Uncertainty Quantification with Lévy-type Random Fields

Von der Fakultät Mathematik und Physik der Universität Stuttgart zur Erlangung der Würde eines Doktors der Naturwissenschaften (Dr. rer. nat.) genehmigte Abhandlung

Vorgelegt von

Andreas Stein

aus Alsfeld

Hauptberichterin:	**Prof. Dr. Andrea Barth**
Mitberichter:	**Prof. Dr. Christian Rohde**
	Prof. Dr. Christoph Schwab

Tag der mündlichen Prüfung: 18.06.2020

Institut für Angewandte Analysis und Numerische Simulation der Universität Stuttgart

2020

Acknowledgements

I would like to express my sincere gratitude to my advisor Prof. Dr. Andrea Barth for her guidance, support and patience during my time at the University of Stuttgart. It is a pleasure to work with her and I could not have imagined having a better advisor and mentor for my doctoral studies.

Also, I would like to thank Prof. Dr. Christian Rohde and Prof. Dr. Christoph Schwab for their willingness to be the co-examiners of this thesis.

Another thank goes to my present and former colleagues from the research group of computational uncertainty quantification at the University of Stuttgart for all the insightful discussions throughout the years. Moreover, I would like to thank Dr. Alexander Grimm for his valuable suggestions that lead to a significant improvement of this thesis.

During my time as doctoral student, I have been partly funded by the German Research Foundation (DFG) as part of the Cluster of Excellence in Simulation Technology (EXC 310/2) at the University of Stuttgart, and it is gratefully acknowledged.

Furthermore, I have to thank Elke Gangl for her administrative support in the submission process of this thesis.

Finally, I would like to thank all my colleagues at the IANS in Stuttgart for creating the pleasant atmosphere at the institute and making my time in Stuttgart enjoyable.

Contents

Abstract

A countless number of models in the natural sciences, engineering and economics are based on partial differential equations (PDEs). Due to insufficient data or measurement errors, certain characteristics of the underlying PDE are subject to uncertainty, and are usually modeled by continuous and/or Gaussian random fields. Although analytically tractable, the applications of continuous or Gaussian random fields are limited: spatial and temporal discontinuities cannot be captured and Gaussian distributions notoriously underestimate the probability of rare events. To this end, the focus of this thesis is on uncertainty quantification with *Lévy-type* random fields, a certain class of discontinuous stochastic objects that provide a significant extension to the existing methodology. In a nutshell, this dissertation explains how to incorporate Lévy-type random fields into PDEs and how the corresponding solutions become accessible by the means of discretization and simulation.

The first main contribution is the introduction of a novel type of random field, consisting of a Gaussian part and a spatially discontinuous jump field. This Lévy-type field serves as a coefficient in advection-diffusion equations and allows to model, for instance, sudden changes in the permeability of a porous medium far more realistically than state-of-the-art continuous models. In contrast to the few examples in the literature, the discontinuous random coefficient in this thesis provides a unique flexibility as it is able to generate virtually any stochastic geometry.

Apart from random PDEs with discontinuous coefficients, hyperbolic transport equations with Lévy noise as source term are considered. The noise processes take values in a (infinite-dimensional) Hilbert space and involve temporal discontinuities. Therefore, heavy-tailed random perturbations are introduced to the transport problem, and the resulting stochastic equation may be utilized, e.g., as a model for the dynamics in commodity forward markets. Due to the lack of tractable discretization schemes for the underlying stochastic PDE, this models have been, up to now, only of theoretical interest. This thesis paves the way to finally apply the forward model with Lévy noise in practice, as it provides the first fully discrete scheme for the corresponding stochastic

transport problem.

In both cases, PDEs with random jump coefficients or with Lévy noise as source term, regularity is inherently low due to the discontinuities and state-of-the-art numerical algorithms are prohibitive. To remedy this issue, several advanced schemes for discontinuous random problems are introduced that outperform standard methods in terms of convergence rates and computational effort. A comprehensive numerical analysis is provided and the superior performance of the new approaches is validated by numerous numerical experiments.

Moreover, an emphasis is put on the approximation of infinite-dimensional, discontinuous random fields, a crucial part in the discretization of stochastic PDEs. A part of this thesis covers the sampling of Hilbert space-valued Lévy processes, and introduces a sampling technique combining truncated Karhunen-Loève expansions with discrete Fourier inversion. This algorithm stands out as the arguably most flexible of a very limited number of methods for the approximation and simulation of Lévy fields.

Zusammenfassung

Unzählige Modelle in Naturwissenschaft, Technik und Ökonomie basieren of partiellen Differentialgleichungen (PDEs). Aufgrund von Messfehlern oder unvollständigen Daten sind gewisse Parameter dieser PDEs mit Unsicherheiten behaftet und werden daher in der Regel durch stetige und/oder Gaußsche Zufallsfelder modelliert. Obwohl diese gute analytische Eigenschaften haben, sind die Anwendungen von stetigen oder Gaußschen Zufallsfeldern limitiert: Sprünge in Zeit oder Ort können nicht abgebildet werden und Gaußsche Verteilungen unterschätzen oft die Wahrscheinlichkeiten von extremen Ereignissen. Aus diesen Gründen thematisiert die vorliegende Arbeit *Uncertainty Quantification* unter Verwendung einer Klasse unstetiger Zufallsfelder, die eine signifikante Erweiterung der existierenden Theorie darstellen. Konkret wird erklärt wie man unstetigen Zufallsfelder in PDEs integriert und wie die entsprechenden Lösungen durch Diskretisierungsmethoden und Simulation approximiert werden können.

Der erste wichtige Beitrag dieser Arbeit ist die Einführung eines neuartigen Zufallsfeldes, das aus einem Gaußschen Anteil, sowie einem unstetigen Sprungfeld besteht. Dieses unstetige Zufallsfeld findet dann Verwendung als Koeffizient in Advektions-Diffusionsgleichungen, beispielsweise um abrupte Änderungen der Permeabilität in porösen Medien zu simulieren. In der Literatur gibt es bisher nur wenige, sehr spezielle Beispiele von unstetigen stochastischen Koeffizienten. Im Gegensatz dazu besticht der Koeffizient in dieser Arbeit durch seine Flexibilität, da er es ermöglicht praktisch jede Art zufälliger Geometrie abzubilden.

Neben zufälligen PDEs mit unstetigen Koeffizienten werden ebenfalls hyperbolische Transportgleichungen mit einem Lévy-Prozess als Quellterm auf der rechten Seite betrachtet. Dieser Prozess ist unstetig in der Zeit und nimmt Werte in einem unendlichdimensionalen Hilbertraum an. Dadurch erhält die Transportgleichung Störungen mit sogenannten *heavy-tails*, also mit hoher Wahrscheinlichkeit für Extremereignisse, und kann zum Beispiel als Modell für Terminkontrakte in Rohstoffmärkten benutzt werden. Da geeignete Diskretisierungsverfahren für deren Implementierung bisher fehlten, waren diese Modelle bis jetzt eher von theoretischem Interesse. In dieser Arbeit wird

zum ersten Mal eine volle Diskretisierung für die zugehörigen stochastischen Transportprobleme vorgestellt, und dadurch ermöglicht die Lévy-Modelle für Terminkontrakte auch erstmals in der Praxis anzuwenden.

Ein Problem, das sowohl bei PDEs mit zufälligen unstetigen Koeffizienten, als auch bei stochastischen PDEs mit Lévy-Rauschen auftritt, ist die niedrige Regularität der Lösung. Dies macht übliche numerische Verfahren zur Lösung von PDEs ineffizient und aus diesem Grund werden mehrere spezielle Methoden für Probleme mit zufälligen Unstetigkeiten entwickelt. Alle Verfahren werden vollständig numerisch analysiert und zahlreiche Experimenten belegen die besseren Performance im Vergleich zu Standard-Methoden.

Ein weiterer Schwerpunkt dieser Arbeit ist die Approximation und Simulation von unendlich-dimensional Lévy-Prozessen, ein essentieller Baustein in der Diskretisierung von stochastischen PDEs. Dazu wird eine Technik vorgestellt die auf einer Kombination von abgeschnittenen Karhunen-Loève-Entwicklungen und diskreter Fourier Inversion besteht. Dieser Algorithmus ragt als die wahrscheinlich flexibelste von sehr wenigen existieren Methoden zur Simulation von Lévy-Feldern heraus.

1 Introduction

A vast range of phenomena in the natural sciences, engineering, finance and economics are modeled by partial differential equations (PDEs). Examples include the description of porous media and subsurface flows ([68, 107])[1], the propagation of aeroacoustic waves ([200]), turbulent flow models ([180]), phase separation in the mixture of fluids ([150]), the heat distribution on CPUs ([55]), optimal control problems in traffic flow networks ([95]), aquifer systems to store thermal energy ([141]), epidemic models for the spreading of diseases ([153]) and parasite populations ([103]), environmental pollution models ([206]), the metabolism of glucose in the human liver ([183]), the valuation of financial derivatives ([42, 111]) and assessment of credit risks ([41]) or models of warfare and pursuit in game theory ([117]). The quantities of interest in these applications are in general functionals of the solution to the PDE. For instance, in a subsurface flow model the solution of the underlying system describes a pressure field, and one may be interested in evaluating the pressure function at specific points in the domain or to know the average pressure in a certain area.

Due to insufficient data, measurement errors or incomplete information, however, certain parameters of the PDE may not be known a-priori. In a subsurface flow model, the permeability of a medium is in general only measured at a discrete set of points and unknown in between. Another common example is the pricing of stock options, where the model parameters have to be estimated statistically based on a time series of quoted market prices. To model this effects more realistically, *uncertainty quantification* has become an increasingly important and popular field of research in the last decades. A common approach in uncertainty quantification is to replace certain characteristics of a PDE model by stochastic objects, for instance, by considering a random differential operator, such as in [1, 14, 15, 25, 29, 53, 59, 63, 85, 90, 96, 148, 164, 165, 190, 194], or to introduce some noise as a function-valued stochastic process, see for instance [23, 24, 28, 35, 67, 69, 86, 121, 132, 151, 152, 173, 198]. The listed references are

[1] All references listed in the introduction, in the articles and in the conclusion of this thesis can be found in the unified bibliography after Chapter 9 at the end of this dissertation.

merely a snapshot of the research activity throughout the recent years and are by far not exhaustive.

Throughout the literature, it is standard practice to model the uncertainty by continuous random fields (see e.g. [53, 59, 63, 85, 96, 164, 194]). Due to continuity, these objects have natural representations in terms of Fourier series that may be exploited in analysis and simulation. The major drawback of continuous random fields is, however, that applications are drastically limited as temporal and/or spatial discontinuities cannot be captured. For example, they are not suitable to model flows through porous media or composite materials where sudden changes of permeability at interfaces occur. Another shortcoming is that time-continuous stochastic processes essentially follow Gaussian distributions, which notoriously underestimate the probabilities of extreme risks. For instance, when applied as a subsurface flow model, Gaussian random fields may not reflect the presence of peaks and asymmetric distributions in the permeability appropriately, see [202]. As another example, Gaussian stock return models fail to match real market data in many cases and are unable to predict the occurrence of rare events, such as financial crisis, properly (see e.g. [89]).

1.1 The contribution of this thesis

To overcome the aforementioned problems, this thesis focuses on uncertainty quantification involving a certain type of discontinuous random objects, which is referred to as *Lévy-type random fields*. In a nutshell, this dissertation explains how to incorporate random discontinuous characteristics into PDEs and how the solutions to this problems become accessible by the means of discretization and simulation. As a consequence, various problems can now be modeled much more realistically, since the severe restrictions imposed by continuous or Gaussian random objects have vanished. The corresponding solutions may be simulated with controlled bias and reasonable effort using the advanced numerical schemes developed in this thesis.

As one of the main contributions, a novel Lévy-type random field with spatial discontinuities is introduced. The decisive feature of this jump field is its unique flexibility, since it allows to model virtually any stochastic geometry. This is achieved by constructing the coefficient as the sum of a Gaussian random field and a discontinuous jump term (for this reason the term *Lévy-type* is borrowed from stochastic analysis). Moreover, temporal Lévy processes taking values in an infinite-dimensional Hilbert space are considered. These objects are function-valued stochastic processes in the classical sense with discontinuous paths in time. Compared to a standard Gaussian noise process, a Lévy process allows to introduce more realistic heavy-tailed perturba-

tions into a dynamical system. Each kind of Lévy-type random field is utilized as a stochastic parameter in a PDE, either as random coefficient or source term.

A fundamental part in the discretization of stochastic PDEs is to obtain tractable closed-form approximations of the underlying random field. While this can be challenging for continuous Gaussian random fields, all difficulties are amplified when dealing with more general Lévy-type random objects. As a further contribution, this thesis introduces a new approximation method for general Hilbert space-valued Lévy noises in Chapter 7. In fact, the proposed algorithm stands out as one among very few methods for general Lévy noises and is arguably the most flexible one. This is somewhat surprising at first glance, since there is a vast literature on the numerical analysis of (parabolic) stochastic PDEs with Lévy noise, see for instance [23, 26, 33, 49, 74, 133]. Therein, however, the proposed noise approximation either suffers from strong assumptions or, as in the majority of cases, is completely neglected. Moreover, certain sampling techniques for the space-time Lévy source term are also applicable for the spatial random jump coefficient, as outlined in Chapters 4 and 5.

Possible applications for PDEs involving Lévy-type random fields are motivated by irregular, discontinuous structures like fractured or porous media, rock strata with spherical inclusions, composite materials and alloys. To model flows, charge- or pressure distributions in this structures, the spatial Lévy-type random field may be utilized as diffusion- or advection coefficient in a PDE. The literature on PDEs with random discontinuous coefficients is sparse, with exceptions being [104, 140, 205], and the analysis is in general tailored to a very specific shape and geometry. In contrast to this, the Lévy-type random field from this thesis has the ability to represent any desired stochastic structure and therefore significantly extends the existing methodology.

Another important example are transport equations with Lévy noise, a popular model for commodity- and interest futures markets (see [24, 52, 173]). So far, this models have mainly been examined from a theoretical perspective and have not been applied to valuate forward contracts, since tractable fully discrete approximations have not been available. Chapter 8 of this thesis closes this gap and stands out as the first (and to this point only) article containing a complete numerical analysis for the full discretization of this type of stochastic transport problem. Additionally, all theoretical findings are confirmed by numerical experiments in scenarios with varying regularity. Similar results have not yet been provided in the literature: in general, the much simpler case of finite-dimensional and/or Gaussian noise is investigated and only semi-discrete schemes are proposed (see e.g. [34, 130, 135, 154] for results on nonlinear hyperbolic PDES with Lévy- respectively Gaussian noise). On a further note, there seems to be a general lack of numerical experiments in the literature on SPDEs with Lévy noise.

For both scenarios, PDEs with random jump coefficients as well as stochastic PDEs with Lévy noise as source term, suitable concepts of solutions are investigated to ensure well-posedness. The discontinuities entail inherently low regularity and, consequently, standard numerical algorithms converge, if at all, at very poor rates. Even if convergence is theoretically ensured, oscillations and instabilities may occur and the corresponding schemes are useless in any application. To overcome the issues of common methods, this thesis introduces a variety of novel algorithms to discretize discontinuous random PDE problems. A comprehensive numerical analysis is provided for each method and stability is ensured to prevent oscillations in the approximated solutions. To this end, several new techniques of proof have been developed, as applicability of existing results is rather limited. For instance, it is crucial to control for stochastic geometries or simulation biases, effects that naturally do not occur in deterministic or simpler stochastic problems. The superior performance of our novel approaches compared to state-of-the-art methods is validated in numerous experiments.

At this point it needs to be emphasized that the considered Lévy-type random objects provide an actual extension to the current state of research, since the aforementioned continuous random fields may be recovered as a special case. The most important examples for this are log-normal diffusion coefficients as in [53, 54, 96, 98, 108, 194] and Q-Wiener processes as space-time driving noise, see for instance [132, 136, 198]. Therefore, all results and proposed algorithms of this thesis are universal and may readily be applied to a large class of important problems in the field of uncertainty quantification.

1.2 The articles in this thesis

The core of this thesis consists of five independent research articles:

- *A study of elliptic partial differential equations with jump diffusion coefficients*,
- *Numerical analysis for time-dependent advection-diffusion problems with random discontinuous coefficients*,
- *A multilevel Monte Carlo algorithm for parabolic advection-diffusion problems with discontinuous coefficients*,
- *Approximation and simulation of infinite-dimensional Lévy processes*, and
- *A stochastic transport problem with Lévy noise: Fully discrete numerical approximation.*

As the articles have been developed partly in parallel, the order is not strictly chronological, but motivated by thematic classification. The first group consists of Chapters 4–6 that consider elliptic/parabolic problems with random Lévy-type coefficients involving spatial discontinuities. They are connected very closely as Chapter 5 and 6 are build on its predecessor(s) and were therefore also finished in the given order. It is common to refer to the type of stochastic equations in Chapters 4–6 as *random PDEs* to emphasize that the source of randomness lies within the PDE's coefficients, and therefore in the corresponding differential operator. This is to obtain a distinction from so-called *stochastic PDEs* (SPDEs) with deterministic coefficients respectively differential operators, but a stochastic source term on the right hand side of the equation. Chapters 7 and 8 constitute the second group of research articles and focus on SPDEs with Lévy noise as external source of uncertainty. The overarching theme of this thesis, however, is to introduce Lévy-type objects as stochastic PDE parameters to allow for more flexibility compared to the state-of-the-art continuous and/or Gaussian random fields. Simulation and discretization of the PDE problem at hand becomes significantly more involved due to the discontinuous random structures. Thus, the scientific literature lacks tractable algorithms and this thesis closes the existing gap for a variety of problems.

In uncertainty quantification, approximation schemes not only need to include the discretization of spatial and temporal domains, but also the *discretization of the stochastic domain*. While the first part is achieved by modifying schemes for deterministic PDEs, the second part involves the approximation of infinite-dimensional random variables and sampling techniques like *Monte Carlo algorithms*. As outlined throughout the central part of this thesis, sampling becomes especially challenging for Lévy-type random objects. Moreover, depending on whether random PDEs or SPDEs are considered, some aspects of discretization may differ drastically. In the first case, *path-wise* error bounds (similar as for deterministic PDEs) can be exploited, where path-wise means that this estimates only hold for a specific realization of the problem. To obtain L^p-estimates with $p \geq 1$ and almost sure convergence, it is then still crucial to control all appearing terms with respect to the entire stochastic domain. When discretizing SPDEs as in Chapter 8, it is necessary to switch from a path-wise to a mean-square-type perspective of discretization. The Lévy-noise on the right hand side of the SPDE is given by an *Itô integral* with each path involving a possibly infinite number of temporal discontinuities. This stochastic integral is in turn defined as the mean-square-limit of a sequence of simple integrals and, therefore, the mean-square theory of the corresponding Itô calculus has to be applied (details are given in Chapter 3 of this thesis). Consequently, path-wise approximation techniques are not

applicable and one is usually bound to L^2-type convergence results.

A short overview on discretization methods with regard to uncertainty quantification is given in Chapter 2, the specific method for each problem in Chapters 4– 8 is sketched in the subsequent summary and is found in detail within the corresponding article.

1.2.1 A. Barth and A. Stein: "A study of elliptic partial differential equations with jump diffusion coefficients"

We consider an elliptic equation with random coefficient, which may for instance account for the uncertain permeability of a given medium in a subsurface flow model. As an extension of this methodology to flows in heterogeneous\fractured\porous media, we incorporate jumps in the diffusion coefficient. More precisely, we consider a second order elliptic problem where the random coefficient is given by the sum of a (continuous) Gaussian random field and a (discontinuous) jump part.

To estimate moments of the solution to the resulting random partial differential equation, we use path-wise finite element approximations combined with multilevel Monte Carlo sampling. In order to account for the discontinuities and improve the convergence of the path-wise approximation, the spatial domain is decomposed with respect to the jump positions in each sample, leading to path-dependent finite element grids. Hence, it is not possible to create a sequence of grids which is suitable for each sample path a-priori. We address this issue by an adaptive multilevel Monte Carlo algorithm, where the discretization on each level is sample-dependent and fulfills given refinement conditions.

1.2.2 A. Barth and A. Stein: "Numerical analysis for time-dependent advection-diffusion problems with random discontinuous coefficients"

In this article we extend the elliptic diffusion problem from Chapter 4 to a time-dependent parabolic advection-diffusion equation. Specifically, a scenario with coupled advection and diffusion coefficients that are modeled as sums of continuous random fields and discontinuous jump components is considered. We employ the same adaptive, path-wise finite element discretization for the numerical approximation of the solution as in Chapter 4. In this previous work, we have merely assumed convergence in mean-square with respect to the $H^1(\mathcal{D})$-norm for the discretization of the spatial domain $\mathcal{D}$ based on our numerical experiments. Here, however, we provide a rigorous proof on the path-wise and mean-squared convergence rate based on a set of suitable assumptions.

As it turns out, this becomes surprisingly involved, since the order of convergence

depends on the shape of the discontinuities. Thus, the random geometry has to be controlled for each sample to obtain bounds on the mean-squared error. To stabilize the numerical approximation and accelerate convergence, the discrete space-time grid is chosen with respect to the varying discontinuities in each sample of the coefficients, leading to a stochastic formulation of the Galerkin projection and the finite element basis. We provide several numerical experiments to verify our theoretical results and show that the regime of assumptions cannot be relaxed significantly.

1.2.3 A. Barth and A. Stein: "A multilevel Monte Carlo algorithm for parabolic advection-diffusion problems with discontinuous coefficients"

In many applications for elliptic or parabolic random PDEs, such as in the previous two articles, the aim is to estimate moments of certain *quantities of interest* (QoIs). Those QoIs are in general functionals of the corresponding path-wise solution, rather than the solution itself. As there are only path-wise numerical solutions at hand to which a given functional may be applied, we obtain a biased version of the true QoI. For instance, in Chapters 4 and 5 this bias is due to the (adaptive) finite element approximation. It is, however, often possible to express this bias in terms of the $L^2(\mathcal{D})$-error of the spatial discretization.

In this article we consider the same problem setting and assumptions as in Chapter 5 and extend the existing a-priori error bounds of the path-wise adaptive discretization from the $H^1(\mathcal{D})$-norm to the $L^2(\mathcal{D})$-norm. To this end, we solve a time-dependent parabolic dual problem, and derive an improved decay rate based on our results for the $H^1(\mathcal{D})$-error in Chapter 5. As expected, the $L^2(\mathcal{D})$-error decays twice as fast on a given spatial grid, allowing us to bound the path-wise bias for functionals of the approximated solution. Based on this result, we introduce a multilevel Monte Carlo algorithm to estimate the moments of a given QoI. A numerical example shows that the adaptive discretization is superior to a standard finite element approach when applied to a multilevel Monte Carlo estimator.

1.2.4 A. Barth and A. Stein: "Approximation and simulation of infinite-dimensional Lévy processes"

The recurring motivation in the first three articles of this thesis was to replace a spatially continuous stochastic object by a more general, discontinuous Lévy-type random field. We now follow the same idea for time-dependent stochastic objects, and shift our attention to infinite-dimensional Lévy processes, also called (time-dependent) Lévy fields. In fact, the random coefficients in Chapters 4–6 can actually be interpreted

as a spatial and stationary analogue to the classical notion of time-dependent Lévy processes from stochastic analysis.

As already indicated in the introduction of this thesis, the approximation of Lévy fields is challenging: For square integrable fields beyond the Gaussian case, it is no longer given that the one-dimensional distributions in the spectral representation with respect to the covariance operator are independent. When simulated via a Karhunen-Loève expansion a set of dependent, but uncorrelated, one-dimensional Lévy processes has to be generated. The dependence structure among the one-dimensional processes ensures that the resulting field exhibits the correct point-wise marginal distributions. To approximate the respective (one-dimensional) Lévy-measures, a numerical method, called discrete Fourier inversion, is developed. For this method, L^p-convergence rates can be obtained and, under certain regularity assumptions, mean-square and L^p-convergence of the approximated field is proved (we emphasize that the Fourier inversion technique has also been applied to sample spatial Lévy-type random coefficients, as shown in the numerical examples from Chapters 4 and 5). Further, the class of *generalized hyperbolic* Lévy fields is introduced, where the point-wise marginal distributions are dependent, but uncorrelated, subordinated Wiener processes. For this specific class one may derive point-wise marginal distributions in closed form. Numerical examples, including *hyperbolic* and *normal-inverse Gaussian* Lévy fields, demonstrate the efficiency of the approach.

1.2.5 A. Barth and A. Stein: "A stochastic transport equation with Lévy noise: Fully discrete numerical approximation"

Using the results from Chapter 7 as foundation, we introduce fully discrete schemes for SPDEs with Lévy noise. A particular interesting example are linear hyperbolic SPDEs, which serve as a model for the dynamics of interest rate and energy forward markets. To be more precise, the forward rate is modeled as the solution to a transport equation with a space-time stochastic process as driving noise. Again, our motivation is to capture discontinuities in time and allow for heavy-tailed distributions, thus we consider Hilbert space-valued Lévy fields as driving noise terms. The numerical discretization of the corresponding SPDE involves several difficulties: Low spatial and temporal regularity of the solution to the problem entails slow convergence rates and instabilities for space/time-discretization schemes.

Even if this problem is resolved, a fully discrete approximation has yet to take the driving noise process into account. The Lévy field admits values in a possibly infinite-dimensional Hilbert space U, hence projections into a finite-dimensional subspace of

U for each discrete point in time are necessary. Finally, as mentioned in the previous Chapter 7, unbiased sampling from the resulting Lévy field is not necessarily possible. We introduce a fully discrete approximation scheme to address the above issues: A discontinuous Galerkin approach for the spatial approximation is coupled with a suitable time stepping scheme to avoid numerical oscillations. Moreover, we use the Fourier inversion technique combined with truncated Karhunen-Loève expansions from Chapter 7 to obtain a suitable approximation of the Lévy field. We provide a rigorous error analysis for the space-time discretization which yields together with the results from Chapter 7 a bound on the mean-squared overall discretization error. As before, we confirm our theoretical results by several numerical examples.

1.2.6 Overall structure

In addition to the five articles, the second chapter of this thesis contains a preliminary discussion on numerical schemes for PDEs and Monte Carlo methods with a view towards uncertainty quantification. Thereafter, Chapter 3 collects some probabilistic results on infinite-dimensional random variables, Lévy processes and stochastic integration. This part provides a theoretical background on random fields and stochastic processes, especially for Chapters 7 and 8 that consider Lévy driving noise. After the two introductory parts on discretization techniques and probability theory, Chapters 4–8 follow as the core of this thesis and contain the articles previously outlined in this section. Finally, this thesis is concluded by some remarks and perspectives for future work in Chapter 9.

2 Discretization techniques in uncertainty quantification

This chapter gives an overview of numerical schemes for PDEs and Monte Carlo methods, the discretization techniques applied in the main part of this thesis. A brief description of the key ideas and a literature review with regard to their applications in uncertainty quantification is provided. Advantages as well as possible shortcomings are highlighted and the algorithms of choice are motivated by comparing them to several alternative approaches. A more detailed and formal description for each scheme is then given in Chapters 4–8 for the specific problem at hand.

2.1 Numerical methods for PDEs

This section discusses some standard numerical approximation schemes for PDEs which are applied in Chapters 4–6 and 8 to PDEs with discontinuous random characteristics. The limitations of each approach are pointed out in this context and it is indicated how this problems have eventually been overcome in the forthcoming chapters.

One of the most popular methods for the numerical approximation of PDEs is the finite element (FE) method, which can be traced back to the works [8, 65, 87, 115, 184, 195]. There is an extensive amount of literature on the FE method with applications to deterministic PDEs, for instance [47, 58, 102, 177] to just name a few. In the recent years, FE-based methods have also successfully been applied in all areas of uncertainty quantification, examples are found in [15, 23, 27, 29, 54, 85].

The basic idea of the FE method may be illustrated by the stationary, elliptic boundary value problem

$$-\nabla\cdot(a(x)\nabla u(x)) = f(x),\ x\in\mathcal{D}, \qquad u(x) = 0,\ x\in\partial\mathcal{D}, \tag{2.1}$$

on a bounded domain $\mathcal{D}\subset\mathbb{R}^d$. For any real $q>0$, let $H^q(\mathcal{D})$ denote the standard Sobolev space over $\mathcal{D}$ with the notational convention $H^0(\mathcal{D}) := L^2(\mathcal{D})$. The suitable

solution space is given by

$$H_0^1(\mathcal{D}) := \{v \in H^1(\mathcal{D}) | \, \gamma v = 0\},$$

with $\gamma : H^1(\mathcal{D}) \to L^2(\partial\mathcal{D})$ being the trace operator. Hence, the weak formulation of Problem (2.1) is to find $u \in V := H_0^1(\mathcal{D})$ such that

$$\int_{\mathcal{D}} a(x)\nabla u(x) \cdot \nabla v(x) dx = \int_{\mathcal{D}} f(x)v(x)dx, \quad v \in V. \tag{2.2}$$

Existence and uniqueness of a weak solution is guaranteed by the Lax-Milgram lemma if a satisfies a uniform ellipticity condition and if f is an element of the dual space of V, i.e. $f \in V^* = H^{-1}(\mathcal{D})$. For the FE approximation of u, the infinite-dimensional space V is replaced by a suitable finite dimensional subspace $V_h \subset V$ associated to some refinement parameter $h > 0$. In general, $h > 0$ is the maximum diameter of a tesselation $\mathcal{K}_h$ of $\mathcal{D}$ and V_h is the space of all *continuous* functions consisting of piecewise polynomials with respect to $\mathcal{K}_h$. The (maximum) degree of the polynomials is given by $p \in \mathbb{N}$ and each basis function of V_h only has a small local support on a few elements of $\mathcal{K}_h$. The discrete version of Problem (2.2) is then to find $u_h \in V_h$ such that

$$\int_{\mathcal{D}} a(x)\nabla u_h(x) \cdot \nabla v_h(x) dx = \int_{\mathcal{D}} f(x)v_h(x)dx, \quad v_h \in V_h,$$

and the discretization error is bounded provided that u is sufficiently regular:

Theorem 2.1.1. *[177, Chapter 6.2] Let u be the solution to Problem (2.2) such that $u \in H^q(\mathcal{D})$ for some $q > 1$. Furthermore, let V_h be the FE space containing all piecewise polynomials up to degree $p \in \mathbb{N}$ and let u_h be the FE approximation of u. Then, there is a constant $C > 0$, independent of u and h, such that*

$$||u - u_h||_{H^m(\mathcal{D})} \leq C\|u\|_{H^q(\mathcal{D})} h^{\min(p+1,q)-m}, \quad m \in \{0, 1\}.$$

In Chapters 4–6, we investigate a version of Problem (2.1), where the diffusion coefficient a is a $L^2(\mathcal{D})$-valued random field. More precisely, a is the sum of a continuous Gaussian part and a stochastic jump field, hence a is discontinuous on $\mathcal{D}$. The corresponding weak solution u then is a random function with very low path-wise regularity, in general only $u \in H^q(\mathcal{D})$ with $q < \frac{3}{2}$ holds almost surely. Therefore, convergence with respect to h is slow and thus reasonably good approximations become computationally expensive. This problem is addressed by introducing a modified FE method, which exploits the higher piecewise regularity of u in between the interfaces of a. We show that this allows us to recover better convergence rates despite the low global regularity

of u seems to permit this at first glance.

There are of course several other discretization schemes for PDEs, such as finite difference (FD)-, finite volume (FV)-, spectral- and collocation methods (see [14, 51, 81, 143, 144, 177]). These are, however, less suited for PDEs with discontinuous coefficients. The FD method is based on uniformly spaced grids, hence an adaptive approach to match the given discontinuities of a as described above is out of reach. Spectral- or collocation methods have in general superior convergence properties compared to FE, but require high regularity and periodicity of the problem. As these assumptions are too restrictive for the setting in Chapters 4–6, it can not be expected that spectral- or collocation approaches outperform the FE method. A discussion of FV methods for hyperbolic PDEs follows shortly.

The class of *hp-finite element* methods, see [189], also needs to be mentioned. In this modification, the FE mesh and/or the polynomial degree is refined/adjusted for certain areas in the domain based on a-posteriori error estimates. This becomes particularly useful if the problem at hand is only irregular at certain areas of the domain, but rather smooth otherwise. Thus, *hp*-FE methods are actually a promising advancement to the methodology in Chapters 4–6. They are, however, analytically and computationally challenging, and their application to PDEs with discontinuous random coefficients is beyond the scope of this thesis (see also the discussion in Chapter 9).

The FE method turns out to be a suitable spatial discretization to solve diffusion-dominated elliptic and parabolic problems. In case of advection-dominated and hyperbolic PDEs as considered in Chapter 8, however, FE methods become inherently unstable and oscillating. To circumvent these issues, *discontinuous Galerkin* (DG) methods have been developed as a generalization of the FE method. Introduced in the pioneering works [142, 179] to solve a neutron transport problem, DG methods are, by now, the state-of-the-art approach for the numerical solution of (non-)linear hyperbolic PDEs. Comprehensive overviews on DG methods and their applications can be found for instance in [61, 109, 185]. Although successfully applied to deterministic PDEs, so far, there are not many applications of the DG method to problems in uncertainty quantification , notable exceptions include [90, 110, 120].

The DG approach is also based on the weak formulation of the PDE and basically the same as for the FE method, but the finite-dimensional test function space V_h now consists of *all* functions built from piecewise polynomials on $\mathcal{K}_h$. This allows especially for discontinuous approximations with more degrees of freedom than their FE counterparts. Since left and right limits of the DG basis differ at the interfaces of $\mathcal{K}_h$, it is necessary to determine a *numerical flux* at the discontinuities. This is the key feature of DG methods, as the numerical flux can be adjusted to the specific problem to

increase stability of the approximation. Consequently, a DG approach is introduced to discretize the linear hyperbolic SPDE in Chapter 8 and it is shown that this is indeed advantageous to standard FE methods.

Another popular discretization scheme for hyperbolic PDEs is the *finite volume* (FV) method, see e.g. [81, 143]. Therein, the corresponding equation is integrated over small volumes and the divergence theorem is applied to obtain integrals over the surface of each volume. This yields a piecewise constant approximation with discontinuities across the surface of each volume, and consequently a numerical flux similar as for the DG method has to be chosen. In contrast to the FD approach, FV schemes may be applied to irregular domains and non-uniform grids. Moreover, FV methods are conservative if the underlying PDE is a hyperbolic conservation law. Applied to the SPDE in Chapter 8, it turns out the FV approach may be recovered as a subclass of DG methods with piecewise constant basis functions and midpoint quadrature. Hence, the DG approach is more general and thus expected to perform at least as good as FV schemes.

To obtain tractable approximations to time-dependent problems, it is still necessary to consider a discretization of the temporal domain. One possibility is to view time as an additional spatial variable and apply a FE approach to the $(d+1)$-dimensional space-time domain. This results in a coupling of all spatio-temporal nodal points and the problem has to be solved simultaneously at all discrete points in time. Thus, the resulting scheme has no time-stepping or iterative character, and is not applicable to a large class of time-dependent problems.

To this end, it is more practical to apply a FD approach for time integration, either based on forward-, backward or symmetric differences. The corresponding time stepping schemes are the *forward* respectively *backward Euler* method and the *Crank-Nicolson* scheme. In general, the latter approach has the best convergence properties, but requires more regularity of the solution, both with respect to time *and* space. The backward Euler scheme is unconditionally stable, while pure forward methods often lead to explosions in the numerical solution. In fact, as linear random/stochastic PDEs are considered, the FD approaches can also be interpreted as a DG discretization of the temporal dimension. For a suitably chosen numerical flux, this is equivalent to using piecewise constant functions in the Euler methods and a piecewise linear DG basis in the Crank-Nicolson scheme.

The solution to the parabolic problem in Chapters 5 and 6 is smooth in time, but non-differentiable in space. Hence, Crank-Nicolson schemes entail numerical oscillations and the only useful time integrator is the backward Euler method. In Chapter 8,

we consider stochastic integrals acting as source term on the right hand side of a linear transport problem. Due to their construction as the L^2-limit of simple integrals (see Section 3.3 in the next chapter), their only reasonable approximation is given by forward differences of the form

$$\int_{t_i}^{t_{i+1}} \Psi(s) dL(s) \approx \Psi(t_i)(L(t_{i+1}) - L(t_i)), \tag{2.3}$$

where L is a Lévy process and Ψ is an admissible integrand for L. Evaluating Ψ on the right hand side in Eq. (2.3) at any other $t \in (t_i, t_{i+1}]$ would result in the approximation of a different type of stochastic integral for which the martingale property and the Itô isometry from Theorem 3.3.2 do not hold anymore. On the other hand, applying a forward integration in time to the differential operator is out of the question due to stability reasons. This results in a mixed backward-forward time stepping scheme, where a backward Euler or Crank-Nicolson approach is applied on the left hand side of the PDE and a forward time stepping on the right hand side.

As an alternative time stepping method, *exponential integrator* schemes have successfully been applied to SPDEs, see [129, 198]. They require however, that the underlying differential operator is essentially a Laplacian to exploit its spectral basis representation. Clearly, this is not given in the hyperbolic setting in Chapter 8 and it turns out that exponential integrators do not perform superior compared to the other discussed schemes. Neither do higher order temporal approximations such as *Runge-Kutta-DG methods*([62]) promise any advantages, as they require high spatio-temporal regularity. In Chapter 8, the solution to the SPDE is only mean-square Hölder-continuous in time, at best Lipschitz-continuous in space and the temporal convergence rate is dominated by the approximation of the stochastic integral on the right hand side anyway. For further research, however, Runge-Kutta-DG methods might be useful when moving from linear to nonlinear hyperbolic problems. This is due to the fact that they rely on fully explicit, stable time stepping schemes and thus avoid solving a nonlinear system of equations in every time step.

2.2 Monte Carlo methods

The quantities of interest in uncertainty quantification are ultimately statistics of the stochastic model at hand, including moments or quantiles of the solution itself as well as of functionals applied to the solution. This problem can always be reformulated to estimate the mean value of a suitably chosen random variable. Arguably, the most popular and intuitive approach to achieve this is the *Monte Carlo method*, dating back

to Ulam, von Neumann and Fermi. The historical development of the Monte Carlo method is outlined in [159], a general introduction can be found in [83, 94, 160].

In short, Monte Carlo approaches estimate the expected value of a random variable X by drawing a large number of independent samples from the distribution of X and then taking the arithmetic mean over all samples. To illustrate this technique, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $X : \Omega \to \mathbb{R}$ be random variable such that $\mathbb{E}(|X|) < +\infty$. For some $M \in \mathbb{N}$, independent samples $X^{(1)}, \dots, X^{(M)}$ are drawn from the distribution of X and the *Monte Carlo estimator* of $\mathbb{E}(X)$ is defined via

$$E_M(X) := \frac{1}{M} \sum_{i=1}^{M} X^{(i)}.$$

Clearly, $\mathbb{E}(E_M(X)) = \mathbb{E}(X)$, and the strong law of large numbers ensures that

$$\lim_{M \to \infty} E_M(X) = \mathbb{E}(X), \quad \mathbb{P}\text{-almost surely.}$$

Moreover, if X is square-integrable, i.e. $\mathbb{E}(X^2) < +\infty$, the independence and identical distribution of the samples $X^{(1)}, \dots, X^{(M)}$ yield

$$\begin{aligned} \mathbb{E}((E_M(X) - \mathbb{E}(X))^2) &= \frac{1}{M^2} \sum_{i,j=1}^{M} \mathbb{E}(X^{(i)} X^{(j)}) - 2\mathbb{E}(X) \frac{1}{M} \sum_{i=1}^{M} \mathbb{E}(X^{(i)}) + \mathbb{E}(X)^2 \\ &= \frac{1}{M} \mathbb{E}(X^2) + \frac{M(M-1) - 2M^2 + M^2}{M^2} \mathbb{E}(X)^2 \\ &= \frac{\mathrm{Var}(X)}{M}. \end{aligned}$$

In this case, the *root-mean-squared error* (RMSE) of the Monte Carlo estimator is

$$\mathrm{RMSE} := \mathbb{E}((E_M(X) - \mathbb{E}(X))^2)^{1/2} = \frac{\mathrm{Var}(X)^{1/2}}{\sqrt{M}}. \tag{2.4}$$

Monte Carlo methods are easy to implement and straightforward to parallelize, very robust and mean-square-convergence is ensured as soon as X is square-integrable from Eq. (2.4). Their major disadvantage is the inherently slow convergence of order $\mathcal{O}(M^{-1/2})$ for the RMSE. For this reason, estimating $\mathbb{E}(X)$ by $E_M(X)$ may become prohibitive if the simulation of X is computationally expensive.

For example, consider the case $X = \Phi(u)$ where u is the solution of a (elliptic) PDE as in Eq. (2.1) with a random diffusion coefficient, i.e. a is a suitable measurable mapping $a : \Omega \to L^\infty(\mathcal{D})$ such that $a(\omega, \cdot) : \mathcal{D} \to (0, \infty)$ holds for almost all $\omega \in \Omega$. Moreover, let $\Phi : V \to \mathbb{R}$ be a given functional on the solution space $V = H_0^1(\mathcal{D})$.

Samples from the exact solution u are in general out of reach, it is, however, possible to obtain approximate samples u_h by the FE method, where $h > 0$ denotes the mesh refinement parameter. Hence, samples of $X_h := \Phi(u_h)$ are available and $\mathbb{E}(X)$ may be estimated via $E_M(X_h)$. Assuming that $\mathbb{E}(\|u\|^2_{H^2(\mathcal{D})}) < +\infty$ and that $\Phi \in V^*$, it follows by Theorem 2.1.1

$$\begin{aligned}
\mathbb{E}((E_M(X_h) - \mathbb{E}(X))^2)^{1/2} &\leq |\mathbb{E}(X_h - X)| + \mathbb{E}((E_M(X_h)) - \mathbb{E}(X_h))^2)^{1/2} \\
&\leq \|\Phi\|_{V^*}\mathbb{E}(\|u - u_h\|_{H^1(\mathcal{D})}) + \frac{\mathrm{Var}(X_h)^{1/2}}{\sqrt{M}} \\
&\leq C\|\Phi\|_{V^*}\Big(\mathbb{E}(\|u\|^2_{H^2(\mathcal{D})})^{1/2}h + \frac{\mathbb{E}(\|u\|^2_{H^2(\mathcal{D})})^{1/2}}{\sqrt{M}}\Big) \\
&\leq C\Big(h + \frac{1}{\sqrt{M}}\Big).
\end{aligned}$$

Hence, to obtain a small RMSE, it is necessary to generate many expensive samples ($M \approx h^{-2}$) on a grid with small refinement parameter $h > 0$. To this end, several techniques have been developed to increase efficiency, a particularly effective example is the *multilevel Monte Carlo method*. Invented by Heinrich in [106] and popularized by Giles in [92] for the pricing of financial derivatives, multilevel Monte Carlo has been applied to a vast range of uncertainty quantification problems in the last decade, see e.g. [28, 29, 54, 66, 120, 194], and Chapters 4 and 6 of this thesis.

To illustrate the main idea behind multilevel Monte Carlo estimation, consider again the aforementioned example where $X = \Phi(u)$ is the functional of a solution to a random PDE. Rather than a fixed parameter h, we now consider a sequence of decreasing refinements $h_0 > \cdots > h_L > 0$ with $L \in \mathbb{N}$ (e.g. $h_\ell := 2^{-\ell-1}$) and the associated hierarchy of approximated random variables $X_0 := \Phi(u_{h_0}), \ldots, X_L := \Phi(u_{h_L})$. With the convention $X_{-1} := 0$, we may express the highest-accuracy approximation X_L by the telescopic sum

$$X_L = \sum_{\ell=0}^{L} X_\ell - X_{\ell-1}.$$

Using this representation, each correction $X_\ell - X_{\ell-1}$ is estimated by the standard Monte Carlo method using a level-dependent number of $M_\ell \in \mathbb{N}$ independent samples. This yields the *multilevel Monte Carlo estimator*

$$E^L(X_L) := \sum_{\ell=0}^{L} E_{M_\ell}(X_\ell - X_{\ell-1}) = \sum_{\ell=0}^{L} \frac{1}{M_\ell} \sum_{i=1}^{M_\ell} X_\ell^{(i,\ell)} - X_{\ell-1}^{(i,\ell)} \tag{2.5}$$

of $\mathbb{E}(X_L)$. In Eq. (2.5), it is crucial to actually sample from the distribution of the

correction term $X_\ell - X_{\ell-1}$, meaning the samples $X_\ell^{(i,\ell)}$ and $X_{\ell-1}^{(i,\ell)}$ *must* be based on the same set of random variables. This dependency is emphasized by the second superscript ℓ, the sampled corrections across the levels are again independent. Since $\mathbb{E}(E^L(X_L)) = \mathbb{E}(X_L)$, the RMSE of the multilevel Monte Carlo estimator is bounded by

$$\mathbb{E}((E^L(X_L) - \mathbb{E}(X))^2)^{1/2} \leq C\Big(h_L + \Big(\sum_{\ell=0}^{L} \frac{\mathrm{Var}(X_\ell - X_{\ell-1})}{M_\ell}\Big)^{1/2}\Big).$$

Given that the variances $\mathrm{Var}(X_\ell - X_{\ell-1})$ decay sufficiently fast in $\ell = 0, \ldots, L$, one may generate many inexpensive samples on the lower levels and, due to the lower variance for large ℓ, it is sufficient to sample only a few expensive corrections on the higher levels. This yields a fast decreasing sequence $M_0 > \cdots > M_L$ and in the example above a RMSE of $\mathcal{O}(2^{-L})$ is achieved with L levels, refinements $h_\ell \approx 2^{-\ell}$ and level-dependent numbers of samples decaying from $M_0 \approx 2^{2L}$ to $M_L \approx L^2$, as for instance shown in Chapter 4. To obtain a similar RMSE with a singlelevel Monte Carlo estimator, $M \approx 2^{2L}$ samples of the high resolution approximation X_L with $h_L \approx 2^{-L}$ are required. Thus, multilevel Monte Carlo approaches can reduce computational time by several orders of magnitudes compared to standard Monte Carlo estimation, from months to days, hours and even to minutes. The precise computational gains from multilevel Monte Carlo are stated under suitable assumptions in the, by now famous, complexity theorem in [92]. Due to their low requirements and good accessibility, multilevel Monte Carlo methods are the algorithms of choice to estimate quantities of interest or moments from PDEs with discontinuous random features.

Apart from sampling-based methods, such as Monte Carlo; other approaches to discretize the stochastic domain, e.g. *stochastic Galerkin* (SG) and *stochastic collocation* (SC) methods have been developed for uncertainty quantification. Here, the random PDEs are regarded as a class of high-dimensional parametrized PDEs to exploit certain structural properties of the problem. A very prominent example is the elliptic diffusion problem as in Eq. (2.1) with a (strictly) positive random field $a : \Omega \to L^\infty(\mathcal{D})$ as coefficient. It can be shown that a admits the Karhunen-Loève expansion

$$a = \mathbb{E}(a) + \sum_{i \in \mathbb{N}} \varphi_i \xi_i \tag{2.6}$$

with respect to a basis $(\varphi_i, i \in \mathbb{N}) \subset L^\infty(\mathcal{D})$ and a sequence $(\xi_i, i \in \mathbb{N})$ of centered random variables (see also Theorem 3.1.6 in Chapter 3). The corresponding parameter space is then formed by the range of $(\xi_i, i \in \mathbb{N})$, for instance by $[-1, 1]^{\mathbb{N}}$ if the ξ_i are uniformly distributed on $[-1, 1]$.

Both, SG and SC methods, have successfully been applied to PDEs with random

coefficients, see for instance [14, 25, 77, 90, 165] and the references therein. In particular, the articles [63, 190] need to be mentioned, where the authors show that SG methods may be superior to sampling-based algorithms, provided the underlying problem is sufficiently smooth with respect to its stochastic domain. Translated to the example above, this means essentially that $\|\varphi_i\|_{L^\infty(\mathcal{D})}$ decays fast enough with respect to the index i. Then, the series in Eq. (2.6) may be cut off after only a few terms to obtain a reasonably good representation of a. In the jump-diffusion setting from Chapters 4–6, however, a involves random jumps and hence the decay of $\|\varphi_i\|_{L^\infty(\mathcal{D})}$ is very slow. Consequently, a large number of terms in the truncated Karhunen-Loève expansion of a are necessary and SG/SC methods are prohibitive. Moreover, it is still an open question to find suitable basis functions φ_i and random variables ξ_i in order to represent discontinuous coefficients as in Eq. (2.6).

Another popular evolution of the Monte Carlo algorithm is the class of *quasi-Monte Carlo* (QMC) methods. In this approach, the pseudo-random samples are replaced by deterministic low discrepancy sequences to achieve a higher convergence rate of the RMSE than $\mathcal{O}(M^{-1/2})$ with respect to the number of samples. The estimation of moments in uncertainty quantification applications is then generally regarded as a quadrature problem in a high-dimensional parameter space and it is possible to recover improved rates of order $\mathcal{O}(M^{-1+\varepsilon})$ for arbitrary small $\varepsilon > 0$. A detailed introduction to QMC for integration in high dimensions can be found in [72]. QMC approaches have been applied to random PDEs with diffusion coefficients as in Eq. (2.6) (see [88, 137]) under similar regularity assumptions on the coefficients as SG/SC methods. More recently, however, QMC approaches were also used in the case that a is log-normal, see [96, 98, 108], and a reasonable truncated series-approximation of a involves a large number of terms. This development indicates that QMC techniques may also be promising for random PDEs with discontinuous coefficients, especially since they may be combined with multilevel Monte Carlo as in [108] (see also the discussion in Chapter 9). In addition, QMC methods are based on the evaluation of inverse cumulative density functions and may thus be utilized in the Fourier inversion algorithm from Chapter 7. Naturally, a further topic for research is then the application of QMC methods to SPDEs with Lévy noise as in Chapter 8.

3 Lévy processes and stochastic integrals in Hilbert spaces

The aim of this chapter is to familiarize the reader with random fields, stochastic processes and -integration in general Hilbert spaces, concepts that are necessary to understand the variety of random objects in the core of this thesis. The first part contains details about covariance operators and spectral expansions, which are recurrently used in all articles of this thesis. Thereafter, some results on Hilbert space-valued Lévy processes and stochastic integration are collected, providing the foundation for the driving noise processes in Chapters 7 and 8. Thus, the style of this chapter is naturally of more mathematical-formal and contrasts somewhat the preceding one, where the key ideas of each discretization method are roughly sketched and followed by a discussion.

The basic concept of Lévy processes in one dimension dates back to Lévy and Khintchine in [127, 145, 146, 147]. Since then, there has been extensive research on Lévy processes, well-known standard works include for instance [6, 40, 187]. The stochastic integrals and the corresponding calculus in Chapter 8 is in the sense of Itô, who introduced a generalization of the Riemann-Stieltjes integral with a Brownian motion as integrator in [118, 119]. For a basic review on Itô calculus with respect to Brownian motions see [125, 168], more general Lévy processes as integrators are discussed in [6]. The stochastic calculus for the corresponding Hilbert-space valued processes is investigated in [67, 158, 173]. Lévy processes and stochastic integration have become two of the most important concepts in probability theory and have been utilized to model various problems in financial mathematics and the natural sciences, examples are given in [17, 67, 94, 173, 188] to name just a few.

Throughout this chapter, let $\mathbb{T} = [0, T]$ for $T > 0$ be a finite time interval and let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \in \mathbb{T}), \mathbb{P})$ be a filtered probability space satisfying the usual conditions, i.e. $\mathcal{F}_0$ is $\mathbb{P}$-complete and $(\mathcal{F}_t, t \in \mathbb{T})$ is right-continuous. For any separable metric space $(E, \|\cdot\|_E)$, the *Borel σ-algebra* with respect to E is defined as the smallest σ-algebra containing all open balls in E and denoted by $\mathcal{B}(E)$. Furthermore, let $(U, (\cdot, \cdot)_U)$ be a separable Hilbert space, let $L(U)$ be the set of all linear operators on U and let

$L_N^+(U) \subset L(U)$ be the set of all nonnegative, symmetric, nuclear operators on U. The space of all Bochner-integrable random variables in U with p-th moment is for any $p \in [1, \infty)$ given by

$$L^p(\Omega; U) := \{X : \Omega \to U \text{ is strongly measurable and } \int_\Omega \|X(\omega)\|_U^p < +\infty\}.$$

3.1 Covariance operators and spectral expansions

As the L^2-theory of random fields is investigated in this chapter, it is natural to consider covariance operators and the associated spectral expansions. To this end, the first two moments for square-integrable, U-valued random variables need to be defined.

Definition 3.1.1. For any $X \in L^1(\Omega; U)$, the *mean* of X is defined by the Lebesgue-Bochner integral

$$\mathbb{E}(X) := \int_\Omega X(\omega) d\mathbb{P}(\omega) \in U.$$

If $X \in L^2(\Omega; U)$, the *covariance operator* of X is the unique $Q \in L_N^+(U)$ such that

$$\mathbb{E}((X - \mathbb{E}(X), \phi)_U (X - \mathbb{E}(X), \psi)_U) = (Q\phi, \psi)_U, \quad \phi, \psi \in U.$$

The following results ensures that the covariance operator $Q \in L_N^+(U)$ is well-defined and unique for each square-integrable X:

Lemma 3.1.2. *For any $X \in L^2(\Omega; U)$, there is a unique covariance operator $Q \in L_N^+(U)$.*

Proof. For the existence of Q see [138, Chapter 2]. Now assume that there exist two covariance operators $Q, \widetilde{Q} \in L_N^+(U)$ for $X \in L^2(\Omega; U)$. By Definition 3.1.1 and the linearity of $Q, \widetilde{Q}$ it follows

$$\begin{aligned} \|Q - \widetilde{Q}\|_{L(U)}^2 &= \sup_{\phi \in U, \phi \neq 0} \frac{\|(Q - \widetilde{Q})\phi\|_U^2}{\|\phi\|_U^2} \\ &= \sup_{\phi \in U, \phi \neq 0} \frac{(Q\phi, (Q - \widetilde{Q})\phi)_U^2 - (\widetilde{Q}\phi, (Q - \widetilde{Q})\phi)_U^2}{\|\phi\|_U^2} \\ &= 0, \end{aligned}$$

and hence $Q = \widetilde{Q}$. □

By the Hilbert-Schmidt theorem, there is a decreasing sequence of non-negative eigenvalues $\eta_1 \geq \eta_2 \geq \cdots \geq 0$ of Q with zero as only accumulation point and the

corresponding eigenfunctions $(e_i, i \in \mathbb{N})$ form an orthonormal basis of U. Moreover, Q is nuclear and therefore of *trace class*, i.e.

$$\mathrm{Tr}(Q) := \sum_{i\in\mathbb{N}} (Qe_i, e_i)_U = \sum_{i\in\mathbb{N}} \eta_i < +\infty.$$

The *square-root* of Q is defined via

$$Q^{1/2}\phi := \sum_{i\in\mathbb{N}} \sqrt{\eta_i}(\phi, e_i)_U e_i, \quad \phi \in U.$$

Since $Q^{1/2}$ is not necessarily injective, the *pseudo-inverse* of Q is given by

$$Q^{-1/2}\varphi := \phi, \quad \text{if } Q^{1/2}\phi = \varphi \text{ and } \|\phi\|_U = \inf_{\varphi\in U:\, Q^{1/2}\varphi=\phi} \{\|\varphi\|_U\}.$$

Definition 3.1.3. Let $X \in L^2(\Omega; U)$ with covariance operator $Q \in L_N^+(U)$. Then, the set $\mathcal{U} := Q^{1/2}(U)$ equipped with the scalar-product

$$(\varphi_1, \varphi_2)_{\mathcal{U}} := (Q^{-1/2}\varphi_1, Q^{-1/2}\varphi_2)_U, \quad \varphi_1, \varphi_2 \in \mathcal{U},$$

is called the *reproducing kernel Hilbert space* (RKHS) of X.

It turns out to be more convenient to study the RKHS $\mathcal{U}$ rather than Q, as restricting U to suitable subspaces does not change the RKHS:

Theorem 3.1.4. *[173, Theorem 7.4] Let $X \in L^2(\Omega; U)$ with covariance operator $Q \in L_N^+(U)$ and let the Hilbert space $(U, (\cdot,\cdot)_U)$ be continuously embedded into another Hilbert space $(\widetilde{U}, (\cdot,\cdot)_{\widetilde{U}})$. If $\widetilde{Q} \in L_N^+(\widetilde{U})$ denotes the covariance operator of X considered as $\widetilde{U}$-valued random variable $X \in L^2(\Omega; \widetilde{U})$, then $Q^{1/2}(U) = \widetilde{Q}^{1/2}(\widetilde{U})$.*

Theorem 3.1.4 becomes particularly useful if X takes values in a Sobolev space and the corresponding embedding theorems are at hand.

Example 3.1.5. In the important case that $U = L^2(\mathcal{D})$ for some compact $\mathcal{D} \subset \mathbb{R}^d$, it can be shown that Q and $Q^{1/2}$ are integral operators with symmetric, nonnegative definite kernel functions $k_Q, k_{Q^{1/2}} : \mathcal{D}^2 \to \mathbb{R}$, i.e.

$$Q\phi = \int_{\mathcal{D}} k_Q(x,\cdot)\phi(x)dx, \quad Q^{1/2}\phi = \int_{\mathcal{D}} k_{Q^{1/2}}(x,\cdot)\phi(x)dx, \qquad \phi \in U,$$

see e.g. [173, Appendix A]. By *Mercer's theorem*, the *reproducing kernel* k_Q may be

represented in terms of the eigenbasis of Q by

$$k(x,y) = \sum_{i\in\mathbb{N}} \eta_i e_i(x) e_i(y), \quad x,y \in \mathcal{D},$$

and hence $k_Q(x,\cdot) = k_Q(\cdot,x) \in \mathcal{U}$. Since any $\varphi \in \mathcal{U}$ is of the form

$$\varphi = Q^{1/2}\phi = \sum_{i\in\mathbb{N}} \sqrt{\eta_i}(\phi, e_i)_U e_i$$

for some $\phi \in U$, Mercer's theorem yields that k_Q satisfies for any $x \in \mathcal{D}$

$$\begin{aligned}(\varphi, k_Q(x,\cdot))_{\mathcal{U}} &= \sum_{i\in\mathbb{N}} (Q^{1/2}\phi, \eta_i e_i)_{\mathcal{U}} e_i(x) \\ &= \sum_{i\in\mathbb{N}} (Q^{1/2}\phi, Q^{1/2}e_i)_{\mathcal{U}} \sqrt{\eta_i} e_i(x) \\ &= \sum_{i\in\mathbb{N}} \sqrt{\eta_i}(\phi, e_i)_U e_i(x) \\ &= \varphi(x),\end{aligned}$$

also called the *reproducing kernel property*. Moreover, the RKHS $\mathcal{U}$ is given by

$$\mathcal{U} = \{\phi \in L^2(\mathcal{D}) | \sum_{i\in\mathbb{N},\, \eta_i\neq 0} \frac{(\phi, e_i)_U^2}{\eta_i} < +\infty\}, \quad (\phi,\psi)_{\mathcal{U}} = \sum_{i\in\mathbb{N},\, \eta_i\neq 0} \frac{(\phi, e_i)_U (\psi, e_i)_U}{\eta_i}.$$

To conclude this section, the following useful spectral representation of X is recorded.

Theorem 3.1.6 (Karhunen-Loève expansion)**.** *Let $X \in L^2(\Omega; U)$ with covariance operator Q and denote by $((\eta_i, e_i), i \in \mathbb{N})$ the (ordered) eigenpairs of Q. Then, X admits the expansion*

$$X = \mathbb{E}(X) + \sum_{i\in\mathbb{N}} \xi_i e_i,$$

where the ξ_i are real-valued, centered random variables with $Cov(\xi_i, \xi_j) = \eta_i \delta_{ij}$ for $i,j \in \mathbb{N}$. The above series converges in $L^2(\Omega; U)$ with truncation error bounded by

$$\| \sum_{i>N}^{\infty} \xi_i e_i \|^2_{L^2(\Omega;U)} := \mathbb{E}\Big(\| \sum_{i>N}^{\infty} \xi_i e_i \|_U^2 \Big) = \sum_{i>N} \eta_i, \quad N \in \mathbb{N}. \tag{3.1}$$

Proof. As $(e_i, i \in \mathbb{N})$ is an orthonormal basis in U, $X - \mathbb{E}(X)$ is expanded via

$$X - \mathbb{E}(X) = \sum_{i\in\mathbb{N}} ((X - \mathbb{E}(X)), e_i)_U e_i.$$

Now define $\xi_i := ((X - \mathbb{E}(X)), e_i)_U$. It is immediate that the ξ_i are centered and satisfy

$$\mathrm{Cov}(\xi_i, \xi_j) = \mathbb{E}((X - \mathbb{E}(X)), e_i)_U (X - \mathbb{E}(X)), e_j)_U) = (Qe_i, e_j)_U = \eta_i \delta_{ij}.$$

To show the convergence in $L^2(\Omega; U)$, let $X_N := \sum_{i=1}^N \xi_i e_i$ for any $N \in \mathbb{N}$ and note that

$$\mathbb{E}(\|X_N\|_U^2) = \mathbb{E}\Big(\|\sum_{i=1}^N \xi_i e_i\|_U^2\Big) = \sum_{i,j=1}^N \mathbb{E}((\xi_i e_i, \xi_j e_j)_U) = \sum_{i=1}^N \eta_i.$$

Since $\mathrm{Tr}(Q) = \sum_{i \in \mathbb{N}} \eta_i < +\infty$, X_N is a $L^2(\Omega; U)$-Cauchy-sequence and thus converges to the limit $X - \mathbb{E}(X)$ by the completeness of $L^2(\Omega; U)$. Consequently,

$$\mathbb{E}(\|X - X_N\|_U^2) = \mathbb{E}\Big(\|\sum_{i>N}^{\infty} \xi_i e_i\|_U^2\Big) = \sum_{i>N} \eta_i.$$

□

Example 3.1.7. If X in Theorem 3.1.6 is Gaussian with mean $\mu \in U$ and covariance operator $Q \in L_N^+(U)$, it holds that

$$X = \mu + \sum_{i \in \mathbb{N}} \sqrt{\eta_i} e_i Z_i, \tag{3.2}$$

where $(Z_i, i \in \mathbb{N})$ is a sequence of *independent*, one-dimensional standard normally distributed random variables (see for instance [3, Chapter 3]). The representation in Eq. (3.2) becomes particularly useful for the simulation of X: the sum may be truncated after a finite number of terms and one then only needs to sample from a one-dimensional standard normal distribution. For an introduction of Gaussian measures on Banach- and Hilbert spaces, see [138] or [173, Chapter 3.5].

Remark 3.1.8. Note that the independence for the random variables in the Karhunen-Loève expansion only holds for the Gaussian case in Example 3.1.7. For general $X \in L^2(\Omega; U)$ as in Theorem 3.1.6, the sequence $(\xi, i \in \mathbb{N})$ consists of *uncorrelated, but not independent* random variables (see also Proposition 3.2.9). As Q is of trace-class, the truncation error in Eq. (3.1) depends on the decay of the eigenvalues and can be made arbitrary small for sufficiently large N. A similar representation as in Eq. (3.2) is also given for Lévy processes in the next subsection, which is the fundamental observation for the approximation algorithm introduced in Chapter 7.

3.2 Lévy random fields

Hilbert space-valued Lévy fields form the basis of the driving noise processes in Chapters 7 and 8. In this section, the class of Lévy processes is introduced and a fundamental decomposition theorem in terms of Gaussian and compound Poisson processes is given.

Definition 3.2.1. A U-valued stochastic process $L = (L(t), t \in \mathbb{T})$ is called *Lévy process* if

- L has stationary and independent increments,
- $L(0) = 0$ $\mathbb{P}$-a.s. and
- L is stochastically continuous, i.e. for all $\varepsilon > 0$ and $t \in \mathbb{T}$ it holds

$$\lim_{s \to t,\, s \in \mathbb{T}} \mathbb{P}(||L(t) - L(s)||_U > \varepsilon) = 0.$$

To obtain a clear distinction from finite-dimensional Lévy processes, L is sometimes called *Lévy field* if $\dim(U) = +\infty$. Lévy processes have in general discontinuous trajectories, regardless of the dimensionality of U. It can be shown, however, that every Lévy process has a modification with càdlàg paths ([173, Theorem 4.3]). The only exceptional case with continuous trajectories are Hilbert-space valued Brownian motions or Wiener processes:

Definition 3.2.2. A zero mean Lévy process W on U with continuous trajectories is called a *Wiener process*.

As subsequently outlined, every Lévy process is the sum of a continuous Wiener process and certain discontinuous jump processes. To this end, the following basic results for Hilbert space-valued Wiener processes are recorded.

Theorem 3.2.3. *[67, Proposition 4.1] A Wiener process W on U is a Gaussian, square-integrable process and $W(t) \sim \mathcal{N}(0, tQ)$, where Q is the covariance operator of $W(1)$. Moreover, if $((\eta_i, e_i), i \in \mathbb{N})$ are the orthonormal eigenpairs of Q, then W admits the series expansion*

$$W(t) = \sum_{i \in \mathbb{N}} (W(t), e_i)_U e_i, \quad t \in \mathbb{T},$$

where $((W(t), e_i)_U, t \in \mathbb{T})$ are (scaled) independent, real-valued Wiener processes with covariance $Cov((W(t), e_i)_U, (W(s), e_i)_U) = \min(t, s)\eta_i$.

Since W is centered Gaussian, the characteristic function of W reads

$$\mathbb{E}(e^{i(W(t),\phi)_U}) = \exp\Big(-\frac{t}{2}(Q\phi,\phi)_U\Big), \quad \phi \in U. \tag{3.3}$$

There is also a U-valued version of a compound Poisson process, which will be the other building block for the Lévy process.

Definition 3.2.4. Let $\widetilde{\nu}$ be a finite measure on $(U, \mathcal{B}(U))$ with $\widetilde{\nu}(\{0\}) = 0$. A *compound Poisson process* with *jump intensity measure* $\widetilde{\nu}$ on U is a càdlàg Lévy process L on U satisfying

$$\mathbb{P}(L(t) \in \widehat{U}) = e^{-\widetilde{\nu}(U)t} \sum_{k\in\mathbb{N}} \frac{t^k}{k!} \widetilde{\nu}^{*k}(\widehat{U}), \quad \widehat{U} \in \mathcal{B}(U),\ t \in \mathbb{T}.$$

Above, $\widetilde{\nu}^{*k}$ denotes the k-fold convolution of $\widetilde{\nu}$, given by

$$\widetilde{\nu}^{*k}(\widehat{U}) := (\widetilde{\nu} * \cdots * \widetilde{\nu})(\widehat{U}), \quad (\widetilde{\nu} * \widetilde{\nu})(\widehat{U}) := \int_U \widetilde{\nu}(\widehat{U} - \phi)\widetilde{\nu}(d\phi).$$

Theorem 3.2.5. *[173, Chapter 4.3] Let L be a compound Poisson process on U with jump intensity measure $\widetilde{\nu}$.*

1. *The characteristic function of L is given by*

$$\mathbb{E}(e^{i(L(t),\phi)_U}) = \exp\Big(-t\int_U 1 - e^{i(\psi,\phi)_U}\widetilde{\nu}(d\psi)\Big), \quad \phi \in U,\ t \in \mathbb{T}.$$

2. *There is a sequence of i.i.d. U-valued random variables $(\Xi, i \in \mathbb{N})$ with law $\Xi_i \sim \frac{\widetilde{\nu}}{\widetilde{\nu}(U)}$ and a Poisson process $N = (N(t), t \in \mathbb{T})$ with intensity $\widetilde{\nu}(U)$, independent of $(\Xi, i \in \mathbb{N})$, such that*

$$L(t) = \sum_{i=1}^{N(t)} \Xi_i.$$

3. *The compound Poisson process L is integrable if and only if*

$$\int_U \|\phi\|_U \widetilde{\nu}(d\phi) < +\infty.$$

If L is integrable,

$$\mathbb{E}(L(t)) = t\int_U \phi\widetilde{\nu}(d\phi), \quad t \in \mathbb{T},$$

and $(L(t) - \mathbb{E}(L(t)), t \in \mathbb{T})$ is called compensated compound Poisson process.

With the notions of Wiener processes and compound Poisson processes on U, the most important representation result for Lévy processes, known as the *Lévy-Khintchine decomposition*, can now be stated.

Theorem 3.2.6. *[173, Theorem 4.23 (Lévy-Khintchine decomposition)] Every Lévy process L has the representation*

$$L(t) = \mu_0 t + W(t) + \sum_{k\in\mathbb{N}} L_{A_k}(t) - t\int_{A_k} \phi\nu(d\phi) + L_{A_0}(t), \quad t \in \mathbb{T}, \tag{3.4}$$

where

- $\mu_0 \in U$ *is a deterministic mean function,*
- W *is a Wiener process with covariance operator* $Q \in L_N^+(U)$*,*
- $A_k := \{\phi \in U \,|\, r_k \leq \|\phi\|_U \leq r_{k-1}\}$ *and* $A_0 := \{\phi \in U \,|\, \|\phi\|_U \geq r_0\}$ *for an arbitrary sequence* $(r_k, k \in \mathbb{N}) \subset \mathbb{R}_{>0}$*, decreasing monotonously to zero,*
- ν *is a measure on* $(U, \mathcal{B}(U))$ *such that* $\int_U \min(\|y\|_U^2, 1)d\nu(y) < +\infty$ *(also called Lévy measure) and* $(L_{A_k}, k \in \mathbb{N})$ *are compound Poisson processes with finite intensity measures* $\widetilde{\nu}_k := \nu|_{A_k}$*, and*
- *all summands in Eq.* (3.4) *are independent stochastic processes on* U*.*

Remark 3.2.7. The Lévy-Khintchine decomposition states that every Lévy process is the sum of a Wiener process W with drift μ_0, a compound Poisson process and a superposition of compensated compound Poisson processes. It needs to be emphasized that changing the cutoff treshold r_0 in the Lévy-Khintchine decomposition affects the mean μ_0. Hence, the above representation depends on r_0 and is a-priori not unique. As seen in Theorem 3.2.8 below, uniqueness can be achieved by fixing $r_0 = 1$, the corresponding representation is also referred to as *Lévy-Itô decomposition*. If $\nu(U) < +\infty$, then L may be represented by a finite number of (compensated) compound Poisson processes. In this case, L has almost surely finitely many jumps in every bounded interval in $\mathbb{T}$ and L is said to be of *finite activity*.

Due to the Lévy-Khintchine decomposition, the following is immediate:

Theorem 3.2.8 (Lévy-Khintchine formula)**.** *Let L be a Lévy process on U. Then, the characteristic function of $L(t)$ is given by*

$$\mathbb{E}[\exp(i(\phi, L(t))_U)] = \exp(t\Psi_L(\phi)), \quad \phi \in U,\, t \in \mathbb{T},$$

with characteristic exponent

$$\Psi_L(\phi) = i(\mu, \phi)_U - \frac{1}{2}(Q\phi, \phi)_U + \int_U e^{i(\phi,\psi)_U} - 1 - i(\phi, \psi)_U \mathbf{1}_{||\psi||_U<1}(\psi)\nu(d\psi).$$

Proof. Due to the independence of all terms in Eq. (3.4), the characteristic exponent is of additive form. The claim follows with the characteristic functions of the Wiener process and the compound Poisson processes from Eq. (3.3) and Theorem 3.2.5, respectively. Using $r_0 = 1$ in Theorem 3.2.6 yields the indicator function $\mathbf{1}_{||\psi||_U<1}$ in Ψ_L. For a more detailed proof, see e.g. [173, Chapter 4] and the references therein. □

Compared with the Lévy-Khintchine decomposition, the Lévy-Khintchine formula assumes a cutoff at $r_0 = 1$ for the non-compensated Poisson process. Hence, the above representation is unique and L is uniquely determined by (μ, Q, ν), also known as the *characteristic triplet*. Similar to the Wiener process, L may be expanded in terms of an orthonormal basis of U and one-dimensional Lévy processes.

Proposition 3.2.9. *[173, Theorem 4.39] If $(\phi_i, i \in \mathbb{N})$ is an orthonormal basis of U, then L has the series expansion*

$$L(t) = \sum_{i\in\mathbb{N}} (L(t), \phi_i)_U \phi_i, \quad t \in \mathbb{T},$$

and the processes $(L(t), \phi_i)_U$ are real-valued Lévy processes with càdlàg paths. The above series converges in probability uniformly in $\mathbb{T}$.

If L is square-integrable, the processes $(L(t), \phi_i)_U$ are uncorrelated and independence only holds in the case that $L = W$ is a Wiener processes on U with the marginals $(L(t), \phi_i)_U$ becoming real-valued Brownian motions. To obtain an expansion with respect to the eigenbasis of the covariance operator of L, it is essential that L is actually square-integrable. As seen in the next result, the integrability of L depends solely on the moments of ν.

Theorem 3.2.10. *Let L be a U-valued Lévy process with characteristic triplet (μ, Q, ν). For any $t \in \mathbb{T}$ and $n \in \mathbb{N}$ it holds that*

$$\mathbb{E}(\|L(t)\|_U^n) < +\infty \quad \Leftrightarrow \quad \int_{\phi\in U, \|\phi\|\geq 1} \|\phi\|_U^n \nu(d\phi) < +\infty.$$

Moreover, if L is square-integrable, there exits $\mu_L \in U$ and $Q_L \in L_N^+(U)$ such that for any $s, t \in \mathbb{T}$ and $\phi, \psi \in U$:

$$\begin{aligned} \mathbb{E}((L(t), \phi)_U) &= t(\mu_L, \phi)_U \\ \mathbb{E}((L(t) - t\mu, \phi)_U (L(s) - s\mu, \psi)_U) &= \min(t, s)(Q_L\phi, \psi)_U \\ \mathbb{E}(\|L(t) - t\mu_L\|_U^2) &= t\, Tr(Q_L). \end{aligned}$$

Proof. The first part of the claim follows from the Lévy-Khintchine decomposition in the same fashion as for finite-dimensional Lévy processes, see e.g. [6, Theorem 2.5.2]. The second part is given in [173, Theorem 4.44]. □

Remark 3.2.11. In general, μ_L and Q_L above are not identical to μ resp. Q from the Lévy triplet, but merely $\mu_L = \mu + \mu_\nu$ and $Q_L = Q + Q_\nu$, with μ_ν, Q_ν stemming from the jump intensity measure ν. However, as Q_L has an orthonormal eigenbasis $(e_i, i \in \mathbb{N})$ with corresponding eigenvalues $\eta_i \geq 0$, L has the series expansion

$$L(t) = \sum_{i \in \mathbb{N}} (L(t), e_i)_U e_i, \quad t \in \mathbb{T},$$

where the one-dimensional marginal Lévy processes $(L(t), e_i)_U$ are uncorrelated and

$$\mathbb{E}((L(t), e_i)_U) = t(\mu_L, e_i)_U, \quad \text{and} \quad \mathrm{Var}((L(t), e_i)_U) = t\eta_i.$$

3.3 Stochastic integration

This section provides a construction of stochastic integrals in Itô's sense with square-integrable Lévy processes on U as integrator. The fundamental property of the corresponding Itô integrals, an infinite-dimensional version of the *Itô isometry*, is stated as the central result at the end of this section. For completeness, it needs to be mentioned that the class of integrators may be generalized to the space of all square-integrable, U-valued martingales, see [173, Chapter 8]. However, as only square-integrable Lévy processes have been considered in Chapters 7 and 8, this section is restricted to the case that L is a centered and square-integrable Lévy process on U for the sake of brevity.

Let Q_L be the covariance operator of L, and observe that by Theorem 3.2.10 the processes $L = (L(t), t \in \mathbb{T})$ and $(\|L(t)\|_U^2 - t\, tr(Q_L), t \in \mathbb{T})$ are U- and real-valued martingales, respectively. Furthermore, let $\mathcal{U} := Q_L^{1/2}(U)$ be the RKHS associated to L and let $(H, (\cdot,\cdot)_H)$ be another separable Hilbert space, not necessarily identical to U. The set of all linear operators from U to H is given by $L(U, H)$ and $L_{HS}(\mathcal{U}, H)$ denotes the space of all Hilbert-Schmidt operators from $\mathcal{U}$ to H.

The construction of the stochastic integral is based on the following class of piecewise constant, $L(U, H)$-valued integrands.

Definition 3.3.1. A $L(U, H)$-valued process $\Psi : \mathbb{T} \times \Omega \to L(U, H)$ is called a *simple process* if there are $t_0, \ldots, t_m \in \mathbb{T}$, operators $\Psi_j \in L(U, H)$ and events $\mathcal{A}_j \in \mathcal{F}_{t_j}$ for

$j = 0, \dots, m-1$ such that

$$\Psi(t) = \sum_{j=0}^{m-1} \mathbf{1}_{\mathcal{A}_j} \mathbf{1}_{[t_j, t_{j+1})}(t) \Psi_j, \quad t \in \mathbb{T}.$$

The set of all $L(U,H)$-valued simple processes is denoted by $\mathcal{S}(U,H)$.

For any simple process $\Psi \in \mathcal{S}(U,H)$, define the stochastic integral

$$I_\Psi(t) = \int_0^t \Psi(s) dL(s) := \sum_{j=1}^{m-1} \mathbf{1}_{\mathcal{A}_j} \Psi_j (L(\min(t, t_{j+1})) - L(\min(t, t_j))).$$

It is straightforward to see that the process $(I_\Psi(t), t \in \mathbb{T})$ is a H-valued, square-integrable martingale with zero mean (as L is centered). Since L has independent increments, there holds the *Itô isometry* for simple integrals given by

$$\|\Psi\|_{L,t}^2 := \mathbb{E}\Big(\| \int_0^t \Psi(s) dL(s) \|_H^2 \Big) = \int_0^t \mathbb{E}(\|\Psi(s)\|_{L_{HS}(\mathcal{U},H)}^2) ds$$

for any $t \in \mathbb{T}$. Note that $\|\cdot\|_{L,T}$ defines a seminorm on $\mathcal{S}(U,H)$ and let $\mathcal{L}_{L,T}^2(H)$ be the completion of $(\mathcal{S}(U,H), \|\cdot\|_{L,T})$. With this at hand, the class of admissible integrands is naturally extended to $\mathcal{L}_{L,T}^2$ via

$$I_\Psi(t) = \int_0^t \Psi(s) dL(s) := \lim_{n\to\infty} \int_0^t \Psi_n(s) dL(s), \quad \Psi \in \mathcal{L}_{L,T}^2(H),$$

where $(\Psi_n, n \in \mathbb{N}) \subset \mathcal{S}(U,H)$ is an approximating sequence of Ψ and the limit is taken in the $L^2(\Omega; H)$-sense. To conclude this chapter, $\mathcal{L}_{L,T}^2(H)$ is characterized as the space of all *predictable*, $L_{HS}(\mathcal{U},H)$-valued processes and the Itô isometry is generalized to the space of all admissible integrands $\mathcal{L}_{L,T}^2(H)$:

Theorem 3.3.2. *[173, Chapter 8.6] Let L be a square-integrable, zero-mean Lévy process on U with covariance operator Q_L and RKHS given by $\mathcal{U} = Q_L^{1/2}(U)$. Then, the space of admissible integrands for L is*

$$\mathcal{L}_{L,T}^2(H) = \{\Psi : \mathbb{T} \times \Omega \to L_{HS}(\mathcal{U},H) \,|\, \Psi \text{ is square-integrable and predictable}\}.$$

Moreover, for any $\Psi \in \mathcal{L}_{L,T}^2(H)$, the process $(I_\Psi(t), t \in T)$ is a centered, square-integrable martingale, and for $t \in \mathbb{T}$ it holds that

$$\mathbb{E}\Big(\| \int_0^t \Psi(s) dL(s) \|_H^2 \Big) = \int_0^t \mathbb{E}(\|\Psi(s) Q_L^{1/2}\|_{L_{HS}(U,H)}^2) ds = \int_0^t \mathbb{E}(\|\Psi(s)\|_{L_{HS}(\mathcal{U},H)}^2) ds.$$

Declaration to the cumulative part

The central part consists of the five research articles in Chapters 4–8 and the publication/review status of each article is:

- Andrea Barth and Andreas Stein, "A study of elliptic partial differential equations with jump diffusion coefficients": published in *ASA/SIAM Journal on Uncertainty Quantification*, 2018, SIAM, volume 6, issue 4, pp. 1707–1742.

- Andrea Barth and Andreas Stein, "Numerical analysis for time-dependent advection-diffusion problems with random discontinuous coefficients": submitted to *Stochastics and Partial Differential Equations: Analysis and Computations* July 2020, currently in the first stage of review, preprint available on ArXiv.

- Andrea Barth and Andreas Stein, "A multilevel Monte Carlo algorithm for parabolic advection-diffusion problems with discontinuous coefficients": published in *Proceedings of the 13th International Conference in Monte Carlo & Quasi-Monte Carlo Methods in Scientific Computing*, 2020, Springer.

- Andrea Barth and Andreas Stein, "Approximation and simulation of infinite-dimensional Lévy processes": published in *Stochastics and Partial Differential Equations: Analysis and Computations*, 2018, Springer, volume 6, issue 2, pp. 286–334.

- Andrea Barth and Andreas Stein, "A stochastic transport problem with Lévy noise: Fully discrete numerical approximation": submitted to *BIT Numerical Mathematics* in October 2019, currently in the second stage of review, preprint available on ArXiv.

I hereby declare that the list of co-authors is exhaustive and that I have not reproduced, without acknowledgement, the work of another. I have majorly contributed to all research articles above, including particularly the phase of problem selection, literature research, the derivation of theoretical results, conduct and evaluation of numerical experiments as well as the writing process.

Compared to the published/submitted version of the articles, I have modified Chapters 4–8 in the following way:

- Some of the figures have been generated newly to obtain a better resolution and a consistent style throughout this thesis. They are, however, based on the exact same data that has been used in the corresponding research article.
- As far as possible, notation has been homogenized throughout Chapters 4–8.
- A short comment on the results has been added at the end of Chapters 4 and 6.
- All articles have been reformatted in the style of this thesis.
- A few typos and grammatical errors have been corrected.
- The bibliography for all articles has been unified and is given at the end of this thesis.

Except for this minor alterations, all articles in Chapters 4–8 are reproductions of their corresponding published/submitted research papers.

Andreas Stein Stuttgart, August 2020

4 A study of elliptic partial differential equations with jump diffusion coefficients

Andreas Stein and Andrea Barth

First published in "ASA/SIAM Journal on Uncertainty Quantification", 2018, SIAM, volume 6, issue 4, pp. 1707–1742.

URL: https://epubs.siam.org/doi/abs/10.1137/17M1148888.

Abstract: As a simplified model for subsurface flows elliptic equations may be utilized. Insufficient measurements or uncertainty in those are commonly modeled by a random coefficient, which then accounts for the uncertain permeability of a given medium. As an extension of this methodology to flows in heterogeneous, fractured or porous media, we incorporate jumps in the diffusion coefficient. These discontinuities then represent transitions in the media. More precisely, we consider a second order elliptic problem where the random coefficient is given by the sum of a (continuous) Gaussian random field and a (discontinuous) jump part. To estimate moments of the solution to the resulting random partial differential equation, we use a path-wise numerical approximation combined with multilevel Monte Carlo sampling. In order to account for the discontinuities and improve the convergence of the path-wise approximation, the spatial domain is decomposed with respect to the jump positions in each sample, leading to path-dependent grids. Hence, it is not possible to create a sequence of grids which is suitable for each sample path a-priori. We address this issue by an adaptive multilevel algorithm, where the discretization on each level is sample-dependent and fulfills given refinement conditions.

4.1 Introduction

Uncertainty quantification plays an increasingly important role in a wide range of problems in the Engineering Sciences and Physics. Examples of sources of uncertainty are imprecise or insufficient measurements and noisy data. In the underlying dynamical system this is modeled via a stochastic operator, stochastic boundary conditions and/or stochastic data. As an example, to model subsurface flow more realistically the coefficient of an (essentially) elliptic equation is assumed to be stochastic. A common approach in the literature is to use (spatially) correlated random fields that are built from uniform distributions or colored log-normal fields. The resulting marginal distributions of the field are (shifted) normally, resp. log-normally distributed. Neither choice is universal enough to accommodate all possible types of permeability, especially not if fractures are incorporated (see [202]), the medium is very heterogeneous or porous.

The last decade has been an active research period on elliptic equations with random data. A non-exhaustive list of publications in this field includes [1, 14, 15, 29, 59, 63, 85, 148, 165, 190, 194]. One can find various ways to approximate the distribution or moments of the solution to the elliptic equation. Next to classical Monte Carlo methods, their multilevel variants and other variance reduction techniques have been applied. The concept of multilevel Monte Carlo simulation has been developed in [106] to calculate parametric integrals and has been rediscovered in [92] to estimate the expected value of functionals of stochastic differential equations. Ever since, multilevel Monte Carlo techniques have been successfully applied to various problems, for instance in the context of elliptic random PDEs in [1, 29, 59, 148, 194] to just name a few. These sampling algorithms are fundamentally different from approaches using Polynomial Chaos. The latter suffer from the fact that one is rather restricted when it comes to possibilities to model the stochastic coefficient. While in the case of fields built from uniform distributions or colored log-normal fields these algorithms can outperform sampling strategies, approaches – like stochastic Galerkin methods – are less promising in our discontinuous setting due to the rather involved structure of the coefficient. In fact, it is even an open problem to define them in the case that a Lévy-field is used.

Our main objective in this paper is to show existence and uniqueness of the solution to the elliptic equation when the coefficient is modeled as a jump-diffusion. By that we mean a field which consists of a deterministic, a Gaussian and non-continuous part. As we show in the numerical examples, this jump-diffusion coefficient can be used to model a wide array of scenarios. This generalizes the work in [148] and uses partly [113]. To approximate the expectation of the solution we develop and test, further, variants of the

multilevel Monte Carlo method which are tailored to our problem: namely adaptive [1] and coupled multilevel Monte Carlo methods. Adaptivity is needed in the jump-diffusion setting, since the coefficient is not continuous. Our analysis shows that the non-adaptivity in a multilevel setting with non-continuous coefficients entails a larger error than when an adaptive algorithm is used. This result is not surprising, since the essence of the multilevel algorithm is that many samples are calculated on coarse grids, where the distributional error from misjudging jump-locations is high. Adaptivity, however, comes to the price that in certain scenarios solving the underlying system of equation becomes computationally expensive. In these settings the advantageous time-to-error performance of adaptive methods may be worse. As a simplified version of *Multifidelity Monte Carlo* sampling (see [171]), we introduce a coupled multilevel Monte Carlo estimator that reuses samples across levels and is preferred when sampling from a certain distribution is computationally expensive. The coupled algorithms outperform, in general, algorithms with a standard sampling strategy of multilevel Monte Carlo, as it actually reduces the mean square error.

In Section 4.2 we introduce the model problem, define *path-wise weak solutions* of random partial differential equations (PDEs) and show almost sure existence and uniqueness under relatively weak assumptions on the model parameters. The main contribution of this section is the existence and uniqueness result in Theorem 4.2.5, which is then readily transferred to the special case of a jump-diffusion problem in the subsequent sections of this article. In Section 4.3 we define the jump-diffusion coefficient and construct suitable approximations. Both stochastic parts of the jump-diffusion coefficient are approximated: The Gaussian one by a standard truncation of its basis representation, and the jump part (if direct sampling from the jump distribution is not possible) by a technique based on Fourier Inversion. We show L^p-type convergence for all existing moments of the approximation. From this result convergence of the approximated solution follows immediately. The approximated solution has still to be discretized to actually estimate moments of it (Section 4.4). The Galerkin-type discretization is directly furthered into an adaptive scheme. Section 4.5 then introduces the sampling methods that are used, namely Monte Carlo, multilevel Monte Carlo and a coupled variant of it. An extensive discussion of numerical examples in one and two dimensions, in Section 4.6, concludes the paper.

[1] Throughout this chapter, the term "adaptive" refers to finite element grids that are aligned *a-priori* to the discontinuities of the diffusion coefficient. This is contrast to the "adaptive finite element method" which is based on *a-posteriori* error estimates and in general includes several stages of mesh refinement. To avoid confusion, the terminology has been changed to "sample-adapted grids" instead of "adaptive grids" in the next two chapters. For this chapter, however, I decided to keep the terminology according to the original article and inserted this footnote to clarify.

4.2 Elliptic boundary value problems and existence of solutions

We consider the following random elliptic equation in a general setting before we specify in Section 4.3 our precise choice of coefficient function. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\mathcal{D} \subset \mathbb{R}^d$, for some $d \in \mathbb{N}$, be a bounded and connected Lipschitz domain. In this paper we consider the linear, random elliptic problem

$$-\nabla \cdot (a(\omega, x)\nabla u(\omega, x)) = f(\omega, x) \quad \text{in } \Omega \times \mathcal{D}, \tag{4.1}$$

where $a : \Omega \times \mathcal{D} \to \mathbb{R}$ is a stochastic jump-diffusion coefficient and $f : \Omega \times \mathcal{D} \to \mathbb{R}$ is a random source function. The Lipschitz boundary $\partial\mathcal{D}$ consists of open $(d-1)$-dimensional manifolds which are grouped into two disjoint subsets Γ_1 and Γ_2 such that $\Gamma_1 \neq \emptyset$ and $\partial\mathcal{D} = \Gamma_1 \cup \Gamma_2$. We impose mixed Dirichlet-Neumann boundary conditions

$$\begin{aligned} u(\omega, x) &= 0 \quad \text{on } \Omega \times \Gamma_1, \\ a(\omega, x)\vec{n} \cdot \nabla u(\omega, x) &= g(\omega, x) \quad \text{on } \Omega \times \Gamma_2, \end{aligned} \tag{4.2}$$

on Eq. (4.1), where $\vec{n}$ is the outward unit normal vector to Γ_2 and $g : \Omega \times \Gamma_2 \to \mathbb{R}$, assuming that the exterior normal derivative $\vec{n} \cdot \nabla u$ on Γ_2 is well-defined for any $u \in C^1(\overline{\mathcal{D}})$. To obtain a path-wise variational formulation of this problem, we use the standard Sobolev space $H^1(\mathcal{D})$ equipped with the norm

$$\|v\|_{H^1(\mathcal{D})} = \left(\int_{\mathcal{D}} |v|^2 + \|\nabla v\|_2^2 dx \right)^{1/2} \quad \text{for } v \in H^1(\mathcal{D}),$$

where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^d$. On the Lipschitz domain $\mathcal{D}$, the existence of a bounded, linear operator

$$\gamma : H^1(\mathcal{D}) \to H^{1/2}(\partial\mathcal{D})$$

with

$$\gamma : H^1(\mathcal{D}) \cap C^\infty(\overline{\mathcal{D}}) \to H^{1/2}(\partial\mathcal{D}), \quad v \mapsto v|_{\partial\mathcal{D}}$$

and

$$\|\gamma v\|_{H^{1/2}(\partial\mathcal{D})} \leq C_{\mathcal{D}} \|v\|_{H^1(\mathcal{D})} \tag{4.3}$$

for $v \in H^1(\mathcal{D})$ and some constant $C_{\mathcal{D}} > 0$, dependent only on $\mathcal{D}$, is ensured by the trace theorem, see for example [73]. At this point, one might argue that the trace operator γ needs to be defined path-wise for any $\omega \in \Omega$, since the Neumann part of the boundary conditions in Eq. (4.2) may contain a random function $g : \Omega \times \Gamma_2 \to \mathbb{R}$.

This is true if one works with γ on $\Gamma_2 \subset \partial\mathcal{D}$, as the trace γv then has to match the boundary condition given by $g(\omega, \cdot)$ on Γ_2 for $\mathbb{P}$-almost all $\omega \in \Omega$ and $v \in H^1(\mathcal{D})$. In our case, for simplicity, we may treat γ independently of $\omega \in \Omega$, since we only consider the trace operator on the homogeneous boundary part Γ_1 to define V as follows: The subspace of $H^1(\mathcal{D})$ with zero trace on Γ_1 is then

$$V := \{v \in H^1(\mathcal{D}) |\ \gamma v|_{\Gamma_1} = 0\},$$

with norm

$$\|v\|_V := \left(\int_{\mathcal{D}} |v|^2 + \|\nabla v(x)\|_2^2 dx\right)^{1/2}.$$

Remark 4.2.1. The condition $\Gamma_1 \neq \emptyset$ implies that V is a closed linear subspace of $H^1(\mathcal{D})$. We may as well work with non-homogeneous boundary conditions on the Dirichlet part, i.e. $u(\omega, x) = g_1(\omega, x)$ for $g_1 : \Omega \times \Gamma_1 \to \mathbb{R}$. The corresponding trace operator γ is still well defined if, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $g_1(\omega, \cdot)$ can be extended to a function $\widetilde{g}_1(\omega, \cdot) \in H^1(\mathcal{D})$. Then, we consider for $\mathbb{P}$-a.e. $\omega \in \Omega$ the problem

$$\begin{aligned} -\nabla \cdot (a(\omega, x)\nabla((u - \widetilde{g}_1)(\omega, x))) &= f + \nabla \cdot (a(\omega, x)\nabla \widetilde{g}_1(\omega, x)) \quad \text{on } \mathcal{D}, \\ (u - \widetilde{g}_1)(\omega, x) &= 0 \quad \text{on } \Gamma_1 \text{ and} \\ a(\omega, x)\vec{n} \cdot \nabla((u - \widetilde{g}_1)(\omega, x)) &= g(\omega, x) - a(\omega, x)\vec{n} \cdot \nabla \widetilde{g}_1(\omega, x) \quad \text{on } \Gamma_2. \end{aligned}$$

But this is in fact a version of Problem (4.1) equipped with Eqs. (4.2) where the source term and Neumann-data have been changed (see also [80, p. 317]).

As the coefficient and the boundary conditions are given by random functions, the solution u is also a random function. Besides path-wise properties, u may also have certain integrability properties with respect to the underlying probability measure. To this end, we introduce the space of *Bochner integrable* random variables resp. random functions (see [67] for an overview).

Definition 4.2.2. Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a Banach space and define the norm $\|\cdot\|_{L^p(\Omega;\mathcal{X})}$ for a $\mathcal{X}$-valued random variable $\varphi : \Omega \to \mathcal{X}$ as

$$\|\varphi\|_{L^p(\Omega,\mathcal{X})} := \begin{cases} \mathbb{E}(\|\varphi\|_{\mathcal{X}}^p)^{1/p} & \text{for } 1 \leq p < +\infty \\ \operatorname{esssup}_{\omega\in\Omega}\|\varphi\|_{\mathcal{X}} & \text{for } p = +\infty \end{cases}.$$

The corresponding space of Bochner-integrable random variables is then given by

$$L^p(\Omega; \mathcal{X}) := \{\varphi : \Omega \to \mathcal{X} \text{ is strongly measurable and } \|\varphi\|_{L^p(\Omega;\mathcal{X})} < +\infty\}.$$

The following set of assumptions on a, f and g allows us to show existence and uniqueness of the solution to Eq. (4.1). Consequently, we denote by $\mathcal{V}'$ the topological dual of a vector space $\mathcal{V}$.

Assumption 4.2.3. Let $H := L^2(\mathcal{D})$. For $\mathbb{P}$-almost all $\omega \in \Omega$ it holds that:

- $a_-(\omega) := \inf_{x\in\mathcal{D}} a(\omega, x) > 0$ and $a_+(\omega) := \sup_{x\in\mathcal{D}} a(\omega, x) < +\infty$.
- $1/a_- \in L^p(\Omega; \mathbb{R})$, $f \in L^q(\Omega; H)$ and $g \in L^q(\Omega; L^2(\Gamma_2))$ for some $p, q \in [1, \infty]$ such that $r := (1/p + 1/q)^{-1} \geq 1$.

Remark 4.2.4. In Assumption 4.2.3, we did not establish a uniform elliptic bound on a in Ω (see for example [148]), neither did we assume a certain spatial regularity as in [54, 194]. The relatively weak assumptions are natural in our context, since in Section 4.3 we model a as a jump-diffusion coefficient and uniform bounds or assumptions on Hölder-continuity are too restrictive. For the investigation of problem (4.1) with piecewise Hölder-continuous coefficients we refer to [194]. We may identify H with its dual and work on the Gelfand triplet $V \subset H \simeq H' \subset V'$. Hence, Assumption 4.2.3 guarantees that $f(\omega, \cdot) \in V'$, and, similarly, $g(\omega, \cdot) \in H^{-1/2}(\Gamma_2)$ for $\mathbb{P}$-almost all $\omega \in \Omega$.

For fixed $\omega \in \Omega$, multiplying the random PDE (4.1) with a test function $v \in V$ and integrating by parts yields the integral equation

$$\int_{\mathcal{D}} a(\omega, x)\nabla u(\omega, x) \cdot \nabla v(x)dx = \int_{\mathcal{D}} f(\omega, x)v(x)dx + \int_{\Gamma_2} g(\omega, x)[Tv](x)dx. \tag{4.4}$$

Consider the bilinear form $B_{a(\omega)}$

$$B_{a(\omega)} : V \times V \to \mathbb{R}, \quad (u, v) \mapsto \int_{\mathcal{D}} a(\omega, x)\nabla u(x) \cdot \nabla v(x)dx$$

and

$$F_\omega : V \to \mathbb{R}, \quad v \mapsto \int_{\mathcal{D}} f(\omega, x)v(x)dx + \int_{\Gamma_2} g(\omega, x)[Tv](x)dx,$$

where the integrals in F_ω are understood as the duality pairings

$$\int_{\mathcal{D}} f(\omega, x)v(x)dx = {}_{V'}\langle f(\omega, \cdot), v\rangle_V \quad \text{and}$$
$$\int_{\Gamma_2} g(\omega, x)[Tv](x)dx = {}_{H^{-1/2}(\Gamma_2)}\langle g(\omega, \cdot), Tv\rangle_{H^{1/2}(\Gamma_2)}.$$

Equation (4.4) then leads to the path-wise variational formulation of Problem (4.1): For $\mathbb{P}$-almost all $\omega \in \Omega$, given $f(\omega, \cdot) \in V'$ and $g(\omega, \cdot) \in H^{-1/2}(\Gamma_2)$, find $u(\omega, \cdot) \in V$ such that

$$B_{a(\omega)}(u(\omega, \cdot), v) = F_\omega(v) \tag{4.5}$$

for all $v \in V$. A function $u(\omega, \cdot) \in V$ that fulfills the path-wise variational formulation is then called *path-wise weak solution* to Problem (4.1).

Theorem 4.2.5. *If Assumption 4.2.3 holds, then there exists a unique path-wise weak solution $u(\omega, \cdot) \in V$ to Problem (4.5) for $\mathbb{P}$-almost all $\omega \in \Omega$. Furthermore, $u \in L^r(\Omega; V)$ and*

$$\|u\|_{L^r(\Omega;V)} \leq C(a_-, \mathcal{D}, p)(\|f\|_{L^q(\Omega;H)} + \|g\|_{L^q(\Omega;L^2(\Gamma_2))}),$$

where $C(a_-, \mathcal{D}, p) > 0$ is a constant depending only on the indicated parameters.

Proof. Choose $\omega \in \Omega$ such that Assumption 4.2.3 is fulfilled. For all $u, v \in V$, we obtain by the Cauchy-Schwarz inequality

$$|B_{a(\omega)}(u, v)| \leq \left(\int_{\mathcal{D}} (a(\omega, x))^2 \|\nabla u(x)\|_2^2 dx \int_{\mathcal{D}} \|\nabla v(x)\|_2^2 dx\right)^{1/2} \leq a_+(\omega)\|u\|_V \|v\|_V.$$

On the other hand,

$$\begin{aligned} B_{a(\omega)}(u, u) &\geq a_-(\omega) \int_{\mathcal{D}} \|\nabla u(x)\|_2^2 dx \\ &= \frac{a_-(\omega)}{2}(\|\nabla u\|_{L^2(\mathcal{D})}^2 + \|\nabla u\|_{L^2(\mathcal{D})}^2) \\ &\geq \frac{a_-(\omega)}{2}(\|\nabla u\|_{L^2(\mathcal{D})}^2 + C_{|\mathcal{D}|}^{-2}\|u\|_{L^2(\mathcal{D})}^2) \\ &\geq \frac{a_-(\omega)}{2} \min(1, C_{|\mathcal{D}|}^{-2})\|u\|_V^2, \end{aligned}$$

where $C_{|\mathcal{D}|}^2 > 0$ stems from the constant in Poincaré's inequality, $C_{|\mathcal{D}|}$, and only depends on $|\mathcal{D}|$. Hence the bilinear form $B_{a(\omega)} : V \times V \to \mathbb{R}$ is continuous and coercive. We use that $H \simeq H' \subset V'$ and the trace theorem (Equation (4.3)) to bound F_ω by

$$\begin{aligned} F_\omega(v) &\leq \|f(\omega, \cdot)\|_{V'}\|v\|_V + \|g(\omega, \cdot)\|_{H^{-1/2}(\Gamma_2)}\|Tv\|_{H^{1/2}(\Gamma_2)} \\ &\leq (\|f(\omega, \cdot)\|_H + C_{\mathcal{D}}\|g(\omega, \cdot)\|_{L^2(\Gamma_2)})\|v\|_V \end{aligned}$$

This shows that F_ω is a bounded linear functional on V (and therefore continuous) for almost all $\omega \in \Omega$. The existence of a unique path-wise weak solution $u(\omega, \cdot)$ is then guaranteed by the Lax-Milgram lemma $\mathbb{P}$-almost surely. If $u(\omega, \cdot)$ is a solution of

Eq. (4.5) for given $f(\omega,\cdot) \in H$ and $g(\omega,\cdot) \in L^2(\Gamma_2)$, then

$$\begin{aligned}\frac{a_-(\omega)}{2}\min(1, C_{|\mathcal{D}|}^{-2})\|u(\omega,\cdot)\|_V^2 &\leq B_{a(\omega)}(u(\omega,\cdot), u(\omega,\cdot)) \\ &= F_\omega(u(\omega,\cdot)) \\ &\leq (\|f(\omega,\cdot)\|_H + C_{\mathcal{D}}\|g(\omega,\cdot)\|_{L^2(\Gamma_2)})\|u(\omega,\cdot)\|_V.\end{aligned}$$

Using Hölder's and Minkowski's inequality together with $r = (1/p + 1/q)^{-1} \geq 1$ yields

$$\begin{aligned}\|u\|_{L^r(\Omega;V)} &\leq \frac{2\max(1, C_{\mathcal{D}})}{\min(1, C_{|\mathcal{D}|}^{-2})}\mathbb{E}\left(a_-^{-p}\right)^{1/p}\mathbb{E}\left((\|f\|_H + \|g\|_{L^2(\Gamma_2)})^q\right)^{1/q} \\ &\leq \underbrace{\frac{2\max(1, C_{\mathcal{D}})}{\min(1, C_{|\mathcal{D}|}^{-2})}\|1/a_-\|_{L^p(\Omega;\mathbb{R})}}_{:=C(a_-,\mathcal{D},p)}(\|f\|_{L^q(\Omega;H)} + \|g\|_{L^q(\Omega;L^2(\Gamma_2))}) < +\infty.\end{aligned}$$

□

In the next section, we introduce the diffusion coefficient a, which allows us to incorporate discontinuities at random points or areas in $\mathcal{D}$. We show the existence and uniqueness of a weak solution to the discontinuous diffusion problem by choosing a such that Assumption 4.2.3 is fulfilled and Theorem 4.2.5 may be applied.

4.3 Discontinuous random elliptic problems

The stochastic coefficient a in a jump-diffusion model should incorporate random discontinuities as well as a Gaussian component. We achieve this characteristic form of a by defining the coefficient as a Gaussian random field with additive discontinuities on random areas of $\mathcal{D}$. Since this usually involves infinite series expansions in the Gaussian component or sampling errors in the jump measure, we also describe how to obtain tractable approximations of a. Subsequently, existence and uniqueness results for weak solutions of the unapproximated resp. approximated jump-diffusion problem based on the results in Section 4.2 are proved. We conclude this section by showing path-wise and L^p-convergence of the approximated solution to the solution $u : \Omega \to V$ to the (unapproximated) discontinuous diffusion problem.

4.3.1 Jump-diffusion coefficients and their approximations

Definition 4.3.1. The *jump-diffusion coefficient* a is defined as

$$a : \Omega \times \mathcal{D} \to \mathbb{R}_{>0}, \quad (\omega, x) \mapsto \overline{a}(x) + \Phi(W(\omega, x)) + P(\omega, x),$$

where

- $\overline{a} \in C^1(\mathcal{D}; \mathbb{R}_{\geq 0})$ is non-negative, continuous and bounded.
- $\Phi \in C^1(\mathcal{D}; \mathbb{R}_{>0})$ is a continuously differentiable, positive mapping.
- $W \in L^2(\Omega; H)$ is a (zero-mean) Gaussian random field associated to a non-negative, symmetric trace class operator $Q : H \to H$.
- λ is a finite measure on $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$ and $\mathcal{T} : \Omega \to \mathcal{B}(\mathcal{D})$, $\omega \mapsto \{\mathcal{T}_1, \dots, \mathcal{T}_\tau\}$ is a random partition of $\mathcal{D}$ with respect to λ. The number τ of elements in $\mathcal{T}$ is a random variable $\tau : \Omega \to \mathbb{N}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}(\tau) = \lambda(\mathcal{D})$.
- $(P_i, i \in \mathbb{N})$ is a sequence of non-negative random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and

$$P : \Omega \times \mathcal{D} \to \mathbb{R}_{\geq 0}, \quad (\omega, x) \mapsto \sum_{i=1}^{\tau(\omega)} \mathbb{1}_{\{\mathcal{T}_i\}}(x) P_i(\omega).$$

  The sequence $(P_i, i \in \mathbb{N})$ is independent of τ (but not necessarily i.i.d.).

Remark 4.3.2. The definition of the measure λ on $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$ in Def. 4.3.1 relates not only to the average number of partition elements $\mathbb{E}(\tau)$, but may further be utilized to concentrate discontinuities of the jump-diffusion coefficient a to certain areas of $\mathcal{D}$. Choosing, for instance, λ as the Lebesgue measure on $\mathcal{D}$ corresponds to uniformly distributed jumps and on average equally sized partition elements $\mathcal{T}_i$. In contrast, if λ is a Gaussian measure on $\mathcal{D}$ around some center point $x_C \in \mathcal{D}$, the number of discontinuities (resp. size of partition elements) will decrease (resp. increase) as one moves away from x_C. We refer to the numerical experiments in Section 4.6, where we give interpretations of λ to model certain characteristics of different jump-diffusion coefficients. On a further note, stochastic independence of W and P is not required.

In general, the structure of a as in Def. 4.3.1 does not allow us to draw samples from the exact distribution of this random function. For an approximation of the Gaussian field, one usually uses truncated Karhunen-Loève expansions: Let $((\eta_i, e_i), i \in \mathbb{N})$ denote the sequence of eigenpairs of Q, where the eigenvalues are given in decaying order $\eta_1 \geq \eta_2 \geq \cdots \geq 0$. Since Q is trace class the Gaussian random field W admits the representation

$$W = \sum_{i \in \mathbb{N}} \sqrt{\eta_i} e_i Z_i,$$

where $(Z_i, i \in \mathbb{N})$ is a sequence of independent and standard normally distributed random variables. The series above converges in $L^2(\Omega; H)$ and $\mathbb{P}$-almost surely (see

i.e. [27]). The truncated Karhunen-Loève expansion W_N of W is then given by

$$W_N := \sum_{i=1}^{N} \sqrt{\eta_i} e_i Z_i,$$

where we call $N \in \mathbb{N}$ the *cut-off index* of W_N. In addition, it may be possible that the sequence of jumps $(P_i, i \in \mathbb{N})$ cannot be sampled exactly but only with an intrinsic bias (see also Remark 4.3.4). The biased samples are denoted by $(\widetilde{P}_i, i \in \mathbb{N})$ and the error which is induced by this approximation is represented by the parameter $\varepsilon > 0$ as in Assumption 4.3.3. To approximate P using the biased sequence $(\widetilde{P}_i, i \in \mathbb{N})$ instead of $(P_i, i \in \mathbb{N})$ we define the jump part approximation

$$P_\varepsilon : \Omega \times \mathcal{D} \to \mathbb{R}, \quad (\omega, x) \mapsto \sum_{i=1}^{\tau(\omega)} \mathbb{1}_{\{\mathcal{T}_i\}}(x) \widetilde{P}_i(\omega).$$

The *approximated jump-diffusion coefficient* $a_{N,\varepsilon}$ is then given by

$$a_{N,\varepsilon}(\omega, x) := \overline{a}(x) + \Phi(W_N(\omega, x)) + P_\varepsilon(\omega, x), \tag{4.6}$$

and the corresponding stochastic PDE with approximated jump-diffusion coefficient reads

$$\begin{aligned} -\nabla \cdot (a_{N,\varepsilon}(\omega, x) \nabla u_{N,\varepsilon}(\omega, x)) &= f(\omega, x) \quad \text{in } \Omega \times \mathcal{D}, \\ u_{N,\varepsilon}(\omega, x) &= 0 \quad \text{on } \Omega \times \Gamma_1, \\ a_{N,\varepsilon}(\omega, x) \vec{n} \cdot \nabla u_{N,\varepsilon}(\omega, x) &= g(\omega, x) \quad \text{on } \Omega \times \Gamma_2. \end{aligned} \tag{4.7}$$

For $\omega \in \Omega$ and given samples $a_{N,\varepsilon}(\omega, \cdot), f(\omega, \cdot)$ and $g(\omega, \cdot)$, we consider the path-wise weak solution $u_{N,\varepsilon}(\omega, \cdot) \in V$ to Problem (4.7) for fixed approximation parameters $N \in \mathbb{N}$ and $\varepsilon > 0$. The variational formulation of Eq. (4.7) is then analogous to Eq. (4.5) given by: For almost all $\omega \in \Omega$ with given $f(\omega, \cdot)$, $g(\omega, \cdot)$, find $u_{N,\varepsilon}(\omega, \cdot) \in V$ such that for all $v \in V$

$$\begin{aligned} B_{a_{N,\varepsilon}(\omega)}(u_{N,\varepsilon}(\omega, \cdot), v) :&= \int_{\mathcal{D}} a_{N,\varepsilon}(\omega, x) \nabla u_{N,\varepsilon}(\omega, x) \cdot \nabla v(x) dx \\ &= \int_{\mathcal{D}} f(\omega, x) v(x) dx + \int_{\Gamma_2} g(\omega, x) [Tv](x) dx \\ &= F_\omega(v). \end{aligned} \tag{4.8}$$

The following assumptions guarantee that we can apply Theorem 4.2.5 also in the jump-diffusion setting and that therefore path-wise solutions u and $u_{N,\varepsilon}$ exist.

Assumption 4.3.3.

(i) The eigenfunctions e_i of Q are continuously differentiable on $\mathcal{D}$ and there exist constants $\alpha, \beta, C_e, C_\eta > 0$ such that for any $i \in \mathbb{N}$

$$\|e_i\|_{L^\infty(\mathcal{D})} \leq 1, \quad \|\nabla e_i\|_{L^\infty(\mathcal{D})} \leq C_e i^\alpha \quad \text{and} \quad \sum_{i=1}^{\infty} \eta_i i^\beta \leq C_\eta < +\infty.$$

(ii) Furthermore, the mapping Φ as in Definition 4.3.1 and its derivative are bounded by

$$\Phi(w) \geq \phi_1 \exp(-\psi_1 w^2), \quad |\Phi'(w)| \leq \phi_2 \exp(\psi_2 |w|), \quad w \in \mathbb{R},$$

where $0 < \psi_1 < (2\,\mathrm{Tr}(Q))^{-1}$ with $\mathrm{Tr}(Q) := \sum_{i\in\mathbb{N}} \eta_i$ and $\phi_1, \phi_2, \psi_2 > 0$ are arbitrary constants.

(iii) Given ψ_1 and $\mathrm{Tr}(Q)$, there exists a $q > (1 - 2\,\mathrm{Tr}(Q)\psi_1)^{-1} =: (1-\eta^*)^{-1} \geq 1$ such that $f \in L^q(\Omega; H)$ and $g \in L^q(\Omega; \Gamma_2)$.

(iv) Finally, for some $\widetilde{s} \in [1, \infty)$, $(P_i, i \in \mathbb{N})$ consists of s-integrable random variables, i.e. $P_i \in L^s(\Omega; \mathbb{R}_{\geq 0})$ for all $i \in \mathbb{N}$ and $s \in [1, \widetilde{s}]$. Further, there exists a sequence of approximations $(\widetilde{P}_i, i \in \mathbb{N})$ so that the sampling error is bounded, for $\varepsilon > 0$, by

$$\mathbb{E}(|\widetilde{P}_i - P_i|^s) \leq \varepsilon, \quad i \in \mathbb{N},\ s \in [1, \widetilde{s}].$$

Remark 4.3.4. Assumption 4.3.3 (i) on the eigenpairs of $((\eta_i, e_i), i \in \mathbb{N})$ is natural. For instance, the case that W is a Brownian-motion-like random field or that Q is a Matérn covariance operator are included. The bounds on Φ and the regularity assumptions on f and g (Assumption 4.3.3 (ii),(iii)) are necessary to ensure that the solution u has at least finite expectation. The sampling error $\mathbb{E}(|\widetilde{P}_i - P_i|^s)$ in Assumption 4.3.3 (iv) may be interpreted in several ways: For instance, it may account for uncertainties in the distribution of P_i, like parameters for which only confidence intervals are available. Another possibility is, that realizations of P_i may not be simulated directly or only at relatively high computational costs, for example by Acceptance Rejection algorithms, see [9, Chapter II]. In this case, it may be favorable to generate the approximation $\widetilde{P}_i$ by a more efficient numerical algorithm (i.e. Fourier inversion techniques, see [30]) instead and to control for the error. The resulting sampling error can then be equilibrated with the truncation error from the Gaussian field to achieve a desired overall accuracy.

4.3.2 Existence of solutions

We first show the existence of a weak solution for both, the jump-diffusion problem (4.1) with a as in Definition 4.3.1, and the approximated problem (4.7).

Lemma 4.3.5. *Let a be a jump-diffusion coefficient (as in Def. 4.3.1). If a, f and g fulfill Assumptions 4.3.3, then the elliptic problem* (4.1) *has a unique weak solution $u \in L^r(\Omega; V)$, where $r \in [1, (1/q + \eta^*)^{-1})$ for $\eta^* := 2\,Tr(Q)\psi_1$.*

Proof. By Theorem 4.2.5, it is sufficient to show that a, f and g fulfill Assumption 4.2.3. Clearly, $0 < a_-(\omega) \leq a_+(\omega) < +\infty$ for almost all $\omega \in \Omega$, as $\overline{a}$ and P are non-negative and we have the lower bound on Φ in Assumption 4.3.3 (ii) by definition. Consequently, it is sufficient to bound the expectation of $(\inf_{x\in\mathcal{D}} \Phi(W(x))^p$, for $1 \leq p < (\eta^*)^{-1}$, from below (see also [53, Section 2.3]).

The random variable $W(x) - W(y)$ follows a centered normal distribution for any $x, y \in \mathcal{D}$. To see this, consider the finite sum

$$\widetilde{W}_M(x, y) := \sum_{i=1}^{M} \sqrt{\eta_i}(e_i(x) - e_i(y))Z_i$$

for $M \in \mathbb{N}$ and a sequence $(Z_i, i = 1, \ldots, M)$ of i.i.d. $\mathcal{N}(0, 1)$-distributed random variables. Clearly, $\widetilde{W}_M(x, y)$ is normally distributed with zero mean and characteristic function

$$\phi_M(t) := \mathbb{E}(\exp(it\widetilde{W}_M(x, y)) = \exp(-\frac{t}{2}\sum_{i=1}^{M} \eta_i(e_i(x) - e_i(y))^2), \quad t \in \mathbb{R}.$$

Using that Q is trace class with $|e_i(z)| \leq 1$ for all $i \in \mathbb{N}$ and $z \in \mathcal{D}$, it follows

$$\sigma^2(x, y) := \sum_{i=1}^{\infty} \eta_i(e_i(x) - e_i(y))^2 \leq 2\mathrm{Tr}(Q) < +\infty$$

and hence

$$\lim_{M\to\infty} \phi_M(t) = \exp(-\frac{t}{2}\sum_{i=1}^{\infty} \eta_i(e_i(x) - e_i(y))^2), \quad t \in \mathbb{R},$$

where the right hand side is the characteristic function of a normal distribution with zero mean and variance $\sigma^2(x, y)$. By the Lévy continuity theorem this implies that

$$W(x) - W(y) \sim \mathcal{N}(0, \sigma^2(x, y))$$

for all $x, y \in \mathcal{D}$. Next we show that W has $\mathbb{P}$-almost surely Hölder continuous paths (see also [53, Proposition 3.4]): Let $0 < b \leq \min(1, \frac{\beta}{2\alpha})$ (where α, β are defined in

Ass. 4.3.3 (i)) and $Z \sim \mathcal{N}(0,1)$. For $x, y, \in \mathcal{D} \subset \mathbb{R}^d$ and any $k \in \mathbb{N}$ we have

$$\begin{aligned}\mathbb{E}(|W(x) - W(y)|^{2k}) &= \mathbb{E}(|\sqrt{\sigma^2(x,y)}Z|^{2k}) \\ &= \frac{(2k)!}{2^k k!}\left(\sum_{i>N} \eta_i (e_i(x) - e_i(y))^{2(1-b)+2b}\right)^k \\ &\leq \frac{(2k)!}{2^k k!} 2^{2(1-b)k} C_e^{2bk} \left(\sum_{i>N} \eta_i i^{2\alpha b}\right)^k \|x-y\|_2^{2bk},\end{aligned}$$

where $0 < b < \beta/(2\alpha)$ and the constant C_e stems from Ass. 4.3.3 (i). The second equality results from the fact that $\mathbb{E}(Z^{2k}) = (2k)!/(2^k k!)$ for all $k \in \mathbb{N}$ and the sum in the inequality is finite because $2\alpha b < \beta$. For any dimension $d \in \mathbb{N}$, we may choose $k > d/(2b)$ and obtain by the Kolmogorov-Chentsov theorem ([67, Theorem 3.5]) that W has a Hölder continuous modification with Hölder exponent $\varrho \in (0, (2bk-d)/2k)$. Hence, W is a centered Gaussian process and almost surely bounded on $\mathcal{D}$. By [3, Theorem 2.1.1] this implies $E := \mathbb{E}(\sup_{x\in\mathcal{D}} W(x)) < +\infty$ and

$$\mathbb{P}(\sup_{x\in\mathcal{D}} W(x) - E \geq c) \leq \exp(-\frac{c^2}{2\overline{\sigma}^2}) \tag{4.9}$$

for all $c > 0$ and $\overline{\sigma}^2 := \sup_{x\in\mathcal{D}} \mathbb{E}(W(x)^2)$. With Assumption 4.3.3 (ii) and since $\|\exp(|W|)\|_{L^\infty(\mathcal{D})} \leq \exp(\|W\|_{L^\infty(\mathcal{D})})$ we further obtain

$$\begin{aligned}\mathbb{E}(1/a_-^p) &\leq \mathbb{E}((\inf_{x\in\mathcal{D}} \Phi(W(\cdot,x))^{-p}) \\ &= \mathbb{E}(\sup_{x\in\mathcal{D}} \Phi(W(\cdot,x))^{-p}) \\ &\leq \frac{1}{\phi_1^p}\mathbb{E}(\sup_{x\in\mathcal{D}} \exp(p\psi_1 |W(\cdot,x)|^2)) \\ &\leq \frac{1}{\phi_1^p}\mathbb{E}(\exp(p\psi_1 \|W\|^2_{L^\infty(\mathcal{D})})).\end{aligned}$$

By Fubini's Theorem and integration by parts we may bound

$$\begin{aligned}\mathbb{E}(\exp(p\psi_1\|W\|^2_{L^\infty(\mathcal{D})})) &= \int_0^\infty 2p\psi_1 c \exp(p\psi_1 c^2)\mathbb{P}(\|W\|_{L^\infty(\mathcal{D})} > c)dc \\ &\leq 2p\psi_1 E \exp(p\psi_1 E^2) \\ &\quad + 2\int_E^\infty p\psi_1 c \exp(p\psi_1 c^2))\mathbb{P}(\|W\|_{L^\infty(\mathcal{D})} > c)dc \\ &\leq 2p\psi_1 E \exp(p\psi_1 E^2) \\ &\quad + 4\int_E^\infty p\psi_1 c \exp((p\psi_1 - \frac{1}{2\overline{\sigma}^2})c^2))dc,\end{aligned}$$

where we have used $\mathbb{P}(\|W(x)\|_{L^\infty(\mathcal{D})} > c) \leq 2\mathbb{P}(\sup_{x\in\mathcal{D}} W(x) > c)$, by the symmetry of W, and Ineq. (4.9) in the last step. The last expectation is finite if and only if $p\psi_1 < \frac{1}{2\overline{\sigma}^2}$. For this to hold, $p < (\eta^*)^{-1}$ is sufficient, since $W(x) \sim N(0, \sum_{i=1}^\infty \eta_i e_i(x))$ and thus $\overline{\sigma}^2 \leq \mathrm{Tr}(Q)$. □

Remark 4.3.6. From Lemma 4.3.5 follows immediately that one cannot expect finite second moments of the solution u for $\eta^* \geq 1/2$ or $q \leq 2$. If we assume that $q = 3$ we need $\eta^* < 1/6$ to have finite second moments (in the case of, for instance, a log-normal Gaussian field). Note that, for all covariance kernels and functionals we use (e.g. log-Gaussian fields with Matérn class covariance or Brownian-motion like covariance kernels), ψ_1 and η^* are much smaller than $1/2$. If one assumes that f, g and a are stochastically independent, then the regularity of the solution (in Ω) is at least the same as the lowest regularity of the data, i.e. f, g or a.

Lemma 4.3.7. *Let $a_{N,\varepsilon}$ be the approximated jump-diffusion coefficient (as in Eq. (4.6)) and define the random variables*

$$a_{N,\varepsilon,-} : \Omega \to \mathbb{R}, \quad \omega \mapsto \inf_{x\in\mathcal{D}} a_{N,\varepsilon}(\omega, x) \quad \textit{and} \quad a_{N,\varepsilon,+} : \Omega \to \mathbb{R}, \quad \omega \mapsto \sup_{x\in\mathcal{D}} a_{N,\varepsilon}(\omega, x)$$

on $(\Omega, \mathcal{F}, \mathbb{P})$. If Assumption 4.3.3 holds, then, for any $N \in \mathbb{N}$ and $\varepsilon > 0$, there exists a unique weak solution $u_{N,\varepsilon} \in L^r(\Omega; V)$ to Problem (4.7), where $r \in [1, (1/q + \eta^*)^{-1})$, for $\eta^* := 2\,Tr(Q)\psi_1$. Furthermore, $\|1/a_{N,\varepsilon,-}\|_{L^p(\Omega,\mathbb{R})}$ is bounded uniformly with respect to ε and N for $p \in [1, (\eta^*)^{-1})$.

Proof. The proof is carried out identically to Lemma 4.3.5, where we replace a_- by $a_{N,\varepsilon,-}$, a_+ by $a_{N,\varepsilon,+}$, $\sigma(x,y)^2$ by $\sum_{i=1}^N \eta_i (e_i(x) - e_i(y))^2$ and $\mathrm{Tr}(Q)$ by $\sum_{i=1}^N \eta_i$. Again, by Eq. (4.6), $a_{N,\varepsilon,+} < +\infty$ $\mathbb{P}$-almost surely. In the case that $\eta_1 = 0$, the random field W is degenerated and equal to zero. Hence, $a_{N,\varepsilon}(\omega, x) \geq \phi_1 > 0$ for all $(\omega, x) \in \Omega \times \mathcal{D}$ and the claim follows immediately. Otherwise, if $\eta_1 > 0$, we obtain in the fashion of Lemma 4.3.5

$$\mathbb{E}(1/a_{N,\varepsilon-}^p) \leq \frac{1}{\phi_1^p}\mathbb{E}\Big(\exp(p\psi_1\|W_N\|^2_{L^\infty(\mathcal{D})})\Big) \leq \frac{1}{\phi_1^p}\mathbb{E}\Big(\exp(p\psi_1\|W\|^2_{L^\infty(\mathcal{D})})\Big),$$

which is again finite if $p < (\eta^*)^{-1}$. The proof is concluded by noting that the last estimate is independent of N and ε. □

4.3.3 Convergence of the approximated diffusion coefficient

The convergence of the approximated solution $u_{N,\varepsilon}$ to u depends on the convergence of the approximated jump-diffusion coefficient $a_{N,\varepsilon}$ to a. We investigate this convergence

by deriving separately convergence rates of the truncated Karhunen-Loève series and the approximation of the jump part.

Theorem 4.3.8. *If Assumption 4.3.3 holds, then for any $p \geq 1$ and $N \in \mathbb{N}$ we have*

$$\|W - W_N\|_{L^p(\Omega; L^\infty(\mathcal{D}))} \leq C_p \Xi_N^{1/2},$$

where $C_p > 0$ is independent of N and $\Xi_N := \sum_{i>N} \eta_i < +\infty$.

In order to prove the bound for the truncation error, we need the following Fernique-type result.

Theorem 4.3.9. *[149, Theorem 2.9] Let $(W(x), x \in \mathcal{D})$ with $\mathcal{D} \subset \mathbb{R}^d$ be a centered Gaussian field. For $\epsilon, \delta > 0$ let*

$$\iota(\epsilon) := \sup_{x \in \mathcal{D},\, \|x-y\|_2 < \epsilon} \mathbb{E}((W(x) - W(y))^2)^{1/2} \quad \text{and} \quad \Theta(\delta) := \int_0^\infty \iota(\delta \exp(-y^2)) dy.$$

Then for all $c > 0$

$$\mathbb{P}(\sup_{x \in \mathcal{D}} W(x) > c) \leq C(\Theta^{-1}(1/c))^{-d} \exp(-\frac{c^2}{2\overline{\sigma}}),$$

where Θ^{-1} is the inverse function of Θ, $\overline{\sigma} := \sup_{x \in \mathcal{D}} \mathbb{E}(W(x)^2)^{1/2}$ and $C > 0$ is an absolute constant.

Proof of Theorem 3.8. With a continuity argument similar to Lemma 4.3.5, we obtain

$$(W - W_N)(x) - (W - W_N)(y) \sim \mathcal{N}(0, \sigma_N^2(x, y)),$$

where $\sigma_N^2(x, y) := \sum_{i>N} \eta_i (e_i(x) - e_i(y))^2$ for all $x, y, \in \mathcal{D}$. Hence

$$\overline{\sigma}_N^2 := \sup_{x \in \mathcal{D}} \mathbb{E}((W - W_N)(x)^2) = \sum_{i>N} \eta_i e_i(x)^2 \leq \sum_{i>N} \eta_i = \Xi_N$$

and we see by the proof of Lemma 4.3.5 that for ι as in Theorem 4.3.9

$$\iota(\epsilon) \leq 2^{1-b} C_e^b \left(\sum_{i \in \mathbb{N}} \eta_i i^{2\alpha b} \right)^{1/2} \epsilon^b =: C_{e,b} \epsilon^b$$

for any $b \in (0, \min(\beta/(2\alpha), 1))$ and each $\epsilon > 0$. We use the estimate on ι to bound Θ for $\delta > 0$ by

$$\Theta(\delta) \leq C_{e,\alpha} \int_0^\infty \delta^b \exp(-y^2 b) dy = \frac{C_{e,b} \delta^b \sqrt{\pi}}{2\sqrt{b}}.$$

Since Θ and Θ^{-1} are monotone increasing this yields with Theorem 4.3.9 for any $c > 0$

$$\begin{aligned}\mathbb{P}(\sup_{x\in\mathcal{D}}(W - W_N)(x) > c) &\leq C\left(\frac{2\sqrt{b}}{C_{e,b}\sqrt{\pi}}\right)^{-d/b} c^{d/b}\exp(-\frac{c^2}{2\overline{\sigma}_N^2}) \\ &:= C_{e,\alpha,b}\, c^{d/b}\exp(-\frac{c^2}{2\overline{\sigma}_N^2}).\end{aligned}$$

Using again that $\mathbb{P}(\sup_{x\in\mathcal{D}}|(W - W_N)|(x) > c) \leq 2\mathbb{P}(\sup_{x\in\mathcal{D}}(W - W_N)(x) > c)$ and Fubini's Theorem, we have for any $p \geq 1$

$$\begin{aligned}\mathbb{E}(\|W - W_N\|^p_{L^\infty(\mathcal{D})}) &= \int_0^\infty pc^{p-1}\mathbb{P}(\sup_{x\in\mathcal{D}}|(W - W_N)(x)| > c)dc \\ &\leq 2C_{e,\alpha,b}\, p\int_0^\infty c^{p-1+d/b}\exp(-\frac{c^2}{2\overline{\sigma}_N^2})dc \\ &= 2^{1+p/2+d/(2b)}C_{e,\alpha,b}\, p\Gamma((p + d/b)/2)\overline{\sigma}_N^{p+d/b} \\ &\leq 2^{1+p/2+d/(2b)}C_{e,\alpha,b}\, p\Gamma((p + d/b)/2)\mathrm{Tr}(Q)^{d/b}\Xi_N^{p/2},\end{aligned}$$

where $\Gamma(\cdot)$ is the Gamma function and we have used the substitution $z = c^2/(2\overline{\sigma}_N^2)$ in the second equality. This proves the claim because the above estimate holds for any $b \in (0, \min(\beta/(2\alpha), 1))$. □

Remark 4.3.10. In [53], the author proves a similar error bound for the truncation error in the Gaussian field, namely

$$\mathbb{E}(\|W - W_N\|_{L^p(\Omega;C^{0,\varrho}(\mathcal{D}))}) \leq C_{b,\varrho,p}\max\left(\sum_{i>N}\eta_i\,,\, C_e^{2b}\sum_{i>N}\eta_i i^{2b}\right)^{1/2}.$$

In the jump-diffusion setting the error bound in Theorem 4.3.8 is advantageous for several reasons:

- In our setting, a will in general not have Hölder continuous paths, but involves discontinuities, hence only the error in $L^p(\Omega; L^\infty(\mathcal{D}))$ is of interest.
- In general it is rather easy to calculate the sum Ξ_N if the first N eigenvalues are known (or can be approximated), whereas this is not necessarily the case for $\sum_{i>N}\eta_i i^{2b}$.

Theorem 4.3.11. *Under Assumption 4.3.3, the sampling error is, for all $s \in [1, \widetilde{s}]$, bounded by*

$$\|P - P_\varepsilon\|_{L^s(\Omega;L^\infty(\mathcal{D}))} \leq (\lambda(\mathcal{D})\varepsilon)^{1/s}.$$

Proof. For any $\omega \in \Omega$, we have

$$\|P(\omega,\cdot) - P_\varepsilon(\omega,\cdot)\|_{L^\infty(\mathcal{D})} = \max_{i=1,\dots,\tau(\omega)} |P_i(\omega) - \widetilde{P}_i(\omega)|^s.$$

By Fubini's Theorem and integration by parts this yields

$$\|P - P_\varepsilon\|^s_{L^s(\Omega;L^\infty(\mathcal{D}))} = \int_0^\infty s c^{s-1} \mathbb{P}\left(\max_{i=1,\dots,\tau} |P_i - \widetilde{P}_i| \geq c\right) dc,$$

since $\lim_{c\to+\infty} \mathbb{P}(\max_{i=1,\dots,\tau} |P_i - \widetilde{P}_i| \geq c) = 0$. For fixed $c > 0$ and $i \in \mathbb{N}$, we define the sets

$$T_i := \{\omega \in \Omega |\, \tau(\omega) = i\} \quad \text{and} \quad A_i := \{\omega \in \Omega |\, |P_i(\omega) - \widetilde{P}_i(\omega)| \geq c\}$$

to obtain the identity

$$\mathbb{P}\left(\max_{i=1,\dots,\tau} |P_i - \widetilde{P}_i| \geq c\right) = \mathbb{P}\left(\bigcup_{i\in\mathbb{N}} T_i \cap (\bigcup_{j=1}^{i} A_j)\right).$$

By the independence of $|P_i - \widetilde{P}_i|$ and τ this yields

$$\begin{aligned}
\mathbb{P}\left(\max_{i=1,\dots,\tau} |P_i - \widetilde{P}_i| \geq c\right) &\leq \sum_{i\in\mathbb{N}} \mathbb{P}\left(T_i \cap (\bigcup_{j=1}^{i} A_j)\right) \\
&= \sum_{i\in\mathbb{N}} \mathbb{P}(T_i)\mathbb{P}(\bigcup_{j=1}^{i} A_j) \\
&\leq \sum_{i\in\mathbb{N}} \mathbb{P}(T_i) \sum_{j=1}^{i} \mathbb{P}(A_j)
\end{aligned}$$

and thus by Fubini's theorem

$$\begin{aligned}
\|P - P_\varepsilon\|^s_{L^s(\Omega;L^\infty(\mathcal{D}))} &= \int_0^\infty \int_0^y s c^{s-1} dc \mathbb{P}(\|P - P_\varepsilon\|_{L^\infty(\mathcal{D})} \in dy) \\
&= \int_0^\infty s c^{s-1} \mathbb{P}(\|P - P_\varepsilon\|_{L^\infty(\mathcal{D})} \geq c)\, dc \\
&\leq \int_0^\infty \sum_{i\in\mathbb{N}} \mathbb{P}(\tau = i) \sum_{j=1}^{i} s c^{s-1} \mathbb{P}(|P_j - \widetilde{P}_j| \geq c)\, dc \\
&= \sum_{i\in\mathbb{N}} \mathbb{P}(\tau = i) \sum_{j=1}^{i} \int_0^\infty s c^{s-1} \mathbb{P}(|P_j - \widetilde{P}_j| \geq c)\, dc \\
&= \sum_{i\in\mathbb{N}} \mathbb{P}(\tau = i) \sum_{j=1}^{i} \mathbb{E}(|P_j - \widetilde{P}_j|^s) \\
&\leq \sum_{i\in\mathbb{N}} \mathbb{P}(\tau = i) i\varepsilon.
\end{aligned}$$

The claim then follows since $\mathbb{E}(\tau) = \lambda(\mathcal{D}) < +\infty$ by Definition 4.3.1.

□

With Theorems 4.3.8 and 4.3.11, follows convergence of the approximated diffusion coefficient:

Theorem 4.3.12. *Let Assumption 4.3.3 hold, then there exists $C(s, \Phi, Q, \lambda, \mathcal{D}) > 0$, depending only on the indicated parameters, such that for any $N \in \mathbb{N}$ and $\varepsilon > 0$*

$$\|a - a_{N,\varepsilon}\|_{L^s(\Omega;L^\infty(\mathcal{D}))} \leq C(s, \Phi, Q, \lambda, \mathcal{D}) \left(\Xi_N^{1/2} + \varepsilon^{1/s}\right).$$

Hence, $a_{N,\varepsilon}$ converges to a in $L^s(\Omega; L^\infty(\mathcal{D}))$ as $N \to \infty$ and $\varepsilon \to 0$.

Proof. Let $N \in \mathbb{N}, \varepsilon > 0$ and $(\omega, x) \in \Omega \times \mathcal{D}$ be fixed. By the mean-value theorem

$$a(\omega, x) - a_{N,\varepsilon}(\omega, x) = \Phi'(\xi_N(\omega, x))(W(\omega, x) - W_N(\omega, x)) + P(\omega, x) - P_\varepsilon(\omega, x),$$

where $\xi_N(\omega, x) \in (W(\omega, x), W_N(\omega, x))$. With Assumption 4.3.3 (ii) on Φ' and by the triangle inequality, we obtain the estimate

$$|\Phi'(\xi_N(\omega, x))| \leq \phi_2 \exp(\psi_2(|W_N(\omega, x)| + |W(\omega, x) - W_N(\omega, x)|)).$$

The random fields W_N and $W - W_N$ are independent, so for any $s \in [1, \infty)$ it holds

$$\begin{aligned}
&\| \exp(\psi_2(|W_N| + |W - W_N|))(W - W_N)\|_{L^s(\Omega;L^\infty(\mathcal{D}))} \\
\leq &\| \exp(\psi_2|W_N|)\|_{L^s(\Omega;L^\infty(\mathcal{D}))} \| \exp(\psi_2|W - W_N|^2)\|_{L^{p_1}(\Omega;L^\infty(\mathcal{D}))} \|(W - W_N)\|_{L^{p_2}(\Omega;L^\infty(\mathcal{D}))}
\end{aligned}$$

where we have used Hölder's inequality with $p_1, p_2 > s$ such that $1/s = 1/p_1 + 1/p_2$. With Young's inequality we obtain

$$\begin{aligned}
\| \exp(\psi_2|W - W_N|)\|^{p_1}_{L^{p_1}(\Omega;L^\infty(\mathcal{D}))} &\leq \mathbb{E}(\exp(\psi_2 p_1 \|W - W_N\|_{L^\infty(\mathcal{D})}) \\
&\leq \exp(\psi_2^2 p_1^2 \mathrm{Tr}(Q))\mathbb{E}\Big(\exp\Big(\frac{\|W - W_N\|^2_{L^\infty(\mathcal{D})}}{4\mathrm{Tr}(Q)}\Big)\Big) \\
&=: C_2(p_1, \psi_2, Q).
\end{aligned}$$

Since $(W - W_N)(\cdot, x) \sim \mathcal{N}(0, \sigma_N^2(x))$ with $\sigma_N^2(x) := \sum_{i>N} \eta_i e_i(x)^2 \leq \mathrm{Tr}(Q)$, we proceed as in Lemma 4.3.5 to conclude by Fernique's theorem that $C_2(p_1, \Phi, Q) < +\infty$ and note

that $C_2(p_1, \Phi, Q)$ is bounded uniformly in N. Similarly,

$$\begin{aligned}\| \exp(\psi_2 |W_N|)\|^s_{L^s(\Omega;L^\infty(\mathcal{D}))} &\leq \exp(\psi_2^2 s^2 \mathrm{Tr}(Q))\mathbb{E}\Big(\exp\big(\frac{\|W_N\|^2_{L^\infty(\mathcal{D})}}{4\mathrm{Tr}(Q)}\big)\Big) \\ &:= C_1(s, \psi_2, Q) < +\infty\end{aligned}$$

for any s and $C_1(s, \psi_2, Q)$ is is bounded uniformly in N. Altogether, this yields

$$\begin{aligned}\|a - a_{N,\varepsilon}\|_{L^s(\Omega;\mathbb{R})} \leq \phi_2 C_1(s, \psi_2, Q))^{1/s} C_2(p_1, \psi_2, Q)^{1/p_1} \|W - W_N\|_{L^{p_2}(\Omega;L^\infty(\mathcal{D})} \\ + \|P - P_\varepsilon\|_{L^s(\Omega;L^\infty(\mathcal{D}))}\end{aligned}$$

and the claim follows by Theorems 4.3.8 and 4.3.11. □

Corollary 4.3.13. *Let Assumption 4.3.3 hold, then there exists a sequence of approximation parameters $((N_i, \varepsilon_i), i \in \mathbb{N})$ in $\mathbb{N}^{\mathbb{N}} \times (0, \infty)^{\mathbb{N}}$, depending only on β and s, such that the error $\|a - a_{N_i,\varepsilon_i}\|_{L^\infty(\mathcal{D})}$ converges to zero $\mathbb{P}$-almost surely as $i \to \infty$.*

Proof. For any $\epsilon > 0$ we get by Markov's inequality

$$\mathbb{P}(\|a_{N_i,\varepsilon_i} - a\|_{L^\infty(\mathcal{D})} \geq \epsilon) \leq \frac{\|a_{N_i,\varepsilon_i} - a\|^s_{L^s(\Omega;L^\infty(\mathcal{D}))}}{\epsilon^s}.$$

Using Theorem 4.3.12 and the inequality $(a + b)^s \leq 2^{s-1}(a^s + b^s)$ for $a, b > 0, s \geq 1$ this leads to

$$\sum_{i\in\mathbb{N}} \mathbb{P}(\|a_{N_i\varepsilon_i} - a\|_{L^\infty(\mathcal{D})} \geq \epsilon) \leq \frac{2^{s-1} C(s, \psi_2, Q, \lambda, \mathcal{D})^s}{\epsilon^s} \left(\sum_{i\in\mathbb{N}} \Xi^{s/2}_{N_i} + \varepsilon_i \right).$$

By Assumption 4.3.3, there exists $\beta > 0$ such that

$$\Xi_{N_i} = \sum_{j>N_i} \eta_j j^\beta j^{-\beta} \leq N_i^{-\beta} \sum_{j>N_i} \eta_j j^\beta \leq N_i^{-\beta} \sum_{j\in\mathbb{N}} \eta_j j^\beta \leq C_\eta N_i^{-\beta}.$$

Now, choosing $\delta > 2$, $N_i := \lceil i^{\delta/\beta s} \rceil$ and $\varepsilon_i := i^{-\delta/2}$ for $i \in \mathbb{N}$ yields the estimate

$$\sum_{i\in\mathbb{N}} \mathbb{P}(\|a_{N_i,\varepsilon_i} - a\|_{L^\infty(\mathcal{D})} \geq \epsilon) \leq \frac{2^{s-1} C(s, \psi_2, Q, \lambda, \mathcal{D})^s}{\epsilon^s} \left(C_\eta^{s/2} + 1 \right) \sum_{i\in\mathbb{N}} i^{-\delta/2} < +\infty,$$

and the sequence $(a_{N_i,\varepsilon_i}, i \in \mathbb{N})$ converges almost surely by the Borel-Cantelli lemma. □

4.3.4 Convergence of the approximated solution

We conclude this section by showing the convergence of $u_{N,\varepsilon}$ towards u, given that $a_{N,\varepsilon} \to a$ in $L^s(\Omega; L^\infty(\mathcal{D}))$.

Theorem 4.3.14. *Let Assumptions 4.3.3 hold such that $t := (2/p + 1/q + 1/s)^{-1} \geq 1$, where $p \in [1, (\eta^*)^{-1})$ for η^* as in Lemma 4.3.5. Then $u_{N,\varepsilon}$ converges to u in $L^t(\Omega; V)$ as $N \to +\infty$ and $\varepsilon \to 0$.*

Proof. Existence and uniqueness of weak solutions $u \in L^r(\Omega; V)$ resp. $u_{N,\varepsilon} \in L^r(\Omega; V)$, for $r \in [1, (1/q + \eta^*)^{-1})$, is guaranteed by Lemma 4.3.5 resp. Lemma 4.3.7 $\mathbb{P}$-almost surely. For notational convenience, we omit the argument $\omega \in \Omega$ in the following path-wise estimates with respect to $\|\cdot\|_V$. With Poincaré's inequality and $a_{N,\varepsilon,-}(\omega) > 0$ $\mathbb{P}$-a.s., we obtain the (path-wise) estimate

$$\begin{aligned}\|u - u_{N,\varepsilon}\|_V^2 &\leq \frac{1}{a_{N,\varepsilon,-}} \int_{\mathcal{D}} a_{N,\varepsilon}(|u - u_{N,\varepsilon}|^2 + \|\nabla u - \nabla u_{N,\varepsilon}\|_2^2)dx \\ &\leq \frac{1 + C_{|\mathcal{D}|}^2}{a_{N,\varepsilon,-}} \int_{\mathcal{D}} a_{N,\varepsilon} \|\nabla u - \nabla u_{N,\varepsilon}\|_2^2 dx\end{aligned}$$

where $C_{|\mathcal{D}|} > 0$ denotes the Poincaré constant. Since u and $u_{N,\varepsilon}$ are weak solutions of Problem (4.1) with Eq. (4.2) resp. Eq. (4.7), we have

$$\int_{\mathcal{D}} a_{N,\varepsilon} \nabla u \cdot \nabla u_{N,\varepsilon} dx = \int_{\mathcal{D}} a \|\nabla u\|_2^2 dx \quad \text{and} \quad \int_{\mathcal{D}} a_{N,\varepsilon} \|\nabla u_{N,\varepsilon}\|_2^2 dx = \int_{\mathcal{D}} a \nabla u \cdot \nabla u_{N,\varepsilon} dx,$$

almost surely, and hence

$$\int_{\mathcal{D}} a_{N,\varepsilon} \|\nabla u - \nabla u_{N,\varepsilon}\|_2^2 dx = \int_{\mathcal{D}} (a_{N,\varepsilon} - a) \nabla u \cdot (\nabla u - \nabla u_{N,\varepsilon}) dx.$$

By Hölder's inequality, $V \subset L^2(\mathcal{D})$, Eq. (4.3) and Theorem 4.2.5 we have

$$\begin{aligned}\|u - u_{N,\varepsilon}\|_V^2 &\leq \frac{1 + C_{|\mathcal{D}|}^2}{a_{N,\varepsilon,-}} \|a_{N,\varepsilon} - a\|_{L^\infty(\mathcal{D})} \|\nabla u\|_{L^2(\mathcal{D})} \|\nabla(u - u_{N,\varepsilon})\|_{L^2(\mathcal{D})} \\ &\leq \frac{1 + C_{|\mathcal{D}|}^2}{a_{N,\varepsilon,-}} \|a_{N,\varepsilon} - a\|_{L^\infty(\mathcal{D})} \|u\|_V \|u - u_{N,\varepsilon}\|_V \\ &\leq \frac{2(1 + C_{|\mathcal{D}|}^2) \max(1, C_{\mathcal{D}})(\|f\|_H + \|g\|_{L^2(\Gamma_2)})}{\min(1, C_{|\mathcal{D}|}^{-2}) a_{N,\varepsilon,-} a_-} \|a_{N,\varepsilon} - a\|_{L^\infty(\mathcal{D})} \|u - u_{N,\varepsilon}\|_V.\end{aligned}$$

Using $t = (2/p + 1/q + 1/s)^{-1} \geq 1$, the t-th moment of the path-wise error is bounded

by

$$\begin{aligned}\|u - u_{N,\varepsilon}\|_{L^t(\Omega;V)} \leq &(1 + C^2_{|\mathcal{D}|})C(a_-, \mathcal{D}, p)\|1/a_{N,\varepsilon,-}\|_{L^p(\Omega;\mathbb{R})} \\ &\cdot (\|f\|_{L^q(\Omega;H)} + \|g\|_{L^q(\Omega;L^2(\Gamma_2))}\|a_{N,\varepsilon} - a\|_{L^s(\Omega;L^\infty(\mathcal{D}))},\end{aligned} \tag{4.10}$$

where $C(a_-, \mathcal{D}, p) > 0$ is as in Theorem 4.2.5. The convergence now follows by Theorem 4.3.12, since $\|a^{-1}_{N,\varepsilon,-}\|_{L^p(\Omega;\mathbb{R})}$ is bounded uniformly in N and ε by Lemma 4.3.7. □

Corollary 4.3.15. *With the Assumptions of Theorem 4.3.14, there exists a sequence of approximation parameters $((N_i, \varepsilon_i), i \in \mathbb{N})$ in $\mathbb{N}^{\mathbb{N}} \times (0, \infty)^{\mathbb{N}}$, depending only on β and t, such that $\|u_{N_i,\varepsilon_i} - u\|_V$ converges to zero $\mathbb{P}$-almost surely as $i \to +\infty$.*

Proof. The proof is analogous to the one of Corollary 4.3.13 with Markov's inequality applied to the mapping $x \mapsto x^t$ and Ineq. (4.10). The sequence $(N_i, \varepsilon_i, i \in \mathbb{N})$ may then also be constructed in the same way as in Corollary 4.3.13 where we simply replace s by t. □

Knowing that $u_{N,\varepsilon}$ converges to u path-wise and in $L^t(\Omega; V)$, we aim to estimate moments of u by drawing samples from the distribution of $u_{N,\varepsilon}$. In general, the distribution of $u_{N,\varepsilon}$ is not known. Further, each path-wise solution is an element of the infinite-dimensional Hilbert space V, which in turn means that we are only able to simulate path-wise approximations of the functions $u_{N,\varepsilon}(\omega, \cdot)$ in a finite-dimensional subspace of V. Next, we show how to construct these approximations in some appropriate subspaces V_ℓ of V and how the discretization error may be controlled.

4.4 Adaptive path-wise discretization

The variational problem to Eq. (4.7) is to find for almost all $\omega \in \Omega$ and given $f(\omega, \cdot)$, $g(\omega, \cdot)$, N and ε a function $u_{N,\varepsilon}(\omega, \cdot) \in V$ such that

$$\begin{aligned}B_{a_{N,\varepsilon}(\omega)}(u_{N,\varepsilon}(\omega, \cdot), v) :&= \int_{\mathcal{D}} a_{N,\varepsilon}(\omega, x)\nabla u_{N,\varepsilon}(\omega, x) \cdot \nabla v(x)dx \\ &= \int_{\mathcal{D}} f(\omega, x)v(x)dx + \int_{\Gamma_2} g(\omega, x)[Tv](x)dx \\ &= F_\omega(v)\end{aligned} \tag{4.8}$$

for all $v \in V$. To find suitable approximations of $u_{N,\varepsilon}(\omega, \cdot)$, we use a standard Galerkin approach with a sequence $\mathcal{V} = (V_\ell, \ell \in \mathbb{N}_0)$ of finite-dimensional subspaces $V_\ell \subset V$. The corresponding family of refinement sizes is given by a sequence $(h_\ell, \ell \in \mathbb{N}_0)$, which decreases monotonically to zero as $\ell \to +\infty$. For any $\ell \in \mathbb{N}$, let $d_\ell \in \mathbb{N}$ and $\{v_1, \ldots, v_{d_\ell}\}$

be a basis of V_ℓ. The discrete version of the variational formulation (4.8) is then: find $u_{N,\varepsilon,\ell}(\omega,\cdot) \in V_\ell$ such that

$$B_{a_{N,\varepsilon}(\omega)}(u_{N,\varepsilon,\ell}(\omega,\cdot), v_\ell) = F_\omega(v_\ell) \quad \text{for all } v_\ell \in V_\ell.$$

The function $u_{N,\varepsilon,\ell}(\omega,\cdot)$ may be expanded with respect to $\{v_1,\dots,v_{d_\ell}\}$ as

$$u_{N,\varepsilon,\ell}(\omega,x) = \sum_{i=1}^{d_\ell} c_i v_i(x),$$

where $c_1,\dots,c_{d_\ell} \in \mathbb{R}$ and $\mathbf{c} := (c_1,\dots,c_{d_\ell})^T$ is the respective coefficient (column-) vector. With this, the discrete variational problem in the finite-dimensional space V_ℓ is equivalent to solving the linear system of equations

$$\mathbf{A}(\omega)\mathbf{c} = \mathbf{F}(\omega),$$

with stochastic stiffness matrix $(\mathbf{A}(\omega))_{ij} = B_{a_{N,\varepsilon}(\omega)}(v_i, v_j)$ and load vector $(\mathbf{F}(\omega))_i = F_\omega(v_i)$ for $i,j \in \{1,\dots,d_\ell\}$. Since the jump-diffusion coefficient is not continuous, in general one would not expect the full convergence rate of the Galerkin approximation.

Example 4.4.1. For a polygonal domain $\mathcal{D} \subset \mathbb{R}^d$, we define by $\mathcal{K} = (\mathcal{K}_\ell, \ell \in \mathbb{N}_0)$ a sequence of triangulations on $\mathcal{D}$. We denote the minimum interior angle of all triangles in $\mathcal{K}_\ell$ by $\vartheta_\ell > 0$ and assume that there exists some $\vartheta > 0$ such that $\inf_{\ell\in\mathbb{N}_0} \vartheta_\ell \geq \vartheta > 0$. The maximum diameter of each triangulation is defined by

$$h_\ell := \max_{K\in\mathcal{K}_\ell} \operatorname{diam}(K), \quad \ell \in \mathbb{N}_0$$

and the finite-dimensional subspaces are given by

$$V_\ell := \{v \in V |\ v|_K \in \mathcal{P}_1, K \in \mathcal{K}_\ell\},$$

where $\mathcal{P}_1$ is the space of all polynomials up to degree one. This yields a sequence $\mathcal{V} = (V_\ell, \ell \in \mathbb{N}_0)$ of subspaces in V with refinement parameters $(h_\ell, \ell \in \mathbb{N}_0)$. For fixed $\ell \in \mathbb{N}_0$, let $\{v_1,\dots,v_{d_\ell}\}$ be a basis of piecewise linear functions of V_ℓ. Given that $u_{N,\varepsilon} \in L^2(\Omega; H^{1+\kappa_a}(\mathcal{D}))$ for some $\kappa_a > 0$, the path-wise discretization error is bounded

by Céa's lemma $\mathbb{P}$-almost surely by

$$\begin{aligned}\|u_{N,\varepsilon}(\omega,\cdot)-u_{N,\varepsilon,\ell}(\omega,\cdot)\|_V &\leq (1+C_{|\mathcal{D}|})\|\nabla(u_{N,\varepsilon}(\omega,\cdot)-u_{N,\varepsilon,\ell}(\omega,\cdot))\|_{L^2(\mathcal{D})}\\ &\leq C_{\vartheta,\mathcal{D}}\frac{a_+(\omega)}{a_-(\omega)}\|u_{N,\varepsilon}(\omega,\cdot)\|_{H^{1+\kappa_a}(\mathcal{D})}h_\ell^{\min(\kappa_a,1)},\end{aligned}$$

where $C_{|\mathcal{D}|}$ is the Poincaré constant and $C_{\vartheta,\mathcal{D}}>0$ is deterministic and only depends on the indicated parameters (see e.g. [102, Chapter 8.3/8.5]). If $\mathcal{K}$ is sample-independent (and thus $(h_\ell,\ell\in\mathbb{N}_0)$ is fixed for any ω), $a_+/a_-\in L^2(\Omega;\mathbb{R})$ and there exists a uniform bound $\|u_{N,\varepsilon}\|_{L^2(\Omega;H^{1+\kappa_a}(\mathcal{D}))}\leq C_u$ in N and ε, we readily obtain

$$\|u_{N,\varepsilon}-u_{N,\varepsilon,\ell}\|_{L^2(\Omega;V)}\leq C_{\vartheta,\mathcal{D}}\mathbb{E}\left(\frac{a_+^2}{a_-^2}\right)^{1/2}C_u h_\ell^{\min(\kappa_a,1)}$$

by Hölder's inequality. We note that a uniform a-priori bound may require higher moments of $1/a_{N,\varepsilon}$, f and g or even essential bounds on $a_{N,\varepsilon}$ which are not ensured by Assumption 4.3.3.

Remark 4.4.2. For jump-diffusion problems, we obtain, in general, a discretization error of order $\kappa_a\in(1/2,1)$. We cannot expect the path-wise "full" order of convergence $\kappa_a=1$ of the finite-dimensional discretization error, since the diffusion coefficient $a_{N,\varepsilon}$ is almost surely discontinuous. Most results that ensure $H^2(\mathcal{D})$-regularity of the path-wise weak solution $u_{N,\varepsilon}(\omega,\cdot)$ need that $a_{N,\varepsilon}(\omega,\cdot)$ is continuously differentiable or that $a_{N,\varepsilon}(\omega,\cdot)\in W^{1,\infty}(\mathcal{D})$, see for instance [80]. The latter would imply that $a_{N,\varepsilon}(\omega,\cdot)$ is continuous by the Sobolev embedding theorem, which contradicts our setting. As we consider path-wise regularity, we can rely on results for deterministic elliptic problems with discontinuous coefficients. We refer to [175] and the references therein, where the author emphasizes that the regularity of the solution depends on the shape and magnitude of the discontinuities. For several examples $H^{1+\kappa_a}$-regularity with $\kappa_a<1$ is shown. In [13] the author states that without special treatment of the interfaces with respect to the triangulation one, in general, cannot expect a better path-wise convergence rate than $\kappa_a=1/2$.

4.4.1 Adaptive triangulations

In view of the previous remark, we aim to increase the order of convergence κ_a with respect to h_ℓ. For this, we employ path-dependent triangulations to match the interfaces generated by the samples of the jump-diffusion coefficient. Hence, we need to reformulate the discrete problem with respect to ω, since the triangulation and matching

basis functions may be sample-dependent. Given a fixed ω and ℓ, we consider a finite-dimensional subspace $\widehat{V}_\ell(\omega) \subset V$ with sample-dependent basis $\{\widehat{v}_1(\omega), \dots, \widehat{v}_{\widehat{d}_\ell}(\omega)\}$ and stochastic dimension $\widehat{d}_\ell(\omega) \in \mathbb{N}$. As before, we denote by $(\widehat{h}_\ell(\omega), \ell \in \mathbb{N}_0)$ the sequence of (random) refinement parameters corresponding to the sequence of subspaces $(\widehat{V}_\ell(\omega), \ell \in \mathbb{N}_0)$. More precisely, for a given random partition $\mathcal{T}(\omega) = (\mathcal{T}_i, i = 1 \dots, \tau(\omega))$ of $\mathcal{D}$, we choose a triangulation $\mathcal{K}_\ell(\omega)$ of $\mathcal{D}$ with respect to $\mathcal{T}(\omega)$ such that

$$\mathcal{T}(\omega) \subset \mathcal{K}_\ell(\omega) \quad \text{and} \quad \widehat{h}_\ell(\omega) := \max_{K \in \mathcal{K}_\ell(\omega)} \operatorname{diam}(K) \leq \overline{h}_\ell \quad \text{for } \ell \in \mathbb{N}_0,$$

where $(\overline{h}_\ell, \ell \in \mathbb{N}_0)$ is a positive sequence of deterministic refinement thresholds, decreasing monotonically to zero. This guarantees that $\widehat{h}_\ell(\omega) \to 0$ almost surely, although the absolute speed of convergence may vary for each ω. Denoting by $\widehat{\vartheta}_\ell(\omega)$ the minimal interior angle within $\mathcal{K}_\ell(\omega)$, we assume similarly to Example 4.4.1 that there exists a $\vartheta > 0$ such that for $\mathbb{P}$-almost all ω

$$\inf_{\ell \in \mathbb{N}_0} \widehat{\vartheta}_\ell(\omega) \geq \vartheta > 0. \tag{4.11}$$

The path-wise discrete variational problem in the sample-adaptive subspace $\widehat{V}_\ell(\omega)$ now reads: Find $\widehat{u}_{N,\varepsilon,\ell}(\omega, \cdot) \in \widehat{V}_\ell(\omega)$ such that

$$B_{a_{N,\varepsilon}(\omega)}(\widehat{u}_{N,\varepsilon,\ell}(\omega, \cdot), \widehat{v}_\ell(\omega)) = F_\omega(\widehat{v}_\ell(\omega)) \quad \text{for all } \widehat{v}_\ell(\omega) \in \widehat{V}_\ell(\omega) \subset V.$$

Since the triangulation is aligned with the discontinuities of $a_{N,\varepsilon}$, this approximation admits an increase of the (path-wise) order of convergence κ_a compared to the non-adaptive, deterministic Galerkin approximations. For piecewise linear basis functions $\widehat{v}_i$, we can expect

$$\begin{aligned} \|u_{N,\varepsilon}(\omega, \cdot) - \widehat{u}_{N,\varepsilon,\ell}(\omega, \cdot)\|_V &\leq C_{\vartheta,\mathcal{D}} \frac{a_+(\omega)}{a_-(\omega)} \Big(\|u_{N,\varepsilon}(\omega, \cdot)\|_V \\ &\qquad + \sum_{i=1}^{\tau(\omega)} \|\nabla \cdot \nabla u_{N,\varepsilon}(\omega, \cdot)\|_{L^2(\mathcal{T}_i(\omega))} \Big) \widehat{h}_\ell(\omega) \\ &=: C_{\vartheta,\mathcal{D}} \widehat{C}_u(\omega) \widehat{h}_\ell(\omega) \end{aligned} \tag{4.12}$$

$\mathbb{P}$-a.s. by results from domain decomposition methods for deterministic elliptic problems (e.g. [36, 39]). We emphasize that with the assumption of Ineq. (4.11) the constant $C_{\vartheta,\mathcal{D}} > 0$ is independent of ω. This estimate holds provided the random partitions are polygonal and $u_{N,\varepsilon}(\omega, \cdot)$ is almost surely piecewise in $H^2(\mathcal{T}_i)$. In this case, the adaptive triangulations perfectly fit the random subdomains in $\mathcal{D}$ and the order of convergence

is the same as if $u_{N,\varepsilon}(\omega,\cdot) \in H^2(\mathcal{D})$. If the random partitions are not polygonal but form C^2-interfaces within $\mathcal{D}$, we obtain log-linear rates:

$$\|u_{N,\varepsilon}(\omega,\cdot) - \widehat{u}_{N,\varepsilon,\ell}(\omega,\cdot)\|_V \leq C_{\vartheta,\mathcal{D}}\widehat{C}_u(\omega)\widehat{h}_\ell(\omega)|\log(\widehat{h}_\ell(\omega))|^{1/2} \tag{4.13}$$

with $\widehat{C}_u(\omega)$ as in Ineq. (4.12), see [56].

Remark 4.4.3. Based on Ass. 4.4.4 and the path-wise estimates in Eqs. (4.12), (4.13), the question arises whether it is possible to derive estimates in a mean-square sense

$$\|u_{N,\varepsilon} - \widehat{u}_{N,\varepsilon,\ell}\|_{L^2(\Omega;V)} \leq \widetilde{C}_u \mathbb{E}(\widehat{h}_\ell^2)^{1/2},$$

where the constant $\widetilde{C}_u$ is independent of N and ε. As the independence of ε is not an issue, a uniform estimate with respect to N requires further summability conditions (i.e. $\beta \geq 2\alpha$) on the eigenvalues in Ass. 4.3.3 and would result in a piecewise differentiable diffusion coefficient a. To illustrate this, we consider the identity

$$\begin{aligned}-\nabla\cdot(a_{N,\varepsilon}(\omega,\cdot)\nabla u_{N,\varepsilon}(\omega,\cdot)) &= -\nabla a_{N,\varepsilon}(\omega,\cdot)\cdot\nabla u_{N,\varepsilon}(\omega,\cdot) - a_{N,\varepsilon}(\omega,\cdot)\nabla\cdot\nabla u_{N,\varepsilon}(\omega,\cdot)\\ &= f(\omega,\cdot)\end{aligned}$$

on some partition element $\mathcal{T}_i(\omega)$. Rearranging terms and taking expectations then yields by Hölder's inequality

$$\begin{aligned}&\mathbb{E}\Big(\sum_{i=1}^{\tau(\omega)}\|\nabla\cdot\nabla u_{N,\varepsilon}(\omega,\cdot)\|_{L^2(\mathcal{T}_i(\omega))}\Big)\\ \leq&\mathbb{E}(\tau)^{1/2}\mathbb{E}\Big(\frac{2\|f\|_H^2 + 2\|\nabla(\overline{a}+\Phi(W_N))\|_{L^\infty(\mathcal{D})}^2\|\nabla u_{N,\varepsilon}\|_H^2}{a_{N,\varepsilon,-}^2}\Big).\end{aligned}$$

If f and $1/a_-$ have sufficiently high moments and the eigenvalues η_i decay fast enough, the right hand side may be bounded uniformly with respect to N, and it is then straightforward to derive the corresponding $L^2(\Omega;V)$-estimates. These assumptions, however, exclude the important cases where the covariance operator of W is of Brownian-motion-type or exponential. Practically, as we show in Section 4.6, the path-wise adaptive convergence rates may also be recovered for the $L^2(\Omega,V)$-error if a has only piecewise Hölder-continuous trajectories.

In view of the preceding remarks and Example 4.4.1, we make the following assumption on the mean-square discretization error:

Assumption 4.4.4. There exist constants $C_{u,a}, C_{u,n}, \kappa_a, \kappa_n > 0$, such that for any $N, \ell \in \mathbb{N}_0$ and $\varepsilon > 0$, the finite-dimensional approximation errors of $u_{N,\varepsilon}$ in the subspaces $\widehat{V}_\ell$ resp. V_ℓ are bounded by

$$\|u_{N,\varepsilon} - \widehat{u}_{N,\varepsilon,\ell}\|_{L^2(\Omega;V)} \leq C_{u,a}\mathbb{E}(\widehat{h}_\ell^{2\kappa_a})^{1/2} \quad \text{resp.} \quad \|u_{N,\varepsilon} - u_{N,\varepsilon,\ell}\|_{L^2(\Omega;V)} \leq C_{u,n}h_\ell^{\kappa_n}.$$

The constants $C_{u,a}, C_{u,n}$ may depend on a, f and g, but are independent of $\widehat{h}_\ell, h_\ell, \kappa_a$ and κ_n.

Note that in general $1 \geq \kappa_a > \kappa_n > 0$ by the previous observations. We consider $\mathbb{E}(\widehat{h}_\ell^{2\kappa_a})^{1/2}$ instead of $\overline{h}_\ell^{\kappa_a}$ as both quantities depend to a great extend on the geometry introduced by a. The parameter $\overline{h}_\ell$ has been merely introduced to ensure that $\lim_{\ell\to+\infty}\mathbb{E}(\widehat{h}_\ell^2) = 0$.

4.5 Estimation of expectations by Monte Carlo methods

As we are able to generate samples from $u_{N,\varepsilon}$ and control for the discretization error in each sample, we may estimate the expected value $\mathbb{E}(u)$ of the weak solution to Eq. (4.1). We focus on multilevel Monte Carlo estimators as they are easily implemented and do not require much regularity of u. Monte Carlo estimators introduce an additional statistical bias besides the error contributions of $a_{N,\varepsilon}$ and the path-wise discretization error from Section 4.4. However, this error can be controlled under natural assumptions and it may be equilibrated according to the other error terms. In this section, we first recall briefly standard Monte Carlo and multilevel Monte Carlo methods to estimate $\mathbb{E}(u)$ and then control for the mean-squared error in both algorithms. We also suggest a modification of the multilevel Monte Carlo estimation to increase computational efficiency before we verify our results on several numerical examples in Section 4.6.

4.5.1 Monte Carlo and multilevel Monte Carlo estimators

Consider a sequence $(u^{(i)}, i \in \mathbb{N})$ of i.i.d. copies of the V-valued random variable u. For $M \in \mathbb{N}$ independent samples, the *Monte Carlo estimator* of $\mathbb{E}(u)$ is defined as

$$E_M(u) := \frac{1}{M}\sum_{i=1}^{M} u^{(i)}.$$

Since we are only able to draw samples of the approximated discrete solution $\widehat{u}_{N,\varepsilon,\ell}$ as introduced in Section 4.4, we aim to control the *root mean-squared error* (RMSE) $\|\mathbb{E}(u) - E_M(\widehat{u}_{N,\varepsilon,\ell})\|_{L^2(\Omega;V)}$. For notational convenience, we focus only on the adaptive

discretization $\widehat{u}_{N,\varepsilon,\ell}$ with mean-square refinement $\mathbb{E}(\widehat{h}_\ell^2)^{1/2}$ and converge rate κ_a in this section. However, all results also hold in the non-adaptive case with $\mathbb{E}(\widehat{h}_\ell^2)^{1/2}$ and κ_a replaced by h_ℓ and κ_n. To derive a bound for the RMSE, we need the following standard result.

Lemma 4.5.1. *Let $M \in \mathbb{N}$ and $u \in L^2(\Omega; V)$. Then*

$$\|\mathbb{E}(u) - E_M(u)\|_{L^2(\Omega;V)} \leq \frac{\|u\|_{L^2(\Omega;V)}}{\sqrt{M}} \quad \text{and} \quad \|E_M(u)\|_{L^2(\Omega;V)} \leq \|u\|_{L^2(\Omega;V)}.$$

Theorem 4.5.2. *Let Assumptions 4.3.3 and 4.4.4 hold such that $t := (2/p + 1/q + 1/s)^{-1} \geq 2$, where $p \in [1, (\eta^*)^{-1})$ for η^* as in Lemma 4.3.5. Then, for any $M, N \in \mathbb{N}$ and $\varepsilon > 0$,*

$$\|\mathbb{E}(u) - E_M(\widehat{u}_{N,\varepsilon,\ell})\|_{L^2(\Omega;V)} \leq C_{MC}\left(\frac{1}{\sqrt{M}} + \Xi_N^{1/2} + \varepsilon^{1/s} + \mathbb{E}(\widehat{h}_\ell^{2\kappa_a})^{1/2}\right),$$

where $C_{MC} > 0$ is a constant independent of M, Ξ_N, ε and $\widehat{h}_\ell$.

Proof. We apply the triangle inequality to obtain

$$\begin{aligned}\|\mathbb{E}(u) - E_M(\widehat{u}_{N,\varepsilon,\ell})\|_{L^2(\Omega;V)} \leq{}& \|\mathbb{E}(u) - E_M(u)\|_{L^2(\Omega;V)} + \|E_M(u - u_{N,\varepsilon})\|_{L^2(\Omega;V)} \\ &+ \|E_M(u_{N,\varepsilon} - \widehat{u}_{N,\varepsilon,\ell})\|_{L^2(\Omega;V)}.\end{aligned}$$

By the first part of Lemma 4.5.1 and Theorem 4.2.5, we bound the first term by

$$\|\mathbb{E}(u) - E_M(u)\|_{L^2(\Omega;V)} \leq C(a_-, \mathcal{D}, p)\frac{\|f\|_{L^q(\Omega;H)} + \|g\|_{L^q(\Omega;\Gamma_2)}}{\sqrt{M}}.$$

The second part of Lemma 4.5.1 yields with the estimate of Theorem 4.3.14

$$\begin{aligned}\|E_M(u - u_{N,\varepsilon})\|_{L^2(\Omega;V)} &\leq \widetilde{C}(a, f, g, \mathcal{D})\|a_{N,\varepsilon}^{-1}\|_{L^p(\Omega;\mathbb{R})}\|a - a_{N,\varepsilon}\|_{L^s(\Omega;L^\infty(\mathcal{D}))} \\ &\leq C(a, f, g, \mathcal{D})(\Xi_N^{1/2} + \varepsilon^{1/s})\end{aligned}$$

where $\widetilde{C}(a, f, g, \mathcal{D}) > 0$ and $C(a, f, g, \mathcal{D}) > 0$ are independent of N and ε, because by Lemma 4.3.7 $\|1/a_{N,\varepsilon,-}\|_{L^p(\Omega;\mathbb{R})}$ is bounded uniformly with respect to these parameters. Finally, Assumption 4.4.4 and Lemma 4.5.1 yield for the third term

$$\|E_M(u_{N,\varepsilon} - \widehat{u}_{N,\varepsilon,\ell})\|_{L^2(\Omega;V)} \leq C_{u,a}\mathbb{E}(\widehat{h}_\ell^{2\kappa_a})^{1/2}$$

where $C_{u,a}$ is also independent of N and ε by assumption.

□

Remark 4.5.3. The estimate in Theorem 4.5.2 suggests that all four error contributions should be equilibrated to obtain a RMSE of order $\mathbb{E}(\widehat{h}_\ell^{2\kappa_a})^{1/2}$. For this, we may choose M, N and ε such that

$$M^{-1} \simeq \Xi_N \simeq \varepsilon^{2/s} \simeq \mathbb{E}(\widehat{h}_\ell^{2\kappa_a}).$$

While this is straightforward for M and usually also for ε, the choice of the truncation index N involves a few difficulties. In general, the eigenvalues $(\eta_i, i \in \mathbb{N})$ of the covariance operator $Q : H \to H$ will not be available in closed form. The decay parameter $\beta > 0$ may be unknown and only be bounded from below, which would result in an overestimation of the term Ξ_N. One possibility to find N is applicable if Q is an integral operator of the form

$$(Q\varphi)[x] = v \int_{\mathcal{D}} k_Q(x, y)\varphi(y)dy, \quad \varphi \in H,$$

with some nonnegative, symmetric and bounded kernel function $k_Q : \mathcal{D}^2 \to \mathbb{R}$ and $v > 0$. In this case, the eigenvalues of Q fulfill the identity $v \int_D dx = \sum_{i\in\mathbb{N}} \eta_i$ (see [79, 203]). Operators with this property are widely used in practice and include for instance Matérn class, Brownian motion and rational quadratic covariance functions (see [178]). The first eigenvalues of Q have to be, in any case, determined (numerically) to approximate the Gaussian field W, so we select N such that

$$\Xi_N = v \int_{\mathcal{D}} dx - \sum_{i=1}^{N} \eta_i \overset{!}{\simeq} \mathbb{E}(\widehat{h}_\ell^{2\kappa_a}).$$

In most cases, the sampling of $a_{N,\varepsilon}$ and $u_{N,\varepsilon,\ell}$ for given boundary data will be computationally expensive: If the eigenvalues of Q decay slowly, it is necessary to include a large number of terms in the Karhunen-Loève expansion to achieve $\Xi_N \simeq \mathbb{E}(\widehat{h}_\ell^{2\kappa_a})$. In addition, sampling of the sequence $(\widetilde{P}_i, i \in \mathbb{N})$ might also be time-consuming if a small error $\mathbb{E}(|\widetilde{P}_i - P_i|^s) \leq \varepsilon$ is desired. Given a sample of $a_{N,\varepsilon}$, one has then to rely on numerical integration schemes to calculate the entries of the stiffness matrix $\mathbf{A}(\omega)$ and the load vector $\mathbf{F}(\omega)$, and solve a possibly large system of linear equations. This motivates the use of advanced Monte Carlo techniques, such as multilevel Monte Carlo, to achieve essentially the same accuracy with reduced computational effort. We briefly recall the idea of the multilevel Monte Carlo sampling in the following.

For $L \in \mathbb{N}$ we consider finite-dimensional subspaces $\widehat{V}_0 \subset \cdots \subset \widehat{V}_L$ of V with refinement sizes $\widehat{h}_0 > \cdots > \widehat{h}_L > 0$ and approximation parameters $N_0 < \cdots < N_L$ and

$\varepsilon_0 > \cdots > \varepsilon_\ell$. We define $u_{N_{-1},\varepsilon_{-1},-1} := 0$ and expand the "finest level approximation" $\widehat{u}_{N_L,\varepsilon_L,L}$ into a telescopic sum to obtain

$$\mathbb{E}(\widehat{u}_{N_L,\varepsilon_L,L}) = \sum_{\ell=0}^{L} \mathbb{E}(\widehat{u}_{N_\ell,\varepsilon_\ell,\ell} - \widehat{u}_{N_{\ell-1},\varepsilon_{\ell-1},\ell-1}).$$

Instead of estimating the left hand side by the ordinary Monte Carlo method, we estimate the expected corrections $\mathbb{E}(\widehat{u}_{N_\ell,\varepsilon_\ell,\ell} - \widehat{u}_{N_{\ell-1},\varepsilon_{\ell-1},\ell-1})$ by generating M_ℓ independent realizations $\widehat{u}^{(i,\ell)}_{N_\ell,\varepsilon_\ell,\ell} - \widehat{u}^{(i,\ell)}_{N_{\ell-1},\varepsilon_{\ell-1},\ell-1}$ on each level and calculating the Monte Carlo estimator $E_{M_\ell}(\widehat{u}_{N_\ell,\varepsilon_\ell,\ell} - \widehat{u}_{N_{\ell-1},\varepsilon_{\ell-1},\ell-1})$. The *multilevel Monte Carlo estimator* of $u_{N_L,\varepsilon_L,L}$ is then defined as

$$\begin{aligned} E^L(\widehat{u}_{N_L,\varepsilon_L,L}) :&= \sum_{\ell=0}^{L} E_{M_\ell}(\widehat{u}_{N_\ell,\varepsilon_\ell,\ell} - \widehat{u}_{N_{\ell-1},\varepsilon_{\ell-1},\ell-1}) \\ &= \sum_{\ell=0}^{L} \frac{1}{M_\ell} \sum_{i=1}^{M_\ell} \widehat{u}^{(i,\ell)}_{N_\ell,\varepsilon_\ell,\ell} - \widehat{u}^{(i,\ell)}_{N_{\ell-1},\varepsilon_{\ell-1},\ell-1} \end{aligned} \tag{4.14}$$

To achieve a desired target RMSE of $\epsilon_{RMSE} > 0$, this estimator requires less computational effort than the standard Monte Carlo approach under certain assumptions. This, by now, classical result was proven in [92, Theorem 3.1]. The proof is rather general and may readily be transferred to the problem of estimating moments of random PDEs (see [29]).

Theorem 4.5.4. *Let Assumptions 4.3.3 and 4.4.4 hold such that* $t := (2/p + 1/q + 1/s)^{-1} \geq 2$, *where* $p \in [1, (\eta^*)^{-1})$ *for* η^* *as in Lemma 4.3.5. For* $L \in \mathbb{N}$, *let* $\widehat{h}_\ell > 0$, $M_\ell, N_\ell \in \mathbb{N}$ *and* $\varepsilon_\ell > 0$ *be the level-dependent approximation parameters for any* $\ell = 0, \ldots, L$ *such that* $\widehat{h}_\ell, \varepsilon_\ell$ *are decreasing and* N_ℓ *is increasing with respect to* ℓ. *Then the multilevel Monte Carlo estimator admits the bound*

$$\begin{aligned} \|\mathbb{E}(u) - E^L(\widehat{u}_{N_L,\varepsilon_L,L})\|_{L^2(\Omega;V)} \leq C\Bigg(&\Xi_{N_L}^{1/2} + \varepsilon_L^{1/s} + \mathbb{E}(\widehat{h}_L^{2\kappa_a})^{1/2} + \frac{1}{\sqrt{M_0}} \\ &+ \sum_{\ell=0}^{L-1} \frac{\Xi_{N_\ell}^{1/2} + \varepsilon_\ell^{1/s} + \mathbb{E}(\widehat{h}_\ell^{2\kappa_a})^{1/2}}{\sqrt{M_{\ell+1}}}\Bigg), \end{aligned}$$

where $C > 0$ *is independent of* L *and the level-dependent approximation parameters.*

Proof. Using the triangle inequality and Jensen's inequality for expectations, we ob-

serve that

$$\begin{aligned}
&\|\mathbb{E}(u) - E^L(\widehat{u}_{N_L,\varepsilon_L,L})\|_{L^2(\Omega;V)} \\
\leq &\|\mathbb{E}(u) - \mathbb{E}(\widehat{u}_{N_L,\varepsilon_L,L})\|_{L^2(\Omega;V)} + \|\mathbb{E}(\widehat{u}_{N_L,\varepsilon_L,L}) - E^L(\widehat{u}_{N_L,\varepsilon_L,L})\|_{L^2(\Omega;V)} \\
\leq &\underbrace{\|u - u_{N_L,\varepsilon_L}\|_{L^2(\Omega;V)} + \|u_{N_L,\varepsilon_L} - \widehat{u}_{N_L,\varepsilon_L,L}\|_{L^2(\Omega;V)}}_{:=I} \\
&+ \underbrace{\|\mathbb{E}(\widehat{u}_{N_L,\varepsilon_L,L}) - E^L(\widehat{u}_{N_L,\varepsilon_L,L})\|_{L^2(\Omega;V)}}_{:=II}.
\end{aligned}$$

Theorem 4.3.14 and Assumption 4.4.4 give a bound for the first term by

$$I \leq C(a,f,g,\mathcal{D})(\Xi_{N_L}^{1/2} + \varepsilon_L^{1/s}) + C_{u,a}\mathbb{E}(\widehat{h}_L^{2\kappa_a})^{1/2},$$

where $C(a,f,g,\mathcal{D}) > 0$ is an independent constant. For the second error term, the definition of E^L in Eq. (4.14) together with Lemma 4.5.1 yield

$$\begin{aligned}
II &\leq \sum_{\ell=0}^{L} \|\mathbb{E}(\widehat{u}_{N_\ell,\varepsilon_\ell,\ell} - \widehat{u}_{N_{\ell-1},\varepsilon_{\ell-1},\ell-1}) - E_{M_\ell}(\widehat{u}_{N_\ell,\varepsilon_\ell,\ell} - \widehat{u}_{N_{\ell-1},\varepsilon_{\ell-1},\ell-1})\|_{L^2(\Omega;V)} \\
&\leq \sum_{\ell=0}^{L} \frac{1}{\sqrt{M_\ell}} \|\widehat{u}_{N_\ell,\varepsilon_\ell,\ell} - \widehat{u}_{N_{\ell-1},\varepsilon_{\ell-1},\ell-1}\|_{L^2(\Omega;V)} \\
&\leq \sum_{\ell=0}^{L} \Big(\frac{1}{\sqrt{M_\ell}} \|\widehat{u}_{N_\ell,\varepsilon_\ell,\ell} - u\|_{L^2(\Omega;V)} + \frac{1}{\sqrt{M_\ell}} \|u - \widehat{u}_{N_{\ell-1},\varepsilon_{\ell-1},\ell-1}\|_{L^2(\Omega;V)} \Big).
\end{aligned}$$

Now, the terms $\|\widehat{u}_{N_\ell,\varepsilon_\ell,\ell} - u\|_{L^2(\Omega;V)}$ may be treated analogously to I:

$$\|\widehat{u}_{N_\ell,\varepsilon_\ell,\ell} - u\|_{L^2(\Omega;V)} \leq C(a,f,g,\mathcal{D})(\Xi_{N_\ell}^{1/2} + \varepsilon_\ell^{1/s}) + C_{u,a}\mathbb{E}(\widehat{h}_\ell^{2\kappa_a})^{1/2}. \tag{4.15}$$

The remaining term $\|u - \widehat{u}_{N_{-1},\varepsilon_{-1},-1}\|_{L^2(\Omega;V)}$ is bounded by Theorem 4.2.5 via

$$\|u - \widehat{u}_{N_{-1},\varepsilon_{-1},-1}\|_{L^2(\Omega;V)} = \|u\|_{L^2(\Omega;V)} \leq C(a_-,\mathcal{D},p)(\|f\|_{L^q(\Omega;H)} + \|g\|_{L^q(\Omega;\Gamma_2)}). \tag{4.16}$$

Substituting the bounds from Ineq. (4.15) and Ineq. (4.16) in II and adding the esti-

mate on I, we finally arrive at

$$\begin{aligned} I + II &\leq C(a, f, g, \mathcal{D})\Big(\Xi_{N_L}^{1/2} + \varepsilon_L^{1/s} + \sum_{\ell=1}^{L} \frac{\Xi_{N_\ell}^{1/2} + \varepsilon_\ell^{1/s} + \Xi_{N_{\ell-1}}^{1/2} + \varepsilon_{\ell-1}^{1/s}}{\sqrt{M_\ell}}\Big) \\ &\quad + C_{u,a}\Big(\mathbb{E}(\widehat{h}_L^{2\kappa_a})^{1/2} + \sum_{\ell=1}^{L} \frac{\mathbb{E}(\widehat{h}_\ell^{2\kappa_a})^{1/2} + \mathbb{E}(\widehat{h}_{\ell-1}^{2\kappa_a})^{1/2}}{\sqrt{M_\ell}}\Big) \\ &\quad + \frac{C(a_-, \mathcal{D}, p)(\|f\|_{L^q(\Omega;H)} + \|g\|_{L^q(\Omega;\Gamma_2)})}{\sqrt{M_0}} \\ &\leq C\Big(\Xi_{N_L}^{1/2} + \varepsilon_L^{1/s} + \mathbb{E}(\widehat{h}_L^{2\kappa_a})^{1/2}\Big) + C\Big(\sum_{\ell=0}^{L-1} \frac{\Xi_{N_\ell}^{1/2} + \varepsilon_\ell^{1/s} + \mathbb{E}(\widehat{h}_\ell^{2\kappa_a})^{1/2}}{\sqrt{M_{\ell+1}}} + \frac{1}{\sqrt{M_0}}\Big). \end{aligned}$$

For the last inequality, we have used $\Xi_{N_\ell} \leq \Xi_{N_{\ell-1}}, \varepsilon_\ell \leq \varepsilon_{\ell-1}$ and $\mathbb{E}(\widehat{h}_\ell^{2\kappa_a})^{1/2} \leq \mathbb{E}(\widehat{h}_{\ell-1}^{2\kappa_a})^{1/2}$ together with the constant

$$C := \max(2C(a, f, g, \mathcal{D}), 2C_{u,a}, C(a_-, \mathcal{D}, p)(\|f\|_{L^q(\Omega;H)} + \|g\|_{L^q(\Omega;\Gamma_2)})).$$

□

Regarding a single realization of the approximation $\widehat{u}_{N_\ell,\varepsilon_\ell,\ell}$, we want the path-wise error $\|u(\omega,\cdot) - \widehat{u}_{N_\ell,\varepsilon_\ell,\ell}(\omega,\cdot)\|_V$ to decrease as ℓ increases, so naturally the parameters ε_ℓ and $\mathbb{E}(\widehat{h}_\ell^{2\kappa_a})^{1/2}$ should decrease in ℓ and N_ℓ increase in ℓ. For example, by using a sequence of refining grids, the refinement parameter $\mathbb{E}(\widehat{h}_\ell^2)^{1/2}$ may be divided roughly by a factor of two in each level, i.e. $2\mathbb{E}(\widehat{h}_\ell^2)^{1/2} \approx \mathbb{E}(\widehat{h}_{\ell-1}^2)^{1/2}$ for any $\ell \in \mathbb{N}$. Similar refining factors may be imposed for the sum of the remaining eigenvalues Ξ_{N_ℓ} and the sampling errors ε_ℓ. One advantage of the multilevel Monte Carlo estimator is that we are now able to even out the error contributions of the sampling bias $\|\widehat{u}_{N_\ell,\varepsilon_\ell,\ell} - u\|_V$ and the statistical error (with respect to M_ℓ) on each level. This is achieved by generating relatively few of the accurate, but expensive, samples for large ℓ and generating more of the cheap, but less accurate, samples on the lower levels.

Corollary 4.5.5. *Let the assumptions of Theorem 4.5.4 hold. For $L \in \mathbb{N}$ and given refinement parameters $\mathbb{E}(\widehat{h}_0^{2\kappa_a})^{1/2} > \cdots > \mathbb{E}(\widehat{h}_L^{2\kappa_a})^{1/2} > 0$ choose $N_\ell \in \mathbb{N}$ and $\varepsilon_\ell > 0$ such that*

$$\Xi_{N_\ell} \simeq \varepsilon_\ell^{2/s} \simeq \mathbb{E}(\widehat{h}_\ell^{2\kappa_a}) \quad \text{for } \ell = 0, \ldots, L,$$

and $M_\ell \in \mathbb{N}$ such that for some arbitrary $\nu > 0$ and $\ell = 1, \ldots, L$

$$M_0^{-1} \simeq \mathbb{E}(\widehat{h}_L^{2\kappa_a}) \quad \text{and} \quad M_\ell^{-1} \simeq \mathbb{E}(\widehat{h}_L^{2\kappa_a})^{-1}\mathbb{E}(\widehat{h}_{\ell-1}^{2\kappa_a})(\ell+1)^{-2(1+\nu)}.$$

The RMSE of the multilevel Monte Carlo estimator is then of order $\mathbb{E}(\widehat{h}_L^{2\kappa_a})^{1/2}$*:*

$$\|\mathbb{E}(u) - E^L(\widehat{u}_{N_L,\varepsilon_L,L})\|_{L^2(\Omega;V)} = \mathcal{O}(\mathbb{E}(\widehat{h}_L^{2\kappa_a})^{1/2}).$$

Proof. With the approximation parameters N_ℓ, ε_ℓ and Theorem 4.5.4, we obtain

$$\begin{aligned}
\|\mathbb{E}(u) - E^L(\widehat{u}_{N_L,\varepsilon_L,L})\|_{L^2(\Omega;V)} &\leq C\Big(4\mathbb{E}(\widehat{h}_L^{2\kappa_a})^{1/2} + \sum_{\ell=0}^{L-1} \frac{\mathbb{E}(\widehat{h}_\ell^{2\kappa_a})^{1/2}}{\sqrt{M_{\ell+1}}}\Big) \\
&\leq C\Big(4\mathbb{E}(\widehat{h}_L^{2\kappa_a})^{1/2} + \mathbb{E}(\widehat{h}_0^{2\kappa_a})^{1/2}\mathbb{E}(\widehat{h}_L^{2\kappa_a})^{1/2} \\
&\quad + \sum_{\ell=1}^{L} \mathbb{E}(\widehat{h}_L^{2\kappa_a})^{1/2}(\ell+1)^{-1-\nu}\Big) \\
&\leq C\Big(3 + \mathbb{E}(\widehat{h}_0^{2\kappa_a})^{1/2} + \zeta(1+\nu)\Big)\mathbb{E}(\widehat{h}_L^{2\kappa_a})^{1/2},
\end{aligned}$$

where $\zeta(\cdot)$ is the Riemann zeta function. □

For the level-dependent choice of N_ℓ and ε_ℓ we refer to Remark 4.5.3. In the remainder of this section, we introduce a modification of the multilevel Monte Carlo method to further reduce computational complexity.

4.5.2 Coupled multilevel Monte Carlo

Recall the multilevel Monte Carlo estimator

$$E^L(\widehat{u}_{N_L,\varepsilon_L,L}) = \sum_{\ell=0}^{L} \frac{1}{M_\ell} \sum_{i=1}^{M_\ell} \Big(\widehat{u}^{(i,\ell)}_{N_\ell,\varepsilon_\ell,\ell} - \widehat{u}^{(i,\ell)}_{N_{\ell-1},\varepsilon_{\ell-1},\ell-1}\Big)$$

of $\mathbb{E}(u)$ as in Eq. (4.14), where the terms in the second sum are independent copies of the corrections $\widehat{u}_{N_\ell,\varepsilon_\ell,\ell} - \widehat{u}_{N_{\ell-1},\varepsilon_{\ell-1},\ell-1}$. In total, one has to generate $M_\ell + M_{\ell+1}$ samples of $\widehat{u}_{N_\ell,\varepsilon_\ell,\ell}$ for each $\ell = 0,\dots,L$ (where we have set $M_{L+1} := 0$). This could be expensive even in low dimensions d. We can reduce this effort if we "recycle" the already available samples and generate the differences

$$\widehat{u}^{(i,\ell)}_{N_\ell,\varepsilon_\ell,\ell} - \widehat{u}^{(i,\ell)}_{N_{\ell-1},\varepsilon_{\ell-1},\ell-1} \quad \text{and} \quad \widehat{u}^{(i,\ell)}_{N_{\ell+1},\varepsilon_{\ell+1},\ell+1} - \widehat{u}^{(i,\ell)}_{N_\ell,\varepsilon_\ell,\ell}$$

based on the same realization $\widehat{u}^{(i,\ell)}_{N_\ell,\varepsilon_\ell,\ell}$ for $\ell = 0,\dots,L$. That is, we drop the second superscript ℓ in $\widehat{u}^{(i,\ell)}_{N_\ell,\varepsilon_\ell,\ell}$ and arrive at the *coupled multilevel Monte Carlo estimator*

$$E_C^L(\widehat{u}_{N_L,\varepsilon_L,L}) := \sum_{\ell=0}^{L} \frac{1}{M_\ell} \sum_{i=1}^{M_\ell} \widehat{u}^{(i)}_{N_\ell,\varepsilon_\ell,\ell} - \widehat{u}^{(i)}_{N_{\ell-1},\varepsilon_{\ell-1},\ell-1}.$$

This entails generating M_ℓ realizations of the random variable $\widehat{u}_{N_\ell,\varepsilon_\ell,\ell}$ instead of $M_\ell + M_{\ell+1}$. The samples $u^{(i)}_{N_\ell,\varepsilon_\ell,\ell}$ are then independent in i, but not anymore across all levels ℓ for a fixed index i, and thus *coupled* with respect to ℓ. This estimator is unbiased, i.e $\mathbb{E}(E^L_C(\widehat{u}_{N_L,\varepsilon_L,L})) = \mathbb{E}(\widehat{u}_{N_L,\varepsilon_L,L})$, and it holds

$$\lim_{L\to+\infty} \mathbb{E}(E^L_C(\widehat{u}_{N_L,\varepsilon_L,L})) = \lim_{L\to+\infty} \mathbb{E}(E^L(\widehat{u}_{N_L,\varepsilon_L,L})) = \lim_{L\to+\infty} \mathbb{E}(u_{N_L,\varepsilon_L}) = \mathbb{E}(u).$$

The introduced modification is a simplified version of the *Multifidelity Monte Carlo estimator* (see [171]). Under suitable assumptions on the variance of u, it is shown in [171] that the Multifidelity Monte Carlo approach achieves the same rate of convergence as the standard multilevel Monte Carlo method with reduced computational effort. The coupled estimator corresponds to a Multifidelity Monte Carlo estimator where the weighting coefficients for all level corrections $\widehat{u}_{N_\ell,\varepsilon_\ell,\ell} - \widehat{u}_{N_{\ell-1},\varepsilon_{\ell-1},\ell-1}$ are set equal to one.

We emphasize that the error bounds derived in Thm. 4.5.4 and Cor. 4.5.5 do not require independence of the sampled differences $\widehat{u}_{N_\ell,\varepsilon_\ell,\ell} - \widehat{u}_{N_{\ell-1},\varepsilon_{\ell-1},\ell-1}$ across the levels ℓ. Thus, the asymptotic order of convergence also holds for the coupled estimator. However, we have now introduced additional correlation across the levels, which may entail higher variance, and thus slower convergence of the coupled method. The variance of the standard multilevel Monte Carlo estimator is easily calculated as

$$\mathrm{Var}(E^L(\widehat{u}_{N_L,\varepsilon_L,L})) = \sum_{\ell=0}^{L} \frac{\mathrm{Var}(\widehat{u}_{N_\ell,\varepsilon_\ell,\ell} - \widehat{u}_{N_{\ell-1},\varepsilon_{\ell-1},\ell-1})}{M_\ell},$$

whereas

$$\begin{aligned}
\mathrm{Var}(E^L_C(\widehat{u}_{N_L,\varepsilon_L,L})) &= \mathrm{Var}\Big(\sum_{\ell=0}^{L} \frac{1}{M_\ell} \sum_{i=1}^{M_\ell} \widehat{u}^{(i)}_{N_\ell,\varepsilon_\ell,\ell} - \widehat{u}^{(i)}_{N_{\ell-1},\varepsilon_{\ell-1},\ell-1} \Big) \\
&= \mathrm{Var}\Big(\sum_{i=1}^{M_0-M_1} \frac{\widehat{u}^{(i)}_{N_0,\varepsilon_0,0}}{M_0} \\
&\quad + \sum_{i=M_0-M_1+1}^{M_0-M_2} \frac{\widehat{u}^{(i)}_{N_1,\varepsilon_1,1} - \widehat{u}^{(i)}_{N_0,\varepsilon_0,0}}{M_1} + \frac{\widehat{u}^{(i)}_{N_0,\varepsilon_0,0}}{M_0} + \dots \\
&\quad + \sum_{i=M_0-M_L+1}^{M_0} \sum_{\ell=0}^{L} \frac{\widehat{u}^{(i)}_{N_\ell,\varepsilon_\ell,\ell} - \widehat{u}^{(i)}_{N_{\ell-1},\varepsilon_{\ell-1},\ell-1}}{M_\ell} \Big) \\
&= \sum_{\ell=0}^{L} (M_\ell - M_{\ell+1}) \mathrm{Var}\left(\sum_{k=0}^{\ell} \frac{\widehat{u}_{N_k,\varepsilon_k,k} - \widehat{u}_{N_{k-1},\varepsilon_{k-1},k-1}}{M_k} \right)
\end{aligned}$$

In case that the differences $\widehat{u}_{N_\ell,\varepsilon_\ell,\ell} - \widehat{u}_{N_{\ell-1},\varepsilon_{\ell-1},\ell-1}$ are positively correlated across the levels, we trade in simulation time for a possibly higher RMSE, where the ratio of this trade is problem-dependent and hard to access beforehand. Nevertheless, we will compare this modified estimator with the standard multilevel Monte Carlo estimator in different scenarios to show its advantages.

4.6 Numerical examples

Throughout this section, we plot the error rates against the smallest estimated root-mean-squared refinement size $\mathbb{E}(\widehat{h}_L^2)^{1/2}$. For the standard, non-adaptive algorithms this corresponds to the preset deterministic refinement size h_L and all error contributions may be equilibrated a-priori as in Corollary 4.5.5. In the adaptive algorithm, to align the discretization error with the error contributions of $\sum_{i>N} \eta_i$, ε and the statistical error, we sample a few realizations of the diffusion coefficient before the start of the Monte Carlo loop. This allows us to estimate the values of $\mathbb{E}(\widehat{h}_\ell^2)^{1/2}$ for each $\ell = 0, \dots, L$ and choose N, ε and the number of samples on each level accordingly. All numerical examples are implemented with MATLAB and calculated on a workstation with 16 GB Memory and Intel quadcore processor with 3.4 GHz.

4.6.1 Numerical examples in 1D

For all test scenarios in this subsection, we consider the diffusion problem (4.1) in the one dimensional domain $\mathcal{D} = (0, 1)$ with homogeneous Dirichlet boundary conditions, i.e. $\Gamma_1 = \partial\mathcal{D}$, and source term $f \equiv 1$. The deterministic part of the diffusion coefficient is $\overline{a} \equiv 0$ and we consider a log-Gaussian component, i.e. $\Phi(w) = \exp(w)$, where the Gaussian field W is characterized by either the *Brownian motion covariance operator* $Q_{BM} : H \to H$ with

$$[Q_{BM}\varphi](y) := \int_{\mathcal{D}} min(x, y)\varphi(x)dx, \quad \varphi \in H$$

or the *squared exponential covariance operator*

$$[Q_{SE}\varphi](y) := \sigma^2 \int_{\mathcal{D}} \exp\left(\frac{-|x-y|^2}{2\rho^2}\right)\varphi(x)dx, \quad \varphi \in H$$

with variance parameter $\sigma^2 > 0$ and correlation length $\rho > 0$. The eigenbasis of Q_{BM} is given by

$$\eta_i = \left(\frac{2\sqrt{2}}{(2i+1)\pi}\right)^2, \quad e_i(x) = \sin\left(\frac{(2i+1)\pi x}{2}\right), \quad i \in \mathbb{N}_0,$$

see for example [6, p. 46 ff.], where the spectral basis of Q_{SE} may be efficiently approximated by Nyström's method, see [178]. The number of partition elements is given by $\tau = \mathcal{P} + 2$, where $\mathcal{P}$ is a Poisson-distributed random variable with intensity parameter 10. On average, this splits the domain in 12 disjoint intervals and the diffusion coefficient has almost surely at least one discontinuity. The random positions of the $\tau - 1$ jumps $x_1, \dots, x_{\tau-1} \subset \mathcal{D}$ in the interior of $\mathcal{D}$ are uniformly distributed over $\mathcal{D}$, generating the random partition $\mathcal{T} = \{(0, x_1), (x_1, x_2), \dots, (x_{\tau-1}, 1)\}$ for each realization of τ and the jump positions $x_1, \dots, x_{\tau-1}$. This fits into our framework of the jump-diffusion coefficient by setting $\lambda = 12\Lambda$, where Λ denotes the Lebesgue measure on $((0,1), \mathcal{B}(0,1))$: The uniform distribution of the discontinuities on $\mathcal{D}$ corresponds to a distribution with respect to Λ on $\mathcal{D}$ and the multiplication with 12 ensures that $\mathbb{E}(\tau)$ is as desired. In the subsequent examples we vary the distribution of the jump heights P_i.

To obtain path-wise approximations of the samples $u_{N,\varepsilon}(\omega, \cdot)$, we use non-adaptive and adaptive piecewise linear elements and compare both approaches. In addition, we combine each discretization method with regular and coupled multilevel Monte Carlo sampling, so in total we compare four different algorithms. The adaptive triangulation is based on each sampled partition $\mathcal{T}(\omega)$ as described in Section 4.4, see Fig. 4.1 and 4.3. In the graphs below, we plot the RMSE of the adaptive algorithms against the inverse estimated refinement size $\mathbb{E}(\widehat{h}_\ell^2)^{-1/2}$, for the non-adaptive algorithms this corresponds to the (deterministic) parameters h_ℓ^{-1}. The entries of the stiffness matrix are approximated by the midpoint rule, which ensures a path-wise error of order h_ℓ^3 on each simplex K. To ensure that this is sufficiently precise, we repeated our experiments with a five-point Gauss-Legendre quadrature, which did not entail significant changes. In the multilevel Monte Carlo algorithm, the non-adaptive triangulations are generated with refinement $h_\ell = 2^{-\ell-1}$, whereas we set the same threshold as maximum refinement size $\overline{h}_\ell = h_\ell = 2^{-\ell-1}$ in the adaptive algorithm. As realized and maximal values of $\widehat{h}_\ell$ may differ significantly, we set $\Xi_{N_\ell} = \varepsilon_\ell = \mathbb{E}(\widehat{h}_\ell^2)$ for $\ell = 0, \dots, L$ and choose the number of samples as $M_0 = \lceil \mathbb{E}(\widehat{h}_L^2)^{-1} \rceil$, $M_\ell = \lceil \mathbb{E}(\widehat{h}_L^2)^{-1} \mathbb{E}(\widehat{h}_\ell^2) \ell^{2(1+\nu)} \rceil$ with $\nu = 0.01$ for $\ell = 1, \dots, L$. The same error equilibration is used for the non-adaptive method, only with $\mathbb{E}(\widehat{h}_\ell^2)$ replaced by h_ℓ^2. To subtract samples $u_{N_\ell, \varepsilon_\ell, \ell}$ and $u_{N_{\ell-1}, \varepsilon_{\ell-1}, \ell-1}$ in the MLMC

estimator and add samples of the same refinement level (but on possibly different stochastic triangulations), we prolong the samples onto a reference grid with refinement size $(\mathbb{E}(\widehat{h}_\ell^2))^{1/2}$. We consider the test cases with $L = 0, \dots, 7$ and use $E^L(\widehat{u}_{N_L,\varepsilon_L,L})$ with $L = 9$, as a reference estimate for $\mathbb{E}(u)$. The RMSE $\|E^L(\widehat{u}_{N_L,\varepsilon_L,L}) - \mathbb{E}(u))\|_{L^2(\Omega;V)}$ is then estimated by averaging 20 samples of the error $\|E^L(\widehat{u}_{N_L,\varepsilon_L,L}) - E^9(u_{N_9,\varepsilon_9,9})\|_V^2$ for $L = 0, \dots, 7$. To calculate the RMSE, we use a reference grid with 10^3 equally spaced points in $\mathcal{D}$, thus the error stemming from the interpolation\prolongation may be neglected.

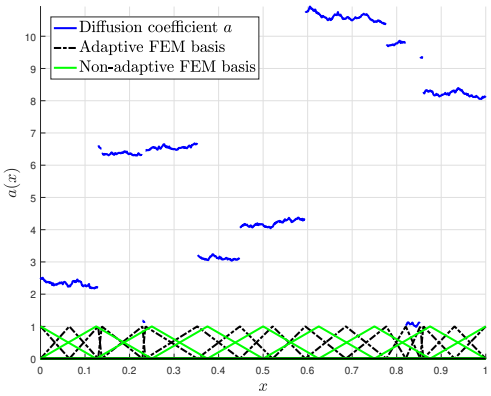

Figure 4.1 Sample of the diffusion coefficient with Brownian motion covariance and uniformly distributed jumps.

As our first numerical example, we use the Brownian motion covariance operator Q_{BM} and i.i.d uniformly distributed jump heights $P_i \sim \mathcal{U}([0, 10])$, hence the sampling error ε_ℓ is equal to zero on every level and may be omitted for this scenario. A sample of the corresponding diffusion coefficient with illustrated adaptive and non-adaptive finite element(FE)-basis is given in Fig. 4.1. Fig. 4.2 indicates that the adaptive algorithm converges considerably faster than the estimator with non-adaptive FE basis. Asymptotically, we see that both adaptive RMSE curves decay with rate nearly one, whereas the non-adaptive methods only show a rate of $\kappa \approx 0.75$. One sees that the adaptive multilevel Monte Carlo estimator also has a better time-to-error ratio, so it is possible to reduce the RMSE significantly using a little more computational effort to adjust the FE basis in each sample. Surprisingly there is little difference in the convergence speed whether or not we use a coupled algorithm combined with adaptive resp. non-adaptive FE. Here one would expect at least a slightly higher RMSE of the coupled algorithms, but in this example, the error of both coupled estimators is even lower compared to their non-coupled alternatives. Naturally, coupling the samples decreases computational time (see Fig. 4.2).

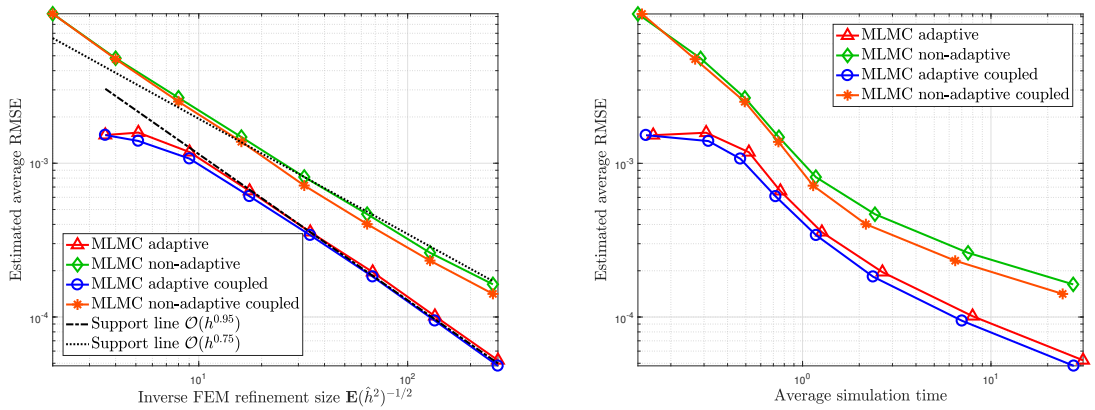

Figure 4.2 Left: RMSE with $\mathcal{U}([0,10])$-distributed jumps, Right: Time-to-error plot.

In the next example, we consider the squared exponential covariance operator Q_{SE} with $\sigma^2 = 0.1, \rho = 0.05$ and a more involved distribution of jump heights, where sampling is rather expensive and may not be realized in a straightforward manner. The jump heights P_i now follow a continuous *generalized inverse Gaussian* (GIG) distribution with density

$$f_{GIG}(x) = \frac{(\psi/\chi)^{\overline{\lambda}/2}}{2K_{\overline{\lambda}}(\sqrt{\psi\chi})} x^{\overline{\lambda}-1} \exp(-\frac{1}{2}(\psi x + \chi x^{-1})), \quad x > 0$$

and parameters $\chi, \psi > 0$, $\overline{\lambda} \in \mathbb{R}$, where $K_{\overline{\lambda}}$ is the modified Bessel function of the second kind with $\overline{\lambda}$ degrees of freedom, see [19, 20]. As shown in [11], sampling this distribution by Acceptance-Rejection is possible, but expensive when $\overline{\lambda} < 0$, since the vast majority of outcomes has to be rejected. We rather generate approximations $\widetilde{P}_i$ of P_i by a Fourier inversion technique such that $\mathbb{E}(|\widetilde{P}_i - P_i|^2) \leq \varepsilon_\ell$ for a given $\varepsilon_\ell > 0$. For details on the Fourier inversion algorithm, the sampling of GIG distributions and the corresponding error bounds we refer to [30]. The GIG parameters are set as $\psi = 0.25, \chi = 9$ and $\overline{\lambda} = -1$, the resulting density f_{GIG} and a sample of the diffusion coefficient are given in Fig. 4.3. The error curves show a similar behavior compared to the first example, with asymptotic error rates of 1 resp. 0.75 for the adaptive resp. non-adaptive algorithms, see Fig. 4.4. Again, coupling tends to produce a slightly lower RMSE. The expensive sampling from the GIG distribution causes increased computational times, which entails that the coupled estimator is even more favorable in a setting with a rather challenging jump height distributions.

The Gaussian random field $W : \Omega \times \mathcal{D} \to \mathbb{R}$ with the Brownian motion covariance operator is almost surely not differentiable in a.e. $x \in \mathcal{D}$, but only Hölder continuous. In

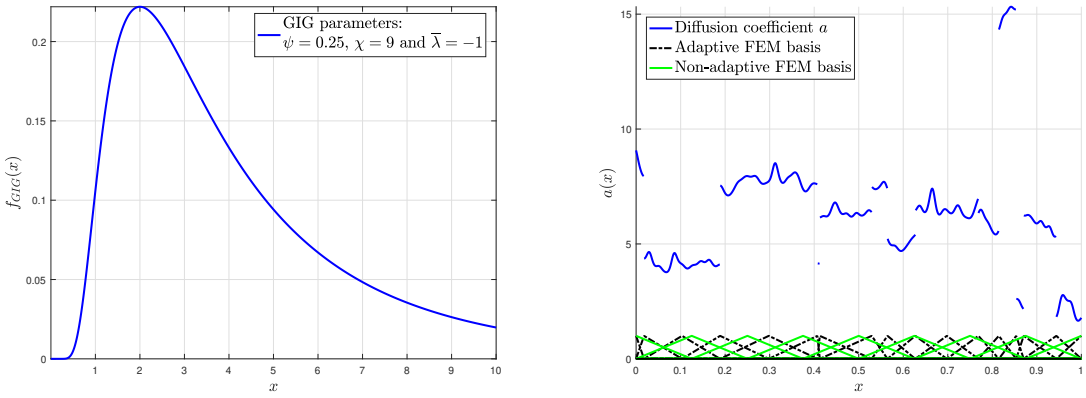

Figure 4.3 Left: GIG density, right: sample of the diffusion coefficient with log-squared exponential covariance and GIG distributed jumps.

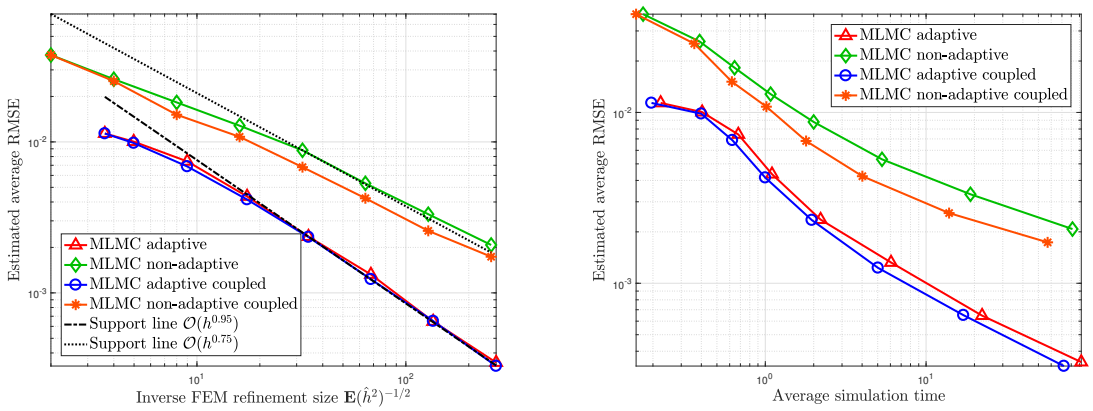

Figure 4.4 Left: RMSE of the example with GIG-distributed jumps, Right: Time-to-error plot.

addition, the covariance of the random variables $W(\cdot, x_1)$ and $W(\cdot, x_2)$, where $x_1, x_2 \in \mathcal{D}$ is given by the kernel function $\min(x_1, x_2)$. For a fixed distance between x_1 and x_2, this implies that the correlation in the random field increases as one moves to the right boundary of the domain. In some applications, however, one might instead want a random field with stationary correlation structure and/or more spatial regularity. This can be achieved with the introduced jump-diffusion coefficient by using, for instance, Q_{SE} or another Matérn class covariance operator. These covariance operators generate random stationary correlated random fields and also increase the regularity of W in $\mathcal{D}$. It is further possible to vary the position and magnitude of the discontinuities of a to model certain desirable characteristics of the diffusion. For example, one could enforce only one jump per sample which is concentrated in some small neighborhood

located around a single point in $\mathcal{D}$. The corresponding jump heights on each partition may then also be chosen concentrated around certain values to model, for instance, transitions in heterogeneous or fractured media.

4.6.2 Numerical results in 2D

In the two-dimensional setting, we work on the domain $\mathcal{D} = (0,1)^2$, with homogeneous Dirichlet or mixed Neumann-Dirichlet boundary conditions and we assume that the deterministic part of the drift coefficient is zero ($\bar{a} \equiv 0$). The Gaussian part of a is given by the Karhunen-Loève expansion with spectral basis

$$\eta_i := \sigma^2 \exp(-\pi^2 i^2 \rho^2), \quad e_i(x) := \sin(\pi i x_1) \sin(\pi i x_2),$$

with correlation length $\rho > 0$ and total variance $\sigma^2 > 0$. This basis is related to the two-dimensional heat kernel

$$G(t,x,y) : [0,\infty) \times \mathcal{D}^2 \to \mathbb{R}^+, \quad (t,x,y) \mapsto \frac{1}{4\pi t} \exp(-\frac{\|x-y\|_2^2}{4t})$$

in the sense that it solves the associated integral equation for $t = \rho^2/2$:

$$\sigma^2 \int_{\mathcal{D}} \exp(-\frac{\|x-y\|_2^2}{2\rho^2}) e_i(y) dy = \eta_i e_i(x), \quad i \in \mathbb{N}$$

with the boundary condition $e_i = 0$ on $\partial\mathcal{D}$, see [99]. Compared with a Gaussian field generated by a squared exponential covariance operator, this field shows a very similar behavior, except that it is zero on the boundary. It, further, has the advantage, that all expressions are available in closed form and we forgo the numerical approximation of the eigenbasis. For all experiments in this section we use the parameters $\sigma^2 = 0.25$ and $\rho = 0.02$. As before, we consider a log-Gaussian random field, meaning $\Phi(w) = \exp(w)$. To illustrate the flexibility of a jump-diffusion coefficient a as in Def. 4.3.1, we vary the random partitioning of $\mathcal{D}$ for each example and give a detailed description below. Again, we approximate the entries of the stiffness matrix by the midpoint rule on each simplex K. To ensure that this is sufficient, we also tested a four-point Gauss-Legendre quadrature rule for triangular domains, which did not change the outcomes significantly. As in the previous subsection, we prolong linearly on a fixed grid to add and subtract the generated samples of the multilevel Monte Carlo estimator. The reference grid for the RMSE consists of 400×400 equally spaced points in the domain $\mathcal{D} = (0,1)^2$, which yields a negligible interpolation error. The multilevel approximation parameters are identical to the 1D example, except that $\overline{h}_\ell = h_\ell = \frac{2}{5} 2^{-\ell}$ and we now

calculate the reference solution on level 7 to estimate the RMSE (by averaging 10 independent multilevel Monte Carlo estimations) up to level $L = 4$ or $L = 5$, depending on the example. All RMSE curves are plotted against the inverse estimated refinement $\mathbb{E}(\widehat{h}^2)^{-1/2}$ on the abscissa.

In the first 2D example, the partitions $\mathcal{T}$ of $\mathcal{D}$ are generated by random lines through the domain. More precisely, we sample independent Poisson random variables $\mathcal{P}_x, \mathcal{P}_y \sim Poi(1)$ and a total of $2(\mathcal{P}_x + \mathcal{P}_y + 2)$ independent $\mathcal{U}([0,1])$-distributed random variables $U_1, \ldots, U_{2(\mathcal{P}_x+\mathcal{P}_y+2)}$. The first $\mathcal{P}_x + 1$ uniform random variables represent the jump positions on $(0,1) \times \{0\}$, the second set $U_{\mathcal{P}_x+2}, \ldots, U_{2\mathcal{P}_x+2}$ are the positions of the discontinuities on the opposing axis $(0,1) \times \{1\}$ in $\partial\mathcal{D}$. We connect opposing points in ascending order by straight lines to obtain a vertical random partition of $\mathcal{D}$. Analogously, the horizontal splitting is achieved by distributing and connecting the remaining $2\mathcal{P}_y + 2$ uniform random variables on the sets $\{0\} \times (0,1)$ and $\{1\} \times (0,1)$. As we obtain an average of $\mathbb{E}((\mathcal{P}_x + 2)(\mathcal{P}_y + 2)) = 9$ partition elements of uniformly distributed size and location, λ may be set as $\lambda = 9\Lambda$ in this example, where Λ is the Lebesgue measure on $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$.

To each of the $(\mathcal{P}_x + 2)(\mathcal{P}_y + 2)$ random quadrangles we assign a jump height P_i, where the sequence $(P_i, i \in \mathbb{N})$ is i.i.d. $\mathcal{U}([0,5])$-distributed. This specific structure of a may be used, for example, to model stationary flows through heterogeneous media, where the hydraulic conductivity varies on sub-domains of the medium. We assume

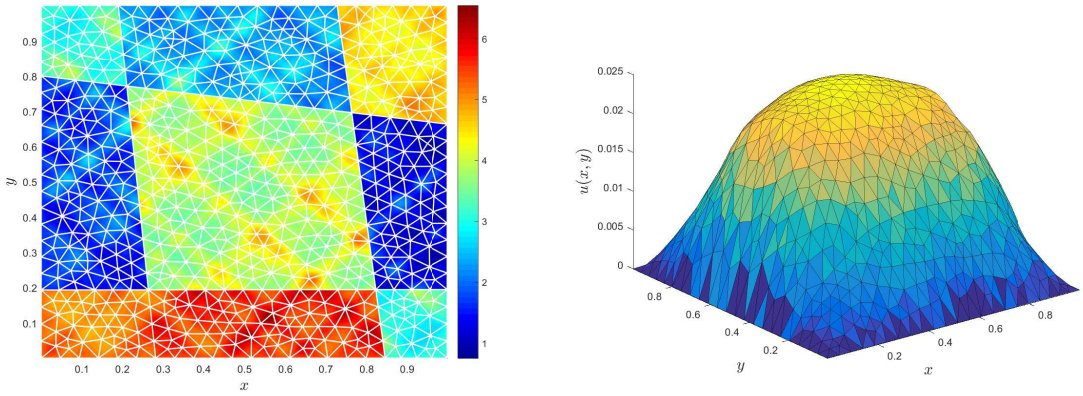

Figure 4.5 Left: Sample of the 2D diffusion coefficient in heterogeneous media. Right: Approximated FE solution to the sample of a.

homogeneous Dirichlet-boundary conditions on $\Gamma_1 = \partial\mathcal{D}$ and $f \equiv 1$ as source term. A sample of the diffusion coefficient and the FE approximation is given in Fig. 4.5. Compared to a solution with constant coefficient, the influence of the discontinuous diffusion is clearly visible in the contour of the approximated solution.

Fig. 4.6 shows that the adaptive multilevel Monte Carlo and non-adaptive multilevel Monte Carlo algorithm perform quite similar, where the asymptotic error rate of the nonadaptive methods is slightly lower with 0.85. Compared to that, we recover a convergence rate of 0.9 and lower absolute errors for the adaptive method. In both cases, coupled and normal multilevel Monte Carlo sampling produces almost identical errors which results in the best time-to-error ratio of the adaptive coupled multilevel Monte Carlo estimator, see Fig. 4.6.

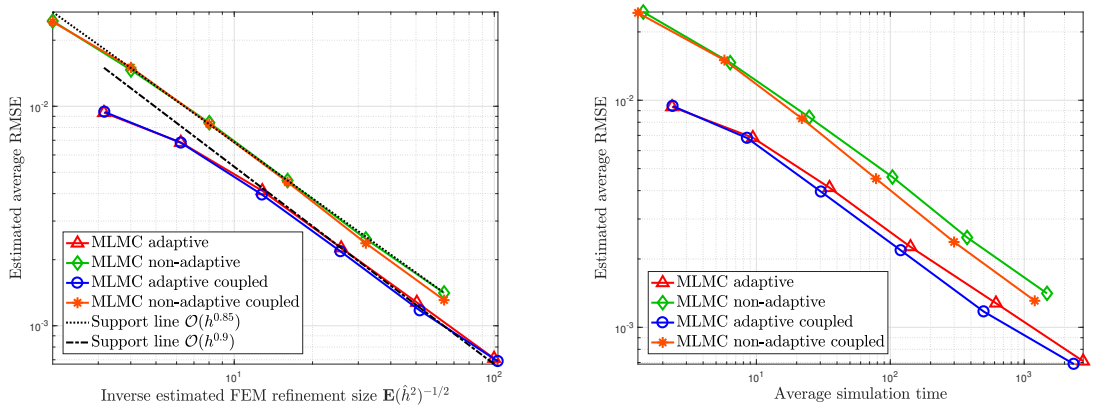

Figure 4.6 Left: RMSE of the heterogeneous media example, Right: Time-to-error plot.

In the second 2D example, we aim to mimic the structure of a fractured porous medium. To this end, we set $\lambda = 5\Lambda$ and sample accordingly $\tau = 5$ uniformly distributed random points $x_1, \ldots, x_5$ on the domain $\mathcal{D}$. Then, for each point x_i, a random length l_i with distribution $\mathcal{U}([0.5, 1.5])$ is generated. We sample, further, five random angles $\alpha_1, \ldots, \alpha_5$ with uniform distribution on the set $[-\frac{\pi}{9}, -\frac{\pi}{36}] \cup [\frac{\pi}{36}, \frac{\pi}{9}]$. We define x_i as the center of a line with length l_i rotated by α_i, where three of the five lines are orientated horizontally and the remaining two lines are vertical. Finally, the line segments outside of $\mathcal{D}$ are removed and each random line is assigned a positive preset width of 0.04, which results in a "trench structure" of the diffusion coefficient as depicted in Fig. 4.7. On the trenches, we set the jump height to $P_1 = 100$, while the jump heights on the remaining quadrangles of the partition is set to 1. Fixing the jump heights captures increasing permeability in the cracks of a certain medium, the Gaussian field still accounts for some uncertainty within each partition element of the domain. The source function is given as $f \equiv 5$. We split $\partial\mathcal{D}$ by $\Gamma_1 := \{0, 1\} \times [0, 1]$ and $\Gamma_2 := (0, 1) \times \{0, 1\}$ and impose the (path-wise) mixed Dirichlet-Neumann boundary

conditions

$$u(\omega,\cdot) = \begin{cases} 0.1 & \text{on } \{0\}\times[0,1] \\ 0.3 & \text{on } \{1\}\times[0,1] \end{cases} \quad \text{and} \quad a(\omega,\cdot)\vec{n}\cdot\nabla u(\omega,\cdot) = 0 \quad \text{on } \Gamma_2 \tag{4.17}$$

for each $\omega \in \Omega$. A sample of the approximated solution is displayed in Fig. 4.7.

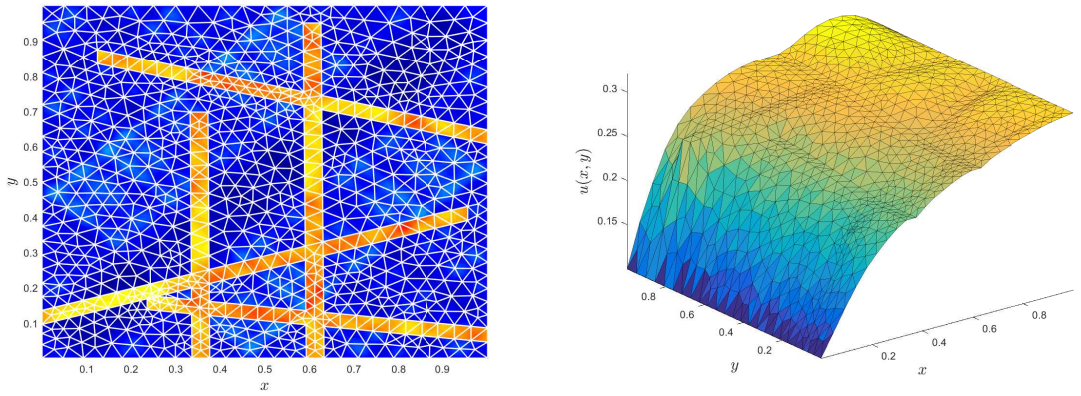

Figure 4.7 Left: Sample of the fractured porous medium diffusion coefficient with adaptive triangulation. Right: Approximated FE solution to the sample of a.

Compared to the first 2D example, there is now a larger gap between the RMSE curves of the adaptive and non-adaptive estimators, see Fig. 4.8. The asymptotic rate for the error is again close to order one for both adaptive methods, while we obtain 0.6 for the non-adaptive algorithms. This is possibly due to the higher magnitude of the discontinuities in the diffusion coefficient compared to the first example. Coupling the samples now leads to a higher RMSE in each algorithm and this effect is more pronounced in the adaptive setting. Asymptotically, the rates of both coupled estimators remain comparable to standard multilevel Monte Carlo. An adaptive triangulation for samples of this particular diffusion coefficient often entails very fine meshes, even if the desired maximum diameter of each triangle is comparably high, as Fig. 4.7 and Fig. 4.8 illustrate. Due to this increase in complexity when using adaptive FE, the simulation times for the respective estimators are now considerably longer as in the previous scenario, see Fig. 4.8. Nevertheless, the adaptive methods still have significantly better time-to-error ratios, but now the standard adaptive method outperforms the corresponding coupled estimator.

As a last 2D example, we discuss a medium with inclusions. To this end, we sample a discrete uniformly-distributed random variable τ where $\tau \in \{1,2,3,4\}$ and define $\lambda := \frac{\mathbb{E}(\tau)}{0.64}\Lambda|_{(0.1,0.9)^2}$. Scaling and restricting the Lebesgue measure on $(0.1,0.9)^2$ means

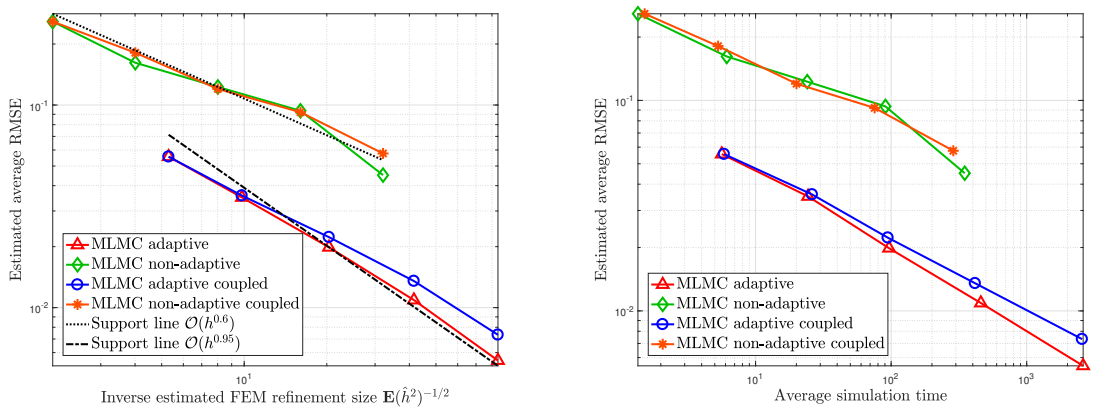

Figure 4.8 Left: RMSE of the fractured porous media example, Right: Time-to-error plot.

that we now draw τ-many uniformly distributed points within the the sub-domain $(0.1, 0.9)^2 \subset \mathcal{D}$. To each of this points we assign a circle of random radius. The radii are $\mathcal{U}([0.075, 0.1])$-distributed. On each circle we assign a jump height of 1, while this parameter is set to 20 outside of the circles. We assume the same Neumann–Dirichlet boundary conditions as in our second 2D-example (see Eq. (4.17)) and set $f \equiv 10$ as source term. A sample of this jump-diffusion coefficient with corresponding FE solution is shown in Fig. 4.9.

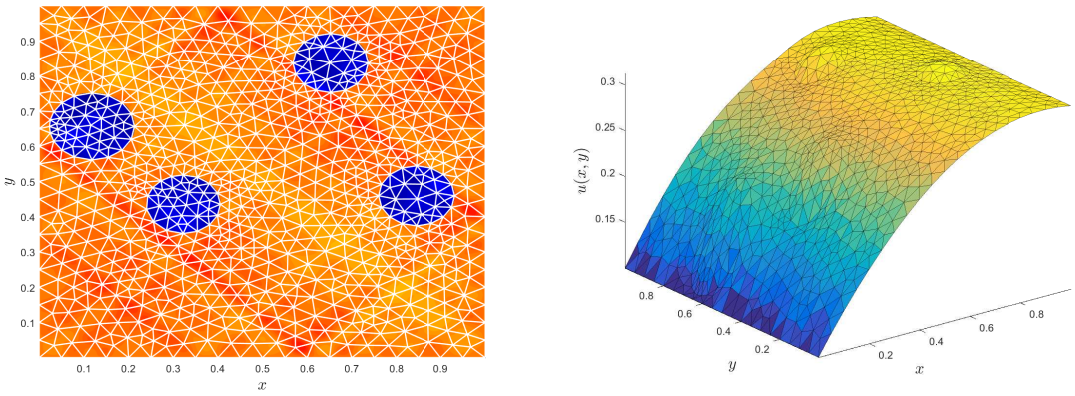

Figure 4.9 Left: Sample of the diffusion coefficient of a medium with inclusions with adaptive triangulation. Right: Approximated FE solution to the sample of a.

As Fig. 4.10 indicates, we obtain a similar behavior of convergence as in the previous example: The RMSE of the adaptive estimators is again significantly lower on all levels and the non-adaptive has again an asymptotic RMSE of order 0.6. For the adaptive methods, we obtain log-linear error decay of order $\mathcal{O}(h|\log(h)|^{\frac{1}{2}})$. This corresponds to the expected path-wise rate for an adaptive FE solution of this diffusion problem,

as the discontinuities have C^2-boundaries. Bootstrapping slightly reduces the RMSE of the non-adaptive method, but has little effect on the adaptive multilevel Monte Carlo estimator. The computational complexity of this scenario is comparable to the heterogeneous media example and thus significantly lower as in case of the fractured porous medium. Finally, the adaptive algorithms again attain better time-to-error with the coupled estimator slightly outperforming the standard adaptive method, see Fig. 4.10.

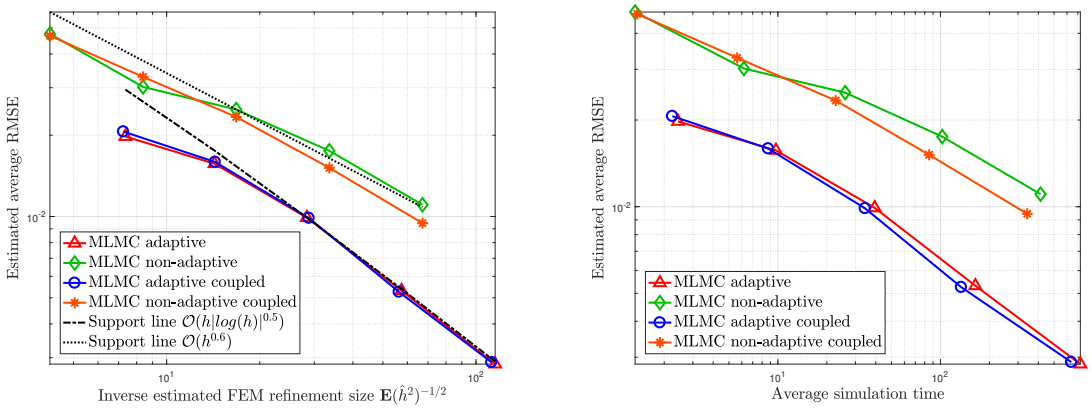

Figure 4.10 Left: RMSE of the example for a medium with inclusions, Right: Time-to-error plot.

Comments by A. Stein on "A study of elliptic partial differential equations with jump diffusion coefficients":

In the first numerical experiment in one dimension from Subsection 4.6.1 we see a converge rate of order $\kappa_a \approx 1$ with respect to the spatial refinement h. These results are theoretically based on Assumption 4.4.4 that requires the same rate of convergence for the strong error in the $L^2(\Omega; V)$-norm, i.e. for the second moment of the FE discretization error. This is somehow puzzling in view of the following chapter, since we have used a log-Brownian motion on $(0, 1)$ as continuous part of the diffusion coefficient. According to Theorem 5.4.3 in Chapter 5, a strong error of order $\kappa_a = 1$ can only be ensured for a piecewise Lipschitz-continuous diffusion coefficient. For a piecewise log-Brownian motion as in Subsection 4.6.1, one would only expect an error decay of order $\kappa_a = 1/2$, which is actually confirmed in the numerical experiments in the next chapter.

To explain this, we keep in mind that we measure the RMSE between the expectation of the solution u and its multilevel Monte Carlo estimator in Subsection 4.6.1. Then, the RMSE more resembles a weak error estimate, i.e. the *norm of the expected bias*, rather than the strong error, i.e. the second moment of the path-wise bias. Under the term *bias*, we have aggregated the truncation error in the Gaussian field *and* the FE discretization error in space. It seems that the low mean-square regularity and corresponding truncation error in the log-Brownian motion does not dominate the overall RMSE in this case, given that all other discretization parameters, i.e. spatial refinement and number of samples, are aligned to order $\kappa_a = 1$. This also indicates that the there might be room for improvement regarding the error bounds that involve the truncation error Ξ_n in the Gaussian random field.

On a further note, we remark that the second 2D-example in Subsection 4.6.2 is in line with the findings from the next chapter, although the stochastic geometry contains non-convex shapes. The reentrant corners of the trenches only affect spatial regularity on comparably small areas of $\mathcal{D}$ in each sample. Therefore, they have only little effect on the overall regularity in $\mathcal{D}$ which yields again a rate of $\kappa_a \approx 1$ in this example.

5 Numerical analysis for time-dependent advection-diffusion problems with random discontinuous coefficients

Andreas Stein and Andrea Barth

Submitted to "Stochastics and Partial Differential Equations: Analysis and Computations" July 2020, Springer, currently in the first stage of review.

Abstract: Subsurface flows are commonly modeled by advection-diffusion equations. Insufficient measurements or uncertain material procurement may be accounted for by random coefficients. To represent, for example, transitions in heterogeneous media, the parameters of the equation are spatially discontinuous. Specifically, a scenario with coupled advection- and diffusion coefficients that are modeled as sums of continuous random fields and discontinuous jump components are considered. For the numerical approximation of the solution, a sample-adapted, path-wise discretization scheme based on a finite element approach is introduced. To stabilize the numerical approximation and accelerate convergence, the discrete space-time grid is chosen with respect to the varying discontinuities in each sample of the coefficients, leading to a stochastic formulation of the Galerkin projection and the finite element basis.

5.1 Introduction

In this paper we are concerned with the well-posedness of a solution to a time-dependent advection-diffusion equation with discontinuous random coefficients and its numerical discretization. The random coefficient function is modeled by a continuous part and a discontinuous part, inspired by the unique characterization of the Lévy-Khinchine formula for Lévy processes. We adopt this idea to spatial domains, meaning we propose jumps occurring on lower-dimensional submanifolds. The numerical discretization method has to account for these discontinuities of the coefficient functions, as otherwise (spatial) convergence rates decline.

This work is a generalization to the elliptic setting which has drawn attention over the last decades. While many publications focus on numerical methods for continuous stochastic coefficients (see, e.g., [1, 12, 14, 15, 16, 29, 59, 63, 85, 112, 148, 164, 165, 190, 194, 204]), the literature on stochastic discontinuous coefficients or stochastic interface problems is sparse (see, e.g., [104, 140, 205]). The reasons are twofold: On one hand a Gaussian random field is a well defined mathematical object and its properties are well studied, on the other hand there is no general definition and approximation method for a (discontinuous) Lévy field. A (centered) Gaussian random field is fully characterized by its covariance operator. Discretization methods range from spectral approximations to Fourier methods (see, e.g., [98, 139, 193]). While we also need an approximation for the continuous (Gaussian) part of the coefficient function, drawing samples from different jump distributions may also introduce a bias. Our main contribution is therefore, to provide a well-posedness result for a parabolic equation with general jump-diffusion and jump-advection coefficient and provide analysis of a numerical approximation. Besides the approximation of the coefficient itself, we prove convergence of a path-wise sample-adapted space-time approximation. Naturally, for path-wise adapted schemes, convergence rates are also random. However, in our setting an upper bound on the mean-squared error can be derived but sampling has to be adopted accordingly.

The paper is structured as follows: In Section 5.2 we state the problem and show a general existence result for path-wise solutions under mild assumptions on the data. In the following section we define the random coefficient functions and show convergence of approximations in appropriate norms. These approximations are used to develop in Section 5.4 path-wise adapted discretization schemes for the solution. Our main contribution is a convergence result for this approximation. We close with one- and two-dimensional numerical experiments, that confirm our theoretical findings.

5.2 Parabolic initial boundary value problems and solutions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, let $\mathbb{T} := [0, T]$ a time interval for some $T > 0$ and let $\mathcal{D} \subset \mathbb{R}^d$, $d \in \{1, 2\}$, be a bounded and convex Lipschitz domain. In this paper we consider the linear, random initial-boundary value problem

$$\begin{aligned} \partial_t u(\omega, x, t) + [Au](\omega, x, t) &= f(\omega, x, t) \quad \text{in } \Omega \times \mathcal{D} \times (0, T], \\ u(\omega, x, 0) &= u_0(\omega, x) \quad \text{in } \Omega \times \mathcal{D} \times \{0\}, \\ u(\omega, x, t) &= 0 \quad \text{on } \Omega \times \partial\mathcal{D} \times \mathbb{T}, \end{aligned} \tag{5.1}$$

where $f : \Omega \times \mathcal{D} \times \mathbb{T} \to \mathbb{R}$ is a random source function and $u_0 : \Omega \times \mathcal{D}$ denotes the random initial condition of the partial differential equation (PDE). Furthermore, A is the second order partial differential operator given by

$$[Au](\omega, x, t) = -\nabla \cdot (a(\omega, x)\nabla u(\omega, x, t)) + b(\omega, x) \cdot \nabla u(\omega, x, t)$$

for $(\omega, x, t) \in \Omega \times \mathcal{D} \times \mathbb{T}$ with

- a stochastic jump-diffusion coefficient $a : \Omega \times \mathcal{D} \to \mathbb{R}$ and
- a discontinuous random convection term $b : \Omega \times \mathcal{D} \to \mathbb{R}^d$.

We could extend the above model problem by including time-dependent diffusion and/or advection coefficients. If a and b are sufficiently smooth with respect to t, i.e. continuously differentiable in $\mathbb{T}$, the temporal convergence rates in Section 5.4 are not affected. The focus of this article, however, is on the numerical analysis of Problem (5.1) with coefficients that involve random spatial discontinuities, hence we assume for the sake of simplicity that a and b are time-independent. We base the analysis of Problem (5.1) on the Sobolev space $H^k(\mathcal{D})$ equipped with the norm

$$\|v\|_{H^k(\mathcal{D})} := \Big(\sum_{|\nu| \leq k} \int_{\mathcal{D}} |D^\nu v(x)|^2 dx \Big)^{1/2} \quad \text{for } k \in \mathbb{N},$$

where $D^\nu := \partial_{x_1}^{\nu_1} \dots \partial_{x_d}^{\nu_d}$ is the mixed partial weak derivative (in space) with respect to the multi-index $\nu \in \mathbb{N}_0^d$. The corresponding seminorm to $H^k(\mathcal{D})$ is denoted by

$$|v|_{H^k(\mathcal{D})} := \Big(\sum_{|\nu| = k} \int_{\mathcal{D}} |D^\nu v(x)|^2 dx \Big)^{1/2}.$$

The fractional order Sobolev spaces $H^s(\mathcal{D})$ for $s > 0$ are defined by the norm

$$\|v\|_{H^s(\mathcal{D})} := \|v\|_{H^{\lfloor s \rfloor}(\mathcal{D})} + |v|_{H^{s-\lfloor s \rfloor}(\mathcal{D})}, \quad |v|^2_{H^{s-\lfloor s \rfloor}(\mathcal{D})} := \int_{\mathcal{D}} \int_{\mathcal{D}} \frac{|v(x) - v(y)|^2}{|x - y|^{d+2(s-\lfloor s \rfloor)}} dxdy,$$

where $|\cdot|_{H^{s-\lfloor s \rfloor}(\mathcal{D})}$ is the the *Gagliardo seminorm*, see [71], and

$$\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}, \ s \mapsto \max(k \in \mathbb{Z}, k \leq s)$$

is the floor operator. Further, we define $H := L^2(\mathcal{D})$ and denote by C a generic positive constant which may change from one line to another. Whenever necessary, the dependence of C on certain parameters is made explicit.

On $\mathcal{D}$, the existence of a bounded, linear operator $\gamma : H^s(\mathcal{D}) \to H^{s-1/2}(\partial\mathcal{D})$ with

$$\gamma : H^s(\mathcal{D}) \cap C^\infty(\overline{\mathcal{D}}) \to H^{s-1/2}(\partial\mathcal{D}), \quad v \mapsto \gamma v = v|_{\partial\mathcal{D}}$$

and

$$\|\gamma v\|_{H^{s-1/2}(\partial\mathcal{D})} \leq C\|v\|_{H^s(\mathcal{D})} \tag{5.2}$$

for $s \in (1/2, 3/2)$, $v \in H^s(\mathcal{D})$ is ensured by the trace theorem, see for example [73]. The constant $C = C(s, \mathcal{D}) > 0$ in Ineq. (5.2) is only dependent on s and $\mathcal{D}$. Since we only consider homogeneous Dirichlet boundary conditions on $\partial\mathcal{D}$, we may treat γ independently of ω and define the suitable solution space V as

$$V := H_0^1(\mathcal{D}) = \{v \in H^1(\mathcal{D}) |\ \gamma v \equiv 0\},$$

equipped with the $H^1(\mathcal{D})$-norm $\|v\|_V := \|v\|_{H^1(\mathcal{D})}$. Due to the homogeneous Dirichlet boundary conditions, the *Poincaré inequality* $\|v\|_H \leq C|v|_{H^1(\mathcal{D})}$ holds with $C = C(|\mathcal{D}|) > 0$ for all $v \in V$, hence the norms $\|\cdot\|_{H^1(\mathcal{D})}$ and $|\cdot|_{H^1(\mathcal{D})}$ are equivalent on V. We work on the Gelfand triplet $V \subset H \subset V' = H^{-1}(\mathcal{D})$, where $\mathcal{V}'$ denotes the topological dual of any vector space $\mathcal{V}$. As the coefficients a and b are given by random functions, any solution u to Problem (5.1) is in general a time-dependent V-valued random variable. To investigate the integrability of u with respect to $\mathbb{T}$ and the underlying probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$, we need to introduce the space of *Bochner integrable* functions.

Definition 5.2.1. Let (Y, Σ, μ) be a σ-finite and complete measure space, let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ a Banach space and define the norm $\|\cdot\|_{L^p(Y;\mathcal{X})}$ for a any function $\varphi : Y \to \mathcal{X}$ by

$$\|\varphi\|_{L^p(Y;\mathcal{X})} := \begin{cases} \left(\int_Y \|\varphi(y)\|_{\mathcal{X}}^p \mu(dy) \right)^{1/p} & \text{for } 1 \leq p < +\infty \\ \operatorname*{ess\,sup}_{y \in Y} \|\varphi(y)\|_{\mathcal{X}} & \text{for } p = +\infty \end{cases}.$$

The corresponding space of Bochner integrable random variables is then given by

$$L^p(Y; \mathcal{X}) := \{\varphi : Y \to \mathcal{X} \text{ is strongly measurable and } \|\varphi\|_{L^p(Y;\mathcal{X})} < +\infty\}.$$

Furthermore, the space of all continuous functions is defined as

$$C(\mathbb{T}; \mathcal{X}) := \{\varphi : \mathbb{T} \to \mathcal{X} \text{ is continuous and } \max_{t \in \mathbb{T}} \|\varphi(t)\|_{\mathcal{X}} < +\infty\}.$$

We are interested in the two particular cases that

- $(Y, \Sigma, \mu) = (\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu_{\mathbb{T}})$, where $\mathcal{B}(\mathbb{T})$ is the Borel σ-algebra over $\mathbb{T}$ and $\mu_{\mathbb{T}}$ is the Lebesgue-measure on $\mathcal{B}(\mathbb{T})$,
- $(Y, \Sigma, \mu) = (\Omega, \mathcal{F}, \mathbb{P})$.

The space $L^p(\Omega; \mathcal{X})$ is commonly referred to as the *space of Bochner integrable random variables*. For $\varphi \in L^1(\mathbb{T}; \mathcal{X})$ we denote by $\partial_t \varphi \in L^1(\mathbb{T}; \mathcal{X})$ the *weak time derivative* of φ if for all $\xi \in C_0^\infty(\mathbb{T}; \mathbb{R})$

$$\int_0^T \partial_t \xi(t) \varphi(t) dt = -\int_0^T \xi(t) \partial_t \varphi(t) dt$$

holds, where $\partial_t \xi$ is the classical time derivative (in a strong sense) of ξ. The set $C_0^\infty(\mathbb{T}; \mathbb{R})$ contains all infinitely differentiable functions $\xi : \mathbb{T} \to \mathbb{R}$ with compact support in $(0, T)$. We record following useful Lemma for the calculus in $L^2(\mathbb{T}; H)$.

Lemma 5.2.2. *[80, Theorem 2, Chapter 5.9] Let $H = L^2(\mathcal{D})$ and $\varphi, \partial_t \varphi \in L^2(\mathbb{T}; H)$. Then, the mapping $\varphi : \mathbb{T} \to H$ is continuous,*

$$\varphi(t_2) = \varphi(t_1) + \int_{t_1}^{t_2} \partial_t \varphi(t) dt, \quad \text{for all } 0 \leq t_1 \leq t_2 \leq T,$$

and it holds for $C = C(T) > 0$ that

$$\max_{t \in \mathbb{T}} \|\varphi(t)\|_H^2 \leq C(\|\varphi\|_{L^2(\mathbb{T}; H)}^2 + \|\partial_t \varphi\|_{L^2(\mathbb{T}; H)}^2).$$

Remark 5.2.3. We may as well consider non-homogeneous boundary conditions, that is $u(\omega, x, t) = g_1(\omega, x, t)$ for $g_1 : \Omega \times \partial\mathcal{D} \times \mathbb{T} \to \mathbb{R}$. The corresponding trace operator γ is still well-defined provided that $g_1(\omega, \cdot, \cdot)$ can be extended almost surely to a function $\widetilde{g}_1(\omega, \cdot, \cdot) \in L^1(\mathbb{T}; H^1(\mathcal{D}))$ with $\partial_t \widetilde{g}_1(\omega, \cdot, \cdot) \in L^1(\mathbb{T}; H^{-1}(\mathcal{D}))$. Then, $u - \widetilde{g}_1 \in L^1(\mathbb{T}; V)$ may be regarded as a solution to the modified problem

$$\begin{aligned}
\partial_t(u - \widetilde{g}_1)(\omega, x, t) + [A(u - \widetilde{g}_1)](\omega, x, t) &= f(\omega, x, t) - [A\widetilde{g}_1](\omega, x, t) \\
&\quad - \partial_t \widetilde{g}_1(\omega, x, t) \quad \text{on } \Omega \times \mathcal{D} \times (0, T], \\
(u - \widetilde{g}_1)(\omega, x, 0) &= u_0(\omega, x) - \widetilde{g}_1(\omega, x, 0) \quad \text{on } \Omega \times \mathcal{D} \times \{0\}, \\
(u - \widetilde{g}_1)(\omega, x, t) &= 0 \quad \text{on } \Omega \times \partial\mathcal{D} \times \mathbb{T}.
\end{aligned}$$

But this is in fact a version of Problem (5.1) with modified source term and initial value (see also [80, Chapter 6.1]).

We introduce the bilinear form associated to A in order to derive a weak formulation of the initial boundary value Problem (5.1). For fixed $\omega \in \Omega$, multiplying Eq. (5.1)

with a test function $v \in V$ and integrating by parts yields the variational equation

$$\int_{\mathcal{D}} \partial_t u(\omega, x, t) v(x) dx + B_\omega(u(\omega, \cdot, t), v) = F_{\omega,t}(v), \quad t \in \mathbb{T}. \tag{5.3}$$

The bilinear form $B_\omega : V \times V \to \mathbb{R}$ is given by

$$\begin{aligned} B_\omega(u, v) &= \int_{\mathcal{D}} a(\omega, x) \nabla u(x) \cdot \nabla v(x) + b(\omega, x) \cdot \nabla u(x) v(x) dx \\ &= (a(\omega, \cdot), \sum_{i=1}^{d} \partial_{x_i} u \partial_{x_i} v) + (b(\omega, \cdot) \cdot \nabla u, v), \end{aligned}$$

where $(\cdot, \cdot)$ denotes the $L^2(\mathcal{D})$-scalar product. The source term is transformed into the right hand side functional

$$F_{\omega,t} : V \to \mathbb{R}, \quad v \mapsto \int_{\mathcal{D}} f(\omega, x, t) v(x) dx, \quad t \in \mathbb{T},$$

and the integrals with respect to $\partial_t u$ and f are understood as the duality pairings

$$\begin{aligned} \int_{\mathcal{D}} \partial_t u(\omega, x, t) v(x) dx &= {}_{V'}\langle \partial_t u(\omega, \cdot, t), v \rangle_V, \\ \int_{\mathcal{D}} f(\omega, x, t) v(x) dx &= {}_{V'}\langle f(\omega, \cdot, t), v \rangle_V. \end{aligned}$$

Definition 5.2.4. For fixed $\omega \in \Omega$, the *path-wise weak solution* to Problem (5.1) is a function $u(\omega, \cdot, \cdot) \in L^2(\mathbb{T}; V)$ with $\partial_t u(\omega, \cdot, \cdot) \in L^2(\mathbb{T}; V')$ such that for $t \in \mathbb{T}$

$${}_{V'}\langle \partial_t u(\omega, \cdot, t), v \rangle_V + B_\omega(u(\omega, \cdot, t), v) = F_{\omega,t}(v), \quad \text{for all } v \in V,$$

and $u(\omega, \cdot, 0) = u_0(\omega, \cdot)$.

The following set of assumptions allows us to show existence and uniqueness of a path-wise weak solution to Eq. (5.1):

Assumption 5.2.5.

- For almost all $\omega \in \Omega$ it holds that

$$a_-(\omega) := \operatorname*{ess\,inf}_{x \in \mathcal{D}} a(\omega, x) > 0 \quad \text{and} \quad a_+(\omega) := \|a(\omega, \cdot)\|_{L^\infty(\mathcal{D})} < +\infty.$$

- $f \in L^p(\Omega; L^2(\mathbb{T}; V'))$, $u_0 \in L^p(\Omega; H)$ and $1/a_- \in L^q(\Omega; \mathbb{R})$, for some $p, q \in [1, \infty]$ such that $1/p + 1/q \leq 1$.

- There are constants $\overline{b}_1, \overline{b}_2 \geq 0$ such that $\|b(\omega, x)\|_\infty \leq \min(\overline{b}_1 a(\omega, x), \overline{b}_2)$ holds for almost all $\omega \in \Omega$ and almost all $x \in \mathcal{D}$.

We note that for any $v \in H^1(\mathcal{D})$ it holds that

$$\Big(\sum_{i=1}^{d} |\partial_{x_i} v(x)|\Big)^2 \leq 2^{d-1} \sum_{i=1}^{d} (\partial_{x_i} v(x))^2, \quad x \in \mathcal{D}, \tag{5.4}$$

and hence $\| \sum_{i=1}^{d} |\partial_{x_i} v| \|_H^2 \leq 2^{d-1} |v|_{H^1(\mathcal{D})}^2$.

Theorem 5.2.6. *Under Assumption 5.2.5 there exists almost surely a unique path-wise weak solution $u(\omega, \cdot, \cdot) \in L^2(\mathbb{T}; V) \cap C(\mathbb{T}; H)$ to Problem (5.1). Furthermore, u is bounded for any $r \in [1, (1/p + 1/q)^{-1}]$ by*

$$\begin{aligned} \mathbb{E}\Big(\sup_{t \in \mathbb{T}} \|u\|_{*,t}^r \Big)^{1/r} &\leq C(1 + \|1/a_-\|_{L^q(\Omega;\mathbb{R})}) \Big(\|u_0\|_{L^p(\Omega;H)} + \|f\|_{L^p(\Omega;L^2(\mathbb{T};V'))} \Big) \\ &< +\infty, \end{aligned} \tag{5.5}$$

with $C = C(\bar{b}, T, q) > 0$ and the path-wise parabolic norm defined by

$$\|u(\omega, \cdot, \cdot)\|_{*,t} := \Big(\|u(\omega, \cdot, t)\|_H^2 + \int_0^t |u(\omega, \cdot, r)|_{H^1(\mathcal{D})}^2 dr \Big)^{1/2}.$$

Moreover, if $f \in L^p(\Omega; L^2(\mathbb{T}; H))$, then for any $r \in [1, (1/p + (1/(2q))^{-1}]$

$$\mathbb{E}\Big(\sup_{t \in \mathbb{T}} \|u\|_{*,t}^r \Big)^{1/r} \leq C(1 + \|1/a_-\|_{L^q(\Omega;\mathbb{R})}^{1/2}) \Big(\|u_0\|_{L^p(\Omega;H)} + \|f\|_{L^p(\Omega;L^2(\mathbb{T};H))} \Big) < +\infty.$$

Proof. For fixed $\omega \in \Omega$, the existence and uniqueness of a path-wise weak solution $u(\omega, \cdot, \cdot) \in L^2(\mathbb{T}; V) \cap C(\mathbb{T}; H)$ to Problem (5.1) is proved identically as for deterministic parabolic problems, see for instance [80, Chapter 7.1] or [177, Chapter 11]. To show the parabolic estimate (5.5), we fix $\omega \in \Omega$, $t \in \mathbb{T}$, test against $v = u(\omega, \cdot, t) \in V$ (in the variational Problem (5.3)) and obtain

$${}_{V'}\langle \partial_t u(\omega, \cdot, t), u(\omega, \cdot, t) \rangle_V + B_\omega(u(\omega, \cdot, t), u(\omega, \cdot, t)) = F_{\omega,t}(u(\omega, \cdot, t)).$$

As $u(\omega, \cdot, \cdot) \in L^2(\mathbb{T}; V)$ it holds that

$${}_{V'}\langle \partial_t u(\omega, \cdot, t), u(\omega, \cdot, t) \rangle_V = \frac{1}{2} \frac{d}{dt} \|u(\omega, \cdot, t)\|_H^2,$$

see i.e. [80, Chapter 5.9]. Rearranging the terms yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(\omega,\cdot,t)\|_H^2 + (a(\omega,\cdot),\sum_{i=1}^{d}(\partial_{x_i}u(\omega,\cdot,t))^2) &= -(b(\omega,\cdot)\cdot\nabla u(\omega,\cdot,t),u(\omega,\cdot,t)) \\ &\quad + F_{\omega,t}(u(\omega,\cdot,t)) \\ &=: I + II.\end{aligned} \tag{5.6}$$

We bound I with Young's inequality, Assumption 5.2.5 and Ineq. (5.4) via

$$\begin{aligned}I &\leq \frac{2^{1-d}}{4\bar{b}_1}\|\|b(\omega,\cdot)\|_\infty^{1/2}\sum_{i=1}^{d}|\partial_{x_i}u(\omega,\cdot,t)|\|_H^2 + 2^{d-1}\bar{b}_1\|\|b(\omega,\cdot)\|_\infty^{1/2}u(\omega,\cdot,t)\|_H^2 \\ &\leq \frac{1}{4}(a(\omega,\cdot),\sum_{i=1}^{d}(\partial_{x_i}u(\omega,\cdot,t))^2) + 2^{d-1}\bar{b}_1\bar{b}_2\|u(\omega,\cdot,t)\|_H^2.\end{aligned}$$

By the Poincaré inequality it holds that $\|u\|_H \leq C|u|_{H^1(\mathcal{D})}$ and we estimate

$$\begin{aligned}II &\leq (1+C^2)\frac{\|f(\omega,\cdot,t)\|_{V'}^2}{a_-(\omega)} + \frac{a_-(\omega)}{4(1+C^2)}\|u(\omega,\cdot,t)\|_V^2 \\ &\leq (1+C^2)\frac{\|f(\omega,\cdot,t)\|_{V'}^2}{a_-(\omega)} + \frac{a_-(\omega)}{4}|u(\omega,\cdot,t)|_{H^1(\mathcal{D})}^2 \\ &\leq (1+C^2)\frac{\|f(\omega,\cdot,t)\|_{V'}^2}{a_-(\omega)} + \frac{1}{4}(a(\omega,\cdot),\sum_{i=1}^{d}(\partial_{x_i}u(\omega,\cdot,t))^2)\end{aligned}$$

Hence, Eq. (5.6) implies

$$\frac{d}{dt}\|u(\omega,\cdot,t)\|_H^2 + (a(\omega,\cdot),\sum_{i=1}^{d}(\partial_{x_i}u(\omega,\cdot,t))^2) \leq C\Big(\frac{\|f(\omega,\cdot,t)\|_H^2}{a_-(\omega)} + \|u(\omega,\cdot,t)\|_H^2\Big).$$

We now use Grönwall's inequality to bound

$$\begin{aligned}&\|u(\omega,\cdot,t)\|_H^2 + a_-(\omega)\int_0^t |u(\omega,\cdot,z)|_{H^1(\mathcal{D})}^2 dz \\ \leq&\|u(\omega,\cdot,t)\|_H^2 + \int_0^t (a(\omega,\cdot),\sum_{i=1}^{d}(\partial_{x_i}u(\omega,\cdot,z))^2)dz \\ \leq&\exp(CT)\Big(\|u_0(\omega,\cdot)\|_H^2 + \frac{\|f(\omega,\cdot,\cdot)\|_{L^2(\mathbb{T};V')}^2}{a_-(\omega)}\Big),\end{aligned}$$

where we emphasize that the last estimate is independent of t. If $a_-(\omega) \leq 1$ holds for

fixed ω, we obtain the path-wise parabolic estimate

$$\sup_{t\in\mathbb{T}} \|u(\omega,\cdot,\cdot)\|^2_{*,t} \leq \exp(CT)\left(\frac{\|u_0(\omega,\cdot)\|^2_H + \|f(\omega,\cdot,\cdot)\|^2_{L^2(\mathbb{T};V')}}{a^2_-(\omega)}\right).$$

On the other hand, if $a_-(\omega) > 1$, it follows that

$$\sup_{t\in\mathbb{T}} \|u(\omega,\cdot,\cdot)\|^2_{*,t} \leq \exp(CT)\Big(\|u_0(\omega,\cdot)\|^2_H + \|f(\omega,\cdot,\cdot)\|^2_{L^2(\mathbb{T};V')}\Big).$$

With the inequalities $\sqrt{c_1+c_2} \leq \sqrt{c_1}+\sqrt{c_2}$ and $(c_1+c_2)^r \leq 2^{r-1}(c_1^r+c_2^r)$ for $c_1, c_2 \geq 0, r \geq 1$, and by taking expectations this yields for any $r \in [1, (1/p+1/q)^{-1}]$

$$\begin{aligned}\mathbb{E}\Big(\sup_{t\in\mathbb{T}} \|u\|^r_{*,t}\Big) &\leq C\mathbb{E}\Big(\frac{\|u_0\|^r_H + \|f\|^r_{L^2(\mathbb{T};H)}}{a^r_-}\mathbb{1}_{\{a_-\leq 1\}} + (\|u_0\|^r_H + \|f\|^r_{L^2(\mathbb{T};V')})\mathbb{1}_{\{a_->1\}}\Big)\\ &\leq C(1+\|1/a_-\|^r_{L^q(\Omega;\mathbb{R})})\Big(\|u_0\|^r_{L^p(\Omega;H)} + \|f\|^r_{L^p(\Omega;L^2(\mathbb{T};V'))}\Big),\end{aligned}$$

where we have used Assumption 5.2.5 and Hölder's inequality for the last estimate.

For the second part of the claim, given that $f \in L^p(\Omega; L^2(\mathbb{T}; H))$, we may bound II via

$$II \leq \frac{1}{2}\|f(\omega,\cdot,t)\|^2_H + \frac{1}{2}\|u(\omega,\cdot,t)\|^2_H$$

and proceed as for the first term, using Grönwall's inequality, to obtain

$$\|u(\omega,\cdot,t)\|^2_H + a_-(\omega)\int_0^t |u(\omega,\cdot,z)|^2_{H^1(\mathcal{D})}dz \leq C\Big(\|u_0(\omega,\cdot)\|^2_H + \|f(\omega,\cdot,\cdot)\|^2_{L^2(\mathbb{T},H)}\Big).$$

Finally, with Hölder's inequality it follows for any $r \in [1, (1/p+1/(2q))^{-1}]$ that

$$\mathbb{E}\Big(\sup_{t\in\mathbb{T}} \|u\|^r_{*,t}\Big)^{1/r} \leq C(1+\|1/a_-\|^{1/2}_{L^q(\Omega;\mathbb{R})})\Big(\|u_0\|_{L^p(\Omega;H)} + \|f\|_{L^p(\Omega;L^2(\mathbb{T};H))}\Big).$$

□

To incorporate discontinuities at random submanifolds of $\mathcal{D}$, we introduce the jump-diffusion coefficient a and jump-advection coefficient b in the following section. The introduced coefficients allow us to apply Theorem 5.2.6 and derive existence, uniqueness and regularity results on the corresponding solution to the parabolic problem with discontinuous coefficients.

5.3 Random parabolic problems with discontinuous coefficients

To obtain a stochastic jump-diffusion coefficient representing the permeability in a subsurface flow model, we use the random coefficient a from the elliptic diffusion problem in [31] consisting of a (spatial) Gaussian random field with additive discontinuities on random submanifolds of $\mathcal{D}$. The specific structure of a may be utilized to model the hydraulic conductivity within heterogeneous and/or fractured media and is thus considered time-independent (see also Remark 5.2.3). The advection term in this model should then be driven by the same random field and inherit the same discontinuous structure as the diffusion term. Thus, we consider the coefficient b as an essentially linear mapping of a. Since the coefficients usually involve infinite series expansions in the Gaussian field and/or sampling errors in the jump measure, we also describe how to obtain tractable approximations of a and b. Subsequently, existence and stability results for weak solutions of the unapproximated resp. approximated parabolic problems based on Theorem 5.2.6 are proved. We conclude this section by showing that the approximated solution converges to the solution u of the (unapproximated) advection-diffusion problem in a suitable norm.

5.3.1 Jump-diffusion coefficients and their approximations

Definition 5.3.1. The *jump-diffusion coefficient* a is defined as

$$a : \Omega \times \mathcal{D} \to \mathbb{R}_{>0}, \quad (\omega, x) \mapsto \overline{a}(x) + \Phi(W(\omega, x)) + P(\omega, x),$$

where

- $\overline{a} \in C^1(\overline{\mathcal{D}}; \mathbb{R}_{\geq 0})$ is non-negative, continuous and bounded.
- $\Phi \in C^1(\mathbb{R}; \mathbb{R}_{>0})$ is a continuously differentiable, positive mapping.
- $W \in L^2(\Omega; H)$ is a (zero-mean) Gaussian random field associated to a non-negative, symmetric trace class operator $Q : H \to H$.
- $\mathcal{T} : \Omega \to \mathcal{B}(\mathcal{D}),\ \omega \mapsto \{\mathcal{T}_1, \ldots, \mathcal{T}_\tau\}$ is a random partition of $\mathcal{D}$, i.e. the $\mathcal{T}_i$ are disjoint open subsets of $\mathcal{D}$ with $\overline{\mathcal{D}} = \bigcup_{i=1}^{\tau} \overline{\mathcal{T}}_i$. The number τ of elements in $\mathcal{T}$ is a random variable $\tau : \Omega \to \mathbb{N}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. Associated to $\mathcal{T}$ is a measure λ on $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$ that controls the position of the random elements $\mathcal{T}_i$.

- $(P_i, i \in \mathbb{N})$ is a sequence of non-negative random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and

$$P : \Omega \times \mathcal{D} \to \mathbb{R}_{\geq 0}, \quad (\omega, x) \mapsto \sum_{i=1}^{\tau(\omega)} \mathbb{1}_{\{\mathcal{T}_i\}}(x) P_i(\omega).$$

The sequence $(P_i, i \in \mathbb{N})$ is independent of τ (but not necessarily i.i.d.).

Based on a, the *jump-advection coefficient* b is given for $\widetilde{b}_1, \widetilde{b}_2 \in L^\infty(\mathcal{D})^d$ by

$$b : \Omega \times \mathcal{D} \to \mathbb{R}^d, \quad (\omega, x) \mapsto \min(a(\omega, x)\widetilde{b}_1(x), \widetilde{b}_2(x)).$$

Remark 5.3.2. The dependence of a and b in Definition 5.3.1 may be interpreted in the way that advection and diffusion are both mainly influenced by the same discontinuous geometry. For instance, in a subsurface flow model, the diffusion coefficient a represents an uncertain permeability that is subject to sudden changes due to cracks, inclusions or other discontinuous structures. Of course, this geometry should also be reflected in the advective forces in the model, and to this end we make the simplifying assumption that b is an (essentially) linear transformation of a. As $\widetilde{b}_1$ is only required to be bounded vector field, there is still some flexibility in modeling of b. This entails particularly that b can admit additional (deterministic) discontinuities. The definition of the advection coefficient immediately implies Assumption 5.2.5 in the sense that $\|b(\omega, x)\|_\infty \leq \min(\overline{b}_1 a(\omega, x), \overline{b}_2)$ holds with suitable constants $\overline{b}_1, \overline{b}_2$ for almost all $\omega \in \Omega$ and almost $x \in \mathcal{D}$. The upper bound with respect to $\overline{b}_2$ is due to technical reasons and not restrictive in practical applications, as $\overline{b}_2$ may be arbitrary large (see also the numerical examples in Section 5.5).

In general, the structure of a as in Def. 5.3.1 does not allow us to draw samples from the exact distribution of this random function. We remark that λ may be used to concentrate the submanifolds that generate $\mathcal{T}$ on certain areas in $\mathcal{D}$, see Section 5.5 for examples. For an approximation of the Gaussian field, one usually uses truncated Karhunen-Loève expansions: Let $((\eta_i, e_i), i \in \mathbb{N})$ denote the sequence of eigenpairs of Q, where $Q : H \to H$ is the covariance operator of the Gaussian field W and the eigenvalues are given in decaying order $\eta_1 \geq \eta_2 \geq \cdots \geq 0$. Since Q is trace class, the Gaussian random field W admits the representation

$$W = \sum_{i \in \mathbb{N}} \sqrt{\eta_i} e_i Z_i,$$

where $(Z_i, i \in \mathbb{N})$ is a sequence of independent and standard normally distributed random variables. The series above converges in $L^2(\Omega; H)$ and almost surely (see

e.g. [27]). The truncated Karhunen-Loève expansion W_N of W is then given by

$$W_N := \sum_{i=1}^{N} \sqrt{\eta_i} e_i Z_i,$$

where we call $N \in \mathbb{N}$ the *cut-off index* of W_N. In addition, it may be possible that the sequence of jumps $(P_i, i \in \mathbb{N})$ cannot be sampled exactly but only with an intrinsic bias (see [31, Remark 3.4]). The biased samples are denoted by $(\widetilde{P}_i, i \in \mathbb{N})$ and the error which is induced by this approximation is represented by the parameter $\varepsilon > 0$ as in Assumption 5.3.3. To approximate P using the biased sequence $(\widetilde{P}_i, i \in \mathbb{N})$ instead of $(P_i, i \in \mathbb{N})$ we define the jump part approximation

$$P_\varepsilon : \Omega \times \mathcal{D} \to \mathbb{R}, \quad (\omega, x) \mapsto \sum_{i=1}^{\tau(\omega)} \mathbb{1}_{\{\mathcal{T}_i\}}(x) \widetilde{P}_i(\omega).$$

The *approximated jump-diffusion coefficient* $a_{N,\varepsilon}$ is then given by

$$a_{N,\varepsilon}(\omega, x) := \overline{a}(x) + \Phi(W_N(\omega, x)) + P_\varepsilon(\omega, x), \tag{5.7}$$

and the *approximated jump-advection coefficient* $b_{N,\varepsilon}$ via

$$b_{N,\varepsilon}(\omega, x) := \min(a_{N,\varepsilon}(\omega, x)\widetilde{b}_1(x), \widetilde{b}_2(x)).$$

Substituting the approximated jump coefficients into the parabolic model problem (5.1) yields

$$\begin{aligned} \partial_t u_{N,\varepsilon}(\omega, x, t) + [A_{N,\varepsilon} u_{N,\varepsilon}](\omega, x, t) &= f(\omega, x, t) \quad \text{in } \Omega \times \mathcal{D} \times (0, T], \\ u_{N,\varepsilon}(\omega, x, 0) &= u_0(\omega, x) \quad \text{in } \Omega \times \mathcal{D} \times \{0\} \\ u_{N,\varepsilon}(\omega, x) &= 0 \quad \text{on } \Omega \times \partial\mathcal{D}, \end{aligned} \tag{5.8}$$

where the approximated second order differential operator $A_{N,\varepsilon}$ is given by

$$[A_{N,\varepsilon} u](\omega, x, t) = -\nabla \cdot (a_{N,\varepsilon}(\omega, x) \nabla u(\omega, x, t)) + b_{N,\varepsilon}(\omega, x) \cdot \nabla u(\omega, x, t).$$

The path-wise variational formulation of Eq. (5.8) is then analogous to Eq. (5.3): For almost all $\omega \in \Omega$ with given $f(\omega, \cdot)$, find $u_{N,\varepsilon}(\omega, \cdot, \cdot) \in L^2(\mathbb{T}; V)$ with weak time derivative $\partial_t u_{N,\varepsilon}(\omega, \cdot, \cdot) \in L^2(\mathbb{T}; V')$ such that it holds, for $t \in \mathbb{T}$

$${}_{V'}\langle \partial_t u_{N,\varepsilon}(\omega, \cdot, t), v \rangle_V + B_\omega^{N,\varepsilon}(u_{N,\varepsilon}(\omega, \cdot, t), v) = F_{\omega,t}(v), \quad \text{for all } v \in V, \tag{5.9}$$

where the approximated bilinear form is given by

$$B_\omega^{N,\varepsilon}(v,w) = \int_{\mathcal{D}} a_{N,\varepsilon}(\omega,x)\nabla v(x)\cdot\nabla w(x) + b_{N,\varepsilon}(\omega,x)\cdot\nabla v(x)w(x)dx, \quad v,w\in V.$$

The following assumptions guarantee that we can apply Theorem 5.2.6 also in the jump-diffusion setting and that therefore path-wise solutions u and $u_{N,\varepsilon}$ exist.

Assumption 5.3.3.

(i) The eigenfunctions e_i of Q are continuously differentiable on $\mathcal{D}$ and there exist constants $\alpha,\beta,C_e,C_\eta>0$ such that for any $i\in\mathbb{N}$

$$\|e_i\|_{L^\infty(\mathcal{D})}\le C_e, \quad \max_{j=1,\dots,d}\|\partial_{x_j}e_i\|_{L^\infty(\mathcal{D})}\le C_e i^\alpha \quad\text{and}\quad \sum_{i=1}^\infty \eta_i i^\beta \le C_\eta < +\infty.$$

(ii) Furthermore, the mapping Φ as in Definition 5.3.1 and its derivative are bounded for any $w\in\mathbb{R}$ by

$$\phi_1\exp(\phi_2 w)\ge\Phi(w)\ge\phi_1\exp(-\phi_2 w), \quad |\frac{d}{dx}\Phi(w)|\le\phi_3\exp(\phi_4|w|),$$

where $\phi_1,\dots,\phi_4>0$ are arbitrary constants.

(iii) There exists $p>1$ such that $f\in L^p(\Omega;L^2(\mathbb{T};V'))$ and $u_0\in L^p(\Omega;H)$. The sequence $(P_i, i\in\mathbb{N})$ consists of nonnegative and bounded random variables $P_i\in[0,\overline{P}]$ for some $\overline{P}>0$. In addition, for $s>1$ such that $1/p+1/s<1$ there exists a sequence of approximations $(\widetilde{P}_i, i\in\mathbb{N})\subset[0,\overline{P}]^{\mathbb{N}}$ so that the sampling error is bounded, for some $\varepsilon>0$, by

$$\mathbb{E}(|\widetilde{P}_i - P_i|^s)\le\varepsilon, \quad i\in\mathbb{N}.$$

Remark 5.3.4. The exponential bounds on Φ and its derivative imply that $u\in L^r(\Omega;L^2(\mathbb{T};V))$ for any $r\in[1,p)$. That is, the integrability of u with respect to Ω only depends on the stochastic regularity of f and u_0. In fact, Theorem 5.2.6 shows that far weaker assumptions on a (resp. Φ) are possible to achieve $u\in L^r(\Omega;L^2(\mathbb{T};V))$, at the cost that r then also depends on the integrability of a_-. At this point we refer to [31], where the regularity of an elliptic diffusion problem with a as in Definition 5.3.1, but less restricted functions Φ and P is investigated. However, Assumption 5.3.3 includes the most important case that $\Phi(W)$ is a log-Gaussian random field and the bounds on Φ are merely imposed for a clear and simplified presentation of the results. On a

further note, the assumptions on the eigenpairs $(\eta_i, e_i, i \in \mathbb{N})$ are natural and include the case that Q is a Matérn-type or Brownian-motion-type covariance function.

Lemma 5.3.5. *Let Assumption 5.3.3 hold and define*

$$a_{N,\varepsilon,-} := \inf_{x\in\mathcal{D}} a_{N,\varepsilon}(\omega, x), \quad a_{N,\varepsilon,+} := \sup_{x\in\mathcal{D}} a_{N,\varepsilon}(\omega, x).$$

Then, $1/a_-, 1/a_{N,\varepsilon,-}, a_+, a_{N,\varepsilon,+} \in L^q(\Omega;\mathbb{R})$ *for any* $q \in [1,\infty)$ *and there exists* $C = C(q,\phi_1,\phi_2) > 0$*, independent of* N *and* ε*, such that*

$$\|1/a_-\|_{L^q(\Omega;\mathbb{R})},\ \|1/a_{N,\varepsilon,-}\|_{L^q(\Omega;\mathbb{R})},\ \|a_+\|_{L^q(\Omega;\mathbb{R})},\ \|a_{N,\varepsilon,+}\|_{L^q(\Omega;\mathbb{R})} \le C < +\infty.$$

Proof. Let the parameters $N \in \mathbb{N}$ and $\varepsilon > 0$ be fixed. From [31, Lemma 3.5], we have that W and W_N are centered, almost surely bounded Gaussian random fields on $\mathcal{D}$ which implies $E := \mathbb{E}(\sup_{x\in\mathcal{D}} W(x)) < +\infty$ as well as

$$\mathbb{P}(\sup_{x\in\mathcal{D}} W(\cdot,x) - E \ge c) \le \exp(-\frac{c^2}{2\overline{\sigma}^2}) \tag{5.10}$$

for all $c > 0$ and $\overline{\sigma}^2 := \sup_{x\in\mathcal{D}} \mathbb{E}(W(\cdot,x)^2) \le \mathrm{Tr}(Q)$. Furthermore,

$$\mathbb{P}(\|W(x)\|_{L^\infty(\mathcal{D})} > c) \le 2\mathbb{P}(\sup_{x\in\mathcal{D}} W(\cdot,x) > c) \tag{5.11}$$

by the symmetry of W. With $\|\exp(|W|)\|_{L^\infty(\mathcal{D})} \le \exp(\|W\|_{L^\infty(\mathcal{D})})$ and Assumption 5.3.3 (ii), we then obtain for arbitrary $q \in [1,\infty)$

$$\begin{aligned}
\mathbb{E}(1/a_-^q) &\le \mathbb{E}\Big(\big(\inf_{x\in\mathcal{D}} \Phi(W(\cdot,x)\big)^{-q}\Big) \\
&= \mathbb{E}\Big(\sup_{x\in\mathcal{D}} \Phi(W(\cdot,x))^{-q}\Big) \\
&\le \frac{1}{\phi_1^q}\mathbb{E}(\sup_{x\in\mathcal{D}} \exp(q\phi_2|W(\cdot,x)|)) \\
&\le \frac{1}{\phi_1^q}\mathbb{E}(\exp(q\phi_2\|W\|_{L^\infty(\mathcal{D})})).
\end{aligned}$$

By Fubini's Theorem, integration by parts and Ineqs. (5.11), (5.10) this yields

$$\begin{aligned}
\mathbb{E}(\exp(q\phi_2\|W\|_{L^\infty(\mathcal{D})})) &= \int_0^\infty q\phi_2 \exp(q\phi_2 c)\mathbb{P}(\|W\|_{L^\infty(\mathcal{D})} > c)dc \\
&\le q\phi_2 \exp(q\phi_2 E) + 2\int_E^\infty q\phi_2 \exp(q\phi_2 c))\mathbb{P}(\sup_{x\in\mathcal{D}} W(\cdot,x) > c)dc \\
&\le q\phi_2 \exp(q\phi_2 E) + 2\int_E^\infty q\phi_2 \exp(q\phi_2 c - \frac{1}{2\overline{\sigma}^2}c^2)dc.
\end{aligned}$$

The last estimate on the right hand side is finite for each $q \in \mathbb{R}$ which proves the claim for a_-. To bound the expectation of a_+, we proceed in the same way by noting that

$$\begin{aligned}\|a_+\|_{L^q(\Omega)} &\leq \|\overline{a}\|_{L^\infty(\mathcal{D})} + \mathbb{E}\Big(|\sup_{x\in\mathcal{D}} \Phi(W(x))|^q\Big) + \overline{P} \\ &\leq \|\overline{a}\|_{L^\infty(\mathcal{D})} + \phi_1 \mathbb{E}\Big(\sup_{x\in\mathcal{D}} \exp(q\phi_2 |W(\cdot,x)|)\Big)^{1/q} + \overline{P}\end{aligned}$$

by Assumption 5.3.3 (ii). Analogously, the claim follows for $a_{N,\varepsilon,-}, a_{N,\varepsilon,+}$ with the same bounds from above as for a_-, a_+ respectively, because

$$\overline{\sigma}_N^2 := \sup_{x\in\mathcal{D}} \mathbb{E}(W_N(x)^2) \leq \sum_{i=1}^N \eta_i \leq \mathrm{Tr}(Q).$$

□

Theorem 5.3.6. *Under Assumption 5.3.3 there exists almost surely a unique weak solution u to Problem* (5.1) *and a unique weak solution $u_{N,\varepsilon}$ to Problem* (5.8) *for each $N \in \mathbb{N}$ and $\varepsilon > 0$. For $r \in [1,p)$, the weak solutions satisfy the parabolic estimate*

$$\mathbb{E}\Big(\sup_{t\in\mathbb{T}} \|u\|_{*,t}^r\Big)^{1/r}, \ \ \mathbb{E}\Big(\sup_{t\in\mathbb{T}} \|u_{N,\varepsilon}\|_{*,t}^r\Big)^{1/r} \leq C\Big(\|u_0\|_{L^p(\Omega;H)} + \|f\|_{L^p(\Omega;L^2(\mathbb{T};V'))}\Big),$$

where $C = C(r,a,b,T) > 0$ is independent of N and ε.

Proof. To apply Theorem 5.2.6, we need Assumption 5.2.5 to hold. By Definition 5.3.1 and Eq. (5.7)

$$a_-(\omega),\ a_{N,\varepsilon,-}(\omega) > 0 \quad \text{and} \quad a_+(\omega),\ a_{N,\varepsilon,+}(\omega) < +\infty$$

holds almost surely. The corresponding advection coefficients are bounded with Remark 5.3.2 via

$$\|b(\omega,x)\|_\infty \leq \min(\overline{b}_1 a(\omega,x), \overline{b}_2) \quad \text{and} \quad \|b_{N,\varepsilon}(\omega,x)\|_\infty \leq \min(\overline{b}_1 a_{N,\varepsilon}(\omega,x), \overline{b}_2),$$

respectively. We further obtain from Lemma 5.3.5 $1/a_-, 1/a_{N,\varepsilon,-} \in L^q(\Omega;\mathbb{R})$ for any $q \in [1,\infty)$ and that $\|1/a_{N,\varepsilon,-}\|_{L^q(\Omega;\mathbb{R})}$ is bounded uniformly with respect to N and ε. For given $r \in [1,p)$, we then choose $q = (1/r - 1/p)^{-1} < +\infty$ and the claim follows by Theorem 5.2.6. □

Having shown the existence and uniqueness of the weak solutions u and $u_{N,\varepsilon}$, we may bound the difference between both solutions in the (expected) parabolic norm with

respect to the approximation parameters N and ε. For this, we record the following estimate on the approximation error $a - a_{N,\varepsilon}$.

Theorem 5.3.7. *[31, Theorem 3.12] Under Assumption 5.3.3, it holds that*

$$\|a - a_{N,\varepsilon}\|_{L^s(\Omega;L^\infty(\mathcal{D}))} \leq C\left(\Xi_N^{1/2} + \varepsilon^{1/s}\right),$$

where $\Xi_N := \sum_{i>N} \eta_i$ *and* $C > 0$ *is independent of* $N \in \mathbb{N}$ *and* $\varepsilon > 0$.

The final result of this section shows the convergence $u_{N,\varepsilon} \to u$ in $L^r(\Omega; L^2(T;V))$ as $N \to +\infty$ and $\varepsilon \to 0$.

Theorem 5.3.8. *Under Assumption 5.3.3, for any* $r \in [1, (1/s + 1/p)^{-1})$*, the approximation error of* u *is bounded in the parabolic norm by*

$$\mathbb{E}\Big(\sup_{t\in\mathbb{T}} \|u - u_{N,\varepsilon}\|_{*,t}^r\Big)^{1/r} \leq C\left(\Xi_N^{1/2} + \varepsilon^{1/s}\right).$$

Proof. By Theorem 5.3.6, existence of solutions u and $u_{N,\varepsilon}$ to the Problems (5.3), (5.9) is guaranteed almost surely, hence for almost all $\omega \in \Omega, t \in \mathbb{T}$ and $v \in V$

$${}_{V'}\langle \partial_t u(\omega,\cdot,t), v\rangle_V + B_\omega(u(\omega,\cdot,t), v) = {}_{V'}\langle \partial_t u_{N,\varepsilon}(\omega,\cdot,t), v\rangle_V + B_\omega^{N,\varepsilon}(u_{N,\varepsilon}(\omega,\cdot,t), v).$$

This identity may be reformulated to the variational problem to find $u - u_{N,\varepsilon} \in L^2(\mathbb{T};V)$ such that for all $t \in \mathbb{T}$ and $v \in V$

$$\begin{aligned}
&{}_{V'}\langle \partial_t(u(\omega,\cdot,t) - u_{N,\varepsilon}(\omega,\cdot,t)), v\rangle_V + B_\omega(u(\omega,\cdot,t) - u_{N,\varepsilon}(\omega,\cdot,t), v)\\
=&{}_{V'}\langle \widetilde{f}(\omega,\cdot,t), v\rangle_V,\\
:=&((a_{N,\varepsilon} - a)(\omega,\cdot), \nabla u_{N,\varepsilon}(\omega,\cdot,t)\cdot\nabla v) + ((b_{N,\varepsilon} - b)(\omega,\cdot)\cdot\nabla u_{N,\varepsilon}(\omega,\cdot,t), v),
\end{aligned}$$

with initial value $(u - u_{N,\varepsilon})(\omega,\cdot,0) \equiv 0$ holds almost surely. Definition 5.3.1 and Remark 5.3.2 imply

$$\|\widetilde{f}(\omega,\cdot,\cdot)\|_{L^2(\mathbb{T};V')} \leq (1 + \bar{b}_1)\|(a - a_{N,\varepsilon})(\omega,\cdot)\|_{L^\infty(\mathcal{D})}\|\sum_{i=1}^d |\partial_{x_i} u_{N,\varepsilon}(\omega,\cdot,\cdot)|\|_{L^2(\mathbb{T};H)},$$

and by Ineq. (5.4) and Theorem 5.3.6 we know that for $\overline{r} \in [1, p)$

$$\begin{aligned}\|\sum_{i=1}^{d} |\partial_{x_i} u_{N,\varepsilon}|\|_{L^{\overline{r}}(\Omega;L^2(\mathbb{T};H))} &\leq 2^{d/2-1/2}\mathbb{E}\Big(\|u_{N,\varepsilon}\|_{*,T}^{\overline{r}}\Big)^{1/\overline{r}} \\ &\leq C\Big(\|u_0\|_{L^p(\Omega;H)} + \|f\|_{L^p(\Omega;L^2(\mathbb{T};V'))}\Big) \\ &< +\infty.\end{aligned}$$

We may now choose $\overline{p} \in [1, (1/s + 1/\overline{r})^{-1}]$ and obtain by Hölder's inequality and Theorem 5.3.7

$$\begin{aligned}\|\widetilde{f}(\omega,\cdot,\cdot)\|_{L^{\overline{p}}(\Omega;L^2(\mathbb{T};V'))} &\leq (1+\overline{b}_1)\|(a-a_{N,\varepsilon})(\omega,\cdot)\|_{L^s(\Omega;L^\infty(\mathcal{D}))}\|\sum_{i=1}^{d}\partial_{x_i}u_{N,\varepsilon}\|_{L^{\overline{r}}(\Omega;L^2(\mathbb{T};H))} \\ &\leq C\left(\Xi_N^{1/2} + \varepsilon^{1/s}\right)\end{aligned}$$

for $C > 0$ independent of N and ε. The claim now follows with Lemma 5.3.5 and by applying Theorem 5.2.6 on $u - u_{N,\varepsilon}$ for

$$q = (1/r - 1/s - 1/\overline{p})^{-1} < (1/r - 1/s - 1/p)^{-1} < +\infty.$$

□

To draw samples of $u_{N,\varepsilon}$, we need to employ further numerical techniques since $u_{N,\varepsilon}(\omega,\cdot,\cdot)$ takes almost surely values in the infinite-dimensional Hilbert space $L^2(\mathbb{T};V)$. Hence, we have to find path-wise approximations of $u_{N,\varepsilon}$ in finite-dimensional subspaces of $L^2(\mathbb{T};V)$ by discretizing the spatial and temporal domain. Next, we construct suitable approximation spaces of V, combine them with a time stepping method and control for the discretization error.

5.4 Path-wise discretization schemes

In the previous section we demonstrated that u may be approximated by $u_{N,\varepsilon}$ for sufficiently large $N \in \mathbb{N}$ resp. small $\varepsilon > 0$. Nevertheless, even $u_{N,\varepsilon}(\omega,\cdot,\cdot)$ will in general not be accessible analytically for fixed ω, N and ε, thus we need to find pathwise finite-dimensional approximations of $u_{N,\varepsilon}(\omega,\cdot,\cdot)$. In the first part of this section we explain how a semi-discrete solution may be obtained by approximating V with a sequence of *sample-adapted* Finite Element (FE) spaces. By sample-adaptedness we mean that the FE mesh is aligned *a-priori* with the discontinuities of P in each sample, i.e. the grid changes with each $\omega \in \Omega$. This is in contrast to *adaptive* FE schemes based

on a-posteriori error estimates that may require several stages of remeshing in each sample, see e.g. [70, 78, 131]. We analyze the discretization error for the pathwise sample-adapted strategy and further emphasize its advantages compared to a standard, sample-independent FE basis. In the second part we combine the spatial discretization with a backward time stepping scheme in $\mathbb{T}$, with the time step chosen accordingly to the sample-dependent FE basis. Finally, we derive the mean-square error between the unbiased solution u and the fully discrete approximation of $u_{N,\varepsilon}$.

5.4.1 Sample-adapted spatial discretization

To find approximations of $u_{N,\varepsilon}(\omega,\cdot,t) \in V$ for fixed $\omega \in \Omega$ and $t \in \mathbb{T}$, we use a standard Galerkin approach based on a sequence $\mathcal{V}_\omega = (V_\ell(\omega), \ell \in \mathbb{N}_0)$ of finite-dimensional and sample-dependent subspaces $V_\ell(\omega) \subset V$. An obvious choice for V_ℓ is the space of piecewise linear Finite Elements with respect to some triangulation of $\mathcal{D}$. We follow the same approach as in [31] and utilize path-dependent meshes to match the interfaces generated by the jump-diffusion and -advection coefficients: For a given random partition $\mathcal{T}(\omega) = (\mathcal{T}_i, i = 1\ldots,\tau(\omega))$ of $\mathcal{D}$, we choose a triangulation $\mathcal{K}_\ell(\omega)$ of $\mathcal{D}$ such that

$$\mathcal{T}(\omega) \subset \mathcal{K}_\ell(\omega) \quad \text{and} \quad h_\ell(\omega) := \max_{K\in\mathcal{K}_\ell(\omega)} \operatorname{diam}(K) \leq \overline{h}_\ell \quad \text{for } \ell \in \mathbb{N}_0.$$

Above, $\operatorname{diam}(K)$ is the longest side length of the triangle K and $(\overline{h}_\ell, \ell \in \mathbb{N}_0)$ is a positive sequence of deterministic refinement thresholds, decreasing monotonically to zero. This guarantees that $h_\ell(\omega) \to 0$ almost surely, although the absolute speed of convergence may vary for each ω. Given that $\mathcal{T}$ splits the domain $\mathcal{D}$ into a finite number of piecewise linear polygons (see Assumption 5.4.1 below), such a triangulation $\mathcal{K}_\ell$ with $\mathcal{T}(\omega) \subset \mathcal{K}_\ell(\omega)$ always exists for any prescribed refinement $\overline{h}_\ell > 0$. Consequently, $V_\ell(\omega)$ is chosen as the space of continuous, piecewise linear functions with respect to $K_\ell(\omega)$:

$$V_\ell(\omega) := \{v_{\ell,\omega} \in C^0(\overline{\mathcal{D}}) \Big|\ v_{\ell,\omega}|_{\partial\mathcal{D}} = 0 \text{ and } v_{\ell,\omega}|_K \in \mathcal{P}_1(K),\ K \in \mathcal{K}_\ell(\omega)\} \subset V.$$

The set $\mathcal{P}_1(K)$ denotes the space of all linear polynomials on the triangle K, and $\{v_{1,\omega},\ldots,v_{d_\ell(\omega),\omega}\}$ is the nodal basis of $V_\ell(\omega)$ that corresponds to the vertices in $\mathcal{K}_\ell$. As discussed in [31, Section 4], the adjustment of $\mathcal{K}_\ell$ to the discontinuities of a and b accelerates convergence of the spatial discretization compared to a fixed, non-adapted FE approach. The semi-discrete version of Problem (5.9) is then to find $u_{N,\varepsilon,\ell}(\omega,\cdot,\cdot) \in$

$L^2(\mathbb{T}; V_\ell(\omega))$ with $\partial_t u_{N,\varepsilon,\ell}(\omega,\cdot,\cdot) \in L^2(\mathbb{T}; (V_\ell(\omega))')$ such that for $t \in \mathbb{T}$ and $v_{\ell,\omega} \in V_\ell(\omega)$

$$\begin{aligned} {}_{V'}\langle \partial_t u_{N,\varepsilon,\ell}(\omega,\cdot,t), v_{\ell,\omega}\rangle_V + B_\omega^{N,\varepsilon}(u_{N,\varepsilon,\ell}(\omega,\cdot,t), v_{\ell,\omega}) &= F_{t,\omega}(v_{\ell,\omega}), \\ u_{N,\varepsilon,\ell}(\omega,\cdot,0) &= u_{0,\ell}(\omega,\cdot), \end{aligned} \tag{5.12}$$

In Eq. (5.12), $u_{0,\ell}(\omega,\cdot) \in V_\ell(\omega)$ is a suitable approximation of $u_0(\omega,\cdot)$, see also Remark 5.4.4. The function $u_{N,\varepsilon,\ell}(\omega,\cdot,t)$ may be expanded with respect to the basis $\{v_{1,\omega},\dots,v_{d_\ell(\omega),\omega}\}$ as

$$u_{N,\varepsilon,\ell}(\omega,x,t) = \sum_{j=1}^{d_\ell(\omega)} c_j(\omega,t) v_{j,\omega}(x), \tag{5.13}$$

where the coefficients $c_1(\omega,t),\dots,c_{d_\ell}(\omega,t) \in \mathbb{R}$ depend on $(\omega,t) \in \Omega\times\mathbb{T}$ and the respective coefficient column-vector is defined as $\mathbf{c}(\omega,\mathbf{t}) := (c_1(\omega,t),\dots,c_{d_\ell}(\omega,t))^T$. With this, the semi-discrete variational problem in the finite-dimensional space $V_\ell(\omega)$ is equivalent to solving the system of ordinary differential equations

$$\frac{d}{dt}\mathbf{c}(\omega,\mathbf{t}) + \mathbf{A}(\omega)\mathbf{c}(\omega,\mathbf{t}) = \mathbf{F}(\omega,\mathbf{t}), \quad \mathbf{t}\in\mathbb{T},$$

for $\mathbf{c}$ with stochastic stiffness matrix $(\mathbf{A}(\omega))_{jk} = B_\omega^{N,\varepsilon}(v_{j,\omega}, v_{k,\omega})$ and time-dependent load vector $(\mathbf{F}(\omega,t))_j = F_{t,\omega}(v_{j,\omega})$ for $j,k \in \{1,\dots,d_\ell(\omega)\}$. To ensure the well-posedness of Eq. (5.12) and derive error bounds of the numerical approximation of u in a mean-square sense, we need to modify Assumption 5.3.3:

Assumption 5.4.1.

(i) The eigenfunctions e_i of Q are continuously differentiable on $\mathcal{D}$ and there exist constants $\alpha,\beta,C_e,C_\eta > 0$ such that $2\alpha \le \beta$ and for any $i\in\mathbb{N}$

$$\|e_i\|_{L^\infty(\mathcal{D})} \le C_e, \quad \max_{j=1,\dots,d} \|\partial_{x_j} e_i\|_{L^\infty(\mathcal{D})} \le C_e i^\alpha \quad \text{and} \quad \sum_{i=1}^\infty \eta_i i^\beta \le C_\eta < +\infty.$$

(ii) Furthermore, the mapping Φ as in Definition 5.3.1 and its derivative are bounded for $w\in\mathbb{R}$ by

$$\phi_1 \exp(\phi_2 w) \ge \Phi(w) \ge \phi_1 \exp(-\phi_2 w), \quad |\frac{d}{dx}\Phi(w)| \le \phi_3 \exp(\phi_4|w|),$$

where $\phi_1,\dots,\phi_4 > 0$ are arbitrary constants.

(iii) There exists $p > 2$ such that $f, \partial_t f \in L^p(\Omega; L^2(\mathbb{T}; H))$ and $u_0 \in L^p(\Omega; V)$. Furthermore, u_0 and f are stochastically independent of $\mathcal{T}$. The sequence $(P_i, i\in\mathbb{N})$

consists of nonnegative and bounded random variables $P_i \in [0, \overline{P}]$ for some $\overline{P} > 0$. In addition, for $s > 2$ such that $1/p + 1/s < 1/2$ there exists a sequence of approximations $(\widetilde{P}_i, i \in \mathbb{N}) \subset [0, \overline{P}]^{\mathbb{N}}$ so that the sampling error is bounded, for some $\varepsilon > 0$, by

$$\mathbb{E}(|\widetilde{P}_i - P_i|^s) \leq \varepsilon, \quad i \in \mathbb{N}.$$

(iv) The partition elements $\mathcal{T}_i(\omega)$ are polygons with piecewise linear boundary for almost all $\omega \in \Omega$ and $\mathbb{E}(\tau^q) < +\infty$ for any $q \in [1, \infty)$.

(v) There is a constant $\kappa \in (0, 1]$ such that for all $N \in \mathbb{N}, \varepsilon > 0, t \in \mathbb{T}$, almost all $\omega \in \Omega$ and $i = 1, \ldots, \tau(\omega)$ it holds that $u_{N,\varepsilon|\mathcal{T}_i}(\omega, \cdot, t) \in H^{1+\kappa}(\mathcal{T}_i)$. Moreover, there is $C > 0$, independent of N, ε and $t \in \mathbb{T}$, such that for some $r_\kappa > 2$

$$\mathbb{E}(\sup_{t\in\mathbb{T}} \max_{i=1,\ldots,\tau} \|u_{N,\varepsilon}(\cdot, \cdot, t)\|^{r_\kappa}_{H^{1+\kappa}(\mathcal{T}_i)}) \leq C < +\infty.$$

(vi) Let 2^V the power set of V. For all $\ell \in \mathbb{N}_0$, the correspondence $\Omega \to 2^V,\ \omega \mapsto V_\ell(\omega)$ admits non-empty values and is *weakly measurable*, i.e. for each open subset $\widetilde{V} \subset V$ it holds that

$$\{\omega \in \Omega |\ V_\ell(\omega) \cap \widetilde{V} \neq \emptyset\} \in \mathcal{F}.$$

(vii) *Conformity*: In dimension $d = 2$, let $K_1, K_2 \in \mathcal{K}_\ell(\omega)$ for some fixed $\ell \in \mathbb{N}_0$ and $\omega \in \Omega$. Then, the intersection $\overline{K}_1 \cap \overline{K}_2$ is either empty, a common edge or a common vertex of $\mathcal{K}_\ell(\omega)$.

(viii) *Shape-regularity*: Let $\rho_{K,out}$ and $\rho_{K,in}$ denote the radius of the outer respectively inner circle of the triangle K. Then, there is a constant $\overline{\rho} > 0$ such that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{\ell\in\mathbb{N}_0} \sup_{K\in\mathcal{K}_\ell(\omega)} \frac{\rho_{K,out}}{\rho_{K,in}} \leq \overline{\rho} < +\infty.$$

Remark 5.4.2. We discuss Assumption 5.4.1 in the following:

- Assumption 5.4.1(i) implies for all $x \in \mathcal{D}$ and $i = 1, \ldots, d$ that

$$\mathbb{E}(|\partial_{x_i} W_N(x)|^2) = \mathbb{E}(|\sum_{j=1}^{n} \sqrt{\eta_j} \partial_{x_i} e_j(x)_j Z_j|^2) \leq C_e \sum_{j=1}^{N} \eta_j j^{2\alpha} \leq C_e \sum_{j=1}^{N} \eta_j j^{\beta},$$

hence there exists a $L^2(\Omega; \mathbb{R})$-limit $\partial_{x_i} W(\cdot, x) := \lim_{N\to+\infty} \partial_{x_i} W_N(\cdot, x)$. Essentially, this means that $2\alpha \leq \beta$ ensures mean-square differentiability (or path-wise

Lipschitz-continuity) of the Gaussian field W. This guarantees that the piecewise regularity parameter $\kappa > 0$ in Assumption 5.4.1(v) is solely influenced by the jump field P. If W only has Hölder continuous paths with Hölder exponent $\varrho < 1$, then the piecewise regularity is at most $u_{N,\varepsilon}(\omega, \cdot, t) \in H^{1+\varrho}(\mathcal{T}_i)$ on each partition element.

- Assumption 5.4.1(iii) essentially ensures that we are able to find a suitable initial data approximation $u_{0,\ell}$ and also control the error of a temporal discretization scheme. The nodal basis functions $v_{j,\omega}$ are solely determined by $\mathcal{T}(\omega)$ and since f, u_0 are stochastically independent of $\mathcal{T}$, we may expand the sample-adapted semi-discrete solution via Eq. (5.13), i.e. obtain a separation of spatial and temporal variables. Furthermore, the condition $1/p + 1/s < 1/2$ enables us to derive all errors in a mean-squared sense.

- Assumption 5.4.1(iv,v) allows us to derive convergence of order $\mathcal{O}(\overline{h}_\ell^\kappa)$ of the sample-adapted FE discretization. In dimension $d = 2$, the regularity parameter $\kappa > 0$ is highly dependent on the geometries generated by P, and a thorough analysis of the parameter κ is beyond the scope of this article. Instead, we refer to a discussion of the related *elliptic interface problems*, for instance in [161, 162, 163, 174, 175], where regularity results in (weighted) Sobolev spaces are derived and illustrated on several examples.

- On a further note, the condition $\mathbb{E}(\tau^q) < +\infty$ allows for the important case that τ is Poisson-distributed. In practical applications, one would rather assume that τ is a bounded random variable to avoid very large values of τ for small probabilities. This, however, has no effect on the theoretical results in this section or the remaining assumptions.

- Assumption 5.4.1(vi) on the correspondence $\omega \mapsto V_\ell(\omega)$ guarantees the (strong) measurability of the approximated solution $u_{N,\varepsilon,\ell} : \Omega \to L^2(\mathbb{T}; V)$, and the corresponding error estimates in the Lebesgue-Bochner spaces are well-defined.

- Finally, conformity and uniform shape-regularity in Ω of the FE mesh allow us to control the constants in the FE error analysis.

Our main result provides an error bound of the semi-discrete sample-adapted FE approximation.

Theorem 5.4.3. *Let Assumption 5.4.1 hold and let $u_{N,\varepsilon,\ell}$ be the sample-adapted FE approximation of $u_{N,\varepsilon}$ as in Eq.* (5.12) *with refinement $\overline{h}_\ell \leq 1$. If the initial data*

approximation $u_{0,\ell}$ *satisfies* $\|u_0 - u_{0,\ell}\|_{L^p(\Omega;H)} \leq C\overline{h}_\ell$ *and* $\|u_{0,\ell}\|_{L^p(\Omega;V)} \leq C\|u_0\|_{L^p(\Omega;V)}$ *with* $C > 0$ *independent of* ℓ*(see Remark 5.4.4), it holds that*

$$\mathbb{E}\Big(\sup_{t\in\mathbb{T}} \|u_{N,\varepsilon} - u_{N,\varepsilon,\ell}\|_{*,t}^2\Big)^{1/2} \leq C\overline{h}_\ell^{\kappa}.$$

Remark 5.4.4. By Assumption 5.4.1, $u_0 \in L^p(\Omega; V)$ and hence $u_0(\omega, \cdot) \in V$ almost surely. One possibility to approximate the initial data is via $u_{0,\ell} := \sum_{i=1}^{d_\ell}(u_0, \widetilde{v}_i)\widetilde{v}_i$, where $\{\widetilde{v}_1, \ldots, \widetilde{v}_{d_\ell(\omega)}\}$ is a H-orthonormal basis of $V_\ell(\omega)$. That is, $u_{\ell,0}$ is the path-wise H-orthogonal projection of $u_0(\omega, \cdot)$ into $V_\ell(\omega)$. On the other hand, if the paths of u_0 are almost surely continuous, we might as well define $u_{0,\ell}$ as the nodal interpolation with respect to the FE basis, i.e. $u_{0,\ell}(\omega, \cdot) := \sum_{i=1}^{d_\ell} u_0(\omega, x_i)v_{i,\omega}$, where $x_1, \ldots, x_{d_\ell(\omega)} \in \mathcal{D}$ are the nodal points corresponding to $\{v_1, \ldots, v_{\ell,\omega}\}$. Either way, we obtain by standard FE theory (see for instance [177, Theorem 3.4.2]) the error bound $\|u_0 - u_{0,\ell}\|_{L^p(\Omega;H)} \leq C\|u_0\|_{L^p(\Omega;V)}\overline{h}_\ell$. We note that this error bound with respect to $\|\cdot\|_H$ remains valid also for a non-adapted standard FE discretization of V. Moreover, in both cases the assumption $\|u_{0,\ell}\|_{L^p(\Omega;V)} \leq C\|u_0\|_{L^p(\Omega;V)}$ holds.

To prove Theorem 5.4.3, we treat the path-wise triangulation as a special case of the *Mortar Finite Element method*, where the basis functions of the approximation spaces $V_\ell(\omega)$ are continuous across the interface of two adjacent partition elements $\mathcal{T}_i$ and $\mathcal{T}_j$. In general, Mortar FE methods for deterministic elliptic and parabolic problems only enforce a "weak continuity condition" on the interfaces, which allows to mesh each partition element of the domain independently but introduces an additional consistency error (see e.g. [36],[39],[201]). To prove Theorem 5.4.3, we need the following intermediate result.

Lemma 5.4.5. *Let Assumption 5.4.1 hold and assume there is a* $C > 0$ *such that* $\|u_{0,\ell}\|_{L^p(\Omega;V)} \leq C\|u_0\|_{L^p(\Omega;V)}$ *for any* $\ell \in \mathbb{N}_0$*. Then, for any* $r \in [1, p)$

$$\Big\|\|\partial_t u_{N,\varepsilon}\|_{L^2(\mathbb{T};H))} + \sup_{t\in\mathbb{T}} \|u_{N,\varepsilon}(\cdot, \cdot, t)\|_V\Big\|_{L^r(\Omega;\mathbb{R})} \leq C\Big(\|u_0\|_{L^p(\Omega;V)} + \|f\|_{L^p(\Omega;L^2(\mathbb{T};H))}\Big),$$
$$\Big\|\|\partial_t u_{N,\varepsilon,\ell}\|_{L^2(\mathbb{T};H))} + \sup_{t\in\mathbb{T}} \|u_{N,\varepsilon,\ell}(\cdot, \cdot, t)\|_V\Big\|_{L^r(\Omega;\mathbb{R})} \leq C\Big(\|u_0\|_{L^p(\Omega;V)} + \|f\|_{L^p(\Omega;L^2(\mathbb{T};H))}\Big),$$

where $C > 0$ *is independent of* N, ε *and* ℓ*.*

Proof. We use the first part of the proof from [80, Chapter 7.1, Theorem 5] to obtain

the path-wise estimate

$$\begin{aligned}
&\|\partial_t u_{N,\varepsilon}(\omega,\cdot,\cdot)\|^2_{L^2(\mathbb{T};H)} + \sup_{t\in\mathbb{T}} \int_{\mathcal{D}} a_{N,\varepsilon}(\omega,x,t)\nabla u_{N,\varepsilon}(\omega,x,t)\cdot\nabla u_{N,\varepsilon}(\omega,x,t)dx \\
\leq& \int_{\mathcal{D}} a_{N,\varepsilon}(\omega,x,t)\nabla u_{N,\varepsilon}(\omega,x,0)\cdot\nabla u_{N,\varepsilon}(\omega,x,0)dx \\
&\quad + \int_0^T \|b_{N,\varepsilon}(\omega,x,t)\cdot\nabla u_{N,\varepsilon}(\omega,\cdot,t)\|^2_H dt + \|f(\omega,\cdot,\cdot)\|^2_{L^2(\mathbb{T};H)} \\
\leq& a_{N,\varepsilon,+}(\omega)\|u_0(\omega,\cdot)\|^2_V + a_{N,\varepsilon,+}(\omega)\bar{b}_1 2^{d-1}\|u(\omega,\cdot,\cdot)\|^2_{T,*} + \|f(\omega,\cdot,\cdot)\|^2_{L^2(\mathbb{T};H)}.
\end{aligned}$$

In the last step, we have used that $\|b_{N,\varepsilon}(\omega,x)\|_\infty \leq \bar{b}_1 a(\omega,x)$ (see Remark 5.3.2) as well as Ineq. (5.4). On the other hand, we have the lower bound

$$\begin{aligned}
&\|\partial_t u_{N,\varepsilon}(\omega,\cdot,\cdot)\|^2_{L^2(\mathbb{T};H)} + \sup_{t\in\mathbb{T}} \int_{\mathcal{D}} a_{N,\varepsilon}(\omega,x)\nabla u_{N,\varepsilon}(\omega,x,t)\cdot\nabla u_{N,\varepsilon}(\omega,x,t)dx \\
\geq& \|\partial_t u_{N,\varepsilon}(\omega,\cdot,\cdot)\|^2_{L^2(\mathbb{T};H)} + a_{N,\varepsilon,-}(\omega)\sup_{t\in\mathbb{T}}|u_{N,\varepsilon}(\omega,\cdot,t)|^2_{H^1(\mathcal{D})}.
\end{aligned}$$

Since the norms $|\cdot|_{H^1(\mathcal{D})}$ and $\|\cdot\|_{H^1(\mathcal{D})} = \|\cdot\|_V$ are equivalent by the Poincaré inequality, we treat $a_{N,\varepsilon,-}$ once more in the fashion of Theorem 5.2.6 to arrive at the estimate

$$\begin{aligned}
&\|\partial_t u_{N,\varepsilon}(\omega,\cdot,\cdot)\|^2_{L^2(\mathbb{T};H)} + \sup_{t\in\mathbb{T}}\|u_{N,\varepsilon}(\omega,x,t)\|^2_V \\
\leq& C(1+1/a_{N,\varepsilon,-}(\omega))a_{N,\varepsilon,+}(\omega)\Big(\|u_0(\omega,\cdot)\|^2_V + \|u(\omega,\cdot,\cdot)\|^2_{T,*} + \|f(\omega,\cdot,\cdot)\|^2_{L^2(\mathbb{T};H)}\Big).
\end{aligned}$$

The first claim follows with $a_{N,\varepsilon,-}, a_{N,\varepsilon,+} \in L^q(\Omega;\mathbb{R})$ for arbitrary large $q \in [1,\infty)$, Hölder's inequality and Theorem 5.2.6. The proof for $u_{N,\varepsilon,\ell}$ follows analogously. □

We conclude this subsection with the proof of our main result.

Proof of Theorem 5.4.3. For fixed $\omega \in \Omega, t \in \mathbb{T}$, Assumption 5.4.1(v) states that $u_{N,\varepsilon}(\omega,\cdot,t) \in H^{1+\kappa}(\mathcal{T}_i)$ for all $i = 1,\dots,\tau(\omega)$ and some $\kappa \in (0,1]$. Since $d \leq 2$ and the partition elements $\mathcal{T}_i$ are polygons with Lipschitz boundary, [174, Lemma 3.1] yields that $u_{N,\varepsilon}(\omega,\cdot,t) \in H^\vartheta(\mathcal{D})$ for $\vartheta = min(1+\kappa, 3/2-\epsilon)$ and any $\epsilon > 0$. This in turn implies with the fractional Sobolev inequality, e.g. [71, Theorem 6.7], that $u_{N,\varepsilon}(\omega,\cdot,t)$ is continuous on $\overline{\mathcal{D}}$. Thus, the interpolation $\mathcal{I}_\ell u_{N,\varepsilon}(\omega,\cdot,t) \in V_\ell(\omega)$ is well-defined, where the linear interpolation operator with respect to $V_\ell(\omega)$ is given by

$$\mathcal{I}_\ell : C^0(\overline{\mathcal{D}}) \to V_\ell(\omega),\ f \mapsto \sum_{i=1}^{d_\ell(\omega)} f(x_i)v_{i,\omega}.$$

We define the error $\theta_\ell := u_{N,\varepsilon} - u_{N,\varepsilon,\ell}$ and observe that Eqs. (5.12) and (5.9) yield

$$
\begin{aligned}
{}_{V'}\langle \partial_t \theta_\ell(\omega,\cdot,t), v_{\ell,\omega}\rangle_V + B^{N,\varepsilon}_\omega(\theta_\ell(\omega,\cdot,t), v_{\ell,\omega}) &= 0 \\
\theta_\ell(\omega,\cdot,0) &= (u_0 - u_{0,\ell})(\omega,\cdot),
\end{aligned}
$$

for all $v_{\ell,\omega} \in V_\ell(\omega)$. We then test against $v_{\ell,\omega} = \mathcal{I}_\ell u_{N,\varepsilon}(\omega,\cdot,t) - u_{N,\varepsilon,\ell}(\omega,\cdot,t)$ and integrate over $[0,t]$ to obtain

$$
\begin{aligned}
&\frac{1}{2}\|\theta_\ell(\omega,\cdot,t)\|_H^2 + \int_0^t (a_{N,\varepsilon}(\omega,\cdot), \sum_{i=1}^d (\partial_{x_i}(\theta_\ell(\omega,\cdot,z))^2) dz \\
=&\frac{1}{2}\|\theta_\ell(\omega,\cdot,0)\|_H^2 + \int_0^t {}_{V'}\langle \partial_t \theta_\ell(\omega,\cdot,z), (1-\mathcal{I}_\ell) u_{N,\varepsilon}(\omega,\cdot,z)\rangle_V dz \\
&+ \int_0^t B^{N,\varepsilon}_\omega(\theta_\ell(\omega,\cdot,z), (1-\mathcal{I}_\ell) u_{N,\varepsilon}(\omega,\cdot,z)) dz \\
&- \int_0^t (b_{N,\varepsilon}(\omega,\cdot)\cdot\nabla\theta_\ell(\omega,\cdot,z), \theta_\ell(\omega,\cdot,z)) dz \\
=:& \frac{1}{2}\|\theta_\ell(\omega,\cdot,0)\|_H^2 + I + II + III.
\end{aligned}
\tag{5.14}
$$

Lemma 5.4.5 implies that $\partial_t \theta_\ell(\omega,\cdot,\cdot) \in L^2(\mathbb{T}; H)$ and with Young's inequality

$$
\begin{aligned}
I &= \int_0^t (\partial_t \theta_\ell(\omega,\cdot,z), (1-\mathcal{I}_\ell) u_{N,\varepsilon}(\omega,\cdot,z)) dz \\
&\le \frac{1}{2}\int_0^t \overline{h}_\ell^{2\kappa} \|\partial_t \theta_\ell(\omega,\cdot,z)\|_H^2 + \overline{h}_\ell^{-2\kappa} \|(1-\mathcal{I}_\ell) u_{N,\varepsilon}(\omega,\cdot,z))\|_H^2 dz.
\end{aligned}
$$

We then use the Cauchy-Schwarz inequality and Ineq. (5.4) to bound the second term

$$
\begin{aligned}
II &= \int_0^t (a_{N,\varepsilon}(\omega,\cdot), \nabla\theta_\ell(\omega,\cdot,z)\cdot\nabla(1-\mathcal{I}_\ell) u_{N,\varepsilon}(\omega,\cdot,z)) dz \\
&\quad + \int_0^t (b_{N,\varepsilon}(\omega,\cdot)\cdot\nabla\theta_\ell(\omega,\cdot,z), (1-\mathcal{I}_\ell) u_{N,\varepsilon}(\omega,\cdot,z)) dz \\
&\le \int_0^t \Big(a_{N,\varepsilon}(\omega,\cdot)\Big(\sum_{i=1}^d (\partial_{x_i}\theta_\ell(\omega,\cdot,z))^2\Big)^{1/2}, \Big(\sum_{i=1}^d (\partial_{x_i}(1-\mathcal{I}_\ell) u_{N,\varepsilon}(\omega,\cdot,z))^2\Big)^{1/2}\Big) dz \\
&\quad + \int_0^t 2^{d/2-1/2}\Big(\|b_{N,\varepsilon}(\omega,\cdot)\|_\infty \Big(\sum_{i=1}^d (\partial_{x_i}\theta_\ell(\omega,\cdot,z))^2\Big)^{1/2}, |(1-\mathcal{I}_\ell) u_{N,\varepsilon}(\omega,\cdot,z)|\Big) dz,
\end{aligned}
$$

and Young's inequality yields

$$\begin{aligned} II &\leq \int_0^t \frac{1}{4}(a_{N,\varepsilon}(\omega,\cdot),\sum_{i=1}^d(\partial_{x_i}\theta_\ell(\omega,\cdot,z))^2) + a_{N,\varepsilon,+}(\omega)|(1-\mathcal{I}_\ell)u_{N,\varepsilon}(\omega,\cdot,z)|^2_{H^1(\mathcal{D})}dz \\ &\quad + \int_0^t \frac{1}{4}(a_{N,\varepsilon}(\omega,\cdot),\sum_{i=1}^d(\partial_{x_i}\theta_\ell(\omega,\cdot,z))^2) + 2^{d-1}\,\bar{b}_1^2 a_{N,\varepsilon,+}(\omega)\|(1-\mathcal{I}_\ell)u_{N,\varepsilon}(\omega,\cdot,z)\|^2_H dz \\ &\leq \frac{1}{2}\int_0^t (a_{N,\varepsilon}(\omega,\cdot),\sum_{i=1}^d(\partial_{x_i}\theta_\ell(\omega,\cdot,z))^2)dz + Ca_{N,\varepsilon,+}(\omega)\int_0^t \|(1-\mathcal{I}_\ell)u_{N,\varepsilon}(\omega,\cdot,z)\|^2_V dz. \end{aligned}$$

Similarly, we bound the last term by

$$III \leq \frac{1}{4}\int_0^t (a_{N,\varepsilon}(\omega,\cdot),\sum_{i=1}^d(\partial_{x_i}\theta_\ell(\omega,\cdot,z))^2)dz + 2^{d-1}\bar{b}_1\bar{b}_2\int_0^t \|\theta_\ell(\omega,\cdot,z)\|^2_H dz.$$

We now plug in the estimates for $I - III$ in Eq. (5.14) and proceed in the fashion of Theorem 5.2.6 with Grönwalls inequality to arrive at

$$\begin{aligned} \sup_{t\in\mathbb{T}}\|\theta_\ell\|^2_{t,*} \leq C(1+1/a_{N,\varepsilon,-}(\omega))\Big(&\|\theta_\ell(\omega,\cdot,0)\|^2_H + \overline{h}_\ell^{2\kappa}\|\partial_t\theta_\ell(\omega,\cdot,\cdot)\|^2_{L^2(\mathbb{T};H)} \\ &+ \overline{h}_\ell^{-2\kappa}\|(1-\mathcal{I}_\ell)u_{N,\varepsilon}(\omega,\cdot,\cdot)\|^2_{L^2(\mathbb{T};H)} \\ &+ a_{N,\varepsilon,+}(\omega)\|(1-\mathcal{I}_\ell)u_{N,\varepsilon}(\omega,\cdot,\cdot)\|^2_{L^2(\mathbb{T};V)}\Big). \end{aligned}$$

Let $r_\kappa > 2$ be as in Assumption 5.4.1(v) and let $q := (1/2 - 1/r)^{-1}$ for some $r \in (2, \min(r_\kappa, p))$. Taking expectations and using Hölder's inequality yields

$$\begin{aligned} \mathbb{E}(\sup_{t\in\mathbb{T}}\|\theta_\ell\|^2_{t,*})^{1/2} &\leq C\Big(\|\theta_\ell(\cdot,\cdot,0)\|_{L^p(\Omega;H)} + \overline{h}_\ell^{\kappa}\|\partial_t\theta_\ell\|_{L^r(\Omega;L^2(\mathbb{T};H))} \\ &\quad + \overline{h}_\ell^{-\kappa}\|(1-\mathcal{I}_\ell)u_{N,\varepsilon}\|_{L^r(\Omega;L^2(\mathbb{T};H))} \\ &\quad + \|(1-\mathcal{I}_\ell)u_{N,\varepsilon}\|_{L^r(\Omega;L^2(\mathbb{T};V))}\Big) \\ &\leq C\Big(\overline{h}_\ell^{\kappa} + \overline{h}_\ell^{-\kappa}\|(1-\mathcal{I}_\ell)u_{N,\varepsilon}\|_{L^r(\Omega;L^2(\mathbb{T};H))} \\ &\quad + \|(1-\mathcal{I}_\ell)u_{N,\varepsilon}\|_{L^r(\Omega;L^2(\mathbb{T};V))}\Big). \end{aligned} \tag{5.15}$$

For the first inequality, we have used Lemma 5.3.5, the second bound follows by Lemma 5.4.5, the assumption $\|u_0 - u_{0,\ell}\|_{L^p(\Omega;H)} \leq C\overline{h}_\ell$ and since $\overline{h}_\ell \leq 1$.

To bound the interpolation error, we first consider for fixed ω and $i = 1, \ldots, \tau(\omega)$ the partition element $\mathcal{T}_i$. Given that $u_{N,\varepsilon}(\omega,\cdot,t) \in H^{1+\kappa}(\mathcal{T}_i)$ for any $t \in \mathbb{T}$ by Assump-

tion 5.4.1(v), the interpolation error is bounded for $m \in \{0, 1\}$ by

$$\begin{aligned}\|(1-\mathcal{I}_\ell)u_{N,\varepsilon}(\omega,\cdot,t)\|_{H^m(\mathcal{T}_i)} &= \sum_{K\in\mathcal{T}_i} \|(1-\mathcal{I}_\ell)u_{N,\varepsilon}(\omega,\cdot,t)\|_{H^m(K)} \\ &\leq Ch^{1+\kappa-m}\|u_{N,\varepsilon}(\omega,\cdot,t)\|_{H^{1+\kappa}(\mathcal{T}_i)},\end{aligned}$$

with $C = C(\overline{\rho}, m, s, d) > 0$ independent of $\mathcal{T}_i$, see for instance [102, Chapter 8.5]. Note that these estimates hold due to the pathwise construction of $V_\ell(\omega)$ and the piecewise regularity assumption $u_{N,\varepsilon}(\omega,\cdot,t) \in H^{1+\kappa}(\mathcal{T}_i)$, and are not valid for arbitrary FE spaces over $\mathcal{D}$. Since $(1-\mathcal{I}_\ell)u_{N,\varepsilon}(\omega,\cdot,t) \in H^1(\mathcal{D})$ we further estimate

$$\begin{aligned}\|(1-\mathcal{I}_\ell)u_{N,\varepsilon}(\omega,\cdot,t)\|_{H^m(\mathcal{D})} &= \sum_{i=1}^{\tau(\omega)} \|(1-\mathcal{I}_\ell)u_{N,\varepsilon}(\omega,\cdot,t)\|_{H^m(\mathcal{T}_i)} \\ &\leq Ch^{1+\kappa-m}\sum_{i=1}^{\tau(\omega)} \|u_{N,\varepsilon}(\omega,\cdot,t)\|_{H^{1+\kappa}(\mathcal{T}_i)}.\end{aligned}$$

Assumption 5.4.1(vi,v) and Hölder's inequality for $q_\kappa := (1/r - 1/r_\kappa)^{-1} < +\infty$ yields

$$\begin{aligned}&\mathbb{E}(\|(1-\mathcal{I}_\ell)u_{N,\varepsilon}\|_{L^r(\Omega;L^2(\mathbb{T};H^m(\mathcal{D})))}) \\ \leq&\mathbb{E}\Big(\Big(\int_0^T C^2\overline{h}_\ell^{2(1+\kappa-m)}\Big(\sum_{i=1}^{\tau(\omega)}\|u_{N,\varepsilon}(\cdot,\cdot,t)\|_{H^{1+\kappa}(\mathcal{T}_i)}\Big)^2\Big)^{r/2}\Big)^{1/r} \\ \leq&C\overline{h}_\ell^{1+\kappa-m}\mathbb{E}\Big(\Big(\int_0^T \tau^2 \max_{i=1,\dots,\tau}\|u_{N,\varepsilon}(\cdot,\cdot,t)\|^2_{H^{1+\kappa}(\mathcal{T}_i)}\Big)^{r/2}\Big)^{1/r} \\ \leq&C\overline{h}_\ell^{1+\kappa-m}\sqrt{T}\mathbb{E}\Big(\tau^r \max_{i=1,\dots,\tau}\|u_{N,\varepsilon}(\cdot,\cdot,t)\|^r_{H^{1+\kappa}(\mathcal{T}_i)}\Big)^{1/r} \\ \leq&C\overline{h}_\ell^{1+\kappa-m}\mathbb{E}(\tau^{q_\kappa})^{1/q_\kappa}\mathbb{E}\Big(\max_{i=1,\dots,\tau}\|u_{N,\varepsilon}(\cdot,\cdot,t)\|^{r_\kappa}_{H^{1+\kappa}(\mathcal{T}_i)}\Big)^{1/r_\kappa}.\end{aligned}$$

Plugging this estimate into Ineq. (5.15) proves the assertion since $\overline{h}_\ell \leq 1$. □

5.4.2 Temporal discretization

In the remainder of this section, we introduce a stable temporal discretization for the semi-discrete Problem (5.12) and derive the corresponding mean-squared error. To this end, we fix $\omega \in \Omega$ and let $u_{N,\varepsilon,\ell}(\omega,\cdot,\cdot)$ again denote the sample-adapted semi-discrete approximation of $u_{N,\varepsilon}(\omega,\cdot,\cdot)$ from Eq. (5.12). For a fully discrete formulation of Problem (5.12), we consider a time grid $0 = t_0 < t_1 < \cdots < t_n = T$ in $\mathbb{T}$ for some

$n \in \mathbb{N}$. The temporal derivative at t_i is approximated by the backward difference

$$\partial_t u_{N,\varepsilon,\ell}(\omega,\cdot,t_i) \approx \frac{u_{N,\varepsilon,\ell}(\omega,\cdot,t_i) - u_{N,\varepsilon,\ell}(\omega,\cdot,t_{i-1})}{t_i - t_{i-1}}, \quad i = 1,\dots,n.$$

This yields the fully discrete problem to find $(u^{(i)}_{N,\varepsilon,\ell}(\omega,\cdot),\, i = 0,\dots,n) \subset V_\ell(\omega)$ such that for all $v_{\ell,\omega} \in V_\ell(\omega)$ and $i = 1,\dots,n$

$$\begin{aligned} \frac{1}{t_i - t_{i-1}}(u^{(i)}_{N,\varepsilon,\ell}(\omega,\cdot) - u^{(i-1)}_{N,\varepsilon,\ell}(\omega,\cdot), v_{\ell,\omega}) + B^{N,\varepsilon}_\omega(u^{(i)}_{N,\varepsilon,\ell}(\omega,\cdot), v_{\ell,\omega}) &= F_{t_i,\omega}(v_{\ell,\omega}) \\ u^{(0)}_{N,\varepsilon,\ell}(\omega,\cdot) &= u_{0,\ell}(\omega,\cdot). \end{aligned} \tag{5.16}$$

For convenience, we assume the temporal grid is equidistant with fixed time step $\Delta t := t_i - t_{i-1} > 0$. The fully discrete solution is now given by

$$u^{(i)}_{N,\varepsilon,\ell}(\omega,x) = \sum_{j=1}^{d_\ell} c_{i,j}(\omega) v_{j,\omega}(x), \quad i = 1,\dots,n,$$

where the coefficient vector $\mathbf{c_i}(\omega) = (c_{i,1}(\omega),\dots,c_{i,d_\ell}(\omega))^T$ solves the linear system of equations

$$(\mathbf{M}(\omega) + \Delta t \mathbf{A}(\omega))\mathbf{c_i}(\omega) = \Delta t \mathbf{F}(\omega,t_i) + \mathbf{M}(\omega)\mathbf{c_{i-1}}(\omega)$$

in every discrete point t_i. The mass matrix consists of the entries $(\mathbf{M}(\omega))_{jk} := (v_{j,\omega}, v_{k,\omega})$, the stiffness matrix and load vector are given by $(\mathbf{A}(\omega))_{jk} = B^{N,\varepsilon}_\omega(v_{j,\omega}, v_{k,\omega})$ and $(\mathbf{F}(\omega,t_i))_j = F_{t_i,\omega}(v_{j,\omega})$, respectively for $j,k \in \{1,\dots,d_\ell(\omega)\}$, as in the semi-discrete case. The initial vector c_0 consists of the basis coefficients of $u_{0,\ell} \in V_\ell$ with respect to $\{v_{1,\omega},\dots,v_{d_\ell(\omega),\omega}\}$. To extend the discrete solution $(u^{(i)}_{N,\varepsilon,\ell}(\omega,\cdot), i = 0,\dots,n)$ to $\mathbb{T}$, we define the linear interpolation

$$\overline{u}_{N,\varepsilon,\ell}(\omega,\cdot,t) := (u^{(i)}_{N,\varepsilon,\ell}(\omega,\cdot) - u^{(i-1)}_{N,\varepsilon,\ell}(\omega,\cdot))\frac{(t - t_{i-1})}{\Delta t} + u^{(i-1)}_{N,\varepsilon,\ell}(\omega,\cdot), \quad t \in [t_{i-1}, t_i],$$

for $i = 1,\dots,n$ and are, therefore, able to estimate the resulting error with respect to the parabolic norm.

Theorem 5.4.6. *Let Assumption 5.4.1 hold, let $(u^{(i)}_{N,\varepsilon,\ell}, i = 0,\dots,n)$ be the fully discrete sample-adapted approximation of $u_{N,\varepsilon}$ as in Eq.* (5.16) *and let $\overline{u}_{N,\varepsilon,\ell}$ be the linear interpolation in $\mathbb{T}$. Then,*

$$\mathbb{E}\Big(\sup_{t\in\mathbb{T}} \|u_{N,\varepsilon,\ell} - \overline{u}_{N,\varepsilon,\ell}\|^2_{*,t}\Big)^{1/2} \le C\Delta t.$$

Proof. We start by investigating the temporal regularity of $u_{N,\varepsilon,\ell}$. For fixed $\omega \in \Omega$

and $0 \leq t_{i-1} < t_i \leq T$ note that $w_i(\omega, \cdot, t) := u_{N,\varepsilon,\ell}(\omega, \cdot, t) - u_{N,\varepsilon,\ell}(\omega, \cdot, t_{i-1})$ solves the variational problem

$$ {}_{V'}\langle \partial_t w_i(\omega, \cdot, t), v_{\ell,\omega} \rangle_V + B^{N,\varepsilon}_\omega(w_i(\omega, \cdot, t), v_{\ell,\omega}) = {}_{V'}\langle f(\omega, \cdot, t) - f(\omega, \cdot, t_{i-1}), v_{\ell,\omega} \rangle_V $$

for $t \in [t_{i-1}, t_i]$ and $v_{\ell,\omega} \in V_\ell(\omega)$ with initial condition $w(\omega, \cdot, t_{i-1}) = 0$. Therefore, in the fashion of Theorem 5.2.6, we obtain the pathwise parabolic estimate

$$\begin{aligned}
&\sup_{t\in[t_{i-1},t_i]} \|w_i(\omega, \cdot, t)\|_H^2 + \int_{t_{i-1}}^{t} |w_i(\omega, \cdot, z)|^2_{H^1(\mathcal{D})} dz \\
\leq & C(1 + 1/a_{N,\varepsilon,-}(\omega)) \int_{t_{i-1}}^{t_i} \|f(\omega, \cdot, z) - f(\omega, \cdot, t_{i-1})\|_H^2 dz \\
= & C(1 + 1/a_{N,\varepsilon,-}(\omega)) \int_{t_{i-1}}^{t_i} \| \int_{t_{i-1}}^{z} \partial_t f(\omega, \cdot, \widetilde{z}) d\widetilde{z}\|_H^2 dz \\
= & C(1 + 1/a_{N,\varepsilon,-}(\omega)) \int_{t_{i-1}}^{t_i} \| \int_{t_{i-1}}^{t_i} \mathbf{1}_{[t_{i-1},t]}(\widetilde{z}) \partial_t f(\omega, \cdot, \widetilde{z}) d\widetilde{z}\|_H^2 dz \\
\leq & C(1 + 1/a_{N,\varepsilon,-}(\omega)) \int_{t_{i-1}}^{t_i} (z - t_{i-1}) dz \int_{t_{i-1}}^{t_i} \|\partial_t f(\omega, \cdot, z)\|_H^2 dz \\
= & C(1 + 1/a_{N,\varepsilon,-}(\omega)) \frac{\Delta t^2}{2} \|\partial_t f(\omega, \cdot, \cdot)\|^2_{L^2([t_i,t_{i-1}];H)}.
\end{aligned} \tag{5.17}$$

For the first identity we have used Lemma 5.2.2, the second estimate follows with Hölder's inequality. Now let $\overline{\overline{u}}_{N,\varepsilon,\ell}$ be the temporal linear interpolation of the semi-discrete solution $u_{N,\varepsilon,\ell}$ at the nodes $t_0, \ldots, t_n$ and consider the splitting

$$\begin{aligned}
\mathbb{E}\Big(\sup_{t\in\mathbb{T}} \|u_{N,\varepsilon,\ell} - \overline{u}_{N,\varepsilon,\ell}\|^2_{*,t} \Big)^{1/2} \leq & \mathbb{E}\Big(\sup_{t\in\mathbb{T}} \|u_{N,\varepsilon,\ell} - \overline{\overline{u}}_{N,\varepsilon,\ell}\|^2_{*,t} \Big)^{1/2} \\
& + \mathbb{E}\Big(\sup_{t\in\mathbb{T}} \|\overline{\overline{u}}_{N,\varepsilon,\ell} - \overline{u}_{N,\varepsilon,\ell}\|^2_{*,t} \Big)^{1/2} \\
=: & \, I + II.
\end{aligned}$$

By Ineq. (5.17) it follows that

$$\begin{aligned}
\sup_{t\in\mathbb{T}} \|u_{N,\varepsilon,\ell} - \overline{\overline{u}}_{N,\varepsilon,\ell}\|^2_{*,t} &\leq \max_{i=1,\ldots,n} \sup_{t\in[t_{i-1},t_i]} \|w_i(\omega, \cdot, t)\|_H^2 + 2 \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|w_i(\omega, \cdot, t)\|^2_{H^1(\mathcal{D})} dt \\
&\leq C(1 + 1/a_{N,\varepsilon,-}(\omega)) \frac{\Delta t^2}{2} \|\partial_t f(\omega, \cdot, \cdot)\|^2_{L^2(\mathbb{T};H)}.
\end{aligned}$$

Thus, by Assumption 5.4.1, Hölder's inequality and Lemma 5.3.5 with $q = (1/2 - 1/p)^{-1}$

$$I \leq C\Delta t (1 + \|1/a_{N,\varepsilon,-}\|_{L^q(\Omega;\mathbb{R})}) \|\partial_t f\|_{L^p(\Omega;L^2(\mathbb{T};H)} \leq C\Delta t.$$

Now let $\theta^{(i)}(\omega,\cdot) := u_{N,\varepsilon,\ell}(\omega,\cdot,t_i) - u^{(i)}_{N,\varepsilon,\ell}(\omega,\cdot)$ denote the pathwise time discretization error at t_i. For any $t \in [t_{i-1}, t_i]$, we observe that $(\overline{u}_{N,\varepsilon,\ell} - \overline{\overline{u}}_{N,\varepsilon,\ell})(\cdot,\cdot,t)$ is a convex combination of θ_i and θ_{i-1}, and therefore

$$II \leq \mathbb{E}(\max_{i=1,\dots,n} \|\theta^{(i)}\|_H^2 + \Delta t \sum_{j=1}^{i} |\theta^{(j)}|^2_{H^1(\mathcal{D})})^{1/2}. \tag{5.18}$$

Hence, it is sufficient to control the error $\theta^{(i)}$ at each node t_i. Combining Eq. (5.16) and Eq. (5.12) yields for $i = 1, \dots, n$

$$\begin{aligned}
&{}_{V'}\langle \theta^{(i)}(\omega,\cdot) - \theta^{(i-1)}(\omega,\cdot), v_{\ell,\omega}\rangle_V + \int_{t_{i-1}}^{t_i} B^{N,\varepsilon}_\omega(\theta^{(i)}(\omega,\cdot), v_{\ell,\omega})dt \\
=&\int_{t_{i-1}}^{t_i} B^{N,\varepsilon}_\omega(u_{N,\varepsilon,\ell}(\omega,\cdot,t_i) - u_{N,\varepsilon,\ell}(\omega,\cdot,t), v_{\ell,\omega}) + {}_{V'}\langle f(\omega,\cdot,t) - f(\omega,\cdot,t_i), v_{\ell,\omega}\rangle_V dt \\
:=&\int_{t_{i-1}}^{t_i} {}_{V'}\langle \overline{f}_i(\omega,\cdot,t), v_{\ell,\omega}\rangle_V dt,
\end{aligned}$$

and initial condition $\theta^{(0)}(\omega,\cdot) = u_{N,\varepsilon,\ell}(\omega,\cdot,0) - u^{(0)}_{N,\varepsilon,\ell}(\omega,\cdot) = 0$. We test against $v_{\ell,\omega} = \theta^{(i)}(\omega,\cdot)$, sum over i and use the discrete Grönwall inequality to obtain (as in Theorem 5.2.6) the discrete estimate

$$\begin{aligned}
&\max_{i=1,\dots,n} \|\theta^{(i)}(\omega,\cdot)\|_H^2 + \Delta t \sum_{j=1}^{i} |\theta^{(j)}(\omega,\cdot)|^2_{H^1(\mathcal{D})} \\
\leq& C(1 + 1/a_{N,\varepsilon,-}(\omega)) \sum_{i=1}^{n} \|\overline{f}_i(\omega,\cdot,\cdot)\|^2_{L^2([t_i,t_{i-1}];V')} \\
\leq& C(1 + 1/a_{N,\varepsilon,-}(\omega)) \Big(a_{N,\varepsilon,+}(\omega)^2 \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |u_{N,\varepsilon,\ell}(\cdot,\cdot,t) - u_{N,\varepsilon,\ell}(\cdot,\cdot,t_i)|^2_{H^1(\mathcal{D})} dt \\
&\quad + \sum_{i=1}^{n} \|f(\omega,\cdot,t) - f(\omega,\cdot,t_1)\|_H^2 dt \Big).
\end{aligned}$$

Proceeding as for Ineq. (5.17), this implies

$$\begin{aligned}
&\max_{i=1,\dots,n} \|\theta^{(i)}(\omega,\cdot)\|_H^2 + \Delta t \sum_{j=1}^{i} |\theta^{(j)}(\omega,\cdot)|^2_{H^1(\mathcal{D})} \\
\leq& C(1 + 1/a_{N,\varepsilon,-}(\omega)) a_{N,\varepsilon,+}(\omega)^2 \Delta t^2 \|\partial_t f(\omega,\cdot,\cdot)\|^2_{L^2(\mathbb{T};H)}.
\end{aligned}$$

We use Assumption 5.4.1, Hölder's inequality and Lemma 5.3.5 for the bound

$$\mathbb{E}\Big(\max_{i=1,\dots,n} \|\theta^{(i)}\|_H^2 + \Delta t \sum_{j=1}^{i} |\theta^{(j)}|^2_{H^1(\mathcal{D})} \Big) \leq C\Delta t^2 \|\partial_t f\|^2_{L^p(\Omega;L^2(\mathbb{T};H))} \leq C\Delta t^2,$$

and the claim finally follows by Ineq. (5.18). □

To conclude this section, we record a bound on the overall approximation error, which is an immediate consequence of Theorems 5.3.8, 5.4.3 and 5.4.6.

Corollary 5.4.7. *Let Assumption 5.4.1 hold, let $\overline{u}_{N,\varepsilon,\ell}$ be the linear interpolation of the fully discrete approximation of $(u^{(i)}_{N,\varepsilon}, i = 0, \dots, n)$ and let $u_{0,\ell}$ satisfy the bounds $\|u_0 - u_{0,\ell}\|_{L^p(\Omega;H)} \leq C\overline{h}_\ell$ and $\|u_{0,\ell}\|_{L^p(\Omega;V)} \leq C\|u_0\|_{L^p(\Omega;V)}$. Then,*

$$\mathbb{E}\Big(\sup_{t\in\mathbb{T}} \|u - \overline{u}_{N,\varepsilon,\ell}\|^2_{*,t}\Big)^{1/2} \leq C\Big(\Xi_N^{1/2} + \varepsilon^{1/s} + \overline{h}_\ell^{\kappa} + \Delta t\Big).$$

5.5 Numerical experiments

In all of our numerical experiments we measure the root mean-squared error

$$RMSE := \mathbb{E}(\|u(\cdot,\cdot,T) - \overline{u}_{N,\varepsilon,\ell}(\cdot,\cdot,T)\|_V^2)^{1/2}.$$

For each given FE discretization parameter $\overline{h}_\ell$, we align the error contributions of N, ε and Δt such that $\Xi_N^{1/2} \simeq \varepsilon^{1/s} \simeq \Delta t \simeq \overline{h}_\ell$. Hence, the dominant source of error is the spatial discretization and Corollary 5.4.7 yields $RMSE \leq C\overline{h}_\ell^{\kappa}$. This allows us to measure the value of κ from Assumption 5.4.1 in the estimated error in our examples. While the choices of Δt and ε are usually straightforward for given $\overline{h}_\ell$, we refer to [31, Remark 5.3], where we describe how to achieve $\Xi_N^{1/2} \simeq \overline{h}_\ell$ for common examples of covariance operators Q. To emphasize the advantage of the sample-adapted FE algorithm introduced in Section 5.4, we also repeat all experiments with a standard FE approach and compare the resulting errors. For the non-adapted FE algorithm, we use for a given triangulation diameter h_ℓ the same approximation parameters $\Delta t, N$ and ε as for the corresponding sample-adapted method. This ensures that the weaker performance of this non-adapted method is due to the mismatch between FE triangulation and the discontinuities of a and b. We approximate the entries of the stiffness matrix for both FE approaches by the midpoint rule on each triangle. If the triangulation is aligned to the discontinuities in a and b, this adds an additional term of order $\overline{h}_\ell$ to the error estimate in Corollary 5.4.7, see for instance [54, Proposition 3.13]. Thus, the bias stemming from the midpoint rule does not dominate the overall order of convergence in the sample-adapted algorithm. In the other case, we cannot quantify the quadrature error due to the discontinuities on certain triangles but suggest based on our experimental observations an error of order $\overline{h}_\ell^{1/2}$.

5.5.1 Numerical examples in 1D

For all test scenarios in this subsection, we consider the advection-diffusion Problem (5.1) in the domain $\mathcal{D} = (0,1)$, with $T = 1$, $u_0(x) = \sin(\pi x)/10$ and source term $f \equiv 1$. The continuous part of the diffusion coefficient a is given by $\overline{a} \equiv 0$ and $\Phi(w) = \exp(w)$, where the Gaussian field W is characterized by the *Matérn covariance operator*

$$Q_M : H \to H, \quad [Q_M\varphi](y) := \int_{\mathcal{D}} \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \Big(\sqrt{2\nu}\frac{|x-y|}{\rho}\Big)^{\nu} K_\nu\Big(\sqrt{2\nu}\frac{|x-y|}{\rho}\Big)\varphi(x)dx$$

for $\varphi \in H$, with smoothness parameter $\nu > 0$, variance $\sigma^2 > 0$ and correlation length $\rho > 0$. Above, Γ denotes the Gamma function and K_ν is the modified Bessel function of the second kind with ν degrees of freedom. It is known that W is mean square differentiable if $\nu > 1$ and, moreover, the paths of W are almost surely in $C^{\lfloor\nu\rfloor,\varrho}(\mathcal{D};\mathbb{R})$ with $\varrho < \nu - \lfloor\nu\rfloor$ for any $\nu \geq 1/2$, see [96, Section 2.2]. The spectral basis of Q_M may be efficiently approximated by Nyström's method, see for instance [193]. In our experiments, we use the covariance parameters $\nu = 3/2$, $\sigma^2 = 1$ and $\rho = 0.05$.

The number of partition elements is given by $\tau = \mathcal{P} + 2$, where $\mathcal{P}$ is Poisson-distributed with intensity parameter 5. On average, this splits the domain in 7 disjoint intervals and the diffusion coefficient has almost surely at least one discontinuity. The position of each jump is sampled according to the measure λ, which we set as the Lebesgue measure λ_L on $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$. More precisely, let $(\widetilde{x}_i, i \in \mathbb{N})$ be a i.i.d. sequence of $\mathcal{U}(\mathcal{D})$-random variables that are independent of τ. We take the first $\tau - 1$ points of this sequence, order them increasingly and denote the ordered subset by $0 < x_1 < \cdots < x_{\tau-1} < 1$. This generates the random partition $\mathcal{T} = \{(0,x_1),(x_1,x_2),\ldots,(x_{\tau-1},1)\}$ for each realization of τ. Conditional on the random variable $\tau = \mathcal{P} + 2 \geq 2$, the distribution of each x_i for $i = 1,\ldots,\tau-1$ is then given by

$$\mathbb{P}(x_i \leq c\,|\tau) = \frac{(\tau-1)!}{(\tau-i)!(i-1)!} c^{\tau-i}(1-c)^{i-1}, \quad c \in \mathcal{D} = (0,1).$$

With the law of iterated expectations, this can be utilized to derive further statistics, such as the average interval width of $\mathcal{T}$ given by

$$\mathbb{E}(\mathbb{E}(x_1|\tau)) = \mathbb{E}(1/\tau) = \sum_{k=0}^{\infty} \frac{5^k e^{-5}}{k!}\frac{1}{k+2} \approx 0.1603$$

with corresponding variance $\mathbb{E}(1/(\tau+1)) - \mathbb{E}(1/\tau)^2 \approx 0.1102$. This also shows that increasing the Poisson parameter in $\mathcal{P}$ resp. τ would yield a longer average computa-

tional time, as more and therefore smaller intervals would be sampled. The order of spatial convergence of the sample-adapted FE scheme on the other hand remains unaffected of the distribution of $\mathcal{T}$. In the subsequent examples we vary the distribution of the jump heights P_i and use the advection coefficient given by

$$b(\omega, x) := 2\sin(2\pi x)a(\omega, x), \quad \omega \in \Omega,\ x \in \mathcal{D}.$$

Note that we did not impose an upper deterministic bound $\bar{b}_2$ on b.

To obtain path-wise approximations of the samples $u_{N,\varepsilon}(\omega, \cdot, \cdot)$, we use non-adapted and sample-adapted piecewise linear elements and compare both approaches. The FE discretization parameter is given by $\overline{h}_\ell = 2^{-\ell}/4$ and we consider the range $\ell = 0, \dots, 6$. We approximate the reference solution u for each sample using sample-adapted FE and set $u_{ref} := \overline{u}_{N_8,\varepsilon_8,8}(\cdot, \cdot, T)$, where we choose $\Delta t_8 \simeq \Xi_{N_8}^{1/2} \simeq \varepsilon_8^{1/2} \simeq 2^{-10}$. The RMSE is estimated by averaging 100 samples of $\|u_{ref} - \overline{u}_{N,\varepsilon,\ell}(\cdot, \cdot, T)\|_V^2$ for $\ell = 0, \dots, 6$. To subtract adapted\non-adapted approximations from the reference solution u_{ref}, we use a fixed grid with $2^{10} + 1$ equally spaced points in $\mathcal{D}$, thus the error stemming from interpolation\prolongation may be neglected.

In our first numerical example, we use i.i.d uniformly distributed jump heights $P_i \sim \mathcal{U}([0, 5])$, hence the sampling error ε is equal to zero and may be omitted for this scenario. A sample of the corresponding PDE coefficients with illustrated adapted\non-adapted FE basis and of the corresponding solution is given in Fig. 5.1. As expected, Fig. 5.1 shows that the sample-adapted FE approximation converges with rate $\kappa \approx 1$, whereas the non-adapted FE method only has rate ≈ 0.55.

In Remark 5.4.2, we stated that the condition $2\alpha \leq \beta$ on the decay of the eigenvalues of Q entails mean square differentiability of W, and thus does not affect the convergence rate of order κ in the sample-adapted method. We suggested that this rate will deteriorate if the paths of W are only Hölder continuous with exponent $\varrho < 1$. To illustrate this, we repeat the first experiment with a changed covariance operator. We now consider the *Brownian motion covariance operator*

$$Q_{BM} : H \to H, \quad [Q_{BM}\varphi](y) := \int_{\mathcal{D}} min(x, y)\varphi(x)dx \quad \text{for } \varphi \in H,$$

with eigenbasis given by $\eta_i = (8/((2i+1)\pi))^2$ and $e_i(x) = \sin((2i+1)\pi x/2)$ for $i \in \mathbb{N}_0$. The paths of W generated with Q_{BM} are Hölder-continuous with $\varrho = 1/2 - \epsilon$ for any $\epsilon > 0$ because $\beta = 1 - \epsilon$ and $\alpha = 1$. A sample of the coefficients and the approximated solution is given in Fig. 5.2. The sample-adapted RMSE is smaller than the non-adapted curve, but both errors now decay at rate $\kappa \approx 0.5$ due to the lack of

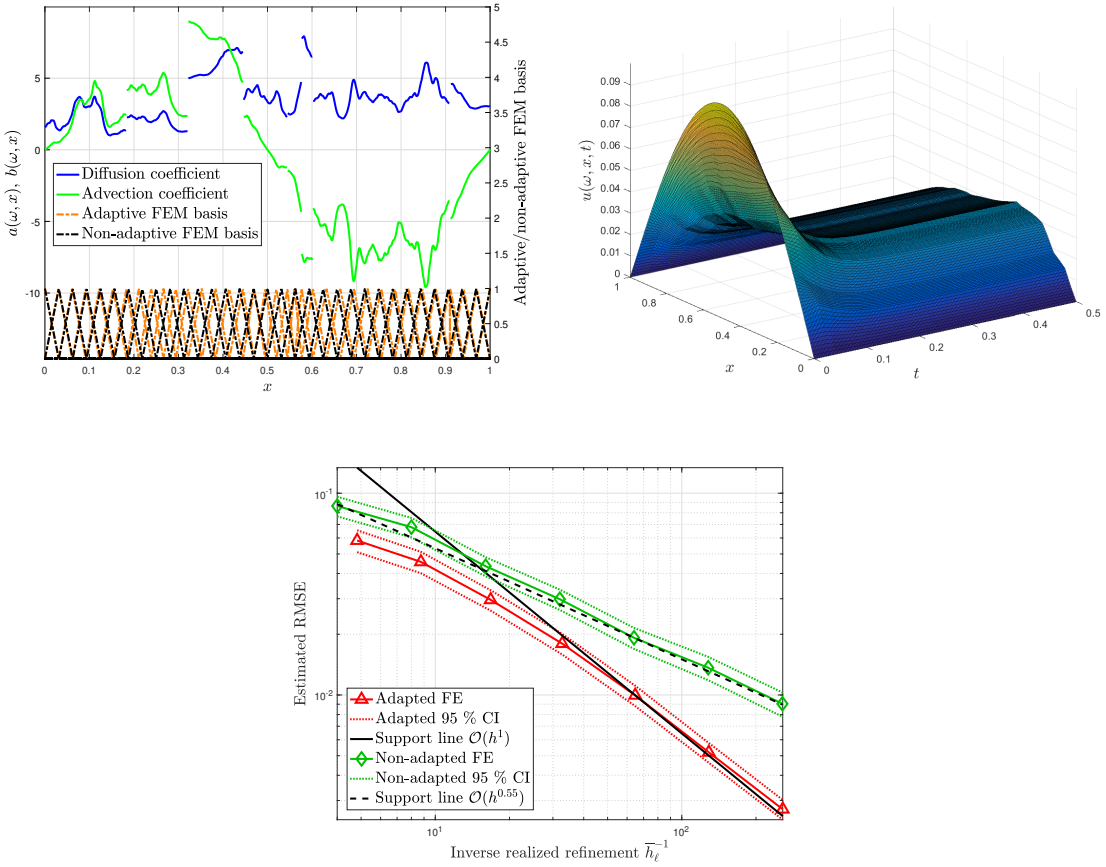

Figure 5.1 First numerical example in 1D with Matérn covariance operator and uniformly distributed jumps. Top left: Diffusion/advection coefficient and adapted/non-adapted FE basis, top right: FE solution corresponding to the sample on the left and the given sample-adapted FE basis, bottom: estimated RMSE vs. inverse spatial refinement size.

(piecewise) spatial regularity of a and b. In general, given that $\varrho \leq 1/2$, it is of course highly problem-dependent if the sample-adapted resp. non-adapted FE algorithm is favorable.

For the last one-dimensional example, we use again the Matérn covariance operator Q_M and consider a more involved distribution of jump heights which entails a positive sampling bias $\varepsilon > 0$. The jump heights P_i now follow a *generalized inverse Gaussian* (GIG) distribution with density

$$f_{GIG}(x) = \frac{(\psi/\chi)^{\lambda/2}}{2K_\lambda(\sqrt{\psi\chi})} x^{\lambda-1} \exp\Big(-\frac{1}{2}(\psi x + \chi x^{-1})\Big), \quad x > 0,$$

and parameters $\chi, \psi > 0$, $\lambda \in \mathbb{R}$, see [19]. Unbiased sampling from this distribution may be rather expensive, hence we generate approximations $\widetilde{P}_i$ of P_i by a Fourier inversion technique which guarantees that $\mathbb{E}(|\widetilde{P}_i - P_i|^2) \leq \varepsilon$ for any desired $\varepsilon > 0$. This

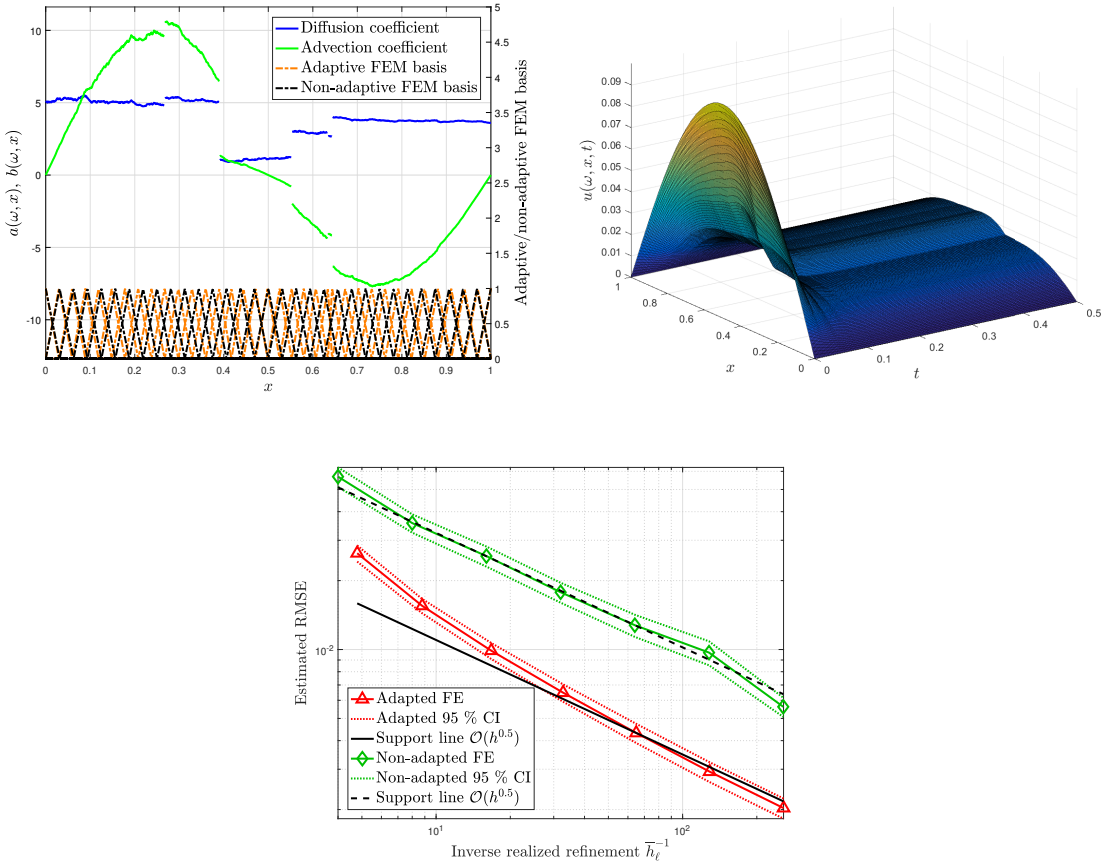

Figure 5.2 Second numerical example in 1D with Brownian motion covariance operator and uniformly distributed jumps. Top left: Diffusion/advection coefficient and adapted/non-adapted FE basis, top right: FE solution corresponding to the sample on the left and the given sample-adapted FE basis, bottom: estimated RMSE vs. inverse spatial refinement size.

allows us to adjust the sampling bias $\varepsilon > 0$ with $\overline{h}_\ell$ (and the corresponding Δt and Ξ_N) for any $\ell \in \mathbb{N}_0$. Details on the Fourier inversion algorithm, the sampling of GIG distributions and the corresponding $L^2(\Omega;\mathbb{R})$-error may be found in [30]. The GIG parameters are set as $\psi = 0.25, \chi = 9$ and $\overline{\lambda} = -1$, the resulting density f_{GIG} and a sample of the coefficients are given in Fig. 5.3. As wee see in Fig. 5.3, the RMSE curves behave similarly as in the first example in this section. The sample-adapted algorithm converges again with rate $\kappa \approx 1$, meaning the sampling error of the GIG jump heights is aligned to the remaining error contributions. Not surprisingly, the non-adapted method again converges with a rate of 0.55.

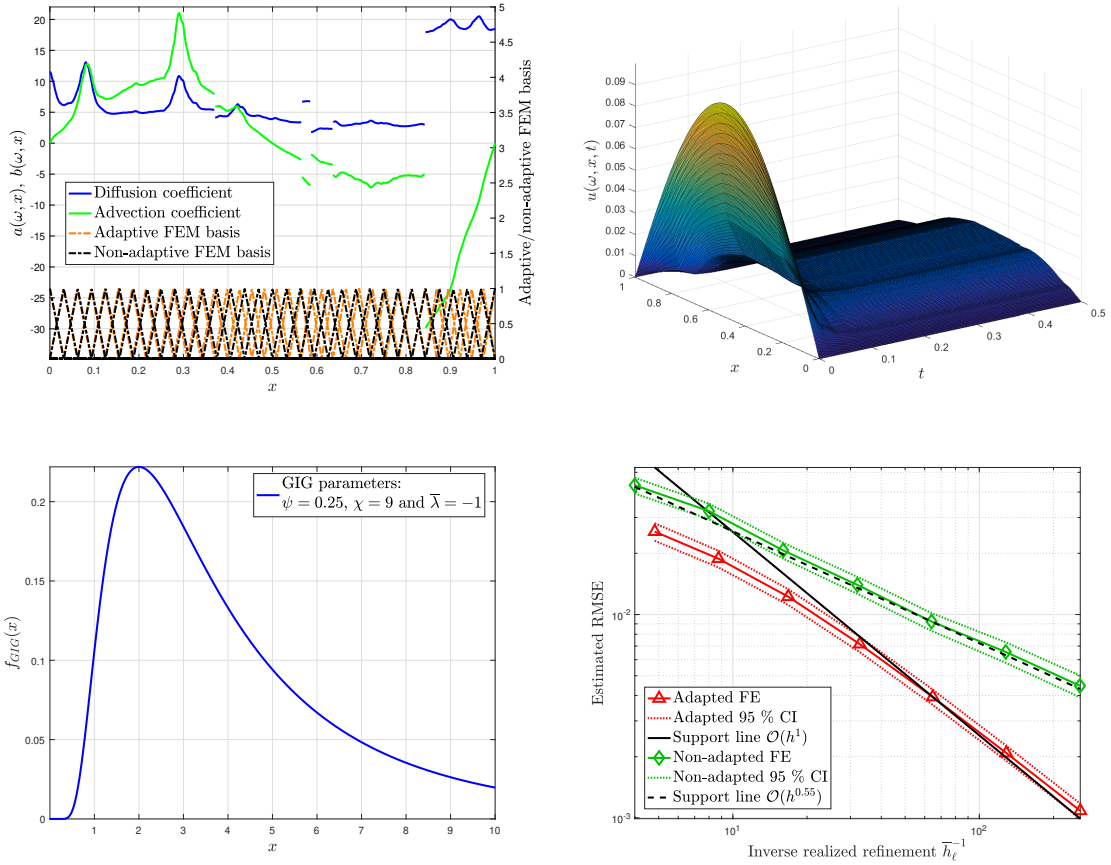

Figure 5.3 Third numerical example in 1D with Matérn covariance operator and GIG distributed jumps. Top left: Diffusion/advection coefficient and adapted/non-adapted FE basis, top right: FE solution corresponding to the sample on the left and the given sample-adapted FE basis, bottom left: GIG density function and parameters, bottom right: estimated RMSE vs. inverse spatial refinement size.

5.5.2 Numerical examples in 2D

In two spatial dimensions, we work on $\mathcal{D} = (0,1)^2$ with $T = 1$, initial data $u_0(x_1, x_2) = \frac{1}{100}\sin(\pi x_1)\sin(\pi x_2)$, source term $f \equiv 1$ and assume again that $\bar{a} \equiv 0$. The Gaussian part of a is given by the Karhunen-Loève expansion

$$W(x) = \sum_{i \in \mathbb{N}} \sqrt{\eta_i} e_i(x) Z_i, \quad x \in \mathcal{D}, \quad Z_i \overset{i.i.d.}{\sim} \mathcal{N}(0,1),$$

with spectral basis given by $\eta_i := \sigma^2 \exp(-\pi^2 i^2 \rho^2)$ and $e_i(x) := \sin(\pi i x_1)\sin(\pi i x_2)$ for $i \in \mathbb{N}$. Again, the parameters $\rho, \sigma^2 > 0$ denote the correlation length and total variance

of W respectively. It can be shown that the above eigenpairs solve the integral equation

$$\sigma^2 \int_{\mathcal{D}} \frac{1}{4\pi t} \exp\Big(- \frac{-\|x-y\|_2^2}{2\rho^2}\Big) e_i(y) dy = \eta_i e_i(x), \quad i \in \mathbb{N},$$

with $e_i = 0$ on $\partial\mathcal{D}$, see [99]. Compared with a Gaussian field generated by a squared exponential covariance operator, this field shows a very similar behavior, except that it is zero on the boundary. It, further, has the advantage, that all expressions are available in closed form and we forgo the numerical approximation of the eigenbasis. The eigenvalues decay exponentially fast with respect to i, hence Assumption 5.4.1 is fulfilled and we use the parameters $\sigma^2 = 0.25$ and $\rho = 0.02$ for all experiments in this section. As before, we consider a log-Gaussian random field, meaning $\Phi(w) = \exp(w)$. To illustrate the flexibility of a jump-diffusion coefficient a as in Def. 5.3.1, we vary the random partitioning of $\mathcal{D}$ for each example and give a detailed description below. We set the spatial discretization parameter to $\overline{h}_\ell = h_\ell = \frac{2}{5}2^{-\ell}$ and consider the cases $\ell = 0, \dots, 5$. To estimate the RMSE, we sample similar to the one-dimensional case the reference solution $u_{ref} := \overline{u}_{N_7,\varepsilon_7,7}(\cdot,\cdot,T)$ with $\Delta t_7 \simeq \Xi_{N_7}^{1/2} \simeq \varepsilon_7^{1/2} \simeq \frac{2}{5}2^{-7}$ and average again 100 independent samples of $\|u_{ref} - \overline{u}_{N,\varepsilon,\ell}(\cdot,\cdot,T)\|_V^2$. For interpolation/prolongation we use a reference grid with $(2^8+1) \times (2^8+1)$ equally spaced points in $\mathcal{D}$. The advection coefficient is in each scenario given by

$$b(\omega, x, y) = 5 \sin(\pi x) \sin(\pi y) a(\omega, x, y) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \omega \in \Omega,\ x \in \mathcal{D}.$$

Again, we did not need to impose an upper bound $\overline{b}_2$ on b for the simulation.

In our first 2D example, we aim to imitate the structure of a heterogeneous medium. For this, we divide the domain by two horizontal and vertical lines. We assume that the horizontal resp. vertical lines do not intersect each other and thus obtain $\tau \equiv 9$. The remaining four intersection points of the lines in $\mathcal{D}$ are uniformly distributed in $(0.2, 0.8)^2$. This is realized by setting λ as the two-dimensional Lebesgue-measure restricted to $(0.2, 0.8)^2 \subset \mathcal{D}$. Finally, we assign i.i.d. jump heights $P_i \sim \mathcal{U}(0, 10)$ to each partition element $\mathcal{T}_i$. Fig. 5.4 shows a sample of the advection- and diffusion coefficient for the heterogeneous medium together with the associated (sample-adapted) FE approximation of u. As before, the sample-adapted method is advantageous and converges with rate $\kappa \approx 1$, suggesting that the choice of P in this example ensures (almost) piecewise H^2-regularity on the partition elements. If we use non-adapted FE, we may still recover a convergence rate of 0.66, which is actually slightly better than the expected rate of 0.5.

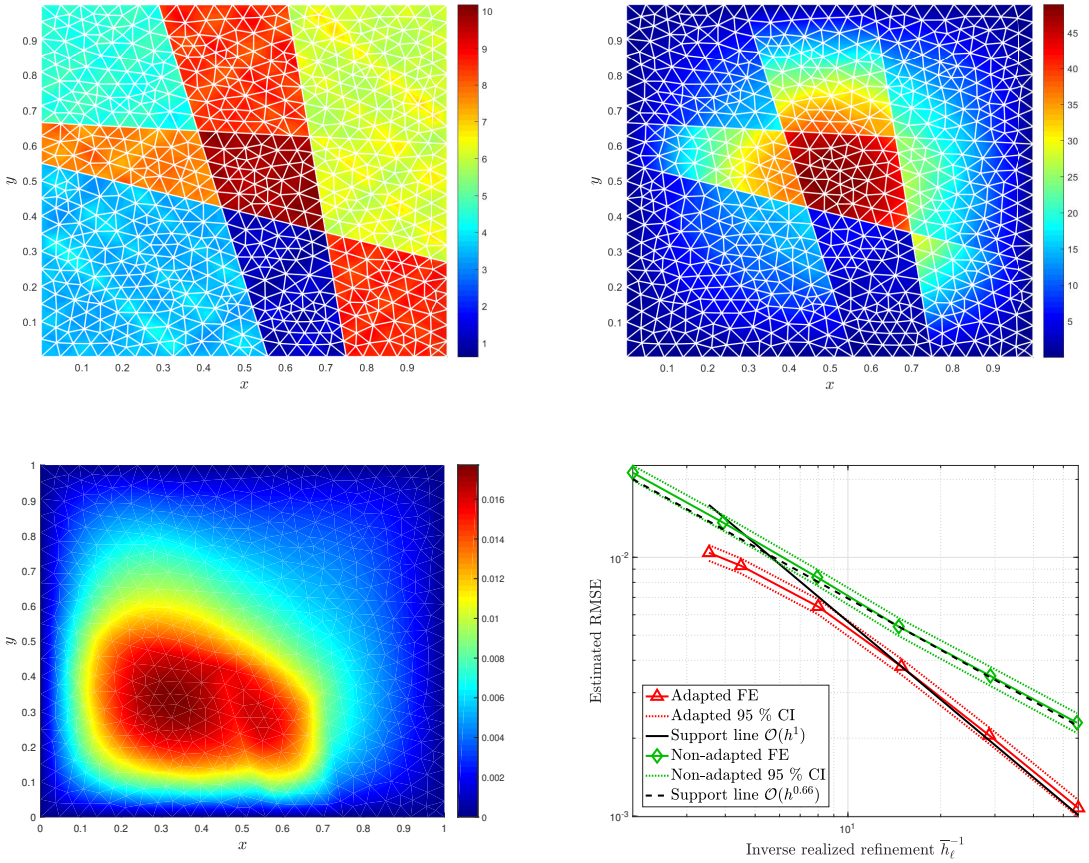

Figure 5.4 First numerical example in 2D (heterogeneous medium). Top left: sample of the diffusion coefficient and sample-adapted triangulation, top right: sample of the advection coefficient with sample-adapted triangulation, bottom left: FE solution at T corresponding to the samples and triangulations on the top, bottom right: estimated RMSE vs. inverse spatial refinement size.

The comparably high value of κ in the previous example may be due to the fact that the jump discontinuities are (on average) not very steep, and in addition all partition elements are convex by construction. As another test for the sample-adapted method, we now investigate an example with reentrant corners and comparably high jumps at the interfaces. To this end, we sample one $\mathcal{U}([0.4, 0.6]^2)$-distributed center point in $\mathcal{D}$ and assign a horizontal and vertical strip with a preset width of 0.3 to this point. Hence, $\tau \equiv 5$ and λ may be interpreted as the Lebesgue-measure restricted to $[0.4, 0.6]^2$ that controls for the position of the center point. This results in a cross-shaped polygon with four reentrant corners and random center in $\mathcal{D}$, see Fig. 5.5). Within the cross we assign a jump height of $P_i = 0$, in the remaining four quadrangles we set $P_i = 20$. A sample of the jump-diffusion- and advection coefficient with corresponding sample-adapted FE solution is shown in Fig. 5.5.

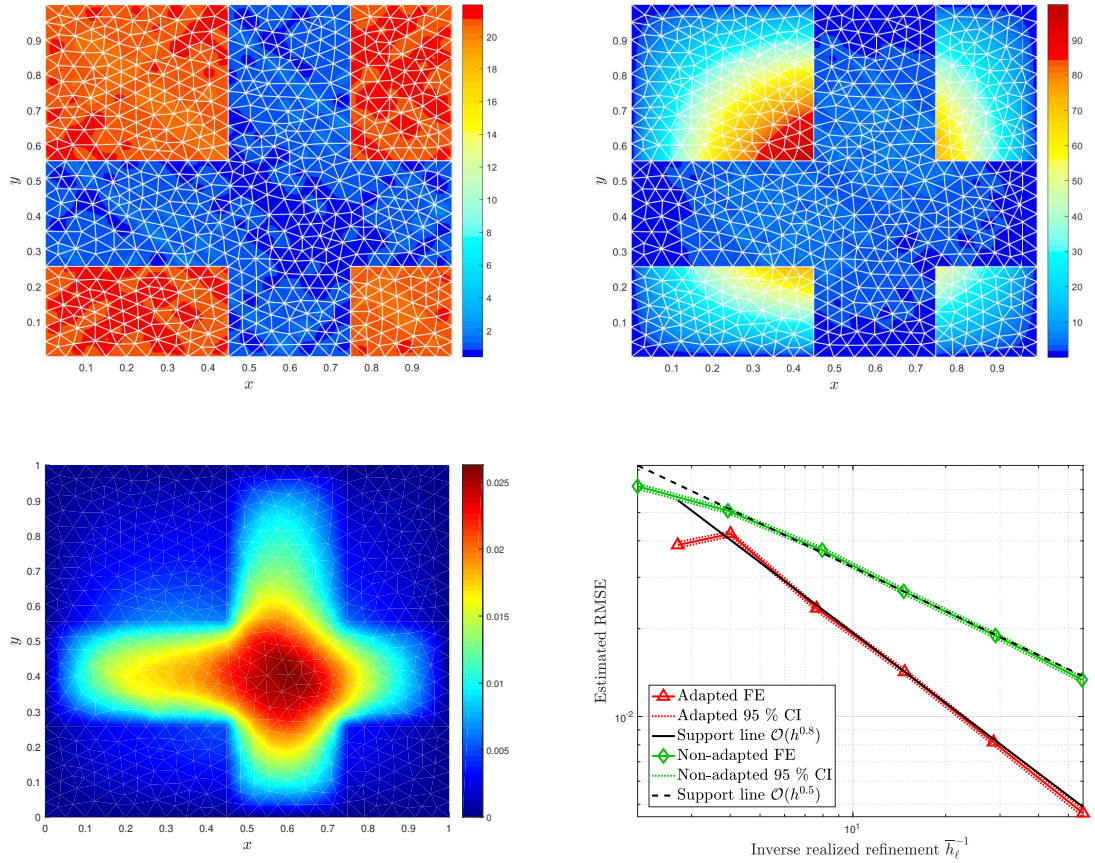

Figure 5.5 Second numerical example in 2D (cross-shaped inclusion). Top left: sample of the diffusion coefficient and sample-adapted triangulation, top right: sample of the advection coefficient with sample-adapted triangulation, bottom left: FE solution at T corresponding to the samples and triangulations on the top, bottom right: estimated RMSE vs. inverse spatial refinement size.

The convergence rate of the sample-adapted FE method now deteriorates, but we obtain a rate of $\kappa \approx 0.8$. This is due to the fact that we can only expect piecewise $H^{5/3}$-regularity of u in the cross-shaped partition element, see [102, Chapter 9.1]. We suggest that a better rate of convergence may be achieved by *adaptive h-finite element methods* (see [187]), i.e. by refining the sample-adapted mesh in the reentrant corners. A thorough analysis of this approach for general random geometries is, however, rather involved and subject to further research. To conclude, we remark that the RMSE-curve of the non-adapted FE discretization in Fig. 5.5 shows a decay rate of only 0.5, hence the sample-adapted approach might also be beneficial in cases where we expect lower piecewise spatial regularity.

6 A multilevel Monte Carlo algorithm for parabolic advection-diffusion problems with discontinuous coefficients

Andreas Stein and Andrea Barth

First published in "The Proceedings of the Thirteenth International Conference on Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing (MCQMC 2018)", 2020, Springer, pp.445-466.

URL:

https://link.springer.com/chapter/10.1007/978-3-030-43465-6_22.

Abstract: The Richards' equation is a model for flow of water in unsaturated soils. The coefficients of this (nonlinear) partial differential equation describe the permeability of the medium. Insufficient or uncertain measurements are commonly modeled by random coefficients. For flows in heterogeneous, fractured or porous media, the coefficients are modeled as discontinuous random fields, where the interfaces along the stochastic discontinuities represent transitions in the media. More precisely, the random coefficient is given by the sum of a (continuous) Gaussian random field and a (discontinuous) jump part. In this work moments of the solution to the random partial differential equation are calculated using a path-wise numerical approximation combined with multilevel Monte Carlo sampling. The discontinuities dictate the spatial discretization, which leads to a stochastic grid. Hence, the refinement parameter and problem-dependent constants in the error analysis are random variables and we derive (optimal) a-priori convergence rates in a mean-square sense.

6.1 Introduction

We consider a linear (diffusion-dominated) advection-diffusion equation with random Lévy fields as coefficients. Adopting the term from stochastic analysis, by a Lévy field we mean a random field which is built from a (continuous) Gaussian random field and a (discontinuous) jump part (following a certain jump measure). In the last decade various ways to approximate the distribution or moments of the solution to a random equation were introduced. Next to classical Monte Carlo methods, their multilevel variants and further variance reduction techniques have been applied. Due to their low regularity constraints, multilevel Monte Carlo techniques have been successfully applied to various problems, for instance in the context of elliptic random PDEs in [1, 29, 31, 59, 148, 194] to just name a few. These sampling approaches differ fundamentally from Polynomial-Chaos-based methods. The latter suffer from high regularity assumptions. While in the case of continuous fields these algorithms can outperform sampling strategies, approaches – like stochastic Galerkin methods – are less promising in our discontinuous setting. In fact, it is even an open problem to define them for Lévy fields. While Richards' equation formulated as a deterministic interface problem was considered in numerous publications (see [68, 82] and the references therein), there is up-to-date no stochastic formulation.

After introducing the necessary basic notation, in this paper we show in Section 6.2 existence and uniqueness of a path-wise weak solution to the random advection-diffusion equation and prove an energy estimate which allows for a moment estimate. Next to space- and time-discretizations, the Lévy field has to be approximated, resulting in an approximated path-wise weak solution. In Section 6.3 we show convergence of this approximated path-wise weak solution, before we introduce a sample-adapted (path-wise) Galerkin approximation. Only if the discretization is adapted to the random discontinuities can we expect full convergence rates. As the main result of this article, we prove the error estimate of the spatial discretization in the L^2-norm. To this end, we utilize the corresponding results with respect to the H^1-norm from [32] and consider the parabolic dual problem. Finally, we combine the sample-adapted spatial discretization with a suitable time stepping method to obtain a fully discrete path-wise scheme. The path-wise approximations are used in Section 6.4 to estimate quantities of interest using a (coupled) multilevel Monte Carlo method. Naturally, the optimal sample numbers on each level depend on the sample-dependent convergence rate. The term *coupled* refers to a simplified version of *Multifidelity Monte Carlo* sampling (see [171]) that reuses samples across levels and is preferred when sampling from a certain distribution is computationally expensive. In Section 6.5, a numerical example

confirms our theoretical results from Section 6.3 and shows that the sample-adapted strategy vastly outperforms a multilevel Monte Carlo estimator with a standard Finite Element discretization in space.

6.2 Parabolic problems with random discontinuous coefficients

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space, $\mathbb{T} = [0, T]$ be a time interval for some $T > 0$ and $\mathcal{D} \subset \mathbb{R}^d$, $d \in \{1, 2\}$, be a polygonal and convex domain. We consider the linear, random initial-boundary value problem

$$\begin{aligned} \partial_t u(\omega, x, t) + [Lu](\omega, x, t) &= f(\omega, x, t) \quad \text{in } \Omega \times \mathcal{D} \times (0, T]), \\ u(\omega, x, 0) &= u_0(\omega, x) \quad \text{in } \Omega \times \mathcal{D} \times \{0\}, \\ u(\omega, x, t) &= 0 \quad \text{on } \Omega \times \partial\mathcal{D} \times \mathbb{T}, \end{aligned} \tag{6.1}$$

where $f : \Omega \times \mathcal{D} \times \mathbb{T} \to \mathbb{R}$ is a random source function and $u_0 : \Omega \times \mathcal{D}$ denotes the initial condition of the above PDE. Furthermore, L is the second order partial differential operator given by

$$[Lu](\omega, x, t) = -\nabla \cdot (a(\omega, x)\nabla u(\omega, x, t)) + b(\omega, x)\mathbf{1}^T \nabla u(\omega, x, t) \tag{6.2}$$

for $(\omega, x, t) \in \Omega \times \mathcal{D} \times \mathbb{T}$ with ∇ operating on the second argument of u. In Eq. (6.2), we set $\mathbf{1} := (1, \ldots, 1)^T \in \mathbb{R}^n$, such that $\mathbf{1}^T \nabla u = \sum_{i=1}^n \partial_{x_i} u$, and consider

- a stochastic jump-diffusion coefficient $a : \Omega \times \mathcal{D} \to \mathbb{R}$ and
- a random discontinuous convection term $b : \Omega \times \mathcal{D} \to \mathbb{R}$ coupled to a.

Throughout this article, we denote by C a generic positive constant which may change from one line to the next. Whenever helpful, the dependence of C on certain parameters is made explicit. To obtain a path-wise variational formulation, we use the standard Sobolev space $H^s(\mathcal{D})$ with norm $\|\cdot\|_{H^s(\mathcal{D})}$ for any $s > 0$, see for instance [2, 71]. Since $\mathcal{D}$ has a Lipschitz boundary, for $s \in (1/2, 3/2)$, the existence of a bounded, linear trace operator $\gamma : H^s(\mathcal{D}) \to H^{s-1/2}(\partial\mathcal{D})$ is ensured by the trace theorem, see [73]. We only consider homogeneous Dirichlet boundary conditions on $\partial\mathcal{D}$, hence we may treat γ independently of $\omega \in \Omega$ and define the suitable solution space V as

$$V := H_0^1(\mathcal{D}) = \{v \in H^1(\mathcal{D}) |\ \gamma v \equiv 0\},$$

equipped with the $H^1(\mathcal{D})$-norm $\|v\|_V := \|v\|_{H^1(\mathcal{D})}$. With $H := L^2(\mathcal{D})$, we work on the Gelfand triplet $V \subset H \subset V' = H^{-1}(\mathcal{D})$, where V' denotes the topological dual

of V, i.e. the space of all bounded, linear functionals on V. In the variational version of Problem (6.1), $\partial_t u$ denotes the weak time derivative of u. Throughout this article, we may as well consider $\partial_t u$ as derivative in a strong sense (also with regard to its approximation at the end of Section 6.3) as we will always assume sufficient temporal regularity. As the coefficients a and b are random functions, any solution u to Problem (6.1) is a time-dependent V-valued random variable. To investigate the regularity of the solution u with respect to $\mathbb{T}$ and the underlying probability measure $\mathbb{P}$ on Ω, we need to introduce the corresponding Lebesgue-Bochner spaces. To this end, let $p \in [1, \infty)$ and$(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be an arbitrary Banach space. For $Y \in \{\mathbb{T}, \Omega\}$, the Lebesgue-Bochner space $L^p(Y; \mathcal{X})$ is defined as

$$L^p(Y;\mathcal{X}) := \{\varphi : Y \to \mathcal{X} \text{ is strongly measurable and } \|\varphi\|_{L^p(Y;\mathcal{X})} < +\infty\},$$

with the norm

$$\|\varphi\|_{L^p(Y;\mathcal{X})} := \begin{cases} \left(\int_{\mathbb{T}} \|\varphi(t)\|_{\mathcal{X}}^p dt \right)^{1/p} & \text{for } Y = \mathbb{T}, \\ \mathbb{E}(\|\varphi\|^p)^{1/p} = \left(\int_{\Omega} \|\varphi(\omega)\|_{\mathcal{X}}^p d\mathbb{P}(d\omega) \right)^{1/p} & \text{for } Y = \Omega. \end{cases}.$$

The bilinear form associated to L is introduced to derive a weak formulation of the initial-boundary value problem (6.1). For fixed $\omega \in \Omega$ and $t \in \mathbb{T}$, multiplying Eq. (6.1) with a test function $v \in V$ and integrating by parts yields

$$ {}_{V'}\langle \partial_t u(\omega, \cdot, t), v\rangle_V + B_\omega(u(\omega, \cdot, t), v) = {}_{V'}\langle f(\omega, \cdot, t), v\rangle_V. \tag{6.3}$$

The bilinear form $B_\omega : V \times V \to \mathbb{R}$ is given by

$$B_\omega(u, v) = \int_{\mathcal{D}} a(\omega, x)\nabla u(x) \cdot \nabla v(x) + b(\omega, x)\mathbf{1}^T \nabla u(x) v(x) dx,$$

and ${}_{V'}\langle \cdot, \cdot\rangle_V$ denotes the (V', V)-duality pairing.

Definition 6.2.1. For fixed $\omega \in \Omega$, the *path-wise weak solution* to Problem (6.1) is a function $u(\omega, \cdot, \cdot) \in L^2(\mathbb{T}; V)$ with $\partial_t u(\omega, \cdot, \cdot) \in L^2(\mathbb{T}; V')$ such that, for $t \in \mathbb{T}$,

$$ {}_{V'}\langle \partial_t u(\omega, \cdot, t), v\rangle_V + B_\omega(u(\omega, \cdot, t), v) = {}_{V'}\langle f(\omega, \cdot, t), v\rangle_V, \quad \text{for all } v \in V$$

and $u(\omega, \cdot, 0) = u_0(\omega, \cdot)$. Furthermore, we define the path-wise parabolic norm by

$$\begin{aligned}\|u(\omega,\cdot,\cdot)\|_{*,t} :&= \left(\|u(\omega,\cdot,t)\|_H^2 + \int_0^t \int_{\mathcal{D}} \nabla u(\omega,x,z)\cdot\nabla u(\omega,x,z)dxdz\right)^{1/2} \\ &= \left(\|u(\omega,\cdot,t)\|_H^2 + \|\|\nabla u(\omega,x,z)\|_2\|^2_{L^2([0,t];H)}\right)^{1/2},\end{aligned}$$

where $\|\cdot\|_2$ is the Euclidean norm on $\mathbb{R}^d$.

To represent the (uncertain) permeability in a subsurface flow model, we use the random jump coefficients a, b from the elliptic/parabolic problems in [31, 32]. The diffusion coefficient is then given by a (spatial) Gaussian random field with additive discontinuities on random areas of $\mathcal{D}$. Its specific structure may be utilized to model the hydraulic conductivity within heterogeneous and/or fractured media and thus a is considered time-independent. The advection term in this model is driven by the same random field and inherits the same discontinuous structure as the diffusion, hence we consider the coefficient b as a linear mapping of a.

Definition 6.2.2. The *jump-diffusion coefficient* a is defined as

$$a : \Omega \times \mathcal{D} \to \mathbb{R}_{>0}, \quad (\omega, x) \mapsto \overline{a}(x) + \Phi(W(\omega, x)) + P(\omega, x),$$

where

- $\overline{a} \in C^1(\overline{\mathcal{D}}; \mathbb{R}_{\geq 0})$ is non-negative, continuous, and bounded.
- $\Phi \in C^1(\mathbb{R}; \mathbb{R}_{>0})$ is a continuously differentiable, positive mapping.
- $W \in L^2(\Omega; H)$ is a (zero-mean) Gaussian random field associated to a non-negative, symmetric trace class operator $Q : H \to H$.
- $\mathcal{T} : \Omega \to \mathcal{B}(\mathcal{D}),\ \omega \mapsto \{\mathcal{T}_1, \ldots, \mathcal{T}_\tau\}$ is a random partition of $\mathcal{D}$, i.e. the $\mathcal{T}_i$ are disjoint open subsets of $\mathcal{D}$ such that $|\mathcal{T}_i| > 0$ and $\overline{\mathcal{D}} = \bigcup_{i=1}^{\tau} \overline{\mathcal{T}}_i$, and $\mathcal{B}(\mathcal{D})$ denotes the Borel-σ-algebra on $\mathcal{D}$. The number of elements in $\mathcal{T}$, τ, is a random variable on $(\Omega, \mathcal{A}, \mathbb{P})$, i.e. $\tau : \Omega \to \mathbb{N}$.
- $(P_i, i \in \mathbb{N})$ is a sequence of non-negative random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ and

$$P : \Omega \times \mathcal{D} \to \mathbb{R}_{\geq 0}, \quad (\omega, x) \mapsto \sum_{i=1}^{\tau(\omega)} \mathbf{1}_{\{\mathcal{T}_i\}}(x) P_i(\omega).$$

  The sequence $(P_i, i \in \mathbb{N})$ is independent of τ (but not necessarily i.i.d.).

Based on a, the *jump-advection coefficient* b is given for $b_1, b_2 \in L^\infty(\mathcal{D})$ by

$$b : \Omega \times \mathcal{D} \to \mathbb{R}, \quad (\omega, x) \mapsto \min(b_1(x)a(\omega, x), b_2).$$

The definition of the random partition $\mathcal{T}$ above is rather general and does not yet assume any structure on the discontinuities. A more specific class of random partitions is considered in our numerical experiment in Section 6.5. We assumed in Definition 6.2.2 that τ and P_i are independent due to technical reasons, i.e. to control for a possible sampling bias in P_i, see [31, Theorem 3.11]. On a further note, we do not require stochastic independence of W and P. In general, our aim is to estimate moments of a *quantity of interest* (QoI) $\Psi(\omega) := \psi(u(\omega, \cdot, \cdot))$ of the weak solution, where $\psi : L^2(\mathbb{T}; V) \to \mathbb{R}$ is a deterministic functional. To ensure existence and a certain regularity of u, and therefore of Ψ, we fix the following set of assumptions.

Assumption 6.2.3.

1. Let $\eta_1 \geq \eta_2 \geq \cdots \geq 0$ denote the eigenvalues of Q in descending order and $(e_i, i \in \mathbb{N}) \subset H$ be the corresponding eigenfunctions. The e_i are continuously differentiable on $\mathcal{D}$ and there exist constants $\alpha, \beta, C_e, C_\eta > 0$ such that $2\alpha \leq \beta$ and for any $i \in \mathbb{N}$

$$\|e_i\|_{L^\infty(\mathcal{D})} \leq C_e, \quad \max_{j=1,\dots,d} \|\partial_{x_j} e_i\|_{L^\infty(\mathcal{D})} \leq C_e i^\alpha \quad \text{and} \quad \sum_{i=1}^{\infty} \eta_i i^\beta \leq C_\eta < +\infty.$$

2. Furthermore, the mapping Φ as in Definition 6.2.2 and its derivative are bounded by

$$\phi_1 \exp(\phi_2|w|) \geq \Phi(w) \geq \phi_1 \exp(-\phi_2|w|), \quad |\frac{d}{dx}\Phi(w)| \leq \phi_3 \exp(\phi_4|w|), \quad w \in \mathbb{R},$$

   where $\phi_1, \dots, \phi_4 > 0$ are arbitrary constants.

3. For some $p > 2$, $f, \partial_t f \in L^p(\Omega; L^2(\mathbb{T}; H)), u_0 \in L^p(\Omega; H^2(\mathcal{D}) \cap V)$ and u_0 and f are stochastically independent of $\mathcal{T}$.

4. The partition elements $\mathcal{T}_i$ are almost surely polygons with piecewise linear boundary and $\mathbb{E}(\tau^n) < +\infty$ for all $n \in \mathbb{N}$.

5. The sequence $(P_i, i \in \mathbb{N})$ consists of nonnegative and bounded random variables $P_i \in [0, \overline{P}]$ for some $\overline{P} > 0$.

6. The functional ψ is Lipschitz continuous on $L^2(\mathbb{T};H)$, i.e. there exists $C_\psi > 0$ such that

$$|\psi(v) - \psi(w)| \leq C_\psi \|v - w\|_{L^2(\mathbb{T};H)} \quad \forall v, u \in L^2(\mathbb{T};H).$$

Remark 6.2.4. The above assumptions are natural and cannot be relaxed significantly to derive the results in Section 6.3. The condition $2\alpha \leq \beta$ implies that W has almost surely Lipschitz continuous paths on $\mathcal{D}$, thus a is piecewise Lipschitz continuous. This is in turn necessary to derive the error estimates of orders $\mathcal{O}(\overline{h}_\ell^{\kappa})$ and $\mathcal{O}(\overline{h}_\ell^{2\kappa})$ in Theorem 6.3.3 and Theorem 6.3.4, respectively, for some $\kappa \in (1/2, 1]$ that is independent of W. The parameter $\overline{h}_\ell$ denotes the Finite Element (FE) refinement and κ should only be influenced by the law of the random jump field P. If any of this assumptions were violated, however, κ may depend on other parameters of the random PDE. For instance, if $\beta/2\alpha < \kappa \leq 1$, we would only obtain an error of approximate order $\mathcal{O}(\overline{h}_\ell^{\beta/2\alpha})$ in Theorem 6.3.3, see [32] for a detailed discussion. The remaining points in Assumption 6.2.3 ensure that all estimates hold in the mean-square sense, i.e. the second moments of all estimates exist and can be bounded with respect to $\overline{h}_\ell$.

We have the following estimate on a and its piecewise Lipschitz norm.

Lemma 6.2.5. *[32, Lemmas 3.6 and 4.8] Let Assumption 6.2.3 hold and define*

$$a_-(\omega) := \operatorname*{ess\,inf}_{x\in\mathcal{D}} a(\omega, x) \quad \text{and} \quad a_+(\omega) := \operatorname*{ess\,sup}_{x\in\mathcal{D}} a(\omega, x).$$

Then, for any $q \in [1, \infty)$

$$1/a_-,\ a_+,\ \max_{i=1,\dots,\tau} \sum_{j=1}^{d} \|\partial_{x_j} a\|_{L^\infty(\mathcal{T}_i)} \in L^q(\Omega;\mathbb{R}).$$

Theorem 6.2.6. *Under Assumption 6.2.3 there exists almost surely a unique path-wise weak solution $u(\omega,\cdot,\cdot) \in L^2(\mathbb{T};V)$ to Problem (6.1) satisfying the estimate*

$$\sup_{t\in\mathbb{T}} \|u(\omega,\cdot,\cdot)\|_{*,t}^2 \leq C/a_-(\omega)\Big(\|u_0(\omega,\cdot)\|_H^2 + \|f(\omega,\cdot,\cdot)\|_{L^2(T;H)}^2\Big) < +\infty. \tag{6.4}$$

In addition, for any $r \in [1, p)$ (with p as in Ass. 6.2.3), u is bounded in expectation by

$$\mathbb{E}\Big(\sup_{t\in\mathbb{T}} \|u\|_{*,t}^r\Big)^{1/r} \leq C\|1/a_-\|_{L^{\widetilde{q}}(\Omega;\mathbb{R})} \Big(\|u_0\|_{L^p(\Omega;H)} + \|f\|_{L^p(\Omega;L^2(\mathbb{T};V'))}\Big) < +\infty. \tag{6.5}$$

with $C = C(r)$ and $\widetilde{q} := (1/r - 1/p)^{-1}$. Furthermore, it holds $\Psi \in L^r(\Omega;\mathbb{R})$.

Proof. The estimates in Ineq. (6.4) and (6.5) follow from [32, Theorem 3.7]. To show that $\Psi \in L^r(\Omega;\mathbb{R})$, we use Assumption 6.2.3 to see that ψ fulfills the linear growth condition $|\psi(v)| \leq C(1 + \|v\|_{L^2(\mathbb{T};H)})$ for some deterministic constant $C = C(\psi) > 0$ and all $v \in L^2(\mathbb{T};H)$. Hence, we have

$$\mathbb{E}(\Psi^r) \leq \mathbb{E}\Big(C^r(1 + \|u\|_{L^2(\mathbb{T};V)})^r\Big) \leq C^r 2^{r-1}\Big(1 + \mathbb{E}\Big(\sup_{t\in\mathbb{T}} \|u\|_{*,t}^r\Big)\Big) < +\infty.$$

□

6.3 Numerical approximation of the solution

In general, the (exact) weak solution u to Problem (6.1) is out of reach and we have to find tractable approximations of u to apply Monte Carlo algorithms for the estimation of $\mathbb{E}(\Psi)$. A common approach is to use a FE discretization of V combined with a time marching scheme to sample path-wise approximations of u. For this, however, it is necessary to evaluate a and b at certain points in $\mathcal{D}$. This is in general infeasible, since the Gaussian field W usually involves an infinite series and/or the jump heights P_i might not be sampled without bias. The latter issue may arise if P_i has non-standard law, e.g. the *generalized inverse Gaussian* distribution, for more details we refer to [31, 32]. We may circumvent this issue by constructing suitable approximations of a and b, for instance by truncated Karhunen-Loève expansions ([53, 54]), circulant embedding methods ([97, 139]) or Fourier inversion techniques for the sampling of P_i ([30, 31]). Hence, we obtain a modified problem with approximated coefficients which may then be discretized in the spatial and temporal domain. To increase the order of convergence in the spatial discretization, we introduce a FE scheme in the second part of this section where we choose the FE grids adapted with respect to the discontinuities in each sample of a and b. Under mild assumptions on the coefficients we then derive errors on the semi- and fully discrete approximations of u.

6.3.1 Approximated diffusion coefficients

As discussed above, there are several methods available to obtain tractable approximations of the diffusion coefficient a, thus we consider a rather general setting here. For some $\epsilon > 0$, let $a_\epsilon : \Omega \times \mathcal{D} \to \mathbb{R}_{>0}$ be an arbitrary approximation of the diffusion coefficient and let (according to Definition 6.2.2)

$$b_\epsilon : \Omega \times \mathcal{D} \to \mathbb{R}, \quad (\omega, x) \mapsto \min(b_1(x)a_\epsilon(\omega, x), b_2(x)),$$

be the canonical approximation of b. Substituting a_ϵ and b_ϵ into Problem (6.1) yields

$$\begin{aligned} \partial_t u_\epsilon(\omega,x,t) + [L_\epsilon u_\epsilon](\omega,x,t) &= f(\omega,x,t) \quad \text{in } \Omega\times\mathcal{D}\times(0,T], \\ u_\epsilon(\omega,x,0) &= u_0(\omega,x) \quad \text{in } \Omega\times\mathcal{D}\times\{0\} \\ u_\epsilon(\omega,x,t) &= 0 \quad \text{on } \Omega\times\partial\mathcal{D}\times\mathbb{T}, \end{aligned} \tag{6.6}$$

where the approximated second order differential operator L_ϵ is given by

$$[L_\epsilon u](\omega,x,t) = -\nabla\cdot(a_\epsilon(\omega,x)\nabla u(\omega,x,t)) + b_\epsilon(\omega,x)\mathbf{1}^T\nabla u(\omega,x,t).$$

The path-wise variational formulation of Eq. (6.6) is then (analogous to Eq. (6.3)) given by: For almost all $\omega\in\Omega$ with given $f(\omega,\cdot,\cdot)$, find $u_\epsilon(\omega,\cdot,\cdot)\in L^2(\mathbb{T};V)$ with $\partial_t u(\omega,\cdot,\cdot)\in L^2(\mathbb{T};V')$ such that, for $t\in\mathbb{T}$,

$${}_{V'}\langle\partial_t u_\epsilon(\omega,\cdot,t),v\rangle_V + B_{\epsilon,\omega}(u_\epsilon(\omega,\cdot,t),v) = F_{\omega,t}(v), \tag{6.7}$$

holds for all $v\in V$ with respect to the approximated bilinear form

$$B_{\epsilon,\omega}(v,w) := \int_{\mathcal{D}} a_\epsilon(\omega,x)\nabla v(x)\cdot\nabla w(x) + b_\epsilon(\omega,x)\mathbf{1}^T\nabla v(x)w(x)dx, \quad v,w\in V.$$

The following assumption guarantees existence and uniqueness of u_ϵ and allows us to bound $u-u_\epsilon$ in a mean-square sense.

Assumption 6.3.1. Let Assumption 6.2.3 hold and let $a_\epsilon:\Omega\times\mathcal{D}\to\mathbb{R}_{>0}$ be an approximation of a for some fixed $\epsilon>0$. Define $a_{\epsilon,-}(\omega) := \operatorname{ess\,inf} a_\epsilon(\omega,x)$ and $a_{\epsilon,+}(\omega) := \operatorname{ess\,sup}_{x\in\mathcal{D}} a_\epsilon(\omega,x)$. Assume that for some $s>(1/2-1/p)^{-1}$ and any $q\in[1,\infty)$, there are constants $C_i>0$, for $i=1,\dots,4$, independent of ϵ, such that

- $\|a-a_\epsilon\|_{L^s(\Omega;L^\infty(\mathcal{D}))}\le C_1\epsilon$,
- $\|1/a_{\epsilon,-}\|_{L^q(\Omega;\mathbb{R})}\le C_2\|1/a_-\|_{L^q(\Omega;\mathbb{R})}<+\infty$,
- $\|a_{\epsilon,+}\|_{L^q(\Omega;\mathbb{R})}\le C_3\|a_+\|_{L^q(\Omega;\mathbb{R})}<+\infty$ and
- $\|\max_{i=1,\dots,\tau}\sum_{j=1}^d\|\partial_{x_j}a_\epsilon\|_{L^\infty(\mathcal{T}_i)}\|_{L^q(\Omega;\mathbb{R})}\le C_4\|\max_{i=1,\dots,\tau}\sum_{j=1}^d\|\partial_{x_j}a\|_{L^\infty(\mathcal{T}_i)}\|_{L^q(\Omega;\mathbb{R})}<+\infty$.

At this point we remark that Assumption 6.3.1 is natural and essentially states that a_ϵ has the same regularity as a. Furthermore, the moments of $a-a_\epsilon$ are controlled by the parameter ϵ and we may achieve an arbitrary good approximation by choosing ϵ sufficiently small. This holds for instance (with $C_2=C_3=C_4=1$) if W is approximated by a truncated Karhunen-Loève expansion (see [31, 32]) or if a_ϵ stems from linear interpolation of discrete sample points of W as we explain in Section 6.5.

Theorem 6.3.2. *Let Assumption 6.3.1 hold and let u_ϵ be the weak solution to Problem* (6.6)*. Then, the root-mean-squared approximation error is bounded by*

$$\mathbb{E}\Big(\sup_{t\in\mathbb{T}}\|u(\cdot,\cdot,t)-u_\epsilon(\cdot,\cdot,t)\|_{*,t}^2\Big)^{1/2}\leq C\epsilon.$$

Proof. By Theorem 6.2.6, we have existence of unique solutions u and u_ϵ to Eqs. (6.3) resp. (6.7) almost surely. Thus, we obtain the variational problem: Find $u-u_\epsilon$ such that

$${}_{V'}\langle\partial_t(u(\omega,\cdot,t)-u_\epsilon(\omega,\cdot,t)),v\rangle_V+B_\omega(u(\omega,\cdot,t)-u_\epsilon(\omega,\cdot,t),v)={}_{V'}\langle\widetilde{f}(\omega,\cdot,t),v\rangle_V$$

for all $t\in\mathbb{T}$ and $v\in V$ with initial condition $(u-u_\epsilon)(\cdot,\cdot,0)\equiv 0$ and right hand side

$$\widetilde{f}(\omega,\cdot,t):=\nabla\cdot((a_\epsilon-a)(\omega,\cdot)\nabla u_\epsilon(\omega,\cdot,t))+(b_\epsilon-b)(\omega,\cdot)\mathbf{1}^T\nabla u_\epsilon(\omega,\cdot,t)\in V'.$$

By Hölder's inequality it holds

$$\begin{aligned}\|\widetilde{f}(\omega,\cdot,\cdot)\|_{L^2(\mathbb{T};V')}&\leq\|(a-a_\epsilon)(\omega,\cdot)\|_{L^\infty(\mathcal{D})}\|\|\nabla u(\omega,\cdot,\cdot)\|_2\|_{L^2(\mathbb{T};H)}\\&\quad+\|(b-b_\epsilon)(\omega,\cdot)\|_{L^\infty(\mathcal{D})}\|\mathbf{1}^T\nabla u(\omega,\cdot,\cdot)\|_{L^2(\mathbb{T};H)}\\&\leq C(1+\|b_1\|_{L^\infty(\mathcal{D})})\|(a-a_\epsilon)(\omega,\cdot)\|_{L^\infty(\mathcal{D})}\|\|\nabla u(\omega,\cdot,\cdot)\|_2\|_{L^2(\mathbb{T};H)},\end{aligned}$$

which yields using Assumption 6.3.1 and Theorem 6.2.6

$$\begin{aligned}\|\widetilde{f}(\omega,\cdot,\cdot)\|_{L^{p_1}(\Omega;L^2(\mathbb{T};V'))}&\leq C(1+\|b_1\|_{L^\infty(\mathcal{D})})\|(a-a_\epsilon)\|_{L^s(\Omega;L^\infty(\mathcal{D}))}\mathbb{E}\Big(\sup_{t\in\mathbb{T}}\|u\|_{*,t}^r\Big)^{1/r}\\&\leq C\epsilon\end{aligned}$$

for $r\in((1/2-1/s)^{-1},p)$ and $p_1:=(1/s+1/r)^{-1}>2$. We may now use Theorem 6.2.6 with $q=(1/2-1/p_1)^{-1}$ to estimate $u-u_\epsilon$ via

$$\mathbb{E}\Big(\sup_{t\in\mathbb{T}}\|u-u_\epsilon\|_{*,t}^2\Big)^{1/2}\leq C\|1/a_-\|_{L^q(\Omega;\mathbb{R})}\|\widetilde{f}\|_{L^{p_1}(\Omega;L^2(\mathbb{T};V'))}\leq C\epsilon.$$

□

6.3.2 Semi-discretization by sample-adapted finite elements

Given a suitable approximation a_ϵ of the diffusion coefficient, we discretize the (approximate) solution u_ϵ in the spatial domain. As a first step, we replace the (infinite-

dimensional) solution space V by a sequence $\mathcal{V} = (V_\ell, \ell \in \mathbb{N}_0)$ of finite dimensional subspaces $V_\ell \subset V$. In general, V_ℓ are standard FE spaces of piecewise linear functions with respect to some given triangulation $\mathcal{K}_\ell$ of $\mathcal{D}$ and h_ℓ represents the maximum diameter of $\mathcal{K}_\ell$. As indicated in [31, 32] using standard FE spaces will not yield the full order of convergence with respect to h_ℓ due to the discontinuities in a_ϵ and b_ϵ. Thus, we follow the same approach as in [31] for Problem (6.7) and utilize path-dependent meshes to match the interfaces generated by the jump-diffusion and -advection coefficients. As this entails changing varying approximation spaces V_ℓ with each sample of a_ϵ resp. b_ϵ, we have to formulate a semi-discrete version of problem (6.7) with respect to $\omega \in \Omega$:

Given a fixed $\omega \in \Omega$ and $\ell \in \mathbb{N}_0$, we consider a (stochastic) finite dimensional subspace $V_\ell(\omega) \subset V$ with sample-dependent basis $\{v_1(\omega), \dots, v_{d_\ell}(\omega)\} \subset V$ and stochastic dimension $d_\ell = d_\ell(\omega) \in \mathbb{N}$. For a given random partition $\mathcal{T}(\omega) = (\mathcal{T}_i, i = 1 \dots, \tau(\omega))$ of polygons on $\mathcal{D}$, we choose a conforming triangulation $\mathcal{K}_\ell(\omega)$ such that

$$\mathcal{T}(\omega) \subset \mathcal{K}_\ell(\omega) \;\text{ and }\; h_\ell(\omega) := \max_{K \in \mathcal{K}_\ell(\omega)} \operatorname{diam}(K) \leq \overline{h}_\ell \;\text{ for } \ell \in \mathbb{N}_0,$$

holds almost surely. The inclusion $\mathcal{T}(\omega) \subset \mathcal{K}_\ell(\omega)$ states that the triangles in $\mathcal{K}_\ell(\omega)$ are chosen to match and fully cover the polygonal partition elements in $\mathcal{T}(\omega)$. Furthermore, $(\overline{h}_\ell, \ell \in \mathbb{N}_0)$ is a sequence of positive, deterministic refinement thresholds, decreasing monotonically to zero. This guarantees that $h_\ell(\omega) \to 0$ for $\ell \to \infty$ almost surely, although the absolute speed of convergence varies for each ω. We assume shape-regularity of the triangulation uniform in Ω, i.e. there exist a $\vartheta \in (0, 1)$ such that

$$0 < \vartheta \leq \sup_{\ell \in \mathbb{N}_0} \sup_{K \in K_\ell(\omega)} \frac{\operatorname{diam}(K)}{\iota_K} \leq \vartheta^{-1} < +\infty \quad \text{almost surely.}$$

In Ineq. (6.3.2), ι_T denotes the diameter of the inscribed circle of the triangle K. For given $\{v_1(\omega), \dots, v_{d_\ell}(\omega)\}$, the semi-discrete version of the variational formulation (6.7) is then to find $u_{\epsilon,\ell}(\omega, \cdot, t) \in V_\ell(\omega)$ such that for $t \in \mathbb{T}$ and $v_\ell(\omega) \in V_\ell(\omega)$

$$\begin{aligned} {}_{V'}\langle \partial_t u_{\epsilon,\ell}(\omega, \cdot, t), v_\ell(\omega)\rangle_V + B_{\epsilon,\omega}(u_{\epsilon,\ell}(\omega, \cdot, t), v_\ell(\omega)) &= {}_{V'}\langle f(\omega, \cdot, t), v_\ell(\omega)\rangle_V, \\ u_{\epsilon,\ell}(\omega, \cdot, 0) &= u_{0,\ell}(\omega, \cdot), \end{aligned} \tag{6.8}$$

where $u_{0,\ell}(\omega, \cdot) \in V_\ell(\omega)$ is a suitable approximation of $u_0(\omega, \cdot)$, for instance the nodal

interpolation of u_0 in $V_\ell(\omega)$. The function $u_{\epsilon,\ell}(\omega,\cdot,t)$ may be expanded as

$$u_{\epsilon,\ell}(\omega,\cdot,t) = \sum_{j=1}^{d_\ell(\omega)} c_j(\omega,t)v_j(\omega),$$

where the coefficients $c_1(\omega,t),\dots,c_{d_\ell}(\omega,t) \in \mathbb{R}$ depend on $(\omega,t) \in \Omega\times\mathbb{T}$ and the respective coefficient (column-)vector is $\mathbf{c}(\omega,\mathbf{t}) := (c_1(\omega,t),\dots,c_{d_\ell}(\omega,t))^T$. With this, the semi-discrete variational problem in the (stochastic) finite dimensional space $V_\ell(\omega)$ is equivalent to solving the system of ordinary differential equations

$$\frac{d}{dt}\mathbf{c}(\omega,\mathbf{t}) + \mathbf{A}(\omega)\mathbf{c}(\omega,\mathbf{t}) = \mathbf{F}(\omega,\mathbf{t}), \quad \mathbf{t}\in\mathbb{T} \tag{6.9}$$

for $\mathbf{c}$ with stochastic stiffness matrix $(\mathbf{A}(\omega))_{jk} = B_{\epsilon,\omega}(v_j(\omega),v_k(\omega))$ and time-dependent load vector $(\mathbf{F}(\omega,t))_j = {}_{V'}\langle f(\omega,\cdot,t),v_j(\omega)\rangle_V$ for $j,k\in\{1,\dots,d_\ell(\omega)\}$. The following result gives an error estimate in the energy norm for $u_\epsilon - u_{\epsilon,\ell}$.

Theorem 6.3.3. *[32, Theorem 4.7] Let Assumption 6.3.1 hold such that for some $\kappa\in(1/2,1]$ it holds that $\mathbb{E}(\max_{i=1,\dots,\tau}\|u\|^2_{H^{1+\kappa}(\mathcal{T}_i)}) < +\infty$. Let $u_{\epsilon,\ell}$ be the semi-discrete sample-adapted approximation of u_ϵ as in Eq. (6.8) and let $\|(u_0-u_{\ell,0})(\omega,\cdot)\|_H \le C\|u_0(\omega,\cdot)\|_V\overline{h}_\ell$ almost surely for all $\ell\in\mathbb{N}_0$. Then, there holds almost surely the path-wise estimate*

$$\sup_{t\in\mathbb{T}}\|(u_\epsilon-u_{\epsilon,\ell})(\omega,\cdot,\cdot)\|_{*,t} \le C/(a_{\epsilon,-}(\omega))^{1/2}\Big(\|f(\omega,\cdot,\cdot)\|_{L^2(\mathbb{T};H)} + \|u_0(\omega,\cdot)\|_V\Big)\overline{h}_\ell^\kappa$$

and, for any $r\in[1,p)$ (with p as in Ass 6.2.3), the expected parabolic estimate

$$\mathbb{E}(\sup_{t\in\mathbb{T}}\|u_\epsilon-u_{\epsilon,\ell}\|^r_{*,t})^{1/r} \le C(\|f\|_{L^p(\Omega;L^2(\mathbb{T};H))} + \|u_0\|_{L^p(\Omega;V)})\overline{h}_\ell^\kappa.$$

The above statement gives a bound on the error in the $L^2(\mathbb{T};V)$-norm. The functional Ψ however is defined on $L^2(\mathbb{T};H)$, thus it is favorable to derive an error bound with respect to the weaker $L^2(\mathbb{T};H)$-norm.

Theorem 6.3.4. *Let Assumption 6.3.1 hold such that for some $\kappa\in(1/2,1]$ there holds $\mathbb{E}(\max_{i=1,\dots,\tau}\|u\|^2_{H^{1+\kappa}(\mathcal{T}_i)}) < +\infty$ and let $\|(u_0-u_{\ell,0})(\omega,\cdot)\|_H \le C\|u_0(\omega,\cdot)\|_{H^2(\mathcal{D})}\overline{h}_\ell^2$ almost surely. Then,*

$$\mathbb{E}(\|u_\epsilon-u_{\ell,\epsilon}\|^2_{L^2(\mathbb{T};H)})^{1/2} \le C\overline{h}_\ell^{2\kappa}.$$

Proof. For fixed ω, we consider the path-wise parabolic dual problem to find $w(\omega,\cdot,\cdot)\in$

$L^2(\mathbb{T};V)$ with $\partial_t w(\omega,\cdot,\cdot) \in L^2(\mathbb{T};V')$ such that, for $t \in \mathbb{T}$,

$$ {}_{V'}\langle \partial_t w(\omega,\cdot,t), v\rangle_V + B_{\epsilon,\omega}(w(\omega,\cdot,t),v) = {}_{V'}\langle g(\omega,\cdot,t), v\rangle_V, \quad \text{for all } v \in V, \tag{6.10}$$

where $w(\omega,\cdot,0) = w_0(\omega,\cdot) := 0$ and $g(\omega,\cdot,t) := (u_\epsilon - u_{\epsilon,\ell})(\omega,\cdot,T-t) \in V$ almost surely for any $t \in \mathbb{T}$ by Theorem 6.2.6. Hence, we may test against $v = g(\omega,\cdot,t)$ in Eq. (6.10) to obtain

$$ \|g(\omega,\cdot,t)\|_H^2 = {}_{V'}\langle \partial_t w(\omega,\cdot,t), g(\omega,\cdot,t)\rangle_V + B_{\epsilon,\omega}(w(\omega,\cdot,t), g(\omega,\cdot,t)). \tag{6.11}$$

Furthermore, for any $v_\ell(\omega) \in V_\ell(\omega)$ it holds by Eqs. (6.7),(6.8)

$$ {}_{V'}\langle \partial_t (u_\epsilon - u_{\epsilon,\ell})(\omega,\cdot,t), v_\ell(\omega)\rangle_V = -B_{\epsilon,\omega}((u_\epsilon - u_{\epsilon,\ell})(\omega,\cdot,t), v_\ell(\omega)) \tag{6.12}$$

and thus

$$\begin{aligned} B_{\epsilon,\omega}(g(\omega,\cdot,t), w(\omega,\cdot,t)) = {}& {}_{V'}\langle \partial_t g(\omega,\cdot,t), v_\ell(\omega) - w(\omega,\cdot,t) + w(\omega,\cdot,t)\rangle_V \\ & + B_{\epsilon,\omega}(g(\omega,\cdot,t), w(\omega,\cdot,t) - v_\ell(\omega)), \end{aligned} \tag{6.13}$$

where we have used the that $\partial_t g(\omega,\cdot,t) = -(\partial_t u_\epsilon - \partial_t u_{\epsilon,\ell})(\omega,\cdot,T-t)$ by the chain rule. Substituting Eq. (6.13) in Eq. (6.11) and integrating over $\mathbb{T}$ yields

$$\begin{aligned} \|g(\omega,\cdot,\cdot)\|_{L^2(\mathbb{T};H)}^2 = {}& \int_0^T {}_{V'}\langle \partial_t w(\omega,\cdot,t), g(\omega,\cdot,t)\rangle_V + {}_{V'}\langle \partial_t g(\omega,\cdot,t), w(\omega,\cdot,t)\rangle_V dt \\ & + \int_0^T {}_{V'}\langle \partial_t g(\omega,\cdot,t), v_\ell(\omega) - w(\omega,\cdot,t)\rangle_V dt \\ & + \int_0^T B_{\epsilon,\omega}(g(\omega,\cdot,t), w(\omega,\cdot,t) - v_\ell(\omega)) dt \\ =: {}& I + II + III. \end{aligned}$$

Integration by parts and the path-wise estimate in Theorem 6.2.6 yield for I

$$\begin{aligned} I &= (w(\omega,\cdot,T), g(\omega,\cdot,T))_H - (w_0(\omega,\cdot), g(\omega,\cdot,0))_H \\ &\le \|w(\omega,\cdot,T)\|_H \|u_0(\omega,\cdot) - u_{0,\ell}(\omega,\cdot)\|_H \\ &\le C \frac{1}{a_{\epsilon,-}(\omega)} \|g(\omega,\cdot,\cdot)\|_{L^2(\mathbb{T};H)} \|u_0(\omega,\cdot)\|_{H^2(\mathcal{D})} \overline{h}_\ell^2, \end{aligned}$$

where we have used $\|(u_0 - u_{\ell,0})(\omega,\cdot)\|_H \le C\|u_0(\omega,\cdot)\|_{H^2(\mathcal{D})}\overline{h}_\ell^2$ in the last step. To bound the second term, we choose $v_\ell = v_\ell(\omega,\cdot,t)$ to be the semi-discrete FE approximation of $w(\omega,\cdot,t)$ in $V_\ell(\omega)$. Since $w_0 \equiv 0$, there is no approximation error in the initial condition

and with the path-wise estimate from Theorem 6.3.3 it follows that

$$\begin{aligned} II &\leq \|\partial_t g(\omega,\cdot,\cdot)\|_{L^2(\mathbb{T};V')}\|v_\ell(\omega,\cdot,\cdot)-w(\omega,\cdot,\cdot)\|_{L^2(\mathbb{T};V)} \\ &\leq C\frac{1}{(a_{\epsilon,-}(\omega))^{1/2}}\|\partial_t g(\omega,\cdot,\cdot)\|_{L^2(\mathbb{T};V')}\|g(\omega,\cdot,\cdot)\|_{L^2(\mathbb{T};H)}\overline{h}_\ell^\kappa. \end{aligned}$$

From Eq. (6.12) and Theorem 6.3.3 we also see that

$$\|\partial_t g(\omega,\cdot,\cdot)\|_{L^2(\mathbb{T};V')} \leq C\frac{a_{\epsilon,+}(\omega)}{(a_{\epsilon,-}(\omega))^{1/2}}\Big(\|f(\omega,\cdot,\cdot)\|_{L^2(\mathbb{T};H)}+\|u_0(\omega,\cdot)\|_V\Big)\overline{h}_\ell^\kappa$$

and thus

$$II \leq C\frac{a_{\epsilon,+}(\omega)}{a_{\epsilon,-}(\omega)}\Big(\|f(\omega,\cdot,\cdot)\|_{L^2(\mathbb{T};H)}+\|u_0(\omega,\cdot)\|_V\Big)\|g(\omega,\cdot,\cdot)\|_{L^2(\mathbb{T};H)}\overline{h}_\ell^{2\kappa}.$$

Similarly, we bound the last term again with Theorem 6.3.3 via

$$\begin{aligned} III &\leq Ca_{\epsilon,+}(\omega)\|g(\omega,\cdot,\cdot)\|_{L^2(\mathbb{T};V)}\|v_\ell(\omega,\cdot,\cdot)-w(\omega,\cdot,\cdot)\|_{L^2(\mathbb{T};V)} \\ &\leq C\frac{a_{\epsilon,+}(\omega)}{a_{\epsilon,-}(\omega)}\Big(\|f(\omega,\cdot,\cdot)\|_{L^2(\mathbb{T};H)}+\|(u_0(\omega,\cdot)\|_V\Big)\|g(\omega,\cdot,\cdot)\|_{L^2(\mathbb{T};H)}\overline{h}_\ell^{2\kappa}. \end{aligned}$$

The estimates on $I-III$ now show that

$$\|g(\omega,\cdot,\cdot)\|_{L^2(\mathbb{T};H)} \leq C\frac{a_{\epsilon,+}(\omega)}{a_{\epsilon,-}(\omega)}\Big(\|f(\omega,\cdot,\cdot)\|_{L^2(\mathbb{T};H)}+\|(u_0(\omega,\cdot)\|_{H^2(\mathcal{D})}\Big)\overline{h}_\ell^{2\kappa}.$$

and the claim follows by Assumption 6.3.1 and Hölder's inequality. □

Remark 6.3.5. We remark that the additional condition on the initial data approximation in Theorem 6.3.4 is fulfilled if u_0 has almost surely continuous paths and $u_{\ell,0}$ is chosen as the path-wise nodal interpolation with respect to the sample-adapted FE basis.

6.3.3 Fully discrete pathwise approximation

For a fully discrete formulation of Problem (6.8), we consider a time grid $0 = t_0 < t_1 < \cdots < t_n = T$ in $\mathbb{T}$ for some $n \in \mathbb{N}$ and assume the grid is equidistant with fixed time step $\Delta t := t_i - t_{i-1} > 0$. The temporal derivative at t_i is approximated by the backward difference

$$\partial_t u_{\epsilon,\ell}(\omega,\cdot,t_i) = (u_{\epsilon,\ell}(\omega,\cdot,t_i) - u_{\epsilon,\ell}(\omega,\cdot,t_{i-1}))/\Delta t, \quad i = 1,\ldots,n.$$

We emphasize again that in our model problem the weak and strong temporal derivative of $u_{\epsilon,\ell}$ coincide due to the temporal regularity of the solution. Hence, the backward difference as an approximation scheme in a strong sense is justified. This yields the fully discrete problem to find $(u_{\epsilon,\ell}^{(i)}(\omega,\cdot), i = 0,\dots,n) \subset V_\ell(\omega)$ such that for all $v_\ell(\omega) \in V_\ell(\omega)$ and $i = 1,\dots,n$

$$\frac{((u_{\epsilon,\ell}^{(i)} - u_{\epsilon,\ell}^{(i-1)})(\omega,\cdot), v_\ell(\omega))_H}{\Delta t} + B_{\epsilon,\omega}(u_{\epsilon,\ell}^{(i)}(\omega,\cdot), v_\ell(\omega)) = {}_{V'}\langle f(\omega,\cdot,t_i), v_\ell(\omega)\rangle_V,$$
$$u_{\epsilon,\ell}^{(0)}(\omega,\cdot) = u_{0,\ell}(\omega,\cdot).$$

The fully discrete solution is given by

$$u_{\epsilon,\ell}^{(i)}(\omega,\cdot) = \sum_{j=1}^{d_\ell(\omega)} c_{i,j}(\omega) v_j(\omega), \quad i = 1,\dots,n,$$

where the coefficient vector $\mathbf{c_i}(\omega) = (c_{i,1}(\omega),\dots,c_{i,d_\ell}(\omega)))$ solves the linear system of equations

$$(\mathbf{M} + \Delta t \mathbf{A}(\omega))\mathbf{c_i}(\omega) = \Delta t \mathbf{F}(\omega, t_i) + \mathbf{M}\mathbf{c_{i-1}}(\omega)$$

in every discrete point in time t_i, and $\mathbf{A}$ and $\mathbf{F}$ are as in Eq. (6.9). The mass matrix is given by $(\mathbf{M})_{jk} := (v_j(\omega), v_k(\omega))_H$ and $\mathbf{c_0}$ consists of the basis coefficients of $u_{0,\ell} \in V_\ell(\omega)$ with respect to $\{v_1(\omega),\dots,v_{d_\ell}(\omega)\}$. We extend the discrete solution to the whole temporal domain by the linear interpolation

$$\overline{u}_{\epsilon,\ell}(\cdot,\cdot,t) := (u_{\epsilon,\ell}^{(i)} - u_{\epsilon,\ell}^{(i-1)})\frac{(t - t_{i-1})}{\Delta t} + u_{\epsilon,\ell}^{(i-1)}, \quad t \in [t_{i-1}, t_i], \quad i = 1,\dots,n.$$

Theorem 6.3.6. *[32, Theorem 4.12] Let Assumption 6.3.1 hold, let $(u_{\epsilon,\ell}^{(i)}, i = 0,\dots,n)$ be the fully discrete sample-adapted approximation of $u_{N,\epsilon}$, and let $\overline{u}_{\epsilon,\ell}$ be the linear interpolation of $(u_{\epsilon,\ell}^{(i)}, i = 0,\dots,n)$ in $\mathbb{T}$. Then, for $C > 0$ independent of ϵ, h_ℓ and Δt, it holds*

$$\mathbb{E}(\sup_{t\in\mathbb{T}} \|u_{\epsilon,\ell} - \overline{u}_{\epsilon,\ell}\|_{*,t}^2)^{1/2} \leq C\Delta t.$$

The final corollary on the overall approximation error is now an immediate consequence of Theorems 6.3.2, 6.3.4 and 6.3.6 and the Lipschitz condition on ψ.

Corollary 6.3.7. *Let Assumption 6.3.1 hold such that for some $\kappa \in (1/2, 1]$ there holds $\mathbb{E}(\max_{i=1,\dots,\tau} \|u\|_{H^{1+\kappa}(\mathcal{T}_i)}^2) < +\infty$ and let $\|(u_0 - u_{\ell,0})(\omega,\cdot)\|_H \leq C\|u_0(\omega,\cdot)\|_{H^2(\mathcal{D})}\overline{h}_\ell^2$ almost surely. The (fully) approximated QoI is defined by $\Psi_{\epsilon,\ell,\Delta t} := \psi(\overline{u}_{\epsilon,\ell})$. Then, there*

holds the error bound

$$\mathbb{E}(|\Psi - \Psi_{\epsilon,\ell,\Delta t}|^2)^{1/2} \leq C(\epsilon + \overline{h}_\ell^{2\kappa} + \Delta t).$$

Given a sequence of discretization tresholds $\overline{h}_\ell > 0$ for $\ell \in \mathbb{N}_0$, one should adjust ϵ and Δt such that $\overline{h}_\ell^{2\kappa} \simeq \epsilon \simeq \Delta t$ to achieve an error equilibrium. Hence, we denote the adjusted parameters on level ℓ by ϵ_ℓ and Δt_ℓ and assume that all errors are equilibrated in the sense that $c\overline{h}_\ell^{2\kappa} \leq \epsilon_\ell, \Delta t_\ell \leq C\overline{h}_\ell^{2\kappa}$ holds for constants $c, C > 0$ independent of ℓ. We further define $\Psi_\ell := \Psi_{\epsilon_\ell,\ell,\Delta t_\ell} = \psi(\overline{u}_{\epsilon_\ell,\ell})$ and obtain with Corollary 6.3.7

$$\mathbb{E}(|\Psi - \Psi_\ell|^2)^{1/2} \leq C\overline{h}_\ell^{2\kappa}. \tag{6.14}$$

6.4 Estimation of moments by multilevel Monte Carlo methods

As we are able to generate samples from $\Psi_\ell = \psi(\overline{u}_{\epsilon_\ell,\ell})$ and control for the discretization error in each sample, we may estimate the expectation $\mathbb{E}(\Psi)$ by Monte Carlo methods. For convenience, we restrict ourselves to the estimation of $\mathbb{E}(\Psi)$, but we note that all results from this section are valid when estimating higher moments of Ψ, given that $u \in L^r(\Omega; L^2(\mathbb{T}; V))$ for sufficiently high r (cf. Theorem 6.2.6). Our focus is on multilevel Monte Carlo (MLMC) estimators, since they are easily implemented, do not require much regularity of Ψ and are significantly more efficient than standard Monte Carlo estimators. The main idea of the MLMC estimation has been developed in [106] and later been rediscovered and popularized in [92]. In this section, we briefly recall the MLMC method and then show how we achieve a desired error rate by adjusting the number of samples on each level to the discretization bias. We also suggest a modification of the MLMC algorithm to increase computational efficiency before we verify our results in Section 6.5.

Let $L \in \mathbb{N}$ be a fixed (maximum) discretization level and assume that the approximation parameters on each level $\ell = 0, \dots, L$ satisfy $\overline{h}_\ell^{2\kappa} \simeq \epsilon_\ell \simeq \Delta t_\ell$ (see Section 6.3). This yields a sequence $\Psi_0, \dots, \Psi_L$ of approximated QoIs, hence the *MLMC estimator* of $\mathbb{E}(\Psi_L)$ is given by

$$E^L(\Psi_L) = \sum_{\ell=0}^{L} \frac{1}{M_\ell} \sum_{i=1}^{M_\ell} \Psi_\ell^{(i,\ell)} - \Psi_{\ell-1}^{(i,\ell)}, \tag{6.15}$$

where we have set $\Psi_{-1} := 0$. Above, $(\Psi_\ell^{(i,\ell)} - \Psi_{\ell-1}^{(i,\ell)}, i \in \mathbb{N})$ is a sequence of independent copies of $\Psi_\ell - \Psi_{\ell-1}$ and $M_\ell \in \mathbb{N}$ denotes the number of samples on each level. To achieve a desired target root mean-squared error (RMSE), this estimator requires less computational effort than the standard Monte Carlo approach under certain assump-

tions. This, by now, classical result was proven in [92, Theorem 3.1] for functionals of stochastic differential equations. The proof is rather general and may readily be transferred to other applications, for instance the estimation of functionals or moments of random PDEs, see [29, 93].

Theorem 6.4.1. *Let Assumption 6.3.1 hold such that for some $\kappa \in (1/2, 1]$ there holds $\mathbb{E}(\max_{i=1,\dots,\tau} \|u\|^2_{H^{1+\kappa}(\mathcal{T}_i)}) < +\infty$ and let $\overline{h}_{\ell-1} \leq C_1 \overline{h}_\ell$ for some $C_1 > 0$ for all $\ell \in \mathbb{N}_0$. For $L \in \mathbb{N}$ and given refinement parameters $\overline{h}_0 > \cdots > \overline{h}_L > 0$ choose $\Delta t_\ell, \epsilon_\ell > 0$ such that $\epsilon_\ell, \Delta t_\ell \leq C_2 \overline{h}_\ell^{2\kappa}$ holds for fixed $C_2 > 0$ and $\ell = 0, \dots, L$. Furthermore, let $(\rho_\ell, \ell = 1, \dots, L) \in (0,1)^L$ be a set of positive weights such that $\sum_{\ell=1}^L \rho_\ell = C_\rho$, with a constant $C_\rho > 0$ independent of L, and set*

$$M_0^{-1} := \lceil \overline{h}_L^{4\kappa} \rceil \quad \text{and} \quad M_\ell^{-1} := \lceil \frac{\overline{h}_L^{4\kappa}}{\overline{h}_\ell^{4\kappa}} \rho_\ell^{-2} \rceil \quad \text{for } \ell = 1, \dots, L.$$

Then, there is a $C > 0$, independent of L and κ, such that

$$\|\mathbb{E}(\Psi) - E^L(\Psi_L)\|_{L^2(\Omega;\mathbb{R})} \leq C \overline{h}_L^{2\kappa}.$$

Proof. As all error contributions $\epsilon_\ell, \Delta t_\ell$ are adjusted to $\overline{h}_\ell$, we obtain by the triangle inequality and Eq. (6.14)

$$\begin{aligned}
\|\mathbb{E}(\Psi) - E^L(\Psi_L)\|_{L^2(\Omega;\mathbb{R})} &\leq \|\mathbb{E}(\Psi) - \mathbb{E}(\Psi_L)\|_{L^2(\Omega;\mathbb{R})} + \|\mathbb{E}(\Psi_L) - E^L(\Psi_L)\|_{L^2(\Omega;\mathbb{R})} \\
&\leq \|\Psi - \Psi_L\|_{L^2(\Omega;\mathbb{R})} \\
&\quad + \|\sum_{\ell=0}^L \mathbb{E}(\Psi_\ell - \Psi_{\ell-1}) - \frac{1}{M_\ell} \sum_{i=1}^{M_\ell} (\Psi_\ell^{(i,\ell)} - \Psi_{\ell-1}^{(i,\ell)})\|_{L^2(\Omega;\mathbb{R})} \\
&\leq C \overline{h}_L^{2\kappa} + \sum_{\ell=0}^L \frac{1}{\sqrt{M_\ell}} \|\Psi_\ell - \Psi_{\ell-1}\|_{L^2(\Omega;\mathbb{R})}.
\end{aligned}$$

At this point we emphasize that we did not use the independence of $\Psi_\ell^{(i,\ell)} - \Psi_{\ell-1}^{(i,\ell)}$ across the levels $\ell = 1, \dots, L$ in the last inequality. We note that

$$\|\Psi_\ell - \Psi_{\ell-1}\|_{L^2(\Omega;\mathbb{R})} \leq \|\Psi - \Psi_\ell\|_{L^2(\Omega;\mathbb{R})} + \|\Psi - \Psi_{\ell-1}\|_{L^2(\Omega;\mathbb{R})} \leq C(1 + C_1) \overline{h}_\ell^{2\kappa}$$

for $\ell \geq 1$ and hence

$$\|\mathbb{E}(\Psi) - E^L(\Psi_L)\|_{L^2(\Omega;\mathbb{R})} \leq C \overline{h}_L^{2\kappa} + \|\Psi_0\|_{L^2(\Omega;\mathbb{R})} \overline{h}_L^{2\kappa} + C(1 + C_1) \overline{h}_L^{2\kappa} \sum_{\ell=1}^L \rho_\ell \leq C \overline{h}_L^{2\kappa}.$$

□

We remark that $C_\rho > 0$ may act as a normalizing constant if MLMC estimators based on different discretization techniques are compared, an example is provided in Section 6.5. To conclude this section, we briefly present a modified MLMC method to accelerate the estimation of $\mathbb{E}(\Psi_L)$. In the definition of the MLMC estimator from Eq. (6.15), the terms in the second sum are independent copies of the corrections $\Psi_\ell - \Psi_{\ell-1}$. Hence, one has to generate a total of $M_\ell + M_{\ell+1}$ samples of Ψ_ℓ for each $\ell = 0, \dots, L$ (where we have set $M_{L+1} := 0$). This effort may be reduced if we "recycle" the already available samples and generate the differences $\Psi_\ell^{(i,\ell)} - \Psi_{\ell-1}^{(i,\ell)}$ and $\Psi_{\ell+1}^{(i,\ell)} - \Psi_\ell^{(i,\ell)}$ based on the same realization $\Psi_\ell^{(i,\ell)}$. That is, we drop the second superscript ℓ above and arrive at the *coupled MLMC estimator*

$$E_C^L(\Psi_L) := \sum_{\ell=0}^{L} \frac{1}{M_\ell} \sum_{i=1}^{M_\ell} \Psi_\ell^{(i)} - \Psi_{\ell-1}^{(i)}. \tag{6.16}$$

Instead of $M_\ell + M_{\ell+1}$ realizations of Ψ_ℓ, the coupled MLMC estimator requires only M_ℓ samples of Ψ_ℓ. The copies $\Psi_\ell^{(i)}$ are still independent in i, but not anymore across all levels ℓ for a fixed index i. Clearly, $\mathbb{E}(E_C^L(\Psi_L)) = \mathbb{E}(\Psi_L)$, and it holds

$$\lim_{L\to+\infty} \mathbb{E}(E_C^L(\Psi_L)) = \lim_{L\to+\infty} \mathbb{E}(E^L(\Psi_L)) = \lim_{L\to+\infty} \mathbb{E}(\Psi_L) = \mathbb{E}(u).$$

The introduced modification is a simplified version of the *Multifidelity Monte Carlo estimator* (see [171]), where the weighting coefficients for all level corrections $\Psi_\ell - \Psi_{\ell-1}$ are set equal to one. An estimator similar to (6.16) with coupled correction terms has also been introduced in the context of SDEs in [181]. As we mentioned in the proof of Theorem 6.4.1, independence of the sampled differences $\Psi_\ell - \Psi_{\ell-1}$ across ℓ is not required for the error estimate, thus, the asymptotic order of convergence also holds for the coupled estimator. To compare RMSEs of the estimators from Eq. (6.15) and

(6.16), we calculate

$$\begin{aligned}
\mathrm{Var}(E_C^L(\Psi_L)) &= \mathrm{Var}\Big(\sum_{\ell=0}^{L}\sum_{i=M_{\ell+1}+1}^{M_\ell}\sum_{k=0}^{\ell}\frac{\Psi_k^{(i)}-\Psi_{k-1}^{(i)}}{M_k}\Big)\\
&= \sum_{\ell=0}^{L}(M_\ell - M_{\ell+1})\mathrm{Var}\left(\sum_{k=0}^{\ell}\frac{\Psi_k-\Psi_{k-1}}{M_k}\right)\\
&= \sum_{\ell=0}^{L}(M_\ell - M_{\ell+1})\Big(\sum_{k=0}^{\ell}\frac{\mathbb{V}_k}{M_k^2} + 2\sum_{k=0}^{\ell}\sum_{j=0}^{k-1}\frac{\mathbb{C}_{j,k}}{M_jM_k}\Big)\\
&= \sum_{k=0}^{L}\left(\frac{\mathbb{V}_k}{M_k^2} + 2\sum_{j=0}^{k-1}\frac{\mathbb{C}_{j,k}}{M_jM_k}\right)\sum_{\ell=k}^{L}(M_\ell - M_{\ell+1})\\
&= \mathrm{Var}(E^L(\Psi_L)) + 2\sum_{k=0}^{L}\sum_{j=0}^{k-1}\frac{\mathbb{C}_{j,k}}{M_j},
\end{aligned}$$

where $\mathbb{V}_k := \mathrm{Var}(\Psi_k - \Psi_{k-1})$ and $\mathbb{C}_{j,k} := \mathrm{Cov}(\Psi_j - \Psi_{j-1}, \Psi_k - \Psi_{k-1})$. Hence, the coupled estimator introduces a higher RMSE if the corrections $\Psi_\ell - \Psi_{\ell-1}$ are positively correlated across the levels. In this case, we trade in variance for simulation time and the ratio of this trade-off is problem-dependent and hard to assess in advance.

6.5 Numerical results

For our numerical experiment we consider $\mathcal{D} = (0,1)^2$ with $T = 1$, initial data $u_0(x_1,x_2) = \frac{1}{10}\sin(\pi x_1)\sin(\pi x_2)$, source term $f \equiv 1$ and set $\bar{a} \equiv 0$. The covariance operator Q of W is given by the by the *Matérn covariance function*

$$[Q\varphi](y) := \int_{\mathcal{D}} \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)}\Big(\sqrt{2\nu}\frac{\|x-y\|_2}{\chi}\Big)^{\nu} K_\nu\Big(\sqrt{2\nu}\frac{\|x-y\|_2}{\chi}\Big)\varphi(x)dx, \quad \varphi \in H,$$

with smoothness parameter $\nu > 0$, variance $\sigma^2 > 0$ and correlation length $\chi > 0$. Above, Γ denotes the Gamma function, $\|\cdot\|_2$ is the Euclidean norm in $\mathbb{R}^2$ and K_ν is the modified Bessel function of the second kind with ν degrees of freedom. We set the covariance parameters as $\nu = 1.5, \sigma = 0.5$ and $\chi = 0.1$, hence Assumption 6.2.3 is fulfilled, see [96]. To approximate the Gaussian field, we use the circulant embedding method from [97] to draw samples of W at a grid of discrete points in $\mathcal{D}$ and then use linear interpolation to obtain an extension to $\overline{\mathcal{D}}$. We choose a maximum distance of $\epsilon > 0$ for the grid points and denote the corresponding approximation by W_ϵ. Furthermore,

we set $\Phi(\cdot) = \exp(\cdot)$ and observe that for any $s \in [1, \infty)$

$$\|\Phi(W) - \Phi(W_\epsilon)\|_{L^s(\Omega;L^\infty(\mathcal{D}))} \leq C\mathbb{E}\Big(\Big(\sum_{j=1}^{d} \|\partial_{x_j}\Phi(W)\|_{L^\infty(\mathcal{D})}\epsilon\Big)^s\Big)^{1/s} \leq C\epsilon$$

holds by the path-wise Lipschitz regularity of W and Lemma 6.2.5 (cf. Assumption 6.3.1).

For the discontinuous random field P, we denote by $\mathcal{U}((c_1, c_2))$ the uniform distribution on the interval $(c_1, c_2) \subset \mathbb{R}$, sample four i.i.d. $\mathcal{U}((0.2, 0.8))$-distributed random variables $U_1, \dots, U_4$ and assign one U_i to each side of the square $\partial\mathcal{D}$. We then connect the points on two opposing edges by a straight line to obtain a random partition $\mathcal{T}$ consisting of $\tau = 4$ convex quadrangles. Finally, we assign independent jump heights $P_1, P_2 \sim \mathcal{U}((0, 1)), P_3 \sim \mathcal{U}((5, 6))$ and $P_4 \sim \mathcal{U}((10, 11))$ to the partition elements, such that two adjacent elements do not have the same jump distribution. This guarantees rather steep discontinuities across the interfaces in $\mathcal{T}$, see Figure 6.1. We do not need any approximation procedure for P and obtain $a_\epsilon := \exp(W_\epsilon) + P$. Clearly, a_ϵ satisfies Assumption 6.3.1 and we define $b_\epsilon := \max(-2a_\epsilon, -5)$. The QoI is given by

$$\Psi(u) := \int_{\mathcal{D}} u(x) \exp(-0.25\|(0.25, 0.75) - x\|_2^2)dx.$$

For the sample-adapted FE approach, we set the refinement parameters to $\overline{h}_\ell^{(a)} = \frac{1}{4}2^{-\ell/2}$ for $\ell \in \mathbb{N}_0$ and choose $\epsilon_\ell^{(a)} = \Delta t_\ell^{(a)} = (\overline{h}_\ell^{(a)})^2$. While this choice gives an error equilibrium for $\kappa = 1$, it ensures that for any $\kappa < 1$ the RMSE is dominated solely by the spatial discretization error. Thus, we may infer the true value of κ from the numerical experiment. We also consider a non-adapted FE method with fixed and deterministic triangulations on $\mathcal{D}$. For given approximation parameters $\epsilon, \overline{h}_\ell^{(na)}$ and Δt in the non-adapted setting, we may not expect a better error bound than

$$\mathbb{E}(|\Psi - \Psi_{\epsilon,\ell,\Delta t}|^2)^{1/2} \leq C(\epsilon + \overline{h}_\ell^{(na)} + \Delta t)$$

in Corollary 6.3.7. This is due to the fact that the standard FE method for elliptic problems with discontinuous coefficients does not converge at a better rate than $\mathcal{O}((\overline{h}^{(na)})^{1/2})$ in the V-norm, see [31, Remark 4.2]. Thus, if we consider again the dual problem as in Theorem 6.3.4, we may not expect a better rate than $\mathcal{O}(\overline{h}^{(na)})$ with respect to the H-norm. We choose the non-adapted FE grid with diameter $\overline{h}_\ell^{(na)} := \frac{1}{4}2^{-\ell}$ and set accordingly $\epsilon_\ell^{(na)} = \Delta t_\ell^{(na)} = \overline{h}_\ell^{(na)}$. In both FE methods, we use the midpoint rule on each triangle to approximate the entries of the stiffness matrix. The resulting quadrature error is of order $\mathcal{O}(\overline{h}_\ell^2)$ with respect to the H-norm in the sample-adapted

case and hence does not dominate the overall approximation error, see [98, Section 2]. For non-adapted FE, no a-priori estimate on the quadrature error is possible due to the discontinuities in a and b, but our results suggest that this bias also in line with the overall approximation error. As $\epsilon_{\ell-1} = 2\epsilon_\ell$, the circulant embedding grids (to sample W_ϵ) are nested and we may achieve the MLMC coupling by first generating the discrete set of points on level ℓ and then taking the appropriate subset of points for level $\ell - 1$.

In the sample-adapted MLMC algorithm, we choose the number of samples via

$$(M_0^{(a)})^{-1} = \lceil (\overline{h}_L^{(a)})^4 \rceil \quad \text{and} \quad (M_\ell^{(a)})^{-1} = \lceil \frac{1}{4} \frac{(\overline{h}_L^{(a)})^4}{(\overline{h}_\ell^{(a)})^4} \Big(\frac{(\ell+1)^{-1.001}}{\sum_{k=1}^L (k+1)^{-1.001}} \Big)^{-2} \rceil$$

for $\ell = 1, \dots, L$, whereas, we choose

$$(M_0^{(na)})^{-1} = \lceil (\overline{h}_L^{(na)})^2 \rceil \quad \text{and} \quad (M_\ell^{(na)})^{-1} = \lceil \frac{(\overline{h}_L^{(na)})^2}{(\overline{h}_\ell^{(na)})^2} \Big(\frac{(\ell+1)^{-1.001}}{\sum_{k=1}^L (k+1)^{-1.001}} \Big)^{-2} \rceil$$

in the non-adapted MLMC approach. Basically, we choose $1/M_\ell$ proportional to $\mathbb{V}_\ell = \mathrm{Var}(\Psi_\ell - \Psi_{\ell-1})$ on each level and thus distribute the errors equally across all levels. Another possibility would be to distribute the computational effort equally (see [93]), which requires estimates on the cost of a single sample on each level. The sequence $(\ell^{-c}, \ell \in \mathbb{N})$ decreases rapidly for $c > 1$ and sums up to $\zeta(c) < +\infty$, where $\zeta(\cdot)$ is the Riemann ζ-function. Hence, the above choice of ρ_i ensures that only a few expensive samples on high levels are necessary and, due to the uniform bound $\sum_{\ell=1}^L \rho_\ell < \zeta(c)$, it is well suited to compare estimators for a varying choice of L. In terms of Theorem 6.4.1, we have chosen $C_\rho = 2$ for the number of samples in the sample-adapted method, whereas $C_\rho = 1$ for standard FE. Similar calculations as in Theorem 6.4.1 show that this choice leads to $\|\Psi - E^L(\Psi_L)\|_{L^2(\Omega,\mathbb{R})} \leq C(2^{-2-L})$ in either case, where the constant C is the same for adapted and non-adapted FE. Hence, C_ρ is merely a normalizing constant and the above choice of M_ℓ ensures that both approaches produce a comparable error for fixed L. Finally, we calculate a reference QoI $\Psi_{ref} := E^L(\Psi_L)$ with $L = 7$ and the sample-adapted method and estimate the relative RMSE $\|\Psi_{ref} - E^L(\Psi_L)\|_{L^2(\Omega,\mathbb{R})}/\Psi_{ref}$ for $L = 0, \dots, 5$ based on 50 independent samples of $E^L(\Psi_L)$ for the sample-adapted and non-adapted MLMC algorithm. For each approach, we use adapted/non-adapted FE combined with a standard/coupled MLMC estimator, thus we compare a total of four algorithms regarding their error decay and efficiency.

Figure 6.1 confirms our theoretical results from Section 6.3, i.e. the sample-adapted spatial discretization yields rate $\mathcal{O}(\overline{h}_\ell^2)$ compared to $\mathcal{O}(\overline{h}_\ell)$ in the non-adapted setting. Hence, we are able to choose coarser spatial grids in the first approach which entails a

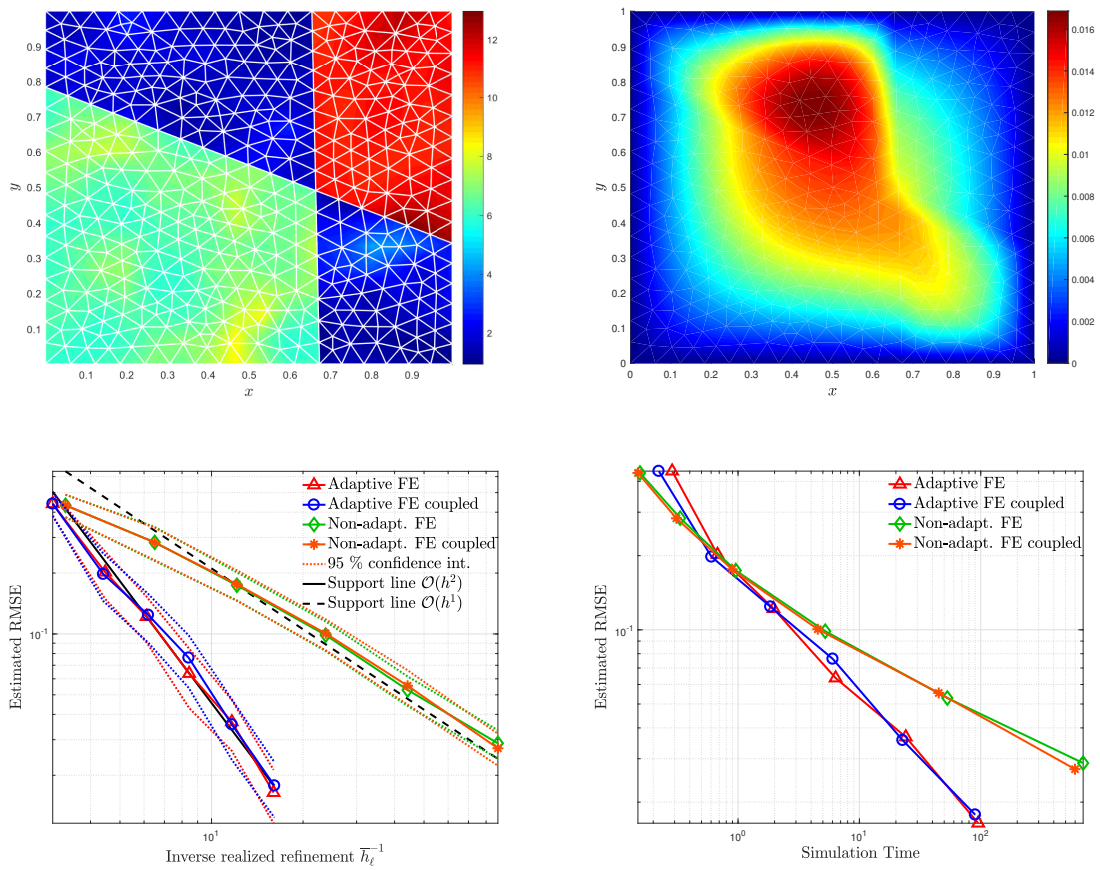

Figure 6.1 Top: Sample of the diffusion coefficient with sample-adapted FE grid (left) and FE solution at $T = 1$ (right). Bottom: RMSE vs. refinement (left) and RMSE vs. simulation time (right).

better time-to-error ratio for both sample-adapted methods. The results also indicate that $\kappa \approx 1$ holds for this particular example, otherwise we would see a lower rate of convergence than $\mathcal{O}(\overline{h}_\ell^2)$ for the sample-adapted methods. While the sample-adapted FE grids have to be generated new for each sample, the $L+1$ deterministic grids for the non-adapted FE method are generated and stored before the Monte Carlo loop. However, as we see from the time-to-error plot, the extra work of renewing the FE meshes for each sample in the sample-adapted method is more than compensated by the increased order of convergence. The computational cost of the sample-adapted MLMC estimators are (roughly) inversely proportional to the squared errors, which is the best possible results one may achieve with MLMC, see [93] and the references therein. To conclude, we remark that the coupled MLMC estimator yields a slight gain in efficiency if combined with non-adapted FE, whereas it produces similar results when using the sample-adapted discretization. We emphasize that there are scenarios

where the coupled estimator outperforms standard MLMC and, on the other hand, there are examples were coupling performs worse due to high correlation terms $\mathbb{C}_{j,k}$ (for both, we refer to numerical examples in [31].) Hence, even though performance is similar to standard MLMC, it makes sense to consider the coupled estimator in our scenario. As we have mentioned at the end of Section 6.4, this behavior may not be expected a-priori.

Comments by A. Stein on "A multilevel Monte Carlo algorithm for parabolic advection-diffusion problems with discontinuous coefficients":

In Definition 6.2.2, we have introduced the advection coefficient b via

$$b : \Omega \times \mathcal{D} \to \mathbb{R}, \quad (\omega, x) \mapsto \min(\max(b_1 a(\omega, x), -\overline{b}_2), \overline{b}_2)$$

for some constants $b_1, \overline{b}_2 > 0$. This ensures the uniform boundedness of b in $\Omega \times \mathcal{D}$ by $\overline{b}_2 > 0$. Therefore, the advection term in the random PDE in this chapter admits the slightly unusual form

$$b\mathbf{1}^T \cdot \nabla u = b \sum_{i=1}^{d} \partial_{x_i} u.$$

This is in contrast to the previous chapter ([32]), which is the foundation of this article and where we have introduced the more general advection coefficient as a random vector

$$b : \omega \times \mathcal{D} \to \mathbb{R}^d, \quad (\omega, x) \mapsto a(\omega, x)\widetilde{b}(x),$$

in Definition 5.3.1 with $\widetilde{b} \in L^\infty(\mathcal{D})^d$. The corresponding advection term in Chapter 5 is then of the common form $b \cdot \nabla u$. After the first stage of review of [32]/Chapter 5, it turned out that the latter representation is sufficient to derive all error estimates and we did not see any changes in the numerical experiments. Hence, changing to a vector-valued coefficient $b(\omega, x) = a(\omega, x)\widetilde{b}(x)$ would not affect the results from this chapter qualitatively, but only allow for a slightly more general advection term. At the point of writing, however, Chapter 6 was already accepted for publication in its current form. Therefore, I decided to keep the contents of this chapter as originally accepted by the conference proceedings and added this comment to explain the different advection coefficients in this chapter and Chapter 5.

7 Approximation and simulation of infinite-dimensional Lévy processes

Andreas Stein and Andrea Barth

First published in "Stochastics and Partial Differential Equations: Analysis and Computations", 2018, Springer, volume 6, issue 2, pp. 286–334.
URL: https://link.springer.com/article/10.1007/s40072-017-0109-2.

Abstract: In this paper approximation methods for infinite-dimensional Lévy processes, also called (time-dependent) Lévy fields, are introduced. For square integrable fields beyond the Gaussian case, it is no longer given that the one-dimensional distributions in the spectral representation with respect to the covariance operator are independent. When simulated via a Karhunen-Loève expansion a set of dependent but uncorrelated one-dimensional Lévy processes has to be generated. The dependence structure among the one-dimensional processes ensures that the resulting field exhibits the correct point-wise marginal distributions. To approximate the respective (one-dimensional) Lévy-measures, a numerical method, called discrete Fourier inversion, is developed. For this method, L^p-convergence rates can be obtained and, under certain regularity assumptions, mean square and L^p-convergence of the approximated field is proved. Further, a class of (time-dependent) Lévy fields is introduced, where the point-wise marginal distributions are dependent but uncorrelated subordinated Wiener processes. For this specific class one may derive point-wise marginal distributions in closed form. Numerical examples, which include hyperbolic and normal-inverse Gaussian fields, demonstrate the efficiency of the approach.

7.1 Introduction

Uncertainty quantification plays an increasingly important role in a wide range of problems in the Engineering Sciences and Physics. Examples of sources of uncertainty are imprecise or insufficient measurements and noisy data. In the underlying dynamical system this is modeled via a stochastic operator, stochastic boundary conditions and/or stochastic data. As an example, to model subsurface flow more realistically the coefficients of an (essentially) elliptic equation are assumed to be stochastic. A common approach in the literature is to use (spatially) correlated random fields that are built from uniform distributions or colored log-normal fields. The resulting point-wise marginal distributions of the field are (shifted) normally, resp. log-normally distributed. Neither choice is universal enough to accommodate all possible types of porosity, especially not if fractures are incorporated (see [202]). In some applications it might even be necessary that the point-wise marginal distribution of the (time-dependent) random field is a pure-jump process (see [24]). Here, we denominate by *point-wise marginal distributions* the distributions resp. processes one obtains by evaluation of the random field at a fixed spatial point. On a note, these are in general the distributions that may be measured in applications.

In the case of a (time-dependent) Gaussian random field, the approximation and simulation via its Karhunen-Loève (KL) expansion is straightforward. Almost sure and L^p-convergence in terms of the decay of the eigenvalues has been shown for truncated KL-expansions in [26]. For infinite-dimensional Lévy processes, also called *Lévy fields*, the approximation may still be attempted via the KL expansions: On a separable Hilbert space $(U, (\cdot,\cdot)_U)$ with orthonormal basis $(e_i, i \in \mathbb{N})$, a square-integrable Lévy field $L = (L(t) \in U, t \geq 0)$ admits the expansion

$$L(t) = \sum_{i\in\mathbb{N}} (L(t), e_i)_U e_i,$$

The sequence $((L(\cdot), e_i)_U, i \in N)$ consists of one-dimensional, real-valued Lévy processes. In contrast to a Gaussian field, the one-dimensional processes $((L(t), e_i)_U, t \geq 0)$ in the spectral representation are not independent but merely uncorrelated. If one were to use independent Lévy processes, the resulting field would not have the desired point-wise marginal distributions and the KL expansion would, therefore, not converge to the desired Lévy field. To circumvent this issue, we approximate L by truncating the series after finite number of terms and generate dependent but uncorrelated processes $((L(t), e_i)_U, t \geq 0)$.

This entails, however, the simulation of one-dimensional Lévy processes. A common

way to do so, is to employ the so called *compound Poisson approximation* (CPA) (see [10, 84, 155, 186, 188] or the references therein). Mean-square convergence results for the CPA are available in some cases, but require rather strong assumptions on the underlying process. In addition, the obtained convergence rates are comparably low with respect to the employed time discretization, which implies that the CPA may not be suitable to sample processes involving computationally expensive components. As one of the main contributions in this paper, we develop a novel approximation method for one-dimensional Lévy processes. This new approach, based on Lévy bridge laws and Fourier inversion, addresses the abovementioned problems. We prove L^p- and almost surely convergence of the approximation under relatively weak assumptions and derive precise error bounds. We show mean-square convergence of the approximation to a given infinite-dimensional Lévy process by combining the Fourier inversion method with an appropriate truncation of the KL expansion.

To obtain a set of dependent but uncorrelated one-dimensional processes, we utilize multi-dimensional time-changed Brownian motions. The underlying variance process is represented by a positive and increasing Lévy process, a so-called *subordinator*. As a class of subordinated processes, we consider *generalized hyperbolic (GH) Lévy processes*, that are based on the generalized hyperbolic distribution and cover for example *normal inverse Gaussian* (NIG) and *hyperbolic* processes. These processes are widely used in applications such as Mathematical Finance, Physics and Biology (see, for instance, [18, 24, 43, 75, 76]). With its fat-tailed distribution a GH-field may also be of value in the modeling of subsurface flows (see [202]). For an overview on subordinated, Hilbert space-valued Lévy processes we refer to [49, 172], where this topic is treated in a rather general setting. Among other subordinated Wiener processes, the construction of an infinite-dimensional NIG process can be found in [38]. As a further contribution of this paper, we approximate the corresponding GH Lévy fields via truncated KL expansions with dependent but uncorrelated GH-distributed one-dimensional processes and show that the approximation converges to an infinite-dimensional GH process. From a simulatory point of view this entails the generation of a certain number of one-dimensional processes with a given set of parameters. Conversely, we introduce a second approach, where we derive the dependence structure of the multi-dimensional GH process to obtain admissible sets of parameters such that the one-dimensional marginal GH processes are decorrelated and follow a desired distribution. Using the Fourier inversion method we are able to simulate GH fields efficiently, even if a large number of one-dimensional GH processes is necessary.

This article is structured as follows: Section 7.2 contains preliminaries on Lévy processes taking values in Hilbert spaces and the main convergence theorem for the

approximation. In Section 7.3, we present a new approach for the approximation of one-dimensional Lévy processes by Lévy bridge laws and prove L^p- and almost sure convergence. To be able to apply the algorithm in a very general setting, we introduce an extension by using Fourier inversion techniques and show how to control the L^p-error. We proceed by investigating the class of GH Lévy processes and state the necessary conditions for the approximated field to have point-wise GH distributed marginals. In Section 7.5, we remark on some implementational details of the algorithm and conclude with NIG- and hyperbolic fields as numerical examples.

7.2 Preliminaries

Throughout this paper, we consider a time interval $\mathbb{T} := [0, T]$, with $T > 0$, and a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$ satisfying the usual conditions. Let $(U, (\cdot, \cdot)_U)$ be a separable Hilbert space and $(U, \mathcal{B}(U))$ a measurable space, where $\mathcal{B}(U)$ denotes the Borel σ-algebra on U. A Lévy process taking values in $(U, (\cdot, \cdot)_U)$ is defined as follows (see [173]):

Definition 7.2.1. A U-valued stochastic process $L = (L(t), t \in \mathbb{T})$ is called Lévy process[1] if

- L has stationary and independent increments,
- $L(0) = 0$ $\mathbb{P}$-almost surely and
- L is stochastically continuous, i.e. for all $\varepsilon > 0$ and $t \in \mathbb{T}$ holds

$$\lim_{s \to t, s \in \mathbb{T}} \mathbb{P}(||L(t) - L(s)||_U > \varepsilon) = 0.$$

The characteristic function of a Lévy process is then given by the *Lévy-Khintchine formula*:

$$\mathbb{E}[\exp(i(h, L(t))_U)] = \exp(t\Psi_L(\psi)), \quad \text{for } \psi \in U,$$

where the exponent is of the form

$$\begin{aligned} \Psi_L(\psi) = i(\iota_U, \psi)_U - \frac{1}{2}(\Sigma_U \psi, \psi)_U \\ + \int_U \exp(i(\psi, y)_U) - 1 - i(\psi, y)_U \mathbf{1}_{||y||_U < 1} \nu_U(dy) \end{aligned} \tag{7.1}$$

[1] In the case that U is an infinite-dimensional Hilbert space, sometimes L is also called *Lévy field* to have a clear distinction from finite-dimensional Lévy processes.

(see [173, Thm. 4.27]). In Eq. (7.1), $\iota_U \in U$, Σ_U is a symmetric, non-negative and nuclear operator on U and $\nu_U : \mathcal{B}(U) \to [0, \infty)$ is a non-negative, σ-finite measure on $\mathcal{B}(U)$ satisfying

$$\nu_U(\{0\}) = 0 \quad \text{and} \quad \int_U \min(1, ||y||_U^2)\, \nu(dy) < +\infty.$$

The triplet $(\iota_U, \Sigma_U, \nu_U)$ is unique for every Lévy process L and called the *characteristic triplet*. For the special case of a one-dimensional Lévy process $\ell = (\ell(t), t \in \mathbb{T})$, the Lévy-Khintchine formula simplifies to

$$\mathbb{E}[\exp(iu\ell(t))] = \exp\Big(t\Big(\iota ui - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} \exp(iuy) - 1 - iuy\mathbf{1}_{|y|<1} d\nu(y)\Big)\Big), \tag{7.2}$$

for $u \in \mathbb{R}$, where $\iota \in \mathbb{R}$, $\sigma^2 > 0$ and ν is a (σ-finite) measure on $\mathcal{B}(\mathbb{R})$ satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} \min(1, y^2)\nu(dy) < +\infty,$$

see for instance [22] or [187]. The notation ℓ for finite-dimensional Lévy processes is introduced to have a clear distinction from the possibly infinite-dimensional Lévy process L.

If W is a U-valued Lévy field with characteristic triplet $(0, \Sigma_U, 0)$, then W is called Σ_U-*Wiener process*. If, further, Σ_U is symmetric, non-negative and nuclear (see Eq. (7.1)) it admits, by the Hilbert-Schmidt theorem, the spectral decomposition

$$\Sigma_U \hat{e}_i = \hat{\eta}_i \hat{e}_i.$$

Here, $((\hat{\eta}_i, \hat{e}_i), i \in \mathbb{N})$ is the sequence of eigenpairs of Σ_U, where the eigenvalues $\hat{\eta}_i$ are positive with zero as their only accumulation point and the sequence $(\hat{e}_i, i \in \mathbb{N})$ forms an orthonormal basis of U. For convenience, we assume that the sequence of eigenvalues $(\hat{\eta}_i, i \in \mathbb{N})$ is given in decaying order. The Σ_U-Wiener process W admits then a unique expansion (also called Karhunen–Loève expansion)

$$W(t) = \sum_{i \in \mathbb{N}} \sqrt{\hat{\eta}_i} \hat{e}_i w_i(t),$$

where $(w_i, i \in \mathbb{N})$ is a sequence of independent, one-dimensional, real-valued Brownian motions. An obvious way to approximate W is, therefore, given by the truncated series

$$W_N(t) := \sum_{i=1}^{N} \sqrt{\hat{\eta}_i} \hat{e}_i w_i(t).$$

It can be shown that the approximations $(W_N, N \in \mathbb{N})$ converge in $L^2(\Omega; U)$ and almost surely to the Σ_U-Wiener process W (see for instance [27]). For the approximation of general (non-continuous) processes L, we aim to apply a similar approach. We assume L is square-integrable, as otherwise L does not admit a KL expansion. For series representations of cylindrical Lévy processes we refer to [7], KL expansions for white noise Lévy fields may be found in [74]. A U-valued stochastic process $(L(t), t \in \mathbb{T})$ is said to be square-integrable if $||L(t)||_{L^2(\Omega;U)} := \mathbb{E}(||L(t)||_U^2) < +\infty$ for all $t \in \mathbb{T}$. Obviously, mean-square convergence can only be well-defined for processes with his property.

Theorem 7.2.2. *([173, Theorem 4.44]) Let L be a square-integrable Lévy process on U. Then there exists a $m \in U$ and a non-negative, symmetric trace class operator Q on U such that for all $\psi_1, \psi_2 \in U$ and $s, t \in (0, T]$*

- $\mathbb{E}((L(t), \psi_1)_U) = t(m, \psi_1)_U$,
- $\mathbb{E}((L(t) - mt, \psi_1)_U (L(s) - ms, \psi_2)_U) = \min(t, s)(Q\psi_1, \psi_2)_U$
- $\mathbb{E}(||L(t) - mt||_U^2) = t\ Tr(Q)$,

where $Tr(Q)$ denotes the trace of Q. The operator Q is also called covariance operator of L and m is called mean.

Note that Q in Theorem 7.2.2 is not necessarily equal to the operator Σ_U from the Lévy-Khintchine formula (7.1). They are only equal if the measure ν_U is zero, meaning the process L has no "jump component"[2]. The operator Q admits a spectral decomposition with a sequence $((\eta_i, e_i), i \in \mathbb{N})$ of orthonormal eigenpairs with non-negative eigenvalues. Thus, L has the spectral expansion

$$L(t) = \sum_{i \in \mathbb{N}} (L(t), e_i)_U e_i,$$

where the one-dimensional Lévy processes $((L(t), e_i)_U, t \in \mathbb{T})$ are not independent, but merely uncorrelated (see [173, Section 4.8.2]). For the approximation of L we employ one-dimensional Lévy processes $(\sqrt{\eta_i}\ell_i, i \in \mathbb{N})$, so that $\sqrt{\eta_i}\ell_i(t)$ is equal to $(L(t), e_i)_U$ in distribution for all $t \in \mathbb{T}$ and all $i \in \mathbb{N}$, and define, for $N \in \mathbb{N}$, the truncated sum

$$L_N(t) := \sum_{i=1}^{N} \sqrt{\eta_i} e_i \ell_i(t).$$

[2]This results in L being a drifted U-valued Gaussian process.

If the spectral basis $((\sqrt{\eta_i}e_i), i \in \mathbb{N})$ of U is given, the approximation of L by L_N reduces to the simulation of dependent but uncorrelated one-dimensional processes ℓ_i. In general, the processes ℓ_i have *infinite activity*, i.e. $\mathbb{P}$-almost all paths of the process $(\ell_i(t), t \in \mathbb{T})$ have an infinite number of jumps in every compact time interval. Popular examples of Lévy processes with infinite activity are normal inverse Gaussian processes or hyperbolic processes, see [75]. As it is not possible to simulate infinitely many jumps, we need to find a suitable approximation $\widetilde{\ell}_i$ of ℓ_i and define

$$\widetilde{L}_N(t) := \sum_{i=1}^{N} \sqrt{\eta_i} e_i \widetilde{\ell}_i(t).$$

In the following, we derive a condition on the approximations $\widetilde{\ell}_i$ that ensures convergence of $\widetilde{L}_N$ to L in $L^2(\Omega; U)$ uniformly on $\mathbb{T}$. Throughout this paper, we construct approximations $\widetilde{\ell}_i$ from a skeleton of discrete realizations at fixed and equidistant points in $\mathbb{T}$. To this end, we introduce, for given $n \in \mathbb{N}$, a time increment $\Delta_n := T/2^n$ and the set $\Theta_n := \{t_j := j\Delta_n,\ j = 0, \dots, 2^n\}$. By $\widetilde{\ell}_i^{(n)}$ we denote some piecewise-constant càdlàg approximation of the process $\widetilde{\ell}_i$ (for a construction see Section 7.3).

Theorem 7.2.3. *Let $L = (L(t), t \in \mathbb{T})$ be a square-integrable, U-valued Lévy process. The covariance operator Q of L admits a spectral decomposition by a sequence of (orthonormal) eigenpairs $((\eta_i, e_i), i \in \mathbb{N})$. Assume that, for $n \in \mathbb{N}$, there exists a sequence of approximations $(\widetilde{\ell}_i^{(n)}, i \in \mathbb{N})$ of the one-dimensional processes $(\ell_i, i \in \mathbb{N})$ on the interval $\mathbb{T}$, such that the $L^2(\Omega; \mathbb{R})$-approximation error can be bounded by*

$$\sup_{t \in \mathbb{T}} \mathbb{E}(|\ell_i(t) - \widetilde{\ell}_i^{(n)}(t)|^2) \leq C_\ell \Delta_n, \tag{7.3}$$

where the constant $C_\ell > 0$ is independent of i. If, for all $i \in \mathbb{N}$, the processes $\sqrt{\eta_i}\ell_i$ are in distribution equal to $(L(\cdot), e_i)_U$ then the sequence of approximations $(\widetilde{L}_N(t), N \in \mathbb{N})$ converges in mean-square-sense to $L(t)$, for each $t \in \mathbb{T}$, and the error is bounded by

$$\sup_{t \in \mathbb{T}} \mathbb{E}(||L(t) - \widetilde{L}_N(t)||_U^2)^{1/2} \leq \Big(T \sum_{i=N+1}^{\infty} \eta_i\Big)^{1/2} + \Big(C_\ell \Delta_n \sum_{i=1}^{N} \eta_i\Big)^{1/2}.$$

Proof. We may assume without loss of generality that the process L has zero mean. Using the triangle inequality, the error term $\mathbb{E}(||L(t) - \widetilde{L}_N(t)||_U^2)$ can be split into

$$\mathbb{E}(||L(t) - \widetilde{L}_N(t)||_U^2)^{1/2} \leq \mathbb{E}(||L(t) - L_N(t)||_U^2)^{1/2} + \mathbb{E}(||L_N(t) - \widetilde{L}_N(t)||_U^2)^{1/2}.$$

The square-integrability of L guarantees that Q is trace class and has positive eigen-

values, i.e. $\mathrm{Tr}(Q) = \sum_{i\in\mathbb{N}} \eta_i < +\infty$. $L(t)$ has covariance tQ, which yields for the first error term

$$\begin{aligned}
\mathbb{E}(||L(t) - L_N(t)||_U^2) &= \mathbb{E}(||L(t)||_U^2) + \mathbb{E}(||L_N(t)||_U^2) - 2\mathbb{E}((L(t), L_N(t))_U) \\
&= t\ \mathrm{Tr}(Q) + \mathbb{E}\Big(\sum_{i,j=1}^{N} \Big((L(t), e_i)_U e_i, (L(t), e_j)_U e_j\Big)_U\Big) \\
&\quad - 2\mathbb{E}\Big(\sum_{i=1}^{N} (L(t), (L(t), e_i)_U e_i)_U\Big) \\
&= t\sum_{i=1}^{\infty} \eta_i + \sum_{i=1}^{N} \mathbb{E}\Big((L(t), e_i)_U^2\Big) - 2\sum_{i=1}^{N} \mathbb{E}\Big((L(t), e_i)_U^2\Big) \\
&= t\sum_{i=1}^{\infty} \eta_i - \sum_{i=1}^{N} \mathbb{E}\Big((L(t), e_i)_U^2\Big).
\end{aligned}$$

With Theorem 7.2.2 we obtain

$$\mathbb{E}((L(t), e_i)_U^2) = t(Qe_i, e_i)_U = t\eta_i,$$

and hence

$$\sup_{t\in\mathbb{T}} \mathbb{E}(||L(t) - L_N(t)||_U^2) = \sup_{t\in\mathbb{T}} t \sum_{i=N+1}^{\infty} \eta_i = T \sum_{i=N+1}^{\infty} \eta_i.$$

As Q is a trace class operator, the sum on the right hand side becomes arbitrary small as $N \to +\infty$. This implies that L_N converges in $L^2(\Omega; U)$ uniformly on $\mathbb{T}$ to L.

For the second error term, we derive with the assumption that $\sqrt{\eta_i}\ell_i \stackrel{\mathcal{L}}{=} (L(\cdot), e_i)_U$ for all $i \in \mathbb{N}$ and Ineq. (7.3)

$$\begin{aligned}
\sup_{t\in\mathbb{T}} \mathbb{E}(||L_N(t) - \widetilde{L}_N(t)||_U^2) &= \sup_{t\in\mathbb{T}} \sum_{i,j=1}^{N} \mathbb{E}\Big(\sqrt{\eta_i\eta_j}(\ell_i(t) - \widetilde{\ell}_i^{(n)}(t))(\ell_j(t) - \widetilde{\ell}_j^{(n)}(t))(e_i, e_j)_U\Big) \\
&= \sum_{i=1}^{N} \eta_i ||e_i||_U^2 \sup_{t\in\mathbb{T}} \mathbb{E}(|\ell_i(t) - \widetilde{\ell}_i^{(n)}(t)|^2) \\
&\leq C_\ell \Delta_n \sum_{i=1}^{N} \eta_i,
\end{aligned}$$

which proves the claim. Above and for the remainder of the paper we express equality in distribution by the relation $\stackrel{\mathcal{L}}{=}$. □

Remark 7.2.4. Theorem 7.2.3 states that the approximation $\widetilde{L}_N$ converges in $L^2(\Omega; U)$ to L uniformly on $\mathbb{T}$, for $N \to +\infty$ and in the case that Ineq. (7.3) holds with a constant C_ℓ in the limit $\Delta_n \to 0$. We may equilibrate both error contributions by

choosing $N \in \mathbb{N}$ such that

$$T \sum_{i=N+1}^{\infty} \eta_i \approx C_\ell \Delta_n \sum_{i=1}^{N} \eta_i. \tag{7.4}$$

The sum of the eigenvalues, $\mathrm{Tr}(Q)$, is often known a priori (for example if Q is a covariance operator of the Matérn class, see Section 7.5). Then, only the first N eigenvalues have to be determined until Eq. (7.4) is fulfilled. Further, optimal values for Δ_n and N may be chosen for given C_ℓ and $(\eta_i, i \in \mathbb{N})$.

Theorem 7.2.3 may be generalized in an L^p-sense (the supremum is omitted for simplicity).

Corollary 7.2.5. *Let the assumptions of Theorem 7.2.3 be fulfilled and, for $p \geq 2$, $\mathbb{E}(||L(t)||^p) < +\infty$ for each $t \in \mathbb{T}$, $\sum_{i\in\mathbb{N}} \eta_i^{p/2} < +\infty$ and*

$$\mathbb{E}(|\ell_i(t) - \widetilde{\ell}_i^{(n)}(t)|^p) \leq C_{p,\ell} \Delta_n,$$

for some $C_{p,\ell} > 0$ independent of i. Then, the $L^p(\Omega; H)$-error is bounded by

$$\begin{aligned}\mathbb{E}(||L(t) - \widetilde{L}_N(t)||_U^p)^{1/p} \leq \Big(\sum_{i>N} \eta_i\Big)^{1/2-1/p} \Big(\sum_{i>N} \eta_i^{p/2} \mathbb{E}(|\ell_i(t)|^p)\Big)^{1/p} \\ + \Big(\sum_{i=1}^{N} \eta_i\Big)^{1/2-1/p} \Big(C_{\ell,p} \Delta_n \sum_{i=1}^{N} \eta_i^{p/2}\Big)^{1/p}.\end{aligned}$$

Proof. The proof follows closely the one of Theorem 7.2.3. We split the error into

$$\mathbb{E}(||L(t) - \widetilde{L}_N(t)||_U^p)^{1/p} \leq \mathbb{E}(||L(t) - L_N(t)||_U^p)^{1/p} + \mathbb{E}(||L_N(t) - \widetilde{L}_N(t)||_U^p)^{1/p}.$$

For the first we obtain

$$\mathbb{E}(||L(t) - L_N(t)||_U^p) = \mathbb{E}((||L(t) - L_N(t)||_U^2)^{p/2}) = \mathbb{E}\Big(\Big(\sum_{i>N} (L(t), e_i)_U^2\Big)^{p/2}\Big).$$

Since $\mathbb{E}(||L(t)||^p) < +\infty$, L_N converges to $L(t)$ in $L^p(\Omega; U)$ by the Monotone Convergence Theorem. Moreover, using $(L(t), e_i) \overset{\mathcal{L}}{=} \sqrt{\eta_i}\ell_i(t)$ and Jensen's inequality we may bound the above error via

$$\mathbb{E}(||L(t) - L_N(t)||_U^p) = \mathbb{E}\Big(\Big(\sum_{i>N} \eta_i \ell_i(t)^2\Big)^{p/2}\Big) \leq \Big(\sum_{i>N} \eta_i\Big)^{p/2-1} \sum_{i>N} \eta_i^{p/2} \mathbb{E}(|\ell_i(t)|^p),$$

where we have used that $\mathbb{E}(|(L(t), e_i)_U|^p) = \eta_i^{p/2}\mathbb{E}(|\ell_i(t)|^p)$ and $\mathbb{E}(||L(t)||^p) < +\infty$. Compared to the case $p = 2$ with $\mathbb{E}(\eta_i^{p/2}|\ell_i(t)|^p) = \eta_i$, one needs additional assumptions

on the p-th moment of ℓ_i to obtain an explicit bound. In a similar fashion, the second error contribution is then bounded by

$$\begin{aligned}\mathbb{E}(||L_N(t)-\widetilde{L}_N(t)||_U^p) &= \mathbb{E}\Big(\Big(\sum_{i=1}^N \eta_i|\ell_i(t)-\widetilde{\ell}_i^{(n)}(t)|^2\Big)^{p/2}\Big)\\ &\leq \Big(\sum_{i=1}^N \eta_i\Big)^{p/2-1}\sum_{i=1}^N \eta_i^{p/2}\mathbb{E}(|\ell_i(t)-\widetilde{\ell}_i^{(n)}(t)|^p)\\ &\leq C_{\ell,p}\Delta_n\Big(\sum_{i=1}^N \eta_i\Big)^{p/2-1}\sum_{i=1}^N \eta_i^{p/2}.\end{aligned}$$

□

By a Borel–Cantelli-type argument almost sure convergence follows from Theorem 7.2.3.

Corollary 7.2.6. *Let the assumptions of Theorem 7.2.3 hold and the eigenvalues of Q fulfill $\sum_{i\in\mathbb{N}}\eta_i(i-1)<+\infty$. If for each $N\in\mathbb{N}$, $n(N)\in\mathbb{N}$ is chosen such that*

$$\Delta_{n(N)} \leq \frac{T\sum_{i>N}\eta_i}{C_\ell\sum_{i=1}^N\eta_i}, \quad N\in\mathbb{N},$$

(see Remark 7.2.4) the approximated Lévy process $\widetilde{L}_N$ converges almost surely to L in U as $N\to+\infty$, where the convergence is uniform in $\mathbb{T}$.

Proof. By Markov's inequality and Theorem 7.2.3, we obtain for any $\varepsilon>0$ and $t\in\mathbb{T}$

$$\begin{aligned}\mathbb{P}(||L(t)-\widetilde{L}_N(t)||_U>\varepsilon) &\leq \frac{\mathbb{E}(||L(t)-\widetilde{L}_N(t)||_U^2)}{\varepsilon^2}\\ &\leq \frac{1}{\varepsilon^2}\Big(\Big(T\sum_{i>N}\eta_i\Big)^{1/2}+\Big(C_\ell\Delta_{n(N)}\sum_{i=1}^N\eta_i\Big)^{1/2}\Big)^2.\end{aligned}$$

With $\Delta_{n(N)}$ as above this yields

$$\sum_{N\in\mathbb{N}}\mathbb{P}(||L(t)-\widetilde{L}_N(t)||_U>\varepsilon)\leq\frac{4T}{\varepsilon^2}\sum_{N\in\mathbb{N}}\sum_{i>N}\eta_i=\frac{4T}{\varepsilon^2}\sum_{i>N}\eta_i(i-1)<+\infty.$$

The claim follows by the Borel-Cantelli Lemma and by the fact that the sum on the right hand side in the inequality is independent of t. □

For the approximation

$$\widetilde{L}_N(t)=\sum_{i=1}^N\sqrt{\eta_i}e_i\widetilde{\ell}_i(t)$$

of L, we required that the one-dimensional Lévy processes $(\widetilde{\ell}_i, i = 1, \dots, N)$ are uncorrelated but not independent. Several questions may arise regarding this truncated sum:

1. How can we efficiently simulate suitable one-dimensional approximations $\widetilde{\ell}_i$ of ℓ_i and determine the constant C_ℓ to apply Theorem 7.2.3?

2. Is L_N again a Lévy field for arbitrary one-dimensional processes $(\ell_i, i \in \mathbb{N})$ and can the point-wise marginal distribution of $L_N(t)$ for a given spectral basis $((\sqrt{\eta}_i e_i), i \in \mathbb{N})$ and fixed $N \in \mathbb{N}$ be determined?

3. Is it possible to construct L_N in a way such that its point-wise marginal processes follow a desired distribution?

In the next chapter we address the first question and present a novel approach for the approximation of arbitrary one-dimensional Lévy processes ℓ_i. We derive explicit error bounds and convergence results in $L^p(\Omega; \mathbb{R})$, hence we are able to determine C_ℓ or at least bound this constant from above. The last two questions on the distribution properties of L_N are then investigated in Section 7.4 for an important subclass of Lévy fields. We discuss distributional features of L_N so as to use the approximation methodology developed in Section 7.3 to efficiently draw samples of the field $\widetilde{L}_N$.

7.3 Simulation of Lévy processes by Fourier inversion

The simulation of an arbitrary one-dimensional Lévy process $\ell = (\ell(t), t \in \mathbb{T})$ is not straightforward, as sufficiently many discrete realizations of ℓ in $\mathbb{T}$ are needed and the distribution of the increment $\ell(t + \Delta_n) - \ell(t)$ for some small time step $\Delta_n > 0$ is not explicitly known in general. A well-known and common way to simulate a Lévy process with characteristic triplet (ι, σ^2, ν) (see Equation (7.2)) is the *compound Poisson approximation* (CPA) suggested in [186] and [188]. All jumps of the process larger than some $\varepsilon > 0$ are approximated by a sum of independent compound Poisson processes and the small jumps by their expected values resp. by a Brownian motion. For details and convergence theorems of this method we refer to [10, 186, 188]. Although the CPA is applicable in a very general setting, in the sense that only the triplet (ι, σ^2, ν) has to be known for simulation, it has several drawbacks. It is possible to show that the CPA converges under certain assumptions in distribution to a Lévy process with characteristic triplet (ι, σ^2, ν), and even strong error rates for CPA-type approximation schemes are given, for instance in [74, 84, 155]. The derived L^p-error rates are, however, rather low with respect to the time discretization, only available for $p \leq 2$ and/or require

strong assumptions on the moments of the Lévy measure ν. Furthermore, if the cumulated density function (CDF) of ν is unknown, numerical integration with respect to ν is necessary. Evaluating the density of ν at sufficiently many points to obtain a good approximation might be time consuming, especially if this involves computationally expensive components (e.g. Bessel functions). It is, further, a-priori not clear how to discretize the measure ν (we refer to a discussion on this matter in [188, Chapter 8]). One could choose for example equidistant or equally weighted points, but this choice might have a significant impact on the precision and the speed of the simulation, and is impossible to be assessed beforehand. The disadvantages of the CPA method motivate the development of an alternative methodology.

In the following, we introduce a new sampling approach which approximates the process ℓ by a refining sequence of piecewise constant càdlàg processes $(\overline{\ell}^{(n)}, n \in \mathbb{N})$. We show its asymptotic convergence in $L^p(\Omega;\mathbb{R})$-sense and almost surely. This approximation suffers from the fact that the necessary conditional densities from which we have to sample are not available for many Lévy processes. For a given refinement parameter n, we develop, therefore, an algorithm to sample an approximation $\widetilde{\ell}^{(n)}$ of $\overline{\ell}^{(n)}$ for which the resulting error may be bounded again in $L^p(\Omega;\mathbb{R})$-sense. This technique is based on the assumption that the characteristic function of ℓ is available in closed form, which is true for a broad class of Lévy processes. We exploit this knowledge by so-called *Fourier inversion* to draw samples of the process' increments over an arbitrary large time step $\Delta_n > 0$. In Section 7.5, we then apply the described method to simulate GH Lévy fields.

7.3.1 A piecewise constant approximation

Throughout this chapter, we consider a one-dimensional Lévy process $\ell = (\ell(t), t \in \mathbb{T})$ with characteristic function $\phi_\ell : \mathbb{R} \to \mathbb{C}$. For any $t \in \mathbb{T}$, we denote by F_t the CDF of $\ell(t)$ and by f_t the corresponding density function, provided that f_t exists. Note that in this case F_t and f_t belong to the probability distribution with characteristic function $(\phi_\ell)^t$. To obtain a refining scheme of approximations of ℓ, we introduce a sampling algorithm for ℓ based on the construction of *Lévy bridges*. In our context, a Lévy bridge is the stochastic process $(\ell(t)|t \in (t_1, t_2))$ pinned to given realizations of the boundary values $\ell(t_1)$ and $\ell(t_2)$ for $0 \leq t_1 < t_2 \leq T$. It has been shown, for instance in [114, Proposition 2.3], that these bridges are Markov processes. Assuming that the density f_t exists for every $t \in \mathbb{T}$ (see also Remark 7.3.8), the distribution of the increment $\ell(t) - \ell(t_1)$ conditional on $\ell(t_2)$ is well-defined whenever $f_{t_2-t_1}(\ell(t_2)) \in (0, +\infty)$. Its

density function is then given as

$$f_t^{t_1,t_2}(x) := \frac{f_{t-t_1}(x) f_{t_2-t}(\ell(t_2) - x)}{f_{t_2-t_1}(\ell(t_2))}, \tag{7.5}$$

with conditional expectation

$$\mathbb{E}(\ell(t)|\ell(t_1), \ell(t_2)) = \frac{\ell(t_2) - \ell(t_1)}{t_2 - t_1}(t - t_1),$$

see [114, 156]. This motivates the following sampling algorithm for a piecewise constant approximation of ℓ:

Algorithm 7.3.1. Let $n \in \mathbb{N}$ and generate a sample of the random variable $\mathcal{X}_{0,1}$ with density f_T. Set $\mathcal{X}_{0,0} := 0$, $i := 1$ and $\Delta_0 := T$.

while $i \leq n$ **do**
  Define $\Delta_i = \frac{T}{2^i}$.
  for $j = 0, 2, \ldots, 2^i$ **do**
    Set $\mathcal{X}_{i,j} = \mathcal{X}_{i-1,j/2}$.
  end for
  for $j = 1, 3, \ldots, 2^i - 1$ **do**
    Generate the (conditional) increment $\mathcal{X}_{i,j} - \mathcal{X}_{i,j-1}$ in $[\frac{(j-1)T}{2^{(i-1)}}, \frac{jT}{2^{(i-1)}}]$.
    That is, sample the random variable $X : \Omega \to \mathbb{R}$ with density

$$x \mapsto \frac{f_{\Delta_i}(x) f_{\Delta_i}(\mathcal{X}_{i,j+1} - x)}{f_{\Delta_{i-1}}(\mathcal{X}_{i,j+1})}$$

    and set $\mathcal{X}_{i,j} := X + \mathcal{X}_{i,j-1}$
  end for
  $i = i + 1$
end while

Define the piecewise constant process

$$\overline{\ell}^{(n)}(t) := \mathcal{X}_{n,2^n} \mathbf{1}_{\{T\}}(t) + \sum_{j=1}^{2^n} \mathcal{X}_{n,j-1} \mathbf{1}_{\{[(j-1)T/2^n, jT/2^n)\}}(t).$$

Eventually, the sequence $(\overline{\ell}^{(n)}, n \in \mathbb{N})$ of càdlàg processes admits a pointwise limit in $L^p(\Omega; \mathbb{R})$ which corresponds to the process ℓ:

Theorem 7.3.2. *Let ϕ_ℓ be a characteristic function of an infinitely divisible probability distribution. For any $t \in \mathbb{T}$, assume the probability density f_t corresponding to $(\phi_\ell)^t$ exists. Further, for $n \in \mathbb{N}$, let $\overline{\ell}^{(n)}$ be the process generated by Algorithm 7.3.1 and f_t*

on $(\Omega, (\mathcal{F}_t, t \geq 0), \mathbb{P})$. *If* $\int_{\mathbb{R}} |x|^p f_1(x)dx < +\infty$ *for some* $p \in [1, +\infty)$, *then*

$$\lim_{n \to +\infty} \mathbb{E}(|\overline{\ell}^{(n)}(t) - \ell(t)|^p) = 0,$$

where ℓ *is a Lévy process with characteristic function* ϕ_ℓ *on* $(\Omega, (\mathcal{F}_t, t \geq 0), \mathbb{P})$.

Proof. For any $n \in \mathbb{N}$ and $t \in \mathbb{T}$ we have that

$$\begin{aligned}
&\mathbb{E}[|\overline{\ell}^{(n+1)}(t) - \overline{\ell}^{(n)}(t)|^p] \\
=&\mathbb{E}\Big(\Big|\sum_{j=1}^{2^{n+1}} \mathcal{X}_{n+1,j-1} \mathbf{1}_{\{[(j-1)T/2^{n+1}, jT/2^{n+1})\}}(t) - \sum_{j=1}^{2^n} \mathcal{X}_{n,j-1} \mathbf{1}_{\{[(j-1)T/2^n, jT/2^n)\}}(t)\Big|^p\Big) \\
=&\mathbb{E}\Big(\Big|\sum_{j=1}^{2^n} (\mathcal{X}_{n+1,2j-1} - \mathcal{X}_{n+1,2j-2}) \mathbf{1}_{\{[(2j-1)T/2^{n+1}, 2jT/2^{n+1})\}}(t)\Big|^p\Big)
\end{aligned}$$

Since the increments $\mathcal{X}_{n+1,j+1} - \mathcal{X}_{n+1,j}$ are i.i.d. with characteristic function $(\phi_\ell)^{T/2^{n+1}}$ by construction, this yields

$$\mathbb{E}[|\overline{\ell}^{(n+1)}(t) - \overline{\ell}^{(n)}(t)|^p] \leq C_{\ell,T} 2^{-n-1} \int_{\mathbb{R}} |x|^p f_1(x)dx = C_{\ell,T,p} 2^{-n-1},$$

where $C_{\ell,T}$ resp. $C_{\ell,T,p}$ are positive constants that are independent of n. Hence, for any $m, n \in \mathbb{N}$ with $m > n$ it follows

$$\mathbb{E}[|\overline{\ell}^{(m)}(t) - \overline{\ell}^{(n)}(t)|^p]^{1/p} \leq C_{\ell,T,p}^{1/p} \sum_{i=n+1}^{m} 2^{-i/p} = C_{\ell,T,p}^{1/p} \frac{2^{-n/p} - 2^{-m/p}}{2^{1/p} - 1},$$

meaning that $(\overline{\ell}^{(n)}(t), n \in \mathbb{N})$ is a $L^p(\Omega; \mathbb{R})$-Cauchy sequence and, therefore, admits a limit. The characteristic function of $\overline{\ell}^{(n)}(t)$ is given by

$$(\phi_\ell)^{\lfloor t2^n/T \rfloor T/2^n} \overset{n \to \infty}{\to} (\phi_\ell)^t.$$

The claim follows since the distribution with characteristic function ϕ_ℓ is infinitely divisible, hence the limit process $\ell = (\ell(t), t \in \mathbb{T})$ is in fact a Lévy process. □

Corollary 7.3.3. *Under the assumptions of Theorem 7.3.2 with* $p = 1$, $\overline{\ell}^{(n)}$ *converges to* ℓ $\mathbb{P}$*-almost surely as* $n \to +\infty$, *uniformly in* $\mathbb{T}$.

Proof. For any $t \in \mathbb{T}$ and $\varepsilon > 0$, we get by Markov's inequality

$$\mathbb{P}(|\overline{\ell}^{(n)}(t) - \ell(t)|) \leq \frac{\mathbb{E}(|\overline{\ell}^{(n)}(t) - \ell(t)|)}{\varepsilon} \leq \frac{C_{\ell,T,p}}{\varepsilon} \sum_{i=n}^{\infty} 2^{-i} = \frac{C_{\ell,T,p} 2^{-n+1}}{\varepsilon}.$$

The claim then follows by the Borel-Cantelli Lemma since

$$\sum_{n=1}^{\infty} \mathbb{P}(|\overline{\ell}^{(n)}(t) - \ell(t)|) \leq \frac{2C_{\ell,T,p}}{\varepsilon} \sum_{n=1}^{\infty} 2^{-n} < +\infty.$$

□

Although Algorithm 7.3.1 has convenient properties in terms of convergence, it may only be applied for a small class of Lévy processes. For a general Lévy process ℓ, the conditional densities in Eq. (7.5) will be unknown and thus simulating from this distributions is impossible. A few exceptions where "bridge sampling" of Lévy processes is feasible include the inverse Gaussian ([182]) and the tempered stable process ([128]). However, if we consider a fixed parameter n, sampling from the bridge distributions is equivalent to the following algorithm:

Algorithm 7.3.4.

1: For $n \in \mathbb{N}$, fix Δ_n, Θ_n as in Section 7.2 and generate 2^n i.i.d random variables $X_1, \ldots, X_{2^n}$ with density f_{Δ_n}.
2: Set $\ell^{(n)}(t) = 0$ if $t \in [0, t_1)$, $\ell^{(n)}(t) = \sum_{k=1}^{j} X_k$ if $t \in [t_j, t_{j+1})$ for $j = 1, \ldots, 2^n - 1$ and $\ell^{(n)}(T) = \sum_{j=1}^{2^n} X_j$.

The equivalence is in the sense that both processes are piecewise constant, càdlàg and all intermediate points follow the same conditional Lévy bridge distributions. Note that $\ell^{(n)}$ coincides with $\overline{\ell}^{(n)}$ from Algorithm 7.3.1 where the initial value has been chosen as $\mathcal{X}_{0,1} = \ell^{(n)}(T) = \sum_{j=1}^{2^n} X_j$. The advantage of Algorithm 7.3.1 is that 2^n independent samples from the same distribution have to be generated, instead of 2^n random variables from (different) conditional distributions. As we will see in the following section, sampling from the distribution with density f_{Δ_n} may be achieved if the characteristic function ϕ_ℓ is available. In addition, we are still able to use the $L^p(\Omega; \mathbb{R})$ error bounds from Theorem 7.3.2 for a fixed $n \in \mathbb{N}$.

7.3.2 Inversion of the characteristic function

For a one-dimensional Lévy process ℓ with characteristic function ϕ_ℓ, the characteristic function of any increment $\ell(t + \Delta_n) - \ell(t)$ can be expressed via

$$\mathbb{E}[\exp(iu(\ell(t + \Delta_n) - \ell(t)))] = \mathbb{E}[\exp(iu(\ell(\Delta_n)))] = (\phi_\ell(u))^{\Delta_n}$$

for any time step $\Delta_n > 0$. If F_{Δ_n} denotes again the CDF of this increment, we obtain by Fourier inversion (see [91])

$$F_{\Delta_n}(x) = \frac{1}{2} - \int_{\mathbb{R}} \frac{(\phi_\ell(u))^{\Delta_n}}{2\pi i u} \exp(-iux) du. \tag{7.6}$$

Using the well-known inverse transformation method (see also [9]) to sample from the CDF, allows us to reformulate Algorithm 7.3.4:

Algorithm 7.3.5.

1: For $n \in \mathbb{N}$, fix Δ_n, Θ_n and generate i.i.d. samples $\mathbb{U}_1, \dots, \mathbb{U}_{2^n}$, where $\mathbb{U}_j \sim \mathcal{U}([0,1])$ on $(\Omega, \mathcal{F}, \mathbb{P})$.
2: Determine $X_j := \inf\{x \in \mathbb{R} | F_{\Delta_n}(x) = \mathbb{U}_j\}$ for $j = 1, \dots, 2^n$.
3: Set $\ell^{(n)}(t) = 0$ if $t \in [0, t_1)$, $\ell^{(n)}(t) = \sum_{k=1}^{j} X_k$ if $t \in [t_j, t_{j+1})$ for $j = 1, \dots, 2^n - 1$ and $\ell^{(n)}(T) = \sum_{j=1}^{2^n} X_j$.

The evaluation of F is crucial and may, in general, only be done numerically. To approximate the integral in Eq. (7.6), we employ the discrete Fourier inversion method introduced in [116]. With this method the approximation error can be controlled with relatively weak assumptions on the characteristic function. Hence, the resulting algorithm is applicable for a broad class of Lévy processes. An alternative algorithm to approximate the CDFs of subordinating processes based on the inversion of Laplace transforms is described in [196]. Although this approach seems promising in terms of computational effort, here we only consider the Fourier inversion technique. The latter is also applicable to Lévy processes without bounded variation and yields uniform error bounds on the approximated CDF.

Assumption 7.3.6. The distribution with characteristic function $(\phi_\ell)^{\Delta_n}$ is continuous with finite variance and CDF F_{Δ_n}. Furthermore,

- there exists a constant $R > 0$ and $\vartheta > 1$ such that $F_{\Delta_n}(-x) \leq R|x|^{-\vartheta}$ and $1 - F_{\Delta_n}(x) \leq R|x|^{-\vartheta}$ for all $x > 0$.
- there exists a constant $B > 0$ and $\theta > 0$ such that $|(\phi_\ell(u))^{\Delta_n}| \leq B|\frac{u}{2\pi}|^{-\theta}$ for all $u \in \mathbb{R}$.

In case of infinite variance, we consider bounds on the density function instead:

Assumption 7.3.7. The distribution with characteristic function $(\phi_\ell)^{\Delta_n}$ is continuous with density f_{Δ_n}. Furthermore,

- there exists a constant $R > 0$ and $\vartheta > 1$ such that $|f_{\Delta_n}(x)| \leq R|x|^{-\vartheta}$ for all $x \in \mathbb{R}$.
- there exists a constant $B > 0$ and $\theta > 0$ such that $|(\phi_\ell(u))^{\Delta_n}| \leq B|\frac{u}{2\pi}|^{-\theta}$ for all $u \in \mathbb{R}$.

Remark 7.3.8. In the case that $\theta > 1$, we have that

$$\int_{\mathbb{R}} |(\phi_\ell(u))^{\Delta_n}| du \leq 2 + 2B \int_1^\infty \left(\frac{u}{2\pi}\right)^{-\theta} < +\infty,$$

which already implies the existence of a continuous density f_{Δ_n} in both scenarios, see for example [187, Proposition 28.1]. Usually, F_{Δ_n} or f_{Δ_n} are unknown, but only the characteristic function $(\phi_\ell)^{\Delta_n}$ is given. To obtain R and ϑ, one can choose

$$R = (-1)^k \frac{d^{2k}}{du^{2k}}((\phi_\ell(u))^{\Delta_n})\Big|_{u=0}$$

and $\vartheta = 2k$ in Ass. 7.3.6, resp.

$$R = \frac{1}{2\pi} \int_{\mathbb{R}} |\frac{d^k}{du^k}((\phi_\ell(u))^{\Delta_n})| du$$

and $\vartheta = k$ in Ass. 7.3.7, where k is any non-negative integer such that the derivatives exist (see [116, Lemma 12 and 13]). For example, in the first set of assumptions, the finite variance ensures that we can use $\vartheta = 2$ and R equal to the second moment of the distribution with characteristic function $(\phi_\ell)^{\Delta_n}$.

As an approximation of F_{Δ_n} as in Eq. (7.6) we introduce the function $\widetilde{F}_{\Delta_n}$ given by

$$\widetilde{F}_{\Delta_n}(x) := \sum_{k=-M/2}^{M/2} q_k \exp(i2\pi kx/J),$$

for $x \in \mathbb{R}$, where

$$q_k := \begin{cases} 1/2 & \text{for } k = 0 \\ \frac{1-\cos(2\pi\kappa k)}{i2\pi k}(\phi_\ell(-2\pi k/J))^{\Delta_n} & \text{for } 0 < |k| < M/2 \\ 0 & \text{for } k = M/2 \end{cases},$$

M is an even integer and $\kappa, J > 0$ are parameters which are determined below. Note that $q_k = \overline{q_{-k}}$, where $\overline{z}$ denotes the complex conjugate for $z \in \mathbb{C}$. The Hermitean symmetry also holds for the function $k \mapsto \exp(i2\pi kx/J)$. This ensures that, for every

$x \in \mathbb{R}$, we have

$$\begin{aligned}\widetilde{F}_{\Delta_n}(x) &= \frac{1}{2} + \sum_{k=1}^{M/2-1} q_k \exp(i2\pi kx/J) + \overline{q_k \exp(i2\pi kx/J)} \\ &= \frac{1}{2} + 2\,\mathrm{Re}\Big(\sum_{k=1}^{M/2-1} q_k \exp(i2\pi kx/J)\Big)\end{aligned}$$

and hence $\widetilde{F}_{\Delta_n}(x) \in \mathbb{R}$ for any real-valued argument x. The last identity should be exploited during the simulation to save computational time as here only half the summation is required. Lastly, we denote by $\zeta(z,s) := \sum_{k=0}^{\infty}(k+s)^{-z}$ for $s, z \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ and $\mathrm{Re}(z) > 1$ the *Hurwitz zeta function* and define as in [116]

$$\begin{aligned}V_1(\kappa,\vartheta) &:= (\kappa/2)^{-\vartheta} + 2\zeta(\vartheta, 1-\frac{\kappa}{2}) + \zeta(\vartheta, 1+\frac{\kappa}{2}) + \zeta(\vartheta, 1-\frac{3\kappa}{2}), \\ V_2(\kappa,\vartheta) &:= \frac{2^{\vartheta-1}\kappa^{1-\vartheta}}{\vartheta-1} + \frac{\kappa}{2}\Big(2\zeta(\vartheta, 1-\frac{\kappa}{2}) + \zeta(\vartheta, 1+\frac{\kappa}{2}) + \zeta(\vartheta, 1-\frac{3\kappa}{2})\Big).\end{aligned}$$

The expressions $V_1(\kappa,\vartheta)$ and $V_2(\kappa,\vartheta)$ establish conditions on the choice of the (not yet determined) parameter κ in Theorem 7.3.9. For a given domain parameter $D > 0$ and accuracy $\varepsilon > 0$ the approximation $\widetilde{F}_{\Delta_n}$ should fulfill the error bound

$$|\widetilde{F}_{\Delta_n}(x) - F_{\Delta_n}(x)| < \varepsilon \quad \text{for } x \in [-D/2, D/2].$$

Once κ is determined, this can be achieved by choosing a sufficiently large parameter J and, based on this J, a sufficiently large number of summands M. Admissible values for κ, J and M depend on D, ε and the constants in Assumption 7.3.6 resp. 7.3.7.

Theorem 7.3.9. *([116, Theorem 10 and 11]) Let $D > 0$ and $\varepsilon > 0$. If Assumption 7.3.6 holds, choose κ, J and M such that*

$$0 < \kappa < \frac{2}{3} \quad \textit{and} \quad \kappa^{\vartheta} V_1(\kappa,\vartheta) \le 2^{\vartheta+1},$$

$$J \ge \frac{D}{\kappa} \quad \textit{and} \quad J \ge \Big(\frac{3RV_1(\kappa,\vartheta)}{2\varepsilon}\Big)^{1/\vartheta},$$

and

$$M \ge 2 + 2J\Big(\frac{6B}{\varepsilon\pi\theta}\Big)^{1/\theta}.$$

If Assumption 7.3.7 holds, choose κ, J and M such that

$$0 < \kappa < \frac{2}{3} \quad \textit{and} \quad \kappa^{\vartheta-1} V_2(\kappa,\vartheta) \le \frac{2^{\vartheta}}{\vartheta-1},$$

$$J \geq \frac{D}{\kappa} \quad \textit{and} \quad J \geq \Big(\frac{3RV_2(\kappa,\vartheta)}{2\varepsilon}\Big)^{1/(\vartheta-1)},$$

and

$$M \geq 2 + 2J\Big(\frac{6B}{\varepsilon\pi\theta}\Big)^{1/\theta}.$$

This yields, for either case, that $|F_{\Delta_n}(x) - \widetilde{F}_{\Delta_n}(x)| < \varepsilon$ *for all* $x \in [-D/2, D/2]$ *and it is always possible to find a* κ *that meets the given conditions.*

Remark 7.3.10. In [116], by

$$J \geq \frac{2}{\kappa}\Big(\frac{3R}{\varepsilon}\Big)^{1/\vartheta} \text{ resp.} \quad J \geq \frac{2}{\kappa}\left(\frac{3R}{\varepsilon(\vartheta-1)}\right)^{1/(\vartheta-1)}$$

in fact stricter conditions are imposed on J. The proofs of Theorems 10 and 11 in [116] still give immediately a proof for Theorem 7.3.9. The advantage of the bounds in Theorem 7.3.9 is that they produce a smaller approximation error in the following analysis (see also Remark 7.3.17). We refer to [116] for an optimal choice of κ depending on ϑ. Once κ is determined, it is favorable to choose D and ε in a way such that none of the parameters has a dominant effect on the resulting number of summations M. This is ensured if the two lower bounds on J are equal, meaning for fixed $D > 0$ we set

$$\varepsilon = \frac{3}{2}RV_1(\kappa,\vartheta)\kappa^{\vartheta}D^{-\vartheta} \quad \text{resp.} \quad \varepsilon = \frac{3}{2}RV_2(\kappa,\vartheta)\kappa^{\vartheta-1}D^{-\vartheta+1}$$

if the first resp. second set of assumptions holds.

Since the error $|\widetilde{F}_{\Delta_n}(x) - F_{\Delta_n}(x)|$ is only bounded for $x \in [-D/2, D/2]$, we have to modify the third step in Algorithm 7.3.5:

Algorithm 7.3.11.

1: For $n \in \mathbb{N}$, fix Δ_n, Θ_n and generate i.i.d. samples $(\mathbb{U}_j \sim \mathcal{U}([0,1]), j = 1, \dots, 2^n)$ on $(\Omega, \mathcal{F}, \mathbb{P})$.

2: Set, for $j = 1, \dots, 2^n$ and $I_{\widetilde{F},D} := \widetilde{F}_{\Delta_n}([-D/2, D/2])$

$$\widetilde{X}_j = \begin{cases} -D/2 & \text{if } \mathbb{U}_j < \min(I_{\widetilde{F},D}) \\ D/2 & \text{if } \mathbb{U}_j > \max(I_{\widetilde{F},D}) \\ \inf\{x \in [-D/2, D/2] \,\big|\, \widetilde{F}_{\Delta_n}(x) = \mathbb{U}_j\} & \text{if } \mathbb{U}_j \in I_{\widetilde{F},D} \end{cases}.$$

3: Set $\widetilde{\ell}^{(n)}(t) = 0$ if $t \in [0, t_1)$, $\widetilde{\ell}^{(n)}(t) = \sum_{k=1}^{j} \widetilde{X}_k$ if $t \in [t_j, t_{j+1})$ for $j = 1, \dots, 2^n - 1$ and $\widetilde{\ell}^{(n)}(T) = \sum_{j=1}^{2^n} \widetilde{X}_j$.

Intuitively, if we choose D large and ε small enough, the atoms in the distribution of $\widetilde{X}_i$ at $\pm D/2$ disappear. The function $\widetilde{F}_{\Delta_n}$ is then sufficiently close to the CDF F_{Δ_n}, hence the generated random variables $\widetilde{X}_i$ will have a distribution similar to F_{Δ_n}. From here on, we define X as the random variable which is generated from $\mathbb{U} \sim \mathcal{U}([0,1])$ by inversion of the (exact) CDF F_{Δ_n} and $\widetilde{X}$ as the random variable generated from $\mathbb{U}$ by inversion of the approximated CDF $\widetilde{F}_{\Delta_n}$.

Theorem 7.3.12. *Let $\widetilde{F}_{\Delta_n}$ be the approximation of F_{Δ_n} which is valid for parameters $D > 0$ and $\varepsilon > 0$ in the sense of Theorem 7.3.9. Then $\widetilde{X}$ converges in distribution to a random variable X with CDF equal to F_{Δ_n} as $D \to +\infty$ and $\varepsilon \to 0$.*

Proof. First, note that $\widetilde{F}_{\Delta_n}$ is not necessarily monotone and might admit arbitrary values outside of $[-D/2, D/2]$, thus cannot be regarded as a CDF. Since $\widetilde{X}$ only admits values in the desired interval, we obtain probability zero for the event that $|\widetilde{X}| > D/2$. With this in mind we construct the CDF of $\widetilde{X}$ and show its convergence in distribution using Portmanteau's theorem. We define the function

$$\widehat{F} : \mathbb{R} \to [0,1], \quad x \mapsto \begin{cases} 0 & \text{if } x < -D/2 \\ \min(1, m_D(x)) \mathbf{1}_{\{m_D(x) > 0\}} & \text{if } x \in [-D/2, D/2] \\ 1 & \text{if } x > D/2 \end{cases},$$

where $m_D(x) := \max_{y \in [-D/2, x]} \widetilde{F}_{\Delta_n}(y)$. The continuity of $\widetilde{F}_{\Delta_n}$ guarantees that $m_D(x)$ is well-defined for each $x \in [-D/2, D/2]$. Clearly, $\widehat{F}$ is monotone increasing and

$$\mathbb{P}(\widetilde{X} \leq x) = \widehat{F}(x)$$

if $|x| > D/2$. For $|x| \leq D/2$ we have that

$$\begin{aligned} \mathbb{P}(\widetilde{X} \leq x) &= \mathbb{P}(\inf\{|y| \leq D/2 \,|\, \widetilde{F}_{\Delta_n}(y) \geq \mathbb{U}\} \leq x) \\ &= \mathbb{P}(\max_{y \in [-D/2, x]} \widetilde{F}_{\Delta_n}(y) \geq \mathbb{U}) \\ &= \min(1, m_D(x)) \mathbf{1}_{\{m_D(x) > 0\}} = \widehat{F}(x), \end{aligned}$$

hence $\widehat{F}$ is the CDF of $\widetilde{X}$. With the monotonicity of F_{Δ_n} and $|F_{\Delta_n} - \widetilde{F}_{\Delta_n}| < \varepsilon$ on

$[-D/2, D/2]$ we get

$$\begin{aligned}\widehat{F}(x) &= \min(1, m_D(x))\mathbf{1}_{\{m_D(x)>0\}} \\ &\leq \min(1, \max_{y\in[-D/2,x]} \widetilde{F}_{\Delta_n}(y)) \\ &\leq \min(1, \max_{y\in[-D/2,x]} F_{\Delta_n}(y) + \varepsilon) \\ &= \min(1, F_{\Delta_n}(x) + \varepsilon),\end{aligned}$$

for $x \in [-D/2, D/2]$ and analogously

$$\widehat{F}(x) \geq \min(1, \max_{y\in[-D/2,x]} F_{\Delta_n}(y) - \varepsilon)\mathbf{1}_{\{\max_{y\in[-D/2,x]} F_{\Delta_n}(y)-\varepsilon>0\}} = \max(F_{\Delta_n}(x) - \varepsilon, 0),$$

thus

$$|\widehat{F}(x) - F_{\Delta_n}(x)| \leq \varepsilon.$$

We choose sequences $(D_k, k \in \mathbb{N})$ with $\lim_{k\to\infty} D_k = +\infty$ and $(\varepsilon_m, m \in \mathbb{N})$ with $\lim_{m\to\infty} \varepsilon_m = 0$ and denote by $\widehat{F}_{k,m}$ the CDF of the random variables $\widetilde{X}_{k,m}$ corresponding to each D_k and ε_m. For every $x \in \mathbb{R}$ there is some $k^* \in \mathbb{N}$ such that $x \in [-D_k/2, D_k/2]$ for all $k \geq k^*$, hence

$$\lim_{m\to\infty} \lim_{k\to\infty} |\widehat{F}_{k,m}(x) - F_{\Delta_n}(x)| \leq \lim_{m\to\infty} \varepsilon_m = 0$$

and the claim follows by Portmanteau's theorem. □

Remark 7.3.13. Before showing the convergence of $\widetilde{X}$ to X in $L^p(\Omega; \mathbb{R})$, we have to make sure that the random variables $\widetilde{X}$ generated by Algorithm 7.3.11 are actually defined on the same probability space $(\Omega, (\mathcal{F}_t, t \geq 0), \mathbb{P})$ as X. Since X represents the increment of a Lévy process ℓ on $(\Omega, (\mathcal{F}_t, t \geq 0), \mathbb{P})$ with CDF F_{Δ_n}, we may define the mapping $\mathbb{U} := F_{\Delta_n} \circ X : \Omega \to [0, 1]$. It is then easily verified that $\mathbb{U}$ is a $\mathcal{U}([0, 1])$-distributed random variable. For fixed parameters $D, \varepsilon > 0$ and an approximation $\widetilde{F}_{\Delta_n}$ of F_{Δ_n} we define the *pseudo inverse* of $\widetilde{F}_{\Delta_n}$ (as in Algorithm 7.3.11 with $I_{\widetilde{F},D} := \widetilde{F}_{\Delta_n}([-D/2, D/2])$) as

$$\widetilde{F}^{-1}_{\Delta_n} : [0, 1] \to \mathbb{R}, \;\; u \mapsto \begin{cases} -D/2 & \text{if } u < \min(I_{\widetilde{F},D}) \\ D/2 & \text{if } u > \max(I_{\widetilde{F},D}) \\ \inf\{x \in [-\frac{D}{2}, \frac{D}{2}] \big| \widetilde{F}_{\Delta_n}(x) = u\} & \text{if } u \in I_{\widetilde{F},D} \end{cases}.$$

We note that $\widetilde{F}^{-1}_{\Delta_n}$ is a piecewise continuous, thus measurable, mapping which implies

that $\widetilde{X} = \widetilde{F}_{\Delta_n}^{-1} \circ F_{\Delta_n} \circ X : \Omega \to [-D/2, D/2]$ is a random variable on $(\Omega, (\mathcal{F}_t, t \geq 0), \mathbb{P})$.

Under additional, but natural, assumptions, it is possible to show stronger convergence results of the approximation for both sets of assumptions.

Theorem 7.3.14. *($L^p(\Omega; \mathbb{R})$-convergence I) Let F_{Δ_n} be continuously differentiable on $\mathbb{R}$ with density f_{Δ_n} (see Remark 7.3.8) and $(\phi_\ell)^{\Delta_n}$ be bounded as in Assumption 7.3.6 with $\vartheta > 1$. Furthermore, assume that the approximation parameters D and ε fulfill $D = C\varepsilon^{-d}$ for $C, d > 0$. If $d < \frac{1}{p}$, then for all $p \in [1, \vartheta)$*

$$\mathbb{E}(|\widetilde{X} - X|^p) \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof. Let $\varepsilon > 0$, $D = C\varepsilon^{-d}$ and $p \in [1, \vartheta)$ be as in the claim. We split the expectation in the following way

$$\mathbb{E}(|\widetilde{X} - X|^p) = \mathbb{E}(|\widetilde{X} - X|^p \mathbf{1}_{\{|X| > D/2\}}) + \mathbb{E}(|\widetilde{X} - X|^p \mathbf{1}_{\{|X| \leq D/2\}}),$$

and show the convergence for each term on the right hand side. Recall that $\widetilde{X} \in [-D/2, D/2]$ by construction. We obtain for the first term

$$\begin{aligned}
\mathbb{E}(|\widetilde{X} - X|^p \mathbf{1}_{\{|X| > D/2\}}) &\leq \int_{D/2}^{\infty} |-D/2 - x|^p f_{\Delta_n}(x) dx + \int_{-\infty}^{-D/2} |D/2 - x|^p f_{\Delta_n}(x) dx \\
&= \int_{D/2}^{\infty} (D/2 + x)^p (f_{\Delta_n}(x) + f_{\Delta_n}(-x)) dx \\
&= \int_{D/2}^{\infty} \int_0^{D/2+x} p y^{p-1} dy (f_{\Delta_n}(x) + f_{\Delta_n}(-x)) dx.
\end{aligned}$$

Using the identity

$$\begin{aligned}
&(x, y) \in (D/2, +\infty) \times (0, D/2 + x) \\
\Leftrightarrow &(x, y) \in \Big((D/2, +\infty) \times (0, D/2)\Big) \cup \Big((y, +\infty) \times (D/2, +\infty)\Big)
\end{aligned}$$

for the domain of interest, we may use Fubini's theorem to exchange the order of

integration and rewrite

$$\begin{aligned}\mathbb{E}(|\widetilde{X}-X|^p\mathbf{1}_{\{|X|>D/2\}}) &\leq \int_{D/2}^{\infty}\int_0^{D/2+x} py^{p-1}dy(f_{\Delta_n}(x)+f_{\Delta_n}(-x))dx\\ &= \int_0^{D/2}\int_{D/2}^{\infty}(f_{\Delta_n}(x)+f_{\Delta_n}(-x))dxpy^{p-1}dy\\ &\quad+\int_{D/2}^{\infty}\int_y^{\infty}(f_{\Delta_n}(x)+f_{\Delta_n}(-x))dxpy^{p-1}dy\\ &= \int_0^{D/2}(1-F_{\Delta_n}(D/2)+F_{\Delta_n}(-D/2))py^{p-1}dy\\ &\quad+\int_{D/2}^{\infty}(1-F_{\Delta_n}(y)+F_{\Delta_n}(-y))py^{p-1}dy.\end{aligned}$$

With the bounds on F_{Δ_n} from Assumption 7.3.6 we then have

$$\begin{aligned}\mathbb{E}(|\widetilde{X}-X|^p\mathbf{1}_{\{|X|>D/2\}}) &\leq 2R(D/2)^{-\vartheta}\int_0^{D/2}py^{p-1}dy+2Rp\int_{D/2}^{\infty}\frac{y^{p-1}}{y^{\vartheta}}dy\\ &= 2R(D/2)^{p-\vartheta}+2Rp\zeta(\vartheta+1-p,D/2).\end{aligned}$$

Note that the Hurwitz zeta function ζ is well-defined (as $\vartheta>p$, $D/2>0$) and converges to 0 as $D\to+\infty$.

For the second term, consider two realizations of the random variables $X(\omega)$ and $\widetilde{X}(\omega)$ for some $\omega\in\Omega$, where $|X(\omega)|\leq D/2$. F_{Δ_n} is continuously differentiable by assumption, hence

$$F_{\Delta_n}(X(\omega))-F_{\Delta_n}(\widetilde{X}(\omega))=f_{\Delta_n}(\xi(\omega))(X(\omega)-\widetilde{X}(\omega)),$$

with $\xi(\omega)$ lying in between $X(\omega)$ and $\widetilde{X}(\omega)$, meaning $|\xi(\omega)|\leq D/2$ and consequently

$$\mathbf{1}_{\{|X(\omega)|\leq D/2\}}(\omega)\leq\mathbf{1}_{\{|\xi(\omega)|\leq D/2\}}(\omega).$$

For $\widetilde{\varepsilon}:=C^{-1}\varepsilon^{d+1}>0$ we split the expectation once more into

$$\begin{aligned}\mathbb{E}(|\widetilde{X}-X|^p\mathbf{1}_{\{|X|\leq D/2\}}) &\leq \mathbb{E}(|\widetilde{X}-X|^p\mathbf{1}_{\{|\xi|\leq D/2\}})\\ &\leq \mathbb{E}(|\widetilde{X}-X|^p\mathbf{1}_{\{|\xi|\leq D/2,f(\xi)\geq\widetilde{\varepsilon}\}})+\mathbb{E}(|\widetilde{X}-X|^p\mathbf{1}_{\{|\xi|\leq D/2,f(\xi)<\widetilde{\varepsilon}\}})\\ &:= I+II.\end{aligned}$$

In case that $f_{\Delta_n}(\xi(\omega)) \geq \widetilde{\varepsilon}$, we can rearrange the terms to

$$|\widetilde{X}(\omega) - X(\omega)|^p = \frac{|F_{\Delta_n}(\widetilde{X}(\omega)) - F_{\Delta_n}(X(\omega))|^p}{f_{\Delta_n}(\xi(\omega))^p}.$$

If X and $\widetilde{X}$ are generated by $\mathbb{U} \sim \mathcal{U}([0,1])$ and $\widehat{F}_{\Delta_n}$ denotes again the CDF of $\widetilde{X}$, this yields

$$|\widetilde{X}(\omega) - X(\omega)|^p = \frac{|F_{\Delta_n}(\widetilde{X}(\omega)) - \widehat{F}_{\Delta_n}(\widetilde{X}(\omega))|^p}{f_{\Delta_n}(\xi(\omega))^p} < \frac{\varepsilon^p}{f_{\Delta_n}(\xi(\omega))^p},$$

where we have used in the second step that $\mathbb{U}(\omega) = F_{\Delta_n}(X(\omega)) = \widehat{F}_{\Delta_n}(\widetilde{X}(\omega))$ and $|F_{\Delta_n}(\widetilde{X}(\omega)) - \widehat{F}_{\Delta_n}(\widetilde{X}(\omega))| < \varepsilon$ (see Theorem 7.3.12). This gives a bound for I:

$$\begin{aligned} I &< \varepsilon^p \, \mathbb{E}(f_{\Delta_n}(\xi)^{-p} \mathbf{1}_{\{|\xi| \leq D/2, f_{\Delta_n}(\xi) \geq \widetilde{\varepsilon}\}}) \\ &= \varepsilon^p \int_{-D/2}^{D/2} \mathbf{1}_{\{f_{\Delta_n}(\xi) \geq \widetilde{\varepsilon}\}} f_{\Delta_n}(\xi)^{1-p} d\xi \\ &\leq \frac{\varepsilon^p}{\widetilde{\varepsilon}^{p-1}} \int_{-D/2}^{D/2} \mathbf{1}_{\{f_{\Delta_n}(\xi) \geq \widetilde{\varepsilon}\}} d\xi. \end{aligned} \tag{7.7}$$

If $f_{\Delta_n}(\xi(\omega)) < \widetilde{\varepsilon}$, we obtain by $|\widetilde{X}(\omega) - X(\omega)| \mathbf{1}_{\{|X(\omega)| \leq D/2\}} \leq D$

$$\begin{aligned} II &\leq D^p \, \mathbb{E}(\mathbf{1}_{\{|\xi| \leq D/2, f_{\Delta_n}(\xi) < \widetilde{\varepsilon}\}}) \\ &= D^p \int_{-D/2}^{D/2} \mathbf{1}_{\{f_{\Delta_n}(\xi) < \widetilde{\varepsilon}\}} f_{\Delta_n}(\xi) d\xi \\ &< D^p \, \widetilde{\varepsilon} \int_{-D/2}^{D/2} \mathbf{1}_{\{f_{\Delta_n}(\xi) < \widetilde{\varepsilon}\}} d\xi \end{aligned} \tag{7.8}$$

and hence by Eqs. (7.7), (7.8) and $\widetilde{\varepsilon} = C^{-1} \varepsilon^{1+d}$

$$\begin{aligned} \mathbb{E}(|\widetilde{X} - X|^p \mathbf{1}_{\{|X| \leq D/2\}}) &\leq I + II \\ &< D^p \, \widetilde{\varepsilon} \Big(\int_{-D/2}^{D/2} \mathbf{1}_{\{f_{\Delta_n}(\xi) \geq \widetilde{\varepsilon}\}} d\xi + \int_{-D/2}^{D/2} \mathbf{1}_{\{f_{\Delta_n}(\xi) < \widetilde{\varepsilon}\}} d\xi \Big) \\ &= D^{p+1} \widetilde{\varepsilon}. \end{aligned}$$

With the estimate for $\mathbb{E}(|\widetilde{X} - X|^p \mathbf{1}_{\{|X| > D/2\}})$, $D = C\varepsilon^{-d}$ and $\widetilde{\varepsilon} = C^{-1}\varepsilon^{1+d}$ this leads to

$$\begin{aligned} \mathbb{E}(|\widetilde{X} - X|^p) &\leq 2Rp\zeta(\vartheta + 1 - p, D/2) + 2R(D/2)^{p-\vartheta} + D^{p+1}\widetilde{\varepsilon} \\ &= 2Rp\zeta(\vartheta + 1 - p, D/2) + 2R(D/2)^{p-\vartheta} + C^{1/d} D^{p-1/d}, \end{aligned}$$

and since $0 < d < \frac{1}{p}$ and $\vartheta > p$ by assumption, $\mathbb{E}(|\widetilde{X} - X|^p) \to 0$ as $\varepsilon \to 0$.

□

Remark 7.3.15. The relation $\widetilde{\varepsilon} = C^{-1}\varepsilon^{1+d}$ is chosen such that the factors preceding the integrals in Eqs. (7.7) and (7.8) are equilibrated. As only the sum of the two integrals is known a-priori, this leads to a better error estimation compared to non-equilibrated factors.

Theorem 7.3.16. *($L^p(\Omega;\mathbb{R})$-convergence II) Let F_{Δ_n} be continuously differentiable on $\mathbb{R}$ with density f_{Δ_n} and $(\phi_\ell)^{\Delta_n}$ be bounded as in Assumption 7.3.7 with $\vartheta > 2$. Furthermore, assume that the approximation parameters D and ε fulfill $D = C\varepsilon^{-d}$ for $C, d > 0$. If $d < \frac{1}{p}$, then for all $p \in [1, \vartheta)$*

$$\mathbb{E}(|\widetilde{X} - X|^p) \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof. Let $\varepsilon > 0$, $D = C\varepsilon^{-d}$ and $p \in [1, \vartheta - 1)$. Again, we split the expectation into

$$\mathbb{E}(|\widetilde{X} - X|^p) = \mathbb{E}(|\widetilde{X} - X|^p \mathbf{1}_{\{|X|>D/2\}}) + \mathbb{E}(|\widetilde{X} - X|^p \mathbf{1}_{\{|X|\leq D/2\}}),$$

and show convergence for the first term only, as the second term can be treated analogously to Theorem 7.3.14. In the same way as in Theorem 7.3.14, we may write for the first term

$$\mathbb{E}(|\widetilde{X} - X|^p \mathbf{1}_{\{|X|>D/2\}}) \leq \int_{D/2}^{\infty} (D/2 + x)^p (f_{\Delta_n}(x) + f_{\Delta_n}(-x))dx,$$

and further, by Assumption 7.3.7, follows

$$\begin{aligned}\int_{D/2}^{\infty} (D/2 + x)^p (f_{\Delta_n}(x) + f_{\Delta_n}(-x))dx &\leq 2R \int_{D/2}^{\infty} (D/2 + x)^p x^{-\vartheta} dx \\ &= 2^{p+1} R \int_0^{\infty} \frac{(D/2 + x/2)^p}{(D/2 + x)^\vartheta} dx \\ &< 2^{p+1} R \zeta(\vartheta - p, C\varepsilon^{-d}/2),\end{aligned}$$

which tends to zero as $\varepsilon \to 0$, because $\vartheta > p + 1$.

□

Remark 7.3.17. As expected, the admissible range of values for d and ϑ narrows as the rate of convergence p increases. For example, to obtain $L^2(\Omega;\mathbb{R})$-convergence, we need $d < \frac{1}{2}$ and $\vartheta > 2$ in Theorem 7.3.14 and $\vartheta > 3$ in Theorem 7.3.16. Recall Remark 7.3.10, where we have concluded that optimal relations between D and ε are

given by

$$D = \kappa \left(3/2\, RV_1(\kappa, \vartheta)\right)^{1/\vartheta} \varepsilon^{-1/\vartheta} \qquad \text{if Assumption 7.3.6 holds and}$$
$$D = \kappa \left(3/2\, RV_2(\kappa, \vartheta)\right)^{1/(\vartheta-1)} \varepsilon^{-1/(\vartheta-1)} \qquad \text{if Assumption 7.3.7 holds.}$$

For $L^p(\Omega;\mathbb{R})$-convergence, we need in both cases $D = C\varepsilon^{-d}$, where $C > 0$ and $d \in (0, 1/p)$. Hence, we can simply use $C = \kappa(3/2\, RV_1(\kappa,\vartheta))^{1/\vartheta}$ and $d = 1/\vartheta < 1/p$ in the first scenario and $C = \kappa(3/2\, RV_2(\kappa,\vartheta))^{1/\vartheta}$ and $d = 1/(\vartheta-1) < 1/p$ for the second set of assumptions. This explains the bounds on J (see also Remark 7.3.10): In Theorem 7.3.14, we obtain the expression $C^\vartheta D^{p-\vartheta}$ as a term of the overall error. If we had used the restrictions on J as in [116], we would have used $C = 2(3R)^{1/\vartheta}$ instead of the choice above and this would have resulted in an error term $C^\vartheta D^{p-\vartheta}$ being nearly twice as large (the argumentation works analogously for Theorem 7.3.16).

Example 7.3.18. The conditions $\vartheta > p$ in Theorem 7.3.14 and $\vartheta > p-1$ in Theorem 7.3.16 can not be relaxed as the following examples show: First, we investigate the Student's t-distribution with 3 degrees of freedom and density function

$$f^{t3}(x) = \frac{\Gamma_G(2)}{\sqrt{3\pi}\Gamma_G(3/2)} \left(1 + \frac{x^2}{3}\right)^{-2},$$

where $x \in \mathbb{R}$ and $\Gamma_G(\cdot)$ is the Gamma function. As shown in [126], this distribution is infinitely divisible and has characteristic function

$$\phi_{t3}(u) := \exp(-\sqrt{3}|u|)(\sqrt{3}|u| + 1),$$

hence we can define a Lévy process $(\ell^{t3}(t), t \in \mathbb{T})$ with $\phi_{t3}(u)$ as characteristic function. For simplicity we set $\Delta_n = 1$. In this case the (symmetric) distribution of the increment $\ell^{t3}(t+\Delta_n) - \ell^{t3}(t)$ has zero mean, finite variance, and its CDF F_1 can be bounded for any $x > 0$ by

$$\begin{aligned} F_1(-x) = 1 - F_1(x) &= \int_{-\infty}^{-x} f^{t3}(y)dy \\ &< \frac{\Gamma_G(2)}{\sqrt{3\pi}\Gamma_G(3/2)} \int_{-\infty}^{-x} \frac{3^2}{y^4} dy \\ &= \frac{\sqrt{3}\Gamma_G(2)}{\sqrt{\pi}\Gamma_G(3/2)} x^{-3} \\ &=: Rx^{-3}. \end{aligned}$$

Thus, this yields $\vartheta = 3$. The bounds for ϕ_{t3} are also straightforward:

$$|\phi_{t3}(u)| \leq (2\pi)^{-1} \max_{\widehat{u}>0} \exp(-\sqrt{3}\widehat{u})(\sqrt{3}\widehat{u}^2 + \widehat{u})|\frac{u}{2\pi}|^{-1} =: B|\frac{u}{2\pi}|^{-1},$$

where the maximum in B is found by differentiation giving $\widehat{u} = (1+\sqrt{5})/(2\sqrt{3})$. Now, all requirements for L^3-convergence except $\vartheta > 3$ are fulfilled. But the t-distribution with 3 degrees of freedom does not admit a third moment, hence we cannot have $L^3(\Omega;\mathbb{R})$-convergence although $\vartheta = 3$.

For the second case we consider the (standard) Cauchy process with characteristic function $(\phi_C(u))^t = \exp(-t|u|)$. It can be shown that the increment over time $\Delta_n > 0$ is again Cauchy-distributed with density

$$f^C_{\Delta_n}(x) = \frac{\Delta_n}{\pi(\Delta_n^2 + x^2)}.$$

This means the CDF of the increment is continuously differentiable and the bounds as in Assumption 7.3.7 are easily found by $f^C_{\Delta_n}(x) \leq (\Delta_n/\pi)|x|^{-2}$ for $x \in \mathbb{R}$ and

$$|(\phi_C(u))^{\Delta_n}| \leq (2\pi)^{-1} \max_{u\in\mathbb{R}} u \exp(-\Delta_n|u|)|\frac{u}{2\pi}|^{-1} = (2\pi\Delta_n)^{-1}\exp(-1)|\frac{u}{2\pi}|^{-1}$$

for $u \in \mathbb{R}$. But clearly, $L^p(\Omega;\mathbb{R})$-convergence in the sense of Theorem 7.3.16 for any $p \geq 1$ is impossible, as the Cauchy process does not have any finite moments.

From L^p-convergence follows almost sure convergence by a Borel–Cantelli-type argument, given ϑ in Assumptions 7.3.6 and 7.3.7 is large enough.

Corollary 7.3.19. *Under the assumptions of Theorem 7.3.14, set*

$$\tau_1 := \min\left(d\vartheta, 1-d\right),$$

let $m \in \mathbb{N}$ and set $\varepsilon = \varepsilon_m = m^{-q}$, with $q > \tau_1^{-1}$. If $(\widetilde{X}_m, m \in \mathbb{N})$ is generated based on the sequence $(\varepsilon_m, m \in \mathbb{N})$ (and the corresponding $D_m = C\varepsilon_m^{-d}$), then $(\widetilde{X}_m, m \in \mathbb{N})$ converges to X $\mathbb{P}$-almost surely.

Proof. If the assumptions of Theorem 7.3.14 with $\vartheta > 1$ hold, we can ensure at least $L^1(\Omega;\mathbb{R})$ convergence. Note that the Hurwitz zeta function

$$\zeta(\vartheta + 1 - p, D/2) = \zeta(\vartheta + 1 - p, C/2\varepsilon^{-d})$$

is of order $\mathcal{O}(\varepsilon^{d(\vartheta+1-p)})$ as $\varepsilon \to 0$. With Markov's inequality, $p = 1$ and the given error

bounds, we get that for each $\widehat{\varepsilon} > 0$ and $m \in \mathbb{N}$

$$\mathbb{P}(|\widetilde{X}_m - X| > \widehat{\varepsilon}) \leq \frac{\mathbb{E}(|\widetilde{X}_m - X|)}{\widehat{\varepsilon}} \leq \frac{\widetilde{C}}{\widehat{\varepsilon}} \left(\varepsilon_m^{d\vartheta} + \varepsilon_m^{1-d}\right) \leq \frac{2\widetilde{C}}{\widehat{\varepsilon}} \varepsilon_m^{\tau_1},$$

(recall that $1 > \tau_1 > 0$ and $\varepsilon_m \leq 1$) where the constant $\widetilde{C} > 0$ depends on R, ϑ and C. But this means

$$\sum_{m=1}^{\infty} \mathbb{P}(|\widetilde{X}_m - X| > \widehat{\varepsilon}) \leq \frac{2\widetilde{C}}{\widehat{\varepsilon}} \sum_{m=1}^{\infty} \varepsilon_m^{\tau_1} = \frac{2\widetilde{C}}{\widehat{\varepsilon}} \sum_{m=1}^{\infty} m^{-q\tau_1} < +\infty,$$

since $q\tau_1 > 1$ by construction. The claim then follows by the Borel-Cantelli lemma. □

Corollary 7.3.20. *Under the assumptions of Theorem 7.3.16, set*

$$\tau_2 := \min\left(d(\vartheta - 1), 1 - d\right),$$

let $m \in \mathbb{N}$ and set $\varepsilon = \varepsilon_m = m^{-q}$, with $q > \tau_2^{-1}$. If $(\widetilde{X}_m, m \in \mathbb{N})$ is generated based on the sequence $(\varepsilon_m, m \in \mathbb{N})$ (and the corresponding $D_m = C\varepsilon_m^{-d}$), then $(\widetilde{X}_m, m \in \mathbb{N})$ converges to X $\mathbb{P}$-almost surely.

We can now combine the error estimates for any increment over time $\Delta_n > 0$ with the piecewise approximation error from Algorithm 7.3.1 to bound the overall error $\ell(t) - \widetilde{\ell}^{(n)}(t)$.

Theorem 7.3.21. *Let ℓ be a Lévy process on $(\Omega, (\mathcal{F}_t, t \geq 0), \mathbb{P})$ with characteristic function ϕ_ℓ, CDF F_t and density f_t for any $t \in \mathbb{T}$. Assume for $n \in \mathbb{N}$ and fixed Δ_n there are constants $R, \vartheta, B, \theta > 0$ such that either Ass 7.3.6 or Ass. 7.3.7 holds. Let $\widetilde{\ell}^{(n)}$ be the piecewise constant approximation of ℓ generated by Algorithm 7.3.11 and the approximation $\widetilde{F}_{\Delta_n}$ of F_{Δ_n}. There are parameters D_n, ε_n for $\widetilde{F}_{\Delta_n}$ such that for any $p \in [1, \vartheta)$ resp. $p \in [1, \vartheta - 1)$ the approximation error is bounded by*

$$\mathbb{E}(|\ell(t) - \widetilde{\ell}^{(n)}(t)|^p)^{1/p} \leq C_{\ell,T,p,R,\vartheta} \Delta_n^{1/p}, \quad t \in \mathbb{T},$$

where the constant $C_{\ell,T,p,R,\vartheta} > 0$ only depends on the indicated parameters.

Proof. By Theorem 7.3.2 we may regard ℓ as the (point-wise) $L^p(\Omega; \mathbb{R})$-limit process of the sequence $(\overline{\ell}^{(n)}, n \in \mathbb{N})$ generated by Algorithm 7.3.1. For fixed n, we may then identify $\overline{\ell}^{(n)}$ with $\ell^{(n)}$ from Algorithm 7.3.5 to obtain

$$\mathbb{E}(|\ell(t) - \widetilde{\ell}^{(n)}(t)|^p)^{1/p} \leq \mathbb{E}(|\ell(t) - \overline{\ell}^{(n)}(t)|^p)^{1/p} + \mathbb{E}(|\ell^{(n)}(t) - \widetilde{\ell}^{(n)}(t)|^p)^{1/p}.$$

The first term is bounded by

$$\mathbb{E}(|\ell(t) - \overline{\ell}^{(n)}(t)|^p)^{1/p} \leq C_{\ell,T,p}^{1/p} \sum_{i=n+1}^{\infty} 2^{-i/p} = \frac{C_{\ell,T,p}^{1/p} 2^{-n/p}}{2^{1/p} - 1}.$$

For the treatment of the second term, we define for $j = 1, \ldots, 2^n$ the random variables $X_j := F_{\Delta_n}^{-1}(\mathbb{U}_j) \stackrel{\mathcal{L}}{=} \ell(\Delta_n)$. Here, $\mathbb{U}_1, \ldots, \mathbb{U}_{2^n}$ is the i.i.d sequence of $\mathcal{U}([0,1])$ random variables on $(\Omega, (\mathcal{F}_t, t \geq 0), \mathbb{P})$ from Algorithm 7.3.5. The increments of the approximation $\widetilde{\ell}^{(n)}$ are then given by $\widetilde{X}_j := \widetilde{F}_{\Delta_n}^{(-1)}(\mathbb{U}_j)$ which yields

$$|\ell^{(n)}(t) - \widetilde{\ell}^{(n)}(t)| \leq \sum_{j=1}^{2^n} |X_j - \widetilde{X}_j|.$$

The differences $(X_j - \widetilde{X}_j, j = 1, \ldots, 2^n)$ are i.i.d. by construction, hence

$$\mathbb{E}\left(|\ell^{(n)}(t) - \widetilde{\ell}^{(n)}(t)|^p\right)^{1/p} \leq \sum_{j=1}^{2^n} \mathbb{E}(|X_j - \widetilde{X}_j|^p)^{1/p} \leq 2^n \mathbb{E}(|X_1 - \widetilde{X}_1|^p)^{1/p}.$$

Now let $\widetilde{F}_{\Delta_n}$ be the approximation of F_{Δ_n} for some $\varepsilon \in (0,1]$ and $D = C\varepsilon^{-d}$. For the first set of assumptions, we apply the error estimates of Theorem 7.3.14 to obtain

$$\begin{aligned}
\mathbb{E}\left(|\ell^{(n)}(t) - \widetilde{\ell}^{(n)}(t)|^p\right) &\leq 2^n \mathbb{E}(|X_1 - \widetilde{X}_1|^p)^{1/p} \\
&\leq 2^n \Big(2Rp\zeta(\vartheta + 1 - p, C\varepsilon^{-d}/2) + 2R(C\varepsilon^{-d}/2)^{p-1/d} + C^p \varepsilon^{1-dp}\Big) \\
&\leq 2^n C_{R,\vartheta,p} \varepsilon^{\tau(p)},
\end{aligned}$$

where $\tau(p) := \min(d(\vartheta + 1 - p), 1 - dp)$ and $C_{R,\vartheta,p} > 0$ depends on C and the indicated parameters. An error of order $\Delta_n^{1/p}$ in the last inequality is then achieved by choosing $\varepsilon = \varepsilon_n = 2^{-(np+n)/\tau(p)}$ and $D_n = C\varepsilon_n^{-d}$. The proof for the second set of assumptions is carried out identically with the only difference that $\tau(p) := \min(d(\vartheta - p), 1 - dp)$. □

Remark 7.3.22. For an efficient simulation one would choose R based on $(\phi_\ell)^{\Delta_n}$ as in Remark 7.3.8 and then C based on R as suggested in Remark 7.3.17. Note that in this case $R = \mathcal{O}(\Delta_n)$ and

$$\begin{aligned}
C &= \kappa\,(3/2\,RV_1(\kappa, \vartheta))^{1/\vartheta} = \mathcal{O}(\Delta_n^{1/\vartheta}) \quad \text{resp.} \\
C &= \kappa\,(3/2\,RV_2(\kappa, \vartheta))^{1/(\vartheta-1)} = \mathcal{O}(\Delta_n^{1/(\vartheta-1)}).
\end{aligned}$$

This has to be considered in the simulation of $\widetilde{\ell}^{(n)}$ for different Δ_n as we point out in the setting of Ass. 7.3.6: As shown in Theorem 7.3.21, the L^p-error $\mathbb{E}(|\ell(t) - \widetilde{\ell}^{(n)}(t)|^p)$ is bounded by

$$\mathbb{E}(|\ell(t) - \widetilde{\ell}^{(n)}(t)|^p)^{1/p} \leq 2^n \Big(2Rp\zeta(\vartheta + 1 - p, C\varepsilon^{-d}/2) + 2R(C\varepsilon^{-d}/2)^{p-1/d} + C^p\varepsilon^{1-dp}\Big)^{1/p} + \frac{C_{\ell,T,p}^{1/p} 2^{-n/p}}{2^{1/p} - 1}.$$

By substituting $\varepsilon = C^{1/d}D^{-1/d}$, $d = 1/\vartheta$ (see Remark 7.3.17) and $2^n = T/\Delta_n$ one obtains

$$\mathbb{E}(|\ell(t) - \widetilde{\ell}^{(n)}(t)|^p)^{1/p} \leq \frac{(2Rp\zeta(\vartheta + 1 - p, D/2) + 2R(D/2)^{p-\vartheta} + C^\vartheta D^{p-\vartheta})^{1/p}}{T^{-1}\Delta_n} + \frac{C_{\ell,T,p}^{1/p} 2^{-n/p}}{2^{1/p} - 1} \tag{7.9}$$

With $R = \mathcal{O}(\Delta_n)$ and $C = \mathcal{O}(\Delta_n^{1/\vartheta})$ this implies with Ineq. (7.9)

$$\mathbb{E}(|\ell(t) - \widetilde{\ell}^{(n)}(t)|^p)^{1/p} = \mathcal{O}(D^{(p-\vartheta)/p}\Delta_n^{1/p-1}) + \mathcal{O}(\Delta_n^{1/p}).$$

To equilibrate both error contributions, one may choose $D := D_n = \Delta_n^{p/(p-\vartheta)}$ in the simulation which leads to an L^p-error of order $\mathcal{O}(\Delta_n^{1/p})$.

As mentioned in the end of Section 7.2, the one-dimensional processes $(\widetilde{\ell}_i, i = 1, \ldots, N)$ in the spectral decomposition are not independent, but merely uncorrelated. In the next section we introduce a class of Lévy fields for which uncorrelated processes can be obtained by subordinating a multi-dimensional Brownian motion. Furthermore, for the simulation of these processes the Fourier inversion method may be employed and a bound for the constant C_ℓ (see Theorem 7.2.3) can be derived.

7.4 Generalized hyperbolic Lévy processes

Distributions which belong to the class of generalized hyperbolic distributions may be used for a wide range of applications. GH distributions have been first introduced in [18] to model mass-sizes in aeolian sand (see also [19]). Since then they have been successfully applied, among others, in Finance and Biology. Giving a broad class the distributions are characterized by six parameters, famous representatives are the Student's t, the normal-inverse Gaussian, the hyperbolic and the variance-gamma distribution. The popularity of GH processes is explained by their flexibility in modeling

various characteristics of a distribution such as asymmetries or heavy tails. A further advantage in our setting is, that the characteristic function is known and, therefore, the Fourier Inversion may be applied to approximate these processes. This section is devoted to investigate several properties of multi-dimensional GH processes which are then used to construct an approximation of an infinite-dimensional GH field. In contrast to the Gaussian case, the sum of two independent and possibly scaled GH processes is in general not again a GH process. We show a possibility to approximate GH Lévy fields via Karhunen-Loève expansions in such a way that the approximated field is itself again a GH Lévy field. This is essential, so as to have convergence of the approximation to a GH Lévy field in the sense of Theorem 7.2.3. Furthermore, we give, for $N \in \mathbb{N}$, a representation of a N-dimensional GH process as a subordinated Brownian motion and show how a multi-dimensional GH process may be constructed from uncorrelated, one-dimensional GH processes with given parameters. This may be exploited by the Fourier inversion algorithm in such a way that the computational expenses to simulate the approximated GH fields are virtually independent of the truncation index N.

Assume, for $N \in \mathbb{N}$, that $\lambda \in \mathbb{R}$, $\alpha > 0$, $\beta \in \mathbb{R}^N$, $\delta > 0$, $\mu \in \mathbb{R}^N$ and Γ is a symmetric, positive definite (spd) $N \times N$-matrix with unit determinant. We denote by $GH_N(\lambda, \alpha, \beta, \delta, \mu, \Gamma)$ the N-dimensional generalized hyperbolic distribution with probability density function

$$f^{GH_N}(x; \lambda, \alpha, \beta, \delta, \mu, \Gamma) = \frac{\gamma^\lambda \alpha^{N/2-\lambda}}{(2\pi)^{N/2}\delta^\lambda K_\lambda(\delta\gamma)} \frac{K_{\lambda-N/2}(\alpha g(x-\mu))}{g(x-\mu)^{N/2-\lambda}} \exp(\beta'(x-\mu))$$

for $x \in \mathbb{R}^N$, where

$$g(x) := \sqrt{\delta^2 + x'\Gamma x}, \quad \gamma^2 := \alpha^2 - \beta'\Gamma\beta$$

and $K_\lambda(\cdot)$ is the modified Bessel-function of the second kind with λ degrees of freedom. The characteristic function of $GH_N(\lambda, \alpha, \beta, \delta, \mu, \Gamma)$ is given by

$$\begin{aligned} \phi_{GH_N}(u; \lambda, \alpha, \beta, \delta, \mu, \Gamma) := \exp(iu'\mu)\Big(\frac{\alpha^2 - \beta'\Gamma\beta}{\alpha^2 - (iu+\beta)'\Gamma(iu+\beta)}\Big)^{\lambda/2} \\ \cdot \frac{K_\lambda(\delta(\alpha^2 - (iu+\beta)'\Gamma(iu+\beta))^{1/2})}{K_\lambda(\delta\gamma)}, \end{aligned} \tag{7.10}$$

where A' denotes the transpose of a matrix or vector A. For simplicity, we assume that the condition

$$\alpha^2 > \beta'\Gamma\beta \tag{7.11}$$

is satisfied[3]. If $N = 1$, clearly, $\Gamma = 1$ is the only possible choice for the "matrix parameter" Γ, thus we omit it in this case and denote the one-dimensional GH distribution by $GH(\lambda, \alpha, \beta, \delta, \mu)$. Barndorff–Nielsen obtains the GH distribution in [19] as a normal variance-mean mixture of a N-dimensional normal distribution and a (one-dimensional) generalized inverse Gaussian (GIG) distribution with density function

$$f^{GIG}(x; a, b, p) = \frac{(b/a)^p}{2K_p(ab)} x^{p-1} \exp(-\frac{1}{2}(a^2 x^{-1} + b^2 x)), \quad x > 0,$$

and parameters $a, b > 0$ and $p \in \mathbb{R}$[4]. To be more precise: Let $w^N(1)$ be a N-dimensional standard normally distributed random vector, Γ a spd $N \times N$-structure matrix with unit determinant and $\ell^{GIG}(1)$ a $GIG(a, b, p)$ random variable, which is independent of $w^N(1)$. For $\mu, \beta \in \mathbb{R}^N$, we set $\delta = a$, $\lambda = p$, $\alpha = \sqrt{b^2 + \beta'\Gamma\beta}$ and define the random variable $\ell^{GH_N}(1)$ as

$$\ell^{GH_N}(1) := \mu + \Gamma\beta\ell^{GIG}(1) + \sqrt{\ell^{GIG}(1)\Gamma} w^N(1). \tag{7.12}$$

Then $\ell^{GH_N}(1)$ is $GH_N(\lambda, \alpha, \beta, \delta, \mu, \Gamma)$-distributed, where $\sqrt{\Gamma}$ denotes the Cholesky decomposition of the matrix Γ. With this in mind, one can draw samples of a GH distribution with given parameters by sampling multivariate normal and GIG-distributed random variables, as $a = \delta > 0$ and $b = \sqrt{\alpha^2 - \beta'\Gamma\beta} > 0$ is guaranteed by the conditions on the GIG parameters (this results in Ineq. (7.11) being fulfilled).

As noted in [75, Section 5], for general $\lambda \in \mathbb{R}$, we cannot assume that the increments of the GH Lévy process (resp. of the subordinating process) over a time length other than one follow a GH distribution (resp. GIG distribution). If $N = 1$, however, the (one-dimensional) GH Lévy process ℓ^{GH} has the representation

$$\ell^{GH}(t) \stackrel{\mathcal{L}}{=} \mu t + \beta\ell^{GIG}(t) + w(\ell^{GIG}(t)), \quad \text{for } t \geq 0,$$

where w is a one-dimensional Brownian motion and ℓ^{GIG} a GIG process independent of w (see [57]). This result yields the following generalization:

Lemma 7.4.1. *For $N \in \mathbb{N}$, the N-dimensional process $\ell^{GH_N} = (\ell^{GH_N}(t),\, t \in \mathbb{T})$, which is $GH_N(\lambda, \alpha, \beta, \delta, \mu, \Gamma)$-distributed, can be represented as a subordinated N-dimensional*

[3] If $\alpha^2 = \beta'\Gamma\beta$ and $\lambda < 0$, the distribution is still well-defined, but one has to consider the limit $\gamma \to 0^+$ in the Bessel functions, see [43, 186].

[4] The notation of the GIG distribution varies throughout the literature, we use the notation from [188].

Brownian motion w^N via

$$\ell^{GH_N}(t) \stackrel{\mathcal{L}}{=} \mu t + \Gamma\beta\ell^{GIG}(t) + \sqrt{\Gamma}w^N(\ell^{GIG}(t)),$$

where $(\ell^{GIG}(t), t \in \mathbb{T})$ is a GIG Lévy process independent of w^N and $\sqrt{\Gamma}$ is the Cholesky decomposition of Γ.

Proof. Since the $GH_N(\lambda, \alpha, \beta, \delta, \mu, \Gamma)$ distribution may be represented as a normal variance-mean mixture (see Eq. (7.12)), we have, that

$$\ell^{GH_N}(1) \stackrel{\mathcal{L}}{=} \mu + \Gamma\beta\ell^{GIG}(1) + \sqrt{\Gamma\ell^{GIG}(1)}w^N(1) \stackrel{\mathcal{L}}{=} \mu + \Gamma\beta\ell^{GIG}(1) + \sqrt{\Gamma}w^N(\ell^{GIG}(1)),$$

where $\ell^{GIG}(1) \sim GIG(\delta, \sqrt{\alpha^2 - \beta'\Gamma\beta}, \lambda)$ and w^N is a N-dimensional Brownian motion independent of $\ell^{GIG}(1)$. The characteristic function of the mixed density is given by

$$\phi_{GH_N}(u; \lambda, \alpha, \beta, \delta, \mu, \Gamma) = e^{iu'\mu}\mathcal{M}_{GIG}(iu'\Gamma\beta - \frac{1}{2}u'\Gamma u; \delta, \sqrt{\alpha^2 - \beta'\Gamma\beta}, \lambda),$$

where $\mathcal{M}_{GIG}$ denotes the moment generating function of $\ell^{GIG}(1)$ (see [21]). The GIG distribution is infinitely divisible, thus this GIG Lévy process $\ell^{GIG} = (\ell^{GIG}(t), t \in \mathbb{T})$ can be defined via its characteristic function for $t \in \mathbb{T}$:

$$\mathbb{E}(\exp(iu\ell^{GIG}(t))) = \Big(\mathcal{M}_{GIG}(iu; \delta, \sqrt{\alpha^2 - \beta'\Gamma\beta}, \lambda)\Big)^t.$$

The infinite divisibility further yields

$$\begin{aligned}
\mathbb{E}\Big(\exp(iu'\ell^{GH_N}(t))\Big) &= \mathbb{E}\Big(\exp(iu'\ell^{GH_N}(1))\Big)^t \\
&= (\phi_{GH}(u; \lambda, \alpha, \beta, \delta, \mu, \Gamma))^t \\
&= e^{iu'\mu t}\Big(\mathcal{M}_{GIG}(iu'\Gamma\beta - \frac{1}{2}u'\Gamma u; \delta, \sqrt{\alpha^2 - \beta'\Gamma\beta}, \lambda)\Big)^t.
\end{aligned}$$

The expression above is the characteristic function of another normal variance-mean mixture, with a GIG-subordinator ℓ^{GIG} and characteristic function

$$\mathbb{E}(\exp(iu\ell^{GIG}(t))) = \Big(\mathcal{M}_{GIG}(iu; \delta, \sqrt{\alpha^2 - \beta'\Gamma\beta}, \lambda)\Big)^t.$$

Hence, $\ell^{GH_N}(t)$ can be expressed as

$$\ell^{GH_N}(t) \stackrel{\mathcal{L}}{=} \mu t + \Gamma\beta\ell^{GIG}(t) + \sqrt{\Gamma}w^N(\ell^{GIG}(t)).$$

□

Remark 7.4.2. In the special case of $\lambda = -\frac{1}{2}$ one obtains the normal inverse Gaussian (NIG) distribution. The mixing density is, in this case, the inverse Gaussian (IG) distribution. We denote the N-dimensional NIG distribution by $NIG_N(\alpha, \beta, \delta, \mu, \Gamma)$. This is the only subclass of GH distributions which is closed under convolutions in the sense that

$$NIG_N(\alpha, \beta, \delta_1, \mu_1, \Gamma) * NIG_N(\alpha, \beta, \delta_2, \mu_2, \Gamma) = NIG_N(\alpha, \beta, \delta_1 + \delta_2, \mu_1 + \mu_2, \Gamma)$$

(see [167]). For $\lambda \in \mathbb{R}$, the sum of independent GH random variables is in general not GH-distributed. This implies further, that one is in general not able to derive bridge laws of these processes in closed form, meaning we need to use the algorithms introduced in Section 7.3.2 for simulation.

As shown in [20], the GH and the GIG distribution are infinitely-divisible, thus we can define the N-dimensional GH Lévy process $\ell^{GH_N} = (\ell^{GH_N}(t), t \in \mathbb{T})$ with characteristic function

$$\mathbb{E}(\exp(iu\ell^{GH_N}(t)) = (\phi_{GH_N}(u; \lambda, \alpha, \beta, \delta, \mu, \Gamma))^t.$$

Remark 7.4.3. If $\lambda = -\frac{1}{2}$, the corresponding NIG Lévy process $(\ell^{NIG_N}(t), t \in \mathbb{T})$ has characteristic function

$$\mathbb{E}[\exp(iu\ell^{NIG_N}(t))] = (\phi_{GH_N}(u; -\frac{1}{2}, \alpha, \beta, \delta, \mu, \Gamma))^t = \phi_{GH_N}(u; -\frac{1}{2}, \alpha, \beta, t\delta, t\mu, \Gamma).$$

This is due to the fact that the characteristic function $\phi_{IG}(u; a, b)$ of the mixing IG distribution fulfills the identity

$$(\phi_{IG}(u; a, b))^t = \phi_{IG}(u; ta, b)$$

for any $t \in \mathbb{T}$ and $a, b > 0$ (see [188]).

We consider the finite time horizon $\mathbb{T} = [0, T]$, for $T < +\infty$, the probability space $(\Omega, (\mathcal{F}_t, t \geq 0), \mathbb{P})$, and a compact domain $\mathcal{D} \subset \mathbb{R}^s$ for $s \in \mathbb{N}$ to define a GH Lévy field as a mapping

$$L^{GH} : \Omega \times \mathcal{D} \times \mathbb{T} \to \mathbb{R}, \quad (\omega, x, t) \mapsto L^{GH}(\omega)(x)(t),$$

such that for each $x \in \mathcal{D}$ the point-wise marginal process

$$L^{GH}(\cdot)(x)(\cdot) : \Omega \times \mathbb{T} \to \mathbb{R}, \quad (\omega, t) \mapsto L^{GH}(\omega)(x)(t),$$

is a one-dimensional GH Lévy process on $(\Omega, (\mathcal{F}_t, t \geq 0), \mathbb{P})$ with characteristic function

$$\mathbb{E}\Big(\exp(iuL^{GH}(x)(t))\Big) = (\phi_{GH}(u; \lambda(x), \alpha(x), \beta(x), \delta(x), \mu(x)))^t,$$

where the indicated parameters are given by continuous functions, i.e. $\lambda, \beta, \mu \in C(\mathcal{D}; \mathbb{R})$ and $\alpha, \delta \in C(\mathcal{D}; \mathbb{R}_{>0})$. We assume that condition (7.11), i.e. $\alpha(x)^2 > \beta(x)^2$, is fulfilled for any $x \in \mathcal{D}$ to ensure that $L^{GH}(x)(\cdot)$ is a well-defined GH Lévy process. This, in turn, means that L^{GH} takes values in the Hilbert space $U = L^2(\mathcal{D})$ and is square integrable as

$$\mathbb{E}(||L^{GH}(t)||_U^2) \leq T\mathbb{E}(||L^{GH}(1)||_U^2) \leq T \max_{x \in \mathcal{D}} \mathbb{E}(L^{GH}(x)(1)^2)V_{\mathcal{D}},$$

where $V_{\mathcal{D}}$ denotes the volume of $\mathcal{D}$. The right hand side is finite since every GH distribution has finite variance (see for example [157, 188]), the parameters of the distribution of $L^{GH}(x)(1)$ depend continuously on x and $\mathcal{D} \subset \mathbb{R}^s$ is compact by assumption. We use the Karhunen-Loève expansion from Section 7.2 to obtain an approximation of a given GH Lévy field. For this purpose, we consider the truncated sum

$$L_N^{GH}(x)(t) := \sum_{i=1}^{N} \varphi_i(x)\ell_i^{GH}(t) \stackrel{\mathcal{L}}{=} \sum_{i=1}^{N} \varphi_i(x)\Big(\mu_i t + \beta_i \ell_i^{GIG}(t) + w_i(\ell_i^{GIG}(t))\Big),$$

where $N \in \mathbb{N}$ and $\varphi_i(x) = \sqrt{\eta_i}e_i(x)$ is the i-th component of the spectral basis evaluated at the spatial point x. For each $i = 1, \ldots, N$, the processes $\ell_i^{GH} := (\ell_i^{GH}(t), t \in \mathbb{T})$ are uncorrelated but dependent $GH(\lambda_i, \alpha_i, \beta_i, \delta_i, \mu_i)$ Lévy process. From Theorem 7.2.3 follows that L_N^{GH} converges in $L^2(\Omega; U)$ to L^{GH} as $N \to +\infty$. With given $\mu_i, \beta_i \in \mathbb{R}$, we have that

$$\ell_i^{GH}(t) \stackrel{\mathcal{L}}{=} \mu_i t + \beta_i \ell_i^{GIG}(t) + w_i(\ell_i^{GIG}(t)), \tag{7.13}$$

where for each i, the process $(\ell_i^{GIG}(t), t \in \mathbb{T})$ is a GIG Lévy process with parameters $a_i = \delta_i, b_i = (\alpha_i^2 - \beta_i^2)^{1/2} > 0$ and $p_i = \lambda_i \in \mathbb{R}$. In addition, $(w_i(t), t \in \mathbb{T})$ is a one-dimensional Brownian motion independent of ℓ_i^{GIG} and all Brownian motions $w_1, \ldots, w_N$ are mutually independent of each other, but the processes $\ell_1^{GIG}, \ldots, \ell_N^{GIG}$ may be correlated. We aim for an approximation $(L_N^{GH}(x)(t), t \in \mathbb{T})$ which is a GH process for arbitrary φ_i and $x \in \mathcal{D}$. Remark 7.4.2 suggests that this cannot be achieved by the summation of independent ℓ_i^{GH}, but rather by using correlated subordinators

$\ell_1^{GIG}, \dots, \ell_N^{GIG}$. Before we determine the correlation structure of the subordinators, we establish a necessary and sufficient condition on the ℓ_i^{GH} to achieve the desired distribution of the approximation.

Lemma 7.4.4. *Let $N \in \mathbb{N}$, $t \in \mathbb{T}$ and $(\ell_i^{GH}, i = 1 \dots, N)$ be GH processes as defined in Eq. (7.13). For a vector $\mathbf{a} = (a_1, \dots, a_N)$ with arbitrary numbers $a_1, \dots, a_N \in \mathbb{R} \setminus \{0\}$, the process $\ell^{GH,\mathbf{a}}$ defined by*

$$\ell^{GH,\mathbf{a}}(t) := \sum_{i=1}^{N} a_i \ell_i^{GH}(t) = \sum_{i=1}^{N} a_i(\mu_i + \beta_i \ell_i^{GIG}(t) + w_i(\ell_i^{GIG}(t)))$$

is a one-dimensional GH process, if and only if the vector

$$\ell^{GH_N}(1) := (\ell_1^{GH}(1), \dots, \ell_N^{GH}(1))'$$

is multivariate $GH_N(\lambda^{(N)}, \alpha^{(N)}, \beta^{(N)}, \delta^{(N)}, \mu^{(N)}, \Gamma)$-distributed with the set of parameters $\lambda^{(N)}$, $\alpha^{(N)}$, $\delta^{(N)} \in \mathbb{R}$, $\beta^{(N)}, \mu^{(N)} \in \mathbb{R}^N$ and structure matrix $\Gamma \in \mathbb{R}^{N \times N}$.

The entries of the coefficient vector $\mathbf{a}$ in $\ell^{GH,\mathbf{a}}$ are later identified with the basis functions $\varphi_i(x)$ for $x \in \mathcal{D}$ to show that $L_N^{GH}(x)(\cdot)$ is a one-dimensional Lévy process and the approximation L_N^{GH} is a U-valued GH Lévy field.

Proof of Lemma 7.4.4. We first consider the case that

$$\ell^{GH_N}(1) \sim GH_N(\lambda^{(N)}, \alpha^{(N)}, \beta^{(N)}, \delta^{(N)}, \mu^{(N)}, \Gamma).$$

It is sufficient to show that $\ell^{GH,\mathbf{a}}(1)$ is a GH-distributed random variable, the infinite divisibility of the GH distribution then implies that $(\ell^{GH,\mathbf{a}}(t), t \in \mathbb{T})$ is a GH process. Since the entries of the coefficient vector $a_1, \dots, a_N$ are non-zero, there exists a non-singular $N \times N$ matrix A, such that $\ell^{GH,\mathbf{a}}(1)$ is the first component of the vector $A\ell^{GH_N}(1)$. If $\ell^{GH_N}(1)$ is multi-dimensional GH-distributed, then follows from [43, Theorem 1], that $A\ell^{GH_N}(1)$ is also multi-dimensional GH-distributed and that the first component of $A\ell^{GH_N}(1)$, namely $\ell^{GH,\mathbf{a}}(1)$, follows a one-dimensional GH distribution (the parameters of the distribution of $\ell^{GH,\mathbf{a}}(1)$ depend on A and on $\lambda^{(N)}, \alpha^{(N)}, \beta^{(N)}, \delta^{(N)}, \mu^{(N)}, \Gamma$ and are explicitly given in [43] and below).
On the other hand, assume that $\ell^{GH,\mathbf{a}}(1)$ is a GH random variable (with arbitrary coefficients), but $\ell^{GH_N}(1)$ is not N-dimensional GH-distributed. This means there is no representation of $\ell^{GH_N}(1)$ such that

$$\ell^{GH_N}(1) \stackrel{\mathcal{L}}{=} \mu + \Gamma\beta\ \ell^{GIG}(1) + \sqrt{\Gamma} w^N(\ell^{GIG}(1))$$

with $\mu, \beta \in \mathbb{R}^N$, $\Gamma \in \mathbb{R}^{N\times N}$ spd with determinant one, a GIG random variable $\ell^{GIG}(1)$ and a N-dimensional Brownian motion w^N independent of $\ell^{GIG}(1)$. This implies that $\ell^{GH,\mathbf{a}}(1) = (A\ell^{GH_N}(1))_1$ has no representation

$$\begin{aligned}\ell^{GH,\mathbf{a}}(1) &= (A\mu)_1 + (A\Gamma\beta)_1\ell^{GIG}(1) + (A\sqrt{\Gamma}w^N(\ell^{GIG}(1)))_1 \\ &\stackrel{\mathcal{L}}{=} (A\mu)_1 + (A\Gamma\beta)_1\ell^{GIG}(1) + \sqrt{\ell^{GIG}(1)A_{[1]}\Gamma A'_{[1]}}w^1(1),\end{aligned}$$

where $A_{[1]}$ denotes the first row of the matrix A and $w^1(1) \sim \mathcal{N}(0,1)$. For the last equality we have used the affine linear transformation property of multi-dimensional normal distributions and that Γ is positive definite. Since $c_A := A_{[1]}\Gamma A'_{[1]} > 0$, we can divide the equation above by $\sqrt{c_A}$ and obtain that $c_A^{-1/2}\,\ell^{GH,\mathbf{a}}(1)$ cannot be a GH-distributed random variable, as it cannot be expressed as a normal variance-mean mixture with a GIG-distribution. But this is a contradiction, since $\ell^{GH,\mathbf{a}}(1)$ is GH-distributed by assumption and the class of GH distributions is closed under regular affine linear transformations (see [43, Theorem 1c]). □

Remark 7.4.5. The condition $a_i \neq 0$ is, in fact, not necessary in Lemma 7.4.4. If, for $k \in \{1,\dots,N-1\}$, k coefficients $a_{i_1} = \cdots = a_{i_k} = 0$, then the summation reduces to

$$\ell^{GH,\mathbf{a}}(t) = \sum_{i=1}^{N} a_i\ell_i^{GH}(t) = \sum_{l=1}^{N-k} a_{j_l}\ell_{j_l}^{GH}(t),$$

where the indices j_l are chosen such that $a_{j_l} \neq 0$ for $l = 1,\dots,N-k$. If $P \in \mathbb{R}^{N\times N}$ is the permutation matrix with

$$\begin{aligned}P\ell^{GH_N}(1) &= P(\ell_1^{GH}(1),\dots,\ell_N^{GH}(1))' \\ &= (\ell_{j_1}^{GH}(1),\dots,\ell_{j_{N-k}}^{GH}(1),\ell_{i_1}^{GH}(1),\dots,\ell_{i_k}^{GH}(1))',\end{aligned}$$

then $P\ell^{GH_N}$ is again N-dimensionally GH-distributed and by [43, Theorem 1a] the vector $(\ell_{j_1}^{GH}(1),\dots,\ell_{j_{N-k}}^{GH}(1))$ admits a $(N-k)$-dimensional GH law. Thus, we only consider the case where all coefficients are non-vanishing.

The previous proposition states that the KL approximation

$$L_N^{GH}(x)(t) = \sum_{i=1}^{N}\varphi_i(x)\ell_i^{GH}(t),$$

can only be a GH process for arbitrary $(\varphi_i(x), i = 1,\dots,N)$ if the ℓ_i^{GH} are correlated in such a way that they form a multi-dimensional GH process. This rules out the possibility of independent processes $(\ell_i^{GH}, i = 1,\dots,N)$, because if $\ell^{GH_N}(1)$ is multi-

dimensional GH-distributed, it is not possible that the marginals $\ell_i^{GH}(1)$ are independent GH-distributed random variables (see [43]). The parameters $\lambda_i, \alpha_i, \beta_i, \delta_i, \mu_i$ of each process ℓ_i^{GH} should remain as unrestricted as possible, so we determine in the next step the parameters of the marginals of a $GH_N(\lambda^{(N)}, \alpha^{(N)}, \beta^{(N)}, \delta^{(N)}, \mu^{(N)}, \Gamma)$ distribution and show how the subordinators $(\ell_i^{GIG}, i = 1, \ldots, N)$ might be correlated. The following result allows us to determine the marginal distributions of a N-dimensional GH distribution.

Lemma 7.4.6. *(Masuda [157], who refers to [44], Lemma A.1.) Let*

$$\ell^{GH_N}(1) = (\ell_1^{GH}(1), \ldots, \ell_N^{GH}(1))' \sim GH_N(\lambda^{(N)}, \alpha^{(N)}, \beta^{(N)}, \delta^{(N)}, \mu^{(N)}, \Gamma),$$

then for each i we have that $\ell_i^{GH}(1) \sim GH(\lambda_i, \alpha_i, \beta_i, \delta_i, \mu_i)$, where

$$\begin{aligned}
&\lambda_i = \lambda^{(N)}, \quad \alpha_i = \Gamma_{ii}^{-1/2} \left[(\alpha^{(N)})^2 - \beta_{-i}' \left(\Gamma_{-i,22} - \Gamma_{-i,21} \Gamma_{ii}^{-1} \Gamma_{-i,12} \right) \beta_{-i} \right]^{1/2} \\
&\beta_i = \beta_i^{(N)} + \Gamma_{ii}^{-1} \Gamma_{-i,12} \beta_{-i}, \quad \delta_i = \sqrt{\Gamma_{ii}} \delta_i^{(N)}, \quad \mu_i = \mu_i^{(N)},
\end{aligned}$$

together with

$$\begin{aligned}
&\beta_{-i} := (\beta_1^{(N)}, \ldots, \beta_{i-1}^{(N)}, \beta_{i+1}^{(N)}, \ldots, \beta_N^{(N)})', \\
&\Gamma_{-i,12} := (\Gamma_{i,1}, \ldots, \Gamma_{i,i-1}, \Gamma_{i,i+1}, \ldots, \Gamma_{i,N}), \quad \Gamma_{-i,21} := \Gamma_{-i,12}'
\end{aligned}$$

and $\Gamma_{-i,22}$ denotes the $(N-1) \times (N-1)$ matrix which is obtained by removing the i-th row and column of Γ.

Assume that $\ell^{GH_N}(1) \sim GH_N(\lambda^{(N)}, \alpha^{(N)}, \beta^{(N)}, \delta^{(N)}, \mu^{(N)}, \Gamma)$, since this is a necessary (and sufficient) condition so that the (truncated) KL expansion is a GH process. Lemma 7.4.6 gives immediately, that for all $i = 1, \ldots, N$, the parameters $\lambda_i = \lambda^{(N)}$ have to be identical, whereas the drift μ_i may be chosen arbitrary for each process ℓ_i^{GH}. Furthermore, the expectation and covariance matrix of $\ell^{GH_N}(1)$ is given by

$$\mathbb{E}(\ell^{GH_N}(1)) = \mu^{(N)} + \frac{\delta^{(N)} K_{\lambda^{(N)}+1}(\delta^{(N)}\gamma^{(N)})}{\gamma^{(N)} K_{\lambda^{(N)}}(\delta^{(N)}\gamma^{(N)})} \Gamma \beta^{(N)} \tag{7.14}$$

and

$$\begin{aligned}
\mathrm{Var}(\ell^{GH_N}(1)) = &\frac{\delta^{(N)} K_{\lambda^{(N)}+1}(\delta^{(N)}\gamma^{(N)})}{\gamma^{(N)} K_{\lambda^{(N)}}(\delta^{(N)}\gamma^{(N)})} \Gamma + \left(\frac{\delta^{(N)}}{\gamma^{(N)}} \right)^2 (\Gamma \beta^{(N)})(\Gamma \beta^{(N)})' \\
&\cdot \left(\frac{K_{\lambda^{(N)}+2}(\delta^{(N)}\gamma^{(N)})}{K_{\lambda^{(N)}}(\delta^{(N)}\gamma^{(N)})} - \frac{K_{\lambda^{(N)}+1}^2(\delta^{(N)}\gamma^{(N)})}{K_{\lambda^{(N)}}^2(\delta^{(N)}\gamma^{(N)})} \right),
\end{aligned} \tag{7.15}$$

where $\gamma^{(N)} := ((\alpha^{(N)})^2 - \beta^{(N)\prime}\Gamma\beta^{(N)})^{1/2}$ (see [157]).

Example 7.4.7. Consider the case that the processes $\ell_1^{GH}, \ldots, \ell_N^{GH}$ are generated by the same subordinating $GIG(a, b, p)$ process ℓ^{GIG}, i.e.

$$\ell_i^{GH}(t) = \mu_i t + \beta_i \ell^{GIG}(t) + w_i(\ell^{GIG}(t)).$$

Then $\ell_i^{GH}(1) \sim GH(\lambda, \alpha_i, \beta_i, \delta, \mu_i)$, where $\lambda = p$, $\delta = a$ are independent of i and $\alpha_i = (b^2 + \beta_i^2)^{1/2}$. If $\mu^{(N)} := (\mu_1 \ldots, \mu_N)'$, $\beta^{(N)} := (\beta_1, \ldots, \beta_N)'$ and Γ is the $N \times N$ identity matrix, then

$$\begin{aligned}\ell^{GH_N}(t) = (\ell_1^{GH}(t), \ldots, \ell_N^{GH}(t))' &\stackrel{\mathcal{L}}{=} \mu t + \beta \ell^{GIG}(t) + w^N(\ell^{GIG}(t)) \\ &= \mu t + \Gamma\beta\ell^{GIG}(t) + \sqrt{\Gamma} w^N(\ell^{GIG}(t)),\end{aligned}$$

where w^N is a N-dimensional Brownian motion independent of ℓ^{GIG}. Hence, $\ell^{GH_N}(t)$ is a multi-dimensional $GH_N(\lambda, \alpha^{(N)}, \beta^{(N)}, \delta, \mu^{(N)}, \Gamma)$ process with $\alpha^{(N)} = \sqrt{b^2 + \beta'\beta}$. One checks using Lemma 7.4.6 that the parameters of the marginals of $\ell^{GH_N}(1)$ and $\ell_i^{GH}(1)$ coincide for each i, and that expectation and covariance of $\ell^{GH_N}(1)$ are given by Eq. (7.14) and Eq. (7.15). By Lemma 7.4.4, the Karhunen-Loève expansion

$$L_N^{GH}(x)(t) = \sum_{i=1}^{N} \varphi_i(x) \ell_i^{GH}(t)$$

in this example is a GH process for each $x \in \mathcal{D}$ and an arbitrary basis $(\varphi_i, i = 1, \ldots, N)$.

Remark 7.4.8. Lemma 7.4.6 dictates that the subordinators $(\ell_i^{GIG}, i = 1, \ldots, N)$ cannot be independent. In Example 7.4.7 fully correlated subordinators were used. A different way to correlate the subordinators, so that Lemma 7.4.6 is fulfilled, would lead to a correlation matrix, just being multiplied with Γ. For simplicity, in the remainder of the paper, especially for the numerical examples in Section 7.5, we use fully correlated subordinators.

As shown in [43, Theorem 1c] the class of N-dimensional GH distributions is closed under regular linear transformations: If $N \in \mathbb{N}$, $\ell^{GH_N}(1) \sim GH_N(\lambda, \alpha, \beta, \delta, \mu, \Gamma)$, A is an invertible $N \times N$-matrix and $b \in \mathbb{R}^N$, then the random vector $A\ell^{GH_N}(1) + b$ has distribution

$$GH_N(\lambda, ||A||^{-1/N}\alpha, (A^{-1})'\beta, ||A||^{1/N}\delta, A\mu + b, ||A||^{-2/N} A\Gamma A'),$$

where $||A||$ denotes the absolute value of the determinant of A. With this and the assumption $\ell^{GH_N}(1) \sim GH_N(\lambda^{(N)}, \alpha^{(N)}, \beta^{(N)}, \delta^{(N)}, \mu^{(N)}, \Gamma)$, we are also able to determine

the point-wise law of L_N^{GH} for given $\varphi_1(x), \ldots, \varphi_N(x)$.

Lemma 7.4.9. *Let $\ell^{GH_N}(1) \sim GH_N(\lambda^{(N)}, \alpha^{(N)}, \beta^{(N)}, \delta^{(N)}, \mu^{(N)}, \Gamma)$ and for $x \in \mathcal{D}$ let $(\varphi_i(x), i = 1, \ldots, N)$ be a sequence of non-zero coefficients (see Remark 7.4.5). Then $(L_N^{GH}(x)(t), t \in \mathbb{T})$ is a GH Lévy process with parameters depending on x.*

Proof. It is again sufficient to show that $L_N^{GH}(x)(1)$ follows a GH law, the resulting parameters are given below. For $x \in \mathcal{D}$, define the $N \times N$ matrix $A(x)$ via

$$A(x)_{ij} := \begin{cases} \varphi_j(x) & \text{if } i = 1 \text{ or if } i = j \\ 0 & \text{elsewhere} \end{cases}.$$

The matrix $A(x)$ is invertible with determinant $\prod_{i=1}^N \varphi_i(x) \neq 0$ and inverse $A(x)^{-1}$ given by

$$A(x)_{ij}^{-1} := \begin{cases} -\varphi_1(x)^{-1} & \text{if } i = 1 \text{ and } j \geq 2 \\ \varphi_i(x)^{-1} & \text{if } i = j \\ 0 & \text{elsewhere} \end{cases}.$$

Then, $L_N^{GH}(x)(1) = \sum_{i=1}^N \varphi_i(x) \ell_i^{GH}(1)$ becomes the first entry of the random vector $A(x)\ell^{GH_N}(1)$. By the affine transformation property of the GH distribution and Lemma 7.4.6 it follows that $L_N^{GH}(x)(1)$ is one-dimensional GH-distributed. Now define $\widetilde{\Gamma} := A(x)\Gamma A(x)'$, the partition

$$\widetilde{\Gamma} = \begin{pmatrix} \widetilde{\Gamma}_{11} & \widetilde{\Gamma}_{2,1}' \\ \widetilde{\Gamma}_{2,1} & \widetilde{\Gamma}_{2,2} \end{pmatrix}$$

such that $\widetilde{\Gamma}_{2,1} \in \mathbb{R}^{N-1}$ and $\widetilde{\Gamma}_{2,2} \in \mathbb{R}^{(N-1)\times(N-1)}$ and the vector

$$\widetilde{\beta} := \left(\beta_2^{(N)}\varphi_2(x)^{-1} - \beta_1^{(N)}\varphi_1(x)^{-1}, \ldots, \beta_N^{(N)}\varphi_N(x)^{-1} - \beta_1^{(N)}\varphi_1(x)^{-1}\right)' \in \mathbb{R}^{N-1}.$$

The parameters $\lambda_L, \alpha_L(x), \beta_L(x), \delta_L(x)$ and $\mu_L(x)$ of $L_N^{GH}(x)$ are then given by

$$\begin{aligned}
\lambda_L &= \lambda^{(N)}, \\
\alpha_L(x) &= \widetilde{\Gamma}_{11}^{-1/2}\left[(\alpha^{(N)})^2 - \widetilde{\beta}'(\widetilde{\Gamma}_{2,2} - \widetilde{\Gamma}_{11}^{-1}\widetilde{\Gamma}_{2,1}\widetilde{\Gamma}_{2,1}')\widetilde{\beta}\right]^{1/2}, \\
\delta_L(x) &= \delta^{(N)}\sqrt{\widetilde{\Gamma}_{11}} = \delta^{(N)}\Big(\sum_{i,j=1}^{N}\varphi_i(x)\varphi_j(x)\Gamma_{ij}\Big)^{1/2}, \\
\beta_L(x) &= \beta_1^{(N)}\varphi_1(x)^{-1} + \widetilde{\Gamma}_{11}^{-1}\widetilde{\Gamma}_{2,1}'\widetilde{\beta} \quad \text{and} \\
\mu_L(x) &= [A(x)\mu^{(N)}]_1 = \sum_{i=1}^{N}\varphi_i(x)\mu_i^{(N)}.
\end{aligned}$$

□

To ensure $L^2(\Omega;\mathbb{R})$ convergence as in Theorem 7.2.3 of the series

$$\widetilde{L}_N^{GH}(x)(t) = \sum_{i=1}^{N}\sqrt{\eta_i}e_i(x)\widetilde{\ell}_i^{GH}(t),$$

we need to simulate approximations of uncorrelated, one-dimensional GH processes ℓ_i^{GH} with given parameters $\ell_i^{GH}(1) \sim GH(\lambda_i, \alpha_i, \beta_i, \delta_i, \mu_i)$. To obtain a sufficiently good approximation of the Lévy field, N is coupled to the time discretization of $\mathbb{T}$ and the decay of the eigenvalues of Q (see Remark 7.2.4). The simulation of a large number N of independent GH processes is computationally expensive, so we focus on a different approach. Instead of generating N dependent but uncorrelated, one-dimensional processes, we generate one N-dimensional process with decorrelated marginals. For this approach to work we need to impose some restrictions on the target parameters $\lambda_i, \alpha_i, \beta_i$ and δ_i.

Theorem 7.4.10. *Let $(\ell_i^{GH}, i = 1\ldots, N)$ be one-dimensional GH processes, where, for $i = 1, \ldots, N$, $\ell_i^{GH}(1) \sim GH(\lambda_i, \alpha_i, \beta_i, \delta_i, \mu_i)$. The vector $\ell^{GH_N} := (\ell_1^{GH}, \ldots, \ell_N^{GH})'$ is only a N-dimensional GH process if there are constants $\lambda \in \mathbb{R}$ and $c > 0$ such that*

$$\lambda_i = \lambda \quad \textit{and} \quad \delta_i(\alpha_i^2 - \beta_i^2)^{1/2} = c$$

for any i. If, in addition, the symmetric matrix $U \in \mathbb{R}^{N\times N}$ defined by

$$U_{ij} := \begin{cases} \delta_i^2 & \text{if } i = j \\ \frac{K_{\lambda+1}(c)^2 - K_{\lambda+2}(c)K_\lambda(c)}{K_{\lambda+1}(c)K_\lambda(c)}\frac{\beta_i\delta_i^2\beta_j\delta_j^2}{c} & \text{if } i \neq j \end{cases},$$

is positive definite, it is possible to construct a N-dimensional GH process $\ell^{GH_N,U}$ with

uncorrelated marginals $\ell_i^{GH,U}$ *and*

$$\ell_i^{GH,U}(1) \stackrel{\mathcal{L}}{=} \ell_i^{GH}(1) \sim GH(\lambda_i, \alpha_i, \beta_i, \delta_i, \mu_i).$$

Proof. We start with the necessary condition to obtain a multi-dimensional GH distribution. Let ℓ^{GH_N} be a N-dimensional GH process with

$$\ell^{GH_N}(1) \sim GH_N(\lambda^{(N)}, \alpha^{(N)}, \beta^{(N)}, \delta^{(N)}, \mu^{(N)}, \Gamma).$$

If the law of the marginals of ℓ^{GH_N} is denoted by $\ell_i^{GH}(1) \sim GH(\lambda_i, \alpha_i, \beta_i, \delta_i, \mu_i)$, then one sees immediately from Lemma 7.4.6 that $\lambda_i = \lambda^{(N)}$ and $\mu_i = \mu_i^{(N)}$ for all $i = 1, \dots, N$. With the equations for β_i and δ_i from Lemma 7.4.6, we derive for $\Gamma\beta^{(N)}$

$$(\Gamma\beta^{(N)})_i = \Gamma_{ii}\beta_i^{(N)} + \sum_{k=1,k\neq i}^{N} \Gamma_{ik}\beta_k^{(N)} = \Gamma_{ii}\beta_i^{(N)} + \Gamma_{ii}(\beta_i - \beta_i^{(N)}) = \Big(\frac{\delta_i}{\delta^{(N)}}\Big)^2 \beta_i, \quad (7.16)$$

which leads to

$$\begin{aligned}
\alpha_i^2 &= \Gamma_{ii}^{-1}(\alpha^{(N)})^2 - \Gamma_{ii}^{-1} \sum_{k=1,k\neq i}^{N} \beta_k^{(N)} \sum_{l=1,l\neq i}^{N} \Gamma_{kl}\beta_l^{(N)} + \Big(\Gamma_{ii}^{-1} \sum_{k=1,k\neq i}^{N} \Gamma_{ik}\beta_k^{(N)}\Big)^2 \\
&= \Big(\frac{\delta^{(N)}\alpha^{(N)}}{\delta_i}\Big)^2 - \Big(\frac{\delta^{(N)}}{\delta_i}\Big)^2 \sum_{k=1,k\neq i}^{N} \beta_k^{(N)}((\Gamma\beta^{(N)})_k - \Gamma_{ik}\beta_i^{(N)}) + (\beta_i - \beta_i^{(N)})^2 \\
&= \Big(\frac{\delta^{(N)}\alpha^{(N)}}{\delta_i}\Big)^2 \\
&\quad - \sum_{k=1,k\neq i}^{N} \beta_k^{(N)} \frac{\delta_k^2}{\delta_i^2}\beta_k + \Big(\frac{\delta^{(N)}}{\delta_i}\Big)^2 \beta_i^{(N)}((\Gamma\beta^{(N)})_i - \Gamma_{ii}\beta_i^{(N)}) + \beta_i^2 - 2\beta_i\beta_i^{(N)} + (\beta_i^{(N)})^2 \\
&= \Big(\frac{\delta^{(N)}\alpha^{(N)}}{\delta_i}\Big)^2 - \sum_{k=1}^{N} \beta_k^{(N)} \frac{\delta_k^2}{\delta_i^2}\beta_k + \beta_i^2.
\end{aligned}$$

The last equation is equivalent to

$$\begin{aligned}
\delta_i^2(\alpha_i^2 - \beta_i^2) &= (\delta^{(N)}\alpha^{(N)})^2 - \sum_{k=1}^{N} \beta_k^{(N)} \underbrace{\delta_k\beta_k}_{=(\delta^{(N)})^2(\Gamma\beta_k^{(N)})} \\
&= (\delta^{(N)})^2 \left((\alpha^{(N)})^2 - \beta^{(N)\prime}\Gamma\beta^{(N)}\right),
\end{aligned} \quad (7.17)$$

and since the right hand side does not depend on i, we get that $\delta_i^2(\alpha_i^2 - \beta_i^2) > 0$ has to be independent of i.

Now assume we have a set of parameters $((\lambda_i, \alpha_i, \beta_i, \delta_i, \mu_i), i = 1, \dots, N)$ with

$$\delta_i \sqrt{\alpha_i^2 - \beta_i^2} = c > 0 \quad \text{and} \quad \lambda_i = \lambda \in \mathbb{R},$$

where c and λ are independent of the index i. Furthermore, let the matrix U as defined in the claim be positive definite. We show how parameters $\lambda^{(U)}, \alpha^{(U)}, \beta^{(U)}, \delta^{(U)}, \mu^{(U)}$ and $\Gamma^{(U)}$ of a N-dimensional GH process $\ell^{GH_N,U}$ may be chosen, such that its marginals are uncorrelated with law $\ell_i^{GH,U}(1) \sim GH(\lambda, \alpha_i, \beta_i, \delta_i, \mu_i)$. Clearly, we have to set $\lambda^{(U)} := \lambda$ and $\mu^{(U)} := (\mu_1, \dots, \mu_N)'$. Eq. (7.16) and Eq. (7.17) yield the conditions

$$(\delta^{(U)})^2 (\Gamma^{(U)} \beta^{(U)})_i = \delta_i^2 \beta_i$$

and

$$\delta^{(U)} \sqrt{(\alpha^{(U)})^2 - \beta^{(U)T} \Gamma^{(U)} \beta^{(U)}} = \delta_i \sqrt{\alpha_i^2 - \beta_i^2} = c.$$

If $(\delta^{(U)})^2 \Gamma^{(U)}$ fulfills the identity $(\delta^{(U)})^2 \Gamma^{(U)} = U$, we get by Eq. (7.15) for $i \neq j$

$$\begin{aligned}
\mathrm{Cov}(\ell_i^{GH,U}(1), \ell_j^{GH,U}(1)) &= \frac{K_{\lambda+1}(c)}{cK_\lambda(c)} (\delta^{(U)})^2 \Gamma_{ij}^{(U)} + \frac{K_{\lambda+2}(c)K_\lambda(c) - K_{\lambda+1}^2(c)}{c^2 K_\lambda(c)^2} \\
&\quad \cdot ((\delta^{(U)})^2 \Gamma^{(U)} \beta^{(U)})_i ((\delta^{(U)})^2 \Gamma^{(U)} \beta^{(U)})_j \\
&= \frac{K_{\lambda+1}(c)}{cK_\lambda(c)} U_{ij} + \frac{K_{\lambda+2}(c)K_\lambda(c) - K_{\lambda+1}^2(c)}{c^2 K_\lambda(c)^2} \delta_i^2 \beta_i \delta_j^2 \beta_j \\
&= 0,
\end{aligned}$$

hence all marginals are uncorrelated. To obtain a well-defined N-dimensional GH distribution, we still have to make sure that $\Gamma^{(U)}$ is spd with unit determinant. If we define $\delta^{(U)} := (\det(U))^{1/(2N)}$, then $\delta^{(U)} > 0$ (since $\det(U) > 0$ by assumption) and $\Gamma^{(U)} = (\delta^{(U)})^{-2} U$ is spd with $\det(\Gamma^{(U)}) = 1$. It remains to determine appropriate parameters $\alpha^{(U)} > 0$ and $\beta^{(U)} \in \mathbb{R}^N$. For $\beta^{(U)}$, we use once again Lemma 7.4.6 to obtain the linear equations

$$\beta_i = \beta_i + (\Gamma_{ii}^{(U)})^{-1} \sum_{k=1, k\neq i} \Gamma_{ik}^{(U)} \beta_k^{(U)},$$

for $i = 1, \dots, N$. The corresponding system of linear equations is given by

$$\begin{pmatrix} (\Gamma_{11}^{(U)})^{-1} & & \\ & \ddots & \\ & & (\Gamma_{NN}^{(U)})^{-1} \end{pmatrix} \Gamma^{(U)} \beta^{(U)} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix},$$

and has a unique solution $\beta^{(U)}$ for any right hand side $(\beta_1, \dots, \beta_N)'$, because $\Gamma^{(U)}$ as constructed above is invertible with positive diagonal entries. Finally, we are able to calculate $\alpha^{(U)}$ via Equation (7.17) as

$$\alpha^{(U)} = \Big(\sum_{k=1}^{N} \delta_k^2 \beta_k \beta_k^{(U)} + \Big(\frac{c}{\delta^{(U)}} \Big)^2 \Big)^{1/2} = \Big(\beta^{(U)\prime} \Gamma^{(U)} \beta^{(U)} + \Big(\frac{c}{\delta^{(U)}} \Big)^2 \Big)^{1/2}$$

and obtain the desired marginal distributions. □

Note that the KL-expansion $L_N^{GH}(x)(\cdot)$ generated by $(\ell_i^{GH,U}, i = 1 \dots, N)$ in Theorem 7.4.10 is a GH process for each $x \in \mathcal{D}$ by Lemma 7.4.9, whereas this is not the case if the processes $(\ell_i^{GH}, i = 1, \dots, N)$ are generated independently of each other: By Lemma 7.4.4 we have that $L_N^{GH}(x)(1)$ is only GH distributed if the vector $(\ell_1^{GH}(1), \dots, \ell_N^{GH}(1))'$ admits a multi-dimensional GH law. As noted in [43] after Theorem 1, this is impossible if the processes (and hence $(\ell_i^{GH}(1), i = 1, \dots, N)$) are independent. Whenever Theorem 7.4.10 is applicable, we are able to approximate a GH Lévy field by generating a N-dimensional GH processes, where N is the truncation index of the KL expansion. To this end, Lemma 7.4.1 suggests the simulation of GIG processes and then subordinating N-dimensional Brownian motions. With this simulation approach the question arises on why we have taken a detour via the subordinating GIG process instead of using the characteristic function a of GH process in Equation (7.10) for a "direct" simulation. This has several reasons: First, the approximation of the inversion formula (7.6) can only be applied for one-dimensional GH processes, where the costs of evaluating ϕ_{GH} or ϕ_{GIG} are roughly the same. In comparison, the costs of sampling a Brownian motion are negligible. Second, in the multi-dimensional case, we need that all marginals of the GH process are generated by the same or correlated subordinator(s), which leaves us no choice but to sample the underlying GIG process. In addition, the simulation of a GH field requires in some cases only one subordinating process to generate a multi-dimensional GH process with uncorrelated marginals (see Theorem 7.4.10). This approach is in general more efficient than sampling a large number of uncorrelated, one-dimensional GH processes for the KL expansion. As we demonstrate in the following section, it is a straightforward application of the Fourier inversion algorithm to approximate a GIG process ℓ^{GIG} with given parameters, since all necessary assumptions are fulfilled and the bounding parameters ϑ, R, θ and B may readily be calculated.

7.5 Numerics

In this section we provide some details on the implementation of the Fourier inversion method. Thereafter, we apply this methodology to approximate a GH Lévy field and conclude with some numerical examples.

7.5.1 Notes on implementation

Suppose we simulate a given one-dimensional Lévy process ℓ which fulfills Assumption 7.3.6 resp. Assumption 7.3.7, using the step size $\Delta_n > 0$ and characteristic function $(\phi_\ell)^{\Delta_n}$. Usually the parameter ϑ cannot be chosen arbitrary high (as for the GIG process), but it may be possible to choose ϑ within a certain range, for instance $\vartheta \in (1, 2]$ for the Cauchy process in Example 7.3.18. As a rule of thumb, ϑ should always be determined as large as possible, as the convergence rates in Theorems 7.3.14 and 7.3.16 directly depend on ϑ. In addition, we concluded in Remark 7.3.22 that $D \simeq \Delta_n^{p/(p-\vartheta)}$ is an appropriate choice to guarantee an L^p-error of order $\mathcal{O}(\Delta_n^{1/p})$. This means that for a given p, D decreases as ϑ increases. Since the number of summations M in Algorithm 7.3.7 depends on D (see Theorem 7.3.9), an increasing parameter ϑ also reduces computational time. Once ϑ is determined, we derive R by differentiation of $(\phi_\ell)^{\Delta_n}$ as in Remark 7.3.8. Similarly to ϑ, it is often possible to choose between several values of $\theta > 0$, but it is difficult to give a-priori a recommendation on how θ should be selected. One rather calculates for several admissible θ the constant $C_\theta := \max_{u\in\mathbb{R}} |u^\theta(\phi_\ell(u))^{\Delta_n}|$ numerically and deducts $B_\theta = (2\pi)^{-\theta} C_\theta$. Each combination of (θ, B_θ) then results in a valid number of summations M_θ in the discrete Fourier Inversion algorithm. Since θ and B_θ are only necessary to determine M_θ, we may simply use the smallest M_θ for the simulation. To find $\widetilde{X}$ with $\widetilde{F}(\widetilde{X}) = U$ in Algorithm 7.3.11, we use a globalized Newton method with *backtracking line search*, also known as *Armijo increment control*. The step lengths during the line search are determined by interpolation, which is a robust technique if combined with a standard Newton method. Details on the globalized Newton method with backtracking may be found, for example, in [166], an example how the algorithm is used is given in [176]. Although convergence of this root finding algorithm is ensured by the increment control, its efficiency depends heavily on the choice of the initial value $\widetilde{X}_0$. Clearly, $\widetilde{X}_0$ should depend on the sampled $U \sim \mathcal{U}([0, 1])$ and be related to the target distribution with characteristic function $(\phi_\ell)^{\Delta_n}$. This means we should determine $\widetilde{X}_0$ implicitly by $F^{(0)}(\widetilde{X}_0) = U$, where $F^{(0)}$ is a CDF of a distribution similar to the target distribution, but which can be inverted efficiently.

7.5.2 Approximation of a GH field

We consider a GH Lévy field on the (separable) Hilbert space $U = L^2(\mathcal{D})$ with a compact spatial domain $\mathcal{D} \subset \mathbb{R}^s$. The operator Q on U is given by a *Matérn covariance operator* with variance $\sigma^2 > 0$, correlation length $\rho > 0$ and a positive parameter $\chi > 0$ defined by

$$[Qh](x) := \sigma^2 \int_{\mathcal{D}} k_\chi(x, y)h(y)dy, \quad \text{for } \psi \in U,$$

where k_χ denotes the *Matérn kernel*. For $\chi = \frac{1}{2}$, we obtain the exponential covariance function and for $\chi \to +\infty$ the squared exponential covariance function. For general $\chi > 0$, the Matérn kernel

$$k_\chi(x, y) := \frac{2^{1-\chi}}{\Gamma_G(\chi)} \Big(\frac{\sqrt{2\chi}|x - y|}{\rho} \Big)^\chi K_\chi \Big(\frac{\sqrt{2\chi}|x - y|}{\rho} \Big)$$

fulfills the limit identity $k_\chi(x, x) = \lim_{y \to x} k_\chi(x, y) = 1$, which can be easily seen by [169, Eq. (10.30.2)]. Here $\Gamma_G(\cdot)$ is the Gamma function. As shown in [79], this implies

$$\mathrm{Tr}(Q) = \sum_{i=1}^{\infty} \eta_i = \sigma^2 \int_{\mathcal{D}} dx, \tag{7.18}$$

where $(\eta_i, i \in \mathbb{N})$ are the eigenvalues of the Matérn covariance operator Q. In general, no analytical expressions for the eigenpairs (η_i, e_i) of Q will be available, but the spectral basis may be approximated by numerically solving a discrete eigenvalue problem and then interpolating by *Nyström's method*. For a general overview of common covariance functions and the approximation of their eigenbasis we refer to [178] and the references therein.

Now let L_N^{GH} be an approximation of a GH field by a N-dimensional GH process $(\ell^{GH_N}(t), t \in \mathbb{T})$ with fixed parameters $\lambda, \alpha, \delta \in \mathbb{R}$, $\beta, \mu \in \mathbb{R}^N$ and $\Gamma \in \mathbb{R}^{N \times N}$. The parameters are chosen in such a way that the multi-dimensional GH process has uncorrelated marginal processes, hence the generated KL expansions

$$L_N^{GH}(x)(t) = \sum_{i=1}^{N} \varphi_i(x)\ell_i^{GH}(t)$$

are again one-dimensional GH processes for any spectral basis $(\varphi_i, i \in \mathbb{N})$ and $x \in \mathcal{D}$. This in turn means, that we may draw samples of ℓ^{GH_N} by simulating a GIG process ℓ^{GIG} with parameters $a = \delta, b = (\alpha^2 - \beta'\Gamma\beta)^{1/2}$ and $p = \lambda$ using Fourier inversion and then subordinating a N-dimensional Brownian motion (see Lemma 7.4.1). The characteristic function of a GIG Lévy process $(\ell^{GIG}(t), t \in \mathbb{T})$ with (fixed) parameters

$a, b > 0$ and $p \in \mathbb{R}$ is given by

$$\phi_{GIG}(u; a, b, p) := \mathbb{E}[\exp(iu\ell^{GIG}(1))] = (1 - 2iub^{-2})^{-p/2} \frac{K_p(ab\sqrt{1 - 2iub^{-2}})}{K_p(ab)}.$$

The GIG distribution corresponding to $(\phi_{GIG})^{\Delta_n}$ with $\Delta_n = 1$ is continuous with finite variance (see [188]), which implies that these properties hold for all distributions with characteristic function $(\phi_{GIG})^{\Delta_n}$, for any $\Delta_n > 0$. The constants as in Assumption 7.3.6 are derived in the following. For $k \in \mathbb{N}$, the k-th moment of the GIG distribution is given as

$$0 < \mathbb{E}\Big((\ell^{GIG}(1))^k\Big) = \Big(\frac{a}{b}\Big)^k \frac{K_{p+k}(ab)}{K_p(ab)} < +\infty.$$

For any $\vartheta = 2k$ we are, therefore, able to calculate the bounding constant R via

$$R = (-1)^k \frac{d^{2k}}{du^{2k}}((\phi_{GIG}(u; a, b, p))^{\Delta_n})\Big|_{u=0},$$

because the derivatives of ϕ_{GIG} evaluated at $u = 0$ are

$$(\phi_{GIG}(0; a, b, p))^{(k)} = i^{-k}\mathbb{E}\Big((\ell^{GIG}(1))^k\Big) = i^{-k}\Big(\frac{a}{b}\Big)^k \frac{K_{p+k}(ab)}{K_p(ab)}.$$

The calculation of the ϑ-th derivative can be implemented easily by using a version of Faà di Bruno's formula containing the Bell polynomials, for details we refer to [124]. The bounding constants θ and B may be determined numerically as described in Section 7.5.1 (e.g. by using the routine `fminsearch` in MATLAB). The derivation of the bounds implies that we can ensure L^p convergence of the approximated GIG process in the sense of Theorem 7.3.21 for any $p \geq 1$, because it is possible to define ϑ as any even integer and then obtain R by differentiation. We observe that the target distribution with characteristic function $(\phi_{GIG}(u; a, b, p))^{\Delta_n}$ and $\Delta_n > 0$ is not necessarily GIG, except for the Inverse Gaussian (IG) case where $p = -1/2$ and $(\phi_{IG}(u; a, b))^{\Delta_n} = \phi_{IG}(u; \Delta_n a, b)$(see Remark 7.4.3). This special feature of the IG distribution is exploited to determine the initial values $\widetilde{X}_0$ in the Newton iteration by moment matching: Consider an $IG(a_0, b_0)$ distribution with mean a_0/b_0 and variance a_0/b_0^3, where the parameters $a_0, b_0 > 0$ are "matched" to the target distribution's mean and variance via

$$\begin{aligned}
\frac{a_0}{b_0} &= i\frac{d}{du}((\phi_{GIG}(u; a, b, p))^{\Delta_n})\Big|_{u=0}, \\
\frac{a_0}{b_0^3} &= (-1)\frac{d^2}{du^2}((\phi_{GIG}(u; a, b, p))^{\Delta_n})\Big|_{u=0} - \Big(i\frac{d}{du}((\phi_{GIG}(u; a, b, p))^{\Delta_n})\Big|_{u=0}\Big)^2.
\end{aligned}$$

If $F^{IG}_{\Delta_n}$ denotes the CDF of this $IG(a_0, b_0)$ distribution, the initial value of the globalized Newton method is given implicitly by $F^{IG}_{\Delta_n}(\widetilde{X}_0) = U$. The inversion of $F^{IG}_{\Delta_n}$ may be executed numerically by many software packages like MATLAB.
With our approach, this results in the approximation of a GIG process $\widetilde{\ell}^{GIG}$ at discrete times $t_j \in \Theta_n$. The N-dimensional GH process ℓ^{GH_N} may then be approximated at t_j for $j = 0, \dots, n$ by the process $\widetilde{\ell}^{GH_N}$ with $\widetilde{\ell}^{GH_N}(t_0) = 0$ and the increments

$$\begin{aligned}\widetilde{\ell}^{GH_N}(t_j) - \widetilde{\ell}^{GH_N}(t_{j-1}) = \mu\Delta_n + \Gamma\beta(\widetilde{\ell}^{GIG}(t_j) - \widetilde{\ell}^{GIG}(t_{j-1})) \\ + \sqrt{(\widetilde{\ell}^{GIG}(t_j) - \widetilde{\ell}^{GIG}(t_{j-1}))}\Gamma w^N_j(1),\end{aligned}$$

for $j = 1, \dots, n$, where the $w^N_j(1)$ are i.i.d. $\mathcal{N}_N(0, \mathbf{1}_{N\times N})$-distributed random vectors. To obtain the process $\widetilde{\ell}^{GH_N}$ at arbitrary times $t \in \mathbb{T}$, we interpolate the samples $(\widetilde{\ell}^{GH_N}(t_j), j = 0 \dots, n)$ piecewise constant as in Algorithm 7.3.11. With this, we are able to generate an approximation of L^{GH}_N at any point $(x, t) \in \mathcal{D} \times \mathbb{T}$ by

$$\widetilde{L}^{GH}_N(x)(t) := \sum_{i=1}^{N} \varphi_i(x)\widetilde{\ell}^{GH}_i(t).$$

The knowledge of $\mathrm{Tr}(Q)$ enables us to determine the truncation index N and the constant C_ℓ as in Remark 7.2.4: For $N \in \mathbb{N}$, let $(\widetilde{\ell}^{GH}_i, i = 1, \dots, N)$ be the approximations of the processes $(\ell^{GH}_i, i = 1, \dots, N)$, where the random vector $(\ell^{GH}_1(1), \dots, \ell^{GH}_N(1))$ is multivariate GH-distributed by assumption. Hence, for every $N \in \mathbb{N}$, we obtain the parameters $a(N), b(N), \lambda(N)$ of a corresponding GIG subordinator $\ell^{GIG,N}$, which is approximated through a piecewise constant process $\widetilde{\ell}^{GIG,N}$ as above. With Eq. (7.9) we calculate the error

$$E^p_{GIG,N} := \sup_{t\in\mathbb{T}} \mathbb{E}(|\ell^{GIG,N}(t) - \widetilde{\ell}^{GIG,N}(t)|^p). \tag{7.19}$$

for $p \in \{1, 2\}$. If $\beta \in \mathbb{R}^N$ and $\Gamma \in \mathbb{R}^{N\times N}$ denote the GH parameters corresponding to $(\ell^{GH}_1(1), \dots, \ell^{GH}_N(1))$, the $L^2(\Omega; \mathbb{R})$ approximation error of each process ℓ^{GH}_i is given by

$$\widetilde{C}_{\ell,i} := \sup_{t\in\mathbb{T}} \frac{\mathbb{E}(|\ell^{GH}_i(t) - \widetilde{\ell}^{GH}_i(t)|^2)}{\Delta_n} = \frac{E^2_{GIG,N}(\Gamma\beta)^2_i + E^1_{GIG,N}\sqrt{\Gamma_{[i]}\Gamma'_{[i]}}}{\Delta_n},$$

where $\Gamma_{[i]}$ indicates the $i-$th row of Γ. Starting with $N = 1$, we compute the first N eigenvalues and the difference

$$T\Big(\mathrm{Tr}(Q) - \sum_{i=1}^{N} \eta_i\Big) - \max_{i=1,\dots,N} \widetilde{C}_{\ell,i}\Delta_n \sum_{i=1}^{N} \eta_i,$$

and increase N by one in every step until this expression is close to zero. If a suitable N is found, we define $C_\ell := \max_{i=1,\dots,N} \widetilde{C}_{\ell,i}$ and thus have equilibrated truncation and approximation errors by ensuring Eq. (7.4). For simplicity, we have implicitly assumed here that the processes ℓ_i^{GH} were normalized in the sense that $\mathrm{Var}(\ell_i^{GH}(t)) = t$. This is due to the fact that $\eta_i \ell_i$ (here with $\ell_i = \ell_i^{GH}$) in Theorem 7.2.3 represents the scalar product $(L(t), e_i)_U$ with variance $\eta_i t$. In case we have unnormalized processes, one can simply divide ℓ_i^{GH} by its standard deviation (see Formula (7.15)) and adjust the constants $\widetilde{C}_{\ell,i}$ and C_ℓ accordingly.

7.5.3 Numerical results

As a test for our algorithm, we generate GH fields on the time interval $\mathbb{T} = [0, 1]$ with step size $\Delta_n = 2^{-6}$, on the spatial domain $\mathcal{D} = [0, 1]$. For practical aspects, one is usually interested in the L^1-error $\mathbb{E}(|\ell(t) - \widetilde{\ell}^{(n)}(t)|)$ and the L^2-error $(\mathbb{E}(|\ell(t) - \widetilde{\ell}^{(n)}(t)|^2))^{1/2}$. Upper bounds for both expressions depend on ϑ and D and are given by Ineq. (7.9). To obtain reasonable errors, we refer to the discussion on the choice of D in Remark 7.3.22 and set $D = \Delta_n^{1/(1-\vartheta)}$. This ensures that the L^1-error is of order $\mathcal{O}(\Delta_n)$ and is a good trade-off between simulation time and the size of the L^2-error for most values of ϑ in the GIG example below. Choosing for example $D = \Delta_n^{2/(2-\vartheta)}$ would reduce the L^2-error to order $\mathcal{O}(\Delta_n)$, but does not have a significant effect on the L^1-error and results in a higher computational time. For the Matérn covariance operator Q we use variance $\sigma^2 = 1$, correlation length $\rho = 0.1$ and $\chi \in \{\frac{1}{2}, \frac{3}{2}\}$, where a higher value of χ increases the regularity of the field along the x-direction. For the fixed GH parameters we choose $\alpha = 5$, $\beta = \mu = 0_N$, $\delta = 4$ and $\Gamma = \mathbf{1}_N$, the shape parameter λ will vary throughout our simulation and admits the values $\lambda \in \{-\frac{1}{2}, 1\}$, which results in NIG resp. hyperbolic GH fields. This parameter setting ensures that the multi-dimensional GH distribution has uncorrelated marginals, hence the truncated KL expansion L_N^{GH} of L^{GH} is itself an infinite-dimensional GH Lévy process. Further, for every $N \in \mathbb{N}$, the constant $\widetilde{C}_{\ell,i}$ from Section 7.5.2 is independent of $i = 1, \dots, N$, thus the truncation index N can easily be determined to balance out the Fourier inversion and truncation error for each combination of λ and χ. To examine the impact of ϑ on the efficiency of the simulation, we set $\vartheta \in \{4, 6, 8, 10\}$ and the constant R as suggested in Section 7.5.2 for each ϑ. For fixed ϑ and R, we choose $\theta \in \{1, 1.5 \dots, 99.5, 100\}$ and calculate for each θ the constant B_θ as in Section 7.5.1. This results in up to 199 different values for the number of summations M_θ, which all guarantee the desired accuracy ε, meaning we can choose the smallest M_θ for our simulation. The optimal value θ_{opt} which leads to the smallest M_θ depends highly on the GH parameters and may vary significantly

with ϑ. For $\lambda = 1$, we found that θ_{opt} ranges from 34 to 68.5, varying with each choice of $\vartheta \in \{4, 6, 8, 10\}$. In contrast, in the second example with $\lambda = -1/2$, we found that $\theta_{opt} = 11$ independent of ϑ. We generate 1.000 approximations $\widetilde{L}_N^{GH}$ for several combinations of λ, χ and ϑ, allowing us to check if the generated samples actually follow the desired target distributions. To this end, we conduct Kolomogorov–Smirnov tests for the subordinating GIG process as well as for the distribution of the GH field at a fixed point in time and space and report on the corresponding p-values.

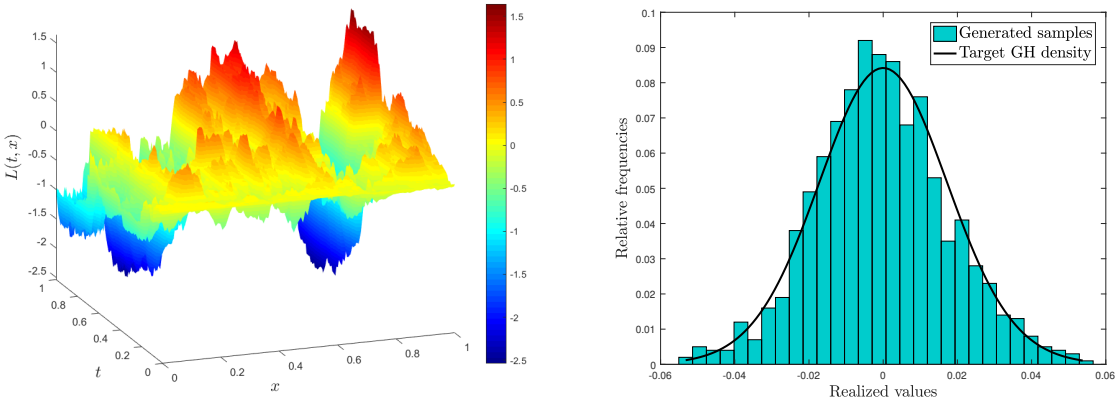

Figure 7.1 Sample (left) and empirical distribution at $x = t = 1$ (right) of a hyperbolic field with parameters $\lambda = 1$, $\chi = 1/2$, $\vartheta = 10$ and truncation after $N = 132$ terms.

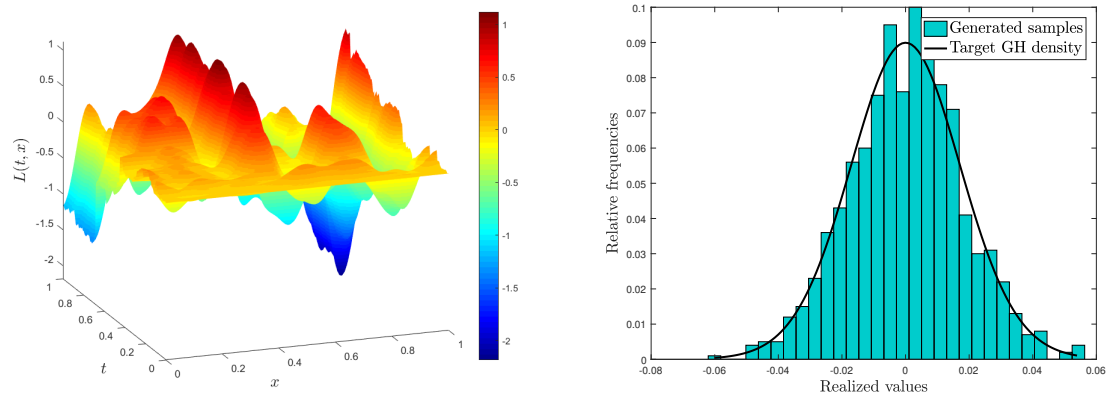

Figure 7.2 Sample (left) and empirical distribution at $x = t = 1$ (right) of a NIG field with parameters $\lambda = -1/2$, $\chi = 3/2$, $\vartheta = 10$ and truncation after $N = 18$ terms.

Figures 7.1 and 7.2 show samples of approximated GH random fields: Along the time axis we see the characteristic behavior of the (pure jump) GH processes for every point $x \in \mathcal{D}$. For a fixed point in time t, the paths along the x-axis vary according to their correlation, depending on the covariance parameter χ. As reported in [178], the eigenvalues of Q decay slower if χ becomes smaller, meaning we need a higher number

of summations N in the KL expansion so that the error contributions are equilibrated. This effect can be seen in Tables 7.1 and 7.2, where the truncation index N changes significantly with χ. If the KL expansion, however, can be sampled by a N-dimensional GH process as suggested in Theorem 7.4.10, the number of summations N has only a minor impact on the computational costs of the KL expansion. This is due to the fact that in this case the time consuming part, namely simulating the subordinator, has to be done only once, regardless of N. Compared to these costs, the costs of subordinating a Brownian motion of any finite dimension are negligible. The histograms in Figures 7.1 and 7.2 show the empirical distribution of the approximation $\widetilde{L}_N^{GH}(x)(t)$ at time $t = 1$ and $x = 1$. The theoretical distribution at time 1 and an arbitrary point $x \in \mathcal{D}$ is again GH, where the parameters are given in Lemma 7.4.9. Obviously, the empirical distributions fit the target GH distributions from Lemma 7.4.9. To be more precise, we have conducted a Kolmogorov-Smirnov test for both, the subordinating GIG process and the GH field at time $t = 1$ and for the latter at $x = 1$. We know the law of both processes at $x \in \mathcal{D}$ and are able to obtain their CDFs sufficiently precise for the tests by numerical integration. The test results for 1.000 samples of the hyperbolic resp. the NIG field with covariance parameters $\chi = \frac{1}{2}$ resp. $\chi = \frac{3}{2}$ are given in Tables 7.1 and 7.2 above and do not suggest that the generated samples follow another distribution than the expected one.

We denote by $E^1_{GIG,N}$ and $E^2_{GIG,N}$ the approximation error of the subordinator as in Eq. (7.19), which we have listed in absolute terms in Tables 7.1 and 7.2. The first error bound is also given relative to Δ_n to show that it is in fact of magnitude $\mathcal{O}(\Delta_n)$. While the $L^1(\Omega;\mathbb{R})$-error $E^1_{GIG,N}$ is relatively constant for each ϑ, the $L^2(\Omega;\mathbb{R})$-error $E^2_{GIG,N}$ is rather high for $\vartheta = 4$, but has an acceptable upper bound for $\vartheta \geq 6$. This is not surprising, since $D = \Delta_n^{1/(1-\vartheta)}$ only guarantees that $\mathbb{E}(|\ell^{GIG}(t) - \widetilde{\ell}^{GIG}(t)|) = \mathcal{O}(\Delta_n)$. We emphasize that the (theoretic) error bounds in Tables 7.1 and 7.2 are very conservative as the triangle inequality and similar "coarse" estimates were used repeatedly in their estimation in Theorem 7.3.14 and 7.3.21. The truncation index N is highly sensitive to χ, but has small or no variations for fixed χ and varying ϑ. Since we choose $t \in [0,1]$, the expression $\mathbb{E}(||L^{GH}(1) - \widetilde{L}_N^{GH}(1)||_U^2)$ in Tables 7.1 and 7.2 is an upper bound for the $L^2(\Omega;U)$-error $\sup_{t\in[0,1]} \mathbb{E}(||L^{GH}(t) - \widetilde{L}_N^{GH}(t)||_U^2)$. Note that this error is small in relative terms, since by our choice of Q and Eq. 7.18 we have $\mathbb{E}(||L^{GH}(1)||_U^2) = \mathrm{Tr}(Q) = 1$.

The p-value of the GH distribution varies if different N are chosen for the KL expansion, which is natural due to statistical fluctuations. More importantly, the null hypothesis, namely that the samples follow a GH distribution with the expected parameters, is never rejected at a 5%-level. As expected, the speed of the simulation heavily depends on ϑ.

ϑ	$E^1_{GIG,N}$	$E^1_{GIG,N}/\Delta_n$	$E^2_{GIG,N}$	$\mathbb{E}[\|\|L^{GH}(1) - \widetilde{L}^{GH}_N(1)\|\|^2_U]$
4	0.0143	0.9166	0.2584	0.0646
6	0.0138	0.8835	0.0749	0.0635
8	0.0138	0.8824	0.0601	0.0634
10	0.0140	0.8975	0.0806	0.0636
ϑ	N	p-value GH	abs. time	rel. time
4	130	0.8246	0.1945 sec.	100.00%
6	133	0.3077	0.1093 sec.	56.19%
8	133	0.3077	0.0851 sec.	43.78%
10	132	0.2873	0.0759 sec.	39.04%

Table 7.1 Errors, p-values and average simulation times per field based on 1.000 simulations. Stepsize $\Delta t = 2^{-6}$ and $D = \Delta t^{1/(1-\vartheta)}$. GH process: $\lambda = 1$, $\alpha = 5, \beta = 0_N$, $\delta = 4, \mu = 0_N, \Gamma = 1_{N\times N}$. Covariance parameters: $\chi = 1/2$, $\rho = 0.1$ and $\sigma^2 = 1$. The KS test for the GIG subordinator returns a p-value of 0.5498 for each $\vartheta \in \{4, 6, 8, 10\}$.

ϑ	$E^1_{GIG,N}$	$E^1_{GIG,N}/\Delta_n$	$E^2_{GIG,N}$	$\mathbb{E}[\|\|L^{GH}(1) - \widetilde{L}^{GH}_N(1)\|\|^2_U]$
4	0.0132	0.8443	0.2079	0.0619
6	0.0128	0.8170	0.0584	0.0608
8	0.0127	0.8155	0.0456	0.0608
10	0.0129	0.8252	0.0589	0.0611
ϑ	N	p-value GH	abs. time	rel. time
4	18	0.9223	0.1039 sec.	100.00%
6	18	0.9223	0.0628 sec.	60.43%
8	18	0.9223	0.0460 sec.	44.29%
10	18	0.9223	0.0380 sec.	38.59%

Table 7.2 Errors, p-values and average simulation times per field based on 1.000 simulations. Stepsize $\Delta t = 2^{-6}$ and $D = \Delta t^{1/(1-\vartheta)}$. GH process: $\lambda = -1/2$, $\alpha = 5, \beta = 0_N$, $\delta = 4, \mu = 0_N, \Gamma = 1_{N\times N}$. Covariance parameters: $\chi = 3/2$, $\rho = 0.1$ and $\sigma^2 = 1$. The KS test for the GIG subordinator returns a p-value of 0.6145 for each $\vartheta \in \{4, 6, 8, 10\}$.

Looking at the results for $\vartheta = 4$, one might argue that the Fourier inversion method is only suitable for processes where this parameter can be chosen high, i.e. for distributions which admit a large number of finite moments. To qualify this objection, we consider once more the t-distribution with three degrees of freedom and the corresponding Lévy process ℓ^{t3} from Example 7.3.18. Since $\mathbb{E}(\ell^{t3}(\Delta_n)) = 0$ and $\mathrm{Var}(\ell^{t3}(\Delta_n)) = \sqrt{3}\Delta_n$, we can choose $\vartheta = 2$ and hence $R = \sqrt{3}\Delta_n$. The characteristic function of $\ell^{t3}(\Delta_n)$ is given by

$$(\phi_{t3}(u))^{\Delta_n} = \exp(-\sqrt{3}\Delta_n|u|)(\sqrt{3}|u| + 1)^{\Delta_n}$$

and B and θ are estimated in the same way as for the GIG process. Using again $\Delta_n = 2^{-6}$ and $D = \Delta_n^{1/(1-\vartheta)}$, the number of summations in the approximation is $M = 12.924$

for $\theta = \frac{19}{2}$. The simulation time for one process $\widetilde{\ell}^{t3}$ with $(\Delta_n)^{-1} = 2^6$ increments in the interval $[0, 1]$ is on average 0.0655 seconds, where the initial values have been approximated by matching the moments of a normal distribution (the Kolmogorov-Smirnov test for a t-distribution at $t = 1$ based on 1.000 samples returns a p-value of 0.5994). In the GIG example, we needed $M = 79.086$ terms in the summation if $\vartheta = 4$ is chosen and still $M = 33.030$ terms for $\vartheta = 10$. This shows that the Fourier Inversion method is also applicable for low values of ϑ and that the GIG (resp. GH) process is a computationally expensive example of a Lévy process.

8 A stochastic transport problem with Lévy noise: Fully discrete numerical approximation

Andreas Stein and Andrea Barth

Submitted to "BIT Numerical Mathematics", October 2019, Springer, currently in the second stage of review.

Abstract: Semilinear hyperbolic stochastic partial differential equations have various applications in the natural and engineering sciences. From a modelling point of view the Gaussian setting can be too restrictive, since phenomena as porous media, pollution models or applications in mathematical finance indicate an influence of noise of a different nature. In order to capture temporal discontinuities and allow for heavy-tailed distributions, Hilbert space valued-Lévy processes (or Lévy fields) as driving noise terms are considered. The numerical discretization of the corresponding SPDE involves several difficulties: Low spatial and temporal regularity of the solution to the problem entails slow convergence rates and instabilities for space/time-discretization schemes. Furthermore, the Lévy process admits values in a possibly infinite-dimensional Hilbert space, hence projections into a finite-dimensional subspace for each discrete point in time are necessary. Finally, unbiased sampling from the resulting Lévy field may not be possible. We introduce a fully discrete approximation scheme that addresses these issues. A discontinuous Galerkin approach for the spatial approximation is coupled with a suitable time stepping scheme to avoid numerical oscillations. Moreover, we approximate the driving noise process by truncated Karhunen-Loéve expansions. The latter essentially yields a sum of scaled and uncorrelated one-dimensional Lévy processes, which may be simulated with controlled bias by Fourier inversion techniques.

8.1 Introduction

In many applications in the natural sciences and financial mathematics partial differential equations (PDEs) are utilized to model dynamics of the underlying system. Often, the dynamical systems are subject to uncertainties for instance due to noisy data, measurement errors or parameter uncertainty. A common approach to capture this behavior is to model the source of uncertainty by continuous Gaussian processes, which are analytically tractable and straightforward to simulate. It turns out, however, that Gaussian random objects are unfit to capture the impact of spatial and temporal discontinuities, for example in flows through fractured porous media or composite materials. Furthermore, Gaussian distributions notoriously underappreciate rare events, thus heavy-tailed, discontinuous Lévy-processes are better suited to model stock returns, interest rate dynamics and energy forward markets. However, replacing Gaussian distributions by a more general class of random objects comes at the cost of lower regularity (both, path-wise and in a mean-square sense) and more advanced sampling techniques are required.

In this article we consider semilinear first order stochastic partial equations (SPDEs) with a random source term. The noise is modeled by a space-time Lévy process taking values in some infinite-dimensional Hilbert space U. Existence and uniqueness of weak solutions to this type of equations is ensured but in general no closed formulas or distributional properties are available. Thus, we need to rely on numerical discretization schemes to estimate moments or statistics of the solution. The numerical approximation of SPDEs has been an active field of research in the last decade. Most publications focus on second order parabolic equations, i.e. stochastic versions of the heat or Allen-Cahn equation, see for instance [27, 28, 69, 100, 101, 121, 129, 132, 136] and the references therein. In this setting, Lévy fields as driving noise of the SPDE have been investigated, among others, in [26, 33, 49, 74, 173]. Results on second order hyperbolic SPDEs may be found, e.g., in [5, 64, 133, 173, 197] and the references therein, nonlinear hyperbolic SPDES are the subject of interest, for example, in [45, 135]. To model the dynamics in financial markets, however, it is more common to consider first order linear hyperbolic SPDEs, for example in the *Heath-Jarrow-Morton model* with *Musiela parametrization* for interest rate forwards, see [46, 52, 105]. Another example can be found in [24, 37], where the authors motivate a stochastic framework to model energy forward markets perturbed by infinite-dimensional noise. The underlying SPDE is a semilinear hyperbolic transport problem, where the nonlinearity stems from a no-arbitrage condition and directly depends on the volatility in the market, meaning the diffusion term of the SPDE. Naturally, the numerical treatment then becomes more

involved than in the parabolic case, as we face lower regularity of the solution and the transport semigroup is not analytic. Consequently, there is very little literature on the numerical analysis of stochastic transport problems as for example [23, 134].

Our contribution is a rigorous regularity analysis and a fully discrete approximation scheme for a stochastic transport equation driven by trace class Lévy noise L. We derive mean-square temporal continuity and spatial regularity in terms of fractional Sobolev norms of the solution under mild assumptions. The degree of spatial smoothness depends on the regularity of L and is made explicit and outlined in detail for the important special case that L is associated to a Matérn covariance function. Furthermore, we consider the transport problem on a bounded domain with suitable inflow boundary conditions rather than on $\mathbb{R}^d$. This is of more practical interest in terms of modeling and simulation, but the boundary naturally limits the maximal regularity of the solution even for smooth noise and initial conditions. To approximate the solution, we couple a stable time stepping scheme with a discontinuous Galerkin approach for the spatial domain. This method has been proven to be more suitable for deterministic hyperbolic problems than continuous finite elements, but, to the best of our knowledge, has not yet been applied in the discretization of SPDEs. Finally, to sample the paths of L and to obtain a fully discrete scheme, we combine truncated Karhunen-Loève expansions with an arbitrary approximation algorithm for the one-dimensional marginal Lévy processes. In each step we provide bounds on the strong mean-squared error and give an estimate of the overall error between the unbiased solution and its fully discrete numerical approximation.

In Section 8.2 we introduce SPDEs with Lévy noise in a rather general setting and state existence and uniqueness results for mild/weak solutions. The next section deals with the stochastic transport equation as a special case in the framework from Section 8.2. We introduce the stochastic transport problem corresponding to a first order differential operator and formulate the necessary assumptions to ensure well-posedness. Thereafter, we establish the spatial Sobolev-regularity as well as the mean-square temporal regularity of the solution, which enables us to provide a rigorous error control in the forthcoming sections. In Section 8.4 we then introduce an Euler-type time stepping scheme which we combine with a discontinuous Galerkin spatial discretization in Section 8.5. Thereafter, we derive the weak problem with respect to the spatio-temporal discretization, estimate the approximation error and outline the advantages of the discontinuous Galerkin approach over regular finite elements. The next part contains the sampling procedure of the infinite-dimensional driving noise and we provide an overall mean-squared error containing temporal, spatial and noise approximation. Finally, we discuss several numerical examples in Section 8.7 to confirm

our theoretical results.

8.2 Stochastic partial differential equations with Lévy noise

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$ be a filtered probability space satisfying the usual conditions and let $\mathbb{T} = [0, T]$ be a finite time interval for some $0 < T < +\infty$. Furthermore, let $(U, (\cdot,\cdot)_U)$ and $(H, (\cdot,\cdot)_H)$ be two separable Hilbert spaces and let $L(U, H)$ and $L(H)$ denote the set of linear bounded operators $O : U \to H$ and $O : H \to H$, respectively. The space of *Hilbert-Schmidt operators* on U is given by

$$L_{HS}(U, H) := \{O \in L(U, H) |\ \|O\|^2_{L_{HS}(U;H)} := \sum_{i\in\mathbb{N}} \|Ou_i\|^2_H < +\infty\},$$

where $(u_i, i \in \mathbb{N})$ is an arbitrary orthonormal basis of U. The Lebesgue-Bochner space of all square-integrable, H-valued random variables is defined as

$$L^2(\Omega; H) := \{Y : \Omega \to H \text{ is strongly measurable, } \|Y\|_{L^2(\Omega;H)} := \mathbb{E}(\|Y\|^2_H)^{1/2} < +\infty\}.$$

For the remainder of this article, we omit the stochastic argument $\omega \in \Omega$ for notational convenience. Solutions to the SPDEs are characterized by path-wise identities that hold almost surely, see Definition 8.2.4 below. Therefore, unless stated otherwise, all appearing equalities and estimates involving stochastic terms are in the path-wise sense and are assumed to hold almost surely. We denote by C a generic positive constant which may change from one line to another. Whenever necessary, the dependency of C on certain parameters is made explicit. Our focus is on stochastic partial differential equations with Lévy noise, meaning the driving noise is a (possibly infinite-dimensional) square-integrable Lévy process defined as follows.

Definition 8.2.1. A U-valued stochastic process $L = (L(t), t \in \mathbb{T})$ is called *Lévy process* if

- L has stationary and independent increments,
- $L(0) = 0$ almost surely and
- L is stochastically continuous, i.e. for all $\varepsilon > 0$ and $t \in \mathbb{T}$ holds

$$\lim_{\substack{s\to t,\\ s\in\mathbb{T}}} \mathbb{P}(\|L(t) - L(s)\|_U > \varepsilon) = 0.$$

L is called *square-integrable* if $\mathbb{E}(\|L(t)\|^2_U) < +\infty$ holds for any $t \in \mathbb{T}$.

We consider the SPDE

$$dX(t) = (AX(t) + F(t, X(t)))dt + G(t, X(t))dL(t), \quad X(0) = X_0, \tag{8.1}$$

on $\mathbb{T}$, where X_0 is a H-valued random variable and $A : D(A) \subset H \to H$ is an unbounded, linear operator generating a semigroup $S = (S(t), t \geq 0) \subset L(H)$ on H. The driving noise is modeled by a square-integrable, U-valued Lévy process L with non-negative, symmetric and trace class covariance operator $Q \in L(U)$, satisfying the identity

$$\mathbb{E}((L(t) - \mathbb{E}(L(t)), \phi)_U (L(t) - \mathbb{E}(L(t)), \psi)_U) = t(Q\phi, \psi)_U, \quad \phi, \psi \in U,\ t \in \mathbb{T}.$$

By the Hilbert-Schmidt theorem, the ordered eigenvalues $\eta_1 \geq \eta_2 \geq \cdots \geq 0$ of Q are non-negative and have zero as their only accumulation point. Moreover, the corresponding eigenfunctions $(e_i, i \in \mathbb{N}) \subset U$ form an orthonormal basis of U and we define the *square-root* of Q via

$$Q^{1/2}\phi := \sum_{i \in \mathbb{N}} \sqrt{\eta_i}(\phi, e_i)_U e_i, \quad \phi \in U.$$

Since $Q^{1/2}$ is not necessarily injective, the *pseudo-inverse* of $Q^{1/2}$ is given by

$$Q^{-1/2}\varphi := \phi, \quad \text{if } Q^{1/2}\phi = \varphi \text{ and } \|\phi\|_U = \inf_{\varphi \in U:\, Q^{1/2}\varphi = \phi} \{\|\varphi\|_U\}.$$

With this, we are able to define the *reproducing kernel Hilbert space* associated to L.

Definition 8.2.2. Let L be a square-integrable, U-valued Lévy process with non-negative, symmetric, trace class covariance operator $Q \in L(U)$. Then, the set $\mathcal{U} := Q^{1/2}(U)$ equipped with the scalar-product

$$(\varphi_1, \varphi_2)_{\mathcal{U}} := (Q^{-1/2}\varphi_1, Q^{-1/2}\varphi_2)_U, \quad \varphi_1, \varphi_2 \in \mathcal{U},$$

is called the *reproducing kernel Hilbert space* (RKHS) of L.

Note that $(\sqrt{\eta_i} e_i, i \in \mathbb{N})$ forms an orthonormal system in the RKHS $\mathcal{U}$ and hence the norm on the space of Hilbert-Schmidt operators $L_{HS}(\mathcal{U}, H)$ is given by

$$\|O\|^2_{L_{HS}(\mathcal{U},H)} = \sum_{i \in \mathbb{N}} \eta_i \|Oe_i\|^2_H, \quad O \in L_{HS}(\mathcal{U}, H).$$

The drift- and diffusion-term in Eq. (8.1) are possibly non-linear and measurable map-

pings $F : \mathbb{T} \times H \to H$ and $G : \mathbb{T} \times H \to L_{HS}(\mathcal{U}, H)$, respectively. Sufficient conditions to ensure that G is actually an admissible integrand for L are discussed below.

Example 8.2.3. An important special case is if $U = L^2(\mathbb{D})$, where $\mathbb{D} \subset \mathbb{R}^d$ is an open and bounded spatial domain for $d \in \mathbb{N}$ and Q is the *Matérn covariance operator* with parameters $\nu, \rho > 0$ given by

$$[Q\phi](x) := \int_{\mathbb{D}} \frac{2^{1-\nu}}{\Gamma(\nu)} \Big(\sqrt{2\nu}\frac{\|x-y\|}{\rho}\Big)^{\nu} K_\nu\Big(\sqrt{2\nu}\frac{\|x-y\|}{\rho}\Big)\phi(y)dy, \quad \phi \in U,\ x \in \mathbb{D}. \tag{8.2}$$

Above, Γ is the Gamma function, K_ν is the modified Bessel function of the second kind with ν degrees of freedom and $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^d$, usually the Euclidean norm. We refer to $\rho > 0$ as the *correlation length* of Q, while $\nu > 0$ controls the spatial regularity of the paths generated by Q. More precisely, it holds that $L(t)(\cdot) \in C^{\lceil \nu \rceil - 1}(\overline{\mathbb{D}})$ almost surely for each $t \in \mathbb{T}$.

To characterize solutions of Problem (8.1), we follow the definitions from [173, Chapter 9]:

Definition 8.2.4. The *predictable σ-algebra* $\mathcal{P}_\mathbb{T}$ is the smallest σ-field on $\Omega \times \mathbb{T}$ containing all sets of the form $\mathcal{A} \times (s,t]$, where $\mathcal{A} \in \mathcal{F}_s$ and $s, t \in \mathbb{T}$ with $s < t$. A H-valued stochastic process $Y : \Omega \times \mathbb{T} \to H$ is called *predictable* if it is a $\mathcal{P}_\mathbb{T}$-$\mathcal{B}(H)$-measurable mapping. The set of all square-integrable, H-valued predictable processes is denoted by

$$\mathcal{X}_\mathbb{T} := \{Y : \Omega \times \mathbb{T} \to H |\, \mathrm{Y \text{ is predictable and }} \sup_{t \in \mathbb{T}} \mathbb{E}(\|Y(t)\|_H^2) < +\infty\}.$$

A process $X \in \mathcal{X}_\mathbb{T}$ is called a *mild solution* to Eq. (8.1) if

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(s, X(s))ds + \int_0^t S(t-s)G(s, X(s))dL(s) \tag{8.3}$$

holds almost surely for all $t \in \mathbb{T}$. In Eq. (8.3) $S : \mathbb{T} \to L(H)$ is the semigroup generated by A, thus $S(t) = e^{tA}$ and Eq. (8.3) may be interpreted as a *variation-of-constants* formula.

Furthermore, $X \in \mathcal{X}_\mathbb{T}$ is called a *weak solution* to Eq. (8.1) if

$$\begin{aligned}(X(t), v)_H = (X_0, v)_H + \int_0^t (X(s), A^*v)_H + (F(s, X(s)), v)_H ds \\ + \int_0^t (G(s, X(s))^* v, dL(s))_\mathcal{U}\end{aligned}$$

holds almost surely for all $v \in D(A^*)$ and $t \in \mathbb{T}$, where $A^* : D(A^*) \to H$, $G(s,v)^* \in$

$L(H,\mathcal{U})$ are the adjoint operators to $A : D(A) \to H$ and $G(s,v) \in L_{HS}(\mathcal{U},H)$, respectively.

In the definition of weak solutions, we use the identification $L_{HS}(\mathcal{U},\mathbb{R}) = \mathcal{U}$. Hence, the integrand $s \mapsto G^*(s,X(s))^*v$ may be interpreted as a $L_{HS}(\mathcal{U},\mathbb{R})$-valued process and we obtain

$$\int_0^t (G(s,X(s))^*v, dL(s))_{\mathcal{U}} := \int_0^t G(s,X(s))^*v dL(s) = \left(v, \int_0^t G(s,X(s))dL(s)\right)_H$$

for any $v \in D(A^*)$, see [173, Chapter 9.3]. The solutions to Problem (8.1) are infinite-dimensional processes, i.e. $X : \Omega \times \mathbb{T} \times \mathbb{D} \to \mathbb{R}$, where $\mathbb{D} \subset \mathbb{R}^d$ for some $d \in \mathbb{N}$. Therefore, in general $H \subset U = L^2(\mathbb{D})$. To ensure that mild resp. weak solutions to (8.1) as in Definition 8.2.4 are well-defined and unique, we fix the following set of assumptions.

Assumption 8.2.5.

i) L is a centered, square integrable, U-valued Lévy process with trace class covariance operator Q.

ii) $X_0 \in L^2(\Omega;H)$ is a $\mathcal{F}_0$-measurable random variable.

iii) $A : D(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup $S = (S(t), t \geq 0)$ of bounded, linear operators on H.

iv) The mappings $F(\cdot,v) : \mathbb{T} \to H$ and $G(\cdot,v) : \mathbb{T} \to L_{HS}(\mathcal{U},H)$ are measurable for each $v \in H$ and there is a constant $C > 0$ such that for all $t \in \mathbb{T}$ and $v,w \in H$

$$\|F(t,v) - F(t,w)\|_H + \|G(t,v) - G(t,w)\|_{L_{HS}(\mathcal{U},H)} \leq C\|v-w\|_H.$$

Remark 8.2.6.

- We focus on mean-square type convergence results in this article and only consider square-integrable processes L. As one sees in Lemma 8.2.9, this enables us to use a version of the Itô isometry for stochastic integrals with respect to Hilbert space-valued Lévy processes. Details on non-square integrable martingales as integrator can be found in [173, Section 8.8].

- If L is of non-zero mean, then $E(L(t)) = t\phi$ for some mean function $\phi \in U$. Hence, we can always assume that $\mathbb{E}(L(t)) = 0$ and incorporate ϕ as part of the nonlinearity F if desired.

- Under Assumption 8.2.5, the Bochner integrals and stochastic integrals appearing in Definition 8.2.4 are well-defined, see [173, Remark 9.6].
- The global Lipschitz-type condition (iv) with respect to the second argument is necessary to ensure existence and uniqueness of mild solutions. Throughout the literature (e.g. in [151],[173]), often slightly weaker assumptions of the form

$$\|S(t)(F(s,v)-F(s,w))\|_H \leq b_F(t,s)\|v-w\|_H, \quad s,t\in(0,T]$$

for a function $b_F \in L^2(\mathbb{T}\times\mathbb{T})$ are imposed. For the numerical analysis in the forthcoming chapters, however, we utilize the weak solution of the SPDE, and it is therefore advantageous to assume Lipschitz continuity of F and G as above. We note that this condition on F and G implies the global linear growth bound

$$\|F(t,v)\|_H + \|G(t,v)\|_{L_{HS}(\mathcal{U},H)} \leq C(1+\|v\|_H), \quad v\in H,\ t\in\mathbb{T}.$$

Theorem 8.2.7. *Under Assumption 8.2.5, there exists a unique mild solution $X\in\mathcal{X}_\mathbb{T}$ to Problem* (8.1)*. Furthermore, X is also the unique weak solution and there exists $C=C(\mathbb{T})>0$, independent of X_0, such that*

$$\|X(t)\|_{L^2(\Omega;H)} \leq C(1+\|X_0\|_{L^2(\Omega;H)}), \quad t\in\mathbb{T}.$$

Proof. Existence and uniqueness of a mild solution as in Eq. (8.3) is proven in detail in [173, Theorem 9.29]. Therefore, we only sketch the main idea here. Let $\beta>0$ be arbitrary and define the norm $\|Y\|_\beta := e^{-\beta T}\sup_{t\in\mathbb{T}}\mathbb{E}(\|Y(t)\|_H^2)^{1/2}$ for any $Y\in\mathcal{X}_\mathbb{T}$. With this, $(\mathcal{X}_\mathbb{T},\|\cdot\|_\beta)$ is a Banach space, and, using X_0 as initial value, on $(\mathcal{X}_\mathbb{T},\|\cdot\|_\beta)$ a sequence of fixed-point iterations is for $n\in\mathbb{N}_0$ given by

$$X_{n+1}=\Psi(X_n) := S(t)X_0 + \int_0^t S(t-s)F(s,X_n(s))ds + \int_0^t S(t-s)G(s,X_n(s))dL(s).$$

Under Assumption 8.2.5, and by choosing $\beta>0$ large enough, it can then be shown that Ψ is a contraction mapping. Hence, existence and uniqueness of mild solutions follow by Banach's fixed-point theorem. The equivalence of weak and mild solutions follows from [173, Theorem 9.15]. □

To conclude this section, we record a lemma on C_0-semigroups and an infinite-dimensional version of the Itô isometry.

Lemma 8.2.8. *[170, Chapter 1.2] Let $S=(S(t),t\geq 0)$ be a C_0-semigroup with infinitesimal generator A on a Banach space $(\mathcal{Y},\|\cdot\|_\mathcal{Y})$. Then, there are constants*

$C_1, C_2 > 0$ such that for all $\phi \in \mathcal{Y}$ and $t \geq 0$

$$\|S(t)\phi\|_{\mathcal{Y}} \leq C_1 e^{C_2 t}\|\phi\|_{\mathcal{Y}}.$$

Lemma 8.2.9. *(Itô isometry, [173, Corollary 8.17]) Let $(\widehat{H}, (\cdot,\cdot)_{\widehat{H}})$ be a separable Hilbert space, let $\kappa : \Omega \times \mathbb{T} \to L_{HS}(\mathcal{U}; \widehat{H})$ be a predictable, square integrable process and let L satisfy Assumption 8.2.5(i). Then, κ is an admissible integrand for L, and for all $t \in \mathbb{T}$ it holds that*

$$\mathbb{E}\left(\|\int_0^t \kappa(s)dL(s)\|_{\widehat{H}}^2\right) = \mathbb{E}\left(\int_0^t \|\kappa(s)\|_{L_{HS}(\mathcal{U};\widehat{H})}^2 ds\right) = \mathbb{E}\left(\int_0^t \sum_{i\in\mathbb{N}} \eta_i \|\kappa(s)e_i\|_{\widehat{H}}^2 ds\right).$$

So far, all results of this section hold in a rather general setting, namely that A is the generator of an arbitrary C_0-semigroup. In the remainder of this article, we investigate the case where A is a first order differential operator and Eq. (8.1) is a (hyperbolic) transport equation with Lévy noise. The next section establishes the spatial and temporal regularity of X in this scenario to pave the way for a numerical analysis of the stochastic transport problem in Sections 8.4-8.6.

8.3 The Stochastic transport equation

Let us regard Eq. (8.1) with respect to a convex spatial domain $\mathbb{D} \subset \mathbb{R}^d$ with $d \in \mathbb{N}$, i.e. the solution X is a H-valued process with $H = L^2(\mathbb{D})$. We denote for $k \in \mathbb{N}$ the standard Sobolev space $H^k(\mathbb{D})$ equipped with the norm resp. seminorm

$$\|v\|_{H^k(\mathbb{D})} := \left(\sum_{|\alpha|\leq k} \int_{\mathbb{D}} |D^\alpha v(x)|^2 dx\right)^{1/2}, \quad |v|_{H^k(\mathbb{D})} := \left(\sum_{|\alpha|=k} \int_{\mathbb{D}} |D^\alpha v(x)|^2 dx\right)^{1/2},$$

where $D^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$ is the mixed partial weak derivative (in space) with respect to the multi-index $\alpha \in \mathbb{N}_0^d$. The fractional order Sobolev spaces $H^q(\mathbb{D})$ for any $q > 0$ are defined by the norm

$$\begin{aligned}\|v\|_{H^q(\mathbb{D})}^2 &= \|v\|_{H^{\lfloor q\rfloor}(\mathbb{D})}^2 + \sup_{|\alpha|=\lfloor q\rfloor} |D^\alpha v|_{H^{(q-\lfloor q\rfloor)}(\mathbb{D})}^2 \\ &:= \|v\|_{H^{\lfloor q\rfloor}(\mathbb{D})}^2 + \sup_{|\alpha|=\lfloor q\rfloor} \int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|D^\alpha v(x) - D^\alpha v(y)|^2}{|x-y|^{d+2(q-\lfloor q\rfloor)}} dxdy,\end{aligned}$$

where the last term is the the *Gagliardo seminorm*, see [71]. Let $A = a \cdot \nabla$, for a fixed vector $a \in \mathbb{R}^d$ in Eq. (8.1) be the first order differential operator such that we obtain

the *stochastic transport problem*

$$dX(t) = (a \cdot \nabla X(t) + F(t, X(t)))dt + G(t, X(t))dL(t), \quad X(0) = X_0. \tag{8.4}$$

The inflow boundary of $\mathbb{D}$ is given by

$$\partial\mathbb{D}^+ := \{x \in \partial\mathbb{D} : a \cdot \vec{n}(x) > 0\},$$

where $\vec{n}$ is the exterior normal vector to $\partial\mathbb{D}$ and the outflow boundary is $\partial\mathbb{D}^- := \partial\mathbb{D} \setminus \partial\mathbb{D}^+$. We equip Eq. (8.4) with the homogeneous inflow boundary condition $X(t) = 0$ on $\partial\mathbb{D}^+$ for all $t \in \mathbb{T}$.

Remark 8.3.1. The restriction to homogeneous inflow boundary conditions is for notational convenience, and not restrictive in our setting. In our numerical examples in Section 8.7, we examine an energy forward model with nonzero, but constant inflow boundary condition $X(t) = c > 0$ on $\partial\mathbb{D}^+$ for all $t \in \mathbb{T}$. To see how this fits in our setting, let X_0, F, G and L be given, and let $X : \Omega \times \mathbb{T} \to H$ be a solution to Eq. (8.5) with $X(t) = 0$ on $\partial\mathbb{D}^+$. For any constant $c \in \mathbb{R}$ we define $X_c(t) := X(t) + c$, as well as the modified coefficients

$$F_c(s, v) := F(s, v - c), \quad G_c(s, v) := G(s, v - c).$$

Note that if F and G satisfy Assumption 8.2.5(iv), the same holds for F_c and G_c. It is then readily verified that X_c satisfies $X(t) = c$ on $\mathbb{T} \times \partial\mathbb{D}^+$ and

$$dX_c(t) = (a \cdot \nabla X_c(t) + F_c(t, X_c(t)))dt + G_c(t, X_c(t))dL(t), \quad X_c(0) = X_0 + c.$$

To derive a weak formulation of Eq. (8.4), we note that for any $v, w \in H^1(\mathbb{D})$ Green's identity yields

$$(-Av, w)_H = (-a \cdot \nabla v, w)_H = (v, a \cdot \nabla w)_H - \int_{\partial\mathbb{D}} a \cdot \vec{n} v w dz = (v, A^* w)_H.$$

Thus, $D(A^*) = H^1(\mathbb{D})$ and A induces for arbitrary small $\varepsilon > 0$ the bilinear form

$$B : H^{1/2+\varepsilon}(\mathbb{D}) \times H^1(\mathbb{D}) \to \mathbb{R}, \quad (v, w) \mapsto (v, a \cdot \nabla w)_H - \int_{\partial\mathbb{D}} a \cdot \vec{n} v w dz.$$

The restriction to $H^{1/2+\varepsilon}(\mathbb{D})$ for the first argument of B is to ensure that its trace on $\partial\mathbb{D}$ is well-defined. Note that B is positive semi-definite on the set $\{v \in H^1(\mathbb{D}) |\ v|_{\partial\mathbb{D}^+} = 0\}$

and defines the seminorm

$$\|v\|^2_{a,\partial\mathbb{D}^-} := 2B(v,v) = -\int_{\partial\mathbb{D}^-} a\cdot\vec{n}v^2(z)dz \geq 0,$$

for all $v \in H^1(\mathbb{D})$ that vanish at the inflow boundary. The weak formulation of Eq. (8.4) is then to find $X : \Omega \times \mathbb{T} \to H$ such that for all $v \in D(A^*)$

$$\begin{aligned}(X(t),v)_H + \int_0^t B(X(s),v)ds = (X_0,v)_H + \int_0^t (F(s,X(s)),v)_H ds \\ + \int_0^t (G(s,X(s))^* v, dL(s))_{\mathcal{U}}.\end{aligned} \tag{8.5}$$

The numerical schemes to approximate X and the corresponding error estimates in this article are mainly based on the weak formulation from Eq. (8.5). As we will see in Theorem 8.3.6, however, mild solutions to Eq. (8.4) are convenient to investigate the spatial regularity of X. The operator $A = a\cdot\nabla$ with homogeneous outflow boundary conditions is the infinitesimal generator of a semigroup S on H, namely the *shift semigroup* given by

$$[S(t)v](x) := \begin{cases} v(at+x) & \text{if } at+x \in \mathbb{D} \\ 0 & \text{if } at+x \notin \mathbb{D} \end{cases}. \tag{8.6}$$

Lemma 8.3.2. *The family of operators $(S(t), t \geq 0)$ defined in Eq. (8.6) forms a C_0-semigroup of bounded linear operators on H. Furthermore, the infinitesimal generator of S is given by $A = a\cdot\nabla$.*

Proof. By the definition of S, it is immediate that $\|S(t)v\|_H \leq \|v\|_H$, $S(0) = I$ and $S(t+s) = S(t)S(s)$ for $t,s \in \mathbb{T}$. Hence, $(S(t), t \geq 0)$ is a semigroup of bounded linear operators on H. To see that $(S(t), t \geq 0)$ is strongly continuous, let $v \in C_c^0(\mathbb{D}) \subset H$ be a compactly supported, continuous function on $\mathbb{D}$. Furthermore, let $\widetilde{v} \in C_c^0(\mathbb{R}^d)$ be the zero-extension of v on $\mathbb{R}^d$ given by

$$\widetilde{v}(x) := \begin{cases} v(x) & \text{if } x \in \mathbb{D} \\ 0 & \text{if } x \in \mathbb{R}^d \setminus \mathbb{D} \end{cases}.$$

This yields

$$\lim_{t\to 0}\|S(t)v - v\|_H^2 = \lim_{t\to 0}\int_{\mathbb{D}} (\widetilde{v}(at+x))) - \widetilde{v}(x))^2 dx = \int_{\mathbb{D}} \lim_{t\to 0}(\widetilde{v}(at+x))) - \widetilde{v}(x))^2 dx = 0.$$

Note that the interchange of limit and integral is justified, since $\widetilde{v}$ is bounded uniformly

on $\mathbb{R}^d$. The last identity holds due to the continuity of $\widetilde{v}$ on $\mathbb{R}^d$. By the density of $C_c^0(\mathbb{D})$ in H, it follows that S is a C_0-semigroup on H.

To see that A is the generator of S, let $v \in C^2(\mathbb{D})$ and note that for any $x \in \mathbb{D}$, there is a $t_x > 0$ such that $at + x \in \mathbb{D}$ for all $t \in (0, t_x)$. Hence, for fixed $x \in \mathbb{D}$, the multidimensional Taylor expansion yields

$$\lim_{t\to 0} \frac{[S(t)v](x) - v(x)}{t} = \lim_{t\to 0} \frac{v(at+x) - v(x)}{t} = a \cdot \nabla v(x) = [Av](x).$$

As $C^2(\mathbb{D})$ is dense in $H^1(\mathbb{D})$, the limit also exists for every $v \in H^1(\mathbb{D})$, in the sense that $\|Av\|_H < +\infty$, and hence $D(A) = H^1(\mathbb{D})$. □

The mild solution to Eq. (8.6) reads

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(s, X(s))ds + \int_0^t S(t-s)G(s, X(s))dL(s). \tag{8.7}$$

From this, we see that the shift by S may introduce (spatial) discontinuities if X_0, F and G do not vanish near the inflow boundary. Hence, Assumption 8.2.5 has to be modified to guarantee a certain regularity of X.

Assumption 8.3.3. Let $q > 1/2$ and let the set $H^q_{0,+}(\mathbb{D})$ contain all $v \in H^q(\mathbb{D})$ such that

- $v|_{\partial\mathbb{D}^+} = 0$,
- there is an extension $\widetilde{v} \in H^q(\mathbb{R}^d)$ of v with $\widetilde{v}|_{\mathbb{D}} = v$ and $\|\widetilde{v}\|_{H^q(\mathbb{R}^d)} \leq C\|v\|_{H^q(\mathbb{D})}$, and
- there exists a bounded, convex set $\mathbb{O} \subset \mathbb{R}^d$ such that $\mathbb{D} \subset \mathbb{O}$, $\partial\mathbb{D}^+ \subset \partial\mathbb{O}$ and $\widetilde{v}|_{\mathbb{O}} \in H^q_0(\mathbb{O})$.

It holds that:

i) L is a square integrable, U-valued Lévy process with zero mean and trace class covariance operator Q. The eigenvalues $(\eta_i, i \in \mathbb{N})$ of Q are given in decreasing order and decay at rate $\eta_i \leq Ci^{-\alpha}$ for some $\alpha > 1$.

ii) $X_0 \in L^2(\Omega; H^q(\mathbb{D}))$ is a $\mathcal{F}_0$-measurable random variable with $X_0(\omega, \cdot) \in H^q_{0,+}(\mathbb{D})$ almost surely.

iii) $F : \mathbb{T} \times H \to H$ and $G : \mathbb{T} \times H \to L_{HS}(\mathcal{U}, H)$ are Hölder continuous with exponent $\frac{1}{2}$ on $\mathbb{T}$ and globally Lipschitz on H, i.e. for all $v, w \in H$ and $s, t \in \mathbb{T}$ it holds that

$$\|F(t,v) - F(s,v)\|_H + \|G(t,v) - G(s,v)\|_{L_{HS}(\mathcal{U},H)} \leq C|t-s|^{\frac{1}{2}}\|v\|_H,$$

and

$$\|F(t,v) - F(t,w)\|_H + \|G(t,v) - G(t,w)\|_{L_{HS}(\mathcal{U},H)} \leq C\|v-w\|_H.$$

iv) Let $(e_i, i \in \mathbb{N}) \subset U$ denote the orthonormal eigenfunctions of Q that correspond to the decreasing sequence of eigenvalues $(\eta_i, i \in \mathbb{N})$. For any $v \in H^q_{0,+}(\mathbb{D})$, $i \in \mathbb{N}$ and $t \in \mathbb{T}$, it holds that $F(t,v), G(t,v)e_i \in H^q_{0,+}(\mathbb{D})$. Moreover, there are constants $0 < \beta < (\alpha - 1)/2\alpha$ (where α is from part i)) and $C > 0$, independent of $v \in H^q_{0,+}(\mathbb{D})$ and $t \in \mathbb{T}$, such that

$$\|F(t,v)\|_{H^q(\mathbb{D})} \leq C(1 + \|v\|_{H^q(\mathbb{D})}),$$

and

$$\|G(t,v)e_i\|_{H^q(\mathbb{D})} \leq C(1 + \|v\|_{H^q(\mathbb{D})})\eta_i^{-\beta}.$$

Remark 8.3.4.

- Since $\mathbb{D}$ is a Lipschitz domain, there is always an extension $\widetilde{v}$ with $\|\widetilde{v}\|_{H^q(\mathbb{R}^d)} \leq C\|v\|_{H^q(\mathbb{D})}$ and $\widetilde{v}|_{\mathbb{O}} \in H^q_0(\mathbb{O})$ with bounded and open $\mathbb{O} \subset \mathbb{R}^d$ (see [71, Theorem 5.4]). The crucial part in the definition of $H^q_{0,+}(\mathbb{D})$ is that $\partial\mathbb{D}^+ \subset \partial\mathbb{O}$ to ensure control of the smoothness at the inflow boundary.

- Assumption 8.3.3(iii) on the Hölder continuity with respect to $\mathbb{T}$ is necessary to ensure the rate of convergence of the time stepping scheme introduced in Section 8.4. Note that this condition also implies that F and G are measurable in $\mathbb{T}$.

- Let us recall Example 8.2.3 with $U = L^2(\mathbb{D})$, Q as the Matérn covariance operator from Eq. (8.2) with smoothness parameter $\nu > 0$ and assume for any $q < \nu$ that

$$\|G(s,v)e_i\|_{H^q(\mathbb{D})} \leq C(1 + \|v\|_{H^q(\mathbb{D})})\|e_i\|_{H^q(\mathbb{D})}. \tag{8.8}$$

  By [96, Proposition 9], Assumption 8.3.3(i) holds with $\alpha = 1 + 2\nu/d > 1$. Hence the parameter β from part (iv) has to satisfy $0 < \beta < (\alpha - 1)/2\alpha = \nu/(d + 2\nu)$. Moreover, if $\nu > d/2$, the proof of [96, Proposition 9] yields for all $q, \widetilde{q}$ such that $0 \leq q \leq \widetilde{q} < d + 2\nu$ the estimate

$$\|e_i\|_{H^q(\mathbb{D})} \leq C\eta_i^{-q/\widetilde{q}}.$$

  Now let $q = \nu - \varepsilon_1$ and $\widetilde{q} = d + 2\nu - \varepsilon_2$, where $\varepsilon_1 > 0$ is arbitrary small and

$\varepsilon_2 \in (0, \varepsilon_1(d/\nu + 2))$. By construction, $\beta := q/\widetilde{q}$ satisfies $0 < \beta < (\alpha - 1)/2\alpha = \nu/(d + 2\nu)$ and Ineq. (8.8) yields

$$\|G(s, v)e_i\|_{H^q(\mathbb{D})} \leq C(1 + \|v\|_{H^q(\mathbb{D})})\eta_i^{-\beta}.$$

Regarding the eigenpairs of Q, Assumption 8.3.3 is therefore satisfied for any $q < \nu$ if $\nu > d/2$ in the Matérn case, and we may infer the (maximum) spatial regularity of X directly from ν (see Theorem 8.3.6).

To derive the spatial regularity of X, we record the following result:

Lemma 8.3.5. *Let $q > 1/2$ and $v \in H^q_{0,+}(\mathbb{D})$, with $H^q_{0,+}(\mathbb{D}) \subset H^q(\mathbb{D})$ defined as in Assumption 8.3.3. Then, there is a constant $C = C(q, \mathbb{D}, d) > 0$ such that for any $t \in \mathbb{T}$ it holds that $\|S(t)v\|_{H^q(\mathbb{D})} \leq C\|v\|_{H^q(\mathbb{D})}$.*

Proof. Let $\widetilde{v} \in H^q(\mathbb{R}^d)$ be the extension of v and $\mathbb{O} \subset \mathbb{R}^d$ be the bounded, convex set such that $\widetilde{v}|_{\mathbb{O}} \in H^q_0(\mathbb{O})$ as in Assumption 8.3.3. Fix $x \in \mathbb{D}$ and $t \in \mathbb{T}$ such that $at + x \notin \mathbb{D}$. Since $\mathbb{D}$ and $\mathbb{O}$ are convex with $\partial\mathbb{D}^+ \subset \partial\mathbb{O}$, it follows that $at + x \notin \mathbb{O}$. As $\widetilde{v}$ vanishes outside of $\mathbb{O}$, this yields $[S(t)v](x) = \widetilde{v}(at + x)$ and therefore

$$\|S(t)v\|_{H^q(\mathbb{D})} \leq \|\widetilde{v}\|_{H^q(\mathbb{R}^d)} \leq C\|v\|_{H^q(\mathbb{D})}.$$

□

Theorem 8.3.6. *Under Assumption 8.3.3, there exist unique solutions X to Eq.* (8.5) *and Eq.* (8.7)*, respectively. Moreover, both solutions coincide almost surely and*

$$\sup_{t\in\mathbb{T}} \|X(t)\|^2_{L^2(\Omega;H^q(\mathbb{D}))} \leq C(1 + \|X_0\|^2_{L^2(\Omega;H^q(\mathbb{D}))}) < +\infty.$$

Proof. Existence, uniqueness and equivalence of a weak resp. mild solution $X : \Omega\times\mathbb{T} \to H$ follow by Theorem 8.2.7, since Assumption 8.3.3 implies Assumption 8.2.5. To derive the spatial regularity, we consider the mild solution X given by the variation-of-constants formula

$$X(t) = S(t)X_0 + \int_0^t S(t - s)F(s, X(s))ds + \int_0^t S(t - s)G(s, X(s))dL(s),$$

and estimate each term on the right hand side separately. By Lemma 8.3.5 and Assumption 8.3.3(ii)

$$\|S(t)X_0\|^2_{L^2(\Omega;H^q(\mathbb{D}))} \leq C\|X_0\|^2_{L^2(\Omega;H^q(\mathbb{D}))} < +\infty.$$

Jensen's inequality and Assumption 8.3.3(iv) yield similarly

$$\begin{aligned}\| \int_0^t S(t-s)F(s,X(s))ds\|^2_{L^2(\Omega;H^q(\mathbb{D}))} &\leq \int_0^t \|S(t-s)F(s,X(s))\|^2_{L^2(\Omega;H^q(\mathbb{D}))}ds \\ &\leq C\int_0^t 1+\|X(s)\|^2_{L^2(\Omega;H^q(\mathbb{D}))}ds.\end{aligned}$$

Moreover, the Itô isometry from Lemma 8.2.9 shows the identity

$$\begin{aligned}\| \int_0^t S(t-s)G(s,X(s))dL(s)\|^2_{L^2(\Omega;H^q(\mathbb{D}))} &= \mathbb{E}\Big(\int_0^t \|S(t-s)G(s,X(s))\|^2_{L_{HS}(\mathcal{U},H^q(\mathbb{D}))}ds\Big) \\ &= \mathbb{E}\Big(\int_0^t \sum_{i\in\mathbb{N}} \eta_i \|S(t-s)G(s,X(s))e_i\|^2_{H^q(\mathbb{D})}ds\Big).\end{aligned}$$

With Lemma 8.3.5 and Assumption 8.3.3(iv) this gives the estimate

$$\begin{aligned}\| \int_0^t S(t-s)G(s,X(s))dL(s)\|^2_{L^2(\Omega;H^q(\mathbb{D}))} &\leq C\sum_{i\in\mathbb{N}} \eta_i^{1-2\beta} \int_0^t 1+\|X(s)\|^2_{L^2(\Omega;H^q(\mathbb{D}))}ds \\ &\leq C\sum_{i\in\mathbb{N}} i^{-\alpha(1-2\beta)} \int_0^t 1+\|X(s)\|^2_{L^2(\Omega;H^q(\mathbb{D}))}ds.\end{aligned}$$

Since $\alpha(1-2\beta)>1$ by Assumption 8.3.3(iv), it holds that $\sum_{i\in\mathbb{N}} i^{-\alpha(1-2\beta)} < +\infty$, and we obtain

$$\|X(t)\|^2_{L^2(\Omega;H^q(\mathbb{D}))} \leq C\Big(1+\|X_0\|^2_{L^2(\Omega;H^q(\mathbb{D}))} + \int_0^t \|X(s)\|^2_{L^2(\Omega;H^q(\mathbb{D}))}ds\Big),$$

where $C>0$ is finite and uniformly bounded in $\mathbb{T}$. The claim then follows by Grönwall's inequality. □

Theorem 8.3.7. *Let Assumption 8.3.3 hold with $q \geq 1$. Then, there is a $C>0$ such that for all $s,t\in\mathbb{T}$*

$$\mathbb{E}(\|X(t)-X(s)\|^2_H) + \mathbb{E}(\|X(t)-X(s)\|^2_{a,\partial\mathbb{D}^-}) \leq C|t-s|.$$

Proof. We first show the claim with respect to $\|\cdot\|_H$. To this end, consider the weak formulation (8.5) to obtain for fixed $s,t\in\mathbb{T}$ with $t\geq s$ and $v\in D(A^*)=H^1(\mathbb{D})$

$$\begin{aligned}(X(t)-X(s),v)_H = &-\int_s^t B(X(r),v)dr + \int_s^t (F(r,X(r)),v)_H ds \\ &+ \int_s^t (G^*(r,X(r))v, dL(r))_{\mathcal{U}}.\end{aligned}$$

By Theorem 8.3.6, $X(t)\in H^1(\mathbb{D})$ almost surely for each $t\in\mathbb{T}$ since $q\geq 1$. Thus, we

test against $v = X(t) - X(s) \in H^1(\mathbb{D})$ and take expectations to obtain

$$\begin{aligned}\mathbb{E}(\|X(t) - X(s)\|_H^2) &= -\int_s^t \mathbb{E}(B(X(r), X(t) - X(s)))dr \\ &\quad + \int_s^t \mathbb{E}((F(r, X(r)), X(t) - X(s))_H)ds \\ &\quad + \mathbb{E}\Big((\int_s^t G^*(r, X(r))(X(t) - X(s)), dL(r))_{\mathcal{U}}\Big) \\ &=: I + II + III.\end{aligned}$$

By definition, $B(v, w) = (-a \cdot \nabla v, w)$ for $v, w \in H^1(\mathbb{D})$ and we estimate the first term with Hölder's inequality and Theorem 8.3.6 via

$$\begin{aligned}I &\leq C\int_s^t \mathbb{E}(\|X(r)\|_{H^1(\mathbb{D})}^2)^{1/2}\mathbb{E}(\|X(t) - X(s)\|_H^2)^{1/2}dr \\ &\leq C(t-s)(\sup_{r\in\mathbb{T}} \mathbb{E}(\|X(r)\|_{H^1(\mathbb{D})}^2)^{1/2})\mathbb{E}(\|X(t) - X(s)\|_H^2)^{1/2} \\ &\leq C(t-s)\mathbb{E}(\|X(t) - X(s)\|_H^2)^{1/2}.\end{aligned}$$

Similarly, we obtain with Assumption 8.3.3(iii)

$$\begin{aligned}II &\leq \int_s^t \mathbb{E}(\|F(r, X(r))\|_H^2)^{1/2}\mathbb{E}(\|X(t) - X(s)\|_H^2)^{1/2}dr \\ &\leq (t-s)(1 + \sup_{r\in\mathbb{T}} \mathbb{E}(\|X(r)\|_H^2)^{1/2})\mathbb{E}(\|X(t) - X(s)\|_H^2)^{1/2} \\ &\leq C(t-s)\mathbb{E}(\|X(t) - X(s)\|_H^2)^{1/2}.\end{aligned}$$

The last term is bounded by Lemma 8.2.9 and Assumption 8.3.3(iii):

$$\begin{aligned}III &\leq \mathbb{E}\Big(\|\int_s^t G(r, X(r))dL(r)\|_H^2\Big)^{1/2}\mathbb{E}(\|X(t) - X(s)\|_H^2)^{1/2} \\ &\leq \Big(\int_s^t \mathbb{E}(\|G(r, X(r))\|_{L_{HS}(\mathcal{U},H)}^2)dr\Big)^{1/2}\mathbb{E}(\|X(t) - X(s)\|_H^2)^{1/2} \\ &\leq (t-s)^{1/2}(1 + \sup_{r\in\mathbb{T}} \mathbb{E}(\|X(r)\|_H^2)^{1/2})\mathbb{E}(\|X(t) - X(s)\|_H^2)^{1/2} \\ &\leq C(t-s)^{1/2}\mathbb{E}(\|X(t) - X(s)\|_H^2)^{1/2},\end{aligned}$$

which shows that

$$\mathbb{E}(\|X(t) - X(s)\|_H^2) \leq C(t-s). \tag{8.9}$$

For the second part, let $(X_n(t) - X_n(s), n \in \mathbb{N}) \subset C^\infty(\overline{\mathbb{D}})$ be a smooth approximating sequence of $X(t) - X(s) \in H^1(\mathbb{D})$. By the density of $C^\infty(\overline{\mathbb{D}})$ in $H^1(\mathbb{D})$, this implies

in particular

$$\mathbb{E}(\|X_n(t) - X_n(s)\|^2_{H^1(\mathbb{D})}) \leq C\mathbb{E}(\|X(t) - X(s)\|^2_{H^1(\mathbb{D})}) < +\infty, \tag{8.10}$$

and by Ineq. (8.9)

$$\mathbb{E}(\|X_n(t) - X_n(s)\|^2_H) \leq C\mathbb{E}(\|X(t) - X(s)\|^2_H) \leq C(t-s),$$

where the constant $C > 0$ is in both cases independent of $n \in \mathbb{N}$. As $X_n(t) - X_n(s)$ is smooth, we may test against $v = -a \cdot \nabla(X_n(t) - X_n(s)) \in H^1(\mathbb{D})$ in Eq. (8.5) to obtain

$$\begin{aligned}
&- \mathbb{E}((X(t) - X(s), a \cdot \nabla(X_n(t) - X_n(s)))_H) \\
&= \int_s^t \mathbb{E}(B(X(r), a \cdot \nabla(X_n(t) - X_n(s)))dr \\
&\quad - \int_s^t \mathbb{E}((F(r, X(r)), a \cdot \nabla(X_n(t) - X_n(s)))_H dr \\
&\quad - \mathbb{E}\Big((\int_s^t (G(r, X(r)))dL(r), a \cdot \nabla(X_n(t) - X_n(s)))_H\Big) \\
&=: IV - V - VI.
\end{aligned} \tag{8.11}$$

It follows immediately by Theorem 8.3.6, Assumption 8.3.3(iii) and Ineq. (8.10) that

$$IV \leq C \int_s^t \mathbb{E}(\|X(r)\|^2_{H^1(\mathbb{D})})^{1/2} dr\, \mathbb{E}(\|X_n(t) - X_n(s)\|^2_{H^1(\mathbb{D})})^{1/2} \leq C(t-s),$$

as well as

$$|V| \leq C \int_s^t \mathbb{E}(1 + \|X(r)\|^2_H)^{1/2} dr\, \mathbb{E}(\|X_n(t) - X_n(s)\|^2_{H^1(\mathbb{D})})^{1/2} \leq C(t-s).$$

To bound the last term, we use Green's identity to rewrite VI as

$$\begin{aligned}
VI &= \mathbb{E}\Big((\int_s^t G(r, X(r))dL(r), a \cdot \nabla(X_n(t) - X_n(s)))_H\Big) \\
&= -\mathbb{E}\Big((a \cdot \nabla \int_s^t G(r, X(r))dL(r), X_n(t) - X_n(s))_H\Big) \\
&\quad + \mathbb{E}\Big(\int_{\partial\mathbb{D}^-} a \cdot \vec{n} \int_s^t G(r, X(r))dL(r)(X_n(t) - X_n(s))dz\Big).
\end{aligned}$$

Applying Young's inequality to the inner products over $\mathbb{D}$ and $\partial\mathbb{D}^+$ yields

$$\begin{aligned}
|VI| \leq C\Big(&\mathbb{E}(\|\int_s^t G(r, X(r))dL(r)\|^2_{H^1(\mathbb{D})}) + \mathbb{E}(\|X_n(t) - X_n(s)\|^2_H)\Big) \\
&+ \mathbb{E}(\|\int_s^t G(r, X(r))dL(r)\|^2_{a,\partial\mathbb{D}^-}) + \frac{1}{4}\mathbb{E}(\|X_n(t) - X_n(s)\|^2_{a,\partial\mathbb{D}^-}).
\end{aligned}$$

Moreover, for any $v \in H^1(\mathbb{D})$, we obtain by the trace theorem

$$\|v\|_{a,\partial\mathbb{D}^-} \leq C\|v\|_{L^2(\partial\mathbb{D})} \leq C\|v\|_{H^{1/2+\varepsilon}(\mathbb{D})} \leq C\|v\|_{H^1(\mathbb{D})}.$$

Together with the Itô isometry from Lemma 8.2.9 this yields

$$\begin{aligned}
|VI| &\leq C\Big(\mathbb{E}(\|\int_s^t G(r, X(r))dL(r)\|^2_{H^1(\mathbb{D})}) + \mathbb{E}(\|X_n(t) - X_n(s)\|^2_H)\Big) \\
&\quad + \frac{1}{4}\mathbb{E}(\|X_n(t) - X_n(s)\|^2_{a,\partial\mathbb{D}^-}) \\
&\leq C\Big(\int_s^t \sum_{i\in\mathbb{N}} \eta_i \mathbb{E}(\|G(r, X(r))e_i\|^2_{H^1(\mathbb{D})})dr + \mathbb{E}(\|X_n(t) - X_n(s)\|^2_H)\Big) \\
&\quad + \frac{1}{4}\mathbb{E}(\|X_n(t) - X_n(s)\|^2_{a,\partial\mathbb{D}^-}) \\
&\overset{(*)}{\leq} C\Big((1 + \sup_{r\in\mathbb{T}} \mathbb{E}(\|X(r)\|^2_{H^1(\mathbb{D})})) \sum_{i\in\mathbb{N}} i^{-\alpha(1-2\beta)} \int_s^t dr + \mathbb{E}(\|X_n(t) - X_n(s)\|^2_H)\Big) \\
&\quad + \frac{1}{4}\mathbb{E}(\|X(t) - X(s)\|^2_{a,\partial\mathbb{D}^-}) \\
&\leq C(t-s) + \frac{1}{4}\mathbb{E}(\|X_n(t) - X_n(s)\|^2_{a,\partial\mathbb{D}^-}).
\end{aligned}$$

For the bound $(*)$, we have used Assumption 8.3.3(iv) on G. The last estimate follows since $\alpha(1-2\beta) > 1$ by Assumption 8.3.3(iv) and from Ineq. (8.10). We substitute the estimates on $IV - VI$ in Eq. (8.11) and rearrange terms to obtain

$$-\mathbb{E}((X(t) - X(s), a \cdot \nabla(X_n(t) - X_n(s)))_H) - \frac{1}{4}\mathbb{E}(\|X_n(t) - X_n(s)\|^2_{a,\partial\mathbb{D}^-}) \leq C(t-s).$$

Note that the constant $C > 0$ on the right hand side is independent of n. Therefore, the claim follows with

$$\begin{aligned}
\|X(t) - X(s)\|^2_{a,\partial\mathbb{D}^-} &= 2B(X(t) - X(s), X(t) - X(s)) \\
&= -2(X(t) - X(s), a \cdot \nabla(X(t) - X(s)))_H
\end{aligned}$$

and a density argument by taking the limit $n \to +\infty$. □

In most cases, it is impossible to access X analytically as the paths of X are time-dependent random functions taking values in the infinite-dimensional Hilbert space H. The time dependency of each sample may be reflected in the coefficients of a suitable basis expansions, but in general no tractable representations are available. Even if closed form solutions with respect to X_0 and a given path of L were known, it would still be unclear how to sample the infinite-dimensional Lévy process L. We

address these issues by introducing suitable time stepping schemes and a discontinuous Galerkin spatial discretization in Sections 8.4 and 8.5, respectively. Moreover, we show in Section 8.6 how to obtain approximate samples of L which finally yields a fully discrete approximation scheme for the stochastic transport problem.

8.4 Temporal discretization

To discretize $\mathbb{T}$, we use $m+1$ equidistant time points $0 = t_0 < \cdots < t_m = T$ and define $\Delta t := T/m > 0$. We employ a *backward Euler* (BE) approximation for the linear part of Eq. (8.5), i.e.

$$\int_{t_{i-1}}^{t_i} B(X(s), v)ds \approx \Delta t B(X(t_i), v), \quad i = 1, \dots, m.$$

The nonlinear part with respect to F and the stochastic integral are approximated by the forward differences

$$\begin{aligned} \int_{t_{i-1}}^{t_i} (F(s, X(s)), v)_H ds &\approx (F(t_{i-1}, X(t_{i-1}))\Delta t, v)_H, \\ \Big(\int_{t_{i-1}}^{t_i} G(s, X(s))dL(s), v \Big)_H &\approx (G(t_{i-1}, X(t_{i-1}))\Delta L^{(i)}, v)_H, \end{aligned} \tag{8.12}$$

where $\Delta L^{(i)} := L(t_i) - L(t_{i-1})$. As the stochastic integral on the left hand side in Eq. (8.12) is an Itô integral, it is crucial to use a forward difference in order to preserve the martingale property of the driving noise. For the nonlinearity F on the other hand, we could have chosen a backward difference or midpoint rule, but with the scheme (8.12) we avoid solving a nonlinear system in every time step and do not affect the overall order of convergence. The time-discrete version of the weak problem is then to find $(X^{(i)}, i = 0, \dots, m) \subset H$ such that $X^{(0)} = X_0$ and for any $v \in D(A^*)$ and $i = 1, \dots, m$

$$(X^{(i)} - X^{(i-1)}, v)_H + \Delta t B(X^{(i)}, v) = \Delta t (F(t_{i-1}, X^{(i-1)}), v)_H + (G(t_{i-1}, X^{(i-1)})\Delta L^{(i)}, v)_H. \tag{8.13}$$

We are able to bound the error of the time-stepping scheme with the results from the previous section:

Theorem 8.4.1. *Let Assumption 8.3.3 hold with $q \geq 1$. Then, for sufficiently small Δt,*

$$\mathbb{E}\Big(\|X(T) - X^{(m)}\|_H^2 + \Delta t \sum_{i=1}^{m} \mathbb{E}(\|X(t_i) - X^{(i)}\|_{a,\partial\mathbb{D}^-}^2) \Big)^{1/2} \leq C\Delta t^{1/2}.$$

Proof. Let $\psi^{(i)} := X(t_i) - X^{(i)}$ and note that by Eqs. (8.5) and (8.13) for all $v \in H^1(\mathbb{D})$

$$\begin{aligned}
&(\psi^{(i)} - \psi^{(i-1)}, v) + \int_{t_{i-1}}^{t_i} B(X(s) - X^{(i)}, v) \\
&= \int_{t_{i-1}}^{t_i} (F(s, X(s)) - F(t_{i-1}, X^{(i-1)}), v)_H ds \\
&\quad + \int_{t_{i-1}}^{t_i} ([G(s, X(s)) - G(t_{i-1}, X^{(i-1)})]^* v, dL(s))_{\mathcal{U}}.
\end{aligned}$$

We test against $v = \psi^{(i)} \in H^1(\mathbb{D})$ and take expectations to obtain

$$\begin{aligned}
&\mathbb{E}\Big((\psi^{(i)} - \psi^{(i-1)}, \psi^{(i)})_H + \int_{t_{i-1}}^{t_i} B(X(s) - X^{(i)}, \psi^{(i)}) ds\Big) \\
&= \mathbb{E}\Big(\int_{t_{i-1}}^{t_i} (F(s, X(s)) - F(t_{i-1}, X^{(i-1)}), \psi^{(i)})_H ds\Big) \\
&\quad + \mathbb{E}\Big(\int_{t_{i-1}}^{t_i} ([G(s, X(s)) - G(t_{i-1}, X^{(i-1)})]^* \psi^{(i)}, dL(s))_{\mathcal{U}}\Big) =: I + II.
\end{aligned} \tag{8.14}$$

Applying the identities

$$\begin{aligned}
(\psi^{(i)} - \psi^{(i-1)}, \psi^{(i)})_H &= \frac{1}{2}\Big(\|\psi^{(i)}\|_H^2 - \|\psi^{(i-1)}\|_H^2 + \|\psi^{(i)} - \psi^{(i-1)}\|_H^2\Big), \\
B(X(s) - X^{(i)}, \psi^{(i)}) &= B(X(s) - X(t_i), \psi^{(i)}) + B(\psi^{(i)}, \psi^{(i)}) \\
&= B(X(s) - X(t_i), \psi^{(i)}) + \frac{1}{2}\|\psi^{(i)}\|_{a,\partial\mathbb{D}^+}^2,
\end{aligned}$$

on the left hand side of Eq. (8.14), and rearranging some terms, yields

$$\begin{aligned}
&\mathbb{E}\Big(\frac{1}{2}(\|\psi^{(i)}\|_H^2 - \|\psi^{(i-1)}\|_H^2) + \Delta t \|\psi^{(i)}\|_{a,\partial\mathbb{D}^-}^2\Big) \\
&= I + II - \mathbb{E}\Big(\int_{t_{i-1}}^{t_i} B(X(s) - X(t_i), \psi^{(i)}) ds\Big) - \frac{1}{2}\mathbb{E}(\|\psi^{(i)} - \psi^{(i-1)}\|_H^2) \\
&=: I + II - III - \frac{1}{2}\mathbb{E}(\|\psi^{(i)} - \psi^{(i-1)}\|_H^2).
\end{aligned} \tag{8.15}$$

For the first term, we obtain with Young's inequality

$$\begin{aligned}
I &\le \frac{1}{2}\int_{t_{i-1}}^{t_i} \mathbb{E}(\|F(s, X(s)) - F(t_{i-1}, X^{(i-1)})\|_H^2) ds + \frac{\Delta t}{2}\mathbb{E}(\|\psi^{(i)}\|_H^2) \\
&=: \frac{1}{2}(I_a + \Delta t \mathbb{E}(\|\psi^{(i)}\|_H^2)),
\end{aligned}$$

and we split I_a once more by Jensen's inequality to obtain

$$\begin{aligned}
I_a &\leq 3\int_{t_{i-1}}^{t_i} \mathbb{E}(\|F(s,X(s)) - F(t_{i-1},X(s)\|_H^2)ds \\
&\quad + 3\int_{t_{i-1}}^{t_i} \mathbb{E}(\|F(t_{i-1},X(s)) - F(t_{i-1},X(t_{i-1})\|_H^2)ds \\
&\quad + 3\int_{t_{i-1}}^{t_i} \mathbb{E}(\|F(t_{i-1},X(t_{i-1})) - F(t_{i-1},X^{(i-1)})\|_H^2)ds \\
&\leq C\Big(\Delta t^2 + \int_{t_{i-1}}^{t_i} \mathbb{E}(\|X(s) - X(t_{i-1})\|_H^2)ds + \Delta t\|\psi^{(i-1)}\|_H^2\Big) \\
&\leq C\Delta t\Big(\Delta t + \|\psi^{(i-1)}\|_H^2\Big).
\end{aligned}$$

We have used Assumption 8.3.3(iii, iv) together with $\psi^{(i-1)} = X(t_{i-1}) - X^{(i-1)}$ for the second inequality and Theorem 8.3.7 for the last estimate. Hence,

$$I \leq C\Delta t\Big(\Delta t + \mathbb{E}(\|\psi^{(i)}\|_H^2) + \mathbb{E}(\|\psi^{(i-1)}\|_H^2)\Big).$$

To bound II, we observe that by Young's inequality, the independent increments of L and Lemma 8.2.9

$$\begin{aligned}
II &= \mathbb{E}\Big(\Big(\int_{t_{i-1}}^{t_i} G(s,X(s)) - G(t_{i-1},X^{(i-1)})dL(s), \psi^{(i)} - \psi^{(i-1)}\Big)_H\Big) \\
&\quad + \mathbb{E}\Big(\Big(\int_{t_{i-1}}^{t_i} G(s,X(s)) - G(t_{i-1},X^{(i-1)})dL(s), \psi^{(i-1)}\Big)_H\Big) \\
&\leq \frac{1}{2}\mathbb{E}\Big(\|\int_{t_{i-1}}^{t_i} G(s,X(s)) - G(t_{i-1},X^{(i-1)})dL(s)\|_H^2\Big) + \frac{1}{2}\mathbb{E}(\|\psi^{(i)} - \psi^{(i-1)}\|_H^2) \\
&\quad + \Big(\mathbb{E}\Big(\int_{t_{i-1}}^{t_i} G(s,X(s)) - G(t_{i-1},X^{(i-1)})dL(s)\Big), \mathbb{E}(\psi^{(i-1)})\Big)_H \\
&= \frac{1}{2}\int_{t_{i-1}}^{t_i} \mathbb{E}(\|G(s,X(s)) - G(t_{i-1},X^{(i-1)})\|_{L_{HS}(\mathcal{U},H)}^2)ds + \frac{1}{2}\mathbb{E}(\|\psi^{(i)} - \psi^{(i-1)}\|_H^2) \\
&\leq C\Delta t\Big(\Delta t + \|\psi^{(i-1)}\|_H^2\Big) + \frac{1}{2}\mathbb{E}(\|\psi^{(i)} - \psi^{(i-1)}\|_H^2).
\end{aligned}$$

In the second equality, the third term has vanished since the Itô integral is of zero mean. The last estimate follows analogously to I using the Lipschitz- and Hölder-type conditions on G from Assumption 8.3.3. The last term is bounded by Young's inequality and Theorem 8.3.7 via

$$|III| \leq \frac{1}{2}\int_{t_{i-1}}^{t_i} \mathbb{E}(\|X(s) - X(t_{i-1})\|_{a,\partial\mathbb{D}^-}^2) + \mathbb{E}(\|\psi^{(i)}\|_{a,\partial\mathbb{D}^-}^2)ds \leq C\Delta t^2 + \frac{\Delta t}{2}\mathbb{E}(\|\psi^{(i)}\|_{a,\partial\mathbb{D}^-}^2).$$

We then substitute the estimates on $I - III$ into Eq. (8.15), observe that $\psi^{(0)} = 0$, and

sum over i to obtain

$$\mathbb{E}\Big(\|\psi^{(m)}\|_H^2 + \Delta t \sum_{i=1}^{m} \|\psi^{(i)}\|_{a,\partial\mathbb{D}^-}^2\Big) \leq C\Delta t\Big(\sum_{i=1}^{m} \Delta t + \mathbb{E}(\|\psi^{(i)}\|_H^2)\Big).$$

Now define $y^{(i)} := \mathbb{E}\Big(\|\psi^{(i)}\|_H^2 + \Delta t \sum_{j=1}^{i} \|\psi^{(i)}\|_{a,\partial\mathbb{D}^-}^2\Big)$ for $i = 1, \ldots, m$ and observe that

$$y^{(m)} \leq C\Delta t\Big(\sum_{i=1}^{m} \Delta t + \mathbb{E}(\|\psi^{(i)}\|_H^2)\Big) \leq C\Delta t\Big(\sum_{i=1}^{m} \Delta t + y^{(i)}\Big).$$

The claim then follows for $\Delta t \in (0, 1/C)$ by the discrete Grönwall inequality. □

Remark 8.4.2. In case that Eq. (8.4) admits *additive noise*, i.e. $G(s, v) = G(s)$, one may replace the BE approach by a Crank-Nicolson (CN) time stepping scheme to achieve a higher order of convergence. That is, we use the trapezoidal rule to approximate the linear term of Eq. (8.5) via

$$\int_{t_{i-1}}^{t_i} B(X(s), v)ds \approx \frac{\Delta t}{2}(B(X(t_i), v) + B(X(t_{i-1}), v)), \quad i = 1, \ldots, m.$$

Under the additional requirement that F and G are also Lipschitz continuous on $\mathbb{T}$, meaning

$$\|F(t, v) - F(s, v)\|_H + \|G(t, v) - G(s, v)\|_{L_{HS}(\mathcal{U},H)} \leq C|t - s|\|X\|_H,$$

we would then expect faster convergence with respect to Δt. Intuitively, the mean-squared error should be of order $\mathcal{O}(\Delta t^\gamma)$ for this discretization, where $\gamma = \gamma(q)$ is increasing in q with $\gamma(q) \in [\frac{1}{2}, 1]$ for any $q \geq 1$. The derivation of γ under suitable assumptions is subject to future work, our numerical experiments in Section 8.7 suggest, however, that $\gamma(q) \approx \frac{2q}{3}$ is possible.

8.5 Discontinuous Galerkin spatial discretization

In this section we discretize Eq (8.13) with respect to the spatial domain. To this end, let $h > 0$ be a refinement parameter and let $\mathcal{K}_h$ be a uniform triangulation of $\mathbb{D}$ with maximum diameter h. For simplicity, we assume that $\mathbb{D}$ is a polygonal domain with piecewise linear boundary and may thus omit errors due to the piecewise linear approximation of $\partial\mathbb{D}$. As a suitable finite-dimensional subspace of H, we choose the corresponding discontinuous Galerkin (DG) space $V_h \subset H$ of piecewise linear polynomials given by

$$V_h := \{v \in H : v|_K \in \mathbf{P_1}(K),\ K \in \mathcal{K}_h\}.$$

The elements of V_h are piecewise linear on the simplices K, but allow for jumps at the interfaces of the triangulation. Hence, the space of continuous, piecewise linear finite elements is contained in V_h, and, as we will see throughout this section, the DG approach yields additional stability with a suitably chosen numerical flux over the discontinuities. In contrast to standard finite element spaces, the weak derivatives $\partial_{x_i} v_h$ are elements of the dual space $H^{-1}(\mathbb{D})$ of $H^1(\mathbb{D})$, but in general $\partial_{x_i} v_h \notin L^2(\mathbb{D})$. Thus, we need to work with the "broken version" of the H-scalar product and the induced norm with respect to $\mathcal{K}_h$ given by

$$(v,w)_{H,h} := \sum_{K\in\mathcal{K}_h} (v,w)_{L^2(K)}, \quad \|v\|_{H,h} := (v,v)_{H,h}. \tag{8.16}$$

This guarantees that $\|\partial_{x_i} v_h\|_{H,h} < +\infty$ even if $\partial_{x_i} v_h \notin L^2(\mathbb{D})$ and, moreover, $(\cdot,\cdot)_{H,h}$ and $(\cdot,\cdot)_H$ coincide on H. Analogously, we define the broken Sobolev norms and semi-norms for any $q > 0$ via

$$\|v\|_{H^q(\mathbb{D}),h} := \sum_{K\in\mathcal{K}_h} \|v\|_{H^q(K)}, \quad |v|_{H^q(\mathbb{D}),h} := \sum_{K\in\mathcal{K}_h} |v|_{H^q(K)}.$$

On each $K \in \mathcal{K}_h$, the space-time discrete weak solution $X_h = (X_h^{(i)}, i = 0,\dots,m) \subset V_h$ should then satisfy

$$\begin{aligned}(X_h^{(i)}, v_h)_{L^2(K)} &= (X_h^{(i-1)}, v_h)_{L^2(K)} + \Delta t(a\cdot\nabla X_h^{(i)}, v_h)_{L^2(K)} \\ &\quad + (F(t_{i-1}, X_h^{(i-1)})\Delta t, v_h)_{L^2(K)} + (G(t_{i-1}, X_h^{(i-1)})\Delta L^{(i)}, v_h)_{L^2(K)} \\ &= (X_h^{(i-1)}, v_h)_{L^2(K)} - \Delta t(X_h^{(i)}, a\cdot\nabla v_h)_{L^2(K)} + \Delta t(\vec{n}\cdot a X_h^{(i)}, v_h)_{L^2(\partial K)} \\ &\quad + (F(t_{i-1}, X_h^{(i-1)})\Delta t, v_h)_{L^2(K)} + (G(t_{i-1}, X_h^{(i-1)})\Delta L^{(i)}, v_h)_{L^2(K)},\end{aligned} \tag{8.17}$$

for any $v_h \in V_h$ and $i = 1,\dots,m$. As $X_h^{(i)}$ and v_h are not uniquely defined on ∂K, we need to introduce a *numerical flux* across each boundary ∂K and denote by $\mathcal{E}_h$ the set of all faces of $\mathcal{K}_h$. Now, let two simplices K^+, K^- share a common interior face $E \in \mathcal{E}_h$ with $E \cap \partial\mathbb{D} = \emptyset$. The outward normal vectors of K^+ and K^- on E are denoted by $\vec{n}^+$ and $\vec{n}^-$, respectively. Similarly, for a scalar\vector-valued function $\psi : K^+ \cup K^- \to \mathbb{R}^d$, we define by ψ^+ the trace of $\psi|_{K^+}$ on E, and ψ^- is the trace of $\psi|_{K^-}$ on E. We denote the jump $[\![\cdot]\!]$ resp. average $\{\{\cdot\}\}$ across E of any scalar\vector-valued

function $\psi : K^+ \cup K^- \to \mathbb{R}^d$ by

$$[\![\psi]\!] := \begin{cases} \vec{n}^+ \cdot \psi^+ + \vec{n}^- \cdot \psi^- & \text{if } d \geq 2 \\ \vec{n}^+ \psi^+ + \vec{n}^- \psi^- & \text{if } d = 1 \end{cases}, \quad \{\{\psi\}\} := \frac{\psi^+ + \psi^-}{2}.$$

As numerical flux on E, we then use the *upwind flux* given by

$$(\vec{n} \cdot av, w)_{L^2(E)} := \int_E (\{\{av\}\} + \frac{|\vec{n} \cdot a|}{2} [\![v]\!])[\![w]\!] dz, \quad v, w \in H, \tag{8.18}$$

and define the scalar product

$$(\vec{n} \cdot av, w)_{\mathcal{E}_h} := (\vec{n} \cdot av, w)_{L^2(\partial \mathbb{D}^-)} + \sum_{E \in \mathcal{E}_h, E \cap \partial \mathbb{D} = \emptyset} (\vec{n} \cdot av, w)_{L^2(E)}.$$

Summing over all K in Eq. (8.17) and using $v_h|_{\partial \mathbb{D}^+} = 0$, we obtain the problem to find a weak solution $X_h : \Omega \times \{t_0, \ldots, t_m\} \to V_h$ such that for all $v_h \in V_h$ and $i = 1, \ldots, m$

$$\begin{aligned} (X_h^{(i)} - X_h^{(i-1)}, v_h)_{H,h} + \Delta t B_h(X_h^{(i)}, v_h) = (F(t_{i-1}, X_h^{(i-1)}) \Delta t, v_h)_{H,h} \\ + (G(t_{i-1}, X_h^{(i-1)}) \Delta L^{(i)}, v_h)_{H,h}, \end{aligned} \tag{8.19}$$

where we have introduced the discrete bilinear form

$$B_h : H_h^{1/2+\varepsilon}(\mathbb{D}) \times H_h^1(\mathbb{D}) \to \mathbb{R}, \quad (v_h, w_h) \mapsto (v_h, a \cdot \nabla w_h)_{H,h} - (\vec{n} \cdot av_h, w_h)_{\mathcal{E}_h}. \tag{8.20}$$

The broken Sobolev spaces are given for any $q \geq 0$ by

$$H_h^q(\mathbb{D}) := \{v \in L^2(\mathbb{D}) \Big| \text{ for all } K \in \mathcal{K}_h \colon v|_K \in H^q(K)\},$$

hence $V_h \times V_h \subset H_h^{1/2+\varepsilon}(\mathbb{D}) \times H_h^1(\mathbb{D})$. Note that the use of the broken scalar product $(\cdot, \cdot)_{H,h}$ in Eq. (8.20) allows us apply B_h on $V_h \times V_h$, despite $V_h \not\subset H^1(\mathbb{D})$. The gradient w_h is considered piecewise on each simplex K via $\| \cdot \|_{H,h}$ in Eq. (8.16) and we account for the discontinuities on the interfaces by the upwind flux from Eq. (8.18). For trial and test functions in $V_h \cap H^1(\mathbb{D})$ the bilinear forms B_h and B correspond since the jump discontinuities vanish on each interior interface E, i.e.

$$B_h(v_h, w_h) = B(v_h, w_h), \quad w_h, v_h \in V_h \cap H^1(\mathbb{D}).$$

Since we consider a DG approximation, however, we have to employ the discrete bilinear form B_h for the numerical analysis in this chapter. For simplicity, we assume that

$X_h^{(0)} := P_h X_0$, where $P_h : H \to V_h$ denotes the orthogonal projection with respect to $(\cdot,\cdot)_{H,h}$ onto V_h. Using once more partial integration in Eq. (8.20) for the first term on the right hand side yields the alternative representation

$$\begin{aligned} B_h(v_h, w_h) &= -(a \cdot \nabla v_h, w_h)_{H,h} + \sum_{K \in \mathcal{K}_h} \int_{\partial K} \vec{n} \cdot a v_h w_h dz - (\vec{n} \cdot a v_h, w_h)_{\mathcal{E}_h} \\ &= -(a \cdot \nabla v_h, w_h)_{H,h} + \sum_{E \in \mathcal{E}_h} \int_E [\![a v_h]\!] \{\{ w_h \}\} + [\![a w_h]\!] \{\{ v_h \}\} dz \\ &\quad - (\vec{n} \cdot a v_h, w_h)_{\mathcal{E}_h} \\ &= -(a \cdot \nabla v_h, w_h)_{H,h} + \sum_{E \in \mathcal{E}_h, E \cap \partial\mathbb{D} = \emptyset} \int_E [\![a w_h]\!] \{\{ v_h \}\} \\ &\quad - \frac{|\vec{n} \cdot a|}{2} [\![v_h]\!] [\![w_h]\!] dz - (\vec{n} \cdot a v, w)_{L^2(\partial\mathbb{D}^-)}. \end{aligned}$$

Thus, with $\vec{n} \cdot a \leq 0$ on $\partial\mathbb{D}^-$, we obtain the discrete seminorm

$$\|v_h\|_{a,\mathcal{E}_h} := 2B_h(v_h, v_h) = \sum_{E \in \mathcal{E}_h} \int_E |\vec{n} \cdot a| [\![v_h]\!]^2 dz \geq 0.$$

Note that $\|v\|_{a,\partial\mathbb{D}^-} = \|v\|_{a,\mathcal{E}_h}$ if $v \in H$ is continuous across all interior faces in $\mathcal{E}_h$. We record an interpolation result in DG spaces and an inverse estimate for V_h as central tools for our error analysis. For a proof we refer to [109, Chapter 4] and the references therein.

Lemma 8.5.1. *Let $v \in H^q(K)$ for $q > 1/2$ and any $K \in \mathcal{K}_h$. Then, for some $C > 0$ independent of h and v it holds that*

$$\|v - P_h v\|_{H,h} \leq C |v|_{H^q(\mathbb{D}),h} \, h^{\min(2,q)}.$$

Moreover, there is some $C > 0$ independent of h and v_h such that for any $v_h \in V_h$

$$\|v_h\|_{L^2(\partial K)} + h^{1/2} \|v_h\|_{H^1(K)} \leq C h^{-1/2} \|v_h\|_{L^2(K)}.$$

Lemma 8.5.1 yields for $v \in H^q(\mathbb{D})$

$$\|v - P_h v\|_{\mathcal{E}_h,a} \leq 2 \sum_{K \in \mathcal{K}_h} \|v - P_h v\|_{L^2(\partial K)} \leq C |v|_{H^q(\mathbb{D}),h} \, h^{\min(2,q)-1/2}, \tag{8.21}$$

which already gives us an indication of the spatial convergence rate with respect to h.

Theorem 8.5.2. *Let Assumption 8.3.3 hold with $q \geq 1$, let $\mathcal{K}_h$ be an arbitrary (fixed) triangulation of $\mathbb{D}$ with meshwidth $h > 0$ and denote by X and $X_h^{(\cdot)}$ the solutions to*

Problem (8.5) *and Problem* (8.19)*, respectively. Then, for sufficiently small* Δt*,*

$$\mathbb{E}\Big(\|X(T) - X_h^{(m)}\|_{H,h}^2 + \Delta t \sum_{i=0}^{m} \|X(t_i) - X_h^{(i)}\|_{\mathcal{E}_h,a}^2\Big)^{1/2} \leq C(\Delta t^{1/2} + h^{q-1/2}).$$

The term $\Delta t \sum_{i=0}^m \|X(t_i) - X_h^{(i)}\|_{\mathcal{E}_h,a}^2$ above represents the additional stability that we achieve by the DG approach with upwind flux in contrast to continuous finite elements. In the latter case, the jump integrals over the interior edges in $\mathcal{E}_h$ vanish, leaving only the outflow boundary term $\|X(t_i) - X_h^{(i)}\|_{a,\partial\mathbb{D}^-}^2$ instead. Therefore, the DG method results in a stable approximation in contrast to a standard finite element method, even though X has continuous spatial paths under the assumption that $q > 1$ for $d = 1, 2$.

of Theorem 8.5.2. Let $i = 1 \ldots, m$ be fixed, define $\phi^{(i)} := X(t_i) - P_h X(t_i)$ and $\psi^{(i)} := P_h X(t_i) - X_h^{(i)}$. Lemma 8.5.1 and Ineq. (8.21) yield with Theorem 8.3.6

$$\mathbb{E}\Big(\|\phi^{(m)}\|_{H,h}^2 + \Delta t \sum_{i=0}^{m} \|\phi^{(i)}\|_{\mathcal{E}_h,a}^2\Big) \leq C \sup_{t\in\mathbb{T}} \mathbb{E}(|X(t)|_{H^q(\mathbb{D})}^2) h^{2q-1} \leq C h^{2q-1}. \tag{8.22}$$

Let $v_h \in V_h$ and consider a simplex $K \in \mathcal{K}_h$ such that $v_h|_K \in H^1(K)$. Since v_h is linear on K, there exists for any $\varepsilon > 0$ a smooth approximation $v_{\varepsilon,K} : \mathbb{R}^d \to \mathbb{R}$ of $v_h|_K$ with compact support $U_{\varepsilon,K} \subset \mathbb{R}^d$ such that $v_h = v_{K,\varepsilon}$ on K. Furthermore, the remaining area of $U_{\varepsilon,K}$ is bounded by $|U_{\varepsilon,K} \setminus K| \leq C\varepsilon$ (where $|\Theta|$ denotes the area of any $\Theta \subset \mathbb{R}^d$), and $v_{\varepsilon,K}$ converges to $v_h|_K$ almost everywhere on K as $\varepsilon \to 0$. For instance, $v_{\varepsilon,K}$ can be chose as the mollification of $v_h|_K$, see [80, Appendix C]. To localize the weak solution X to each simplex $K \in \mathcal{K}_h$, we test against $v_{\varepsilon,K}|_{\mathbb{D}} \in H^1(\mathbb{D})$ in Eq. (8.5), and take the limit $\varepsilon \to 0$ with respect to $\|\cdot\|_H$. As $v_{K,\varepsilon} \to v_h$ almost everywhere on K, and $v_{K,\varepsilon} \to 0$ almost everywhere on $\mathbb{R}^d \setminus K$ for $\varepsilon \to 0$, this yields for any $v_h \in V_h$ and $i = 1, \ldots, m$

$$\begin{aligned}
(X(t_i), v_h)_{L^2(K)} &= (X(t_{i-1}), v_h)_{L^2(K)} + \int_{t_{i-1}}^{t_i} (a \cdot \nabla X(s), v_h)_{L^2(K)} ds \\
&\quad + \int_{t_{i-1}}^{t_i} (F(s, X(s)), v_h)_{L^2(K)} ds + \Big(\int_{t_{i-1}}^{t_i} G(s, X(s)) dL(s), v_h\Big)_{L^2(K)} \\
&= (X(t_{i-1}), v_h)_{L^2(K)} - \int_{t_{i-1}}^{t_i} (X(s), a \cdot \nabla v_h)_{L^2(K)} ds \\
&\quad - \int_{t_{i-1}}^{t_i} (\vec{n} \cdot aX(s), v_h)_{L^2(\partial K)} ds \\
&\quad + \int_{t_{i-1}}^{t_i} (F(s, X(s)), v_h)_{L^2(K)} ds + \Big(\int_{t_{i-1}}^{t_i} G(s, X(s)) dL(s), v_h\Big)_{L^2(K)}.
\end{aligned}$$

Summing over all $K \in \mathcal{K}_h$ then shows conformity of the DG formulation, i.e., for all $v_H \in V_h$ we have

$$\begin{aligned}(X(t_i) - X(t_{i-1}), v_h)_{H,h} = &- \int_{t_{i-1}}^{t_i} B_h(X(s), v_h) ds + \int_{t_{i-1}}^{t_i} (F(s, X(s)), v_h)_{H,h} ds \\ &+ \Big(\int_{t_{i-1}}^{t_i} G(s, X(s)) dL(s), v_h \Big)_{H,h}.\end{aligned} \tag{8.23}$$

To estimate $\psi^{(i)}$, we combine Eqs. (8.19) and (8.23) to obtain

$$\begin{aligned}(\psi^{(i)} - \psi^{(i-1)}, v_h)_{H,h} = &-(\phi^{(i)} - \phi^{(i-1)}, v_h)_{H,h} \\ &- \int_{t_{i-1}}^{t_i} B_h(X(s) - X(t_i) + \phi^{(i)} + \psi^{(i)}, v_h) ds \\ &+ \int_{t_{i-1}}^{t_i} (F(s, X(s)) - F(t_{i-1}, X_h^{(i-1)}), v_h)_{H,h} ds \\ &+ \Big(\int_{t_{i-1}}^{t_i} (G(s, X(s)) - G(t_{i-1}, X_h^{(i-1)})) dL(s), v_h \Big)_{H,h}.\end{aligned}$$

Since $\phi^{(\cdot)} = (1 - P_h)X(\cdot)$, it holds that $(\phi^{(i)} - \phi^{(i-1)}, v_h)_{H,h} = 0$ for all $v_h \in V_h$. Testing against $v_h = \psi^{(i)} \in V_h$, taking expectations, and rearranging terms yields

$$\begin{aligned}&\frac{1}{2}\mathbb{E}\Big(\|\psi^{(i)}\|_{H,h}^2 - \|\psi^{(i-1)}\|_{H,h}^2 + \Delta t \|\psi^{(i)}\|_{a,\mathcal{E}_h}^2 \Big) \\ = &- \int_{t_{i-1}}^{t_i} \mathbb{E}(B_h(X(s) - X(t_i) + \phi^{(i)}, \psi^{(i)})) ds \\ &+ \int_{t_{i-1}}^{t_i} \mathbb{E}((F(s, X(s)) - F(t_{i-1}, X(t_{i-1})), \psi^{(i)})_{H,h}) ds \\ &\quad + \int_{t_{i-1}}^{t_i} \mathbb{E}((F(t_{i-1}, X(t_{i-1})) - F(t_{i-1}, X_h^{(i-1)}), \psi^{(i)})_{H,h}) ds \\ &+ \mathbb{E}\Big(\Big(\int_{t_1}^{t_2} G(s, X(s)) - G(t_{i-1}, X(t_{i-1})), dL(s), \psi^{(i)} \Big)_{H,h} \Big) \\ &\quad + \mathbb{E}\Big(\Big(\int_{t_1}^{t_2} G(t_{i-1}, X(t_{i-1})) - G(t_{i-1}, X_h^{(i-1)}) dL(s), \psi^{(i)} \Big)_{H,h} \Big) \\ &- \frac{1}{2}\mathbb{E}(\|\psi^{(i)} - \psi^{(i-1)}\|_{H,h}^2) \\ = &: I + II + III - \frac{1}{2}\mathbb{E}(\|\psi^{(i)} - \psi^{(i-1)}\|_{H,h}^2).\end{aligned}$$

We now bound the terms $I - III$ for each i, and then use a telescopic sum to estimate $\psi^{(m)}$, using that $\psi^{(0)} = 0$. To bound I, we use Young's inequality to together with

Theorem 8.3.7 and Ineq. (8.21)

$$\begin{aligned} I &\leq C\int_{t_{i-1}}^{t_i} \mathbb{E}(\|X(s)-X(t_i)\|^2_{a,\partial\mathbb{D}^-})ds + C\Delta t\mathbb{E}(\|\phi^{(i)}\|^2_{a,\mathcal{E}_h}) + \frac{\Delta t}{4}\mathbb{E}(\|\psi^{(i)}\|^2_{a,\mathcal{E}_h}) \\ &\leq C\Big(\Delta t^2 + \Delta t\mathbb{E}(|X(t_i)|^2_{H^q(\mathbb{D})})h^{2q-1}\Big) + \frac{\Delta t}{4}\mathbb{E}(\|\psi^{(i)}\|^2_{a,\mathcal{E}_h}) \\ &\leq C\Delta t(\Delta t + h^{2q-1}) + \frac{\Delta t}{4}\mathbb{E}(\|\psi^{(i)}\|^2_{a,\mathcal{E}_h}) \end{aligned}$$

We proceed in the same fashion as in the proof of Theorem 8.4.1 and bound the remaining terms by the triangle inequality and Assumption 8.3.3(iii) via

$$II \leq C\Delta t\Big(\Delta t + \mathbb{E}(\|X(t_{i-1}) - X_h^{(i-1)}\|^2_{H,h}) + \mathbb{E}(\|\psi^{(i)}\|^2_{H,h})\Big),$$

and

$$III \leq C\Delta t\Big(\Delta t + \mathbb{E}(\|X(t_{i-1}) - X_h^{(i-1)}\|^2_{H,h})\Big) + \frac{1}{2}\mathbb{E}(\|\psi^{(i)} - \psi^{(i-1)}\|^2_{H,h}).$$

Hence, substituting the estimates on $I - III$ and summing over i yields

$$\begin{aligned} \mathbb{E}\Big(\|\psi^{(m)}\|^2_{H,h} + \Delta t\sum_{i=0}^{m}\|\psi^{(i)}\|^2_{\mathcal{E}_h,a}\Big) \leq C\Big(\Delta t + h^{2q-1} + \Delta t\sum_{i=0}^{m-1}\mathbb{E}(\|X(t_i) - X_h^{(i)}\|^2_{H,h}) \\ + \Delta t\sum_{i=0}^{m}\|\psi^{(i)}\|^2_{H,h}\Big). \end{aligned}$$

By applying the discrete Grönwall inequality as in the proof of Theorem 8.4.1 we obtain for any $\Delta t \in (0, 1/C)$

$$\mathbb{E}\Big(\|\psi^{(m)}\|^2_{H,h} + \Delta t\sum_{i=0}^{m}\|\psi^{(m)}\|^2_{\mathcal{E}_h,a}\Big) \leq C\Big(\Delta t + h^{2q-1} + \Delta t\sum_{i=0}^{m-1}\mathbb{E}(\|X(t_i) - X_h^{(i)}\|^2_{H,h})\Big). \tag{8.24}$$

The claim follows by substituting the estimates (8.22) and (8.24) in the right hand side of

$$\begin{aligned} &\mathbb{E}\Big(\|X(T) - X_h^{(m)}\|^2_{H,h} + \Delta t\sum_{i=0}^{m}\|X(t_i) - X_h^{(i)}\|^2_{\mathcal{E}_h,a}\Big) \\ \leq &2\mathbb{E}\Big(\|\phi^{(m)}\|^2_{H,h} + \Delta t\sum_{i=0}^{m}\|\phi^{(i)}\|^2_{\mathcal{E}_h,a}\Big) + 2\mathbb{E}\Big(\|\psi^{(m)}\|^2_{H,h} + \Delta t\sum_{i=0}^{m}\|\psi^{(i)}\|^2_{\mathcal{E}_h,a}\Big), \end{aligned}$$

using that $X_h^{(0)} = P_h X_0$ with Lemma 8.5.1 and applying once more the discrete Grön-

wall inequality. □

Remark 8.5.3. To conclude this section we remark that the estimate of $\mathcal{O}(h^{q-1/2})$ in Theorem 8.5.2 corresponds to the rate of convergence for the linear DG method applied to a deterministic transport problem, first shown in [123] for $d = 2$. In fact, for a general triangulation of the domain, this is the best result one may achieve. In [60], however, the authors show that for deterministic transport problems a rate of $\mathcal{O}(h^k)$ is possible, provided the solution is in $H^k(\mathbb{D})$ with $k \in \mathbb{N}$ and the meshes satisfy certain conditions with respect to the flow vector a. If $d = 1$, these conditions are automatically fulfilled, hence, we expect to see mean-squared errors of order $\mathcal{O}(h^q)$ for $q \geq 1$, which is confirmed by our experiments in Section 8.7.

8.6 Noise approximation

After discretizing the temporal and spatial domain of Problem (8.5), it is in general necessary to derive a numerically tractable approximation of the infinite-dimensional driving noise L. For this, we will utilize a series representation of L and truncate the expansion after a finite number of terms. Since the covariance operator Q of L is symmetric and of trace class, L admits the *Karhunen-Loève expansion*

$$L(t) = \sum_{i \in \mathbb{N}} (L(t), e_i)_U e_i, \quad t \in \mathbb{T}. \tag{8.25}$$

The scalar products $(L(t), e_i)_H$ are one-dimensional *uncorrelated, but not independent*, Lévy processes with zero mean and variance η_i (see [173]). In general, infinitely many of the eigenvalues η_i will be strictly greater than zero, hence we truncate the series in Eq. (8.25) after $N \in \mathbb{N}$ terms to obtain the truncated Karhunen-Loève expansion

$$L_N(t) := \sum_{i=1}^{N} (L(t), e_i)_U e_i, \quad t \in \mathbb{T}.$$

It can be shown, see for example [30], that L_N converges to L in mean-square uniformly on $\mathbb{T}$ with the truncation error bounded by

$$\mathbb{E}(\|L_N(t) - L(t)\|_U^2) \leq T \sum_{i>N} \eta_i, \quad t \in \mathbb{T}.$$

When simulating L_N, it is vital to generate $(L(t), e_1), \ldots, (L(t), e_n)$ as uncorrelated, but stochastically dependent Lévy processes for fixed N. Besides the truncation, another bias may occur when sampling the one-dimensional processes $((L(t), e_i)_H, t \in \mathbb{T})$. For

$\eta_i > 0$, consider the normalized processes

$$\ell_i = (\ell_i(t), t \in \mathbb{T}) := \frac{((L(t), e_i)_U, t \in \mathbb{T})}{\sqrt{\eta_i}}, \tag{8.26}$$

with unit variance such that the identity

$$L_N(t) \stackrel{L}{=} \sum_{i=1}^{N} \sqrt{\eta_i} \ell_i(t) e_i$$

holds with respect to probability law of $L_N(t)$. For a general one-dimensional Lévy process ℓ_i, it is not possible to sample from the exact distribution of $\ell_i(t)$ for arbitrary $t \in \mathbb{T}$. There are a few important exceptions, for instance normal-inverse Gaussian (NIG) or variance Gamma (VG) processes (see [188]), in any other case, however, one is forced to use approximate simulation algorithms. The most popular technique is the compound Poisson approximation (CPA), see for instance [10, 74, 84, 186], which usually guarantees weak convergence. A drawback of the CPA methods is that it requires rather strong assumptions on the one-dimensional Lévy processes ℓ_i to bound the approximation error in a mean-square sense and is difficult to implement. Another approach is to use the Fourier inversion (FI) technique introduced in [30], which ensures error control in a $L^p(\Omega; \mathbb{R})$-sense under relatively weak assumptions on ℓ_i. With the FI method, we are able to approximate very general types of Lévy noise and control the mean-squared error, for instance if L stems from the important class of *generalized hyperbolic* (GH) Lévy processes introduced in [18, 19]. To allow for arbitrary approximation techniques, we formulate the following assumption.

Assumption 8.6.1. Let $\widetilde{\ell}_i$ be arbitrary approximations of ℓ_i (based on CPA, FI, etc.) such that the processes $(\widetilde{\ell}_i, i \in \mathbb{N})$ are jointly uncorrelated, but stochastically dependent, and let

$$\widetilde{L}_N(t) := \sum_{i=1}^{N} \sqrt{\eta_i} \widetilde{\ell}_i(t) e_i$$

be the approximated U-valued Lévy field. There is a constant $\varepsilon_L > 0$ such that for all $i \in \mathbb{N}$ and $t \in \mathbb{T}$

$$\mathbb{E}(|\widetilde{\ell}_i(t) - \ell_i(t)|^2) \leq \varepsilon_L.$$

Remark 8.6.2. Assumption 8.6.1 yields that the overall noise approximation error is bounded by

$$\mathbb{E}(\|L(t) - \widetilde{L}_N(t)\|_U^2) \leq T \sum_{i>N} \eta_i + \varepsilon_L \sum_{i=1}^{N} \eta_i, \quad t \in \mathbb{T}, \tag{8.27}$$

hence we have a separation between the truncation error with respect to N and the

simulation bias ε_L. Often, an arbitrary small error ε_L may be achieved with sufficient computational effort and it is possible to reduce the noise approximation error in Eq. (8.27) to any desired amount by increasing the number of terms in the expansion and decreasing ε_L. This is for instance the case for GH Lévy fields approximated by FI as in [30]. Moreover, we are able to achieve an equilibration between both types of errors in the sense that

$$\mathbb{E}(\|L(t) - L_N(t)\|_U^2) = T \sum_{i>N} \eta_i \approx \varepsilon_L \sum_{i=1}^{N} \eta_i = \mathbb{E}(\|L_N(t) - \widetilde{L}_N(t)\|_U^2)$$

Substituting L by $\widetilde{L}_N$ in Eq. (8.19) yields the problem to find $(\widetilde{X}_{h,N}^{(i)}, i = 1, \dots, m) \subset V_h$ such that for all $v_h \in V_h$ and $i = 1, \dots, n$ it holds

$$\begin{aligned}(\widetilde{X}_{h,N}^{(i)} - \widetilde{X}_{h,N}^{(i-1)}, v_h)_{H,h} + \Delta t B_h((\widetilde{X}_{h,N}^{(i)}, v_h) &= (F(t_{i-1}, \widetilde{X}_{h,N}^{(i-1)})\Delta t, v_h)_{H,h} \\ &\quad + (G(t_{i-1}, \widetilde{X}_{h,N}^{(i-1)})\Delta \widetilde{L}_N^{(i)}, v_h)_{H,h},\end{aligned} \tag{8.28}$$

where $X_{h,N}^{(0)} := P_h X_0$ and $\Delta \widetilde{L}_N^{(i)} := \widetilde{L}_N(t_i) - \widetilde{L}_N(t_{i-1})$. To complete the error analysis, we derive the overall approximation error between $\widetilde{X}_{h,N}^{(\cdot)}$ and the unbiased weak solution X to Eq. (8.5).

Theorem 8.6.3. *Under Assumption 8.3.3 with $q \geq 1$ and Assumption 8.6.1, it holds for sufficiently small Δt that*

$$\mathbb{E}(\|\widetilde{X}_{h,N}^{(m)} - X(T)\|_{H,h}^2)^{1/2} \leq C\Big(\Delta t^{1/2} + h^{q-1/2} + \Big(\sum_{i>N} \eta_i\Big)^{1/2} + \Big(\varepsilon_L \sum_{i=1}^{N} \eta_i\Big)^{1/2}\Big).$$

Proof. We start by estimating the error $\mathbb{E}(\|\widetilde{X}_{h,N}^{(m)} - X_h^{(m)}\|_H^2)$ and define $\widetilde{\psi}_N^{(i)} := \widetilde{X}_{h,N}^{(i)} - X_h^{(i)} \in V_h$. Testing against $\widetilde{\psi}_N^{(i)}$ in Eqs. (8.19) and (8.28), we obtain

$$\begin{aligned}&\mathbb{E}(\|\widetilde{\psi}_N^{(i)}\|_{H,h}^2 - \|\widetilde{\psi}_N^{(i-1)}\|_{H,h}^2 + \Delta t \|\widetilde{\psi}_N^{(i)}\|_{\mathcal{E}_h,a}^2) \\ &= 2\Delta t \mathbb{E}\Big((F(t_{i-1}, X_h^{(i-1)}) - F(t_{i-1}, \widetilde{X}_{h,N}^{(i-1)}), \widetilde{\psi}_N^{(i)})_{H,h}\Big) \\ &\quad + 2\mathbb{E}\Big((G(t_{i-1}, X_h^{(i-1)})\Delta L_i - G(t_{i-1}, \widetilde{X}_{h,N}^{(i-1)})\Delta \widetilde{L}_N^{(i)}, \widetilde{\psi}_N^{(i)})_{H,h}\Big) \\ &\quad - \mathbb{E}(\|\widetilde{\psi}_N^{(i)} - \widetilde{\psi}_N^{(i-1)}\|_{H,h}^2) \\ &=: 2(I + II) - \mathbb{E}(\|\widetilde{\psi}_N^{(i)} - \widetilde{\psi}_N^{(i-1)}\|_{H,h}^2).\end{aligned}$$

We use Assumption 8.3.3(iii) and Young's inequality to bound the first term by

$$I \leq C\Delta t \mathbb{E}\Big(\|X_h^{(i-1)} - \widetilde{X}_{h,N}^{(i-1)}\|_{H,h} \|\widetilde{\psi}_N^{(i)}\|_{H,h}\Big) \leq C\Delta t \mathbb{E}(\|\widetilde{\psi}_N^{(i-1)}\|_H^2 + \|\widetilde{\psi}_N^{(i)}\|_H^2).$$

Similar to Theorem 8.4.1, we use for II that $\Delta L^{(i)}$ and $\Delta\widetilde{L}_N^{(i)}$ are independent of $\widetilde{\psi}_N^{(i)}$ and that

$$\mathbb{E}\Big(G(t_{i-1},X_h^{(i-1)})(\Delta L_i-\Delta\widetilde{L}_N^{(i)})\Big)=\mathbb{E}\Big((G(t_{i-1},X_h^{(i-1)})-G(t_{i-1},\widetilde{X}_{h,N}^{(i-1)}))\Delta\widetilde{L}_N^{(i)}\Big)=0$$

to obtain

$$\begin{aligned}
II&=\mathbb{E}\Big((G(t_{i-1},X_h^{(i-1)})(\Delta L_i-\Delta\widetilde{L}_N^{(i)}),\widetilde{\psi}_N^{(i)})_{H,h}\Big)\\
&\quad+\mathbb{E}\Big(((G(t_{i-1},X_h^{(i-1)})-G(t_{i-1},\widetilde{X}_{h,N}^{(i-1)}))\Delta\widetilde{L}_N^{(i)},\widetilde{\psi}_N^{(i)})_{H,h}\Big)\\
&=\mathbb{E}\Big((G(t_{i-1},X_h^{(i-1)})(\Delta L_i-\Delta\widetilde{L}_N^{(i)}),\widetilde{\psi}_N^{(i)}-\widetilde{\psi}_N^{(i-1)})_{H,h}\Big)\\
&\quad+\mathbb{E}\Big(((G(t_{i-1},X_h^{(i-1)})-G(t_{i-1},\widetilde{X}_{h,N}^{(i-1)}))\Delta\widetilde{L}_N^{(i)},\widetilde{\psi}_N^{(i)}-\widetilde{\psi}_N^{(i-1)})_{H,h}\Big)\\
&\le\mathbb{E}\Big(\|G(t_{i-1},X_h^{(i-1)})\|^2_{L_{HS}(\mathcal{U},H)}\|\Delta L_i-\Delta\widetilde{L}_N^{(i)}\|_U^2\Big)+\frac{1}{4}\mathbb{E}(\|\widetilde{\psi}_N^{(i)}-\widetilde{\psi}_N^{(i-1)}\|_{H,h}\Big)\\
&\quad+\mathbb{E}\Big(\|G(t_{i-1},X_h^{(i-1)})-G(t_{i-1},\widetilde{X}_{N,h}^{(i-1)})\|^2_{L_{HS}(\mathcal{U},H)}\|\Delta\widetilde{L}_N^{(i)}\|_U^2\Big)\\
&\quad+\frac{1}{4}\mathbb{E}(\|\widetilde{\psi}_N^{(i)}-\widetilde{\psi}_N^{(i-1)}\|_{H,h}\Big).
\end{aligned}$$

Then, the independence of $X_h^{(i-1)}$ from $\Delta L^{(i)}$ and $\Delta\widetilde{L}_N^{(i)}$ yields together with Assumption 8.3.3(iii), Ineq. (8.27) and Theorem 8.2.7

$$\begin{aligned}
II&\le\mathbb{E}(\|G(t_{i-1},X_h^{(i-1)})\|^2_{L_{HS}(\mathcal{U},H)})\mathbb{E}(\|\Delta L_i-\Delta\widetilde{L}_N^{(i)}\|_U^2)\\
&\quad+\mathbb{E}(\|G(t_{i-1},X_h^{(i-1)})-G(t_{i-1},\widetilde{X}_{N,h}^{(i-1)})\|^2_{L_{HS}(\mathcal{U},H)})\mathbb{E}(\|\Delta\widetilde{L}_N^{(i)}\|_U^2)\\
&\quad+\frac{1}{2}\mathbb{E}(\|\widetilde{\psi}_N^{(i)}-\widetilde{\psi}_N^{(i-1)}\|_{H,h})\\
&\le C(1+\sup_{i=1,\dots,m}\mathbb{E}(\|X_h^{(i-1)}\|_H^2))\Delta t(\sum_{i>N}\eta_i+\varepsilon_L\sum_{i=1}^N\eta_i)+\Delta tC\mathbb{E}(\|\widetilde{\psi}_N^{(i-1)}\|^2_{H,h})\\
&\quad+\frac{1}{2}\mathbb{E}(\|\widetilde{\psi}_N^{(i)}-\widetilde{\psi}_N^{(i-1)}\|_{H,h})\\
&\le C\Delta t(\sum_{i>N}\eta_i+\varepsilon_L\sum_{i=1}^N\eta_i)+\Delta tC\mathbb{E}(\|\widetilde{\psi}_N^{(i-1)}\|^2_{H,h})+\frac{1}{2}\mathbb{E}(\|\widetilde{\psi}_N^{(i)}-\widetilde{\psi}_N^{(i-1)}\|_{H,h}).
\end{aligned}$$

Summing over i and the discrete Grönwall inequality then give the final estimate

$$\mathbb{E}(\|\widetilde{X}_{h,N}^{(m)}-X_h^{(m)}\|_H^2)\le C\Big(\sum_{i>N}\eta_i+\varepsilon_L\sum_{i=1}^N\eta_i\Big),$$

and the claim follows with the triangle inequality and Theorem 8.5.2. □

8.7 Numerical results

For our numerical examples we consider the spatial domain $\mathbb{D} = (0,1)$ with time interval $\mathbb{T} = [0,1]$, take $U = L^2((0,1))$ and let Q be given by the Matérn covariance operator from Example 8.2.3

$$[Q\phi](x) := \int_0^1 \frac{2^{1-\nu}}{\Gamma(\nu)} \Big(\sqrt{2\nu}\frac{|x-y|}{\rho}\Big)^{\nu} K_\nu\Big(\sqrt{2\nu}\frac{|x-y|}{\rho}\Big)\phi(y)dy, \quad \phi \in U,\ x \in (0,1).$$

We fix the correlation length to $\rho = 1/8$ and vary the smoothness parameter $\nu > 0$ throughout our experiments. The eigenpairs $((\eta_i, e_i), i \in \mathbb{N})$ of Q may be approximated by solving a discrete eigenvalue problem and interpolating, see [178, Chapter 4.3]. Remark 8.3.4, that relates the spectral basis of the Matérn kernel to Assumption 8.3.3, shows that $\nu > 1$ is required to achieve $q = 1$, which is in turn necessary to apply the error estimates from Sections 8.4 – 8.6. As supplement to the theoretical framework that holds for $\nu > 1$, we will also investigate cases where $\nu \leq 1$. We consider GH Lévy fields, i.e. the one-dimensional processes $(\ell_i, i \in \mathbb{N})$ from Eq. 8.26 are uncorrelated GH Lévy processes. More importantly, for each $N \in \mathbb{N}$ the vector-valued process $(GH_N(t), t \in \mathbb{T}) := ((\ell_1(t), \dots, \ell_N(t)), t \in \mathbb{T})$ is a N-dimensional GH Lévy process with parameters $\widehat{\lambda} \in \mathbb{R}, \widehat{\alpha} > 0, \widehat{\delta} > 0, \widehat{\beta} \in \mathbb{R}^n, \widehat{\mu} \in \mathbb{R}^N$ and $\widehat{\Gamma} \in \mathbb{R}^{N\times N}$, where $\widehat{\alpha}^2 > \widehat{\beta} \cdot \widehat{\Gamma}\widehat{\beta}$ and the matrix $\widehat{\Gamma}$ is symmetric, positive definite with unit variance. The characteristic function of GH_N is given for $u \in \mathbb{R}^N$ by

$$\mathbb{E}(e^{iu\cdot GH(t)}) = e^{iu\cdot\widehat{\mu}t}\Big(\frac{\widehat{\alpha}^2 - \widehat{\beta}\cdot\widehat{\Gamma}\widehat{\beta}}{\widehat{\alpha}^2 - (iu+\widehat{\beta})\cdot\widehat{\Gamma}(iu+\beta)}\Big)^{\widehat{\lambda}t/2} \frac{K_{\widehat{\lambda}}(\widehat{\delta}(\widehat{\alpha}^2 - (iu+\widehat{\beta})\cdot\widehat{\Gamma}(iu+\widehat{\beta}))^{1/2})^t}{K_{\widehat{\lambda}}(\widehat{\delta}(\widehat{\alpha}^2 - \widehat{\beta}\cdot\widehat{\Gamma}\widehat{\beta})^{1/2})^t}.$$

We achieve a zero-mean process by setting $\widehat{\beta} = \widehat{\mu} = (0, \dots, 0)$. An important class of the GH family are *normal inverse Gaussian* (NIG) processes, where $\widehat{\lambda} = -1/2$. For more details on multidimensional GH distributions and the simulation of GH Lévy fields we refer again to [30] and the references therein. In all subsequent experiments, we use a NIG Lévy field with $\widehat{\alpha} = 10$, $\widehat{\delta} = 1$, $\widehat{\beta} = \widehat{\mu} = (0, \dots, 0)$ and $\widehat{\Gamma} = \mathbf{1}_N$ for each truncation index N. The choice of NIG fields is motivated by the results from [4], where the authors pointed out that this class of Lévy fields is well-suited to fit empirical log-returns in electricity forward markets. We are able to simulate multidimensional NIG processes without bias, i.e. Assumption 8.6.1 holds with $\varepsilon_L = 0$. To measure the error in each example, we generate a reference solution X_{ref} on a very fine spatio-temporal resolution and with sufficiently high cutoff index N, and then sample approximations $\widetilde{X}^{(m)}_{h,N}$ based on the same path of L. The overall root-mean-squared

error (RMSE) from Theorem 8.6.3 is estimated by averaging 500 independent samples of $X_{ref}(T) - \widetilde{X}_{h,N}^{(m)}$, i.e.,

$$\mathbb{E}(\|X(T) - \widetilde{X}_{h,N}^{(m)}\|_H^2) \approx \frac{1}{500} \sum_{k=1}^{500} \|(X_{ref}(T) - \widetilde{X}_{h,N}^{(m)})_k\|_H^2,$$

where the subscript k denotes the k-th Monte Carlo sample. The actual approximation parameters $\Delta t, h$ and N for $\widetilde{X}_{h,N}^{(m)}$ and X_{ref} vary for each example and are given below.

As our first experiment, we consider the stochastic transport problem

$$dX(t,x) = \frac{1}{2}\partial_x X(t,x)dt + \frac{1}{10}\Big(\frac{X(t,x)^3}{X(t,x)^2+1} + (1-x)\Big)dL(t,x), \quad x \in \mathbb{D},\ t \in \mathbb{T}, \tag{8.29}$$

with $X_0(x) := x(1-x)$ and homogeneous inflow boundary conditions $X(t,1) = 0$. This SPDE has a genuine nonlinearity in the diffusion coefficient, given by

$$[G(t, X(t))\varphi](x) = \frac{1}{10}\Big(\frac{X(t,x)^3}{X(t,x)^2+1} + (1-x)\Big)\varphi(x), \quad \varphi \in \mathcal{U}.$$

Note that $G(t,\cdot)$ is Lipschitz in $L_{HS}(\mathcal{U}; H)$, where the Lipschitz constant is bounded uniformly in $\mathbb{T}$. Let $\widetilde{X_0}$ be the zero-extension of X_0 to $\mathbb{R}$ and observe that $\widetilde{X_0} \in H^{3/2-\varepsilon}(\mathbb{R})$ for any $\varepsilon > 0$. Moreover, for any $q \in (1/2, 3/2)$ and $v \in H^q_{0,+}$ it holds that $G(t,v) \in H^q_{0,+}$. The linear growth bound in Assumption 8.3.3(iv) on G for $q \in (1/2, 3/2)$ is derived in Appendix 8.8 for the reader's convenience. Hence, Assumption 8.3.3 is satisfied with $q < \nu$ for any given $\nu \in (1/2, 3/2)$, see Remark 8.3.4. Thus, spatial regularity of $q \in (1/2, \nu)$ as in Theorem 8.3.6 is ensured. Since the noise is multiplicative (G depends on X), we couple the DG spatial discretization with a BE time stepping scheme. We use $\nu \in \{0.5, 0.75, 1, 1.1, 1.2\}$ to investigate also cases in which Assumption 8.3.3 with $q \geq 1$ is violated. A sample of the driving noise and the corresponding approximation of X for $\nu = 1.1$ is given in the top row of Figure 8.1. In the lower left plot in Figure 8.1, we clearly see the discontinuities in time if we plot X at the outflow boundary, in contrast to the (spatially) continuous paths of $X(T,\cdot) : \mathbb{D} \to \mathbb{R}$. Remark 8.5.3 suggests that the overall discretization error satisfies

$$\mathbb{E}(\|\widetilde{X}_{h,N}^{(m)} - X(T)\|_{H,h}^2)^{1/2} \leq C\Big(\Delta t^{1/2} + h^q + \sum_{i>N} \eta_i\Big)$$

since we gain additional convergence of order $h^{1/2}$ as $d = 1$. We equilibrate all errors

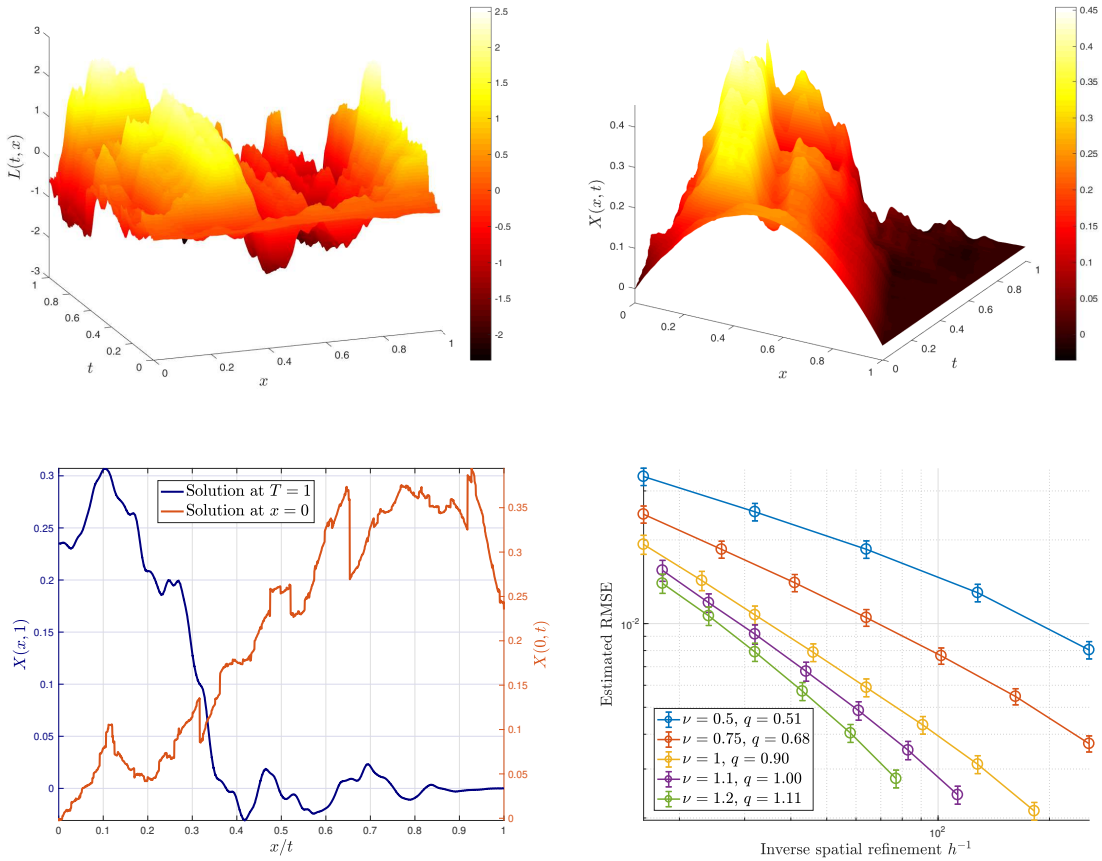

Figure 8.1 First numerical example (sine function): driving noise with $\nu = 1.1$ (top left) and the corresponding BE-DG-solution to Eq. (8.29) on $[0,1]^2$ (top right). The path of solution at T resp. $\partial\mathbb{D}^-$ is given at the bottom left and the RMSE vs. the inverse spatial refinement h^{-1} as well as the estimated convergence rates q are depicted on the bottom right. In the bottom right plot, the bars on each RMSE curve indicate the 95-% confidence interval of the estimated error.

for given Δt using $q \approx \nu$ and by choosing h and N such that

$$\Delta t^{1/2} = \Big(\sum_{i>N} \eta_i \Big)^{1/2} = h^{\nu}. \tag{8.30}$$

The time steps are of size

$$\Delta t = 2^{-m}, \quad m = \lceil 2\log_2(16)\nu \rceil, \ldots, min(\lfloor 16\nu \rfloor, 15),$$

and we use $\Delta t_{ref} = 2^{-min(\lfloor 20\nu \rfloor, 18)}$ with N_{ref}, h_{ref} according to Eq. (8.30) for the reference solution. To measure the actual rate of convergence q, we observe that with

Eq. (8.30)

$$\log\left(\mathbb{E}(\|X(T) - \widetilde{X}_{h,N}^{(m)}\|_{H,h}^2)^{1/2}\right) \approx \log(Ch^q) = q\log(h) + \log(C).$$

Thus, we estimate q by a linear regression with response variable $\log(\mathbb{E}(\|X(T) - \widetilde{X}_{h,N}^{(m)}\|_{H,h}^2)^{1/2})$ and $\log(h)$ as regressor, the results are depicted in Figure 8.1 (bottom right). First of all, for $\nu \in \{1, 1.1, 1.2\}$, we observe converge rates of $q = \nu - \epsilon$ with $\epsilon \approx 0.1$, which confirms our theoretical findings from Section 8.5 as well as Remark 8.5.3. More surprisingly, we also observe convergence rates close to $q \approx \nu$ for $\nu \in \{0.5, 0.75\}$, meaning that the BE Euler scheme also seems to converge with rate $\Delta t^{1/2}$ in practice if $q < 1$. As G is multiplicative, we cannot expect better temporal convergence by using CN times stepping instead of BE, but only if we used a higher order approximation of the stochastic integral, for instance a Milstein scheme as in [26].

To investigate an example with additive noise, we use the energy forward model from [24] given by

$$dX(t,x) = \partial_x X(t,x) + e^{-2\alpha x}\sigma(t,x)^2 dt + e^{-\alpha x}\sigma(t,x)dL(t,x), \quad x \in \mathbb{D},\, t \in \mathbb{T}. \quad (8.31)$$

We set $\alpha = 2$, choose the volatily function

$$\sigma : \mathbb{T} \times \overline{\mathbb{D}} \to \mathbb{R}_+^0, \quad (x,t) \mapsto (1-x)(x+t),$$

and matching initial/inflow boundary conditions given by

$$X_0(x) = e^{-\alpha x} + \frac{\sigma(0,x)^2 K_0(\widehat{\alpha})}{\alpha\pi}(1 - e^{-\alpha x}), \quad X(t,1) = e^{-\alpha}.$$

The coefficients $F(t,X) := e^{-2\alpha\cdot}\sigma(\cdot,t)^2$ and $G(t,X) := e^{-\alpha\cdot}\sigma(\cdot,t)$ in Eq. (8.31) are independent of X and, moreover, globally Lipschitz continuous with respect to $\mathbb{T}$. If $\widetilde{\sigma}$ denotes the zero-extension of $\sigma(t,\cdot)$ on $(0,\infty)$ for some $t \in \mathbb{T}$, then $\widetilde{\sigma} \in H^{3/2-\varepsilon}((0,+\infty))$ for any $\varepsilon > 0$ and $\widetilde{\sigma}, \partial_x\widetilde{\sigma} \in L^\infty((0,+\infty))$. Therefore, it holds that $F(t,v), G(t,v)e_i \in H_{0,+}^q$ for any $q \in (1/2, 3/2)$ and $v \in H_{0,+}^q(\mathbb{D})$. In addition, $e^{-\alpha\cdot}\sigma(\cdot,t) \in C^\infty(\overline{\mathbb{D}})$ and $\|e^{-\alpha\cdot}\sigma(\cdot,t)\|_{C^\infty(\mathbb{D})}$ is bounded uniformly in $\mathbb{T}$. Hence, Assumption 8.3.3 is satisfied for any $q \in (1/2, \min(\nu, 3/2))$. Samples of the driving noise and the approximated solution for $\nu = 1$ and $\nu = 2$ are given in Figure 8.2.

We use the CN scheme as in Remark 8.4.2 in order to achieve a convergence rate of $\mathcal{O}(\Delta t^{\gamma(q)})$ with $\gamma(q) > 1/2$ for varying $\nu \in \{0.5, 0.75, \ldots, 4\}$, time steps $\Delta t = 2^{-4}, \ldots, 2^{-9}$ and $\Delta t_{ref} = 2^{-12}$. As we cannot (yet) adjust the DG discretization to Δt

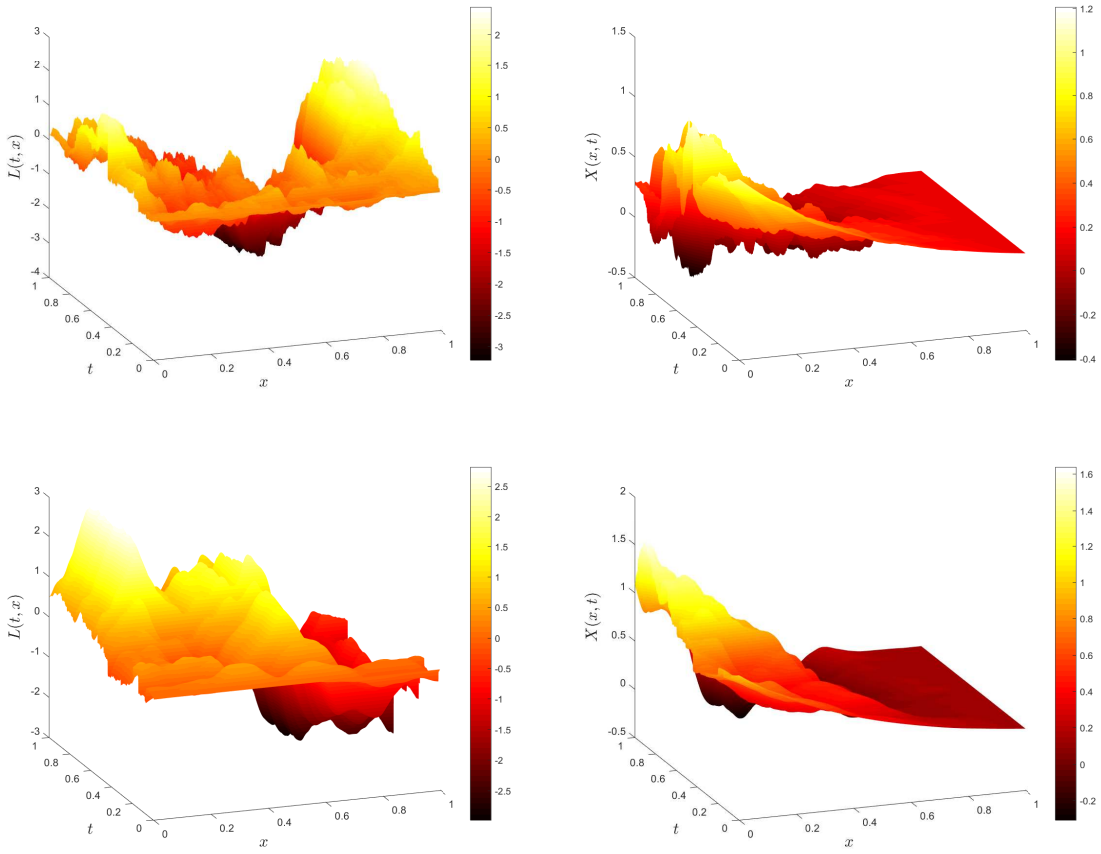

Figure 8.2 Second numerical example (forward model): the driving noises (left column) and CN-DG-solutions (right column) to Eq. (8.31) with $\nu = 1$ (top) and $\nu = 2$ (bottom).

to obtain equilibrated errors, we use a fixed spatial grid with $h = h_{ref} = 2^{-11}$ in every scenario. Furthermore, the maximal rate of convergence is expected to be $\gamma(q) \approx 1$, thus we choose the truncation index N such that $\sum_{i>N} \eta_i \leq \Delta t^2$ for each Δt and ν. The decay rate γ is estimated by linear regression of the RMSE against Δt as in the first example and we plot the RMSE vs. the inverse temporal refinement Δt^{-1} as well as the estimated γ vs. ν in Figure 8.3

We see that $\gamma(q)$ seems to be growing linearly in q with roughly $\gamma(q) \approx 2/3q$, where $q := \min(\nu, 3/2) - \varepsilon$ for arbitrary small $\varepsilon > 0$. This confirms our expectations from Remark 8.4.2, where we stated that γ should be increasing in q, and reach the maximum of one for any $\nu > \frac{3}{2}$. This observation allows us to equilibrate again all error terms and we repeat the above experiment with $\Delta t = 2^{-7}, \ldots, 2^{-12}, \Delta t_{ref} = 2^{-14}$ and by setting h and N such that

$$\Delta t^{2q/3} = h^q = (\sum_{i>N} \eta_i)^{1/2} \Rightarrow \quad h = \Delta t^{2/3}, \ \sum_{i>N} \eta_i = \Delta t^{4q/3}.$$

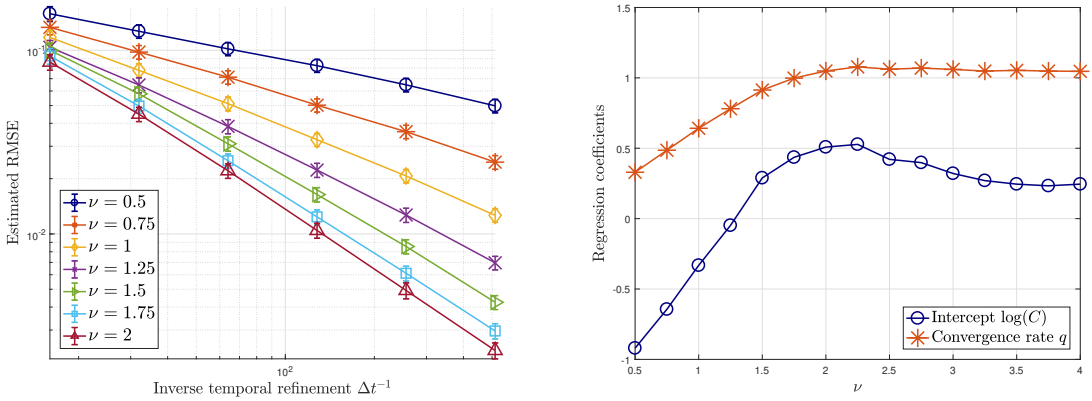

Figure 8.3 Second numerical example (forward model): RMSE vs. inverse temporal refinement Δt^{-1} (left) and estimated convergence rates of CN time stepping scheme for Eq. (8.31) (right). In the left plot, the bars on each RMSE curve indicate the 95-% confidence interval of the estimated error. The results have been generated with a fixed spatial discretization of $h = 2^{-11}$ and dimension truncation error $\sum_{i>N} \eta_i \leq \Delta t^2$

We estimate q by regression of the RMSE on h and plot the results for every ν in Figure 8.4.

Again, the error decay with respect to h is of order $q \approx min(\nu, 3/2)$ as in the first example with multiplicative noise. Hence, for problems with additive noise it is advantageous to use a CN-DG discretization scheme to reduce computational complexity. If, for instance, $q = 1$ and we want to achieve an RMSE of magnitude ε, we need to employ a DG grid with roughly ε^{-2d} nodal points and choose N such that $\sum_{i>N} \eta_i = \mathcal{O}(\varepsilon^2)$, regardless of the time stepping scheme. In the BE method, however, the number of timesteps needs to be of order ε^{-2}, while we require only $\varepsilon^{-3/2}$ time steps in the CN method. In addition, this number can be further reduced if we have higher spatial regularity of $q > 1$, whereas we are bound to order $\Delta t^{1/2}$ for any $q \geq 1$ in the BE scheme.

8.8 Appendix

In this appendix, we derive the linear growth bound on G as in Assumption 8.3.3(iv) for the first numerical example in Section 8.7. To this end, let $i \in \mathbb{N}$ and e_i be any eigenfunction of Q. Define the rational function $g(x) := \frac{x^3}{x^2+1}$ for $x \in \mathbb{R}$, and observe that the derivative $g'(x) = \frac{x^4+3x^2}{(x^2+1)^2}$ is bounded on $\mathbb{R}$. First, we consider the case $q \in (1/2, 1)$. As $d = 1$, we have the Sobolev embedding $H^q(\mathbb{D}) \hookrightarrow C^{0,q-1/2}(\overline{\mathbb{D}})$ (see,

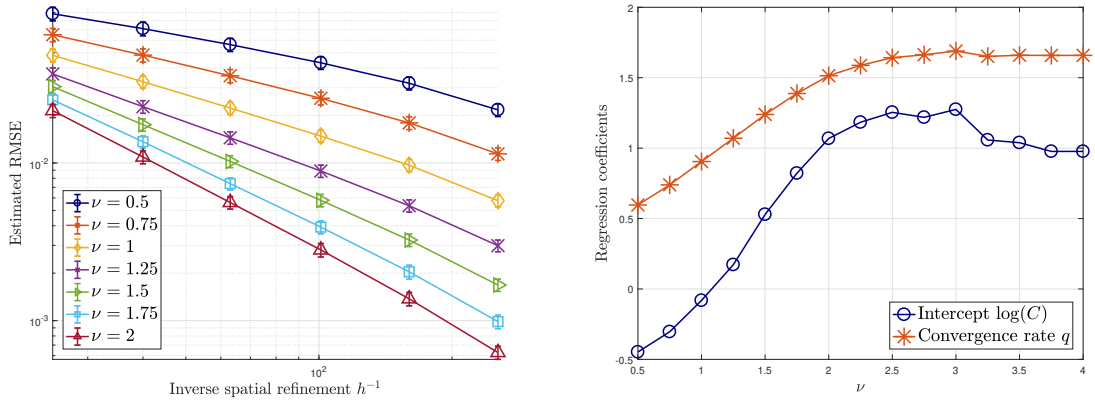

Figure 8.4 Second numerical example (forward model): RMSE vs. inverse spatial refinement Δt^{-1} (left) and estimated convergence rates of the CN-DG discretization for Eq. (8.31) (right). In the left plot, the bars on each RMSE curve indicate the 95-% confidence interval of the estimated error. The results have been generated with adjusted spatio temporal grid $h = \Delta t^{2/3}$ and dimension truncation error $\sum_{i>N} \eta_i \leq \Delta t^{4q/3}$.

e.g., Theorem 8.2 in [71]), which yields with the triangle inequality

$$\begin{aligned}
&\|G(t, X(t))e_i\|^2_{H^q(\mathbb{D})} \\
\leq& C \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{([G(t, X(t))e_i](x) - [G(t, X(t))e_i](y))^2}{|x-y|^{d+2q}} dxdy \\
\leq& C \Big(\|g(X(t,\cdot)) + (1-\cdot)\|^2_{H^q(\mathbb{D})} \|e_i\|^2_{L^\infty(\mathbb{D})} + \|g(X(t,\cdot)) + (1-\cdot)\|^2_{L^\infty(\mathbb{D})} \|e_i\|^2_{H^q(\mathbb{D})} \Big) \\
\leq& C \Big(1 + \|g(X(t,\cdot))\|^2_{H^q(\mathbb{D})} + \|X(t,\cdot)\|^2_{L^\infty(\mathbb{D})} \Big) \|e_i\|^2_{H^q(\mathbb{D})}.
\end{aligned}$$

In the last estimate, we have used that $|g(x)| \leq x$ for any $x \in \mathbb{R}$, $\|\cdot\|_{L^\infty(\mathbb{D})} = \|\cdot\|_{C^0(\overline{\mathbb{D}})}$ on $C^0(\overline{\mathbb{D}})$ and $H^q(\mathbb{D}) \hookrightarrow C^{0,q-1/2}(\overline{\mathbb{D}})$. As $|g'(x)| \leq C$ with $C > 0$ independent of x, the the mean-value theorem yields

$$\|g(X(t,\cdot))\|^2_{H^q(\mathbb{D})} = \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|g(X(t,x)) - g(X(t,y))|^2}{|x-y|^{d+2q}} dxdy \leq C \|X(t,\cdot)\|^2_{H^q(\mathbb{D})},$$

which shows that for $q \in (1/2, 1)$

$$\|G(t, X(t))e_i\|_{H^q(\mathbb{D})} \leq C(1 + \|X(t,\cdot)\|_{H^q(\mathbb{D})}) \|e_i\|^2_{H^q(\mathbb{D})}.$$

To treat the case $q = 1$, we use the product rule to take the derivative of $G(t, X(t))e_i$:

$$\begin{aligned}\partial_x\Big(G(t,X(t))e_i\Big) &= \partial_x\Big((g(X(t,\cdot)) + (1-\cdot))e_i\Big)\\ &= (g(X(t,\cdot)) + (1-\cdot))\partial_x e_i + (\partial_x X(t,\cdot)g'(X(t,\cdot)) - 1)e_i.\end{aligned} \tag{8.32}$$

Hölder's inequality then yields

$$\begin{aligned}&\|G(t,X(t))e_i\|_{H^1(\mathbb{D})}\\ \leq& C\Big(\|g(X(t,\cdot)) + (1-\cdot)\|_{L^\infty(\mathbb{D})}\|e_i\|_{H^1(\mathbb{D})} + \|g(X(t,\cdot)) + (1-\cdot)\|_{H^1(\mathbb{D})}\|e_i\|_{L^\infty(\mathbb{D})}\Big)\\ \leq& C\Big((1 + \|X(t,\cdot)\|_{L^\infty(\mathbb{D})})\|e_i\|_{H^1(\mathbb{D})}\\ &\quad + (1 + \|X(t,\cdot)\|_H + \|\partial_x X(t,\cdot)g'(X(t,\cdot))\|_H)\|e_i\|_{L^\infty(\mathbb{D})}\Big)\\ \leq& C(1 + \|X(t,\cdot)\|_{H^1(\mathbb{D})})\|e_i\|_{H^1(\mathbb{D})}.\end{aligned}$$

The last estimate holds again since $\|\cdot\|_{L^\infty(\mathbb{D})} = \|\cdot\|_{C^0(\overline{\mathbb{D}})}$ on $C^0(\overline{\mathbb{D}})$ and $H^1(\mathbb{D}) \hookrightarrow C^{0,1/2}(\overline{\mathbb{D}})$ for $d = 1$.

For the case $q \in (1, 3/2)$, there are no $L^\infty(\mathbb{D})$-estimates at hand for $\partial_x X(t,\cdot), \partial_x e_i \in H^{q-1}(\mathbb{D})$. Instead, we make use of the following Lemma on compositions and products in fractional Sobolev spaces

Lemma 8.8.1. *[48, Theorem 1.1. and Corollary 6.2], 1.) Let $f \in C^2(\mathbb{R})$ such that $f(0) = 0$ and $f', f'' \in L^\infty(\mathbb{R})$. Then, for any $1 \leq q < +\infty$ and $v \in H^q(\mathbb{D}) \cap L^{2q}(\mathbb{D})$ it holds that $f(v) \in H^q(\mathbb{D})$ and there is a $C = C(f, q, \mathbb{D})$ such that*

$$\|f(v)\|_{H^q(\mathbb{D})} \leq C\|v\|_{H^q(\mathbb{D})}.$$

2.) Let $1 < q < +\infty$, $f_1 \in H^q(\mathbb{D}) \cap L^\infty(\mathbb{D})$ and let $f_2 \in H^{q-1}(\mathbb{D}) \cap L^{2q}(\mathbb{D})$. Then $f_1 f_2 \in H^{q-1}(\mathbb{D})$ and

$$\|f_1 f_2\|_{H^{q-1}(\mathbb{D})} \leq C\Big(\|f_1\|_{L^\infty(\mathbb{D})}\|f_2\|_{H^{q-1}(\mathbb{D})} + \|f_1\|^{1-1/q}_{H^q(\mathbb{D})}\|f_1\|^{1/q}_{L^\infty(\mathbb{D})}\|f_2\|_{L^{2q}(\mathbb{D})}\Big).$$

Remark 8.8.2. The estimates in Lemma 8.8.1 are originally stated with respect to $H^q(\mathbb{R}^d)$ instead of $H^q(\mathbb{D})$ in [48]. Nevertheless, the above statement remains true since $\mathbb{D} \subset \mathbb{R}^d$ is a bounded Lipschitz domain. For any $v \in H^q(\mathbb{D})$, this in turn guarantees the existence of an extension $\widetilde{v} \in H^q(\mathbb{R}^d)$ such that $v = \widetilde{v}|_{\mathbb{D}}$ and $\|\widetilde{v}\|_{H^q(\mathbb{R}^d)} \leq C\|v\|_{H^q(\mathbb{D})}$, see [71, Theorem 5.4]. The last estimate then allows to consider Sobolev spaces with respect to the domain $\mathbb{D} \subset \mathbb{R}^d$.

Since $q > 1/2$, we have that $X(t,\cdot) \in L^\infty(\mathbb{D}) \subset L^{2q}(\mathbb{D})$. As all derivatives of g are

bounded on $\mathbb{R}$, the first part of Lemma 8.8.1 yields

$$\|g(X(t,\cdot))\|_{H^q(\mathbb{D})} + \|g'(X(t,\cdot))\|_{H^q(\mathbb{D})} \leq C\|X(t,\cdot)\|_{H^q(\mathbb{D})}. \tag{8.33}$$

In addition, by taking $f_1 = g'(X(t,\cdot))$ and $f_2 = \partial_x X(t,\cdot)$, we obtain by the second part of Lemma 8.8.1

$$\begin{aligned}\|\partial_x X(t,\cdot)g'(X(t,\cdot))\|_{H^{q-1}(\mathbb{D})} &\leq C\|X(t,\cdot)\|_{H^q(\mathbb{D})}\Big(\|\partial_x X(t,\cdot)\|_{H^{q-1}(\mathbb{D})} + \|\partial_x X(t,\cdot)\|_{L^{2q}(\mathbb{D})}\Big)\\ &\leq C\|X(t,\cdot)\|_{H^q(\mathbb{D})}\|\partial_x X(t,\cdot)\|_{H^{q-1}(\mathbb{D})}.\end{aligned} \tag{8.34}$$

In the last estimate, we have used the fractional Sobolev inequality from Theorem 6.5 in [71] that shows $H^{q-1}(\mathbb{D}) \hookrightarrow L^{2q}(\mathbb{D})$ for $q \in (1, 3/2)$ and $d = 1$. Finally, we apply Lemma 8.8.1 to both terms in Eq. (8.32) and obtain with Ineqs. (8.33) and (8.34)

$$\begin{aligned}\|\partial_x G(t,X(t))e_i\|_{H^{q-1}(\mathbb{D})} &= \|g(X(t,\cdot)) + (1-\cdot))\partial_x e_i\|_{H^{q-1}(\mathbb{D})}\\ &\quad + \|\partial_x X(t,\cdot)g'(X(t,\cdot)) - 1)e_i\|_{H^{q-1}(\mathbb{D})}\\ &\leq C(1 + \|X(t,\cdot)\|_{H^q(\mathbb{D})})\|e_i\|_{H^q(\mathbb{D})}.\end{aligned}$$

9 Conclusions and outlook

This thesis has motivated the approach of uncertainty quantification with Lévy-type random fields from a perspective of modeling, computation and simulation. The potential of models with discontinuous stochastic structures has been unfolded by tractable algorithms and, therefore, the results from this thesis mark the starting point for the exploration of a still underappreciated topic in the field of uncertainty quantification.

The main contributions can be summarized as follows: A new type of discontinuous Lévy-type random field, capable of modeling arbitrary stochastic geometries, has been introduced. Utilized in stationary and time-dependent random PDEs, this stochastic coefficient paves the way for a broad range of possible applications. As any state-of-the-art continuous random coefficient may be recovered as a special case of this model, this is a significant extension of the common methodology in uncertainty quantification. Moreover, an algorithm for the approximation and simulation of general Lévy fields has been developed. The Fourier inversion approach in Chapter 7 differs drastically from the standard simulation algorithms and is, therefore, more flexible and easier to implement. Finally, a fully discrete scheme for a stochastic transport problem with Lévy noise has been presented, the first of his kind for this particular problem. Thus, it is finally possible to apply models based on transport equations with Lévy noise in practice, for instance to valuate commodity forward contracts. For each of the various random PDEs and SPDEs in this thesis, a well-posed analysis has been provided, tractable approximations of the random field have been discussed and suitable algorithms for the numerical discretization have been derived. All of the steps have turned out to be rather challenging, since existing results could only be applied to a limited extend.

It is my opinion that the contents of this thesis will shift the attention towards the novel approach of uncertainty quantification with Lévy-type random fields. The research on random PDEs with (log-) Gaussian coefficients tends towards its saturation and the discontinuous fields of this thesis provide the natural extension to this terminology. In fact, the results from Chapter 4–6 have already motivated ongoing research

on the probabilistic structure of Lévy-type random fields, as well as their applications in nonlinear conservation laws and inverse problems. Also, I believe that Chapters 7 and 8 form the basis for the actual implementation of forward models with heavy-tailed source terms. So far, the existing models have only been recognized from a theoretical perspective and their application has been, until this point, neglected.

There are, of course, still several open questions and various topics for further research. First of all, as indicated in Chapter 2, the proposed schemes could be improved by advanced numerical algorithms. For instance, the adaptive, sample dependent grids in Chapters 4–6 may become computationally expensive for very detailed and irregular stochastic geometries. Here, *hp*-finite element methods ([189]) may be a valid alternative, since the refinement is not based on the sample of the diffusion coefficient but on a-posteriori-error estimates in each sample. This *hp*-adaptive schemes could then be incorporated in the recently developed *continuous level Monte Carlo* estimator from [70]. As a generalization of the multilevel Monte Carlo method, this approach allows for stochastic refinement parameters that occur naturally in the jump-diffusion setting.

Furthermore, coupling multilevel Monte Carlo with quasi-Monte Carlo (QMC) sampling (see Section 2.2 and the references therein) seems promising for random PDEs with discontinuous coefficients. In the recent work [108], the authors introduce a novel multilevel QMC algorithm based on so-called *product weights* for random PDEs with log-normal diffusion coefficients. The key assumption therein is that the basis functions in the Karhunen-Loève expansion of the underlying random field are locally supported. However, this assumption holds by definition for the jump part of the coefficients in Chapters 4–6, so a natural next step is to investigate similar algorithms in the general jump-diffusion setting. Moreover, QMC methods have not yet been applied to estimate moments or functionals of SPDEs, thus there is a lot of potential for further research.

The contemporary successes of *Deep Learning* shows that deep neural networks (DNNs) may change the field of scientific computing, not only with regard to uncertainty quantification. Of particular interest are the approaches in [122, 191, 192, 199], where the authors show that DNNs are able to solve very high-dimensional (parametric) PDEs. This immediately applies to the solution of random PDEs (see [191]), but DNNs have yet to be tested for problems with discontinuous Lévy-type objects. Finally, the latest development to use DNNs for the sampling of complex discontinuous microstructures, such as porous rock or alloys, has to be mentioned (see [50] and the references therein). In the setting of this thesis, it would be interesting to investigate the possibility of sampling Lévy-type objects by DNNs. This could open the door to generate even more complex geometries and eliminate several bottlenecks of current techniques, such as approximating the eigenbasis in a Karhunen-Loève expansion.

Bibliography

[1] A. Abdulle, A. Barth, and C. Schwab, *Multilevel Monte Carlo methods for stochastic elliptic multiscale PDEs*, Multiscale Modeling & Simulation, 11 (2013), pp. 1033–1070.

[2] R. A. Adams and J. J. Fournier, *Sobolev Spaces*, vol. 140, Elsevier, 2003.

[3] R. J. Adler and J. E. Taylor, *Random Fields and Geometry*, Springer Science & Business Media, 2009.

[4] A. Andresen, S. Koekebakker, and S. Westgaard, *Modeling electricity forward prices using the multivariate normal inverse gaussian distribution*, Journal of Energy Markets, 3 (2010), pp. 1–23.

[5] R. Anton, D. Cohen, S. Larsson, and X. Wang, *Full discretization of semilinear stochastic wave equations driven by multiplicative noise*, SIAM Journal on Numerical Analysis, 54 (2016), pp. 1093–1119.

[6] D. Applebaum, *Lévy Processes and Stochastic Calculus*, Cambridge University Press, 2009.

[7] D. Applebaum and M. Riedle, *Cylindrical Lévy processes in Banach spaces*, Proceedings of the London Mathematical Society, 101 (2010), pp. 697–726.

[8] J. H. Argyris and S. Kelsey, *Energy Theorems and Structural Analysis*, Springer, 1960.

[9] S. Asmussen and P. W. Glynn, *Stochastic Simulation: Algorithms and Analysis*, vol. 57, Springer Science & Business Media, 2007.

[10] S. Asmussen and J. Rosiński, *Approximations of small jumps of Lévy processes with a view towards simulation*, Journal of Applied Probability, 38 (2001), pp. 482–493.

[11] A. Atkinson, *The simulation of generalized inverse Gaussian and hyperbolic random variables*, SIAM Journal on Scientific and Statistical Computing, 3 (1982), pp. 502–515.

[12] C. Audouze and P. B. Nair, *Some a priori error estimates for finite element approximations of elliptic and parabolic linear stochastic partial differential equations*, Int. Journal for Uncertainty Quant., 4 (2014).

[13] I. Babuška, *The finite element method for elliptic equations with discontinuous coefficients*, Computing, 5 (1970), pp. 207–213.

[14] I. Babuška, F. Nobile, and R. Tempone, *A stochastic collocation method for elliptic partial differential equations with random input data*, SIAM J. Numer. Anal., 45 (2007), pp. 1005–1034.

[15] I. Babuška, R. Tempone, and G. E. Zouraris, *Galerkin finite element approximations of stochastic elliptic partial differential equations*, SIAM J. Numer. Anal., 42 (2004), pp. 800–825.

[16] D. A. Barajas-Solano and D. M. Tartakovsky, *Stochastic collocation methods for nonlinear parabolic equations with random coefficients*, SIAM/ASA Journal on Uncertainty Quant., 4 (2016), pp. 475–494.

[17] V. Barbu, G. Da Prato, and M. Röckner, *Stochastic Porous Media Equations*, vol. 2163, Springer, 2016.

[18] O. E. Barndorff-Nielsen, *Exponentially decreasing distributions for the logarithm of particle size*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, 353 (1977), pp. 401–419.

[19] ——, *Hyperbolic distributions and distributions on hyperbolae*, Scandinavian Journal of Statistics, 5 (1978), pp. 151–157.

[20] O. E. Barndorff-Nielsen and C. Halgreen, *Infinite divisibility of the hyperbolic and generalized inverse Gaussian distributions*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, 38 (1977), pp. 309–311.

[21] O. E. Barndorff-Nielsen, J. Kent, and M. Sørensen, *Normal variance-mean mixtures and z distributions*, International Statistical Review / Revue Internationale de Statistique, 50 (1982), pp. 145–159.

[22] O. E. Barndorff-Nielsen, T. Mikosch, and S. I. Resnick, *Lévy Processes: Theory and Applications*, Springer Science & Business Media, 2012.

[23] A. Barth, *A finite element method for martingale-driven stochastic partial differential equations*, Communications on Stochastic Analysis, 4 (2010), pp. 355–375.

[24] A. Barth and F. E. Benth, *The forward dynamics in energy markets — infinite-dimensional modelling and simulation*, Stochastics: An International Journal of Probability and Stochastic Processes, 86 (2014), pp. 932–966.

[25] A. Barth, R. Bürger, I. Kröker, and C. Rohde, *Computational uncertainty quantification for a clarifier-thickener model with several random perturbations: A hybrid stochastic Galerkin approach*, Computers & Chemical Engineering, 89 (2016), pp. 11–26.

[26] A. Barth and A. Lang, *Milstein approximation for advection-diffusion equations driven by multiplicative noncontinuous martingale noises*, Appl. Math. Optim., 66 (2012), pp. 387–413.

[27] ——, *Simulation of stochastic partial differential equations using finite element methods*, Stochastics, 84 (2012), pp. 217–231.

[28] A. Barth, A. Lang, and C. Schwab, *Multilevel Monte Carlo method for parabolic stochastic partial differential equations*, BIT Numerical Mathematics, 53 (2013), pp. 3–27.

[29] A. Barth, C. Schwab, and N. Zollinger, *Multi-level Monte Carlo Finite Element method for elliptic PDEs with stochastic coefficients*, Numerische Mathematik, 119 (2011), pp. 123–161.

[30] A. Barth and A. Stein, *Approximation and simulation of infinite-dimensional Lévy processes*, Stochastics and Partial Differential Equations: Analysis and Computations, 6 (2018), pp. 286–334.

[31] ——, *A study of elliptic partial differential equations with jump diffusion coefficients*, SIAM/ASA Journal on Uncertainty Quantification, 6 (2018), pp. 1707–1743.

[32] ——, *Numerical analysis for time-dependent advection-diffusion problems with random discontinuous coefficients*, arXiv preprint arXiv:1902.02130, (2019).

[33] A. Barth and T. Stüwe, *Weak convergence of Galerkin approximations of stochastic partial differential equations driven by additive Lévy noise*, Mathematics and Computers in Simulation, 143 (2018), pp. 215–225.

[34] C. Bauzet, J. Charrier, and T. Gallouët, *Convergence of monotone finite volume schemes for hyperbolic scalar conservation laws with multiplicative noise*, Stochastics and Partial Differential Equations: Analysis and Computations, 4 (2016), pp. 150–223.

[35] S. Becker, B. Gess, A. Jentzen, and P. E. Kloeden, *Strong convergence rates for explicit space-time discrete numerical approximations of stochastic Allen-Cahn equations*, arXiv preprint arXiv:1711.02423, (2017).

[36] F. B. Belgacem, *The mortar finite element method with Lagrange multipliers*, Numerische Mathematik, 84 (1999), pp. 173–197.

[37] F. E. Benth and S. Koekebakker, *Stochastic modeling of financial electricity contracts*, Energy Economics, 30 (2008), pp. 1116–1157.

[38] F. E. Benth and P. Krühner, *Integrability of multivariate subordinated Lévy processes in Hilbert space*, Stochastics, 87 (2015), pp. 458–476.

[39] C. Bernardi, Y. Maday, and F. Rapetti, *Basics and some applications of the mortar element method*, GAMM-Mitteilungen, 28 (2005), pp. 97–123.

[40] J. Bertoin, *Lévy Processes*, vol. 121, Cambridge University Press, 1996.

[41] T. R. Bielecki, M. Jeanblanc, and M. Rutkowski, *PDE approach to valuation and hedging of credit derivatives*, Quantitative Finance, 5 (2005), pp. 257–270.

[42] F. Black and M. Scholes, *The pricing of options and corporate liabilities*, Journal of Political Economy, 81 (1973), pp. 637–654.

[43] P. Blæsild, *The two-dimensional hyperbolic distribution and related distributions, with an application to Johannsen's bean data*, Biometrika, 68 (1981), pp. 251–263.

[44] P. Blæsild and J. Jensen, *Multivariate distributions of hyperbolic type*, Springer Netherlands, Dordrecht, 1981, pp. 45–66.

[45] D. Blomker and A. Jentzen, *Galerkin approximations for the stochastic burgers equation*, SIAM Journal on Numerical Analysis, 51 (2013), pp. 694–715.

[46] A. Brace and M. Musiela, *A multifactor Gauss Markov implementation of Heath, Jarrow, and Morton*, Mathematical Finance, 4 (1994), pp. 259–283.

[47] S. Brenner and R. Scott, *The Mathematical Theory of Finite Element Methods*, vol. 15, Springer Science & Business Media, 2007.

[48] H. Brezis and P. Mironescu, *Gagliardo-nirenberg, composition and products in fractional sobolev spaces*, Journal of Evolution Equations, 1 (2001), pp. 387–404.

[49] Z. Brzeźniak and J. Zabczyk, *Regularity of Ornstein–Uhlenbeck processes driven by a Lévy white noise*, Potential Analysis, 32 (2010), pp. 153–188.

[50] R. Cang, Y. Xu, S. Chen, Y. Liu, Y. Jiao, and M. Yi Ren, *Microstructure representation and reconstruction of heterogeneous materials via deep belief network for computational material design*, Journal of Mechanical Design, 139 (2017).

[51] C. Canuto, M. Y. Hussaini, A. Quarteroni, A. Thomas Jr, et al., *Spectral Methods in Fluid Dynamics*, Springer Science & Business Media, 2012.

[52] R. Carmona and M. R. Tehranchi, *Interest Rate Models: An Infinite Dimensional Stochastic Analysis Perspective*, Springer Science & Business Media, 2007.

[53] J. Charrier, *Strong and weak error estimates for elliptic partial differential equations with random coefficients*, SIAM Journal on Numerical Analysis, 50 (2012), pp. 216–246.

[54] J. Charrier, R. Scheichl, and A. L. Teckentrup, *Finite element error analysis of elliptic PDEs with random coefficients and its application to multilevel Monte Carlo methods*, SIAM Journal on Numerical Analysis, 51 (2013), pp. 322–352.

[55] C.-T. Chen, C.-K. Wu, and C. Hwang, *Optimal design and control of cpu heat sink processes*, IEEE Transactions on components and Packaging Technologies, 31 (2008), pp. 184–195.

[56] Z. Chen and J. Zou, *Finite element methods and their convergence for elliptic and parabolic interface problems*, Numerische Mathematik, 79 (1998), pp. 175–202.

[57] A. Cherny and A. Shiryaev, *Change of time and measure for Lévy Processes*, Lectures for the Summer School "From Lévy Processes to Semimartingales - Recent Theoretical Developments and Applications to Finance", (2002).

[58] P. Ciarlet, *The Finite Element Method for Elliptic Problems*, Classics in Applied Mathematics, Society for Industrial and Applied Mathematics, 2002.

[59] K. A. Cliffe, M. B. Giles, R. Scheichl, and A. L. Teckentrup, *Multilevel Monte Carlo methods and applications to elliptic PDEs with random coefficients*, Comput. Vis. Sci., 14 (2011), pp. 3–15.

[60] B. Cockburn, B. Dong, and J. Guzmán, *Optimal convergence of the original DG method for the transport-reaction equation on special meshes*, SIAM Journal on Numerical Analysis, 46 (2008), pp. 1250–1265.

[61] B. Cockburn, G. E. Karniadakis, and C.-W. Shu, *Discontinuous Galerkin Methods: Theory, Computation and Applications*, vol. 11, Springer Science & Business Media, 2000.

[62] B. Cockburn and C.-W. Shu, *The Runge-Kutta local projection-discontinuous-Galerkin finite element method for scalar conservation laws*, ESAIM: Mathematical Modelling and Numerical Analysis, 25 (1991), pp. 337–361.

[63] A. Cohen, R. DeVore, and C. Schwab, *Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDEs*, J. Analysis and Applications, 9 (2011), pp. 11–47.

[64] D. Cohen and L. Quer-Sardanyons, *A fully discrete approximation of the one-dimensional stochastic wave equation*, IMA Journal of Numerical Analysis, 36 (2016), pp. 400–420.

[65] R. Courant, *Variational methods for the solution of problems of equilibrium and vibrations*, 1943.

[66] M. Croci, M. B. Giles, M. E. Rognes, and P. E. Farrell, *Efficient white noise sampling and coupling for multilevel monte carlo with nonnested meshes*, SIAM/ASA Journal on Uncertainty Quantification, 6 (2018), pp. 1630–1655.

[67] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge university press, second ed., 2014.

[68] G. Dagan, U. Hornung, and P. Knabner, *Mathematical Modeling for Flow and Transport Through Porous Media*, Springer, 1991.

[69] A. Debussche and J. Printems, *Weak order for the discretization of the stochastic heat equation*, Mathematics of Computation, 78 (2009), pp. 845–863.

[70] G. Detommaso, T. Dodwell, and R. Scheichl, *Continuous level monte carlo and sample-adaptive model hierarchies*, SIAM/ASA Journal on Uncertainty Quantification, 7 (2019), pp. 93–116.

[71] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bulletin des Sciences Mathématiques, 136 (2012), pp. 521–573.

[72] J. Dick, F. Y. Kuo, and I. H. Sloan, *High-dimensional integration: the quasi-Monte Carlo way*, Acta Numerica, 22 (2013), pp. 133–288.

[73] Z. Ding, *A proof of the trace theorem of Sobolev spaces on Lipschitz domains*, Proceedings of the American Mathematical Society, 124 (1996), pp. 591–600.

[74] T. Dunst, E. Hausenblas, and A. Prohl, *Approximate Euler method for parabolic stochastic partial differential equations driven by space-time Lévy noise*, SIAM Journal on Numerical Analysis, 50 (2012), pp. 2873–2896.

[75] E. Eberlein, *Lévy processes: theory and applications*, Birkhäuser Boston, Boston, MA, 2001, ch. Application of Generalized Hyperbolic Lévy Motions to Finance, pp. 319–336.

[76] E. Eberlein and U. Keller, *Hyperbolic distributions in finance*, Bernoulli, 1 (1995), pp. 281–299.

[77] M. Eigel, C. J. Gittelson, C. Schwab, and E. Zander, *Adaptive stochastic Galerkin FEM*, Computer Methods in Applied Mechanics and Engineering, 270 (2014), pp. 247–269.

[78] M. Eigel, C. Merdon, and J. Neumann, *An adaptive multilevel monte carlo method with stochastic bounds for quantities of interest with uncertain data*, SIAM/ASA Journal on Uncertainty Quantification, 4 (2016), pp. 1219–1245.

[79] O. G. Ernst, C. E. Powell, D. J. Silvester, and E. Ullmann, *Efficient solvers for a linear stochastic Galerkin mixed formulation of diffusion problems with random data*, SIAM Journal on Scientific Computing, 31 (2009), pp. 1424–1447.

[80] L. Evans, *Partial Differential Equations*, Graduate studies in mathematics, American Mathematical Society, 2010.

[81] R. Eymard, T. Gallouët, and R. Herbin, *Finite volume methods*, Handbook of Numerical Analysis, 7 (2000), pp. 713–1018.

[82] M. W. Farthing and F. L. Ogden, *Numerical solution of Richards' equation: a review of advances and challenges*, Soil Science Society of America Journal, 81 (2017), pp. 1257–1269.

[83] G. Fishman, *Monte Carlo: Concepts, Algorithms, and Applications*, Springer Science & Business Media, 2013.

[84] N. Fournier, *Simulation and approximation of Lévy-driven stochastic differential equations*, ESAIM: Probability and Statistics, 15 (2011), pp. 233–248.

[85] P. Frauenfelder, C. Schwab, and R. A. Todor, *Finite elements for elliptic problems with stochastic coefficients*, Computer Methods in Applied Mechanics and Engineering, 194 (2005), pp. 205–228.

[86] D. Furihata, M. Kovács, S. Larsson, and F. Lindgren, *Strong convergence of a fully discrete finite element approximation of the stochastic Cahn–Hilliard equation*, SIAM Journal on Numerical Analysis, 56 (2018), pp. 708–731.

[87] B. G. Galerkin, *Series solution of some problems of elastic equilibrium of rods and plates*, Vestn. Inzh. Tekh, 19 (1915), pp. 897–908.

[88] R. N. Gantner, L. Herrmann, and C. Schwab, *Quasi–Monte Carlo integration for affine-parametric, elliptic pdes: Local supports and product weights*, SIAM Journal on Numerical Analysis, 56 (2018), pp. 111–135.

[89] R. Gençay and F. Selçuk, *Extreme value theory and value-at-risk: Relative performance in emerging markets*, International Journal of Forecasting, 20 (2004), pp. 287–303.

[90] J. Giesselmann, F. Meyer, and C. Rohde, *A posteriori error analysis for random scalar conservation laws using the stochastic Galerkin method.*, IMA J. Numer. Anal., (2019).

[91] J. Gil-Pelaez, *Note on the inversion theorem*, Biometrika, 38 (1951), pp. 481–482.

[92] M. B. Giles, *Multilevel Monte Carlo path simulation*, Operations Research, 56 (2008), pp. 607–617.

[93] M. B. Giles, *Multilevel Monte Carlo methods*, Acta Numerica, 24 (2015), pp. 259–328.

[94] P. Glasserman, *Monte Carlo Methods In Financial Engineering*, vol. 53, Springer Science & Business Media, 2013.

[95] P. Goatin, S. Göttlich, and O. Kolb, *Speed limit and ramp meter control for traffic flow networks*, Engineering Optimization, 48 (2016), pp. 1121–1144.

[96] I. G. Graham, F. Y. Kuo, J. A. Nichols, R. Scheichl, C. Schwab, and I. H. Sloan, *Quasi-Monte Carlo finite element methods for elliptic PDEs with lognormal random coefficients*, Numerische Mathematik, 131 (2015), pp. 329–368.

[97] I. G. Graham, F. Y. Kuo, D. Nuyens, R. Scheichl, and I. H. Sloan, *Analysis of circulant embedding methods for sampling stationary random fields*, SIAM Journal on Numerical Analysis, 56 (2018), pp. 1871–1895.

[98] I. G. Graham, F. Y. Kuo, D. Nuyens, R. Scheichl, and I. H. Sloan, *Circulant embedding with QMC: analysis for elliptic PDE with lognormal coefficients*, Numerische Mathematik, (2018), pp. 1–33.

[99] D. S. Grebenkov and B.-T. Nguyen, *Geometrical structure of Laplacian eigenfunctions*, SIAM Review, 55 (2013), pp. 601–667.

[100] I. Gyöngy and A. Millet, *On discretization schemes for stochastic evolution equations*, Potential Analysis, 23 (2005), pp. 99–134.

[101] I. Gyöngy, S. Sabanis, and D. Šiška, *Convergence of tamed Euler schemes for a class of stochastic evolution equations*, Stochastics and Partial Differential Equations: Analysis and Computations, 4 (2016), pp. 225–245.

[102] W. Hackbusch, *Elliptic Differential Equations: Theory and Numerical Treatment*, Springer, 2. softcover print. ed., 2017.

[103] K. Hadeler and K. Dietz, *Nonlinear hyperbolic partial differential equations for the dynamics of parasite populations*, in Hyperbolic Partial Differential Equations, Elsevier, 1983, pp. 415–430.

[104] H. Harbrecht and J. Li, *First order second moment analysis for stochastic interface problems based on low-rank approximation*, ESAIM: Mathematical Modelling and Num. Analysis, 47 (2013), pp. 1533–1552.

[105] D. Heath, R. Jarrow, and A. Morton, *Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation*, Econometrica: Journal of the Econometric Society, (1992), pp. 77–105.

[106] S. Heinrich, *Multilevel Monte Carlo methods*, Third International Conference on Large Scale Scientific Computing, 1 (2001), pp. 58–67.

[107] R. Helmig, *Multiphase Flow and Transport Processes in the Subsurface: a Contribution to the Modeling of Hydrosystems.*, Springer-Verlag, 1997.

[108] L. Herrmann and C. Schwab, *QMC integration for lognormal-parametric, elliptic pdes: local supports and product weights*, Numerische Mathematik, 141 (2019), pp. 63–102.

[109] J. S. Hesthaven and T. Warburton, *Nodal Discontinuous Galerkin Methods: Algorithms, Analysis, and Applications*, Springer Science & Business Media, 2007.

[110] J. S. Hesthaven, T. Warburton, C. Chauviere, and L. Wilcox, *High-order discontinuous Galerkin methods for computational electromagnetics and uncertainty quantification*, in Scientific Computing in Electrical Engineering SCEE 2008, Springer, 2010, pp. 403–412.

[111] S. L. Heston, *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, The Review of Financial Studies, 6 (1993), pp. 327–343.

[112] V. H. Hoang and C. Schwab, *Sparse tensor Galerkin discretization of parametric and random parabolic pdes—analytic regularity and generalized polynomial chaos approximation*, SIAM Journal on Mathematical Analysis, 45 (2013), pp. 3050–3083.

[113] H. Hoel, E. Von Schwerin, A. Szepessy, and R. Tempone, *Adaptive multilevel Monte Carlo simulation*, in Numerical Analysis of Multiscale Computations, Springer, 2012, pp. 217–234.

[114] E. Hoyle, L. P. Hughston, and A. Macrina, *Lévy random bridges and the modelling of financial information*, Stochastic Processes and their Applications, 121 (2011), pp. 856 – 884.

[115] A. Hrennikoff, *Solution of problems of elasticity by the framework method*, J. Appl. Mech., (1941).

[116] P. Hughett, *Error bounds for numerical inversion of a probability characteristic function*, SIAM Journal on Numerical Analysis, 35 (1998), pp. 1368–1392.

[117] R. Isaacs, *Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization*, Courier Corporation, 1999.

[118] K. Itô, *Stochastic integral*, Proceedings of the Imperial Academy, 20 (1944), pp. 519–524.

[119] ——, *On Stochastic Differential Equations*, no. 4, American Mathematical Soc., 1951.

[120] A. Jasra, K. J. Law, and Y. Zhou, *Forward and inverse uncertainty quantification using multilevel Monte Carlo algorithms for an elliptic nonlocal equation*, International Journal for Uncertainty Quantification, 6 (2016).

[121] A. Jentzen and P. E. Kloeden, *The numerical approximation of stochastic partial differential equations*, Milan Journal of Mathematics, 77 (2009), pp. 205–244.

[122] A. Jentzen, D. Salimova, and T. Welti, *A proof that deep artificial neural networks overcome the curse of dimensionality in the numerical approximation of Kolmogorov partial differential equations with constant diffusion and nonlinear drift coefficients*, arXiv preprint arXiv:1809.07321, (2018).

[123] C. Johnson and J. Pitkäranta, *An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation*, Mathematics of Computation, 46 (1986), pp. 1–26.

[124] W. P. Johnson, *The curious history of Faà di Bruno's formula*, The American Mathematical Monthly, 109 (2002), pp. 217–234.

[125] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer, 1998.

[126] D. Kelker, *Infinite divisibility and variance mixtures of the normal distribution*, The Annals of Mathematical Statistics, 42 (1971), pp. 802–808.

[127] A. Y. Khintchine, *Zur Theorie der unbeschränkt teilbaren Verteilungsgesetze*, Rec. Math. Moscou, n. Ser, 2 (1937), pp. 79–117.

[128] K.-K. Kim and S. Kim, *Simulation of tempered stable Lévy bridges and its applications*, Operations Research, 64 (2016), pp. 495–509.

[129] P. E. Kloeden, G. J. Lord, A. Neuenkirch, and T. Shardlow, *The exponential integrator scheme for stochastic partial differential equations: pathwise error bounds*, Journal of Computational and Applied Mathematics, 235 (2011), pp. 1245–1260.

[130] U. Koley, A. K. Majee, and G. Vallet, *A finite difference scheme for conservation laws driven by lévy noise*, IMA Journal of Numerical Analysis, 38 (2017), pp. 998–1050.

[131] R. Kornhuber and E. Youett, *Adaptive multilevel monte carlo methods for stochastic variational inequalities*, SIAM Journal on Numerical Analysis, 56 (2018), pp. 1987–2007.

[132] M. Kovács, S. Larsson, and F. Lindgren, *On the backward Euler approximation of the stochastic Allen-Cahn equation*, Journal of Applied Probability, 52 (2015), pp. 323–338.

[133] M. Kovács, F. Lindner, and R. L. Schilling, *Weak convergence of finite element approximations of linear stochastic evolution equations with additive Lévy noise*, SIAM/ASA Journal on Uncertainty Quantification, 3 (2015), pp. 1159–1199.

[134] M. Krivko and M. V. Tretyakov, *Numerical integration of the Heath–Jarrow–Morton model of interest rates*, IMA Journal of Numerical Analysis, 34 (2014), pp. 147–196.

[135] I. Kröker and C. Rohde, *Finite volume schemes for hyperbolic balance laws with multiplicative noise*, Applied Numerical Mathematics, 62 (2012), pp. 441–456.

[136] R. Kruse, *Optimal error estimates of Galerkin finite element methods for stochastic partial differential equations with multiplicative noise*, IMA Journal of Numerical Analysis, 34 (2014), pp. 217–251.

[137] F. Y. Kuo, C. Schwab, and I. H. Sloan, *Quasi-Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficients*, SIAM Journal on Numerical Analysis, 50 (2012), pp. 3351–3374.

[138] H.-H. Kuo, *Gaussian measures in Banach spaces*, Springer, 1975.

[139] A. Lang and J. Potthoff, *Fast simulation of Gaussian random fields*, Monte Carlo Methods and Applications, 17 (2011), pp. 195–214.

[140] C. Lang, A. Sharma, A. Doostan, and K. Maute, *Heaviside enriched extended stochastic FEM for problems with uncertain material interfaces*, Computational Mechanics, 56 (2015), pp. 753–767.

[141] K. S. Lee, *A review on concepts, applications, and models of aquifer thermal energy storage systems*, Energies, 3 (2010), pp. 1320–1334.

[142] P. Lesaint and P.-A. Raviart, *On a finite element method for solving the neutron transport equation*, Publications Mathématiques et Informatique de Rennes, (1974), pp. 1–40.

[143] R. J. LeVeque, *Finite Volume Methods for Hyperbolic Problems*, vol. 31, Cambridge University Press, 2002.

[144] R. J. LeVeque, *Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-state and Time-dependent Problems*, vol. 98, SIAM, 2007.

[145] P. Lévy, *Sur les intégrales dont les éléments sont des variables aléatoires indépendantes*, Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 3 (1934), pp. 337–366.

[146] P. Levy, *L'addition des variables aléatoires définies sur une circonférence*, Bulletin de la Société mathématique de France, 67 (1939), pp. 1–41.

[147] P. Lévy, *Théorie de l'Addition des Variables Aléatoires*, 1 (1954).

[148] J. Li, X. Wang, and K. Zhang, *Multi-level Monte Carlo weak Galerkin method for elliptic equations with stochastic jump coefficients*, Applied Mathematics and Computation, 275 (2016), pp. 181–194.

[149] W. V. Li and Q.-M. Shao, *Gaussian processes: inequalities, small ball probabilities and applications*, Handbook of Statistics, 19 (2001), pp. 533–597.

[150] C. Liu and J. Shen, *A phase field model for the mixture of two incompressible fluids and its approximation by a Fourier-spectral method*, Physica D: Nonlinear Phenomena, 179 (2003), pp. 211–228.

[151] W. Liu and M. Röckner, *Stochastic Partial Differential Equations: An Introduction*, Springer, 2015.

[152] G. J. Lord, C. E. Powell, and T. Shardlow, *An Introduction to Computational Stochastic PDEs*, vol. 50, Cambridge University Press, 2014.

[153] E. M. Lotfi, M. Maziane, K. Hattaf, and N. Yousfi, *Partial differential equations of an epidemic model with spatial diffusion*, International Journal of Partial Differential Equations, 2014 (2014).

[154] A. K. Majee, *Convergence of a flux-splitting finite volume scheme for conservation laws driven by lévy noise*, Applied Mathematics and Computation, 338 (2018), pp. 676–697.

[155] R. Maller and A. Szimayer, *Finite approximation schemes for Lévy processes, and their application to optimal stopping problems*, Stochastic Processes and their Applications, 117 (2007), pp. 1422–1447.

[156] R. Mansuy and M. Yor, *Harnesses, Lévy bridges and Monsieur Jourdain*, Stochastic Processes and their Applications, 115 (2005), pp. 329 – 338.

[157] H. Masuda, *On multidimensional Ornstein-Uhlenbeck processes driven by a general Lévy process*, Bernoulli, 10 (2004), pp. 97–120.

[158] M. Métivier and J. Pellaumail, *Stochastic Integration*, Academic Press, 2014.

[159] N. Metropolis, *The beginning of the Monte Carlo method*, Los Alamos Science, 15 (1987), pp. 125–130.

[160] T. Müller-Gronbach, E. Novak, and K. Ritter, *Monte Carlo-Algorithmen*, Springer-Verlag, 2012.

[161] S. Nicaise, *Polygonal interface problems: higher regularity results*, Communications in Partial Differential Equations, 15 (1990), pp. 1475–1508.

[162] S. Nicaise and A.-M. Sändig, *General interface problems—i*, Mathematical Methods in the Applied Sciences, 17 (1994), pp. 395–429.

[163] S. Nicaise and A.-M. Sändig, *General interface problems—ii*, Mathematical Methods in the Applied Sciences, 17 (1994), pp. 431–450.

[164] F. Nobile and R. Tempone, *Analysis and implementation issues for the numerical approximation of parabolic equations with random coefficients*, Int. J. for Num. Methods in Engineering, 80 (2009), pp. 979–1006.

[165] F. Nobile, R. Tempone, and C. G. Webster, *A sparse grid stochastic collocation method for partial differential equations with random input data*, SIAM J. Numer. Anal., 46 (2008), pp. 2309–2345.

[166] J. Nocedal and S. Wright, *Numerical Optimization*, Springer Science & Business Media, 2006.

[167] T. A. Øigård, A. Hanssen, and R. E. Hansen, *The multivariate normal inverse Gaussian distribution: EM-estimation and analysis of synthetic aperture sonar data*, in Proceedings of the 12th European Conference on Signal Processing Conference, Sept 2004, pp. 1433–1436.

[168] B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications*, Springer Science & Business Media, 2013.

[169] F. W. Olver, *NIST Handbook of Mathematical Functions*, Cambridge University Press, 2010.

[170] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, vol. 44, Springer Science & Business Media, Applied Mathematical Sciences, 2 ed., 1983.

[171] B. Peherstorfer, K. Willcox, and M. Gunzburger, *Optimal model management for multifidelity Monte Carlo estimation*, SIAM Journal on Scientific Computing, 38 (2016), pp. A3163–A3194.

[172] V. Pérez-Abreu and A. Rocha-Arteaga, *Covariance-parameter Lévy processes in the space of trace class operators*, Infinite Dimensional Analysis, Quantum Probability and Related Topics, 08 (2005).

[173] S. Peszat and J. Zabczyk, *Stochastic Partial Differential Equations with Lévy Noise*, Cambridge University Press, 2007.

[174] M. Petzoldt, *Regularity results for laplace interface problems in two dimensions*, Zeitschrift für Analysis und ihre Anwendungen, 20 (2001), pp. 431–455.

[175] ——, *A posteriori error estimators for elliptic equations with discontinuous coefficients*, Advances in Computational Mathematics, 16 (2002), pp. 47–75.

[176] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in C*, vol. 2, Cambridge University Press, 1996.

[177] A. Quarteroni and A. Valli, *Numerical Approximation of Partial Differential Equations*, Springer Science & Business Media, 2 ed., 1997.

[178] C. E. Rasmussen and C. K. Williams, *Gaussian Processes for Machine Learning*, the MIT Press, 2 (2006).

[179] W. H. Reed and T. Hill, *Triangular mesh methods for the neutron transport equation*, tech. rep., Los Alamos Scientific Lab., N. Mex.(USA), 1973.

[180] W. C. Reynolds, *Computation of turbulent flows*, Annual Review of Fluid Mechanics, 8 (1976), pp. 183–208.

[181] C.-h. Rhee and P. W. Glynn, *Unbiased estimation with square root convergence for SDE models*, Operations Research, 63 (2015), pp. 1026–1043.

[182] C. Ribeiro and N. Webber, *A Monte Carlo method for the normal inverse Gaussian option valuation model using an inverse Gaussian bridge*, Preprint, City University, (2003).

[183] T. Ricken, N. Waschinsky, and D. Werner, *Simulation of steatosis zonation in liver lobule—a continuummechanical bi-scale, tri-phasic, multi-component approach*, in Biomedical Technology, Springer, 2018, pp. 15–33.

[184] W. Ritz, *Über eine neue Methode zur Lösung gewisser Variationsprobleme der mathematischen Physik*, Journal fur Mathematik, 135 (1909).

[185] B. Riviere, *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation*, SIAM, 2008.

[186] T. H. Rydberg, *The normal inverse Gaussian Lévy process: simulation and approximation*, Comm. Statist. Stochastic Models, 13 (1997), pp. 887–910.

[187] K. Sato, *Lévy Processes and Infinite Divisibility*, Cambridge University Press, 1999.

[188] W. Schoutens, *Lévy Processes in Finance: Pricing Financial Derivatives*, John Wiley and Sons, 2003.

[189] C. Schwab, *p-and hp-Finite Element Methods: Theory and Applications in Solid and Fluid Mechanics (Numerical Mathematics and Scientific Computation)*, Oxford University Press, New York, 1999.

[190] C. Schwab and C. J. Gittelson, *Sparse tensor discretizations of high-dimensional parametric and stochastic PDEs*, Acta Numerica, 20 (2011), pp. 291–467.

[191] C. Schwab and J. Zech, *Deep learning in high dimension: neural network expression rates for generalized polynomial chaos expansions in UQ*, Analysis and Applications, 17 (2019), pp. 19–55.

[192] J. Sirignano and K. Spiliopoulos, *Dgm: A deep learning algorithm for solving partial differential equations*, Journal of Computational Physics, 375 (2018), pp. 1339–1364.

[193] G. Stefanou and M. Papadrakakis, *Assessment of spectral representation and Karhunen–Loève expansion methods for the simulation of Gaussian stochastic fields*, Computer Methods in Applied Mechanics and Engineering, 196 (2007), pp. 2465–2477.

[194] A. L. Teckentrup, R. Scheichl, M. B. Giles, and E. Ullmann, *Further analysis of multilevel Monte Carlo methods for elliptic PDEs with random coefficients*, Numerische Mathematik, 125 (2013), pp. 569–600.

[195] M. Turner, R. Clough, H. Martin, and L. Topp, *Stiffness and deflection analysis of complex structures*, Journal of the Aeronautical Sciences, 23 (1956), pp. 805–823.

[196] M. S. Veillette and M. S. Taqqu, *A technique for computing the PDFs and CDFs of nonnegative infinitely divisible random variables*, Journal of Applied Probability, 48 (2011), p. 217–237.

[197] J. B. Walsh, *On numerical solutions of the stochastic wave equation*, Illinois Journal of Mathematics, 50 (2006), pp. 991–1018.

[198] X. Wang, *An exponential integrator scheme for time discretization of nonlinear stochastic wave equation*, Journal of Scientific Computing, 64 (2015), pp. 234–263.

[199] E. Weinan, J. Han, and A. Jentzen, *Deep learning-based numerical methods for high-dimensional parabolic partial differential equations and backward stochastic differential equations*, Communications in Mathematics and Statistics, 5 (2017), pp. 349–380.

[200] J. E. Williams and D. L. Hawkings, *Sound generation by turbulence and surfaces in arbitrary motion*, Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences, 264 (1969), pp. 321–342.

[201] B. I. Wohlmuth, *A mortar finite element method using dual spaces for the lagrange multiplier*, SIAM Journal on Numerical Analysis, 38 (2000), pp. 989–1012.

[202] D. Zhang and Q. Kang, *Pore scale simulation of solute transport in fractured porous media*, Geophysical Research Letters, 31 (2004). L12504.

[203] D. Zhang and Z. Lu, *An efficient, high-order perturbation approach for flow in random porous media via Karhunen–Loève and polynomial expansions*, Journal of Computational Physics, 194 (2004), pp. 773–794.

[204] G. Zhang and M. Gunzburger, *Error analysis of a stochastic collocation method for parabolic partial differential equations with random input data*, SIAM Journal on Numerical Analysis, 50 (2012), pp. 1922–1940.

[205] T. Zhou, *Stochastic alerkin methods for elliptic interface problems with random input*, Journal of Computational and Applied Mathematics, 236 (2011), pp. 782–792.

[206] Z. Zlatev and I. Dimov, *Computational and Numerical Challenges in Environmental Modelling*, vol. 13, Elsevier, 2006.