Virtual Levels of Multi-Particle Quantum Systems and Their Implications for the Efimov Effect

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Abstract

In this thesis we investigate Schrödinger operators corresponding to N-particle quantum systems in dimension $d \geq 3$. First, we study the lower spectral threshold of the essential spectrum of the operator. We assume that it coincides with the half-line $[0, \infty)$ and we consider the case that the system is in a "critical state", i.e. where a negative eigenvalue of the operator is created as soon as an additional arbitrarily small part of the potential is added. In this case there is a solution of the Schrödinger equation corresponding to the spectral threshold zero, which is either an eigenvalue or a resonance of the operator. We are concerned with questions regarding the behaviour of such solutions at infinity and we provide estimates on their corresponding decay rates for a class of long- and short-range potentials. They depend on the underlying dimension, the number of quantum particles and their respective masses. Furthermore, we show that the obtained estimates are optimal by providing the concrete asymptotic behavior of the solutions in the case of short-range pair interactions.

Based on these results we then investigate the discrete spectrum of multi-particle Schrödinger operators with regard to the Efimov effect. In case of $d \geq 3$ and $N \geq 3$ or $d \geq 5$ and $N \geq 2$ the solutions described above are eigenfunctions corresponding to the eigenvalue zero. For such systems we prove by variational methods that the Schrödinger operator corresponding to the $(N+1)$-body system has only a finite number of negative eigenvalues. The case of three particles in dimension four is fundamentally different, because in this case the two-body subsystems can have zero-energy resonances. For this reason, we choose a different approach based on the method of the Faddeev equations to prove that the Efimov effect cannot exist for three-body systems in dimension four.

Zusammenfassung

In dieser Dissertation untersuchen wir Schrödingeroperatoren, die Systeme von N Quantenteilchen in Dimension $d \geq 3$ beschreiben. Zunächst beschäftigen wir uns mit der unteren Spektralkante des essentiellen Spektrums des Operators. Hierbei nehmen wir an, dass es mit der Halbgeraden $[0, \infty)$ zusammenfällt und betrachten den Fall, dass das System sich im "kritischen Zustand" befindet, d.h. bei dem der Operator stets einen negativen Eigenwert generiert, sobald ein noch so kleiner Teil des Potentials hinzuaddiert wird. In diesem Fall existiert eine Lösung der Schrödingergleichung zu der Spektralkante Null, die sowohl ein Eigenwert, als auch eine Resonanz des Operators sein kann. Wir untersuchen das Verhalten der entsprechenden Lösung im Unendlichen und liefern für eine Klasse von lang- und kurzreichweitigen Potentialen Abschätzungen für die entsprechenden Abfallraten. Diese hängen von der zugrundeliegenden Dimension, der Anzahl der Quantenteilchen und ihren jeweiligen Massen ab. Wir zeigen außerdem, dass die Abschätzungen scharf sind, indem wir das konkrete asymptotische Verhalten von solchen Lösungen im Fall von kurzreichweitigen Wechselwirkungen beweisen.

Darauf aufbauend untersuchen wir anschließend im Hinblick auf den Efimov-Effekt das diskrete Spektrum von Mehrteilchen-Schrödingeroperatoren. Im Fall von $d \geq 3$ und $N \geq 3$ oder $d \geq 5$ und $N \geq 2$ handelt es sich bei den oben beschriebenen Lösungen um Eigenfunktionen zum Eigenwert Null. Für solche Systeme zeigen wir mit variationellen Methoden, dass der zum $(N+1)$-Teilchensystem gehörende Operator nur eine endliche Anzahl an negativen Eigenwerten besitzen kann. Der Fall von drei Teilchen in Dimension vier ist grundlegend anders, da hier die Teilsysteme mit zwei Teilchen Resonanzen in der Null haben können. Aus diesem Grund wählen wir hierzu eine auf den Faddeev-Gleichungen basierende Methode, um zu zeigen, dass es für Systeme bestehend aus drei Teilchen keinen Efimov-Effekt geben kann.

Contents

1 Introduction

The theory of the Schrödinger equation has grown into a large area in both physics and mathematics. Especially in quantum mechanics, the existence of eigenvalues and the behaviour of the corresponding eigenfunctions of Schrödinger operators have been studied for many years. It is well known, that eigenfunctions of the Schrödinger operator corresponding to eigenvalues below its essential spectrum have an exponential decay at infinity [Agm82]. At the edge of the essential spectrum, however, the situation is quite different. Consider the Schrödinger operator

$$h(\lambda) = -\Delta + \lambda V \tag{1.0.1}$$

acting in $L^2(\mathbb{R}^d)$ with a coupling constant $\lambda > 0$ and where the real valued potential V satisfies $V(x) \to 0$ as $|x| \to \infty$. It is known that for such potentials the essential spectrum of $h(\lambda)$ is given by $\sigma_{\text{ess}}(h(\lambda)) = [0, \infty)$. Assume that $E(\lambda) < 0$ is an eigenvalue of $h(\lambda)$, which for some fixed $\lambda_0 > 0$ satisfies $E(\lambda) \to 0$ as $\lambda \searrow \lambda_0$. In other words, the eigenvalue is absorbed into the continuous spectrum. On the other hand, if $\lambda \nearrow \lambda_0 + \varepsilon$ for some $\varepsilon > 0$, then a negative eigenvalue is created from the essential spectrum. This situation is known as the coupling constant threshold, see [KS80a]. In literature, the spectral threshold $E(\lambda_0) = 0$ is also known as the virtual level of the Schrödinger operator, see for example [Yaf75]. The term virtual level is to be understood here as a generic term, since it can be both an eigenvalue and a zero-energy resonance of the operator. Roughly speaking, zero is a resonance if there exists a solution $\psi : \mathbb{R}^d \to \mathbb{R}$ of the Schrödinger equation

$$-\Delta\psi + V\psi = 0, \tag{1.0.2}$$

which for $d \geq 3$ decays in some sense to zero as $|x| \to \infty$ but is not an element of $L^2(\mathbb{R}^d)$. There is a large body of literature in both physics and mathematics studying zero-energy resonances of Schrödinger operators in various contexts, e.g. [Pin88, Yaf75, Wei99, JK79, Jen84, Jen80].

For example, in dimension $d = 3$ it is known that for non-positive short-range potentials without negative eigenvalues of $h(\lambda_0)$ and without symmetry restrictions of the domain of the operator the virtual level is a resonance, where the corresponding solution ψ satisfies

$$\psi(x) \sim c|x|^{-1} \qquad \text{as} \qquad |x| \to \infty \tag{1.0.3}$$

and the constant c is proportional to the integral over the function $V\psi$, see [Yaf00]. However, with certain symmetry restrictions the virtual level can be a zero-energy eigenvalue of the operator, which is precisely when $c = 0$, see for example [KS80a]. Furthermore, in case of long-range potentials even sub-exponential decay of the solution can occur [HOHOS83, GG07, HJL19b].

The decay rate of the solution corresponding to the zero-energy resonance in (1.0.3) is the same as that of the fundamental solution of the Laplace operator, which for dimension $d \geq 3$ is given by $G(x) = c_d|x|^{2-d}$. This is due to the structure of the Poisson's equation (1.0.2), which under certain restrictions on the potential V in dimension $d = 3$ can be rewritten as

$$\psi(x) = \left(G * (V\psi)\right)(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{V(y)\psi(y)}{|x-y|}\,\mathrm{d}y. \tag{1.0.4}$$

Therefore, based on the behaviour of the fundamental solution of the Laplace operator, it is evident that for short-range potentials virtual levels in dimension $d = 4$ are resonances, where the corresponding solution is on the edge of being square integrable. In dimension $d \geq 5$, however, virtual levels must be zero-energy eigenvalues. The approach of Green's function formalism in order to study the behaviour of such solutions can be found for various individual cases in a wide range of literature, see for example [GG07], [Mur86] and the references contained therein.

In quantum mechanics, neglecting certain constants, the operator (1.0.1) determines the evolution of states of one particle in a potential or the relative motion of two interacting particles. In general, a system of N quantum particles of masses

$m_1, \dots, m_N > 0$ and position vectors $x_1, \dots x_N \in \mathbb{R}^d$ is described by the N-body Schrödinger operator

$$H_N = -\sum_{i=1}^{N} \frac{1}{2m_i} \Delta_{x_i} + \sum_{1 \le i < j \le N} V_{ij}(x_i - x_j) \tag{1.0.5}$$

acting on $L^2(\mathbb{R}^{dN})$, where V_{ij} are the corresponding pair interactions. Spectral properties of such operators differ fundamentally from the one- and two-particle operators described above. Especially the existence of virtual levels, and in particular the asymptotic behaviour of the corresponding solutions, lead to a number of non-intuitive phenomena, one of which is the so-called Efimov effect. It was originally discovered by the physicist V. Efimov in the 1970's [Efi70], which can be described as follows: The three-body Schrödinger operator of three-dimensional particles interacting via short-range potentials has an infinite number of negative eigenvalues, if the Hamiltonians of the two-body subsystems have no negative eigenvalues and at least two of them have a zero-energy resonance. It is surprising because although the interactions are short-range, the three-body system behaves as if there is a long-range attraction.

The first rigorous mathematical proof was provided by D. R. Yafaev [Yaf74], which is based on the study of the known Faddeev equations for three-body systems together with the low-energy behaviour of the resolvents corresponding to the two-particle Schrödinger operators. The proof consists of three major steps:

(i) Applying the Birman-Schwinger principle to characterize the number of eigenvalues of the three-particle Schrödinger operator.

(ii) For the proof of the infiniteness of the discrete spectrum, any compact perturbation of the kernel can be neglected.

(iii) Analysing the spectral asymptotics of the operator in step **(ii)**, which does not depend on the particular form of the pair potentials.

Years later, A. Sobolev completed the proof of D. R. Yafaev by providing the low energy asymptotics of the counting function of the three-body Schrödinger operator [Sob93]. Namely, by the use of the behaviour of the resolvents of the corresponding two-body Hamiltonians [JK79] together with the calculation of the distribution of a Toeplitz operator it was proved that the number of eigenvalues of the three-body Hamiltonian increases logarithmically as soon as one approaches the spectral threshold zero. Precisely, the counting function $N(z)$ of eigenvalues that are less than $z < 0$ admits the asymptotics

$$\lim_{z \to 0^-} \frac{N(z)}{|\ln |z||} = \mathcal{U}_0 > 0. \tag{1.0.6}$$

The constant $\mathcal{U}_0$ depends only on the mass rations of the particles and not on the pair potentials, which also underlines the universality of the Efimov effect.

This phenomenon, originally coming from physics, has inspired many other mathematical results in this field. A variational approach was presented by Y. Ovchinnikov and I. Sigal [OS79], where the authors used the technique of the Born-Oppenheimer approximation to prove the Efimov effect. Later, H. Tamura further developed the method and provided the proof under more general conditions on the potentials, see the works [Tam91, Tam93]. An interesting question from a mathematical point of view is under which circumstances the effect no longer occurs and which conditions can be weakened. In the work [Yaf76] D. R. Yafaev has shown that the three-particle Schrödinger operator can only have a finite discrete spectrum if the two-particle subsystems have negative eigenvalues or if at least two of them have no resonances. In this regard see also the work [Zhi74] of G. M. Zhislin. In the work [VZ83] S. A. Vugalter and G. M. Zhislin considered systems of thee particles restricted to certain symmetry subspaces and proved that the corresponding Hamiltonian has only a finite number of negative eigenvalues. The key argument in their work is that virtual levels of the two-body operators for such systems are eigenvalues at the threshold of the essential spectrum. It is the first result which shows that zero-energy eigenfunctions in the subsystems cannot produce the Efimov effect. Further results regarding the absence of the Efimov effect are for example [Vug96, Vug98, VZ84, VZ83].

Despite the mathematical results concerning the existence or non-existence of the Efimov effect, from a physical point of view it was not clear for about 35 years whether the Efimov effect could be observed experimentally. It had become a remarkable challenge to do so. In 2006, quantum states were observed in an ultracold gas of caesium atoms, indicating that they are related to the Efimov effect

[KMW+06]. Another experimental observation of the effect, for example, was made in bosonic quantum gases [ZDD+09]. As a result, research interest in this field has increased considerably in the last years. For an overview with more references and also further details on the physical aspect of the Efimov effect we refer to [NE17].

In the year 2013 the physicists Y. Nishida, S. Moroz and D. T. Son discovered the so-called super Efimov effect [NMT13]. They showed that in case of three nonrelativistic spinless fermions in dimension two, where every two-body subsystem has a resonance at zero, the three-body system has an infinite number of negative bound states. Moreover, in this case the counting function $N(z)$ of negative eigenvalues less than $z < 0$ satisfies

$$\lim_{z \to 0^-} \frac{N(z)}{|\ln|\ln|z|||} = \frac{8}{3\pi}. \tag{1.0.7}$$

The first mathematical proof of this was later provided by D. K. Gridnev [Gri14] using techniques similar to [Yaf74] and [Sob93]. Comparing this result with the result [VZ83] of S. A. Vugalter and G. M. Zhislin mentioned above shows that the same systems in different dimensions lead to completely different structures of the discrete spectra of the corresponding Schrödinger operators. The main reason for this is the sensitive behaviour of the virtual levels corresponding to the Schrödinger operators of the subsystems. In view of the result of S. A. Vugalter and G. M. Zhislin one could assume that the Efimov effect depends on whether the threshold energy of the subsystems is a resonance or an eigenvalue only. As already mentioned above, in dimension four the two-body virtual level is a resonance and the corresponding solution behaves like $c|x|^{-2}$ as $|x| \to \infty$. In physical literature, however, it is not expected that an Efimov-type effect can possibly occur in case of three quantum particles in dimension four [NT11]. Furthermore, in 2017 Y. Nishida has shown with a series of physical arguments that an Efimov-type effect is possible for four two-dimensional bosons [Nis17], if the three-body subsystems are at zero-energy resonance. Here it should be noted that the underlying configuration space of the three-body system of two-dimensional particles has dimension four as well. This indicates that the characteristics of virtual levels in multi-particle systems are different. However, in contrast to the cases $N = 1$ and $N = 2$, the study of virtual levels of Schr̈dinger operators corresponding to $N \geq 3$ particles is much more complicated.

Consider the N-body Schrödinger operator after separation of the center of mass, i.e.

$$H(\lambda) = H_0 + \lambda \sum_{1 \le i < j \le N} V_{ij} \tag{1.0.8}$$

with a coupling constant $\lambda > 0$. Here H_0 is the operator of the kinetic energy of relative motion and V_{ij} are the pair interactions of the particles. Suppose that for some critical $\lambda_0 > 0$ the operator $H(\lambda)$ has a bound state $\psi(\lambda)$ with a bound state energy $E(\lambda) < \inf \sigma_{\text{ess}}(H(\lambda))$ for all λ close to λ_0. Let $E(\lambda) \to \inf \sigma_{\text{ess}}(H(\lambda_0))$ as $\lambda \to \lambda_0$. Whether $E(\lambda_0)$ is a bound state energy of $H(\lambda_0)$ and how the corresponding function $\psi(\lambda_0)$ behaves has been considered in the literature for many individual cases, e.g. [KS80b, BFLS14, Kar87, HJL19a, HOHOS83, GG07, BGS85, Gri12a, Gri12b].

With regard to the Efimov effect the case $\inf \sigma_{\text{ess}}(H(\lambda)) = 0$ is of most interest, since according to the HVZ theorem in case of negative eigenvalues in the subsystems, the bottom of the essential spectrum of the operator corresponding to the whole system coincides with the lowest eigenvalue of the subsystems. In 2011 D. K. Gridnev proved under the assumption $\sigma_{\text{ess}}(H(\lambda)) = [0, \infty)$ that for systems of $N = 3$ particles in dimension $d = 3$ virtual levels are zero-energy eigenvalues, provided the two-body subsystems do not have a zero-energy resonance and the pair interactions V_{ij} are non-positive and belong to $L^1(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$, see [Gri12a]. Later the result was extended to systems consisting of $N \ge 4$ particles in dimension $d = 3$, where in addition the assumption of non-positivity of the potentials was dropped [Gri12b]. The proof is based on the analysis of integral equations corresponding to the zero-energy solution of the Schrödinger equation.

However, the study of the behaviour of the corresponding eigenfunctions faces a number of difficulties compared to the cases of one- and two-particle systems. The heuristic approach with the Green's function formalism described above in the case of a regular Schrödinger equation (1.0.4) might suggest that the eigenfunction could have similar behaviour as the fundamental solution of the Laplace operator in the configuration space of the system. In case of $N \ge 3$ particles in dimension $d = 3$ it decays as $C|x|^{-3(N-5)}$ for $|x| \to \infty$. However, even if one assumes that every potential V_{ij} is compactly supported, the sum $V = \sum V_{ij}$ does not necessarily have to tend to zero at infinity, which considerably complicates the implementation of the method of such integral equations. In addition, the kinetic energy operator

H_0 in (1.0.8), which also depends on the masses of the respective particles, must have an effect on the behaviour of the corresponding solutions as well. Furthermore, in case of Coulomb-potentials, i.e. potentials of the form $c|x|^{-1}$, and some other long-range interactions, it is known that such threshold eigenvalues might have a sub-exponential decay, see [HOHOS83, HJL19a, GG07]. The question is how the corresponding solutions behave for different classes of potentials in different dimensions and how their decay rates depend on the respective quantum particles and their masses.

As described above this also has a fundamental impact on the existence and non-existence of the Efimov effect for such systems. Although there are still many open problems, both from a mathematical and a physical point of view regarding when an effect similar to the Efimov effect exists for multi-particle systems, the two physicists R. D. Amado and F. C. Greenwood made the following claim in 1976 [AG73]: Systems consisting of $N \geq 4$ bosons in dimension three cannot produce the Efimov effect, assuming that only the $(N-1)$-body subsystems have virtual levels. In 2013, D. K. Gridnev provided the first mathematical proof of this claim, see [Gri13].

1.1 Summary of the main results

The results of this thesis are based upon the following three articles:

(i) S. Barth and A. Bitter. *On the virtual level of two-body interactions and applications to three-body systems in higher dimensions.*
Journal of Mathematical Physics, 60 (11):113504, 2019.

(ii) S. Barth and A. Bitter. *Decay rates of bound states at the spectral threshold of multi-particle Schrödinger operators.*
Doc. Math., 25:721-735, 2020.

(iii) S. Barth, A. Bitter, and S. Vugalter. *Decay properties of zero-energy resonances of multi-particle Schrödinger operators and why the Efimov effect does not exist for systems of $N \geq 4$ particles.*
arXiv: 1910.04139, 2020.

This work consists of two main parts. The first part deals with the existence and the behaviour of solutions corresponding to virtual levels of one-body and many-body Schrödinger operators. In the second part the results from the first part are applied to prove several statements regarding the Efimov effect.

At first we study virtual levels of general Schrödinger operators at the threshold zero. We assume that the potentials are relatively form-bounded with relative bound zero, i.e. we consider real-valued functions $V : \mathbb{R}^d \to \mathbb{R}$ in dimension $d \geq 3$, such that for every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ with

$$\langle |V|\psi, \psi\rangle \leq \varepsilon \|\nabla\psi\|^2 + C(\varepsilon)\|\psi\|^2 \qquad \text{for any} \quad \psi \in H^1(\mathbb{R}^d). \tag{1.1.1}$$

This will allow us to overcome the difficulty that the sum of potentials in the multi-particle case does not converge to zero at infinity. Furthermore, it also provides a variational approach to the investigation of virtual levels of one-body Schrödinger operators

$$h = -\Delta + V \qquad \text{in} \quad L^2(\mathbb{R}^d). \tag{1.1.2}$$

We consider the case where

$$h \geq 0 \qquad \text{and} \qquad \inf \sigma(-(1-\varepsilon)\Delta + V) < 0 \tag{1.1.3}$$

is satisfied for any $\varepsilon \in (0, 1)$ and where

$$\langle h\psi, \psi\rangle - \gamma_0\|\nabla\psi\|^2 - \alpha_0^2\langle |x|^{-\beta_0}\psi, \psi\rangle \geq 0 \tag{1.1.4}$$

holds for some constants $\alpha_0, \gamma_0 > 0$, $\beta_0 \in (0, 2]$ and all functions ψ supported outside a ball of a fixed radius. We prove that in this case there exists a unique zero-energy solution $\varphi_0 : \mathbb{R}^d \to \mathbb{R}$ satisfying

$$\|\nabla\varphi_0\|^2 + \langle V\varphi_0, \varphi_0\rangle = 0. \tag{1.1.5}$$

We show that the decay rate of φ_0 depends on the constants α_0 and β_0 in (1.1.4). In the case $\beta_0 = 2$ the solution φ_0 satisfies

$$\nabla\left(|\cdot|^{\alpha_0}\varphi_0\right) \in L^2(\mathbb{R}^d) \qquad \text{and} \qquad (1 + |\cdot|)^{\alpha_0 - 1}\varphi_0 \in L^2(\mathbb{R}^d). \tag{1.1.6}$$

In the case $\beta \in (0,2)$ we prove that φ_0 then satisfies

$$\exp\left(\alpha_0 \kappa^{-1} |\cdot|^{\kappa}\right) \varphi_0 \in L^2(\mathbb{R}^d), \qquad \text{where} \quad \kappa = 1 - \frac{\beta_0}{2}. \tag{1.1.7}$$

As a simple conclusion we apply the result to different classes of potentials and determine in connection with the Hardy constant $C_d = \frac{(d-2)^2}{4}$ in dimension $d \geq 3$ critical cases when it is a resonance and when it is an eigenvalue. Since the method is purely variational, we allow potentials with local singularities.

The main applications of this result are virtual levels of multi-particle Schrödinger operators of the form (1.0.5) corresponding to $N \geq 3$ quantum particles in dimension $d \geq 3$. By introducing the space

$$R_0 = \left\{ x = (x_1 \ldots, x_N) \in \mathbb{R}^{dN} : \sum_{i=1}^{N} m_i x_i = 0 \right\} \tag{1.1.8}$$

and the scalar product

$$\langle x, y \rangle_1 = \sum_{i=1}^{N} 2m_i \langle x_i, y_i \rangle, \qquad |x|_1 = \sqrt{\langle x, x \rangle}, \tag{1.1.9}$$

the Hamiltonian of the system after separation of the center of mass is given by

$$H = -\Delta_0 + V, \tag{1.1.10}$$

where $V = \sum_{1 \leq i < j \leq N} V_{ij}$ and $-\Delta_0$ is the Laplace-Beltrami operator on R_0 with respect to the metric $\langle \cdot, \cdot \rangle_1$, cf. [SS70]. We consider potentials of the form

$$V_{ij} = V_{ij}^{(1)} + V_{ij}^{(2)}, \tag{1.1.11}$$

where $V_{ij}^{(1)}$ are short-ranged, decaying as $c|x|^{-2-\nu}$ for some $\nu > 0$ and are allowed to have local singularities. The functions $V_{ij}^{(2)}$ are assumed to be non-negative, bounded and tend to zero as $|x| \to \infty$. We prove that in case of $H \geq 0$ and

$$\sigma_{\text{ess}}\left(-(1-\varepsilon)\Delta_0 + V\right) = [0, \infty) \quad \text{and} \quad \sigma_{\text{disc}}\left(-(1-\varepsilon)\Delta_0 + V\right) \neq \emptyset \tag{1.1.12}$$

for any sufficiently small $\varepsilon > 0$ zero is an eigenvalue of H. We show that a corresponding eigenfunction φ_0 satisfies

$$\nabla_0 \left(|\cdot|_1^{\alpha_0} \varphi_0 \right) \in L^2(R_0) \qquad \text{and} \qquad (1 + |\cdot|_1)^{\alpha_0 - 1} \varphi_0 \in L^2(R_0) \tag{1.1.13}$$

for any $0 \leq \alpha_0 < \frac{d(N-1)-2}{2}$. Furthermore, if $V_{ij}^{(2)}(x_{ij}) \geq \alpha_{ij} |x_{ij}|^{-\beta}$ holds for some constants $\alpha_{ij} > 0$ and $\beta \in (0, 2)$, then φ_0 decays sub-exponentially with

$$\exp\left(\mu |\cdot|_1^{\kappa}\right) \varphi_0 \in L^2(R_0), \qquad \text{where} \quad \kappa = 1 - \frac{\beta}{2} \tag{1.1.14}$$

and $\mu > 0$ depends on the coefficients α_{ij} and on the masses of the particles only.

The obtained estimate on the decay rate of φ_0 in (1.1.13) is close to optimal. We prove this by using

$$\nabla_0 \left(|\cdot|_1^{\alpha_0} \varphi_0 \right) \in L^2(R_0) \tag{1.1.15}$$

as an a-priori estimate, which then allows us to obtain the asymptotic behaviour of φ_0 by studying its integral representation corresponding to the zero-energy eigenvalue equation. Precisely, we prove that in case of short-range potentials V_{ij} decaying as

$$|V_{ij}(x)| \leq C|x|^{-2-\nu} \qquad \text{for} \quad |x| \geq A \tag{1.1.16}$$

with constants $\nu > 0$ and $A, C > 0$, the solution φ_0 satisfies

$$\varphi_0(x) = \frac{C_0}{|x|_1^{\beta}} + g(x) \quad \text{as} \quad |x|_1 \to \infty, \tag{1.1.17}$$

where $\beta = d(N-1) - 2$ and the remainder g belongs to $L^p(R_0)$ for any p satisfying

$$\frac{\beta + 2}{\beta + \frac{\gamma^*}{1+\gamma^*}} < p < \frac{\beta + 2}{\beta} \qquad \text{with} \quad \gamma^* = \min\left\{ \frac{d}{2} - 1, \nu \right\}. \tag{1.1.18}$$

Furthermore, the constant C_0 is given by

$$C_0 = -\frac{1}{(\beta - 2)|\mathbb{S}^{\beta-1}|} \int\limits_{R_0} \sum_{1 \leq i < j \leq N} V_{ij} \varphi_0 \, \mathrm{d}x, \tag{1.1.19}$$

where $|\mathbb{S}^{\beta-1}|$ is the volume of the unit sphere in $\mathbb{R}^\beta$. We discuss the importance of the constant C_0 by providing examples of systems where the constant is always non-zero and where it vanishes.

In the second part of the thesis we apply the obtained results to prove statements about the discrete spectrum of multi-particle Schrödinger operators. We show that the Efimov effect does not occur for systems of $N \geq 4$ particles in dimension $d \geq 3$ by giving a purely variational proof, which is based on the method developed by S. A. Vugalter and G. A. Zhsilin in [VZ83]. The key argument in the proof is that virtual levels of Hamiltonians corresponding to $(N-1)$-body subsystems satisfying (1.1.12) are eigenvalues. This allows us to prove that for the corresponding N-body system there exist constants $\varepsilon > 0$ and $b > 0$, such that

$$\|\nabla_0\varphi\|^2 + \sum_{1\leq i<j\leq N} \langle V_{ij}\varphi, \varphi\rangle - \varepsilon\| |x|_1^{-1}\varphi\|^2 \geq 0 \tag{1.1.20}$$

holds for all $\varphi \in H^1(R_0)$ with $\operatorname{supp}\varphi \subset \{x \in R_0,\ |x|_1 \geq b\}$. This leads to the finiteness of the discrete spectrum of the N-body Hamiltonian. The approach is fundamentally different from the method developed in [Gri13] and generalizes it in the sense that we consider a larger class of potentials and go beyond dimension three. Furthermore, we apply the result to systems with a fixed permutation symmetry.

In case of three particles in dimension four virtual levels of the two-body subsystems are resonances and as already mentioned the corresponding zero-energy solutions decay like $c|x|^{-2}$ as $|x| \to \infty$, which is the boundary between square integrable and non-square integrable. The approach described above is no longer suitable in this case because it relies on certain estimates regarding the localization error, which do not hold here. We prove the finiteness of the discrete spectrum of three-body Schrödinger operators with short-range potentials in dimension four by adapting the techniques of A. Sobolev [Sob93] and D. R. Yafaev [Yaf74] to this case, which involves the investigation of the kernel of a Birman-Schwinger type operator. Since this method was originally used to prove the Efimov effect for three particles in dimension three, it also allows us to compare both systems and provide a precise reason why in this case resonances in every two-body subsystem do not lead to the Efimov effect.

1.2 Outline of the thesis

The thesis is structured as follows.

In Chapter 2 we summarize basic concepts of Schrödinger operators corresponding to multi-particle quantum systems and introduce the notation we need for the following chapters.

In Chapter 3 we study the concept of virtual levels of Schrödinger operators with different classes of potentials in the case where the bottom of the essential spectrum of the operator is zero. We prove the existence of solutions corresponding to the zero-energy eigenvalue equation and give their decay rates at infinity. Furthermore, we extend the results to systems of a fixed permutation symmetry. We also provide the asymptotic behaviour of these solutions in dependence of the number of particles, their masses and the corresponding dimension.

In Chapter 4 we apply the results from Chapter 3 to prove statements about the discrete spectrum of multi-particle Schrödinger operators in terms of the Efimov effect, starting with the special case of three particles in dimension four and then moving on to $N \geq 4$ particles in dimension $d \geq 3$.

2 Basic Concepts of Multi-Particle Quantum Systems

As preparation for the study of multi-particle systems in the next chapters we briefly introduce the concept of Schrödinger operators and summarize properties that will be important for the later course of the thesis. The contents in this chapter are based on the presentations given in [Sim71], [RS75], [Dav95], [CFKS87] and [GS11]. We start with the case of one particle.

2.1 One-particle Schrödinger operators

Schrödinger operators corresponding to one particle in a potential acting on a subspace of $L^2(\mathbb{R}^d)$ are formally given by

$$(hf)(x) = -\Delta f(x) + V(x)f(x), \tag{2.1.1}$$

where $-\Delta$ is the Laplacian on $\mathbb{R}^d$ and $V : \mathbb{R}^d \to \mathbb{R}$ is a real-valued function, called the potential of the Hamiltonian h. The total energy of the system is determined by the quadratic form $f \mapsto \langle hf, f\rangle$, which is split into the kinetic energy $\langle -\Delta f, f\rangle$ and the potential energy $\langle V f, f\rangle$. It is therefore more accessible to work with the quadratic form of the operator h.

2.1.1 The quadratic form of a Schrödinger operator

Throughout the following we work in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$. We denote by $D(A)$ the domain of an operator A and by $Q(A)$ the form domain of the corresponding quadratic form of A. The following theorem is known as the KLMN theorem, where the letters stand for Kato, Lions, Lax-Milgram and Nelson.

Theorem 2.1.1 (cf. [Sim71, Theorem 2]). *Let A be a positive self-adjoint operator and suppose that β is a symmetric bilinear form with the corresponding form domain $Q(\beta) \supset Q(A)$ so that for some $\tau < 1$ and $b \in \mathbb{R}$,*

$$|\beta(x,x)| \leq \tau \langle Ax, y \rangle + b\|x\|^2 \tag{2.1.2}$$

for all $x \in Q(A)$. Then the quadratic form

$$x \mapsto \langle Ax, x \rangle + \beta(x,x) \tag{2.1.3}$$

defined on $Q(A) \cap Q(\beta) = Q(A)$ is the form of a self-adjoint operator, which is bounded below.

The operator version of Theorem 2.1.1 is the Kato-Rellich theorem.

Theorem 2.1.2 (cf. [Sim71, Theorem 1]). *Suppose that A is a self-adjoint operator and B is symmetric with domain $D(B) \supset D(A)$, such that for $\tau < 1$ and $b \in \mathbb{R}$,*

$$\|Bx\| \leq \tau \|Ax\| + b\|x\| \tag{2.1.4}$$

for all $x \in D(A)$. Then $A + B$ is a self-adjoint operator on $D(A)$ and essentialy self-adjoint on any core of A.

Let h_0 be the closure of

$$-\Delta : C_0^\infty(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \qquad (-\Delta f)(x) = -\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} f(x) \tag{2.1.5}$$

and denote by q_0 the quadratic form of h_0 with the corresponding form domain $Q(q_0) = H^1(\mathbb{R}^d)$. For all $\psi \in H^1(\mathbb{R}^d)$ it is given by

$$q_0[\psi] = \int_{\mathbb{R}^d} |\nabla \psi(x)|^2 \, \mathrm{d}x. \tag{2.1.6}$$

According to the KLMN-Theorem 2.1.1 for any potential $V : \mathbb{R}^d \to \mathbb{R}$ satisfying

$$\langle |V|\psi, \psi\rangle \leq \tau\|\nabla\psi\|^2 + b\|\psi\| \qquad \text{for every} \quad \psi \in H^1(\mathbb{R}^d) \tag{2.1.7}$$

with $\tau < 1$ and $b \in \mathbb{R}$ we obtain a self-adjoint operator corresponding to the quadratic form q with form domain $H^1(\mathbb{R}^d)$ and

$$q[\psi] = \|\nabla\psi\|^2 + \langle V\psi, \psi\rangle, \qquad \psi \in H^1(\mathbb{R}^d). \tag{2.1.8}$$

For convenience, we denote this operator by

$$h = -\Delta + V \tag{2.1.9}$$

and its domain by

$$D(h) = \{\psi \in H^1(\mathbb{R}^d) : -\Delta\psi + V\psi \in L^2(\mathbb{R}^d)\}. \tag{2.1.10}$$

It should be noted that (2.1.9) is a form sum and not an operator sum. Moreover, for $\psi \in H^1(\mathbb{R})$ the expression $-\Delta\psi + V\psi \in L^2(\mathbb{R}^d)$ is to be understood in the distributional sense, which means that there exists a function $f \in L^2(\mathbb{R}^d)$, such that

$$\int_{\mathbb{R}^d} (\nabla\psi \cdot \nabla\varphi + \psi\varphi V)\, \mathrm{d}x = \int_{\mathbb{R}^d} f\varphi \,\mathrm{d}x \tag{2.1.11}$$

is satisfied for all $\varphi \in C_0^\infty(\mathbb{R}^d)$. However, if we assume that the potential V is $-\Delta$-bounded with relative bound $\tau < 1$, i.e., when

$$\|V\psi\| \leq \tau\|\Delta\psi\| + b\|\psi\|, \qquad \psi \in H^2(\mathbb{R}^d) \tag{2.1.12}$$

is satisfied, then by the Kato-Rellich-Theorem 2.1.2 the operator h is self-adjoint on

$$D(h) = D(-\Delta) = H^2(\mathbb{R}^d). \tag{2.1.13}$$

In the following we denote by $\sigma(h)$, $\sigma_{\text{ess}}(h)$ and $\sigma_{\text{disc}}(h)$ the spectrum, the essential spectrum and the discrete spectrum of h, respectively.

In this thesis we study the lower edge of the essential spectrum of the Schrödinger operator h. Assume there exists a minimizer $\varphi_0 \in H^1(\mathbb{R}^d)$ of the quadratic form q, which means that φ_0 satisfies

$$\inf_{\psi \in H^1(\mathbb{R}^d),\ \|\psi\|=1} q[\psi] = q[\varphi_0] > -\infty. \tag{2.1.14}$$

Then by the variational principle φ_0 is an eigenfunction of h corresponding to the lowest eigenvalue $\lambda = q[\varphi_0] = \inf \sigma(h)$, e.g. [LL01, Theorem 11.8]. Such an eigenvalue of h is called the ground state energy and the corresponding eigenfunction φ_0 is called the ground state, which represents the configuration of the system with the lowest total energy. The concept of zero-energy resonances and resonance states of Schrödinger operators mentioned in the introduction is closely related to this, as these are minimizers of q in the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^d)$, defined as the closure of $C_0^\infty(\mathbb{R}^d)$ with respect to the gradient-norm

$$\left(\int_{\mathbb{R}^d} |\nabla(\cdot)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}, \qquad d \geq 3. \tag{2.1.15}$$

However, here the conditions on the potential must be adjusted accordingly so that the quadratic form q is well defined on $\dot{H}^1(\mathbb{R}^d)$. This will be discussed in the next chapter. Note that the assumption $d \geq 3$ in (2.1.15) is essential. Furthermore, in this case every $\psi \in \dot{H}^1(\mathbb{R}^d)$ satisfies the well known Hardy inequality [FLW]

$$\int_{\mathbb{R}^d} |\nabla \psi(x)|^2 \, \mathrm{d}x \geq \left(\frac{d-2}{2} \right)^2 \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2} \, \mathrm{d}x. \tag{2.1.16}$$

2.1.2 Relatively form-bounded potentials

We will mainly focus on a class of potentials, which is slightly smaller than the one consisting of potentials satisfying (2.1.7).

Definition 2.1.3. A potential $V : \mathbb{R}^d \to \mathbb{R}$ is relatively form-bounded with relative bound zero, if for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ with

$$\langle |V|\psi, \psi \rangle \leq \varepsilon \|\nabla \psi\|^2 + C(\varepsilon) \|\psi\|^2 \quad \text{for every } \psi \in H^1(\mathbb{R}^d). \tag{2.1.17}$$

Any bounded potential satisfying

$$|V(x)| \leq C \qquad \text{for all} \quad x \in \mathbb{R}^d \tag{2.1.18}$$

and some $C > 0$ is obviously relatively form-bounded with relative bound zero. The advantage of form-bounded potentials is that it is not required that $V(x) \to 0$ as $|x| \to \infty$, which will play a crucial role in case of multi-particle systems. Furthermore, stronger local singularities can be allowed. For example, let $d = 3$ and consider

$$V(x) = |x|^{-2} |\ln |x||^{-\delta}, \qquad \delta > 0. \tag{2.1.19}$$

Then for every $\delta > 0$ the potential V is relatively form-bounded with relative bound zero, see p.8 in [CFKS87]. In the following we give further examples, some of which will be important in the next chapters.

In dimension three so-called Rollnik potentials $V \in \mathcal{R} + L^\infty(\mathbb{R}^3)$ with

$$V \in \mathcal{R} \qquad \Leftrightarrow \qquad \iint \frac{|V(x)||V(y)|}{|x-y|^2} \, \mathrm{d}x\mathrm{d}y < \infty \tag{2.1.20}$$

are relatively form-bounded with relative bound zero, see [RS75, Theorem X.19]. In the case of other dimensions we will later refer to the following theorem.

Theorem 2.1.4 (cf. [RS75, Theorem X.19] and [RS75, Theorem X.20])**.** *Assume that $d \geq 3$ and $V : \mathbb{R}^d \to \mathbb{R}$ belongs to $L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$, where*

$$\begin{cases} p = \frac{d}{2}, & \text{if } d \neq 4, \\ p > 2, & \text{if } d = 4. \end{cases} \tag{2.1.21}$$

Then V is relatively form-bounded with relative bound zero.

Remark. For $d \geq 4$ the potential V is not only relatively form-bounded, but also $-\Delta$-bounded with relative bound zero. In dimension three this is the case if condition (2.1.21) is replaced with $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$. In general, every $-\Delta$-bounded potential with relative bound zero is also relatively form-bounded with relative bound zero, see [RS75, Theorem X.18].

As a concluding example we consider $d \geq 3$ and assume that V is a real-valued measurable function on $\mathbb{R}^d$ satisfying

$$\lim_{\alpha \to 0} \left(\sup_{x \in \mathbb{R}^d} \int_{\{|x-y| \leq \alpha\}} |x-y|^{2-d} |V(y)| \, \mathrm{d}y \right) = 0. \tag{2.1.22}$$

Then V is relatively form-bounded with relative bound zero, see [CFKS87] and [Kat72]. Note that the potential defined in (2.1.19) belongs to this class, if and only if $\delta > 1$.

2.1.3 The localization error

An important tool for this thesis is the concept of the so-called localization error. It is related to a formula, which in the literature is known as the IMS localization formula. First we introduce the notion of a partition of unity, since in quantum mechanics it differs from the usual one.

Definition 2.1.5 (cf. [CFKS87, Definition 3.1])**.** A family of smooth functions $\{\chi_\alpha\}_{\alpha \in I}$ indexed by a set I is called a partition of unity if

(i) $0 \leq \chi_\alpha(x) \leq 1$ and $\sum_{\alpha \in I} \chi_\alpha^2(x) = 1$ for all $x \in \mathbb{R}^d$,

(ii) $\{\chi_\alpha\}_{\alpha \in I}$ is locally finite, i.e. on any compact set K we have $\chi_\alpha = 0$ for all but finitely many $\alpha \in I$,

(ii) $\sup_{x \in \mathbb{R}^d} \sum_{\alpha \in I} |\nabla \chi_\alpha(x)|^2 < \infty$.

Remark. The smoothness of the functions is not necessary but is often required in the literature. Later we will always consider a finite partition of unity, i.e., where the index set I is finite. In this case we simply say that $\{\chi_\alpha\}_{\alpha=1}^n$ is a partition of unity.

Theorem 2.1.6 (cf. [Sim83, Lemma 3.1.])**.** *Consider $h = -\Delta + V$, where the potential V satisfies* (2.1.17) *and let $\{\chi_\alpha\}_{\alpha=1}^n$ be a partition of unity. Then*

$$h = \sum_{\alpha=1}^n \chi_\alpha h \chi_\alpha - \sum_{\alpha=1}^n |\nabla \chi_\alpha|^2. \tag{2.1.23}$$

The term $\sum_{\alpha=1}^n |\nabla \chi_\alpha|^2$ is called the localization error.

In the next chapters we will apply the following estimate of the localization error in combination with the Hardy inequality. It is a modified variant of [VZ83, Lemma 5.1], which was proved in [BBV20].

Lemma 2.1.7. *For any $\varepsilon > 0$ and any fixed $b > 0$ one can find $\tilde{b} > b$ and real valued functions $\chi_1, \chi_2 : \mathbb{R}^d \to \mathbb{R}$ with piecewise continuous derivatives, such that*

$$\chi_1^2 + \chi_2^2 = 1, \qquad \chi_1(x) = \begin{cases} 1, & |x| \leq b \\ 0, & |x| > \tilde{b} \end{cases} \tag{2.1.24}$$

and

$$|\nabla \chi_1(x)|^2 + |\nabla \chi_2(x)|^2 \leq \varepsilon |x|^{-2}. \tag{2.1.25}$$

Proof. Let $\varepsilon > 0$ and $b > 0$ be fixed. We can always find a function $u \in C^1(\mathbb{R}_+)$ satisfying the following conditions.

(i) $u(t) = 1$ for $t \leq b$,

(ii) u is non-increasing on $[b, \infty)$,

(iii) The derivative u' satisfies

$$u'(t) \left(1 - u^2(t)\right)^{-\frac{1}{2}} \to 0 \qquad \text{as} \qquad t \to b. \tag{2.1.26}$$

Now since by definition $u \leq 1$, we can define the function $v := \sqrt{1 - u^2}$. Let

$$\chi_1 : \mathbb{R}^d \to \mathbb{R}, \ \chi_1(x) := u\left(|x|\right) \quad \text{and} \quad \chi_2 : \mathbb{R}^d \to \mathbb{R}, \ \chi_2(x) := v\left(|x|\right). \tag{2.1.27}$$

By $\chi_1^2 + \chi_2^2 = 1$ it follows

$$|\nabla \chi_1|^2 + |\nabla \chi_2|^2 = \frac{|\nabla \chi_1|^2}{(1 - \chi_1^2)} = \frac{(u'(|x|))^2}{1 - (u(|x|))^2}. \tag{2.1.28}$$

Since $u'(|x|)\,(1 - (u(|x|))^2)^{-\frac{1}{2}} \to 0$ as $|x| \to b$, we can take $b' > b$ so close to b that

$$\frac{(u'(|x|))^2}{1 - (u(|x|))^2} \leq \varepsilon |x|^{-2}, \quad |x| \in [b, b']. \tag{2.1.29}$$

This, together with (2.1.28) implies

$$\big(|\nabla\chi_1|^2 + |\nabla\chi_2|^2\big) \le \varepsilon |x|^{-2}, \quad |x| \in [b, b']. \tag{2.1.30}$$

Let $\tilde{b} > b$ and $b' \in (b, \tilde{b})$. By an abuse of notation we redefine the function u for arguments $t \ge b'$ as

$$u(t) = u(b') \ln\left(\frac{t}{\tilde{b}}\right)\left(\ln\left(\frac{b'}{\tilde{b}}\right)\right)^{-1}, \quad t \in [b', \tilde{b}] \quad \text{and} \quad u(t) = 0, \quad t \ge \tilde{b}. \tag{2.1.31}$$

Although the value $u(b')$ is close to 1, it is always strictly less than 1. As before we set

$$\chi_1(x) = u(|x|) \qquad \text{and} \qquad \chi_2(x) = v(|x|), \quad |x| \ge b'. \tag{2.1.32}$$

Then for $|x| \ge b'$ we obtain

$$|\nabla\chi_1|^2 + |\nabla\chi_2|^2 \le \frac{u^2(b')}{1 - u^2(b')}\left(\ln\left(\frac{b'}{\tilde{b}}\right)\right)^{-2} |x|^{-2}. \tag{2.1.33}$$

For any fixed b' we can choose $\tilde{b}$ sufficiently large, such that

$$\left(\ln\left(\frac{b'}{\tilde{b}}\right)\right)^{2} \ge \frac{1 - u^2(b')}{\varepsilon u^2(b')}. \tag{2.1.34}$$

This yields (2.1.25). □

Remark. By Lemma 2.1.7 and the Hardy inequality (2.1.16), for any $\varepsilon > 0$ we can find functions χ_1, χ_2 with

$$\int_{\mathbb{R}^d} |\nabla\chi_i(x)|^2 |\psi(x)|^2 \,\mathrm{d}x \le \varepsilon \int_{\mathbb{R}^d} |\nabla\psi(x)|^2 \,\mathrm{d}x \tag{2.1.35}$$

for $i = 1, 2$ and every function $\psi \in \dot{H}^1(\mathbb{R}^d)$, $d \ge 3$.

2.2 Multi-particle quantum systems

In the following we briefly introduce Schrödinger operators of N interacting quantum particles by adapting the presentation of [CFKS87].

The positions of N quantum particles of masses $m_1, \dots, m_N > 0$ each moving in $\mathbb{R}^d$ is represented by a vector $x = (x_1, \dots, x_N) \in \mathbb{R}^{dN}$, where each entry $x_i \in \mathbb{R}^d$ is the position vector of the ith particle. The operator of the kinetic energy of the system is given by

$$\tilde{H}_0 = -\sum_{i=1}^{N} \frac{1}{2m_i} \Delta_{x_i}, \tag{2.2.1}$$

where Δ_{x_i} is the Laplacian with respect to the variable $x_i = (x_{i1}, \dots, x_{id}) \in \mathbb{R}^d$. The operator of the potential energy of particle pair interactions is given by the multiplication of the function

$$V(x) = \sum_{1 \le i < j \le N} V_{ij}(x_i - x_j), \tag{2.2.2}$$

where each V_{ij} describes the interaction of the particles of masses m_i and m_j and position vectors x_i and x_j, respectively. The Hamiltonian of the whole system is given by

$$H_N = -\sum_{i=1}^{N} \frac{1}{2m_i} \Delta_{x_i} + \sum_{1 \le i < j \le N} V_{ij}(x_i - x_j). \tag{2.2.3}$$

In this section we assume that each V_{ij} is relatively bounded with relative bound zero.

Remark. For simplicity we always set the Planck's constant $\hbar = 1$ in our considerations.

2.2.1 Separation of the center of mass

In order to investigate spectral properties of H_N, it is convenient to eliminate the center of mass. In that sense, for each $x = (x_1, \dots, x_N) \in \mathbb{R}^{dN}$ we consider the center of mass of the system, which is given by

$$R(x) = \frac{1}{M} \sum_{i=1}^{N} m_i x_i, \qquad M = \sum_{i=1}^{N} m_i. \tag{2.2.4}$$

The $d(N-1)$-dimensional subspace

$$R_0 = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^{dN} : R(x) = 0\} \tag{2.2.5}$$

of $\mathbb{R}^{dN}$ is the space of relative positions of the particles. One can choose linear mappings

$$y_1, \ldots, y_{N-1} : \mathbb{R}^{dN} \to \mathbb{R}^d, \tag{2.2.6}$$

such that

$$y : \mathbb{R}^{dN} \to \mathbb{R}^{d(N-1)}, \quad y(x) = (y_1(x), \ldots, y_{N-1}(x)) \tag{2.2.7}$$

is an isomorphism of $R_0 \subset \mathbb{R}^{dN}$ and $\mathbb{R}^{d(N-1)}$ with $y_i(x) = 0$ if $x_1 = \ldots = x_N$. By computing the Laplacian on $\mathbb{R}^{dN}$ in terms of the coordinates $y_1, \ldots, y_{N-1}$ and R, one obtains cross terms of the form $\nabla_{y_i}\nabla_{y_j},\ i \neq j$, but not of the form $\nabla_{y_i}\nabla_R$. Hence, the free Hamiltonian (2.2.1) splits into

$$\tilde{H}_0 = \left(-\frac{1}{2M}\Delta_R\right) \otimes \mathbb{1} + \mathbb{1} \otimes H_0, \tag{2.2.8}$$

where Δ_R is the Laplacian with respect to the variable R acting on $L^2(\mathbb{R}^d)$ and H_0 depends on the choice of $y_1, \ldots, y_{N-1}$ and acts on $L^2(R_0) \cong L^2(\mathbb{R}^{d(N-1)})$. Since the potential V does not depend on the variable R the Hamiltonian H_N splits into

$$H_N = -\left(\frac{1}{2M}\Delta_R\right) \otimes \mathbb{1} + \mathbb{1} \otimes H, \tag{2.2.9}$$

where

$$H = H_0 + V. \tag{2.2.10}$$

Equation (2.2.9) shows that the center of mass of the whole system moves like a free particle, whereas the relative motion of the particles is described by the operator H.

In the following we describe a different approach of introducing the Hamiltonian H in (2.2.10), which is due to A. G. Sigalov and I. M. Sigal [SS70]. We rely on several results from the works [VZ84], [Zhi74] and [VZ83], which are presented in the spirit of [SS70]. Therefore, we will adapt the notation and use it in the further course of the thesis.

For this purpose, we introduce the scalar product $\langle\cdot,\cdot\rangle_1 : \mathbb{R}^{dN}\times\mathbb{R}^{dN}\to\mathbb{R}$ by

$$\langle x,y\rangle_1 := \sum_{i=1}^{N} 2m_i\langle x_i,y_i\rangle, \qquad |x|_1^2 = \langle x,x\rangle_1. \tag{2.2.11}$$

Here $\langle\cdot,\cdot\rangle$ is the standard scalar product on $\mathbb{R}^d$. Let P_0 be the projection operator from $\mathbb{R}^{dN}$ to R_0, which is explicitly given by

$$P_0 : x = (x_1,\dots,x_N) \;\mapsto\; (x_1 - R(x),\dots,x_N - R(x)) \tag{2.2.12}$$

with $R(x)$ being defined by (2.2.4). The d-dimensional subspace of $\mathbb{R}^{dN}$ orthogonal to R_0 is

$$R_c = \left\{ x = (x_1,\dots,x_N)\in\mathbb{R}^{dN} : x_i = \frac{1}{M}\sum_{j=1}^{N} m_j x_j,\; i = 1,\dots,N \right\}, \tag{2.2.13}$$

which is the configuration space of the center-of-mass position of the system. The orthogonality of R_0 and R_c is understood in the sense of the scalar product $\langle\cdot,\cdot\rangle_1$. Indeed, assume that $x\in R_0$ and $y\in R_c$, then due to $y_1 = y_j$ for all $j = 2,\dots,N$ it follows

$$\langle x,y\rangle_1 = \sum_{i=1}^{N} 2m_i\langle x_i,y_i\rangle = \left\langle \sum_{i=1}^{N} 2m_i x_i, y_1 \right\rangle = 0. \tag{2.2.14}$$

Hence, $\mathbb{R}^{dN} = R_0\oplus R_c$ and

$$L^2(\mathbb{R}^{dN}) \cong L^2(R_0)\otimes L^2(R_c). \tag{2.2.15}$$

Since by definition of P_0 we have

$$(P_0x)_i - (P_0x)_j = x_i - R(x) - (x_j - R(x)) = x_i - x_j, \tag{2.2.16}$$

the potential V in (2.2.2) satisfies

$$V(x) = V(P_0x). \tag{2.2.17}$$

Therefore, the operator H_N is decomposed as in (2.2.9), where H_0 is the Laplace-

Beltrami operator on R_0 with respect to the metric $\langle \cdot, \cdot \rangle_1$, which we denote by $-\Delta_0$. Let $B_{R_0} = \{g_1, \dots, g_{d(N-1)}\}$ and $B_{R_c} = \{f_1, \dots, f_d\}$ be bases of R_0 and R_c, both orthonormal with respect to $\langle \cdot, \cdot \rangle_1$. Then $B_{R_0} \cup B_{R_c}$ is an orthonormal basis of $\mathbb{R}^{dN}$. If the coordinate of a vector in $\mathbb{R}^{dN}$ with respect to the basis vector g_i is denoted by ν_i and its coordinate with respect to the basis vector f_k by η_k, then

$$\tilde{H}_0 = \Delta_0 + \Delta_c, \tag{2.2.18}$$

where

$$\Delta_0 = \sum_{i=1}^{d(N-1)} \frac{\partial^2}{\partial \nu_i^2} \qquad \text{and} \qquad \Delta_c = \sum_{k=1}^{d} \frac{\partial^2}{\partial \eta_k^2}, \tag{2.2.19}$$

see [SS70]. We denote by ∇_0 the gradient corresponding to Δ_0. Finally, our main object of study is the Hamiltonian

$$H = -\Delta_0 + V. \tag{2.2.20}$$

There is a significant difference between one- and two-particle systems and systems consisting of $N \geq 3$ particles. Even if each potential $V_{ij}(x)$ tends to zero as $|x| \to \infty$, the potential V does not necessarily have to. In case of two particles after separation of the center of mass this difficulty disappears. This is in contrast to the case $N \geq 3$, where the investigation of such systems is significantly more complicated.

2.2.2 Partition of the system

In the following we introduce the concept of the so-called partition of the system into clusters. We follow the presentation of [BBV20] and [VZ84].

An arbitrary non-empty subset $C \subseteq \{1, \dots, N\}$ is called a subsystem or cluster of the system and we denote by $|C|$ the number of its particles. Let

$$R_0[C] = \left\{ x \in \mathbb{R}^{dN} \ : \ \sum_{i \in C} m_i x_i = 0, \ x_j = 0, \ j \notin C \right\} \tag{2.2.21}$$

be the corresponding subspace of the relative positions of the particles within the cluster C. Denote by $-\Delta_0[C]$ the Laplace-Beltrami operator on $R_0[C]$ with respect to the scalar product $\langle\cdot,\cdot\rangle_1$ and denote by

$$V[C] = \sum_{i,j\in C,\ i<j} V_{ij} \tag{2.2.22}$$

the potential of interactions of the particles in C. Then for $1 < |C| < N$ the corresponding cluster Hamiltonian with its center of mass removed is given by

$$H[C] = -\Delta_0[C] + V[C]. \tag{2.2.23}$$

The operator $H[C]$ acts on $L^2(R_0[C])$ and it describes the relative motion of the particles within the cluster C ignoring all the other particles of the system. Note that for $C = \{1,\dots,N\}$ we have $R_0[C] = R_0$, so we set $H[C] = H$. For $|C| = 1$ we have $R_0[C] = \{0\}$ and $L^2(\{0\}) = \mathbb{C}$, so in this case we set $H[C] = 0$.
We say that $Z_p = (C_1,\dots,C_p)$ is a partition or a cluster decomposition of the system of order $|Z_p| = p$, if and only if for all $i,j = 1,\dots,p$ with $i \neq j$ we have

$$C_i \cap C_j = \emptyset \qquad \text{and} \qquad \bigcup_{j=1}^{p} C_j = \{1,\dots,N\}. \tag{2.2.24}$$

We refer to $C \subset Z_p$ as a cluster of the partition $Z_p = (C_1,\dots,C_p)$, if $C = C_i$ for some $i = 1,\dots,p$. The only unique partitions are Z_1 and Z_N, where Z_1 corresponds to the whole system and Z_N is the case when each cluster consists of a single particle. Let

$$R_0(Z_p) = \bigoplus_{C_k\subset Z_p} R_0[C_k] \qquad \text{and} \qquad R_c(Z_p) = R_0 \ominus R_0(Z_p). \tag{2.2.25}$$

This yields the decomposition

$$L^2(R_0(Z_p)) = L^2(R_0[C_1]) \otimes \cdots \otimes L^2(R_0[C_p]). \tag{2.2.26}$$

By abuse of notation we use the same symbols $-\Delta_0[C_i]$ and $H[C_i]$ for the operators

$$I\otimes\cdots\otimes I\otimes(-\Delta_0[C_i])\otimes I\otimes\cdots\otimes I \quad \text{and} \quad I\otimes\cdots\otimes I\otimes H[C_i]\otimes I\otimes\cdots\otimes I, \tag{2.2.27}$$

which both act on $L^2(R_0(Z_p))$. Now the cluster decomposition Hamiltonian acting on $L^2(R_0(Z_p))$ is given by

$$H(Z_p) = \sum_{C_k \subset Z_p} H[C_k]. \tag{2.2.28}$$

The operator $H(Z_p)$ describes the joint internal dynamics of the non-interacting clusters $C_1, \dots, C_p$. Let $-\Delta_0(Z_p)$ be the Laplace-Beltrami operator on $R_0(Z_p)$, then

$$-\Delta_0(Z_p) = -\sum_{C_k \subset Z_p} \Delta_0[C_k]. \tag{2.2.29}$$

Corresponding to the decomposition $L^2(R_0) = L^2(R_0(Z_p)) \otimes L^2(R_c(Z_p))$ we will sometimes use the same symbols $H[C_i]$ and $H(Z_p)$ for the operators acting on $L^2(R_0)$ as

$$H[C_i] \otimes I \qquad \text{and} \qquad H(Z_p) \otimes I, \tag{2.2.30}$$

respectively. We denote the intercluster interaction by

$$I(Z_p) = V - \sum_{C_k \subset Z_p} V[C_k]. \tag{2.2.31}$$

Here $I(Z_p)$ is the sum of pair potentials of particles from different clusters of the partition $Z_p = (C_1, \dots, C_p)$. Then the Hamiltonian H defined in (2.2.20) can be written as

$$H = H(Z_p) \otimes I + I \otimes (-\Delta_c(Z_p)) + I(Z_p), \tag{2.2.32}$$

where $-\Delta_c(Z_p)$ is the Laplace-Beltrami operator on $R_c(Z_p)$.
Now we introduce coordinates in spaces $R_0(Z_p)$ and $R_c(Z_p)$ which we will work with in the following. Denote by $P_0(Z_p)$ and $P_c(Z_p)$ the projections in R_0 on $R_0(Z_p)$ and $R_c(Z_p)$, respectively. For $x \in R_0$ let

$$q(Z_p) = P_0(Z_p)x \qquad \text{and} \qquad \xi(Z_p) = P_c(Z_p)x. \tag{2.2.33}$$

To emphasize the dependence of the respective coordinates $q(Z_p)$ and $\xi(Z_p)$ we will simply write

$$-\Delta_{q(Z_p)} = -\Delta_0(Z_p) \qquad \text{and} \qquad -\Delta_{\xi(Z_p)} = -\Delta_c(Z_p). \tag{2.2.34}$$

Thus we can write the Hamiltonian H in (2.2.32) as

$$H = -\Delta_{q(Z_p)} - \Delta_{\xi(Z_p)} + V \qquad \text{or} \qquad H = H(Z_p) - \Delta_{\xi(Z_p)} + I(Z_p). \tag{2.2.35}$$

Note that $q(Z_p) = (q_1^{Z_p}, \dots, q_N^{Z_p})$ corresponds to the positions of the particles within the clusters of the partition Z_p and $\xi(Z_p) = (\xi_1^{Z_p}, \dots, \xi_N^{Z_p})$ describes the center-of-mass positions of the corresponding clusters. For the purpose of illustrating the coordinates $q(Z_p)$ and $\xi(Z_p)$ we give the following

Example. Consider a system of $N = 5$ particles of masses $m_1, \dots, m_5 > 0$ and a partition $Z_2 = (C_1, C_2)$, where C_1 is a subsystem of two particles with masses m_1, m_2 and position vectors x_1, x_2 and C_2 is a subsystem of three particles with masses m_3, m_4, m_5 and position vectors x_3, x_4, x_5, respectively. Let

$$M_1 = m_1 + m_2, \qquad M_2 = m_3 + m_4 + m_5 \tag{2.2.36}$$

and denote

$$R_1(x) = \frac{1}{M_1} \sum_{i=1}^{2} m_i x_i \qquad \text{and} \qquad R_2(x) = \frac{1}{M_2} \sum_{i=3}^{5} m_i x_i. \tag{2.2.37}$$

Then the coordinates $q(Z_2)$ and $\xi(Z_2)$ are given by

$$q(Z_2) = (x_1 - R_1(x), x_2 - R_1(x), x_3 - R_2(x), x_4 - R_2(x), x_5 - R_2(x)), \tag{2.2.38}$$

$$\xi(Z_2) = (R_1(x), R_1(x), R_2(x), R_2(x), R_2(x)). \tag{2.2.39}$$

The Euclidean distance of the clusters C_1 and C_2 can be computed with the help of $|\cdot|_1 = \sqrt{\langle \cdot, \cdot \rangle_1}$. Indeed, for every $x \in R_0$ we have $|R(x)|^2 = 0$, which is equivalent to

$$M_1^2 |R_1(x)|^2 + M_2^2 |R_2(x)|^2 + 2 M_1 M_2 \langle R_1(x), R_2(x) \rangle = 0. \tag{2.2.40}$$

By adding the term

$$M_1 M_2 |R_1(x)|^2 + M_1 M_2 |R_2(x)|^2 \tag{2.2.41}$$

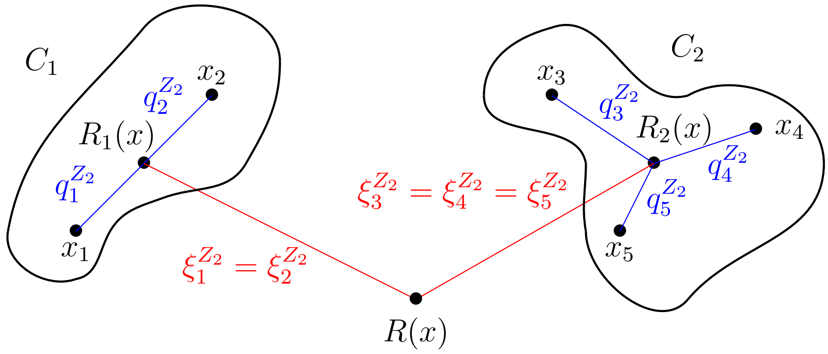

Figure 2.1: Coordinates $q(Z_2)$ and $\xi(Z_2)$

on both sides of (2.2.40) it yields

$$M_1|R_1(x)|^2 + M_2|R_2(x)|^2 = \frac{M_1M_2}{M_1+M_2}|R_1(x) - R_2(x)|^2. \tag{2.2.42}$$

Hence, by (2.2.11) we obtain

$$|\xi(Z_2)|_1 = \left(\frac{2M_1M_2}{M_1+M_2}\right)^{\frac{1}{2}} |R_1(x) - R_2(x)|. \tag{2.2.43}$$

For $q(Z_2) = x - \xi(Z_2)$ we have

$$\begin{aligned} |q(Z_2)|_1^2 =& 2m_1|x_1 - R_1(x)|^2 + 2m_2|x_2 - R_1(x)|^2 \\ &+ 2m_3|x_1 - R_2(x)|^2 + 2m_4|x_1 - R_2(x)|^2 + 2m_5|x_1 - R_2(x)|^2. \end{aligned} \tag{2.2.44}$$

Now consider the case when

$$|q(Z_2)|_1 \leq \kappa|\xi(Z_2)|_1 \qquad \text{for some small} \quad \kappa > 0. \tag{2.2.45}$$

In view of (2.2.43) and (2.2.44) this means that particles belonging to the same cluster are relatively close to each other compared to the distance of the respective clusters.

Motivated by the example above we now define regions in R_0, which we will refer to as cones in the following chapters. See for example [VZ84], p. 51 for more details on the characteristics of such cones.

Definition 2.2.1. For a partition Z_p and constants $\kappa, R > 0$ denote

$$K(Z_p, \kappa) = \{x \in R_0 \ : \ |q\,(Z_p)\,|_1 \leq \kappa |\xi\,(Z_p)\,|_1\}\,. \tag{2.2.46}$$

In the further course of the thesis we will study the spectrum of a multi-particle Schrödinger operator by making a partition of the unity of the configuration space R_0, corresponding to different partitions Z_p of the system into different clusters. By doing so we can systematically separate the cones $K_R(Z_p, \kappa)$, where for sufficiently small $\kappa > 0$ particles belonging to the same cluster in the partition Z_p are close to each other and the other clusters are far away.

2.2.3 Exponential decay of bound states

In the next chapter we will study solutions of the eigenvalue equation corresponding to the lower threshold of the essential spectrum of the Hamiltonian H. The following theorem identifies the location of the essential spectrum. It is known as the HVZ theorem, proved by Zhislin [Zhi60], van Winter [vW64] and Hunziker [Hun66]. Although it is known under more general conditions on the pair potentials, we formulate it in the way we will use it. We consider potentials V_{ij}, where for some constants $A, C, \nu > 0$

$$|V_{ij}(x)| \leq C|x|^{-2-\nu} \quad \text{if } |x| \geq A \qquad \text{and} \qquad V_{ij} \in L^p_{\text{loc}}(\mathbb{R}^d) \tag{2.2.47}$$

with $p > 2$ for $d = 4$ and $p = \frac{d}{2}$ for $d \geq 3,\ d \neq 4$.

Theorem 2.2.2 (cf. [RS78, Theorem XIII.17 (the HVZ theorem)])**.** *Let H be the N-body Hamiltonian defined by* (2.2.20)*, where the pair potentials V_{ij} satisfy* (2.2.47)*. Denote*

$$\Sigma_{Z_p} = \inf\left(\sigma(H(Z_p))\right) \qquad \textit{and} \qquad \Sigma = \min_{Z_p,\ p>1} \Sigma_{Z_p}. \tag{2.2.48}$$

Then the essential spectrum of the operator H is given by

$$\sigma_{\text{ess}}(H) = [\Sigma, \infty)\,. \tag{2.2.49}$$

Our approach in the coming chapter to investigate solutions corresponding to

Σ is based on the method developed by S. Agmon [Agm82], which was used to prove that eigenfunctions corresponding to eigenvalues below the threshold Σ decay exponentially. Here we also refer to the earlier work [O'C73] by A. J. O'Connor. Similar to the HVZ theorem we only formulate a weaker version that fits our framework.

Theorem 2.2.3 (cf. [Agm82, Teorem 4.13])**.** *Let H be the N-body Hamiltonian defined by* (2.2.20)*, where the pair potentials V_{ij} satisfy* (2.2.47)*. Assume that*

$$\sigma_{\mathrm{ess}}(H) = [\Sigma, \infty). \tag{2.2.50}$$

If H has an eigenvalue $E < \Sigma$, then the corresponding eigenfunction ψ satisfies

$$\int |\psi(x)|^2 \mathrm{e}^{2\alpha|x|}\,\mathrm{d}x < \infty \qquad \textit{for every} \quad \alpha < \sqrt{\Sigma - E}. \tag{2.2.51}$$

Theorem 2.2.3 shows in particular that as long as the eigenvalue E has a positive distance to the threshold Σ of the essential spectrum of the operator H, the corresponding eigenfunction decays exponentially and the decay rate increases as the eigenvalue is further away from the threshold. It turns out that in the borderline case $E = \Sigma$ the decay rates represent the entire remaining range, from sub-exponential to polynomial ones. This will be the main subject of the next chapter.

3 Virtual Levels of Schrödinger Operators

As already mentioned in the introduction, the investigation of the solutions of the Schrödinger equation corresponding to the threshold of the essential spectrum of the operator has taken many directions in both physics and mathematics. As a result, many different terms have become established and there are several related definitions in the literature; Critically bound [Ric03], coupling constant threshold [KS80a, KS80b], binding threshold [BFLS14], resonances [Sob93, Tam93], virtual levels [Yaf75], etc. We will focus on the concept of virtual levels, which we use as a generic term for threshold eigenvalues and resonances. The main subject of this chapter is the study of decay properties of zero-energy resonances and eigenfunctions of multi-particle Schrödinger operators. First, we present a variational approach that allows us to prove the existence and in some cases uniqueness of solutions corresponding to virtual levels of the one-body Schrödinger operator. With this method we also obtain estimates on their rates of decay at infinity. We choose the conditions for the potentials in such a way that we are able to apply the results to systems of $N \geq 3$ quantum particles in dimension $d \geq 3$. In case of short-range potentials the obtained estimates on the decay rates allow us to prove the corresponding asymptotic behaviour of the solutions. With regard to the Efimov effect, we only consider the case where the essential spectrum of the operator coincides with the half-line $[0, \infty)$.

3.1 Zero-energy solutions of the Schrödinger equation

First we introduce the concept of virtual levels for one-particle Schrödinger operators and prove a general theorem, which we will apply to different systems later in the thesis. Although our main goal is multi-particle systems, we also apply it to one-particle operators for both short-range and long-range potentials, which partly coincides with already known results, cf. [Mur86, Pin88, HJL19b, Yaf75, Yaf00], but also provides new insights. We follow the presentation of [BBV20].

3.1.1 Threshold resonances and bound states

In the following we consider the Schrödinger operator

$$h = -\Delta + V \qquad \text{in} \ \ L^2(\mathbb{R}^d), \ d \geq 3, \tag{3.1.1}$$

which was introduced in section 2.1. We assume that the potential V is relatively form-bounded with relative bound zero, i.e. for every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ with

$$\langle |V|\psi, \psi\rangle \leq \varepsilon \|\nabla\psi\|^2 + C(\varepsilon)\|\psi\|^2, \qquad \psi \in H^1(\mathbb{R}^d). \tag{3.1.2}$$

The corresponding quadratic form q with form domain $H^1(\mathbb{R}^d)$ is given by

$$q[\psi] = \|\nabla\psi\|^2 + \langle V\psi, \psi\rangle, \qquad \psi \in H^1(\mathbb{R}^d). \tag{3.1.3}$$

For any $\varepsilon \in (0, 1)$ we denote

$$h_\varepsilon = h + \varepsilon\Delta \tag{3.1.4}$$

and let $\dot{H}^1(\mathbb{R}^d)$ be the homogeneous Sobolev space defined in section 2.1.

Definition 3.1.1. We say that the operator h has a virtual level at zero, if $h \geq 0$ and for any $\varepsilon \in (0, 1)$ we have

$$\inf \sigma_{\mathrm{ess}}(h_\varepsilon) = 0 \qquad \text{and} \qquad \inf \sigma\left(h_\varepsilon\right) < 0. \tag{3.1.5}$$

Our main result in this section is the following theorem, which is the basis for further investigations.

Theorem 3.1.2. *Assume that the operator h has a virtual level at zero. If there exist constants $\alpha_0 > 0$, $b > 0$ and $\gamma_0 \in (0,1)$, such that for any $\psi \in H^1(\mathbb{R}^d)$ satisfying* $\operatorname{supp}(\psi) \subset \{x \in \mathbb{R}^d : \ |x| \geq b\}$ *we have*

$$\langle h\psi, \psi\rangle - \gamma_0 \|\nabla\psi\|^2 - \alpha_0^2 \langle |x|^{-2}\psi, \psi\rangle \geq 0, \tag{3.1.6}$$

then the following assertions hold true:

(i) *If $\alpha_0 > 1$, then zero is a simple eigenvalue of h and the corresponding eigenfunction φ_0 satisfies*

$$\nabla\left(|\cdot|^{\alpha_0}\varphi_0\right) \in L^2(\mathbb{R}^d) \qquad \text{and} \qquad (1+|\cdot|)^{\alpha_0-1}\varphi_0 \in L^2(\mathbb{R}^d). \tag{3.1.7}$$

Moreover, there exists $\delta_0 > 0$, such that for any function $\psi \in H^1(\mathbb{R}^d)$ with $\langle\nabla\psi, \nabla\varphi_0\rangle = 0$ we have

$$\langle h\psi, \psi\rangle \geq \delta_0 \|\nabla\psi\|^2. \tag{3.1.8}$$

(ii) *If $\alpha_0 \in (0,1)$ and in addition*

$$\langle |V|\psi, \psi\rangle \leq C\|\nabla\psi\|^2 \tag{3.1.9}$$

is satisfied for any function $\psi \in \dot{H}^1(\mathbb{R}^d)$ and some constant $C > 0$, then there exists a non-vanishing function $\varphi_1 \in \dot{H}^1(\mathbb{R}^d)$ with

$$\|\nabla\varphi_1\|^2 + \langle V\varphi_1, \varphi_1\rangle = 0. \tag{3.1.10}$$

Moreover, we have

$$\nabla\left(|\cdot|^{\alpha_0}\varphi_1\right) \in L^2(\mathbb{R}^d) \qquad \text{and} \qquad (1+|\cdot|)^{\alpha_0-1}\varphi_1 \in L^2(\mathbb{R}^d). \tag{3.1.11}$$

If we assume that for some $C > 0$

$$\|V\psi\|^2 \leq C\left(\|\nabla\psi\|^2 + \|\psi\|^2\right) \tag{3.1.12}$$

for every function $\psi \in C_0^\infty(\mathbb{R}^d)$*, then the solution* $\varphi_1 \in \dot{H}^1(\mathbb{R}^d)$ *of* (3.1.10) *is unique. Furthermore, there exists* $\delta_1 > 0$*, such that for any function* $\psi \in \dot{H}^1(\mathbb{R}^d)$ *with* $\langle \nabla\psi, \nabla\varphi_1\rangle = 0$ *we have*

$$\langle h\psi, \psi\rangle \geq \delta_1 \|\nabla\psi\|^2. \tag{3.1.13}$$

(iii) *If instead of* (3.1.6) *a stronger inequality*

$$\langle h\psi, \psi\rangle - \gamma_0\|\nabla\psi\|^2 - \alpha_0^2\langle |x|^{-\beta}\psi, \psi\rangle \geq 0 \tag{3.1.14}$$

is satisfied for some constants $\alpha_0, \gamma_0 > 0$ *and* $\beta \in (0,2)$*, then the function* φ_0 *in part* ***(i)*** *of the theorem satisfies*

$$\exp\left(\alpha_0\kappa^{-1}|\cdot|^\kappa\right)\varphi_0 \in L^2(\mathbb{R}^d), \qquad \text{where} \quad \kappa = 1 - \frac{\beta}{2}. \tag{3.1.15}$$

Remark. **(i)** We will see later that estimates (3.1.7) and (3.1.11) of the decay of the zero-energy solutions are close to optimal. It will follow from the proof that for those to hold we do not need the assumption that the operator h has no negative eigenvalues.

(ii) Function φ_1 in part **(ii)** of the theorem is not necessarily an eigenfunction of h, since it may not belong to $L^2(\mathbb{R}^d)$. In this case zero is a resonance of h.

In order to prove Theorem 3.1.2 we need several lemmas. The following lemma is based on a part of the proof of the main theorem in [Zhi74].

Lemma 3.1.3. *Consider* $h = -\Delta + V$ *in* $L^2(\mathbb{R}^d)$ *with* $d \geq 3$*, where* V *satisfies* (3.1.2)*. Assume there exist* $\varepsilon > 0$ *and* $b > 0$*, such that*

$$\langle h\psi, \psi\rangle - \varepsilon\langle |x|^{-2}\psi, \psi\rangle \geq 0 \tag{3.1.16}$$

holds for any $\psi \in H^1(\mathbb{R}^d)$ *with* $\operatorname{supp}(\psi) \subset \{x \in \mathbb{R}^d,\ |x| \geq b\}$*. Then*

(i) $\inf \sigma_{\text{ess}}(h) \geq 0$*,*

(ii) *the operator* h *has at most a finite number of negative eigenvalues,*

(iii) *zero is not an infinitely degenerate eigenvalue of* h,

(iv) *for* V *with* (3.1.9) *the space* W *of functions* $\varphi \in \dot{H}^1(\mathbb{R}^d)$ *satisfying*

$$\int_{\mathbb{R}^d} \nabla\varphi(x) \cdot \nabla\psi(x)\,\mathrm{d}x + \int_{\mathbb{R}^d} V(x)\varphi(x)\psi(x)\,\mathrm{d}x = 0 \tag{3.1.17}$$

for all functions $\psi \in \dot{H}^1(\mathbb{R}^d)$ *is at most finite-dimensional.*

Proof. We construct a finite-dimensional subspace $M \subset L^2(\mathbb{R}^d)$, such that

$$\langle h\psi, \psi\rangle > 0 \qquad \text{for all } \psi \in H^1(\mathbb{R}^d) \quad \text{with } \psi \perp M. \tag{3.1.18}$$

Due to Lemma 2.1.7 we have

$$\langle h\psi, \psi\rangle \geq L[\psi\chi_1] + L[\psi\chi_2], \tag{3.1.19}$$

where the functional $L : H^1(\mathbb{R}^d) \to \mathbb{R}$ is given by

$$L[\psi] = \langle h\psi, \psi\rangle - \varepsilon\langle |x|^{-2}\psi, \psi\rangle. \tag{3.1.20}$$

Since the function $\psi\chi_2$ is supported outside the ball of radius $b > 0$, condition (3.1.16) implies $L[\psi\chi_2] \geq 0$. Hence, it suffices to show that we have $L[\psi\chi_1] > 0$ for any $\psi \perp M$ with M being a finite-dimensional space. By Hardy's inequality and (3.1.2) we have

$$L[\psi\chi_1] \geq (1 - 5\varepsilon)\|\nabla(\chi_1\psi)\|^2 - C(\varepsilon)\|\chi_1\psi\|^2. \tag{3.1.21}$$

For $k \in \mathbb{N}$ let

$$M_k := \{\varphi_1\chi_1, \ldots, \varphi_k\chi_1\}, \tag{3.1.22}$$

where $\{\varphi_1, \ldots, \varphi_k\}$ is an orthonormal set of eigenfunctions corresponding to the k lowest eigenvalues of the Laplacian, acting on $L^2\left(\{x \in \mathbb{R}^d : |x| \leq b\}\right)$ with Dirichlet boundary conditions. For $\psi \perp M_k$ we have $\psi\chi_1 \perp \varphi_1, \ldots, \varphi_k$, which for sufficiently large k implies

$$\|\nabla(\psi\chi_1)\|^2 \geq 2\,(1 - \varepsilon)^{-1}\,C(\varepsilon)\|\psi\chi_1\|^2. \tag{3.1.23}$$

Therefore, we conclude $L[\psi\chi_1] > 0$. This proves statements **(i)**-**(iii)**.

In order to prove statement **(iv)** we consider $\tilde{h} = h - (1+|x|)^{-3}$. The operator $\tilde{h}$ satisfies (3.1.19) for $b > 0$ sufficiently large and we can use similar arguments as above. If the space W is not finite-dimensional, then $\tilde{h}$ has an infinite number of negative eigenvalues. This is a contradiction to **(ii)**. □

Proof of statement (i) of Theorem 3.1.2. By Lemma 3.1.3 there exists a sequence of eigenfunctions $\psi_n \in H^1(\mathbb{R}^d)$, corresponding to eigenvalues $E_n < 0$ of the operator $h_{n^{-1}}$, i.e. we have

$$-\left(1 - n^{-1}\right)\Delta\psi_n + V\psi_n = E_n\psi_n. \tag{3.1.24}$$

We normalize the sequence $(\psi_n)_{n\in\mathbb{N}}$ by $\|\nabla\psi_n\| = 1$ and take a weakly convergent subsequence, also denoted by $(\psi_n)_{n\in\mathbb{N}}$, which has a weak limit $\varphi_0 \in \dot{H}^1(\mathbb{R}^d)$. Note that by the Rellich–Kondrachov theorem $(\psi_n)_{n\in\mathbb{N}}$ converges to φ_0 in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$. In the following we will prove that φ_0 satisfies all the assertions of statement **(i)** of the theorem.

Lemma 3.1.4. *The weak limit $\varphi_0 \in \dot{H}^1(\mathbb{R}^d)$ of the sequence $(\psi_n)_{n\in\mathbb{N}}$ does not vanish.*

Proof. We consider the functional

$$L[\psi,\varepsilon] := (1-\varepsilon)\|\nabla\psi\|^2 + \langle V\psi,\psi\rangle, \tag{3.1.25}$$

where $\psi \in H^1(\mathbb{R}^d)$ and $\varepsilon > 0$. Let $b > 0$, such that (3.1.6) is satisfied. We fix $\varepsilon_1 > 0$ and construct functions χ_1, χ_2 in accordance with Lemma 2.1.7, which implies

$$L[\psi,\varepsilon] \geq L[\psi\chi_1, \varepsilon+\varepsilon_1] + L[\psi\chi_2, \varepsilon+\varepsilon_1] \tag{3.1.26}$$

for every $\psi \in H^1(\mathbb{R}^d)$ independently of ε. Since the operator h is non-negative, we have

$$\begin{aligned} L[\psi\chi_1, \varepsilon+\varepsilon_1] &= (1-\varepsilon-\varepsilon_1)\|\nabla(\psi\chi_1)\|^2 + \langle V\psi\chi_1, \psi\chi_1\rangle \\ &\geq -(\varepsilon+\varepsilon_1)\|\nabla(\psi\chi_1)\|^2. \end{aligned} \tag{3.1.27}$$

In addition, since $\operatorname{supp}(\psi\chi_2) \subset \{x \in \mathbb{R}^d : |x| \geq b\}$ we conclude by (3.1.6) that

$$\begin{aligned} L[\psi\chi_2, \varepsilon + \varepsilon_1] &= (1 - \varepsilon - \varepsilon_1)\|\nabla(\psi\chi_2)\|^2 + \langle V\psi\chi_2, \psi\chi_2\rangle \\ &= (1 - \gamma_0)\|\nabla(\psi\chi_2)\|^2 + \langle V\psi\chi_2, \psi\chi_2\rangle + (\gamma_0 - \varepsilon - \varepsilon_1)\|\nabla(\psi\chi_2)\|^2 \\ &\geq (\gamma_0 - \varepsilon - \varepsilon_1)\|\nabla(\psi\chi_2)\|^2. \end{aligned} \tag{3.1.28}$$

Hence, (3.1.27) and (3.1.28) imply

$$L[\psi, \varepsilon] \geq -(\varepsilon + \varepsilon_1)\|\nabla(\psi\chi_1)\|^2 + (\gamma_0 - \varepsilon - \varepsilon_1)\|\nabla(\psi\chi_2)\|^2. \tag{3.1.29}$$

For $\psi = \psi_n$ and $\varepsilon = n^{-1}$, estimate (3.1.29) yields

$$-(\varepsilon_1 + n^{-1})\|\nabla(\psi_n\chi_1)\|^2 + (\gamma_0 - \varepsilon_1 - n^{-1})\|\nabla(\psi_n\chi_2)\|^2 < 0, \tag{3.1.30}$$

which implies

$$(\gamma_0 - \varepsilon_1 - n^{-1})\left(\|\nabla(\psi_n\chi_1)\|^2 + \|\nabla(\psi_n\chi_2)\|^2\right) < \gamma_0\|\nabla(\psi_n\chi_1)\|^2. \tag{3.1.31}$$

By the normalization of ψ_n we have

$$\|\nabla(\psi_n\chi_1)\|^2 + \|\nabla(\psi_n\chi_2)\|^2 \geq \|\nabla\psi_n\|^2 = 1 \tag{3.1.32}$$

for every $n \in \mathbb{N}$. Hence, by (3.1.31) we obtain

$$\|\nabla(\psi_n\chi_1)\|^2 \geq \frac{\gamma_0 - \varepsilon_1 - n^{-1}}{\gamma_0} \geq 1 - \varepsilon_2, \tag{3.1.33}$$

where $\varepsilon_2 > 0$ can be chosen arbitrarily small by choosing $\varepsilon_1 > 0$ sufficiently small and $n \in \mathbb{N}$ sufficiently large. Due to (3.1.28) with $\varepsilon = n^{-1}$ we have

$$L[\psi_n\chi_2, n^{-1} + \varepsilon_1] > 0. \tag{3.1.34}$$

This, together with (3.1.26) and $L[\psi_n, n^{-1}] < 0$ implies

$$\begin{aligned} 0 > L[\psi_n\chi_1, n^{-1} + \varepsilon_1] &= \left(1 - n^{-1} - \varepsilon_1\right) \|\nabla(\psi_n\chi_1)\|^2 + \langle V\psi_n\chi_1, \psi_n\chi_1\rangle \\ &\geq \left(1 - n^{-1} - 2\varepsilon_1\right) \|\nabla(\psi_n\chi_1)\|^2 - C(\varepsilon_1)\|\psi_n\chi_1\|^2, \end{aligned} \tag{3.1.35}$$

where in the last inequality we used (3.1.2). By combining (3.1.35) and (3.1.33) we arrive at

$$\|\psi_n\chi_1\|^2 \geq \frac{(1 - n^{-1} - 2\varepsilon_1)(1 - \varepsilon_2)}{C(\varepsilon_1)}. \tag{3.1.36}$$

Since χ_1 is compactly supported, $|\chi_1| \leq 1$ and $(\psi_n)_{n\in\mathbb{N}}$ converges to φ_0 in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$, the last inequality proves the Lemma. □

Remark. Note that

$$\|\nabla(\chi_1\psi_n)\|^2 + \|\nabla(\chi_2\psi_n)\|^2 = \|\nabla\psi_n\|^2 + \int \left(|\nabla\chi_1|^2 + |\nabla\chi_2|^2\right) |\psi_n|^2 \, \mathrm{d}x. \tag{3.1.37}$$

By inequality (2.1.35) the last term on the r.h.s. of (3.1.37) can be estimated by $\varepsilon\|\nabla\psi_n\|^2 = \varepsilon$, which implies

$$\|\nabla(\chi_2\psi_n)\|^2 \leq (1 + \varepsilon) - \|\nabla(\chi_1\psi_n)\|^2. \tag{3.1.38}$$

Combining (3.1.38) with (3.1.33) yields

$$\|\nabla(\chi_2\psi_n)\|^2 \leq \tilde{\varepsilon}, \tag{3.1.39}$$

where $\tilde{\varepsilon} > 0$ can be chosen arbitrarily small for large $\tilde{b}$ and n. We will use this estimate later.

Lemma 3.1.5. *Assume that* (3.1.5) *and* (3.1.6) *hold for some* $\alpha_0 > 1$. *Then there exists a constant* $C > 0$, *such that for any eigenfunction* $\psi_n \in H^1(\mathbb{R}^d)$ *corresponding to a negative eigenvalue of the operator* $h_{n^{-1}}$, *normalized by* $\|\nabla\psi_n\| = 1$, *we have*

$$\|\nabla(|\cdot|^{\alpha_0}\psi_n)\| \leq C \qquad \text{and} \qquad \|(1 + |\cdot|)^{\alpha_0 - 1}\psi_n\| \leq C. \tag{3.1.40}$$

Remark. Recall that eigenfunctions ψ_n of the operators $h_{n^{-1}}$ decay exponentially and their decay rates depend on the distances of the corresponding eigenvalues to the bottom of the essential spectrum, see Theorem 2.2.3. Since for $n \to \infty$ the negative eigenvalues of $h_{n^{-1}}$ converge to zero, these estimates are not uniform in $n \in \mathbb{N}$. However, Lemma 3.1.5 shows that if condition (3.1.6) is satisfied, a uniform estimate exists. This estimate is of the polynomial type and the corresponding power depends on the parameter α_0 in (3.1.6) only.

Proof of Lemma 3.1.5. For any $\varepsilon > 0$ and $R > 0$ we define the function

$$G_\varepsilon(x) = \frac{|x|^{\alpha_0}}{1 + \varepsilon |x|^{\alpha_0}} \chi_R(|x|), \tag{3.1.41}$$

where $\chi_R \in C^\infty(\mathbb{R})$ with

$$\chi_R(|x|) = \begin{cases} 0, & |x| \le R, \\ 1, & |x| \ge 2R. \end{cases} \tag{3.1.42}$$

A simple calculation shows

$$\nabla G_\varepsilon(x) = \frac{x}{|x|} \left(\chi_R'(|x|) G_\varepsilon(x) + \chi_R(|x|) \alpha_0 |x|^{-1} G_\varepsilon(x) (1 + \varepsilon |x|^{\alpha_0})^{-1} \right). \tag{3.1.43}$$

By (3.1.42) we have $\chi_R(|x|) = 1$ and $\chi_R'(|x|) = 0$ for $|x| \ge 2R$. Hence, for such arguments we can estimate

$$|\nabla G_\varepsilon(x)| = \frac{\alpha_0 |x|^{\alpha_0 - 1}}{(1 + \varepsilon |x|^{\alpha_0})^2} \le \alpha_0 |x|^{-1} |G_\varepsilon(x)|. \tag{3.1.44}$$

Furthermore, for $|x| \in [R, 2R]$ the function $|\nabla G_\varepsilon|$ is uniformly bounded in ε and for $|x| \le R$ we obviously have $G_\varepsilon(x) = 0$. Since every function ψ_n satisfies

$$-(1 - n^{-1}) \Delta \psi_n + V \psi_n = E_n \psi_n \tag{3.1.45}$$

with $E_n < 0$ and each ψ_n decays exponentially, we can multiply (3.1.45) with $G_\varepsilon^2 \overline{\psi_n}$ and integrate by parts to obtain

$$\left(1 - n^{-1}\right) \langle \nabla \psi_n, \nabla \left(G_\varepsilon^2 \psi_n \right) \rangle + \langle V \psi_n, G_\varepsilon^2 \psi_n \rangle = E_n \| G_\varepsilon \psi_n \|^2 < 0. \tag{3.1.46}$$

Furthermore, by

$$\operatorname{Re}\langle V\psi_n, G_\varepsilon^2\psi_n\rangle = \langle V\psi_n, G_\varepsilon^2\psi_n\rangle \quad \text{and} \quad \operatorname{Re} E_n\|G_\varepsilon\psi_n\|^2 = E_n\|G_\varepsilon\psi_n\|^2 \tag{3.1.47}$$

we conclude

$$\operatorname{Re}\langle\nabla\psi_n, \nabla\left(G_\varepsilon^2\psi_n\right)\rangle = \langle\nabla\psi_n, \nabla\left(G_\varepsilon^2\psi_n\right)\rangle. \tag{3.1.48}$$

Note that

$$\begin{aligned}
\operatorname{Re}\langle\nabla\psi_n, \nabla(G_\varepsilon^2\psi_n)\rangle &= \operatorname{Re}\langle\nabla\psi_n, G_\varepsilon\psi_n\nabla G_\varepsilon\rangle + \operatorname{Re}\langle(\nabla\psi_n)G_\varepsilon, \nabla(G_\varepsilon\psi_n)\rangle \\
&= \operatorname{Re}\langle\nabla(\psi_n G_\varepsilon), \psi_n\nabla G_\varepsilon\rangle - \operatorname{Re}\langle\psi_n\nabla G_\varepsilon, \psi_n\nabla G_\varepsilon\rangle \\
&\quad + \operatorname{Re}\langle\nabla(\psi_n G_\varepsilon), \nabla(\psi_n G_\varepsilon)\rangle - \operatorname{Re}\langle\psi_n\nabla G_\varepsilon, \nabla(\psi_n G_\varepsilon)\rangle \\
&= \operatorname{Re}\langle\nabla(\psi_n G_\varepsilon), \nabla(\psi_n G_\varepsilon)\rangle - \operatorname{Re}\langle\psi_n\nabla G_\varepsilon, \psi_n\nabla G_\varepsilon\rangle.
\end{aligned} \tag{3.1.49}$$

This implies

$$\langle\nabla\psi_n, \nabla(G_\varepsilon^2\psi_n)\rangle = \|\nabla(\psi_n G_\varepsilon)\|^2 - \|\psi_n\nabla G_\varepsilon\|^2, \tag{3.1.50}$$

which together with (3.1.46) yields

$$\left(1 - \frac{1}{n}\right)\left(\|\nabla(\psi_n G_\varepsilon)\|^2 - \int |\psi_n|^2|\nabla G_\varepsilon|^2 \,\mathrm{d}x\right) + \int V|\psi_n G_\varepsilon|^2 \,\mathrm{d}x < 0. \tag{3.1.51}$$

By Hardy's inequality we get

$$\int\limits_{\{R\le|x|\le 2R\}} |\nabla G_\varepsilon|^2|\psi_n|^2 \,\mathrm{d}x \le C \int\limits_{\{R\le|x|\le 2R\}} |\psi_n|^2 \,\mathrm{d}x \le \widetilde{C}R^2 \int |\nabla\psi_n|^2 \,\mathrm{d}x =: C_0. \tag{3.1.52}$$

Hence, substituting (3.1.44) and (3.1.52) into (3.1.51) yields

$$\left(1 - n^{-1}\right)\|\nabla(\psi_n G_\varepsilon)\|^2 + \langle VG_\varepsilon\psi_n, G_\varepsilon\psi_n\rangle - \alpha_0^2 \int_{\{|x|>2R\}} \frac{|G_\varepsilon\psi_n|^2}{|x|^2} \,\mathrm{d}x \le C_1, \tag{3.1.53}$$

where $C_1 > 0$ does not depend on n or ε. Note that the function $G_\varepsilon\psi_n$ is supported outside the ball of radius $R > 0$. For $R > b$ it therefore satisfies (3.1.6), i.e. we have

$$(1-\gamma_0)\|\nabla(G_\varepsilon\psi_n)\|^2 + \langle VG_\varepsilon\psi_n, G_\varepsilon\psi_n\rangle - \alpha_0^2\langle|x|^{-2}G_\varepsilon\psi_n, G_\varepsilon\psi_n\rangle \ge 0. \tag{3.1.54}$$

For $n > 2\gamma_0^{-1}$ estimates (3.1.53) and (3.1.54) imply

$$\frac{\gamma_0}{2}\|\nabla(G_\varepsilon\psi_n)\|^2 \leq C. \tag{3.1.55}$$

Taking $\varepsilon \to 0$ yields $\|\nabla\left(|\cdot|^{\alpha_0}\psi_n\right)\| \leq C$, which together with Hardy's inequality completes the proof. □

Lemma 3.1.6. *Assume that* (3.1.5) *and* (3.1.6) *hold for some* $\alpha_0 > 1$*. Then zero is an eigenvalue of* h *and the corresponding eigenfunction* φ_0 *satisfies*

$$\nabla\left(|\cdot|^{\alpha_0}\varphi_0\right) \in L^2(\mathbb{R}^d) \qquad \textit{and} \qquad (1+|\cdot|)^{\alpha_0-1}\varphi_0 \in L^2(\mathbb{R}^d). \tag{3.1.56}$$

Proof. Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of eigenfunctions of the operators $h_{n^{-1}}$ normalized by $\|\nabla\psi_n\| = 1$. This sequence has a subsequence, also denoted by $(\psi_n)_{n\in\mathbb{N}}$, with a weak limit $\varphi_0 \in \dot{H}^1(\mathbb{R}^d)$. According to Lemma 3.1.4 we have $\varphi_0 \not\equiv 0$. Furthermore, the sequence $(\psi_n)_{n\in\mathbb{N}}$ converges to φ_0 in $L^2_{\text{loc}}(\mathbb{R}^d)$ and by Lemma 3.1.5 we have $\|(1+|\cdot|)^{\alpha_0-1}\psi_n\| \leq C$ for $\alpha_0 > 1$ and C independent of $n \in \mathbb{N}$. Therefore, we conclude

$$(1+|\cdot|)^{\alpha_0-1}\varphi_0 \in L^2(\mathbb{R}^d). \tag{3.1.57}$$

This also shows that $\langle V\varphi_0, \varphi_0\rangle$ is well defined. Our next goal is to prove that φ_0 satisfies

$$\langle V\varphi_0, \varphi_0\rangle = -1. \tag{3.1.58}$$

For this purpose we write

$$\begin{aligned}\langle V\varphi_0, \varphi_0\rangle &= \langle V\varphi_0, \varphi_0 - \psi_n\rangle + \langle V\varphi_0, \psi_n\rangle \\ &= \langle V\varphi_0, \varphi_0 - \psi_n\rangle + \langle V(\varphi_0 - \psi_n), \psi_n\rangle + \langle V\psi_n, \psi_n\rangle.\end{aligned} \tag{3.1.59}$$

Due to (3.1.2) the first term on the r.h.s. of (3.1.59) can be estimated by

$$\begin{aligned}|\langle V\varphi_0, \varphi_0 - \psi_n\rangle| &\leq \langle |V|^{\frac{1}{2}}\varphi_0, |V|^{\frac{1}{2}}|\varphi_0 - \psi_n|\rangle \\ &\leq \left(\|\nabla\varphi_0\|^2 + C(1)\|\varphi_0\|^2\right)^{\frac{1}{2}}\left(\varepsilon\|\nabla(\varphi_0 - \psi_n)\|^2 + C(\varepsilon)\|\varphi_0 - \psi_n\|^2\right)^{\frac{1}{2}} \\ &\leq C\left(2\varepsilon\left(\|\nabla\varphi_0\|^2 + \|\nabla\psi_n\|^2\right) + C(\varepsilon)\|\varphi_0 - \psi_n\|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{3.1.60}$$

Note that by the semi-continuity of the norm we have $\|\nabla\varphi_0\| \leq 1$. Since

$$\|\psi_n - \varphi_0\| \to 0 \qquad \text{as} \qquad n \to \infty, \tag{3.1.61}$$

choosing $\varepsilon > 0$ sufficiently small and $n \in \mathbb{N}$ sufficiently large we can get the r.h.s. of (3.1.60) arbitrarily small. Similar arguments show that the second term on the r.h.s. of (3.1.59) can be done arbitrarily small as well. Consequently, we have $\langle V\psi_n, \psi_n\rangle \to \langle V\varphi_0, \varphi_0\rangle$ as $n \to \infty$. By

$$(1 - n^{-1})\|\nabla\psi_n\|^2 + \langle V\psi_n, \psi_n\rangle \leq 0 \qquad \text{and} \qquad \|\nabla\psi_n\| = 1 \tag{3.1.62}$$

we conclude $\langle V\varphi_0, \varphi_0\rangle = -1$. Since $\|\nabla\varphi_0\| \leq 1$, we have

$$\|\nabla\varphi_0\|^2 + \langle V\varphi_0, \varphi_0\rangle \leq 0. \tag{3.1.63}$$

This implies $\|\nabla\varphi_0\| = 1$. Hence, φ_0 is a minimizer of the quadratic form of h and it is therefore an eigenfunction of h, corresponding to the eigenvalue zero. Finally, repeating the same arguments for φ_0, which we used in Lemma 3.1.6 to get (3.1.55) for the eigenfunctions ψ_n, we obtain $\nabla(|\cdot|^{\alpha_0}\varphi_0) \in L^2(\mathbb{R}^d)$. □

Our next goal is to prove inequality (3.1.8) and the non-degeneracy of φ_0. We will do it in Lemma 3.1.7 - Lemma 3.1.9.

Lemma 3.1.7. *For any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$, such that for any $n \geq n_0$ and any eigenfunction ψ_n with $\|\nabla\psi_n\| = 1$, corresponding to some negative eigenvalue of the operator $h_{n^{-1}}$, we have $\|\psi_n - \varphi_0\| < \varepsilon$.*

Proof. We assume that we have eigenfunctions $\psi_n \in H^1(\mathbb{R}^d)$ with $\|\nabla\psi_n\| = 1$, corresponding to some negative eigenvalues of the operator $h_{n^{-1}}$ for $n \in \mathbb{N}$. Furthermore, we assume that $\|\psi_n - \varphi_0\| \geq C > 0$. Proceeding as in the proof of Lemmas 3.1.4 and 3.1.6 we can find a subsequence, also denoted by $(\psi_n)_{n\in\mathbb{N}}$, such that $(\psi_n)_{n\in\mathbb{N}}$ converges to some function $\tilde{\varphi}_0 \in H^1(\mathbb{R}^d)$ with $\tilde{\varphi}_0 \not\equiv 0$, $\|\nabla\tilde{\varphi}_0\| = 1$ and

$$\|\nabla\tilde{\varphi}_0\|^2 + \langle V\tilde{\varphi}_0, \tilde{\varphi}_0\rangle = 0. \tag{3.1.64}$$

By $\|\nabla\varphi_0\| = \|\nabla\tilde{\varphi}_0\| = 1$ and $\|\psi_n - \varphi_0\| \geq C > 0$ we conclude that φ_0 and $\tilde{\varphi}_0$ are linearly independent. Due to [Goe77] the ground state is always non-degenerate. Consequently, φ_0 and $\tilde{\varphi}_0$ cannot be linearly independent. $\square$

Lemma 3.1.8. *For any sufficiently small $\varepsilon > 0$ the operator h_ε has one non-degenerate negative eigenvalue.*

Proof. Assume there is a sequence $a(n) \in (0,1)$ with $a(n) \to 0$ as $n \to \infty$, such that for any $n \in \mathbb{N}$ the operator $h_{a(n)} = -(1-a(n))\Delta + V$ has at least two eigenvalues. Recall that the lowest eigenvalue of $h_{a(n)}$ is non-degenerate. We consider two eigenfunctions $\psi_n^{(1)}$ and $\psi_n^{(2)}$ of $h_{a(n)}$, normalized by $\|\psi_n^{(1)}\| = \|\psi_n^{(2)}\| = 1$, where $\psi_n^{(1)}$ corresponds to the lowest eigenvalue. Now $\psi_n^{(1)}$ and $\psi_n^{(2)}$ are orthogonal in $L^2(\mathbb{R}^d)$ and by Lemma 3.1.7 $\psi_n^{(1)}$ and $\psi_n^{(2)}$ both converge to $\varphi_0 \in L^2(\mathbb{R}^d)$, which is a contradiction. $\square$

Lemma 3.1.9. *There exists $\delta_0 > 0$, such that for every function $\psi \in H^1(\mathbb{R}^d)$ with $\langle \nabla\psi, \nabla\varphi_0\rangle = 0$ we have*

$$(1-\delta_0)\|\nabla\psi\|^2 + \langle V\psi, \psi\rangle \geq 0. \tag{3.1.65}$$

Proof. We prove the lemma by contradiction. Assume there is no such constant $\delta_0 > 0$. Then there exists a sequence of functions $g_n \in H^1(\mathbb{R}^d)$ with

$$\langle \nabla g_n, \nabla\varphi_0\rangle = 0 \qquad \text{and} \qquad \langle h_{n^{-1}} g_n, g_n\rangle < 0. \tag{3.1.66}$$

Note that for $c_1, c_2 \in \mathbb{C}$ we have

$$\begin{aligned}\langle h_{n^{-1}}(c_1 g_n + c_2\varphi_0), (c_1 g_n + c_2\varphi_0)\rangle = c_1^2\langle h_{n^{-1}} g_n, g_n\rangle + c_2^2\langle h_{n^{-1}}\varphi_0, \varphi_0\rangle \\ + 2\operatorname{Re} c_1\overline{c_2}\langle h_{n^{-1}} g_n, \varphi_0\rangle.\end{aligned} \tag{3.1.67}$$

Furthermore, it is easy to see that

$$\operatorname{Re}\langle h_{n^{-1}} g_n, \varphi_0\rangle = \operatorname{Re}\langle g_n, h\varphi_0\rangle - n^{-1}\operatorname{Re}\langle \nabla g_n, \nabla\varphi_0\rangle = 0 \tag{3.1.68}$$

and

$$\langle h_{n^{-1}}\varphi_0, \varphi_0\rangle = \langle h\varphi_0, \varphi_0\rangle - n^{-1}\|\nabla\varphi_0\|^2 = -n^{-1}. \tag{3.1.69}$$

Hence, we conclude that for any linear combination $c_1 g_n + c_2\varphi_0$ we have

$$\langle h_{n^{-1}}(c_1 g_n + c_2\varphi_0), (c_1 g_n + c_2\varphi_0)\rangle < 0. \tag{3.1.70}$$

Since by (3.1.66) the functions φ_0 and g_n are linearly independent, for any $n \in \mathbb{N}$ we can find a linear combination f_n of φ_0 and g_n, such that f_n is orthogonal to the ground state of $h_{n^{-1}}$. According to Lemma 3.1.8 for sufficiently large $n \in \mathbb{N}$ the operator $h_{n^{-1}}$ has only one negative eigenvalue, which yields $\langle h_{n^{-1}} f_n, f_n\rangle \geq 0$. This is a contradiction to (3.1.70). □

Combining Lemma 3.1.6 and Lemma 3.1.9 proves Theorem 3.1.2 **(i)**. □

Proof of statement (ii) of Theorem 3.1.2. Note that for $\alpha_0 \in (0,1)$ the sequence of eigenfunctions ψ_n of the operators $h_{n^{-1}}$, normalized by $\|\nabla\psi_n\| = 1$, does not necessarily converge in $L^2(\mathbb{R}^d)$. To ensure that the quadratic form q is well defined for the weak limit $\varphi_1 \in \dot{H}^1(\mathbb{R}^d)$ and that $\langle V\psi_n, \psi_n\rangle$ converges to $\langle V\varphi_1, \varphi_1\rangle$ as $n \to \infty$, we assume that the potential V satisfies (3.1.9). We will prove part **(ii)** of Theorem 3.1.2 in two steps. In Lemma 3.1.10 we prove the existence of a function φ_1 satisfying (3.1.10). Then, in Lemma 3.1.11 we prove the uniqueness of φ_1 and the inequality (3.1.13).

Lemma 3.1.10. *Assume that* (3.1.5) *and* (3.1.6) *hold for* $\alpha_0 \in (0,1)$ *and in addition*

$$\langle |V|\psi, \psi\rangle \leq C\|\nabla\psi\|^2 \tag{3.1.71}$$

is satisfied for any function $\psi \in \dot{H}^1(\mathbb{R}^d)$ *and some constant* $C > 0$. *Then, there exists a function* $\varphi_1 \in \dot{H}^1(\mathbb{R}^d)$ *with*

$$\|\nabla\varphi_1\|^2 + \langle V\varphi_1, \varphi_1\rangle = 0. \tag{3.1.72}$$

Moreover, φ_1 *satisfies*

$$\nabla\left(|\cdot|^{\alpha_0}\varphi_1\right) \in L^2(\mathbb{R}^d) \qquad \textit{and} \qquad (1 + |\cdot|)^{\alpha_0 - 1}\varphi_1 \in L^2(\mathbb{R}^d). \tag{3.1.73}$$

Proof. By assumption (3.1.5) there exists a sequence of functions $\psi_n \in \dot{H}^1(\mathbb{R}^d)$

satisfying

$$\left(1 - n^{-1}\right) \|\nabla \psi_n\|^2 + \langle V \psi_n, \psi_n \rangle < 0 \qquad \text{and} \qquad \|\nabla \psi_n\| = 1. \tag{3.1.74}$$

Repeating the same arguments as in Lemma 3.1.4 shows that there is a subsequence, also denoted by $(\psi_n)_{n\in\mathbb{N}}$, which converges in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ to some function $\varphi_1 \in \dot{H}^1(\mathbb{R}^d)$. Let us prove that φ_1 is a minimizer of the quadratic form of h in $\dot{H}^1(\mathbb{R}^d)$ by showing $\langle V\varphi_1, \varphi_1\rangle = -1$. We fix the constant $b > 0$ and construct functions χ_1, χ_2 according to Lemma 2.1.7. By $\chi_1^2 + \chi_2^2 = 1$ we have

$$\langle V\varphi_1, \varphi_1\rangle = \langle V\varphi_1, \varphi_1\chi_1^2\rangle + \langle V\varphi_1, \varphi_1\chi_2^2\rangle. \tag{3.1.75}$$

Note that

$$\begin{aligned} \langle V\varphi_1, \varphi_1\chi_1^2\rangle &= \langle V(\varphi_1 - \psi_n), \varphi_1\chi_1^2\rangle + \langle V\psi_n, \varphi_1\chi_1^2\rangle \\ &= \langle V(\varphi_1 - \psi_n), \varphi_1\chi_1^2\rangle + \langle V\psi_n, \psi_n\chi_1^2\rangle + \langle V\psi_n, (\varphi_1 - \psi_n)\chi_1^2\rangle. \end{aligned} \tag{3.1.76}$$

We estimate the first term on the r.h.s. of (3.1.76) by

$$\begin{aligned} |\langle V(\varphi_1 - \psi_n), \varphi_1\chi_1^2\rangle| &\le \||V|^{\frac{1}{2}}\chi_1(\varphi_1 - \psi_n)\| \cdot \||V|^{\frac{1}{2}}\varphi_1\| \\ &\le C\||V|^{\frac{1}{2}}\chi_1(\varphi_1 - \psi_n)\| \cdot \|\nabla\varphi_1\| \\ &\le C\left(\varepsilon\|\nabla\left(\chi_1(\varphi_1 - \psi_n)\right)\|^2 + C(\varepsilon)\|\chi_1(\varphi_1 - \psi_n)\|^2\right)^{\frac{1}{2}}. \end{aligned} \tag{3.1.77}$$

Here we used (3.1.2), (3.1.9), $|\chi_1| \le 1$ and $\|\nabla\varphi_1\| \le 1$. Moreover, we have

$$\|\nabla(\chi_1(\varphi_1 - \psi_n))\|^2 \le 2\|\nabla\chi_1\|^2\|\varphi_1 - \psi_n\|^2_{\mathrm{supp}\,(\chi_1)} + 2\|\nabla(\varphi_1 - \psi_n)\|^2. \tag{3.1.78}$$

Since $\psi_n \to \varphi_1$ in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and χ_1 is compactly supported, for fixed $\varepsilon_1 > 0$ and large $n \in \mathbb{N}$ we get

$$\|\nabla(\chi_1(\varphi_1 - \psi_n))\|^2 \le 2\varepsilon_1 + 4\|\nabla\varphi_1\|^2 + 4\|\nabla\psi_n\|^2 \le 9. \tag{3.1.79}$$

For any fixed $\tilde{\varepsilon} > 0$ and large n this implies

$$|\langle V(\varphi_1 - \psi_n), \chi_1^2 \varphi_1 \rangle| \le C \left(9\varepsilon + C(\varepsilon)\|\chi_1(\varphi_1 - \psi_n)\|^2\right)^{\frac{1}{2}} \le \tilde{\varepsilon}. \tag{3.1.80}$$

Applying similar arguments to the last term on the r.h.s. of (3.1.76) yields

$$|\langle V\psi_n\chi_1, (\varphi_1 - \psi_n)\chi_1 \rangle| \le \tilde{\varepsilon}. \tag{3.1.81}$$

Hence, we get

$$\langle V\varphi_1\chi_1, \varphi_1\chi_1 \rangle \le \langle V\psi_n\chi_1, \psi_n\chi_1 \rangle + 2\tilde{\varepsilon}. \tag{3.1.82}$$

Further, by (3.1.71) we have

$$\langle V\varphi_1\chi_2, \varphi_1\chi_2 \rangle \le C\|\nabla(\varphi_1\chi_2)\|^2 \le 2C\|(\nabla\varphi_1)\chi_2\|^2 + 2C\|(\nabla\chi_2)\varphi_1\|^2. \tag{3.1.83}$$

Since φ_1 belongs to the space $\dot{H}^1(\mathbb{R}^d)$ and χ_2 is bounded and supported in the region $\{x \in \mathbb{R}^d : |x| \ge b\}$, the first term on the r.h.s. of (3.1.83) is arbitrarily small if b is sufficiently large. Due to (2.1.35) we have

$$\|(\nabla\chi_2)\varphi_1\|^2 \le \varepsilon\|\nabla\varphi_1\|^2 = \varepsilon \tag{3.1.84}$$

for $\tilde{b} > 0$ sufficiently large. This shows that the second term on the r.h.s. of (3.1.83) can be done arbitrarily small. Therefore, we obtain

$$\langle V\varphi_1\chi_2, \varphi_1\chi_2 \rangle \le 2\tilde{\varepsilon}. \tag{3.1.85}$$

Collecting estimates (3.1.82) and (3.1.85) yields

$$\langle V\varphi_1, \varphi_1 \rangle \le \langle V\psi_n\chi_1, \psi_n\chi_1 \rangle + 4\tilde{\varepsilon} \tag{3.1.86}$$

for $n \in \mathbb{N}$ sufficiently large.

Let us estimate the r.h.s. of (3.1.86). Assumption (3.1.9) implies

$$\begin{aligned}\langle V\psi_n\chi_1, \psi_n\chi_1\rangle &= \langle V\psi_n, \psi_n\rangle - \langle V\psi_n\chi_2, \psi_n\chi_2\rangle \\ &\le \langle V\psi_n, \psi_n\rangle + C\|\nabla(\psi_n\chi_2)\|^2 \\ &\le -\left(1-n^{-1}\right) + C\|\nabla(\psi_n\chi_2)\|^2.\end{aligned} \tag{3.1.87}$$

Due to the remark after Lemma 3.1.4 we can choose $n \in \mathbb{N}$ and $\tilde{b} > 0$, such that $\|\nabla(\psi_n\chi_2)\| \le \varepsilon$. Therefore, we conclude

$$\langle V\varphi_1, \varphi_1\rangle = -1 \qquad \text{and} \qquad \|\nabla\varphi_1\|^2 + \langle V\varphi_1, \varphi_1\rangle = 0. \tag{3.1.88}$$

Now we prove that

$$\nabla\left(|\cdot|^{\alpha_0}\varphi_1\right) \in L^2(\mathbb{R}^d) \qquad \text{and} \qquad (1+|\cdot|)^{\alpha_0-1}\varphi_1 \in L^2(\mathbb{R}^d). \tag{3.1.89}$$

Let G_ε be the function defined by (3.1.41). Since φ_1 is a minimizer of the quadratic form of (3.1.88) in $\dot{H}^1(\mathbb{R}^d)$, it satisfies the Euler-Lagrange equation in a generalized sense, i.e.,

$$\langle \nabla\varphi_1, \nabla\psi\rangle + \langle V\varphi_1, \psi\rangle = 0, \qquad \psi \in \dot{H}^1(\mathbb{R}^d). \tag{3.1.90}$$

For $\psi = G_\varepsilon^2\varphi_1$ we therefore obtain

$$\langle \nabla\varphi_1, \nabla\left(G_\varepsilon^2\varphi_1\right)\rangle + \langle V\varphi_1, G_\varepsilon^2\varphi_1\rangle = 0. \tag{3.1.91}$$

Computations similar to (3.1.49), together with (3.1.91), imply

$$\|\nabla(\varphi_1 G_\varepsilon)\|^2 - \int |\varphi_1|^2|\nabla G_\varepsilon|^2\,\mathrm{d}x + \int V|\varphi_1 G_\varepsilon|^2\,\mathrm{d}x = 0. \tag{3.1.92}$$

By (3.1.44) we can rewrite (3.1.92) as

$$\|\nabla(\varphi_1 G_\varepsilon)\|^2 + \langle V\varphi_1 G_\varepsilon, \varphi_1 G_\varepsilon\rangle - \alpha_0^2\int\limits_{\{|x|\ge 2R\}} \frac{|G_\varepsilon\varphi_1|^2}{|x|^2}\,\mathrm{d}x \le \int\limits_{\{R\le|x|\le 2R\}} |\varphi_1|^2|\nabla G_\varepsilon|^2\,\mathrm{d}x. \tag{3.1.93}$$

Since the function $|\nabla G_\varepsilon|$ is uniformly bounded in ε for $|x| \in [R, 2R]$, we conclude

$$\begin{aligned}\int_{\{R\le|x|\le 2R\}} |\varphi_1|^2 |\nabla G_\varepsilon|^2 \,\mathrm{d}x \le C \int_{\{R\le|x|\le 2R\}} |\varphi_1|^2 \,\mathrm{d}x \le C_1 R^2 \int \frac{|\varphi_1|^2}{(2|x|)^2} \,\mathrm{d}x \\ \le C_2 \int |\nabla\varphi_1|^2 \,\mathrm{d}x \le C_2, \qquad (3.1.94)\end{aligned}$$

where the constant $C_2 > 0$ does not depend on $\varepsilon > 0$. Similar to the proof of Lemma 3.1.5, assumption (3.1.6) implies

$$\|\nabla(\varphi_1 G_\varepsilon)\| \le C.$$

Taking $\varepsilon \to 0$ yields $\|\nabla(|x|^{\alpha_0}\varphi_1)\| < \infty$, which together with the Hardy inequality implies

$$(1 + |\cdot|)^{\alpha_0 - 1}\varphi_1 \in L^2(\mathbb{R}^d). \qquad (3.1.95)$$

This completes the proof. □

Lemma 3.1.11. *Assume that*

$$\|V\psi\|^2 \le C\left(\|\nabla\psi\|^2 + \|\psi\|^2\right) \qquad (3.1.96)$$

for some $C > 0$ and every function $\psi \in C_0^\infty(\mathbb{R}^d)$. Then the solution $\varphi_1 \in \dot{H}^1(\mathbb{R}^d)$ in Lemma 3.1.10 is unique. Moreover, there exists a constant $\delta_1 > 0$, such that for any function $\psi \in \dot{H}^1(\mathbb{R}^d)$ with $\langle\nabla\psi, \nabla\varphi_1\rangle = 0$ we have

$$\langle h\psi, \psi\rangle \ge \delta_1 \|\nabla\psi\|^2. \qquad (3.1.97)$$

Proof. We will prove the lemma by contradiction. Assume that there is no such constant $\delta_1 > 0$, then there exists a sequence of functions $(\psi_n^{(1)})_{n\in\mathbb{N}}$ in $\dot{H}^1(\mathbb{R}^d)$, such that $\|\nabla\psi_n^{(1)}\| = 1$, $\langle\nabla\psi_n^{(1)}, \nabla\varphi_1\rangle = 0$ and

$$\left(1 - n^{-1}\right)\|\nabla\psi_n^{(1)}\|^2 + \langle V\psi_n^{(1)}, \psi_n^{(1)}\rangle < 0. \qquad (3.1.98)$$

Moreover, there exists a subsequence, which by abuse of notation is denoted by $(\psi_n^{(1)})_{n\in\mathbb{N}}$ again, and a function $\tilde{\varphi}_1 \in \dot{H}^1(\mathbb{R}^d)$, such that $\psi_n^{(1)} \rightharpoonup \tilde{\varphi}_1$ in $\dot{H}^1(\mathbb{R}^d)$ and therefore $\psi_n^{(1)} \to \tilde{\varphi}_1$ in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$. Obviously, both functions φ_1 and $\tilde{\varphi}_1$ are linearly independent and $\tilde{\varphi}_1$ is also a minimizer of the quadratic form q of the operator h. Due to (3.1.90) with $\psi = \tilde{\varphi}_1$, any linear combination of φ_1 and $\tilde{\varphi}_1$ is also a minimizer of the quadratic form q. By Hardy's inequality both functions φ_1 and $\tilde{\varphi}_1$ belong to the weighted L^2-space with the weight $(1 + |\cdot|)^{-2}$. Since the subspace of linear combinations of φ_1 and $\tilde{\varphi}_1$ is two-dimensional, it contains two functions which are orthogonal with respect to the weighted scalar product. At least one of these functions, say f, has a non-trivial positive part f_+ and a non-trivial negative part f_-, which are also minimizers of the quadratic form of the operator h and satisfy the corresponding Schrödinger equation. Functions f_+ and f_- are zero on some open sets. Since V satisfies (3.1.96), the unique continuation theorem [SS80, Theorem 2.1] implies $f_+ = f_- = 0$. This contradiction proves Lemma 3.1.11. $\square$

Combining Lemma 3.1.10 and Lemma 3.1.11 completes the proof of Theorem 3.1.2 **(ii)**. $\square$

Proof of statement (iii) of Theorem 3.1.2. The proof is similar to the proof of Lemma 3.1.5 and Lemma 3.1.6 after replacing the function G_ε in (3.1.41) with the function

$$J_\varepsilon = \exp\left(\alpha_0 \kappa^{-1} \frac{|x|^\kappa}{1 + \varepsilon |x|^\kappa}\right) \chi_R(x), \quad \kappa = 1 - \frac{\beta}{2}, \tag{3.1.99}$$

where $\chi_R(x)$ is defined by (3.1.42) and where α_0, β are the constants in (3.1.14). Indeed, in this case we have

$$\nabla J_\varepsilon(x) = \frac{x}{|x|}\left(\chi_R'(|x|) J_\varepsilon(x) + \chi_R(|x|)\alpha_0 J_\varepsilon(x) \frac{|x|^{\kappa-1}}{(1 + \varepsilon|x|^\kappa)^2}\right), \tag{3.1.100}$$

which implies

$$|\nabla J_\varepsilon(x)| \leq \alpha_0 |x|^{\kappa-1} |J_\varepsilon(x)| \qquad \text{for } |x| \geq 2R. \tag{3.1.101}$$

Furthermore, we have $\nabla J_\varepsilon(x) = 0$ for $|x| \in [0, R]$. For arguments $x \in \mathbb{R}^d$ with $|x| \in [R, 2R]$ the function $|\nabla J_\varepsilon|$ is uniformly bounded in ε. Repeating the same steps as in the proof of Lemma 3.1.5 with the corresponding eigenfunctions ψ_n of

h_{n-1} and with J_ε instead of G_ε we arrive at

$$\left(1 - n^{-1}\right) \|\nabla(J_\varepsilon \psi_n)\|^2 + \langle V J_\varepsilon \psi_n, J_\varepsilon \psi_n\rangle - \alpha_0^2 \int\limits_{\{|x|>2R\}} \frac{|J_\varepsilon \psi_n|^2}{|x|^{2(1-\kappa)}} \,\mathrm{d}x \leq C, \qquad (3.1.102)$$

where C does not depend on $n \in \mathbb{N}$ or $\varepsilon > 0$. The rest of the proof now follows analogously as in the proofs of Lemma 3.1.5 and Lemma 3.1.6, together with the assumption (3.1.14) with $\beta = 2(1-\kappa)$. □

3.1.2 Connection to Hardy's inequality

Before we proceed to multi-particle systems, we will apply Theorem 3.1.2 to different classes of potentials in the one-particle case. The following two theorems can be found in [BBV20].

Theorem 3.1.12. *Let $d \geq 3$ and $h = -\Delta + V$, where the potential V satisfies*

$$\begin{cases} V \in L^{\frac{d}{2}}(\mathbb{R}^d), & \text{if } d \neq 4, \\ V \in L^2(\mathbb{R}^4) \cap L^{2+\mu}(\mathbb{R}^4) \text{ for some } \mu > 0, & \text{if } d = 4. \end{cases} \qquad (3.1.103)$$

If h has a virtual level, then there exists a non-vanishing solution $\varphi_0 \in \dot{H}^1(\mathbb{R}^d)$ of the equation

$$\|\nabla \varphi_0\|^2 + \langle V \varphi_0, \varphi_0\rangle = 0. \qquad (3.1.104)$$

For any $0 \leq \alpha_0 < \frac{d-2}{2}$ the function φ_0 satisfies

$$\nabla\left(|\cdot|^{\alpha_0} \varphi_0\right) \in L^2(\mathbb{R}^d) \qquad \text{and} \qquad (1 + |\cdot|)^{\alpha_0 - 1} \varphi_0 \in L^2(\mathbb{R}^d). \qquad (3.1.105)$$

Remark. **(i)** Note that Theorem 3.1.12 shows in particular that for any dimension $d \geq 5$ virtual levels are eigenfunctions and not resonances, cf. [Yaf00]. This follows from the fact that the Hardy constant $C_d = \left(\frac{d-2}{2}\right)^2$ is greater than one for all $d \geq 5$.

(ii) With regard to the estimate on the decay rate of φ_0 in (3.1.105), see also [Mur86].

Proof. By Theorem 2.1.4 the assumption (3.1.103) implies that the potential V is relatively form-bounded with relative bound zero, i.e. for every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ with

$$\langle |V|\psi, \psi\rangle \leq \varepsilon \|\nabla\psi\|^2 + C(\varepsilon)\|\psi\|^2 \qquad \text{for any } \psi \in H^1(\mathbb{R}^d). \tag{3.1.106}$$

In order to be able to apply Theorem 3.1.2 we need to show that assumption (3.1.6) is fulfilled. We have to prove that there exist constants $\alpha_0 > 0$, $b > 0$ and $\gamma_0 > 0$, such that for any function $\psi \in H^1(\mathbb{R}^d)$ with $\operatorname{supp}(\psi) \subset \{x \in \mathbb{R}^d : \ |x| \geq b\}$ we have

$$\langle h\psi, \psi\rangle - \gamma_0\|\nabla\psi\|^2 - \alpha_0^2\langle |x|^{-2}\psi, \psi\rangle \geq 0. \tag{3.1.107}$$

Furthermore, in case of $\alpha_0 \in (0, 1)$ we need to show that (3.1.9) is satisfied. Therefore, let $\psi \in H^1(\mathbb{R}^d)$ with $\operatorname{supp}(\psi) \subset \{x \in \mathbb{R}^d : \ |x| \geq b\}$. We set $p = \frac{d}{2}$ and $q = \frac{d}{d-2}$, which implies $\frac{1}{p} + \frac{1}{q} = 1$. Hence, by the Hölder inequality we obtain

$$\int_{\{|x|\geq b\}} |V(x)||\varphi(x)|^2 \,\mathrm{d}x \leq \bigg(\int_{\{|x|\geq b\}} |V(x)|^{\frac{d}{2}} \,\mathrm{d}x\bigg)^{\frac{2}{d}} \bigg(\int_{\{|x|\geq b\}} |\psi(x)|^{\frac{2d}{d-2}} \,\mathrm{d}x\bigg)^{\frac{d-2}{d}}. \tag{3.1.108}$$

Applying Sobolev's inequality yields

$$\bigg(\int_{\{|x|\geq b\}} |\psi(x)|^{\frac{2d}{d-2}} \,\mathrm{d}x\bigg)^{\frac{d-2}{d}} \leq C^*\|\nabla\psi\|^2 \tag{3.1.109}$$

for some constant $C^* > 0$ independent of ψ. Now since by assumption we have $V \in L^{\frac{d}{2}}(\mathbb{R}^d)$, for any $\gamma_0 > 0$ we can choose $b > 0$ sufficiently large, such that

$$\bigg(\int_{\{|x|\geq b\}} |V(x)|^{\frac{d}{2}} \,\mathrm{d}x\bigg)^{\frac{2}{d}} \leq \frac{\gamma_0}{C^*}. \tag{3.1.110}$$

By (3.1.108) we obtain

$$\langle V\psi, \psi\rangle \geq -\langle |V|\psi, \psi\rangle \geq -\gamma_0\|\nabla\psi\|^2. \tag{3.1.111}$$

Hence, we arrive at

$$\begin{aligned} \langle h\psi, \psi\rangle - \gamma_0\|\nabla\psi\|^2 &= \|\nabla\psi\|^2 + \langle V\psi, \psi\rangle - \gamma_0\|\nabla\psi\|^2 \\ &\geq (1-2\gamma_0)\|\nabla\psi\|^2. \end{aligned} \tag{3.1.112}$$

Let

$$0 < \alpha_0 < \sqrt{C_d} = \frac{d-2}{2} \qquad \text{and} \qquad 0 < \gamma_0 < \frac{C_d - \alpha_0^2}{2C_d}. \tag{3.1.113}$$

Then, by (2.1.16) we obtain

$$\langle h\psi, \psi\rangle - \gamma_0\|\nabla\psi\|^2 - \alpha_0^2\langle |x|^{-2}\psi, \psi\rangle \geq (C_d - \alpha_0^2 - 2\gamma_0 C_d)\langle |x|^{-2}\psi, \psi\rangle. \tag{3.1.114}$$

Since by (3.1.113) we have $(C_d - \alpha_0^2 - 2\gamma_0 C_d) \geq 0$, it remains to apply Theorem 3.1.2 to complete the proof. □

Obviously, any short-range potential belongs to $L^{\frac{d}{2}}(\mathbb{R}^d)$. Even though Theorem 3.1.12 covers a larger class than short-range potentials, it cannot be applied to potentials decaying as $c|x|^{-2}$ or $c|x|^{-\beta}$ for some $\beta \in (0,2)$ as $|x| \to \infty$. It is to be expected that a slower decay rate of the potential will result in a faster decay of the solution φ_0. In dimensions $d = 3$ and $d = 4$ resonances can become threshold-eigenvalues and the decay rate of the corresponding eigenfunctions in all dimensions $d \geq 3$ can change into the sub-exponential one, if the potential decays slower than $|x|^{-2}$. However, in order to be able to apply Theorem 3.1.2 for such cases, we consider potentials positive at infinity.

Theorem 3.1.13. *Let $d \geq 3$ and $h = -\Delta + V$, where the potential V satisfies*

(i) $V \in L^{\frac{d}{2}}_{\mathrm{loc}}(\mathbb{R}^d)$ *for $d \neq 4$ and $V \in L^{2+\mu}_{\mathrm{loc}}(\mathbb{R}^d)$ for some $\mu > 0$ if $d = 4$,*

(ii) *there exist constants $A_1, A_2 \geq 0, \beta_1 > 0$ and $\beta_2 \in (0,2]$ with*

$$\beta_1 |x|^{-\beta_2} \leq V(x) \leq A_1 \qquad \text{for} \qquad |x| \geq A_2, \tag{3.1.115}$$

(iii) $V(x) \to 0$ *as $|x| \to \infty$.*

Assume that h has a virtual level, then there exists a solution $\varphi_0 \in \dot{H}^1(\mathbb{R}^d), \varphi_0 \neq 0$

of the equation

$$\|\nabla\varphi_0\|^2 + \langle V\varphi_0, \varphi_0\rangle = 0. \tag{3.1.116}$$

If $\beta_2 = 2$*, then for any* $0 \le \alpha_0 < \sqrt{\beta_1 + 4^{-1}(d-2)^2}$ *the function* φ_0 *satisfies*

$$\nabla\left(|\cdot|^{\alpha_0}\varphi_0\right) \in L^2(\mathbb{R}^d) \qquad \textit{and} \qquad (1+|\cdot|)^{\alpha_0-1}\varphi_0 \in L^2(\mathbb{R}^d). \tag{3.1.117}$$

If $\beta_2 < 2$*, then* φ_0 *satisfies*

$$\exp\left(\beta_1|\cdot|^{\kappa}\right)\varphi_0 \in L^2(\mathbb{R}^d), \qquad \textit{where} \quad \kappa = 1 - \frac{\beta_2}{2}. \tag{3.1.118}$$

Remark. **(i)** Theorem 3.1.13 implies in particular that for $d = 3$ zero is an eigenvalue of h for $\beta_1 > \frac{3}{4}$ and in case $d = 4$ zero is an eigenvalue of h for any $\beta_1 > 0$.

(ii) For similar results we refer to [GG07, HJL19b].

Proof. The proof follows from similar arguments as in the proof of Theorem 3.1.12 but with the difference that here we need to apply Theorem 3.1.2 **(iii)**. □

3.2 Virtual levels of multi-particle quantum systems

In this section we apply the results of Theorem 3.1.2 to multi-particle systems and we additionally prove that the estimate of the decay rate is very close to optimal by providing the concrete asymptotic behaviour of such solutions in case of short-range potentials. We stick to the notation introduced in section 2.2 and follow the presentation of [BBV20] and [BB20].

3.2.1 Zero-energy solutions of the N-body Schrödinger equation

We consider a system of $N \ge 3$ quantum particles in dimension $d \ge 3$ with masses $m_i > 0,\ i = 1, \dots, N$, and position vectors $x_i \in \mathbb{R}^d,\ i = 1, \dots, N$. The corresponding

Hamiltonian H_N acting on $L^2\left(\mathbb{R}^{dN}\right)$ is given by

$$H_N = -\sum_{i=1}^{N} \frac{1}{2m_i}\Delta_{x_i} + \sum_{1\le i<j\le N} V_{ij}(x_{ij}), \quad x_{ij} = x_i - x_j, \tag{3.2.1}$$

where the potentials V_{ij} describe the particle pair interactions. In the following we assume that $V_{ij} = V_{ij}^{(1)} + V_{ij}^{(2)}$, such that for some constants $A, C, \nu > 0$

$$|V_{ij}^{(1)}(x_{ij})| \le C|x_{ij}|^{-2-\nu}, \text{ if } |x_{ij}| \ge A \qquad \text{and} \qquad V_{ij}^{(1)} \in L^p_{\text{loc}}(\mathbb{R}^d), \tag{3.2.2}$$

where $p > 2$ for $d = 4$ and $p = \frac{d}{2}$ for $d \neq 4$. Furthermore, we assume that

$$V_{ij}^{(2)} \ge 0 \quad \text{is bounded and} \quad V_{ij}^{(2)}(x_{ij}) \to 0 \quad \text{as} \quad |x_{ij}| \to \infty. \tag{3.2.3}$$

By Theorem 2.1.4 the potentials V_{ij} are relatively form-bounded with relative bound zero. We will consider the operator H_N in the center-of-mass frame, i.e. we consider the operator

$$H = -\Delta_0 + V \tag{3.2.4}$$

defined in (2.2.20).

Definition 3.2.1. For an arbitrary cluster C we say that the corresponding operator $H[C] = -\Delta_0[C] + V[C]$ has a virtual level at zero, if $H[C] \ge 0$ and for all sufficiently small $\varepsilon > 0$ we have

$$\sigma_{\text{ess}}\left(-(1-\varepsilon)\Delta_0[C] + V[C]\right) = [0, \infty) \tag{3.2.5}$$

and

$$\sigma_{\text{disc}}\left(-(1-\varepsilon)\Delta_0[C] + V[C]\right) \neq \emptyset. \tag{3.2.6}$$

Remark. Assume that H has a virtual level, i.e. $C = \{1, \ldots, N\}$ in the upper definition. Then condition (3.2.5) together with the HVZ theorem imply that there exists $\varepsilon > 0$, such that for any cluster C' with $1 < |C'| < N$ we have

$$\sigma\left(-(1-\varepsilon)\Delta_0[C'] + V[C']\right) = [0, \infty). \tag{3.2.7}$$

In particular, (3.2.7) implies that if H has a virtual level, then every cluster Hamiltonian $H[C']$ does not have resonances or eigenvalues at zero.

Theorem 3.2.2. *Consider a system of $N \geq 3$ particles in dimension $d \geq 3$. Suppose that the potentials V_{ij} satisfy (3.2.2) and (3.2.3). Assume that H has a virtual level at zero. Then*

(i) *zero is an eigenvalue of H and the corresponding eigenfunction φ_0 satisfies*

$$\nabla_0 (|\cdot|_1^{\alpha_0} \varphi_0) \in L^2(R_0) \qquad \text{and} \qquad (1 + |\cdot|_1)^{\alpha_0 - 1} \varphi_0 \in L^2(R_0) \tag{3.2.8}$$

for any $0 \leq \alpha_0 < \frac{d(N-1)-2}{2}$.

(ii) *There exists a constant $\delta_0 > 0$, such that for every function $\psi \in H^1(R_0)$ satisfying $\langle \nabla_0 \psi, \nabla_0 \varphi_0 \rangle = 0$ we have*

$$(1 - \delta_0)\|\nabla_0 \psi\|^2 + \langle V\psi, \psi \rangle \geq 0. \tag{3.2.9}$$

(iii) *If $V_{ij}^{(2)}$ satisfies $V_{ij}^{(2)}(x_{ij}) \geq \alpha_{ij}|x_{ij}|^{-\beta}$ for $\alpha_{ij} > 0$ and $\beta \in (0,2)$, then zero is an eigenvalue of H and the corresponding eigenfunction φ_0 satisfies*

$$\exp\left(\mu |\cdot|_1^{\kappa}\right) \varphi_0 \in L^2(R_0), \tag{3.2.10}$$

where $\kappa = 1 - \frac{\beta}{2}$ and $\mu > 0$ depends on the coefficients α_{ij} and on the masses of the particles only.

Remark. Theorem 3.2.2 shows in particular that for systems of d-dimensional particles with $d \geq 3$ virtual levels can be resonances only in case of two-body cluster Hamiltonians.

To emphasize the main idea of the proof, we first prove the statement for $N = 3$ in dimension $d = 3$ and then extend the proof to the remaining cases afterwards. We will use the following two lemmas.

Lemma 3.2.3 (cf. [Zhi74, Lemma 2.1])**.** *Suppose that Z_2 and Z_2' are arbitrary partitions of the system into two clusters with $Z_2 \neq Z_2'$. Let $K(Z_2, \kappa)$ and $K(Z_2', \kappa)$*

be the regions defined in (2.2.46)*. Then there exists* $\kappa_0 > 0$*, such that for all* $0 < \kappa < \kappa_0$ *we have*

$$K(Z_2, \kappa) \cap K(Z_2', \kappa) = \{0\}. \tag{3.2.11}$$

Lemma 3.2.4 (cf. [VZ83, Lemma 5.1])**.** *Given* $\varepsilon > 0$ *and* $\kappa > 0$*, for each partition* Z_p *one can find* $0 < \kappa' < \kappa$ *and functions* $u_{Z_p},\ v_{Z_p}\ :\ R_0 \to \mathbb{R}$*, such that*

$$u_{Z_p}^2 + v_{Z_p}^2 = 1, \qquad u_{Z_p}(x) = \begin{cases} 1, & x \in K\left(Z_p, \kappa'\right) \\ 0, & x \notin K\left(Z_p, \kappa\right) \end{cases} \tag{3.2.12}$$

and

$$|\nabla_0 u_{Z_p}|^2 + |\nabla_0 v_{Z_p}|^2 < \varepsilon \left[|v_{Z_p}|^2 |x|_1^{-2} + |u_{Z_p}|^2 |q\left(Z_p\right)|_1^{-2} \right] \tag{3.2.13}$$

for $x \in K\left(Z_p, \kappa\right) \setminus K\left(Z_p, \kappa'\right)$*.*

Remark. Using Lemma 3.2.4 we will make a partition of unity to separate the cones $K(Z_p, \kappa)$ for all partitions Z_p, starting with $p = 2$. We note at this point that due to $u_{Z_p}(0) = 1$ and Lemma 3.2.3 formally the functions in Lemma 3.2.4 do not form a partition of unity. However, since we always consider the region $\{x \in R_0 : |x|_1 \geq R\}$ with $R > 0$ only, this will not be a problem. For $p > 2$ we can avoid this technical difficulty in a similar way.

Proof of Theorem 3.2.2 for $N = 3$ *particles in dimension* $d = 3$*.* We show that all conditions of statement **(i)** of Theorem 3.1.2 are fulfilled. Note that in this case we have to prove (3.2.8) with $\alpha_0 \in (0, 2)$ in dimension $\dim R_0 = d(N-1)$. In addition show that if $V_{ij}^{(2)}(x_{ij}) \geq \alpha_{ij} |x_{ij}|^{-\beta}$ for some constants $\alpha_{ij} > 0$ and $\beta \in (0, 2)$, then (3.2.10) follows from statement **(iii)** of Theorem 3.1.2.

Since $V_{ij} \in L^{\frac{3}{2}}_{\text{loc}}(\mathbb{R}^3)$ and it decays at infinity, by Theorem 2.1.4 for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ with

$$\langle |V_{ij}| \varphi, \varphi \rangle \leq \varepsilon \|\nabla_{x_{ij}} \varphi\|^2 + C(\varepsilon) \|\varphi\|^2 \qquad \text{for any } \varphi \in H^1(R_0), \tag{3.2.14}$$

which implies (3.1.2) for $V = \sum_{1 \leq i < j \leq N} V_{ij}$. In order to prove statements **(i)** and **(ii)** of the theorem it is sufficient to prove that we have

$$L[\varphi] := (1 - \gamma_0) \left\|\nabla_0 \varphi\right\|^2 + \langle V\varphi, \varphi \rangle - \alpha_0^2 \||x|_1^{-1} \varphi\|^2 \geq 0 \tag{3.2.15}$$

for some constant $\gamma_0 > 0$, any $\alpha_0 \in (1,2)$ and any function $\varphi \in H^1(R_0)$ satisfying $\operatorname{supp}(\varphi) \subset \{x \in R_0 : |x|_1 \geq R\}$ for some sufficiently large $R > 0$.
The proof of (3.2.15) is based on ideas of the proof of an estimate from below of the quadratic form of a multi-particle operator in [VZ84]. The main difference is that in [VZ84] it was sufficient to prove a similar inequality with an arbitrary small $\alpha_0 > 0$ in the case when the operators of the subsystems do not have any virtual levels. In our case we need to prove (3.2.15) for $\alpha_0 \in (1,2)$. Similar to [VZ83] we will make a partition of unity of the configuration space R_0 of the system, separating the cones $K(Z_2,\kappa)$, corresponding to different partitions into two clusters. We will choose $\kappa > 0$ sufficiently small to compensate the term $-\alpha_0|x|_1^{-1}$ with a small part of the kinetic energy.
Let u_{Z_2} be the localization functions defined by (3.2.12). Recall that u_{Z_2} is supported in the cone in the configuration space, where two particles belonging to the same cluster in Z_2 are close one to another and the third particle is very far away from this cluster. Applying Lemma 3.2.3 and Lemma 3.2.4 yields

$$L[\varphi] \geq \sum_{Z_2} L_1\left[\varphi u_{Z_2}\right] + L_2\left[\varphi \mathcal{V}\right], \tag{3.2.16}$$

where $\mathcal{V} = \sqrt{1 - \sum_{Z_2} u_{Z_2}^2}$ and the functionals $L_1, L_2 : H^1(R_0) \to \mathbb{R}$ are defined by

$$\begin{aligned} L_1[\psi] &:= (1-\gamma_0)\left\|\nabla_0\psi\right\|^2 + \langle V\psi,\psi\rangle - \|\alpha_0|x|_1^{-1}\psi\|^2 - \varepsilon\left\||q(Z_2)|_1^{-1}\psi\right\|^2, \\ L_2[\psi] &:= (1-\gamma_0)\left\|\nabla_0\psi\right\|^2 + \langle V\psi,\psi\rangle - \|\alpha_0|x|_1^{-1}\psi\|^2 - \varepsilon\left\||x|_1^{-1}\psi\right\|^2. \end{aligned} \tag{3.2.17}$$

We will prove that $L_1[\varphi u_{Z_2}] \geq 0$ and $L_2[\varphi\mathcal{V}] \geq 0$, if $\varepsilon, \gamma_0 > 0$ and $\kappa > 0$ are sufficiently small and $R > 0$ is sufficiently large. Here, $\kappa > 0$ is the parameter in Definition 2.2.1 of the cone

$$K(Z_2,\kappa) = \{x \in R_0 :\ |q(Z_2)|_1 \leq \kappa|\xi(Z_2)|_1\}. \tag{3.2.18}$$

At first we estimate $L_1[\varphi u_{Z_2}]$ for an arbitrary partition $Z_2 = (C_1, C_2)$. Note that

$$L_1[\varphi u_{Z_2}] = \langle H(Z_2)\varphi u_{Z_2}, \varphi u_{Z_2}\rangle - \gamma_0 \left\|\nabla_{q(Z_2)}(\varphi u_{Z_2})\right\|^2 + (1-\gamma_0)\left\|\nabla_{\xi(Z_2)}(\varphi u_{Z_2})\right\|^2 + \langle I(Z_2)\varphi u_{Z_2}, \varphi u_{Z_2}\rangle - \|\alpha_0|x|_1^{-1}\varphi u_{Z_2}\|^2 - \varepsilon \left\||q(Z_2)|_1^{-1}\varphi u_{Z_2}\right\|^2. \tag{3.2.19}$$

Without loss of generality we assume that in $Z_2 = (C_1, C_2)$ the cluster C_1 has two particles and C_2 has only one particle. Applying (3.2.7) we get

$$\langle H(Z_2)\varphi u_{Z_2}, \varphi u_{Z_2}\rangle \geq \mu_0 \|\nabla_{q(Z_2)}(\varphi u_{Z_2})\|^2 \tag{3.2.20}$$

for some $\mu_0 > 0$ independent of φ. For sufficiently small $\varepsilon > 0$ and $\gamma_0 > 0$ this yields

$$\langle H(Z_2)\varphi u_{Z_2}, \varphi u_{Z_2}\rangle - \gamma_0\left\|\nabla_{q(Z_2)}(\varphi u_{Z_2})\right\|^2 - \varepsilon\left\||q(Z_2)|_1^{-1}\varphi u_{Z_2}\right\|^2 \geq \frac{\mu_0}{2}\|\nabla_{q(Z_2)}(\varphi u_{Z_2})\|^2. \tag{3.2.21}$$

Therefore, we arrive at

$$L_1[\varphi u_{Z_2}] \geq \frac{\mu_0}{2}\|\nabla_{q(Z_2)}(\varphi u_{Z_2})\|^2 + (1-\gamma_0)\left\|\nabla_{\xi(Z_2)}(\varphi u_{Z_2})\right\|^2 + \langle I(Z_2)\varphi u_{Z_2}, \varphi u_{Z_2}\rangle - \|\alpha_0|x|_1^{-1}\varphi u_{Z_2}\|^2. \tag{3.2.22}$$

On the support of u_{Z_2} we have $|q(Z_2)|_1 \leq \kappa|\xi(Z_2)|_1$, which by Hardy's inequality implies

$$\frac{\mu_0}{2}\|\nabla_{q(Z_2)}(\varphi u_{Z_2})\|^2 \geq \frac{\mu_0}{8\kappa^2}\left\||\xi(Z_2)|_1^{-1}\varphi u_{Z_2}\right\|^2. \tag{3.2.23}$$

Since $\operatorname{supp}(\varphi u_{Z_2}) \subset K(Z_2,\kappa)\setminus B(R)$ with $B(R)$ being a ball in the space R_0 with radius R, it holds $|x_{ij}| \geq C|\xi(Z_2)|_1$ for $i \in C_1,\ j \in C_2$ and some $C > 0$. Therefore, by $V_{ij} \geq V_{ij}^{(1)}$ and $|V_{ij}^{(1)}(x_{ij})| \leq C|\xi(Z_2)|_1^{-2-\nu}$ we can estimate the r.h.s. of (3.2.22) from below by

$$\frac{\mu_0}{8\kappa^2}\||\xi(Z_2)|_1^{-1}\varphi u_{Z_2}\|^2 - C\||\xi(Z_2)|_1^{-1}\varphi u_{Z_2}\|^2 - \alpha_0^2\||\xi(Z_2)|_1^{-1}\varphi u_{Z_2}\|^2 \geq 0 \tag{3.2.24}$$

for sufficiently small $\kappa > 0$. Now to prove part **(i)** and part **(ii)** of the theorem in the case of $N = 3$ it suffices to show $L_2[\mathcal{V}\varphi] \geq 0$. Note that on the support of $\mathcal{V}$

all the distances between the particles are large. Note that $V_{ij} \geq V_{ij}^{(1)}$ and on the support of $\mathcal{V}\varphi$ we have

$$|V_{ij}^{(1)}(x_{ij})| \leq C|x|_1^{-2-\nu} \leq \varepsilon |x|_1^{-2}, \qquad i,j = 1,2,3,\ i \neq j, \tag{3.2.25}$$

where $\varepsilon > 0$ can be chosen arbitrarily small by choosing $R > 0$ sufficiently large. This yields

$$L_2[\mathcal{V}\varphi] \geq (1-\gamma_0)\|\nabla_0\left(\mathcal{V}\varphi\right)\|^2 - \left(\alpha_0^2 - 2\varepsilon\right)\||x|_1^{-1}\varphi\mathcal{V}\|^2. \tag{3.2.26}$$

Since $\dim R_0 = 6$, Hardy's inequality implies

$$\|\nabla_0(\mathcal{V}\varphi)\|^2 \geq 4\||x|_1^{-1}\mathcal{V}\varphi\|^2. \tag{3.2.27}$$

For $\alpha_0 < 2$ we can choose $0 < \varepsilon < \frac{4-\alpha_0^2}{2}$ and $\gamma_0 > 0$ sufficiently small, such that $L_2[\varphi\mathcal{V}] \geq 0$, which completes the proof of statement **(i)** and **(ii)** for $d = 3$ and $N = 3$.
In order to prove statement **(iii)** it suffices to note that for $\beta \in (0,2)$ and $\alpha_{ij} > 0$ we have $\sum_{1\leq i<j\leq N} V_{ij}^{(2)}(x_{ij}) \geq C|x|_1^{-\beta}$. Applying statement **(iii)** of Theorem 3.1.2 completes the proof for $N = 3$. □

In order to prove Theorem 3.2.2 for the case $d = 3$ and $N \geq 4$ we need the following Lemma, see [BBV20]. It goes back to the work of M. A. Antonets, G. M. Zhislin, and I. A. Shereshevskij [AZS].

Lemma 3.2.5 (cf. [BBV20, Theorem B.2])**.** *For $N \geq 3$ there exist functions $\kappa, \kappa' : \mathbb{N} \to (0,\infty)$ with $\kappa'(m) < \kappa(m)$ and where for any $2 \leq m \leq N-1$ and any cluster decompositions Z_m, Z_m' with $|Z_m| = |Z_m'| = m$ and $Z_m \neq Z_m'$ we have*

$$K\left(Z_m, \kappa(m)\right) \cap K\left(Z_m', \kappa(m)\right) \subset \bigcup_{Z_n:\, n<m} K\left(Z_n, \kappa'(n)\right). \tag{3.2.28}$$

*Proof of Theorem 3.2.2 **(i)**-**(ii)** for $N \geq 4$ and $d = 3$.* For the sake of convenience, in this part we assume that $V_{ij}^{(2)} = 0$ for every $i,j = 1,\dots,N$ with $i \neq j$. Let $L[\cdot]$ be the functional defined in (3.2.15). In the following we will show that we have $L[\varphi] \geq 0$ for every $0 \leq \alpha_0 < \frac{3N-5}{2}$ and every $\varphi \in H^1(R_0)$ with $\text{supp}(\varphi) \subset R_0 \setminus B(R)$, where $R > 0$ is sufficiently large. Analogously to the case $N = 3$ we get

$$L[\varphi] \geq \sum_{Z_2} L_1 [\varphi u_{Z_2}] + L_2 [\varphi \mathcal{V}_2], \tag{3.2.29}$$

where the functionals L_1, L_2 are defined in (3.2.17) and $\mathcal{V}_2 = \sqrt{1 - \sum_{Z_2} u_{Z_2}^2}$. By repeating the same arguments as in the case $N = 3$, one can easily show that we have $L_1[\varphi u_{Z_2}] \geq 0$ for all two-cluster partitions Z_2. Hence, it suffices to prove $L_2[\mathcal{V}_2 \varphi] \geq 0$.

Due to Lemma 3.2.5 we can find $\kappa(3) > 0$, such that on the support of $\mathcal{V}_2\varphi$ the cones $K(Z_3, \kappa(3))$ and $K(Z_3', \kappa(3))$ do not overlap for $Z_3 \neq Z_3'$. Applying Lemma 3.2.4 yields

$$L_2 [\mathcal{V}_2 \varphi] \geq \sum_{Z_3} L_1'[u_{Z_3} \mathcal{V}_2 \varphi] + L_2'[\mathcal{V}_3 \mathcal{V}_2 \varphi], \tag{3.2.30}$$

where $\mathcal{V}_3 = \sqrt{1 - \sum_{Z_3} u_{Z_3}^2}$ on the support of $\mathcal{V}_2\varphi$ and

$$\begin{aligned} L_1'[\psi] = \langle H(Z_3)\psi, \psi\rangle - \gamma_0 \left\|\nabla_{q(Z_3)}\psi\right\|^2 + (1-\gamma_0) \left\|\nabla_{\xi(Z_3)}\psi\right\|^2 \\ + \langle I(Z_3)\psi, \psi\rangle - \left(\alpha_0^2 + \varepsilon\right) \||x|_1^{-1}\psi\|^2 - \varepsilon \left\|\psi|q(Z_3)|_1^{-1}\right\|^2, \end{aligned} \tag{3.2.31}$$

$$L_2'[\psi] = (1-\gamma_0) \|\nabla_0\psi\|^2 + \langle V\psi, \psi\rangle - \|\alpha_0|x|_1^{-1}\psi\|^2 - 2\varepsilon \left\||x|_1^{-1}\psi\right\|^2. \tag{3.2.32}$$

Since for each cluster C_j in the partition Z_3 we have (3.2.7), it follows

$$\langle H(Z_3)\psi, \psi\rangle \geq \mu_0 \|\nabla_{q(Z_3)}\psi\|^2 \tag{3.2.33}$$

for some $\mu_0 > 0$ independent of ψ. In addition, on the support of $u_{Z_3}\mathcal{V}_2$ we can estimate $|V_{ij}(x_{ij})| \leq c|\xi(Z_3)|_1^{-2-\nu}$ for i, j belonging to different clusters in Z_3. Consequently, by the same arguments as in the estimate of $L_1[u_{Z_2}\varphi]$ we get $L_1'[u_{Z_3}\mathcal{V}_2\varphi] \geq 0$. Repeating this process, we see that to prove the theorem it suffices to show

$$L_3[\psi] := (1-\gamma_0) \|\nabla_0\psi\|^2 + \langle V\psi, \psi\rangle - \|\alpha_0|x|_1^{-1}\psi\|^2 - \varepsilon \left\|\psi|x|_1^{-1}\right\|^2 \geq 0 \tag{3.2.34}$$

for small $\varepsilon, \gamma_0 > 0$ and for functions $\psi \in H^1(R_0)$, which are supported outside the ball of the radius R in the space R_0 in the region, where for any pair of particles i, j it holds $|x_{ij}| \geq \tilde{c}|x|_1$ for some constant $\tilde{c} > 0$. In this region we have

$$|V_{ij}(x_{ij})| \leq c|x|_1^{-2-\nu}. \tag{3.2.35}$$

We choose $0 < \varepsilon < \frac{(3(N-1)-2)^2}{4} - \alpha_0^2$, such that Hardy's inequality in dimension $3(N-1)$ implies (3.2.34). Now we can apply Theorem 3.1.2 to conclude that zero is a simple eigenvalue of H and the corresponding eigenfunction φ_0 satisfies

$$\nabla_0 \left(| \cdot |_1^{\alpha_0} \varphi_0 \right) \in L^2(R_0) \qquad \text{and} \qquad (1 + |\cdot|_1)^{\alpha_0 - 1} \varphi_0 \in L^2(R_0) \tag{3.2.36}$$

for every $\alpha_0 < \frac{3N-5}{2}$. This completes the proof of statement **(i)** and **(ii)** of Theorem 3.2.2 in the case $d = 3$ and $N \geq 4$. Finally, since Hardy's inequality holds true for every $d \geq 3$, the proof of the theorem can be adapted to the case $d \geq 4$ by replacing the Hardy constant in the corresponding dimension. Statement **(iii)** of the theorem follows from statement **(iii)** of Theorem 3.1.2 similar to the case of $N = 3$. □

3.2.2 Virtual levels in systems of a fixed permutation symmetry

Since the proof of Theorem 3.2.2 is based on variational methods only, we can extend the result to systems of a fixed permutation symmetry. At first we explain what is meant by permutation symmetry in multi-particle systems by following the presentation of [SS70].

Consider a system of N particles containing $K \leq N$ identical particles, where $x_1, \ldots, x_K$ denote the position vectors of the identical particles. We denote by S_K the group of permutations of the identical particles and for each $g \in S_K$ let

$$T_g \psi(x_1, \ldots, x_N) = \psi(x_{g^{-1}(1)}, \ldots, x_{g^{-1}(K)}, x_{K+1}, \ldots, x_N) \tag{3.2.37}$$

be the operator in L^2, which realizes the permutation of the corresponding identical particles. Let π be an irreducible representation of S_K and denote by P^π the corresponding projection, i.e. $P^\pi L^2$ is a subspace of functions transformed under the action of T_g with respect to the representation π. For any partition $Z_p = (C_1, \ldots, C_p)$

we define

$$S_K(Z_p) = S_K(C_1) \times \ldots \times S_K(C_p) \rtimes S_0, \tag{3.2.38}$$

where each $S_K(C_i)$ is the subgroup of permutations of the identical particles within the subsystems C_i and S_0 is the subgroup of S_K consisting of permutations $g_0 \in S_K$ that reverse the positions of the identical subsystems C_i in the partition Z_p. Here $\rtimes$ is the semidirect product of subgroups for a group. If there are no identical subsystems, then S_0 is the trivial group $S_0 = \{e\}$. Denote by $\pi(Z_p)$ an irreducible representation of $S_K(Z_p)$. The relation $\pi(Z_p) \prec \pi$ means that $\pi(Z_p)$ is contained in the restriction of π to $S_K(Z_p)$. The operators H and $H(Z_p)$ are invariant under the action of the groups S_K and $S_K(Z_p)$, respectively. This allows us to define H^π and $H^{\pi(Z_p)}$ as the restrictions of H and $H(Z_p)$ to the subspaces of functions transformed according to π and $\pi(Z_p)$, respectively.

Similar to the case of Theorem 3.1.2 and Theorem 3.2.2, we first formulate a statement of a general nature and then apply it to multi-particle systems. Therefore, let

$$h = -\Delta + V \qquad \text{in} \quad L^2(\mathbb{R}^d) \tag{3.2.39}$$

be invariant under action of a symmetry group G and let π be an irreducible representation of G. Denote by P^π the projection in $L^2(\mathbb{R}^d)$ onto the subspace of functions transformed according to the representation π. In the following we assume that for every function $\psi \in L^2(\mathbb{R}^d)$ and $\chi \in C_0(\mathbb{R}^d)$ with $\chi(x) = \chi(|x|)$ the condition $P^\pi\psi = \psi$ implies $P^\pi(\chi\psi) = \chi\psi$. We denote $h^\pi = P^\pi h$, $h^\pi_\varepsilon = P^\pi h_\varepsilon$, $\mathcal{H}^\pi = P^\pi H^1(\mathbb{R}^d)$ and $\dot{\mathcal{H}}^\pi = P^\pi \dot{H}^1(\mathbb{R}^d)$. The following theorem is a straightforward generalization of Theorem 3.1.2. However, in this case the minimizers of the quadratic form of the operator h^π are not necessarily unique, but it follows from Lemma 3.1.3 that the corresponding subspaces are still finite-dimensional.

Theorem 3.2.6. *Suppose that V satisfies* (3.1.2)*. Furthermore, assume that we have*

$$h^\pi \geq 0 \qquad \textit{and} \qquad \inf \sigma\left(h^\pi_\varepsilon\right) < 0 \tag{3.2.40}$$

for any $\varepsilon \in (0,1)$. If there exist constants $\alpha_0 > 0$, $b > 0$ and $\gamma_0 \in (0,1)$, such that *for any function* $\psi \in \mathcal{H}^\pi$ *with* $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^d : \ |x| \geq b\}$ *we have*

$$\langle h^\pi \psi, \psi \rangle - \gamma_0 \|\nabla \psi\|^2 - \alpha_0^2 \langle |x|^{-2} \psi, \psi \rangle \geq 0, \tag{3.2.41}$$

then the following assertions hold:

(i) *If* $\alpha_0 > 1$, *then zero is an eigenvalue of* h^π *with finite degeneracy. Denote by* $\mathcal{W}_0$ *the corresponding eigenspace. Then for any* $\varphi_0 \in \mathcal{W}_0$ *we have*

$$\nabla\left(|\cdot|^{\alpha_0}\varphi_0\right) \in L^2(\mathbb{R}^d) \qquad \textit{and} \qquad (1 + |\cdot|)^{\alpha_0 - 1}\varphi_0 \in L^2(\mathbb{R}^d). \tag{3.2.42}$$

Moreover, there exists a constant $\delta_0 > 0$, *such that for any function* $\psi \in \mathcal{H}^\pi$ *satisfying* $\langle \nabla\psi, \nabla\varphi_0 \rangle = 0$ *for all* $\varphi_0 \in \mathcal{W}_0$ *we have*

$$\langle h^\pi \psi, \psi \rangle \geq \delta_0 \|\nabla \psi\|^2. \tag{3.2.43}$$

(ii) *If* $\alpha_0 \in (0, 1)$ *and in addition*

$$\langle |V| \psi, \psi \rangle \leq C \|\nabla \psi\|^2 \tag{3.2.44}$$

for any function $\psi \in \dot{\mathcal{H}}^\pi$ *and some constant* $C > 0$, *then there exists a finite-dimensional subspace* $\mathcal{W}_1 \subset \dot{\mathcal{H}}^\pi$, *such that for any function* $\varphi_1 \in \mathcal{W}_1$ *we have*

$$\|\nabla \varphi_1\|^2 + \langle V \varphi_1, \varphi_1 \rangle = 0. \tag{3.2.45}$$

Moreover, φ_1 *satisfies*

$$\nabla\left(|\cdot|^{\alpha_0}\varphi_1\right) \in L^2(\mathbb{R}^d) \qquad \textit{and} \qquad (1 + |\cdot|)^{\alpha_0 - 1}\varphi_1 \in L^2(\mathbb{R}^d) \tag{3.2.46}$$

and there exists a constant $\delta_1 > 0$, *such that for any function* $\psi \in \dot{\mathcal{H}}^\pi$ *satisfying the condition* $\langle \nabla\psi, \nabla\varphi_1 \rangle = 0$ *for all* $\varphi_1 \in \mathcal{W}_1$ *we have*

$$\langle h^\pi \psi, \psi \rangle \geq \delta_1 \|\nabla \psi\|^2. \tag{3.2.47}$$

(iii) *If instead of* (3.2.41) *a stronger inequality*

$$\langle h^\pi \psi, \psi\rangle - \gamma_0 \|\nabla\psi\|^2 - \langle \alpha_0^2 |x|^{-\beta}\psi, \psi\rangle \geq 0 \tag{3.2.48}$$

holds for some constant $\alpha_0 > 0$ *and* $\beta \in (0,2)$, *then each function* $\varphi_0 \in \mathcal{W}_0$ *in part* ***(i)*** *of the theorem satisfies*

$$\exp\left(\alpha_0 \kappa^{-1} |\cdot|^\kappa\right) \varphi_0 \in L^2(\mathbb{R}^d), \qquad \textit{where} \quad \kappa = 1 - \frac{\beta}{2}. \tag{3.2.49}$$

Now we consider a system of N particles containing $K \leq N$ identical particles.

Definition 3.2.7. We say that for an irreducible representation π of S_K the operator H^π has a virtual level of symmetry π, if $H^\pi \geq 0$ and for all sufficiently small $\varepsilon > 0$

$$\sigma_{\text{ess}}\left(P^\pi(H + \varepsilon\Delta_0)\right) = [0, \infty) \tag{3.2.50}$$

and

$$\sigma_{\text{disc}}\left(P^\pi(H + \varepsilon\Delta_0)\right) \neq \emptyset. \tag{3.2.51}$$

Similar to the remark after Definition 3.2.1, condition (3.2.50) together with the HVZ theorem imply that for any partition Z_p with $p > 1$ and any irreducible representation $\pi(Z_p) \prec \pi$ we have for sufficiently small $\varepsilon > 0$

$$P^{\pi(Z_p)}\left(H(Z_p) + \varepsilon\Delta_0(Z_p)\right) \geq 0. \tag{3.2.52}$$

The following theorem is a straightforward generalization of Theorem 3.2.2. However, since the decay rate of φ_0 depends on the Hardy constant, which can be larger for certain representations π, this can result in a stronger decay rate.

Theorem 3.2.8. *Suppose that* $N \geq 3$ *and consider the operator* H^π, *where the potentials* V_{ij} *satisfy* (3.2.2) *and* (3.2.3). *Assume that* H^π *has a virtual level of symmetry* π. *Then*

(i) *zero is an eigenvalue of* H^π *with finite degeneracy. Let* $\mathcal{W}_0$ *be the corresponding eigenspace, then for any* $\varphi_0 \in \mathcal{W}_0$ *we have*

$$\nabla_0\left(|\cdot|_1^{\alpha_0}\varphi_0\right) \in L^2(R_0) \qquad \textit{and} \qquad (1 + |\cdot|_1)^{\alpha_0 - 1}\varphi_0 \in L^2(R_0) \tag{3.2.53}$$

for any $0 \leq \alpha_0 < \frac{d(N-1)-2}{2}$.

(ii) *There exists a constant* $\delta_0 > 0$*, such that for any function* $\psi \in P^\pi H^1(R_0)$ *satisfying* $\langle \nabla_0 \psi, \nabla_0 \varphi_0 \rangle = 0$ *for all* $\varphi_0 \in \mathcal{W}_0$*, we have*

$$(1-\delta_0)\|\nabla_0 \psi\|^2 + \langle V\psi, \psi \rangle \geq 0. \tag{3.2.54}$$

(iii) *If* $V_{ij}^{(2)}$ *satisfies* $V_{ij}^{(2)}(x) \geq \alpha_{ij}|x|^{-\beta}$ *for some constants* $\alpha_{ij} > 0$ *and* $\beta \in (0,2)$*, then for every function* $\varphi_0 \in \mathcal{W}_0$ *we have*

$$\exp\left(\mu|\cdot|_1^\kappa\right)\varphi_0 \in L^2(R_0), \tag{3.2.55}$$

where $\kappa = 1 - \frac{\beta}{2}$ *and* $\mu > 0$ *depends on the coefficients* α_{ij} *and on the masses of the particles only.*

Remark. The group S_K has two one-dimensional representations. The trivial one corresponds to bosons and the antisymmetric representation corresponds to fermions. The other higher-dimensional representations have no immediate interpretation in quantum mechanics but are of independent mathematical interest.

3.3 Asymptotics of multi-particle bound states at the threshold

As we have seen in Theorem 3.2.2 in the last section, in case of a virtual level of the N-body Hamiltonian H of $N \geq 3$ particles in dimension $d \geq 3$, zero is a simple eigenvalue of H and the corresponding eigenfunction φ_0 satisfies

$$(1+|\cdot|_1)^\alpha \varphi_0 \in L^2(R_0) \qquad \text{for any} \quad \alpha < \frac{d(N-1)}{2} - 2. \tag{3.3.1}$$

Since we can choose α arbitrarily close to $\frac{d(N-1)}{2} - 2$, we get an estimate from below on the rate of decay of φ_0. Indeed, recall that the configuration space R_0 has dimension $d(N-1)$. For simplicity, assume that

$$\varphi_0(x) \sim c|x|_1^{-\tau} \quad \text{as} \quad |x|_1 \to \infty. \tag{3.3.2}$$

Let $\alpha = \frac{d(N-1)}{2} - 2 - \varepsilon$ for some $\varepsilon > 0$, then (3.3.1) yields

$$2(\tau - \alpha) > d(N-1) \iff \tau > d(N-1) - 2 - \varepsilon. \tag{3.3.3}$$

For $\varepsilon \to 0$ we get $d(N-1)-2$ as the lower bound of the decay rate of φ_0. Note that the fundamental solution of the Laplace operator in R_0 decays like $c|x|_1^{-(d(N-1)-2)}$ as $|x|_1 \to \infty$, which corresponds to the same lower bound (3.3.3). Heuristically, by the Green's function formalism, one could conclude from this that virtual levels of multi-particle systems with short-range potentials always admit such a behaviour at infinity. In this section we answer this question by giving an asymptotic behaviour of eigenfunctions corresponding to zero-energy eigenvalues of H, which provides an upper bound of the estimate (3.3.1). We follow the presentation of [BB20].

Similar to the last section, we consider $d, N \geq 3$ and we assume that the potentials V_{ij} satisfy

$$|V_{ij}(x_{ij})| \leq c|x_{ij}|^{-2-\nu}, \quad x_{ij} \in \mathbb{R}^d,\ |x_{ij}| \geq A \tag{3.3.4}$$

for some constants $c, \nu, A > 0$ and we allow singularities of the type

$$\begin{cases} V_{ij} \in L^2_{\text{loc}}(\mathbb{R}^d), & \text{if } d = 3, \\ V_{ij} \in L^p_{\text{loc}}(\mathbb{R}^d) \text{ for some } p > 2, & \text{if } d = 4, \\ V_{ij} \in L^{\frac{d}{2}}_{\text{loc}}(\mathbb{R}^d), & \text{if } d \geq 5. \end{cases} \tag{3.3.5}$$

However, compared to the last section we also allow systems in which H can have negative eigenvalues. Precisely, in terms of Definition 3.2.1 our main assumption here is that for all sufficiently small $\varepsilon > 0$ we have that

$$\sigma_{\text{ess}}\left(-(1-\varepsilon)\Delta_0 + V\right) = [0, \infty). \tag{3.3.6}$$

This is in particular the situation where for any cluster C with $1 < |C| < N$ the Hamiltonian $H[C]$ has neither resonances nor eigenvalues at the threshold zero. However, the operator H of the whole system might have eigenvalues $E \leq 0$. Taking into account the remark after Theorem 3.1.2, even in case of negative eigenvalues of H the eigenfunction corresponding to the eigenvalue zero satisfies (3.2.8).

In accordance with 2.2.21, for a fixed pair of particles $i \neq j$ and $k \neq i, j$ we use the abbreviation

$$R_{ij} = \{x = (x_1, \dots, x_N) \in R_0 \,:\, m_i x_i + m_j x_j = 0,\ x_k = 0\} \tag{3.3.7}$$

and $R_{ij}^{\perp} = R_0 \ominus R_{ij}$. In the same spirit we denote by P_{ij} and $P_{ij}^{\perp}$ the projections in R_0 with respect to the scalar product $\langle \cdot, \cdot \rangle_1$ onto R_{ij} and $R_{ij}^{\perp}$, respectively. For $x \in R_0$ we set

$$q_{ij} = P_{ij} x \qquad \text{and} \qquad \xi_{ij} = P_{ij}^{\perp} x. \tag{3.3.8}$$

Computations similar to (2.2.43) show that for any $1 \le i < j \le N$ we have

$$|q_{ij}|_1 = \left(\frac{2 m_i m_j}{m_i + m_j} \right)^{\frac{1}{2}} |x_{ij}|, \tag{3.3.9}$$

which together with (3.3.4) implies

$$|V_{ij}(x_{ij})| \le C |q_{ij}|_1^{-2-\nu} \quad \text{for some constant } C > 0 \text{ and all } |x_{ij}| \ge A. \tag{3.3.10}$$

Theorem 3.3.1. *Assume that H satisfies the conditions described above. Suppose that φ_0 is an eigenfunction of H corresponding to the eigenvalue zero. Then*

(i) *for all $1 \le i < j \le N$ we have*

$$V_{ij}(x_{ij}) \varphi_0(x) \in L^1(R_0). \tag{3.3.11}$$

(ii) *Let $\beta = d(N-1) - 2$ and denote by $|\mathbb{S}^{\beta-1}|$ the volume of the unit sphere in $\mathbb{R}^{\beta}$. Furthermore, let*

$$C_0 = -\frac{1}{(\beta - 2)|\mathbb{S}^{\beta-1}|} \int_{R_0} \sum_{1 \le i < j \le N} V_{ij}(x_{ij}) \varphi_0(x) \, \mathrm{d}x. \tag{3.3.12}$$

Then the function φ_0 has the following asymptotics:

$$\varphi_0(x) = \frac{C_0}{|x|_1^{\beta}} + g(x) \quad \text{as} \quad |x|_1 \to \infty, \tag{3.3.13}$$

where the remainder g belongs to $L^p(R_0)$ for any p satisfying

$$\frac{\beta+2}{\beta+\frac{\gamma^*}{1+\gamma^*}} < p < \frac{\beta+2}{\beta} \qquad \textit{with} \quad \gamma^* = \min\left\{\frac{d}{2}-1, \nu\right\}. \tag{3.3.14}$$

Remark. **(i)** The most interesting question with regard to (3.3.13) is whether the constant C_0 is zero or not. In the next section we discuss the importance of this constant by presenting examples where the constant is always zero and where it is never zero.

(ii) The relation (3.3.13) also shows that the decay rate of the eigenfunction φ_0 does not depend on the potentials as long as they are short-range and the constant C_0 is not zero. However, due to

$$|x|_1 = \left(\sum_{i=1}^{N} 2m_i|x_i|^2\right)^{\frac{1}{2}} \tag{3.3.15}$$

for not identical masses the decay of $\varphi_0(x)$ depends on the direction of x.

(iii) The theorem is formulated for $N \geq 3$ particles only. By adjusting the notation it can be proved for one- and two-body Schrödinger operators

$$-\frac{1}{2m}\Delta + V \qquad \text{in} \quad L^2(\mathbb{R}^d) \tag{3.3.16}$$

as well. However, for $d=3$ and $d=4$ we must speak of general zero-energy solutions, since in these cases it could be resonances. The zero-energy solution φ_0 in any dimension $d \geq 3$ then satisfies

$$\varphi_0(x) = -\frac{m\cdot\Gamma\left(\frac{d}{2}\right)}{(d-2)\pi^{\frac{d}{2}}}|x|^{2-d}\int V(y)\varphi_0(y)\,\mathrm{d}y + g(x) \qquad \text{as } |x|\to\infty, \tag{3.3.17}$$

where

$$g \in L^p(\mathbb{R}^d) \qquad \text{for} \qquad \frac{d}{d-2+\frac{\nu}{1+\nu}} < p < \frac{d}{d-2}. \tag{3.3.18}$$

We prove the theorem successively through a series of propositions. The first statement follows from

Proposition 3.3.2. *For all* $1 \le i < j \le N$ *and any* $0 < \gamma < \gamma^*$ *we have*

$$(1 + |x|_1)^{\gamma} V_{ij}(x_{ij})\varphi_0(x) \in L^1(R_0). \tag{3.3.19}$$

Proof. By (3.2.8), together with $|\nabla_{q_{ij}}| \le |\nabla_0|$ and Hardy's inequality in the space $H^1(R_{ij})$ we have

$$(1 + |q_{ij}|_1)^{-1} (1 + |x|_1)^{\alpha} \varphi_0 \in L^2(R_0). \tag{3.3.20}$$

Note that this estimate is crucial. In fact, using the Hardy inequality in x instead of q_{ij} at this point would not be enough. For any fixed $0 < \gamma < \gamma^*$ we write

$$(1 + |x|_1)^{\gamma} V_{ij}(x_{ij})\varphi_0(x) = (1 + |q_{ij}|_1)^{-1} (1 + |x|_1)^{\alpha} \varphi_0(x) f(x), \tag{3.3.21}$$

where

$$f(x) := (1 + |x|_1)^{-\alpha+\gamma} (1 + |q_{ij}|_1) V_{ij}(x_{ij}). \tag{3.3.22}$$

In view of (3.3.20) to prove Proposition 3.3.2 it suffices to show that f belongs to $L^2(R_0)$. Note that by definition of R_{ij} and $R_{ij}^{\perp}$ we have

$$L^2(R_0) = L^2(R_{ij}) \otimes L^2(R_{ij}^{\perp}). \tag{3.3.23}$$

Hence, we write

$$f(x) = f(x)\chi_{\{|x_{ij}|<A\}} + f(x)\chi_{\{|x_{ij}|\ge A\}} \tag{3.3.24}$$

and estimate $f\chi_{\{|x_{ij}|<A\}}$ and $f\chi_{\{|x_{ij}|\ge A\}}$ separately, starting with $f\chi_{\{|x_{ij}|<A\}}$.
Due to (3.3.9) and (3.3.5) we have

$$(1 + |q_{ij}|_1) V_{ij}(x_{ij})\chi_{\{|x_{ij}|<A\}} \in L^2(R_{ij}). \tag{3.3.25}$$

Therefore, in order to prove $f\chi_{\{|x_{ij}|<A\}} \in L^2(R_0)$ we only need to show that the function $(1 + |x|_1)^{-\alpha+\gamma}$ belongs to $L^2(R_{ij}^{\perp})$. Since

$$\dim(R_{ij}^{\perp}) = d(N - 2) \tag{3.3.26}$$

we conclude that

$$(1 + |\xi_{ij}|_1)^{-\alpha+\gamma} \in L^2(R_{ij}^{\perp}) \qquad \iff \qquad \alpha - \gamma > \frac{d(N-2)}{2}. \tag{3.3.27}$$

Recall that $\gamma < \gamma^*$, which in particular implies $\gamma < \frac{d}{2} - 1$. Therefore, the condition in (3.3.27) is satisfied if we choose α close enough to $\frac{d(N-1)-2}{2}$. By the relation $|x|_1^2 = |q_{ij}|_1^2 + |\xi_{ij}|_1^2$ we have

$$(1 + |x|_1)^{-1} \le (1 + |\xi_{ij}|_1)^{-1} \tag{3.3.28}$$

and therefore $(1 + |x|_1)^{-\alpha+\gamma} \in L^2(R_{ij}^{\perp})$. This implies

$$f\chi_{\{|x_{ij}|<A\}} \in L^2(R_0). \tag{3.3.29}$$

In order to prove that the function $f\chi_{\{|x_{ij}|\ge A\}}$ belongs to $L^2(R_0)$ we show that it can be estimated by

$$|f(x)\chi_{\{|x_{ij}|\ge A\}}| \le C|f_1(q_{ij})| \cdot |f_2(\xi_{ij})|, \tag{3.3.30}$$

where $f_1 \in L^2(R_{ij})$ and $f_2 \in L^2(R_{ij}^{\perp})$. Here we will use the assumption that the potential $V_{ij}(x_{ij})$ decays faster than $|q_{ij}|_1^{-2}$ as $|x_{ij}| \to \infty$. By

$$\dim(R_{ij}) = d \qquad \text{and} \qquad \dim(R_{ij}^{\perp}) = d(N-2) \tag{3.3.31}$$

for any $0 < \varepsilon < \nu - \gamma$ we have

$$f_1(q_{ij}) := (1 + |q_{ij}|_1)^{-\frac{d}{2}-\varepsilon} \in L^2(R_{ij}) \tag{3.3.32}$$

and

$$f_2(\xi_{ij}) := (1 + |\xi_{ij}|_1)^{-\alpha+\gamma-\nu+\varepsilon+\frac{d}{2}-1} \in L^2(R_{ij}^{\perp}). \tag{3.3.33}$$

Note that we can always assume $\nu < \frac{d}{2} - 1$. Hence, by the use of $|q_{ij}|_1, |\xi_{ij}|_1 \le |x|_1$ we obtain

$$(1 + |x|_1)^{-\alpha+\gamma} \le (1 + |\xi_{ij}|_1)^{-\alpha+\gamma-\nu+\varepsilon+\frac{d}{2}-1} (1 + |q_{ij}|_1)^{1-\frac{d}{2}+\nu-\varepsilon}. \tag{3.3.34}$$

This, together with (3.3.10) yields

$$|f(x)\chi_{\{|x_{ij}|\geq A\}}| \leq C|f_1(q_{ij})| \cdot |f_2(\xi_{ij})| \tag{3.3.35}$$

and therefore

$$f\chi_{\{|x_{ij}|\geq A\}} \in L^2(R_0). \tag{3.3.36}$$

This completes the proof of Proposition 3.3.2. □

Now we turn to the proof of statement **(ii)** of Theorem 3.3.1. Recall that by assumption we have

$$H\varphi_0 = (-\Delta_0 + V)\,\varphi_0 = 0 \tag{3.3.37}$$

and according to Proposition 3.3.2 we also have $V\varphi_0 \in L^1(R_0)$. Therefore, we can apply [LL01, Theorem 6.21] to conclude

$$\varphi_0(x) = \frac{-1}{(\beta-2)|\mathbb{S}^{\beta-1}|}\int_{R_0} |x-y|_1^{-\beta} V(y)\varphi_0(y)\,\mathrm{d}y. \tag{3.3.38}$$

We derive the asymptotics (3.3.13) by studying the integral representation of φ_0 in (3.3.38). We will see that only certain regions of the configuration space R_0 contribute to the leading term of φ_0. We write

$$\varphi_0(x) = \frac{-1}{(\beta-2)|\mathbb{S}^{\beta-1}|}\left(I_1(x) + I_2(x)\right), \tag{3.3.39}$$

where

$$\begin{aligned} I_1(x) &= \int\limits_{\{|x-y|_1\leq 1\}} |x-y|_1^{-\beta} V(y)\varphi_0(y)\,\mathrm{d}y, \\ I_2(x) &= \int\limits_{\{|x-y|_1> 1\}} |x-y|_1^{-\beta} V(y)\varphi_0(y)\,\mathrm{d}y. \end{aligned} \tag{3.3.40}$$

First we show that the function I_1 belongs to the remainder g in (3.3.13).

Proposition 3.3.3. *The function I_1 is an element of $L^p(R_0)$ for all $1 \leq p < \frac{\beta+2}{\beta}$.*

Proof. Due to $\dim(R_0) = d(N-1)$ and $\beta = d(N-1) - 2$ we have

$$|x|_1^{-\beta} \chi_{\{|x|_1 \le 1\}} \in L^p(R_0) \quad \text{for all} \quad 1 \le p < \frac{\beta + 2}{\beta}. \tag{3.3.41}$$

Applying Young's inequality yields the claim of Proposition 3.3.3. ☐

Now we show that only a part of I_2 gives the leading term in (3.3.13). Let $\eta = \frac{1}{1+\gamma^*}$. For $x \in R_0$ we define

$$\begin{aligned} \Omega_1(x) &= \{y \in R_0 \;:\; |x-y|_1 > 1,\; |y|_1 > |x|_1^\eta\}, \\ \Omega_2(x) &= \{y \in R_0 \;:\; |x-y|_1 > 1,\; |y|_1 \le |x|_1^\eta\} \end{aligned} \tag{3.3.42}$$

and

$$I_{2,k}(x) = \int_{\Omega_k(x)} |x-y|_1^{-\beta} V(y) \varphi_0(y) \, \mathrm{d}y, \qquad k = 1, 2. \tag{3.3.43}$$

At first we prove that the function $I_{2,1}$ belongs to the remainder g in (3.3.13).

Proposition 3.3.4. *Let $I_{2,1}$ be given by (3.3.42) and (3.3.43), then we have*

$$I_{2,1} \in L^p(R_0) \quad \textit{for all} \quad \frac{\beta+2}{\beta + \frac{\gamma^*}{1+\gamma^*}} < p < \frac{\beta+2}{\beta}. \tag{3.3.44}$$

Proof. Let $\gamma < \gamma^*$. By the use of $|y|_1 > |x|_1^\eta$ for $y \in \Omega_1(x)$ we get

$$\begin{aligned} |I_{2,1}(x)| &\le \int_{\Omega_1(x)} |x-y|_1^{-\beta} |V(y)\varphi_0(y)| \, \mathrm{d}y \\ &\le (1 + |x|_1^\eta)^{-\gamma} \int_{\Omega_1(x)} |x-y|_1^{-\beta} (1 + |y|_1)^\gamma \, |V(y)\varphi_0(y)| \, \mathrm{d}y \\ &= (1 + |x|_1^\eta)^{-\gamma} \, \tilde{I}_{2,1}(x). \end{aligned} \tag{3.3.45}$$

We show that for any fixed p satisfying (3.3.44) we find a constant $\gamma < \gamma^*$, such that the function on the r.h.s. of (3.3.45) belongs to $L^p(R_0)$. Note that $\frac{\gamma^*}{1+\gamma^*} = \eta\gamma^*$, which for γ sufficiently close to γ^* implies

$$p > \frac{\beta+2}{\beta + \eta\gamma}. \tag{3.3.46}$$

By Proposition 3.3.2 and Young's inequality we have

$$\tilde{I}_{2,1}(x) = \int_{\Omega_1(x)} |x-y|_1^{-\beta}(1+|y|_1)^{\gamma}|V(y)\varphi_0(y)|\,\mathrm{d}y \in L^s(R_0) \tag{3.3.47}$$

for $s > \frac{d(N-1)}{d(N-1)-2}$. Now we apply Hölder's inequality to the r.h.s. of (3.3.45). For this purpose we fix a constant $s > \frac{d(N-1)}{d(N-1)-2}$ and define

$$t_1 = \frac{s}{s-p} \geq 1 \quad \text{and} \quad t_2 = \frac{s}{p} \geq 1 \quad \text{with} \quad \frac{1}{t_1} + \frac{1}{t_2} = 1. \tag{3.3.48}$$

Then we formally get

$$\int_{R_0} \frac{|\tilde{I}_{2,1}(x)|^p}{(1+|x|_1^{\eta})^{\gamma p}}\,\mathrm{d}x \leq \left(\int_{R_0}(1+|x|_1^{\eta})^{-\gamma p t_1}\mathrm{d}x\right)^{\frac{1}{t_1}}\left(\int_{R_0}|\tilde{I}_{2,1}(x)|^{pt_2}\,\mathrm{d}x\right)^{\frac{1}{t_2}}. \tag{3.3.49}$$

Since $pt_2 = s$ and $\tilde{I}_{2,1} \in L^s(R_0)$, the second integral on the r.h.s of (3.3.49) is finite. Due to $\dim(R_0) = d(N-1)$, to prove the finiteness of the first integral on the r.h.s of (3.3.49) it suffices to show that $\eta\gamma p t_1 > d(N-1)$. By definition of t_1 this is equivalent to

$$\begin{aligned} \eta\gamma s p > d(N-1)(s-p) \quad &\Leftrightarrow \quad p(\eta\gamma s + d(N-1)) > d(N-1)s \\ &\Leftrightarrow \quad \frac{1}{p} < \frac{\eta s\gamma + d(N-1)}{sd(N-1)} = \frac{\eta\gamma}{d(N-1)} + \frac{1}{s}. \end{aligned} \tag{3.3.50}$$

Since $p > \frac{d(N-1)}{d(N-1)-2+\gamma\eta}$, we see that the condition in (3.3.50) is fulfilled if s is chosen sufficiently close to $\frac{d(N-1)}{d(N-1)-2}$. It remains to use the relation $\beta = d(N-1)-2$ to complete the proof. □

Now we finally show that $I_{2,2}$ yields the leading term of φ_0 in (3.3.13).

Proposition 3.3.5. *Let $I_{2,2}$ be given by* (3.3.43)*, then we have*

$$I_{2,2}(x) = |x|_1^{-\beta}\int_{\Omega_2(x)} V(y)\varphi_0(y)\,\mathrm{d}y + h(x) \quad \text{as} \quad |x|_1 \to \infty, \tag{3.3.51}$$

where

$$h \in L^p(R_0) \quad \textit{for all} \quad p > \frac{\beta + 2}{\beta + \frac{\gamma^*}{1+\gamma^*}}. \tag{3.3.52}$$

Proof. For $y \in \Omega_2(x)$ and $|x|_1 \gg 1$ sufficiently large we have

$$|x-y|_1^{-1} = |x|_1^{-1} \left| \frac{x}{|x|_1} - \frac{y}{|x|_1} \right|_1^{-1} \geq |x|_1^{-1} \left(1 - |x|_1^{\eta-1}\right) \tag{3.3.53}$$

and on the other hand

$$|x-y|_1^{-1} \leq (|x|_1 - |y|_1)^{-1} = |x|_1^{-1} \sum_{k=0}^{\infty} \left(\frac{|y|_1}{|x|_1} \right)^k \leq |x|_1^{-1} \left(1 + c|x|_1^{\eta-1}\right) \tag{3.3.54}$$

for some $c > 0$. This results in

$$|x|_1^{-1}\big(1 - |x|_1^{\eta-1}\big) \leq |x-y|_1^{-1} \leq |x|_1^{-1}\big(1 + c|x|_1^{\eta-1}\big). \tag{3.3.55}$$

We apply this inequality to the positive and the negative part of the integrand of $I_{2,2}$. Therefore, let

$$(V\varphi_0)_+(x) = \max\{V(x)\varphi_0(x), 0\} \tag{3.3.56}$$

and

$$(V\varphi_0)_- = -(V\varphi_0 - (V\varphi_0)_+). \tag{3.3.57}$$

Then we have

$$|x|_1^{-\beta}\big(1 - |x|_1^{\eta-1}\big)^\beta \int_{\Omega_2(x)} \big(V\varphi_0\big)_\pm(y)\, \mathrm{d}y \leq \int_{\Omega_2(x)} \frac{\big(V\varphi_0\big)_\pm(y)}{|x-y|_1^\beta}\, \mathrm{d}y \tag{3.3.58}$$

and

$$\int_{\Omega_2(x)} \frac{\big(V\varphi_0\big)_\pm(y)}{|x-y|_1^\beta}\, \mathrm{d}y \leq |x|_1^{-\beta}\big(1 + c|x|_1^{\eta-1}\big)^\beta \int_{\Omega_2(x)} \big(V\varphi_0\big)_\pm(y)\, \mathrm{d}y. \tag{3.3.59}$$

Since $\dim(R_0) = d(N-1)$ we conclude from (3.3.58) and (3.3.59) that there exist functions

$$h_\pm \in L^p(R_0), \qquad p > \frac{d(N-1)}{d(N-1) - 2 + 1 - \eta}, \tag{3.3.60}$$

such that for sufficiently large $|x|_1$ we have

$$\int\limits_{\Omega_2(x)} \frac{\big(V\varphi_0\big)_\pm(y)}{|x-y|_1^\beta}\,\mathrm{d}y = |x|_1^{-\beta} \int\limits_{\Omega_2(x)} (V\varphi_0(y))_\pm\,\mathrm{d}y + h_\pm(x). \tag{3.3.61}$$

Hence, we obtain

$$I_{2,2}(x) = |x|_1^{-\beta} \int\limits_{\Omega_2(x)} V(y)\varphi_0(y)\,\mathrm{d}y + h(x) \quad \text{as} \quad |x|_1 \to \infty, \tag{3.3.62}$$

where $h = h_+ - h_-$ belongs to $L^p(R_0)$ for p given in (3.3.60). By $1-\eta = \frac{\gamma^*}{1+\gamma^*}$ and $\beta = d(N-1)-2$ we conclude the proof of Proposition 3.3.5. □

Proof of Theorem 3.3.1. By Propositions 3.3.3, 3.3.4 and 3.3.5 we have

$$\varphi_0(x) = \frac{-|x|_1^{-\beta}}{(\beta-2)|\mathbb{S}^{\beta-1}|} \int\limits_{\Omega_2(x)} V(y)\varphi_0(y)\,\mathrm{d}y + g(x) \quad \text{as} \quad |x|_1 \to \infty \tag{3.3.63}$$

with

$$g \in L^p(R_0) \qquad \text{for} \qquad \frac{\beta+2}{\beta+\frac{\gamma}{1+\gamma}} < p < \frac{\beta+2}{\beta}. \tag{3.3.64}$$

Note that the integral on the r.h.s of (3.3.63) is over the set $\Omega_2(x)$, which is in contrast to (3.3.13), where the integral is over the whole space R_0. Therefore, to complete the proof of Theorem 3.3.1 it remains to show that

$$|x|_1^{-\beta} \int\limits_{R_0\setminus\Omega_2(x)} V(y)\varphi_0(y)\,\mathrm{d}y \tag{3.3.65}$$

does not contribute to the leading term in the asymptotic estimate of φ_0. Due to Proposition 3.3.2 it is easy to see that for any $\gamma < \gamma^*$ we have

$$\Big|\int\limits_{R_0\setminus\Omega_2(x)} V(y)\varphi_0(y)\,\mathrm{d}y\Big| \le C\,(1+|x|_1)^{-\eta\gamma} \tag{3.3.66}$$

for $|x|_1$ sufficiently large. This implies

$$|x|_1^{-\beta} \int\limits_{R_0 \setminus \Omega_2(x)} V(y)\varphi_0(y)\,\mathrm{d}y \in L^p(R_0) \quad \text{for} \quad p > \frac{\beta+2}{\beta+\frac{\gamma}{1+\gamma}}. \tag{3.3.67}$$

Choosing $\gamma < \gamma^*$ sufficiently close to γ^* and combining (3.3.63) and (3.3.67) completes the proof of the theorem. □

3.3.1 Examples of systems with different asymptotics of threshold bound states

In this section we give some examples that recapitulate some of the main results of this chapter and highlight their differences. In particular, we compare the statements of Theorem 3.2.2 and Theorem 3.2.8 for certain systems by examining the constant

$$C_0 = -\frac{1}{(\beta-2)|\mathbb{S}^{\beta-1}|} \int\limits_{R_0} \sum_{1 \le i < j \le N} V_{ij}(x_{ij})\varphi_0(x)\,\mathrm{d}x \tag{3.3.68}$$

in the asymptotics (3.3.13) of a zero-energy eigenfunction. Precisely, we give examples of systems where C_0 is never zero and where it is always zero. We follow [BB20].

One-particle case

We start with the case of a one-particle Schrödinger operator

$$h = -\Delta + V \qquad \text{in} \qquad L^2(\mathbb{R}^d), \tag{3.3.69}$$

which will show what is to be expected in the multi-particle case. In order to speak about eigenvalues and not resonances, we assume that $d \ge 5$. Suppose that the potential V has a compact support $K \subset \mathbb{R}^d$ and is spherically symmetric, i.e. it satisfies

$$V(x) = V(|x|) \qquad \text{for all} \;\; x \in \mathbb{R}^d. \tag{3.3.70}$$

Then $\sigma_{\text{ess}}(h) = [0, \infty)$. Assume that zero is an eigenvalue of h and φ_0 is a corresponding spherically symmetric eigenfunction. Since V has a compact support, we have

$$(h\varphi_0)(x) = -\Delta\varphi_0(x) = 0 \qquad \text{for all } x \in \mathbb{R}^d \backslash K. \tag{3.3.71}$$

Here one can explicitly calculate the function φ_0 by reducing the problem to the radial equation in one dimension. For arguments outside of the support of V, as a spherically symmetric harmonic function, φ_0 is given by

$$\varphi_0(x) = c_1 + c_2|x|^{-(d-2)}. \tag{3.3.72}$$

Due to $\varphi_0(x) \to 0$ as $|x| \to \infty$ we obviously have $c_1 = 0$. In case of $c_2 = 0$ the function φ_0 would vanish outside of the support of V. This is not possible for solutions of a second order differential equation. Hence, we have $c_2 \neq 0$ and by (3.3.17) we conclude $\langle V, \varphi_0 \rangle \neq 0$. Note that in the one-particle case the scalar product $\langle V, \varphi_0 \rangle$ corresponds to the constant C_0 in (3.3.68), which in the multi-particle case cannot be formally written as a scalar product, because the sum of potentials does not belong to $L^2(R_0)$. If φ_0 is a function of angular momentum $l \geq 1$, then we always have $\langle V, \varphi_0 \rangle = 0$, cf. [KS80a]. We emphasize that in both cases there is no restriction on whether the operator h has negative eigenvalues, i.e. zero need not necessarily be the ground state.

Multi-particle case

Now we move to systems of $N \geq 3$ particles in dimension $d \geq 3$ with the corresponding Hamiltonian

$$H = -\Delta_0 + \sum_{1 \leq i < j \leq N} V_{ij} \qquad \text{in} \quad L^2(R_0). \tag{3.3.73}$$

Similar to one-particle systems, with regard to the constant C_0 both cases can occur in multi-particle systems as well. Assume that all conditions of Theorem 3.2.2 are fulfilled. Then a zero-energy eigenfunction φ_0 of the operator H satisfies

$$\varphi_0(x) = C_0|x|_1^{-d(N-1)+2} + g(x) \qquad \text{as} \qquad |x|_1 \to \infty, \tag{3.3.74}$$

where $g \in L^p(R_0)$ for

$$\frac{d(N-1)}{d(N-1)-2+\frac{\nu}{1+\nu}} < p < \frac{d(N-1)}{d(N-1)-2} \tag{3.3.75}$$

with a sufficiently small $\nu > 0$. Now assume that H has no negative eigenvalues and the potential of the system is non-positive, i.e.

$$\sum_{1\le i<j\le N} V_{ij}(x_{ij}) \le 0 \qquad \text{for all} \qquad x_{ij} \in \mathbb{R}^d. \tag{3.3.76}$$

Since in this case zero is the ground state energy of the system, the eigenfunction φ_0 does not change its sign and can therefore be chosen to be strictly positive or negative. This implies $C_0 \neq 0$ and thus determines the asymptotic behaviour of φ_0 by (3.3.74). Note that with regard to (3.3.75) the function

$$w : R_0 \to \mathbb{R},\ w(x) = (1+|x|_1)^{-d(N-1)+2} \tag{3.3.77}$$

belongs to $L^q(R_0)$, only if $q > \frac{d(N-1)}{d(N-1)-2}$.

As already mentioned in the remark of Theorem 3.2.8, in case that certain symmetries are involved, the decay rate of φ_0 can only increase. For example, assume that all potentials V_{ij} are spherically symmetric. Then the operator H is invariant under the action of the group $\mathrm{SO}(\mathbb{R}^d)$. Consider the operator H on a subspace of functions transformed according to a fixed irreducible representation of degree $l = 0, 1, \ldots$ of the group, see for example [Ham62] . Assume that zero is an eigenvalue and φ_0 is a corresponding eigenfunction of H with rotational symmetry of degree $l \geq 1$. Then, due to the orthogonality of functions corresponding to different irreducible representations we have $C_0 = 0$. Therefore, in this case by (3.3.74) the eigenfunction φ_0 always decays at least as fast as $C|x|_1^{-\theta}$, where $\theta > d(N-1)-2$.

Bosons and fermions

According to the example above we conclude that for $N \geq 3$ identical bosons in dimension $d \geq 3$ with non-positive short-range interactions, the zero-energy ground state of the corresponding Hamiltonian H behaves like $C_0|x|_1^{-d(N-1)+2}$ as $|x|_1 \to \infty$.

However, if we consider a system of $N \geq 3$ particles in dimension $d \geq 3$, which contains at least $K \geq 3$ identical fermions, then the decay rate of φ_0 increases. To this end, assume that zero is a bound state of the Hamiltonian H, then φ_0 is orthogonal to all functions symmetric with respect to permutations of each pair of fermions. This implies $C_0 = 0$. For such systems it does not matter whether the potentials are non-positive and if zero is the ground state. Hence, by (3.3.74) the eigenfunction φ_0 always decays at least as fast as $C|x|_1^{-\theta}$ with $\theta > d(N-1) - 2$.

4 The Efimov Effect

As described in the introduction, the Efimov effect is strongly related to the existence and properties of virtual levels. In order to give a mathematical description of the Efimov effect, we present the proof of A. Sobolev by briefly summarizing the work [Sob93]. Since our method for the investigation of the three-body system in dimension four is based on the same method as developed in [Sob93], we will explain the techniques in the proof, which will be important for us later, in more detail. This, together with the results obtained in the previous chapter will allow us to compare the same systems but in different dimensions. We prove that in contrast to dimension three the two-body resonances in dimension four do not lead to the infinite number of negative eigenvalues of the three-body system. Due to the chosen method we can provide a precise reason for this. Furthermore, we prove the finiteness of the three-particle Hamiltonian in dimension four restricted to the subspace of functions antisymmetric with respect to the permutation of particles. We also extend the result to three-body systems in dimension $d \geq 5$. We then move on to many-body systems with $N \geq 4$ particles in dimension $d \geq 3$. In contrast to the case of three particles in dimension four, the main reason for the absence of the Efimov effect for such systems is that virtual levels of the corresponding Hamiltonians are eigenvalues, see Theorem 3.2.2. Here we will use the technique of S. A. Vugalter and G. M. Zhislin [VZ83] and extend it to arbitrary multi-particle systems. We combine this with the results of the previous chapter and show that the discrete spectrum of Schrödinger operators of such systems is always finite. Furthermore, we apply the result to systems with a fixed permutation symmetry.

4.1 Three quantum particles in dimension three

In this section we present a short proof of the Efimov effect by giving a brief summary of the work [Sob93].

Consider a system of three quantum particles of masses $m_1, m_2, m_3 > 0$ in dimension three with position vectors $x_1, x_2, x_3 \in \mathbb{R}^3$ and potentials $v_{12}, v_{23}, v_{31} : \mathbb{R}^3 \to \mathbb{R}$. The Hamiltonian of the system in coordinate representation is given by

$$-\frac{1}{2m_1}\Delta_{x_1} - \frac{1}{2m_2}\Delta_{x_2} - \frac{1}{2m_3}\Delta_{x_3} + v_{12}(x_1 - x_2) + v_{23}(x_2 - x_3) + v_{31}(x_3 - x_1). \quad (4.1.1)$$

Assume that the potentials v_{ij} satisfy

$$v_{ij}(x) \leq 0 \qquad \text{and} \qquad |v_{ij}(x)| \leq C(1 + |x|)^{-b}, \ b > 3. \quad (4.1.2)$$

After separation of the center of mass, the Hamiltonian of relative motion is denoted by

$$H = H_0 + \sum_{1 \leq i < j \leq 3} v_{ij}. \quad (4.1.3)$$

The configuration space R_0 is a six-dimensional subspace of $\mathbb{R}^9$ and by assumptions (4.1.2) the Hamiltonian H is essentially self-adjoint. In contrast to Hamiltonians of arbitrary N particles, in case of $N = 3$ it is usual to work with a fixed set of coordinates, mostly with the so-called Jacobi coordinates. In this sense, denote by $\alpha = ij$ an arbitrary pair of particles and for $l \neq i, j$ set

$$x_\alpha = x_i - x_j \qquad \text{and} \qquad y_\alpha = \frac{m_i x_i + m_j x_j}{m_i + m_j} - x_l. \quad (4.1.4)$$

Furthermore, let

$$m_\alpha = \frac{m_i m_j}{m_i + m_j} \qquad \text{and} \qquad n_\alpha = \frac{m_l(m_i + m_j)}{m_i + m_j + m_l}. \quad (4.1.5)$$

Then,

$$H_0 = -\frac{1}{2m_\alpha}\Delta_{x_\alpha} - \frac{1}{2n_\alpha}\Delta_{y_\alpha}. \quad (4.1.6)$$

Every two-body subsystem corresponding to the subscript $\alpha \in \{12, 23, 31\}$ is then described in the center of mass frame by the Schrödinger operator

$$h_\alpha = -\frac{1}{2m_\alpha}\Delta_{x_\alpha} + v_\alpha(x_\alpha) \qquad \text{in} \quad L^2(\mathbb{R}^3), \tag{4.1.7}$$

where m_α is the so-called reduced mass. For the sake of simplicity we leave out the index x_α and write $-\frac{1}{2m_\alpha}\Delta$.

Denote by $N(z)$ the number of eigenvalues of the operator H below $z < 0$. The following theorem describes the Efimov effect.

Theorem 4.1.1. *Let the pair potentials v_α satisfy* (4.1.2). *Suppose that $h_\alpha \geq 0$ for all $\alpha \in \{12, 23, 31\}$ and that one of the following conditions is fulfilled:*

(i) *Zero is a resonance for all two-particle subsystems;*

(ii) *Zero is a resonance for two-particle subsystems α, β and is neither a resonance nor an eigenvalue of h_γ, where $\alpha \neq \beta \neq \gamma$.*

Then the operator H has an infinite negative discrete spectrum and the counting function $N(z)$ obeys the relation

$$\lim_{z \to 0^-} |\log|z||^{-1} N(z) = \mathcal{U}_0, \tag{4.1.8}$$

where the constant $\mathcal{U}_0 > 0$ depends only on the mass ratios and not on the pair potentials v_α.

In order to simplify the representation of H, the computations are carried out in the momentum space. For $i = 1, 2, 3$, denote by k_i the conjugate variable of x_i and introduce the set of variables (k_α, p_α), conjugate with respect to the Jacobi-coordinates (x_α, y_α). For respectively different $i, j, l \in \{1, 2, 3\}$ they are explicitly given by

$$k_{ij} = \frac{m_j k_i - m_i k_j}{m_i + m_j}, \qquad p_{ij} = \frac{m_l(k_i + k_j) - (m_i + m_j)k_l}{m_i + m_j + m_l}. \tag{4.1.9}$$

Furthermore, it holds

$$k_{ij} = -p_{jl} - \frac{m_i}{m_i + m_j} p_{ij} = p_{li} + \frac{m_j}{m_i + m_j} p_{ij}. \tag{4.1.10}$$

See [Fad63] for more details. Hence, one can write

$$k_\alpha = d_{\alpha\beta} p_\alpha + e_{\alpha\beta} p_\beta, \tag{4.1.11}$$

where the constants $d_{\alpha\beta}$ and $e_{\alpha\beta}$ can be computed by the relation (4.1.10). The shift from x_α to k_α is carried out by the partial Fourier transform

$$(\Phi_\alpha f)(k_\alpha, \cdot) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}\langle k_\alpha, x_\alpha \rangle} f(x_\alpha, \cdot)\, \mathrm{d}x_\alpha. \tag{4.1.12}$$

In this setting, by abuse of notation, the three-body Hamiltonian has the form

$$H = H_0 + \sum_\alpha V_\alpha, \tag{4.1.13}$$

where the interactions V_α are given by

$$V_\alpha = \Phi_\alpha v_\alpha \Phi_\alpha^* \tag{4.1.14}$$

and H_0 is the multiplication-operator

$$(H_0 f)(k, p) = H^0(k, p) \cdot f(k, p). \tag{4.1.15}$$

Here H^0 is given by

$$H^0(k, p) = \frac{k_\alpha^2}{2m_\alpha} + \frac{p_\alpha^2}{2n_\alpha} = \frac{k_\beta^2}{2m_\beta} + \frac{p_\beta^2}{2n_\beta} = \frac{k_\gamma^2}{2m_\gamma} + \frac{p_\gamma^2}{2n_\gamma}. \tag{4.1.16}$$

The function H^0 can be expressed in terms of p_α and p_β, where in such a case it is denoted by $H^0_{\alpha\beta}(p, q)$. It is explicitly given by

$$\begin{aligned} H^0_{\alpha\beta}(p, q) &= \frac{p_{12}^2}{2m_{23}} + \frac{\langle p_{12}, p_{23} \rangle}{m_2} + \frac{p_{23}^2}{2m_{12}} = \frac{p_{23}^2}{2m_{31}} + \frac{\langle p_{23}, p_{31} \rangle}{m_3} + \frac{p_{31}^2}{2m_{23}} \\ &= \frac{p_{31}^2}{2m_{23}} + \frac{\langle p_{31}, p_{12} \rangle}{m_1} + \frac{p_{12}^2}{2m_{31}}. \end{aligned} \tag{4.1.17}$$

Hence, one can write

$$H^0_{\alpha\beta}(p,q) = \frac{p^2}{2m_\beta} + \frac{\langle p,q\rangle}{l_\gamma} + \frac{p^2}{2m_\alpha}, \tag{4.1.18}$$

where $l_\gamma \in \{m_1, m_2, m_3\}$ is given by the relation (4.1.17). It is easy to see that

$$H^0_{\alpha\beta}(p,q) \geq \frac{p^2}{2m_\alpha} + \frac{q^2}{2m_\beta}. \tag{4.1.19}$$

The coefficient $\mathcal{U}_0$ in (4.1.8) is expressed by means of the self-adjoint integral operator $\hat{S}(\lambda)$, $\lambda \in \mathbb{R}$ in the space $L^{2(3)}(\mathbb{S}^2)$, whose kernel depends on the scalar product $t = \langle \xi, \nu\rangle$ with $\xi, \nu \in \mathbb{S}^2$ and has the form

$$\begin{cases} \hat{S}_{\alpha\alpha}(t,\lambda) = 0, \\ \hat{S}_{\alpha\beta}(t,\lambda) = (2\pi)^{-1} a_{\alpha\beta} \mathrm{e}^{\mathrm{i}b_{\alpha\beta}\lambda} \frac{\sinh\left(\lambda(\arccos(c_{\alpha\beta}t)\right)}{\sinh(\pi\lambda)\sqrt{(1-c^2_{\alpha\beta}t^2)}}, \end{cases} \tag{4.1.20}$$

where

$$a_{\alpha\beta} = \kappa_{\alpha\beta}\left(\frac{n_\alpha n_\beta}{m_\alpha m_\beta}\right)^{\frac{1}{4}}, \qquad b_{\alpha\beta} = \frac{1}{2}\log\left(\frac{m_\alpha}{m_\beta}\right), \qquad c_{\alpha\beta} = \frac{(m_\alpha m_\beta)^{\frac{1}{2}}}{l_\gamma}. \tag{4.1.21}$$

Here $\kappa_{\alpha\beta} = 1$ if both systems α and β have a zero-energy resonance and $\kappa_{\alpha\beta} = 0$ otherwise. Let $n(\mu, \hat{S}(\lambda))$ be the number of eigenvalues of $\hat{S}(\lambda)$ greater than μ. Note that $c_{\alpha\beta} < 1$, which implies that the integral

$$\mathcal{U}(\mu) := (4\pi)^{-1}\int_{-\infty}^{\infty} n(\mu, \hat{S}(\lambda))\,\mathrm{d}\lambda \tag{4.1.22}$$

is finite. Moreover, the function $\mathcal{U}(\mu)$ is continuous in $\mu > 0$, see [Sob93, Lemma 3.4]. It turns out that $\mathcal{U}_0 = \mathcal{U}(1)$. The proof of Theorem 4.1.1 is based on the investigation of a symmetrized form of the Faddeev equations.

4.1.1 Faddeev equations

We briefly sketch the derivation of the Faddeev equations developed in [Mot08] and adapt the notation from [Sob93] accordingly.

Consider the eigenvalue equation

$$\left(H_0 + \sum_\alpha V_\alpha\right) u = zu, \tag{4.1.23}$$

where $z < 0$ and $\alpha \in \{12, 23, 31\}$. Denote $R_0(z) = (H_0 - z)^{-1}$, then by (4.1.23)

$$u = -R_0(z) \sum_\alpha V_\alpha u. \tag{4.1.24}$$

Note that

$$u = \sum_\alpha u_\alpha, \qquad u_\alpha = -R_0(z) V_\alpha u. \tag{4.1.25}$$

We denote

$$H_\alpha = H_0 + V_\alpha \qquad \text{and} \qquad R_\alpha(z) = (H_\alpha - z)^{-1}, \tag{4.1.26}$$

then it follows

$$u_\alpha = -R_0(z) V_\alpha \sum_\alpha u_\alpha \Longleftrightarrow u_\alpha + R_0(z) V_\alpha u_\alpha = -R_0(z) V_\alpha (u_\beta + u_\gamma) \tag{4.1.27}$$

$$\Longleftrightarrow u_\alpha = -R_\alpha(z) V_\alpha (u_\beta + u_\gamma). \tag{4.1.28}$$

By $V_\alpha \leq 0$ and the resolvent identity

$$R_\alpha(z) = R_0(z) - R_0(z) V_\alpha R_\alpha(z), \tag{4.1.29}$$

one arrives at

$$u_\alpha = R_0(z)(I + |V_\alpha| R_\alpha(z)) |V_\alpha| (u_\beta + u_\gamma). \tag{4.1.30}$$

For $z < 0$ let

$$W_\alpha(z) = I + |V_\alpha|^{\frac{1}{2}} R_\alpha(z) |V_\alpha|^{\frac{1}{2}}. \tag{4.1.31}$$

Then equation (4.1.30) yields

$$u_\alpha = R_0(z) |V_\alpha|^{\frac{1}{2}} W_\alpha(z) |V_\alpha|^{\frac{1}{2}} (u_\beta + u_\gamma). \tag{4.1.32}$$

Substituting

$$f_\alpha = W_\alpha^{\frac{1}{2}}(z)|V_\alpha|^{\frac{1}{2}}(u_\beta + u_\gamma) \tag{4.1.33}$$

for every $z < 0$ one arrives at the system of equations

$$f_{12} = W_{12}^{\frac{1}{2}}|V_{12}|^{\frac{1}{2}}R_0|V_{23}|^{\frac{1}{2}}W_{23}^{\frac{1}{2}}f_{23} + W_{12}^{\frac{1}{2}}|V_{12}|^{\frac{1}{2}}R_0|V_{31}|^{\frac{1}{2}}W_{31}^{\frac{1}{2}}f_{31}, \tag{4.1.34}$$

$$f_{23} = W_{23}^{\frac{1}{2}}|V_{23}|^{\frac{1}{2}}R_0|V_{12}|^{\frac{1}{2}}W_{12}^{\frac{1}{2}}f_{12} + W_{23}^{\frac{1}{2}}|V_{23}|^{\frac{1}{2}}R_0|V_{31}|^{\frac{1}{2}}W_{31}^{\frac{1}{2}}f_{31}, \tag{4.1.35}$$

$$f_{31} = W_{31}^{\frac{1}{2}}|V_{31}|^{\frac{1}{2}}R_0|V_{12}|^{\frac{1}{2}}W_{12}^{\frac{1}{2}}f_{12} + W_{31}^{\frac{1}{2}}|V_{31}|^{\frac{1}{2}}R_0|V_{23}|^{\frac{1}{2}}W_{23}^{\frac{1}{2}}f_{23}. \tag{4.1.36}$$

In other words, the eigenvalue equation (4.1.23) is equivalent to

$$A(z)F = F, \qquad \text{where} \quad F = (f_{12}, f_{23}, f_{31}) \tag{4.1.37}$$

and $A(z)$ is a $3 \times 3-$matrix with the entries

$$A_{\alpha\beta}(z) = W_\alpha^{\frac{1}{2}}(z)|V_\alpha|^{\frac{1}{2}}R_0(z)|V_\beta|^{\frac{1}{2}}W_\beta^{\frac{1}{2}}(z). \tag{4.1.38}$$

Hence, we give the following

Definition 4.1.2. Let $z < 0$ and

$$A(z) = W^{\frac{1}{2}}(z)K(z)W^{\frac{1}{2}}(z), \tag{4.1.39}$$

where

$$W(z) = \begin{pmatrix} W_{12}(z) & 0 & 0 \\ 0 & W_{23}(z) & 0 \\ 0 & 0 & W_{31}(z) \end{pmatrix}, \tag{4.1.40}$$

such that

$$W_\alpha(z) = I + |V_\alpha|^{\frac{1}{2}}R_\alpha(z)|V_\alpha|^{\frac{1}{2}}, \qquad R_\alpha(z) = (H_0 + V_\alpha - z)^{-1} \tag{4.1.41}$$

and

$$K(z) = \begin{pmatrix} 0 & K_{12|23}(z) & K_{12|31}(z) \\ K_{23|12}(z) & 0 & K_{23|31}(z) \\ K_{31|12}(z) & K_{32|23}(z) & 0 \end{pmatrix}, \tag{4.1.42}$$

where

$$K_{\alpha\beta}(z) = |V_\alpha|^{\frac{1}{2}} R_0(z) |V_\beta|^{\frac{1}{2}}, \qquad R_0(z) = (H_0 - z)^{-1}. \tag{4.1.43}$$

The following statement relates the spectra of the operators H and $A(z)$.

Theorem 4.1.3. *For $z < 0$ the operator $A(z)$ is compact and continuous in z and*

$$N(z) = n(1, A(z)), \tag{4.1.44}$$

where $N(z)$ is the number of eigenvalues of H below $z < 0$ and $n(1, A(z))$ is the number of eigenvalues of $A(z)$ greater than one.

We will only present the proof of the compactness and the continuity of $A(z)$ for $z < 0$. In $z = 0$ the compactness is lost due to a singularity of $W(z)$, which results from the resonances of the two-body systems. This is the reason for the infinity of the negative discrete spectrum of H. To see this, one has to study the integral kernel of the resolvent corresponding to the two-body Schrödinger operator.

4.1.2 Zero-energy resonances in two-body subsystems

With regard to definition (4.1.2) it is necessary to study the behaviour of the resolvents $r_\alpha(z)$ of the two-body Schrödinger operators

$$h_\alpha = -\frac{1}{2m_\alpha}\Delta + v_\alpha \tag{4.1.45}$$

for $z \to 0$ in the presence of resonances. It should be noted here that the technique developed in the work [JK79] was important to this part of the proof in [Sob93].

Definition 4.1.4. Let $z < 0$ and $\alpha \in \{12, 23, 31\}$. Denote by $r_\alpha(z)$ the resolvent of the two-body Schrödinger operator h_α and define the operator

$$w_\alpha(z) = I + |v_\alpha|^{\frac{1}{2}} r_\alpha(z) |v_\alpha|^{\frac{1}{2}}. \tag{4.1.46}$$

Further, denote by G^0_α, G^1_α the operators with the kernels

$$G^0_\alpha(x,y) = \frac{m_\alpha}{2\pi}\frac{|v_\alpha(x)|^{\frac{1}{2}}|v_\alpha(y)|^{\frac{1}{2}}}{|x-y|}, \qquad G^1_\alpha(x,y) = \frac{m_\alpha^{\frac{3}{2}}}{\sqrt{2}\pi}|v_\alpha(x)|^{\frac{1}{2}}|v_\alpha(y)|^{\frac{1}{2}}. \tag{4.1.47}$$

Lemma 4.1.5. *Let $z = -k^2, k > 0$ and denote by $r_\alpha^0(z)$ the resolvent of $-\frac{1}{2m_\alpha}\Delta$. Further, let v_α satisfy* (4.1.2)*. Then, for any positive $\delta < \min\left\{1, \frac{(b-3)}{2}\right\}$ the relation holds*

$$|v_\alpha|^{\frac{1}{2}} r_\alpha^0(z) |v_\alpha|^{\frac{1}{2}} = G_\alpha^0 - kG_\alpha^1 + k^{1+\delta} G_\alpha^2(k), \tag{4.1.48}$$

where $G_\alpha^2(k)$ is continuous in $k \geq 0$.

Using (4.1.48) one can determine the behavior of $w_\alpha(z)$ for $z \to 0$. Indeed, by the resolvent identity it follows

$$r_\alpha(z) = r_\alpha^0(z) - r_\alpha^0(z) v_\alpha r_\alpha(z) = r_\alpha^0(z) - r_\alpha(z) v_\alpha r_\alpha^0(z), \tag{4.1.49}$$

which implies

$$\begin{aligned} I &= \left(I - |v_\alpha|^{\frac{1}{2}} r_\alpha^0(z) |v_\alpha|^{\frac{1}{2}}\right)\left(I + |v_\alpha|^{\frac{1}{2}} r_\alpha(z) |v_\alpha|^{\frac{1}{2}}\right) \\ &= \left(I + |v_\alpha|^{\frac{1}{2}} r_\alpha(z) |v_\alpha|^{\frac{1}{2}}\right)\left(I - |v_\alpha|^{\frac{1}{2}} r_\alpha^0(z) |v_\alpha|^{\frac{1}{2}}\right). \end{aligned} \tag{4.1.50}$$

This yields

$$w_\alpha(z) = I + |v_\alpha|^{\frac{1}{2}} r_\alpha(z) |v_\alpha|^{\frac{1}{2}} = \left(I - |v_\alpha|^{\frac{1}{2}} r_\alpha^0(z) |v_\alpha|^{\frac{1}{2}}\right)^{-1}. \tag{4.1.51}$$

Now assume that h_α has a resonance at zero and denote by f_α the corresponding resonance function. Recall that in case of no negative eigenvalues of h_α such zero-energy resonances are non-degenerate, the function f_α decays like $c|x|^{-1}$ as $|x| \to \infty$ and it satisfies

$$\int_{\mathbb{R}^3} v_\alpha(x) f_\alpha(x)\, \mathrm{d}x \neq 0. \tag{4.1.52}$$

These properties, together with Lemma 4.1.5 provide a certain behaviour of the operator $w_\alpha(z)$ for $z \to 0$, which plays a fundamental role for the behaviour of the operator $A(z)$. In the following, let

$$\varphi_\alpha = |v_\alpha|^{\frac{1}{2}} f_\alpha. \tag{4.1.53}$$

Note that $\varphi_\alpha \in L^2(\mathbb{R}^3)$ and $\langle \varphi_\alpha, |v_\alpha|^{\frac{1}{2}} \rangle \neq 0$.

Lemma 4.1.6. *Assume that v_α satisfies* (4.1.2). *Further, let $z = -k^2$, where $k > 0$ is small enough, such that $w_\alpha(z)$ is defined.*

(i) *If zero is neither a resonance nor an eigenvalue, then $w_\alpha(-k^2)$ is continuous in $k \geq 0$.*

(ii) *If zero is a resonance, then for any positive $\delta < \frac{1}{2}\min\{1, b-3\}$ the representation*

$$w_\alpha(-k^2) = \frac{\langle\cdot,\varphi_\alpha\rangle\varphi_\alpha}{k} + k^{-1+\delta} w^{(\delta)}(k) \tag{4.1.54}$$

is valid, where the operator $w^{(\delta)}(k)$ is continuous in $k \geq 0$. If, in addition, $h_\alpha \geq 0$ then $w_\alpha \geq 0$ and

$$\big(w_\alpha(-k^2)\big)^{\frac{1}{2}} = \|\varphi_\alpha\|^{-1}\frac{\langle\cdot,\varphi_\alpha\rangle\varphi_\alpha}{k^{\frac{1}{2}}} + k^{-\frac{1-\delta}{2}}\tilde{w}_\alpha^{(\delta)}(k), \tag{4.1.55}$$

where the operator $\tilde{w}_\alpha^{(\delta)}(k)$ is continuous in $k \geq 0$.

By the assumptions of Lemma 4.1.6 the representation (4.1.55) of $w_\alpha^{\frac{1}{2}}(z)$ only applies to $z < 0$ with $|z|$ sufficiently small. Even if this representation is only needed for small $|z|$, it will be convenient to write down the operator $w_\alpha^{\frac{1}{2}}(z)$ in this form for all $z \leq 0$. Namely, let $\zeta \in C^\infty(\mathbb{R}_+)$ be a function with $\zeta(t) > 0$ for all $t > 0$, $\zeta(z) = t$ for $t \leq 1$, and $\zeta(t) = 1$ for $t \geq 2$. Then for all $k \geq 0$ it holds

$$\big(w_\alpha(-k^2)\big)^{\frac{1}{2}} = \|\varphi_\alpha\|^{-1}\zeta(k)^{-\frac{1}{2}}\langle\cdot,\varphi_\alpha\rangle\varphi_\alpha + \zeta(k)^{-\frac{1-\delta}{2}}\tilde{w}_\alpha^{(\delta)}(k), \tag{4.1.56}$$

where the operator $\tilde{w}_\alpha^{(\delta)}(k)$ is continuous in $k \geq 0$.

4.1.3 Two-body resonances in three-body systems

It is easy to see that for $W_\alpha(z)$ defined in (4.1.41) and $w_\alpha(z)$ defined in (4.1.46), the following relation holds:

$$W_\alpha(z) = \Phi_\alpha w_\alpha\left(z - \frac{p_\alpha^2}{2n_\alpha}\right)\Phi_\alpha^*. \tag{4.1.57}$$

Here Φ_α is the partial Fourier transform defined in (4.1.12). Since $w_\alpha(z')$ is bounded in $L^2(\mathbb{R}^3)$ uniformly in $z' \leq z < 0$, the operator $W_\alpha(z)$ is bounded. Further properties of the operator $w_\alpha(z)$, such as (4.1.56) can now be transferred $W_\alpha(z)$. The compactness and continuity of $A(z)$ for $z < 0$ are now a consequence of the following

Lemma 4.1.7. *Let $\Gamma_\alpha(z)$ be the multiplication by the function $\zeta\left(\frac{p_\alpha^2}{2n_\alpha} - z\right)$ and $\Gamma(z) = \operatorname{diag}\{\Gamma_{12}(z), \Gamma_{23}(z), \Gamma_{31}(z)\}$. For $\mu, \nu \geq 0$ define*

$$K^{\mu,\nu}(z) = (\Gamma(z))^{-\mu} K(z) (\Gamma(z))^{-\nu}, \tag{4.1.58}$$

where $K(z)$ is defined by (4.1.42). Then the operator $K^{\mu,\nu}(z)$ is continuous in z for $z < 0$ and compact for all $\mu \geq 0$ and $\nu \geq 0$. If $\nu \leq \frac{1}{4}$, $\mu \leq \frac{1}{4}$ and $\mu + \nu < \frac{1}{2}$, then $K^{\mu,\nu}(z)$ is continuous up to $z = 0$.

Proof. Set $\Phi = \operatorname{diag}\{\Phi_{12}, \Phi_{23}, \Phi_{31}\}$, where Φ_α is the partial Fourier transform defined in (4.1.12). It is sufficient to consider the operator

$$\tilde{K}(z) = \Phi^* K^{\mu,\nu}(z)\Phi, \tag{4.1.59}$$

where each entry $\tilde{K}_{\alpha\beta}(z)$ has the kernel

$$\frac{1}{(2\pi)^3} e^{ixpd_{\alpha\beta}} \frac{|v_\alpha(x)|^{\frac{1}{2}} e^{ixp'e_{\alpha\beta}} e^{-ix'pd_{\beta\alpha}} |v_\beta(x')|^{\frac{1}{2}}}{\left(\zeta\left(\frac{p^2}{2n_\alpha} - z\right)\right)^\mu \left(\zeta\left(\frac{p'^2}{2n_\beta} - z\right)\right)^\nu \left(H^0_{\alpha\beta}(p,p') - z\right)} e^{-ix'p'e_{\beta\alpha}} \tag{4.1.60}$$

with $x = x_\alpha, x' = x_\beta, p = p_\alpha, p' = p_\beta$. Denote by χ_R the multiplication by the characteristic function of the ball $\{p \in \mathbb{R}^3 : |p| \leq R\}$. Then

$$\tilde{K}_{\alpha\beta}(z) = Z^R_{\alpha\beta}(z) + Y^R_{\alpha\beta}(z), \tag{4.1.61}$$

where

$$Z^R_{\alpha\beta}(z) = \chi_R \tilde{K}_{\alpha\beta}\chi_R + (I - \chi_R)\tilde{K}_{\alpha\beta}\chi_R + \chi_R\tilde{K}_{\alpha\beta}(I - \chi_R), \tag{4.1.62}$$

$$Y^R_{\alpha\beta}(z) = (I - \chi_R)\tilde{K}_{\alpha\beta}(I - \chi_R). \tag{4.1.63}$$

The kernel of $Z^R_{\alpha\beta}(z)$ is obviously square-integrable over its arguments for $z < 0$, which shows that it belongs to the Hilbert-Schmidt class. Hence, suppose that

$\mu+\nu<\frac{1}{2}$, $\mu\leq\frac{1}{4}$, $\nu\leq\frac{1}{4}$ and $z\leq 0$. Inequality (4.1.19) implies

$$H^0_{\alpha\beta}(p,q)\geq cp^{2\kappa}(p')^{2\kappa'},\qquad \kappa+\kappa'=1. \tag{4.1.64}$$

Therefore, the kernel of $Z^R_{\alpha\beta}(z)$ can be estimated by

$$|v_\alpha(x)|^{\frac{1}{2}}\chi_R(p)p^{-2(\mu+\nu)}(p')^{-2(\mu+\nu)}\chi_R(p')|v_\beta(x')|^{\frac{1}{2}}. \tag{4.1.65}$$

Choosing $\mu+\kappa<\frac{3}{4}$ and $\nu+\kappa'<\frac{3}{4}$ shows that the first term in (4.1.62) belongs to the Hilbert-Schmidt class and is therefore continuous in $z\leq 0$. The second operator in (4.1.62) is Hilbert-Schmidt, since its kernel is bounded by

$$C|v_\alpha(x)|^{\frac{1}{2}}(1-\chi_R(p))p^{-2}\left(\frac{p'^2}{2n_\beta}-z\right)^{-\nu}\chi_R(p')|v_\beta(x')|^{\frac{1}{2}}. \tag{4.1.66}$$

Using the same argument for the last term in (4.1.62) proves the compactness of $Z^R_{\alpha\beta}(z)$. The norm of the operator (4.1.63) is bounded by CR^{-2} for all $z\leq 0$, since

$$\chi_R(p)\left(H^0_{\alpha\beta}(p,p')-z\right)\chi_R(p')\geq CR^2. \tag{4.1.67}$$

By (4.1.61) this implies

$$\|Z^R_{\alpha\beta}(z)-\tilde{K}_{\alpha\beta}(z)\|\to 0\qquad \text{as } R\to\infty. \tag{4.1.68}$$

The compactness and continuity of $\tilde{K}_{\alpha\beta}(z)$ now follow from those of $Z^R_{\alpha\beta}(z)$. □

The following lemma shows that for the asymptotics (4.1.8) any compact perturbation can be neglected.

Lemma 4.1.8. *Let $T(z)=T_0(z)+T_1(z)$, where $T_0(z)$ ($T_1(z)$) is compact and continuous in $z<0$ ($z\leq 0$). Assume that for some function f with $f(z)\to 0$ as $z\to 0^-$ there exists the limit*

$$\lim_{z\to 0^-}f(z)n(\lambda,T_0(z))=l(\lambda), \tag{4.1.69}$$

continuous in $\lambda > 0$*. Then the same limit exists for* $T(z)$ *and*

$$\lim_{z\to 0^-} f(z)n(\lambda, T(z)) = l(\lambda). \tag{4.1.70}$$

Now we can give a sketch of the proof of Theorem 4.1.1. For a complete proof see [Sob93].

Proof of Theorem 4.1.1. Since the function $\mathcal{U}(\cdot)$ is continuous, by Lemma 4.1.8 any perturbation of $A(z)$, which is compact and continuous up to $z = 0$, does not contribute to the asymptotics (4.1.8). Suppose that condition **(i)** of the theorem is fulfilled.

Let $\Gamma_\alpha(z)$ be the multiplication by the function $\zeta\left(\frac{p_\alpha^2}{2n_\alpha} - z\right)$ and Π_α be the operator such that

$$(\Pi_\alpha f)(k_\alpha, p_\alpha) = \|\varphi_\alpha\|^{-1} (\Phi_\alpha\varphi_\alpha) \int f(k'_\alpha, p_\alpha)\overline{(\Phi_\alpha\varphi_\alpha)(k'_\alpha)}\, \mathrm{d}k'_\alpha. \tag{4.1.71}$$

Then

$$\begin{aligned}(W_\alpha(z))^{\frac{1}{2}} &= (\Gamma_\alpha(z))^{-\frac{1}{4}}\,\Pi_\alpha + (\Gamma_\alpha(z))^{-\frac{1-\delta}{4}}\,\tilde{W}_\alpha^{(\delta)}(z)\\ &= \Pi_\alpha\,(\Gamma_\alpha(z))^{-\frac{1}{4}} + \tilde{W}_\alpha^{(\delta)}(z)\,(\Gamma_\alpha(z))^{-\frac{1-\delta}{4}},\end{aligned} \tag{4.1.72}$$

where

$$\tilde{W}_\alpha^{(\delta)}(z) = \Phi_\alpha\tilde{w}_\alpha^{(\delta)}\left(z - \frac{p_\alpha^2}{2n_\alpha}\right)\Phi_\alpha^* \tag{4.1.73}$$

is bounded and continuous in $z \le 0$. Thus

$$\begin{aligned}(W(z))^{\frac{1}{2}} &= (\Gamma(z))^{-\frac{1}{4}}\,\Pi + (\Gamma(z))^{-\frac{1-\delta}{4}}\,\tilde{W}^{(\delta)}(z)\\ &= \Pi\,(\Gamma(z))^{-\frac{1}{4}} + \tilde{W}^{(\delta)}(z)\,(\Gamma(z))^{-\frac{1-\delta}{4}},\end{aligned} \tag{4.1.74}$$

where $\Gamma(z), W(z), \tilde{W}^{(\delta)}(z)$ and Π are diagonal matrices with the corresponding entries $\Gamma_\alpha(z), W_\alpha(z), \tilde{W}_\alpha^{(\delta)}(z)$ and Π_α, respectively. Therefore, one can decompose $A(z) = A^0(z) + Y(z)$, where

$$A^0(z) = \Pi K^{\frac{1}{4},\frac{1}{4}}(z)\Pi \tag{4.1.75}$$

and

$$Y(z) = \Pi K^{\mu,\nu}(z)\tilde{W}^{(\delta)}(z) + \tilde{W}^{(\delta)}(z)K^{\nu,\mu}(z)\Pi + \tilde{W}^{(\delta)}(z)K^{\nu,\nu}(z)\tilde{W}^{(\delta)}(z) \quad (4.1.76)$$

with $\mu = \frac{1}{4}$ and $\nu = \frac{1-\delta}{4}$. By Lemma 4.1.7 the operators $K^{\mu,\nu}(z)$, $K^{\nu,\mu}(z)$ and $K^{\nu,\nu}(z)$ are compact and continuous in $z \leq 0$. Hence, by Lemma 4.1.8 the operator $Y(z)$ does not contribute to the asymptotics of $A(z)$ for $z \to 0$. Therefore, it suffices to study the operator $A^0(z)$. Let χ be the characteristic function of the ball $\{p : |p| \leq 1\}$ and χ_α the multiplication by the function $\chi(p_\alpha)$. Define the matrix $\Xi = \operatorname{diag}\{\chi_{12}, \chi_{23}, \chi_{31}\}$. It is easy to see that $A(z) - \Xi A^0(z)\Xi$ is compact and continuous in z up to $z = 0$. Further, let $F = \operatorname{diag}\{F_{12}, F_{23}, F_{31}\}$, where

$$F_\alpha : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^6), \quad (F_\alpha f)(k_\alpha, p_\alpha) = (\Phi_\alpha \varphi_\alpha)(k_\alpha) f(p_\alpha). \quad (4.1.77)$$

Then

$$(F_\alpha^* f)(p_\alpha) = \int \overline{(\Phi_\alpha \varphi_\alpha)(k_\alpha)} f(k_\alpha, p_\alpha)\, \mathrm{d}k_\alpha. \quad (4.1.78)$$

Since $\Pi_\alpha^2 = \langle \cdot, \Phi_\alpha \varphi_\alpha \rangle \Phi_\alpha \varphi_\alpha = F_\alpha F_\alpha^*$, the non-trivial eigenvalues of $\Xi A^0(z)\Xi$ coincide with those of the operator

$$S(z) = \Xi K^{\frac{1}{4},\frac{1}{4}}(z)\Xi\Pi^2. \quad (4.1.79)$$

The operator $S(z)$ acts on functions in $L^2(B_1)$, $B_r = \{p : |p| \leq r\}$, and its kernel is given by

$$\frac{\overline{\psi_\alpha(d_{\alpha\beta}p + e_{\alpha\beta})}\psi_\beta(d_{\beta\alpha}q + e_{\beta\alpha}p)}{\zeta\left(\frac{p^2}{2n_\alpha} - z\right)^{\frac{1}{4}} (H^0_{\alpha\beta}(p,q) - z)\zeta\left(\frac{q^2}{2n_\beta} - z\right)^{\frac{1}{4}}}, \quad (4.1.80)$$

where $\psi_\alpha(k) = \left(\Phi_\alpha |v_\alpha|^{\frac{1}{2}} \varphi_\alpha\right)(k)$ and the constants $e_{\alpha\beta}, e_{\beta\alpha}$ and $d_{\alpha\beta}, d_{\beta\alpha}$ are given by (4.1.10). Due to $\langle \varphi_\alpha, |v_\alpha|^{\frac{1}{2}} \rangle \neq 0$ one can always normalize

$$(2\pi)^{\frac{3}{2}}\, \psi_\alpha(0) = \langle \varphi_\alpha, |v_\alpha|^{\frac{1}{2}} \rangle = 2^{\frac{1}{4}} \pi^{\frac{1}{2}} m_\alpha^{-\frac{3}{4}}. \quad (4.1.81)$$

By the use of $|\mathrm{e}^{-\langle k,x\rangle}-1|\leq|k|^{\delta}|x|^{\delta}$ it follows

$$|\psi_{\alpha}(k)-\psi(0)|\leq C_{\delta}|k|^{\delta},\qquad 0<\delta<\frac{b-3}{2}. \tag{4.1.82}$$

This, together with $H^0_{\alpha\beta}(p,q)\geq Cp^{2\kappa}q^{2\kappa'}$ for any $\kappa,\kappa'\geq 0$ with $\kappa+\kappa'=1$ shows that the difference is bounded by

$$\begin{aligned}\frac{C(|p|^{\delta}+|q|^{\delta})}{\left(\frac{p^2}{2n_\alpha}-z\right)^{\frac14}\left(H^0_{\alpha\beta}(p,q)-z\right)\left(\frac{q^2}{2n_\beta}-z\right)^{\frac14}}\leq & C'|p|^{\delta-2\kappa-\frac12}|q|^{-2\kappa'-\frac12}\\ & +C'|p|^{-2\kappa'-\frac12}|q|^{\delta-2\kappa-\frac12}.\end{aligned} \tag{4.1.83}$$

Choosing $\kappa\in\left(\frac12,\frac{1+\delta}{2}\right)$ shows that this operator is Hilbert-Schmidt up to $z=0$. Hence, one can replace the functions $\zeta\left(\frac{p^2}{2n_\alpha}-z\right)$ and $\zeta\left(\frac{q^2}{2n_\beta}-z\right)$ by $\frac{p^2}{2n_\alpha}-z$ and $\frac{q^2}{2n_\beta}-z$, respectively. One arrives at the operator with the kernel

$$2^{-\frac52}\pi^{-2}\left(m_\alpha m_\beta\right)^{-\frac34}\left(\frac{p^2}{2n_\alpha}-z\right)^{-\frac14}\left(\frac{q^2}{2n_\beta}-z\right)^{-\frac14}\left(H^0_{\alpha\beta}(p,q)-z\right)^{-1}. \tag{4.1.84}$$

For $r>0$ let $U=\operatorname{diag}\{U_r,U_r,U_r\}$, where

$$U_r: L^2(B_1)\to L^2(B_r),\quad (U_rf)(p)=r^{-\frac32}f\left(r^{-1}p\right). \tag{4.1.85}$$

Using U one can show that the operator with the kernel (4.1.84) is unitary equivalent to that with the kernel

$$2^{-\frac52}\pi^{-2}\left(m_\alpha m_\beta\right)^{-\frac34}\left(\frac{p^2}{2n_\alpha}+1\right)^{-\frac14}\left(\frac{q^2}{2n_\beta}+1\right)^{-\frac14}\left(H^0_{\alpha\beta}(p,q)+1\right)^{-1}, \tag{4.1.86}$$

acting on $L^2(B_r)$ with $r=|z|^{-\frac12}$. Further, one can replace

$$\left(\frac{p^2}{2n_\alpha}+1\right)^{-1},\quad\left(\frac{q^2}{2n_\beta}+1\right)^{-1}\qquad\text{and}\qquad H^0_{\alpha\beta}(p,q)+1 \tag{4.1.87}$$

by

$$2n_\alpha p^{-2}(1-\chi(p)),\quad 2n_\beta q^{-2}(1-\chi(q))\qquad\text{and}\qquad H^0_{\alpha\beta}(p,q), \tag{4.1.88}$$

respectively. Indeed, it is easy to see that the difference is a Hilbert-Schmidt operator continuous up to $z = 0$. Using (4.1.17) one arrives at the operator in $L^2(B_r \backslash B_1)$ with the kernel

$$(2\pi)^{-2}(m_\alpha m_\beta)^{-\frac{3}{4}}(n_\alpha n_\beta)^{\frac{1}{4}}|p|^{-\frac{1}{2}}|q|^{-\frac{1}{2}}\left(\frac{p^2}{2m_\beta}+\frac{\langle p,q\rangle}{l_\gamma}+\frac{q^2}{2m_\alpha}\right)^{-1}. \tag{4.1.89}$$

For $R > 0$ let $M = \operatorname{diag}\{M_R, M_R, M_R\}$, where

$$\begin{aligned} M_R &: L^2(B_r\backslash B_1) \to L^2((0,R)\times\mathbb{S}^2, \mathrm{d}x\otimes\mathrm{d}\Omega), \\ (M_R f)(x,\omega) &= \mathrm{e}^{\frac{3x}{2}} f(\mathrm{e}^x,\omega), \quad x\in(0,R),\ \omega\in\mathbb{S}^2. \end{aligned} \tag{4.1.90}$$

Using M one can show that the operator with the kernel (4.1.89) is unitary equivalent to the operator S_R, $R = \frac{1}{2}|\log|z||$, defined in $L^2((0,R)\times\mathbb{S}^2, \mathrm{d}x\otimes\mathrm{d}\Omega)$ with the kernel $S_{\alpha\beta}(x-x', \langle\xi,\nu\rangle), \xi,\nu\in\mathbb{S}^2$, where

$$\begin{cases} S_{\alpha\alpha}(x,t) = 0, \\ S_{\alpha\beta}(x,t) = (2\pi)^{-2}\frac{a_{\alpha\beta}}{\cosh(x+b_{\alpha\beta})+c_{\alpha\beta}t}. \end{cases} \tag{4.1.91}$$

Here the constants $a_{\alpha\beta}, b_{\alpha\beta}, c_{\alpha\beta}$ are given by (4.1.21). By an argument known as the calculation of the canonical distribution of a Toeplitz operator one can show that

$$\lim_{R\to\infty} R^{-1} n(\mu, S_R) = 2\mathcal{U}(\mu). \tag{4.1.92}$$

See [Sob93, Theorem 4.5] for a detailed proof of (4.1.92). By Theorem 4.1.3 this implies (4.1.8) with $\mathcal{U}_0 = \mathcal{U}(1)$.

Now assume that condition **(ii)** of Theorem 4.1.1 is fulfilled. Without loss of generality suppose that operators h_{12} and h_{23} have a zero-energy resonance and for h_{31} zero is neither a resonance nor an eigenvalue. By Lemma 4.1.6 the operator $W_{31}(z)$ is continuous in $z \le 0$. Hence, setting $\varphi_{31} = 0$ shows that $W_{31}(z)$ satisfies (4.1.72), where

$$\tilde{W}_{31}^{(\delta)}(z) = (\Gamma_{31}(z))^{\frac{1-\delta}{4}} W_{31}(z). \tag{4.1.93}$$

Repeating the same arguments from above completes the proof. □

4.2 Absence of the Efimov effect in dimension four

In this section we apply the technique presented in the previous section to systems of three particles in dimension four and prove that an Efimov-type effect does not exist in such systems. We follow the presentation of [BB19].

4.2.1 Systems of three four-dimensional quantum particles

We adapt the notation introduced in the last section, but adjust it to dimension four accordingly. Consider a system of three quantum particles of masses $m_1, m_2, m_3 > 0$ and pair interactions $v_{12}, v_{23}, v_{31} : \mathbb{R}^4 \to \mathbb{R}$ with the corresponding Hamiltonian in coordinate representation

$$-\frac{1}{2m_1}\Delta_{x_1} - \frac{1}{2m_2}\Delta_{x_2} - \frac{1}{2m_3}\Delta_{x_3} + v_{12}(x_1 - x_2) + v_{23}(x_2 - x_3) + v_{31}(x_3 - x_1). \quad (4.2.1)$$

For every subscript $\alpha \in \{12, 23, 31\}$ we assume that the potential v_α satisfies

$$v_\alpha \leq 0 \qquad \text{and} \qquad |v_\alpha(x)| \leq C(1 + |x|)^{-b}, \; b > 4. \quad (4.2.2)$$

The Hamiltonian of relative motion is given by

$$H = H_0 + \sum_\alpha v_\alpha, \quad (4.2.3)$$

where H_0 is the free Hamiltonian of the system. The corresponding configuration space R_0 in this case is an eight-dimensional subspace of $\mathbb{R}^{12}$. Under assumptions (4.2.2) on the potentials v_α the operator H is essentially self-adjoint. Every two-body subsystem corresponding to the subscript α is described in the center of mass frame by the Hamiltonian

$$h_\alpha = -\frac{1}{2m_\alpha}\Delta + v_\alpha \qquad \text{in } L^2(\mathbb{R}^4), \quad (4.2.4)$$

where m_α is the reduced mass. Denote $\mu = \min_\alpha \inf \sigma(h_\alpha)$, then by the HVZ theorem

$$\sigma_{\text{ess}}(H) = [\mu, \infty). \quad (4.2.5)$$

The case $\mu < 0$ in dimension three was studied earlier in [Zhi74] and [Yaf76] and can be adapted to the case $d = 4$. We consider the case $\mu = 0$.

Theorem 4.2.1. *Assume that all potentials v_α satisfy* (4.2.2)*. Then $\sigma_{\text{disc}}(H)$ is finite.*

In case of three identical particles the corresponding pair interactions satisfy

$$v_{ij}(x_i - x_j) = v_{ij}(x_j - x_i), \qquad i \neq j \tag{4.2.6}$$

and the operator H is invariant under the action of the group S_3 of permutatation of particles. Denote by π_1, π_2 and π_3 the three irreducible representations of S_3, where π_1 is the trivial representation, π_2 the antisymmetric representation and π_3 the two-dimensional irreducible representation, respectively. Denote by P^{π_i} with $i \in \{1, 2, 3\}$ the corresponding projection. In case of π_2 we denote the two-body Hamiltonians on $P^{\pi_2} L^2(\mathbb{R}^4)$ by h_α^{as} and the corresponding three-body Hamiltonian by H^{as}.

Theorem 4.2.2. *Assume that all potentials v_α satisfy* (4.2.2) *and $v_\alpha(x) = v_\alpha(-x)$. Then $\sigma_{\text{disc}}(H^{\text{as}})$ is finite.*

Remark. Consequently, an Efimov-type effect does not exist in dimension four.

The strategy of the proof of Theorem 4.2.1 is to adapt the technique of [Sob93] summarized in Section 4.1 to simplify the representation of H and carry out the computations in the momentum space. Analogous to the three-dimensional case in [Sob93] we denote by k_i the conjugate variable of x_i and introduce the set of variables (k_α, p_α), conjugate with respect to the Jacobi-coordinates (x_α, y_α). They are explicitly given by (4.1.9). In dimension $d = 4$ the shift from x_α to k_α is done by the partial Fourier transform

$$\left(\Phi_\alpha f\right)(k_\alpha, \cdot) = (2\pi)^{-2} \int_{\mathbb{R}^4} \mathrm{e}^{-\mathrm{i}\langle k_\alpha, x_\alpha \rangle} f(x_\alpha, \cdot)\, \mathrm{d}x_\alpha. \tag{4.2.7}$$

The relation of (k_α, p_α) and (p_α, p_β) is described by

$$k_\alpha = d_{\alpha\beta} p_\alpha + e_{\alpha\beta} p_\beta, \tag{4.2.8}$$

where the coefficients $d_{\alpha\beta}$ and $e_{\alpha\beta}$ can be expressed via the masses m_1, m_2 and m_3, see (4.1.10). The Hamiltonian has the form

$$H = H_0 + \sum_\alpha V_\alpha, \tag{4.2.9}$$

where the interactions are given by

$$V_\alpha = \Phi_\alpha v_\alpha \Phi_\alpha^* \tag{4.2.10}$$

and H_0 is the multiplication operator

$$(H_0 f)(k,p) = H^0(k,p) f(k,p), \tag{4.2.11}$$

where the function $H^0(k,p)$ is given by (4.1.16). H^0 expressed in terms of p_α, p_β is denoted by $H^0_{\alpha\beta}$ and it takes the form

$$H^0_{\alpha\beta}(p_\alpha, p_\beta) = \frac{p_\alpha^2}{2m_\beta} + \frac{\langle p_\alpha, p_\beta \rangle}{l_\gamma} + \frac{p_\beta^2}{2m_\alpha}, \tag{4.2.12}$$

where the constants m_α, n_α and l_α are the same as in section 4.1. Finally, we will use the same symmetrized form of Faddeev equations to study the discrete spectrum of H, i.e. for $z < 0$ we consider the matrix

$$A(z) = W^{\frac{1}{2}}(z) K(z) W^{\frac{1}{2}} \tag{4.2.13}$$

from Definition 4.1.2 but where every entry

$$A_{\alpha\beta}(z) = W_\alpha^{\frac{1}{2}}(z) |V_\alpha|^{\frac{1}{2}} (H_0 - z)^{-1} |V_\beta|^{\frac{1}{2}} W_\beta^{\frac{1}{2}} \tag{4.2.14}$$

is an operator acting in $L^2(\mathbb{R}^4)$. In Theorem 4.1.3 the Birman-Schwinger principle (4.1.44) also applies in dimension four. Hence, the proof of the following proposition is the same as in [Sob93].

Proposition 4.2.3. *Let $N(z)$ be the number of eigenvalues of the operator H below $z < 0$ and let $n(1, A(z))$ be the number of eigenvalues of the operator $A(z)$ greater* *than one. Then*

$$N(z) = n(1, A(z)). \tag{4.2.15}$$

Remark. As we have seen in the previous section, in case of three-dimensional particles the corresponding operator $A(z)$ is compact for $z < 0$ and due to resonances in the two-body subsystems the compactness is lost for $z \to 0$. We will see that in dimension four the singularity in $z = 0$ caused by the resonances in the two-body subsystems is not strong enough to break the compactness of $A(z)$ for $z \to 0$. To this end, we study the the operator $W_\alpha(z)$ in the frame of two-body subsystems.

4.2.2 Resonance interaction of two four-dimensional particles

The technique developed by A. Jensen in [JK79] was used by A. Sobolev in [Sob93] to obtain the representation (4.1.54) of the operator $w_\alpha(z)$ for $z \to 0$. In [Jen84] this technique was extended to dimension four. We will use this method in a similar way to derive a representation of $w_\alpha(z)$ for $z \to 0$ in our case. However, the difficulty is that dimension four is even and therefore Hankel functions are involved.

Definition 4.2.4. For $z < 0$ let $r_\alpha(z)$ be the resolvent of $h_\alpha = -\frac{1}{2m_\alpha}\Delta + v_\alpha$ and

$$w_\alpha(z) = I + |v_\alpha|^{\frac{1}{2}} r_\alpha(z) |v_\alpha|^{\frac{1}{2}}. \tag{4.2.16}$$

Similar to (4.1.51) for $r^0_\alpha(z) = \left(-\frac{1}{2m_\alpha}\Delta - z\right)^{-1}$ we have that

$$w_\alpha(z) = \left(I - |v_\alpha|^{\frac{1}{2}} r^0_\alpha(z) |v_\alpha|^{\frac{1}{2}}\right)^{-1}. \tag{4.2.17}$$

The next two lemmas together with (4.2.17) show how a zero-energy resonance of the two-body operator h_α affects $w_\alpha(z)$.

Lemma 4.2.5. *Let G_α be the integral operator in $L^2(\mathbb{R}^4)$ with the kernel*

$$G_\alpha(x, y) = \frac{m_\alpha}{2\pi^2} \frac{|v_\alpha(x)|^{\frac{1}{2}} |v_\alpha(y)|^{\frac{1}{2}}}{|x - y|^2}. \tag{4.2.18}$$

If zero is a resonance of h_α, then $\mu = 1$ is a simple eigenvalue of G_α.

Proof. By assumption (4.2.5) with $\mu = 0$, together with (4.2.2) we have that $\sigma(h_\alpha) = [0, \infty)$. Therefore, the zero-energy resonance is non-degenerate. Let f_α be a resonance state of h_α and set $\varphi_\alpha = |v_\alpha|^{\frac{1}{2}} f$. Since $f_\alpha \in \dot{H}^1(\mathbb{R}^4)$ we conclude $\varphi_\alpha \in L^2(\mathbb{R}^4)$ and

$$\begin{aligned}(G_\alpha \varphi_\alpha)(x) &= \frac{m_\alpha}{2\pi^2} \int_{\mathbb{R}^4} \frac{|v_\alpha(x)|^{\frac{1}{2}} |v_\alpha(y)|^{\frac{1}{2}}}{|x-y|^2} \varphi_\alpha(y) \, \mathrm{d}y \\ &= |v_\alpha(x)|^{\frac{1}{2}} \left(-\frac{m_\alpha}{2\pi^2} \int_{\mathbb{R}^4} \frac{v_\alpha(y) f_\alpha(y)}{|x-y|^2} \, \mathrm{d}y \right) \\ &= |v_\alpha(x)|^{\frac{1}{2}} f_\alpha(x) \\ &= \varphi_\alpha(x).\end{aligned} \tag{4.2.19}$$

□

Lemma 4.2.6. *Let G_α be the operator defined by the kernel* (4.2.18)*. For $z < 0$, $|z|$ sufficiently small, there exist compact operators $G_\alpha^{(1)}$, $G_\alpha^{(2)}$ and a constant $\delta > 0$, such that*

$$|v_\alpha|^{\frac{1}{2}} r_\alpha^0(z) |v_\alpha|^{\frac{1}{2}} = G_\alpha + z G_\alpha^{(1)} + z \ln|z| G_\alpha^{(2)} + |z|^{1+\delta} G_\alpha^{(\delta)}(z), \tag{4.2.20}$$

where $G_\alpha^{(\delta)}(z)$ is an operator with $\|G_\alpha^{(\delta)}(z)\|_{HS} \leq C_\delta |z|^\delta$. Here $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm.

Proof. In the following we consider $|z| < 1$. The kernel of $(-\Delta - z)^{-1}$ is given by

$$(-\Delta - z)^{-1}(|x-y|) = \frac{\mathrm{i}\sqrt{z}}{8\pi |x-y|} H_1^{(1)}\left(\sqrt{z}|x-y|\right), \quad x, y, \in \mathbb{R}^4, \tag{4.2.21}$$

where $H_1^{(1)}$ is the first Hankel function, see for example in [AS64]. Hence, it follows

$$\begin{aligned} r_\alpha^0(z, |x-y|) &= \left(-(2m_\alpha)^{-1}\Delta - z\right)^{-1}(|x-y|) \\ &= \frac{m_\alpha \mathrm{i}\sqrt{2m_\alpha z}}{4\pi|x-y|} H_1^{(1)}\left(\sqrt{2m_\alpha z}|x-y|\right). \end{aligned} \tag{4.2.22}$$

According to [AS64], p.360, one has $H_1^{(1)}(\zeta) = J_1(\zeta) + \mathrm{i}Y_1(\zeta)$ and

$$\begin{aligned} J_1(\zeta) &= \frac{\zeta}{2}\sum_{k=0}^{\infty} \frac{\left(-\frac{1}{4}\zeta^2\right)^k}{k!(k+1)!}, \\ Y_1(\zeta) &= -\frac{2}{\pi\zeta} + \frac{2}{\pi}\ln\left(\frac{\zeta}{2}\right) J_1(\zeta) - \frac{\zeta}{2\pi}\sum_{k=0}^{\infty} \frac{\left(-\frac{1}{4}\zeta^2\right)^k (\psi(k+1)+\psi(k+2))}{k!\,(k+1)!}, \end{aligned} \tag{4.2.23}$$

where

$$\psi(1) = -1, \qquad \psi(k) = \sum_{j=1}^{k-1}\frac{1}{j} - \gamma,\ k \geq 2 \tag{4.2.24}$$

and γ is the Euler–Mascheroni constant. Hence, we obtain

$$H_1^{(1)}(\zeta) = -\frac{2\mathrm{i}}{\pi\zeta} + \left(\frac{\zeta}{2} + \frac{\zeta\mathrm{i}}{\pi}\ln\left(\frac{\zeta}{2}\right)\right)\sum_{k=0}^{\infty} a_k\left(\zeta^2\right)^k - \frac{\zeta\mathrm{i}}{2\pi}\sum_{k=0}^{\infty} b_k\left(\zeta^2\right)^k, \tag{4.2.25}$$

where

$$a_k = \frac{(-1)^k}{4^k k!(k+1)!} \qquad \text{and} \qquad b_k = (\psi(k+1)+\psi(k+2))\,a_k. \tag{4.2.26}$$

Note that both series in (4.2.25) converge for every $\zeta \in \mathbb{C}$. By (4.2.22) the kernel of $|v_\alpha(x)|^{\frac{1}{2}} r_\alpha^0(z)|v_\alpha(y)|^{\frac{1}{2}}$ is given by

$$|v_\alpha(x)|^{\frac{1}{2}}|v_\alpha(y)|^{\frac{1}{2}}\frac{m_\alpha\mathrm{i}\sqrt{2m_\alpha z}}{4\pi|x-y|}H_1^{(1)}\left(\sqrt{2m_\alpha z}|x-y|\right), \tag{4.2.27}$$

which by (4.2.25) can be decomposed as

$$G_\alpha + zG_\alpha^{(1)} + z\ln|z|G_\alpha^{(2)} + G, \tag{4.2.28}$$

where for $\tilde{v}_\alpha(x,y) = |v_\alpha(x)|^{\frac{1}{2}}|v_\alpha(y)|^{\frac{1}{2}}$ the kernels $G_\alpha, G_\alpha^{(1)}$ and $G_\alpha^{(2)}$ are given by

$$G_\alpha(x,y) = \frac{m_\alpha}{2\pi^2}\frac{\tilde{v}_\alpha(x,y)}{|x-y|^2}, \tag{4.2.29}$$

$$G_\alpha^{(1)}(x,y) = \frac{m_\alpha^2}{4\pi^2}\tilde{v}_\alpha(x,y)\left(\psi(1)+\psi(2)-\ln(2m_\alpha)-2\ln\left(\frac{|x-y|}{2}\right)\right), \tag{4.2.30}$$

$$G_\alpha^{(2)}(x,y) = -\frac{m_\alpha^2}{4\pi^2}\tilde{v}_\alpha(x,y). \tag{4.2.31}$$

We will show that the remainder $G(x,y,z)$ is a Hilbert-Schmidt kernel and that the Hilbert-Schmidt norm is of order $\mathcal{O}\left(|z|^{1+\delta}\right)$ as $z \to 0$, where $\delta > 0$ is sufficiently small.

Let $\sqrt{2m_\alpha|z|}|x-y| > 1$. By [AS64], p.364, we have

$$\left|H_1^{(1)}(\zeta)\right| \leq c|\zeta|^{-\frac{1}{2}}, \qquad |\zeta| \geq 1. \tag{4.2.32}$$

Relations (4.2.22) and (4.2.28) imply that $|G(x,y,z)|\chi_{\{\sqrt{2m_\alpha|z|}|x-y|>1\}}$ can be estimated by

$$c\frac{|z|^{\frac{1}{2}}\tilde{v}_\alpha(x,y)}{|x-y|}\left|H_1^{(1)}(\sqrt{2m_\alpha z}|x-y|)\right| + |G_\alpha| + |z||G_\alpha^{(1)}| + |z\ln|z|||G_\alpha^{(2)}|. \tag{4.2.33}$$

Hence, by definition of the kernels (4.2.29)-(4.2.31), together with (4.2.32) we obtain

$$\begin{aligned} &|G(x,y,z)|\chi_{\{\sqrt{2m_\alpha|z|}|x-y|>1\}} \\ &\quad\leq |z||\ln|z||\tilde{v}_\alpha(x,y)\left(c_1 + c_2\left|\ln\left(\frac{|x-y|}{2}\right)\right|\right) \\ &\quad\leq |z||\ln|z||\tilde{v}_\alpha(x,y)\frac{(1+|x|)^{4\delta}(1+|y|)^{4\delta}}{(1+|x-y|)^{4\delta}}\left(c_1 + c_2\left|\ln\left(\frac{|x-y|}{2}\right)\right|\right) \\ &\quad\leq c|z|^{1+2\delta}|\ln|z||\tilde{v}_\alpha(x,y)(1+|x|)^{4\delta}(1+|y|)^{4\delta}\left(c_1 + c_2\left|\ln\left(\frac{|x-y|}{2}\right)\right|\right). \end{aligned} \tag{4.2.34}$$

Now let $\sqrt{2m_\alpha|z|}|x-y| \leq 1$. Note that in view of (4.2.25) we have

$$G(x,y,z) = z|v_\alpha(x)|^{\frac{1}{2}}\left(\sum_{j=1}^{\infty}\sum_{k=0}^{1} z^j(\ln|z|)^k G_j^k(x,y)\right)|v_\alpha(y)|^{\frac{1}{2}}, \tag{4.2.35}$$

where the kernels G_j^k are defined by

$$G_j^1(x,y) = -(2m_\alpha)^j \alpha_j |x-y|^{2j}, \tag{4.2.36}$$

$$G_j^0(x,y) = (2m_\alpha)^j \alpha_j |x-y|^{2j} \left(\beta_j - 2\ln\left(\frac{|x-y|}{2}\right)\right) \tag{4.2.37}$$

and the constants α_j, β_j are given by

$$\alpha_j = \frac{(-1)^j m_\alpha^2}{4^{j+1}\pi^2 j!(j+1)!}, \qquad \beta_j = \psi(j+1) + \psi(j+2) - \ln(2m_\alpha). \tag{4.2.38}$$

By definition of the kernels G_j^k we have

$$G(x,y,z) = z|v_\alpha(x)|^{\frac{1}{2}} \left(\sigma_1(x,y,z) + \sigma_2(x,y,z) + \sigma_3(x,y,z)\right) |v_\alpha(y)|^{\frac{1}{2}}, \tag{4.2.39}$$

where

$$\sigma_1(x,y,z) = \sum_{j=1}^{\infty} \alpha_j \beta_j \left(\sqrt{2m_\alpha z}|x-y|\right)^{2j}, \tag{4.2.40}$$

$$\sigma_2(x,y,z) = -2\ln\left(\frac{|x-y|}{2}\right) \sum_{j=1}^{\infty} \alpha_j \left(\sqrt{2m_\alpha z}|x-y|\right)^{2j}, \tag{4.2.41}$$

$$\sigma_3(x,y,z) = -\ln|z| \sum_{j=1}^{\infty} \alpha_j \left(\sqrt{2m_\alpha z}|x-y|\right)^{2j}. \tag{4.2.42}$$

We are going to estimate σ_1, σ_2 and σ_3 separately. Let $0 < \delta < 2^{-1}$. Since by assumption we have $\sqrt{2m_\alpha|z|}|x-y| \leq 1$, it holds

$$\begin{aligned} |\sigma_1(x,y,z)| &\leq \left(\sqrt{2m_\alpha|z|}|x-y|\right)^{4\delta} \sum_{j=1}^{\infty} |\alpha_j\beta_j| \left(\sqrt{2m_\alpha|z|}|x-y|\right)^{2(j-2\delta)} \\ &\leq C|z|^{2\delta}|x-y|^{4\delta} \sum_{j=1}^{\infty} |\alpha_j\beta_j| \\ &\leq C_1|z|^{2\delta}(1+|x|)^{4\delta}(1+|y|)^{4\delta}. \end{aligned} \tag{4.2.43}$$

In the last inequality we used the fact that $\sum\limits_{j=1}^{\infty}|\alpha_j\beta_j|<\infty$. Analogously we obtain

$$\begin{aligned}|\sigma_2(x,y,z)| &\leq 2\left|\ln\left(\frac{|x-y|}{2}\right)\right|\sum_{j=1}^{\infty}|\alpha_j|\left(\sqrt{2m_\alpha|z|}|x-y|\right)^{2j}\\ &\leq C_2|z|^{2\delta}\left|\ln\left(\frac{|x-y|}{2}\right)\right|(1+|x|)^{4\delta}(1+|y|)^{4\delta}\end{aligned}\tag{4.2.44}$$

and also

$$|\sigma_3(x,y,z)|\leq C_3|z|^{2\delta}|\ln|z||(1+|x|)^{4\delta}(1+|y|)^{4\delta}.\tag{4.2.45}$$

Hence, by collecting estimates (4.2.43)-(4.2.45), together with (4.2.39), we see that for $|z|<1$ sufficiently small $|G(x,y,z)|\chi_{\{\sqrt{2m_\alpha|z|}|x-y|\leq 1\}}$ can be estimated by

$$|z|^{1+2\delta}|\ln|z||\tilde{v}_\alpha(x,y)(1+|x|)^{4\delta}(1+|y|)^{4\delta}\left(c_3+c_4\left|\ln\left(\frac{|x-y|}{2}\right)\right|\right).\tag{4.2.46}$$

By combining estimates (4.2.46) and (4.2.34) we obtain

$$\begin{aligned}|G(x,y,z)| &\leq |G(x,y,z)|\chi_{\{\sqrt{2m_\alpha|z|}|x-y|\leq 1\}}+|G(x,y,z)|\chi_{\{\sqrt{2m_\alpha|z|}|x-y|>1\}}\\ &\leq C|z|^{1+2\delta}|\ln|z||\tilde{v}_\alpha(x,y)\times\\ &\qquad\times(1+|x|)^{4\delta}(1+|y|)^{4\delta}\left(1+\left|\ln\left(\frac{|x-y|}{2}\right)\right|\right).\end{aligned}\tag{4.2.47}$$

Since

$$\left|\ln\left(\frac{|x-y|}{2}\right)\right|\leq C\max\left\{|x-y|^{\varepsilon},|x-y|^{-\varepsilon}\right\},\ \varepsilon>0\tag{4.2.48}$$

and

$$|v_\alpha(x)|\leq C(1+|x|)^{-b},\ b>4,\tag{4.2.49}$$

we can choose $\varepsilon,\delta>0$, such that $0<\delta<\frac{b-4-2\varepsilon}{8}$. This implies that the remainder $G(z)$ belongs to the Hilbert-Schmidt class and that the operator norm is of order $\mathcal{O}\left(|z|^{1+2\delta}|\ln|z||\right)$. Hence, the operator $G_\alpha^{(\delta)}(z)=|z|^{-1-\delta}G(z)$ is bounded up to $z\leq 0$. Furthermore, we have that

$$|v_\alpha|^{\frac{1}{2}}r_\alpha^0(z)|v_\alpha|^{\frac{1}{2}}=G_\alpha+zG_\alpha^{(1)}+z\ln|z|G_\alpha^{(2)}+|z|^{1+\delta}G_\alpha^{(\delta)}(z),\ \delta>0.\tag{4.2.50}$$

This completes the proof. □

Remark. We used similar arguments as in [Jen84], where it was shown that for

$$|v_\alpha(x)| \leq C(1+|x|)^{-b}, \ b > 8, \tag{4.2.51}$$

$G(z)$ is of order $\mathcal{O}(|z|^2 \ln |z|)$. We allow weaker assumptions on the potential and obtain a weaker estimate as a result.

Lemma 4.2.7. *If zero is a resonance of h_α, then for $z < 0$, $|z|$ sufficiently small, the operator $w_\alpha(z)$ has the representation*

$$w_\alpha(z) = (z(\ln|z| - \tau_\alpha))^{-1} \langle \cdot, \varphi_\alpha \rangle \varphi_\alpha + (z(\ln|z| - \tau_\alpha))^{-1+\delta} \, w_\alpha^{(\delta)}(z), \tag{4.2.52}$$

where $\delta > 0$ is sufficiently small, φ_α is an eigenfunction of the operator G_α corresponding to the eigenvalue $\mu = 1$ and $\tau_\alpha \in \mathbb{R}$ is a constant, which depends on the potential v_α and on the mass m_α. In addition, the operator $w_\alpha^{(\delta)}(z)$ is bounded for $z \leq 0$.

Proof. Let

$$s_\alpha(z) = I - |v_\alpha|^{\frac{1}{2}} r_\alpha^0(z) |v_\alpha|^{\frac{1}{2}}. \tag{4.2.53}$$

We will use expansion (4.2.20) of Lemma 4.2.6 in order to compute the inverse

$$s_\alpha^{-1}(z) = w_\alpha(z). \tag{4.2.54}$$

Let P_0 be the one-dimensional projection on the subspace associated with the eigenfunction φ_α of the operator G_α corresponding to the eigenvalue $\mu = 1$ and denote by P_1 the projection onto the orthogonal complement of the eigenspace of μ in $L^2(\mathbb{R}^4)$. Following [Jen84], for every $\psi \in L^2(\mathbb{R}^4)$ we have the unique partition $\psi = P_0\psi + P_1\psi$, which allows us to write $s_\alpha(z)\psi$ as $S(z)(P_0\psi, P_1\psi)^\top$, where

$$S(z) = \begin{pmatrix} P_0 s_\alpha(z) P_0 & P_0 s_\alpha(z) P_1 \\ P_1 s_\alpha(z) P_0 & P_1 s_\alpha(z) P_1 \end{pmatrix}. \tag{4.2.55}$$

Furthermore, let

$$P(z)=\begin{pmatrix}|z|^{-\frac{1}{2}}P_0 & 0\\ 0 & P_1\end{pmatrix}\qquad\text{and}\qquad B(z)=P(z)S(z)P(z). \tag{4.2.56}$$

The entries of $B(z)$ are given by

$$b_{00}(z)=|z|^{-1}P_0(I-|v_\alpha|^{\frac{1}{2}}r^0_\alpha(z)|v_\alpha|^{\frac{1}{2}})P_0, \tag{4.2.57}$$
$$b_{01}(z)=|z|^{-\frac{1}{2}}P_0(I-|v_\alpha|^{\frac{1}{2}}r^0_\alpha(z)|v|^{\frac{1}{2}})P_1, \tag{4.2.58}$$
$$b_{10}(z)=|z|^{-\frac{1}{2}}P_1(I-|v_\alpha|^{\frac{1}{2}}r^0_\alpha(z)|v_\alpha|^{\frac{1}{2}})P_0, \tag{4.2.59}$$
$$b_{11}(z)=P_1(I-|v_\alpha|^{\frac{1}{2}}r^0_\alpha(z)|v_\alpha|^{\frac{1}{2}})P_1. \tag{4.2.60}$$

By the use of Lemma 4.2.6 we have

$$B(z)=C(z)+D(z), \tag{4.2.61}$$

where the matrix $C(z)$ is given by

$$C(z)=\begin{pmatrix}P_0\left(G^{(1)}_\alpha+\ln|z|G^{(2)}_\alpha\right)P_0 & 0\\ 0 & P_1\left(I-G_\alpha\right)P_1\end{pmatrix} \tag{4.2.62}$$

and the matrix $D(z)$ is given by

$$D(z)=\begin{pmatrix}d_{00}(z) & d_{01}(z)\\ d_{10}(z) & d_{11}(z)\end{pmatrix} \tag{4.2.63}$$

with the corresponding entries

$$d_{00}(z)=-|z|^\delta P_0G^{(\delta)}_\alpha(z)P_0, \tag{4.2.64}$$
$$d_{01}(z)=|z|^{\frac{1}{2}}P_0\left(G^{(1)}_\alpha+\ln|z|G^{(2)}_\alpha-|z|^\delta G^{(\delta)}_\alpha(z)\right)P_1, \tag{4.2.65}$$
$$d_{10}(z)=|z|^{\frac{1}{2}}P_1\left(G^{(1)}_\alpha+\ln|z|G^{(2)}_\alpha-|z|^\delta G^{(\delta)}_\alpha(z)\right)P_0, \tag{4.2.66}$$
$$d_{11}(z)=|z|P_1\left(G^{(1)}_\alpha+\ln|z|G^{(2)}_\alpha-|z|^\delta G^{(\delta)}_\alpha(z)\right)P_0. \tag{4.2.67}$$

By abuse of notation we write

$$D(z) = \mathcal{O}\left(|z|^{\delta}\right). \tag{4.2.68}$$

Since P_1 projects onto the subspace of functions orthogonal to φ_α, the operator $P_1(I - G_\alpha)P_1$ is invertible. Now since $\langle |v_\alpha|^{\frac{1}{2}}, \varphi_\alpha\rangle \neq 0$, we can normalize φ_α by

$$\langle |v_\alpha|^{\frac{1}{2}}, \varphi_\alpha\rangle = \frac{2\pi}{m_\alpha}. \tag{4.2.69}$$

Then we have

$$\langle G_\alpha^{(2)}\varphi_\alpha, \varphi_\alpha\rangle = -1 \qquad \text{and} \qquad \langle G_\alpha^{(1)}\varphi_\alpha, \varphi_\alpha\rangle = \tau_\alpha, \tag{4.2.70}$$

where due to (4.2.30) the constant τ_α is given by

$$\frac{m_\alpha^2}{4\pi^2}\iint\left(C_\alpha - 2\ln\left(\frac{|x-y|}{2}\right)\right)|v_\alpha(x)|^{\frac{1}{2}}|v_\alpha(y)|^{\frac{1}{2}}\varphi_\alpha(x)\varphi_\alpha(y)\,\mathrm{d}x\mathrm{d}y \tag{4.2.71}$$

with $C_\alpha = \psi(1) + \psi(2) - \ln(2m_\alpha)$. Using the relation

$$P_0 = \|\varphi_\alpha\|^{-2}\langle\cdot, \varphi_\alpha\rangle\varphi_\alpha \tag{4.2.72}$$

we obtain

$$P_0\left(G_\alpha^{(1)} + \ln|z|G_\alpha^{(2)}\right)P_0 = \frac{(\tau_\alpha - \ln|z|)}{\|\varphi_\alpha\|^2}P_0 \tag{4.2.73}$$

and therefore

$$C^{-1}(z) = \begin{pmatrix} \frac{\langle\cdot,\varphi_\alpha\rangle\varphi_\alpha}{(\tau_\alpha - \ln|z|)} & 0 \\ 0 & K \end{pmatrix}, \tag{4.2.74}$$

where $K = (P_1(I - G_\alpha)P_1)^{-1}$. Now we can write

$$B(z) = C(z) + D(z) = \left(I + D(z)C^{-1}(z)\right)C(z). \tag{4.2.75}$$

By (4.2.68) we have

$$\|D(z)C^{-1}(z)\| \xrightarrow{z\to 0} 0. \tag{4.2.76}$$

Therefore, we obtain the inverse of $B(z)$ by the Neumann series

$$B^{-1}(z) = C^{-1}(z)\left(I - \left(-D(z)C^{-1}(z)\right)\right)^{-1} \tag{4.2.77}$$

$$= C^{-1}(z) + C^{-1}(z)\sum_{n=1}^{\infty}\left(-D(z)C^{-1}(z)\right)^{n}. \tag{4.2.78}$$

Note that

$$\sum_{n=1}^{\infty}\|D(z)C^{-1}(z)\|^{n} \leq \frac{\|D(z)C^{-1}(z)\|}{1-\|D(z)C^{-1}(z)\|}, \tag{4.2.79}$$

which together with (4.2.68) yields

$$B^{-1}(z) = \begin{pmatrix} \frac{\langle\cdot,\varphi_\alpha\rangle\varphi_\alpha}{(\tau_\alpha-\ln|z|)} & 0 \\ 0 & K \end{pmatrix} + \mathcal{O}\left(|z|^{\delta}\right). \tag{4.2.80}$$

Furthermore, we have that

$$S^{-1}(z) = P(z)B^{-1}(z)P(z) \tag{4.2.81}$$

and $|z|(\tau_\alpha - \ln|z|) = z(\ln|z| - \tau_\alpha)$ for $|z|$ sufficiently small. This completes the proof. □

Lemma 4.2.8. *For $z<0$, $|z|$ sufficiently small, the operator $w_\alpha(z)$ is positive and we have*

$$w_\alpha^{\frac{1}{2}}(z) = \frac{\langle\cdot,\varphi_\alpha\rangle\varphi_\alpha}{\|\varphi_\alpha\|\sqrt{z\left(\ln|z|-\tau_\alpha\right)}} + \left(z(\ln|z|-\tau_\alpha)\right)^{-\frac{1-\delta}{2}}\tilde{w}_\alpha^{(\delta)}(z), \tag{4.2.82}$$

where $\tilde{w_\alpha}^{(\delta)}(z)$ is bounded for $z \leq 0$.

Proof. For $|z|$ sufficiently small we have $z(\ln|z|-\tau_\alpha) > 0$ Hence, by $P_0 = P_0^2$ we obtain

$$\left(\frac{\langle\cdot,\varphi_\alpha\rangle\varphi_\alpha}{z(\ln|z|-\tau_\alpha)}\right)^{\frac{1}{2}} = \left(\frac{\|\varphi_\alpha\|^2 P_0^2}{z(\ln|z|-\tau_\alpha)}\right)^{\frac{1}{2}} = \frac{\langle\cdot,\varphi_\alpha\rangle\varphi_\alpha}{\|\varphi_\alpha\|\sqrt{z(\ln|z|-\tau_\alpha)}}. \tag{4.2.83}$$

By $r_\alpha(z) \geq 0$ for $h_\alpha \geq 0$ we have that $w_\alpha(z) \geq 0$. Using

$$\|A^{\frac{1}{2}} - B^{\frac{1}{2}}\| \leq \|A - B\|^{\frac{1}{2}} \tag{4.2.84}$$

for positive operators A, B, Lemma 4.2.7 implies

$$\left\| w_\alpha^{\frac{1}{2}}(z) - \frac{\langle \cdot, \varphi_\alpha \rangle \varphi_\alpha}{\|\varphi_\alpha\| \sqrt{z\,(\ln|z| - \tau_\alpha)}} \right\| \leq C(z(\ln|z| - \tau_\alpha))^{-\frac{1-\delta}{2}}. \tag{4.2.85}$$

This completes the proof. □

4.2.3 Finiteness of the discrete spectrum

Now we move to the three-body system. In this section we prove that every entry $A_{\alpha\beta}(z)$ of the matrix $A(z)$ is a compact operator for every $z \leq 0$. By Definition 4.1.2 we have

$$A_{\alpha\beta}(z) = W_\alpha^{\frac{1}{2}}(z) K_{\alpha\beta}(z) W_\beta^{\frac{1}{2}}(z). \tag{4.2.86}$$

Due to the partial Fourier transform Φ_α, Φ_β, defined by (4.2.7), and the structure of the operator $A_{\alpha\beta}(z)$, we will make use of the mixed coordinates $(x_\alpha, p_\alpha), (x_\beta, p_\beta)$. We start with the proof of the compactness of $K_{\alpha\beta}(z)$, which can be proved with similar arguments as in the proof of Lemma 4.1.7. However, the proof cannot be applied directly due to different dimensions. For the sake of completeness we give a proof by adjusting it to our case.

Lemma 4.2.9. *The operator $K_{\alpha\beta}(z)$, defined by*

$$K_{\alpha\beta}(z) = |V_\alpha|^{\frac{1}{2}} R_0(z) |V_\beta|^{\frac{1}{2}}, \qquad R_0(z) = (H_0 - z)^{-1} \tag{4.2.87}$$

is compact for every $z \leq 0$.

Proof. It is sufficient to consider the operator

$$\tilde{K}_{\alpha\beta}(z) = \Phi_\alpha^* K_{\alpha\beta}(z) \Phi_\beta. \tag{4.2.88}$$

For $R \geq 1$ let

$$\chi_R : \mathbb{R}^4 \to \mathbb{R},\ \chi_R(p) = \begin{cases} 1, & |p| \leq R, \\ 0, & |p| > R. \end{cases} \tag{4.2.89}$$

We decompose $\tilde{K}_{\alpha\beta}(z) = Z^R_{\alpha\beta}(z) + Y^R_{\alpha\beta}(z)$, where in contrast to the decomposition (4.1.61) we set

$$Z^R_{\alpha\beta}(z) = \chi_R \tilde{K}_{\alpha\beta} \chi_R, \tag{4.2.90}$$
$$Y^R_{\alpha\beta}(z) = (I - \chi_R)\tilde{K}_{\alpha\beta}\chi_R + \chi_R \tilde{K}_{\alpha\beta}(I - \chi_R) + (I - \chi_R)\tilde{K}_{\alpha\beta}(I - \chi_R). \tag{4.2.91}$$

The kernel of the operator $Z^R_{\alpha\beta}(z)$ is square-integrable for $z \leq 0$. Indeed, by the relation $V_\alpha = \Phi_\alpha v_\alpha \Phi^*_\alpha$ and by the use of (4.2.8) it follows

$$\begin{aligned} &\left(\tilde{K}_{\alpha\beta}(z) f\right)(x_\alpha, p_\alpha) \\ &= \int_{\mathbb{R}^4} \mathrm{d}k_\alpha \frac{\mathrm{e}^{\mathrm{i}k_\alpha x_\alpha} |v_\alpha(x_\alpha)|^{\frac{1}{2}}}{(H^0(k_\alpha, p_\alpha) - z)} \int_{\mathbb{R}^4} \mathrm{d}x_\beta \mathrm{e}^{-\mathrm{i}k_\beta x_\beta} |v_\beta(x_\beta)|^{\frac{1}{2}} f(x_\beta, p_\beta) \\ &= C \int_{\mathbb{R}^4} \mathrm{d}p_\beta \frac{\mathrm{e}^{\mathrm{i}x_\alpha\left(d_{\alpha\beta}p_\alpha + e_{\alpha\beta}p_\beta\right)} |v_\alpha(x_\alpha)|^{\frac{1}{2}}}{(H^0_{\alpha\beta}(p_\alpha, p_\beta) - z)} \int_{\mathbb{R}^4} \mathrm{d}x_\beta \mathrm{e}^{-\mathrm{i}x_\beta\left(d_{\beta\alpha}p_\alpha + e_{\beta\alpha}p_\beta\right)} |v_\beta(x_\beta)|^{\frac{1}{2}} f(x_\beta, p_\beta). \end{aligned} \tag{4.2.92}$$

Hence, the kernel of $\tilde{K}_{\alpha\beta}(z)$ is of the form

$$\tilde{K}_{\alpha\beta}\left((x,p),(x',p')\right) = c\mathrm{e}^{\mathrm{i}xpd_{\alpha\beta}} \frac{|v_\alpha(x)|^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}xp'e_{\alpha\beta}} \mathrm{e}^{-\mathrm{i}x'pd_{\beta\alpha}} |v_\beta(x')|^{\frac{1}{2}}}{(H^0_{\alpha\beta}(p,p') - z)} \mathrm{e}^{-\mathrm{i}x'p'e_{\beta\alpha}}. \tag{4.2.93}$$

By the estimate

$$H^0_{\alpha\beta}(p_\alpha, p_\beta) \geq c|p_\alpha|^{2\kappa}|p_\beta|^{2\kappa'} \tag{4.2.94}$$

with $\kappa = \kappa' = \frac{1}{2}$ it follows that $Z^R_{\alpha\beta}(z)$ belongs to the Hilbert-Schmidt class for every $z \leq 0$. Using estimate (4.1.19) one can see that the norm of the operator $Y^R_{\alpha\beta}(z)$ is bounded by CR^{-2} for every $z \leq 0$, where C does not depend on z. Hence, we have

$$\|\tilde{K}_{\alpha\beta}(z) - Z^R_{\alpha\beta}(z)\| = \|Y^R_{\alpha\beta}(z)\| \to 0 \tag{4.2.95}$$

as $R \to \infty$, which completes the proof. □

Recall that $w_\alpha(z)$ is uniformly bounded in $L^2(\mathbb{R}^4)$ for every $z \le z_0 < 0$, where $|z_0| > 0$ can be chosen arbitrarily small. Furthermore, in accordance with Definition 4.1.2 we have

$$W_\alpha(z) = \Phi_\alpha w_\alpha \left(z - \frac{p_\alpha^2}{2n_\alpha} \right) \Phi_\alpha^*. \tag{4.2.96}$$

Hence, the operator $W_\alpha(z)$ is bounded for every $z < 0$, which together with Lemma 4.2.9 and Proposition 4.2.3 implies that the counting function satisfies $N(z) < \infty$ for every $z < 0$. The critical case is the existence of a zero-energy resonance of the two-body Hamiltonian h_α, which by Lemma 4.2.8 affects the behaviour of the operator $w_\alpha(z)$ and therefore $W_\alpha(z)$ as $z \to 0$. In this case the operator $w_\alpha^{\frac{1}{2}}(z)$ has the representation

$$w_\alpha^{\frac{1}{2}}(z) = \frac{\langle \cdot, \varphi_\alpha \rangle \varphi_\alpha}{\|\varphi_\alpha\| \sqrt{z\left(\ln|z| - \tau_\alpha\right)}} + \left(z(\ln|z| - \tau_\alpha)\right)^{-\frac{1-\delta}{2}} \tilde{w}_\alpha^{(\delta)}(z), \tag{4.2.97}$$

where $|z| < 1$ is sufficiently small and $\ln|z| - \tau_\alpha < 0$. Our goal is to use this representation of combination with (4.2.96) to study the integral kernel of

$$A_{\alpha\beta}(z) = W_\alpha^{\frac{1}{2}}(z) K_{\alpha\beta}(z) W_\beta^{\frac{1}{2}}(z) \qquad \text{for} \quad z \to 0. \tag{4.2.98}$$

However, we can use (4.2.97) only if $\left| z - \frac{p_\alpha^2}{2n_\alpha} \right|$ is sufficiently small and when $\ln\left(z - \frac{p_\alpha^2}{2n_\alpha} \right) - \tau_\alpha < 0$ holds. Therefore, similar to (4.1.56) for every α we introduce the following auxiliary function $\zeta_\alpha : (-\infty, 0) \to \mathbb{R}$, where $\zeta_\alpha \in C^\infty$, $\zeta_\alpha(t) > 0$ for all $t < 0$ and

$$\zeta_\alpha(t) = \begin{cases} \sqrt{t\left(\ln|t| - \tau_\alpha\right)}, & t \in (\mu_\alpha, 0), \\ 1, & t \le -1. \end{cases} \tag{4.2.99}$$

The constant $\mu_\alpha \in (-1, 0)$ is chosen so that we have $\ln|t| - \tau_\alpha < 0$ for all $t \in [\mu_\alpha, 0)$. This will allow us to use (4.2.97) not only for small z but for every $z < 0$. To do so, we introduce the operator

$$\tilde{u}_\alpha^{(\delta)}(z) = \begin{cases} \tilde{w}_\alpha^{(\delta)}(z), & z \in (\mu_\alpha, 0), \\ \zeta_\alpha(z)^{-\delta} \left(\zeta_\alpha(z) w_\alpha^{\frac{1}{2}}(z) - \|\varphi\|^{-1} \langle \cdot, \varphi_\alpha \rangle \varphi_\alpha \right), & z \in (-\infty, \mu_\alpha]. \end{cases} \tag{4.2.100}$$

Since $w_\alpha^{\frac{1}{2}}(z)$ is uniformly bounded for $z \leq \mu_\alpha < 0$ and $\tilde{w}_\alpha^{(\delta)}(z)$ is continuous up to $z = 0$, it follows

$$w_\alpha^{\frac{1}{2}}(z) = \zeta_\alpha(z)^{-1}\|\varphi_\alpha\|^{-1}\langle\cdot,\varphi_\alpha\rangle\varphi_\alpha + \zeta_\alpha(z)^{-1+\delta}\tilde{u}_\alpha^{(\delta)}(z), \tag{4.2.101}$$

where the operator $\tilde{u}_\alpha^{(\delta)}(z)$ is continuous up to $z = 0$. From (4.2.96) and (4.2.98) it is clear that for $z = 0$ the kernel of $A_{\alpha\beta}(z)$ admits a singularity in $p_\alpha = 0$ and $p_\beta = 0$. For this reason, we will study the kernel in four disjoint areas. Simply put, we will cut the region with respect to the variables p_α, p_β, where both $|p_\alpha|, |p_\beta|$ are small, both $|p_\alpha|, |p_\beta|$ are large and the other two cases where $|p_\alpha|$ is small and $|p_\beta|$ is large, and vice versa. Here, it should be noted that in dimension four the mixed cases of one of the variables $|p_\alpha|$,$|p_\beta|$ being small and the other one being large is more complicated compared to the three-dimensional case. The reason for that is that after squaring the kernel the resolvent provides in both cases the decay $|p_\alpha|^{-4}$ and $|p_\beta|^{-4}$, which in dimension three yields the Hilbert-Schmidt property. This argument cannot be adapted to the four-dimensional case.

Lemma 4.2.10. *Let $\Gamma_\alpha(z)$ be the operator of multiplication by $\zeta_\alpha\left(z - \frac{p_\alpha^2}{2n_\alpha}\right)$, i.e.*

$$(\Gamma_\alpha(z)f)(k_\alpha, p_\alpha) = \zeta_\alpha\left(z - \frac{p_\alpha^2}{2n_\alpha}\right)\cdot f(k_\alpha, p_\alpha). \tag{4.2.102}$$

Then the operator

$$M_{\alpha\beta}(z) = \Gamma_\alpha(z)^{-1}K_{\alpha\beta}(z)\Gamma_\beta(z)^{-1} \tag{4.2.103}$$

is compact for every $z \leq 0$.

Proof. We consider the operator

$$\tilde{M}_{\alpha\beta}(z) = \Phi_\alpha^* M_{\alpha\beta}(z)\Phi_\beta. \tag{4.2.104}$$

The compactness for $z < 0$ follows from Lemma 4.2.9. We only need to consider the case $z = 0$. Similar to (4.2.93) the kernel of $\tilde{M}_{\alpha\beta}(0)$ is formally given by

$$\tilde{M}_{\alpha\beta}\left((x,p),(x',p')\right) = c\mathrm{e}^{\mathrm{i}xpd_{\alpha\beta}}\frac{|v_\alpha(x)|^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}xp'e_{\alpha\beta}}\mathrm{e}^{-\mathrm{i}x'pd_{\beta\alpha}}|v_\beta(x')|^{\frac{1}{2}}}{\zeta_\alpha\left(-\frac{p^2}{2n_\alpha}\right)H^0_{\alpha\beta}(p,p')\zeta_\beta\left(-\frac{p'^2}{2n_\beta}\right)}\mathrm{e}^{-\mathrm{i}x'p'e_{\beta\alpha}}. \tag{4.2.105}$$

Let $\mu_\alpha, \mu_\beta < 0$ be in accordance with (4.2.99) and $0 < r < \min(|\mu_\alpha|, |\mu_\beta|\}$. Denote by $\chi_r(p)$ the multiplication by the characteristic function of the area

$$\left\{ p \in \mathbb{R}^4 \ : \frac{|p|^2}{2n} < r \right\} \qquad \text{with} \quad n = \min\{n_\alpha, n_\beta\}. \tag{4.2.106}$$

We decompose

$$\tilde{M}_{\alpha\beta} = \tilde{M}^1_{\alpha\beta} + \tilde{M}^2_{\alpha\beta} + \tilde{M}^3_{\alpha\beta}, \tag{4.2.107}$$

such that

$$\tilde{M}^1_{\alpha\beta} = \chi_r(p)\tilde{M}_{\alpha\beta}\chi_r(p'), \tag{4.2.108}$$

$$\tilde{M}^2_{\alpha\beta} = \chi_r(p)\tilde{M}_{\alpha\beta}(I - \chi_r(p')) + (I - \chi_r(p))\tilde{M}_{\alpha\beta}\chi_r(p'), \tag{4.2.109}$$

$$\tilde{M}^3_{\alpha\beta} = (I - \chi_r(p))\tilde{M}_{\alpha\beta}(I - \chi_r(p')). \tag{4.2.110}$$

The compactness of $\tilde{M}^3_{\alpha\beta}$ follows from Lemma 4.2.9.

Let us prove that $\tilde{M}^2_{\alpha\beta}$ is compact. We consider only the first term, the second one can be treated analogously. Let $R > r > 0$ be fixed. Then the first term of $\tilde{M}^2_{\alpha\beta}$ can be written as

$$\chi_r(p)\tilde{M}_{\alpha\beta}(I - \chi_r(p')) = X_{\alpha\beta} + Y_{\alpha\beta}, \tag{4.2.111}$$

where

$$X_{\alpha\beta} = \chi_r(p)\tilde{M}_{\alpha\beta}(\chi_R(p') - \chi_r(p')), \qquad Y_{\alpha\beta} = \chi_r(p)\tilde{M}_{\alpha\beta}(I - \chi_R(p')). \tag{4.2.112}$$

By the use of

$$H^0_{\alpha\beta}(p, p') \geq cp'^2 \tag{4.2.113}$$

the absolute value of the kernel of $X_{\alpha\beta}$ can be estimated from above by

$$c\chi_r(p)\frac{|v_\alpha(x)|^{\frac{1}{2}}|v_\beta(x')|^{\frac{1}{2}}}{|p||p'|^2}(\chi_R(p') - \chi_r(p')), \tag{4.2.114}$$

which is square-integrable with respect to the arguments x, x', p, p'. The kernel of

$Y_{\alpha\beta}$ is given by

$$c\chi_r(p)(1-\chi_R(p'))\mathrm{e}^{\mathrm{i}xpd_{\alpha\beta}}\frac{|v_\alpha(x)|^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}xp'e_{\alpha\beta}}\mathrm{e}^{-\mathrm{i}x'pd_{\beta\alpha}}|v_\beta(x')|^{\frac{1}{2}}}{\zeta_\alpha\left(-\frac{p^2}{2n_\alpha}\right)H^0_{\alpha\beta}(p,p')}\mathrm{e}^{-\mathrm{i}x'p'e_{\beta\alpha}}. \tag{4.2.115}$$

We will show that $Y^*_{\alpha\beta}Y_{\alpha\beta}$ is continuous and that its operator norm tends to zero as $R\to\infty$. The kernel $Y^*_{\alpha\beta}Y_{\alpha\beta}\left((x'',p''),(x',p')\right)$ is given by

$$\iint \overline{Y_{\alpha\beta}\left((x,p),(x',p')\right)}Y_{\alpha\beta}\left((x,p),(x'',p'')\right)\mathrm{d}x\mathrm{d}p. \tag{4.2.116}$$

Hence, $|Y^*_{\alpha\beta}Y_{\alpha\beta}\left((x'',p''),(x',p')\right)|$ can be estimated from above by

$$c|\widehat{v_\alpha}(p'-p'')||v_\beta(x')|^{\frac{1}{2}}|v_\beta(x'')|^{\frac{1}{2}}J(p',p'')(I-\chi_R(p''))(1-\chi_R(p')), \tag{4.2.117}$$

where

$$\widehat{v_\alpha}(p'-p'') = \int |v_\alpha(x)|\mathrm{e}^{-\mathrm{i}e_{\alpha\beta}x(p'-p'')}\,\mathrm{d}x, \tag{4.2.118}$$

$$J(p',p'') = \int_{\left\{|p|<\sqrt{2nr}\right\}}\frac{1}{p^2(p^2+p'^2)(p^2+p''^2)}\,\mathrm{d}p. \tag{4.2.119}$$

Due to the characteristic functions $(I-\chi_R(p''))$ and $(1-\chi_R(p'))$ in (4.2.117) we can assume that $|p'|,|p''|\geq c>0$ for some fixed c, which yields

$$J(p',p'')\leq\frac{C}{p'^2p''^2}. \tag{4.2.120}$$

This implies

$$\begin{aligned}&|Y^*_{\alpha\beta}Y_{\alpha\beta}\left((x'',p''),(x',p')\right)|\\&\qquad\leq C\frac{|\widehat{v_\alpha(p'-p'')|}}{p'^2p''^2}|v_\beta(x')|^{\frac{1}{2}}|v_\beta(x'')|^{\frac{1}{2}}(I-\chi_R(p''))(1-\chi_R(p')).\end{aligned} \tag{4.2.121}$$

For $\xi_1,\xi_2\in\mathbb{R}^4\backslash\{0\}$ and $b>4$ we define the function

$$y(\xi_1,\xi_2) = (1+|\xi_1|)^{-\frac{b}{2}}|\xi_2|^{-2}. \tag{4.2.122}$$

By assumption (4.2.2) we have

$$v_\alpha \in L^1(\mathbb{R}^4) \cap L^\infty(\mathbb{R}^4) \qquad \text{and} \qquad \widehat{v_\alpha} \in L^2(\mathbb{R}^4) \cap L^\infty(\mathbb{R}^4). \tag{4.2.123}$$

Hence, by the use of (4.2.121) and $(1+|\cdot|)^{-\frac{b}{2}}|v_\beta(\cdot)|^{\frac{1}{2}} \in L^1(\mathbb{R}^4)$ we obtain

$$\begin{aligned} \int \left| Y^*_{\alpha\beta} Y_{\alpha\beta} \left((x'',p''),(x',p') \right) y(x',p') \right| \, \mathrm{d}x' \mathrm{d}p' &\leq \frac{|v_\beta(x'')|^{\frac{1}{2}}}{|p''|^2} C_R \\ &\leq y(x'',p'') C_R, \end{aligned} \tag{4.2.124}$$

where $C_R \to 0$ as $R \to \infty$. By symmetry we also have

$$\int \left| Y^*_{\alpha\beta} Y_{\alpha\beta} \left((x'',p''),(x',p') \right) y(x'',p'') \right| \, \mathrm{d}x'' \mathrm{d}p'' \leq y(x',p') C_R. \tag{4.2.125}$$

Hence, we can apply the Schur test, see for example [HS78], to conclude that $Y_{\alpha\beta}$ is a bounded operator on L^2, where the operator norm tends to zero as $R \to \infty$. By applying the same arguments to the second kernel of (4.2.109) we conclude that $\tilde{M}^2_{\alpha\beta}$ is compact.

It remains to show that $\tilde{M}^1_{\alpha\beta}$ is compact. By definition of the function (4.2.99) and in view of the characteristic functions $\chi_r(p), \chi_r(p')$ with

$$0 < r < \mu < \min\{|\mu_\alpha|, |\mu_\beta|\}, \tag{4.2.126}$$

it is sufficient to show that the integral

$$\int_{\{|p|<\mu\}} \int_{\{|p'|<\mu\}} K(p,p') \, \mathrm{d}p' \mathrm{d}p \tag{4.2.127}$$

is finite, where $\mu > 0$ is sufficiently small and the kernel K is given by

$$K(p,p') = \frac{1}{|p|^2 \, |\ln|p|| \left(H^0_{\alpha\beta}(p,p') \right)^2 |p'|^2 \, |\ln|p'||}. \tag{4.2.128}$$

Note that

$$\left(H^0_{\alpha\beta}(p,p') \right)^2 \geq c|p|^{4\kappa}|p'|^{4\kappa'} \qquad \text{with} \qquad \kappa + \kappa' = 1. \tag{4.2.129}$$

We set $\kappa = 0$ and use spherical coordinates $p = (\omega, \rho),\ p' = (\omega', \rho')$ to obtain

$$\begin{aligned}
\int\limits_{\{|p|<\mu\}} \int\limits_{\{|p'|<\mu\}} K(p,p')\,\mathrm{d}p'\mathrm{d}p &= \int\limits_{\{|p|<\mu\}} \left(\int\limits_{\{|p'|\le|p|\}} K(p,p')\,\mathrm{d}p' + \int\limits_{\{|p|<|p'|<\mu\}} K(p,p')\,\mathrm{d}p' \right) \mathrm{d}p \\
&\le C \int\limits_{\{|p|<\mu\}} \frac{1}{p^2\,|\ln|p||} \left(\int\limits_{\{|p|\le|p'|<\mu\}} \frac{1}{p'^6\,|\ln|p'||}\,\mathrm{d}p' \right) \mathrm{d}p \\
&\le C' \int_0^\mu \frac{\rho}{|\ln\rho|} \left(\int_\rho^\mu \frac{1}{\rho'^3\,|\ln\rho'|} \mathrm{d}\rho' \right) \mathrm{d}\rho \\
&= C' \int_0^\mu \frac{\rho}{|\ln\rho|} F(\mu,\rho)\,\mathrm{d}\rho, \qquad (4.2.130)
\end{aligned}$$

where the function F is given by

$$F(\mu,\rho) = \int_\rho^\mu \frac{1}{\rho'^3\,|\ln\rho'|} \mathrm{d}\rho' = -\frac{1}{2\rho'^2\,|\ln\rho'|}\Bigg|_\rho^\mu - \int_\rho^\mu \frac{1}{2\rho'^3\,|\ln\rho'|^2}. \qquad (4.2.131)$$

For $\mu > 0$ sufficiently small we have that $|\ln\rho'| \ge 1$, which implies

$$|F(\mu,\rho)| \le C(\mu) + \frac{1}{2\rho^2\,|\ln\rho|} + \frac{1}{2}\,|F(\mu,\rho)|\,. \qquad (4.2.132)$$

Hence, by inserting (4.2.132) into (4.2.130) we obtain

$$\begin{aligned}
\int\limits_{\{|p|<\mu\}} \int\limits_{\{|p'|<\mu\}} K(p,p')\,\mathrm{d}p'\mathrm{d}p &\le C' \int_0^\mu \frac{\rho}{|\ln\rho|} F(\mu,\rho)\,\mathrm{d}\rho \qquad (4.2.133) \\
&\le C_1 + C_2 \int_0^\mu \frac{1}{\rho(\ln\rho)^2}\,\mathrm{d}\rho < \infty. \qquad (4.2.134)
\end{aligned}$$

This completes the proof of Lemma 4.2.10. □

Now we are ready to prove Theorem 4.2.1 and Theorem 4.2.2.

Proof of Theorem 4.2.1. At first we assume that for every $\alpha \in \{12, 23, 31\}$ the two-body Hamiltonian h_α has a virtual level at zero. By assumption (4.2.5) with $\mu = 0$ and Theorem 3.3.1, zero is a resonance of h_α. By (4.2.101) every entry $A_{\alpha\beta}(z)$ of

$A(z)$ defined in (4.2.14) can then be written as

$$\begin{aligned}A_{\alpha\beta}(z) = \Pi_\alpha \Gamma_\alpha(z)^{-1} K_{\alpha\beta}(z) \Gamma_\beta(z)^{-1} \Pi_\beta + \tilde{U}_\alpha^{(\delta)}(z) \Gamma_\alpha(z)^{-1+\delta} K_{\alpha\beta}(z) \Gamma_\beta(z)^{-1} \Pi_\beta \\ + \Pi_\alpha \Gamma_\alpha(z)^{-1} K_{\alpha\beta}(z) \left(\Gamma_\beta(z)\right)^{-1+\delta} \tilde{U}_\beta^{(\delta)}(z) \\ + \tilde{U}_\alpha^{(\delta)}(z) \Gamma_\alpha(z)^{-1+\delta} K_{\alpha\beta}(z) \Gamma_\beta(z)^{-1+\delta} \tilde{U}_\beta^{(\delta)}(z),\end{aligned} \tag{4.2.135}$$

where the operator Π_α is defined by

$$(\Pi_\alpha f)(k_\alpha, p_\alpha) = \|\varphi_\alpha\|^{-1} (\Phi_\alpha \varphi_\alpha)(k_\alpha) \int f(k'_\alpha, p_\alpha) \overline{(\Phi_\alpha \varphi_\alpha)(k'_\alpha)} \, \mathrm{d}k'_\alpha \tag{4.2.136}$$

and $\tilde{U}_\alpha^{(\delta)}(z)$ is given by

$$\tilde{U}_\alpha^{(\delta)}(z) = \Phi_\alpha \tilde{u}_\alpha^{(\delta)} \left(z - \frac{p_\alpha^2}{2n_\alpha} \right) \Phi_\alpha^*. \tag{4.2.137}$$

The operator $\tilde{u}_\alpha^{(\delta)}(z)$ is defined by (4.2.100). Now since the operators Π_α, Π_β and $\tilde{U}_\alpha^{(\delta)}(z), \tilde{U}_\beta^{(\delta)}(z)$ are bounded for $z \leq 0$, the finiteness of $\sigma_{\mathrm{disc}}(H)$ follows from Lemma 4.2.10 and Proposition 4.2.3.

Assume that for one subsystem, say α, the operator h_α does not have a resonance. In this case the operator $w_\alpha(z)$ is continuous up to $z = 0$. Indeed, one can easily see that $\mu = 1$ is not an eigenvalue of the operator with the kernel

$$G_\alpha(x, y) = \frac{m_\alpha}{2\pi^2} \frac{|v_\alpha(x)|^{\frac{1}{2}} |v_\alpha(y)|^{\frac{1}{2}}}{|x - y|^2}. \tag{4.2.138}$$

Similar to the proof of Lemma 4.2.7 we then have

$$w_\alpha(z) = (I - G_\alpha + o(1))^{-1} = (I - G_\alpha)^{-1} + o(1), \; z \to 0. \tag{4.2.139}$$

This implies the finiteness of $\sigma_{\mathrm{disc}}(H)$ in this case as well. This concludes the proof. □

Proof of Theorem 4.2.2. By Theorem 3.3.1 a virtual level of h_α^{as} is an eigenvalue and by Theorem 3.1.3 such an eigenvalue has always finite multiplicity. Let E_0 be the corresponding eigenspace. By Theorem 3.2.6 there exists a constant $\mu_{E_0} > 0$, such that for every function $g \perp E_0$ in $\dot{H}^1(\mathbb{R}^4)\backslash\{0\}$ we have

$$\langle h_\alpha^{\mathrm{as}} g, g\rangle \geq \mu_{E_0} |||\nabla g|||^2. \tag{4.2.140}$$

The finiteness of $\sigma_{\mathrm{disc}}(H^{\mathrm{as}})$ now follows by the same arguments as in the proof of [VZ83, Theorem 2.1]. □

Remark. The main difference between the two systems in Theorem 4.2.1 and Theorem 4.2.2 is that virtual levels in the two-particle subsystems in Theorem 4.2.1 are not eigenvalues but resonances. By the assumptions of Theorem 4.2.2, however, they are always eigenvalues, which therefore requires a completely different method of proof. By the remark after Theorem 3.1.12 virtual levels of two-body Schrödinger operators in dimension $d \geq 5$ are eigenvalues. By combining this with the technique of [VZ83], we can generalize the statement of Theorem 4.2.1 to dimension $d \geq 4$. In the next section we extend this technique to arbitrary N-body systems with $N \geq 4$ quantum particles in dimension $d \geq 3$.

4.3 On the Efimov effect for more than three quantum particles

As already mentioned in the introduction, it is an interesting question from both a mathematical and physical point of view whether there is an Efimov-type effect in case of more than three particles. One result of the general case of N particles is the assertion [AG73] of the two physicists R. D. Amado and F. C. Greenwood from 1973, which was first mathematically proven by D. K. Gridnev in 2013, see [Gri13].

4.3.1 The assertion of Amado and Greenwood

Since our results are related to this, in the following we will give a brief summary of the main result of the work [Gri13].

Consider the Schrödinger operator of $N \geq 4$ particles in dimension $d = 3$, i.e.

$$H = H_0 + V, \tag{4.3.1}$$

where H_0 is the operator of kinetic energy with the center of mass removed and $V = \sum_{1\le i<j\le N} v_{ij}$ with the pair-interactions v_{ij} satisfying

$$v_{ij} \in L^1(\mathbb{R}^3) \cap L^3(\mathbb{R}^3) \qquad \text{for all} \;\; i \neq j. \tag{4.3.2}$$

Let $V_{\{j\}}$ be the sum of pair-interactions where particle $j \in \{1, \dots, N\}$ is removed and let $V_{\{j,s\}}$ be the sum of pair-interactions where particles j and s are removed. Furthermore, let $V^+_{j,s}$ and $V^-_{j,s}$ be the sums of the corresponding positive and negative parts of the potentials. Assume that

$$\sigma_{\text{ess}}(H) = [0, \infty) \tag{4.3.3}$$

and that there exists $\omega > 0$ with

$$H_0 + V^+_{j,s} - (1+\omega)V^-_{j,s} \geq 0 \qquad \text{for all } 1 \le j < s \le N. \tag{4.3.4}$$

Theorem 4.3.1 (cf. [Gri13, Theorem 3.])**.** *Assume that the operator H satisfies* (4.3.2)*,* (4.3.3) *and* (4.3.4)*. Then the discrete spectrum $\sigma_{\text{disc}}(H)$ of H is finite.*

The method of the proof of Theorem 4.3.1 is based on the techniques developed in [Yaf74] and [Sob93], presented in section 4.1. Here again the main idea is to reduce the problem to the analysis of an integral operator. However, in this case the operator is created by applying the Birman-Schwinger principle N times. Due to the typical structure of the Birman-Schwinger operator it requires an investigation of the resolvent of the operators corresponding to the respective subsystems. The main focus of the proof of Theorem 4.3.1 is the investigation of the case when a $(N-1)$-particle subsystem is at critical coupling, e.g. [KS80b]. An important ingredient is the following theorem, which was shown in [Gri12b] using similar methods as described above.

Theorem 4.3.2 (cf. [Gri12b, Theorem 2.])**.** *Let $N \geq 3$ and assume that the operator*

$$H(\lambda) = H_0 + \lambda \sum_{1\le i<j\le N} v_{ij} \tag{4.3.5}$$

satisfies $\sigma_{\text{ess}}(H(\lambda)) = [0, \infty)$ and $v_{ij} \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. Assume that $H(\lambda)$ is at *critical coupling* λ_{cr}*, i.e.* $H(\lambda_{cr}) \geq 0$ *and* $H(\lambda_{cr}) + \varepsilon \sum_{1\leq i<j\leq N} v_{ij} \ngeq 0$ *for any* $\varepsilon > 0$*. Suppose that* $H(\lambda_{cr})$ *has no subsystems, which have a bound state with* $E \leq 0$*, and no particle pairs at critical coupling. Then there exists a normalized* ψ_0 *in the domain* $D(H_0)$ *of* H_0 *such that* $H(\lambda_{cr})\psi_0 = 0$*.*

In the previous chapter in Theorem 3.2.2 and Theorem 3.3.1 we have established decay rates of the solutions corresponding to the virtual levels. In particular, it implies that for $N \geq 3$ particles these are eigenvalues in all dimensions $d \geq 3$. Theorem 3.2.2 can therefore be seen as a generalization of Theorem 4.3.2 with respect to the restrictions of both the potentials and the dimension. Since the method of proof of Theorem 4.3.1 depends strongly on the behaviour of the resolvents corresponding to subsystems of the three-dimensional particles, it cannot be directly applied to systems of other dimensions. In the next section we take a different approach, based on variational methods, to present a different proof and extend the statement of Theorem 4.3.1 to other systems.

4.3.2 Absence of the Efimov effect in many-body systems

In this section we prove the absence of the Efimov effect for $N \geq 4$ quantum particles in dimension $d \geq 3$ by generalizing the method developed in [VZ83] to any N-particle system and combining it with the results of Theorem 3.2.2. Furthermore, with the help of Theorem 3.2.8 we generalize this result to systems with fixed permutation symmetries. We follow the presentation of [BBV20].

In the following we stick to the notation introduced in section 2.2 and consider the operator H defined in (2.2.20). We assume that the pair interactions V_{ij} are of the form $V_{ij} = V_{ij}^{(1)} + V_{ij}^{(2)}$, such that for some constants $A, C, \nu > 0$ we have

$$|V_{ij}^{(1)}(x_{ij})| \leq C|x_{ij}|^{-2-\nu}, \text{ if } |x_{ij}| \geq A \qquad \text{and} \qquad V_{ij}^{(1)} \in L^p_{\mathrm{loc}}(\mathbb{R}^d), \tag{4.3.6}$$

where

$$\begin{cases} p = 2, & \text{if } d = 3, \\ p > 2, & \text{if } d = 4, \\ p = \frac{d}{2}, & \text{if } d \geq 5. \end{cases} \tag{4.3.7}$$

Furthermore, we assume that

$$V_{ij}^{(2)}(x_{ij}) \geq 0 \quad \text{is bounded and} \quad V_{ij}^{(2)}(x_{ij}) \to 0 \quad \text{as} \quad |x_{ij}| \to \infty. \tag{4.3.8}$$

Theorem 4.3.3. *Consider the operator H with $d \geq 3$ and $N \geq 4$, where the potentials V_{ij} satisfy* (4.3.6) *and* (4.3.8)*. Assume that for any subsystem C with $|C| = N-1$ the operator $H[C]$, defined in* (2.2.23)*, satisfies*

$$H[C] \geq 0 \qquad \textit{and} \qquad \sigma_{\text{ess}}\left(-(1-\varepsilon)\Delta_0[C] + V[C]\right) = [0, \infty) \tag{4.3.9}$$

for any $\varepsilon \in (0,1)$. Then the discrete spectrum of H is finite.

Remark. We emphasize that in Theorem 4.3.3 the operator $H[C]$ corresponding to subsystems C with $|C| = N-1$ may have a virtual level. On the other hand, operators $H[C']$ corresponding to subsystems C' with $|C'| < N-1$ do not have have virtual levels.

Proof. Consider the functional $L_1 : H^1(R_0) \to \mathbb{R}$, defined by

$$L_1[\varphi] := \langle H\varphi, \varphi \rangle - \varepsilon \| |x|_1^{-1} \varphi \|^2. \tag{4.3.10}$$

Due to Lemma 3.1.3, in order to prove the theorem it suffices to show that there exist constants $\varepsilon > 0$ and $b > 0$, such that $L_1[\varphi] \geq 0$ holds for all functions $\varphi \in H^1(R_0)$ with $\operatorname{supp} \varphi \subset \{x \in R_0,\ |x|_1 \geq b\}$. Applying Lemma 3.2.4 yields

$$L_1[\varphi] \geq \sum_{Z_2} L_2[\varphi u_{Z_2}] + L_3[\varphi \mathcal{V}], \tag{4.3.11}$$

where $\mathcal{V} = \sqrt{1 - \sum_{Z_2} u_{Z_2}^2}$ and the functionals $L_2, L_3 : H^1(R_0) \to \mathbb{R}$ are defined by

$$L_2[\psi] := \langle H\psi, \psi \rangle - \varepsilon \| |x|_1^{-1} \psi \|^2 - \varepsilon_1 \| |q(Z_2)|_1^{-1} \psi \|^2_{\Omega(Z_2)}, \tag{4.3.12}$$

$$L_3[\psi] := \langle H\psi, \psi \rangle - (\varepsilon + \varepsilon_1) \| |x|_1^{-1} \psi \|^2, \tag{4.3.13}$$

with

$$\Omega(Z_2) \subset \{x \in R_0 : |x|_1 \geq b,\ \kappa' |\xi(Z_2)|_1 \leq |q(Z_2)|_1 \leq \kappa |\xi(Z_2)|_1\}. \tag{4.3.14}$$

The constants $\varepsilon_1 > 0$ and $\kappa > 0$ can be chosen arbitrarily small and $\kappa' > 0$ depends on ε_1 and κ. At first we prove that $L_2[\varphi u_{Z_2}] \geq 0$. We need to distinguish between two different types of partitions $Z_2 = (C_1, C_2)$:

(i) $|C_1| < N - 1$ and $|C_2| < N - 1$,

(ii) $|C_1| = N - 1$ or $|C_2| = N - 1$.

As it was mentioned in the remark after Theorem 4.3.3 in case **(i)** the operators $H[C_1]$ and $H[C_2]$ do not have virtual levels, i.e. there exists a constant $\mu_0 > 0$, such that

$$\langle H(Z_2)\varphi u_{Z_2}, \varphi u_{Z_2}\rangle \geq \mu_0 \|\nabla_0(\varphi u_{Z_2})\|^2 \tag{4.3.15}$$

holds for any $\varphi \in H^1(R_0)$. In this case analogously to the proof of Theorem 3.2.2 we conclude that $L_2[\varphi u_{Z_2}] \geq 0$.

We turn to case **(ii)**, where the Hamiltonians of the subsystems may have virtual levels. Suppose that $|C_1| = N - 1$ and that $H[C_1]$ has a virtual level. Then, according to Theorem 3.2.2, zero is a simple eigenvalue of $H[C_1]$. Let φ_0 be the corresponding eigenfunction with $\|\varphi_0\| = 1$. Let

$$\varphi u_{Z_2}(q(Z_2), \xi(Z_2)) = \varphi_0\big(q(Z_2)\big) f\big(\xi(Z_2)\big) + g\big(q(Z_2), \xi(Z_2)\big), \tag{4.3.16}$$

where

$$f(\xi(Z_2)) = \|\nabla_{q(Z_2)}\varphi_0\|^{-2} \langle \nabla q_{Z_2}(\varphi u_{Z_2}), \nabla_{q(Z_2)}\varphi_0\rangle_{q(Z_2)} \tag{4.3.17}$$

and

$$\langle \nabla_{q(Z_2)} g(\cdot, \xi(Z_2)), \nabla_{q(Z_2)}\varphi_0\rangle = 0 \tag{4.3.18}$$

for almost every $\xi(Z_2)$. Note that

$$\begin{aligned} L_2[\varphi u_{Z_2}] = \langle H[C_1]\ g, g\rangle + \langle H[C_1]\ \varphi_0 f, \varphi_0 f\rangle + 2\,\mathrm{Re}\langle H[C_1]\ g, \varphi_0 f\rangle \\ + \|\nabla_{\xi(Z_2)}(\varphi u_{Z_2})\|^2 + \langle I(Z_2)\varphi u_{Z_2}, \varphi u_{Z_2}\rangle \\ - \varepsilon \||x|_1^{-1}\varphi u_{Z_2}\|^2 - \varepsilon_1 \||q(Z_2)|_1^{-1}\varphi u_{Z_2}\|^2_{\Omega(Z_2)}. \end{aligned} \tag{4.3.19}$$

Since $H[C_1]\varphi_0 = 0$, the second term and the third term on the r.h.s. of (4.3.19) are zero. Due to the orthogonality condition (4.3.18), Theorem 3.2.2 yields

$$\langle H[C_1]g, g\rangle \geq \delta_0 \|\nabla_{q(Z_2)} g\|^2 \tag{4.3.20}$$

for some $\delta_0 > 0$. Hence, we arrive at

$$\begin{aligned} L_2[\varphi u_{Z_2}] \geq & \delta_0 \|\nabla_{q(Z_2)} g\|^2 + \|\nabla_{\xi(Z_2)}(\varphi u_{Z_2})\|^2 + \langle I(Z_2)\varphi u_{Z_2}, \varphi u_{Z_2}\rangle \\ & - \varepsilon \||x|_1^{-1} \varphi u_{Z_2}\|^2 - \varepsilon_1 \||q(Z_2)|_1^{-1} \varphi u_{Z_2}\|^2_{\Omega(Z_2)}. \end{aligned} \tag{4.3.21}$$

Now since $V_{ij} \geq V_{ij}^{(1)}$, we have

$$\begin{aligned} \langle I(Z_2)\varphi u_{Z_2}, \varphi u_{Z_2}\rangle \geq & \sum_{i\in C_1, j\in C_2} \langle V_{ij}^{(1)} \varphi u_{Z_2}, \varphi u_{Z_2}\rangle \geq - \sum_{i\in C_1, j\in C_2} \langle |V_{ij}^{(1)}| \varphi u_{Z_2}, \varphi u_{Z_2}\rangle \\ \geq & -C \||\xi(Z_2)|_1^{-1-\frac{\nu}{2}} \varphi u_{Z_2}\|^2 \geq -\varepsilon_2 \|\nabla_{\xi(Z_2)} \varphi u_{Z_2}\|^2, \end{aligned} \tag{4.3.22}$$

where $\varepsilon_2 > 0$ can be chosen arbitrarily small by choosing $b > 0$ sufficiently large. Here we used the fact that on the support of φu_{Z_2} we have

$$|V_{ij}^{(1)}(x_{ij})| \leq C|\xi(Z_2)|_1^{-2-\nu} \leq \frac{\varepsilon_2}{4} |\xi(Z_2)|_1^{-2} \tag{4.3.23}$$

for i, j belonging to different clusters. This implies

$$\begin{aligned} L_2[\varphi u_{Z_2}] \geq \delta_0 \|\nabla_{q(Z_2)} g\|^2 + & (1-\varepsilon_2) \|\nabla_{\xi(Z_2)}(\varphi u_{Z_2})\|^2 \\ & - \varepsilon \||x|_1^{-1} \varphi u_{Z_2}\|^2 - \varepsilon_1 \||q(Z_2)|_1^{-1} \varphi u_{Z_2}\|^2_{\Omega(Z_2)}. \end{aligned} \tag{4.3.24}$$

Since on the support of φu_{Z_2} we have $|x|_1^{-1} \leq |\xi(Z_2)|_1^{-1}$, applying Hardy's inequality yields

$$(1-\varepsilon_2)\|\nabla_{\xi(Z_2)}(\varphi u_{Z_2})\|^2 - \varepsilon \||x|_1^{-1} \varphi u_{Z_2}\|^2 \geq (1-\varepsilon_3)\|\nabla_{\xi(Z_2)}(\varphi u_{Z_2})\|^2, \tag{4.3.25}$$

where $\varepsilon_3 = \varepsilon_2 + 4\varepsilon$. This implies

$$\begin{aligned} L_2[\varphi u_{Z_2}] \geq \delta_0 \|\nabla_{q(Z_2)} g\|^2 + (1-\varepsilon_3) \|\nabla_{\xi(Z_2)}(\varphi u_{Z_2})\|^2 & \\ - \varepsilon_1 \||q(Z_2)|_1^{-1} \varphi u_{Z_2}\|^2_{\Omega(Z_2)}. & \end{aligned} \tag{4.3.26}$$

Let us estimate the last term on the r.h.s. of (4.3.26). Note that

$$\||q(Z_2)|_1^{-1}\varphi u_{Z_2}\|^2_{\Omega(Z_2)} \leq 2\||q(Z_2)|_1^{-1}\varphi_0 f\|^2_{\Omega(Z_2)} + 2\||q(Z_2)|_1^{-1} g\|^2_{\Omega(Z_2)}. \tag{4.3.27}$$

By combining the terms $\delta_0\|\nabla_{q(Z_2)} g\|^2$ and $2\varepsilon_1\||q(Z_2)|_1^{-1} g\|^2_{\Omega(Z_2)}$ and applying Hardy's inequality we get for small $\varepsilon_1 > 0$

$$L_2[\varphi u_{Z_2}] \geq (1-\varepsilon_3)\|\nabla_{\xi(Z_2)}(\varphi u_{Z_2})\|^2 - 2\varepsilon_1\||q(Z_2)|_1^{-1}\varphi_0 f\|^2_{\Omega(Z_2)}. \tag{4.3.28}$$

Now we estimate the last term on the r.h.s. of (4.3.28). Note that for $\kappa > 0$ sufficiently small and $x \in \Omega(Z_2)$ it holds $|\xi(Z_2)|_1 \geq \frac{b}{2}$ and

$$\begin{aligned}
\||q(Z_2)|_1^{-1}\varphi_0 f\|^2_{\Omega(Z_2)} &\leq \int\limits_{\{|\xi(Z_2)|_1 \geq \frac{b}{2}\}} |f|^2 \mathrm{d}\xi(Z_2) \int\limits_{\tilde{\Omega}(Z_2,\xi(Z_2))} |\varphi_0|^2 |q(Z_2)|_1^{-2}\, \mathrm{d}q(Z_2) \\
&\leq (\kappa')^{-2} \int\limits_{\{|\xi(Z_2)|_1 \geq \frac{b}{2}\}} \Phi |f|^2 |\xi(Z_2)|_1^{-2}\, \mathrm{d}\xi(Z_2),
\end{aligned} \tag{4.3.29}$$

where $\tilde{\Omega}(Z_2, \xi(Z_2)) = \{q(Z_2) :\ \kappa'|\xi(Z_2)|_1 \leq |q(Z_2)|_1 \leq \kappa|\xi(Z_2)|_1\}$ and

$$\Phi\left(\xi(Z_2)\right) = \int_{\tilde{\Omega}(Z_2,\xi(Z_2))} |\varphi_0(q(Z_2))|^2\, \mathrm{d}q(Z_2). \tag{4.3.30}$$

Since φ_0 is square-integrable in $q(Z_2)$, for fixed $\kappa' > 0$ and any $\delta > 0$ one can find $b > 0$, such that $\Phi\left(\xi(Z_2)\right) < \delta$ holds uniformly in $|\xi(Z_2)|_1 \geq \frac{b}{2}$. Hence, for any fixed $\kappa' > 0$ and $\varepsilon_4 > 0$ we can choose $b > 0$ sufficiently large, such that

$$\||q(Z_2)|_1^{-1}\varphi_0 f\|^2_{\Omega(Z_2)} \leq \varepsilon_4 \int |\xi(Z_2)|_1^{-2} |f(\xi(Z_2))|^2\, \mathrm{d}\xi(Z_2). \tag{4.3.31}$$

This, together with (4.3.28) yields

$$L_2[\varphi u_{Z_2}] \geq (1-\varepsilon_3)\|\nabla_{\xi(Z_2)}(\varphi u_{Z_2})\|^2 - 2\varepsilon_1\varepsilon_4\||\xi(Z_2)|_1^{-1} f\|^2. \tag{4.3.32}$$

In the following we will estimate the first term on the r.h.s. of (4.3.32). By Hardy's inequality we have

$$\|\nabla_{\xi(Z_2)}(\varphi u_{Z_2})\|^2 \geq \frac{1}{4}\|\varphi u_{Z_2}|\xi(Z_2)|_1^{-1}\|^2 = \frac{1}{4}\|\varphi_0 f|\xi(Z_2)|_1^{-1} + g|\xi(Z_2)|_1^{-1}\|^2. \quad (4.3.33)$$

Hence, $\|\nabla_{\xi(Z_2)}(\varphi u_{Z_2})\|^2$ can be estimated from below by

$$\frac{1}{4}\left(\|\varphi_0 f|\xi(Z_2)|_1^{-1}\|^2 + \|g|\xi(Z_2)|_1^{-1}\|^2 - 2\left|\langle\varphi_0 f|\xi(Z_2)|_1^{-1}, g|\xi(Z_2)|_1^{-1}\rangle\right|\right). \quad (4.3.34)$$

Note that functions f and g are supported in the region $|\xi(Z_2)|_1 \geq (1+\kappa^2)^{-\frac{1}{2}}|x|_1$, where $|x|_1 \geq b > 0$. Hence, $f|\xi(Z_2)|_1^{-1} \in L^2(R_c(Z_2))$ and $g|\xi(Z_2)|_1^{-1} \in L^2(R_0)$. By the assumptions on the potentials V_{ij} we have $\varphi_0 \in H^2(R_0(Z_2))$. Therefore, $\langle\nabla_{q(Z_2)}\varphi_0, \nabla_{q(Z_2)}g|\xi(Z_2)|^{-1}\rangle = 0$ and [VZ83, Lemma 5.3] imply

$$\left|\langle\varphi_0 f, g|\xi(Z_2)|_1^{-2}\rangle\right| \leq \frac{(1-\omega)}{2}\left(\|\varphi_0 f|\xi(Z_2)|_1^{-1}\|^2 + \|g|\xi(Z_2)|_1^{-1}\|^2\right), \quad (4.3.35)$$

where $\omega > 0$ depends on $\|\varphi_0\|, \|\nabla_0\varphi_0\|$ and $\|\Delta_0\varphi_0\|$ only. By combining (4.3.35) and (4.3.34) we get

$$\begin{aligned}\|\nabla_{\xi(Z_2)}(\varphi u_{Z_2})\|^2 &\geq \frac{\omega}{2}\left(\|\varphi_0 f|\xi(Z_2)|_1^{-1}\|^2 + \|g|\xi(Z_2)|_1^{-1}\|^2\right)\\ &\geq \frac{\omega}{2}\|f|\xi(Z_2)|^{-1}\|^2.\end{aligned} \quad (4.3.36)$$

This, together with (4.3.32) implies $L_2[\varphi u_{Z_2}] \geq 0$.
It remains to prove that $L_3[\varphi\mathcal{V}] \geq 0$ holds for every function $\varphi \in H^1(R_0)$ satisfying $\operatorname{supp}\varphi \subset \{x \in R_0,\ |x|_1 \geq b\}$. For any partition $Z_p = (C_1, \dots, C_p)$ with $p \geq 3$ the corresponding operators $H[C_i]$ do not have virtual levels. Therefore, we can estimate the functional $L_3[\varphi\mathcal{V}]$ in cones corresponding to partitions Z_p into $3 \leq p \leq N-1$ clusters, similarly to the proof of Theorem 3.2.2. In the region, which remains after the separation of cones corresponding to all Z_p with $p \leq N-1$ it holds $|V_{ij}^{(1)}(x_{ij})| \leq c|x|_1^{-2-\nu}$ for all $i \neq j$. Applying Hardy's inequality completes the proof. □

The difficult part in the proof of Theorem 4.3.3 is the case when the $(N-1)$-body subsystems have virtual levels at zero. Here the key ingredient is the partition (4.3.16), where by Theorem 3.2.2 φ_0 is the eigenfunction corresponding to the eigenvalue zero. In case of $N = 3$ particles in dimension $d = 3$ or $d = 4$ with short-range potentials virtual levels of two-body subsystems are resonances. Here the term (4.3.29) can no longer be controlled. According to Theorem 3.2.8 virtual levels of the operator H restricted to a subspace with a fixed permutation symmetry are eigenvalues as well. However, in this case these eigenvalues no longer have to be simple, but still have finite multiplicity.

Therefore, we can generalize Theorem 4.3.3 to multi-particle systems with permutation symmetries. Let Z_1 be a system of $N \geq 4$ particles containing $K \leq N$ identical particles. Let the operators H^π, $H^\pi(Z_p)$, the group $S_K(Z_p)$ and $\pi_K(Z_p) \prec \pi$ be defined as in subsection 3.2.2. By Theorem 3.2.8 and the proof of Theorem 4.3.3 we conclude

Theorem 4.3.4. *Consider the operator H^π with $d \geq 3$ and $N \geq 4$, where the potentials V_{ij} satisfy* (4.3.6) *and* (4.3.8)*. Assume there exists $\varepsilon > 0$, such that for all partitions Z_2 into two clusters C_1 and C_2 with $|C_1| = N-1$ or $|C_2| = N-1$ we have*

$$P^{\pi(Z_2)}H(Z_2) \geq 0 \quad \text{and} \quad \sigma_{\text{ess}}\left(P^{\pi(Z_2)}\left(H(Z_2) + \varepsilon\Delta_0(Z_2)\right)\right) = [0, \infty) \tag{4.3.37}$$

for all $\pi(Z_2) \prec \pi$. Moreover, we assume that for all partitions Z_2 into two clusters C_1 and C_2 with $|C_1| \neq N-1$ and $|C_2| \neq N-1$ we have

$$\sigma\left(P^{\pi(Z_2)}\left(H(Z_2) + \varepsilon\Delta_0(Z_2)\right)\right) = [0, \infty) \tag{4.3.38}$$

for all $\pi(Z_2) \prec \pi$. Then the discrete spectrum of H^π is finite.

Further Developments and Open Problems

Lower dimensions

In the thesis we have considered only dimension $d \geq 3$ and have excluded the lower dimensions one and two. An important tool for many arguments in the proofs was the Hardy inequality

$$\int_{\mathbb{R}^d} |\nabla \psi(x)|^2 \, \mathrm{d}x \geq \left(\frac{d-2}{2}\right)^2 \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2} \, \mathrm{d}x, \tag{4.3.39}$$

which is known not to hold in dimension $d = 1$ and $d = 2$. This also has as a consequence that the space $\dot{H}^1(\mathbb{R}^d)$ cannot be defined in the same way. Furthermore, in dimension $d = 1$ and $d = 2$ the fundamental solutions of the Laplace operator are of the type $c_1|x|$ and $c_2 \ln |x|$, respectively. Hence, most results cannot simply be applied to lower dimensions, because the behaviour of virtual levels is significantly different in these cases. This is already apparent from the different properties of the Laplace operator in dimension one and two compared to dimension three or higher. With regard to the existence and the behaviour at infinity of the solutions corresponding to the virtual levels it requires a somewhat different approach. This also applies to the Efimov effect of one- and two-dimensional multi-particle systems. In this regard we refer to the joint project with S. Barth and S. Vugalter, which is currently in preparation and will be published soon.

Virtual levels in the subsystems

In Theorem 4.3.3 we assumed that for any subsystem C with $|C| = N - 1$ the operators $H[C]$ defined in (2.2.23) may have virtual levels. However, the operators $H[C']$ corresponding to subsystems C' with $|C'| < N - 1$ do not have have virtual levels. This condition was essential in order to prove that zero-energy solutions of Hamiltonians of $(N-1)$-body subsystems decay sufficiently fast, which implies that the corresponding virtual levels are eigenvalues at the threshold of the essential spectrum. This is due to the assumption that we can subtract a small part of the total kinetic energy of the quadratic form and it remains positive. In the spirit of [KS80b] this corresponds to the critical coupling constant λ being multiplied by the sum of the total pair interactions. However, one can also consider Hamiltonians where each potential V_{ij} is multiplied by its own coupling constant λ_{ij}. For example, in case of three particles in dimension three it was discussed in [Gri15] that one can tune the three-body system to have a zero-energy resonance. Here the question arises what overall properties such N-body systems have and how the corresponding solutions behave at infinity. Furthermore, with regard to the discrete spectrum of the operator, it is not clear what implications such subsystems can have. In this context we also refer to the conjecture formulated in [Gri13], where it is expected that for such systems there may be an Efimov-type effect for four particles in dimension three.

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