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On the Eigenvalues of the Non-Self-Adjoint Robin Laplacian on Bounded Domains and Compact Quantum Graphs

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Abstract

This thesis addresses several questions about properties and asymptotic behaviours of eigenvalues of the non-self-adjoint Robin Laplacian $-\Delta_\Omega^\alpha$, that is, of the eigenvalue problem

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \alpha u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where Ω is either a bounded (smooth or Lipschitz) domain in $\mathbb{R}^d$, $d \geq 1$, or a compact quantum graph and α is a parameter. In recent years a large body of literature has developed around these questions in the self-adjoint case, that is, for a real Robin parameter $\alpha \in \mathbb{R}$, and it is a natural question to ask whether those results can be generalised for $\alpha \in \mathbb{C}$; and this is the question we want to pursue here. According to the vast amount of recent literature, there seems to be a burgeoning interest in the spectral properties of the Robin Laplacian, which is why we want to give a comprehensive overview on this subject. After a brief summary on the results for the self-adjoint Robin Laplacian, presented to gain insight into what to expect for $\alpha \in \mathbb{C}$, we start by proving regularity results on the eigenvalues λ as (meromorphic) functions of α. Besides, we answer the question whether one can find an orthonormal basis (or weaker types of bases) of $L^2(\Omega)$ consisting of Robin eigenfunctions. Our main interests, however, are the localisation of the spectrum as well as the asymptotic behaviour of λ as functions of α as $\alpha \to \infty$ in $\mathbb{C}$. The tools commonly used to study spectral properties of self-adjoint Laplace operators are totally inapplicable as they rely on the variational min-max characterisation of eigenvalues, test-function arguments, and Dirichlet-Neumann bracketing techniques. However, all these techniques become useless if non-self-adjoint operators are considered; and this is the case as soon as $\operatorname{Im} \alpha \neq 0$. To this end, we use two different approaches in order to study properties of the eigenvalues as well as their asymptotic behaviour as functions of α as $\alpha \to \infty$ in $\mathbb{C}$.

Firstly, we establish trace-type inequalities for functions on smooth and non-smooth domains Ω to prove a localisation theorem for the eigenvalues. This allows us to show

that, for fixed α, the entire spectrum (or more precisely the numerical range of $-\Delta_\Omega^\alpha$) is contained in a parabolic region in $\mathbb{C}$. What is more, this localisation theorem gives us control over both the real and imaginary parts of any eigenvalue in terms of the real and imaginary parts of α, which then implies eigenvalue estimates for Lipschitz domains that are new even in the real case. This approach is further applied to compact quantum graphs, that is, metric graphs on which the Laplace operator acts. To this end, we consider such graphs, where some (or all of the) vertices are equipped with, possibly different, Robin parameters $\alpha_j \in \mathbb{C}$ (also called δ couplings) and on the remaining vertices continuity-Kirchhoff (also called Neumann–Kirchhoff) conditions are imposed.

Secondly, we exploit a duality result for the Dirichlet-to-Neumann operator $M(\lambda)$ which is already known in the self-adjoined case: a number $\lambda \in \mathbb{C}$ is an eigenvalue of the Robin Laplacian with parameter $\alpha \in \mathbb{C}$ if and only if α is an eigenvalue of the Dirichlet-to-Neumann operator with parameter λ. As it turns out, it is often more convenient to study the eigenvalues α of $M(\lambda)$ instead of the eigenvalues λ of $-\Delta_\Omega^\alpha$ and this is the path we take. This approach yields several results for domains and quantum graphs, respectively: on the one hand, we consider explicit examples of domains $\Omega \subset \mathbb{R}^d$, such as intervals and higher dimensional hyperrectangles and balls. We use explicit calculations in order to obtain a detailed analysis of the problem; we give asymptotic error terms for the eigenvalue asymptotics and we compare their behaviour to what happens in the real case. On the other hand, the same Dirichlet-to-Neumann duality approach applied to quantum graphs not only allows us to prove a dichotomy result on what happens to λ when $\alpha \to \infty$ in certain regimes of the complex plane, but since $M(\lambda)$ can be calculated more or less explicitly, we give an (almost) complete answer to the question of the asymptotic behaviour as the Robin parameters $\alpha_j \to \infty$ in certain regimes.

Zusammenfassung

Diese Dissertation befasst sich mit den spektralen Eigenschaften des nicht-selbstadjungierten Robin Laplace Operators $-\Delta_\Omega^\alpha$, d.h. mit den Eigenwerten des Randwertproblems

$$\begin{aligned} -\Delta u &= \lambda u \qquad && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \alpha u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Dabei ist Ω entweder ein beschränktes (glattes oder Lipschitz) Gebiet im $\mathbb{R}^d$ oder ein kompakter Quantengraph und die Zahl α ist ein Parameter. Insbesondere von Interesse ist dabei das asymptotische Verhalten der Eigenwerte in Abhängigkeit vom Robin-Parameter α. In den letzten Jahren entwickelte sich eine große Anzahl an Artikeln, die sich mit dem Robin-Laplace im selbstadjungierten Fall, d.h. für reelle Robin-Parameter α, beschäftigen. Wir stellen uns die Frage, ob die Resultate auf den Fall $\alpha \in \mathbb{C}$ verallgemeinert und ob neue Erkenntnisse aus dieser Untersuchung gezogen werden können; und dies wollen wir als Anlass nutzen, um einen umfassenden Überblick über das Thema zu geben und dabei neue Resultate zu präsentieren. Um ein Gefühl für das Problem selbst und die Erwartungen für den komplexen Fall zu bekommen, fassen wir zunächst die bekannten Resultate im selbstadjungierten Fall zusammen. Für komplexes α untersuchen wir die Eigenwerte λ in Abhängigkeit vom Parameter α und zeigen, dass es sich dabei um meromorphe Funktionen handelt. Außerdem beantworten wir die Frage, wann man eine Orthonormalbasis des $L^2(\Omega)$ aus Eigenfunktionen finden kann und betrachten im Zuge dessen auch schwächere Basis-Begriffe. Unser Hauptinteresse besteht jedoch zum einen in der Lokalisierung des Spektrums und zum anderen im asymptotischen Verhalten der Eigenkurven λ, wenn α in $\mathbb{C}$ gegen Unendlich strebt. Üblicherweise wird im Kontext selbstadjungierter Operatoren auf Werkzeuge wie die variationelle min-max Charakterisierung von Eigenwerten, Testfunktionsargumente oder das Dirichlet-Neumann-Bracketing zurückgegriffen. All diese Techniken basieren jedoch darauf, dass der zugrundeliegende Operator selbstadjungiert ist. Da dies nicht mehr der Fall ist, sobald $\operatorname{Im}\alpha \neq 0$ gilt, müssen wir auf alternative Techniken ausweichen und wir

verwenden daher zwei verschiedene Ansätze, um die Eigenschaften von Eigenwerten sowie ihr asymptotisches Verhalten zu untersuchen.

Zunächst stellen wir Spurungleichungen für Funktionen auf glatten und Lipschitz Gebieten auf und beweisen damit ein Lokalisierungstheorem. Dieses besagt, dass für feste Parameter α das gesamte Spektrum (oder genauer, der numerische Wertebereich) innerhalb eines parabolischen Bereichs in $\mathbb{C}$ liegt. Darüberhinaus lässt es uns ebenso den Real- und Imaginärteil von λ mithilfe des Real- und Imaginärteils von α kontrollieren. Dies wiederum liefert Eigenwertabschätzungen auf Lipschitz Gebieten, die selbst im reellen Fall neu sind. Den Ansatz, den numerischen Wertebereich zu untersuchen, wenden wir ebenfalls auf Quantengraphen an. Dabei handelt es sich um metrische Graphen bestehend aus Knoten und Kanten, wobei der Laplace Operator auf letzteren wirkt und die Knoten als Rand des Graphen interpretiert werden können. Zu diesem Zweck betrachten wir Graphen, bei welchen einige (oder alle) Knoten mit (möglicherweise unterschiedlichen) Robin-Parametern $\alpha_j \in \mathbb{C}$ versehen sind. Die restlichen Knoten genügen dabei der Stetigkeits- und Kirchhoff-Bedingung (auch Neumann–Kirchhoff-Bedingung genannt).

Zweitens nutzen wir eine aus dem selbstadjungierten Fall bekannte Dualitätsaussage zwischen dem Robin-Laplace und dem Dirichlet-zu-Neumann Operator $M(\lambda)$. Dieses besagt, dass eine Zahl $\lambda \in \mathbb{C}$ genau dann Eigenwert des Robin-Laplace mit Parameter $\alpha \in \mathbb{C}$ ist, wenn α ein Eigenwert des Dirichlet-zu-Neumann Operators zum Parameter λ ist: Eine Analyse der Eigenwerte α von $M(\lambda)$ ist oft günstiger als die der Robin Eigenwerte λ. Dieser Ansatz liefert mehrere Resultate, sowohl im Bezug auf Gebiete als auch für Quantengraphen. Einerseits betrachten wir konkrete Modelfälle, wie etwa Intervalle und höherdimensionale Quader und Kugeln. Dabei nutzen wir explizit durchgeführte Berechnungen, um möglichst viele Details und Zusammenhänge herauszuarbeiten; wir geben im Zuge dessen auch asymptotische Fehlerterme der Eigenwertasymptotik(en) an und vergleichen diese erneut mit den Resultaten des selbstadjungierten Problems. Andererseits liefert der selbe Dirichlet-zu-Neumann Ansatz für Quantengraphen ein Dichotomie-Resultat mit Informationen darüber, wie sich λ verhalten kann, wenn α auf eine gewisse Art und Weise gegen Unendlich divergiert. Da sich in diesem Fall $M(\lambda)$ mehr oder minder explizit berechnen lässt, geben wir ebenfalls eine (fast) vollständige Antwort auf die Frage nach dem asymptotischen Verhalten der Eigenwerte auf beliebigen kompakten Quantengraphen, wenn α unter gegebenen Voraussetzungen gegen Unendlich strebt.

Contents

Preprints and publications

This thesis is based on the following publications.

- *On the eigenvalues of the Robin Laplacian with a complex parameter*
  Sabine Bögli, James B. Kennedy, and **Robin Lang**
  submitted on October 31, 2019
  arXiv: 1908.06041 [math.SP] (2019)
  Reference: [30]
- *On the eigenvalues of quantum graph Laplacians with large complex δ couplings*
  James B. Kennedy and **Robin Lang**
  accepted by *Portugaliae Mathematica*
  arXiv: 2001.10244 [math.SP] (2020)
  Reference: [76]

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Let me start back in 2011. When you hear your first lectures in mathematics, it is almost certain that you are completely overwhelmed by the strict formalisms, unfamiliar idioms, and rigorous proofs: none of the skills needed seem even close to anything you got taught in school. But, after occasionally appearing thoughts about resignation, you grit your teeth and together with your fellow students you learn to enjoy the rigour of the *language maths* that you are about to appropriate. Here, I want to make my first acknowledgement to this group of smart people, especially to Andreas Bitter, Simon Barth, Jonas Brinker, Thomas Hamm, and Jonas Hetz, some of whose helped me to keep going since day one. (I should mention that, in the moment of writing this paragraph, every single one of them works on their dissertation – I feel lucky and grateful to have had this group of people around me.) A few lectures later you finally start to speak *maths* fluently, that is, you are now used to the *common maths slang* and you wonder how definite, explicit, and clear communication was possible before. The feeling rises that you finally somewhat *understood mathematics*: today, my humble thought on this slight miscalculation is: "No, you don't." My way to the point of writing my master's thesis mostly consisted of studying partial differential equations sprinkled with extraordinary lectures by Timo Weidl on spectral theory, which particularly lead me, that needs saying, to starting my work on this very thesis under his guidance.

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Robin Lang

Basic notation

Numbers, vectors, and sets

$:=$	equal to by definition
$\varnothing$	empty set
$A \subset B$	A is a subset of B
$A \subsetneq B$	$A \neq B$ is a subset of B
$\#A$	the cardinality of a (possibly infinite) set A
$\mathbb{R}$	set of real numbers $(-\infty, +\infty)$
$\mathbb{R}_+$ / $\mathbb{R}_+^0$	set of positive / non-negative real numbers
$\mathbb{R}_-$ / $\mathbb{R}_-^0$	set of negative / non-positive real numbers
$\mathbb{S}^{d-1}$	$(d-1)$-sphere $\{x \in \mathbb{R}^d : \lvert x\rvert = 1\}$ in $\mathbb{R}^d$
χ_A	the characteristic function $\mathbb{R} \to \mathbb{R}$ on a set $A \subset R$
$\mathbb{Z}$	set of integers
$\mathbb{N}$ / $\mathbb{N}_0$ / $\mathbb{N}_0^-$	set of positive / non-negative / non-positive integers
$B_\delta(x)$	open ball with radius δ and centre x
$D_{\delta,\delta'}(x)$	open annulus with inner and outer radii $0 < \delta < \delta'$ and centre x
$\mathbb{C}$ / i	set of complex numbers / imaginary unit in $\mathbb{C}$
$\overline{\mathbb{R}}$ / $\overline{\mathbb{C}}$	$\mathbb{R} \cup \{\pm\infty\}$ / $\mathbb{C} \cup \{\infty\}$
$\operatorname{Re} z$ / $\operatorname{Im} z$	real part / imaginary part of $z \in \mathbb{C}$
$\overline{z}$ / M^*	complex conjugate of $z \in \mathbb{C}$ and $M \subset \mathbb{C}$
Γ	complex Gamma function $\mathbb{C} \setminus \mathbb{N}_0^- \to \mathbb{C}$
$S_\theta^\pm$ / $T_\theta^\pm$	sectors in $\mathbb{C}$, see Definition 3.6.1
$\mathbb{C}^{m\times n}$	set of complex $(m \times n)-$matrices
I_n	identity matrix in $\mathbb{C}^{n\times n}$
x^T	transpose of a vector $x \in \mathbb{C}^n$
$\check{x}$	$(x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$
δ_{ij}	Kronecker delta for $i, j \in \mathbb{Z}$
$\operatorname{diag}\{e_1, \ldots, e_n\}$	$(n \times n)$-matrix M with entries $M_{ij} = \delta_{ii} e_i$
Ω / $\partial\Omega$	open domain in $\mathbb{R}^d$ / its boundary
$\operatorname{cl}(M) = \overline{M}$	closure of a set M
M_X^c	complement $X \setminus M$ of a subset $M \subset X$
$\operatorname{conv}(M)$	convex hull of a discrete set M
$\operatorname{acc}(M)$	points of accumulation of a set $M \subset N$ for a metric space N

Functions, spaces, norms, and convergence

$\operatorname{supp} u$	support $\operatorname{cl}\{x \in \Omega : u(x) \neq 0\}$ of a mapping u
$\operatorname{Res}_z(f)$	the residue of a function f in $z \in \mathbb{C}$
∂^β	partial derivative for a multi-index $\beta \in \mathbb{R}^d$
$C^k(\Omega)$	k times continuously differentiable functions on Ω
$C_c^k(\Omega)$	functions in $C^k(\Omega)$ which have compact support
$\mathcal{S}(\mathbb{R}^d)$	Schwartz space
$\mathcal{S}^*(\mathbb{R}^d)$	Topological dual space of $\mathcal{S}(\mathbb{R}^d)$
$\dim X$	dimension of a linear space X
$\operatorname{codim} Y$	dimension of X/Y for a subspace $Y \subset X$
V^*	continuous dual space of a topological vector space V
$\ell^2(\mathbb{C})$	square summable sequences in $\mathbb{C}$
B	Banach space
H	Hilbert space
$\oplus$ / $\ominus$	orthogonal sum / difference
$M^\perp$	orthogonal complement to a set M
$(f,g)_H$	inner product of f and g in a (pre) Hilbert space H
I_X	identity operator $I : X \to X$
$\Vert f\Vert_X$	X-norm of f
$\Vert f\Vert$	$L^2(\Omega)$-norm of f
$f_k \to f$ in H	convergence of f_k to f w.r.t. $\Vert\cdot\Vert_H$
$f_k \rightharpoonup f$ in H	weak convergence of f_k to f w.r.t. $(\cdot\,,\cdot)_H$
$X \hookrightarrow Y$	the embedding $I : X \to Y$ is bounded
$X \overset{\Subset}{\hookrightarrow} Y$	the embedding $I : X \to Y$ is compact
$\operatorname{tr} f$	trace of a function f

Generic abbreviation

cf.	confer ("compare")
e.g.	exempli gratia ("for example")
i.e.	id est ("that is")
viz.	videlicet ("namely")
w.r.t.	with respect to
a.e.	almost everywhere (w.r.t. a measure μ)
$\square$	end of proof

Operators and their spectra

$\mathcal{L}(H)$	set of bounded (linear) operators on H
$S_\infty(H)$	set of compact (linear) operators on H
$\mathcal{A}$	generic linear operator
$\mathcal{A}(\alpha)$	family of linear operators $\{\mathcal{A}(\alpha) : \alpha \in \mathbb{C}\}$
$a[\,\cdot\,,\,\cdot\,]$	sesquilinear form $D(a) \times D(a) \to \mathbb{C}$
$a[u]$	quadratic form $a[u,u]$
$D(\mathcal{A})$ / $D(a)$	domain of a $\mathcal{A}$ / a
$R(\mathcal{A})$	range of $\mathcal{A}$
$N(\mathcal{A})$	kernel of $\mathcal{A}$
$G(\mathcal{A})$	graph of $\mathcal{A}$
$W(\mathcal{A})$ / $W(a)$	numerical range of a $\mathcal{A}$ / a
$\mathcal{A}^*$ / $\mathcal{A}^{-1}$	adjoint and inverse of $\mathcal{A}$
$\rho(\mathcal{A})$	resolvent set of $\mathcal{A}$
$\sigma(\mathcal{A})$	spectrum of a closed operator $\mathcal{A}$
$\sigma_p(\mathcal{A})$ / $\sigma_{ess}(\mathcal{A})$	point / essential spectrum of a closed operator $\mathcal{A}$
$R_z(\mathcal{A})$	resolvent $(\mathcal{A} - zI)^{-1}$ of $\mathcal{A}$
$M(\lambda)$	Dirichlet-to-Neumann operator
∇	Nabla operator $(\partial_1, \dots, \partial_d)$
Δ	Laplace operator $\sum_{j=1}^d \partial_j^2$
Δ_w	Laplace-Beltrami operator on $\mathbb{S}^{d-1}$
$-\Delta_\Omega^D$ / $-\Delta_\Omega^N$	Dirichlet / Neumann Laplacian on Ω
$-\Delta_\Omega^\alpha$	Robin Laplacian on Ω with parameter $\alpha \in \mathbb{C}$

Quantum graphs

$\mathcal{E}$	set of edges of a graph
$\mathcal{V}$	set of vertices of a graph
$\mathcal{V}_R$	subset of vertices equipped with a Robin boundary condition
$\mathcal{V}_N$	set $\mathcal{V} \setminus \mathcal{V}_R$
$\mathcal{G}(\mathcal{V}, \mathcal{E})$	graph consisting of $\mathcal{V}$ and $\mathcal{E}$
$\deg v$	number of edges incident with a vertex v
$\mathfrak{D}$	smallest degree of all $v \in \mathcal{V}_R$
$\ell_\mathcal{G}$	smallest length of all edges of a (sub)graph $\mathcal{G}$
$\mathfrak{m}(M)$	smallest modulus of all elements of a finite set (or vector) M

Chapter 1

Introduction

1.1 Motivation

The *Laplace operator* (or *Laplacian*) is a differential operator named after the French mathematician Pierre-Simon de Laplace (1749–1827). He used it to describe motions of objects in outer space (so called *celestial mechanics*) by applying it to the gravitational potential

$$V_G(x) = -\frac{GM}{x} \tag{1.1.1}$$

in $\mathbb{R}$, where G is the gravitational constant and $M > 0$ is the mass of a single point mass in the origin. This gravitational potential $V_G(x)$ describes the work that needs to be done to move a unit mass from infinity to x. More generally in $\mathbb{R}^3$, if we assume that V_G does not originate from a point mass but comes from a continuous mass distribution $\rho(y) \geq 0$ at $y \in \mathbb{R}^3$, then V_G reads

$$V_G(x) = -\int_{\mathbb{R}^3} \frac{G}{|x-y|} \rho(y) \, \mathrm{d}V(y). \tag{1.1.2}$$

Consequently, we can recover the mass distribution ρ from V_G by using the Laplace operator Δ, that is,

$$\rho(x) = \frac{1}{4\pi G} \Delta V_G(x). \tag{1.1.3}$$

However, gravitational fields are just the tip of the iceberg when it comes to physical phenomena whose descriptions are based on differential equations or, more precisely, on the Laplace operator, such as electric potentials, diffusion equations to describe thermal conduction, and the propagation of (e.g. electromagnetic) waves, to name just a few. Since often bounded geometries are considered when describing physical phenomena, it is necessary to equip Poisson's equation

$$\Delta u = f \tag{1.1.4}$$

for suitable functions u and f with further conditions. As a first (simple) example we want to consider a finite vibrating string:

(1) if we control the end points, that is, attach them to some fixed component, the obtained vibration can be described by the wave equation equipped with *Dirichlet boundary conditions*[1];

(2) controlling the vertical forces on the end points of the string results in *Neumann boundary conditions*[2];

(3) supposing an elastic attachment, that is, both ends of the string are attached to springs: then the vertical forces on the end points are not controlled by some given function, but they are proportional to the displacements of the end points. The boundary condition used to describe this motion is a mixture of the two conditions above and called *Robin boundary condition*[3].

When we consider $f = \lambda u$ as the right-hand side of Poisson's equation (1.1.4) we obtain the eigenvalue equation $\Delta u = \lambda u$, known as the *Helmholtz equation*[4], which leads us to the mathematical field of spectral theory (we want to refer to Section 2.3 for more details on the spectral theory of Dirichlet and Neumann Laplacians). In spectral theory one wants to study the spectrum of an eigenvalue problem, that is, the numbers λ such that there exists a non-zero function u with $Pu = \lambda u$ for some differential operator P. In this thesis we want to consider $P = -\Delta$ as the differential operator and equip it with the Robin boundary condition on a fixed domain Ω, that is, a sufficiently smooth, bounded, connected open set in $\mathbb{R}^d$, $d \geq 1$. This problem

[1] named after the German mathematician *Peter Gustav Lejeune Dirichlet*, 1805–1859

[2] named after the German mathematician *Carl Gottfried Neumann*, 1832–1925

[3] named after the French mathematician *Victor Gustave Robin*, 1855–1897. However, Robin never used the third boundary condition and it is still unclear who first attached his name to it; for more information on Robin's work and the third boundary condition, we refer to the article *The Third Boundary Condition - Was it Robin's?* [66].

[4] named after the German physicist *Hermann Ludwig Ferdinand von Helmholtz*, 1821–1894.

reads

$$-\Delta u = \lambda u \qquad \text{in } \Omega, \tag{1.1.5a}$$
$$\frac{\partial u}{\partial \nu} + \alpha u = 0 \qquad \text{on } \partial\Omega, \tag{1.1.5b}$$

for a *Robin parameter* α appearing in the boundary condition. Note that here and throughout ν denotes the *outer* unit normal to the boundary $\partial\Omega$, and if $d = 1$, then we understand $\Omega \subset \mathbb{R}$ to be a bounded open interval. From now on, we will be partially following [30, Section 1].

In recent years a large body of literature has developed around the asymptotic behaviour of the eigenvalues of (1.1.5) as the parameter $\alpha \in \mathbb{R}$ tends to $\pm\infty \in \overline{\mathbb{R}}$. To this end, we denote the solutions λ of

$$-\Delta u = \lambda u \qquad \text{in } \Omega, \tag{1.1.6a}$$
$$u = 0 \qquad \text{on } \partial\Omega, \tag{1.1.6b}$$

numbered in ascending order, by

$$\lambda_1 \le \lambda_2 \le \cdots \to \infty, \tag{1.1.7}$$

that is, the eigenvalues of the boundary value problem for the Dirichlet Laplacian. Since the solutions of the Robin problem (1.1.4) depend smoothly on the Robin parameter α, we similarly denote them by

$$\lambda_1(\alpha) \le \lambda_2(\alpha) \le \cdots \to \infty, \tag{1.1.8}$$

and interpret each $\lambda_j(\alpha)$ as a function of α clarifying what we meant by writing "asymptotic behaviour of the eigenvalues": in contrast to Weyl's asymptotics, where, inter alia, the behaviour of the k-th eigenvalue is studied as $k \to \infty$, we want to study both single eigenvalues and the whole spectrum as the boundary parameter tends to infinity. We want to mention that the following paragraph is a brief summary and we refer to Section 2.4 for more details and formal statements. In the real case, i.e.,

$\alpha \in \mathbb{R}$, it is known that, for each $k \in \mathbb{N}$,

$$\lambda_k(\alpha) \to \lambda_k \tag{1.1.9}$$

from below (that is, $\lambda_k(\alpha)$ are monotonically increasing in α) as $\alpha \to +\infty$; the corresponding rate of convergence was studied and proved in [57, 58]. On the other hand, if $\alpha \to -\infty$, the situation is much more complicated. Consider $\Omega \in \mathbb{R}^d$ to be smooth and fix any $k \in \mathbb{N}$. Then, the k-th eigencurve $\lambda_k(\alpha)$ satisfies

$$\lambda_k(\alpha) \sim -\alpha^2 \tag{1.1.10}$$

as $\alpha \to -\infty$. Note that the enumeration of the eigenvalues in the sense of (1.1.8) does not respect the analyticity of the curves – if we follow the analytic branches of the eigenvalues, then the numbering is permuted as $-\alpha$ increases. The existence of such *crossing points* allows the existence of further curves of eigenvalues which converge to eigenvalues of the Dirichlet Laplacian from above [3, 38, 45, 63, 64, 85, 90]. Based

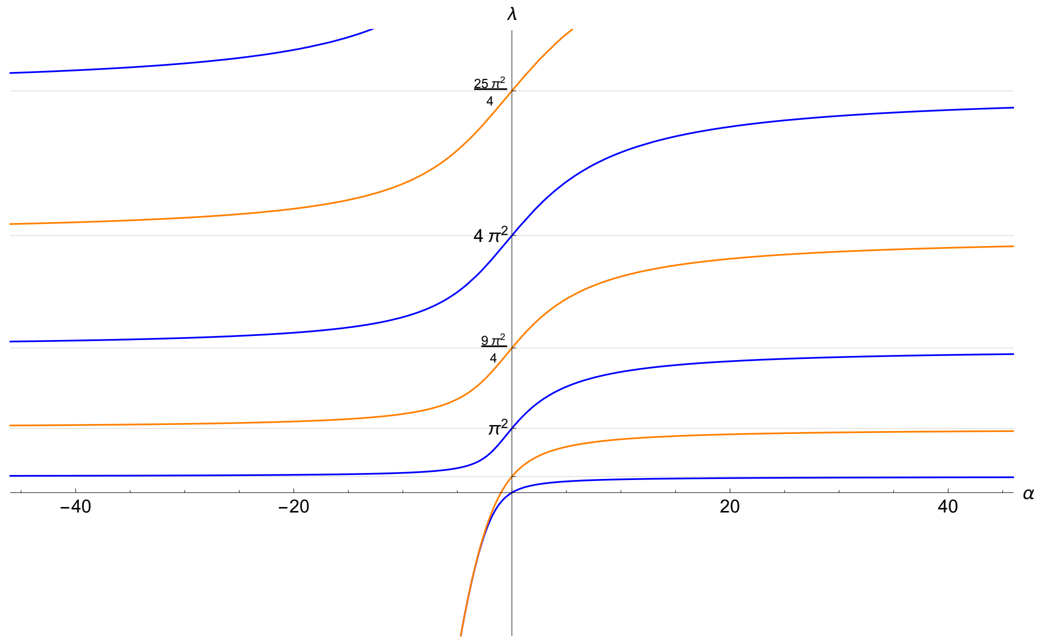

Figure 1.1.1: Plot of the eigenvalues λ of the one dimensional Robin problem as functions of $\alpha \in \mathbb{R}$. Note that in this case there are exactly two eigenvalues which behave like $-\alpha^2$ as $\alpha \to -\infty$. Here, the eigenvalues λ_k are globally analytic functions of α and there are no crossings at all.

on these statements several question arise:

(1) Can one determine more precise asymptotics, that is, higher (or in this case lower) terms in the asymptotic expansion of $\lambda_k(\alpha)$ as $\alpha \to -\infty$?

(2) What happens if we no longer require Ω to be smooth (for example by allowing corners in the boundary $\partial\Omega$)?

(3) If we no longer restrict α to be real but allow complex numbers $\alpha \in \mathbb{C}$ as parameters, can we make any statement about the asymptotics of complex eigencurves as $|\alpha| \to \infty$ in $\mathbb{C}$?

The first question has been extensively studied and answered in [54, 59, 68, 82, 100] who proved statements where the coefficients of further terms heavily depend on the geometry of Ω, more particular on its curvature. The second question, i.e., the case of less regularity, has also been considered in several articles [35, 78, 79, 80, 88]: to this end, a finite number of "model corners" – that is, corners which can be mapped to a model cone of fixed angle of opening by a smooth diffeomorphism – is considered. In this model case they prove that the asymptotics is mainly driven by the "most acute" corner(s) of the domain; or in other words: the sharper the corner(s), the larger the leading coefficient C of the asymptotics $\lambda \sim -C\alpha^2$ as $\alpha \to -\infty$.

However, the third question of considering complex Robin parameters is the one this thesis is devoted to. But why are (complex) Robin parameters interesting in the first place? Due to the large body of literature (partially cited above) it is a natural question to ask for generalisations of this problem; especially out of intrinsic interest as someone who is interested in spectral theory. The road to working on the spectra non-self-adjoint operators while having a background in self-adjoint theory is long and rocky; this matter itself is challenging and thus worthwhile and desirable. Even though there is no somewhat *comprehensive* work on the eigenvalue asymptotics of the non-self-adjoint Robin Laplacian, there seems to be a burgeoning interest in this topic in more specific contexts. When one speaks of *scientific interest*, however, an application reference is usually inevitable. The physical term describing boundary conditions of the third type is often called *impedance boundary condition*, which plays a fundamental role in some of the articles listed below:

Firstly, we want to mention *half-spaces* where the Robin boundary condition is confined to a straight axis and the domain on which the Helmholtz equation is studied is unbounded. If the Robin parameter is allowed to be variable, that is, α is a real function of the location $x \in \partial\Omega$, the problem models outdoor sound propagation over inhomogeneous flat terrain and acts as a model of rough surface scattering [40, 87]. Mathematically, in this setting, sufficient conditions on the function α have been studied which guarantee the total absence of eigenvalues in the spectrum [41]. Note that in this half-plane case, the Robin problem 1.1.5 still admits a discrete set of eigenvalues outside the essential spectrum, that is, studying this problem should give a more complete picture of the eigenvalue behaviour even in the real case. Secondly, another field where impedance boundary conditions are crucial is scattering theory of electromagnetic waves that collide with an object with its surface structure being related to the real and complex parts of the Robin parameter α [6, 69]. Also mathematically, inverse scattering problems have been studied, where the scattered field satisfies mixed Dirichlet-impedance conditions on the boundary of the scatterer [39]; for works on similar fields see [86, 87]. Furthermore, there are more various contexts such as metric quantum graphs [71] (this field will be considered in the last chapter of this thesis), thin layers [32, 84], triangles [93, 110], and especially waveguides [31, 97, 98, 99, 107]. It seems astonishing that despite the great interest in complex Robin (or impedance) boundary conditions, the literature still lacks of a comprehensive overview of the spectral properties of the non-self-adjoint Robin Laplacian on (bounded) Lipschitz domains. To this end, we will treat this issue systematically; here, we want to give a brief summary of the results obtained therein.

1.2 Main results

The following theorem combines statements from Theorems 3.1.2, 3.2.11, 3.2.18, 3.3.1, and 3.3.8; we refer to Chapter 3 for the individual theorems and their respective proofs. Note that this theorem allows us to speak of analytical curves – up to possible crossing points – when considering the Robin eigenvalues as functions of the parameter α. This especially motivates the study of the question regarding the asymptotics of these curves in the complex plane as $\alpha \to \infty$ in $\mathbb{C}$.

Theorem 1.2.1 (see [30, Theorem 1.1])**.** *Suppose $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is a bounded Lipschitz domain. Then we have the following statements.*

(1) Each eigenvalue has finite algebraic multiplicity and depends locally analytically on $\alpha \in \mathbb{C}$*: more precisely, if* $(\lambda_k(\alpha_0))_{k\in\mathbb{N}}$ *is an enumeration of the eigenvalues (each repeated according to its finite algebraic multiplicity) for some* $\alpha_0 \in \mathbb{R}$*, then each* $\lambda_k(\alpha_0)$ *may be extended to a meromorphic function* $\lambda_k(\alpha)$ *such that for any* $\alpha \in \mathbb{C}$*, these eigenvalues form the spectrum of the corresponding Robin Laplacian.*

(2) Away from crossing points of eigenvalues, each eigenvalue $\lambda_k(\alpha)$ *and the corresponding eigenprojection are holomorphic functions of* α*, whereas at the crossing points the weighted eigenvalue mean and the total projection are holomorphic.*

(3) If $\lambda_k(\alpha)$ *is simple with eigenfunction* $\psi = \psi(\alpha)$*, then* $\lambda_k'(\alpha)$ *is given by*

$$\lambda_k'(\alpha) = \frac{\int_{\partial\Omega} \psi^2 \,\mathrm{d}\sigma(x)}{\int_\Omega \psi^2 \,\mathrm{d}x} \tag{1.2.1}$$

(where the right-hand side is to be interpreted as a holomorphic continuation in the event that the denominator is zero, as any singularities are removable).

(4) For any $\alpha \in \mathbb{C}$*, the set of eigenfunctions and generalised eigenfunctions corresponding to the eigenvalues* $\{\lambda_k(\alpha) : k \in \mathbb{N}\}$ *can be chosen to form an Abel basis of* $L^2(\Omega)$*, of order*

$$\frac{d-1}{2} + \delta \tag{1.2.2}$$

for any $\delta > 0$*, and even a Riesz basis if* $d = 1$*.*

(5) However, for any $\alpha \in \mathbb{C} \setminus \mathbb{R}$*, the eigenfunctions can* not *be chosen to form an orthonormal basis of* $L^2(\Omega)$*. For the definitions of both bases, we refer to Definitions 3.3.3, 3.3.4, and 3.3.5.*

In the self-adjoint case it is known that the set of eigenfunctions form an orthonormal basis of $L^2(\Omega)$ for all $\alpha \in \mathbb{R}$ and it is a natural question to ask whether this property remains valid in the complex case. We can answer this question with the negative result of Theorem 3.3.1 (i.e., Theorem 1.2.1 (5)), although, there are weaker basis concepts to give the positive results mentioned in (4). For the sake of completeness and to build a foundation for our research, we give an overview of the results known in the real (self-adjoint) case in Section 2.4. To give our expectations based on these results (and on the explicit calculations for model domains, such as intervals, cuboids and balls), we refer to the brief summary above and give the following conjecture, which, however, will not be answered completely in this work.

Conjecture 1.2.2 (see [30, Conjecture 1.2])**.** *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain, and suppose $\alpha \in \mathbb{C}$, $|\alpha| \to \infty$.*

(1) If $\operatorname{Re}\alpha \to -\infty$, then there exists a sequence of absolutely divergent eigenvalues. Any limit point of non-divergent analytic eigenvalue curves of eigenvalues is an eigenvalue of the Dirichlet Laplacian (that is, a solution of (1.1.6)*).*

(i) If Ω has C^1 boundary, then each divergent eigenvalue behaves asymptotically like $-\alpha^2 + o(\alpha^2)$.

(ii) If Ω has Lipschitz boundary, then for any divergent analytic curve of eigenvalues $\lambda = \lambda_k(\alpha)$, there is a constant $C_{\Omega,k} \in [1, \infty)$ depending only on k from Theorem 1.2.1, such that $\lambda_k(\alpha) = -C_{\Omega,k}\alpha^2 + o(\alpha^2)$.

(2) If $\operatorname{Re}\alpha$ remains bounded from below, then each eigenvalue converges to an eigenvalue of the Dirichlet Laplacian.

We also want to note that most statements of Conjecture 1.2.2 are already known in the self-adjoint case. In spite of that, the asymptotics on Lipschitz domains, i.e. (ii), is still a question to be answered even for $\alpha \in \mathbb{R}$, see [36, Open Problems 4.17 and 4.20]. It is worth noting that if $|\operatorname{Im}\alpha|$ grows faster than $|\operatorname{Re}\alpha|$ as $\alpha \to \infty$ in $\mathbb{C}$, for the divergent eigenvalues it is now possible to have large *positive* real part.

Unsurprisingly, the main problem of studying the complex case is that the techniques commonly used in the articles where only self-adjoint operators are considered are completely inapplicable in the non-self-adjoint case as they rely on the variational min-max characterisation of eigenvalues and associated tools, such as test function arguments, as in [45, 63, 64, 85] or Dirichlet-Neumann bracketing techniques (cf. [54, 88, 100]) used for the decomposition of the Robin Laplacian. Thus, it is necessary to revise and extend our toolbox which, besides acclimatisation, provides us with the pleasant advantage that results for complex parameters maintain valid if restricted to $\alpha \in \mathbb{R}$.

Estimates on the numerical range I

This is the case, for example, for the following theorem which will be proved as a slightly stronger version, namely for the numerical range – estimates on this set build first pillar of this thesis – of the associated sesquilinear Robin form (to which our Laplacian is associated): we refer to Section 3.4 for details, including a description of the parabolic-type region $\Lambda_{\Omega,\alpha}$ as depicted in Figure 1.2.1, and in particular to Theorem 3.4.1 for the stronger version and its proof.

Theorem 1.2.3 (see [30, Theorem 1.3]). *Suppose $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded Lipschitz domain. Then there exist constants $C_1 \geq 2$ and $C_2 > 0$ depending only on Ω, such that for any $\alpha \in \mathbb{C}$, any corresponding eigenvalue $\lambda \in \mathbb{C}$ of* (1.1.5) *is contained in the set*

$$\Lambda_{\Omega,\alpha} := \Big\{ t + \alpha \cdot s \in \mathbb{C} : t \geq 0,\ s \in [0, C_1\sqrt{t} + C_2] \Big\}; \tag{1.2.3}$$

in particular, we have the estimate

$$\operatorname{Re} \lambda \geq -\frac{C_1^2}{4} |\operatorname{Re} \alpha|^2 - C_2 |\operatorname{Re} \alpha|. \tag{1.2.4}$$

If Ω has C^2 boundary, then we may choose $C_1 = 2$.

The statements given above are based on sharp trace-type estimates on the boundary integral of the Robin eigenfunctions. Since the Robin form simply reads

$$a_\alpha[u, v] = \int_\Omega \nabla u \overline{\nabla v} \, \mathrm{d}x + \alpha \int_{\partial\Omega} u \overline{v} \, \mathrm{d}\sigma(x), \tag{1.2.5}$$

this boundary term is the only term in the expression for λ with possibly non-zero imaginary part and this allows us to control the location of the whole spectrum (more precisely the numerical range) for fixed $\alpha \in \mathbb{C}$ inside an explicitly specified parabolic-type region $\Lambda_{\Omega,\alpha}$ of the complex plane.

As mentioned before, Theorem 1.2.3 might be restricted to the real case, more precisely for $\operatorname{Re} \alpha = \alpha < 0$, and from (1.2.4) we obtain a new result even in the self-adjoint case.

Corollary 1.2.4 (see [30, Corollary 1.4]). *Suppose $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded Lipschitz domain. Then there exist constants $c_1 \geq 1$ and $c_2 > 0$ depending only on Ω*

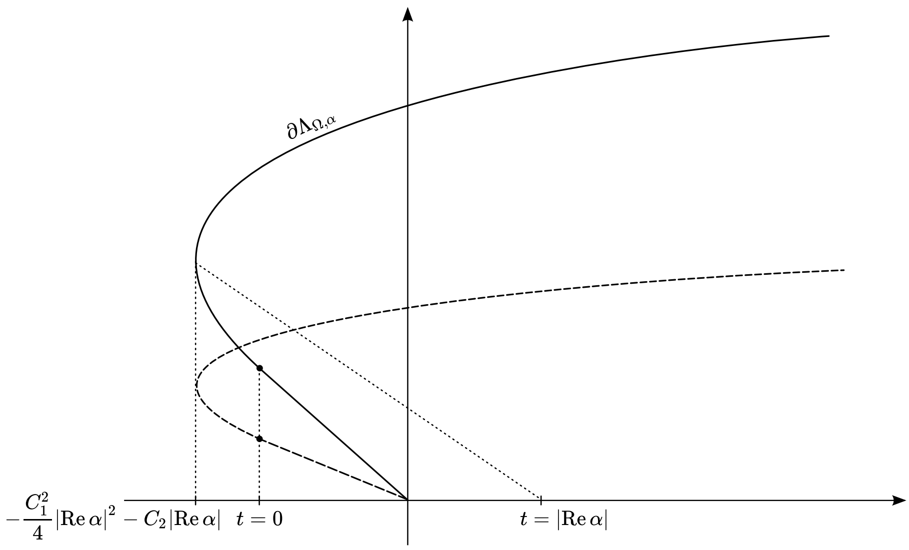

Figure 1.2.1: The set $\Lambda_{\Omega,\alpha}$ for $\mathrm{Re}\,\alpha < 0$ and two different choices of $\mathrm{Im}\,\alpha > 0$. As $\mathrm{Im}\,\alpha \to 0$, the region collapses to the part of the real axis from $-\frac{C_1^2}{4}|\mathrm{Re}\,\alpha|^2 - C_2|\mathrm{Re}\,\alpha|$ to $+\infty$.

such that for any $\alpha < 0$ and any corresponding eigenvalue $\lambda \in \mathbb{R}$ we have

$$\lambda \geq -c_1\alpha^2 + c_2\alpha. \tag{1.2.6}$$

If Ω has C^2 boundary, then we may choose $c_1 = 1$.

At this point there are two things worth mentioning: firstly, Corollary 1.2.4 answers Open Problem 4.17 from [36], namely that for any bounded Lipschitz domain Ω, there exists a constant $c_1 > 0$ depending only on Ω such that

$$\lambda_1(\alpha) \geq -c_1\alpha^2 \tag{1.2.7}$$

asymptotically as $\alpha \to -\infty$. Secondly, this also seems to be the first time that for any general C^2 domain there exists a constant $c_2 > 0$ depending only on Ω such that

$$\lambda(\alpha) \geq -\alpha^2 + c_2\alpha \tag{1.2.8}$$

holds for all $\alpha < 0$. The constant c_2 might be estimated explicitly in terms of the geometry of $\partial\Omega$: since the proof is based on expressions where the (maximal) mean curvature of the domain plays a major role, it is apparent why we require Ω to be of (at least) class C^2. For more details on c_2 we refer to Remark 3.4.13.

The Dirichlet-to-Neumann approach

Another approach – the second pillar of this thesis – is based on the duality between the Robin Laplacian on $L^2(\Omega)$ and the *Dirichlet-to-Neumann operator* on $L^2(\partial\Omega)$. To this end, suppose that (i) $\lambda = \lambda(\alpha) \in \mathbb{C}$ is an eigenvalue of the Robin Laplacian for some given parameter $\alpha \in \mathbb{C}$ and (ii) that this eigenvalue is in the resolvent set of the Dirichlet Laplacian, i.e., λ is not in the spectrum of the eigenvalue problem (1.1.6). The Dirichlet-to-Neumann operator $M(\lambda)$ is the operator which maps g to the (negative of the) outer normal derivative $-\partial_\nu u$ of the solution (if one exists) of the Dirichlet problem

$$-\Delta u = \lambda u \qquad \text{in } \Omega, \tag{1.2.9a}$$

$$u = g \qquad \text{on } \partial\Omega. \tag{1.2.9b}$$

Then, α is an eigenvalue of $M(\lambda)$ for this value of λ. What is more, $M(\lambda)$ is defined in such a way that $\lambda \in \mathbb{C}$ is an eigenvalue of the Robin problem (1.1.5) for a given parameter $\alpha \in \mathbb{C}$ if and only if α is an eigenvalue of the Dirichlet-to-Neumann operator for the spectral parameter λ. Consequently, studying the properties of Robin eigenvalues $\lambda = \lambda(\alpha)$ as functions of α is equivalent to studying the Dirichlet-to-Neumann eigenvalues $\alpha = \alpha(\lambda)$ as functions of λ. Since it turns out that it is often more convenient to study $\alpha(\lambda)$ instead of $\lambda(\alpha)$ (even though some kind of *inversion* is needed to translate the results from one picture into the other), this is the approach we will take. This technique is standard and well known for real α [12, Theorem 3.1], and it has been exploited on a regular basis in various other contexts, see for example [12, 15, 43, 61, 92]. Besides, there are also results on the duality between elliptic differential operators and operators of Dirichlet-to-Neumann type in the non-self-adjoint case [34, Theorem 4.10], but in this thesis we give a direct proof not only for the duality result of the complex eigenvalues but also for the corresponding eigenfunctions of both operators: u is an eigenfunction of the Robin problem (1.1.5) for a given parameter $\alpha \in \mathbb{C}$ if and only if $\operatorname{tr} u$ is an eigenfunction of $M(\lambda)$. One of the results obtained by exploiting this equivalence is the following dichotomy theorem which does not previously (with respect to [30]) seem to have been formally proved (as stated in [36, Open Problem 4.11]). For the proof and more details we refer to two paragraphs of this work: for the one-dimensional case $d = 1$, see Theorem 4.1.1 and for $d \geq 2$, see Section 3.5.

Theorem 1.2.5 (see [30, Theorem 1.5])**.** *Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a bounded Lipschitz domain and $\alpha \in \mathbb{C}$. Then, each analytic eigenvalue curve $\lambda = \lambda(\alpha)$ of the Robin Laplacian $-\Delta_\Omega^\alpha$ either converges to a point in the Dirichlet spectrum or diverges to ∞ in $\mathbb{C}$ as $\alpha \to \infty$ in $\mathbb{C}$.*

In contrast to the preceding theorem where individual eigencurves are considered, we conversely study the entirety of the spectrum as $\alpha \to \infty$ in $\mathbb{C}$. To be more precise, we analyse the possible points of accumulation of the Robin eigenvalues. It turns out that if α diverges away from the negative real semi-axis, the only points of accumulation lie in the Dirichlet spectrum. On the other hand, if α is contained in a neighbourhood of the negative real semi-axis, the situation is much more complicated since the crossing points of the eigencurves might accumulate.

Theorem 1.2.6 (see [30, Theorem 1.6])**.** *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain and $\alpha \in \mathbb{C}$.*

(1) If $\alpha \to \infty$ in $\mathbb{C}$ in such a way that either $\operatorname{Re}\alpha$ remains bounded from below or $\left|\frac{\operatorname{Re}\alpha}{\operatorname{Im}\alpha}\right|$ remains bounded, then the only points of accumulation of the Robin Laplacian eigenvalues as $\alpha \to \infty$ are eigenvalues of the Dirichlet Laplacian.

(2) However, any $\lambda \in \mathbb{C}$ is a point of accumulation of the eigenvalues of the Robin Laplacian if $\alpha \in \mathbb{C}$ is allowed to be arbitrary. More precisely, given any $\lambda \in \mathbb{C}$ there exist $\alpha_k \in \mathbb{C}$, $k \in \mathbb{N}$, $|\alpha_k| \to \infty$, such that λ is an eigenvalue of the Robin Laplacian with parameter α_k, for all $k \in \mathbb{N}$.

We prove a slightly more precise version of Theorem 1.2.6 (1), namely Theorem 3.6.3 plus Remark 3.6.4. For the proof and more details on the contrast between both statements of Theorem 1.2.6 we refer to Section 3.6.

Furthermore, we give a detailed analysis of the asymptotic behaviour of the Robin eigenvalues where Ω is chosen to be a concrete example. To this end, we can (more or less explicitly) calculate and analyse the corresponding Dirichlet-to-Neumann operators on intervals, d-dimensional rectangles (in this work often called cuboids or hyperrectangles), and balls in $d \geq 2$ dimensions. Studying the asymptotics of $M(\lambda)$ on these model domains and exploiting the aforementioned duality – we devote the whole Chapter 4 to this topic – supports Conjecture 1.2.2. Besides, we expect that many of the ideas drawn from this analysis could be transferred to more general settings. As it turns out, this expectation is true for compact quantum graphs which might be interpreted as a generalisation of the bounded interval. The setting of metric graphs is the subject of Chapter 5 which, in particular, provides a proof of a version of Conjecture 1.2.2.

By *quantum graphs* we mean metric graphs (for a definition and more details, see Section 5.1) on which the Laplace operator acts. To this end, we consider compact metric graphs $\mathcal{G}(\mathcal{V}, \mathcal{E})$ consisting of a finite set of edges $\mathcal{E} = \{e_1, \ldots, e_m\}$, $m \in \mathbb{N}$, joined in a certain way at a finite set of vertices $\mathcal{V} = \{v_1, \ldots, v_n\}$, $n \in \mathbb{N}$.

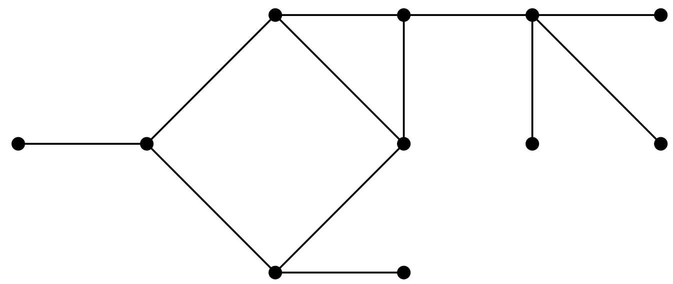

Figure 1.2.2: A compact graph with $|\mathcal{V}| = 11$ vertices and $|\mathcal{E}| = 12$ edges.

We equip $k \leq n$ of the vertices (ordered such that they are denoted by $v_1, \ldots, v_k$) with Robin boundary conditions and the $n - k$ remaining vertices $v_{k+1}, \ldots, v_n$ of $\mathcal{V}$ with continuity-Kirchhoff (also called Neumann–Kirchhoff) conditions. We then define a differential operator on $\mathcal{G} = \mathcal{G}(\mathcal{V}, \mathcal{E})$ by taking the negative of the second derivative (that is, $-\Delta$) on each edge $e_1, \ldots, e_m$. To this end, each $e \in \mathcal{E}$ is identified with a compact interval $[0, \ell_e] \subseteq \mathbb{R}$ of length $\ell_e = |e| > 0$. At the same time we equip the vertices, that is, the endpoints of the edges $e \in \mathcal{E}$, with the vertex conditions mentioned above. More precisely, we assume that the domain of the corresponding operator $-\Delta^{\alpha}_{\mathcal{V}_R}$ consists of $L^2(\mathcal{G})$ functions f such that

(i) continuity at all vertices $v \in \mathcal{V}$,

(ii) the δ condition

$$\sum_{e \sim v_j} \frac{\partial}{\partial \nu} f|_e(v_j) + \alpha_j f(v_j) = 0, \tag{1.2.10}$$

$\alpha_j \in \mathbb{C}$, $j = 1, \dots, k$, at a distinguished set $\mathcal{V}_R := \{v_1, \dots, v_k\} \subset \mathcal{V}$ of Robin vertices (here $f|_e$ is the restriction of the function f on $\mathcal{G}$ to the edge e, $\frac{\partial}{\partial \nu} f|_e(v)$ is the derivative of f at the endpoint of e pointing into v_j, and the summation is over all edges e incident with v_j), and

(iii) the usual Kirchhoff condition (also known as current conservation, see [27, eq. (1.4.4)]), corresponding to $\alpha = 0$, at all vertices in $\mathcal{V} \setminus \mathcal{V}_R$.

For brevity, we will write $\alpha = (\alpha_1, \dots, \alpha_k)^T \in \mathbb{C}^k$ for the vector containing the Robin parameters on the vertices of $\mathcal{V}_R$ ordered such that α_j represents the parameter at v_j. To put (ii) into perspective, the boundary $\partial\Omega$ of a domain $\Omega \in \mathbb{R}^d$ corresponds to the distinguished set $\mathcal{V}_R$ of Robin vertices at which the vertex condition (1.2.10) is imposed; conditions of this type are also called δ coupling or δ interaction. We note that, similarly to Theorem 1.2.1, all eigenvalues of $-\Delta^{\alpha}_{\mathcal{V}_R}$ are at least piecewise analytic functions of $\alpha \in \mathbb{C}^k$, and for any fixed α they form an at most countable set. We obtain the following result similar to Theorem 1.2.5; here, we denote by $\mathfrak{m}(\alpha)$ the smallest of the moduli of its components, viz.

$$\mathfrak{m}(\alpha) = \min_{j=1,\dots,k} |\alpha_j|. \tag{1.2.11}$$

Theorem 1.2.7 (see [76, Corollary 4.3])**.** *Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ be a compact metric graph with $\mathcal{V}_R = \{v_1, \dots, v_k\}$ as its set of Robin vertices and let $\alpha \in \mathbb{C}^k$ be the vector containing the Robin parameters $\alpha_j \in \mathbb{C}$. Then, each analytic eigenvalue curve $\lambda = \lambda(\alpha)$ of the Robin Laplacian $-\Delta^{\alpha}_{\mathcal{V}_R}$ either converges to a point in the Dirichlet spectrum or diverges to ∞ in $\mathbb{C}$ as $\mathfrak{m}(\alpha) \to \infty$.*

For the proof, see Theorem 5.3.2 and Corollary 5.3.3. Before we come to our main result for quantum graphs, we want to emphasise the importance of this topic in spectral theory and applications: since each edge corresponds to a bounded interval and each compact graph can be interpreted as finitely many such intervals "glued together" at (some or all of) their endpoints, detailed analyses of quantum graphs is often more easily accessible compared to the case where higher dimensional objects are studied. Moreover, even though graphs sometimes even allow explicit computations to verify abstract theories or, on the other hand, to gain insight into what to expect for domains or manifolds, they often turn out to be non-trivial. This occurs for example with problems such as the Anderson localisation (the absence of diffusion of waves in a disordered medium), the field of quantum chaos (the description of chaotic classical dynamical systems in terms of quantum theory) or geometric spectral theory [109, 65, 24, 77], to name just a few. Additionally, the vertex condition studied in this thesis appears frequently in the literature on quantum graphs; for a description, see [27, Section 1.4], and for literature featuring these conditions we refer to [25, 28, 51, 55, 70, 71, 105], among many others. We also refer to the book [27, Preface and Chapters 1 and 7] as well as to the articles [25, 55, 81] for even more references and information on these topics.

To state our main result, a version of Conjecture 1.2.2 for compact quantum graphs, we need to introduce another Laplace operator on $\mathcal{G}$. To this end, we denote by $-\Delta^D_{\mathcal{V}_R}$ the (Dirichlet) Laplace operator on $L^2(\mathcal{G})$, where the δ condition (ii) of $-\Delta^\alpha_{\mathcal{V}_R}$ is replaced by the Dirichlet (zero) condition at the distinguished set $\mathcal{V}_R \subset \mathcal{V}$. Note that, if v_j is such a vertex equipped with the Dirichlet condition, the graph $\mathcal{G}$ *decouples* to a disjoint union of $\deg v_j$ subgraphs, where $\deg v_j$ is the number of edges incident with v_j, cf. Figure 1.2.3.

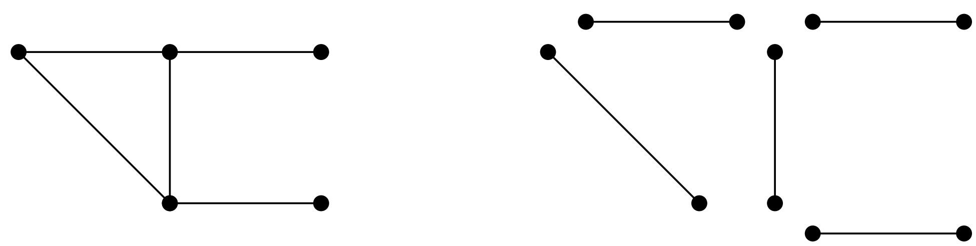

Figure 1.2.3: Dirichlet conditions at a vertex v_j imply that the graph decouples to a disjoint union of $\deg v_j$ subgraphs. If every vertex is a Dirichlet vertex, then we arrive at $|\mathcal{E}|$ subgraphs, each of which is an interval.

Theorem 1.2.8 (see [76, Theorem 1.2])**.** *Suppose $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a compact metric graph, and for the set of Robin vertices $\mathcal{V}_R = \{v_1, \dots, v_k\} \subset \mathcal{V}$, suppose that each $v_j \in \mathcal{V}_R$ is equipped with the Robin parameter $\alpha_j \in \mathbb{C}$, $j = 1, \dots, k$, and set $\alpha := (\alpha_1, \dots, \alpha_k) \in \mathbb{C}^k$. We suppose that for some $m \in \{0, 1, \dots, k\}$*

(1) $\alpha_0 := \alpha_1 = \dots = \alpha_m \to \infty$ in a sector fully contained in the open left half-plane;

(2) $\alpha_{m+1}, \ldots, \alpha_k \to \infty$ *in such a way that* $\operatorname{Re}\alpha_j$ *remains bounded from below as* $\alpha_j \to \infty$*, for all* $m+1 \le j \le k$*.*

Then, as $\alpha \to \infty$*, counting multiplicities there are exactly* m *eigenvalues* λ *of* $-\Delta^{\alpha}_{\mathcal{V}_R}$ *which diverge away from the positive real semi-axis (that is, whose distance to the positive real semi-axis grows to* ∞*); these satisfy the asymptotics*

$$\lambda = -\frac{\alpha_0^2}{(\deg v_j)^2} + \mathcal{O}\left(\alpha_0^2 \mathrm{e}^{\ell_{\mathcal{G}} \operatorname{Re}\alpha_0}\right) \tag{1.2.12}$$

as $\alpha \to \infty$*, where* $\ell_{\mathcal{G}}$ *is the length of the shortest edge of* $\mathcal{G}$*. Every eigenvalue of* $-\Delta^{\alpha}_{\mathcal{V}_R}$ *which does not diverge to* ∞ *in* $\mathbb{C}$ *converges to an eigenvalue of* $-\Delta^{D}_{\mathcal{V}_R}$*.*

In other words, we prove two possible behaviours of Robin eigenvalues as $\alpha_j \to \infty$ in $\mathbb{C}$ in the following regimes. On the one hand, for each Robin vertex $v_j \in \mathcal{V}_R$ equipped with α_0 such that the real part of the Robin parameter satisfies $\operatorname{Re}\alpha_0 \to -\infty$ sufficiently quickly, we obtain a single divergent eigenvalue λ of $-\Delta^{\alpha}_{\mathcal{V}_R}$. On the other hand, if $\operatorname{Re}\alpha_j$ remains bounded from below as $\alpha_j \to \infty$, then we end up with a Dirichlet vertex condition in v_j. Before proceeding, we want to make a couple of comments on the latter theorem: First of all, we remark that the statement is obtained by exploiting the duality of $-\Delta^{\alpha}_{\mathcal{V}_R}$ and the associated Dirichlet-to-Neumann operator $M(\lambda)$. Since the spectrum of the self-adjoint Dirichlet Laplacian $-\Delta^{D}_{\mathcal{V}_R}$ is purely real, so is the set of singularities of $M(\lambda)$. Consequently, in this regime the duality, or more precisely the relationship between α and λ, is far more complicated and this is why in Theorem 1.2.8 we deliberately avoid considering any potential eigenvalue curves which diverge within any fixed strip around the real positive semi-axis. For more information in the much simpler setting of one-dimensional intervals, we refer to Section 4.1.3 and especially to Proposition 4.1.6.

Secondly, it appears to be a natural question to ask whether the result remains true if the non-Robin vertices are equipped with some other self-adjoint condition(s) than the standard (continuity) Kirchhoff condition (iii). While our proof makes explicit use of the Kirchhoff conditions on $\mathcal{V}_N$, we expect that certain generalisations, that is, replacing (iii) by a different vertex condition, would be possible; see Remark 5.3.7 for more details.

Thirdly, there is a noteworthy generalisation in the statement of Theorem 1.2.8 compared to Conjecture 1.2.2 (and the theorems for domains): here, the Robin parameter α is variable in the sense that it can be seen as a function $\alpha : \mathcal{V} \to \mathbb{C}$ with

$$\alpha(v_j) = \begin{cases} \alpha_j & \text{for } j = 1, \dots, k, \\ 0 & \text{for } j = k+1, \dots, n, \end{cases} \tag{1.2.13}$$

while in the statements for domains (namely Theorem 1.2.3 and Corollary 1.2.4) we fix $\alpha \in \mathbb{C}$ to be independent of the position $x \in \partial\Omega$. However, some of our results remain valid in essentially the same form even if α is considered a function $\alpha \in L^\infty(\partial\Omega, \mathbb{C})$ instead of a constant $\alpha \in \mathbb{C}$. This generalisation is possible especially for two subject areas in particular: roughly speaking, for basic operator-theoretic properties in Section 3.1 as well as for the estimates on the numerical range in Section 3.4 due to the robust trace-type inequalities the estimates therein are based on. We refer to Remarks 3.1.4 and 3.4.15, respectively, for more details. For most other cases, however, this kind of generalisation introduces significant complications (see Remark 3.2.16 for the Dirichlet-to-Neumann operator as such an example) and we expect that many results would need heavy modifications in order to remain valid.

Fourth, in addition to supporting Conjecture 1.2.2, Theorem 1.2.8 should be of independent interest: as mentioned before, vertex conditions of Robin type are frequently studied (even though it is mostly the self-adjoint case examined in the literature), however, there are further works in the field of spectral theory of quantum graphs where the relationship between the parameter α and the eigenvalue λ (in other words the function $\alpha \mapsto \lambda(\alpha)$) plays a significant role [25, 51]. In addition to basic spectral and generation properties of graph Laplacians with complex δ couplings [70, 71], just recently a Weyl law for the eigenvalue asymptotics of star graphs for fixed complex $\alpha \in \mathbb{C}$ was established [105]. Moreover, there is a huge body of literature on the eigenvalue asymptotics for Laplacians on $\mathbb{R}^d$, where large α are often used to model potentials supported on a lower-dimensional manifold; we refer to [47, 52, 53] for more information on this topic.

Fifth, the key to the proof of Theorem 1.2.8 is a well-chosen representation (see Lemma 5.2.1) of the Dirichlet-to-Neumann operator: since there are only finitely many Robin vertices $v_1, \dots, v_k$, the Dirichlet-to-Neumann operator – also known as (Titchmarsh-Weyl) M-function – takes the form of a $k \times k$ matrix in a more or less explicit representation. This provides an enormous advantage compared to corresponding abstract results for domains. Now, the asymptotic behaviour of M

with respect to its spectral parameter $\lambda = \lambda(\alpha)$ only needs to be translated into a description for $\lambda(\alpha)$. To this end, we exploit the (in the real case well known) duality result of Theorem 5.2.2 and, similarly to the interval case, this allows us to give an essentially complete answer to the question of the asymptotic behaviour of Robin eigenvalues for large complex Robin parameters.

And sixth, it is worth mentioning that the asymptotics described in (1.2.12) is, to the best of our knowledge, even in the case of real α new and since there is an active interest in eigenvalue estimates on quantum graphs (see [25] and the references therein), we explicitly state this special case.

Theorem 1.2.9 (see [76, Theorem 1.3])**.** *Keep the assumptions of Theorem 1.2.8. Suppose now that $\alpha := \alpha_1 = \ldots = \alpha_k$ is real and negative and all vertices in $\mathcal{V}_R$ are equipped with the common Robin parameter α, and that*

$$\deg v_1 \leq \deg v_2 \leq \ldots \leq \deg v_k \tag{1.2.14}$$

Then for

$$\alpha < -2 \max_{j=1,\ldots,k} \left\{ \frac{\deg v_j}{\ell_j} \right\} \tag{1.2.15}$$

the self-adjoint operator $-\Delta_{\mathcal{V}_R}^{\alpha}$ has exactly k negative eigenvalues (here ℓ_j is the length of the shortest edge incident with v_j). Moreover, for each $j = 1, \ldots, k$, the jth eigenvalue $\lambda_j = \lambda_j(\alpha)$ behaves like

$$\lambda_j(\alpha) = -\frac{\alpha^2}{(\deg v_j)^2} + \mathcal{O}\left(\alpha^2 \mathrm{e}^{\ell_{\mathcal{G}} \alpha}\right) \tag{1.2.16}$$

as $\alpha \to -\infty$. Every other eigenvalue $\lambda_j(\alpha)$, $j \geq k+1$, converges to an eigenvalue of $-\Delta_{\mathcal{V}_R}^{D}$.

In the real (self-adjoint) setting, delta conditions can sometimes play a role in the *surgery* methods used for quantum graphs which is why it can be useful to understand how the eigenvalues depend on these conditions, see [25, Section 3]. We will remain for a moment in the real case: there, one can show that the j-th Robin eigenvalue $\lambda_j(\alpha)$ converges to the $(j-k)th$ Dirichlet eigenvalue from above, for any $j \geq k+1$; we refer to [27, Theorem 3.1.13] for the proof when $k = 1$ (the general case is analogous). We draw explicit attention to the coefficient C of the leading term asymptotics $-C\alpha^2$: in Theorem 1.2.9 we prove that $C = (\deg v_j)^{-2}$, i.e., we have $C < 1$ as soon as more than one edge is incident with v_j. This is a major difference to the behaviour on domains in $\mathbb{R}^d$: if the domain Ω is smooth, then we always have $C = 1$ [45, 90]; if $\partial\Omega$ contains corners, then, depending on the *sharpness* of those corners, we obtain $C > 1$. For more details and the actual theorems we refer internally to Section 2.4.1, and externally to [78, 80, 88]. However, the first observation of the leading coefficient depending on the geometry of the boundary goes back to [85] which might be seen as the pioneer article for eigenvalue asymptotics of the Robin Laplacian.

Back to the numerical range

We exploit the same overarching idea to localise the spectrum of the non-self-adjoint Robin Laplacian as described in the prior paragraph *Estimates on the numerical range* in order to prove statements similar to Theorem 1.2.3 and Corollary 1.2.4 in the setting of compact quantum graphs. We prove an adapted version of Theorem 1.2.3 to obtain a similar parabolic region $\Lambda_{\mathcal{G},\alpha} \subset \mathbb{C}$, cf. Figure 1.2.1, containing the whole of the spectrum. Notationally, for the fixed set $\mathcal{V}_R = \{v_1, \dots, v_k\}$ of Robin vertices we will always write

$$\mathfrak{D} := \min_{j=1,\dots,k} \deg v_j \tag{1.2.17}$$

for the smallest number of edges being incident with any Robin vertex. We also denote by $\ell_{\mathcal{G}} = \min\{\ell_e : e \in \mathcal{E}\}$ the length of the shortest edge in the whole of $\mathcal{G}$.

Theorem 1.2.10 (see [76, Theorem 5.1]). *(1) Let $\alpha \in \mathbb{C}^k$. Then the numerical range $W(a_\alpha)$, and in particular every eigenvalue of $-\Delta^{\alpha}_{\mathcal{V}_R}$, is contained in the set*

$$\Lambda_{\mathcal{G},\alpha} := \left\{ t + \sum_{j=1}^{k} \alpha_j s_j \in \mathbb{C} : t \geq 0,\ s_j \in \left[0, \frac{2}{\mathfrak{D}}\sqrt{\tau_j} + \frac{2}{\mathfrak{D}\ell_{\mathcal{G}}}\right] \right\}, \tag{1.2.18}$$

where the numbers $0 \leq \tau_j \leq t$ satisfy $\sum_{j=1}^{k} \tau_j \leq t$.

(2) If $\alpha_1 = \ldots = \alpha_k =: \alpha \in \mathbb{C}$ is independent of $j = 1, \dots, k$, then $W(a_\alpha)$ is

contained in

$$\Lambda_{\mathcal{G},\alpha} := \left\{ t + \alpha \cdot s \in \mathbb{C} : t \geq 0, s \in \left[0, \frac{2}{\mathfrak{D}}\sqrt{t} + \frac{1}{\mathfrak{D}\ell_{\mathcal{G}}} \right] \right\}. \tag{1.2.19}$$

This localisation theorem gives us control over both the real and imaginary party of any eigenvalue in terms of the real and imaginary party of α, thus, as a direct consequence, we obtain the following corollary.

Corollary 1.2.11 (see [76, Corollary 5.2])**.** *Let* $\alpha \in \mathbb{C}$ *such that* $\operatorname{Re}\alpha < 0$*. Then any eigenvalue* $\lambda \in \sigma(-\Delta^{\alpha}_{\mathcal{V}_R})$ *satisfies*

$$\operatorname{Re}\lambda \geq -\frac{(\operatorname{Re}\alpha)^2}{\mathfrak{D}^2} + \frac{\operatorname{Re}\alpha}{\mathfrak{D}\ell_{\mathcal{G}}}. \tag{1.2.20}$$

Furthermore, we do not only give estimates on the real part but also on the (modulus of) the imaginary part of λ and we refer to Section 5.4 and especially Theorem 5.4.15 for more details. We want to remark that if the components of $\alpha \in \mathbb{C}^k$ have a sufficiently large negative real part, these bounds (for both $\operatorname{Re}\lambda$ and $\operatorname{Im}\lambda$) are essentially asymptotically optimal. Let us therefore briefly consider the simplest case where $\alpha < 0$ is independent of the k Robin vertices $v_1, \ldots, \deg v_k$, which we order such that $\deg v_1 \leq \deg v_2 \leq \cdots \leq v_k$. Let $|\mathcal{G}|$ be the total length of $\mathcal{G}$ (that is, the sum of all edge lengths). Then, we obtain the following two-sided bound on the lowest eigenvalue $\lambda_1(\alpha)$,

$$\lambda_1(\alpha) \geq -\frac{\alpha^2}{(\deg v_1)^2} + \frac{\alpha}{\ell_{\mathcal{G}} \deg v_1}, \tag{1.2.21a}$$

$$\lambda_1(\alpha) < \min\left\{ -\frac{\alpha^2}{(\deg v_1)^2} - \frac{2\alpha}{\ell_{\mathcal{G}} \deg v_1} - \frac{1}{\ell_{\mathcal{G}}^2}, \frac{k\alpha}{|\mathcal{G}|} \right\}. \tag{1.2.21b}$$

This statement is proved in Corollary 5.4.10 and Remark 5.4.11.

1.3 Structure of the thesis

This thesis is organised as follows. To establish a formal basis for future sections, we start in Chapter 2 by introducing preliminaries and basics about boundary value problems. In Sections 2.1 and 2.2 we provide fundamental ideas and theorems from functional analysis and spectral theory, such as Sobolev spaces on Lipschitz domains and their boundaries, trace operators, and the variational characterisation of eigenvalues. After a short summary of Dirichlet, Neumann, and Robin boundary conditions and their corresponding forms and operators in Section 2.3, we briefly sketch the case of the Robin eigenvalues (or eigencurves) in the special case when Ω is a bounded interval where everything can be calculated explicitly. Section 2.4 is devoted to the history of Robin eigenvalue asymptotics: we summarise the crucial milestones of what has been achieved so far in the field of spectral asymptotics of the self-adjoint Robin Laplacian on both smooth and non-smooth (that is, Lipschitz) domains.

In Chapter 3 we start with a detailed spectral analysis of the Robin Laplacian on domains $\Omega \subset \mathbb{R}^d$. In Sections 3.1 and 3.2 we generalise results in the self-adjoint case to the case where α is no longer real (see Theorem 3.1.2). Moreover, we prove that $\{\mathcal{A}(\alpha) := -\Delta_\Omega^\alpha : \alpha \in \mathbb{C}\}$ forms a self-adjoint holomorphic family of operators, that is, $\mathcal{A}(\alpha)^* = \mathcal{A}(\overline{\alpha})$, as well as that each Robin eigenvalue $\lambda_k(\alpha)$ can be extended to a meromorphic function with at most algebraic singularities at non-real crossing-points of eigenvalues, cf. Theorem 3.2.11, and the same is proved for the corresponding eigennilpotents. Another generalisation is given in Theorem 3.2.18: for $\alpha \in \mathbb{R}$ there is a known formula for the derivative $\lambda'(\alpha)$ of a simple eigenvalue λ with respect to the Robin parameter and we prove that this representation is indeed locally meromorphic with at most removable singularities. In view of the analytic dependence of the eigenfunctions of the Robin Laplacian for $\alpha \in \mathbb{C}$, Section 3.3 is devoted to the question whether these eigenfunctions also still have reasonable basis properties of $L^2(\Omega)$: we prove the negative result of Theorem 3.3.1, which says that the eigenfunctions of $-\Delta_\Omega^\alpha$ form an orthonormal basis of $L^2(\Omega)$ if and only if $\alpha \in \mathbb{R}$, cf. Theorem 1.2.1 (5). Besides, we introduce *weaker* basis concepts (such as *Riesz*, *Bari*, and *Abel* bases) in order to prove a positive result which corresponds to Theorem 1.2.1 (4). The analysis of the numerical range in Section 3.4 leads us to our main localisation Theorem 3.4.1: to obtain that the numerical range (and hence the spectrum) of the Robin Laplacian is contained in a parabolic region $\Lambda_{\Omega,\alpha}$ we prove the crucial (and sharp) trace-type inequality of Lemma 3.4.7 for Lipschitz domains. In Section 3.5 we introduce and analyse the Dirichlet-to-Neumann operator, including a proof of Theorem 1.2.5, as well as the "duality" between the Robin and Dirichlet-to-Neumann eigenvalue problems, which is well known in the real case. In a later section this will form the (abstract) foundation of the explicit analysis of model domains in Chapter 4. However, before proceeding with explicit calculations, we study the question of which values $\lambda \in \mathbb{C}$ can be reached as points of accumulation of the Robin eigenvalues as $\alpha \to \infty$ in $\mathbb{C}$; up to a sector in which α diverges, we answer this question in Theorem 3.6.3. The last section of this chapter, namely Section 3.7, then rounds off the spectral analysis of the Robin Laplacian by recalling and applying recent results from [21, Sections 2, and 3] to our operator.

In Chapter 4, we give three concrete examples (model cases) of Ω. Here, we additionally pay attention to the error estimates appearing in the asymptotic expansions. Firstly, start with the interval in Section 4.1, where we not only calculate the Dirichlet-to-Neumann operator explicitly, but we give a detailed analysis of what happens in the picture of the Dirichlet-to-Neumann operator and the Robin Laplacian, respectively, including a consideration of the relation between the eigenvalues diverging near the positive real semi-axis and the parameter α. Divergence of the eigenvalues λ outside an arbitrarily small sector around the positive real semi-axis is shown to be possible only if $\operatorname{Re}\alpha \to -\infty$: the only two divergent eigenvalues λ behave like $-\alpha^2$, while the rest converge to the Dirichlet spectrum. If $\operatorname{Re}\alpha$ remains bounded from below, then, at least outside such a sector, all eigenvalues are convergent (see Theorems 4.1.1 and 4.1.4, as well as Proposition 4.1.6). Besides, in Section 4.1.2, we describe the procedure of *inverting* the asymptotical expansions from $\alpha(\lambda)$ to $\lambda(\alpha)$ while the information about the error terms is preserved. Here and throughout the text we call this the *Rouché inversion technique*. Furthermore, we prove a version of Theorem 1.2.1 (5) by explicit calculations. Secondly, in Section 4.2 we use our results on the interval to study d-dimensional rectangles (also called hyperrectangles or cuboids), see Theorem 4.2.2, as they are constructed as d intervals orthogonally "glued together". And thirdly, we treat d-dimensional balls in Section 4.3, where the asymptotic behaviour of complex Bessel functions are used to determine the asymptotics of the Robin eigenvalues; see in particular Theorems 4.3.5 and 4.3.6.

Finally, in Chapter 5 we study all the aspects and properties mentioned above but in the setting where the domain Ω is replaced by a compact quantum graph $\mathcal{G}$. In Section 5.1 we give a brief summary of the basic notation of metric graph, where we define suitable function spaces and introduce different vertex conditions (especially the so called δ coupling) and the corresponding Laplace operators. We continue in Section 5.2 by introducing Dirichlet-to-Neumann matrices $M(\lambda)$ to derive the block matrix representation of Lemma 5.2.1. This allows us to establish the asymptotic behaviour of M in Section 5.3 which is key for proving Theorem 1.2.8. Finally, in Section 5.4 we give estimates on the numerical range of the Robin Laplacian and hence inequalities for the real and imaginary parts of its eigenvalues.

1.4 How to read this thesis

Even though this doctoral thesis is designed to be a coherent text and the order of the chapters and sections are chosen to serve this purpose, there are some remarks on how to read this dissertation.

(1) This thesis is based on the two articles [30] (joint work with James B. Kennedy and Sabine Bögli) and [76] (joint work with James B. Kennedy) as listed on page 11. However, since the problems studied here are generalisations of a topic already (at least partially) covered by former papers, there are sections which contain notation and results from textbooks or articles we have not proved or introduced ourselves. To clarify and distinguish these different cases, mainly two "citing methods" are used: the first indicator is the brief introduction preceding each chapter or (sub)section, while the second one is a short text preceding the statement in question. If there is no indicator whatsoever, the statements are due to the two articles mentioned above, that is, new results proved by my co-authors and myself. To give a rough idea of the subject areas falling into the last category, in Figure 1.4.1 they are marked with an asterisk. Chapters 3 and 4 are mainly based on [30, Sections 3-8 and Sections 2, 9], the results in Chapter 5 are due to [76].

(2) The underlying papers [30, 76] form a proper subset of this thesis and the main results of both papers and this thesis coincide. However, in this thesis there are many additional paragraphs, remarks, and statements which support the main results with further information and details: besides a higher level of detail in almost every proof, many additional figures sketch or illustrate the ideas of proof or clarify given notation. Furthermore, especially when considering examples in Chapter 4, there are supplementary statements (e.g. Lemma 4.1.7) which might follow from the abstract theory of Chapter 3: explicit calculations,

however, appeared to be useful for gaining insight into what to expect in the general case and so they are included as well.

(3) The previous item leads additionally to a last remark, namely, that there are two possible (and canonical) ways to read this thesis. (Before actually reading this paragraph, for reasons of clarity, we highly recommend taking a look at the corresponding Figure 1.4.1 first.) As an alternative to reading the thesis in the order presented (that is, interpreting Chapters 4 and 5 as applications of the theory established in Chapter 3), one can start out with reading Chapter 4 in order to gain intuition as well as more insight into the approaches from Chapter 3. This is the path we, the authors of [30, 76], took. Since (parts of) Chapter 5 might be interpreted as a generalisation of the discrete calculations in the case where Ω is an interval, one could also read Sections 5.1- 5.3 as a comprehensive generalisation of Section 4.1.

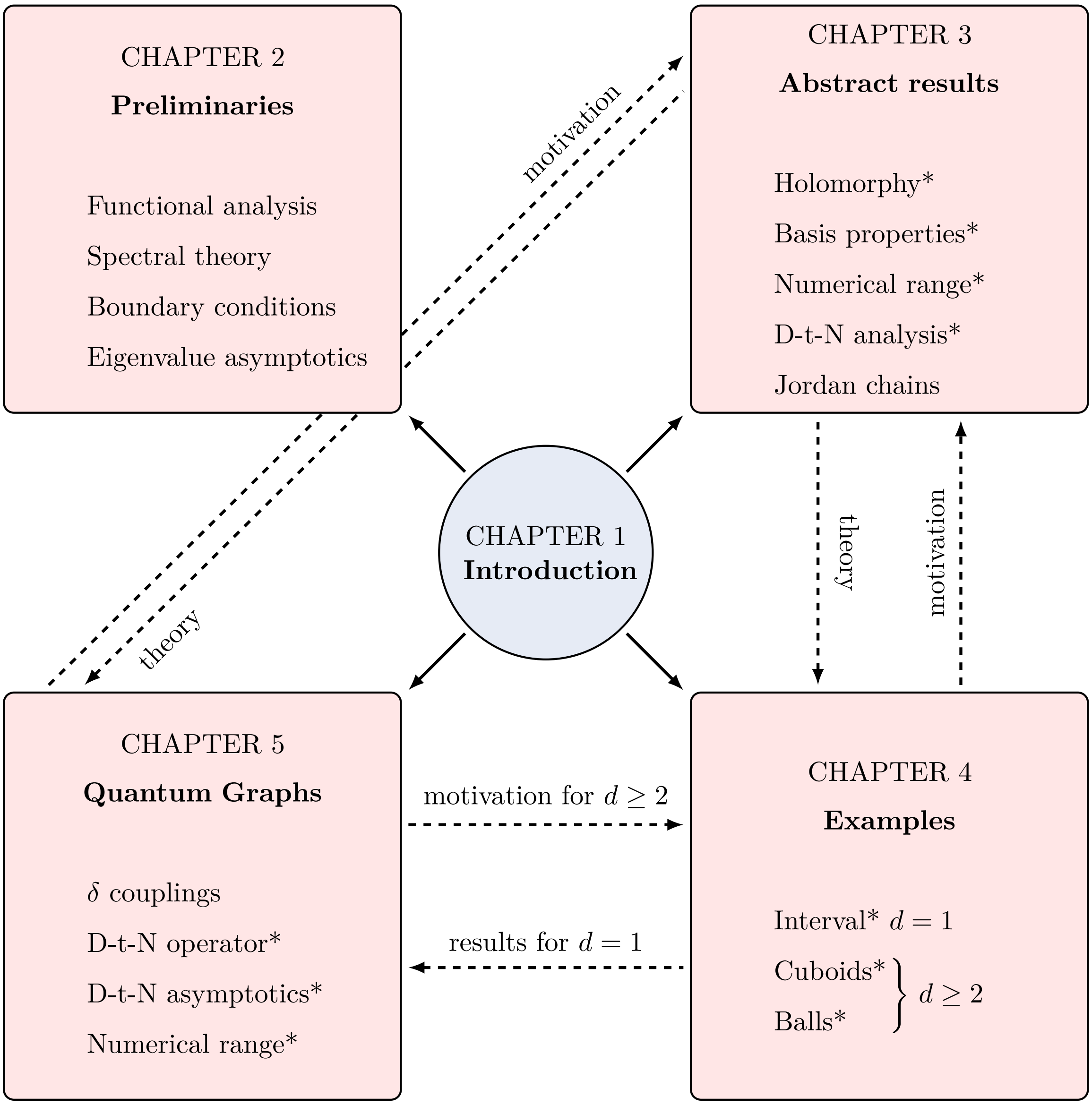

Figure 1.4.1: The structure of this thesis and the relation between the chapters. Sections marked with * are due to [30, 76]. The dashed arrows indicate the "flow of information". The abbreviation D-t-N stands for "Dirichlet-to-Neumann".

Chapter 2

Preliminaries and boundary value problems

Contents

In this chapter we will introduce function spaces, corresponding embedding theorems, and collect a number of basic properties which we will use to introduce and understand the Laplacian and some of its boundary value problems. The boundary conditions in this context, namely *Dirichlet*, *Neumann*, and *Robin* conditions, will be introduced in Section 2.3. Since our aim is to make statements that are as general as possible (at least with respect to the boundary regularity), we will be using the framework of McLean [94, Chapter 3] and Kato [74, Chapters V and VII], and we start by recalling some definitions from there. We assume throughout that (1) H is a Hilbert space with inner product $(\,\cdot\,,\,\cdot\,)_H$ and norm $\|\cdot\|_H$, and (2) that, if not stated otherwise, $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain.

2.1 Some functional analysis

In this section we build the foundations for considering linear operators and their boundary value problems on bounded Lipschitz domains $\Omega \subset \mathbb{R}^d$. We start by introducing *Sobolev spaces* both on domains and on their boundaries $\partial\Omega$, respectively. A complete introduction can be found in many textbooks, e.g. [2, 62, 5]. We, however, use the framework, definitions and theorems of [94, Section 3] since the statements therein (Sobolev embedding theorems as well as compactness of the trace operator) are given in a generality to even hold for Lipschitz domains.

2.1.1 Sobolev spaces on Lipschitz domains

We start by defining what exactly we mean by the term *Lipschitz domain*; heuristically, this means that the boundary $\partial\Omega$ of a domain Ω can *locally* (after an elementary transformation) be described by the graph of a Lipschitz continuous function.

Definition 2.1.1. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be an open set. If there is a Lipschitz continuous function $\zeta : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$\Omega = \left\{ x \in \mathbb{R}^{d-1} : x_d < \zeta(\check{x}) \text{ for all } \check{x} = (x_1, \dots, x_{d-1})^T \in \mathbb{R}^{d-1} \right\}, \tag{2.1.1}$$

then Ω is called a *Lipschitz hypograph*.

Definition 2.1.2. The open set Ω is a *Lipschitz domain* if its boundary $\partial\Omega$ is compact and if there exist finite families $\{W_j\}$ and $\{\Omega_j\}$ such that the following conditions hold.

(1) The family $\{W_j\}$ is a finite open cover of $\partial\Omega$: each $W_j \subset \mathbb{R}^d$ is open and we have $\partial\Omega \subset \bigcup_j W_j$.

(2) Each Ω_j can be transformed to a Lipschitz hypograph by a rigid motion, that is, by a rotation plus a translation.

(3) The set Ω satisfies

$$W_j \cap \Omega = W_j \cap \Omega_j \tag{2.1.2}$$

for each j.

Definition 2.1.3. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, a non-empty open subset.

(1) The *Sobolev space* $W^{k,p}(\Omega)$ of order k based on $L^p(\Omega)$ is defined by

$$W^{k,p}(\Omega) = \left\{ u \in L^p(\Omega) : \partial^\beta u \in L^p(\Omega) \text{ for all } |\beta| \leq k \right\}, \tag{2.1.3}$$

where $\beta \in \mathbb{N}^d$ is a multi-index and $\partial^\beta u$ is viewed as a distribution on Ω, that is, $\partial^\beta u \in L^p(\Omega)$ means that there exists a function $g_\beta \in L^p(\Omega)$ such that

$$(u, \partial^\beta \varphi)_{L^2(\Omega)} = (-1)^{|\beta|} (g_\beta, \varphi)_{L^2(\Omega)} \tag{2.1.4}$$

for all test functions $\varphi \in \mathcal{D}(\Omega)$. The function g_β is then often called *weak partial derivative* of u.

(2) The norm $\|\cdot\|_{W^{k,p}(\Omega)}$ defined by

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\beta| \leq k} \int_\Omega \left| \partial^\beta u(x) \right|^p \, \mathrm{d}x \right)^{1/p} \tag{2.1.5}$$

is called *Sobolev norm.* The completeness of the space $L^p(\Omega)$ implies that $W^{k,p}(\Omega)$ together with the latter norm is a Banach space.

Since the canonical setting of trace operators are Sobolev spaces of fractional order, we need the following definition.

Definition 2.1.4. (1) For $0 < \mu < 1$ we denote by $|\cdot|_{\mu,p,\Omega}$ the *Slobodeckij semi-norm*[1]

$$|u|_{\mu,p,\Omega} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{d+p\mu}} \, \mathrm{d}x \, \mathrm{d}y \right). \tag{2.1.6}$$

(2) For $s = k + \mu \in (k, k+1)$, we define

$$W^{s,p}(\Omega) := \left\{ u \in W^{k,p}(\Omega) : |\partial^\beta u|_{\mu,p,\Omega} < \infty \text{ for all } |\beta| = k \right\}. \tag{2.1.7}$$

[1] This norm is sometimes also called *Gagliardo semi-norm*, see [46, Section 2].

This space equipped with the norm $\|\cdot\|_{W^{s,p}(\Omega)}$ given by

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{W^{k,p}(\Omega)} + \sum_{|\beta|=k} \left|\partial^\beta u\right|^p_{\mu,p,\Omega} \right) \tag{2.1.8}$$

is called *Sobolev space of fractional order s.*

Remark 2.1.5. The latter Definition 2.1.4 also holds for $p = \infty$: note that the integrand of (2.1.6) is the pth power of

$$\frac{|u(x) - u(y)|}{|x - y|^{\mu+d/p}}. \tag{2.1.9}$$

Consequently, if $p = \infty$, we get the usual Hölder semi-norm of order μ.

Remark 2.1.6. For $p = 2$ we write $H^s(\Omega)$ instead of $W^{s,2}(\Omega)$. Due to the extension theorem of Stein, cf. [111, Chapter VI.3: Theorem 5' and Section 3.3], for $s \geq 0$, the space $H^s(\Omega)$ defined in this way coincides with the space of functions from $H^s(\mathbb{R}^d)$ restricted to Ω. In the second case, for $s \in \mathbb{R}$, we define the so called *Bessel potential* of order s,

$$\mathcal{J}^s : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d) : u(x) \mapsto \int_{\mathbb{R}^d} (1 + |\xi|^2)^{s/2} \hat{u}(\xi) \mathrm{e}^{2\pi \mathrm{i}\, \xi \cdot x} \, \mathrm{d}\xi, \tag{2.1.10}$$

for $x \in \mathbb{R}^d$, where $\hat{u}$ is the Fourier transform of u in the *Schwartz space* $\mathcal{S}(\mathbb{R}^d)$.

We arrive at the following definition.

Definition 2.1.7. For any $s \in \mathbb{R}$ we define the Sobolev space of order s on $\mathbb{R}^d$ by

$$H^s(\mathbb{R}^d) := \left\{ u \in \mathcal{S}^*(\mathbb{R}^d) : \mathcal{J}^s u \in L^2(\mathbb{R}^d) \right\} \tag{2.1.11}$$

and equip this space with the inner product

$$(u, v)_{H^s(\mathbb{R}^d)} := (\mathcal{J}^s u, \mathcal{J}^s v)_{L^2(\mathbb{R}^d)} \tag{2.1.12}$$

and the induced norm

$$\|u\|_{H^s(\mathbb{R}^d)} := \sqrt{(u, u)_{H^s(\mathbb{R}^d)}} = \|\mathcal{J}^s u\|_{L^2(\mathbb{R}^d)}. \tag{2.1.13}$$

Remark 2.1.8. Note that the Bessel potential

$$\mathcal{J}^s : H^s(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \tag{2.1.14}$$

introduced in (2.1.10) is a unitary isomorphism, and $\mathcal{J}^0 u = u$ implies that

$$H^0(\mathbb{R}^d) = L^2(\mathbb{R}^d). \tag{2.1.15}$$

Definition 2.1.9. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain and let $s > 0$. If we denote by $H_0^s(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^s(\Omega)$, then $H^{-s}(\Omega)$ is defined as the dual space $H_0^s(\Omega)^*$.

We want to give an important compactness result which originated in the well known article of Rellich [104].

Theorem 2.1.10. *Let $\Omega \in \mathbb{R}^d$ be a bounded, open subset and assume that*

$$-\infty < t < s < \infty. \tag{2.1.16}$$

Then, the embedding $H^s(\Omega) \hookrightarrow H^t(\Omega)$ is compact. In particular, for $s > 0$, every $H^s(\Omega)$ is compactly embedded in $L^2(\Omega)$.

2.1.2 Sobolev spaces on the boundary

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz hypograph, cf. Definition 2.1.1, and let $0 \leq s \leq 1$. The corresponding Sobolev space $H^s(\partial\Omega)$ is constructed in terms of Sobolev spaces on $\mathbb{R}^{d-1}$. Let $u \in L^2(\partial\Omega)$ and define

$$u_\zeta(\check{x}) = u(\check{x}, \zeta(\check{x})) \tag{2.1.17}$$

for $\check{x} \in \mathbb{R}^{d-1}$.

Definition 2.1.11. We define the space $H^s(\partial\Omega)$ by

$$H^s(\partial\Omega) := \left\{ u \in L^2(\Omega) : u_\zeta \in H^s(\mathbb{R}^{d-1}) \right\} \tag{2.1.18}$$

and equip it with the inner product

$$(u, v)_{H^s(\partial\Omega)} := (u_\zeta, v_\zeta)_{H^s(\mathbb{R}^{d-1})}. \tag{2.1.19}$$

Definition 2.1.12. The space $H^{-s}(\partial\Omega)$ is defined as the completion of $L^2(\partial\Omega)$ with respect to the norm given by

$$\|u\|_{H^{-s}(\partial\Omega)} := \left\| u_\zeta \sqrt{1 + |\nabla\zeta|^2} \right\|_{H^{-s}(\mathbb{R}^{d-1})}. \tag{2.1.20}$$

It can be followed that $H^{-s}(\partial\Omega)$ is a realisation of the dual space $H^s(\partial\Omega)^*$.

To define $H^s(\partial\Omega)$ for Lipschitz domains Ω, recall the notation of Definition 2.1.2, we choose a partition of unity $\{\Phi_j\}$ subordinate to the open cover $\{W_j\}$ of $\partial\Omega$. To this end, we choose Φ_j to satisfy $\sum_j \Phi_j(x) = 1$ for all $x \in \partial\Omega$.

Definition 2.1.13. The space $H^s(\partial\Omega)$ is defined as in Definition 2.1.11 together with the inner product

$$(u, v)_{H^s(\partial\Omega)} := \sum_j (\Phi_j u, \Phi_j v)_{H^s(\partial\Omega_j)}. \tag{2.1.21}$$

2.1.3 The trace operator

When working with boundary value problems it is natural to consider $u|_{\partial\Omega}$ as an element of a Sobolev space on $\partial\Omega$ while u itself is an element of a Sobolev space on Ω. The function which maps u to $u|_{\partial\Omega}$ is then called *trace operator*; we dedicate this short section to the operator just mentioned and its properties. Let $s > 1/2$. As before, unless otherwise stated, for the proofs of the following statements we refer to [94, Section 3, esp. pp. 100-106]: the trace operator $\mathrm{tr} : \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}(\mathbb{R}^{d-1})$ given by

$$\mathrm{tr}\, u(x) := u(\check{x}, 0) \tag{2.1.22}$$

for $\check{x} \in \mathbb{R}^{d-1}$ can be uniquely extended to a bounded linear operator

$$\mathrm{tr} : H^s(\mathbb{R}^d) \to H^{s-1/2}(\mathbb{R}^{d-1}). \tag{2.1.23}$$

Since tr has a continuous right inverse, we have

$$H^{s-1/2}(\mathbb{R}^{d-1}) = \left\{ \operatorname{tr} u : u \in H^s(\mathbb{R}^d) \right\}. \tag{2.1.24}$$

If we now consider Lipschitz domains, then the trace operator

$$\operatorname{tr} : \mathcal{D}(\overline{\Omega}) \to \mathcal{D}(\partial\Omega) \tag{2.1.25}$$

defined by $\operatorname{tr} u = u|_{\partial\Omega}$ satisfies the following extension property.

Theorem 2.1.14. *For* $1/2 < s < 3/2$ *the operator* tr *from* (2.1.25) *has a unique extension to a bounded linear operator*

$$\operatorname{tr} : H^s(\Omega) \to H^{s-1/2}(\partial\Omega). \tag{2.1.26}$$

For a proof and more details we refer to [42, Lemma 3.6].

2.2 Some spectral theory

For the sake of completeness we give some basic terms and theorems of spectral theory of (unbounded) operators and forms on Hilbert spaces. We will be using the framework of *Spectral Theory of Self-Adjoint Operators in Hilbert Space* by M. S. Birman and M Z. Solomjak [29, Chapter 3].

Definition 2.2.1. A linear mapping

$$\mathcal{A} : D(\mathcal{A}) \subset H \to H \tag{2.2.1}$$

is called *linear operator* on a Hilbert space H. By $D(\mathcal{A})$ we denote its *domain* and by $R(\mathcal{A}) = \mathcal{A}D(\mathcal{A})$ its *range* (or image). With respect to the inner product

$$(u, v)_{\mathcal{A}} = (u, v)_H + (\mathcal{A}u, \mathcal{A}v)_H, \qquad u, v \in D(\mathcal{A}) \tag{2.2.2}$$

the domain $D(\mathcal{A})$ becomes a pre-Hilbert space, denoted by $H_{\mathcal{A}}$; the corresponding norm $\|\cdot\|_{\mathcal{A}}$ given by

$$\|u\|_{\mathcal{A}}^2 = \|u\|_H^2 + \|\mathcal{A}u\|_H^2, \qquad u \in D(\mathcal{A}) \tag{2.2.3}$$

is called $\mathcal{A}$-*norm*.

Definition 2.2.2. Let $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ be a linear operator. $\mathcal{A}$ is called *closed* if $H_{\mathcal{A}}$ is complete, i.e. $H_{\mathcal{A}}$ is a pre-Hilbert space with respect to (2.2.2).

There are well known properties which are equivalent to $\mathcal{A}$ being closed: The following lemma is due to [29, Theorem 3.2.1.].

Lemma 2.2.3. *Let $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ be a linear operator. Then, the following properties are equivalent:*

(1) $\mathcal{A}$ is closed.

(2) The graph

$$G(\mathcal{A}) := \{(u, \mathcal{A}u) \in H \times H : u \in D(\mathcal{A})\} \tag{2.2.4}$$

is closed in $H \times H$.

(3) Whenever a sequence $(x_n)_{n\in\mathbb{N}}$ in $D(\mathcal{A})$ converges in H, i.e., $x_n \to x \in H$, and $\mathcal{A}x_n \to y$ in H, then we have $x \in D(\mathcal{A})$ and $y = Ax$.

Definition 2.2.4. Let $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ be a closed, densely defined linear operator, that is, $D(\mathcal{A}) \subset H$ is dense with respect to $\| \cdot \|_H$.

(1) The set

$$\rho(\mathcal{A}) := \{z \in \mathbb{C} : \mathcal{A} - zI : D(\mathcal{A}) \to H \text{ has a bounded inverse}\} \tag{2.2.5}$$

is called *resolvent set* of $\mathcal{A}$;

(2) for $z \in \rho(\mathcal{A})$ we call

$$R_z(\mathcal{A}) := (\mathcal{A} - zI)^{-1} : H \to D(\mathcal{A}) \tag{2.2.6}$$

the *resolvent* of $\mathcal{A}$.

(3) The complement of the resolvent set $\sigma(\mathcal{A}) := \mathbb{C} \setminus \rho(\mathcal{A})$ is called *spectrum* of $\mathcal{A}$;

(4) we denote by $\sigma_{\mathrm{p}}(\mathcal{A})$ the *point spectrum* $\mathcal{A}$, that is

$$\sigma_{\mathrm{p}}(\mathcal{A}) := \{z \in \sigma(\mathcal{A}) : \mathcal{A} - zI : D(\mathcal{A}) \to H \text{ is not injective}\}; \tag{2.2.7}$$

(5) and we call the set

$$\sigma_{\text{ess}}(\mathcal{A}) := \{z \in \sigma(\mathcal{A}) : (\mathcal{A} - zI) : D(\mathcal{A}) \to H \\ \text{is not a Fredholm operator}\} \tag{2.2.8}$$

the *essential spectrum* of $\mathcal{A}$.

Note that in the literature the definition of *essential spectra* of non-self-adjoint operators might vary. Our definition (often denoted by σ_{e3}) using *Fredholm operators* is due to Wolf, cf. [114]. For five distinct definitions and their respective characteristics, see [49, Chapter IX].

Definition 2.2.5. Let $a : D(a) \times D(a) \subset H \times H \to \mathbb{C}$ be a densely defined sesquilinear form and let $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ be a linear operator. We call the set

$$W(\mathcal{A}) := \{(\mathcal{A}u, u) : u \in D(\mathcal{A}) \text{ and } \|u\|_H = 1\} \subset \mathbb{C} \tag{2.2.9}$$

the *numerical range of* $\mathcal{A}$ and, likewise, the set

$$W(a) := \{a[u, u] : u \in D(a) \text{ and } \|u\|_H = 1\} \subset \mathbb{C} \tag{2.2.10}$$

the *numerical range of* a. If $\mathcal{A}$ is the operator associated with a, that is, if $\mathcal{A}$ is defined by

$$D(\mathcal{A}) = \{u \in D(a) : \exists\, h \in H \text{ with } a[u, v] = (h, v) \;\; \forall v \in D(a)\}, \tag{2.2.11a}$$

$$\mathcal{A}u = h, \tag{2.2.11b}$$

then it follows immediately from the definitions that

$$\sigma_p(\mathcal{A}) \subset W(\mathcal{A}) \subset W(a). \tag{2.2.12}$$

Definition 2.2.6. We call $\mathcal{A}$ *m-sectorial* (of semi-angle θ) if there exist a vertex $\gamma \in \mathbb{R}$ and an angle $0 \le \theta < \pi/2$ such that

$$W(\mathcal{A}) \subset \{z \in \mathbb{C} : |\arg(z - \gamma)| \le \theta\} \tag{2.2.13}$$

(where the principal argument of a complex number is taken to be between $-\pi$ and π) and for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < \gamma$ we have that $\lambda \in \rho(\mathcal{A})$ satisfies the resolvent norm estimate

$$\|(\mathcal{A} - \lambda I)^{-1}\|_{H \to H} \leq \frac{1}{|\gamma - \operatorname{Re}\lambda|}. \tag{2.2.14}$$

The form a is likewise called *sectorial* (of semi-angle θ) if (2.2.13) holds for $W(a)$ instead of $W(\mathcal{A})$. Neglecting the resolvent estimate (2.2.14), $\mathcal{A}$ and a, respectively, satisfying (2.2.13) are called *sectorial.*

The following property, cf. [74, Corollary VI.2.3], of the numerical range of an operator and its associated sesquilinear form is an immediate conclusion of Kato's *first representation theorem* [74, Theorem VI.2.1].

Lemma 2.2.7. *The numerical range $W(\mathcal{A})$ of $\mathcal{A}$ is a dense subset of the numerical range $W(a)$ of a.*

The following statement is known as the *Riesz–Schauder* theorem or simply *spectral theorem for compact operators.* For its proof we refer to [102, Theorem VI.15].

Theorem 2.2.8. *Let $\mathcal{A} \in S_\infty(H)$ be a compact operator on H. Then, its spectrum $\sigma(A)$ is a discrete set having no limit points except perhaps $\lambda = 0$. Further, any $\lambda \in \sigma(A)\backslash\{0\}$ is an eigenvalue of finite multiplicity (i.e. the corresponding space of eigenvectors is finite dimensional).*

With the assistance of the *spectral mapping theorem* we obtain the following corollary.

Corollary 2.2.9. *Let $\mathcal{A}$ be a linear operator with compact resolvent $R_z(\mathcal{A}) \in S_\infty(H)$. Then,*

(1) for the spectrum of $\mathcal{A}$ we have $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$;

(2) either $\sigma(\mathcal{A})$ is finite or there exists a sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ such that

$$\lim_{k\to\infty} |\lambda_k| = \infty \qquad \text{and} \qquad \sigma(\mathcal{A}) = \{\lambda_k \mid k \in \mathbb{N}\}; \tag{2.2.15}$$

(3) $\lambda \in \sigma_p(A)$ implies that $\dim N(\lambda I - \mathcal{A}) < \infty$.

The next statement is often called *Hilbert–Schmidt theorem*. Its proof can be found in [102, Theorem VI.16].

Theorem 2.2.10. *Let $\mathcal{A} \in S_\infty(H)$ be a self-adjoint compact operator on H. Then there is a complete orthonormal basis $(\psi_k)_{k\in\mathbb{N}}$ of H consisting of eigenfunctions ψ_k of $\mathcal{A}$. The associated sequence of eigenvalues $(\lambda_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ satisfies $\lim_{k\to\infty} \lambda_k = 0$.*

We arrive at the following version of the spectral theorem for m-dissipative operators; for both the statement itself and its proof, see [9, Theorem 1.4.8].

Theorem 2.2.11. *Let $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ be a linear operator on a separable and infinite dimensional Hilbert space H. Assume that $\mathcal{A}$*

(1) is m-dissipative, i.e., $\operatorname{Re}(\mathcal{A}u, u)_H \leq 0$ for all $u \in D(\mathcal{A})$ and $(I - \mathcal{A})$ is surjective;

(2) is symmetric, i.e., $(\mathcal{A}u, u)_H = (u, \mathcal{A}u)_H$ for all $u \in D(\mathcal{A})$;

(3) has compact resolvent, i.e., $R_z(\mathcal{A}) \in S_\infty(H)$ for every $z \in \rho(A)$.

Then, there exists an orthonormal basis $(\psi_k)_{k\in\mathbb{N}} \subseteq D(A)$ of H and an associated sequence of eigenvalues $(\lambda_k)_{k\in\mathbb{N}} \subseteq (-\infty, 0]$, such that

$$\mathcal{A}\psi_k = \lambda_k \psi_k \qquad \text{and} \qquad \lim_{k\to\infty} \lambda_k = -\infty. \tag{2.2.16}$$

Furthermore, $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \{u \in H \mid (\lambda_n(u, \psi_k))_{k\in\mathbb{N}} \in \ell^2(\mathbb{C})\}, \tag{2.2.17a}$$

$$\mathcal{A}u = \sum_{k=1}^{\infty} \lambda_k (u, \psi_k)\psi_k. \tag{2.2.17b}$$

2.2.1 Variational characterisation

A major tool for estimating the eigenvalues of self-adjoint operators is the so called *min-max principle* or the *variational characterisation of eigenvalues*. This technique allows us to use test functions to obtain upper estimates for the eigenvalues. Since we will need the notation in Chapter 5, we give a brief introduction; the definitions and statements in this section are well known and can be found in almost every textbook about spectral theory, e.g. [62, Section 8.12], [103, Chapter XIII.1], or [74, Section I.6.10].

Definition 2.2.12. Let $a : D(a) \times D(a) \to \mathbb{R}$ be a real sesquilinear form. The quotient defined by

$$R[a](u) := \frac{a[u,u]}{\|u\|_{L^2(\Omega)}} \tag{2.2.18}$$

is called the *Rayleigh quotient* of a.

We adapt the general min-max (or more precisely *max-min*) principle to the case of the Robin Laplacian.

Theorem 2.2.13. *Let $\mathcal{A} : H \to H$ be a self-adjoint operator which is semi-bounded from below and associated to the form $a : D(a) \times D(a) \to \mathbb{R}$. Then, its nth eigenvalue $\lambda_n(\mathcal{A})$ can be characterised by*

$$\lambda_n(\mathcal{A}) = \sup_{\substack{N \subset D(a) \\ \operatorname{codim} N = n-1}} \inf_{\substack{u \in N \\ u \neq 0}} R[a](u). \tag{2.2.19}$$

In particular, the first eigenvalue of the real Robin Laplacian $\lambda_1(\alpha)$ reads

$$\lambda_1(\alpha) = \inf_{\substack{u \in H^1(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \alpha \int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma}{\int_\Omega |u|^2 \,\mathrm{d}x}. \tag{2.2.20}$$

2.3 Dirichlet, Neumann, and Robin Laplacians

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a bounded domain, that is, a bounded open set with a finite number of connected components, and (if $d \geq 2$) assume that it is a Lipschitz domain, cf. Definition 2.1.2. We will proceed as follows: firstly, we define a sesquilinear form on $H^1(\Omega)$,

$$a_\alpha : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}, \tag{2.3.1}$$

for arbitrary $\alpha \in \mathbb{C}$. By restricting the form domain $D(a_\alpha) = H^1(\Omega) \times H^1(\Omega)$ to $H_0^1(\Omega) \times H_0^1(\Omega)$ and by setting the Robin parameter $\alpha = 0$, we then obtain the Dirichlet and Neumann Laplacian, respectively. To this end, let $\alpha \in \mathbb{C}$ and define

$$a_\alpha[u,v] = \int_\Omega \nabla u \cdot \overline{\nabla v} \,\mathrm{d}x + \alpha \int_{\partial\Omega} u\overline{v} \,\mathrm{d}\sigma(x). \tag{2.3.2}$$

We will refer to this form as the *Robin form* (for the parameter α) and we want to use the abbreviation $a_\alpha[u] := a_\alpha[u,u]$ for $u = v$; the form is then called *quadratic*. If $a = a_\alpha$ is the Robin form in Definition 2.2.12, then the Rayleigh quotient (see Definition 2.2.12) reads

$$R[a_\alpha](u) = \frac{\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \alpha \int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma}{\int_\Omega |u|^2 \,\mathrm{d}x}. \tag{2.3.3}$$

Recall that Ω is a Lipschitz domain. We use the framework of [12, Section 2] in order to clarify what we mean by Δu and $\partial_\nu u$ of $u \in H^1(\Omega)$.

Definition 2.3.1. Let $\Omega \subset \mathbb{R}^d$ be any bounded Lipschitz domain and let $u \in H^1(\Omega)$.

(1) We say that $\Delta u \in L^2(\Omega)$ if there exists $f \in L^2(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \overline{\nabla v} \,\mathrm{d}x = \int_\Omega f \overline{v} \,\mathrm{d}x \tag{2.3.4}$$

for all $v \in H_0^1(\Omega)$.

(2) Let additionally $\Delta u \in L^2(\Omega)$. We say that $\frac{\partial u}{\partial \nu} = \partial_\nu u \in L^2(\partial\Omega)$ if there exists $g \in L^2(\partial\Omega)$ such that

$$\int_\Omega \Big(\nabla u \cdot \overline{\nabla v} - (\Delta u)\overline{v}\Big) \,\mathrm{d}x = \int_{\partial\Omega} g\overline{v} \,\mathrm{d}\sigma \tag{2.3.5}$$

for all $v \in H^1(\Omega)$. We then write $\partial_\nu u = g$.

Before proceeding, we want to make a few remarks: firstly, the *usual* normal derivative $\partial_\nu u$ is generally not defined for all $u \in H^1(\Omega)$ since Ω is assumed to be a Lipschitz domain. To this end, we explicitly remark on the Definition 2.3.1(2); we say that the (outer) normal derivative $\partial_\nu u$ exists, if it exists, in the *weak sense* and we will not speak of the "distributional sense" here owing to the low regularity of the boundary. Secondly, both (1) and (2) of Definition 2.3.1 are chosen in a way such that Green's formula

$$\int_\Omega \Big(\nabla u \cdot \overline{\nabla v} - (\Delta u)\overline{v}\Big) \,\mathrm{d}x = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\,\overline{v} \,\mathrm{d}\sigma \tag{2.3.6}$$

holds for all $v \in H^1(\Omega)$ whenever $u \in H^1(\Omega)$, $\Delta u \in L^2(\Omega)$, and $\partial_\nu u \in L^2(\Omega)$. By the trace theorem (cf. Theorem 2.1.14 for $s = 1$) the Robin form a_α is coercive, that is, there exists $c > 0$ such that $\operatorname{Re} a_\alpha[u] \geq c\|u\|^2_{H^1(\Omega)}$ for all $u \in D(a_\alpha) = H_1(\Omega)$. Consequently, a_α is closed, which then implies that there exists an associated operator $\tilde{\mathcal{A}}(\alpha)$ on $L^2(\Omega)$ (see almost every textbook about operator theory, e.g. [102]). By *associated* we mean that $D(\tilde{\mathcal{A}}(\alpha)) \subset D(a_\alpha)$ and $a_\alpha[u,v] = (\tilde{\mathcal{A}}(\alpha)u, v)_{L^2(\Omega)}$ for every $u \in D(\tilde{\mathcal{A}}(\alpha))$ and every $v \in D(a_\alpha)$. The domain of the associated operator reads

$$D(\tilde{A}(\alpha)) = \Big\{u \in D(a_\alpha) : \exists\, h \in L^2(\Omega) \text{ such that } a_\alpha[u,v] = (\tilde{A}(\alpha)h, v) \;\; \forall\, v \in D(a_\alpha)\Big\}, \tag{2.3.7}$$

cf. [14, Section 1].

Definition 2.3.2. We define the following operator denoted by $-\Delta^\alpha_\Omega$ on $L^2(\Omega)$ by

$$D(-\Delta^\alpha_\Omega) = \left\{u \in H^1(\Omega) \;:\; \Delta u \in L^2(\Omega) \text{ and } \frac{\partial u}{\partial \nu} \in L^2(\partial\Omega) \text{ with } \frac{\partial u}{\partial \nu} + \alpha u = 0\right\}, \tag{2.3.8a}$$

$$-\Delta^\alpha_\Omega u = -\Delta u. \tag{2.3.8b}$$

Here,

$$\Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} \tag{2.3.9}$$

is the positive distributional Laplacian and $\frac{\partial u}{\partial \nu}$ is to be understood in the sense of Definition 2.3.1(2). The associated eigenvalue problem reads

$$-\Delta u = \lambda u \qquad \text{in } \Omega, \tag{2.3.10a}$$

$$\frac{\partial u}{\partial \nu} + \alpha u = 0 \qquad \text{on } \partial\Omega; \tag{2.3.10b}$$

the Robin boundary condition is also referred to as *third type*.

We want to mention that the mathematical community does not agree on whether this boundary condition should in fact be called *Robin condition* since it was never used by the namesake Victor Gustave Robin, see *Third Boundary Condition - Was it Robin's?* [66]. We wish to prove that the operator $\mathcal{A}(\alpha) = -\Delta^\alpha_\Omega$ from Definition 2.3.2 is indeed the one associated to a_α.

Lemma 2.3.3. *The operator on* $L^2(\Omega)$ *associated to* a_α *is defined by*

$$D(-\Delta_\Omega^\alpha) = \Big\{u \in H^1(\Omega) \;:\; \Delta u \in L^2(\Omega) \quad \text{and } \frac{\partial u}{\partial \nu} \in L^2(\partial\Omega) \text{ with } \frac{\partial u}{\partial \nu} + \alpha u = 0\Big\}, \tag{2.3.11a}$$

$$-\Delta_\Omega^\alpha u = -\Delta u. \tag{2.3.11b}$$

For the proof, we follow the steps of [14, Proof of Theorem 2.3].

Proof. Let $\tilde{\mathcal{A}}(\alpha)$ be the operator on $L^2(\Omega)$ associated to a_α. *Step 1: We show that* $D(\tilde{\mathcal{A}}(\alpha)) \subset D(\mathcal{A}(\alpha))$ *and* $\mathcal{A}(\alpha)u = \tilde{\mathcal{A}}(\alpha)$ *on* $D(\tilde{\mathcal{A}}(\alpha))$*.* Fix any $u \in D(\tilde{\mathcal{A}}(\alpha))$, let $\tilde{\mathcal{A}}(\alpha)u = f \in L^2(\Omega)$, and let $v \in H^1(\Omega)$ be arbitrary. Then we have

$$\int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x + \alpha \int_{\partial\Omega} u\overline{v}\,\mathrm{d}\sigma(x) = a_\alpha[u, v] \tag{2.3.12a}$$

$$= (\tilde{\mathcal{A}}(\alpha)u, v) \tag{2.3.12b}$$

$$= \int_\Omega f\overline{v}\,\mathrm{d}x \tag{2.3.12c}$$

for all $v \in H^1(\Omega)$ by definition of the relation $a_\alpha \sim \tilde{\mathcal{A}}(\alpha)$. We may choose any $v \in C_c^\infty(\Omega)$ in (2.3.12) to obtain $f = -\Delta u$. Using this, we immediately obtain

$$\int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x + \int_\Omega (\Delta u)\overline{v}\,\mathrm{d}x = \int_{\partial\Omega} (-\alpha u)\overline{v}\,\mathrm{d}\sigma(x) \tag{2.3.13}$$

for all $v \in H^1(\Omega)$, that is, $\partial_\nu u$ exists in the weak sense of Definition 2.3.1(2), and it satisfies $\partial_\nu u = -\alpha u$, or equivalently,

$$\frac{\partial u}{\partial \nu} + \alpha u = 0. \tag{2.3.14}$$

Step 2 (converse of Step 1): we prove that $D(\mathcal{A}(\alpha)) \subset D(\tilde{\mathcal{A}}(\alpha))$ *and* $\tilde{\mathcal{A}}(\alpha)u = \mathcal{A}(\alpha)$ *on* $D(\mathcal{A}(\alpha))$*.* To do this, let $u \in D(\mathcal{A}(\alpha))$ as in Definition 2.3.2; we observe that

$$\int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x + \int_\Omega (\Delta u)\overline{v}\,\mathrm{d}x = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\,\overline{v}\,\mathrm{d}\sigma(x) = \int_{\partial\Omega} (-\alpha u)\overline{v}\,\mathrm{d}\sigma(x) \tag{2.3.15}$$

for all $v \in H^1(\Omega)$. Rearrangement yields

$$a_\alpha[u, v] = -\int_\Omega (\Delta u)\overline{v}\,\mathrm{d}x \tag{2.3.16}$$

for all $v \in H^1(\Omega)$ and that is, by definition 2.3.7, exactly $u \in D(\tilde{\mathcal{A}}(\alpha))$, as well as

$$\tilde{A}(\alpha)u = -\Delta u = -\Delta_\Omega^\alpha u = \mathcal{A}(\alpha)u. \tag{2.3.17}$$

□

For more details on this topic we refer to [12, Section 2] or [33, Section 1]. We use the Robin form to define the Dirichlet and Neumann Laplacians, respectively.

Definition 2.3.4. If $\alpha = 0$, then we write $-\Delta_\Omega^N$ in place of $-\Delta_\Omega^0$ for the operator on $L^2(\Omega)$ associated with the *Dirichlet form* a_0 on its form domain $D(a_0) = H^1(\Omega)$. The operator $-\Delta_\Omega^N$ is called the *Neumann Laplacian*. The associated eigenvalue problem reads

$$-\Delta u = \lambda u \qquad \text{in } \Omega, \tag{2.3.18a}$$

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega; \tag{2.3.18b}$$

the Neumann boundary condition is also referred to as *second type*.

In fact, the following definition is usually presented as a theorem where (in a similar manner is for the Robin Laplacian) it is proved that the given domain coincides with the one of the operator that is associated to the particular form. However, the Dirichlet Laplacian is well known and we omit the proof (cf. [17, Section 3]).

Definition 2.3.5. If a_0 is restricted to $H_0^1(\Omega) \times H_0^1(\Omega)$, then we call the associated operator the *Dirichlet Laplacian*, which we denote by $-\Delta_\Omega^D$; that is,

$$D(-\Delta_\Omega^D) = \left\{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\right\}, \tag{2.3.19a}$$

$$-\Delta_\Omega^D u = -\Delta u. \tag{2.3.19b}$$

The associated eigenvalue problem reads

$$-\Delta u = \lambda u \qquad \text{in } \Omega, \tag{2.3.20a}$$

$$u = 0 \qquad \text{on } \partial\Omega; \tag{2.3.20b}$$

the Dirichlet boundary condition is also referred to as *first type*.

The following theorem is well known and we omit its proof. The properties listed mostly follow from the fact that the resolvent $R_z(\mathcal{A})$ (for $\mathcal{A} = -\Delta_\Omega^D$ or $\mathcal{A} = -\Delta_\Omega^N$) is a compact operator and thus, the spectral theorem, namely Theorem 2.2.11, is applicable.

Theorem 2.3.6. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a bounded Lipschitz domain. The operators $-\Delta_\Omega^D$ and $-\Delta_\Omega^N$*

(1) both have compact resolvent,

(2) are m-accretive, self-adjoint and semi-bounded from below in $L^2(\Omega)$.

(3) Their spectra

$$\sigma(-\Delta_\Omega^D) \subset (0, \infty) \qquad \text{and} \qquad \sigma(-\Delta_\Omega^N) \subset [0, \infty) \tag{2.3.21}$$

are discrete, consisting only of eigenvalues of finite multiplicity, whose algebraic and geometric multiplicities always coincide, and with $+\infty$ as their only point of accumulation.

Remark 2.3.7. Due to the fact that

$$D(a_0) = H_0^1(\Omega) \subset H_0^1(\Omega') \tag{2.3.22}$$

(extend any function $H_0^1(\Omega)$ to Ω' by zero) for any $\Omega \subset \Omega'$, it is immediate by the max-min principle, see Theorem 2.2.13, that the Dirichlet eigenvalues satisfy the so called *domain monotonicity*

$$\lambda_k^D(\Omega') \leq \lambda_k^D(\Omega) \tag{2.3.23}$$

for all $k \in \mathbb{N}$.

2.3.1 Robin eigencurves in one dimension

To gain insight into what to expect in general, we want to consider the one dimensional case of the Laplace eigenvalue problem with real Robin boundary conditions. Since there are only two boundary points $\{\pm 1\}$ of $\Omega = (-1, 1)$, the Robin problem reads

$$-u'' = \lambda u \qquad \text{on } (-1,1), \tag{2.3.24a}$$
$$-u'(-1) + \alpha u(-1) = 0, \tag{2.3.24b}$$
$$+u'(+1) + \alpha u(+1) = 0. \tag{2.3.24c}$$

Note that we always consider the outer normal derivative. It is well known that the general solution of (2.3.24) is given by

$$u(x) = C_+ \cos(\sqrt{\lambda} x) + C_- \sin(\sqrt{\lambda} x) \tag{2.3.25}$$

with coefficients $C_-, C_+ \in \mathbb{R}$. The first derivatives with respect to x

$$u'(x) = -C_+ \sqrt{\lambda} \sin(\sqrt{\lambda} x) + C_- \sqrt{\lambda} \cos(\sqrt{\lambda} x) \tag{2.3.26}$$

at the boundary points $x = \mp 1$ read

$$-u'(-1) = -C_+ \sqrt{\lambda} \sin(\sqrt{\lambda}) - C_- \sqrt{\lambda} \cos(\sqrt{\lambda}) \tag{2.3.27a}$$
$$\text{and} \qquad u'(+1) = -C_+ \sqrt{\lambda} \sin(\sqrt{\lambda}) + C_- \sqrt{\lambda} \cos(\sqrt{\lambda}), \tag{2.3.27b}$$

respectively. Then the boundary conditions (2.3.24b) and (2.3.24c) are equivalent to

$$\sin(\sqrt{\lambda}) \Big[- C_+ \sqrt{\lambda} - \alpha C_- \Big] + \cos(\sqrt{\lambda}) \Big[- C_- \sqrt{\lambda} + \alpha C_+ \Big] = 0, \tag{2.3.28a}$$
$$\sin(\sqrt{\lambda}) \Big[- C_+ \sqrt{\lambda} + \alpha C_- \Big] + \cos(\sqrt{\lambda}) \Big[C_- \sqrt{\lambda} + \alpha C_+ \Big] = 0, \tag{2.3.28b}$$

which, by addition and subtraction, respectively, implies

$$-C_+ \sqrt{\lambda} \sin(\sqrt{\lambda}) + \alpha C_+ \cos(\sqrt{\lambda}) = 0, \tag{2.3.29a}$$
$$-\alpha C_- \sin(\sqrt{\lambda}) - C_- \sqrt{\lambda} \cos(\sqrt{\lambda}) = 0. \tag{2.3.29b}$$

We arrive at

$$\alpha = \sqrt{\lambda}\tan\sqrt{\lambda} \qquad \text{for } \lambda \neq \frac{\pi^2}{4}(2j+1)^2, \tag{2.3.30a}$$

$$\text{or} \qquad \alpha = -\sqrt{\lambda}\cot\sqrt{\lambda} \qquad \text{for } \lambda \neq \pi^2 j^2 \tag{2.3.30b}$$

for any $j \in \mathbb{N}_0$ (for $j = 0$, that is, $\lambda \to 0^+$, equation (2.3.30b) can be continuously extended to $\alpha = -1$). We consider the following two cases.

(1) If $\lambda = \pi^2 j^2$ for some $j \in \mathbb{N}_0$, or in other words $\sin(\sqrt{\lambda}) = 0$ and $|\cos(\sqrt{\lambda})| = 1$, then the equations (2.3.28) lead to

$$-C_-\sqrt{\lambda} + \alpha C_+ = 0 = C_-\sqrt{\lambda} + \alpha C_+, \tag{2.3.31}$$

which implies $C_- = 0$ due to the *outer* equations and consequently $C_+ = 0$ for $\alpha \neq 0$ due to the *inner* equation. We arrive at $u \equiv 0$, a contradiction.

(2) If $\lambda = (2j+1)^2\pi^2/4$ for some $j \in \mathbb{N}_0$, or in other words $\cos(\sqrt{\lambda}) = 0$ and $|\sin(\sqrt{\lambda})| = 1$, then a similar argument yields $u \equiv 0$, as well.

This approach allows us to make some observations based on the latter calculations and on Figure 2.3.1.

(1) For each value of λ, which is not an eigenvalue of the corresponding Dirichlet problem, that is,

$$\lambda \neq \lambda_j^D = \frac{j^2\pi^2}{4} \tag{2.3.32}$$

for $j \in \mathbb{N}$, there are exactly two corresponding values of α.

(2) The eigenvalues are (strictly) monotonically increasing in α for all $\alpha \in \mathbb{R}$.

(3) There are no crossings of the positive eigenvalue curves.

(4) If $\alpha < 0$ is sufficiently large, then there are exactly two (mostly referred to as $\lambda_1(\alpha) \leq \lambda_2(\alpha)$) negative eigenvalues, which seem to diverge to $-\infty$ as $\alpha \to -\infty$ and these are the only branches which are unbounded in λ; for more details and generalisations, see Section 2.4.1 for real parameters and Section 4.1.2 for the complex case. Every other eigenvalue converges to some point of the Dirichlet spectrum as $\alpha \to -\infty$.

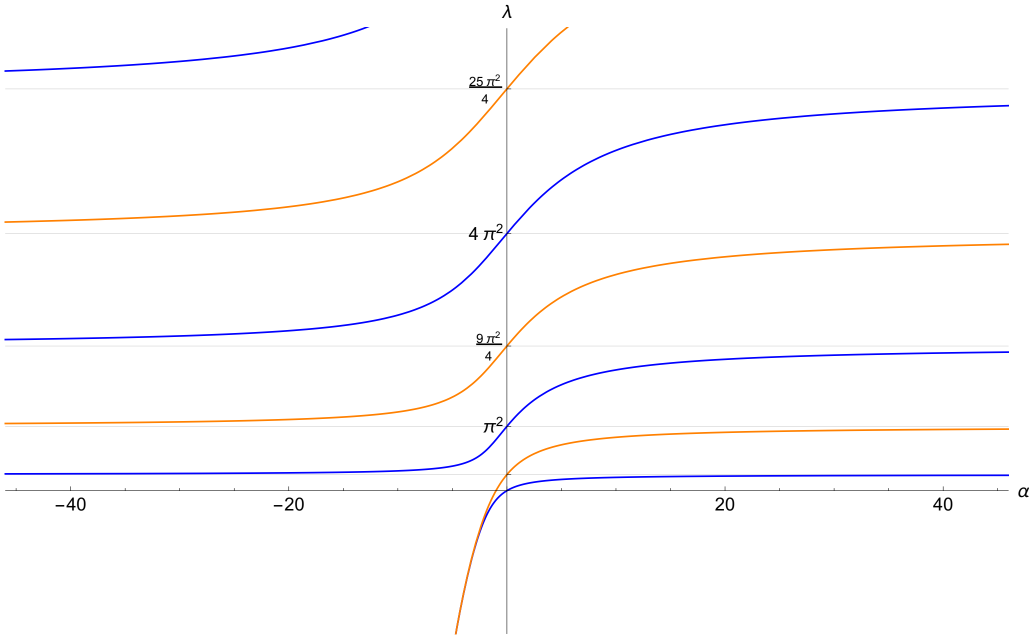

Figure 2.3.1: Plot of the eigenvalues λ of the Robin problem (2.3.28) as functions of $\alpha \in \mathbb{R}$: the blue graph originates from (2.3.30a), the orange one from (2.3.30b); the horizontal grid lines represent the corresponding exception points (2.3.30).

(5) If $\alpha = 0$, then the eigenvalues $\lambda(0)$ are given by $(\pi j)^2/4$ for $j \in \mathbb{N}_0$, that is, exactly the eigenvalues of the Neumann Laplacian on $(-1, 1)$.

(6) If $\alpha > 0$, then every eigenvalue is positive and each of them converges to some point of the Dirichlet spectrum.

2.4 Eigenvalue asymptotics

For a fixed domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, we consider the Robin eigenvalue problem for a real parameter $\alpha \in \mathbb{R}$ and ask the question as to what happens to the eigenvalues in $\sigma(-\Delta_\Omega^\alpha) = \{\lambda_k(\alpha) : k \in \mathbb{N}\}$ as α tends to $\pm\infty$ in $\mathbb{R}$.

2.4.1 Large negative parameter

We want to ignore the term *large* for a brief moment and assume that $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. If we use the constant test function $u \equiv 1$ (note that this cannot be an eigenfunction for $\alpha < 0$) in the variational max-min characterisation of the first eigenvalue (2.2.20), we immediately obtain

$$\lambda_1(\alpha) < \alpha \frac{|\partial\Omega|}{|\Omega|} \tag{2.4.1}$$

for any $\alpha < 0$, that is, a first trivial estimate from above; in particular, we have $\lambda_1(\alpha) < 0$ for any $\alpha < 0$. Assume that $\Omega \subset \{x_1 > 0\}$ is contained in the upper half-space. D. Daners and J. B. Kennedy [45, Lemma 2.1 and Remark 2.2] used the test function $u(x) = \mathrm{e}^{-\alpha x}$ to obtain an estimate from above which originated in the article [63] by T. Giorgi and R. Smits from 2007: for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, we have

$$\lambda_1(\alpha) < -\alpha^2 \tag{2.4.2}$$

for any $\alpha \leq 0$. We wish to point out that there is a fundamental difference when considering the nth eigencurve $\lambda_n(\alpha)$ (counted accordingly to multiplicity) or an analytic branch: in general the nth eigenvalue may be *overtaken* by the $(n+1)$st eigenvalue as $\alpha \to -\infty$. These *crossing points* permute the order of the eigenvalues $\lambda_1(\alpha) \leq \lambda_2(\alpha) \leq \ldots$ as exemplified in Figure 2.4.2; however, when considering the (possibly) non-smooth eigencurves, we have the following theorem [36, Proposition 4.8].

Theorem 2.4.1. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain. Then for any fixed $n \geq 1$, we have $\lambda_n(\alpha) \to -\infty$ as $\alpha \to -\infty$.*

Smooth domains

So, to obtain an upper estimate on the first eigenvalue with a test function argument is straight forward, however, proving the corresponding lower estimate is profoundly harder: the first approach to get an asymptotic expansion of the principal eigenvalue as $\alpha \to -\infty$ is by A. A. Lacey, J. R. Ockendon, and J. Sabina from 1998 [85]. They proved that $\lambda_1(\alpha)$ behaves asymptotically like $-\alpha^2$ if the domain Ω is somewhat similar to a ball in $\mathbb{R}^d$; to be more precise, let $\Omega \in \mathbb{R}^d$ be a bounded C^k domain for $k \geq 2$ such that there exists a C^k diffeomorphism h_0 (that is, h_0^{-1} exists and is also of class C^k),

$$h_0 : S^{d-1} \to \partial\Omega, \tag{2.4.3}$$

where $S^{d-1} \subset \mathbb{R}^d$ is the $(d-1)$-sphere in d dimensions. In particular, this assumption implies that the annular region

$$\Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \varepsilon\} \tag{2.4.4}$$

can be mapped onto the spherical annulus

$$D_\varepsilon := B_1(0) \setminus \overline{B_{1-\varepsilon}(0)} \tag{2.4.5}$$

for some small $\varepsilon > 0$: for example, let $y = rw$ be the representation of $y \in D_\varepsilon$ in spherical coordinates, that is,

$$1 - \varepsilon < r < 1 \qquad \text{and} \qquad \omega \in S^{d-1}, \tag{2.4.6}$$

and let $\nu(x)$ be the outer unit normal to Ω, as well as h_0 as in 2.4.3. Then we use the transformation $x = h(y)$ given by

$$h(y) = h_0(\omega) + (r-1)\nu(h_0(\omega)). \tag{2.4.7}$$

The transformation is depicted in Figure 2.4.1.

For more details on the following theorem and its proof, we refer to [85, Theorem 2.2].

Theorem 2.4.2. *Let $\Omega \in \mathbb{R}^d$, $d \geq 2$, be equivalent to a sphere as described above. Then, we have $\lambda_1(\alpha) = -\alpha^2 + o\left(\alpha^2\right)$ as $\alpha \to -\infty$.*

Note that the proof of Theorem 2.4.2 can be used verbatim if Ω consists of finitely many connected components, each of which being equivalent to a sphere. Several years later in 2004, the result was generalised by J. Lou and M. Zhu. Not only did they state Theorem 2.4.2 for generic domains of class C^1 [90, Theorem 1.1], but they allowed α to depend on the position of the boundary, that is, if we replace α by $\alpha b(x)$ for some continuous function $b : \partial\Omega \to \mathbb{R}$ and denote by b_+ the non-negative

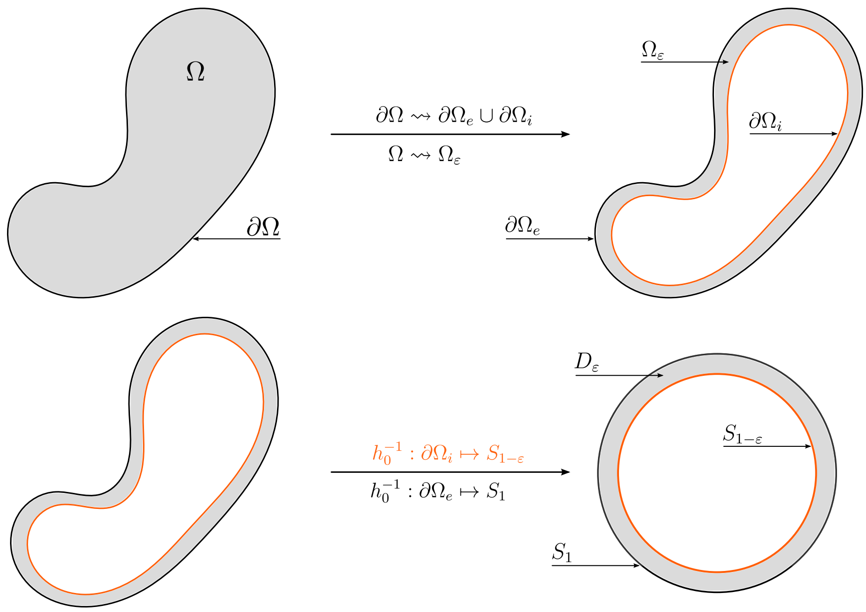

Figure 2.4.1: We use the C^k diffeomorphism h_0 to map the annular region Ω_ε onto the spherical annulus D_ε.

part of b, we have

$$\lambda_1(\alpha) = -\alpha^2 \left(\max_{x \in \partial\Omega} b_+(x) \right)^2 + o\left(\alpha^2\right) \tag{2.4.8}$$

as $\alpha \to -\infty$ [90, Remark 1.1]. The interest in this topic did not slacken and so the main result of Lou and Zhu was further generalised by D. Daners and J. B. Kennedy in 2010 to be applicable for higher eigenvalues [45, Theorem 1.1]:

Theorem 2.4.3. *Let $\Omega \in \mathbb{R}^d$, $d \geq 2$, be a bounded domain of class C^1. Then for every $n \in \mathbb{N}$ we have*

$$\lambda_n(\alpha) = -\alpha^2 + o\left(\alpha^2\right) \tag{2.4.9}$$

as $\alpha \to -\infty$.

Remark 2.4.4. Note that in this theorem we do not follow what we will later on call *analytic eigencurves*: the nth eigenvalue might be *overtaken* by the λ_{n-1} at some distinct threshold $\alpha = \alpha_0$, that is

$$\lambda_n(\alpha_0) = \lambda_{n-1}(\alpha_0). \tag{2.4.10}$$

These possible crossing points imply a permutation of the elements of $\sigma(-\Delta_\Omega^\alpha) = \{\lambda_k(\alpha) : k \in \mathbb{N}\}$ as depicted in Figure 2.4.2.

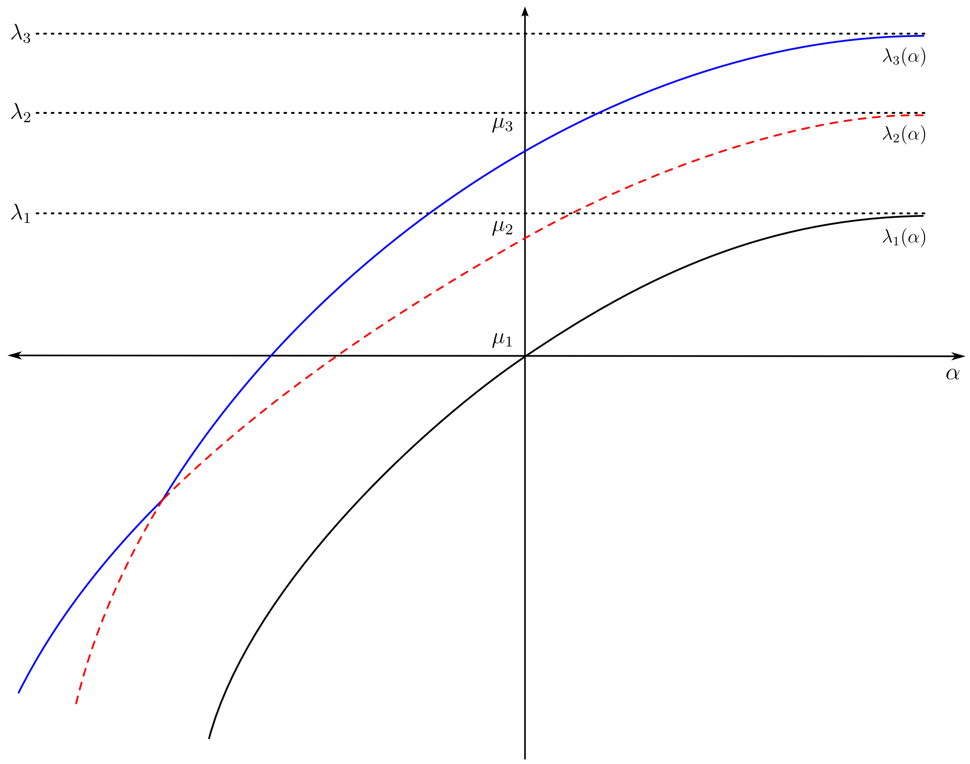

Figure 2.4.2: Crossing points of the eigencurves, where λ_i, μ_i, and $\lambda_i(\alpha)$ denote the Dirichlet, Neumann, and Robin eigenvalues, respectively.

Besides, there are results for the second term asymptotics of the principal eigenvalue for $\alpha \to -\infty$: P. Freitas and D. Krejčiřík proved the following version for balls and spherical shells (annular regions), see [59, Theorem 3].

Theorem 2.4.5. *Given positive numbers $0 < r < R$ we denote by*

$$D_{r,R} := \{x \in \mathbb{R}^d : r < |x| < R\} \tag{2.4.11}$$

the spherical annulus of width $R - r$ and centre 0. Then we have the asymptotics

$$\lambda_1[D_{r,R}](\alpha) = -\alpha^2 + \frac{d-1}{R}\alpha + o(\alpha), \tag{2.4.12a}$$

$$\lambda_1[B_R](\alpha) = -\alpha^2 + \frac{d-1}{R}\alpha + o(\alpha) \tag{2.4.12b}$$

as $\alpha \to -\infty$.

This result for the second term asymptotics was further generalised by P. Exner, A. Minakov, and L. Parnovski in 2014 [54, Theorem 1.3] and again in 2015 by K. Pankrashkin and N. Popoff [100, Theorem 1]. Here, we want to give the latter result for C^3 and C^4 domains, respectively. We denote by $\kappa_{\max} = \kappa_{\max}(\Omega)$ the maximum mean curvature of the boundary $\partial\Omega$ of a C^3 domain. Since the sign of the mean curvature depends on the choice of the normal vector to $\partial\Omega$ as well as on the sign convention for the Weingarten tensor, we specify that here a convex domain does always have positive mean curvature.

Theorem 2.4.6. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be of class C^3, then for any fixed $k \in \mathbb{N}$ we have*

$$\lambda_k(\alpha) = -\alpha^2 + (d-1)\kappa_{\max}\alpha + \mathcal{O}\left(\alpha^{2/3}\right) \tag{2.4.13}$$

as $\alpha \to -\infty$. If, additionally, the domain is of class C^4, then the remainder estimate can be replaced by $\mathcal{O}\left(\alpha^{1/2}\right)$.

This result may be used to consider conjectures regarding a *reverse Faber-Krahn inequality*; M. Bareket conjectured that for any $\alpha < 0$ the ball maximises the first eigenvalue of the Robin problem among all domains of the same volume [19, Section 1]. Considering a counterexample from [59], let B_r be a ball of radius $r > 0$ and $D_{r,R}$ be as defined in Theorem 2.4.5 of the same volume as B_r, we have

$$\kappa_{\max}(B_r) = \frac{1}{r} > \frac{1}{R} = \kappa_{\max}(D_{r,R}) \tag{2.4.14}$$

and hence

$$\lambda_1[B_r](\alpha) < \lambda_1[D_{r,R}](\alpha) \tag{2.4.15}$$

for sufficiently large $\alpha < 0$ by Theorem 2.4.6.

Lipschitz domains

In this section we want to discuss the effect of the domain having corners on the first term of the asymptotics as $\alpha \to -\infty$. For the half-angle $0 < \delta \leq \pi$ we define the planar sector

$$U_\delta := \left\{(r, \theta) = z \in \mathbb{R}^2 : |\theta| < \delta\right\}, \tag{2.4.16}$$

where the functions $u(x, y) = u(r, \theta)$ shall be denoted by the common notation in Cartesian coordinates for $r > 0$ and $\theta \in [0, 2\pi)$. The two major cases, namely sectors of acute and obtuse angles, are depicted in Figure 2.4.3. If $\Omega = U_\delta$, M. Levitin and

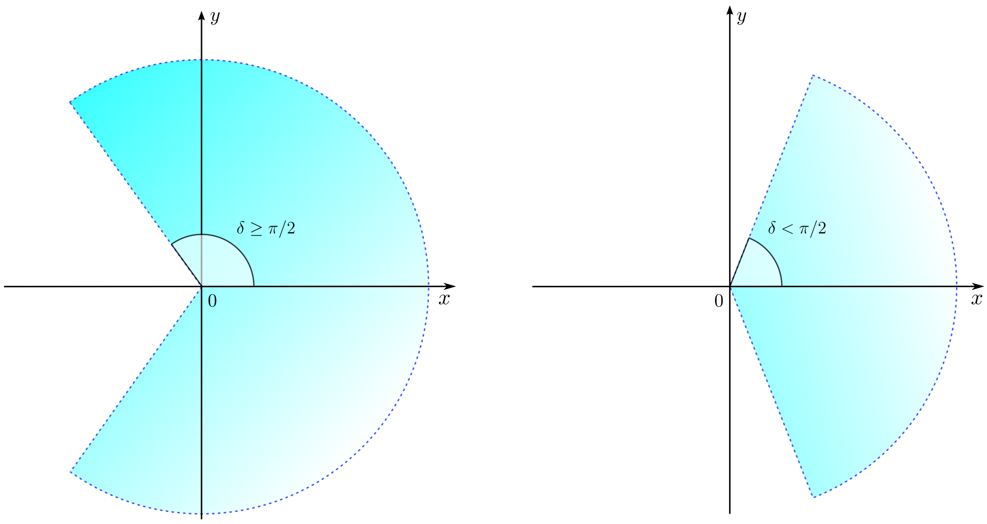

Figure 2.4.3: Planar sectors U_δ for two different choices of δ. Note that, unlike depicted here, U_δ is not truncated but an infinite sector in $\mathbb{R}^2$.

L. Parnovski showed in [88] by construction of the corresponding eigenfunction that the principal eigenvalue satisfies

$$\lambda_1[U_\delta](\alpha) = -\frac{\alpha^2}{\sin^2\delta} \quad \text{if } 0 < \delta < \frac{\pi}{2}, \tag{2.4.17a}$$

$$\lambda_1[U_\delta](\alpha) = -\alpha^2 \quad \text{if } \frac{\pi}{2} \leq \delta < \pi. \tag{2.4.17b}$$

Their main result, however, is the following asymptotical expression where the leading coefficient equals the principal eigenvalue of a model cone K_y which is C^∞-diffeomorphic to the "most acute" corner(s) of the domain Ω, cf. Figure 2.4.4. To this end, Ω is assumed to be piecewise smooth and to satisfy a *uniform interior cone condition* and for reasons of clarity we refer to [88, Theorem 3.2] for more details on the assumptions on Ω and for the proof.

Theorem 2.4.7. *Assume that the domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$,*

(1) is piecewise smooth,

(2) satisfies the interior cone condition,

(3) satisfies the existence of a C^∞-diffeomorphism f_y for each $y \in \partial\Omega$ as described above and depicted in Figure 2.4.4.

Then we have

$$\lambda_1(\alpha) = -C_\Omega \alpha^2 + o(\alpha^2) \tag{2.4.18}$$

as $\alpha \to -\infty$, where

$$C_\Omega := \sup_{y \in \partial\Omega} C_y = \sup_{y \in \partial\Omega} -\lambda_1[K_y](-1) > 0. \tag{2.4.19}$$

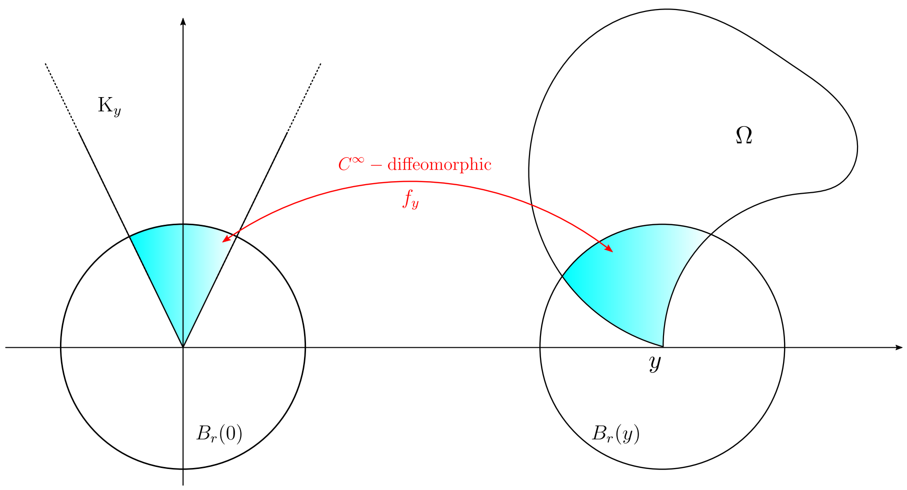

Figure 2.4.4: C^∞-diffeomorphism f_y from the model cone K_y onto the corresponding corner of Ω.

If y is a corner point but the boundary is somewhat *bent inside* such that there exists a hyperplane $H_y \ni y$ and a radius $r > 0$ with

$$B_r(y) \cap H_y \subset \overline{\Omega}, \tag{2.4.20}$$

cf. Figure 2.4.4, then we have $C_y = 1$. The same holds if Ω is smooth at y [88, Theorem 3.5]. Furthermore, the same authors generalise this result for a variable Robin parameter: if α is replaced by $\alpha w(y)$ for a boundary weight

$$w : \partial\Omega \to \mathbb{R} \qquad \text{with} \qquad \sup_{y\in\partial\Omega} w(y) > 0, \tag{2.4.21}$$

then the coefficient C_Ω of Theorem 2.4.7 is replaced by

$$C_\Omega := \sup_{\substack{y\in\partial\Omega \\ w(y)>0}} w(y)^2 C_y, \tag{2.4.22}$$

see [88, Remark 3.3].

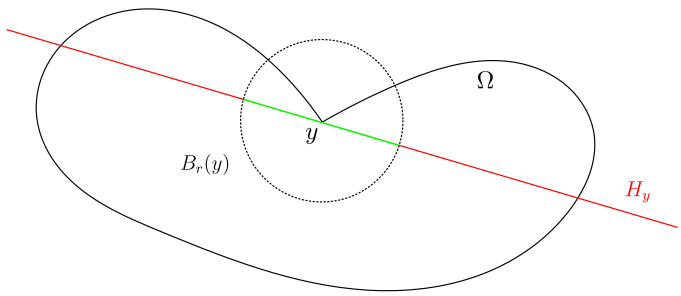

Figure 2.4.5: A domain Ω where $\partial\Omega$ is bent inwardly at a corner point $y \in \partial\Omega$ such that the green segment of the hyperplane H_y is fully contained in $\overline{\Omega}$.

2.4.2 Large positive parameter

From the variational characterisation, see Theorem 2.2.13, it is clear that each $\lambda_n(\,\cdot\,)$ is an increasing function of $\alpha \in \mathbb{R}$ (in fact, the eigenvalues are strictly increasing, cf. [106, Theorem 3.2]): if we assume that $\lambda_n(\alpha)$ is an algebraically simple (that is, the dimension of the associated spectral projection) eigenvalue for some α and if we denote the corresponding eigenfunction by ψ_n, then the derivative of the eigenvalue with respect to α is given by

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\lambda_n(\alpha) = \frac{\int_{\partial\Omega} \psi_n^2 \,\mathrm{d}\sigma}{\int_{\Omega} \psi_n^2 \,\mathrm{d}x} \geq 0. \tag{2.4.23}$$

This expression for the derivative is due to [7, Lemma 11] for $\alpha > 0$, however, the proof works verbatim for all $\alpha \in \mathbb{R}$. We will generalise this result in Section 3.2.2. Furthermore, by [57, Theorem 1] and the following arguments we can follow that there exists an infinite number of crossing points of the Robin eigenvalues with the α axis, cf. Figure 2.4.2. Indeed, let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain. It is known that, as long as the trace operator $H^1(\Omega) \to L^2(\partial\Omega)$ is compact, the spectrum of the *Steklov problem*

$$-\Delta u = 0 \quad \text{in } \Omega, \tag{2.4.24a}$$

$$\frac{\partial u}{\partial \nu} - \tau u = 0 \quad \text{on } \partial\Omega \tag{2.4.24b}$$

is purely discrete and consists only of eigenvalues $\{\tau_j : j \in \mathbb{N}\}$ [13, Section 2]. Each τ_j corresponds to a zero eigenvalue of the Robin problem for the parameter $\alpha = -\tau_j$ which proves the assertion. This together with the monotonicity (2.4.23) implies that each of these eigenvalues must remain negative for $\alpha < -\tau_j$ and positive for $\alpha > -\tau_j$. Back to the case of positive α, if we assume that Ω is of class C^2, then each Robin eigenvalue converges to some point of the Dirichlet spectrum, that is, the spectrum of the Dirichlet operator $\sigma(-\Delta_\Omega^D)$, as $\alpha \to +\infty$, cf. [57, Theorem 2].

Theorem 2.4.8. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain of class C^2. Then the Robin eigenvalues $\lambda_k(\alpha)$, $k = 1, 2, \ldots$, satisfy*

$$0 \leq \lambda_k^D - \lambda_k(\alpha) \leq \frac{C\left(\lambda_k^D\right)^2}{\sqrt{\alpha}} \tag{2.4.25}$$

for all $\alpha > 0$. The constant C does not depend on k.

In particular, we have $\lambda_k(\alpha) \to \lambda_k^D$ as $\alpha \to +\infty$ which justifies the regime of Figure 2.4.2 for $\alpha > 0$.

Chapter 3

Spectral analysis and basis properties of the Robin Laplacian

Contents

In this chapter we will formally introduce the Robin Laplacian, denoted by $-\Delta_\Omega^\alpha$, on bounded domains $\Omega \subset \mathbb{R}^d$ and establish regularity and basis properties of the eigenvalues (as functions of the complex parameter α) and eigenfunctions, respectively. Moreover, we will prove a localisation theorem for the numerical range in order to give estimates on the whole Robin spectrum and hence on the eigenvalues of $-\Delta_\Omega^\alpha$. The Dirichlet-to-Neumann operator will be introduced whose duality to the Robin eigenvalue problem will be exploited to (1) prove statements on the points of accumulation of the Robin eigenvalues and (2) to give explicit calculations in Chapter 4.

3.1 The Laplace operator and its boundary value problems

The Robin Laplacian, as it will turn out, acts as a cross between both the Dirichlet and the Neumann Laplacian. The following lemma is key to establishing its properties.

Lemma 3.1.1. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a bounded Lipschitz domain and $\alpha \in \mathbb{C}$. The Robin form a_α given by* (2.3.2) *is bounded in $H^1(\Omega)$ and sectorial of semi-angle θ for any $0 < \theta < \pi/2$.*

Proof. Fix any $\alpha \neq 0$ (if $\alpha = 0$, the proof is trivial since the form a_0 is associated to the self-adjoint Neumann Laplacian). Due to the trace theorem 2.1.14 for $s = 3/4$ and the Sobolev embedding theorem by Rellich 2.1.10 we have that the map given by the composition

$$H^1(\Omega) \overset{\Subset}{\hookrightarrow} H^{3/4}(\Omega) \xrightarrow{\text{tr}} H^{1/4}(\partial\Omega) \hookrightarrow L^2(\partial\Omega) \tag{3.1.1}$$

is compact and thus bounded. This implies

$$|a_\alpha[u]| = \left| \int_\Omega |\nabla u|^2 \, \mathrm{d}x + \alpha \int_{\partial\Omega} |u|^2 \, \mathrm{d}\sigma(x) \right| \tag{3.1.2a}$$

$$\leq \|\nabla u\|^2_{L^2(\Omega)} + |\alpha| \|u\|^2_{L^2(\partial\Omega)} \tag{3.1.2b}$$

$$\leq C_\alpha \|u\|^2_{H^1(\Omega)} \tag{3.1.2c}$$

for all $u \in H^1(\Omega)$, that is, the Robin form a_α is well defined and bounded on $H^1(\Omega) \times H^1(\Omega)$. Sectoriality then follows immediately from the definition of a_α. □

The following theorem now follows from Kato's first representation theorem, see Theorem 3.2.2, [74, Corollary VI.2.3], plus the fact that the form domain $H^1(\Omega)$ is densely and compactly embedded in $L^2(\Omega)$ since Ω is a bounded Lipschitz domain: see Theorem 2.1.10 for the compactness and for the density statement we use the fact that the space of test functions $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$. The following theorem is due to [30, Theorem 3.3].

Theorem 3.1.2. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a bounded Lipschitz domain and $\alpha \in \mathbb{C}$. The operator $-\Delta_\Omega^\alpha$ is*

(1) semi-bounded from below in $L^2(\Omega)$,

(2) locally uniformly (in $\alpha \in \mathbb{C}$) m-sectorial of semi-angle θ for any $0 < \theta < \pi/2$,

(3) densely defined on $H^1(\Omega)$, and

(4) its spectrum $\sigma(-\Delta_\Omega^\alpha)$ is discrete, consisting of eigenvalues of finite algebraic multiplicity, with their only point of accumulation being $\infty \in \overline{\mathbb{C}}$;

(5) it is self-adjoint if and only if $\alpha \in \mathbb{R}$ and

(6) for any given $\alpha \in \mathbb{R}$, its eigenfunctions may be chosen to form an orthonormal basis of $L^2(\Omega)$.

We also briefly state for the record a result (see [30, Theorem 3.4]) on the generation properties of holomorphic semigroups. We will not need this here, so we do not go into any details.

Theorem 3.1.3. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a bounded Lipschitz domain and $\alpha \in \mathbb{C}$. The operator Δ_Ω^α generates a holomorphic C_0-semigroup of operators of semi-angle θ, for any $0 < \theta < \pi/2$.*

For more on holomorphic semigroups, including their definition, see [11, Chapter 3].

Proof. This follows immediately from the resolvent estimate contained in the m-sectoriality assertion of Theorem 3.1.2, combined with Proposition 3.7.4 and Theorem 3.7.11 of [11]. □

Since we are interested in the spectral properties of the Robin Laplacian and for future reference, we state explicitly the weak form of the eigenvalue equation: $\lambda \in \mathbb{C}$ is an eigenvalue of the operator $-\Delta_\Omega^\alpha$, that is, of the boundary value problem

$$-\Delta u = \lambda u \qquad \text{in } \Omega, \tag{3.1.3a}$$

$$\frac{\partial u}{\partial \nu} + \alpha u = 0 \qquad \text{on } \partial\Omega, \tag{3.1.3b}$$

with eigenfunction ψ, if and only if

$$a_\alpha[\psi, v] = \int_\Omega \nabla\psi \cdot \overline{\nabla v}\,\mathrm{d}x + \int_{\partial\Omega} \alpha\psi\,\overline{v}\,\mathrm{d}\sigma(x) = \lambda \int_\Omega \psi\,\overline{v}\,\mathrm{d}x \tag{3.1.4}$$

for all $v \in H^1(\Omega)$. Finally, we briefly summarise what happens if α is allowed to be a function $\alpha : \partial\Omega \to \mathbb{C}$ instead of a constant. Even though this generalisation will not be deepened for domains in this work, it will still play a greater role as we consider quantum graphs in Section 5; there we will be considering compact quantum graphs, that is, α is a function from the set of vertices into $\mathbb{C}$.

Remark 3.1.4. If $\alpha \in L^\infty(\partial\Omega, \mathbb{C})$, then the sesquilinear form a_α may be defined in the same way (see (2.3.2)) and maintains its properties (in particular Lemma 3.1.1) due to the continued validity of the trace theorem and hence the estimate

$$\left|\int_{\partial\Omega} \alpha |u|^2 \,\mathrm{d}\sigma(x)\right| \leq \|\alpha\|_{L^\infty(\partial\Omega)} \|u\|^2_{L^2(\partial\Omega)} \tag{3.1.5a}$$

$$\leq \varepsilon \|\nabla u\|^2_{L^2(\Omega)} + C(\varepsilon, \|\alpha\|_{L^\infty(\partial\Omega)}) \|u\|^2_{L^2(\Omega)}. \tag{3.1.5b}$$

Consequently, Theorem 3.1.2 remains valid with the obvious modifications that $-\Delta_\Omega^\alpha$ is self-adjoint if and only if $\alpha(x) \in \mathbb{R}$ for all $x \in \partial\Omega$. The local uniform sectoriality of Theorem 3.1.2 depends only on $\|\alpha\|_{L^\infty(\partial\Omega)}$: indeed, for given semi-angle θ the vertex in the sectoriality estimate can be chosen in dependence only on the estimate given in (3.1.5). Theorem 3.1.3 then holds verbatim.

3.2 Holomorphic dependence of the Robin eigenvalues and eigenfunctions on the parameter

In this section we wish to study the dependence of the eigenvalues $\lambda(\alpha)$ and eigenprojections Q_λ of the Robin Laplacian $-\Delta_\Omega^\alpha$ on the parameter $\alpha \in \mathbb{C}$. This segment splits into two parts: in Section 3.2.1 we apply Kato's theory of *holomorphic families of operators* to show that there is a family of eigencurves (as functions of $\alpha \in \mathbb{C}$), each of them analytic apart from potential crossing points, which describe the totality of the spectrum for any fixed α. Moreover, we prove that the eigenprojections as operators on $L^2(\Omega)$ likewise depend analytically on α, again except at the crossing points. However, here caution is recommended: the *normalised* eigenfunctions themselves do not change analytically: see Theorem 3.2.14. Then, in Section 3.2.2, we obtain a formula for the derivative of an eigencurve with respect to α, at any point where the corresponding eigenprojection is one-dimensional, that is, where the eigenvalue is simple.

3.2.1 A holomorphic family of operators

As mentioned, we will start by applying Kato's theory, see [74, Chapter VII], to study the behaviour of the eigenvalues and eigenprojections of the Robin Laplacians $-\Delta_\Omega^\alpha$ in dependence on the parameter $\alpha \in \mathbb{C}$. As before, $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is a fixed bounded Lipschitz domain. We first recall some more basic theory. To this end, we use the framework of Section 2.2 to give important theorems about the relation between linear operators and their associated forms and we start with the *first representation theorem*, cf. [74, Theorem VI.2.1].

Definition 3.2.1. We call a linear submanifold $D' \subset D(a)$ a *core* of a sesquilinear form $a : D(a) \times D(a) \to \mathbb{C}$, if the restriction

$$a' : D' \times D' \subset D(a) \times D(a) \to \mathbb{C} \tag{3.2.1}$$

of a has the closure $\overline{a'} = a$.

Theorem 3.2.2. *Let $a : D(a) \times D(a) \subset H \to \mathbb{C}$ be a densely defined, closed, sectorial sesquilinear form on H. Then there exists an m-sectorial operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ such that*

(1) $D(\mathcal{A}) \subset D(a)$ and

$$a[u, v] = (\mathcal{A}u, v)_H \tag{3.2.2}$$

for all $u \in D(\mathcal{A})$ and $v \in D(a)$;

(2) if $u \in D(a)$, $w \in H$ and

$$a[u, v] = (w, v)_H \tag{3.2.3}$$

holds for every v belonging to a core of a, then $u \in D(\mathcal{A})$ and $\mathcal{A}u = w$. The m-sectorial operator $\mathcal{A}$ is uniquely determined by the condition (1).

The following two corollaries are due to [74, Corollary VI.2.2 and VI.2.3].

Corollary 3.2.3. *If we define the form $a' : D(\mathcal{A}) \to H$ for the operator $\mathcal{A} : D(\mathcal{A}) \to H$ by*

$$a'[u, v] := (\mathcal{A}u, v)_H, \tag{3.2.4}$$

then we have $\overline{a'} = a$.

Corollary 3.2.4. *The numerical range* $W(\mathcal{A})$ *is a dense subset of* $W(a)$.

Furthermore, we need to specify what we will be meaning by a holomorphic families of operators or forms. The following definitions are due to [74, Section VII.2.1].

Definition 3.2.5. Let X, Y be Banach spaces. A family of closed operators $\mathcal{A}(\alpha) : X \to Y$ for $\alpha \in D_0 \subseteq \mathbb{C}$ is called a *holomorphic family of type (A)* if

(1) the domain $D(\mathcal{A}(\alpha)) = D(\mathcal{A})$ is independent of α,

(2) $\mathcal{A}(\alpha)u$ is holomorphic for $\alpha \in D_0$ for every fixed $u \in D(\mathcal{A})$.

Definition 3.2.6. Let a_α be a family of sesquilinear forms on $D(a)$ for $\alpha \in D_0 \subseteq \mathbb{C}$. This family is said to be *holomorphic of type (a)* if

(1) each a_α is sectorial and closed with $D(a_\alpha) = d(a)$ independent of α and dense in H,

(2) $a_\alpha[u, u]$ is holomorphic for $\alpha \in D_0$ for each fixed $u \in d(a)$.

Remark 3.2.7. (1) If $\mathcal{A}(\alpha)$ is holomorphic of type (A), it is actually holomorphic, i.e. representable as a convergent power series.

(2) By using the polarisation formula, (2) from Definition 3.2.6 implies that $a_\alpha[u, v]$ is holomorphic in α for each fixed pair $u, v \in D(a)$.

Theorem 3.2.8. *Let* a_α *be a holomorphic family of forms of type (a). For each* α *let* $\mathcal{A}(\alpha) \sim a_\alpha$ *be the associated m-sectorial operator. Then* $\mathcal{A}(\alpha)$ *form a holomorphic family of operators and* $\mathcal{A}(\alpha)$ *are locally uniformly sectorial.*

Definition 3.2.9. We call the family $\mathcal{A}(\alpha)$ a *holomorphic family of type (B)* if $\mathcal{A}(\alpha)$ is holomorphic, m-sectorial, and $\mathcal{A}(\alpha) \sim a_\alpha$, where a_α is a holomorphic family of forms of type (a).

To emphasise the dependence on α and for ease of notation, from now on we will write

$$\mathcal{A}(\alpha) := -\Delta_\Omega^\alpha. \tag{3.2.5}$$

For an isolated eigenvalue λ of a linear operator $\mathcal{A}$ on a Hilbert space H, its *eigenprojection* Q_λ is defined as follows (see [74, Section III.6.5]).

Definition 3.2.10. Take a closed curve $\Gamma_\lambda \subset \rho(\mathcal{A})$ enclosing λ but no other point of $\sigma(\mathcal{A})$. Then,

$$Q_\lambda = -\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_\lambda} (\mathcal{A} - zI)^{-1} \, \mathrm{d}z \tag{3.2.6}$$

is a projection onto the algebraic eigenspace of λ in H. Besides, Q_λ is independent of the choice of Γ_λ.

We arrive at the following results; see [30, Theorem 4.1].

Theorem 3.2.11. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a bounded, Lipschitz domain and let $\mathcal{A}(\alpha)$, $\alpha \in \mathbb{C}$, be given by* (3.2.5).

(1) The operator family $\mathcal{A}(\alpha)$, $\alpha \in \mathbb{C}$, is self-adjoint holomorphic, i.e., $\mathcal{A}(\alpha)^* = \mathcal{A}(\overline{\alpha})$.

(2) Each eigenvalue $\lambda_k(\alpha)$ can be extended to a meromorphic function with at most algebraic singularities at non-real crossing points of eigenvalues. There are only finitely many eigenvalue curves meeting at locally finitely many crossing points. The same is true of the corresponding eigenprojections Q_λ and eigennilpotents $(\mathcal{A}(\alpha) - \lambda(\alpha))Q_{\lambda(\alpha)}$.

Before proving the latter theorem we follow [30, Section 4] in order to give two remarks on crossing points of the eigencurves and on the analytic continuation property of the spectrum for a fixed $\alpha_0 \in \mathbb{C}$.

Remark 3.2.12. Suppose that two different eigenvalue curves $\lambda_1(\alpha)$ and $\lambda_2(\alpha)$ meet at λ for $\alpha = \alpha_0$, i.e.

$$\lambda = \lambda_1(\alpha_0) = \lambda_2(\alpha_0). \tag{3.2.7}$$

Then, the corresponding separating curves $\Gamma_{\lambda_1(\alpha)}$, $\Gamma_{\lambda_2(\alpha)}$ in (3.2.6) do not exist in the limit $\alpha \to \alpha_0$. However, the holomorphic continuation of the total projection $\widehat{Q}_\lambda(\alpha) := Q_{\lambda_1(\alpha)} + Q_{\lambda_2(\alpha)}$ exists in α_0 and is equal to the eigenprojection for λ of $\mathcal{A}(\alpha_0)$. In addition, let m_1 and m_2 denote the respective algebraic multiplicities (that is, the dimensions of the associated spectral projections) and $m = m_1 + m_2$ the multiplicity at α_0. By [74, Sections VII.4.5, II.2], the weighted eigenvalue mean

$$\widehat{\lambda}(\alpha) := \frac{1}{m} \left(m_1 \lambda_1(\alpha) + m_2 \lambda_2(\alpha) \right) \tag{3.2.8}$$

is holomorphic in α_0. A corresponding statement holds in the case of more than two curves meeting at λ, but in general the eigennilpotents may be discontinuous in α_0.

Remark 3.2.13. Theorem 3.2.11 proves in particular parts (1) and (2) of Theorem 1.2.1. Let us briefly explain in particular how we obtain the fact that the analytic continuation $\mathfrak{E}(\alpha)$ of the eigenvalues

$$\mathfrak{E}(\alpha_0) := \{\lambda_k(\alpha_0) : k \in \mathbb{N}\} \tag{3.2.9}$$

for given $\alpha_0 \in \mathbb{C}$ exhaust the spectrum $\sigma(-\Delta_\Omega^\alpha)$ for any $\alpha \in \mathbb{C}$. Indeed, suppose there were some $\alpha \in \mathbb{C}$ and an eigenvalue $\lambda(\alpha)$ which did not lie on any of the eigencurves $\lambda_k(\alpha)$, that is, $\lambda(\alpha) \notin \mathfrak{E}(\alpha)$. Then, said eigenvalue $\lambda(\alpha)$ could itself be extended to an analytic eigenvalue curve on $\mathbb{C}$ by Theorem 3.2.11(2), and in particular we would have an eigenvalue $\lambda(\alpha_0)$ not included among the the $\lambda_k(\alpha_0)$, that is, $\lambda(\alpha_0) \notin \mathfrak{E}(\alpha_0)$, a contradiction to the assumption that $(\lambda_k(\alpha_0))_{k\in\mathbb{N}}$ (counting multiplicities) is the totality of the spectrum at α_0.

Proof of Theorem 3.2.11. (1) By [74, Theorem VII.4.2], $\mathcal{A}(\alpha)$, $\alpha \in \mathbb{C}$, is a holomorphic family of operators, and by [74, Remark VII.4.7], it is a self-adjoint holomorphic family. (2) Then it follows from [74, Theorem VII.1.8] that the eigenvalues and eigenprojections depend (locally) holomorphically on α, and hence so do the eigennilpotents. Since the operator family is self-adjoint holomorphic, there are no singularities at real crossing points of eigenvalues, see [74, Section VII.3.1]. The finiteness of the number of eigenvalue curves meeting at a crossing point, and of the local number of crossing points, follows from $\mathcal{A}(\alpha)$ having compact resolvent and from the holomorphy of the eigenvalue curves. More precisely, since the total projection (see Remark 3.2.12) is locally holomorphic, [74, Problem III.3.21] implies that the dimension of its range is locally constant and thus finite. This also implies that if there were infinitely many crossing points in a compact set, then finitely many eigenvalue curves meet at infinitely many points which have an accumulation point; now the identity theorem implies that the eigenvalue curves have to be identical.

It remains to prove that for any fixed $\alpha_0 \in \mathbb{C}$ each eigenvalue $\lambda_k(\alpha_0)$ can be extended to a function which is holomorphic on $\mathbb{C}$ except at the crossing points. We fix such a $\lambda_k(\alpha_0)$ and take an arbitrary compact subset $K \subset \mathbb{C}$ that is the closure of an open, connected set. It suffices to prove that if K contains α_0 in its interior, then there is a bounded holomorphic (except for crossing points) eigenvalue curve $\lambda_k(\alpha)$, $\alpha \in K$, which coincides with $\lambda_k(\alpha_0)$ at $\alpha = \alpha_0$.

To this end we consider the resolvent of $\mathcal{A}(\alpha)$ for $\alpha \in K$. We set

$$\rho_\Omega^K := \bigcap_{\alpha \in K} \rho(-\Delta_\Omega^\alpha). \tag{3.2.10}$$

Note that $\rho_\Omega^K \neq \varnothing$ since the operator family $\mathcal{A}(\alpha)$, $\alpha \in K$, is uniformly sectorial, see Theorem 3.1.2 (2). Fix $z \in \rho_\Omega^K$; then the resolvent family

$$R_z(\alpha) = (\mathcal{A}(\alpha) - zI)^{-1}, \quad \alpha \in K, \tag{3.2.11}$$

is not only compact but bounded-holomorphic [74, Theorem VII.1.3]. Thus, the point spectrum

$$\sigma_p(R_z(\alpha)) = \sigma(R_z(\alpha)) \setminus \{0\} \tag{3.2.12}$$

consists of eigenvalues of finite algebraic multiplicity, and with 0 as their only point of accumulation. Denote the eigenvalues of $R_z(\alpha_0)$ by $\mu_j(\alpha_0)$, where the ordering is chosen in such a way that

$$\lambda_j(\alpha_0) = \frac{1}{\mu_j(\alpha_0)} + z \tag{3.2.13}$$

for all j. Now the eigenvalue $\mu_k(\alpha_0)$ may be extended to a holomorphic eigenvalue curve, first to a neighbourhood of α_0. This curve $\mu_k(\alpha)$ cannot take on the value 0 for any $\alpha \in K$, since otherwise $R_z(\alpha)$ would not be invertible; hence its modulus has a non-zero minimum on any compact set. Together with the bounded-holomorphy of $\mathcal{A}(\alpha)$, $\alpha \in K$, we obtain that $\mu_k(\alpha)$ can be extended holomorphically to all of K except at only finitely many crossing points with other eigenvalue curves. Via the identification

$$\lambda_k(\alpha) = \frac{1}{\mu_k(\alpha)} + z \tag{3.2.14}$$

we obtain that $\lambda_k(\alpha)$ is well defined and holomorphic on all of K except at the crossing points. Since $\alpha_0 \in \mathbb{C}$ and k were arbitrary, this completes the proof. □

Even though Theorem 3.2.11 establishes that the eigenprojections can be continued holomorphically (away from possible crossing points), the eigenfunctions lose this property when normalised to have $L^2(\Omega)$-norm one (see [30, Theorem 4.4]):

Theorem 3.2.14. *Let H be a separable Hilbert space and let $D \subset \mathbb{C}$ be an open, connected set. Let $A(\alpha)$ be an operator family on H such that its eigenfunctions $u(\alpha)$ depend holomorphically on $\alpha \in D$. Then the norm $\|u(\alpha)\|_H$ is non-constant on D or u does not depend on $\alpha \in D$.*

Proof. Let $\alpha \in D$, assume the family of normalised eigenfunctions $u(\alpha)$ of $\mathcal{A}(\alpha)$ to be holomorphic and fix an arbitrary $\alpha_0 \in D$. Then, the function $f : D \to \mathbb{C}$ defined by $f(\alpha) = (u(\alpha_0), u(\alpha))$ satisfies

$$|f(\alpha)| \leq \|u(\alpha_0)\|_H \|u(\alpha)\|_H = 1, \tag{3.2.15}$$

that is, f is contractive on D. Now, since $f(\alpha_0) = 1$, the maximum principle yields that $|f| \equiv 1$ is constant and by $f(\alpha_0) = 1$ we conclude $f \equiv 1$. Furthermore, for any $\alpha \in D$ we have

$$\|u(\alpha) - u(\alpha_0)\|_H^2 = (u(\alpha) - u(\alpha_0), u(\alpha) - u(\alpha_0)) \tag{3.2.16a}$$

$$= \|u(\alpha)\|_H^2 + \|u(\alpha_0)\|_H^2 - 2\mathrm{Re}\,(u(\alpha_0), u(\alpha)) = 0. \tag{3.2.16b}$$

Consequently, $u(\alpha) = u(\alpha_0)$ and the family of eigenfunctions is independent of α, a contradiction. □

The question whether the eigenfunctions of $-\Delta^{\alpha}_{(-a,a)}$ are orthogonal in $L^2((-a,a))$ will be clarified in Section 3.3.

Remark 3.2.15. In the case of such domains, where one can describe the eigenvalues more or less explicitly (that is, as solutions of transcendental equations), namely intervals, balls and (hyper-) rectangles, it is possible to show that the eigennilpotents are always zero; see Remark 4.2.1 for the case of hyperrectangles and Remark 4.3.4 for the case of the ball. It thus seems reasonable to expect that the eigennilpotents are zero on any Lipschitz domain.

Remark 3.2.16. However, it is easy to see that there can be nontrivial eigennilpotents if α is allowed to be a function on the boundary. Take the simplest possible case of an interval $\Omega = (-a, a)$ and suppose $\alpha : \{-a, a\} \to \mathbb{C}$ is a function. Then for some values of α the eigennilpotents are non-zero: indeed, following the approach of [83, Section 3], let $t \in \mathbb{R}$ and consider purely imaginary $\alpha_t(x)$ of the form

$$\alpha_t(x) = \begin{cases} -\mathrm{i}t & \text{for } x = -a, \\ +\mathrm{i}t & \text{for } x = +a. \end{cases} \tag{3.2.17}$$

Then the spectrum of the Robin Laplacian $\mathcal{A}(\alpha_t) = -\Delta^{\alpha_t}_{(-a,a)}$ reads

$$\sigma\left(\mathcal{A}(\alpha_t)\right) = \left\{t^2\right\} \cup \left\{k_j^2\right\}_{j\in\mathbb{N}}, \tag{3.2.18}$$

where $k_j := \frac{\pi j}{2a}$, $j \geq 1$. That is, the spectrum consists of the eigenvalues of the Neumann Laplacian independently of t, plus the eigenvalue t^2. This eigenvalue (interpreted as the case $j = 0$) corresponds to the eigenfunction

$$u_0(x) = -\mathrm{e}^{-\mathrm{i}tx}, \tag{3.2.19}$$

while the rest of the eigenfunctions for k_j^2 read

$$u_j(x) = \cos(k_j(x + a)) - \frac{\mathrm{i}t}{k_j}\sin(k_j(x + a)). \tag{3.2.20}$$

Note that each u_j, $j \geq 1$, is, like its eigenvalue, independent of t. Fix $j \in \mathbb{N}$. The eigenvalue curves t^2 and k_j^2 obviously cross at $t = k_j$, meaning that the algebraic multiplicity at this point should be two. However, the eigenfunctions u_0 and u_j converge to the same function as $t \to k_j$: indeed, we check that for

$$g(x) = \frac{\mathrm{i}}{2t}x\mathrm{e}^{-\mathrm{i}tx} - \frac{\mathrm{e}^{2\mathrm{i}ta}}{4t^2}\mathrm{e}^{\mathrm{i}tx} \tag{3.2.21}$$

when $t = k_j$ the corresponding eigennilpotent satisfies

$$\left(\mathcal{A}(\alpha_t) - t^2\right) g(x) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} g(x) - t^2 g(x) \tag{3.2.22a}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{i}}{2t}\mathrm{e}^{-\mathrm{i}tx} + \frac{1}{2}x\mathrm{e}^{-\mathrm{i}tx} - \frac{\mathrm{e}^{2\mathrm{i}ta}\mathrm{i}}{4t}\mathrm{e}^{\mathrm{i}tx}\right) - t^2\left(\frac{\mathrm{i}}{2t}x\mathrm{e}^{-\mathrm{i}tx} - \frac{\mathrm{e}^{2\mathrm{i}ta}}{4t^2}\mathrm{e}^{\mathrm{i}tx}\right) \tag{3.2.22b}$$

$$= -\left(\frac{1}{2}\mathrm{e}^{-\mathrm{i}tx} + \frac{1}{2}\mathrm{e}^{-\mathrm{i}tx} - \frac{\mathrm{i}t}{2}x\mathrm{e}^{-\mathrm{i}tx} + \frac{\mathrm{e}^{2\mathrm{i}ta}}{4}\mathrm{e}^{-\mathrm{i}tx}\right) - \frac{\mathrm{i}t}{2}x\mathrm{e}^{-\mathrm{i}tx} + \frac{\mathrm{e}^{2\mathrm{i}ta}}{4}\mathrm{e}^{\mathrm{i}tx} \tag{3.2.22c}$$

$$= -\mathrm{e}^{-\mathrm{i}tx} = u_0(x) \tag{3.2.22d}$$

as well as

$$\left(\mathcal{A}(\alpha_t) - t^2\right)^2 g = \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - t^2\right) u_0(x) = -t^2\mathrm{e}^{-\mathrm{i}tx} + t^2\mathrm{e}^{-\mathrm{i}tx} = 0. \tag{3.2.23}$$

Consequently, we obtained

$$\left(\mathcal{A}(\alpha_t) - t^2\right) g = u_0 \neq 0 \qquad \text{and} \qquad \left(\mathcal{A}(\alpha_t) - t^2\right)^2 g = 0, \tag{3.2.24}$$

that is, g is a root vector and the geometric and algebraic eigenspaces do not coincide at $t = k_j$.

However, the focus of this work is on the eigenvalue asymptotics and it would take us too far afield to explore the question of the eigennilpotents here. So we leave it as an open problem to investigate them in the case that α is independent of $x \in \partial\Omega$.

Open Problem 3.2.17. *Given any bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, suppose that $\lambda = \lambda(\alpha)$ is a repeated eigenvalue of $\mathcal{A}(\alpha) = -\Delta_\Omega^\alpha$ for some $\alpha \in \mathbb{C}$. Is the eigennilpotent $(\mathcal{A}(\alpha) - \lambda(\alpha))Q_{\lambda(\alpha)}$ necessarily equal to zero?*

3.2.2 The derivative with respect to α

Let $\lambda(\alpha)$ be a simple eigenvalue with corresponding eigenfunction ψ. We wish to give a formula for the derivative $\lambda'(\alpha)$ along its corresponding eigencurve, which by

Theorem 3.2.11 always exists. Especially but not only in the special case $\alpha = 0$ a corresponding formula (see [36, eq. (4.12)]),

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\lambda(\alpha) = \frac{\int_{\partial\Omega} \psi^2 \,\mathrm{d}\sigma(x)}{\int_\Omega \psi^2 \,\mathrm{d}x}, \tag{3.2.25}$$

is reasonably well known; for more details see [36, Section 4.3.2] and the references therein. The proof of the following theorem (which is due to [30, Theorem 4.8]) is given after the proof of Lemma 3.2.20.

Theorem 3.2.18. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a bounded, Lipschitz domain, let $\alpha_0 \in \mathbb{C}$, and let $\lambda = \lambda(\alpha)$ be any holomorphic family of eigenvalues. Suppose that there exists a $\delta > 0$ such that for all α in the neighbourhood $B_\delta(\alpha_0)$ of α_0, $\lambda(\alpha)$ is a simple eigenvalue of $-\Delta_\Omega^\alpha$, with eigenfunction $\psi(\alpha)$ which is chosen to be holomorphic in α. Then in a neighbourhood of α the function*

$$\alpha \mapsto \frac{\int_{\partial\Omega} \psi(\alpha)^2 \,\mathrm{d}\sigma(x)}{\int_\Omega \psi(\alpha)^2 \,\mathrm{d}x} \tag{3.2.26}$$

is meromorphic with at most removable singularities. Its holomorphic continuation is equal to $\lambda'(\alpha)$ at every point in $B_\delta(\alpha_0)$.

This justifies writing simply

$$\lambda'(\alpha) = \frac{\int_{\partial\Omega} \psi(\alpha)^2 \,\mathrm{d}\sigma(x)}{\int_\Omega \psi(\alpha)^2 \,\mathrm{d}x} \tag{3.2.27}$$

for all $\alpha \in B_\delta(\alpha_0)$, and in particular Theorem 3.2.18 implies Theorem 1.2.1(3).

We leave it as an open problem to determine whether the mapping (3.2.26) can actually have (removable) singularities, or whether the denominator never vanishes.

Open Problem 3.2.19. *Let $\lambda(\alpha)$ be any algebraically simple eigenvalue of $-\Delta_\Omega^\alpha$ for some $\Omega \subset \mathbb{R}^d$ bounded and Lipschitz and $\alpha \in \mathbb{C}$, and denote by $\psi(\alpha)$ its eigenfunction, scaled arbitrarily. Does it follow that*

$$\int_\Omega \psi(\alpha)^2 \,\mathrm{d}x \neq 0 \ ? \tag{3.2.28}$$

We first prove that under the assumptions of the theorem the derivative of the eigenfunction ψ with respect to α, which we denote by $\psi'(\alpha)$ (and which exists as an element of $L^2(\Omega)$ by another application of Theorem 3.2.11) is actually in $H^1(\Omega)$. Notationally, we will take $z \in \mathbb{C}$ to be small enough that $\alpha + z \in B_\delta(\alpha_0)$, that is, $|\alpha + z - \alpha_0| < \delta$.

Lemma 3.2.20. *Under the assumptions of Theorem 3.2.18, we have $\psi'(\alpha) \in H^1(\Omega)$ for all $\alpha \in B_\delta(\alpha_0)$.*

Proof. We will show that

$$\limsup_{z\to 0} \frac{\|\nabla\psi(\alpha+z) - \nabla\psi(\alpha)\|_2^2}{|z|^2} < \infty. \tag{3.2.29}$$

Since we already know that $\nabla\psi'(\alpha)$ exists in the distributional sense (as $\psi'(\alpha) \in L^2(\Omega)$), it will then follow from (3.2.29) that actually $\nabla\psi'(\alpha) \in L^2(\Omega)$.

To prove (3.2.29), we fix $z \in \mathbb{C}$ sufficiently small (as explained above) and use the weak form of the equation for both $\lambda(\alpha+z)$ and $\lambda(\alpha)$ to obtain (with $(\cdot,\cdot)$ the inner product on $L^2(\Omega)$)

$$\begin{aligned}
&\|\nabla(\psi(\alpha+z) - \psi(\alpha))\|_2^2 && \text{(3.2.30a)}\\
&\quad = \operatorname{Re} \int_\Omega (\nabla\psi(\alpha+z) - \nabla\psi(\alpha)) \cdot \overline{(\nabla\psi(\alpha+z) - \nabla\psi(\alpha))}\, \mathrm{d}x && \text{(3.2.30b)}\\
&\quad = \operatorname{Re} \Big[\lambda(\alpha+z)\big(\psi(\alpha+z), \psi(\alpha+z) - \psi(\alpha)\big)\Big]\\
&\qquad - \operatorname{Re}\left[(\alpha+z)\int_{\partial\Omega} \psi(\alpha+z)\overline{(\psi(\alpha+z) - \psi(\alpha))}\, \mathrm{d}\sigma(x)\right]\\
&\qquad\quad - \operatorname{Re}\Big[\lambda(\alpha)\big(\psi(\alpha), \psi(\alpha+z) - \psi(\alpha)\big)\Big] && \text{(3.2.30c)}\\
&\qquad\qquad + \operatorname{Re}\left[\alpha\int_{\partial\Omega} \psi(\alpha)\overline{(\psi(\alpha+z) - \psi(\alpha))}\, \mathrm{d}\sigma(x)\right]\\
&\quad = \operatorname{Re}\Big((\lambda(\alpha+z)\psi(\alpha+z) - \lambda(\alpha)\psi(\alpha)), \psi(\alpha+z) - \psi(\alpha)\Big)\\
&\qquad - \operatorname{Re}\int_{\partial\Omega} \Big((\alpha+z)\psi(\alpha+z) && \text{(3.2.30d)}\\
&\qquad\qquad - \alpha\psi(\alpha))\overline{(\psi(\alpha+z) - \psi(\alpha))}\Big)\, \mathrm{d}\sigma(x).
\end{aligned}$$

We next estimate the integrand in the boundary integral (3.2.30d) as follows:

$$
\begin{aligned}
&-\operatorname{Re}\left[\Big((\alpha+z)\psi(\alpha+z)-\alpha\psi(\alpha)\Big)\overline{(\psi(\alpha+z)-\psi(\alpha))}\right] && (3.2.31a)\\
&\quad= -\operatorname{Re}(\alpha+z)|\psi(\alpha+z)-\psi(\alpha)|^2 \\
&\qquad\qquad + \operatorname{Re}\left[z\psi(\alpha)\overline{(\psi(\alpha+z)-\psi(\alpha))}\right] && (3.2.31b)\\
&\quad\le -\operatorname{Re}(\alpha+z)|\psi(\alpha+z)-\psi(\alpha)|^2 \\
&\qquad\qquad + \frac{1}{2}|\psi(\alpha+z)-\psi(\alpha)|^2 + \frac{|z|^2}{2}|\psi(\alpha)|^2. && (3.2.31c)
\end{aligned}
$$

Applying the trace inequality in the form

$$
\int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma(x) \le \varepsilon\|\nabla u\|^2 + C_\varepsilon\|u\|^2 \tag{3.2.32}
$$

for all $u \in H^1(\Omega)$, where $C_\varepsilon > 0$ depends only on $\varepsilon > 0$, to each of the two integrals

$$
\left|-\operatorname{Re}(\alpha+z)+\frac{1}{2}\right| \int_{\partial\Omega} |\psi(\alpha+z)-\psi(\alpha)|^2 \,\mathrm{d}\sigma(x) \tag{3.2.33}
$$

and

$$
\frac{|z|^2}{2}\int_{\partial\Omega} |\psi(\alpha)|^2 \,\mathrm{d}x, \tag{3.2.34}
$$

and choosing $\varepsilon > 0$ small enough that

$$
\eta := \varepsilon\left[-\operatorname{Re}(\alpha+z)+\frac{1}{2}\right] < 1 \tag{3.2.35}
$$

leads us to

$$
\begin{aligned}
&\|\nabla(\psi(\alpha+z)-\psi(\alpha))\|_2^2 \\
&\quad\le \operatorname{Re}\Big((\lambda(\alpha+z)\psi(\alpha+z)-\lambda(\alpha)\psi(\alpha)), \psi(\alpha+z)-\psi(\alpha)\Big) \\
&\quad\quad + \eta\|\nabla(\psi(\alpha+z)-\psi(\alpha))\|_2^2 \\
&\quad\quad + C_\varepsilon\left(-\operatorname{Re}(\alpha+z)+\frac{1}{2}\right)\|\psi(\alpha+z)-\psi(\alpha)\|_2^2 \\
&\quad\quad + |z|^2\left(\frac{\varepsilon}{2}\|\nabla\psi(\alpha)\|_2^2 + \frac{C_\varepsilon}{2}\|\psi(\alpha)\|_2^2\right).
\end{aligned} \tag{3.2.36}
$$

Now ε can be chosen independently of $\alpha \in B_\delta(\alpha_0)$; in particular, with such a choice, the coefficient of $|z|^2$ depends only on α, that is, we may write

$$C_\alpha := \frac{\varepsilon}{2}\|\nabla\psi(\alpha)\|_2^2 + \frac{C_\varepsilon}{2}\|\psi(\alpha)\|_2^2 \tag{3.2.37}$$

for this coefficient. We now divide by $|z|^2$ and pass to the limit as $z \to 0$ to obtain

$$\limsup_{z\to 0} \frac{\|\nabla(\psi(\alpha+z)-\psi(\alpha))\|^2}{|z|^2} \tag{3.2.38a}$$

$$\begin{aligned} &\le \frac{1}{1-\eta}\operatorname{Re}\left(\lambda'(\alpha)\psi(\alpha)+\lambda(\alpha)\psi'(\alpha),\psi'(\alpha)\right) \\ &\qquad + \frac{1}{1-\eta}C_\varepsilon\left(-\operatorname{Re}\alpha+\frac{1}{2}\right)\|\psi'(\alpha)\|^2 + C_\alpha. \end{aligned} \tag{3.2.38b}$$

Since we already know that $\psi'(\alpha) \in L^2(\Omega)$, the right-hand side of the above inequality is finite. This establishes (3.2.29) and hence completes the proof of the lemma. □

Proof of Theorem 3.2.18 and hence of Theorem 1.2.1(3). We choose $\psi(\alpha) \in H^1(\Omega)$ as a test function in the weak form of the eigenvalue equation for $\lambda(\alpha)$:

$$\int_\Omega (\nabla\psi(\alpha))^2\,\mathrm{d}x + \alpha\int_{\partial\Omega}\psi(\alpha)^2\,\mathrm{d}\sigma(x) - \lambda(\alpha)\int_\Omega \psi(\alpha)^2\,\mathrm{d}x = 0. \tag{3.2.39}$$

The left-hand side clearly depends holomorphically on α. Moreover, since $\psi'(\alpha) \in H^1(\Omega)$ by Lemma 3.2.20, we may calculate its derivative as

$$\begin{aligned} 2\int_\Omega \nabla\psi'(\alpha)\cdot\nabla\psi(\alpha)\,\mathrm{d}x &+ \int_{\partial\Omega}\psi(\alpha)^2\,\mathrm{d}\sigma(x) \\ &+ 2\alpha\int_{\partial\Omega}\psi'(\alpha)\psi(\alpha)\,\mathrm{d}\sigma(x) - \lambda'(\alpha)\int_\Omega\psi(\alpha)^2\,\mathrm{d}x \\ &\qquad - 2\lambda(\alpha)\int_\Omega\psi'(\alpha)\psi(\alpha)\,\mathrm{d}x = 0. \end{aligned} \tag{3.2.40}$$

But the weak form (3.1.4) of the eigenvalue equation for $\lambda(\alpha)$ also implies that

$$\begin{aligned} 2\int_\Omega \nabla\psi'(\alpha)\cdot\nabla\psi(\alpha)\,\mathrm{d}x &+ 2\alpha\int_{\partial\Omega}\psi'(\alpha)\psi(\alpha)\,\mathrm{d}\sigma(x) \\ &= 2\lambda(\alpha)\int_\Omega\psi'(\alpha)\psi(\alpha)\,\mathrm{d}x, \end{aligned} \tag{3.2.41}$$

whence

$$\lambda'(\alpha)\int_\Omega\psi(\alpha)^2\,\mathrm{d}x = \int_{\partial\Omega}\psi(\alpha)^2\,\mathrm{d}\sigma(x). \tag{3.2.42}$$

This yields (3.2.27) in the case that $\int_\Omega \psi(\alpha)^2 \,\mathrm{d}x \neq 0$. But since we know that $\lambda'(\alpha)$ is holomorphic in $B_\delta(\alpha_0)$, as are the mappings

$$\alpha \mapsto \int_\Omega \psi(\alpha)^2 \,\mathrm{d}x, \qquad \alpha \mapsto \int_{\partial\Omega} \psi(\alpha)^2 \,\mathrm{d}\sigma(x), \tag{3.2.43}$$

if the left-hand side of (3.2.42) vanishes at some point, then the right-hand side must vanish as well, and to the same order. It follows that any singularities of the mapping (3.2.26) in $B_\delta(\alpha_0)$ are removable. This completes the proof of the theorem. □

3.3 Basis properties of the eigenfunctions

Given the analytic dependence of the eigenfunctions $\{u_k(\alpha)\}_{k\geq 1}$ of the Robin Laplacian for $\alpha \in \mathbb{C}$, cf. Theorem 3.2.11, it is a natural question to ask whether they also still have reasonable basis properties. The best case, that is, the eigenfunctions can be chosen to form an orthonormal basis of $L^2(\Omega)$, is well known for real α due to the self-adjointness of the corresponding operator, cf. [36, Section 4.2]. We will show that the set of eigenfunctions lacks this property as soon as α is no longer real. This negative result of Theorem 3.3.1 seems devastating, however, there are other (weaker) basis concepts (such as *Riesz*, *Bari*, and *Abel* bases, see Definitions 3.3.3, 3.3.4, and 3.3.5) to investigate. In this section we will explore this question and, in particular, prove parts (4) and (5) of Theorem 1.2.1.

We start with the negative result (5), that the eigenfunctions do not generally form an orthonormal basis (see [30, Theorem 5.1]).

Theorem 3.3.1. *Let $\Omega \in \mathbb{R}^d$, $d \geq 1$, be a bounded Lipschitz domain and $\alpha \in \mathbb{C}$. Then the eigenfunctions $e_k(\alpha)$, $k \in \mathbb{N}$, of $-\Delta_\Omega^\alpha$ can be chosen to form an orthonormal basis of $L^2(\Omega)$ if and only if $\alpha \in \mathbb{R}$.*

Proof. For ease of notation, in this section we will write $\mathcal{A}(\alpha) := -\Delta_\Omega^\alpha$. For $\alpha \in \mathbb{R}$ the claim follows from the self-adjointness of $\mathcal{A}(\alpha)$.

Let $\alpha \in \mathbb{C} \setminus \mathbb{R}$ and assume that the eigenfunctions $\{e_k(\alpha)\}_{k=1}^\infty$ of $\mathcal{A}(\alpha)$ do form an orthonormal basis of $L^2(\Omega)$. To distinguish the notation from the complex conjugation $\overline{z}$ of $z \in \mathbb{C}$ and

$$M^* = \{\overline{z} \in \mathbb{C} : z \in M\} \tag{3.3.1}$$

of $M \subset \mathbb{C}$, let $\mathrm{cl}(M)$ be the closure of M. Let

$$u \in D\left(\mathcal{A}(\alpha)^*\right) = D(\mathcal{A}(\overline{\alpha})) \subset L^2(\Omega) \tag{3.3.2}$$

have $L^2(\Omega)$-norm one. Then there is a unique representation of u,

$$u = \sum_{k=1}^{\infty} (u, e_k(\alpha))\, e_k(\alpha), \qquad 1 = \|u\|_2^2 = \sum_{k=1}^{\infty} |(u, e_k(\alpha))|^2, \tag{3.3.3}$$

which we use to calculate

$$\begin{aligned}
\overline{(u, \mathcal{A}(\alpha)^* u)} &= (\mathcal{A}(\alpha)^* u, u) && (3.3.4a)\\
&= \sum_{k=1}^{\infty} \overline{(u, e_k(\alpha))}\, ((-\Delta_\Omega^\alpha)^* u, e_k(\alpha)) && (3.3.4b)\\
&= \sum_{k=1}^{\infty} (e_k(\alpha), u)\, (u, \lambda_k(\alpha) e_k(\alpha)) && (3.3.4c)\\
&= \sum_{k=1}^{\infty} |(u, e_k(\alpha))|^2\, \overline{\lambda_k(\alpha)}. && (3.3.4d)
\end{aligned}$$

By definition of the numerical range (2.2.10) and the identity

$$\mathrm{cl}(W(a_\alpha)) = \mathrm{cl}(W(\mathcal{A}(\alpha))), \tag{3.3.5}$$

cf. Lemma 2.2.7, we obtain

$$\mathrm{cl}(W(\mathcal{A}(\alpha)^*)) = \mathrm{cl}(W(a_\alpha^*)) = \mathrm{cl}(W(a_\alpha))^* = \mathrm{cl}(W(\mathcal{A}(\alpha)))^*. \tag{3.3.6}$$

Note that due to the normalisation of u we have

$$\sum_{k=1}^{\infty} |(u, e_k(\alpha))|^2 = 1 \tag{3.3.7}$$

and (3.3.4d) can be interpreted as an infinite convex combination of the complex conjugated elements $\lambda_k(\alpha) \in \sigma(\mathcal{A}(\alpha))$. The convex hull of the whole spectrum will be denoted by

$$\mathrm{conv}\left(\sigma(\mathcal{A}(\alpha))\right) = \mathrm{conv}\left\{\lambda_k(\alpha) : k \in \mathbb{N}\right\}. \tag{3.3.8}$$

Due to (3.3.4) and (3.3.6) we obtain

$$\mathrm{cl}(W(\mathcal{A}(\alpha)))^* = \mathrm{cl}(W(\mathcal{A}(\alpha)^*)) = \mathrm{cl}\left(\mathrm{conv}\left(\sigma(\mathcal{A}(\alpha))\right)^*\right) \tag{3.3.9}$$

and by complex conjugation of both sides we arrive at

$$\mathrm{cl}(W(a_\alpha)) = \mathrm{cl}(W(\mathcal{A}(\alpha))) = \mathrm{cl}\left(\mathrm{conv}(\sigma(\mathcal{A}(\alpha)))\right). \tag{3.3.10}$$

This equation leads us to a contradiction as follows: since $\mathcal{A}(\alpha)$ is sectorial and its resolvent is compact, for any sufficiently large $r > 0$ the *truncated* convex hull

$$P_r(\alpha) := \mathrm{conv}\left\{\lambda_k(\alpha) : k \in \mathbb{N},\ |\lambda_k(\alpha)| \le r\right\} \subset \mathbb{C} \tag{3.3.11}$$

is a polygon which contains at most finitely many eigenvalues of $\mathcal{A}(\alpha)$, see Figure 3.3.1.

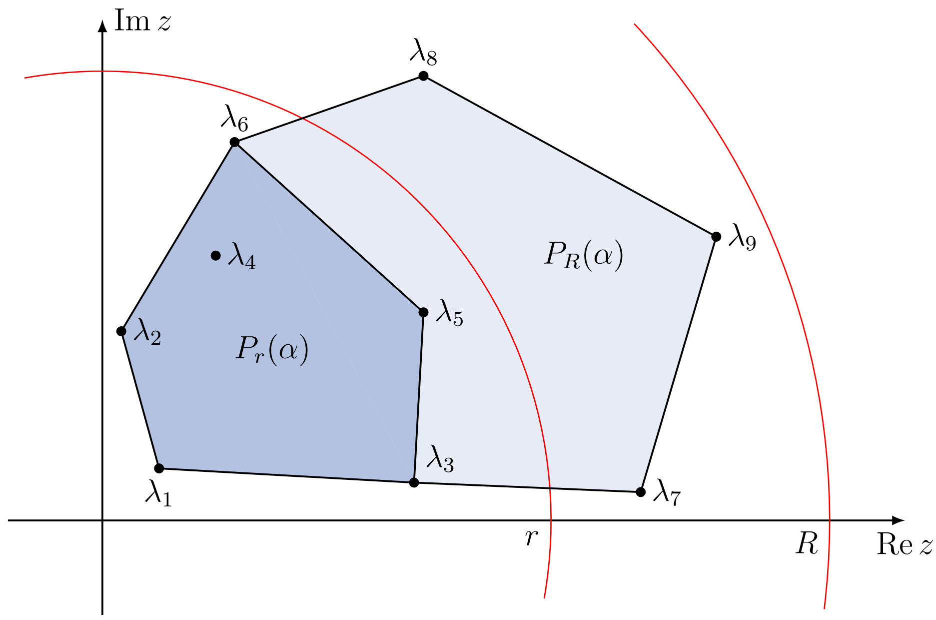

Figure 3.3.1: The truncated convex hull $P_r(\alpha)$ containing all eigenvalues $\lambda_j = \lambda_j(\alpha)$ in $B_r(0)$. Here we see two convex hulls for different radii $r < R$. Note that there are at most finitely many eigenvalues in each $P_r(\alpha)$, that is, this set is polygon-shaped.

We show that $P_r(\alpha)$ is contained in the upper half-plane, or equivalently $\mathrm{Im}\,\lambda_k(\alpha) >$

0 for all $k \in \mathbb{N}$: due to

$$\sigma(A(\alpha)) \subset W(a_\alpha) \subset \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\} \tag{3.3.12}$$

it is clear that $\operatorname{Im} \lambda_k(\alpha) \geq 0$ for all $k \in \mathbb{N}$. Now assume that there exists an eigenvalue $\lambda \in P_r(\alpha) \cap \mathbb{R}$, that is, we find a corresponding (normalised) eigenfunction $u \in D(\mathcal{A}(\alpha))$ such that

$$\lambda = (\mathcal{A}(\alpha)u, u) = \int_\Omega |\nabla u|^2 \mathrm{d}x + \alpha \int_{\partial\Omega} |u|^2 \mathrm{d}\sigma(x) \in \mathbb{R}. \tag{3.3.13}$$

This holds if and only if $u|_{\partial\Omega} = 0$, that is, u is an eigenfunction of the Dirichlet Laplacian $\mathcal{A}^D := -\Delta_\Omega^D$. Furthermore, $u \in D(\mathcal{A}(\alpha))$ yields

$$0 = \partial_\nu u + \alpha u = \partial_\nu u \tag{3.3.14}$$

on $\partial\Omega$ and u is additionally a Neumann eigenfunction, a contradiction. In other words, we have shown that

$$\mathrm{cl}\,(\mathrm{conv}(\sigma(\mathcal{A}(\alpha)))) \cap \mathbb{R} = \varnothing. \tag{3.3.15}$$

Since the principal eigenvalue $\lambda_1 = \min \sigma(\mathcal{A}^D)$ can be represented by the variational max-min characterisation, there exists a normalised minimising function (the associated eigenfunction) $u_1 \in H_0^1(\Omega)$ such that

$$\lambda_1 = \int_\Omega |\nabla u_1|^2 \, \mathrm{d}x. \tag{3.3.16}$$

Recall that the domains of the Dirichlet Laplacian and the Robin form satisfy

$$D(\mathcal{A}^D) \subset H_0^1(\Omega) \subset H^1(\Omega) = D(a_\alpha), \tag{3.3.17}$$

see Section 2.3. Consequently, $\lambda_1 \in \mathrm{cl}(W(a_\alpha))$, a contradiction to (3.3.10) and (3.3.15). Hence the eigenfunctions $\{e_k(\alpha)\}_{k=1}^\infty$ of $\mathcal{A}(\alpha)$ do not form an orthonormal basis of $L^2(\Omega)$. □

One may show by explicit calculation that, even on the interval $\Omega = (-a, a)$, consistent with Theorem 3.3.1, for given $\alpha \in \mathbb{C} \setminus \mathbb{R}$ the eigenfunctions of the Robin

Laplacian $-\Delta_\Omega^\alpha$ belonging to different eigenspaces are not in general orthogonal to each other: this statement is studied in Section 4.1.4, more precisely as the statement of Lemma 4.1.7. Hence, for our positive result, we necessarily need to introduce "weaker" notions of basis. Here we will consider three: Bari, Riesz and Abel bases. The definitions of the first two of these, namely Definitions 3.3.3 and 3.3.4, are taken from [72, 3.6.16-19]. For what follows we assume $(H, \|\cdot\|_H)$ to be a separable complex Hilbert space. We start by defining what we mean by a *basis* of H.

Definition 3.3.2. A set $\mathcal{B} = \{e_k\}_{k=1}^\infty \subset H$ is called a *basis* (or *Schauder basis*) of H if for each $h \in H$ there exists a unique, convergent series representation

$$h = \sum_{k=1}^\infty h_k e_k \tag{3.3.18}$$

with coefficients $h_k = h_k(h) \in \mathbb{C}$.

We denote by $\ell^2(\mathbb{C})$ the space of square summable sequences, namely

$$\ell^2(\mathbb{C}) := \left\{ (x_k)_k \subset \mathbb{C} : \|(x_k)_k\|_{\ell^2(\mathbb{C})} := \left(\sum_{k=1}^\infty |x_k|^2 \right)^{1/2} < \infty \right\}. \tag{3.3.19}$$

Definition 3.3.3. Let $\mathcal{B} = \{e_k\}_{k=1}^\infty$ be a basis of H. Then $\mathcal{B}$ is called a *Riesz basis* if there are constants $0 < m \leq M$ such that

$$m\|(h_k)_k\|_{\ell^2} \leq \|h\|_H \leq M\|(h_k)_k\|_{\ell^2} \tag{3.3.20}$$

holds for any $h = \sum_{k=1}^\infty h_k e_k \in H$.

Definition 3.3.4. Let $\mathcal{B}' = \{e'_k\}_{k=1}^\infty$ be an orthonormal basis of H. A set $\mathcal{B} = \{e_k\}_{k=1}^\infty \subset H$ is called a *Bari basis* of H if $\mathcal{B}$ is quadratically near $\mathcal{B}'$, that is,

$$\sum_{k=1}^\infty \|e_k - e'_k\|_H^2 < \infty. \tag{3.3.21}$$

An Abel basis, as first introduced in [89] and also defined for example in [115, Section 1.2.13], is always defined with respect to the eigenvectors and generalised eigenvectors (for short, generalised eigenvectors) of a densely defined sectorial operator

$\mathcal{A}$. The intuitive idea is that the formal series expansion

$$\sum_{k=1}^{\infty} h_k e_k \tag{3.3.22}$$

of an element $h \in H$ in the generalised eigenvectors e_k of A may not converge. The goal is to force it into convergence by multiplying each coefficient h_k by a weight $\mathrm{e}^{-\lambda_k^\gamma t}$ (where λ_k is the eigenvalue corresponding to e_k), such that

$$\sum_{k=1}^{\infty} h_k \mathrm{e}^{-\lambda_k^\gamma t} e_k \tag{3.3.23}$$

converges for each fixed $t > 0$, and this series then converges to h as $t \to 0$, then $\{e_k\}_{k=1}^{\infty}$ is an Abel basis of order $\gamma \geq 0$. (Note that an Abel basis will not generally be a basis in the sense of Definition 3.3.2, since the - series expansion is explicitly not required to converge.) The general idea is based on the *Abel summability* of (divergent) series by *Abelian means*; we refer to [67, Section 4.7] for more details. To give the definition of Abel bases we copy the one from [115].

Definition 3.3.5. Suppose $\mathcal{A} : H \supset D(A) \to H$ to be a densely defined operator with purely discrete spectrum, such that all but finitely many of its eigenvalues lie in the sector

$$T_\theta^+ = \{z \in \mathbb{C} : |\arg z| < \theta\} \tag{3.3.24}$$

for some $\theta \in (0, \pi)$. Then we say that the generalised eigenvectors of A form an *Abel basis of H of order* $\gamma \geq 0$ if $\gamma\theta < \pi/2$ and if there exists an enumeration of the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ (with $\{e_k\}_{k=1}^{\infty}$ the corresponding enumeration of the generalised eigenvectors) such that for this fixed enumeration, for each $h \in H$, there exists a sequence of coefficients $h_k \in \mathbb{C}$ for which the series

$$h(t) := \sum_{k=1}^{\infty} h_k \mathrm{e}^{-\lambda_k^\gamma t} e_k \tag{3.3.25}$$

is convergent for all $t > 0$, and $h(t) \to h$ in H as $t \to 0^+$.

For the eigenvalues λ which do not lie in T_θ^+, the weight $\mathrm{e}^{-\lambda_k^\gamma t}$ in (3.3.25) is to be interpreted as 1, while if some λ is a repeated eigenvalue and $\{e_j, \ldots, e_{j+\ell}\}$ is a basis of its eigenspace, then the corresponding terms in the series (3.3.25) are to be interpreted in terms of the eigenprojection, that is,

$$\sum_{k=j}^{j+\ell} h_k \mathrm{e}^{-\lambda^\gamma t} e_k \tag{3.3.26}$$

is to be replaced by

$$\frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_\lambda} \mathrm{e}^{-\lambda^\gamma t} (\mathcal{A} - zI)^{-1} h \,\mathrm{d}z, \tag{3.3.27}$$

where Γ_λ is any closed path in $\mathbb{C}$ separating λ from the rest of the spectrum.

The definition can be extended to allow $\gamma\theta < \pi$ in place of $\gamma\theta < \pi/2$; we refer, again, to [115, Section 1.2.13].

Remark 3.3.6. One may derive from the definitions that an orthonormal basis is always a Bari basis, a Bari basis is always a Riesz basis, and a Riesz basis, if it consists of the generalised eigenfunctions of a suitable operator, is always an Abel basis of order zero. The latter, in turn, is an Abel basis of any positive order $\gamma > 0$, provided only that the sectoriality estimate $\gamma\theta < \pi$ still holds.

Our goal is to show that the eigenfunctions of $\mathcal{A}(\alpha)$ form (at least) an Abel basis of $L^2(\Omega)$, for any $\alpha \in \mathbb{C}$. This is based on a theorem of M. S. Agranovich (the main theorem of [4]), which we recall here for ease of reference.

Theorem 3.3.7. *Suppose H and V are separable complex Hilbert spaces such that $V \hookrightarrow H$ is compact, and suppose that $a : V \times V \to \mathbb{C}$ is a bounded, coercive sesquilinear form. Denote by*

$$b := \operatorname{Re} a = \frac{a + \overline{a}}{2} \qquad \text{and} \qquad c := \mathrm{i}\operatorname{Im} a = a - b \tag{3.3.28}$$

the real and imaginary forms, respectively, which add to give a. Denote by A and B the operators on H associated with a and b, respectively. Suppose that

(i) there exist $0 \le q \le 1$ and $m > 0$ such that

$$|c[u,u]| \le m \|B^{1/2} u\|_H^{2q} \|u\|_H^{2-2q} \tag{3.3.29}$$

for all $u \in V$, and

(ii) there exists $p > 0$ such that the sequence of eigenvalues $\lambda_k(B)$, $k \geq 1$, of B (bounded from below by assumption), repeated according to their multiplicities, has the asymptotic behaviour

$$\limsup_{k\to\infty} \frac{\lambda_k(B)}{k^p} > 0. \tag{3.3.30}$$

Then A has discrete spectrum, the invariant subspaces of A are all finite dimensional, and the corresponding eigenfunctions and generalised eigenfunctions of A constitute

(1) a Bari basis of H if $p(1-q) > 1$, or

(2) a Riesz basis of H if $p(1-q) = 1$, or

(3) an Abel basis of H, of order $1/p + (q-1) + \delta$ for any (sufficiently small) $\delta > 0$, if $p(1-q) < 1$.

Theorem 3.3.7 was already used for a similar purpose in [71, Section 5] to prove a corresponding one-dimensional result; more precisely, for the Laplacian on a compact metric graph, equipped with complex δ conditions at one or more of the vertices (corresponding to a complex Robin condition), one can apply (2) to obtain a Riesz basis. With this background, we can now state our main positive result, which corresponds to Theorem 1.2.1(4) and hence to [30, Theorem 5.7].

Theorem 3.3.8. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a bounded Lipschitz domain and $\alpha \in \mathbb{C}$.*

(1) If $d = 1$, then there is a Riesz basis of $L^2(\Omega)$ consisting of the eigenfunctions and generalised eigenfunctions of $-\Delta_\Omega^\alpha$;

(2) If $d \geq 2$, then there is an Abel basis of $L^2(\Omega)$ of order $(d-1)/2 + \delta$ for any (sufficiently small) $\delta > 0$, consisting of the eigenfunctions and generalised eigenfunctions of $-\Delta_\Omega^\alpha$.

The conditions on d in the latter theorem are by no means sharp; we leave the following question as an open problem.

Open Problem 3.3.9. *Do the eigenfunctions of $-\Delta_\Omega^\alpha$ still form a Riesz basis of $L^2(\Omega)$ if $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain in dimension $d \geq 2$, as they do when $d = 1$ and when Ω is rectangular (see Remark 4.2.1)?*

Furthermore, here we repeat the open problem from Remark 3.2.15 to establish that the eigennilpotents are always trivial, that is, that all generalised eigenfunctions are in fact eigenfunctions.

Proof of Theorem 3.3.8. We only need to apply Theorem 3.3.7, as was done in [71, Section 5] for $d = 1$, and in fact we refer there for the proof in this case.

So suppose that $d \geq 2$. Obviously, we choose $H = L^2(\Omega)$ and $V = H^1(\Omega)$. Given $\alpha \in \mathbb{C}$, which will be fixed throughout, we suppose $\omega \geq 0$ to be such that

$$a_\alpha[u,u] + \omega(u,u) \tag{3.3.31}$$

is coercive $H^1(\Omega)$, which we may always do by the trace inequality, cf. Lemma 3.1.1. We then choose

$$a[u,v] := a_\alpha[u,v] + \omega(u,v), \tag{3.3.32}$$

so that

$$b = a_{\operatorname{Re}\alpha} + \omega(\,\cdot\,,\,\cdot\,)_2, \tag{3.3.33a}$$

$$A = -\Delta_\Omega^\alpha + \omega I, \tag{3.3.33b}$$

$$B = -\Delta_\Omega^{\operatorname{Re}\alpha} + \omega I, \tag{3.3.33c}$$

and to apply Theorem 3.3.7 we only need to check the conditions (i) and (ii). We will show that (i) holds for $q = 1/2$ and (ii) holds for any $p \leq d/2$, leading in particular to the order of the Abel basis claimed in the theorem.

For (i), first note that for any operator B satisfying the assumptions of the theorem, we have that

$$\left(B^{1/2}u, B^{1/2}u\right) = (Bu,u) = b[u,u] \tag{3.3.34}$$

for all $u \in D(B)$, and in particular $\|B^{1/2}u\|_2 = b[u,u]^{1/2}$ for all $u \in V = H^1(\Omega)$. Up to a possibly different constant, the form b defines an equivalent norm on $H^1(\Omega)$. Thus, in our setting, and with $q = 1/2$, (3.3.29) reduces to the question of the existence of a constant $m > 0$ such that, for all $u \in H^1(\Omega)$

$$|\mathrm{Im}\,\alpha| \int_\Omega |u|^2 \,\mathrm{d}\sigma \le m\|u\|_{H^1(\Omega)}\|u\|_2. \tag{3.3.35}$$

But this, in turn, follows immediately from the trace inequality (3.4.52) of Remark 3.4.11, to be proved below.

For (ii), note that the constant ω has no effect on the asymptotic behaviour of the eigenvalues; thus we may assume without loss of generality that $\omega = 0$. We are thus interested in the smallest $p > 0$ such that

$$\limsup_{k\to\infty} \frac{\lambda_k(-\Delta_\Omega^{\mathrm{Re}\,\alpha})}{k^p} > 0. \tag{3.3.36}$$

However, the Weyl asymptotics for the Robin Laplacian (see, for example, [7, Section 1] or [73, 112]), valid for any $\mathrm{Re}\,\alpha$, namely

$$\lambda_k(-\Delta_\Omega^{\mathrm{Re}\,\alpha}) = C_d(|\Omega|)k^{d/2} + o(k^{d/2}) \tag{3.3.37}$$

as $k \to \infty$ for a constant $C_d(|\Omega|) > 0$, leads us to $p \le d/2$. □

3.4 On the numerical range

Since $-\Delta_\Omega^\alpha$ ceases to be self-adjoint for $\alpha \in \mathbb{C} \setminus \mathbb{R}$, techniques commonly used to obtain estimates for localising the spectrum fail. The idea is to localise the numerical range $W(a_\alpha)$ of the Robin form which (since it contains the whole spectrum of our operator) particularly provides estimates for the eigenvalues. In particular, we obtain estimates on $\mathrm{Re}\,\lambda$ for complex (and as a special case for real) α for both Lipschitz domains and those having C^2 boundary. Even though both cases are based on trace-type inequalities, cf. Lemma 3.4.7, we obtain somewhat different results, and require a different method of proof: the reason why we use C^2 boundary to describe the smooth case (compared with the non-smooth Lipschitz boundary) are curvature estimates used to obtain constants for the eigenvalue inequalities of Lemma 3.4.7. In the case of general Lipschitz domains the function describing the distance from the boundary $\partial\Omega$ does no longer enjoy the same regularity properties and a different, local argument is needed.

As mentioned before, this section is devoted to give bounds on the numerical range of a_α associated with the operator $-\Delta_\Omega^\alpha$ on a general Lipschitz domain $\Omega \subset \mathbb{R}^d$, which we recall is given by

$$W(a_\alpha) = \left\{ \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_{\partial\Omega} \alpha |u|^2 \,\mathrm{d}\sigma \text{ such that } u \in H^1(\Omega) \text{ with } \|u\|_2 = 1 \right\} \subset \mathbb{C}. \tag{3.4.1}$$

The spectrum satisfies

$$\sigma(-\Delta_\Omega^\alpha) = \sigma_p(-\Delta_\Omega^\alpha) \subset W(-\Delta_\Omega^\alpha) \subset W(a_\alpha), \tag{3.4.2}$$

and in addition to giving an independent proof of the sectoriality of the form and the operator claimed in Section 3.1 (more precisely, Lemma 3.1.1), these bounds will more importantly provide us with an estimate on the rate at which any eigenvalues $\lambda(\alpha)$ can diverge in the regime $\operatorname{Re}\alpha < 0$. Moreover, these bounds allow us to control the imaginary part of the eigenvalue. In particular, the following theorem contains Theorem 1.2.3 (see [30, Theorem 6.1]).

Theorem 3.4.1. *Suppose $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded Lipschitz domain. Then there exist constants $C_1 \geq 2$ and $C_2 > 0$ depending only on Ω such that for $\alpha \in \mathbb{C}$ the set $W(a_\alpha)$ is contained in*

$$\Lambda_{\Omega,\alpha} = \left\{ t + \alpha \cdot s \in \mathbb{C} : t \geq 0,\ s \in [0, C_1\sqrt{t} + C_2] \right\}. \tag{3.4.3}$$

In particular, we have the estimate

$$\operatorname{Re}\lambda \geq -\frac{C_1^2}{4}|\operatorname{Re}\alpha|^2 - C_2|\operatorname{Re}\alpha| \tag{3.4.4}$$

for all $\lambda \in \sigma(-\Delta_\Omega^\alpha)$. If Ω has C^2 boundary, then we may choose $C_1 = 2$.

The regions $\Lambda_{\Omega,\alpha}$ for different values of $\alpha \in \mathbb{C}$ are depicted in Figure 3.4.1 for $\operatorname{Re}\alpha > 0$ and $\operatorname{Im}\alpha > 0$ and in Figure 3.4.2 for $\operatorname{Re}\alpha < 0$ and $\operatorname{Im}\alpha > 0$. The constants $C_1, C_2 > 0$ from (3.4.3) heavily depend on the geometry of $\partial\Omega$, and with our method of proof it should be possible to give an estimate on them, at least in principle; see Remark 3.4.13 for a discussion of the meaning of C_2 in the case of smooth domains, where we obtain an expression for C_2 related to the curvature of $\partial\Omega$, cf. (3.4.34).

It does not seem clear whether or not we should expect $C_1 = 2$ in Theorem 3.4.1 for domains of class C^1, not just C^2; cf. Remark 3.4.3.

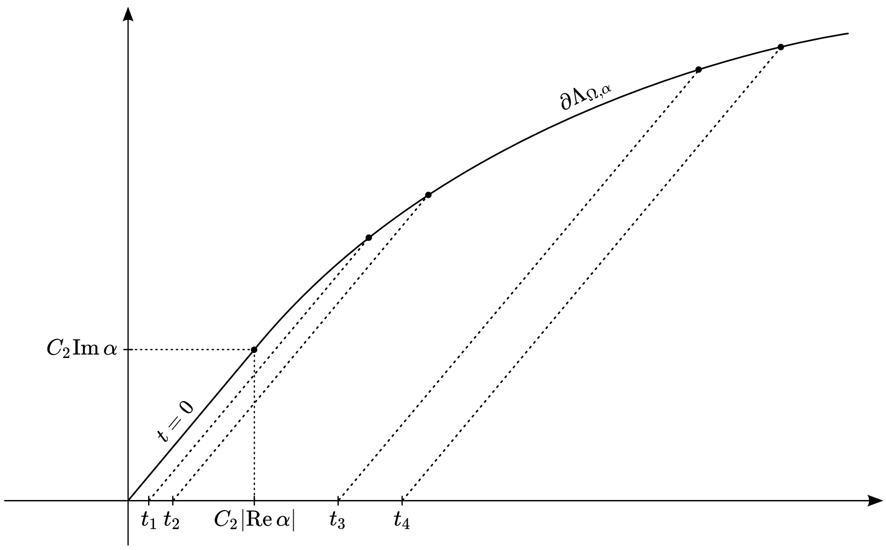

Figure 3.4.1: The set $\Lambda_{\Omega,\alpha}$, which contains the numerical range $W(a_\alpha)$, for a representative choice of $\operatorname{Re}\alpha > 0$ and $\operatorname{Im}\alpha > 0$, corresponding to the region between the curve $\partial\Lambda_{\Omega,\alpha}$ and the real axis. The region is composed of the union of segments of the form $\{t + \alpha \cdot s \in \mathbb{C} : s \in [0, C_1\sqrt{t} + C_2]\}$, each of slope $\operatorname{Im}\alpha/\operatorname{Re}\alpha$, for different values of $t \geq 0$; the dotted lines show these segments for selected values of $t_1, \ldots, t_4 > 0$. Their endpoints form a parabolic section of $\partial\Lambda_{\Omega,\alpha}$ open to the right.

Given the continuing interest in the self-adjoint case and since it is the first time a bound of the following form valid for all $\alpha < 0$ and for general C^2 domains has been found, we cover this special case in the following proposition.

Proposition 3.4.2. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain. Then, there exist constants $c_1 \geq 1$ and $c_2 > 0$ depending only on Ω such that for any $\alpha < 0$ and any corresponding eigenvalue $\lambda \in \mathbb{R}$ we have*

$$\lambda \geq -c_1\alpha^2 + c_2\alpha. \tag{3.4.5}$$

If Ω has C^2 boundary, then we may choose $c_1 = 1$.

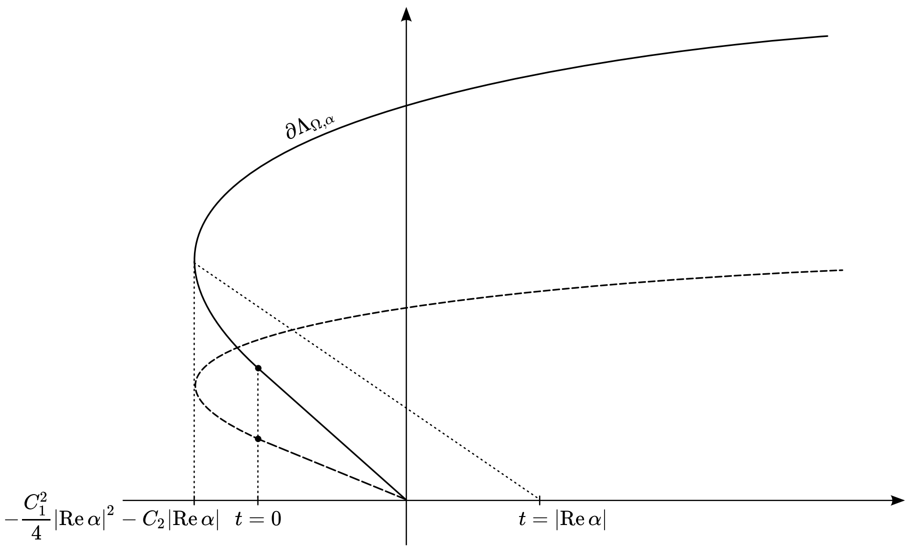

Figure 3.4.2: The set $\Lambda_{\Omega,\alpha}$ for $\mathrm{Re}\,\alpha < 0$ and two different choices of $\mathrm{Im}\,\alpha > 0$ (whose upper boundaries correspond to the solid and dashed curves, respectively). As $\mathrm{Im}\,\alpha \to 0$, the region collapses to the part of the real axis from $-\frac{C_1^2}{4}|\mathrm{Re}\,\alpha|^2 - C_2|\mathrm{Re}\,\alpha|$ to $+\infty$.

Furthermore, we wish to discuss how they fit in with known results before we turn to the proofs. To this end, we start with a discussion of the real case for smooth domains.

Remark 3.4.3. We recall the well known bound

$$\lambda_1(\alpha) < -|\alpha|^2 \tag{3.4.6}$$

on the principal (smallest) Robin eigenvalue $\lambda_1(\alpha)$ of $-\Delta_\Omega^\alpha$ for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ and $\alpha < 0$, which may be obtained by a simple variational argument (we refer to [63, Theorem 2.3] or [36, Proposition 4.12] for external references and internally to Section 2.4.1). Together with this bound, Proposition 3.4.2 in the case of a C^2 domain gives a new, simpler proof of the asymptotic behaviour

$$\lambda_1(\alpha) = -|\alpha|^2 + \mathcal{O}(\alpha) \tag{3.4.7}$$

as $\alpha \to -\infty$. The only other proof of this fact available for C^2 – actually C^1 – domains, which is completely different and involves a blow-up argument, is the principal result of the article [90] from 2004. All other proofs (which give more terms in the expansion) require more boundary regularity: indeed, if Ω is C^3, then, as $\alpha \to -\infty$,

$$\lambda_1(\alpha) = -|\alpha|^2 - (d-1)\bar{\kappa}_{\max}|\alpha| + \mathcal{O}(\alpha^{2/3}), \tag{3.4.8}$$

where $\bar{\kappa}_{\max}$ denotes the maximal mean curvature of $\partial\Omega$, see [36, Section 4.4.2.1] for a discussion and further references.

It is interesting to note that in the case of smooth Ω, that is, $\lambda_1 \sim -\alpha^2$, the constant C_2 appearing in Theorem 3.4.1 is likewise related to the curvature of $\partial\Omega$ (see Remark 3.4.13 for more details); the presence of the curvature suggests that the "smooth" version of Theorem 3.4.1 does not hold under significantly weaker regularity assumptions than C^2. We leave the following question as an open problem.

Open Problem 3.4.4. *Determine whether a better bound than the one in Theorem 3.4.1 is possible for C^1 domains.*

After studying the case of smooth domains in the previous remark, we now consider a less regular boundary of Ω.

Remark 3.4.5. We recall that for domains Ω with piecewise smooth boundary and a finite number of "model corners", the asymptotic behaviour of the principal eigenvalue reads

$$\lambda_1(\alpha) = -C|\alpha|^2 + o(\alpha^2) \tag{3.4.9}$$

as $\alpha \to -\infty$, for a constant $C \geq 1$ depending on the opening angle(s) of the "most acute" corner(s) of Ω (we refer to [36, pp. 94–95] for details and references); it is an open problem to show that (3.4.9) also holds on general Lipschitz domains [36, Open Problem 4.17]. The lower bound of Theorem 3.4.1 in the form of Corollary 1.2.4, together with (3.4.6), at least implies a two-sided asymptotic bound of this form.

The following remark goes beyond our topic, but for the sake of completeness we still want to mention it.

Remark 3.4.6. We incidentally note that the bound of Theorem 3.4.1 on the numerical range of a_α and therefore on the spectrum $-\Delta_\Omega^\alpha$ implies that, for any Lipschitz domain $\Omega \subset \mathbb{R}^d$ and any $\alpha \in \mathbb{C}$, the operator Δ_Ω^α generates a *cosine* *function*, that is, the corresponding wave equation is well posed (see [11, Section 3.14] for more details on cosine functions of operators). In fact, it is known that an operator generates a cosine function if and only if its numerical range and spectrum are contained in a parabolic region (see [11, Theorems 3.17.4 and 3.17.5] or [10, Theorem 5.3]). Here, we see that $-\Lambda_{\Omega,\alpha}$ is contained in the parabolic region described in that theorem for sufficiently large $\omega > 0$. How large ω has to be depends on α, C_1 and C_2.

The proof of Theorem 3.4.1 is based on the following trace-type inequality.

Lemma 3.4.7. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain. Then there exist constants $C_1 \geq 2$ and $C_2 > 0$, both depending only on Ω, such that*

$$\int_{\partial\Omega} |u|^2 \, \mathrm{d}\sigma \leq C_1 \|\nabla u\|_2 + C_2 \tag{3.4.10}$$

for all $u \in H^1(\Omega)$ with $\|u\|_2 = 1$. If Ω has C^2 boundary, then we may choose $C_1 = 2$.

The constants C_1 and C_2 will be the same as the ones appearing in the statements of Theorems 3.4.1 and 1.2.3. The proofs for the cases of C^2 and Lipschitz boundaries are completely different. For the smooth case, which we treat first, we need a technical lemma involving the geometry of Ω near its boundary, where we will heavily rely on the assumption that $\partial\Omega$ is C^2. We first introduce some notation:

Definition 3.4.8. For a bounded domain $\Omega \subset \mathbb{R}^d$, we define

(1) the signed distance function to $\partial\Omega$, namely $d_\Omega \in C(\mathbb{R}^d; \mathbb{R})$, by

$$d_\Omega(x) := \begin{cases} \operatorname{dist}(x, \partial\Omega) = \inf_{z \in \partial\Omega} |x - z| & \text{if } x \in \overline{\Omega}, \\ -\operatorname{dist}(x, \partial\Omega) & \text{if } x \in \mathbb{R}^d \setminus \overline{\Omega}. \end{cases} \tag{3.4.11}$$

(2) Given any $\varepsilon > 0$, we also set

$$\Omega_\varepsilon := \{x \in \mathbb{R}^d : |d_\Omega(x)| < \varepsilon\} \tag{3.4.12}$$

to be the (open) "strip" around (or neighbourhood of) $\partial\Omega$ of width 2ε, where we also write

$$\begin{aligned} \Omega_\varepsilon^+ &:= \Omega_\varepsilon \cap \Omega = \{x \in \Omega : d_\Omega(x) < \varepsilon\}, \\ \Omega_\varepsilon^- &:= \Omega_\varepsilon \cap \Omega^c_{\mathbb{R}^d} = \{x \in \Omega^c_{\mathbb{R}^d} : -d_\Omega(x) < \varepsilon\}, \end{aligned} \tag{3.4.13}$$

cf. Figure 3.4.3.

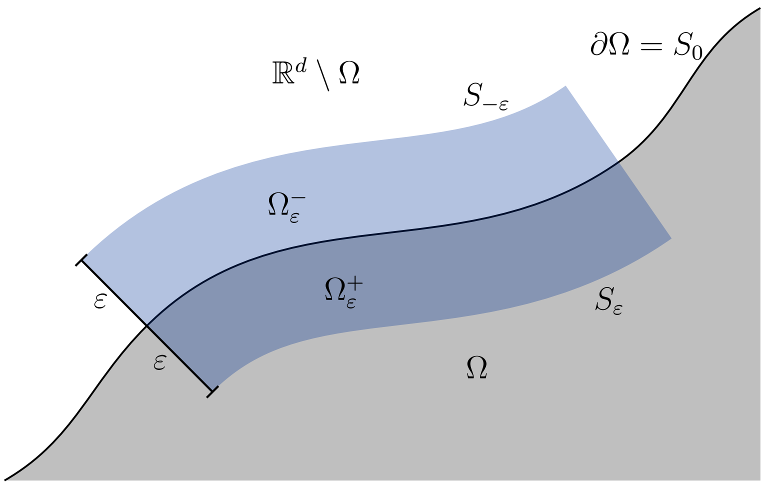

Figure 3.4.3: Depiction of the neighbourhoods Ω_ε, Ω_ε^-, and Ω_ε^+ for an exemplary section of $\partial\Omega$.

(3) Finally, we define the level surfaces of d_Ω in Ω_ε, viz.

$$S_t := \{x \in \mathbb{R}^d : d_\Omega(x) = t\} \tag{3.4.14}$$

for $t \in \mathbb{R}$ to be

$$\Omega_\varepsilon = \bigcup_{t \in (-\varepsilon, \varepsilon)} S_t. \tag{3.4.15}$$

Lemma 3.4.9. *Suppose $\Omega \subset \mathbb{R}^d$ is a bounded domain of class C^2. Then there exists $\varepsilon > 0$ such that*

(1) $d_\Omega|_{\overline{\Omega}_\varepsilon} \in C^2(\overline{\Omega}_\varepsilon)$*;*

(2) for each $x \in \overline{\Omega}_\varepsilon$ *there exists a unique minimiser* $z_x \in \partial\Omega$ *such that*

$$d_\Omega(x) = |x - z_x|; \tag{3.4.16}$$

(3) for each $x \in \overline{\Omega}_\varepsilon \setminus \partial\Omega$*,*

$$\nabla d_\Omega(x) = \frac{x - z_x}{|x - z_x|} \tag{3.4.17}$$

with z_x *as in (2). In particular,* $|\nabla d_\Omega(x)| = 1$ *for all* $x \in \overline{\Omega}_\varepsilon$*;*

(4) for each $t \in [-\varepsilon, \varepsilon]$*,* S_t *is a compact manifold of class* C^1*; and*

(5) for each $f \in C^1(\overline{\Omega}_\varepsilon)$ *the function*

$$t \mapsto \int_{S_t} f \, \mathrm{d}\sigma \tag{3.4.18}$$

is differentiable at every $t \in (-\varepsilon, \varepsilon)$*, and its derivative, given by*

$$\int_{S_t} \partial_t f + f \Delta d_\Omega \, \mathrm{d}\sigma, \tag{3.4.19}$$

is in $C([-\varepsilon, \varepsilon])$*. In particular, for any* $f \in C^1(\overline{\Omega^+_\varepsilon})$ *and any* $\varepsilon_1 \in [0, \varepsilon)$*,*

$$\int_{S_{\varepsilon_1}} f \, \mathrm{d}\sigma - \int_{\partial\Omega} f \, \mathrm{d}\sigma = \int_{\Omega^+_{\varepsilon_1}} \partial_t f + f \Delta d_\Omega \, \mathrm{d}x. \tag{3.4.20}$$

Proof. (1) Since $\partial\Omega$ is assumed to be C^2, this statement is contained in [62, Appendix, Lemma 1], see also [18, Lemma 2.4.2]. (2) follows (possibly for a different ε) from [56, Lemma 4.11] where we need that $\partial\Omega$ is of class $C^{1,1}$. However, a covering argument using the fact that $\partial\Omega$ is compact, that is, we can consider finitely many balls covering $\partial\Omega$, is sufficient to apply said Lemma on each such ball (in the language of [56], (2) means that $\mathrm{reach}(\partial\Omega) > 0$). (3) then follows from [56, Theorem 4.8], where we note that $\nabla d_\Omega \in C^1(\overline{\Omega}_\varepsilon)$ and $|\nabla d_\Omega| = 1$ in $\overline{\Omega}_\varepsilon \setminus \partial\Omega$ implies that $|\nabla d_\Omega| = 1$ everywhere in $\overline{\Omega}_\varepsilon$. (4) follows from (1) using the Implicit Function Theorem and the fact that ∇d_Ω never vanishes on $\overline{\Omega}_\varepsilon$ by (3), together with a covering argument since S_t is clearly compact. For (5), fix $f \in C^1(\overline{\Omega}_\varepsilon)$ and for brevity write

$$F(t) := \int_{S_t} f \, \mathrm{d}\sigma. \tag{3.4.21}$$

Firstly, we claim that (3.4.19) is the distributional derivative of F. Indeed, for any test function $\varphi \in C_c^\infty(-\varepsilon, \varepsilon)$, we have

$$\int_{-\varepsilon}^{\varepsilon} F(t)\varphi(t) \, \mathrm{d}t = \int_{-\varepsilon}^{\varepsilon} \int_{S_t} f\varphi(t) \, \mathrm{d}\sigma \, \mathrm{d}t = \int_{\Omega_\varepsilon} f\varphi \circ d_\Omega \, \mathrm{d}x \tag{3.4.22}$$

by the coarea formula in the form of [50, Section 3.4.3], using the fact that the S_t are the level surfaces of d_Ω and $|\nabla d_\Omega| = 1$ everywhere by (3). In particular,

$$\begin{aligned}
\int_{-\varepsilon}^{\varepsilon} F(t)\varphi'(t) \, \mathrm{d}t = \int_{\Omega_\varepsilon} f\varphi' \circ d_\Omega \, \mathrm{d}x &= \int_{\Omega_\varepsilon} f\nabla d_\Omega \cdot \nabla(\varphi \circ d_\Omega) \, \mathrm{d}x && (3.4.23a)\\
&= -\int_{\Omega_\varepsilon} \varphi \circ d_\Omega \operatorname{div}(f\nabla d_\Omega) \, \mathrm{d}x && (3.4.23b)\\
&= -\int_{-\varepsilon}^{\varepsilon} \varphi(t) \int_{S_t} \operatorname{div}(f\nabla d_\Omega) \, \mathrm{d}x \, \mathrm{d}t, && (3.4.23c)
\end{aligned}$$

where for the second last equality we have used the divergence theorem (integration by parts) and the compact support of φ; and the last equality follows from another application of the coarea formula. The claim now follows from the short calculation

$$\operatorname{div}(f\nabla d_\Omega) = \nabla f \cdot \nabla d_\Omega + f\Delta d_\Omega = \partial_t f + f\Delta d_\Omega, \tag{3.4.24}$$

valid pointwise in $\overline{\Omega}_\varepsilon$ since d_Ω is C^2 by (1), and using the fact that ∇d_Ω points in the direction of t by (3). We next note that the integrand in (3.4.19) is in $C(\overline{\Omega}_\varepsilon)$ and hence a short argument using the compactness of S_t and the uniform continuity of the integrand shows that the integral in (3.4.19) is in fact in $C([-\varepsilon, \varepsilon])$; in particular, it is the pointwise derivative of F at every point in $(-\varepsilon, \varepsilon)$.

Finally, for (3.4.20), by what we have just shown we may apply the Fundamental Theorem of Calculus in the form of [108, Theorem 7.21] to the function F on the interval $[0, \varepsilon_1]$ (for any $\varepsilon_1 < \varepsilon$) to obtain

$$F(\varepsilon_1) - F(0) = \int_0^{\varepsilon_1} \int_{S_t} \partial_t f + f\Delta d_\Omega \, \mathrm{d}\sigma. \tag{3.4.25}$$

A final application of the coarea formula to the integral on the right-hand side, together with the definition of F, yields (3.4.20). $\square$

Proof of Lemma 3.4.7. The case of C^2 boundary. We keep the notation of d_Ω, $\Omega_\varepsilon^\pm$, and S_t from Lemma 3.4.8 and note that it suffices to prove

$$\int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma \leq C_1 \|\nabla u\|_2 + C_2 \tag{3.4.26}$$

for all $u \in C^1(\overline{\Omega})$, by density of the latter set in $H^1(\Omega)$ for bounded Ω of class C^2 (cf. [62, Section 7.6]) and the trace theorem. We let $\varepsilon > 0$ be as in Lemma 3.4.9 (in particular, by making ε a little smaller if necessary we can always assume that (3.4.20) holds with ε in place of ε_1) and choose a cut-off function $\varphi \in C^1(\overline{\Omega})$ such that $0 \leq \varphi(x) \leq 1$ for all $x \in \overline{\Omega}$ and

$$\begin{cases} \varphi(x) = 0 & \text{for all } x \in \Omega \setminus \Omega_\varepsilon, \\ \varphi|_{S_t} = \text{const} & \text{for all fixed } t \in [0, \varepsilon], \\ \varphi|_{\partial\Omega} = 1. \end{cases} \tag{3.4.27}$$

The existence of such a function is guaranteed by the regularity statements in Lemma 3.4.9: indeed, if we let $\psi \in C^\infty([0,\infty))$ be any smooth function satisfying $\psi(0) = 1$ and $\psi(t) = 0$ for all $t \geq \varepsilon$, then we may take $\varphi = \psi \circ d_\Omega$. Now fix $u \in C^1(\overline{\Omega})$ normalised such that $\|u\|_2 = 1$. Then

$$f := |u|^2 \varphi \in C^1(\overline{\Omega}) \tag{3.4.28}$$

and we apply formula (3.4.20) from Lemma 3.4.9 to obtain the following equation (3.4.30). Using that $\varphi = 1$ on $\partial\Omega$ and $\varphi = 0$ on S_ε in (3.4.29), we obtain

$$-\int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma = \int_{S_\varepsilon} |u|^2 \varphi \,\mathrm{d}\sigma - \int_{\partial\Omega} |u|^2 \varphi \,\mathrm{d}\sigma \tag{3.4.29}$$

$$= \int_{\Omega_\varepsilon^+} \partial_t(|u|^2\varphi) + |u|^2 \varphi \Delta d_\Omega \,\mathrm{d}x \tag{3.4.30}$$

$$= \int_{\Omega_\varepsilon} 2\varphi \mathrm{Re}\,(\overline{u}\partial_t u) + |u|^2 \partial_t \varphi + |u|^2 \varphi \Delta d_\Omega \,\mathrm{d}x. \tag{3.4.31}$$

Due to $\varphi = 0$ on $\Omega \setminus \Omega_\varepsilon^+$, we may estimate

$$\begin{aligned}\int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma &\le 2\|\varphi\|_\infty \|u\|_2 \|\nabla u\|_2 + \|u\|_2^2 \|\nabla\varphi\|_\infty \\ &\qquad + \|u\|_2^2 \|\varphi\|_\infty \cdot \max_{x\in\overline{\Omega}_\varepsilon} |\Delta d_\Omega(x)| \end{aligned} \tag{3.4.32a}$$

$$= 2\|\nabla u\|_2 + \|\nabla\varphi\|_\infty + \max_{x\in\overline{\Omega_\varepsilon^+}} |\Delta d_\Omega(x)| \tag{3.4.32b}$$

using the normalisation $\|u\|_2 = 1$ as well as $\|\varphi\|_\infty = 1$ (where all norms are over Ω). This proves the assertion

$$\int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma \le C_1 \|\nabla u\|_2 + C_2 \tag{3.4.33}$$

with

$$C_2 := \|\nabla\varphi\|_\infty + \max_{x\in\overline{\Omega_\varepsilon^+}} |\Delta d_\Omega(x)|. \tag{3.4.34}$$

□

If Ω is a Lipschitz domain, the proof of Lemma 3.4.7 is based on a very different, local argument. To this end, we need another auxiliary statement on the normal vector to $\partial\Omega$, which needs some additional notation: fix $z \in \partial\Omega$ and a neighbourhood $\mathcal{U}_z$ of z such that within $\mathcal{U}_z$, $\partial\Omega$ is given by the graph of a Lipschitz function $\zeta : \mathbb{R}^{d-1} \to \mathbb{R}$ such that $\Omega_z := \Omega \cap \mathcal{U}_z$ lies in the region

$$\left\{ x \in \mathbb{R}^d : x_d < \zeta\left(\check{x}\right) \right\}, \tag{3.4.35}$$

where we use the notation

$$x = (x_1, \dots, x_d)^T = (\check{x}, x_d)^T \in \mathbb{R}^d \simeq \mathbb{R}^{d-1} \times \mathbb{R}. \tag{3.4.36}$$

Such a neighbourhood $\mathcal{U}_z$ exists by Definition 2.1.2. From now on, we will write

$$(\partial\Omega)_z := \partial\Omega \cap \mathcal{U}_z \tag{3.4.37}$$

for the part of the boundary of Ω which lies inside the just defined neighbourhood $\mathcal{U}_z$, cf. Figure 3.4.4. Then in the coordinate system (3.4.35), the normal vector to

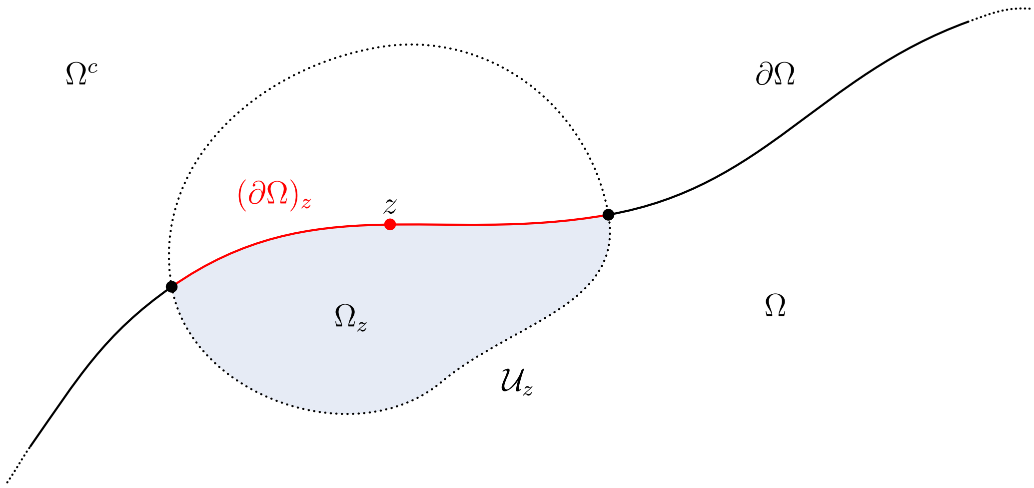

Figure 3.4.4: Depiction of the neighbourhood $\mathcal{U}_z$ for some boundary point $z \in \partial\Omega$ plus the corresponding sets Ω_z and $(\partial\Omega)_z$, respectively.

$\partial\Omega$ given by

$$\nu = (\nu_1, \dots, \nu_d)^T : \partial\Omega \to \mathbb{R}^d, \tag{3.4.38}$$

which is an L^∞-function since $\partial\Omega$, is Lipschitz (see [37, Section 1]). We omitted the proof of the following statement in [30, Proof of Lemma 6.5]; thus, for completeness' sake, we give in this thesis.

Proposition 3.4.10. *After the preceding procedure the d-th component ν_d of the normal vector ν satisfies*

$$\operatorname{ess\,inf}\{\nu_d(y) : y \in (\partial\Omega)_z\} > 0. \tag{3.4.39}$$

Proof. In the neighbourhood $\mathcal{U}_z$ the boundary $\partial\Omega$ is locally given by the graph of a Lipschitz function $\zeta : \mathbb{R}^{d-1} \to \mathbb{R}$ (cf. Definitions 2.1.1 and 2.1.2). From now on, every statement is to be understood locally in this neighbourhood. It is well known that $\nabla\zeta$ exists a.e. and $\nabla\zeta \in L^\infty(\mathbb{R}^{d-1})$ is bounded, as well; and the same holds for the (outer) normal vector

$$\nu = (\nu_1, \dots, \nu_d)^T : \partial\Omega \to \mathbb{R}^d. \tag{3.4.40}$$

Since Ω_z lies in the region (3.4.35), that is, Ω lies on *one side* of ζ, we have $\nu_d \geq 0$ a.e. Besides, if we denote by $T_\zeta(\check{x})$ the tangential hyperplane to ζ (at those points $\check{x}$ where it exists), then at $x = (\check{x}, x_d)^T \in \mathbb{R}^d$ the plane $T_\zeta(\check{x})$ is given by

$$\operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \frac{\partial\zeta}{\partial x_1}(\check{x}) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ \frac{\partial\zeta}{\partial x_2}(\check{x}) \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ \frac{\partial\zeta}{\partial x_{d-1}}(\check{x}) \end{pmatrix}\right\}. \tag{3.4.41}$$

By construction, we have $\nu \cdot \tau = 0$ for all $\tau \in T_\zeta(\check{x})$ (where it exists). We now rewrite $\nu \cdot \tau = 0$ in terms of the $d-1$ vectors from (3.4.41) and obtain the system of equations

$$\nu_1(x) + \nu_d(x)\frac{\partial\zeta}{\partial x_1}(\check{x}) \quad = 0 \tag{3.4.42a}$$

$$\nu_2(x) + \nu_d(x)\frac{\partial\zeta}{\partial x_2}(\check{x}) \quad = 0 \tag{3.4.42b}$$

$$\vdots$$

$$\nu_{d-1}(x) + \nu_d(x)\frac{\partial\zeta}{\partial x_{d-1}}(\check{x}) = 0 \tag{3.4.42c}$$

which hold a.e. The following argument leads us to $\nu_d > 0$ a.e.: we already know that $\nu_d \geq 0$ a.e. If we assume $\nu_d = 0$, then the n-th equation of (3.4.42) implies that $\nu_n = 0$ for $n = 1, \dots, d-1$, hence $\nu = 0$, a contradiction. Furthermore, since $\nabla\zeta$ is bounded, there exists $C_1 > 0$ such that $|\partial_j\zeta(\check{x})| \leq C_1$ for all $j = 1, \dots, d-1$ and for almost every $\check{x}$. Consequently, recall that $\nu_d \neq 0$, we obtain

$$\left|\frac{\nu_j(x)}{\nu_d(x)}\right| \leq C_2 \qquad \Leftrightarrow \qquad |\nu_j(x)|^2 \leq C_3|\nu_d(x)|^2 \tag{3.4.43}$$

a.e. for a constant $C_3 = C_2^2 > 0$ and for all $j = 1, \dots, d-1$. By the latter inequality it follows from the normalisation of ν that

$$1 = |\nu(x)|^2 = \sum_{j=1}^{d} |\nu_j(x)|^2 = |\nu_d(x)|^2 + \sum_{j=1}^{d-1} |\nu_j(x)|^2 \tag{3.4.44a}$$

$$\leq [1 + (d-1)C_3]|\nu_d(x)|^2. \tag{3.4.44b}$$

Together with $\nu_d > 0$ a.e. we arrive at

$$\nu_d(x) \geq \frac{1}{\sqrt{1 + (d-1)C_3}} > 0 \tag{3.4.45}$$

a.e. in the given neighbourhood $\mathcal{U}_z$. □

We now turn to the remaining part of the proof of Lemma 3.4.7.

Proof of Lemma 3.4.7. The case of Lipschitz boundary. Since in the case of general Lipschitz domains the corresponding parametrisation of Ω_ε does not enjoy the same regularity properties, we give a different, local argument. We recall the notation preceding Proposition 3.4.10 and we fix a test function $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $0 \leq \varphi \leq 1$ on $\mathbb{R}^d$ as well as

$$\begin{cases} \varphi(y) = 1 & \text{for all } y \in (\partial\Omega)_z, \\ \varphi(y) = 0 & \text{for all } y \in \partial\Omega \ \text{ with } \ \nu_d(y) \leq 0. \end{cases} \tag{3.4.46}$$

Since $\partial\Omega$ is Lipschitz (see Definition 2.1.2), by shrinking the neighbourhood $\mathcal{U}_z$ if necessary, we can always guarantee the existence of such a φ. Then for a given function $u \in H^1(\Omega)$ normalised to $\|u\|_2 = 1$, we have

$$\operatorname*{ess\,inf}_{y \in (\partial\Omega)_z} \nu_d(y) \int_{(\partial\Omega)_z} |u|^2 \,\mathrm{d}\sigma \leq \int_{\partial\Omega} \varphi |u|^2 \nu_d \,\mathrm{d}\sigma = \int_\Omega \frac{\partial}{\partial x_d}(\varphi|u|^2)\,\mathrm{d}x \tag{3.4.47}$$

by the divergence theorem applied to the function

$$F = (0, \dots, 0, \varphi|u|^2) \in W^{1,1}(\Omega) \tag{3.4.48}$$

and the Lipschitz domain Ω (see [96, Théorème 3.1.1]). The latter integral may be estimated by

$$\int_\Omega \frac{\partial}{\partial x_d}(\varphi|u|^2)\,\mathrm{d}x \le \|\nabla\varphi\|_\infty \|u\|_2^2 + 2\|\varphi\|_\infty \|\nabla u\|_2 \|u\|_2. \tag{3.4.49}$$

Using the normalisations $\|u\|_2 = 1$, $\|\varphi\|_\infty = 1$, the estimate (3.4.47) may be written as

$$\int_{(\partial\Omega)_z} |u|^2\,\mathrm{d}\sigma \le C_{1,z}\|\nabla u\|_2 + C_{2,z} \tag{3.4.50}$$

for suitable constants $C_{1,z}, C_{2,z} > 0$ depending on $z \in \partial\Omega$. Since $\partial\Omega$ is compact, a simple covering argument now yields the assertion (3.4.10). Note that for every $z \in \partial\Omega$ we have

$$C_{1,z} = \frac{2}{\operatorname*{ess\,inf}_{y\in(\partial\Omega)_z} \nu_d(y)} \ge 2 \tag{3.4.51}$$

since $|\nu| = 1$ and $C_{1,z} < \infty$ by Proposition 3.4.10; hence also $C_1 \ge 2$. □

Remark 3.4.11. If Ω is a Lipschitz domain, the above proof also yields the slightly different trace inequality

$$\int_{\partial\Omega} |u|\,\mathrm{d}\sigma \le C(\Omega)\|u\|_{H^1(\Omega)}\|u\|_2 \tag{3.4.52}$$

for all $u \in H^1(\Omega)$, needed in the proof of Theorem 3.3.8: Recall that $(\partial\Omega)_z = \partial\Omega \cap \mathcal{U}_z$. Then, by (3.4.49), we have

$$\operatorname*{ess\,inf}_{y\in(\partial\Omega)_z} \nu_d(y) \int_{\partial\Omega\cap\mathcal{U}_z} |u|^2\,\mathrm{d}\sigma \le \int_\Omega \frac{\partial}{\partial x_d}(\varphi|u|^2)\,\mathrm{d}x \tag{3.4.53a}$$

$$\le \|\nabla\varphi\|_\infty \|u\|_2^2 + 2\|\varphi\|_\infty \|\nabla u\|_2 \|u\|_2 \tag{3.4.53b}$$

$$\le (\|\nabla\varphi\|_\infty + 2\|\varphi\|_\infty)\,\|u\|_{H^1(\Omega)}\|u\|_2, \tag{3.4.53c}$$

leading to

$$\int_{(\partial\Omega)_z} |u|^2\,\mathrm{d}\sigma \le C_z\|u\|_{H^1(\Omega)}\|u\|_2 \tag{3.4.54}$$

for all $u \in H^1(\Omega)$, for a constant $C_z > 0$ depending only on $z \in \partial\Omega$. A covering argument as in the above lemma then yields (3.4.52).

In preparation for the proof of Theorem 3.4.1 we want to give one last lemma.

Proposition 3.4.12. *Let* $\alpha \neq 0$ *and* $C > 0$. *Then*

$$C\|\nabla u\|_2 \leq \frac{C^2}{4}|\operatorname{Re}\alpha| + \frac{\|\nabla u\|_2^2}{|\operatorname{Re}\alpha|}. \tag{3.4.55}$$

Proof. By multiplying the rearranged inequality (3.4.55) by $4|\operatorname{Re}\alpha| > 0$, we obtain

$$C^2|\operatorname{Re}\alpha|^2 - 4C|\operatorname{Re}\alpha|\|\nabla u\|_2 + 4\|\nabla u\|_2^2 = \Big(C|\operatorname{Re}\alpha| - 2\|\nabla u\|_2\Big)^2 \geq 0. \tag{3.4.56}$$

□

Proof of Theorem 3.4.1. Let $C_1 \geq 2$, $C_2 > 0$ be the constants from Lemma 3.4.7; in particular, we assume $C_1 = 2$ if Ω is of class C^2. Fix $u \in H^1(\Omega)$ with $\|u\|_2 = 1$ and set

$$\lambda := \|\nabla u\|_2^2 + \alpha \int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma \in W(a_\alpha). \tag{3.4.57}$$

Then, for

$$t := \|\nabla u\|_2^2 \geq 0 \qquad \text{and} \qquad s := \int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma \geq 0 \tag{3.4.58}$$

we have

$$\operatorname{Re}\lambda = t + \operatorname{Re}\alpha \cdot s \qquad \text{and} \qquad \operatorname{Im}\lambda = \operatorname{Im}\alpha \cdot s. \tag{3.4.59}$$

Moreover, by Lemma 3.4.7, we obtain that $s \leq C_1\sqrt{t} + C_2$; thus $\lambda \in \Lambda_{\Omega,\alpha}$.

To see that every $\lambda \in W(a_\alpha)$, and hence every $\lambda \in \sigma(-\Delta_\Omega^\alpha)$, satisfies the estimate (3.4.4), we first remark that if $\operatorname{Re}\alpha \geq 0$, then clearly

$$\Lambda_{\Omega,\alpha} \subset \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}. \tag{3.4.60}$$

Hence we may assume without loss of generality that $\operatorname{Re}\alpha < 0$. Then by definition (cf. (3.4.57)) and both Lemma 3.4.7 and Proposition 3.4.12 we have

$$\begin{aligned}
\operatorname{Re}\lambda &= \|\nabla u\|_2^2 + \operatorname{Re}\alpha \int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma && (3.4.61a)\\
&= \|\nabla u\|_2^2 - |\operatorname{Re}\alpha| \int_{\partial\Omega} |u|^2 \,\mathrm{d}\sigma && (3.4.61b)\\
&\geq \|\nabla u\|_2^2 - |\operatorname{Re}\alpha| \Big[C_1 \|\nabla u\|_2 + C_2 \Big] && (3.4.61c)\\
&\geq \|\nabla u\|_2^2 - |\operatorname{Re}\alpha| \left[\frac{C_1^2}{4} |\operatorname{Re}\alpha| + \frac{\|\nabla u\|_2^2}{|\operatorname{Re}\alpha|} + C_2 \right]; && (3.4.61d)
\end{aligned}$$

thus, we arrive at

$$\operatorname{Re}\lambda \geq -\left(\frac{C_1}{2}\right)^2 |\operatorname{Re}\alpha|^2 - C_2 |\operatorname{Re}\alpha| \tag{3.4.62}$$

which completes the proof. □

Remark 3.4.13. Suppose that $\partial\Omega$ is of class C^2. We recall that the constant $C_2 = C_2(\Omega)$ appearing in Theorem 3.4.1 and Lemma 3.4.7, as noted in (3.4.34), may in this case be taken

$$C_2 = \|\nabla\varphi\|_\infty + \max_{x \in \overline{\Omega_\varepsilon^+}} |\Delta d_\Omega(x)|, \tag{3.4.63}$$

where $\varepsilon > 0$ is as in Lemma 3.4.9 and φ is chosen to have support in $\overline{\Omega_\varepsilon^+}$. Let us be a bit more specific. We may take $\|\nabla\varphi\|_\infty$ to be $1/\varepsilon$, corresponding to a linear function of $t \in [0, \varepsilon]$ extended by 0 at $t = \varepsilon$ (which can be approximated arbitrarily well in the ∞-norm by C^1 functions), while for $x \in \overline{\Omega_\varepsilon^+}$, it is known that the Hessian of the signed distance function is equal to the Weingarten map of the (unique) surface S_t passing through x, at x. In particular,

$$|\Delta d_\Omega(x)| = \left| \sum_{j=1}^{d-1} \kappa_j^{S_t}(x) \right| = (d-1) \left| \bar{\kappa}^{S_t}(x) \right| \tag{3.4.64}$$

where $\kappa_1^{S_t}(\,\cdot\,), \ldots, \kappa_d^{S_t}(\,\cdot\,)$ are the principal curvatures at a given point of S_t and $\bar{\kappa}^{S_t}$ is its mean curvature [18, Lemma 2.4.2 and Remark 2.4.4]. This means that the essentially optimal form of the constant C_2 coming from our proof – to be compared with the coefficient of α in (3.4.8) – is

$$C_2 = \frac{1}{\varepsilon} + (d-1) \max_{t\in[0,\varepsilon]} \max_{x\in S_t} \left|\bar{\kappa}^{S_t}(x)\right|, \tag{3.4.65}$$

where $\varepsilon > 0$ is any constant for which Lemma 3.4.9(2) holds; in the language of [56], we may take any $\varepsilon \in (0, \mathrm{reach}(\partial\Omega)]$. (Note that we explicitly do not claim that C_2 from (3.4.65) is *optimal* in the general sense.) As a simple example we consider a ball $B \subset \mathbb{R}^d$.

Example 3.4.14. In the case of a ball $B_R \subset \mathbb{R}^d$, $d \geq 2$, of radius $R > 0$, for each principal curvature $\kappa_j^{S_t}$ on S_t (cf. Definition 3.4.8) we have

$$\kappa_j^{S_t} = \frac{1}{R-t} \tag{3.4.66}$$

for all $j = 1, \dots, d$, and thus the same holds for the mean curvature $\bar{\kappa}^{S_t}$, where $t = t(x)$ is chosen such that $x \in S_t$. Consequently, for $\varepsilon < R$, (3.4.65) becomes

$$C_2 = \frac{1}{\varepsilon} + (d-1) \max_{t\in[0,\varepsilon]} \left(\frac{1}{R-t}\right) = \frac{1}{\varepsilon} + \frac{d-1}{R-\varepsilon}. \tag{3.4.67}$$

We may take any $\varepsilon < R$ and due to the localisation Theorem 3.4.1 and (3.4.66) we arrive at the optimisation problem

$$\mathrm{Re}\,\lambda \geq -|\mathrm{Re}\,\alpha|^2 - \min_{\varepsilon\in(0,R)} \left[\frac{1}{R-\varepsilon} + \frac{d-1}{\varepsilon}\right] |\mathrm{Re}\,\alpha|. \tag{3.4.68}$$

To solve this, we interpret C_2 as a function of ε,

$$C_2(\varepsilon) := \frac{1}{\varepsilon} + \frac{d-1}{R-\varepsilon}, \tag{3.4.69}$$

to be minimised for $0 < \varepsilon < R$. Consequently, for a minimising $0 < \varepsilon_0 < R$, we require

$$0 = C_2'(\varepsilon_0) = -\frac{1}{\varepsilon_0^2} + \frac{d-1}{(R-\varepsilon_0)^2} \qquad \Leftrightarrow \qquad \varepsilon_0 = \frac{R}{\sqrt{d-1}+1}, \tag{3.4.70}$$

which, after a short calculation, gives

$$C_2(\varepsilon_0) = \frac{d + 2\sqrt{d-1}}{R}. \tag{3.4.71}$$

($C_2(\varepsilon_0)$ is indeed a minimum since $C_2''(\varepsilon) > 0$ for all $\varepsilon < R$). This value of C_2 may be compared with the known bound and asymptotics for real negative α

$$-|\alpha|^2 - \frac{d-1}{R}|\alpha| > \lambda_1(-\Delta_B^\alpha) = -|\alpha|^2 - \frac{d-1}{R}|\alpha| + o(\alpha) \tag{3.4.72}$$

where the inequality is valid for all $\alpha < 0$ and the asymptotic expansion is for $\alpha \to -\infty$, see [8, Theorem 3 and eq. (1.2)].

Remark 3.4.15. If we allow variable $\alpha \in L^\infty(\partial\Omega, \mathbb{C})$, then it is clear that similar results hold since the key trace estimate, Lemma 3.4.7, does not depend on α, although the region $\Lambda_{\Omega,\alpha}$ can no longer be described explicitly in general. However, (3.4.4) has a direct equivalent: if we set

$$\|\operatorname{Re}\alpha\|_\infty := \operatorname*{ess\,sup}_{x\in\partial\Omega} |\operatorname{Re}\alpha(x)|, \qquad \|\operatorname{Im}\alpha\|_\infty := \operatorname*{ess\,sup}_{x\in\partial\Omega} |\operatorname{Im}\alpha(x)|, \tag{3.4.73}$$

then, mimicking the arguments of the proof of Theorem 3.4.1 we obtain the estimate

$$\operatorname{Re}\lambda \geq -\frac{C_1^2}{4}\|\operatorname{Re}\alpha\|_\infty^2 - C_2\|\operatorname{Re}\alpha\|_\infty \tag{3.4.74}$$

for all $\lambda \in \sigma(-\Delta_\Omega^\alpha)$, or more generally all $\lambda \in W(a_\alpha)$, where

$$\begin{cases} C_1 \geq 2,\ C_2 > 0 & \text{for Lipschitz domains,} \\ C_1 \geq 2,\ C_2 = 2 & \text{for domains of class } C^2. \end{cases} \tag{3.4.75}$$

Even in the case of real-valued α, this may be viewed as a partial generalisation of [90, Remark 1.1], which establishes the asymptotics for variable $\alpha \in \mathbb{R}$ of the form $\alpha = tb(x)$, $t \to -\infty$, for a fixed continuous function $b \in C(\partial\Omega)$. Moreover, we can still obtain parabolic estimates on the numerical range of the type necessary to ensure that Δ_Ω^α generates a cosine function (cf. Remark 3.4.6). For simplicity assume that $\operatorname{Re}\alpha(x) \geq 0$ almost everywhere (whence also $\operatorname{Re}\lambda \geq 0$ for any $\lambda \in W(a_\alpha)$);

then, with C_1, C_2 as above,

$$|\mathrm{Im}\,\lambda| = \left|\int_{\partial\Omega} \mathrm{Im}\,\alpha\,|u|^2\,\mathrm{d}\sigma(x)\right| \leq \|\mathrm{Im}\,\alpha\|_\infty (C_1 \|\nabla u\|_2 + C_2) \tag{3.4.76a}$$

$$\leq \|\mathrm{Im}\,\alpha\|_\infty (C_1 \sqrt{\mathrm{Re}\,\lambda} + C_2), \tag{3.4.76b}$$

independently of $\mathrm{Re}\,\alpha \geq 0$.

3.5 The Dirichlet-to-Neumann operator

From now on, we will be interested in the asymptotic behaviour of the eigenvalues $\lambda(\alpha)$ of $-\Delta_\Omega^\alpha$ as $\alpha \to \infty$ in $\mathbb{C}$. To this end, we will exploit the duality between the Robin eigenvalue problem (1.1.5) and the eigenvalue problem

$$M(\lambda)g = \alpha g \tag{3.5.1}$$

of the *Dirichlet-to-Neumann operator* $M(\lambda)$ acting on $\partial\Omega$. This operator is defined for λ in the resolvent set $\rho(-\Delta_\Omega^D)$ of the Dirichlet Laplacian; its formal definition can be found in (3.5.9), the duality result just mentioned is contained in Theorem 3.5.8. For more information on this operator, we refer to, e.g., [12, 15, 21, 20, 43, 61, 92]; its relationship to the Robin Laplacian (at least for real α) is explored in [12, Section 2] and [15, Section 8], for example, and for complex α see for example [61, Section 3]. In order to define $M(\lambda)$, we first need to recall a solubility result for the inhomogeneous Dirichlet boundary value problem. Here and in what follows we fix a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$ and write $\mathrm{tr}\,u = u|_{\partial\Omega}$ for the trace of a function $u \in H^1(\Omega)$. However, if there is no ambiguity, we will tend to omit the "tr" notation. Moreover, recall that every $g \in H^{1/2}(\partial\Omega)$ is the trace of a function $u \in H^1(\Omega)$.

Lemma 3.5.1. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain and let $\lambda \in \rho(-\Delta_\Omega^D) \subset \mathbb{C}$.*

(1) For each $g \in H^{1/2}(\partial\Omega)$, the Dirichlet boundary value problem

$$\begin{aligned} -\Delta u &= \lambda u \qquad && \textit{in } \Omega, \\ u &= g && \textit{on } \partial\Omega, \end{aligned} \tag{3.5.2}$$

interpreted in the usual weak sense, has a unique solution $u_\lambda \in H^1(\Omega)$, that is, u_λ solves

$$\int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x = \lambda \int_\Omega u\overline{v}\,\mathrm{d}x \tag{3.5.3}$$

for all $v \in H^1_0(\Omega)$, and $\operatorname{tr} u = g$.

(2) For such λ, if

$$H^1(\lambda) := \{u \in H^1(\Omega) : -\Delta u = \lambda u \text{ as in } (3.5.3)\}, \tag{3.5.4}$$

then we have the direct sum decomposition $H^1(\Omega) = H^1_0(\Omega) \oplus H^1(\lambda)$.

Proof. For $\lambda \in \left(\mathbb{R} \cap \rho(-\Delta_\Omega^D)\right)$, this follows immediately from [12, Lemma 2.2], together with the fact that $H^{1/2}(\partial\Omega) = \operatorname{tr} H^1(\Omega)$; for general $\lambda \in \rho(-\Delta_\Omega^D)$ the same proof works verbatim. □

We denote by $P(\lambda) : H^{1/2}(\partial\Omega) \to H^1(\Omega)$ the Poisson operator given by

$$P(\lambda) : g \mapsto u_\lambda, \tag{3.5.5}$$

where u_λ solves (3.5.2), which is well defined for any $\lambda \in \rho(-\Delta_\Omega^D)$; indeed, one may show that $P(\lambda)$ is a bijection from $H^{1/2}(\partial\Omega)$ onto $H^1(\lambda)$ as defined in (3.5.4) and in fact a right inverse of the trace operator. We can now define the Dirichlet-to-Neumann operator. For $\lambda \in \rho(-\Delta_\Omega^D)$, we first define a sesquilinear form

$$q_\lambda : H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega) \to \mathbb{C} \tag{3.5.6}$$

by

$$q_\lambda[g,h] = \int_\Omega \nabla P(\lambda)g \cdot \overline{\nabla P(\lambda)h} - \lambda P(\lambda)g\,\overline{P(\lambda)h}\,\mathrm{d}x. \tag{3.5.7}$$

The (negative) *Dirichlet-to-Neumann operator*

$$M(\lambda) : D(M(\lambda)) \subset L^2(\partial\Omega) \to L^2(\partial\Omega) \tag{3.5.8}$$

is then the operator in $L^2(\partial\Omega)$ associated with $-q_\lambda$, which is given by

$$\begin{aligned} D(M(\lambda)) &= \left\{ g \in H^{1/2}(\partial\Omega) : \frac{\partial}{\partial\nu} P(\lambda) g \in L^2(\partial\Omega) \right\}, \\ M(\lambda) &= -\frac{\partial}{\partial\nu} P(\lambda), \end{aligned} \tag{3.5.9}$$

see [21]. In words, the Dirichlet-to-Neumann operator maps given *Dirichlet data* $g = \operatorname{tr} u$ to the *Neumann data* $-\frac{\partial u}{\partial\nu}$ of the same solution $u = P(\lambda)g$ of $-\Delta u = \lambda u$.

Lemma 3.5.2. *Let $\lambda \in \rho(-\Delta_\Omega^D)$. The operator $-M(\lambda)$ is*

(1) closed,

(2) densely defined,

(3) m-sectorial,

(4) and has compact resolvent in $L^2(\partial\Omega)$.

In particular, its spectrum consists of eigenvalues of finite algebraic multiplicity and is denoted by

$$\sigma(M(\lambda)) = \{\alpha_k \in \mathbb{C} : k \in \mathbb{N}\}. \tag{3.5.10}$$

Proof. Everything except the sectoriality follows immediately since $H^{1/2}(\partial\Omega)$ is densely and compactly embedded in $L^2(\partial\Omega)$, and q_λ is closed on $H^{1/2}(\partial\Omega)$ (see e.g. [101, Theorem 3.8]). For the sectoriality of the operator, it suffices to show that q_λ is sectorial, that is, that there exist constants $\omega, \mu \in \mathbb{R}$ such that

$$\operatorname{Re} q_\lambda[g, g] + \omega \|g\|^2_{L^2(\partial\Omega)} \geq \mu \|g\|^2_{H^{1/2}(\partial\Omega)} \tag{3.5.11}$$

for all $g \in H^{1/2}(\partial\Omega)$. To prove (3.5.11), by the fact that the trace map is bounded from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$, cf. Theorem 2.1.14, it certainly suffices to show that for any $\lambda \in \mathbb{C}$ there exists $\omega \geq 0$ such that the square root of η, given by

$$\eta(u) := \int_\Omega \left(|\nabla u|^2 - \operatorname{Re}\lambda |u|^2 \right) \mathrm{d}x + \omega \int_{\partial\Omega} |u|^2 \, \mathrm{d}\sigma, \tag{3.5.12}$$

$u \in H^1(\Omega)$, defines an equivalent norm on $H^1(\Omega)$. But this, in turn, follows immediately from Maz'ya's inequality in the form of [15, eq. (4)]. We conclude that

q_λ and $M(\lambda)$ are sectorial. For a slightly different (but equivalent) approach in the case of real λ, we refer to [16, Corollary 2.2 and Section 4.4]. However, the proof can be carried over verbatim to complex λ. □

Remark 3.5.3. It may be shown that the domain of the Dirichlet-to-Neumann operator is indeed $D(M(\lambda)) = H^1(\partial\Omega)$ for any $\lambda \in \rho(-\Delta_\Omega^D)$. However, since we will not need this, we refer to [94, Theorem 4.25 for $s = 1/2$] for both the proof and more details.

Remark 3.5.4. (1) It is also possible to define the Dirichlet-to-Neumann operator for $\lambda \in \sigma(-\Delta_\Omega^D)$, either as a multi-valued operator, or by factoring out the Dirichlet eigenfunctions corresponding to λ from $H^1(\Omega)$. We will not need this here, so we do not go into the details, which may be found in [12].

(2) In dimension $d = 1$, that is, for a bounded, non-degenerate interval, the Dirichlet-to-Neumann operator can be represented by the 2×2-matrix given by (4.1.16). Obviously, Lemma 3.5.2 continues to hold in this case. For more details we refer to Section 4.1.

Lemma 3.5.5. *The Dirichlet-to-Neumann operator $M(\lambda)$*

(1) is meromorphic with respect to the spectral parameter $\lambda \in \mathbb{C}$ and

(2) its singularities are poles of finite order and coincide with the eigenvalues of the corresponding Dirichlet Laplacian, i.e., the set of singularities of $\lambda \mapsto M(\lambda)$ is $\sigma(-\Delta_\Omega^D)$.

(3) For $\lambda \in \rho(-\Delta_\Omega^D)$, $M(\lambda)$ is a self-adjoint holomorphic operator family and the corresponding quadratic forms are holomorphic of type (a) (in the sense of Kato [74, Section VI.4.2], see Definition 3.2.6).

In the proof of the previous lemma we will use a perturbation formula for $M(\lambda)$ in terms of the fixed operator $M(0)$; for its proof we refer to [20, Lemma 2.4 for $\mu = 0$]. To give this statement, we note that for $\lambda \in \rho(-\Delta_\Omega^D)$ the Poisson operator $P(\lambda)$ given by (3.5.5) and its adjoint

$$P^*(\lambda) : H_0^{-1}(\Omega) \to H^{-1/2}(\partial\Omega) \tag{3.5.13}$$

are well defined.

Lemma 3.5.6. *If $\lambda \in \rho(-\Delta_\Omega^D)$, then we have*

$$M(\lambda) = M(0) + \lambda P(0)^* \left(I + \lambda(-\Delta_\Omega^D - \lambda I)^{-1}\right) P(0). \tag{3.5.14}$$

Proof of Lemma 3.5.5. The perturbation formula of Lemma 3.5.6 depends polynomially on λ and on the resolvent of the Dirichlet Laplacian which is known to be a meromorphic function with poles of finite order, as follows from Theorem 2.3.6. This proves that $M(\lambda)$, $\lambda \in \rho(-\Delta_\Omega^D)$, is a holomorphic operator family. It is self-adjoint holomorphic, i.e. $M(\overline{\lambda}) = (M(\lambda))^*$, by (3.5.14) and using that $M(0)$ is self-adjoint and that $\rho(-\Delta_\Omega^D)$ is symmetric about the real axis. The corresponding quadratic forms q_λ are holomorphic of type (a) (see Definition 3.2.5), where the sectoriality was proved in Lemma 3.5.2. ☐

Remark 3.5.7. One can show by exactly the same argument as in the proof of Theorem 3.2.11 that the corresponding eigenprojections can be chosen to depend holomorphically on $\lambda \in \rho(-\Delta_\Omega^D)$.

We can now state the following duality result linking the eigenvalues α of the operator $M(\lambda)$ and λ of $-\Delta_\Omega^\alpha$. In the case of real α this is standard and well known (see [12, Theorem 3.1]); however, for completeness' sake we give a proof in the complex case. In fact, the duality between elliptic differential operators and operators of Dirichlet-to-Neumann type, or so-called *Titchmarsh–Weyl M-functions*, is also known in the non-selfadjoint case, e.g. see [34, Theorem 4.10], but here we give a direct proof including a corresponding duality result for the eigenfunctions of both operators, that is, a function u is an eigenfunction of the Robin Laplacian if and only if $\operatorname{tr} u$ is an eigenfunction of the Dirichlet-to-Neumann operator.

Theorem 3.5.8. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain. For any $\alpha \in \mathbb{C}$ and any $\lambda \in \rho(-\Delta_\Omega^D)$, we have that*

(1) $\lambda \in \sigma(-\Delta_\Omega^\alpha)$ if and only if $\alpha \in \sigma(M(\lambda))$ and

(2) u is an eigenfunction of $-\Delta_\Omega^\alpha$ corresponding to the eigenvalue λ if and only if $\operatorname{tr} u$ is an eigenfunction of $M(\lambda)$ for its eigenvalue α.

Proof. Note first that the spectra of $M(\lambda)$ and $-\Delta_\Omega^\alpha$ consist only of eigenvalues of finite multiplicity (see Lemma 3.5.2 and Theorem 3.1.2, respectively). On the one hand, we have $\lambda \in \sigma(-\Delta_\Omega^\alpha)$ for given $\alpha \in \mathbb{C}$ with eigenfunction $u \in H^1(\Omega)$ if and only if

$$\int_\Omega \nabla u \cdot \overline{\nabla v} \,\mathrm{d}x - \lambda \int_\Omega u\overline{v} \,\mathrm{d}x = -\alpha \int_{\partial\Omega} u\overline{v} \,\mathrm{d}\sigma \tag{3.5.15}$$

for all $v \in H^1(\Omega)$, see (3.1.4). On the other hand, by (3.5.7) we have $\alpha \in \sigma(M(\lambda))$ for given $\lambda \in \rho(-\Delta_\Omega^D)$ with eigenfunction $g \in H^{1/2}(\partial\Omega)$ if and only if

$$\int_\Omega \nabla P(\lambda)g \cdot \overline{\nabla P(\lambda)h} - \lambda P(\lambda)g \,\overline{P(\lambda)h} \,\mathrm{d}x = -\alpha \int_{\partial\Omega} g\overline{h} \,\mathrm{d}\sigma \tag{3.5.16}$$

for all $h \in H^{1/2}(\partial\Omega)$. Using the fact that $P(\lambda)g$ satisfies the Dirichlet boundary condition in the weak sense (3.5.3), which we recall by

$$\int_\Omega \nabla P(\lambda)g \cdot \overline{\nabla v} \,\mathrm{d}x = \lambda \int_\Omega P(\lambda)g\overline{v} \,\mathrm{d}x \tag{3.5.17}$$

for all $v \in H_0^1(\Omega)$, together with the direct sum decomposition $H^1(\Omega) = H_0^1(\Omega) \oplus H^1(\lambda)$ of Lemma 3.5.1, it follows that the eigenfunction g of $M(\lambda)$ satisfies

$$\int_\Omega \Big(\nabla P(\lambda)g \cdot \overline{\nabla v} - \lambda P(\lambda)\,\overline{v}g\Big) \,\mathrm{d}x = -\alpha \int_{\partial\Omega} g\,\overline{\operatorname{tr} v} \,\mathrm{d}\sigma \tag{3.5.18}$$

for all $v \in H^1(\Omega)$. Comparing (3.5.15) and (3.5.18) leads immediately to the statement $\lambda \in \sigma(-\Delta_\Omega^\alpha)$ if and only if $\alpha \in \sigma(M(\lambda))$ (as long as $\lambda \in \rho(-\Delta_\Omega^D)$), with $g = \operatorname{tr} u$, or, equivalently, $u = P(\lambda)g$.

□

Remark 3.5.9. A corresponding statement holds for any generalised eigenfunctions, as shown very recently in [22]. For more details on this topic, see Section 3.7.

Finally, we turn to the proof of the dichotomy result.

Proof of Theorem 1.2.5. By Lemma 3.5.5 the Dirichlet-to-Neumann operator $M(\lambda)$ is a meromorphic operator family whose set of singularities consists of poles of finite order and coincides with the spectrum $\sigma(-\Delta_\Omega^D)$ of the corresponding Dirichlet Laplacian. Now let $(\alpha_k)_{k\in\mathbb{N}}$ be any complex sequence with $\alpha_k \to \infty$ as $k \to \infty$. Assume that the eigenvalues $\lambda_k := \lambda(\alpha_k)$ corresponding to α_k on a common analytic branch (for any fixed choice of slicing) remain bounded as $k \to \infty$; without loss of generality we may suppose that $\lambda_k \to \lambda_0 \in \mathbb{C}$ as $k \to \infty$. Then by Theorem 3.5.8, for each k we may write $\alpha_k = \alpha(\lambda_k)$ for the Dirichlet-to-Neumann eigenvalues, which likewise belong to a common analytic branch. For this branch we have $\alpha_k \to \infty$ as $\lambda_k \to \lambda_0$. By definition, this means that λ_0 must be a singularity of the operator family $M(\lambda)$. The only possibility is that $\lambda_0 \in \sigma(-\Delta_\Omega^D)$. □

3.6 The points of accumulation of the Robin eigenvalues

In this section we study the question of which values $\lambda \in \mathbb{C}$ can be reached as points of accumulation of the eigenvalues of $-\Delta_\Omega^\alpha$ as $\alpha \to \infty$. However, the answer of this question depends on *how* $\alpha \to \infty$ in $\mathbb{C}$, that is, in which regime the considered *α-path* runs to ∞ in $\mathbb{C}$. To this end, we confine ourselves to complex sectors having their vertex in the origin. We start by dividing the complex plane in the following fashion; here we assume that the principal argument is always between $-\pi$ and π. Furthermore, throughout this section we suppose $\Omega \subset \mathbb{R}^d$, $d \geq 2$, to be a fixed bounded Lipschitz domain; and for any set $A \subseteq \mathbb{C}$ the set of points of accumulation of A is denoted by $\operatorname{acc}(A)$.

Definition 3.6.1. (1) Let $0 < \theta < \pi/2$ be an (arbitrarily small) angle and define the two open sectors

$$S_\theta^+ := \{z \in \mathbb{C} \ : \ \theta < \arg z < \pi - \theta\} \tag{3.6.1}$$

and

$$T_\theta^+ := \{z \in \mathbb{C} \ : \ |\arg z| < \theta\} \tag{3.6.2}$$

in the upper and right-hand half-planes, respectively. We then define

$$S_\theta^- := -S_\theta^+ \qquad \text{and} \qquad T_\theta^- := -T_\theta^+ \tag{3.6.3}$$

to be the corresponding sectors reflected in the real and imaginary axes, respectively. Consequently, the complex plane is, up to two straight lines mutually crossing in $z = 0$, symmetrically partitioned into four sectors; see Figure 3.6.1.

(2) If $\theta = \pi/2$, both lower and upper sectors $S_\theta^\pm$ vanish and $T_\theta^\pm$ are defined as an extension of (3.6.2); and as before we define $T_{\pi/2}^- := -T_{\pi/2}^+$.

(3) If $\pi/2 < \theta' < \pi$, we define $T_{\theta'}^+$ by (3.6.2), that is, we have a partition of the complex plane in two asymmetric sectors $T_{\theta'}^+$ and $T_{\pi-\theta'}^-$.

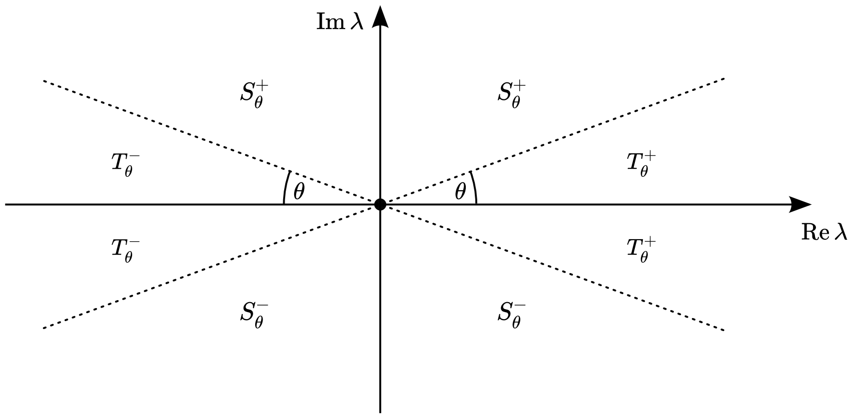

Figure 3.6.1: The four sectors $S_\theta^\pm$ and $T_\theta^\pm$ for $0 < \theta < \pi/2$.

Our principal aim is to prove Theorem 1.2.6. As mentioned in the introduction, the Dirichlet-to-Neumann operator will be used in the proof, more precisely of part (2). For (1), we will draw on some ideas similar to the ones of [38] for the case of real negative $\alpha \to -\infty$; in particular, the following lemma, which we will use repeatedly, recalls [38, Lemma 2.1].

Lemma 3.6.2. *Let $(\alpha_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ be any divergent sequence in $\mathbb{C}$ and for each $k \in \mathbb{N}$ select a Robin eigenvalue $\lambda_k := \lambda(\alpha_k) \in \sigma(-\Delta_\Omega^{\alpha_k})$ (we do* not *require the λ_k to belong to the same analytic eigenvalue branch). Suppose that*

(1) the sequence $(\lambda_k)_{k\in\mathbb{N}}$ is bounded, and

(2) for each $k \in \mathbb{N}$ there exists an associated eigenfunction ψ_k normed to $\|\psi_k\|_{L^2(\Omega)} = 1$, such that the sequence $(\|\psi_k\|_{H^1(\Omega)})_{k\in\mathbb{N}}$ of $H^1(\Omega)$-norms is bounded.

Then

$$\operatorname{acc}\{\lambda_k : k \in \mathbb{N}\} \subseteq \sigma(-\Delta_\Omega^D). \tag{3.6.4}$$

Moreover, if up to a subsequence $\lambda_k \to \lambda^D \in \sigma(-\Delta_\Omega^D)$, then up to a further subsequence the ψ_k converge weakly in $H^1(\Omega)$ to a Dirichlet eigenfunction associated with λ^D.

Proof. Let λ be any point of accumulation; without loss of generality we suppose that $\lim_{k\to\infty} \lambda_k = \lambda$. We first claim that the corresponding eigenfunctions ψ_k satisfy

$$\int_{\partial\Omega} |\psi_k|^2 \, \mathrm{d}\sigma \to 0 \tag{3.6.5}$$

as $k \to \infty$: in fact, this follows since an eigenfunction ψ_k satisfies $a_\alpha[\psi_k] = \lambda_k$, cf. (3.1.4), viz.

$$\int_{\partial\Omega} |\psi_k|^2 \, \mathrm{d}\sigma = \frac{1}{\alpha_k} \left[\lambda_k - \int_\Omega |\nabla \psi_k|^2 \, \mathrm{d}x \right] \tag{3.6.6}$$

for $\alpha_k \neq 0$. The fact that the both λ_k and the integrals

$$\int_\Omega |\nabla \psi_k|^2 \, \mathrm{d}x \tag{3.6.7}$$

are bounded by assumption together with $1/\alpha_k \to 0$ as $k \to \infty$ implies (3.6.5). Next, since the ψ_k are bounded in $H^1(\Omega)$, up to a subsequence there exists a weak limit $\psi \in H^1(\Omega)$ such that

$$\psi_k \rightharpoonup \psi \text{ in } H^1(\Omega) \tag{3.6.8}$$

as $k \to \infty$. By the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, cf. Theorem 2.1.10, we have

$$\psi_k \rightharpoonup \psi \text{ in } L^2(\Omega). \tag{3.6.9}$$

Additionally, using the compactness of Theorem 2.1.10 for $t = 3/4 < 1 = s$, Theorem 2.1.14 for $s = 3/4$, and the boundedness of the boundary $\partial\Omega$, we have that the map given by the composition

$$H^1(\Omega) \overset{\Subset}{\hookrightarrow} H^{3/4}(\Omega) \overset{\mathrm{tr}}{\longrightarrow} H^{1/4}(\partial\Omega) \hookrightarrow L^2(\partial\Omega) \tag{3.6.10}$$

is compact, that is, we have the sequence $\operatorname{tr} \psi_k$ converges in $L^2(\partial\Omega)$. In particular, ψ

has zero trace, so $\psi \in H_0^1(\Omega)$. Finally, using the eigenvalue equation for λ_k and the weak H^1-convergence of the ψ_k, for all $\varphi \in H_0^1(\Omega)$ (whose traces vanish) we have

$$\int_\Omega \nabla\psi \cdot \overline{\nabla\varphi}\,\mathrm{d}x = \lim_{k\to\infty} \int_\Omega \nabla\psi_k \cdot \overline{\nabla\varphi}\,\mathrm{d}x \tag{3.6.11a}$$

$$= \lim_{k\to\infty} \lambda_k \int_\Omega \psi_k\overline{\varphi}\,\mathrm{d}x = \lambda \int_\Omega \psi\overline{\varphi}\,\mathrm{d}x. \tag{3.6.11b}$$

Since $\psi \in H_0^1(\Omega)$, this says exactly that λ is an eigenvalue, and ψ a corresponding eigenfunction, of the Dirichlet Laplacian. □

We can now give the proof of Theorem 1.2.6(1). We will in fact prove the following slightly more precise version, which also allows us to conclude convergence of the eigenfunctions not mentioned in Theorem 1.2.6. As before, note that we do not require our eigenvalues to belong to the same analytic curve and recall that the sector T_θ^- is the sector in the left half-plane with semi-angle $0 < \theta < \pi/2$ introduced in Definition 3.6.1.

Theorem 3.6.3. *Let $(\alpha_k)_{k\in\mathbb{N}}$ be any divergent sequence in the sector $\mathbb{C} \setminus T_\theta^-$ for some $\theta > 0$ such that $\alpha_k \to \infty$ in $\mathbb{C}$, and for each $k \in \mathbb{N}$ let*

$$\lambda_k := \lambda(\alpha_k) \in \sigma(-\Delta_\Omega^{\alpha_k}) \tag{3.6.12}$$

be any corresponding eigenvalue. Then

$$\mathrm{acc}\{\lambda_k : k \in \mathbb{N}\} \subseteq \sigma(-\Delta_\Omega^D). \tag{3.6.13}$$

Moreover, if up to a subsequence $\lambda_k \to \lambda \in \sigma(-\Delta_\Omega^D)$, then there exist eigenfunctions for λ_k which, possibly up to a further subsequence, converge weakly to a Dirichlet eigenfunction for λ.

Proof of Theorem 3.6.3 and hence of Theorem 1.2.6(1). The goal is to directly apply Lemma 3.6.2 in order to obtain the conclusion of the theorem. For this we need to show that under the stated assumptions if the λ_k (or any subsequence thereof) are bounded, then they always admit corresponding eigenfunctions ψ_k which (under the normalisation $\|\psi_k\|_2 = 1$) are bounded in $H^1(\Omega)$. To this end, first assume without loss of generality that the sequence $(\lambda_k)_{k\in\mathbb{N}}$ actually converges to some $\lambda \in \mathbb{C}$; we distinguish between two possibilities, which together completely cover the sector $\mathbb{C} \setminus T_\theta^-$: (i) $\operatorname{Re} \alpha_k \geq 0$ for all $k \in \mathbb{N}$ and (ii) $\left|\frac{\operatorname{Re} \alpha_k}{\operatorname{Im} \alpha_k}\right|$ remains bounded, respectively.

(i) Suppose that, for each λ_k, ψ_k is any associated eigenfunction such that $\|\psi_k\|_2 = 1$. We observe that the weak form (3.1.4) of the eigenvalue equation implies

$$\int_\Omega |\nabla \psi_k|^2 \, \mathrm{d}x + \operatorname{Re} \alpha_k \int_{\partial\Omega} |\psi_k|^2 \, \mathrm{d}\sigma = \operatorname{Re} \lambda_k \to \operatorname{Re} \lambda \tag{3.6.14}$$

as $k \to \infty$. Since $\operatorname{Re} \alpha_k \geq 0$, this is only possible if the first summand, that is, the sequence $(\|\nabla \psi_k\|_2^2)_{k \in \mathbb{N}}$, remains bounded, which in turn means that the ψ_k form a bounded sequence in $H^1(\Omega)$ and Lemma 3.6.2 is applicable.

(ii) Let the ψ_k be as before but now we consider the imaginary part of (3.1.4) to obtain

$$\operatorname{Im} \alpha_k \int_{\partial\Omega} |\psi_k|^2 \, \mathrm{d}\sigma = \operatorname{Im} \lambda_k \to \operatorname{Im} \lambda \tag{3.6.15}$$

by assumption. Now the condition (ii)

$$\sup_{k \in \mathbb{N}} \left|\frac{\operatorname{Re} \alpha_k}{\operatorname{Im} \alpha_k}\right| < \infty \tag{3.6.16}$$

implies that

$$\operatorname{Re} \alpha_k \int_{\partial\Omega} |\psi_k|^2 \, \mathrm{d}\sigma \tag{3.6.17}$$

and hence also

$$\alpha_k \int_{\partial\Omega} |\psi_k|^2 \, \mathrm{d}\sigma \tag{3.6.18}$$

must in particular remain bounded as $k \to \infty$. Since λ_k was also assumed bounded, we conclude that

$$\int_\Omega |\nabla \psi_k|^2 \, \mathrm{d}x = \lambda_k - \alpha_k \int_{\partial\Omega} |\psi_k|^2 \, \mathrm{d}\sigma \tag{3.6.19}$$

likewise remains bounded (recall that $\|\psi_k\|_2 = 1$), meaning that the ψ_k are bounded in $H^1(\Omega)$.

□

We next turn to the proof of Theorem 1.2.6(2). This is in fact an immediate implication of the fact that the Dirichlet-to-Neumann operator is unbounded.

Proof of Theorem 1.2.6(2). First suppose that $\lambda \in \rho(-\Delta_\Omega^D)$ and let $M(\lambda)$ be the Dirichlet-to-Neumann operator introduced in Section 3.5 (see (3.5.9)). Then by Lemma 3.5.2, $M(\lambda)$ admits a sequence of eigenvalues $\alpha_k \in \mathbb{C}$ such that $|\alpha_k| \to \infty$. By Theorem 3.5.8, for each such $\alpha_k \in \mathbb{C}$, we have that $\lambda \in \sigma(-\Delta_\Omega^{\alpha_k})$.

For $\lambda \in \sigma(-\Delta_\Omega^D)$ the argument is the same except that $M(\lambda)$ becomes a multivalued operator; see Remark 3.5.4 (2). □

Remark 3.6.4. We draw explicit attention to the marked contrast between parts (1) and (2) of Theorem 1.2.6: on the one hand, for α diverging away from the negative real semiaxis (more precisely outside the sector T_θ^- for arbitrarily small $\theta > 0$), all eigenvalues either diverge absolutely or converge to points in the Dirichlet spectrum. This is not just true of the individual analytic branches of eigenvalues but for any arbitrary sequence of eigenvalues in this region. On the other hand, for *any* $\lambda \in \mathbb{C}$ we can find an infinite sequence of parameters α_k, which must end up "close" to the negative real semi-axis, for which λ is a Robin eigenvalue (this is where the sufficiently large eigenvalues of the Dirichlet-to-Neumann operator $M(\lambda)$ are to be found, for any λ). Thus the whole of $\mathbb{C}$ can be obtained as points of accumulation if we place no restriction on α. The reason why this is not inconsistent with Theorem 1.2.5 is that there we are interested in the behaviour of the *analytic curves of eigenvalues* (rather than sequences of α_k which may be drawn from different analytic curves).

3.7 Jordan chains

The following results are due to [21, Sections 2, and 3], where characterisations of Jordan chains of m-sectorial second-order elliptic partial differential operators with measurable coefficients and (local or non-local) Robin boundary conditions are studied. We, however, focus on the application of this theory on the Robin Laplacian, c.f. [21, Section 4]. Throughout this section we denote by

$$\mathcal{A} : D(\mathcal{A}) \subset B \to B \tag{3.7.1}$$

a linear operator in a Banach space B. Furthermore, let B_1, B_2 be Banach spaces, let $F \subset \mathbb{C}$ be an open complex set and for all $\lambda \in F$ let $M(\lambda) \in \mathcal{L}(B_1, B_2)$ be holomorphic on F. We denote by $M^l(\lambda)$ the lth derivative of $M(\lambda)$ for $l \in \mathbb{N}_0$.

Definition 3.7.1. Let $k \in \mathbb{N}_0$, $f_{-1} = 0$ and $\lambda_0 \in \mathbb{C}$.

(1) The set of vectors $J_{\mathcal{A},k} := \{f_0, \dots, f_k\} \subset B$ (for $f_0 \neq 0$) is called *Jordan chain* for $\mathcal{A}$ at λ_0 if $J_{\mathcal{A},k} \subset D(\mathcal{A})$ and

$$(\mathcal{A} - \lambda_0) f_j = f_{j-1} \tag{3.7.2}$$

for all $0 \leq j \leq k$. The vector f_0 is called *eigenvector* of $\mathcal{A}$ at the *eigenvalue* λ_0; the other vectors $f_1, \dots, f_k$ are called *generalised eigenvectors* (or *root vectors*) of $\mathcal{A}$ at λ_0.

(2) The set of vectors $J_{M(\lambda_0),k} := \{\varphi_0, \dots, \varphi_k\}$ (for $\varphi_0 \neq 0$) is called *Jordan chain* for $M(\,\cdot\,)$ at $\lambda_0 \in F$ if $J_{M(\lambda_0),k} \subset B_1$ and

$$\sum_{l=0}^{j} \frac{1}{l!} M^{(l)}(\lambda_0) \varphi_{j-l} = 0 \tag{3.7.3}$$

for all $0 \leq j \leq k$. The vector φ_0 is called *eigenvector* of $M(\,\cdot\,)$ at the *eigenvalue* λ_0; the other vectors $\varphi_1, \dots, \varphi_k$ are called *generalised eigenvectors* (or *root vectors*) of $M(\,\cdot\,)$ at λ_0.

Jordan chains for holomorphic operator functions originated in [75], however, for more details we refer to [91, Section II.11]. The following theorem (that is, [21, Theorem 4.1] adapted to our needs) characterises Jordan chains for Robin Laplacians by those of the corresponding Dirichlet-to-Neumann operator, and vice versa.

Theorem 3.7.2. *Let $\lambda_0 \in \rho(-\Delta_\Omega^D)$ and $f_{-1} = 0$. Consider the holomorphic function*

$$\lambda \mapsto M[\alpha](\lambda) := M(\lambda) - \alpha \tag{3.7.4}$$

from $\rho(-\Delta_\Omega^D)$ into $\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$. Then the following holds.

(1) Let $J_{-\Delta_\Omega^\alpha,k} = \{f_0, \dots, f_k\}$ be a Jordan chain for $-\Delta_\Omega^\alpha$ at λ_0. For all $m \in$

$\{0, \ldots, k\}$ *define*

$$\varphi_m := \operatorname{tr} f_m. \tag{3.7.5}$$

Then, $J_{M[\alpha](\lambda_0),k} = \{\varphi_0, \ldots, \varphi_k\}$ *is a Jordan chain for* $M[\alpha](\,\cdot\,)$ *at* λ_0*.*

(2) Let $\{\varphi_0, \ldots, \varphi_k\}$ *be a Jordan chain for* $M[\alpha](\,\cdot\,)$ *at* λ_0*. For all* $m \in \{0, \ldots, k\}$ *let* $f_m \in H^1(\Omega)$ *be the unique solution of the boundary value problem*

$$(-\Delta - \lambda_0) f_m = f_{m-1}, \qquad \operatorname{tr} f_m = \varphi_m. \tag{3.7.6}$$

Then $\{f_0, \ldots, f_k\}$ *is a Jordan chain for* $-\Delta_\Omega^\alpha$ *at* λ_0*.*

Chapter 4

Examples: Intervals, Cuboids, and Balls

Contents

As already mentioned in the paragraph *How to read this thesis*, this chapter can fulfil two different purposes: on the one hand as a *preparation* for the whole subject of complex Robin Laplacians, that is, to gain insight into what to expect in general, or, on the other hand, to apply the theory introduced and developed in the previous chapters, especially in Chapter 3. Here, we start by studying the case of simple geometries, where everything can be computed explicitly. In the first section we will consider the Robin Laplacian on the symmetric interval $\Omega = (-a, a)$ for $a > 0$ to explicitly construct the Dirichlet-to-Neumann matrix $M(\lambda)$ and compute its two eigenvalues $\alpha_\pm$ for any spectral parameter $\lambda \in \rho(-\Delta_\Omega^D)$. We further examine the spectral and asymptotical properties of $M(\lambda)$ to exploit the duality between Robin eigenvalues $\lambda(\alpha)$ and the eigenvalues $\alpha(\lambda)$ of the Dirichlet-to-Neumann operator; and by this we obtain the asymptotical behaviour of the Robin eigenvalues as $\alpha \to \infty$ in $\mathbb{C}$. In Section 4.2, we use these ideas obtained for intervals to conclude results for higher-dimensional cuboids in order to generalise the one-dimensional results. In Section 4.3 we consider d-dimensional balls as another canonical example: we separate the radial part of the Laplacian from the spherical Laplace-Beltrami operator to conclude a somewhat closed representation of the Dirichlet-to-Neumann matrix. Its entries consist of scaled Bessel functions whose asymptotical behaviour is used to obtain the asymptotics of its eigenvalues $\alpha(\lambda)$. As before, we exploit the duality of $\lambda(\alpha)$ and $\alpha(\lambda)$ to finally obtain the spectral asymptotics of the Robin eigenvalues as $\alpha \to \infty$.

4.1 The interval

We start by fixing $a > 0$ and consider the interval $\Omega = (-a, a) \subset \mathbb{R}$ of length $2a$; here the Robin boundary value problem for any given $\alpha \in \mathbb{C}$ reads

$$-\Delta u = -u'' = \lambda u \qquad \text{on } (-a, a), \tag{4.1.1a}$$

$$-u'(-a) + \alpha u(-a) = 0, \tag{4.1.1b}$$

$$+u'(+a) + \alpha u(+a) = 0. \tag{4.1.1c}$$

Note that the sign of $u'(\pm a)$ corresponds to the outer normal derivative at $\pm a$. We wish to find an explicit matrix representation of the Dirichlet-to-Neumann operator introduced in Section 3.5. To this end, we start with the inhomogeneous Dirichlet eigenvalue problem on $\Omega = (-a, a)$, namely

$$-u'' = \lambda u \qquad \text{on } (-a, a), \tag{4.1.2a}$$

$$u(-a) = g_1, \tag{4.1.2b}$$

$$u(+a) = g_2, \tag{4.1.2c}$$

for given Dirichlet data $g := (g_1, g_2)^T \in \mathbb{C}^2$, which allows us to study the Robin problem (4.1.1) with the same methods described in Section 3.5: any $\lambda \in \mathbb{C}$ solving the Dirichlet problem (4.1.2) for given $g \in \mathbb{C}^2$ is an eigenvalue of the Laplacian with complex Robin boundary conditions (4.1.1) if and only if there is a solution u of (4.1.2) such that

$$u'(-a) = \alpha g_1 \qquad \text{and} \qquad -u'(a) = \alpha g_2. \tag{4.1.3}$$

Let us write $M(\lambda)$ for the function which maps

$$(g_1, g_2)^T \mapsto (u'(-a), -u'(a))^T, \tag{4.1.4}$$

that is, $M(\lambda) \in \mathbb{C}^{2\times 2}$ is the *Dirichlet-to-Neumann operator*. In the case of $\Omega = (-a, a)$, the operator in its representation as a 2×2-matrix maps given Dirichlet data $(g_1, g_2)^T$ to the associated Neumann data. Dirichlet-to-Neumann operators in general are considered in Section 3.5. Thus a Robin eigenvalue λ of $-\Delta^{\alpha}_{(-a,a)}$ for given α corresponds to an eigenvalue α of the eigenvalue equation

$$M(\lambda)g = \alpha g = \alpha \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \tag{4.1.5}$$

for given λ. In anticipation of our later strategy, to study the behaviour of the Robin eigenvalues, we will in fact study the eigenvalues α of the matrix $M(\lambda)$. To this end, we start with the general solution u of (4.1.2) given by

$$u(x) = C_+ \cos(\sqrt{\lambda}x) + C_- \sin(\sqrt{\lambda}x). \tag{4.1.6}$$

The coefficients C_+ and C_- depend on the (half) interval length a, the square root of the spectral parameter $\sqrt{\lambda}$, and the Dirichlet data g: adding and subtracting both equations

$$u(-a) = C_+ \cos(\sqrt{\lambda}a) - C_- \sin(\sqrt{\lambda}a) = g_1 \tag{4.1.7a}$$

$$\text{and} \qquad u(+a) = C_+ \cos(\sqrt{\lambda}a) + C_- \sin(\sqrt{\lambda}a) = g_2, \tag{4.1.7b}$$

respectively, immediately implies

$$C_+ := \frac{g_2 + g_1}{2\cos(\sqrt{\lambda}a)} \qquad \text{and} \qquad C_- := \frac{g_2 - g_1}{2\sin(\sqrt{\lambda}a)}, \tag{4.1.8}$$

where $C_+ = 0$ if u is odd and $C_- = 0$ if u is even. Using these constants, the (outer) normal derivatives of u read

$$-u'(-a) = \sqrt{\lambda}\left(-\frac{g_2+g_1}{2}\tan\sqrt{\lambda}a - \frac{g_2-g_1}{2}\cot\sqrt{\lambda}a\right), \tag{4.1.9}$$

and similarly

$$u'(+a) = \sqrt{\lambda}\left(-\frac{g_2+g_1}{2}\tan\sqrt{\lambda}a + \frac{g_2-g_1}{2}\cot\sqrt{\lambda}a\right). \tag{4.1.10}$$

Consequently, for λ such that

$$\lambda \notin S_a := \{z \in \mathbb{C} : \sin(\sqrt{z}a)\cos(\sqrt{z}a) = 0\} \tag{4.1.11}$$

we map

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \mapsto -\begin{pmatrix} -u'(-a) \\ u'(+a) \end{pmatrix} = \sqrt{\lambda}\begin{pmatrix} \frac{g_2+g_1}{2}\tan\sqrt{\lambda}a + \frac{g_2-g_1}{2}\cot\sqrt{\lambda}a \\ \frac{g_2+g_1}{2}\tan\sqrt{\lambda}a - \frac{g_2-g_1}{2}\cot\sqrt{\lambda}a \end{pmatrix} \tag{4.1.12}$$

and the corresponding matrix representation of this linear map, namely the Dirichlet-to-Neumann matrix,

$$M(\lambda) = \frac{\sqrt{\lambda}}{2}\begin{pmatrix} \tan\sqrt{\lambda}a - \cot\sqrt{\lambda}a & \tan\sqrt{\lambda}a + \cot\sqrt{\lambda}a \\ \tan\sqrt{\lambda}a + \cot\sqrt{\lambda}a & \tan\sqrt{\lambda}a - \cot\sqrt{\lambda}a \end{pmatrix}, \tag{4.1.13}$$

is well defined (for $\lambda \notin S_a$). Then, by construction, we have that u is an eigenvector for the eigenvalue α if and only if

$$-\partial_\nu u(\pm a) = M(\lambda)u(\pm a) = \alpha u(\pm a), \tag{4.1.14}$$

that is, exactly the Robin boundary condition (4.1.1). By the two identities

$$\tan z - \cot z = -2\cot 2z \qquad \text{and} \qquad \tan z + \cot z = 2\csc 2z \tag{4.1.15}$$

for $z \in \mathbb{C}$ with $\sin z \neq 0$ we arrive at

$$M(\lambda) = \sqrt{\lambda} \begin{pmatrix} -\cot 2\sqrt{\lambda}a & \csc 2\sqrt{\lambda}a \\ \csc 2\sqrt{\lambda}a & -\cot 2\sqrt{\lambda}a \end{pmatrix}. \tag{4.1.16}$$

To calculate the eigenvalues α_+ and α_- of (4.1.16) we apply the identity $\csc^2 z - \cot^2 z = 1$ to

$$\begin{aligned} \det(M(\lambda) - \alpha I_2) = \alpha^2 + 2\alpha\sqrt{\lambda}\cot(2\sqrt{\lambda}a) \\ + \lambda\cot^2(2\sqrt{\lambda}a) - \lambda\csc^2(2\sqrt{\lambda}a) \end{aligned} \tag{4.1.17}$$

to obtain

$$\alpha_\pm = \sqrt{\lambda}\left(\pm\csc(2a\sqrt{\lambda}) - \cot(2a\sqrt{\lambda})\right) \tag{4.1.18}$$

except at the singularities of cot and csc, that is, except at $z \in \mathbb{C}$ such that $\sin(2a\sqrt{z}) = 0$; these singularities, however, correspond exactly to the numbers

$$\lambda = \frac{\pi^2 j^2}{4a^2}, \qquad j \in \mathbb{Z}, \tag{4.1.19}$$

i.e., the eigenvalues of the Dirichlet Laplacian $-\Delta^D_{(-a,a)}$. From this representation we can also deduce that the eigenvalues $\lambda(\alpha)$ of (4.1.5) depend analytically on $\lambda \neq \pi^2 j^2/(4a^2)$, or, equivalently, away from possible crossing points $\lambda(\alpha)$ can be considered as analytic curves. This is proved formally, and in a more general setting, in Section 3.2. Moreover, to establish what types of behaviour of $\lambda(\alpha)$ are possible as $\alpha \to \infty$, we may equally ask what conditions on λ guarantee that the eigenvalues α of the matrix $M(\lambda)$ diverge. It occurs that there are three different situations in which this can happen. We classify them as follows:

(1) $\sqrt{\lambda}$ approaches a pole of cot or csc, that is, zeros of sin, which represent the Dirichlet eigenvalues. In this case the Robin eigenvalue λ converges to a Dirichlet eigenvalue as $\alpha \to \infty$;

(2) λ diverges to ∞ in $\mathbb{C}$ away from the positive real axis, where the poles of cot and csc are located. In this case, as we shall see, both eigenvalues of $M(\lambda)$ diverge as $\pm\mathrm{i}\sqrt{\lambda}$, corresponding to two divergent Robin eigenvalues $\lambda = -\alpha^2 + o(\alpha^2)$;

(3) λ diverges to ∞ but remains within a finite distance of the real axis, say some strip parallel to $\operatorname{Im}\lambda = 0$. While it is clear that the eigenvalues of $M(\lambda)$ must also diverge in this case, the relationship between α and λ appears to be more complicated owing to the proximity of $\sqrt{\lambda}$ to the poles of $M(\lambda)$.

Let us examine each situation a little more closely, that is, the next three sections are dedicated to the three cases (1) – (3).

4.1.1 Convergence to the Dirichlet spectrum

For the real case $\alpha \in \mathbb{R}$ it is known from Theorem 2.4.8 (for the original article, see [57, Theorem 2]) that on any bounded domain of class C^2 the jth Robin eigenvalue $\lambda_j(\alpha) \in \mathbb{R}$ (numbered ascendingly) converges to the associated Dirichlet eigenvalue λ_j^D, that is, there exists some constant $C > 0$ such that

$$0 \le \lambda_j^D - \lambda_j(\alpha) \le C \frac{(\lambda_j^D)^2}{\sqrt{\alpha}} \tag{4.1.20}$$

as $\alpha \to +\infty$, where C does not depend on j. Again the proofs make heavily use of the self-adjointness of the problem. For the interval one finds a similar behaviour as $\operatorname{Re}\alpha \to +\infty$ as shown in the next theorem.

Consider the behaviour of the eigenvalues $\alpha(\lambda)$ of $M(\lambda)$ as $\sqrt{\lambda}$ approaches a singularity of cot or csc, that is, λ approaches an eigenvalue of the Dirichlet Laplacian: this is the only case in which α may diverge while λ remains bounded. Inverting this statement by writing λ as a function of α leads to the following theorem (see [30, Theorem 2.1]).

Theorem 4.1.1. *Suppose the analytic eigencurve $\lambda = \lambda(\alpha)$ remains bounded as $\alpha \to \infty$ in $\mathbb{C}$. Then it converges to some eigenvalue of the Dirichlet Laplacian, that is, there exists some $j \in \mathbb{Z}$ such that*

$$\lambda(\alpha) \to \frac{\pi^2 j^2}{4a^2} \tag{4.1.21}$$

as $\alpha \to \infty$.

Proof. The poles of cot and csc are of order one, and thus so are the poles of the meromorphic Dirichlet-to-Neumann operator $M(\lambda)$ given by (4.1.16). If $\lambda(\alpha)$ remains bounded as $\alpha \to \infty$ in $\mathbb{C}$, the only possibility for this behaviour is that $\sqrt{\lambda}$ approaches one of said poles, namely $\sqrt{\lambda} \to \pi j/(2a)$ for any $j \in \mathbb{Z}$, that is, $\lambda \to \pi^2 j^2/(4a^2)$ as $\alpha \to \infty$. □

Remark 4.1.2. In principle, one could derive additional terms in the asymptotic expansion of $\lambda(\alpha)$ as $\alpha \to \infty$, in powers of α^{-1}; let us sketch briefly how one might get further information. The fact that the poles of $M(\lambda)$ coincide with the Dirichlet eigenvalues λ_j^D allows us to obtain a partial fraction decomposition

$$M(\lambda) = \frac{1}{\sqrt{\lambda} A_j - \sqrt{\lambda_j^D}} + G_j(\sqrt{\lambda}) \tag{4.1.22}$$

for a matrix-valued function G_j which is holomorphic (thus bounded) in a neighbourhood of $\sqrt{\lambda_j^D}$, and matrices A_j. Calculating the residues $\pm\pi j/(2a^2)$ of the on- and off-diagonal components of $M(\lambda)$, we can write down A_j explicitly, which, together with the bounded G_j terms, may yield a more detailed statement.

4.1.2 Divergent eigenvalues away from the positive real axis

Suppose now that $\lambda \to \infty$ in $\mathbb{C}$ in such a way that its distance to the positive real axis diverges. For simplicity, we will actually suppose that λ diverges in a non-trivial sector $\mathbb{C} \setminus T_\theta^+$ away from the positive real axis (recall Definition 3.6.1 and Figure 3.6.1). We then make the following assumption.

Assumption 4.1.3. We suppose that λ diverges in the sector

$$\mathbb{C} \setminus T_{2\theta}^+ = \{z \in \mathbb{C} : 2\theta < \arg z < 2\pi - 2\theta\} \tag{4.1.23}$$

for some small $\theta \in (0, \pi/2)$.

This assumption clearly ensures that λ does not approach any eigenvalue $\lambda_j^D \in \sigma(-\Delta_\Omega^D) \subset \mathbb{R}$ of the Dirichlet Laplacian. However, as we shall see, it is not necessary for the asymptotical behaviour $\lambda = -\alpha^2 + o(\alpha^2)$. Moreover, the assumption is equivalent to $\sqrt{\lambda}$ diverging to ∞ in one of the sectors $S_\theta^\pm$. But this implies in particular that $\operatorname{Im} \sqrt{\lambda} \to \pm\infty$, and for such $\sqrt{\lambda}$ we can determine the asymptotic behaviour of the Dirichlet-to-Neumann matrix (4.1.16), based on the asymptotics of its entries

$$\cot z = \mathrm{i}\left(1 + \frac{2}{\mathrm{e}^{2\mathrm{i}z} - 1}\right) = \mp\mathrm{i} + \mathcal{O}\left(\mathrm{e}^{\mp 4 \operatorname{Im} z}\right) \tag{4.1.24}$$

and

$$\csc z = \frac{2\mathrm{i}}{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}} = \mathcal{O}\left(\mathrm{e}^{\mp 2 \operatorname{Im} z}\right) \tag{4.1.25}$$

as $\operatorname{Im} z \to \pm\infty$, independently of $\operatorname{Re} z$. Indeed, Assumption 4.1.3 allows us to choose $z = a\sqrt{\lambda}$, which leads to

$$M(\lambda) = \mathrm{i}\sqrt{\lambda}\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} + \mathcal{O}\left(\sqrt{\lambda}\mathrm{e}^{\mp 2a \operatorname{Im}\sqrt{\lambda}}\right). \tag{4.1.26}$$

Considering the leading coefficients to be $\mathrm{i}I_2$ and $-\mathrm{i}I_2$, respectively, and recalling the eigenvalue problem for $M(\lambda)$ (4.1.5), in each of the cases $\operatorname{Im}\sqrt{\lambda} \to +\infty$ and $\operatorname{Im}\sqrt{\lambda} \to -\infty$ we obtain the respective existence of exactly one diverging eigenvalue behaving like $\alpha = \alpha(\lambda)$, whose square satisfies the behaviour

$$\alpha^2 = -\lambda + \mathcal{O}\left(\lambda \mathrm{e}^{\mp 2a \operatorname{Im}\sqrt{\lambda}}\right) \tag{4.1.27}$$

as $\operatorname{Im}\sqrt{\lambda} \to \pm\infty$. Inverting the equation from $\alpha(\lambda)$ to $\lambda(\alpha)$ and noting that these eigenvalues always correspond to $\operatorname{Re}\alpha \to -\infty$ (more precisely, we want $\alpha \to \infty$ in the left half-plane away from the imaginary axis, in order to guarantee that $-\alpha^2$ remains away from the positive real axis), we arrive at the following result (see [30, Theorem 2.4]).

Theorem 4.1.4. *For the interval $\Omega = (-a, a)$, if $\alpha \to \infty$ in a sector of the form T_φ^- for any $\varphi \in (0, \pi/2)$ (see Definition 3.6.1), then for any $\theta \in (0, \pi - 2\varphi)$ there are exactly two divergent eigenvalues of the Robin Laplacian in the sector $\mathbb{C} \setminus T_\theta^+$; these satisfy the asymptotics*

$$\lambda(\alpha) = -\alpha^2 + \mathcal{O}\left(\alpha^2 \mathrm{e}^{2a \operatorname{Re}\alpha}\right) \tag{4.1.28}$$

as $\alpha \to \infty$ in T_φ^-. If $\alpha \to \infty$ in such a way that $\operatorname{Re}\alpha$ remains bounded from below, then the Robin Laplacian has no divergent eigenvalues in $\mathbb{C} \setminus T_\theta^+$, for any $\theta > 0$.

A special case and immediate implication (see [30, Corollary 2.5]) of the latter theorem is α diverging on any ray (half-line) in the sector T_φ^- for some given $\varphi \in (0, \pi/2)$: we suppose α may be written as a function

$$\alpha : (0, \infty) \ni t \mapsto t\mathrm{e}^{\mathrm{i}\vartheta} \in \mathbb{C} \tag{4.1.29}$$

for some fixed $\pi/2 < \vartheta < 3\pi/2$, which in particular means that $\alpha(t) \in T_\varphi^-$ for all $t > 0$.

Corollary 4.1.5. *For the interval $\Omega = (-a, a)$, if $\alpha(t) = t\mathrm{e}^{\mathrm{i}\vartheta} \to \infty$ for any fixed $\pi/2 < \vartheta < 3\pi/2$, then for any $\theta \in (0, \pi - 2\vartheta)$, for sufficiently large $t > 0$ there are exactly two eigenvalues λ of the Robin Laplacian in the sector $\mathbb{C} \setminus T_\theta^+$, and these both satisfy the asymptotics*

$$\lambda(\alpha(t)) = -t^2\mathrm{e}^{2\mathrm{i}\vartheta} + \mathcal{O}\left(t^2\mathrm{e}^{2\cos(\vartheta)at}\right) \tag{4.1.30}$$

as $t \to \infty$.

The eigenvalue behaviour described in Theorems 4.1.1 and 4.1.4, and our approach taken here, might be compared with the corresponding case of real α discussed in Section 2.3.1 or, for even more details, with [36, Section 4.3.1].

The proof of Theorem 4.1.4 and hence of Corollary 4.1.5

We recall from Section 4.1 that as $\operatorname{Im}\sqrt{\lambda} \to \pm\infty$ the matrix $M(\lambda)$ has two divergent eigenvalues α whose squares both behave like

$$\alpha^2 = -\lambda + \mathcal{O}_\pm\left(\lambda\mathrm{e}^{\mp 2a\operatorname{Im}\sqrt{\lambda}}\right). \tag{4.1.31}$$

This equation implies the anticipated asymptotic behaviour $\lambda \sim -\alpha^2$, however, we are also interested in the asymptotic remainder term, that is, a function f such that

$$\lambda = -\alpha^2 + \mathcal{O}(f(\alpha)) \tag{4.1.32}$$

as $\alpha \to \infty$ in T_φ^-. Consequently, our next goal is to invert the asymptotic equation (4.1.31) from $\alpha(\lambda)$ to obtain the asymptotic equation for $\lambda(\alpha)$ and thus prove Theorem 4.1.4. We first sketch the idea behind our inversion, namely an application of

Rouché's theorem, because we will also use this again in Section 4.3 when considering d-dimensional balls. Let $\tau \geq 0$ and let $h : \mathbb{C} \to \mathbb{C}$ be a continuous function such that $h(z) \to 0$ as $\operatorname{Im} z \to +\infty$. Suppose that $\alpha = \alpha(\lambda)$, as a holomorphic function of λ, satisfies the asymptotics

$$\alpha(\lambda) = \mathrm{i}\sqrt{\lambda} + \tau + g(\sqrt{\lambda}) \tag{4.1.33}$$

as $\operatorname{Im} \sqrt{\lambda} \to +\infty$ for a certain error term $g(\sqrt{\lambda})$ which is $\mathcal{O}(h(\sqrt{\lambda}))$; for the choice of h see the corresponding asymptotics for the Dirichlet-to-Neumann operator, that is, the remainder term g of (4.1.26) for the interval and (4.3.37) for the ball, respectively. For given λ and hence $\alpha = \alpha(\lambda)$, we define a new holomorphic function f_α by

$$f_\alpha(z) := \mathrm{i}z + \tau - \alpha, \tag{4.1.34}$$

whose only zero is given by $z_\alpha := \mathrm{i}(\tau - \alpha)$. Then (4.1.33) becomes

$$f_\alpha(z) + g(z) = 0$$

if and only if $z = \sqrt{\lambda(\alpha)}$. Let $B^\alpha := B_{r_\alpha}(z_\alpha)$ be a ball with centre z_α and some given radius $r_\alpha > 0$, cf. Figure 4.1.1. Then, by Rouché's theorem, if α is sufficiently large

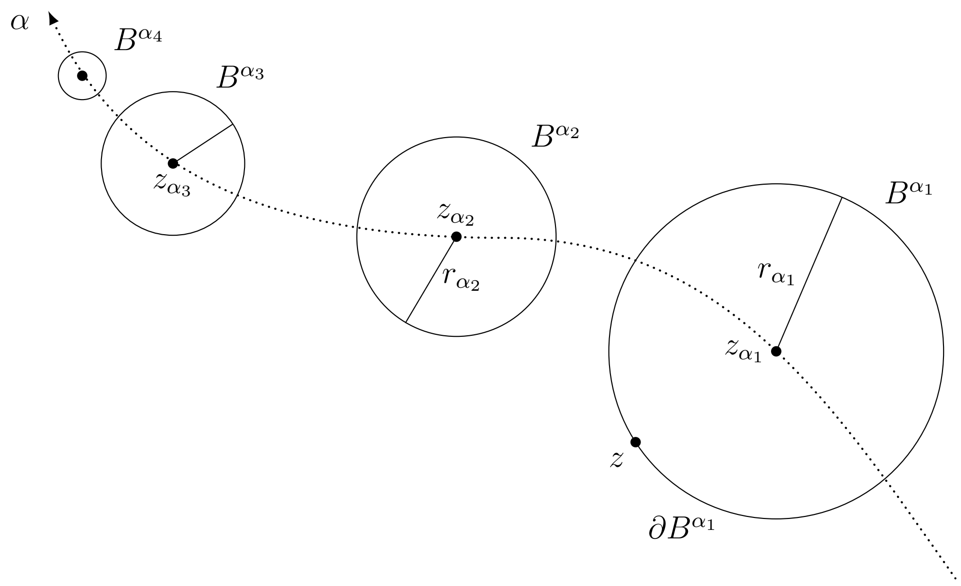

Figure 4.1.1: Sequence of balls B^α as $\operatorname{Re} \alpha \to -\infty$. For each $z \in \partial B^\alpha$ we have $|f_\alpha(z)| = r_\alpha$ which is used to compare the moduli of g and f_α on ∂B^α.

and

$$|g(z)| < |f_\alpha(z)| \tag{4.1.35}$$

for all $z \in \partial B^\alpha$, both f_α and $f_\alpha + g$ have exactly one zero in B^α. This technique proves not only the existence of an eigenvalue of the Robin Laplacian that satisfies said asymptotics, but gives an error term in the asymptotic expansion of $\lambda(\alpha)$ as follows. By construction, for each $z \in \partial B^\alpha$ we have $|f_\alpha(z)| = r_\alpha$. Moreover, $g = \mathcal{O}(h)$ as $\operatorname{Im} z \to +\infty$ implies the existence of some constant $\delta > 0$, such that

$$|g(z)| \leq \delta |h(z)| \tag{4.1.36}$$

on ∂B^α for all sufficiently large α. For all such α we want r_α to satisfy

$$\delta\,|h(z)| < r_\alpha, \quad z \in \partial B^\alpha. \tag{4.1.37}$$

To ensure this inequality, the decay of h is crucial: if it is too slow, then the method fails. This will be clarified in the following proof.

Proof of Theorem 4.1.4. The eigenvalues $\alpha_\pm$ of the Dirichlet-to-Neumann matrix (4.1.16) read

$$\alpha_\pm = \sqrt{\lambda}\left(\pm\csc(2a\sqrt{\lambda}) - \cot(2a\sqrt{\lambda})\right). \tag{4.1.38}$$

As $\alpha \to \infty$ in $\mathbb{C}$ we have either $|\sqrt{\lambda}| \to \infty$ or $\sqrt{\lambda}$ is forced to approach a zero of $\sin(2a\,\cdot\,)$, which corresponds to a Dirichlet eigenvalue. The second case, in particular, requires λ to remain bounded and thus is covered by Theorem 4.1.1 (alternatively, one could adapt the proof of Theorem 1.2.6 to dimension $d = 1$). We divide the proof into four steps:

Step 1: We assume that for some given $\theta \in (0, \pi)$ some Robin eigenvalue λ diverges to ∞ away from the real axis, inside the sector S_θ^+ which, in particular, yields $\operatorname{Im}\sqrt{\lambda} \to +\infty$. In Section 4.1.2 we saw that for this behaviour of $\sqrt{\lambda}$ we obtain

$$\alpha = \pm\mathrm{i}\sqrt{\lambda} + \mathcal{O}\left(\sqrt{\lambda}\mathrm{e}^{-2a\operatorname{Im}\sqrt{\lambda}}\right) \tag{4.1.39}$$

as $\operatorname{Im}\sqrt{\lambda} \to \infty$; for more details see (4.1.24)-(4.1.27). (The other case $\operatorname{Im}\sqrt{\lambda} \to -\infty$ will be discussed in Step 3.)

Step 2: It remains to invert this asymptotical behaviour by means of the Rouché inversion technique sketched above. This will prove (based on the assumed asymptotic behaviour of α) the existence of exactly two (see Step 3 and 4) divergent eigenvalues λ which obey (4.1.39) away from the real axis. Here we deal with the inversion; as mentioned before, we will only consider the case $\operatorname{Im}\sqrt{\lambda} \to +\infty$ in detail. So let $\tau = 0$, that is $f_\alpha(z_\alpha) = 0$ for $z_\alpha = -\mathrm{i}\alpha$. Here we take $h(z) := z\mathrm{e}^{-2a\operatorname{Im} z}$, which satisfies $h(z) \to 0$ as $\operatorname{Im} z \to +\infty$. By construction, every point $z \in \partial B^\alpha$ is represented by

$$z = z_\alpha + r_\alpha \mathrm{e}^{\mathrm{i}\varphi} = -\mathrm{i}\alpha + r_\alpha \mathrm{e}^{\mathrm{i}\varphi} \tag{4.1.40}$$

for some $\varphi \in [0, 2\pi)$. Our goal is to estimate h as in (4.1.37): a short calculation using (4.1.40) gives

$$\begin{aligned}
|h(z)| &= \left|z\mathrm{e}^{-2a\operatorname{Im} z}\right| && (4.1.41\text{a})\\
&= \left|\left(-\mathrm{i}\alpha + r_\alpha \mathrm{e}^{\mathrm{i}\varphi}\right)\exp\left[-2a\operatorname{Im}\left(-\mathrm{i}\alpha + r_\alpha \mathrm{e}^{\mathrm{i}\varphi}\right)\right]\right| && (4.1.41\text{b})\\
&\le (|\alpha| + r_\alpha)\exp\left[-2a\operatorname{Re}(-\alpha) - 2a\operatorname{Im}\left(r_\alpha \mathrm{e}^{\mathrm{i}\varphi}\right)\right] && (4.1.41\text{c})\\
&= (|\alpha| + r_\alpha)\exp\left[2a\operatorname{Re}\alpha - 2ar_\alpha\sin\varphi\right] && (4.1.41\text{d})\\
&\le (|\alpha| + r_\alpha)\,\mathrm{e}^{2ar_\alpha}\mathrm{e}^{2a\operatorname{Re}\alpha}. && (4.1.41\text{e})
\end{aligned}$$

We now choose $r_\alpha > 0$ to ensure (4.1.37) on ∂B^α. To this end, it suffices to find r_α such that

$$\delta\left(|\alpha| + r_\alpha\right)\mathrm{e}^{2a\operatorname{Re}\alpha} < r_\alpha \mathrm{e}^{-2ar_\alpha} \tag{4.1.42}$$

for sufficiently large α; we make the *ansatz*

$$r_\alpha = C|\alpha|\mathrm{e}^{2a\operatorname{Re}\alpha} \tag{4.1.43}$$

for a suitable constant $C > 0$ (in fact we may take any $C > \delta$). Then, for such an r_α, (4.1.42) is equivalent to

$$\delta\left(|\alpha| + C|\alpha|\mathrm{e}^{2a\operatorname{Re}\alpha}\right)\mathrm{e}^{2a\operatorname{Re}\alpha} < C|\alpha|\mathrm{e}^{2a\operatorname{Re}\alpha}\exp\left[-2aC|\alpha|\mathrm{e}^{2a\operatorname{Re}\alpha}\right] \tag{4.1.44}$$

and after rearrangement we obtain

$$\delta\left(1+C\mathrm{e}^{2a\mathrm{Re}\,\alpha}\right)<C\exp\left[-2aC|\alpha|\mathrm{e}^{2a\mathrm{Re}\,\alpha}\right], \tag{4.1.45}$$

that is,

$$\delta\left(1+C\mathrm{e}^{2a\mathrm{Re}\,\alpha}\right)\mathrm{e}^{2Ca|\alpha|\mathrm{e}^{2a\mathrm{Re}\,\alpha}}<C. \tag{4.1.46}$$

Since

$$C\mathrm{e}^{2a\mathrm{Re}\,\alpha}\to 0 \qquad \text{and} \qquad \mathrm{e}^{2Ca|\alpha|\mathrm{e}^{2a\mathrm{Re}\,\alpha}}\to 1 \tag{4.1.47}$$

as $\mathrm{Re}\,\alpha\to-\infty$, the left-hand side of (4.1.46) converges to δ and hence (4.1.42) is satisfied whenever $\mathrm{Re}\,\alpha$ is sufficiently large negative, how large depending only on a, δ and C. In particular, for the *ansatz* (4.1.43), the inequality (4.1.42) is then valid. We arrive at

$$\sqrt{\lambda}(\alpha)=-\mathrm{i}\alpha+\mathcal{O}\left(\alpha\mathrm{e}^{2a\mathrm{Re}\,\alpha}\right) \tag{4.1.48}$$

as $\mathrm{Re}\,\alpha\to-\infty$, and thus

$$\lambda(\alpha)=-\alpha^2+\mathcal{O}\left(\alpha^2\mathrm{e}^{2a\mathrm{Re}\,\alpha}\right). \tag{4.1.49}$$

Step 3: We want to sketch how to adapt the proof to the assumption $\mathrm{Im}\,\sqrt{\lambda}\to-\infty$: one chooses $f_\alpha(z)=-\mathrm{i}z-\alpha$ which vanishes only for $z_\alpha=+\mathrm{i}\alpha$. Similar calculations as above lead to

$$\sqrt{\lambda(\alpha)}=+\mathrm{i}\alpha+\mathcal{O}\left(\alpha\mathrm{e}^{2a\mathrm{Re}\,\alpha}\right) \tag{4.1.50}$$

as $\mathrm{Re}\,\alpha\to-\infty$.

Step 4 – Conclusion: We obtain that in both cases $\mathrm{Im}\,\sqrt{\lambda}\to\pm\infty$ the real part $\mathrm{Re}\,\alpha$ is always negative and divergent, that is, each divergent $\sqrt{\lambda}$ within a sector of the form S_θ^+ or S_θ^- (note that $\sqrt{\lambda}\in S_\theta^+$ if and only if $-\sqrt{\lambda}\in S_\theta^-$) corresponds to

$$\sqrt{\lambda}=\mathrm{i}\alpha++\mathcal{O}\left(\alpha\mathrm{e}^{2a\mathrm{Re}\,\alpha}\right) \tag{4.1.51}$$

and

$$\sqrt{\lambda} = -\mathrm{i}\alpha + \mathcal{O}\left(\alpha \mathrm{e}^{2a \operatorname{Re} \alpha}\right), \tag{4.1.52}$$

respectively. We conclude that, under the assumption that α diverges in a sector T_φ^- with $\varphi \in (0, \pi/2)$, there are exactly two divergent eigenvalues $\lambda \in \mathbb{C} \setminus T_{2\theta}^+$, and both of them satisfy (4.1.49) as $\operatorname{Re}\alpha \to -\infty$. Moreover, this implies that if $\operatorname{Re}\alpha$ remains bounded from below, then there are no divergent eigenvalues $\lambda \to \infty$ in $\mathbb{C} \setminus T_\theta^+$ for any $0 < \theta < \pi/2$. □

4.1.3 Divergent eigenvalues near the positive real axis

While the previous Section 4.1.3 we studied the divergence of the eigenvalues away from the real axis, we will now consider the other case of $\lambda \to \infty$ such that, in particular, there is no $\theta > 0$ such that λ diverges in $\mathbb{C} \setminus T_\theta^+$. The calculations of Section 4.1.2 show that the language of sectors is not necessary to obtain the asymptotical form (4.1.26) of the Dirichlet-to-Neumann matrix $M(\lambda)$, indeed, it is sufficient to assume that $\operatorname{Im}\sqrt{\lambda} \to \infty$ and $\operatorname{Im}\sqrt{\lambda} \to -\infty$, respectively. To this end, we assume that $\operatorname{Im}\lambda \neq 0$ (and hence $\sqrt{\lambda}$) remains bounded. In this case the asymptotics of $M(\lambda)$ is less obvious: we have already seen that each of its entries is meromorphic with poles of order one on the real axis. While the off-diagonal entries vanish as $\operatorname{Im}\sqrt{\lambda} \to \pm\infty$, this is not the case if $\operatorname{Im}\sqrt{\lambda}$ remains bounded and the analysis becomes more difficult. Note that the poles of $M(\lambda)$ correspond to Dirichlet and Neumann eigenvalues, respectively, and each of them is a discrete point located on the non-negative real half-axis. If λ diverges in some strip (of fixed width) parallel to the real axis, the chosen path will pass arbitrarily closely to every single one of them. The question arises which associated paths of α in the complex plane correspond to such λ paths.

It would appear that any such λ path requires $\operatorname{Re}\alpha$ to be unbounded from below, cf. Figure 4.1.2. The explicit form of $M(\lambda)$, which we recall by

$$M(\lambda) = \sqrt{\lambda} \begin{pmatrix} -\cot 2\sqrt{\lambda}a & \csc 2\sqrt{\lambda}a \\ \csc 2\sqrt{\lambda}a & -\cot 2\sqrt{\lambda}a \end{pmatrix}, \tag{4.1.53}$$

allows us to calculate its two eigenvalues $\alpha_\pm$ explicitly, that is, for $z := 2\sqrt{\lambda}a$,

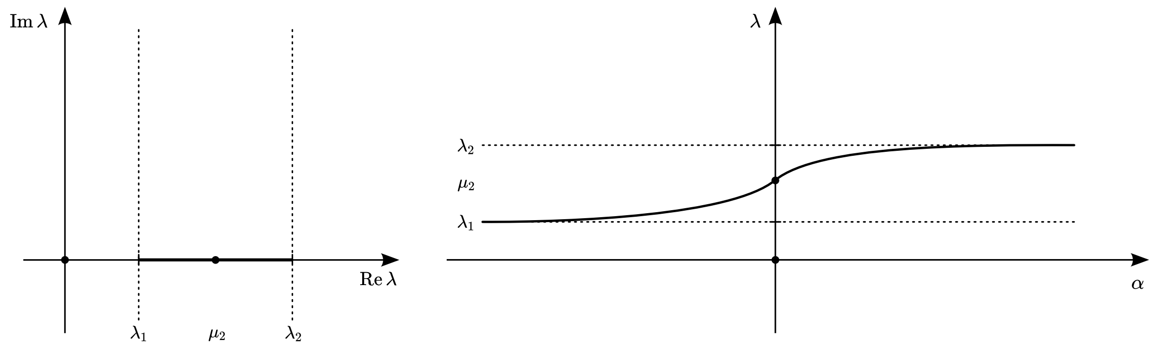

Figure 4.1.2: On the left a path in the λ-plane from one pole λ_1 of the Dirichlet-to-Neumann operator to the next one λ_2 while passing a zero (the second Neumann eigenvalue μ_2). On the right-hand side the real curve $\lambda(\alpha) \in \mathbb{R}$ increasing from λ_1 to λ_2 as $\alpha \in \mathbb{R}$ tends from $-\infty$ to $+\infty$, cf. Figure 2.3.1 or [36, Section 4.3].

$$\alpha_\pm = \sqrt{\lambda}\,(-\cot z \pm \csc z) \tag{4.1.54a}$$

$$= -\mathrm{i}\sqrt{\lambda}\left(\frac{\mathrm{e}^{2\mathrm{i}z}+1}{\mathrm{e}^{2\mathrm{i}z}-1} \mp \frac{2\mathrm{e}^{\mathrm{i}z}}{\mathrm{e}^{2\mathrm{i}z}-1}\right) = \pm\mathrm{i}\sqrt{\lambda}\,\frac{\mathrm{e}^{\mathrm{i}z}\mp 1}{\mathrm{e}^{\mathrm{i}z}\pm 1}, \tag{4.1.54b}$$

see (4.1.18). From this equation it follows that $\alpha \sim \pm\mathrm{i}\sqrt{\lambda}$ as $\operatorname{Im}\sqrt{\lambda} \to \infty$ in $\mathbb{C}$, cf. (4.1.26). However, let us consider $\sqrt{\lambda} = \sqrt{\lambda}(\tau)$ following some (continuous) path described by

$$\sqrt{\lambda} = x(\tau) + \mathrm{i}y(\tau). \tag{4.1.55}$$

Here, the functions x and y possess the properties that

(1) $x : [0,1) \to \mathbb{R}$ is unbounded with $x(\tau) \to \infty$ as $\tau \to 1$;

(2) $y : [0,1) \to \mathbb{R}$ is bounded.

Then we observe the following: firstly, since the imaginary part y of $\sqrt{\lambda}$ remains bounded, for each such path and each sector $T_\theta^\pm$ there exists some $\tau_\theta \in [0,1)$ such that

$$\sqrt{\lambda} \in T_\theta^\pm \qquad \text{for all } \tau_\theta < \theta < 1; \tag{4.1.56}$$

up to a possibly different $\theta > 0$ so too does λ. Secondly, the boundedness of $\operatorname{Im}\lambda$ implies

$$\lambda = x^2 + 2\mathrm{i}xy - y^2 \quad \Rightarrow \quad |2xy| = |\operatorname{Im}\lambda| < c \tag{4.1.57}$$

for a constant $c > 0$, that is, $y = \mathcal{O}(1/x)$. Without loss of generality, we consider α_+; from

$$\alpha_+ = (\mathrm{i}x - y)\frac{\mathrm{e}^{2\mathrm{i}ax} - \mathrm{e}^{2ay}}{\mathrm{e}^{2\mathrm{i}ax} + \mathrm{e}^{2ay}} \tag{4.1.58}$$

by a somewhat tedious calculation we arrive at

$$\operatorname{Re}\alpha_+ = \frac{1}{2}\,\frac{y\,(1-\mathrm{e}^{4ay})}{\cosh(2ay) - \cos(2ax)} - \frac{x\sin(2ax)}{\cosh(2ay) - \cos(2ax)}. \tag{4.1.59}$$

Both denominators are bounded (and since $\cosh \geq 1$ and $\cosh t = 1$ if and only if $t = 0$ they can only vanish if $y = 0$) and so is the numerator of the first quotient by assertion. The second numerator, however, diverges (indefinitely) as $x = \operatorname{Re}\sqrt{\lambda} \to \infty$. This proves Proposition 4.1.6.

Proposition 4.1.6. *Suppose λ diverges along a path within a strip of fixed width around the positive real axis. Then the corresponding eigenvalues $\alpha(\lambda)$ of $M(\lambda)$ satisfy $|\operatorname{Im}\alpha(\lambda)| \to \infty$ and $\operatorname{Re}\alpha(\lambda)$ oscillates and diverges indefinitely.*

Among other things, this intimates, when combined with the proof of Theorem 4.1.4, that the Robin Laplacian can only have divergent eigenvalues in the regime $\operatorname{Re}\alpha \to -\infty$: indeed, if $\operatorname{Re}\alpha$ remains bounded from below, then the conjecture rules out divergent eigenvalues λ such that $\operatorname{Im}\sqrt{\lambda}$ remains bounded; while by Theorem 4.1.4 and its proof there can be no divergent eigenvalues λ such that $\operatorname{Im}\sqrt{\lambda} \to \pm\infty$.

4.1.4 On the orthogonality of the eigenfunctions

In this section we continue with the analysis of the one dimensional Robin problem. It is a natural question to ask whether for a fixed parameter $\alpha \in \mathbb{C}$ the set of eigenfunctions can – just like in the real case – still be chosen to form an orthonormal basis of $L^2(-a, a)$. In contrast to the tools used to obtain the general results of Section 3.3 here we rely on explicit calculations. So, for $j \in \{1, 2\}$ let $\lambda_j \in \mathbb{C}$ be two different eigenvalues of the Robin Laplacian $-\Delta_\Omega^\alpha$ on the interval $\Omega = (-a, a)$ for some fixed $a > 0$. Even though we used the representation of eigenfunctions being sums of sin and cos functions, here we will consider the eigenfunctions $u_j : \Omega \to \mathbb{C}$ (corresponding to $\lambda_j = k_j^2$) represented by

$$u_j(x) = A_j \mathrm{e}^{\mathrm{i}k_j x} + B_j \mathrm{e}^{-\mathrm{i}k_j x}. \tag{4.1.60}$$

Without loss of generality we choose each k_j such that $\operatorname{Im} k_j \geq 0$. Since both functions u_j satisfy the Robin boundary condition

$$-\Delta u = -u'' = \lambda u \qquad \text{on } (-a, a), \tag{4.1.61a}$$

$$-u'(-a) + \alpha u(-a) = 0, \tag{4.1.61b}$$

$$+u'(+a) + \alpha u(+a) = 0. \tag{4.1.61c}$$

we obtain

$$A_j \mathrm{e}^{-\mathrm{i}k_j a}(\alpha - \mathrm{i}k_j) + B_j \mathrm{e}^{\mathrm{i}k_j a}(\alpha + \mathrm{i}k_j) = 0 \tag{4.1.62a}$$

$$A_j \mathrm{e}^{\mathrm{i}k_j a}(\alpha + \mathrm{i}k_j) + B_j \mathrm{e}^{-\mathrm{i}k_j a}(\alpha - \mathrm{i}k_j) = 0 \tag{4.1.62b}$$

for constants $A_j, B_j \in \mathbb{C} \setminus \{0\}$. Rearrangement of both equations in (4.1.62) yields

$$\mathrm{e}^{2\mathrm{i}k_j a} = \frac{B_j}{A_j} \frac{\mathrm{i}k_j - \alpha}{\mathrm{i}k_j + \alpha} = \frac{A_j}{B_j} \frac{\mathrm{i}k_j - \alpha}{\mathrm{i}k_j + \alpha}. \tag{4.1.63}$$

Hence, we need to consider the two cases (i) $A_j = B_j$ and (ii) $A_j = -B_j$. In case (i) we have

$$\mathrm{e}^{2\mathrm{i}k_j a} = \frac{\mathrm{i}k_j - \alpha}{\mathrm{i}k_j + \alpha} \qquad \Longleftrightarrow \qquad \mathrm{e}^{2\mathrm{i}\overline{k_j} a} = \frac{\mathrm{i}\overline{k_j} - \overline{\alpha}}{\mathrm{i}\overline{k_j} + \overline{\alpha}}; \tag{4.1.64}$$

case (ii) similarly implies

$$\mathrm{e}^{2\mathrm{i}k_j a} = -\frac{\mathrm{i}k_j - \alpha}{\mathrm{i}k_j + \alpha} \qquad \Longleftrightarrow \qquad \mathrm{e}^{2\mathrm{i}\overline{k_j} a} = -\frac{\mathrm{i}\overline{k_j} - \overline{\alpha}}{\mathrm{i}\overline{k_j} + \overline{\alpha}}. \tag{4.1.65}$$

Consequently, this dichotomy corresponds to only having even and odd eigenfunctions, respectively, and any eigenfunction of $-\Delta_\Omega^\alpha$ can be written as

$$u_\pm(x) = C\left(\mathrm{e}^{\mathrm{i}x\sqrt{\lambda}} \pm \mathrm{e}^{-\mathrm{i}x\sqrt{\lambda}}\right) = \begin{cases} C_+ \cos\left(x\sqrt{\lambda}\right) & \text{for } (+), \\ C_- \sin\left(x\sqrt{\lambda}\right) & \text{for } (-) \end{cases} \tag{4.1.66}$$

for constants $C_\pm = C_\pm(\lambda) \in \mathbb{C} \setminus \{0\}$. This allows us to prove the following lemma.

Lemma 4.1.7. *The eigenfunctions of* $-\Delta_{(-a,a)}^\alpha$ *are* L^2*-orthogonal if and only if* $\alpha \in \mathbb{R}$.

Proof. We have already seen that each eigenfunction u of $-\Delta_\Omega^\alpha$ is either even, that is $u = u_+$ in the sense of (4.1.66), or odd. The Robin boundary problem on the interval $(-a, a)$ is symmetric (note again that its definition is with respect to the outer normal derivative), the operator itself is invariant under reflection, and the subspaces of even and odd eigenfunctions are mutually orthogonal. Thus, it suffices to prove that $(u_1, u_2) = 0$ if and only if $\alpha \in \mathbb{R}$ for u_1 and u_2 being simultaneously even or odd. So, for $j \in \{1, 2\}$, let $k_j = \sqrt{\lambda_j} \neq 0$ be the roots (with $\operatorname{Im} k_j \geq 0$) of two different eigenvalues corresponding to even eigenfunctions $u_1 \neq u_2$ (scaled such that $A_j = B_j = 1$ for $j = 1, 2$), that is,

$$\begin{aligned}
(u_1, u_2) &= \int_{-a}^{a} \left(\mathrm{e}^{\mathrm{i}xk_1} + \mathrm{e}^{-\mathrm{i}xk_1}\right)\left(\mathrm{e}^{-\mathrm{i}x\overline{k_2}} + \mathrm{e}^{\mathrm{i}x\overline{k_2}}\right) \mathrm{d}x && (4.1.67\text{a}) \\
&= \int_{-a}^{a} \left(\mathrm{e}^{\mathrm{i}x(k_1-\overline{k_2})} + \mathrm{e}^{\mathrm{i}x(k_1+\overline{k_2})} + \mathrm{e}^{-\mathrm{i}x(k_1+\overline{k_2})} + \mathrm{e}^{-\mathrm{i}x(k_1-\overline{k_2})}\right) \mathrm{d}x && (4.1.67\text{b}) \\
&= \frac{2}{\mathrm{i}(k_1-\overline{k_2})}\left(\mathrm{e}^{\mathrm{i}a(k_1-\overline{k_2})} - \mathrm{e}^{-\mathrm{i}a(k_1-\overline{k_2})}\right) \\
&\qquad + \frac{2}{\mathrm{i}(k_1+\overline{k_2})}\left(\mathrm{e}^{\mathrm{i}a(k_1+\overline{k_2})} - \mathrm{e}^{-\mathrm{i}a(k_1+\overline{k_2})}\right) && (4.1.67\text{c}) \\
&= \frac{2\mathrm{e}^{\mathrm{i}a(k_1-\overline{k_2})}}{\mathrm{i}(k_1-\overline{k_2})}\left(1 - \mathrm{e}^{-2\mathrm{i}a(k_1-\overline{k_2})}\right) \\
&\qquad + \frac{2\mathrm{e}^{\mathrm{i}a(k_1+\overline{k_2})}}{\mathrm{i}(k_1+\overline{k_2})}\left(1 - \mathrm{e}^{-2\mathrm{i}a(k_1+\overline{k_2})}\right). && (4.1.67\text{d})
\end{aligned}$$

By (4.1.64) and some rearrangements we see that $(u_1, u_2) = 0$ if and only if

$$
\begin{aligned}
& 0 = \frac{1}{k_1 - \overline{k_2}} \left(1 - \mathrm{e}^{-2\mathrm{i}a(k_1 - \overline{k_2})}\right) + \frac{\mathrm{e}^{2\mathrm{i}a\overline{k_2}}}{k_1 + \overline{k_2}} \left(1 - \mathrm{e}^{-2\mathrm{i}a(k_1 + \overline{k_2})}\right) && \text{(4.1.68a)} \\
\Leftrightarrow \quad & 0 = \left(k_1 + \overline{k_2}\right) \left(1 - \frac{\mathrm{i}k_1 + \alpha}{\mathrm{i}k_1 - \alpha} \cdot \frac{\mathrm{i}\overline{k_2} - \overline{\alpha}}{\mathrm{i}\overline{k_2} + \overline{\alpha}}\right) \\
& \quad + \left(k_1 - \overline{k_2}\right) \left(\frac{\mathrm{i}\overline{k_2} - \overline{\alpha}}{\mathrm{i}\overline{k_2} + \overline{\alpha}}\right) \left(1 - \frac{\mathrm{i}k_1 + \alpha}{\mathrm{i}k_1 - \alpha} \cdot \frac{\mathrm{i}\overline{k_2} + \overline{\alpha}}{\mathrm{i}\overline{k_2} - \overline{\alpha}}\right). && \text{(4.1.68b)}
\end{aligned}
$$

By an elementary but somewhat tedious calculation we arrive at

$$0 = 2k_1\overline{k_2}\,(\overline{\alpha} - \alpha) = 4\mathrm{i}k_1\overline{k_2}\mathrm{Im}\,\alpha,$$

or in other words $\mathrm{Im}\,\alpha = 0$. In the case of both u_j being odd, by using (4.1.65) instead of (4.1.64), in (4.1.68b) the changed signs cancel each other out and we arrive at the exact same result. □

4.2 Cuboids

Based on our understanding of the interval we can easily obtain results for d-dimensional cuboids (sometimes called *hyperrectangles*): fix the dimension $d \geq 2$ and choose d intervals $(-a_j, a_j)$ for $a_1, \ldots, a_d > 0$. We denote by

$$Q := (-a_1, a_1) \times \cdots \times (-a_d, a_d)$$

the d-dimensional cuboid of edge lengths $2a_1, \ldots, 2a_d$ with its centre in $0 \in \mathbb{R}^d$. Let

$$\mathcal{A}_j(\alpha) := -\Delta^{\alpha}_{e_j}, \tag{4.2.1}$$

$j = 1, \ldots, d$, be the one-dimensional Robin Laplacian on the edge $e_j \simeq (-a_j, a_j)$. Then there exists a sequence of eigenvalues

$$\lambda_1(\alpha), \lambda_2(\alpha), \ldots \in \sigma(-\Delta^{\alpha}_{Q}) \subset \mathbb{C} \tag{4.2.2}$$

of the Robin Laplacian $-\Delta_Q^\alpha$ on Q such that each of these eigenvalues $\lambda_k(\alpha)$, $k \in \mathbb{N}$, is given by a sum of eigenvalues of the constituent operators $\mathcal{A}_j(\alpha)$, that is,

$$\lambda_k(\alpha) = \sum_{j=1}^{d} \lambda^{(j)}(\alpha), \tag{4.2.3}$$

where $\lambda^{(j)} \in \sigma(\mathcal{A}_j(\alpha))$.

Remark 4.2.1. By Theorem 3.3.8 (1) we know that in $d = 1$ dimension and for each $\alpha \in \mathbb{C}$ there exists a Riesz basis of $L^2((-a,a))$ consisting of the eigenfunctions of $-\Delta^\alpha_{(-a,a)}$. Now let $d \geq 2$; for the d-dimensional hyperrectangle $Q \subset \mathbb{R}^d$ it follows by separation of variables that there exists at least a Riesz basis of $L^2(Q)$ consisting of products of the one-dimensional eigenfunctions of the operators $\mathcal{A}_j(\alpha)$ for $j = 1, \dots, d$, which correspond to the eigenvalues of the form (4.2.3).

Theorems 4.1.1 and 4.1.4 state that, if $\alpha \to \infty$ in a sector of the form T_φ^- for some $0 < \varphi < \pi/2$, i.e., a sector completely contained in the left half-plane, on each e_j there are two eigenvalues of $\mathcal{A}_j$, call them $\lambda_1^{(j)}, \lambda_2^{(j)}$, both of which diverge like $-\alpha^2$ as $\alpha \to \infty$. Consequently, if we start with $j = 1$, since a single diverging eigenvalue of $\mathcal{A}_1$ can be added to $d-1$ non-divergent eigenvalues on the remaining edges $e_2, \dots, e_d$, and for each divergent one we have infinitely many choices, there are infinitely many divergent eigenvalues of $-\Delta_Q^\alpha$ which behave asymptotically like $-\alpha^2$. In the next step we choose two divergent eigenvalues $\lambda^{(1)}$ of $\mathcal{A}_1$ and $\lambda^{(2)}$ of $\mathcal{A}_2$ and $d-2$ non-divergent eigenvalues of $\mathcal{A}_3, \dots, \mathcal{A}_d$. Adding everything up we obtain infinitely many eigenvalues of $-\Delta_Q^\alpha$ behaving like $-2\alpha^2$. We proceed successively up to step $(d-1)$ to obtain infinitely many eigenvalues that behave like $-(d-1)\alpha^2$. However, the final step is different: since there are two divergent eigenvalues for each $\mathcal{A}_j$, $j = 1, \dots, d$, we obtain not infinitely many but 2^d possibilities for an eigenvalue of $-\Delta_Q^\alpha$ to satisfy the asymptotics $-d\alpha^2$. This results in the following theorem which is due to [30, Theorem 9.3].

Theorem 4.2.2. *Let $Q \subset \mathbb{R}^d$, $d \geq 2$, be a hyperrectangle and suppose that $\alpha \to \infty$ in a sector of the form T_φ^- for some $\varphi \in (0, \pi/2)$ (see Definition 3.6.1). Then for each $j = 1, \dots, d-1$ there are infinitely many divergent eigenvalues $\lambda(\alpha)$ of $-\Delta_Q^\alpha$ such that the leading term asymptotics reads $\lambda(\alpha) \sim -j\alpha^2$ and (at least) 2^d eigenvalues which behave like $\lambda(\alpha) \sim -d\alpha^2$.*

Remark 4.2.3. As already mentioned in Chapter 1 there are several results on the eigenvalue asymptotics for domains with less regularity and real parameter α. However, there are no results for general Lipschitz domains but only for those having a finite number of "model corners". Just like in the case of real α [36] we expect that the asymptotics is mainly driven by the "most acute" corner(s) of the domain – the sharper the corner(s), the larger the (negative) leading coefficient of the asymptotics.

4.3 d-dimensional balls

We next consider the model case of higher dimensional balls

$$\Omega = B := B_1(0) \subset \mathbb{R}^d \tag{4.3.1}$$

in dimension $d \geq 2$. We will use the notation $\partial B = \mathbb{S}^{d-1}$ interchangeably. The idea to study Laplacians on spherical domains is standard: we separate the radial and spherical parts ∂_r and Δ_w from the Laplacian Δ. The spherical part Δ_w, called *Laplace–Beltrami operator*, is introduced in Section 4.3.2 and has a multitude of well known (spectral) properties, which allow us to write functions in $L^2(B)$ in their series representations with respect to the eigenfunctions of the Laplace-Beltrami operator. We start by introducing basic notation for spherical harmonics, i.e, as we will later see, eigenfunctions of Δ_w.

4.3.1 On spherical harmonics

Laplace's spherical harmonics are special functions on $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ and a basic tool for studying partial differential equations satisfying a given spherical symmetry in higher dimensions. The set of Laplace's spherical harmonics is complete and all functions within are pairwise orthogonal: they form an orthonormal basis of $L^2(\mathbb{S}^{d-1})$, i.e., each function defined on a sphere $\mathbb{S}^{d-1}$ can be written as a sum of spherical harmonics. This can be interpreted as a generalisation of the fact that each periodic function defined on a circle can be written as a sum of sines and cosines, or in other words in its Fourier series representation. We briefly recall a few properties of the eigenvalues and eigenfunctions of the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$, which will be useful in the sequel. For more details and further explanations we refer to [48, Section 2.2] which is also the literature both this and the next Section 4.3.1 are based on.

Definition 4.3.1. For $l \in \mathbb{N}_0$ let $\mathcal{P}_l^{\mathbb{C}}(d)$ denote the space of all homogeneous polynomials of degree l in d variables with complex coefficients and define

(1) $\mathcal{P}_l^{\mathbb{C}}(\mathbb{S}^{d-1}) := \left\{ P|_{\mathbb{S}^{d-1}} : P \in \mathcal{P}_l^{\mathbb{C}}(d) \right\}$,

(2) $\mathcal{H}_l^{\mathbb{C}}(d) := \left\{ P \in \mathcal{P}_l^{\mathbb{C}}(d) : \Delta P = 0 \right\}$,

(3) $\mathcal{H}_l^{\mathbb{C}}(\mathbb{S}^{d-1}) := \left\{ P|_{\mathbb{S}^{d-1}} : P \in H_l^{\mathbb{C}}(d) \right\}$.

Theorem 4.3.2. *Let E_l, $l \in \mathbb{N}_0$, denote the complex eigenspaces of the Laplace-Beltrami operator Δ_ω on $\mathbb{S}^{d-1}$. Then we have $E_l = \mathcal{H}_l^{\mathbb{C}}(\mathbb{S}^{d-1})$. Furthermore, they are of dimension*

$$M_l^d := \dim E_l = \dim \mathcal{H}_l^{\mathbb{C}}(\mathbb{S}^{d-1}) = \binom{d+l-1}{l-1} - \binom{d+l-3}{l-1}, \tag{4.3.2}$$

and we have

$$L^2_{\mathbb{C}}(\mathbb{S}^{d-1}) = \bigoplus_{l=0}^{\infty} E_l. \tag{4.3.3}$$

The E_l are eigenspaces of the Laplace-Beltrami operator and one can calculate that the corresponding eigenvalues μ_l^2 of the corresponding eigenvalue problem

$$-\Delta_\omega Y_{l,j} = \mu_l^2 Y_{l,j} \tag{4.3.4}$$

are given by

$$\mu_l^2 = l(d+l-2), \tag{4.3.5}$$

where we denote by $Y_{l,j}$ the eigenfunctions for $j = 0, \dots, M_l^d$ for fixed $l \in \mathbb{N}_0$, viz. the spherical harmonics.

4.3.2 The Dirichlet Laplacian and Dirichlet-to-Neumann operator on balls

Let u solve the Dirichlet eigenvalue problem

$$-\Delta u = \lambda u \qquad \text{on } B, \tag{4.3.6a}$$

$$u = g \qquad \text{on } \partial B, \tag{4.3.6b}$$

for given $g \in L^2(\partial B)$. Then we can write $u \in L^2(B)$ and $g \in L^2(\partial B)$ in their unique series representations

$$u(r,\omega) = \sum_{l=0}^{\infty} \sum_{j=0}^{M_l^d} u_{l,j}(r) Y_{l,j}(\omega) \tag{4.3.7}$$

and

$$g(\omega) = \sum_{l=0}^{\infty} \sum_{j=0}^{M_l^d} g_{l,j} Y_{l,j}(\omega), \tag{4.3.8}$$

respectively. Here, we denote by $\omega \in \mathbb{S}^{d-1}$ and $r \geq 0$ the angle and the radius of a point $x \in B$, and $Y_{l,j} \in \mathcal{H}_l^{\mathbb{C}}(\mathbb{S}^{d-1})$ is the jth spherical harmonic in the lth eigenspace of the Laplace-Beltrami operator. Note that the continuity condition requires $u_{l,j}(1) = g_{l,j}$.

The Laplace operator in polar coordinates reads

$$\Delta = \partial_r^2 + \frac{d-1}{r}\partial_r + \frac{1}{r^2}\Delta_\omega \tag{4.3.9}$$

and with this (and the eigenvalues (4.3.5)) in mind we say that a function u is a solution of the Dirichlet problem (4.3.6) if and only if there exists a bounded sequence $(u_{0,j})_{j\geq 0}$ in $\mathbb{C}$ such that

$$\lambda^2 u_{l,j} = -u''_{l,j}(r) - \frac{d-1}{r} u'_{l,j}(r) + \frac{l(d+l-2)}{\lambda^2} u_{l,j}(r), \tag{4.3.10a}$$

$$u_{l,j}(1) = g_{l,j}, \tag{4.3.10b}$$

$$u_{l,j}(0) = \delta_{0l} u_{0,j} \tag{4.3.10c}$$

hold for all $j = 0, \dots, M_l^d$ and for each $l \in \mathbb{N}_0$. Here $\delta_{0j} \in \{0,1\}$ is the Kronecker delta. If we fix l and j, this differential equation is equivalent to

$$r^2 v'' + (d-1) r v' - \lambda^2 \left[\frac{l(d+l-2)}{\lambda^2} - r^2 \right] v = 0. \tag{4.3.11}$$

This is called *Bessel's differential equation* and since we require $u_{l,j}(1) = g_{l,j}$ one can show that the ansatz

$$u_{l,j}(r) = \frac{g_{l,j}}{J_{\frac{d}{2}+l-1}(\lambda)} \, r^{1-\frac{d}{2}} J_{\frac{d}{2}+l-1}(\lambda r) \tag{4.3.12}$$

solves the boundary value problem (4.3.10). However, this calculation is tedious but standard and we omit it. Here and from now on J_m is the *Bessel function of the first kind* of order $m \in \mathbb{R}$: applying the Frobenius method on Bessel's differential equation implies

$$J_m(z) = \left(\frac{z}{2}\right)^m \sum_{k=0}^{\infty} \frac{\left(-\frac{z^2}{4}\right)^k}{k!\Gamma(m+k+1)}, \tag{4.3.13}$$

see [1, 9.1.10, p. 360]; however, we will not need this series representation. Using the identity

$$J_m'(\sqrt{\lambda}) = \frac{m}{\lambda} J_m(\sqrt{\lambda}) - J_{m+1}(\sqrt{\lambda}), \tag{4.3.14}$$

also called (second) *recurrence relation*, for all $m \in \mathbb{C}$, see [113, Chap. XVII, 17.21

(B)], we take the normal derivative ∂_r,

$$-\frac{\partial_r \left(r^{1-\frac{d}{2}} J_{\frac{d}{2}+l-1}(\lambda r)\right)}{J_{\frac{d}{2}+l-1}(\lambda)}\Bigg|_{r=1} = -\frac{\left(1-\frac{d}{2}\right) J_{\frac{d}{2}+l-1}(\lambda)}{J_{\frac{d}{2}+l-1}(\lambda)} + \frac{\lambda\left(\frac{\frac{d}{2}+l-1}{\lambda} J_{\frac{d}{2}+l-1}(\lambda) - J_{\frac{d}{2}+l}(\lambda)\right)}{J_{\frac{d}{2}+l-1}(\lambda)} \tag{4.3.15a}$$

$$= -\lambda \frac{-J_{\frac{d}{2}+k}(\lambda)}{J_{\frac{d}{2}+k-1}(\lambda)} - \left(1-\frac{d}{2}+\frac{d}{2}+l-1\right)\frac{J_{\frac{d}{2}+l-1}(\lambda)}{J_{\frac{d}{2}+l-1}(\lambda)} \tag{4.3.15b}$$

$$= \lambda \frac{J_{\frac{d}{2}+l}(\lambda)}{J_{\frac{d}{2}+l-1}(\lambda)} - l, \tag{4.3.15c}$$

to obtain that for any given $l \in \mathbb{N}_0$ the Dirichlet-to-Neumann operator $M^{(l)}(\lambda)$ (with respect to the l-th subspace $\mathcal{H}_l^{\mathbb{C}}$) maps the Dirichlet data $g_{l,j}$ onto (4.3.15c). We conclude that this *part* $M^{(l)}(\lambda)$ of Dirichlet-to-Neumann operator on ∂B in the subspace $\mathcal{H}_l^{\mathbb{C}}$, identified in the canonical way with $\mathbb{C}^{M_l^d}$ via the eigenfunctions of $\Delta_\omega|_{\mathcal{H}_l^{\mathbb{C}}}$, is representable by an $M_l^d \times M_l^d$ diagonal matrix each of whose diagonal entries is equal to

$$M^{(l)}(\lambda) = \sqrt{\lambda}\frac{J_{\frac{d}{2}+l}(\sqrt{\lambda})}{J_{\frac{d}{2}+l-1}(\sqrt{\lambda})} - l. \tag{4.3.16}$$

The Dirichlet-to-Neumann operator $M(\lambda)$ is then obtained by summing over all subspaces $\mathcal{H}_l^{\mathbb{C}}$, that is, it may be represented as a diagonal matrix. In particular, for each l, there are exactly M_l^d eigenvalues α of the Dirichlet-to-Neumann operator $M(\lambda)$ equal to $M^{(l)}(\lambda)$, and for our purposes it suffices to consider the $M^{(l)}(\lambda)$ individually.

Remark 4.3.3. Since, for orders $m \in \{\pm\frac{1}{2}, \pm\frac{3}{2}, \dots\}$, the Bessel functions J_m can be expressed through trigonometric functions multiplied by rational and square root functions – here if and only if d is odd, in particular for $d = 3$ – the radial parts $u_{l,j}$ appear as these spherical Bessel functions. Note that, e.g. for $d = 3$, the Bessel functions $J_{l+\frac{1}{2}}$ in the ansatz (4.3.12) are multiplied by $1/\sqrt{r}$ which makes them smooth in $r \to 0$.

Remark 4.3.4. Since each $M^{(l)}(\lambda)$ is diagonal and the Jordan chains of the Robin Laplacian and the corresponding Dirichlet-to-Neumann operator are of the same length, see Section 3.7, root vectors and eigenfunctions coincide and the eigennilpotents are always zero.

Observe that the zeros of the denominator in (4.3.16) are simple and so are the poles of the whole operator. The numerator does not cancel any of the poles, which follows from (4.3.14). If we assume that

$$J_m(\sqrt{\lambda_0}) = 0 = J_{m+1}(\sqrt{\lambda_0}) \tag{4.3.17}$$

for some $m \in \mathbb{C}$ and some $\lambda_0 \in \mathbb{C}\backslash\{0\}$ (we only need $m \in \frac{1}{2}\mathbb{N}_0$), then this implies $J_m'(\sqrt{\lambda_0}) = 0$ and J_m has a zero of order 2, a contradiction. It follows that $M^{(l)}$ is a meromorphic function having only simple, real poles and an essential singularity in $\pm\infty$. We have that

$$M^{(l)}(\lambda)g = \alpha g \qquad \Rightarrow \qquad |\alpha| = \left|\sqrt{\lambda}\frac{J_{\frac{d}{2}+l}(\sqrt{\lambda})}{J_{\frac{d}{2}+l-1}(\sqrt{\lambda})} - l\right| < \infty \tag{4.3.18}$$

implies that for $|\alpha| \to \infty$ the right-hand side is forced to diverge as well. Using (4.3.16), we are led via an explicit formula to the same dichotomy we saw in the general case in Theorem 1.2.6. Namely, there are two possibilities: either λ converges to the Dirichlet spectrum or diverges absolutely. In the latter case, as with the interval, we may further distinguish between eigenvalues λ diverging away from the positive real axis or in the vicinity of it. This leads to the following three cases.

(1) $\sqrt{\lambda}$ approaches a pole of $M^{(l)}(\lambda)$, i.e. a zero of $J_{\frac{d}{2}+l-1}$, meaning the eigenvalue λ converges to some element of the Dirichlet spectrum;

(2) $\lambda \to \infty$ in a sector of the form $\mathbb{C}\backslash T_{2\theta}^{+}$ for some small $\theta > 0$ (see Definition 3.6.1), that is, Assumption 4.1.3 holds. In this case, $\sqrt{\lambda}$ remains in $S_\theta^{\pm}$ and the quotient of the Bessel functions in the expression for $M^{(l)}$ remains bounded, see (4.3.30c);

(3) the more complicated case of divergence, where $\lambda \to \infty$ in a sector $T_{2\theta}^{+}$.

We analyse the three cases separately.

4.3.3 The convergent eigenvalues

We start with the convergent eigenvalues; we are interested in establishing the rate of convergence. As we intimated for the interval, we may consider the residues of the Dirichlet-to-Neumann operator, which also in the case of balls can be reduced to a scalar problem. For $m \in \mathbb{R}$ and $p \in \mathbb{N}_0$, we denote the pth zero of the Bessel function J_m of order m by $j_{m,p} \in \mathbb{R}$. The main result for the converging eigenvalues on d-dimensional balls is the following theorem (see [30, Theorem 9.8]).

Theorem 4.3.5. *Fix $l, p \in \mathbb{N}_0$. The eigenvalues $\lambda = \lambda(\alpha)$ converging to the Dirichlet spectrum satisfy*

$$\lambda(\alpha) = j^2_{\frac{d}{2}-l+1,p} - \frac{2j^2_{\frac{d}{2}-l+1,p}}{\alpha} + \mathcal{O}\left(\frac{1}{\alpha^2}\right) \tag{4.3.19}$$

as $|\alpha| \to \infty$.

Proof. The statement is proved by calculation of the residues of $M^{(l)}$. Indeed, setting

$$m_l := \frac{d}{2} - l + 1 \tag{4.3.20}$$

and using (4.3.14), for the pth pole we calculate

$$\operatorname{Res}_{j_{m_l,p}}\left(M^{(l)}\right) = \lim_{\sqrt{\lambda}\to j_{m_l,p}} \left[(\sqrt{\lambda} - j_{m_l,p})\left(\sqrt{\lambda}\frac{J_{m_l+1}(\sqrt{\lambda})}{J_{m_l}(\sqrt{\lambda})} - l\right)\right] \tag{4.3.21a}$$

$$= \lim_{\sqrt{\lambda}\to j_{m_l,p}} \left[\sqrt{\lambda}J_{m_l+1}(\sqrt{\lambda})\,\frac{\sqrt{\lambda} - j_{m_l,p}}{J_{m_l}(\sqrt{\lambda}) - J_{m_l}(j_{m_l,p})}\right] \tag{4.3.21b}$$

$$= j_{m_l,p}J_{m_l+1}(j_{m_l,p}) \times \lim_{\sqrt{\lambda}\to j_{m_l,p}} \left(\frac{J_{m_l}(\sqrt{\lambda}) - J_{m_l}(j_{m_l,p})}{\sqrt{\lambda} - j_{m_l,p}}\right)^{-1} \tag{4.3.21c}$$

$$= j_{m_l,p}J_{m_l+1}(j_{m_l,p}) \times \left(J'_{m_l}(j_{m_l,p})\right)^{-1} \tag{4.3.21d}$$

$$= j_{m_l,p}J_{m_l+1}(j_{m_l,p}) \times \left(\frac{m_l}{j_{m_l,p}}J_{m_l}(j_{m_l,p}) - J_{m_l+1}(j_{m_l,p})\right)^{-1} \tag{4.3.21e}$$

$$= -j_{m_l,p}. \tag{4.3.21f}$$

Since α is an eigenvalue of $M^{(l)}(\lambda)$ we obtain

$$(\sqrt{\lambda} - j_{m_l,p})\alpha = \mathrm{Res}_{j_{m_l,p}}\left(M^{(l)}\right) + \mathcal{O}(\sqrt{\lambda} - j_{m_l,p}) \tag{4.3.22a}$$

$$= -j_{m_l,p} + \mathcal{O}(\sqrt{\lambda} - j_{m_l,p}) \tag{4.3.22b}$$

as $\sqrt{\lambda} \to j_{m_l,p}$, it follows that

$$\sqrt{\lambda(\alpha)} = j_{m_l,p} - \frac{j_{m_l,p}}{\alpha} + \mathcal{O}\left(\frac{1}{\alpha^2}\right) \tag{4.3.23}$$

and hence for the eigenvalue λ

$$\lambda(\alpha) = j_{m_l,p}^2 - 2\frac{j_{m_l,p}^2}{\alpha} + \mathcal{O}\left(\frac{1}{\alpha^2}\right) \tag{4.3.24}$$

as $\alpha \to \infty$. □

4.3.4 Divergence away from the positive real axis

We next study those divergent eigenvalues λ which remain away from the positive real axis, that is, we now apply Assumption 4.1.3. It will turn out that (unlike for the interval) the assumption of divergence in a sector, that is, that $\mathrm{Im}\sqrt{\lambda}$ grows sufficiently rapidly compared with $\mathrm{Re}\sqrt{\lambda}$, is important. However, the approach to determine the asymptotics of α as functions of λ is similar: we first need an asymptotic expansion of $M^{(l)}$ for large λ, which in turn requires knowledge of the asymptotics of the Bessel functions appearing in (4.3.16). To this end, let $H_m^{(1)}, H_m^{(2)}$ be the *Hankel functions* of the first and second kind, that is,

$$2J_m(\sqrt{\lambda}) = H_m^{(1)}(\sqrt{\lambda}) + H_m^{(2)}(\sqrt{\lambda}), \tag{4.3.25}$$

and set

$$P(m,\sqrt{\lambda}) := \sum_{l=0}^{\infty}(-1)^l \frac{\Gamma\left(m+2l+\frac{1}{2}\right)}{(2l)!\Gamma\left(m-2l+\frac{1}{2}\right)(2\sqrt{\lambda})^{2l}} \tag{4.3.26a}$$

$$\begin{aligned} &= 1 - \frac{(4m^2-1)(4m^2-9)}{2!(8\sqrt{\lambda})^2} \\ &\quad + \frac{(4m^2-1)(4m^2-9)(4m^2-25)(4m^2-49)}{4!(8\sqrt{\lambda})^4} - \dots \end{aligned} \tag{4.3.26b}$$

as well as

$$Q(m,\sqrt{\lambda}) := \sum_{l=0}^{\infty}(-1)^l \frac{\Gamma\left(m+(2l+1)+\frac{1}{2}\right)}{(2l+1)!\Gamma\left(m-(2l+1)+\frac{1}{2}\right)(2\sqrt{\lambda})^{2l+1}} \tag{4.3.27a}$$

$$= \frac{4m^2-1}{8\sqrt{\lambda}} - \frac{(4m^2-1)(4m^2-9)(4m^2-25)}{3!(8\sqrt{\lambda})^3} + \dots . \tag{4.3.27b}$$

It is known that (see [1, 9.2, p. 364])

$$H_m^{(1)}(\sqrt{\lambda}) = \sqrt{\frac{2}{\pi\sqrt{\lambda}}}\left(P(m,\sqrt{\lambda}) + \mathrm{i}Q(m,\sqrt{\lambda})\right)\mathrm{e}^{\mathrm{i}\sqrt{\lambda}-\frac{\mathrm{i}\pi}{4}(2m+1)} \tag{4.3.28}$$

for $-\pi < \arg\sqrt{\lambda} < 2\pi$ and

$$H_m^{(2)}(\sqrt{\lambda}) = \sqrt{\frac{2}{\pi\sqrt{\lambda}}}\left(P(m,\sqrt{\lambda}) - \mathrm{i}Q(m,\sqrt{\lambda})\right)\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}+\frac{\mathrm{i}\pi}{4}(2m+1)} \tag{4.3.29}$$

for $-2\pi < \arg\sqrt{\lambda} < \pi$. Let $P_l(m,\sqrt{\lambda})$ and $Q_l(m,\sqrt{\lambda})$ be the sums (4.3.26a) and (4.3.27a), respectively, up to the lth summand. To obtain the order $1/\sqrt{\lambda}$, we only have to consider the first terms in the expansions of P and Q to obtain

$$J_m(\sqrt{\lambda}) = \frac{1}{2}\left(H_m^{(1)}(\sqrt{\lambda}) + H_m^{(2)}(\sqrt{\lambda})\right) \tag{4.3.30a}$$

$$\sim \frac{1}{2}\sqrt{\frac{2}{\pi\sqrt{\lambda}}}\Big[\left(P_0(m,\sqrt{\lambda}) + \mathrm{i}Q_0(m,\sqrt{\lambda})\right)\mathrm{e}^{\mathrm{i}\sqrt{\lambda}-\frac{\mathrm{i}\pi}{4}(2m+1)} + \left(P_0(m,\sqrt{\lambda}) - \mathrm{i}Q_0(m,\sqrt{\lambda})\right)\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}+\frac{\mathrm{i}\pi}{4}(2m+1)}\Big] \tag{4.3.30b}$$

$$= \sqrt{\frac{1}{2\pi\sqrt{\lambda}}}\left[\left(1 + \mathrm{i}\frac{4m^2-1}{8\sqrt{\lambda}}\right)\mathrm{e}^{\mathrm{i}\sqrt{\lambda}-\frac{\mathrm{i}\pi}{4}(2m+1)} + \left(1 - \mathrm{i}\frac{4m^2-1}{8\sqrt{\lambda}}\right)\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}+\frac{\mathrm{i}\pi}{4}(2m+1)}\right]. \tag{4.3.30c}$$

The non-(fractional-order)polynomial terms $\mathrm{e}^{\mathrm{i}\sqrt{\lambda}}$ and $\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}}$ of $H_m^{(1)}(\sqrt{\lambda})$ and $H_m^{(2)}(\sqrt{\lambda})$ yield exponential decrease and increase in S_θ^+ and S_θ^-, respectively. In a neighbourhood of the real axis, however, the remainder terms of the increasing expansion dominate the leading terms of the decreasing expansion on the other side of the real axis. This is why we want to add up both terms to obtain [1, 9.2.1, p. 364], viz.

$$J_m(\sqrt{\lambda}) = \frac{1}{\sqrt{2\pi\sqrt{\lambda}}}\left(2\cos\left(\sqrt{\lambda} - \frac{\mathrm{i}\pi}{4}[2m+1]\right) + \mathrm{e}^{|\mathrm{Im}\sqrt{\lambda}|}\mathcal{O}\left(|\sqrt{\lambda}|^{-1}\right)\right) \tag{4.3.31}$$

as $\sqrt{\lambda} \to \infty$ outside T_θ^- (in particular in T_θ^+). This expansion outside T_θ^+ (in particular in T_θ^-) is obtained by point reflection of $J_m(\sqrt{\lambda})$ in zero.

Considering the two cases where $\sqrt{\lambda} \to \infty$ in S_θ^- and S_θ^+, separately, we arrive at

$$J_m(\sqrt{\lambda}) = \frac{1}{\sqrt{2\pi\sqrt{\lambda}}}\left(1 - \mathrm{i}\frac{4m^2-1}{8\sqrt{\lambda}}\right)\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}(1+2m)} + \mathcal{O}\left(\frac{1}{\lambda}\right) \tag{4.3.32}$$

as $\sqrt{\lambda} \to \infty$ in S_θ^- and

$$J_m(\sqrt{\lambda}) = \frac{1}{\sqrt{2\pi\sqrt{\lambda}}}\left(1 + \mathrm{i}\frac{4m^2-1}{8\sqrt{\lambda}}\right)\mathrm{e}^{+\mathrm{i}\sqrt{\lambda}}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}(1+2m)} + \mathcal{O}\left(\frac{1}{\lambda}\right) \tag{4.3.33}$$

as $\sqrt{\lambda} \to \infty$ in S_θ^+, respectively. Using (4.3.32) and (4.3.33) and recalling the relation

$m = \frac{d}{2} + l - 1$, we calculate

$$\frac{J_{m+1}(\sqrt{\lambda})}{J_m(\sqrt{\lambda})} = \frac{\left(1 - \mathrm{i}\frac{4(m+1)^2-1}{8\sqrt{\lambda}}\right)\exp\left(\frac{\mathrm{i}\pi}{4}\left[1+2(m+1)\right]\right)}{\left(1 - \mathrm{i}\frac{4m^2-1}{8\sqrt{\lambda}}\right)\exp\left(\frac{\mathrm{i}\pi}{4}\left[1+2m\right]\right)} + \mathcal{O}\left(\frac{1}{\lambda}\right) \tag{4.3.34a}$$

$$= \frac{8\sqrt{\lambda} - \mathrm{i}\left[(4m^2-1) + 4(2m+1)\right]}{8\sqrt{\lambda} - \mathrm{i}(4m^2-1)}\exp\left(\frac{\mathrm{i}\pi}{2}\right) + \mathcal{O}\left(\frac{1}{\lambda}\right) \tag{4.3.34b}$$

$$= \left(1 - \mathrm{i}\frac{4(2m+1)}{8\sqrt{\lambda} - \mathrm{i}(2m+1)(2m-1)}\right)\mathrm{i} + \mathcal{O}\left(\frac{1}{\lambda}\right) \tag{4.3.34c}$$

$$= \left(1 - \mathrm{i}\frac{1}{\frac{2\sqrt{\lambda}}{2m+1} - \frac{\mathrm{i}}{2}(2m-1)}\right)\mathrm{i} + \mathcal{O}\left(\frac{1}{\lambda}\right) \tag{4.3.34d}$$

$$= \mathrm{i} + \frac{2m+1}{\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{\lambda}\right) \tag{4.3.34e}$$

$$= \mathrm{i} + \frac{d-1}{2\sqrt{\lambda}} + \frac{l}{\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{\lambda}\right). \tag{4.3.34f}$$

The calculations for the second case $\sqrt{\lambda} \to \infty$ in S_θ^+ works similarly and results in

$$\frac{J_{m+1}(\sqrt{\lambda})}{J_m(\sqrt{\lambda})} = -\mathrm{i} + \frac{d-1}{2\sqrt{\lambda}} + \frac{l}{\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{\lambda}\right) \tag{4.3.35}$$

and we arrive at

$$\frac{J_{m+1}(\sqrt{\lambda})}{J_m(\sqrt{\lambda})} = \pm\mathrm{i} + \frac{d-1}{2\sqrt{\lambda}} + \frac{l}{\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{\lambda}\right) \tag{4.3.36}$$

in $S_\theta^\pm$. Recalling (4.3.16), this means that for each $l \in \mathbb{N}_0$ we obtain an $\alpha = \alpha(\lambda)$ with the behaviour

$$\alpha = \mp\mathrm{i}\sqrt{\lambda} + \frac{d-1}{2} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right) \tag{4.3.37}$$

as $\sqrt{\lambda} \to \infty$ in $S_\theta^\pm$, respectively. This leads us to the following statement, which is due to [30, Theorem 9.9].

Theorem 4.3.6. *Let $\Omega = B_1(0) \subseteq \mathbb{R}^d$, $d \geq 2$, and let $\alpha \to \infty$ in a sector of the form T_φ^- for any $0 < \varphi < \pi/2$. Then there are infinitely many Robin eigenvalues*

$\lambda(\alpha)$ *such that*

$$\lambda(\alpha) = -\alpha^2 + \alpha(d-1) + \mathcal{O}(1) \tag{4.3.38}$$

as $\alpha \to \infty$ *in* T_φ^-.

Proof. We invert (4.3.37) to obtain (4.3.38) using Rouché's theorem, as explained in Section 4.1.2. First, let

$$\tau = \frac{d-1}{2} \tag{4.3.39}$$

and let $\alpha \in \mathbb{C}$ be large in some non-trivial sector T_φ^- for any large $0 < \varphi < \pi/2$. Following the approach of the aforementioned Section 4.1.2 and restricting ourselves to the *positive* case of $\operatorname{Im}\sqrt{\lambda} \to +\infty$, we have

$$f_\alpha(z) := \frac{d-1}{2} + \mathrm{i}z - \alpha \tag{4.3.40}$$

and

$$g(\sqrt{\lambda}) = \mathcal{O}\left(h(\sqrt{\lambda})\right) \tag{4.3.41}$$

for the error term

$$h(\sqrt{\lambda}) = \frac{1}{\sqrt{\lambda}} \to 0 \tag{4.3.42}$$

as $\operatorname{Im}\sqrt{\lambda} \to +\infty$. Then the unique zero z_α of f_α reads

$$z_\alpha = \mathrm{i}\left(\frac{d-1}{2} - \alpha\right). \tag{4.3.43}$$

Let $r_\alpha > 0$ be the radius of the Ball $B^\alpha = B_{r_\alpha}(z_\alpha)$. Then for $z \in B^\alpha$ and sufficiently large α we can estimate

$$|h(z)| \le \frac{c}{\left|z_\alpha + \mathrm{e}^{\mathrm{i}\varphi}\frac{p}{|\alpha|}\right|} = \frac{c}{\left|\mathrm{i}\frac{d-1}{2} - \mathrm{i}\alpha + \mathrm{e}^{\mathrm{i}\varphi}\frac{p}{|\alpha|}\right|} < \frac{c'}{|\alpha|} \tag{4.3.44}$$

for some constants $c, c' > 0$. We make the ansatz

$$|f_\alpha(z)| = r_\alpha = \frac{C}{|\alpha|} \tag{4.3.45}$$

for a suitable constant $C > 0$. Consequently, since (4.1.36) holds for some $\delta > 0$ for sufficiently large α, a similar calculation as in the proof of Theorem 4.1.4 yields

$$|g(z)| < |f_\alpha(z)| \quad \text{on} \quad \partial B_{\frac{C}{|\alpha|}}(z_\alpha). \tag{4.3.46}$$

We end up with the existence of exactly one eigenvalue λ which behaves like

$$\sqrt{\lambda(\alpha)} = -\mathrm{i}\alpha + \frac{d-1}{2} + \mathcal{O}\left(\frac{1}{\alpha}\right), \tag{4.3.47}$$

that is, together with the case $\operatorname{Im}\sqrt{\lambda} \to -\infty$, there are exactly two eigenvalues which behave like

$$\lambda(\alpha) = -\alpha^2 + (d-1)\alpha + \mathcal{O}(1) \tag{4.3.48}$$

as $\operatorname{Re}\alpha \to -\infty$. □

Remark 4.3.7. For any divergent eigenvalue curve $\lambda(\alpha)$ on the ball, there is a complete asymptotic expansion in powers of α, and the above method might be used to obtain arbitrarily many terms of it. Indeed, the asymptotics of the Bessel functions (or more precisely the Hankel functions) provides us with everything needed to determine the asymptotics of the Dirichlet-to-Neumann operators $M^{(l)}(\lambda)$, and thus of α as $\operatorname{Im}\sqrt{\lambda} \to \pm\infty$: taking more terms in (4.3.30a) results in more terms in (4.3.32) and (4.3.33), and so too in (4.3.37), which can then again be inverted.

4.3.5 Divergence near the positive real axis

We reconsider the asymptotics in (4.3.31) and observe the oscillating nature of the cosine part as $\operatorname{Re}\sqrt{\lambda}$ increases – the summand $\frac{\mathrm{i}\pi}{4}[2m+1]$ appearing in the argument is simply a phase shift. Suppose that $\operatorname{Im}\sqrt{\lambda}$ remains bounded as $\sqrt{\lambda} \to \infty$ in T_θ^+, i.e., we explicitly do not apply Assumption 4.1.3. Then it appears that $J_m(\sqrt{\lambda})$ is dominated by $\lambda^{1/4}\cos(\sqrt{\lambda})$. However, the cosine having zeros on the real axis might be, in a neighbourhood of said zeros, dominated by the $\mathcal{O}(|\sqrt{\lambda}|^{-1})$ remainder term:

thus this asymptotic expansion cannot be used. Furthermore, a calculation of the asymptotics as in (4.3.36) then leads to more complicated behaviour.

Chapter 5

Quantum Graph Laplacians

Contents

5.1 The Robin problem on compact quantum graphs

This last chapter follows the article *On the eigenvalues of quantum graph Laplacians with large complex δ couplings* [76] by James B. Kennedy and myself - not always in structure but in content. In similar ways to the previous chapters on the Robin Laplacian on bounded domains we exploit the duality of the Dirichlet-to-Neumann operator $M(\lambda)$ and the Robin Laplacian $-\Delta_{\mathcal{G}}^{\alpha}$ in order to give asymptotic formulas for the eigenvalues, see Sections 5.2 and 5.3 for the construction of the Dirichlet-to-Neumann operator and its asymptotics, respectively. In Section 5.4 we analyse the numerical range of $-\Delta_{\mathcal{G}}^{\alpha}$ to obtain (similarly to the localisation theorem 3.4.1) that the numerical range and hence the eigenvalues are localised in a parabolic region. This allows us to control both the real and imaginary parts of the Robin eigenvalues in terms of the real and imaginary party of α.

5.1.1 On quantum graphs

We first need to introduce some basic terminology; we refer to the monographs [27, 95] or the elementary introduction [23] for more details. The theory of *quantum graphs* considers differential operators on so called metric graphs, that is, a set of points (vertices) connected by segments (edges). For the differential operator the edges are no different from one-dimensional intervals: the Laplacian, for example, acts as the second derivative along each edge; the vertices can be interpreted as the *boundary* of the graph which in our case gets equipped with the Robin condition, that is, δ couplings in some (or all) of the vertices.

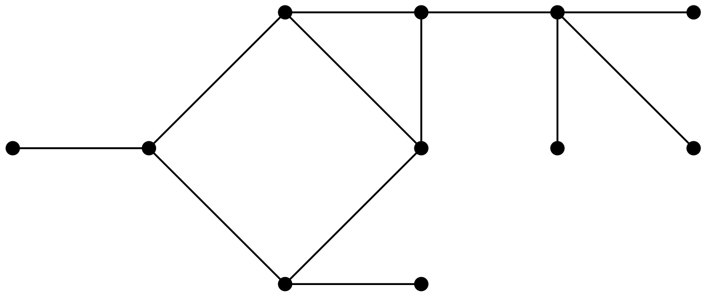

Figure 5.1.1: A (connected) compact graph with 11 vertices and 12 edges.

Definition 5.1.1. A *compact metric graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of

(1) a finite vertex set $\mathcal{V} = \{v_1, \dots, v_n\}$

(2) and a finite edge set $\mathcal{E} = \{e_1, \dots, e_m\}$,

where each edge e is identified with a compact interval $[0, \ell_e] \subset \mathbb{R}$ of length $\ell_e > 0$, denoted by

$$e \simeq [0, \ell_e]. \tag{5.1.1}$$

The endpoints 0 and ℓ_e correspond to the vertices which are incident with the edge e and we write $v \sim e$ (or interchangeably $e \sim v$), if a vertex v is incident with e.

While the notation with respect to the endpoints 0 and ℓ_e implicitly presupposes an orientation on e, we will see that since the sesquilinear forms are invariant under the transformation

$$[0, \ell_e] \ni x \mapsto \ell_e - x, \tag{5.1.2}$$

the associated differential operators we will be considering do not depend on this choice of orientation.

Definition 5.1.2. (1) The degree of a vertex $v \in \mathcal{V}$, denoted by $\deg v \geq 1$, is the number of edges with which v is incident, that is,

$$\deg v := \# \{e \in \mathcal{E} : e \sim v\} \tag{5.1.3}$$

for $v \in \mathcal{V}$.

(2) We explicitly allow our graphs to have loops (edges both of whose endpoints correspond to the same vertex; in this case the edge is counted twice when computing the degree of the vertex) and we allow multiple edges between any given pair of vertices, cf. Figure 5.1.2.

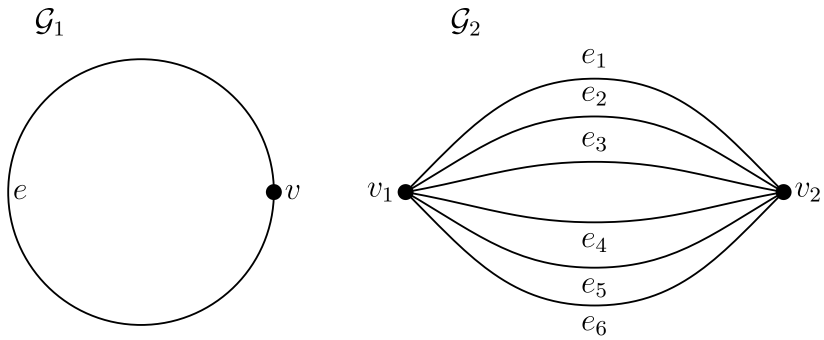

Figure 5.1.2: The graph $\mathcal{G}_1$ is called *loop* and $\mathcal{G}_2$ is a *pumpkin graph* with parallel edges $e_1, \dots, e_6$.

Equipped with the usual metric corresponding to the shortest Euclidean path between two points, $\mathcal{G}$ is a compact metric space. The graph is connected if and only if it is connected as a metric space. We will *always* assume $\mathcal{G}$ to be such a connected compact metric graph. On $\mathcal{G}$, as customary, we can define the space $L^2(\mathcal{G})$ of square integrable functions, the space $C(\mathcal{G}) \hookrightarrow L^2(\mathcal{G})$ of continuous functions, and the Sobolev space $H^1(\mathcal{G}) \hookrightarrow C(\mathcal{G})$, respectively:

Definition 5.1.3. We define

(1) the following function spaces

$$L^2(\mathcal{G}) = \bigoplus_{e\in\mathcal{E}} L^2(e) \simeq \bigoplus_{e\in\mathcal{E}} L^2((0,\ell_e)), \tag{5.1.4a}$$

$$\begin{aligned} C(\mathcal{G}) = \{f : \mathcal{G} \to \mathbb{C} \ : \ f|_e \in C(e) \text{ for all } e \in \mathcal{E} \\ \text{and } f \text{ is continuous at each } v \in \mathcal{V}\}, \end{aligned} \tag{5.1.4b}$$

$$H^1(\mathcal{G}) = \{f \in C(\mathcal{G}) \ : \ f|_e \in H^1(e) \text{ for all } e \in \mathcal{E}\}; \tag{5.1.4c}$$

(2) the integral of a function f over $\mathcal{G}$ by

$$\int_{\mathcal{G}} f \, \mathrm{d}x := \sum_{e\in\mathcal{E}} \int_e f|_e \, \mathrm{d}x; \tag{5.1.5}$$

(3) the expression $\frac{\partial}{\partial\nu} f|_e(v)$ as the derivative of f along the edge e at v, pointing into v (which may be thought of as the outer normal derivative to the edge e at v); this exists if $f|_e \in C^1(e)$.

5.1.2 The Robin Laplacian: complex δ couplings

To define our operator, we first need to identify a distinguished set of vertices, which will be equipped with our Robin-type condition: we fix an arbitrary set

$$\mathcal{V}_R = \{v_1, \dots, v_k\} \subset \mathcal{V} \tag{5.1.6}$$

with cardinality $k \leq n := |\mathcal{V}|$ and call $\mathcal{V}_R$ the *set of Robin vertices*, consistent with the nomenclature in [27, Section 1.4.1]. We further fix a vector

$$\alpha = (\alpha_1, \dots, \alpha_k)^T \in \mathbb{C}^k \tag{5.1.7}$$

with $\alpha_j = \alpha(v_j)$ for $j = 1, \dots, k$, that is, the vector of Robin parameters where each $v_1, \dots, v_k$ gets equipped with its corresponding parameter $\alpha_1, \dots, \alpha_k$.

Definition 5.1.4. We define a sesquilinear form $a_\alpha : H^1(\mathcal{G}) \times H^1(\mathcal{G}) \to \mathbb{C}$ by

$$a_\alpha[f,g] := \int_{\mathcal{G}} f' \cdot \overline{g}' \,\mathrm{d}x + \sum_{j=1}^{k} \alpha_j f(v_j)\overline{g(v_j)} \tag{5.1.8}$$

for $f, g \in H^1(\mathcal{G})$ in the sense of (5.1.5).

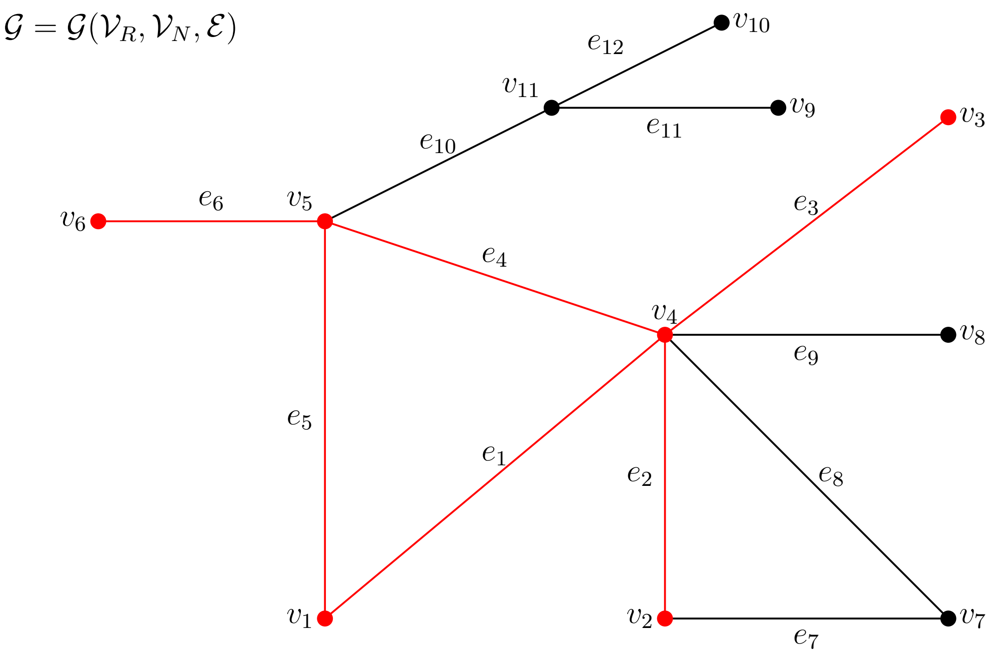

Figure 5.1.3: A connected graph $\mathcal{G}(\mathcal{V}_R, \mathcal{V}_N, \mathcal{E})$ with six Robin vertices $v_1, \dots, v_6$, five Neumann–Kirchhoff vertices $v_7, \dots, v_{12}$, and 12 edges $e_1, \dots, e_{12}$: the red edges $e_1, \dots, e_6$ are intervals of the type R–R (cf. Section 4.1), $e_7, \dots, e_{10}$ are of type R–N and e_{11}, e_{12} of type N–N.

Integration by parts (each occurring integral is simply 1−dimensional) shows that the operator on $L^2(\mathcal{G})$ associated with a_α is the Laplacian, i.e., $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on each edge, whose domain consists of those functions $f \in H^1(\mathcal{G})$ such that

(1) $f|_e \in H^2(e) \hookrightarrow C^1(e)$ for all $e \in \mathcal{E}$,

(2) f is continuous at every vertex $v \in \mathcal{V}$,

(3) f satisfies the following vertex conditions:

(3.1) if $v_j \in \mathcal{V}_R$, then

$$\sum_{e\sim v_j} \frac{\partial}{\partial\nu} f|_e(v_j) + \alpha_j f(v_j) = 0; \tag{5.1.9}$$

(3.2) if $v_j \in \mathcal{V} \setminus \mathcal{V}_R = \{v_{k+1}, \dots, v_n\}$, then

$$\sum_{e\sim v_j} \frac{\partial}{\partial\nu} f|_e(v_j) = 0. \tag{5.1.10}$$

By way of analogy with its counterparts on domains and manifolds, we will call the unbounded operator on $L^2(\mathcal{G})$ associated with the form a_α from (5.1.8) the *Robin Laplacian* and denote it by $-\Delta^{\alpha}_{\mathcal{V}_R}$. Note that as mentioned in the introduction this Robin condition is most commonly known as a δ condition (or δ coupling) in the literature. We call and note that condition (5.1.10) on non-Robin vertices is the condition usually known as Kirchhoff, which corresponds to the Robin condition (5.1.9) with $\alpha = 0$. The Kirchhoff condition together with continuity in each

$$v \in \mathcal{V}_N := \mathcal{V} \setminus \mathcal{V}_R = \{v_{k+1}, \dots, v_n\} \tag{5.1.11}$$

is then known variously in the literature as natural, standard, or even sometimes just Neumann or Neumann–Kirchhoff; it is for this reason that we will use the letter "N" as an index for the corresponding vertex set $\mathcal{V}_N = \mathcal{V} \setminus \mathcal{V}_R$. For the construction of the Dirichlet-to-Neumann matrix, we will need the Dirichlet Laplacian on $\mathcal{G}$.

Definition 5.1.5. We will say that there is a Dirichlet condition at a vertex $v_j \in \mathcal{V}$ if all functions in the domain of the form or operator are simply equal to zero at v_j; no further conditions on the functions are imposed at v_j. We will denote by $-\Delta^{D}_{\mathcal{V}_0}$ the Laplacian satisfying

(1) Dirichlet conditions at every vertex in a subset $\mathcal{V}_0 \subset \mathcal{V}$,

(2) continuity plus Kirchhoff conditions at all vertices of $\mathcal{V}_N = \mathcal{V} \setminus \mathcal{V}_0$.

At the level of sesquilinear forms, the form associated with this operator is given by

$$a_0[f, g] = \int_{\mathcal{G}} f' \cdot \overline{g}' \, \mathrm{d}x \tag{5.1.12}$$

for f, g in its form domain $D(a_0)$ which coincides with

$$H_0^1(\mathcal{G}, \mathcal{V}_0) := \{f \in H^1(\mathcal{G}) : f(v_j) = 0 \text{ for all } v_j \in \mathcal{V}_0\}. \tag{5.1.13}$$

We will correspondingly write $-\Delta_{\mathcal{V}}^D$ for the Laplacian on $L^2(\mathcal{G})$ satisfying Dirichlet conditions at every vertex of $\mathcal{V}$. In this case $\mathcal{G}$ decouples to a disjoint union of $m = |\mathcal{E}|$ intervals, each equipped with Dirichlet conditions at both endpoints.

We refer in particular to [27, Section 1.4] for more details on these operators and vertex conditions.

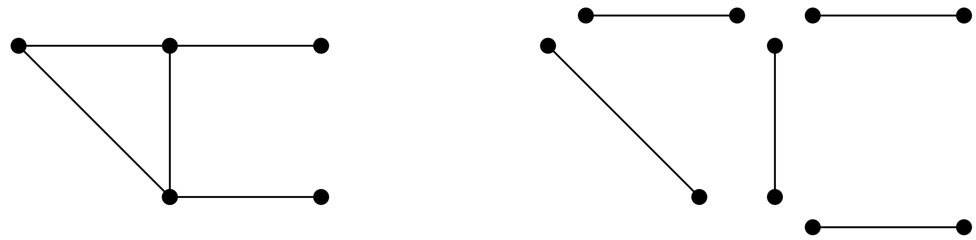

Figure 5.1.4: Dirichlet conditions at a vertex v imply that the graph decouples to a disjoint union of $\deg v$ subgraphs. If every vertex is a Dirichlet vertex, then we arrive at $|\mathcal{E}|$ subgraphs, each of which being an interval.

All the operators $-\Delta_{\mathcal{V}_R}^{\alpha}$, $-\Delta_{\mathcal{V}_R}^{D}$ are seen to have compact resolvent (since the embedding of $H^1(\mathcal{G})$ into $L^2(\mathcal{G})$ is compact), and hence discrete spectrum, for any $\alpha \in \mathbb{C}^k$. For real α or Dirichlet conditions, this is contained in [27, Theorem 3.1.1]. For complex α, this may be deduced from [26, Section 3] or [71, Sections 3.5 and 5], or proved directly using the compactness of the embedding $H^1(\mathcal{G}) \hookrightarrow L^2(\mathcal{G})$ and the fact that $-\Delta_{\mathcal{V}_R}^{\alpha}$ must have non-empty resolvent set, e.g., by Theorem 5.4.1. For each eigenvalue $\lambda \in \sigma(-\Delta_{\mathcal{V}_R}^{\alpha})$, there exists an eigenfunction $\psi \in H^1(\mathcal{G})$ which satisfies

$$\int_{\mathcal{G}} \psi' \cdot \overline{\varphi}' \,\mathrm{d}x + \sum_{j=1}^{k} \alpha_j \psi(v_j)\overline{\varphi(v_j)} = \lambda \int_{\mathcal{G}} \psi\overline{\varphi} \,\mathrm{d}x \tag{5.1.14}$$

for all $\varphi \in H^1(\mathcal{G})$. Throughout, we will assume the connected, compact graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and set $\mathcal{V}_R \subset \mathcal{V}$ of Robin vertices to be fixed. Before continuing, we first note the following basic property of the dependence of the eigenvalues of $-\Delta_{\mathcal{V}_R}^{\alpha}$ on $\alpha \in \mathbb{C}^k$ (see [76, Lemma 2.1]).

Lemma 5.1.6. *The operator family* $\mathcal{A}(\alpha) = -\Delta^{\alpha}_{\mathcal{V}_R}$, $\alpha \in \mathbb{C}$, *is self-adjoint holomorphic; in particular,* $\mathcal{A}(\alpha)^* = \mathcal{A}(\overline{\alpha})$ *for all* $\alpha \in \mathbb{C}$, *and up to possible crossing points, each eigenvcurve* $\lambda(\alpha)$ *depends holomorphically on* α.

The proof is similar as the one for Theorem 3.2.11, that is, [74, Theorems VII.4.2 and VII.1.8, Remark VII.4.7]. Analyticity results can also be found in [26, Section 3.4].

5.2 The Dirichlet-to-Neumann operator

We now turn to the proof of Theorem 1.2.8. It is based on the Dirichlet-to-Neumann operator $M(\lambda)$, cf. Section 3.5, and its asymptotical behaviour as its spectral parameter λ tends to infinity in $\mathbb{C}$. To construct the Dirichlet-to-Neumann matrix on quantum graphs we follow a similar approach as in Section 4.1, that is, we consider the inhomogeneous Dirichlet problem where a vertex v is equipped with the Dirichlet condition if and only if $v \in \mathcal{V}_R$; every other vertex satisfies the Neumann–Kirchhoff condition. We start with the Dirichlet-to-Neumann matrix $M_{e_{ij}}(\lambda)$ on a single edge $e_{ij} \in \mathcal{E}$ connecting two distinct vertices v_i and v_j, and extend it by zero to all the vertices in $\mathcal{V}_N$ to obtain an $n \times n$ matrix $\widetilde{M}_{e_{ij}}(\lambda)$. Then, by summing over all these matrices we obtain $M_{\mathcal{V}}(\lambda)$, that is, the operator acting on all of $\mathcal{G}$. If we remove all *phantom entries*, namely each entry A_{ij} of $M_{\mathcal{V}}(\lambda)$ such that there is no edge joining v_i and v_j, a representation of $M_{\mathcal{V}}$ in block form allows us to give a formula for the Dirichlet-to-Neumann matrix $M(\lambda)$ acting on the *correct* vertex set $\mathcal{V}_R$. We want to perform these steps in detail: given a vector (Dirichlet data)

$$g = (g_1, \dots, g_k)^T \in \mathbb{C}^k \approx \mathcal{V}_R \tag{5.2.1}$$

and $\lambda \notin \sigma(-\Delta^{D}_{\mathcal{V}_R})$, there exists a unique weak solution $f \in H^1(\mathcal{G})$ of the Dirichlet problem

$$-f'' = \lambda f \qquad \text{edgewise,} \tag{5.2.2a}$$

$$f|_{\mathcal{V}_R} = g, \tag{5.2.2b}$$

$$\sum_{e \sim v_j} \frac{\partial}{\partial \nu} f|_e(v_j) = 0 \qquad \text{for all } v_j \in \mathcal{V}_N. \tag{5.2.2c}$$

The Dirichlet-to-Neumann operator $M(\lambda)$ maps given Dirichlet data $g = f|_{\mathcal{V}_R}$ to the corresponding Neumann data $-\frac{\partial}{\partial \nu} f|_e(v_j)$ of the same solution f of the problem (5.2.2), that is, a map from $\mathcal{V}_R$ to itself. If we fix the order $v_1, \dots, v_k$ of the vertices in $\mathcal{V}_R$, then $M(\lambda)$ is canonically identifiable with a matrix in $\mathbb{C}^{k\times k}$. In future we shall make this identification without further comment. We now wish to analyse this operator in more detail: we first note that we may assume without loss of generality that $\mathcal{G}$ does not have any loops nor multiple parallel edges (i.e., between any two distinct vertices there is at most one edge); indeed, if this is not the case, then we may insert a new, artificial vertex of degree two in the middle of each affected edge as it is depicted in Figure 5.2.1. When these vertices are equipped with continuity and Kirchhoff conditions, the Laplacian on the resulting graph is unitarily equivalent to the one on the unaltered graph (see [24, Section 3]), and so the Dirichlet-to-Neumann operator on the unaffected set $\mathcal{V}_R$ of Robin vertices is equally unaffected.

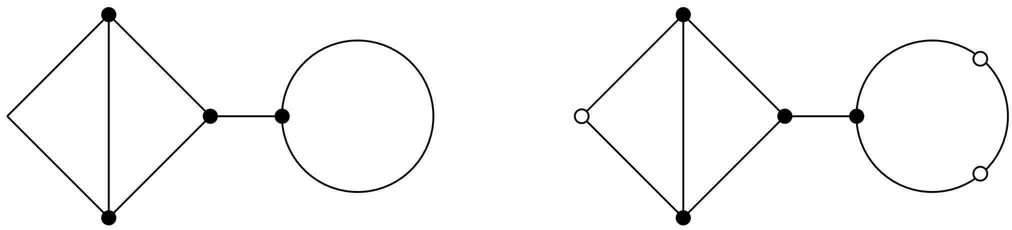

Figure 5.2.1: To avoid loops and parallel edges, that is, edges e_1 and e_2 and vertices v_1 and v_2 with $v_1 \sim e_j \sim v_2$ for $j = 1, 2$ (e.g. as $\mathcal{G}_2$ depicted in Figure 5.1.2, artificial Neumann–Kirchhoff vertices of degree 2 are inserted; they do not have any effect on the spectral properties of our differential operators. Note that the loop requires not one but two artificial vertices to avoid parallel edges.

Now let $v_i, v_j \in \mathcal{V}$ be any two distinct vertices and suppose they are joined by a (unique) edge e_{ij} having length $\ell_{ij} > 0$. It is known, and a short calculation shows, that the Dirichlet-to-Neumann operator associated with the graph consisting just of this edge (that is, an interval of length ℓ_{ij}) and the parameter

$$\lambda \in \mathbb{C} \setminus \left\{ \frac{\pi^2 n^2}{\ell_{ij}^2} : n \in \mathbb{N} \right\} \tag{5.2.3}$$

may be represented by the matrix

$$M_{e_{ij}}(\lambda) = \sqrt{\lambda} \begin{pmatrix} -\cot\sqrt{\lambda}\ell_{ij} & \csc\sqrt{\lambda}\ell_{ij} \\ \csc\sqrt{\lambda}\ell_{ij} & -\cot\sqrt{\lambda}\ell_{ij} \end{pmatrix}. \tag{5.2.4}$$

For more details and a derivation of this representation, we refer to Section 4.1. Fix $\lambda \in \mathbb{C}$, to be specified precisely later. We denote by $\widetilde{M}_{e_{ij}} \in \mathbb{C}^{n\times n}$ the matrix corresponding to the operator (5.2.4) extended by zero to the $n-k$ other vertices in $\mathcal{V}_N$. That is, for fixed $1 \le i, j \le n$, the (i,i)- and (j,j)-entries of $\widetilde{M}_{e_{ij}}$ are given by

$$\sqrt{\lambda}A_{ij} := -\sqrt{\lambda}\cot\sqrt{\lambda}\ell_{ij}\,; \tag{5.2.5}$$

the (i,j)- and (j,i)-entries of $\widetilde{M}_{e_{ij}}$ are given by

$$\sqrt{\lambda}B_{ij} := \sqrt{\lambda}\csc\sqrt{\lambda}\ell_{ij}. \tag{5.2.6}$$

All other entries are zero, that is,

$$\widetilde{M}_{e_{ij}}(\lambda) = \sqrt{\lambda} \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & A_{ij} & 0 & B_{ij} & \vdots \\ \vdots & 0 & \cdots & 0 & \vdots \\ \vdots & B_{ij} & 0 & A_{ij} & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \in \mathbb{C}^{n\times n}. \tag{5.2.7}$$

Note that there are always $(|i-j|-1)$ zeroes between the 4 non-zero entries of $\widetilde{M}_{e_{ij}}(\lambda)$, that is, $\widetilde{M}_{e_{ij}}(\lambda)$ does always (and only) consist of the quadratic and symmetric block

$$\begin{pmatrix} A_{ij} & 0 & B_{ij} \\ 0 & \cdots & 0 \\ B_{ij} & 0 & A_{ij} \end{pmatrix} \in \mathbb{C}^{(|i-j|+1)\times(|i-j|+1)} \tag{5.2.8}$$

embedded in $\mathbb{C}^{n\times n}$: here, A_{ij} are always diagonal entries of $\widetilde{M}_{e_{ij}}(\lambda)$. We may then write the Dirichlet-to-Neumann operator $M_{\mathcal{V}}(\lambda)$ acting on *all* n vertices $\mathcal{V}$, that is,

$M_{\mathcal{V}}(\lambda) \in \mathbb{C}^{n\times n}$, by summing over all the *localised* matrices (5.2.7),

$$M_{\mathcal{V}}(\lambda) = \sum_{e\in\mathcal{E}} \widetilde{M}_e(\lambda) \in \mathbb{C}^{n\times n}, \tag{5.2.9}$$

which is well defined as long as λ is not in $\sigma(-\Delta_{\mathcal{V}}^D)$, i.e., not in the Dirichlet spectrum of any of the decoupled edges considered as a collection of disjoint intervals. Note that not each pair (v_i, v_j) is connected by some e_{ij}: if we set $A_{ij} = B_{ij} = 0$ whenever there is no edge joining v_i and v_j, then each diagonal entry of $M_{\mathcal{V}}(\lambda)$ consists of exactly n addends (some of them being 0) and we may explicitly write the (i, j)-entry of $M_{\mathcal{V}}(\lambda)$ as

$$(M_{\mathcal{V}}(\lambda))_{ij} = \sqrt{\lambda} \begin{cases} \sum_{p=1}^{n} A_{ip} & \text{if } i = j, \\ B_{ij} & \text{if } i \neq j. \end{cases} \tag{5.2.10}$$

By this representation it is immediate that $M_{\mathcal{V}}(\lambda)$ depends analytically on λ, with isolated singularities at the discrete set $\sigma(-\Delta_{\mathcal{V}}^D)$, that is, the spectrum of the Laplacian on $\mathcal{G}$ where each vertex $v \in \mathcal{V}$ is equipped with the Dirichlet condition. Importantly, the Dirichlet-to-Neumann matrix $M_{\mathcal{V}}(\lambda)$ acting on $\mathcal{V}_R$ can be written in a natural way in terms of $M_{\mathcal{V}}(\lambda)$. We recall that $\mathcal{V}$ consists of the (ordered) vertices $v_1, \dots, v_n$, such that the first k entries $\mathcal{V}_R = \{v_1, \dots, v_k\}$ are equipped with the Robin boundary condition. With this ordering, we write $M_{\mathcal{V}}(\lambda)$ in block form as

$$M_{\mathcal{V}}(\lambda) = \begin{pmatrix} R & C^T \\ C & K \end{pmatrix}, \tag{5.2.11}$$

where

(1) $R \in \mathbb{C}^{k\times k}$ represents the restriction of $M_{\mathcal{V}}$ to the Robin vertices $v_1, \dots, v_k$,

(2) $K \in \mathbb{C}^{(n-k)\times(n-k)}$ is the restriction to the remaining $n-k$ Neumann–Kirchhoff vertices $v_{k+1}, \dots, v_n$,

(3) and $C \in \mathbb{C}^{(n-k)\times k}$ and its transpose C^T give the interaction ("coupling") between these two groups of vertices.

The following representation (see [76, Lemma 3.1]) is adapted from [44], although we expect it is known elsewhere.

Lemma 5.2.1. *With the representation* (5.2.11), *the matrix K is invertible if and only if $\lambda \notin \sigma(-\Delta^D_{\mathcal{V}_R})$. Whenever K is invertible, the operator $M(\lambda)$ is well defined and may be represented in matrix form by*

$$M(\lambda) = R - C^T K^{-1} C. \tag{5.2.12}$$

Proof. Let

$$x^R := (x_1, \dots, x_k)^T = g \in \mathbb{C}^k \simeq \mathcal{V}_R \tag{5.2.13}$$

be the Dirichlet data from (5.2.2b). We write

$$x = \begin{pmatrix} x^R \\ x^N \end{pmatrix} = \begin{pmatrix} g \\ x^N \end{pmatrix} \in \mathbb{C}^n \tag{5.2.14}$$

for the components

$$x^N := (x_{k+1}, \dots, x_n)^T := f|_{\mathcal{V}_N} \in \mathbb{C}^{n-k} \simeq \mathcal{V}_N, \tag{5.2.15}$$

of values of f in the non-Robin vertices. Then x^N is well defined and thus uniquely determined by the first k entries x^R since $\lambda \notin \sigma(-\Delta^D_{\mathcal{V}_R})$. By construction we have

$$M_{\mathcal{V}}(\lambda)x = \begin{pmatrix} R & C^T \\ C & K \end{pmatrix} \begin{pmatrix} x^R \\ x^N \end{pmatrix} = \begin{pmatrix} \sum_{e \sim v} \frac{\partial}{\partial \nu} f|_e(v) \\ 0 \end{pmatrix} = \begin{pmatrix} M(\lambda)x^N \\ 0 \end{pmatrix} \tag{5.2.16}$$

That is,

$$M(\lambda)x^N = Rx^R + C^T x^N, \tag{5.2.17}$$

where

$$Cx^R + Kx^N = 0. \tag{5.2.18}$$

Since x^N is uniquely determined by $x^R \in \mathbb{C}^k$ arbitrary, we must have that K is invertible. This implies

$$x^N = -K^{-1}Cx^R, \tag{5.2.19}$$

and thus (5.2.12) follows if $\lambda \notin \sigma(-\Delta^D_{\mathcal{V}_R})$. If on the other hand $\lambda \in \sigma(-\Delta^D_{\mathcal{V}_R})$, then since x^N is no longer uniquely determined by x^R in general (if ψ is an eigenfunction of $-\Delta^D_{\mathcal{V}_R}$, then $x^N + \psi|_{\mathcal{V}_N}$ is also a solution), K cannot be invertible. $\square$

We can now state the central duality result linking the eigenvalues of $M(\lambda)$ and $-\Delta^\alpha_{\mathcal{V}_R}$. Here we will still suppose that the vector $\alpha = (\alpha_1, \ldots, \alpha_k)^T \in \mathbb{C}^k$ is given and assume that the (ordered) vertices $v_j \in \mathcal{V}_R = \{v_1, \ldots, v_k\}$ are equipped with the corresponding Robin parameter α_j; for brevity we will then write

$$I_\alpha := \operatorname{diag}\{\alpha_1, \ldots, \alpha_k\} \in \mathbb{C}^{k\times k}. \tag{5.2.20}$$

The next statement is well known in the case of real $\alpha \in \mathbb{R}$ (see [27, Theorem 3.5.2]); the proof in the complex vector case $\alpha \in \mathbb{C}^k$ is identical, and we omit it.

Theorem 5.2.2. *Let $\lambda \in \rho(-\Delta^D_{\mathcal{V}_R})$. Then $\lambda \in \sigma(-\Delta^\alpha_{\mathcal{V}_R})$ if and only if*

$$\det(M(\lambda) - I_\alpha) = 0. \tag{5.2.21}$$

5.3 Asymptotics of the Dirichlet-to-Neumann operator

We now investigate what happens to $M(\lambda)$ when $\lambda \to \infty$. We first note the following trivial but useful implication of Lemma 5.2.1, see [76, Lemma 4.1].

Lemma 5.3.1. *The Dirichlet-to-Neumann matrix $M(\lambda)$ is a meromorphic function of λ. It is well defined for all $\lambda \in \rho(-\Delta^D_{\mathcal{V}_R})$, and each $\lambda \in \sigma(-\Delta^D_{\mathcal{V}_R})$ is a pole of finite order of $M(\lambda)$.*

For a vector $z = (z_1, \ldots, z_k) \in \mathbb{C}^k$ we denote by $\mathfrak{m}(z)$ the smallest of the moduli of its components z_j, $j = 1, \ldots, k$, i.e.

$$\mathfrak{m}(z) = \min_{j=1,\ldots,k} |z_j|. \tag{5.3.1}$$

Using the duality between the eigenvalues $\lambda \in \sigma(-\Delta^{\alpha}_{\mathcal{V}_R})$ of the Robin Laplacian and the eigenvalues $\alpha \in \sigma(M(\lambda))$ of the Dirichlet-to-Neumann matrix (see Theorem 5.2.2), we obtain the following statement, see [76, Theorem 4.2].

Theorem 5.3.2. *For any compact graph $\mathcal{G}$ and any bounded set $\Omega \subset \mathbb{C}$ such that*

$$\operatorname{dist}(\Omega, \sigma(-\Delta^{D}_{\mathcal{V}_R})) > 0 \tag{5.3.2}$$

there exists a number $\hat{\alpha} > 0$ depending only on Ω, $\mathcal{G}$, and $\mathcal{V}_R$ such that

$$\sigma(-\Delta^{\alpha}_{\mathcal{V}_R}) \cap \Omega = \varnothing \tag{5.3.3}$$

for all $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{C}^k$ such that $\mathfrak{m}(\alpha) > \hat{\alpha}$.

In other words, if we fix any bounded set Ω (strictly) away from the Dirichlet spectrum, as $\mathfrak{m}(\alpha)$ grows and exceeds a certain threshold, each single eigenvalue curve will have left Ω. Since Ω can be chosen to be arbitrarily close to any of the discrete points of $\sigma(-\Delta^{D}_{\mathcal{V}_R})$ the latter theorem immediately implies the following dichotomy.

Corollary 5.3.3. *Suppose $\mathfrak{m}(\alpha) \to \infty$ and $\lambda = \lambda(\alpha)$ is an analytic branch of eigenvalues of $-\Delta^{\alpha}_{\mathcal{V}_R}$. Then either $\lambda \to \infty$ in $\mathbb{C}$ or λ converges to a point in $\sigma(-\Delta^{D}_{\mathcal{V}_R})$ as $\mathfrak{m}(\alpha) \to \infty$.*

Proof of Theorem 5.3.2 and hence of Corollary 5.3.3. For $\mathfrak{m}(\alpha) > 0$ we consider the invertibility of

$$M(\lambda) - I_\alpha = I_\alpha \left(I_{\alpha^{-1}} M(\lambda) - I \right), \tag{5.3.4}$$

where $I_{\alpha^{-1}} := \operatorname{diag}\{\alpha_1^{-1}, \ldots, \alpha_k^{-1}\}$ is well defined (since $\mathfrak{m}(\alpha) > 0$) and satisfies $I_{\alpha^{-1}} = I_\alpha^{-1}$ by definition. Now the matrix $M(\lambda) \in \mathbb{C}^{k \times k}$ is a meromorphic function of λ with singularities at $\sigma(-\Delta^{D}_{\mathcal{V}_R})$ (see Lemma 5.3.1), and hence its norm is uniformly bounded on $\Omega \subset\subset \rho(-\Delta^{D}_{\mathcal{V}_R})$. Hence for each such Ω there exists a constant $c_\Omega > 0$ independent of $\alpha \in \mathbb{C}^k$ such that

$$\sup_{\lambda \in \Omega} \|I_{\alpha^{-1}} M(\lambda)\|_{\mathbb{C}^k \to \mathbb{C}^k} \leq \sup_{\lambda \in \Omega} \|I_{\alpha^{-1}}\|_{\mathbb{C}^k \to \mathbb{C}^k} \, \|M(\lambda)\|_{\mathbb{C}^k \to \mathbb{C}^k} \tag{5.3.5}$$

$$= c_\Omega \, \|I_{\alpha^{-1}}\|_{\mathbb{C}^k \to \mathbb{C}^k} \;\longrightarrow\; 0 \tag{5.3.6}$$

as $\mathfrak{m}(\alpha) \to \infty$. This convergence implies that $(I_{\alpha^{-1}}M(\lambda) - I)$ is invertible (as a Neumann series) and the right-hand side of (5.3.4) is, too. That is, there exists a constant $\hat{\alpha} > 0$ such that

$$\mathfrak{m}(\alpha) > \hat{\alpha} \quad \Longrightarrow \quad M(\lambda) - I_\alpha \text{ is invertible.} \tag{5.3.7}$$

In particular, the kernel of $M(\lambda) - I_\alpha$ is trivial for $\mathfrak{m}(\alpha) > \hat{\alpha}$, and thus there exist no eigenvalues of the Robin Laplacian in Ω by Theorem 5.2.2. □

It remains to analyse what divergent behaviour is possible, and under what circumstances. To this end, we use the representations (5.2.10), (5.2.11), and (5.2.12) together with ideas drawn from Section 4.1 for the interval. We first note that since the coefficients A_{ij} and B_{ij} given by (5.2.5) and (5.2.6), respectively, only consist of sine and cosine functions of $\sqrt{\lambda}$. Thus they are periodic in $\operatorname{Re}\sqrt{\lambda}$ and we only need to consider the case $\operatorname{Im}\sqrt{\lambda} \to \pm\infty$, in which case we have the asymptotics

$$\cot z = \mathrm{i}\left(1 + \frac{2}{\mathrm{e}^{2\mathrm{i}z} - 1}\right) = \mp\mathrm{i} + \mathcal{O}\left(\mathrm{e}^{\mp 4 \operatorname{Im} z}\right) \tag{5.3.8}$$

and

$$\csc z = \frac{2\mathrm{i}}{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}} = \mathcal{O}\left(\mathrm{e}^{\mp 2 \operatorname{Im} z}\right) \tag{5.3.9}$$

as $\operatorname{Im} z \to \pm\infty$, independently of $\operatorname{Re} z$. For

$$z = \frac{\ell_{ij}}{2}\sqrt{\lambda} \tag{5.3.10}$$

this gives the following asymptotic expansion of $M(\lambda) \in \mathbb{C}^{k\times k}$. In what follows, for brevity we will set

$$D := \operatorname{diag}\{\deg v_1, \dots, \deg v_k\} \in \mathbb{N}^{k\times k} \tag{5.3.11}$$

as well as

$$\tilde{D} := \operatorname{diag}\{\deg v_{k+1}, \dots, \deg v_n\} \in \mathbb{N}^{(n-k)\times(n-k)}. \tag{5.3.12}$$

The following lemma corresponds to [76, Lemma 4.4].

Lemma 5.3.4. *Suppose* $\lambda \to \infty$ *in* $\mathbb{C}$ *in such a way that* $\operatorname{Im}\sqrt{\lambda} \to \pm\infty$, *and recall the definition* $\ell_{\mathcal{G}} := \min\{\ell_e : e \in \mathcal{E}\} > 0$. *Then* $M(\lambda)$ *has the asymptotic expansion*

$$M(\lambda) = \pm\mathrm{i}\sqrt{\lambda}D + \mathcal{O}\left(\sqrt{\lambda}\mathrm{e}^{\mp\ell_{\mathcal{G}}\operatorname{Im}\sqrt{\lambda}}\right). \tag{5.3.13}$$

as $\lambda \to \infty$.

Proof of Lemma 5.3.4. Recall the matrices R, C and K introduced in (5.2.11). Then the expression

$$(M_{\mathcal{V}}(\lambda))_{ij} = \sqrt{\lambda}\begin{cases}\sum_{p=1}^{n} A_{ip} & \text{if } i = j,\\ B_{ij} & \text{if } i \neq j.\end{cases} \tag{5.3.14}$$

for the coefficients of these matrices, cf. (5.2.10), plus the asymptotics

$$A_{ij} = \pm\mathrm{i} + \mathcal{O}\left(\mathrm{e}^{\mp2\ell_{ij}\operatorname{Im}\sqrt{\lambda}}\right) \quad \text{and} \quad B_{ij} = \mathcal{O}\left(\mathrm{e}^{\mp\ell_{ij}\operatorname{Im}\sqrt{\lambda}}\right), \tag{5.3.15}$$

respectively, as $\operatorname{Im}\sqrt{\lambda} \to \pm\infty$, which follow from (5.3.8) and (5.3.9), imply that

$$R = \pm\mathrm{i}\sqrt{\lambda}D + \mathcal{O}\left(\sqrt{\lambda}\mathrm{e}^{\mp\ell_{\mathcal{G}}\operatorname{Im}\sqrt{\lambda}}\right). \tag{5.3.16}$$

Note that D is the coefficient matrix since $A_{ij} = 0$ if and only if there is no edge joining v_i and v_j. To be more precise, if there are $\deg v_j$ edges joining the Robin vertex v_j, then there are exactly $\deg v_j$ non-zero addends in the sum (5.3.14), each of which behaves like $\pm\mathrm{i}\sqrt{\lambda} + \mathcal{O}\left(\sqrt{\lambda}\mathrm{e}^{\mp2\ell_{ij}\operatorname{Im}\sqrt{\lambda}}\right)$. Similarly, we obtain

$$C,\, C^T = \mathcal{O}\left(\sqrt{\lambda}\mathrm{e}^{\mp\ell_{\mathcal{G}}\operatorname{Im}\sqrt{\lambda}}\right) \tag{5.3.17}$$

and

$$K = \pm\mathrm{i}\sqrt{\lambda}\tilde{D} + \mathcal{O}\left(\sqrt{\lambda}\mathrm{e}^{\mp\ell_{\mathcal{G}}\operatorname{Im}\sqrt{\lambda}}\right). \tag{5.3.18}$$

From the latter, we obtain (the rough but sufficient estimate) that $K^{-1} = \mathcal{O}(1/\sqrt{\lambda})$

and hence also

$$C^T K^{-1} C = \mathcal{O}\left(\sqrt{\lambda}\mathrm{e}^{\mp 2\ell_{\mathcal{G}} \operatorname{Im}\sqrt{\lambda}}\right). \tag{5.3.19}$$

Combined with the asymptotic expansion for R and the representation

$$M(\lambda) = R - C^T K^{-1} C \tag{5.3.20}$$

(cf. (5.2.12)) of $M(\lambda)$, this immediately yields the assertion (5.3.13). □

Our next step is to use the asymptotics of $M(\lambda)$ to analyse the asymptotical behaviour of its eigenvalues. To this end, we use the *Gershgorin circle theorem*, see [60], which is particularly useful for diagonally dominant matrices.

Theorem 5.3.5. *Let $A = (a_{ij}) \in \mathbb{C}^{k\times k}$ and define the radii*

$$r_i := \sum_{\substack{j=1\\ j\neq i}} |a_{ij}|, \qquad i = 1, \dots k. \tag{5.3.21}$$

(1) The spectrum of $A = (a_{ij}) \in \mathbb{C}^{k\times k}$ satisfies

$$\sigma(A) \subseteq \bigcup_{i=1}^{k} \overline{B_{r_i}(a_{ii})} =: G_A. \tag{5.3.22}$$

(2) Moreover, every connected component of G_A contains exactly as many eigenvalues as diagonal elements of A.

Corollary 5.3.6. *For the Dirichlet-to-Neumann matrix $M(\lambda)$ on a compact graph with Robin vertices $v_1, \dots, v_k$ we have that its eigenvalues β_j, $j = 1, \dots, k$, satisfy*

$$\beta_j = \mathrm{i}\sqrt{\lambda}\deg v_j + \mathcal{O}\left(\sqrt{\lambda}\mathrm{e}^{-\ell_{\mathcal{G}} \operatorname{Im}\sqrt{\lambda}}\right) \tag{5.3.23}$$

as $\operatorname{Im}\sqrt{\lambda} \to +\infty$.

Proof. By the statement

$$M(\lambda) = \mathrm{i}\sqrt{\lambda} D + \mathcal{O}\left(\sqrt{\lambda}\mathrm{e}^{-\ell_{\mathcal{G}} \operatorname{Im}\sqrt{\lambda}}\right) \tag{5.3.24}$$

of Theorem 5.3.4 it is immediate that

$$r_i(\lambda) = \mathcal{O}\left(\sqrt{\lambda}\mathrm{e}^{-\ell_{\mathcal{G}}\mathrm{Im}\sqrt{\lambda}}\right) \tag{5.3.25}$$

as $\mathrm{Im}\sqrt{\lambda} \to +\infty$. Thus, the Gershgorin disks (at this point it is irrelevant if they are open or closed)

$$B_{r_i(\lambda)}\left(M_{ii}(\lambda)\right) = B_{r_i(\lambda)}\left(\mathrm{i}\sqrt{\lambda}\deg v_i + \mathcal{O}\left(\sqrt{\lambda}\mathrm{e}^{-\ell_{\mathcal{G}}\mathrm{Im}\sqrt{\lambda}}\right)\right), \tag{5.3.26}$$

$i = 1, \ldots, k$, get exponentially smaller as $\mathrm{Im}\sqrt{\lambda} \to +\infty$. Additionally, if λ is sufficiently large, then the connected components of $G_{M(\lambda)}$ are exactly

$$k' := |\{\deg v_1, \ldots, \deg v_k\}| \leq k \tag{5.3.27}$$

disjoint disks and their union contains all the eigenvalues $\beta_1, \ldots, \beta_k$. The statement of the corollary is immediate after a possible rearrangement of their order. □

To prove Theorem 1.2.8, it remains to "invert" these asymptotics, that is, express these curves as functions of α. For this part of the argument, we may essentially appeal to the proof given in [30, Section 9.1.3] for the corresponding statement on the interval.

Proof of Theorem 1.2.8.
Firstly, we sketch the general idea: assume that $\alpha = (\alpha_1, \ldots, \alpha_k) \to \infty$ in $\mathbb{C}^k$ and recall the following two cases that for some $m = 0, 1, \ldots, k$ the vector α can be reordered such that

(1) $\alpha_0 := \alpha_1 = \cdots = \alpha_m \to \infty$ in a sector fully contained in the open left half-plane;

(2) $\alpha_{m+1}, \ldots, \alpha_k \to \infty$ such that $\mathrm{Re}\,\alpha_j$ remains bounded from below, for all $m + 1 \leq j \leq k$.

Note that the first case is allowed to be empty, which is the case if and only if $m = 0$. Here and throughout the proof we understand "eigenvalue" to mean "analytic curve of eigenvalues". We wish to show that for each $v_1, \ldots, v_m$ (each of which is equipped with α_0) there exists a corresponding eigenvalue λ_j which behaves like

$$\lambda_j = -\frac{\alpha_0^2}{(\deg v_j)^2} + \mathcal{O}\left(\alpha_0^2 e^{\ell_{\mathcal{G}} \operatorname{Re} \alpha_0}\right) \tag{5.3.28}$$

as $\alpha_0 \to \infty$, and that these m eigenvalues $\lambda_1, \dots, \lambda_m$ are the only ones which diverge away from the positive real semi-axis.

Step 1: There are at least m such eigenvalues. Suppose first that λ is such an eigenvalue diverging away from the positive real semi-axis; then necessarily $\operatorname{Im}\sqrt{\lambda} \to \pm\infty$. We obtain k eigenvalues of $M(\lambda)$, namely $\beta_1, \dots, \beta_k$, each of which behaves like (5.3.23). Now fix $j = 1, \dots, m$. By the same inversion argument based on Rouché's theorem that was used in Section 4.1, there exists an eigenvalue λ of $-\Delta_{\mathcal{V}_R}^{\alpha}$ which satisfies

$$\lambda = -\frac{\alpha_0^2}{(\deg v_j)^2} + \mathcal{O}(f(\alpha_0)) \tag{5.3.29}$$

with the asymptotical error term

$$f(\alpha_0) = \alpha_0^2 e^{\ell_{\mathcal{G}} \operatorname{Re} \alpha_0} \tag{5.3.30}$$

as $\alpha_0 \to \infty$. We arrive at m divergent eigenvalues, each of which satisfies (5.3.28).

Step 2: There are at most m such eigenvalues. Suppose now that there is an additional, $(m+1)$st divergent eigenvalue $\lambda = \lambda(\alpha)$ which satisfies $\operatorname{Im}\sqrt{\lambda} \to \pm\infty$. Then, again, the matrix $M(\lambda)$ has k eigenvalues satisfying (5.3.23). By assumption, λ is not an eigenvalue of $-\Delta_{\mathcal{V}_R}^{\alpha}$ corresponding to the m curves $\lambda_1, \dots, \lambda_m$ found above, that is, it does not correspond to the parameters α_0. Hence, applying the same inversion procedure, there must be some $q \in \{m+1, \dots, k\}$ such that λ corresponds to the eigenvalue $\alpha_q \leftrightarrow \lambda$ described asymptotically by (5.3.23). But now Step 4 of the proof of Theorem 4.1.4 shows that the condition $\operatorname{Im}\sqrt{\lambda} \to \pm\infty$ together with the relation (5.3.23) implies that necessarily $\operatorname{Re}\alpha_q \to -\infty$ as $\lambda(\alpha_q) \to \infty$. This contradicts assumption (2), and we conclude that no such divergent eigenvalue λ can exist which is not already among the m found above. Finally, we already know from Corollary 5.3.3 that each eigenvalue of $-\Delta_{\mathcal{V}_R}^{\alpha}$ which does not diverge to ∞ converges to some eigenvalue of the Dirichlet Laplacian $-\Delta_{\mathcal{V}_R}^{D}$ as $\alpha \to \infty$. This completes the proof. □

Remark 5.3.7. It is already mentioned in the introduction that we always assume the standard (continuity-Kirchhoff) conditions to hold on the set $\mathcal{V}_N = \mathcal{V} \setminus \mathcal{V}_R$ of non-Robin vertices. The fact that we use these vertex conditions plays a major role in two places: (1) the block matrix representation $M(\lambda) = R - C^T K^{-1} C$ of the Dirichlet-to-Neumann operator from Lemma 5.2.1, and (2), in the subsequent asymptotics thereof in Lemma 5.3.4, more precisely in (5.3.19).

(1) Let us suppose for a moment that the functions in the domain of $-\Delta^{\alpha}_{\mathcal{V}_R}$ satisfy other (local) vertex conditions for all vertices in $\mathcal{V}_N$. In general, this would result in a different matrix $\widetilde{K}$ in both equations (5.2.11) and (5.2.16). Consequently, the last $n-k$ components $Cx^R + \widetilde{K}x^N$ of (5.2.16) may no longer vanish and the argument that x^N is easily expressible as a function of x^R fails. Moreover, if the vertex condition assumed to hold on $\mathcal{V}_N$ implies that $Cx^R + \widetilde{K}x^N$ depends on f (in our setting, the canonical example is any Robin condition), then x^N may no longer be uniquely determined by x^R, that is, $\widetilde{K}$ is no longer invertible. Thus for our proof, we require the boundary condition to satisfy $Cx^R + \widetilde{K}x^N = 0$.

(2) The asymptotic behaviour (5.3.13) of the Dirichlet-to-Neumann matrix does indeed depend on the asymptotics of $\widetilde{K}$, however, we only require that $C^T \widetilde{K}^{-1} C$ does not influence the leading term. In other words, we only need that $\widetilde{K}$ does not decrease too rapidly as $\operatorname{Im} \sqrt{\lambda} \to \pm\infty$.

However, since this consideration is negligible for our main subject, will omit the question of other vertex conditions for which our proof would still work.

5.4 Estimates on the numerical range and the eigenvalues

In this section we want to give estimates on the location of the eigenvalues to complement the asymptotic results of the divergent eigenvalues described by Theorem 1.2.8. To this end, we study the numerical range $W(a_\alpha)$ of the Robin form a_α with (respect to the set $\mathcal{V}_R = \{v_1, \dots, v_k\} \subset \mathcal{V}$ of Robin vertices and the vector of Robin parameters $\alpha = (\alpha_1, \ldots, \alpha_k)^T \in \mathbb{C}^k$), namely

$$W(a_\alpha) = \{a_\alpha[f,f] : \|f\|_2 = 1\} \tag{5.4.1a}$$
$$= \left\{ \int_{\mathcal{G}} |f'|^2 \,\mathrm{d}x + \sum_{j=1}^k \alpha_j |f(v_j)|^2 : \int_{\mathcal{G}} |f|^2 \,\mathrm{d}x = 1 \right\} \subset \mathbb{C}, \tag{5.4.1b}$$

noting that every eigenvalue of $-\Delta^\alpha_{\mathcal{V}_R}$ is an element of $W(a_\alpha)$. For every fixed $\alpha \in \mathbb{C}^k$ we will present three sets of results which, while perhaps not surprising, give a fairly complete picture of the location spectrum: our first results, namely Theorem 5.4.1, give an estimate on the location of the set $W(a_\alpha)$ in the complex plane,

$$\sigma(-\Delta^\alpha_{\mathcal{G}}) = \sigma_p(-\Delta^\alpha_{\mathcal{G}}) \subset W(-\Delta^\alpha_{\mathcal{G}}) \subset W(a_\alpha), \tag{5.4.2}$$

analogous to those of Section 3.4 for the complex Robin Laplacian on a domain Ω in $\mathbb{R}^d$. This leads to bounds on the real part of the eigenvalues which are, in particular, sharp up to the first term of the asymptotics as $\alpha \to \infty$ in $\mathbb{C}^k$. In addition to these bounds, we also consider more precise estimates on the imaginary part of the eigenvalues afterwards. For the numerical range, we consider the case of $\alpha \in \mathbb{C}^k$ and the case of vertex-independent

$$\alpha := \alpha_1 = \ldots = \alpha_k \in \mathbb{C} \tag{5.4.3}$$

separately. Notationally, for the fixed set $\mathcal{V}_R = \{v_1, \ldots, v_k\}$ of Robin vertices we will always write

$$\mathfrak{D} := \min_{j=1,\ldots,k} \deg v_j. \tag{5.4.4}$$

We also recall that $\ell_{\mathcal{G}} = \min\{\ell_e : e \in \mathcal{E}\}$ is the length of the shortest edge in $\mathcal{G}$. The following statement is due to [76, Theorem 5.1].

Theorem 5.4.1. *(1) Let $\alpha \in \mathbb{C}^k$. Then the numerical range $W(a_\alpha)$, and in particular every eigenvalue of $-\Delta^\alpha_{\mathcal{V}_R}$, is contained in the set*

$$\Lambda_{\mathcal{G},\alpha} := \left\{ t + \sum_{j=1}^k \alpha_j s_j \in \mathbb{C} : t \geq 0,\ s_j \in \left[0, \frac{2}{\mathfrak{D}}\sqrt{\tau_j} + \frac{2}{\mathfrak{D}\ell_{\mathcal{G}}}\right] \right\}, \tag{5.4.5}$$

where the numbers $0 \leq \tau_j \leq t$ *satisfy* $\sum_{j=1}^{k} \tau_j \leq t$.

(2) *If* $\alpha := \alpha_1 = \ldots = \alpha_k \in \mathbb{C}$ *is independent of* $j = 1, \ldots, k$, *then* $W(a_\alpha)$ *is contained in*

$$\Lambda_{\mathcal{G},\alpha} := \left\{ t + \alpha s \in \mathbb{C} : t \geq 0, s \in \left[0, \frac{2}{\mathfrak{D}}\sqrt{t} + \frac{1}{\mathfrak{D}\ell_{\mathcal{G}}} \right] \right\}. \tag{5.4.6}$$

Remark 5.4.2. (1) By the definition of the numerical range (5.4.1) (or more fundamentally, by Definition 5.1.4) we have that if $\operatorname{Re}\alpha_j = \operatorname{Re}\alpha(v_j) \geq 0$ for all $v_j \in \mathcal{V}_R$, then $\operatorname{Re}\lambda \geq 0$ automatically as well, whereas if the components of $\operatorname{Re}\alpha$ are all negative or of indefinite sign, then $\operatorname{Re}\lambda$ may be negative.

(2) The set $\Lambda_{\mathcal{G},\alpha}$ is depicted in Figure 5.4.1 in the simple case that $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha, \operatorname{Im}\alpha > 0$; one might compare this depiction with Figures 3.4.1 and 3.4.2 for the complex Robin Laplacian on a domain in $\mathbb{R}^d$.

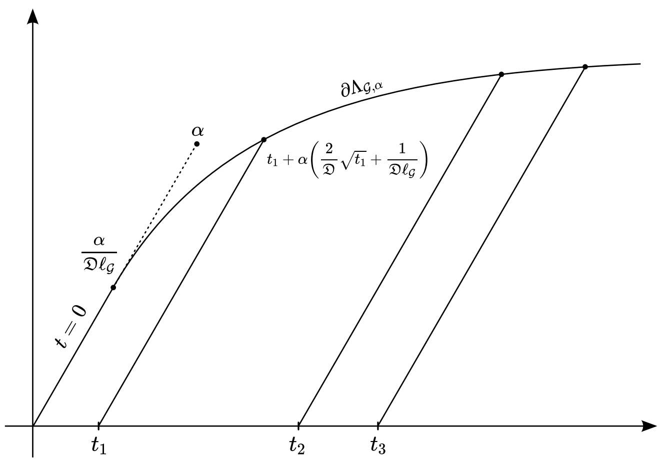

Figure 5.4.1: The set $\Lambda_{\mathcal{G},\alpha}$ from Theorem 5.4.1(2), which contains the numerical range $W(a_\alpha)$, for a representative choice of $\operatorname{Re}\alpha > 0$ and $\operatorname{Im}\alpha > 0$, corresponding to the region between the curve $\partial\Lambda_{\mathcal{G},\alpha}$ and the real axis. The region is composed of the union of segments of the form $\{t + \alpha \cdot s \in \mathbb{C} : s \in [0, 2\sqrt{t}/\mathfrak{D} + 1/\mathfrak{D}\ell_G]\}$, each of slope $\operatorname{Im}\alpha/\operatorname{Re}\alpha$, for different values of $t \geq 0$; the parallel lines show these segments for selected values of $t_1, t_2, t_3 > 0$. Their endpoints form a parabolic section of $\partial\Lambda_{\mathcal{G},\alpha}$ open to the right.

The statement of Theorem 5.4.1 will follow from a kind of "trace-type" inequality which allows us to control precisely the value that an arbitrary H^1-function takes at a given vertex in terms of certain subgraphs (star shaped graphs having the given vertex as their centre) around it. This might be compared with [30, Lemma 6.5]. To this end, we first require some notation regarding these subgraphs.

Definition 5.4.3. (1) Let $\xi : \mathcal{E} \to (0,1]$ be an edge-dependent length scaling factor. Given any vertex $v_j \in \mathcal{V}$, we denote by

$$\mathcal{S}_j^\xi := \bigcup_{e \sim v_j} \xi(e)e \tag{5.4.7}$$

the star subgraph of $\mathcal{G}$ whose central vertex is v_j and whose pendant edges are the edges e incident with v_j, scaled by the factor $\xi(e) \in (0,1]$.

(2) If $\xi \equiv 1$ on $\mathcal{E}$, that is, none of the edges is shortened and $\mathcal{S}_j^\xi$ is exactly the subgraph which is formed by deleting each single edge $e \not\sim v_j$, then we write $\mathcal{S}_j$ instead of $\mathcal{S}_j^1$ and call it the *spanning star* at v_j.

We will always make the identification that $\mathcal{S}_j^\xi$ is a subgraph of $\mathcal{G}$ (denoted by $\mathcal{S}_j^\xi \subset \mathcal{G}$); in particular, we will treat the scaled edge $\xi(e)e \subset \mathcal{S}_j^\xi$ as a subset of the edge $e \subset \mathcal{G}$.

Definition 5.4.4. Let $\mathcal{G} = \mathcal{G}(\mathcal{V}, \mathcal{E})$ be a compact metric graph and let $\mathcal{V}_0 \subset \mathcal{V}$ be an arbitrary collection of its vertices. We denote by $\mathcal{G}_0$ the subgraph of $\mathcal{G}$ consisting of the union of all spanning stars of all vertices $v \in \mathcal{V}_0$.

Example 5.4.5. Assume that $\mathcal{G}$ is already a star with vertex set $\mathcal{V} = \{v_0, v_1, \ldots, v_n\}$ centred in v_0. We further call the edges $e_j \sim v_j$ for $j = 1, \ldots, n$.

(1) If $\mathcal{V}_0 = \{v_0\}$ consists only of the central vertex v_0, then the corresponding spanning star $\mathcal{G}_0$ is the whole of $\mathcal{G}$.

(2) If $\mathcal{V}_0 = \{v_j\}$ for any $j = 1, \ldots, n$, then $\mathcal{G}_0$ is just the single edge e_j. The spanning star corresponding to any vertex set $\mathcal{V}_0 \subset \mathcal{V}$ such that $v_0 \notin \mathcal{V}_0$, then $\mathcal{G}_0$ is, again, a star missing all edges e_j with $v_j \notin \mathcal{V}_0$.

Definition 5.4.6. For any subgraph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ of $\mathcal{G}$ we define

$$\ell_{\mathcal{G}'} := \min_{e \in \mathcal{E}'} \ell_e \tag{5.4.8}$$

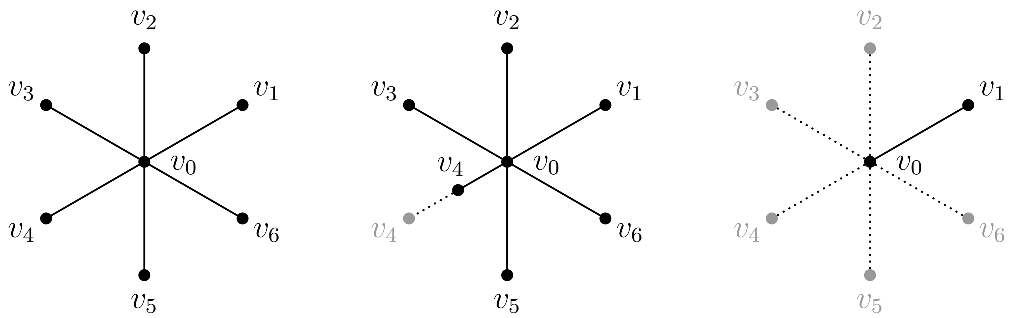

Figure 5.4.2: Left: star graph $\mathcal{G}$ as in Example 5.4.5 for $n = 6$ and $\ell_{e_i} = \ell_{e_j}$ for all $i \neq j$; this graph is also $\mathcal{G}_0$ for $\mathcal{V}_0 = \{v_0\}$. Centre: S_0^ξ for $\xi(e_j) = 1$, $j = 1, 2, 3, 5, 6$, and $\xi(e_4) = 0.5$. Right: spanning star $\mathcal{G}_0$ for $\mathcal{V}_0 = \{v_1\}$.

to be the length of the shortest edge e of $\mathcal{G}'$. As usual, we set $\|f\|_{\mathcal{G}'} := \|f\|_{\mathcal{G}',2}$ to be the $L^2(\mathcal{G}')$-norm of f.

The following lemma (see [76, Lemma 6.1]) is key to prove Theorem 5.4.1.

Lemma 5.4.7. *Let $\xi : \mathcal{E} \to (0, 1]$ and $v_j \in \mathcal{V}$ be arbitrary and denote by $\mathcal{S}_j^\xi$ the scaled star at v_j as described in Definition 5.4.4. Then*

$$\deg v_j |f(v_j)|^2 \leq 2\|f\|_{\mathcal{S}_j^\xi} \|f'\|_{\mathcal{S}_j^\xi} + \frac{1}{\ell_{\mathcal{S}_j^\xi}} \|f\|^2_{\mathcal{S}_j^\xi} \tag{5.4.9}$$

for all $f \in H^1(\mathcal{G})$. Moreover, if $\mathcal{V}_0 \subset \mathcal{V}$ is an arbitrary set of vertices of $\mathcal{G}$ and $\mathcal{G}_0$ is the subgraph union of spanning stars for $\mathcal{V}_0$ as described above, then we have the estimate

$$\sum_{v_j \in \mathcal{V}_0} \deg v_j |f(v_j)|^2 \leq 2\|f\|_{\mathcal{G}_0} \|f'\|_{\mathcal{G}_0} + \frac{1}{\ell_{\mathcal{G}_0}} \|f\|^2_{\mathcal{G}_0}. \tag{5.4.10}$$

For the proof of Lemma 5.4.7, we will use the following cut-off functions $\varphi_j \in \mathbb{C}(\mathcal{G})$, which are supported in a certain neighbourhood of v_j as depicted in Figure 5.4.3: for each v_j we define $\varphi_j \in H^1(\mathcal{G})$ with support in $\mathcal{S}_j^\xi$ by setting

$$\varphi_j^\xi(x) = \begin{cases} 1 - \frac{\operatorname{dist}(x, v_j)}{\xi(e)\ell_e} & \text{if } x \in \xi(e)e \subset \mathcal{S}_j^\xi \\ 0 & \text{otherwise.} \end{cases} \tag{5.4.11}$$

Then clearly $0 \leq \varphi_j^\xi \leq 1$; moreover, since we are assuming that $\mathcal{G}$ does not have any loops, if $\xi(e) = 1$ for all e then the collection $(\varphi_j^\xi)_{j=1}^n$ is a partition of unity. Note that, if the scaling factor $\xi \equiv 1$ is trivial, we write φ_j instead of φ_j^1.

Lemma 5.4.8. *The cut-off function φ_j^ξ fulfils the following properties.*

(1) φ_j^ξ has norm $\|\varphi_j^\xi\|_\infty = 1$ and it is weakly differentiable with norm

$$\|(\varphi_j^\xi)'\|_\infty = \frac{1}{\ell_{\mathcal{S}_j^\xi}}. \tag{5.4.12}$$

(2) If $\xi \equiv 1$ and $v_i \sim e \sim v_j$, we have

$$\varphi_i = 1 - \varphi_j \qquad \textit{and} \qquad \varphi_i' = -\varphi_j'. \tag{5.4.13}$$

Proof. (1) The first statement is clear by construction and the weak differentiability follows from the fact that the function is one-dimensional and Lipschitz continuous on $\mathcal{G}$. Note that the *steepest decent* of φ_j^ξ occurs on the shortest edge of $\mathcal{S}_j^\xi$; and since the function is linear on each edge $\xi(e)e$, we have

$$\left\|(\varphi_j^\xi)'\right\|_{\xi(e)e,\infty} = \frac{1}{\xi(e)\ell_e} = \frac{1}{\ell_{\xi(e)e}} \leq \frac{1}{\ell_{\mathcal{S}_j^\xi}} \tag{5.4.14}$$

for all $e \sim v_j$, and equality if and only if $\xi(e)e$ is the shortest edge of $\mathcal{S}_j^\xi$.

(2) Let $\xi \equiv 1$ be constant and let $e \simeq [0, \ell_e]$ be the edge incident with v_i and v_j, where 0 corresponds to v_i; here the functions φ_i and φ_j have the explicit representations

$$\varphi_i(x) = \left(1 - \frac{1}{\xi(e)\ell_e}x\right)\chi_{[0,\xi(e)\ell_e]}(x) = 1 - \frac{1}{\ell_e}x \tag{5.4.15}$$

and

$$\varphi_j(x) = \left(1 - \frac{1}{\xi(e)} + \frac{1}{\xi(e)\ell_e}x\right)\chi_{[\ell_e(1-\xi(e)),\ell_e]}(x) = \frac{1}{\ell_e}x \tag{5.4.16}$$

respectively, that is, $\varphi_i = 1 - \varphi_j$. The last statement follows by differentiation of this equation. This completes the proof.

□

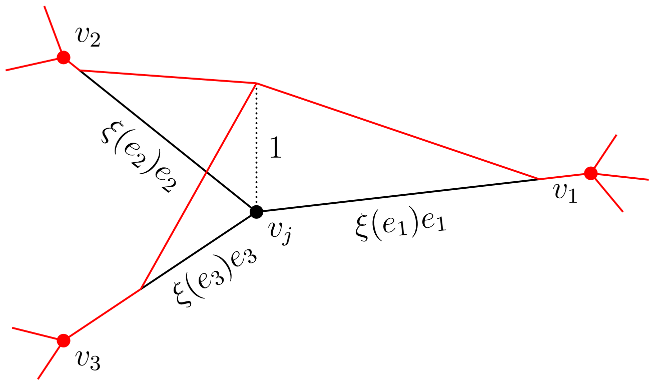

Figure 5.4.3: Exemplary excerpt of a quantum graph. The function φ_j^ξ is supported on $\mathcal{S}_j^\xi$ (here, $\xi < 1$ on e_1, e_2, e_3); in particular, we have $\varphi_j^\xi(x) = 0$ on the rest of $\mathcal{G}$.

The last tool needed to prove Lemma 5.4.7 is the following auxiliary statement.

Lemma 5.4.9. *Let* $(a, b) \subset \mathbb{R}$ *be a non-degenerate interval. If* u, v *are (weakly) differentiable on* (a, b)*, then we have (weakly)*

$$\left(|u|^2 v\right)' = 2v\mathrm{Re}\,(\overline{u}u') + |u|^2 v'. \tag{5.4.17}$$

Proof. Due to $|u|^2 = u\overline{u}$ it suffices to prove $(u\overline{u})' = 2\mathrm{Re}\,(u\overline{u})$. To this end, we introduce real functions a, b, c, d such that

$$u = a + \mathrm{i}b \qquad \text{and} \qquad u' = c + \mathrm{i}d. \tag{5.4.18}$$

Elementary calculations give

$$(u\overline{u})' = u'\overline{u} + u(\overline{u})' = 2(ac + bd), \tag{5.4.19}$$

which, together with

$$\mathrm{Re}\,(ac + \mathrm{i}ad - \mathrm{i}bc + bd) = \mathrm{Re}\,((a - \mathrm{i}b)(c + \mathrm{i}d)) = \mathrm{Re}\,(\overline{u}u'), \tag{5.4.20}$$

proves the assertion. □

Proof of Lemma 5.4.7. Step 1: Proof of (5.4.9). We start by fixing any arbitrary $v_j \in \mathcal{V}$. Let the spanning star $\mathcal{S}_j^\xi$ and the cut-off function φ_j^ξ be as described above, and let $f \in H^1(\mathcal{G})$ be arbitrary. Each of the edges $\xi(e)e$ of $\mathcal{S}_j^\xi$ is identified with the interval $[0, \ell_e]$, where the right boundary point ℓ_e corresponds to the *centre* vertex v_j, and 0 is the *first* zero of φ_j^ξ on each edge to reach, that is, $\varphi_j^\xi(0) = 0$ on each $\xi(e)e$. This fact plus the fundamental theorem of calculus (note the differentiability statement of Lemma 5.4.8) on that interval applied to the function $|f|^2\varphi_j^\xi$ gives

$$|f(v_j)|^2 = \int_0^{\xi(e)\ell_e} (|f|^2\varphi_j^\xi)' \,\mathrm{d}x \tag{5.4.21a}$$

$$= \int_0^{\xi(e)\ell_e} 2\varphi_j^\xi \mathrm{Re}\,(\bar{f}\, f') + |f|^2(\varphi_j^\xi)' \,\mathrm{d}x, \tag{5.4.21b}$$

where we used Lemma 5.4.9. Summing over all $\deg v_j$ edges $\xi(e)e \sim v_j$ yields

$$\deg v_j |f(v_j)|^2 = \int_{\mathcal{S}_j^\xi} 2\varphi_j^\xi \mathrm{Re}\,(\bar{f}\, f') + |f|^2(\varphi_j^\xi)' \,\mathrm{d}x \tag{5.4.22a}$$

$$\leq 2\|\varphi_j^\xi\|_{\mathcal{S}_j^\xi,\infty}\|f\|_{\mathcal{S}_j^\xi}\|f'\|_{\mathcal{S}_j^\xi} + \|(\varphi_j^\xi)'\|_{\mathcal{S}_j^\xi,\infty}\|f\|^2_{\mathcal{S}_j^\xi}. \tag{5.4.22b}$$

Using Lemma 5.4.8 implies

$$\deg v_j |f(v_j)|^2 \leq 2\|f\|_{\mathcal{S}_j^\xi}\|f'\|_{\mathcal{S}_j^\xi} + \frac{1}{\ell_{\mathcal{S}_j^\xi}}\|f\|^2_{\mathcal{S}_j^\xi}, \tag{5.4.23}$$

that is, (5.4.9).

Step 2: Proof of (5.4.10). We argue similarly but distinguish edges which are incident with two vertices of $\mathcal{V}_0$. More precisely, if $v_i, v_j \in \mathcal{V}_0$ are two distinct vertices and $v_i \sim e \sim v_j$, then we obtain the estimate

$$\begin{aligned} |f(v_i)|^2 + |f(v_j)|^2 = \int_0^{\ell_e} & 2\varphi_i \mathrm{Re}\,(\bar{f}\, f') + |f|^2\varphi_i' \\ & + 2\varphi_j \mathrm{Re}\,(\bar{f}\, f') + |f|^2\varphi_j' \,\mathrm{d}x. \end{aligned} \tag{5.4.24}$$

But since $\varphi_i = 1 - \varphi_j$ and $\varphi_i' = -\varphi_j'$ by statement (2) of Lemma 5.4.8, this reduces to

$$|f(v_i)|^2 + |f(v_j)|^2 \le 2\|f\|_e\|f'\|_e. \tag{5.4.25}$$

We now sum over all edges

$$e \subset \mathcal{G}_0 = \bigcup_{j\in\mathbb{N}\,:\,v_j\in\mathcal{V}_0} \mathcal{S}_j \tag{5.4.26}$$

both of whose endpoints are in $\mathcal{V}_0$. To these we also sum the estimates

$$|f(v_i)|^2 \le 2\|f\|_e\|f'\|_e + \frac{1}{\ell_e}\|f\|_e^2, \tag{5.4.27}$$

as obtained above, over all edges e in $\mathcal{E}_0$ which have only one incident vertex v_i in $\mathcal{V}_0$. Since each edge in the union $\mathcal{G}_0$ of the spanning stars of $\mathcal{V}_0$ is counted only once, this yields

$$\sum_{v_j\in\mathcal{V}_0} \deg v_j |f(v_j)|^2 \le 2\|f\|_{\mathcal{G}_0}\|f'\|_{\mathcal{G}_0} + \frac{1}{\ell_{\mathcal{G}_0}}\|f\|_{\mathcal{G}_0}^2, \tag{5.4.28}$$

that is, (5.4.10). □

We can now give the proofs of Theorems 5.4.1 and 5.4.15.

Proof of Theorem 5.4.1. Proof of (1). Let

$$\lambda = \|f'\|_{\mathcal{G}}^2 + \sum_{j=1}^{k} \alpha_j |f(v_j)|^2, \tag{5.4.29}$$

$f \in H^1(\mathcal{G})$ normed to $\|f\|_{\mathcal{G}} = 1$, be any point in $W(a_\alpha)$. We set

$$t := \|f'\|_{\mathcal{G}}^2 \qquad \text{and} \qquad s_j := |f(v_j)|^2 \tag{5.4.30}$$

for each Robin vertex $v_j \in \mathcal{V}_R$ and we consider $\mathcal{S}_j^\xi$ for $\xi(e) = 1/2$ for each $e \sim v_j$. Then the stars $\mathcal{S}_j^\xi$ are all pairwise disjoint for $j = 1, \ldots, k$. By (5.4.30) λ has the form

$$\lambda = t + \sum_{j=1}^{k} \alpha_j s_j, \tag{5.4.31}$$

and by the first statement of Lemma 5.4.7, that is, the trace estimate for a single spanning star, we obtain

$$\begin{aligned} s_j &\leq \frac{2}{\deg v_j} \|f'\|_{\mathcal{S}_j^{1/2}} \|f\|_{\mathcal{S}_j^{1/2}} + \frac{1}{\ell_{\mathcal{S}_j^{1/2}} \deg v_j} \|f\|^2_{\mathcal{S}_j^{1/2}} && (5.4.32a) \\ &\leq \frac{2}{\mathfrak{D}} \|f'\|_{\mathcal{S}_j^{1/2}} \|f\|_{\mathcal{G}} + \frac{2}{\mathfrak{D}\ell_{\mathcal{G}}} \|f\|^2_{\mathcal{G}} && (5.4.32b) \\ &= \frac{2}{\mathfrak{D}} \sqrt{\tau_j} + \frac{2}{\mathfrak{D}\ell_{\mathcal{G}}} && (5.4.32c) \end{aligned}$$

for each $j = 1, \ldots, k$, where we want to make two notes: in (5.4.32b) we used that every edge of every star $\mathcal{S}_j^{1/2}$ has length at least $\ell_{\mathcal{G}}/2$ and in (5.4.32c) the $\tau_j = \|f\|^2_{\mathcal{S}_j^{1/2}}$ are as in the statement of the theorem.

Proof of (2). Here we set $t := \|f'\|^2_{\mathcal{G}}$ as before, but now

$$s := \sum_{j=1}^{k} |f(v_j)|^2. \tag{5.4.33}$$

Thus λ has the form $\lambda = t + \alpha s$, and the estimate (5.4.10) from Lemma 5.4.7 together with the same procedure as in the previous step implies that

$$s \leq \frac{2}{\mathfrak{D}} \|f'\|_{\mathcal{G}_0} \|f\|_{\mathcal{G}_0} + \frac{1}{\ell_{\mathcal{G}_0} \mathfrak{D}} \|f\|^2_{\mathcal{G}_0} \leq \frac{2}{\mathfrak{D}} \sqrt{t} + \frac{1}{\mathfrak{D}\ell_{\mathcal{G}}}, \tag{5.4.34}$$

which completes the proof. □

We now turn to the estimates on the real part of the eigenvalues announced above, that is, an estimate on $\operatorname{Re}\lambda$ by an α-dependent term (for $\operatorname{Re}\alpha < 0$) from above. They also demonstrate the asymptotic optimality of the bounds on $\Lambda_{\mathcal{G},\alpha}$ (see Remark 5.4.11). For simplicity, in what follows, we will assume that $\alpha \in \mathbb{C}$ is independent of the vertices; a similar statement holds in the general case. The following statement is due to [76, Corollary 5.2].

Corollary 5.4.10. *Let* $\operatorname{Re}\alpha < 0$*. Then any eigenvalue* $\lambda \in \sigma(-\Delta^{\alpha}_{\mathcal{V}_R})$ *satisfies*

$$\operatorname{Re}\lambda \geq -\frac{(\operatorname{Re}\alpha)^2}{\mathfrak{D}^2} + \frac{\operatorname{Re}\alpha}{\mathfrak{D}\ell_{\mathcal{G}}}. \tag{5.4.35}$$

Proof of Corollary 5.4.10. This follows directly from Theorem 5.4.1(2); indeed, for any eigenvalue λ

$$\operatorname{Re}\lambda = t + \operatorname{Re}\alpha \cdot s \geq t + \operatorname{Re}\alpha \left(\frac{2\sqrt{t}}{\mathfrak{D}} + \frac{1}{\mathfrak{D}\ell_{\mathcal{G}}} \right). \tag{5.4.36}$$

To minimise the right-hand side of the latter term for all possible $t > 0$ we set

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(t + \frac{2\operatorname{Re}\alpha\sqrt{t}}{\mathfrak{D}} + \frac{\operatorname{Re}\alpha}{\mathfrak{D}\ell_{\mathcal{G}}} \right) = 1 + \frac{\operatorname{Re}\alpha}{\mathfrak{D}\sqrt{t}} \tag{5.4.37}$$

which is, as a elementary calculation shows, true for

$$\sqrt{t} = -\frac{\operatorname{Re}\alpha}{\mathfrak{D}} \quad \Rightarrow \quad t = \frac{(\operatorname{Re}\alpha)^2}{\mathfrak{D}^2} > 0. \tag{5.4.38}$$

Indeed, this is a minimiser of (5.4.36) since the right-hand side of (5.4.37) is strictly monotonically increasing in t, that is, a change of sign from minus to plus. If we use this minimiser t in (5.4.36), we immediately obtain (5.4.35). □

Remark 5.4.11. (1) If $m \geq 1$, that is, at least one $\alpha_j \to \infty$ in a sector fully contained in the open left half-plane, then Theorem 1.2.8 implies the existence of an eigenvalue behaving like $-\alpha^2/\mathfrak{D}^2$ as $\operatorname{Re}\alpha \to -\infty$, meaning that the first term of the estimate of Corollary 5.4.10 is correct in this regime.

(2) Actually, in the case of real negative α a test function argument can be used to give a complementary upper bound on the smallest (real) eigenvalue

$$\lambda_1(\alpha) := \min \sigma(-\Delta^{\alpha}_{\mathcal{V}_R}). \tag{5.4.39}$$

To be more precise we have the following lemma.

Lemma 5.4.12. *Let* $\mathcal{G}$ *be a compact metric graph of total length* $|\mathcal{G}|$ *and with* $k \in \mathbb{N}$

Robin vertices. For real $\alpha < 0$ we have that the first Robin eigenvalue satisfies

$$\lambda_1(\alpha) \leq \begin{cases} -\frac{\alpha^2}{\mathfrak{D}^2} - \frac{2\alpha}{\mathfrak{D}\ell_{\mathcal{G}}} - \frac{1}{\ell_{\mathcal{G}}^2} = -\left[\frac{\alpha}{\mathfrak{D}} + \frac{1}{\ell_{\mathcal{G}}}\right]^2 & \text{if } \alpha < -\frac{\mathfrak{D}}{\ell_{\mathcal{G}}} < 0, \\ \frac{k\alpha}{|\mathcal{G}|} & \text{for all } \alpha < 0. \end{cases} \tag{5.4.40}$$

The proof of this lemma will rely on the variational (min-max) characterisation of the eigenvalues, valid for all real α (see Theorem 2.2.13), as well as the following eigenvalue estimate for stars, see [76, Lemma 6.2].

Lemma 5.4.13. *Let $\mathcal{S}$ be a star with a Robin parameter of strength α at its central vertex of degree $\mathfrak{D}$ and Dirichlet conditions at all other vertices. Then its first eigenvalue $\lambda_1^D(\alpha, \mathcal{S})$ satisfies*

$$\lambda_1^D(\alpha, \mathcal{S}) \leq -\left(\frac{\alpha}{\mathfrak{D}} + \frac{1}{\ell_{\mathcal{S}}}\right)^2 < 0 \tag{5.4.41}$$

if $\alpha < -\frac{\mathfrak{D}}{\ell_{\mathcal{S}}}$, where as before $\ell_{\mathcal{S}}$ denotes the length of the shortest edge of $\mathcal{S}$.

Proof. We observe that the secular equation for $-\lambda_1^D(\alpha, \mathcal{S}) > 0$ reads

$$\sqrt{\lambda} \coth(\sqrt{\lambda}\ell_{\mathcal{G}}) = -\frac{\alpha}{\mathfrak{D}}, \tag{5.4.42}$$

that is, $-\lambda_1^D(\alpha, \mathcal{S})$ is the smallest solution $\lambda > 0$ of this equation. This follows from the fact that the vertex conditions and the symmetry property that the eigenfunction must be invariant under permutations of the $\mathfrak{D}$ equal edges of $\mathcal{S}$ (cf., e.g., [25, Section 5]). Now the elementary inequality

$$\coth(x) \leq \frac{1}{x} + 1, \qquad x > 0, \tag{5.4.43}$$

applied to the left-hand side of (5.4.42) gives

$$\frac{1}{\ell_{\mathcal{S}}} + \sqrt{\lambda} \geq -\frac{\alpha}{\mathfrak{D}}. \tag{5.4.44}$$

This is nontrivial if and only if $\alpha < -\mathfrak{D}/\ell_{\mathcal{S}}$. In this case, rearranging gives (5.4.41). □

We now turn to the proof of the upper bound in Lemma 5.4.12.

Proof Lemma 5.4.12. The *universal* bound

$$\lambda_1(\alpha,\mathcal{G}) \le \frac{k\alpha}{|\mathcal{G}|} \tag{5.4.45}$$

valid for all $\alpha < 0$ follows immediately from taking $f \equiv 1$, that is, the (Neumann) eigenfunction corresponding to $\alpha = 0$, as a test function in the variational characterisation. The other estimate will follow immediately from Lemma 5.4.13 and the inequality

$$\lambda_1(\alpha,\mathcal{G}) \le \lambda_1^D(\alpha,\mathcal{S}), \tag{5.4.46}$$

where $\mathcal{S} = \mathcal{S}_1^1$ is the star subgraph of $\mathcal{G}$ with central vertex v_1, which we recall has degree

$$\deg v_1 = \mathfrak{D} = \min_{j=1,\dots,k} \deg v_j, \tag{5.4.47}$$

as introduced above. To obtain inequality (5.4.46) we argue as follows: consider the first Dirichlet eigenvalue $\lambda_1^D(\alpha,\mathcal{S})$ on the star $\mathcal{S}$ and extend its associated eigenfunction ψ by zero to the rest of $\mathcal{G}$, that is, the extension of ψ may be canonically identified with a function $\tilde{\psi} \in H^1(\mathcal{G})$. Using $\tilde{\psi}$ as a test function in the Rayleigh quotient $R[a_\alpha]$ (see Definition 2.2.12) is exactly equal to $\lambda_1^D(\alpha,\mathcal{S})$, viz.

$$R[a_\alpha](\tilde{\psi}) = \frac{\int_{\mathcal{G}} |\tilde{\psi}'|^2 \,\mathrm{d}x + \sum_{j=1}^k \alpha_j |\tilde{\psi}(v_j)|^2}{\|\tilde{\psi}\|_G} = \lambda_1^D(\alpha,\mathcal{S}); \tag{5.4.48}$$

equivalently, we may appeal directly to [25, Theorem 3.10(1)]. □

While only the first term of the asymptotics has been considered in Remark 5.4.11 and Lemma 5.4.12, we can observe the following behaviour regarding the second term.

Remark 5.4.14. We observe that as $\alpha \to -\infty$, we have

$$\lambda_1(\alpha) = -\frac{\alpha^2}{\mathfrak{D}^2} + o(\alpha^{-N}) \tag{5.4.49}$$

for all $N \in \mathbb{N}$; while as $\alpha \to 0$, since $\lambda_1'(0) = 1/|\mathcal{G}|$ (see [27, Proposition 3.1.6] and use that the eigenfunctions for the principal Neumann eigenvalue $\lambda_1(0) = 0$ are constant),

$$\lambda_1(\alpha) = \frac{k\alpha}{|\mathcal{G}|} + \mathcal{O}(\alpha^2) \tag{5.4.50}$$

as $\alpha \to 0$. Hence there can be no "correct" coefficient $c \in \mathbb{R}$ of α in any (upper or lower) bound of the form $-\alpha^2/\mathfrak{D}^2 + c\alpha$ which is valid for all $\alpha < 0$ and asymptotically sharp for $\alpha \to 0$ and $\alpha \to -\infty$.

We finish with a more precise statement (which is due to [76, Theorem 5.4]) about the imaginary parts of the eigenvalues.

Theorem 5.4.15. *Let* $\alpha \in \mathbb{C}^k$.

(1) If $\operatorname{Re}\alpha_j \geq 0$ *for all* $j = 1, \dots, k$, *then any eigenvalue* λ *of* $-\Delta^{\alpha}_{\mathcal{V}_R}$ *satisfies*

$$|\operatorname{Im}\lambda| \leq \max_{j=1,\dots,k} \frac{|\operatorname{Im}\alpha_j|}{\deg v_j}\left[2\sqrt{\operatorname{Re}\lambda} + \frac{1}{\mathfrak{D}\ell_{\mathcal{G}}}\right]. \tag{5.4.51}$$

(2) If $\operatorname{Re}\alpha_j < 0$ *for at least one* $j = 1, \dots, k$, *then for every* $0 < \varepsilon < 1$ *there exists a constant* $C = C(\varepsilon) > 0$ *depending on* $\mathcal{G}$ *and each* $\operatorname{Re}\alpha_j < 0$ *such that*

$$|\operatorname{Im}\lambda| \leq \max_{j=1,\dots,k} \frac{|\operatorname{Im}\alpha_j|}{\deg v_j}\left[2(1-\varepsilon)\sqrt{\operatorname{Re}\lambda + C} + \frac{1}{\mathfrak{D}\ell_{\mathcal{G}}}\right]. \tag{5.4.52}$$

Proof. (1) We simply note that, if $f \in H^1(\mathcal{G})$ is an eigenfunction corresponding to λ, normalised so that $\|f\|_{\mathcal{G}} = 1$, then by the second statement of Lemma 5.4.7 applied to the union $\mathcal{G}_0$ of the stars $\mathcal{S}^1_j$, $j = 1, \dots, k$, whose total length we estimate from below by $\ell_{\mathcal{G}}$,

$$|\operatorname{Im}\lambda| = \left|\sum_{j=1}^{k} \operatorname{Im}\alpha_j |f(v_j)|^2\right| \leq \sum_{j=1}^{k} \frac{|\operatorname{Im}\alpha_j|}{\deg v_j}\deg v_j |f(v_j)|^2 \tag{5.4.53a}$$

$$\leq \max_{j=1,\dots,k} \frac{|\operatorname{Im}\alpha_j|}{\deg v_j}\left[2\|f'\|_{\mathcal{G}} + \frac{1}{\mathfrak{D}\ell_{\mathcal{G}}}\right] \tag{5.4.53b}$$

$$\leq \max_{j=1,\dots,k} \frac{|\operatorname{Im}\alpha_j|}{\deg v_j}\left[2\sqrt{\operatorname{Re}\lambda} + \frac{1}{\mathfrak{D}\ell_{\mathcal{G}}}\right], \tag{5.4.53c}$$

where the last inequality follows from taking the real part of the quadratic form (5.1.8) for $\lambda \in \mathbb{C}$ since $\operatorname{Re}\alpha$ was assumed non-negative.

(2) We use the following weighted trace inequality: fix $k' \in \{1,\dots,k\}$ such that (after relabelling the $v_1,\dots,v_k$ if necessary) $\operatorname{Re}\alpha_j < 0$ if and only if $j \le k'$. Then for every $\delta > 0$ there exists a constant

$$C = C(\mathcal{G}, \operatorname{Re}\alpha_1,\dots,\operatorname{Re}\alpha_{k'},\delta) \tag{5.4.54}$$

such that

$$0 \le \sum_{j=1}^{k'} (-\operatorname{Re}\alpha_j)|f(v_j)|^2 \le \delta\|f'\|_{\mathcal{G}}^2 + C\|f\|_{\mathcal{G}}^2 \tag{5.4.55}$$

for all $f \in H^1(\mathcal{G})$, which can be obtained from the usual trace inequality by a standard ε-C_ε argument (as in (3.2.32)). Since for the eigenfunction f, normalised so that $\|f\|_{\mathcal{G}} = 1$,

$$\operatorname{Re}\lambda = \|f'\|_{\mathcal{G}}^2 + \sum_{j=1}^{k} \operatorname{Re}\alpha_j|f(v_j)|^2 \ge \|f'\|_{\mathcal{G}}^2 - \delta\|f'\|_{\mathcal{G}}^2 - C \tag{5.4.56}$$

it follows that

$$\|f'\|_{\mathcal{G}} \le \frac{\sqrt{\operatorname{Re}\lambda + C}}{\sqrt{1-\delta}}. \tag{5.4.57}$$

If we now substitute the factor $1/\sqrt{1-\delta}$ by $1-\varepsilon$, then the same argument as in (1), viz.

$$|\operatorname{Im}\lambda| = \left|\sum_{j=1}^{k} \operatorname{Im}\alpha_j|f(v_j)|^2\right| \le \max_{j=1,\dots,k} \frac{|\operatorname{Im}\alpha_j|}{\deg v_j}\left[2\|f'\|_{\mathcal{G}} + \frac{1}{\mathfrak{D}\ell_{\mathcal{G}}}\right], \tag{5.4.58}$$

leads to (5.4.52). □

We finish with the proof of Theorem 1.2.9.

Proof of Theorem 1.2.9. The existence of k eigenvalues with the claimed asymptotics, and the fact that non-divergent eigenvalues converge to points in $\sigma(-\Delta_{\mathcal{V}_R}^D)$ follow immediately from Theorem 1.2.8. We next show that there are no more than k divergent eigenvalues. This follows from a standard interlacing statement: denoting the kth eigenvalue of $-\Delta_{\mathcal{V}_R}^D$ (counted with multiplicities) by λ_k^D, since the forms associated with $-\Delta_{\mathcal{V}_R}^{\alpha}$ and $-\Delta_{\mathcal{V}_R}^{D}$ coincide on the form domain $H_0^1(\mathcal{G}, \mathcal{V}_R)$ of the latter, and the quotient space

$$H^1(\mathcal{G})/H_0^1(\mathcal{G}, \mathcal{V}_R) \tag{5.4.59}$$

has dimension k, it follows from the min-max characterisation of the eigenvalues that

$$\lambda_{j-k}^D \leq \lambda_j(\alpha) \leq \lambda_j^D \tag{5.4.60}$$

for all $\alpha \in \mathbb{R}$ and all $j \geq k+1$ (see also, e.g., [27, Section 3.1.6] or [25, Sections 3.1 and 4.1]). Hence $\lambda_j(\alpha)$ remains bounded whenever $j \geq k+1$, and so by Corollary 5.3.3 converges to an eigenvalue of the Dirichlet Laplacian. It remains to prove that for

$$\alpha < -2 \max_{j=1,\dots,k} \left\{ \frac{\deg v_j}{\ell_j} \right\} \tag{5.4.61}$$

the Robin Laplacian has exactly k negative eigenvalues: by the above reasoning, it suffices to find one fixed α for which it has at least k such negative eigenvalues. To this end, for each $j = 1, \dots, k$, we consider each star $\mathcal{S}_j^{1/2}$ subgraph of $\mathcal{G}$ with Robin condition at its central vertex v_j; denote by ψ_j the test function equal to the eigenfunction for the Dirichlet eigenvalue $\lambda_1^D(\alpha, \mathcal{S}_j^{1/2})$ on $\mathcal{S}_j^{1/2}$, extended by zero to a function in $H^1(\mathcal{G})$, and whose Rayleigh quotient equals $\lambda_1^D(\alpha, \mathcal{S}_j^{1/2})$. Then, since the supports of ψ_j are pairwise disjoint (which is due to the scaling factor $\xi \equiv 1/2$), we can define the k-dimensional space

$$\mathcal{H}_k := \bigoplus_{j=1}^{k} \psi_j \subset H^1(\mathcal{G}) \tag{5.4.62}$$

as a space of test functions for $\lambda_k(\alpha, \mathcal{G})$. If we choose any

$$\alpha < -2 \max_{j=1,\dots,k} \left\{ \frac{\deg v_j}{\ell_{\mathcal{S}_j}} \right\}, \tag{5.4.63}$$

then by Lemma 5.4.13 each function ψ_j, and thus every function in $\mathcal{H}_k$ has a negative Rayleigh quotient (where one should not forget the scaling factor $2\ell_{\mathcal{S}_j}^{1/2} = \ell_{\mathcal{S}_j}$). It follows from the min-max characterisation that $\lambda_k(\alpha, \mathcal{G}) < 0$ for such α. $\square$

List of Figures

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