mu_n)$ *and* $\mu_n \rightharpoonup \mu, n \to \infty$*, where* $\rightharpoonup$ *denotes weak measure convergence.*

We denote the distribution function of μ_n by F_n for $n \in \mathbb{N}$.

First, we give convergence results for the generalized hyperbolic functions introduced in Section 3.1. Let $p_k,\ q_k,\ k \in \mathbb{N}$ be defined by μ and $p_{k,n}, q_{k,n},\ k \in \mathbb{N}$ be defined by μ_n for $n \in \mathbb{N}$.

Lemma 3.13: *For $x \in [0,1]$ and $k, n \in \mathbb{N}$ we have*

$$\begin{aligned} |q_{2k}(x) - q_{2k,n}(x)| &\leq 2\frac{\|F - F_n\|_\infty x^k}{(k-1)!}, \\ |p_{2k}(x) - p_{2k,n}(x)| &\leq 2\frac{\|F - F_n\|_\infty x^k}{(k-1)!}, \\ |q_{2k+1}(x) - q_{2k+1,n}(x)| &\leq 2\frac{\|F - F_n\|_\infty x^k}{(k-1)!}, \\ |p_{2k+1}(x) - p_{2k+1,n}(x)| &\leq 2\frac{\|F - F_n\|_\infty x^k}{(k-1)!}. \end{aligned}$$

Proof. The distribution function of μ is continuous. Hence, weak measure convergence implies uniform convergence of the corresponding distribution functions (see e.g. [12, Section 8.1]). We can thus apply [30, Lemma 3.1]. □

For $z \in \mathbb{R}$ let $\cosh_z$, $\sinh_z$ be defined by μ and $\cosh_{z,n}$, $\sinh_{z,n}$ be defined by μ_n for $n \in \mathbb{N}$. We obtain a convergence result for the generalized hyperbolic functions, comparable to that for generalized trigonometric functions in [30].

Lemma 3.14: *Let $z \in \mathbb{R}$. Then,*

$$\begin{aligned} \|\cosh_z - \cosh_{z,n}\|_\infty &\leq 2z^2 e^{z^2} \|F - F_n\|_\infty, \\ \left\|\cosh_z' - \cosh_{z,n}'\right\|_\infty &\leq \left(z^2 + 2z^4 e^{z^2}\right) \|F - F_n\|_\infty, \\ \|\sinh_z - \sinh_{z,n}\|_\infty &\leq 2z^3 e^{z^2} \|F - F_n\|_\infty. \end{aligned}$$

Proof. Let $x \in [0,1]$ and $n \in \mathbb{N}$. Then,

$$\begin{aligned} |\cosh_z(x) - \cosh_{z,n}(x)| &\leq \sum_{k=1}^{\infty} |p_{2k}(x) - p_{2k,n}(x)| \, z^{2k} \\ &\leq 2\sum_{k=1}^{\infty} \frac{\|F - F_n\|_\infty}{(k-1)!} z^{2k} \\ &= 2\sum_{k=0}^{\infty} \frac{\|F - F_n\|_\infty}{k!} z^{2k+2} \\ &= 2z^2 e^{z^2} \|F - F_n\|_\infty. \end{aligned}$$

Further, note that

$$\cosh'_z(x) = \sum_{k=1}^{\infty} p_{2k-1}(x) z^{2k}$$

and

$$|p_1(x) - p_{1,n}(x)| = |\mu([0,x]) - \mu_n([0,x])| \leq \|F - F_n\|_\infty .$$

With that,

$$\begin{aligned}
\left|\cosh'_z(x) - \cosh'_{z,n}(x)\right| &\leq \sum_{k=1}^{\infty} |p_{2k-1}(x) - p_{2k-1,n}(x)|\, z^{2k} \\
&\leq \left(z^2 + 2\sum_{k=2}^{\infty} \frac{z^{2k}}{(k-2)!} \right) \|F - F_n\|_\infty \\
&= \left(z^2 + 2z^4 e^{z^2} \right) \|F - F_n\|_\infty .
\end{aligned}$$

Finally,

$$\begin{aligned}
|\sinh_z(x) - \sinh_{z,n}(x)| &\leq \sum_{k=1}^{\infty} |q_{2k+1}(x) - q_{2k+1,n}(x)|\, z^{2k+1} \\
&\leq 2\sum_{k=1}^{\infty} \frac{z^{2k+1}}{(k-1)!} \|F - F_n\|_\infty \\
&= 2\sum_{k=0}^{\infty} \frac{z^{2k+3}}{k!} \|F - F_n\|_\infty \\
&= 2z^3 e^{z^2} \|F - F_n\|_\infty . \qquad \square
\end{aligned}$$

We turn to the main result of this section. For $b \in \{N, D\}$, $\lambda > 0$ and $n \in \mathbb{N}$, let R^b_λ be defined by μ, $R^b_{\lambda,n}$ be defined by μ_n and let the resolvent densities be analogously defined. We assume $\operatorname{supp}(\mu) \subseteq \operatorname{supp}(\mu_n)$ for all $n \in \mathbb{N}$. Then, the mapping

$$\pi_n : (C[0,1])^b_\mu \to (C[0,1])^b_{\mu_n}, \ f \mapsto f \tag{40}$$

defines an embedding, where $f \in (C[0,1])^b_{\mu_n}$ denotes the $L^2([0,1], \mu_n)$-equivalence class of the representative of $f \in (C[0,1])^b_\mu$ that is linear on each interval $I \subseteq \operatorname{supp}(\mu_n) \setminus \operatorname{supp}(\mu)$.

Theorem 3.15: *Let $\lambda > 0$. Then, for all $f \in (C[0,1])^b_\mu$,*

$$\lim_{n\to\infty} \left\| R^b_{\lambda,n}\pi_n f - \pi_n R^b_\lambda f \right\|_\infty = 0.$$

Proof. We simplify the notation in this proof by omitting all embeddings. If we evaluate on $\operatorname{supp}(\mu_n) \setminus \operatorname{supp}(\mu)$, we always evaluate the representative that is linear on each interval $I \subseteq \operatorname{supp}(\mu_n) \setminus \operatorname{supp}(\mu)$. First, we consider the case $b = N$. Let $\lambda > 0$, $n \in \mathbb{N}$, $x, y \in [0,1]$ with $x \le y$. Using the triangle inequality,

$$\begin{aligned}
&\left|\rho^N_\lambda(x,y) - \rho^N_{\lambda,n}(x,y)\right| \\
&\le \left|\left(\cosh'_{\sqrt{\lambda}}(1)\right)^{-1} - \left(\cosh'_{\sqrt{\lambda},n}(1)\right)^{-1}\right| \left|\cosh_{\sqrt{\lambda}}(x)\cosh_{\sqrt{\lambda}}(1-y)\right| \\
&+ \left|\cosh_{\sqrt{\lambda}}(x) - \cosh_{\sqrt{\lambda},n}(x)\right| \left|\left(\cosh'_{\sqrt{\lambda},n}(1)\right)^{-1}\cosh_{\sqrt{\lambda}}(1-y)\right| \\
&+ \left|\cosh_{\sqrt{\lambda}}(1-y) - \cosh_{\sqrt{\lambda},n}(1-y)\right| \left|\left(\cosh'_{\sqrt{\lambda},n}(1)\right)^{-1}\cosh_{\sqrt{\lambda},n}(x)\right|.
\end{aligned} \tag{41}$$

We have

$$\cosh'_{\sqrt{\lambda}}(1) = \sum_{k=1}^{\infty} \lambda^k p_{2k-1}(1) \ge \lambda p_1(1) = \lambda \tag{42}$$

and similarly $\cosh'_{\sqrt{\lambda},n}(1) \ge \lambda$. Applying this along with Lemma 3.14, we get

$$\begin{aligned}
\left|\left(\cosh'_{\sqrt{\lambda}}(1)\right)^{-1} - \left(\cosh'_{\sqrt{\lambda},n}(1)\right)^{-1}\right| &= \left|\frac{\cosh'_{\sqrt{\lambda},n}(1) - \cosh'_{\sqrt{\lambda}}(1)}{\cosh'_{\sqrt{\lambda}}(1)\cosh'_{\sqrt{\lambda},n}(1)}\right| \\
&\le \frac{\left(\lambda + 2\lambda^2 e^\lambda\right)\|F - F_n\|_\infty}{\lambda^2}
\end{aligned}$$

and thus with (37)

$$\begin{aligned}
&\left|\left(\cosh'_{\sqrt{\lambda}}(1)\right)^{-1} - \left(\cosh'_{\sqrt{\lambda},n}(1)\right)^{-1}\right| \left|\cosh_{\sqrt{\lambda}}(x)\cosh_{\sqrt{\lambda}}(1-y)\right| \\
&\le \frac{\left(e^{2\lambda} + 2\lambda e^{3\lambda}\right)\|F - F_n\|_\infty}{\lambda}.
\end{aligned}$$

For the second term on the right-hand side of inequality (41), we calculate

$$\left|\cosh_{\sqrt{\lambda}}(x) - \cosh_{\sqrt{\lambda},n}(x)\right| \left|\left(\cosh'_{\sqrt{\lambda},n}(1)\right)^{-1}\cosh_{\sqrt{\lambda}}(1-y)\right| \le 2e^{2\lambda}\|F - F_n\|_\infty.$$

Treating the third term analogously and plugging the above calculations into (41) yields

$$
\begin{aligned}
&\lim_{n\to\infty}\max_{x\in[0,1]}\left|\rho_\lambda^N(x,y)-\rho_{\lambda,n}^N(x,y)\right| \\
&\qquad \le \lim_{n\to\infty}\frac{\left(e^{2\lambda}+2\lambda e^{3\lambda}\right)\|F-F_n\|_\infty}{\lambda}+4e^{2\lambda}\|F-F_n\|_\infty \\
&\qquad = \lim_{n\to\infty}\left(\frac{1}{\lambda}+2e^\lambda+4\right)e^{2\lambda}\|F-F_n\|_\infty \\
&\qquad = 0.
\end{aligned}
$$

Further, by (37) and (42),

$$
\begin{aligned}
&\left|\int_0^1\rho_\lambda^N(x,y)f(y)d\mu(y)-\int_0^1\rho_\lambda^N(x,y)f(y)d\mu_n(y)\right| \\
&\le \left|\left(\cosh'_{\sqrt{\lambda}}(1)\right)^{-1}\cosh_{\sqrt{\lambda}}(x)\right|\left|\int_0^1\cosh_{\sqrt{\lambda}}(1-y)f(y)d\mu(y)\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left.-\int_0^1\cosh_{\sqrt{\lambda}}(1-y)f(y)d\mu_n(y)\right| \\
&\le \frac{e^\lambda}{\lambda}\left|\int_0^1\cosh_{\sqrt{\lambda}}(1-y)f(y)d\mu(y)-\int_0^1\cosh_{\sqrt{\lambda}}(1-y)f(y)d\mu_n(y)\right|.
\end{aligned}
$$

Due to weak measure convergence,

$$
\lim_{n\to\infty}\int_0^1\cosh_{\sqrt{\lambda}}(1-y)f(y)d\mu_n(y)-\int_0^1\cosh_{\sqrt{\lambda}}(1-y)f(y)d\mu(y)=0
$$

and consequently,

$$
\lim_{n\to\infty}\max_{x\in[0,1]}\left|\int_0^1\rho_\lambda^N(x,y)f(y)d\mu(y)-\int_0^1\rho_\lambda^N(x,y)f(y)d\mu_n(y)\right|=0.
$$

We get the same result for $x\ge y$ and obtain

$$
\begin{aligned}
&\lim_{n\to\infty}\max_{x\in[0,1]}\left|R_{\lambda,n}^N f(x)-R_\lambda^N f(x)\right| \\
&\le \lim_{n\to\infty}\max_{x\in[0,1]}\left|\int_0^1\rho_\lambda^N(x,y)f(y)d\mu(y)-\int_0^1\rho_\lambda^N(x,y)f(y)d\mu_n(y)\right| \\
&\quad+\lim_{n\to\infty}\max_{x\in[0,1]}\left|\int_0^1\left(\rho_\lambda^N(x,y)-\rho_{\lambda,n}^N(x,y)\right)f(y)d\mu_n\right| \\
&= 0.
\end{aligned}
$$

Now, let $b = D$. Again by the triangle inequality, for $n \in \mathbb{N}$ and $x, y \in [0,1]$, $x \leq y$,

$$
\begin{aligned}
&\left|\rho_\lambda^D(x,y) - \rho_{\lambda,n}^D(x,y)\right| \\
&\leq \frac{1}{\sqrt{\lambda}}\Bigg(\left|\left(\sinh_{\sqrt{\lambda}}(1)\right)^{-1} - \left(\sinh_{\sqrt{\lambda},n}(1)\right)^{-1}\right| \left|\sinh_{\sqrt{\lambda}}(x)\sinh_{\sqrt{\lambda}}(1-y)\right| \\
&\quad + \left|\sinh_{\sqrt{\lambda}}(x) - \sinh_{\sqrt{\lambda},n}(x)\right| \left|\left(\sinh_{\sqrt{\lambda},n}(1)\right)^{-1}\sinh_{\sqrt{\lambda}}(1-y)\right| \\
&\quad + \left|\sinh_{\sqrt{\lambda}}(1-y) - \sinh_{\sqrt{\lambda},n}(1-y)\right| \left|\left(\sinh'_{\sqrt{\lambda},n}(1)\right)^{-1}\sinh_{\sqrt{\lambda},n}(x)\right|\Bigg).
\end{aligned}
\tag{43}
$$

We have

$$
\sinh_{\sqrt{\lambda}}(1) = \sum_{k=0}^{\infty} \lambda^{k+\frac{1}{2}} q_{2k+1}(1) \geq \sqrt{\lambda} q_1(1) = \sqrt{\lambda}
$$

and thus

$$
\left|\left(\sinh_{\sqrt{\lambda}}(1)\right)^{-1} - \left(\sinh_{\sqrt{\lambda},n}(1)\right)^{-1}\right| \leq 2\sqrt{\lambda}e^{\lambda}\left\|F - F_n\right\|_\infty .
$$

Arguing in the same way as before, we get

$$
\begin{aligned}
\lim_{n\to\infty}\max_{x\in[0,1]}\left|\rho_\lambda^D(x,y) - \rho_{\lambda,n}^D(x,y)\right| &\leq \lim_{n\to\infty}\frac{2}{\sqrt{\lambda}}\sqrt{\lambda}e^{\lambda}\left\|F - F_n\right\|_\infty \lambda e^{2\lambda} \\
&\quad + \lim_{n\to\infty}\frac{4}{\sqrt{\lambda}}\lambda^{\frac{3}{2}}e^{\lambda}\left\|F - F_n\right\|_\infty e^{\lambda} \\
&= \lim_{n\to\infty}\left(2e^{\lambda} + 4\right)\lambda e^{2\lambda}\left\|F - F_n\right\|_\infty \\
&= 0.
\end{aligned}
$$

Further,

$$
\begin{aligned}
&\max_{x\in[0,1]}\left|\int_0^1 \rho_\lambda^D(x,y)f(y)d\mu(y) - \int_0^1 \rho_\lambda^D(x,y)f(y)d\mu_n(y)\right| \\
&\leq \max_{x\in[0,1]}\Bigg| \left(\sqrt{\lambda}\sinh_{\sqrt{\lambda}}(1)\right)^{-1}\sinh_{\sqrt{\lambda}}(x)\int_0^1 \sinh_{\sqrt{\lambda}}(1-y)f(y)d\mu(y) \\
&\qquad\qquad - \int_0^1 \sinh_{\sqrt{\lambda}}(1-y)f(y)d\mu_n(y)\Bigg| \\
&\leq \left|\left(\sqrt{\lambda}\sinh_{\sqrt{\lambda}}(1)\right)^{-1}\right| \left\|\sinh_{\sqrt{\lambda}}\right\|_\infty \Bigg|\int_0^1 \sinh_{\sqrt{\lambda}}(1-y)f(y)d\mu(y) \\
&\qquad\qquad - \int_0^1 \sinh_{\sqrt{\lambda}}(1-y)f(y)d\mu_n(y)\Bigg|.
\end{aligned}
$$

Due to the weak measure convergence, this goes to zero as n tends to ∞. Deducing the same result for $x \geq y$ and combining the above inequalities,

$$\begin{aligned}
&\lim_{n\to\infty} \max_{x\in[0,1]} \left|R_{\lambda,n}^D f(x) - R_\lambda^D f(x)\right| \\
&\quad \leq \lim_{n\to\infty} \max_{x\in[0,1]} \left| \int_0^1 \rho_\lambda^D(x,y) f(y) d\mu(y) - \int_0^1 \rho_\lambda^D(x,y) f(y) d\mu_n(y) \right| \\
&\quad\quad + \lim_{n\to\infty} \max_{x\in[0,1]} \left| \int_0^1 \left(\rho_\lambda^D(x,y) - \rho_{\lambda,n}^D(x,y)\right) f(y) d\mu_n \right| \\
&\quad = 0.
\end{aligned}$$

□

3.3.2 Graph norm convergence

Let μ be defined as before and let $\lambda > 0$. Analogously to the restricted semigroup, we define the restricted resolvent operator by

$$\begin{aligned}
\bar{R}_\lambda^N &: (C[0,1])_\mu^N \to (C[0,1])_\mu^N,\ \bar{R}_\lambda^N f := R_\lambda^N f, \\
\bar{R}_\lambda^D &: (C[0,1])_\mu^D \to (C[0,1])_\mu^D,\ \bar{R}_\lambda^D f := R_\lambda^D f.
\end{aligned}$$

Further, we define the operators $\bar{\Delta}_\mu^N$ and $\bar{\Delta}_\mu^D$ by

$$\begin{aligned}
\bar{\Delta}_\mu^N f := \Delta_\mu^N f,\ \ \mathcal{D}\left(\bar{\Delta}_\mu^N\right) := \left\{ f \in \mathcal{D}\left(\Delta_\mu^N\right) : \Delta_\mu^N f \in (C[0,1])_\mu^N \right\}, \\
\bar{\Delta}_\mu^D f := \Delta_\mu^D f,\ \ \mathcal{D}\left(\bar{\Delta}_\mu^D\right) := \left\{ f \in \mathcal{D}\left(\Delta_\mu^D\right) : \Delta_\mu^D f \in (C[0,1])_\mu^D \right\},
\end{aligned}$$

which are called the part of the operator Δ_μ^N in $C[0,1])_\mu^N$ and the part of the operator Δ_μ^D in $C[0,1])_\mu^D$, respectively. The following Lemma shows how the restricted semigroup, the restricted resolvent and the part of the operator are connected. For that, let $b \in \{N, D\}$.

Lemma 3.16: *(i) The infinitesimal generator of the strongly continuous contraction semigroup $\left(\bar{T}_t^b\right)_{t\geq 0}$ is $\bar{\Delta}_\mu^b$.*

(ii) $\bar{R}_\lambda^b$ is the resolvent of $\bar{\Delta}_\mu^b$.

Proof. For all $f \in L^2([0,1],\mu)$, we have that $\|f\|_\infty \geq \|f\|_\mu$, therefore the inclusion map $i : (C[0,1])_\mu^b \to L^2([0,1],\mu)$, $f \mapsto f$ is continuous. Moreover, $\left(\bar{T}_t^b\right)_{t\geq 0}$ defines a strongly continuous contraction semigroup on $(C[0,1])_\mu^b$ and $(C[0,1])_\mu^b$ is $\left(\bar{T}_t^b\right)_{t\geq 0}$-invariant (see Corollary 3.11). We thus can apply [21, 2.3 Proposition] to verify (i).

We turn to part (ii). Let $\lambda > 0$ and let $\widetilde{R}^b_\lambda$ be the resolvent of $\bar{\Delta}^b_\mu$. By part (i) and [21, 1.10 Theorem], this operator is well-defined and given by

$$\widetilde{R}^b_\lambda f = \int_0^\infty e^{-\lambda s} \bar{T}^b_s f ds, \;\; f \in (C[0,1])^b_\mu.$$

Further, by definition of $\left(\bar{T}^b_t\right)_{t \geq 0}$ and $\bar{R}^b_\lambda$,

$$\bar{R}^b_\lambda f = R^b_\lambda f = \int_0^\infty e^{-\lambda s} T^b_s f ds = \int_0^\infty e^{-\lambda s} \bar{T}^b_s f ds, \;\; f \in (C[0,1])^b_\mu.$$

It follows $\widetilde{R}^b_\lambda = \bar{R}^b_\lambda$ on $(C[0,1])^b_\mu$. □

We are now able to establish graph norm convergence. To this end, let $(\mu_n)_{n \in \mathbb{N}}$ satisfy Assumption 3.12 and we assume $\text{supp}(\mu) \subseteq \text{supp}(\mu_n)$ for all $n \in \mathbb{N}$.

Theorem 3.17: *Let $b \in \{N, D\}$. For $f \in \mathcal{D}\left(\bar{\Delta}^b_\mu\right)$ there exists $(f_n)_{n \in \mathbb{N}}$ with $f_n \in \mathcal{D}\left(\bar{\Delta}^b_{\mu_n}\right)$ such that for $n \in \mathbb{N}$*

$$\lim_{n \to \infty} \|\pi_n f - f_n\|_\infty + \left\|\pi_n \bar{\Delta}^b_\mu f - \bar{\Delta}^b_{\mu_n} f_n\right\|_\infty = 0.$$

Proof. Let $\lambda > 0$, $f \in \mathcal{D}\left(\bar{\Delta}^b_\mu\right)$ and $g := \left(\lambda - \bar{\Delta}^b_\mu\right) f$. Then, $f = \bar{R}^b_\lambda g$ and we define $f_n := \bar{R}^b_{\lambda,n} \pi_n g$. Applying Theorem 3.15,

$$\lim_{n \to \infty} \|\pi_n f - f_n\|_\infty = 0. \tag{44}$$

Further,

$$\bar{\Delta}^b_\mu f = \lambda f - \left(\lambda - \bar{\Delta}^b_\mu f\right) f = \lambda f - g$$

and

$$\bar{\Delta}^b_{\mu_n} f_n = \lambda f_n - \left(\lambda - \bar{\Delta}^b_{\mu_n}\right) f_n = \lambda f_n - \pi_n g.$$

It follows

$$\left\|\pi_n \bar{\Delta}^b_\mu f - \bar{\Delta}^b_{\mu_n} f_n\right\|_\infty = \lambda \|\pi_n f - f_n\|_\infty$$

and thus, by (44),

$$\lim_{n \to \infty} \left\|\pi_n \bar{\Delta}^b_\mu f - \bar{\Delta}^b_{\mu_n} f_n\right\|_\infty = 0.$$ □

3.3.3 Strong semigroup convergence

For $b \in \{N, D\}$, let $\left(T_t^b\right)_{t\geq 0}$ be defined by μ, $\left(T_{t,n}^b\right)_{t\geq 0}$ be defined by μ_n and analogously the restricted semigroups $\left(\bar{T}_t^b\right)_{t\geq 0}$ and $\left(\bar{T}_{t,n}^b\right)_{t\geq 0}$ be defined by μ and μ_n, respectively. The main result of this chapter is a direct consequence of the previous results.

Theorem 3.18: *Let $f \in (C[0,1])_\mu^b$. Then, for all $t \geq 0$*

$$\lim_{n\to\infty} \left\|\pi_n \bar{T}_t^b f - \bar{T}_{t,n}^b \pi_n f\right\|_\infty = 0,$$

uniformly on bounded time intervals.

Proof. For $n \in \mathbb{N}$, π_n is a bounded linear transformation between Banach spaces. Further, $\left(\bar{T}_t^b\right)_{t\geq 0}$ and $\left(\bar{T}_{t,n}^b\right)_{t\geq 0}$, $n \in \mathbb{N}$ are strongly continuous contraction semigroups on their respective spaces (see Corollary 3.11). Hence, due to [22, Theorem 6.1], the assertion is a direct consequence of Theorem 3.17. □

Strong semigroup convergence can be interpreted as convergence of solutions to heat equations. The connection is given as follows (see [21, Proposition 6.2]).

Lemma 3.19: *Let A be the generator of a strongly continuous semigroup $(S_t)_{t\geq 0}$ on a Banach space X. Then, for each $f \in \mathcal{D}(A)$ the abstract heat equation*

$$\begin{aligned} \frac{\partial u}{\partial t}(t) &= Au(t), \\ u(0) &= f \end{aligned} \tag{45}$$

for $t \geq 0$ has a unique classical solution in X given by

$$u : [0,\infty) \to X, \ t \mapsto S_t f,$$

meaning that u is continuously differentiable with respect to X, $u(t) \in \mathcal{D}(A)$ and (45) holds for all $t \geq 0$.

Let $T > 0$ and $f \in \mathcal{D}\left(\bar{\Delta}_\mu^b\right)$. Theorem 3.18 implies that the classical solution to

$$\begin{aligned} \frac{\partial u_n}{\partial t}(t) &= \bar{\Delta}_{\mu_n}^b u_n(t), \\ u_n(0) &= \pi_n f \end{aligned}$$

converges uniformly for $(t,x) \in [0,T] \times [0,1]$ to the classical solution to

$$\begin{aligned} \frac{\partial u}{\partial t}(t) &= \bar{\Delta}_\mu^b u(t), \\ u(0) &= f \end{aligned}$$

as $n \to \infty$, assuming that $\pi_n f \in \mathcal{D}\left(\bar{\Delta}_{\mu_n}^b\right)$. However, the assumption $f \in \mathcal{D}\left(\bar{\Delta}_\mu^b\right)$ and $\pi_n f \in \mathcal{D}\left(\bar{\Delta}_{\mu_n}^b\right)$ for all $n \in \mathbb{N}$ is very restrictive, as the following example illustrates.

Example 3.20: Let μ be a measure according to the given conditions such that $\operatorname{supp}(\mu)$ is a λ^1-zero set and assume that $\operatorname{supp}(\mu_n) = [0,1]$ for all $n \in \mathbb{N}$. Further, let $f \in \mathcal{D}\left(\bar{\Delta}_\mu^b\right)$. Then, for $i \geq 1$, $\pi_n f$ is linear on $[a_i, b_i]$. Now, if we assume that $\pi_n f \in \mathcal{D}\left(\bar{\Delta}_{\mu_n}^b\right)$, then $\bar{\Delta}_{\mu_n}^b f(x) = 0$, $x \in [a_i, b_i]$ and thus $\bar{\Delta}_{\mu_n}^b f = 0 \in (C[0,1])_{\mu_n}^b$. If $b = D$, we obtain $\pi_n f = 0 \in (C[0,1])_{\mu_n}^D$ and thus $f = 0 \in (C[0,1])_\mu^D$ and if $b = N$, $(\pi_n f)' = 0 \in C[0,1]$ and thus $f' = 0 \in (C[0,1])_\mu^N$.

This motivates the following solution concept (compare [21, Definition 6.3]).

Definition 3.21: *Let X be a Banach space, $A : X \to X$ and $f \in X$. We call a map $u : [0,\infty) \to X,\ t \mapsto u(t)$ solution to the abstract heat equation*

$$\begin{aligned} \frac{du}{dt}(t) &= Au(t), \\ u(0) &= f \end{aligned} \tag{46}$$

for $t \geq 0$ if u is continuous with respect to X for $t \geq 0$, $u(t) \in \mathcal{D}(A)$ for all $t > 0$ and $\lim_{h\to 0} \frac{u(t+h)-u(t)}{h} = Au(t)$ with respect to X for $t > 0$.

Using this solution concept, we can establish the desired convergence for any initial condition with respect to the uniform norm.

Theorem 3.22: *Let $f \in (C[0,1])_\mu^b$ and let $(\mu_n)_{n\in\mathbb{N}}$ satisfy Assumption 3.12. Further, let $\{u(t) : t \geq 0\}$ be the unique solution to*

$$\begin{aligned} \frac{du}{dt}(t) &= \bar{\Delta}_\mu^b u(t),\ t \geq 0, \\ u(0) &= f \end{aligned} \tag{47}$$

and let for $n \geq 1$ $\{u_n(t) : t \geq 0\}$ be the unique solution to

$$\begin{aligned} \frac{du_n}{dt}(t) &= \bar{\Delta}^b_{\mu_n} u_n(t), \ t \geq 0, \\ u_n(0) &= \pi_n f. \end{aligned} \tag{48}$$

Then,

$$\lim_{n\to\infty} \|\pi_n u(t) - u_n(t)\|_\infty = 0, \tag{49}$$

uniformly on bounded time intervals.

Proof. First, we show that $t \mapsto \bar{T}^b_t f$ is a solution to (47). Let $t > 0$. By (39) we have for any $k \in \mathbb{N}$

$$u(t) = \bar{T}^b_t f = T^b_t f \in \mathcal{D}\left(\left(\Delta^b_\mu\right)^k\right).$$

It follows that $\Delta^b_\mu u(t) \in \mathcal{D}\left(\Delta^b_\mu\right)$ and especially $\Delta^b_\mu u(t) \in (C[0,1])^b_\mu$, which implies $u(t) \in \mathcal{D}\left(\bar{\Delta}^b_\mu\right)$. From the strong continuity of $\left(\bar{T}^b_t\right)_{t\geq 0}$ along with the semigroup property we get the continuity of u with respect to $(C[0,1])^b_\mu$. Further, since $\bar{\Delta}^b_\mu$ is the infinitesimal generator of $\left(\bar{T}^b_t\right)_{t\geq 0}$,

$$\lim_{h\to 0} \frac{u(t+h) - u(t)}{h} = \lim_{h\to 0} \frac{\bar{T}^b_h \bar{T}^b_t f - \bar{T}^b_t f}{h} = \bar{\Delta}^b_\mu \bar{T}^b_t f = \bar{\Delta}^b_\mu u(t).$$

For the proof of uniqueness, first note that the unique solution to

$$\begin{aligned} \frac{dv}{dt}(t) &= \Delta^b_\mu v(t), \ t \geq 0, \\ v(0) &= f \end{aligned} \tag{50}$$

on the Hilbert space $L^2([0,1],\mu)$ is given by $v(t) = T^b_t f$ (see [48, Theorem B.2.6]). We now show that a solution to (47), which we denote by u, is also a solution to (50). The continuity with respect to $L^2([0,1],\mu)$ follows from

$$\|u(t) - u(s)\|_\mu \leq \|u(t) - u(s)\|_\infty, \quad s,t \geq 0.$$

Let $t > 0$. We have $u(t) \in \mathcal{D}\left(\bar{\Delta}^b_\mu\right)$, which by definition implies that $u(t) \in \mathcal{D}\left(\Delta^b_\mu\right)$.

Further,

$$\begin{aligned}
\lim_{h\to 0}\left\|\frac{u(t+h)-u(t)}{h}-\Delta_\mu^b u(t)\right\|_\mu &= \lim_{h\to 0}\left\|\frac{u(t+h)-u(t)}{h}-\bar{\Delta}_\mu^b u(t)\right\|_\mu \\
&\leq \lim_{h\to 0}\left\|\frac{u(t+h)-u(t)}{h}-\bar{\Delta}_\mu^b u(t)\right\|_\infty \\
&= 0.
\end{aligned}$$

Therefore, u is a solution to (50). This proves the uniqueness. We can follow the same arguments to verify that $\bar{T}_{t,n}^b\pi_n f$ is the unique solution to (48) for $n\in\mathbb{N}$. Then, (49) is a direct consequence of Theorem 3.18. $\square$

3.4 Applications

Example 3.23: As a first application, we consider a non-atomic Borel probability measure μ on $[0,1]$ such that $0,1\in\operatorname{supp}(\mu)$ and $\operatorname{supp}(\mu)\neq[0,1]$. We define for $\varepsilon\in(0,1)$ the approximating probability measure μ_ε by

$$\mu_\varepsilon := \frac{\mu+\varepsilon\lambda^1}{1+\varepsilon}.$$

It is elementary that μ_ε converges weakly to μ as $\varepsilon\to 0$ and Theorem 3.22 is applicable. Let $b\in\{N,D\}$ and $f\in(C[0,1])_\mu^b$. Then, the unique solution $\{u_\varepsilon(t): t\geq 0\}$ to

$$\begin{aligned}
\frac{du_\varepsilon}{dt}(t) &= \bar{\Delta}_{\mu_\varepsilon}^b u_\varepsilon(t),\ t\geq 0,\\
u_\varepsilon(0) &= \pi_\varepsilon f,
\end{aligned}$$

where $\pi_\varepsilon:(C[0,1])_\mu^b\to(C[0,1])_{\mu_\varepsilon}^b$ is an embedding as previously defined (see (40)), converges to the unique solution $\{u(t): t\geq 0\}$ to

$$\begin{aligned}
\frac{du}{dt}(t) &= \bar{\Delta}_\mu^b u(t),\\
u(0) &= f
\end{aligned}$$

for each $t\geq 0$ with respect to the uniform norm as ε tends to zero.

In the previous example, μ could be chosen to be an absolutely continuous measure,

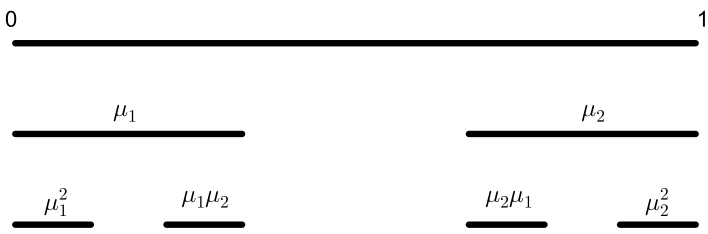

Figure 5: Approximating Cantor measures of levels $n = 1, 2$.

for example $\lambda^1_{|[0,\frac{1}{3}]\cup[\frac{2}{3},1]}$, or to be a singular measure, as a self-similar measure on the Cantor set. Furthermore, it is not required that the approximating measures have full support.

Example 3.24: Let μ be the unique invariant Borel probability measure on $[0,1]$ given by the IFS consisting of $S_1(x) = \frac{x}{3}$ and $S_2(x) = \frac{2}{3} + \frac{x}{3}$, $x \in [0,1]$ and weights $\mu_1, \mu_2 \in (0,1)$, i.e. μ is a Cantor measure. Following [30], for $n \in \mathbb{N}$ we define the approximating Cantor measures of level n by

$$\mu_n(B) \coloneqq 3^n \sum_{x\in\{1,2\}^n} \lambda^1_{|I_x} \prod_{i=1}^{n} \mu_{x_i}, \;\; B \in B([0,1]),$$

where $I_x \coloneqq (S_{x_1} \circ \ldots \circ S_{x_n})([0,1])$, $x \in \{1,2\}^n$. Figure 5 illustrates the approximating Cantor measures of levels $n = 1, 2$. We denote the distribution function of μ by F and the distribution function of μ_n by F_n for $n \in \mathbb{N}$. Then, $\|F - F_n\|_\infty \to 0$ (see [30, Proposition 4.2]) as well as $\operatorname{supp}(\mu) \subset \operatorname{supp}(\mu_n)$ for $n \in \mathbb{N}$ and Theorem 3.22 can be applied. Hence, for $f \in (C[0,1])^b_\mu$, the unique solution $\{u_n(t) : t \geq 0\}$ to

$$\begin{aligned} \frac{du_n}{dt}(t) &= \bar{\Delta}^b_{\mu_n} u_n(t), \\ u_n(0) &= \pi_n f \end{aligned}$$

converges to the unique solution $\{u(t) : t \geq 0\}$ to

$$\begin{aligned} \frac{du}{dt}(t) &= \bar{\Delta}^b_{\mu} u(t), \\ u(0) &= f \end{aligned}$$

for each $t \geq 0$ with respect to the uniform norm as n tends to infinity.

Finally, we connect both applications.

Example 3.25: Let $\varepsilon > 0$, $n \in \mathbb{N}$ and let μ, μ_n, $\{u(t) : t \geq 0\}$ and $\{u_n(t) : t \geq 0\}$ be defined as in Example 3.24. We define $\mu_{n,\varepsilon}$ by

$$\mu_{n,\varepsilon} := \frac{\mu_n + \varepsilon\lambda^1}{1+\varepsilon},$$

i.e. analogously to Example 3.23, and $\{u_{n,\varepsilon}(t) : t \geq 0\}$ to be the solution to

$$\begin{aligned} \frac{du_{n,\varepsilon}}{dt}(t) &= \bar{\Delta}^b_{\mu_{n,\varepsilon}} u_{n,\varepsilon}(t), \\ u_{n,\varepsilon}(0) &= \pi_{n,\varepsilon} f, \end{aligned}$$

where $\pi_{n,\varepsilon}$ is an embedding as previously defined. Further, let $t \in [0, \infty)$ and $\delta > 0$. By Example 3.24, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have that

$$\|u(t) - u_n(t)\|_\infty < \frac{\delta}{2}.$$

By Example 3.23, for each $n \geq n_0$ there exists $\varepsilon_n > 0$ such that for all $\varepsilon < \varepsilon_n$ we have that

$$\|u_n(t) - u_{n,\varepsilon}(t)\|_\infty < \frac{\delta}{2}.$$

Hence, for all $n \geq n_0$, $\varepsilon < \varepsilon_n$ it holds

$$\|u(t) - u_{n,\varepsilon}(t)\|_\infty < \delta.$$

Hence, the heat on a rod with mass distribution given by a Cantor measure diffuses approximately like the heat on a rod possessing a strictly positive mass density which is small off the Cantor set.

4 Analysis of Measure Theoretic Stochastic Heat Equations

Let μ be a self-similar measure on a Cantor-like set K according to Section 2.2. Further, let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space and let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfy the usual conditions. The object of study in this chapter is the stochastic PDE

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= \Delta_\mu^b u_t(x) + f(t, u(t,x)) + g(t, u(t,x))\xi(t,x), \\ u(0,x) &= u_0(x) \end{aligned} \tag{51}$$

for $(t,x) \in [0,T] \times [0,1]$, where $T > 0$, $b \in \{N, D\}$, $u_0 : \Omega \times [0,1] \to \mathbb{R}$, $f, g : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$. Further, ξ denotes a $\mathbb{F}$-space-time white noise based on μ according to Definition 2.10.

Before we elaborate on the mild solution to (51), we need to have a closer look on the resolvent density and the heat kernel of Δ_μ^b.

4.1 Approximation of the resolvent density

We develop a method to approximate the Delta functional on a Cantor-like set K, in particular to approximate the resolvent density, which will then again be used to approximate point evaluations of heat kernels.

For $n \geq 1$, let Λ_n be the partition of the word space $\mathbb{W}$ defined by

$$\Lambda_n = \{\omega = \omega_1 ... \omega_m \in \mathbb{W}^* : r_{\omega_1} \cdots r_{\omega_{m-1}} > r_{\max}^n \geq r_\omega\},$$

where $r_{\max} := \max_{i=1,...,N} r_i$. Further, let d_H be the Hausdorff dimension of K, $\nu_i := \frac{\mu_i}{r_i^{d_H}}$ for $i \in \{1, ..., N\}$ and $K_\omega := S_\omega(K)$ for $\omega \in \mathbb{W}$. $|A|$ denotes the cardinality of a set A if A is countable and the diameter if A is uncountable.

Lemma 4.1: *Let $n \in \mathbb{N}$. Then,*

(i) $|\Lambda_n| < \infty$ *and* $\bigcup_{\omega \in \Lambda_n} K_\omega = K$.

(ii) *For* $\omega \in \Lambda_n$ *there exists a subset* $\Lambda' \subseteq \Lambda_{n+1}$ *such that* $K_\omega = \bigcup_{\nu \in \Lambda'} K_\nu$.

(iii) *For* $\omega, \nu \in \Lambda_n$, $\omega \neq \nu$, *it holds* $|K_\omega \cap K_\nu| \in \{0, 1\}$.

(iv) For $\omega \in \Lambda_n$*, it holds* $\mu(K_\omega) > r_{\max}^{nd_H} r_{\min}^{d_H} \nu_{\min}^n$*, where* $r_{\min} := \min_{i=1,...,N} r_i$*,* $\nu_{\min} := \min_{i=1,...,N} \nu_i$*.*

(v) For $w \in \mathbb{W}^*$ *there exists* $n \in \mathbb{N}$ *such that* $w \in \Lambda_n$*. Consequently, for all* $m \geq n$ *there exists* $\Lambda'_m \subseteq \Lambda_m$ *such that* $K_w = \cup_{\nu \in \Lambda'_m} K_\nu$*.*

If the measure μ is given by $\mu_i = r_i^{d_H}$ and thus $\nu_i = 1,\ i = 1, ..., N$, the estimate in (iv) coincides with that in [41, Lemma 3.5(iv)].

Proof. (i) The first claim is obvious. For the second, note that $\cup_{w \in \mathbb{W}} K_w = K$ and that $\cup_{w \in \Lambda_n} \Sigma_w = \mathbb{W}$ and thus $\cup_{v \in \Sigma_w, w \in \Lambda_n} K_v = K$. It remains to show that $K_w = \cup_{v \in \Sigma_w} K_v$ for $w \in \Lambda_n$. This follows by applying S_w to both sides of equation $\cup_{\nu \in \mathbb{W}} K_\nu = K$.

(ii) Let $\omega \in \Lambda_n$. We know by part (i) that $K_w = \cup_{\nu \in \Sigma_w} K_\nu$. If $r_w \leq r_{\max}^{n+1}$, choose $\Lambda' = \{w\}$. Now, let $r_w > r_{\max}^{n+1}$ and $i \in \{1, ..., N\}$. We have $r_\omega \leq r_{\max}^n$, which implies $r_\omega r_i \leq r_{\max}^{n+1}$ and thus $wi \in \Lambda_{n+1}$. Hence, choosing $\Lambda' = \{w1, ...wN\}$ verifies the statement.

(iii) The assertion follows directly from the fact that $|S_i([0,1]) \cap S_j([0,1])| \in \{0, 1\}$ for $i \neq j$ along with the injectivity of S_i for $i \in \{1, ..., N\}$.

(iv) Let $\omega \in \Lambda_n$ and $m := |\omega|$. By definition of Λ_n we have $r_{\omega_1} \cdots r_{\omega_{m-1}} > r_{\max}^n$ and therefore $r_\omega > r_{\max}^n r_{\min}$. Then,

$$\begin{aligned}
\mu_\omega &= r_{\omega_1}^{d_H} \frac{\mu_{\omega_1}}{r_{\omega_1}^{d_H}} \cdots r_{\omega_m}^{d_H} \frac{\mu_{\omega_m}}{r_{\omega_m}^{d_H}} \\
&\geq r_\omega^{d_H} \nu_{\min}^m \\
&> r_{\max}^{nd_H} r_{\min}^{d_H} \nu_{\min}^m \\
&\geq r_{\max}^{nd_H} r_{\min}^{d_H} \nu_{\min}^n.
\end{aligned}$$

In the last inequality, we have used that $m \leq n$ and $\nu_{\min} \leq 1$.

(v) Let $w = w_1 .... w_m \in \mathbb{W}^*$. Choose $n \in \mathbb{N}$ such that $r_w \leq r_{\max}^n$ and $r_w > r_{\max}^{n+1}$. We have $r_{w_1} ... r_{w_{m-1}} r_{\max} > r_{w_1} ... r_{w_m}$ and thus

$$r_{w_1} ... r_{w_{m-1}} > r_{w_1} ... r_{w_m} r_{\max}^{-1} > r_{\max}^{n+1} r_{\max}^{-1} = r_{\max}^n.$$

We have found an $n \in \mathbb{N}$ such that $w \in \Lambda_n$. For the second part, we can argue as in (ii) by induction.

□

We introduce a sequence of functions approximating the Delta functional. Hereby, we use the notation from [41]. We prepare this by defining the n-neighbourhood of $x \in K$ for $n \in \mathbb{N}$ by

$$D_n^0(x) \coloneqq \bigcup_{w \in \Lambda_n, x \in K_w} K_w.$$

Note that $D_n^0(x)$ consists of at least one element of $\{K_w : w \in \Lambda_n\}$, which follows from Lemma 4.1(i), and of at most two elements since pairs of these elements intersect in at most one point. From the latter and the definition of Λ_n it follows that

$$|D_n^0(x)| \leq 2r_{\max}^n \tag{52}$$

and we define the approximating functions for $x \in K$ and $n \geq 1$ by

$$f_n^x(y) \coloneqq \mu(D_n^0(x))^{-1} \mathbb{1}_{D_n^0(x)}(y), \; y \in [0,1]. \tag{53}$$

By Lemma 4.1(iv),

$$||f_n^x||_\mu^2 = \mu(D_n^0(x))^{-1} < r_{\max}^{-nd_H} r_{\min}^{-d_H} \nu_{\min}^{-n}. \tag{54}$$

Lemma 4.2: *If $x \in K$ and $g \in L^2([0,1],\mu)$ is continuous, then $\lim_{n\to\infty} \langle f_n^x, g\rangle_\mu = g(x)$.*

Proof. Let $x \in K$ be fixed. For $n \in \mathbb{N}$ and $\omega \in \Lambda_n$, we have $|K_\omega| \leq r_{\max}^n$ and thus, for $y \in D_n^0(x)$, $|x-y| \leq r_{\max}^n$. Now, let $\varepsilon > 0$. Since g is continuous, there exists $\delta > 0$ such that $|g(x) - g(y)| < \varepsilon$ for $y \in [0,1]$ with $|y-x| < \delta$. Choose $n \in \mathbb{N}$ such that $r_{\max}^n < \delta$. Then,

$$\begin{aligned} |\langle f_n^x, g\rangle_\mu - g(x)| &= \frac{1}{\mu(D_n^0(x))} \left| \int_{D_n^0(x)} g(y) d\mu(y) - g(x) \right| \\ &\leq \frac{1}{\mu(D_n^0(x))} \int_{D_n^0(x)} |g(y) - g(x)| d\mu(y) \\ &\leq \frac{1}{\mu(D_n^0(x))} \mu(D_n^0(x)) \cdot \varepsilon = \varepsilon. \end{aligned}$$

□

Let $b \in \{N, D\}$. The resolvent density of Δ_μ^b is a product of eigenfunctions on Δ_μ^b (see Corollary 3.6). Therefore, it is elementary to check that there exists a constant

L_1 such that for $(x, y, z) \in [0,1]^3$

$$\left|\rho_1^b(x,z) - \rho_1^b(y,z)\right| \leq L_1|x-y|. \tag{55}$$

Lemma 4.3: *Let $x_1, x_2 \in K$ and $m, n \geq 1$. Then,*

$$\left|\int_0^1 \int_0^1 \rho_1^b(y,z) f_m^{x_1}(y) f_n^{x_2}(z) d\mu(y) d\mu(z) - \rho_1^b(x_1,x_2)\right| \leq L_1(r_{\max}^n + r_{\max}^m).$$

Proof. Using the Lipschitz continuity of ρ_1^b and (52),

$$\begin{aligned}
&\left|\int_0^1 \int_0^1 \left(\rho_1^b(y,z) - \rho_1^b(x_1,x_2)\right) f_m^{x_1}(y) f_n^{x_2}(z) d\mu(y) d\mu(z)\right| \\
&\leq \int_0^1 \int_0^1 \left(\left|\rho_1^b(y,z) - \rho_1^b(x_1,z)\right| + \left|\rho_1^b(x_1,z) - \rho_1^b(x_1,x_2)\right|\right) f_m^{x_1}(y) f_n^{x_2}(z) d\mu(y) d\mu(z) \\
&= \frac{1}{\mu(D_m^0(x_1))\mu(D_n^0(x_2))} \Bigg(\int_{D_m^0(x_2)} \int_{D_n^0(x_1)} \left|\rho_1^b(y,z) - \rho_1^b(x_1,z)\right| \\
&\qquad\qquad + \left|\rho_1^b(x_1,z) - \rho_1^b(x_1,x_2)\right| d\mu(y) d\mu(z) \Bigg) \\
&\leq \frac{1}{\mu(D_m^0(x_1))\mu(D_n^0(x_2))} \int_{D_m^0(x_2)} \int_{D_n^0(x_1)} L_1 \left(r_{\max}^m + r_{\max}^n\right) d\mu(y) d\mu(z) \\
&= L_1 \left(r_{\max}^m + r_{\max}^n\right).
\end{aligned}$$

□

4.2 Heat kernel properties

In this section, we recap basic properties of heat kernels and prove a couple of continuity properties. Recall Definition 3.7 for the definition of the heat kernel. Moreover, we define for $h \in L^2([0,1],\mu)$ $\int_0^1 p_0^b(x,y)h(y)d\mu(y) := h(x)$. Part (iii) of the subsequent lemma shows that this is a meaningful definition. Further, let $b \in \{N, D\}$.

Lemma 4.4: *Let $T > 0$, $h \in L^2([0,1],\mu)$ and $(T_t^b)_{t\geq 0}$ be the strongly continuous semigroup associated to Δ_μ^b.*

(i) For all $(t,x,y) \in [T,\infty) \times [0,1]^2$, there exists $K_T > 0$ such that $\left|p_t^b(x,y)\right| < K_T$.

(ii) $(t,x,y) \mapsto p_t^b(x,y)$ is continuous on $(0,\infty) \times [0,1]^2$.

(iii) For $t > 0, s \geq 0$, $x, y \in [0,1]$, $\int_0^1 p_s^b(x,z)p_t^b(z,y)d\mu(z) = p_{t+s}^b(x,y)$.

(iv) *For* $(t,x,y) \in (0,\infty) \times [0,1]^2$*, let* $p^b_{t,x}(y) := p^b_t(x,y)$*. Then,* $p^b_{t,x} \in \mathcal{D}\left(\Delta^b_\mu\right)$ *and for all* $t \in (0,\infty)$*,* $x, y \in [0,1]$*,*

$$\frac{\partial}{\partial t} p^b_t(x,y) = \Delta^b_\mu p^b_{t,x}(y).$$

(v) *For* $(t,x,y) \in (0,\infty) \times [0,1]^2$*,* $p^b_t(x,y) \geq 0$*.*

(vi) *For* $(t,x) \in (0,\infty) \times [0,1]$*,* $\int_0^1 p^N_t(x,y) d\mu(y) = 1$ *and* $\int_0^1 p^D_t(x,y) d\mu(y) \leq 1$ *.*

(vii) *For* $t \in (0,\infty)$*,* $T^b_t h(x) = \left\langle p^b_{t,x}, h \right\rangle_\mu$ *in* $L^2([0,1],\mu)$*.*

(viii) *For* $t > 0$*,* $\sup_{x,y\in[0,1]} p^b_t(x,y) = \|T^b_t\|_{1\to\infty}$*, where* $\|A\|_{p\to q}$ *denotes the operator norm of an operator* $A : L^p \to L^q$*.*

Proof. (i)-(vi) are well-known for Neumann boundary conditions (see e.g. [54]). (i)-(vi) for Dirichlet boundary conditions can be checked in the exact same way as for heat kernels on p.c.f. self-similar sets in [48, Chapter 5].

The proof of (viii) is a standard argument. Let $t \in (0,\infty)$ be fixed and $C := \sup_{x,y\in[0,1]} \left|p^b_t(x,y)\right|$. Further, let $f \in L^1([0,1],\mu)$. Then, for $x \in [0,1]$,

$$\left|T^b_t f(x)\right| = \left|\int_0^1 p^b_t(x,y) f(y) d\mu(y)\right| \leq C \int_0^1 |f(y)|\, d\mu(y)$$

and thus $\left\|T^b_t\right\|_{1\to\infty} \leq C$.

Recall that $[0,1]\backslash K = \bigcup_{i=1}^\infty (a_i, b_i)$ (see (18)). Since p^b_t is symmetric and continuous on $[0,1]^2$ and we evaluate the representative that is for $i \geq 1$, $y \in [0,1]$ linear on $[a_i, b_i] \times \{y\}$, there exist $x_0, y_0 \in K$ such that $p^b_t(x_0,y_0) = C$. Let $n \in \mathbb{N}$ and $f^{x_0}_n$ be defined as in (53). We have $\|f^{x_0}_n\|_1 = 1$. By Lemma 4.2,

$$\lim_{n\to\infty} \left\langle f^{x_0}_n(\cdot), p^b_t(\cdot, y_0) \right\rangle_\mu = p^b_t(x_0,y_0) = C.$$

Hence, for all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that

$$\begin{aligned} \left|T^b_t f^{x_0}_n(y_0)\right| &= \left\langle f^{x_0}_n(\cdot), p^b_t(\cdot, y_0) \right\rangle_\mu \\ &\geq C - \varepsilon. \end{aligned}$$

It follows that $\left\|T^b_t f^{x_0}_n\right\|_\infty \geq C - \varepsilon$, which implies $\left\|T^b_t\right\|_{1\to\infty} \geq C$ as ε can be chosen arbitrarily small. $\square$

Hambly and Yang [41, Lemma 6.6] investigated continuity properties of heat kernels associated to Laplacians on connected p.c.f. self-similar sets. We give a comparable result for Cantor-like sets, where we additionally take all self-similar measures according to the given conditions into account. Recall that γ is defined to be the spectral exponent of Δ_μ^b and $\delta = \max_{1\le i\le N} \frac{\log \mu_i}{\log((\mu_i r_i)^\gamma)}$.

Proposition 4.5: *Let $T > 0$.*

(i) There exists $C_0(T) > 0$ such that for all $(t, x, y) \in (0, T] \times [0, 1]^2$

$$p_t^b(x, y) \le C_0(T) t^{-\gamma\delta}.$$

(ii) There exists $C_1(T) > 0$ such that for all $(t, x, x', y) \in (0, T] \times [0, 1]^3$

$$\left|p_t^b(x, y) - p_t^b(x', y)\right| \le C_1(T)|x - x'|^{\frac{1}{2}} t^{-\frac{1}{2}-\frac{\gamma\delta}{2}}.$$

(iii) There exists $C_2(T) > 0$ such that for all $(s, t, x) \in (0, T]^2 \times [0, 1]$ with $s \le t$

$$\left|p_s^b(x, x) - p_t^b(x, x)\right| \le C_2(T) \left(s^{-\gamma\delta} - t^{-\gamma\delta}\right).$$

Proof. We follow the proof of [41, Lemma 6.6].

(i) We can apply [48, Proposition B.3.7] as in (30) to obtain the existence of a constant $C_0(1)$ such that $\left\|T_t^b\right\|_{1\to\infty} \le C_0(1)t^{-\gamma\delta}$ for $t \in (0, 1]$. By Lemma 4.4(viii),

$$\sup_{x,y\in[0,1]} p_t^b(x, y) \le C_0(1)t^{-\gamma\delta}, \quad t \in (0, 1].$$

If $T > 1$, the assertion follows from the previous inequality and the fact that p^b is continuous and thus bounded on $[1, T] \times [0, 1]^2$.

(ii) Let $(t, y) \in (0, T] \times [0, 1]$ be fixed. By part (i),

$$\left\|p_{\frac{t}{2}}^b(\cdot, y)\right\|_\mu^2 = \int_0^1 p_{\frac{t}{2}}^b(x, y)^2 d\mu(x) = p_t^b(y, y) \le C_0(T)t^{-\gamma\delta}.$$

Let $u(x) := p_{\frac{t}{2}}^b(x, y)$, $x \in [0, 1]$. With [38, Lemma 1.3.3(i)] and the contractivity of $T^b_{\frac{t}{2}}$, it follows that

$$\begin{aligned}\mathcal{E}\left(T^b_{\frac{t}{2}}u, T^b_{\frac{t}{2}}u\right) &\le \frac{1}{t}\left(\|u\|_\mu^2 - \left\|T^b_{\frac{t}{2}}u\right\|_\mu^2\right)\\ &\le \frac{1}{t}\|u\|_\mu^2\\ &\le C_0(T)t^{-1-\gamma\delta}.\end{aligned}$$

Since $p^b_t(\cdot, y)$ is continuously differentiable, we can apply the Cauchy-Schwarz inequality and get for $x, x' \in [0,1]$

$$\begin{aligned}\left|p^b_t(x,y) - p^b_t(x',y)\right| &\le \int_x^{x'} \left|\frac{\partial}{\partial z}p^b_t(z,y)\right| dz\\ &\le |x-x'|^{\frac{1}{2}}\left(\int_0^1 \left|\frac{\partial}{\partial z}p^b_t(z,y)\right|^2 dz\right)^{\frac{1}{2}}.\end{aligned}$$

Note that $T^b_{\frac{t}{2}}u = p^b_t(\cdot, y)$. We obtain

$$\begin{aligned}\left|p^b_t(x,y) - p^b_t(x',y)\right|^2 &\le |x-x'|\mathcal{E}\left(p^b_t(\cdot,y), p^b_t(\cdot,y)\right)\\ &= |x-x'|\mathcal{E}\left(T^b_{\frac{t}{2}}u, T^b_{\frac{t}{2}}u\right)\\ &\le |x-x'|\, C_0(T)t^{-1-\gamma\delta}\end{aligned}$$

and therefore
$$\sup_{x,x'\in[0,1]} \frac{\left|p^b_t(x,y) - p^b_t(x',y)\right|}{|x-x'|^{\frac{1}{2}}} \le \sqrt{C_0(T)}t^{-\frac{1}{2}-\frac{\gamma\delta}{2}}.$$

(iii) Let $t \in (0,T]$, $x, y \in [0,1]$ and $p^b_{\frac{t}{2},x}(y) := p^b_{\frac{t}{2}}(x,y)$. We have

$$\begin{aligned}\frac{\partial}{\partial t}p^b_t(x,x) &= \frac{\partial}{\partial t}\int_0^1 p^b_{\frac{t}{2}}(x,y)^2 d\mu(y)\\ &= \int_0^1 \frac{\partial}{\partial t}p^b_{\frac{t}{2}}(x,y)^2 d\mu(y)\\ &= 2\int_0^1 p^b_{\frac{t}{2},x}(y)\frac{\partial}{\partial t}p^b_{\frac{t}{2},x}(y) d\mu(y),\end{aligned}$$

where we can interchange integral and derivative since $\frac{\partial}{\partial t}p^b_{\frac{t}{2}}(x,y)^2$ is bounded on $[t-\varepsilon, t+\varepsilon]\times[0,1]^2$ for $\varepsilon > 0$ sufficiently small. By Lemma 4.4(iv) along with identity (19) and (20), respectively,

$$\int_0^1 p^b_{\frac{t}{2},x}(y)\frac{\partial}{\partial t}p^b_{\frac{t}{2},x}(y)d\mu(y) = \int_0^1 p^b_{\frac{t}{2},x}(y)\Delta^b_\mu p^b_{\frac{t}{2},x}(y)d\mu(y)$$
$$= -\mathcal{E}\left(p^b_{\frac{t}{2},x}, p^b_{\frac{t}{2},x}\right).$$

We obtain the existence of a constant $C_2'(T)$ such that

$$\left|\frac{\partial}{\partial t}p^b_t(x,x)\right| = \left|2\mathcal{E}\left(p^b_{\frac{t}{2},x}\right)\right| \leq C_2'(T)t^{-1-\gamma\delta},$$

where the last step can be checked in the same way as in the proof of (ii). We conclude for all $x \in [0,1]$, $s,t \in (0,T]$

$$\left|p^b_s(x,x) - p^b_t(x,x)\right| \leq C_2'(T)\int_s^t z^{-1-\gamma\delta}dz = \frac{C_2'(T)}{\gamma\delta}\left(s^{-\gamma\delta} - t^{-\gamma\delta}\right). \qquad \square$$

4.3 Heat kernel approximation

We develop a way to approximate point evaluations of heat kernels. First, we provide upper estimates of functionals of the heat kernel using the resolvent density.

Lemma 4.6: *Let $h \in L^2([0,1],\mu)$ and $t \in (0,\infty)$. Then,*

$$\int_0^t \int_0^1 \left(\int_0^1 p^b_s(x,y)h(y)d\mu(y)\right)^2 d\mu(x)ds \leq \frac{e^{2t}}{2}\int_0^1\int_0^1 \rho_1(x,y)h(x)h(y)d\mu(x)d\mu(y).$$

Proof. Let $h \in L^2([0,1],\mu)$ and $t \in (0,\infty)$. We adapt ideas from [42, Lemma 4.6]. We first note that T^b_t is self-adjoint, which follows directly by the spectral representation (21) along with the functional calculus theory (see [64, Theorem VIII.5]). Using this and Lemma 4.4(vii),

$$\int_0^t \int_0^1 \left(\int_0^1 p^b_s(x,y)h(y)d\mu(y)\right)^2 d\mu(x)ds = \int_0^t \left\langle T^b_s h, T^b_s h\right\rangle_\mu ds$$
$$= \int_0^t \left\langle T^b_{2s} h, h\right\rangle_\mu ds$$
$$\leq e^{2t}\int_0^t e^{-2s}\left\langle T^b_{2s} h, h\right\rangle_\mu ds.$$

Further, we have the following connection of semigroup and resolvent: If $\lambda > 0$, then $R_\lambda^b h = \int_0^\infty e^{-\lambda t} T_t^b h dt$ (see e.g. [21, Theorem 1.10]). With that,

$$\begin{aligned} e^{2t} \int_0^t e^{-2s} \left\langle T_{2s}^b h, h \right\rangle_\mu ds &\leq e^{2t} \left\langle \int_0^\infty e^{-2s} T_{2s}^b h ds, h \right\rangle_\mu \\ &= \frac{e^{2t}}{2} \left\langle \int_0^1 \rho_1^b(\cdot, y) h(y) d\mu(y), h \right\rangle_\mu \\ &= \frac{e^{2t}}{2} \int_0^1 \int_0^1 \rho_1^b(x,y) h(x) h(y) d\mu(x) d\mu(y). \qquad \square \end{aligned}$$

We derive a way to approximate $(t, y) \mapsto p_t^b(x, y)$ for fixed $x \in K$.

Lemma 4.7: *Let $t \in (0, \infty)$ and $x \in K$. Then,*

$$\int_0^t \int_0^1 \left(\left\langle p_s^b(\cdot, y), f_n^x \right\rangle_\mu - p_s^b(x,y) \right)^2 d\mu(y) ds \leq 2 L_1 e^{2t} r_{\max}^n.$$

Proof. Let $x \in K$ and $m, n \geq 1$. By Lemma 4.3,

$$\begin{aligned} &\left| \int_0^1 \int_0^1 \rho_1^b(z,y)(f_m^x(z) - f_n^x(z))(f_m^x(y) - f_n^x(y)) d\mu(z) d\mu(y) \right| \\ &= \Big| \int_0^1 \int_0^1 \rho_1^b(z,y) f_m^x(z) f_m^x(y) - \rho_1^b(x,x) - \rho_1^b(z,y) f_m^x(z) f_n^x(y) + \rho_1^b(x,x) \\ &\quad - \rho_1^b(z,y) f_n^x(z) f_m^x(y) + \rho_1^b(x,x) + \rho_1^b(z,y) f_n^x(z) f_n^x(y) - \rho_1^b(x,x) d\mu(z) d\mu(y) \Big| \\ &\leq 4 L_1 (r_{\max}^m + r_{\max}^n). \end{aligned} \tag{56}$$

Now, let $s \in (0, \infty)$ and $x \in K$. Then,

$$\begin{aligned} &\int_0^1 \left(\left\langle p_s^b(\cdot, y), f_n^x \right\rangle_\mu - p_s^b(x,y) \right)^2 d\mu(y) \\ &= \int_0^1 \left(\sum_{k=1}^\infty e^{-\lambda_k^b s} \varphi_k^b(y) \left\langle \varphi_k^b, f_n^x \right\rangle_\mu - p_s^b(x,y) \right)^2 d\mu(y) \\ &= \int_0^1 \left(\sum_{k=1}^\infty e^{-\lambda_k^b s} \left[\left\langle \varphi_k^b, f_n^x \right\rangle_\mu - \varphi_k^b(x) \right] \varphi_k^b(y) \right)^2 d\mu(y) \\ &= \sum_{k=1}^\infty e^{-2\lambda_k^b s} \left[\left\langle \varphi_k^b, f_n^x \right\rangle_\mu - \varphi_k^b(x) \right]^2. \end{aligned}$$

By Lemma 4.2 and Fatou's lemma,

$$\begin{aligned}
&\sum_{k=1}^{\infty} e^{-2\lambda_k^b s}\left[\left\langle \varphi_k^b, f_n^x\right\rangle_\mu - \varphi_k^b(x)\right]^2 \\
&= \sum_{k=1}^{\infty} e^{-2\lambda_k^b s}\left[\left\langle \varphi_k^b, f_n^x\right\rangle_\mu - \lim_{m\to\infty}\left\langle \varphi_k^b, f_m^x\right\rangle_\mu\right]^2 \\
&= \sum_{k=1}^{\infty} \lim_{m\to\infty} e^{-2\lambda_k^b s}\left[\left\langle \varphi_k^b, f_n^x\right\rangle_\mu - \left\langle \varphi_k^b, f_m^x\right\rangle_\mu\right]^2 \\
&\leq \liminf_{m\to\infty} \sum_{k=1}^{\infty} e^{-2\lambda_k^b s}\left[\left\langle \varphi_k^b, f_n^x\right\rangle_\mu - \left\langle \varphi_k^b, f_m^x\right\rangle_\mu\right]^2.
\end{aligned}$$

Further, using Fatou's Lemma again,

$$\begin{aligned}
&\int_0^t \int_0^1 \left(\left\langle p_s^b(\cdot, y), f_n^x\right\rangle_\mu - p_s^b(x,y)\right)^2 d\mu(y)ds \\
&\leq \liminf_{m\to\infty} \int_0^t \sum_{k=1}^{\infty} e^{-2\lambda_k^b s}\left\langle \varphi_k^b, f_n^x - f_m^x\right\rangle_\mu^2 ds \\
&= \liminf_{m\to\infty} \int_0^t \left\|\sum_{k=1}^{\infty} e^{-\lambda_k^b s}\left\langle \varphi_k^b, f_n^x - f_m^x\right\rangle_\mu \varphi_k^b\right\|_\mu^2 ds \\
&= \liminf_{m\to\infty} \int_0^t \int_0^1 \left(\int_0^1 p_s^b(y,z)(f_n^x(z) - f_m^x(z))d\mu(z)\right)^2 d\mu(y)ds.
\end{aligned}$$

Finally, by Lemma 4.6 and estimate (56),

$$\begin{aligned}
&\int_0^t \int_0^1 \left(\int_0^1 p_s^b(y,z)(f_n^x(z) - f_m^x(z))d\mu(z)\right)^2 d\mu(y)ds \\
&\leq \frac{e^{2t}}{2}\int_0^1\int_0^1 \rho_1^b(x,y)(f_n^x(y) - f_m^x(y))(f_n^x(z) - f_m^x(z))d\mu(y)d\mu(z) \\
&\leq 2L_1 e^{2t}(r_{\max}^n + r_{\max}^m).
\end{aligned}$$

We conclude

$$\begin{aligned}
\int_0^t \int_0^1 \left(\left\langle p_s^b(\cdot, y), f_n^x\right\rangle_\mu - p_s^b(x,y)\right)^2 d\mu(y)ds &= \liminf_{m\to\infty} 2L_1 e^{2t}\left(r_{\max}^n + r_{\max}^m\right) \\
&= 2L_1 e^{2t} r_{\max}^n.
\end{aligned}$$

□

4.4 Existence, uniqueness and continuity

Let $b \in \{N, D\}$ and $T > 0$ be fixed. We define the concept of a solution to (51) that will be the object of study in the present chapter.

Definition 4.8: *A **mild solution** to the SPDE (51) is defined as a predictable $[0,T] \times [0,1]$-indexed process $u(t,x)$ such that for every $(t,x) \in [0,T] \times [0,1]$ it holds almost surely*

$$\begin{aligned} u(t,x) = &\int_0^1 p_t^b(x,y) u_0(y) d\mu(y) + \int_0^t \int_0^1 p_{t-s}^b(x,y) f(s,u(s,y)) d\mu(y) ds \\ &+ \int_0^t \int_0^1 p_{t-s}^b(x,y) g(s,u(s,y)) \xi(s,y) d\mu(y) ds, \end{aligned} \tag{57}$$

where the last term is a stochastic integral in the sense of Walsh (see Section 2.3).

We define the spaces in which we will search for a solution.

Definition 4.9: *Let $q \geq 2$ and let $S_{q,T}$ be the space of $[0,T] \times [0,1]$-indexed predictable processes v that satisfy*

$$||v||_{q,T} := \sup_{t\in[0,T]} \sup_{x\in[0,1]} \left(\mathbb{E}\left[|v(t,x)|^q\right]\right)^{\frac{1}{q}} < \infty.$$

Further, we define $\mathcal{S}_{q,T}$ as the space of equivalence classes of elements of $S_{q,T}$, where two processes v_1, v_2 are equivalent if $v_1(t,x) = v_2(t,x)$ almost surely for all $(t,x) \in [0,T] \times [0,1]$.

Note that $\mathcal{S}_{q,T}$ is a Banach space. The proof works by standard arguments, so we skip it here.

We allow for random initial data u_0, but since u_0 is not time-dependent, we need to introduce a further space.

Definition 4.10: *Let $q \geq 2$. We define S_q as the space of $[0,1]$-indexed processes v that are measurable from $\mathcal{F}_0 \otimes \mathcal{B}([0,1])$ into $\mathcal{B}(\mathbb{R})$ and satisfy*

$$\|v\|_q := \sup_{x\in[0,1]} \left(\mathbb{E}\left[|v(x)|^q\right]\right)^{\frac{1}{q}} < \infty.$$

Further, we define $\mathcal{S}_q$ as the space of equivalence classes of elements of S_q, where two processes v_1, v_2 are equivalent if $v_1(x) = v_2(x)$ almost surely for all $x \in [0,1]$.

$\mathcal{S}_q$ is a Banach space as well. The proof is a standard argument and we skip it.

Throughout this section, we make the following assumptions, which are adapted from [41, Hypothesis 6.2].

Assumption 4.11: *There exists $q \geq 2$ such that*

(i) $u_0 \in \mathcal{S}_q$,

(ii) f *and* g *are predictable and satisfy the following Lipschitz and linear growth conditions: There exists* $L > 0$ *and a predictable process* $M : \Omega \times [0,T] \to \mathbb{R}$ *such that* $\|M\|_{q,T} := \sup_{s\in[0,T]} \|M(s)\|_{L^q(\Omega)} < \infty$ *and for all* $(w,t,x,y) \in \Omega \times [0,T] \times \mathbb{R}^2$

$$|f(\omega,t,x) - f(\omega,t,y)| + |g(\omega,t,x) - g(\omega,t,y)| \leq L|x-y|,$$
$$|f(\omega,t,x)| + |g(\omega,t,x)| \leq M(w,t) + L|x|.$$

Predictability of a process $f : \Omega \times [0,T] \times \mathbb{R}$ can be defined in the same way as in Section 2.3 (see also [69]).

We are now able to prove stochastic continuity properties of v_1 and v_2, which are defined for $(t,x) \in [0,T] \times [0,1]$ and $v_0 \in \mathcal{S}_{q,T}$ by

$$v_1(t,x) := \int_0^t \int_0^1 p_{t-s}^b(x,y) g(s, v_0(s,y)) \xi(s,y) d\mu(y) ds, \tag{58}$$
$$v_2(t,x) := \int_0^t \int_0^1 p_{t-s}^b(x,y) f(s, v_0(s,y)) d\mu(y) ds. \tag{59}$$

Proposition 4.12: *Let $q \geq 2$ be fixed. Then, there exists a constant $C_3 > 0$ such that for $v_0 \in \mathcal{S}_{q,T}$ v_1 and v_2 are well-defined and we have for all $s,t \in [0,T]$, $x,y \in [0,1]$, $i \in \{1,2\}$*

$$\mathbb{E}\left[|v_i(t,x) - v_i(t,y)|^q\right] \leq C_3 \left(1 + \|v_0\|_{q,T}^q\right) |x-y|^{\frac{q}{2}},$$
$$\mathbb{E}\left[|v_i(s,x) - v_i(t,x)|^q\right] \leq C_3 \left(1 + \|v_0\|_{q,T}^q\right) |s-t|^{q\left(\frac{1}{2} - \frac{\gamma\delta}{2}\right)}.$$

Remark 4.13: Note that $\gamma\delta < 1$ is equivalent to

$$\max_{i=1,\ldots,N} \frac{\log \mu_i}{\log(\mu_i r_i)} < 1,$$

which is satisfied as we assume $0 < \mu_i, r_i < 1$.

Proof. We adapt ideas from [41, Proposition 6.7]. First, we consider v_1. For $x \in [0,1]$, $(t,y) \to p_t^b(x,y)$ is continuous on $(0,T] \times [0,1]$ and g and v_0 are predictable. The integrand is thus predictable. By Hypothesis 4.11, for $q \geq 2$, $(t,x) \in [0,T] \times [0,1]$,

$$\mathbb{E}\left[|g(t,v_0(t,x))|^q\right] \leq 2^{q-1}\left(\mathbb{E}\left[|M(t)|^q\right] + L^q \mathbb{E}\left[|v_0(t,x)|^q\right]\right). \tag{60}$$

Hence, by Lemmas 4.4(iii) and 4.5,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \int_0^1 p_{t-s}^b(x,y)^2 g(s,v_0(s,y))^2 d\mu(y)ds\right] \\
&\leq \left(2 \sup_{s\in[0,T]} \|M(s)\|^2_{L^2(\Omega)} + 2L^2 \|v_0\|^2_{q,T}\right) \int_0^t \int_0^1 p_{t-s}^b(x,y)^2 d\mu(y)ds \\
&= \left(2 \sup_{s\in[0,T]} \|M(s)\|^2_{L^2(\Omega)} + 2L^2 \|v_0\|^2_{q,T}\right) \int_0^t p_{2s}^b(x,x)ds \\
&\leq \left(2 \sup_{s\in[0,T]} \|M(s)\|^2_{L^2(\Omega)} + 2L^2 C_0(2T) \|v_0\|^2_{q,T}\right) \int_0^t (2s)^{-\gamma\delta} ds,
\end{aligned}$$

which is finite due to $\gamma\delta < 1$. Hence, $v_1(t,x)$ is well-defined.

In order to prove the spatial estimate for v_1, let $(t,x,y) \in [0,T] \times [0,1]^2$ be fixed. There exists a constant $C(q)$ such that

$$\begin{aligned}
&\mathbb{E}\left[|v_1(t,x) - v_1(t,y)|^q\right] \\
&= \mathbb{E}\left[\left|\int_0^t \int_0^1 \left(p_{t-s}^b(x,z) - p_{t-s}^b(y,z)\right) g(s,v_0(s,z))\xi(s,z)d\mu(z)ds\right|^q\right] \\
&\leq C(q)\mathbb{E}\left[\left|\int_0^t \int_0^1 \left(p_{t-s}^b(x,z) - p_{t-s}^b(y,z)\right)^2 g(s,v_0(s,z))^2 d\mu(z)ds\right|^{\frac{q}{2}}\right] && (61) \\
&\leq C(q)\left|\int_0^t \int_0^1 \left|\left(p_{t-s}^b(x,z) - p_{t-s}^b(y,z)\right)^q \mathbb{E}\left[|g(s,v_0(s,z))|^q\right]\right|^{\frac{2}{q}} d\mu(z)ds\right|^{\frac{q}{2}} && (62) \\
&= C(q)\left|\int_0^t \int_0^1 \left(p_{t-s}^b(x,z) - p_{t-s}^b(y,z)\right)^2 \left(\mathbb{E}\left[|g(s,v_0(s,z))|^q\right]\right)^{\frac{2}{q}} d\mu(z)ds\right|^{\frac{q}{2}} \\
&\leq 2^{q-1}C(q)\left(\|M\|^q_{q,T} + L^q\|v_0\|^q_{q,T}\right)\left|\int_0^t \int_0^1 \left(p_{t-s}^b(x,z)\right.\right. \\
&\qquad\qquad \left.\left. - p_{t-s}^b(y,z)\right)^2 d\mu(z)ds\right|^{\frac{q}{2}}, && (63)
\end{aligned}$$

where we have used the Burkholder-Davis-Gundy inequality (see [49, Theorem B.1]) in (61), Minkowski's integral inequality in (62) and relation (60) in (63). We proceed by dealing with the integral term in (63), whereby we first treat the case $x, y \in K$. Applying Lemma 4.3, Lemma 4.6, Lemma 4.7 and the Lipschitz continuity of the resolvent density (see (55)),

$$
\begin{aligned}
&\int_0^t \int_0^1 \left(p_{t-s}^b(x,z) - p_{t-s}^b(y,z)\right)^2 d\mu(z)ds \\
&= \lim_{n\to\infty} \int_0^t \int_0^1 \left\langle p_{t-s}^b(\cdot,z), f_n^x - f_n^y \right\rangle_\mu^2 d\mu(z)ds \\
&\leq \frac{e^{2t}}{2} \lim_{n\to\infty} \int_0^1 \int_0^1 \rho_1^b(z_1,z_2)(f_n^x(z_1) - f_n^y(z_1))(f_n^x(z_2) - f_n^y(z_2))d\mu(z_1)d\mu(z_2) \\
&= \frac{e^{2t}}{2} \left|\rho_1^b(x,x) - 2\rho_1^b(x,y) + \rho_1^b(y,y)\right| \\
&\leq L_1 e^{2t}|x-y|.
\end{aligned}
$$

For $i \in \mathbb{N}$ and $x \in (a_i, b_i)$, recall that we evaluate the $\mathcal{D}_\mu^2$-representative of $p_t^b(\cdot, y)$ for $(t,y) \in (0,T] \times [0,1]$. We thus have

$$
p_t^b(x,y) = p_t^b(a_i,y) + \left(\frac{x-a_i}{b_i-a_i}\right)\left(p_t^b(b_i,y) - p_t^b(a_i,y)\right), \; y \in [0,1]
$$

and consequently,

$$
\begin{aligned}
v_1(t,x) &= \int_0^t \int_0^1 p_{t-s}^b(x,y)g(s,v_0(s,y))\xi(s,y)d\mu(y)ds \\
&= v_1(t,a_i) + \left(\frac{x-a_i}{b_i-a_i}\right)(v_1(t,b_i) - v_1(t,a_i))
\end{aligned}
$$

almost surely. Consequently, for all $x, y \in [0,1]$

$$
\left|\int_0^t \int_0^1 \left(p_{t-s}^b(x,z) - p_{t-s}^b(y,z)\right)^2 d\mu(z)ds\right|^{\frac{q}{2}} \leq (L_1)^{\frac{q}{2}} e^{qt}|x-y|^{\frac{q}{2}}.
$$

We conclude

$$
\mathbb{E}\left[|v_1(t,x) - v_1(t,y)|^q\right] \leq 2^{q-1}e^{qt}C(q)\,(L_1)^{\frac{q}{2}}\left(\|M\|_{q,T}^q + L^q\|v_0\|_{q,T}^q\right)|x-y|^{\frac{q}{2}}.
$$

This proves the spatial estimate.

We turn to the temporal estimate. Let $s, t \in [0,T]$ with $s < t$ and $x \in [0,1]$. Then, by using the Burkholder-Davis-Gundy inequality, Minkowski's integral inequality and inequality (60) in the same way as before, we get

$$\begin{aligned}
&\mathbb{E}\left[|v_1(t,x) - v_1(s,x)|^q\right] \\
&\le C(q)\left|\int_0^t \int_0^1 \left|\left(p^b_{t-u}(x,y) - p^b_{s-u}(x,y)\mathbb{1}_{[0,s]}(u)\right)^2 \mathbb{E}\left[|g(s,v_0(s,y))|^q\right]\right|^{\frac{2}{q}} d\mu(y)du\right|^{\frac{q}{2}} \\
&\le 2^{q-1}C(q)\left(\|M\|^q_{q,T} + L^q\|v_0\|^q_{q,T}\right)\left|\int_0^t \int_0^1 \left(p^b_{t-u}(x,y)\right.\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left.\left. - p^b_{s-u}(x,y)\mathbb{1}_{[0,s]}(u)\right)^2 d\mu(y)du\right|^{\frac{q}{2}}.
\end{aligned}$$

We split the latter integral in the time intervals $[0,s]$ and $(s,t]$ and get for the first part by Proposition 4.5(iii)

$$\begin{aligned}
&\int_0^s \int_0^1 \left(p^b_{t-u}(x,y) - p^b_{s-u}(x,y)\right)^2 d\mu(y)du \\
&\quad= \int_0^s \int_0^1 \left(p^b_u(x,y) - p^b_{u+t-s}(x,y)\right)^2 d\mu(y)du \\
&\quad= \int_0^s p^b_{2u}(x,x) - 2p^b_{2u+t-s}(x,x) + p^b_{2(u+t-s)}(x,x)du \\
&\quad\le 2^{-\gamma\delta}C_2(2T)\int_0^s u^{-\gamma\delta} - 2\left(u + \frac{t-s}{2}\right)^{-\gamma\delta} + (u+t-s)^{-\gamma\delta}\,du \qquad (64) \\
&\quad\le 2^{-\gamma\delta+1}C_2(2T)\int_0^s u^{-\gamma\delta} - \left(u + \frac{t-s}{2}\right)^{-\gamma\delta} du \\
&\quad= \frac{2^{-\gamma\delta+1}}{1-\gamma\delta}C_2(2T)\left(s^{1-\gamma\delta} - \left(\frac{s}{2} + \frac{t}{2}\right)^{1-\gamma\delta} + \left(\frac{t-s}{2}\right)^{1-\gamma\delta}\right) \\
&\quad\le \frac{C_2(2T)}{1-\gamma\delta}(t-s)^{1-\gamma\delta}.
\end{aligned}$$

For the second part by Proposition 4.5(i)

$$\begin{aligned}
\int_s^t \int_0^1 p^b_{t-u}(x,y)^2 d\mu(y)du &= \int_0^{t-s} p^b_{2u}(x,x)du \\
&\le 2^{-\gamma\delta}C_0(2T)\int_0^{t-s} u^{-\gamma\delta}du \\
&\le \frac{2^{-\gamma\delta}}{1-\gamma\delta}C_0(2T)(t-s)^{1-\gamma\delta}.
\end{aligned}$$

The statements about v_2 can be shown in the same way using Jensen's inequality instead of the Burkholder-Davis-Gundy inequality. $\square$

Corollary 4.14: *Let $q \geq 2$ and $v_0 \in \mathcal{S}_{q,T}$. Then, v_1 and v_2 defined as in (58)-(59) are elements of $\mathcal{S}_{q,T}$.*

Proof. By setting $s = 0$ in Proposition 4.12 we obtain $\|v_i\|_{q,T} < \infty$, $i = 1, 2$. We need to show that v_1 is predictable. Let $n \in \mathbb{N}$, $(t, x) \in [0, T] \times [0, 1]$ and

$$v_1^n(t,x) := \sum_{i,j=0}^{2^n-1} v_1\left(\frac{i}{2^n}T, \frac{j}{2^n}\right) \mathbb{1}_{\left(\frac{i}{2^n}T, \frac{i+1}{2^n}T\right]}(t) \mathbb{1}_{\left(\frac{j}{2^n}, \frac{j+1}{2^n}\right]}(x).$$

Evidently, $\|v_1^n\|_{q,T} < \infty$. To prove that v_1^n is predictable, we show that v_1^n is the $\mathcal{S}_{q,T}$-limit of a sequence of simple processes. To this end, let $N \geq 1$, $(t, x) \in [0, T] \times [0, 1]$ and

$$v_1^{n,N}(t,x) := ((-N) \vee v_1^n(t,x)) \wedge N.$$

This defines a simple process since $\left((-N) \vee v_1\left(\frac{i}{2^n}T, \frac{j}{2^n}\right)\right) \wedge N$ is $\mathcal{F}_{\frac{iT}{2^n}}$-measurable and bounded for $0 \leq i, j \leq 2^{n-1}$. Further, it converges in $\mathcal{S}_{q,T}$ to v_1^n as $N \to \infty$. Indeed, let X be a set, $f : X \to \mathbb{R}$. We define $f^+(x) := f(x) \vee 0$ and $f^-(x) := f(x) \wedge 0$, $x \in X$. With that,

$$\begin{aligned}
&\lim_{N\to\infty} \left\| v_1^n(t,x) - v_1^{n,N}(t,x) \right\|_{q,T} \\
&\leq \lim_{N\to\infty} \sum_{i,j=0}^{2^n-1} \left\| v_1^+\left(\frac{i}{2^n}T, \frac{j}{2^n}\right) - \left(v_1^+\left(\frac{i}{2^n}T, \frac{j}{2^n}\right) \wedge N\right) \right\|_{q,T} \\
&\qquad\qquad + \left\| v_1^-\left(\frac{i}{2^n}T, \frac{j}{2^n}\right) - \left((-N) \vee v_1^-\left(\frac{i}{2^n}T, \frac{j}{2^n}\right)\right) \right\|_{q,T} \\
&= \lim_{N\to\infty} \sum_{i,j=0}^{2^n-1} \left\| v_1^+\left(\frac{i}{2^n}T, \frac{j}{2^n}\right) - \left(v_1^+\left(\frac{i}{2^n}T, \frac{j}{2^n}\right) \wedge N\right) \right\|_{L^q(\Omega)} \\
&\qquad\qquad + \left\| v_1^-\left(\frac{i}{2^n}T, \frac{j}{2^n}\right) - \left((-N) \vee v_1^-\left(\frac{i}{2^n}T, \frac{j}{2^n}\right)\right) \right\|_{L^q(\Omega)} \\
&= 0,
\end{aligned}$$

where the last equation follows by the monotone convergence theorem. We conclude that v_1^n is predictable for $n \in \mathbb{N}$. By Proposition 4.12,

$$\begin{aligned}
\|v_1 - v_1^n\|_{q,T} &\leq \sup_{|s-t|<\frac{T}{n}} \sup_{|x-y|<\frac{1}{n}} \|v_1(s,x) - v_1(t,y)\|_{L^q(\Omega)} \\
&\leq \sup_{|s-t|<\frac{T}{n}} \sup_{x\in[0,1]} \|v_1(s,x) - v_1(t,x)\|_{L^q(\Omega)} \\
&\quad + \sup_{t\in[0,T]} \sup_{|x-y|<\frac{1}{n}} \|v_1(t,x) - v_1(t,y)\|_{L^q(\Omega)} \\
&\leq \left(C_3 \left(1 + \|v_0\|_{q,T}^q\right)\right)^{\frac{1}{q}} \left(\left(\frac{T}{n}\right)^{\frac{1}{2}-\frac{\gamma\delta}{2}} + \left(\frac{1}{n}\right)^{\frac{1}{2}}\right) \to 0, \ n \to \infty.
\end{aligned}$$

Hence, v_1 is predictable. The predictability of v_2 follows in the same way. $\square$

We can now follow the methods in [41, Theorem 6.9] to establish existence and uniqueness.

Theorem 4.15: *Assume Condition 4.11 with $q \geq 2$. Then, SPDE (51) has a unique mild solution in $\mathcal{S}_{q,T}$.*

Proof. Uniqueness: Let $u, \widetilde{u} \in \mathcal{S}_{q,T}$ be mild solutions to (51). Then $v := u - \widetilde{u} \in \mathcal{S}_{2,T}$. By setting $G(t) := \sup_{x\in[0,1]} \mathbb{E}\left[v^2(t,x)\right]$, $t \in [0,T]$, we calculate for $(t,x) \in [0,T] \times [0,1]$

$$\begin{aligned}
&\mathbb{E}\left[v(t,x)^2\right] \\
&\leq 2T\mathbb{E}\left[\int_0^t \int_0^1 \left(p_{t-s}^b(x,y)\right)^2 \left(f(s,u(s,y)) - f\left(s,\widetilde{u}(s,y)\right)\right)^2 d\mu(y)ds\right] \\
&\quad + 2\mathbb{E}\left[\int_0^t \int_0^1 \left(p_{t-s}^b(x,y)\right)^2 \left(g(s,u(s,y)) - g\left(s,\widetilde{u}(s,y)\right)\right)^2 d\mu(y)ds\right] \qquad (65) \\
&\leq 2(T+1)L^2\mathbb{E}\left[\int_0^t \int_0^1 v^2(s,y)\left(p_{t-s}^b(x,y)\right)^2 d\mu(y)ds\right] \\
&\leq 2(T+1)L^2 \int_0^t G(s) \int_0^1 \left(p_{t-s}^b(x,y)\right)^2 d\mu(y)ds \\
&= 2(T+1)L^2 \int_0^t G(s) p_{2(t-s)}^b(x,x)ds \\
&\leq 2^{1-\gamma\delta}(T+1)C_0(2T)L^2 \int_0^t G(s)(t-s)^{-\gamma\delta}ds, \qquad (66)
\end{aligned}$$

where we have used Walsh's isometry (see Section 2.3) and Hölder's inequality in

(65) and Proposition 4.5(i) in (66). For $t \in [0,T]$, it follows

$$G(t) \leq 2^{1-\gamma\delta}(T+1)C_0(2T)L^2 \int_0^t G(s)(t-s)^{-\gamma\delta} ds.$$

Setting $h_n = G$, $n \in \mathbb{N}$ in [69, Lemma 3.3] gives $G(t) = 0$ for $t \in [0,T]$. We conclude $u(t,x) = \widetilde{u}(t,x)$ almost surely for every $(t,x) \in [0,T] \times [0,1]$.

Existence: As usual, we apply Picard iteration to find a solution. For that, let $u_1 = 0 \in \mathcal{S}_{q,T}$ and for $n \geq 1$, $(t,x) \in [0,T] \times [0,1]$,

$$\begin{aligned} u_{n+1}(t,x) = \int_0^1 p_t^b(x,y)u_0(y)d\mu(y) + \int_0^t \int_0^1 p_{t-s}^b(x,y)f(s,u_n(s,y))d\mu(y)ds \\ + \int_0^t \int_0^1 p_{t-s}^b(x,y)g(s,u_n(s,y))\xi(s,y)d\mu(y)ds. \end{aligned} \tag{67}$$

Let $n \geq 1$, assume that $u_n \in \mathcal{S}_{q,T}$ and define u_{n+1} as in (67). The last two terms on the right-hand side are elements of $\mathcal{S}_{q,T}$ by Corollary 4.14. The first term is predictable because it is $\mathcal{F}_0$-measurable and thus adapted and almost surely continuous due to the dominated convergence theorem and Proposition 4.5(i). Furthermore, by Minkowski's integral inequality and Lemma 4.4(vi)

$$\begin{aligned} \mathbb{E}\left[\left|\int_0^1 p_t^b(x,y)u_0(y)d\mu(y)\right|^q\right] &\leq \left|\int_0^1 p_t^b(x,y)\mathbb{E}\left[|u_0(y)|^q\right]^{\frac{1}{q}} d\mu(y)\right|^q \\ &\leq \|u_0\|_q^q \left|\int_0^1 p_t^b(x,y)d\mu(y)\right|^q \\ &\leq \|u_0\|_q^q. \end{aligned}$$

It follows $u_{n+1} \in \mathcal{S}_{q,T}$.

We prove that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}_{q,T}$. For $n \in \mathbb{N}$, let $v_n \coloneqq u_{n+1} - u_n \in \mathcal{S}_{q,T}$. By Hölder's and the Burkholder-Davis-Gundy inequality, for $(t,x) \in [0,T] \times [0,1]$

$$\begin{aligned} &\mathbb{E}\left[|v_{n+1}(t,x)|^q\right] \\ &\leq 2^{q-1}T^{\frac{q}{2}}\mathbb{E}\left[\left|\int_0^t \int_0^1 p_{t-s}^b(x,y)^2 \left(f(s,u_{n+1}(s,y)) - f\left(s,u_n(s,y)\right)\right)^2 d\mu(y)ds\right|^{\frac{q}{2}}\right] \\ &\quad + 2^{q-1}C(q)\mathbb{E}\left[\left|\int_0^t \int_0^1 p_{t-s}^b(x,y)^2 \left(g(s,u_{n+1}(s,y)) - g\left(s,u_n(s,y)\right)\right)^2 d\mu(y)ds\right|^{\frac{q}{2}}\right] \end{aligned}$$

and using the Lipschitz property of f and g and Minkowski's integral inequality,

$$\begin{aligned}
&\mathbb{E}\left[|v_{n+1}(t,x)|^q\right] \\
&\le 2^{q-1}\left(T^{\frac{q}{2}}+C(q)\right)L^q\mathbb{E}\left[\left|\int_0^t\int_0^1 p^b_{t-s}(x,y)^2v_n^2(s,y)d\mu(y)ds\right|^{\frac{q}{2}}\right] \\
&\le 2^{q-1}\left(T^{\frac{q}{2}}+C(q)\right)L^q\left(\int_0^t\int_0^1 p^b_{t-s}(x,y)^2\left(\mathbb{E}\left[|v_n(s,y)|^q\right]\right)^{\frac{2}{q}}d\mu(y)ds\right)^{\frac{q}{2}}.
\end{aligned}$$

Let $n\in\mathbb{N}$ and $H_n(s):=\sup_{y\in[0,1]}\left(\mathbb{E}\left[|v_n(s,y)|^q\right]\right)^{\frac{2}{q}}$, $s\in[0,T]$. Then, there exists a constant $\widetilde{c}_n$ such that $|H_n(t)|\le\widetilde{c}_n$ for each $t\in[0,T]$. By setting $C:=\left(2^{q-1}\left(T^{\frac{q}{2}}+C(q)\right)L^q\right)^{\frac{2}{q}}$ we get for $(t,x)\in[0,T]\times[0,1]$ and $n\in\mathbb{N}$

$$\begin{aligned}
\left(\mathbb{E}\left[|v_{n+1}(t,x)|^q\right]\right)^{\frac{2}{q}} &\le C\int_0^t H_n(s)p^b_{2(t-s)}(x,x)ds \\
&\le 2^{-\gamma\delta}C_0(2T)C\int_0^t H_n(s)(t-s)^{-\gamma\delta}ds,
\end{aligned}$$

where we have used Proposition 4.5(i) and thus

$$H_{n+1}(t)\le 2^{-\gamma\delta}C_0(2T)C\int_0^t H_n(s)(t-s)^{-\gamma\delta}ds.$$

By [69, Lemma 3.3], there exists a constant $C'>0$ and an integer $k\ge 1$ such that for $n,m\ge 1$, $t\in[0,T]$

$$H_{n+mk}(t)\le\frac{C'^m}{(m-1)!}\int_0^t H_n(s)(t-s)ds.$$

Therefore, $\sum_{m\ge1}\sqrt{H_{n+mk}(t)}$ converges uniformly for $t\in[0,T]$, which is straight forward to check by the ratio test using that $\sqrt{\frac{H_{n+(m+1)k}(t)}{H_{n+mk}(t)}}\le\sqrt{\frac{C'}{m}}$ for $n,m\ge1$. We conclude

$$\sup_{t\in[0,T]}\sqrt{H_n(t)}\to 0,\ \ n\to\infty,$$

which implies the same for $\|v_n\|_{q,T}$. Hence, $(u_n)_{n\ge1}$ is a Cauchy sequence in $\mathcal{S}_{q,T}$ with limit, say u. To show that u satisfies (57), let $(t,x)\in[0,T]\times[0,1]$ be fixed and take the limit in $L^q(\Omega)$ for $n\to\infty$ on both sides of (67). We get $u(t,x)$ on the left-hand side. For the right-hand side, note that there is a constant $C''\ge 0$ such that

$$\mathbb{E}\left[\left|\int_0^t \int_0^1 p^b_{t-s}(x,y)\left(f(s,u(s,y)) - f\left(s,u_n(s,y)\right)\right)\xi(s,y)d\mu(y)ds\right|^q\right]$$
$$+\,\mathbb{E}\left[\left|\int_0^t \int_0^1 p^b_{t-s}(x,y)\left(g(s,u(s,y)) - g\left(s,u_n(s,y)\right)\right)d\mu(y)ds\right|^q\right]$$
$$\leq C''\left(\int_0^t \int_0^1 p^b_{t-s}(x,y)^2\left(\mathbb{E}\left[|u(s,y)-u_n(s,y)|^q\right]\right)^{\frac{2}{q}} d\mu(y)ds\right)^{\frac{q}{2}},$$

which goes to zero as n tends to infinity with the same argumentation as before. $\square$

Before stating the main result, we define $u^{\mathrm{sto}} \in \mathcal{S}_{q,T}$ by

$$u^{\mathrm{sto}}(t,x) := \int_0^t \int_0^1 p^b_{t-s}(x,y)f(s,u(s,y))d\mu(y)ds$$
$$+ \int_0^t \int_0^1 p^b_{t-s}(x,y)g(s,u(s,y))\xi(s,y)d\mu(y)ds$$

almost surely for $(t,x) \in [0,T]\times[0,1]$. That is,

$$u^{\mathrm{sto}}(t,x) = u(t,x) - \int_0^1 p^b_t(x,y)u_0(y)d\mu(y)$$

almost surely for $(t,x) \in [0,T]\times[0,1]$

Theorem 4.16: *Assume Condition 4.11 with $q \geq 2$. Then, there exists a version of u^{sto}, denoted by $\tilde{u}^{\mathrm{sto}}$, such that the following holds:*

(i) If $q > 2$ and $t \in [0,T]$, $\tilde{u}^{\mathrm{sto}}(t,\cdot)$ is essentially $\frac{1}{2}-\frac{1}{q}$-Hölder continuous on $[0,1]$.

(ii) If $q > \left(\frac{1}{2}-\frac{\gamma\delta}{2}\right)^{-1}$ and $x \in [0,1]$, $\tilde{u}^{\mathrm{sto}}(\cdot,x)$ is essentially $\frac{1}{2}-\frac{\gamma\delta}{2}-\frac{1}{q}$-Hölder continuous on $[0,T]$.

(iii) If $q > 2\left(\frac{1}{2}-\frac{\gamma\delta}{2}\right)^{-1}$, $\tilde{u}^{\mathrm{sto}}$ is essentially $\left(\frac{1}{2}-\frac{\gamma\delta}{2}-\frac{2}{q}\right)$-Hölder continuous on $[0,T]\times[0,1]$.

Proof. The continuity properties in part (i) and (ii) of a version of u^{sto} are direct consequences of Proposition 4.12 by setting $v_0 := u$ and Kolmogorov's continuity theorem (see e.g. [52, Proposition 21.6]).

Now, let $(s,t,x,y) \in [0,T]^2\times[0,1]^2$ and without loss of generality, we assume

$|s-t| \leq 1$. Then,

$$\begin{aligned}
&\mathbb{E}\left[\left|u^{\text{sto}}(s,x)-u^{\text{sto}}(t,y)\right|^q\right] \\
&\leq 2^{q-1}\left(\mathbb{E}\left[\left|u^{\text{sto}}(s,x)-u^{\text{sto}}(t,x)\right|^q\right]+\mathbb{E}\left[\left|u^{\text{sto}}(t,x)-u^{\text{sto}}(t,y)\right|^q\right]\right) \\
&\leq 2^q C_3\left(1+\|u\|_{q,T}^q\right)\left(|x-y|^{\frac{q}{2}}+|s-t|^{q\left(\frac{1}{2}-\frac{\gamma\delta}{2}\right)}\right) \\
&\leq 2^q C_3\left(1+\|u\|_{q,T}^q\right)\max\left\{|x-y|^{\frac{q}{2}},|s-t|^{q\left(\frac{1}{2}-\frac{\gamma\delta}{2}\right)}\right\} \\
&\leq 2^q C_3\left(1+\|u\|_{q,T}^q\right)\max\left\{|x-y|,|s-t|\right\}^{q\left(\frac{1}{2}-\frac{\gamma\delta}{2}\right)}.
\end{aligned}$$

The result follows by Kolmogorov's continuity theorem in two dimensions (see e.g. [52, Remark 21.7]). □

Remark 4.17: We consider u^{sto} because the regularity of u is, in general, restricted by the regularity of $u-u^{\text{sto}}$. However, we can formulate the following: There is a version of $u-u^{\text{sto}}$ such that this version is for fixed $t \in (0,T]$ $\frac{1}{2}$-Hölder continuous on $[0,1]$ and for fixed $x \in [0,1]$ and $T_1 > 0$ $\frac{1}{2}$-Hölder continuous on $[T_1,T]$. This can be checked in the same way as [41, Lemma 6.10] using Proposition 4.5. Consequently, there exists a version $\tilde{u}$ of u such that

(i) If $q>2$ and $t \in (0,T]$, $\tilde{u}(t,\cdot)$ is essentially $\frac{1}{2}-\frac{1}{q}$-Hölder continuous on $[0,1]$.

(ii) If $q > \left(\frac{1}{2}-\frac{\gamma\delta}{2}\right)^{-1}$ and $x \in [0,1]$, $\tilde{u}(\cdot,x)$ is essentially $\frac{1}{2}-\frac{\gamma\delta}{2}-\frac{1}{q}$-Hölder continuous on $[T_1,T]$.

Example 4.18: (i) If u_0, f and g satisfy Assumption 4.11 for some $q \geq 2$ and, in addition, are uniformly bounded, q can be chosen arbitrarily large such that we obtain $\frac{1}{2}$ for the ess. spatial and $\frac{1}{2}-\frac{\gamma\delta}{2}$ for the ess. temporal Hölder exponent. If, moreover, the measure μ is the natural measure on K, then

$$\frac{1}{2}-\frac{\gamma\delta}{2}=\frac{1}{2}-\frac{1}{2}\max_{1\leq i\leq N}\frac{\log\left(r_i^{d_H}\right)}{\log\left(r_i^{d_H+1}\right)}=\frac{1}{2}-\frac{d_H}{2d_H+2}=\frac{1}{2d_H+2}.$$

The left-hand side of Figure 6 visualizes that for $0<d_H\leq 1$.

For $d_H=1$ and the natural measure on K, we meet the well-known Hölder continuity properties for the stochastic heat equation defined by the standard Laplacian. Further, in case of Δ_μ^b being a Laplacian on a p.c.f. self-similar set with $1 \leq d_H < 2$, Hambly and Yang [41, Theorem 6.14]) established the

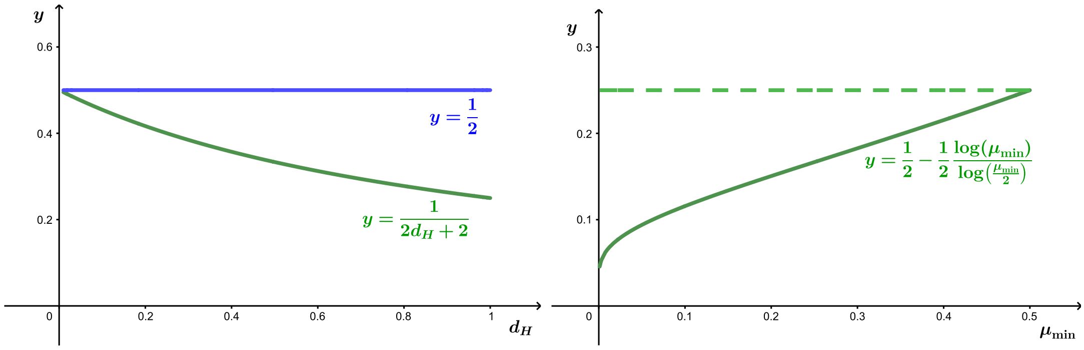

Figure 6: Hölder exponent graphs for the stochastic heat equation

essential spatial Hölder exponent $\frac{1}{2}$ and the essential temporal Hölder exponent $\frac{1}{2d_H+2}$. Therefore, our results can be understood as an extension of their results to the case of $d_H < 1$.

(ii) If μ is not the natural measure on K, then it can be easily checked that

$$\frac{1}{2} - \frac{\gamma\delta}{2} < \frac{1}{2d_H + 2}.$$

As an example, consider the weighted IFS given by $S_1, S_2 : [0,1] \to \mathbb{R}$, $S_1(x) = \frac{x}{2}$, $S_2(x) = \frac{x}{2} + \frac{1}{2}$, $x \in [0,1]$ and weights $\mu_1, \mu_2 \in (0,1)$. Then,

$$\frac{1}{2} - \frac{\gamma\delta}{2} = \frac{1}{2} - \frac{1}{2}\max_{i=1,2}\frac{\log \mu_i}{\log(\mu_i r_i)} = \frac{1}{2} - \frac{1}{2}\frac{\log \mu_{\min}}{\log\left(\frac{\mu_{\min}}{2}\right)},$$

which goes to zero as $\mu_{\min} \to 0$. The right-hand side of Figure 6 indicates that.

4.5 Weak intermittency

Let $b \in \{N, D\}$. We consider the process u given by

$$\begin{aligned} u(t,x) = &\int_0^1 p_t^b(x,y)u_0(y)d\mu(y) + \int_0^t \int_0^1 p_{t-s}^b(x,y)f(s,u(s,y))d\mu(y)ds \\ &+ \int_0^t \int_0^1 p_{t-s}^b(x,y)g(s,u(s,y))\xi(s,y)d\mu(y)ds \end{aligned} \tag{68}$$

almost surely for $(t,x) \in [0,\infty) \times [0,1]$.

According to [49, Definition 7.7], we call u weakly intermittent on $[0,1]$ if the lower and the upper moment Lyapunov exponents, which are respectively the functions γ and $\bar{\gamma}$ defined for $p \in (0,\infty),\ x \in [0,1]$ by

$$\gamma(p,x) := \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[|u(t,x)|^p\right], \quad \bar{\gamma}(p,x) := \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[|u(t,x)|^p\right],$$

satisfy

$$\gamma(2,x) > 0, \quad \bar{\gamma}(p,x) < \infty, \quad p \in [2,\infty),\ x \in [0,1].$$

In this section, we make the following additional assumption:

Assumption 4.19: *We assume that $u_0 \in \mathcal{S}_q$ for some $q \geq 2$. Moreover, we assume that f and g are predictable and satisfy the following Lipschitz and linear growth condition: There exists a constant $L > 0$ such that for all $(w,t,x,y) \in \Omega \times [0,\infty) \times \mathbb{R}^2$*

$$\begin{aligned} |f(\omega,t,x) - f(\omega,t,y)| + |g(\omega,t,x) - g(\omega,t,y)| &\leq L|x-y|, \\ |f(\omega,t,x)| + |g(\omega,t,x)| &\leq L(1+|x|). \end{aligned}$$

Predictability of a process $f : \Omega \times [0,\infty) \times \mathbb{R}$ can be defined in the same way as in Section 2.3 (compare [69]).

First, we establish the upper bound.

Theorem 4.20: *There exists $C_4 > 0$ such that for all $p \in [1,q]$, $(t,x) \in [0,\infty) \times [0,1]$*

$$\left(\mathbb{E}\left[|u(t,x)|^p\right]\right)^{\frac{1}{p}} \leq \left(2\left\|u_0\right\|_q + 1\right) e^{C_4 p^{\frac{1}{1-\gamma\delta}} t}.$$

Remark 4.21: If μ is the natural measure on K, we have $\gamma = \frac{d_H}{d_H+1}$ and thus $\frac{1}{1-\gamma} = d_H + 1$. Then, the above inequality reads as

$$\left(\mathbb{E}\left[|u(t,x)|^p\right]\right)^{\frac{1}{p}} \leq \left(2\left\|u_0\right\|_q + 1\right) e^{C_4 p^{d_H+1} t}.$$

Before proving Theorem 4.20, we need to extend our heat kernel estimates.

Lemma 4.22: *Let $b \in \{N,D\}$. There exists $C_5 > 0$ such that for all $(t,x) \in (0,\infty) \times [0,1]$*

$$p_t^b(x,x) \leq C_5 \left(1 + t^{-\gamma\delta}\right).$$

Proof. Let $T > 0$. By (23) and (24), for each $(t,x) \in (0,\infty)\times \in [0,1]$

$$\left|p_t^b(x,x)\right| \leq \sum_{k\geq 1} e^{-c_0 k^{\frac{1}{\gamma}} t} c_2^2 k^{\delta}.$$

The term on the right-hand side converges uniformly on $[T,\infty)$ (see [48, Lemma 5.1.4]). Further, we have $\lambda_1^N = 0$, $\varphi_1^N \equiv 1$, $\lambda_1^D > 0$. The dominated convergence theorem gives

$$\lim_{t\to\infty} p_t^N(x,x) = 1, \ \lim_{t\to\infty} p_t^D(x,x) = 0,$$

uniformly for all $x \in [0,1]$. This along with Proposition 4.5(i) implies the result. □

Proof of Theorem 4.20. We follow the methods of [41, Theorem 7.5]. Let $p \in [2,q]$, $\alpha > 0$, $(t,x) \in [0,\infty) \times [0,1]$ be fixed. By the Burkholder-Davis-Gundy inequality,

$$\begin{aligned}
&e^{-\alpha t}\left(\mathbb{E}\left[|u(t,x)|^p\right]\right)^{\frac{1}{p}} \\
&\leq \|u_0\|_p + \left(\mathbb{E}\left[\left|\int_0^t \int_0^1 e^{-\alpha t} p_{t-s}^b(x,y) f(s,u(s,y)) d\mu(y) ds\right|^p\right]\right)^{\frac{1}{p}} \\
&\quad + 2\sqrt{p}\left(\mathbb{E}\left[\left|\int_0^t \int_0^1 e^{-2\alpha t} p_{t-s}^b(x,y)^2 g(s,u(s,y))^2 ds\right|^{\frac{p}{2}}\right]\right)^{\frac{1}{p}}.
\end{aligned}$$

To estimate the first integral, we apply Minkowski's integral inequality and obtain

$$\begin{aligned}
&\left(\mathbb{E}\left[\left|\int_0^t \int_0^1 e^{-\alpha t} p_{t-s}^b(x,y) f(s,u(s,y)) d\mu(y) ds\right|^p\right]\right)^{\frac{1}{p}} \\
&\leq \int_0^t \int_0^1 e^{-\alpha t} p_{t-s}^b(x,y) \left(\mathbb{E}\left[|f(s,u(s,y))|^p\right]\right)^{\frac{1}{p}} d\mu(y) ds \\
&\leq L \int_0^t \int_0^1 e^{-\alpha t} p_{t-s}^b(x,y) \left(1 + \left(\mathbb{E}\left[|u(s,y)|^p\right]\right)^{\frac{1}{p}}\right) d\mu(y) ds \\
&\leq L\left(te^{-\alpha t} + \int_0^t \int_0^1 e^{-\alpha t} p_{t-s}^b(x,y) \sup_{z\in[0,1]} \left(\left(\mathbb{E}\left[|u(s,z)|^p\right]\right)^{\frac{1}{p}}\right) d\mu(y) ds\right) \\
&\leq L\left(\frac{1}{\alpha} + \sup_{(s,z)\in[0,t]\times[0,1]} \left(e^{-\alpha s}\mathbb{E}\left[|u(s,z)|^p\right]\right)^{\frac{1}{p}} \int_0^t e^{-\alpha(t-s)} ds\right) \\
&\leq \frac{L}{\alpha}\left(1 + \sup_{(s,z)\in[0,t]\times[0,1]} \left(e^{-\alpha s}\mathbb{E}\left[|u(s,z)|^p\right]\right)^{\frac{1}{p}}\right).
\end{aligned}$$

In the second last inequality, we have used that $t \to te^{-\alpha t}$ reaches its maximum at $t = \frac{1}{\alpha}$. We turn to the second integral. Applying Minkowski's integral inequality and Lemma 4.22,

$$
\begin{aligned}
&\left(\mathbb{E}\left[\left|\int_0^t \int_0^1 e^{-2\alpha t} p_{t-s}^b(x,y)^2 g(s,u(s,y))^2 d\mu(y) ds\right|^{\frac{p}{2}}\right]\right)^{\frac{1}{p}} \\
&\leq \left(\int_0^t \int_0^1 e^{-2\alpha t} p_{t-s}^b(x,y)^2 \left(\mathbb{E}\left[|g(s,u(s,y))|^p\right]\right)^{\frac{2}{p}} d\mu(y) ds\right)^{\frac{1}{2}} \\
&\leq L \left(\int_0^t e^{-2\alpha t} p_{2(t-s)}^b(x,x) \sup_{z\in[0,1]} \left(1 + \left(\mathbb{E}\left[|u(s,z)|^p\right]\right)^{\frac{1}{p}}\right)^2 ds\right)^{\frac{1}{2}} \\
&= L \left(\int_0^t e^{-2\alpha(t-s)} p_{2(t-s)}^b(x,x) \sup_{z\in[0,1]} \left(e^{-\alpha s} + e^{-\alpha s}\left(\mathbb{E}\left[|u(s,z)|^p\right]\right)^{\frac{1}{p}}\right)^2 ds\right)^{\frac{1}{2}} \\
&\leq L \sup_{(s,z)\in[0,t]\times[0,1]} \left(e^{-\alpha s} + e^{-\alpha s}\left(\mathbb{E}\left[|u(s,z)|^p\right]\right)^{\frac{1}{p}}\right) \left(\int_0^t e^{-2\alpha(t-s)} p_{2(t-s)}^b(x,x) ds\right)^{\frac{1}{2}} \\
&\leq 2^{-\frac{1}{2}} L \sup_{(s,z)\in[0,t]\times[0,1]} \left(1 + e^{-\alpha s}\left(\mathbb{E}\left[|u(s,z)|^p\right]\right)^{\frac{1}{p}}\right) \left(\int_0^\infty e^{-\alpha s} p_s^b(x,x) ds\right)^{\frac{1}{2}} \\
&\leq 2^{-\frac{1}{2}} L C_5^{\frac{1}{2}} \left(\int_0^\infty e^{-\alpha s}(1 + s^{-\gamma\delta}) ds\right)^{\frac{1}{2}} \sup_{(s,z)\in[0,t]\times[0,1]} \left(1 + e^{-\alpha s}\left(\mathbb{E}\left[|u(s,z)|^p\right]\right)^{\frac{1}{p}}\right).
\end{aligned}
$$

We denote the gamma function by Γ and calculate

$$
\begin{aligned}
\int_0^\infty e^{-\alpha s}(1 + s^{-\gamma\delta}) ds &= \int_0^\infty e^{-\alpha s} ds + \int_0^\infty e^{-\alpha s} s^{-\gamma\delta} ds \\
&= \frac{1}{\alpha} + \frac{1}{\alpha}\int_0^\infty e^{-s}\left(\frac{s}{\alpha}\right)^{-\gamma\delta} ds \\
&= \frac{1 + \alpha^{\gamma\delta}\Gamma(1-\gamma\delta)}{\alpha}.
\end{aligned}
$$

This implies that a constant $C_4' > 0$ exists such that for all $\alpha \geq 1$

$$
\int_0^\infty e^{-\alpha s}(1 + s^{-\gamma\delta}) ds \leq C_4' \alpha^{\gamma\delta - 1}.
$$

Then,

$$
\begin{aligned}
&e^{-\alpha t}\left(\mathbb{E}\left[|u(t,x)|^p\right]\right)^{\frac{1}{p}} \\
&\le \|u_0\|_p + \left(\frac{L}{\alpha} + L\sqrt{2C_5C_4'\alpha^{\gamma\delta-1}p}\right) \sup_{(s,z)\in[0,t]\times[0,1]} \left(1 + e^{-\alpha s}\left(\mathbb{E}\left[|u(s,z)|^p\right]\right)^{\frac{1}{p}}\right).
\end{aligned}
$$

Now, let $\alpha := C_4''p^{\frac{1}{1-\gamma\delta}}$, where $C_4'' \ge 1$ does not depend on p. Then,

$$
\frac{L}{\alpha} + L\sqrt{2C_5C_4'\alpha^{\gamma\delta-1}p} = \frac{L}{C_4''p^{\frac{1}{1-\gamma\delta}}} + L\sqrt{2C_5C_4'}\left(C_4''\right)^{\frac{\gamma\delta}{2}-\frac{1}{2}},
$$

which goes to zero as $C_4'' \to \infty$. Consequently, we can choose $C_4'' \ge 1$ such that

$$
\frac{L}{\alpha} + L\sqrt{2C_5C_4'\alpha^{\gamma\delta-1}p} \le \frac{1}{2}.
$$

This leads to

$$
\left(\mathbb{E}\left[|u(t,x)|^p\right]\right)^{\frac{1}{p}} \le \left(2\|u_0\|_p + 1\right) e^{C_4''p^{\frac{1}{1-\gamma\delta}}t}.
$$

If $1 \le p < 2$, we have for $(t,x) \in [0,\infty)\times[0,1]$

$$
\begin{aligned}
\left(\mathbb{E}\left[|u(t,x)|^p\right]\right)^{\frac{1}{p}} &\le \left(\mathbb{E}\left[|u(t,x)|^2\right]\right)^{\frac{1}{2}} \\
&\le \left(2\|u_0\|_p + 1\right) e^{C_4''2^{\frac{1}{1-\gamma\delta}}t} \\
&= \left(2\|u_0\|_p + 1\right) e^{C_4''\left(\frac{2}{p}\right)^{\frac{1}{1-\gamma\delta}}p^{\frac{1}{1-\gamma\delta}}t}
\end{aligned}
$$

and obtain the assertion for all $p \in [1,q]$ by setting $C_4 := C_4''2^{\frac{1}{1-\gamma\delta}}$. □

The above proposition immediately implies for $p \in [1,q]$

$$
\bar{\gamma}(p,x) \le \limsup_{t\to\infty} \frac{1}{t} \sup_{x\in[0,1]} \log \mathbb{E}\left[|u(t,x)|^p\right] = C_4p^{1+\frac{1}{1-\gamma\delta}}.
$$

We now establish a lower bound for the second moment of u defined by

$$
u(t,x) = \int_0^1 p_t^N(x,y)u_0(y)d\mu(y) + \int_0^t\int_0^1 p_{t-s}^N(x,y)g(u(s,y))\xi(s,y)d\mu(y)ds \tag{69}
$$

almost surely for $(t,x) \in [0,\infty)\times[0,1]$. That is, we let $f := 0$ and g be not time-dependent in (68). Furthermore, let the following conditions hold.

Assumption 4.23: *(i) $u_0 : [0,1] \to \mathbb{R}$ is measurable and bounded.*

(ii) $g : \mathbb{R} \to \mathbb{R}$ satisfies the following Lipschitz and linear growth conditions: There exists $L > 0$ such that for all $(x, y) \in \mathbb{R}^2$

$$|g(x) - g(y)| \leq L|x - y|,$$
$$|g(x)| \leq +L(1 + |x|).$$

Proposition 4.24: *Define $L_g \coloneqq \inf_{x \in \mathbb{R}\setminus\{0\}} \left|\frac{g(x)}{x}\right|$ and assume $\inf_{x\in[0,1]} u_0(x) > 0$. Then, $\gamma(2, x) \geq L_g^2$ for all $x \in [0,1]$.*

Proof. Let $(t, x) \in (0, \infty) \times [0, 1]$. By the non-negativity of u_0 and Lemma 4.4(vi)

$$\int_0^1 p_t^N(x, y)u_0(y)d\mu(y) \geq \inf_{y\in[0,1]} u_0(y) \int_0^1 p_t^N(x, y)d\mu(y) = \inf_{y\in[0,1]} u_0(y).$$

Using Walsh's isometry, the zero-mean property of the stochastic integral and the previous estimate, we get

$$\begin{aligned}
&\mathbb{E}\left[u(t,x)^2\right] \\
&= \left(\int_0^1 p_t^N(x,y)u_0(y)d\mu(y)\right)^2 + \int_0^t \int_0^1 p_{t-s}^N(x,y)^2 \mathbb{E}\left[g(u(s,y))^2\right] d\mu(y)ds \\
&\quad + \int_0^1 p_t^N(x,y)u_0(y)d\mu(y)\mathbb{E}\left[\int_0^t \int_0^1 p_{t-s}^N(x,y)g(u(s,y))\xi(s,y)d\mu(y)ds\right] \\
&\geq \inf_{z\in[0,1]} u_0(z)^2 + \int_0^t \int_0^1 L_g^2 p_{t-s}^N(x,y)^2 \mathbb{E}\left[u(s,y)^2\right] d\mu(y)ds.
\end{aligned}$$

It holds $\varphi_1^N = 1$ and $\lambda_1^N = 0$ and consequently,

$$p_t^N(x, x) = \sum_{k\geq 1} e^{-\lambda_k^N t} \varphi_k^N(x)^2 \geq e^{-\lambda_1^N t} \varphi_1^N(x)^2 = 1.$$

With $I(t) \coloneqq \inf_{x\in[0,1]} \mathbb{E}\left[u(t,x)^2\right],\ t \geq 0$, we obtain

$$\begin{aligned}
I(t) &\geq \inf_{z\in[0,1]} u_0(z)^2 + \int_0^t \int_0^1 L_g^2 p_{t-s}^N(x,y)^2 I(s)d\mu(y)ds \\
&= \inf_{z\in[0,1]} u_0(z)^2 + \int_0^t L_g^2 p_{2(t-s)}^N(x,x)I(s)ds \\
&\geq \inf_{z\in[0,1]} u_0(z)^2 + \int_0^t L_g^2 I(s)ds. \qquad (70)
\end{aligned}$$

It follows

$$
\begin{aligned}
I(t) &\geq \inf_{z\in[0,1]} u_0(z)^2 + \int_0^t L_g^2 \inf_{z\in[0,1]} u_0(z)^2 ds + \int_0^t L_g^2 \int_0^s L_g^2 I(u) du ds \\
&= \inf_{z\in[0,1]} u_0(z)^2 + L_g^2 \inf_{z\in[0,1]} u_0(z)^2 t + \int_0^t L_g^2 \int_0^s L_g^2 I(u) du ds
\end{aligned}
$$

and by iterating this

$$
I(t) \geq \inf_{z\in[0,1]} u_0(z)^2 \sum_{n=0}^{\infty} \frac{L_g^{2n} t^n}{n!} = \inf_{z\in[0,1]} u_0(z)^2 e^{L_g^2 t},
$$

which yields

$$
\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[u(t,x)^2\right] \geq \liminf_{t\to\infty} \frac{\log\left(\inf_{z\in[0,1]} u_0(z)^2\right)}{t} + L_g^2 = L_g^2. \qquad \square
$$

The main result of this section follows directly.

Corollary 4.25: *Let* $\inf_{x\in[0,1]} u_0(x) > 0$ *and* $L_g > 0$. *Then,* u *given by* (69) *is weakly intermittent on* $[0,1]$.

5 Analysis of Measure Theoretic Stochastic Wave Equations

5.1 Preliminaries

Let $b \in \{N, D\}$, $T > 0$ and let μ be a self-similar measure on a Cantor-like set K according to Section 2.2. Further, let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space and let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfy the usual conditions. In this chapter, we address the hyperbolic stochastic PDE

$$\begin{aligned}\frac{\partial^2}{\partial t^2}u(t,x) &= \Delta_\mu^b u_t(x) + f(t,u(t,x))\xi(t,x),\\ u(0,x) &= u_0(x),\\ \frac{\partial}{\partial t}u(0,x) &= u_1(x)\end{aligned} \tag{71}$$

for $(t,x) \in [0,T] \times [0,1]$, $u_0, u_1 : [0,1] \to \mathbb{R}$, $f : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$. Further, ξ denotes a space-time white noise based on μ according to Definition 2.10.

Before defining the concept of a mild solution to (71), we need to have a closer look on deterministic wave equations.

To this end, let $u_0 \in \mathcal{D}\left(\Delta_\mu^b\right)$, $u_1 \in \mathcal{D}\left(\left(-\Delta_\mu^b\right)^{\frac{1}{2}}\right)$ and let $u_{0,k}^b, u_{1,k}^b, k \geq 1$ such that $u_0 = \sum_{k\geq 1} u_{0,k}^b \varphi_k^b$ and $u_1 = \sum_{k\geq 1} u_{1,k}^b \varphi_k^b$. In particular, $\sum_{k\geq 1}\left(\lambda_k^b\right)^2\left(u_{0,k}^b\right)^2 < \infty$ and $\sum_{k\geq 1}\lambda_k^b\left(u_{1,k}^b\right)^2 < \infty$. For $(t,x,y) \in [0,\infty) \times [0,1]^2$ let

$$P_t^N(x,y) := t + \sum_{k\geq 2}\frac{\sin\left(\sqrt{\lambda_k^N}t\right)}{\sqrt{\lambda_k^N}}\varphi_k^N(x)\varphi_k^N(y)$$

and

$$P_t^D(x,y) := \sum_{k\geq 1}\frac{\sin\left(\sqrt{\lambda_k^D}t\right)}{\sqrt{\lambda_k^D}}\varphi_k^D(x)\varphi_k^D(y).$$

This is called wave propagator of Δ_μ^b, $b \in \{N, D\}$. As Δ_μ^b is a self-adjoint, dissipative operator on $L^2([0,1],\mu)$, the wave equation

$$\begin{aligned}\frac{\partial^2}{\partial t^2}u(t,x) &= \Delta_\mu^b u_t(x),\\ u(0,x) &= u_0(x),\\ \frac{\partial u(0,x)}{\partial t} &= u_1(x)\end{aligned}$$

for $t \geq 0$ on $L^2([0,1],\mu)$ has a unique solution, which is for $(t,x) \in [0,\infty) \times [0,1]$ given by

$$u(t,x) = C_t^b u_0(x) + S_t^b u_1(x),$$

where the operators $S_t^b,\ C_t^b : L^2([0,1],\mu) \to L^2([0,1],\mu)$ are defined by

$$\begin{aligned}S_t^N &: \sum_{k\geq 1} f_k^N \varphi_k^N \mapsto t f_1^N + \sum_{k\geq 2} \frac{\sin\left(\sqrt{\lambda_k^N} t\right)}{\sqrt{\lambda_k^N}} f_k^N \varphi_k^N,\\ S_t^D &: \sum_{k\geq 1} f_k^D \varphi_k^D \mapsto \sum_{k\geq 1} \frac{\sin\left(\sqrt{\lambda_k^D} t\right)}{\sqrt{\lambda_k^D}} f_k^D \varphi_k^D,\\ C_t^N &: \sum_{k\geq 1} f_k^N \varphi_k^N \mapsto \sum_{k\geq 2} \cos\left(\sqrt{\lambda_k^N} t\right) f_k^N \varphi_k^N,\\ C_t^D &: \sum_{k\geq 1} f_k^D \varphi_k^D \mapsto \sum_{k\geq 1} \cos\left(\sqrt{\lambda_k^D} t\right) f_k^D \varphi_k^D.\end{aligned}$$

for $t \geq 0$. The connection to the wave propagator is given by

$$\int_0^1 P_t^b(x,y)u_1(y)d\mu(y) = S_t^b u_1(x),\ \ (t,x) \in [0,\infty) \times [0,1].$$

The operator families $\{S_t^b : t \geq 0\}$ and $\{C_t^b : t \geq 0\}$ are called strongly continuous sine and cosine family, respectively (see also [68]).

We are now able to define the concept of a solution to (71) that we will investigate in this chapter.

Definition 5.1: *A* ***mild solution*** *to the SPDE* (71) *is defined as a predictable* $[0,T] \times [0,1]$*-indexed process such that for each* $(t,x) \in [0,T] \times [0,1]$ *it holds almost* *surely*

$$u(t,x) = C_t^b u_0(x) + S_t^b u_1(x) + \int_0^t \int_0^1 P_{t-s}^b(x,y) f(s,u(s,y))\xi(s,y) d\mu(y) ds. \quad (72)$$

We now give a set of conditions, which we assume to hold throughout this chapter.

Assumption 5.2: *(i)* $u_0 \in \mathcal{D}\left(\Delta_\mu^b\right)$, $u_1 \in \mathcal{D}\left((-\Delta_\mu^b)^{\frac{1}{2}}\right)$.

(ii) There exists $q \geq 2$ such that f is predictable and satisfies the following Lipschitz and linear growth conditions: There exists $L > 0$ and a predictable process $M : \Omega \times [0,T] \to \mathbb{R}$ with $\|M\|_{q,T} := \sup_{s\in[0,T]} \|M(s)\|_{L^q(\Omega)} < \infty$ such that for all $(w,t,x,y) \in \Omega \times [0,T] \times \mathbb{R}^2$,

$$|f(\omega,t,x) - f(\omega,t,y)| \leq L|x-y|,$$
$$|f(\omega,t,x)| \leq M(w,t) + L|x|.$$

Recall that γ is defined to be the spectral exponent of Δ_μ^b. Note that the self-adjointness of Δ_μ^b along with the assumption $u_0 \in \mathcal{D}\left(\Delta_\mu^b\right)$ implies that there is a constant $C_6 < \infty$ such that

$$\left|\left\langle u_0, \varphi_k^b \right\rangle_\mu\right| \leq C_6 k^{-\frac{1}{\gamma}}, \quad k \geq 1.$$

Indeed, for $k \geq 2$ we have

$$\left|\left\langle u_0, \varphi_k^N \right\rangle_\mu\right| = \frac{1}{\lambda_k^N} \left|\left\langle u_0, -\Delta_\mu^N \varphi_k^N \right\rangle_\mu\right| = \frac{1}{\lambda_k^N} \left|\left\langle -\Delta_\mu^N u_0, \varphi_k^N \right\rangle_\mu\right| \leq c_0^{-1} \left\|\Delta_\mu^N u_0\right\|_\mu k^{-\frac{1}{\gamma}}.$$

Analogously, since $u_1 \in \mathcal{D}\left((-\Delta_\mu^b)^{\frac{1}{2}}\right)$, there is a constant $C_7 < \infty$ such that

$$\left|\left\langle u_1, \varphi_k^b \right\rangle_\mu\right| \leq C_7 k^{-\frac{1}{2\gamma}}, \quad k \geq 1.$$

5.2 Wave propagator properties and approximation

Let $T > 0$ and $t \in [0, T]$ be fixed. Obviously, $P_t^b : (x, y) \mapsto P_t^b(x, y)$ is an element of $L^2\left([0,1]^2, \mu \otimes \mu\right)$. We provide a way to approximate $P_t^b(x, \cdot) : y \mapsto P_t^b(x, y)$ by $L^2([0,1], \mu)$-functions for fixed $x \in K$. Then, we deduce that $P_t^b(x, \cdot) \in L^2\left([0,1], \mu\right)$ for $x \in [0,1]$.

Let $A : L^2([0,1], \mu) \to L^2([0,1], \mu)$ be a linear operator. We denote the operator norm of A by $\|A\|$. This is defined by

$$\|A\| \coloneqq \sup_{\substack{g \in L^2([0,1],\mu): \\ \|g\|_\mu = 1}} \|Ag\|_\mu .$$

We show that the map $t \mapsto S_t^b$ is Lipschitz continuous on $[0, \infty)$ with respect to the operator norm.

Lemma 5.3: *Let* $0 \leq s < t$. *Then,*

$$\left\|S_t^b - S_s^b\right\| \leq t - s.$$

Proof. Let $g = \sum_{k \geq 1} g_k^N \varphi_k^N$ such that $\|g\|_\mu = 1$. Then,

$$\begin{aligned} &\left\|(S_t^N - S_s^N) g\right\|_\mu^2 \\ &= (t-s)^2 \left(g_k^N\right)^2 + \sum_{k \geq 2} \frac{\left(\sin\left(\sqrt{\lambda_k^N} t\right) - \sin\left(\sqrt{\lambda_k^N} s\right)\right)^2}{\lambda_k^N} \left(g_k^N\right)^2 . \end{aligned}$$

We have $\lambda_2^N \geq 1$ (see e.g. [1, Section 3.3.1]). Hence,

$$\begin{aligned} (t-s)^2 \left(g_1^N\right)^2 &+ \sum_{k \geq 2} \frac{\left(\sin\left(\sqrt{\lambda_k^N} t\right) - \sin\left(\sqrt{\lambda_k^N} s\right)\right)^2}{\lambda_k^N} \left(g_k^N\right)^2 \\ &\leq (t-s)^2 \left(g_1^N\right)^2 + \sup_{\lambda \geq 1} \frac{\left(\sin\left(\sqrt{\lambda} t\right) - \sin\left(\sqrt{\lambda} s\right)\right)^2}{\lambda} \sum_{k \geq 2} \left(g_k^N\right)^2 \\ &\leq (t-s)^2 \sum_{k \geq 1} \left(g_k^N\right)^2 \\ &= (t-s)^2. \end{aligned}$$

Further, using $\lambda_1^D \geq 1$ (see e.g. [58, Lemma 4.9]), we obtain the assertion for Dirichlet boundary conditions in the same way. □

The following lemma provides a way to find upper estimates of functionals of the wave propagator using the resolvent density.

Lemma 5.4: *There is a constant $\bar{C}_8 \geq 0$ such that for all $g \in L^2([0,1],\mu)$ and all $t \in [0,T]$,*

$$\int_0^1 \left(S_t^b g(x)\right)^2 d\mu(x) \leq \bar{C}_8 \int_0^1 \int_0^1 \rho_1^b(x,y)g(x)g(y)d\mu(x)d\mu(y).$$

Proof. Let $g = \sum_{k=1}^{\infty} g_k^N \varphi_k^N$ and $t \in [0,T]$. Then,

$$\begin{aligned}
&\int_0^1 \left(S_t^b g(x)\right)^2 d\mu(x) \\
&= \left\| t g_0^N + \sum_{k=2}^{\infty} \frac{\sin\left(\sqrt{\lambda_k^N} t\right)}{\sqrt{\lambda_k^N}} g_k^N \varphi_k^N \right\|_\mu^2 \\
&= t^2 \left(g_0^N\right)^2 + \sum_{k=2}^{\infty} \frac{\sin^2\left(\sqrt{\lambda_k^N} t\right)}{\lambda_k^N} \left(g_k^N\right)^2 \\
&\leq (T^2 \vee 1) \left(\left(g_0^N\right)^2 + \sum_{k=2}^{\infty} \frac{1}{\lambda_k^N} \left(g_k^N\right)^2 \right) \\
&\leq (T^2 \vee 1) \frac{1+\lambda_2^N}{\lambda_2^N} \sum_{k=1}^{\infty} \frac{1}{1+\lambda_k^N} \left(g_k^N\right)^2 \\
&= (T^2 \vee 1) \frac{1+\lambda_2^N}{\lambda_2^N} \left\langle g, \left(1-\Delta_\mu^N\right)^{-1} g \right\rangle_\mu .
\end{aligned} \tag{73}$$

By definition of the resolvent density,

$$\left(1-\Delta_\mu^N\right)^{-1} g(x) = \int_0^1 \rho_1^N(x,y)g(y)d\mu(y), \ x \in [0,1].$$

Plugging this into (73), the assertion for $b = N$ follows. The case $b = D$ works similarly. □

This leads to an approximation of $y \mapsto P_t^b(x,y)$ for fixed $(t,x) \in [0,T] \times K$.

Lemma 5.5: *There is a constant $C_8 \geq 0$ such that for all $x \in K$, $t \in [0,T]$ and $n \in \mathbb{N}$*

$$\int_0^1 \left(S_t^b f_n^x(y) - P_t^b(x,y)\right)^2 d\mu(y) \leq C_8 r_{\max}^n.$$

Proof. Let $x \in K$, $t \in [0,T]$ and $n \in \mathbb{N}$. Note that $\langle 1, f_n^x\rangle_\mu = \int_0^1 f_n^x(y) d\mu(y) = 1$. Then,

$$\begin{aligned}
&\int_0^1 \left(S_t^N f_n^x(y) - P_t^N(x,y)\right)^2 d\mu(y) \\
&= \int_0^1 \left(t\,\langle 1, f_n^x\rangle_\mu + \sum_{k=2}^\infty \frac{\sin\left(\sqrt{\lambda_k^N} t\right)}{\sqrt{\lambda_k^N}} \varphi_k^N(y) \left\langle \varphi_k^N, f_n^x\right\rangle_\mu - P_t^N(x,y)\right)^2 d\mu(y) \\
&= \int_0^1 \left(t\left(\langle 1, f_n^x\rangle_\mu - 1\right) + \sum_{k=2}^\infty \frac{\sin\left(\sqrt{\lambda_k^N} t\right)}{\sqrt{\lambda_k^N}} \left[\left\langle \varphi_k^N, f_n^x\right\rangle_\mu - \varphi_k^N(x)\right] \varphi_k^N(y)\right)^2 d\mu(y) \\
&= \int_0^1 \left(\sum_{k=2}^\infty \frac{\sin\left(\sqrt{\lambda_k^N} t\right)}{\sqrt{\lambda_k^N}} \left[\left\langle \varphi_k^N, f_n^x\right\rangle_\mu - \varphi_k^N(x)\right] \varphi_k^N(y)\right)^2 d\mu(y).
\end{aligned}$$

By the Riesz-Fischer theorem, it holds

$$\begin{aligned}
&\int_0^1 \left(\sum_{k=2}^\infty \frac{\sin\left(\sqrt{\lambda_k^N} t\right)}{\sqrt{\lambda_k^N}} \left[\left\langle \varphi_k^N, f_n^x\right\rangle_\mu - \varphi_k^N(x)\right] \varphi_k^N(y)\right)^2 d\mu(y) \\
&= \sum_{k=2}^\infty \frac{\sin^2\left(\sqrt{\lambda_k^N} t\right)}{\lambda_k^N} \left[\left\langle \varphi_k^N, f_n^x\right\rangle_\mu - \varphi_k^N(x)\right]^2,
\end{aligned}$$

provided that the latter series converges. By Fatou's lemma and Lemma 4.2,

$$\begin{aligned}
&\sum_{k=2}^\infty \frac{\sin^2\left(\sqrt{\lambda_k^N} t\right)}{\lambda_k^N} \left[\left\langle \varphi_k^N, f_n^x\right\rangle_\mu - \varphi_k^N(x)\right]^2 \\
&= \sum_{k=2}^\infty \frac{\sin^2\left(\sqrt{\lambda_k^N} t\right)}{\lambda_k^N} \left[\left\langle \varphi_k^N, f_n^x\right\rangle_\mu - \lim_{m\to\infty} \left\langle \varphi_k^N, f_m^x\right\rangle_\mu\right]^2 \\
&\leq \liminf_{m\to\infty} \sum_{k=2}^\infty \frac{\sin^2\left(\sqrt{\lambda_k^N} t\right)}{\lambda_k^N} \left[\left\langle \varphi_k^N, f_n^x\right\rangle_\mu - \left\langle \varphi_k^N, f_m^x\right\rangle_\mu\right]^2.
\end{aligned}$$

Further, by Lemma 4.3 and Lemma 5.4, for $m \in \mathbb{N}$

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{\sin^2\left(\sqrt{\lambda_k^N}t\right)}{\lambda_k^N} \left[\left\langle \varphi_k^N, f_n^x \right\rangle_\mu - \left\langle \varphi_k^N, f_m^x \right\rangle_\mu\right]^2 \\
&= \int_0^1 \left(S_t^N(f_n^x(y) - f_m^x(y))\right)^2 d\mu(y) \\
&\leq \bar{C}_8 \int_0^1 \int_0^1 \rho_1^N(z,y)(f_n^x(y) - f_m^x(y))(f_n^x(z) - f_m^x(z))d\mu(y)d\mu(z) \\
&= \bar{C}_8 \int_0^1 \int_0^1 \rho_1^N(z,y)\big(f_m^x(z)f_m^x(y) - f_m^x(z)f_n^x(y) \\
&\qquad\qquad - f_n^x(z)f_m^x(y) + f_n^x(z)f_n^x(y)\big)d\mu(z)d\mu(y) \\
&= \bar{C}_8 \int_0^1 \int_0^1 \rho_1^N(z,y)f_m^x(z)f_m^x(y) - \rho_1^N(x,x) - \rho_1^N(z,y)f_m^x(z)f_n^x(y) + \rho_1^N(x,x) \\
&\quad - \rho_1^N(z,y)f_n^x(z)f_m^x(y) + \rho_1^N(x,x) + \rho_1^N(z,y)f_n^x(z)f_n^x(y) - \rho_1^N(x,x)d\mu(z)d\mu(y) \\
&\leq 4\bar{C}_8 L_1(r_{\max}^m + r_{\max}^n).
\end{aligned}$$

We conclude

$$\begin{aligned}
\int_0^1 \left(S_t^N f_n^x(y) - P_t^N(x,y)\right)^2 d\mu(y) &\leq \liminf_{m\to\infty} 4\bar{C}_8 L_1(r_{\max}^n + r_{\max}^m) \\
&= 4\bar{C}_8 L_1 r_{\max}^n.
\end{aligned}$$

The estimates for Dirichlet boundary conditions can be found similarly. □

Finally, we are able to establish that $P_t^b(x,\cdot) \in L^2([0,1],\mu)$ for fixed $(t,x) \in [0,T] \times [0,1]$.

Lemma 5.6: *There exists a constant $C_9 > 0$ such that*

$$\sup_{t\in[0,T]} \sup_{x\in[0,1]} \left\|P_t^b(x,\cdot)\right\|_\mu < C_9.$$

Proof. Let $h_1, h_2 : [0,1] \to \mathbb{R} \cup \{-\infty, \infty\}$ be measurable. By Minkowski's inequality,

$$\begin{aligned}
\left(\int_0^1 h_1^2(x)d\mu(x)\right)^{\frac{1}{2}} &= \left(\int_0^1 (h_1(x) - h_2(x) + h_2(x))^2 d\mu(x)\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^1 (h_1(x) - h_2(x))^2 d\mu(x)\right)^{\frac{1}{2}} + \left(\int_0^1 h_2^2(x)d\mu(x)\right)^{\frac{1}{2}},
\end{aligned}$$

where it is not required that the integrals are finite (see e.g. [20, Theorem VI 1.8]). Further, let $t \in [0,T]$. For $f \in L^2([0,1],\mu)$ we have

$$\left\|S_t^b f\right\|_\mu \leq (T \vee 1) \left\|f\right\|_\mu ,$$

which follows directly by the definition. Using this, Lemma 5.5 and inequality (54), we obtain for $x \in K$, $n \in \mathbb{N}$,

$$\begin{aligned}
&\left(\int_0^1 \left(P_t^b(x,y)\right)^2 d\mu(y)\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^1 \left(S_t^b f_n^x(y) - P_t^b(x,y)\right)^2 d\mu(y)\right)^{\frac{1}{2}} + \left(\int_0^1 \left(S_t^b f_n^x(y)\right)^2 d\mu(y)\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^1 \left(S_t^b f_n^x(y) - P_t^b(x,y)\right)^2 d\mu(y)\right)^{\frac{1}{2}} + (T \vee 1) \left(\int_0^1 \left(f_n^x(y)\right)^2 d\mu(y)\right)^{\frac{1}{2}} \\
&\leq C_8^{\frac{1}{2}} r_{\max}^{\frac{n}{2}} + (T \vee 1)\, r_{\max}^{-\frac{nd_H}{2}} r_{\min}^{-\frac{d_H}{2}} \nu_{\min}^{-\frac{n}{2}}.
\end{aligned}$$

Choose $n = 1$. As the right-hand side of the last inequality does not depend on $(t,x) \in [0,T] \times K$, the assertion follows for $x \in K$.

Recall that $[0,1] \setminus K = \bigcup_{i=1}^{\infty} (a_i, b_i)$ (see (18)). For $x \in [0,1] \setminus K$, let $i \in \mathbb{N}$ such that $x \in (a_i, b_i)$. By the linearity of φ_k^b on (a_i, b_i) for $k \in \mathbb{N}$ it follows that

$$P_t^b(x,y) = \frac{x - a_i}{b_i - a_i} \left(P_t^b(b_i, y) - P_t^b(a_i, y)\right), \ (t,y) \in [0,T] \times [0,1].$$

Since $a_i, b_i \in K$, we obtain the result. $\square$

5.3 Existence, uniqueness and continuity

Let $b \in \{N, D\}$, $T > 0$ and $q \geq 2$. We prove continuity properties of v_i, $i = 1,2,3$, which are defined for $v_0 \in \mathcal{S}_{q,T}$ and $(t,x) \in [0,T] \times [0,1]$ by

$$v_1(t,x) := \int_0^t \int_0^1 P_{t-s}^b(x,y) f(s, v_0(s,y)) \xi(s,y) d\mu(y) ds, \tag{74}$$

$$v_2(t,x) := S_t^b u_1(x), \tag{75}$$

$$v_3(t,x) := C_t^b u_0(x). \tag{76}$$

We will use the following lemma.

Lemma 5.7: *Let $a < 0, b > a$. Then, there is a constant $C_{a,b} \geq 0$ such that for all $t \in [0, \infty)$*

$$\sum_{k \in \mathbb{N}} k^{a-1} \wedge t k^{b-1} \leq C_{a,b} t^{\frac{-a}{b-a} \wedge 1}.$$

Proof. See [42, Lemma 5.2]. □

We start with proving Hölder continuity properties of v_2 and v_3.

Proposition 5.8: *There exists a constant $C_{10} > 0$ such that for all $i \in \{2, 3\}$, $s, t \in [0, T]$, $x, y \in [0, 1]$, we have*

$$|v_i(t,x) - v_i(t,y)| \leq C_{10} |x - y|^{\frac{1}{2}},$$
$$|v_i(s,x) - v_i(t,x)| \leq C_{10} |s - t|^{(2-(2+\delta)\gamma) \wedge 1}.$$

Proof. First, we treat the spatial continuity. Let $t \in [0, T]$, $x, y \in K$. By Lemma 4.3, Lemma 5.4, Lemma 5.5 and the Lipschitz continuity of the resolvent density,

$$\begin{aligned}
&\left| \int_0^1 P_t^b(x,z) u_1(z) d\mu(z) - \int_0^1 P_t^b(y,z) u_1(z) d\mu(z) \right|^2 \\
&\leq \int_0^1 \left(P_t^b(x,z) - P_t^b(y,z) \right)^2 u_1^2(z) d\mu(z) \\
&\leq \sup_{z \in [0,1]} \left(u_1^2(z) \right) \int_0^1 \left(P_t^b(x,z) - P_t^b(y,z) \right)^2 d\mu(z) \\
&= \sup_{z \in [0,1]} \left(u_1^2(z) \right) \lim_{n \to \infty} \int_0^1 \left(S_t^b \left(f_n^x - f_n^y \right)(z) \right)^2 d\mu(z) \\
&\leq \sup_{z \in [0,1]} \left(u_1^2(z) \right) \bar{C}_8 \lim_{n \to \infty} \int_0^1 \int_0^1 \rho_1^b(z_1, z_2)(f_n^x(z_1) - f_n^y(z_1))(f_n^x(z_2) \\
&\qquad\qquad - f_n^y(z_2)) d\mu(z_1) d\mu(z_2) \\
&= \sup_{z \in [0,1]} \left(u_1^2(z) \right) \bar{C}_8 \left(\rho_1^b(x,x) - 2\rho_1^b(x,y) + \rho_1^b(y,y) \right) \\
&\leq 2 L_1 \bar{C}_8 \sup_{z \in [0,1]} \left(u_1^2(z) \right) |x - y|. && (77)
\end{aligned}$$

For $i \in \mathbb{N}$ and $x \in (a_i, b_i)$, recall that we evaluate the $\mathcal{D}_\mu^2$-representative of φ_k^b for each $k \in \mathbb{N}$. We thus have

$$P_t^b(x,y) = P_t^b(a_i, y) + \left(\frac{x - a_i}{b_i - a_i} \right) \left(P_t^b(b_i, y) - P_t^b(a_i, y) \right), \; y \in [0,1]$$

and therefore

$$
\begin{aligned}
v_2(t,x) &= \int_0^1 P_t^b(x,z)u_1(z)d\mu(z) \\
&= \int_0^1 \left(P_t^b(a_i,y) + \left(\frac{x-a_i}{b_i-a_i}\right)\left(P_t^b(b_i,y) - P_t^b(a_i,y)\right)\right) u_1(z)d\mu(z) \\
&= v_2(t,a_i) + \left(\frac{x-a_i}{b_i-a_i}\right)\left(v_2(t,b_i) - v_2(t,a_i)\right).
\end{aligned}
$$

Hence, for all $t \in [0,T]$, $x,y \in [0,1]$

$$
|v_2(t,x) - v_2(t,y)|^2 \leq 2L_1\bar{C}_8 \sup_{z\in[0,1]} \left(u_1^2(z)\right)|x-y|.
$$

To deal with v_3, for $k \geq 2$, we set $\widetilde{u}_{0,k}^N \coloneqq \sqrt{\lambda_k^N}u_{0,k}^N$. Then, for $t \in [0,T]$, $x \in [0,1]$,

$$
\begin{aligned}
C_t^N u_0(x) &= u_{0,1}^N + \sum_{k\geq 2} \cos\left(\sqrt{\lambda_k^N}t\right)u_{0,k}^N\varphi_k^N(x) \\
&= u_{0,1}^N + \sum_{k\geq 2} \frac{\cos\left(\sqrt{\lambda_k^N}t\right)}{\sqrt{\lambda_k^N}}\widetilde{u}_{0,k}^N\varphi_k^N(x).
\end{aligned}
$$

It holds $\sum_{k\geq 2}\widetilde{u}_{0,k}^N\varphi_k^N \in \mathcal{D}\left(\left(-\Delta_\mu^b\right)^{\frac{1}{2}}\right)$. We can now proceed in the same way as in the proof for v_2. For Dirichlet boundary conditions, the proof works similarly.

For the temporal estimate, let $s,t \in [0,T]$ with $s < t$ and $x \in [0,1]$. Then, there is a constant $C_{10}' > 0$ such that

$$
\begin{aligned}
&\left|\left(S_t^N - S_s^N\right)u_1(x)\right| \\
&\leq (t-s)|u_{1,0}^N| + \sum_{k=2}^{\infty}\left(\frac{\left|\sin\left(\sqrt{\lambda_k^N}t\right) - \sin\left(\sqrt{\lambda_k^N}s\right)\right|}{\sqrt{\lambda_k^N}}\right)|u_{1,k}^N\varphi_k^N(x)| \\
&\leq (t-s)|u_{1,0}^N| + \sum_{k=2}^{\infty}\left(\frac{2\wedge\left(\sqrt{\lambda_k^N}t - \sqrt{\lambda_k^N}s\right)}{\sqrt{\lambda_k^N}}\right)|u_{1,k}^N\varphi_k^N(x)| \\
&\leq C_{10}'\sum_{k=2}^{\infty} k^{\frac{\delta}{2}-\frac{1}{\gamma}} \wedge \left(|t-s|k^{\frac{\delta}{2}-\frac{1}{2\gamma}}\right).
\end{aligned}
$$

Recall that $\gamma\delta < 1$. We have

$$\frac{\delta}{2} - \frac{1}{\gamma} + 1 < \frac{1}{2\gamma} - \frac{1}{\gamma} + 1 < 1 - \frac{1}{2\gamma} \leq 0. \tag{78}$$

We can thus choose $a \coloneqq \frac{\delta}{2} - \frac{1}{\gamma} + 1$ and $b \coloneqq \frac{\delta}{2} - \frac{1}{2\gamma} + 1$ in Lemma 5.7 to get

$$\left|\left(S_t^N - S_s^N\right) u_1(x)\right| \leq C'_{10} C_{a,b} |t-s|^{(2-(2+\delta)\gamma)\wedge 1}.$$

By similar methods, there is a constant $C''_{10} > 0$ such that

$$\left|\left(C_t^N - C_s^N\right) u_0(x)\right| \leq C''_{10} \sum_{k=2}^{\infty} k^{\frac{\delta}{2}-\frac{1}{\gamma}} \wedge \left(|s-t| k^{\frac{\delta}{2}-\frac{1}{2\gamma}}\right),$$

which leads to

$$\left|\left(C_t^N - C_s^N\right) u_0(x)\right| \leq C''_{10} C_{a,b} |t-s|^{(2-(2+\delta)\gamma)\wedge 1}.$$

The calculation for Dirichlet boundary conditions can be done in the same way. $\square$

The estimates for v_1 are more complicated to derive.

Proposition 5.9: *Assume Condition 5.2 with $q \geq 2$. Then, there exists a constant $C_{11} > 0$ such that for all $v_0 \in \mathcal{S}_{q,T}$ it holds that v_1 is well-defined, predictable and for all $s, t \in [0,T]$, $x, y \in [0,1]$,*

$$\begin{aligned} \mathbb{E}\left[|v_1(t,x) - v_1(t,y)|^q\right] &\leq C_{11}\left(1 + \|v_0\|_{q,T}^q\right) |x-y|^{\frac{q}{2}}, \\ \mathbb{E}\left[|v_1(s,x) - v_1(t,x)|^q\right] &\leq C_{11}\left(1 + \|v_0\|_{q,T}^q\right) |s-t|^{q\left(d_H + 1 + \frac{\log \nu_{\min}}{\log r_{\max}}\right)^{-1}}. \end{aligned}$$

Proof. For fixed $x \in [0,1]$, $(t,y) \mapsto P_t^b(x,y)$ is measurable and deterministic and thus predictable. Further, f and v_0 are predictable, according to the assumption. Consequently, the integrand in (74) is predictable. By Hypothesis 5.2, for $(t,x) \in [0,T] \times [0,1]$, $q \geq 2$,

$$|f(t, v_0(t,x))|^q \leq 2^{q-1} |M(t)|^q + 2^{q-1} L^q |v_0(t,x)|^q, \quad (t,x) \in [0,T] \times [0,1] \tag{79}$$

and thus

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^t \int_0^1 P_{t-s}^b(x,y)^2 \left(f(s,v_0(s,y))\right)^2 d\mu(y)ds\right] \\
&\leq 2 \sup_{s\in[0,T]} \|M(s)\|_{L^2(\Omega)}^2 + 2L^2 \|v_0\|_{q,T}^2 \int_0^t \int_0^1 P_{t-s}^b(x,y)^2 d\mu(y)ds \\
&\leq 2 \sup_{s\in[0,T]} \|M(s)\|_{L^2(\Omega)}^2 + 2L^2 T \|v_0\|_{q,T}^2 \sup_{(s,x)\in[0,T]\times[0,1]} \int_0^1 P_s^b(x,y)^2 d\mu(y),
\end{aligned}
$$

which is uniformly bounded for $(t,x) \in [0,T] \times [0,1]$, due to Lemma 5.6. Consequently, $v_1(t,x)$ is well-defined.

Spatial estimate: Let $t \in [0,T]$, $x, y \in [0,1]$. Using (79), the Burkholder-Davis-Gundy inequality and Minkowski's integral inequality, there exists $C(q) > 0$ such that

$$
\begin{aligned}
&\mathbb{E}\left[|v_1(t,x) - v_1(t,y)|^q\right] \\
&= \mathbb{E}\left[\left|\int_0^t \int_0^1 \left(P_{t-s}^b(x,z) - P_{t-s}^b(y,z)\right) f(s,v_0(s,z))\xi(s,z)d\mu(z)ds\right|^q\right] \\
&\leq C(q)\mathbb{E}\left[\left|\int_0^t \int_0^1 \left(P_{t-s}^b(x,z) - P_{t-s}^b(y,z)\right)^2 f(s,v_0(s,z))^2 d\mu(z)ds\right|^{\frac{q}{2}}\right] \\
&\leq C(q)\left|\int_0^t \int_0^1 \left(P_{t-s}^b(x,z) - P_{t-s}^b(y,z)\right)^2 \left(\mathbb{E}\left[|f(s,v_0(s,z))|^q\right]\right)^{\frac{2}{q}} d\mu(z)ds\right|^{\frac{q}{2}} \\
&\leq 2^{q-1}C(q)\left(\|M\|_{q,T}^q + L^q\|v_0\|_{q,T}^q\right)\left|\int_0^t \int_0^1 \left(P_s^b(x,z) - P_s^b(y,z)\right)^2 d\mu(z)ds\right|^{\frac{q}{2}}. \quad (80)
\end{aligned}
$$

We proceed by estimating the integral term in (80). As in (77), we get

$$
\int_0^t \int_0^1 \left(P_s^b(x,z) - P_s^b(y,z)\right)^2 d\mu(z)ds \leq 2TL_1\bar{C}_8|x-y|ds
$$

and thus

$$
\mathbb{E}\left[|v_1(t,x) - v_1(t,y)|^q\right] \leq 2^{\frac{3q}{2}-1}T^{\frac{q}{2}}C(q)\left(L_1\right)^{\frac{q}{2}}\left(\|M\|_{q,T}^q + L^q\|v_0\|_{q,T}^q\right)|x-y|^{\frac{q}{2}}.
$$

This proves the spatial estimate.

Temporal estimate: We adapt ideas from [43, Proposition 4.3]. Let $s,t \in [0,T]$ with $s<t$ and $x \in K$ be fixed. As before, we get

$$\begin{aligned}
&\mathbb{E}\left[|v_1(t,x)-v_1(s,x)|^q\right] \\
&\leq 2^{q-1}C(q)\left(\|M\|_{q,T}^q + L^q\|v_0\|_{q,T}^q\right)\left|\int_0^t\int_0^1 \left(P^b_{t-u}(x,y)\right.\right. \\
&\qquad\qquad\left.\left. - P^b_{s-u}(x,y)\mathbb{1}_{[0,s]}(u)\right)^2 d\mu(y)du\right|^{\frac{q}{2}}.
\end{aligned} \tag{81}$$

We split the above integral into its parts $[0,s]$ and $(s,t]$ and consider the first part. Let $n \in \mathbb{N}$. By Lemma 5.5 and the triangle inequality for $u \in [0,s]$,

$$\begin{aligned}
&\left(\int_0^1 \left(P^b_{t-u}(x,y)-P^b_{s-u}(x,y)\right)^2 d\mu(y)\right)^{\frac{1}{2}} \\
&- \left(\int_0^1 \left(\left(S^b_{t-u}-S^b_{s-u}\right)f_n^x(y)\right)^2 d\mu(y)\right)^{\frac{1}{2}} \\
&\quad \leq \left(\int_0^1 \left(P^b_{t-u}(x,y)-P^b_{s-u}(x,y)-\left(S^b_{t-u}-S^b_{s-u}\right)f_n^x(y)\right)^2 d\mu(y)\right)^{\frac{1}{2}} \\
&\quad \leq \left(\int_0^1 \left(S^b_{t-u}f_n^x(y)-P^b_{t-u}(x,y)\right)^2 d\mu(y)\right)^{\frac{1}{2}} \\
&\qquad + \left(\int_0^1 \left(S^b_{s-u}f_n^x(y)-P^b_{s-u}(x,y)\right)^2 d\mu(y)\right)^{\frac{1}{2}} \\
&\quad \leq 2C_8^{\frac{1}{2}} r_{\max}^{\frac{n}{2}}
\end{aligned}$$

and by resorting and squaring,

$$\begin{aligned}
&\int_0^1 \left(P^b_{t-u}(x,y)-P^b_{s-u}(x,y)\right)^2 d\mu(y) \\
&\leq 2\int_0^1 \left(\left(S^b_{t-u}-S^b_{s-u}\right)f_n^x(y)\right)^2 d\mu(y) + 8C_8 r_{\max}^n.
\end{aligned}$$

Hence, by integration,

$$\begin{aligned}
&\int_0^s\int_0^1 \left(P^b_{t-u}(x,y)-P^b_{s-u}(x,y)\right)^2 d\mu(y)du \\
&\leq 2\int_0^s\int_0^1 \left(\left(S^b_{t-u}-S^b_{s-u}\right)f_n^x(y)\right)^2 d\mu(y)du + 8C_8 s r_{\max}^n.
\end{aligned}$$

We consider the integral term on the right-hand side of the previous inequality. Applying Lemma 5.3,

$$\begin{aligned}\int_0^s \int_0^1 \left(S^b_{t-u} f^x_n(y) - S^b_{s-u} f^x_n(y)\right)^2 d\mu(y)du &= \int_0^s \left\| \left(S^b_{t-u} - S^b_{s-u}\right) f^x_n \right\|_\mu^2 du \\ &\le T \left\| f^x_n \right\|_\mu^2 (t-s)^2.\end{aligned}$$

We turn to the second part and get in the same way as before,

$$\begin{aligned}&\int_s^t \int_0^1 \left(P^b_{t-u}(x,y)\right)^2 d\mu(y)du \\ &\le 2\int_s^t \int_0^1 \left(S^b_{t-u} f^x_n(y)\right)^2 d\mu(y) + 2C_8 r_{\max}^n du \\ &= 2\int_s^t \int_0^1 \left(S^b_{t-u} f^x_n(y)\right)^2 d\mu(y)du + 2C_8 (t-s) r_{\max}^n.\end{aligned}$$

Again, we give an upper bound for the integral term. By Lemma 5.3,

$$\begin{aligned}\int_s^t \left\| S^b_{t-u} f^x_n \right\|_\mu^2 du &= \int_s^t \left\| \left(S^b_{t-u} - S^b_0\right) f^x_n \right\|_\mu^2 du \\ &\le \left\| f^x_n \right\|_\mu^2 \int_s^t (t-u)^2 du \\ &= \frac{1}{3} \left\| f^x_n \right\|_\mu^2 (t-s)^3.\end{aligned}$$

Further, by (54),

$$\left\| f^x_n \right\|_\mu^2 < r_{\min}^{-d_H} r_{\max}^{-nd_H} \nu_{\min}^{-n}.$$

Consequently, there exist $C > 0$ and $C' > 0$ such that for all $s, t \in [0, T]$, $x \in K$, $n \in \mathbb{N}$

$$\begin{aligned}\int_0^t \int_0^1 \left(P^b_{t-u}(x,y) - P^b_{s-u}(x,y) \mathbb{1}_{[0,s]}(u)\right)^2 d\mu(y)du &\le C(t-s)^2 r_{\max}^{-nd_H} \nu_{\min}^{-n} + C' r_{\max}^n \\ &\le C(t-s)^2 r_{\max}^{-nd_H} \nu_{\min}^{-n} + C'' r_{\max}^n,\end{aligned}$$

where $C'' := \max\left\{C', C(t-s)^2 \left(d_H + \frac{\log \nu_{\min}}{\log r_{\max}}\right)\right\}$. In order to find the minimum in n, we define

$$f(y) := C(t-s)^2 e^{y \log\left(\frac{1}{r_{\max}}\right)\left(d_H + \frac{\log \nu_{\min}}{\log r_{\max}}\right)} + C'' e^{-\log\left(\frac{1}{r_{\max}}\right) y}.$$

By differentiating,

$$f'(y) = C(t-s)^2 \log\left(\frac{1}{r_{\max}}\right)\left(d_H + \frac{\log(\nu_{\min})}{\log(r_{\max})}\right) e^{y\log\left(\frac{1}{r_{\max}}\right)\left(d_H + \frac{\log(\nu_{\min})}{\log(r_{\max})}\right)}$$
$$- C'' \log\left(\frac{1}{r_{\max}}\right) e^{-\log\left(\frac{1}{r_{\max}}\right)y}$$

and by setting zero,

$$e^{y\log\left(\frac{1}{r_{\max}}\right)\left(d_H + \frac{\log(\nu_{\min})}{\log(r_{\max})}+1\right)}$$
$$= \frac{C'' \log\left(\frac{1}{r_{\max}}\right)}{C(t-s)^2 \log\left(\frac{1}{r_{\max}}\right)\left(d_H + \frac{\log(\nu_{\min})}{\log(r_{\max})}\right)} = \frac{C''}{C(t-s)^2\left(d_H + \frac{\log(\nu_{\min})}{\log(r_{\max})}\right)}.$$

By logarithmising we obtain

$$y\log\left(\frac{1}{r_{\max}}\right)\left(d_H + \frac{\log(\nu_{\min})}{\log(r_{\max})} + 1\right) = \log\left(\frac{C''}{C(t-s)^2\left(d_H + \frac{\log(\nu_{\min})}{\log(r_{\max})}\right)}\right).$$

Solving this equation for y yields

$$y = \frac{1}{\log\left(\frac{1}{r_{\max}}\right)\left(d_H + \frac{\log(\nu_{\min})}{\log(r_{\max})} + 1\right)} \log\left(\frac{C''}{C(t-s)^2\left(d_H + \frac{\log(\nu_{\min})}{\log(r_{\max})}\right)}\right),$$

which we denote by y_0. This value does not need to be an integer, but there is an integer n such that $n \in [y_0, y_0+1)$. It is elementary to see that y_0 is the unique minimum on $\mathbb{R}$. Hence, f is increasing on $[y_0, \infty)$. It follows that there exists $C''' > 0$ such that

$$\int_0^t \int_0^1 \left(P^b_{t-u}(x,y) - P^b_{s-u}(x,y)\mathbb{1}_{[0,s]}(u)\right)^2 d\mu(y)du \le f\left(y_0+1\right).$$

We calculate

$$
\begin{aligned}
&f\left(y_0+1\right)\\
&= C(t-s)^2\left(\frac{1}{r_{\max}}\right)^{\left(d_H+\frac{\log(\nu_{\min})}{\log(r_{\max})}\right)}\left(\frac{1}{r_{\max}}\right)^{\frac{\log\left(\frac{C''}{C(t-s)^2\left(d_H+\frac{\log(\nu_{\min})}{\log(r_{\max})}\right)}\right)}{\log\left(\frac{1}{r_{\max}}\right)}\frac{d_H+\frac{\log(\nu_{\min})}{\log(r_{\max})}}{d_H+1+\frac{\log(\nu_{\min})}{\log(r_{\max})}}}\\
&\quad + C'' r_{\max}\left(\frac{1}{r_{\max}}\right)^{\frac{\log\left(\frac{C''}{C(t-s)^2\left(d_H+\frac{\log(\nu_{\min})}{\log(r_{\max})}\right)}\right)}{\log\left(\frac{1}{r_{\max}}\right)}\frac{-1}{d_H+1+\frac{\log(\nu_{\min})}{\log(r_{\max})}}}\\
&= C(t-s)^2\left(\frac{1}{r_{\max}}\right)^{\left(d_H+\frac{\log(\nu_{\min})}{\log(r_{\max})}\right)}\left(\frac{C''}{C(t-s)^2\left(d_H+\frac{\log(\nu_{\min})}{\log(r_{\max})}\right)}\right)^{\frac{d_H+\frac{\log(\nu_{\min})}{\log(r_{\max})}}{d_H+1+\frac{\log(\nu_{\min})}{\log(r_{\max})}}}\\
&\quad + C'' r_{\max}\left(\frac{C''}{C(t-s)^2\left(d_H+\frac{\log(\nu_{\min})}{\log(r_{\max})}\right)}\right)^{\frac{-1}{d_H+1+\frac{\log(\nu_{\min})}{\log(r_{\max})}}}\\
&= C'''(t-s)^{\frac{2}{d_H+1+\frac{\log(\nu_{\min})}{\log(r_{\max})}}}.
\end{aligned}
$$

The case $x \in [0,1]\setminus K$ follows as before by using the linearity of φ_k^b on $[a_i, b_i]$ for $i, k \geq 1$. $\square$

Corollary 5.10: *Assume Condition 5.2 with $q \geq 2$ and let $v_0 \in \mathcal{S}_{q,T}$. Then, v_i, $i = 1,2,3$, defined as in (74)-(76), are elements of $\mathcal{S}_{q,T}$.*

Proof. By setting $s = 0$ in Proposition 5.8 and Proposition 5.9, we obtain $\|v_i\|_{q,T} < \infty$, $i = 1,2,3$. We need to show that v_1 is predictable. For $n \in \mathbb{N}$, $(t,x) \in [0,T]\times[0,1]$, let

$$
v_1^n(t,x) := \sum_{i,j=0}^{2^n-1} v_1\left(\frac{i}{2^n}T, \frac{j}{2^n}\right)\mathbb{1}_{\left(\frac{i}{2^n}T, \frac{i+1}{2^n}T\right]}(t)\mathbb{1}_{\left(\frac{j}{2^n}, \frac{j+1}{2^n}\right]}(x).
$$

Evidently, $\|v_1^n\|_{q,T} < \infty$. To prove that v_1^n is predictable, we show that v_1^n is the $\mathcal{S}_{q,T}$-limit of a sequence of simple processes. To this end, for $N \geq 1$, $(t,x) \in [0,T]\times[0,1]$, let

$$
v_1^{n,N}(t,x) := ((-N)\vee v_1^n(t,x))\wedge N, \quad t\in[0,T],\; x\in[0,1].
$$

This defines a simple process as $\left((-N) \vee v_1\left(\frac{i}{2^n}T, \frac{j}{2^n}\right)\right) \wedge N$ is $\mathcal{F}_{\frac{iT}{2^n}}$-measurable and bounded for $0 \leq i, j \leq 2^{n-1}$. Further, it converges in $\mathcal{S}_{q,T}$ to v_1^n as $N \to \infty$. Indeed, we have

$$\begin{aligned}
&\lim_{N\to\infty} \left\| v_1^n(t,x) - v_1^{n,N}(t,x) \right\|_{q,T} \\
&\leq \lim_{N\to\infty} \sum_{i,j=0}^{2^n-1} \left\| v_1^+\left(\frac{i}{2^n}T, \frac{j}{2^n}\right) - \left(v_1^+\left(\frac{i}{2^n}T, \frac{j}{2^n}\right) \wedge N\right) \right\|_{q,T} \\
&\qquad\qquad + \left\| v_1^-\left(\frac{i}{2^n}T, \frac{j}{2^n}\right) - \left((-N) \vee v_1^-\left(\frac{i}{2^n}T, \frac{j}{2^n}\right)\right) \right\|_{q,T} \\
&= \lim_{N\to\infty} \sum_{i,j=0}^{2^n-1} \left\| v_1^+\left(\frac{i}{2^n}T, \frac{j}{2^n}\right) - \left(v_1^+\left(\frac{i}{2^n}T, \frac{j}{2^n}\right) \wedge N\right) \right\|_{L^q(\Omega)} \\
&\qquad\qquad + \left\| v_1^-\left(\frac{i}{2^n}T, \frac{j}{2^n}\right) - \left((-N) \vee v_1^-\left(\frac{i}{2^n}T, \frac{j}{2^n}\right)\right) \right\|_{L^q(\Omega)} \\
&= 0,
\end{aligned}$$

where the last equation follows by the monotone convergence theorem. We conclude that v_1^n is predictable for $n \in \mathbb{N}$. By Proposition 5.9,

$$\begin{aligned}
\|v_1 - v_1^n\|_{q,T} &\leq \sup_{|s-t|<\frac{T}{n}} \sup_{|x-y|<\frac{1}{n}} \|v_1(s,x) - v_1(t,y)\|_{L^q(\Omega)} \\
&\leq \sup_{|s-t|<\frac{T}{n}} \sup_{x\in[0,1]} \|v_1(s,x) - v_1(t,x)\|_{L^q(\Omega)} \\
&\quad + \sup_{t\in[0,T]} \sup_{|x-y|<\frac{1}{n}} \|v_1(t,x) - v_1(t,y)\|_{L^q(\Omega)} \\
&\leq \left(C_{11}\left(1 + \|v_0\|_{q,T}^q\right)\right)^{\frac{1}{q}} \left(\left(\frac{T}{n}\right)^{\frac{1}{d_H + 1 + \frac{\log \nu_{\min}}{\log r_{\max}}}} + \left(\frac{1}{n}\right)^{\frac{1}{2}} \right) \to 0, \; n \to \infty.
\end{aligned}$$

Hence, v_1 is predictable. The predictability of v_2 and v_3 follows from the fact that they are measurable and deterministic. □

Theorem 5.11: *Assume Condition 5.2 with $q \geq 2$. Then the SPDE (71) has a unique mild solution in $\mathcal{S}_{q,T}$.*

Proof. Uniqueness: Let $u, \widetilde{u} \in \mathcal{S}_{q,T}$ be mild solutions to (71). Then $v := u - \widetilde{u} \in \mathcal{S}_{2,T}$. Setting $G(t) := \sup_{x\in[0,1]} \mathbb{E}\left[v^2(t,x)\right]$, $t \in [0,T]$ and using Walsh's isometry yields for

$(t,x) \in [0,T] \times [0,1]$

$$
\begin{aligned}
\mathbb{E}\left[v(t,x)^2\right] &= \mathbb{E}\left[\left(\int_0^t \int_0^1 P^b_{t-s}(x,y)\left(f(s,u(s,y)) - f\left(s,\widetilde{u}(s,y)\right)\right)\xi(s,y)d\mu(y)ds\right)^2\right] \\
&= \mathbb{E}\left[\int_0^t \int_0^1 P^b_{t-s}(x,y)^2\left(f(s,u(s,y)) - f\left(s,\widetilde{u}(s,y)\right)\right)^2 d\mu(y)ds\right] \\
&\leq L^2\mathbb{E}\left[\int_0^t \int_0^1 P^b_{t-s}(x,y)^2 v(s,y)^2 d\mu(y)ds\right] \\
&\leq L^2\left(\int_0^t \sup_{z\in[0,1]} \mathbb{E}\left[v^2(s,z)\right]\int_0^1 P^b_{t-s}(x,y)^2 d\mu(y)ds\right) \\
&\leq L^2 T \sup_{t\in[0,T]} \left\|P^b_t(x,\cdot)\right\|^2_\mu \int_0^t \sup_{z\in[0,1]} \mathbb{E}\left[v^2(s,z)\right]ds \\
&\leq L^2 T \sup_{t\in[0,T]} \left\|P^b_t(x,\cdot)\right\|^2_\mu \int_0^t G(s)ds
\end{aligned}
$$

and thus

$$
G(t) \leq L^2 T \sup_{t\in[0,T]} \left\|P^b_t(x,\cdot)\right\|^2_\mu \int_0^t G(s)ds.
$$

Since G is continuous on $[0,T]$ (by Proposition 5.9 with $v_0 := v$), we can apply Gronwall's lemma to derive $G(s) = 0$ for $s \in [0,T]$. Therefore, $u(t,x) = \widetilde{u}(t,x)$ almost surely for every $(t,x) \in [0,T] \times [0,1]$.

Existence: We follow the methods in the proof of [41, Theorem 7.5] and use Picard iteration to find a solution. For that, let $u_2 := 0 \in \mathcal{S}_{q,T}$ and for $n \geq 2$, $(t,x) \in [0,T] \times [0,1]$,

$$
\begin{aligned}
u_{n+1}(t,x) &:= C_t u_0(x) + S_t u_1(x) \\
&\qquad + \int_0^t \int_0^1 P^b_{t-s}(x,y) f(s,u_n(s,y))\xi(s,y)d\mu(y)ds.
\end{aligned}
\tag{82}
$$

Proposition 5.8 and Proposition 5.9 imply that $u_n \in \mathcal{S}_{q,T}$ for each $n \geq 3$. We prove that $(u_n)_{n\geq 2}$ is a Cauchy sequence in $\mathcal{S}_{q,T}$. Let $n \geq 2$, $(t,x) \in [0,T] \times [0,1]$ and let $w_n := u_{n+1} - u_n \in \mathcal{S}_{q,T}$. Using the Burkholder-Davis-Gundy inequality, the Lipschitz property of f as well as Minkowski's integral inequality leads to

$$\begin{aligned}
&\mathbb{E}\left[|w_{n+1}(t,x)|^q\right] \\
&= \mathbb{E}\left[\left|\int_0^t \int_0^1 P^b_{t-s}(x,y)\left(f(s,u_{n+1}(s,y)) - f\left(s,u_n(s,y)\right)\right)\xi(s,y)d\mu(y)ds\right|^q\right] \\
&\le C(q)\mathbb{E}\left[\left|\int_0^t \int_0^1 P^b_{t-s}(x,y)^2\left(f(s,u_{n+1}(s,y)) - f\left(s,u_n(s,y)\right)\right)^2 d\mu(y)ds\right|^{\frac{q}{2}}\right] \\
&\le C(q)L^q\mathbb{E}\left[\left|\int_0^t \int_0^1 P^b_{t-s}(x,y)^2 w_n^2(s,y)d\mu(y)ds\right|^{\frac{q}{2}}\right] \\
&\le C(q)L^q\left(\int_0^t \int_0^1 P^b_{t-s}(x,y)^2\left(\mathbb{E}\left[|w_n(s,y)|^q\right]\right)^{\frac{2}{q}} d\mu(y)ds\right)^{\frac{q}{2}} \\
&\le C(q)L^q \sup_{s\in[0,T]}\left\|P^b_s(x,\cdot)\right\|^q_\mu\left(\int_0^t \sup_{y\in[0,1]}\left(\mathbb{E}\left[|w_n(s,y)|^q\right]\right)^{\frac{2}{q}} ds\right)^{\frac{q}{2}}.
\end{aligned}$$

Let $n \ge 2$ and $H_n(t) := \sup_{x\in[0,1]}\left(\mathbb{E}\left[|w_n(t,x)|^q\right]\right)^{\frac{2}{q}}$. Then, there is a constant κ_n such that $|H_n(t)| \le \kappa_n$ for all $t \in [0,T]$. By Lemma 5.6, for $(t,x) \in [0,T]\times[0,1]$,

$$\left(\mathbb{E}\left[|w_{n+1}(t,x)|^q\right]\right)^{\frac{2}{q}} \le C(q)^{\frac{2}{q}} L^2 C_9^2 \sup_{t\in[0,T]}\int_0^t H_n(s)ds$$

and thus

$$H_{n+1}(t) \le C(q)^{\frac{2}{q}} L^2 C_9^2 \sup_{t\in[0,T]}\int_0^t H_n(s)ds.$$

With $\kappa := C(q)^{\frac{2}{q}} L^2 C_9^2$, we see that $H_3(t) \le \kappa\kappa_2 t$ and deduce inductively

$$H_{n+2}(t) \le \kappa_2\frac{(\kappa t)^n}{n!}, \quad n \ge 1.$$

The series $\sum_{n\ge2} H_n^{\frac{1}{2}}(t)$ is uniformly convergent on $[0,T]$, which can be verified by the ratio test using that $\sqrt{\frac{H_{n+1}(t)}{H_n(t)}} \le \sqrt{\frac{\kappa t}{n+1}}$ for $n \ge 2$. We conclude

$$\sup_{t\in[0,T]} H_n^{\frac{1}{2}}(t) \to 0, \ n \to \infty,$$

which implies the same for $\|w_n\|_{q,T}$. Hence, $(u_n)_{n\ge2}$ is a Cauchy sequence in $\mathcal{S}_{q,T}$ and we denote the limit by u. To check that u satisfies (72), let $(t,x) \in [0,T]\times[0,1]$ be fixed and take the limit in $L^q(\Omega)$ for $n\to\infty$ on both sides of (82). We get $u(t,x)$

on the left-hand side. For the right-hand side, note that

$$\mathbb{E}\left[\left|\int_0^t \int_0^1 P^b_{t-s}(x,y)\left(f(s,u(s,y)) - f\left(s,u_n(s,y)\right)\right)\xi(s,y)d\mu(y)ds\right|^q\right]$$
$$\leq C(q)L^q\left(\int_0^t \int_0^1 P^b_{t-s}(x,y)^2\left(\mathbb{E}\left[|u(s,y)-u_n(s,y)|^q\right]\right)^{\frac{2}{q}} d\mu(y)ds\right)^{\frac{q}{2}},$$

which goes to zero as n tends to infinity with the same argumentation as before. □

Propositions 5.8 and 5.9 provide different temporal Hölder exponents, so they need to be compared. Recall that $\delta = \max_{i=1,\dots,N} \frac{\log \mu_i}{\log((\mu_i r_i)^\gamma)}$ and $\nu_{\min} = \min_{i=1,\dots,N} \frac{\mu_i}{r_i^{d_H}}$.

Lemma 5.12: *We have*

$$\left(d_H + 1 + \frac{\log \nu_{\min}}{\log r_{\max}}\right)^{-1} \leq 2 - (2+\delta)\gamma.$$

Proof. We calculate

$$\begin{aligned}
&\frac{\min_{i=1,\dots,N} \log \mu_i - \log r_i^{d_H}}{\max_{i=1,\dots,N} \log r_i} - (1-d_H) + 2 \\
&\geq \max_{i=1,\dots,N} \frac{\log \mu_i - \log r_i^{d_H}}{\log r_i} - (1-d_H) + 2 \\
&= \max_{i=1,\dots,N} \frac{\log \mu_i - \log r_i + (1-d_H)\log r_i}{\log r_i} - (1-d_H) + 2 \\
&= \max_{i=1,\dots,N} \frac{\log \mu_i - \log r_i}{\log r_i} + 2.
\end{aligned}$$

It follows

$$\begin{aligned}
\left(d_H + 1 + \frac{\log \nu_{\min}}{\log r_{\max}}\right)^{-1} &\leq \left(\max_{i=1,\dots,N} \frac{\log \mu_i - \log r_i}{\log r_i} + 2\right)^{-1} \\
&= \min_{i=1,\dots,N} \left(\frac{\log \mu_i - \log r_i}{\log r_i} + 2\right)^{-1} \\
&= \min_{i=1,\dots,N} \frac{\log r_i}{\log \mu_i + \log r_i} \\
&= \min_{i=1,\dots,N} \left(1 - \frac{\log \mu_i}{\log \mu_i + \log r_i}\right) \\
&= 1 - \max_{i=1,\dots,N} \frac{\log \mu_i}{\log \mu_i + \log r_i} \\
&= (1 - \gamma\delta)
\end{aligned}$$

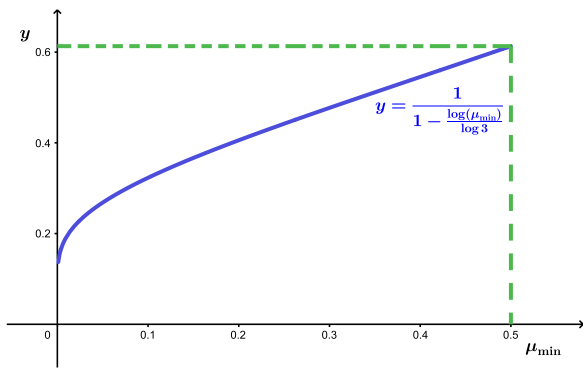

Figure 7: Temporal Hölder exponent graph for the stochastic wave equation

and since $\gamma \leq \frac{1}{2}$

$$\left(d_H + 1 + \frac{\log \nu_{\min}}{\log r_{\max}}\right)^{-1} \leq (1 - \gamma\delta) \leq 2 - 2\gamma - \gamma\delta. \qquad \square$$

Using this lemma and the established Hölder continuity properties, the main result of this chapter is a direct consequence of Kolmogorov's continuity theorem.

Theorem 5.13: *Assume Condition 5.2 with $q \geq 2$. Then, the mild solution u to the SPDE (71) has a version $\widetilde{u}$ such that the following holds:*

(i) If $q > 2$ and $t \in [0, T]$, $\widetilde{u}(t, \cdot)$ is ess. $\frac{1}{2} - \frac{1}{q}$-Hölder continuous on $[0, 1]$.

(ii) If $q > 2 \vee \left(d_H + 1 + \frac{\log \nu_{\min}}{\log r_{\max}}\right)$ and $x \in [0, 1]$, $\widetilde{u}(\cdot, x)$ is ess. $\left(d_H + 1 + \frac{\log \nu_{\min}}{\log r_{\max}}\right)^{-1}$ $-\frac{1}{q}$-Hölder continuous on $[0, T]$.

(iii) If $q > 4 \vee \left(2\left(d_H + 1 + \frac{\log \nu_{\min}}{\log r_{\max}}\right)\right)$, $\widetilde{u}$ is ess. $\left(\left(d_H + 1 + \frac{\log \nu_{\min}}{\log r_{\max}}\right)^{-1} \wedge \frac{1}{2}\right) - \frac{2}{q}$-Hölder continuous on $[0, T] \times [0, 1]$.

Proof. Using Propositions 5.8 and 5.9 and Lemma 5.12, this can be proven in the same way as Theorem 4.16. $\square$

Example 5.14: If μ is not the natural measure on K, then $\nu_{\min} \neq 0$. As an example, consider the classical Cantor set given by the IFS consisting of $S_1(x) = \frac{x}{3}$, $S_2(x) = \frac{2}{3} + \frac{x}{3}$, $x \in [0, 1]$ with weights $\mu_1, \mu_2 \in (0, 1)$. If u_0, u_1 and f satisfy Assumption 5.2 and f is uniformly bounded, q can be chosen arbitrarily large. We obtain

$$\left(d_H + 1 + \frac{\log \nu_{\min}}{\log r_{\max}}\right)^{-1} = \left(\frac{\log 2}{\log 3} + 1 - \frac{\log \mu_{\min} + \log 2}{\log 3}\right)^{-1} = \left(1 - \frac{\log \mu_{\min}}{\log 3}\right)^{-1}$$

for the essential temporal Hölder exponent. Figure 7 shows the corresponding graph.

5.4 Weak intermittency

Let $b \in \{N, D\}$. We consider

$$u(t,x) = C_t^b u_0(x) + \int_0^t \int_0^1 P_{t-s}^b(x,y) f(s,u(s,y))\xi(s,y)d\mu(y)ds \tag{83}$$

for $(t,x) \in [0,\infty) \times [0,1]$. Recall that the upper moment Lyapounov exponents are defined by

$$\bar{\gamma}(p,x) = \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[|u(t,x)|^p\right], \quad p \in (0,\infty),\ x \in [0,1].$$

Let $\varepsilon \geq 0$. According to [13], we call u weakly intermittent on $[\varepsilon, 1-\varepsilon]$ if

$$\bar{\gamma}(2,x) > 0, \quad \bar{\gamma}(p,x) < \infty, \quad p \in [2,\infty),\ x \in [\varepsilon, 1-\varepsilon].$$

Henceforth, we make additional assumptions.

Assumption 5.15: *We assume that Condition 5.2(i) holds. Furthermore, we assume that f is predictable and fulfills the following Lipschitz and linear growth condition: There exists a constant $L > 0$ such that for all $(w,t,x,y) \in \Omega \times [0,T] \times \mathbb{R}^2$*

$$\begin{aligned} |f(\omega,t,x) - f(\omega,t,y)| &\leq L|x-y|, \\ |f(\omega,t,x)| &\leq L(1+|x|). \end{aligned}$$

First, we establish an upper bound.

Proposition 5.16: *There exist constants $C_{12}, C_{13} > 0$ such that for all $p \in [1,q]$, $(t,x) \in [0,\infty) \times [0,1]$*

$$\mathbb{E}\left[|u(t,x)|^p\right] \leq C_{12} e^{C_{13}p^2 t}.$$

Proof. v_3 is uniformly bounded on $[0,\infty) \times [0,1]$. This can be verified by the same methods as in the proof of Proposition 5.8. For example, for $b = D$ and $(t,x) \in [0,\infty) \times [0,1]$,

$$C_t^D u_0(x) = \left| \sum_{k=1}^{\infty} \cos\left(\sqrt{\lambda_k^D} t\right) \varphi_k^D(x) u_{0,k}^D \right| \leq \sum_{k=1}^{\infty} c_2 C_6 k^{\frac{\delta}{2}} k^{-\frac{1}{\gamma}} = c_2 C_6 \sum_{k=1}^{\infty} k^{\frac{\delta}{2} - \frac{1}{\gamma}},$$

where the sum is finite due to $\frac{\delta}{2} - \frac{1}{\gamma} < 0$ (see (78)). Hence, there is a constant $K > 0$ such that $\sup_{(t,x)\in[0,\infty)\times[0,1]} |v_3(t,x)| = K$. Now, let $p \in [2,q]$. By the Burkholder-

Davis-Gundy inequality and Minkowski's integral inequality, for $(t, x) \in [0, \infty) \times [0, 1]$

$$
\begin{aligned}
& e^{-\alpha t} \left(\mathbb{E}\left[|u(t,x)|^p\right]\right)^{\frac{1}{p}} \\
& \leq e^{-\alpha t} K + \left(\mathbb{E}\left[\left|\int_0^t \int_0^1 e^{-\alpha t} P^b_{t-s}(x,y) f(s,u(s,y))\xi(s,y) d\mu(y) ds\right|^p\right]\right)^{\frac{1}{p}} \\
& \leq e^{-\alpha t} K + 2\sqrt{p} \left(\int_0^t \int_0^1 e^{-2\alpha t} P^b_{t-s}(x,y)^2 \left(\mathbb{E}\left[|f(s,u(s,y))|^p\right]\right)^{\frac{2}{p}} d\mu(y) ds\right)^{\frac{1}{2}} \\
& \leq e^{-\alpha t} K + L2\sqrt{p} \left(\int_0^t \int_0^1 e^{-2\alpha t} P^b_{t-s}(x,y)^2 \sup_{z\in[0,1]} \left(1 + \left(\mathbb{E}\left[|u(s,z))|^p\right]\right)^{\frac{1}{p}}\right)^2 d\mu(y) ds\right)^{\frac{1}{2}}
\end{aligned}
$$

and further, by Lemma 5.6,

$$
\begin{aligned}
& \int_0^t \int_0^1 e^{-2\alpha(t-s)} P^b_{t-s}(x,y)^2 \sup_{z\in[0,1]} \left(e^{-\alpha s} + e^{-\alpha s} \left(\mathbb{E}\left[|u(s,z))|^p\right]\right)^{\frac{1}{p}}\right)^2 d\mu(y) ds \\
& \leq \sup_{(s,z)\in[0,T]\times[0,1]} \left(e^{-\alpha s} + e^{-\alpha s} \left(\mathbb{E}\left[|u(s,z))|^p\right]\right)^{\frac{1}{p}}\right)^2 \int_0^t \int_0^1 e^{-2\alpha(t-s)} P^b_{t-s}(x,y)^2 d\mu(y) ds \\
& \leq \left(1 + \sup_{(s,z)\in[0,T]\times[0,1]} e^{-\alpha s} \left(\mathbb{E}\left[|u(s,z))|^p\right]\right)^{\frac{1}{p}}\right)^2 C_9^2 \int_0^t e^{-2\alpha(t-s)} ds \\
& \leq \frac{C_9^2}{2\alpha} \left(1 + \sup_{(s,z)\in[0,T]\times[0,1]} e^{-\alpha s} \left(\mathbb{E}\left[|u(s,z))|^p\right]\right)^{\frac{1}{p}}\right)^2 .
\end{aligned}
$$

Let $\alpha := 8C_9^2 L^2 p$. Then, $\frac{L2\sqrt{p}C_9}{\sqrt{2\alpha}} = \frac{1}{2}$ and thus

$$
\left(\mathbb{E}\left[|u(t,x)|^p\right]\right)^{\frac{1}{p}} \leq 2K + e^{\alpha t} = 2K + e^{8C_9^2 L^2 pt}.
$$

It remains to check the case $p \in [1, 2)$. For $(t, x) \in [0, \infty) \times [0, 1]$ and $p \in [1, 2)$, we have

$$
\left(\mathbb{E}\left[|u(t,x)|^p\right]\right)^{\frac{1}{p}} \leq \left(\mathbb{E}\left[|u(t,x)|^2\right]\right)^{\frac{1}{2}} \leq 2K + e^{16C_9^2 L^2 t} \leq 2K + e^{16C_9^2 L^2 pt}. \qquad \square
$$

We immediately obtain for $p \geq 1$

$$
\bar{\gamma}(p) = \limsup_{t\to\infty} \frac{1}{t} \sup_{x\in[0,1]} \log \mathbb{E}\left[|u(t,x)|^p\right] \leq \limsup_{t\to\infty} \frac{\log C_{12}}{t} + C_{13} p^2 = C_{13} p^2.
$$

For the lower bound, we deal with

$$u(t,x) \coloneqq C_t^b u_0(x) + \int_0^t \int_0^1 P_{t-s}^b(x,y) f(u(s,y)) \xi(s,y) d\mu(y) ds \tag{84}$$

for $(t,x) \in [0,\infty) \times [0,1]$. That is, we let $u_1 = 0$ and f be not time-dependent in (71). We assume the following conditions.

Assumption 5.17: *Let Condition 5.2(i) hold. Furthermore, let $f : \mathbb{R} \to \mathbb{R}$ satisfy the following Lipschitz and linear growth conditions: There exists $L > 0$ such that for all $(x,y) \in \mathbb{R}^2$*

$$|f(x) - f(y)| \le L|x-y|,$$
$$|f(x)| \le L(1 + |x|).$$

Proposition 5.18: *Assume $L_f \coloneqq \inf_{x \in \mathbb{R}\setminus\{0\}} \left| \frac{f(x)}{x} \right| > 0$.*

(i) Let $b = N$ and $\inf_{x\in[0,1]} u_0(x) > 0$. Then, there exists a constant $\kappa > 0$ such that $\bar{\gamma}(2,x) \ge \kappa$ for all $x \in [0,1]$.

(ii) Let $b = D$, $\varepsilon > 0$ and $\inf_{x\in[\varepsilon,1-\varepsilon]} u_0(x) > 0$. Then, there exists a constant $\kappa_\varepsilon > 0$ such that $\bar{\gamma}(2,x) \ge \kappa_\varepsilon$ for all $x \in [\varepsilon, 1-\varepsilon]$.

Proof. Let $\varepsilon \ge 0$, $\inf_{x\in[\varepsilon,1-\varepsilon]} u_0(x) > 0$, $x \in [\varepsilon, 1-\varepsilon]$. It suffices to find a constant $\beta_\epsilon > 0$ such that

$$\int_0^\infty e^{-\beta t} \mathbb{E}\left[u(t,x)^2\right] dt = \infty \quad \text{for all} \quad \beta \le \beta_\epsilon \tag{85}$$

(compare the proof of [13, Theorem 3.2]). Using Walsh's isometry and the zero-mean property of the stochastic integral yields for $t \in [0,\infty)$

$$\begin{aligned}\mathbb{E}\left[u(t,x)^2\right] dt &= v_3(t,x)^2 + \int_0^t \int_0^1 P_{t-s}^b(x,y)^2 \mathbb{E}\left[f(u(s,y))^2\right] d\mu(y) ds \\ &\quad + v_3(t,x) \mathbb{E}\left[\int_0^t \int_0^1 P_{t-s}^b(x,y) f(u(s,y)) \xi(s,y) d\mu(y) ds\right] \\ &= v_3(t,x)^2 + \int_0^t \int_0^1 P_{t-s}^b(x,y)^2 \mathbb{E}\left[f(u(s,y))^2\right] d\mu(y) ds\end{aligned}$$

and thus, by Laplace transformation for $\beta > 0$,

$$\int_0^\infty e^{-\beta t}\mathbb{E}\left[u(t,x)^2\right]dt = \int_0^\infty e^{-\beta t} v_3(t,x)^2 dt \\ + \int_0^\infty e^{-\beta t}\int_0^t\int_0^1 P^b_{t-s}(x,y)^2\mathbb{E}\left[f(u(s,y))^2\right]d\mu(y)dsdt.$$

In order to bound the first term on the right-hand side from below, note that $v_3(0,x) = u_0(x) \geq \inf_{y\in[\varepsilon,1-\varepsilon]} u_0(y) > 0$. By Proposition 5.8, v_3 is Hölder continuous in t uniformly for all $x \in [0,1]$. We thus obtain the existence of a constant $t_0 > 0$ such that $v_3(t,x) > \frac{u_0}{2}$, $(t,x) \in [0,t_0]\times[\varepsilon,1-\varepsilon]$. Let $K_\beta := \frac{u_0^2}{16\beta}$. Then, for $x \in [\varepsilon, 1-\varepsilon]$

$$\int_0^\infty e^{-\beta t}\mathbb{E}\left[u(t,x)^2\right]dt \\ \geq K_\beta + L_f^2\int_0^\infty e^{-\beta t}\int_0^t\int_0^1 P^b_{t-s}(x,y)^2\mathbb{E}\left[u(s,y)^2\right]d\mu(y)dsdt$$

and for $(t,y) \in [0,\infty)\times[\varepsilon,1-\varepsilon]$ with $P^b(x,y) : [0,\infty) \to \mathbb{R},\ t \mapsto P^b_t(x,y)$,

$$\int_0^t P^b_{t-s}(x,y)^2\mathbb{E}\left[(u(s,y)^2\right]ds = \left(P^b(x,y) * \mathbb{E}\left[u(\cdot,y)^2\right]\right)(t),$$

where $*$ denotes the time convolution. It holds $\mathcal{L}_\beta(f*g) = \mathcal{L}_\beta f \cdot \mathcal{L}_\beta g$, where $\mathcal{L}$ denotes the Laplace transformation. Hence,

$$\int_0^\infty e^{-\beta t}\int_0^t\int_0^1 P^b_{t-s}(x,y)^2\mathbb{E}\left[u(s,y)^2\right]d\mu(y)dsdt \\ = \int_0^1\int_0^\infty e^{-\beta t}\int_0^t P^b_{t-s}(x,y)^2\mathbb{E}\left[u(s,y)^2\right]dsdtd\mu(y) \\ = \int_0^1\int_0^\infty e^{-\beta t}P^b_t(x,y)^2dt\int_0^\infty e^{-\beta s}\mathbb{E}\left[u(s,y)^2\right]dsd\mu(y).$$

Let $M_\beta(x) := \int_0^\infty e^{-\beta s}\mathbb{E}\left[u(s,x)^2\right]ds$. Then,

$$M_\beta(x) \geq K_\beta + L_f^2\int_0^1\int_0^\infty e^{-\beta t}P^b_t(x,y)^2 M_\beta(y)dtd\mu(y). \tag{86}$$

If $b = N$, we set $\varepsilon := 0$ and have for all $(t, x) \in [0, \infty) \times [0, 1]$

$$\begin{aligned}\left\|P_t^N(x,\cdot)\right\|_\mu^2 &= t^2 + \sum_{k\geq 2} \frac{\sin^2\left(\sqrt{\lambda_k^N} t\right)}{\lambda_k^N} \left(\varphi_k^N\right)^2(x) \\ &\geq t^2\end{aligned}$$

and thus

$$\begin{aligned}\int_0^\infty \int_0^1 e^{-\beta t} P_t^N(x,y)^2 K_\beta d\mu(y) dt &= K_\beta \int_0^\infty e^{-\beta t} \left\|P_t^N(x,\cdot)\right\|_\mu^2 dt \\ &\geq K_\beta \int_0^\infty e^{-\beta t} t^2 dt \\ &= 2K_\beta \beta^{-3}.\end{aligned}$$

By iterating this in (86), we obtain for all $x \in [0, 1]$

$$M_\beta(x) \geq K_\beta \sum_{n=0}^{\infty} \left(2L_f^2 \beta^{-3}\right)^n.$$

This sum diverges if and only if $\beta \leq \sqrt[3]{2L_f^2}$, which verifies (85).

Now, let $b = D$, $\varepsilon > 0$ and $c' := \inf_{x\in[\varepsilon,1-\varepsilon]} \varphi_1^D(x)^2$. As $\varphi_1^D(x) \neq 0$, $x \in (0, 1)$ (see [29, Proposition 2.5]), we have $c' > 0$. Then,

$$\begin{aligned}\int_0^\infty \int_0^1 e^{-\beta t} P_t^D(x,y)^2 K_\beta d\mu(y) dt &\geq K_\beta \int_0^\infty e^{-\beta t} \sum_{k=1}^{\infty} \frac{\sin^2\left(\sqrt{\lambda_k^D} t\right)}{\lambda_k^D} \left(\varphi_k^D\right)^2(x) dt \\ &\geq K_\beta \int_0^\infty e^{-\beta t} \frac{\sin^2\left(\sqrt{\lambda_1^D} t\right)}{\lambda_1^D} \left(\varphi_1^D\right)^2(x) dt \\ &\geq K_\beta \int_0^\infty e^{-\beta t} \frac{\sin^2\left(\sqrt{\lambda_1^D} t\right)}{\lambda_1^D} c' dt \\ &= \frac{K_\beta c'}{(\lambda_1^D)^{\frac{3}{2}}} \int_0^\infty e^{-\frac{\beta}{\sqrt{\lambda_1^D}} t} \sin^2(t) dt \\ &= \frac{K_\beta c'}{(\lambda_1^D)^{\frac{3}{2}}} \frac{2}{\left(\frac{\beta}{\sqrt{\lambda_1^D}}\right)^3 + 4\left(\frac{\beta}{\sqrt{\lambda_1^D}}\right)} > 0.\end{aligned}$$

By iterating this in (86) we obtain

$$M_\beta(x) \geq K_{\beta,\varepsilon} \sum_{n=0}^{\infty} \left(\frac{2c' \left(\lambda_1^D\right)^{-\frac{3}{2}} L_f^2}{\left(\frac{\beta}{\sqrt{\lambda_1^D}}\right)^3 + 4\left(\frac{\beta}{\sqrt{\lambda_1^D}}\right)} \right)^n .$$

Let $\bar{\beta} \coloneqq \frac{\beta}{\sqrt{\lambda_1^D}}$. The above sum is equal to ∞ for all β such that $\bar{\beta}^3 + 4\bar{\beta} \leq 2c' \left(\lambda_1^D\right)^{-\frac{3}{2}} L_f^2$. This verifies (85). $\square$

We directly obtain the main result of this section.

Corollary 5.19: *Let $L_f > 0$ and let u be given by (84).*

(i) Let $b = N$ and $\inf_{x\in[0,1]} u_0(x) > 0$. Then, u is weakly intermittent on $[0,1]$.

(ii) Let $b = D$, $\varepsilon > 0$, $\inf_{x\in[\varepsilon,1-\varepsilon]} u_0(x) > 0$. Then, u is weakly intermittent on $[\varepsilon, 1-\varepsilon]$.

A Some Technical Details

Let μ be an atomless Borel probability measure on $[0,1]$ such that $0,1 \in \operatorname{supp}(\mu)$ and let $K \coloneqq \operatorname{supp}(\mu) \neq [0,1]$. Recall that $[0,1] \setminus K = \bigcup_{i=1}^{\infty}(a_i,b_i)$ (see (18)),

$$\mathcal{D}^1 = \Big\{ f : [0,1] \to \mathbb{R} : \text{ there exists } f' \in L^2\left([0,1],\lambda^1\right) : \\ f(x) = f(0) + \int_0^x f'(y)dy,\ x \in [0,1] \Big\}$$

and that $\mathcal{F}$ is defined to be the space of all $L^2([0,1],\mu)$-equivalence classes possessing a $\mathcal{D}^1$-representative.

Lemma A.1: *Let $f \in \mathcal{F}$. Then, there exists a unique continuous representative g in the equivalence class of f such that g is for $i \geq 1$ linear on $[a_i,b_i]$ and $g \in \mathcal{D}^1$.*

Proof. The uniqueness is clear. For the proof of existence, let $\bar{f}$ be a $\mathcal{D}^1$-representative of f. We define a function $h : [0,1] \to \mathbb{R}$ by

$$h(x) \coloneqq \begin{cases} \bar{f}'(x) & \text{if } x \in K, \\ \frac{\bar{f}(b_i)-\bar{f}(a_i)}{b_i-a_i} & \text{if } x \in (a_i,b_i),\ i \geq 1 \end{cases}$$

and $g : [0,1] \to \mathbb{R}$ by

$$g(x) \coloneqq \bar{f}(0) + \int_0^x h(y)dy,\ x \in [0,1].$$

For $i \geq 1$, g is obviously linear on $[a_i,b_i]$. Further, let $x \in K$ and $J_x \coloneqq \{i \geq 1 : b_i \leq x\}$. Then, for $x \in K$,

$$\begin{aligned} \int_0^x h(y)dy &= \int_{K\cap[0,x]} h(y)dy + \sum_{i\in J_x} \int_{a_i}^{b_i} h(y)dy \\ &= \int_{K\cap[0,x]} \bar{f}'(y)dy + \sum_{i\in J_x} \bar{f}(b_i) - \bar{f}(a_i) \\ &= \int_{K\cap[0,x]} \bar{f}'(y)dy + \sum_{i\in J_x} \int_{a_i}^{b_i} \bar{f}'(y)dy \\ &= \int_0^x \bar{f}'(y)dy. \end{aligned}$$

For $x \in K$, it follows that $g(x) = \bar{f}(x)$. It remains to show that $h \in L^2\left([0,1],\lambda^1\right)$.

By the Cauchy-Schwarz inequality, we have for $a, b \in [0,1]$

$$|\bar{f}(b) - \bar{f}(a)|^2 \leq |b-a| \int_a^b \left(\bar{f}')(x)\right)^2 dx$$

and thus, for $i \geq 1$,

$$\int_{a_i}^{b_i} h^2(x)dx = \frac{(\bar{f}(b_i) - \bar{f}(a_i))^2}{(b_i - a_i)} \leq \int_{a_i}^{b_i} \left(\bar{f}'(x)\right)^2 dx.$$

We conclude

$$\begin{aligned}
\int_0^1 h^2(x)dx &= \sum_{i\in\mathbb{N}} \int_{a_i}^{b_i} h^2(x)dx + \int_K h^2(x)dx \\
&\leq \sum_{i\in\mathbb{N}} \int_{a_i}^{b_i} \left(\bar{f}'(x)\right)^2 dx + \int_K \left(\bar{f}'(x)\right)^2 dx \\
&= \int_0^1 \left(\bar{f}'(x)\right)^2 dx,
\end{aligned}$$

which is finite due to $\bar{f} \in \mathcal{D}^1$. $\square$

Lemma A.2: *Let $b \in \{N, D\}$, $\psi : L^2([0,1], \mu) \to L^2(K, \mu)$, $u \mapsto u|_K$ and*

$$\widetilde{\Delta}_\mu^b(u) := \psi \Delta_\mu^b \psi^{-1} u, \quad \mathcal{D}\left(\widetilde{\Delta}_\mu^b\right) := \psi\left(\mathcal{D}\left(\Delta_\mu^b\right)\right).$$

Then,

(i) $\left(\widetilde{\Delta}_\mu^b, \mathcal{D}\left(\widetilde{\Delta}_\mu^b\right)\right)$ is self-adjoint and dissipative. In particular, $\widetilde{\Delta}_\mu^b$ is the generator of a unique strongly continuous semigroup $\left(\widetilde{T}_t^b\right)_{t\geq 0}$. Further, u is an eigenfunction of $\widetilde{\Delta}_\mu^b$ for the eigenvalue λ if and only if $\psi^{-1}u$ is an eigenfunction of Δ_μ^b for the eigenvalue λ.

(ii) $\widetilde{\mathcal{E}}\left(\widetilde{u}, \widetilde{v}\right) := \mathcal{E}(\psi^{-1}\widetilde{u}, \psi^{-1}\widetilde{v})$, $\widetilde{u}, \widetilde{v} \in \widetilde{\mathcal{F}} := \psi(\mathcal{F})$ defines a Dirichlet form, which is associated to $\widetilde{\Delta}_\mu^N$ and $\widetilde{\mathcal{E}}\left(\widetilde{u}, \widetilde{v}\right), \widetilde{u}, \widetilde{v} \in \widetilde{\mathcal{F}_0} := \psi(\mathcal{F}_0)$ defines a Dirichlet form associated to $\widetilde{\Delta}_\mu^D$.

Proof. (i) First, we show that $\widetilde{\Delta}_\mu^b$ is self-adjoint. We denote the inner product on $L^2(K, \mu)$ also by $\langle \cdot, \cdot \rangle_\mu$. Let $\widetilde{u} \in \mathcal{D}\left(\widetilde{\Delta}_\mu^b\right)$ and $u := \psi^{-1}\widetilde{u}$. $\mathcal{D}\left(\Delta_\mu^b\right)$ is dense in $L^2([0,1], \mu)$. Therefore, for any $u \in L^2([0,1], \mu)$, there is a sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n \in \mathcal{D}\left(\Delta_\mu^b\right)$, $n \in \mathbb{N}$ such that $\|u_n - u\|_\mu \to 0$ for $n \to \infty$. Because of

$\|u_n - u\|_\mu = \|\psi u_n - \widetilde{u}\|_\mu$ for all $n \in \mathbb{N}$ and $\psi u_n \in \mathcal{D}\left(\widetilde{\Delta}_\mu^b\right)$, $\mathcal{D}\left(\widetilde{\Delta}_\mu^b\right)$ is dense in $L^2(K,\mu)$.

Now, let $\widetilde{u}, \widetilde{v} \in \mathcal{D}\left(\widetilde{\Delta}_\mu^b\right)$ and $u := \psi^{-1}\widetilde{u},\ v := \psi^{-1}\widetilde{v}$. It is straight forward to check that $v \mapsto \left\langle u, \Delta_\mu^b v\right\rangle_\mu$ is a linear continuous mapping on $\mathcal{D}\left(\Delta_\mu^b\right)$ if and only if $\widetilde{v} \mapsto \left\langle \widetilde{u}, \widetilde{\Delta}_\mu^b \widetilde{v}\right\rangle_\mu$ is linear and continuous on $\mathcal{D}\left(\widetilde{\Delta}_\mu^b\right)$, which yields $\mathcal{D}\left(\widetilde{\Delta}_\mu^b\right) = \mathcal{D}\left(\left(\widetilde{\Delta}_\mu^b\right)^*\right)$. Further, for all $\widetilde{u}, \widetilde{v} \in \mathcal{D}\left(\widetilde{\Delta}_\mu^b\right)$

$$\begin{aligned}
\left\langle \widetilde{\Delta}_\mu^b \widetilde{u}, \widetilde{v}\right\rangle_\mu &= \left\langle \psi \Delta_\mu^b \psi^{-1} \psi u, \psi v\right\rangle_\mu \\
&= \left\langle \psi \Delta_\mu^b u, \psi v\right\rangle_\mu \\
&= \left\langle \Delta_\mu^b u, v\right\rangle_\mu \\
&= \left\langle u, \Delta_\mu^b v\right\rangle_\mu \\
&= \left\langle \psi u, \psi \Delta_\mu^b \psi^{-1} \psi v\right\rangle_\mu \\
&= \left\langle \widetilde{u}, \widetilde{\Delta}_\mu^b \widetilde{v}\right\rangle_\mu .
\end{aligned}$$

The self-adjointness of Δ_μ^b follows. The dissipativity of $\widetilde{\Delta}_\mu^b$ implies the dissipativity of Δ_μ^b since

$$\left\langle \widetilde{\Delta}_\mu^b \widetilde{u}, \widetilde{u}\right\rangle_\mu = \left\langle \Delta_\mu^b u, u\right\rangle_\mu \leq 0.$$

The self-adjointness along with the dissipativity implies that $\widetilde{\Delta}_\mu^b$ generates a strongly continuous semigroup $\left(\widetilde{T}_t^b\right)_{t\geq 0}$ (see [48, Theorem B.2.2]). It remains to verify the statement about eigenvalues and eigenfunctions of $\widetilde{\Delta}_\mu^b$. For that, let $\lambda < 0$, $u \in \mathcal{D}\left(\Delta_\mu^b\right)$. The bijectivity of ψ implies that $\left(\Delta_\mu^b - \lambda\right) u = 0$ if and only if $\psi\left(\Delta_\mu^b - \lambda\right) u = 0$. The results about eigenvalues and eigenfunctions follow.

(ii) Again, let $\widetilde{u}, \widetilde{v} \in \mathcal{D}\left(\widetilde{\Delta}_\mu^N\right)$ and $u = \psi^{-1}\widetilde{u},\ v = \psi^{-1}\widetilde{v}$. The density of $\widetilde{\mathcal{F}}$ in $L^2(K,\mu)$ can be checked exactly like the density of $\mathcal{D}\left(\widetilde{\Delta}_\mu^N\right)$ in $L^2([0,1],\mu)$. Further, it is obvious that $\widetilde{\mathcal{E}}$ defines a positive definite, symmetric bilinear form. We verify that, with $\alpha > 0$ and $\widetilde{\mathcal{E}}_\alpha(\widetilde{u}, \widetilde{v}) := \widetilde{\mathcal{E}}(\widetilde{u}, \widetilde{v}) + \alpha \langle \widetilde{u}, \widetilde{v}\rangle_\mu$, $\left(\widetilde{F}, \widetilde{\mathcal{E}}_\alpha\right)$ is a Hilbert space. Note that $\widetilde{\mathcal{E}}_\alpha(\widetilde{u}, \widetilde{v}) = \mathcal{E}_\alpha(u, v)$, which implies that $\widetilde{\mathcal{E}}_\alpha$ defines an inner product. Now, let $(\widetilde{u}_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\widetilde{\mathcal{F}}$. Then, $u_n = \psi^{-1}\widetilde{u}_n,\ n \in \mathbb{N}$ is a Cauchy sequence in $\mathcal{F}$ with limit, say u. Since

$\|\widetilde{u}_n - \psi u\|_\mu = \|u_n - u\|_\mu$ for all n, ψu is the limit of $(\widetilde{u}_n)_{n\in\mathbb{N}}$ in $\widetilde{\mathcal{F}}$. For the Markov property, we calculate

$$\widetilde{\mathcal{E}}\left(0 \vee \widetilde{u} \wedge 1\right) = \mathcal{E}\left(0 \vee u \wedge 1\right) \leq \mathcal{E}(u) = \widetilde{\mathcal{E}}\left(\widetilde{u}\right).$$

To verify that $\widetilde{\Delta}_\mu^N$ is associated to $\widetilde{\mathcal{E}}$, we apply the correspondence between Δ_μ^N and $\mathcal{E}$ to get

$$-\left\langle \widetilde{\Delta}_\mu^N \widetilde{u}, \widetilde{v} \right\rangle_\mu = -\left\langle \Delta_\mu^N u, v \right\rangle_\mu = \mathcal{E}(u,v) = \widetilde{\mathcal{E}}\left(\widetilde{u}, \widetilde{v}\right).$$

The case $b = D$ works similarly.

$\square$

B Directions for Further Research

Remark B.1: Consider the heat equation (9) with initial value given by the Delta distribution $\delta_y : g \mapsto g(y)$ for $y \in \operatorname{supp}(\mu)$. Then, the heat kernel

$$p_t^b(x,y) = \sum_{k\geq 1} e^{-\lambda_k^b t} \varphi_k^b(x)\varphi_k^b(y),\ \ (t,x) \in [0,\infty) \times [0,1]$$

solves the equation in the distributional sense. The heat kernel is of particular importance in the context of the associated Markov process (compare the remark below) and stochastic partial differential equations (compare Section 4). It is an open question whether weak measure convergence implies convergence of the corresponding heat kernels in an appropriate sense.

Remark B.2: The operator Δ_μ^b on $L^2([0,1],\mu)$ is the infinitesimal generator of a Markov process, called a quasi-diffusion (compare e.g. [47, 53–55]). Convergence of semigroups raises the question whether the associated Markov processes also converge weakly. If $\mu_n \rightharpoonup \mu$, our results imply that for each $f \in (C[0,1])_\mu^b$, $t \in [0,\infty)$ and each starting point $x \in [0,1]$

$$\mathbb{E}\left[f\left(X_n^b(t)\right)\right] = T_{t,n}^b f(x) \to T_t^b f(x) = \mathbb{E}\left[f\left(X^b(t)\right)\right],\ n \to \infty,$$

where X^b is associated to Δ_μ^b and X_n^b is associated to $\Delta_{\mu_n}^b$. A direct argument extends this to all continuous functions on $[0,1]$. Then, a modification of the corresponding proof in [14] gives convergence of all finite-dimensional distributions. The tightness would be required to establish that $X_n^b \to X^b$ weakly in the Skorokhod space of càdlàg functions.

Remark B.3: Let μ be of full support. Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2}(t) = \Delta_\mu^b u(t), \quad t \in [0, \infty) \tag{87}$$

on $L^2([0,1],\mu)$. This hyperbolic equation describes the motion of a vibrating string with mass distribution μ such that, if it is deflected, a tension force drives it back towards its state of equilibrium. If μ were not of full support, the string would have massless parts. It is not clear how to interpret massless parts of a string. We suppose that the motion of such a string behaves approximately like the motion of a string with very little mass on these gaps, analogous to our results about the diffusion of heat.

Assume that $u(0) \in \mathcal{D}\left(\Delta_\mu^b\right)$ and, for reasons of simplicity, that the initial velocity vanishes. Then, there exists a unique solution to (87) in $L^2([0,1],\mu)$ given by $u(t) = C_t^b u(0)$, $t \geq 0$, where $\left(C_t^b\right)_{t\geq 0}$ denotes the strongly continuous cosine family of Δ_μ^b (compare Section 5.1). We have already shown that $\mu_n \rightharpoonup \mu$ implies strong resolvent convergence of the corresponding operators restricted to continuous functions. It is well-known that this implies convergence of the corresponding cosine families $\left(C_{t,n}^b\right)_{t\geq 0}$, which implies convergence of the solutions to the corresponding wave equations, provided that there exist $M > 0$ and $w \geq 0$ such that for all $n \geq 1$, $t \geq 0$ $\left\|C_{t,n}^b\right\| \leq Me^{wt}$ (see [39]). Proving that the restriction of C_t^b to $(C[0,1])_\mu^b$ for $t \geq 0$ is the cosine family of $\bar{\Delta}_\mu^b$ (and analogously for μ_n) and verifying the above estimate would be a way to establish the desired convergence of solutions to the wave equations.

Remark B.4: We have shown that under some regularity conditions, the mild solution to a stochastic heat equation given by (68) satisfies the upper moment bound

$$\left(\mathbb{E}\left[|u(t,x)|^p\right]\right)^{\frac{1}{p}} \leq \left(2\left\|u_0\right\|_q + 1\right) e^{C_4 p^{\frac{1}{1-\gamma\delta}} t}.$$

for a constant $C_4 > 0$. We conjecture that there are constants $C_{13}, C_{14} > 0$ such that for all $(t,x) \in [0,\infty) \times [0,1]$, $p \geq 1$ we have

$$\left(\mathbb{E}\left[|u(t,x)|^p\right]\right)^{\frac{1}{p}} \geq C_{13} e^{C_{14} p^{\frac{1}{1-\gamma\delta}} t},$$

where we probably have to assume further regularity conditions, as $f(t,x) = 0,\ g(t,x) = x,\ u_0(x) = 1$, $(t,x) \in [0,\infty) \times [0,1]$, which leads to the so-called parabolic Anderson model. Comparable results are known for the parabolic Anderson model defined by the standard Laplacian (compare e.g. [3, Theorem 2.6]). The proof relies on the fact that the moments of the mild solution can be expressed in terms of the local times of Brownian motion. We suppose that a generalization of this concept to Cantor-like sets would lead to the desired lower moment bound.

Treating the same problem for the stochastic wave equation is probably even more difficult as there is no such lower bound known for the stochastic wave equation defined by the standard one-dimensional Laplacian, according to the knowledge of the author.

Remark B.5: The investigation of stochastic heat and wave equations raises the question of further stochastic PDEs defined by Δ_μ^b, such as the stochastic Burgers equation

$$\frac{\partial}{\partial t}u(t,x) = \Delta_\mu^b u(t,x) + f(t,u(t,x)) - u(t,x)\frac{\partial}{\partial x}u(t,x) + \xi(t,x). \tag{88}$$

for $(t,x) \in [0,T] \times [0,1]$. It is known (see e.g. [4]) that, assuming sufficient regularity and appropriate initial data, the mild solution to (88) for $\mu = \lambda^1$ possesses a version that is essentially $\frac{1}{2}$-Hölder continuous in space and essentially $\frac{1}{4}$-Hölder continuous in time, which coincides with our results concerning the regularity of the stochastic heat equation. This suggests the assumption that one could establish the same Hölder exponents for the mild solution to (88) as for the stochastic heat equation defined by Δ_μ^b. However, as the proof in [4] makes use of stochastic calculus, it seems that our results are not trivially generalizable to this equation.

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