Basic Representation Theory of Crossed Modules

Master's Thesis

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Contents

Introduction

Crossed modules

A crossed module $V = \big(M, G, \gamma, f\big)$ consists of groups M and G, an action $\gamma\colon G \to \mathrm{Aut}\,(M)$, $g \mapsto (m \mapsto m^g)$ and a group morphism $f\colon M \to G$ that satisfies

$$(m^g)f = (mf)^g \quad \text{and} \quad m^n = m^{nf}$$

for m, $n \in M$ and $g \in G$.

We write $V\pi_1 := \ker f$ and $V\pi_0 := G/Mf$.

Appearance of crossed modules in general

Groups appear as follows. Each object in each category has an automorphism group.

Similarly, crossed modules appear as follows. Each object in each 2-category has an automorphism crossed module.

As starting point, we take the automorphism crossed module of an object of the 2-category of categories, i.e. the automorphism crossed module of a category $\mathcal{X}$, called the symmetric crossed module $\mathrm{S}_{\mathcal{X}}$ on $\mathcal{X}$; cf. Lemma 48. ([1])

[1] Here, 'symmetric' is not used in the sense of 'braided'.

Crossed modules and topology

The category of groups is equivalent to the homotopy category of CW-spaces for which only the first homotopy group is allowed to be nontrivial. Similarly, the category of crossed modules has a homotopy category which is equivalent to the category of CW-spaces for which only the first and the second homotopy group are allowed to be nontrivial.

To achieve this, J.H.C. Whitehead attached to a CW-complex with 1-skeleton A and 2-skeleton X a crossed module $\big(M, G, \gamma, f\big)$, where M is the second relative homotopy group of the pair (X, A) and where G is the first homotopy group of A; cf. [**4,** §2.2, p. 41], [**16,** Thm. 2.4.8].

Crossed modules and invertible monoidal categories

A monoidal category is a category $\mathcal{C}$ together with a unit object I and an associative tensor product $(\otimes)$ on the objects $\mathrm{Ob}(\mathcal{C})$ and on the morphisms $\mathrm{Mor}(\mathcal{C})$. This is to be understood in a strict sense; cf. Definition 12. Note that a monoidal category can be viewed as a 2-category with a single object.

An invertible monoidal category $\mathcal{C}$ is a monoidal category in which the objects and the morphisms are invertible with respect to the tensor product $(\otimes)$.

To a crossed module we may attach an invertible monoidal category via the construction Cat; cf. Definition 21, Lemma 39. Conversely, to an invertible monoidal category we may attach a crossed module via the construction CM; cf. Lemma 42.

Therefore, a crossed module is essentially the same as an invertible monoidal category; cf. Proposition 43.

This correspondence is due to Brown and Spencer [**5,** Thm. 1], who state that it has been independently discovered beforehand, but not published by Verdier and Duskin.

Cayley for crossed modules

For each category $\mathcal{X}$, we have a symmetric crossed module $\mathrm{S}_{\mathcal{X}} = \big(\mathrm{M}_{\mathcal{X}}, \mathrm{G}_{\mathcal{X}}, \gamma_{\mathcal{X}}, \mathrm{f}_{\mathcal{X}}\big)$, where $\mathrm{G}_{\mathcal{X}}$ consists of the autofunctors of $\mathcal{X}$ and where $\mathrm{M}_{\mathcal{X}}$ consists of the isotransformations from the identity $\mathrm{id}_{\mathcal{X}}$ to some autofunctor of $\mathcal{X}$; cf. Lemma 48.

In particular, for a crossed module V, we obtain a symmetric crossed module $\mathrm{S}_{V\mathrm{Cat}}$. An analogue to Cayley's Theorem holds, namely that there is a canonical injective crossed module morphism ρ_V^{Cayley} from V to $\mathrm{S}_{V\mathrm{Cat}}$, for which both $\rho_V^{\mathrm{Cayley}}\pi_1$ and $\rho_V^{\mathrm{Cayley}}\pi_0$ are injective; cf. §0.4 items 2, 4 and 6, Theorem 62.

For example, if V is the crossed module with $M = C_4 = \langle b\rangle$, $G = C_4 = \langle a\rangle$, $bf = a$ and $b^a = b^-$, then we have

$$|M| = 4\,,\ |G| = 4\,,\ |V\pi_1| = 2\,,\ |V\pi_0| = 2\,.$$

For the symmetric crossed module $\mathrm{S}_{V\mathrm{Cat}}$, we have

$$|\,\mathrm{M}_{V\mathrm{Cat}}\,| = 64\,,\ |\,\mathrm{G}_{V\mathrm{Cat}}\,| = 32\,,\ |\,\mathrm{S}_{V\mathrm{Cat}}\,\pi_1| = 4\,,\ |\,\mathrm{S}_{V\mathrm{Cat}}\,\pi_0| = 2\,.$$

Cf. §A.9, §A.7.

R-linear extension and units

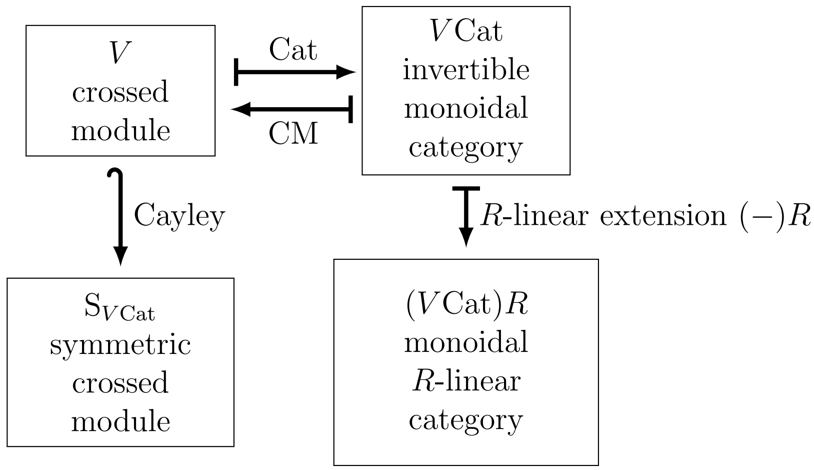
To each category $\mathcal{C}$, we may attach its R-linear extension $\mathcal{C}R$, which is an R-linear category.

For a monoidal category $\mathcal{C}$, its R-linear extension $\mathcal{C}R$ is a monoidal R-linear category; cf. Lemma 85.

For each monoidal category $\mathcal{D}$ we have the unit invertible monoidal category $\mathcal{D}\mathrm{U}$, whose objects are the tensor invertible objects of $\mathcal{D}$ and whose morphisms are the tensor invertible morphisms of $\mathcal{D}$.

Let $\mathcal{C}$ be an invertible monoidal category. Let $\mathcal{D}$ be an R-linear monoidal category. We have a bijective correspondence between monoidal R-linear functors from $\mathcal{C}R$ to $\mathcal{D}$ on the on hand, and monoidal functors from $\mathcal{C}$ to $\mathcal{D}\mathrm{U}$ on the other hand; cf. Lemma 95.

Summary: Constructions for a crossed module V



The functor $\mathrm{Real}_{\mathcal{M}}$

For an R-linear category $\mathcal{M}$, we have the monoidal R-linear category $\mathrm{End}_R(\mathcal{M})$ whose objects are the R-linear functors from $\mathcal{M}$ to $\mathcal{M}$, and whose morphisms are the transformations between such functors. The tensor product on the objects is given by composition of functors, and the tensor product on the morphisms is given by horizontal composition of transformations; cf. Lemma 80.

Using the construction U, we obtain an invertible monoidal category

$$\mathrm{Aut}_R(\mathcal{M}) := \big(\mathrm{End}_R(\mathcal{M})\big)\mathrm{U} \subseteq \mathrm{End}_R(\mathcal{M})\,.$$

On the other hand, we have the crossed submodule

$$\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M}) = \big(\,\mathrm{M}_{\mathcal{M}}^R, \mathrm{G}_{\mathcal{M}}^R, \gamma_{\mathcal{M}}^R, \mathrm{f}_{\mathcal{M}}^R\,\big) \leqslant \mathrm{S}_{\mathcal{M}}\,,$$

where $\mathrm{G}_{\mathcal{M}}^R \leqslant \mathrm{G}_{\mathcal{M}}$ is the subgroup consisting of the R-linear autofunctors of $\mathcal{M}$ and where $\mathrm{M}_{\mathcal{M}}^R \leqslant \mathrm{M}_{\mathcal{M}}$ is the subgroup consisting of the isotransformations from the identity $\mathrm{id}_{\mathcal{M}}$ to some R-linear autofunctor of $\mathcal{X}$; cf. Lemma 81.

Using the construction Cat, we obtain an invertible monoidal category

$$\big(\,\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\mathrm{Cat}\ .$$

It turns out that we have a monoidal isofunctor

$$\mathrm{Real}_{\mathcal{M}} \colon \big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\,\mathrm{Cat} \xrightarrow{\sim} \mathrm{Aut}_R(\mathcal{M})\,;$$

cf. Theorem 99.

The entire situation concerning an R-linear category $\mathcal{M}$ can be depicted as follows.

$$\begin{tikzcd}
\boxed{\begin{array}{c} \mathrm{S}_{\mathcal{M}} \\ \text{symmetric} \\ \text{crossed module} \end{array}} & \\
\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M}) \arrow[u, hook] \arrow[r, mapsto, "\mathrm{Cat}"] \arrow[d, equal, "\text{Lemma 98}"] & \boxed{\begin{array}{c} \big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\,\mathrm{Cat} \\ \text{invertible monoidal} \\ \text{category} \end{array}} \arrow[d, "\mathrm{Real}_{\mathcal{M}}\,,\ \text{Theorem 99}", "\wr"'] \\
\begin{array}{c} \big(\mathrm{Aut}_R(\mathcal{M})\big)\,\mathrm{CM} \\ \text{crossed module} \end{array} & \boxed{\begin{array}{c} \mathrm{Aut}_R(\mathcal{M}) := \big(\mathrm{End}_R(\mathcal{M})\big)\,\mathrm{U} \\ \text{invertible monoidal category} \end{array}} \arrow[l, mapsto, "\mathrm{CM}"] \arrow[d, hook] \\
 & \boxed{\begin{array}{c} \mathrm{End}_R(\mathcal{M}) \\ \text{monoidal } R\text{-linear} \\ \text{category} \end{array}}
\end{tikzcd}$$

Modules and representations of crossed modules

Modules

Classically, given an R-algebra A, an A-module can be given as an R-module M together with an R-algebra morphism $A \to \mathrm{End}_R(M)$. This action is usually written as an exterior multiplication action, defining $m \cdot x$ for $m \in M$ and $x \in A$.

Now suppose given a monoidal R-linear category $\mathcal{A}$. An $\mathcal{A}$-module is an R-linear category $\mathcal{M}$ together with a monoidal R-linear functor $\mathcal{A} \to \mathrm{End}_R(\mathcal{M})$; cf. Definition 100. This action is usually written as an exterior tensor product action, defining $M \otimes X$ for $M \in \mathrm{Ob}(\mathcal{M})$ and $X \in \mathrm{Ob}(\mathcal{A})$, and likewise for morphisms.

Given $\mathcal{A}$-modules $\mathcal{M}$ and $\mathcal{N}$, an R-linear functor $F \colon \mathcal{M} \to \mathcal{N}$ is called $\mathcal{A}$-linear if $(M \otimes X)F = MF \otimes X$ for $M \in \mathrm{Ob}(\mathcal{M})$ and $X \in \mathrm{Ob}(\mathcal{A})$, and likewise for morphisms.

Representations

Classically, given a group G and an R-module M, a representation of G on M is given by a group morphism $G \to \mathrm{Aut}_R(M)$.

It gives rise to an R-algebra morphism $RG \to \mathrm{End}_R(M)$, and thus M becomes an RG-module. Conversely, from an RG-algebra morphism $RG \to \mathrm{End}_R(M)$ we can obtain a representation $G \to \mathrm{Aut}_R(M)$ of G on M.

So a representation of G is essentially the same as an RG-module.

Now, let V be a crossed module and let $\mathcal{M}$ be an R-linear category. A crossed module morphism $\rho \colon V \to \mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})$ is called a representation of V on $\mathcal{M}$.

For a representation $\rho \colon V \to \mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})$, we can construct a monoidal R-linear functor $\Phi_\rho \colon (V\mathrm{Cat})R \to \mathrm{End}_R(\mathcal{M})$. So $\mathcal{M}$ becomes a $(V\mathrm{Cat})R$-module; cf. Lemma 121. Conversely, from a monoidal R-linear functor $\Phi \colon (V\mathrm{Cat})R \to \mathrm{End}_R(\mathcal{M})$, we can obtain a representation $\rho_\Phi \colon V \to \mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})$ of V on $\mathcal{M}$; cf. Lemma 122.

So a representation of V is essentially the same as a $(V\mathrm{Cat})R$-module.

A first step towards Maschke

Let $V = \big(M, G, \gamma, f\big)$ be a crossed module. We have the crossed module

$$\bar{V} := \big(Mf, G, c, \mathrm{id}_G\big|_{Mf}\big)\,,$$

with $c\colon G \to \mathrm{Aut}(Mf)$, $g \mapsto (x \mapsto x^g)$. We have a surjective crossed module morphism $V \to \bar{V}$ given as follows.

$$\begin{tikzcd} M \arrow[r, "f|^{Mf}"] \arrow[d, "f"'] & Mf \arrow[d, "\mathrm{id}_G|_{Mf}"] \\ G \arrow[r, "\mathrm{id}_G"'] & G \end{tikzcd}$$

This induces a monoidal R-linear functor $F\colon (V\mathrm{Cat})R \to (\bar{V}\mathrm{Cat})R$.

From that, we obtain an R-linear functor $\Theta_F\colon (V\mathrm{Cat})R \to \mathrm{End}_R\big((\bar{V}\,\mathrm{Cat})R\big)$. So $(\bar{V}\,\mathrm{Cat})R$ becomes a $(V\mathrm{Cat})R$-module. Moreover, $(V\mathrm{Cat})R$ carries the structure as a regular $(V\mathrm{Cat})R$-module.

Then $F\colon (V\mathrm{Cat})R \to (\bar{V}\mathrm{Cat})R$ is a $(V\mathrm{Cat})R$-linear functor.

We want to investigate under which conditions on R the $(V\mathrm{Cat})R$-linear functor F is a retraction. This question can be answered in a reasonable way if we extend the scope by admitting prefunctors:

A prefunctor P from a category $\mathcal{C}$ to a category $\mathcal{D}$ is defined to be a pair of maps $(\mathrm{Ob}(P)\,, \mathrm{Mor}(P))$ with $\mathrm{Ob}(P):\ \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and $\mathrm{Mor}(P):\ \mathrm{Mor}(C) \to \mathrm{Mor}(D)$, where $\mathrm{Mor}(P)$ is compatible with composition but *not* necessarily with identities; cf. Definition 127.

If $\mathcal{C}$ and $\mathcal{D}$ are $(V\mathrm{Cat})R$-modules, then the notion of a $(V\mathrm{Cat})R$-linear prefunctor from $\mathcal{C}$ to $\mathcal{D}$ is defined analogously to that of a $(V\mathrm{Cat})R$-linear functor, again omitting compatibility with identities; cf. Definition 129.

Then the $(V\mathrm{Cat})R$-linear functor F has a $(V\mathrm{Cat})R$-linear prefunctor P as a coretraction if the order $|\ker f|$ is finite and invertible in $R\,$; cf. Proposition 135.

$$\begin{tikzcd} (V\mathrm{Cat})R \arrow[r, "F"'] & (\bar{V}\mathrm{Cat})R \arrow[l, bend right, "P"'] \end{tikzcd}$$

A dictionary

Classical case	Case treated here	Reference
X: set	$\mathcal{X}$: category	§0.2, item 1
G: group	V: crossed module	§0.4, item 1
	$V\mathrm{Cat}$: invertible monoidal category	D21, R29
group morphism $\varphi\colon G \to H$	crossed module morphism $\rho\colon V \to W$	§0.4, item 2
	monoidal functor $\rho\,\mathrm{Cat}\colon V\mathrm{Cat} \to W\mathrm{Cat}$	D31, L39.(1)
M: R-module	$\mathcal{M}$: R-linear category	D65
A: R-algebra	$\mathcal{A}$: monoidal R-linear category	D73
$\mathrm{U}(A)$: unit group of the R-algebra A	$\mathcal{A}\mathrm{U}$: unit invertible monoidal category of the monoidal R-linear category $\mathcal{A}$	L91
S_X: symmetric group on X	$\mathrm{S}_{\mathcal{X}}$: symmetric crossed module on $\mathcal{X}$	L48
a group morphism $\varphi\colon G \to \mathrm{S}_X$ defines a G-set X	a crossed module morphism $\rho\colon V \to \mathrm{S}_{\mathcal{X}}$ defines a strong V-crossed category $\mathcal{X}$, also called V-category $\mathcal{X}$	L55

Classical case	Case treated here	Reference
the injective group morphism $\varphi\colon G \to \mathrm{S}_G$ given by Cayley's Theorem for groups	the injective crossed module morphism $\rho\colon V \to \mathrm{S}_{V\mathrm{Cat}}$ which is also injective on π_1 and π_0	T62
R-algebra morphism $\sigma\colon A \to B$	monoidal R-linear functor $F\colon \mathcal{A} \to \mathcal{B}$	D74
$\mathrm{End}_R(M)$: endomorphism R-algebra	$\mathrm{End}_R(\mathcal{M})$: endomorphism monoidal R-linear category	L80
$\mathrm{Aut}_R(M)$: automorphism group	$\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})$: automorphism crossed module	L81
	$\big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\,\mathrm{Cat} \simeq \big(\mathrm{End}_R(\mathcal{M})\big)\,\mathrm{U}$: invertible monoidal category	T99
an R-algebra morphism $\sigma\colon A \to \mathrm{End}_R(M)$ defines an A-module $M = (M, \sigma)$	a monoidal R-linear functor $\Phi\colon \mathcal{A} \to \mathrm{End}_R(\mathcal{M})$ defines an $\mathcal{A}$-module $\mathcal{M} = (\mathcal{M}, \Phi)$	D100
RG: R-algebra, called group algebra of G over R	$(V\mathrm{Cat})R$: monoidal R-linear category	L85, R115

Classical case	Case treated here	Reference
a group morphism $\varphi\colon G \to \mathrm{Aut}_R(M)$ defines a representation of G on M, which yields an RG-module M	a crossed module morphism $\rho\colon V \to \mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})$ defines a representation of V on $\mathcal{M}$, which yields a $(V\mathrm{Cat})R$-module $\mathcal{M}$	D120, L121
a G-set X yields a permutation module RX over RG	a V-category $\mathcal{X}$ yields a permutation module $\mathcal{X}R$ over $(V\mathrm{Cat})R$	P125

Related approaches

Miemietz and Mazorchuk consider 2-representations, defined as 2-functors from a 2-category to the 2-category of module categories over finite dimensional algebras over fields [**12,** §2.2]. The definition of a representation of a crossed modules used here in §8.3 essentially fits into their framework, since a crossed module V corresponds to a invertible monoidal category $V\mathrm{Cat}$, which in turn can be seen as a 2-category with a single object. From this point of view, a representation of V is a 2-functor from $V\mathrm{Cat}$ to the 2-category of R-linear categories.

In contrast, Forrester-Barker defines a representation of a crossed module V as a 2-functor from from $V\mathrm{Cat}$ to the 2-category of complexes of R-modules concentrated in positions 1 and 0 [**7,** Def. 2.4.1].

Similarly, Barrett and Mackaay define a representation of a crossed module to be a variant of a 2-functor from $V\mathrm{Cat}$ to a bicategory called 2-Vect, defined directly using matrices [**2,** Def. 3.14, Def. 4.1.(a)].

Still another approach has been taken by Bantay, who defines a representation of a crossed module (M, G, γ, f) to be a group representation of G on a complex vector space V, together with an extra map from M to $\mathrm{End}_{\mathbb{C}}(V)$ compatible with that action [**1,** §3]. This has been pursued further by Maier and Schweigert [**11**] and by Dehghani and Davvaz [**6,** §6], who develop a character theory in this context. Lebed and Wagemann interpret a representation in the sense of Bantay as a certain Yetter-Drinfel'd-module with respect to the group algebras of M and G [**9,** Ex. 2.13]. An interpretation of a representation in this sense as a 2-functor from VCat to a suitable 2-category seems to be nonobvious.

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I would also like to thank Sebastian Thomas, for providing us with the construction of the symmetric crossed module on which this master thesis is based.

Chapter 0

Conventions

0.1 Sets

Let X, Y be sets.

1. In general, we write maps on the right, i.e. the map $X \xrightarrow{f} Y$ maps $x \in X$ to $xf \in Y$.
   We make some exceptions for standard constructions, such as Ob, Mor, Aut, etc.

2. Suppose given a subset $Z \subseteq Y$. Let $X \xrightarrow{f} Y$ be a map.
   We write $f^{-}(Z) := \{x \in X \colon f(x) \in Z\}$ for the preimage of Z under f.

0.2 Categories and functors

Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories.

1. By a *category* $\mathcal{C}$, we understand a small category (with respect to a given universe). I.e. we stipulate that $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$ are sets.

   So a category is given by $\mathcal{C} = \big(\mathrm{Mor}(\mathcal{C})\,, \mathrm{Ob}(\mathcal{C})\,, (s, i, t), \blacktriangle\big)$, where $\mathrm{Mor}(\mathcal{C})$ is the set of morphisms, $\mathrm{Ob}(\mathcal{C})$ is the set of objects, $s \colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$ is the source map, $i \colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C})$ is the map sending an object to its identity morphism, $t \colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$ is the target map, and $(\blacktriangle)$ is the composition of morphisms.

To use the symbol ($\blacktriangle$) for composition is somewhat unusual, but it serves to distinguish composition and multiplication. Cf. e.g. Definition 2. The symbol ($\blacktriangle$) should remind of a commutative diagram.

2. By writing $X \xrightarrow{u} Y \xrightarrow{v} Z$ in $\mathcal{C}$, we implicitly suppose given objects X, Y, $Z \in \mathrm{Ob}(\mathcal{C})$ and morphisms u, $v \in \mathrm{Mor}(\mathcal{C})$ with $u\boldsymbol{s} = X$, $u\boldsymbol{t} = Y$ and $v\boldsymbol{s} = Y$, $v\boldsymbol{t} = Z$.

3. A morphism $(X \xrightarrow{u} Y) \in \mathrm{Mor}(\mathcal{C})$ is called *isomorphism* if there exists a morphism $v \in \mathrm{Mor}(\mathcal{C})$ such that $u \blacktriangle v = \mathrm{id}_X$ and $v \blacktriangle u = \mathrm{id}_Y$ hold. Then we write $v := u^-$ and we call u^- the inverse of u.

4. Let $X, Y \in \mathrm{Ob}(\mathcal{C})$. We write ${}_{\mathcal{C}}(X,Y) := \{a \in \mathrm{Mor}(\mathcal{C}) : a\boldsymbol{s} = X,\ a\boldsymbol{t} = Y\}$ for the set of morphisms from X to Y.

5. A *functor* from $\mathcal{C}$ to $\mathcal{D}$ is given by $F := (\mathrm{Mor}(F)\,, \mathrm{Ob}(F))$ where

$$\mathrm{Ob}(F) : \ \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D}) \quad \text{and} \quad \mathrm{Mor}(F) : \ \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})\,.$$

   A functor is required to satisfy

$$\begin{aligned} u\boldsymbol{s}\,\mathrm{Ob}(F) &= u\,\mathrm{Mor}(F)\,\boldsymbol{s} \\ u\boldsymbol{t}\,\mathrm{Ob}(F) &= u\,\mathrm{Mor}(F)\,\boldsymbol{t} \\ X\boldsymbol{i}\,\mathrm{Mor}(F) &= X\,\mathrm{Ob}(F)\,\boldsymbol{i} \end{aligned}$$

   for $u \in \mathrm{Mor}(\mathcal{C})$, $X \in \mathrm{Ob}(\mathcal{C})$, and

$$(u \blacktriangle v)\,\mathrm{Mor}(F) = u\,\mathrm{Mor}(F) \,\blacktriangle v\,\mathrm{Mor}(F)\ ,$$

   for $X \xrightarrow{u} Y \xrightarrow{v} Z$ in $\mathcal{C}$.

   For $X \in \mathrm{Ob}(\mathcal{C})$, we write $XF := X\,\mathrm{Ob}(F) \in \mathrm{Ob}(\mathcal{D})$. For $u \in \mathrm{Mor}(\mathcal{C})$, we write $uF := u\,\mathrm{Mor}(F) \in \mathrm{Mor}(\mathcal{D})$.

6. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be a functors. We write $(F * G)\colon \mathcal{C} \to \mathcal{E}$ for the composite of F and G. If unambiguous, we sometimes write for short $FG := F * G$.

7. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called *isofunctor* from $\mathcal{C}$ to $\mathcal{D}$ if there exist a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $FG = \mathrm{id}_{\mathcal{C}}$ and $GF = \mathrm{id}_{\mathcal{D}}$ hold. Then we write $F^- := G$.

   If $\mathcal{C} = \mathcal{D}$ then an isofunctor $F\colon \mathcal{C} \to \mathcal{C}$ is called an *autofunctor*.

8. By $\mathrm{Aut}\,(\mathcal{C}) := \{\mathcal{C} \xrightarrow{F} \mathcal{C} \colon F$ is an autofunctor$\}$ we denote the set of autofunctors from $\mathcal{C}$ to $\mathcal{C}$. For $F \in \mathrm{Aut}\,(\mathcal{C})$, we also write $\big(\mathcal{C} \xrightarrow[\sim]{F} \mathcal{C}\big) := \big(\mathcal{C} \xrightarrow{F} \mathcal{C}\big)$. The set $\mathrm{Aut}\,(\mathcal{C})$ is actually a group; cf. Lemma 45.(1) below.

9. For $F, G \in \mathrm{Aut}(\mathcal{C})$, we write $F^G := G^- F G$.

10. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Suppose given subcategories $\mathcal{C}' \subseteq \mathcal{C}$ and $\mathcal{D}' \subseteq \mathcal{D}$. Suppose given a functor $F'\colon \mathcal{C}' \to \mathcal{D}'$ such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \\ \mathcal{C}' \arrow[u, hook] \arrow[r, "F'"'] & \mathcal{D}' \arrow[u, hook] \end{tikzcd}$$

    I.e., for $X \in \mathrm{Ob}(\mathcal{C}')$ and $u \in \mathrm{Mor}(\mathcal{C}')$, we have $XF' = XF$ and $uF' = uF$. Then we write $F\big|_{\mathcal{C}'}^{\mathcal{D}'} := F'\colon \mathcal{C}' \to \mathcal{D}'$.

11. If $\mathcal{C}$ is a subcategory of $\mathcal{D}$ then we write $J_{\mathcal{C},\mathcal{D}}\colon \mathcal{C} \to \mathcal{D}$ for the *embedding functor* from $\mathcal{C}$ to $\mathcal{D}$. We often abbreviate $J := J_{\mathcal{C},\mathcal{D}}\colon \mathcal{C} \to \mathcal{D}$.

0.3 Functors and transformations

Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{K}$ be categories.

1. We write $[\mathcal{C}, \mathcal{D}]$ for the category of functors from $\mathcal{C}$ to $\mathcal{D}$. The set of objects $\mathrm{Ob}\big([\mathcal{C}, \mathcal{D}]\big)$ of this category consists of the functors from $\mathcal{C}$ to $\mathcal{D}$. The set of morphisms $\mathrm{Mor}\big([\mathcal{C}, \mathcal{D}]\big)$ consists of the transformations between such functors.

2. Let $F, G \in \mathrm{Ob}\big([\mathcal{C}, \mathcal{D}]\big)$ be functors from $\mathcal{C}$ to $\mathcal{D}$.

    A *transformation* $\big(F \xrightarrow{a} G\big) \in \mathrm{Mor}([\mathcal{C}, \mathcal{D}])$ from F to G is a tuple of morphisms $\big(XF \xrightarrow{Xa} XG\big)_{X \in \mathrm{Ob}(\mathcal{C})}$ with the property that the following diagram is commutative for $\big(X \xrightarrow{u} Y\big) \in \mathrm{Mor}(\mathcal{C})$.

$$\begin{tikzcd} XF \arrow[r, "Xa"] \arrow[d, "uF"'] & XG \arrow[d, "uG"] \\ YF \arrow[r, "Ya"'] & YG \end{tikzcd}$$

Sometimes we write

$$a = \left(XF \xrightarrow{Xa} XG\right)_{X\in\mathrm{Ob}(\mathcal{C})} =: \left(\begin{tikzcd} X \arrow[d, "u"] & \longmapsto & XF \arrow[r, "Xa"] \arrow[d, "uF"'] & XG \arrow[d, "uG"] \\ Y & & YF \arrow[r, "Ya"'] & YG \end{tikzcd}\right)$$

for the transformation a from F to G.

Recall that in fact such a transformation may be viewed as a functor from $\mathcal{C}$ to $[\Delta_1, \mathcal{D}]$ yielding F on 0, and G on 1, respectively, where Δ_1 is the poset $\{0, 1\}$, regarded as a category.

3. For transformations $\left(F \xrightarrow{a} F'\right) \in \mathrm{Mor}([\mathcal{C}, \mathcal{D}])$ and $\left(G \xrightarrow{b} G'\right) \in \mathrm{Mor}([\mathcal{D}, \mathcal{E}])$, their *horizontal composite* is given by

$$a * b = \left(XFG \xrightarrow{X(a*b)} XF'\,G'\right)_{X\in\mathrm{Ob}(\mathcal{C})} := (aG) \blacktriangle (F'\,b) = (Fb) \blacktriangle (aG').$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\downarrow a"] & \mathcal{D} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\downarrow b"] & \mathcal{E} \end{tikzcd}$$

Note that for $X \in \mathrm{Ob}(\mathcal{C})$, we have the following commutative diagram.

$$\begin{tikzcd} XFG \arrow[r, "XFb"] \arrow[d, "XaG"'] \arrow[dr, "X(a*b)"] & XFG' \arrow[d, "XaG'"] \\ XF'\,G \arrow[r, "XF'\,b"'] & XF'\,G' \end{tikzcd}$$

Horizontal composition $(*)$ is associative:

For $\left(F \xrightarrow{a} F'\right) \in \mathrm{Mor}([\mathcal{C}, \mathcal{D}])$, $\left(G \xrightarrow{b} G'\right) \in \mathrm{Mor}([\mathcal{D}, \mathcal{E}])$, $\left(H \xrightarrow{c} H'\right) \in \mathrm{Mor}([\mathcal{E}, \mathcal{K}])$,

we have

$$\begin{aligned}(a * b) * c &= (a * b)H \blacktriangle (F'G')c \\ &= (aG \blacktriangle F'b)H \blacktriangle (F'G')c \\ &= (aGH) \blacktriangle (F'bH) \blacktriangle (F'G'c) \\ &= (aGH) \blacktriangle F'(bH \blacktriangle G'c) \\ &= a(GH) \blacktriangle F'(c * b) \\ &= a * (b * c)\,.\end{aligned}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow a"] & \mathcal{D} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow b"] & \mathcal{E} \arrow[r, bend left, "H"] \arrow[r, bend right, "H'"'] \arrow[r, phantom, "\Downarrow c"] & \mathcal{K} \end{tikzcd}$$

4. We say that $\big(F \xrightarrow{a} G\big) \in \mathrm{Mor}([\mathcal{C}, \mathcal{D}])$ is an *isotransformation* if $Xa \in \mathrm{Mor}(\mathcal{C})$ is an isomorphism for $X \in \mathrm{Ob}(\mathcal{C})$.

5. For transformations $\big(F \xrightarrow{a} F'\big)$, $\big(F' \xrightarrow{b} F''\big) \in \mathrm{Mor}([\mathcal{C}, \mathcal{D}])$, their *vertical composite* is given by

$$a \blacktriangle b := \big(XF \xrightarrow{(Xa)(Xb)} XF''\big)_{X \in \mathrm{Ob}(\mathcal{C})}\ .$$

   Vertical composition ($\blacktriangle$) is associative:

   Suppose given $\big(E \xrightarrow{a} F\big)$, $\big(F \xrightarrow{b} G\big)$, $\big(G \xrightarrow{c} H\big) \in \mathrm{Mor}([\mathcal{C}, \mathcal{D}])$.

   For $X \in \mathrm{Ob}(\mathcal{C})$, we have

$$\begin{aligned}X\big((a \blacktriangle b) \blacktriangle c\big) &= \big(X(a \blacktriangle b)\big) \blacktriangle (Xc) = \big((Xa) \blacktriangle (Xb)\big) \blacktriangle (Xc) = (Xa) \blacktriangle \big((Xb) \blacktriangle (Xc)\big) \\ &= (Xa) \blacktriangle \big(X(b \blacktriangle c)\big) = X\big(a \blacktriangle (b \blacktriangle c)\big)\,.\end{aligned}$$

0.4 Crossed modules

1. Let G and M be groups. Let $\gamma\colon G \to \mathrm{Aut}(M)$ and $f\colon M \to G$ be group morphisms.

   For $m \in M$ and $g \in G$, we write $m^g := m(g\gamma)$.

Then $V := \big(M, G, \gamma, f\big)$ is a *crossed module* if the conditions (CM1) and (CM2) are satisfied.

(CM1) For $m \in M$ and $g \in G$, we have
$$(m^g)f = (mf)^g\,.$$

(CM2) For $m,\, n \in M$, we have
$$m^{nf} = m^n\,.$$

Cf. [**15,** Def. 5].

For an example of a crossed module, cf. §A.1.

2. Let $V = \big(M, G, \gamma, f\big)$ and $W = \big(N, H, \beta, k\big)$ be crossed modules. Suppose given group morphisms $\lambda\colon M \to N$ and $\mu\colon G \to H$.

   Suppose that the following conditions (1) and (2) hold.

   (1) We have
   $$f \blacktriangle \mu = \lambda \blacktriangle k\,,$$
   i.e. the following diagram is commutative.
   $$\begin{tikzcd} M \arrow[r, "\lambda"] \arrow[d, "f"'] & N \arrow[d, "k"] \\ G \arrow[r, "\mu"'] & H \end{tikzcd}$$

   (2) For $m \in M$ and $g \in G$, we have
   $$(m^g)\lambda = (m\lambda)^{g\mu}\,.$$

   Then $\rho := (\lambda, \mu)\colon V \to W$ is a *crossed module morphism*; cf. [**15,** Def. 13].

3. The category having as objects crossed modules and as morphisms crossed module morphisms is called the *category of crossed modules*, and is denoted by $\mathcal{CRMod}$.

   For $\big(V \xrightarrow{(\lambda,\mu)} W \xrightarrow{(\tilde\lambda,\tilde\mu)} X\big)$ in $\mathcal{CRMod}$, their composite is given by
   $$(\lambda, \mu) \blacktriangle (\tilde\lambda, \tilde\mu) = (\lambda \blacktriangle \tilde\lambda, \mu \blacktriangle \tilde\mu)\,.$$

4. Let $V = \big(M, G, \gamma, f\big)$ and $W = \big(N, H, \beta, k\big)$ be crossed modules. Let $(\lambda, \mu)\colon V \to W$ be a crossed module morphism.

   We say that (λ, μ) is *injective* if the group morphisms $\lambda\colon M \to N$ and $\mu\colon G \to H$ are injective.

   We say that (λ, μ) is *surjective* if the group morphisms $\lambda\colon M \to N$ and $\mu\colon G \to H$ are surjective.

5. Let $V = \big(M, G, \gamma, f\big)$ be a crossed module. We have $Mf \trianglelefteqslant G$; cf. [**15,** Lemma 7.(2)].

   We write $V\pi_1 := \ker f$ and $V\pi_0 := G/Mf$. We have the following exact sequence of groups, where the morphism on the right hand side maps g to $g(MF)$.

$$V\pi_1 \hookrightarrow M \xrightarrow{f} G \twoheadrightarrow V\pi_0$$

6. Let $V = \big(M, G, \gamma, f\big)$ and $W = \big(N, H, k, \beta\big)$ be crossed modules.

   Suppose given a crossed module morphism $\rho := (\lambda, \mu)\colon V \to W$. We have the group morphisms

$$\rho\pi_1\colon V\pi_1 \to W\pi_1,\ m \mapsto m\lambda\,.$$

   and

$$\rho\pi_0\colon V\pi_0 \to W\pi_0,\ g(Mf) \mapsto g\mu(Nk)\,.$$

   So we have the following commutative diagram.

$$\begin{tikzcd}
V\pi_1 \arrow[r, hook] \arrow[d, dotted, "\rho\pi_1"'] & M \arrow[r, "f"] \arrow[d, "\lambda"'] & G \arrow[r, two heads] \arrow[d, "\mu"] & V\pi_0 \arrow[d, dotted, "\rho\pi_0"] \\
W\pi_1 \arrow[r, hook] & N \arrow[r, "k"'] & H \arrow[r, two heads] & W\pi_0
\end{tikzcd}$$

Chapter 1

Crossed modules and crossed categories

In [**15,** Def. 71], we have introduced the notion of a V-crossed category $\mathcal{C}$, satisfying certain properties (CC1) and (CC2), formalising the situation in which a crossed module V acts on a category $\mathcal{C}$.

In Lemma 48 below, we will construct the symmetric crossed module $\mathrm{S}_{\mathcal{C}}$ on the category $\mathcal{C}$.

Then we want to use a crossed module morphism $V \to \mathrm{S}_{\mathcal{C}}$ to formalise this situation.

But to obtain equivalent formalisations, it turned out that in [**15,** Def. 71], we missed a property.

To remedy this, we introduce the notion of a strong V-crossed category $\mathcal{C}$ in Definition 2 below, adding a property (CC3). Then we shall right away abbreviate the notion of a strong V-crossed category to just a V-category.

The result will be that to have a V-category $\mathcal{C}$ is the same as to have a crossed module morphism $V \to \mathrm{S}_{\mathcal{C}}$; cf. Proposition 57 below.

Let $V = \big(M, G, \gamma, f\big)$ be a crossed module.

Recall that a V-crossed category defined as in [**15,** Def. 71] satisfy the properties (CC1) and (CC2) for the composition of the morphisms $(\,\blacktriangle\,)$.

Reminder 1 (V-crossed sets)

Using $\gamma\colon G \to \mathrm{Aut}(M)$, we have the semidirect product $G \ltimes M$; cf. [**15,** Def. 56].

We have group morphisms

$$\begin{array}{rlll} s\colon & (G \ltimes M) \to G\,, & (g,m) & \mapsto & g\,, \\ i\colon & (G \ltimes M) \leftarrow G\,, & (g,1) & \leftarrow\!\shortmid & g\,, \\ t\colon & (G \ltimes M) \to G\,, & (g,m) & \mapsto & g \cdot mf\,. \end{array}$$

Cf. [**15,** Lem. 58].

Recall that an V-crossed set $[\![U,W]\!]_{\mathrm{set}} = \big(U,W,(\sigma,\iota,\tau)\big)$ consists of a $G \ltimes M$-set U, a G-set W and maps

$$\sigma\colon U \to W$$
$$\iota\colon U \leftarrow W$$
$$\tau\colon U \to W$$

that satisfy the properties (CS1) and (CS2).

(CS1) (i) $\iota\sigma = \mathrm{id}_W$

(ii) $\iota\tau = \mathrm{id}_W$

(CS2) (i) $\big(u \cdot (g,m)\big)\sigma = u\sigma \cdot (g,m)s \quad \forall u \in U,\ (g,m) \in G \ltimes M$

(ii) $\big(u \cdot (g,m)\big)\tau = u\tau \cdot (g,m)t \quad \forall u \in U,\ (g,m) \in G \ltimes M$

(iii) $(w \cdot g)\iota = w\iota \cdot gi \quad \forall w \in W,\ g \in G.$

Cf. [**15,** Def. 59].

Definition 2 (Strong V-crossed category)

Let $\mathcal{C} = \big(\mathrm{Mor}(\mathcal{C}), \mathrm{Ob}(\mathcal{C}), (\boldsymbol{s}, \boldsymbol{i}, \boldsymbol{t}), (\,\blacktriangle\,)\big)$ be a category together with the structure of an V-crossed set on

$$[\![\mathrm{Mor}(\mathcal{C}), \mathrm{Ob}(\mathcal{C})]\!]_{\mathrm{set}} = \big(\mathrm{Mor}(\mathcal{C}), \mathrm{Ob}(\mathcal{C}), (\boldsymbol{s}, \boldsymbol{i}, \boldsymbol{t})\big)\,.$$

We call $\mathcal{C}$ a *strong V-crossed category* or *V-category* if (CC1), (CC2) and (CC3) hold.

(CC1) For $X \xrightarrow{a} Y \xrightarrow{b} Z$ in $\mathcal{C}$ and $g \in G$, we have

$$(a \blacktriangle b) \cdot (g,1) = \big(a \cdot (g,1)\big) \blacktriangle \big(b \cdot (g,1)\big)\,.$$

(CC2) For $X \xrightarrow{a} Y \xrightarrow{b} Z$ in $\mathcal{C}$ and $m \in M$, we have

$$(a \blacktriangle b) \cdot (1, m) = a \blacktriangle \big(b \cdot (1, m)\big)\,.$$

(CC3) For $X \xrightarrow{a} Y \xrightarrow{b} Z$ in $\mathcal{C}$ and $m \in M$, we have

$$(a \blacktriangle b) \cdot (m^{-}f, m) = \big(a \cdot (m^{-}f, m)\big) \blacktriangle b\,.$$

So (CC2) treats multiplication with elements of $G \ltimes M$ in the kernel of s, whereas (CC3) treats multiplication with elements of $G \ltimes M$ in the kernel of t.

Remark 3 Let $\mathcal{C}$ be a V-crossed category. Suppose given $X \xrightarrow{a} Y \xrightarrow{b} Z$ in $\mathcal{C}$ and $m \in M$. Then, (CC3) and (CC3') are equivalent.

$$\begin{array}{rl} \text{(CC3)} & (a \blacktriangle b) \cdot (m^{-}f, m) = \big(a \cdot (m^{-}f, m)\big) \blacktriangle b \\ \text{(CC3')} & (a \blacktriangle b) \cdot (1, m) = \big(a \cdot (1, m)\big) \blacktriangle \ \big(b \cdot (mf, 1)\big) \end{array}$$

Proof. Suppose given $X \xrightarrow{a} Y \xrightarrow{b} Z$ in $\mathcal{C}$ and $m \in M$.

We have

$$\begin{array}{rlcl} & (a \blacktriangle b) \cdot (m^{-}f, m) & = & \big(a \cdot (m^{-}f, m)\big) \blacktriangle b \\ \Leftrightarrow & (a \blacktriangle b) \cdot (1, m^{mf}) & = & \big(a \cdot (m^{-}f, m) \blacktriangle b\big) \cdot (mf, 1) \\ \overset{\text{(CM2)}}{\Leftrightarrow} & (a \blacktriangle b) \cdot (1, m) & = & \big(a \cdot (m^{-}f, m) \blacktriangle b\big) \cdot (mf, 1) \\ \overset{\text{(CC1)}}{\Leftrightarrow} & (a \blacktriangle b) \cdot (1, m) & = & \big(a \cdot (1, m^{mf})\big) \blacktriangle \big(b \cdot (mf, 1)\big) \\ \overset{\text{(CM2)}}{\Leftrightarrow} & (a \blacktriangle b) \cdot (1, m) & = & \big(a \cdot (1, m)\big) \blacktriangle \big(b \cdot (mf, 1)\big)\,. \end{array}$$

□

Remark 4 Consider the V-crossed category $\mathcal{C}V = \big(G \ltimes M, G, (s, i, t), \blacktriangle\,\big)$ defined as in [**15,** Rem. 73.(0)].

From now on we shall write

$$V\mathrm{Cat} := \mathcal{C}V = \big(G \ltimes M, G, (s, i, t), (\,\blacktriangle\,)\big)\,;$$

cf. Lemma 39 below.

The composition in the category $V\mathrm{Cat}$ is given by

$$(g, m) \blacktriangle (g \cdot mf, m') = (g, mm')\,,$$

for $g \xrightarrow{(g,m)} g \cdot mf \xrightarrow{(g \cdot mf, m')} g \cdot (mm')f$ in $V\mathrm{Cat}$.

We shall revise the results of [**15,** §4.3] and verify that they remain valid in the context of strong V-crossed categories.

Remark 5 *Let* $W := \big(N, H, \beta, k\big) \leqslant V$ *be a crossed submodule; cf.* [**15,** Def. 17].

(1) *Consider the* V*-crossed category*

$$W \underset{\mathcal{C}}{\backslash\!\!\backslash} V = \big((H \ltimes N)\backslash(G \ltimes M), H\backslash G, (\bar{s}, \bar{i}, \bar{t}), \blacktriangle\,\big)\,;$$

cf. [**15,** Lem. 76].

Recall that the composition is given by

$$(H \ltimes N)(g, m) \blacktriangle (H \ltimes N)(g \cdot mf, \tilde{m}) = (H \ltimes N)(g, m\tilde{m})\,,\ \textit{for } g \in G, m, \tilde{m} \in M\,.$$

Recall that the action of $G \ltimes M$ *on* $(H \ltimes N)\backslash(G \ltimes M)$ *is given by*

$$\big((H \ltimes N)(\tilde{g}, \tilde{m})\big) \cdot (g, m) := (H \ltimes N)(\tilde{g}g, \tilde{m}^g\, m)\,,\ \textit{for } g, \tilde{g} \in G, \tilde{m}, m \in M\,,$$

and that the action of G *on* $H\backslash G$ *is given by*

$$(H\tilde{g}) \cdot g := H(\tilde{g}g)\,,\ \textit{for } g, \tilde{g} \in G\,.$$

The V*-crossed category* $W \underset{\mathcal{C}}{\backslash\!\!\backslash} V$ *is a strong* V*-crossed category.*

(2) *Consider the* V*-crossed category*

$$V\mathrm{Cat} = \big(G \ltimes M, G, (s, i, t), \blacktriangle\,\big)$$

where the action of $G \ltimes M$ *on* $G \ltimes M$ *is given by the right multiplication in* $G \ltimes M$ *and where the action of* G *on* G *is given by the right multiplication in* G*; cf.* [**15,** Rem. 73.(1)].

The V*-crossed category* $V\mathrm{Cat}$ *is a strong* V*-crossed category.*

(3) *Consider the* V*-crossed category*

$$V\mathrm{Cat} = \big(G \ltimes M, G, (s, i, t), \blacktriangle\,\big)$$

where the action of $G \ltimes M$ *on* $G \ltimes M$ *is given by the conjugation of* $G \ltimes M$ *on* $G \ltimes M$ *and where the action of* G *on* G *is given by the conjugation of* G *on* G*; cf.* [**15,** Rem. 73.(2)].

The V*-crossed category* $V\mathrm{Cat}$ *is a strong* V*-crossed category.*

(4) *Let* $\mathcal{C} = \big(\mathrm{Mor}(\mathcal{C})\,, \mathrm{Ob}(\mathcal{C})\,, (\boldsymbol{s}, \boldsymbol{i}, \boldsymbol{t}), \blacktriangle\,\big)$ *be a strong V-crossed category, i.e. the category* $\mathcal{C}$ *carries the structure of a strong V-crossed set on* $\big(\mathrm{Mor}(\mathcal{C})\,, \mathrm{Ob}(\mathcal{C})\,, (\boldsymbol{s}, \boldsymbol{i}, \boldsymbol{t})\big)$*; cf. Definition 2. Suppose given* $x \in \mathrm{Ob}(\mathcal{C})$*.*

Consider the V-crossed category

$$xV = \big((x\boldsymbol{i})(G \ltimes M), xG, (\boldsymbol{s}, \boldsymbol{i}, \boldsymbol{t}), \blacktriangle\,\big) \leqslant \mathcal{C}\,,$$

also called the orbit of x under V; cf. [**15,** Lem. 81].

The action of $G \ltimes M$ on $(x\boldsymbol{i})(G \ltimes M)$ is given by

$$\big((x\boldsymbol{i}) \cdot (\tilde{g}, \tilde{m})\big) \cdot (g, m) := (x\boldsymbol{i}) \cdot (\tilde{g}g, \tilde{m}^g\, m)\,, \ \textit{for } g, \tilde{g} \in G, m, \tilde{m} \in M\,.$$

The action of G on xG is given by

$$(x \cdot \tilde{g}) \cdot g := x \cdot (\tilde{g}g)\,, \ \textit{for } g, \tilde{g} \in G\,.$$

The V-crossed category xV is a strong V-crossed category.

Proof. Ad (1). We have only to verify the property (CC3); cf. [**15,** Lem. 76].

Suppose given $(H \ltimes N)(g, m)$, $(H \ltimes N)(\tilde{g}, \tilde{m}) \in \mathrm{Mor}\Big(W \mathbin{{}_{\mathcal{C}}\!\backslash\!\backslash} V\Big) = (H \ltimes N) \mathbin{{}_{\mathcal{C}}\!\backslash\!\backslash} (G \ltimes M)$ with

$$\big((H \ltimes M)(g, m)\big)\bar{t} = \big((H \ltimes N)(\tilde{g}, \tilde{m})\big)\bar{s}\,.$$

Then it follows that

$$H\tilde{g} = \big((H \ltimes N)(\tilde{g}, \tilde{m})\big)\bar{s} = \big((H \ltimes M)(g, m)\big)\bar{t} = H(g \cdot mf)\,.$$

So there exists some $h \in H$ such that

$$\tilde{g} = h \cdot g \cdot mf\,.$$

Therefore

$$(H \ltimes N)(\tilde{g}, \tilde{m}) = (H \ltimes N)\big((h, 1) \cdot (g \cdot mf, \tilde{m})\big) = (H \ltimes N)(g \cdot mf, \tilde{m})\,.$$

So we have $Hg \xrightarrow{(H \ltimes N)(g,m)} H(g \cdot mf) \xrightarrow{(H \ltimes N)(g \cdot mf, \tilde{m})} H\big(g \cdot (m\tilde{m}f)\big)$ in $H \mathbin{{}_{\mathcal{C}}\!\backslash\!\backslash} G$.

Suppose given $y \in M$.

Note that

$$\big((H \ltimes N)(g, m)\big) \cdot (y^- f, y) = (H \ltimes N)(g \cdot y^- f, m^{y^- f} \cdot y) \overset{\text{(CM2)}}{=} (H \ltimes N)(g \cdot y^- f, ym)\,,$$

and that

$$\big((H \ltimes N)(g \cdot y^- f, ym)\big)\bar{t} = H(g \cdot y^- f \cdot (ym)f) = H(g \cdot mf) = \big((H \ltimes N)(g \cdot mf, \tilde{m})\big)\bar{s}\,.$$

So we have

$$\begin{array}{cl} & \big((H \ltimes N)(g,m) \cdot (x^- f, x)\big) \blacktriangle (H \ltimes N)(g \cdot mf, \tilde{m}) \\ = & (H \ltimes N)(g \cdot x^- f, xm) \blacktriangle (H \ltimes N)(g \cdot mf, \tilde{m}) \\ = & (H \ltimes N)(g \cdot x^- f, xm\tilde{m}) \\ \overset{\text{(CM2)}}{=} & (H \ltimes N)(g \cdot x^- f, (m\tilde{m})^{x^- f} \cdot x) \\ = & (H \ltimes N)(g, m\tilde{m}) \cdot (x^- f, x) \\ = & \big((H \ltimes N)(g,m) \blacktriangle (H \ltimes N)(g \cdot mf, \tilde{m})\big) \cdot (x^- f, x)\,. \end{array}$$

This shows (CC3).

Ad (2). This follows from (1) with $W = 1$.

Ad (3). We only have to verify the property (CC3); cf. [**15,** Rem. 73.(2)].

Recall that conjugation in $G \ltimes M$ is denoted by $(\star)$, i.e. for $(g,m), (\tilde{g}, \tilde{m}) \in G \ltimes M$ we write

$$(g,m) \star (\tilde{g}, \tilde{m}) := (\tilde{g}, \tilde{m})^- \cdot (g,m) \cdot (\tilde{g}, \tilde{m})\,.$$

Suppose given $g \xrightarrow{(g,m)} g \cdot mf \xrightarrow{(g \cdot mf, \tilde{m})} g \cdot (m\tilde{m})f$ in $V\mathrm{Cat}$ and suppose given $y \in M$.

Note that

$$\begin{array}{rcl} (g,m) \star (y^- f, y) & = & (yf, y^-) \cdot (g,m) \cdot (y^- f, y) \\ & = & \Big(yf \cdot g \cdot y^- f, \big((y^-)^g \cdot m\big)^{y^- f} \cdot y\Big) \\ & \overset{\text{(CM2)}}{=} & \big(yf \cdot g \cdot y^- f, y \cdot (y^-)^g \cdot m\big)\,, \end{array}$$

and that

$$\begin{aligned}\big(yf \cdot g \cdot y^- f, y \cdot (y^-)^g \cdot m\big)t &= yf \cdot g \cdot \big((y^-)^g\big) f \cdot mf \\ &\overset{\text{(CM1)}}{=} xf \cdot g \cdot g^- \cdot (x^- f) \cdot g \cdot mf \\ &= g \cdot mf \\ &= (g \cdot mf, \tilde{m})s\,.\end{aligned}$$

So we have

$$\begin{aligned}\big((g,m) \star (y^- f, y)\big) \blacktriangle (g \cdot mf, \tilde{m}) &= \big(yf \cdot g \cdot y^- f, y \cdot (y^-)^g \cdot m\big) \blacktriangle (g \cdot mf, \tilde{m}) \\ &= \big(yf \cdot g \cdot y^- f, y \cdot (y^-)^g \cdot m\tilde{m}\big) \\ &= \big(yf \cdot g \cdot y^- f, y \cdot (y^-)^g \cdot m\tilde{m} \cdot y^- \cdot y\big) \\ &\overset{\text{(CM2)}}{=} \Big(yf \cdot g \cdot y^- f, \big((y^-)^g \cdot m\tilde{m}\big)^{y^- f} \cdot y\Big) \\ &= (yf, y^-) \cdot (g, m\tilde{m}) \cdot (y^- f, y) \\ &= \big((g,m) \blacktriangle (g \cdot mf, \tilde{m})\big) \star (x^- f, x)\,.\end{aligned}$$

This shows (CC3).

Ad (4). We only have to verify the property (CC3); cf. [**15,** Lem. 81].

Suppose given $(x\boldsymbol{i}) \cdot (g,m)$, $(x\boldsymbol{i}) \cdot (\tilde{g},\tilde{m}) \in \operatorname{Mor}(xV) = (x\boldsymbol{i})(G \ltimes M)$ with

$$\big((x\boldsymbol{i}) \cdot (g,m)\big)\boldsymbol{t} = \big((x\boldsymbol{i}) \cdot (\tilde{g},\tilde{m})\big)\boldsymbol{s}\,.$$

Then it follows that

$$\begin{aligned}x \cdot \tilde{g} &\overset{\text{(CS1)}}{=} x\boldsymbol{i}\boldsymbol{s} \cdot (\tilde{g},\tilde{m})s \\ &\overset{\text{(CS2)}}{=} \big((x\boldsymbol{i}) \cdot (\tilde{g},\tilde{m})\big)\boldsymbol{s} \\ &= \big((x\boldsymbol{i}) \cdot (g,m)\big)\boldsymbol{t} \\ &\overset{\text{(CS2)}}{=} x\boldsymbol{i}\boldsymbol{t} \cdot (g,m)t \\ &\overset{\text{(CS1)}}{=} x \cdot (g \cdot mf)\,.\end{aligned}$$

So $x = x \cdot (g \cdot mf \cdot \tilde{g}^-)$.

Therefore, we have

$$\begin{aligned}(x\boldsymbol{i}) \cdot (\tilde{g},\tilde{m}) &= \Big(\big(x \cdot (g \cdot mf \cdot \tilde{g}^-)\big)\boldsymbol{i}\Big) \cdot (\tilde{g},\tilde{m}) \\ &\overset{\text{(CS2)}}{=} \big((x\boldsymbol{i}) \cdot (g \cdot mf \cdot \tilde{g}^-, 1)\big) \cdot (\tilde{g},\tilde{m}) \\ &= (x\boldsymbol{i}) \cdot (g \cdot mf, \tilde{m})\,.\end{aligned}$$

So we have $x \cdot g \xrightarrow{(x\boldsymbol{i})\cdot(g,m)} x \cdot (g \cdot mf) \xrightarrow{(x\boldsymbol{i})\cdot(g\cdot mf,\tilde{m})} x \cdot \big(g \cdot (m\tilde{m})f\big)$ in xV.

Suppose given $y \in M$.

Note that

$$\big((x\boldsymbol{i}) \cdot (g,m)\big) \cdot (y^{-}f, y) = (x\boldsymbol{i}) \cdot (g \cdot y^{-}f, m^{y^{-}f} \cdot y) \overset{\text{(CM1)}}{=} (x\boldsymbol{i}) \cdot (g \cdot y^{-}f, y \cdot m)\,,$$

and that

$$\begin{array}{rcl} \big((x\boldsymbol{i}) \cdot (g \cdot y^{-}f, y \cdot m)\big)\boldsymbol{t} & \overset{\text{(CS2)}}{=} & x\boldsymbol{i}\boldsymbol{t} \cdot (g \cdot y^{-}f, y \cdot m)t \\ & \overset{\text{(CS1)}}{=} & x \cdot (g \cdot y^{-}f \cdot yf \cdot mf) \\ & = & x \cdot (g \cdot mf) \\ & \overset{\text{(CS1)}}{=} & x\boldsymbol{i}\boldsymbol{s} \cdot (g \cdot mf, \tilde{m})s \\ & \overset{\text{(CS2)}}{=} & \big((x\boldsymbol{i}) \cdot (g \cdot mf, \tilde{m})\big)\boldsymbol{s}\,. \end{array}$$

So we have

$$\begin{array}{rl} & \Big(\big((x\boldsymbol{i}) \cdot (g,m)\big) \cdot (y^{-}f, y)\Big) \blacktriangle \big((x\boldsymbol{i}) \cdot (g \cdot mf, \tilde{m})\big) \\ = & \big((x\boldsymbol{i}) \cdot (g \cdot y^{-}f, y \cdot m)\big) \blacktriangle \big((x\boldsymbol{i})(g \cdot mf, \tilde{m})\big) \\ = & (x\boldsymbol{i}) \cdot (g \cdot y^{-}f, y \cdot m \cdot \tilde{m}) \\ \overset{\text{(CM2)}}{=} & (x\boldsymbol{i}) \cdot (g \cdot y^{-}f, (m\tilde{m})^{y^{-}f} \cdot y) \\ = & \big((x\boldsymbol{i}) \cdot (g, m\tilde{m})\big) \cdot (y^{-}f, y) \\ = & \Big(\big((x\boldsymbol{i}) \cdot (g,m)\big) \blacktriangle \big((x\boldsymbol{i}) \cdot (g \cdot mf, \tilde{m})\big)\Big) \cdot (y^{-}f, y)\,. \end{array}$$

This shows (CC3). □

Remark 6 In the proof of Remark 5.(4), there was no need to assume (CC3) for $\mathcal{C}$. But in [**15,** Lem. 81], (CS2) is used.

The assumptions on $\mathcal{C}$ made in Remark 5.(4) thus is not the most general possible. So the assertion remains valid if $\mathcal{C}$ is assumed to be a only V-crossed category.

Remark 7 *Let $\mathcal{C}$ be a strong V-crossed category.*

Suppose given an V-crossed subcategory $\mathcal{D} \leqslant \mathcal{C}$.

Then, $\mathcal{D}$ is a strong V-crossed category.

Proof. We only have to verify the property (CC3); cf. [**15,** Rem. 75].

We have $\big(\operatorname{Mor}(\mathcal{D}), \operatorname{Ob}(\mathcal{D}), (\underline{s}, \underline{i}, \underline{t}), \blacktriangle\big) \leqslant \big(\operatorname{Mor}(\mathcal{C}), \operatorname{Ob}(\mathcal{C}), (s, i, t), \blacktriangle\big)$.

So, for $u, v \in \operatorname{Mor}(\mathcal{D}) \subseteq \operatorname{Mor}(\mathcal{C})$, and $m \in M$, we have

$$(u \blacktriangle v)(m^- f, m) = \big(u \cdot (m^- f, m)\big) \blacktriangle v\,.$$

This shows (CC3). $\square$

Reminder 8 (V-crossed category morphism)

Suppose given V-crossed categories

$$\begin{aligned} \mathcal{C} &= \big(\operatorname{Mor}(\mathcal{C}), \operatorname{Ob}(\mathcal{C}), (s, i, t), \blacktriangle\big) \\ \mathcal{D} &= \big(\operatorname{Mor}(\mathcal{D}), \operatorname{Ob}(\mathcal{D}), (\underline{s}, \underline{i}, \underline{t}), (\blacktriangle)\big)\,. \end{aligned}$$

Suppose given maps $\zeta\colon \operatorname{Mor}(\mathcal{C}) \to \operatorname{Mor}(\mathcal{D})$ and $\eta\colon \operatorname{Ob}(\mathcal{C}) \to \operatorname{Ob}(\mathcal{D})$.

We say that $(\zeta, \eta)\colon \mathcal{C} \to \mathcal{D}$ is a *V-crossed category morphism* if the properties (1-6) are satisfied; cf. [**15,** Def. 64, 77].

(1) We have $s \blacktriangle \eta = \zeta \blacktriangle \underline{s} : \operatorname{Mor}(\mathcal{C}) \to \operatorname{Ob}(\mathcal{D})$.

(2) We have $i \blacktriangle \zeta = \eta \blacktriangle \underline{i} : \operatorname{Ob}(\mathcal{C}) \to \operatorname{Mor}(\mathcal{D})$.

(3) We have $t \blacktriangle \eta = \zeta \blacktriangle \underline{t} : \operatorname{Ob}(\mathcal{C}) \to \operatorname{Mor}(\mathcal{D})$.

(4) For $u \in \operatorname{Mor}(\mathcal{D})$, $(g, m) \in G \ltimes M$, we have $\big(u \cdot (g, m)\big)\zeta = u\zeta \cdot (g, m)$.

(5) For $X \in \operatorname{Ob}(\mathcal{C})$, $g \in G$, we have $(X \cdot g)\eta = X\eta \cdot g$.

(6) For $X \xrightarrow{u} Y \xrightarrow{v} Z$ in $\mathcal{C}$, we have $(u \blacktriangle v)\zeta = u\zeta \blacktriangle v\zeta$.

In particular, $(\zeta, \eta)\colon \mathcal{C} \to \mathcal{D}$ is a functor.

Given V-crossed categories $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ and V-crossed category morphisms $(\zeta, \eta)\colon \mathcal{C} \to \mathcal{D}$ and $(\zeta', \eta')\colon \mathcal{D} \to \mathcal{E}$, we let

$$(\zeta, \eta) \blacktriangle (\zeta', \eta') := (\zeta \blacktriangle \zeta', \eta \blacktriangle \eta')\,.$$

Definition 9 (The category of V-categories)

The *category of strong V-crossed categories* is the full subcategory of the category of V-crossed categories having as objects the strong V-crossed categories.

The category of strong V-crossed categories is also called the *category of V-categories*.

Proposition 10 (Orbit Lemma for strong V-crossed categories)
Let $\mathcal{C}$ be a strong V-crossed category. Suppose given $w \in \mathrm{Ob}(\mathcal{C})$.

Let
$$N_{\mathrm{C}}(w) = \{m \in M : (w\boldsymbol{i}) \cdot (1, m) = w\boldsymbol{i}\} \quad \text{and} \quad H_{\mathrm{C}}(w) = \{g \in G : w \cdot g = w\}\,.$$

Consider the centralizer $\mathrm{C}_V(w) = [\![N_{\mathrm{C}}(w), H_{\mathrm{C}}(w)]\!]$ of w in V. Recall that $\mathrm{C}_V(w) \leqslant V$ is a crossed submodule of V; cf. [**15,** Lem. 69.(2)].

Consider the strong V-crossed category $\mathrm{C}_V(w) \,{}_{\mathcal{C}}\!\!\setminus\!\!\setminus V$; cf. Remark 5.(1). *Consider the strong V-crossed category wV; cf. Remark 5.*(4).

Then we have an isomorphism in the category of strong V-crossed categories given by

$$(\zeta, \eta) \colon \quad \mathrm{C}_V(w) \,{}_{\mathcal{C}}\!\!\setminus\!\!\setminus V \quad \longrightarrow \quad wV\ ,$$

where
$$\begin{array}{rcrcl} \zeta & : & \big(\mathrm{C}_{G \ltimes M}(w\boldsymbol{i})\big)\backslash(G \ltimes M) & \longrightarrow & (w\boldsymbol{i})(G \ltimes M) \\ & & \big(\mathrm{C}_{G \ltimes M}(w\boldsymbol{i})\big)(g, m) & \longmapsto & (w\boldsymbol{i}) \cdot (g, m) \end{array}$$

and
$$\begin{array}{rcrcl} \eta & : & \mathrm{C}_G(w)\backslash G & \longrightarrow & wG \\ & & \big(\mathrm{C}_G(w)\big)g & \longmapsto & w \cdot g\ . \end{array}$$

Proof. By [**15,** Prop. 82], (ζ, η) is a V-crossed category isomorphism. By Remark 5.(1,4), $\mathrm{C}_V(w) \,{}_{\mathcal{C}}\!\!\setminus\!\!\setminus V$ and wV are strong V-crossed categories. □

Remark 11 *Let W be a crossed module. Let $(\lambda, \mu) \colon V \to W$ be a crossed module morphism. Then, (λ, μ) is injective if and only if $\ker(\lambda, \mu) = 1$.*

Proof. We may conclude as follows.
$$\begin{aligned} (\lambda, \mu) \text{ is injective } &\Leftrightarrow \lambda,\ \mu \text{ are injective group morphisms} \\ &\Leftrightarrow \ker \lambda = 1 \text{ and } \ker \mu = 1 \\ &\Leftrightarrow \ker\,(\lambda, \mu) = 1\,. \end{aligned}$$

□

Chapter 2

Crossed modules and invertible monoidal categories

Let $\mathcal{C} = \big(\operatorname{Mor}(\mathcal{C})\,,\operatorname{Ob}(\mathcal{C})\,,(s_{\mathcal{C}}, i_{\mathcal{C}}, t_{\mathcal{C}}), \blacktriangle\big)$ and $\mathcal{D} = \big(\operatorname{Mor}(\mathcal{D})\,,\operatorname{Ob}(\mathcal{D})\,,(s_{\mathcal{D}}, i_{\mathcal{D}}, i_{\mathcal{D}}), \blacktriangle\big)$ be categories.

If unambiguous, we write

$$
\begin{aligned}
\big(\operatorname{Mor}(\mathcal{C})\,,\operatorname{Ob}(\mathcal{C})\,,(s, i, t), \blacktriangle\big) &:= \big(\operatorname{Mor}(\mathcal{C})\,,\operatorname{Ob}(\mathcal{C})\,,(s_{\mathcal{C}}, i_{\mathcal{C}}, t_{\mathcal{C}}), \blacktriangle\big)\\
\big(\operatorname{Mor}(\mathcal{D})\,,\operatorname{Ob}(\mathcal{D})\,,(s, i, t), \blacktriangle\big) &:= \big(\operatorname{Mor}(\mathcal{D})\,,\operatorname{Ob}(\mathcal{D})\,,(s_{\mathcal{D}}, i_{\mathcal{D}}, i_{\mathcal{D}}), \blacktriangle\big)\,.
\end{aligned}
$$

2.1 Monoidal categories

Definition 12 (Monoidal category) Suppose we have a functor

$$
\begin{array}{rccccl}
(\otimes)\colon & \mathcal{C} \times \mathcal{C} & \longrightarrow & \mathcal{C} & \\
 & (\ X\ ,\ Y\ ) & \longmapsto & X \otimes Y & \text{for } X, Y \in \operatorname{Ob}(\mathcal{C}) \\
 & \left(\begin{array}{c} X \\ \downarrow a \\ X' \end{array},\ \begin{array}{c} Y \\ \downarrow b \\ Y' \end{array}\right) & \longmapsto & \left(\begin{array}{c} X\otimes Y \\ \downarrow a\otimes b \\ X'\otimes Y' \end{array}\right) & \text{for } a, b \in \operatorname{Mor}(\mathcal{C})
\end{array}
$$

and an object $I \in \operatorname{Ob}(\mathcal{C})$ such that the following conditions (1) and (2) hold.

(1) For $X \in \operatorname{Ob}(\mathcal{C})$, $u \in \operatorname{Mor}(\mathcal{C})$, we have

$$X \otimes I = X = I \otimes X \quad \text{and} \quad u \otimes \mathrm{id}_I = u = \mathrm{id}_I \otimes u\,.$$

(2) We have
$$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z) \quad \text{for } X, Y, Z \in \mathrm{Ob}(\mathcal{C}) \ ,$$
and
$$(a \otimes b) \otimes c = a \otimes (b \otimes c) \quad \text{for } a, b, c \in \mathrm{Mor}(\mathcal{C}) \ .$$

Then we call $(\mathcal{C}, I, \otimes)$ a *monoidal category.* ([2])

The functor $(\otimes)$ is called its *tensor product.*

Further, for $a \in \mathrm{Mor}(\mathcal{C})$ and $X \in \mathrm{Ob}(\mathcal{C})$, we shall often write
$$a \otimes X := a \otimes X\boldsymbol{i} = a \otimes \mathrm{id}_X$$
and
$$X \otimes a := X\boldsymbol{i} \otimes a = \mathrm{id}_X \otimes a \, .$$

Remark 13 Suppose given $X \xrightarrow{a} Y \xrightarrow{b} Z$ and $X' \xrightarrow{a'} Y' \xrightarrow{b'} Z'$ in $\mathcal{C}$.

Functoriality of $\otimes$ in Definition 12 means that we have
$$(a \blacktriangle b) \otimes (a' \blacktriangle b') = (a \otimes a') \blacktriangle (b \otimes b'), \quad \text{and} \quad \mathrm{id}_X \otimes \mathrm{id}_Y = \mathrm{id}_{X \otimes Y} \, .$$

We also have
$$(a \otimes b)\boldsymbol{s} = a\boldsymbol{s} \otimes b\boldsymbol{s} \ , \ (a \otimes b)\boldsymbol{t} = a\boldsymbol{t} \otimes b\boldsymbol{t} \quad \text{and} \quad (X \otimes Y)\boldsymbol{i} = X\boldsymbol{i} \otimes Y\boldsymbol{i} \, .$$

Remark 14 *Suppose given a category $\mathcal{C}$ together with a functor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ such that* (1, 2) *hold.*

(1) *We have an object $I \in \mathrm{Ob}(\mathcal{C})$ such that*
$$u \otimes \mathrm{id}_I = u = \mathrm{id}_I \otimes u \quad \textit{for } u \in \mathrm{Mor}(\mathcal{C}) \ .$$

(2) *We have*
$$(a \otimes b) \otimes c = a \otimes (b \otimes c) \quad \textit{for } a, b, c \in \mathrm{Mor}(\mathcal{C}) \ .$$

Then, $(\mathcal{C}, I, \otimes)$ is a monoidal category.

So to show that a category $\mathcal{C}$ is a monoidal category, it suffices to show that the morphisms possess the required properties.

[2] In the literature, monoidal categories are often defined as involving compatibility isomorphisms. We demand these compatibility isomorphism to be identities. So our notion of monoidal categories is the strict version.

Proof. Suppose given $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$.

We have

$$\begin{aligned} X \otimes I &= (\mathrm{id}_X\, s) \otimes (\mathrm{id}_I\, s) = (\mathrm{id}_X \otimes \mathrm{id}_I)s = (\mathrm{id}_X)s = X \\ I \otimes X &= (\mathrm{id}_I\, s) \otimes (\mathrm{id}_X\, s) = (\mathrm{id}_I \otimes \mathrm{id}_X)s = (\mathrm{id}_X)s = X\,. \end{aligned}$$

Further, we have

$$\begin{aligned} (X \otimes Y) \otimes Z &= \big((\mathrm{id}_X\, s) \otimes (\mathrm{id}_Y\, s)\big) \otimes (\mathrm{id}_Z\, s) = \big((\mathrm{id}_X \otimes \mathrm{id}_Y) \otimes \mathrm{id}_Z\big)s \\ &= \big(\mathrm{id}_X \otimes (\mathrm{id}_Y \otimes \mathrm{id}_Z)\big)s = (\mathrm{id}_X\, s) \otimes \big((\mathrm{id}_Y\, s) \otimes (\mathrm{id}_Z\, s)\big) \\ &= X \otimes (Y \otimes Z)\,. \end{aligned}$$

□

Remark 15 (Unit object) Suppose given a monoidal category $(\mathcal{C}, I, \otimes)$.

Suppose we have an object $\tilde{I} \in \mathrm{Ob}(\mathcal{C})$ such that $X \otimes \tilde{I} = X = \tilde{I} \otimes X$ holds for $X \in \mathrm{Ob}(\mathcal{C})$.

Then, $I = \tilde{I}$.

So the object I is uniquely determined by its property (1) in Definition 12. We call I the *unit object* of $\mathcal{C}$.

Proof. We have $\quad I = I \otimes \tilde{I} = \tilde{I}\,.$ □

Definition 16 (Monoidal subcategory)

Suppose given monoidal categories $(\mathcal{C}, I, \otimes)$ and $(\mathcal{D}, \tilde{I}, \tilde{\otimes})$.

We say that $(\mathcal{D}, \tilde{I}, \tilde{\otimes})$ is a *monoidal subcategory* of $(\mathcal{C}, I, \otimes)$ if the conditions $(1, 2, 3, 4)$ hold.

(1) The category $\mathcal{D}$ is a subcategory of $\mathcal{C}$.

(2) We have $I = \tilde{I}$.

(3) For $X, Y \in \mathrm{Ob}(\mathcal{D}) \subseteq \mathrm{Ob}(\mathcal{C})$ we have

$$X \otimes Y = X \,\tilde{\otimes}\, Y\,.$$

(4) For $u, v \in \mathrm{Mor}(\mathcal{D}) \subseteq \mathrm{Mor}(\mathcal{C})$ we have

$$u \otimes v = u \,\tilde{\otimes}\, v\,.$$

Then we often write $(\mathcal{D}, I, \otimes) := (\mathcal{D}, \tilde{I}, \tilde{\otimes})$.

We then often just say that $\mathcal{D}$ is a monoidal subcategory of $\mathcal{C} = (\mathcal{C}, I, \otimes)$, using the fact that there $\tilde{I}$ and $\tilde{\otimes}$ are uniquely determined by $\mathcal{D}$.

Lemma 17 *Let $\mathcal{C} = (\mathcal{C}, I, \otimes)$ be a monoidal category. Suppose given a subcategory $\mathcal{D} \subseteq \mathcal{C}$. Suppose that the conditions $(1, 2)$ hold.*

(1) *We have $I \in \mathrm{Ob}(\mathcal{D})$.*

(2) *For $u, v \in \mathrm{Mor}(\mathcal{D})$ we have $u \otimes v \in \mathrm{Mor}(\mathcal{D})$.*

Then $\mathcal{D}$ is a monoidal subcategory of $\mathcal{C}$.

Proof. Suppose given X, $Y \in \mathrm{Ob}(\mathcal{D})$.

We have

$$\mathrm{id}_{X \otimes Y} = \mathrm{id}_X \otimes \mathrm{id}_Y \overset{(2)}{\in} \mathrm{Mor}(\mathcal{D}) \ .$$

This shows $X \otimes Y \in \mathrm{Ob}(\mathcal{D})$.

Let $\tilde{\otimes} := \otimes\big|_{\mathcal{D} \times \mathcal{D}}^{\mathcal{D}}$.

We show that $(\mathcal{D}, I, \tilde{\otimes})$ is a monoidal category.

For $u \in \mathrm{Mor}(\mathcal{D})$, we have

$$u \,\tilde{\otimes}\, \mathrm{id}_I = u \otimes \mathrm{id}_I = u = \mathrm{id}_I \otimes u = \mathrm{id}_I \,\tilde{\otimes}\, u \, .$$

For a, b, $c \in \mathrm{Mor}(\mathcal{D})$, we have

$$(a \,\tilde{\otimes}\, b) \,\tilde{\otimes}\, c = (a \otimes b) \otimes c = a \otimes (b \otimes c) = a \,\tilde{\otimes}\, (b \,\tilde{\otimes}\, c) \, .$$

So, by Remark 14, $(\mathcal{D}, I, \tilde{\otimes})$ is a monoidal category.

Properties (3) and (4) in Definition 16 hold by construction of $\tilde{\otimes}$.

So $\mathcal{D}$ is a monoidal subcategory of $\mathcal{C}$. □

Corollary 18 *Let $(\mathcal{C}, I, \otimes)$ be a monoidal category. Let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory. Then $\mathcal{D}$ is a monoidal subcategory of $\mathcal{C}$ if and only if $(1, 2)$ hold.*

(1) *We have $I \in \mathrm{Ob}(\mathcal{D})$.*

(2) *For* $X,\, Y \in \mathrm{Ob}(\mathcal{D})$, *we have* $X \otimes Y \in \mathrm{Ob}(\mathcal{D})$.

Proof. Ad $\Rightarrow$. Suppose that $\mathcal{D}$ is a monoidal subcategory of $\mathcal{C}$.

Then, in particular, we have $I \in \mathrm{Ob}(\mathcal{D})$ and we have $X \otimes Y \in \mathrm{Ob}(\mathcal{D})$, for $X,\, Y \in \mathrm{Ob}(\mathcal{D})$.

Ad $\Leftarrow$. Suppose that conditions $(1, 2)$ hold.

So in particular, we have $I \in \mathrm{Ob}(\mathcal{D})$. Moreover, for $\big(X \xrightarrow{u} X'\big)$, $\big(Y \xrightarrow{v} Y'\big) \in \mathrm{Mor}(\mathcal{D})$, we have $X \otimes Y \in \mathrm{Ob}(\mathcal{D})$ and $X' \otimes Y' \in \mathrm{Ob}(\mathcal{D})$. Since $\mathcal{D}$ is a full subcategory we also have $\big(X \otimes Y \xrightarrow{u \otimes v} X' \otimes Y'\big) \in \mathrm{Mor}(\mathcal{D})$.

Therefore, by Lemma 17, $\mathcal{D}$ is a monoidal subcategory of $\mathcal{C}$. $\square$

Definition 19 (Tensor invertibility) Let $(\mathcal{C}, I, \otimes)$ be a monoidal category.

(1) We say that an object $X \in \mathrm{Ob}(\mathcal{C})$ is *tensor invertible* if there exists an object $Y \in \mathrm{Ob}(\mathcal{C})$ such that

$$X \otimes Y = I = Y \otimes X$$

holds.

(2) We say that a morphism $u \in \mathrm{Mor}(\mathcal{C})$ is *tensor invertible* if there exists a morphism $v \in \mathrm{Mor}(\mathcal{C})$ such that

$$u \otimes v = \mathrm{id}_I = v \otimes u$$

holds.

Remark 20 (Tensor inverses) Let $(\mathcal{C}, I, \otimes)$ be a monoidal category.

(1) Suppose given $X \in \mathrm{Ob}(\mathcal{C})$. Suppose we have objects $Y,\, \tilde{Y} \in \mathrm{Ob}(\mathcal{C})$ such that

$$X \otimes Y = I = Y \otimes X \quad \text{and} \quad X \otimes \tilde{Y} = I = \tilde{Y} \otimes X$$

holds. Then $Y = \tilde{Y}$.

We write $X^{\otimes -} := Y$ and call $X^{\otimes -}$ the *tensor inverse* of X in $\mathrm{Ob}(\mathcal{C})$.

(2) Suppose given $u \in \mathrm{Mor}(\mathcal{C})$. Suppose we have morphisms $v,\, \tilde{v} \in \mathrm{Mor}(\mathcal{C})$ such that

$$u \otimes v = \mathrm{id}_I = v \otimes u \quad \text{and} \quad u \otimes \tilde{v} = \mathrm{id}_I = \tilde{v} \otimes u$$

holds. Then $v = \tilde{v}$.

We write $u^{\otimes -} := v$ and call $u^{\otimes -}$ the *tensor inverse* of u in $\mathrm{Mor}(\mathcal{C})$.

(3) The unit object I is tensor invertible. We have $I^{\otimes-} = I$.

(4) Suppose given $X, Y \in \mathrm{Ob}(\mathcal{C})$. Suppose that X and Y are tensor invertible.
Then, $X \otimes Y$ is tensor invertible, and we have

$$(X \otimes Y)^{\otimes-} = Y^{\otimes-} \otimes X^{\otimes-} .$$

(5) Suppose given $u, v \in \mathrm{Mor}(\mathcal{C})$. Suppose that u and v are tensor invertible.
Then $u \otimes v$ is tensor invertible, and we have

$$(u \otimes v)^{\otimes-} = v^{\otimes-} \otimes u^{\otimes-} .$$

(6) Suppose given $X \in \mathrm{Ob}(\mathcal{C})$. Suppose that X is tensor invertible.
Then $X^{\otimes-}$ is tensor invertible, and we have

$$(X^{\otimes-})^{\otimes-} = X .$$

(7) Suppose given $u \in \mathrm{Mor}(\mathcal{C})$. Suppose that u is tensor invertible.
Then $u^{\otimes-}$ is tensor invertible, and we have

$$(u^{\otimes-})^{\otimes-} = u .$$

(8) Suppose given $\big(X \xrightarrow{u} Y\big) \in \mathrm{Mor}(\mathcal{C})$. Suppose that u is tensor invertible.
Then X and Y are tensor invertible, and we have $\big(X^{\otimes-} \xrightarrow{u^{\otimes-}} Y^{\otimes-}\big)$, i.e.

$$(u^{\otimes-})\boldsymbol{s} = (u\boldsymbol{s})^{\otimes-} \text{ and } (u^{\otimes-})\boldsymbol{t} = (u\boldsymbol{t})^{\otimes-} .$$

(9) Suppose given $X \in \mathrm{Ob}(\mathcal{C})$.
Then X is tensor invertible if and only if id_X is tensor invertible.
In this case, we have $\mathrm{id}_{X^{\otimes-}} = (\mathrm{id}_X)^{\otimes-}$.

(10) Suppose given $(X \xrightarrow{u} Y \xrightarrow{v} Z)$ in $\mathcal{C}$ such that u and v are tensor invertible.
Then $u \blacktriangle v$ is tensor invertible and we have

$$(u \blacktriangle v)^{\otimes-} = u^{\otimes-} \blacktriangle v^{\otimes-} \colon X^{\otimes-} \to Z^{\otimes-} .$$

Proof. Ad (1). We have

$$Y = I \otimes Y = \tilde{Y} \otimes X \otimes Y = \tilde{Y} \otimes I = \tilde{Y}\,.$$

Ad (2). We have

$$v = \mathrm{id}_I \otimes v = \tilde{v} \otimes u \otimes v = \tilde{v} \otimes \mathrm{id}_I = \tilde{v}\,.$$

Ad (3). We have

$$I \otimes I = I\,.$$

Ad (4). We have

$$(X \otimes Y) \otimes (Y^{\otimes -} \otimes X^{\otimes -}) = I = (Y^{\otimes -} \otimes X^{\otimes -}) \otimes (X \otimes Y)\,.$$

Ad (5). We have

$$(u \otimes v) \otimes (v^{\otimes -} \otimes u^{\otimes -}) = \mathrm{id}_I = (v^{\otimes -} \otimes u^{\otimes -}) \otimes (u \otimes v)\,.$$

Ad (6). We have

$$X^{\otimes -} \otimes X = I = X \otimes X^{\otimes -}\,.$$

By (1), we have $(X^{\otimes -})^{\otimes -} = X$.

Ad (7). We have

$$u^{\otimes -} \otimes u = \mathrm{id}_I = u \otimes u^{\otimes -}\,.$$

By (2), we have $(u^{\otimes -})^{\otimes -} = u$.

Ad (8). Consider $\big(X' \xrightarrow{u^{\otimes -}} Y'\big)$.

We have

$$X \otimes X' = u\boldsymbol{s} \otimes (u^{\otimes -})\boldsymbol{s} = (u \otimes u^{\otimes -})\boldsymbol{s} = (\mathrm{id}_I)\boldsymbol{s} = I\,.$$

Similarly, we have $X' \otimes X = I$.

This shows $X' = X^{\otimes -}$.

We have

$$Y \otimes Y' = u\boldsymbol{t} \otimes (u^{\otimes -})\boldsymbol{t} = (u \otimes u^{\otimes -})\boldsymbol{t} = (\mathrm{id}_I)\boldsymbol{t} = I\,.$$

Similarly, we have $Y' \otimes Y = I$.

This shows $Y' = Y^{\otimes -}$.

Ad (9). Ad $\Rightarrow$. Suppose that X is tensor invertible.

We have

$$\mathrm{id}_X \otimes \mathrm{id}_{X^{\otimes -}} = \mathrm{id}_{X \otimes X^{\otimes -}} = \mathrm{id}_I \, .$$

Similarly, we have $\mathrm{id}_{X^{\otimes -}} \otimes \mathrm{id}_X = \mathrm{id}_I$.

This shows $(\mathrm{id}_X)^{\otimes -} = \mathrm{id}_{X^{\otimes -}}$.

Ad $\Leftarrow$. Suppose that id_X is tensor invertible. Then, by (8), X is tensor invertible.

Ad (10). We have

$$(u \blacktriangle v) \otimes (u^{\otimes -} \blacktriangle v^{\otimes -}) = (u \otimes u^{\otimes -}) \blacktriangle (v \otimes v^{\otimes -}) = \mathrm{id}_I \blacktriangle \mathrm{id}_I = \mathrm{id}_I \, .$$

Similarly, we have $(u^{\otimes -} \blacktriangle v^{\otimes -}) \otimes (u \blacktriangle v) = \mathrm{id}_I$.

This shows $(u \blacktriangle v)^{\otimes -} = u^{\otimes -} \blacktriangle v^{\otimes -}$. □

Definition 21 (Invertible monoidal category)

Let $(\mathcal{C}, I, \otimes)$ be a monoidal category. Suppose that the following conditions $(1, 2)$ hold.

(1) Each $X \in \mathrm{Ob}(\mathcal{C})$ is tensor invertible; cf. Definition 19.(1).

(2) Each $a \in \mathrm{Mor}(\mathcal{C})$ is tensor invertible; cf. Definition 19.(2).

Then, we call $(\mathcal{C}, I, \otimes)$ an *invertible monoidal category.* ([3])

Remark 22 *Suppose given a monoidal category* $(\mathcal{C}, I, \otimes)$. *Suppose that property* (2) *from Definition* 21 *holds for* $\mathcal{C}$.

Then $(\mathcal{C}, I, \otimes)$ *is an invertible monoidal category.*

So condition (1) *in Definition* 21 *may be dropped without changing the definition.*

Proof. Suppose given $X \in \mathrm{Ob}(\mathcal{C})$.

For $\mathrm{id}_X \in \mathrm{Mor}(\mathcal{C})$ there exists a morphism $\big(Y \xrightarrow{b} Z\big) \in \mathrm{Mor}(\mathcal{C})$ such that we have

$$\mathrm{id}_X \otimes b = \mathrm{id}_I = b \otimes \mathrm{id}_X \, .$$

[3] In the literature, an invertible monoidal category is also called a *categorical group* or a *category in groups*. Cf. Remark 23 below.

So we have

$$X \otimes Y = (\mathrm{id}_X \boldsymbol{s}) \otimes b\boldsymbol{s} = (\mathrm{id}_X \otimes b)\boldsymbol{s} = (\mathrm{id}_I)\boldsymbol{s} = I$$
$$Y \otimes X = b\boldsymbol{s} \otimes (\mathrm{id}_X \boldsymbol{s}) = (b \otimes \mathrm{id}_X)\boldsymbol{s} = (\mathrm{id}_I)\boldsymbol{s} = I\,.$$

This shows $Y = X^{\otimes -}$. So X is tensor invertible. □

Remark 23 Let $(\mathcal{C}, I, \otimes)$ be a monoidal category.

Then, $(\mathcal{C}, I, \otimes)$ is an invertible monoidal category if and only if $(1, 2)$ hold.

(1) The set of objects $\mathrm{Ob}(\mathcal{C})$ together with the operation $(\otimes)$ is a group with neutral element I.

(2) The set of morphism $\mathrm{Mor}(\mathcal{C})$ together with the operation $(\otimes)$ is a group with neutral element id_I.

Remark 24 Let $(\mathcal{C}, I, \otimes)$ be an invertible monoidal category.

The source map $\boldsymbol{s}\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$, the target map $\boldsymbol{t}\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$ and the identity map $\boldsymbol{i}\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C})$ are group morphisms.

In particular, we have normal subgroups

$$\ker \boldsymbol{s} = \{u \in \mathrm{Mor}(\mathcal{C}) : u\boldsymbol{s} = I\} \trianglelefteq \mathrm{Mor}(\mathcal{C})$$
$$\ker \boldsymbol{t} = \{u \in \mathrm{Mor}(\mathcal{C}) : u\boldsymbol{t} = I\} \trianglelefteq \mathrm{Mor}(\mathcal{C})\ .$$

Lemma 25 *Let $(\mathcal{C}, I, \otimes)$ be an invertible monoidal category. Suppose given $u \in \ker \boldsymbol{s}$ and $v \in \ker \boldsymbol{t}$.*

Then $\quad u \otimes v = v \otimes u\,.$

Proof. We have

$$\begin{aligned} u \otimes v &= (\mathrm{id}_I \blacktriangle u) \otimes (v \blacktriangle \mathrm{id}_I) = (\mathrm{id}_I \otimes v) \blacktriangle (u \otimes \mathrm{id}_I) = v \blacktriangle u = (v \otimes \mathrm{id}_I) \blacktriangle (\mathrm{id}_I \otimes u) \\ &= (v \blacktriangle \mathrm{id}_I) \otimes (\mathrm{id}_I \blacktriangle u) = v \otimes u\,. \end{aligned}$$

□

Lemma 26 *Let $(\mathcal{C}, I, \otimes)$ be a monoidal category.*

(1) *Suppose given an isomorphism $\big(X \xrightarrow{u} Y\big)$ in $\mathcal{C}$ such that X and Y are tensor invertible. Then u is tensor invertible and its tensor inverse is given by*

$$u^{\otimes -} = Y^{\otimes -} \otimes u^{-} \otimes X^{\otimes -}\colon\ X^{\otimes -} \to Y^{\otimes -}\,.$$

(2) *Suppose given a tensor invertible morphism* $\big(X \xrightarrow{u} Y\big)$ *in* $\mathcal{C}$ *.*

Then u *is an isomorphism and its inverse is given by*

$$u^- = Y \otimes u^{\otimes -} \otimes X\colon\ Y \to X\,.$$

Proof. Ad (1). We have

$$\begin{aligned}
u \otimes (Y^{\otimes -} \otimes u^- \otimes X^{\otimes -}) &= (u \blacktriangle \mathrm{id}_Y) \otimes \big((\mathrm{id}_{X^{\otimes -}}) \blacktriangle (Y^{\otimes -} \otimes u^- \otimes X^{\otimes -})\big) \\
&= (u \otimes X^{\otimes -}) \blacktriangle (Y \otimes Y^{\otimes -} \otimes u^- \otimes X^{\otimes -}) \\
&= (u \otimes X^{\otimes -}) \blacktriangle (u^- \otimes X^{\otimes -}) = (u \blacktriangle u^-) \otimes X^{\otimes -} \\
&= \mathrm{id}_X \otimes \mathrm{id}_{X^{\otimes -}} = \mathrm{id}_I\,, \\
(Y^{\otimes -} \otimes u^- \otimes X^{\otimes -}) \otimes u &= \big((Y^{\otimes -} \otimes u^- \otimes X^{\otimes -}) \blacktriangle \mathrm{id}_{Y^{\otimes -}}\big) \otimes (\mathrm{id}_X \blacktriangle u) \\
&= (Y^{\otimes -} \otimes u^- \otimes X^{\otimes -} \otimes X) \blacktriangle (Y^{\otimes -} \otimes u) \\
&= (Y^{\otimes -} \otimes u^-) \blacktriangle (Y^{\otimes -} \otimes u) \\
&= Y^{\otimes -} \otimes (u^- \blacktriangle u) = \mathrm{id}_{Y^{\otimes -}} \otimes \mathrm{id}_Y = \mathrm{id}_I\,.
\end{aligned}$$

Ad (2). We have

$$\begin{aligned}
u \blacktriangle (Y \otimes u^{\otimes -} \otimes X) &= (u \otimes I) \blacktriangle (Y \otimes u^{\otimes -} \otimes X) = (u \blacktriangle \mathrm{id}_Y) \otimes \big(\mathrm{id}_I \blacktriangle (u^{\otimes -} \otimes X)\big) \\
&= u \otimes u^{\otimes -} \otimes X = \mathrm{id}_I \otimes X = \mathrm{id}_X\,, \\
(Y \otimes u^{\otimes -} \otimes X) \blacktriangle u &= (Y \otimes u^{\otimes -} \otimes X) \blacktriangle (I \otimes u) = \big((Y \otimes u^{\otimes -}) \blacktriangle \mathrm{id}_I\big) \otimes (\mathrm{id}_X \blacktriangle u) \\
&= Y \otimes u^{\otimes -} \otimes u = Y \otimes \mathrm{id}_I = \mathrm{id}_Y\,.
\end{aligned}$$

□

Corollary 27 *Let* $(\mathcal{C}, I, \otimes)$ *be an invertible monoidal category.*

Then every morphism in $\mathcal{C}$ *is an isomorphism.*

Proof. Suppose given $u \in \mathrm{Mor}(\mathcal{C})$. Since u is tensor invertible we have that u is an isomorphism by Lemma 26.(2). □

Remark 28 *Let* $(\mathcal{C}, I, \otimes)$ *be an invertible monoidal category.*

Then, for $(X \xrightarrow{u} Y \xrightarrow{v} Z)$ *in* $\mathcal{C}$ *, we have*

$$u \blacktriangle v = u \otimes Y^{\otimes -} \otimes v\,.$$

Proof. For $(X \xrightarrow{u} Y \xrightarrow{v} Z)$ in $\mathcal{C}$, we have

$$\begin{aligned} u \otimes Y^{\otimes-} \otimes v &= (u \otimes Y^{\otimes-} \otimes v) \blacktriangle \mathrm{id}_Z \overset{27}{=} (u \otimes Y^{\otimes-} \otimes v) \blacktriangle (\mathrm{id}_I \otimes v^-) \blacktriangle v \\ &= \Big(\big((u \otimes Y^{\otimes-}) \blacktriangle \mathrm{id}_I\big) \otimes (v \blacktriangle v^-) \Big) \blacktriangle v = (u \otimes Y^{\otimes-} \otimes Y) \blacktriangle v \\ &= u \blacktriangle v\,. \end{aligned}$$

□

Remark 29 (The invertible monoidal category $V\mathrm{Cat}$)
Let $V = \big(M, G, \gamma, f\big)$ be a crossed module.

Consider the category $V\mathrm{Cat} = \big(G \ltimes M, G, (s,i,t), \blacktriangle\big)$; cf. Remark 4.

(1) *We have the functor*

$$\begin{array}{rcccccl} (\,\cdot\,): & V\mathrm{Cat} & \times & V\mathrm{Cat} & \longrightarrow & V\mathrm{Cat} & \\ & (\quad g & , & h \quad) & \longmapsto & g \cdot h & \text{for}\, g, h \in G \\ & \left(\begin{array}{c} g \\ \downarrow (g,m) \\ g \cdot mf \end{array} \right. & , & \left. \begin{array}{c} h \\ \downarrow (h,n) \\ h \cdot nf \end{array} \right) & \longmapsto & \left(\begin{array}{c} g \cdot h \\ \downarrow (g,m)\cdot(h,n) \\ g \cdot mf \cdot h \cdot nf \end{array} \right) & \text{for}\, (g,m),(h,n) \in G \ltimes M. \end{array}$$

Here, $g \cdot h$ is the product in the group G and $(g,m)\cdot(h,n)$ is the product in the group $G \ltimes M$.

(2) *We have the monoidal category $(V\mathrm{Cat}, 1_G, \,\cdot\,)$.*

(3) *The monoidal category $(V\mathrm{Cat}, 1_G, \,\cdot\,)$ is an invertible monoidal category.*

Proof. Ad (1). For $(g,m),(h,n) \in G \ltimes M$, we have

$$\begin{aligned} \big((g,m)\cdot(h,n)\big)s &= (g\cdot h, m^h \cdot n)s = g \cdot h = (g,m)s \cdot (n,h)s \\ (g\cdot h)i &= (g\cdot h, 1) = (g,1)\cdot(h,1) = gi \cdot hi \\ \big((g,m)\cdot(h,n)\big)t &= (g\cdot h, m^h \cdot n)t = g \cdot h \cdot (m^h \cdot n)f \overset{\text{(CM1)}}{=} g \cdot h \cdot (mf)^h \cdot nf \\ &= g \cdot mf \cdot h \cdot nf = (g,m)t \cdot (h,n)t\,. \end{aligned}$$

Suppose given $g \xrightarrow{(g,m)} g \cdot mf \xrightarrow{(g\cdot mf, m')} g \cdot (mm')f$ and $h \xrightarrow{(h,n)} h \cdot nf \xrightarrow{(h\cdot nf, n')} h \cdot (nn')f$ in $V\mathrm{Cat}$.

We have

$$\begin{aligned}
& \big((g,m) \blacktriangle (g \cdot mf, m')\big) \cdot \big((h,n) \blacktriangle (h \cdot nf, n')\big) \\
= \;& (g, mm') \cdot (h, nn') \\
= \;& \big(g \cdot h, (mm')^h \cdot nn'\big) \\
= \;& \big(g \cdot h, m^h \cdot (m')^h \cdot nn'\big) \\
= \;& \big(g \cdot h, m^h \cdot n \cdot n^- \cdot (m')^h \cdot n \cdot n'\big) \\
\overset{\text{(CM2)}}{=} \;& \big(g \cdot h, m^h \cdot n \cdot (m')^{h \cdot nf} \cdot n'\big) \\
= \;& (g \cdot h, m^h \cdot n) \blacktriangle \big(g \cdot h \cdot (m^h \cdot n)f, (m')^{h \cdot nf} \cdot n'\big) \\
\overset{\text{(CM1)}}{=} \;& (g \cdot h, m^h \cdot n) \blacktriangle \big((g \cdot mf) \cdot (h \cdot nf), (m')^{h \cdot nf} \cdot n'\big) \\
= \;& \big((g,m) \cdot (h,n)\big) \blacktriangle \big((g \cdot mf, m') \cdot (h \cdot nf, n')\big)\,.
\end{aligned}$$

So $(\,\cdot\,)$ is a functor.

Ad (2). Suppose given (g, m), (g', m') and $(g'', m'') \in \mathrm{Mor}(V\mathrm{Cat}) = G \ltimes M$.

We have

$$(g,m) \cdot \mathrm{id}_{1_G} = (g,m) \cdot (1,1) = (g,m) = (1,1) \cdot (g,m) = \mathrm{id}_{1_G} \cdot (g,m)\,.$$

Moreover, we have

$$\big((g,m) \cdot (g',m')\big) \cdot (g'',m'') = (g,m) \cdot \big((g',m') \cdot (g'',m'')\big)$$

since the group multiplication $(\,\cdot\,)$ in $G \ltimes M$ is associative.

So, by Remark 14, $V\mathrm{Cat}$ is a monoidal category.

Ad (3). Suppose given $(g, m) \in G \ltimes M = \mathrm{Mor}(V\mathrm{Cat})$. The latter being a group, we recall that

$$(g,m) \cdot \big(g^-, (m^-)^{g^-}\big) = (1,1) = \big(g^-, (m^-)^{g^-}\big) \cdot (g,m)\,.$$

So (g, m) is invertible with respect to $(\,\cdot\,)$.

Thus, by Remark 22, $V\mathrm{Cat}$ is an invertible monoidal category. □

Example 30 Let H be an abelian group.

We have a category $H\mathcal{C}$ with $\mathrm{Ob}(H\mathcal{C}) := \{H\}$ and $\mathrm{Mor}(H\mathcal{C}) := \{h \colon h \in H\} = H$. Composition in $H\mathcal{C}$ is given by

$$h \blacktriangle h' := h \cdot h'\,,$$

for h, $h' \in \mathrm{Mor}(H\mathcal{C})$. We have $\mathrm{id}_H = 1$.

We have a functor

$$(\,\cdot\,)\colon H\mathcal{C} \times H\mathcal{C} \to H\mathcal{C}\,,\ \left(\begin{array}{c} H \\ \downarrow h \\ H \end{array},\ \begin{array}{c} H \\ \downarrow \tilde{h} \\ H \end{array}\right) \mapsto \left(\begin{array}{c} H \\ \downarrow h \cdot \tilde{h} \\ H \end{array}\right).$$

Note that

$$(h \cdot h') \cdot (\tilde{h} \cdot \tilde{h}') = (h \cdot \tilde{h}) \cdot (h' \cdot \tilde{h}')$$

for h, h', $\tilde{h}$, $\tilde{h}' \in H$ since H is abelian.

We have an invertible monoidal category $(H\mathcal{C}, H, \,\cdot\,)$.

2.2 Monoidal functors

Let $(\mathcal{C}, I_\mathcal{C}\,, \underset{\mathcal{C}}{\otimes})$, $(\mathcal{D}, I_\mathcal{D}\,, \underset{\mathcal{D}}{\otimes})$ and $(\mathcal{E}, I_\mathcal{E}, \underset{\mathcal{E}}{\otimes})$ be monoidal categories.

Definition 31 (Monoidal functor)

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

We call F a *monoidal functor* if $(1, 2, 3)$ hold.

(1) We have $\quad (X \underset{\mathcal{C}}{\otimes} Y)F = XF \underset{\mathcal{D}}{\otimes} YF \quad$ for $X, Y \in \mathrm{Ob}(\mathcal{C})$.

(2) We have $\quad (u \underset{\mathcal{C}}{\otimes} v)F = uF \underset{\mathcal{D}}{\otimes} vF \quad$ for $u, v \in \mathrm{Mor}(\mathcal{C})$.

(3) We have $\quad I_\mathcal{C} F = I_\mathcal{D}$.

For an example of how to calculate monoidal functors, cf. §A.3.

Remark 32

(1) *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor satisfying the conditions* (2) *and* (3) *of Definition* 31.
Then F is a monoidal functor.

(2) *Suppose that $\mathcal{D}$ is an invertible monoidal category; cf. Definition* 21.
Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor satisfying condition (2) *of Definition* 31.
Then F is a monoidal functor.

Proof. We show that in Definition 31, (2) implies (1).

Suppose given $X, Y \in \mathrm{Ob}(\mathcal{C})$. We have

$$\begin{aligned}(X \underset{\mathcal{C}}{\otimes} Y) &= \big((\mathrm{id}_X \underset{\mathcal{C}}{\otimes} \mathrm{id}_Y)\mathfrak{s}\big)F = \big((\mathrm{id}_X \underset{\mathcal{C}}{\otimes} \mathrm{id}_Y)F\big)\mathfrak{s} = \big((\mathrm{id}_X F) \underset{\mathcal{D}}{\otimes} (\mathrm{id}_Y F)\big)\mathfrak{s} \\ &= (\mathrm{id}_{XF}\, \mathfrak{s}) \otimes (\mathrm{id}_{YF}\, \mathfrak{s}) = XF \otimes YF\,.\end{aligned}$$

We show that in Definition 31, (1) implies (3) if $\mathcal{D}$ is invertible.

We have

$$I_{\mathcal{C}}F = I_{\mathcal{C}}F \underset{\mathcal{D}}{\otimes} I_{\mathcal{C}}F \underset{\mathcal{D}}{\otimes} (I_{\mathcal{C}}F)^{-} = (I_{\mathcal{C}} \underset{\mathcal{C}}{\otimes} I_{\mathcal{C}})F \underset{\mathcal{D}}{\otimes} (I_{\mathcal{C}}F)^{-} = I_{\mathcal{C}}F \underset{\mathcal{D}}{\otimes} (I_{\mathcal{C}}F)^{-} = I_{\mathcal{D}}\,.$$

Now both assertions of Remark 32 follows. □

Remark 33 *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a monoidal functor.*

(1) *Let $X \in \mathrm{Ob}(\mathcal{C})$ be tensor invertible in $\mathcal{C}$. Then XF is tensor invertible in $\mathcal{D}$ and we have $(XF)^{\otimes-} = (X^{\otimes-})F$.*

(2) *Let $u \in \mathrm{Mor}(\mathcal{C})$ be tensor invertible in $\mathcal{C}$. Then uF is tensor invertible in $\mathcal{D}$ and we have $(uF)^{\otimes-} = (u^{\otimes-})F$.*

Proof. Ad (1). For $X \in \mathrm{Ob}(\mathcal{C})$, we have

$$(X^{\otimes-})F \underset{\mathcal{D}}{\otimes} XF = (X^{\otimes-} \underset{\mathcal{C}}{\otimes} X)F = I_{\mathcal{C}}F = I_{\mathcal{D}}\,.$$

Ad (2). For $u \in \mathrm{Mor}(\mathcal{C})$, we have

$$(u^{\otimes-})F \underset{\mathcal{D}}{\otimes} uF = (u^{\otimes-} \underset{\mathcal{C}}{\otimes} u)F = \mathrm{id}_{I_{\mathcal{C}}}F = \mathrm{id}_{I_{\mathcal{D}}}\,.$$

□

Lemma 34 (Identity and composition of monoidal functors)

(1) *The identity map $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ is a monoidal functor.*

(2) *Suppose given monoidal functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$. Then their composite $F * G\colon \mathcal{C} \to \mathcal{E}$ is a monoidal functor.*

(3) *Suppose that $F\colon \mathcal{C} \to \mathcal{D}$ is a monoidal isofunctor. Then $F^{-}\colon \mathcal{D} \to \mathcal{C}$ is a monoidal functor.*

Proof. We use Remark 32.

Ad (1). We have

$$I_{\mathcal{C}}\,\mathrm{id}_{\mathcal{C}} = I_{\mathcal{C}}\,,$$

and for u, $v \in \mathrm{Mor}(\mathcal{C})$, we have

$$(u \underset{\mathcal{C}}{\otimes} v)\mathrm{id}_{\mathcal{C}} = u \underset{\mathcal{C}}{\otimes} v = u\,\mathrm{id}_{\mathcal{C}} \underset{\mathcal{C}}{\otimes} v\,\mathrm{id}_{\mathcal{C}}\,.$$

Ad (2). We have

$$I_{\mathcal{C}}FG = I_{\mathcal{D}}G = I_{\mathcal{E}}\,,$$

and for u, $v \in \mathrm{Mor}(\mathcal{C})$, we have

$$(u \underset{\mathcal{C}}{\otimes} v)FG = (uF \underset{\mathcal{D}}{\otimes} vF)G = uFG \underset{\mathcal{E}}{\otimes} vFG\,.$$

Ad (3). We have

$$I_{\mathcal{D}}\,F^{-} = I_{\mathcal{C}}FF^{-} = I_{\mathcal{C}}\,.$$

For u, $v \in \mathrm{Mor}(\mathcal{D})$, we have

$$(u \underset{\mathcal{D}}{\otimes} v)F^{-} = (uF^{-}F \underset{\mathcal{D}}{\otimes} vF^{-}F)F^{-} = (uF^{-} \underset{\mathcal{C}}{\otimes} vF^{-})FF^{-} = uF^{-} \underset{\mathcal{C}}{\otimes} vF^{-}\,.$$

□

2.3 Monoidal transformations

Let $(\mathcal{B}, I_{\mathcal{B}}, \underset{\mathcal{B}}{\otimes})$, $(\mathcal{C}, I_{\mathcal{C}}, \underset{\mathcal{C}}{\otimes})$, $(\mathcal{D}, I_{\mathcal{D}}, \underset{\mathcal{D}}{\otimes})$ and $(\mathcal{E}, I_{\mathcal{E}}, \underset{\mathcal{E}}{\otimes})$ be monoidal categories.

Definition 35 (Monoidal transformation)

Suppose given monoidal functors $F, G\colon \mathcal{C} \to \mathcal{D}$. Suppose given a transformation $\eta\colon F \to G$. We say that η is a *monoidal transformation* from F to G if $(1, 2)$ are satisfied.

(1) We have

$$I_{\mathcal{C}}\,\eta = \mathrm{id}_{I_{\mathcal{D}}}\colon I_{\mathcal{C}}F = I_{\mathcal{D}} \to I_{\mathcal{D}} = I_{\mathcal{C}}G\,.$$

(2) For $X,\, Y \in \mathrm{Ob}(\mathcal{C})$, we have

$$(X \underset{\mathcal{C}}{\otimes} Y)\eta = X\eta \underset{\mathcal{D}}{\otimes} Y\eta\,.$$

$$\begin{tikzcd}
(X \underset{\mathcal{C}}{\otimes} Y)F \arrow[r, "(X \underset{\mathcal{C}}{\otimes} Y)\eta"] \arrow[d, "\mathrm{id}"'] & (X \underset{\mathcal{C}}{\otimes} Y)G \arrow[d, "\mathrm{id}"] \\
XF \underset{\mathcal{D}}{\otimes} YF \arrow[r, "X\eta \underset{\mathcal{D}}{\otimes} Y\eta"'] & XG \underset{\mathcal{D}}{\otimes} YG
\end{tikzcd}$$

For an example of how to calculate monoidal transformations cf. §A.4. For a further calculation example of a monoidal transformation, cf. Example 44 below.

Remark 36 *Suppose that the monoidal category* $(\mathcal{D}, I_{\mathcal{D}}, \underset{\mathcal{D}}{\otimes})$ *is an invertible monoidal category. Suppose given monoidal functors* $F,\, G\colon \mathcal{C} \to \mathcal{D}$ *.*

Suppose given a transformation $\eta\colon F \to G$ *satisfying* (2) *in Definition* 35*.*

Then η *is a monoidal transformation.*

Proof. We have

$$\begin{aligned}
I_{\mathcal{C}}\,\eta &= I_{\mathcal{C}}\,\eta \underset{\mathcal{D}}{\otimes} \mathrm{id}_{I_{\mathcal{D}}} = I_{\mathcal{C}}\,\eta \underset{\mathcal{D}}{\otimes} I_{\mathcal{C}}\,\eta \underset{\mathcal{D}}{\otimes} \big((I_{\mathcal{C}})\eta\big)^{\otimes -} = (I_{\mathcal{C}} \underset{\mathcal{C}}{\otimes} I_{\mathcal{C}})\eta \underset{\mathcal{D}}{\otimes} (I_{\mathcal{C}}\,\eta)^{\otimes -} \\
&= I_{\mathcal{C}}\,\eta \underset{\mathcal{D}}{\otimes} (I_{\mathcal{C}}\,\eta)^{\otimes -} = \mathrm{id}_{I_{\mathcal{D}}}\,.
\end{aligned}$$

□

Remark 37 Suppose given monoidal functors $H\colon \mathcal{B} \to \mathcal{C}$, $F,\ F',\ F''\colon \mathcal{C} \to \mathcal{D}$ and $G,\ G'\colon \mathcal{D} \to \mathcal{E}$. Suppose given monoidal transformations $\eta\colon F \to F'$, $\eta'\colon F' \to F''$ and $\vartheta\colon G \to G'$.

$$\begin{tikzcd}[column sep=large]
\mathcal{B} \arrow[r, "H"] & \mathcal{C} \arrow[r, bend left=50, "F"{name=A}] \arrow[r, "F'"{name=B, description}] \arrow[r, bend right=50, "F''"'{name=C}] & \mathcal{D} \arrow[r, bend left=50, "G"{name=D}] \arrow[r, bend right=50, "G'"'{name=E}] & \mathcal{E}
\arrow[from=A, to=B, "\eta", shorten=2pt]
\arrow[from=B, to=C, "\eta'", shorten=2pt]
\arrow[from=D, to=E, "\vartheta", shorten=2pt]
\end{tikzcd}$$

(1) The transformation $\mathrm{id}_F\colon F \to F$ is monoidal.

(2) The vertical composite $\eta \blacktriangle \eta'\colon F \to F''$ is a monoidal transformation.

(3) We have monoidal transformations $H\eta\colon HF \to HF'$ and $\eta G\colon FG \to F'G$.

(4) The horizontal composite $\eta * \vartheta\colon FG \to F'G'$ is a monoidal transformation.

Proof. Ad (1). We have

$$I_{\mathcal{C}}\,\mathrm{id}_F = \mathrm{id}_{I_{\mathcal{C}}F} = \mathrm{id}_{I_{\mathcal{D}}}\,.$$

For $X, Y \in \mathrm{Ob}(\mathcal{C})$, we have

$$(X \underset{\mathcal{C}}{\otimes} Y)\mathrm{id}_F = \mathrm{id}_{(X \underset{\mathcal{C}}{\otimes} Y)F} = \mathrm{id}_{XF \underset{\mathcal{D}}{\otimes} YF} = \mathrm{id}_{XF} \underset{\mathcal{D}}{\otimes} \mathrm{id}_{YF} = X\mathrm{id}_F \underset{\mathcal{D}}{\otimes} Y\mathrm{id}_F\,.$$

Ad (2). We have

$$I_{\mathcal{C}}\,(\eta \blacktriangle \eta') = I_{\mathcal{C}}\,\eta \blacktriangle I_{\mathcal{C}}\,\eta' = \mathrm{id}_{I_{\mathcal{D}}} \blacktriangle \mathrm{id}_{I_{\mathcal{D}}} = \mathrm{id}_{I_{\mathcal{D}}}\,.$$

For X, $Y \in \mathrm{Ob}(\mathcal{C})$, we have

$$\begin{aligned}(X \underset{\mathcal{C}}{\otimes} Y)(\eta \blacktriangle \eta') &= (X \underset{\mathcal{C}}{\otimes} Y)\eta \blacktriangle (X \underset{\mathcal{C}}{\otimes} Y)\eta' = (X\eta \underset{\mathcal{D}}{\otimes} Y\eta) \blacktriangle (X\eta' \underset{\mathcal{D}}{\otimes} Y\eta') \\ &= (X\eta \blacktriangle X\eta') \underset{\mathcal{D}}{\otimes} (Y\eta \blacktriangle Y\eta') = X(\eta \blacktriangle \eta') \underset{\mathcal{D}}{\otimes} Y(\eta \blacktriangle \eta')\,.\end{aligned}$$

Ad (3). We have

$$I_{\mathcal{B}}\,H\eta = I_{\mathcal{C}}\,\eta = \mathrm{id}_{I_{\mathcal{D}}}\,,$$

and similarly we get

$$I_{\mathcal{C}}\,\eta G = \mathrm{id}_{I_{\mathcal{D}}}\,G = \mathrm{id}_{I_{\mathcal{E}}}\,,$$

For $A\,, B \in \mathrm{Ob}(\mathcal{B})$, we have

$$(A \underset{\mathcal{B}}{\otimes} B)H\eta = (AH \underset{\mathcal{C}}{\otimes} BH)\eta = AH\eta \underset{\mathcal{D}}{\otimes} BH\eta\,.$$

For X, $Y \in \mathrm{Ob}(\mathcal{C})$, we have

$$(X \underset{\mathcal{C}}{\otimes} Y)\eta G = (X\eta \underset{\mathcal{D}}{\otimes} Y\eta)G = X\eta G \underset{\mathcal{E}}{\otimes} Y\eta G\,.$$

Ad (4). We have $\eta * \vartheta = (F\vartheta) \blacktriangle (\eta G')\colon FG \to F'G'$.

By (3), the transformations $F\vartheta\colon FG \to FG'$ and $\eta G'\colon FG \to F'G'$ are monoidal. Then, by (2), $\eta * \vartheta = (F\vartheta) \blacktriangle (\eta G')$ is a monoidal transformation. $\square$

2.4 The functors Cat and CM

Definition 38 (Category of invertible monoidal categories)

(1) The category having as objects monoidal categories and as morphisms monoidal functors is called the *category of monoidal categories*, and is denoted by $\mathcal{M}on\mathcal{C}at$.

(2) The full subcategory of $\mathcal{M}on\mathcal{C}at$ that consists of invertible monoidal categories is called the *category of invertible monoidal categories*, and is denoted by $\mathit{Inv}\mathcal{M}on\mathcal{C}at$.

Lemma 39 (The functor Cat)

Suppose given crossed modules $V = \big(M, G, \gamma, f\big)$ *and* $W = \big(N, H, \beta, k\big)$*. Recall that we have an invertible monoidal category given by*

$$V\mathrm{Cat} = \Big(\big(G \ltimes M,\, G, (s,i,t),\, {\scriptstyle\blacktriangle}\big),\, 1_G\,,\, \cdot\Big)\,;$$

cf. Remark 29*.*

(1) *Suppose given a crossed module morphism* $\rho = (\lambda, \mu)\colon V \to W$*; cf.* §0.4 *item* 2*.*

We have a monoidal functor given by

$$\begin{array}{rrcll} \rho\,\mathrm{Cat}\colon & V\mathrm{Cat} & \longrightarrow & W\mathrm{Cat} & \\ & g & \longmapsto & g\mu & \text{for } g \in \mathrm{Ob}(V\mathrm{Cat}) \\ & (g,m) & \longmapsto & (g\mu, m\lambda) & \text{for } (g,m) \in \mathrm{Mor}(V\mathrm{Cat})\ . \end{array}$$

(2) *We have a functor*

$$\begin{array}{rrcll} \mathrm{Cat}\colon & \mathcal{CR}\mathcal{M}od & \longrightarrow & \mathit{Inv}\mathcal{M}on\mathcal{C}at & \\ & V & \longmapsto & V\mathrm{Cat} & \text{for } V \in \mathrm{Ob}(\mathcal{CR}\mathcal{M}od) \\ & \rho & \longmapsto & \rho\,\mathrm{Cat} & \text{for } \rho \in \mathrm{Mor}(\mathcal{CR}\mathcal{M}od)\ . \end{array}$$

Proof. Ad (1). We show that $\rho\,\mathrm{Cat}$ is a functor.

For $g \in G = \mathrm{Ob}(V\mathrm{Cat})$, we have

$$(\mathrm{id}_g)(\rho\,\mathrm{Cat}) = (g,1)(\rho\,\mathrm{Cat}) = (g\mu, 1\lambda) = \mathrm{id}_{g\mu} = \mathrm{id}_{(g)(\rho\,\mathrm{Cat})}$$

Suppose given $g \xrightarrow{(g,m)} g \cdot mf \xrightarrow{(g \cdot mf, m')} g \cdot (mm')f$ in $V\mathrm{Cat}$.

Write $a := (g, m)$, $b := (g \cdot mf, m')$.

We have

$$\begin{aligned}\big((a)(\rho\,\mathrm{Cat})\big)\,\boldsymbol{t} &= \big((g,m)(\rho\,\mathrm{Cat})\big)\,\boldsymbol{t} = (g\mu, m\lambda)\,\boldsymbol{t} = g\mu \cdot m\lambda k = g\mu \cdot mf\mu = (g \cdot mf)\mu \\ &= \big((g \cdot mf)\mu, m'\lambda\big)\,\boldsymbol{s} = \big((g \cdot mf, m')(\rho\,\mathrm{Cat})\big)\,\boldsymbol{s} = \big((b)(\rho\,\mathrm{Cat})\big)\,\boldsymbol{s}\,.\end{aligned}$$

So, $(a)(\rho\,\mathrm{Cat})$ and $(b)(\rho\,\mathrm{Cat})$ are composable.

We have

$$\begin{aligned}(a)(\rho\,\mathrm{Cat}) \blacktriangle (b)(\rho\,\mathrm{Cat}) &= (g\mu, m\lambda) \blacktriangle \big((g \cdot mf)\mu, m'\lambda\big) = (g\mu, m\lambda \cdot m'\lambda) = \big(g\mu, (mm')\lambda\big) \\ &= (g, mm')(\rho\,\mathrm{Cat}) = \big((g,m) \blacktriangle (g \cdot mf, m')\big)(\rho\,\mathrm{Cat}) = (a \blacktriangle b)(\rho\,\mathrm{Cat})\,.\end{aligned}$$

So, $\rho\,\mathrm{Cat}$ is a functor.

We show that $\rho\,\mathrm{Cat}$ is a monoidal functor.

For (g, m) and $(g', m') \in \mathrm{Mor}(V\,\mathrm{Cat})$, we have

$$\begin{aligned}\big((g,m)\cdot(g',m')\big)(\rho\,\mathrm{Cat}) &= (g \cdot g', m^{g'} \cdot m')(\rho\,\mathrm{Cat}) = \big((g \cdot g')\mu, (m^{g'} \cdot m)\lambda\big) \\ &= \big(g\mu \cdot g'\mu, (m\lambda)^{g'\mu} \cdot m\lambda\big) = (g\mu, m\lambda) \cdot (g'\mu, m'\lambda) \\ &= (g,m)(\rho\,\mathrm{Cat}) \cdot (g', m')(\rho\,\mathrm{Cat})\,.\end{aligned}$$

Thus, by Remark 32.(2), $\rho\,\mathrm{Cat}$ is a monoidal functor.

Ad (2). Suppose given $V \xrightarrow{(\lambda,\mu)} V' \xrightarrow{(\lambda',\mu')} V''$ in $\mathcal{CRMod}$. We write $\rho := (\lambda, \mu)$ and $\rho' := (\lambda', \mu')$.

By (1), we have $V\,\mathrm{Cat} \xrightarrow{\rho\,\mathrm{Cat}} V'\,\mathrm{Cat} \xrightarrow{\rho'\,\mathrm{Cat}} V''\,\mathrm{Cat}$ in $\mathit{InvMonCat}$.

First, note that

$$\mathrm{id}_V\,\mathrm{Cat} = (\mathrm{id}_M, \mathrm{id}_G)\,\mathrm{Cat} = \mathrm{id}_{V\,\mathrm{Cat}}\,;$$

cf. (1).

So, we have

$$\begin{aligned}(\rho\,\mathrm{Cat})\boldsymbol{s} &= V\,\mathrm{Cat} = (\rho\boldsymbol{s})\,\mathrm{Cat}\,, \\ (V\,\mathrm{Cat})\boldsymbol{i} &= \mathrm{id}_{V\,\mathrm{Cat}} = (V\boldsymbol{i})\,\mathrm{Cat}\,, \\ (\rho\,\mathrm{Cat})\boldsymbol{t} &= V'\,\mathrm{Cat} = (\rho\boldsymbol{t})\,\mathrm{Cat}\;.\end{aligned}$$

Now suppose given $u := (g, m) \in \mathrm{Mor}(V\mathrm{Cat}) = G \ltimes M$. We have

$$\begin{aligned}(u)\big((\rho \blacktriangle \rho')\mathrm{Cat}\big) &= (g, m)\big((\lambda \blacktriangle \lambda', \mu \blacktriangle \mu')\mathrm{Cat}\big) = (g\mu\mu', m\lambda\lambda') = (g\mu, m\lambda)\big((\lambda', \mu')\mathrm{Cat}\big) \\ &= (g, m)\big((\lambda, \mu)\mathrm{Cat}\big)\big((\lambda', \mu')\mathrm{Cat}\big) = (u)\big((\rho\,\mathrm{Cat}) * (\rho'\,\mathrm{Cat})\big) .\end{aligned}$$

Hence, Cat is functor. □

Lemma 40 (Crossed module from an invertible monoidal category)

Suppose given an invertible monoidal category $(\mathcal{C}, I, \otimes)$.

Recall that we have groups $\big(\mathrm{Ob}(\mathcal{C}), \otimes\big)$ and $\big(\mathrm{Mor}(\mathcal{C}), \otimes\big)$; cf. Remarks 23 *and* 24.

Consider the groups

$$\begin{aligned}\tilde{G} &:= \mathrm{Ob}(\mathcal{C}) \\ \tilde{M} &:= \ker \boldsymbol{s} = \{u \in \mathrm{Mor}(\mathcal{C}) : u\boldsymbol{s} = I\} \trianglelefteq \mathrm{Mor}(\mathcal{C}) .\end{aligned}$$

Consider the maps

$$\begin{aligned}&\tilde{\gamma} \colon \tilde{G} \to \mathrm{Aut}(\tilde{M}), \; X \mapsto \big(u \mapsto X^{\otimes-} \otimes u \otimes X\big) \\ &\tilde{f} := \boldsymbol{t}|_{\tilde{M}} : \tilde{M} \to \tilde{G}, \; u \mapsto u\boldsymbol{t} .\end{aligned}$$

(1) *The maps $\tilde{\gamma}$ and $\tilde{f}$ are group morphisms.*

(2) *We have a crossed module given by $\tilde{V} = \big(\tilde{M}, \tilde{G}, \tilde{\gamma}, \tilde{f}\big)$.*

Cf. [**10,** Lem. 2.2].

By the construction given above, we obtain a crossed module $\tilde{V}$ from an invertible monoidal category $\mathcal{C}$. We shall write

$$\mathcal{C}\,\mathrm{CM} := \tilde{V} = \big(\tilde{M}, \tilde{G}, \tilde{\gamma}, \tilde{f}\big) .$$

Proof. Ad (1). Suppose given $X, Y \in \tilde{G}$ and $u, v \in \tilde{M}$.

We have

$$\begin{aligned}(u \otimes v)(X\tilde{\gamma}) &= X^{\otimes-} \otimes (u \otimes v) \otimes X = (X^{\otimes-} \otimes u \otimes X) \otimes (X^{\otimes-} \otimes v \otimes X) \\ &= (u)(X\tilde{\gamma}) \otimes (v)(X\tilde{\gamma}) .\end{aligned}$$

So, $X\tilde{\gamma}$ is a group morphism.

We have

$$\begin{aligned}(u)(X\tilde{\gamma})\big((X^{\otimes -})\tilde{\gamma}\big) &= (X^{\otimes -} \otimes u \otimes X)\big((X^{\otimes -})\tilde{\gamma}\big)\\ &= (X^{\otimes -})^{\otimes -} \otimes X^{\otimes -} \otimes u \otimes X \otimes X^{\otimes -}\\ &= X \otimes X^{\otimes -} \otimes u \otimes X \otimes X^{\otimes -}\\ &= u\,.\end{aligned}$$

Likewise, we have $(u)\big((X^{\otimes -})\tilde{\gamma}\big)(X\tilde{\gamma}) = u$. Therefore, $(X^{\otimes -})\tilde{\gamma}$ is the inverse of $X\tilde{\gamma}$.

So, $X\tilde{\gamma} \in \operatorname{Aut}(\tilde{M})$. Hence, $\tilde{\gamma}$ is well-defined.

We have

$$\begin{aligned}(u)\big((X \otimes Y)\tilde{\gamma}\big) &= (X \otimes Y)^{\otimes -} \otimes u \otimes (X \otimes Y) = Y^{\otimes -} \otimes X^{\otimes -} \otimes u \otimes X \otimes Y\\ &= Y^{\otimes -} \otimes (u)(X\tilde{\gamma}) \otimes Y = (u)(X\tilde{\gamma})(Y\tilde{\gamma})\,.\end{aligned}$$

So, $\tilde{\gamma}$ is a group morphism.

Suppose given $u, v \in \tilde{M}$. We have

$$(u \otimes v)\tilde{f} = (u \otimes v)\boldsymbol{t} = u\boldsymbol{t} \otimes v\boldsymbol{t} = u\tilde{f} \otimes v\tilde{f}\,.$$

So, $\tilde{f}$ is a group morphism.

Ad (3). Ad (CM1). Suppose given $u \in \tilde{M}$ and $X \in \tilde{G}$. We have

$$\begin{aligned}(u^X)\tilde{f} &= (X^{\otimes -} \otimes u \otimes X)\tilde{f} = (X^{\otimes -} \otimes u \otimes X)\boldsymbol{t} = X^{\otimes -} \otimes u\boldsymbol{t} \otimes X\\ &= (u\boldsymbol{t})^X = (u\tilde{f})^X\,.\end{aligned}$$

Ad (CM2). Suppose given $u \in \tilde{M}$. Suppose given $(I \xrightarrow{v} Y) \in \tilde{M}$.

Note that we have

$$(v \otimes Y^{\otimes -})\boldsymbol{t} = v\boldsymbol{t} \otimes Y^{\otimes -} = Y \otimes Y^{\otimes -} = I\,.$$

Therefore, $(v \otimes Y^{\otimes -}) \in \ker \boldsymbol{t}$.

Then, by Lemma 25, it follows that $v \otimes Y^{\otimes -} \otimes u = u \otimes v \otimes Y^{\otimes -}$.

So, we have

$$\begin{aligned}u^{v\tilde{f}} &= u^Y = Y^{\otimes -} \otimes u \otimes Y = v^{\otimes -} \otimes (v \otimes Y^{\otimes -} \otimes u) \otimes Y = v^{\otimes -} \otimes (u \otimes v \otimes Y^{\otimes -}) \otimes Y\\ &= v^{\otimes -} \otimes u \otimes v = u^v\,.\end{aligned}$$

$\square$

Example 41

Let H be an abelian group.

(1) We have the crossed module $W := \big(H, 1, \iota, \kappa\big)$ with

$$\begin{aligned} \iota &\colon 1 \to \mathrm{Aut}\,(H)\,,\ 1 \mapsto \mathrm{id}_H \\ \kappa &\colon H \to 1\,,\ h \mapsto 1\,; \end{aligned}$$

cf. [**15,** Ex. 11].

We consider the invertible monoidal category $W\mathrm{Cat}$; cf. Remark 29. Then

$$\begin{aligned} \mathrm{Ob}(W\mathrm{Cat}) &= 1 \\ \mathrm{Mor}(W\mathrm{Cat}) &= 1 \ltimes H\,. \end{aligned}$$

The tensor multiplication in $\mathrm{Ob}(W\mathrm{Cat}) = 1$ is given by the group multiplication in 1, and the tensor multiplication in $\mathrm{Mor}(W\mathrm{Cat}) = 1 \ltimes H$ is given by the group multiplication in $1 \ltimes H$.

The composition in $W\mathrm{Cat}$ is given by

$$(1, h) \blacktriangle (1, h') = (1, h \cdot h')\,,$$

for $h,\, h' \in H$.

(2) Consider the invertible monoidal category $(H\mathcal{C}, H, \,\cdot\,)$ from Example 30. Recall that

$$\begin{aligned} \mathrm{Ob}(H\mathcal{C}) &= \{H\} \\ \mathrm{Mor}(H\mathcal{C}) &= \{h\colon h \in H\} = H\,. \end{aligned}$$

We want to to show that $H\mathcal{C}$ is isomrphic to $W\mathrm{Cat}$ via the monoidal isofunctor

$$\begin{array}{rccccl} F\colon & H\mathcal{C} & \longrightarrow & W\mathrm{Cat} & & \\ & H & \longmapsto & 1 & \text{for} & H \in \mathrm{Ob}(H\mathcal{C}) \\ & h & \longmapsto & (1, h) & \text{for} & h \in \mathrm{Mor}(H\mathcal{C}) \end{array}$$

Moreover, we show that its inverse is given by the monoidal isofunctor

$$\begin{array}{rccccl} F^{-}\colon & W\mathrm{Cat} & \longrightarrow & H\mathcal{C} & & \\ & 1 & \longmapsto & H & \text{for} & 1 \in \mathrm{Ob}(W\mathrm{Cat}) \\ & (1, h) & \longmapsto & h & \text{for} & (1, h) \in \mathrm{Mor}(W\mathrm{Cat})\,. \end{array}$$

We show that F is a functor:

For $H \xrightarrow{h} H \in \mathrm{Mor}(H\mathcal{C})$ and $H \in \mathrm{Ob}(H\mathcal{C})$, we have

$$\begin{aligned} h\boldsymbol{s}F &= HF = 1 = (1,h)\boldsymbol{s} = hF\boldsymbol{s} \\ H\boldsymbol{i}F &= 1_H F = (1,1) = 1\boldsymbol{i} = HF\boldsymbol{i} \\ h\boldsymbol{t}F &= HF = 1 = (1,h)\boldsymbol{t} = hF\boldsymbol{t}\,. \end{aligned}$$

For $(H \xrightarrow{h} H \xrightarrow{h'} H)$ in $H\mathcal{C}$, we have

$$(h \blacktriangle h')F = (h \cdot h')F = (1, h \cdot h') = (1,h) \cdot (1,h') = hF \cdot h'F = hF \blacktriangle h'F\,.$$

So F is a functor.

We show that F is monoidal:

For h, $h' \in \mathrm{Mor}(H\mathcal{C})$, we have

$$(h \cdot h')F = (1, h \cdot h') = (1,h) \cdot (1,h') = hF \cdot h'F\,.$$

Then, by Remark 32.(2), F is monoidal.

Consider

$$G\colon W\mathrm{Cat} \to H\mathcal{C}\,,\ \big(1 \xrightarrow{(1,h)} 1\big) \mapsto \big(H \xrightarrow{h} H\big)\,.$$

We show that G is a functor:

For $(1 \xrightarrow{(1,h)} 1) \in \mathrm{Mor}(W\mathrm{Cat})$ and $1 \in \mathrm{Ob}(W\mathrm{Cat})$, we have

$$\begin{aligned} (1,h)\boldsymbol{s}G &= 1G = H = h\boldsymbol{s} = (1,h)G\boldsymbol{s} \\ 1\boldsymbol{i}G &= (1,1)G = 1 = H\boldsymbol{i} = 1G\boldsymbol{i} \\ (1,h)\boldsymbol{t}G &= 1G = H = h\boldsymbol{t} = (1,h)G\boldsymbol{t}\,. \end{aligned}$$

For $1_1 \xrightarrow{(1,h)} 1 \xrightarrow{(1,h')} 1)$ in $W\mathrm{Cat}$, we have

$$\big((1,h) \blacktriangle (1,h')\big)G = (1, h \cdot h')G = h \cdot h' = (1,h)G \cdot (1,h')G = (1,h)G \blacktriangle (1,h')G\,.$$

So G is a functor.

We show that $F^- \overset{!}{=} G$:

For $h \in \mathrm{Mor}(H\mathcal{C})$, we have

$$h(F * G) = (1,h)G = h\,.$$

This shows $F * G = \mathrm{id}_{H\mathcal{C}}$.

For $(1, h) \in \mathrm{Mor}(W\mathrm{Cat})$, we have

$$(1, h)(G * F) = hF = (1, h)\,.$$

This shows $G * F = \mathrm{id}_{W\mathrm{Cat}}$.

So we have $F^{-} = G$.

Altogether, F is a monoidal isofunctor.

Note that F^{-} is also a isomonoidal functor; cf. Lemma 34.(3).

Lemma 42 (The functor CM)

Suppose given invertible monoidal categories $(\mathcal{C}, I_{\mathcal{C}}, \underset{\mathcal{C}}{\otimes})$ *and* $(\mathcal{D}, I_{\mathcal{D}}, \underset{\mathcal{D}}{\otimes})$*. Recall that we have a crossed module given by*

$$\mathcal{C}\,\mathrm{CM} = \left(\ker \mathsf{s}_{\mathcal{C}}\,, \mathrm{Ob}(\mathcal{C})\,, \tilde{\gamma}, \mathsf{t}_{\mathcal{C}}|_{\ker \mathsf{s}_{\mathcal{C}}}\right),$$

where $\tilde{\gamma}\colon\ \mathrm{Ob}(\mathcal{C}) \to \mathrm{Aut}\,(\ker \mathsf{s}_{\mathcal{C}})$, $X \mapsto (u \mapsto X^{\otimes -} \underset{\mathcal{C}}{\otimes} u \underset{\mathcal{C}}{\otimes} X)$*; cf. Lemma* 40.(2).

(1) *Suppose given a monoidal functor* $F\colon \mathcal{C} \to \mathcal{D}$ *.*

We have a crossed module morphism $F\,\mathrm{CM} := (\lambda_F, \mu_F)\colon \mathcal{C}\,\mathrm{CM} \to \mathcal{D}\,\mathrm{CM}$ *given by*

$$\begin{array}{lcclccc} \mu_F\colon & \mathrm{Ob}(\mathcal{C}) & \longrightarrow & \mathrm{Ob}(\mathcal{D}) & , & X & \longmapsto & XF \\ \lambda_F\colon & \ker \mathsf{s}_{\mathcal{C}} & \longmapsto & \ker \mathsf{s}_{\mathcal{D}} & , & (I_{\mathcal{C}} \xrightarrow{u} X) & \longmapsto & (I_{\mathcal{D}} \xrightarrow{uF} XF)\,. \end{array}$$

(2) *We have a functor*

$$\begin{array}{llcll} \mathrm{CM}\colon & \mathit{InvMonCat} & \longrightarrow & \mathit{CRMod} & \\ & \mathcal{C} & \longmapsto & \mathcal{C}\,\mathrm{CM} & \text{for } \mathcal{C} \in \mathrm{Ob}(\mathit{InvMonCat}) \\ & F & \longmapsto & F\,\mathrm{CM} & \text{for } F \in \mathrm{Mor}(\mathit{InvMonCat})\,. \end{array}$$

Proof. Ad (1). We show that μ_F is a group morphism.

Suppose given $X, Y \in \mathrm{Ob}(\mathcal{C})$. We have

$$(X \underset{\mathcal{C}}{\otimes} Y)\mu_F = (X \underset{\mathcal{C}}{\otimes} Y)F = XF \underset{\mathcal{D}}{\otimes} YF = X\mu_F \underset{\mathcal{D}}{\otimes} Y\mu_F\,.$$

We show that λ_F is well-defined.

Suppose given $(I_{\mathcal{C}} \xrightarrow{u} X) \in \ker s_{\mathcal{C}}$.

We have

$$(uF)s_{\mathcal{D}} = (us_{\mathcal{C}})F = I_{\mathcal{C}}\,F = I_{\mathcal{D}}\,.$$

So, $uF \in \ker s_{\mathcal{D}}$.

We show that λ_F is a group morphism.

Suppose given $(I_{\mathcal{C}} \xrightarrow{u} X)$, $(I_{\mathcal{C}} \xrightarrow{v} Y) \in \ker s_{\mathcal{C}}$.

We have

$$(u \underset{\mathcal{C}}{\otimes} v)\lambda_F = (u \underset{\mathcal{C}}{\otimes} v)F = uF \underset{\mathcal{D}}{\otimes} vF = u\lambda_F \underset{\mathcal{D}}{\otimes} v\lambda_F\,.$$

We show that (λ_F, μ_F) is a crossed module morphism.

Suppose given $X \in \mathrm{Ob}(\mathcal{C})$ and $(I_{\mathcal{C}} \xrightarrow{u} Y) \in \ker s_{\mathcal{C}}$. Write $\bar{t}_{\mathcal{C}} := t_{\mathcal{C}}|_{\ker s_{\mathcal{C}}}$ and $\bar{t}_{\mathcal{D}} := t_{\mathcal{D}}|_{\ker s_{\mathcal{D}}}$.

We have

$$(u)\lambda_F\,\bar{t}_{\mathcal{D}} = uF\bar{t}_{\mathcal{D}} = u\bar{t}_{\mathcal{C}}\,F = (u)\bar{t}_{\mathcal{C}}\,\mu_F\,.$$

$$\begin{tikzcd}
\ker s_{\mathcal{C}} \arrow[r, "\lambda_F"] \arrow[d, "\bar{t}_{\mathcal{C}}"'] & \ker s_{\mathcal{D}} \arrow[d, "\bar{t}_{\mathcal{D}}"] \\
\mathrm{Ob}(\mathcal{C}) \arrow[r, "\mu_F"'] & \mathrm{Ob}(\mathcal{D})
\end{tikzcd}$$

We have

$$\begin{aligned}
(u^X)\lambda_F &= (X^{\otimes -} \underset{\mathcal{C}}{\otimes} u \underset{\mathcal{C}}{\otimes} X)\lambda_F = (X^{\otimes -} \underset{\mathcal{C}}{\otimes} u \underset{\mathcal{C}}{\otimes} X)F = (XF)^{\otimes -} \underset{\mathcal{D}}{\otimes} uF \underset{\mathcal{D}}{\otimes} XF \\
&= (X\mu_F)^{\otimes -} \underset{\mathcal{D}}{\otimes} u\lambda_F \underset{\mathcal{D}}{\otimes} X\mu_F = (u\lambda_F)^{X\mu_F}\,.
\end{aligned}$$

Ad (2). Suppose given $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ in $\mathit{InvMonCat}$.

By (1), we have $\mathcal{C}\,\mathrm{CM} \xrightarrow{F\,\mathrm{CM}} \mathcal{D}\,\mathrm{CM} \xrightarrow{G\,\mathrm{CM}} \mathcal{E}\,\mathrm{CM}$ in $\mathcal{CRMod}$, where $F\,\mathrm{CM} = (\lambda_F, \mu_F)$, $G\,\mathrm{CM} = (\lambda_G, \mu_G)$.

First note that

$$\mathrm{id}_{\mathcal{C}}\,\mathrm{CM} = (\lambda_{\mathrm{id}_{\mathcal{C}}}, \mu_{\mathrm{id}_{\mathcal{C}}}) = \mathrm{id}_{\mathcal{C}\,\mathrm{CM}}\,.$$

So we have

$$\begin{aligned}(F\,\mathrm{CM})\boldsymbol{s} &= \mathcal{C}\,\mathrm{CM} = (F\boldsymbol{s})\,\mathrm{CM}\,,\\ (\mathcal{C}\,\mathrm{CM})\boldsymbol{i} &= \mathrm{id}_{\mathcal{C}\,\mathrm{CM}} = (\mathcal{C}\boldsymbol{i})\,\mathrm{CM}\,,\\ (F\,\mathrm{CM})\boldsymbol{t} &= \mathcal{D}\,\mathrm{CM} = (F\boldsymbol{t})\,\mathrm{CM}\ .\end{aligned}$$

Now suppose given $X \in \mathrm{Ob}(\mathcal{C})$ and $(I_{\mathcal{C}} \xrightarrow{u} Y) \in \ker \boldsymbol{s}_{\mathcal{C}}$.

We have

$$(X)\big((FG)\,\mathrm{CM}\big) = XFG = (XF)G = \big((X)(F\,\mathrm{CM})\big)G = (X)\big((F\,\mathrm{CM})(G\,\mathrm{CM})\big)\,.$$

We have

$$\begin{aligned}(u)\big((FG)\,\mathrm{CM}\big) &= (I_{\mathcal{E}} \xrightarrow{uFG} YFG) = (I_{\mathcal{D}} \xrightarrow{uF} YF)(G\,\mathrm{CM}) = \big((I_{\mathcal{C}} \xrightarrow{u} Y)(F\,\mathrm{CM})\big)(G\,\mathrm{CM})\\ &= (u)\big((F\,\mathrm{CM})(G\,\mathrm{CM})\big)\,.\end{aligned}$$

So, $(FG)\,\mathrm{CM} = (F\,\mathrm{CM})(G\,\mathrm{CM})$.

Hence, CM is a functor. $\square$

The following proposition is essentially a reformulation of [**5**, Thm. 1] of Brown and Spencer.

Proposition 43

(1) *Suppose given a crossed module* $V = \big(G, M, \gamma, f\big)$.

Consider $\ker s = \{(1,m) \in G \ltimes M \colon m \in M\}$, *the kernel of the group morphism* $s\colon G \ltimes M \to G$, $(g,m) \mapsto g$; *cf. Reminder* 1.

Consider the group isomorphism

$$\pi_M\colon\ \ker s \xrightarrow{\sim} M,\ \ (1,m) \mapsto m\,.$$

We have a crossed module isomorphism given by

$$(\pi_M, \mathrm{id}_G)\colon\ V\,\mathrm{Cat}\,\mathrm{CM} \xrightarrow{\sim} V\ .$$

(2) *Suppose given an invertible monoidal category* $\mathcal{C} = \Big(\big(\,\mathrm{Mor}(\mathcal{C})\,, \mathrm{Ob}(\mathcal{C})\,, (\boldsymbol{s}, \boldsymbol{i}, \boldsymbol{t}),\ \blacktriangle\ \big),\ I\,, \otimes\Big)$.

We have the monoidal isofunctor of invertible monoidal categories

$$\begin{array}{lcccl} F\colon & \mathcal{C}\,\mathrm{CM}\,\mathrm{Cat} & \xrightarrow{\sim} & \mathcal{C} & \\ & X & \longmapsto & X & \textit{for } X \in \mathrm{Ob}(\mathcal{C}\,\mathrm{CM}\,\mathrm{Cat})\ , \\ & \Big(X \xrightarrow{(X,\, I \xrightarrow{u} Y)} X \otimes Y\Big) & \longmapsto & \big(X \xrightarrow{X \otimes u} X \otimes Y\big) & \textit{for } (X,u) \in \mathrm{Mor}(\mathcal{C}\,\mathrm{CM}\,\mathrm{Cat})\ , \end{array}$$

with inverse monoidal functor

$$\begin{array}{llll} F^{-}\colon & \mathcal{C} & \longrightarrow & \mathcal{C}\,\mathrm{CM}\,\mathrm{Cat} \\ & X & \longmapsto & X & \textit{for } X \in \mathrm{Ob}(\mathcal{C})\,, \\ (X \xrightarrow{u} Y) & & \longmapsto & \Big(X \xrightarrow{(X,\, I \xrightarrow{X^{\otimes -} \otimes u} X^{\otimes -} \otimes Y)} Y\Big) & \textit{for } (X \xrightarrow{u} Y) \in \mathrm{Mor}(\mathcal{C})\,. \end{array}$$

Proof. Ad (1). Recall that we have the invertible monoidal category $(V\mathrm{Cat}, 1_G, \cdot\,)$ with

$$\mathrm{Ob}(V\mathrm{Cat}) = G$$
$$\mathrm{Mor}(V\mathrm{Cat}) = G \ltimes M\,;$$

cf. Remark 29.

Then we have

$$V\,\mathrm{Cat}\,\mathrm{CM} = \big(\ker s, G, \tilde{\gamma}, \tilde{f}\,\big)$$

with

$$\tilde{\gamma}\colon G \to \mathrm{Aut}\,(\ker s)\,,\ g \mapsto \big((1,m) \mapsto (g,1)^{-} \cdot (1,m) \cdot (g,1) = (1, m^{g})\big)$$
$$\tilde{f} = t|_{\ker s}\colon\ \ker s \to G\,,\ (1,m) \mapsto mf\,;$$

cf. Lemma 40.

Suppose given $g \in \mathrm{Ob}(V\mathrm{Cat}\,\mathrm{CM}) = G$ and $(1,m) \in \ker s$, where $m \in M$.

We have

$$(1,m)(\tilde{f} \blacktriangle \mathrm{id}_G) = (mf)\mathrm{id}_G = mf = (1,m)(\pi_M \blacktriangle f)\,.$$

$$\begin{tikzcd} \ker s \arrow[r, "\pi_M"] \arrow[d, "\tilde{f}"'] & M \arrow[d, "f"] \\ \mathrm{Ob}(V\mathrm{Cat}\,\mathrm{CM}) \arrow[r, "\mathrm{id}_G"'] & G \end{tikzcd}$$

We have

$$\big((1,m)^{g}\big)\pi_M = (1,m^{g})\pi_M = m^{g} = \big((1,m)\pi_M\big)^{g\mathrm{id}_G}\,.$$

So (π_M, id_G) is a crossed module morphism.

Since id_G and π_M are group isomorphisms, (π_M, id_G) is a crossed module isomorphism.

Ad (2). We have $\mathcal{C}\,\mathrm{CM} = \big(\ker \boldsymbol{s}, \mathrm{Ob}(\mathcal{C})\,, \tilde{\gamma}, \bar{\boldsymbol{t}}\big)$ with

$$\tilde{\gamma}\colon\ \mathrm{Ob}(\mathcal{C}) \to \mathrm{Aut}\,(\ker \boldsymbol{s}),\ X \mapsto (u \mapsto X^{\otimes -} \otimes u \otimes X),$$

and with $\bar{\boldsymbol{t}} := \boldsymbol{t}|_{\ker \boldsymbol{s}}$; cf. Lemma 40.

Recall from Reminder 1 that we have

$$\mathcal{C}\,\mathrm{CM\,Cat} = \Big(\big(\,\mathrm{Ob}(\mathcal{C}) \ltimes \ker \boldsymbol{s},\ \mathrm{Ob}(\mathcal{C})\,, (s', i', t'),\ \blacktriangle\big), I, \otimes\Big)$$

with

$$\begin{aligned}
\tilde{s}\colon\ & \mathrm{Ob}(\mathcal{C}) \ltimes \ker \boldsymbol{s} \to \mathrm{Ob}(\mathcal{C})\,,\ (X, u) \mapsto X\,,\\
\tilde{i}\colon\ & \mathrm{Ob}(\mathcal{C}) \ltimes \ker \boldsymbol{s} \leftarrow \mathrm{Ob}(\mathcal{C})\,,\ (X, \mathrm{id}_I) \leftarrow\!\shortmid X\,,\\
\tilde{t}\colon\ & \mathrm{Ob}(\mathcal{C}) \ltimes \ker \boldsymbol{s} \to \mathrm{Ob}(\mathcal{C})\,,\ (X, u) \mapsto X \otimes u\bar{\boldsymbol{t}}\,.
\end{aligned}$$

The composition in $\mathcal{C}\,\mathrm{CM\,Cat}$ is given by

$$(X, I \xrightarrow{u} Y) \blacktriangle (X \otimes Y, I \xrightarrow{v} Z) = (X, I \xrightarrow{u \otimes v} Y \otimes Z)\,,$$

where $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and where $u,\ v \in \ker \boldsymbol{s}$.

The tensor multiplication in $\mathcal{C}\,\mathrm{CM\,Cat}$ is given by the tensor product on $\mathrm{Ob}(\mathcal{C}\,\mathrm{CM\,Cat}) = \mathrm{Ob}(\mathcal{C})$ and by

$$(X, I \xrightarrow{u} Y) \otimes (X', I \xrightarrow{u'} Y') = (X \otimes X', I \xrightarrow{(X')^{\otimes -} \otimes u \otimes X' \otimes u'} (X')^{\otimes -} \otimes Y \otimes X' \otimes Y')$$

on $\mathrm{Mor}(\mathcal{C}\,\mathrm{CM\,Cat}) = \mathrm{Ob}(\mathcal{C}) \ltimes \ker \boldsymbol{s}$, where $X,\ X',\ Y,\ Y' \in \mathrm{Ob}(\mathcal{C})$ and where $u,\ u' \in \ker \boldsymbol{s}$.

We show that F is a functor.

Suppose given $X \in \mathrm{Ob}(\mathcal{C})$, $(I \xrightarrow{u} Y) \in \ker \boldsymbol{s}$. We have

$$\begin{aligned}
\big((X, u)F\big)\boldsymbol{s} &= (X \otimes u)\boldsymbol{s} = X \otimes u\boldsymbol{s} = X \otimes I = X = XF = \big((X, u)\tilde{s}\big)F\,,\\
(XF)\boldsymbol{i} &= X\boldsymbol{i} = X \otimes \mathrm{id}_I = (X, \mathrm{id}_I)F = (X\tilde{i})F\,,\\
\big((X, u)F\big)\boldsymbol{t} &= (X \otimes u)\boldsymbol{t} = X \otimes u\boldsymbol{t} = X \otimes Y = (X \otimes Y)F = \big((X, u)\tilde{t}\big)F\,.
\end{aligned}$$

Suppose given $\big(X \xrightarrow{(X, I \xrightarrow{u} Y)} X \otimes Y \xrightarrow{(X \otimes Y, I \xrightarrow{v} Z)} X \otimes Y \otimes Z\big)$ in $\mathcal{C}\,\mathrm{CM\,Cat}$.

We write $a := (X, u)$, $b := (X \otimes Y, v)$. Note that $a \blacktriangle b = (X, I \xrightarrow{u \otimes v} X \otimes Y)$.

We have

$$\begin{aligned} aF \blacktriangle bF &= (X,u)F \blacktriangle (X\otimes Y, v)F = (X\otimes u) \blacktriangle (X \otimes Y \otimes v) \\ &= (\mathrm{id}_X \otimes u) \blacktriangle \big(\mathrm{id}_X \otimes (Y\otimes v)\big) \overset{13}{=} (\mathrm{id}_X \blacktriangle \mathrm{id}_X) \otimes \big(u \blacktriangle (Y\otimes v)\big) \\ &= X \otimes \big((u\otimes I) \blacktriangle (Y\otimes v)\big) \overset{13}{=} X \otimes \big((u \blacktriangle \mathrm{id}_Y) \otimes (\mathrm{id}_I \blacktriangle v)\big) = X\otimes (u\otimes v) \\ &= (X, u\otimes v)F = (a \blacktriangle b)F\,. \end{aligned}$$

So F is a functor.

We show that F is monoidal.

Suppose given $(X, I \xrightarrow{u} Y)$, $(X', I \xrightarrow{u'} Y') \in \mathrm{Mor}(\mathcal{C}\,\mathrm{CM}\,\mathrm{Cat}) = \mathrm{Ob}(\mathcal{C}) \ltimes \ker \boldsymbol{s}$.

We have

$$\begin{aligned} \big((X,u)\otimes (X',u')\big)F &= \big(X\otimes X', (X')^{\otimes -} \otimes u \otimes X' \otimes u'\big)F \\ &= X\otimes X' \otimes (X')^{\otimes -} \otimes u \otimes X' \otimes u' = (X\otimes u)\otimes (X'\otimes u') \\ &= (X,u)F \otimes (X',u')F\,. \end{aligned}$$

Thus, by Remark 32.(2), F is a monoidal functor.

Consider

$$\begin{array}{rccccl} G\colon & \mathcal{C} & \longrightarrow & \mathcal{C}\,\mathrm{CM}\,\mathrm{Cat} & \\ & X & \longmapsto & X & \text{for } X \in \mathrm{Ob}(\mathcal{C}) \\ & (X \xrightarrow{u} Y) & \longmapsto & (X,\ I \xrightarrow{X^{\otimes -}\otimes u} X^{\otimes -}\otimes Y) & \text{for } (X \xrightarrow{u} Y) \in \mathrm{Mor}(\mathcal{C})\,. \end{array}$$

We show that G is a functor.

Suppose given $Z \in \mathrm{Ob}(\mathcal{C})$ and $(X \xrightarrow{u} Y) \in \mathrm{Mor}(\mathcal{C})$.

We have

$$(uG)\tilde{s} = (X, X^{\otimes -}\otimes u)\tilde{s} = X = XG = (u\boldsymbol{s})G\,,$$

$$(ZG)\tilde{i} = Z\tilde{i} = (Z, \mathrm{id}_I) = (Z, Z^{\otimes -}\otimes \mathrm{id}_Z) = (Z \xrightarrow{Z\boldsymbol{i}} Z)G = (Z\boldsymbol{i})G\,,$$

$$(uG)\tilde{t} = (X, X^{\otimes -}\otimes u)\tilde{t} = X \otimes (X^{\otimes -}\otimes u)\bar{\boldsymbol{t}} = X\otimes X^{\otimes -}\otimes Y = Y = YG = (u\boldsymbol{t})G\,.$$

For $X \xrightarrow{u} Y \xrightarrow{v} Z$ in $\mathcal{C}$, we have

$$\begin{aligned} uG \blacktriangle vG &= (X, X^{\otimes -}\otimes u) \blacktriangle (Y, Y^{\otimes -}\otimes v) = (X, X^{\otimes -} \otimes u \otimes Y^{\otimes -}\otimes v) \\ &= \Big(X, X^{\otimes -} \otimes \big((u\otimes Y^{\otimes -}) \blacktriangle \mathrm{id}_I\big) \otimes (\mathrm{id}_Y \blacktriangle v)\Big) \\ &\overset{13}{=} \Big(X, X^{\otimes -}\otimes \big((u\otimes Y^{\otimes -})\otimes Y\big) \blacktriangle (I\otimes v)\Big) = \big(X, X^{\otimes -}\otimes (u\blacktriangle v)\big) = (u\blacktriangle v)G\,. \end{aligned}$$

So G is a functor.

We show that $G = F^-$.

Suppose given $\big(X \xrightarrow{(X, I \xrightarrow{u} Y)} X \otimes Y\big) \in \operatorname{Mor}(\mathcal{C}\,\mathrm{CM}\,\mathrm{Cat})$.

We have

$$(X,u)(F * G) = (X \otimes u)G = (X, X^{\otimes -} \otimes X \otimes u) = (X,u)\,.$$

So $F * G = \mathrm{id}_{\mathcal{C}\,\mathrm{CM}\,\mathrm{Cat}}$.

Suppose given $(X \xrightarrow{u} Y) \in \operatorname{Mor}(\mathcal{C})$.

We have

$$u(G * F) = (X, X^{\otimes -} \otimes u)F = X \otimes X^{\otimes -} \otimes u = u\,.$$

So $G * F = \mathrm{id}_{\mathcal{C}}$.

This shows $F^- = G$.

By Lemma 34.(3), F^- is monoidal. □

2.5 An example for a monoidal transformation: a homotopy

Example 44 Suppose given crossed modules $V := \big(M, G, \gamma, f\big)$ and $W := \big(N, H, \beta, k\big)$. Suppose given crossed module morphisms $\rho := (\lambda, \mu)\colon V \to W$ and $\tilde{\rho} := (\tilde{\lambda}, \tilde{\mu})\colon V \to W$.

Consider the invertible monoidal categories $\mathcal{C} := V\mathrm{Cat}$ and $\mathcal{D} := W\mathrm{Cat}$; cf. Remark 29.

We recall that

$$\operatorname{Ob}(\mathcal{C}) = G,\ \operatorname{Mor}(\mathcal{C}) = G \ltimes M,\ \operatorname{Ob}(\mathcal{D}) = H,\ \operatorname{Mor}(\mathcal{D}) = H \ltimes N\,.$$

Consider the monoidal functors

$$\begin{aligned} F &:= \rho\,\mathrm{Cat}\colon\ \mathcal{C} \to \mathcal{D}\,,\quad \big(g \xrightarrow{(g,m)} g \cdot mf\big) \mapsto \big(g\mu \xrightarrow{(g\mu, m\lambda)} (g \cdot mf)\mu\big) \\ \tilde{F} &:= \tilde{\rho}\,\mathrm{Cat}\colon\ \mathcal{C} \to \mathcal{D}\,,\quad \big(g \xrightarrow{(g,m)} g \cdot mf\big) \mapsto \big(g\tilde{\mu} \xrightarrow{(g\tilde{\mu}, m\tilde{\lambda})} (g \cdot mf)\tilde{\mu}\big)\,; \end{aligned}$$

cf. Lemma 39.(1).

A map $\chi\colon G \to N$ is called a *homotopy* from ρ to $\tilde{\rho}$ if the following conditions $(1,2,3)$ are satisfied; cf. [**16,** §4].

(1) We have $(g\mu)^- \cdot g\tilde{\mu} = (g)(\chi \blacktriangle k)$ for $g \in G$.

(2) We have $(m\lambda)^- \cdot m\tilde{\lambda} = (m)(f \blacktriangle \chi)$ for $m \in M$.

(3) We have $(g \cdot g')\chi = (g\chi)^{g'\mu} \cdot g'\chi$ for $g, g' \in G$.

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, bend right, "\lambda"'] \arrow[d, bend left, "\tilde{\lambda}"] & G \arrow[dl, "\chi"'] \arrow[d, bend right, "\mu"'] \arrow[d, bend left, "\tilde{\mu}"] \\
N \arrow[r, "k"'] & H
\end{tikzcd}$$

Suppose given a homotopy $\chi\colon G \to N$ from ρ to $\tilde{\rho}$.

Then the tuple of morphisms given by

$$\eta := \chi\,\mathrm{Cat} = \big((g)(\chi\,\mathrm{Cat})\big)_{g\in G} := \ \big(gF \xrightarrow{(g\mu,\, g\chi)} g\tilde{F}\big)_{g\in G}$$

is a monoidal transformation from F to $\tilde{F}$.

The tuple η is well-defined:

For $g \in G$, we have

$$\begin{aligned}
(g\eta)s &= (g\mu, g\chi)s = g\mu = gF\,,\\
(g\eta)t &= (g\mu, g\chi)t = g\mu \cdot \big((g)(\chi \blacktriangle k)\big) \overset{(1)}{=} g\tilde{\mu} = g\tilde{F}\,.
\end{aligned}$$

The tuple η is a transformation from F to $\tilde{F}$:

Suppose given $\big(g \xrightarrow{(g,m)} g \cdot mf\big)$ in $\mathcal{C}$.

Note that

$$\begin{aligned}
(g \cdot mf)\chi &\overset{(3)}{=} (g\chi)^{mf\mu} \cdot \big((m)(f \blacktriangle \chi)\big) \overset{(2)}{=} (g\chi)^{mf\mu} \cdot (m\lambda)^- \cdot m\tilde{\lambda} = (g\chi)^{m\lambda k} \cdot (m\lambda)^- \cdot m\tilde{\lambda}\\
&\overset{\text{(CM2)}}{=} (m\lambda)^- \cdot g\chi \cdot m\lambda \cdot (m\lambda)^- \cdot m\tilde{\lambda} = (m\lambda)^- \cdot g\chi \cdot m\tilde{\lambda}\,.
\end{aligned}$$

So we have

$$\begin{aligned}\big((g,m)F\big) \blacktriangle \big((g\cdot mf)\eta\big) &= (g\mu, m\lambda) \blacktriangle \big((g\cdot mf)\mu, (g\cdot mf)\chi\big) = \big(g\mu,\, m\lambda\cdot(g\cdot mf)\chi\big) \\ &= (g\mu,\, g\chi\cdot m\tilde{\lambda}) = (g\mu, g\chi) \blacktriangle (g\tilde{\mu}, m\tilde{\lambda}) = (g\eta) \blacktriangle \big((g,m)\tilde{F}\big)\,.\end{aligned}$$

$$\begin{tikzcd} gF \arrow[r, "g\eta"] \arrow[d, "(g,m)F"'] & g\tilde{F} \arrow[d, "(g,m)\tilde{F}"] \\ (g\cdot mf)F \arrow[r, "(g\cdot mf)\eta"'] & (g\cdot mf)\tilde{F} \end{tikzcd}$$

The transformation η is monoidal:

Concerning the tensor products on $\mathcal{C}$ and on $\mathcal{D}$, cf. Remark 29.

For $g, g' \in G$, we have

$$\begin{aligned}(g\cdot g')\eta &= \big((g\cdot g')\mu,\, (g\cdot g')\chi\big) \overset{(3)}{=} \big(g\mu\cdot g'\mu,\, (g\chi)^{g'\mu}\cdot g'\chi\big) = (g\mu, g\chi)\cdot(g'\mu, g'\chi) \\ &= g\eta\cdot g'\eta\,.\end{aligned}$$

Then, by Remark 36, η is monoidal.

Chapter 3

The symmetric crossed module on a category

Let $\mathcal{X} = \big(\operatorname{Mor}(\mathcal{X}), \operatorname{Ob}(\mathcal{X}), (s, i, t), \blacktriangle\big)$ be a category.
Let $V = \big(M, G, \gamma, f\big)$ be a crossed module.

3.1 Definition of the symmetric crossed module on a category

Lemma 45 (The groups $\mathrm{G}_{\mathcal{X}}$ and $\mathrm{M}_{\mathcal{X}}$)

(1) *Consider the set*

$$\mathrm{G}_{\mathcal{X}} := \operatorname{Aut}(\mathcal{X}) = \{\big(\mathcal{X} \xrightarrow{F} \mathcal{X}\big) \colon F \text{ is an autofunctor}\}$$

together with the composition of functors $(*)$*. Then* $(\mathrm{G}_{\mathcal{X}}, *)$ *is a group and its neutral element is given by* $\mathrm{id}_{\mathcal{X}}$ *.*

(2) *Consider the set*

$$\mathrm{M}_{\mathcal{X}} := \{\big(\mathrm{id}_{\mathcal{X}} \xrightarrow{a} F\big) \colon F \in \operatorname{Aut}(\mathcal{X}) \text{ and } a \text{ is an isotransformation}\}.$$

On $\mathrm{M}_{\mathcal{X}}$ *, we define a multiplication by*

$$\big(\mathrm{id}_{\mathcal{X}} \underset{\sim}{\xrightarrow{a}} F\big) * \big(\mathrm{id}_{\mathcal{X}} \underset{\sim}{\xrightarrow{b}} G\big) := \big(\mathrm{id}_{\mathcal{X}} \underset{\sim}{\xrightarrow{a*b}} FG\big) = a \blacktriangle (Fb) = b \blacktriangle (aG)\,;$$

cf. §0.3 item 3.

$$\begin{tikzcd} \mathrm{id}_{\mathcal{X}} \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, "a*b"] & G \arrow[d, "aG"] \\ F \arrow[r, "Fb"'] & FG \end{tikzcd}$$

Then $(\mathrm{M}_{\mathcal{X}}, *)$ *is a group.*

Its neutral element is $\big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{\mathrm{id}_{\mathrm{id}_{\mathcal{X}}}} \mathrm{id}_{\mathcal{X}}\big)$. *The inverse of* $\big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{a} F\big) \in \mathrm{M}_{\mathcal{X}}$ *with respect to* $(*)$ *is given by* $a^{*-} := a^- F^- = \big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{a^- F^-} F^-\big)$.

For an example how to calculate $\mathrm{G}_{\mathcal{X}}$ and $\mathrm{M}_{\mathcal{X}}$ in case $\mathcal{X} = V\mathrm{Cat}$ for a crossed module V, cf. §A.2, §A.5, §A.6.

Proof. Ad (1). The composition of functors is associative, and therefore, the multiplication in $\mathrm{G}_{\mathcal{X}}$ is associative.

Suppose given $F, G \in \mathrm{G}_{\mathcal{X}}$.

We have $F * G \in \mathrm{G}_{\mathcal{X}}$, since FG is an autofunctor.

We have $F * \mathrm{id}_{\mathcal{X}} = F$ and $\mathrm{id}_{\mathcal{X}} * F = F$. Therefore, $1_{\mathrm{G}_{\mathcal{X}}} = \mathrm{id}_{\mathcal{X}}$.

We have $F * F^- = \mathrm{id}_{\mathcal{X}}$ and $F^- * F = \mathrm{id}_{\mathcal{X}}$. Therefore, the inverse for $F \in \mathrm{G}_{\mathcal{X}}$ is given by F^-.

Ad (2). Note that the multiplication $(*)$ is the horizontal composition of transformations; cf. §0.3 item 3. So in particular, $(*)$ is associative.

Suppose given $\big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{a} F\big), \big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{b} G\big) \in \mathrm{M}_{\mathcal{X}}$.

We have $a * b = a \blacktriangle Fb = b \blacktriangle aG \colon \mathrm{id}_{\mathcal{X}} \to FG$. So, $a * b$ is an isotransformation to an autofunctor FG. Therefore, $a * b \in \mathrm{M}_{\mathcal{X}}$.

We have

$$a * \mathrm{id}_{\mathrm{id}_{\mathcal{X}}} = \big(\mathrm{id}_{\mathcal{X}} \xrightarrow{a} F\big) * \big(\mathrm{id}_{\mathcal{X}} \xrightarrow{\mathrm{id}_{\mathrm{id}_{\mathcal{X}}}} \mathrm{id}_{\mathcal{X}}\big) = \mathrm{id}_{\mathrm{id}_{\mathcal{X}}} \blacktriangle (a\, \mathrm{id}_{\mathcal{X}}) = a\,,$$

and we have

$$\mathrm{id}_{\mathrm{id}_{\mathcal{X}}} * a = \big(\mathrm{id}_{\mathcal{X}} \xrightarrow{\mathrm{id}_{\mathrm{id}_{\mathcal{X}}}} \mathrm{id}_{\mathcal{X}}\big) * \big(\mathrm{id}_{\mathcal{X}} \xrightarrow{a} F\big) = a \blacktriangle (\mathrm{id}_{\mathrm{id}_{\mathcal{X}}} F) = a \blacktriangle \mathrm{id}_F = a\,.$$

Therefore, $1_{\mathrm{M}_\mathcal{X}} = \mathrm{id}_{\mathrm{id}_\mathcal{X}}$.

We have $a^{*-} = a^- F^- \colon \mathrm{id}_\mathcal{X} \to F^-$. So a^{*-} is an isotransformation where F^- is an isofunctor. Therefore, $a^{*-} \in \mathrm{M}_\mathcal{X}$.

Further, we have

$$\begin{aligned} a * a^{*-} &= \big(\mathrm{id}_\mathcal{X} \xrightarrow{a} F\big) * \big(\mathrm{id}_\mathcal{X} \xrightarrow{a^- F^-} F^-\big) = (a^- F^-) \blacktriangle (aF^-) = (a^- \blacktriangle a)F^- \\ &= \mathrm{id}_F F^- = \mathrm{id}_{FF^-} = \mathrm{id}_{\mathrm{id}_\mathcal{X}} = 1_{\mathrm{M}_\mathcal{X}} , \end{aligned}$$

and we have

$$a^{*-} * a = \big(\mathrm{id}_\mathcal{X} \xrightarrow{a^- F^-} F^-\big) * \big(\mathrm{id}_\mathcal{X} \xrightarrow{a} F\big) = a \blacktriangle (a^- F^- F) = a \blacktriangle a^- = \mathrm{id}_{\mathrm{id}_\mathcal{X}} = 1_{\mathrm{M}_\mathcal{X}} .$$

Therefore, a^{*-} is the inverse of a. □

Remark 46 (Inverses in $\mathrm{M}_\mathcal{X}$) *Suppose given* $a = \big(\mathrm{id}_\mathcal{X} \xrightarrow[\sim]{a} F\big) \in \mathrm{M}_\mathcal{X}$ *; cf. Lemma* 45.(2). *We have*

$$a^{*-} = a^- F^- = F^- a^- .$$

Proof. We have

$$a * (F^- a^-) = a \blacktriangle (FF^- a^-) = a \blacktriangle a^- = \mathrm{id}_{\mathrm{id}_\mathcal{X}} .$$

Therefore, $F^- a^- = a^{*-} = a^- F^-$; cf. Lemma 45.(2). □

Remark 47 *Suppose given functors* $F, G \colon \mathcal{X} \to \mathcal{X}$.

Suppose given transformations $\big(\mathrm{id}_\mathcal{X} \xrightarrow{a} F\big)$ *and* $\big(\mathrm{id}_\mathcal{X} \xrightarrow{b} G\big)$ *such that* $a * b = b * a = \mathrm{id}_{\mathrm{id}_\mathcal{X}}$ *holds.*

Then we have the following statements $(1, 2)$.

(1) *We have* $F,\, G \in \mathrm{Aut}\,(\mathcal{X})$*, i.e. the functors* F *and* G *are autofunctors. Moreover, we have* $G = F^-$.

(2) *The transformations* a *and* b *are isotransformations.*

Proof. Ad (1). We have the following commutative diagram.

$$\begin{tikzcd} \mathrm{id}_\mathcal{X} \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, "a * b"] & G \arrow[d, "aG"] \\ F \arrow[r, "Fb"'] & FG \end{tikzcd}$$

Since we have $a * b = \mathrm{id}_{\mathrm{id}_{\mathcal{X}}} \colon \mathrm{id}_{\mathcal{X}} \to \mathrm{id}_{\mathcal{X}}$ by assumption, we have $FG = \mathrm{id}_{\mathcal{X}}$.

Likewise, we have $GF = \mathrm{id}_{\mathcal{X}}$.

Ad (2). From (1), we know that $G = F^-$.

We have

$$a \blacktriangle (Fb) = a * b = \mathrm{id}_{\mathrm{id}_{\mathcal{X}}} ,$$

and

$$(Fb) \blacktriangle a = (Fb) \blacktriangle \big((FF^-)a\big) = F\big(b \blacktriangle (F^- a)\big) = F(b * a) = F\,\mathrm{id}_{\mathrm{id}_{\mathcal{X}}} = \mathrm{id}_F .$$

Therefore, $a^- = Fb$ and a is an isotransformation.

Likewise, we have $b^- = Ga$ and b is an isotransformation. □

Lemma 48 (Symmetric crossed module)

Consider the groups

$$\begin{aligned} \mathrm{G}_{\mathcal{X}} &= \mathrm{Aut}\,(\mathcal{X}) \\ \mathrm{M}_{\mathcal{X}} &= \{\big(\mathrm{id}_{\mathcal{X}} \xrightarrow{a} F\big) \colon F \in \mathrm{Aut}\,(\mathcal{X}) \text{ and } a \text{ is an isotransformation } \} \end{aligned}$$

from Lemma 45*.*

We have an action of $\mathrm{G}_{\mathcal{X}}$ *on* $\mathrm{M}_{\mathcal{X}}$ *given by the group morphism*

$$\begin{aligned} \gamma_{\mathcal{X}} \colon\ \mathrm{G}_{\mathcal{X}} &\longrightarrow \mathrm{Aut}\,(\mathrm{M}_{\mathcal{X}}) \\ G &\longmapsto \Big(\big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{a} F\big) \mapsto \big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{G^- a G} G^- F G\big)\Big) , \end{aligned}$$

and a group morphism

$$\mathrm{f}_{\mathcal{X}} \colon\ \mathrm{M}_{\mathcal{X}} \to \mathrm{G}_{\mathcal{X}} ,\ \ \big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{a} F\big) \mapsto F\ .$$

Then, $\big(\,\mathrm{M}_{\mathcal{X}}, \mathrm{G}_{\mathcal{X}}, \gamma_{\mathcal{X}}, \mathrm{f}_{\mathcal{X}}\,\big)$ *is a crossed module, called the* symmetric crossed module *on* $\mathcal{X}$*. We write*

$$\mathrm{S}_{\mathcal{X}} := \big(\,\mathrm{M}_{\mathcal{X}}, \mathrm{G}_{\mathcal{X}}, \gamma_{\mathcal{X}}, \mathrm{f}_{\mathcal{X}}\,\big) .$$

For $G \in \mathrm{G}_{\mathcal{X}}$ *and* $\big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{a} F\big) \in \mathrm{M}_{\mathcal{X}}$*, we write*

$$a^G := (a)(G\,\gamma_{\mathcal{X}}) = G^- a G \colon \mathrm{id}_{\mathcal{X}} \to F^G = G^- F G$$

for the action of G *on* a*.*

For examples of symmetric crossed modules, cf. Example 54 and §A.2−§A.9.

Proof. We show that $\gamma_{\mathcal{X}}$ is well-defined.

Suppose given $G \in \mathrm{G}_{\mathcal{X}}$ and $\big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{a} F\big)$, $\big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{b} H\big) \in \mathrm{M}_{\mathcal{X}}$.

We have

$$G^{-}aG\colon \mathrm{id}_{\mathcal{X}} \to G^{-}FG\,,$$

where $G^{-}FG$ is an autofunctor of $\mathcal{X}$ and $G^{-}aG$ is an isotransformation. So $G^{-}aG \in \mathrm{M}_{\mathcal{X}}$.

We have

$$\begin{aligned} G^{-}(a * b)G &= G^{-}(a \blacktriangle Fb)G = G^{-}aG \blacktriangle G^{-}FbG = G^{-}aG \blacktriangle (G^{-}FG)(G^{-}bG) \\ &= (G^{-}aG) * (G^{-}bG)\,. \end{aligned}$$

Moreover, we have

$$(G^{-})^{-}(G^{-}aG)G^{-} = GG^{-}aGG^{-} = a\,,$$

and

$$G^{-}\big((G^{-})^{-}aG^{-}\big)G = G^{-}GaG^{-}G = a\,.$$

This shows that $\gamma_{\mathcal{X}}$ is a well-defined map from $\mathrm{G}_{\mathcal{X}}$ to $\mathrm{Aut}(\mathrm{M}_{\mathcal{X}})$.

We show that $\gamma_{\mathcal{X}}$ is a group morphism.

Suppose given $G, H \in \mathrm{G}_{\mathcal{X}}$ and suppose given $\big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{a} F\big) \in \mathrm{M}_{\mathcal{X}}$.

We have

$$\begin{aligned} (a)\big((GH)\gamma_{\mathcal{X}}\big) &= \big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{(GH)^{-}a\,(GH)} (GH)^{-}F(GH)\big) = \big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{H^{-}G^{-}a\,GH} H^{-}G^{-}FGH\big) \\ &= \big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{G^{-}aG} G^{-}FG\big)(H\,\gamma_{\mathcal{X}}) = \big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{a} F\big)(G\,\gamma_{\mathcal{X}})(H\,\gamma_{\mathcal{X}}) \\ &= (a)(G\,\gamma_{\mathcal{X}})(H\,\gamma_{\mathcal{X}})\,. \end{aligned}$$

Thus, $\gamma_{\mathcal{X}}$ is a group morphism.

We show that $\mathrm{f}_{\mathcal{X}}$ is a group morphism. Suppose given $\big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{a} F\big)$, $\big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{b} G\big) \in \mathrm{M}_{\mathcal{X}}$.

We have

$$(a * b)\,\mathrm{f}_{\mathcal{X}} = \big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{a*b} FG\big)\,\mathrm{f}_{\mathcal{X}} = FG = \big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{a} F\big)\,\mathrm{f}_{\mathcal{X}} * \big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{b} G\big)\,\mathrm{f}_{\mathcal{X}} = (a\,\mathrm{f}_{\mathcal{X}}) * (b\,\mathrm{f}_{\mathcal{X}})\,.$$

Ad (CM1). Suppose given $\big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{a} F\big) \in \mathrm{M}_{\mathcal{X}}$ and $G \in \mathrm{G}_{\mathcal{X}}$. We have

$$(a^G)\,\mathrm{f}_{\mathcal{X}} = \big(\mathrm{id}_X \xrightarrow[\sim]{G^- aG} G^- FG\big)\,\mathrm{f}_{\mathcal{X}} = G^- FG = F^G = (a\,\mathrm{f}_{\mathcal{X}})^G .$$

Ad (CM2). Suppose given $\big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{a} F\big)$, $\big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{b} G\big) \in \mathrm{M}_{\mathcal{X}}$. We have

$$\begin{aligned} a^b = b^{*-} * a * b &= \big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{b^- G^-} G^-\big) * \big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{a} F\big) * \big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{b} G\big) \\ &= \big((b^- G^-) \blacktriangle (G^- a)\big) * \big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{b} G\big) = b \blacktriangle \big((b^- G^-) \blacktriangle (G^- a)\big)G = b \blacktriangle (b^- G^- G) \blacktriangle (G^- aG) \\ &= b \blacktriangle\ b^- \blacktriangle (G^- aG) = G^- aG = a^G = a^{b\,\mathrm{f}_{\mathcal{X}}} . \end{aligned}$$

□

3.2 Inner automorphisms of a category

Lemma 49 (Construction of isotransformations)
Suppose given bijective maps φ, ψ: $\mathrm{Ob}(\mathcal{X}) \to \mathrm{Ob}(\mathcal{X})$. Suppose given tuples of isomorphisms $a = \big(X \xrightarrow[\sim]{Xa} X\varphi\big)_{X \in \mathrm{Ob}(\mathcal{X})}$ and $b = \big(X \xrightarrow[\sim]{Xb} X\psi\big)_{X \in \mathrm{Ob}(\mathcal{X})}$.

(1) *We can define a functor $F_a \colon \mathcal{X} \to \mathcal{X}$ by letting*

$$\begin{array}{lcll} XF_a & := & X\varphi & \text{for } X \in \mathrm{Ob}(\mathcal{X}) , \\ uF_a & := & (Xa)^- \blacktriangle u \blacktriangle Ya \colon X\varphi \to Y\varphi & \text{for } \big(X \xrightarrow{u} Y\big) \in \mathrm{Mor}(\mathcal{X}) . \end{array}$$

Then F_a is an autofunctor of $\mathcal{X}$.

Its inverse is given as follows. Consider the tuple of isomorphisms

$$\tilde{a} := \big(X \xrightarrow[\sim]{X\tilde{a} := ((X\varphi^-)a)^-} X\varphi^-\big)_{X \in \mathrm{Ob}(\mathcal{X})} .$$

Then $F_{\tilde{a}}$ is the inverse of F_a.

For $\big(X \xrightarrow{u} Y\big) \in \mathrm{Mor}(\mathcal{X})$, we also write $u^a := uF_a = (Xa)^- \blacktriangle u \blacktriangle Ya$.

(2) *The tuple a is an isotransformation from* $\mathrm{id}_{\mathcal{X}}$ *to F_a.*

(3) *We have* $F_a * F_b = F_{a*b}$.

Proof. Ad (1). We show that F_a is a functor.

Suppose given $X \xrightarrow{u} Y \xrightarrow{v} Z$ in $\mathcal{X}$.

We have

$$\begin{aligned} (uF_a)\boldsymbol{s} &= X\varphi = XF_a = (u\boldsymbol{s})F_a \\ (XF_a)\boldsymbol{i} &= \mathrm{id}_{X\varphi} = (Xa)^- \blacktriangle Xa = (Xa)^- \blacktriangle \mathrm{id}_X \blacktriangle Xa = \mathrm{id}_X F_a = (X\boldsymbol{i})F_a \\ (uF_a)\boldsymbol{t} &= Y\varphi = YF_a = (u\boldsymbol{t})F_a \,. \end{aligned}$$

Further, we have

$$(u \blacktriangle v)F_a = (Xa)^- \blacktriangle u \blacktriangle v \blacktriangle Za = (Xa)^- \blacktriangle u \blacktriangle Ya \blacktriangle (Ya)^- \blacktriangle v \blacktriangle Za = (uF_a) \blacktriangle (vF_a)\,.$$

So, $F_a \colon \mathcal{X} \to \mathcal{X}$ is a functor.

Then $F_{\tilde{a}} \colon \mathcal{X} \to \mathcal{X}$ is a functor as well.

We show that $F_{\tilde{a}}$ is the inverse of F_a .

For $\big(X \xrightarrow{u} Y\big) \in \mathrm{Mor}(\mathcal{X})$, we have

$$\begin{aligned} u(F_a * F_{\tilde{a}}) &= \big((Xa)^- \blacktriangle u \blacktriangle Ya\big)F_{\tilde{a}} = \big((X\varphi)\tilde{a}\big)^- \blacktriangle (Xa)^- \blacktriangle u \blacktriangle Ya \blacktriangle (Y\varphi)\tilde{a} \\ &= \Big(\Big(\big((X\varphi)\varphi^-\big)a\Big)^-\Big)^- \blacktriangle (Xa)^- \blacktriangle u \blacktriangle Ya \blacktriangle \Big(\big((Y\varphi)\varphi^-\big)a\Big)^- \\ &= Xa \blacktriangle (Xa)^- \blacktriangle u \blacktriangle Ya \blacktriangle (Ya)^- = u\,. \end{aligned}$$

This shows $F_a * F_{\tilde{a}} = \mathrm{id}_{\mathcal{X}}$.

For $\big(X \xrightarrow{u} Y\big) \in \mathrm{Mor}(\mathcal{X})$, we have

$$\begin{aligned} u(F_{\tilde{a}} * F_a) &= \big((X\tilde{a})^- \blacktriangle u \blacktriangle Y\tilde{a}\big)F_a = \big((X\varphi^-)a\big)^- \blacktriangle (X\tilde{a})^- \blacktriangle u \blacktriangle Y\tilde{a} \blacktriangle (Y\varphi^-)a \\ &= X\tilde{a} \blacktriangle (X\tilde{a})^- \blacktriangle u \blacktriangle Y\tilde{a} \blacktriangle (Y\tilde{a})^- = u\,. \end{aligned}$$

This shows $F_{\tilde{a}} * F_a = \mathrm{id}_{\mathcal{X}}$.

Therefore $F_{\tilde{a}} = (F_a)^-$. In particular, F_a is an autofunctor.

Ad (2). We show that $a = \big(X \xrightarrow{Xa} X\varphi\big)_{X \in \mathrm{Ob}(\mathcal{X})}$ is a transformation from $\mathrm{id}_{\mathcal{X}}$ to F_a .

For $\big(X \xrightarrow{u} Y\big) \in \mathrm{Mor}(\mathcal{X})$, we have

$$Xa \blacktriangle uF_a = Xa \blacktriangle (Xa)^- \blacktriangle u \blacktriangle Ya = u \blacktriangle Ya\,.$$

Therefore, we have the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "Xa"] \arrow[d, "u"'] & XF_a \arrow[d, "uF_a"] \\ Y \arrow[r, "Ya"'] & YF_a \end{tikzcd}$$

So a is a transformation from $\mathrm{id}_{\mathcal{X}}$ to F_a . Since it consists of isomorphisms, it is an isotransformation.

Ad (3). By (2), we know that $a = \big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{a} F_a\big)$ is an isotransformation from $\mathrm{id}_{\mathcal{X}}$ to F_a and $b = \big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{b} F_b\big)$ is an isotransformation from $\mathrm{id}_{\mathcal{X}}$ to F_b .

Recall that $a * b = a \blacktriangle F_a\, b \colon \mathrm{id}_{\mathcal{X}} \to F_a * F_b$; cf. Lemma 45.(2). Note that, for $X \in \mathrm{Ob}(\mathcal{X})$, we have

$$X(F_a * F_b) = (X\varphi)F_b = X\varphi\psi\,.$$

Moreover,

$$X(a * b) = X(a \blacktriangle F_a\, b) = Xa \blacktriangle (X\varphi)b$$

is an isomorphism, for $X \in \mathrm{Ob}(\mathcal{X})$.

Consider the isotransformation

$$a * b = \big(X \xrightarrow[\sim]{X(a \blacktriangle F_a b)} X(F_a * F_b)\big)_{X \in \mathrm{Ob}(\mathcal{X})} = \big(X \xrightarrow[\sim]{X(a \blacktriangle F_a b)} X\varphi\psi\big)_{X \in \mathrm{Ob}(\mathcal{X})}\,.$$

By (1), the functor $F_{a*b} \colon \mathcal{X} \to \mathcal{X}$ is defined by the following construction.

$$\begin{array}{rcll} XF_{a*b} & = & X\varphi\psi & \text{for } X \in \mathrm{Ob}(\mathcal{X}) \\ uF_{a*b} & = & \big(X(a*b)\big)^{-} \blacktriangle u \blacktriangle \big(Y(a*b)\big) \colon X\varphi\psi \to Y\varphi\psi & \text{for } \big(X \xrightarrow{u} Y\big) \in \mathrm{Mor}(\mathcal{X}) \end{array}$$

For $\big(X \xrightarrow{u} Y\big) \in \mathrm{Mor}(\mathcal{X})$, we have

$$\begin{aligned} (u)F_{a*b} &= \big(X(a*b)\big)^{-} \blacktriangle u \blacktriangle Y(a*b) = \big(X(a \blacktriangle F_a\, b)\big)^{-} \blacktriangle u \blacktriangle Y(a \blacktriangle F_a\, b) \\ &= \big(Xa \blacktriangle XF_a\, b\big)^{-} \blacktriangle u \blacktriangle Ya \blacktriangle YF_a\, b \\ &= \big(Xa \blacktriangle (X\varphi)\, b\big)^{-} \blacktriangle u \blacktriangle Ya \blacktriangle (Y\varphi)\, b \\ &= \big((X\varphi)\, b\big)^{-} \blacktriangle (Xa)^{-} \blacktriangle u \blacktriangle Ya \blacktriangle (Y\varphi)\, b = \big((Xa)^{-} \blacktriangle u \blacktriangle Ya\big)\, F_b \\ &= (u)\, F_a\, F_b\,. \end{aligned}$$

So, $F_{a*b} = F_a * F_b$. $\square$

Definition 50 (Inner automorphism) Let $F \in \mathrm{Aut}(\mathcal{X})$.

If $\mathrm{id}_{\mathcal{X}} \simeq F$ then we call F an *inner automorphism*. We write

$$\mathrm{Inn}(\mathcal{X}) := \{F \in \mathrm{Aut}(\mathcal{X}) \colon F \text{ is an inner automorphism}\} = \{F \in \mathrm{Aut}(\mathcal{X}) \colon \mathrm{id}_{\mathcal{X}} \simeq F\};$$

cf. Lemma 52 below.

Remark 51 Let $F \in \mathrm{Aut}(\mathcal{X})$ be an inner automorphism.

Suppose given an isotransformation $\big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{a} F\big)$.

We have $F = F_a$; cf. Lemma 49.

Proof. Since F is an automorphism, we have the bijection $\varphi\colon \mathrm{Ob}(\mathcal{X}) \to \mathrm{Ob}(\mathcal{X})$ given by $X\varphi := XF$ for $X \in \mathrm{Ob}(\mathcal{X})$.

Note that $a = \big(X \xrightarrow{Xa} X\varphi\big)_{X \in \mathrm{Ob}(\mathcal{X})}$. So, $XF = X\varphi = XF_a$ for $X \in \mathrm{Ob}(\mathcal{X})$.

For $\big(X \xrightarrow{u} Y\big) \in \mathrm{Mor}(\mathcal{X})$, we have the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "Xa"] \arrow[d, "u"'] & XF \arrow[d, "uF"] \\ Y \arrow[r, "Ya"'] & YF \end{tikzcd}$$

Hence, $uF = (Xa)^{-}\, u\, (Ya) = u^a = uF_a$. $\square$

Lemma 52 (Inner automorphism group)

Consider the symmetric crossed module $\mathrm{S}_{\mathcal{X}} = \big(\mathrm{M}_{\mathcal{X}}, \mathrm{G}_{\mathcal{X}}, \gamma_{\mathcal{X}}, \mathrm{f}_{\mathcal{X}}\big)$*; cf. Lemma* 48.

We have $\mathrm{Inn}(\mathcal{X}) = \mathrm{M}_{\mathcal{X}}\,\mathrm{f}_{\mathcal{X}} \trianglelefteqslant \mathrm{G}_{\mathcal{X}} = \mathrm{Aut}(\mathcal{X})$*; cf. Definition* 50.

We call $\mathrm{Inn}(\mathcal{X})$ *the* inner automorphism group *of the category* $\mathcal{X}$.

Proof. Let $F \in \mathrm{Aut}(\mathcal{X})$. Then

$$F \in \mathrm{Inn}(\mathcal{X}) \quad \Leftrightarrow \quad \text{We have } \mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{a} F \text{ for an isotransformation } a \quad \Leftrightarrow \quad F \in \mathrm{M}_{\mathcal{X}}\,\mathrm{f}_{\mathcal{X}}.$$

Therefore, $\mathrm{Inn}(\mathcal{X}) = \mathrm{M}_{\mathcal{X}}\,\mathrm{f}_{\mathcal{X}}$.

We have $\mathrm{M}_{\mathcal{X}}\,\mathrm{f}_{\mathcal{X}} \leqslant \mathrm{G}_{\mathcal{X}}$ since $\mathrm{f}_{\mathcal{X}}$ is a group morphism.

Further, we have $\mathrm{M}_{\mathcal{X}}\,\mathrm{f}_{\mathcal{X}} \trianglelefteqslant \mathrm{G}_{\mathcal{X}}$; cf. e.g. [**15,** Lem. 7.(2)]. $\square$

Remark 53 *Consider the symmetric crossed module* $\mathrm{S}_{\mathcal{X}} = \big(\mathrm{M}_{\mathcal{X}}, \mathrm{G}_{\mathcal{X}}, \gamma_{\mathcal{X}}, \mathrm{f}_{\mathcal{X}}\big)$.

(1) *We have* $\mathrm{S}_{\mathcal{X}}\,\pi_1 = \mathrm{Aut}\,(\mathrm{id}_{\mathcal{X}})$.

(2) *We have* $\mathrm{S}_{\mathcal{X}}\,\pi_0 = \mathrm{Aut}\,(\mathcal{X})/\,\mathrm{Inn}(\mathcal{X})$.

Cf. §0.4 *item* 5.

Proof. Ad (1). Recall that $\mathrm{S}_{\mathcal{X}}\,\pi_1 = \ker(\mathrm{f}_{\mathcal{X}})$.
Suppose given $\big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{a} F\big) \in \mathrm{M}_{\mathcal{X}}$. We have

$$a \in \ker \mathrm{f}_{\mathcal{X}} \quad\Leftrightarrow\quad a\,\mathrm{f}_{\mathcal{X}} = \mathrm{id}_{\mathcal{X}} \quad\Leftrightarrow\quad F = \mathrm{id}_{\mathcal{X}} \quad\Leftrightarrow\quad a \in \mathrm{Aut}\,(\mathrm{id}_{\mathcal{X}})\,.$$

So, $\ker \mathrm{f}_{\mathcal{X}} = \mathrm{Aut}\,(\mathrm{id}_{\mathcal{X}})$.
Ad (2). Recall that $\mathrm{S}_{\mathcal{X}}\,\pi_0 = \mathrm{G}_{\mathcal{X}}\,/\,\mathrm{M}_{\mathcal{X}}\,\mathrm{f}_{\mathcal{X}}$.
We have $\mathrm{G}_{\mathcal{X}} = \mathrm{Aut}\,(\mathcal{X})$ and $\mathrm{M}_{\mathcal{X}}\,\mathrm{f}_{\mathcal{X}} = \mathrm{Inn}(\mathcal{X})$; cf. Lemma 45 and Lemma 52.
So, $\mathrm{S}_{\mathcal{X}}\,\pi_0 = \mathrm{Aut}\,(\mathcal{X})/\,\mathrm{Inn}(\mathcal{X})$. □

3.3 An example for a symmetric crossed module

Example 54 Let G be a group. We have a category $G\mathcal{C}$ with $\mathrm{Ob}(G\mathcal{C}) := \{G\}$ and $\mathrm{Mor}(G\mathcal{C}) := \{g\colon g \in G\} = G$. Composition is given by multiplication in G.

Consider the symmetric crossed module $\mathrm{S}_{G\mathcal{C}} = \big(\mathrm{M}_{G\mathcal{C}}, \mathrm{G}_{G\mathcal{C}}, \mathrm{f}_{G\mathcal{C}}, \gamma_{G\mathcal{C}}\big)$; cf. Lemma 48.

Consider the crossed module $\big(G, \mathrm{Aut}\,(G), \mathrm{id}_{\mathrm{Aut}\,(G)}, c\big)$ with

$$c\colon G \to \mathrm{Aut}\,(G)\,,\ \ g \mapsto (x \mapsto x^g)\ ;$$

cf. e.g. [**15,** Ex. 8].

We want to show that $\big(G, \mathrm{Aut}\,(G), \mathrm{id}_{\mathrm{Aut}\,(G)}, c\big) \overset{!}{\simeq} \mathrm{S}_{G\mathcal{C}}$.

We show that $\mathrm{G}_{G\mathcal{C}} \overset{!}{\simeq} \mathrm{Aut}(G)$.

Suppose given $F \in \mathrm{G}_{G\mathcal{C}}$, i.e. $F\colon G\mathcal{C} \to G\mathcal{C}$ is an autofunctor.

Then we have $\mathrm{Ob}(F) : \{G\} \to \{G\}$, $G \mapsto G$. Moreover, we have the group isomorphism $\mathrm{Mor}(F) : G \to G$, $g \mapsto gF$.

Conversely, each group isomorphism $\varphi\colon G \to G$ yields an autofunctor $\varphi\mathcal{C}$ that consists $\mathrm{Ob}(\varphi\mathcal{C}) : \{G\} \to \{G\}$, $G \mapsto G$ and of $\mathrm{Mor}(\varphi\mathcal{C}) : G \to G$, $g \mapsto g\varphi$.

So we have the group isomorphism

$$\mu\colon\ \mathrm{Aut}(G) \to \mathrm{G}_{G\mathcal{C}}\,,\ \varphi \mapsto (\varphi\mathcal{C}\colon g \mapsto g\varphi)\,.$$

Hence, we have $\mathrm{Aut}\,(G) \simeq \mathrm{G}_{G\mathcal{C}}$.

We show that $\mathrm{M}_{G\mathcal{C}} \overset{!}{\simeq} G$.

Consider the map

$$\lambda\colon G \to \mathrm{M}_{V\mathrm{Cat}}\,,\ x \mapsto a_x = \big(G \xrightarrow[\sim]{Ga_x} G\big)_{G\in\mathrm{Ob}(G\mathcal{C})} := \big(G \xrightarrow[\sim]{x} G\big)\,.$$

Suppose given $x \in G$. By Lemma 49.(1,2), we have the isotransformation

$$\big(\mathrm{id}_{G\mathcal{C}} \xrightarrow[\sim]{a_x} F_x\big) = \big(G \xrightarrow[\sim]{x} G\big)\,,$$

where the autofunctor F_x maps a morphism $u \in \mathrm{Mor}(G\mathcal{C}) = G$ to $u^x = x^- u x$. So λ is a well-defined map.

Moreover, for $x, y \in G$, we have

$$\begin{aligned}(G)(x\lambda * y\lambda) &= (G)(a_x * a_y) = (G)\big(a_x \blacktriangle (F_x a_y)\big) = Ga_x \blacktriangle (GF_x a_y) = Ga_x \blacktriangle Ga_y = xy \\ &= Ga_{xy} = (G)(xy)\lambda\,.\end{aligned}$$

This shows $x\lambda * y\lambda = (xy)\lambda$.

So λ is a group morphism.

Consider the map $\lambda'\colon\ \mathrm{M}_{G\mathcal{C}} \to G$, $\big(\mathrm{id}_{G\mathcal{C}} \xrightarrow[\sim]{a} F\big) \mapsto Ga$.

We show that $\lambda' \overset{!}{=} \lambda^-$.

For $x \in G$, we have

$$(x)(\lambda \blacktriangle \lambda') = \big(\mathrm{id}_{G\mathcal{C}} \xrightarrow[\sim]{a_x} F_x\big)\lambda' = Ga_x = x\,.$$

This shows $\lambda \blacktriangle \lambda' = \mathrm{id}_G$.

For $\big(\mathrm{id}_{G\mathcal{C}} \xrightarrow[\sim]{a} F\big) = \big(G \xrightarrow[\sim]{Ga} G\big) \in \mathrm{M}_{G\mathcal{C}}$, we have

$$(a)(\lambda' \blacktriangle \lambda) = (Ga)\lambda = \big(\mathrm{id}_{G\mathcal{C}} \xrightarrow[\sim]{a_{Ga}} F_{Ga}\big) = \big(G \xrightarrow[\sim]{Ga} G\big) = \big(\mathrm{id}_{G\mathcal{C}} \xrightarrow[\sim]{a} F\big) = a\,.$$

This shows $\lambda' \blacktriangle \lambda = \mathrm{id}_{\mathrm{M}_{G\mathcal{C}}}$.

So we have $\lambda' = \lambda^-$ and we have the group isomorphism

$$\lambda\colon G \to \mathrm{M}_{G\mathcal{C}}\,,\ x \mapsto \big(\mathrm{id}_{G\mathcal{C}} \xrightarrow[\sim]{a_x} F_x\big)\,.$$

Thus, we have $\mathrm{M}_{G\mathcal{C}} \simeq G$.

Recall the group morphisms

$$\mathrm{f}_{G\mathcal{C}}\colon\ \mathrm{M}_{G\mathcal{C}} \to \mathrm{G}_{G\mathcal{C}},\ \big(\mathrm{id}_{G\mathcal{C}} \xrightarrow[\sim]{a_x} F_x\big) \mapsto F_x$$

and

$$\begin{array}{rcl}
\gamma_{G\mathcal{C}}\colon\ \mathrm{G}_{G\mathcal{C}} & \to & \mathrm{Aut}\,(\mathrm{M}_{G\mathcal{C}}) \\
\varphi\mathcal{C} & \mapsto & \Big(\big(\mathrm{id}_{G\mathcal{C}} \xrightarrow[\sim]{a_x} F_x\big) \mapsto \big(\mathrm{id}_{G\mathcal{C}} \xrightarrow[\sim]{(\varphi\mathcal{C})^- a_x(\varphi\mathcal{C})} (\varphi\mathcal{C})^- F_x(\varphi\mathcal{C})\big)\Big) \\
& & = \Big(a_x \mapsto \big(G \xrightarrow[\sim]{G(\varphi\mathcal{C})^- a_x(\varphi\mathcal{C})} G\big)\Big) \\
& & = \Big(a_x \mapsto \big(G \xrightarrow[\sim]{x\varphi} G\big)\Big) \\
& & = (a_x \mapsto a_{x\varphi})\,,
\end{array}$$

where $\varphi \in \mathrm{Aut}(G)$ and $x \in G$; cf. Lemma 48.

We show that $(\lambda, \mu)\colon \big(G, \mathrm{Aut}\,(G), \mathrm{id}_{\mathrm{Aut}\,(G)}, c\big) \to \mathrm{S}_{G\mathcal{G}}$ is a crossed module isomorphism; cf. §0.4 item 2, [**15,** Lem. 15].

For $g \in G$ and $\varphi \in \mathrm{Aut}\,(G)$, we have

$$(g^\varphi)\lambda = (g\varphi)\lambda = a_{g\varphi} = (a_g)^{\varphi\mathcal{C}} = (g\lambda)^{\varphi\mu}\,,$$

and

$$(g)c\mu = (x \mapsto x^g)\,\mu = F_g = (a_g)\,\mathrm{f}_{G\mathcal{C}} = (g)\lambda\,\mathrm{f}_{G\mathcal{C}}\ .$$

$$\begin{tikzcd}
G \arrow[r, "\lambda", "\sim"'] \arrow[d, "c"'] & \mathrm{M}_{G\mathcal{C}} \arrow[d, "\mathrm{f}_{G\mathcal{C}}"] \\
\mathrm{Aut}\,(G) \arrow[r, "\sim", "\mu"'] & \mathrm{G}_{G\mathcal{C}}
\end{tikzcd}$$

So, altogether, we have

$$(\lambda, \mu)\colon \big(G, \mathrm{Aut}\,(G), \mathrm{id}_{\mathrm{Aut}\,(G)}, c\big) \xrightarrow{\sim} \mathrm{S}_{G\mathcal{C}} \ .$$

3.4 Action of a crossed module on a category

Lemma 55 (V-category from a crossed module morphism)

Consider $\mathrm{S}_{\mathcal{X}} = \big(\mathrm{M}_{\mathcal{X}}, \mathrm{G}_{\mathcal{X}}, \boldsymbol{\gamma}_{\mathcal{X}}, \mathrm{f}_{\mathcal{X}}\big)$.

Suppose we have a crossed module morphism $(\lambda, \mu)\colon V \to \mathrm{S}_{\mathcal{X}}$ *. So we have* $\lambda\colon M \to \mathrm{M}_{\mathcal{X}}$ *and* $\mu\colon G \to \mathrm{G}_{\mathcal{X}}$ *.*

(1) *The set* $\mathrm{Ob}(\mathcal{X})$ *is a* G*-set via*

$$X \cdot g := (X)(g\mu) \in \mathrm{Ob}(\mathcal{X})$$

for $X \in \mathrm{Ob}(\mathcal{X})$*,* $g \in G$*.*

(2) *The set* $\mathrm{Mor}(\mathcal{X})$ *is a* $G \ltimes M$*-set via*

$$u \cdot (g, m) := u(g\mu) \blacktriangle (u\boldsymbol{t})(g\mu)(m\lambda) \in \mathrm{Mor}(\mathcal{X})$$

for $u \in \mathrm{Mor}(\mathcal{X})$*,* $g \in G$*,* $m \in M$*.*

(3) *We have an* V*-crossed set given by*

$$[\![\mathrm{Mor}(\mathcal{X})\,, \mathrm{Ob}(\mathcal{X})]\!]_{\mathrm{set}} = \big(\,\mathrm{Mor}(\mathcal{X})\,, \mathrm{Ob}(\mathcal{X})\,, (\boldsymbol{s}, \boldsymbol{i}, \boldsymbol{t})\big)$$

together with the group actions from (1) *and* (2)*; cf. Reminder* 1*.*

(4) *The category* $\mathcal{X} = \big(\,\mathrm{Mor}(\mathcal{X})\,, \mathrm{Ob}(\mathcal{X})\,, (\boldsymbol{s}, \boldsymbol{i}, \boldsymbol{t}), \blacktriangle\,\big)$ *together with the structure of a* V*-crossed set given as in* (3) *is a* V*-category; cf. Definition* 2*.*

Proof. Ad (1). Suppose given $X \in \mathrm{Ob}(\mathcal{X})$, $g, h \in G$.

We have

$$X \cdot 1 = (X)(1\mu) = X\,.$$

We have

$$(X \cdot g) \cdot h = \big((X)(g\mu)\big)(h\mu) = (X)\big((g\mu)(h\mu)\big) = (X)\big((gh)\mu\big) = X \cdot (gh)\,.$$

Ad (2). Suppose given $\big(X \xrightarrow{u} Y\big) \in \operatorname{Mor}(\mathcal{X})$, $g, h \in G$, $m, n \in M$.

We have

$$u \cdot (1,1) = u(1\mu) \blacktriangle Y(1\mu)(1\lambda) = u \blacktriangle Y \mathrm{id}_{\mathrm{id}_{\mathcal{X}}} = u \blacktriangle \mathrm{id}_Y = u\,.$$

Note that $m\lambda\, \mathrm{f}_{\mathcal{X}} = mf\mu$, for $m \in M$, since (λ, μ) is a crossed module morphism.

So, $\big((X)(m\lambda)\big)\boldsymbol{t} = (X)(mf\mu)$ for $m \in M$, $X \in \operatorname{Ob}(\mathcal{X})$.

We have

$$
\begin{array}{cll}
& u \cdot \big((g,m)\cdot(h,n)\big) & \\
= & u \cdot (gh, m^h \cdot n) & \big(\text{multiplication in } G \ltimes M\big) \\
= & u\big((gh)\mu\big) \blacktriangle \Big(Y\big((gh)\mu\big)\Big)\big((m^h \cdot n)\lambda\big) & \big(\text{definition of } (\,\cdot\,)\big) \\
= & u\big((gh)\mu\big) \blacktriangle \Big(Y\big((gh)\mu\big)\Big)\big((m^h)\lambda * n\lambda\big) & \big(\lambda \text{ group morphism}\big) \\
= & u\big((gh)\mu\big) \blacktriangle \Big(Y\big((gh)\mu\big)\Big)\Big((m\lambda)^{h\mu} * n\lambda\Big) & \big((\lambda,\mu) \text{ crossed module morphism}\big) \\
\overset{48}{=} & u\big((gh)\mu\big) \blacktriangle \Big(Y\big((gh)\mu\big)\Big)\Big(\big((h\mu)^-\,(m\lambda)(h\mu)\big) * n\lambda\Big) & \\
\overset{45.(2)}{=} & u\big((gh)\mu\big) \blacktriangle \Big(Y\big((gh)\mu\big)\Big)\Big(\big((h\mu)^-\,(m\lambda)(h\mu)\big) \blacktriangle \big((h\mu)^-\,(mf\mu)(h\mu)\big)(n\lambda)\Big) & \\
= & u\big((gh)\mu\big) \blacktriangle \Big(Y\big((gh)\mu\big)\Big)\big((h\mu)^-\,(m\lambda)(h\mu)\big) \blacktriangle \Big(Y\big((gh)\mu\big)\Big)\big((h\mu)^-\,(mf\mu)(h\mu)\big)(n\lambda) & \\
= & u(g\mu)(h\mu) \blacktriangle Y(g\mu)(h\mu)(h\mu)^-\,(m\lambda)(h\mu) \blacktriangle Y(g\mu)(h\mu)(h\mu)^-\,(mf\mu)(h\mu)(n\lambda) & \\
= & u(g\mu)(h\mu) \blacktriangle Y(g\mu)(m\lambda)(h\mu) \blacktriangle Y(g\mu)(mf\mu)(h\mu)(n\lambda) & \\
= & \big(u(g\mu) \blacktriangle Y(g\mu)(m\lambda)\big)(h\mu) \blacktriangle \big(Y(g\mu)(mf\mu)\big)(h\mu)(n\lambda) & \big(h\mu \text{ functor}\big) \\
= & \big(u(g\mu) \blacktriangle Y(g\mu)(m\lambda)\big) \cdot (h,n) & \big(\text{definition of } (\,\cdot\,)\big) \\
= & \big(u \cdot (g,m)\big) \cdot (h,n) & \big(\text{definition of } (\,\cdot\,)\big)
\end{array}
$$

Ad (3). Suppose given $X \in \operatorname{Ob}(\mathcal{X})$, $\big(X \xrightarrow{u} Y\big) \in \operatorname{Mor}(\mathcal{X})$ and $g \in G$, $m \in M$.

Ad (CS1). We have $X\boldsymbol{is} = X$ and $X\boldsymbol{it} = X$.

Ad (CS2). We have

$$\big(u \cdot (g,m)\big)\boldsymbol{s} = \big(u(g\mu) \blacktriangle Y(g\mu)(m\lambda)\big)\boldsymbol{s} = X(g\mu) = X \cdot g = u\boldsymbol{s} \cdot (g,m)s\,.$$

We have

$$\begin{aligned}\big(u\cdot(g,m)\big)\boldsymbol{t} &= \big(u(g\mu)\blacktriangle Y(g\mu)(m\lambda)\big)\boldsymbol{t} = Y(g\mu)(mf\mu) = Y\big((g\cdot mf)\mu\big) = Y\cdot(g\cdot mf)\\ &= u\boldsymbol{t}\cdot(g,m)t\,.\end{aligned}$$

We have

$$\begin{aligned}(X\cdot g)\boldsymbol{i} &= \big(X(g\mu)\big)\boldsymbol{i} = \mathrm{id}_{X(g\mu)} = \mathrm{id}_{X(g\mu)}\blacktriangle\mathrm{id}_{X(g\mu)} = \mathrm{id}_X(g\mu)\blacktriangle X(g\mu)\mathrm{id}_{\mathrm{id}_{\mathcal{X}}} = \mathrm{id}_X\cdot(g,1)\\ &= X\boldsymbol{i}\cdot gi\,.\end{aligned}$$

Ad (4). By (3), it suffices to show the properties (CC1), (CC2) and (CC3).

For $\big(X \xrightarrow{u} Y\big)\in\mathrm{Mor}(\mathcal{X})$, $g\in G$, $m\in M$, note that we have

$$u\cdot(g,1) = u(g\mu)\blacktriangle Y(g\mu)\mathrm{id}_{\mathrm{id}_{\mathcal{X}}} = u(g\mu)\blacktriangle\mathrm{id}_{Y(g\mu)} = u(g\mu)\,,$$

and

$$u\cdot(1,m) = u\,\mathrm{id}_{\mathcal{X}}\blacktriangle Y\mathrm{id}_{\mathcal{X}}(m\lambda) = u\blacktriangle Y(m\lambda)\,.$$

Suppose given $X\xrightarrow{u}Y\xrightarrow{v}Z$ in $\mathcal{X}$ and suppose given $g\in G$, $m\in M$.

Ad (CC1). We have

$$(u\blacktriangle v)\cdot(g,1) = (u\blacktriangle v)(g\mu) = u(g\mu)\blacktriangle v(g\mu) = \big(u\cdot(g,1)\big)\blacktriangle\big(v\cdot(g,1)\big)\,.$$

Ad (CC2). We have

$$(u\blacktriangle v)\cdot(1,m) = (u\blacktriangle v)\blacktriangle Z(m\lambda) = u\blacktriangle\big(v\blacktriangle Z(m\lambda)\big) = u\blacktriangle\big(v\cdot(1,m)\big)\,.$$

Ad (CC3). By Remark 3, it suffices to show that

$$(u\blacktriangle v)\cdot(1,m) \overset{!}{=} \big(u\cdot(1,m)\big)\blacktriangle\big(v\cdot(mf,1)\big)\,.$$

Since $\big(\mathrm{id}_{\mathcal{X}} \xrightarrow[\sim]{m\lambda} mf\mu\big)$ is an isotransformation, we have the following commutative diagram.

$$\begin{tikzcd} Y \arrow[r, "Y(m\lambda)"] \arrow[d, "v"'] & Y(mf\mu) \arrow[d, "v(mf\mu)"] \\ Z \arrow[r, "Z(m\lambda)"'] & Z(mf\mu) \end{tikzcd}$$

So we have

$$(u \blacktriangle v) \cdot (1, m) = u \blacktriangle v \blacktriangle Z(m\lambda) = u \blacktriangle Y(m\lambda) \blacktriangle v(mf\mu) = \big(u \cdot (1, m)\big) \blacktriangle \big(v \cdot (mf, 1)\big)\,.$$

$\square$

Lemma 56 (Crossed module morphism from a V-category)

Suppose that $\mathcal{X}$ is a V-category.

We have a crossed module morphism $(\lambda_\mathcal{X}, \mu_\mathcal{X})\colon V \to \mathrm{S}_\mathcal{X}$ given by

$$\mu_\mathcal{X}\colon G \to \mathrm{G}_\mathcal{X}\,,\quad g \mapsto g\mu_\mathcal{X} := \big((X \xrightarrow{u} Y) \mapsto (X \cdot g \xrightarrow{u\cdot(g,1)} Y \cdot g)\big)$$

$$\lambda_\mathcal{X}\colon M \to M_\mathcal{X}\,,\quad m \mapsto m\lambda_\mathcal{X} := \left(\begin{array}{c} X \\ \downarrow u \\ Y \end{array} \;\longmapsto\; \begin{tikzcd} X \arrow[r, "\mathrm{id}_X \cdot (1,m)"] \arrow[d, "u"'] & X \cdot mf \arrow[d, "u\cdot(mf,1)"] \\ Y \arrow[r, "\mathrm{id}_Y \cdot (1,m)"'] & Y \cdot mf \end{tikzcd}\right)$$

Hence, $m\lambda_\mathcal{X}$ is an isotransformation from $\mathrm{id}_\mathcal{X}$ *to $mf\mu_\mathcal{X}$. Note that $X \cdot mf = (X)(mf\mu_\mathcal{X})$ for $X \in \mathrm{Ob}(\mathcal{X})$.*

So, as formulas, we have

$$\begin{array}{rcll} (u)(g\mu_\mathcal{X}) & = & u \cdot (g, 1) & \textit{for } u \in \mathrm{Mor}(\mathcal{X})\,,\ g \in G\,, \\ (X)(g\mu_\mathcal{X}) & = & X \cdot g & \textit{for } X \in \mathrm{Ob}(\mathcal{X})\,,\ g \in G\,, \\ (X)(m\lambda_\mathcal{X}) & = & \mathrm{id}_X \cdot (1, m) & \textit{for } X \in \mathrm{Ob}(\mathcal{X})\,,\ m \in M\,. \end{array}$$

We also have

$$u(g\mu_\mathcal{X}) \blacktriangle (u\boldsymbol{t})(g\mu_\mathcal{X})(m\lambda_\mathcal{X}) = u \cdot (g, m) \quad \textit{for } u \in \mathrm{Mor}(\mathcal{X})\,,\ g \in G,\ m \in M\,.$$

Proof. We shall abbreviate $\mu := \mu_\mathcal{X}$ and $\lambda := \lambda_\mathcal{X}$.

We show that μ is a well-defined map.

Suppose given $X \xrightarrow{u} Y \xrightarrow{v} Z$ in $\mathcal{X}$ and $g \in G$.

We have

$$\begin{aligned}\big(u(g\mu)\big)\boldsymbol{s} &= \big(u\cdot(g,1)\big)\boldsymbol{s} \overset{\text{(CS2)}}{=} u\boldsymbol{s}\cdot(g,1)s = X\cdot g = u\boldsymbol{s}\cdot g = (u\boldsymbol{s})(g\mu)\,,\\ \big(u(g\mu)\big)\boldsymbol{t} &= \big(u\cdot(g,1)\big)\boldsymbol{t} \overset{\text{(CS2)}}{=} u\boldsymbol{t}\cdot(g,1)t = Y\cdot g = u\boldsymbol{t}\cdot g = (u\boldsymbol{t})(g\mu)\,,\\ \big(X(g\mu)\big)\boldsymbol{i} &= (X\cdot g)\boldsymbol{i} \overset{\text{(CS2)}}{=} X\boldsymbol{i}\cdot gi = X\boldsymbol{i}\cdot(g,1) = (X\boldsymbol{i})(g\mu)\,,\end{aligned}$$

and we have

$$(u \blacktriangle v)(g\mu) = (u \blacktriangle v)\cdot(g,1) \overset{\text{(CC1)}}{=} \big(u\cdot(g,1)\big) \blacktriangle \big(v\cdot(g,1)\big) = u(g\mu) \blacktriangle v(g\mu)\,.$$

So $g\mu$ is a functor.

For $u \in \mathrm{Mor}(\mathcal{X})$, we have

$$u(g\mu)(g^-\mu) = \big(u\cdot(g,1)\big)(g^-\mu) = u\cdot(g,1)\cdot(g^-,1) = u\cdot(1,1) = u\,.$$

Likewise, we also have $u(g^-\mu)(g\mu) = u$.

So, $g^-\mu$ is the inverse of $g\mu$.

Altogether, we have $g\mu \in \mathrm{Aut}\,(\mathcal{X}) = \mathrm{G}_{\mathcal{X}}$. Therefore, μ is a well-defined map.

We show that μ is a group morphism.

Suppose given $u \in \mathrm{Mor}(\mathcal{X})$ and $g, h \in G$. We have

$$u\big((gh)\mu\big) = u\cdot(gh,1) = u\cdot(g,1)\cdot(h,1) = \big(u\cdot(g,1)\big)h\mu = (u)\big((g\mu)\,(h\mu)\big)\,.$$

Therefore, $(gh)\mu = (g\mu)(h\mu)$.

We show that λ is a well-defined map.

Suppose given $m \in M$ and $\big(X \xrightarrow{u} Y\big) \in \mathrm{Mor}(\mathcal{X})$.

We have

$$\begin{aligned}\big(\mathrm{id}_X\cdot(1,m)\big)\boldsymbol{s} &\overset{\text{(CS2)}}{=} (\mathrm{id}_X)\boldsymbol{s}\cdot(1,m)s = X\cdot 1 = X\,,\\ \big(\mathrm{id}_X\cdot(1,m)\big)\boldsymbol{t} &\overset{\text{(CS2)}}{=} (\mathrm{id}_X)\boldsymbol{t}\cdot(1,m)t = X\cdot mf = (X)(mf\mu)\,.\end{aligned}$$

We have

$$u \blacktriangle \big(\mathrm{id}_Y\cdot(1,m)\big) \overset{\text{(CC2)}}{=} (u \blacktriangle \mathrm{id}_Y)\cdot(1,m) = (\mathrm{id}_X \blacktriangle u)\cdot(1,m) \overset{3}{=} \big(\mathrm{id}_X\cdot(1,m)\big) \blacktriangle \big(u\cdot(mf,1)\big)\,.$$

So we have the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X \cdot (1,m)"] \arrow[d, "u"'] & X \cdot mf \arrow[d, "u \cdot (mf,1)"] \\ Y \arrow[r, "\mathrm{id}_Y \cdot (1,m)"'] & Y \cdot mf \end{tikzcd}$$

Therefore, $m\lambda$ is a transformation from $\mathrm{id}_{\mathcal{X}}$ to $mf\mu$ for $m \in M$.

To show that $m\lambda$ is a well-defined map, it remains to show that $m\lambda \in \mathrm{M}_{\mathcal{X}}$ for $m \in M$.

We have

$$\begin{array}{rcl} X(m\lambda * m^-\lambda) & = & X\big(m\lambda \blacktriangle (mf\mu)(m^-\lambda)\big) \\ & = & X(m\lambda) \blacktriangle X(mf\mu)(m^-\lambda) \\ & = & X(m\lambda) \blacktriangle (X \cdot mf)(m^-\lambda) \\ & = & \big(\mathrm{id}_X \cdot (1,m)\big) \blacktriangle \big(\mathrm{id}_{X\cdot mf} \cdot (1,m^-)\big) \\ & = & \big(\mathrm{id}_X \cdot (1,m)\big) \blacktriangle \Big(\big(\mathrm{id}_{X\cdot mf} \cdot (1,m^-) \cdot (m^-f,1)\big) \cdot (mf,1)\Big) \\ & \overset{3}{=} & \Big(\mathrm{id}_X \blacktriangle \big(\mathrm{id}_{X\cdot mf} \cdot (1,m^-) \cdot (m^-f,1)\big)\Big) \cdot (1,m) \\ & = & \Big(\mathrm{id}_{X\cdot mf} \cdot \big(m^-f, (m^-)^{m^-f}\big)\Big) \cdot (1,m) \\ & \overset{\text{(CM2)}}{=} & \mathrm{id}_{X\cdot mf} \cdot (m^-f, m^-) \cdot (1,m) \\ & = & \mathrm{id}_{X\cdot mf} \cdot (m^-f, 1) \\ & = & \mathrm{id}_{X(mf\mu)}(m^-f\mu) \\ & = & \mathrm{id}_X\,(mf\mu)(mf\mu)^- \\ & = & \mathrm{id}_X\,. \end{array}$$

Therefore, $m\lambda * m^-\lambda = \mathrm{id}_{\mathrm{id}_{\mathcal{X}}}$.

Likewise, we have $m^-\lambda * m\lambda = \mathrm{id}_{\mathrm{id}_{\mathcal{X}}}$.

Thus, by Remark 47.(2), the transformations $m\lambda$ and $m^-\lambda$ are isotransformations.

Altogether, we have $m\lambda \in \mathrm{M}_{\mathcal{X}}$. Therefore, λ is well-defined.

We show that λ is a group morphism.

For $m, n \in M$ and $X \in \mathrm{Ob}(X)$, we have

$$\begin{array}{rcl}
X(m\lambda * n\lambda) & = & X\big((m\lambda) \blacktriangle (mf\mu)(n\lambda)\big) \\
& = & X(m\lambda) \blacktriangle (X)(mf\mu)(n\lambda) \\
& = & X(m\lambda) \blacktriangle (X \cdot mf)(n\lambda) \\
& = & \big(\mathrm{id}_X \cdot (1,m)\big) \blacktriangle \big(\mathrm{id}_{X \cdot mf} \cdot (1,n)\big) \\
& = & \big(\mathrm{id}_X \cdot (1,m)\big) \blacktriangle \Big(\big(\mathrm{id}_{X \cdot mf} \cdot (1,n) \cdot (m^- f, 1)\big) \cdot (mf, 1)\Big) \\
& \overset{3}{=} & \Big(\mathrm{id}_X \blacktriangle \big(\mathrm{id}_{X \cdot mf} \cdot (1,n) \cdot (m^- f, 1)\big)\Big) \cdot (1,m) \\
& = & \mathrm{id}_{X \cdot mf} \cdot (1,n) \cdot (m^- f, 1) \cdot (1,m) \\
& = & \mathrm{id}_{X \cdot mf} \cdot (m^- f, n^{m^- f} \cdot m) \\
& \overset{\text{(CM2)}}{=} & \mathrm{id}_{X \cdot mf} \cdot (m^- f, mn) \\
& = & \mathrm{id}_{X \cdot mf} \cdot (m^- f, 1) \cdot (1, mn) \\
& = & \mathrm{id}_{X \cdot (mf) \cdot (m^- f)} \cdot (1, mn) \\
& = & \mathrm{id}_X \cdot (1, mn) \\
& = & X\big((mn)\lambda\big) .
\end{array}$$

So λ is a group morphism.

We show that (λ, μ) is a crossed module morphism.

Suppose given $X \in \mathrm{Ob}(\mathcal{X})$ and $m \in M$. We have

$$X(m\lambda\, \mathrm{f}_{\mathcal{X}}) = \big(X(m\lambda)\big)\, \mathrm{f}_{\mathcal{X}} = \big(\mathrm{id}_X \cdot (1,m)\big)\, \mathrm{f}_{\mathcal{X}} = X \cdot mf = X(mf\mu) .$$

So, $\lambda\, \mathrm{f}_{\mathcal{X}} = f\mu$.

Suppose given $X \in \mathrm{Ob}(\mathcal{X})$ and $m \in M$, $g \in G$. We have

$$\begin{aligned}
X\big((m^g)\lambda\big) &= \mathrm{id}_X \cdot (1, m^g) = \mathrm{id}_X \cdot (g^-, 1) \cdot (1,m) \cdot (g,1) = \big((\mathrm{id}_X)(g^-\mu)\big) \cdot (1,m) \cdot (1,g) \\
&= \mathrm{id}_{(X)(g^-\mu)} \cdot (1,m) \cdot (g,1) = \big(X(g^-\mu)(m\lambda)\big) \cdot (g,1) \\
&= X\big((g^-\mu)(m\lambda)(g\mu)\big) = X\big((m\lambda)^{g\mu}\big) .
\end{aligned}$$

So, $(m^g)\lambda = (m\lambda)^{g\mu}$.

Therefore, (λ, μ) is a crossed module morphism.

Finally, suppose given $(X \xrightarrow{u} Y) \in \mathrm{Mor}(\mathcal{X})$, $g \in G$ and $m \in M$.

We have

$$\begin{aligned} u \cdot (g, m) &= \big(u \cdot (g,1)\big) \cdot (1,m) = \big(u \cdot (g,1) \blacktriangle \mathrm{id}_{Y \cdot g}\big) \cdot (1,m) \\ &\overset{\text{(CC2)}}{=} u \cdot (g,1) \blacktriangle \big(\mathrm{id}_{Y \cdot g} \cdot (1,m)\big) = u(g\mu) \blacktriangle Y(g\mu)(m\lambda)\,. \end{aligned}$$

$\square$

Proposition 57

(1) *Recall that we are given a category* $\mathcal{X} = \big(\operatorname{Mor}(\mathcal{X})\,, \operatorname{Ob}(\mathcal{X})\,, (s, i, t), \blacktriangle\big)$.

Suppose given the structure of a V-category on $\mathcal{X}$*; cf. Definition* 2.

Recall that $\operatorname{Mor}(\mathcal{X})$ *is a* $G \ltimes M$*-set and that* $\operatorname{Ob}(\mathcal{X})$ *is a* G*-set. Let us denote the action of* $G \ltimes M$ *on* $\operatorname{Mor}(\mathcal{X})$ *by* $\beta \colon G \ltimes M \to \mathrm{S}_{\operatorname{Mor}(\mathcal{X})}$ *and the action of* G *on* $\operatorname{Ob}(\mathcal{X})$ *by* $\delta \colon G \to \mathrm{S}_{\operatorname{Ob}(\mathcal{X})}$ *.*

From the V-category $\mathcal{X}$ *we obtain the crossed module morphism* $(\lambda, \mu) \colon V \to \mathrm{S}_{\mathcal{X}}$ *given in Lemma* 56*.*

In turn, by Lemma 55*, the morphism* (λ, μ) *induces the structure of a V-category on the category* $\mathcal{X} = \big(\operatorname{Mor}(\mathcal{X})\,, \operatorname{Ob}(\mathcal{X})\,, (s, i, t), \blacktriangle\big)$*. In particular, we obtain actions* $\beta' \colon G \ltimes M \to \mathrm{S}_{\operatorname{Mor}(\mathcal{X})}$ *and* $\delta' \colon G \to \mathrm{S}_{\operatorname{Ob}(\mathcal{X})}$ *.*

Then, we have

$$\big(\operatorname{Mor}(\mathcal{X}), \operatorname{Ob}(\mathcal{X}), (s, i, t), (\blacktriangle), \beta, \delta\big) = \big(\operatorname{Mor}(\mathcal{X}), \operatorname{Ob}(\mathcal{X}), (s, i, t), (\blacktriangle), \beta', \delta'\big).$$

(2) *Suppose given a crossed module morphism* $(\lambda, \mu) \colon V \to \mathrm{S}_{\mathcal{X}}$ *.*

By Lemma 55*, we obtain the structure of a V-category on the category* $\mathcal{X}$*.*

In turn, by Lemma 56*, the V-category* $\mathcal{X}$ *gives a crossed module morphism*

$$(\lambda', \mu') \colon V \to \mathrm{S}_{\mathcal{X}}\ .$$

Then, we have

$$(\lambda, \mu) = (\lambda', \mu')\,.$$

Proof. Ad (1). Suppose given $X \in \operatorname{Ob}(\mathcal{X})$, $(X \xrightarrow{u} Y) \in \operatorname{Mor}(\mathcal{X})$, $g \in G$ and $m \in M$.

We have

$$X(g\delta) \overset{56}{=} X(g\mu) \overset{55}{=} X(g\delta')\,.$$

Therefore, $g\delta = g\delta'$, and so $\delta = \delta'$.

We have

$$u\big((g,m)\beta\big) \overset{56}{=} u(g\mu) \blacktriangle Y(g\mu)(m\lambda) \overset{55}{=} u\big((g,m)\beta'\big) .$$

Therefore, $(g,m)\beta = (g,m)\beta'$, and so $\beta = \beta'$.

Ad (2). Suppose given $\big(X \xrightarrow{u} Y\big) \in \mathrm{Mor}(\mathcal{X})$ and $g \in G$. We have

$$u(g\mu) = u(g\mu) \blacktriangle Y(g\mu)\mathrm{id}_{\mathrm{id}_{\mathcal{X}}} = u(g\mu) \blacktriangle Y(g\mu)(1\lambda) \overset{55}{=} u \cdot (g,1) \overset{56}{=} u(g\mu') .$$

Therefore, $g\mu = g\mu'$, and so $\mu = \mu'$.

Suppose given $X \in \mathrm{Ob}(\mathcal{X})$ and $m \in M$. We have

$$X(m\lambda) = \mathrm{id}_X \blacktriangle X(m\lambda) = \mathrm{id}_X\,(1\mu) \blacktriangle (\mathrm{id}_X\, \boldsymbol{t})(1\mu)(m\lambda) \overset{55}{=} \mathrm{id}_X \cdot (1,m) \overset{56}{=} (X)\,(m\lambda') .$$

Therefore, $m\lambda = m\lambda'$, and so $\lambda = \lambda'$. $\square$

3.5 The Cayley embedding

3.5.1 Mapping into a symmetric crossed module

Lemma 58 *Let $\mathcal{X}$ be a V-category.*

Consider the crossed module morphism $(\lambda_{\mathcal{X}}, \mu_{\mathcal{X}})\colon V \to \mathrm{S}_{\mathcal{X}}$ given in Lemma 56.

$$\begin{aligned} \mu_{\mathcal{X}}\colon\ \ G &\longrightarrow \mathrm{G}_{\mathcal{X}}\,, \quad g \mapsto g\mu_{\mathcal{X}} := \big((X \xrightarrow{u} Y) \mapsto (X \cdot g \xrightarrow{u\cdot(g,1)} Y \cdot g)\big) \\ \lambda_{\mathcal{X}}\colon\ \ M &\longrightarrow \mathrm{M}_{\mathcal{X}}\,, \quad m \mapsto m\lambda_{\mathcal{X}} := \big(X \xrightarrow[\sim]{\mathrm{id}_X\cdot(1,m)} X \cdot mf\big)_{X \in \mathrm{Ob}(\mathcal{X})} \end{aligned}$$

Recall that $\mathrm{Mor}(\mathcal{X})$ is a $(G \ltimes M)$-set.

The crossed module morphism $(\lambda_{\mathcal{X}}, \mu_{\mathcal{X}})$ is injective if and only if the action

$$\beta\colon G \ltimes M \to \mathrm{S}_{\mathrm{Mor}(\mathcal{X})}$$

is injective.

Proof. We write $\lambda := \lambda_{\mathcal{X}}$ and $\mu := \mu_{\mathcal{X}}$.

By Lemma 56, we have

$$u \cdot (g, m) = u(g\mu) \blacktriangle (Y \cdot g)(m\lambda)$$

for $g \in G$, $m \in M$ and $(X \xrightarrow{u} Y) \in \mathrm{Mor}(\mathcal{X})$.

Ad $\Rightarrow$. Suppose that (λ, μ) is injective; cf. §0.4 item 4.

Suppose given $g \in G$, $m \in M$ such that $u \cdot (g, m) = u$ holds for $(X \xrightarrow{u} Y) \in \mathrm{Mor}(\mathcal{X})$. We have to show that $(g, m) \overset{!}{=} (1, 1)$.

For $X \in \mathrm{Ob}(\mathcal{X})$, note that we have

$$X = (\mathrm{id}_X)\mathsf{s} = \big(\mathrm{id}_X \cdot (g, m)\big)\mathsf{s} \overset{\text{(CS2)}}{=} (\mathrm{id}_X)\mathsf{s} \cdot (g, m)s = X \cdot g\,,$$

and

$$\mathrm{id}_X = \mathrm{id}_X \cdot (g, m) = \mathrm{id}_X\,(g\mu) \blacktriangle (X \cdot g)(m\lambda) = \mathrm{id}_X \blacktriangle X(m\lambda) = \mathrm{id}_X \cdot (1, m)\,.$$

Suppose given $(X \xrightarrow{u} Y) \in \mathrm{Mor}(\mathcal{X})$. We have

$$u = u \cdot (g, m) = u(g\mu) \blacktriangle (Y \cdot g)(m\lambda) \overset{56}{=} u(g\mu) \blacktriangle \big(\mathrm{id}_Y \cdot (1, m)\big) = u(g\mu) \blacktriangle \mathrm{id}_Y = u(g\mu)\,.$$

So, $u(g\mu) = u$ for $u \in \mathrm{Mor}(\mathcal{X})$. Therefore $g\mu = \mathrm{id}_{\mathcal{X}}$. Since μ is injective we conclude that $g = 1$.

We have

$$m\lambda = \big(X \xrightarrow{\mathrm{id}_X \cdot (1,m)} X \cdot mf\big)_{X \in \mathrm{Ob}(\mathcal{X})} = \big(X \xrightarrow{\mathrm{id}_X} X\big)_{X \in \mathrm{Ob}(\mathcal{X})} = \mathrm{id}_{\mathrm{id}_{\mathcal{X}}}\,.$$

Since λ is injective we conclude that $m = 1$.

Hence, we have $(g, m) = (1, 1)$.

Ad $\Leftarrow$. Suppose that $\beta\colon G \ltimes M \to \mathrm{S}_{\mathrm{Mor}(\mathcal{X})}$ is injective.

We show that μ is injective.

Suppose given $g \in G$ such that $g\mu = \mathrm{id}_{\mathcal{X}}$. For $\big(X \xrightarrow{u} Y\big) \in \mathrm{Mor}(\mathcal{X})$, we have

$$(u)\big((g, 1)\beta\big) = u \cdot (g, 1) = u(g\mu) = u\mathrm{id}_{\mathcal{X}} = u\,.$$

So, $(g, 1)\beta = \mathrm{id}_{\mathrm{Mor}(\mathcal{X})}$. Since β is injective it follows that $(g, 1) = (1, 1)$. So, $g = 1$.

Therefore, μ is injective.

We show that λ is injective.

Suppose given $m \in M$ such that $m\lambda = \mathrm{id}_{\mathrm{id}_{\mathcal{X}}}$. For $\big(X \xrightarrow{u} Y\big) \in \mathrm{Mor}(\mathcal{X})$, we have

$$(u)\big((1,m)\beta\big) = u \cdot (1,m) \overset{56}{=} u(1\mu) \blacktriangle (Y \cdot 1)(m\lambda) = u\mathrm{id}_{\mathcal{X}} \blacktriangle Y\mathrm{id}_{\mathrm{id}_{\mathcal{X}}} = u \blacktriangle \mathrm{id}_Y = u\,.$$

So, $(1,m)\beta = \mathrm{id}_{\mathrm{Mor}(\mathcal{X})}$. Since β is injective it follows that $(1,m) = (1,1)$. So, $m = 1$.
Therefore, λ is injective.
So, (λ, μ) is injective. $\square$

The following proposition is a crossed module analogue of Cayley's Theorem for groups.

Proposition 59 *We have an injective crossed module morphism*

$$\rho_V^{\mathrm{Cayley}} \;=\; (\lambda_V^{\mathrm{Cayley}}, \mu_V^{\mathrm{Cayley}}) \;:=\; (\lambda_{V\mathrm{Cat}}, \mu_{V\mathrm{Cat}}) \;:\; V \;\to\; \mathrm{S}_{V\mathrm{Cat}}$$

called Cayley embedding*, where*

$$\begin{aligned} \mu_V^{\mathrm{Cayley}}\colon\;\; G &\longrightarrow \mathrm{G}_{V\mathrm{Cat}}\,, & x &\mapsto x\mu_V^{\mathrm{Cayley}} := \Big(\big(g \xrightarrow{(g,m)} g\cdot mf\big) \mapsto \big(gx \xrightarrow{(gx,m^x)} (g\cdot mf)x\big)\Big) \\ \lambda_V^{\mathrm{Cayley}}\colon\;\; M &\longrightarrow \mathrm{M}_{V\mathrm{Cat}}\,, & m &\mapsto n\lambda_V^{\mathrm{Cayley}} := \big(g \xrightarrow[\sim]{(g,n)} g\cdot nf\big)_{g\in G}\,; \end{aligned}$$

cf. Lemma 56*.*
So, for $n \in M$, we have

$$n\lambda_V^{\mathrm{Cayley}} = \left(\begin{array}{c} g \\ \Big\downarrow (g,m) \\ g\cdot mf \end{array} \;\longmapsto\; \begin{tikzcd} g \arrow[r, "{(g,n)}"] \arrow[d, "{(g,m)}"'] & g\cdot nf \arrow[d, "{(g\cdot nf,\, n^{-}mn)}"] \\ g\cdot mf \arrow[r, "{(g\cdot mf,\, n)}"'] & g\cdot (mn)f \end{tikzcd} \right)$$

*So the crossed module V is isomorphic to a crossed submodule of the symmetric crossed module $\mathrm{S}_{V\mathrm{Cat}}$ on the category V*Cat*.*
We often write $\lambda^{\mathrm{Cayley}} := \lambda_V^{\mathrm{Cayley}}$ and $\mu^{\mathrm{Cayley}} := \mu_V^{\mathrm{Cayley}}$.

For an example of the Cayley embedding cf. §A.10.

Proof. The category $V\mathrm{Cat}$ is a V-category; cf. Remark 5.(2).

Lemma 56 yields the crossed module morphism $(\lambda, \mu) := (\lambda_{V\mathrm{Cat}}, \mu_{V\mathrm{Cat}})\colon V \to \mathrm{S}_{V\mathrm{Cat}}$ given as follows.

We have

$$\mu\colon G \to \mathrm{G}_{V\mathrm{Cat}}$$
$$x \mapsto x\mu = \Big((g \xrightarrow{(g,m)} g \cdot mf) \mapsto \big(gx \xrightarrow{(g,m)\cdot(x,1)} (g \cdot mf)x\big) = \big(gx \xrightarrow{(gx,m^x)} (g \cdot mf)x\big)\Big)\,.$$

We have

$$\lambda\colon M \to \mathrm{M}_{V\mathrm{Cat}}\,,\ n \mapsto n\lambda\,,$$

where $n\lambda$ maps a morphism $\left(\begin{array}{c} g \\ \downarrow{\scriptstyle (g,m)} \\ g\cdot mf \end{array}\right) \in \mathrm{Mor}(V\mathrm{Cat})$ to the diagram morphism

$$\left(\begin{tikzcd} g \arrow[r, "{(g,1)\cdot(1,nf)}"] \arrow[d, "{(g,m)}"'] & g\cdot nf \arrow[d, "{(g,m)\cdot(1,nf)}"] \\ g\cdot mf \arrow[r, "{(g\cdot mf,1)\cdot(1,nf)}"'] & g\cdot mf\cdot nf \end{tikzcd}\right)$$

$$= \left(\begin{tikzcd} g \arrow[r, "{(g,nf)}"] \arrow[d, "{(g,m)}"'] & g\cdot nf \arrow[d, "{(g\cdot nf,m^{nf})}"] \\ g\cdot mf \arrow[r, "{(g\cdot mf,nf)}"'] & g\cdot (mn)f \end{tikzcd}\right)$$

$$\overset{\text{(CM2)}}{=} \left(\begin{tikzcd} g \arrow[r, "{(g,nf)}"] \arrow[d, "{(g,m)}"'] & g\cdot nf \arrow[d, "{(g\cdot nf,n^{-}mn)}"] \\ g\cdot mf \arrow[r, "{(g\cdot mf,nf)}"'] & g\cdot (mn)f \end{tikzcd}\right).$$

The action of $G \ltimes M$ on $\mathrm{Mor}(V\mathrm{Cat}) = G \ltimes M$ is given by the right multiplication of $G \ltimes M$ on $G \ltimes M$, i.e.

$$\beta\colon G \ltimes M \to \mathrm{S}_{G\ltimes M},\ (g,m) \mapsto \big((h,n) \mapsto (h,n)\cdot(g,m) = (hg, n^g \cdot m)\big)\,.$$

Since this action β is injective we conclude that (λ,μ) is injective; cf. Lemma 58.

By Remark 11, we have $\ker(\lambda,\mu) = 1$. Therefore, V is isomorphic to $\mathrm{im}(\lambda,\mu) \leqslant \mathrm{S}_{V\mathrm{Cat}}$; cf. [**15,** Lem. 27]. □

3.5.2 Comparison with Cayley for G/Mf

Suppose given a crossed module $V = \big(M, G, \gamma, f\big)$.

Recall that

$$\begin{aligned} V\pi_0 &= G/Mf \\ V\pi_1 &= \ker f\,; \end{aligned}$$

cf. §0.4 item 5.

Consider the category $V\mathrm{Cat}$; cf. Remark 4.

Recall that

$$\begin{aligned} \mathrm{Ob}(V\mathrm{Cat}) &= G \\ \mathrm{Mor}(V\mathrm{Cat}) &= G \ltimes M\,. \end{aligned}$$

Consider the symmetric crossed module $\mathrm{S}_{V\mathrm{Cat}} = \big(\,\mathrm{G}_{V\mathrm{Cat}}, \mathrm{M}_{V\mathrm{Cat}}, \gamma_{V\mathrm{Cat}}, \mathrm{f}_{V\mathrm{Cat}}\,\big)$; cf. Lemma 48.

Recall that

$$\begin{aligned} \mathrm{G}_{V\mathrm{Cat}} &= \{F\colon V\mathrm{Cat} \to V\mathrm{Cat}\colon F \text{ is an autofunctor}\} \\ \mathrm{M}_{V\mathrm{Cat}} &= \{\mathrm{id}_{V\mathrm{Cat}} \xrightarrow[\sim]{a} F\colon a \text{ is an isotransformation}, F \in \mathrm{G}_{V\mathrm{Cat}}\} \\ \gamma_{V\mathrm{Cat}}\colon\ & \mathrm{G}_{V\mathrm{Cat}} \to \mathrm{Aut}(\mathrm{M}_{V\mathrm{Cat}}),\ H \mapsto (a \mapsto H^- aH) \\ \mathrm{f}_{V\mathrm{Cat}}\colon\ & \mathrm{M}_{V\mathrm{Cat}} \to \mathrm{G}_{V\mathrm{Cat}}\,, \big(\mathrm{id}_{V\mathrm{Cat}} \xrightarrow[\sim]{a} F\big) \mapsto F\,. \end{aligned}$$

We write $\mathrm{Inn} := \mathrm{Inn}(V\mathrm{Cat}) \overset{52}{=} \mathrm{M}_{V\mathrm{Cat}}\,\mathrm{f}_{V\mathrm{Cat}}$. Then $\mathrm{S}_{V\mathrm{Cat}}\,\pi_0 = \mathrm{G}_{V\mathrm{Cat}}\,/\,\mathrm{Inn}$.

Lemma 60 *We have the group morphism*

$$\begin{aligned} \mathrm{S}_{V\mathrm{Cat}}\,\pi_0 = \mathrm{G}_{V\mathrm{Cat}}\,/\,\mathrm{Inn} &\overset{\varphi}{\to} \mathrm{S}_{G/Mf} = \mathrm{S}_{V\pi_0}, \\ F\,\mathrm{Inn} &\mapsto \big((F\,\mathrm{Inn})\varphi\colon G/Mf \to G/Mf,\ g(Mf) \mapsto gF(Mf)\big)\,. \end{aligned}$$

Proof. Suppose given $F \in \mathrm{G}_{V\mathrm{Cat}}$.

The map $u_F\colon G/Mf \to G/Mf,\ g(Mf) \mapsto gF(Mf)$ is well-defined:

Suppose given $g, \tilde{g} \in G$ such that $g(Mf) = \tilde{g}(Mf)$. Then $\tilde{g} = g \cdot mf$ for some $m \in M$. We show that $gF(Mf) \overset{!}{=} \tilde{g}F(Mf)$, i.e. $(gF)^{-} \cdot (g \cdot mf)F \overset{!}{\in} Mf$.

Consider the morphism $\big(g \xrightarrow{(g,m)} g \cdot mf\big) \in \mathrm{Mor}(V\mathrm{Cat}) = G \ltimes M$. Then the morphism $(g,m)F = \big(gF \xrightarrow{(g,m)F} (g \cdot mf)F\big) \in \mathrm{Mor}(V\mathrm{Cat}) = G \ltimes M$ is of the form $(g,m)F = (gF, n)$ for some $n \in M$. We have

$$(g \cdot mf)F = \big((g,m)F\big)t = (gF,n)t = gF \cdot nf\,.$$

So, $(gF)^{-} \cdot (g \cdot mf)F = nf \in Mf$.

Therefore, u_F is well-defined.

We *claim* that u_F is bijective.

Consider $F^{-} \in \mathrm{G}_{V\mathrm{Cat}}$. Then the composite map

$$u_F \blacktriangle u_{F^-}\colon G/Mf \to G/Mf,\ g(Mf) \mapsto gFF^{-}(Mf) = g(Mf)$$

is the identity. Similarly, the composite

$$u_{F^-} \blacktriangle u_F\colon G/Mf \to G/Mf,\ g(Mf) \mapsto gF^{-}F(Mf) = g(Mf)$$

is the identity.

This proves the *claim*.

This defines a map

$$\begin{array}{rlll} \tilde{\varphi}\colon & \mathrm{G}_{V\mathrm{Cat}} & \to & \mathrm{S}_{G/Mf} \\ & F & \mapsto & F\tilde{\varphi} := \big(u_F\colon G/Mf \to G/Mf\,,\ g(Mf) \mapsto gF(Mf)\big)\,. \end{array}$$

We show that $\tilde{\varphi}$ is a group morphism.

Suppose given $F, F' \in \mathrm{G}_{V\mathrm{Cat}}$.

For $g \in G$, we have

$$\big(g(Mf)\big)\big((FF')\tilde{\varphi}\big) = gFF'(Mf) = \big(gF(Mf)\big)(F'\tilde{\varphi}) = \big(g(Mf)\big)(F\tilde{\varphi})(F'\tilde{\varphi})\,.$$

So $(FF')\tilde{\varphi} = (F\tilde{\varphi})(F'\tilde{\varphi})$.

Therefore, $\tilde{\varphi}$ is a group morphism.

We show that $\tilde{\varphi}$ maps Inn to the trivial subgroup.

Suppose given $H \in \mathrm{Inn}$. We show that $H\tilde{\varphi} \overset{!}{=} \mathrm{id}_{G/Mf}$.

Since $H \in \mathrm{Inn} = \mathrm{M}_{V\mathrm{Cat}}\,\mathrm{f}_{V\mathrm{Cat}}$, there exists an isotransformation $a \in \mathrm{M}_{V\mathrm{Cat}}$ such that $H = a\,\mathrm{f}_{V\mathrm{Cat}}$, i.e. such that $a = \big(\mathrm{id}_{V\mathrm{Cat}} \xrightarrow[\sim]{a} H\big) \in \mathrm{M}_{V\mathrm{Cat}}$.

Suppose given $g \in G = \mathrm{Ob}(V\mathrm{Cat})$. Consider the morphism

$$\big(g \xrightarrow[\sim]{ga} gH\big) \in \mathrm{Mor}(V\mathrm{Cat}) = G \ltimes M\,.$$

Then ga is of the form $ga = (g, x)$ for some $x \in M$.

We have

$$gH = (ga)t = (g, x)t = g \cdot xf\,.$$

So we get

$$\big(g(Mf)\big)(H\tilde{\varphi}) = gH(Mf) = (g \cdot xf)(Mf) = g(Mf)\,.$$

Therefore, $H\tilde{\varphi} = \mathrm{id}_{G/Mf}$.

So we have the group morphism

$$\begin{aligned} \varphi\colon\ & \mathrm{S}_{V\mathrm{Cat}}\,\pi_0 \to \mathrm{S}_{G/Mf} \\ & F\,\mathrm{Inn} \mapsto (F\,\mathrm{Inn})\varphi := \big(F\tilde{\varphi}\colon G/Mf \to G/Mf,\ g(Mf) \mapsto gF(Mf)\big)\,. \end{aligned}$$

□

Lemma 61 *Consider the injective group morphism*

$$\begin{aligned} V\pi_0 = G/Mf &\xrightarrow{\Psi} \mathrm{S}_{G/Mf} = \mathrm{S}_{V\pi_0} \\ x(Mf) &\mapsto \big(x(Mf)\big)\Psi = \big(g(Mf) \mapsto gx(Mf)\big) \end{aligned}$$

given by Cayley's Theorem for groups.

Consider the group morphism

$$\mu^{\mathrm{Cayley}}\colon G \to \mathrm{G}_{V\mathrm{Cat}}\,,\ x \mapsto x\mu^{\mathrm{Cayley}} = \Big(\big(g \xrightarrow{(g,m)} g \cdot mf\big) \mapsto \big(gx \xrightarrow{(gx,m^x)} (g \cdot mf)x\big)\Big)$$

from Proposition 59.

Consider the group morphism

$$(\lambda^{\mathrm{Cayley}}, \mu^{\mathrm{Cayley}})\pi_0\colon V\pi_0 \to \mathrm{S}_{V\mathrm{Cat}}\,\pi_0,\ x(Mf) \mapsto x\mu^{\mathrm{Cayley}}\,\mathrm{Inn}\,.$$

Consider the group morphism

$$\varphi\colon\ \mathrm{S}_{V\mathrm{Cat}}\,\pi_0 \to V\pi_0\,,\ \ F\,\mathrm{Inn} \mapsto \big(g(Mf) \mapsto gF(Mf)\big)$$

from Lemma 60*.*

Then we have

$$\Psi = (\lambda^{\mathrm{Cayley}}, \mu^{\mathrm{Cayley}})\pi_0 \blacktriangle \varphi\,,$$

i.e. we have the following commutative diagram.

$$\begin{tikzcd} V\pi_0 \arrow[d, "(\lambda^{\mathrm{Cayley}}, \mu^{\mathrm{Cayley}})\pi_0"'] \arrow[dr, "\Psi"] & \\ \mathrm{S}_{V\mathrm{Cat}}\,\pi_0 \arrow[r, "\varphi"'] & \mathrm{S}_{V\pi_0} \end{tikzcd}$$

In particular, $(\lambda^{\mathrm{Cayley}}, \mu^{\mathrm{Cayley}})\pi_0$ *is injective.*

Proof. Suppose given $x \in G$. For $g \in G$, we have

$$\begin{aligned} \big(g(Mf)\big)\Big(\big(x(Mf)\big)(\lambda^{\mathrm{Cayley}}, \mu^{\mathrm{Cayley}})\pi_0\ \varphi\Big) &= \big(g(Mf)\big)\big((x\mu^{\mathrm{Cayley}}\,\mathrm{Inn})\varphi\big) \\ &= \big(g(x\mu^{\mathrm{Cayley}})\big)(Mf) \\ &= gx(Mf) \\ &= \big(g(Mf)\big)\Big(\big(x(Mf)\big)\Psi\Big)\,. \end{aligned}$$

So $\big(x(Mf)\big)(\lambda^{\mathrm{Cayley}}, \mu^{\mathrm{Cayley}})\pi_0\ \varphi = \big(x(Mf)\big)\Psi$ and therefore $(\lambda^{\mathrm{Cayley}}, \mu^{\mathrm{Cayley}})\pi_0 \blacktriangle \varphi = \Psi$. □

Theorem 62 *Recall from Proposition* 59 *that for our crossed module* V *we have the Cayley embedding, i.e. the injective crossed module morphism*

$$\rho_V^{\mathrm{Cayley}}\colon V \to \mathrm{S}_{V\mathrm{Cat}}\ .$$

The group morphisms $\rho_V^{\mathrm{Cayley}}\pi_0$ *and* $\rho_V^{\mathrm{Cayley}}\pi_1$ *are injective.*

In particular, every crossed module is isomorphic to a crossed submodule of a symmetric crossed module on a category such that the inclusion morphism is injective on π_0 *and* π_1 *.*

Proof. This follows from Proposition 59 and from Lemma 61, observing that ρ_V^{Cayley} injective implies that $\rho_V^{\mathrm{Cayley}}\pi_1$ is injective. □

The following example shows that the group morphism φ from Lemma 60 is not injective in general.

Example 63 Suppose given an abelian group M.

Consider the crossed module $V := \big(M, 1, \iota, \kappa\big)$, where

$$\begin{aligned} \kappa &\colon M \to 1,\ m \mapsto 1 \\ \iota &\colon 1 \to \mathrm{Aut}(M),\ 1 \mapsto \mathrm{id}_M\,; \end{aligned}$$

cf. [**15,** Ex. 11].

We consider the category $V\mathrm{Cat}$. Then

$$\begin{aligned} \mathrm{Ob}(V\mathrm{Cat}) &= 1 \\ \mathrm{Mor}(V\mathrm{Cat}) &= 1 \ltimes M \xrightarrow[\sim]{p} M\,,\ (1,m) \mapsto m\,. \end{aligned}$$

Moreover, $(1,m) \blacktriangle (1,m') = (1,mm')$ for m, $m' \in M$.

In particular, we have

$$\big((1,m) \blacktriangle (1,m')\big)p = mm' = (1,m)p \cdot (1,m')p\,.$$

We want to determine the symmetric crossed module $\mathrm{S}_{V\mathrm{Cat}} = \big(\,\mathrm{M}_{V\mathrm{Cat}}, \mathrm{G}_{V\mathrm{Cat}}, \gamma_{V\mathrm{Cat}}, \mathrm{f}_{V\mathrm{Cat}}\,\big)$.

Step 1. We *claim* that we have the mutually inverse group isomorphisms

$$\begin{aligned} \xi &\colon\ \mathrm{G}_{V\mathrm{Cat}} \xrightarrow[\sim]{} \mathrm{Aut}(M),\ F \mapsto \big(F\xi \colon m \mapsto (1,m)F\,p\big) \\ \xi' &\colon\ \mathrm{Aut}(M) \xrightarrow[\sim]{} \mathrm{G}_{V\mathrm{Cat}}\,,\ \phi \mapsto \big(\phi\xi' \colon \big(1 \xrightarrow{(1,m)} 1\big) \mapsto \big(1 \xrightarrow{(1,m\phi)} 1\big)\big)\,. \end{aligned}$$

Construction of ξ .

Suppose given $F \in \mathrm{G}_{V\mathrm{Cat}}$.

The map $v_F \colon M \to M,\ m \mapsto (1,m)Fp$ is a group morphism:

For m, $m' \in M$, we have

$$\begin{aligned} (m \cdot m')v_F &= (1,mm')Fp = \big((1,m) \blacktriangle (1,m')\big)Fp = \big((1,m)F \blacktriangle (1,m')F\big)p \\ &= (1,m)Fp \cdot (1,m')Fp = (m)v_F \cdot (m')v_F\,. \end{aligned}$$

Therefore, v_F is a group morphism.

The map v_F is bijective:

Consider $F^- \in \mathrm{G}_{V\mathrm{Cat}}$. For $m \in M$, we have

$$m(v_F \blacktriangle v_{F^-}) = \big((1,m)Fp\big)v_{F^-} = \big(1,(1,m)Fp\big)F^-p = (1,m)FF^-p = (1,m)p = m\,.$$

Therefore, $v_F \blacktriangle v_{F^-} = \mathrm{id}_M$.

Likewise, we have $v_{F^-} \blacktriangle v_F = \mathrm{id}_M$.

So v_F is bijective.

This defines a map

$$\xi\colon\ \mathrm{G}_{V\mathrm{Cat}} \to \mathrm{Aut}(M),\ \ F \mapsto F\xi := \big(v_F\colon M \to M,\ \ m \mapsto (1,m)Fp\big)\,.$$

The map ξ is a group morphism:

Suppose given $F,\ F' \in \mathrm{G}_{V\mathrm{Cat}}$. For $m \in M$, we have

$$\begin{aligned}(m)\big((F\xi)(F'\xi)\big) &= \big((1,m)Fp\big)(F'\xi) = \Big(\big(1,(1,m)Fp\big)\Big)F'p = (1,m)FF'p \\ &= (m)\big((FF')\xi\big)\,.\end{aligned}$$

So $(F\xi)(F'\xi) = (FF')\xi$.

Therefore, ξ is a group morphism.

Construction of ξ'.

Suppose given $\phi \in \mathrm{Aut}(M)$.

We show that $\quad v'_\phi\colon V\mathrm{Cat} \to V\mathrm{Cat},\ \big(1 \xrightarrow{(1,m)} 1\big) \mapsto \big(1 \xrightarrow{(1,m\phi)} 1\big) \quad$ is a functor:

We have

$$(\mathrm{id}_1)v'_\Phi = \big(1 \xrightarrow{(1,1\Phi)} 1\big) = \big(1 \xrightarrow{(1,1)} 1\big) = \mathrm{id}_1\,.$$

For $\big(1 \xrightarrow{(1,m)} 1 \xrightarrow{(1,m')} 1\big)$ in $V\mathrm{Cat}$, we have

$$\begin{aligned}\big((1,m) \blacktriangle (1,m')\big)v'_\phi &= (1,mm')v'_\phi = \big(1,(mm')\phi\big) = \big(1,(m\phi)(m'\phi)\big) = (1,m\phi) \blacktriangle (1,m'\phi) \\ &= \big((1,m)v'_\phi\big) \blacktriangle \big((1,m')v'_\phi\big)\,.\end{aligned}$$

So v'_ϕ is a functor.

We show that v'_ϕ is an autofunctor:

Consider $\phi^- \in \mathrm{Aut}(M)$. For $(1, m) \in \mathrm{Mor}(V\mathrm{Cat})$, we have
$$(1, m)(v'_\phi \blacktriangle v'_{\phi^-}) = (1, m\phi)v'_{\phi^-} = (1, m\phi\phi^-) = (1, m)\,.$$

So $v'_\phi \blacktriangle v'_{\phi^-} = \mathrm{id}_{V\mathrm{Cat}}$. Likewise, we have $v'_{\phi^-} \blacktriangle v'_\phi = \mathrm{id}_{V\mathrm{Cat}}$.

Therefore, v'_ϕ is an autofunctor. So $v'_\phi \in \mathrm{G}_{V\mathrm{Cat}}$.

This defines a map
$$\xi'\colon\ \mathrm{Aut}(M) \to \mathrm{G}_{V\mathrm{Cat}}\,,\ \phi \mapsto \phi\xi' := \big(v'_\phi\colon V\mathrm{Cat} \to V\mathrm{Cat},\ \big(1 \xrightarrow{(1,m)} 1\big) \mapsto \big(1 \xrightarrow{(1,m\phi)} 1\big)\big)\,.$$

The map ξ' is a group morphism:

Suppose given ϕ, $\phi' \in \mathrm{Aut}(M)$. For $(1, m) \in \mathrm{Mor}(V\mathrm{Cat})$, we have
$$(1, m)\big((\phi\,\phi')\xi'\big) = (1, m\phi\,\phi') = (1, m\phi)(\phi'\xi') = (1, m)\big((\phi\xi')(\phi'\xi')\big)\,.$$

So $(\phi\,\phi')\xi' = (\phi\xi')(\phi'\xi')$. Therefore, ξ' is a group morphism.

We show that $\xi' \overset{!}{=} \xi^-$.

Suppose given $F \in \mathrm{G}_{V\mathrm{Cat}}$. For $(1, m) \in \mathrm{Mor}(V\mathrm{Cat})$, we have
$$(1, m)(F\xi\,\xi') = \big(1, m(F\xi)\big) = \big(1, (1, m)Fp\big) = (1, m)F\,.$$

This shows $\xi \blacktriangle \xi' = \mathrm{id}_{\mathrm{G}_{V\mathrm{Cat}}}$.

Suppose given $\phi \in \mathrm{Aut}(M)$. For $m \in M$, we have
$$(m)(\phi\xi'\,\xi) = (1, m)(\phi\xi')p = (1, m\phi)p = m\phi\,.$$

This shows $\xi' \blacktriangle \xi = \mathrm{id}_{\mathrm{Aut}\,M}$.

So $\xi' = \xi^-$.

Altogether, we have the mutually inverse group isomorphisms $\xi\colon\ \mathrm{G}_{V\mathrm{Cat}} \to \mathrm{Aut}(M)$ and $\xi'\colon\ \mathrm{Aut}\,M \to \mathrm{G}_{V\mathrm{Cat}}$, which shows the *claim*.

Step 2. We *claim* that we have the mutually inverse group isomorphisms
$$\zeta'\colon\ \mathrm{M}_{V\mathrm{Cat}} \xrightarrow[\sim]{} M\,,\ \big(1 \xrightarrow[\sim]{(1,m)} 1\big) \mapsto m$$
$$\zeta\colon M \xrightarrow[\sim]{} \mathrm{M}_{V\mathrm{Cat}}\,,\ x \mapsto (x\zeta\colon \mathrm{id}_{V\mathrm{Cat}} \to \mathrm{id}_{V\mathrm{Cat}})\,,$$

where

$$x\zeta = \left(\begin{tikzcd} 1 \arrow[d, "{(1,m)}"] \\ 1 \end{tikzcd} \longmapsto \begin{tikzcd} 1 \arrow[r, "{(1,x)}"] \arrow[d, "{(1,m)}"'] & 1 \arrow[d, "{(1,m)}"] \\ 1 \arrow[r, "{(1,x)}"'] & 1 \end{tikzcd} \right).$$

So $x\zeta = \big(1 \xrightarrow[\sim]{(1,x)} 1\big)_{1\in 1}$.

Construction of ζ.

Suppose given $x \in M$.

We show that $\big(1 \xrightarrow[\sim]{(1,x)} 1\big)_{1\in 1}$ is an isotransformation:

Suppose given $\big(1 \xrightarrow{(1,m)} 1\big) \in \operatorname{Mor}(V\mathrm{Cat})$. We have

$$(1,m) \blacktriangle (1,x) = (1,mx) = (1,xm) = (1,x) \blacktriangle (1,m)\,,$$

since M is abelian.

So the following diagram is commutative.

$$\begin{tikzcd} 1 \arrow[r, "{(1,x)}"] \arrow[d, "{(1,m)}"'] & 1 \arrow[d, "{(1,m)}"] \\ 1 \arrow[r, "{(1,x)}"'] & 1 \end{tikzcd}$$

This defines a map

$$\zeta\colon M \to \mathrm{M}_{V\mathrm{Cat}}\,,\ x \mapsto x\zeta := \big(1 \xrightarrow[\sim]{(1,x)} 1\big)_{1\in 1}\,.$$

We show that ζ is a group morphism:

For $x,\, x' \in M$, we have

$$\begin{aligned}(x\zeta) * (x'\zeta) &= x\zeta \blacktriangle \mathrm{id}_{V\mathrm{Cat}}(x'\zeta) = \big(1 \xrightarrow[\sim]{(1,x)} 1\big) \blacktriangle \big(1 \xrightarrow[\sim]{(1,x')} 1\big) = \big(1 \xrightarrow[\sim]{(1,x)\,\blacktriangle\,(1,x')} 1\big) \\ &= \big(1 \xrightarrow[\sim]{(1,xx')} 1\big) = (x \cdot x')\zeta\,.\end{aligned}$$

Therefore, ζ is a group morphism.

Construction of ζ'.

Suppose given $a \in \mathrm{M}_{V\mathrm{Cat}}$. Then $a = \big(\mathrm{id}_{V\mathrm{Cat}} \xrightarrow[\sim]{a} F\big) = \big(1 \xrightarrow[\sim]{(1,x)} 1\big)_{1\in 1}$ for some $F \in \mathrm{G}_{V\mathrm{Cat}}$ and some $x \in M$.

Suppose given $(1, m) \in \mathrm{Mor}(V\mathrm{Cat})$. Then we have the following commutative diagram.

$$\begin{tikzcd} 1 \arrow[r, "{(1,x)}"] \arrow[d, "{(1,m)}"'] & 1 \arrow[d, "{(1,m)F}"] \\ 1 \arrow[r, "{(1,x)}"'] & 1 \end{tikzcd}$$

So

$$(1, m)F = (1, x)^- \blacktriangle (1, m) \blacktriangle (1, x) = (1, x^- m x) = (1, m)$$

since M is abelian.

Therefore $F = \mathrm{id}_{V\mathrm{Cat}}$.

So $a \in \mathrm{Mor}(V\mathrm{Cat})$ is of the form $a = \big(\mathrm{id}_{V\mathrm{Cat}} \xrightarrow[\sim]{a} \mathrm{id}_{V\mathrm{Cat}}\big) = \big(1 \xrightarrow[\sim]{(1,x)} 1\big)_{1\in 1} = x\zeta$ for some $x \in M$.

We show that $\zeta' \overset{!}{=} \zeta^-$.

For $x \in M$, we have

$$(x)(\zeta \blacktriangle \zeta') = \Big(\big(1 \xrightarrow[\sim]{(1,x)} 1\big)_{1\in 1}\Big)\zeta' = x\,.$$

This shows $\zeta \blacktriangle \zeta' = \mathrm{id}_M$.

For $\big(1 \xrightarrow[\sim]{(1,x)} 1\big)_{1\in 1} \in \mathrm{M}_{V\mathrm{Cat}}$, we have

$$\Big(\big(1 \xrightarrow[\sim]{(1,x)} 1\big)_{1\in 1}\Big)(\zeta' \blacktriangle \zeta) = x\zeta = \big(1 \xrightarrow[\sim]{(1,x)} 1\big)_{1\in 1}\,.$$

This shows $\zeta' \blacktriangle \zeta = \mathrm{id}_{\mathrm{M}_{V\mathrm{Cat}}}$.

So $\zeta' = \zeta^-$.

Altogether, we have mutually inverse group morphisms $\zeta \colon M \to \mathrm{M}_{V\mathrm{Cat}}$ and $\zeta' \colon \mathrm{M}_{V\mathrm{Cat}} \to M$, which shows the *claim.*

Step 3. For $\big(\mathrm{id}_{V\mathrm{Cat}} \xrightarrow[\sim]{a} \mathrm{id}_{V\mathrm{Cat}}\big) \in \mathrm{Mor}(V\mathrm{Cat})$, we have $a\,\mathrm{f}_{V\mathrm{Cat}} = \mathrm{id}_{V\mathrm{Cat}}$.

Therefore

$$\mathrm{f}_{V\mathrm{Cat}} \colon\ \mathrm{M}_{V\mathrm{Cat}} \to \mathrm{G}_{V\mathrm{Cat}}\,,\ \ \big(1 \xrightarrow[\sim]{(1,m)} 1\big)_{1\in 1} \mapsto \mathrm{id}_{V\mathrm{Cat}}\,.$$

Step 4. We have

$$\begin{aligned} V\pi_0 &= 1/Mf = 1/1 \simeq 1 \\ \mathrm{S}_{V\mathrm{Cat}}\,\pi_0 &= \mathrm{G}_{V\mathrm{Cat}}\,/\,\mathrm{M}_{V\mathrm{Cat}}\,\mathrm{f}_{V\mathrm{Cat}} \simeq \mathrm{Aut}\,(M)/(M_{V\mathrm{Cat}}) \simeq \mathrm{Aut}\,(M) \\ \mathrm{S}_{V\pi_0} &\simeq \mathrm{S}_{\{1\}} \simeq 1\,. \end{aligned}$$

So the commutative diagram

$$\begin{tikzcd} V\pi_0 \arrow[d, "(\lambda^{\mathrm{Cayley}},\,\mu^{\mathrm{Cayley}})\pi_0"'] \arrow[dr, "\Psi"] & \\ \mathrm{S}_{V\mathrm{Cat}}\,\pi_0 \arrow[r, "\varphi"'] & \mathrm{S}_{V\pi_0} \end{tikzcd}$$

from Lemma 61, where φ is given in Lemma 60, can be replaced isomorphically by the following diagram.

$$\begin{tikzcd} 1 \arrow[d] \arrow[dr] & \\ \mathrm{Aut}\,(M) \arrow[r] & 1 \end{tikzcd}$$

Step 5. For instance, for $M := C_3$, we have $\mathrm{Aut}\,(M) \simeq C_2 \not\simeq 1$. So the group morphism φ from Lemma 60 is not injective in general.

Chapter 4

R-linear categories

Let R be a commutative ring with identity $1 = 1_R$.

We shall recall some basic facts on R-linear categories; cf. [**13,** §1.4].

4.1 Definition of an R-linear category

Definition 64 (Preadditive category) A category $\mathcal{M}$ together with maps

$$(+) = (+_{X,Y})\colon {}_{\mathcal{M}}(X,Y) \times {}_{\mathcal{M}}(X,Y) \to {}_{\mathcal{M}}(X,Y),\ (m,n) \mapsto m+n$$

for $X, Y \in \mathrm{Ob}(\mathcal{M})$ is called a *preadditive category* if $(1, 2, 3)$ hold.

(1) For X, $Y \in \mathrm{Ob}(\mathcal{M})$, we have an abelian group $\big({}_{\mathcal{M}}(X,Y),\ +_{X,Y}\big)$.

We often write $\quad 0 = 0_{X,Y} := 0_{{}_{\mathcal{M}}(X,Y)} \quad$ for X, $Y \in \mathrm{Ob}(\mathcal{M})$.

(2) For $W \xrightarrow{a} X \overset{b_1}{\underset{b_2}{\rightrightarrows}} Y \xrightarrow{c} Z$ in $\mathcal{M}$, we have

$$a \blacktriangle (b_1 + b_2) \blacktriangle c = a \blacktriangle b_1 \blacktriangle c + a \blacktriangle b_2 \blacktriangle c\,.$$

Definition 65 (R-linear category) A preadditive category $\mathcal{M}$ together with a ring morphism $\varepsilon\colon R \to \mathrm{End}(\mathrm{id}_{\mathcal{M}})$ is called an *R-linear category.*

For $r \in R$ and $(X \xrightarrow{u} Y) \in \mathrm{Mor}(\mathcal{M})$, we write

$$ur := u \blacktriangle Y(r\varepsilon) = X(r\varepsilon) \blacktriangle u \colon X \to Y.$$

$$\begin{tikzcd} X \arrow[r, "X(r\varepsilon)"] \arrow[d, "u"'] & X \arrow[d, "u"] \\ Y \arrow[r, "Y(r\varepsilon)"'] & Y \end{tikzcd}$$

In particular, we have $X(r\varepsilon) = \mathrm{id}_X\, r \colon X \to X$, for $X \in \mathrm{Ob}(\mathcal{M})$ and $r \in R$.

We often write $\mathcal{M} := (\mathcal{M}, \varepsilon)$.

Remark 66 *Let $\mathcal{M} = (\mathcal{M}, \varepsilon)$ be an R-linear category.*

Suppose given r, $r' \in R$. Suppose given $X \underset{\tilde{u}}{\overset{u}{\rightrightarrows}} Y \xrightarrow{v} Z$ in $\mathcal{M}$.

(1) *We have* $(u \blacktriangle v)r = u \blacktriangle vr = ur \blacktriangle v$.

(2) *We have* $(u + \tilde{u})r = ur + \tilde{u}r$.

(3) *We have* $u(r + r') = ur + ur'$.

(4) *We have* $u(rr') = (ur)r'$.

(5) *We have* $u1_R = u$.

In particular, ${}_{\mathcal{M}}(X, Y)$ is an R-module.

Proof. Ad (1). We have

$$\begin{aligned} (u \blacktriangle v)r &= (u \blacktriangle v) \blacktriangle Z(r\varepsilon) = u \blacktriangle \big(v \blacktriangle Z(r\varepsilon)\big) = u \blacktriangle vr\,, \\ u \blacktriangle vr &= u \blacktriangle \big(v \blacktriangle Z(r\varepsilon)\big) = u \blacktriangle \big(Y(r\varepsilon) \blacktriangle v\big) = \big(u \blacktriangle Y(r\varepsilon)\big) \blacktriangle v = ur \blacktriangle v\,. \end{aligned}$$

Ad (2). We have

$$(u + \tilde{u})r = (u + \tilde{u}) \blacktriangle Y(r\varepsilon) = u \blacktriangle Y(r\varepsilon) + \tilde{u} \blacktriangle Y(r\varepsilon) = ur + \tilde{u}r\,.$$

Ad (3). We have

$$\begin{aligned} u(r + r') &= u \blacktriangle Y\big((r + r')\varepsilon\big) = u \blacktriangle Y\big((r\varepsilon) + (r'\varepsilon)\big) = u \blacktriangle \big(Y(r\varepsilon) + Y(r'\varepsilon)\big) \\ &= u \blacktriangle Y(r\varepsilon) + u \blacktriangle Y(r'\varepsilon) = ur + ur'\,. \end{aligned}$$

Ad (4). We have

$$u(rr') = u \blacktriangle Y\big((rr')\varepsilon\big) = u \blacktriangle Y\big((r\varepsilon) \blacktriangle (r'\varepsilon)\big) = u \blacktriangle Y(r\varepsilon) \blacktriangle Y(r'\varepsilon) = (ur)r' .$$

Ad (5). We have

$$u1_R = u \blacktriangle Y(1_R\varepsilon) = u \blacktriangle Y \mathrm{id}_{\mathrm{id}_{\mathcal{M}}} = u .$$

□

Definition 67 (R-linear subcategory)

Suppose given R-linear categories $\mathcal{M} = (\mathcal{M}, \varepsilon)$ and $\mathcal{N} = (\mathcal{N}, \varepsilon')$.

We say that $\mathcal{N}$ is an *R-linear subcategory* of $\mathcal{M}$ if the conditions $(1, 2, 3)$ are satisfied.

(1) The category $\mathcal{N}$ is a subcategory of $\mathcal{M}$.

(2) For X, $Y \in \mathrm{Ob}(\mathcal{N})$, we have the following commutative diagram.

$$\begin{tikzcd}
{}_{\mathcal{M}}(X,Y) \times_{\mathcal{M}} (X,Y) \arrow[r, "(+)"] & {}_{\mathcal{M}}(X,Y) \\
{}_{\mathcal{N}}(X,Y) \times_{\mathcal{N}} (X,Y) \arrow[r, "(+)"'] \arrow[u, hook] & {}_{\mathcal{N}}(X,Y) \arrow[u, hook]
\end{tikzcd}$$

(3) For $r \in R$ and $X \in \mathrm{Ob}(\mathcal{N})$, we have

$$X(r\varepsilon') = X(r\varepsilon) \colon X \to X .$$

Remark 68 Let $\mathcal{M} = (\mathcal{M}, \varepsilon)$ be an R-linear category.

Suppose given a subcategory $\mathcal{N}$ of $\mathcal{M}$. Suppose that the conditions $(1, 2)$ hold.

(1) For X, $Y \in \mathrm{Ob}(\mathcal{N})$, we have

$$\big(X \xrightarrow{0_{\mathcal{M}(X,Y)}} Y\big) \in \mathrm{Mor}(\mathcal{N}) .$$

(2) For r, $r' \in R$ and $X \overset{u}{\underset{u'}{\rightrightarrows}} Y$ in $\mathcal{N}$, we have

$$\big(X \xrightarrow{ur+u'r'} Y\big) \in \mathrm{Mor}(\mathcal{N}) .$$

Then, for X, $Y \in \mathrm{Ob}(\mathcal{N})$, the map

$$(+)\colon {}_{\mathcal{M}}(X,Y) \times {}_{\mathcal{M}}(X,Y) \to_{\mathcal{M}} (X,Y)$$

restricts to the map

$$(+)\colon {}_{\mathcal{N}}(X,Y) \times {}_{\mathcal{N}}(X,Y) \to_{\mathcal{N}} (X,Y)\,.$$

Let

$$\varepsilon'\colon R \to \mathrm{End}(\mathrm{id}_{\mathcal{N}})\,,\ r \mapsto \big(X(r\varepsilon)\big)_{X \in \mathrm{Ob}(\mathcal{N})}\,.$$

Then $r\varepsilon'$ is in fact a transformation, for $r \in R$.

Moreover, ε' is a ring morphism.

Finally, $(\mathcal{N}, \varepsilon')$ is an R-linear subcategory of $(\mathcal{M}, \varepsilon)$.

In particular, every full subcategory of $\mathcal{M}$ is an R-linear subcategory of $\mathcal{M}$.

4.2 R-linear functors

Definition 69 (Additive functor) Let $\mathcal{M}$, $\mathcal{N}$ be preadditive categories. Let $F\colon \mathcal{M} \to \mathcal{N}$ be a functor.

We call F *additive* if

$$(u+v)F = uF + vF$$

holds for $\big(X \xrightarrow{u} Y\big), \big(X \xrightarrow{v} Y\big) \in \mathrm{Mor}(\mathcal{M})$, $X, Y \in \mathrm{Ob}(\mathcal{M})$.

Definition 70 (R-linear functor) Let $\mathcal{M} = (\mathcal{M}, \varepsilon)$ and $\mathcal{N} = (\mathcal{N}, \varepsilon')$ be R-linear categories. Let $F\colon \mathcal{M} \to \mathcal{N}$ be a functor.

We say that F is *R-linear* if it is additive and if

$$F(r\varepsilon') = (r\varepsilon)F$$

holds for $r \in R$.

Remark 71 *Let $\mathcal{M} = (\mathcal{M}, \varepsilon)$ and $\mathcal{N} = (\mathcal{N}, \varepsilon')$ be R-linear categories. Let $F\colon \mathcal{M} \to \mathcal{N}$ be a functor. Then* (1) *and* (2) *are equivalent.*

(1) *The functor F is R-linear.*

(2) *For* $r, s \in R$ *and* $\big(X \xrightarrow{u} Y\big), \big(X \xrightarrow{v} Y\big) \in \mathrm{Mor}(\mathcal{M})$*, we have*
$$(ur + vs)F = (uF)r + (vF)s\,.$$

Recall that we have $ur = u \blacktriangle Y(r\varepsilon) = X(r\varepsilon) \blacktriangle u$ *for* $r \in R$, $\big(X \xrightarrow{u} Y\big) \in \mathrm{Mor}(\mathcal{M})$*; cf. Definition* 65*.*

Proof. Ad (1) $\Rightarrow$ (2). Suppose given $r, s \in R$ and $\big(X \xrightarrow{u} Y\big), \big(X \xrightarrow{v} Y\big) \in \mathrm{Mor}(\mathcal{M})$.

We have
$$\begin{aligned}(ur + vs)F &= \big(u \blacktriangle Y(r\varepsilon) + v \blacktriangle Y(s\varepsilon)\big)F \\ &= \big(u \blacktriangle Y(r\varepsilon)\big)F + \big(v \blacktriangle Y(s\varepsilon)\big)F \\ &= uF \blacktriangle Y(r\varepsilon)F + vF \blacktriangle Y(s\varepsilon)F \\ &= uF \blacktriangle YF(r\varepsilon') + vF \blacktriangle YF(s\varepsilon') \\ &= (uF)r + (vF)s\,.\end{aligned}$$

Ad (2) $\Rightarrow$ (1). For u, $v \in \mathrm{Mor}(\mathcal{M})$, we have
$$(u + v)F = uF + vF\,.$$

So F is additive.

Suppose given $X \in \mathrm{Ob}(\mathcal{M})$. Suppose given $r, s \in R$.

Note that the map
$$_{\mathcal{M}}(X, Y) \to {}_{\mathcal{N}}(XF, YF)\,,\ a \mapsto aF$$
is R-linear.

We have
$$\begin{aligned}X\big((r\varepsilon)F\big) &= \big(X(r\varepsilon)\big)F \overset{65}{=} (\mathrm{id}_X\, r)F = (\mathrm{id}_X\, F)r = \mathrm{id}_X\, F \blacktriangle (XF)(r\varepsilon) = (XF)(r\varepsilon) \\ &= X\big(F(r\varepsilon)\big)\,.\end{aligned}$$

This shows $(r\varepsilon)F = F(r\varepsilon)$. $\square$

Lemma 72 *Let* $\mathcal{M} = (\mathcal{M}, \varepsilon)$*,* $\mathcal{N} = (\mathcal{N}, \varepsilon')$ *and* $\mathcal{P} = (\mathcal{P}, \varepsilon'')$ *be* R*-linear categories. Let* $F \colon \mathcal{M} \to \mathcal{N}$ *and* $G \colon \mathcal{N} \to \mathcal{P}$ *be* R*-linear functors.*

(1) *The functor* $\mathrm{id}_M \colon \mathcal{M} \to \mathcal{M}$ *is an* R*-linear functor.*

(2) *The composite $F * G \colon \mathcal{M} \to \mathcal{P}$ is an R-linear functor.*

(3) *Suppose that $F \colon \mathcal{M} \to \mathcal{N}$ is an R-linear isofunctor. The inverse $F^{-} \colon \mathcal{N} \to \mathcal{M}$ is an R-linear functor.*

Proof. We use Remark 71.

Suppose given $X \overset{u_1}{\underset{u_2}{\rightrightarrows}} Y$ in $\mathcal{M}$. Suppose given $r, s \in R$.

Ad (1). We have

$$(u_1 r + u_2\, s)\mathrm{id}_{\mathcal{M}} = u_1 r + u_2\, s = (u_1\, \mathrm{id}_{\mathcal{M}})r + u_2\, \mathrm{id}_{\mathcal{M}})s\,.$$

Ad (2). We have

$$\begin{aligned}(u_1 r + u_2\, s)(F * G) &= \big((u_1 F)r + (u_2 F)s\big)G = (u_1 FG)r + (u_2 FG)s \\ &= \big(u_1(F * G)\big)r + \big(u_2(F * G)\big)s\,.\end{aligned}$$

Ad (3). We have

$$\begin{aligned}(u_1 r + u_2\, s)F^{-} &= \big((u_1 F^{-} F)r + (u_2 F^{-} F)s\big)F^{-} = \big((u_1 F^{-})r + (u_2 F^{-})s\big)FF^{-} \\ &= (u_1 F^{-})r + (u_2 F^{-})s\,.\end{aligned}$$

□

4.3 Monoidal R-linear categories

Definition 73 (Monoidal R-linear category)

Suppose given a preadditive category $\mathcal{A}$.

Let $(\mathcal{A}, \varepsilon)$ be an R-linear category; cf. Definition 65.

Let $(\mathcal{A}, I, \otimes)$ be a monoidal category; cf. Definition 12.

Suppose that (1,2) hold.

(1) For $r \in R$ and $u, v \in \mathrm{Mor}(\mathcal{A})$ we have

$$(u \otimes v)r = u \otimes vr = ur \otimes v\,.$$

(2) For $X \overset{u_1}{\underset{u_2}{\rightrightarrows}} Y$ in $\mathcal{A}$ and $v \in \operatorname{Mor}(\mathcal{A})$ we have

$$\begin{aligned}(u_1 + u_2) \otimes v &= (u_1 \otimes v) + (u_2 \otimes v)\\ v \otimes (u_1 + u_2) &= (v \otimes u_1) + (v \otimes u_2)\,.\end{aligned}$$

Then we call $(\mathcal{A}, I, \otimes, \varepsilon)$ a *monoidal R-linear category.* We often write $\mathcal{A} = (\mathcal{A}, I, \otimes, \varepsilon)$.

Definition 74 (Monoidal R-linear functor) Suppose given monoidal R-linear categories $\mathcal{A}$ and $\mathcal{B}$. Suppose given a functor $F\colon \mathcal{A} \to \mathcal{B}$.

We say that F is a *monoidal R-linear functor* if F is monoidal and R-linear; cf. Definitions 31, 70.

Remark 75 *Suppose given a monoidal R-linear category $\mathcal{A} = (\mathcal{A}, I, \otimes, \varepsilon)$.*
For $\big(A \xrightarrow{a} B\big) \in \operatorname{Mor}(\mathcal{A})$ *and* $X,\, Y \in \operatorname{Ob}(\mathcal{A})$, *we have*

$$a \otimes 0_{X,Y} = 0_{A\otimes X,\, B\otimes Y}\,.$$

Proof. Suppose given $\big(A \xrightarrow{a} B\big) \in \operatorname{Mor}(\mathcal{A})$ and $X,\, Y \in \operatorname{Ob}(\mathcal{A})$.
Note that

$$a \otimes 0_{X,Y} \in {}_{\mathcal{A}}(A \otimes X, B \otimes Y)\,.$$

We have

$$\begin{aligned}a \otimes 0_{X,Y} &= (a \otimes 0_{X,Y}) + (a \otimes 0_{X,Y}) - (a \otimes 0_{X,Y}) = \big(a \otimes (0_{X,Y} + 0_{X,Y})\big) - (a \otimes 0_{X,Y})\\ &= (a \otimes 0_{X,Y}) - (a \otimes 0_{X,Y}) = 0_{A\otimes X,\, B\otimes Y}\,.\end{aligned}$$

□

Chapter 5

$\mathrm{End}_R(\mathcal{M})$ and $\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})$ of an R-linear category $\mathcal{M}$

Let $\mathcal{M}$ an R-linear category.

5.1 The monoidal R-linear category $\mathrm{End}_R(\mathcal{M})$

Lemma 76 (The preadditive category $[\mathcal{B},\mathcal{C}[)$

Let $\mathcal{B}$ be a category. Let $\mathcal{C}$ be a preadditive category. Consider the category of functors $[\mathcal{B},\mathcal{C}]$; cf. §0.3 item 1.

For $X \in \mathrm{Ob}(\mathcal{B})$ and $\big(F \xrightarrow{a} G\big)$, $\big(F \xrightarrow{b} G\big) \in \mathrm{Mor}([\mathcal{B},\mathcal{C}])$ let

$$X(a+b) := Xa + Xb\,.$$

Endowed with this addition, $[\mathcal{B},\mathcal{C}]$ is a preadditive category.

In particular, we have

$$0_{F,G} = (0_{XF,\,XG})_{X\in\mathrm{Ob}(\mathcal{B})}$$

for F, $G \in \mathrm{Ob}([\mathcal{B},\mathcal{C}])$.

Proof. Suppose given F, $G \in \mathrm{Ob}([\mathcal{B},\mathcal{C}])$.

Then $\big({}_{[\mathcal{B},\mathcal{C}]}(F,G),+\big)$ is an abelian group with neutral element

$$0_{F,G} = (XF \xrightarrow{0_{XF,\,XG}} XG)_{X\in\mathrm{Ob}(\mathcal{B})}\,,$$

in which, for $a \in {}_{[\mathcal{B},\mathcal{C}]}(F,G)$, its inverse is given by

$$-a = (XF \xrightarrow{-Xa} XG)_{X\in\mathrm{Ob}(\mathcal{B})}\,.$$

Suppose given $F \xrightarrow{a} G \overset{b_1}{\underset{b_2}{\rightrightarrows}} H \xrightarrow{c} K$ in $[\mathcal{B},\mathcal{C}]$. For $X \in \mathrm{Ob}(\mathcal{B})$ we have

$$\begin{aligned} X\big(a \blacktriangle (b_1+b_2) \blacktriangle c\big) &= Xa \blacktriangle \big(X(b_1+b_2)\big) \blacktriangle Xc = Xa \blacktriangle (Xb_1 + Xb_2) \blacktriangle Xc \\ &= Xa \blacktriangle Xb_1 \blacktriangle Xc + Xa \blacktriangle Xb_2 \blacktriangle Xc = X(a \blacktriangle b_1 \blacktriangle c) + X(a \blacktriangle b_2 \blacktriangle c) \\ &= X(a \blacktriangle b_1 \blacktriangle c + a \blacktriangle b_2 \blacktriangle c)\,. \end{aligned}$$

Thus, $a \blacktriangle (b_1+b_2) \blacktriangle c = a \blacktriangle b_1 \blacktriangle c + a \blacktriangle b_2 \blacktriangle c$. □

Lemma 77 (The preadditive category ${}_{\mathrm{add}}[\mathcal{B},\mathcal{C}]$)

Suppose given preadditive categories $\mathcal{B}$, $\mathcal{C}$.

We have the full subcategory ${}_{\mathrm{add}}[\mathcal{B},\mathcal{C}] \subseteq [\mathcal{B},\mathcal{C}]$ given by

$$\mathrm{Ob}({}_{\mathrm{add}}[\mathcal{B},\mathcal{C}]) := \{\mathcal{B} \xrightarrow{F} \mathcal{C} \colon F \text{ is additive }\}\,.$$

Then ${}_{\mathrm{add}}[\mathcal{B},\mathcal{C}]$ is a preadditive category.

Proof. By Lemma 76, $[\mathcal{B},\mathcal{C}]$ is a preadditive category. Since ${}_{\mathrm{add}}[\mathcal{B},\mathcal{C}] \subseteq [\mathcal{B},\mathcal{C}]$ is a full subcategory, ${}_{\mathrm{add}}[\mathcal{B},\mathcal{C}]$ is also a preadditive category. □

Corollary 78 (The preadditive category $\mathrm{End}_{\mathrm{add}}(\mathcal{A})$)

Suppose given a preadditive category $\mathcal{A}$. Let $\mathrm{End}_{\mathrm{add}}(\mathcal{A}) := {}_{\mathrm{add}}[\mathcal{A},\mathcal{A}]$.

Then $\mathrm{End}_{\mathrm{add}}(\mathcal{A})$ *is a preadditive category.*

Proof. This is Lemma 77 with $\mathcal{A} = \mathcal{B} = \mathcal{C}$. □

Definition 79 (The category $\mathrm{End}_R(\mathcal{M})$)

Consider the functor category $[\mathcal{M},\mathcal{M}]$; cf. §0.3 item 5.

By $\mathrm{End}_R(\mathcal{M})$ we denote the full subcategory $\mathrm{End}_R(\mathcal{M}) \subseteq [\mathcal{M},\mathcal{M}]$ given by

$$\mathrm{Ob}(\mathrm{End}_R(\mathcal{M})) := \{\mathcal{M} \xrightarrow{F} \mathcal{M} \colon F \text{ is an } R\text{-linear functor }\}\,;$$

cf. Definition 70.

Moreover, we have

$$\mathrm{End}_R(\mathcal{M}) \subseteq \mathrm{End}_{\mathrm{add}}(\mathcal{M}) \subseteq [\mathcal{M}, \mathcal{M}]\,;$$

cf. Corollary 78.

Lemma 80 (The endomorphism monoidal R-linear category $\mathrm{End}_R(\mathcal{M})$)

Recall that $\mathcal{M} = (\mathcal{M}, \varepsilon)$ *is an* R*-linear category, where* $\varepsilon\colon R \to \mathrm{End}(\mathrm{id}_{\mathcal{M}})$ *is a ring morphism.*

Consider the category $\mathrm{End}_R(\mathcal{M}) \subseteq [\mathcal{M}, \mathcal{M}]$ *from Definition* 79.

(1) *We have the preadditive category* $\mathrm{End}_R(\mathcal{M})$.

(2) *We have a ring morphism*

$$\begin{array}{rcl} \epsilon\colon\; R & \to & \mathrm{End}\left(\mathrm{id}_{\mathrm{End}_R(\mathcal{M})}\right) \\ r & \mapsto & r\epsilon = \left(F \xrightarrow{F(r\epsilon)} F\right)_{F \in \mathrm{Ob}\left(\,\mathrm{End}_R(\mathcal{M})\right)}\,, \end{array}$$

with

$$\begin{array}{rcl} F(r\epsilon) = \left(XF \xrightarrow{X\left(F(r\epsilon)\right)} XF\right)_{X \in \mathrm{Ob}(\mathcal{M})} & := & \left(XF \xrightarrow{\left(X(r\varepsilon)\right)F} XF\right)_{X \in \mathrm{Ob}(\mathcal{M})} \\ & = & \left(XF \xrightarrow{(XF)(r\varepsilon)} XF\right)_{X \in \mathrm{Ob}(\mathcal{M})}\,, \end{array}$$

for $F \in \mathrm{Ob}(\mathrm{End}_R(\mathcal{M}))$*; cf. Definition* 70.

(3) *We have a functor*

$$\begin{array}{rccccl} (\,*\,)\colon & \mathrm{End}_R(\mathcal{M}) \times \mathrm{End}_R(\mathcal{M}) & \longrightarrow & \mathrm{End}_R(\mathcal{M}) & \\ & (\; F \;,\; G \;) & \longmapsto & FG & \text{for } F, G \in \mathrm{Ob}(\mathrm{End}_R(\mathcal{M})) \\ & \left(\begin{array}{c} F \\ \downarrow a \\ F' \end{array} ,\; \begin{array}{c} G \\ \downarrow b \\ G' \end{array} \right) & \longmapsto & \left(\begin{array}{c} F * G \\ \downarrow a*b \\ F' * G' \end{array} \right) & \text{for } a, b \in \mathrm{Mor}(\mathrm{End}_R(\mathcal{M}))\,. \end{array}$$

(4) *We have an* R*-linear category given by* $\left(\mathrm{End}_R(\mathcal{M}),\, \epsilon\right)$*; cf. Definition* 73.

For $r \in R$ *and* $\left(F \xrightarrow{a} G\right) \in \mathrm{Mor}(\mathrm{End}_R(\mathcal{M}))$*, we have*

$$ar = a \blacktriangle G(r\epsilon) = F(r\epsilon) \blacktriangle a\,.$$

So $X(ar) = (Xa)r$ *for* $X \in \mathrm{Ob}(\mathcal{M})$, $a \in \mathrm{Mor}\left(\mathrm{End}_R(\mathcal{M})\right)$, $r \in R$.

(5) *We have a monoidal category given by* $\big(\mathrm{End}_R(\mathcal{M}), \mathrm{id}_{\mathcal{M}}, *\big)$*; cf. Definition* 12.

(6) *We have a monoidal R-linear category given by* $\big(\mathrm{End}_R(\mathcal{M}), \mathrm{id}_{\mathcal{M}}, *, \epsilon\big)$*; cf. Definition* 73.

We call

$$\mathrm{End}_R(\mathcal{M}) = \big(\mathrm{End}_R(\mathcal{M}), \mathrm{id}_{\mathcal{M}}, *, \epsilon\big)$$

the endomorphism monoidal R-linear category *of* $\mathcal{M}$*.*

Proof. Ad (1). By Corollary 78, $\mathrm{End}_{\mathrm{add}}(\mathcal{M})$ is a preadditive category. Since $\mathrm{End}_R(\mathcal{M})$ is a full subcategory of $\mathrm{End}_{\mathrm{add}}(\mathcal{M})$, we have the preadditive category $\mathrm{End}_R(\mathcal{M})$.

Ad (2). *We show that* ϵ *is a well-defined map.*

Suppose given $r \in R$. We have to show that $r\epsilon$ is a transformation from $\mathrm{id}_{\mathrm{End}_R(\mathcal{M})}$ to $\mathrm{id}_{\mathrm{End}_R(\mathcal{M})}$.

Suppose given $F \in \mathrm{Ob}\big(\mathrm{End}_R(\mathcal{M})\big)$. We have to show that $F(r\epsilon)$ is a transformation from F to F.

Suppose given $\big(X \xrightarrow{u} Y\big) \in \mathrm{Mor}(\mathcal{M})$. Consider the transformation $r\varepsilon = \big(X \xrightarrow{X(r\varepsilon)} X\big)_{X \in \mathrm{Ob}(\mathcal{M})}$ from $\mathrm{id}_{\mathcal{M}}$ to $\mathrm{id}_{\mathcal{M}}$. Then we have

$$X(r\varepsilon) \blacktriangle u = u \blacktriangle Y(r\varepsilon)\,.$$

$$\begin{tikzcd} X \arrow[r, "X(r\varepsilon)"] \arrow[d, "u"'] & X \arrow[d, "u"] \\ Y \arrow[r, "Y(r\varepsilon)"'] & Y \end{tikzcd}$$

Therefore

$$\begin{aligned} X\big(F(r\epsilon)\big) \blacktriangle uF &= \big(X(r\varepsilon)\big)F \blacktriangle uF = \big(X(r\varepsilon) \blacktriangle u\big)F = \big(u \blacktriangle Y(r\varepsilon)\big)F = uF \blacktriangle \big(Y(r\varepsilon)\big)F \\ &= uF \blacktriangle Y\big(F(r\epsilon)\big)\,. \end{aligned}$$

This shows that the following diagram is commutative.

$$\begin{tikzcd} XF \arrow[r, "X\big(F(r\epsilon)\big)"] \arrow[d, "uF"'] & XF \arrow[d, "uF"] \\ YF \arrow[r, "Y\big(F(r\epsilon)\big)"'] & YF \end{tikzcd}$$

So $F(r\epsilon)$ is a transformation from F to F.

Suppose given $\big(F \xrightarrow{a} G\big) \in \mathrm{Mor}\,\big(\,\mathrm{End}_R(\mathcal{M})\big)$.

Suppose given $X \in \mathrm{Ob}(\mathcal{M})$.

Consider the transformation $r\varepsilon = \big(Y \xrightarrow{Y(r\varepsilon)} Y\big)_{Y\in\mathrm{Ob}(\mathcal{M})}$ from $\mathrm{id}_{\mathcal{M}}$ to $\mathrm{id}_{\mathcal{M}}$. Consider the morphism $\big(XF \xrightarrow{Xa} XG\big) \in \mathrm{Mor}(\mathcal{M})$. Then we have the following commutative diagram.

$$\begin{tikzcd} XF \arrow[r, "(XF)(r\varepsilon)"] \arrow[d, "Xa"'] & XF \arrow[d, "Xa"] \\ XG \arrow[r, "(XG)(r\varepsilon)"'] & XG \end{tikzcd}$$

So we have

$$\begin{aligned} X\big(F(r\epsilon) \blacktriangle a\big) &= X\big(F(r\epsilon)\big) \blacktriangle Xa = \big((XF)(r\varepsilon)\big) \blacktriangle Xa = Xa \blacktriangle \big((XG)(r\varepsilon)\big) = Xa \blacktriangle X\big(G(r\epsilon)\big) \\ &= X\big(a \blacktriangle G(r\epsilon)\big)\,. \end{aligned}$$

This shows $F(r\epsilon) \blacktriangle a = a \blacktriangle G(r\epsilon)$.

Therefore, we have the following commutative diagram.

$$\begin{tikzcd} F \arrow[r, "F(r\epsilon)"] \arrow[d, "a"'] & F \arrow[d, "a"] \\ G \arrow[r, "G(r\epsilon)"'] & G \end{tikzcd}$$

So $r\epsilon = \big(F \xrightarrow{F(r\epsilon)} F\big)_{F\in\mathrm{Ob}\,\left(\,\mathrm{End}_R(\mathcal{M})\right)}$ is a transformation from $\mathrm{id}_{\mathrm{End}_R(\mathcal{M})}$ to $\mathrm{id}_{\mathrm{End}_R(\mathcal{M})}$.

Therefore, ϵ is a well-defined map.

We show that ϵ is a ring morphism.

Suppose given $r,\ s \in R$.

For $F \in \mathrm{Ob}(\mathrm{End}_R(\mathcal{M}))$ and $X \in \mathrm{Ob}(\mathcal{M})$, we have

$$X\big(F(1\epsilon)\big) = (XF)(1\varepsilon) = (XF)\mathrm{id}_{\mathrm{id}_{\mathcal{M}}} = \mathrm{id}_{XF} = X\mathrm{id}_F = X(F\,\mathrm{id}_{\mathrm{id}_{\mathrm{End}_R(\mathcal{M})}})\,.$$

So, $F(1\epsilon) = F\,\mathrm{id}_{\mathrm{id}_{\mathrm{End}_R(\mathcal{M})}}$ for $F \in \mathrm{Ob}(\mathcal{M})$. Therefore $1\epsilon = \mathrm{id}_{\mathrm{id}_{\mathrm{End}_R(\mathcal{M})}}$.

For $F \in \mathrm{Ob}(\mathrm{End}_R(\mathcal{M}))$ and $X \in \mathrm{Ob}(\mathcal{M})$, we have

$$\begin{aligned} X\Big(F\big((r+s)\epsilon\big)\Big) &= (XF)\big((r+s)\varepsilon\big) = (XF)(r\varepsilon + s\varepsilon) = (XF)(r\varepsilon) + (XF)(s\varepsilon) \\ &= X\big(F(r\epsilon)\big) + X\big(F(s\epsilon)\big) = X\big(F(r\epsilon) + F(s\epsilon)\big) = X\big(F(r\epsilon + s\epsilon)\big)\,. \end{aligned}$$

So, $F\big((r+s)\epsilon\big) = F(r\epsilon + s\epsilon)$ for $F \in \mathrm{Ob}(\mathcal{M})$. Therefore $(r+s)\epsilon = r\epsilon + s\epsilon$.

For $F \in \mathrm{Ob}(\mathrm{End}_R(\mathcal{M}))$ and $X \in \mathrm{Ob}(\mathcal{M})$, we have

$$\begin{aligned} X\Big(F\big((rs)\epsilon\big)\Big) &= (XF)\big((rs)\varepsilon\big) = (XF)(r\varepsilon \blacktriangle s\varepsilon) = (XF)(r\varepsilon) \blacktriangle (XF)(s\varepsilon) \\ &= X\big(F(r\epsilon)\big) \blacktriangle X\big(F(s\epsilon)\big) = X\big(F(r\epsilon) \blacktriangle F(s\epsilon)\big) = X\big(F(r\epsilon \blacktriangle s\epsilon)\big)\,. \end{aligned}$$

So, $F\big((rs)\epsilon\big) = F(r\epsilon \blacktriangle s\epsilon)$ for $F \in \mathrm{Ob}(\mathcal{M})$. Therefore $(rs)\epsilon = r\epsilon \blacktriangle s\epsilon$.

This shows that ϵ is a ring morphism.

Ad (3). Suppose given $F \xrightarrow{a} F' \xrightarrow{a'} F''$ and $G \xrightarrow{b} G' \xrightarrow{b'} G'$ in $\mathrm{End}_R(\mathcal{M})$. Note that the composite $F * G$ is an R-linear functor since F and G are R-linear; cf. Lemma 72.

We have

$$\mathrm{id}_F * \mathrm{id}_G = (\mathrm{id}_F\ G) \blacktriangle (F\,\mathrm{id}_G) = \mathrm{id}_{F*G}\,.$$

We have

$$\begin{aligned} (a \blacktriangle a') * (b \blacktriangle b') &= (a \blacktriangle a')G \blacktriangle F''(b \blacktriangle b') = aG \blacktriangle (a'G \blacktriangle F''b) \blacktriangle F''b' = aG \blacktriangle (a' * b) \blacktriangle F''b' \\ &= (aG \blacktriangle F'b) \blacktriangle (a'G' \blacktriangle F''b') = (a * b) \blacktriangle (a' * b')\,. \end{aligned}$$

Ad (4). By (1), $\mathrm{End}_R(\mathcal{M})$ is a preadditive category.

By (2), $\epsilon\colon R \to \mathrm{End}(\mathrm{id}_{\mathrm{End}_R(\mathcal{M})})$ is a ring morphism.

So, $(\mathrm{End}_R(\mathcal{M}),\ \epsilon\,)$ is an R-linear category.

Suppose given $r \in R$ and $\big(F \xrightarrow{a} G\big) \in \mathrm{Mor}(\mathrm{End}_R(\mathcal{M}))$.

We have
$$ar = a \blacktriangle G(r\epsilon) = F(r\epsilon) \blacktriangle a\,;$$
cf. Definition 65.

For $X \in \mathrm{Ob}(\mathcal{M})$, we have
$$X(ar) = X\big(a \blacktriangle G(r\epsilon)\big) = Xa \blacktriangle X\big(G(r\epsilon)\big) = Xa \blacktriangle (XG)(r\varepsilon) = (Xa)r\,.$$

Ad (5). For $\big(F \xrightarrow{a} G\big) \in \mathrm{Mor}(\mathrm{End}_R(\mathcal{M}))$, we have
$$\begin{aligned} \mathrm{id}_{\mathrm{id}_{\mathcal{M}}} * a &= (\mathrm{id}_{\mathrm{id}_{\mathcal{M}}}\, F) \blacktriangle (\mathrm{id}_{\mathcal{M}}\ a) = a \\ a * \mathrm{id}_{\mathrm{id}_{\mathcal{M}}} &= a\,\mathrm{id}_{\mathcal{M}} \blacktriangle F\,\mathrm{id}_{\mathrm{id}_{\mathcal{M}}} = a\,. \end{aligned}$$

Recall that the horizontal composition of transformations $(*)$ are associative; cf. §0.3 item 3. This shows that $\big(\,\mathrm{End}_R(\mathcal{M}), \mathrm{id}_{\mathcal{M}}, *\,\big)$ is a monoidal category; cf. Remark 14.

Ad (6). Suppose given $r \in R$. Suppose given $\big(F \xrightarrow{a} F'\big)$, $\big(G \xrightarrow{b} G'\big) \in \mathrm{Mor}(\mathrm{End}_R(\mathcal{M}))$. We have
$$\begin{aligned} (a*b)r &= (a*b) \blacktriangle (F'G')(r\epsilon) = aG \blacktriangle F'b \blacktriangle F'G'(r\epsilon) \\ a * br &= aG \blacktriangle F'(br) = aG \blacktriangle F'\big(b \blacktriangle G'(r\epsilon)\big) = aG \blacktriangle F'b \blacktriangle F'G'(r\epsilon) \\ ar * b &= (ar)G \blacktriangle F'b = \big(a \blacktriangle F'(r\epsilon)\big)G \blacktriangle F'b = aG \blacktriangle F'\big((r\epsilon)G \blacktriangle b\big) = aG \blacktriangle F'\big((r\epsilon) * b\big) \\ &= aG \blacktriangle F'\big(b \blacktriangle (r\epsilon)G'\big) = aG \blacktriangle F'b \blacktriangle F'(r\epsilon)G' \overset{G'\ R\text{-linear}}{=} aG \blacktriangle F'b \blacktriangle F'G'(r\epsilon)\,. \end{aligned}$$

$\square$

5.2 The crossed module $\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})$

Lemma 81 (The crossed module $\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})$)
Consider the symmetric crossed module $\mathrm{S}_{\mathcal{M}} = \big(\,\mathrm{G}_{\mathcal{M}}, \mathrm{M}_{\mathcal{M}}, \mathrm{f}_{\mathcal{M}}, \gamma_{\mathcal{M}}\big)$ *on* $\mathcal{M}$*; cf. Lemma* 48.

(1) *We have subgroups*
$$\begin{aligned} \mathrm{G}_{\mathcal{M}}^R &:= \{\mathcal{M} \xrightarrow{F} \mathcal{M} \colon F \text{ is an } R\text{-linear autofunctor }\} = \{F \in \mathrm{G}_{\mathcal{M}} \colon F \text{ is } R\text{-linear }\} \leqslant \mathrm{G}_{\mathcal{M}} \\ \mathrm{M}_{\mathcal{M}}^R &:= \{\big(\mathrm{id}_{\mathcal{M}} \xrightarrow{a} F\big) \colon F \in \mathrm{G}_{\mathcal{M}}^R \text{ and } a \text{ is an isotransformation}\} \leqslant \mathrm{M}_{\mathcal{M}}\ . \end{aligned}$$

(2) *Consider the maps*

$$\mathrm{f}_{\mathcal{M}}^R\colon\ \mathrm{M}_{\mathcal{M}}^R \to \mathrm{G}_{\mathcal{M}}^R\,,\ \left(\mathrm{id}_{\mathcal{M}} \xrightarrow[\sim]{a} F\right) \mapsto F$$

$$\gamma_{\mathcal{M}}^R\colon\ \mathrm{G}_{\mathcal{M}}^R \to \mathrm{Aut}\left(\mathrm{M}_{\mathcal{M}}^R\right),\ G \mapsto \left(\left(\mathrm{id}_{\mathcal{M}} \xrightarrow[\sim]{a} F\right) \mapsto \left(\mathrm{id}_{\mathcal{M}} \xrightarrow[\sim]{G^- aG} G^- FG\right)\right).$$

We have a crossed submodule

$$\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M}) := \left(\,\mathrm{M}_{\mathcal{M}}^R, \mathrm{G}_{\mathcal{M}}^R, \gamma_{\mathcal{M}}^R, \mathrm{f}_{\mathcal{M}}^R\,\right) \leqslant \mathrm{S}_{\mathcal{M}}\ .$$

We call $\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})$ *the* automorphism crossed module *of* $\mathcal{M}$*.*

The upper index CM in $\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})$ should merely indicate that $\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})$ is a crossed module. However, cf. Lemma 98 below.

Proof. Ad (1). By Lemma 72.(1), we have $\mathrm{id}_{\mathcal{M}} \in \mathrm{G}_{\mathcal{M}}^R$. Suppose given $F, G \in \mathrm{G}_{\mathcal{M}}^R$. Then, by Lemma 72.(2, 3), we have $G^- \in \mathrm{G}_{\mathcal{M}}^R$ and $FG^- \in \mathrm{G}_{\mathcal{M}}^R$. So, we have a subgroup $\mathrm{G}_{\mathcal{M}}^R \leqslant \mathrm{G}_{\mathcal{M}}$.

Consider the group morphism $\mathrm{M}_{\mathcal{M}} \xrightarrow{\mathrm{f}_{\mathcal{M}}} \mathrm{G}_{\mathcal{M}}$. Since we have $\mathrm{G}_{\mathcal{M}}^R \leqslant \mathrm{G}_{\mathcal{M}}$, we have a subgroup $\mathrm{f}_{\mathcal{M}}^{-1}(\mathrm{G}_{\mathcal{M}}^R) = \{\mathrm{id}_{\mathcal{M}} \xrightarrow[\sim]{a} F\colon F \in \mathrm{G}_{\mathcal{M}}^R\} = \mathrm{M}_{\mathcal{M}}^R \leqslant \mathrm{M}_{\mathcal{M}}$.

Ad (2). Suppose given $\left(\mathrm{id}_{\mathcal{M}} \xrightarrow[\sim]{a} F\right) \in \mathrm{M}_{\mathcal{M}}^R$. Then

$$(a)\,\mathrm{f}_{\mathcal{M}}^R = F = (a)\,\mathrm{f}_{\mathcal{M}} \in \mathrm{G}_{\mathcal{M}}^R\ .$$

So, $\mathrm{f}_{\mathcal{M}}^R = \mathrm{f}_{\mathcal{M}}\big|_{\mathrm{M}_{\mathcal{M}}^R}^{\mathrm{G}_{\mathcal{M}}^R}$.

$$\begin{tikzcd}
\mathrm{M}_{\mathcal{M}} \arrow[r, "\mathrm{f}_{\mathcal{M}}"] & \mathrm{G}_{\mathcal{M}} \\
\mathrm{M}_{\mathcal{M}}^R \arrow[u, hook] \arrow[r, "\mathrm{f}_{\mathcal{M}}^R"'] & \mathrm{G}_{\mathcal{M}}^R \arrow[u, hook]
\end{tikzcd}$$

Suppose given $\left(\mathrm{id}_{\mathcal{M}} \xrightarrow[\sim]{a} F\right) \in \mathrm{M}_{\mathcal{M}}^R$ and $G \in \mathrm{G}_{\mathcal{M}}^R$. We have

$$a^G = \left(\mathrm{id}_{\mathcal{M}} \xrightarrow[\sim]{G^- aG} G^- FG\right).$$

By (1), we have $G^- FG \in \mathrm{G}_{\mathcal{M}}^R$. So, $a^G \in \mathrm{M}_{\mathcal{M}}^R$.

Therefore, $\mathrm{Aut}_R(\mathcal{M}) \leqslant \mathrm{S}_{\mathcal{M}}$ is a crossed submodule; cf. [**15,** Def. 17].

$\square$

Chapter 6

The operations $\mathrm{L} = (-)R$ and U

We will construct in § 6.1 an operation $\mathrm{L} = (-)R$ that maps from monoidal categories, monoidal functors and monoidal transformations to monoidal R-linear categories, monoidal R-linear functors and monoidal transformations by R-linear extension.

This could be summarized by saying that L is a 2-functor from the 2-category of monoidal categories to the 2-category of monoidal R-linear categories.

We will construct in § 6.2 an operation U that maps from monoidal categories, monoidal functors and monoidal transformations to invertible monoidal categories, monoidal functors and monoidal transformations.

This could be summarized by saying that U is a 2-functor from the 2-category of monoidal categories to the 2-category of invertible monoidal categories.

We will show that in § 6.3 that L and U are related in a way that could be called a 2-adjunction.

We hope that the reader who wishes to use the language of 2-categories will be able to rephrase our assertions accordingly.

6.1 The operation $\mathrm{L} = (-)R$

Definition 82 (The category $\mathcal{C}R$)

Recall that we are given a category $\mathcal{C}$ and a commutative ring R with identity.

We have a category $\mathcal{C}R$ given as follows.

We set

$$\mathrm{Ob}(\mathcal{C}R) := \mathrm{Ob}(\mathcal{C}) \,.$$

For $X, Y \in \mathrm{Ob}(\mathcal{C}R)$, the set of morphism from X to Y is given by the free module over R with basis ${}_{\mathcal{C}}(X,Y)$,

$${}_{\mathcal{C}R}(X,Y) := \big({}_{\mathcal{C}}(X,Y)\big)R\,.$$

Writing a morphism of $\mathcal{C}R$ in the form $\sum\limits_{i\in S} u_i r_i \colon X \to Y$, we implicitly suppose given a finite set S indexing this formal sum, and implicitly suppose $(u_i \colon X \to Y) \in \mathrm{Mor}(\mathcal{C})$ and $r_i \in R$ for $i \in S$. Often, we also write $\sum\limits_{i} u_i r_i = \sum\limits_{i\in S} u_i r_i$.

For $X \xrightarrow{u} Y \xrightarrow{v} Z$ in $\mathcal{C}R$, where $u = \sum\limits_{i\in S} u_i r_i$, $v = \sum\limits_{j\in T} v_j s_j$, the composite is given by

$$u \blacktriangle v = \Big(\sum_{i\in S} u_i r_i\Big) \blacktriangle \Big(\sum_{j\in T} v_j s_j\Big) := \sum_{(i,j)\in S\times T} (u_i \blacktriangle v_j) r_i s_j\,.$$

Lemma 83 (The R-linear category $\mathcal{C}R$)

(1) *The category $\mathcal{C}R$ is a preadditive catgory; cf. Definition* 64.

(2) *We have a ring morphism $\varphi_R \colon R \to \mathrm{End}(\mathrm{id}_{\mathcal{C}R})$, $r \mapsto r\varphi_R$, with*

$$X(r\varphi_R) := \mathrm{id}_X r$$

for $X \in \mathrm{Ob}(\mathcal{C}R)$. I.e. for $r \in R$, we have

$$r\varphi_R = \left(\begin{array}{ccc} \begin{tikzcd} X \arrow[d, "u"] \\ Y \end{tikzcd} & \longmapsto & \begin{tikzcd} X \arrow[r, "\mathrm{id}_X r"] \arrow[d, "u"'] & X \arrow[d, "u"] \\ Y \arrow[r, "\mathrm{id}_Y r"'] & Y \end{tikzcd} \end{array} \right) .$$

So, $(\mathcal{C}R, \varphi_R)$ is an R-linear category; cf. Definition 65.

Proof. Ad (1). Suppose given $W \xrightarrow{a} X \overset{b}{\underset{b'}{\rightrightarrows}} Y \xrightarrow{c} Z$ in $\mathcal{C}R$. Without loss of generality, we may write $a = \sum\limits_{k\in K} a_k r_k$, $b = \sum\limits_{l\in L} b_l s_l$, $b' = \sum\limits_{l\in L} b_l s'_l$, $c = \sum\limits_{p\in P} c_p t_p$.

We have

$$\begin{aligned}
a \blacktriangle (b+b') \blacktriangle c &= \big(\sum_{k\in K} a_k r_k\big) \blacktriangle \Big(\big(\sum_{l\in L} b_l s_l\big) + \big(\sum_{l\in L} b_l s'_l\big)\Big) \blacktriangle \big(\sum_{p\in P} c_p t_p\big) \\
&= \big(\sum_{k\in K} a_k r_k\big) \blacktriangle \Big(\sum_{l\in L} b_l (s_l + s'_l)\Big) \blacktriangle \big(\sum_{p\in P} c_p t_p\big) \\
&= \sum_{(k,l,p)\in K\times L\times P} (a_k \blacktriangle b_l \blacktriangle c_p) r_k (s_l + s'_l) t_p \\
&= \sum_{(k,l,p)\in K\times L\times P} (a_k \blacktriangle b_l \blacktriangle c_p) r_k s_l t_p + (a_k \blacktriangle b_l \blacktriangle c_p) r_k s'_l t_p \\
&= \big(\sum_{(k,l,p)\in K\times L\times P} (a_k \blacktriangle b_l \blacktriangle c_p) r_k s_l t_p\big) + \big(\sum_{(k,l,p)\in K\times L\times P} (a_k \blacktriangle b_l \blacktriangle c_p) r_k s'_l t_p\big) \\
&= \Big(\big(\sum_{k\in K} a_k r_k\big) \blacktriangle \big(\sum_{l\in L} b_l s_l\big) \blacktriangle \big(\sum_{p\in P} c_p t_p\big)\Big) + \Big(\big(\sum_{k\in K} a_k r_k\big) \blacktriangle \big(\sum_{l\in L} b_l s'_l\big) \blacktriangle \big(\sum_{p\in P} c_p t_p\big)\Big) \\
&= a \blacktriangle b \blacktriangle c + a \blacktriangle b' \blacktriangle c\,.
\end{aligned}$$

So $\mathcal{C}R$ is a preadditive category.

Ad (2). *We show that φ_R is well-defined.*

Suppose given $r \in R$ and $u = \big(X \xrightarrow{\sum_{i\in S} u_i r_i} Y\big) \in \mathrm{Mor}(\mathcal{C}R)$.

We have

$$\begin{aligned}
X(r\varphi_R) \blacktriangle u &= \mathrm{id}_X r \blacktriangle \big(\sum_{i\in S} u_i s_i\big) = \sum_{i\in S} (\mathrm{id}_X \blacktriangle u_i) r s_i = \sum_{i\in S} (u_i \blacktriangle \mathrm{id}_Y) r s_i = \big(\sum_{i\in S} u_i s_i\big) \blacktriangle \mathrm{id}_Y r \\
&= u \blacktriangle Y(r\varphi_R)\,.
\end{aligned}$$

Therefore, $r\varphi_R$ is a transformation from $\mathrm{id}_{\mathcal{C}R}$ to $\mathrm{id}_{\mathcal{C}R}$.

We show that φ_R is a ring morphism.

Suppose given $r, s \in R$.

For $X \in \mathrm{Ob}(\mathcal{C}R)$, we have

$$X(1\varphi_R) = \mathrm{id}_X\, 1_R = \mathrm{id}_X\,.$$

Therefore, $1\varphi_R = \mathrm{id}_{\mathrm{id}_{\mathcal{C}R}}$.

For $X \in \mathrm{Ob}(\mathcal{C}R)$, we have

$$X\big((r+s)\varphi_R\big) = \mathrm{id}_X (r+s) = \mathrm{id}_X r + \mathrm{id}_X s = X(r\varphi_R) + X(s\varphi_R) = X(r\varphi_R + s\varphi_R)\,.$$

Therefore, $(r+s)\varphi_R = r\varphi_R + s\varphi_R$.

For $X \in \mathrm{Ob}(\mathcal{C}R)$, we have

$$\begin{aligned} X\big((rs)\varphi_R\big) &= \mathrm{id}_X(rs) = (\mathrm{id}_X \blacktriangle \mathrm{id}_X)(rs) = (\mathrm{id}_X r) \blacktriangle (\mathrm{id}_X s) = X(r\varphi_R) \blacktriangle X(s\varphi_R) \\ &= X\big((r\varphi_R) \blacktriangle (s\varphi_R)\big) \end{aligned}$$

Therefore, $(rs)\varphi_R = (r\varphi_R) \blacktriangle (s\varphi_R)$.

So φ_R is a ring morphism. $\square$

Lemma 84 (The monoidal category $\mathcal{C}R$)

Suppose given a monoidal category $(\mathcal{C}, I_{\mathcal{C}}, \underset{\mathcal{C}}{\otimes})$*; cf. Definition* 12*.*

Consider the category $\mathcal{C}R$*; cf. Definition* 82*.*

(1) *We have a functor*

$$\begin{array}{llll} (\underset{\mathcal{C}R}{\otimes}) : & \mathcal{C}R \times \mathcal{C}R & \longrightarrow & \mathcal{C}R \\ & (X,Y) & \longmapsto & X \underset{\mathcal{C}R}{\otimes} Y := X \underset{\mathcal{C}}{\otimes} Y \qquad \textit{for } X, Y \in \mathrm{Ob}(\mathcal{C}R) \\ & \big(\sum_i u_i r_i\,,\, \sum_j v_j s_j\big) & \longmapsto & \big(\sum_i u_i r_i\big) \underset{\mathcal{C}R}{\otimes} \big(\sum_j v_j s_j\big) := \sum_{i,j} (u_i \underset{\mathcal{C}}{\otimes} v_j) r_i s_j \\ & & & \qquad \textit{for } \sum_i u_i r_i\,,\, \sum_j v_j s_j \in \mathrm{Mor}(\mathcal{C}R)\,. \end{array}$$

(2) *We have a monoidal category* $(\mathcal{C}R, I_{\mathcal{C}}, \underset{\mathcal{C}R}{\otimes})$*.*

Proof. Ad (1). For $u = \big(X \xrightarrow{\sum_i u_i r_i} X'\big)$, $v = \big(Y \xrightarrow{\sum_j v_j s_j} Y'\big) \in \mathrm{Mor}(\mathcal{C}R)$, we have

$$\begin{aligned} (u \underset{\mathcal{C}R}{\otimes} v)\boldsymbol{s} &= \Big(\sum_{i,j} (u_i \underset{\mathcal{C}}{\otimes} v_j) r_i s_j\Big)\boldsymbol{s} = X \underset{\mathcal{C}}{\otimes} Y = X \underset{\mathcal{C}R}{\otimes} Y = u\boldsymbol{s} \underset{\mathcal{C}R}{\otimes} v\boldsymbol{s} \\ (u \underset{\mathcal{C}R}{\otimes} v)\boldsymbol{t} &= \Big(\sum_{i,j} (u_i \underset{\mathcal{C}}{\otimes} v_j) r_i s_j\Big)\boldsymbol{t} = X' \underset{\mathcal{C}}{\otimes} Y' = X' \underset{\mathcal{C}R}{\otimes} Y' = u\boldsymbol{t} \underset{\mathcal{C}R}{\otimes} v\boldsymbol{t} \\ (X \underset{\mathcal{C}R}{\otimes} Y)\boldsymbol{i} &= (X \underset{\mathcal{C}}{\otimes} Y)\boldsymbol{i} = \mathrm{id}_{X \underset{\mathcal{C}}{\otimes} Y} = \mathrm{id}_X \underset{\mathcal{C}}{\otimes} \mathrm{id}_Y = \mathrm{id}_X \underset{\mathcal{C}R}{\otimes} \mathrm{id}_Y = X\boldsymbol{i} \underset{\mathcal{C}R}{\otimes} Y\boldsymbol{i}\,. \end{aligned}$$

Moreover, for $X \xrightarrow{\sum_i u_i r_i} Y \xrightarrow{\sum_j v_j s_j} Z$ and $X' \xrightarrow{\sum_k u'_k t_k} Y' \xrightarrow{\sum_l v'_l p_l} Z'$ in $\mathcal{C}R$, we have

$$\begin{aligned}(u \blacktriangle v) \underset{\mathcal{C}R}{\otimes} (u' \blacktriangle v') &= \sum_{i,j} (u_i \blacktriangle v_j) r_i s_j \underset{\mathcal{C}R}{\otimes} \sum_{k,l} (u'_k \blacktriangle v'_l) t_k p_l = \sum_{i,j,k,l} \big((u_i \blacktriangle v_j) \underset{\mathcal{C}}{\otimes} (u'_k \blacktriangle v'_l)\big) r_i s_j t_k p_l \\ &= \sum_{i,k,j,l} \big((u_i \underset{\mathcal{C}}{\otimes} u'_k) \blacktriangle (v_i \underset{\mathcal{C}}{\otimes} v'_l)\big) r_i t_k s_j p_l = \sum_{i,k} (u_i \underset{\mathcal{C}}{\otimes} u'_k) r_i t_k \blacktriangle \sum_{j,l} (v_j \underset{\mathcal{C}}{\otimes} v'_l) s_j p_l \\ &= (u \underset{\mathcal{C}R}{\otimes} u') \blacktriangle (v \underset{\mathcal{C}R}{\otimes} v') \,.\end{aligned}$$

So $(\underset{\mathcal{C}R}{\otimes})$ is a functor.

Ad (2). Suppose given $u = \sum_i u_i r_i$, $v = \sum_j v_j s_j$ and $w = \sum_k w_k t_k \in \mathrm{Mor}(\mathcal{C}R)$. Write $I := I_{\mathcal{C}}$.

We have

$$\begin{aligned}u \underset{\mathcal{C}R}{\otimes} \mathrm{id}_I &= \big(\sum_i u_i r_i\big) \underset{\mathcal{C}R}{\otimes} \mathrm{id}_I = \sum_i (u_i \underset{\mathcal{C}}{\otimes} \mathrm{id}_I) r_i = \sum_i u_i r_i = u \\ \mathrm{id}_I \underset{\mathcal{C}R}{\otimes} u &= \mathrm{id}_I \underset{\mathcal{C}R}{\otimes} \big(\sum_i u_i r_i\big) = \sum_i (\mathrm{id}_I \underset{\mathcal{C}}{\otimes} u_i) r_i = \sum_i u_i r_i = u \,.\end{aligned}$$

Further, we have

$$\begin{aligned}(u \underset{\mathcal{C}R}{\otimes} v) \underset{\mathcal{C}R}{\otimes} w &= \big(\sum_{i,j} (u_i \underset{\mathcal{C}}{\otimes} v_j) r_i s_j\big) \underset{\mathcal{C}R}{\otimes} \big(\sum_k w_k t_k\big) = \sum_{i,j,k} \big((u_i \underset{\mathcal{C}}{\otimes} v_j) \underset{\mathcal{C}}{\otimes} w_k\big)(r_i s_j) t_k \\ &= \sum_{i,j,k} \big(u_i \underset{\mathcal{C}}{\otimes} (v_j \underset{\mathcal{C}}{\otimes} w_k)\big) r_i (s_j t_k) = \big(\sum_i u_i r_i\big) \underset{\mathcal{C}R}{\otimes} \big(\sum_{j,k} (v_j \underset{\mathcal{C}}{\otimes} w_k) s_j t_k\big) \\ &= u \underset{\mathcal{C}R}{\otimes} (v \underset{\mathcal{C}R}{\otimes} w) \,.\end{aligned}$$

So, by Remark 14, $(\mathcal{C}R, I_{\mathcal{C}}, \underset{\mathcal{C}R}{\otimes})$ is a monoidal category. □

Lemma 85 (The monoidal R-linear category $\mathcal{C}R$)

Suppose given a monoidal category $(\mathcal{C}, I_{\mathcal{C}}, \underset{\mathcal{C}}{\otimes})$; cf. Definition 12.

Consider the R-linear category $(\mathcal{C}R, \varphi_R)$; cf. Lemma 83.

Consider the monoidal category $(\mathcal{C}R, I_{\mathcal{C}}, \underset{\mathcal{C}R}{\otimes})$; cf. Lemma 84.

Then $(\mathcal{C}R, I_{\mathcal{C}}, \underset{\mathcal{C}R}{\otimes}, \varphi_R)$ is a monoidal R-linear category; cf. Definition 73.

Proof. Suppose given $t \in R$ and $u = \sum_i u_i r_i$, $v = \sum_j v_j s_j \in \mathrm{Mor}(\mathcal{C}R)$.

We have

$$\begin{aligned}
(u \underset{\mathcal{C}R}{\otimes} v)t &= \big(\sum_{i,j}(u_i \underset{\mathcal{C}}{\otimes} v_j)r_i s_j\big)t = \sum_{i,j}(u_i \underset{\mathcal{C}}{\otimes} v_j)r_i s_j t = \sum_{i,j}(u_i \underset{\mathcal{C}}{\otimes} v_j)r_i(s_j t) \\
&= \big(\sum_i u_i r_i\big) \underset{\mathcal{C}R}{\otimes} \big(\sum_j v_j s_j t\big) = u \underset{\mathcal{C}R}{\otimes} vt\,, \\
(u \underset{\mathcal{C}R}{\otimes} v)t &= \big(\sum_{i,j}(u_i \underset{\mathcal{C}}{\otimes} v_j)r_i s_j\big)t = \sum_{i,j}(u_i \underset{\mathcal{C}}{\otimes} v_j)r_i s_j t = \sum_{i,j}(u_i \underset{\mathcal{C}}{\otimes} v_j)(r_i t)s_j \\
&= \big(\sum_i u_i r_i t\big) \underset{\mathcal{C}R}{\otimes} \big(\sum_j v_j s_j\big) = ut \underset{\mathcal{C}R}{\otimes} v\,.
\end{aligned}$$

So, $(\mathcal{C}R, I_\mathcal{C}, \underset{\mathcal{C}R}{\otimes}, \varphi_R)$ is a monoidal R-linear category. $\square$

Lemma 86 (The functor FR)

Suppose given categories $\mathcal{C}$ and $\mathcal{D}$. Suppose given a functor $F\colon \mathcal{C} \to \mathcal{D}$.

(1) *We have an R-linear functor given by*

$$\begin{array}{rrcll}
FR\colon & \mathcal{C}R & \longrightarrow & \mathcal{D}R & \\
& X & \longmapsto & XFR := XF & \text{for } X \in \mathrm{Ob}(\mathcal{C}R) \\
& u = \sum\limits_{k\in K} u_k r_k & \longmapsto & uFR := \sum\limits_{k\in K}(u_k F)r_k & \text{for } u \in \mathrm{Mor}(\mathcal{C}R)\,.
\end{array}$$

(2) *Suppose that $\mathcal{C}$ and $\mathcal{D}$ are monoidal categories. Suppose that $F\colon \mathcal{C} \to \mathcal{D}$ is a monoidal functor.*

Then the functor $FR\colon \mathcal{C}R \to \mathcal{D}R$ given in (1) is a monoidal R-linear functor.

Proof. Ad (1). *We show that FR is a functor.*

Suppose given $X \xrightarrow{\sum_i u_i r_i} Y \xrightarrow{\sum_j v_j s_j} Z$ in $\mathcal{C}R$. Write $u := \sum_i u_i r_i$ and $v := \sum_j v_j s_j$.

We have

$$\begin{aligned}
(uFR)\boldsymbol{s} &= \big(\sum_i (u_i F)r_i\big)\boldsymbol{s} = XF = XFR = (u\boldsymbol{s})FR \\
(XFR)\boldsymbol{i} &= (XF)\boldsymbol{i} = (X\boldsymbol{i})F \\
(uFR)\boldsymbol{t} &= \big(\sum_i (u_i F)r_i\big)\boldsymbol{t} = YF = YFR = (u\boldsymbol{t})F\,,
\end{aligned}$$

and we have

$$\begin{aligned}(u \blacktriangle v)FR &= \big(\sum_{i,j}(u_i \blacktriangle v_j)r_i s_j\big)FR = \sum_{i,j}\big((u_i \blacktriangle v_j)F\big)r_i s_j = \sum_{i,j}(u_i F \blacktriangle v_j F)r_i s_j \\ &= \sum_{i,j}(u_i F)r_i \blacktriangle (v_j F)s_j = \big(\sum_i (u_i F)r_i\big) \blacktriangle \big(\sum_j (v_j F)s_j\big) = uFR \blacktriangle vFR\,.\end{aligned}$$

So FR is a functor.

We show that FR is R-linear.

Suppose given $s,\, t \in R$ and suppose give $X \overset{u}{\underset{u'}{\rightrightarrows}} Y \in \mathrm{Mor}(\mathcal{C}R)$. Without loss of generality, we may write $u =: \sum_i u_i r_i$ and $u' =: \sum_i u_i r_i'$.

We have

$$\begin{aligned}(ur + u's)FR &= \Big(\big(\sum_i u_i r_i s\big) + \big(\sum_i u_i r_i' t\big)\Big)FR = \big(\sum_i u_i(r_i s + r_i' t)\big)FR \\ &= \sum_i (u_i F)(r_i s + r_i' t) = \sum_i \big((u_i F)r_i s + (u_i F)r_i' t\big) \\ &= \big(\sum_i (u_i F)r_i\big)s + \big(\sum_i (u_i F)r_i'\big)t = (uFR)s + (u'FR)t\,.\end{aligned}$$

Thus, by Remark 71, FR is R-linear.

Ad (2). By (1), FR is an R-linear functor. We have to show that FR is monoidal.

We have

$$(I_{\mathcal{C}})FR = (I_{\mathcal{C}})F = I_{\mathcal{D}}\,.$$

Suppose given $u = \sum_i u_i r_i\,,\ v = \sum_j v_j s_j \in \mathrm{Mor}(\mathcal{C}R)$.

We have

$$\begin{aligned}(u \otimes v)FR &= \big(\sum_{i,j}(u_i \otimes v_j)r_i s_j\big)FR = \sum_{i,j}\big((u_i \otimes v_j)F\big)r_i s_j = \sum_{i,j}\big((u_i F) \otimes (v_j F)\big)r_i s_j \\ &= \big(\sum_i (u_i F)r_i\big) \otimes \big(\sum_j (v_j F)s_j\big) = (uFR) \otimes (vFR)\,.\end{aligned}$$

Thus, by Remark 32.(1), FR is monoidal. $\square$

Lemma 87 (The transformation aR)

Suppose given categories $\mathcal{C}$ and $\mathcal{D}$. Suppose given functors F, $G\colon \mathcal{C} \to \mathcal{D}$. Suppose given a transformation $a\colon F \to G$.

Consider the R-linear categories $\mathcal{C}R$ and $\mathcal{D}R$; cf. Lemma 83. Consider the R-linear functors FR, $GR\colon \mathcal{C}R \to \mathcal{D}R$; cf. Lemma 86.

(1) *Then we have a transformation $aR\colon FR \to GR$ given by*

$$aR = \big((X)FR \xrightarrow{(X)aR} (X)GR\big)_{X\in\mathrm{Ob}(\mathcal{C}R)} := \big(XF \xrightarrow{Xa} XG\big)_{X\in\mathrm{Ob}(\mathcal{C})}\,.$$

(2) *Suppose that $\mathcal{C}$ and $\mathcal{D}$ are monoidal categories. Suppose that F, $G\colon \mathcal{C} \to \mathcal{D}$ are monoidal functors. Suppose that $a\colon F \to G$ is a monoidal transformation.*

Then the transformation $aR\colon FR \to GR$ given in (1) is a monoidal transformation.

Proof. Ad (1). For $X \xrightarrow{u} Y$ in $\mathcal{C}R$, where $u = \sum_i u_i r_i$, we have

$$\begin{aligned}
\big((X)aR\big) \blacktriangle \big((u)GR\big) &= Xa \blacktriangle \Big(\sum_i (u_iG)r_i\Big) = \sum_i \big(Xa \blacktriangle (u_iG)r_i\big) = \sum_i (Xa \blacktriangle u_iG)r_i \\
&= \sum_i \big((u_iF) \blacktriangle Ya\big)r_i = \sum_i \big((u_iF)r_i \blacktriangle Ya\big) = \Big(\sum_i (u_iF)r_i\Big) \blacktriangle Ya \\
&= \big((u)FR\big) \blacktriangle \big((Y)aR\big)\,.
\end{aligned}$$

$$\begin{tikzcd}
(X)FR \arrow[r, "(X)aR"] \arrow[d, "(u)FR"'] & (X)GR \arrow[d, "(u)GR"] \\
(Y)FR \arrow[r, "(Y)aR"'] & (Y)GR
\end{tikzcd}$$

Ad (2). We have

$$(I_{\mathcal{C}R})aR = (I_{\mathcal{C}})aR = (I_{\mathcal{C}})a = \mathrm{id}_{I_{\mathcal{D}}} = \mathrm{id}_{I_{\mathcal{D}R}}\,.$$

For $X \in \mathrm{Ob}(\mathcal{C}R) = \mathrm{Ob}(\mathcal{C})$, we have

$$(X \underset{\mathcal{C}R}{\otimes} Y)aR = (X \underset{\mathcal{C}}{\otimes} Y)a = (Xa) \underset{\mathcal{D}}{\otimes} (Ya) = \big((X)aR\big) \underset{\mathcal{D}R}{\otimes} \big((Y)aR\big)\,.$$

$\square$

Lemma 88 *Suppose given categories $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{K}$.*
Suppose given functors $F, F', F'' \colon \mathcal{C} \to \mathcal{D}$ and $G, G' \colon \mathcal{D} \to \mathcal{E}$ and $H \colon \mathcal{E} \to \mathcal{K}$.
Suppose given transformations $a \colon F \to F'$ and $a' \colon F' \to F''$ and $b \colon G \to G'$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F"] \arrow[r, "F'"] \arrow[r, bend right=50, "F''"'] \arrow[r, phantom, shift left=3ex, "\Downarrow a"] \arrow[r, phantom, shift right=3ex, "\Downarrow a'"] & \mathcal{D} \arrow[r, bend left=50, "G"] \arrow[r, bend right=50, "G'"'] \arrow[r, phantom, "\Downarrow b"] & \mathcal{E} \arrow[r, "H"] & \mathcal{K}
\end{tikzcd}$$

(1) *We have* $\quad (\mathrm{id}_{\mathcal{C}})R = \mathrm{id}_{\mathcal{C}R}$ *.*

(2) *We have* $\quad (F * G)R = FR * GR$ *.*

(3) *We have* $\quad (\mathrm{id}_F)R = \mathrm{id}_{FR}$ *.*

(4) *We have* $\quad (a \blacktriangle a')R = aR \blacktriangle a'R$ *.*

(5) *We have* $\quad (a * b)R = aR * bR$ *.*

(6) *We have* $\quad (FbH)R = (FR)(bR)(HR)$ *.*

$$\begin{tikzcd}
\mathcal{C}R \arrow[r, bend left=50, "FR"] \arrow[r, "F'R"] \arrow[r, bend right=50, "F''R"'] \arrow[r, phantom, shift left=3ex, "\Downarrow aR"] \arrow[r, phantom, shift right=3ex, "\Downarrow a'R"] & \mathcal{D}R \arrow[r, bend left=50, "GR"] \arrow[r, bend right=50, "G'R"'] \arrow[r, phantom, "\Downarrow bR"] & \mathcal{E}R \arrow[r, "HR"] & \mathcal{K}R
\end{tikzcd}$$

Proof. Ad (1). For $u = \sum_i u_i r_i \in \mathrm{Mor}(\mathcal{C}R)$, we have

$$(u)\big((\mathrm{id}_{\mathcal{C}})R\big) = \sum_i (u_i \,\mathrm{id}_{\mathcal{C}}) r_i = \sum_i u_i r_i = u\,.$$

So $(\mathrm{id}_{\mathcal{C}})R = \mathrm{id}_{\mathcal{C}R}$.

Ad (2). For $u = \sum_i u_i r_i \in \mathrm{Mor}(\mathcal{C}R)$, we have

$$\begin{aligned}(u)\big((F * G)R\big) &= \sum_i \big(u_i(F \blacktriangle G)\big)r_i = \sum_i \big((u_iF)G\big)r_i = \big(\sum_i (u_iF)r_i\big)(GR)\\ &= \Big(\big(\sum_i u_ir_i\big)(FR)\Big)(GR) = (u)\big((FR) * (GR)\big)\,.\end{aligned}$$

So $(F * G)R = FR * GR$.

Ad (3). For $X \in \mathrm{Ob}(\mathcal{C}R)$, we have

$$(X)\big((\mathrm{id}_{FR})R\big) = (X)\mathrm{id}_F = \mathrm{id}_{XF} = \mathrm{id}_{XFR} = (X)\mathrm{id}_{FR}\,.$$

So $(\mathrm{id}_F)R = \mathrm{id}_{FR}$.

Ad (4). For $X \in \mathrm{Ob}(\mathcal{C})\,R$, we have

$$(X)\big((a \blacktriangle a')R\big) = X(a \blacktriangle a') = Xa \blacktriangle Xa' = \big((X)aR\big) \blacktriangle \big((X)a'R\big) = (X)(aR \blacktriangle a'R)\,.$$

So $(a \blacktriangle a')R = aR \blacktriangle a'R$.

Ad (5). For $X \in \mathrm{Ob}(\mathcal{C}R)$, we have

$$\begin{aligned}(X)\big((a * b)R\big) &= (X)(a * b)\\ &= (X)(aG \blacktriangle F'b)\\ &= \big((X)(aG)\big) \blacktriangle \big((X)(F'b)\big)\\ &= \Big(\big((X)aR\big)G\Big) \blacktriangle \Big(\big((X)F'R\big)b\Big)\\ &= \Big(\big((X)aR\big)GR\Big) \blacktriangle \Big(\big((X)F'R\big)bR\Big)\\ &= \Big((X)\big((aR)(GR)\big)\Big) \blacktriangle \Big((X)\big((F'R)(bR)\big)\Big)\\ &= (X)\Big(\big((aR)(GR)\big) \blacktriangle \big((F'R)(bR)\big)\Big)\\ &= (X)(aR * bR)\,.\end{aligned}$$

So $(a * b)R = aR * bR$.

Ad (6). For $X \in \mathrm{Ob}(\mathcal{C})$, we have

$$\begin{aligned}(X)\big((FbH)R\big) &= (X)(FbH) = \big((X)(FR)\big)(bH) = \big((X)(FR)(bR)\big)H\\ &= (X)(FR)(bR)(HR) = (X)\big((FR)(bR)(HR)\big)\,.\end{aligned}$$

So $(FbH)R = (FR)(bR)(HR)$. $\square$

Remark 89 Suppose given categories $\mathcal{C}$ and $\mathcal{D}$. Suppose given a functor $F\colon \mathcal{C} \to \mathcal{D}$.

(1) Suppose that F is an isofunctor. Then $FR\colon \mathcal{C}R \to \mathcal{D}R$ is an R-linear isofunctor and its inverse is given by $F^{-}R\colon \mathcal{D}R \to \mathcal{C}R$. This follows by Lemma 88.(2).

(2) Suppose that $\mathcal{C}$ and $\mathcal{D}$ are monoidal categories. Suppose that F is a monoidal functor. Then $FR\colon \mathcal{C}R \to \mathcal{D}R$ is a monoidal R-linear isofunctor and its inverse is given by $F^{-}R\colon \mathcal{D}R \to \mathcal{C}R$. This follows by (1) and Lemma 86.(2).

Remark 90 Suppose given a category $\mathcal{C}$. Consider the category $\mathcal{C}R$; cf. Definition 82.

(1) We have a faithful functor given by

$$\begin{array}{rccll} P\colon & \mathcal{C} & \longrightarrow & \mathcal{C}R & \\ & X & \longmapsto & XP := X & \text{for } X \in \mathrm{Ob}(\mathcal{C}) \\ & u & \longmapsto & uP := u\,1_R & \text{for } u \in \mathrm{Mor}(\mathcal{C})\ . \end{array}$$

So we may identify the category $\mathcal{C}$ with its image under the functor P, and thus, we may consider $\mathcal{C}$ as a subcategory of $\mathcal{C}R$. Hence, we write $J_{\mathcal{C},\mathcal{C}R} := P$.

(2) Suppose that the category $\mathcal{C}$ is monoidal.

Then, by Lemma 84, $\mathcal{C}R$ is monoidal, and so, $\mathcal{C}$ is a monoidal subcategory of $\mathcal{C}R$; cf. Definition 16.

6.2 The construction U

Lemma 91 (The invertible monoidal category $\mathcal{C}\mathrm{U}$)

Let $(\mathcal{C}, I, \otimes)$ be a monoidal category.

(1) *We may define a subcategory $\mathcal{C}\mathrm{U}$ of $\mathcal{C}$ as follows.*

$$\mathrm{Ob}(\mathcal{C}\mathrm{U}) := \{X \in \mathrm{Ob}(\mathcal{C}) : X \text{ is tensor invertible}\}$$
$$\mathrm{Mor}(\mathcal{C}\mathrm{U}) := \{\big(X \xrightarrow{u} Y\big) \in \mathrm{Mor}(\mathcal{C}) : u \text{ is tensor invertible}\};$$

cf. Definition 19.

(2) *Then $(\mathcal{C}\mathrm{U}, I, \otimes)$ is a monoidal subcategory of $(\mathcal{C}, I, \otimes)$; cf. Definition* 16.

(3) *The monoidal category* $(\mathcal{C}\mathrm{U}, I, \otimes)$ *is an invertible monoidal category; cf. Definition* 21.

Note that in general $\mathcal{C}\mathrm{U}$ *is not a full subcategory of* $\mathcal{C}$.

Proof. Ad (1). By Remark 20.(8, 9), $\mathcal{C}\mathrm{U}$ is closed under source, target and identity. By Remark 20.(10), $\mathcal{C}\mathrm{U}$ is closed under composition.

So $\mathcal{C}\mathrm{U}$ is a subcategory of $\mathcal{C}$.

Ad (2). By Remark 20.(3), the unit object $I \in \mathrm{Ob}(\mathcal{C})$ is tensor invertible. Therefore $I \in \mathrm{Ob}(\mathcal{C}\mathrm{U})$.

Suppose given $u, v \in \mathrm{Mor}(\mathcal{C}\mathrm{U})$, i.e. u and v are tensor invertible morphisms in $\mathcal{C}$.

By Remark 20.(5), $u \otimes v \in \mathrm{Mor}(\mathcal{C})$ is tensor invertible. Therefore $u \otimes v \in \mathrm{Mor}(\mathcal{C}\mathrm{U})$.

So, by Lemma 17, $(\mathcal{C}\mathrm{U}, I, \otimes)$ is a monoidal subcategory of $(\mathcal{C}, I, \otimes)$.

Ad (3). Suppose given $u \in \mathrm{Mor}(\mathcal{C}\mathrm{U})$.

By Remark 20.(7), we have $u^{\otimes -} \in \mathrm{Mor}(\mathcal{C}\mathrm{U})$.

Therefore, $(\mathcal{C}\mathrm{U}, I, \otimes)$ is an invertible monoidal category; cf. Remark 22. □

Lemma 92 (The monoidal functor $F\mathrm{U}$)

Suppose given monoidal categories $\mathcal{C}$ *and* $\mathcal{D}$. *Suppose given a monoidal functor* $F\colon \mathcal{C} \to \mathcal{D}$; *cf. Definition* 31.

Consider the invertible monoidal categories $\mathcal{C}\mathrm{U}$ *and* $\mathcal{D}\mathrm{U}$; *cf. Lemma* 91.

We have the monoidal functor $F\mathrm{U} := F\big|_{\mathcal{C}\mathrm{U}}^{\mathcal{D}\mathrm{U}}$.

Proof. Suppose given $X \in \mathrm{Ob}(\mathcal{C}\mathrm{U})$ and $u \in \mathrm{Mor}(\mathcal{C}\mathrm{U})$.

By Remark 33, we have $(X^{\otimes -})F = (XF)^{\otimes -} \in \mathrm{Ob}(\mathcal{D})$ and $(u^{\otimes -})F = (uF)^{\otimes -} \in \mathrm{Mor}(\mathcal{D})$. So, $XF \in \mathrm{Ob}(\mathcal{D}\mathrm{U})$ and $uF \in \mathrm{Mor}(\mathcal{D}\mathrm{U})$.

Thus the functor $F\mathrm{U} := F\big|_{\mathcal{C}\mathrm{U}}^{\mathcal{D}\mathrm{U}} : \mathcal{C}\mathrm{U} \to \mathcal{D}\mathrm{U}$ exists.

Moreover, for $u, v \in \mathrm{Mor}(\mathcal{C}\mathrm{U})$, we have

$$(u \otimes v)F\mathrm{U} = (u \otimes v)F = uF \otimes vF = uF\mathrm{U} \otimes vF\mathrm{U} \ .$$

So, by Remark 32.(2), $F\mathrm{U}$ is a monoidal functor. □

Remark 93 Suppose given $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ in MonCat.

Consider $\mathcal{C}\mathrm{U} \xrightarrow{F\mathrm{U}} \mathcal{D}\mathrm{U} \xrightarrow{G\mathrm{U}} \mathcal{E}\mathrm{U}$ in $\mathit{InvMonCat}$.

(1) We have $\quad (\mathrm{id}_{\mathcal{C}})\mathrm{U} = \mathrm{id}_{\mathcal{C}\mathrm{U}}$.

(2) We have $\quad (F * G)\mathrm{U} = (F\mathrm{U}) * (G\mathrm{U})$.

Remark 94 The invertible monoidal category $\mathcal{C}\mathrm{U}$ is a monoidal subcategory of $\mathcal{C}$; cf. Definition 16.

So we have the monoidal embedding functor $J_{\mathcal{C}\mathrm{U},\mathcal{C}} \colon \mathcal{C}\mathrm{U} \to \mathcal{C}$, $\big(X \xrightarrow{u} Y\big) \mapsto \big(X \xrightarrow{u} Y\big)$.

6.3 The relation between L and U

Lemma 95 *Suppose given an invertible monoidal category $\mathcal{C}$. Suppose given a monoidal R-linear category $\mathcal{D}$.*

Consider the monoidal R-linear category $\mathcal{C}R$; cf. Lemma 85. Consider the invertible monoidal category $\mathcal{D}U$; cf. Lemma 91.

(1) *Suppose given a monoidal functor $F \colon \mathcal{C} \to \mathcal{D}\mathrm{U}$.*

Then there exists a unique monoidal R-linear functor $\hat{F} \colon \mathcal{C}R \to \mathcal{D}$ such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, hook, "J_{\mathcal{C},\mathcal{C}R}"'] & \mathcal{D}\mathrm{U} \arrow[d, hook, "J_{\mathcal{D}\mathrm{U},\mathcal{D}}"] \\ \mathcal{C}R \arrow[r, "\hat{F}"'] & \mathcal{D} \end{tikzcd}$$

This functor $\hat{F}$ is given by

$$\begin{array}{rrcll} \hat{F}\colon & \mathcal{C}R & \longrightarrow & \mathcal{D} & \\ & X & \longmapsto & X\hat{F} := XF & \text{for } X \in \mathrm{Ob}(\mathcal{C}R) \\ & u = \sum\limits_{k\in K} u_k r_k & \longmapsto & u\hat{F} := \sum\limits_{k\in K} (u_k F) r_k & \text{for } u \in \mathrm{Mor}(\mathcal{C}R)\ ; \end{array}$$

cf. Definition 82.

(2) *Suppose given a monoidal R-linear functor $G\colon \mathcal{C}R \to \mathcal{D}$.*

Then there exists a unique monoidal functor $\check{G}\colon \mathcal{C} \to \mathcal{D}U$ such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\check{G}"] \arrow[d, hook, "J_{\mathcal{C},\mathcal{C}R}"'] & \mathcal{D}U \arrow[d, hook, "J_{\mathcal{D}\mathrm{U},\mathcal{D}}"] \\ \mathcal{C}R \arrow[r, "G"'] & \mathcal{D} \end{tikzcd}$$

This functor $\check{G}$ is given by

$$\begin{array}{rrcll} \check{G}\colon & \mathcal{C} & \longrightarrow & \mathcal{D}\mathrm{U} & \\ & X & \longmapsto & X\check{G} := XG & \text{for } X \in \mathrm{Ob}(\mathcal{C}) \\ & u & \longmapsto & u\check{G} := uG & \text{for } u \in \mathrm{Mor}(\mathcal{C}) \; . \end{array}$$

I.e. we have $\check{G} = G\big|_{\mathcal{C}}^{\mathcal{D}\mathrm{U}}$.

(3) *Suppose given a monoidal functor $F\colon \mathcal{C} \to \mathcal{D}\mathrm{U}$. Consider the monoidal R-linear functor $\hat{F}\colon \mathcal{C}R \to \mathcal{D}$ from* (1).

Then

$$\check{\hat{F}} = F \, .$$

(4) *Suppose given a monoidal R-linear functor $G\colon \mathcal{C}R \to \mathcal{D}$. Consider the monidal functor $\check{G}\colon \mathcal{C} \to \mathcal{D}\mathrm{U}$ from* (2).

Then

$$\hat{\check{G}} = G \, .$$

Proof. Ad (1). $\hat{F}$ is a functor:

For $X \in \mathrm{Ob}(\mathcal{C}R)$, we have

$$(\mathrm{id}_X)\hat{F} = (\mathrm{id}_X)F = \mathrm{id}_{XF} = \mathrm{id}_{X\hat{F}} \, .$$

Suppose given $X \xrightarrow{\sum_i u_i r_i} Y \xrightarrow{\sum_j v_j s_j} Z$ in $\mathcal{C}R$. Write $u := \sum\limits_i u_i r_i$ and $v := \sum\limits_j v_j s_j$.

We have

$$(u \blacktriangle v)\hat{F} = \Big(\big(\sum_i u_i r_i\big) \blacktriangle \big(\sum_j v_j s_j\big)\Big)\hat{F} = \big(\sum_{i,j} (u_i \blacktriangle v_j) r_i s_j\big)\hat{F} = \sum_{i,j} \big((u_i \blacktriangle v_j)F\big) r_i s_j$$

$$\begin{aligned}
&= \sum_{i,j} \big((u_i F) \blacktriangle (v_j F)\big) r_i s_j = \big(\sum_i (u_i F) r_i\big) \blacktriangle \big(\sum_j (v_j F) s_j\big) \\
&= \Big(\big(\sum_i u_i r_i\big)\hat{F}\Big) \blacktriangle \Big(\big(\sum_j v_j s_j\big)\hat{F}\Big) = u\hat{F} \blacktriangle v\hat{F}\,.
\end{aligned}$$

So, $\hat{F}$ is a functor.

The functor $\hat{F}$ is monoidal:

Suppose given $u = \sum\limits_i u_i r_i\,,\ v = \sum\limits_j v_j s_j \in \operatorname{Mor}(\mathcal{C}R)$.

We have

$$\begin{aligned}
(u \otimes v)\hat{F} &= \Big(\big(\sum_i u_i r_i\big) \otimes \big(\sum_j v_j s_j\big)\Big)\hat{F} = \big(\sum_{i,j} (u_i \otimes v_j) r_i s_j\big)\hat{F} = \sum_{i,j} \big((u_i \otimes v_j)F\big) r_i s_j \\
&= \sum_{i,j} (u_i F \otimes v_j F) r_i s_j = \big(\sum_i (u_i F) r_i\big) \otimes \big(\sum_j (v_j F) s_j\big) \\
&= \big(\sum_i u_i r_i\big)\hat{F} \otimes \big(\sum_j v_j s_j\big)\hat{F} = u\hat{F} \otimes v\hat{F}\,,
\end{aligned}$$

and

$$(I_{\mathcal{C}R})\hat{F} = (I_{\mathcal{C}})\hat{F} = (I_{\mathcal{C}})F = I_{\mathcal{D}} = I_{\mathcal{D}\mathrm{U}}\,.$$

So, by Remark 32.(1), $\hat{F}$ is monoidal.

The functor $\hat{F}$ is R-linear:

Suppose given $r, s \in R$.

Suppose given $u := \big(\sum\limits_i u_i r_i \colon X \to Y\big)$, $v := \big(\sum\limits_j v_j s_j \colon X \to Y\big) \in \operatorname{Mor}(\mathcal{C}R)$. We have

$$\begin{aligned}
(ur + vs)\hat{F} &= \Big(\big(\sum_i u_i r_i\big) r + \big(\sum_j v_j s_j\big) s\Big)\hat{F} = \big(\sum_i u_i r_i\, r + \sum_j v_j s_j\, s\big)\hat{F} \\
&= \sum_i (u_i F) r_i\, r + \sum_j (v_j F) s_j\, s = \big(\sum_i (u_i F) r_i\big) r + \big(\sum_j (v_j F) s_j\big) s \\
&= (u\hat{F}) r + (v\hat{F}) s\,.
\end{aligned}$$

By Remark 71, $\hat{F}$ is R-linear.

The diagram in (1) is commutative:

Suppose given $u \in \mathrm{Mor}(\mathcal{C})$. We have

$$(u)J_{\mathcal{C},\mathcal{C}R}\,\hat{F} = u\hat{F} = uF = (uF)J_{\mathcal{D}\mathrm{U},\mathcal{D}} = (u)F\,J_{\mathcal{D}\mathrm{U},\mathcal{D}}\,.$$

Therefore $J_{\mathcal{C},\mathcal{C}R}\,\hat{F} = F\,J_{\mathcal{D}\mathrm{U},\mathcal{D}}$.

The functor $\hat{F}$ is unique with respect to this commutativity:

Suppose given an R-linear monoidal functor $\tilde{F}\colon \mathcal{C}R \to \mathcal{D}$ such that $J_{\mathcal{C},\mathcal{C}R}\,\tilde{F} = F\,J_{\mathcal{D}\mathrm{U},\mathcal{D}}$ holds.

Suppose given $u = \sum_i u_i r_i \in \mathrm{Mor}(\mathcal{C}R)$. We have

$$\begin{aligned} u\tilde{F} &= \big(\sum_i u_i r_i\big)\tilde{F} = \sum_i (u_i\tilde{F})r_i = \sum_i (u_i\,J_{\mathcal{C},\mathcal{C}R}\,\tilde{F})r_i = \sum_i (u_i F\,J_{\mathcal{D}\mathrm{U},\mathcal{D}})r_i \\ &= \sum_i (u_i\,J_{\mathcal{C},\mathcal{C}R}\,\hat{F})r_i = \sum_i (u_i\hat{F})r_i = \big(\sum_i u_i r_i\big)\hat{F} = u\hat{F}\,. \end{aligned}$$

So, $\tilde{F} = \hat{F}$.

Ad (2). The functor $\check{G}$ is well-defined:

Suppose given $X \in \mathrm{Ob}(\mathcal{C})$. By Remark 33.(1), XG is tensor invertible in $\mathcal{D}$. Therefore $XG \in \mathrm{Ob}(\mathcal{D}\mathrm{U})$.

Suppose given $u \in \mathrm{Mor}(\mathcal{C})$. By Remark 33.(2), uG is tensor invertible in $\mathcal{D}$. Therefore $uG \in \mathrm{Mor}(\mathcal{D}\mathrm{U})$.

Thus, $G\big|_{\mathcal{C}}^{\mathcal{D}\mathrm{U}}$ exists and we may let $\check{G} := G\big|_{\mathcal{C}}^{\mathcal{D}\mathrm{U}}$.

The functor $\check{G}$ is monoidal since G is monoidal, and $\mathcal{C} \subseteq \mathcal{C}R$ and $\mathcal{D}\mathrm{U} \subseteq \mathcal{D}$ are monoidal subcategories; cf. Remark 32.(2).

The diagram in (2) is commutative:

Suppose given $u \in \mathrm{Mor}(\mathcal{C})$. We have

$$(u)J_{\mathcal{C},\mathcal{C}R}\,G = uG = (u)G\,J_{\mathcal{D}\mathrm{U},\mathcal{D}} = (u)\check{G}\,J_{\mathcal{D}\mathrm{U},\mathcal{D}}\,.$$

Therefore $J_{\mathcal{C},\mathcal{C}R}\,G = \check{G}\,J_{\mathcal{D}\mathrm{U},\mathcal{D}}$.

The functor $\check{G}$ is unique with respect to this commutativity:

Suppose given a monoidal functor $\tilde{G}\colon \mathcal{C} \to \mathcal{D}\mathrm{U}$ such that $J_{\mathcal{C},\mathcal{C}R}\,G = \tilde{G}\,J_{\mathcal{D}\mathrm{U},\mathcal{D}}$. Suppose given $u \in \mathrm{Mor}(\mathcal{C})$.

We have

$$(u)\tilde{G} = (u)\tilde{G}\, J_{\mathcal{D}\mathrm{U},\,\mathcal{D}} = (u)J_{\mathcal{C},\,\mathcal{C}R}\, G = (u)\check{G}\, J_{\mathcal{D}\mathrm{U},\,\mathcal{D}} = u\check{G}\,.$$

So, $\tilde{G} = \check{G}$.

Ad (3). By (1), we have the following commutative diagram.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, hook, "J_{\mathcal{C},\,\mathcal{C}R}"'] & \mathcal{D}U \arrow[d, hook, "J_{\mathcal{D}\mathrm{U},\,\mathcal{D}}"] \\
\mathcal{C}R \arrow[r, "\hat{F}"'] & \mathcal{D}
\end{tikzcd}$$

By (2), the functor $\check{\hat{F}}$ is the unique monoidal functor from $\mathcal{C}$ to $\mathcal{D}$U such that the following diagram is commutative.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\check{\hat{F}}"] \arrow[d, hook, "J_{\mathcal{C},\,\mathcal{C}R}"'] & \mathcal{D}U \arrow[d, hook, "J_{\mathcal{D}\mathrm{U},\,\mathcal{D}}"] \\
\mathcal{C}R \arrow[r, "\hat{F}"'] & \mathcal{D}
\end{tikzcd}$$

So we conclude that $\check{\hat{F}} = F$.

Ad (4). By (2), we have the following commutative diagram.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\check{G}"] \arrow[d, hook, "J_{\mathcal{C},\,\mathcal{C}R}"'] & \mathcal{D}U \arrow[d, hook, "J_{\mathcal{D}\mathrm{U},\,\mathcal{D}}"] \\
\mathcal{C}R \arrow[r, "G"'] & \mathcal{D}
\end{tikzcd}$$

By (1), the functor $\hat{\check{G}}$ is the unique monoidal R-linear functor from $\mathcal{C}R$ to $\mathcal{D}$ such that the following diagram is commutative.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\check{G}"] \arrow[d, hook, "J_{\mathcal{C},\,\mathcal{C}R}"'] & \mathcal{D}U \arrow[d, hook, "J_{\mathcal{D}\mathrm{U},\,\mathcal{D}}"] \\
\mathcal{C}R \arrow[r, "\hat{\check{G}}"'] & \mathcal{D}
\end{tikzcd}$$

So we conclude that $\hat{\check{G}} = G$. $\square$

Lemma 96 *Suppose given an invertible monoidal category $\mathcal{C}$. Suppose given a monoidal R-linear category $\mathcal{D}$.*

Consider the monoidal R-linear category $\mathcal{C}R$; cf. Lemma 85. *Consider the invertible monoidal category $\mathcal{D}U$; cf.* Lemma 91.

Recall that $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{C}R)$*; cf.* Definition 82.

(1) *Suppose given monoidal functors $F, F' \colon \mathcal{C} \to \mathcal{D}\mathrm{U}$. Suppose given a monoidal transformation $a \colon F \to F'$.*

Consider the monoidal R-linear functors $\hat{F}, \hat{F}' \colon \mathcal{C}R \to \mathcal{D}$ given in Lemma 95.(1).

Then there exists a unique monoidal transformation $\hat{a} \colon \hat{F} \to \hat{F}'$ such that

$$J_{\mathcal{C},\mathcal{C}R}\ \hat{a} = a\, J_{\mathcal{D}\mathrm{U},\mathcal{D}}\,.$$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow a"] \arrow[d, hook, "J_{\mathcal{C},\mathcal{C}R}"'] & \mathcal{D}\mathrm{U} \arrow[d, hook, "J_{\mathcal{D}\mathrm{U},\mathcal{D}}"] \\
\mathcal{C}R \arrow[r, bend left, "\hat{F}"] \arrow[r, bend right, "\hat{F}'"'] \arrow[r, phantom, "\Downarrow \hat{a}"] & \mathcal{D}
\end{tikzcd}$$

This transformation $\hat{a}$ is given by

$$\hat{a} = \big(X\hat{F} \xrightarrow{X\hat{a}} X\hat{F}'\big)_{X \in \mathrm{Ob}(\mathcal{C}R)} := \big(XF \xrightarrow{Xa} XF'\big)_{X \in \mathrm{Ob}(\mathcal{C})}\,.$$

(2) *Suppose given monoidal R-linear functors $G, G' \colon \mathcal{C}R \to \mathcal{D}$. Suppose given a monoidal transformation $b \colon G \to G'$.*

Consider the monoidal functors $\check{G}, \check{G}' \colon \mathcal{C} \to \mathcal{D}\mathrm{U}$ from Lemma 95.(2).

Then there exists a unique monoidal transformation $\check{b} \colon \check{G} \to \check{G}'$ such that

$$J_{\mathcal{C},\mathcal{C}R}\ b = \check{b}\, J_{\mathcal{D}\mathrm{U},\mathcal{D}}\,.$$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "\check{G}"] \arrow[r, bend right, "\check{G}'"'] \arrow[r, phantom, "\downarrow \check{b}"] \arrow[d, hook, "J_{\mathcal{C},\mathcal{C}R}"'] & \mathcal{D}\mathrm{U} \arrow[d, hook, "J_{\mathcal{D}\mathrm{U},\mathcal{D}}"] \\
\mathcal{C}R \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\downarrow b"] & \mathcal{D}
\end{tikzcd}$$

This transformation $\check{b}$ is given by

$$\check{b} = \big(X\check{G} \xrightarrow{X\check{b}} X\check{G}'\big)_{X \in \mathrm{Ob}(\mathcal{C})} := \big(XG \xrightarrow{Xb} XG'\big)_{X \in \mathrm{Ob}(\mathcal{C}R)}\,.$$

(3) *Suppose given monoidal functors F, $F' \colon \mathcal{C} \to \mathcal{D}\mathrm{U}$. Suppose given a monoidal transformation $a \colon F \to F'$. Consider the monoidal transformation $\hat{a} \colon \hat{F} \to \hat{F}'$ from* (1)*.*

Then

$$\check{\hat{a}} = a\,.$$

(4) *Suppose given monoidal R-linear functors G, $G' \colon \mathcal{C}R \to \mathcal{D}$. Suppose given a monoidal transformation $b \colon G \to G'$. Consider the monoidal transformation $\check{b} \colon \check{G} \to \check{G}'$ from* (2)*.*

Then

$$\hat{\check{b}} = b\,.$$

Proof. Ad (1). We show that $\hat{a}$ is a transformation:

Suppose given $u := \sum_i u_i r_i \colon X \to Y$ in $\mathcal{C}R$.

We have

$$\begin{aligned} u\hat{F} \blacktriangle Ya &= \Big(\sum_i u_i r_i\Big)\hat{F} \blacktriangle Ya = \Big(\sum_i (u_i F) r_i\Big) \blacktriangle Ya = \sum_i (u_i F \blacktriangle Ya) r_i \\ &= \sum_i (Xa \blacktriangle u_i F') r_i = Xa \blacktriangle \Big(\sum_i (u_i F') r_i\Big) = Xa \blacktriangle \Big(\sum_i u_i r_i\Big)\hat{F}' \\ &= Xa \blacktriangle u\hat{F}'\,. \end{aligned}$$

So $\hat{a}$ is a transformation.

The transformation $\hat{a}$ is monoidal:

We have

$$(I_{\mathcal{C}R})\hat{a} = (I_{\mathcal{C}})\hat{a} = (I_{\mathcal{C}})a = \mathrm{id}_{I_{\mathcal{D}\mathrm{U}}} = \mathrm{id}_{I_{\mathcal{D}}}\,.$$

For $X, Y \in \mathrm{Ob}(\mathcal{C}R)$, we have

$$(X \underset{\mathcal{C}R}{\otimes} Y)\hat{a} = (X \underset{\mathcal{C}}{\otimes} Y)a = Xa \underset{\mathcal{D}\mathrm{U}}{\otimes} Ya = X\hat{a} \underset{\mathcal{D}}{\otimes} Y\hat{a}\,.$$

So $\hat{a}$ is monoidal.

The transformation $\hat{a}$ satisfies the equation given in (1):

For $X \in \mathrm{Ob}(\mathcal{C})$, we have

$$(X)J_{\mathcal{C},\mathcal{C}R}\ \hat{a} = X\hat{a} = Xa = (X)a\ J_{\mathcal{D}\mathrm{U},\mathcal{D}}\,.$$

So, $J_{\mathcal{C},\mathcal{C}R}\ \hat{a} = a\ J_{\mathcal{D}\mathrm{U},\mathcal{D}}\,.$

The transformation $\hat{a}$ is unique with respect to this equation:

Suppose given a monoidal transformation $\tilde{a}\colon \hat{F} \to \hat{F}'$ satisfying $J_{\mathcal{C},\mathcal{C}R}\ \tilde{a} = a\ J_{\mathcal{D}\mathrm{U},\mathcal{D}}\,.$

Then, for $X \in \mathrm{Ob}(\mathcal{C})$, we have

$$X\tilde{a} = (X)J_{\mathcal{C},\mathcal{C}R}\ \tilde{a} = (X)a\ J_{\mathcal{D}\mathrm{U},\mathcal{D}} = (X)J_{\mathcal{C},\mathcal{C}R}\ \hat{a} = X\hat{a}\,.$$

So, $\tilde{a} = \hat{a}$.

Ad (2). We show that $\check{b}$ is well-defined:

For $X \in \mathrm{Ob}(\mathcal{C})$, we have

$$(Xb) \underset{\mathcal{D}}{\otimes} (X^{\otimes-}b) = (X \underset{\mathcal{C}}{\otimes} X^{\otimes-})b = (I_{\mathcal{C}})b = \mathrm{id}_{I_{\mathcal{D}}}\,.$$

Likewise, we have $(X^{\otimes-}b) \underset{\mathcal{D}}{\otimes} (Xb) = \mathrm{id}_{I_{\mathcal{D}}}\,.$

Therefore, the morphism Xb is tensor invertible in $\mathcal{D}$. Hence the tuple

$$\check{b} = \big(X\check{G} \xrightarrow{X\check{b}} X\check{G}'\big)_{X\in\mathrm{Ob}(\mathcal{C})} = \big(XG \xrightarrow{XG} XG'\big)_{X\in\mathrm{Ob}(\mathcal{C}R)}$$

has entries in $\mathcal{D}\mathrm{U}$. It is a transformation from $\check{G}$ to $\check{G}'$ since given $X \xrightarrow{u} Y$ in $\mathcal{C}$, we obtain

$$u\check{G} \blacktriangle Y\check{b} = uG \blacktriangle Y\check{b} = Xb \blacktriangle uG' = X\check{b} \blacktriangle uG'\,.$$

The transformation $\check{b}$ is monoidal since b is monoidal; cf. Remark 36.

The transformation $\check{b}$ satisfies the equation given in (2):

For $X \in \mathrm{Ob}(\mathcal{C})$, we have
$$(X)J_{\mathcal{C},\mathcal{C}R}\ b = Xb = X\check{b} = (X)\check{b}\ J_{\mathcal{D}\mathrm{U},\mathcal{D}}\,.$$

The transformation $\check{b}$ is unique with respect to this equation:

Suppose given a monoidal transformation $\tilde{b}\colon \check{G} \to \check{G}'$ satisfying $J_{\mathcal{C},\mathcal{C}R}\ b = \tilde{b}\ J_{\mathcal{D}\mathrm{U},\mathcal{D}}\,.$

Then, for $X \in \mathrm{Ob}(\mathcal{C})$, we have
$$X\tilde{b} = X\tilde{b}\ J_{\mathcal{D}\mathrm{U},\mathcal{D}} = X J_{\mathcal{C},\mathcal{C}R}\ b = X\check{b}\ J_{\mathcal{D}\mathrm{U},\mathcal{D}} = X\check{b}\,.$$

So, $\tilde{b} = \check{b}$.

Ad (3). We have $\check{\hat{a}}\colon \check{\hat{F}} \to \check{\hat{F}}'$. From Lemma 95.(3), we know that $\check{\hat{F}} = F$ and that $\check{\hat{F}}' = F'$. So $\check{\hat{a}}\colon F \to F'$.

By (1), we have
$$J_{\mathcal{C},\mathcal{C}R}\ \hat{a} = a\, J_{\mathcal{D}\mathrm{U},\mathcal{D}}\,.$$

By (2), $\check{\hat{a}}$ is the unique monoidal transformation from F to F' that satisfies the following equation.
$$J_{\mathcal{C},\mathcal{C}R}\ \hat{a} = \check{\hat{a}}\, J_{\mathcal{D}\mathrm{U},\mathcal{D}}\,.$$

So we conclude that $\check{\hat{a}} = a$.

Ad (4). We have $\hat{\check{b}}\colon \hat{\check{G}} \to \hat{\check{G}}'$. From Lemma 95.(4), we know that $\hat{\check{G}} = G$ and that $\hat{\check{G}}' = G'$. So $\hat{\check{b}}\colon G \to G'$.

By (2), we have
$$J_{\mathcal{C},\mathcal{C}R}\ b = \check{b}\, J_{\mathcal{D}\mathrm{U},\mathcal{D}}\,.$$

By (1), $\hat{\check{b}}$ is the unique monoidal transformation from G to G' that satisfies the following equation.
$$J_{\mathcal{C},\mathcal{C}R}\ \hat{\check{b}} = \check{b}\, J_{\mathcal{D}\mathrm{U},\mathcal{D}}\,.$$

So we conclude that $\hat{\check{b}} = b$. $\square$

Chapter 7

The isomorphism between $\mathrm{Aut}_R(\mathcal{M})$ and $\big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\mathrm{Cat}$

Remark 97 (The invertible monoidal categories $\big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\mathrm{Cat}$ and $\mathrm{Aut}_R(\mathcal{M})$)

(1) Recall that we have the invertible monoidal category $\big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\mathrm{Cat}$ with

$$\begin{aligned}
\mathrm{Ob}\big(\big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\,\mathrm{Cat}\big) &= \mathrm{G}_{\mathcal{M}}^R \\
&= \{\mathcal{M} \xrightarrow{F} \mathcal{M} \colon F \text{ is } R\text{-linear isofunctor}\} \\
\mathrm{Mor}\big(\big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\,\mathrm{Cat}\big) &= \mathrm{G}_{\mathcal{M}}^R \ltimes \mathrm{M}_{\mathcal{M}}^R \\
&= \{(G, \mathrm{id}_{\mathcal{M}} \xrightarrow{a} F) \colon G, F \in \mathrm{G}_{\mathcal{M}}^R \text{ and } a \text{ is an isotransformation}\},
\end{aligned}$$

$$\begin{array}{llll}
\boldsymbol{s}\colon & \mathrm{G}_{\mathcal{M}}^R \ltimes \mathrm{M}_{\mathcal{M}}^R \to \mathrm{G}_{\mathcal{M}}^R\,, & (G, \mathrm{id}_{\mathcal{M}} \xrightarrow{a} F) & \mapsto \quad G \\
\boldsymbol{i}\colon & \mathrm{G}_{\mathcal{M}}^R \ltimes \mathrm{M}_{\mathcal{M}}^R \leftarrow \mathrm{G}_{\mathcal{M}}^R\,, & (G, \mathrm{id}_{\mathcal{M}} \xrightarrow{\mathrm{id}_{\mathrm{id}_{\mathcal{M}}}} \mathrm{id}_{\mathcal{M}}) & \hookleftarrow \quad G \\
\boldsymbol{t}\colon & \mathrm{G}_{\mathcal{M}}^R \ltimes \mathrm{M}_{\mathcal{M}}^R \to \mathrm{G}_{\mathcal{M}}^R\,, & (G, \mathrm{id}_{\mathcal{M}} \xrightarrow{a} F) & \mapsto \quad GF\,.
\end{array}$$

For $G \xrightarrow{(G, \mathrm{id}_{\mathcal{M}} \xrightarrow{a} F)} GF \xrightarrow{(GF, \mathrm{id}_{\mathcal{M}} \xrightarrow{a'} F')} GFF'$ in $\big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\mathrm{Cat}$, their composite is given by

$$(G,a) \blacktriangle (GF,a') = (G, \mathrm{id}_{\mathcal{M}} \xrightarrow{a} F) \blacktriangle (GF, \mathrm{id}_{\mathcal{M}} \xrightarrow{a'} F') = (G, \mathrm{id}_{\mathcal{M}} \xrightarrow{a*a'} FF') = (G, a*a')\,.$$

Note that each morphism from G to GF is of the form $(G, \mathrm{id}_{\mathcal{M}} \xrightarrow{a} F)$ since G is an isofunctor.

Cf. Lemma 39 and Lemma 81.

(2) Recall from Lemma 91, 80 that we have the invertible monoidal category $\big(\mathrm{End}_R(\mathcal{M})\big)\mathrm{U}$ with

$$\begin{array}{rcl} \mathrm{Ob}\big(\big(\mathrm{End}_R(\mathcal{M})\big)\mathrm{U}\big) & = & \{F \in \mathrm{Ob}(\mathrm{End}_R(\mathcal{M})) : F \text{ is tensor invertible}\} \\ & \overset{80,\,72.(3)}{=} & \{\mathcal{M} \xrightarrow{F} \mathcal{M} \colon F \text{ is an } R\text{-linear isofunctor}\} \\ \mathrm{Mor}\big(\big(\mathrm{End}_R(\mathcal{M})\big)\mathrm{U}\big) & = & \{\big(F \xrightarrow{a} G\big) \in \mathrm{Mor}(\mathrm{End}_R(\mathcal{M})) : a \text{ is tensor invertible}\} \\ & \overset{20.(8),\,26}{=} & \{\big(F \xrightarrow{a} G\big) \colon F, G \text{ are } R\text{-linear isofunctors,} \\ & & \quad \text{and } a \text{ is an isotransformation}\}\,, \end{array}$$

$$\begin{array}{rlcrcl} \boldsymbol{s}\colon & \mathrm{Mor}\big(\big(\mathrm{End}_R(\mathcal{M})\big)\mathrm{U}\big) \to \mathrm{Ob}\big(\big(\mathrm{End}_R(\mathcal{M})\big)\mathrm{U}\big) & , & \big(F \xrightarrow{a} G\big) & \mapsto & F \\ \boldsymbol{i}\colon & \mathrm{Mor}\big(\big(\mathrm{End}_R(\mathcal{M})\big)\mathrm{U}\big) \leftarrow \mathrm{Ob}\big(\big(\mathrm{End}_R(\mathcal{M})\big)\mathrm{U}\big) & , & \big(F \xrightarrow{\mathrm{id}_F} F\big) & \leftarrow\!\shortmid & F \\ \boldsymbol{t}\colon & \mathrm{Mor}\big(\big(\mathrm{End}_R(\mathcal{M})\big)\mathrm{U}\big) \to \mathrm{Ob}\big(\big(\mathrm{End}_R(\mathcal{M})\big)\mathrm{U}\big) & , & \big(F \xrightarrow{a} G\big) & \mapsto & G\,. \end{array}$$

For $F \xrightarrow{a} G \xrightarrow{b} H$ in $\big(\mathrm{End}_R(\mathcal{M})\big)\mathrm{U}$, their composite is given by

$$\big(F \xrightarrow{a} G\big) \blacktriangle \big(G \xrightarrow{b} H\big) = \big(F \xrightarrow{a \blacktriangle b} H\big)\,.$$

We write

$$\mathrm{Aut}_R(\mathcal{M}) := \big(\mathrm{End}_R(\mathcal{M})\big)\mathrm{U}\ .$$

Lemma 98 *Consider the functor* CM: $\mathcal{CRMod} \to \boldsymbol{\mathcal{InvMonCat}}$*; cf. Lemma* 42*. Consider the automorphism crossed module* $\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})$*; cf. Lemma* 81*.*

We have

$$\big(\mathrm{Aut}_R(\mathcal{M})\big)\mathrm{CM} = \mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\,.$$

Proof. We write $\big(M', G', \gamma', f'\big) := \big(\mathrm{Aut}_R(\mathcal{M})\big)\,\mathrm{CM}\,.$

We have

$$\begin{aligned} M' &\overset{40}{=} \{a \in \mathrm{Mor}\,\big(\mathrm{Aut}_R(\mathcal{M})\big) \colon a\boldsymbol{s} = \mathrm{id}_{\mathcal{M}}\} \\ &= \{\big(\mathrm{id}_{\mathcal{M}} \xrightarrow[\sim]{a} H\big) \colon H \in \mathrm{G}_{\mathcal{M}}^R \text{ and } a \text{ is an isotransformation }\} \\ &= \mathrm{M}_{\mathcal{M}}^R\ . \end{aligned}$$

The multiplication in $\mathrm{M}_{\mathcal{M}}^R \leqslant \mathrm{M}_{\mathcal{M}}$ is given by the group multiplication $(*)$ of $\mathrm{M}_{\mathcal{M}}$ restricted to $\mathrm{M}_{\mathcal{M}}^R$, where $(*)$ is the horizontal composition of transformations; cf. Lemma 45.(2). The multiplication in M' is the horizontal composition $(*)$ in $\mathrm{Aut}_R(\mathcal{M})$ inherited from $\mathrm{End}_R(\mathcal{M})$, and subsequently restricted to M'; cf. Lemma 40.

So $(M', *) = (\mathrm{M}^R_{\mathcal{M}}, *)$.

We have

$$G' \overset{40}{=} \mathrm{Ob}\big(\,\mathrm{Aut}_R(\mathcal{M})\big) = \mathrm{Ob}\Big(\big(\,\mathrm{End}_R(\mathcal{M})\big)\mathrm{U}\Big) = \mathrm{G}^R_{\mathcal{M}}\ .$$

The multiplication in $\mathrm{G}^R_{\mathcal{M}}$ is given by the composition of functors $(*)$ in $\mathrm{G}_{\mathcal{M}}$ restricted to $\mathrm{G}^R_{\mathcal{M}}$; cf. Lemma 45. The multiplication in G' is the composition $(*)$ of functors in $\mathrm{Ob}\big(\,\mathrm{End}_R(\mathcal{M})\big)$.

So $(G', *) = (\mathrm{G}^R_{\mathcal{M}}, *)$.

For $F \in G'$ and $\big(\mathrm{id}_{\mathcal{M}} \xrightarrow[\sim]{a} H\big) \in M' = \mathrm{M}^R_{\mathcal{M}}$, we have

$$\begin{aligned} a(F\gamma') &\overset{40}{=} \big(F^- \xrightarrow[\sim]{\mathrm{id}_{F^-}} F^-\big) * \big(\mathrm{id}_{\mathcal{M}} \xrightarrow[\sim]{a} H\big) * \big(F \xrightarrow[\sim]{\mathrm{id}_F} F\big) \\ &= \big(\mathrm{id}_{\mathcal{M}} \xrightarrow[\sim]{F^- a F} F^- H F\big) \\ &\overset{81}{=} a(F\boldsymbol{\gamma}^R_{\mathcal{M}})\,. \end{aligned}$$

This shows $\gamma' = \boldsymbol{\gamma}^R_{\mathcal{M}}$.

For $\big(\mathrm{id}_{\mathcal{M}} \xrightarrow[\sim]{a} F\big) \in M' = \mathrm{M}^R_{\mathcal{M}}$, we have

$$af' \overset{40}{=} a\boldsymbol{t} = F \overset{81}{=} a\,\mathrm{f}^R_{\mathcal{M}}\ .$$

This shows $f' = \mathrm{f}^R_{\mathcal{M}}$.

Altogether, we have $\big(\,\mathrm{Aut}_R(\mathcal{M})\big)\,\mathrm{CM} = \mathrm{Aut}^{\mathrm{CM}}_R(\mathcal{M})$. $\square$

Theorem 99 (The isofunctor $\mathrm{Real}_{\mathcal{M}}$) *Suppose given an R-linear category $\mathcal{M}$.*

Consider the invertible monoidal categories $\big(\,\mathrm{Aut}^{\mathrm{CM}}_R(\mathcal{M})\big)\mathrm{Cat}$ and $\mathrm{Aut}_R(\mathcal{M})$; cf. Remark 97.

We have the monoidal isofunctor

$$\begin{array}{llcl} \mathrm{Real}_{\mathcal{M}}\colon & \big(\,\mathrm{Aut}^{\mathrm{CM}}_R(\mathcal{M})\big)\mathrm{Cat} & \overset{\sim}{\longrightarrow} & \mathrm{Aut}_R(\mathcal{M}) \\ & G & \longmapsto & G\,\mathrm{Real}_{\mathcal{M}} := G \\ & & & \quad \textit{for } G \in \mathrm{Ob}\Big(\big(\,\mathrm{Aut}^{\mathrm{CM}}_R(\mathcal{M})\big)\,\mathrm{Cat}\Big)\ , \\ & \Big(G \xrightarrow{(G,\, \mathrm{id}_{\mathcal{M}} \xrightarrow{a} F)} GF\Big) & \longmapsto & (G,a)\,\mathrm{Real}_{\mathcal{M}} := \big(G \xrightarrow{Ga} GF\big) \\ & & & \quad \textit{for } (G,a) \in \mathrm{Mor}\Big(\big(\,\mathrm{Aut}^{\mathrm{CM}}_R(\mathcal{M})\big)\,\mathrm{Cat}\Big)\ . \end{array}$$

Its inverse is given by the monoidal isofunctor

$$\begin{array}{rccll}
(\mathrm{Real}_{\mathcal{M}})^-\colon & \mathrm{Aut}_R(\mathcal{M}) & \longrightarrow & \big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\mathrm{Cat} & \\
& G & \longmapsto & G(\mathrm{Real}_{\mathcal{M}})^- := G & \text{for } G \in \mathrm{Ob}(\mathrm{Aut}_R(\mathcal{M})) \\
& (F \xrightarrow{a} G) & \longmapsto & a(\mathrm{Real}_{\mathcal{M}})^- := \Big(F \xrightarrow{(F,\, \mathrm{id}_{\mathcal{M}} \xrightarrow{F^-\, a} F^-\, G)} G\Big) & \\
& & & & \text{for } (F \xrightarrow{a} G) \in \mathrm{Mor}(\mathrm{Aut}_R(\mathcal{M}))\,.
\end{array}$$

The situation can be depicted as follows.

$$\begin{tikzcd}
\big(\mathrm{S}_{\mathcal{M}}\big)\mathrm{Cat} & & & \mathrm{End}_R(\mathcal{M}) \\
\big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\mathrm{Cat} \arrow[u, hook] \arrow[r, "\mathrm{Real}_{\mathcal{M}}", "\sim"'] & \mathrm{Aut}_R(\mathcal{M}) \arrow[rr, equal] & & \big(\mathrm{End}_R(\mathcal{M})\big)\mathrm{U} \arrow[u, hook]
\end{tikzcd}$$

Proof. By Lemma 98, we have $\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M}) = \big(\mathrm{Aut}_R(\mathcal{M})\big)\,\mathrm{CM}$.

Then, by Proposition 43.(2), for $\mathcal{C} = \mathrm{Aut}_R(\mathcal{M})$, we have the monoidal isofunctor

$$\begin{array}{rccll}
\mathrm{Real}_{\mathcal{M}}\colon & \big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\mathrm{Cat} & \longrightarrow & \mathrm{Aut}_R(\mathcal{M}) & \\
& G & \longmapsto & G\,\mathrm{Real}_{\mathcal{M}} & = G \\
& & & \multicolumn{2}{l}{\text{for } G \in \mathrm{Ob}\Big(\big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\,\mathrm{Cat}\Big)\ ,} \\
& \Big(G \xrightarrow{(G,\, \mathrm{id}_{\mathcal{M}} \xrightarrow{a} F)} GF\Big) & \longmapsto & (G,a)\,\mathrm{Real}_{\mathcal{M}} & = \big(G \xrightarrow{\mathrm{id}_G * a} G * F\big) \\
& & & & = \big(G \xrightarrow{Ga} GF\big) \\
& & & \multicolumn{2}{l}{\text{for } (G,a) \in \mathrm{Mor}\Big(\big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\,\mathrm{Cat}\Big)}
\end{array}$$

with inverse

$$\begin{array}{rcclll}
(\mathrm{Real}_{\mathcal{M}})^-\colon & \mathrm{Aut}_R(\mathcal{M}) & \longrightarrow & \big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\mathrm{Cat} & & \\
& G & \longmapsto & G(\mathrm{Real}_{\mathcal{M}})^- & = G & \text{for } G \in \mathrm{Ob}(\mathrm{Aut}_R(\mathcal{M}))\ , \\
& \big(F \xrightarrow{a} G\big) & \longmapsto & a(\mathrm{Real}_{\mathcal{M}})^- & = \Big(F \xrightarrow{(F,\, \mathrm{id}_{\mathcal{M}} \xrightarrow{\mathrm{id}_{F^{\otimes -}} * a} F^{\otimes -} * G)} G\Big) & \\
& & & & = \Big(F \xrightarrow{(F,\, \mathrm{id}_{\mathcal{M}} \xrightarrow{F^- a} F^- G)} G\Big) & \\
& & & & & \text{for } \big(F \xrightarrow{a} G\big) \in \mathrm{Mor}(\mathrm{Aut}_R(\mathcal{M}))\,.
\end{array}$$

□

Chapter 8

Modules over a monoidal R-linear category

Let $\mathcal{A} = (\mathcal{A}, I_{\mathcal{A}}, \tilde{\otimes}, \varphi)$ be a monoidal R-linear category.

8.1 $\mathcal{A}$-modules, $\mathcal{A}$-linear functors and $\mathcal{A}$-linear transformations

8.1.1 $\mathcal{A}$-modules

Definition 100 ($\mathcal{A}$-module)

Suppose given an R-linear category $\mathcal{M}$. Suppose given a monoidal R-linear functor

$$\begin{array}{rccl} \Phi\colon & \mathcal{A} & \to & \operatorname{End}_R(\mathcal{M})\,, \\ & \big(A \xrightarrow{a} B\big) & \mapsto & \left(\begin{array}{c} M \\ \Big\downarrow m \\ N \end{array} \quad \longmapsto \quad \begin{array}{ccc} (M)(A\Phi) & \xrightarrow{(M)(a\Phi)} & (M)(B\Phi) \\ {\scriptstyle (m)(A\Phi)}\Big\downarrow & & \Big\downarrow{\scriptstyle (m)(B\Phi)} \\ (N)(A\Phi) & \xrightarrow[(N)(a\Phi)]{} & (N)(B\Phi) \end{array} \right); \end{array}$$

cf. Definition 70 and Lemma 80.

Then $(\mathcal{M}, \Phi)$ is called an $\mathcal{A}$-*module* or a *module over* $\mathcal{A}$.

We often write $\mathcal{M} := (\mathcal{M}, \Phi)$.

For $M, N \in \mathrm{Ob}(\mathcal{M})$, $A, B \in \mathrm{Ob}(\mathcal{A})$, $\big(M \xrightarrow{m} N\big) \in \mathrm{Mor}(\mathcal{M})$ and $\big(A \xrightarrow{a} B\big) \in \mathrm{Mor}(\mathcal{A})$, we write

$$
\begin{aligned}
M \otimes A &:= (M)(A\Phi) \in \mathrm{Ob}(\mathcal{M}) \\
m \otimes A &:= (m)(A\Phi)\colon (M)(A\Phi) \to (N)(A\Phi) \\
M \otimes a &:= (M)(a\Phi)\colon (M)(A\Phi) \to (M)(B\Phi) \\
m \otimes a &:= (m)(A\Phi) \blacktriangle (N)(a\Phi) = (M)(a\Phi) \blacktriangle (m)(B\Phi)\colon (M)(A\Phi) \to (N)(B\Phi)\,.
\end{aligned}
$$

So we obtain the following commutative diagram in $\mathcal{M}$.

$$
\begin{tikzcd}
M \otimes A \arrow[r, "M \otimes a"] \arrow[d, "m \otimes A"'] \arrow[dr, "m \otimes a" description] & M \otimes B \arrow[d, "m \otimes B"] \\
N \otimes A \arrow[r, "N \otimes a"'] & N \otimes B
\end{tikzcd}
$$

We call $(\otimes)$ the *action tensor product* of $\mathcal{A}$ on $\mathcal{M}$.

Remark 101 *Suppose given an* $\mathcal{A}$*-module* $(\mathcal{M}, \Phi)$*.*

Suppose given $M \xrightarrow{m} M' \xrightarrow{m'} M''$ *in* $\mathcal{M}$*.*

Suppose given $A \xrightarrow{a} A' \xrightarrow{a'} A''$ *in* $\mathcal{A}$ *and* $B \xrightarrow{b} B'$ *in* $\mathcal{A}$*.*

(1) *We have* $\quad \mathrm{id}_M \otimes A = \mathrm{id}_{M \otimes A}\,.$

(2) *We have* $\quad (m \blacktriangle m') \otimes A = (m \otimes A) \blacktriangle (m' \otimes A)\,.$

(3) *We have* $\quad M \otimes I_{\mathcal{A}} = M\,.$

(4) *We have* $\quad m \otimes I_{\mathcal{A}} = m\,.$

(5) *We have* $\quad M \otimes \mathrm{id}_A = \mathrm{id}_{M \otimes A}\,.$

(6) *We have* $\quad m \otimes \mathrm{id}_A = m \otimes A\,.$

(7) *We have* $\quad \mathrm{id}_M \otimes a = M \otimes a\,.$

(8) *We have* $\quad M \otimes (a \blacktriangle a') = (M \otimes a) \blacktriangle (M \otimes a')\,.$

(9) *We have* $\quad (m \otimes a) \otimes b = m \otimes (a \otimes b)\,.$

(10) *We have* $\quad (m \blacktriangle m') \otimes (a \blacktriangle a') = (m \otimes a) \blacktriangle (m' \otimes a')\,.$

Proof. Ad (1). We have

$$\mathrm{id}_M \otimes A = (\mathrm{id}_M)(A\Phi) = \mathrm{id}_{M(A\Phi)} = \mathrm{id}_{M\otimes A}\,.$$

Ad (2). We have

$$(m \blacktriangle m') \otimes A = (m \blacktriangle m')(A\Phi) = \big(m(A\Phi)\big) \blacktriangle \big(m'(A\Phi)\big) = (m \otimes A) \blacktriangle (m' \otimes A)\,.$$

Ad (3). We have

$$M \otimes I_{\mathcal{A}} = (M)(I_{\mathcal{A}}\,\Phi) = (M)\,I_{\mathrm{End}_R(\mathcal{M})} = (M)\mathrm{id}_{\mathcal{M}} = M\,.$$

Ad (4). We have

$$m \otimes I_{\mathcal{A}} = (m)(I_{\mathcal{A}}\,\Phi) = (m)\,I_{\mathrm{End}_R(\mathcal{M})} = (m)\mathrm{id}_{\mathcal{M}} = m\,.$$

Ad (5). We have

$$M \otimes \mathrm{id}_A = (M)(\mathrm{id}_A\,\Phi) = (M)(\mathrm{id}_{A\Phi}) = \mathrm{id}_{M(A\Phi)} = \mathrm{id}_{M\otimes A}\,.$$

Ad (6). We have

$$\begin{aligned} m \otimes \mathrm{id}_A &= (m)(A\Phi) \blacktriangle (M')(\mathrm{id}_A\,\Phi) = (m)(A\Phi) \blacktriangle (M' \otimes \mathrm{id}_A) \overset{(5)}{=} (m)(A\Phi) \blacktriangle \mathrm{id}_{M'\otimes A} \\ &= (m)(A\Phi) = m \otimes A\,. \end{aligned}$$

Ad (7). We have

$$\mathrm{id}_M \otimes a = (\mathrm{id}_M)(A\Phi) \blacktriangle (M)(a\Phi) \overset{(1)}{=} \mathrm{id}_{M\otimes A} \blacktriangle \big((M)(a\Phi)\big) = (M)(a\Phi) = M \otimes a\,.$$

Ad (8). We have

$$\begin{aligned} M \otimes (a \blacktriangle a') &= (M)\big((a \blacktriangle a')\Phi\big) = (M)(a\Phi \blacktriangle a'\Phi) = (M)(a\Phi) \blacktriangle (M)(a'\Phi) \\ &= (M \otimes a) \blacktriangle (M \otimes a')\,. \end{aligned}$$

Ad (9). We have

$$\begin{aligned}(m \otimes a) \otimes b &= \big((m)(A\Phi) \blacktriangle (M')(a\Phi)\big) \otimes b \\ &= \big((m)(A\Phi) \blacktriangle (M')(a\Phi)\big)(B\Phi) \blacktriangle \big((M')(A'\Phi)\big)(b\Phi) \\ &= (m)(A\Phi)(B\Phi) \blacktriangle (M')(a\Phi)(B\Phi) \blacktriangle (M')(A'\Phi)(b\Phi) \\ &= (m)\big((A \otimes B)\Phi\big) \blacktriangle (M')\big((a\Phi)(B\Phi) \blacktriangle (A'\Phi)(b\Phi)\big) \\ &= (m)\big((A \otimes B)\Phi\big) \blacktriangle (M')(a\Phi * b\Phi) \\ &= (m)\big((A \otimes B)\Phi\big) \blacktriangle (M')\big((a \otimes b)\Phi\big) \\ &= m \otimes (a \otimes b)\,.\end{aligned}$$

Ad (10). We have

$$\begin{aligned}(m \blacktriangle m') \otimes (a \blacktriangle a') &= \big((m \blacktriangle m') \otimes A\big) \blacktriangle \big(M'' \otimes (a \blacktriangle a')\big) \\ &\overset{(2,8)}{=} (m \otimes A) \blacktriangle (m' \otimes A) \blacktriangle (M'' \otimes a) \blacktriangle (M'' \otimes a') \\ &= (m \otimes A) \blacktriangle (m' \otimes a) \blacktriangle (M'' \otimes a') \\ &= (m \otimes A) \blacktriangle (M' \otimes a) \blacktriangle (m' \otimes A') \blacktriangle (M'' \otimes a') \\ &= (m \otimes a) \blacktriangle (m' \otimes a')\,.\end{aligned}$$

□

Definition 102 ($\mathcal{A}$-submodule)

Suppose given $\mathcal{A}$-modules $(\mathcal{M}, \Phi)$ and $(\mathcal{N}, \Phi')$.

We say that $(\mathcal{N}, \Phi')$ is an $\mathcal{A}$-*submodule* of $(\mathcal{M}, \Phi)$ if $(1, 2, 3)$ hold.

(1) The category $\mathcal{N}$ is an R-linear subcategory of $\mathcal{M}$; cf. Definition 67.

(2) For $A \in \mathrm{Ob}(\mathcal{A})$, we have $A\Phi' = A\Phi\big|_{\mathcal{N}}^{\mathcal{N}}$.

(3) For $N \in \mathrm{Ob}(\mathcal{N})$ and $\big(A \xrightarrow{a} B\big) \in \mathrm{Mor}(\mathcal{A})$, we have

$$(N)(a\Phi') = (N)(a\Phi) \colon (N)(A\Phi) \to (N)(B\Phi)\,.$$

Lemma 103 (The functor Θ_F) *Suppose given monoidal R-linear categories $(\mathcal{A}, I_{\mathcal{A}}, \underset{\mathcal{A}}{\otimes}, \varphi)$ and $(\mathcal{B}, I_{\mathcal{B}}, \underset{\mathcal{B}}{\otimes}, \tilde{\varphi})$. Suppose given a monoidal R-linear functor $F \colon \mathcal{A} \to \mathcal{B}$; cf. Definition* 74.

Consider the monoidal R-linear category $\mathrm{End}_R(\mathcal{B})$*; cf. Lemma* 80.

Then, we have a monoidal R-linear functor Θ_F given by

$$\begin{array}{rcl} \Theta_F\colon & \mathcal{A} & \to \ \mathrm{End}_R(\mathcal{B})\,, \\ \left(A \xrightarrow{a} A'\right) & \mapsto & \left(\begin{tikzcd} X \arrow[dd, "u"] & & X \underset{\mathcal{B}}{\otimes} (AF) \arrow[r, "X \underset{\mathcal{B}}{\otimes} (aF)"] \arrow[dd, "u \underset{\mathcal{B}}{\otimes} (AF)"'] & X \underset{\mathcal{B}}{\otimes} (A'F) \arrow[dd, "u \underset{\mathcal{B}}{\otimes} (A'F)"] \\ & \longmapsto & & \\ X' & & X' \underset{\mathcal{B}}{\otimes} (AF) \arrow[r, "X' \underset{\mathcal{B}}{\otimes} (aF)"'] & X' \underset{\mathcal{B}}{\otimes} (A'F) \end{tikzcd} \right) \end{array}$$

In particular, $(\mathcal{B}, \Theta_F)$ is an $\mathcal{A}$-module.

Proof. Write $\Theta := \Theta_F$ and $\otimes := \underset{\mathcal{B}}{\otimes}$.

We show that Θ is well-defined.

We show that $A\Theta$ and $a\Theta$ are well-defined for $A \in \mathrm{Ob}(\mathcal{A})$ and $a \in \mathrm{Mor}(\mathcal{A})$.

Suppose given $A \in \mathrm{Ob}(\mathcal{A})$.

We show that $A\Theta \overset{!}{\in} \mathrm{Ob}(\mathrm{End}_R(\mathcal{B}))$, i.e. that $A\Theta$ is an R-linear functor.

Suppose given $X \xrightarrow{u} X' \xrightarrow{u'} X''$ in $\mathcal{B}$. We have

$$\begin{aligned} (u\boldsymbol{s})(A\Theta) &= X(A\Theta) = X \otimes AF = u\boldsymbol{s} \otimes \mathrm{id}_{AF}\,\boldsymbol{s} = (u \otimes \mathrm{id}_{AF})\boldsymbol{s} \\ &= \big(u(A\Theta)\big)\boldsymbol{s} \\ (u\boldsymbol{t})(A\Theta) &= (X')(A\Theta) = X' \otimes AF = u\boldsymbol{t} \otimes \mathrm{id}_{AF}\,\boldsymbol{t} = (u \otimes \mathrm{id}_{AF})\boldsymbol{t} \\ &= \big(u(A\Theta)\big)\boldsymbol{t} \\ (X\boldsymbol{i})(A\Theta) &= (\mathrm{id}_X)(A\Theta) = \mathrm{id}_X \otimes \mathrm{id}_{AF} = \mathrm{id}_{X \otimes AF} = \big(X(A\Theta)\big)\boldsymbol{i}\,. \end{aligned}$$

Further, we have

$$\begin{aligned} (u \blacktriangle u')(A\Theta) &= (u \blacktriangle u') \otimes \mathrm{id}_{AF} = (u \otimes \mathrm{id}_{AF}) \blacktriangle (u' \otimes \mathrm{id}_{AF}) \\ &= \big(u(A\Theta)\big) \blacktriangle \big(u'(A\Theta)\big)\,. \end{aligned}$$

So $A\Theta$ is a functor.

Suppose given $r, s \in R$ and $X \underset{u_2}{\overset{u_1}{\rightrightarrows}} X'$ in $\mathcal{B}$. We have

$$\begin{aligned}(u_1\, r + u_2\, s)(A\Theta) &= (u_1\, r + u_2\, s) \otimes \mathrm{id}_{AF} = (u_1 \otimes \mathrm{id}_{AF})\, r + (u_2 \otimes \mathrm{id}_{AF})\, s \\ &= \big(u_1(A\Theta)\big)r + \big(u_2(A\Theta)\big)s\,.\end{aligned}$$

So $A\Theta$ is R-linear.

Hence, $A\Theta \in \mathrm{Ob}(\mathrm{End}_R(\mathcal{B}))$.

Suppose given $\big(A \xrightarrow{a} A'\big) \in \mathrm{Mor}(\mathcal{A})$. We show that $a\Theta \overset{!}{\in} \mathrm{Mor}(\mathrm{End}_R(\mathcal{B}))$, i.e. that $a\Theta$ is a transformation.

Suppose given $(X \xrightarrow{u} X')$ in $\mathrm{Mor}(\mathcal{B})$. We have

$$\begin{aligned}\big(X(a\Theta)\big) \blacktriangle \big(u(A'\Theta)\big) &= \big(\mathrm{id}_X \otimes aF\big) \blacktriangle (u \otimes \mathrm{id}_{A'F}) \overset{101.(10)}{=} (\mathrm{id}_X \blacktriangle u) \otimes \big(aF \blacktriangle \mathrm{id}_{A'F}\big) \\ &= u \otimes aF = (u \blacktriangle \mathrm{id}_{X'}) \otimes \big(\mathrm{id}_{AF} \blacktriangle aF\big) \overset{101.(10)}{=} (u \otimes \mathrm{id}_{AF}) \blacktriangle \big(\mathrm{id}_{X'} \blacktriangle aF\big) \\ &= \big(u(A\Theta)\big) \blacktriangle \big(X'(a\Theta)\big)\,.\end{aligned}$$

$$\begin{tikzcd} X(A\Theta) \arrow[r, "X(a\Theta)"] \arrow[d, "u(A\Theta)"'] & X(A\Theta) \arrow[d, "u(A'\Theta)"] \\ X'(A\Theta) \arrow[r, "X'(a\Theta)"'] & X'(A\Theta) \end{tikzcd}$$

So $a\Theta$ is a transformation.

We show that Θ is a functor.

Suppose given $A \xrightarrow{a} A' \xrightarrow{a'} A''$ in $\mathcal{A}$. Suppose given $X \in \mathrm{Ob}(\mathcal{B})$. We have

$$\begin{aligned}(a\boldsymbol{s})\Theta &= A\Theta = (a\Theta)\boldsymbol{s} \\ (a\boldsymbol{t})\Theta &= A'\Theta = (a\Theta)\boldsymbol{t} \\ (A\boldsymbol{i})\Theta &= (\mathrm{id}_A)\Theta = \big(X \otimes (\mathrm{id}_A)F\big)_{X \in \mathrm{Ob}(\mathcal{B})} = (X \otimes \mathrm{id}_{AF})_{X \in \mathrm{Ob}(\mathcal{B})} = (\mathrm{id}_{X \otimes AF})_{X \in \mathrm{Ob}(\mathcal{B})} \\ &= (\mathrm{id}_{X(A\Theta)})_{X \in \mathrm{Ob}(\mathcal{B})} = \mathrm{id}_{A\Theta} = (A\Theta)\boldsymbol{i}\,.\end{aligned}$$

Further, we have

$$\begin{aligned} X\big((a \blacktriangle a')\Theta\big) &= \mathrm{id}_X \otimes \big((a \blacktriangle a')F\big) = \mathrm{id}_X \otimes (aF \blacktriangle a'F) = (\mathrm{id}_X \blacktriangle \mathrm{id}_X) \otimes (aF \blacktriangle a'F) \\ &\overset{101.(10)}{=} (\mathrm{id}_X \otimes aF) \blacktriangle (\mathrm{id}_X \otimes a'F) = X(a\Theta) \blacktriangle X(a'\Theta) \\ &= X(a\Theta \blacktriangle a'\Theta)\,. \end{aligned}$$

This shows $(a \blacktriangle a')\Theta = a\Theta \blacktriangle a'\Theta$. So Θ is a functor.

We show that Θ is R-linear.

Suppsose given $A \overset{a_1}{\underset{a_2}{\rightrightarrows}} A'$ in $\mathcal{A}$. Suppose given $r, s \in R$. Suppose given $X \in \mathrm{Ob}(\mathcal{B})$.

We have

$$\begin{aligned} X\big((a_1 r + a_2 s)\Theta\big) &= \mathrm{id}_X \otimes \big((a_1 r + a_2 s)F\big) = \mathrm{id}_X \otimes \big((a_1 F)r + (a_2 F)s\big) \\ &= \big(\mathrm{id}_X \otimes (a_1 F)\big)r + \big(\mathrm{id}_X \otimes (a_2 F)\big)s \\ &= X\big((a_1\Theta)r\big) + X\big((a_2\Theta)s\big) \\ &= X\big((a_1\Theta)r + (a_2\Theta)s\big)\,. \end{aligned}$$

So Θ is R-linear; cf. Remark 71.

We show that Θ is monoidal.

Suppose given $\big(A \xrightarrow{a} A'\big)$, $\big(\tilde{A} \xrightarrow{\tilde{a}} \tilde{A}'\big) \in \mathrm{Mor}(\mathcal{A})$.

For $X \in \mathrm{Ob}(\mathcal{B})$, we have

$$\begin{aligned} X(a\Theta * \tilde{a}\Theta) &= X\big((a\Theta)(\tilde{A}\Theta) \blacktriangle (A'\Theta)(\tilde{a}\Theta)\big) \\ &= X\big((a\Theta)(\tilde{A}\Theta)\big) \blacktriangle X\big((A'\Theta)(\tilde{a}\Theta)\big) \\ &= (\mathrm{id}_X \otimes aF)(\tilde{A}\Theta) \blacktriangle (X \otimes A'F)(\tilde{a}\Theta) \\ &= (\mathrm{id}_X \otimes aF \otimes \mathrm{id}_{\tilde{A}F}) \blacktriangle (\mathrm{id}_{X \otimes A'F} \otimes \tilde{a}F) \\ &= (\mathrm{id}_X \otimes aF \otimes \mathrm{id}_{\tilde{A}F}) \blacktriangle (\mathrm{id}_X \otimes \mathrm{id}_{A'F} \otimes \tilde{a}F) \\ &\overset{101.(10)}{=} (\mathrm{id}_X \blacktriangle \mathrm{id}_X) \otimes \big((aF \otimes \mathrm{id}_{\tilde{A}F}) \blacktriangle (\mathrm{id}_{A'F} \otimes \tilde{a}F)\big) \\ &= \mathrm{id}_X \otimes \big((aF \blacktriangle \mathrm{id}_{A'F}) \otimes (\mathrm{id}_{\tilde{A}F} \blacktriangle \tilde{a}F)\big) \\ &= \mathrm{id}_X \otimes (aF \otimes \tilde{a}F) \\ &= \mathrm{id}_X \otimes \big((a \otimes \tilde{a})F\big) \\ &= X\big((a \otimes \tilde{a})\Theta\big)\,. \end{aligned}$$

This shows $a\Theta * \tilde{a}\Theta = (a \otimes \tilde{a})\Theta$.

For $u \in \mathrm{Mor}(\mathcal{B})$, we have

$$u(I_{\mathcal{A}}\Theta) = u \otimes I_{\mathcal{A}}F = u \otimes I_{\mathcal{B}} \overset{101.(4)}{=} u = u\mathrm{id}_{\mathcal{B}}\,.$$

This shows $I_{\mathcal{A}}\Theta = \mathrm{id}_{\mathcal{B}}$.

So Θ is monoidal; cf. Remark 32.(3).

Altogether, Θ is a monoidal R-linear functor. $\square$

Definition 104 (The regular $\mathcal{A}$-module)

Suppose given a monoidal R-linear category $(\mathcal{A}, I_{\mathcal{A}}, \underset{\mathcal{A}}{\otimes}, \varphi)$. Consider the monoidal R-linear functor $\Theta := \Theta_{\mathrm{id}_{\mathcal{A}}} \colon \mathcal{A} \to \mathrm{End}_R(\mathcal{A})$ from Lemma 103 .

$$\begin{array}{rcl} \Theta\colon \quad \mathcal{A} & \to & \mathrm{End}_R(\mathcal{A})\,, \\ \left(A \xrightarrow{a} A'\right) & \mapsto & \left(\begin{tikzcd} X \arrow[d, "u"] \\ X' \end{tikzcd} \longmapsto \begin{tikzcd} X \underset{\mathcal{A}}{\otimes} A \arrow[r, "X \underset{\mathcal{A}}{\otimes} a"] \arrow[d, "u \underset{\mathcal{A}}{\otimes} A"'] & X \underset{\mathcal{A}}{\otimes} A' \arrow[d, "u \underset{\mathcal{A}}{\otimes} A'"] \\ X' \underset{\mathcal{A}}{\otimes} A \arrow[r, "X' \underset{\mathcal{A}}{\otimes} a"'] & X' \underset{\mathcal{A}}{\otimes} A' \end{tikzcd} \right) \end{array}$$

Then $(\mathcal{A}, \Theta)$ is an $\mathcal{A}$-module, called the *regular $\mathcal{A}$-module.*

8.1.2 $\mathcal{A}$-linear functors

Definition 105 ($\mathcal{A}$-linear functor)

Suppose given $\mathcal{A}$-modules $(\mathcal{M}, \Phi)$ and $(\mathcal{N}, \Phi')$.

An R-linear functor $F\colon \mathcal{M} \to \mathcal{N}$ is called *$\mathcal{A}$-linear* if we have

$$(m \otimes a)F = mF \otimes a$$

for $m \in \mathrm{Mor}(\mathcal{M})$ and $a \in \mathrm{Mor}(\mathcal{A})$.

Lemma 106 *Suppose given $\mathcal{A}$-modules $(\mathcal{M}, \Phi)$ and $(\mathcal{N}, \Phi')$. Suppose given an R-linear functor $F\colon \mathcal{M} \to \mathcal{N}$.*

Then F is an $\mathcal{A}$-linear functor if and only if the conditions $(1, 2)$ *hold.*

(1) *For* $m \in \operatorname{Mor}(\mathcal{M})$ *and* $A \in \operatorname{Mor}(\mathcal{A})$*, we have*

$$(m \otimes A)F = mF \otimes A\,.$$

I.e. we have

$$(A\Phi)F = F(A\Phi')$$

for $A \in \operatorname{Ob}(\mathcal{A})$ *.*

(2) *For* $M \in \operatorname{Ob}(\mathcal{M})$ *and* $a \in \operatorname{Mor}(\mathcal{A})$*, we have*

$$(M \otimes a)F = MF \otimes a\,.$$

I.e. we have

$$(a\Phi)F = F(a\Phi')$$

for $a \in \operatorname{Mor}(\mathcal{A})$ *.*

So, for an $\mathcal{A}$*-linear functor, we have*

$$\left(\begin{tikzcd}
(M \otimes A)F \arrow[r, "(M \otimes a)F"] \arrow[d, "(m \otimes A)F"'] \arrow[dr, "(m \otimes a)F" description] & (M \otimes B)F \arrow[d, "(m \otimes B)F"] \\
(M' \otimes A)F \arrow[r, "(M' \otimes a)F"'] & (M' \otimes B)F
\end{tikzcd}\right)$$

$$= \quad \left(\begin{tikzcd}
MF \otimes A \arrow[r, "MF \otimes a"] \arrow[d, "mF \otimes A"'] \arrow[dr, "mF \otimes a" description] & MF \otimes B \arrow[d, "mF \otimes B"] \\
M'F \otimes A \arrow[r, "M'F \otimes a"'] & M'F \otimes B
\end{tikzcd}\right)$$

for $\left(M \xrightarrow{m} M'\right) \in \operatorname{Mor}(\mathcal{M})$ *and* $\left(A \xrightarrow{a} B\right) \in \operatorname{Mor}(\mathcal{A})$ *.*

Proof. Ad $\Rightarrow$. Suppose that $F\colon \mathcal{M} \to \mathcal{N}$ is an $\mathcal{A}$-linear functor.
Suppose given $\left(A \xrightarrow{a} B\right) \in \operatorname{Mor}(\mathcal{A})$ and $m = \left(M \xrightarrow{m} M'\right) \in \operatorname{Mor}(\mathcal{M})$.

We have
$$(m)\big((A\Phi)F\big) \overset{101.(6)}{=} (m \otimes \mathrm{id}_A)F = (mF) \otimes \mathrm{id}_A = (m)\big(F(A\Phi')\big)\,.$$

Therefore, $(A\Phi)F = F(A\Phi')$.

We have
$$(M)\big((a\Phi)F\big) \overset{101.(7)}{=} (\mathrm{id}_M \otimes a)F = (\mathrm{id}_M\, F) \otimes a = \mathrm{id}_{MF} \otimes a \overset{101.(7)}{=} MF \otimes a = (M)\big(F(a\Phi')\big)\,.$$

Therefore, $(a\Phi)F = F(a\Phi')$.

Ad $\Leftarrow$. Suppose that (1,2) hold.

For $\big(M \xrightarrow{m} M'\big) \in \mathrm{Mor}(\mathcal{M})$ and $\big(A \xrightarrow{a} B\big) \in \mathrm{Mor}(\mathcal{A})$, we have
$$\begin{aligned}(m \otimes a)F &= \big((m \otimes A) \blacktriangle (M' \otimes a)\big)F = (m \otimes A)F \blacktriangle (M' \otimes a)F \\ &= (mF \otimes A) \blacktriangle (M'F \otimes a) = mF \otimes a\,.\end{aligned}$$

So F is $\mathcal{A}$-linear. □

Lemma 107 *Suppose given $\mathcal{A}$-modules $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{P}$. Suppose given $\mathcal{A}$-linear functors $F\colon \mathcal{M} \to \mathcal{N}$ and $G\colon \mathcal{N} \to \mathcal{P}$.*

(1) *The identity* $\mathrm{id}_{\mathcal{M}}\colon \mathcal{M} \to \mathcal{M}$ *is an $\mathcal{A}$-linear functor.*

(2) *The composite* $F * G\colon \mathcal{M} \to \mathcal{P}$ *is an $\mathcal{A}$-linear functor.*

Proof. We use Lemma 106.

Ad (1). For $m \in \mathrm{Mor}(\mathcal{M})$ and $a \in \mathrm{Mor}(\mathcal{A})$, we have
$$(m \otimes a)\mathrm{id}_{\mathcal{M}} = m \otimes a = (m\,\mathrm{id}_{\mathcal{M}}) \otimes a\,.$$

Ad (2). For $m \in \mathrm{Ob}(\mathcal{M})$ and $a \in \mathrm{Mor}(\mathcal{A})$, we have
$$(m \otimes a)FG = (mF \otimes a)G = mFG \otimes a\,.$$

□

Lemma 108 *Suppose given monoidal R-linear categories $\mathcal{B}$ and $\mathcal{C}$. Suppose given monoidal R-linear functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$.*

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$$

By Lemma 103, the R-linear category $\mathcal{B}$ is an $\mathcal{A}$-module via

$$\begin{aligned}
\Theta_F\colon \quad & \mathcal{A} \;\to\; \mathrm{End}_R(\mathcal{B})\,, \\
& \big(A \xrightarrow{a} A'\big) \mapsto \left(\begin{tikzcd} X \arrow[d, "u"] \\ X' \end{tikzcd} \longmapsto \begin{tikzcd} X \underset{\mathcal{B}}{\otimes} (AF) \arrow[r, "X \underset{\mathcal{B}}{\otimes} (aF)"] \arrow[d, "u \underset{\mathcal{B}}{\otimes} (AF)"'] & X \underset{\mathcal{B}}{\otimes} (A'F) \arrow[d, "u \underset{\mathcal{B}}{\otimes} (A'F)"] \\ X' \underset{\mathcal{B}}{\otimes} (AF) \arrow[r, "X' \underset{\mathcal{B}}{\otimes} (aF)"'] & X' \underset{\mathcal{B}}{\otimes} (A'F) \end{tikzcd}\right)
\end{aligned}$$

and the R-linear category $\mathcal{C}$ is an $\mathcal{A}$-module via

$$\begin{aligned}
\Theta_{FG}\colon \quad & \mathcal{A} \;\to\; \mathrm{End}_R(\mathcal{C})\,, \\
& \big(A \xrightarrow{a} A'\big) \mapsto \left(\begin{tikzcd} Y \arrow[d, "v"] \\ Y' \end{tikzcd} \longmapsto \begin{tikzcd} Y \underset{\mathcal{C}}{\otimes} (AFG) \arrow[r, "Y \underset{\mathcal{C}}{\otimes} (aFG)"] \arrow[d, "v \underset{\mathcal{C}}{\otimes} (AFG)"'] & Y \underset{\mathcal{C}}{\otimes} (A'FG) \arrow[d, "v \underset{\mathcal{C}}{\otimes} (A'FG)"] \\ Y' \underset{\mathcal{C}}{\otimes} (AFG) \arrow[r, "Y' \underset{\mathcal{C}}{\otimes} (aFG)"'] & Y' \underset{\mathcal{C}}{\otimes} (A'FG) \end{tikzcd}\right).
\end{aligned}$$

Then the R-linear functor $G\colon \mathcal{B} \to \mathcal{C}$ is $\mathcal{A}$-linear.

Proof. Suppose given $\big(M \xrightarrow{m} N\big) \in \mathrm{Mor}(\mathcal{B})$ and $\big(A \xrightarrow{a} B\big) \in \mathrm{Mor}(\mathcal{A})$.
We have

$$\begin{aligned}
(m \otimes a)G &= \big((m)(A\Theta_F) \blacktriangle (N)(a\Theta_F)\big)G = \big((m \underset{\mathcal{B}}{\otimes} AF) \blacktriangle (N \underset{\mathcal{B}}{\otimes} aF)\big)G \\
&= (m \underset{\mathcal{B}}{\otimes} AF)G \blacktriangle (N \underset{\mathcal{B}}{\otimes} aF)G = (mG \underset{\mathcal{C}}{\otimes} AFG) \blacktriangle (NG \underset{\mathcal{C}}{\otimes} aFG) \\
&= (mG)(A\Theta_{FG}) \blacktriangle (NG)(a\Theta_{FG}) = (mG) \otimes a\,.
\end{aligned}$$

$\square$

8.1.3 $\mathcal{A}$-linear transformations

Definition 109 ($\mathcal{A}$-linear transformations)

Suppose given $\mathcal{A}$-modules $(\mathcal{M}, \Phi)$, $(\mathcal{N}, \Phi')$. Suppose given $\mathcal{A}$-linear functors $F, G \colon \mathcal{M} \to \mathcal{N}$.

A transformation $\eta \colon F \to G$ is called $\mathcal{A}$-*linear* if we have

$$\big((M \otimes A)F \xrightarrow{(M\otimes A)\eta} (M \otimes A)G\big) = \big((MF) \otimes A \xrightarrow{(M\eta)\otimes A} (MG) \otimes A\big)$$

for $M \in \mathrm{Ob}(\mathcal{M})$ and $A \in \mathrm{Ob}(\mathcal{A})$.

I.e. we have

$$(A\Phi)\eta = \eta(A\Phi')$$

for $A \in \mathrm{Ob}(\mathcal{A})$.

Lemma 110 *Suppose given $\mathcal{A}$-modules $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{P}$. Suppose given $\mathcal{A}$-linear functors $H \colon \mathcal{L} \to \mathcal{M}$, $F, F', F'' \colon \mathcal{M} \to \mathcal{N}$ and $G, G' \colon \mathcal{N} \to \mathcal{P}$. Suppose given $\mathcal{A}$-linear transformations $\eta \colon F \to F'$, $\eta' \colon F' \to F''$ and $\vartheta \colon G \to G'$.*

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "H"] & \mathcal{M} \arrow[r, bend left=50, "F"] \arrow[r, "F'"] \arrow[r, bend right=50, "F''"'] \arrow[r, phantom, shift left=3, "\Downarrow\eta"] \arrow[r, phantom, shift right=3, "\Downarrow\eta'"] & \mathcal{N} \arrow[r, bend left=50, "G"] \arrow[r, bend right=50, "G'"'] \arrow[r, phantom, "\Downarrow\vartheta"] & \mathcal{P}
\end{tikzcd}$$

(1) *The transformation* $\mathrm{id}_F \colon F \to F$ *is $\mathcal{A}$-linear.*

(2) *The vertical composite $\eta \blacktriangle \eta' \colon F \to F''$ is an $\mathcal{A}$-linear transformation.*

(3) *We have $\mathcal{A}$-linear transformations $H\eta \colon HF \to HF'$ and $\eta G \colon FG \to F'G$.*

(4) *The horizontal composite $\eta * \vartheta \colon FG \to F'G'$ is an $\mathcal{A}$-linear transformation.*

Proof. Suppose given $L \in \mathrm{Ob}(\mathcal{L})$, $M \in \mathrm{Ob}(\mathcal{M})$ and $A \in \mathrm{Ob}(\mathcal{A})$.

Ad (1). We have

$$(M \otimes A)\mathrm{id}_F = \mathrm{id}_{(M\otimes A)F} = \mathrm{id}_{(MF)\otimes A} = \mathrm{id}_{MF} \otimes A = (M\mathrm{id}_F) \otimes A\,.$$

Ad (2). We have

$$\begin{aligned}(M \otimes A)(\eta \blacktriangle \eta') &= \big((M \otimes A)\eta\big) \blacktriangle \big((M \otimes A)\eta'\big) = \big((M\eta) \otimes A\big) \blacktriangle \big((M\eta') \otimes A\big)\\ &= \big((M\eta) \blacktriangle (M\eta')\big) \otimes A = \big((M)(\eta \blacktriangle \eta')\big) \otimes A\,.\end{aligned}$$

Ad (3). We have

$$(L \otimes A)H\eta = \big((LH) \otimes A\big)\eta = \big((LH)\eta\big) \otimes A = \big((L)(H\eta)\big) \otimes A\,,$$

and similarly

$$(M \otimes A)\eta G = \big((M\eta) \otimes A\big)G = \big((M\eta)G\big) \otimes A = \big((M)(\eta G)\big) \otimes A\,.$$

Ad (4). Recall that $\eta * \vartheta = (F\vartheta) \blacktriangle (\eta G')$; cf. §0.3 item 3. By (3), $F\vartheta$ and $\eta G'$ are $\mathcal{A}$-linear, and then by (2), $\eta * \vartheta = (F\vartheta) \blacktriangle (\eta G')$ is $\mathcal{A}$-linear. □

8.2 The monoidal R-linear category $\mathrm{End}_{\mathcal{A}}(\mathcal{A})$

Let $(\mathcal{A}, I, \otimes, \varepsilon)$ be a monoidal R-linear category.

Lemma 111 *Consider the monoidal R-linear category $\mathrm{End}_R(\mathcal{A})$; cf. Lemma 80.*

We have the subcategory $\mathrm{End}_{\mathcal{A}}(\mathcal{A}) \subseteq \mathrm{End}_R(\mathcal{A})$ with

$$\begin{aligned}\mathrm{Ob}\big(\mathrm{End}_{\mathcal{A}}(\mathcal{A})\big) &:= \{\mathcal{A} \xrightarrow{F} \mathcal{A} \colon (a \otimes b)F = a \otimes (bF) \text{ for } a,\, b \in \mathrm{Mor}(\mathcal{A})\}\\ \mathrm{Mor}\big(\mathrm{End}_{\mathcal{A}}(\mathcal{A})\big) &:= \{F \xrightarrow{a} G \colon F,\, G \in \mathrm{Ob}\big(\mathrm{End}_{\mathcal{A}}(\mathcal{A})\big)\\ &\qquad\qquad \text{and } (X \otimes Y)a = X \otimes (Ya) \text{ for } X,\, Y \in \mathrm{Ob}(\mathcal{A})\}\,.\end{aligned}$$

So here we can consider an action of $\mathcal{A}$ on $\mathcal{A}$ from the left. The functors appearing in $\mathrm{End}_{\mathcal{A}}(\mathcal{A})$ are to be compared with Definition 105. The transformations appearing in $\mathrm{End}_{\mathcal{A}}(\mathcal{A})$ are to compared with Definition 109.

Proof. Suppose given $F \xrightarrow{a} G \xrightarrow{b} H$ in $\mathrm{End}_{\mathcal{A}}(\mathcal{A})$.

Suppose given $X,\, Y \in \mathrm{Ob}(\mathcal{A})$.

We have

$$(X \otimes Y)\mathrm{id}_F = X \otimes Y = X \otimes (Y\mathrm{id}_F)\,.$$

This shows $\mathrm{id}_F \in \mathrm{Mor}\big(\mathrm{End}_{\mathcal{A}}(\mathcal{A})\big)$.

We have

$$\begin{aligned}(X \otimes Y)(a \blacktriangle b) &= (X \otimes Y)a \blacktriangle (X \otimes Y)b = \big(X \otimes (Ya)\big) \blacktriangle \big(X \otimes (Yb)\big) = X \otimes (Ya \blacktriangle Yb) \\ &= X \otimes \big(Y(a \blacktriangle b)\big)\,.\end{aligned}$$

This shows $a \blacktriangle b \in \mathrm{Mor}\big(\,\mathrm{End}_{\mathcal{A}}(\mathcal{A})\big)$.

So $\mathrm{End}_{\mathcal{A}}(\mathcal{A}))$ is a subcategory of $\mathrm{End}_R(\mathcal{A})$. □

Remark 112 *Suppose given* $F \in \mathrm{End}_{\mathcal{A}}(\mathcal{A})$ *. For* $X,\ Y \in \mathrm{Ob}(\mathcal{A})$ *, we have*

$$(X \otimes Y)F = X \otimes (YF)\,.$$

Proof. For $X,\ Y \in \mathrm{Ob}(\mathcal{A})$, we have

$$(X \otimes Y)F = \big((\mathrm{id}_X \otimes \mathrm{id}_Y)F\big)\mathfrak{s} = \big(\mathrm{id}_X \otimes (\mathrm{id}_Y F)\big)\mathfrak{s} = X \otimes (YF)\,.$$

□

Remark 113 (The monoidal R-linear category $\mathrm{End}_{\mathcal{A}}(\mathcal{A})$)

(1) *The category* $\mathrm{End}_{\mathcal{A}}(\mathcal{A})$ *is a monoidal subcategory of* $\big(\,\mathrm{End}_R(\mathcal{A}), \mathrm{id}_{\mathcal{A}}, *\,\big)$ *; cf. Definition* 16*.*

(2) *The category* $\mathrm{End}_{\mathcal{A}}(\mathcal{A})$ *is an R-linear subcategory of* $\big(\,\mathrm{End}_R(\mathcal{A}), \epsilon\,\big)$ *; cf. Definitions* 65*,* 67*.*

(3) *We have the monoidal R-linear category* $\mathrm{End}_{\mathcal{A}}(\mathcal{A})$*; cf. Definition* 73*.*

Proof. Ad (1). For $a,\ b \in \mathrm{Mor}(\mathcal{A})$, we have

$$(a \otimes b)\mathrm{id}_{\mathcal{A}} = a \otimes b = a \otimes (b\,\mathrm{id}_{\mathcal{A}})\,.$$

This shows that the unit object $\mathrm{id}_{\mathcal{A}}$ is contained in $\mathrm{Ob}\big(\,\mathrm{End}_{\mathcal{A}}(\mathcal{A})\big)$.

Suppose given $\big(F \xrightarrow{a} F'\big)$, $\big(G \xrightarrow{b} G'\big) \in \mathrm{Mor}\big(\,\mathrm{End}_{\mathcal{A}}(\mathcal{A})\big)$. For $X,\ Y \in \mathrm{Ob}(\mathcal{A})$, we have

$$\begin{aligned}(X \otimes Y)(a * b) &= (X \otimes Y)(aG \blacktriangle F'b) = (X \otimes Y)(aG) \blacktriangle (X \otimes Y)(F'b) \\ &= \big(X \otimes (Ya)\big)G \blacktriangle \big(X \otimes (YF')\big)b = \big(X \otimes (YaG)\big) \blacktriangle \big(X \otimes (YF'b)\big) \\ &= X \otimes \big((YaG) \blacktriangle (YF'b)\big) = X \otimes \big(Y(aG \blacktriangle F'b)\big) = X \otimes \big(Y(a * b)\big)\,.\end{aligned}$$

This shows $a * b \in \mathrm{Mor}(\mathrm{End}_{\mathcal{A}}(\mathcal{A}))$.

Then, by Lemma 17, $\mathrm{End}_{\mathcal{A}}(\mathcal{A})$ is a monoidal subcategory of $\mathrm{End}_R(\mathcal{A})$.

Ad (2). Suppose given $F,\, G \in \mathrm{Ob}\big(\,\mathrm{End}_{\mathcal{A}}(\mathcal{A})\big)$.

For $X,\, Y \in \mathrm{Ob}(\mathcal{A})$, we have

$$(X \otimes Y)0_{F,G} \overset{76}{=} 0_{(X\otimes Y)F,\,(X\otimes Y)G} = 0_{X\otimes(YF),\,X\otimes(YG)} \overset{75}{=} X \otimes 0_{YF,\,YG} \overset{76}{=} X \otimes (Y\,0_{F,G})\,.$$

This shows $0_{F,G} \in \mathrm{Mor}\big(\,\mathrm{End}_{\mathcal{A}}(\mathcal{A})\big)$.

Suppose given $r,\, r' \in R$ and $F \underset{a'}{\overset{a}{\rightrightarrows}} G$ in $\mathrm{End}_{\mathcal{A}}(\mathcal{A})$.

For $X,\, Y \in \mathrm{Ob}(\mathcal{A})$, we have

$$\begin{aligned}(X \otimes Y)(ar + a'r') &= \big((X \otimes Y)a\big)r + \big((X \otimes Y)a'\big)r' = \big(X \otimes (Ya)\big)r + \big(X \otimes (Ya')\big)r' \\ &= \Big(X \otimes \big((Ya)r\big)\Big) + \Big(X \otimes \big((Ya')r'\big)\Big) = X \otimes \big((Ya)r + (Ya')r'\big) \\ &= X \otimes \big(Y(ar + a'r')\big)\,.\end{aligned}$$

This shows $ar + a'r' \in \mathrm{Mor}\big(\,\mathrm{End}_{\mathcal{A}}(\mathcal{A})\big)$.

Then, by Remark 68, $\mathrm{End}_{\mathcal{A}}(\mathcal{A})$ is an R-linear subcategory of $\big(\,\mathrm{End}_R(\mathcal{A}), \epsilon\,\big)$.

Ad (3). The category $\mathrm{End}_{\mathcal{A}}(\mathcal{A})$ is a monoidal R-linear category since $\mathrm{End}_R(\mathcal{A})$ is a monoidal R-linear category and since (1) and (2) hold; cf. Definition 73. $\square$

Lemma 114 *We have the monoidal R-linear isofunctor*

$$\begin{array}{rccll} \Psi\colon & \mathcal{A} & \longrightarrow & \mathrm{End}_{\mathcal{A}}(\mathcal{A}) & \\ & A & \longmapsto & A\Psi := \big((X \xrightarrow{u} Y) \mapsto (X \otimes A \xrightarrow{u\otimes A} Y \otimes A)\big) & \text{for } A \in \mathrm{Ob}(\mathcal{A}) \\ & (A \xrightarrow{a} A') & \longmapsto & a\Psi & \text{for } a \in \mathrm{Mor}(\mathcal{A}) \end{array}$$

with

$$a\Psi := \left(\begin{tikzcd} X \arrow[d, "u"] \\ Y \end{tikzcd} \longmapsto \begin{tikzcd} X \otimes A \arrow[r, "X \otimes a"] \arrow[d, "u \otimes A"'] & X \otimes A' \arrow[d, "u \otimes A'"] \\ Y \otimes A \arrow[r, "Y \otimes a"'] & Y \otimes A' \end{tikzcd} \right)\,.$$

Its inverse is given by the monoidal R-linear isofunctor

$$\begin{array}{rccll} \Psi^-\colon & \mathrm{End}_{\mathcal{A}}(\mathcal{A}) & \longrightarrow & \mathcal{A} & \\ & F & \longmapsto & IF & \text{for } F \in \mathrm{Ob}(\mathrm{End}_{\mathcal{A}}(\mathcal{A})) \\ & (F \xrightarrow{b} G) & \longmapsto & (IF \xrightarrow{Ib} IG) & \text{for } b \in \mathrm{Mor}(\mathrm{End}_{\mathcal{A}}(\mathcal{A}))\,. \end{array}$$

Proof. Consider the monoidal R-linear functor $\Theta_{\mathrm{id}_\mathcal{A}} \colon \mathcal{A} \to \mathrm{End}_R(\mathcal{A})$ from Definition 104.

$$\begin{array}{rcl} \Theta_{\mathrm{id}_\mathcal{A}} \colon & \mathcal{A} \; \to & \mathrm{End}_R(\mathcal{A})\,, \\ \left(A \xrightarrow{a} A'\right) \mapsto & & \left(\begin{array}{c} X \\ \downarrow u \\ Y \end{array} \;\mapsto\; \begin{array}{ccc} X \otimes A & \xrightarrow{X \otimes a} & X \otimes A' \\ {\scriptstyle u \otimes A}\downarrow & & \downarrow{\scriptstyle u \otimes A'} \\ Y \otimes A & \xrightarrow[Y \otimes a]{} & Y \otimes A' \end{array} \right) \end{array}$$

We show that $\Theta_{\mathrm{id}_\mathcal{A}}\big|^{\mathrm{End}_\mathcal{A}(\mathcal{A})}$ exists.

Suppose given $A \in \mathrm{Ob}(\mathcal{A})$. We have to show that $A\Theta_{\mathrm{id}_\mathcal{A}} \overset{!}{\in} \mathrm{Ob}(\mathrm{End}_\mathcal{A}(\mathcal{A}))$.

For a, $\tilde{a} \in \mathrm{Mor}(\mathcal{A})$, we have

$$(a \otimes \tilde{a})(A\Theta_{\mathrm{id}_\mathcal{A}}) = (a \otimes \tilde{a}) \otimes A = a \otimes (\tilde{a} \otimes A) = a \otimes \big(\tilde{a}(A\Theta_{\mathrm{id}_\mathcal{A}})\big)\,.$$

This shows $A\Theta_{\mathrm{id}_\mathcal{A}} \in \mathrm{Ob}(\mathrm{End}_\mathcal{A}(\mathcal{A}))$.

Suppose given $a \in \mathrm{Mor}(\mathcal{A})$. We have to show that $a\Theta_{\mathrm{id}_\mathcal{A}} \overset{!}{\in} \mathrm{Mor}(\mathrm{End}_\mathcal{A}(\mathcal{A}))$.

For X, $Y \in \mathrm{Ob}(\mathcal{A})$, we have

$$(X \otimes Y)(a\Theta_{\mathrm{id}_\mathcal{A}}) = (X \otimes Y) \otimes a = X \otimes (Y \otimes a) = X \otimes \big(Y(a\Theta_{\mathrm{id}_\mathcal{A}})\big)\,.$$

This shows $a\Theta_{\mathrm{id}_\mathcal{A}} \in \mathrm{Mor}(\mathrm{End}_\mathcal{A}(\mathcal{A}))$.

So let $\Psi := \Theta_{\mathrm{id}_\mathcal{A}}\big|^{\mathrm{End}_\mathcal{A}(\mathcal{A})}$.

Then $\Psi \colon \mathcal{A} \to \mathrm{End}_\mathcal{A}(\mathcal{A})$ is a monoidal R-linear functor.

We show that $\Psi' \colon \mathrm{End}_\mathcal{A}(\mathcal{A}) \to \mathcal{A}\,, \left(F \xrightarrow{b} G\right) \mapsto \left(IF \xrightarrow{Ib} IG\right)$ is a functor.

Suppose given $F \xrightarrow{b} G \xrightarrow{c} H$ in $\mathrm{End}_\mathcal{A}(\mathcal{A})$.

We have

$$\begin{aligned} b\Psi'\boldsymbol{s} &= (Ib)\boldsymbol{s} = IF = b\boldsymbol{s}\Psi' \\ F\Psi'\boldsymbol{i} &= IF\boldsymbol{i} = \mathrm{id}_{IF} = I\mathrm{id}_F = \mathrm{id}_F\Psi' = F\boldsymbol{i}\Psi' \\ b\Psi'\boldsymbol{t} &= (Ib)\boldsymbol{t} = IG = b\boldsymbol{t}\Psi'\,. \end{aligned}$$

We have

$$(b \blacktriangle c)\Psi' = I(b \blacktriangle c) = Ib \blacktriangle Ic = b\Psi' \blacktriangle c\Psi' \,.$$

So Ψ' is a functor.

We show that Ψ' is the inverse isofunctor of Ψ.

For $\big(A \xrightarrow{a} A'\big) \in \mathrm{Mor}(\mathcal{A})$, we have

$$a(\Psi * \Psi') = I(a\Psi) = I \otimes a = a \,.$$

This shows $\Psi * \Psi' = \mathrm{id}_{\mathcal{A}}$.

Conversely, suppose given $\big(F \xrightarrow{b} G\big) \in \mathrm{Mor}(\mathrm{End}_{\mathcal{A}}(\mathcal{A}))$. For $X \in \mathrm{Ob}(\mathcal{A})$, we have

$$X\big(b(\Psi' * \Psi)\big) = X\big((Ib)\Psi\big) = X \otimes (Ib) = (X \otimes I)b = Xb \,.$$

This shows $b(\Psi' * \Psi) = b$.

So $\Psi' * \Psi = \mathrm{id}_{\mathrm{End}_{\mathcal{A}}(\mathcal{A})}$.

Therefore, we have $\Psi' = \Psi^-$.

Then, by Lemma 34.(3), Ψ^- is a monoidal functor, and by Lemma 72, Ψ^- is an R-linear functor.

So Ψ^- is monoidal R-linear functor. □

8.3 Representations of a crossed module V

8.3.1 The monoidal R-linear category $(V\mathrm{Cat})R$

Remark 115 (The monoidal R-linear category $(V\mathrm{Cat})R$)

Suppose given a crossed module $V = \big(M, G, \gamma, f\big)$. Consider the invertible monoidal category $V\mathrm{Cat}$; cf. Remark 29. Recall that

$$\mathrm{Ob}(V\mathrm{Cat}) = G \,,\ \mathrm{Mor}(V\mathrm{Cat}) = G \ltimes M \,.$$

For $g \xrightarrow{(g,m)} g \cdot mf \xrightarrow{(g \cdot mf, m')} g \cdot (mm')f$ in $V\mathrm{Cat}$, their composite is given by

$$(g, m) \blacktriangle (g \cdot mf, m') = (g, mm') \,.$$

Consider the monoidal R-linear category $(V\mathrm{Cat})R$; cf. Lemma 85. Then

$$\mathrm{Ob}((V\mathrm{Cat})R) = \mathrm{Ob}(V\mathrm{Cat}) = G\,,$$

and for $g \in G$ and $m \in M$, the set of morphisms from g to $g \cdot mf$ in $(V\mathrm{Cat})R$ is given by

$${}_{(V\mathrm{Cat})R}(g, g\cdot mf) = \big({}_{V\mathrm{Cat}}(g, g\cdot mf)\big)R\,.$$

For $g \in G$ and $m, m' \in M$, note that

$$g\cdot mf = g\cdot m'f \;\Leftrightarrow\; mf = m'f \;\Leftrightarrow\; m(m')^- \in \ker f \;\Leftrightarrow\; \exists k' \in \ker f\colon m = m'\,k'\,.$$

So, for $g \in G$ and $m \in M$, the set of morphisms from g to $g\cdot mf$ is given by

$$\begin{aligned}{}_{(V\mathrm{Cat})R}(g, g\cdot mf) &= \Big\{\sum_i (g, m_i)r_i\colon r_i \in R,\ m_i \in M,\ \text{where } m_i f = mf\Big\}\\ &= \Big\{\sum_i (g, mk_i)r_i\colon r_i \in R,\ k_i \in \ker f\Big\}\,.\end{aligned}$$

Writing a morphism of $(V\mathrm{Cat})R$ in the form $\sum_i (g, m\,k_i)r_i\colon g \to g\cdot mf$, we implicitly suppose $k_i \in \ker f$, $r_i \in R$ with $i \in I$, where I is a finite set.

Example 116 Suppose given crossed modules $V := \big(M, G, \gamma, f\big)$ and $W := \big(N, H, \beta, \ell\big)$. Consider the invertible monoidal categories $V\mathrm{Cat}$ and $W\mathrm{Cat}$; cf. Remark 29.

Recall that

$$\begin{array}{llll}\mathrm{Ob}(V\mathrm{Cat}) & = G, & \mathrm{Ob}(W\mathrm{Cat}) & = H\\ \mathrm{Mor}(V\mathrm{Cat}) & = G \ltimes M, & \mathrm{Mor}(W\mathrm{Cat}) & = H \ltimes N\,.\end{array}$$

Consider the monoidal R-linear categories $\mathcal{A} := (V\mathrm{Cat})R$ and $\mathcal{B} := (W\mathrm{Cat})R$; cf. Lemma 85. Then

$$\begin{aligned}\mathrm{Ob}(\mathcal{A}) &= \mathrm{Ob}(V\mathrm{Cat}) = G\,,\\ \mathrm{Ob}(\mathcal{B}) &= \mathrm{Ob}(W\mathrm{Cat}) = H\,.\end{aligned}$$

For $g \in G$ and $m \in M$, the set of morphisms from g to $g\cdot mf$ is given by

$${}_{\mathcal{A}}(g, g\cdot mf) = \Big\{\sum_i (g, m_i)r_i\colon r_i \in R,\ m_i \in M,\ \text{where } m_i f = mf\Big\}\,.$$

Similarly, for $h \in H$ and $n \in N$, the set of morphisms from h to $h\cdot n\ell$ is given by

$${}_{\mathcal{B}}(h, h\cdot n\ell) = \Big\{\sum_j (h, n_j)r_j\colon r_j \in R,\ n_j \in N,\ \text{where } n_j\ell = n\ell\Big\}\,.$$

This situation now yields an example for the action tensor product.

(1) Suppose given a crossed module morphism $\rho := (\lambda, \mu)\colon V \to W$. Consider the monoidal R-linear functor $F := (\rho\,\mathrm{Cat})R\colon \mathcal{A} \to \mathcal{B}$; cf. Lemma 39, 86.

Consider the monoidal R-linear functor $\Theta_F\colon \mathcal{A} \to \mathrm{End}_R(\mathcal{B})$ from Lemma 103. Then $\mathcal{B}$ is an $\mathcal{A}$-module via

$$\Theta_F\colon \mathcal{A} \to \mathrm{End}_R(\mathcal{B}),\, \big(g \xrightarrow{(g,m)} g \cdot mf\big) \mapsto \big(g\Theta_F \xrightarrow{(g,m)\Theta_F} (g \cdot mf)\Theta_F\big),$$

where $\big(g\Theta_F \xrightarrow{(g,m)\Theta_F} (g \cdot mf)\Theta_F\big)$ maps a morphism $\left(\begin{array}{c} h \\ \downarrow{\scriptstyle (h,n)} \\ h\cdot n\ell \end{array}\right) \in \mathrm{Mor}(\mathcal{B})$ to the diagram morphism

$$\left(\begin{tikzcd} h \cdot gF \arrow[r, "hi \cdot (g,m)F"] \arrow[d, "(h,n)\cdot (gF)i"'] & h \cdot (g \cdot mf)F \arrow[d, "(h,n)\cdot \big((g\cdot mf)F\big)i"] \\ (h \cdot n\ell)\cdot gF \arrow[r, "(h\cdot n\ell)i \cdot (g,m)F"'] & (h \cdot n\ell)\cdot (g \cdot mf)F \end{tikzcd}\right)$$

$$= \left(\begin{tikzcd} h \cdot g\mu \arrow[r, "(h \cdot g\mu,\, m\lambda)"] \arrow[d, "(h \cdot g\mu,\, n^{g\mu})"'] & h \cdot (g \cdot mf)\mu \arrow[d, "\big(h \cdot (g\cdot mf)\mu,\, n^{(g\cdot mf)\mu}\big)"] \\ (h \cdot n\ell)\cdot g\mu \arrow[r, "(h\cdot n\ell \cdot g\mu,\, m\lambda)"'] & (h \cdot n\ell)\cdot (g \cdot mf)\mu \end{tikzcd}\right);$$

cf. Lemma 108.

Suppose given $h \in \mathrm{Ob}(\mathcal{B})$ and $\big(h \xrightarrow{(h,n)} h \cdot n\ell\big) \in \mathrm{Mor}(\mathcal{B})$. Suppose given $g \in \mathrm{Ob}(\mathcal{A})$ and $\big(g \xrightarrow{(g,m)} g \cdot mf\big) \in \mathrm{Mor}(\mathcal{A})$.

The action tensor product of $\mathcal{A}$ on $\mathcal{B}$ works as follows; cf. Definition 100.

$$\begin{array}{rcl} h \otimes g & = & h(g\Theta_F) = h \cdot g\mu \\ (h,n) \otimes g & = & (h,n)(g\Theta_F) = (h,n)\cdot (g\mu)i = (h,n)\cdot (g\mu,1) = (h \cdot g\mu, n^{g\mu}) \\ h \otimes (g,m) & = & h\big((g,m)\Theta_F\big) = (hi)\cdot (g,m)F = (h,1)\cdot (g\mu, m\lambda) = (h \cdot g\mu, m\lambda) \end{array}$$

$$
\begin{aligned}
(h,n)\otimes(g,m) &= \big((h,n)(g\Theta_F)\big) \blacktriangle \Big((h\cdot n\ell)\big((g,m)\Theta_F\big)\Big) \\
&= \big((h,n)\cdot(g\mu,1)\big) \blacktriangle \big((h\cdot n\ell,1)\cdot(g\mu,m\lambda)\big) \\
&= (h\cdot g\mu, n^{g\mu}) \blacktriangle (h\cdot n\ell\cdot g\mu, m\lambda) \\
&= (h\cdot g\mu, n^{g\mu}\cdot m\lambda) = (h,n)\cdot(g\mu,m\lambda) \\
&= (h,n)\cdot(g,m)F\,.
\end{aligned}
$$

(2) Consider in particular the identity crossed module morphism

$$\mathrm{id}_V := (\mathrm{id}_M, \mathrm{id}_G)\colon V \to V\,.$$

Write $\mathcal{A} := (V\mathrm{Cat})R$. Consider the monoidal R-linear functor

$$\mathrm{id}_{\mathcal{A}} = \mathrm{id}_{(V\,\mathrm{Cat})R} = \big((\mathrm{id}_V)\mathrm{Cat}\big)R\colon \mathcal{A}\to\mathcal{A}\,;$$

cf. Lemma 86.

Consider the monoidal R-linear functor $\Theta := \Theta_{\mathrm{id}_{\mathcal{A}}}\colon \mathcal{A}\to\mathrm{End}_R(\mathcal{A})$ from Lemma 103. Then $\mathcal{A}$ is a regular $\mathcal{A}$-module via

$$\Theta\colon \mathcal{A}\to\mathrm{End}_R(\mathcal{A})\,,\big(g \xrightarrow{(g,m)} g\cdot mf\big)\mapsto \big(g\Theta \xrightarrow{(g,m)\Theta} (g\cdot mf)\Theta\big)\,,$$

where $\big(g\Theta \xrightarrow{(g,m)\Theta} (g\cdot mf)\Theta\big)$ maps a morphism $\left(\begin{smallmatrix} h \\ \downarrow{\scriptstyle (h,n)} \\ h\cdot nf\end{smallmatrix}\right)\in\mathrm{Mor}(\mathcal{A})$ to the diagram morphism

$$
\left(\begin{tikzcd}
h\cdot g \arrow[r, "hi\cdot(g,m)"] \arrow[d, "(h,n)\cdot gi"'] & h\cdot(g\cdot mf) \arrow[d, "(h,n)\cdot(g\cdot mf)i"] \\
(h\cdot nf)\cdot g \arrow[r, "(h\cdot nf)i\cdot(g,m)"'] & (h\cdot nf)\cdot(g\cdot mf)
\end{tikzcd}\right)
$$

$$
= \left(\begin{tikzcd}
h\cdot g \arrow[r, "(h\cdot g,m)"] \arrow[d, "(h\cdot g,n^g)"'] & h\cdot(g\cdot mf) \arrow[d, "\big(h\cdot(g\cdot mf),n^{(g\cdot mf)}\big)"] \\
(h\cdot nf)\cdot g \arrow[r, "(h\cdot nf\cdot g,m)"'] & (h\cdot nf)\cdot(g\cdot mf)
\end{tikzcd}\right);
$$

cf. (1) and Definition 104.

Suppose given $h \in \mathrm{Ob}(\mathcal{A})$ and $\big(h \xrightarrow{(h,n)} h \cdot nf\big) \in \mathrm{Mor}(\mathcal{A})$. Suppose given $g \in \mathrm{Ob}(\mathcal{A})$ and $\big(g \xrightarrow{(g,m)} g \cdot mf\big) \in \mathrm{Mor}(\mathcal{A})$.

The action tensor product of $\mathcal{A}$ on $\mathcal{A}$ works as follows; cf. Definition 100.

$$\begin{array}{rcl}
h \otimes g & = & h(g\Theta) = h \cdot g \\
(h,n) \otimes g & = & (h,n)(g\Theta) = (h,n) \cdot gi = (h,n) \cdot (g,1) = (h \cdot g, n^g) \\
h \otimes (g,m) & = & h\big((g,m)\Theta\big) = (hi) \cdot (g,m) = (h,1) \cdot (g,m) = (h \cdot g, m) \\
(h,n) \otimes (g,m) & = & \big((h,n)(g\Theta)\big) \blacktriangle \Big((h \cdot nf)\big((g,m)\Theta\big)\Big) \\
& = & \big((h,n) \cdot (g,1)\big) \blacktriangle \big((h \cdot nf, 1) \cdot (g,m)\big) \\
& = & (h \cdot g, n^g) \blacktriangle (h \cdot nf \cdot g, m) \\
& = & (h \cdot g, n^g \cdot m) \\
& = & (h,n) \cdot (g,m)\,.
\end{array}$$

Example 117 Suppose given crossed modules

$$V := \big(M, G, \gamma, f\big)\,,\ V' := \big(M', G', \gamma', f'\big) \text{ and } V'' := \big(M'', G'', \gamma'', f''\big)\,.$$

Suppose given crossed modules morphisms

$$\rho := (\lambda, \mu)\colon V \to V' \text{ and } \rho := (\lambda', \mu')\colon V' \to V''\,.$$

Consider the monoidal R-linear categories

$$\mathcal{C} := (V\mathrm{Cat})R\,,\ \mathcal{C}' := (V'\mathrm{Cat})R \text{ and } \mathcal{C}'' := (V''\mathrm{Cat})R\,;$$

cf. Remark 115.

Consider the monoidal R-linear functors

$$F := (\rho\,\mathrm{Cat})R\colon \mathcal{C} \to \mathcal{C}' \text{ and } F' := (\rho'\,\mathrm{Cat})R\colon \mathcal{C}' \to \mathcal{C}''\,;$$

cf. Lemma 86.

So we have

$$\mathcal{C} \xrightarrow{F} \mathcal{C}' \xrightarrow{F'} \mathcal{C}''\,.$$

Then, by Lemma 107, F' is an $\mathcal{C}$-linear functor.

In fact, for $\big(g' \xrightarrow{(g',m')} g' \cdot m'f\big) \in \mathrm{Mor}(\mathcal{C}')$ and $\big(g \xrightarrow{(g,m)} g \cdot mf\big) \in \mathrm{Mor}(\mathcal{C})$, we have

$$\begin{array}{rcl}
\big((g'm') \otimes (g,m)\big)F' & \overset{116.(1)}{=} & \big(g' \cdot g\mu, (m')^{g\mu} \cdot m\lambda\big)F' = \Big((g' \cdot g\mu)\mu', \big((m')^{g\mu} \cdot m\lambda\big)\lambda'\Big) \\
& = & \big(g'\mu' \cdot g\mu\mu', (m'\lambda')^{g\mu\mu'} \cdot m\lambda\lambda'\big) \\
& = & (g'\mu', m'\lambda') \cdot (g\mu\mu', m\lambda\lambda') = (g',m')F' \otimes (g,m)\,.
\end{array}$$

8.3.2 Representations of V and modules over $(V\mathrm{Cat})R$

Let $V = \big(M, G, \gamma, f\big)$ be a crossed module. Let $\mathcal{M} = (\mathcal{M}, \varepsilon)$ be an R-linear category.

Reminder 118 Consider the functor $\mathrm{Cat}\colon \mathcal{CRMod} \to \mathcal{InvMonCat}$ from the category of crossed modules to the category of invertible monoidal categories; cf. Lemma 39.

Consider the functor $\mathrm{CM}\colon \mathcal{InvMonCat} \to \mathcal{CRMod}$ from the category of invertible monoidal categories to the category of crossed modules; cf. Lemma 42.

Then we have the crossed module isomorphism $(\pi_M, \mathrm{id}_G)\colon V\mathrm{Cat}\,\mathrm{CM} \underset{\sim}{\longrightarrow} V$, where

$$\pi_M\colon 1 \ltimes M \underset{\sim}{\longrightarrow} M\,,\ (1,m) \mapsto m\,;$$

cf. Proposition 43.(1).

Note that $(\pi_M, \mathrm{id}_G)^- = (\iota_M, \mathrm{id}_G)\colon V \to V\mathrm{Cat}\,\mathrm{CM}$ with

$$\iota_M\colon M \to 1 \ltimes M\,,\ m \mapsto (1,m)\,.$$

Remark 119 (The invertible monoidal category $\big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\mathrm{Cat}$)

Recall that the automorphism crossed module $\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M}) = \big(\mathrm{M}_R^{\mathcal{M}}, \mathrm{G}_{\mathcal{M}}^R, \gamma_{\mathcal{M}}^R, \mathrm{f}_{\mathcal{M}}^R\big)$ of $\mathcal{M}$ is given as follows; cf. Lemma 81.

$$\begin{aligned}
\mathrm{G}_{\mathcal{M}}^R &= \{\mathcal{M} \xrightarrow{G} \mathcal{M}\colon G \text{ is an } R\text{-linear autofunctor}\}\\
\mathrm{M}_{\mathcal{M}}^R &= \{(\mathrm{id}_{\mathcal{M}} \xrightarrow{a} F)\colon F \in \mathrm{G}_{\mathcal{M}}^R \text{ and } a \text{ is an isotransformation}\}\\
\mathrm{f}_{\mathcal{M}}^R\colon\ \mathrm{M}_{\mathcal{M}}^R &\to \mathrm{G}_{\mathcal{M}}^R\,,\ \big(\mathrm{id}_{\mathcal{M}} \xrightarrow{a} F\big) \mapsto F\\
\gamma_{\mathcal{M}}^R\colon\ \mathrm{G}_{\mathcal{M}}^R &\to \mathrm{Aut}(\mathrm{M}_{\mathcal{M}}^R)\,,\ G \mapsto \Big(\big(\mathrm{id}_{\mathcal{M}} \xrightarrow{a} F\big) \mapsto \big(\mathrm{id}_{\mathcal{M}} \xrightarrow{G^-aG} G^-FG\big)\Big)
\end{aligned}$$

Consider the invertible monoidal category $\big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\mathrm{Cat}$. Recall from Remark 97.(1) that

$$\begin{aligned}
\mathrm{Ob}\Big(\big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\,\mathrm{Cat}\Big) &= \mathrm{G}_{\mathcal{M}}^R\\
\mathrm{Mor}\Big(\big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\,\mathrm{Cat}\Big) &= \mathrm{G}_{\mathcal{M}}^R \ltimes \mathrm{M}_{\mathcal{M}}^R\,,
\end{aligned}$$

and that

$$\begin{array}{rlcl}
\boldsymbol{s}\colon & (\mathrm{G}_{\mathcal{M}}^R \ltimes \mathrm{M}_{\mathcal{M}}^R) \to \mathrm{G}_{\mathcal{M}}^R\,, & (G, \mathrm{id}_{\mathcal{M}} \xrightarrow{a} F) & \mapsto & G\\
\boldsymbol{i}\colon & (\mathrm{G}_{\mathcal{M}}^R \ltimes \mathrm{M}_{\mathcal{M}}^R) \leftarrow \mathrm{G}_{\mathcal{M}}^R\,, & (G, \mathrm{id}_{\mathcal{M}} \xrightarrow{\mathrm{id}_{\mathrm{id}_{\mathcal{M}}}} \mathrm{id}_{\mathcal{M}}) & \leftarrow\!\shortmid & G\\
\boldsymbol{t}\colon & (\mathrm{G}_{\mathcal{M}}^R \ltimes \mathrm{M}_{\mathcal{M}}^R) \to \mathrm{G}_{\mathcal{M}}^R\,, & (G, \mathrm{id}_{\mathcal{M}} \xrightarrow{a} F) & \mapsto & GF\,.
\end{array}$$

Composition in $\big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\mathrm{Cat}$ is given by

$$(G, \mathrm{id}_{\mathcal{M}} \xrightarrow{a} F) \blacktriangle (GF, \mathrm{id}_{\mathcal{M}} \xrightarrow{a'} F') = (G, \mathrm{id}_{\mathcal{M}} \xrightarrow{a * a'} FF'),$$

where $G, F, F' \in \mathrm{G}_{\mathcal{M}}^R$ and where a and a' are isotransformations.

Definition 120 (Representation of a crossed module)

A crossed module morphism $\rho\colon V \to \mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})$ is called a *representation of* V *on* $\mathcal{M}$.

Lemma 121 *Suppose given a representation* $\rho =: (\lambda, \mu)\colon V \to \mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})$.

Consider the monoidal R-linear categories $(V\mathrm{Cat})R$ *and* $\mathrm{End}_R(\mathcal{M})$*; cf. Remark* 115 *and Lemma* 80.

Then we have a monoidal R-linear functor given by

$$\begin{array}{rrcll} \hat{\Phi}_\rho\colon & (V\mathrm{Cat})R & \to & \mathrm{End}_R(\mathcal{M}) & \\ & g & \mapsto & g\hat{\Phi}_\rho := g\mu & \textit{for } g \in \mathrm{Ob}(V\mathrm{Cat}) \\ & z := \sum\limits_i (g, mk_i)r_i & \mapsto & z\hat{\Phi}_\rho & \textit{for } z \in \mathrm{Mor}(V\mathrm{Cat}) \end{array}$$

with

$$z\hat{\Phi}_\rho := \left(\begin{tikzcd} X \arrow[d, "u"] \\ Y \end{tikzcd} \longmapsto \begin{tikzcd}[column sep=huge] X(g\mu) \arrow[r, "\sum_i (X(g\mu))((mk_i)\lambda)r_i"] \arrow[d, "u(g\mu)"'] & X\big((g \cdot mf)\mu\big) \arrow[d, "u((g \cdot mf)\mu)"] \\ Y(g\mu) \arrow[r, "\sum_i (Y(g\mu))((mk_i)\lambda)r_i"'] & Y\big((g \cdot mf)\mu\big) \end{tikzcd} \right).$$

So altogether, we obtain a $(V\mathrm{Cat})R$*-module* $(\mathcal{M}, \hat{\Phi}_\rho)$*; cf. Definition* 100.

Proof. By applying the functor $\mathrm{Cat}\colon \mathcal{CRMod} \to \mathcal{InvMonCat}$ to the crossed module morphism ρ, we obtain the monoidal functor

$$\begin{array}{rrcll} \rho\,\mathrm{Cat}\colon & V\mathrm{Cat} & \to & \big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\mathrm{Cat} & \\ & g & \mapsto & g(\rho\,\mathrm{Cat}) = g\mu & \text{for } g \in \mathrm{Ob}(V\mathrm{Cat}) \\ & (g, m) & \mapsto & (g, m)(\rho\,\mathrm{Cat}) = (g\mu, \mathrm{id}_{\mathcal{M}} \xrightarrow{m\lambda} mf\mu) & \text{for } (g, m) \in \mathrm{Mor}(V\mathrm{Cat})\,; \end{array}$$

cf. Lemma 39.(1).

Consider the monoidal isofunctor

$$\begin{array}{rrcll} \mathrm{Real}_{\mathcal{M}}\colon & \big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\mathrm{Cat} & \xrightarrow{\sim} & \big(\mathrm{End}_R(\mathcal{M})\big)\mathrm{U} & \\ & F & \mapsto & F\,\mathrm{Real}_{\mathcal{M}} := F & \text{for } F \in \mathrm{G}_{\mathcal{M}}^R \\ & (F, \mathrm{id}_{\mathcal{M}} \xrightarrow{a} H) & \mapsto & (F,a)\,\mathrm{Real}_{\mathcal{M}} & \text{for } (F,a) \in \mathrm{G}_{\mathcal{M}}^R \ltimes \mathrm{M}_{\mathcal{M}}^R \end{array}$$

with

$$(F,a)\,\mathrm{Real}_{\mathcal{M}} = Fa = \left(\begin{array}{c} X \\ \downarrow u \\ Y \end{array} \longmapsto \begin{tikzcd} XF \arrow[r, "XFa"] \arrow[d, "uF"'] & XFH \arrow[d, "uFH"] \\ YF \arrow[r, "YFa"'] & YFH \end{tikzcd} \right);$$

cf. Theorem 99.

So we have the composite monoidal functor $\Phi_\rho := \rho\,\mathrm{Cat} * \mathrm{Real}_{\mathcal{M}}$

$$\begin{array}{rrcll} \Phi_\rho\colon & V\mathrm{Cat} & \to & \big(\mathrm{End}_R(\mathcal{M})\big)\mathrm{U} & \\ & g & \mapsto & g\Phi_\rho = g\mu & \text{for } g \in \mathrm{Ob}(V\mathrm{Cat}) \\ & (g,m) & \mapsto & (g,m)\Phi_\rho & \text{for } (g,m) \in \mathrm{Mor}(V\mathrm{Cat}) \end{array}$$

with

$$(g,m)\Phi_\rho = \left(\begin{array}{c} X \\ \downarrow u \\ Y \end{array} \longmapsto \begin{tikzcd} X(g\mu) \arrow[r, "X\big((g\mu)(m\lambda)\big)"] \arrow[d, "u(g\mu)"'] & X\big((g\cdot mf)\mu\big) \arrow[d, "u\big((g\cdot mf)\mu\big)"] \\ Y(g\mu) \arrow[r, "Y\big((g\mu)(m\lambda)\big)"'] & Y\big((g\cdot mf)\mu\big) \end{tikzcd} \right).$$

Then, by Lemma 95.(1), we have a monoidal R-linear functor

$$\begin{array}{rrcll} \hat{\Phi}_\rho\colon & (V\mathrm{Cat})R & \to & \mathrm{End}_R(\mathcal{M}) & \\ & g & \mapsto & g\hat{\Phi}_\rho = g\Phi_\rho = g\mu & \text{for } g \in \mathrm{Ob}(V\mathrm{Cat}) \\ & z := \sum_i (g, mk_i) r_i & \mapsto & z\hat{\Phi}_\rho = \sum_i (g, mk_i)\Phi_\rho\, r_i & \text{for } z \in \mathrm{Mor}(V\mathrm{Cat}) \end{array}$$

with

$$
z\hat{\Phi}_\rho = \left(\begin{array}{c} X \\ \Big\downarrow u \\ Y \end{array} \quad \longmapsto \quad \begin{tikzcd} X(g\mu) \arrow[r, "\sum_i \left(X(g\mu)\right)\left((mk_i)\lambda\right)r_i"] \arrow[d, "u(g\mu)"'] & X\left((g\cdot mf)\mu\right) \arrow[d, "u\left((g\cdot mf)\mu\right)"] \\ Y(g\mu) \arrow[r, "\sum_i \left(Y(g\mu)\right)\left((mk_i)\lambda\right)r_i"'] & Y\left((g\cdot mf)\mu\right) \end{tikzcd}\right) .
$$

□

Lemma 122 *Consider the invertible monoidal category* $V\mathrm{Cat}$*; cf. Remark* 29*.*

Suppose given a monoidal R*-linear functor* $\Phi\colon (V\mathrm{Cat})R \to \mathrm{End}_R(\mathcal{M})$ *, i.e. we have a* $(V\mathrm{Cat})R$*-module* $(\mathcal{M}, \Phi)$ *; cf. Definition* 100*.*

Then we have a representation $\rho_\Phi = (\lambda_\Phi, \mu_\Phi)\colon V \to \mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})$ *of* V *on* $\mathcal{M}$*, where*

$$
\begin{aligned}
\lambda_\Phi &\colon M \to \mathrm{M}_{\mathcal{M}}^R\,,\ m \mapsto (1,m)\Phi \\
\mu_\Phi &\colon G \to \mathrm{G}_{\mathcal{M}}^R\,,\ g \mapsto g\Phi\,.
\end{aligned}
$$

Proof. By Lemma 95.(2), we have the monoidal functor

$$
\begin{array}{rrcll}
\check{\Phi}\colon & V\mathrm{Cat} & \longrightarrow & \big(\mathrm{End}_R(\mathcal{M})\big)\mathrm{U} = \mathrm{Aut}_R(\mathcal{M}) & \\
& g & \longmapsto & g\check{\Phi} := g\Phi & \text{for } g \in \mathrm{Ob}(V\mathrm{Cat}) \\
& (g,m) & \longmapsto & (g,m)\check{\Phi} := (g,m)\Phi & \text{for } (g,m) \in \mathrm{Mor}(V\mathrm{Cat})\ .
\end{array}
$$

By applying the functor $\mathrm{CM}\colon \mathit{InvMonCat} \to \mathcal{CRMod}$ to $\check{\Phi}$, we obtain the crossed module morphism

$$
(\lambda, \mu) := \check{\Phi}\,\mathrm{CM}\colon V\mathrm{Cat}\,\mathrm{CM} \to \big(\mathrm{Aut}_R(\mathcal{M})\big)\,\mathrm{CM} \overset{98}{=} \mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\,,
$$

where

$$
\lambda\colon 1 \ltimes M \to \mathrm{M}_{\mathcal{M}}^R\,,\ (1,m) \mapsto \big(\mathrm{id}_{\mathcal{M}} \xrightarrow{(1,m)\Phi} mf\Phi\big)\,,
$$

and where

$$
\mu\colon G \to \mathrm{G}_{\mathcal{M}}^R\,,\ g \mapsto g\Phi\,;
$$

cf. Lemma 39.

Consider the crossed module isomorphism

$$(\iota_{\mathcal{M}}, \mathrm{id}_G)\colon V \xrightarrow[\sim]{} V\mathrm{Cat}\,\mathrm{CM}\,;$$

cf. Reminder 118.

Then we obtain the crossed module morphism

$$(\lambda_\Phi, \mu_\Phi) := (\iota_{\mathcal{M}}, \mathrm{id}_G) \blacktriangle \check{\Phi}\,\mathrm{CM}\,,\colon V \to \mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})$$

with

$$\lambda_\Phi\colon M \to \mathrm{M}_{\mathcal{M}}^R\,,\ m \mapsto m\lambda_\Phi = (m)(\iota_M \blacktriangle \lambda) = (1,m)(\lambda) = (1,m)\Phi$$

and with

$$\mu_\Phi\colon G \to \mathrm{G}_{\mathcal{M}}^R\,,\ g \mapsto g\mu_\Phi = (g)(\mathrm{id}_G \blacktriangle \mu) = g\Phi$$

as desired. □

Lemma 123

(1) *Suppose given a representation* $\rho\colon V \to \mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})$.

By Lemma 121*, we have the monoidal* R*-linear functor* $\hat{\Phi}_\rho\colon (V\mathrm{Cat})R \to \mathrm{End}_R(\mathcal{M})$.

In turn, by Lemma 122*, we obtain the representation*

$$\rho_{\hat{\Phi}_\rho}\colon V \to \mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\,.$$

Then

$$\rho_{\hat{\Phi}_\rho} = \rho\,.$$

(2) *Suppose given a monoidal* R*-linear functor* $\Phi\colon (V\mathrm{Cat})R \to \mathrm{End}_R(\mathcal{M})$*, i.e. we have the* $(V\mathrm{Cat})R$*-module* $(\mathcal{M}, \Phi)$.

By Lemma 122*, we have the representation* $\rho_\Phi\colon V \to \mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})$.

In turn, by Lemma 121*, we obtain a monoidal* R*-linear functor*

$$\hat{\Phi}_{\rho_\Phi}\colon (V\mathrm{Cat})R \to \mathrm{End}_R(\mathcal{M})\,.$$

Then

$$\hat{\Phi}_{\rho_\Phi} = \Phi\,.$$

Proof. Ad (1). We write $(\lambda_{\hat{\Phi}_\rho}, \mu_{\hat{\Phi}_\rho}) := \rho_{\hat{\Phi}_\rho}$ and we write $(\lambda, \mu) := \rho$.

For $m \in M$, we have

$$m\lambda_{\hat{\Phi}_\rho} \overset{122}{=} (1, m)\hat{\Phi}_\rho \overset{121}{=} m\lambda\,.$$

For $g \in G$, we have

$$g\mu_{\hat{\Phi}_\rho} \overset{122}{=} g\hat{\Phi}_\rho \overset{121}{=} g\mu\,.$$

This shows $\rho_{\hat{\Phi}_\rho} = (\lambda_{\hat{\Phi}_\rho}, \mu_{\hat{\Phi}_\rho}) = (\lambda, \mu) = \rho$.

Ad (2). We write $(\lambda_\Phi, \mu_\Phi) := \rho_\Phi$.

For $\sum\limits_i (g, mk_i)r_i \in \mathrm{Mor}\big((V\mathrm{Cat})R\big)$, we have

$$\begin{aligned}
(g, mk_i)\Phi &= \big((g,1)\cdot(1, mk_i)\big)\Phi \overset{\Phi \text{ monoidal}}{=} (g,1)\Phi * (1, mk_i)\Phi \\
&= (g\Phi)\big((1, mk_i)\Phi\big) \blacktriangle \big((g,1)\Phi\big)\big((mf)\Phi\big) = (g\Phi)\big((1, mk_i)\Phi\big) \blacktriangle (\mathrm{id}_g\Phi)(mf)\Phi \\
&= (g\Phi)(1, mk_i)\Phi\,,
\end{aligned}$$

and thus

$$\begin{aligned}
\Big(\sum_i (g, mk_i)r_i\Big)\hat{\Phi}_{\rho_\Phi} &\overset{121}{=} \sum_i (g\mu_\Phi)\big((mk_i)\lambda_\Phi\big)r_i \overset{122}{=} \sum_i (g\Phi)\big((1, mk_i)\Phi\big)r_i = \sum_i \big((g, mk_i)\Phi\big)r_i \\
&= \Big(\sum_i (g, mk_i)r_i\Big)\Phi\,.
\end{aligned}$$

This shows $\hat{\Phi}_\rho = \Phi$. □

8.3.3 Permutation modules

Let $V = \big(M, G, \gamma, f\big)$ be a crossed module.

Let $\mathcal{X} = \big(\mathrm{Mor}(\mathcal{X}), \mathrm{Ob}(\mathcal{X}), (\boldsymbol{s}, \boldsymbol{i}, \boldsymbol{t}), \blacktriangle\big)$ be a category. Consider the R-linear category $\mathcal{X}R$; cf. Definition 82. Recall that

$$\begin{aligned}
\mathrm{Ob}(\mathcal{X}R) &= \mathrm{Ob}(\mathcal{X}) \\
{}_{\mathcal{X}R}(X, Y) &= \big({}_{\mathcal{X}}(X, Y)\big)R \quad \text{ for } \; X, Y \in \mathrm{Ob}(\mathcal{X}R)\,.
\end{aligned}$$

Consider the symmetric crossed module $\mathrm{S}_{\mathcal{X}} = \big(\mathrm{M}_{\mathcal{X}}, \mathrm{G}_{\mathcal{X}}, \gamma_{\mathcal{X}}, \mathrm{f}_{\mathcal{X}}\big)$ on $\mathcal{X}$ and the symmetric crossed module $\mathrm{S}_{\mathcal{X}R} = \big(\mathrm{M}_{\mathcal{X}R}, \mathrm{G}_{\mathcal{X}R}, \gamma_{\mathcal{X}R}, \mathrm{f}_{\mathcal{X}R}\big)$ on $\mathcal{X}R$; cf. Lemma 48.

Recall that

$$\begin{aligned}
\mathrm{G}_{\mathcal{X}} &= \mathrm{Aut}(\mathcal{X}) = \{\mathcal{X} \xrightarrow{F} \mathcal{X} \colon F \text{ is an autofunctor}\} \\
\mathrm{M}_{\mathcal{X}} &= \{\big(\mathrm{id}_{\mathcal{X}} \xrightarrow{a} F\big) \colon F \in \mathrm{G}_{\mathcal{X}} \text{ and } a \text{ is an isotransformation}\} \\
\mathrm{G}_{\mathcal{X}R} &= \mathrm{Aut}(\mathcal{X}R) = \{\mathcal{X}R \xrightarrow{F'} \mathcal{X}R \colon F' \text{ is an autofunctor}\} \\
\mathrm{M}_{\mathcal{X}R} &= \{\big(\mathrm{id}_{\mathcal{X}R} \xrightarrow{a'} F'\big) \colon F' \in \mathrm{G}_{\mathcal{X}R} \text{ and } a' \text{ is an isotransformation}\}\,.
\end{aligned}$$

The symmetric crossed module $\mathrm{S}_{\mathcal{X}R}$ has the crossed submodule

$$\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{X}R) = \big(\,\mathrm{M}_{\mathcal{X}R}^R, \mathrm{G}_{\mathcal{X}R}^R, \gamma_{\mathcal{X}R}^R, \mathrm{f}_{\mathcal{X}R}^R\,\big) \leqslant \mathrm{S}_{\mathcal{X}R}\ .$$

Recall that

$$\begin{aligned}
\mathrm{G}_{\mathcal{X}R}^R &= \{\mathcal{X}R \xrightarrow{F} \mathcal{X}R \colon F \text{ is an } R\text{-linear autofunctor}\} \\
\mathrm{M}_{\mathcal{X}R}^R &= \{\big(\mathrm{id}_{\mathcal{X}R} \xrightarrow{a} F\big) \colon F \in \mathrm{G}_{\mathcal{X}R}^R \text{ and } a \text{ is an isotransformation}\};
\end{aligned}$$

cf. Lemma 81.

Lemma 124

(1) *We have a crossed module morphism*

$$\rho_{\mathcal{X},R} := (\lambda_{\mathcal{X},R}, \mu_{\mathcal{X},R}) \colon\ \mathrm{S}_{\mathcal{X}} \to \mathrm{S}_{\mathcal{X}R}$$

with

$$\begin{aligned}
\mu_{\mathcal{X},R} \colon\ \mathrm{G}_{\mathcal{X}} \to \mathrm{G}_{\mathcal{X}R}\,, &\ \big(\mathcal{X} \xrightarrow{F} \mathcal{X}\big) \mapsto \big(\mathcal{X}R \xrightarrow{FR} \mathcal{X}R\big) \\
\lambda_{\mathcal{X},R} \colon\ \mathrm{M}_{\mathcal{X}} \to \mathrm{M}_{\mathcal{X}R}\,, &\ \big(\mathrm{id}_{\mathcal{X}} \xrightarrow{a} F\big) \mapsto \big(\mathrm{id}_{\mathcal{X}R} \xrightarrow{aR} FR\big)\,;
\end{aligned}$$

cf. Lemma 86 *and Lemma* 87.

(2) *We have* $\mathrm{im}(\rho_{\mathcal{X},R}) \leqslant \mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{X}R)$, *i.e. we have*

$$\mathrm{im}(\lambda_{\mathcal{X},R}) \leqslant \mathrm{G}_{\mathcal{X}R}^R \quad \textit{and} \quad \mathrm{im}(\mu_{\mathcal{X},R}) \leqslant \mathrm{M}_{\mathcal{X}R}^R\ .$$

So we have group morphisms

$$\begin{aligned}
\check{\mu}_{\mathcal{X},R} := \mu_{\mathcal{X},R}\big|^{\mathrm{G}_{\mathcal{X}R}^R} \colon\ \mathrm{G}_{\mathcal{X}} \to \mathrm{G}_{\mathcal{X}R}^R\,, &\ \big(\mathcal{X} \xrightarrow{F} \mathcal{X}\big) \mapsto \big(\mathcal{X}R \xrightarrow{FR} \mathcal{X}R\big) \\
\check{\lambda}_{\mathcal{X},R} := \lambda_{\mathcal{X},R}\big|^{\mathrm{M}_{\mathcal{X},R}^R} \colon\ \mathrm{M}_{\mathcal{X}} \to \mathrm{M}_{\mathcal{X}R}^R\,, &\ \big(\mathrm{id}_{\mathcal{X}} \xrightarrow{a} F\big) \mapsto \big(\mathrm{id}_{\mathcal{X}R} \xrightarrow{aR} FR\big)\,.
\end{aligned}$$

We obtain a crossed module morphism

$$\check{\rho}_{\mathcal{X},R} := (\check{\lambda}_{\mathcal{X},R}, \check{\mu}_{\mathcal{X}R}) \colon\ \mathrm{S}_{\mathcal{X}} \to \mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{X}R)\,;$$

cf. [**15,** Lem. 25.(2), Rem. 20, 19].

Proof. Ad (1). We write $\mu_R := \mu_{\mathcal{X},R}$ and $\lambda_R := \lambda_{\mathcal{X},R}$.

We show that μ_R and λ_R are group morphisms.

For $F, G \in \mathrm{G}_{\mathcal{X}}$, we have

$$(F * G)\mu_R = (F * G)R = FR * GR = F\mu_R * G\mu_R\,;$$

cf. Lemma 88.(2).

So, μ_R is a group morphism.

For $a, b \in \mathrm{M}_{\mathcal{X}}$, we have

$$(a * b)\lambda_R = (a * b)R = aR * bR = a\lambda_R * b\lambda_R\,;$$

cf. Lemma 88.(5).

So, λ_R is a group morphism.

We show that (λ_R, μ_R) is a crossed module morphism.

Suppose given $\big(\mathrm{id}_{\mathcal{X}} \xrightarrow{a} F\big) \in \mathrm{M}_{\mathcal{X}}$ and $G \in \mathrm{G}_{\mathcal{X}}$.

We have

$$\begin{aligned} (a^G)\lambda_R &= \big(\mathrm{id}_{\mathcal{X}} \xrightarrow{G^- aG} G^- FG\big)\lambda_R = \big(\mathrm{id}_{\mathcal{X}} \xrightarrow{(G^- aG)R} (G^- FG)R\big) \\ &= \big(\mathrm{id}_{\mathcal{X}} \xrightarrow{(G^- R)(aR)(GR)} (G^- R)(FR)(GR)\big) \\ &= \big(\mathrm{id}_{\mathcal{X}} \xrightarrow{(GR)^-\,(aR)(GR)} (GR)^-\,(FR)(GR)\big) \\ &= \big(\mathrm{id}_{\mathcal{X}} \xrightarrow{aR} FR\big)^{GR} = (aR)^{GR} = (a\lambda_R)^{G\mu_R}\,; \end{aligned}$$

cf. Lemma 88.

We have

$$\begin{aligned} (a)\lambda_R\,\mathrm{f}_{\mathcal{X}R} &= \big(\mathrm{id}_{\mathcal{X}} \xrightarrow{a} F\big)\lambda_R\,\mathrm{f}_{\mathcal{X}R} = \big(\mathrm{id}_{\mathcal{X}R} \xrightarrow{aR} FR\big)\,\mathrm{f}_{\mathcal{X}R} = FR = (F)\mu_R \\ &= \big(\mathrm{id}_{\mathcal{X}} \xrightarrow{a} F\big)\,\mathrm{f}_{\mathcal{X}}\,\mu_R = (a)\,\mathrm{f}_{\mathcal{X}}\,\mu_R\,. \end{aligned}$$

$$\begin{tikzcd} \mathrm{M}_{\mathcal{X}} \arrow[r, "\lambda_R"] \arrow[d, "\mathrm{f}_{\mathcal{X}}"'] & \mathrm{M}_{\mathcal{X}R} \arrow[d, "\mathrm{f}_{\mathcal{X}R}"] \\ \mathrm{G}_{\mathcal{X}} \arrow[r, "\mu_R"'] & \mathrm{G}_{\mathcal{X}R} \end{tikzcd}$$

So, (λ_R, μ_R) is a crossed module morphism.

Ad (2). We show that $\mathrm{im}(\mu_R) \overset{!}{\leqslant} \mathrm{G}^R_{\mathcal{X},R}$.

Suppose given $F \in \mathrm{G}_{\mathcal{X}}$. Then $F\mu_R = FR$ is an R-linear functor; cf. Lemma 86.(1). So $F\mu_R \in \mathrm{G}^R_{\mathcal{X},R}$.

We show that $\mathrm{im}(\lambda_R) \overset{!}{\leqslant} \mathrm{M}^R_{\mathcal{X},R}$.

Suppose given $a = (\mathrm{id}_{\mathcal{X}} \xrightarrow{a} H) \in \mathrm{M}_{\mathcal{X}}$. Then $a\lambda_R = \big(\mathrm{id}_{\mathcal{X}} \xrightarrow{a} H\big)(\lambda_R) = \big(\mathrm{id}_{\mathcal{X}} \xrightarrow{aR} HR\big)$, where $HR \in \mathrm{G}^R_{\mathcal{X},R}$. So $a\lambda_R \in \mathrm{M}^R_{\mathcal{X},R}$.

□

Proposition 125 (Permutation modules) *Suppose given a V-category $\mathcal{X}$; cf. Definition* 2. *Consider the crossed module morphism from Lemma* 56

$$\rho_{\mathcal{X}} := (\lambda_{\mathcal{X}}, \mu_{\mathcal{X}})\colon V \to \mathrm{S}_{\mathcal{X}}\,;$$

cf. also Proposition 57.

Recall from Lemma 56 *that for $X \in \mathrm{Ob}(\mathcal{X})$, we have $X \cdot g = X(g\mu_{\mathcal{X}})$ and that for $\big(X \xrightarrow{u} Y\big) \in \mathrm{Mor}(\mathcal{X})$ we have $u \cdot (g, m) = u(g\mu_{\mathcal{X}}) \blacktriangle \big(Y(g\mu_{\mathcal{X}})(m\lambda_{\mathcal{X}})\big)$ for $g \in G$ and $m \in M$.*

Recall that then

$$\begin{aligned} \mu_{\mathcal{X}}\colon\;\; & G \to \mathrm{G}_{\mathcal{X}}\,, \\ & g \;\mapsto \big(g\mu_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}\,,\, (X \xrightarrow{u} Y) \mapsto (X \cdot g \xrightarrow{u\cdot(g,1)} Y \cdot g)\big) \end{aligned}$$

and

$$\begin{aligned} \lambda_{\mathcal{X}}\colon M &\to \mathrm{M}_{\mathcal{X}}\,, \\ m \mapsto m\lambda_{\mathcal{X}} &= \left(\begin{array}{ccc} \begin{array}{c} X \\ \downarrow u \\ Y \end{array} & \longmapsto & \begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X \cdot (1,m)} & X \cdot mf \\ u\downarrow & & \downarrow u\cdot(mf,1) \\ Y & \xrightarrow[\mathrm{id}_Y \cdot (1,m)]{} & Y \cdot mf \end{array} \end{array} \right). \end{aligned}$$

Then $(\mathcal{X}R, \Phi)$ *is a* $(V\mathrm{Cat})R$*-module via*

$$\begin{array}{rcll} (V\mathrm{Cat})R & \xrightarrow{\Phi} & \mathrm{End}_R(\mathcal{X}R) & \\ g & \longmapsto & g\Phi := g\mu_{\mathcal{X}} & \textit{for } g \in \mathrm{Ob}((V\mathrm{Cat})R) \\ z := \left(g \xrightarrow{\sum_i (g, m\,k_i) r_i} g \cdot mf\right) & \longmapsto & z\Phi & \textit{for } z \in \mathrm{Mor}((V\mathrm{Cat})R) \end{array}$$

where the transformation $z\Phi$ *maps a morphism* $\left(\begin{smallmatrix} X \\ \downarrow \sum_j u_j s_j \\ Y \end{smallmatrix}\right) \in \mathrm{Mor}(\mathcal{X}R)$ *to the diagram morphism*

$$\left(\begin{tikzcd} X \cdot g \arrow[r, "\sum_i \mathrm{id}_X \cdot (g, mk_i) r_i"] \arrow[d, "\sum_j (u_j \cdot (g,1)) s_j"'] & X \cdot (g \cdot mf) \arrow[d, "\sum_j \left(u_j \cdot \left(g(mf), 1\right)\right) s_j"] \\ Y \cdot g \arrow[r, "\sum_i \mathrm{id}_Y \cdot (g, mk_i) r_i"'] & Y \cdot (g \cdot mf) \end{tikzcd}\right).$$

We shall call $(\mathcal{X}R, \Phi)$ *the* permutation module *over* $(V\mathrm{Cat})R$ *corresponding to the* V*-category* $\mathcal{X}$*.*

Proof. Consider the crossed module morphism from Lemma 124.(2)

$$\check{\rho}_{\mathcal{X},R} = (\check{\lambda}_{\mathcal{X},R}, \check{\mu}_{\mathcal{X},R})\colon \mathrm{S}_{\mathcal{X}} \to \mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{X}R) \leqslant \mathrm{S}_{\mathcal{X}R}$$

with

$$\check{\mu}_{\mathcal{X},R}\colon \mathrm{G}_{\mathcal{X}} \to \mathrm{G}_{\mathcal{X}R}^{R}\,, \left(\mathcal{X} \xrightarrow{F} \mathcal{X}\right) \mapsto \left(\mathcal{X}R \xrightarrow{FR} \mathcal{X}R\right)$$
$$\check{\lambda}_{\mathcal{X},R}\colon \mathrm{M}_{\mathcal{X}} \to \mathrm{M}_{\mathcal{X}R}^{R}\,, \left(\mathrm{id}_{\mathcal{X}} \xrightarrow{a} F\right) \mapsto \left(\mathrm{id}_{\mathcal{X}R} \xrightarrow{aR} FR\right).$$

So we have a crossed module morphism from V to $\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{X}R)$ given by

$$\rho := \rho_{\mathcal{X}} \blacktriangle \check{\rho}_{\mathcal{X},R} = (\lambda_{\mathcal{X}} \blacktriangle \check{\lambda}_{\mathcal{X},R}, \mu_{\mathcal{X}} \blacktriangle \check{\mu}_{\mathcal{X},R})\colon V \to \mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{X}R)\,.$$

We write $\mu := \mu_{\mathcal{X}} \blacktriangle \check{\mu}_{\mathcal{X},R}$ and $\lambda := \lambda_{\mathcal{X}} \blacktriangle \check{\lambda}_{\mathcal{X},R}$.

By Lemma 121, we have a monoidal R-linear functor

$$\begin{array}{rrcll} \Phi := \hat{\hat{\Phi}}_\rho\colon & (V\mathrm{Cat})R & \to & \mathrm{End}_R(\mathcal{X}R) & \\ & g & \mapsto & g\hat{\hat{\Phi}}_\rho = g\mu & \text{for } g \in \mathrm{Ob}(V\mathrm{Cat}) \\ & z := \sum_i (g, mk_i) r_i & \mapsto & z\hat{\hat{\Phi}}_\rho & \text{for } z \in \mathrm{Mor}(V\mathrm{Cat}) \end{array}$$

where the transformation $z\hat{\Phi}_\rho$ maps a morphism $\left(\begin{smallmatrix} X \\ \downarrow \sum_j u_j s_j \\ Y \end{smallmatrix}\right) \in \mathrm{Mor}(\mathcal{X}R)$ to the diagram morphism

$$\left(\begin{tikzcd} X(g\mu) \arrow[r, "\sum_i \left(X(g\mu)\right)\left((mk_i)\lambda\right)r_i"] \arrow[d, "\left(\sum_j u_j s_j\right)(g\mu)"'] & X\big((g\cdot mf)\mu\big) \arrow[d, "\left(\sum_j u_j s_j\right)\left((g\cdot mf)\mu\right)"] \\ Y(g\mu) \arrow[r, "\sum_i \left(Y(g\mu)\right)\left((mk_i)\lambda\right)r_i"'] & Y\big((g\cdot mf)\mu\big) \end{tikzcd}\right)$$

$$= \left(\begin{tikzcd} (X)(g\mu_{\mathcal{X}}) \arrow[r, "\sum_i \left(X(g\mu_{\mathcal{X}})\right)\left((mk_i)\lambda_{\mathcal{X}}\right)r_i"] \arrow[d, "\sum_j \left((u_j)(g\mu_{\mathcal{X}})\right)s_j"'] & (X)\big((g\cdot mf)\mu_{\mathcal{X}}\big) \arrow[d, "\sum_j \left((u_j)\left((g\cdot mf)\mu_{\mathcal{X}}\right)\right)s_j"] \\ (Y)(g\mu_{\mathcal{X}}) \arrow[r, "\sum_i \left(Y(g\mu_{\mathcal{X}})\right)\left((mk_i)\lambda_{\mathcal{X}}\right)r_i"'] & (Y)\big((g\cdot mf)\mu_{\mathcal{X}}\big) \end{tikzcd}\right)$$

$$= \left(\begin{tikzcd} X\cdot g \arrow[r, "\sum_i \mathrm{id}_X\cdot(g, mk_i)r_i"] \arrow[d, "\sum_j \left(u_j\cdot(g,1)\right)s_j"'] & X\cdot(g\cdot mf) \arrow[d, "\sum_j \left(u_j\cdot(g\cdot mf,1)\right)s_j"] \\ Y\cdot g \arrow[r, "\sum_i \mathrm{id}_Y\cdot(g, mk_i)r_i"'] & Y\cdot(g\cdot mf) \end{tikzcd}\right).$$

□

Example 126

(1) Recall that we have a V-category given by $V\mathrm{Cat} = \big(G \ltimes M, G, (s,i,t), \blacktriangle\big)$; cf. Remark 5.(2).

Recall that morphisms in $(V\mathrm{Cat})R$ are of the form $\big(\sum_i (g, mk_i)r_i\colon g \to g\cdot mf\big)$. Often, it suffices to consider R-linear generators of the form (g,m).

Consider the permutation module $((V\mathrm{Cat})R, \Phi)$ of V over $V\mathrm{Cat}$ from Proposition 125, with

$$\begin{array}{rcl} (V\mathrm{Cat})R & \xrightarrow{\Phi} & \mathrm{End}_R\big((V\mathrm{Cat})R\big) \\ \big(g \xrightarrow{(g,m)} g\cdot mf\big) & \longmapsto & \big(g\Phi \xrightarrow{(g,m)\Phi} (g\cdot mf)\Phi\big)\,, \end{array}$$

where the transformation $(g,m)\Phi$ maps a morphism $\left(\begin{smallmatrix} h \\ \downarrow {\scriptstyle (h,n)} \\ h\cdot nf \end{smallmatrix}\right) \in \mathrm{Mor}(\mathcal{X}R)$ to the diagram morphism

$$\left(\begin{tikzcd} h\cdot g \arrow[r, "{(h\cdot g,m)}"] \arrow[d, "{(h\cdot g,n^g)}"'] & h\cdot g\cdot mf \arrow[d, "{(h\cdot g\cdot mf,n^{g\cdot mf})}"] \\ h\cdot nf\cdot g \arrow[r, "{(h\cdot nf\cdot g,m)}"'] & h\cdot nf\cdot g\cdot mf \end{tikzcd}\right).$$

(2) Consider the regular $(V\mathrm{Cat})R$-module $((V\mathrm{Cat})R,\Theta)$ from Example 116.(2), with

$$\begin{array}{rcl} (V\mathrm{Cat})R & \xrightarrow{\Theta} & \mathrm{End}_R\big((V\mathrm{Cat})R\big) \\ \big(g \xrightarrow{(g,m)} g\cdot mf\big) & \longmapsto & \big(g\Theta \xrightarrow{(g,m)\Theta} (g\cdot mf)\Theta\big)\,, \end{array}$$

where the transformation $(g,m)\Theta$ maps a morphism $\left(\begin{smallmatrix} h \\ \downarrow {\scriptstyle (h,n)} \\ h\cdot nf \end{smallmatrix}\right) \in \mathrm{Mor}(\mathcal{X}R)$ to the diagram morphism

$$\left(\begin{tikzcd} h\cdot g \arrow[r, "{(h\cdot g,m)}"] \arrow[d, "{(h\cdot g,n^g)}"'] & h\cdot g\cdot mf \arrow[d, "{(h\cdot g\cdot mf,n^{g\cdot mf})}"] \\ h\cdot nf\cdot g \arrow[r, "{(h\cdot nf\cdot g,m)}"'] & h\cdot nf\cdot g\cdot mf \end{tikzcd}\right).$$

(3) The permutation module of V over $(V\mathrm{Cat})R$ is the regular $\big((V\mathrm{Cat})R\big)$-module since $\Phi=\Theta$; cf. $(1,2)$.

Chapter 9

Maschke: a first step

9.1 Prefunctors

Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories.

Let $\mathcal{A}$ be a monoidal R-linear category.

Definition 127 (Prefunctors)

Suppose given a pair of maps $P := \big(\operatorname{Mor}(P)\,,\,\operatorname{Ob}(P)\big)$ where

$$\operatorname{Ob}(P):\ \operatorname{Ob}(\mathcal{C}) \to \operatorname{Ob}(\mathcal{D}) \quad \text{and} \quad \operatorname{Mor}(P):\ \operatorname{Mor}(\mathcal{C}) \to \operatorname{Mor}(\mathcal{D})\ .$$

We call P a *prefunctor* from $\mathcal{C}$ to $\mathcal{D}$ if the conditions $(1,2)$ are satisfied.

(1) For $u \in \operatorname{Mor}(\mathcal{C})$, we have

$$(u)\big(\boldsymbol{s} \blacktriangle \operatorname{Ob}(P)\big) = (u)\big(\operatorname{Mor}(P) \blacktriangle \boldsymbol{s}\big) \quad \text{and} \quad (u)\big(\boldsymbol{t} \blacktriangle \operatorname{Ob}(P)\big) = (u)\big(\operatorname{Mor}(P) \blacktriangle \boldsymbol{t}\big)\,.$$

(2) For $(X \xrightarrow{u} Y \xrightarrow{v} Z)$ in $\mathcal{C}$, we have

$$(u \blacktriangle v)\operatorname{Mor}(P) = u\operatorname{Mor}(P) \blacktriangle v\operatorname{Mor}(P)\ .$$

For $X \in \operatorname{Ob}(\mathcal{C})$, we write $XP := (X)\operatorname{Ob}(P)$. For $u \in \operatorname{Mor}(\mathcal{C})$, we write $uP := (u)\operatorname{Mor}(P)$.

So a prefunctor is a functor if it respects identities.

Definition 128 (R-linear prefunctors)

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are R-linear categories; cf. Definition 65.

Let $P\colon \mathcal{C} \to \mathcal{D}$ be a prefunctor.

We call P an *R-linear prefunctor* if

$$(ur + vs)P = (uP)r + (vP)s$$

holds for $r, s \in R$, $X, Y \in \mathrm{Ob}(\mathcal{C})$ and $u, v \in {}_{\mathcal{C}}(X, Y)$. Cf. also Remark 71.

Definition 129 ($\mathcal{A}$-linear prefunctor)

Let $(\mathcal{M}, \Phi_{\mathcal{M}})$ and $(\mathcal{N}, \Phi_{\mathcal{N}})$ be $\mathcal{A}$-modules; cf. Definition 100.

An R-linear prefunctor $P\colon \mathcal{M} \to \mathcal{N}$ is called *$\mathcal{A}$-linear* if we have

$$(m \otimes a)P = mP \otimes a$$

for $m \in \mathrm{Mor}(\mathcal{M})$ and $a \in \mathrm{Mor}(\mathcal{A})$. Cf. also Definition 105.

Lemma 130 *Let $(\mathcal{M}, \Phi_{\mathcal{M}})$ and $(\mathcal{N}, \Phi_{\mathcal{N}})$ be $\mathcal{A}$-modules; cf. Definition* 100.

Suppose given an R-linear prefunctor $P\colon \mathcal{M} \to \mathcal{N}$.

Then P is an $\mathcal{A}$-linear prefunctor if and only if the conditions $(1, 2)$ hold.

(1) *For $A \in \mathrm{Ob}(A)$ and $m \in \mathrm{Mor}(\mathcal{M})$, we have*

$$(m \otimes A)P = mP \otimes A\,.$$

I.e. we have

$$(A\Phi_{\mathcal{M}}) * P = P * (A\Phi_{\mathcal{N}})$$

for $A \in \mathrm{Ob}(\mathcal{A})$.

(2) *For $a \in \mathrm{Mor}(\mathcal{A})$ and $M \in \mathrm{Ob}(\mathcal{M})$, we have*

$$(M \otimes a)P = \mathrm{id}_M P \otimes a\,.$$

I.e. we have

$$M(a\Phi_{\mathcal{M}})P = \big((\mathrm{id}_M P)(A\Phi_{\mathcal{N}})\big) \blacktriangle \big(MP(a\Phi_{\mathcal{N}})\big) = \big(MP(a\Phi_{\mathcal{N}})\big) \blacktriangle \big((\mathrm{id}_M P)(B\Phi_{\mathcal{N}})\big)$$

for $\big(A \xrightarrow{a} B\big) \in \mathrm{Mor}(\mathcal{A})$ and $M \in \mathrm{Ob}(\mathcal{M})$.

Cf. also Lemma 106*.*

Proof. Ad $\Rightarrow$. Suppose that P is an $\mathcal{A}$-linear functor.

Suppose given $m \in \mathrm{Mor}(\mathcal{M})$ and $A \in \mathrm{Ob}(\mathcal{A})$.

We have

$$(m \otimes A)P \overset{101.(6)}{=} (m \otimes \mathrm{id}_A)P = mP \otimes \mathrm{id}_A \overset{101.(6)}{=} mP \otimes A\,.$$

Suppose given $M \in \mathrm{Ob}(\mathcal{M})$ and $a \in \mathrm{Mor}(\mathcal{A})$.

We have

$$(M \otimes a)P \overset{101.(7)}{=} (\mathrm{id}_M \otimes a)P = \mathrm{id}_M P \otimes a\,.$$

Ad $\Leftarrow$. Suppose that $(1, 2)$ hold.

Suppose given $\big(A \xrightarrow{a} A'\big) \in \mathrm{Mor}(\mathcal{A})$ and $\big(M \xrightarrow{m} M'\big) \in \mathrm{Mor}(\mathcal{M})$.

We have

$$\begin{aligned} (m \otimes a)P &= \big((m \otimes A) \blacktriangle (M' \otimes a)\big)P = (m \otimes A)P \blacktriangle (M' \otimes a)P \\ &= (mP \otimes A) \blacktriangle (\mathrm{id}_{M'}P \otimes a) \overset{101.(6)}{=} (mP \otimes \mathrm{id}_A) \blacktriangle (\mathrm{id}_{M'}P \otimes a) \\ &\overset{101.(10)}{=} (mP \blacktriangle \mathrm{id}_{M'}P) \otimes (\mathrm{id}_A \blacktriangle a) = (m \blacktriangle \mathrm{id}_{M'})P \otimes a = mP \otimes a\,. \end{aligned}$$

$\square$

Remark 131 *Suppose that $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are $\mathcal{A}$-modules.*

(1) *An $\mathcal{A}$-linear functor $F \colon \mathcal{C} \to \mathcal{D}$ is an $\mathcal{A}$-linear prefunctor.*

(2) *Suppose given $\mathcal{A}$-linear prefunctors $P \colon \mathcal{C} \to \mathcal{D}$ and $P' \colon \mathcal{D} \to \mathcal{E}$.*
Then the composite $P * P' \colon \mathcal{C} \to \mathcal{E}$ is an $\mathcal{A}$-linear prefunctor.

Proof. Ad (1). Let $F \colon \mathcal{C} \to \mathcal{D}$ be an $\mathcal{A}$-linear functor.

For $m \in \mathrm{Mor}(\mathcal{C})$ and $a \in \mathrm{Mor}(\mathcal{A})$, we have

$$(m \otimes a)F = mF \otimes a\,;$$

cf. Lemma 106.

So F is an $\mathcal{A}$-linear prefunctor.

Ad (2). For $m \in \operatorname{Mor}(\mathcal{C})$ and $a \in \operatorname{Mor}(\mathcal{A})$, we have

$$(m \otimes a)(P * P') = (mP \otimes a)P' = m(P * P') \otimes a\,.$$

Then, by Lemma 130, $P * P'$ is an $\mathcal{A}$-linear prefunctor. □

Remark 132 *Let $(\mathcal{M}, \Phi_{\mathcal{M}})$ and $(\mathcal{N}, \Phi_{\mathcal{N}})$ be $\mathcal{A}$-modules.*

Suppose given an $\mathcal{A}$-linear prefunctor $P\colon \mathcal{M} \to \mathcal{N}$ that is not a functor.

Then there exists some $M \in \operatorname{Ob}(\mathcal{M})$ and some $a \in \operatorname{Mor}(\mathcal{A})$ such that

$$(M \otimes a)P \neq MP \otimes a\,.$$

Proof. Since P is not a functor there exists an $M \in \operatorname{Ob}(\mathcal{M})$ such that

$$\mathrm{id}_M P \neq \mathrm{id}_{MP}\,.$$

Then, for example, for $a = \mathrm{id}_I \in \operatorname{Mor}(\mathcal{A})$, we have

$$(M \otimes a)P \overset{101.(7)}{=} (\mathrm{id}_M \otimes a)P = \mathrm{id}_M P \otimes a = \mathrm{id}_M P \neq \mathrm{id}_{MP} = \mathrm{id}_{MP} \otimes a \overset{101.(7)}{=} MP \otimes a\,.$$

□

9.2 A first step towards Maschke

Suppose given a crossed module $V = \big(M, G, \gamma, f\big)$.

Consider the invertible monoidal category $V\mathrm{Cat}$; cf. Remark 29.

Remark 133 *We have the crossed module $\bar{V} = \big(Mf, G, c, \dot{f}\big)$ with*

$$c\colon G \to \operatorname{Aut}(Mf)\,,\ g \mapsto \big(mf \mapsto g^{-}(mf)g \overset{\text{(CM1)}}{=} (m^g)f\big)$$
$$\dot{f} := \mathrm{id}_G\big|_{Mf}\colon Mf \to G\,,\ g \mapsto g\,,$$

and we have the surjective crossed module morphism

$$(\bar{f}, \mathrm{id}_G)\colon V \to \bar{V}$$

with

$$\bar{f} := f\big|^{Mf}\colon M \to Mf\,,\ m \mapsto mf\,;$$

cf. [**15,** Lem. 37.(1)].

$$\begin{tikzcd} M \arrow[r, "\bar{f}"] \arrow[d, "f"'] & Mf \arrow[d, "\dot{f}"] \\ G \arrow[r, "\mathrm{id}_G"'] & G \end{tikzcd}$$

Consider the invertible monoidal category $\bar{V}\,\mathrm{Cat}$ *; cf. Remark* 29*. Recall that*

$$\begin{aligned} \mathrm{Ob}\big(\bar{V}\,\mathrm{Cat}\big) &= G \\ \mathrm{Mor}\big(\bar{V}\,\mathrm{Cat}\big) &= G \ltimes Mf\,. \end{aligned}$$

Then, for $g, h \in \mathrm{Ob}\big(\bar{V}\mathrm{Cat}\big) = G$*, the set of morphisms from* g *to* h *is given as follows.*

$$_{\bar{V}\,\mathrm{Cat}}(g,h) = \begin{cases} \{(g, g^{-}h)\} & \textit{if } g^{-}h \in Mf \\ \emptyset & \textit{if } g^{-}h \notin Mf \end{cases}$$

Proof. Suppose given $g, h \in \mathrm{Ob}\big(\bar{V}\,\mathrm{Cat}\big) = G$.

We consider the case $g^{-}h \in Mf$.

Suppose given a morphism $u \in {}_{\bar{V}\,\mathrm{Cat}}(g,h)$. Note that u is of the form $u = (x, y)$ for some $x \in G$ and for some $y \in Mf$.

Then we have

$$g = (x, y)s = x\,,$$

and we have

$$h = (x, y)t = x \cdot y\dot{f} = g \cdot y\,.$$

So $g^{-}h = y$.

This shows ${}_{\bar{V}\,\mathrm{Cat}}(g,h) = \{(g, g^{-}h)\}$.

We consider the case $g^{-}h \notin Mf$.

If we *assume* that there exists a morphism $u = (x, y) \in {}_{\bar{V}\mathrm{Cat}}(g,h)$, where $x \in G$ and $y \in Mf$, then we have $x = g$ and $h = xy = gy$, and so we have $g^{-}h = y \in Mf$ which *contradicts* the assumption.

This shows ${}_{\bar{V}\,\mathrm{Cat}}(g,h) = \emptyset$. $\square$

Remark 134 Consider the crossed module $\bar{V} = \big(Mf, G, c, \dot{f}\big)$, the crossed module morphism $(\bar{f}, \mathrm{id}_G)\colon V \to \bar{V}$ and the invertible monoidal category $\bar{V}\,\mathrm{Cat}$ from Remark 133.

Consider the monoidal functor

$$\begin{aligned}(\bar{f}, \mathrm{id}_G)\mathrm{Cat}\colon V\mathrm{Cat} &\to \bar{V}\,\mathrm{Cat}\\ \big(g \xrightarrow{(g,m)} g \cdot mf\big) &\mapsto \big(g\,\mathrm{id}_G \xrightarrow{(g\,\mathrm{id}_G,\, m\bar{f})} (g \cdot mf)\mathrm{id}_G\big) = \big(g \xrightarrow{(g,mf)} g \cdot mf\big);\end{aligned}$$

cf. Lemma 39.

By R-linear extension, we obtain the monoidal R-linear functor

$$\begin{aligned}F := \big((\bar{f}, \mathrm{id}_G)\mathrm{Cat}\,\big)R\colon (V\mathrm{Cat})R &\to (\bar{V}\,\mathrm{Cat})R\\ \big(g \xrightarrow{\sum_i (g,mk_i)r_i} g \cdot mf\big) &\mapsto \big(g \xrightarrow{\sum_i (g,mf)r_i} g \cdot mf\big) = \big(g \xrightarrow{(g,mf)r} g \cdot mf\big)\,,\end{aligned}$$

where $r := \sum_i r_i$; cf. Lemma 86 and Remark 115.

By Lemma 108, $(\bar{V}\,\mathrm{Cat})R$ is a $(V\mathrm{Cat})R$-module via

$$\begin{aligned}\Theta_F\colon (V\mathrm{Cat})R &\to \mathrm{End}_R\big((\bar{V}\,\mathrm{Cat})R\big)\\ u := \big(g \xrightarrow{\sum_i (g,mk_i)r_i} g \cdot mf\big) &\mapsto u\Theta_F\,,\end{aligned}$$

where $u\Theta_F$ maps a morphism $\left(\begin{array}{c} h \\ \downarrow{\scriptstyle (h,\, h^- h')r'} \\ h' \end{array}\right) \in \mathrm{Mor}\big((\bar{V}\,\mathrm{Cat})R\big)$ to the diagram morphism

$$\left(\begin{array}{ccc} h \cdot g & \xrightarrow{(h,1)\cdot(g,mf)r} & h \cdot (g \cdot mf) \\ {\scriptstyle (h,h^-h')r' \cdot (g,1)}\big\downarrow & & \big\downarrow{\scriptstyle (h,h^-h')r' \cdot (g \cdot mf,1)} \\ h' \cdot g & \xrightarrow[(h',1)\cdot(g,mf)r]{} & h' \cdot (g \cdot mf) \end{array}\right)$$

with $r := \sum_i r_i \in R$, i.e. to

$$\left(\begin{tikzcd} h \cdot g \arrow[r, "(hg, mf)r"] \arrow[d, "\big(hg, g^{-}(h^{-}h')g\big)r'"'] & h \cdot (g \cdot mf) \arrow[d, "\big(h(g\cdot mf), (g\cdot mf)^{-}(h^{-}h')(g\cdot mf)\big)r'"] \\ h' \cdot g \arrow[r, "(h'g, mf)r"'] & h' \cdot (g \cdot mf) \end{tikzcd}\right).$$

Moreover, recall that we have a $(V\mathrm{Cat})R$-module $\big((V\mathrm{Cat})R, \Theta\big)$; cf. Example 116.(2).

Proposition 135 *Suppose that* $\ker f$ *is finite and suppose that* $|\ker f|$ *is invertible in* R.

Consider the monoidal R*-linear category* $(\bar{V}\,\mathrm{Cat})R$ *from Remark* 133 *and the monoidal* R*-linear functor* $F = \big((\bar{f}, \mathrm{id}_G)\mathrm{Cat}\big)R\colon (V\mathrm{Cat})R \to (\bar{V}\,\mathrm{Cat})R$ *from Remark* 134.

Consider the $(V\mathrm{Cat})R$*-module* $\big((\bar{V}\,\mathrm{Cat})R, \Theta_F\big)$ *from Remark* 134.

We have the $\mathcal{A}$*-linear prefunctor*

$$\begin{aligned} P\colon (\bar{V}\,\mathrm{Cat})R &\to (V\mathrm{Cat})R \\ \Big(g \xrightarrow{(g, g^{-}h)r} h\Big) &\mapsto \Bigg(g \xrightarrow{\Big(\sum\limits_{\substack{m\in M\\ mf = g^{-}h}} (g,m)r\Big)\frac{1}{|\ker f|}} h\Bigg). \end{aligned}$$

Moreover, we have

$$P * F = \mathrm{id}_{(\bar{V}\,\mathrm{Cat})R}\,.$$

Proof. We write $K := \ker f$.

We show that P is a prefunctor.

Suppose given $\big(g \xrightarrow{(g,g^{-}h)r} h \xrightarrow{(h,h^{-}l)r'} l\big)$ in $(\bar{V}\,\mathrm{Cat})R$.

We have

$$\begin{aligned} \big((g, g^{-}h)r\big)\boldsymbol{s}P = gP = g = \Big(\Big(\sum_{\substack{m\in M\\ mf = g^{-}h}} (g,m)r\Big)\frac{1}{|K|}\Big)\boldsymbol{s} = \big((g, g^{-}h)r\big)P\boldsymbol{s} \\ \big((g, g^{-}h)r\big)\boldsymbol{t}P = hP = h = g \cdot g^{-}h = \Big(\Big(\sum_{\substack{m\in M\\ mf = g^{-}h}} (g,m)r\Big)\frac{1}{|K|}\Big)\boldsymbol{t} = \big((g, g^{-}h)r\big)P\boldsymbol{t}\,. \end{aligned}$$

We have

$$
\begin{array}{rcl}
\big((g,g^-h)r\big)P \blacktriangle \big((h,h^-l)r'\big)P & = & \bigg(\Big(\sum_{\substack{n\in M\\ nf=g^-h}}(g,n)r\Big)\dfrac{1}{|K|}\bigg) \blacktriangle \bigg(\Big(\sum_{\substack{n'\in M\\ n'f=h^-l}}(h,n')r'\Big)\dfrac{1}{|K|}\bigg) \\
& = & \Big(\sum_{\substack{n,n'\in M\\ nf=g^-h\\ n'f=h^-l}}\big((g,n)\blacktriangle(h,n')\big)rr'\Big)\dfrac{1}{|K|^2} \\
& = & \Big(\sum_{\substack{n,n'\in M\\ nf=g^-h\\ n'f=h^-l}}(g,nn')rr'\Big)\dfrac{1}{|K|^2} \\
& \overset{\substack{m=nn'\\ n'=n^-m}}{=} & \Big(\sum_{\substack{m\in M\\ mf=g^-l}}\sum_{\substack{n\in M\\ nf=g^-h}}(g,m)rr'\Big)\dfrac{1}{|K|^2} \\
& = & \Big(\sum_{\substack{m\in M\\ mf=g^-l}}(g,m)rr'\Big)\dfrac{1}{|K|} \\
& = & \big((g,g^-l)rr'\big)P \\
& = & \big((g,g^-h)r\blacktriangle(h,h^-l)r'\big)P\,.
\end{array}
$$

So P is a prefunctor.

We show that P is R-linear.

Suppose given $g \overset{u}{\underset{v}{\rightrightarrows}} h$ in $(\bar{V}\,\mathrm{Cat})R$ and suppose given $s, t \in R$.

Note that we have $u = (g,g^-h)r$ and $v = (g,g^-h)r'$ for some $r,\ r' \in R$; cf. Remark 133.

Then

$$
\begin{aligned}
(us+vt)P &= \big((g,g^-h)rs + (g,g^-h)r't\big)P = \big((g,g^-h)(rs+r't)\big)P \\
&= \Big(\sum_{\substack{m\in M\\ mf=g^-h}}(g,m)(rs+r't)\Big)\frac{1}{|K|} \\
&= \Big(\sum_{\substack{m\in M\\ mf=g^-h}}(g,m)r\Big)\frac{s}{|K|} + \Big(\sum_{\substack{m\in M\\ mf=g^-h}}(g,m)r'\Big)\frac{t}{|K|} \\
&= (uP)s + (vP)t\,.
\end{aligned}
$$

So P is R-linear.

We show that P is $\mathcal{A}$-linear.

Suppose given $u := (g,g^-h)r \in \mathrm{Mor}\big((\bar{V}\,\mathrm{Cat})R\big)$ and $a := \sum_i (x,mk_i)r'_i \in \mathrm{Mor}((V\mathrm{Cat})R)$. We write $r' := \sum_i r'_i\,$.

Since $(g, g^- h) \in \mathrm{Mor}\big((\bar{V}\,\mathrm{Cat})R\big)$ we have $g^- h \in Mf$. So there exists some $m_0 \in M$ such that $m_0 f = g^- h$. Therefore

$$f^{-1}(g^- h) = m_0 K = \{m_0 k \colon k \in K\}\,.$$

Moreover, we have

$$\big((m_0)^x\, m\big) f \overset{\text{(CM1)}}{=} x^- \cdot m_0 f \cdot x \cdot mf = x^- \cdot g^- h \cdot x \cdot mf\,.$$

So $(m_0)^x\, m \in f^{-1}(x^- \cdot g^- h \cdot x \cdot mf)$. Therefore

$$f^{-1}(x^- \cdot g^- h \cdot x \cdot mf) = (m_0)^x\, mK = \{(m_0)^x\, mk \colon k \in K\}\,.$$

We have

$$\begin{aligned}
uP \cdot a &= \big((g, g^- h) r\big) P \cdot \textstyle\sum_i (x, mk_i) r_i' \\
&= \Big(\sum_{\substack{n \in M \\ nf = g^- h}} (g, n) r \Big) \frac{1}{|K|} \cdot \textstyle\sum_i (x, mk_i) r_i' \\
&= \Big(\sum_{\substack{n \in M \\ nf = g^- h}} \textstyle\sum_i (g, n) \cdot (x, mk_i) r r_i' \Big) \frac{1}{|K|} \\
&= \Big(\textstyle\sum_i \sum_{n \in m_0 K} (gx, n^x \cdot mk_i) r r_i' \Big) \frac{1}{|K|} \\
&= \Big(\textstyle\sum_i \sum_{k \in K} \big(gx, (m_0 k)^x \cdot mk_i\big) r r_i' \Big) \frac{1}{|K|} \\
&= \Big(\textstyle\sum_i \sum_{k \in K} \big(gx, (m_0)^x\, m \cdot (k^x)^m \cdot k_i\big) r r_i' \Big) \frac{1}{|K|} \\
&\overset{k' := (k^x)^m \cdot k_i}{=} \Big(\textstyle\sum_i \sum_{k' \in K} \big(gx, (m_0)^x\, m \cdot k'\big) r r_i' \Big) \frac{1}{|K|} \\
&\overset{r' := \sum_i r_i'}{=} \Big(\sum_{k' \in K} \big(gx, (m_0)^x\, mk'\big) r r' \Big) \frac{1}{|K|} \\
&= \Big(\sum_{n \in (m_0)^x\, mK} (gx, n) r r' \Big) \frac{1}{|K|} \\
&= \Big(\sum_{\substack{n \in M \\ nf = x^- \cdot g^- h \cdot x \cdot mf}} (gx, n) r r' \Big) \frac{1}{|K|} \\
&= \Big(\big(gx, (g^- h)^x \cdot mf\big) r r' \Big) P
\end{aligned}$$

$$\begin{aligned}
&= \big((g, g^{-}h)r \cdot (x, mf)r'\big)P \\
&= \Big((g, g^{-}h)r \cdot \big(\textstyle\sum_i (x, mk_i)r'_i\big)F\Big)P \\
&= (u \cdot a)P\,.
\end{aligned}$$

So P is an $\mathcal{A}$-linear prefunctor.

We show that $P * F \overset{!}{=} \mathrm{id}_{(\bar{V}\,\mathrm{Cat})R}\,.$

For $(g, g^{-}h)r \in \mathrm{Mor}\big(\bar{V}\,\mathrm{Cat}\big)$, we have

$$\begin{aligned}
\big((g, g^{-}h)r\big)(P * F) &= \Bigg(\Big(\sum_{\substack{m\in M\\ mf=g^{-}h}} (g, m)r\Big)\frac{1}{|K|}\Bigg)F \\
&= \Big(\sum_{\substack{m\in M\\ mf=g^{-}h}} (g, mf)r\Big)\frac{1}{|K|} \\
&= (g, g^{-}h)r\,|K|\,\frac{1}{|K|} \\
&= (g, g^{-}h)r\,.
\end{aligned}$$

This shows $P * F = \mathrm{id}_{(\bar{V}\,\mathrm{Cat})R}\,.$ $\square$

Remark 136 Consider the $(V\mathrm{Cat})R$-modules $\big(\bar{V}\,\mathrm{Cat})R, \Theta_F\big)$ and $\big((V\mathrm{Cat})R, \Theta\big)$ cf. Remark 134.

Consider the $\mathcal{A}$-linear prefunctor $P\colon (\bar{V}\,\mathrm{Cat})R \to (V\mathrm{Cat})R$ from Proposition 135.

(1) In general, the prefunctor $P\colon (\bar{V}\,\mathrm{Cat})R \to (V\mathrm{Cat})R$ given in Proposition 135 is not a functor.

For example, if $R \neq 0$ and if f is not injective then we have

$$\mathrm{id}_g P = (g, 1)P = \Big(\sum_{m\in\ker f} (g, m)\Big)\frac{1}{|\ker f|} \neq (g, 1) = \mathrm{id}_{gP}$$

for $g \in \mathrm{Ob}\big(\bar{V}\mathrm{Cat}\big) = G$.

(2) Suppose that $R \neq 0$ and suppose that $\ker f \neq 1$.

Let $u := (g,m) \in \operatorname{Mor}\big((\bar{V}\,\mathrm{Cat})R\big)$. Then, for $x \in \operatorname{Ob}\big((\bar{V}\,\mathrm{Cat})R\big) = G$, we have

$$\begin{aligned}(x \cdot u)P &= \big((x,1)\cdot(g,mf)\big)P \\ &= (xg,mf)P = \Big(\sum_{\substack{n\in M\\ nf=mf}} (xg,n)\Big)\frac{1}{|\ker f|} \\ &= \Big(\sum_{k\in\ker f} (xg,mk)\Big)\frac{1}{|\ker f|}\,,\end{aligned}$$

and we have

$$xP \cdot u = (x,1)\cdot(g,m) = (xg,m)\,.$$

So in general, we have $(x\cdot u)P \neq xP\cdot u$ for $x \in \operatorname{Ob}\big((\bar{V}\,\mathrm{Cat})R\big)$ and $u \in \operatorname{Mor}\big((\bar{V}\,\mathrm{Cat})R\big)$; cf. Remark 132.

Appendix A

Calculation of a Cayley embedding

We shall consider the assertion of Theorem 62, i.e. the analogue of Cayley's Theorem, in an example. To perform the necessary calculations, we use Magma [**3**].

A.1 An example of a crossed module V

We consider the crossed module $V = (M, G, \gamma, f)$ with $M = \langle b : b^4 \rangle$, with $G = \langle a : a^4 \rangle$, with $\gamma : G \to \mathrm{Aut}(M)$, $a \mapsto (b \mapsto b^-)$ and with $f : M \to G$, $b \mapsto a^2$; cf. [**14,** §1.5.6], [**15,** Ex. 30].

```
T := SymmetricGroup(4);
M := sub<T | T!(1,2,3,4) >;
G := sub<T | T!(1,2,3,4) >;
f := hom<M -> G | m :-> G!(m^2) > ;
xi := hom<M -> M | x :-> x^-1 >;
gamma := hom< G -> AutomorphismGroup(M) | <G.1, xi>>;
Mor := CartesianProduct(Set(G),Set(M));
// testing (CM1) and (CM2)
print &and[(m@(g@gamma))@f eq (m@f)^g : g in G, m in M];
print &and[m^n eq m@(n@f@gamma) : m in M, n in M];
```

Magma chooses the generators `G.1 = (1,2,3,4)` and `M.1 = (1,2,3,4)`.

A.2 Preparations for the symmetric crossed module $\mathrm{S}_{V\mathrm{Cat}}$

To calculate the underlying sets of both groups for the symmetric crossed module $\mathrm{S}_{V\mathrm{Cat}}$, we make use of the following program developed in [**8,** Alg. 34], which we document here for sake of completeness.

```
SymmetricCrossedModule := function(M,G,f,gamma);
 Ob  := Set(G);
 Gseq := [x : x in G];
 Mor := CartesianProduct(Set(G),Set(M));
 invert := map< Mor -> Mor | x :-> <x[1] * (x[2]@f) , x[2]^-1> >;
 MFP,xi := FPGroup(M);
 numb0 := map<{-1,0,1} -> {1,2} | [<-1,1>,<0,2>,<1,2>]>;
 numb := map< Integers() -> {1,2} | z :-> Sign(z)@numb0 >;
 m_seq := function(m)
  return ElementToSequence(m@@xi);
 end function;
 nog := NumberOfGenerators(MFP);
 M_gen := [(MFP.i)@xi : i in [1..nog]];
 Mor_gen := CartesianProduct(Set(G),{1..nog});
 Mf := Image(f);
 Kf := Kernel(f);
 Tr0 := Transversal(G,Mf);
 Tr := [x^-1 : x in Tr0]; // left coset representatives
 TrRep := map<G -> Tr | g :-> [x : x in Tr | g^-1 * x in Mf][1]>;
 MfRep := map<G -> Mf | g :-> [x^-1 * g : x in Tr | g^-1 * x in Mf][1]>;
 sect := map<Mf -> M | n :-> [m : m in M | m@f eq n][1]>;
 STr := SymmetricGroup(Set(Tr));
 phi := Action(GSet(STr));
 SMf := SymmetricGroup(Set(Mf));
 psi := Action(GSet(SMf));
 DPSMf := CartesianProduct([SMf : i in [1..#Tr]]);
 SKf := SymmetricGroup(Set(Kf));
 eta := Action(GSet(SKf));
 DPSKf_inner := CartesianProduct([SKf : i in [1..#M_gen]]);
 DPSKf_outer := CartesianProduct([DPSKf_inner : i in [1..#G]]);
```

```
DPKf := CartesianProduct([Kf : i in [1..#G]]);
counter := 0;
ListOfAutofunctors := [];
ListOfIsotrafos := [];
for s in STr do
 for smftup in DPSMf do
  F_Ob := map<Ob->Ob | g:-><g@TrRep,s>@phi*<g@MfRep,smftup[Index(Tr,g@TrRep)]>@psi>;
  for skftup in DPSKf_outer do
   counter +:= 1;
   sk := map< Mor_gen -> SKf | x :-> skftup[Index(Gseq,x[1])][x[2]] >;
   F_Mor_gen_plus  := map< Mor_gen -> Mor | x :->
          < x[1]@F_Ob , ( (x[1]@F_Ob)^-1 * (x[1] * M_gen[x[2]]@f)@F_Ob )@sect
        * < Kf!((M_gen[x[2]]@f@sect)^-1 * M_gen[x[2]]), x@sk >@eta > >;
   F_Mor_gen_minus := map< Mor_gen -> Mor | x :->
          < x[1] * (M_gen[x[2]]@f)^-1 , x[2]>@F_Mor_gen_plus@invert >;
   F_Mor_gen := [F_Mor_gen_minus,F_Mor_gen_plus];
   F_Mor := map< Mor -> Mor | x :->
          < x[1]@F_Ob , &*([Id(M)] cat [(< x[1] * &*([Id(M)] cat
          [ M_gen[Abs(i)]^Sign(i) where i is m_seq(x[2])[j] : j in [1..l-1]])@f,
          Abs(m_seq(x[2])[l]) >@F_Mor_gen[m_seq(x[2])[l]@numb])[2]
          : l in [1..#m_seq(x[2])]]) > >;
   is_functor := true;
   for y in CartesianProduct([G,M,M]) do
    if not (<y[1], y[2] * y[3]>@F_Mor)[2] eq
           (<y[1],y[2]>@F_Mor)[2] * (<y[1] * y[2]@f, y[3]>@F_Mor)[2] then
     is_functor := false;
     break y;
    end if;
   end for;
   if is_functor and #[x@F_Mor : x in Mor] eq #{x@F_Mor : x in Mor} then
    print "autofunctor", counter;
    ListOfAutofunctors cat:= [<F_Ob,F_Mor>];
    if s eq Id(STr) then  // now searching for isotransformations
     for k_tup in DPKf do
      candidate_trafo := map<Ob -> Mor | g :->
          < g, (Mf!(g^-1 * g@F_Ob))@sect * k_tup[Index(Gseq,g)]> >;
          // so this candidate transformation at g actually
          // has value g@candidate_trafo
```

```
      is_trafo := true;
      for z in Mor do
       if not (z[1]@candidate_trafo)[2] * (z@F_Mor)[2] eq
               z[2] * ((z[1]*(z[2]@f))@candidate_trafo)[2] then
        is_trafo := false;
        break z;
       end if;
      end for;
      if is_trafo then
       print "isotransformation", counter;
       ListOfIsotrafos cat:= [<candidate_trafo,<F_Ob,F_Mor>>];
      end if;
     end for;
    end if;
   end if;
  end for;
 end for;
end for;
return <ListOfAutofunctors, ListOfIsotrafos>;
end function;
```

We define

$$\texttt{SCM := SymmetricCrossedModule(M,G,f,gamma)}\ .$$

This yields `#SCM[1] = 32` autofunctors of $V\mathrm{Cat}$ and `#SCM[2] = 64` isotransformations from the identity on $V\mathrm{Cat}$ to an autofunctor of $V\,\mathrm{Cat}$. In other words, we have $|\,\mathrm{G}_{V\mathrm{Cat}}\,| = 32$ and $|\,\mathrm{M}_{V\mathrm{Cat}}\,| = 64$.

The program neither calculates the action of $\mathrm{G}_{V\mathrm{Cat}}$ on $\mathrm{M}_{V\mathrm{Cat}}$ nor the group morphism from $\mathrm{M}_{V\mathrm{Cat}}$ to $\mathrm{G}_{V\mathrm{Cat}}$. We will calculate both below; cf. §A.5, A.6.

A.3 Monoidal autofunctors of $V\mathrm{Cat}$

We want to determine which of the autofunctors $F \in \mathrm{G}_{V\mathrm{Cat}}$ are monoidal. Since $V\mathrm{Cat}$ is an invertible monoidal category it suffices to verify that an autofunctor F is compatible with the tensor product of morphisms; cf. Remark 32.(2).

```
IsMonoidal := function(F);
// F: autofunctor
 is_monoidal := true;
 for x in Mor do
  u := x@(F[2]);
  for y in Mor do
   v := y@(F[2]);
   z := Mor!<x[1] * y[1], (x[2])@(y[1]@gamma) * y[2]>;
   w := z@(F[2]);
   if not w eq <u[1] * v[1], (u[2])@(v[1]@gamma) * v[2]> then
    is_monoidal := false;
    break x;
   end if;
  end for;
 end for;
 return is_monoidal;
end function;

for i in [1..#SCM[1]] do
 print i, IsMonoidal(SCM[1][i]);
end for;
```

The program yields `IsMonoidal(SCM[1][i]) = true` for $i \in \{1,4,5,8\}$. So we have 4 monoidal autofunctors in $\mathrm{G}_{V\mathrm{Cat}}$.

A.4 Monoidal isotransformations of VCat

Now we want to determine which of the isotransformations $a \in \mathrm{M}_{V\mathrm{Cat}}$ are monoidal. To that end, we consider the monoidal autofunctors and the isotransformations from $\mathrm{id}_{V\mathrm{Cat}}$ to F. We investigate whether the isotransformation a is compatible with the evaluation on the objects in $\mathrm{Ob}(V\mathrm{Cat}) = G$.

```
IsMonoidalIsotrafo := function(a);
// a : isotransformation from id to a[2]
 F := a[2];
 is_monoidal := true;
```

```
 if IsMonoidal(F) then
  for g in G do
   u := g@(a[1]);
   for h in G do;
    v := h@(a[1]);
    if not (g*h)@(a[1]) eq <u[1] * v[1], (u[2])@(v[1]@gamma) * v[2]> then
     is_monoidal := false;
     break g;
    end if;
   end for;
  end for;
  return is_monoidal;
 else
  return false;
 end if;
end function;

for i in [1..#SCM[2]] do
 print i, IsMonoidalIsotrafo(SCM[2][i]);
end for;
```

The program yields `IsMonoidalTrafo(SCM[2][j]) = true` for $j \in \{1, 2, 17, 18\}$. So we have 4 monoidal isotransformations in $\mathrm{M}_{V\mathrm{Cat}}$.

A.5 The group $\mathrm{G}_{V\mathrm{Cat}}$

In the following, we want to determine the group $\mathrm{G}_{V\mathrm{Cat}}$ as a permutation group.

To that end, we use the faithful action of $\mathrm{G}_{V\mathrm{Cat}}$ on the set of morphisms in $V\mathrm{Cat}$, listed in `Mor_list`. The resulting permutation group will be called `GroupOfAutofunctors`.

```
Mor_list := [x : x in CartesianProduct(Set(G),Set(M))];
// morphisms in VCat
S := SymmetricGroup(#Mor_list);
// GroupOfAutofunctors will be a subgroup of S
GroupOfAutofunctors := sub<S | [S![Index(Mor_list,x@SCM[1][k][2]) : x in Mor_list] :
                                 k in [1..#SCM[1]]] >;
```

Further, we calculate which group of order 32 from the Small-Group-Library is isomorphic to `GroupOfAutofunctors`.

```
Index([IsIsomorphic(GroupOfAutofunctors,PermutationGroup(FPGroup(SG))) :
            SG in SmallGroups(32)],true);
```

The comparison yields the index `27`. So $\mathrm{G}_{V\mathrm{Cat}}$ is isomorphic to the group `SmallGroups(32)[27]` .

Now we want to turn `GroupOfAutofunctors` into a permutation group of smaller degree.

For that, we first turn `GroupOfAutofunctors` into the finitely presented group `GAFP`, where `phiA` is a group isomorphism from `GAFP` to `GroupOfAutofunctors` . Then we turn `GAFP` into the permutation group `GA`, where `psiA` is a group isomorphism from `GAFP` to `GA` .

$$\begin{tikzcd} & \texttt{GAFP} \arrow[dl, "\texttt{psiA}"'] \arrow[dr, "\texttt{phiA}"] & \\ \texttt{GA} & & \texttt{GroupOfAutofunctors} \end{tikzcd}$$

```
GAFP,phiA := FPGroup(GroupOfAutofunctors);
GA,psiA := PermutationGroup(GAFP);
```

Then `GA` is a permutation group of degree 8 . We have `GA` $=$ $\langle$ `F1, F2, F3` $\rangle$, where the generators `F1, F2, F3` are defined as follows.

```
F1 := GA!(1, 5)(2, 6)(3, 7)(4, 8);
F2 := GA!(1, 3)(2, 4);
F3 := GA!(1, 2)(3, 4);
```

One may verify that we have indeed `Order(sub< GA | F1, F2, F3 >) = 32` .

A.6 The group $\mathrm{M}_{V\mathrm{Cat}}$

In the following, we want to determine the group $\mathrm{M}_{V\mathrm{Cat}}$ as a permutation group. For this purpose, we embed $\mathrm{M}_{V\mathrm{Cat}}$ into the symmetric group on the set $\mathrm{M}_{V\mathrm{Cat}}$ via the Cayley embedding. The resulting permutation group will be called `GroupOfIsotrafos`.

```
Ob := Set(G);
Mor := CartesianProduct(Set(G),Set(M));

SI := SymmetricGroup(#SCM[2]);
// GroupOfIsotrafos will be a subgroup of SI

MultOfIsotrafos := function(i,j)
// this calculates SCM[2][i] * SCM[2][j], multiplication in M_VCat
 a := SCM[2][i][1];
 // the transformation as a map from Ob to Mor
 F := SCM[2][i][2];
 // target functor of a
 b := SCM[2][j][1];
 // the transformation as a map from Ob to Mor
 H := SCM[2][j][2];
 // target functor of b
 ab := < map< Ob -> Mor | [<g, <g,(g@a)[2] * ( g@(F[1])@b )[2]> > : g in Ob] >,
                               <F[1] * H[1], F[2] * H[2]> >;
 for i in [1..#SCM[2]] do
  if &and[g@ab[1] eq g@SCM[2][i][1] : g in Ob] then
   return i;
   break i;
  end if;
 end for;
end function;

GroupOfIsotrafos := sub<SI | [SI![ MultOfIsotrafos(i,j) : i in [1..#SCM[2]] ] :
                               j in [1..#SCM[2]] ] >;
```

Now we want to turn `GroupOfIsotrafos` into a permutation group of smaller degree.

For that, we first turn `GroupOfIsotrafos` into the finitely presented group `GIFP`, where `phiI` is a group isomorphism from `GIFP` to `GroupOfIsotrafos`. Then we turn `GIFP` into the permutation group `GI`, where `psiI` is a group isomorphism from `GIFP` to `GI`.

$$\begin{tikzcd} & \mathtt{GIFP} \arrow[dl, "\mathtt{psiI}"'] \arrow[dr, "\mathtt{phiI}"] & \\ \mathtt{GI} & & \mathtt{GroupOfIsotrafos} \end{tikzcd}$$

```
GIFP,phiI := FPGroup(GroupOfIsotrafos);
GI,psiI := PermutationGroup(GIFP);
```

Then `GI` is a permutation group of degree 16. We have `GI` $= \langle$ `a1, a2, a3, a4` $\rangle$ where the generators `a1, a2, a3, a4` are defined as follows.

```
a1 := GI!(1, 9, 3, 11)(2, 10, 4, 12)(5, 13, 7, 15)(6, 14, 8, 16);
a2 := GI!(1, 6)(2, 5)(3, 8)(4, 7)(9, 14)(10, 13)(11, 16)(12, 15);
a3 := GI!(1, 5, 2, 6)(3, 7, 4, 8)(9, 13, 10, 14)(11, 15, 12, 16);
a4 := GI!(1, 3)(2, 4)(5, 7)(6, 8);
```

One may verify that we have indeed `Order(sub< GI | a1, a2, a3, a4 >) = 64`.

A.7 The group morphism $\mathrm{f}_{V\mathrm{Cat}}\colon\ \mathrm{M}_{V\mathrm{Cat}} \to \mathrm{G}_{V\mathrm{Cat}}$

In §A.5 and §A.6 we have constructed the groups `GA` and `GI` that are isomorphic to $\mathrm{G}_{V\mathrm{Cat}}$ and $\mathrm{M}_{V\mathrm{Cat}}$, respectively.

Now we want to find a group morphism `fPerm` : `GI` $\to$ `GA` as an isomorphic replacement for $\mathrm{f}_{V\mathrm{Cat}}\colon\ \mathrm{M}_{V\mathrm{Cat}} \to \mathrm{G}_{V\mathrm{Cat}}$.

First, we want to determine to which elements from the list `SCM[2]` the generators `a1,a2,a3,a4` of the group `GI` correspond. For that, we map `a1, a2, a3, a4` to `GroupOfIsotrafos` via the group morphism `psiI^-1` $\blacktriangle$ `phiI`. Then we map these elements to elements of the list `SCM[2]`, using that `SCM[2][1]` is the identity of the group $\mathrm{M}_{V\mathrm{Cat}}$.

We get the following result.

```
actI := Action(GSet(GroupOfIsotrafos));
// action of an isotransformation in GroupOfIsotrafos
// on the elements of GroupOfIsotrafos via Cayley

<1,a1@(psiI^-1)@phiI>@actI;
// 33
<1,a2@(psiI^-1)@phiI>@actI;
// 22
<1,a3@(psiI^-1)@phiI>@actI;
// 17
<1,a4@(psiI^-1)@phiI>@actI;
// 11
```

We obtain the following correspondences.

$$\begin{aligned} \texttt{a1} &\leftrightarrow \texttt{SCM[2][33]} \\ \texttt{a2} &\leftrightarrow \texttt{SCM[2][22]} \\ \texttt{a3} &\leftrightarrow \texttt{SCM[2][17]} \\ \texttt{a4} &\leftrightarrow \texttt{SCM[2][11]} \end{aligned}$$

The following function yields the number of a given morphism from the list `Mor_list`.

```
ind_ml := function(x)
// x: morphism; yields the number of the morphism x from the list Mor_list
 return Index(Mor_list,x);
end function;
```

Note that $\mathrm{f}_{V\mathrm{Cat}}$ maps the element `SCM[2][33]` to its target functor $F :=$ `SCM[2][33][2]`. Moreover, we have `SCM[2][33][2][2]` $= \mathrm{Mor}(F)$. So, for $x \in \mathrm{Mor}(V\mathrm{Cat})$,

```
(x@SCM[2][33][2][2])@ind_ml
```

is the number of the morphism xF in the list `Mor_list`. So the map $\mathrm{Mor}(F)$ can be written as the permutation

```
S![(x@SCM[2][33][2][2])@ind_ml : x in Mor_list] ,
```

where `S = SymmetricGroup(#Mor_list)`. It is an element in `GroupOfAutofunctors`; cf. §A.5. Finally, we map it to the group `GA` via the group morphism `phiA^-1` ▴ `psiA`.

We obtain the following.

```
(S![(x@SCM[2][33][2][2])@ind_ml : x in Mor_list])@(phiA^-1)@psiA;
// (1, 3)(2, 4), this is F2
(S![(x@SCM[2][21][2][2])@ind_ml : x in Mor_list])@(phiA^-1)@psiA;
// (5, 8)(6, 7), this is F1 * F2 * F3 * F1
(S![(x@SCM[2][17][2][2])@ind_ml : x in Mor_list])@(phiA^-1)@psiA;
// (5, 7)(6, 8), this is F1 * F2 * F1
(S![(x@SCM[2][11][2][2])@ind_ml : x in Mor_list])@(phiA^-1)@psiA;
// (1, 2)(3, 4), this is F3
```

We may now construct `fPerm` as follows.

```
FI<A1,A2,A3,A4> := FreeGroup(4);
eta := hom<FI -> GI | [a1,a2,a3,a4]>;
xi := hom<FI -> GA | [F2, F1*F2*F3*F1, F1*F2*F1, F3]>;
fPerm := hom<GI -> GA | x :-> x@@eta@xi >;
```

Altogether, the group morphism `fPerm` maps as follows.

$$
\begin{array}{rcl}
\mathtt{GI} & \to & \mathtt{GA} \\
\mathtt{a1} & \mapsto & \mathtt{a1@fPerm} = \mathtt{(1,\ 3)(2,\ 4)} = \mathtt{F2} \\
\mathtt{a2} & \mapsto & \mathtt{a2@fPerm} = \mathtt{(5,\ 8)(6,\ 7)} = \mathtt{F1 * F2 * F3 * F1} \\
\mathtt{a3} & \mapsto & \mathtt{a3@fPerm} = \mathtt{(5,\ 7)(6,\ 8)} = \mathtt{F1 * F2 * F1} \\
\mathtt{a4} & \mapsto & \mathtt{a4@fPerm} = \mathtt{(1,\ 2)(3,\ 4)} = \mathtt{F3}
\end{array}
$$

Note that `fPerm : GI` $\to$ `GA` is not injective since we have `#GI = 64` and `#GA = 32`. Moreover, `fPerm` is not surjective since we have `Order(Image(fPerm)) = 16`.

In particular, the kernel of `fPerm` has order 4 and cokernel of `fPerm` has order 2. In other words, we have $|\,\mathrm{S}_{V\mathrm{Cat}}\,\pi_1| = 4$ and $|\,\mathrm{S}_{V\mathrm{Cat}}\,\pi_0| = 2$.

$$
\begin{tikzcd}
 & \mathrm{M}_{V\mathrm{Cat}} \arrow[r, "\mathrm{f}_{V\mathrm{Cat}}"] \arrow[d, equal] & \mathrm{G}_{V\mathrm{Cat}} \arrow[d, equal] & \\
 & \texttt{SCM[2]} & \texttt{SCM[1]} \arrow[d] & \\
 & \texttt{GroupOfIsotrafos} \arrow[u] & \texttt{GroupOfAutofunctors} \arrow[dr, "\texttt{phiA\^{}-1}"] & \\
\texttt{GIFP} \arrow[ur, "\texttt{phiI}"] & & & \texttt{GAFP} \arrow[dl, "\texttt{psiA}"] \\
 & \texttt{GI} \arrow[ul, "\texttt{psiI\^{}-1}"] \arrow[r, dotted, "\texttt{fPerm}"] & \texttt{GA} &
\end{tikzcd}
$$

A.8 The group action $\gamma_{V\mathrm{Cat}}\colon \mathrm{G}_{V\mathrm{Cat}} \to \mathrm{Aut}(\mathrm{M}_{V\mathrm{Cat}})$

Now we want to determine the action of `GroupOfAutofunctors` on `GroupOfIsotrafos`, isomorphically replaced by an action of `GA` on `GI`. It suffices to determine the action on generators.

For that, we need the following functions.

```
actA := Action(GSet(GroupOfAutofunctors));
// action of an autofunctor in GroupOfAutofunctors on a morphism in VCat
IsEqualIsotrafoSmall := function(a,b)
// a, b: maps from Ob to Mor
// Compares two isotransformations at every object.
 return &and[g@a eq g@b : g in Ob];
end function;
```

Recall that `GI` = ⟨ `a1, a2, a3, a4` ⟩ and that we have the following correspondance; cf. §A.6.

```
a1 ↔ SCM[2][33]
a2 ↔ SCM[2][22]
a3 ↔ SCM[2][17]
a4 ↔ SCM[2][11]
```

Recall that we have `GA` = ⟨ `F1, F2, F3` ⟩ , and that we have the following situation; cf. §A.5.

$$\begin{tikzcd} & \mathtt{GAFP} \arrow[dl, "\mathtt{psiA}"'] \arrow[dr, "\mathtt{phiA}"] & \\ \mathtt{GA} & & \mathtt{GroupOfAutofunctors} \end{tikzcd}$$

We map the generators `F1, F2, F3` to `GroupOfAutofunctors` .

```
F1M := F1@(psiA^-1)@phiA;
F2M := F2@(psiA^-1)@phiA;
F3M := F3@(psiA^-1)@phiA;
```

Now we want to determine the action of `F1` on `a1` , i.e. we want to calculate the isotransformation `a1^F1` .

Suppose given `g` $\in \mathrm{Ob}(V\mathrm{Cat}) = G$. We have the following.

```
Mor_list[<<g,Id(M)>@ind_ml,F1M^-1>@actA];
// image of the morphism (g,1) under the functor F1M^-1

Mor_list[<<g,Id(M)>@ind_ml,F1M^-1>@actA][1];
// image of the object g under the functor F1M^-1

Mor_list[<<g,Id(M)>@ind_ml,F1M^-1>@actA][1]@SCM[2][33][1];
// isotransformation corresponding to a1, precomposed with F1M^-1,
// evaluated at the object g
```

```
Mor_list[<Mor_list[<<g,Id(M)>@ind_ml,F1M^-1>@actA][1]@SCM[2][33][1]
                                                   @ind_ml,F1M>@actA];
// isotransformation corresponding to a1, precomposed with F1M^-1,
// postcomposed with F1M, evaluated at the object g
```

So the transformation `a1^F1` corresponds to the following element of the list `SCM[2]`.

```
a1toF1 := map< Ob -> Mor | g :-> Mor_list[ <Mor_list[
           <<g,Id(M)>@ind_ml,F1M^-1>@actA ][1]@SCM[2][33][1]@ind_ml,F1M>@actA ]>;
```

Further,

```
a1toF1Nr := [i : i in [1..#SCM[2]] |
                         IsEqualIsotrafoSmall(a1toF1,SCM[2][i][1])] [1];
```

yields the index `17` of the isotransformation `a1toF1` in the list `SCM[2]`.

So `a1^F1` is given by the following element in `GI`.

```
(SI![MultOfIsotrafos(j,a1toF1Nr) : j in [1..#SCM[2]]])@(phiI^-1)@psiI;
```

We calculate the following.

```
a1toF1 := map< Ob -> Mor | g :-> Mor_list[ <Mor_list[
            <<g,Id(M)>@ind_ml,F1M^-1>@actA ][1]@SCM[2][33][1]@ind_ml,F1M>@actA ]>;
a1toF1Nr := [i : i in [1..#SCM[2]] | IsEqualIsotrafoSmall(a1toF1,SCM[2][i][1])][1];
// 17
(SI![MultOfIsotrafos(j,a1toF1Nr) : j in [1..#SCM[2]]])@(phiI^-1)@psiI;
// (1, 5, 2, 6)(3, 7, 4, 8)(9, 13, 10, 14)(11, 15, 12, 16) , this is a3

a1toF2 := map< Ob -> Mor | g :-> Mor_list[ <Mor_list[
            <<g,Id(M)>@ind_ml,F2M^-1>@actA ][1]@SCM[2][33][1]@ind_ml,F2M>@actA ]>;
a1toF2Nr := [i : i in [1..#SCM[2]] | IsEqualIsotrafoSmall(a1toF2,SCM[2][i][1])][1];
// 33
(SI![MultOfIsotrafos(j,a1toF2Nr) : j in [1..#SCM[2]]])@(phiI^-1)@psiI;
// (1, 9, 3, 11)(2, 10, 4, 12)(5, 13, 7, 15)(6, 14, 8, 16), this is a1
```

```
a1toF3 := map<Ob -> Mor | g :-> Mor_list[ <Mor_list[
             <<g,Id(M)>@ind_ml,F3M^-1>@actA ][1]@SCM[2][33][1]@ind_ml,F3M>@actA ]>;
a1toF3Nr := [i : i in [1..#SCM[2]] | IsEqualIsotrafoSmall(a1toF3,SCM[2][i][1])][1];
// 35
(SI![MultOfIsotrafos(j,a1toF3Nr) : j in [1..#SCM[2]]])@(phiI^-1)@psiI;
// (1, 11, 3, 9)(2, 12, 4, 10)(5, 15, 7, 13)(6, 16, 8, 14), this is a1^-1

a2toF1 := map<Ob -> Mor | g :-> Mor_list[ <Mor_list[
             <<g,Id(M)>@ind_ml,F1M^-1>@actA ][1]@SCM[2][22][1]@ind_ml,F1M>@actA ]>;
a2toF1Nr := [i : i in [1..#SCM[2]] | IsEqualIsotrafoSmall(a2toF1,SCM[2][i][1])][1];
// 41
(SI![MultOfIsotrafos(j,a2toF1Nr) : j in [1..#SCM[2]]])@(phiI^-1)@psiI;
// (1, 9)(2, 10)(3, 11)(4, 12)(5, 13)(6, 14)(7, 15)(8, 16), this is a1 * a4

a2toF2 := map<Ob -> Mor | g :-> Mor_list[ <Mor_list[
             <<g,Id(M)>@ind_ml,F2M^-1>@actA ][1]@SCM[2][22][1]@ind_ml,F2M>@actA ]>;
a2toF2Nr := [i : i in [1..#SCM[2]] | IsEqualIsotrafoSmall(a2toF2,SCM[2][i][1])][1];
// 22
(SI![MultOfIsotrafos(j,a2toF2Nr) : j in [1..#SCM[2]]])@(phiI^-1)@psiI;
// (1, 6)(2, 5)(3, 8)(4, 7)(9, 14)(10, 13)(11, 16)(12, 15), this is a2

a2toF3 := map<Ob -> Mor | g :-> Mor_list[ <Mor_list[
             <<g,Id(M)>@ind_ml,F3M^-1>@actA ][1]@SCM[2][22][1]@ind_ml,F3M>@actA ]>;
a2toF3Nr := [i : i in [1..#SCM[2]] | IsEqualIsotrafoSmall(a2toF3,SCM[2][i][1])][1];
// 22
(SI![MultOfIsotrafos(j,a2toF3Nr) : j in [1..#SCM[2]]])@(phiI^-1)@psiI;
// (1, 6)(2, 5)(3, 8)(4, 7)(9, 14)(10, 13)(11, 16)(12, 15), this is a2

a3toF1 := map<Ob -> Mor | g :-> Mor_list[ <Mor_list[
             <<g,Id(M)>@ind_ml,F1M^-1>@actA ][1]@SCM[2][17][1]@ind_ml,F1M>@actA ]>;
a3toF1Nr := [i : i in [1..#SCM[2]] | IsEqualIsotrafoSmall(a3toF1,SCM[2][i][1])][1];
// 33
(SI![MultOfIsotrafos(j,a3toF1Nr) : j in [1..#SCM[2]]])@(phiI^-1)@psiI;
// (1, 9, 3, 11)(2, 10, 4, 12)(5, 13, 7, 15)(6, 14, 8, 16), this is a1
```

```
a3toF2 := map<Ob -> Mor | g :-> Mor_list[ <Mor_list[
            <<g,Id(M)>@ind_ml,F2M^-1>@actA ][1]@SCM[2][17][1]@ind_ml,F2M>@actA ]>;
a3toF2Nr := [i : i in [1..#SCM[2]] | IsEqualIsotrafoSmall(a3toF2,SCM[2][i][1])][1];
// 17
(SI![MultOfIsotrafos(j,a3toF2Nr) : j in [1..#SCM[2]]])@(phiI^-1)@psiI;
// (1, 5, 2, 6)(3, 7, 4, 8)(9, 13, 10, 14)(11, 15, 12, 16), this is a3

a3toF3 := map<Ob -> Mor | g :-> Mor_list[ <Mor_list[
            <<g,Id(M)>@ind_ml,F3M^-1>@actA ][1]@SCM[2][17][1]@ind_ml,F3M>@actA ]>;
a3toF3Nr := [i : i in [1..#SCM[2]] | IsEqualIsotrafoSmall(a3toF3,SCM[2][i][1])][1];
// 17
(SI![MultOfIsotrafos(j,a3toF3Nr) : j in [1..#SCM[2]]])@(phiI^-1)@psiI;
// (1, 5, 2, 6)(3, 7, 4, 8)(9, 13, 10, 14)(11, 15, 12, 16), this is a3

a4toF1 := map<Ob -> Mor | g :-> Mor_list[ <Mor_list[
            <<g,Id(M)>@ind_ml,F1M^-1>@actA ][1]@SCM[2][11][1]@ind_ml,F1M>@actA ]>;
a4toF1Nr := [i : i in [1..#SCM[2]] | IsEqualIsotrafoSmall(a4toF1,SCM[2][i][1])][1];
// 5
(SI![MultOfIsotrafos(j,a4toF1Nr) : j in [1..#SCM[2]]])@(phiI^-1)@psiI;
// (5, 6)(7, 8)(13, 14)(15, 16), this is a2 * a3

a4toF2 := map<Ob -> Mor | g :-> Mor_list[ <Mor_list[
            <<g,Id(M)>@ind_ml,F2M^-1>@actA ][1]@SCM[2][11][1]@ind_ml,F2M>@actA ]>;
a4toF2Nr := [i : i in [1..#SCM[2]] | IsEqualIsotrafoSmall(a4toF2,SCM[2][i][1])][1];
// 9
(SI![MultOfIsotrafos(j,a4toF2Nr) : j in [1..#SCM[2]]])@(phiI^-1)@psiI;
// (9, 11)(10, 12)(13, 15)(14, 16), this is a1^2 * a4

a4toF3 := map<Ob -> Mor | g :-> Mor_list[ <Mor_list[
            <<g,Id(M)>@ind_ml,F3M^-1>@actA ][1]@SCM[2][11][1]@ind_ml,F3M>@actA ]>;
a4toF3Nr := [i : i in [1..#SCM[2]] | IsEqualIsotrafoSmall(a4toF3,SCM[2][i][1])][1];
// 11
(SI![MultOfIsotrafos(j,a4toF3Nr) : j in [1..#SCM[2]]])@(phiI^-1)@psiI;
// (1, 3)(2, 4)(5, 7)(6, 8), this is a4
```

We may now construct `gammaPerm` as follows. Recall that

```
FI<A1,A2,A3,A4> = FreeGroup(4);
eta = hom<FI -> GI | [a1,a2,a3,a4]>;
```

cf. A.7. We have

```
zeta1 := hom<FI -> GI | [a3,a1*a4,a1,a2*a3]>;
F1aut := hom<GI -> GI | x :-> x@@eta@zeta1 >;
// action of F1 on the generators of GI

zeta2 := hom<FI -> GI | [a1,a2,a3,a1^2 * a4]>;
F2aut := hom<GI -> GI | x :-> x@@eta@zeta2 >;
// action of F2 on the generators of GI

zeta3 := hom<FI -> GI | [a1^-1,a2,a3,a4]>;
F3aut := hom<GI -> GI | x :-> x@@eta@zeta3 >;
// action of F3 on the generators of GI

FA<FF1,FF2,FF3> := FreeGroup(3);
etaA := hom<FA -> GA | [F1,F2,F3]>;
xiA := hom<FA -> AutomorphismGroup(GI) | [F1aut,F2aut,F3aut]>;

gammaPerm := hom< GA -> AutomorphismGroup(GI) | x :-> x@@etaA@xiA >;
```

So, the isomorphic replacement `gammaPerm` of $\gamma_{V\mathrm{Cat}}$ acts as follows.

$$
\begin{aligned}
&\texttt{a1\^{}F1} = \texttt{a1@(F1@gammaPerm)} = \texttt{a3}\\
&\texttt{a1\^{}F2} = \texttt{a1@(F2@gammaPerm)} = \texttt{a1}\\
&\texttt{a1\^{}F3} = \texttt{a1@(F3@gammaPerm)} = \texttt{a1\^{}-1}\\
&\texttt{a2\^{}F1} = \texttt{a2@(F1@gammaPerm)} = \texttt{a1 * a4}\\
&\texttt{a2\^{}F2} = \texttt{a2@(F2@gammaPerm)} = \texttt{a2}\\
&\texttt{a2\^{}F3} = \texttt{a2@(F3@gammaPerm)} = \texttt{a2}
\end{aligned}
$$

a3^F1 = a3@(F1@gammaPerm) = a1
a3^F2 = a3@(F2@gammaPerm) = a3
a3^F3 = a3@(F3@gammaPerm) = a3
a4^F1 = a4@(F1@gammaPerm) = a2 * a3
a4^F2 = a4@(F2@gammaPerm) = a1^2 * a4
a4^F3 = a4@(F3@gammaPerm) = a4

A.9 The crossed module $\mathrm{S}_{V\mathrm{Cat}}$, isomorphically replaced

We summarize.

Given V as in §A.1, the crossed module $\mathrm{S}_{V\mathrm{Cat}}$ is isomorphic to the crossed module (GA, GI, gammaPerm, fPerm).

From §A.5, we have GA = ⟨ F1, F2, F3 ⟩, where

```
F1 = GA!(1, 5)(2, 6)(3, 7)(4, 8);
F2 = GA!(1, 3)(2, 4);
F3 = GA!(1, 2)(3, 4);
```

Moreover, from §A.6, GI = ⟨ a1, a2, a3, a4 ⟩, where

```
a1 = GI!(1, 9, 3, 11)(2, 10, 4, 12)(5, 13, 7, 15)(6, 14, 8, 16);
a2 = GI!(1, 6)(2, 5)(3, 8)(4, 7)(9, 14)(10, 13)(11, 16)(12, 15);
a3 = GI!(1, 5, 2, 6)(3, 7, 4, 8)(9, 13, 10, 14)(11, 15, 12, 16);
a4 = GI!(1, 3)(2, 4)(5, 7)(6, 8);
```

The group morphism fPerm : GI → GA maps as follows; cf. §A.7.

a1 ↦ a1@fPerm = (1, 3)(2, 4) = F2
a2 ↦ a2@fPerm = (5, 8)(6, 7) = F1 * F2 * F3 * F1
a3 ↦ a3@fPerm = (5, 7)(6, 8) = F1 * F2 * F1
a4 ↦ a4@fPerm = (1, 2)(3, 4) = F3

The group morphism `gammaPerm` : `GA` $\to$ Aut(`GI`) maps as follows; cf. §A.8.

```
a1^F1 = a1@(F1@gammaPerm) = a3
a1^F2 = a1@(F2@gammaPerm) = a1
a1^F3 = a1@(F3@gammaPerm) = a1^-1
a2^F1 = a2@(F1@gammaPerm) = a1 * a4
a2^F2 = a2@(F2@gammaPerm) = a2
a2^F3 = a2@(F3@gammaPerm) = a2
a3^F1 = a3@(F1@gammaPerm) = a1
a3^F2 = a3@(F2@gammaPerm) = a3
a3^F3 = a3@(F3@gammaPerm) = a3
a4^F1 = a4@(F1@gammaPerm) = a2 * a3
a4^F2 = a4@(F2@gammaPerm) = a1^2 * a4
a4^F3 = a4@(F3@gammaPerm) = a4
```

A.10 The Cayley embedding

Consider the embedding $(\lambda^{\text{Cayley}}, \mu^{\text{Cayley}})\colon V \to \mathrm{S}_{V\mathrm{Cat}}$, where

$$\begin{aligned} \mu^{\text{Cayley}}\colon\ & G \longrightarrow \mathrm{G}_{V\mathrm{Cat}}\,, \quad x \mapsto x\mu^{\text{Cayley}} := \Big(\big(g \xrightarrow{(g,m)} g(mf)\big) \mapsto \big(gx \xrightarrow{(gx,m^x)} g(mf)x\big)\Big) \\ \lambda^{\text{Cayley}}\colon\ & M \longrightarrow \mathrm{M}_{V\mathrm{Cat}}\,, \quad m \mapsto m\lambda^{\text{Cayley}} := \big(g \xrightarrow[\sim]{(g,m)} g(mf)\big)_{g\in G}\,; \end{aligned}$$

cf. Proposition 59. We want to calculate the isomorphic replacements for μ^{Cayley} and for λ^{Cayley}.

A.10.1 The group morphism μ^{Cayley}

Recall that we have `G` $= \langle\, a \,\rangle$ with $a =$ `G.1` $=$ `(1, 2, 3, 4)`; cf. §A.1.

Here, we calculate the group morphism `muCayley` : `G` $\to$ `GA`, the isomorphic replacement of $\mu^{\text{Cayley}}\colon G \to \mathrm{G}_{V\mathrm{Cat}}$.

```
muCayley := function(x)
// x : element in G
// group morphism from G to GA
 Ob  := Set(G);
 Mor := CartesianProduct(Set(G),Set(M));
 F_Ob := map<Ob -> Ob | g :-> g*x >;
 F_Mor := map<Mor -> Mor | u :-> <u[1]*x,u[2]@(x@gamma)> >;
 if not &and[(<y[1], y[2] * y[3]>@F_Mor)[2] eq
      (<y[1],y[2]>@F_Mor)[2] * (<y[1] * y[2]@f, y[3]>@F_Mor)[2] :
      y in CartesianProduct([G,M,M])] then
  print "Not a functor";
 end if;
 return <F_Ob,F_Mor>;
end function;
```

Then the image of `G.1` under `muCayley` is given as follows.

```
autofunctor_a :=
(GroupOfAutofunctors![(x@(muCayley(G.1)[2]))@ind_ml : x in Mor_list])@@phiA@psiA;
//(1, 8, 3, 6)(2, 7, 4, 5), this is F2 * F3 * F1 * F3
```

So `muCayley` maps as follows.

$$\begin{array}{rcl} \mathtt{G} & \to & \mathtt{GA} \\ \mathtt{G.1} & \mapsto & \mathtt{autofunctor\_a = F2 * F3 * F1 * F3} \end{array}$$

A.10.2 The group morphism λ^{Cayley}

Recall that we have $\mathtt{M} = \langle\, b \,\rangle$ with $b = \mathtt{M.1} = \mathtt{(1,\ 2,\ 3,\ 4)}$; cf. §A.1.

Here, we calculate the group morphism `lambdaCayley` $:\ \mathtt{M} \to \mathtt{GI}$, the isomorphic replacement of $\lambda^{\text{Cayley}} \colon M \to \mathrm{M}_{V\mathrm{Cat}}$.

```
lambdaCayley := function(y)
// y : element in M
// group morphism from M to GI
 Ob  := Set(G);
```

```
 Mor := CartesianProduct(Set(G),Set(M));
 F := muCayley(y@f);
 trafo := map<Ob -> Mor | g :-> <g,y> >;
 if not &and[(z[1]@trafo)[2] * (z@F[2])[2]
          eq z[2] * ((z[1]*(z[2]@f))@trafo)[2] : z in Mor] then
   print "Not a transformation";
 end if;
 return <trafo, F>;
end function;
```

We calculate the following.

```
lambdaCayley(M.1);
// image of M.1 under lambdaCayley, M.1 as an isotransformation in GI

lambdaNr := [i : i in [1..#SCM[2]] |
         IsEqualIsotrafoSmall(lambdaCayley(M.1)[1], SCM[2][i][1])][1];
// index of the transformation lambdaCayley(M.1) from the list SCM[2],
// this yields 49

isotrafo_b :=
(SI![MultOfIsotrafos(j,lambdaNr) : j in [1..#SCM[2]]])@(phiI^-1)@psiI;
// M.1 as a permutation in GI,
// this yields (1, 13, 4, 16)(2, 14, 3, 15)(5, 10, 8, 11)(6, 9, 7, 12),
// which is a1 * a3
```

So `lambdaCayley` maps as follows.

$$\begin{aligned} \texttt{M} &\to \texttt{GI} \\ \texttt{M.1} &\mapsto \texttt{isotrafo\_b := a1 * a3} \end{aligned}$$

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Zusammenfassung

Ein verschränkter Modul $V := \big(M, G, \gamma, f\big)$ besteht aus Gruppen M und G, einer Operation $\gamma\colon G \to \mathrm{Aut}(M)$, $g \mapsto (m \mapsto m^g)$ und einem Gruppenmorphismus $f\colon M \to G$, der

$$(m^g)f = (mf)^g \quad \text{und} \quad m^n = m^{nf}$$

erfüllt für m, $n \in M$ und $g \in G$.

Eine monoidale Kategorie ist eine Kategorie $\mathcal{C}$ zusammen mit einem Einheitsobjekt I und einem assoziativen Tensorprodukt $(\otimes)$ auf den Objekten $\mathrm{Ob}(\mathcal{C})$ und den Morphismen $\mathrm{Mor}(\mathcal{C})$.

Eine invertierbar monoidale Kategorie ist eine monoidale Kategorie $\mathcal{C}$, deren Objekte und Morphismen bezüglich des Tensorproduktes $(\otimes)$ invertierbar sind.

Die Kategorie der verschränkten Moduln und die Kategorie der invertierbaren monoidalen Kategorien sind äquivalent vermöge des Isofunktors Cat. Zu einem verschränkten Modul V haben wir also eine invertierbar monoidale Kategorie $V\mathrm{Cat}$.

Zu einer Menge X gibt es die symmetrische Gruppe S_X. Analog können wir auf einer Kategorie $\mathcal{X}$ den symmetrischen verschränkten Modul $\mathrm{S}_{\mathcal{X}}$ definieren, der aus den Autofunktoren von $\mathcal{X}$ und zugehörigen Isotransformationen besteht.

Nach dem Satz von Cayley gibt es zu einer Gruppe G einen injektiven Gruppenmorphismus $G \to \mathrm{S}_G$. Analog gibt es die folgende Aussage: Zu einem verschränkten Modul V gibt es einen injektiven verschränkten Modulmorphismus $V \to \mathrm{S}_{V\mathrm{Cat}}$.

Zu einem R-Modul M gibt es die Endomorphismen-Algebra $\mathrm{End}_R(M)$ und die Automorphismengruppe $\mathrm{Aut}_R(M)$. Analog dazu gibt es zu einer R-linearen Kategorie $\mathcal{M}$ die monoidale R-lineare Kategorie $\mathrm{End}_R(\mathcal{M})$. Diese hat die invertierbare monoidale Teilkategorie $\big(\mathrm{End}_R(\mathcal{M})\big)\mathrm{U}$, bestehend aus den tensorinvertierbaren Objekten und Morphismen.

Desweiteren haben wir den verschränkten Teilmodul $\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})$ von $\mathrm{S}_{\mathcal{M}}$, bestehend aus den R-linearen Autofunktoren von $\mathcal{M}$ und zugehörigen Isotransformationen.

Wir haben einen monoidalen Isofunktor zwischen invertierbaren monoidalen Kategorien

$$\mathrm{Real}_{\mathcal{M}}\colon \big(\mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})\big)\mathrm{Cat} \xrightarrow{\sim} \big(\mathrm{End}_R(\mathcal{M})\big)\mathrm{U}\,.$$

Klassisch lässt sich ein A-Modul, für eine R-Algebra A, angeben durch einen R-Modul M und einen R-Algebrenmorphismus $A \to \mathrm{End}_R(M)$.

Für eine monoidale R-lineare Kategorie $\mathcal{A}$ lässt sich ein $\mathcal{A}$-Modul angeben durch eine R-lineare Kategorie $\mathcal{M}$ und einen monoidalen R-linearen Funktor $\mathcal{A} \to \mathrm{End}_R(\mathcal{M})$.

Klassisch ist ein Darstellung von G auf M, für eine Gruppe G und einen R-Modul M, definiert als ein Gruppenmorphismus $G \to \mathrm{Aut}_R(M)$.

Für einen verschränkten Modul V und eine R-lineare Kategorie $\mathcal{M}$ ist eine Darstellung von V auf M definiert als ein verschränkter Modulmorphismus $V \to \mathrm{Aut}_R^{\mathrm{CM}}(\mathcal{M})$. Einer solchen Darstellung entspricht ein monoidaler R-linearer Funktor $(V\mathrm{Cat})R \to \mathrm{End}_R(\mathcal{M})$. Somit spielt $(V\mathrm{Cat})R$ die analoge Rolle zum Gruppenring im klassischen Fall.

Ferner können wir ein Analogon zu einem Permutationsmodul konstruieren. Sei $\mathcal{X}$ eine Kategorie. Sei V ein verschränkter Modul. Sei $V \to \mathrm{S}_{\mathcal{X}}$ ein verschränkter Modulmorphismus, d.h. V operiere auf $\mathcal{X}$. Sei $\mathcal{X}R$ die R-lineare Hülle von $\mathcal{X}$. Wir konstruieren den monoidalen R-linearen Funktor $(V\mathrm{Cat})R \to \mathrm{End}_R(\mathcal{X}R)$, welcher $\mathcal{X}R$ zu einem $(V\mathrm{Cat})R$-Modul macht, genannt Permutationsmodul auf der V-Kategorie $\mathcal{X}$.

Erklärung

Hiermit versichere ich, dass ich meine Arbeit selbstständig verfasst und keine andere als die angegebenen Quellen benutzt habe. Alle wörtlich oder sinngemäß aus anderen Werken übernommenen Aussagen habe ich als solche gekennzeichnet. Weiterhin versichere ich, dass die eingereichte Arbeit weder vollständig noch in wesentlichen Teilen Gegenstand eines anderen Prüfungsverfahrens gewesen ist und dass das elektronische Exemplar mit den anderen Exemplaren übereinstimmt.

Stuttgart, Oktober 2018

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