Gendo-Frobenius algebras
and
comultiplication

Von der Fakultät Mathematik und Physik der Universität Stuttgart
zur Erlangung der Würde eines Doktors der Naturwissenschaften
(Dr. rer. nat.)
genehmigte Abhandlung

Vorgelegt von
Cigdem Yirtici
aus Edirne, Türkei

Hauptberichter:	Prof. Dr. Steffen König
Mitberichterin:	Prof. Dr. Anne Henke
Mitberichter:	Prof. Dr. Ming Fang

Tag der mündlichen Prüfung: 01.12.2020

Institut für Algebra und Zahlentheorie der Universität Stuttgart

2020

Acknowledgements

First of all, I would like to thank my supervisor Steffen Koenig. I am grateful for his guidance, continuous support and patience, for proofreading and valued suggestions, and for providing a friendly work environment.

I would like to thank Claus Michael Ringel for his valuable advice and suggestions which led to starting my PhD in this wonderful environment.

I also thank the working group in Stuttgart for providing a great working atmosphere.

My PhD studies were financially supported by DFG. I had also financial support by DAAD for a while. I am grateful for all the support which allowed me to fully focus on my research.

My deep and sincere gratitude also goes to Bahadir Gizlici for his support and encouragement.

And finally, I would like to thank my family for their support.

Eigenständigkeitserklärung

Hiermit bestätige ich, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Hilfsmittel benutzt habe. Die Stellen der Arbeit, die dem Wortlaut oder dem Sinn nach anderen Werken entnommen sind, wurden unter Angabe der Quelle kenntlich gemacht.

Cigdem Yirtici

Contents

Abstract

Two large classes of algebras, Frobenius algebras and gendo-symmetric algebras, are characterised by the existence of a comultiplication with some special properties. Symmetric algebras are both Frobenius and gendo-symmetric. In [20], Kerner and Yamagata investigated two variations of gendo-symmetric algebras and in fact these two variations contain gendo-symmetric and Frobenius algebras. We call one of these variations gendo-Frobenius algebras. In this thesis, we construct a comultiplication for gendo-Frobenius algebras, which specialises to the known comultiplications on Frobenius algebras and on gendo-symmetric algebras. Moreover, we show that Frobenius algebras are precisely those gendo-Frobenius algebras that have a counit compatible with this comultiplication.

Zusammenfassung

Zwei große Klassen von Algebren, Frobenius Algebren und gendo-symmetrische Algebren, sind charakterisiert durch die Existenz einer Komultiplikation mit einigen besonderen Eigenschaften. Symmetrische Algebren sind sowohl Frobenius als auch gendo-symmetrisch. In [20] untersuchten Kerner und Yamagata zwei Varianten von gendo-symmetrischen Algebren und die beiden Varianten enthalten tatsächlich die gendo-symmetrischen und die Frobenius Algebren. Eine dieser Varianten nennen wir gendo-Frobenius Algebren. In dieser Arbeit konstruieren wir eine Komultiplikation für Gendo-Frobenius Algebren, die die bekannten Komultiplikationen bei Frobenius Algebren und bei gendo-symmetrischen Algebren als Spezialfälle enthält. Darüber hinaus zeigen wir, dass Frobenius Algebren genau jene Gendo-Frobenius Algebren sind, deren Koeins mit dieser Komultiplikation kompatibel ist.

Chapter 1

Introduction

Representation theory is an important branch of mathematics which has applications in many other areas of mathematics, also in physics, chemistry and computer science. Significant building blocks of representation theory are algebraic structures. An algebraic structure consists of a set and a collection of operations on this set, which obey certain axioms. For example, groups which have just one operation (such as multiplication or composition) and rings which have two operations (addition and multiplication) are basic algebraic structures. The definition of an algebraic structure can have any number of sets and any number of axioms. For instance, vector space structure has two sets (an abelian group and a field) and two operations (vector addition and scalar multiplication), which satisfies some axioms. Another fundamental algebraic structure is algebra over a field. Roughly speaking, an algebra A over a field k is a vector space over k equipped with an additional operation (multiplication in A) and A is simply called k-algebra. Briefly, in the standard definition of k-algebra, it has three operations (addition, scalar multiplication and multiplication in A), which satisfies some axioms. Under some conditions, this situation can be improved further by a fourth operation, comultiplication, and some classes of k-algebras can be characterised by existence of a comultiplication with particular properties.

Let us explain this situation by an example. Let $G = \{g_1, ..., g_n\}$ be a finite group written multiplicatively. The group algebra kG over k is defined as the set of linear combinations $\sum_{i=1}^{n} c_i g_i$ (where $c_i \in k$) with multiplication given by linearly extending the multiplication in G. Indeed, the group algebra kG has a comultiplication $\Delta : kG \to kG \otimes_k kG$ as a fourth operation such that $\Delta(g) = \sum_{i=1}^{n} gg_i \otimes_k g_i^{-1}$ for any $g \in G$. The comultiplication Δ is a kG-bimodule morphism. The group algebra kG actually also has another, different comultiplication $\widetilde{\Delta} : kG \to kG \otimes_k kG$ which sends g to $g \otimes_k g$ for any $g \in G$. This group algebra kG admits a Hopf algebra structure over k with the comultiplication $\widetilde{\Delta}$, the counit $f : kG \to k$ such that $f(g) = 1$ for any $g \in G$ and some other special linear maps. For more detail about Hopf algebras, see [31], Chapter VI and for more information about Δ and $\widetilde{\Delta}$, see Remark 2.2.6 in Chapter 2. However, we consider Δ as our main comultiplication for the group algebra kG since it is more suitable to our context and in this thesis we use this comultiplication. Here, it is natural to ask the following question.

Why does a group algebra kG have comultiplication?

Answer: Because it is a symmetric algebra and symmetric algebras are characterised by the existence of a comultiplication with certain properties. Roughly speaking, a k-algebra A is symmetric if there is an isomorphism $\lambda : A \cong \mathrm{D}(A)$ of A-bimodules, where D denotes the usual k-duality functor $\mathrm{Hom}_k(-, k)$. Dualising the multiplication map $\mu : A \otimes_k A \to A$ gives a map $\mu^* : \mathrm{D}(A) \to \mathrm{D}(A) \otimes_k \mathrm{D}(A)$. Therefore, by using λ, we obtain the comultiplication $\Delta : A \to A \otimes_k A$. Here, $\lambda(1_A)$ serves as a counit for Δ.

This answer leads to another natural question:

Are there any classes of nonsymmetric algebras which are characterised by the existence of a comultiplication with different properties?

Answer: Yes. The class of Frobenius algebras and the class of gendo-symmetric algebras are characterised by the existence of a comultiplication with some special properties. Moreover, in this thesis we call gendo-Frobenius algebras a special class of Morita algebras and construct a comultiplication for gendo-Frobenius algebras, which specialises to the known comultiplications on Frobenius algebras and on gendo-symmetric algebras.

Now we are going into more detail and give some information on symmetric algebras, Frobenius algebras, gendo-symmetric algebras and our main topic gendo-Frobenius algebras.

Symmetric algebras. A symmetric algebra can be characterised equivalently as: a finite dimensional algebra A equipped with an associative nondegenerate symmetric bilinear form, or equipped with a central linear form whose kernel does not contain a nonzero one-sided ideal, or equipped with a left (or right) A-isomorphism from A to the dual space $\mathrm{D}(A)$ which is also right (or left) A-linear, respectively. Symmetric algebras are special class of Frobenius algebras and contain well-known classes of algebras: matrix algebras, group algebras of finite groups and some quantum groups. For more information about symmetric algebras, see Subsection 2.1.1.

Frobenius algebras. A Frobenius algebra can be characterised equivalently as: a finite dimensional algebra A equipped with an associative nondegenerate bilinear form, or equipped with a linear form whose kernel contains no nonzero ideals, or equipped with an A-linear isomorphism from A to the dual space $\mathrm{D}(A)$. Frobenius algebras were first studied by Frobenius [16] around 1900 and later by Brauer, Nesbitt [4] and Nakayama [26, 27] in 1937–1941. Later a significant characterisation of Frobenius algebras in terms of comultiplication appeared. As stated in [21], the characterisation of Frobenius algebras in terms of comultiplication goes back at least to Lawvere [23] (1967), and it was rediscovered by Quinn [28] and Abrams [1] in the 1990's.

Frobenius algebras are algebras and also coalgebras, with compatibility between multiplication and comultiplication. Examples are matrix rings, group rings and the ring of characters of a representation. Hopf algebras are Frobenius algebras as well. In recent years, Frobenius algebras started to become more popular because of their connection with computer science and theoretical physics. In computer science Frobenius algebras appeared in concurrent programming, control theory, quantum computing, etc [11]. In physics Frobenius algebras have an outstanding connection with topological quantum field theory. More clearly, in [1], Abrams showed that the category of commutative Frobenius algebras is equivalent to the category of two dimensional topological quantum field theories. Moreover, in the same paper, he proved that commutative Frobenius algebras are characterised by the existence of a comultiplication with properties like counit and coassociative. Later he showed that these characterisations work for noncommutative Frobenius algebras as well, that is, in [2], he proved that noncommutative Frobenius algebras are characterised by the existence of a comultiplication as same properties with commutative Frobenius algebras.

In this thesis, we especially focus on the studies of Abrams on Frobenius algebras with respect to comultiplication [1, 2] and obtain new results. Therefore, we devote Chapter 2 to Frobenius algebras.

Now we are going to give information on gendo-symmetric algebras. Therefore, we need the following concept: dominant dimension.

Dominant dimension. Let A be a finite dimensional k-algebra. The dominant dimension of A is at least d (written by dom.dim$A \geq d$) if there is an injective coresolution

$$0 \longrightarrow A \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{d-1} \longrightarrow I_d \longrightarrow \cdots$$

such that all modules I_i where $0 \leq i \leq d-1$ are also projective.

There is another homological invariant which is defined as follows.

Global dimension. Let A be a finite dimensional k-algebra. The global dimension of A is defined the supremum of the set of projective dimensions of all A-modules.

A finite dimensional left A-module M is said to have *double centralizer property* if the canonical homomorphism of algebras $f : A \to \mathrm{End}_B(M)$ is an isomorphism for $B = \mathrm{End}_A(M)^{op}$.

If dom.dim$A \geq 1$, then I_0 in the definition of dominant dimension is projective-injective and up to isomorphism it is the unique minimal faithful right A-module. Therefore, it is of the form eA for some idempotent e in A. Note that eA is a generator-cogenerator as a left eAe-module. If further dom.dim$A \geq 2$, then eA has double centraliser property, namely, $A \cong \mathrm{End}_{eAe}(eA)$ canonically.

One of the most important examples of double centralizer property is classical Schur-Weyl duality between Schur algebras $S_k(n,r)$ for $n \geq r$ and group algebras of symmetric groups Σ_r. Let us explain this example in more detail.

Let k be an infinite field of any characteristic and E be an n-dimensional k-vector space. Let Σ_r be the symmetric group on r letters. Then group algebra $k\Sigma_r$ operates naturally on $E^{\otimes r}$ from the right. By definition, the Schur algebra $S_k(n,r) = \mathrm{End}_{k\Sigma_r}(E^{\otimes r})$. Let $n \geq r$. There is an outstanding theorem which is called *Schur-Weyl duality.* This theorem relates the representation theories of general linear and symmetric groups and states that there is a double centralizer property, namely, $S_k(n,r) \cong \mathrm{End}_{k\Sigma_r}(E^{\otimes r})$ and $k\Sigma_r \cong \mathrm{End}_{S_k(n,r)}(E^{\otimes r})$. Therefore, the Schur algebra $S_k(n,r)$ has dominant dimension at least two. Indeed, in this case, the tensor space $E^{\otimes r}$ is a faithful projective and injective module.

We now ask the following question:

What properties are the two algebras $S_k(n,r)$ and $k\Sigma_r$ sharing?

Since $k\Sigma_r$ is a symmetric algebra, it has comultiplication. Surprisingly, Schur algebra $S_k(n,r)$ has comultiplication as well, that is, the two algebras $S_k(n,r)$ and $k\Sigma_r$ are sharing the same property, comultiplication. In fact, we have general version of this situation which is called gendo-symmetric algebras.

Gendo-symmetric algebras. A new class of algebras called gendo-symmetric algebras have been introduced by Fang and Koenig [13, 14]. The construction of gendo-symmetric algebras comes by using symmetric algebras and the Morita-Tachikawa correspondence which is the general form of Auslander's correspondence. Let us give more detail.

Let Λ be an Artin algebra and M be an Λ-module which is generator-cogenerator. This means M contains each indecomposable projective module and each indecomposable injective module as a direct summand, up to isomorphism. And let A be any Artin algebra.

Morita-Tachikawa Correspondence. There is a correspondence between the class of all pairs (Λ, M)

and the class of all algebras A of dominant dimension at least two.

This correspondence sends the algebra A of dominant dimension at least two to $(\Lambda = eAe, M = eA)$. Conversely, it sends the pair (Λ, M) to the endomorphism ring $A = \mathrm{End}_\Lambda(M)$. Therefore, this correspondence states that every endomorphism algebra of generator-cogenerator has dominant dimension at least 2, and is characterised by this property. To turn this correspondence into a bijection, we only need to require Λ to be basic in the pair (Λ, M).

Now, restrict the Morita-Tachikawa correspondence to more special case. Here, assume that Λ has finite representation type, that is the number of isomorphism classes of indecomposable representations of Λ is finite. Therefore, we can choose M to be a full direct sum of indecomposable Λ-modules that each indecomposable module occurs at least once as a summand, up to isomorphism. This restriction is the very famous Auslander's correspondence. It can be stated as follows:

Auslander's Correspondence. There is a bijection between the algebras Λ of finite-representation type and the algebras A with dominant dimension at least two and global dimension at most two.

In [18], Iyama established higher Auslander's correspondence. Here, on the right hand side is the class of algebras of dominant dimension at least n and global dimension at most n for a natural number $n \geq 2$. On the left hand side, for $n \geq 3$ there are new objects, which have turned out to be important in cluster theory. This higher Auslander's correspondence gives a new direction to research and also has many applications.

Auslander's correspondence shows how to apply Morita-Tachikawa correspondence by making a particular choice of (Λ, M). In [14], Fang and Koenig provided a new correspondence in the same style with Auslander's correspondence, where the algebra Λ is now restricted to symmetric algebras:

There is a bijection between the class of all pairs (Λ, M) where Λ is finite-dimensional symmetric algebra and M a generator in Λ-mod and the class of all algebras A which is finite-dimensional and $\mathrm{Hom}_A(\mathrm{D}(A), A) \cong A$ as (A, A)-bimodules, where D denotes the duality over the ground field. Since Λ is symmetric, generator M over Λ is same as cogenerator. Therefore, only generators are mentioned here. The algebras A in this bijection are called *gendo-symmetric algebras*.

The term 'gendo-symmetric' is meant to indicate that one characterisation of these algebras is as endomorphism rings of generators (module containing each indecomposable projective module at least once as a direct summand) over a symmetric algebra.

Gendo-symmetric algebras are characterised by the existence of a comultiplication and have the properties used for defining the bar cocomplex. The exactness of this bar cocomplex is used to determine the dominant dimension of gendo-symmetric algebras [13]. Gendo-symmetric algebras extend the subclass $\mathcal{A}$ of quasi-hereditary algebras introduced in [15]. These include the algebras on both sides of classical Schur-Weyl duality and of Soergel's structure theorem for the BGG-category $\mathcal{O}$. Moreover, the class of gendo-symmetric algebras contains many other examples from algebraic Lie theory as well as symmetric algebras and Auslander algebras of symmetric algebras.

In this thesis, we focus on the studies of Fang and Koenig on gendo-symmetric algebras with respect to comultiplication [13, 14] and obtain new results. Therefore, we devote Chapter 3 to gendo-symmetric algebras.

Both Frobenius algebras and gendo-symmetric algebras are characterised by the existence of a comultiplication with some special properties. However, these two classes of algebras have differences. For example, Frobenius algebras have a counit compatible with their comultiplication but gendo-symmetric algebras do not, in general. Here, it is natural to ask whether there are other properties distinguishing Frobenius algebras from gendo-symmetric algebras. More precisely, which properties of Frobenius algebras do gendo-symmetric algebras fail to have? At this point, the following question appears:

Question 1. What are the differences between Frobenius algebras and gendo-symmetric algebras with respect to comultiplication?

In Section 3.4, we answer this question and clarify these differences between Frobenius algebras and gendo-symmetric algebras with respect to comultiplication.

Besides these differences, there are also important similarities. For example, both are characterised by the existence of a comultiplication with some special properties as we mentioned before. Moreover, both contain symmetric algebras. Here, it is natural to ask the following question.

Question 2. Is there a common generalisation of Frobenius algebras and gendo-symmetric algebras such that this generalisation has a comultiplication, which specialises to the known comultiplications on Frobenius algebras and on gendo-symmetric algebras?

Answering this question leads to introducing a new class of algebras which we called gendo-Frobenius algebras.

Gendo-Frobenius algebras. In [20], Kerner and Yamagata investigated two variations of gendo-symmetric algebras and in fact these two variations contain gendo-symmetric and Frobenius algebras. First variation is motivated by Morita [24] and they called a finite dimensional algebra A *Morita algebra*, if A is the endomorphism ring of a generator–cogenerator over a self-injective algebra. In other words, from Morita-Tachikawa correspondence by making a particular choice of (Λ, M), where Λ is finite-dimensional selfinjective algebra and M a generator-cogenerator in Λ-mod and the class of all algebras A which is finite-dimensional, we obtain Morita algebras. Morita algebras form a class of algebras properly containing all self-injective algebras and Auslander algebras of self-injective algebras of finite representation type. They are also properly contained in the class of algebras with dominant dimension at least 2. Second variation is defined by relaxing the condition on the bimodule isomorphism in the definition of gendo-symmetric algebras and we call these algebras *gendo-Frobenius algebras*. In this thesis, we construct a comultiplication for gendo-Frobenius algebras, which specialises to the known comultiplications on Frobenius algebras and on gendo-symmetric algebras and so they are the common generalisation that we asked in Question 2. Therefore, in Chapter 4, we mainly focus on introducing gendo-Frobenius algebras and their comultiplication.

Inspired by Fang and Koenig, we mean the term 'gendo-Frobenius' to indicate that one characterisation of these algebras is as endomorphism rings of generators-cogenerators (module containing each indecomposable projective and injective module at least once as a direct summand) over a Frobenius algebra.

We may visualize the hierarchy of the finite dimensional algebras on which we work in this study as follows.

$$\begin{array}{ccccc}
 & & \text{Morita algebras} & & \\
 & & \downarrow & & \\
 & & \text{Gendo-Frobenius algebras} & & \\
 & \swarrow & & \searrow & \\
\text{Frobenius algebras} & & & & \text{Gendo-symmetric algebras} \\
 & \searrow & & \swarrow & \\
 & & \text{Symmetric algebras} & &
\end{array}$$

In the above diagram, an arrow means the class on top contains the class below.

This thesis is organized as follows. Second chapter begins with giving the definition of Frobenius algebras and providing basic properties as well as some necessary results. We also introduce the Nakayama automorphism of Frobenius algebras which is fundamental for further considerations. Moreover, we introduce symmetric algebras which are special class of Frobenius algebras. We additionally give some examples for Frobenius algebras. Frobenius Nakayama algebras which are essential for Chapter 4 are also introduced in this chapter. Later we recall the definition of k-algebras and formulate its axioms in terms of commutative diagrams. By reversing all arrows in these diagrams, we also give the definition of coalgebras. After that we introduce an important characterisation of Frobenius algebras in terms of comultiplication and give main results of this chapter. In particular, we emphasize that Proposition 2.1.4 in [3] given by Abrams in commutative case is also satisfied for all finite dimensional Frobenius algebras over k (Theorem 2.2.9) and then inspired by a result of Fang and Koenig ([13], Lemma 2.6) we give a theorem which shows the structure of the comultiplication of Frobenius algebras and plays a crucial role to clarify differences between Frobenius and gendo-symmetric algebras (Theorem 2.2.11).

Chapter 3 is devoted to introducing gendo-symmetric algebras and their characterisation in terms of comultiplication. We first give the definition of gendo-symmetric algebras and then exhibit some examples of these algebras. Later we introduce the construction of gendo-symmetric algebras' comultiplication and some results which were obtained by Fang and Koenig [13]. We additionally give new results on gendo-symmetric algebras with respect to comultiplication.

We have already mentioned that the Schur algebra $A = S_k(n, r)$ for $n \geq r$ and the symmetric algebra $k\Sigma_r$ are sharing the same property, comultiplication. Since A is gendo-symmetric, $\mathrm{End}_{eAe}(eA) \cong A$ and $\mathrm{End}_A(eA) \cong eAe$ where eA is a basic faithful projective-injective A-module for an idempotent e of A such that eAe is symmetric. Moreover, $A \cong \mathrm{End}_{k\Sigma_r}(E^{\otimes r})$ and $k\Sigma_r \cong \mathrm{End}_A(E^{\otimes r})$, where $E^{\otimes r}$ is a faithful projective-injective A-module. Since all endomorphism rings of faithful projective-injective A-modules are Morita equivalent (see Lemma 2.3 in [22]), the symmetric algebra eAe is Morita equivalent to the group algebra $k\Sigma_r$. The following theorem relates the comultiplication on the Schur algebra A with the comultiplication on the symmetric algebra eAe. Moreover, it gives also the general situation, that is, gives the relation between the comultiplication of any gendo-symmetric algebra A and comultiplication of the symmetric algebra eAe.

Theorem A (Theorem 3.2.10) *Let A be a gendo-symmetric algebra with a basic faithful projective-injective A-module Ae for an idempotent e of A such that eAe is symmetric. Let $\pi : A \to eAe$ be the k-linear map such that $\pi(a) = eae$ for $a \in A$. Suppose that Δ_A is a comultiplication of A. Then there exists a comultiplication Δ_{eAe} of eAe such that $(\pi \otimes \pi)\Delta_A = \Delta_{eAe}\pi$.*

The following theorem also shows the relation between the comultiplication of any gendo-symmetric algebra A and comultiplication of the gendo-symmetric quotient algebra $B = A/I$, where I is a two-sided ideal of A.

Theorem B (Theorem 3.2.13) *Let A and B be gendo-symmetric algebras such that $B = A/I$, where I is a two-sided ideal of A, and $\pi : A \to B$ be the canonical surjection. Suppose that Ae and Bf are basic faithful projective-injective A-module and B-module, respectively, where $e = e' + e''$ is an orthogonal decomposition and $f = e' + I$. Let Δ_A be a comultiplication of A. Then there exists a comultiplication Δ_B of B such that $(\pi \otimes \pi)\Delta_A = \Delta_B \pi$.*

We also visit gendo-symmetric Schur algebras and obtain some new results in terms of comultiplication in Chapter 3. More clearly, we first give the definition of Schur algebras and indicate in which case the Schur algebras are gendo-symmetric. At the same time, we give information about their dominant dimension. Later we introduce the Schur-Weyl duality and give some related examples. We introduce some results from [33] to show the motivation of the main results related to gendo-symmetric Schur algebras. In particular, there are some remarkable algebras related to gendo-symmetric Schur algebras by [33] and we give some results which compute the comultiplication of these algebras (Proposition 3.2.25 & Proposition 3.2.26).

Moreover, in Chapter 3, we introduce a result given by Fang and Koenig [13] on the characterisation of gendo-symmetric algebras and their dominant dimension by using bar cocomplex. At this point, it is time to mention a major homological conjecture and also a major open problem in representation theory, which is called *Nakayama conjecture*. It states that if A is finite-dimensional algebra over a field and dom.dim$A = \infty$, then A is self-injective. In this chapter, we give a hypothesis by using the characterisation of gendo-symmetric algebras in terms of bar cocomplex. Proving this hypothesis may lead to prove *Nakayama conjecture for gendo-symmetric algebras* which states that if A is a gendo-symmetric algebra and dom.dim$A = \infty$, then A is symmetric. This hypothesis is stated as follows.

Hypothesis *Let A be a gendo-symmetric algebra with comultiplication Δ. Then there exists a bar cocomplex for A, using the comultiplication Δ. Suppose that this bar cocomplex is exact. Then there exists a counit of (A, Δ).*

The connection to the Nakayama conjecture uses the following implications from [13]:

A gendo-symmetric algebra A with comultiplication Δ has a counit $\Rightarrow$ The bar cocomplex of A is exact $\Rightarrow$ domdim$A = \infty$.

Now let us consider the reverse of these implications. Let A be a gendo-symmetric algebra with comultiplication Δ. Then

dom.dim$A = \infty \overset{1}{\Rightarrow}$ The bar cocomplex of A is exact $\overset{(*)}{\Rightarrow}$ There exists a counit of $(A, \Delta) \overset{2}{\Rightarrow}$ A is symmetric.

The implications $\overset{1}{\Rightarrow}$ and $\overset{2}{\Rightarrow}$ are known by [13]. If the hypothesis is proved, then the implication (*) is satisfied and Nakayama conjecture for gendo-symmetric algebras is proved.

First main problem discussed in this study was Question 1. In Section 3.4, we answer Question 1 and clarify the differences between Frobenius algebras and gendo-symmetric algebras with respect to comultiplication.

Chapter 4 is devoted to introducing gendo-Frobenius algebras and constructing their comultiplication. We first collect some necessary results obtained by Kerner and Yamagata [20] for background and then define the gendo-Frobenius algebras by using a result of them ([20], Theorem 3). We additionally give some examples for gendo-Frobenius algebras. Later inspired by Fang and Koenig [13], we construct a coassociative comultiplication (possibly without a counit) for gendo-Frobenius algebras and give its properties. In particular, we show that:

Theorem C (Theorem 4.2.3 & Proposition 4.2.13) *Let A be a gendo-Frobenius algebra. Then A has a coassociative comultiplication which is a map of A-bimodules. In addition, there is a compatible counit if and only if A is Frobenius.*

Indeed, in this thesis we also show that there are further constructions possible that yield comultiplications on gendo-Frobenius algebras. In Subsection 4.2.3, we investigate three such constructions and show that they are lacking crucial properties such as being coassociative. However, since comultiplication of Frobenius algebras and comultiplication of gendo-symmetric algebras which have been introduced by Abrams [2] and Fang and Koenig [13], respectively, are coassociative, we mainly focus on the comultiplication given in Theorem C. We also mentioned that gendo-Frobenius algebras contain both Frobenius and gendo-symmetric algebras. Therefore, Theorem C answers Question 2 which is second main problem discussed in this study. Moreover, we compare the comultiplication in Theorem C with the comultiplication of Frobenius algebras which is given by Abrams [2] by assuming that the finite dimensional algebra is Frobenius.

Finally, we give some results on comultiplication of Frobenius Nakayama algebras and their compatible counit. In particular, we give a comultiplication formula for Frobenius Nakayama algebras (Theorem 4.3.5) and also a result which is related to compatible counit of these algebras (Theorem 4.3.9).

In this study some classes of examples such as Schur algebras or Frobenius Nakayama algebras especially play important role and exhibit the connection between the chapters. We discuss these examples at several places by showing and computing their different properties. Let us reveal some of these examples by mentioning what we are studying and what we are showing there, with references to the relevant sections. For the definition of Frobenius Nakayama algebra N_n^m $(m, n \geq 1)$, see Subsection 2.3.1.

Frobenius Nakayama algebra N_3^2. In Subsection 2.1.3, we give the Nakayama automorphism and the Nakayama permutation of N_3^2. In Section 2.2, we compute its comultiplication by using Theorem 2.2.9 and obtain its counit. We mention that it is a gendo-Frobenius algebra in Subsection 4.1.1 and give a detailed computation of its comultiplication in Section 4.2. Moreover, after computing comultiplication of this algebra, we obtain again its counit compatible with that comultiplication. We use this algebra also in Section 4.3 and we again compute its comultiplication by using the formula in Corollary 4.3.6. Indeed, in this study, we compute the comultiplication of N_3^2 by using three different methods.

Schur algebra $S_k(2,2)$. In fact, we generally use an algebra A which is Morita equivalent to $S_k(2,2)$ if k is an infinite field of characteristic 2 (see Subsection 3.2.1). In Subsection 3.1.1, we show that A is gendo-symmetric and compute the dominant dimension of A. Since being gendo-symmetric and dominant dimension are Morita invariant properties, at the same time we actually show $S_k(2,2)$ is gendo-symmetric and compute the dominant dimension of $S_k(2,2)$. In Section 3.2, we again use the algebra A in the example which is an application of Theorem B.

A block of a quantised Schur algebra in quantum characteristic 4. In Subsection 3.1.1, we use an algebra A which is Morita equivalent to a block of a quantised Schur algebra in quantum characteristic 4 (see Subsection 3.2.1). We show that A is gendo-symmetric and compute the dominant dimension of A. In Section 3.2, as an application of Theorem A, we show the relation between the comultiplication of A and the comultiplication of eAe for a suitable choice of the idempotent $e \in A$.

Morita algebras with associated Frobenius Nakayama algebra N_2^1. Let $B = N_2^1$ and M be a faithful right B-module which satisfies a certain condition. In Subsection 4.1.1, we show that the algebra $A = \mathrm{End}_B(M)$ is gendo-Frobenius under this condition. In Section 4.2, we compute the comultiplication of A in detail. We also mention that the algebra A does not have a counit compatible with its comultiplication since it is not Frobenius. In Subsection 4.1.1, we state that the algebra A may not be gendo-Frobenius even if it is Morita algebra when the right B-module M does not satisfy the condition mentioned above. We explain this condition there as well. Moreover, we use the gendo-Frobenius algebra A to show that other approaches to new comultiplications given in Subsection 4.2.3 are lacking crucial properties such as being coassociative.

Chapter 2

Frobenius algebras

Frobenius algebras were first studied by Frobenius [16] around 1900 and later by Brauer, Nesbitt [4] and Nakayama [26, 27] in 1937–1941. As stated in [21], the characterisation of Frobenius algebras in terms of comultiplication goes back at least to Lawvere [23] (1967), and it was rediscovered by Quinn [28] and Abrams [1] in the 1990's. In this study, we generally use a number of results of Abrams [1, 2]. To learn more about Frobenius algebras, see [31] and [34]. Moreover, to learn more detail about commutative Frobenius algebras and its relation with topological quantum field theory, see [1] and [21].

In this chapter, we introduce a main object of this study, Frobenius algebras, as well as give some important results with respect to comultiplication.

More clearly, first section is devoted to defining Frobenius algebras. We also introduce the Nakayama automorphism of Frobenius algebras which is fundamental for further considerations. Moreover, we introduce symmetric algebras which are special class of Frobenius algebras and Frobenius Nakayama algebras which are essential for the last chapter. We additionally give some examples for Frobenius algebras.

In the second section, an important characterisation of Frobenius algebras in terms of comultiplication is introduced and main results of this chapter are given.

Throughout this chapter, all algebras and modules are finite dimensional over an arbitrary field k unless stated otherwise. By mod-A, we denote the category of finite dimensional right A-modules and by D the usual k-duality functor $\mathrm{Hom}_k(-,k)$. For simplicity, we denote $\otimes_k$ by $\otimes$.

2.1 Definition and examples of Frobenius algebras

Definition and basic properties

Definition 2.1.1. A finite dimensional k-algebra A is called *Frobenius* if it satisfies one of the following equivalent conditions:

(i) There exists a nondegenerate bilinear form $\beta : A\otimes A \to k$ which is associative, that is, $\beta(ab\otimes c) = \beta(a\otimes bc)$ for all $a, b, c \in A$.

(ii) There exists a linear form $\varepsilon : A \to k$ whose kernel does not contain a nonzero left ideal of A.

(iii) There exists an isomorphism $\lambda_L : A \to \mathrm{D}(A)$ of left A-modules.

(iv) There exists a linear form $\varepsilon' : A \to k$ whose kernel does not contain a nonzero right ideal of A.

(v) There exists an isomorphism $\lambda_R : A \to \mathrm{D}(A)$ of right A-modules.

The above definition is based on [31], Theorem IV.2.1, which provides the equivalence of the five conditions.

Remark 2.1.2. The linear form $\varepsilon : A \to k$ in Definition 2.1.1 is called *Frobenius form.*

At this point, we give some parts of the proof which provides the equivalence of the five conditions in Definition 2.1.1 and plays an important role to prove especially Theorem 2.2.9. Let $\mu : A \otimes A \to A$ be the multiplication map and $\overline{\mu} : A \to \mathrm{End}(A)$ be the map such that $\overline{\mu}(a)(b) := \mu(b \otimes a)$. Here, we note that given $\lambda_L : A \cong \mathrm{D}(A)$ satisfying condition (iii), the linear form $\varepsilon = \lambda_L(1_A)$ satisfies condition (ii). Given the linear form $\varepsilon : A \to k$ satisfying condition (ii), the bilinear form $\beta = \varepsilon \circ \mu$ satisfies condition (i). Given $\beta : A \otimes A \to k$ satisfying the condition (i), the linear form $\varepsilon = \beta(1_A \otimes -) = \beta(- \otimes 1_A)$ satisfies condition (ii). And the linear map $\lambda_L = \varepsilon \circ \overline{\mu}$ satisfies condition (iii).

Proposition 2.1.3. *([21], Lemma 2.2.8) If A is a k-algebra with Frobenius form ε, then every other Frobenius form on A is given by $c \cdot \varepsilon$, where c is an invertible element of A. Equivalently, for a given fixed left A-module isomorphism $\lambda_L : A \cong D(A)$, the elements in $D(A)$ which are Frobenius forms are precisely the images of the invertible elements in A.*

The following proposition gives a general method to find a bilinear form giving the structure of a Frobenius algebra to a finite dimensional k-algebra.

Proposition 2.1.4. *([35], Proposition 1.10.18) Let $A = kQ/I$ be a Frobenius algebra over k given by a quiver Q and ideal of relations I, and fix a k-basis $\mathfrak{B}$ of A consisting of pairwise distinct nonzero paths of the quiver Q. Assume that $\mathfrak{B}$ contains a basis of the socle $soc(A)$ of A. Define a k-linear mapping ε on the basis elements by*

$$\varepsilon(b) = \begin{cases} 1 & \text{if } b \in soc(A) \\ 0 & \text{otherwise} \end{cases}$$

for $b \in \mathfrak{B}$. Then an associative nondegenerate k-bilinear form $\beta : A \otimes A \to k$ for A is given by $\beta(x \otimes y) := \varepsilon(xy)$.

The following theorem shows that the construction in Proposition 2.1.4 is the only possible construction.

Theorem 2.1.5. *([35], Proposition 3.6.14) Let A be a finite dimensional Frobenius k-algebra and suppose $A = kQ/I$ for a quiver Q and an admissible ideal I and an algebraically closed field k. Then for every nondegenerate associative bilinear form $\beta : A \otimes A \to k$, there is a k-basis $\mathfrak{B}$ containing a k-basis of the socle such that $\beta(x \otimes y) = \varepsilon(xy)$, where ε is defined by*

$$\varepsilon(b) = \begin{cases} 1 & \text{if } b \in soc(A) \cap \mathfrak{B} \\ 0 & \text{if } b \in \mathfrak{B} \setminus soc(A). \end{cases}$$

Definition 2.1.6. A finite dimensional k-algebra A is called *self-injective* if the modules A_A and ${}_AA$ are injective.

Therefore, a finite dimensional k-algebra A is self-injective if the projective modules in mod-A (respectively, in mod-A^{op}) coincide with the injective modules.

Let A be a finite dimensional self-injective algebra over k. Then 1_A has a decomposition $1_A = \sum_{i=1}^{n} \sum_{j=1}^{m_i} e_{ij}$, where the e_{ij} are the pairwise orthogonal idempotents, with $e_{ij}A \cong e_{rs}A$ if and only if $i = r$. Therefore, $e_{11}A$, $e_{21}A$,...,$e_{n1}A$ is a complete set of pairwise nonisomorphic indecomposable right A-modules. The socle of a projective-injective right A-module is simple, but not necessarily isomorphic to the top and each simple module occurs once as a top and once as a socle. Hence, there exists a permutation ν of $\{1, ..., n\}$, called the *Nakayama permutation*, such that

$$\mathrm{top}(e_{\nu(i)1}A) \cong \mathrm{soc}(e_{i1}A) \text{ for all } i \in \{1, ..., n\}.$$

The following selected results show the close relation between Frobenius and self-injective algebras.

Proposition 2.1.7. *([31], Proposition IV.3.8) Let A be a Frobenius algebra over a field k. Then A is a self-injective algebra.*

Proposition 2.1.8. *([31], Proposition IV.3.9) Let A be a basic self-injective finite dimensional algebra over a field k. Then A is a Frobenius algebra.*

Corollary 2.1.9. *([31], Corollary IV.3.12) Let Q be a finite quiver, k a field, I an admissible ideal of the path algebra kQ, and $A = kQ/I$ the associated bound quiver algebra, and assume A is a self-injective algebra. Then A is a Frobenius algebra.*

Corollary 2.1.10. *([31], Corollary IV.3.11) Let A be a finite dimensional self-injective algebra over a field k. Then A is Morita equivalent to a Frobenius algebra.*

Note that the class of all finite dimensional self-injective algebras over k is closed under Morita equivalences (see [31], Proposition IV.3.10). But, the class of Frobenius algebras is not (see [31], Chapter IV). In fact, the class of all Frobenius algebras over a field k is a proper subclass of the class of all finite dimensional self-injective algebras, and the class of all finite dimensional self-injective algebras is the smallest class of finite dimensional algebras which contains the class of all Frobenius algebras and is closed under the Morita equivalences.

Now we first give the definition of Nakayama automorphism which plays an essential role in this study, and later observe a prominent result which is related to Nakayama automorphism.

Definition 2.1.11. For a Frobenius algebra A and a nondegenerate associative bilinear form $\beta : A \otimes A \to k$, the k-algebra automorphism $\nu_A : A \to A$ with $\beta(x \otimes y) = \beta(y \otimes \nu_A(x))$ for all $x, y \in A$ is said to be the *Nakayama automorphism* of A associated to β.

For existence of Nakayama automorphism, see [31], Proposition IV.3.1. Note that every Frobenius algebra A has a Nakayama automorphism which is unique up to inner automorphisms ([31], Corollary IV.3.5).

Remark 2.1.12. By the proof of Proposition 2.2 in [30], if A is a Frobenius algebra, and $1_A = \sum_{i=1}^{n} \sum_{j=1}^{m_i} e_{ij}$ is the decomposition of 1_A into the sum of pairwise orthogonal primitive idempotents, then we have

$$\mathrm{top}(\nu_A(e_{ij})A) \cong \mathrm{soc}(e_{ij}A).$$

Then, in particular, the Nakayama automorphism ν_A induces a Nakayama permutation ν of $\{1, ..., n\}$.

For more information about properties of Nakayama automorphism, see [25] and [31]. We now continue with the promised result which is related to Nakayama automorphism. Let ν_A be the Nakayama automorphism associated to a nondegenerate associative bilinear form $\beta : A \otimes A \to k$. Take the associated linear form $\varepsilon = \beta(- \otimes 1_A) = \beta(1_A \otimes -) : A \to k$. Then we have the isomorphism of left A-modules

$$\lambda_L : {}_AA \to {}_A\mathrm{D}(A)$$

such that $\lambda_L(x)(y) = \varepsilon(yx)$ for $x, y \in A$. Moreover, for $x, y, z \in A$, we have

$$\begin{aligned}\lambda_L(xz)(y) &= \varepsilon(y(xz)) = \beta(yxz \otimes 1_A) \\ &= \beta(yx \otimes z) = \beta(\nu_A^{-1}(z) \otimes yx) = \beta(\nu_A^{-1}(z)yx \otimes 1_A) \\ &= \varepsilon(\nu_A^{-1}(z)yx) = \lambda_L(x)(\nu_A^{-1}(z)y) = (\lambda_L(x)\nu_A^{-1}(z))(y)\end{aligned}$$

and so $\lambda_L(xz) = \lambda_L(x)\nu_A^{-1}(z)$. This shows that λ_L defines an isomorphism

$$\lambda_L : A \to \mathrm{D}(A)_{\nu_A^{-1}}$$

of A-bimodules.

2.1.1 Symmetric algebras

In this part, we introduce symmetric algebras which are special Frobenius algebras. Symmetric algebras contain well-known classes of algebras: group algebras of finite groups and some quantum groups. Note that symmetric algebras are also gendo-symmetric which is introduced in the next chapter.

Definition 2.1.13. A finite dimensional k-algebra A is called *symmetric* if it satisfies one of the following equivalent conditions:

(i) There exists a nondegenerate associative bilinear form $\beta : A \otimes A \to k$ such that $\beta(a \otimes b) = \beta(b \otimes a)$ for all $a, b \in A$.

(ii) There exists a linear form $\varepsilon : A \to k$ such that $\varepsilon(ab) = \varepsilon(ba)$ for all $a, b \in A$, and whose kernel does not contain a nonzero one-sided ideal of A.

(iii) There exists an isomorphism $\lambda : A \to \mathrm{D}(A)$ of A-bimodules.

The above definition is based on [31], Theorem IV.2.2, which provides the equivalence of the three conditions.

Proposition 2.1.14. *([21], Lemma 2.2.11) Let (A, ε) be a symmetric Frobenius algebra (i.e. ε is central). Then every other central Frobenius form on A is given by $c \cdot \varepsilon$, where c is a central invertible element of A.*

Note that a Frobenius algebra A is symmetric if and only if ν_A is inner ([34], Therorem 2.4.1). In this case, we may take the identity automorphism as a Nakayama automorphism.

Proposition 2.1.15. *([31], Corollary IV.4.3) Let A and B be Morita equivalent finite dimensional algebras over a field k. Then A is a symmetric algebra if and only if B is a symmetric algebra.*

Therefore, the class of all symmetric algebras over k is closed under Morita equivalences.

A finite dimensional algebra A over a field k is called *weakly symmetric* if $\mathrm{soc}(P) \cong \mathrm{top}(P)$ for any indecomposable projective module P in mod-A. Note that finite dimensional weakly symmetric algebras over k are Frobenius ([31], Corollary IV.6.3) and symmetric algebras over k are weakly symmetric ([31], Corollary IV.6.4).

Remark 2.1.16. Commutative finite dimensional self-injective algebras are symmetric. See [31], Proposition IV.4.6.

2.1.2 Examples

In this part, some examples of Frobenius algebras are exhibited.

Example 2.1.17. The field of complex numbers $\mathbb{C}$ over $\mathbb{R}$ is a Frobenius algebra over $\mathbb{R}$. The obvious Frobenius form is the following

$$\begin{aligned} \mathbb{C} &\to \mathbb{R} \\ a+bi &\to a. \end{aligned}$$

Example 2.1.18. Let $A = \mathrm{Mat}_n(k)$ be the ring of all n-by-n matrices over a field k. The usual trace map of this matrix ring is defined as follows:

$$\begin{aligned} \mathrm{Tr} : \mathrm{Mat}_n(k) &\to k \\ (a_{ij}) &\mapsto \sum_i a_{ii}. \end{aligned}$$

Then A is a Frobenius algebra with the usual trace map. Therefore, Tr is the Frobenius form of A. Moreover, it is a symmetric algebra since the two matrix products CD and DC in A have the same trace.

Example 2.1.19. Finite dimensional semisimple algebras over k are symmetric Frobenius. See [31], Corollary IV.5.17.

Example 2.1.20. Let kG be the group algebra of a finite group G over a field k. Then kG is a Frobenius algebra with the linear form

$$\begin{aligned} \varepsilon : kG &\to k \\ \sum_{g\in G} \lambda_g g &\mapsto \lambda_{1_G}. \end{aligned}$$

Here, ε is the Frobenius form of kG. We observe that $\varepsilon(ab) = \sum_{g\in G} \lambda_g \mu_{g^{-1}}$ and $\varepsilon(ba) = \sum_{g\in G} \mu_g \lambda_{g^{-1}}$, where $a = \sum_{g\in G} \lambda_g g$, $b = \sum_{g\in G} \mu_g g$ in kG. Hence, $\varepsilon(ab) = \varepsilon(ba)$. Then the Frobenius algebra kG is symmetric.

Example 2.1.21. Finite dimensional Hopf algebras over k are Frobenius. For more information, see [31], Chapter IV.

Example 2.1.22. Let k be a field and Q be a quiver given as follows:

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha_1"] & 2 \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\alpha_2"] & 3 \arrow[l, shift left, "\beta_2"] \end{tikzcd}$$

Let I be the ideal in the path algebra kQ generated by $\alpha_1\beta_1$, $\beta_2\alpha_2$, $\beta_1\alpha_1 - \alpha_2\beta_2$ and $A = kQ/I$ be the associated bound quiver algebra. We observe that $\mathrm{D}(Ae_3) \cong e_1A$, $\mathrm{D}(Ae_2) \cong e_2A$ and $\mathrm{D}(Ae_1) \cong e_3A$. This means that every indecomposable projective module is also injective, that is, A is self-injective. Then by using Corollary 2.1.9, we say that A is Frobenius. Since $\mathrm{top}(e_1A) \ncong \mathrm{soc}(e_1A)$ and $\mathrm{top}(e_3A) \ncong \mathrm{soc}(e_3A)$, A is not weakly symmetric and so A is a nonsymmetric Frobenius algebra (see Subsection 2.1.1). A has a k-basis $\{e_1, e_2, e_3, \alpha_1, \alpha_2, \beta_1, \beta_2, \alpha_1\alpha_2, \beta_2\beta_1, \beta_1\alpha_1\}$ so $\mathrm{D}(A)$ has the dual basis

$\{e_1^*, e_2^*, e_3^*, \alpha_1^*, \alpha_2^*, \beta_1^*, \beta_2^*, (\alpha_1\alpha_2)^*, (\beta_2\beta_1)^*, (\beta_1\alpha_1)^*\}$. We observe that there is a left A-module isomorphism λ_L which is explicitly defined on the basis elements by

$$\begin{aligned} \lambda_L : A &\cong \mathrm{D}(A) \\ e_1 &\mapsto (\beta_2\beta_1)^* \\ e_2 &\mapsto (\beta_1\alpha_1)^* \\ e_3 &\mapsto (\alpha_1\alpha_2)^* \\ \alpha_1 &\mapsto \beta_1^* \\ \alpha_2 &\mapsto \alpha_1^* \\ \beta_1 &\mapsto \beta_2^* \\ \beta_2 &\mapsto \alpha_2^* \\ \alpha_1\alpha_2 &\mapsto e_1^* \\ \beta_2\beta_1 &\mapsto e_3^* \\ \beta_1\alpha_1 &\mapsto e_2^*. \end{aligned}$$

Here, the Frobenius form of A is $\varepsilon = \lambda_L(1_A)$. Since $1_A = e_1 + e_2 + e_3$, we obtain that $\varepsilon = (\alpha_1\alpha_2)^* + (\beta_2\beta_1)^* + (\beta_1\alpha_1)^*$.

Example 2.1.23. Let k be a field and Q be a quiver given as follows:

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"] \end{tikzcd}$$

Let I be the ideal in the path algebra kQ generated by $\alpha\beta\alpha$ and $\beta\alpha\beta$, and $A = kQ/I$ be the associated bound quiver algebra. A has a k-basis $\{e_1, e_2, \alpha, \beta, \alpha\beta, \beta\alpha\}$ so $\mathrm{D}(A)$ has the dual basis $\{e_1^*, e_2^*, \alpha^*, \beta^*, (\alpha\beta)^*, (\beta\alpha)^*\}$. Observe that there is an A-bimodule isomorphism $\lambda : A \to \mathrm{D}(A)$ which is explicitly defined on the basis elements by

$$\begin{aligned} \lambda : A &\cong \mathrm{D}(A) \\ e_1 &\mapsto (\alpha\beta)^* \\ e_2 &\mapsto (\beta\alpha)^* \\ \alpha &\mapsto \beta^* \\ \beta &\mapsto \alpha^* \\ \alpha\beta &\mapsto e_1^* \\ \beta\alpha &\mapsto e_2^*. \end{aligned}$$

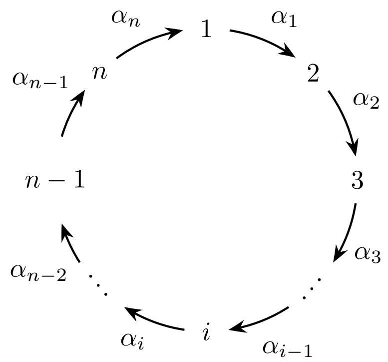
Then A is symmetric by Definition 2.1.13. Since the Frobenius form of A is $\varepsilon = \lambda(1_A)$ and $1_A = e_1 + e_2$, we obtain that $\varepsilon = (\alpha\beta)^* + (\beta\alpha)^*$.

2.1.3 Frobenius Nakayama algebras

A finite dimensional k-algebra A over a field k is called a *Nakayama algebra* if all indecomposable projective modules and all indecomposable injective modules in mod-A are uniserial modules. For more information about Nakayama algebras, see [31], Chapter I.10. In this study, we focus on Frobenius

Nakayama algebras on which we give some results in the last chapter. Therefore, in this part, we introduce some prominent results on Frobenius and symmetric Nakayama algebras which we use in Chapter 4.

Let $N_n^m = kQ/I$ $(m, n \geq 1)$ be the algebra of the following quiver



such that I is the ideal in the path algebra kQ generated by all compositions of $m+1$ consecutive arrows in Q.

Theorem 2.1.24. *([31], Corollary IV.3.12 & Theorem IV.6.15) Let Q be a finite connected quiver with nonempty set of arrows, k a field, I an admissible ideal of the path algebra kQ, and $A = kQ/I$ the associated bound quiver algebra. Then the following are equivalent.*

(i) A is a Frobenius Nakayama algebra.

(ii) $A = N_n^m$ for some positive integers m and n.

Theorem 2.1.25. *([31], Corollary IV.6.16) Let Q be a finite connected quiver with nonempty set of arrows, k a field, I an admissible ideal of the path algebra kQ, and $A = kQ/I$ the associated bound quiver algebra. Then the following are equivalent.*

(i) A is a symmetric Nakayama algebra.

(ii) A is a weakly symmetric Nakayama algebra.

(iii) $A = N_n^m$ for some positive integers m and n, with n dividing m.

Example 2.1.26. Let $A = N_3^2$ and ν_A be a Nakayama automorphism of A. A has a k-basis $\{e_1, e_2, e_3, \alpha_1, \alpha_2, \alpha_3, \alpha_1\alpha_2, \alpha_2\alpha_3, \alpha_3\alpha_1\}$. Then the Nakayama automorphism ν_A can be explicitly defined on the basis elements by

$$\begin{aligned}
\nu_A : A &\to A \\
e_1 &\mapsto e_3 \\
e_2 &\mapsto e_1 \\
e_3 &\mapsto e_2 \\
\alpha_1 &\mapsto \alpha_3 \\
\alpha_2 &\mapsto \alpha_1 \\
\alpha_3 &\mapsto \alpha_2 \\
\alpha_1\alpha_2 &\mapsto \alpha_3\alpha_1 \\
\alpha_2\alpha_3 &\mapsto \alpha_1\alpha_2 \\
\alpha_3\alpha_1 &\mapsto \alpha_2\alpha_3.
\end{aligned}$$

Observe that the Nakayama automorphism ν_A induces a Nakayama permutation ν of A which is the cyclic permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

2.2 Frobenius algebras and comultiplication

In this section, we introduce an important characterisation of Frobenius algebras in terms of comultiplication and give the main results of this chapter. For this aim, we first recall the definition of k-algebras and formulate its axioms in terms of commutative diagrams. By reversing all arrows in these diagrams, we also give the definition of coalgebras.

Remember that all algebras and modules are finite dimensional over an arbitrary field k in this study.

Definition 2.2.1. A *k-algebra A* is defined with two k-linear maps

$$\mu : A \otimes A \to A \text{ and } \eta : k \to A$$

such that the following three diagrams commute:

$$\begin{tikzcd}
& A\otimes A\otimes A \arrow[dl, "\mu\otimes\mathrm{id}_A"'] \arrow[dr, "\mathrm{id}_A\otimes\mu"] & \\
A\otimes A \arrow[dr, "\mu"'] & & A\otimes A \arrow[dl, "\mu"] \\
& A &
\end{tikzcd}$$

$$\begin{tikzcd}
k\otimes A \arrow[r, "\eta\otimes\mathrm{id}_A"] \arrow[dr] & A\otimes A \arrow[d, "\mu"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes A \arrow[d, "\mu"'] & A\otimes k \arrow[l, "\mathrm{id}_A\otimes\eta"'] \arrow[dl] \\
A &
\end{tikzcd}$$

Here, $\mathrm{id}_A : A \to A$ is the identity map and μ and η are called *multiplication* and *unit* map, respectively. Also, first diagram satisfies associativity condition, second and third diagrams satisfy unity condition.

Now, we give the structure of coalgebra over k, that is the dual of the structure of k-algebra. We define the coalgebra by reversing all arrows in the above maps given in the definition of k-algebras.

Definition 2.2.2. A *coalgebra* over k is defined with two k-linear maps

$$\alpha : A \to A \otimes A \text{ and } \varepsilon : A \to k$$

such that the following three diagrams commute:

$$\begin{tikzcd}
& A\otimes A\otimes A & \\
A\otimes A \arrow[ur, "\alpha\otimes\mathrm{id}_A"] & & A\otimes A \arrow[ul, "\mathrm{id}_A\otimes\alpha"'] \\
& A \arrow[ul, "\alpha"] \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
k\otimes A & A\otimes A \arrow[l, "\varepsilon\otimes\mathrm{id}_A"'] \\
& A \arrow[ul] \arrow[u, "\alpha"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes A \arrow[r, "\mathrm{id}_A\otimes\varepsilon"] & A\otimes k \\
A \arrow[u, "\alpha"] \arrow[ur] &
\end{tikzcd}$$

Here, again $\mathrm{id}_A : A \to A$ is the identity map and α and ε are called *comultiplication* and *counit* map, respectively. The satisfied condition in the first diagram is called coassociativity, second and third diagrams are called counity condition.

We denoted the Frobenius form and the counit with same notation ε. This is not a coincidence. In ([21], Proposition 2.3.22), it is proved that every Frobenius algebra has a unique coalgebra structure for which the Frobenius form is the counit, and which is A-linear. Conversely, in ([21], Proposition 2.3.24), it is also proved that given a k-algebra equipped with an A-linear coalgebra structure, then the counit is a Frobenius form. So, this leads to a very important characterisation for Frobenius algebras. In [1], Abrams did it for commutative Frobenius algebras and also in [2], he proved that all of these characterisations work for non-commutative Frobenius algebras. He also established the following important theorem:

Theorem 2.2.3. *([2], Theorem 2.1) An algebra A is a Frobenius algebra if and only if it has a coassociative counital comultiplication $\alpha : A \to A \otimes_k A$ which is a map of A-bimodules.*

Here, we will not give the complete proof of this theorem. But it is necessary to give the construction of the comultiplication α which is given in [2], Theorem 2.1.

Construction of the comultiplication $\alpha : A \to A \otimes_k A$.

Let A be a Frobenius algebra and $\mu : A \otimes_k A \to A$ be the multiplication map. Since A is Frobenius, there is a left A-module isomorphism $\lambda_L : A \cong \mathrm{D}(A)$. Define the comultiplication map $\alpha_L : A \to A \otimes_k A$ to be the composition $(\lambda_L^{-1} \otimes_k \lambda_L^{-1}) \circ \mu^* \circ \lambda_L$:

$$\begin{tikzcd} A \arrow[r, "\alpha_L"] \arrow[d, "\lambda_L"'] & A \otimes_k A \\ \mathrm{D}(A) \arrow[r, "\mu^*"'] & \mathrm{D}(A) \otimes_k \mathrm{D}(A) \arrow[u, "\lambda_L^{-1} \otimes_k \lambda_L^{-1}"'] \end{tikzcd}$$

Note that there is a canonical isomorphism $\zeta : \mathrm{D}(A) \otimes_k \mathrm{D}(A) \xrightarrow{\sim} \mathrm{D}(A \otimes_k A)$ which is described in 2.1.17 in [21]. The dual of $\mu : A \otimes_k A \to A$ is actually a map $\mu^* : \mathrm{D}(A) \to \mathrm{D}(A \otimes_k A)$. Since ζ is an isomorphism, we can compose μ^* with the inverse of ζ and write $\mu^* : \mathrm{D}(A) \to \mathrm{D}(A) \otimes_k \mathrm{D}(A)$. Abrams showed that α_L is a map of left A-modules.

Using the right A-module isomorphism $\lambda_R : A \cong \mathrm{D}(A)$, it is analogous to define α_R and Abrams also showed that this comultiplication map α_R is a map of right A-modules. Moreover, he proved that $\alpha_L = \alpha_R$. Then we define $\alpha := \alpha_L = \alpha_R$. Therefore, this map $\alpha : A \to A \otimes_k A$ is a map of A-bimodules.

Furthermore, let $\varepsilon : A \to k$ denote $\lambda_R(1_A)$. Note that $\lambda_L(1_A) = \lambda_R(1_A)$ and thus that ε serves as a counit for α.

We mentioned that there is a left A-module isomorphism $\lambda_L : A \cong \mathrm{D}(A)$. Here we note that A is viewed as the left regular module over itself, and $\mathrm{D}(A)$ is made a left A-module by the action $(a \cdot f)(b) := f(ba)$ for any $a, b \in A$ and $f \in \mathrm{D}(A)$. Since $\varepsilon = \lambda_L(1_A)$, all elements of $\mathrm{D}(A)$ are of the form $a \cdot \varepsilon$ for any $a \in A$. The isomorphism $\lambda_L : A \cong \mathrm{D}(A)$ allows us to define a multiplication φ_L in $\mathrm{D}(A)$ by

$$\varphi_L(a \cdot \varepsilon \otimes b \cdot \varepsilon) := ab \cdot \varepsilon.$$

To see it more clearly, observe the following.

$$\begin{array}{ccccccc} \varphi_L : \mathrm{D}(A) \otimes_k \mathrm{D}(A) & \xrightarrow{\lambda_L^{-1} \otimes \lambda_L^{-1}} & A \otimes_k A & \xrightarrow{\mu} & A & \xrightarrow{\lambda_L} & \mathrm{D}(A) \\ a \cdot \varepsilon \otimes_k b \cdot \varepsilon & \longmapsto & a \otimes_k b & \longmapsto & ab & \longmapsto & ab \cdot \varepsilon \end{array}$$

Note also that α_R can be used to define the multiplication φ_L such that

$$\varphi_L(a \cdot \varepsilon \otimes b \cdot \varepsilon) = (b \cdot \varepsilon \otimes a \cdot \varepsilon) \circ \alpha_R = ab \cdot \varepsilon.$$

Dually, the isomorphism $\lambda_R : A \cong \mathrm{D}(A)$ allows us to define a multiplication φ_R in $\mathrm{D}(A)$ by

$$\varphi_R(\varepsilon \cdot a \otimes \varepsilon \cdot b) := \varepsilon \cdot ab.$$

To see it more clearly, observe the following.

$$\begin{array}{ccccccc} \varphi_R : \mathrm{D}(A) \otimes_k \mathrm{D}(A) & \xrightarrow{\lambda_R^{-1} \otimes \lambda_R^{-1}} & A \otimes_k A & \xrightarrow{\mu} & A & \xrightarrow{\lambda_R} & \mathrm{D}(A) \\ \varepsilon \cdot a \otimes_k \varepsilon \cdot b & \longmapsto & a \otimes_k b & \longmapsto & ab & \longmapsto & \varepsilon \cdot ab \end{array}$$

Note also that α_L can be used to define the multiplication φ_R such that

$$\varphi_R(\varepsilon \cdot a \otimes \varepsilon \cdot b) = (\varepsilon \cdot b \otimes \varepsilon \cdot a) \circ \alpha_L = \varepsilon \cdot ab.$$

Remark 2.2.4. Commutative Frobenius algebras have a very attractive application to two-dimensional topological quantum field theories. In [1], it is proved that the category of commutative Frobenius algebras is equivalent to the category of two-dimensional topological quantum field theories.

Example 2.2.5. Let $G = \{g_1, ..., g_n\}$ be a finite group written multiplicatively and kG be the group algebra over k. Then the group algebra kG has a comultiplication $\alpha : kG \to kG \otimes_k kG$ such that $\alpha(g) = \sum_{i=1}^n gg_i \otimes_k g_i^{-1} = \sum_{i=1}^n g_i \otimes_k g_i^{-1}g$ for any $g \in G$. The counit ε of (kG, α) was given in Example 2.1.20 as Frobenius form of kG.

Remark 2.2.6. The group algebra kG has actually another comultiplication $\widetilde{\alpha} : kG \to kG \otimes_k kG$ which sends g to $g \otimes_k g$ for any $g \in G$. This group algebra kG admits a Hopf algebra structure over k with the comultiplication $\widetilde{\alpha}$, the counit $f : kG \to k$ such that $f(g) = 1$ for any $g \in G$ and some other special linear maps. For more detail about Hopf algebras, see [31], Chapter VI. However, the counit f may not be a Frobenius form of kG since the kernel of f may contain a nonzero left ideal of kG. Therefore, we may not say that (kG, f) is a Frobenius algebra. However, since the counit ε of (kG, α) given in Example 2.2.5 is also a Frobenius form of kG, we construct the symmetric Frobenius algebra structure of kG by using this ε (see Example 2.1.20).

Lemma 2.2.7. *Let A be a Frobenius algebra with the comultiplication α and the compatible counit ε. Suppose that $\alpha(1_A) = \sum_{i=1}^n x_i \otimes y_i$. Then $\sum_{i=1}^n \varepsilon(ax_i)y_i = a = \sum_{i=1}^n x_i\varepsilon(y_ia)$ for all $a \in A$.*

Proof. Let A be a Frobenius algebra. By Theorem 2.2.3, it is known that the comultiplication α is a map of A-bimodules. Therefore, $a\alpha(1_A) = \alpha(a) = \alpha(1_A)a$. Since $\alpha(1_A) = \sum_{i=1}^n x_i \otimes y_i$ by assumption, we obtain that $\alpha(a) = \sum_{i=1}^n ax_i \otimes y_i = \sum_{i=1}^n x_i \otimes y_ia$. We know that A is also a coalgebra since it is Frobenius. Then by using the counity condition of the coalgebra given in Definition 2.2.2, we obtain that $(\varepsilon \otimes \mathrm{id}_A)\alpha(a) = a = (\mathrm{id}_A \otimes \varepsilon)\alpha(a)$, this means that $\sum_{i=1}^n \varepsilon(ax_i)y_i = a = \sum_{i=1}^n x_i\varepsilon(y_ia)$ for all $a \in A$. $\square$

Proposition 2.2.8. *Let A be a Frobenius algebra with the comultiplication α. Suppose that $\alpha(1_A) = \sum_{i=1}^{n} x_i \otimes y_i$. Then*

$$\alpha(1_A) = \sum_{i=1}^{n} x_i \otimes y_i = \sum_{i=1}^{n} y_i \otimes \nu_A^{-1}(x_i),$$

where ν_A is a Nakayama automorphism of A.

Proof. Let A be a Frobenius algebra with a nondegenerate associative bilinear form $\beta : A \otimes A \to k$ and ε be its corresponding Frobenius form. Recall that $\beta(x \otimes y) = \varepsilon(xy)$ for all $x, y \in A$. Then by using the previous lemma and Definition 2.1.11, we obtain that

$$\begin{aligned}\alpha(1_A) = \sum_{i=1}^{n} x_i \otimes y_i &= \sum_{i,j=1}^{n} \varepsilon(x_i x_j) y_j \otimes y_i \\ &= \sum_{i,j=1}^{n} y_j \otimes \varepsilon(x_i x_j) y_i \\ &= \sum_{i,j=1}^{n} y_j \otimes \varepsilon(\nu_A^{-1}(x_j) x_i) y_i \\ &= \sum_{j=1}^{n} y_j \otimes \nu_A^{-1}(x_j).\end{aligned}$$

□

The following theorem shows that Proposition 2.1.4 in [3] which was given by Abrams in commutative case is also satisfied for all finite dimensional Frobenius algebras over k. Indeed, in the proof of Proposition 4.3 in [2], Abrams mentioned and used this result. However, we give an explicit proof by using Abrams' results (see the proofs of Theorem 1 in [1] and Theorem 2.1.4 in [3]) and by taking into consideration the adjustments made by Abrams in the proof of Theorem 2.1 in [2] for noncommutative case. Here our aim is to emphasize this result since it is very useful for computing the comultiplication of Frobenius algebras.

Theorem 2.2.9. *Let A be a Frobenius algebra with the left A-module isomorphism $\lambda_L : A \cong D(A)$ and the comultiplication $\alpha : A \to A \otimes_k A$. Suppose that $v_1, ..., v_n$ is a basis for A and $v_i' = \lambda_L^{-1}(v_i^*)$ such that $v_i^*(v_j) = \delta_{ij}$. Then the following are satisfied.*

(i) $\alpha(1_A) = \sum_i v_i' \otimes v_i$.

(ii) $\alpha(a) = \sum_i a v_i' \otimes v_i = \sum_i v_i' \otimes v_i a$ for $a \in A$.

Proof. (i) Let A be Frobenius and $\varepsilon : A \to k$ be the counit. To prove it, we give the construction which is given by Abrams in the proof of Theorem 1, [1]. So, let us define $\beta := \varepsilon \circ \mu : A \otimes A \to k$, where μ is the multiplication map. And also, we define $\psi := \alpha \circ \eta : k \to A \otimes A$ such that $\eta : k \to A$ is the unit of A. Abrams showed that the following diagram commutes:

$$\begin{tikzcd} & & A \otimes A \otimes A \arrow[dr, "id_A \otimes \mu"] & & \\ k \otimes A \arrow[r, "\eta \otimes id_A"] & A \otimes A \arrow[ur, "\alpha \otimes id_A"] \arrow[dr, "\mu"'] & & A \otimes A \arrow[r, "id_A \otimes \varepsilon"] & A \otimes k \\ & & A \arrow[ur, "\alpha"'] & & \end{tikzcd}$$

Thus, the top line shows that $(id_A \otimes \beta) \circ (\psi \otimes id_A)$ is the identity map on A. Now, we choose a basis

$v_1, ..., v_n$ for A. Then for any $a \in A$, this composition maps as follows:

$$a \mapsto \psi(1_k) \otimes a = (\sum_j u_j \otimes v_j) \otimes a \mapsto \sum_j u_j \beta(v_j \otimes a) = a,$$

where u_j are some elements in A. In fact, these u_j form a basis for A, since they clearly span A, and there are at most $(A : k)$ of them. Let $a = u_i$. Then we see that $u_i = \sum_j u_j \beta(v_j \otimes u_i)$, so $\beta(v_j \otimes u_i) = \delta_{ij}$.

Let $v'_1, ..., v'_n$ denote the dual basis relative to β for a given basis $v_1, ..., v_n$. In other words, v'_i satisfy $\beta(v_j \otimes v'_i) = \delta_{ij}$. Therefore, we have $\alpha(1_A) = \psi(1_k) = \sum_j u_j \otimes v_j$ and $u_j = v'_j$.

In fact, since

$$\lambda_L(v'_i)(v_j) = \varepsilon \circ \overline{\mu}(v'_i)(v_j) = \varepsilon \circ \mu(v_j \otimes v'_i) = \beta(v_j \otimes v'_i) = \delta_{ij},$$

we have that $v'_i = \lambda_L^{-1}(v_i^*)$ such that $v_i^*(v_j) = \delta_{ij}$. Therefore, we obtain that

$$\alpha(1_A) = \sum_i v'_i \otimes v_i.$$

(ii) Since α is a map of A-bimodules, we say that $a\alpha(1_A) = \alpha(a) = \alpha(1_A)a$ for any $a \in A$. □

Example 2.2.10. Let k be a field and Q be a quiver given as follows:

$$\begin{tikzcd}
& 1 \arrow[dr, "\alpha_1"] & \\
3 \arrow[ur, "\alpha_3"] & & 2 \arrow[ll, "\alpha_2"]
\end{tikzcd}$$

Let I be the ideal in the path algebra kQ generated by $\alpha_1\alpha_2\alpha_3$, $\alpha_2\alpha_3\alpha_1$, $\alpha_3\alpha_1\alpha_2$ and $A = kQ/I$ be the associated bound quiver algebra. A has a k-basis $\{e_1, e_2, e_3, \alpha_1, \alpha_2, \alpha_3, \alpha_1\alpha_2, \alpha_2\alpha_3, \alpha_3\alpha_1\}$ so $\mathrm{D}(A)$ has the dual basis $\{e_1^*, e_2^*, e_3^*, \alpha_1^*, \alpha_2^*, \alpha_3^*, (\alpha_1\alpha_2)^*, (\alpha_2\alpha_3)^*, (\alpha_3\alpha_1)^*\}$. It is a nonsymmetric Frobenius Nakayama algebra by Section 2.1.3. We observe that the left A-module isomorphism λ_L can be explicitly defined on the basis elements by

$$\begin{aligned}
\lambda_L : A &\cong \mathrm{D}(A) \\
e_1 &\mapsto (\alpha_2\alpha_3)^* \\
e_2 &\mapsto (\alpha_3\alpha_1)^* \\
e_3 &\mapsto (\alpha_1\alpha_2)^* \\
\alpha_1 &\mapsto \alpha_3^* \\
\alpha_2 &\mapsto \alpha_1^* \\
\alpha_3 &\mapsto \alpha_2^* \\
\alpha_1\alpha_2 &\mapsto e_1^* \\
\alpha_2\alpha_3 &\mapsto e_2^* \\
\alpha_3\alpha_1 &\mapsto e_3^*.
\end{aligned}$$

Then let $e_1 = v_1$, $e_2 = v_2$, $e_3 = v_3$, $\alpha_1 = v_4$, $\alpha_2 = v_5$, $\alpha_3 = v_6$, $\alpha_1\alpha_2 = v_7$, $\alpha_2\alpha_3 = v_8$ and $\alpha_3\alpha_1 = v_9$. Therefore, by using Theorem 2.2.9, we write that $v'_1 = \alpha_1\alpha_2$, $v'_2 = \alpha_2\alpha_3$, $v'_3 = \alpha_3\alpha_1$, $v'_4 = \alpha_2$, $v'_5 = \alpha_3$, $v'_6 = \alpha_1$, $v'_7 = e_3$, $v'_8 = e_1$ and $v'_9 = e_2$ since $v'_i = \lambda_L^{-1}(v_i^*)$ such that $v_i^*(v_j) = \delta_{ij}$. Then by using the formula given in Theorem 2.2.9 (i), we obtain that

$$\alpha(1_A) = \sum_i v_i' \otimes v_i = \alpha_1\alpha_2 \otimes e_1 + \alpha_2\alpha_3 \otimes e_2 + \alpha_3\alpha_1 \otimes e_3 + \alpha_2 \otimes \alpha_1$$
$$+ \alpha_3 \otimes \alpha_2 + \alpha_1 \otimes \alpha_3 + e_3 \otimes \alpha_1\alpha_2 + e_1 \otimes \alpha_2\alpha_3 + e_2 \otimes \alpha_3\alpha_1.$$

Moreover, by using Theorem 2.2.9 (ii), we obtain that

$$\begin{aligned}
\alpha(e_1) &= \alpha_1\alpha_2 \otimes e_1 + \alpha_1 \otimes \alpha_3 + e_1 \otimes \alpha_2\alpha_3\\
\alpha(e_2) &= \alpha_2\alpha_3 \otimes e_2 + \alpha_2 \otimes \alpha_1 + e_2 \otimes \alpha_3\alpha_1\\
\alpha(e_3) &= \alpha_3\alpha_1 \otimes e_3 + \alpha_3 \otimes \alpha_2 + e_3 \otimes \alpha_1\alpha_2\\
\alpha(\alpha_1) &= \alpha_1\alpha_2 \otimes \alpha_1 + \alpha_1 \otimes \alpha_3\alpha_1\\
\alpha(\alpha_2) &= \alpha_2\alpha_3 \otimes \alpha_2 + \alpha_2 \otimes \alpha_1\alpha_2\\
\alpha(\alpha_3) &= \alpha_3\alpha_1 \otimes \alpha_3 + \alpha_3 \otimes \alpha_2\alpha_3\\
\alpha(\alpha_1\alpha_2) &= \alpha_1\alpha_2 \otimes \alpha_1\alpha_2\\
\alpha(\alpha_2\alpha_3) &= \alpha_2\alpha_3 \otimes \alpha_2\alpha_3\\
\alpha(\alpha_3\alpha_1) &= \alpha_3\alpha_1 \otimes \alpha_3\alpha_1.
\end{aligned}$$

Here, the counit $\varepsilon = \lambda_L(1_A)$. Since $1_A = e_1 + e_2 + e_3$, we obtain that $\varepsilon = (\alpha_1\alpha_2)^* + (\alpha_2\alpha_3)^* + (\alpha_3\alpha_1)^*$.

Theorem 2.2.11. *Let A be a Frobenius algebra with the comultiplication $\alpha : A \to {}_AA \otimes_k A_A$. Then*

$$Im(\alpha) = \{\sum u_i \otimes v_i \mid \sum u_i x \otimes v_i = \sum u_i \otimes \nu_A^{-1}(x) v_i, \ \ \forall x \in A\},$$

where ν_A is a Nakayama automorphism of A.

Proof. We postpone this proof until Section 4.2 where it will be part of a more general result. $\square$

Chapter 3

Gendo-symmetric algebras

A new class of algebras called gendo-symmetric algebras have been introduced by Fang and Koenig [13, 14]. Gendo-symmetric algebras are defined by a special case of the Morita-Tachikawa correspondence, which shows that algebras of dominant dimension at least two are exactly the endomorphism rings of generator-cogenerators over an algebra (which in our case is assumed to be symmetric). A generator-cogenerator is a module that up to isomorphism contains each indecomposable projective or injective module at least once as a direct summand. An algebra A of dominant dimension at least two has a faithful projective-injective module, say Ae, and also there is a double centraliser property on this (A, eAe)-bimodule Ae, that is, $A \cong \mathrm{End}_{eAe}(Ae)$. There are important examples of this situation. Classical Schur-Weyl duality between Schur algebras $S_k(n,r)$ for $n \geq r$ and group algebras of symmetric groups Σ_r as well as Soergel's structure theorem for the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ are such examples. Hence the class of gendo-symmetric algebras contains these and many other examples from algebraic Lie theory as well as symmetric algebras and their Auslander algebras. Gendo-symmetric algebras are characterised by the existence of a comultiplication and have the properties used for defining the bar cocomplex. The exactness of this bar cocomplex is used to determine the dominant dimension of gendo-symmetric algebras. For more information about gendo-symmetric algebras, see [13] and [14].

This chapter is devoted to introducing gendo-symmetric algebras and their characterisation in terms of comultiplication. We first give the definition of gendo-symmetric algebras and then give some examples of these algebras. In the second section, we revisit the comultiplication of gendo-symmetric algebras and give new results on it. As subsection we visit gendo-symmetric Schur algebras and give new results on the existence of a comultiplication. Next we introduce a result given by Fang and Koenig [13] on the characterisations of gendo-symmetric algebras and their dominant dimension by using bar cocomplex and we give a hypothesis which may lead to prove Nakayama conjecture for gendo-symmetric algebras.

Both Frobenius and gendo-symmetric algebras contain symmetric algebras and both are characterised by the existence of a comultiplication with some special properties. However, these two classes of algebras have differences. For example, Frobenius algebras have counit compatible with their comultiplication but gendo-symmetric algebras do not, in general. More clearly, a gendo-symmetric algebra has a counit compatible with its comultiplication if and only if it is a symmetric algebra. Here, it is natural to ask whether there are other properties distinguishing Frobenius algebras from gendo-symmetric algebras. More precisely, what are the differences of gendo-symmetric and Frobenius algebras with respect to comultiplication? In the last subsection, we answer this question in a different way.

Throughout, all algebras and modules are finite dimensional over an arbitrary field k unless stated otherwise. By D, we denote the usual k-duality functor $\mathrm{Hom}_k(-, k)$. For simplicity, we denote $\otimes_k$ by $\otimes$.

3.1 Definition and examples of gendo-symmetric algebras

Definition 3.1.1. A finite dimensional k-algebra A is called *gendo-symmetric* if it satisfies one of the following equivalent conditions:

(i) A is the endomorphism algebra of a generator over a symmetric algebra.

(ii) $\mathrm{Hom}_A({}_A\mathrm{D}(A), {}_AA) \cong A$ as A-bimodules.

(iii) $\mathrm{D}(A) \otimes_A \mathrm{D}(A) \cong \mathrm{D}(A)$ as A-bimodules.

(iv) $\mathrm{dom.dim}A \geq 2$ and $\mathrm{D}(Ae) \cong eA$ as (eAe, A)-bimodules, where Ae is a basic faithful projective-injective A-module.

From condition (iv) in Definition 3.1.1, we see that symmetric algebras are gendo-symmetric by choosing $e = 1_A$.

The above definition is based on [14], Theorem 3.2, which provides the equivalence of the four conditions.

3.1.1 Examples

In this part, some examples of gendo-symmetric algebras are exhibited.

Example 3.1.2. Let $B = k[x]/(x^2)$ and $M = B \oplus k$. Note that B is symmetric and M is a generator. Suppose that $A = \mathrm{End}_B(B \oplus k)$. Then A is given by the following quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

such that $\beta\alpha = 0$. We see that A is a gendo-symmetric algebra by using Definition 3.1.1 (i).

We now describe modules by Jordan-Hoelder series. The algebra A has two simple modules $S(1) = 1$ and $S(2) = 2$. The indecomposable projective modules are $P(1) = \begin{smallmatrix}1\\2\\1\end{smallmatrix}$ and $P(2) = \begin{smallmatrix}2\\1\end{smallmatrix}$, and the indecomposable injective modules are $I(1) = \begin{smallmatrix}1\\2\\1\end{smallmatrix}$ and $I(2) = \begin{smallmatrix}1\\2\end{smallmatrix}$.

Observe that an injective resolution of A is

$$0 \to A = P(1) \oplus P(2) \to P(1) \oplus P(1) \to P(1) \to I(2) \to 0.$$

Since $I(1) = P(1)$ is projective, but $I(2)$ is not, we obtain that $\mathrm{dom.dim}A = 2$. Also, since B is symmetric, it is self-injective and $\mathrm{dom.dim}B = \infty$.

Example 3.1.3. Let k be a field and Q be a quiver given as follows:

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3 \underset{\beta_3}{\overset{\alpha_3}{\rightleftarrows}} 4$$

Let I be the ideal in the path algebra kQ generated by $\alpha_1\alpha_2$, $\alpha_2\alpha_3$, $\beta_3\beta_2$, $\beta_2\beta_1$, $\beta_1\alpha_1 - \alpha_2\beta_2$, $\beta_2\alpha_2 - \alpha_3\beta_3$ and $\beta_3\alpha_3$ and $A = kQ/I$ be the associated bound quiver algebra.

Let $e = e_1 + e_2 + e_3$. The algebra A has four indecomposable projective modules

$$P(1) = \begin{matrix} 1 \\ 2 \\ 1 \end{matrix} \quad P(2) = \begin{matrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{matrix} \quad P(3) = \begin{matrix} & 3 & \\ 2 & & 4 \\ & 3 & \end{matrix} \quad P(4) = \begin{matrix} 4 \\ 3 \end{matrix}.$$

The first three of these are injective, too, and their direct sum Ae is a faithful projective-injective A-module.

Two algebras A and $B = eAe$ also are in a double centraliser situation, on a faithfully balanced bimodule Ae. This means that $A \cong \mathrm{End}_{eAe}(Ae)$. Observe that B is symmetric and Ae is a generator. Therefore, A is gendo-symmetric.

Since $P(1) = I(1)$, $P(2) = I(2)$ and $P(3) = I(3)$ are injective, it is enough to resolve $P(4)$:

$$0 \to P(4) \to I(3) \to I(2) \to I(1) \to I(1) \to I(2) \to I(3) \to I(4) \to 0.$$

Since $I(4) = \begin{matrix} 3 \\ 4 \end{matrix}$ is not projective, we obtain that $\mathrm{dom.dim} A = 6$. Also, since $B = eAe$ is symmetric, it is self-injective and $\mathrm{dom.dim} B = \infty$.

Example 3.1.4. Let $N_3^3 = kQ/I$ be the algebra of the following quiver

$$\begin{tikzcd} & 1 \arrow[dr, "\alpha_1"] & \\ 3 \arrow[ur, "\alpha_3"] & & 2 \arrow[ll, "\alpha_2"] \end{tikzcd}$$

such that I is the ideal of the path algebra kQ generated by all compositions of 4 consecutive arrows. By using Theorem 2.1.25, we obtain that N_3^3 is a symmetric Nakayama algebra. Therefore, it is gendo-symmetric and $\mathrm{dom.dim} N_3^3 = \infty$.

Remark 3.1.5. The class of gendo-symmetric algebras includes the subclass $\mathcal{A}$ of quasi-hereditary algebras introduced in [15]. These include the algebras on both sides of classical Schur-Weyl duality and of Soergel's structure theorem for the BGG-category $\mathcal{O}$. More information on Schur-Weyl duality can be found in Subsection 3.2.1.

3.2 Gendo-symmetric algebras and comultiplication

In this part, we revisit the comultiplication of gendo-symmetric algebras and give new results on it. We start by giving the construction of the comultiplication of gendo-symmetric algebras which has been obtained by Fang and Koenig in [13].

Let A be a gendo-symmetric algebra. Fix an (eAe, A)-bimodule isomorphism $\iota : eA \cong \mathrm{D}(Ae)$. By the double centralizer property $\mathrm{End}_{eAe}(eA) \cong A$, there is an A-bimodule isomorphism $\gamma : Ae \otimes_{eAe} eA \cong D(A)$ such that

$$\gamma(ae \otimes_{eAe} eb)(x) = \iota(ebx)(ae) \tag{3.1}$$

for a, b, x in A. Hence there is an isomorphism in Definition 3.1.1 (iii)

$$\mathrm{D}(A) \otimes_A \mathrm{D}(A) \cong (Ae \otimes_{eAe} eA) \otimes_A (Ae \otimes_{eAe} eA) \cong Ae \otimes_{eAe} eA \xrightarrow{\gamma} \mathrm{D}(A)$$

where the first isomorphism is $\gamma^{-1} \otimes_A \gamma^{-1}$.

Let m be the composition of the canonical A-bimodule morphism with the above isomorphism such that

$$m : \mathrm{D}(A) \otimes_k \mathrm{D}(A) \to \mathrm{D}(A) \otimes_A \mathrm{D}(A) \cong \mathrm{D}(A),$$

where

$$m(\gamma(ae \otimes_{eAe} eb) \otimes_k \gamma(ce \otimes_{eAe} ed)) = \gamma(ae \otimes_{eAe} eb \otimes_A ce \otimes_{eAe} ed) = \gamma(aebce \otimes_{eAe} ed). \tag{3.2}$$

Dualising m yields

$$\Delta : A \to {}_AA \otimes_k A_A$$

such that

$$(f \otimes g)\Delta(a) = m(g \otimes f)(a) \tag{3.3}$$

for any f, g in $\mathrm{D}(A)$ and a in A.

Theorem 3.2.1. *([13], Theorem 2.4) Let A be a gendo-symmetric algebra. Then $\Delta : A \to {}_AA \otimes_k A_A$ is a coassociative comultiplication on A.*

The following two results show that Δ is coassociative and also a map of A-bimodules.

Lemma 3.2.2. *([13], Lemma 2.5) The map m satisfies*

$$m(1 \otimes m) = m(m \otimes 1)$$

as k-morphisms from $D(A) \otimes_k D(A) \otimes_k D(A)$ to $D(A)$.

Lemma 3.2.3. *([13], Lemma 2.6) Let $\Delta : A \to {}_AA \otimes_k A_A$ be as above. Then*
(i) Δ is an A-bimodule morphism.
(ii) $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$.
(iii) $Im(\Delta) = \{\sum u_i \otimes v_i \mid \sum u_i x \otimes v_i = \sum u_i \otimes x v_i, \ \ \forall x \in A\}$.

Corollary 3.2.4. *([13], Corollary 2.7) The comultiplication Δ is unique up to precomposing it with multiplication by an invertible central element.*

Thus Δ is called the *canonical comultiplication* attached to the gendo-symmetric algebra A.

Proposition 3.2.5. *([13], Proposition 2.8) Let A be a gendo-symmetric k-algebra with the canonical comultiplication $\Delta : A \to {}_AA \otimes_k A_A$. Then*
(i) (A, Δ) has a counit iff A is symmetric.
(ii) Let $\Delta(1) = \sum x_i \otimes y_i$. Then $\Delta(1) = \sum y_i \otimes x_i$.

Proposition 3.2.6. *([13], Proposition 2.10 (2)) Let A be a gendo-symmetric k-algebra. If B is Morita equivalent to A, then B is gendo-symmetric.*

Procedure for obtaining the comultiplication Δ of any gendo-symmetric algebra A.

(1) Choose an idempotent $e \in A$ so that Ae a basic faithful projective-injective A-module.

(2) Write the (eAe, A)-bimodule isomorphism $\iota : eA \to \mathrm{D}(Ae)$ explicitly on a choice of basis elements.

(3) Write the A-bimodule isomorphism $\gamma : Ae \otimes_{eAe} eA \cong \mathrm{D}(A)$ by using (3.1) to obtain the basis elements of $\mathrm{D}(A)$ in terms of the elements of $\mathrm{Im}\gamma$.

(4) Obtain the multiplication table of $\mathrm{D}(A)$ by using (3.2).

(5) Dualise m by using (3.3) and obtain Δ on the basis elements of A.

(6) By using the linearity of Δ, obtain Δ on any element $a \in A$.

Let us apply this procedure on the following examples.

Example 3.2.7. Let A be the gendo-symmetric algebra in Example 3.1.2. A has a k-basis $\{e_1, e_2, \alpha, \beta, \alpha\beta\}$ so $\mathrm{D}(A)$ has the dual basis $\{e_1^*, e_2^*, \alpha^*, \beta^*, (\alpha\beta)^*\}$.

(1) We choose $e = e_1$ since Ae_1 is a basic faithful projective-injective A-module.

(2) The (eAe, A)-bimodule isomorphism ι is explicity defined on the basis elements by

$$\begin{aligned} \iota : eA &\cong \mathrm{D}(Ae) \\ e_1 &\mapsto (\alpha\beta)^* \\ \alpha &\mapsto \beta^* \\ \alpha\beta &\mapsto e_1^*. \end{aligned}$$

(3) The A-bimodule isomorphism γ is explicitly defined by

$$\begin{aligned} \gamma : Ae \otimes_{eAe} eA &\cong \mathrm{D}(A) \\ e_1 \otimes \alpha\beta &\mapsto e_1^* \\ \beta \otimes \alpha &\mapsto e_2^* \\ \beta \otimes e_1 &\mapsto \alpha^* \\ e_1 \otimes \alpha &\mapsto \beta^* \\ e_1 \otimes e_1 &\mapsto (\alpha\beta)^*. \end{aligned}$$

(4) We obtain the multiplication table of the basis elements of $\mathrm{D}(A)$ as follows:

m	e_1^*	e_2^*	α^*	β^*	$\alpha\beta^*$
e_1^*	0	0	0	0	e_1^*
e_2^*	0	0	0	0	0
α^*	0	0	0	e_2^*	α^*
β^*	0	0	e_1^*	0	0
$\alpha\beta^*$	e_1^*	0	0	β^*	$(\alpha\beta^*)$

(5) Dualising m yields

$$\Delta : A \to {}_AA \otimes_k A_A$$

such that $(f \otimes g)\Delta(a) = m(g \otimes f)(a)$ for any $f, g \in \mathrm{D}(A)$ and $a \in A$. So let

$$\begin{aligned} f &= \lambda_1 e_1^* + \lambda_2 e_2^* + \lambda_3 \alpha^* + \lambda_4 \beta^* + \lambda_5 (\alpha\beta)^* \\ g &= \mu_1 e_1^* + \mu_2 e_2^* + \mu_3 \alpha^* + \mu_4 \beta^* + \mu_5 (\alpha\beta)^*, \end{aligned}$$

where $\lambda_i, \mu_i \in k$ for $1 \leq i \leq 5$. By using the table in the previous step, we get

$$m(g \otimes f) = (\mu_1\lambda_5 + \mu_4\lambda_3 + \mu_5\lambda_1)e_1^* + \mu_3\lambda_4 e_2^* + \mu_3\lambda_5\alpha^* + \mu_5\lambda_4\beta^* + \mu_5\lambda_5(\alpha\beta)^*.$$

Then

$$m(g \otimes f)(e_1) = \mu_1\lambda_5 + \mu_4\lambda_3 + \mu_5\lambda_1$$

$$
\begin{aligned}
m(g\otimes f)(e_2) &= \mu_3\lambda_4\\
m(g\otimes f)(\alpha) &= \mu_3\lambda_5\\
m(g\otimes f)(\beta) &= \mu_5\lambda_4\\
m(g\otimes f)(\alpha\beta) &= \mu_5\lambda_5.
\end{aligned}
$$

Since $(f\otimes g)\Delta(a) = m(g\otimes f)(a)$ for all $a \in A$, we obtain that

$$
\begin{aligned}
\Delta(e_1) &= \alpha\beta\otimes e_1 + \alpha\otimes\beta + e_1\otimes\alpha\beta\\
\Delta(e_2) &= \beta\otimes\alpha\\
\Delta(\alpha) &= \alpha\beta\otimes\alpha\\
\Delta(\beta) &= \beta\otimes\alpha\beta\\
\Delta(\alpha\beta) &= \alpha\beta\otimes\alpha\beta.
\end{aligned}
$$

(6) Let $a \in A$. Then we can write $a = a_1e_1 + a_2e_2 + a_3\alpha + a_4\beta + a_5\alpha\beta$, where $a_i \in k$ for $1 \le i \le 5$. The linearity of Δ gives that

$$\Delta(a) = a_1\Delta(e_1) + a_2\Delta(e_2) + a_3\Delta(\alpha) + a_4\Delta(\beta) + a_5\Delta(\alpha\beta).$$

Example 3.2.8. Let k be a field and Q be a quiver given as follows:

$$
\begin{tikzcd}
1 \arrow[r, shift left, "\alpha_1"] & 2 \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\alpha_2"] & 3 \arrow[l, shift left, "\beta_2"]
\end{tikzcd}
$$

Let I be the ideal in the path algebra kQ generated by $\alpha_1\alpha_2$, $\beta_2\beta_1$, $\beta_2\alpha_2$, $\beta_1\alpha_1 - \alpha_2\beta_2$ and $A = kQ/I$ be the associated bound quiver algebra. A has a k-basis $\{e_1, e_2, e_3, \alpha_1, \alpha_2, \beta_1, \beta_2, \alpha_1\beta_1, \alpha_2\beta_2\}$ so $\mathrm{D}(A)$ has the dual basis $\{e_1^*, e_2^*, e_3^*, \alpha_1^*, \alpha_2^*, \beta_1^*, \beta_2^*, (\alpha_1\beta_1)^*, (\alpha_2\beta_2)^*\}$.

(1) We choose $e = e_1 + e_2$ since $A(e_1 + e_2)$ is a basic faithful projective-injective A-module.

(2) The (eAe, A)-bimodule isomorphism ι is explicity defined on the basis elements by

$$
\begin{aligned}
\iota : eA &\cong \mathrm{D}(Ae)\\
e_1 &\mapsto (\alpha_1\beta_1)^*\\
e_2 &\mapsto (\alpha_2\beta_2)^*\\
\alpha_1 &\mapsto \beta_1^*\\
\beta_1 &\mapsto \alpha_1^*\\
\alpha_2 &\mapsto \beta_2^*\\
\alpha_1\beta_1 &\mapsto e_1^*\\
\alpha_2\beta_2 &\mapsto e_2^*.
\end{aligned}
$$

(3) The A-bimodule isomorphism γ is explicitly defined by

$$
\begin{aligned}
\gamma : Ae\otimes_{eAe} eA &\cong \mathrm{D}(A)\\
e_1\otimes\alpha_1\beta_1 &\mapsto e_1^*\\
e_2\otimes\alpha_2\beta_2 &\mapsto e_2^*
\end{aligned}
$$

$$\beta_2 \otimes \alpha_2 \mapsto e_3^*$$
$$\beta_1 \otimes e_1 \mapsto \alpha_1^*$$
$$\beta_2 \otimes e_2 \mapsto \alpha_2^*$$
$$e_1 \otimes \alpha_1 \mapsto \beta_1^*$$
$$e_2 \otimes \alpha_2 \mapsto \beta_2^*$$
$$e_1 \otimes e_1 \mapsto (\alpha_1\beta_1)^*$$
$$e_2 \otimes e_2 \mapsto (\alpha_2\beta_2)^*.$$

(4) We obtain the multiplication table of the basis elements of $\mathrm{D}(A)$ as follows:

m	e_1^*	e_2^*	e_3^*	α_1^*	α_2^*	β_1^*	β_2^*	$(\alpha_1\beta_1)^*$	$(\alpha_2\beta_2)^*$
e_1^*	0	0	0	0	0	0	0	e_1^*	0
e_2^*	0	0	0	0	0	0	0	0	e_2^*
e_3^*	0	0	0	0	0	0	0	0	0
α_1^*	0	0	0	0	0	e_2^*	0	α_1^*	0
α_2^*	0	0	0	0	0	0	e_3^*	0	α_2^*
β_1^*	0	0	0	e_1^*	0	0	0	0	β_1^*
β_2	0	0	0	0	e_2^*	0	0	0	0
$(\alpha_1\beta_1)^*$	e_1^*	0	0	0	0	β_1^*	0	$(\alpha_1\beta_1)^*$	0
$(\alpha_2\beta_2)^*$	0	e_2^*	0	α_1^*	0	0	β_2^*	0	$(\alpha_2\beta_2)^*$

(5) Dualising m yields
$$\Delta : A \to {}_AA \otimes_k A_A$$
such that $(f \otimes g)\Delta(a) = m(g \otimes f)(a)$ for any $f, g \in \mathrm{D}(A)$ and $a \in A$. So let

$$f = \lambda_1 e_1^* + \lambda_2 e_2^* + \lambda_3 e_3^* + \lambda_4 \alpha_1^* + \lambda_5 \alpha_2^* + \lambda_6 \beta_1^* + \lambda_7 \beta_2^* + \lambda_8 (\alpha_1\beta_1)^* + \lambda_9 (\alpha_2\beta_2)^*$$
$$g = \mu_1 e_1^* + \mu_2 e_2^* + \mu_3 e_3^* + \mu_4 \alpha_1^* + \mu_5 \alpha_2^* + \mu_6 \beta_1^* + \mu_7 \beta_2^* + \mu_8 (\alpha_1\beta_1)^* + \mu_9 (\alpha_2\beta_2)^*,$$

where $\lambda_i, \mu_i \in k$ for $1 \leq i \leq 9$. By using the table in the previous step, we get

$$\begin{aligned} m(g \otimes f) = {} & (\mu_1\lambda_8 + \mu_6\lambda_4 + \mu_8\lambda_1)e_1^* + (\mu_2\lambda_9 + \mu_4\lambda_6 + \mu_7\lambda_5 + \mu_9\lambda_2)e_2^* + \mu_5\lambda_7 e_3^* \\ & + (\mu_4\lambda_8 + \mu_9\lambda_4)\alpha_1^* + \mu_5\lambda_9\alpha_2^* \\ & + (\mu_6\lambda_9 + \mu_8\lambda_6)\beta_1^* + \mu_9\lambda_7\beta_2^* \\ & + \mu_8\lambda_8(\alpha_1\beta_1)^* + \mu_9\lambda_9(\alpha_2\beta_2)^*. \end{aligned}$$

Then

$$m(g \otimes f)(e_1) = \mu_1\lambda_8 + \mu_6\lambda_4 + \mu_8\lambda_1$$
$$m(g \otimes f)(e_2) = \mu_2\lambda_9 + \mu_4\lambda_6 + \mu_7\lambda_5 + \mu_9\lambda_2$$
$$m(g \otimes f)(e_3) = \mu_5\lambda_7$$
$$m(g \otimes f)(\alpha_1) = \mu_4\lambda_8 + \mu_9\lambda_4$$
$$m(g \otimes f)(\alpha_2) = \mu_5\lambda_9$$
$$m(g \otimes f)(\beta_1) = \mu_6\lambda_9 + \mu_8\lambda_6$$

$$\begin{aligned}
m(g \otimes f)(\beta_2) &= \mu_9 \lambda_7 \\
m(g \otimes f)(\alpha_1\beta_1) &= \mu_8 \lambda_8 \\
m(g \otimes f)(\alpha_2\beta_2) &= \mu_9 \lambda_9.
\end{aligned}$$

Since $(f \otimes g)\Delta(a) = m(g \otimes f)(a)$ for all $a \in A$, we obtain that

$$\begin{aligned}
\Delta(e_1) &= \alpha_1\beta_1 \otimes e_1 + \alpha_1 \otimes \beta_1 + e_1 \otimes \alpha_1\beta_1 \\
\Delta(e_2) &= \alpha_2\beta_2 \otimes e_2 + \beta_1 \otimes \alpha_1 + \alpha_2 \otimes \beta_2 + e_2 \otimes \alpha_2\beta_2 \\
\Delta(e_3) &= \beta_2 \otimes \alpha_2 \\
\Delta(\alpha_1) &= \alpha_1\beta_1 \otimes \alpha_1 + \alpha_1 \otimes \alpha_2\beta_2 \\
\Delta(\alpha_2) &= \alpha_2\beta_2 \otimes \alpha_2 \\
\Delta(\beta_1) &= \alpha_2\beta_2 \otimes \beta_1 + \beta_1 \otimes \alpha_1\beta_1 \\
\Delta(\beta_2) &= \beta_2 \otimes \alpha_2\beta_2 \\
\Delta(\alpha_1\beta_1) &= \alpha_1\beta_1 \otimes \alpha_1\beta_1 \\
\Delta(\alpha_2\beta_2) &= \alpha_2\beta_2 \otimes \alpha_2\beta_2.
\end{aligned}$$

(6) Let $a \in A$. Then we can write

$$a = a_1e_1 + a_2e_2 + a_3e_3 + a_4\alpha_1 + a_5\alpha_2 + a_6\beta_1 + a_7\beta_2 + a_8\alpha_1\beta_1 + a_9\alpha_2\beta_2,$$

where $a_i \in k$ for $1 \leq i \leq 9$. The linearity of Δ gives that

$$\Delta(a) = a_1\Delta(e_1)+a_2\Delta(e_2)+a_3\Delta(e_3)+a_4\Delta(\alpha_1)+a_5\Delta(\alpha_2)+a_6\Delta(\beta_1)+a_7\Delta(\beta_2)+a_8\Delta(\alpha_1\beta_1)+a_9\Delta(\alpha_2\beta_2).$$

Definition 3.2.9. Let (A, Δ) be a gendo-symmetric algebra. A *coideal* of the gendo-symmetric algebra A is a k-vector subspace C of A such that $\Delta(C) \subseteq C \otimes A + A \otimes C$.

Since a gendo-symmetric algebra (A, Δ) may not have a counit, in general, we can not use this property in the above definition.

Theorem 3.2.10. *Let A be a gendo-symmetric algebra with a basic faithful projective-injective A-module Ae for an idempotent e of A such that eAe is symmetric. Let $\pi : A \to eAe$ be the k-linear map such that $\pi(a) = eae$ for $a \in A$. Suppose that Δ_A is a comultiplication of A. Then there exists a comultiplication Δ_{eAe} of eAe such that $(\pi \otimes \pi)\Delta_A = \Delta_{eAe}\pi$.*

Proof. Since A is gendo-symmetric and eAe is symmetric, there are multiplications

$$m_{\mathrm{D}(A)} : \mathrm{D}(A) \otimes_k \mathrm{D}(A) \to \mathrm{D}(A)$$

and

$$m_{\mathrm{D}(eAe)} : \mathrm{D}(eAe) \otimes_k \mathrm{D}(eAe) \to \mathrm{D}(eAe).$$

Let $\pi : A \to eAe$ be the k-linear map which is given in the assumption. Take the dual of π. Then we have $\pi^* : \mathrm{D}(eAe) \to \mathrm{D}(A)$ such that $\pi^*(f)(a) = f \circ \pi(a) = f(eae)$ for any $f \in \mathrm{D}(eAe)$ and $a \in A$. In fact, there is an A-bimodule isomorphism $\gamma : Ae \otimes_{eAe} eA \cong \mathrm{D}(A)$. Multiplying by e on the left and right implies an (eAe, eAe)-bimodule isomorphism $\overline{\gamma} : eAe \otimes_{eAe} eAe \cong \mathrm{D}(eAe)$ and we define $m_{\mathrm{D}(eAe)}$

by using $\overline{\gamma}$. Here, for any $a, b, x \in A$ $\gamma(eae \otimes_{eAe} ebe)(x) = \gamma(eae \otimes_{eAe} ebe)(exe)$ since γ is an A-bimodule isomorphism. We may suppose that $\gamma(eae \otimes_{eAe} ebe)(exe) = \overline{\gamma}(eae \otimes_{eAe} ebe)(exe)$.

We now observe the following diagram

$$\begin{tikzcd}
\mathrm{D}(eAe) \otimes_k \mathrm{D}(eAe) \arrow[r, "m_{\mathrm{D}(eAe)}"] \arrow[d, "\pi^* \otimes \pi^*"'] & \mathrm{D}(eAe) \arrow[d, "\pi^*"] \\
\mathrm{D}(A) \otimes_k \mathrm{D}(A) \arrow[r, "m_{\mathrm{D}(A)}"] & \mathrm{D}(A)
\end{tikzcd}$$

Observe that

$$\begin{aligned}
\pi^* \circ m_{\mathrm{D}(eAe)}(\overline{\gamma}(eae \otimes_{eAe} ebe) \otimes_k \overline{\gamma}(ece \otimes_{eAe} ede))(x) &= \pi^*(\overline{\gamma}(eaebece \otimes_{eAe} ede))(x) \\
&= \overline{\gamma}(eaebece \otimes_{eAe} ede) \circ \pi(x) \\
&= \overline{\gamma}(eaebece \otimes_{eAe} ede)(exe)
\end{aligned}$$

for any $x \in A$. Let $\Delta_A(x) = \sum_i u_i \otimes_k v_i$. Then

$$\begin{aligned}
& m_{\mathrm{D}(A)} \circ (\pi^* \otimes \pi^*)(\overline{\gamma}(eae \otimes_{eAe} ebe) \otimes_k \overline{\gamma}(ece \otimes_{eAe} ede))(x) \\
&= m_{\mathrm{D}(A)}(\overline{\gamma}(eae \otimes_{eAe} ebe) \circ \pi \otimes_k \overline{\gamma}(ece \otimes_{eAe} ede) \circ \pi)(x) \\
&= (\overline{\gamma}(ece \otimes_{eAe} ede) \circ \pi \otimes_k \overline{\gamma}(eae \otimes_{eAe} ebe) \circ \pi)\Delta_A(x) \\
&= (\overline{\gamma}(ece \otimes_{eAe} ede) \circ \pi \otimes_k \overline{\gamma}(eae \otimes_{eAe} ebe) \circ \pi) \sum_i u_i \otimes_k v_i \\
&= \sum_i (\overline{\gamma}(ece \otimes_{eAe} ede) \circ \pi(u_i) \otimes_k \overline{\gamma}(eae \otimes_{eAe} ebe) \circ \pi(v_i)) \\
&= \sum_i (\overline{\gamma}(ece \otimes_{eAe} ede)(eu_ie) \otimes_k \overline{\gamma}(eae \otimes_{eAe} ebe)(ev_ie)) \\
&= \sum_i (\gamma(ece \otimes_{eAe} ede)(eu_ie) \otimes_k \gamma(eae \otimes_{eAe} ebe)(ev_ie)) \\
&= \sum_i (\gamma(ece \otimes_{eAe} ede)(eu_i) \otimes_k \gamma(eae \otimes_{eAe} ebe)(v_ie) \text{ (since } \gamma \text{ is an } A\text{-bimodule isomorphism)} \\
&= \gamma(ece \otimes_{eAe} ede) \otimes_k \gamma(eae \otimes_{eAe} ebe)(\sum_i eu_i \otimes_k v_ie) \\
&= \gamma(ece \otimes_{eAe} ede) \otimes_k \gamma(eae \otimes_{eAe} ebe)\Delta_A(exe) \text{ (since } \Delta_A \text{ is an } A\text{-bimodule morphism)} \\
&= m_{\mathrm{D}(A)}(\gamma(eae \otimes_{eAe} ebe) \otimes_k \gamma(ece \otimes_{eAe} ede))(exe) \\
&= \gamma(eaebece \otimes_{eAe} ede)(exe) \\
&= \overline{\gamma}(eaebece \otimes_{eAe} ede)(exe).
\end{aligned}$$

Thus we obtain that $\pi^* \circ m_{\mathrm{D}(eAe)} = m_{\mathrm{D}(A)} \circ (\pi^* \otimes \pi^*)$. Therefore, the above diagram is commutative. By dualising the above diagram, we have the following commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\Delta_A"] \arrow[d, "\pi"'] & A \otimes_k A \arrow[d, "\pi \otimes \pi"] \\
eAe \arrow[r, "\Delta_{eAe}"] & eAe \otimes_k eAe
\end{tikzcd}$$

In other words,

$$(\pi \otimes \pi)\Delta_A = \Delta_{eAe}\pi.$$

□

We keep the notations introduced in the above theorem.

Corollary 3.2.11. *Kernel of π is a coideal of the gendo-symmetric algebra A.*

Proof. By the above theorem, we have $(\pi\otimes\pi)\Delta_A = \Delta_{eAe}\pi$. Since $\Delta_{eAe}\pi(\mathrm{Ker}(\pi)) = 0$, $(\pi\otimes\pi)\Delta_A(\mathrm{Ker}(\pi)) = 0$, that is, $\Delta_A(\mathrm{Ker}(\pi)) \subseteq \mathrm{Ker}(\pi\otimes\pi)$. Observe that

$$\mathrm{Ker}(\pi\otimes\pi) = \mathrm{Ker}(\pi)\otimes A + A\otimes \mathrm{Ker}(\pi).$$

Therefore, $\Delta_A(\mathrm{Ker}(\pi)) \subseteq \mathrm{Ker}(\pi)\otimes A + A\otimes\mathrm{Ker}(\pi)$. By definition of coideal, $\mathrm{Ker}(\pi)$ is a coideal of A. □

Example 3.2.12. Let A be the path algebra of the following quiver

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3 \underset{\beta_3}{\overset{\alpha_3}{\rightleftarrows}} 4$$

such that $\alpha_1\alpha_2 = 0$, $\alpha_2\alpha_3 = 0$, $\beta_3\beta_2 = 0$, $\beta_2\beta_1 = 0$, $\beta_1\alpha_1 = \alpha_2\beta_2$, $\beta_2\alpha_2 = \alpha_3\beta_3$ and $\beta_3\alpha_3 = 0$. Observe that A is a gendo-symmetric algebra. We choose $e = e_1 + e_2 + e_3$ so that Ae is a basic faithful projective-injective A-module. Then eAe becomes a path algebra of the following quiver

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3$$

such that $\alpha_1\alpha_2 = 0$, $\beta_2\beta_1 = 0$, $\alpha_1\beta_1\alpha_1 = 0$, $\beta_2\alpha_2\beta_2 = 0$ and $\alpha_2\beta_2 = \beta_1\alpha_1$. The map π is explicitly defined on the basis elements by

$$\begin{aligned}
\pi : A &\to eAe\\
e_1 &\mapsto e_1\\
e_2 &\mapsto e_2\\
e_3 &\mapsto e_3\\
e_4 &\mapsto 0\\
\alpha_1 &\mapsto \alpha_1\\
\alpha_2 &\mapsto \alpha_2\\
\alpha_3 &\mapsto 0\\
\beta_1 &\mapsto \beta_1\\
\beta_2 &\mapsto \beta_2\\
\beta_3 &\mapsto 0\\
\alpha_1\beta_1 &\mapsto \alpha_1\beta_1\\
\beta_1\alpha_1 &\mapsto \beta_1\alpha_1\\
\beta_2\alpha_2 &\mapsto \beta_2\alpha_2.
\end{aligned}$$

Suppose that Δ_A is the comultiplication of A which is defined on the basis elements by

$$\Delta_A(e_1) = \alpha_1\beta_1\otimes e_1 + \alpha_1\otimes\beta_1 + e_1\otimes\alpha_1\beta_1$$

$$\begin{aligned}
\Delta_A(e_2) &= \beta_1\alpha_1 \otimes e_2 + \beta_1 \otimes \alpha_1 + \alpha_2 \otimes \beta_2 + e_2 \otimes \beta_1\alpha_1 \\
\Delta_A(e_3) &= \beta_2\alpha_2 \otimes e_3 + \beta_2 \otimes \alpha_2 + \alpha_3 \otimes \beta_3 + e_3 \otimes \beta_2\alpha_2 \\
\Delta_A(e_4) &= \beta_3 \otimes \alpha_3 \\
\Delta_A(\alpha_1) &= \alpha_1\beta_1 \otimes \alpha_1 + \alpha_1 \otimes \beta_1\alpha_1 \\
\Delta_A(\alpha_2) &= \beta_1\alpha_1 \otimes \alpha_2 + \alpha_2 \otimes \beta_2\alpha_2 \\
\Delta_A(\alpha_3) &= \beta_2\alpha_2 \otimes \alpha_3 \\
\Delta_A(\beta_1) &= \beta_1\alpha_1 \otimes \beta_1 + \beta_1 \otimes \alpha_1\beta_1 \\
\Delta_A(\beta_2) &= \beta_2\alpha_2 \otimes \beta_2 + \beta_2 \otimes \beta_1\alpha_1 \\
\Delta_A(\beta_3) &= \beta_3 \otimes \beta_2\alpha_2 \\
\Delta_A(\alpha_1\beta_1) &= \alpha_1\beta_1 \otimes \alpha_1\beta_1 \\
\Delta_A(\beta_1\alpha_1) &= \beta_1\alpha_1 \otimes \beta_1\alpha_1 \\
\Delta_A(\beta_2\alpha_2) &= \beta_2\alpha_2 \otimes \beta_2\alpha_2.
\end{aligned}$$

Then there exists a comultiplication Δ_{eAe} of eAe which is defined on the basis elements by

$$\begin{aligned}
\Delta_{eAe}(e_1) &= \alpha_1\beta_1 \otimes e_1 + \alpha_1 \otimes \beta_1 + e_1 \otimes \alpha_1\beta_1 \\
\Delta_{eAe}(e_2) &= \beta_1\alpha_1 \otimes e_2 + \beta_1 \otimes \alpha_1 + \alpha_2 \otimes \beta_2 + e_2 \otimes \beta_1\alpha_1 \\
\Delta_{eAe}(e_3) &= \beta_2\alpha_2 \otimes e_3 + \beta_2 \otimes \alpha_2 + e_3 \otimes \beta_2\alpha_2 \\
\Delta_{eAe}(\alpha_1) &= \alpha_1\beta_1 \otimes \alpha_1 + \alpha_1 \otimes \beta_1\alpha_1 \\
\Delta_{eAe}(\alpha_2) &= \beta_1\alpha_1 \otimes \alpha_2 + \alpha_2 \otimes \beta_2\alpha_2 \\
\Delta_{eAe}(\beta_1) &= \beta_1\alpha_1 \otimes \beta_1 + \beta_1 \otimes \alpha_1\beta_1 \\
\Delta_{eAe}(\beta_2) &= \beta_2\alpha_2 \otimes \beta_2 + \beta_2 \otimes \beta_1\alpha_1 \\
\Delta_{eAe}(\alpha_1\beta_1) &= \alpha_1\beta_1 \otimes \alpha_1\beta_1 \\
\Delta_{eAe}(\beta_1\alpha_1) &= \beta_1\alpha_1 \otimes \beta_1\alpha_1 \\
\Delta_{eAe}(\beta_2\alpha_2) &= \beta_2\alpha_2 \otimes \beta_2\alpha_2
\end{aligned}$$

such that $(\pi \otimes \pi)\Delta_A = \Delta_{eAe}\pi$.

The following result shows the relation between the comultiplications of gendo-symmetric algebra A and gendo-symmetric quotient algebra $B = A/I$, where I is a two-sided ideal of A.

Theorem 3.2.13. *Let A and B be gendo-symmetric algebras such that $B = A/I$, where I is a two-sided ideal of A, and $\pi : A \to B$ be the canonical surjection. Suppose that Ae and Bf are basic faithful projective-injective A-module and B-module, respectively, where $e = e' + e''$ is an orthogonal decomposition and $f = e' + I$. Let Δ_A be a comultiplication of A. Then there exists a comultiplication Δ_B of B such that $(\pi \otimes \pi)\Delta_A = \Delta_B\pi$.*

Proof. Since A and B are gendo-symmetric, there are multiplications $m_{\mathrm{D}(A)} : \mathrm{D}(A) \otimes_k \mathrm{D}(A) \to \mathrm{D}(A)$ and $m_{\mathrm{D}(B)} : \mathrm{D}(B) \otimes_k \mathrm{D}(B) \to \mathrm{D}(B)$, respectively. Note that $m_{\mathrm{D}(A)}$ is a A-bimodule and $m_{\mathrm{D}(B)}$ is a B-bimodule morphism. Let $\pi : A \to B$ be the canonical surjection. By dualizing this surjective morphism, we obtain an injective morphism $\bar{i} : \mathrm{D}(B) \to \mathrm{D}(A)$. Observe that there is an algebra isomorphism $\phi : \mathrm{D}(B) = D(A/I) \cong I^0$, where $I^0 = \{f \in \mathrm{D}(A) \mid f(x) = 0, \text{ for all } x \in I\}$. Then we obtain that $\bar{i} : \mathrm{D}(B) \cong I^0 \hookrightarrow \mathrm{D}(A)$, that is $\bar{i} = i\phi$, where $i : I^0 \hookrightarrow \mathrm{D}(A)$ is an inclusion.

There are also A-bimodule isomorphism $\gamma : Ae \otimes_{eAe} eA \cong \mathrm{D}(A)$ and B-bimodule isomorphism $\overline{\gamma} : Bf \otimes_{fBf} fB \cong \mathrm{D}(B)$. We can write $\overline{\gamma}$ as $\overline{\gamma} : Ae'/I \otimes_{e'Ae'/I} e'A/I \cong \mathrm{D}(B)$.

Therefore, the injective morphism $\bar{i}$ can be defined as

$$\bar{i}(\overline{\gamma}((ae'+I) \otimes_{e'Ae'/I} (e'b+I))) = i\phi(\overline{\gamma}((ae'+I) \otimes_{e'Ae'/I} (e'b+I))) = i(\gamma(ae' \otimes_{eAe} e'b)) = \gamma(ae' \otimes_{eAe} e'b)$$

such that $\gamma(ae' \otimes_{eAe} e'b)(x) = 0$ for all $x \in I$. Therefore, we obtain the following commutative diagram.

$$\begin{array}{ccc} \mathrm{D}(B) \otimes_k \mathrm{D}(B) & \xrightarrow{m_{\mathrm{D}(B)}} & \mathrm{D}(B) \\ {\scriptstyle \bar{i}\otimes\bar{i}}\downarrow & & \downarrow{\scriptstyle \bar{i}} \\ \mathrm{D}(A) \otimes_k \mathrm{D}(A) & \xrightarrow{m_{\mathrm{D}(A)}} & \mathrm{D}(A) \end{array}$$

Let us check this commutativity. From Section 3.2, it is known that $m_{\mathrm{D}(A)}$ and $m_{\mathrm{D}(B)}$ are defined by

$$m_{\mathrm{D}(A)} : \mathrm{D}(A) \otimes_k \mathrm{D}(A) \to \mathrm{D}(A) \otimes_A \mathrm{D}(A) \cong \mathrm{D}(A) \qquad (*)$$

and

$$m_{\mathrm{D}(B)} : \mathrm{D}(B) \otimes_k \mathrm{D}(B) \to \mathrm{D}(B) \otimes_B \mathrm{D}(B) \cong \mathrm{D}(B), \quad (**)$$

respectively. Therefore,

$$\begin{aligned} & \bar{i}m_{\mathrm{D}(B)}(\overline{\gamma}((ae'+I) \otimes_{e'Ae'/I} (e'b+I)) \otimes_k \overline{\gamma}((ce'+I) \otimes_{e'Ae'/I} (e'd+I))) \\ \overset{(1)}{=} & \bar{i}(\overline{\gamma}((ae'+I) \otimes_{e'Ae'/I} (e'b+I) \otimes_{A/I} (ce'+I) \otimes_{e'Ae'/I} (e'd+I))) \\ = & \bar{i}(\overline{\gamma}((ae'bce'+I) \otimes_{e'Ae'/I} (e'd+I))) \\ = & i\phi(\overline{\gamma}((ae'bce'+I) \otimes_{e'Ae'/I} (e'd+I))) \\ = & i(\gamma(ae'bce' \otimes_{eAe} e'd)) \\ = & \gamma(ae'bce' \otimes_{eAe} e'd), \end{aligned}$$

where (1) is obtained by $(**)$, and

$$\begin{aligned} & m_{\mathrm{D}(A)}(\bar{i} \otimes \bar{i})(\overline{\gamma}((ae'+I) \otimes_{e'Ae'/I} (e'b+I)) \otimes_k \overline{\gamma}((ce'+I) \otimes_{e'Ae'/I} (e'd+I))) \\ = & m_{\mathrm{D}(A)}(\bar{i}(\overline{\gamma}((ae'+I) \otimes_{e'Ae'/I} (e'b+I))) \otimes_k \bar{i}(\overline{\gamma}((ce'+I) \otimes_{e'Ae'/I} (e'd+I)))) \\ = & m_{\mathrm{D}(A)}(i\phi(\overline{\gamma}((ae'+I) \otimes_{e'Ae'/I} (e'b+I))) \otimes_k i\phi(\overline{\gamma}((ce'+I) \otimes_{e'Ae'/I} (e'd+I)))) \\ = & m_{\mathrm{D}(A)}(i(\gamma(ae' \otimes_{eAe} e'b)) \otimes_k i(\gamma(ce' \otimes_{eAe} e'd))) \\ = & m_{\mathrm{D}(A)}(\gamma(ae' \otimes_{eAe} e'b) \otimes_k \gamma(ce' \otimes_{eAe} e'd)) \\ \overset{(2)}{=} & \gamma(ae' \otimes_{eAe} e'b \otimes_A ce' \otimes_{eAe} e'd) \\ = & \gamma(ae'bce' \otimes_{eAe} e'd), \end{aligned}$$

where (2) is obtained by $(*)$. Hence, we obtain that $\bar{i}m_{\mathrm{D}(B)} = m_{\mathrm{D}(A)}(\bar{i} \otimes \bar{i})$. By dualising the above diagram, we have the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\Delta_A} & A \otimes_k A \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi\otimes\pi} \\ B & \xrightarrow{\Delta_B} & B \otimes_k B \end{array}$$

where $\pi : A \to B$ is canonical surjection. Then we obtain that

$$(\pi \otimes \pi)\Delta_A = \Delta_B \pi.$$

□

Let us illustrate this result by an example.

Example 3.2.14. Let A be the path algebra of the following quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha_1"] & 2 \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\alpha_2"] & 3 \arrow[l, shift left, "\beta_2"] \end{tikzcd}$$

such that $\alpha_1\alpha_2 = 0$, $\beta_2\beta_1 = 0$, $\beta_2\alpha_2 = 0$, $\beta_1\alpha_1 = \alpha_2\beta_2$ which is given in Example 3.2.8, and B be the path algebra of following quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha_1"] & 2 \arrow[l, shift left, "\beta_1"] \end{tikzcd}$$

such that $\beta_1\alpha_1 = 0$ which is given in Example 3.2.7. Observe that $B \cong A/Ae_3A$. Let $\pi : A \to B$ be the canonical surjection. Let Δ_A be the comultiplication of A which is computed in Example 3.2.8. Then there exists a comultiplication Δ_B of B, which is computed in Example 3.2.7, such that $(\pi \otimes \pi)\Delta_A = \Delta_B \pi$.

In Theorem 3.2.13, we assumed that $e = e' + e''$ is an orthogonal decomposition and $f = e' + I$. Let us see why this assumption is necessary by an example.

Example 3.2.15. Let A be the algebra given in previous example and $e = e_1 + e_2$. Let $I = \mathrm{rad}A$. Then $B \cong A/\mathrm{rad}A$ is semisimple and so symmetric. Here we can consider B as the path algebra of the quiver which has 3 vertices and no arrows. Since B is symmetric, $f = 1_B = e_1 + e_2 + e_3$. We see that this example does not satisfy the assumption in the theorem. We now determine the invertible central elements of B. Any element $x \in B$ can be written by $x = c_1e_1 + c_2e_2 + c_3e_3$ where $c_1, c_2, c_3 \in k$. Suppose that x is an invertible element of B such that $xy = 1_B$, where $y = d_1e_1 + d_2e_2 + d_3e_3$. Then

$$\begin{aligned} xy &= (c_1e_1 + c_2e_2 + c_3e_3)(d_1e_1 + d_2e_2 + d_3e_3) \\ &= c_1d_1e_1 + c_2d_2e_2 + c_3d_3e_3 \\ &= e_1 + e_2 + e_3. \end{aligned}$$

Hence, $c_1d_1 = c_2d_2 = c_3d_3 = 1_k$. This means that c_1, c_2, c_3 are nonzero elements of the field k. Moreover, for any $b = b_1e_1 + b_2e_2 + b_3e_3 \in B$, we observe that

$$\begin{aligned} xb &= (c_1e_1 + c_2e_2 + c_3e_3)(b_1e_1 + b_2e_2 + b_3e_3) \\ &= c_1b_1e_1 + c_2b_2e_2 + c_3b_3e_3 \\ &= b_1c_1e_1 + b_2c_2e_2 + b_3c_3e_3 \\ &= (b_1e_1 + b_2e_2 + b_3e_3)(c_1e_1 + c_2e_2 + c_3e_3) = bx. \end{aligned}$$

Then we obtain that the central invertible elements of B are of the form $x = c_1e_1 + c_2e_2 + c_3e_3$ where c_1, c_2, c_3 are nonzero elements of the field k.

By Corollary 3.2.4, any comultiplication Δ_B of B on the basis elements is of the form

$$\Delta_B(e_1) = c_1e_1 \otimes e_1$$

$$\Delta_B(e_2) = c_2 e_2 \otimes e_2$$
$$\Delta_B(e_3) = c_3 e_3 \otimes e_3.$$

Let $\pi : A \to B$ be the canonical surjection. Then we observe that

$$(\pi \otimes \pi)\Delta_A(e_1) = (\pi \otimes \pi)(\alpha_1\beta_1 \otimes e_1 + \alpha_1 \otimes \beta_1 + e_1 \otimes \alpha_1\beta_1) = 0$$

and

$$\Delta_B \pi(e_1) = \Delta_B(e_1) = c_1 e_1 \otimes e_1$$

are not equal and the conclusion of Theorem 3.2.13 does not hold.

3.2.1 Gendo-symmetric Schur algebras

Schur algebras, named after Issai Schur who was a student of Frobenius, are certain finite dimensional algebras which relate representation theories of general linear and symmetric groups. In fact, this relation is obtained by using an important theorem which is called Schur-Weyl duality. Some Schur algebras are also contained in the class of gendo-symmetric algebras. Therefore, we will give a place to Schur algebras in this study.

In this part, we first give the definition of Schur algebras and indicate in which case the Schur algebras are gendo-symmetric. At the same time, we give information about their dominant dimension. Later we introduce the Schur-Weyl duality and give some related examples. Lastly, we give some results from [33] to show the motivation of the main results of this subsection.

For more information about Schur algebras, see [7] and [17].

Let k be an infinite field, n and r be two natural numbers. Let $I(n,r)$ be the set of multi-indices $(i_1, ..., i_r)$ with $i_\rho \in \{1, ..., n\}$ for $1 \le \rho \le r$. Let $A_k(n,r)$ be the k-space of homogeneous polynomials of degree r in the n^2 indeterminants $\{c_{i,j} \mid 1 \le i,j \le n\}$. This k-space $A_k(n,r)$ is clearly spanned by the monomials $c_{\underline{i},\underline{j}} = c_{i_1,j_1}...c_{i_r,j_r}$ where $\underline{i}, \underline{j} \in I(n,r)$. The *Schur algebra* $S_k(n,r)$ is defined to be the k-dual of $A_k(n,r)$.

Let $n \ge r$. Then $S_k(n,r)$ is gendo-symmetric (see [13]). Therefore, any faithful projective and injective left $S_k(n,r)$-module satisfies the double centralizer property and the dominant dimension of $S_k(n,r)$ equals

$$\max\{d \mid \mathrm{H}^i(\mathcal{C}^{\bullet}_{S_k(n,r)}) = 0 \mid 0 \le i \le d\} + 1,$$

by Theorem 3.3.1 which is given in the next section. Moreover, if $r \ge p = \mathrm{char}(k) > 0$, then $\mathrm{dom.dim} S_k(n,r) = 2(p-1)$ ([15], Theorem 5.1); otherwise, $S_k(n,r)$ is semisimple and so $\mathrm{dom.dim} S_k(n,r) = \infty$.

Let $n < r$. In this case, the Schur algebra $S_k(n,r)$ does not always have a faithful projective-injective module and may have dominant dimension zero. Instead of projective-injective modules, in [22], Koenig, Slungard and Xi used (partial) tilting modules, that is, self-dual modules which are filtered by Weyl modules.

However, in [12], Fang made some further investigation and more results under the mild condition $r \le n(p-1)$; and he proved the following theorem.

Theorem 3.2.16. *([12], Theorem 3.10) Let k be an infinite field of characteristic $p > 0$, n and r be two natural numbers. If $r \le n(p-1)$, then $S_k(n,r)$ is gendo-symmetric with dominant dimension equal to* *the largest number t (or ∞) such that the following complex is exact*

$$A_k(n,r)^{\otimes(t+1)} \overset{\partial_t}{\to} A_k(n,r)^{\otimes t} \to ... \overset{\partial_2}{\to} A_k(n,r) \otimes_k A_k(n,r) \overset{\partial_1}{\to} A_k(n,r) \to 0$$

where $\partial_i(a_0 \otimes ... \otimes a_i) = \sum_{j=0}^{i-1}(-1)^j a_0 \otimes ... \otimes a_{j-1} \otimes \Theta(a_j, a_{j+1}) \otimes a_{j+2} \otimes ... \otimes a_i$ *for* $a_0, ..., a_i \in A_k(n,r)$ *and* Θ *is the multiplication map defined in [12], Theorem 3.2.*

The condition $r \le n(p-1)$ in the last theorem is sufficient but not necessary for $S_k(n,r)$ to be gendo-symmetric. For example, if k is an infinite field of characteristic 2, then $S_k(2,3)$ is semisimple by Theorem 2 in [8], hence it has dominant dimension ∞. $S_k(2,4)$ has dominant dimension 0 by [22]; and $S_k(2,7)$ has dominant dimension 2 since it is Morita equivalent to $S_k(2,2) \times k \times k$. Here both $S_k(2,3)$ and $S_k(2,7)$ are gendo-symmetric.

Remark 3.2.17. The q-Schur algebras are q-analogues of the classical Schur algebras, in which the symmetric group is replaced by the corresponding Hecke algebra and the general linear group by an appropriate quantum group.

Now we introduce an outstanding result which is called Schur-Weyl duality.

Theorem 3.2.18. *(Schur-Weyl duality) Let n and r be two natural numbers and let k be an infinite field of any characteristic. Let the general linear group $GL_n(k)$ act diagonally from the left on $E = (k^n)^{\otimes r}$ (with natural action on k^n) and let the symmetric group Σ_r act from the right by place permutations. Denote by $S_k(n,r)$ the algebra generated by the image of the GL_n-action (the "Schur algebra").*

(a) Suppose $n \ge r$. Then there is a double centralizer property

$$S_k(n,r) \cong End_{k\Sigma_r}(E)$$
$$k\Sigma_r \cong End_{S_k(n,r)}(E).$$

(b) Suppose $n < r$. Denote by B the quotient of $k\Sigma_r$ modulo the kernel of the action of $k\Sigma_r$ on E. Then there is a double centralizer property

$$S_k(n,r) \cong End_B(E)$$
$$B \cong End_{S_k(n,r)}(E).$$

(c) Parts (a) and (b) remain true if one replaces the Schur algebra $S_k(n,r)$ by the quantized Schur algebra $S_q(n,r)$ and the group algebra $k\Sigma_r$ of the symmetric group by the Hecke algebra $\mathcal{H}_q(r)$ ($q \neq 0$) of type A.

Schur [29] proved (a) and (b) at least in characteristic zero in order to relate the representation theories of the general linear and the symmetric groups. There are various proofs for the general case. For different approaches and cases, see [5], [6], [9], [10] and [19]. In [22], there is also a computation-free proof. In addition, see [17].

Theorem 3.2.19. *([22], Theorem 1.3) (a) A (classical or quantized) Schur algebra $S(n,r)$ with $n \ge r$ has dominant dimension at least two.*

(b) For a (classical or quantized) Schur algebra $S(n,r)$ with $n < r$ there is a tilting module T such that $S(n,r)$ has T-dominant dimension at least two.

Example 3.2.20. The algebra A in Example 3.1.2 is Morita equivalent to the Schur algebra $S_k(2,2)$ if k is an infinite field of characteristic 2. Then $B = eAe$ is the corresponding group algebra $k\Sigma_2$ occuring in Schur-Weyl duality.

Example 3.2.21. The algebra A in Example 3.1.3 is Morita equivalent to a block of a quantised Schur algebra in quantum characteristic four. Then $B = eAe$ is the corresponding block of a Hecke algebra occuring in quantised Schur-Weyl duality.

From now on, we underline some remarkable parts of [33] by Xi. So until the next section we assume that k is an algebraically closed field and $n \geq r$. We consider the case $\text{char}(k) = p > 0$ and $r = p$.

Theorem 3.2.22. *([33], Theorem 2.8) Let A be a connected basic symmetric algebra and M the socle of an indecomposable projective left ideal of A such that A/M is quasi-hereditary. Then A is a simple algebra, or isomorphic to $k[x]/(x^2)$ where $k[x]$ is the polynomial ring in one variable x, or there is a natural number $n \geq 2$, such that A is the path algebra of the following quivers*

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3 \cdots n-1 \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} n$$

modulo the ideal generated by

$$\alpha_{i-1}\alpha_i,\ \beta_i\beta_{i-1},\ \alpha_i\beta_i - \beta_{i-1}\alpha_{i-1} \textit{ for } i = 2, \cdots, n-1;$$

$$\alpha_1\beta_1\alpha_1,\ \beta_{n-1}\alpha_{n-1}\beta_{n-1}.$$

The converse of the above theorem holds.

Note that an algebra A is called *quadratic* if the basic algebra of A can be written by quiver and relations with all relations of degree 2. The above theorem equivalently says that if there is an indecomposable module M over a symmetric algebra A such that $E := \text{End}_A({}_AA \oplus M)$ is quasi-hereditary, then the algebra E is quadratic. By Proposition in [33], the algebra $S_k(n,p)$ is Morita equivalent to an algebra of the form $\text{End}_A({}_AA \oplus M)$ with $A = k\Sigma_p$ and M an indecomposable module. Since the Schur algebra is quasi-hereditary (see [15]), by Theorem 3.2.22 the basic algebra of $S_k(n,p)$ is of the form in following theorem.

Theorem 3.2.23. *([33], Theorem) Let k be an algebraically closed field with characteristic $p > 0$. Then each block of the Schur algebra $S_k(n,p)$ with $n \geq p$ is either simple or Morita equivalent to the path algebra P (over k) of*

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3 \cdots n-1 \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} n$$

modulo the ideal generated by

$$\alpha_i\alpha_{i+1},\ \beta_{i+1}\beta_i,\ \alpha_{i+1}\beta_{i+1} - \beta_i\alpha_i,\ \textit{for } 1 \leq i \leq n-1;$$

$$\beta_{n-1}\alpha_{n-1},$$

where $n \geq 1$ and depends only on p. Moreover, there is only one non-simple block. Thus, in particular, $S_k(n,p)$ is a quadratic algebra.

Remark 3.2.24. In [32], Wen considered the case where A is self-injective and obtained a complete list of self-injective algebras with an indecomposable A-module M such that the k-algebra $\text{End}_A(A \oplus M)$ is quasi-hereditary, which properly includes the algebras in Theorem 3.2.22.

We observe that the algebra in Theorem 3.2.23 is gendo-symmetric Schur algebra. Now we are ready to give its comultiplication with following result.

Proposition 3.2.25. *Let A be the path algebra (over k) of the following quiver*

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3 \cdots n-1 \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} n$$

modulo the ideal generated by

$$\alpha_i\alpha_{i+1},\ \beta_{i+1}\beta_i,\ \alpha_{i+1}\beta_{i+1} - \beta_i\alpha_i \text{ for } 1 \leq i \leq n-2;$$

$$\beta_{n-1}\alpha_{n-1},$$

where $n \geq 2$. Then A has a comultiplication Δ which is defined on basis elements by

$$\begin{aligned}
\Delta(e_1) &= \alpha_1\beta_1 \otimes e_1 + \alpha_1 \otimes \beta_1 + e_1 \otimes \alpha_1\beta_1\\
\Delta(e_n) &= \beta_{n-1} \otimes \alpha_{n-1}\\
\Delta(e_{i+1}) &= \alpha_{i+1}\beta_{i+1} \otimes e_{i+1} + \beta_i \otimes \alpha_i + \alpha_{i+1} \otimes \beta_{i+1} + e_{i+1} \otimes \alpha_{i+1}\beta_{i+1}\\
\Delta(\alpha_i) &= \alpha_i\beta_i \otimes \alpha_i + \alpha_i \otimes \alpha_{i+1}\beta_{i+1}\\
\Delta(\alpha_{n-1}) &= \alpha_{n-1}\beta_{n-1} \otimes \alpha_{n-1}\\
\Delta(\beta_i) &= \alpha_{i+1}\beta_{i+1} \otimes \beta_i + \beta_i \otimes \alpha_i\beta_i\\
\Delta(\beta_{n-1}) &= \beta_{n-1} \otimes \alpha_{n-1}\beta_{n-1}\\
\Delta(\alpha_j\beta_j) &= \alpha_j\beta_j \otimes \alpha_j\beta_j,
\end{aligned}$$

where $1 \leq i \leq n-2$ and $1 \leq j \leq n-1$.

Proof. Let $e = e_1 + e_2 + \cdots + e_{n-1}$ since $A(e_1 + e_2 + \cdots + e_{n-1})$ is a basic faithful projective-injective A-module. There exists an (eAe, A)-bimodule isomorphism ι which is explicity defined on the basis elements by

$$\begin{aligned}
\iota : eA &\cong \mathrm{D}(Ae)\\
e_i &\mapsto (\alpha_i\beta_i)^*\\
\alpha_i &\mapsto \beta_i^*\\
\beta_j &\mapsto \alpha_j^*\\
\alpha_i\beta_i &\mapsto e_i^*,
\end{aligned}$$

where $1 \leq i \leq n-1$ and $1 \leq j \leq n-2$.

By using (3.1), the A-bimodule isomorphism γ is explicitly defined by

$$\begin{aligned}
\gamma : Ae \otimes_{eAe} eA &\cong \mathrm{D}(A)\\
e_i \otimes \alpha_i\beta_i &\mapsto e_i^*\\
\beta_{n-1} \otimes \alpha_{n-1} &\mapsto e_n^*\\
\beta_i \otimes e_i &\mapsto \alpha_i^*\\
e_i \otimes \alpha_i &\mapsto \beta_i^*\\
e_i \otimes e_i &\mapsto (\alpha_i\beta_i)^*,
\end{aligned}$$

where $1 \leq i \leq n-1$.

Let $f, g \in \mathrm{D}(A)$ such that

$$f = \sum_{i=1}^{n} a_i e_i^* + \sum_{i=1}^{n-1} b_i \alpha_i^* + \sum_{i=1}^{n-1} c_i \beta_i^* + \sum_{i=1}^{n-1} d_i (\alpha_i \beta_i)^*$$

and

$$g = \sum_{i=1}^{n} a'_i e_i^* + \sum_{i=1}^{n-1} b'_i \alpha_i^* + \sum_{i=1}^{n-1} c'_i \beta_i^* + \sum_{i=1}^{n-1} d'_i (\alpha_i \beta_i)^*.$$

By using γ and (3.2), we obtain that

$$\begin{aligned} m(g \otimes f) = (a'_1 d_1 + b'_1 c_1 + d'_1 a_1) e_1^* + (a'_{i+1} d_{i+1} + b'_i c_i + c'_{i+1} b_{i+1} + d'_{i+1} a_{i+1}) e_{i+1}^* + (b'_{n-1} c_{n-1}) e_n^* \\ +(b'_i d_i + d'_{i+1} b_i) \alpha_i^* + (b'_{n-1} d_{n-1}) \alpha_{n-1}^* \\ +(c'_i d_{i+1} + d'_i c_i) \beta_i^* + (d'_{n-1} c_{n-1}) \beta_{n-1}^* \\ +(d'_j d_j)(\alpha_j \beta_j)^*, \end{aligned}$$

where $1 \leq i \leq n-2$ and $1 \leq j \leq n-1$. Then

$$\begin{aligned} m(g \otimes f)(e_1) &= a'_1 d_1 + b'_1 c_1 + d'_1 a_1 \\ m(g \otimes f)(e_{i+1}) &= a'_{i+1} d_{i+1} + b'_i c_i + c'_{i+1} b_{i+1} + d'_{i+1} a_{i+1} \\ m(g \otimes f)(e_n) &= b'_{n-1} c_{n-1} \\ m(g \otimes f)(\alpha_i) &= b'_i d_i + d'_{i+1} b_i \\ m(g \otimes f)(\alpha_{n-1}) &= b'_{n-1} d_{n-1} \\ m(g \otimes f)(\beta_i) &= c'_i d_{i+1} + d'_i c_i \\ m(g \otimes f)(\beta_{n-1}) &= d'_{n-1} c_{n-1} \\ m(g \otimes f)(\alpha_j \beta_j) &= d'_j d_j, \end{aligned}$$

where $1 \leq i \leq n-2$ and $1 \leq j \leq n-1$.

Since $(f \otimes g)\Delta(a) = m(g \otimes f)(a)$ for all $a \in A$, dualising m gives the desired comultiplication. □

There is a remarkable symmetric algebra given in Theorem 3.2.22. The following results give comultiplication of this algebra and its compatible counit.

Proposition 3.2.26. *Let A be the path algebra (over k) of the following quiver*

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3 \cdots n-1 \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} n$$

modulo the ideal generated by

$$\alpha_i \alpha_{i+1},\ \beta_{i+1} \beta_i,\ \alpha_{i+1} \beta_{i+1} - \beta_i \alpha_i \text{ for } 1 \leq i \leq n-2;$$

$$\alpha_1 \beta_1 \alpha_1,\ \beta_{n-1} \alpha_{n-1} \beta_{n-1},$$

where $n \geq 2$. Then A has a comultiplication Δ which is defined on basis elements by

$$\Delta(e_1) = \alpha_1 \beta_1 \otimes e_1 + \alpha_1 \otimes \beta_1 + e_1 \otimes \alpha_1 \beta_1$$

$$
\begin{aligned}
\Delta(e_n) &= \beta_{n-1}\alpha_{n-1} \otimes e_n + \beta_{n-1} \otimes \alpha_{n-1} + e_n \otimes \beta_{n-1}\alpha_{n-1} \\
\Delta(e_{i+1}) &= \alpha_{i+1}\beta_{i+1} \otimes e_{i+1} + \beta_i \otimes \alpha_i + \alpha_{i+1} \otimes \beta_{i+1} + e_{i+1} \otimes \alpha_{i+1}\beta_{i+1} \\
\Delta(\alpha_i) &= \alpha_i\beta_i \otimes \alpha_i + \alpha_i \otimes \alpha_{i+1}\beta_{i+1} \\
\Delta(\alpha_{n-1}) &= \alpha_{n-1}\beta_{n-1} \otimes \alpha_{n-1} + \alpha_{n-1} \otimes \beta_{n-1}\alpha_{n-1} \\
\Delta(\beta_i) &= \alpha_{i+1}\beta_{i+1} \otimes \beta_i + \beta_i \otimes \alpha_i\beta_i \\
\Delta(\beta_{n-1}) &= \beta_{n-1}\alpha_{n-1} \otimes \beta_{n-1} + \beta_{n-1} \otimes \alpha_{n-1}\beta_{n-1} \\
\Delta(\alpha_j\beta_j) &= \alpha_j\beta_j \otimes \alpha_j\beta_j \\
\Delta(\beta_{n-1}\alpha_{n-1}) &= \beta_{n-1}\alpha_{n-1} \otimes \beta_{n-1}\alpha_{n-1},
\end{aligned}
$$

where $1 \leq i \leq n-2$ *and* $1 \leq j \leq n-1$.

Proof. Since A is symmetric, we choose $e = 1_A$. Then there exists an (A, A)-bimodule isomorphism ι which is explicity defined on the basis elements by

$$
\begin{aligned}
\iota : A &\cong \mathrm{D}(A) \\
e_i &\mapsto (\alpha_i\beta_i)^* \\
e_n &\mapsto (\beta_{n-1}\alpha_{n-1})^* \\
\alpha_i &\mapsto \beta_i^* \\
\beta_i &\mapsto \alpha_i^* \\
\alpha_i\beta_i &\mapsto e_i^* \\
\beta_{n-1}\alpha_{n-1} &\mapsto e_n^*
\end{aligned}
$$

where $1 \leq i \leq n-1$.

By using the similar way with the proof of Proposition 3.2.25, we complete the proof. □

Proposition 3.2.27. *Let A and Δ be as in Proposition 3.2.26. Then the counit δ of (A, Δ) is explicitly defined on the basis elements by*

$$
\begin{aligned}
\delta : \alpha_j\beta_j &\mapsto 1 \textit{ for } 1 \leq j \leq n-1 \\
\beta_{n-1}\alpha_{n-1} &\mapsto 1 \\
\textit{otherwise} &\mapsto 0.
\end{aligned}
$$

Proof. Here, A is a symmetric algebra with the A-bimodule isomorphism ι which is given in the proof of Proposition 3.2.26. By taking into account Proposition 3.4.1 which is proved in Section 3.4, and Chapter 2, we see that the counit δ of (A, Δ) corresponds to the Frobenius form of A which is equal to $\iota(1_A)$. Since $1_A = e_1 + \cdots + e_n$, we obtain that

$$
\delta = \iota(1_A) = \sum_{j=1}^{n-1} (\alpha_j\beta_j)^* + (\beta_{n-1}\alpha_{n-1})^*.
$$

□

3.3 Characterisations of gendo-symmetric algebras and their dominant dimension

In this section, we first introduce the main result of [13] which gives the characterisation of the dominant dimension of a gendo-symmetric algebra in terms of exactness of the bar cocomplex. At the same time, this result shows that the class of gendo-symmetric algebras is characterised by the existence of this bar cocomplex. Later we give a hypothesis which leads to prove Nakayama conjecture for gendo-symmetric algebras by using this mentioned result.

Theorem 3.3.1. *([13], Theorem 3.6) Let A be a finite dimensional k-algebra and $n \geq 2$ an integer. Then A is a gendo-symmetric algebra with* dom.dim$A \geq n$ *if and only if there is an A-bimodule morphism $\Delta : A \to {}_AA \otimes_k A_A$ satisfying*

(1) Δ is injective;

(2) $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$ and

(3) $\mathrm{Im}\Delta \subseteq \{\sum u_i \otimes v_i \in A \otimes_k A \mid \sum u_i a \otimes v_i = \sum u_i \otimes a v_i, \ \forall a \in A\}$ *such that the complex*

$$\mathcal{C}_A^{\bullet}: \ 0 \to A \xrightarrow{\Delta} A \otimes_k A \xrightarrow{\delta^1} A \otimes_k A \otimes_k A \to ... \xrightarrow{\delta^{n-1}} A^{\otimes n+1} \to ...$$

has cohomologies $H^i(\mathcal{C}_A^{\bullet}) = 0$ for $0 \leq i \leq n-1$, where the differential $\delta^r : A^{\otimes r+1} \to A^{\otimes r+2}$ is given by: for any $a_0, ..., a_r \in A$

$$\delta^r(a_0 \otimes ... \otimes a_r) = \sum_{i=0}^{r} (-1)^i a_0 \otimes ... \otimes a_{i-1} \otimes \Delta(a_i) \otimes a_{i+1} \otimes ... \otimes a_r.$$

Remark 3.3.2. In the next section, this characterisation of gendo-symmetric algebras is compared with Theorem 2.2.3 which is proved by Abrams.

Nakayama conjecture is a major homological conjecture and also a major open problem in representation theory. It states that if A is finite-dimensional algebra over a field and dom.dim$A = \infty$, then A is self-injective. Nakayama conjecture for gendo-symmetric algebras states that if A is a gendo-symmetric algebra and dom.dim$A = \infty$, then A is symmetric. The above characterisation of gendo-symmetric algebras may lead to prove Nakayama conjecture for gendo-symmetric algebras by first verifying the following hypothesis.

Hypothesis 3.3.3. Let A be a gendo-symmetric algebra with the comultiplication Δ. Suppose that the bar cocomplex $\mathcal{C}_{\mathcal{A}}^{\bullet}$ is exact. Then there exists a counit of (A, Δ).

The connection to the Nakayama conjecture uses the following implications from [13]:

A gendo-symmetric algebra A with the comultiplication Δ has a counit $\Rightarrow$ The bar cocomplex of A is exact $\Rightarrow$ domdim$A = \infty$.

Now let us consider the reverse of these implications. Let A be a gendo-symmetric algebra with the comultiplication Δ. Then

dom.dim$A = \infty \overset{1}{\Rightarrow}$ The bar cocomplex of A is exact $\overset{(*)}{\Rightarrow}$ There exists a counit of $(A, \Delta) \overset{2}{\Rightarrow}$ A is symmetric.

The implications $\overset{1}{\Rightarrow}$ and $\overset{2}{\Rightarrow}$ are known by Theorem 3.3.1 and Proposition 3.2.5, respectively. If the hypothesis is proved, then the implication (*) is satisfied and Nakayama conjecture for gendo-symmetric algebras is proved.

3.4 Differences between Frobenius and gendo-symmetric algebras with respect to comultiplication

Two large classes of algebras, Frobenius algebras and gendo-symmetric algebras, are characterised by the existence of a comultiplication with some special properties and both contain symmetric algebras. However, there are differences between them. In this section, we clarify these differences between Frobenius algebras and gendo-symmetric algebras with respect to comultiplication by collecting the related results given in Chapter 2 and in this chapter. But first we give a result on symmetric algebras with respect to comultiplication.

Recall that $\alpha : A \to A \otimes_k A$ is the comultiplication given in Section 2.2 when we assume that A is Frobenius and $\Delta : A \to A \otimes_k A$ is the comultiplication given in Section 3.2 when we assume that A is gendo-symmetric.

Let A be a symmetric algebra over a field k. We keep the notations introduced in Chapter 2 and this chapter. Since A is symmetric, we choose $e = 1_A$ and have the A-bimodule isomorphism $\iota : A \cong \mathrm{D}(A)$ by Section 3.2.

Proposition 3.4.1. *Let A be a symmetric algebra with the A-bimodule isomorphism $\lambda : A \cong D(A)$. Suppose that $\lambda = \iota$. Then α is equal to Δ.*

Proof. Let A be symmetric and $\iota : A \cong \mathrm{D}(A)$ be the A-bimodule isomorphism. There is an A-bimodule isomorphism

$$\mathrm{Hom}_A(\mathrm{D}(A), A) \cong \mathrm{Hom}_A(A \otimes_A A, A) \cong \mathrm{Hom}_A(A, A) \cong A,$$

where the first isomorphism is $\mathrm{Hom}_A(\gamma, A)$. Let

$$\Theta : \mathrm{D}(A) \cong A$$

be the inverse image of $1 \in A$ under the above isomorphism. Then $(\Theta \circ \gamma)(a \otimes b) = ab$ for $a, b \in A$. In particular, $\Theta = \iota^{-1}$.

Since the Frobenius form ε of A is equal to $\lambda(1_A)$, all elements of $\mathrm{D}(A)$ are of the form $a \cdot \varepsilon$ for any $a \in A$. By Section 2.2, it is known that the isomorphism $\lambda : A \cong \mathrm{D}(A)$ allows us to define a multiplication φ such that $\varphi(a \cdot \varepsilon \otimes b \cdot \varepsilon) = (b \cdot \varepsilon \otimes a \cdot \varepsilon) \circ \alpha = ab \cdot \varepsilon$.

Let $\vartheta : A \otimes_A A \cong A$ be the A-bimodule isomorphism such that $\vartheta(a \otimes_A b) = ab$ and $\mu' : A \otimes_k A \to A \otimes_A A$ be the map such that $\mu'(a \otimes_k b) = a \otimes_A b$ for any $a, b \in A$. Suppose that $\lambda' := \lambda \circ \vartheta$ and $\varphi' := \lambda' \circ \mu' \circ \lambda^{-1} \otimes \lambda^{-1}$. Observe that $\varphi = \varphi'$.

Clearly, there are also isomorphisms of A-bimodules $\lambda \otimes_k \lambda : A \otimes_k A \cong \mathrm{D}(A) \otimes_k \mathrm{D}(A)$ and $\lambda \otimes_A \lambda : A \otimes_A A \cong \mathrm{D}(A) \otimes_A \mathrm{D}(A)$.

On the other hand, there is a multiplication $m : \mathrm{D}(A) \otimes_k \mathrm{D}(A) \to \mathrm{D}(A) \otimes_A \mathrm{D}(A) \cong \mathrm{D}(A)$ which is given in this chapter such that $m(g \otimes f)(a) = (f \otimes g)\Delta(a)$ for any $f, g \in \mathrm{D}(A)$ and $a \in A$. Then we obtain the following commutative diagram

$$\begin{tikzcd}
D(A) \otimes_k D(A) \arrow[r, "\lambda^{-1}\otimes_k\lambda^{-1}"] \arrow[dr, "\mathrm{id}"'] & A \otimes_k A \arrow[r, "\mu'"] \arrow[d, "\lambda\otimes_k\lambda"] & A \otimes_A A \arrow[r, "\lambda'"] \arrow[d, "\lambda\otimes_A\lambda"] & D(A) \\
 & D(A) \otimes_k D(A) \arrow[r] & D(A) \otimes_A D(A) \arrow[ur] &
\end{tikzcd}$$

where id is the identity map of $D(A) \otimes_k D(A)$. Therefore, we have $\varphi = m \circ \mathrm{id}$, that is, $\varphi = m$. Dualising this commutative diagram gives $\alpha = \Delta$.

For the detailed computation which shows that the above diagram is commutative, see the proof of Proposition 4.2.19 which will be a more general result than Proposition 3.4.1. □

We have already mentioned that two classes of algebras, Frobenius algebras and gendo-symmetric algebras, have differences. For example, Frobenius algebras have counit property but gendo-symmetric algebras do not, in general. Here, it is natural to ask whether there are other properties distinguishing Frobenius algebras from gendo-symmetric algebras. More precisely, the following question appears.

Question. What are the differences of gendo-symmetric and Frobenius algebras with respect to comultiplication?

Suppose that A is a finite-dimensional k-algebra. Let **FA** denote the Frobenius algebras and **GA** denote the gendo-symmetric algebras. When we write these abbreviations **FA** and **GA**, we assume that the finite dimensional algebra A is Frobenius and gendo-symmetric, respectively. From now on, we answer the above question by clarifying the differences step by step.

(1) **FA.** There is a left (or right) A-module isomorphism $\mathrm{Hom}_A(D(A), A) \cong A$ (Theorem 4.1.7). Moreover, there is an A-bimodule isomorphism $\mathrm{Hom}_A(D(A), A)_{\nu_A} \cong A$, where ν_A is a Nakayama automorphism of A (Proposition 4.1.11).

GA. There is an A-bimodule isomorphism $\mathrm{Hom}_A(D(A), A) \cong A$ (Definition 3.1.1).

Dually,

FA. There is a right (or left) A-module isomorphism $D(A) \otimes_A D(A) \cong D(A)$. Moreover, there is an A-bimodule isomorphism ${}_{\nu_A}D(A) \otimes_A D(A) \cong D(A)$.

GA. There is an A-bimodule isomorphism $D(A) \otimes_A D(A) \cong D(A)$.

(2) **FA.** $\alpha : A \to A \otimes_k A$ is a coassociative counital comultiplication on A (Theorem 2.2.3).

GA. $\Delta : A \to A \otimes_k A$ is a coassociative comultiplication on A. However, there is a compatible counit if and only if A is symmetric (Theorem 3.2.1 & Proposition 3.2.5).

(3) **FA.** $\mathrm{Im}(\alpha) = \{\sum u_i \otimes v_i \mid \sum u_i x \otimes v_i = \sum u_i \otimes \nu_A^{-1}(x) v_i, \ \forall x \in A\}$, where ν_A is a Nakayama automorphism of A (Theorem 2.2.11).

GA. $\mathrm{Im}(\Delta) = \{\sum u_i \otimes v_i \mid \sum u_i x \otimes v_i = \sum u_i \otimes x v_i, \ \forall x \in A\}$ (Lemma 3.2.3).

(4) **FA.** Let $\alpha(1_A) = \sum_{i=1}^n x_i \otimes y_i$. Then $\alpha(1_A) = \sum_{i=1}^n y_i \otimes \nu_A^{-1}(x_i)$, where ν_A is a Nakayama automorphism of A (Proposition 2.2.8).

GA. Let $\Delta(1_A) = \sum x_i \otimes y_i$. Then $\Delta(1_A) = \sum y_i \otimes x_i$ (Proposition 3.2.5).

We see that Nakayama automorphism plays crucial role in these differences.

Remark 3.4.2. The class of gendo-symmetric algebras is closed under Morita equivalences (Proposition 3.2.6), but the class of Frobenius algebras is not ([31], Chapter IV).

We now compare the two important results of Frobenius and gendo-symmetric algebras. Let us first remember them.

Theorem 1. *([2], Theorem 2.1) An algebra A is a Frobenius algebra if and only if it has a coassociative counital comultiplication $\alpha : A \to A \otimes_k A$ which is a map of A-bimodules.*

Theorem 2. *([13], Theorem 3.6) Let A be a finite dimensional k-algebra and $n \geq 2$ an integer. Then A is a gendo-symmetric algebra with* dom.dim$A \geq n$ *if and only if there is an A-bimodule morphism $\Delta : A \to {}_AA \otimes_k A_A$ satisfying*
(1) Δ is injective;
(2) $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$ and
(3) $\mathrm{Im}\Delta \subseteq \{\sum u_i \otimes v_i \in A \otimes_k A \mid \sum u_i a \otimes v_i = \sum u_i \otimes a v_i, \ \forall a \in A\}$ *such that the complex*

$$\mathcal{C}_A^\bullet : \ 0 \to A \overset{\Delta}{\to} A \otimes_k A \overset{\delta^1}{\to} A \otimes_k A \otimes_k A \to ... \overset{\delta^{n-1}}{\to} A^{\otimes n+1} \to ...$$

has cohomologies $\mathrm{H}^i(\mathcal{C}_A^\bullet) = 0$ for $0 \leq i \leq n-1$, *where the differential $\delta^r : A^{\otimes r+1} \to A^{\otimes r+2}$ is given by: for any $a_0, ..., a_r \in A$*

$$\delta^r(a_0 \otimes ... \otimes a_r) = \sum_{i=0}^{r} (-1)^i a_0 \otimes ... \otimes a_{i-1} \otimes \Delta(a_i) \otimes a_{i+1} \otimes ... \otimes a_r.$$

By using Theorem 2.2.11 which is proved in Subsection 4.2.2 and Theorem 1, we obtain the following corollary which is similar to Theorem 2.

Corollary 3.4.3. *Let A be a finite dimensional k-algebra. Then A is a Frobenius algebra if and only if there is a counital comultiplication $\alpha : A \to A \otimes_k A$ which is a map of A-bimodules satisfying*
(i) α is injective;
(ii) $(\alpha \otimes 1)\alpha = (1 \otimes \alpha)\alpha$ and
(iii) Im(α) = $\{\sum u_i \otimes v_i \mid \sum u_i x \otimes v_i = \sum u_i \otimes \nu_A^{-1}(x) v_i, \ \forall x \in A\}$, where ν_A is a Nakayama automorphism of A.
Moreover, if A is Frobenius, then dom.dim$A = \infty$.

We see that Theorem 2 and the above corollary is similar but also they have some differences. First difference is Nakayama automorphism. Another one is that the comultiplication α is counital. In addition, if A is Frobenius, it is self-injective and so dom.dim$A = \infty$. Therefore, the characterisation of dominant dimension of a Frobenius algebra in terms of exactness of the bar cocomplex is unneeded. However, by using Theorem 2 and Corollary 3.4.3, it would be good to have this kind of characterisation in terms of bar cocomplex for gendo-Frobenius algebras which are defined in the next chapter. Having this kind of characterisation for gendo-Frobenius algebras would be a big step to prove Nakayama conjecture. Hence, it is a good idea to work on this characterisation for further research.

Chapter 4

Gendo-Frobenius algebras

Two large classes of algebras, Frobenius algebras and gendo-symmetric algebras, are characterised by the existence of a comultiplication with some special properties. Moreover, both contain symmetric algebras. Here, it is natural to ask the following question.

Question. Is there a common generalisation of Frobenius algebras and gendo-symmetric algebras such that this generalisation has a comultiplication, which specialises to the known comultiplications on Frobenius algebras and on gendo-symmetric algebras?

Answering this question leads to introducing a new class of algebras which we called gendo-Frobenius algebras.

In [20], Kerner and Yamagata investigated two variations of gendo-symmetric algebras and in fact these two variations contain gendo-symmetric and Frobenius algebras. First variation is motivated by Morita [24] and they called a finite dimensional algebra A *Morita algebra*, if A is the endomorphism ring of a generator–cogenerator over a self-injective algebra. Second one is defined by relaxing the condition on the bimodule isomorphism in Definition 3.1.1 (ii) and we call these algebras *gendo-Frobenius algebras*. The class of gendo-Frobenius algebras is the common generalisation that we asked in the above question.

This chapter is devoted to introducing gendo-Frobenius algebras and constructing their comultiplication. We first give the preliminary results, define the gendo-Frobenius algebras and give some examples of these algebras. In the second section, we construct the comultiplication of gendo-Frobenius algebras and give the main results of this chapter. We also compare this comultiplication with the comultiplication of Frobenius algebras which is given by Abrams (Theorem 2.2.3) by assuming that the finite dimensional algebra is Frobenius. Moreover, we show that there are other approaches to new comultiplications and carry out these constructions as well. However, it turns out that they all lead to comultiplications which are lacking crucial properties such as being coassociative.

Finally, we give some results on comultiplication of Frobenius Nakayama algebras and their compatible counit.

Throughout, all algebras and modules are finite dimensional over an arbitrary field k unless stated otherwise. By D, we denote the usual k-duality functor $\mathrm{Hom}_k(-, k)$.

4.1 Definition and examples of gendo-Frobenius algebras

In [24], Morita studied endomorphism algebras of a generator-cogenerator over a self-injective algebra, and in fact he gave the following theorem.

Theorem 4.1.1. *([24]) Let A be a finite dimensional k-algebra. Then the following statements are equivalent:*

(i) A is isomorphic to the endomorphism algebra of a finite dimensional faithful module over a self-injective k-algebra B.

(ii) There is an idempotent e of A such that Ae and eA are injective faithful (left and right respectively) A-modules and the (A, eAe)-bimodule Ae has the double centralizer property.

(iii) There is an idempotent e of A such that Ae and eA are injective faithful (left and right respectively) A-modules and the (eAe, A)-bimodule eA has the double centralizer property.

Note that if B is a self-injective algebra, then a finite dimensional faithful B-module M is a generator–cogenerator, since there is a (split) monomorphism $B \hookrightarrow M^r$ for some $r \geq 1$. Motivated by Morita, in [20], Kerner and Yamagata gave the following definition.

Definition 4.1.2. A finite dimensional k-algebra A is called *Morita algebra* with associated idempotent e and associated self-injective algebra B if it satisfies the conditions in Theorem 4.1.1.

Morita algebras form a class of algebras properly containing all self-injective algebras and Auslander algebras of self-injective algebras of finite representation type. They are properly contained in the class of algebras with dominant dimension at least 2.

In [20], Kerner and Yamagata investigated two variations of gendo-symmetric algebras. First one is the algebras which is given in the above theorem, that is, Morita algebras and cleary it contains both gendo-symmetric and Frobenius algebras. Moreover, they gave some important results on Morita algebras (see [20]). Second one is defined by relaxing the condition on the bimodule isomorphism in Definition 3.1.1 (ii), and in fact we focus on this variation. Before explaining this, let us give some preliminary results.

Let A and B be finite dimensional k-algebras and ${}_AX_B$ be an (A, B)-bimodule. Then the right multiplication map

$$\begin{aligned} r_X : B &\to \mathrm{End}_A(X)^{op} \\ b &\mapsto r_b \end{aligned}$$

and the left multiplication map

$$\begin{aligned} l_X : A &\to \mathrm{End}_B(X) \\ a &\mapsto l_a \end{aligned}$$

are algebra homomorphisms. X is said to have *double centralizer property* if both r_X and l_X are bijective.

Lemma 4.1.3. *([20], Lemma 1.1) Let ${}_AX_B$ and ${}_AY_B$ be finite dimensional (A, B)-bimodules.*

(i) If ${}_\alpha X \cong Y$ as left A-modules for some $\alpha \in Aut(A)$, and r_X as well as r_Y are isomorphisms, then there is $\beta \in Aut(B)$ such that ${}_\alpha X \cong Y_\beta$ as (A, B)-bimodules.

(ii) If $X \cong Y_\beta$ as right B-modules for some $\beta \in Aut(B)$, and l_X as well as l_Y are isomorphisms, then there is $\alpha \in Aut(A)$ such that ${}_\alpha X \cong Y_\beta$ as (A, B)-bimodules.

Remark 4.1.4. The following lemma is obtained from the Lemma 2.4 in [20]. In Lemma 2.4, we see that for a finite dimensional k-algebra A such that $\mathrm{D}(Ae) \cong eA$ as right A-modules for an idempotent e of A, there is an (eAe, A)-bimodule isomorphism ${}_{\nu_{eAe}}eA \cong \mathrm{D}(Ae)$, where ν_{eAe} is a Nakayama automorphism of eAe. But in the following lemma, under the same assumption, there is an (eAe, A)-bimodule isomorphism

${}_{\nu_{eAe}^{-1}} eA \cong \mathrm{D}(Ae)$. This difference comes from the definition of the Nakayama automorphism. We use Definition 2.1.11, but Kerner and Yamagata use the following definition.

For a Frobenius algebra A and a nondegenerate associative k-bilinear form $(-,-) : A \times A \to k$, k-algebra automorphism $\nu : A \to A$ with $(\nu(x), y) = (y, x)$ for all $x, y \in A$ is said to be the Nakayama automorphism of A associated to $(-,-)$. As a result of this, they use the fact that for a Frobenius algebra A, there is an (A, A)-bimodule isomorphism $\mathrm{D}(A)_{\nu_A} \cong A$.

But in this work, it is more convenient to use Definition 2.1.11 of Nakayama automorphism. Therefore, some of the results which are given in [20] will be rearranged by considering these differences.

Lemma 4.1.5. *Let A be a finite dimensional k-algebra and $D(Ae) \cong eA$ as right A-modules for an idempotent e of A. Then eAe is Frobenius and ${}_{\nu_{eAe}^{-1}} eA \cong D(Ae)$ as (eAe, A)-bimodules, where ν_{eAe} is a Nakayama automorphism of eAe.*

Proof. Observe that (eAe, A)-bimodules $\mathrm{D}(Ae)$ and eA are faithful eAe-modules and $l_{eA} : eAe \to \mathrm{End}_A(eA)$ and $l_{\mathrm{D}(Ae)} : eAe \to \mathrm{End}_A(\mathrm{D}(Ae))^{op}$ are isomorphisms. Let us apply Lemma 4.1.3 (ii) to the right A-module isomorphism $eA \cong \mathrm{D}(Ae)$. Then we obtain an (eAe, A)-bimodule isomorphism ${}_{\alpha}eA \cong \mathrm{D}(Ae)$ where α is an automorphism of eAe. Multiplying e on the right implies an (eAe, eAe)-bimodule isomorphism ${}_{\alpha}eAe \cong \mathrm{D}(eAe)$. By taking the dual of this isomorphism, we obtain that $\mathrm{D}(eAe)_{\alpha} \cong eAe$ as (eAe, eAe)-bimodules. Therefore, eAe is a Frobenius algebra and α^{-1} is a Nakayama automorphism of eAe (see Chapter 2). Thus, we get ${}_{\nu_{eAe}^{-1}} eA \cong \mathrm{D}(Ae)$ as (eAe, A)-bimodules. □

Definition 4.1.6. Let A be a finite dimensional k-algebra. An idempotent e of A is called *self-dual* if $\mathrm{D}(eA) \cong Ae$ as left A-modules, and *faithful* if both Ae and eA are faithful A-modules.

Observe that self-duality of an idempotent is left–right symmetric. Obviously, an algebra A is a Frobenius algebra if and only if the identity 1_A of A is a self-dual idempotent.

Now, we are ready to explain the second variation which we mentioned before. Inspired by [14], in [20], Kerner and Yamagata considered the case, when the module $\mathrm{Hom}_A(D(A), A)$ is isomorphic to A, at least as a one-sided module and they obtained the following result.

Theorem 4.1.7. *([20], Theorem 3) For a finite dimensional k-algebra A, the following statements are equivalent:*

(i) $Hom_A(D(A), A) \cong A$ as left A-modules.

(ii) $Hom_A(D(A), A) \cong A$ as right A-modules.

(iii) A is a Morita algebra with an associated idempotent e such that eAe is a Frobenius algebra with Nakayama automorphism ν_{eAe} and $Ae \cong Ae_{\nu_{eAe}}$ as right eAe-modules.

(iv) A is a Morita algebra with an associated idempotent e such that eAe is a Frobenius algebra with Nakayama automorphism ν_{eAe} and $eA \cong {}_{\nu_{eAe}}eA$ as left eAe-modules.

(v) A is isomorphic to the endomorphism algebra of a finite dimensional faithful right module M over a Frobenius algebra B such that $M \cong M_{\nu_B}$ as right B-modules.

(vi) A is isomorphic to the opposite endomorphism algebra of a finite dimensional faithful left module N over a Frobenius algebra B such that $N \cong {}_{\nu_B}N$ as left B-modules.

Remark 4.1.8. The idempotent e of A in Theorem 4.1.7 is self-dual and faithful by the proof of Theorem 3 in [20].

Definition 4.1.9. A finite dimensional k-algebra A is called **gendo-Frobenius** if it satisfies one of the equivalent conditions in Theorem 4.1.7.

From conditions (iii) and (iv) in Theorem 4.1.7, we see that Frobenius algebras are gendo-Frobenius by choosing $e = 1_A$.

Theorem 4.1.10. *([20], Theorem 4.2) Let A be a Morita algebra. If B is Morita equivalent to A, then B is a Morita algebra.*

But the class of gendo-Frobenius algebras is not closed under Morita equivalences since the property $\mathrm{Hom}_A(\mathrm{D}(A), A) \cong A$ as left (or right) A-modules is not Morita invariant.

We may visualize the hierarchy of the finite dimensional algebras which are given in this study as follows.

$$\begin{tikzcd}
 & \text{Morita algebras} \arrow[d] & \\
 & \text{Gendo-Frobenius algebras} \arrow[dl] \arrow[dr] & \\
\text{Frobenius algebras} \arrow[dr] & & \text{Gendo-symmetric algebras} \arrow[dl] \\
 & \text{Symmetric algebras} &
\end{tikzcd}$$

In the above diagram, an arrow means the class on top contains the class below.

The following proposition is the rearranged version of Proposition 3.5 in [20] by taking into account Remark 4.1.4.

Proposition 4.1.11. *Let $Hom_A(D(A), A) \cong A$ as left A-modules for an algebra A and $D(Ae) \cong eA$ as right A-modules, where Ae and eA are faithful. Then there is an automorphism $\sigma \in Aut(A)$ such that*

(i) $Hom_A(D(A), A)_\sigma \cong A$ as (A, A)-bimodules and σ is uniquely determined up to an inner automorphism.

(ii) $eA \cong {}_{\nu_{eAe}} eA_\sigma$ as (eAe, A)-bimodules.

(iii) Moreover, in case e is basic, we can choose the σ such that $\sigma(e) = e$ and the restriction of σ to eAe is a Nakayama automorphism of eAe.

Proof. (i) The proof is similar to the proof of Proposition 3.5 (i) in [20]. But here we apply Lemma 4.1.3 to the isomorphism ${}_A A \cong {}_A\mathrm{Hom}_A(\mathrm{D}(A), A)$. So we obtain that there is an automorphism σ such that $A \cong \mathrm{Hom}_A(\mathrm{D}(A), A)_\sigma$ as (A, A)-bimodules.

(ii) By applying e on the left side of the (A, A)-bimodule isomorphism $A \cong \mathrm{Hom}_A(\mathrm{D}(A), A)_\sigma$, we obtain the following (eAe, A)-bimodule isomorphisms

$$\begin{aligned}
eA &\cong e\mathrm{Hom}_A(\mathrm{D}(A), A)_\sigma = \mathrm{Hom}_A(\mathrm{D}(A)e, A)_\sigma \\
&= \mathrm{Hom}_A(\mathrm{D}(eA), A)_\sigma \cong \mathrm{Hom}_A(Ae_{\nu_{eAe}}, A)_\sigma \\
&= {}_{\nu_{eAe}}\mathrm{Hom}_A(Ae, A)_\sigma \cong {}_{\nu_{eAe}} eA_\sigma
\end{aligned}$$

since $\mathrm{D}(eA) \cong Ae_{\nu_{eAe}}$ as (A, eAe)-bimodules.

(iii) We first replace σ in the proof of Proposition 3.5 (iii) in [20] with σ^{-1}. Then by using the same proof, we obtain that there is a $\theta \in \mathrm{Aut}(A)$ with $\theta(x) = cxc^{-1}$ for all $x \in A$, where c is an invertible element in A such that $(\theta\sigma^{-1})(e) = e$ and $\theta\sigma^{-1} \in \mathrm{Aut}(A)$. Observe that $\mathrm{Hom}_A(\mathrm{D}(A), A) \cong A_{\sigma^{-1}} \cong A_{\theta\sigma^{-1}}$ as (A, A)-bimodules, because $A \cong A_\theta$ as (A, A)-bimodules. By replacing σ^{-1} with $\theta\sigma^{-1}$, we obtain that $\sigma^{-1}(e) = e$, that is, $\sigma(e) = e$. Now, we multiply e on right side of the isomorphism $eA_\sigma \cong {}_{\nu_{eAe}^{-1}} eA$ given in (ii). Then we obtain (eAe, eAe)-bimodule isomorphisms $eAe_{\sigma_e} \cong {}_{\nu_{eAe}^{-1}} eAe \cong eAe_{\nu_{eAe}}$, where σ_e

denotes the restriction of σ to eAe. By using Lemma II.7.15 and Corollary IV.3.5 in [31], we obtain that $\sigma_e = \theta_e \nu_{eAe}$ for some inner automorphism θ_e of the algebra eAe, which shows that σ_e is a Nakayama automorphism of eAe. $\square$

4.1.1 Examples

In this part, some examples of gendo-Frobenius algebras are exhibited.

Example 4.1.12. Let k be a field and Q be a quiver given as follows:

$$\begin{tikzcd} & 1 \arrow[dr, "\alpha_1"] & \\ 3 \arrow[ur, "\alpha_3"] & & 2 \arrow[ll, "\alpha_2"] \end{tikzcd}$$

Let I be the ideal in the path algebra kQ generated by $\alpha_1\alpha_2\alpha_3$, $\alpha_2\alpha_3\alpha_1$ and $\alpha_3\alpha_1\alpha_2$; and $A = kQ/I$ be the associated bound quiver algebra. By Theorem 2.1.24, A is Frobenius and so gendo-Frobenius.

Example 4.1.13. Let B be the path algebra of the following quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "\beta_1"] & 2 \arrow[l, shift left, "\beta_2"] \end{tikzcd}$$

such that $\beta_1\beta_2 = 0 = \beta_2\beta_1$. Then B is a nonsymmetric Frobenius algebra (see Subsection 2.1.3) and it has a Nakayama automorphism ν_B such that $\nu_B(e_1) = e_2$, $\nu_B(e_2) = e_1$, $\nu_B(\beta_1) = \beta_2$ and $\nu_B(\beta_2) = \beta_1$. Let $M = B \oplus S_1 \oplus S_2$, where S_1 and S_2 are simple modules corresponding to e_1 and e_2, respectively; and $A = \mathrm{End}_B(M)$. Then M is a faithful right B-module, and A is a Morita algebra with associated Frobenius algebra B. Moreover, $M_{\nu_B} \cong M$ as right B-modules. Hence, by Theorem 4.1.7, we obtain that A is a gendo-Frobenius algebra and also $\mathrm{Hom}_A(\mathrm{D}(A), A) \cong A$ as one-sided A-modules. Note that A is isomorphic to the path algebra of the following quiver

$$\begin{tikzcd} & 1 \arrow[dl, "\alpha_1"'] & \\ 3 \arrow[dr, "\alpha_3"'] & & 4 \arrow[ul, "\alpha_4"'] \\ & 2 \arrow[ur, "\alpha_2"'] & \end{tikzcd}$$

such that $\alpha_3\alpha_2 = 0 = \alpha_4\alpha_1$.

Remark 4.1.14. Even if A is a basic Morita algebra, it does not need to be a gendo-Frobenius algebra. Let us consider the algebra B in Example 4.1.13. Let $M = B \oplus S_1$. Then M_B is faithful and $A = \mathrm{End}_B(M)$ is a Morita algebra with associated Frobenius algebra B. However, $M_{\nu_B} \ncong M$ as right B-modules. Hence, by Theorem 4.1.7, A is not gendo-Frobenius, and also $\mathrm{Hom}_A(\mathrm{D}(A), A) \ncong A$ as one-sided A-modules.

Since the class of gendo-Frobenius algebras contains the classes of gendo-symmetric and Frobenius algebras, all examples given in Chapter 2 and Chapter 3 are also examples for gendo-Frobenius algebras.

4.2 Gendo-Frobenius algebras and comultiplication

In this section, inspired by [13], we construct a coassociative comultiplication (possibly without a counit) for gendo-Frobenius algebras and give its properties. For this aim, we first give some preliminary results.

For simplicity, we keep the notations, which we will introduce throughout this part, until the next subsection.

Let A be a gendo-Frobenius algebra with a faithful and self-dual idempotent e. By following Lemma 4.1.5, fix an (eAe, A)-bimodule isomorphism $\iota : {}_{\nu_{eAe}^{-1}} eA \cong \mathrm{D}(Ae)$, where ν_{eAe} is a Nakayama automorphism of the Frobenius algebra eAe. Since A is gendo-Frobenius, there is a left A-module isomorphism $\mathrm{Hom}_A(\mathrm{D}(A), A) \cong A$. Then by Proposition 4.1.11 (ii), fix an (eAe, A)-bimodule isomorphism $\eta : eA_\sigma \cong {}_{\nu_{eAe}^{-1}} eA$, where $\sigma \in \mathrm{Aut}(A)$ and it is uniquely determined up to an inner automorphism.

Lemma 4.2.1. *Let A be a gendo-Frobenius algebra with a faithful and self-dual idempotent e. Then $Ae \otimes_{eAe} eA_\sigma \cong D(A)$ as A-bimodules.*

Proof. By using the double centralizer property of Ae and the isomorphisms ι and η, we obtain the following A-bimodule isomorphism

$$\begin{aligned} A \cong \mathrm{Hom}_{eAe}(Ae, Ae) &\cong \mathrm{Hom}_{eAe}(\mathrm{D}(Ae), \mathrm{D}(Ae)) \\ &\cong \mathrm{Hom}_{eAe}({}_{\nu_{eAe}^{-1}} eA, \mathrm{D}(Ae)) \\ &\cong \mathrm{Hom}_k(Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA, k) \\ &\cong \mathrm{Hom}_k(Ae \otimes_{eAe} eA_\sigma, k). \end{aligned}$$

Let us fix an (eAe, A)-bimodule isomorphism $\tau : eA_\sigma \cong \mathrm{D}(Ae)$ by using ι and η. Then by dualising $\mathrm{Hom}_k(Ae \otimes_{eAe} eA_\sigma, k) \cong A$, we obtain that there is an A-bimodule isomorphism $\gamma : Ae \otimes_{eAe} eA_\sigma \cong D(A)$ such that

$$\gamma(ae \otimes_{eAe} eb)(x) = \tau(eb\sigma(x))(ae) \tag{4.1}$$

for all $a, b, x \in A$. □

Proposition 4.2.2. *Let A be a finite dimensional k-algebra. Then A is gendo-Frobenius if and only if there exists an automorphism $\omega \in Aut(A)$ such that $D(A)_{\omega^{-1}} \otimes_A D(A) \cong D(A)$ as A-bimodules.*

Proof. Let A be gendo-Frobenius. By using the isomorphism γ, observe that there is an A-bimodule isomorphism $\gamma' : Ae \otimes_{eAe} eA \cong D(A)_{\sigma^{-1}}$. Hence, there is an A-bimodule isomorphism

$$\begin{aligned} \epsilon : \mathrm{D}(A)_{\sigma^{-1}} \otimes_A \mathrm{D}(A) &\overset{(1)}{\cong} (Ae \otimes_{eAe} eA) \otimes_A (Ae \otimes_{eAe} eA_\sigma) \\ &\cong Ae \otimes_{eAe} eAe \otimes_{eAe} eA_\sigma \\ &\cong Ae \otimes_{eAe} eA_\sigma \\ &\cong \mathrm{D}(A), \end{aligned} \tag{4.2}$$

where (1) is $\gamma'^{-1} \otimes_A \gamma^{-1}$, and it is explicitly defined by

$$\begin{aligned} \epsilon : \gamma'(ae \otimes_{eAe} eb) \otimes_A \gamma(ce \otimes_{eAe} ed) &\mapsto (ae \otimes_{eAe} eb) \otimes_A (ce \otimes_{eAe} ed) \\ &\mapsto ae \otimes_{eAe} ebce \otimes_{eAe} ed \\ &\mapsto aebce \otimes_{eAe} ed \\ &\mapsto \gamma(aebce \otimes_{eAe} ed), \end{aligned}$$

for any $a, b, c, d \in A$. Here, $\omega = \sigma$.

Now let $\mathrm{D}(A)_{\omega^{-1}} \otimes_A \mathrm{D}(A) \cong \mathrm{D}(A)$ as A-bimodules. Taking the dual of this isomorphism gives the A-bimodule isomorphism $\mathrm{Hom}_A({}_\omega\mathrm{D}(A), A) \cong A$. Then we obtain the following isomorphisms of A-bimodules

$$A \cong \mathrm{Hom}_A({}_\omega\mathrm{D}(A), A) \cong \mathrm{Hom}_A(\mathrm{D}(A), A)_\omega.$$

It means that there is a left A-module isomorphism $\mathrm{Hom}_A(\mathrm{D}(A), A) \cong A$ and by Definition 4.1.9, A is gendo-Frobenius. Here, for any $a \in A$, $\omega(a) = \sigma(uau^{-1})$, where u is an invertible element of A by Proposition 4.1.11. $\square$

Let m_1 be the composition of the canonical A-bimodule morphism

$$\phi : \mathrm{D}(A)_{\sigma^{-1}} \otimes_k \mathrm{D}(A) \to \mathrm{D}(A)_{\sigma^{-1}} \otimes_A \mathrm{D}(A)$$

with the isomorphism ϵ given in the proof of Proposition 4.2.2 such that

$$m_1 : \mathrm{D}(A)_{\sigma^{-1}} \otimes_k \mathrm{D}(A) \xrightarrow{\phi} \mathrm{D}(A)_{\sigma^{-1}} \otimes_A \mathrm{D}(A) \overset{\epsilon}{\cong} \mathrm{D}(A),$$

where

$$\begin{aligned} m_1 : \gamma'(ae \otimes_{eAe} eb) \otimes_k \gamma(ce \otimes_{eAe} ed) &\mapsto \gamma'(ae \otimes_{eAe} eb) \otimes_A \gamma(ce \otimes_{eAe} ed) \\ &\mapsto \gamma(aebce \otimes_{eAe} ed). \end{aligned}$$

Let $m_2 : \mathrm{D}(A) \otimes_k \mathrm{D}(A) \to \mathrm{D}(A)_{\sigma^{-1}} \otimes_k \mathrm{D}(A)$ be the map which is defined by

$$m_2(\gamma(ae \otimes_k eb) \otimes_k \gamma(ce \otimes ed)) = \gamma'(ae \otimes eb) \otimes_k \gamma(ce \otimes ed),$$

where $\gamma(ae \otimes eb), \gamma(ce \otimes ed) \in \mathrm{D}(A)$ and $\gamma'(ae \otimes eb) \in \mathrm{D}(A)_{\sigma^{-1}}$.

Claim. The map m_2 is an A-bimodule morphism.

Proof of Claim. It is enough to check that

$$m_2(x\gamma(ae \otimes eb) \otimes_k \gamma(ce \otimes ed)) = xm_2(\gamma(ae \otimes eb) \otimes_k \gamma(ce \otimes ed))$$

and

$$m_2(\gamma(ae \otimes eb) \otimes_k \gamma(ce \otimes ed)y) = m_2(\gamma(ae \otimes eb) \otimes_k \gamma(ce \otimes ed))y$$

for any $x, y \in A$. We observe that

$$\begin{aligned} m_2(x\gamma(ae \otimes_{eAe} eb) \otimes_k \gamma(ce \otimes_{eAe} ed)) &= m_2(\gamma(xae \otimes_{eAe} eb) \otimes_k \gamma(ce \otimes_{eAe} ed)) \\ &= \gamma'(xae \otimes_{eAe} eb) \otimes_k \gamma(ce \otimes_{eAe} ed) \\ xm_2(\gamma(ae \otimes_{eAe} eb) \otimes_k \gamma(ce \otimes_{eAe} ed)) &= x\gamma'(ae \otimes_{eAe} eb) \otimes_k \gamma(ce \otimes_{eAe} ed) \\ &= \gamma'(xae \otimes_{eAe} eb) \otimes_k \gamma(ce \otimes_{eAe} ed). \end{aligned}$$

Therefore, $m_2(x\gamma(ae \otimes eb) \otimes_k \gamma(ce \otimes ed)) = xm_2(\gamma(ae \otimes eb) \otimes_k \gamma(ce \otimes ed))$. Also,

$$\begin{aligned} m_2(\gamma(ae \otimes_{eAe} eb) \otimes_k \gamma(ce \otimes_{eAe} ed)y) &= m_2(\gamma(ae \otimes_{eAe} eb) \otimes_k \gamma(ce \otimes_{eAe} ed\sigma(y))) \\ &= \gamma'(ae \otimes_{eAe} eb) \otimes_k \gamma(ce \otimes_{eAe} ed\sigma(y)) \end{aligned}$$

$$m_2(\gamma(ae \otimes_{eAe} eb) \otimes_k \gamma(ce \otimes_{eAe} ed))y = \gamma'(ae \otimes_{eAe} eb) \otimes_k \gamma(ce \otimes_{eAe} ed)y$$
$$= \gamma'(ae \otimes_{eAe} eb) \otimes_k \gamma(ce \otimes_{eAe} ed\sigma(y)).$$

Hence, $m_2(\gamma(ae \otimes eb) \otimes_k \gamma(ce \otimes ed)y) = m_2(\gamma(ae \otimes eb) \otimes_k \gamma(ce \otimes ed))y$. This means that m_2 is an A-bimodule morphism. □

Let m be the following composition map

$$m : \mathrm{D}(A) \otimes_k \mathrm{D}(A) \xrightarrow{m_2} \mathrm{D}(A)_{\sigma^{-1}} \otimes_k \mathrm{D}(A) \xrightarrow{m_1} \mathrm{D}(A),$$

where

$$m : \gamma(ae \otimes_{eAe} eb) \otimes_k \gamma(ce \otimes_{eAe} ed) \mapsto \gamma'(ae \otimes_{eAe} eb) \otimes_k \gamma(ce \otimes_{eAe} ed) \qquad (4.3)$$
$$\mapsto \gamma(aebce \otimes_{eAe} ed).$$

Dualising m yields an A-bimodule morphism

$$\Delta : A \to {}_A A \otimes_k A_A$$

such that

$$(f \otimes g)\Delta(x) = m(g \otimes f)(x) \qquad (4.4)$$

for any f, g in $\mathrm{D}(A)$ and x in A.

Theorem 4.2.3. *Let A be a gendo-Frobenius algebra. Then*

$$\Delta : A \to {}_A A \otimes_k A_A$$

is a coassociative comultiplication which is a map of A-bimodules.

The proof of Theorem 4.2.3 consists of the following two lemmas.

Lemma 4.2.4. *The map m satisfies*

$$m(1 \otimes m) = m(m \otimes 1)$$

as k-morphisms from $D(A) \otimes_k D(A) \otimes_k D(A)$ to $D(A)$.

Proof. For any $a, b, c, d, x, y \in A$, the definition of m imply that

$$m(\gamma(ae \otimes eb) \otimes_k \gamma(ce \otimes ed)) = m_1 m_2(\gamma(ae \otimes eb) \otimes_k \gamma(ce \otimes ed))$$
$$= m_1(\gamma'(ae \otimes eb) \otimes_k \gamma(ce \otimes ed))$$
$$= \gamma(aebce \otimes ed).$$

Then

$$m(1 \otimes m)(\gamma(ae \otimes eb) \otimes_k \gamma(ce \otimes ed) \otimes_k \gamma(xe \otimes ey)) = m(\gamma(ae \otimes eb) \otimes_k \gamma(cedxe \otimes ey))$$
$$= \gamma(aebcedxe \otimes ey)$$
$$m(m \otimes 1)(\gamma(ae \otimes eb) \otimes_k \gamma(ce \otimes ed) \otimes_k \gamma(xe \otimes ey)) = m(\gamma(aebce \otimes ed) \otimes_k \gamma(xe \otimes ey))$$

$$= \gamma(aebcedxe \otimes ey)$$

This means that $m(1 \otimes m) = m(m \otimes 1)$. $\square$

Lemma 4.2.5. *Let* $\Delta : A \to {}_AA \otimes_k A_A$ *be as above. Then*

(i) Δ *is an* A*-bimodule morphism.*

(ii) $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$.

Proof. (i) By definition of Δ, there are the following equalities for $a, b, c, d, x, y \in A$.

$$\begin{aligned}(\gamma(ae \otimes eb) \otimes \gamma(ce \otimes ed))\Delta(xy) &= m(\gamma(ce \otimes ed) \otimes \gamma(ae \otimes eb))(xy)\\ &= \gamma(cedae \otimes eb)(x1y) = \gamma(ycedae \otimes eb\sigma(x))(1)\\ (\gamma(ae \otimes eb) \otimes \gamma(ce \otimes ed))x\Delta(y) &= \gamma(ae \otimes eb\sigma(x)) \otimes \gamma(ce \otimes ed)\Delta(y)\\ &= \gamma(cedae \otimes eb\sigma(x))(y) = \gamma(ycedae \otimes eb\sigma(x))(1)\\ (\gamma(ae \otimes eb) \otimes \gamma(ce \otimes ed))\Delta(x)y &= (\gamma(ae \otimes eb) \otimes \gamma(yce \otimes ed))\Delta(x)\\ &= \gamma(ycedae \otimes eb)(x) = \gamma(ycedae \otimes eb\sigma(x))(1)\end{aligned}$$

Therefore, $\Delta(xy) = x\Delta(y) = \Delta(x)y$, that is, Δ is an A-bimodule morphism.

(ii) Let $\Delta(u) = \sum u_i \otimes v_i$ for $u \in A$. Then

$$\begin{aligned}(\gamma(ae \otimes eb) \otimes \gamma(ce \otimes ed) \otimes \gamma(xe \otimes ey))(1 \otimes \Delta)\Delta(u) &= \sum \gamma(ae \otimes eb)(u_i)(\gamma(ce \otimes ed) \otimes \gamma(xe \otimes ey))\Delta(v_i)\\ &= \sum \gamma(ae \otimes eb)(u_i)\gamma(xeyce \otimes ed)(v_i)\\ &= \gamma(ae \otimes eb) \otimes \gamma(xeyce \otimes ed)\Delta(u)\\ &= \gamma(xeycedae \otimes eb)(u)\\ (\gamma(ae \otimes eb) \otimes \gamma(ce \otimes ed) \otimes \gamma(xe \otimes ey))(\Delta \otimes 1)\Delta(u) &= \sum \gamma(ae \otimes eb) \otimes \gamma(ce \otimes ed)\Delta(u_i)\gamma(xe \otimes ey)(v_i)\\ &= \sum \gamma(cedae \otimes eb)(u_i)\gamma(xe \otimes ey)(v_i)\\ &= \gamma(cedae \otimes eb) \otimes \gamma(xe \otimes ey)\Delta(u)\\ &= \gamma(xeycedae \otimes eb)(u)\end{aligned}$$

This means that $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$. $\square$

Proposition 4.2.6. *Let* A *be a gendo-Frobenius algebra and* $\Delta : A \to {}_AA \otimes_k A_A$ *be as above. Then*

$$Im(\Delta) = \{\sum u_i \otimes v_i \mid \sum u_i x \otimes v_i = \sum u_i \otimes \sigma^{-1}(x)v_i, \ \ \forall x \in A\}.$$

Proof. Let $\Sigma = \{\sum u_i \otimes v_i \mid \sum u_i x \otimes v_i = \sum u_i \otimes \sigma^{-1}(x)v_i, \ \ \forall x \in A\}$. Let $\Delta(u) = \sum u_i \otimes v_i$, for any $u \in A$. Then for any $f, g \in \mathrm{D}(A)$ and $x \in A$,

$$(f \otimes g)(\sum u_i x \otimes v_i) = (xf \otimes g)\Delta(u) = m(g \otimes xf)(u)$$

$$(f \otimes g)(\sum u_i \otimes \sigma^{-1}(x)v_i) = (f \otimes g\sigma^{-1}(x))\Delta(u) = m(g\sigma^{-1}(x) \otimes f)(u).$$

By definition of m, there is an equality $m(g \otimes_k xf) = m(g\sigma^{-1}(x) \otimes_k f)$. Because, let $f = \gamma(ae \otimes eb)$ and $g = \gamma(ce \otimes ed)$, then

$$m(g \otimes_k xf) = m_1 m_2(\gamma(ce \otimes ed) \otimes_k x\gamma(ae \otimes eb)) = m_1 m_2(\gamma(ce \otimes ed) \otimes_k \gamma(xae \otimes eb))$$

$$
\begin{aligned}
&= m_1(\gamma'(ce\otimes ed)\otimes_k \gamma(xae\otimes eb)) = \gamma(cedxae\otimes eb)\\
m(g\sigma^{-1}(x)\otimes_k f) &= m_1m_2(\gamma(ce\otimes ed)\sigma^{-1}(x)\otimes_k \gamma(ae\otimes eb)) = m_1m_2(\gamma(ce\otimes edx)\otimes_k \gamma(ae\otimes eb))\\
&= m_1(\gamma'(ce\otimes edx)\otimes_k \gamma(ae\otimes eb)) = \gamma(cedxae\otimes eb).
\end{aligned}
$$

Thus $\Delta(u)\in\Sigma$ and so $\mathrm{Im}(\Delta)\subseteq\Sigma$.

Conversely, for each $\theta=\sum u_i\otimes v_i\in\Sigma$, there is a k-linear map $\mathrm{D}(A)\to A$, denoted by $\overline{\theta}$, such that $\overline{\theta}(f)=\sum f(u_i)v_i$ for any $f\in\mathrm{D}(A)$. Since for any $x\in A$, $\sum u_ix\otimes v_i=\sum u_i\otimes\sigma^{-1}(x)v_i$, it follows

$$
\overline{\theta}(xf)=\sum(xf)(u_i)v_i=\sum f(u_ix)v_i=\sum f(u_i)\sigma^{-1}(x)v_i=\sigma^{-1}(x)\overline{\theta}(f).
$$

Then $\overline{\theta}$ is a left A-module morphism, that is, $\overline{\theta}\in\mathrm{Hom}_A({}_\sigma\mathrm{D}(A),A)\cong\mathrm{Hom}_A(\mathrm{D}(A),{}_{\sigma^{-1}}A)$. Since $\mathrm{D}(A)_{\sigma^{-1}}\otimes_A\mathrm{D}(A)\cong\mathrm{D}(A)$ as A-bimodules, by taking the dual of this isomorphism, we obtain that $\mathrm{Hom}_A(\mathrm{D}(A),{}_{\sigma^{-1}}A)\cong A$ as A-bimodules. Therefore, $\mathrm{Hom}_A({}_\sigma\mathrm{D}(A),A)\cong A$ as A-bimodules. Now, observe that the map $\xi:\Sigma\to\mathrm{Hom}_A({}_\sigma\mathrm{D}(A),A)$ which sends θ to $\overline{\theta}$ is injective. To show that it is enough to prove $\mathrm{Ker}\xi=\{0\}$. In fact, $\xi(\theta)=\xi(\sum u_i\otimes v_i)=\overline{\theta}=0$ means that $\overline{\theta}(f)=\sum f(u_i)v_i=0$ for any $f\in\mathrm{D}(A)$. So we obtain that $u_i=0$ or $v_i=0$. Therefore, $\theta=0$. Also, since m is surjective, Δ is injective. Then by using $\mathrm{Im}\Delta\subseteq\Sigma$ and previous facts, we obtain the composition of following injective maps

$$
\mathrm{Im}(\Delta)\to\Sigma\to\mathrm{Hom}_A({}_\sigma\mathrm{D}(A),A)\cong A\to\mathrm{Im}(\Delta).
$$

Therefore, $\mathrm{Im}\Delta=\Sigma$. $\square$

Remark 4.2.7. Let A be a gendo-Frobenius k-algebra with a faithful and self-dual idempotent e. To obtain a comultiplication $\widetilde{\Delta}$ which is different from Δ by using the same construction given in this chapter, we first fix an (eAe,A)-bimodule isomorphism $\widetilde{\tau}:eA_\omega\cong\mathrm{D}(Ae)$ which is different from τ. Here, $\omega\in\mathrm{Aut}(A)$ and by Proposition 4.1.11, for any $a\in A$, $\omega(a)=\sigma(uau^{-1})$, where u is an invertible element of A. Then we have an A-bimodule isomorphism

$$
\widetilde{\gamma}:Ae\otimes_{eAe}eA_\omega\cong\mathrm{D}(A)
$$

such that $\widetilde{\gamma}(ae\otimes eb)(x)=\widetilde{\tau}(eb\omega(x))(ae)$ for any $a,b,x\in A$. By using the same construction, we obtain the following A-bimodule morphism

$$
\widetilde{m}:\mathrm{D}(A)\otimes_k\mathrm{D}(A)\to\mathrm{D}(A)_{\omega^{-1}}\otimes_k\mathrm{D}(A)\to\mathrm{D}(A)_{\omega^{-1}}\otimes_A\mathrm{D}(A)\cong\mathrm{D}(A).
$$

Dualising $\widetilde{m}$ gives an A-bimodule morphism

$$
\widetilde{\Delta}:A\to{}_AA\otimes_kA_A
$$

such that $(f\otimes g)\widetilde{\Delta}(x)=\widetilde{m}(g\otimes f)(x)$ for any $f,g\in\mathrm{D}(A)$ and $x\in A$.

Corollary 4.2.8. *Let A be a gendo-Frobenius k-algebra with a faithful and self-dual idempotent e. Suppose that $\widetilde{\Delta}$ is as given in Remark 4.2.7. Then Im$(\Delta)\cong$ Im$(\widetilde{\Delta})$ as A-bimodules.*

Proof. By the proof of Proposition 4.2.6, we obtain that

$$
\mathrm{Im}(\Delta)=\mathrm{Hom}_A({}_\sigma\mathrm{D}(A),A)\cong A
$$

as A-bimodules. Rearranging Proposition 4.2.6 by taking into account Remark 4.2.7 gives that

$$\mathrm{Im}(\widetilde{\Delta}) = \mathrm{Hom}_A({}_\omega\mathrm{D}(A), A) \cong A$$

as A-bimodules. □

Procedure for obtaining the comultiplication Δ of any gendo-Frobenius algebra A.

(1) Choose a faithful and self-dual idempotent e of A.

(2) Write the (eAe, A)-bimodule isomorphism $\tau : eA_\sigma \to \mathrm{D}(Ae)$ explicitly on a choice of basis elements.

(3) Write the A-bimodule isomorphism $\gamma : Ae \otimes_{eAe} eA_\sigma \cong \mathrm{D}(A)$ by using (4.1) to obtain the basis elements of $\mathrm{D}(A)$ in terms of the elements of $\mathrm{Im}\gamma$.

(4) Obtain the multiplication table of $\mathrm{D}(A)$ by using (4.3).

(5) Dualise m by using (4.4) and obtain Δ on the basis elements of A.

(6) By using the linearity of Δ, obtain Δ on any element $a \in A$.

Let us apply this procedure on the following examples.

Example 4.2.9. Let A be the gendo-Frobenius algebra in Example 4.1.12. A has a k-basis $\{e_1, e_2, e_3, \alpha_1, \alpha_2, \alpha_3, \alpha_1\alpha_2, \alpha_2\alpha_3, \alpha_3\alpha_1\}$ so $\mathrm{D}(A)$ has the dual basis $\{e_1^*, e_2^*, e_3^*, \alpha_1^*, \alpha_2^*, \alpha_3^*, (\alpha_1\alpha_2)^*, (\alpha_2\alpha_3)^*, (\alpha_3\alpha_1)^*\}$.

(1) We choose $e = 1_A$ since 1_A is a faithful and self-dual idempotent of A.

(2) The (A, A)-bimodule isomorphism τ is explicity defined on the basis elements by

$$\begin{aligned}
\tau : A_\sigma &\cong \mathrm{D}(A)\\
e_1 &\mapsto (\alpha_2\alpha_3)^*\\
e_2 &\mapsto (\alpha_3\alpha_1)^*\\
e_3 &\mapsto (\alpha_1\alpha_2)^*\\
\alpha_1 &\mapsto \alpha_3^*\\
\alpha_2 &\mapsto \alpha_1^*\\
\alpha_3 &\mapsto \alpha_2^*\\
\alpha_1\alpha_2 &\mapsto e_1^*\\
\alpha_2\alpha_3 &\mapsto e_2^*\\
\alpha_3\alpha_1 &\mapsto e_3^*.
\end{aligned}$$

(3) The A-bimodule isomorphism γ is explicitly defined by

$$\begin{aligned}
\gamma : A \otimes_A A_\sigma &\cong \mathrm{D}(A)\\
e_1 \otimes e_1 &\mapsto (\alpha_2\alpha_3)^*\\
e_1 \otimes \alpha_1 &\mapsto \alpha_3^*\\
e_1 \otimes \alpha_1\alpha_2 &\mapsto e_1^*\\
e_2 \otimes e_2 &\mapsto (\alpha_3\alpha_1)^*\\
e_2 \otimes \alpha_2 &\mapsto \alpha_1^*\\
e_2 \otimes \alpha_2\alpha_3 &\mapsto e_2^*\\
e_3 \otimes e_3 &\mapsto (\alpha_1\alpha_2)^*
\end{aligned}$$

$$e_3 \otimes \alpha_3 \mapsto \alpha_2^*$$
$$e_3 \otimes \alpha_3\alpha_1 \mapsto e_3^*.$$

(4) We obtain the multiplication table of the basis elements of $\mathrm{D}(A)$ as follows:

m	e_1^*	e_2^*	e_3^*	α_1^*	α_2^*	α_3^*	$(\alpha_1\alpha_2)^*$	$(\alpha_2\alpha_3)^*$	$(\alpha_3\alpha_1)^*$
e_1^*	0	0	0	0	0	0	e_1^*	0	0
e_2^*	0	0	0	0	0	0	0	e_2^*	0
e_3^*	0	0	0	0	0	0	0	0	e_3^*
α_1^*	0	0	0	0	e_2^*	0	α_1^*	0	0
α_2^*	0	0	0	0	0	e_3^*	0	α_2^*	0
α_3^*	0	0	0	e_1^*	0	0	0	0	α_3^*
$(\alpha_1\alpha_2)^*$	0	0	e_3^*	0	α_2^*	0	$(\alpha_1\alpha_2)^*$	0	0
$(\alpha_2\alpha_3)^*$	e_1^*	0	0	0	0	α_3^*	0	$(\alpha_2\alpha_3)^*$	0
$(\alpha_3\alpha_1)^*$	0	e_2^*	0	α_1^*	0	0	0	0	$(\alpha_3\alpha_1)^*$

(5) Dualising m yields
$$\Delta : A \to {}_AA \otimes_k A_A$$
such that $(f \otimes g)\Delta(a) = m(g \otimes f)(a)$ for any $f, g \in \mathrm{D}(A)$ and $a \in A$. So let

$$\begin{aligned}
f &= \lambda_1 e_1^* + \lambda_2 e_2^* + \lambda_3 e_3^* + \lambda_4 \alpha_1^* + \lambda_5 \alpha_2^* + \lambda_6 \alpha_3^* + \lambda_7 (\alpha_1\alpha_2)^* + \lambda_8 (\alpha_2\alpha_3)^* + \lambda_9 (\alpha_3\alpha_1)^* \\
g &= \mu_1 e_1^* + \mu_2 e_2^* + \mu_3 e_3^* + \mu_4 \alpha_1^* + \mu_5 \alpha_2^* + \mu_6 \alpha_3^* + \mu_7 (\alpha_1\alpha_2)^* + \mu_8 (\alpha_2\alpha_3)^* + \mu_9 (\alpha_3\alpha_1)^*,
\end{aligned}$$

where $\lambda_i, \mu_i \in k$ for $1 \le i \le 9$. By using the table in the previous step, we get

$$\begin{aligned}
m(g \otimes f) = {} & (\mu_1\lambda_7 + \mu_6\lambda_4 + \mu_8\lambda_1)e_1^* + (\mu_2\lambda_8 + \mu_4\lambda_5 + \mu_9\lambda_2)e_2^* + (\mu_3\lambda_9 + \mu_5\lambda_6 + \mu_7\lambda_3)e_3^* \\
& + (\mu_4\lambda_7 + \mu_9\lambda_4)\alpha_1^* + (\mu_5\lambda_8 + \mu_7\lambda_5)\alpha_2^* + (\mu_6\lambda_9 + \mu_8\lambda_6)\alpha_3^* \\
& + (\mu_7\lambda_7)(\alpha_1\alpha_2)^* + (\mu_8\lambda_8)(\alpha_2\alpha_3)^* + (\mu_9\lambda_9(\alpha_3\alpha_1)^*.
\end{aligned}$$

Then

$$\begin{aligned}
m(g \otimes f)(e_1) &= \mu_1\lambda_7 + \mu_6\lambda_4 + \mu_8\lambda_1 \\
m(g \otimes f)(e_2) &= \mu_2\lambda_8 + \mu_4\lambda_5 + \mu_9\lambda_2 \\
m(g \otimes f)(e_3) &= \mu_3\lambda_9 + \mu_5\lambda_6 + \mu_7\lambda_3 \\
m(g \otimes f)(\alpha_1) &= \mu_4\lambda_7 + \mu_9\lambda_4 \\
m(g \otimes f)(\alpha_2) &= \mu_5\lambda_8 + \mu_7\lambda_5 \\
m(g \otimes f)(\alpha_3) &= \mu_6\lambda_9 + \mu_8\lambda_6 \\
m(g \otimes f)(\alpha_1\alpha_2) &= \mu_7\lambda_7 \\
m(g \otimes f)(\alpha_2\alpha_3) &= \mu_8\lambda_8 \\
m(g \otimes f)(\alpha_3\alpha_1) &= \mu_9\lambda_9.
\end{aligned}$$

Since $(f \otimes g)\Delta(a) = m(g \otimes f)(a)$ for all $a \in A$, we obtain that

$$\begin{aligned}
\Delta(e_1) &= \alpha_1\alpha_2 \otimes e_1 + \alpha_1 \otimes \alpha_3 + e_1 \otimes \alpha_2\alpha_3 \\
\Delta(e_2) &= \alpha_2\alpha_3 \otimes e_2 + \alpha_2 \otimes \alpha_1 + e_2 \otimes \alpha_3\alpha_1 \\
\Delta(e_3) &= \alpha_3\alpha_1 \otimes e_3 + \alpha_3 \otimes \alpha_2 + e_3 \otimes \alpha_1\alpha_2 \\
\Delta(\alpha_1) &= \alpha_1\alpha_2 \otimes \alpha_1 + \alpha_1 \otimes \alpha_3\alpha_1 \\
\Delta(\alpha_2) &= \alpha_2\alpha_3 \otimes \alpha_2 + \alpha_2 \otimes \alpha_1\alpha_2 \\
\Delta(\alpha_3) &= \alpha_3\alpha_1 \otimes \alpha_3 + \alpha_3 \otimes \alpha_2\alpha_3 \\
\Delta(\alpha_1\alpha_2) &= \alpha_1\alpha_2 \otimes \alpha_1\alpha_2 \\
\Delta(\alpha_2\alpha_3) &= \alpha_2\alpha_3 \otimes \alpha_2\alpha_3 \\
\Delta(\alpha_3\alpha_1) &= \alpha_3\alpha_1 \otimes \alpha_3\alpha_1.
\end{aligned}$$

(6) Let $a \in A$. Then we can write $a = a_1e_1 + a_2e_2 + a_3e_3 + a_4\alpha_1 + a_5\alpha_2 + a_6\alpha_3 + a_7\alpha_1\alpha_2 + a_8\alpha_2\alpha_3 + a_9\alpha_3\alpha_1$, where $a_i \in k$ for $1 \leq i \leq 9$. The linearity of Δ gives that

$$\Delta(a) = a_1\Delta(e_1) + a_2\Delta(e_2) + a_3\Delta(e_3) + a_4\Delta(\alpha_1) + a_5\Delta(\alpha_2) + a_6\Delta(\alpha_3) + a_7\Delta(\alpha_1\alpha_2) + a_8\Delta(\alpha_2\alpha_3) + a_9\Delta(\alpha_3\alpha_1).$$

Moreover, since A is Frobenius, by Proposition 4.2.19 which is proved later, the counit of (A, Δ) is $\delta = \tau(1_A)$ and we obtain that

$$\delta = (\alpha_1\alpha_2)^* + (\alpha_2\alpha_3)^* + (\alpha_3\alpha_1)^*.$$

Example 4.2.10. Let A be the gendo-Frobenius algebra in Example 4.1.13. A has a k-basis $\{e_1, e_2, e_3, e_4, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1\alpha_3, \alpha_2\alpha_4\}$ so $\mathrm{D}(A)$ has the dual basis $\{e_1^*, e_2^*, e_3^*, e_4^*, \alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*, (\alpha_1\alpha_3)^*, (\alpha_2\alpha_4)^*\}$.

(1) We choose $e = e_1 + e_2$ since $e_1 + e_2$ is a faithful and self-dual idempotent of A.

(2) The (eAe, A)-bimodule isomorphism τ is explicity defined on the basis elements by

$$\begin{aligned}
\tau : eA_\sigma &\cong \mathrm{D}(Ae) \\
e_1 &\mapsto (\alpha_2\alpha_4)^* \\
e_2 &\mapsto (\alpha_1\alpha_3)^* \\
\alpha_1 &\mapsto \alpha_4^* \\
\alpha_2 &\mapsto \alpha_3^* \\
\alpha_1\alpha_3 &\mapsto e_1^* \\
\alpha_2\alpha_4 &\mapsto e_2^*.
\end{aligned}$$

(3) The A-bimodule isomorphism γ is explicitly defined by

$$\begin{aligned}
\gamma : Ae \otimes_{eAe} eA_\sigma &\cong \mathrm{D}(A) \\
e_2 \otimes e_2 &\mapsto (\alpha_1\alpha_3)^* \\
e_2 \otimes \alpha_2 &\mapsto \alpha_3^* \\
e_2 \otimes \alpha_2\alpha_4 &\mapsto e_2^* \\
e_1 \otimes e_1 &\mapsto (\alpha_2\alpha_4)^* \\
e_1 \otimes \alpha_1 &\mapsto \alpha_4^* \\
e_1 \otimes \alpha_1\alpha_3 &\mapsto e_1^*
\end{aligned}$$

$$
\begin{aligned}
\alpha_3 \otimes e_2 &\mapsto \alpha_1^* \\
\alpha_3 \otimes \alpha_2 &\mapsto e_3^* \\
\alpha_4 \otimes e_1 &\mapsto \alpha_2^* \\
\alpha_4 \otimes \alpha_1 &\mapsto e_4^*.
\end{aligned}
$$

(4) We obtain the multiplication table of the basis elements of $\mathrm{D}(A)$ as follows:

m	e_1^*	e_2^*	e_3^*	e_4^*	α_1^*	α_2^*	α_3^*	α_4^*	$(\alpha_1\alpha_3)^*$	$(\alpha_2\alpha_4^*)$
e_1^*	0	0	0	0	0	0	0	0	e_1^*	0
e_2^*	0	0	0	0	0	0	0	0	0	e_2^*
e_3^*	0	0	0	0	0	0	0	0	0	0
e_4^*	0	0	0	0	0	0	0	0	0	0
α_1^*	0	0	0	0	0	0	e_3^*	0	α_1^*	0
α_2^*	0	0	0	0	0	0	0	e_4^*	0	α_2^*
α_3^*	0	0	0	0	0	e_2^*	0	0	0	0
α_4^*	0	0	0	0	e_1^*	0	0	0	0	0
$(\alpha_1\alpha_3)^*$	0	e_2^*	0	0	0	0	α_3^*	0	$(\alpha_1\alpha_3)^*$	0
$(\alpha_2\alpha_4)^*$	e_1^*	0	0	0	0	0	0	α_4^*	0	$(\alpha_2\alpha_4)^*$

(5) Dualising m yields

$$\Delta : A \to {}_AA \otimes_k A_A$$

such that $(f \otimes g)\Delta(a) = m(g \otimes f)(a)$ for any $f, g \in \mathrm{D}(A)$ and $a \in A$. So let

$$
\begin{aligned}
f &= \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + \lambda_5 \alpha_1 + \lambda_6 \alpha_2 + \lambda_7 \alpha_3 + \lambda_8 \alpha_4 + \lambda_9 \alpha_1\alpha_3 + \lambda_{10}\alpha_2\alpha_4 \\
g &= \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 + \mu_4 e_4 + \mu_5 \alpha_1 + \mu_6 \alpha_2 + \mu_7 \alpha_3 + \mu_8 \alpha_4 + \mu_9 \alpha_1\alpha_3 + \mu_{10}\alpha_2\alpha_4,
\end{aligned}
$$

where $\lambda_i, \mu_i \in k$ for $1 \leq i \leq 10$. By using the table in the previous step, we get

$$
\begin{aligned}
m(g \otimes f) = {} & (\mu_1\lambda_9 + \mu_8\lambda_5 + \mu_{10}\lambda_1)e_1^* + (\mu_2\lambda_{10} + \mu_7\lambda_6 + \mu_9\lambda_2)e_2^* + (\mu_5\lambda_6)e_3^* + (\mu_6\lambda_7)e_4^* \\
& + (\mu_5\lambda_9)\alpha_1^* + (\mu_6\lambda_{10})\alpha_2^* + (\mu_9\lambda_7)\alpha_3^* + (\mu_{10}\lambda_8)\alpha_4^* \\
& + (\mu_9\lambda_9)(\alpha_1\alpha_3)^* + (\mu_{10}\lambda_{10})(\alpha_2\alpha_4)^*.
\end{aligned}
$$

Then

$$
\begin{aligned}
m(g \otimes f)(e_1) &= \mu_1\lambda_9 + \mu_8\lambda_5 + \mu_{10}\lambda_1 \\
m(g \otimes f)(e_2) &= \mu_2\lambda_{10} + \mu_7\lambda_6 + \mu_9\lambda_2 \\
m(g \otimes f)(e_3) &= \mu_5\lambda_6 \\
m(g \otimes f)(e_4) &= \mu_6\lambda_7 \\
m(g \otimes f)(\alpha_1) &= \mu_5\lambda_9 \\
m(g \otimes f)(\alpha_2) &= \mu_6\lambda_{10} \\
m(g \otimes f)(\alpha_3) &= \mu_9\lambda_7 \\
m(g \otimes f)(\alpha_4) &= \mu_{10}\lambda_8 \\
m(g \otimes f)(\alpha_1\alpha_3) &= \mu_9\lambda_9
\end{aligned}
$$

$$m(g \otimes f)(\alpha_2\alpha_4) = \mu_{10}\lambda_{10}.$$

Since $(f \otimes g)\Delta(a) = m(g \otimes f)(a)$ for all $a \in A$, we obtain that

$$\begin{aligned}
\Delta(e_1) &= \alpha_1\alpha_3 \otimes e_1 + \alpha_1 \otimes \alpha_4 + e_1 \otimes \alpha_2\alpha_4 \\
\Delta(e_2) &= \alpha_2\alpha_4 \otimes e_2 + \alpha_2 \otimes \alpha_3 + e_2 \otimes \alpha_1\alpha_3 \\
\Delta(e_3) &= \alpha_3 \otimes \alpha_1 \\
\Delta(e_4) &= \alpha_4 \otimes \alpha_2 \\
\Delta(\alpha_1) &= \alpha_1\alpha_3 \otimes \alpha_1 \\
\Delta(\alpha_2) &= \alpha_2\alpha_4 \otimes \alpha_2 \\
\Delta(\alpha_3) &= \alpha_3 \otimes \alpha_1\alpha_3 \\
\Delta(\alpha_4) &= \alpha_4 \otimes \alpha_2\alpha_4 \\
\Delta(\alpha_1\alpha_3) &= \alpha_1\alpha_3 \otimes \alpha_1\alpha_3 \\
\Delta(\alpha_2\alpha_4) &= \alpha_2\alpha_4 \otimes \alpha_2\alpha_4.
\end{aligned}$$

(6) Let $a \in A$. Then we can write $a = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5\alpha_1 + a_6\alpha_2 + a_7\alpha_3 + a_8\alpha_4 + a_9\alpha_1\alpha_3 + a_{10}\alpha_2\alpha_4$, where $a_i \in k$ for $1 \leq i \leq 10$. The linearity of Δ gives that

$$\begin{aligned}
\Delta(a) = {} & a_1\Delta(e_1) + a_2\Delta(e_2) + a_3\Delta(e_3) + a_4\Delta(e_4) \\
& + a_5\Delta(\alpha_1) + a_6\Delta(\alpha_2) + a_7\Delta(\alpha_3) + a_8\Delta(\alpha_4) \\
& + a_9\Delta(\alpha_1\alpha_3) + a_{10}\Delta(\alpha_2\alpha_4).
\end{aligned}$$

Observe that the algebra A in Example 4.2.10 is not Frobenius. Therefore, it is natural to ask whether the algebra A has a counit compatible with Δ or not. Indeed, (A, Δ) does not have a counit. After giving a preliminary result, we give a proposition which explains why (A, Δ) does not have a counit and describes a general situation.

Remark 4.2.11. Let us consider the following A-bimodule isomorphism

$$\begin{aligned}
\mathrm{Hom}_A(\mathrm{D}(A), A_\sigma) &\cong \mathrm{Hom}_A(\mathrm{D}(A)_{\sigma^{-1}}, A) \\
&\cong \mathrm{Hom}_A(Ae \otimes_{eAe} eA, A) \\
&\cong \mathrm{Hom}_{eAe}(eA, eA) \\
&\cong A
\end{aligned}$$

where the second isomorphism is $\mathrm{Hom}_A(\gamma', A)$. Let $\Theta : \mathrm{D}(A) \to A_\sigma$ be the inverse image of $1 \in A$ under the above isomorphism. Then $(\Theta \circ \gamma)(ae \otimes eb) = aeb$ for $a, b \in A$. Actually, Θ is an A-bimodule morphism with $e\Theta = \tau^{-1}$.

The following observation will be used to prove Proposition 4.2.19.

Instead of $\tau : eA_\sigma \cong \mathrm{D}(Ae)$, we can write $\tau' : eA \cong \mathrm{D}(Ae)_{\sigma^{-1}}$. Let us now consider the following A-bimodule isomorphism

$$\begin{aligned}
\mathrm{Hom}_A(\mathrm{D}(A)_{\sigma^{-1}}, A) &\cong \mathrm{Hom}_A(Ae \otimes_{eAe} eA, A) \\
&\cong \mathrm{Hom}_{eAe}(eA, eA) \\
&\cong A
\end{aligned}$$

where the first isomorphism is $\mathrm{Hom}_A(\gamma', A)$. Let $\Theta' : \mathrm{D}(A)_{\sigma^{-1}} \to A$ be the inverse image of $1 \in A$ under the above isomorphism. Then $(\Theta' \circ \gamma')(ae \otimes eb) = aeb$ for $a, b \in A$. Actually, Θ' is an A-bimodule morphism with $e\Theta' = \tau'^{-1}$.

Lemma 4.2.12. *Let A be a gendo-Frobenius k-algebra and $m : D(A) \otimes_k D(A) \to D(A)$ as before. Then*

$$\Theta(m(f \otimes g)) = \Theta(f)\Theta(g)$$

for any $f, g \in D(A)$.

Proof. Let $f = \gamma(ae \otimes eb)$ and $g = \gamma(ce \otimes ed)$. Then observe that

$$(\Theta \circ m)(\gamma(ae \otimes eb) \otimes \gamma(ce \otimes ed)) = \Theta(\gamma(aebce \otimes ed)) = aebced = (aeb)(ced)$$

$$\Theta(\gamma(ae \otimes eb))\Theta(\gamma(ce \otimes ed)) = (aeb)(ced).$$

□

Proposition 4.2.13. *Let A be a gendo-Frobenius k-algebra with the comultiplication $\Delta : A \to {}_AA \otimes_k A_A$. Then (A, Δ) has a counit if and only if A is Frobenius.*

Proof. Let $\delta \in \mathrm{D}(A)$ be a counit of (A, Δ). Then $m(\delta \otimes f)(a) = (f \otimes \delta)\Delta(a) = f(1 \otimes \delta)\Delta(a) = f(a)$, and similarly $m(f \otimes \delta)(a) = (\delta \otimes f)\Delta(a) = f(a)$ for any $a \in A$. Therefore, δ is a unit of $(\mathrm{D}(A), m)$. Now, let u be the image of δ under $\Theta : \mathrm{D}(A) \to A_\sigma$. Then $\Theta m(\delta \otimes \gamma(ae \otimes eb)) = \Theta(\gamma(ae \otimes eb))$. So, we obtain that $uaeb = aeb$ for any $a, b \in A$ by Lemma 4.2.12. Hence, we obtain that $u = 1$ since AeA is a faithful left A-module. As a result, Θ is surjective as an A-bimodule morphism and thus an isomorphism by comparing dimensions. So A is Frobenius. In fact, σ is a Nakayama automorphism of A.

Conversely, let A be Frobenius. Then by Theorem 2.2.3 and Proposition 4.2.19 which is given later, (A, Δ) has a counit. □

In particular, the case when A is Frobenius, which is considered in Theorem 2.2.3, is now obtained as a special case of Theorem 4.2.3 and Proposition 4.2.13.

Corollary 4.2.14. *Let A be a Frobenius k-algebra. Then it has a coassociative counital comultiplication $\Delta : A \to {}_AA \otimes_k A_A$ which is a map of A-bimodules.*

Moreover, the case A is gendo-symmetric, which is considered in Theorem 3.2.1, is obtained as a special case of Theorem 4.2.3.

Corollary 4.2.15. *Let A be a gendo-symmetric k-algebra. Then it has a coassociative comultiplication $\Delta : A \to {}_AA \otimes_k A_A$ which is a map of A-bimodules.*

Remark 4.2.16. If we assume that the finite dimensional algebra A is gendo-symmetric, we can choose σ as identity automorphism. Therefore, the comultiplication given in this section and the comultiplication given in Section 3.2 are equal.

4.2.1 Alternative approach to the proof of Lemma 4.2.4 and to writing the comultiplication

In this subsection, we give an alternative approach to the proof of Lemma 4.2.4 and also to writing the comultiplication $\Delta : A \to A \otimes_k A$ by using $Ae \otimes_{eAe} eA_\sigma$ instead of $\mathrm{D}(A)$ since $Ae \otimes_{eAe} eA_\sigma \cong \mathrm{D}(A)$ as A-bimodules (see Lemma 4.2.1).

Let A be a gendo-Frobenius algebra with a faithful and self-dual idempotent e. Let us construct a multiplication map

$$m' : (Ae \otimes_{eAe} eA_\sigma) \otimes_k (Ae \otimes_{eAe} eA_\sigma) \to Ae \otimes_{eAe} eA_\sigma$$

which is the composition of the following maps

$$\begin{aligned} m' : (Ae \otimes_{eAe} eA_\sigma) \otimes_k (Ae \otimes_{eAe} eA_\sigma) &\xrightarrow{m'_2} (Ae \otimes_{eAe} eA) \otimes_k (Ae \otimes_{eAe} eA_\sigma) \\ &\xrightarrow{\phi'} (Ae \otimes_{eAe} eA) \otimes_A (Ae \otimes_{eAe} eA_\sigma) \\ &\xrightarrow{\epsilon'} Ae \otimes_{eAe} eA_\sigma, \end{aligned}$$

where $m'_1 = \epsilon'\phi'$. This composition is explicitly defined by

$$\begin{aligned} m' : (ae \otimes eb) \otimes_k (ce \otimes ed) &\mapsto (ae \otimes eb) \otimes_k (ce \otimes ed) \\ &\mapsto (ae \otimes eb) \otimes_A (ce \otimes ed) \\ &\mapsto aebce \otimes ed \end{aligned}$$

for any $a, b, c, d \in A$. The map m'_2 can be defined by

$$m'_2 : (ae \otimes eb) \otimes_k (ce \otimes ed) \mapsto (ae \otimes eb) \otimes_k (ce \otimes ed)$$

since $Ae \otimes_{eAe} eA_\sigma$ and $Ae \otimes_{eAe} eA$ are same as k-vector spaces and the tensor product is over k.

Lemma 4.2.17. *Let m' be as above. Then m' is an A-bimodule morphism.*

Proof. It is enough to check that $m'(x(ae \otimes eb) \otimes_k (ce \otimes ed)) = xm'((ae \otimes eb) \otimes_k (ce \otimes ed))$ for any $x \in A$ and $m'((ae \otimes eb) \otimes_k (ce \otimes ed)y) = m'((ae \otimes eb) \otimes_k (ce \otimes ed))y$ for any $y \in A$. We observe that

$$\begin{aligned} m'(x(ae \otimes eb) \otimes_k (ce \otimes ed)) &= m'_1 m'_2(x(ae \otimes eb) \otimes_k (ce \otimes ed)) \\ &= m'_1 m'_2((xae \otimes eb) \otimes_k (ce \otimes ed)) \\ &= m'_1((xae \otimes eb) \otimes_k (ce \otimes ed)) \\ &= xaebce \otimes ed \\ xm'((ae \otimes eb) \otimes_k (ce \otimes ed)) &= xm'_1 m'_2((ae \otimes eb) \otimes_k (ce \otimes ed)) \\ &= xm'_1((ae \otimes eb) \otimes_k (ce \otimes ed)) \\ &= xaebce \otimes ed \end{aligned}$$

Then $m'(x(ae \otimes eb) \otimes_k (ce \otimes ed)) = xm'((ae \otimes eb) \otimes_k (ce \otimes ed))$ for any $x \in A$. Also,

$$\begin{aligned} m'((ae \otimes eb) \otimes_k (ce \otimes ed)y) &= m'_1 m'_2((ae \otimes eb) \otimes_k (ce \otimes ed\sigma(y))) \\ &= m'_1((ae \otimes eb) \otimes_k (ce \otimes ed\sigma(y))) \\ &= aebce \otimes ed\sigma(y) \\ m'((ae \otimes eb) \otimes_k (ce \otimes ed))y &= m'_1 m'_2((ae \otimes eb) \otimes_k (ce \otimes ed))y \\ &= m'_1((ae \otimes eb) \otimes_k (ce \otimes ed))y \\ &= aebce \otimes ed\sigma(y) \end{aligned}$$

Then $m'((ae \otimes eb) \otimes_k (ce \otimes ed)y) = m'((ae \otimes eb) \otimes_k (ce \otimes ed))y$ for any $y \in A$. Therefore, m' is an

A-bimodule morphism. $\square$

Proposition 4.2.18. *Let A be a gendo-Frobenius algebra, $m : D(A) \otimes_k D(A) \to D(A)$ be the multiplication map defined by (4.3) and m' be as above. Then m' coincides with m.*

Proof. Let $\gamma' : Ae \otimes_{eAe} eA \cong \mathrm{D}(A)_{\sigma^{-1}}$ be the A-bimodule isomorphism which is given in the proof of Proposition 4.2.2. Consider the following diagram

$$\begin{tikzcd}
(Ae \otimes_{eAe} eA_\sigma) \otimes_k (Ae \otimes_{eAe} eA_\sigma) \arrow[r, "\gamma\otimes_k\gamma"] \arrow[d, "m_2'"] & \mathrm{D}(A) \otimes_k \mathrm{D}(A) \arrow[d, "m_2"] \\
(Ae \otimes_{eAe} eA) \otimes_k (Ae \otimes eA_\sigma) \arrow[r, "\gamma'\otimes_k\gamma"] \arrow[d, "\phi'"] & \mathrm{D}(A)_{\sigma^{-1}} \otimes_k \mathrm{D}(A) \arrow[d, "\phi"] \\
(Ae \otimes_{eAe} eA) \otimes_A (Ae \otimes eA_\sigma) \arrow[r, "\gamma'\otimes_A\gamma"] \arrow[d, "\epsilon'"] & \mathrm{D}(A)_{\sigma^{-1}} \otimes_A \mathrm{D}(A) \arrow[d, "\epsilon"] \\
Ae \otimes_{eAe} eA_\sigma \arrow[r, "\gamma"] & \mathrm{D}(A)
\end{tikzcd}$$

This diagram is commutative since the maps $\gamma \otimes_k \gamma$, $\gamma' \otimes_k \gamma$, $\gamma' \otimes_A \gamma$ and γ are isomorphisms. Therefore, it gives that

$$m(\gamma \otimes_k \gamma)((ae \otimes eb) \otimes_k (ce \otimes ed)) = m(\gamma(ae \otimes eb) \otimes_k \gamma(ce \otimes ed)) = \gamma(aebce \otimes ed)$$

and

$$\gamma m'((ae \otimes eb) \otimes_k (ce \otimes ed)) = \gamma(aebce \otimes ed)$$

are equal. This means that m' coincides with m. $\square$

Alternative Proof to Lemma 4.2.4. For any $a, b, c, d, x, y \in A$,

$$\begin{aligned}
m'(1 \otimes m')((ae \otimes_{eAe} eb) \otimes_k (ce \otimes_{eAe} ed) \otimes_k (xe \otimes_{eAe} ey)) &= m'((ae \otimes_{eAe} eb) \otimes_k (cedxe \otimes_{eAe} ey)) \\
&= aebcedxe \otimes_{eAe} ey \\
m'(m' \otimes 1)((ae \otimes_{eAe} eb) \otimes_k (ce \otimes_{eAe} ed) \otimes_k (xe \otimes_{eAe} ey)) &= m'((aebce \otimes_{eAe} ed) \otimes_k (xe \otimes_{eAe} ey)) \\
&= aebcedxe \otimes_{eAe} ey.
\end{aligned}$$

Proposition 4.2.18 completes the proof. $\square$

4.2.2 Comparison with Abrams' comultiplication

Let A be a Frobenius algebra over a field k. In this subsection, we compare the comultiplication $\Delta : A \to A \otimes_k A$ given in Section 4.2 and the comultiplication $\alpha : A \to A \otimes_k A$ given by Abrams (Theorem 2.2.3). Moreover, we give the proof of Theorem 2.2.11 as promised.

We keep the notations introduced in Chapter 2 and this chapter. Since A is Frobenius, we choose $e = 1_A$ and have the A-bimodule isomorphism $\tau : A_\sigma \cong \mathrm{D}(A)$ by Section 4.2 such that σ is a Nakayama automorphism of A.

Proposition 4.2.19. *Let A be a Frobenius algebra with the left A-module isomorphism $\lambda_L : A \cong D(A)$ which defines an isomorphism $\lambda_L : A_\sigma \cong D(A)$ of A-bimodules, where σ is a Nakayama automorphism of A. Suppose that $\lambda_L = \tau$. Then α is equal to Δ.*

Proof. Let A be Frobenius and $\tau : A_\sigma \cong \mathrm{D}(A)$ be the A-bimodule isomorphism. We can consider τ as $\tau' : A \to \mathrm{D}(A)_{\sigma^{-1}}$ such that $\tau(a) = \tau'(a)$ for any $a \in A$. Therefore, $\lambda_L(a) = \tau'(a)$ for any $a \in A$. Moreover, there is an A-bimodule isomorphism $\gamma : A \otimes_A A_\sigma \cong \mathrm{D}(A)$ by Lemma 4.2.1 and so $\gamma' : A \otimes_A A \cong \mathrm{D}(A)_{\sigma^{-1}}$. By Remark 4.2.11, we have an A-bimodule isomorphism $\Theta' : \mathrm{D}(A)_{\sigma^{-1}} \to A$ with $\Theta' = \tau'^{-1}$. By following the same remark, we write $\tau'^{-1}(\gamma'(x \otimes y)) = xy$ for any $x, y \in A$.

Since the Frobenius form ε of A is equal to $\lambda_L(1_A)$, all elements of $\mathrm{D}(A)$ are of the form $a \cdot \varepsilon$ for any $a \in A$. By Section 2.2, it is known that the isomorphism $\lambda_L : A \cong \mathrm{D}(A)$ allows us to define a multiplication φ_L such that $\varphi_L(a \cdot \varepsilon \otimes b \cdot \varepsilon) = (b \cdot \varepsilon \otimes a \cdot \varepsilon) \circ \alpha_R = ab \cdot \varepsilon$.

Let $\vartheta : A \otimes_A A \cong A$ be the A-bimodule isomorphism such that $\vartheta(a \otimes_A b) = ab$ and $\mu' : A \otimes_k A \to A \otimes_A A$ be the map such that $\mu'(a \otimes_k b) = a \otimes_A b$ for any $a, b \in A$. Suppose that $\lambda'_L := \lambda_L \circ \vartheta$ and $\varphi'_L := \lambda'_L \circ \mu' \circ \lambda_L^{-1} \otimes \lambda_L^{-1}$. Then observe the following

$$\begin{array}{ccccccc} \varphi'_L : \mathrm{D}(A) \otimes_k \mathrm{D}(A) & \xrightarrow{\lambda_L^{-1} \otimes \lambda_L^{-1}} & A \otimes_k A & \xrightarrow{\mu'} & A \otimes_A A & \xrightarrow{\lambda'_L} & \mathrm{D}(A) \\ a \cdot \varepsilon \otimes_k b \cdot \varepsilon & \longmapsto & a \otimes_k b & \longmapsto & a \otimes_A b & \longmapsto & ab \cdot \varepsilon \end{array}$$

Therefore, $\varphi_L = \varphi'_L$.

Observe that there are isomophisms of left A-modules $\tau' \otimes_k \lambda_L : A \otimes_k A \cong \mathrm{D}(A)_{\sigma^{-1}} \otimes_k \mathrm{D}(A)$ and $\tau' \otimes_A \lambda_L : A \otimes_A A \cong \mathrm{D}(A)_{\sigma^{-1}} \otimes_A \mathrm{D}(A)$. We now observe the following diagram

$$\begin{tikzcd} \mathrm{D}(A) \otimes_k \mathrm{D}(A) \arrow[r, "\lambda_L^{-1} \otimes_k \lambda_L^{-1}"] \arrow[dr, "m_2"'] & A \otimes_k A \arrow[r, "\mu'"] \arrow[d, "\tau' \otimes_k \lambda_L"] & A \otimes_A A \arrow[r, "\lambda'_L"] \arrow[d, "\tau' \otimes_A \lambda_L"'] & \mathrm{D}(A) \\ & \mathrm{D}(A)_{\sigma^{-1}} \otimes_k \mathrm{D}(A) \arrow[r, "\phi"] & \mathrm{D}(A)_{\sigma^{-1}} \otimes_A \mathrm{D}(A) \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

Since $\gamma : A \otimes_A A_\sigma \cong \mathrm{D}(A)$ as A-bimodules and $\varepsilon \in \mathrm{D}(A)$, we can write $\varepsilon = \gamma(x \otimes_A y)$ for suitable $x, y \in A$. Since $\mathrm{D}(A) = \mathrm{D}(A)_{\sigma^{-1}}$ as k-vector spaces, we can consider ε as $\varepsilon = \gamma'(x \otimes_A y)$ when we need to use it. Then any $a \cdot \varepsilon$ of $\mathrm{D}(A)$ can be written as $a \cdot \varepsilon = \gamma(ax \otimes_A y)$ and any $a \cdot \varepsilon$ of $\mathrm{D}(A)_{\sigma^{-1}}$ can be written as $a \cdot \varepsilon = \gamma'(ax \otimes_A y)$. Therefore, $\lambda_L^{-1}(\gamma(ax \otimes_A y)) = \lambda_L^{-1}(a \cdot \varepsilon) = a$. Then $\tau'^{-1}(\gamma'(ax \otimes_A y)) = axy = a$ by definition of τ'^{-1} given above. Since A is faithful A-module, $xy = 1$. Moreover, $(\tau' \otimes_k \lambda_L)(a \otimes_k b) = \gamma'(ax \otimes_A y) \otimes_k \gamma(bx \otimes_A y)$ and $(\tau' \otimes_A \lambda_L)(a \otimes_A b) = \gamma'(ax \otimes_A y) \otimes_A \gamma(bx \otimes_A y)$. Recall that $m = \epsilon \circ \phi \circ m_2$. For the definitions of ϵ, ϕ and m_2, see (4.2) and page 65.

Then by using the above information, first observe that

$$\begin{aligned} (\tau' \otimes_k \lambda_L) \circ (\lambda_L^{-1} \otimes_k \lambda_L^{-1})(a \cdot \varepsilon \otimes_k b \cdot \varepsilon) &= (\tau' \otimes_k \lambda_L)(a \otimes_k b) \\ &= \gamma'(ax \otimes_A y) \otimes_k \gamma(bx \otimes_A y) \\ m_2(a \cdot \varepsilon \otimes_k b \cdot \varepsilon) &= m_2(\gamma(ax \otimes_A y) \otimes_k \gamma(bx \otimes_A y)) \\ &= \gamma'(ax \otimes_A y) \otimes_k \gamma(bx \otimes_A y). \end{aligned}$$

It means that left side of the above diagram is commutative.

Also, we see that

$$(\tau' \otimes_A \lambda_L) \circ \mu'(a \otimes_k b) = (\tau' \otimes_A \lambda_L)(a \otimes_A b)$$

$$
\begin{aligned}
&= \gamma'(ax \otimes_A y) \otimes_A \gamma(bx \otimes_A y) \\
\phi \circ (\tau' \otimes_k \lambda_L)(a \otimes_k b) &= \phi(\gamma'(ax \otimes_A y) \otimes_k \gamma(bx \otimes_A y)) \\
&= \gamma'(ax \otimes_A y) \otimes_A \gamma(bx \otimes_A y).
\end{aligned}
$$

Hence, middle part of the diagram is commutative.

Moreover, we have

$$
\begin{aligned}
\epsilon \circ (\tau' \otimes_A \lambda_L)(a \otimes_A b) &= \epsilon(\gamma'(ax \otimes_A y) \otimes_A \gamma(bx \otimes_A y)) \\
&= \gamma(axybx \otimes_A y) \\
&= \gamma(abx \otimes_A y) \\
&= ab \cdot \varepsilon \\
\lambda'_L(a \otimes_A b) &= ab \cdot \varepsilon.
\end{aligned}
$$

Therefore, right side of the diagram is commutative. This means that $\varphi'_L = m$ and so $\varphi_L = m$. Then dualising gives that $\alpha_R = \Delta$.

There is also a comultiplication α_L which is a map of left A-modules and in [2], Abrams proved that $\alpha_L = \alpha_R$ and defined $\alpha := \alpha_L = \alpha_R$. Hence, we obtain that $\alpha = \Delta$. □

Proof of Theorem 2.2.11. Let A be a Frobenius algebra with the left A-module isomorphism $\lambda_L : A \cong \mathrm{D}(A)$. By Section 4.2, there is a comultiplication Δ and an A-bimodule isomorphism $\tau : A_\sigma \cong \mathrm{D}(A)$. Since A is Frobenius, σ is a Nakayama automorphism of A. The left A-module isomorpism λ_L defines an isomorphism $\lambda_L : A_{\nu_A} \cong \mathrm{D}(A)$ of A-bimodules, where ν_A is a Nakayama automorphism of A. Since Nakayama automorphism is unique up to inner automorphisms, $\sigma = \theta\nu_A$ for some inner automorphism θ of the algebra A. We may choose θ as identity automorphism and so $\sigma = \nu_A$. We may also suppose that $\lambda_L = \tau$. Then by Proposition 4.2.19, $\alpha = \Delta$ and by Lemma 4.2.5, $\mathrm{Im}(\Delta) = \{\sum u_i \otimes v_i \mid \sum u_i x \otimes v_i = \sum u_i \otimes \sigma^{-1}(x) v_i, \ \forall x \in A\}$. Therefore, we obtain that

$$
\mathrm{Im}(\alpha) = \{\sum u_i \otimes v_i \mid \sum u_i x \otimes v_i = \sum u_i \otimes \nu_A^{-1}(x) v_i, \ \ \forall x \in A\}.
$$

□

4.2.3 More comultiplications which are not coassociative

There are further constructions possible that yield comultiplications on gendo-Frobenius algebras. In this subsection, we investigate three such constructions and show that they are lacking crucial properties such as being coassociative. Throughout this subsection, we assume that A is a gendo-Frobenius k-algebra with a faithful and self-dual idempotent e.

Construction I. Fix an (eAe, A)-bimodule isomorphism $\iota : {}_{\nu_{eAe}^{-1}} eA \cong \mathrm{D}(Ae)$, where ν_{eAe} is a Nakayama automorphism of the Frobenius algebra eAe. Then by using the double centralizer property of Ae and the isomorphism ι, we obtain the following A-bimodule isomorphism

$$
\begin{aligned}
A \cong \mathrm{Hom}_{eAe}(Ae, Ae) &\cong \mathrm{Hom}_{eAe}(\mathrm{D}(Ae), \mathrm{D}(Ae)) \\
&\cong \mathrm{Hom}_{eAe}({}_{\nu_{eAe}^{-1}} eA, \mathrm{D}(Ae)) \\
&\cong \mathrm{Hom}_k(Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA, k).
\end{aligned}
$$

By dualising $\mathrm{Hom}_k(Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA, k) \cong A$, we obtain that there is an A-bimodule isomorphism

$$\psi : Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA \cong \mathrm{D}(A)$$

such that $\psi(ae \otimes_{eAe} eb)(x) = \iota(ebx)(ae)$ for all $a, b, x \in A$. Hence, there is a left A-module isomorphism

$$\begin{aligned}
\epsilon_1^L : \mathrm{D}(A) \otimes_A \mathrm{D}(A) &\overset{(1)}{\cong} (Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA) \otimes_A (Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA) \\
&\cong Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eAe \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA \\
&\overset{(2)}{\cong} Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eAe \otimes_{eAe} eA \\
&\cong Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA \\
&\cong \mathrm{D}(A)
\end{aligned}$$

where (1) is $\psi^{-1} \otimes_A \psi^{-1}$ and (2) is obtained from Theorem 4.1.7 (iv) which states that there is a left eAe-module isomorphism $\omega_L : {}_{\nu_{eAe}^{-1}} eA \cong eA$. (Instead of the left eAe-module isomorphism ω_L, we could use the (eAe, A)-bimodule isomorphism $\eta : eA_\sigma \cong {}_{\nu_{eAe}^{-1}} eA$ and obtain the (A, A)-bimodule isomorphism $\mathrm{D}(A) \otimes_A \mathrm{D}(A) \cong \mathrm{D}(A)_\sigma$. But, we discuss it in Construction II). The map ϵ_1^L is explicitly defined by

$$\begin{aligned}
\epsilon_1^L : \psi(ae \otimes_{eAe} eb) \otimes_A \psi(ce \otimes_{eAe} ed) &\mapsto (ae \otimes_{eAe} eb) \otimes_A (ce \otimes_{eAe} ed) \\
&\mapsto ae \otimes_{eAe} ebce \otimes_{eAe} ed \\
&\mapsto ae \otimes_{eAe} ebce \otimes_{eAe} ed' \\
&\mapsto ae \otimes_{eAe} ebced' \\
&\mapsto \psi(ae \otimes_{eAe} ebced')
\end{aligned}$$

such that $\omega_L(ed) = ed'$. Let m_1^L be the composition of the canonical left A-module morphism with the above isomorphism such that

$$m_1^L : \mathrm{D}(A) \otimes_k \mathrm{D}(A) \to \mathrm{D}(A) \otimes_A \mathrm{D}(A) \overset{\epsilon_1^L}{\cong} \mathrm{D}(A),$$

where

$$\begin{aligned}
m_1^L(\psi(ae \otimes_{eAe} eb) \otimes_k \psi(ce \otimes_{eAe} ed)) &= \epsilon_1^L(\psi(ae \otimes_{eAe} eb) \otimes_A \psi(ce \otimes_{eAe} ed)) \\
&= \psi(ae \otimes_{eAe} ebced').
\end{aligned}$$

However, m_1^L is not associative. We will show this in the next example by using the algebra in Example 4.1.13. Then dualising m_1^L gives the following non-coassociative comultiplication

$$\Delta_1^R : A \to A \otimes_k A$$

which is a map of right A-modules such that $m_1^L(g \otimes f)(a) = (f \otimes g)\Delta_1^R(a)$ for any $f, g \in \mathrm{D}(A)$ and $a \in A$.

Moreover, there is a right A-module isomorphism

$$\epsilon_1^R : \mathrm{D}(A) \otimes_A \mathrm{D}(A) \overset{(1)}{\cong} (Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA) \otimes_A (Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA)$$

$$
\begin{aligned}
&\cong Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eAe \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA \\
&\cong Ae_{\nu_{eAe}} \otimes_{eAe} eAe \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA \\
&\overset{(2)}{\cong} Ae \otimes_{eAe} eAe \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA \\
&\cong Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA \\
&\cong \mathrm{D}(A)
\end{aligned}
$$

where (1) is $\psi^{-1} \otimes_A \psi^{-1}$ and (2) is obtained from Theorem 4.1.7 (iii) which states that there is a right eAe-module isomorphism $\omega_R : Ae_{\nu_{eAe}} \cong Ae$. The map ϵ_1^R is explicitly defined by

$$
\begin{aligned}
\epsilon_1^R : \psi(ae \otimes_{eAe} eb) \otimes_A \psi(ce \otimes_{eAe} ed) &\mapsto (ae \otimes_{eAe} eb) \otimes_A (ce \otimes_{eAe} ed) \\
&\mapsto ae \otimes_{eAe} ebce \otimes_{eAe} ed \\
&\mapsto ae \otimes_{eAe} ebce \otimes_{eAe} ed \\
&\mapsto a'e \otimes_{eAe} ebce \otimes_{eAe} ed \\
&\mapsto a'ebce \otimes_{eAe} ed \\
&\mapsto \psi(a'ebce \otimes_{eAe} ed)
\end{aligned}
$$

such that $\omega_R(ae) = a'e$.

Let m_1^R be the composition of the canonical right A-module morphism with the above isomorphism such that

$$
m_1^R : \mathrm{D}(A) \otimes_k \mathrm{D}(A) \to \mathrm{D}(A) \otimes_A \mathrm{D}(A) \overset{\epsilon_1^R}{\cong} \mathrm{D}(A),
$$

where

$$
\begin{aligned}
m_1^R(\psi(ae \otimes_{eAe} eb) \otimes_k \psi(ce \otimes_{eAe} ed)) &= \epsilon_1^R(\psi(ae \otimes_{eAe} eb) \otimes_A \psi(ce \otimes_{eAe} ed)) \\
&= \psi(a'ebce \otimes_{eAe} ed).
\end{aligned}
$$

However, m_1^R is not associative. We will show this in the next example by using the algebra in Example 4.1.13. Then dualising m_1^R gives the following non-coassociative comultiplication

$$
\Delta_1^L : A \to A \otimes_k A
$$

which is a map of left A-modules such that $m_1^R(g \otimes f)(a) = (f \otimes g)\Delta_1^L(a)$ for any $f, g \in \mathrm{D}(A)$ and $a \in A$.

Example 4.2.20. Let A be the gendo-Frobenius algebra given in Example 4.1.13. A has a k-basis $\{e_1, e_2, e_3, e_4, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1\alpha_3, \alpha_2\alpha_4\}$ so $\mathrm{D}(A)$ has the dual basis $\{e_1^*, e_2^*, e_3^*, e_4^*, \alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*, (\alpha_1\alpha_3)^*, (\alpha_2\alpha_4)^*\}$. We choose $e = e_1 + e_2$ since $e_1 + e_2$ is a faithful and self-dual idempotent of A. The (eAe, A)-bimodule isomorphism $\iota : {}_{\nu_{eAe}^{-1}} eA \cong \mathrm{D}(Ae)$ is explicitly defined on the basis elements of A by

$$
\begin{aligned}
\iota : {}_{\nu_{eAe}^{-1}} eA &\cong \mathrm{D}(Ae) \\
e_1 &\mapsto (\alpha_1\alpha_3)^* \\
e_2 &\mapsto (\alpha_2\alpha_4)^* \\
\alpha_1 &\mapsto \alpha_3^* \\
\alpha_2 &\mapsto \alpha_4^*
\end{aligned}
$$

$$\alpha_1\alpha_3 \mapsto e_2^*$$
$$\alpha_2\alpha_4 \mapsto e_1^*.$$

Since $\psi(ae \otimes_{eAe} eb)(x) = \iota(ebx)(ae)$ for all $a, b, x \in A$, the A-bimodule isomorphism $\psi : Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA \cong \mathrm{D}(A)$ is explicitly defined by

$$\begin{aligned}
\psi : Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA &\cong \mathrm{D}(A) \\
e_1 \otimes e_2 &\mapsto (\alpha_2\alpha_4)^* \\
e_1 \otimes \alpha_2 &\mapsto \alpha_4^* \\
e_1 \otimes \alpha_2\alpha_4 &\mapsto e_1^* \\
e_2 \otimes e_1 &\mapsto (\alpha_1\alpha_3)^* \\
e_2 \otimes \alpha_1 &\mapsto \alpha_3^* \\
e_2 \otimes \alpha_1\alpha_3 &\mapsto e_2^* \\
\alpha_3 \otimes e_1 &\mapsto \alpha_1^* \\
\alpha_3 \otimes \alpha_1 &\mapsto e_3^* \\
\alpha_4 \otimes e_2 &\mapsto \alpha_2^* \\
\alpha_4 \otimes \alpha_2 &\mapsto e_4^*.
\end{aligned}$$

Then we obtain the table of multiplication m_1^L as follows:

m_1^L	e_1^*	e_2^*	e_3^*	e_4^*	α_1^*	α_2^*	α_3^*	α_4^*	$(\alpha_1\alpha_3)^*$	$(\alpha_2\alpha_4^*)$
e_1^*	0	0	0	0	0	0	0	0	0	e_1^*
e_2^*	0	0	0	0	0	0	0	0	e_2^*	0
e_3^*	0	0	0	0	0	0	0	0	0	0
e_4^*	0	0	0	0	0	0	0	0	0	0
α_1^*	0	0	0	0	0	0	0	e_3^*	0	α_1^*
α_2^*	0	0	0	0	0	0	e_4^*	0	α_2^*	0
α_3^*	0	0	0	0	e_2^*	0	0	0	0	0
α_4^*	0	0	0	0	0	e_1^*	0	0	0	0
$(\alpha_1\alpha_3)^*$	e_2^*	0	0	0	0	0	0	α_3^*	0	$(\alpha_1\alpha_3)^*$
$(\alpha_2\alpha_4)^*$	0	e_1^*	0	0	0	0	α_4^*	0	$(\alpha_2\alpha_4)^*$	0

By the same process as previous examples, dualising m_1^L yields

$$\begin{aligned}
\Delta_1^R(e_1) &= \alpha_2\alpha_4 \otimes e_1 + \alpha_2 \otimes \alpha_4 + e_2 \otimes \alpha_2\alpha_4 \\
\Delta_1^R(e_2) &= \alpha_1\alpha_3 \otimes e_2 + \alpha_1 \otimes \alpha_3 + e_1 \otimes \alpha_1\alpha_3 \\
\Delta_1^R(e_3) &= \alpha_4 \otimes \alpha_1 \\
\Delta_1^R(e_4) &= \alpha_3 \otimes \alpha_2 \\
\Delta_1^R(\alpha_1) &= \alpha_2\alpha_4 \otimes \alpha_1 \\
\Delta_1^R(\alpha_2) &= \alpha_1\alpha_3 \otimes \alpha_2 \\
\Delta_1^R(\alpha_3) &= \alpha_4 \otimes \alpha_1\alpha_3 \\
\Delta_1^R(\alpha_4) &= \alpha_3 \otimes \alpha_2\alpha_4 \\
\Delta_1^R(\alpha_1\alpha_3) &= \alpha_2\alpha_4 \otimes \alpha_1\alpha_3
\end{aligned}$$

$$\Delta_1^R(\alpha_2\alpha_4) = \alpha_1\alpha_3 \otimes \alpha_2\alpha_4.$$

In fact, the multiplication m_1^L is not associative, because, for example

$$m_1^L(m_1^L \otimes 1)(\alpha_1^* \otimes_k \alpha_2\alpha_4^* \otimes \alpha_4^*) = m_1^L(\alpha_1^* \otimes_k \alpha_4^*) = e_3^*$$

and

$$m_1^L(1 \otimes m_1^L)(\alpha_1^* \otimes_k \alpha_2\alpha_4^* \otimes \alpha_4^*) = m_1^L(\alpha_1^* \otimes 0) = 0$$

are not equal.

Moreover, we obtain the table of multiplication m_1^R as follows:

m_1^R	e_1^*	e_2^*	e_3^*	e_4^*	α_1^*	α_2^*	α_3^*	α_4^*	$(\alpha_1\alpha_3)^*$	$(\alpha_2\alpha_4^*)$
e_1^*	0	0	0	0	0	0	0	0	0	e_2^*
e_2^*	0	0	0	0	0	0	0	0	e_1^*	0
e_3^*	0	0	0	0	0	0	0	0	0	0
e_4^*	0	0	0	0	0	0	0	0	0	0
α_1^*	0	0	0	0	0	0	0	e_4^*	0	α_2^*
α_2^*	0	0	0	0	0	0	e_3^*	0	α_1^*	0
α_3^*	0	0	0	0	e_1^*	0	0	0	0	0
α_4^*	0	0	0	0	0	e_2^*	0	0	0	0
$(\alpha_1\alpha_3)^*$	e_1^*	0	0	0	0	0	0	α_4^*	0	$(\alpha_2\alpha_4)^*$
$(\alpha_2\alpha_4)^*$	0	e_2^*	0	0	0	0	α_3^*	0	$(\alpha_1\alpha_3)^*$	0

By the same process as before, dualising m_1^R yields

$$\begin{aligned}
\Delta_1^L(e_1) &= \alpha_1\alpha_3 \otimes e_2 + \alpha_1 \otimes \alpha_3 + e_1 \otimes \alpha_1\alpha_3 \\
\Delta_1^L(e_2) &= \alpha_2\alpha_4 \otimes e_1 + \alpha_2 \otimes \alpha_4 + e_2 \otimes \alpha_2\alpha_4 \\
\Delta_1^L(e_3) &= \alpha_3 \otimes \alpha_2 \\
\Delta_1^L(e_4) &= \alpha_4 \otimes \alpha_1 \\
\Delta_1^L(\alpha_1) &= \alpha_1\alpha_3 \otimes \alpha_2 \\
\Delta_1^L(\alpha_2) &= \alpha_2\alpha_4 \otimes \alpha_1 \\
\Delta_1^L(\alpha_3) &= \alpha_3 \otimes \alpha_2\alpha_4 \\
\Delta_1^L(\alpha_4) &= \alpha_4 \otimes \alpha_1\alpha_3 \\
\Delta_1^L(\alpha_1\alpha_3) &= \alpha_1\alpha_3 \otimes \alpha_2\alpha_4 \\
\Delta_1^L(\alpha_2\alpha_4) &= \alpha_2\alpha_4 \otimes \alpha_1\alpha_3.
\end{aligned}$$

In fact, the multiplication m_1^R is not associative, because, for example

$$m_1^R(m_1^R \otimes 1)(\alpha_1^* \otimes_k \alpha_2\alpha_4^* \otimes \alpha_3^*) = m_1^R(\alpha_2^* \otimes_k \alpha_3^*) = e_3^*$$

and

$$m_1^R(1 \otimes m_1^R)(\alpha_1^* \otimes_k \alpha_2\alpha_4^* \otimes \alpha_3^*) = m_1^R(\alpha_1^* \otimes \alpha_3^*) = 0$$

are not equal.

And, moreover, we observe that $\Delta_1^L \neq \Delta_1^R$. Indeed, we obtain the following two commutative diagrams

$$\begin{tikzcd} & A \arrow[dl, "\Delta_1^L"'] \arrow[dr, "\Delta"] & \\ A \otimes_k A \arrow[rr, "id_A \otimes_k \sigma^{-1}"'] & & A \otimes_k A \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "\Delta_1^R"'] \arrow[dr, "\Delta"] & \\ A \otimes_k A \arrow[rr, "\sigma \otimes_k id_A"'] & & A \otimes_k A \end{tikzcd}$$

Construction II. By Construction I, there is an A-bimodule isomorphism

$$\psi : Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA \cong D(A)$$

such that $\psi(ae \otimes_{eAe} eb)(x) = \iota(ebx)(ae)$, for all $a, b, x \in A$. Hence, there is an A-bimodule isomorphism

$$\begin{aligned} \epsilon_2 : \mathrm{D}(A) \otimes_A \mathrm{D}(A) &\overset{(1)}{\cong} (Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA) \otimes_A (Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA) \\ &\cong Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eAe \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA \\ &\overset{(2)}{\cong} Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eAe \otimes_{eAe} eA_\sigma \\ &\cong Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA_\sigma \\ &\cong \mathrm{D}(A)_\sigma \end{aligned}$$

where (1) is $\psi^{-1} \otimes_A \psi^{-1}$ and (2) is obtained from Proposition 4.1.11 (ii) which states that there is an (eAe, A)-bimodule isomorphism $\eta : eA_\sigma \cong {}_{\nu_{eAe}^{-1}} eA$. This map is explicitly defined by

$$\begin{aligned} \epsilon_2 : \psi(ae \otimes eb) \otimes_A \psi(ce \otimes ed) &\overset{(1)}{\mapsto} (ae \otimes_{eAe} eb) \otimes_A (ce \otimes_{eAe} ed) \\ &\mapsto ae \otimes_{eAe} ebce \otimes_{eAe} ed \\ &\overset{(2)}{\mapsto} ae \otimes_{eAe} ebce \otimes_{eAe} ed' \\ &\mapsto ae \otimes_{eAe} ebced' \\ &\mapsto \psi(ae \otimes_{eAe} ebced') \end{aligned}$$

such that $\eta^{-1}(ed) = ed'$. Let m_2 be the composition of the canonical A-bimodule morphism with the above isomorphism such that

$$m_2 : \mathrm{D}(A) \otimes_k \mathrm{D}(A) \to \mathrm{D}(A) \otimes_A \mathrm{D}(A) \overset{\epsilon_2}{\cong} \mathrm{D}(A)_\sigma,$$

where

$$\begin{aligned} m_2(\psi(ae \otimes_{eAe} eb) \otimes_k \psi(ce \otimes_{eAe} ed)) &= \epsilon_2(\psi(ae \otimes_{eAe} eb) \otimes_A \psi(ce \otimes_{eAe} ed)) \\ &\cong \psi(ae \otimes_{eAe} ebced'). \end{aligned}$$

In fact, $m_2 = m_1^L$ as left A-module morphisms, and m_2 is not associative because of the same reason with Construction I. Dualising m_2 gives

$$\Delta_2 : {}_\sigma A \to A \otimes_k A.$$

Also, $\Delta_2 = \Delta_1^R$ as right A-module morphisms. Indeed, Δ_2 is not in the form of comultiplication that we wished as an A-bimodule morphism since it twists with automorphism σ. Moreover, it is non-coassociative since m_2 is not associative.

Construction III. By the proof of Proposition 4.2.2, there is an A-bimodule isomorphism $\gamma : Ae \otimes_{eAe} eA_\sigma \cong \mathrm{D}(A)$ and (eAe, A)-bimodule isomorphism $\tau : eA_\sigma \cong \mathrm{D}(A)$ such that $\gamma(ae \otimes_{eAe} eb)(x) = \tau(eb\sigma(x))(ae)$ for any $a, b, x \in A$. By Proposition 4.1.11 (ii), there is an (eAe, A)-bimodule isomorphism $\eta : eA_\sigma \cong {}_{\nu_{eAe}^{-1}} eA$, where ν_{eAe} is a Nakayama automorphism of eAe.

Note that there is an eAe-bimodule isomorphism $\chi : eA_\sigma \otimes_A Ae \cong eA\sigma(e)_{\sigma_e}$, where σ_e is the restriction of σ to eAe and χ is explicitly defined by $\chi : eb \otimes ce \to eb\sigma(c)\sigma(e)$, for any $b, c \in A$.

Suppose that $\sigma(e) = e$. By Proposition 4.1.11 (iii), we obtain that σ_e is a Nakayama automorphism of eAe. Then let $\sigma_e = \nu_{eAe}$. All information above gives the following A-bimodule isomorphism

$$\begin{aligned}
\epsilon_3 : \mathrm{D}(A) \otimes_A \mathrm{D}(A) &\cong (Ae \otimes_{eAe} eA_\sigma) \otimes_A (Ae \otimes_{eAe} eA_\sigma) \\
&\cong Ae \otimes_{eAe} eA\sigma(e)_{\sigma_e} \otimes_{eAe} eA_\sigma \\
&\cong Ae \otimes_{eAe} eAe_{\nu_{eAe}} \otimes_{eAe} eA_\sigma \\
&\cong Ae_{\nu_{eAe}} \otimes_{eAe} eA_\sigma \\
&\cong Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}} eA_\sigma \\
&\cong Ae \otimes_{eAe} eA_{\sigma^2} \\
&\cong \mathrm{D}(A)_\sigma
\end{aligned}$$

which is explicitly defined by

$$\begin{aligned}
\epsilon_3 : \gamma(ae \otimes_{eAe} eb) \otimes_A \gamma(ce \otimes_{eAe} ed) &\cong (ae \otimes_{eAe} eb) \otimes_A (ce \otimes_{eAe} ed) \\
&\cong ae \otimes_{eAe} eb\sigma(c)\sigma(e) \otimes_{eAe} ed \\
&\cong ae \otimes_{eAe} eb\sigma(c)e \otimes_{eAe} ed \\
&\cong aeb\sigma(c)e \otimes_{eAe} ed \\
&\cong aeb\sigma(c)e \otimes_{eAe} ed \\
&\cong aeb\sigma(c)e \otimes_{eAe} ed' \\
&\cong \gamma(aeb\sigma(c)e \otimes_{eAe} ed'),
\end{aligned}$$

where $\eta^{-1}(ed) = ed'$.

Let m_3 be the composition of the canonical A-bimodule morphism with the above isomorphism such that

$$m_3 : \mathrm{D}(A) \otimes_k \mathrm{D}(A) \to \mathrm{D}(A) \otimes_A \mathrm{D}(A) \overset{\epsilon_3}{\cong} \mathrm{D}(A)_\sigma,$$

where

$$\begin{aligned}
m_3(\gamma(ae \otimes_{eAe} eb) \otimes_k \gamma(ce \otimes_{eAe} ed)) &= \epsilon_3(\gamma(ae \otimes_{eAe} eb) \otimes_A \gamma(ce \otimes_{eAe} ed)) \\
&\cong \gamma(aeb\sigma(c)e \otimes_{eAe} ed').
\end{aligned}$$

Dualising m_3 gives the following A-bimodule morphism

$$\Delta_3 : {}_\sigma A \to A \otimes_k A$$

such that $m_3(g \otimes f)(a) = (f \otimes g)\Delta_3(a)$ for any $f, g \in \mathrm{D}(A)$ and $a \in A$. Let us consider $\Delta_3 : A \to A \otimes_k A$

only as right A-module morphism to make it in the form of comultiplication that we wished. However, Δ_3 is not coassociative. Let us see it on the following example.

Example 4.2.21. Let A be the gendo-Frobenius algebra in Example 4.1.13. A has a k-basis $\{e_1, e_2, e_3, e_4, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1\alpha_3, \alpha_2\alpha_4\}$ so $\mathrm{D}(A)$ has the dual basis $\{e_1^*, e_2^*, e_3^*, e_4^*, \alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*, (\alpha_1\alpha_3)^*, (\alpha_2\alpha_4)^*\}$. We choose $e = e_1 + e_2$ since $e_1 + e_2$ is a faithful and self-dual idempotent of A. Before, we computed the (eAe, A)-bimodule isomorphism τ explicitly on the basis elements by

$$\begin{aligned} \tau : eA_\sigma &\cong \mathrm{D}(Ae) \\ e_1 &\mapsto (\alpha_2\alpha_4)^* \\ e_2 &\mapsto (\alpha_1\alpha_3)^* \\ \alpha_1 &\mapsto \alpha_4^* \\ \alpha_2 &\mapsto \alpha_3^* \\ \alpha_1\alpha_3 &\mapsto e_1^* \\ \alpha_2\alpha_4 &\mapsto e_2^* \end{aligned}$$

and the A-bimodule isomorphism γ by

$$\begin{aligned} \gamma : Ae \otimes_{eAe} eA_\sigma &\cong \mathrm{D}(A) \\ e_2 \otimes e_2 &\mapsto (\alpha_1\alpha_3)^* \\ e_2 \otimes \alpha_2 &\mapsto \alpha_3^* \\ e_2 \otimes \alpha_2\alpha_4 &\mapsto e_2^* \\ e_1 \otimes e_1 &\mapsto (\alpha_2\alpha_4)^* \\ e_1 \otimes \alpha_1 &\mapsto \alpha_4^* \\ e_1 \otimes \alpha_1\alpha_3 &\mapsto e_1^* \\ \alpha_3 \otimes e_2 &\mapsto \alpha_1^* \\ \alpha_3 \otimes \alpha_2 &\mapsto e_3^* \\ \alpha_4 \otimes e_1 &\mapsto \alpha_2^* \\ \alpha_4 \otimes \alpha_1 &\mapsto e_4^*. \end{aligned}$$

Then we obtain the table of multiplication m_3 as follows:

m_3	e_1^*	e_2^*	e_3^*	e_4^*	α_1^*	α_2^*	α_3^*	α_4^*	$(\alpha_1\alpha_3)^*$	$(\alpha_2\alpha_4^*)$
e_1^*	0	0	0	0	0	0	0	0	0	e_1^*
e_2^*	0	0	0	0	0	0	0	0	e_2^*	0
e_3^*	0	0	0	0	0	0	0	0	0	0
e_4^*	0	0	0	0	0	0	0	0	0	0
α_1^*	0	0	0	0	0	0	0	e_3^*	0	α_1^*
α_2^*	0	0	0	0	0	0	e_4^*	0	α_2^*	0
α_3^*	0	0	0	0	e_2^*	0	0	0	0	0
α_4^*	0	0	0	0	0	e_1^*	0	0	0	0
$(\alpha_1\alpha_3)^*$	e_2^*	0	0	0	0	0	0	α_3^*	0	$(\alpha_1\alpha_3)^*$
$(\alpha_2\alpha_4)^*$	0	e_1^*	0	0	0	0	α_4^*	0	$(\alpha_2\alpha_4)^*$	0

By the same process as previous examples, dualising m_3 yields

$$\begin{aligned}
\Delta_3(e_1) &= \alpha_2\alpha_4 \otimes e_1 + \alpha_2 \otimes \alpha_4 + e_2 \otimes \alpha_2\alpha_4 \\
\Delta_3(e_2) &= \alpha_1\alpha_3 \otimes e_2 + \alpha_1 \otimes \alpha_3 + e_1 \otimes \alpha_1\alpha_3 \\
\Delta_3(e_3) &= \alpha_4 \otimes \alpha_1 \\
\Delta_3(e_4) &= \alpha_3 \otimes \alpha_2 \\
\Delta_3(\alpha_1) &= \alpha_2\alpha_4 \otimes \alpha_1 \\
\Delta_3(\alpha_2) &= \alpha_1\alpha_3 \otimes \alpha_2 \\
\Delta_3(\alpha_3) &= \alpha_4 \otimes \alpha_1\alpha_3 \\
\Delta_3(\alpha_4) &= \alpha_3 \otimes \alpha_2\alpha_4 \\
\Delta_3(\alpha_1\alpha_3) &= \alpha_2\alpha_4 \otimes \alpha_1\alpha_3 \\
\Delta_3(\alpha_2\alpha_4) &= \alpha_1\alpha_3 \otimes \alpha_2\alpha_4.
\end{aligned}$$

In fact, the multiplication m_3 is not associative, because, for example

$$m_3(m_3 \otimes 1)(\alpha_1^* \otimes_k \alpha_2\alpha_4^* \otimes \alpha_4^*) = m_3(\alpha_1^* \otimes_k \alpha_4^*) = e_3^*$$

and

$$m_3(1 \otimes m_3)(\alpha_1^* \otimes_k \alpha_2\alpha_4^* \otimes \alpha_4^*) = m_3(\alpha_1^* \otimes 0) = 0$$

are not equal.

Indeed, $\Delta_3 = \Delta_2$ as A-bimodule morphisms and $\Delta_3 = \Delta_1^R$ as right A-module morphisms under the assumption $\sigma(e) = e$. Without assuming $\sigma(e) = e$, alternatively we can take the (eAe, eAe)-bimodule isomorphism $\eta \otimes_A \mathrm{id}_{Ae} : eA_\sigma \otimes_A Ae \cong {}_{\nu_{eAe}^{-1}}eA \otimes_A Ae$ instead of χ. Then we obtain the following A-bimodule isomorphism

$$\begin{aligned}
\epsilon_4 : \mathrm{D}(A) \otimes_A \mathrm{D}(A) &\cong (Ae \otimes_{eAe} eA_\sigma) \otimes_A (Ae \otimes_{eAe} eA_\sigma) \\
&\cong (Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}}eA) \otimes_A (Ae \otimes_{eAe} eA_\sigma) \\
&\cong Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}}eAe \otimes_{eAe} eA_\sigma \\
&\cong Ae \otimes_{eAe} {}_{\nu_{eAe}^{-1}}eA_\sigma \\
&\cong Ae \otimes_{eAe} eA_{\sigma^2} \\
&\cong \mathrm{D}(A)_\sigma.
\end{aligned}$$

In fact, this is same as the isomorphism ϵ_2 and making the same computations gives the non-coassociative comultiplication Δ_2.

4.3 Comultiplication of Frobenius Nakayama algebras and their compatible counit

In this subsection, we give some results on Frobenius Nakayama algebras with respect to comultiplication. More clearly, we give a comultiplication formula for the Frobenius Nakayama algebras and their compatible counit.

Let Q be a finite connected quiver with nonempty set of arrows, k a field, I an admissible ideal of the path algebra kQ, and $A = kQ/I$ the associated bound quiver algebra.

Definition 4.3.1. A path of length $n \geq 1$ in Q is a sequence of arrows $p = \alpha_1\alpha_2 \cdots \alpha_n$ such that $t(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i \leq n-1$, where $s, t : Q_1 \to Q_0$ are source and target maps, respectively. The length of p is denoted by $\ell(p) = n$.

Notation: Let $A = kQ/I$ be a Frobenius algebra and p_i be a path in Q. Throughout this subsection, the index i of p_i will denote the starting point of p_i, that is, $s(p_i) = i$ for $i \in Q_0$. If $\ell(p_i) = k$, for simplicity, we will denote this path by p_{i_k}, where k only denotes the length of p_i. In addition, $p_{\nu(i)_k} := \nu_A(p_{i_k})$ where ν_A is a Nakayama automorphism of A and ν is the Nakayama permutation of A induced by ν_A.

We now focus on Frobenius Nakayama algebras. By Section 2.1.3, $A = kQ/I$ is a Frobenius Nakayama algebra if and only if $A = N_n^m$.

Throughout this subsection, we assume that $\Delta : A \to A \otimes_k A$ is the comultiplication which is introduced in Section 4.2.

Example 4.3.2. Let $A = N_3^2$. A has a k-basis $\{e_1, e_2, e_3, \alpha_1, \alpha_2, \alpha_3, \alpha_1\alpha_2, \alpha_2\alpha_3, \alpha_3\alpha_1\}$. See Example 2.1.26 for the Nakayama automorphism ν_A and the Nakayama permutation ν of A induced by ν_A. By using the above notation, we have $p_{i_0} = e_i$, $p_{i_1} = \alpha_i$ for $1 \leq i \leq 3$, $p_{1_2} = \alpha_1\alpha_2$, $p_{2_2} = \alpha_2\alpha_3$ and $p_{3_2} = \alpha_3\alpha_1$. Therefore, $p_{\nu^{-1}(1)_0} = e_2$, $p_{\nu^{-1}(2)_0} = e_3$, $p_{\nu^{-1}(3)_0} = e_1$, $p_{\nu^{-1}(1)_1} = \alpha_2$, $p_{\nu^{-1}(2)_1} = \alpha_3$, $p_{\nu^{-1}(3)_1} = \alpha_1$, $p_{\nu^{-1}(1)_2} = \alpha_2\alpha_3$, $p_{\nu^{-1}(2)_2} = \alpha_3\alpha_1$ and $p_{\nu^{-1}(3)_2} = \alpha_1\alpha_2$.

Proposition 4.3.3. *Let $A = N_n^m$, ν_A be a Nakayama automorphism of A and ν be the Nakayama permutation of $\{1, ..., n\}$ induced by ν_A. Suppose that $\{e_1, \cdots, e_n\}$ is the set of idempotents, and for each e_i there exists a path x_i which is an element of socA, where $s(x_i) = i$, $t(x_i) = \nu(i)$ and $\ell(x_i) = m$ for all $i \in Q_0$. Then there is an A-bimodule isomorphism $\tau : A_{\nu_A} \cong D(A)$ which is explicitly defined on the following basis elements of A by*

$$\begin{aligned}\tau : e_i &\mapsto x^*_{\nu^{-1}(i)}\\ x_i &\mapsto e_i^*\\ p_{i_k} &\mapsto (p_{\nu^{-1}(j)_{m-k}})^* \text{ for } 1 \leq k \leq m-1,\end{aligned}$$

where $p_{i_k}p_{j_{m-k}} = x_i$, for all $i \in Q_0$ and for suitable choices of j which depends on k.

Proof. Since A is a Frobenius algebra, there is an A-bimodule isomorphism $\tau : A_{\nu_A} \cong \mathrm{D}(A)$, where ν_A is a Nakayama automorphism of A. Now, we need to write this isomorphism explicitly.

Assume that τ is defined as in the proposition. It is necessary to show that $\tau(ab) = a\tau(b) = \tau(a)\nu_A^{-1}(b)$ for all $a, b \in A$. So, by using the linearity of τ, it is enough to check this condition on every basis element of A. Then firstly observe the following cases for $\tau(ab)$.

<u>Case 1</u>:

$$\begin{aligned}\tau(e_ie_i) &= x^*_{\nu^{-1}(i)}\\ \tau(e_ix_i) &= e_i^*\\ \tau(e_ip_{i_k}) &= (p_{\nu^{-1}(j)_{m-k}})^*,\end{aligned}$$

where $1 \leq k \leq m-1$.
Case 2:

$$\tau(x_i e_{\nu(i)}) = e_i^*.$$

Case 3:

$$\tau(p_{i_k} e_j) = (p_{\nu^{-1}(j)_{m-k}})^*.$$

For $s < m-k$,

$$\tau(p_{i_k} p_{j_s}) = \tau(p_{i_{k+s}}) = (p_{\nu^{-1}(r)_{m-(k+s)}})^*,$$

where $t(p_{i_{k+s}}) = t(p_{j_s}) = r$.
For $s = m-k$,

$$\tau(p_{i_k} p_{j_s}) = \tau(p_{i_{k+s}}) = \tau(x_i) = e_i^*$$

since $\ell(x_i) = m$.
Note that if $\tau(ab)$ is not in the above cases for any basis elements a, b of A, then $\tau(ab) = 0$.

We now observe these cases for $a\tau(b)$.

Case 1:

$$\begin{aligned} e_i \tau(e_i) &= e_i x^*_{\nu^{-1}(i)} = x^*_{\nu^{-1}(i)} \\ e_i \tau(x_i) &= e_i e_i^* = e_i^* \\ e_i \tau(p_{i_k}) &= e_i (p_{\nu^{-1}(j)_{m-k}})^* = (p_{\nu^{-1}(j)_{m-k}})^*, \end{aligned}$$

where $1 \leq k \leq m-1$.

Case 2:

$$x_i \tau(e_{\nu(i)}) = x_i x_i^* = e_i^*.$$

Case 3:

$$p_{i_k} \tau(e_j) = p_{i_k} x^*_{\nu^{-1}(j)} = (p_{\nu^{-1}(j)_{m-k}})^*.$$

For $s < m-k$,

$$p_{i_k} \tau(p_{j_s}) = p_{i_k} p^*_{\nu^{-1}(r)_{m-s}} = (p_{\nu^{-1}(r)_{m-(k+s)}})^*$$

since $t(p_{j_s}) = r$.
For $s = m-k$,

$$p_{i_k} \tau(p_{j_s}) = p_{i_k} p^*_{i_{m-s}} = e_i^*$$

since in this case $t(p_{j_s}) = \nu(i)$.
Note that if $a\tau(b)$ is not in the above cases for any basis elements a, b of A, then $a\tau(b) = 0$.

We lastly observe these cases for $\tau(a)\nu_A^{-1}(b)$.

Case 1:

$$\tau(e_i)\nu_A^{-1}(e_i) = x^*_{\nu^{-1}(i)}e_{\nu^{-1}(i)} = x^*_{\nu^{-1}(i)}$$
$$\tau(e_i)\nu_A^{-1}(x_i) = x^*_{\nu^{-1}(i)}x_{\nu^{-1}(i)} = e_i^*$$
$$\tau(e_i)\nu_A^{-1}(p_{i_k}) = x^*_{\nu^{-1}(i)}p_{\nu^{-1}(i)_k} = (p_{\nu^{-1}(j)_{m-k}})^*,$$

where $1 \leq k \leq m-1$.
Case 2:

$$\tau(x_i)\nu_A^{-1}(e_{\nu(i)}) = e_i^*e_i = e_i^*.$$

Case 3:

$$\tau(p_{i_k})\nu_A^{-1}(e_j) = (p_{\nu^{-1}(j)_{m-k}})^*e_{\nu^{-1}(j)} = (p_{\nu^{-1}(j)_{m-k}})^*.$$

For $s < m-k$,

$$\tau(p_{i_k})\nu_A^{-1}(p_{j_s}) = (p_{\nu^{-1}(j)_{m-k}})^*p_{\nu^{-1}(j)_s} = (p_{\nu^{-1}(r)_{m-(k+s)}})^*$$

since $t(p_{j_s}) = r$.

For $s = m-k$,

$$\tau(p_{i_k})\nu_A^{-1}(p_{j_s}) = (p_{\nu^{-1}(j)_{m-k}})^*p_{\nu^{-1}(j)_s} = e_i^*$$

since in this case $t(p_{j_s}) = \nu(i)$.

Note that if $\tau(a)\nu_A^{-1}(b)$ is not in the above cases for any basis elements a, b of A, then $\tau(a)\nu_A^{-1}(b) = 0$.

Then by the above cases, we obtain that $\tau(ab) = a\tau(b) = \tau(a)\nu_A^{-1}(b)$ for all $a, b \in A$. □

Proposition 4.3.4. *Let $A = N_n^m$, ν_A be a Nakayama automorphism of A and ν be the Nakayama permutation of $\{1, ..., n\}$ induced by ν_A. Suppose that $\{e_1, \cdots, e_n\}$ is the set of idempotents, and for each e_i there exists a path x_i which is an element of socA, with $s(x_i) = i$, $t(x_i) = \nu(i)$ and $\ell(x_i) = m$ for all $i \in Q_0$. Then there is an A-bimodule isomorphism $\gamma : A \otimes_A A_{\nu_A} \cong D(A)$ which is explicitly defined by*

$$\begin{aligned} \gamma : e_i \otimes e_i &\mapsto x^*_{\nu^{-1}(i)} \\ e_i \otimes x_i &\mapsto e_i^* \\ e_i \otimes p_{i_k} &\mapsto (p_{\nu^{-1}(j)_{m-k}})^* \text{ for } 1 \leq k \leq m-1, \end{aligned}$$

where $p_{i_k}p_{j_{m-k}} = x_i$, for all $i \in Q_0$ and for suitable choices of j which depends on k.

Proof. Since A is a Frobenius algebra, there exists an (A, A)-bimodule isomorphism $\tau : A_{\nu_A} \cong \mathrm{D}(A)$. From Proposition 4.3.3, τ is defined by

$$\begin{aligned} \tau : e_i &\mapsto x^*_{\nu^{-1}(i)} \\ x_i &\mapsto e_i^* \\ p_{i_k}e_j &\mapsto (p_{\nu^{-1}(j)_{m-k}})^* \text{ for } 1 \leq k \leq m-1, \end{aligned}$$

where $p_{i_k}p_{j_{m-k}} = x_i$, for all $i \in Q_0$ and for suitable choices of j which depends on k.

Moreover, there is an A-bimodule isomorphism $\gamma : A \otimes_A A_{\nu_A} \cong \mathrm{D}(A)$ such that $\gamma(a \otimes b)(x) = \tau(b\nu_A(x))(a)$ for all $a, b, x \in A$. By using the isomorphism τ and the last equality, the isomorphism γ is explicitly defined by

$$\begin{aligned}\gamma : e_i \otimes e_i &\mapsto x^*_{\nu^{-1}(i)}\\ e_i \otimes x_i &\mapsto e_i^*\\ e_i \otimes p_{i_k} &\mapsto (p_{\nu^{-1}(j)_{m-k}})^* \text{ for } 1 \le k \le m-1,\end{aligned}$$

where $p_{i_k} p_{j_{m-k}} = x_i$, for all $i \in Q_0$ and for suitable choices of j which depends on k. □

Theorem 4.3.5. *Let $A = N_n^m$, ν_A be a Nakayama automorphism of A and ν be the Nakayama permutation of $\{1, ..., n\}$ induced by ν_A. Suppose that $\{e_1, \cdots, e_n\}$ is the set of idempotents, and for each e_i there exists a path x_i which is an element of socA, with $s(x_i) = i$, $t(x_i) = \nu(i)$ and $\ell(x_i) = m$ for all $i \in Q_0$. Then there exists a comultiplication Δ of A which is given for e_i by*

$$\Delta(e_i) = \sum_{\substack{0 \le k \le m\\ p_{i_k} p_{j_{m-k}} = x_i}} p_{i_k} \otimes p_{\nu^{-1}(j)_{m-k}},$$

for suitable choices of j which depends on k.

Note that $p_{i_0} = e_i$ and $p_{i_m} = x_i$.

Proof. By Lemma 4.2.1, there is an A-bimodule isomorphism $\gamma : Ae \otimes_{eAe} eA_\sigma \cong \mathrm{D}(A)$ and by Section 4.2, there is a multiplication map

$$m : \mathrm{D}(A) \otimes_k \mathrm{D}(A) \to \mathrm{D}(A)$$

such that $m(\gamma(ae \otimes_{eAe} eb) \otimes_k \gamma(ce \otimes_{eAe} ed)) = \gamma(aebce \otimes_{eAe} ed)$ for any $a, b, c, d \in A$. Since A is Frobenius, we choose $e = 1_A$. By Proposition 4.1.11, we can let $\sigma = \nu_A$. Therefore, we can consider the A-bimodule isomorphism γ as γ in Proposition 4.3.4.

Let $t(x_i) = s$. By Proposition 4.3.4, we obtain that

$$\begin{aligned} m(e_i^* \otimes_k x_i^*) &= m(\gamma(e_i \otimes_A x_i e_s) \otimes_k \gamma(e_s \otimes_A e_s))\\ &= \gamma(e_i \otimes_A x_i) = e_i^*\\ m(x^*_{\nu^{-1}(i)} \otimes_k e_i^*) &= m(\gamma(e_i \otimes_A e_i) \otimes_k \gamma(e_i \otimes_A x_i e_s))\\ &= \gamma(e_i \otimes_A x_i) = e_i^* \\ m((p_{\nu^{-1}(j)_{m-k}})^* \otimes_k (p_{i_k})^*) &= m(\gamma(e_i \otimes_A p_{i_k}) \otimes_k \gamma(e_j \otimes_A p_{j_{m-k}}))\\ &= \gamma(e_i \otimes_A p_{i_k} p_{j_{m-k}}) = \gamma(e_i \otimes_A x_i) = e_i^*. \end{aligned} \tag{4.5}$$

Observe that the only way of writing e_i^* as a product of basis elements is given above.

Now, let $f, g \in \mathrm{D}(A)$ such that

$$\begin{aligned} f &= \lambda_0 e_i^* + \lambda_1 p_{i_1}^* + \lambda_2 p_{i_2}^* + \cdots + \lambda_{m-1} p^*_{i_{m-1}} + \lambda_m x_i^* + X\\ g &= \mu_0 e_i^* + \mu_1 p^*_{\nu^{-1}(j)_1} + \mu_2 p^*_{\nu^{-1}(j)_2} + \cdots + \mu_{m-1} p^*_{\nu^{-1}(j)_{m-1}} + \mu_m x^*_{\nu^{-1}(i)} + Y, \end{aligned} \tag{4.6}$$

where p_{i_k} ($p_{\nu^{-1}(j)_{m-k}}$) denotes the path starting (ending) at vertex i such that $p_{i_k} p_{j_{m-k}} = x_i$ for $1 \le k \le m-1$ and for suitable choices of j which depends on k. Here, X, Y denote the linear combination of the remaining basis elements of $D(A)$.

By using (4.5), we obtain that

$$m(g \otimes f)(e_i) = \mu_0\lambda_m + \mu_1\lambda_{m-1} + \cdots + \mu_{m-1}\lambda_1 + \mu_m\lambda_0.$$

Since $m(g \otimes f)(e_i) = (f \otimes g)\Delta(e_i)$,

$$(f \otimes g)\Delta(e_i) = \mu_0\lambda_m + \mu_1\lambda_{m-1} + \cdots + \mu_{m-1}\lambda_1 + \mu_m\lambda_0. \tag{4.7}$$

Let $\Delta(e_i) = \sum_k u_k \otimes v_k$. Then by using (4.6) and (4.7), we obtain that

$$\begin{aligned}(f \otimes g)\Delta(e_i) &= (\lambda_0\mu_m(e_i^* \otimes x^*_{\nu^{-1}(i)}) + \lambda_1\mu_{m-1}(p^*_{i_1} \otimes p^*_{\nu^{-1}(j)_{m-1}}) + ... + \lambda_m\mu_0(x_i^* \otimes e_i^*))\sum_k u_k \otimes v_k \\ &= (\lambda_0 e_i^*(u_0)\mu_m x^*_{\nu^{-1}(i)}(v_0) + \lambda_1 p^*_{i_1}(u_1)\mu_{m-1}p^*_{\nu^{-1}(j)_{m-1}}(v_1) + ... + \lambda_m x_i^*(u_m)\mu_0 e_i^*(v_m)) \\ &= \mu_0\lambda_m + \mu_1\lambda_{m-1} + \cdots + \mu_{m-1}\lambda_1 + \mu_m\lambda_0.\end{aligned}$$

So, solving the above equation gives that

$$\begin{aligned}&u_0 = e_i,\ v_0 = x_{\nu^{-1}(i)},\\ &u_k = p_{i_k},\ v_k = p_{\nu^{-1}(j)_{m-k}} \text{ for } 1 \le k \le m-1,\\ &u_m = x_i,\ v_m = e_i.\end{aligned}$$

Then we obtain the following formula

$$\Delta(e_i) = \sum_{\substack{0 \le k \le m \\ p_{i_k}p_{j_{m-k}} = x_i}} p_{i_k} \otimes p_{\nu^{-1}(j)_{m-k}},$$

where j depends on k. □

Observe that $\Delta(1_A) = \Delta(e_1 + e_2 + \cdots + e_n) = \Delta(e_1) + \Delta(e_2) + \cdots + \Delta(e_n)$ for $n = |Q_0|$. Hence, as a result of this fact and the above theorem, we obtain the following corollary.

Corollary 4.3.6. *Let $A = N_n^m$, ν_A be a Nakayama automorphism of A and ν be the Nakayama permutation of $\{1, ..., n\}$ induced by ν_A. Suppose that $\{e_1, \cdots, e_n\}$ is the set of idempotents, and for each e_i there exists a path x_i which is an element of soc$A$, with $s(x_i) = i$, $t(x_i) = \nu(i)$ and $\ell(x_i) = m$ for all $i \in Q_0$. Then there exists a comultiplication Δ of A which is given for 1_A by*

$$\Delta(1_A) = \sum_{\substack{1 \le i \le n \\ 0 \le k \le m \\ p_{i_k}p_{j_{m-k}} = x_i}} p_{i_k} \otimes p_{\nu^{-1}(j)_{m-k}}$$

for suitable choices of j which depends on k.

Note that $p_{i_0} = e_i$ and $p_{i_m} = x_i$.

Remark 4.3.7. Let A and Δ be as given in Corollary 4.3.6. Since Δ is an A-bimodule morphism, we obtain that $\Delta(a) = a\Delta(1_A) = \Delta(1_A)a$ for any $a \in A$.

Example 4.3.8. Let us consider again the algebra N_3^2. In Example 4.3.2, all p_{i_k} for $1 \le i \le 3$ and $0 \le k \le 2$ were computed. By using Corollary 4.3.6, we compute all appropriate $p_{\nu^{-1}(j)_{2-k}}$ depending on p_{i_k} and obtain the following

$$\begin{aligned}\Delta(1_A) = &p_{1_0} \otimes p_{\nu^{-1}(1)_2} + p_{1_1} \otimes p_{\nu^{-1}(2)_1} + p_{1_2} \otimes p_{\nu^{-1}(3)_0} \\ &+ p_{2_0} \otimes p_{\nu^{-1}(2)_2} + p_{2_1} \otimes p_{\nu^{-1}(3)_1} + p_{2_2} \otimes p_{\nu^{-1}(1)_0} \\ &+ p_{3_0} \otimes p_{\nu^{-1}(3)_2} + p_{3_1} \otimes p_{\nu^{-1}(1)_1} + p_{3_2} \otimes p_{\nu^{-1}(2)_0}.\end{aligned}$$

Then by Example 4.3.2, we obtain that

$$\begin{aligned}\Delta(1_A) = &e_1 \otimes \alpha_2\alpha_3 + \alpha_1 \otimes \alpha_3 + \alpha_1\alpha_2 \otimes e_1 \\ &+ e_2 \otimes \alpha_3\alpha_1 + \alpha_2 \otimes \alpha_1 + \alpha_2\alpha_3 \otimes e_2 \\ &+ e_3 \otimes \alpha_1\alpha_2 + \alpha_3 \otimes \alpha_2 + \alpha_3\alpha_1 \otimes e_3.\end{aligned}$$

Since Δ is an A-bimodule morphism, for example,

$$\Delta(\alpha_1) = \Delta(1_A)\alpha_1 = \alpha_1 \otimes \alpha_3\alpha_1 + \alpha_1\alpha_2 \otimes \alpha_1.$$

Theorem 4.3.9. *Let $A = N_n^m$, ν_A be a Nakayama automorphism of A and ν be the Nakayama permutation of $\{1, ..., n\}$ induced by ν_A. Suppose that $\{e_1, \cdots, e_n\}$ is the set of idempotents, and for each e_i there exists a path x_i which is an element of soc$A$, with $s(x_i) = i$ and $t(x_i) = \nu(i)$ for all $i \in Q_0$. Then $\delta = x_1^* + \cdots + x_n^*$ is the counit of (A, Δ), where Δ is as given in Corollary 4.3.6.*

Proof. Here, A is a Frobenius algebra with the A-bimodule isomorphism $\tau : A_{\nu_A} \cong \mathrm{D}(A)$ which is explicitly defined by

$$\begin{aligned}\tau : e_i &\mapsto x^*_{\nu^{-1}(i)} \\ x_i &\mapsto e_i^* \\ p_{i_k} &\mapsto (p_{\nu^{-1}(j)_{m-k}})^* \text{ for } 1 \leq k \leq m-1,\end{aligned}$$

where $p_{i_k}p_{j_{m-k}} = x_i$, for all $i \in Q_0$ and for suitable choices of j which depends on k. See Proposition 4.3.3.

By taking into account Proposition 4.2.19 and Chapter 2, we say that the counit δ of (A, Δ) corresponds to the Frobenius form of A which is equal to $\tau(1_A)$. Therefore, we obtain that

$$\delta = \tau(1_A) = x_1^* + \cdots + x_n^*.$$

$\square$

Corollary 4.3.10. *Let $A = N_n^m$. Suppose that $\{e_1, \cdots, e_n\}$ is the set of idempotents, and for each e_i there exists a path x_i which is an element of soc$A$. Then the counit δ of (A, Δ), where Δ is as given in Corollary 4.3.6, is explicitly defined by*

$$\begin{aligned}\delta : x_i &\mapsto 1 \textit{ for all } i \in Q_0, \\ \textit{otherwise} &\mapsto 0.\end{aligned}$$

By Section 2.1.3, $A = kQ/I$ is a symmetric Frobenius Nakayama algebra if and only if $A = N_n^m$, with n dividing m. Moreover, since A is symmetric, we can take the identity automorphism as Nakayama automorphism of A. Then we obtain the following result by using Corollary 4.3.6.

Corollary 4.3.11. *Let $A = N_n^m$, where n divides m. Suppose that $\{e_1, \cdots, e_n\}$ is the set of idempotents, and for each e_i there exists a path x_i which is an element of soc$A$, with $s(x_i) = t(x_i) = i$ and $\ell(x_i) = m$ for all $i \in Q_0$. Then there exists a comultiplication Δ of A which is given for 1_A by*

$$\Delta(1_A) = \sum_{\substack{1 \leq i \leq n \\ 0 \leq k \leq m \\ p_{i_k} p_{j_{m-k}} = x_i}} p_{i_k} \otimes p_{j_{m-k}}$$

for suitable choices of j which depends on k.

Note that $p_{i_0} = e_i$ and $p_{i_m} = x_i$.

Bibliography

[1] Abrams, L., Two-dimensional topological quantum field theories and Frobenius algebras, J. Knot Theory Ramifications 5 (1996), 569-587.

[2] Abrams, L., Modules, Comodules and Cotensor Products over Frobenius Algebras. J. Algebra 219 (1999), 201-213.

[3] Abrams, L., Frobenius Algebra Structures in Topological Quantum Field Theory and Quantum Cohomology, Doctoral thesis, The Johns Hopkins University, 1997.

[4] Brauer, R., Nesbitt, C., On the regular representations of algebras. Proc. Nat. Acad. Sci. USA 23 (1937), 236–240.

[5] Concini, C.D., Procesi, C., A characteristic free approach to invariant theory, Adv. Math. 21 (1976), 330-354.

[6] Dipper, R., James, G., The q-Schur algebra, Proc. London Math. Soc. 59 (1989), 23-50.

[7] Donkin, S., The q-Schur Algebra, London Math. Soc. Lecture Notes, Vol.253, Cambridge University Press, 1998.

[8] Doty, S., Nakano, D.K., Semisimple Schur algebras, Math. Proc. Cambridge Philos. Soc. 124 (1998) 15–20.

[9] Du, J., A note on quantized Weyl reciprocity at roots of unity, Algebra Colloq. 2 (1995), 363-372.

[10] Du, J., Parshall, B., Scott, L., Quantum Weyl reciprocity and tilting modules, Comm. Math. Phys. 195 (1998), 321-352.

[11] Duncan, R., Dunne, K., Interacting Frobenius algebras are Hopf. Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science (2016), 535–544.

[12] Fang, M., Permanents, Doty coalgebras and dominant dimension of Schur algebras. Adv. Math. 264 (2014), 155–182.

[13] Fang, M. and Koenig, S., Gendo-symmetric Algebras, Canonical Comultiplication, Bar Cocomplex and Dominant Dimension. Trans. Amer. Math. Soc. 368 (2016), no. 7, 5037-5055.

[14] Fang, M. and Koenig, S., Endomorphism Algebras of Generators over Symmetric Algebras. J. Algebra 332 (2011), 428-433.

[15] Fang, M. and Koenig, S., Schur functors and dominant dimension, Trans. Amer. Math. Soc. 363 (2011) 1555-1576.

[16] Frobenius, G., Theorie der hyperkomplexen Grössen. Sitzungsber. K. Preuss. Akad. Wiss. 24 (1903), 504–537; 634–645.

[17] Green, J.A., Polynomial Representations of GL_n, Lecture Notes in Math., Vol.830, Springer-Verlag, New York, 1980.

[18] Iyama, O., Auslander Correspondence, Advances in Mathematics 210 (2007), 51-82.

[19] Jimbo, M., A q-analogue of $U(gl(N+1))$, Hecke algebra, and the Yang Baxter equation, Lett. Math. Phys. 11 (1986), 247-252.

[20] Kerner, O. and Yamagata, K., Morita Algebras. J. Algebra 382 (2013), 185-202.

[21] Kock, J., Frobenius Algebras and 2D Topological Quantum Field Theories, London Mathematical Society Student Texts, 59. Cambridge University Press, 2003.

[22] Koenig, S., Slungard, I.H, Xi, C.C., Double centralizer properties, dominant dimension and tilting modules, J. Algebra 240 (2001) 393–412.

[23] Lawvere, F.W., Ordinal sums and equational doctrines. In B. Eckmann, editor, Seminar on Triples and Categorical Homology Theory, ETH 1966/67. No. 80 in Lecture Notes in Mathematics, pp. 141–155. Springer-Verlag, New York, 1969.

[24] Morita, K., Duality for modules and its applications to the theory of rings with minimum condition, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A 6 (1958) 83–142.

[25] Murray, W., Nakayama automorphisms of Frobenius algebras, Journal of Algebra 269 (2003) 599–609.

[26] Nakayama, T., On Frobeniusean algebras I. Ann. of Math. 40 (1939), 611–633.

[27] Nakayama, T., On Frobeniusean algebras II. Ann. of Math. 42 (1941), 1–21.

[28] Quinn, F., Lectures on axiomatic topological quantum field theory. In Geometry and Quantum Field Theory (Park City, UT, 1991), pp. 323–453. Amer. Math. Soc., Providence, RI, 1995.

[29] Schur, I., Über die rationalen Darstellungen der allgemeinen linearen Gruppe, Sitzungsber. Akad. Berlin (1927), 58–75. Reprinted as pp. 68–85 in Gesammelte Abhandlungen, III, Springer, Berlin, 1973.

[30] Skowroński, A., Periodicity in representation theory of algebras, ICTP Lecture Notes 2006 available online at http://webusers.imj-prg.fr/bernhard.keller/ictp2006/lecturenotes/skowronski.pdf.

[31] Skowroński, A., Yamagata, K., Frobenius Algebras I, EMS Textbk. Math., European Mathematical Society, Zürich, 2011.

[32] Wen, D.W., Self-injective algebras with an indecomposable quasi-heredity complement, Comm. Alg., 30 (9) (2002), pp. 4417-4423.

[33] Xi, C.C., The Structure of Schur Algebras $S_k(n,p)$ for $n \geq p$, Can. J. Math.Vol. 44 (3), 1992 pp. 665-672.

[34] Yamagata, K., Frobenius Algebras, in: Handbook of Algebra I, Elsevier, North-Holland, Amsterdam, 1996, pp. 841-887.

[35] Zimmermann, A., Representation Theory, A Homological Algebra Point of View, Springer, International Publishing Switzerland, 2014.