Invasion phenomena in pattern-forming systems admitting a conservation law structure

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Abstract

Pattern-forming systems admitting a conservation law structure are omnipresent in physical problems. A particular class of interest are free boundary problems in fluid dynamics such as the Bénard-Marangoni problem. In these systems patterns often arise in the wake of an invading heteroclinic front which connects the unstable ground state to a spatially periodic pattern. The main objective of this thesis is to understand this transition process by studying these so-called *modulating traveling fronts*.

In the first part we show the existence of modulating traveling fronts in two model problems. First we consider a one-dimensional Swift-Hohenberg equation coupled to a conservation law. Then, in a second step we study the effect of adding dispersion to the model, which breaks the reflexion symmetry. In both cases we observe that as a bifurcation parameter increases beyond a threshold the system undergoes a Turing bifurcation. We study the dynamics near this bifurcation. Initially, we show that stationary, periodic solutions bifurcate from a homogeneous ground state. Following that, we construct modulating traveling fronts for the systems, which provide a mechanism of pattern formation. The existence proof uses spatial dynamics and center manifold theory for a reduction to a finite-dimensional problem.

In the non-dispersive case, the main challenge lies in the center manifold reduction due to the presence of infinitely many imaginary eigenvalues for vanishing bifurcation parameter. However, we show that center manifold theory is still applicable since the eigenvalues leave the imaginary axis with different velocities as the bifurcation parameter increases. Compared to non-conservative systems, we address new difficulties arising from an additional neutral mode at Fourier wave number $k = 0$ by exploiting that the amplitude of the conserved variable is small compared to the other variables.

In the dispersive case, the center manifold reduction can be performed using standard results. However, the main difficulty shifts to the analysis of the reduced dynamics on the center manifold since the presence of dispersive effects gives rise to genuinely complex coefficients, while we obtained real coefficients in the non-dispersive case.

In the second part of this thesis, we study the nonlinear stability of invading fronts in a Ginzburg-Landau equation with an additional conservation law. This system appears as a generic amplitude equation for pattern-forming systems admitting a conservation law structure. In particular, the invading fronts approximately describe the amplitude of modulating traveling fronts discussed in the first part, which motivates the investigation of their stability. We prove the nonlinear stability of sufficiently fast invading fronts with respect to perturbations which are exponentially localized ahead of the front. The proof is based on the use of exponential weights ahead of the front to stabilize the ground state. The main challenges are the lack of a comparison principle and the fact that the invading state is only diffusively stable, i.e. perturbations of the invading state decay polynomially in time.

Zusammenfassung

Musterbildende Systeme mit Erhaltungsgröße sind allgegenwärtig in physikalischen Problemen. Von besonderem Interesse sind hierbei fluiddynamische Probleme mit freier Oberfläche wie das Bénard-Marangoni Problem. In diesen Systemen bilden sich Muster oft durch eine heterokline Front, die einen instabilen Grundzustand mit einem räumlich periodischen Muster verbindet. Das zentrale Ziel dieser Arbeit ist es, diesen Übergangsprozess zu verstehen, indem diese sogenannten *modulierenden Fronten* untersucht werden.

Im ersten Teil zeigen wir die Existenz von modulierenden Fronten für zwei Modellprobleme. Zuerst betrachten wir eine eindimensionale Swift-Hohenberg Gleichung, die mit einer Erhaltungsgleichung gekoppelt ist. In einem weiteren Schritt untersuchen wir dann den Effekt von zusätzlichen dispersiven Termen im Modell. In beiden Fällen beobachten wir, dass eine Turing-Bifurkation im System stattfindet, wenn ein Bifukationsparameter über einen kritischen Wert hinaus erhöht wird. Wir untersuchen die Dynamik nahe dieser Bifurkation. Zunächst zeigen wir, dass stationäre, periodische Lösungen von einem homogenen Grundzustand verzweigen. Danach konstruieren wir modulierende Fronten für das System, die einen Mechanismus für Musterbildung darstellen. Im Existenzbeweis nutzen wir räumliche Dynamik und die Theorie der Zentrumsmannigfaltigkeiten, um eine Reduktion auf ein endlichdimensionales Problem durchzuführen.

Im nicht-dispersiven Fall liegt die zentrale Schwierigkeit in der Zentrumsmannigfaltigkeitenreduktion, da am Bifurkationspunkt unendlich viele Eigenwerte auf der imaginären Achse liegen. Wir zeigen jedoch, dass die Theorie der Zentrumsmannigfaltigkeiten trotzdem anwendbar ist, da die Eigenwerte die imaginäre Achse mit unterschiedlichen Geschwindigkeiten verlassen wenn der Bifurkationsparameter erhöht wird. Verglichen zu musterbildenden Systemen ohne Erhaltungsgröße treten hierbei neue Herausforderungen auf, die durch eine zusätzliche, neutrale Mode an der Fourierwellenzahl $k = 0$ entstehen. Zur Lösung dieser Schwierigkeiten nutzen wir aus, dass die Amplitude der Erhaltungsgröße klein ist im Vergleich zu den anderen Variablen.

Im dispersiven Fall können wir die Zentrumsmannigfaltigkeitenreduktion mittels Stan-

dartresultaten durchführen. Die Schwierigkeit in diesem Fall liegt in der Analyse der reduzierten Dynamik auf der Zentrumsmannigfaltigkeit, da die dispersiven Effekte zu komplexen Koeffizienten im reduzierten System führen, während wir im nicht-dispersiven Fall reelle Koefizienten erhalten haben.

Im zweiten Teil dieser Arbeit untersuchen wir die nichtlineare Stabilität von invasiven Fronten in einer Ginzburg-Landau Gleichung mit einer zusätzlichen Erhaltungsgröße. Dieses System tritt generisch als Amplitudengleichung bei musterbildenden Systemen mit Erhaltungsgröße auf. Insbesondere approximieren die invasiven Fronten die Amplitude der modulierenden Fronten, die im ersten Teil diskutiert wurden, was die Untersuchung ihrer Stabilität motiviert. Wir zeigen die nichtlineare Stabilität von hinreichend schnellen invasiven Fronten bezüglich Störungen, die vor der Front exponentiell lokalisiert sind. Im Beweis verwenden wir exponentielle Gewichte vor der Front, um den Grundzustand zu stabilisieren. Schwierigkeiten entstehen hierbei dadurch, dass das System kein Vergleichsprinzip erfüllt und dadurch, dass der invasive Zustand nur diffusive Stabilität aufweist und Störungen des invasiven Zustandes daher nur polynomiell in der Zeit abfallen.

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My colleagues at the IADM also deserve my special thanks for creating such an enjoyable and inspiring work atmosphere. Finally, I would like to thank my family and friends. Without their ongoing support this endeavor would not have been possible.

I hereby certify that this thesis has been composed by myself and describes my own work unless otherwise acknowledged in the text. All references and verbatim extracts have been quoted and all sources of information have been specifically acknowledged.

Stuttgart, 2021 Bastian Hilder

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1. Introduction

The dynamics of pattern-forming systems on unbounded domains has been an important part of research for the last decades since such phenomena can be observed in many real-world examples from biology, chemistry and fluid dynamics, see [CH93]. Prototypical examples of such systems are the Taylor-Couette problem and the Rayleigh-Bénard problem. Typically, pattern formation occurs as follows. When an external control parameter, e.g. the temperature for the Reyleigh-Bénard problem, increases beyond a critical value the spatially homogeneous equilibrium gets unstable and spatially periodic solutions bifurcate.

A typical mechanism causing this instability is the Turing instability, where the spatially homogeneous ground state destabilizes via a spectral curve with positive real part at Fourier wave number $\pm k_c$, see Figure 1.1. This instability mechanism was found by Turing [Tur52] in reaction-diffusion systems who observed that different diffusion coefficients can lead to instability.

A slightly different mechanism can be found in pattern-forming systems admitting a conservation law structure. Here, in addition to the instability at Fourier wave number $\pm k_c$ there is an additional spectral curve which vanishes at Fourier wave number $k = 0$ originating from the conservation law structure, see Figure 1.2. This scenario is typical for pattern-formation in fluid problems with a free surface such as the Bénard-Marangoni

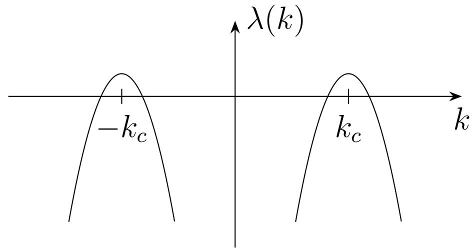

Figure 1.1.: Classical instability mechanism

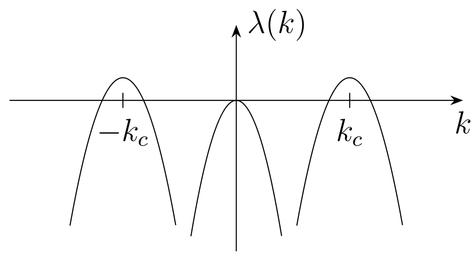

Figure 1.2.: New instability mechanism with additional curve at $k = 0$.

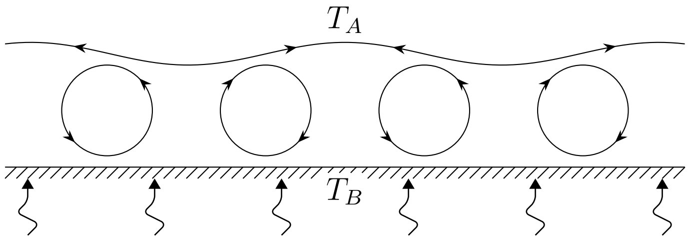

Figure 1.3.: Schematic depiction of the Bénard-Marangoni convection close to the first instability, with $T_B > T_A$.

problem, which describes a free-surface-fluid which is heated from below, see Figure 1.3. Although the instability is fairly similar to a classical Turing instability, different dynamical behavior can be found in pattern-forming systems admitting a conservation law structure, see e.g. [Kno16].

One particularly important difference concerns the amplitude equation which refers to a reduced equation that is used to describe the dynamical behavior of pattern-forming systems close to the onset of instability (see e.g. [SU17]). For classical pattern-forming systems, such as the Taylor-Couette and Reyleigh-Bénard problem, it is well known that the Ginzburg-Landau equation generically appears as an amplitude equation and we refer [SU17] for an overview. However, due to the additional neutral mode at Fourier wave number $k = 0$ present in conserved pattern-forming systems, the Ginzburg-Landau equation does not govern the dynamics close to the onset of instability. Instead, a modified Ginzburg-Landau system, that is, a Ginzburg-Landau equation coupled to a conservation law, can be derived as an amplitude equation, see e.g. [MC00, HSZ11, SZ13, Zim14].

1.1. Modulating traveling fronts and related concepts

Of particular interest for these systems is the transition from the unstable homogeneous ground state to the spatially periodic pattern. It turns out that these patterns often arise in the wake of an invading heteroclinic front which connects the unstable ground state to the periodic pattern. This behavior is captured by *modulating traveling fronts*, which are solutions of the form

$$u(t, x) = U(x - ct, x - \omega t),$$

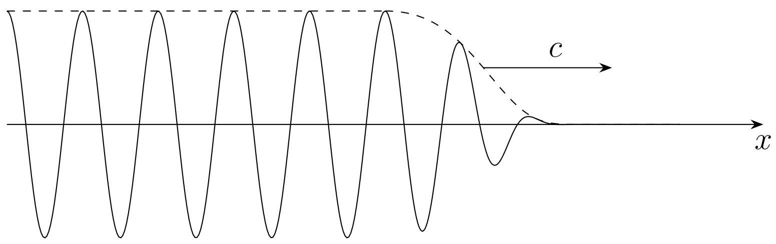

Figure 1.4.: Modulating traveling front

where U is periodic with respect to its second argument and satisfies

$$\lim_{\xi \to -\infty} U(\xi, p) = u_{\text{per}}(p) \text{ and } \lim_{\xi \to \infty} U(\xi, p) = u_0,$$

with $\xi = x - ct$ and $p = x - \omega t$, see Figure 1.4. Here, u_0 is the homogeneous ground state and u_{per} the spatially periodic pattern. Additionally, $x \in \mathbb{R}$ denotes the unbounded spatial coordinate, c the velocity of the front, $t \geq 0$ the time and ω the phase velocity. Solutions of this type have already been established for non-conservative pattern-forming systems. These results include the case of cubic nonlinearities such as the Swift-Hohenberg equation [CE86, EW91] as well as quadratic nonlinearities such as the Taylor-Couette problem in an infinte cylinder [HCS99] and a nonlocal Fisher-KPP equation [FH15]. In a 2-dimensional setting, similar solutions have been explored for a modified Swift-Hohenberg equation [DSSS03], which in particular includes the invasion of a ground state by a hexagonal pattern.

A related type of solutions to explore the transition from the ground state to the pattern is by concidering so-called triggered fronts, see [GS14, GS16]. Although similar to modulating traveling fronts, triggered fronts are constructed in a slightly modified setting, where a triggering effect travels through the domain. The trigger destabilizes the ground state, i.e. ahead of the trigger the ground state is stable and behind the trigger the ground state is unstable.

1.2. Nonlinear stability of invading fronts

Apart from their existence, another key question regarding modulating traveling fronts is if they are stable and if they are, under which conditions. From a physical point of view, stability is crucial as only stable effects have a chance of being observable in natural systems. Also from a phenomenological point of view it is an interesting question since modulating traveling fronts describe an invasion process of an unstable stationary state. Hence, it cannot be expected that they are stable with respect to localized perturbation in any translationally invariant norm. Nevertheless, for sufficiently fast modulating traveling fronts in the Swift-Hohenberg equation and the Taylor-Couette problem one can show that they are stable with respect to perturbations which are exponentially localized ahead of the front, see [ES00, ES02]. It turns out that both the sufficiently fast spreading speed as well as the exponential localization of the perturbations ahead of the front are crucial.

These ideas are in fact related to the stability of invading fronts in reaction-diffusion equations such as the Fisher-KPP equation. Invading fronts are a specific type of traveling front solutions, where an unstable ground state is invaded by a stable invading state. That is, a solution $u(t,x) = U(\xi)$, with $\xi = x - ct$ and

$$\lim_{\xi \to -\infty} U(\xi) = u_{\text{invading}} \text{ and } \lim_{\xi \to +\infty} U(\xi) = u_{\text{ground}},$$

with u_{ground} being unstable. Starting with the work of Sattinger [Sat76, Sat77] the nonlinear stability of these fronts has been extensively studied using various different techniques such as renormalization groups [BK92, BK94, BKL94, Gal94] and, more recently, estimates of the pointwise Green's function [FH19a, FH19b]. The common ground in these works is the use of exponential weights ahead of the front to stabilize the (unstable) ground state, which is in line with the stability results for modulating traveling fronts.

Depending on the invasion speed of the fronts, the stability results can be classified into two categories. The first one, e.g. [Sat76, Sat77, ES00, ES02], deals with stability of supercritical fronts, which move strictly faster than the linear spreading speed of the problem, which is determined by the linearization about the unstable ground state (see [vS03] for details). The second group, e.g. [Gal94, FH19a, FH19b], considers critical fronts which spread with the linear spreading speed and stability results for this setting are usually much harder to obtain.

Notably, while a lot is known for scalar equations or for systems, which satisfy a comparison principle, for systems without a comparison principle a lot of mathematical questions are still open.

1.3. Brief summary of results

We now give an overview of the specific settings studied in this thesis and outline the obtained results.

1.3.1. Existence of modulating traveling fronts

We study the existence of modulating traveling fronts in pattern-forming systems admitting a conservation law structure in two scenarios.

A Swift-Hohenberg equation with an additional conservation law. The first model is a Swift-Hohenberg equation which is coupled to a conservation law, namely

$$\partial_t u = -(1 + \partial_x^2)^2 u + \varepsilon^2 \alpha_0 u + uv - u^3, \tag{1.1a}$$

$$\partial_t v = \partial_x^2 v + \gamma \partial_x^2 (u^2), \tag{1.1b}$$

with $u(t,x), v(t,x) \in \mathbb{R}$, $x \in \mathbb{R}$, $t \geq 0$ and parameters $\gamma \in \mathbb{R}$, $\alpha_0 > 0$ and $0 < \varepsilon \ll 1$. Since $\alpha_0 > 0$, we are in a spectral situation as depicted in Figure 1.2, that is, there is a spectral curve which vanishes at Fourier wave number $k = 0$ and there are spectral curves with positive real part locally around a critical Fourier wave number $\pm k_c$.

Although the model (1.1) is purely phenomenological, it shares important properties with the Bénard-Marangoni problem. In particular, for both models the modified Ginzburg-Landau system can be derived as an amplitude equation, see also [SZ17].

The existence proof is based on spatial dynamics and center manifold reduction. The main challenge is that at the bifurcation point, i.e. for $\varepsilon = 0$, there are inifinitly many eigenvalues on the imaginary axis, which prevents the application of standard center manifold theory. However, if $\alpha_0 > 0$ and $\varepsilon > 0$ and thus beyond the onset of instability, the eigenvalues leave the imaginary axis with different speeds. A finite number is of order ε, while the remaining ones are of order $\sqrt{\varepsilon}$. This opens up an ε-dependent spectral gap and allows the application of center manifold theory. However, the size of the center manifold is also ε-dependent. Hence, we prove that the center manifold is large enough (with respect to ε) to contain the modulating traveling fronts. We conclude the proof by establishing the existence of heteroclinic orbits in the reduced dynamics on the center manifold, using perturbation theory. *The details of the construction are contained in Chapter 2. The results have also been published in [Hil20a].*

A dispersive Swift-Hohenberg equation with an additional conservation law. In the second setting, we add dispersive terms to (1.1), which break the reflexion symmetry, i.e. the invariance with respect to $x \mapsto -x$. More specifically, we consider the model

$$\partial_t u = -(1+\partial_x^2)^2 u + \varepsilon^2 \alpha_0 u + c_u \partial_x^3 u + uv + u\partial_x u - u^3, \tag{1.2a}$$

$$\partial_t v = \partial_x^2 v + c_v \partial_x v + \gamma_1 \partial_x^2(u^2) + \gamma_2 \partial_x(u^2), \tag{1.2b}$$

with $u(t,x), v(t,x) \in \mathbb{R}$, $x \in \mathbb{R}$, $t \geq 0$ and parameters $c_u, c_v, \gamma_1, \gamma_2, \alpha_0 \in \mathbb{R}$ and $0 < \varepsilon \ll 1$. Again we try to understand the behavior of (1.2) close to the onset of instability, that is, for $\alpha_0 > 0$. Due to the symmetry breaking dispersive terms, we now expect that a modulating traveling front has non-zero phase velocity ω.

After establishing a family of periodic solutions for (1.2), we proceed to construct modulating traveling fronts as for the non-dispersive system (1.1). It turns out that for a large class of parameters there is a spectral gap around the imaginary axis in the linear part of the spatial dynamics formulation. Therefore, we apply center manifold theory to derive a reduced equation, which has complex coefficients due to the dispersive terms in (1.2). The main challenge is then to establish the existence of heteroclinic orbits in the reduced equation, which correspond to modulating traveling fronts in the full equation. Depending one the parameter regime this can be done fully analytically, or under some assumptions, which can be established numerically. *The precise parameter regimes and the corresponding results are discussed in Chapter 3.*

1.3.2. Nonlinear stability of invading fronts in the amplitude equation

Motivated by the fact that a Ginzburg-Landau equation coupled to a conservation law,

$$\partial_t A = \partial_x^2 A + A + AB - A|A|^2, \tag{1.3a}$$

$$\partial_t B = \mu \partial_x^2 B + \gamma \partial_x^2(|A|^2), \tag{1.3b}$$

with $A(t,x) \in \mathbb{C}$, $B(t,x) \in \mathbb{R}$ and $\mu > 0, \gamma \in \mathbb{R}$, appears generically as an amplitude equation for pattern-forming systems admitting a conservation law structure, we study the nonlinear stability of invading fronts in (1.3). That is, we study the stability of fronts

$(A_{\text{front}}, B_{\text{front}})(\xi)$ with $\xi = x - ct$, satisfying

$$\lim_{\xi \to -\infty} (A_{\text{front}}, B_{\text{front}})(\xi) = (1, 0) \text{ and } \lim_{\xi \to +\infty} (A_{\text{front}}, B_{\text{front}})(\xi) = (0, 0)$$

for some speed $c > 0$. These fronts in particular determine the amplitude of the modulating front solutions of (1.1) to lowest order.

We use the rotational symmetry (i.e. $A \mapsto Ae^{i\chi}$) of (1.3) and write the system in polar coordinates. Then, we first show the nonlinear stability of the invading state $(A, B) = (1, 0)$ with respect to sufficiently small and sufficiently localized perturbations. The main challenge in the proof is that the invading state is only diffusively stable on the linear level, that is, perturbations only exhibit a polynomial decay rate in time. Building on this result, we show the nonlinear stability of sufficiently fast (i.e. supercritical) invading fronts $(A_{\text{front}}, B_{\text{front}})$ with respect to perturbations which are exponentially localized ahead of the front. Here, the main difficulty originates from the lack of a comparison principle, which makes it more difficult to control the spectrum of the exponentially weighted operator. *The details for these results are given in Chapter 4, see also [Hil20b].*

Remark. For the reader's convenience we recall the setting at the beginning of each chapter.

Part I.

Existence of modulating traveling fronts

2. Modulating traveling fronts for a Swift-Hohenberg equation in the case of an additional conservation law

2.1. Introduction

We are interested in the mechanism which drives the pattern formation in parabolic evolution equations on the real line. It turns out that in such systems patterns often arise in the wake of an invading heteroclinic front which connects the unstable ground state u_0 to the periodic pattern u_{per}. To model this behavior, we construct *modulating traveling fronts*. These are solutions of the form

$$u(t,x) = U(x - ct, x - \omega t), \tag{2.1}$$

where U is periodic with respect to its second argument and satisfies

$$\lim_{\xi \to -\infty} U(\xi, p) = u_{\mathrm{per}}(p) \text{ and } \lim_{\xi \to \infty} U(\xi, p) = u_0(p), \tag{2.2}$$

with $\xi = x - ct$ and $p = x - \omega t$, see Figure 2.1. Here, $x \in \mathbb{R}$ denotes the unbounded spatial direction of the problem, c the velocity of the front, $t \geq 0$ the time and ω the phase velocity. Solutions of this type are already established for non-conservative systems which exhibit a Turing bifurcation. These results include the case of cubic nonlinearities such as the Swift-Hohenberg equation [CE86, EW91] as well as quadratic nonlinearities such as the Taylor-Couette problem in an infinite cylinder [HCS99] and a nonlocal Fisher-KPP equation [FH15].

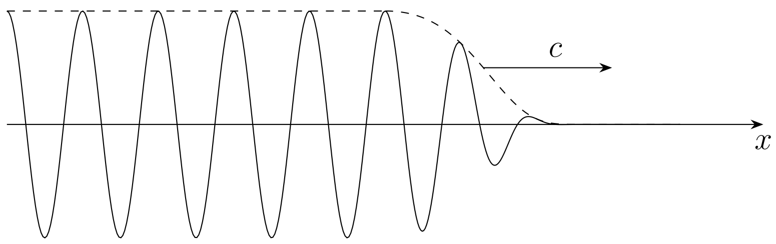

Figure 2.1.: Modulating travelling front

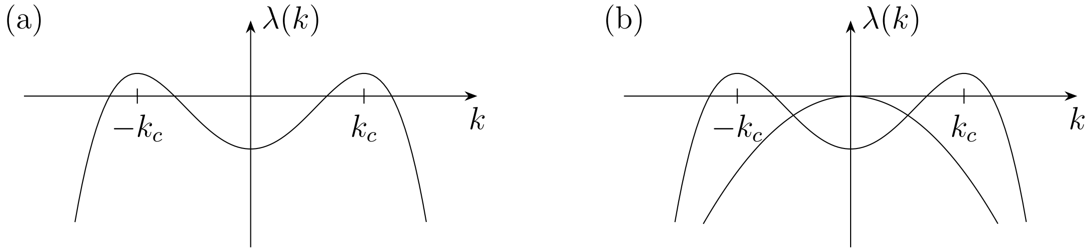

Figure 2.2.: Curves of eigenvalues over the Fourier wave number k for a classical Turing instability (a) and the new type of instability with marginally stable modes at $k = 0$ (b).

In this chapter, we show the existence of modulating traveling fronts for a pattern-forming system with a conserved quantity which presents a nontrivial extension of the current results for non-conservative systems as we outline below. It turns out that the behavior of these systems differs from the one of classical pattern-forming systems (see e.g. [Kno16]). In particular, the Ginzburg-Landau equation cannot be justified as an amplitude equation as opposed to the classical situation where this is generically the case (see [SU17] for details). Instead, a modified Ginzburg-Landau system, that is, a Ginzburg-Landau equation coupled to a conservation law, see (2.6a)–(2.6b), appears as amplitude equation, see [MC00]. A closer analysis reveals that the additional conservation law in the amplitude equation comes from the presence of an additional critical spectral curve of the linearisation about the ground state touching the imaginary axis at Fourier wave number $k = 0$, see Figure 2.2. This presents new difficulties in the rigorous justification of the amplitude equation which is done in [HSZ11, SZ13, DKSZ16] but also the dynamics of the modified Ginzburg-Landau system is less well understood as opposed to the classical Ginzburg-Landau equation (see [SZ17] for a discussion). Indeed, we will see that both issues lead to new difficulties for the problem at hand.

2.1.1. The problem and main result

We consider the Swift-Hohenberg equation with an additional conservation law of the form

$$\partial_t u = -(1+\partial_x^2)^2 u + \alpha u + uv - u^3, \tag{2.3a}$$

$$\partial_t v = \partial_x^2 v + \gamma \partial_x^2 (u^2), \tag{2.3b}$$

with $\alpha, \gamma, u(t,x), v(t,x) \in \mathbb{R}$ and $t \geq 0$ (see [CM03, SZ17]). Here α takes the role of an external control parameter. The system (2.3a)–(2.3b) has a family of spatially homogeneous equilibria $(0, v_0)$ with $v_0 \in \mathbb{R}$ which destabilizes when $\alpha > v_0$. However, we restrict the following analysis to $v_0 = 0$, since the general case can be brought back to this case by replacing α by $\alpha + v_0$. Linearisation of (2.3a)–(2.3b) about $(0,0)$ and the Fourier transform yield that the spectral situation is indeed as depicted in Figure 2.2b. Hence, at $\alpha = 0$ there is a Turing bifurcation and we will prove that a family of periodic solutions bifurcates. Close to the first instability, namely for $\alpha = \varepsilon^2 \alpha_0$, these solutions are of the form

$$u_{\mathrm{per}}(x) = \varepsilon A_u \cos(x + x_0) + \mathcal{O}(\varepsilon^2) \tag{2.4a}$$

$$v_{\mathrm{per}}(x) = \varepsilon^2 A_v \cos(2(x + x_0)) + \mathcal{O}(\varepsilon^3), \tag{2.4b}$$

with $x_0 \in [0, 2\pi)$ and $A_u, A_v \in \mathbb{R}$ (see Lemma 2.2.1).

Note that although (2.3a)–(2.3b) is a purely phenomenological model, as discussed in [SZ17], it shares important properties with the Bérnard-Marangoni problem. In particular, both models show the same kind of Turing instability and both are reflection symmetric. Moreover, the (formal) amplitude equation of (2.3a)–(2.3b) is of the same form as the one for the Bénard-Marangoni problem [Zim14, Theorem 4.3.3] (see Section 2.2). Thus, we expect that both models exhibit similar behavior close to the first instability.

As mentioned before, we aim to rigorously construct a modulating traveling front for the above system. In particular, we are interested in slow moving fronts with velocity $c = \varepsilon c_0$, which is the natural scaling corresponding to the scaling of α. Furthermore, since the system (2.3a)–(2.3b) possesses reflection symmetry and the periodic solutions (2.4a)–(2.4b) are stationary, we set the phase velocity $\omega = 0$. The idea of the existence proof is to use center manifold theory to reduce the problem to finding a heteroclinic orbit in a finite-dimensional system. We now briefly outline the strategy, describe the challenges therein and formulate our main result. Finally, we comment on the differences to previous results and novelties in our work.

Center manifold theory is a well-established tool to reduce the dimension of a studied system. We refer to [HI11] for a detailed overview. In the case at hand, it can be applied in the following, non-standard way. Inserting the modulating traveling front ansatz into (2.3a)–(2.3b) we find that the linearisation about the ground state has infinitely many imaginary eigenvalues for $\varepsilon = 0$. Therefore, standard center manifold results are not applicable, since these require that only finitely many eigenvalues lie on the imaginary axis. However, as ε gets positive we find similar to [EW91] that the eigenvalues depart from the imaginary axis with different velocities and thus, a spectral gap of size $\mathcal{O}(\sqrt{\varepsilon})$ between finitely many eigenvalues close to the imaginary axis and the rest of the spectrum arises. We use this to construct a 6-dimensional center manifold although, since the size of the spectral gap depends on ε, it is not a priori clear whether small (with respect to ε) solutions are contained in the center manifold. Therefore, we prove that the constructed center manifold is large enough to contain the perodic solutions (2.4a)–(2.4b). Finally, we establish the existence of heteroclinic connections for the finite-dimensional system on the center manifold and arrive at the main result.

Theorem 2.1.1. *For $c_0^2 > 16\alpha_0 > 0$ there exist $\varepsilon_0 > 0$ and $\gamma_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $|\gamma| < \gamma_0$ the system* (2.3a)–(2.3b) *has modulating traveling front solutions of the form* (2.1) *satisfying the boundary conditions* (2.2)*. Furthermore, the solution is of the form*

$$
\begin{aligned}
u_f(t,x) &= \varepsilon U(\varepsilon(x - \varepsilon c_0 t)) \cos(x + x_0) + \mathcal{O}(\varepsilon^2) \\
v_f(t,x) &= \varepsilon^2 \left[V_0(\varepsilon(x - \varepsilon c_0 t)) + V_1(\varepsilon(x - \varepsilon c_0 t)) \cos(2(x + x_0))\right] + \mathcal{O}(\varepsilon^3)
\end{aligned}
$$

for any $x_0 \in [0, 2\pi)$, with $U(y), V_0(y), V_1(y) \in \mathbb{R}$.

Remark 2.1.2. We point out that our rigorous result, Theorem 2.1.1, only covers the case that γ is contained in a small neighbourhood of zero. However, numerical experiments presented in Section 2.4 show that we can expect that the result also holds for $\gamma \in (-3, \infty)$. Notably, we cannot expect that either small periodic patterns or small modulating fronts exist for $\gamma \leq -3$ since the reduced system does not have nontrivial fixed points in this parameter region. We refer to Section 2.2 for a detailed discussion. □

Remark 2.1.3. We note that a similar instability occurs in the models studied in [MC00, SZ13, Suk16]. In fact, with a modulating front ansatz (2.1) we find a similar spectral situation as for (2.3a)–(2.3b), see Section 3.1. Namely, for $\varepsilon = 0$ infinitely many eigenvalues lie on the imaginary axis and depart with different velocities for $\varepsilon > 0$. Thus, we expect that similar results hold true in these cases. □

Although the strategy used to establish Theorem 2.1.1 is similar to the one used in [CE86, EW91, HCS99] the following new challenges occur. First, the proof of the center manifold result, Theorem 2.3.6, is more involved because of the additional spectral curve due to the conservation law, see Figure 2.2b. Second, as can be expected from the corresponding amplitude equation, the reduced system on the center manifold is 3-dimensional instead of 2-dimensional which increases the difficulty of establishing a heteroclinic orbit.

We now comment on the differences and new ideas in the center manifold result in more detail. Compared to the pure Swift-Hohenberg case [CE86, EW91], new difficulties arise from the presence of quadratic nonlinearities instead of cubic ones. Therefore, as discussed in [HCS99], the estimates for the semigroups and projections established in [EW91] seem insufficient to establish a large enough center manifold. Haragus and Schneider [HCS99] address this issue by rescaling the central and hyperbolic part with different powers of ε, i.e. $u = \varepsilon^\beta u_c + \varepsilon^\gamma u_h$, with $0 < \beta < 1 < \gamma < 2$ chosen appropriately, which is motivated by the fact that the corresponding periodic equilibrium has the same expansion. Finally, they remove the critical quadratic terms using a normal form transformation.

However, it turns out that this is not a viable strategy in our case since not all critical quadratic terms can be eliminated using normal form transformation, see Remark 2.3.3. A closer analysis reveals that the resonating quadratic contributions come from the fact that quadratic interactions of central modes, i.e. modes with non-negative growth rate, give again central modes. This is due to the additional spectral curve touching zero at $k = 0$ by virtue of the conserved quantity present in the system. Note that this is also one of the main issues in the justification of the modified Ginzburg-Landau approximation of equations with conserved quantities in [SZ13] since the quadratic interactions of central modes are not exponentially damped in contrast to pattern-forming systems without conserved quantity.

We solve these issues related to the quadratic contributions by using a similar rescaling as used by Haragus and Schneider [HCS99]. However, we additionally exploit $u_{\text{per}} = \mathcal{O}(\varepsilon)$ and $v_{\text{per}} = \mathcal{O}(\varepsilon^2)$ and thus we rescale u and v differently in terms of ε. Moreover, we use the transformation $w = v + \gamma u^2$ which gives an additional ε in the quadratic nonlinearity of the conservation law. Note that this transformation does not eliminate the quadratic nonlinearites and hence is not a normal form transform. It turns out that this is enough to treat all nonlinearities similar to cubic ones and therefore, normal form transform is not necessary, see also Remark 2.3.8.

2.1.2. Outline

This chapter is organized as follows. In Section 2.2 we formally derive the amplitude equation of (2.3a)–(2.3b) using a multiple scaling ansatz and show the existence of periodic solutions. Following, we formulate the modulating front problem and prove a center manifold result including bounds on the size of the manifold in Section 2.3. Afterwards, Section 2.4 contains the derivation of the reduced equations on the center manifold and the construction of heteroclinic orbits establishing the existence of modulating fronts for (2.3a)–(2.3b).

2.2. Amplitude equation and spatially periodic equilibria

We want to derive an amplitude equation for (2.3a)–(2.3b). For this, we note that the linearisation of (2.3a)–(2.3b) about the trivial solution $(u, v) = (0, 0)$ has solutions of the form

$$u(t, x) = e^{ikx} e^{\lambda_u(k)t} \text{ and } v(t, x) = e^{ikx} e^{\lambda_v(k)t},$$

where $\lambda_u(k) = -(1 - k^2)^2 + \alpha$ and $\lambda_v(k) = -k^2$, with $k \in \mathbb{R}$, which corresponds to the spectrum depicted in Figure 2.2b. Therefore for $\alpha \leq 0$ the ground state $(u, v) = (0, 0)$ is stable while for $\alpha > 0$ it gets unstable. Since we are interested in solutions bifurcating from the ground state when α gets positive, we set $\alpha = \varepsilon^2 \alpha_0$ for some $\alpha_0 > 0$. We now derive the formal amplitude equation close to the instability with the ansatz

$$\begin{aligned}
u(t, x) &= \varepsilon A(T, X) e^{ix} + \varepsilon \overline{A}(T, X) e^{-ix}, \\
v(t, x) &= \varepsilon^2 B_0(T, X) + \varepsilon^2 B_1(T, X) e^{2ix} + \varepsilon^2 \overline{B}_1(T, X) e^{-2ix},
\end{aligned}$$

where $X = \varepsilon x$ and $T = \varepsilon^2 t$. Inserting this ansatz into the system (2.3a)–(2.3b) then yields

$$\begin{aligned}
0 &= \varepsilon^3 E(-\partial_T A + 4\partial_X^2 A + \alpha_0 A + AB_0 + \bar{A}B_1 - 3|A|^2 A) + \varepsilon^3 E^3(AB_1 + A^3) \\
&\quad + c.c. + \mathcal{O}(\varepsilon^4), \qquad (2.5a) \\
0 &= \varepsilon^4(-\partial_T B_0 + \partial_X^2 B_0 + 2\gamma\partial_X^2(|A|^2)) + \varepsilon^4 E^2(-\partial_T B_1 + \partial_X^2 B_1 + \gamma\partial_X^2(A^2)) \\
&\quad + 4i\varepsilon^3 E^2(\partial_X B_1 + \gamma\partial_X(A^2)) - 4\varepsilon^2 E^2(B_1 + \gamma A^2) + c.c. + \mathcal{O}(\varepsilon^5), \qquad (2.5b)
\end{aligned}$$

where $E = e^{ix}$. Equating the $\varepsilon^2 E^2$ contributions in (2.5b) to zero then yields $B_1 = -\gamma A^2$. With this, equating the $\varepsilon^3 E$-terms in (2.5a) and the ε^4-terms in (2.5b) to zero leads to the formal amplitude equations

$$\begin{aligned}
\partial_T A &= 4\partial_X^2 A + \alpha_0 A + AB_0 - (3+\gamma)A|A|^2, \qquad (2.6a) \\
\partial_T B_0 &= \partial_X^2 B_0 + 2\gamma\partial_X^2|A|^2. \qquad (2.6b)
\end{aligned}$$

Note that the extended ansatz for v includes noncritical modes, i.e., modes which correspond to negative eigenvalues. However, these additional terms are necessary in the derivation to equate the coefficient in front of $\varepsilon^2 e^{2ix}$ in (2.5b) to zero. This contribution then leads to a nontrivial contribution to the cubic term in (2.6a). Note that this system deviates from the one derived in [MC00] since the sign of the cubic coefficient in (2.6a) depends on the coupling parameter γ. Specifically, for $\gamma < -3$ the sign is positive which leads to the non-existence of nontrivial equilibria in (2.6a)–(2.6b). Indeed this formal prediction is reflected in the full system as for $\gamma \leq -3$ the full system (2.3a)–(2.3b) does not have stationary periodic solutions of small amplitude. Making these formal ideas rigorous we obtain the following result.

Lemma 2.2.1. *For all* $\gamma_0 > -3$ *and* $\alpha_0 > 0$ *there exists an* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *and* $\gamma \in (\gamma_0, \infty)$ *the system* (2.3a)–(2.3b) *has stationary periodic solutions of the form*

$$u_{per}(x) = \varepsilon 2\sqrt{\frac{\alpha_0 - q_0^2}{3+\gamma}}\cos(k_c(x + x_0)) + \mathcal{O}(\varepsilon^2), \qquad (2.7a)$$

$$v_{per}(x) = -\varepsilon^2 2\frac{(\alpha_0 - q_0^2)\gamma}{3+\gamma}\cos(2k_c(x + x_0)) + \mathcal{O}(\varepsilon^3), \qquad (2.7b)$$

where $x_0 \in [0, 2\pi/k_c)$ *and* $k_c^2 = 1 + \varepsilon q_0$. *Here,* k_c *is the critical wave number and* $q_0 \in (-\sqrt{\alpha_0}, \sqrt{\alpha_0})$. *Furthermore,* $\int_0^{2\pi/k_c} v_{per}\, dx = 0$.

Proof. The proof is based on center manifold reduction. We define the space

$$H^l_{\text{per}} := \left\{ u(x) = \sum_{n\in\mathbb{Z}} u_n e^{ink_cx} \;:\; \|u\|^2_{H^l_{\text{per}}} = \sum_{n\in\mathbb{Z}} (1+n^2)^l |u_n|^2 < \infty \right\}$$

and

$$\begin{aligned}
\mathcal{Z} &= \left\{ (u,v) \in H^4_{\text{per}} \times H^2_{\text{per}} \;:\; u_n = \bar{u}_{-n},\, v_n = \bar{v}_{-n},\, v_0 = 0 \right\}, \\
\mathcal{Y} &= \left\{ (u,v) \in H^2_{\text{per}} \times H^2_{\text{per}} \;:\; u_n = \bar{u}_{-n},\, v_n = \bar{v}_{-n},\, v_0 = 0 \right\}, \\
\mathcal{X} &= \left\{ (u,v) \in H^0_{\text{per}} \times H^0_{\text{per}} \;:\; u_n = \bar{u}_{-n},\, v_n = \bar{v}_{-n},\, v_0 = 0 \right\},
\end{aligned}$$

where u_n denotes the n-th Fourier coefficient of u, $n \in \mathbb{Z}$ and $\bar{u}$ is the complex conjugate of u. Furthermore, note that the condition $v_0 = \int_0^{2\pi/k_c} v(x)\, dx = 0$ is conserved by the dynamic of (2.3a)–(2.3b). Then, we write (2.3a)–(2.3b) as

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = L_\varepsilon \begin{pmatrix} u \\ v \end{pmatrix} + R(u,v),$$

where L_ε is the linear part and R the nonlinear part. Here, the linear operator L_ε is given by

$$L_\varepsilon = \begin{pmatrix} -(1+\partial_x^2)^2 + \varepsilon^2\alpha_0 & 0 \\ 0 & \partial_x^2 \end{pmatrix} \in \mathcal{L}(\mathcal{Z};\mathcal{X}),$$

where $\mathcal{L}(\mathcal{Z};\mathcal{X})$ is the space of bounded, linear operators from $\mathcal{Z}$ to $\mathcal{X}$. Furthermore, the nonlinearity is given by

$$R(u,v) = \begin{pmatrix} uv - u^3 \\ \gamma\partial_x^2(u^2) \end{pmatrix},$$

which is a smooth map from $\mathcal{Z}$ to $\mathcal{Y}$. Note in particular that $R(0,0) = 0$ and $DR(0,0) = 0$. The eigenvalues of L_ε can be explicitly calculated using Fourier transform as

$$\lambda_{u,n} = -(1 - n^2k_c^2)^2 + \varepsilon^2\alpha_0, \quad n \in \mathbb{Z}$$

$$\lambda_{v,n} = -n^2 k_c^2, \quad n \in \mathbb{Z} \setminus \{0\}.$$

Note, that $\lambda_{v,0} = 0$ is not an eigenvalue since we require $v_0 = 0$. Thus, the spectrum can be decomposed in the central part $\sigma_c(L_\varepsilon) = \{\lambda_{u,\pm 1}\}$ and the remaining hyperbolic part $\sigma_h(L_\varepsilon)$. Finally, we have for $|\omega| \geq \omega_0 > 0$ that

$$\left\|(i\omega I - L_\varepsilon)^{-1}\right\|_{\mathcal{L}(\mathcal{X};\mathcal{X})} \leq C(\sup_{n\in\mathbb{Z}} |i\omega + (1-n^2k_c^2)^2 - \varepsilon^2\alpha_0|^{-1} + \sup_{n\in\mathbb{Z}} |i\omega - n^2k_c^2|^{-1}) \leq \frac{C(\omega_0)}{|\omega|}.$$

Thus, since $\mathcal{Z}$, $\mathcal{Y}$ and $\mathcal{X}$ are Hilbert spaces, we can apply the center manifold theorem [HI11, Theorem 3.3] using [HI11, Remark 3.6]. This gives a smooth map $h = h(u_1, u_{-1}, \varepsilon) = h_u \times h_v$, which defines the center manifold. We then introduce the following coordinates

$$\begin{aligned} u(t) &= \varepsilon A(\varepsilon^2 t)e^{ik_cx} + \varepsilon\bar{A}(\varepsilon^2 t)e^{-ik_cx} + h_u(\varepsilon A(\varepsilon^2 t), \varepsilon\bar{A}(\varepsilon^2 t), \varepsilon), \\ v(t) &= h_v(\varepsilon A(\varepsilon^2 t), \varepsilon\bar{A}(\varepsilon^2 t), \varepsilon). \end{aligned}$$

To determine an approximation of h_u, h_v we use [HI11, Corollary 2.12] and obtain

$$\begin{aligned} \varepsilon^2 D_A h_u \partial_T A = &-(1+\partial_x^2)^2 h_u + \varepsilon^2\alpha_0 h_u \\ &+ P_{h,u}\left((\varepsilon Ae^{ik_cx} + \varepsilon\bar{A}e^{-ik_cx} + h_u)h_v + (\varepsilon Ae^{ik_cx} + \varepsilon\bar{A}e^{-ik_cx} + h_u)^3\right), \end{aligned}$$

where $P_{h,u}$ is the projection onto the hyperbolic eigenspace of the u-equation, $D_A h_u$ denotes the derivative of h_u with respect to A and $T = \varepsilon^2 t$. Using that $h_u, h_v = \mathcal{O}(\varepsilon^2)$ since they are at least quadratic in A, the above equation yields that $h_u = \mathcal{O}(\varepsilon^3)$. For h_v we find

$$\varepsilon^3 D_A h_v \partial_T A = \partial_x^2 h_v + \gamma\partial_x^2((\varepsilon Ae^{ik_cx} + \varepsilon\bar{A}e^{-ik_cx} + h_u)^2),$$

where we used that all $\lambda_{v,n}$ are in the hyperbolic part of the spectrum. Since $h_u = \mathcal{O}(\varepsilon^3)$ we find by equating the ε^2-terms in the above equation to zero that

$$h_v = -\gamma\varepsilon^2(A^2e^{2ik_cx} + \bar{A}^2e^{-2ik_cx}) + \mathcal{O}(\varepsilon^4).$$

Thus, the reduced equation on the center manifold for A is given by

$$\partial_T A = (\alpha_0 - q_0^2)A - (3+\gamma)A|A|^2 + g(A,\varepsilon), \tag{2.8}$$

with $g(A,\varepsilon) = \mathcal{O}(\varepsilon^2)$. For $\varepsilon = 0$ and $\gamma > -3$, (2.8) has a nontrivial stationary state $|A|^2 = \frac{\alpha_0 - q_0^2}{3+\gamma}$. It remains to show that there exists a stationary state nearby for $\varepsilon > 0$ small. We proceed similar to [SU17, Theorem 13.2.2] and use the translational invariance of (2.3a)–(2.3b), which yields that (2.8) is invariant with respect to $A \mapsto Ae^{iy}$, $y \in \mathbb{R}$. Therefore, $g(A,\varepsilon) = A\tilde{g}(|A|^2,\varepsilon)$ and hence, we may write A in polar coordinates, i.e. $A = re^{i\phi}$, and obtain

$$\partial_T r = (\alpha_0 - q_0^2)r - (3+\gamma)r^3 + r\tilde{g}(r^2,\varepsilon), \quad \partial_T \phi = 0.$$

Thus, any stationary state satisfies

$$G(r,\varepsilon) := (\alpha_0 - q_0^2)r - (3+\gamma)r^3 + r\tilde{g}(r^2,\varepsilon) = 0.$$

Since

$$G\left(\pm\sqrt{(\alpha_0 - q_0^2)/(3+\gamma)}, 0\right) = 0 \text{ and } \partial_r G\left(\pm\sqrt{(\alpha_0 - q_0^2)/(3+\gamma)}, 0\right) \neq 0$$

we can apply the implicit function theorem to obtain stationary states

$$r_\pm(\varepsilon) = \pm\sqrt{(\alpha_0 - q_0^2)/(3+\gamma)} + \mathcal{O}(\varepsilon^2).$$

Using the coordinates on the center manifold, this concludes the proof. □

Remark 2.2.2. We conjecture that the periodic solutions in Lemma 2.2.1 are stable in a suitable sense. To back this up, we perform a formal analysis of the spectral stability using the corresponding amplitude equation in Section 2.B. We then discuss in Chapter 5 how a rigorous result could be obtained. □

2.3. Formulation of the problem and center manifold reduction

We now turn to the existence of modulating front solutions of the form (2.1) connecting the trivial solution $(u, v) = (0, 0)$ to the periodic solution established in Lemma 2.2.1. The aim of this section is to establish the spatial dynamics formulation and show that there is a center manifold which contains the periodic solutions. Thus, we make the ansatz

$$u(t, x) = \mathcal{U}(\xi, p) \text{ and } v(t, x) = \mathcal{V}(\xi, p),$$

where $\xi = x - ct$ and $p = x$, and $\mathcal{U}, \mathcal{V}$ are $2\pi/k_c$-periodic with respect to their second argument and satisfy

$$\lim_{\xi \to +\infty} (\mathcal{U}, \mathcal{V})(\xi, p) = (0, 0) \text{ and } \lim_{\xi \to -\infty} (\mathcal{U}, \mathcal{V})(\xi, p) = (u_{\text{per}}, v_{\text{per}})(p).$$

Inserting this ansatz into (2.3a)–(2.3b) and using $\partial_t = -c\partial_\xi$ and $\partial_x = \partial_\xi + \partial_p$ we obtain

$$\begin{aligned} -c\partial_\xi \mathcal{U} &= -(1 + (\partial_\xi + \partial_p)^2)^2 \mathcal{U} + \alpha \mathcal{U} + \mathcal{U}\mathcal{V} - \mathcal{U}^3, \\ -c\partial_\xi \mathcal{V} &= (\partial_\xi + \partial_p)^2 \mathcal{V} + \gamma (\partial_\xi + \partial_p)^2 (\mathcal{U}^2). \end{aligned}$$

Note that the nonlinearity contains p-derivatives. To avoid this, we introduce the transformation $\mathcal{W} := \mathcal{V} + \gamma \mathcal{U}^2$ which gives the equivalent transformed system

$$-c\partial_\xi \mathcal{U} = -(1 + (\partial_\xi + \partial_p)^2)^2 \mathcal{U} + \alpha \mathcal{U} + \mathcal{U}\mathcal{W} - (1 + \gamma)\mathcal{U}^3, \tag{2.9a}$$

$$-c\partial_\xi \mathcal{W} = (\partial_\xi + \partial_p)^2 \mathcal{W} - \gamma c \partial_\xi (\mathcal{U}^2). \tag{2.9b}$$

Another consequence of the transformation $\mathcal{W} := \mathcal{V} + \gamma \mathcal{U}^2$ is the presence of the front velocity c in the nonlinear term in (2.9b). Especially with the choice $c = \varepsilon c_0$ this depends on ε. It turns out this is crucial for the construction of a center manifold of sufficient size, see Remark 2.3.8.

2.3.1. Spatial dynamics formulation and spectrum

Since $\mathcal{U}(\xi,\cdot), \mathcal{W}(\xi,\cdot)$ are $2\pi/k_c$-periodic, we make a Fourier ansatz

$$\mathcal{U}(\xi,p) = \sum_{n\in\mathbb{Z}} \mathcal{U}_n(\xi) e^{ink_c p} \text{ and } \mathcal{W}(\xi,p) = \sum_{n\in\mathbb{Z}} \mathcal{W}_n(\xi) e^{ink_c p},$$

where $\mathcal{U}_n = \bar{\mathcal{U}}_{-n}$ and $\mathcal{W}_n = \bar{\mathcal{W}}_{-n}$ since $\mathcal{U}, \mathcal{W}$ are real-valued. Inserting this ansatz into (2.9a)–(2.9b) gives an infinite dimensional ODE system with respect to the dynamic variable ξ

$$-c\partial_\xi \mathcal{U}_n = -(1 + (\partial_\xi + ink_c)^2)^2 \mathcal{U}_n + \alpha \mathcal{U}_n + \sum_{l+k=n} \mathcal{U}_l \mathcal{W}_k - (1+\gamma) \sum_{k+l+m=n} \mathcal{U}_k \mathcal{U}_l \mathcal{U}_m, \tag{2.10a}$$

$$-c\partial_\xi \mathcal{W}_n = (\partial_\xi + ink_c)^2 \mathcal{W}_n - \gamma c \partial_\xi \sum_{k+l=n} \mathcal{U}_k \mathcal{U}_l, \tag{2.10b}$$

with $n \in \mathbb{Z}$. We write (2.10a) and (2.10b) as a first order system

$$\partial_\xi \begin{pmatrix} U_n \\ W_n \end{pmatrix} = L_n \begin{pmatrix} U_n \\ W_n \end{pmatrix} + \mathcal{N}_n(U,W), \tag{2.11}$$

where $(U_n, W_n)^T = (U_{n0}, U_{n1}, U_{n2}, U_{n3}, W_{n0}, W_{n1})^T \in \mathbb{C}^6$ with $U_{nj} = \partial_\xi^j \mathcal{U}_n$ for $j = 0,1,2,3$ and $W_{nj} = \partial_\xi^j \mathcal{W}_n$ for $j = 0,1$. Since all coupling terms are nonlinear, the linear part L_n is given by

$$L_n = \begin{pmatrix} L_n^{\mathrm{SH}} & 0 \\ 0 & L_n^{\mathrm{con}} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ A_n & B_n & C_n & D_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & G_n & H_n \end{pmatrix} \in \mathbb{C}^{6\times 6},$$

where $A_n = -(1-\mu^2)^2 + \alpha$, $B_n = -4i\mu(1-\mu^2) + c$, $C_n = 6\mu^2 - 2$, $D_n = -4i\mu$, $G_n = \mu^2$ and $H_n = 2i\mu - c$. Here, we used the abbreviation $\mu = nk_c$ for notational simplicity. The nonlinearity $\mathcal{N}_n$ is given by

$$\mathcal{N}_n(U,W) = \left(0,0,0,\sum_{p+q=n} U_{p0}W_{q0} - (1+\gamma)\sum_{p+q+r=n} U_{p0}U_{q0}U_{r0}, 0, 2c\gamma \sum_{p+q=n} U_{p0}U_{q1}\right)^T$$

for all $n \in \mathbb{Z}$.

Spectrum of the L_n

Following [EW91], we set $k_c = 1$ for simplicity in what follows. This yields $\mu = n$. However, the results remain true for $k_c = 1 + \varepsilon^2 q_0$, with $q_0 \in (-\sqrt{\alpha_0}, \sqrt{\alpha_0})$ similar to Lemma 2.2.1. Recall that the L_n have a blockmatrix structure, i.e.

$$L_n = \begin{pmatrix} L_n^{\mathrm{SH}} & 0 \\ 0 & L_n^{\mathrm{con}} \end{pmatrix},$$

with L_n^{SH} originates from (2.10a) and L_n^{con} originates from (2.10b). Thus, we can analyse the spectrum for each block individually. To do so, we set $\alpha = \varepsilon^2\alpha_0$ and $c = \varepsilon c_0$ for some $\alpha_0, c_0 > 0$. We find for $\varepsilon = 0$ that L_n^{SH} has two double eigenvalues $\lambda_{n,\pm} = -i(n \pm 1)$ corresponding to a Jordan block of size 2. For $\varepsilon > 0$, we find with $\Delta = \sqrt{c_0^2 - 16\alpha_0}$ that

$$\lambda_{n,+}^{\pm} = \begin{cases} \varepsilon\dfrac{-c_0 \pm \Delta}{8} + \mathcal{O}(\varepsilon^2), & n = -1, \\[2ex] -i(n+1) \pm \varepsilon^{1/2} i^{3/2} \dfrac{\sqrt{c_0(n+1)}}{2} + \mathcal{O}(\varepsilon), & n \neq -1, \end{cases} \tag{2.12}$$

$$\lambda_{n,-}^{\pm} = \begin{cases} \varepsilon\dfrac{-c_0 \pm \Delta}{8} + \mathcal{O}(\varepsilon^2), & n = 1, \\[2ex] -i(n-1) \pm \varepsilon^{1/2} i^{3/2} \dfrac{\sqrt{c_0(n-1)}}{2} + \mathcal{O}(\varepsilon), & n \neq 1. \end{cases} \tag{2.13}$$

Similarly, we find for L_n^{con} that for $n \neq 0$ the eigenvalues are given by

$$\nu_{n,\pm} = in \pm \varepsilon^{\frac{1}{2}} i^{\frac{3}{2}} \sqrt{nc_0} + \mathcal{O}(\varepsilon) \tag{2.14}$$

while for $n = 0$ we have two eigenvalues $\nu_{0,+} = 0$ and $\nu_{0,-} = -\varepsilon c_0$. Here, the zero eigenvalue at $n = 0$ comes from the fact that (2.10b) is a conservation law. The calculation leading to (2.12)–(2.14) are given in Appendix 2.A.

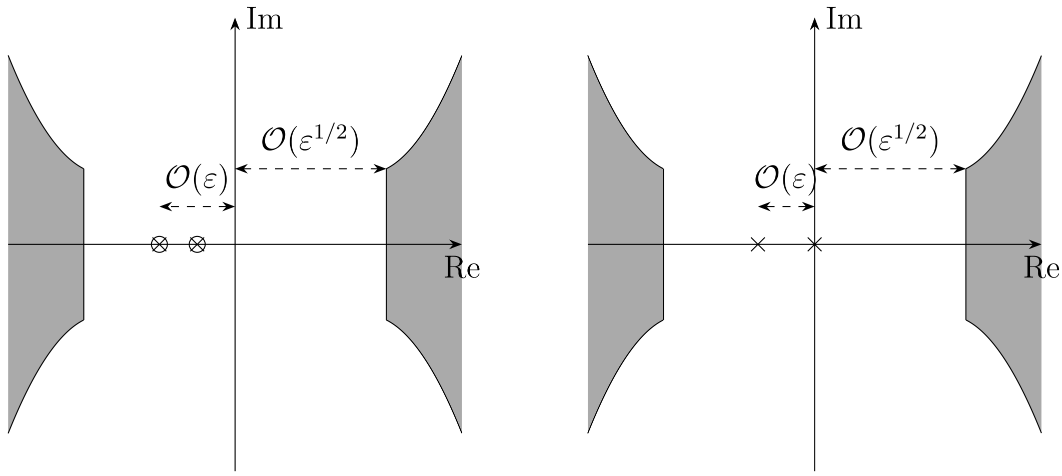

Figure 2.3.: Left: spectrum of L^{SH} with double eigenvalues, Right: spectrum of L^{con}.

Summarizing, there are 6 "central" eigenvalues with real part $\mathcal{O}(\varepsilon)$ while the rest of the spectrum has real part $\mathcal{O}(\varepsilon^{1/2})$. This is illustrated in Figure 2.3. We thus have a similar mechanism as for the pure Swift-Hohenberg equation. For $\varepsilon = 0$ we have a purely imaginary spectrum and as ε gets positive the eigenvalues depart from the imaginary axis with different velocities which allows us to construct a center manifold. However, the size of this center manifold will depend on ε and the remainder of this section is devoted to the construction of a center manifold that is large enough to contain the spatially periodic equilibria from Lemma 2.2.1.

The function space and spectrum of L

We follow [EW91] and define the function space

$$H^l(\mathbb{C}^m) := \left\{ U(p) = \sum_{n\in\mathbb{Z}} U_n e^{ink_c p} \in \mathbb{C}^m : \|U\|_{H^l} < \infty \right\}, \tag{2.15}$$

where $l > 0$, k_c the critical wave number as in Lemma 2.2.1 and $\|\cdot\|_{H^l}$ is defined by

$$\|U\|_{H^l}^2 := \sum_{n\in\mathbb{Z}} (1+n^2)^l |U_n|^2.$$

This space is a Hilbert space for $l > 0$ and a Banach algebra for $l > 1/2$. To simplify the notation, we define $\mathcal{E}_{l_1,l_2} := H^{l_1}(\mathbb{C}^4) \times H^{l_2}(\mathbb{C}^2)$ which is equipped with the norm

$\|(U,W)\|_{\mathcal{E}_{l_1,l_2}} = \|U\|_{H^{l_1}} + \|W\|_{H^{l_2}}$ for $U \in H^{l_1}(\mathbb{C}^4)$ and $W \in H^{l_2}(\mathbb{C}^2)$. Additionally, we introduce extra notation for the case $l_1 = l_2$, i.e. we denote $\mathcal{E}_l := \mathcal{E}_{l,l}$. Using this notation, we can write (2.9a)–(2.9b) equivalently as

$$\partial_\xi \begin{pmatrix} U \\ W \end{pmatrix} = L \begin{pmatrix} U \\ W \end{pmatrix} + \mathcal{N}(U,W), \tag{2.16}$$

where $(U,W) \in \mathcal{E}_l$, and the linear operator $L : \mathcal{E}_{l+4,l+2} \to \mathcal{E}_l$ and the nonlinearity $\mathcal{N} : \mathcal{E}_l \to \mathcal{E}_l$ are defined as

$$L \begin{pmatrix} U \\ W \end{pmatrix} = \sum_{n\in\mathbb{Z}} L_n \begin{pmatrix} U_n \\ W_n \end{pmatrix} e^{ink_cp} \text{ and } \mathcal{N}(U,W) = \sum_{n\in\mathbb{Z}} \mathcal{N}_n(U,W) e^{ink_cp}.$$

Here L_n and $\mathcal{N}_n$ are given in (2.11). In particular we note that to calculate the spectrum of L it is sufficient to calculate the spectra of L_n for $n \in \mathbb{Z}$ since $\sigma(L) = \bigcup_{n\in\mathbb{Z}} \sigma(L_n)$. Furthermore, since $\mathcal{E}_l$ is a Banach algebra for $l > 1/2$ and the nonlinearity $\mathcal{N}$ has a polynomial structure we have the following result.

Corollary 2.3.1. *Let $r > 0$ and $l > 1/2$ be arbitrary and $\gamma \in \mathbb{R}$ the coupling parameter in* (2.3a)–(2.3b)*. Then $\mathcal{N} : \mathcal{E}_l \to \mathcal{E}_l$ is Lipschitz continuous on $B_r(0) \subset \mathcal{E}_l$, the open ball around* 0 *with radius $r > 0$, with*

$$\|\mathcal{N}(U_1,W_1) - \mathcal{N}(U_2,W_2)\|_{\mathcal{E}_l} \le C(1+|\gamma|)r \,\|(U_1,W_1) - (U_2,W_2)\|_{\mathcal{E}_l}$$

for some constant $C < \infty$ independent of γ and r.

2.3.2. Center manifold theorem

Due to the spectral situation, cf. Figure 2.3, the size of the center manifold depends on ε. Therefore, in order to estimate its size, we have to recall the proof of the center manifold theorem. As mentioned in the introduction, our proof is inspired by the one in [HCS99]. First, we introduce the projections on the $\mathcal{O}(\varepsilon)$-center part of the spectrum using the Dunford integral

$$P_{n,c} = \frac{1}{2\pi i} \int_{\Gamma_{n,c}} (\lambda I - L_n)^{-1} \, d\lambda,$$

where $\Gamma_{n,c}$ is a smooth, simple curve surrounding the central spectrum of L_n which is well defined since $L_n \in \mathbb{C}^{6\times 6}$. Especially, we have $P_{n,c} = 0$ for $n \notin \{0, \pm 1\}$. Furthermore, we define the projection on the $\mathcal{O}(\varepsilon^{1/2})$-hyperbolic part of the spectrum $P_{n,h} = I - P_{n,c}$. Using that the central and the hyperbolic spectra are $\mathcal{O}(1)$-separated – they have a different imaginary part of distance $\mathcal{O}(1)$ – these operators can be bounded independently of ε and n. We define for $(U, W) \in \mathcal{E}_l$

$$\mathcal{P}_c \begin{pmatrix} U \\ W \end{pmatrix} = \sum_{n\in\mathbb{Z}} P_{n,c} \begin{pmatrix} U_n \\ W_n \end{pmatrix} e^{ink_c p}, \tag{2.17}$$

which satisfies the following corollary.

Corollary 2.3.2. *Let $l > 1/2$. There exists an $\varepsilon_0 > 0$ and $C < \infty$ independent of ε_0 such that*

$$\|\mathcal{P}_c\|_{\mathcal{L}(\mathcal{E}_l)} < C$$

for all $\varepsilon \in (0, \varepsilon_0)$, where $\mathcal{L}(\mathcal{E}_l)$ denotes the space of linear, bounded maps from $\mathcal{E}_l$ into itself.

With this, we introduce the central part of the operator $L_c := \mathcal{P}_c L$ and $L_{n,c} := P_{n,c} L_n$ and we define L_h and $L_{n,h}$ analogously. Furthermore, we introduce $(U_c, W_c) = \mathcal{P}_c(U, W)$ and $(U_h, W_h) = \mathcal{P}_h(U, W)$. We recall that the periodic solutions are of different order in terms of ε, namely $u_{\mathrm{per}} = \mathcal{O}(\varepsilon)$ and $w_{\mathrm{per}} = v_{\mathrm{per}} + \gamma u_{\mathrm{per}}^2 = \mathcal{O}(\varepsilon^2)$. Keeping this in mind, we introduce the rescaling

$$U_c = \varepsilon^{\gamma_u} \underline{U}_c,\ U_h = \varepsilon^{\beta_u} \underline{U}_h,\ W_c = \varepsilon^{\gamma_w} \underline{W}_c,\ W_h = \varepsilon^{\beta_w} \underline{W}_h, \tag{2.18}$$

with $\gamma_u < 1$, $\beta_u, \gamma_w < 2$ and $\beta_w < 3$. Hence, by projecting (2.16) onto the center and hyperbolic eigenspaces and inserting the rescaling (2.18) we obtain

$$\partial_\xi \begin{pmatrix} \underline{U}_c \\ \underline{W}_c \end{pmatrix} = L_c \begin{pmatrix} \underline{U}_c \\ \underline{W}_c \end{pmatrix} + \underline{\mathcal{N}}_c^{\mathrm{quad}}(\underline{U}_c, \underline{U}_h, \underline{W}_c, \underline{W}_h) + \underline{\mathcal{N}}_c^{\mathrm{cub}}(\underline{U}_c, \underline{U}_h, \underline{W}_c, \underline{W}_h), \tag{2.19a}$$

$$\partial_\xi \begin{pmatrix} \underline{U}_h \\ \underline{W}_h \end{pmatrix} = L_h \begin{pmatrix} \underline{U}_h \\ \underline{W}_h \end{pmatrix} + \underline{\mathcal{N}}_h^{\mathrm{quad}}(\underline{U}_c, \underline{U}_h, \underline{W}_c, \underline{W}_h) + \underline{\mathcal{N}}_h^{\mathrm{cub}}(\underline{U}_c, \underline{U}_h, \underline{W}_c, \underline{W}_h), \tag{2.19b}$$

with the rescaled nonlinearities given by

$$\begin{aligned}
\underline{\mathcal{N}}_c^{\text{quad}} &= \mathcal{P}_c(0,0,0,\varepsilon^{-\gamma_u} N_U, 0, \varepsilon^{1-\gamma_w} 2\gamma c_0 N_W)^T, \\
\underline{\mathcal{N}}_h^{\text{quad}} &= \mathcal{P}_h(0,0,0,\varepsilon^{-\beta_u} N_U, 0, \varepsilon^{1-\beta_w} 2\gamma c_0 N_W)^T, \\
\underline{\mathcal{N}}_c^{\text{cub}} &= \mathcal{P}_c(0,0,0,-(1+\gamma)\varepsilon^{-\gamma_u}(\varepsilon^{\gamma_u}(\underline{U}_c)_0 + \varepsilon^{\beta_u}(\underline{U}_h)_0)^3, 0, 0)^T, \\
\underline{\mathcal{N}}_h^{\text{cub}} &= \mathcal{P}_h(0,0,0,-(1+\gamma)\varepsilon^{-\beta_u}(\varepsilon^{\gamma_u}(\underline{U}_c)_0 + \varepsilon^{\beta_u}(\underline{U}_h)_0)^3, 0, 0)^T,
\end{aligned}$$

where

$$\begin{aligned}
N_U &= \varepsilon^{\gamma_u+\gamma_w}(\underline{U}_c)_0(\underline{W}_c)_0 + \varepsilon^{\gamma_u+\beta_w}(\underline{U}_c)_0(\underline{W}_h)_0 + \varepsilon^{\beta_u+\gamma_w}(\underline{U}_h)_0(\underline{W}_c)_0 + \varepsilon^{\beta_u+\beta_w}(\underline{U}_h)_0(\underline{W}_h)_0, \\
N_W &= \varepsilon^{2\gamma_u}(\underline{U}_c)_0(\underline{U}_c)_1 + \varepsilon^{\gamma_u+\beta_u}((\underline{U}_c)_0(\underline{U}_h)_1 + (\underline{U}_h)_0(\underline{U}_c)_1) + \varepsilon^{2\beta_u}(\underline{U}_h)_0(\underline{U}_h)_1.
\end{aligned}$$

Here, we used that $c = \varepsilon c_0$.

Remark 2.3.3. Note that in contrast to [HCS99], the quadratic interaction of the central modes does not vanish in $\mathcal{N}_c^{\text{quad}}$. Furthermore, the novel term cannot be eliminated via normal form transform with an $\mathcal{O}(1)$-bound for $\varepsilon \to 0$. To see this we recall the nonresonance condition

$$|\lambda_1 + \lambda_2 - \lambda_3| > C$$

uniformly for all small $\varepsilon > 0$ for some C independent of ε, see [HCS99] and the references therein. Here λ_3 corresponds to the eigenvalue of the equation in which the quadratic term should be eliminated and λ_1, λ_2 correspond to the eigenvalues belonging to the variables in the quadratic term. However, since we want to eliminate quadratic terms of the form $(\underline{U}_c)_0(\underline{U}_c)_1$ in the equation for $\underline{W}_c$ and $(\underline{U}_c)_0(\underline{W}_c)_0$ in the equation for $\underline{U}_c$ we find that $\lambda_i \to 0$ for $\varepsilon \to 0$ and $i = 1, 2, 3$ since all central eigenvalues vanish for $\varepsilon = 0$. In particular, their imaginary part vanishes. Thus the nonresonance condition is violated and we cannot eliminate these terms with a bounded normal form transform. □

From Corollary 2.3.1, we then obtain the following result.

Lemma 2.3.4. *Let $r > 0$, $l > 1/2$, $\varepsilon > 0$ and $\gamma \in \mathbb{R}$ the coupling parameter in* (2.3a)–(2.3b)*. Then, $\underline{\mathcal{N}}_c := \underline{\mathcal{N}}_c^{quad} + \underline{\mathcal{N}}_c^{cub} : \mathcal{E}_l \times \mathcal{E}_l \to \mathcal{E}_l$ and $\underline{\mathcal{N}}_h := \underline{\mathcal{N}}_h^{quad} + \underline{\mathcal{N}}_h^{cub} : \mathcal{E}_l \times \mathcal{E}_l \to \mathcal{E}_l$*

are Lipschitz continuous on $B_r(0) \subset \mathcal{E}_l$ with

$$\|\underline{\mathcal{N}}_c(\underline{X}_1) - \underline{\mathcal{N}}_c(\underline{X}_2)\|_{\mathcal{E}_l \times \mathcal{E}_l} \leq C(1+|\gamma|) r \varepsilon^{\min(\gamma_w, \beta_w, \beta_u + \gamma_w - \gamma_u, 1 + 2\gamma_u - \gamma_w)} \|\underline{X}_1 - \underline{X}_2\|_{\mathcal{E}_l \times \mathcal{E}_l},$$
$$\|\underline{\mathcal{N}}_h(\underline{X}_1) - \underline{\mathcal{N}}_h(\underline{X}_2)\|_{\mathcal{E}_l \times \mathcal{E}_l} \leq C(1+|\gamma|) r \varepsilon^{\min(\gamma_u + \gamma_w - \beta_u, \gamma_w, \beta_w, 1 + 2\gamma_u - \beta_w)} \|\underline{X}_1 - \underline{X}_2\|_{\mathcal{E}_l \times \mathcal{E}_l},$$

where $\underline{X} = (\underline{U}_c, \underline{U}_h, \underline{W}_c, \underline{W}_h)$ and $C < \infty$ independent of ε and γ.

It turns out that this is enough to prove that the system (2.19a)–(2.19b) has a center manifold of size $\mathcal{O}(1)$ for a certain choice of $\gamma_u, \gamma_w, \beta_u, \beta_w$. This then yields the desired result. To show this, we define the semigroup corresponding to the stable, unstable and central part of L_n by

$$S_{n,c}(t) = \frac{1}{2\pi i} \int_{\Gamma_{n,c}} (\lambda I - L_n)^{-1} e^{\lambda t} \, d\lambda,$$
$$S_{n,s}(t) = \frac{1}{2\pi i} \int_{\Gamma_{n,s}} (\lambda I - L_n)^{-1} e^{\lambda t} \, d\lambda,$$
$$S_{n,u}(t) = \frac{1}{2\pi i} \int_{\Gamma_{n,u}} (\lambda I - L_n)^{-1} e^{\lambda t} \, d\lambda,$$

where $\Gamma_{n,c}, \Gamma_{n,s}$ and $\Gamma_{n,u}$ are smooth, simple curves around the central, stable and unstable part of the spectrum of L_n, respectively. And similarly to $\mathcal{P}_c$ we define

$$S_j(t)X = \sum_{n \in \mathbb{Z}} S_{n,j}(t) X_n e^{ink_c p}$$

for $X \in \mathcal{E}_l$ and $j \in \{c, s, u\}$. We now provide estimates for these operators.

Lemma 2.3.5. *There exist constants $C_1, C_2 < \infty$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following estimates hold*

$$\|S_c(t)\|_{\mathcal{L}(\mathcal{E}_l)} \leq C_1(1+|t|) e^{C_2 \varepsilon |t|},$$
$$\|S_u(t)\|_{\mathcal{L}(\mathcal{E}_l)} \leq C_1 \frac{1}{\sqrt{\varepsilon}} e^{-C_2 \sqrt{\varepsilon} |t|} \quad \textit{for } t < 0,$$
$$\|S_s(t)\|_{\mathcal{L}(\mathcal{E}_l)} \leq C_1 \frac{1}{\sqrt{\varepsilon}} e^{-C_2 \sqrt{\varepsilon} |t|} \quad \textit{for } t > 0.$$

Proof. For the estimate of the central semigroup we remark that the central spectrum is $\mathcal{O}(1)$-bounded away from the rest of the spectrum for ε sufficiently small. Hence, for any small $\varepsilon_0 > 0$ we can fix the curves $\Gamma_{n,c}$ for all $\varepsilon \in (0, \varepsilon_0)$. Now, we estimate the semigroup $S_{n,c}$ for $n \in \{\pm 1, 0\}$. This is sufficient since $\sigma(L_n)$ does not contain central eigenvalues for $n \notin \{0, \pm 1\}$. Since L_n has a block structure, we can decompose the residual $(\lambda - L_n)^{-1}$ into $(\lambda - L_n^{\mathrm{SH}})^{-1}$ and $(\lambda - L_n^{\mathrm{con}})^{-1}$ and thus, we can estimate both parts separately. For $n = 0$, the Swift-Hohenberg part has no central eigenvalues and we focus on $L_0^{\mathrm{con}} \in \mathbb{C}^{2\times 2}$ which has only central eigenvalues. We rewrite the residual as $(\lambda - L_0^{\mathrm{con}})^{-1} = \det(\lambda - L_0^{\mathrm{con}})^{-1} \operatorname{adj}(\lambda - L_0^{\mathrm{con}})$, where $\operatorname{adj}(A)$ denotes the adjunct of A and obtain by explicitly calculating the adjunct

$$\begin{aligned}\left\| \frac{1}{2\pi i} \int_{\Gamma_{0,c}} \frac{1}{\lambda(\lambda + \varepsilon c_0)} \operatorname{adj}(\lambda - L_0^{\mathrm{con}}) e^{\lambda t} \, d\lambda \right\|_{\mathbb{C}^{2\times 2}} &\leq \frac{1}{2\pi i} \left| \int_{\Gamma_{0,c}} \left[\frac{e^{\lambda t}}{\lambda} + \frac{e^{\lambda t}}{\lambda + \varepsilon c_0} + \frac{e^{\lambda t}}{\lambda(\lambda + \varepsilon c_0)} \right] d\lambda \right| \\ &\leq C_1(1 + |t|) e^{C_2 \varepsilon |t|}.\end{aligned}$$

By similar calculations, we obtain the same estimate for $n = \pm 1$ since L_n^{SH} has two central and two hyperbolic eigenvalues. Thus, since the estimates are independent of $n \in \mathbb{Z}$, we obtain the upper bound

$$\|S_c(t)X\|_{\mathcal{E}_l} = \left(\sum_{n\in\mathbb{Z}} (1 + n^2)^l |S_{n,c}(t)X|^2 \right)^{1/2} \leq C(1 + |t|) e^{C_2 \varepsilon |t|} \, \|X\|_{\mathcal{E}_l}$$

for any $X \in \mathcal{E}_l$ which yields the estimate for $S_c(t)$.

Next, we prove the estimate for the stable and unstable semigroup, respectively. We focus on the stable semigroup since the proofs are identical. Note that we can decompose the spectrum of L_n into three pairs of eigenvalues which are $\mathcal{O}(1)$ apart from each other, see (2.12)–(2.13) and (2.14). Thus, for any small $\varepsilon > 0$ we can decompose the curve $\Gamma_{n,s}$ into three separate curves. However, these cannot be chosen independent of ε since the eigenvalues are symmetric with respect to the imaginary axis and their real part vanishes for $\varepsilon \to 0$. We now estimate the contribution of L_n^{con} and remark that the contribution of L_n^{SH} can be handled analogously since there are only two relevant eigenvalues as mentioned above. Hence, for $n \neq 0$ we calculate for $t > 0$

$$\begin{aligned}\left\| \frac{1}{2\pi i} \int_{\Gamma_{n,s}} (\lambda I - L_n^{\mathrm{con}})^{-1} e^{\lambda t} \, d\lambda \right\|_{\mathbb{C}^{2\times 2}} &= \left\| \frac{1}{2\pi i} \int_{\Gamma_{n,s}} \frac{1}{\lambda(\lambda + \varepsilon c_0)} \operatorname{adj}(\lambda - L_0^{\mathrm{con}}) e^{\lambda t} \, d\lambda \right\|_{\mathbb{C}^{2\times 2}} \\ &\leq \frac{1}{2\pi i} \left\| \int_{\Gamma_{n,s}} \left[\frac{e^{\lambda t}}{(\lambda - \nu_{n,+})(\lambda - \nu_{n,-})} \operatorname{adj}(\lambda - L_n^{\mathrm{SH}}) \right] d\lambda \right\|_{\mathbb{C}^{2\times 2}} \\ &\leq \left\| \operatorname{adj}(\nu_{n,-} - L_n^{\mathrm{SH}}) \right\|_{C^{2\times 2}} \frac{1}{|\nu_{n,-} - \nu_{n,+}|} e^{\nu_{n,-} t}\end{aligned}$$

$$\leq C_1 \frac{1}{\sqrt{\varepsilon}} e^{-C_2\sqrt{\varepsilon}t},$$

where we used that $(\nu_{n,\pm}) = \mathcal{O}(\varepsilon^{1/2})$ and $(\nu_{n,-}) < 0$. Since this estimate is independent of $n \in \mathbb{Z}$ we obtain the desired estimate for S_s in $\mathcal{L}(\mathcal{E}_l)$. □

We now introduce a smooth cut-off function $\chi_r \in [0,1]$ for some $r > 0$ which we define by $\chi_r(x) = 1$ for $|x| < r/2$, $\chi_r = 0$ for $|x| > r$ and $\chi_r \in (0,1)$ for $|x| \in [r/2, r]$. Using this cut-off function, we define $\underline{\tilde{\mathcal{N}}}_j := \chi_r \underline{\mathcal{N}}_j$ for $j \in \{c, h\}$ which is a globally Lipschitz continuous mapping with the estimates given in Lemma 2.3.5. We now split $\mathcal{P}_h = \mathcal{P}_s + \mathcal{P}_u$ into a projection onto the stable eigenspaces $\mathcal{P}_s$ and a projection onto the unstable eigenspaces $\mathcal{P}_u$ which we define similarly to $\mathcal{P}_c$. Then, we follow the standard procedure for the construction of center manifold results and show that the mapping

$$\begin{aligned} X_c(t) &= S_c(t)X_c(0) + \int_0^t S_c(t-\tau)\underline{\tilde{\mathcal{N}}}_c(X(\tau))\,d\tau \\ X_u(t) &= -\int_t^\infty S_u(t-\tau)\underline{\tilde{\mathcal{N}}}_h(X(\tau))\,d\tau \\ X_s(t) &= \int_{-\infty}^t S_s(t-\tau)\underline{\tilde{\mathcal{N}}}_h(X(\tau))\,d\tau \end{aligned} \tag{2.20}$$

has a fixed point in

$$\mathcal{X}_\eta := \left\{ X \in C^0(\mathbb{R}, \mathcal{E}_l) : \|X\|_\eta := \sup_{t\in\mathbb{R}} e^{-\eta|t|} \|X\|_{\mathcal{E}_l} < \infty \right\}$$

with $\eta = \frac{C_2}{2}\sqrt{\varepsilon}$ where we use the decomposition $X_j = \mathcal{P}_j X$ for $j \in \{c, s, u\}$. Using Lemma 2.3.4 we now estimate

$$\left\|\underline{\tilde{\mathcal{N}}}_c(X_1) - \underline{\tilde{\mathcal{N}}}_c(X_2)\right\|_\eta \leq C(1+|\gamma|) r \varepsilon^{\min(\gamma_w, \beta_w, \beta_u + \gamma_w - \gamma_u, 1 + 2\gamma - \gamma_w)} \|X_1 - X_2\|_\eta, \tag{2.21a}$$

$$\left\|\underline{\tilde{\mathcal{N}}}_h(X_1) - \underline{\tilde{\mathcal{N}}}_h(X_2)\right\|_\eta \leq C(1+|\gamma|) r \varepsilon^{\min(\gamma_u + \gamma_w - \beta_u, \gamma_w, \beta_w, 1 + 2\gamma_u - \beta_w)} \|X_1 - X_2\|_\eta, \tag{2.21b}$$

for any $X_1, X_2 \in \mathcal{X}_\eta$. Furthermore, for any nonlinear mapping $V : \mathcal{E}_l \to \mathbb{R}$ we have

$$\begin{aligned} \left\| \int_0^t S_c(t-\tau) V(X(\tau))\,d\tau \right\|_\eta &\leq \sup_{t\in\mathbb{R}} e^{-\eta|t|} \int_0^t C_1(1+|t|)e^{C_2\varepsilon|t-\tau|} e^{\eta\tau}\,d\tau\, \|V(X)\|_\eta \\ &\leq C\varepsilon^{-1} \|V(X)\|_\eta, \\ \left\| \int_t^\infty S_u(t-\tau) V(X(\tau))\,d\tau \right\|_\eta &\leq \sup_{t\in\mathbb{R}} e^{-\eta|t|} \int_t^\infty C_1 \frac{1}{\sqrt{\varepsilon}} e^{-C_2\sqrt{\varepsilon}|t-\tau|} e^{\eta|\tau|}\,d\tau\, \|V(X)\|_\eta \end{aligned}$$

$$\leq C\varepsilon^{-1} \|V(X)\|_\eta ,$$

$$\left\| \int_{-\infty}^{t} S_s(t-\tau)V(X(\tau))\, d\tau \right\|_\eta \leq C\varepsilon^{-1} \|V(X)\|_\eta ,$$

where we used $\int_0^\infty e^{\sqrt{\varepsilon}t} t\, dt = \mathcal{O}(\varepsilon^{-1})$ and $\eta < C_2\sqrt{\varepsilon}$. Combining these estimates we find that the system (2.20) maps $\mathcal{X}_\eta$ into itself and that we can estimate the Lipschitz constant of (2.20) which is of order

$$C(1+|\gamma|)r\varepsilon^{\min(\kappa_1,\kappa_2)-1},$$

where κ_1, κ_2 are the exponents in (2.21a) and (2.21b), respectively. Finally, we set

$$\gamma_u = 2/3+\delta,\ \beta_u = 1+\delta/2,\ \gamma_w = 4/3 \text{ and } \beta_w = 4/3+\delta \tag{2.22}$$

for some $\delta > 0$. This choice yields that $\kappa_1, \kappa_2 > 1$ and thus, (2.20) is a contraction in $\mathcal{X}_\eta$ provided that ε is small enough. Especially, this does not impose a restriction on the cut-off radius r. Thus following the standard proof of the center manifold theorem, see, e.g. [HI11], we have a center manifold of size $\mathcal{O}(1)$ for the rescaled system (2.19a)–(2.19b). Finally, by reverting the rescaling (2.18) we arrive at the main result of this section.

Theorem 2.3.6. *Let $l > 1/2$, $\delta \in (0, 1/3)$ and $\gamma \in \mathbb{R}$. There exists an $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there exists a neighbourhood $O_c = O_u \times O_w \subset \mathcal{E}_c := \mathcal{P}_c\mathcal{E}_l$ of the origin and a mapping $h = (h_u, h_w) : O_c \to \mathcal{E}_h := \mathcal{P}_h\mathcal{E}_l$ such that the following holds.*

- *The neighbourhood O_u is of size $\mathcal{O}(\varepsilon^{2/3+\delta})$ and O_w is of size $\mathcal{O}(\varepsilon^{4/3})$.*
- *The center manifold*

$$\mathcal{M}_c = \{(U,W) = (U_c,W_c) + h(U_c,W_c) : (U_c,W_c) \in O_c\}$$

contains all small bounded solutions of (2.16)*.*

- *Every solution of the reduced system*

$$\partial_\xi \begin{pmatrix} U_c \\ W_c \end{pmatrix} = L_c \begin{pmatrix} U_c \\ W_c \end{pmatrix} + \mathcal{P}_c\mathcal{N}(U_c + h_u(U_c,W_c), W_c + h_w(U_c,W_c)) \tag{2.23}$$

gives a solution to the full system (2.16) *via $(U,W) = (U_c,W_c) + h(U_c,W_c)$.*

Remark 2.3.7. We point out that due to the ansatz (2.18), the center manifold in Theorem 2.3.6 has different sizes in u and w-direction, respectively. However, this is sufficient for the construction of modulating fronts. To see this, we recall that the periodic solution established in Lemma 2.2.1 satisfies $u_{\text{per}} = \mathcal{O}(\varepsilon)$ and $w_{\text{per}} := v_{\text{per}} + \gamma u_{\text{per}}^2 = \mathcal{O}(\varepsilon^2)$. We will see that the modulating front solutions have the same property and thus, the center manifold constructed above is sufficiently large to contain modulating fronts. □

Remark 2.3.8. As seen above, the construction of the center manifold does not require a normal form transform as in [HCS99]. The reason for this lies both in the different rescaling of the U and W contribution in (2.18) and the transformation $w = v + \gamma u^2$ which gives an additional ε in front of the quadratic term in (2.9b). Using this we see that $U_c W_c = \mathcal{O}(\varepsilon^{\gamma_u + \gamma_w})$ and $c\partial_\xi(U_c^2) = \mathcal{O}(\varepsilon^{1+2\gamma_u})$. Therefore, both critical quadratic nonlinearities scale better than ε^2 for γ_u, γ_w chosen in (2.22) and can be treated without further issues. □

2.4. Reduced equations and heteroclinic connections

We now derive the reduced equations on the center manifold constructed in Theorem 2.3.6 and establish the existence of heteroclinic orbits connecting a circle of non-trivial fixed points to the origin. The reduction mainly follows [EW91], however, the construction of the heteroclinic orbits requires more work since we obtain an additional equation corresponding to the conservation law. The main idea for this construction is to handle first the case $\gamma = 0$ which corresponds to the pure Swift-Hohenberg equation and afterwards use perturbation arguments to establish the persistence of these orbits for $\gamma \neq 0$ close to 0. Finally we numerically investigate the case for large γ in which we also find heteroclinic orbits.

2.4.1. Derivation of the reduced equations

We start by introducing the following coordinates for the central modes

$$\begin{pmatrix} U_c(\xi, p) \\ W_c(\xi) \end{pmatrix} = \varepsilon \left(A(\varepsilon\xi)\varphi_1^+ + B(\varepsilon\xi)\varphi_1^- \right) e^{ik_c p} + \varepsilon \left(\overline{A}(\varepsilon\xi)\overline{\varphi_1^+} + \overline{B}(\varepsilon\xi)\overline{\varphi_1^-} \right) e^{-ik_c p}$$
$$+ (0, 0, 0, 0, \varepsilon^2 W_{c0}(\varepsilon\xi), \varepsilon^3 W_{c1}(\varepsilon\xi))^T,$$

where $A, B \in \mathbb{C}$, $W_{c0}, W_{c1} \in \mathbb{R}$ and $\varphi_1^+, \varphi_1^- \in \mathbb{C}^6$ are the eigenvectors corresponding to the eigenvalues $\lambda_{1,-}^{\pm}$ of L_1 computed in (2.13). Here, we normalize the eigenvectors $\varphi_1^{\pm}$ such that the first component is equal to one. Note that since L_1 has a block diagonal structure, the last two components of $\varphi_1^{\pm}$ are zero. Additionally, we used that $U_1 = \bar{U}_{-1}$ – this gives the complex conjugated terms in U_c – and that all eigenvalues of L_0^{con} are central. Furthermore, we find $h_u \in \mathcal{O}(\varepsilon^3)$ and $h_w \in \mathcal{O}(\varepsilon^4)$. To see this, we recall that

$$
\begin{aligned}
Dh_u \partial_\xi X_c &= L_h^{\text{SH}} h_u + \mathcal{P}_h^{\text{SH}}(\mathcal{N}^{\text{SH}}(X_c + h(X_c))),\\
Dh_w \partial_\xi X_c &= L_h^{\text{con}} h_w + \mathcal{P}_h^{\text{con}}(\mathcal{N}^{\text{con}}(X_c + h(X_c)),
\end{aligned}
$$

where $X_c = (U_c, W_c)^T$ and

$$
\begin{aligned}
\mathcal{N}^{\text{SH}}(X_c + h(X_c)) &= (0, 0, 0, (U_c + h_u(X_c))_0 (W_c + h_w(X_c))_0 - (1+\gamma)(U_c + h_u(X_c))_0^3)^T,\\
\mathcal{N}^{\text{con}}(X_c + h(X_c)) &= (0, 2c\gamma(U_c + h_u(X_c))_0 (U_c + h_u(X_c))_1)^T.
\end{aligned}
$$

Now, recalling that $c = \varepsilon c_0$ and noting that h_u, h_w are at least quadratic in X_c we find $\mathcal{P}_h^{\text{SH}}(\mathcal{N}^{\text{SH}}(X_c + h(X_c))) = \mathcal{O}(\varepsilon^3)$ and $\mathcal{P}_h^{\text{con}}(\mathcal{N}^{\text{con}}(X_c + h(X_c)) = \mathcal{O}(\varepsilon^4)$. Finally, since the hyperbolic eigenvalues have an imaginary part which is uniformly bounded away from zero for all $\varepsilon > 0$, i.e. L_h has a bounded inverse we find $h_u = \mathcal{O}(\varepsilon^3)$ and $h_w = \mathcal{O}(\varepsilon^4)$ as claimed.

To derive the reduced equations on the center manifold, we recall that the reduced dynamic is determined by

$$
\partial_\xi \begin{pmatrix} U_c \\ W_c \end{pmatrix} = L_c \begin{pmatrix} U_c \\ W_c \end{pmatrix} + \mathcal{P}_c(\mathcal{N}(U_c + h_u(U_c, W_c), W_c + h_w(U_c, W_c))), \tag{2.24}
$$

where $\mathcal{P}_c$ is the projection onto the central eigenspace, see (2.17), and the nonlinearity $\mathcal{N}$ is given in (2.16). To obtain an equation for A, B we now project onto the eigenspaces spanned by $\varphi_1^{\pm}$ respectively, using the projection

$$
P_{\varphi_1^{\pm}}((U_1, W_1)^T) = \frac{\left\langle \psi_1^{\pm}, (U_1, W_1)^T \right\rangle}{\left\langle \psi_1^{\pm}, \varphi_1^{\pm} \right\rangle} \varphi_1^{\pm}.
$$

Here, $\langle \cdot, \cdot \rangle$ is the euclidean scalar product on $\mathbb{C}^6 \times \mathbb{C}^6$ and $\psi_1^{\pm}$ is the eigenvector of the adjoint matrix L_1^* corresponding to the eigenvector $\overline{\lambda_{1,-}^{\pm}}$. We normalize $\psi_1^{\pm} \in \mathbb{C}^6$ given by

$$\psi_1^{\pm} = ((\psi_1^{\pm})_0, (\psi_1^{\pm})_1, (\psi_1^{\pm})_2, (\psi_1^{\pm})_3, 0, 0)^T$$

such that $(\psi_1^{\pm})_3 = 1$, which yields that

$$\begin{aligned} &P_{\varphi_1^{\pm}}(\mathcal{N}_1(U_c + h_u(U_c, W_c), W_c + h_w(U_c, W_c))) \\ &\quad = \frac{\varepsilon^3}{\left\langle \psi_1^{\pm}, \varphi_1^{\pm} \right\rangle} \left((A+B)W_{c0} - 3(1+\gamma)(A+B)|A+B|^2 + \mathcal{O}(\varepsilon^2) \right) \varphi_1^{\pm}, \end{aligned}$$

where we also used again that the first component of $\varphi_1^{\pm}$ is equal to one. Therefore, applying $P_{\varphi_1^{\pm}}$ to (2.24) we find

$$\begin{aligned} \varepsilon^2 \partial_{\tilde{\xi}} A &= \varepsilon \lambda_1^+ A + \frac{\varepsilon^3}{\left\langle \psi_1^+, \varphi_1^+ \right\rangle} \left((A+B)W_{c0} - 3(1+\gamma)(A+B)|A+B|^2 + \mathcal{O}(\varepsilon^2) \right), \\ \varepsilon^2 \partial_{\tilde{\xi}} B &= \varepsilon \lambda_1^- B + \frac{\varepsilon^3}{\left\langle \psi_1^-, \varphi_1^- \right\rangle} \left((A+B)W_{c0} - 3(1+\gamma)(A+B)|A+B|^2 + \mathcal{O}(\varepsilon^2) \right), \end{aligned}$$

where $\tilde{\xi} = \varepsilon \xi$. Next, we calculate the normalizing constant $\left\langle \psi_1^{\pm}, \varphi_1^{\pm} \right\rangle^{-1}$. For that we use that $\left\langle \psi_1^{\pm}, \varphi_1^{\pm} \right\rangle = -p_1'(\lambda_{1,-}^{\pm})$, the derivative of the negative characteristic polynomial of L_1^{SH}, see [EW91, Appendix C] and recall that $\psi_1^{\pm}$ is normalized such that the last non-zero component is one. Using the characteristic polynomial given in Appendix 2.A and that $\lambda_{1,-}^{\pm} = \varepsilon(-c_0 \pm \Delta)/8 + \mathcal{O}(\varepsilon)$ with $\Delta = \sqrt{c_0^2 - 16\alpha_0}$, we find

$$\left\langle \psi_1^{\pm}, \varphi_1^{\pm} \right\rangle = \mp \varepsilon \Delta + \mathcal{O}(\varepsilon^2),$$

and thus,

$$\left\langle \psi_1^{\pm}, \varphi_1^{\pm} \right\rangle^{-1} = \frac{\mp 1}{\varepsilon \Delta}(1 + \mathcal{O}(\varepsilon)).$$

Then, the reduced equations for A, B are given by

$$\partial_{\tilde{\xi}} A = \frac{-c_0 + \Delta}{8} A - \frac{1}{\Delta} \left((A+B)W_{c0} - 3(1+\gamma)(A+B)|A+B|^2 \right) + \mathcal{O}(\varepsilon^2), \tag{2.25a}$$

$$\partial_{\tilde{\xi}} B = \frac{-c_0 - \Delta}{8} B + \frac{1}{\Delta} \left((A+B)W_{c0} - 3(1+\gamma)(A+B)|A+B|^2 \right) + \mathcal{O}(\varepsilon^2) \tag{2.25b}$$

To derive an equation for W_{c0}, W_{c1} we use that the two central eigenvalues originating from the conservation law are the two eigenvalues of L_0^{con}. Next, we recall from (2.11) that

$$\partial_\xi \begin{pmatrix} W_{00} \\ W_{01} \end{pmatrix} = L_0^{\text{con}} \begin{pmatrix} W_{00} \\ W_{01} \end{pmatrix} + \begin{pmatrix} 0 \\ 2\varepsilon c_0 \gamma \sum_{p+q=0} U_{p0} U_{q1}. \end{pmatrix}$$

Furthermore, we recall that $U_{nj} = \partial_\xi^j \mathcal{U}_n$ and $\mathcal{U}_n = \bar{\mathcal{U}}_{-n}$, which yields

$$\sum_{p+q=0} U_{p0} U_{q1} = 2\mathcal{U}_0 \partial_\xi \mathcal{U}_0 + \sum_{p \in \mathbb{N}} (\mathcal{U}_p \overline{\partial_\xi \mathcal{U}_p} + \bar{\mathcal{U}}_p \partial_\xi \mathcal{U}_p) = \partial_\xi \sum_{p \in \mathbb{N}_0} |U_{p0}|^2,$$

with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. This yields that

$$\partial_\xi^2 W_{00} = -\varepsilon c_0 \partial_\xi W_{00} + 2\varepsilon c_0 \gamma \partial_\xi \sum_{p \in \mathbb{N}_0} |U_{p0}|^2,$$

which we integrate once with respect to ξ to obtain

$$\partial_\xi W_{00} = -\varepsilon c_0 W_{00} + 2\varepsilon c_0 \gamma \sum_{p \in \mathbb{N}_0} |U_{p0}|^2.$$

Note that using this integration we have eliminated the central eigenvalue at zero. Therefore, in the above coordinates the equation on the center manifold for W_{c0} then reads as

$$\varepsilon^3 \partial_{\tilde{\xi}} W_{c0} = -\varepsilon^3 c_0 W_{c0} + 2\varepsilon^3 c_0 \gamma |A + B|^2 + \mathcal{O}(\varepsilon^6).$$

Now, following [EW91, Section 3], we substitute A, B in (2.25a)–(2.25b) by

$$A = \frac{1}{2}\left(\hat{A} + \frac{c_0}{\Delta}\hat{A} + \frac{8}{\Delta}\tilde{B}\right), \quad B = \frac{1}{2}\left(\hat{A} - \frac{c_0}{\Delta}\hat{A} - \frac{8}{\Delta}\tilde{B}\right),$$

and obtain, by additionally defining $\hat{B} := \partial_{\tilde{\xi}} \hat{A} = \tilde{B} + \mathcal{O}(\varepsilon^2)$ and $\hat{W}_0 := W_{c0}$, that

$$\partial_{\tilde{\xi}} \hat{A} = \hat{B}, \tag{2.26a}$$

$$\partial_{\tilde{\xi}} \hat{B} = \frac{1}{4}\left(-\alpha_0 \hat{A} - c_0 \hat{B} - \hat{A}\hat{W}_0 + 3(1+\gamma)\hat{A}|\hat{A}|^2\right) + \mathcal{O}(\varepsilon^2), \tag{2.26b}$$

$$\partial_{\tilde{\xi}} \hat{W}_0 = -c_0 \hat{W}_0 + 2c_0 \gamma |\hat{A}|^2 + \mathcal{O}(\varepsilon^3). \tag{2.26c}$$

This system then determines the dynamic on the center manifold.

Remark 2.4.1. Note that although we have 6 central eigenvalues, the reduced system (2.26a)–(2.26c) is 3-dimensional. As explained in the derivation, by integrating the equation for W_{00} once with respect to ξ we removed the zero eigenvalue by using the conservation property. Furthermore, since the solutions of (2.3a)–(2.3b) are supposed to be real, we have $U_j = \bar{U}_{-j}$ for $j \in \mathbb{Z}$. This means that the remaining two central eigenvalues $\lambda^{\pm}_{-1,+}$ yield the complex conjugated equations of (2.26a)–(2.26b). □

2.4.2. Construction of a heteroclinic orbit

We now show that the ODE system (2.26a)–(2.26c) has a circle of nontrivial fixed points. We also show that for $(\varepsilon, \gamma) = (0, 0)$ the system exhibits heteroclinic orbits connecting the nontrivial fixed points to the origin. Finally, we prove that these orbits are persistent for (ε, γ) in a sufficiently small neighborhood of zero, which gives the main result, Theorem 2.4.2.

First, we study the fixed points of (2.26a)–(2.26c), provided that $c_0^2 > 16\alpha_0$. The system has the trivial fixed point $(0, 0, 0)$. Furthermore, for $\varepsilon = 0$ and $\gamma > -3$ we find a circle of nontrivial fixed points at

$$\left(\sqrt{\frac{1}{3+\gamma}\alpha_0}e^{i\phi}, 0, \frac{2\gamma}{3+\gamma}\alpha_0\right),$$

where $\phi \in [0, 2\pi)$. Using the implicit function theorem similar to the proof of Lemma 2.2.1 these fixed points also persist for $\varepsilon > 0$ sufficiently small. Note that these fixed points correspond to the periodic solutions in Lemma 2.2.1 since inserting the fixed points into the transformation for v gives

$$v = w - \gamma u^2 = \varepsilon^2 \hat{W}_0 - 2\gamma\varepsilon^2 |\hat{A}|^2 + \mathcal{O}(\varepsilon^3) = \mathcal{O}(\varepsilon^3).$$

Next, we set $(\gamma, \varepsilon) = (0, 0)$. Then, the origin is a hyperbolic stable fixed point since the linearization has real, negative spectrum. Additionally, splitting the system in real and imaginary parts – recall that $\hat{W}_0$ is real – we find that the circle of nontrivial fixed points forms a normally hyperbolic invariant set with one positive, one zero and three negative eigenvalues. The existence of a heteroclinic orbit connecting the circle of nontrivial fixed points with the origin follows from the fact that $(\hat{A}, \hat{B}, 0)$ is an invariant set and thus, the system reduces to the system studied in [EW91]. Therefore, the heterclinic orbit can be found by phase plane analysis.

We now show that these orbits persist for (γ, ε) in a small neighborhood of zero. The idea is to show that for $(\gamma, \varepsilon) = (0, 0)$ the unstable manifold of the nontrivial fixed point and the stable manifold of the origin intersect transverally. Therefore, the intersection is stable with respect to small perturbations and hence, the intersection persists for (γ, ε) in a small neighborhood of zero. Therefore, we define

$$H(\hat{A}, \hat{B}) := 2|\hat{B}|^2 + \frac{\alpha_0}{2}|\hat{A}|^2 - \frac{3}{4}|\hat{A}|^4,$$

which is a Lyapunov function for the system (2.26a)–(2.26b) for $(\gamma, \varepsilon) = (0, 0)$ and $\hat{W}_0 = 0$. Differentiation with respect to ξ of H along a solution yields $\partial_\xi H = -c_0|\hat{B}|^2 \leq 0$. Furthermore, we remark that the system has no fixed points with $|\hat{A}|^2 < \alpha_0/3$ except for the origin. Thus, the stable manifold of the origin, denoted by $\mathcal{M}_s(0, 0, 0)$, contains the set

$$S_0 = \left\{(\hat{A}, \hat{B}, \hat{W}_0) \in \mathbb{C}^3 : \hat{W}_0 = 0 \text{ and } H(s\hat{A}, s\hat{B}) < H\left(\sqrt{\frac{\alpha_0}{3}}, 0\right) \text{ for all } s \in [0, 1]\right\}.$$

We now show that the unstable manifold of the nontrivial fixed point $\mathcal{M}_u(A_f, B_f, W_f)$ intersects $\mathcal{M}_s(0, 0, 0)$ transversally. Therefore, we note that $\mathcal{M}_u(A_f, B_f, W_f)$ lies in the set $(\hat{A}, \hat{B}, 0) \in \mathbb{C}^3$ since this is invariant and $\mathcal{M}_u(A_f, B_f, W_f)$ is tangential to the unstable eigenspace of the nontrivial fixed points which has no contribution in $\hat{W}_0$-direction for $(\gamma, \varepsilon) = (0, 0)$. Moreover, the unstable eigenspace of the nontrivial fixed point intersects S_0 transversally in $\mathbb{C}^2 \times \{0\}$. Using the stable manifold theorem (see e.g. [Per01]) and the fact that the stable space of the linearization about the origin is three dimensional, we conclude that $\mathcal{M}_s(0, 0, 0)$ is a smooth, three dimensional manifold in $\mathbb{C}^3$. Therefore, for every point $(\hat{A}, \hat{B}, 0) \in S_0$ there exists a small neighborhood in $\mathbb{C}^3$ which is contained in $\mathcal{M}_s(0, 0, 0)$. Hence, $\mathcal{M}_s(0, 0, 0)$ and $\mathcal{M}_u(A_f, B_f, W_f)$ intersect transversally and we obtain the persistence of the heteroclinic connections for small perturbations. In particular, this includes (γ, ε) in a small neighborhood of zero.

Now using the transformation $W = V + \gamma U^2$, we obtain the existence of modulating traveling fronts for (2.3a)–(2.3b). In particular, this also yields an approximate representation of these solutions. Thus, we proved the following theorem.

Theorem 2.4.2. *Let $c_0^2 > 16\alpha_0 > 0$. Then there exists $\varepsilon_0 > 0$, $\gamma_0 > 0$ such that for*

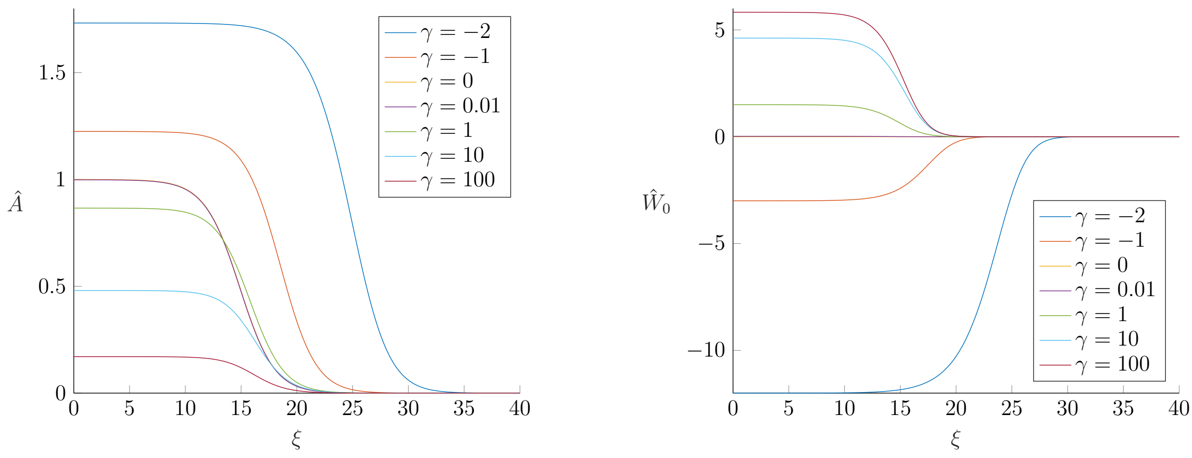

Figure 2.4.: Numerical solutions of reduced system (2.26a)–(2.26c) for $c_0 = 7$, $\alpha_0 = 3$.

all $\varepsilon \in (0, \varepsilon_0)$ *and* $|\gamma| < \gamma_0$ *the system* (2.3a)–(2.3b) *has a modulating travelling front solution* (u_f, v_f). *Furthermore, it holds that*

$$u_f(t,x) = \varepsilon 2|\hat{A}(y)| \cos(x + x_0) + \mathcal{O}(\varepsilon^2),$$
$$v_f(t,x) = \varepsilon^2 \left[\hat{W}_0(y) - 2\gamma|\hat{A}(y)|^2 - 2\gamma|\hat{A}(y)|^2 \cos(2(x+x_0))\right] + \mathcal{O}(\varepsilon^3),$$

in L^∞ *where* $y = \varepsilon x - \varepsilon^2 c_0 t$, $x_0 \in [0, 2\pi)$ *and* $(\hat{A}, \hat{W}_0)$ *are a heteroclinic solution of* (2.26a)–(2.26c).

Remark 2.4.3. Recall that we set $k_c = 1$ in the calculations above. However, the stationary, periodic solutions exist also for k_c close to 1. Therefore, we expect that a similar results holds in the setting of Lemma 2.2.1. □

We stress that the above result is only valid for γ close to zero. However, we expect that this is a restriction only imposed due to the use of perturbation arguments. A first intuition comes from the fact that the circle of nontrivial fixed points exists for all $\gamma > -3$. This is backed by Figure 2.4 which shows a numerical simulation of heteroclinic orbits for different values of γ. For this we approximated an initial point on the unstable manifold of the nontrivial fixed point using the unstable eigenvector of the linearization about this fixed point. Then, using this approximation, we solved the forward dynamics given by the reduced system (2.26a)–(2.26c). It turns out that the spectral stability of the nontrivial fixed point does not change, i.e. the circle of nontrivial fixed points is normally hyperbolic with exactly one unstable direction. Furthermore, the numerical calculation gives heteroclinic orbits of the system which hints the existence of such solutions.

Finally, the numerical calculation hints that for large γ the solution is close to a heteroclinic orbit of a limiting system which can be constructed by introducing the rescaled variables

$$\begin{pmatrix} \tilde{A} \\ \tilde{B} \end{pmatrix} := \sqrt{\frac{3+\gamma}{\alpha_0}} \begin{pmatrix} \hat{A} \\ \hat{B} \end{pmatrix} \text{ and } \tilde{W}_0 := \frac{3+\gamma}{2\alpha_0\gamma}\hat{W}_0.$$

Inserting this into (2.26a)–(2.26c) and formally passing to $\gamma \to \infty$ we find the formal asymptotic system

$$\begin{aligned} \partial_\xi \tilde{A} &= \tilde{B} \\ \partial_\xi \tilde{B} &= \frac{1}{4}\left(-c_0\tilde{B} - \alpha_0\tilde{A} - 2\alpha_0\tilde{A}\tilde{W}_0 + 3\alpha_0\tilde{A}|\tilde{A}|\right) \\ \partial_\xi \tilde{W}_0 &= -c_0\tilde{W}_0 + c_0|\tilde{A}|^2. \end{aligned}$$

As expected this system has a trivial fixed point at $(0,0,0)$ and a circle of nontrivial fixed points at $(e^{i\phi},0,1)$ for $\phi \in [0,2\pi)$. However, we note that this asymptotic model has limited use for the description of the full dynamics (2.3a)–(2.3b) since the size of the center manifold vanishes for $\gamma \to \infty$.

Appendix

2.A. Calculation of eigenvalues

We now give the calculations, which lead to the expansions (2.12)–(2.14) of the eigenvalues of L_n, $n \in \mathbb{Z}$. Therefore, recall that L_n is a block-diagonal matrix with

$$L_n^{\mathrm{SH}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(1+n^2)^2 + \varepsilon^2\alpha_0 & -4in(1-n^2) + \varepsilon c_0 & 6n^2 - 2 & -4in \end{pmatrix}$$

and

$$L_n^{\mathrm{con}} = \begin{pmatrix} 0 & 1 \\ n^2 & 2in - \varepsilon c_0 \end{pmatrix}.$$

2.A.1. Eigenvalues of L_n^{SH}

We first calculate the approximation of the eigenvalues of L_n^{SH}, see (2.12)–(2.13). According to [EW91, Appendix C] the characteristic polynomial is given by

$$\begin{aligned} p_n^{\mathrm{SH}}(\lambda, \varepsilon) &= -\lambda^4 - 4in\lambda^3 + \lambda^2(6n^2 - 2) + \lambda(-4in(1-n^2) + \varepsilon c_0) - (1+n^2)^2 + \varepsilon^2\alpha_0 \\ &= -(\lambda + i(n+1))^2(\lambda + i(n-1))^2 + \varepsilon c_0\lambda + \varepsilon^2\alpha_0. \end{aligned}$$

Hence for $\varepsilon = 0$, L_n^{SH} has two double eigenvalues $\lambda_{n,\pm} = -i(n \pm 1)$.

We start by considering the case $\lambda_{1,-} = 0$ and define

$$\tilde{p}_{1,-}^{\mathrm{SH}}(\delta, \varepsilon) := \varepsilon^{-2} p_1^{\mathrm{SH}}(\lambda_{1,-} + \varepsilon\delta, \varepsilon) = -\delta^2(\varepsilon\delta + 2i)^2 + c_0\delta + \alpha_0.$$

Setting $\varepsilon = 0$ and solving for δ yields

$$\delta_\pm = \frac{-c_0 \pm \Delta}{8}, \quad \Delta = \sqrt{c_0^2 - 16\alpha_0}.$$

Since $\tilde{p}^{\mathrm{SH}}_{1,-}$ is continuously differentiable,

$$\tilde{p}^{\mathrm{SH}}_{1,-}(\delta_\pm, 0) = 0 \text{ and } \partial_\delta \tilde{p}^{\mathrm{SH}}_{1,-}(\delta_\pm, 0) \neq 0$$

we can apply the implicit function theorem, which yields that for $\varepsilon > 0$ small there exists a root of p^{SH}_1 of the form

$$\lambda^\pm_{1,-} = \varepsilon \frac{-c_0 \pm \Delta}{8} + \mathcal{O}(\varepsilon^2).$$

The same calculation in the case $\lambda_{-1,+} = 0$ yields

$$\lambda^\pm_{-1,+} = \varepsilon \frac{-c_0 \pm \Delta}{8} + \mathcal{O}(\varepsilon^2).$$

Next, we consider the case that $\lambda_{n,\pm} \neq 0$. Similar to the case above, we define

$$\tilde{p}^{\mathrm{SH}}_{n,\pm}(\delta, \tilde{\varepsilon}) = \tilde{\varepsilon}^{-2} p^{\mathrm{SH}}_{n,\pm}(-i(n \pm 1) + \tilde{\varepsilon}\delta, \tilde{\varepsilon}^2) = -\delta^2(\mp 2i + \tilde{\varepsilon}\delta)^2 + c_0(-i(n \pm 1) + \tilde{\varepsilon}\delta) + \tilde{\varepsilon}^2\alpha_0,$$

where we set $\varepsilon = \tilde{\varepsilon}^2$ so that $\tilde{p}_{n,\pm}$ is continuously differentiable in both arguments. Setting $\tilde{\varepsilon} = 0$ and solving for δ then gives

$$\delta^+_\pm = i^{3/2} \frac{\sqrt{(n \pm 1)c_0}}{2} \text{ and } \delta^-_\pm = -\delta^+_\pm.$$

Again the implicit function theorem is applicable and yields that for $\varepsilon > 0$ small the roots of $p^{\mathrm{SH}}_{n,\pm}$ are of the form

$$\lambda^+_{n,\pm} = -i(n \pm 1) + \varepsilon^{1/2} i^{3/2} \frac{\sqrt{(n \pm 1)c_0}}{2} + \mathcal{O}(\varepsilon),$$
$$\lambda^-_{n,\pm} = -i(n \pm 1) - \varepsilon^{1/2} i^{3/2} \frac{\sqrt{(n \pm 1)c_0}}{2} + \mathcal{O}(\varepsilon)$$

Hence, we obtained (2.12)–(2.13).

2.A.2. Eigenvalues of L_n^{con}

Let $n \neq 0$. We now give the calculations leading up to (2.14). In this case, the characteristic polynomial is given by

$$p_n^{\mathrm{con}}(\nu, \varepsilon) = (\nu - in)^2 + \varepsilon c_0 \nu.$$

For $\varepsilon = 0$, we find a double root at $\nu_n = in$. For $\varepsilon > 0$, we define

$$\tilde{p}_n^{\mathrm{con}}(\delta, \tilde{\varepsilon}) = \tilde{\varepsilon}^{-2} p_n^{\mathrm{con}}(in + \tilde{\varepsilon}\delta, \tilde{\varepsilon}^2) = \delta^2 + \tilde{\varepsilon} c_0 \delta + c_0 in,$$

which is continuously differentiable in $(\delta, \tilde{\varepsilon})$. Next, solving $\tilde{p}_n^{\mathrm{con}}(\delta, 0) = 0$ for δ gives two solutions $\delta_\pm = \pm i^{3/2}\sqrt{nc_0}$. The implicit function theorem then yields that for $\varepsilon > 0$ small the eigenvalues of L_n^{con} for $n \neq 0$ are approximated by

$$\nu_{n,\pm} = in \pm \varepsilon^{1/2} i^{3/2} \sqrt{nc_0} + \mathcal{O}(\varepsilon),$$

which is (2.14).

2.B. Spectral stability of periodic solutions

This section is devoted to study the stability of periodic solutions established in Lemma 2.2.1. Since a rigorous proof of stability can be done by following the recent works [Suk16] we restrict the treatment here to adapting the formal calculations in [Suk16, SZJV18] to our setting in order to establish the parameter regimes for which stability can be expected. The approach is similar to the Ginzburg-Landau formalism presented in [Sch96].

Let now $(u_{\mathrm{per}}, v_{\mathrm{per}})$ be the periodic solutions from Lemma 2.2.1. The linearisation about $(u_{\mathrm{per}}, v_{\mathrm{per}})$ is given by

$$\begin{aligned} \partial_t u &= -(1 + \partial_x^2)^2 u + \alpha u + u_{\mathrm{per}} v + v_{\mathrm{per}} u - 3u_{\mathrm{per}}^2 u, \\ \partial_t v &= \partial_x^2 v + 2\gamma \partial_x^2 (u_{\mathrm{per}} u), \end{aligned}$$

which we write as

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = L_{\mathrm{per}} \begin{pmatrix} u \\ v \end{pmatrix}.$$

We note that $L_{\text{per}} = L_{\text{per}}(x)$ is a periodic operator in x and thus, Floquet-Bloch theory can be used to determine its spectrum in $L^2(\mathbb{R};\mathbb{R}^2)$, see [DLP+11] for more details. Then, following e.g. [JZ11a], for any $\ell \in [-k_c/2, k_c/2]$ we define

$$L_\ell[w] := e^{-i\ell x} L_{\text{per}}(x)[w] e^{i\ell x}$$

as operators on $L^2_{\text{per}}(0, 2\pi/k_c)$, which denotes the space of $2\pi/k_c$-periodic L^2-functions. Then, the L^2 spectrum of L is given by the union of the L^2_{per} spectra of the L_ℓ, that is,

$$\sigma_{L^2}(L) = \bigcup_{\ell} \sigma_{L^2_{\text{per}}}(L_\ell).$$

Note that in the subsequent analysis we drop the explicit reference to the space and just write $\sigma(\cdot)$ for the spectrum when the space is clear from the context. It is the aim of this section to study for which parameters (α_0, γ) we can formally verify the *diffusive spectral stability* assumptions (see e.g. [JNRZ13]).

(A1) It holds that $\sigma(L) \subset \{\lambda \in \mathbb{C} | \operatorname{Re}(\lambda) < 0\} \cup \{0\}$.

(A2) There exists a $\sigma > 0$ such that for all $\ell \in [-k_c/2, k_c/2]$ we have $\sigma(L_\ell) \subset \{\lambda \in \mathbb{C} | \operatorname{Re}(\lambda) < -\sigma|\ell|^2\}$.

(A3) $\lambda = 0$ is a semi-simple eigenvalue of L_0 whose eigenspace has dimension 2.

Before we start with the formal calculations, we motivate why $\lambda = 0$ is a semi-simple eigenvalue with multiplicity 2 in our case. Therefore, recall that (2.3a)–(2.3b) is translational invariant, i.e., if $(u, v)(t, x)$ is a solution then so is $(u, v)(t, x + x_0)$ for any $x_0 \in \mathbb{R}$. In this case, it is well-known that $\lambda = 0$ is in the point spectrum of L since $\partial_x(u_{\text{per}}, v_{\text{per}})^T$ solves the corresponding eigenvalue equation. The second eigenfunction is given by $\partial_{v_0}(u_{\text{per}}, v_{\text{per}})^T|_{v_0=0}$. Intuitively, this is because the system (2.3a)–(2.3b) conserves v_0 and thus, the difference between two solutions with slightly different masses v_0 cannot decay to zero. This can be made rigorous by first slightly extending the argument presented in Lemma 2.2.1 to include a general non-zero mass v_0 of v_{per} and then differentiating the equation with respect to v_0 to obtain

$$0 = L_0 \partial_{v_0} \begin{pmatrix} u_{\text{per}} \\ v_{\text{per}} \end{pmatrix}.$$

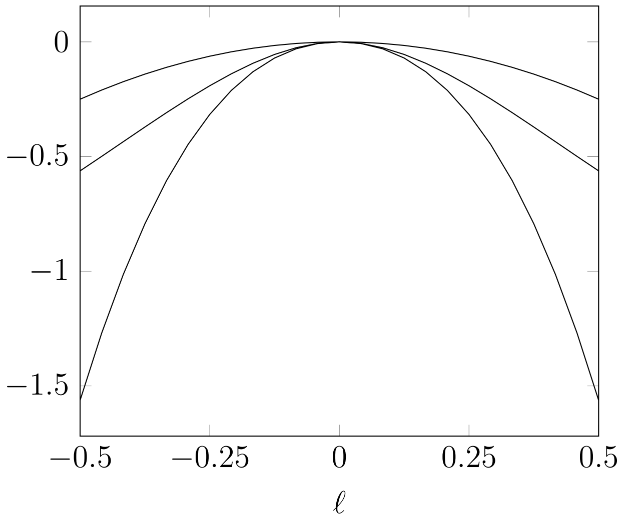

Figure 2.B.1.: Spectrum of L_ℓ^0 for $k \in \{0, \pm 1\}$.

Here, we especially used that the phase velocity of all periodic solutions vanishes independently of the mass v_0.

Remark 2.B.1. We note that the fact that the phase velocity of the periodic solutions does not depend on v_0 is crucial. In fact, it turns out that $\lambda = 0$ is a semi-simple eigenvalue if and only if this is true. Otherwise, $\lambda = 0$ corresponds to a non-trivial Jordan block which further complicates the analysis. For a detailed discussion of this in the case of systems of conservation laws, we refer to [JNRZ14]. $\square$

Since (A3) is therefore satisfied, we now deal with the remaining assumptions (A1)–(A2). The discussion is split into two parts. The first part deals with the spectrum for ℓ away from zero and the second one deals with the situation for ℓ close to zero, which is the critical case as the spectrum touches the imaginary axis for $\ell = 0$.

2.B.1. Spectrum for ℓ away from zero

First we note that $u_{\text{per}}, v_{\text{per}}$ and $\alpha = \varepsilon^2 \alpha_0$ vanish if ε tends to zero. Therefore, we follow [Suk16] and write

$$L_\ell = \begin{pmatrix} -(1 + (\partial_x + i\ell)^2)^2 & 0 \\ 0 & (\partial_x + i\ell)^2 \end{pmatrix} + \begin{pmatrix} \alpha + v_{\text{per}} - 3u_{\text{per}} & u_{\text{per}} \\ 2\gamma(\partial_x + i\ell)^2(u_{\text{per}}[\cdot]) & 0 \end{pmatrix} := L_\ell^0 + L_\ell^1.$$

Note that for $\varepsilon = 0$ we have $L_\ell = L_\ell^0$ and the spectrum is given by

$$\sigma(L_\ell^0) = \{-(1-(k+\ell)^2)^2 | k \in \mathbb{Z}, \ell \in [-k_c/2, k_c/2]\} \\ \cup \{-(k+\ell)^2 | k \in \mathbb{Z}, \ell \in [-k_c/2, k_c/2]\},$$

see Figure 2.B.1. Using perturbation theory, the spectrum of L_ℓ is close to the one of L_ℓ^0 for ε small. Therefore, outside a neighborhood of $\ell = 0$, (A2) is satisfied and it remains to study the spectrum of L_ℓ for ℓ close to zero.

2.B.2. Small frequency spectrum via amplitude equations

To control the spectrum for ℓ close to zero, we derive an asymptotic expansion of the eigenvalues of L_ℓ. For the formal calculation, we follow [Suk16, SZJV18] and use the fact that the modified Ginzburg-Landau system

$$\partial_T A = 4\partial_X^2 A + \alpha_0 A + AB - (3+\gamma)A|A|^2 \tag{2.27a}$$

$$\partial_T B = \partial_X^2 B + 2\gamma\partial_X^2 |A|^2 \tag{2.27b}$$

is a formal amplitude equation corresponding to (2.3a)–(2.3b) (see also [HSZ11, SZ13, Zim14]). This equation can be derived from (2.3a)–(2.3b) using the multiple scaling ansatz

$$u(x,t) = \varepsilon A(X,T)e^{ix} + c.c. + h.o.t \\ v(x,t) = \varepsilon^2 B_0(X,T) + \varepsilon^2 B_1(X,T)e^{2ix} + c.c. + h.o.t.,$$

see Section 2.2. We note that for $\gamma > -3$ and $q_0 \in (-\sqrt{\alpha_0}, \sqrt{\alpha_0})$, (2.27) has solutions of the form

$$A_{q_0,\gamma}(X) = \sqrt{\frac{\alpha_0 - q_0^2}{3+\gamma}} e^{i(q_0/2)X}, \\ B_{q_0,\gamma}(X) = 0,$$

which correspond to the periodic wave solutions found in Lemma 2.2.1 for $v_0 = 0$, i.e.,

$$u_{\text{per}}(x) = \varepsilon A_{q_0,\gamma}(X)e^{ix} + c.c. + h.o.t,$$

$$v_{\text{per}}(x) = \varepsilon^2 C_{q_0,\gamma}(X)e^{2ix} + c.c. + h.o.t, \quad C_{q_0,\gamma} = -A^2_{q_0,\gamma}.$$

To obtain the spectrum of L_ℓ locally around $\ell = 0$, we thus insert the ansatz

$$\begin{aligned} u(x,t) &= (a_r + ia_i)(X,T)e^{i((q_0/2)X+x)} + c.c. + h.o.t. \\ v(x,t) &= \varepsilon b + \varepsilon e^{i(q_0X+2x)}(\Phi_r + i\Phi_i + c.c.) + h.o.t. \end{aligned}$$

into the linearisation of (2.3a)–(2.3b) about the periodic solutions. Here, $a_r, a_i, b, \Phi_r, \Phi_i \in \mathbb{R}$. Equating the the terms of order ε to zero we find

$$\Phi_r = -2\gamma\sqrt{\frac{\alpha_0 - q_0^2}{3+\gamma}} \text{ and } \Phi_i = -2\gamma\sqrt{\frac{\alpha_0 - q_0^2}{3+\gamma}}.$$

Using this, we find for the terms of order ε^3 that

$$\begin{aligned} \partial_T a_r &= 4\partial_X^2 a_r - 4q_0\partial_X a_i - 2(\alpha_0 - q_0^2)a_r + \sqrt{\frac{\alpha_0 - q_0^2}{3+\gamma}}b, \\ \partial_T a_i &= 4\partial_X^2 a_i + 4q_0\partial_X a_r, \\ \partial_T b &= 4\gamma\sqrt{\frac{\alpha_0 - q_0^2}{3+\gamma}}\partial_X^2 a_r + \partial_X^2 b. \end{aligned}$$

The corresponding eigenvalue problem in Fourier space is given by

$$\lambda\begin{pmatrix} a_r \\ a_i \\ b \end{pmatrix} = \mathcal{L}\begin{pmatrix} a_r \\ a_i \\ b \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} -4k^2 - 2(\alpha_0 - q_0^2) & -4iq_0k & \sqrt{\frac{\alpha_0 - q_0^2}{3+\gamma}} \\ 4iq_0k & -4k^2 & 0 \\ -4\gamma\sqrt{\frac{\alpha_0 - q_0^2}{3+\gamma}}k^2 & 0 & -k^2 \end{pmatrix}$$

For $|k| \ll 1$ we thus calculate the eigenvalues of $\mathcal{L}$ as

$$\lambda_{1/2} = \lambda_\pm k^2 + \mathcal{O}(|k|^3), \text{ and } \lambda_3 = -2(\alpha_0 + q_0^2) - \frac{6q_0^2(2+\gamma) + 2\alpha_0(6+\gamma)}{(\alpha_0 - q_0^2)(3+\gamma)}k^2, \tag{2.28}$$

with

$$\lambda_\pm = \frac{3q_0^2(13+5\gamma) - \alpha_0(15+7\gamma)}{2(\alpha_0 - q_0^2)(3+\gamma)}$$

$$\pm\frac{\sqrt{\alpha_0^2(9+\gamma)^2-2q_0^2\alpha_0(297+\gamma(210+410\gamma))+q_0^4(1089+\gamma(786+145\gamma))}}{2(\alpha_0-q_0^2)(3+\gamma)}$$

Utilizing that $q_0^2<\alpha_0$ and $\gamma>-3$ as in Lemma 2.2.1, we find that $\lambda_3<0$. Thus, the stability condition is satisfied if $\lambda_\pm<0$. We now discuss this condition for two cases of particular interest.

Case $q_0=0$. For vanishing q_0, the expression for $\lambda_\pm$ simplifies to

$$\lambda_+=\frac{-3(1+\gamma)}{3+\gamma}\text{ and }\lambda_-=-\frac{12+4\gamma}{3+\gamma}.$$

Using $\lambda_-<\lambda_+$, we find that in this case $\lambda_\pm<0$ if and only if $\gamma>-1$. In particular, the stability condition is independent of $\alpha_0>0$. We highlight that if the system (2.27) is rescaled to the form of (4.1) we find that the coefficients are given by

$$\mu=\frac{1}{4}\text{ and }\tilde{\gamma}=\frac{1}{2}\frac{\gamma}{3+\gamma},$$

where μ is the diffusion coefficient and $\tilde{\gamma}$ is the coupling parameter. Therefore, $\gamma>-1$ is equivalent to $\tilde{\gamma}>-\mu$, which is the stability condition for the stability of the invading state $(A,B)=(1,0)$, see Theorem 4.1.1.

Case $\gamma=0$. In the case of a decoupled conservation law, that is $\gamma=0$, we find that

$$\lambda_\pm=\frac{39q_0^2-15\alpha_0\pm3(3\alpha_0-11q_0^2)}{6(\alpha_0-q_0^2)}$$

and thus,

$$\lambda_-=-1\text{ and }\lambda_+=\frac{12q_0^2-4\alpha_0}{\alpha_0-q_0^2}.$$

Hence $\lambda_+<0$ if and only if $q_0^2<\alpha_0/3$, which recovers the well-known Eckhaus stability boundary for stability of stationary, periodic solutions of the real Ginzburg-Landau equation, see [Eck65].

3. Modulating traveling fronts in the case of additional dispersion

3.1. Introduction

In the previous Chapter 2 we dealt with the invasion of a homogeneous ground state by a stationary periodic state. The purpose of the following chapter is to extend this result to the case of invasion by traveling periodic solutions with non-zero phase velocity. To this end, we extend our toy model by adding dispersive terms to both the Swift-Hohenberg equation and the conservation law, which break the reflexion symmetry $x \mapsto -x$ of the equations. That is, we consider the model

$$\partial_t u = -(1+\partial_x^2)^2 u + \varepsilon^2 \alpha_0 u + c_u \partial_x^3 u + uv + u\partial_x u - u^3, \tag{3.1a}$$
$$\partial_t v = \partial_x^2 v + c_v \partial_x v + \gamma_1 \partial_x^2 (u^2) + \gamma_2 \partial_x (u^2), \tag{3.1b}$$

with $u(t,x), v(t,x), c_u, c_v, \gamma_1, \gamma_2 \in \mathbb{R}$. This system corresponds to a thin-film flow on an inclined, heated surface. The goal is to construct modulating traveling fronts of the form

$$(u,v)(t,x) = (U,V)(\xi,p)$$

with $\xi = x - ct$ and $p = x - \omega t$. Furthermore, U and V are assumed to be periodic with respect to their second argument and satisfy the asymptotic conditions

$$\lim_{\xi\to-\infty} (U,V)(\xi,p) = (u_{\text{per}}, v_{\text{per}})(p) \text{ and } \lim_{\xi\to+\infty} (U,V)(\xi,p) = (0,0).$$

Here $(u_{\text{per}}, v_{\text{per}})$ denotes a traveling periodic solution of (3.1a)–(3.1b), which we establish in Section 3.2. While in Chapter 2 the phase velocity ω vanishes due to the reflexion symmetry, we expect that $\omega = c_u + \mathcal{O}(\varepsilon)$ in (3.1a)–(3.1b). We highlight that this setting

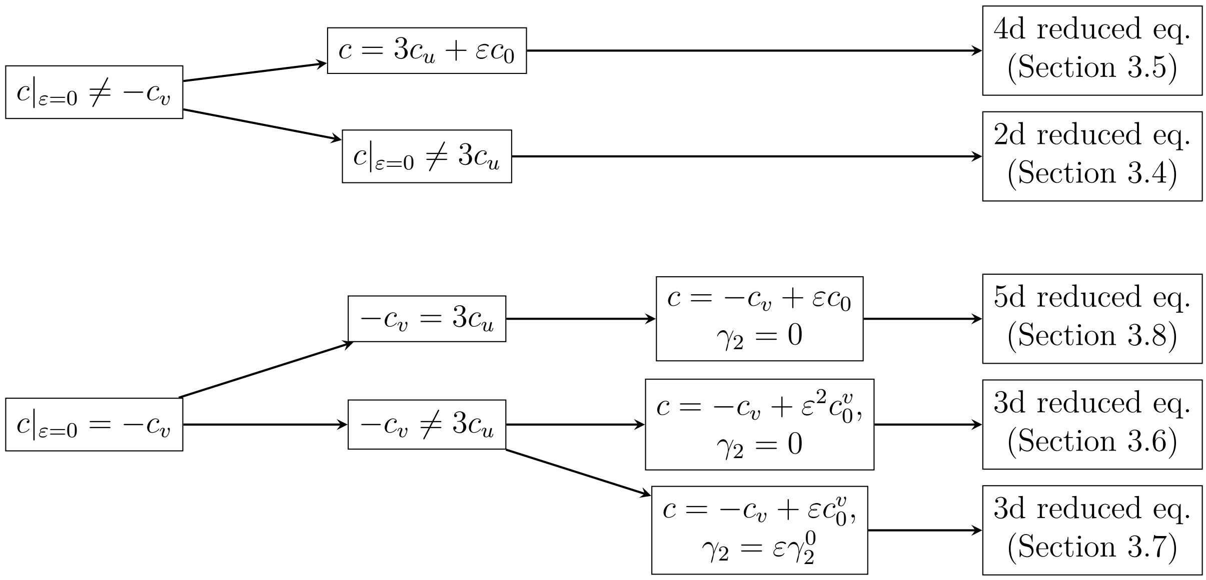

Figure 3.1.: An overview of the different scenarios in this chapter. Starting from the cases that either $c|_{\varepsilon=0}$ equals $-c_v$ or not, we depict how different parameter choices effect the dimension of the reduced equation on the center manifold. Here, we count dimensions after splitting complex variables into their real and imaginary parts. In particular, this means that in all cases with even dimension the conserved variable is slaved on the center manifold and does not contribute an additional equation.

resembles the Taylor-Couette problem in the case of counter-rotating cylinders, see [IM91, HCS99].

It turns out that the challenges compared to the stationary case in Chapter 2 are quite different. First, the spectral analysis of the linear part in the spatial dynamics formulation shows that there is an $\mathcal{O}(1)$ spectral gap, i.e. its size is independent of ε, between the central and the hyperbolic part of the spectrum. Therefore, the construction of a center manifold can be done by standard results. However, the presence of dispersive terms in the equations yields a reduced equation with complex coefficients instead of real coefficients as in the stationary case. Hence, the main difficulty in this chapter is the construction of heteroclinic connections in the reduced equations on the center manifold as well as their persistence.

Depending on the parameters in (3.1a)–(3.1b) and the invading speed c, we encounter different situations, which are summarized in Figure 3.1. Namely, we discuss the case that c is away from the linear group velocity, which is given by $3c_u$. In this case the reduced equation is given by a scalar equation with complex coefficients, which can be dealt with by passing to polar coordinates. Then, we consider the case that c is close to the linear group velocity, that is $c = 3c_u + \varepsilon c_0$. For this choice, the reduced equation corresponds to a complex Ginzburg-Landau equation and we establish the existence of heterolinic orbits in this model using numerical methods.

We note that in the first two cases the conservation law does not contribute to the reduced dynamics. In fact, the conserved variable is slaved by the Swift-Hohenberg part on the center manifold. Therefore, we additionally consider the case that c is close to $-c_v$ for which we obtain a non-trivial central eigenvalue from the conservation law. Furthermore, we assume that $\gamma_2 = 0$ since $\partial_x(u^2)$ in (3.1b) leads to terms in the reduced equations which blow up at least like ε^{-1} for $\varepsilon \to 0$ in the natural scaling. We then have to distinguish two subcases. The first one is that $c = -c_v + \varepsilon^2 c_0^v$ and $-c_v \neq 3c_u$. In this case we obtain a two-dimensional reduced equation, where one equation originates from the Swift-Hohenberg equation and the other one from the conservation law. The second subcase is $c = -c_v + \varepsilon c_0$ and $-c_v = 3c_u$ for which we obtain an additional pair of complex conjugated central eigenvalues from the Swift-Hohenberg equation. This leads to a three-dimensional reduced equation on the center manifold. In both cases we proceed similarly, that is, we first remove the nonlinear coupling in (3.1b) by setting $\gamma_1 = 0$ for which we can analyze the problem without considering the conservation law. Then, we use the existence of heteroclinic connections in the first two cases – in which the conservation quantity was slaved – and show their persistence if the coupling in (3.1b) is sufficiently weak, i.e. γ_1 close to zero. Again in the case $-c_v \neq 3c_u$ we obtain an analytical existence result, while in the case $-c_v = 3c_u$ we rely on numerical methods to obtain the existence of heteroclinic orbits.

Finally, we consider a slight modification of the setting that c is close to $-c_v \neq 3c_u$, that is, we set make the ansatz $c = -c_v + \varepsilon c_0^v$ instead of $c = -c_v + \varepsilon^2 c_0^v$. This modification allows to include a non-vanishing, but small coefficient $\gamma_2 = \varepsilon \gamma_2^0$ in the analysis. For this choice the central eigenvalues are of different order with respect to ε and we obtain a fast-slow system on the center manifold, i.e. a system of the form

$$\begin{aligned} \dot{f} &= F(f, g, \varepsilon), \\ \varepsilon \dot{g} &= G(f, g, \varepsilon), \end{aligned}$$

where $\dot{(\cdot)}$ denotes a “time” derivative and F, G are suitable smooth functions. We then obtain the existence and persistence of a heteroclinic orbit for the reduced system by using geometric singular perturbation theory (see e.g. [Kue15]).

3.1.1. Outline

This chapter is organized as follows. We begin by establishing the existence of traveling periodic solutions of (3.1a)–(3.1b) in Section 3.2. In Section 3.3 we then derive the spatial dynamics formulation and discuss the spectral properties of the resulting linear operator. This section also includes the center manifold result. In the subsequent Sections 3.4–3.8 we then discuss the existence of modulating traveling fronts in different parameter regimes as outlined above, see also Figure 3.1.

3.2. Existence of periodic traveling wave solutions

We start our analysis by establishing the existence of patterns. In contrast to Chapter 2, we expect that the periodic solutions travel with a non-zero phase velocity due to the symmetry breaking terms in (3.1a)–(3.1b). However, the idea of applying a center manifold reduction still works in this case. The main difference compared to Lemma 2.2.1 is the construction of non-trivial, real fixed points on the center manifold, which requires a specific choice of the phase velocity.

Lemma 3.2.1. *Let $B \in \mathbb{R}$ and $\alpha_0 \in \mathbb{R}$ such that $B + \alpha_0 > 0$. Then, there exist $\varepsilon_0 > 0$, $\gamma^* > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $\gamma_1, \gamma_2 \in (-\gamma^*, \gamma^*)$ there exist $A^* \in \mathbb{R}$, $\omega_0^* \in \mathbb{R}$ given by*

$$
\begin{aligned}
A^* &= \pm\sqrt{\frac{(9+4c_u^2)(B+\alpha_0)}{4(7+3c_u^2)}} + \mathcal{O}(\varepsilon^2 + |\gamma_1| + |\gamma_2|),\\
\omega_0^* &= -\frac{c_u(B+\alpha_0)}{6(7+3c_u^2)} + \mathcal{O}(\varepsilon^2 + |\gamma_1| + |\gamma_2|),
\end{aligned}
$$

such that (3.1a)–(3.1b) *has a periodic traveling wave solution of the form*

$$
\begin{aligned}
u_{per}(p) &= \varepsilon A^* e^{ip} + c.c. + \mathcal{O}(\varepsilon^2 + |\gamma_1| + |\gamma_2|),\\
v_{per}(p) &= \varepsilon^2 B + \varepsilon^2 \frac{-4\gamma_1 + 2i\gamma_2}{4 - 2i(c_u + c_u)}(A^*)^2 e^{2ip} + c.c. + \mathcal{O}(\varepsilon^3 + |\gamma_1| + |\gamma_2|),
\end{aligned}
$$

with $p = x - \omega t$ and phase velocity $\omega = c_u + \varepsilon^2\omega_0^*$.

Proof. We proceed similar to the proof of Lemma 2.2.1. In a co-moving frame with velocity $\omega = c_u + \varepsilon^2\omega_0$, the system (3.1a)–(3.1b) reads as

$$\begin{aligned}\partial_t u &= -(1+\partial_p^2)^2 u + \varepsilon^2\alpha_0 u + c_u\partial_p^3 u + \omega\partial_p u + uv + \frac{1}{2}\partial_p(u^2) - u^3,\\ \partial_t v &= \partial_p^2 v + (c_v+\omega)\partial_p v + \gamma_1\partial_p^2(u^2) + \gamma_2\partial_p(u^2),\end{aligned}$$

with $p = x - \omega t$. We now apply center manifold theory using the spaces

$$\begin{aligned}\mathcal{Z}_B &= \left\{(u,v)\in H^4_{\text{per}}\times H^2_{\text{per}} \,:\, u_n = \bar{u}_{-n}, v_n = \bar{v}_{-n}, v_0 = \varepsilon^2 B\right\},\\ \mathcal{Y}_B &= \left\{(u,v)\in H^2_{\text{per}}\times H^2_{\text{per}} \,:\, u_n = \bar{u}_{-n}, v_n = \bar{v}_{-n}, v_0 = \varepsilon^2 B\right\},\\ \mathcal{X}_B &= \left\{(u,v)\in H^0_{\text{per}}\times H^0_{\text{per}} \,:\, u_n = \bar{u}_{-n}, v_n = \bar{v}_{-n}, v_0 = \varepsilon^2 B\right\},\end{aligned}$$

with $B \in \mathbb{R}$ and H^n_{per} the space of 2π-periodic H^n functions for $n \in \mathbb{N}$. We highlight that these spaces are invariant under the dynamics of (3.1a)–(3.1b). The linear part L is given by

$$L = \begin{pmatrix} -(1+\partial_p^2)^2 + \varepsilon^2\alpha_0 + c_u\partial_p^3 + \omega\partial_p & 0 \\ 0 & \partial_p^2 + (c_v+\omega)\partial_p \end{pmatrix}$$

and its spectrum can be calculated explicitly using Fourier transform as

$$\begin{aligned}\lambda_{u,n} &= -(1-n^2)^2 + \varepsilon^2\alpha_0 + i(n-n^3)c_u + in\varepsilon^2\omega_0, \quad n\in\mathbb{Z},\\ \lambda_{v,n} &= -n^2 + in(c_u + c_v + \varepsilon^2\omega_0), \quad n \in \mathbb{Z}\setminus\{0\}.\end{aligned}$$

Note that since we consider spaces with fixed mode $v_0 = \varepsilon^2 B$, we find that $\lambda_{v,0} = 0$ is not part of the spectrum. Thus, the spectrum can be decomposed in a central part $\sigma_c(L) = \{\lambda_{u,\pm 1}\}$ and a remaining hyperbolic part $\sigma_h(L)$. Since we also have a resolvent estimate by explicit calculation in Fourier space and $\mathcal{Z}_B$, $\mathcal{Y}_B$ and $\mathcal{X}_B$ are Hilbert spaces the center manifold results available in [HI11] are applicable. Therefore, we introduce the following coordinates on the center manifold

$$\begin{aligned}u(t) &= \varepsilon A(\varepsilon^2 t)e^{ip} + c.c. + h_u(\varepsilon A),\\ v(t) &= \varepsilon^2 B + h_v(\varepsilon A).\end{aligned}$$

Proceeding similar to the proof of Lemma 2.2.1 we find

$$h_u = \varepsilon^2 \frac{i}{9+6ic_u} A^2 e^{2ip} + c.c. + \mathcal{O}(\varepsilon^3),$$
$$h_v = \varepsilon^2 \frac{-2\gamma_1 + i\gamma_2}{2 - i(c_u + c_v)} + \mathcal{O}(\varepsilon^3).$$

Therefore, we arrive at a reduced equation on the center manifold, which is given by

$$\partial_T A = (\alpha_0 + i\omega_0)A + AB + \left(-3 - \frac{1}{9+6ic_u} + \frac{-2\gamma_1 + i\gamma_2}{2 - i(c_u + c_v)}\right) A|A|^2 + g(A,\varepsilon),$$

with $g(A,\varepsilon) = \mathcal{O}(\varepsilon^2)$. For $(\varepsilon, \gamma_1, \gamma_2) = 0$, this equation has a family of stationary solutions if and only if

$$|A|^2 = \left(3 + \frac{1}{9+6ic_u}\right)^{-1} (\alpha_0 + B + i\omega_0)$$

is real and positive. Therefore, we calculate

$$\mathrm{Im}\left[\left(3 + \frac{1}{9+6ic_u}\right)^{-1} (\alpha_0 + B + i\omega_0)\right] = \frac{3c_u(\alpha_0 + B) + 3(42 + 18c_u^2)\omega_0}{392 + 161c_u^2}.$$

This expression vanishes if and only if

$$\omega_0 = \omega_0^* := -\frac{c_u(B + \alpha_0)}{6(7 + 3c_u^2)}.$$

Then, the expression for $|A|^2$ simplifies to

$$|A^*|^2 := \frac{(9 + 4c_u^2)(B + \alpha_0)}{4(7 + 3c_u^2)},$$

which is positive since $B + \alpha_0 > 0$ by assumption. Therefore, we have a non-trivial, stationary solution of the reduced equation for $(\varepsilon, \gamma_1, \gamma_2) = (0,0,0)$.

It remains to show that this solution persists for $(\varepsilon, \gamma_1, \gamma_2)$ close to zero. Recalling that the system (3.1a)–(3.1b) is translationally invariant, the reduced dynamic is invariant with respect to $A \mapsto Ae^{i\phi}$. Therefore, we may assume that the solution is real and positive and satisfies

$$\begin{cases} 0 = (\alpha_0 + B)A + a^{\mathrm{cub}}(\gamma_1, \gamma_2)A^3 + \mathrm{Re}(g(A, \varepsilon)) =: G_1(A, \omega_0, \varepsilon, \gamma_1, \gamma_2), \\ 0 = \omega_0 A + b^{\mathrm{cub}}(\gamma_1, \gamma_2)A^3 + \mathrm{Im}(g(A, \varepsilon)) =: G_2(A, \omega_0, \varepsilon, \gamma_1, \gamma_2), \end{cases}$$

where the coefficients are given by

$$\begin{aligned} a^{\mathrm{cub}}(\gamma_1, \gamma_2) &= \mathrm{Re}\left(-3 - \frac{1}{9 + 6ic_u} + \frac{-2\gamma_1 + i\gamma_2}{2 - i(c_u + c_v)}\right), \\ b^{\mathrm{cub}}(\gamma_1, \gamma_2) &= \mathrm{Im}\left(-3 - \frac{1}{9 + 6ic_u} + \frac{-2\gamma_1 + i\gamma_2}{2 - i(c_u + c_v)}\right). \end{aligned}$$

Additionally, we define $G := (G_1, G_2)^T$. Then, by construction it holds that $G(A^*, \omega_0^*, 0, 0, 0) = 0$ and we have

$$D_{(A,\omega_0)}G|_{(A,\omega_0,\varepsilon,\gamma_1,\gamma_2)=(A^*,\omega_0^*,0,0,0)} = \begin{pmatrix} \alpha_0 + B + 3a^{\mathrm{cub}}(0,0)(A^*)^2 & 0 \\ \omega_0^* + 3b^{\mathrm{cub}}(0,0)(A^*)^2 & A^* \end{pmatrix}.$$

Since $A^* > 0$, this matrix is invertible if and only if

$$\alpha_0 + B + 3a^{\mathrm{cub}}(0,0)(A^*)^2 = -2(B + \alpha_0) \neq 0,$$

which is true since $B + \alpha_0 > 0$ by assumption. Therefore, the solutions persist by the implicit function theorem and we obtain the statement of the lemma. □

3.3. Spatial dynamics formulation, spectral analysis and center manifold theorem

Similar to Chapter 2, we proceed in the construction of modulating front solutions by using spatial dynamics techniques. We therefore make the following ansatz

$$(u, v)(t, x) = (U, V)(x - ct, x - \omega t) = (U, V)(\xi, p),$$

which we assume to be 2π-periodic with respect to its second argument. Here, ω is the phase velocity from Lemma 3.2.1. Inserting this into (3.1a)–(3.1b), we find

$$\begin{aligned}
-c\partial_\xi U - \omega\partial_p U &= -(1+(\partial_\xi+\partial_p)^2)^2 U + \varepsilon^2\alpha_0 U + c_u(\partial_\xi+\partial_p)^3 U \\
&\qquad\qquad\qquad\qquad + UV + \frac{1}{2}(\partial_\xi+\partial_p)(U^2) - U^3, \\
-c\partial_\xi V - \omega\partial_p V &= (\partial_\xi+\partial_p)^2 V + c_v(\partial_\xi+\partial_p)V + \gamma_1(\partial_\xi+\partial_p)^2(U^2) + \gamma_2(\partial_\xi+\partial_p)(U^2).
\end{aligned}$$

Using the periodicity with respect to p we can make a Fourier series ansatz and by writing the resulting equations as a first order system we obtain the corresponding spatial dynamics formulation

$$\partial_\xi \begin{pmatrix} U_n \\ V_n \end{pmatrix} = \begin{pmatrix} L_n^{\mathrm{SH}} & 0 \\ 0 & L_n^{\mathrm{con}} \end{pmatrix} \begin{pmatrix} U_n \\ V_n \end{pmatrix} + \mathcal{N}_n(U,V),$$

where $U_n = (U_{nj})_{j=0,\dots,3}$ and $V_n = (V_{nj})_{j=0,1}$ with $U_{nj} = \partial_\xi^j U_n$ and $V_{nj} = \partial_\xi^j V_n$. Here, the linear part is given by

$$L_n^{\mathrm{SH}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ A_n & B_n & C_n & D_n \end{pmatrix}, \quad L_n^{\mathrm{con}} = \begin{pmatrix} 0 & 1 \\ E_n & F_n \end{pmatrix}$$

with

$$\begin{aligned}
A_n &= \varepsilon^2\alpha_0 - ic_u n^3 + in\omega - (1-n^2)^2, \\
B_n &= -3n^2 c_u + c + 4i(n^3 - n), \\
C_n &= 3inc_u + 6n^2 - 2, \\
D_n &= c_u - 4in, \\
E_n &= -inc_v + n^2 - in\omega \\
F_n &= -(c + c_v + 2in).
\end{aligned}$$

Additionally, the nonlinearity $\mathcal{N}_n(U,V)$ is of the form $(0,0,0,\mathcal{N}_n^{\mathrm{SH}}(U,V),0,\mathcal{N}_n^{\mathrm{con}}(U,V))^T$ with

$$
\begin{aligned}
\mathcal{N}_n^{\mathrm{SH}}(U,V) &= \sum_{k+j=n} U_{k0}V_{j0} + \sum_{k+j=n} U_{k0}\left(U_{j1} + ijU_{j0}\right) - \sum_{k+j+l=n} U_{k0}U_{j0}U_{l0} \\
\mathcal{N}_n^{\mathrm{con}}(U,V) &= -2\gamma_1 \sum_{k+j=n} \left(U_{k0}U_{j2} + U_{k1}U_{j1}\right) - 2\gamma_1 in \sum_{k+j=n} U_{k0}U_{j1} + n^2 2\gamma_1 \sum_{k+j=n} U_{k0}U_{j0} \\
&\quad - 2\gamma_2 \sum_{k+j=n} U_{k0}U_{j1} - \gamma_2 in \sum_{k+j=n} U_{k0}U_{j0}
\end{aligned}
$$

Similar to Chapter 2 we can then write (3.1a)–(3.1b) equivalently as an infinite dimensional dynamical system on $\mathcal{E}_l = H^l(\mathbb{C}^4) \times H^l(\mathbb{C}^2)$, with H^l defined in (2.15). The system then reads as

$$
\partial_\xi \begin{pmatrix} U \\ V \end{pmatrix} = L \begin{pmatrix} U \\ V \end{pmatrix} + \mathcal{N}(U,V) \tag{3.2}
$$

with $(U,V) \in \mathcal{E}_l$ and the linear operator $L : \mathcal{E}_{l+4,l+2} \to \mathcal{E}_l$ and the nonlinearity $\mathcal{N} : \mathcal{E}_{l+4,l+2} \to \mathcal{E}_{l+2}$ are given by

$$
L \begin{pmatrix} U \\ V \end{pmatrix} = \sum_{n\in\mathbb{Z}} L_n \begin{pmatrix} U_n \\ V_n \end{pmatrix} e^{inp} \text{ and } \mathcal{N}(U,V) = \sum_{n\in\mathbb{Z}} \mathcal{N}_n(U,V)e^{inp}.
$$

Here, recall that $\mathcal{E}_{l_1,l_2} := H^{l_1}(\mathbb{C}^4) \times H^{l_2}(\mathbb{C}^2)$. To apply center manifold theory, we need to analyze the spectrum of L, which is given by the union of the spectra of L_n^{SH} and L_n^{con} using the diagonal structure of L_n.

Lemma 3.3.1. *Let $c_u \neq -c_v$ and $c|_{\varepsilon=0} \neq c_u$. Then, there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ the following holds.*

- *If $c|_{\varepsilon=0} \neq -c_v$, the matrix L_n^{con} has one simple, central eigenvalue $\lambda = 0$ for $n = 0$.*
- *If $c = -c_v + \mathcal{O}(\varepsilon)$, the matrix L_n^{con} has one simple eigenvalue $\lambda_0 = 0$ and one simple eigenvalue $\lambda_1 = -c - c_v = \mathcal{O}(\varepsilon)$ for $n = 0$.*

All other eigenvalues are bounded away from the imaginary axis uniformly in ε and $n \in \mathbb{Z}$.

Proof. We begin by proving the statement for $\varepsilon = 0$ and then show that it persists for

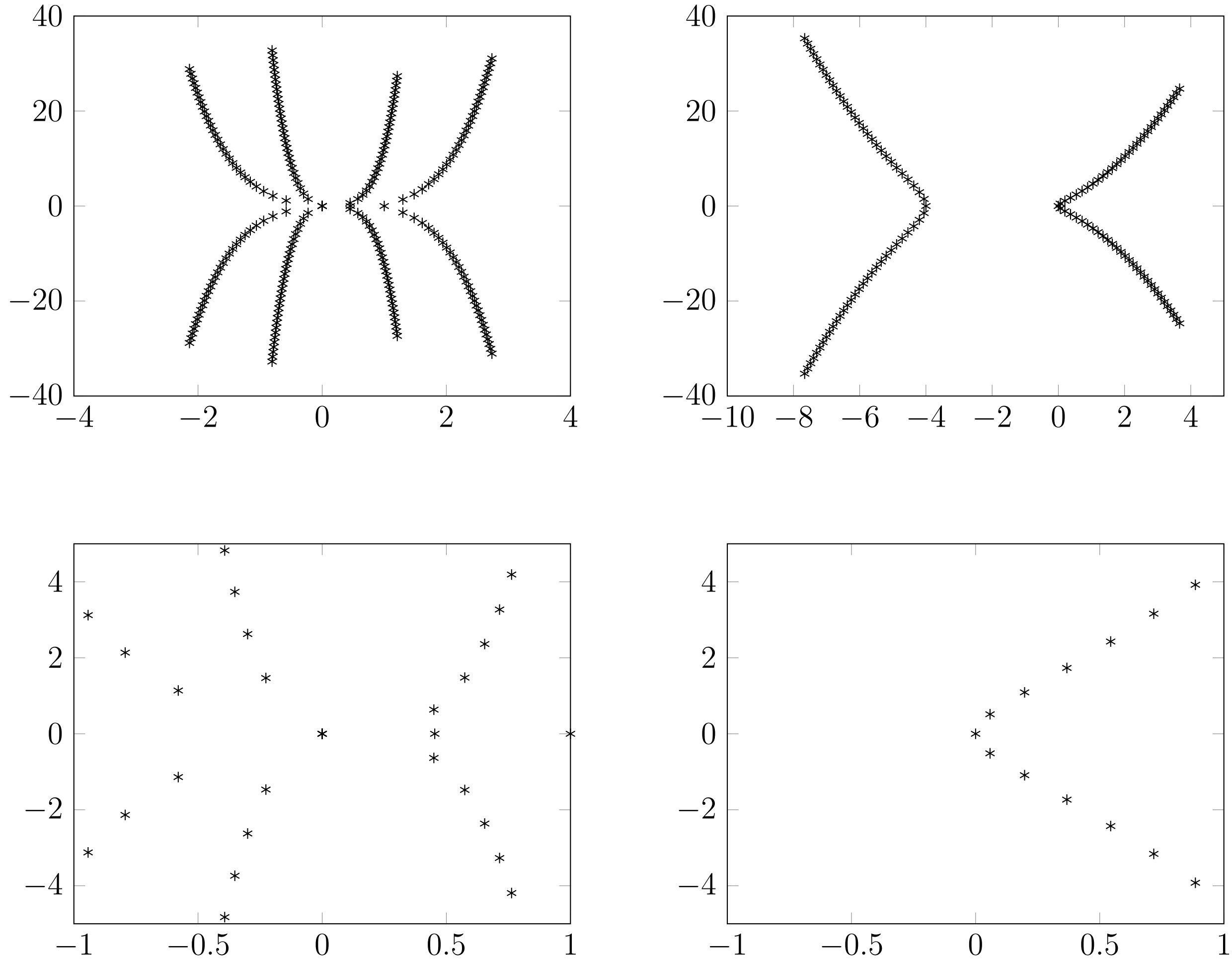

Figure 3.2.: Numerical calculation of the eigenvalues of L_n^{SH} and L_n^{con}, respectively, for $\alpha_0 = 1$, $c_u, c_v = 1$, $c = 3$ and $\varepsilon = 0$. Top left and right show the eigenvalues of L_n^{SH} and L_n^{con} for $-30 \leq n \leq 30$, respectively. Bottom left and right are rescaled versions of the top left and top right plots, respectively, centered around the origin.

$\varepsilon > 0$ small by using perturbation theory. Thus, setting $\varepsilon = 0$ we obtain

$$\det(\lambda - L_n^{\mathrm{con}}) = i(c_u + c_v)n - n^2 + (2in + c + c_v)\lambda + \lambda^2.$$

For $n \in \mathbb{Z}$ with $|n|$ large any eigenvalue has the expansion $\lambda = -in \pm (|n|)^{1/2}(-1)^{1/4}\sqrt{c - c_u} + \mathcal{O}(n^\gamma)$ for $|n| \to \infty$ and $\gamma < 1/2$. Therefore, since $c|_{\varepsilon=0} \neq c_u$ it holds $\mathrm{Re}(\lambda) = \mathcal{O}(\sqrt{|n|})$ for $n \to \infty$.

Next, we show that L_n^{con} has no purely imaginary eigenvalues for $n \neq 0$. For that, we set $\lambda \in \mathbb{R}$ and solve $\mathrm{Re}(\det(i\lambda - L_n^{\mathrm{con}})) = 0$ which gives $\lambda = -n$. Inserting this into the imaginary part, we obtain

$$\mathrm{Im}(i\lambda - L_n^{\mathrm{con}})|_{\lambda=-n} = (c_u - c)n.$$

Since by assumption $c_u - c|_{\varepsilon=0} \neq 0$ the matrix L_n^{con} has no purely imaginary eigenvalues for $n \neq 0$. For $n = 0$, we find the eigenvalues explicitly as $\lambda = 0$ and $\lambda = -c - c_v$ which proves the statement for $\varepsilon = 0$.

Finally, for the persistence we use that the eigenvalues of a matrix depend continuously on its entries. Thus, for any $N \in \mathbb{N}$ there exists a $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $|n| < N$ the spectrum of L_n^{con} is bounded away from the imaginary axis, except for one simple eigenvalue at zero. Furthermore, the coefficients of the asymptotic expansion in n depend continuously on ε and thus, for large N the real part of any eigenvalue of L_N^{con} grows like $\sqrt{|N|}$ which yields that we can bound all eigenvalues uniformly away from the imaginary axis for $|N|$ sufficiently large. Therefore, the number of eigenvalues close to the imaginary axis does not change and we know by explicit calculations that L_0^{con} still has a simple eigenvalue $\lambda = 0$ and a simple eigenvalue $\lambda = -c - c_v$ for any small $\varepsilon > 0$ which proves the statement. □

Lemma 3.3.2. *Let $c|_{\varepsilon=0} \neq c_u$. Then, there exists an $\varepsilon_0 > 0$ such that the following results hold for all $\varepsilon \in (0, \varepsilon_0)$.*

- *If $c = 3c_u + \mathcal{O}(\varepsilon)$, the matrix L_n^{SH} for $n = \pm 1$ has two eigenvalue $\lambda_{c,\pm 1}^{1/2}$, which vanish for $\varepsilon \to 0$.*
- *If $c|_{\varepsilon=0} \neq 3c_u$, the matrix L_n^{SH} for $n = \pm 1$ has a simple eigenvalue $\lambda_{c,\pm 1}$, which vanishes for $\varepsilon \to 0$.*

All other eigenvalues are bounded away from the imaginary axis uniformly in ε and $n \in \mathbb{Z}$.

Proof. We proceed similarly to the previous proof. The determinant for $\varepsilon = 0$ is given by

$$\begin{aligned}\det(\lambda - L_n^{\mathrm{SH}}) = 1 - ic_u n - 2n^2 + ic_u n^3 + n^4 + \lambda(-c + 4in + 3c_u n^2 - 4in^3) \\ + \lambda^2(2 - 3ic_u n - 6n^2) + \lambda^3(-c_u + 4in) + \lambda^4.\end{aligned}$$

For $n \in \mathbb{Z}$ with $|n|$ large we find that the four eigenvalues of L_n^{SH} have the expansion

$$\lambda = \begin{cases} -in \pm |n|^{1/4}(-1)^{3/8}(c - c_u)^{1/4} + \mathcal{O}(|n|^\gamma) \\ -in \pm |n|^{1/4}(-1)^{7/8}(c - c_u)^{1/4} + \mathcal{O}(|n|^\gamma) \end{cases},$$

with $\gamma < 1/4$. Hence, the real part of λ grows at least with $|n|^{1/4}$ for $|n| \to \infty$ since $c|_{\varepsilon=0} \neq c_u$ by assumption.

Next, we prove that L_n^{SH} has no purely imaginary eigenvalues if $n \neq \pm 1$. For that we concider

$$\begin{aligned}\mathrm{Re}(i\lambda - L_n^{\mathrm{SH}}) &= (n^2 - 1)^2 + (-4n + 4n^3)\lambda - \lambda^2(2 - 6n^2) + 4n\lambda^3 + \lambda^4 \\ &= (-1 + (n + \lambda)^2)^2 = 0,\end{aligned}$$

whose real solutions are given by $\lambda = -n \pm 1$ which are double roots. For the imaginary part we obtain

$$\mathrm{Im}(i\lambda - L_n^{\mathrm{SH}}) = nc_u(n^2 - 1) + \lambda(-c + 3c_u n^2) + \lambda^2 3c_u n + \lambda^3 c_u.$$

For $n = \pm 1$ this equation has a simple root at $\lambda = 0$ if $3c_u \neq c$ and a double root at $\lambda = 0$ if $3c_u = c$. Inserting $\lambda = -n \pm 1$ we find $\mathrm{Im}(i(-n \pm 1) - L_n^{\mathrm{SH}}) = (c_u - c)(-n \pm 1)$ which does not vanish unless $n \neq \pm 1$ since $c|_{\varepsilon=0} \neq c_u$ by assumption.

Thus, it remains to show the persistence for $\varepsilon > 0$. We proceed similar to Lemma 3.3.1 to conclude that the situation persists in the sense that the number of eigenvalues close to the imaginary axis does not change. However, we note that in general $\lambda = 0$ will not be an eigenvalue. Instead, there are two non-zero eigenvalues close (with respect to ε) to the imaginary axis for L_n^{SH}, $n = \pm 1$. □

With the above results on the spectrum and by utilizing the block-diagonal structure of L, we can apply the center manifold theorem in [SU17]. Following Chapter 2 and denoting the spectral projection onto the central and hyperbolic eigenspaces by $\mathcal{P}_c$ and $\mathcal{P}_h$, respectively, we thus obtain the following result.

Theorem 3.3.3. *Let $l > 1/2$, $c_u \neq -c_v$ and $c|_{\varepsilon=0} \neq c_u$. Then there exists an $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ exists a neighbourhood $O_c \subset \mathcal{E}_c := \mathcal{P}_c\mathcal{E}_l$ of the origin and a mapping $h = (h_u, h_v) : O_c \to \mathcal{E}_h := \mathcal{P}_h\mathcal{E}_l$ such that the center manifold*

$$\mathcal{M}_c := \{(U, V) = (U_c, V_c) + h(U_c, V_c) \, : \, (U_c, V_c) \in O_c)\}$$

is invariant and contains all small bounded solutions of (3.2)*. Furthermore, every solution of the reduced system*

$$\partial_\xi \begin{pmatrix} U_c \\ V_c \end{pmatrix} = L_c \begin{pmatrix} U_c \\ V_c \end{pmatrix} + \mathcal{P}_c\mathcal{N}(U_c + h_u(U_c, W_c), V_c + h_v(U_c, V_c))$$

gives a solution to the full system (3.2) *via $(U, V) = (U_c, V_c) + h(U_c, V_c)$.*

We now proceed by discussing the different cases as outlined in the introduction in Sections 3.4–3.8.

3.4. Modulating traveling fronts in the case $c|_{\varepsilon=0} \neq 3c_u$

We consider the case that the modulating front moves with speed c such that $c|_{\varepsilon=0} \neq 3c_u$, that is the speed of the front is different from the group velocity. Following the results in Lemmas 3.3.1–3.3.2, the central spectrum of the linear operator L from (3.2) then consists of a simple eigenvalue at $\lambda_c^{\mathrm{con}} = 0$ coming from the conservation law and a pair of simple eigenvalues $\lambda_{c,\pm 1}^{\mathrm{SH}}$ coming from the L^{SH}. To calculate an approximation of $\lambda_{c,1}^{\mathrm{SH}}$, the central eigenvalue of L_1^{SH} we use that

$$\det(\lambda - L_1^{\mathrm{SH}}) = -\lambda^4 + \lambda^3(c_u - 4i) + \lambda^2(3ic_u + 4) + \lambda(-3c_u + c) + \varepsilon^2(\alpha_0 + i\omega_0^*),$$

utilizing that the phase velocity in Lemma 3.2.1 is given by $\omega = c_u + \varepsilon^2\omega_0^*$. Setting $\lambda = \varepsilon^2\delta$ in the above characteristic polynomial and letting ε tend to zero, we find that

$$\delta^* = \frac{\alpha_0 + i\omega_0^*}{3c_u - c}.$$

Then, the implicit function theorem gives the persistence for $\varepsilon > 0$ small and the expansion

$$\lambda_{c,1}^{\mathrm{SH}} = \varepsilon^2 \frac{\alpha_0 + i\omega_0^*}{3c_u - c} + \mathcal{O}(\varepsilon^3).$$

3.4.1. Reduced equation on the center manifold

We derive the reduced equation on the center manifold similar to Section 2.4.1. Therefore, let ϕ_c^{SH} be the eigenvector corresponding to $\lambda_{c,1}^{\mathrm{SH}}$, which is normalized such that its first component is equal to one. Similarly, let ϕ_c^{con} be the eigenvector corresponding to $\lambda_c^{\mathrm{con}} = 0$. Then, let

$$\begin{aligned} U_c &= \varepsilon A(\varepsilon^2 \xi) \phi_c^{\mathrm{SH}} e^{ip} + c.c., \\ V_c &= \varepsilon^2 B(\varepsilon^2 \xi) \phi_c^{\mathrm{con}}, \end{aligned}$$

and we additionally introduce the notation $X_c = (U_c, V_c)^T$. Then, we write functions on the center manifold as

$$\begin{aligned} U &= U_c + h_u(X_c), \\ V &= V_c + h_c(X_c), \end{aligned}$$

with h given in Theorem 3.3.3. We proceed by determining the functions h_u and h_v, respectively. To this end, we use that

$$\begin{aligned} Dh_u \partial_\xi X_c &= L_h^{\mathrm{SH}} h_u + \mathcal{P}_h^{\mathrm{SH}}(\mathcal{N}^{\mathrm{SH}}(X_c + h(X_c))), \\ Dh_v \partial_\xi X_c &= L_h^{\mathrm{con}} h_v + \mathcal{P}_h^{\mathrm{con}}(\mathcal{N}^{\mathrm{con}}(X_c + h(X_c))), \end{aligned}$$

where $\mathcal{P}_h^{\mathrm{SH}}$ and $\mathcal{P}_h^{\mathrm{con}}$ are the projections onto the hyperbolic spectrum of L^{SH} and L^{con}, respectively. Using that h_u, h_v are at least quadratic in ε since they are at least quadratic in X_c, we find

$$\begin{aligned} \mathcal{P}_h \mathcal{N}^{\mathrm{SH}}(X_c + h(X_c)) &= (0, 0, 0, 2\varepsilon^2 i A^2 e^{2ip} + c.c.)^T + \mathcal{O}(\varepsilon^3), \\ \mathcal{P}_h \mathcal{N}^{\mathrm{con}}(X_c + h(X_c)) &= (0, 4\varepsilon^2 \gamma_1 A^2 e^{2ip} - 2i\gamma_2 \varepsilon^2 A^2 e^{2ip} + c.c.)^T + \mathcal{O}(\varepsilon^3). \end{aligned}$$

Thus, we obtain the lowest order terms of h_u, h_v by solving

$$
\begin{aligned}
0 &= L_h^{\mathrm{SH}} h_u + (0,0,0,\varepsilon i A^2 e^{2ip} + c.c.)^T, \\
0 &= L_h^{\mathrm{con}} h_v + (0, 4\varepsilon^2\gamma_1 A^2 e^{2ip} - 2i\gamma_2 \varepsilon^2 A^2 e^{2ip} + c.c.)^T,
\end{aligned}
$$

which yields

$$
\begin{aligned}
h_u &= \left(\varepsilon^2 \frac{iA^2 e^{2ip}}{9+6ic_u} + c.c., 0, 0, 0\right)^T + \mathcal{O}(\varepsilon^3), \\
h_v &= \left(\varepsilon^2 A^2 \frac{-2\gamma_1 + i\gamma_2}{2 - i(c_u + c_v)} e^{2ip} + c.c., 0\right)^T + \mathcal{O}(\varepsilon^3).
\end{aligned}
$$

Utilizing this approximation, we can now derive the reduced equations on the center manifold, which are in general given by

$$
\begin{aligned}
\varepsilon^3 \partial_{\tilde{\xi}} A &= \varepsilon \lambda_c A + \mathcal{P}_{\phi_c^{\mathrm{SH}}} \mathcal{N}^{\mathrm{SH}}(X_c + h(X_c)), \\
\varepsilon^4 \partial_{\tilde{\xi}} B &= \mathcal{P}_{\phi_c^{\mathrm{con}}} \mathcal{N}^{\mathrm{con}}(X_c + h(X_c)),
\end{aligned}
$$

where $\tilde{\xi} = \varepsilon^2 \xi$ and $\mathcal{P}_{\phi_c^{\mathrm{SH}}}$ and $\mathcal{P}_{\phi_c^{\mathrm{con}}}$ are the projections onto the eigenspaces spanned by ϕ_c^{SH} and ϕ_c^{con}, respectively. Following [EW91], the projections are given by

$$
\mathcal{P}_{\phi_c^{\mathrm{SH}}}(U_1) = \frac{\left\langle \psi_c^{\mathrm{SH}}, U_1 \right\rangle}{\left\langle \psi_c^{\mathrm{SH}}, \phi_c^{\mathrm{SH}} \right\rangle}, \qquad \mathcal{P}_{\phi_c^{\mathrm{con}}}(V_0) = \frac{\left\langle \psi_c^{\mathrm{con}}, V_0 \right\rangle}{\left\langle \psi_c^{\mathrm{con}}, \phi_c^{\mathrm{con}} \right\rangle},
$$

where ψ_c^{SH} and ψ_c^{con} are the eigenvectors of the adjoint eigenvalue problem and they are normalized such that their last component is equal to one. Therefore, we find

$$
\begin{aligned}
\mathcal{P}_{\phi_c^{\mathrm{SH}}} \mathcal{N}^{\mathrm{SH}}(X_c + h(X_c)) &= \frac{\varepsilon^3}{\left\langle \psi_c^{\mathrm{SH}}, \phi_c^{\mathrm{SH}} \right\rangle} \left[A \left(B + \frac{-2\gamma_1 + i\gamma_2}{2 - i(c_u + c_v)} |A|^2 \right) - \frac{A|A|^2}{9 + 6ic_u} - 3A|A|^2 \right] \\
&\quad + \mathcal{O}(\varepsilon^4), \\
\mathcal{P}_{\phi_c^{\mathrm{con}}} \mathcal{N}^{\mathrm{con}}(X_c + h(X_c)) &= \frac{-2\gamma_2 \varepsilon^4}{\left\langle \psi_c^{\mathrm{con}}, \phi_c^{\mathrm{con}} \right\rangle} \partial_{\tilde{\xi}} |A|^2 + \mathcal{O}(\varepsilon^5).
\end{aligned}
$$

We obtained the latter expression by using that

$$
\sum_{k+j=0} U_{k0} U_{j1} = \frac{1}{2} \partial_\xi \sum_{k+j=0} U_{k0} U_{j0},
$$

since $U_{j1} = \partial_\xi U_{j0}$. Finally, we calculate following again [EW91] that

$$\begin{aligned}
\left\langle \psi_c^{\mathrm{SH}}, \phi_c^{\mathrm{SH}} \right\rangle &= -\partial_\lambda \det(\lambda - L_1^{\mathrm{SH}})|_{\lambda=\lambda_{c,1}^{\mathrm{SH}}} = 3c_u - c + \mathcal{O}(\varepsilon^2),\\
\langle \psi_c^{\mathrm{con}}, \phi_c^{\mathrm{con}} \rangle &= -(c + c_v).
\end{aligned}$$

Hence, to lowest order in ε, the dynamics on the center manifold are determined by

$$\begin{aligned}
\partial_{\tilde{\xi}} A &= \frac{\alpha_0 + i\omega_0^*}{3c_u - c} A + \frac{1}{3c_u - c}\left(AB + \left(-3 - \frac{1}{9 + 6ic_u} + \frac{-2\gamma_1 + i\gamma_2}{2 - i(c_u + c_v)}\right) A|A|^2\right),\\
\partial_{\tilde{\xi}} B &= \frac{2\gamma_2}{c_v + c}\partial_{\tilde{\xi}}(|A|^2).
\end{aligned}$$

By integrating the second equation with respect to $\tilde{\xi}$ we thus find that B is to lowest order slaved to $|A|^2$. Therefore, we arrive at

$$\partial_{\tilde{\xi}} A = \frac{\alpha_0 + i\omega_0^*}{3c_u - c} A + \frac{A|A|^2}{3c_u - c}\left(\frac{2\gamma_2}{c_v + c} - 3 - \frac{1}{9 + 6ic_u} + \frac{-2\gamma_1 + i\gamma_2}{2 - i(c_u + c_v)}\right) \tag{3.3}$$

3.4.2. Existence of heteroclinic orbits

We now show that the reduced equation on the center manifold (3.3) exhibits heteroclinic connections. As in the construction of periodic solutions, see Lemma 3.2.1, we first discuss the case that $\gamma_1 = \gamma_2 = 0$. Thus, we consider

$$\partial_{\tilde{\xi}} A = \frac{\alpha_0 + i\omega_0^*}{3c_u - c} A + \frac{A|A|^2}{3c_u - c}\left(-3 - \frac{1}{9 + 6ic_u}\right).$$

Using the invariance with respect to $A \mapsto Ae^{i\phi}$, we write this equation in polar coordinates $A = r_A e^{i\phi_A}$, which yields

$$\partial_{\tilde{\xi}} r_A = \frac{\alpha_0}{3c_u - c} r_A + \frac{r_A^3}{3c_u - c}\left(-3 - \frac{1}{9 + 4c_u^2}\right), \tag{3.4a}$$

$$\partial_{\tilde{\xi}} \phi_A = \frac{\omega_0^*}{3c_u - c} + \frac{r_A^2}{3c_u - c}\frac{2c_u}{27 + 12c_u^2}. \tag{3.4b}$$

Since (3.4a) is independent of the angle ϕ_A, it is sufficient to construct heteroclinic orbits in the r_A-equation connecting $r_A = A^*$ and $r_A = 0$ with A^* from Lemma 3.2.1. Therefore, linearizing about $r_A = 0$ leads to

$$L_0 = \frac{\alpha_0}{3c_u - c}$$

and thus, the origin is stable for $c > 3c_u$ and unstable for $c < 3c_u$. Similarly, the linearization about $r_A = A^*$ is given by

$$L_{A^*} = -\frac{4\alpha_0}{3c_u - c}.$$

Hence, $r_A = A^*$ is unstable for $c > 3c_u$ and stable for $c < 3c_u$. Since (3.4a) possesses no stationary states $\tilde{r}_A \in (0, A^*)$, there exists a heteroclinic solution $r_{c,\mathrm{het}}$ of (3.4a) such that

1. if $c > 3c_u$ it holds that

$$\lim_{\tilde{\xi}\to-\infty} r_{c,\mathrm{het}}(\tilde{\xi}) = A^* \text{ and } \lim_{\tilde{\xi}\to\infty} r_{c,\mathrm{het}}(\tilde{\xi}) = 0,$$

2. if $c < 3c_u$ it holds that

$$\lim_{\tilde{\xi}\to-\infty} r_{c,\mathrm{het}}(\tilde{\xi}) = 0 \text{ and } \lim_{\tilde{\xi}\to\infty} r_{c,\mathrm{het}}(\tilde{\xi}) = A^*.$$

3.4.3. Persistence of heteroclinic orbits and final result

It remains to show that these heteroclinic connections persist if the system is perturbed with higher order terms in $(\varepsilon, \gamma_1, \gamma_2)$. Using the conservative structure of (3.1b) we find that the full reduced equation for B on the center manifold, including higher order terms, is of the form

$$\partial_{\tilde{\xi}} B = \frac{2\gamma_2}{c + c_v} \partial_{\tilde{\xi}} |A|^2 + \partial_{\tilde{\xi}} \tilde{g}(\varepsilon, \gamma_1, \gamma_2, A, B),$$

with $\tilde{g} = \mathcal{O}(\varepsilon)$. An application of the implicit function theorem thus yields

$$B = \frac{2\gamma_2}{c + c_v} |A|^2 + \mathcal{O}(\varepsilon(1 + |\gamma_1| + |\gamma_2|)|A|^2).$$

Next, using the translational symmetry of (3.1a)–(3.1b) similarly to the proof of Lemma 3.2.1 we obtain the persistence of the nontrivial stationary state $r_A = A^*$ for a specific choice of ω_0^*. Then, the continuous dependence of λ_c^{SH}, ϕ^{SH} and ψ_c^{SH} on $(\varepsilon, \gamma_1, \gamma_2)$ proves the following result.

Theorem 3.4.1. *Let $c_u \neq -c_v$ and $c|_{\varepsilon=0} \notin \{c_u, -c_v, 3c_u\}$. Then, there exist $\varepsilon_0 > 0$ and $\gamma^* > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $\gamma_1, \gamma_2 \in (-\gamma^*, \gamma^*)$ the system* (3.1a)–(3.1b) *has the following solutions.*

1. *If $c > 3c_u$, there exists a family of modulating front solutions $(u_{front}, v_{front}) = (u_{front}, v_{front})(\xi, p)$ with*

$$\lim_{\xi \to -\infty} (u_{front}, v_{front})(\xi, p) = (u_{per}, v_{per})(p) \text{ and } \lim_{\xi \to +\infty} (u_{front}, v_{front})(\xi, p) = (0, 0).$$

2. *If $c < 3c_u$, there exists a family of modulating front solutions $(u_{front}, v_{front}) = (u_{front}, v_{front})(\xi, p)$ with*

$$\lim_{\xi \to -\infty} (u_{front}, v_{front})(\xi, p) = (0, 0) \text{ and } \lim_{\xi \to +\infty} (u_{front}, v_{front})(\xi, p) = (u_{per}, v_{per})(p)$$

Here, $\xi = x - ct$ and $p = x - \omega t$ with the phase velocity ω given in Lemma 3.2.1.

3.5. Modulating traveling fronts in the case $c = 3c_u + \varepsilon c_0$

We now consider the case that the spreading speed is close to the linear group velocity, that is, we assume $c = 3c_u + \varepsilon c_0$ for $c_0 \in \mathbb{R}$ and $c_v \neq -3c_u$. Recalling Lemmas 3.3.2 and 3.3.1, the central part of the spectrum consists of two eigenvalues from $L_{\pm 1}^{\mathrm{SH}}$, respectively, and a simple zero eigenvalue from L_0^{con}.

Proceeding similar to the previous Section 3.4 we calculate the two central eigenvalues of L_1^{SH}, denoted by $\lambda_\pm$ by approximating the roots of

$$\det(\lambda - L_1^{\mathrm{SH}}) = -\lambda^4 + \lambda^3(c_u - 4i) + \lambda^2(3ic_u + 4) + \lambda\varepsilon c_0 + \varepsilon^2(\alpha_0 + i\omega_0^*),$$

where ω_0^* is given in Lemma 3.2.1 and we used $c = 3c_u + \varepsilon c_0$. Inserting the ansatz $\lambda = \varepsilon\delta$ and dividing by ε^2 then yields

$$\varepsilon^{-2} \det(\varepsilon\delta - L_1^{\mathrm{SH}}) = \delta^2(3ic_u + 4) + c_0\delta + \alpha_0 + i\omega_0^* + \mathcal{O}(\varepsilon).$$

Equating the lowest order contributions to zero, we find that the central eigenvalues of L_1^{SH} are given by

$$\begin{aligned}
\lambda_\pm &= \varepsilon\delta_\pm + \mathcal{O}(\varepsilon^2),\\
\delta_\pm &= \frac{-c_0 \pm \sqrt{c_0^2 - 4(3ic_u + 4)(\alpha_0 + i\omega_0^*)}}{8 + 6ic_u} =: -\frac{c_0}{8 + 6ic_u} \pm \frac{\Delta}{8 + 6ic_u}.
\end{aligned}$$

3.5.1. Reduced equations on the center manifold

We now derive the reduced equations, which capture the dynamics on the center manifold. Therefore, we introduce the coordinates

$$\begin{aligned}
U_c(\xi) &= \varepsilon\left(A_+(\varepsilon\xi)\phi_+^{\mathrm{SH}} + A_-(\varepsilon\xi)\phi_-^{\mathrm{SH}}\right)e^{ip} + c.c.,\\
V_c(\xi) &= \varepsilon^2 B(\varepsilon\xi)\phi_c^{\mathrm{con}},
\end{aligned}$$

where $\phi_\pm^{\mathrm{SH}}, \phi_c^{\mathrm{con}}$ are the eigenvectors corresponding the central eigenvalues of $L_1^{\mathrm{SH}}, L_0^{\mathrm{con}}$, respectively. Again, we normalize the first component of the eigenvectors to one. Furthermore, let $h = (h_u, h_v)$ be the map defining the center manifold such that

$$\begin{aligned}
U(\xi) &= U_c(\xi) + h_u(A_+, A_-, B),\\
V(\xi) &= V_c(\xi) + h_v(A_+, A_-, B),
\end{aligned}$$

see Theorem 3.3.3. Redoing the computations in the previous Section 3.4.1 we obtain again that

$$\begin{aligned}
h_u &= \left(\varepsilon^2\frac{i(A_+ + A_-)}{9 + 6ic_u}e^{2ip} + c.c., 0, 0, 0\right)^T + \mathcal{O}(\varepsilon^3),\\
h_v &= \left(\varepsilon^2\frac{-2\gamma_1 + i\gamma_2}{2 - i(c_u + c_v)}(A_+ + A_-)e^{2ip} + c.c., 0\right)^T + \mathcal{O}(\varepsilon^3).
\end{aligned}$$

Furthermore, let again $\psi_\pm^{\mathrm{SH}}$ be the eigenvectors corresponding to the adjoint problem, where we normalize the last component to one. Then, we calculate

$$\left\langle \psi_\pm^{\mathrm{SH}}, \phi_\pm^{\mathrm{SH}} \right\rangle = -\partial_\lambda \det(\lambda - L_1^{\mathrm{SH}})|_{\lambda=\lambda_\pm} = \mp\varepsilon\Delta + \mathcal{O}(\varepsilon^2).$$

Proceeding further along the lines of Section 3.4.1 we find that, to lowest order in ε, B is slaved by $A_+ + A_-$, that is,

$$B = \frac{2\gamma_1}{c + c_v}|A_+ + A_-|^2.$$

Thus, up to higher order terms the reduced equations on the center manifold are given by

$$\begin{aligned}\partial_{\tilde{\xi}} A_+ &= \delta_+ A_+ + a_{\text{cub}}(A_+ + A_-)|A_+ + A_-|^2,\\ \partial_{\tilde{\xi}} A_- &= \delta_- A_- - a_{\text{cub}}(A_+ + A_-)|A_+ + A_-|^2,\end{aligned}$$

with $\tilde{\xi} = \varepsilon\xi$ and

$$a_{\text{cub}} = -\frac{1}{\Delta}\left(\frac{2\gamma_1}{c + c_v} - 3 - \frac{1}{9 + 6ic_u} + \frac{-2\gamma_1 + i\gamma_2}{2 - i(c_u + c_v)}\right) =: -\frac{\tilde{a}_{\text{cub}}}{\Delta}.$$

Finally, we rewrite this equation by substituting

$$A := A_+ + A_- \text{ and } \tilde{A} := -\frac{c_0}{8 + 6ic_u}(A_+ + A_-) + \frac{\Delta}{8 + 6ic_u}(A_+ - A_-), \tag{3.5}$$

which yields

$$\partial_{\tilde{\xi}} A = \tilde{A}, \tag{3.6a}$$

$$\partial_{\tilde{\xi}} \tilde{A} = -\frac{2c_0}{8 + 6ic_u}\tilde{A} + \frac{\Delta^2 - c_0^2}{(8 + 6ic_u)^2}A - \frac{2\tilde{a}_{\text{cub}}}{8 + 6ic_u}A|A|^2. \tag{3.6b}$$

3.5.2. Numerical analysis of the the reduced system

We now study the dynamics of the reduced equations (3.6). First, note that if $(A(\tilde{\xi}), \tilde{A}(\tilde{\xi}))$ is a solutions of (3.6) for c_0, then $(A(-\tilde{\xi}), -\tilde{A}(-\tilde{\xi}))$ is a solution for $-c_0$. Therefore, we can restrict the discussion to the case that $c_0 \geq 0$. Since we are interested in the existence of heteroclinic orbits connecting the fixed point $(A^*, 0)$ to the origin, we consider the stability of the respective fixed points. Therefore, we split the system (3.6) in real and imaginary part, which yields the four-dimensional system

$$\partial_{\tilde{\xi}} A_r = \tilde{A}_r, \tag{3.7a}$$

$$\partial_{\tilde{\xi}} A_i = \tilde{A}_i, \tag{3.7b}$$

$$\partial_{\tilde{\xi}} \tilde{A}_r = a_r A_r - a_i A_i + b_r \tilde{A}_r - b_i \tilde{A}_i + c_r A_r (A_r^2 + A_i^2) - c_i A_i (A_r^2 + A_i^2), \tag{3.7c}$$

$$\partial_{\tilde{\xi}} \tilde{A}_i = a_r A_i + a_i A_r + b_r \tilde{A}_i + b_i \tilde{A}_r + c_r A_i (A_r^2 + A_i^2) + c_i A_r (A_r^2 + A_i^2), \tag{3.7d}$$

where the subscripts r and i denote the real and imaginary part of a variable, respectively. Furthermore, we used the notation

$$a := \frac{\Delta^2 - c_0^2}{(8 + 6ic_u)^2},\ b := -\frac{2c_0}{8 + 6ic_u},\ c := -\frac{2\tilde{a}_{\text{cub}}}{8 + 6ic_u}.$$

Recalling the construction of A^* in Lemma 3.2.1 we find that the linearization about $(A^*, 0, 0, 0)$ reads as

$$L_{\text{invading}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2c_r(A^*)^2 & 0 & b_r & -b_i \\ 2c_i(A^*)^2 & 0 & b_i & b_r \end{pmatrix}.$$

Therefore, L_{invading} has one zero eigenvalue. Furthermore, numerical computations show that L_{invading} additionally has one unstable and two stable eigenvalues, independently of the value of $c_0 > 0$, see Figure 3.3.

Next, we study the linearization about the origin, denoted by L_0. Numerically, we find the following behavior. There exists some $c_0^* > 0$ such that for all $c_0 > c_0^*$, the origin is a stable hyperbolic fixed point, that is, all eigenvalues of L_0 have negative real part. At $c_0 = c_0^*$ a pair of complex conjugated eigenvalues crosses the imaginary axis, which gives rise to a Hopf bifurcation. Hence, periodic solutions bifurcate. We study these periodic solutions further using AUTO [DCD+07]. It turns out that there exists a $c_0^{**} \in (0, c_0^*)$, such that the periodic solutions are (spectrally) stable for $c_0 \in (c_0^{**}, c_0^*)$, see Figure 3.4. Recall here, that a periodic solution is (spectrally) stable if all its Floquet multipliers have absolute value less or equal to one and if any Floquet multiplier with absolute value one does not have a nontrivial Jordan block, see [SU17]. Finally, at $c_0 = c_0^{**}$ we find a Torus bifurcation and the family of periodic solutions destabilizes. The findings are summarized in the bifurcation diagram Figure 3.5. In the depicted case $\alpha_0 = c_u = 1$ we find that $c_0^* \approx 1.55172$ and $c_0^{**} \approx 1.01550$.

We now turn to the numerical study of heteroclinic connections originating from the

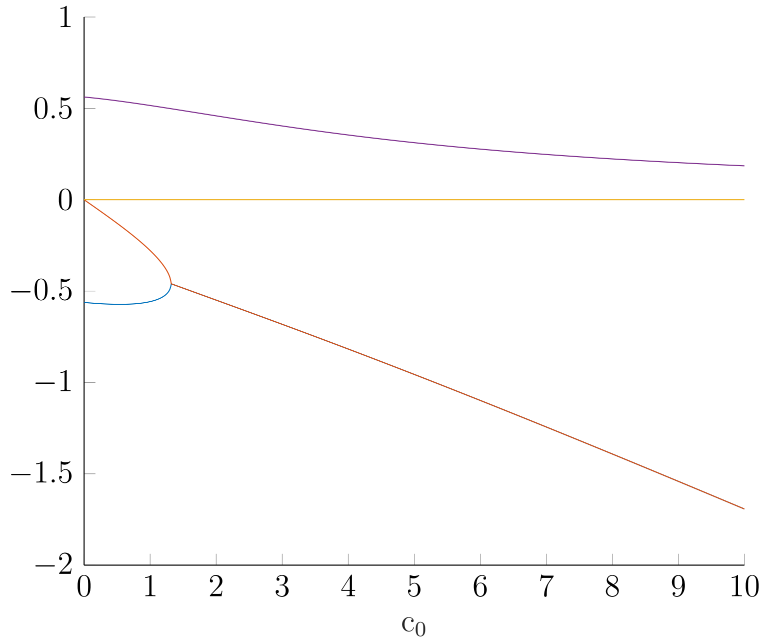

Figure 3.3.: Real part of the spectrum of L_{invading} for $\alpha_0 = c_u = 1$ and $0 \leq c_0 \leq 10$.

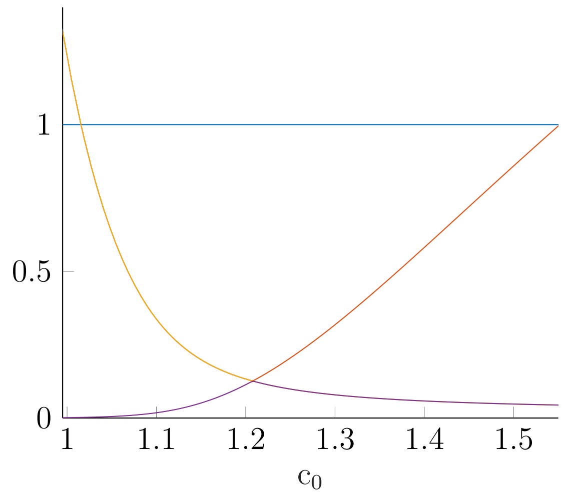

Figure 3.4.: Absolute values of Floquet multipliers corresponding to the bifurcating periodic solution for $\alpha_0 = c_u = 1$.

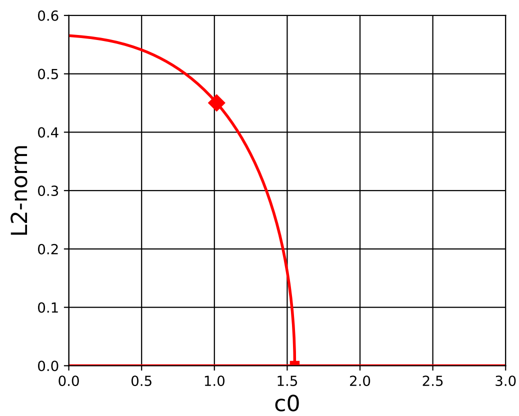

Figure 3.5.: Bifurcation diagram of the origin for $\alpha_0 = c_u = 1$. The squares mark the bifurcation points.

invading state $(A^*, 0, 0, 0)$ in (3.7). Since $(A^*, 0, 0, 0)$ has a one-dimensional unstable manifold, we employ a shooting-type algorithm, that is, we choose an initial condition on the unstable manifold and compute the forward orbit. In practice, the initial condition is chosen on the unstable eigenspace close to the fixed point, since the unstable eigenspace is tangential to the unstable manifold. We find the following behavior. For $c_0 > c_0^*$ there is a heteroclinic orbit connecting $(A^*, 0, 0, 0)$ to the origin. However, as opposed to the real case discussed in Section 2.4 the corresponding front is not necessarily monotonically decreasing to zero since the eigenvalues of the origin are genuinely complex. As c_0 decreases below c_0^*, the heteroclinic orbit connects to the bifurcating periodic orbit instead of the origin. Finally, for $c_0 < c_0^{**}$ but close to c_0^{**} the heteroclinic orbit connects to an oscillating periodic solution. Further decreasing c_0 leads to instability and no heterolinics are found. We plot the amplitude $\mathrm{Re}(A)$ (recall the ansatz for the central mode and that the eigenvalues are normalized such that the first component equals one) for each cases in Figure 3.6.

3.5.3. Persistence of heterolinic orbits and final result

We now turn to the persistence of heteroclinic connections with respect to small perturbations in $(\varepsilon, \gamma_1, \gamma_2)$. First, proceeding similarly to Section 3.4.3 we obtain that

$$B = \frac{2\gamma_1}{c + c_v}|A_+ + A_-|^2 + \mathcal{O}(\varepsilon(1 + |\gamma_1| + |\gamma_2|)|A_+ + A_-|^2),$$

that is, B is slaved by $|A_+ + A_-|^2$. We focus on the persistence of heteroclinic connections to the origin as found numerically provided that c_0 is sufficiently large. Therefore, we make the following assumptions about the spectra and the existence of heteroclinic orbits, which is in line with the numerical observations in the previous section.

(A1) There exists a c_0^* such that for all $c_0 > c_0^*$ the origin is a (spectrally) stable, hyperbolic fixed point with respect to the dynamics given by (3.7) and the linearization about the invading state $(A^*, 0, 0, 0)$ has a one-dimensional unstable eigenspace.

(A2) There exists a c_0^* such that for all $c_0 > c_0^*$, the system (3.6) exhibits a heteroclinic orbit $(A_h, \tilde{A}_h)$ such that

$$\lim_{\tilde{\xi} \to -\infty} (A_h, \tilde{A}_h) = (A^*, 0) \text{ and } \lim_{\tilde{\xi} \to +\infty} (A_h, \tilde{A}_h) = (0, 0).$$

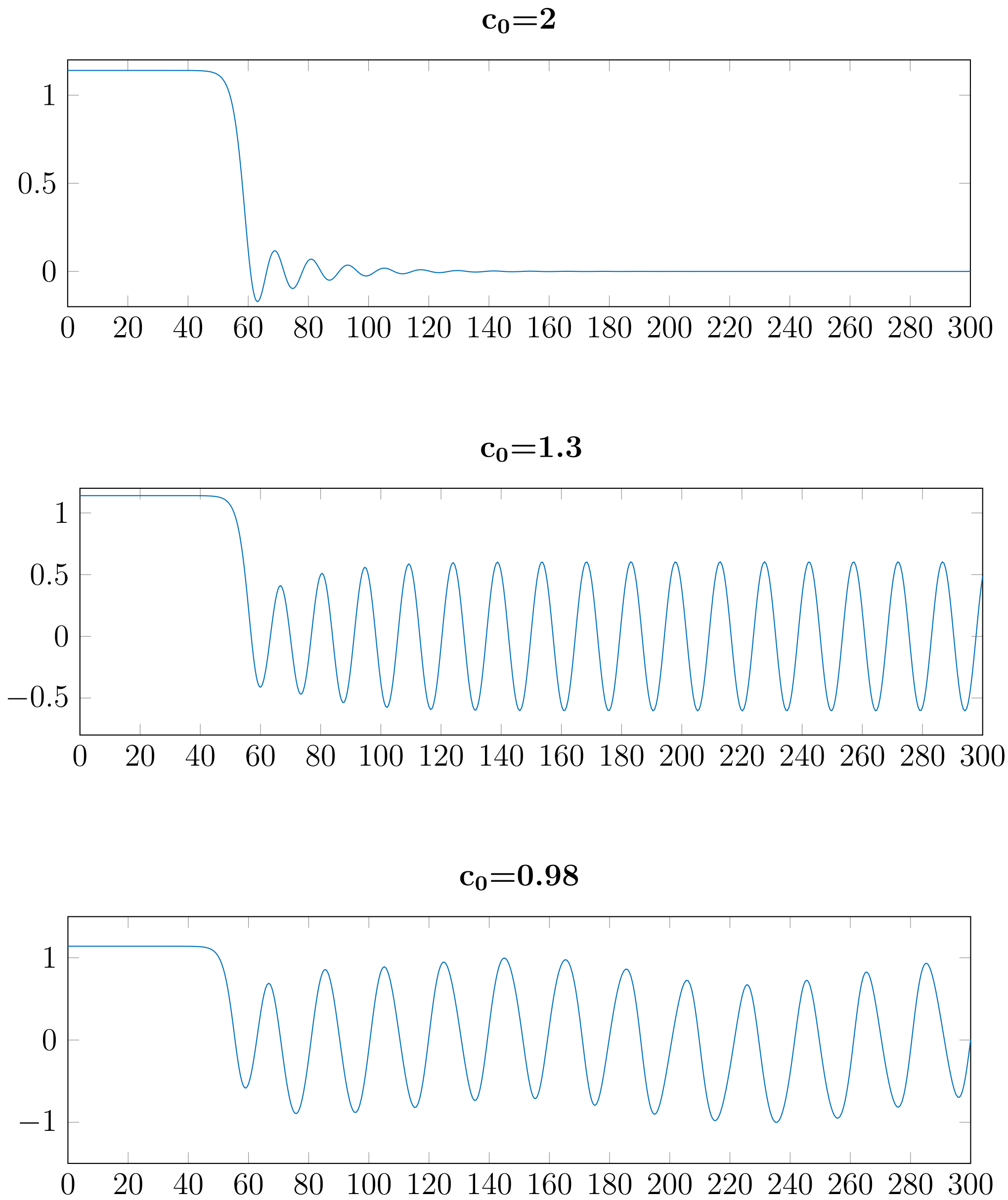

Figure 3.6.: Plot of amplitude of the modulating front $\mathrm{Re}(A)$ for $\alpha_0 = c_u = 1$ and $c_0 = 2, 1.3, 0.98$.

Using the translational invariance of (3.1a)–(3.1b) we obtain the persistence of the fixed point $(A^*, 0, 0, 0)$ for a specific choice of ω^* with respect to higher order perturbation similar to Lemma 3.2.1. Then, since the origin is a stable, hyperbolic fixed point, the corresponding stable manifold is 4-dimensional. Therefore, using Assumption (A2) the one-dimensional unstable manifold of the invading state $(A^*, 0, 0, 0)$ and the stable manifold of the origin have to intersect transversally. Hence, the intersection persists under small perturbations in $(\varepsilon, \gamma_1, \gamma_2)$ and we obtain the following result.

Theorem 3.5.1. *Assume that (A1)–(A2) are satisfied and that $c_u \neq -c_v$, $c_v \neq -3c_u$ and $c = 3c_u + \varepsilon c_0$ with $c_0 > c_0^*$. Then, there exist $\varepsilon_0 > 0$ and $\gamma^* > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $\gamma_1, \gamma_2 \in (-\gamma^*, \gamma^*)$ the system* (3.1a)–(3.1b) *has a family of modulating front solutions $(u_{front}, v_{front})(\xi, p)$ such that*

$$\lim_{\xi \to -\infty} (u_{front}, v_{front})(\xi, p) = (u_{per}, v_{per})(p) \text{ and } \lim_{\xi \to +\infty} (u_{front}, v_{front})(\xi, p) = (0, 0),$$

where $\xi = x - ct$ and $p = x - \omega t$ with $\omega = c_u + \varepsilon^2 \omega_0^*$ given in Lemma 3.2.1.

3.6. Modulating traveling fronts in the case $c = -c_v + \varepsilon^2 c_0^v$

In the previous Sections 3.4–3.5 the presence of the conservation law (3.1b) was largely irrelevant due to the fact that the conserved part was slaved by the other parts. We now consider the case that $c = -c_v + \varepsilon^2 c_0^v$ and $c_v \neq -3c_u$ for some $c_0^v \in \mathbb{R}$ and $\varepsilon > 0$. Thus, following Lemma 3.3.1, the central spectrum originating from the conservation law consists of a zero eigenvalue and one eigenvalue of order ε. Since the argument in the previous cases relied on the fact that we can remove all the central eigenvalues from the conservation law by integration, we expect that in this case the conservation law contributes to the reduced dynamics.

3.6.1. Reduced equations on the center manifold

First, we calculate the central eigenvalues as in the previous sections. Since the system (3.1a)–(3.1b) is linearly decoupled, we can perform the calculation for the Swift-Hohenberg part and the conservation law part separately. The central eigenvalues originating from L^{SH} have already been calculated in Section 3.4 and are given by

$$\lambda_{c,1}^{\mathrm{SH}} = \varepsilon^2\delta^* + \mathcal{O}(\varepsilon^3), \qquad \delta^* = \frac{\alpha_0 + i\omega_0^*}{3c_u - c}.$$

Furthermore, the non-zero central eigenvalue from the conservation law is given by $\lambda_{c,0}^{\mathrm{con}} = -c - c_v = -\varepsilon^2 c_0^v$, according to Lemma 3.3.1.

Let now ϕ_c^{SH} be the eigenvector corresponding to $\lambda_{c,1}^{\mathrm{SH}}$ of L_1^{SH}. Then, we introduce the following notation

$$U_c = \varepsilon A(\varepsilon^2\xi)\phi_c^{\mathrm{SH}} e^{ip} + c.c., \tag{3.8a}$$

$$V_c = \varepsilon^2 B(\varepsilon^2\xi), \tag{3.8b}$$

with $A(\tilde{\xi}) \in \mathbb{C}$, $B(\tilde{\xi}) = (B_1, B_2)^T(\tilde{\xi}) \in \mathbb{R}^2$ and $\tilde{\xi} = \varepsilon^2\xi$. Thus, we write any function on the center manifold as

$$U = U_c + h_u(X_c), \tag{3.9a}$$

$$V = V_c + h_v(X_c), \tag{3.9b}$$

with $X_c = (U_c, V_c)$ and h given in Theorem 3.3.3. Repeating the calculations from Section 3.4 we obtain

$$\begin{aligned} h_u(X_c) &= \left(\varepsilon^2 \frac{iA^2}{9 + 6ic_u} e^{2ip} + c.c., 0, 0, 0\right)^T + \mathcal{O}(\varepsilon^3), \\ h_v(X_c) &= \left(\varepsilon^2 A^2 \frac{-2\gamma_1 + i\gamma_2}{2 - i(c_u + c_v)} + c.c, 0\right)^T + \mathcal{O}(\varepsilon^3). \end{aligned}$$

Then, the reduced equation on the center manifold takes the form

$$\begin{aligned} \varepsilon^3 \partial_{\tilde{\xi}} A &= \varepsilon \lambda_{c,1}^{\mathrm{SH}} A + \mathcal{P}_{\phi_c^{\mathrm{SH}}} \mathcal{N}^{\mathrm{SH}}(X_c + h(X_c)), \\ \varepsilon^4 \partial_{\tilde{\xi}} B &= \varepsilon^2 L_0^{\mathrm{con}} B + \mathcal{N}_0^{\mathrm{con}}(X_c + h(X_c)). \end{aligned}$$

Here, $\mathcal{P}_{\phi_c^{\mathrm{SH}}}$ is the projection onto the eigenspace spanned by ϕ_c^{SH} and is given by

$$\mathcal{P}_{\phi_c^{\mathrm{SH}}}(U_1) = \frac{\left\langle \psi_c^{\mathrm{SH}}, U_1 \right\rangle}{\langle \psi_c^{\mathrm{SH}}, \phi_c^{\mathrm{SH}} \rangle} = \frac{\left\langle \psi_c^{\mathrm{SH}}, U_1 \right\rangle}{3c_u - c + \mathcal{O}(\varepsilon^2)},$$

where ψ_c^{SH} is the eigenvector of the adjoint problem.

Since the equation for A has already been derived in Section 3.4, we focus on the equation for B. Therefore, we recall that

$$L_0^{\mathrm{con}} = \begin{pmatrix} 0 & 1 \\ 0 & -\varepsilon^2 c_0^v \end{pmatrix}$$

and

$$\mathcal{N}_0^{\mathrm{con}} = \left(0, -2\gamma_1 \sum_{k+j=0} (U_{k0}U_{j2} + U_{k1}U_{j1}) - 2\gamma_2 \sum_{k+j=0} U_{k0}U_{j1}\right)^T.$$

With this, we obtain

$$\begin{aligned} \partial_\xi B_1 &= B_2, \\ \partial_\xi B_2 &= -\varepsilon^2 c_0^v B_2 + \varepsilon^{-4}\left(-\gamma_1 \partial_\xi^2 \sum_{k+j=0} U_{k0}U_{j0} - \gamma_2 \partial_\xi \sum_{k+j=0} U_{k0}U_{j0}\right). \end{aligned}$$

Writing this system as a second order equation and integrating once with respect to ξ we find that

$$\partial_\xi B_1 = -\varepsilon^2 c_0^v B_1 + \varepsilon^{-4}\left(-\gamma_1 \partial_\xi \sum_{k+j=0} U_{k0}U_{j0} - \gamma_2 \sum_{k+j=0} U_{k0}U_{j0}\right).$$

It turns out that any $\gamma_2 \neq 0$ leads to contributions of order ε^{-2} in the chosen scaling. Thus, we set $\gamma_2 = 0$ in what follows. With that, we obtain

$$\begin{aligned} \partial_\xi B_1 &= -\varepsilon^2 c_0^v B_1 - 2\gamma_1 \varepsilon^{-4} \sum_{k+j=0} U_{k0}U_{j1} \\ &= -\varepsilon^2 c_0^v B_1 - 4\gamma_1 \operatorname{Re}(\lambda_{c,1}^{\mathrm{SH}})|A|^2 + \mathcal{O}(\varepsilon^3), \end{aligned} \tag{3.10}$$

where we used that $\phi_c^{\mathrm{SH}} = (1, \lambda_{c,1}^{\mathrm{SH}}, (\lambda_{c,1}^{\mathrm{SH}})^2, (\lambda_{c,1}^{\mathrm{SH}})^3)^T$ following [EW91] and that $U_1 = \varepsilon A \phi_c^{\mathrm{SH}} + \mathcal{O}(\varepsilon^3)$ on the center manifold. Finally, applying the chain rule and inserting the approximation of $\lambda_{c,1}^{\mathrm{SH}}$ yields

$$\partial_{\tilde{\xi}} B_1 = -c_0^v B_1 - 4\gamma_1 \operatorname{Re}(\delta^*)|A|^2 + \mathcal{O}(\varepsilon) = -c_0^v B_1 - 4\gamma_1 \frac{\alpha_0}{3c_u - c} + \mathcal{O}(\varepsilon).$$

Summarizing, the reduced equations on the center manifold read to lowest order as

$$\partial_{\tilde{\xi}} A = \frac{\alpha_0 + i\omega_0^*}{3c_u - c} A + \frac{1}{3c_u - c}\left(AB_1 + \left(-3 - \frac{1}{9 + 6ic_u} + \frac{-2\gamma_1}{2 - i(c_u + c_v)} \right) A|A|^2 \right), \tag{3.11a}$$

$$\partial_{\tilde{\xi}} B_1 = -c_0^v B_1 - 4\gamma_1 \frac{\alpha_0}{3c_u - c} |A|^2. \tag{3.11b}$$

3.6.2. Existence and persistence of heteroclinic orbits

We now establish the existence of heteroclinic orbits connecting $(A^*, 0)$ to the origin in (3.11a)–(3.11b) for γ_1 close to zero. Note that for $\gamma_1 = 0$, the set $\{A \in \mathbb{C}, B_1 \in \mathbb{R} : B_1 = 0\}$ is invariant. Thus, the problem reduces to finding heteroclinic orbits in (3.11a) for $\gamma_1 = B_1 = 0$, which has been done in Section 3.4. Hence, it remains to show that the orbit persists for γ_1 close to zero and $\varepsilon > 0$ small.

Recalling the existence proof from Section 3.4, we use polar coordinates and write $A = r_A e^{i\phi_A}$. Then, the system reads as

$$\partial_{\tilde{\xi}} r_A = \frac{\alpha_0}{3c_u - c} r_A + \frac{1}{3c_u - c}\left(r_A B_1 + \left(-3 - \frac{1}{9 + 4c_u^2} \right) r_A^3 \right), \tag{3.12a}$$

$$\partial_{\tilde{\xi}} \phi_A = \frac{\omega_0^*}{3c_u - c} + \frac{r_A^2}{3c_u - c} \frac{2c_u}{27 + 12c_u^2}, \tag{3.12b}$$

$$\partial_{\tilde{\xi}} B_1 = -c_0^v B_1 - 4\gamma_1 \frac{\alpha_0}{3c_u - c} r_A^2. \tag{3.12c}$$

Since (3.12a) and (3.12c) are independent of ϕ_A we can analyze them separately. We note that the system is invariant with respect to $(\tilde{\xi}, c_0^v, 3c_u - c) \mapsto (-\tilde{\xi}, -c_0^v, -(3c_u - c))$. Hence, we restrict the analysis to $3c_u - c < 0$ or, equivalently, $-c_v > 3c_u$ and $\varepsilon > 0$ sufficiently small. Linearizing about the origin we find

$$L_0 = \begin{pmatrix} \frac{\alpha_0}{3c_u - c} & 0 \\ 0 & -c_0^v \end{pmatrix}$$

Thus, if $c_0^v > 0$, the origin is a stable fixed point, while for $c_0^v < 0$, the origin is a saddle point. Next, we set $\gamma_1 = 0$ and linearize about the invading state $(A^*, 0)$, which yields

$$L_{\text{invading}} = \begin{pmatrix} -\frac{4\alpha_0}{3c_u - c} & \frac{A^*}{3c_u - c} \\ 0 & -c_0^v \end{pmatrix}.$$

Hence, if $c_0^v > 0$, the invading state is a saddle point with a one-dimensional unstable eigenspace and if $c_0^v < 0$, the invading state is unstable.

In the case that $c_0^v > 0$ the setting is comparable to the ones in Sections 2.4 and 3.4, that is, the invading state has a one-dimensional unstable manifold and the origin is a stable, hyperbolic fixed point. Therefore, we establish existence and persistence of heteroclinic orbits similarly to the previous cases. For $B_1 = 0$ the existence has been obtained in Section 3.4. Since the stable manifold of the origin is two-dimensional, the intersection of the heteroclinic orbit and the stable manifold of the origin is transversal. Furthermore, since the heteroclinic orbit is a subset of the one-dimensional unstable manifold of the invading state, this unstable manifold intersects transversally with the stable manifold of the origin. Hence, the intersection persists under small perturbations. This gives the existence of heteroclinic connections for $\varepsilon > 0$ small and $\gamma_1 \neq 0$ close to zero, that is, we obtain a solution $(A_{\text{het}}, B_{1,\text{het}})(\tilde{\xi})$ such that

$$\lim_{\tilde{\xi} \to -\infty} (A_{\text{het}}, B_{1,\text{het}})(\tilde{\xi}) = (A^*, 0) \text{ and } \lim_{\tilde{\xi} \to +\infty} (A_{\text{het}}, B_{1,\text{het}})(\tilde{\xi}) = (0, 0). \tag{3.13}$$

In the case $c_0^v < 0$, we consider the time-reversed system, i.e., $\tilde{\xi} \mapsto -\tilde{\xi} =: \hat{\xi}$. Then, the origin has a one-dimensional unstable manifold and the invading state is a stable fixed point. Therefore, we can repeat the above argument to find a persistent heteroclinic orbit connecting $(0, 0)$ for $\hat{\xi} \to -\infty$ to $(A^*, 0)$ for $\hat{\xi} \to +\infty$. Reversing the coordinate change then gives a persistent heteroclinic orbit satisfying (3.13). Combining both cases we obtain the following result.

Theorem 3.6.1. *Let $c = -c_v + \varepsilon^2 c_0^v$ with $c_0^v \neq 0$. Furthermore, assume that $-c_v > 3c_u$ and $c_u \notin \{-c_v, 3c_u\}$ and $\gamma_2 = 0$. Then, there exist $\varepsilon_0 > 0$ and $\gamma_1^* > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $\gamma_1 \in (-\gamma_1^*, \gamma_1^*)$ the system* (3.1a)–(3.1b) *has a family of modulating front solutions $(u_{\mathit{front}}, v_{\mathit{front}})(\xi, p)$ such that*

$$\lim_{\xi \to -\infty} (u_{\mathit{front}}, v_{\mathit{front}})(\xi, p) = (u_{\mathit{per}}, v_{\mathit{per}})(p) \text{ and } \lim_{\xi \to +\infty} (u_{\mathit{front}}, v_{\mathit{front}})(\xi, p) = (0, 0),$$

where $\xi = x - ct$ and $p = x - \omega t$ with $\omega = c_u + \varepsilon^2 \omega_0^*$ given in Lemma 3.2.1.

Remark 3.6.2. Due to the symmetry $(\tilde{\xi}, c_0^v, 3c_u - c) \mapsto (-\tilde{\xi}, -c_0^v, -(3c_u - c))$ in the reduced equations (3.12a)–(3.12c) we obtain a “reflected” modulating front in the case

$-c_v < 3c_u$. That is, a modulating front solution $(u_{\text{front}}, v_{\text{front}})(\xi, p)$ such that

$$\lim_{\xi\to-\infty}(u_{\text{front}}, v_{\text{front}})(\xi, p) = (0,0) \text{ and } \lim_{\xi\to+\infty}(u_{\text{front}}, v_{\text{front}})(\xi, p) = (u_{\text{per}}, v_{\text{per}})(p).$$

□

3.7. Modulating traveling fronts the case $c = -c_v + \varepsilon c_0^v$ with $\gamma_2 = \varepsilon\gamma_2^0$ and $c_v \neq -3c_u$

Above, we discussed the case that $c = -c_v + \varepsilon^2 c_0^v$ and $c_v \neq -3c_u$. We highlight that the difference of the front speed c and $-c_v$ is of order ε^2. The reason to use this particular scaling is that the central eigenvalues of both the Swift-Hohenberg part L^{SH} and the conservation law part L^{con} are of the same order, that is they are of order ε^2. Here, we discuss the case $c = -c_v + \varepsilon c_0^v$, that is, c differs from $-c_v$ by a term of order ε instead of ε^2.

Since we consider a parameter regime where $c_v \neq -3c_u$, the number of central eigenvalues of L^{SH} does not change as long as the front speed c is close to $-c_v$, see Lemma 3.3.2. Hence we still have that $\lambda_{c,1}^{\text{SH}} = \mathcal{O}(\varepsilon^2)$. In contrast, the non-zero central eigenvalue from the conservation law is now given by $\lambda_{c,0}^{\text{con}} = -\varepsilon c_0^v$. Therefore, the central eigenvalues are of different order with respect to ε. To capture the relevant dynamics on the center manifold, we again make the ansatz

$$\begin{aligned} U &= \varepsilon A(\varepsilon^2\xi)\phi_c^{\text{SH}} + c.c. + h_u(X_c), \\ V &= \varepsilon^2 B(\varepsilon^2\xi) + h_v(X_c), \end{aligned}$$

see (3.8), (3.9). Then, proceeding as above, we obtain after removing the zero eigenvalue of L_0^{con} by integration that the dynamics on the center manifold is captured by

$$\partial_\xi A = \frac{\alpha_0 + i\omega_0^*}{3c_u - c}A + \frac{1}{3c_u - c}\left(AB_1 + \left(-3 - \frac{1}{9+6ic_u} + \frac{-2\gamma_1 + i\gamma_2}{2 - i(c_u + c_v)}\right)A|A|^2\right), \tag{3.14a}$$

$$\partial_\xi B_1 = -\varepsilon^{-1} c_0^v B_1 + \varepsilon^{-4}\left(-\gamma_1\partial_\xi \sum_{k+j=0} U_{k0}U_{j0} - \gamma_2 \sum_{k+j=0} U_{k0}U_{j0}\right). \tag{3.14b}$$

We obtain that the γ_1-nonlinearity is of order ε^4 due to $U \sim \varepsilon$ and $\partial_\xi \sim \varepsilon^2$, which balances with ε^{-4} in front of the nonlinearities (see also (3.10)). In contrast, due to the missing derivative, the γ_2-nonlinearity is of order ε^2, which leads to an ε^{-2}-contribution in the equation (3.14b) for B_1. To balance this with the linear part of (3.14b), which is of order ε^{-1}, we set $\gamma_2 = \varepsilon\gamma_2^0$ for $\gamma_2^0 \in \mathbb{R}$. Thus, we obtain on the center manifold, up to higher order terms, that

$$\partial_\xi A = \frac{\alpha_0 + i\omega_0^*}{3c_u - c} A + \frac{1}{3c_u - c}\left(AB_1 + \left(-3 - \frac{1}{9 + 6ic_u} + \frac{-2\gamma_1 + i\varepsilon\gamma_2^0}{2 - i(c_u + c_v)}\right) A|A|^2\right), \tag{3.15a}$$

$$\partial_\xi B_1 = -\varepsilon^{-1}\left(c_0^v B_1 + 2\gamma_2^0|A|^2\right) - 4\gamma_1 \frac{\alpha_0}{3c_u - c}|A|^2. \tag{3.15b}$$

Formally equating the ε^{-1}-contributions in (3.15b) to zero yields an algebraic equation for B_1, this is,

$$B_1 = -\frac{2\gamma_2^0}{c_0^v}|A|^2.$$

We point out that we obtained a similar relation in Sections 3.4 and 3.5, where B_1 was slaved by A, if one accounts for $c + c_v = \varepsilon c_0^v$ and $\gamma_2 = \varepsilon\gamma_2^0$. Therefore, we might expect that to lowest order the dynamics of B_1 is determined by A, although B_1 is likely not slaved by A for $\varepsilon > 0$.

For a more rigorous treatment, we write A in polar coordinates, that is $A = r_A e^{i\phi_A}$ and obtain

$$\partial_\xi r_A = \frac{\alpha_0}{3c_u - c} r_A + \frac{1}{3c_u - c}\left(r_A B_1 + \left(-3 - \frac{1}{9 + 4c_u^2} - \frac{4\gamma_1}{4 + (c_u + c_v)^2}\right) r_A^3\right), \tag{3.16a}$$

$$\partial_\xi \phi_A = \frac{\omega_0^*}{3c_u - c} - \frac{r_A^2}{3c_u - c}\left(\frac{2c_u}{27 + 12c_u^2} + \frac{2\gamma_1(c_u + c_v)}{4 + (c_u + c_v)^2}\right), \tag{3.16b}$$

$$\varepsilon\partial_\xi B_1 = c_0^v B_1 + 2\gamma_2^0 r_A^2 - \varepsilon 4\gamma_1 \frac{\alpha_0}{3c_u - c} r_A^2. \tag{3.16c}$$

Since $\varepsilon > 0$ is small, this is a fast-slow system. Therefore, we use geometric singular perturbation theory (see e.g. [Kue15] for further details) to establish the existence of heterolinic connections from $(A^*, 0, B_1^*)$ to the origin. Here $B_1^* = -\frac{2\gamma_1^0}{c_0^v}|A^*|^2 + \mathcal{O}(\varepsilon)$.

We first note that the r_A- and B_1-equation are independent of the angle ϕ_A and thus, it is sufficient to construct a heteroclinic orbit for the (r_A, B_1)-system determined by (3.16a) and (3.16c). For the application of geometric singular perturbation theory to construct persistent heteroclinic orbits, we follow [Kue15, Section 6.1]. First, the slow subsystem is given by

$$\partial_{\tilde{\xi}} r_A = \frac{\alpha_0}{3c_u - c} r_A + \frac{1}{3c_u - c}\left(r_A B_1 + \left(-3 - \frac{1}{9+4c_u^2} - \frac{4\gamma_1}{4+(c_u+c_v)^2} \right) r_A^3 \right), \tag{3.17a}$$

$$0 = -c_0^v B_1 - 2\gamma_2^0 r_A^2. \tag{3.17b}$$

and thus, the critical manifold in this case is given by

$$C_0 := \left\{ (r_A, B_1) \in \mathbb{R}^2 \ : \ B_1 = -\frac{2\gamma_2^0}{c_0^v} r_A^2 \right\}.$$

Next, the fast subsystem reads as

$$\partial_{\hat{\xi}} r_A = 0, \tag{3.18a}$$

$$\partial_{\hat{\xi}} B_1 = -c_0^v B_1 - 2\gamma_2^0 r_A^2, \tag{3.18b}$$

where $\hat{\xi} = \tilde{\xi}/\varepsilon$. Note that the critical manifold C_0 is globally attractive with respect to the flow of the fast subsystem (3.18), that is, for all initial data the solution of (3.18) tends to C_0 for $\hat{\xi} \to +\infty$. Additionally, we introduce the unstable manifold of the invading state $p_f = (A^*, -2\gamma_2^0 (c_0^v)^{-1} (A^*)^2)$ and the stable manifold of the origin $p_0 = (0,0)$ as

$$W^u(p_f) := \{ p \in C_0 \ : \ \Phi_{\tilde{\xi}}^{\text{slow}}(p) \to p_f \text{ for } \tilde{\xi} \to -\infty \},$$
$$W^s(p_0) := \{ p \in C_0 \ : \ \Phi_{\tilde{\xi}}^{\text{slow}}(p) \to p_0 \text{ for } \tilde{\xi} \to +\infty \},$$

where $\Phi_{\tilde{\xi}}^{\text{slow}}$ denotes the flow of the slow subsystem (3.17). Finally, we define

$$N_{p_f} := \bigcup_{p \in W^u(p_f)} \{ q \in \mathbb{R}^2 \ : \ \Phi_{\hat{\xi}}^{\text{fast}}(q) \to p \text{ for } \hat{\xi} \to -\infty \},$$
$$N_{p_0} := \bigcup_{p \in W^s(p_0)} \{ q \in \mathbb{R}^2 \ : \ \Phi_{\hat{\xi}}^{\text{fast}}(q) \to p \text{ for } \hat{\xi} \to +\infty \},$$

where $\Phi_{\hat{\xi}}^{\text{fast}}$ denotes the flow of the fast subsystem (3.18). Using this notation, we aim to apply [Kue15, Theorem 6.1.1] which we recapitulate – adapted to our setting – for sake of completeness.

Theorem 3.7.1 ([Kue15, Theorem 6.1.1])**.** *Assuming that the fixed points p_f and p_0 are hyperbolic in the slow flow and that the manifolds N_{p_f} and N_{p_0} intersect transversally. Then, provided that the slow subsystem* (3.17) *has a heteroclinic orbit from p_f to p_0, there exists a $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the system* (3.16a), (3.16c) *has a heteroclinic orbit connecting p_f to p_0.*

Similar to the case $c = -c_v + \varepsilon^2 c_0^v$, see Section 3.6, we restrict to $-c_v > 3c_u$ in the subsequent analysis since the case $-c_v < 3c_u$ can be dealt with using symmetry arguments (see Remark 3.6.2). Additionally, we assume that γ_1 is close to zero and that

$$\gamma_2^0 > -c_0^v \left(1 + \frac{1}{3(9 + 4c_u^2)}\right). \tag{3.19}$$

Then we find that p_0 is a stable fixed point and that p_f is an unstable fixed point similar to Section 3.4. Here we use (3.19) to prove that p_f is unstable. Therefore, the (one-dimensional) slow subsystem has a heteroclinic orbit connecting p_f to p_0. Furthermore, we obtain that the set $\{(r_A, B_1) \in \mathbb{R}^2 \ : \ r_A \in (0, A^*),\ B_1 = -2\gamma_1^0 (c_0^v)^{-1} r_A^2\}$ is contained in both $W^u(p_f)$ and $W^s(p_0)$. Finally, since the critical manifold C_0 is globally attractive with respect to the fast flow we find that $N_{p_f} = W^u(p_f)$ and that $(0, A^*) \times \mathbb{R} \subset N_{p_0}$. Hence, N_{p_f} and N_{p_0} intersect transversally and applying Theorem 3.7.1 yields the following result.

Theorem 3.7.2. *Let $c = -c_v + \varepsilon c_0^v$ with $c_0^v \neq 0$. Furthermore, assume that $-c > 3c_u$ and $c_u \notin \{-c_v, 3c_u\}$ and $\gamma_2 = \varepsilon\gamma_2^0$ with γ_2^0 satisfying* (3.19)*. Then, there exist $\varepsilon_0 > 0$ and $\gamma_1^* > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $\gamma_1 \in (-\gamma_1^*, \gamma_1^*)$ the system* (3.1a)–(3.1b) *has a family of modulating front solutions $(u_{front}, v_{front})(\xi, p)$ such that*

$$\lim_{\xi \to -\infty} (u_{front}, v_{front})(\xi, p) = (u_{per}, v_{per})(p) \text{ and } \lim_{\xi \to +\infty} (u_{front}, v_{front}) = (0, 0),$$

where $\xi = x - ct$ and $p = x - \omega t$ with $\omega = c_u + \varepsilon^2 \omega_0^*$ given in Lemma 3.2.1.

Remark 3.7.3. As an alternative, similar to Chapter 2, we could in principle use a different decomposition of the spectrum into central and hyperbolic part in order to establish modulating front solutions in this case. That is, we consider the $\mathcal{O}(\varepsilon)$-eigenvalue as a hyperbolic one instead of a central one, since the eigenvalues are of different order.

However, with this splitting it would be necessary to prove that the center manifold is sufficiently large (similar to Theorem 2.3.6) and therefore, the use of the fast-slow structure of (3.15b) gives a more straightforward approach. □

3.8. Modulating traveling fronts in the case $c = -c_v + \varepsilon c_0$ and $c_v = -3c_u$

Finally, we discuss the case that $c = -c_v + \varepsilon c_0$ under the condition that $c_v = -3c_u$. In this case, we obtain six central eigenvalues, which were calculated in the previous sections. The four central eigenvalues corresponding to the Swift-Hohenberg part given by

$$\lambda_{\pm}^{\mathrm{SH}} = \varepsilon\delta_{\pm} + \mathcal{O}(\varepsilon^2), \quad \delta_{\pm} = -\frac{c_0}{8+6ic_u} \pm \frac{\Delta}{8+6ic_u}$$

and originate from L_1^{SH} and their complex conjugates corresponding to L_{-1}^{SH}. Here, recall that $\Delta = \sqrt{c_0^2 - 4(3ic_u+4)(\alpha_0 + i\omega^*)}$. Additionally, we find two central eigenvalues form the conservation law

$$\lambda_0^{\mathrm{con}} = 0 \text{ and } \lambda_1^{\mathrm{con}} = -\varepsilon c_0,$$

see Lemma 3.3.1. The derivation of the corresponding reduced equations on the center manifold is very similar to the previous Sections 3.5 and 3.6. Therefore, we refrain from repeating the calculations in detail and only comment on the differences. Note that as in Section 3.6 we set $\gamma_2 = 0$, since $\gamma_2 \neq 0$ leads to unbounded coefficients in the nonlinearity as $\varepsilon \to 0$. First, the central mode of the conservation law is no longer slaved by the Swift-Hohenberg part and thus, we obtain a nonlinear coupling as in (3.11a). Second, for the nonlinearity of the conservation law, we use that the second component of $U_c = (U_{c,0}, U_{c,1}, U_{c,2}, U_{c,3})$ can be approximated by

$$\begin{aligned} U_{c,1} &= \varepsilon^2(A_+\delta_+ + A_-\delta_-)e^{ip} + c.c. + \mathcal{O}(\varepsilon^3) \\ &= \varepsilon^2\left(-\frac{c_0}{8+6ic_u}(A_+ + A_-) + \frac{\Delta}{8+6ic_u}(A_+ - A_-)\right)e^{ip} + c.c. + \mathcal{O}(\varepsilon^3) \\ &= \varepsilon^2\tilde{A}e^{ip} + c.c. + \mathcal{O}(\varepsilon^3) \end{aligned}$$

with $\tilde{A}$ defined in (3.5). Hence, we obtain the reduced system

$$\partial_{\tilde{\xi}} A = \tilde{A}, \tag{3.20a}$$

$$\partial_{\tilde{\xi}} \tilde{A} = -\frac{2c_0}{8 + 6ic_u}\tilde{A} + \frac{\Delta^2 - c_0^2}{(8 + 6ic_u)^2}A - \frac{AB_1}{8 + 6ic_u} - \frac{2\hat{a}_{\text{cub}}}{8 + 6ic_u}A|A|^2, \tag{3.20b}$$

$$\partial_{\tilde{\xi}} B_1 = -c_0 B_1 - 4\gamma_1 \operatorname{Re}(A\bar{\tilde{A}}), \tag{3.20c}$$

where

$$\hat{a}_{\text{cub}} = -3 - \frac{1}{9 + 6ic_u} + \frac{-2\gamma_1}{2 - i(c_u + c_v)}.$$

3.8.1. Existence and persistence of heteroclinic orbits

We proceed similar to Section 3.6, that is, we use that for $\gamma_1 = 0$ the set $\{(A, \tilde{A}, B_1) \in \mathbb{C}^2 \times \mathbb{R} : B_1 = 0\}$ is invariant. On this set, the dynamics on the center manifold reduces to (3.6). Recall that we studied this equation using numerical methods and found a heteroclinic connection from $(A^*, 0)$ to $(0, 0)$ if $c_0 > c_0^*$ for some c_0^*, see Section 3.5.2. For the persistence we make the assumptions (A1)–(A2), that is we assume that for $c_0 > c_0^*$ the linearization about the invading state has a one-dimensional unstable eigenspace, the origin is a stable, hyperbolic fixed point and that there exists a heteroclinic orbit. Furthermore, we note that adding (3.20c) does not change the spectral properties of both fixed points since (3.20c) only contributes negative (i.e. stable) eigenvalues if $\gamma_1 = 0$.

Since we assumed the existence of a heterolinic connection and that the linearization about $(A^*, 0, 0)$ has a one-dimensional unstable eigenspace, the heterolinic orbit is a subset of the unstable manifold of $(A^*, 0, 0)$. Furthermore, the orbit connects to the origin and since the origin has a 3-dimensional stable manifold – all eigenvalues are stable – the unstable manifold of $(A^*, 0, 0)$ and the stable manifold of the origin have to intersect and the intersection has to be transversal. Thus, the heteroclinic connection persists under small perturbation, in particular for small (ε, γ_1). By using the center manifold theorem, we then obtain the following result.

Theorem 3.8.1. *Let $c_0^* > 0$ such that (A1)–(A2) are true. Additionally, let $c_v = -3c_u$, $c = 3c_u + \varepsilon c_0$ with $c_0 > c_0^*$ and $\gamma_2 = 0$. Then, there exist $\varepsilon_0 > 0$ and $\gamma_1^* > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $\gamma_1 \in (-\gamma_1^*, \gamma_1^*)$ the system* (3.1a)–(3.1b) *has a family of modulating* *front solutions* $(u_{front}, v_{front})(\xi, p)$ *such that*

$$\lim_{\xi \to -\infty} (u_{front}, v_{front})(\xi, p) = (u_{per}, v_{per})(p) \text{ and } \lim_{\xi \to +\infty} (u_{front}, v_{front})(\xi, p) = (0, 0),$$

where $\xi = x - ct$ *and* $p = x - \omega t$ *with* $\omega = c_u + \varepsilon^2 \omega_0^*$ *given in Lemma 3.2.1.*

Part II.

Nonlinear stability

4. Nonlinear stability of fast invading fronts in a Ginzburg-Landau equation coupled to an additional conservation law

4.1. Introduction

We have seen in Section 2.2 that the Ginzburg-Landau equation coupled to an additional conservation law

$$\partial_t A = \partial_x^2 A + A + AB - A|A|^2, \tag{4.1a}$$

$$\partial_t B = \mu \partial_x^2 B + \gamma \partial_x^2 (|A|^2), \tag{4.1b}$$

appears as an amplitude equation for the Swift-Hohenberg equation in the case of an additional conservation law. Here, $A(t,x) \in \mathbb{C}$, $B(t,x) \in \mathbb{R}$ and $\mu > 0$, $\gamma \in \mathbb{R}$. More generally, this model arises generically as an amplitude equation for Turing pattern-forming systems admitting a conservation law structure, see also [MC00, HSZ11, SZ13, SZ17] for examples. For these systems, the amplitude equation (4.1) can be formally derived either using the symmetries of the physical system [MC00] or by a multiple scaling analysis [SZ13, Zim14] as in Section 2.2.

The system (4.1) has real front solutions $(A,B)(x,t) = (A_{\text{front}}, B_{\text{front}})(x-ct)$ connecting an invading state $(A_f, B_f) = (1,0)$ to the origin, see Figure 4.1. The existence of these fronts with velocity $c \geq 2$ can be established for γ close to zero using perturbation arguments, following the ideas used in Section 2.4.1. In particular, that these fronts approximately determine the amplitude of the modulating front solutions in a Swift-

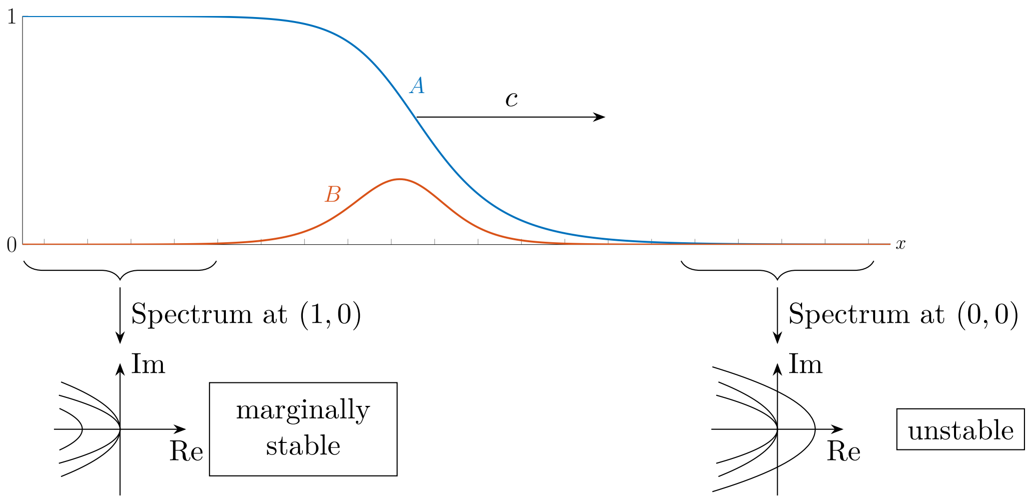

Figure 4.1.: Plot of the front in (4.1) and spectral stability of the rest states.

Hohenberg equation with an additional conservation law,

$$\begin{aligned}\partial_t u &= -(1+\partial_x^2)^2 u + \varepsilon^2 \alpha_0 u + uv - u^3,\\ \partial_t v &= \partial_x^2 v + \gamma \partial_x^2 (u^2),\end{aligned}$$

with γ close to zero, $\varepsilon > 0$ small and $\alpha_0 > 0$, which we constructed in Chapter 2. Therefore, as a first step in the understanding of the stability of the modulating traveling fronts, we consider the stability of invading fronts in (4.1).

4.1.1. Main results and challenges

We now formulate the main results and discuss arising challenges. The set-up is as follows. Writing (4.1) in polar coordinates, i.e. $A = Re^{i\phi}$ and introducing the local wave number $\psi := \partial_x \phi$ we obtain

$$\partial_t R = \partial_x^2 R + R + RB - R^3 - R\psi^2, \tag{4.2a}$$

$$\partial_t B = \mu \partial_x^2 B + \gamma \partial_x^2 (R^2), \tag{4.2b}$$

$$\partial_t \psi = \partial_x^2 \psi + 2\partial_x \frac{(\partial_x R)\psi}{R}. \tag{4.2c}$$

Since the front is real, it holds that $\psi_{\text{front}} = 0$ and the front connects the invading state $(A_f, B_f, \psi_f) = (1, 0, 0)$ to the origin. Linearising about the origin, we find essential spectrum with positive real part and thus, the origin is spectrally unstable, see Figure 4.1. In contrast, the linearisation about $(1, 0, 0)$ in Fourier space is given by

$$\hat{\tilde{L}}(k) = \begin{pmatrix} -k^2 - 2 & 1 & 0 \\ -2\gamma k^2 & -\mu k^2 & 0 \\ 0 & 0 & -k^2 \end{pmatrix},$$
$$\text{i.e. } \sigma(\hat{\tilde{L}}(k)) = \left\{ \frac{1}{2} \left(k^2(-(\mu+1)) \pm \sqrt{k^4(\mu-1)^2 - 4k^2(2\gamma+\mu-1) + 4} - 2 \right), -k^2 \right\},$$

for $k \in \mathbb{R}$ and the L^2-spectrum of $\tilde{L}$ is given by the union of $\sigma(\hat{\tilde{L}}(k))$ over all $k \in \mathbb{R}$. In particular, if $\gamma > -\mu$, the state $(1, 0, 0)$ is spectrally stable with spectrum touching the imaginary axis as depicted in Figure 4.1.

Nonlinear stability of the invading state $(1, 0, 0)$

We start by establishing the nonlinear stability of the invading state $(1, 0, 0)$ with respect to localized perturbations. This is indeed a nontrivial task since the linearization has essential spectrum touching the imaginary axis. Let (r_A, r_B, r_ψ) be a perturbation of $(A_f, B_f, \psi_f) = (1, 0, 0)$, which satisfies

$$\begin{aligned}
\partial_t r_A &= \partial_x^2 r_A - 2r_A + r_B + r_A r_B - 3r_A^2 - r_A^3 - (1 + r_A) r_\psi^2, \\
\partial_t r_B &= \mu \partial_x^2 r_B + \gamma \partial_x^2 (2r_A + r_A^2), \\
\partial_t r_\psi &= \partial_x^2 r_\psi + 2\partial_x \frac{(\partial_x r_A) r_\psi}{1 + r_A}.
\end{aligned}$$

We rewrite this by introducing the dummy variable $r_C := \partial_x r_A$ and using $\partial_x^2(r_A^2) = 2\partial_x(r_A r_C)$. Then, we consider the extended system for $r = (r_A, r_C, r_B, r_\psi)$, which we abbreviate by

$$\partial_t r = Lr + \mathcal{N}(r). \tag{4.3}$$

Here, L denotes the linearisation about the invading state and $\mathcal{N}$ the nonlinearity. In particular, the system is locally well-posed in the Sobolev space H^m for $m > 1/2$ using standard methods, see e.g. [Hen81].

As expected, the L^2-spectrum of L is given by two curves touching the imaginary axis, while the remaining spectrum has a strictly negative real part, see Section 4.2.1. Moreover, we find that L is diagonizable in Fourier space locally about $k = 0$, see Lemma 4.2.1. Thus, locally diagonalizing the system leads to two polynomially decaying variables and two exponentially decaying ones. We then use this separation to prove the nonlinear stability of the invading state if the perturbation is initially sufficiently small.

Theorem 4.1.1. *Let $\mu > 0$, $\gamma > -\mu$ and $m > 1/2$. There exists a $\varepsilon > 0$ such that for all perturbations (r_A, r_B, r_ψ) of the invading state $(A_f, B_f, \psi_f) = (1, 0, 0)$ satisfying*

$$\|(r_A, r_B, r_\psi)|_{t=0}\|_{(W^{1,1}\cap H^{m+1})\times(L^1\cap H^m)\times(L^1\cap H^m)} < \varepsilon$$

holds that

$$\|(r_A, r_B, r_\psi)(t)\|_{H^{m+1}\times H^m\times H^m} \lesssim (1+t)^{-1/4}$$

for all $t \geq 0$.

Remark 4.1.2 (Notation and spaces)**.** In Theorem 4.1.1, we denote L^2-based Sobolev spaces by H^m and L^1-based Sobolev spaces by $W^{1,m}$, see [SU17]. Additionally, to simplify the notation, in the above Theorem 4.1.1 and throughout the chapter we use the notation $a \lesssim b$, i.e. there exists a constant $C < \infty$ such that $a \leq Cb$. □

Remark 4.1.3. We note that the use of polar coordinates is crucial in the stability analysis. It turns out that when using cartesian coordinates the analysis is much more subtile since the nonlinearities are of a less favorable form. This is to some extent not surprising since polar coordinates exploit the specific properties of the Ginzburg-Landau nonlinearity $A|A|^2$. □

Remark 4.1.4. Note that the difference in regularity, i.e. $r_A \in H^{m+1}$ and $r_B \in H^m$, originates in the introduction of the dummy variable $r_C = \partial_x r_A$. This reflects that the original system (4.1) is well-posed in $H^{m+1} \times H^m$ since it is semilinear in these spaces, see also [Zim14, Gau17]. □

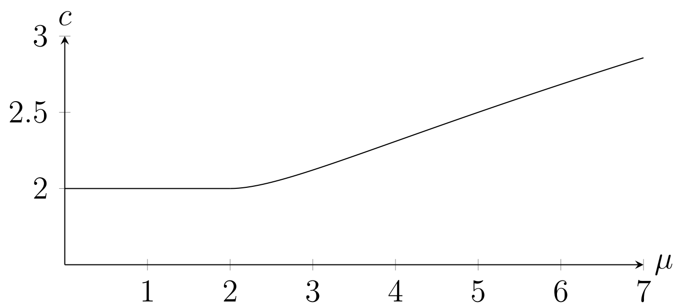

Figure 4.2.: Plot of the minimal velocity $c_{\min}(\mu)$, which is sufficient to prove nonlinear stability.

Nonlinear stability of the traveling front $(A_{\mathsf{front}}, B_{\mathsf{front}}, \psi_{\mathsf{front}})$

We now outline the expected mechanism for the stability of sufficiently fast fronts. Here, a front is deemed sufficiently fast if its spreading speed c is strictly larger than

$$c_{\min}(\mu) = \begin{cases} 2, & 0 < \mu \leq 2, \\ \dfrac{\mu}{\sqrt{\mu - 1}}, & \mu > 2, \end{cases} \tag{4.4}$$

where μ is the diffusion parameter in (4.1), see Figure 4.2. As will be discussed below, this minimal velocity is necessary to stabilize the origin in a co-moving frame with speed $c > c_{\min}(\mu)$ using exponential weights, see also Remarks 4.1.7 and 4.4.2.

In the scalar case such as the Fischer-KPP equation, we observe the following behavior, see Figure 4.3. For a sufficiently fast front, the front spreads faster than the perturbations. Therefore, if a perturbation is behind the front, it remains in this region and since the invading state is stable, the perturbation decays. Now, consider a localized perturbation ahead of the front. Since the origin is unstable, the perturbation grows exponentially fast in time as long as it is small. However, since the front is faster than the perturbation, the perturbation is transported to the stable region behind the front. If the perturbation is still sufficiently small at that point in time, it then decays. This idea for stability was first proposed by Sattinger [Sat76, Sat77] and then later used for example by Eckmann and Schneider [ES02]. It turns out that a similar mechanism also is applicable in our setting.

Next, we outline how to turn this idea into a mathematical proof. Let (r_A, r_B, r_ψ) be a perturbation of the front $(A_{\text{front}}, B_{\text{front}}, \psi_{\text{front}})$ and $r_C := \partial_x r_A$ as above. Then,

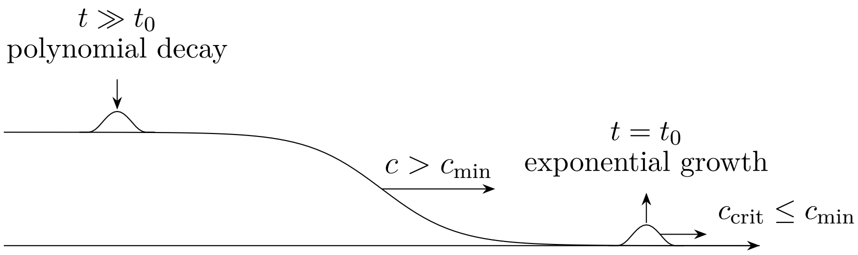

Figure 4.3.: Expected behavior of small perturbations ahead of the bulk of the front. At time $t = t_0$, the perturbation is located ahead of the front and grows exponentially. However, if the perturbation is small enough, the front reaches it for $t \gg t_0$ and behind the bulk of the front the perturbation shows diffusive decay.

$r = (r_A, r_C, r_B, r_\psi)$ satisfies

$$\partial_t r = L_{\text{front}} r + \mathcal{N}_2(r), \tag{4.5}$$

where $L_{\text{front}} = L_{\text{front}}(\xi)$, with $\xi = x - ct$, denotes the linear part and $\mathcal{N}_2$ the nonlinear part. Since we are interested in perturbations, which are exponentially decaying ahead of the front, we additionally introduce an exponentially weighted variable $w_j(\xi, t) := r_j(\xi + ct, t)e^{\beta_j \xi}$ for $j \in \{A, C, B, \psi\}$ and some $\beta_j > 0$ to be determined later. This variable satisfies the weighted problem

$$\partial_t w = L_\beta w + \mathcal{N}_\beta(w, r), \tag{4.6}$$

see (4.18), with $w = (w_A, w_C, w_B, w_\psi)$, where $\mathcal{N}_\beta(w, r)$ is linear in w but nonlinear as a function of both w and r.

Remark 4.1.5. It turns out that we can choose $\beta_A = \beta_C = \beta_B$ in the weighted variable. Therefore, we only denote β_ψ seperately and otherwise write $\beta = \beta_j$, $j \in \{A, C, B\}$. $\square$

Since the origin is unstable, L_{front} will possess unstable spectrum in any translationally invariant space. Therefore, only an interplay between the non-weighted variable r and the weighted variable w can lead to stability. To use this, we rewrite (4.5) as

$$\partial_t r = Lr + \mathcal{N}_1(r) + \mathcal{N}_2(r), \tag{4.7}$$

where L is the linearisation about the (extended) invading state $(1,0,0,0)$ and $\mathcal{N}_1(r) := L_{\text{front}}r - Lr$. In particular, L is now a constant-coefficient operator, independent of (x,t). Furthermore, we highlight that the coefficients in $\mathcal{N}_1$ vanish exponentially fast as $\xi \to -\infty$, since $(A_{\text{front}}, B_{\text{front}}, \psi_{\text{front}}) \to (1,0,0)$ for $\xi \to -\infty$. Additionally using that we expect that the perturbation is transported behind the front, see Figure 4.3, we can expect that $\mathcal{N}_1(r)$ shows exponential temporal decay. In fact this can be made rigorous by considering an interplay between the non-weighted and weighted variable and thus, we consider the coupled (r,w)-system (4.7)–(4.6) in our analysis. In particular, despite being linear, we do not consider $\mathcal{N}_1$ in the linear analysis, but instead we can include it in the nonlinear analysis of the coupled system, see Section 4.3.2.

The major benefit of this strategy is that it is not necessary to control the spectrum of L_{front}. Instead, the difficulty shifts to the spectral analysis of the weighted operator L_β. However, provided that $c > c_{\min}(\mu)$, it turns out that the spectrum of L_β has strictly negative real part for properly chosen $\beta, \beta_\psi > 0$ and γ close to zero, see the subsequent Theorem 4.1.6.

Although this idea has already been used by Sattinger [Sat76, Sat77] and Eckmann and Schneider [ES02], the implementation in these cases relied heavily on the use of the comparison principle to show that w is linearly exponentially stable. However, in our setting, a comparison principle is not available, see Remark 4.1.11. Instead, we use that for $\gamma = 0$, for which the conservation law decouples, we can establish estimates for the spectrum of L_β. Then, considering γ close to zero as a small perturbation we establish spectral stability, see Lemma 4.3.1. Using this spectral stability we can prove the nonlinear stability of the invading fronts, cf. Theorem 4.3.9.

Theorem 4.1.6. *Fix $m \in \mathbb{N}$ and let $\mu > 0$, $\gamma > -\mu$ and $c > c_{min}(\mu)$, see* (4.4)*. Then, there exists a $\beta_\psi^0 > 0$ such that for $\beta > 0$ and $\beta_\psi > 0$ satisfying*

$$\begin{aligned}
&\max(\beta^2 - c\beta + 1, \mu\beta - c) < 0,\\
&\beta_\psi \in (0, \beta_\psi^0),\\
&\beta \leq \frac{1}{2}(c - \sqrt{c^2-4}) + \beta_\psi,
\end{aligned}$$

exists a $\varepsilon > 0$, $\gamma_0 > 0$ and a $\bar{\kappa} > 0$ such that if

$$\|(r_A, r_B, r_\psi)|_{t=0}\|_{(W^{1,1}\cap H^{m+1})\times(L^1\cap H^m)\times(L^1\cap H^m)} < \varepsilon,$$

$$\left\|(r_A, r_B)|_{t=0} e^{\beta x}\right\|_{H^{m+1}\times H^m} + \left\|r_\psi|_{t=0} e^{\beta_\psi x}\right\|_{H^m} < \varepsilon,$$

and $\gamma \in (-\gamma_0, \gamma_0)$ *it holds that*

$$\begin{aligned}
&\|(r_A, r_B, r_\psi)(t)\|_{H^{m+1}\times H^m\times H^m} \lesssim (1+t)^{-1/4},\\
&\left\|(r_A, r_B)(t) e^{\beta(x-ct)}\right\|_{H^{m+1}\times H^m} + \left\|r_\psi(t) e^{\beta_\psi(x-ct)}\right\|_{H^m} \lesssim e^{-\tilde{\kappa} t}
\end{aligned}$$

for all $t \geq 0$*.*

Remark 4.1.7. Note that the existence of $\beta, \beta_\psi > 0$ in the above theorem is guaranteed assuming that $c > c_{\min}(\mu)$. However, we want to stress that $c_{\min}(\mu)$ is not necessarily the critical spreading speed of the problem, i.e. the speed with which steep enough inital data spreads. In fact the existence of such a critical spreading speed and its value, in particular for $\mu > 2$, is an open question. We refer to Chapter 5 for a more detailed discussion. □

Remark 4.1.8. Since all fronts for $c > c_{\min}(\mu)$ are stable, a natural question is which speed is chosen given some initial data. A closer analysis of the weights yields that in order to obtain stability of a front with some speed $c > c_{\min}(\mu)$ we have to require that the perturbation decays faster than the front for $\xi \to \infty$. Therefore, a criterion for the selected speed is the decay rate of the initial data for $\xi \to \infty$. □

We point out that the above Theorem 4.1.6 establishes nonlinear stability only for γ close to zero. Since the existence of invading fronts, using perturbation theory as in Section 2.4, requires a similar restriction it is reasonable to make this assumption. However, there is numerical evidence that the invading fronts exist even if γ is not close to zero, see Section 2.4 and Figure 2.4. Assuming the existence of fronts in (4.1), we find that the restriction hinges purely on the spectral stability of the weighted operator, see (A2) for the precise formulation. In particular, the restriction to γ close to zero is not required in the nonlinear analysis. Therefore, additionally assuming spectral stability, we obtain the nonlinear stability also for large $|\gamma|$, which we summarize in the following result.

Theorem 4.1.9. *Fix* $m \in \mathbb{N}$ *and let* $\mu > 0$*,* $\gamma > -\mu$ *and* $c > 0$*. Additionally, assume the following.*

(A1) The system (4.1) *possesses real invading front solutions* $(A_{\mathit{front}}, B_{\mathit{front}})(\xi)$ *with* $\xi = x - ct$*, which satisfy*

$$\lim_{\xi\to-\infty}(A_{\mathit{front}}, B_{\mathit{front}})(\xi) = (1,0) \text{ and } \lim_{\xi\to+\infty}(A_{\mathit{front}}, B_{\mathit{front}}) = (0,0).$$

(A2) There exists weights $\beta, \beta_\psi > 0$ *such that the operator* L_β *is a densely defined, closed, sectorial operator in* L^2 *and there exists a* $\kappa > 0$ *such that* $\mathrm{Re}(\sigma(L_\beta)) \leq -\kappa$*, where* $\sigma(L_\beta)$ *denotes the* L^2*-spectrum of* L_β*.*

Then, there exists a $\varepsilon > 0$ *and* $\bar{\kappa} > 0$ *such that if*

$$\begin{aligned}
&\|(r_A, r_B, r_\psi)|_{t=0}\|_{(W^{1,1}\cap H^{m+1})\times(L^1\cap H^m)\times(L^1\cap H^m)} < \varepsilon,\\
&\left\|(r_A, r_B)|_{t=0}e^{\beta x}\right\|_{H^{m+1}\times H^m} + \left\|r_\psi|_{t=0}e^{\beta_\psi x}\right\|_{H^m} < \varepsilon,
\end{aligned}$$

it holds that

$$\begin{aligned}
&\|(r_A, r_B, r_\psi)(t)\|_{H^{m+1}\times H^m\times H^m} \lesssim (1+t)^{-1/4},\\
&\left\|(r_A, r_B)(t)e^{\beta(x-ct)}\right\|_{H^{m+1}\times H^m} + \left\|r_\psi(t)e^{\beta_\psi(x-ct)}\right\|_{H^m} \lesssim e^{-\bar{\kappa}t}
\end{aligned}$$

for all $t \geq 0$*.*

Remark 4.1.10. In Theorem 4.1.6 we assume that the front is sufficiently fast, i.e. $c > c_{\min}(\mu)$ and although such an assumption is not explicitly made in Theorem 4.1.9, a similar restriction is likely required to guarantee the spectral stability (A2). In the scalar case, e.g. $\partial_t u = \partial_x^2 u + u - u^2$, stability has also been proven for fronts with minimal velocity, also called critical fronts, for which L_β has essential spectrum touching the imaginary axis, see e.g. [Kir92, Gal94, FH19a]. In particular, the stability for critical fronts in the Ginzburg-Landau equation, $\partial_t A = \partial_x^2 A + A - A|A|^2$, can be obtained using renormalization groups [BK94]. For systems, similar results have only been proven in special cases for which a comparison principle holds such as a Lotka-Volterra competition model [FH19b]. However, if the system lacks a comparison principle, as it is for the system in our setting (see Remark 4.1.11), similar stability questions are still open, see Chapter 5 for a discussion. □

Remark 4.1.11. We note that even in the case of real perturbations, where the Ginzburg-Landau equation degenerates to a KPP equation, the system (4.1) lacks a comparison principle. For a counter example, note that $(0, B)$, $B \in \mathbb{R}$ is an invariant set of (4.1). Thus, for any $\gamma \neq 0$ such that a front $(A_{\text{front}}, B_{\text{front}})$ with some speed $c > 2$ exists, take initial data $(0, B_0)$ such that $|B_{\text{front}}| \geq |B_0|$. The solution to this initial data is explicitly given as a solution of the heat equation $\partial_t B = \mu \partial_x^2 B$. Hence, B decays polynomially to zero as time tends to infinity. However, since the front moves with speed $c > 2$ and $B_{\text{front}}(\xi)$ tends to zero exponentially fast as $\xi \to -\infty$, see Figure 4.1, there exists $(t_0, x_0) \in \mathbb{R}^2$ such that $|B_{\text{front}}(x_0 - ct_0)| \leq |B(t_0, x_0)|$. Thus, the comparison principle cannot hold. $\square$

4.1.2. Outline

The plan for this chapter is as follows. In Section 4.2 we proof the nonlinear stability of the invading state $(1, 0, 0)$ of the system (4.2). Following, we establish nonlinear stability of the front $(A_{\text{front}}, B_{\text{front}}, \psi_{\text{front}})$, starting from a linear stability result Lemma 4.3.1. Section 4.4 then presents the proof of the aforementioned spectral and linear stability result.

4.2. Nonlinear stability of the invading state

We now study the nonlinear stability of the invading state. A perturbation $(r_A, r_B, r_\psi) := (A, B, \psi) - (1, 0, 0)$ satisfies the equation

$$\partial_t r_A = \partial_x^2 r_A - 2r_A + r_B + r_A r_B - 3r_A^2 - r_A^3 - (1 + r_A) r_\psi^2, \tag{4.8a}$$

$$\partial_t r_B = \mu \partial_x^2 r_B + \gamma \partial_x^2 (2r_A + r_A^2), \tag{4.8b}$$

$$\partial_t r_\psi = \partial_x^2 r_\psi + 2\partial_x \left(\frac{(\partial_x r_A) r_\psi}{1 + r_A} \right). \tag{4.8c}$$

Note that this system is ill-posed when considered in $H^m \times H^m \times H^m$ since the nonlinearity contains a second derivative. However, we can use the special structure of the nonlinearity by considering the system in $H^{m+1} \times H^m \times H^m$, see also [Zim14, Gau17]. In particular, in this setting, the system is semilinear and thus locally well-posed by standard methods.

More specifically, in the subsequent analysis we exploit this structure by introducing a dummy variable $r_C := \partial_x r_A$ and use

$$\partial_x^2 (2r_A + r_A^2) = 2\partial_x (r_C + r_A r_C)$$

in order to turn (4.8) into a semilinear system in H^m. Additionally, we replace $\partial_x r_A$ in (4.8c) by r_C. By differentiating (4.8a), r_C satisfies

$$\partial_t r_C = \partial_x^2 r_C - 2\alpha r_C - 2(1-\alpha)\partial_x r_A + \partial_x(r_B + r_A r_B - 3r_A^2 - r_A^3 - (1+r_A)r_\psi). \quad (4.9)$$

We abbreviate the four dimensional system (4.8)–(4.9) as

$$\partial_t r = Lr + \mathcal{N}(r), \quad (4.10)$$

with $r = (r_A, r_C, r_B, r_\psi)^T \in \{r \in \mathbb{R}^4 \,|\, \partial_x r_A = r_C\}$. Here L denotes the linear part and $\mathcal{N}$ the nonlinearities. In Fourier space, the system reads as

$$\begin{aligned}\partial_T \hat{r} = &\begin{pmatrix} -k^2-2 & 0 & 1 & 0 \\ -2(1-\alpha)ik & -k^2-2\alpha & ik & 0 \\ 0 & 2\gamma ik & -\mu k^2 & 0 \\ 0 & 0 & 0 & -k^2 \end{pmatrix} \hat{r} + \begin{pmatrix} \hat{r}_A * \hat{r}_B - 3\hat{r}_A^{*2} - \hat{r}_A^{*3} - (1+\hat{r}_A) * \hat{r}_\psi^{*2} \\ ik(\hat{r}_A * \hat{r}_B - 3\hat{r}_A^{*2} - \hat{r}^{*3} - (1+\hat{r}_A) * \hat{r}_\psi^{*2}) \\ 2\gamma ik(\hat{r}_A * \hat{r}_C) \\ 2ik\mathcal{F}(r_C r_\psi (1+r_A)^{-1}) \end{pmatrix} \\ &=: \hat{L}\hat{r} + \hat{\mathcal{N}}(\hat{r}).\end{aligned}$$

Here $\hat{(\cdot)}$ and $\mathcal{F}(\cdot)$ denote Fourier transform and $f * g$ denotes the convolution of the functions f and g. Additionally f^{*q} denotes q-times convolution of f with itself. Finally, we note that the $\hat{r}_\psi$-equation is linearly decoupled from the rest and thus, we write

$$\hat{L} = \begin{pmatrix} \hat{\tilde{L}} & 0 \\ 0 & -k^2 \end{pmatrix}.$$

4.2.1. Linear stability analysis

We begin the analysis by studying the linearized system. Therefore, we calculate the spectrum of L. Since one eigenvalue of $\hat{L}$ is trivially given by $\lambda_\psi(k) = -k^2$, we focus on the spectrum of $\tilde{L}$, which is given by

$$\begin{aligned}\sigma(\tilde{L}) = \bigcup_{k\in\mathbb{R}} \Bigg\{ &-\frac{1}{2}(\mu+1)k^2 - \frac{1}{2}\sqrt{(\mu-1)^2k^4 - 4(2\gamma+\mu-1)k^2+4} - 1, -k^2-2\alpha, \\ &\frac{1}{2}\left(-(\mu+1)k^2 + \sqrt{(\mu-1)^2k^4 - 4(2\gamma+\mu-1)k^2+4} - 2\right)\Bigg\}\end{aligned}$$

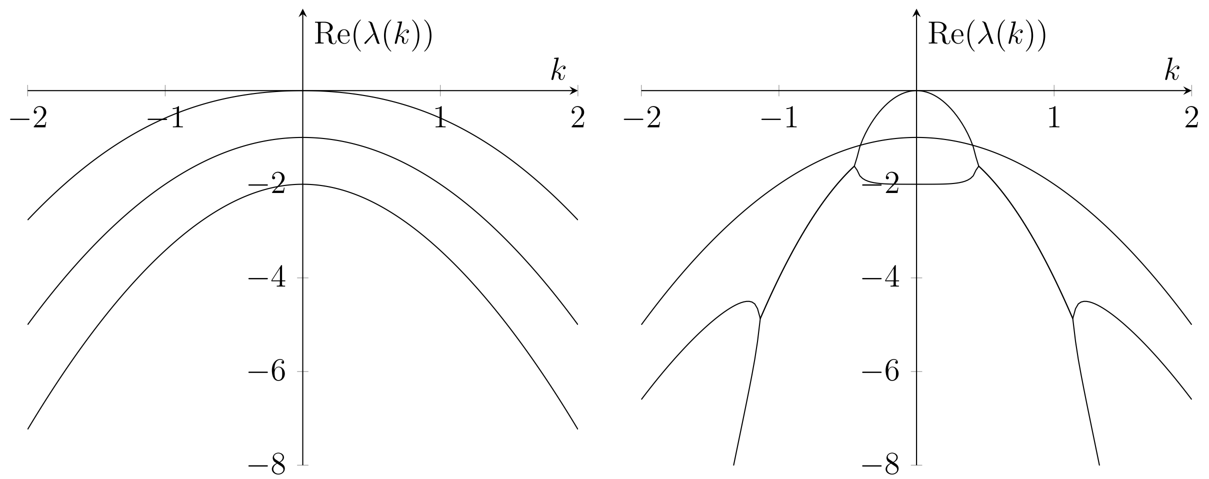

Figure 4.4.: Eigenvalue curves of $\tilde{L}$ for $\alpha = 1/2$, $\gamma = -1/2$, $\mu = 1$ (left) and $\alpha = 1/2$, $\gamma = 1$, $\mu = 5$ (right), which are parameterized by Fourier wave number k.

$$= \bigcup_{k\in\mathbb{R}} \{\lambda_A(k), \lambda_C(k), \lambda_B(k)\}.$$

Note that for $\gamma > -\mu$ there exists a $\theta_j > 0, j \in \{A, C, B\}$ such that

$$\begin{aligned} \mathrm{Re}(\lambda_A(k)) &\leq -\theta_A k^2 - 1, \\ \mathrm{Re}(\lambda_C(k)) &\leq -\theta_C k^2 - 2\alpha, \\ \mathrm{Re}(\lambda_B(k)) &\leq -\theta_B k^2 \end{aligned}$$

see Figure 4.4. Thus, the system has two eigenspaces which correspond to exponential decay in time and two which correspond to polynomial decay in time. To extract this behavior we diagonalize the system in Fourier space using the following result.

Lemma 4.2.1. *Let $\mu > 0$, $\gamma > -\mu$ and $\alpha \in (0, 1)$. Then there exists a $k_0 > 0$ such that the eigenvalue curves of $\hat{\tilde{L}}(k)$ do not intersect for any $k \in (-k_0, k_0)$ and thus, $\hat{\tilde{L}}(k)$ is diagonizable via an invertible transformation $\hat{\tilde{S}}(k)$, i.e.*

$$\hat{\tilde{\Lambda}}(k) := \mathrm{diag}(\lambda_A(k), \lambda_C(k), \lambda_B(k)) = \hat{\tilde{S}}(k)^{-1}\hat{\tilde{L}}(k)\hat{\tilde{S}}(k)$$

for all $k \in (-k_0, k_0)$. Furthermore, the transformation can be chosen such that, locally *about $k = 0$, the following element-wise estimates hold*

$$|\hat{\hat{S}}(k)| \lesssim \begin{pmatrix} 1 & |k| & 1 \\ |k| & 1 & |k| \\ k^2 & |k| & 1 \end{pmatrix} \quad \text{and} \quad |\hat{\hat{S}}^{-1}(k)| \lesssim \begin{pmatrix} 1 & |k| & 1 \\ |k| & 1 & 0 \\ k^2 & |k| & 1 \end{pmatrix}.$$

Here, both the inequality as well as the absolute value are defined by element-wise application.

Proof. Using the given formulas for λ_j, $j \in \{A, C, B\}$ it is straightforward to check that there is no intersection at $k = 0$. Thus, by continuity of eigenvalues there exists a $k_0 > 0$ such that $\hat{L}(k)$ is diagonizable for all $k \in (-k_0, k_0)$. Finally, we verify the estimates by explicitly calculating the eigenvectors and noting that these are only unique up to normalization. □

Remark 4.2.2. Note that the parameter $\alpha \in (0, 1)$ separates the eigenvalue curves λ_A and λ_C. □

Motivated by the above result, we introduce a smooth and symmetric cut-off function $\chi_{k_0} : \mathbb{R} \to [0, 1]$ such that $\chi_{k_0}(k) = 1$ if $k \in (-k_0/2, k_0/2)$ and $\chi_{k_0}(k) = 0$ if $k \notin (-k_0, k_0)$. Then, we define

$$\hat{S}(k) := \chi_{k_0} \begin{pmatrix} \hat{\hat{S}}(k) & 0 \\ 0 & 1 \end{pmatrix} + (1 - \chi_{k_0})I, \tag{4.11}$$

where I is the 4×4 identity matrix. Furthermore, since $\hat{\hat{S}}(k = 0)$ is an upper triangular matrix and the diagonal elements can be chosen arbitrarily by a different normalization of the eigenvectors, there is a choice such that $\zeta\hat{\hat{S}}(k = 0) + (1 - \zeta)I$ is invertible for all $\zeta \in [0, 1]$. Hence, $\hat{S}(k)$ is invertible for all $k \in \mathbb{R}$ if $|k_0|$ is chosen small enough. Therefore, we define $\hat{S}^{-1}(k)$ as the inverse of $\hat{S}(k)$ for every $k \in \mathbb{R}$. Since for k close to zero, $\chi_{k_0} = 1$, this transformation diagonalizes $\hat{L}(k)$ locally around $k = 0$. Thus, it extracts the different linear behavior from the system. Next, we introduce $\hat{p} = (\hat{p}_A, \hat{p}_C, \hat{p}_B, \hat{p}_\psi) := \hat{S}^{-1}\hat{r}$, which then satisfies $\partial_T \hat{p} = \hat{\Lambda}\hat{p} + \hat{S}^{-1}\hat{\mathcal{N}}(\hat{S}\hat{p})$, with $\hat{\Lambda} := \hat{S}^{-1}\hat{L}\hat{S}$. Furthermore, we expect that $\hat{p}_A$ and $\hat{p}_C$ decay exponentially fast on the linear level, while $\hat{p}_B$ and $\hat{p}_\psi$ show polynomial decay, since $\lambda_B(k = 0) = \lambda_\psi(k = 0) = 0$.

Remark 4.2.3. Note that $p_\psi = r_\psi$ and in particular, p_ψ is does not depend on r_A, r_C and

r_B via the diagonalisation. Vice versa, p_A, p_C, p_B are determined by a linear combination of r_A, r_C, r_B. $\square$

4.2.2. Function spaces and semigroup estimates

Starting from the operators $\hat{\Lambda}, \hat{S}$ and $\hat{S}^{-1}$, which are defined in Fourier space, we define the corresponding operators in the physical space as Fourier multipliers, e.g. $S = \mathcal{F}^{-1}\hat{S}\mathcal{F}$, where $\mathcal{F}$ denotes the Fourier transform. Therefore, the Sobolev spaces H^m (see e.g. [SU17]) are a natural setting since H^m is isomorphic to $\widehat{H_m}$ via Fourier transform. Here $\widehat{H_m}$ is a weighted L^2 space defined by

$$\widehat{H_m} := \left\{ u \in L^2 : \|\hat{\rho}(k)^m u(k)\|_{L^2(k)} < \infty \right\},$$

where $\hat{\rho}(k) := (1 + k^2)^{1/2}$, see [SU17]. Furthermore, we note that for $m > 1/2$, we have

$$\begin{aligned} \|fg\|_{H^m} &\lesssim \|f\|_{H^m} \|g\|_{H^m}, \\ \left\|\hat{f} * \hat{g}\right\|_{\widehat{H_m}} &\lesssim \left\|\hat{f}\right\|_{\widehat{H_m}} \|\hat{g}\|_{\widehat{H_m}}, \end{aligned}$$

that is, H^m and $\widehat{H_m}$ are Banach algebras with respect to pointwise multiplication and convolution, respectively.

Now, let P_j, $j \in \{A, C, B, \psi\}$ be the projection on the j-th component. Then, we can prove the following linear stability estimates.

Corollary 4.2.4. *Let $\mu > 0$, $\gamma > -\mu$, $m \in \mathbb{N}_0$ and $\alpha \in (0, 1)$. Furthermore, let*

$$|\hat{\rho}_{n_1,n_2}(k)| \lesssim |k|^{n_1} (|k|/\sqrt{1 + k^2})^{n_2}$$

for $n_1, n_2 \in \mathbb{N}_0$. Then, there exists a $\theta > 0$ such that for any $p = (p_A, p_C, p_B, p_\psi) \in H^m$ holds that

$$\begin{aligned} \left\|P_A e^{t\Lambda} \rho_{n_1,n_2} p\right\|_{H^m} &\lesssim t^{-n_1/2} e^{-t} \|p_A\|_{H^m} + t^{-n_1/2} e^{-\theta t} \|(p_A, p_C, p_B)\|_{H^m}, \\ \left\|P_C e^{t\Lambda} \rho_{n_1,n_2} p\right\|_{H^m} &\lesssim t^{-n_1/2} e^{-2\alpha t} \|p_C\|_{H^m} + t^{-n_1/2} e^{-\theta t} \|(p_A, p_C, p_B)\|_{H^m}, \\ \left\|P_B e^{t\Lambda} p\right\|_{H^m} &\lesssim (1 + t)^{-1/4} \|p_B\|_{L^1 \cap H^m} + e^{-\theta t} \|(p_A, p_C, p_B)\|_{H^m}, \\ \left\|P_B e^{t\Lambda} \rho_{n_1,n_2} p\right\|_{H^m} &\lesssim (1 + t)^{-n_2/2} t^{-n_1/2} \|p_B\|_{H^m} + t^{-n_1/2} e^{-\theta t} \|(p_A, p_C, p_B)\|_{H^m}, \\ \left\|P_\psi e^{t\Lambda} p\right\|_{H^m} &\lesssim (1 + t)^{-1/4} \|p_\psi\|_{L^1 \cap H^m}, \end{aligned}$$

$$\left\| P_\psi e^{t\Lambda} \rho_{n_1,n_2} p \right\|_{H^m} \lesssim (1+t)^{-n_2/2} t^{-n_1/2} \left\| p_\psi \right\|_{H^m}.$$

Proof. We use the isomorphism of H^m and $\widehat{H_m}$ and the construction of the transformation S, see (4.11). Let $j \in \{A, C, B\}$. Then for some $\theta > 0$ holds

$$\left\| P_j e^{t\Lambda} p \right\|_{H^m} \lesssim \left\| P_j e^{t\hat{\Lambda}} \hat{p} \right\|_{\widehat{H_m}} \lesssim \left\| \chi_{k_0/2} e^{\lambda_j t} \hat{p}_j \right\|_{\widehat{H_m}} + e^{-\theta t} \left\| (\hat{p}_A, \hat{p}_C, \hat{p}_B) \right\|_{\widehat{H_m}},$$

since all eigenvalues of $\hat{\Lambda}(k)$ have strictly negative real part and are uniformly bounded away from the imaginary axis for $k \notin (-k_0/4, k_0/4)$. Furthermore, the first term separates the different behavior of the eigenvalues about $k = 0$. Thus, for $j = A, C$ we obtain the exponential bounds as states in the corollary by noting that $\mathrm{Re}(\lambda_A) \leq -1$ and $\mathrm{Re}(\lambda_C) \leq -2\alpha$, respectively. Finally, for $j = B$ we can calculate

$$\begin{aligned} \left\| \chi_{k_0/2} e^{\lambda_B t} \hat{p}_B \right\|_{\widehat{H_m}} &\lesssim \left\| e^{-\theta_B k^2 t} \hat{p}_B \right\|_{\widehat{H_m}} \lesssim \min \left(\left\| e^{-\theta_B k^2 t} \right\|_{L^\infty} \left\| \hat{p}_B \right\|_{\widehat{H_m}}, \left\| e^{-\theta_B k^2 t} \right\|_{\widehat{H_m}} \left\| \hat{p}_B \right\|_{L^\infty} \right) \\ &\lesssim (1+t)^{-1/4} \left\| p_B \right\|_{L^1 \cap H^m}. \end{aligned}$$

The estimates in case that $n_1, n_2 \geq 1$ can be obtain in a similar manner, using the behavior of λ_j locally about $k = 0$ and that the eigenvalues behave like $-k^2$ for $k \to \pm\infty$. Finally, since $P_\psi e^{t\Lambda} p = e^{t\partial_x^2} p_\psi$, the remaining estimates follow with the same techniques. □

4.2.3. Nonlinear stability

We now show nonlinear stability of the invading state by studying the temporal decay of the transformed perturbation $p := S^{-1} r$ satisfying the equation

$$\partial_t p = \Lambda p + S^{-1} \mathcal{N}(Sp). \tag{4.12}$$

That is, we prove the following result.

Lemma 4.2.5. *Let $\mu > 0$, $\gamma > -\mu$ and $m > 1/2$. There exists a $\varepsilon > 0$ such that for all solutions p of* (4.12) *with*

$$\left\| (p_A, p_C, p_B, p_\psi)|_{t=0} \right\|_{H^m \times H^m \times (L^1 \cap H^m) \times (L^1 \cap H^m)} < \varepsilon$$

it holds that

$$\|p_A(t)\|_{H^m}, \|p_C(t)\|_{H^m} \lesssim (1+t)^{-1/2} \text{ and } \|p_B(t)\|_{H^m}, \|p_\psi(t)\|_{H^m} \lesssim (1+t)^{-1/4}. \quad (4.13)$$

From this result we can deduce the nonlinear stability of the nontrivial fixed point of (4.8), see Theorem 4.1.1, which we restate here.

Theorem 4.2.6. *Let $\mu > 0$, $\gamma > -\mu$ and $m > 1/2$. There exists a $\varepsilon > 0$ such that for all perturbations (r_A, r_B, r_ψ) of the invading state $(1, 0, 0)$ satisfying*

$$\|(r_A, r_B, r_\psi)|_{t=0}\|_{(W^{1,1}\cap H^{m+1})\times(L^1\cap H^m)\times(L^1\cap H^m)} < \varepsilon$$

holds that

$$\|(r_A, r_B, r_\psi)(t)\|_{H^{m+1}\times H^m\times H^m} \lesssim (1+t)^{-1/4}$$

for all $t \geq 0$.

Proof. We define $r_0 = (r_A, \partial_x r_A, r_B, r_\psi)|_{t=0}$ and $p_0 := S^{-1} r_0$. Furthermore, we use that the Fourier transform is an isomorphism between H^m and $\widehat{H_m}$ and obtain

$$\|p_0\|_{H^m} \lesssim \|\hat{p}_0\|_{\widehat{H_m}} \lesssim \left\|\hat{S}^{-1}\hat{r}_0\right\|_{\widehat{H_m}} \lesssim \|\hat{r}_0\|_{\widehat{H_m}} \lesssim \|r_0\|_{H^m} \lesssim \|(r_A, r_B, r_\psi)|_{t=0}\|_{H^{m+1}\times H^m\times H^m},$$

with constants independent of r_0. To bound the L^1-norm we use that $\|S^{-1}f\|_{L^1} \lesssim \|f\|_{L^1}$ for any $f \in L^1$. To prove this, we first note that any compactly supported C^2-function g defines a L^1-Fourier multiplier; it holds that

$$\begin{aligned}\left\|\mathcal{F}^{-1}(g\hat{f})\right\|_{L^1} &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} g(k) e^{-ik(y-x)}\, dk \right) dy \right| dx \\ &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(y)|\, |\hat{g}(y-x)|\, dy\, dx \leq \|f\|_{L^1} \|\hat{g}\|_{L^1}\end{aligned}$$

and since g is C^2 and compactly supported, its Fourier transform $\hat{g}$ is well-defined and an element of L^1. Then, we note that $\hat{S}$ can be written as a summation of a compactly supported and a constant, diagonal matrix and hence, its (point-wise in Fourier space) inverse $\hat{S}^{-1}$ has the same decomposition. Therefore, the conjectured L^1-estimate holds and we have

$$\|p_0\|_{L^1} = \left\|S^{-1} r_0\right\|_{L^1} \lesssim \|r_0\|_{L^1}.$$

Hence, for ε sufficiently small the conditions of Lemma 4.2.5 are satisfied. Furthermore, we obtain with $r_C := \partial_x r_A$ that

$$\begin{aligned}\|(r_A(t), r_C(t), r_B(t), r_\psi(t))\|_{H^m} &\lesssim \left\|\hat{S}(\hat{p}_A(t), \hat{p}_C(t), \hat{p}_B(t), \hat{p}_\psi(t))\right\|_{\widehat{H_m}} \\ &\lesssim \|(p_A(t), p_C(t), p_B(t), p_\psi(t))\|_{H^m}\end{aligned}$$

where the latter term decays to zero like $(1+t)^{-1/4}$. In particular, since we control r_A as well as $r_C = \partial_x r_A$ in H^m, we obtain decay of r_A in H^{m+1} which proves the statement. □

Remark 4.2.7. Since H^m is continuously embedded into L^∞ for $m > 1/2$ by Sobolev embedding, we obtain from Theorem 4.2.6 that the invading state is also $((W^{1,1} \cap H^{m+1}) \times (L^1 \cap H^m) \times (L^1 \cap H^m), L^\infty)$ asymptotically stable. However, we expect that the perturbation exhibits faster decay in L^∞, similar to solutions of the heat equation. □

Remark 4.2.8. We highlight that the stability result requires that $r_A|_{t=0} \in W^{1,1} \cap H^{m+1}$ and $r_B|_{t=0} \in L^1 \cap H^m$, i.e., $r_A|_{t=0}$ is of higher regularity than $r_B|_{t=0}$. As discussed above this meshes well with the results on existence and uniqueness for the modified Ginzburg-Landau system (4.1) in [Gau17, Zim14]. □

We now outline the plan to prove Lemma 4.2.5. Recall that $\hat{L}(k)$ has four eigenvalue curves $\lambda_A(k)$, $\lambda_C(k)$, $\lambda_B(k)$ and $\lambda_\psi(k)$. Furthermore, recall that only the spectral curves λ_B and λ_ψ touch the imaginary axis, while both λ_A and λ_C are bounded away from the imaginary axis. Therefore, we expect that the main challenge in the stability proof comes from the stability of p_B and p_ψ, the linearly polynomially decaying variables. However, a closer inspection of the nonlinearity in (4.12) reveals that

$$P_B \hat{S}^{-1} \hat{\mathcal{N}}(\hat{S}\hat{p}) = \mathcal{O}(|k|) \text{ and } P_\psi \hat{S}^{-1} \hat{\mathcal{N}}(\hat{S}\hat{p}) = \mathcal{O}(|k|) \text{ for } k \to 0,$$

where we used Lemma 4.2.1. Since $k = 0$ is the Fourier mode with no decay – the eigenvalue curves touch the imaginary axis for $k = 0$ – the influence of the critical modes in a neighborhood of $k = 0$ is small and thus, stability is more likely. This can also be observed in the example $\partial_t u = \partial_x^2 u - \partial_x^n(u^2)$, where the origin is unstable for $n = 0$ but stable for $n = 1$ (see e.g. [Wei81, SU17]).

Next, we recall that a solution of (4.12) is given by the Duhamel formula

$$p(t) = e^{t\Lambda}p(t=0) + \int_0^t e^{(t-s)\Lambda}S^{-1}\mathcal{N}(Sp(s))\,ds.$$

Our goal is to show that

$$\eta_A(t) := \sup_{0\le s\le t}(1+s)^{1/2}\left\|p_A(s)\right\|_{H^m}, \tag{4.14a}$$

$$\eta_C(t) := \sup_{0\le s\le t}(1+s)^{1/2}\left\|p_C(s)\right\|_{H^m}, \tag{4.14b}$$

$$\eta_B(t) := \sup_{0\le s\le t}(1+s)^{1/4}\left\|p_B(s)\right\|_{H^m}, \tag{4.14c}$$

$$\widetilde{\eta_B}(t) = \sup_{0\le s\le t}(1+s)^{1/2}\left\|S_{CB}p_B(s)\right\|_{H^m}, \tag{4.14d}$$

$$\eta_\psi(t) := \sup_{0\le s\le t}(1+s)^{1/4}\left\|p_\psi(s)\right\|_{H^m}, \tag{4.14e}$$

are bounded uniformly for all time, which in turn proves Lemma 4.2.5. To be precise, we prove that $E(t) := \eta_A(t) + \eta_C(t) + \eta_B(t) + \widetilde{\eta_B}(t) + \eta_\psi(t)$ is uniformly bounded, provided the initial data is small in a suitable function space. The main result to show this is the following estimate.

Lemma 4.2.9. *Let $\mu > 0$, $\gamma > -\mu$ and $m > 1/2$. Then, if*

$$\|(p_A, p_C, p_B, p_\psi)|_{t=0}\|_{H^m\times H^m\times(L^1\cap H^m)\times(L^1\cap H^m)} < \varepsilon$$

for a small $\varepsilon > 0$, there exists a constant $K < \infty$ independent of ε such that

$$E(t) \le K(\varepsilon + E(t)^2) \tag{4.15}$$

for all $t \ge 0$ such that the solution $(p_A, p_C, p_B, p_\psi)(t)$ of (4.12) *exists in H^m and is sufficiently small.*

Proof. First, we note that the integral terms containing an exponentially decaying function in time are harmless, since the nonlinearity contains at most one derivative, the transformation $\hat{S}^{-1}$ is uniformly bounded from $\widehat{H_m}$ to $\widehat{H_m}$ and the eigenvalues of $\hat{L}$ decay like $-k^2$ for $k \to \pm\infty$. Furthermore, we have that the estimate

$$\|p_A(s)\|_{H^m} + \|p_C(s)\|_{H^m} + \|p_B(s)\|_{H^m} \lesssim (1+s)^{-1/4}E(t)$$

holds for any $0 \leq s \leq t$. Therefore, since the nonlinearity is at least quadratic in p, we can establish estimates on η_A, η_C using Corollary 4.2.4, the fact that the problem is semilinear and thus, all appearing integrals are finite and that

$$\int_0^t e^{-\kappa(t-s)}(t-s)^{-n_1}(1+t-s)^{-n_2}(1+s)^{-1/2}\, ds \lesssim (1+t)^{-1/2}$$

for all $n_1 = 0, 1$ and $n_2 = 0, 1, 2$ and $\kappa > 0$.

To estimate p_ψ we note that since $m > 1/2$, H^m is continuously embedded into C_b and thus, since the H^m-norm of p is small, we can bound $1 + (Sp)_A$ away from zero uniformly in space. Using Corollary 4.2.4, this yields

$$\begin{aligned}
\|p_\psi(t)\|_{H^m} &\lesssim (1+t)^{-1/4} \|p_\psi|_{t=0}\|_{L^1\cap H^m} \\
&\quad + \int_0^t \left\| e^{-k^2(t-s)} k \mathcal{F}((Sp)_C(s) p_\psi(s)(1 + (Sp)_A(s))^{-1}) \right\|_{\widehat{H_m}} ds \\
&\lesssim (1+t)^{-1/4} \|p_\psi|_{t=0}\|_{L^1\cap H^m} \\
&\quad + \int_0^t (t-s)^{-1/2} \|p_\psi(s)\|_{H^m} (\|p_A(s)\|_{H^m} + \|p_C(s)\|_{H^m} + \|S_{CB} p_B(s)\|_{H^m})\, ds \\
&\lesssim (1+t)^{-1/4} (\|p_\psi|_{t=0}\|_{L^1\cap H^m} + E(t)^2),
\end{aligned}$$

as long as $p(t)$ is small in H^m. Here, we used that $S_{CB} p_B$ shows better decay rates than p_B, i.e. $(1+t)^{-1/2}$ compared to $(1+t)^{-1/4}$. Note that this improved decay rate originates from the fact that $\hat{S}_{CB}(k) = \mathcal{O}(|k|)$ locally about $k = 0$.

Finally, we estimate p_B and $S_{CB} p_B$. Using Lemma 4.2.1 and Corollary 4.2.4 we obtain for p_B that

$$\begin{aligned}
\|p_B(t)\|_{H^m} \lesssim (1+t)^{-1/4} \|p_B|_{t=0}\|_{L^1\cap H^m} + e^{-\theta t} \|(p_A, p_C, p_B)|_{t=0}\|_{H^m} \\
+ \int_0^t \left\| P_B e^{(t-s)\Lambda} S^{-1} \mathcal{N}(Sp(s)) \right\|_{H^m} ds.
\end{aligned}$$

To estimate the integral contribution we need the following estimate,

$$|(\hat{S}_{AB} \hat{p}_B * \hat{S}_{CB} \hat{p}_B)(k)| \lesssim |k| |\hat{p}_B|^{*2}(k) \tag{4.16}$$

for all $k \in (-k_0/2, k_0/2)$. To prove this, we note that $\hat{\tilde{S}}_{CB}(-k) = -\hat{\tilde{S}}_{CB}(k)$ and $\hat{\tilde{S}}_{AB}(-k) = \hat{\tilde{S}}_{AB}(k)$ by explicit calculation of the eigenvectors and appropriate normalization. Since χ_{k_0} is a symmetric function, $\hat{S}_{CB}$ and $\hat{S}_{AB}$, respectively, have the same property. Therefore, it holds that $(\hat{S}_{AB}\hat{p}_B * \hat{S}_{CB}\hat{p}_B)(k = 0) = 0$. Thus, for k in a neighborhood of zero, we use that

$$\begin{aligned}
&\sup_{\omega\in\mathbb{R}} |\hat{S}_{CB}(k-\omega) - \hat{S}_{CB}(-\omega)| \leq C|k|, \\
&\sup_{\omega\in\mathbb{R}} |\hat{S}_{AB}(k-\omega) - \hat{S}_{AB}(-\omega)| \leq C|k|
\end{aligned}$$

for $|k| \ll 1$, which can be checked using the definition of $\hat{S}$ given in (4.11). Using this, we obtain

$$\begin{aligned}
(\hat{S}_{AB}\hat{p}_B * \hat{S}_{CB}\hat{p}_B)(k) &= \int_{\mathbb{R}} \hat{S}_{AB}(k-\omega)\hat{p}_B(k-\omega)\hat{S}_{CB}(\omega)\hat{p}_B(\omega)\, d\omega \\
&= \int_{\mathbb{R}} \hat{S}_{AB}(\omega)\hat{p}_B(k-\omega)\hat{S}_{CB}(\omega)\hat{p}_B(\omega)\, d\omega \\
&\qquad + \int_{\mathbb{R}} (\hat{S}_{AB}(k-\omega) - \hat{S}_{AB}(-\omega))\hat{p}_B(k-\omega)\hat{S}_{CB}(\omega)\hat{p}_B(\omega)\, d\omega
\end{aligned}$$

and similarly using $f * g = g * f$,

$$\begin{aligned}
(\hat{S}_{AB}\hat{p}_B * \hat{S}_{CB}\hat{p}_B)(k) = &- \int_{\mathbb{R}} \hat{S}_{CB}(\omega)\hat{p}_B(k-\omega)\hat{S}_{AB}(\omega)\hat{p}_B(\omega)\, d\omega \\
&+ \int_{\mathbb{R}} (\hat{S}_{CB}(k-\omega) - \hat{S}_{CB}(-\omega))\hat{p}_B(k-\omega)\hat{S}_{AB}(\omega)\hat{p}_B(\omega)\, d\omega
\end{aligned}$$

Putting all above identities and estimates together then gives

$$|(\hat{S}_{AB}\hat{p}_B * \hat{S}_{CB}\hat{p}_B)(k)| \leq K|k||\hat{p}_B|^{*2}(k)$$

for $|k| \ll 1$, which proves (4.16).

Then, we use (4.16), the behavior of S^{-1} close to $k = 0$ (see Lemma 4.2.1) and Corollary 4.2.4 to obtain

$$\begin{aligned}
&\int_0^t \left\| P_B e^{(t-s)\Lambda} S^{-1} \mathcal{N}(Sp(s)) \right\|_{H^m} ds \\
&\quad \lesssim \int_0^t e^{-\theta(t-s)} \max(1, (t-s)^{-1/2}) \left\|(p_A, p_C, p_B)(s)\right\|_{H^m}^2 ds \\
&\qquad + \int_0^t ((1+t-s)^{-1} + (1+t-s)^{-1/2}(t-s)^{-1/2}) \left\|(p_A, p_C, p_B, p_\psi)(s)\right\|_{H^m}^2 ds \\
&\qquad + 2|\gamma| \int_0^t (t-s)^{-1/2} (\|p_A(s)\|_{H^m} + \|p_C(s)\|_{H^m})^2\, ds \\
&\qquad + 2|\gamma| \int_0^t (t-s)^{-1/2} \left(\|p_B(s)\|_{H^m} \left(\|p_A(s)\|_{H^m} + \|p_C(s)\|_{H^m}\right)\right) ds
\end{aligned}$$

$$
\begin{aligned}
&+ 2|\gamma| \int_0^t (t-s)^{-1/2}(1+t-s)^{-1/2} \, \|p_B(s)\|_{H^m}^2 \, ds \\
&\lesssim (1+t)^{-1/4} E(t)^2,
\end{aligned}
$$

as long as $p(t)$ is small in H^m. Therefore, we have

$$
\eta_B(t) \lesssim \varepsilon + E(t)^2
$$

for all $t \geq 0$ such that $p(t)$ is small in H^m. Similarly, we obtain the estimate for $S_{CB} p_B$ using that $\hat{S}_{CB}(k) = \mathcal{O}(|k|)$ for $k \to 0$, which gives an additional $(1+t-s)^{-1/2}$ using Corollary 4.2.4. This proves the conjectured estimate. □

Proof of Lemma 4.2.5. We obtain the short-time existence of solutions of (4.12) using standard local existence and uniqueness theory since the system is semilinear. Therefore, the solution exists as long as (p_A, p_C, p_B, p_ψ) remain small in H^m. Then, the desired stability result follows from Lemma 4.2.9 by continuous induction provided that $\varepsilon > 0$ is sufficiently small, e.g. following [JZ11b], which we demonstrate below.

We write the energy estimate (4.15) equivalently as

$$
0 \leq K(\varepsilon + E(t)^2) - E(t) =: \mathcal{P}_\varepsilon(E(t)),
$$

where $K < \infty$. Note that $E(t)$ is non-negative and depends continuously on time, since (p_A, p_C, p_B, p_ψ) are continuous in H^m with respect to time. By choosing $\varepsilon > 0$ sufficiently small the polynomial $\mathcal{P}_\varepsilon(x)$ has two zeros for $x \geq 0$. Furthermore, $\mathcal{P}_\varepsilon(0) > 0$ and $\mathcal{P}_\varepsilon(x)$ tends to $+\infty$ as x tends to infinity, see Figure 4.5. Therefore, we find two intervals $[0, x_0], [x_1, +\infty) \subset \mathbb{R}^+$ with $x_0 < x_1$ such that $\mathcal{P}_\varepsilon$ is non-negative on $[0, x_0] \cup [x_1, +\infty)$ and negative on the complement. Here, x_0 is approximately given by $K\varepsilon$ for ε sufficiently small.

Since $\mathcal{P}_\varepsilon$ is required to be non-negative by the energy estimate and $E(t)$ depends continuously on time, this yields that either $E(t) \in [0, x_0]$ or $E(t) \in [x_1, +\infty)$ for all $t \geq 0$ such that the solution exists. Thus, by choosing $\varepsilon > 0$ small enough we find $E(0) \leq x_0$ and thus $E(t) \leq x_0$ for all $t \geq 0$ such that the solution exists. In particular, this gives a uniform bound on the H^m-norm of (p_A, p_C, p_B, p_ψ) for all time of their existence. Thus, by employing continuous induction, this implies that the solution exists in H^m for all $t \geq 0$ and especially that the energy $E(t)$ is uniformly bounded for all $t \geq 0$. This proves the result. □

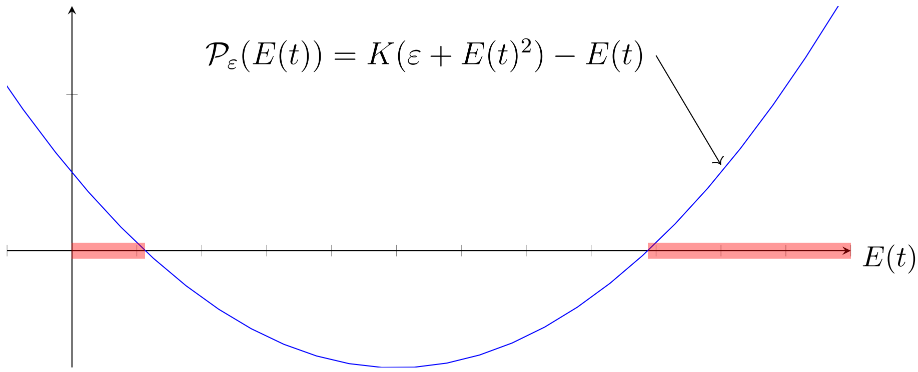

Figure 4.5.: Plot of the polynomial $\mathcal{P}_\varepsilon$ for $\varepsilon > 0$ small. Here, the intervals $[0, x_0]$ and $[x_1, +\infty)$ on which $\mathcal{P}_\varepsilon \geq 0$ are highlighted.

4.3. Nonlinear stability of an invading front

In Theorem 4.2.6, we proved the nonlinear stability of the invading state $(1, 0, 0)$ of the modified Ginzburg-Landau system in polar coordinates (4.2) with respect to localized perturbations. Next, we study the stability of an invading front connecting this state to the origin, i.e. $(A_{\text{front}}, B_{\text{front}}, \psi_{\text{front}})(\xi)$ with $\xi = x - ct$ and

$$\lim_{\xi \to -\infty} (A_{\text{front}}, B_{\text{front}}, \psi_{\text{front}})(\xi) = (1, 0, 0) \text{ and } \lim_{\xi \to +\infty} (A_{\text{front}}, B_{\text{front}}, \psi_{\text{front}})(\xi) = (0, 0, 0).$$

As discussed in the introduction, such a front exists for $c \geq 2$ and γ close to zero. As highlighted above, this front connects a diffusively stable state, the invading state, to an unstable one, the origin.

We now follow the strategy outlined in the introduction. A perturbation (r_A, r_B, r_ψ) of the front $(A_{\text{front}}, B_{\text{front}}, \psi_{\text{front}})$ satisfies

$$\begin{aligned}
\partial_t r_A &= \partial_x^2 r_A + r_A + r_A B_{\text{front}} + r_B A_{\text{front}} + r_A r_B - 3 r_A A_{\text{front}}^2 - 3 r_A^2 A_{\text{front}} - r_A^3 \\
&\qquad - (A_{\text{front}} + r_A) r_\psi^2, \\
\partial_t r_B &= \mu \partial_x^2 r_B + \gamma \partial_x^2 (2 r_A A_{\text{front}} + r_A^2), \\
\partial_t r_\psi &= \partial_x^2 r_\psi + 2 \partial_x \left(\frac{(\partial_x A_{\text{front}} + \partial_x r_A) r_\psi}{A_{\text{front}} + r_A} \right).
\end{aligned}$$

Again, we introduce the additional variable $r_C = \partial_x r_A$ and $C_{\text{front}} = \partial_\xi A_{\text{front}}$. Then, the extended system with the unknown $r = (r_A, r_C, r_B, r_\psi)^T$ is given by

$$\begin{aligned}
\partial_t r_A &= \partial_x^2 r_A + r_A + r_A B_{\text{front}} + r_B A_{\text{front}} + r_A r_B - 3r_A A_{\text{front}}^2 - 3r_A^2 A_{\text{front}} - r_A^3 \\
&\quad - (A_{\text{front}} + r_A) r_\psi^2, \\
\partial_t r_C &= \partial_x^2 r_C + r_C + \partial_x (r_A B_{\text{front}} + r_B A_{\text{front}} + r_A r_B - 3r_A A_{\text{front}}^2 - 3r_A^2 A_{\text{front}} - r_A^3) \\
&\quad - \partial_x ((A_{\text{front}} + r_A) r_\psi^2), \\
\partial_t r_B &= \mu \partial_x^2 r_B + 2\gamma \partial_x (r_C A_{\text{front}} + r_A C_{\text{front}} + r_A r_C) \\
\partial_t r_\psi &= \partial_x^2 r_\psi + 2\partial_x \left(\frac{(\partial_x A_{\text{front}} + r_C) r_\psi}{A_{\text{front}} + r_A} \right).
\end{aligned}$$

We write the system as

$$\partial_t r = Lr + \mathcal{N}_1(r) + \mathcal{N}_2(r), \tag{4.17}$$

where L is the same linear operator defined on $\{r = (r_A, r_C, r_B, r_\psi)^T \in \mathbb{R}^4 \,|\, \partial_x r_A = r_C\}$ as in the stability of the invading state in system (4.10) and $\mathcal{N}_1(r) = L_{\text{front}} r - Lr$, where L_{front} is the linearisation about the front. Finally, $\mathcal{N}_2$ contains the remaining nonlinearities. Explicitly, $\mathcal{N}_1$ and $\mathcal{N}_2$ read as

$$\mathcal{N}_1(r) = \begin{pmatrix} r_A B_{\text{front}} + r_B (A_{\text{front}} - 1) - 3r_A (A_{\text{front}}^2 - 1) \\ \partial_x (r_A B_{\text{front}} + r_B (A_{\text{front}} - 1) - 3r_A (A_{\text{front}}^2 - 1)) \\ 2\gamma \partial_x (r_C (A_{\text{front}} - 1) + r_A C_{\text{front}}) \\ 2\partial_x \left(\dfrac{\partial_\xi A_{\text{front}}}{A_{\text{front}}} r_\psi \right) \end{pmatrix},$$

$$\mathcal{N}_2(r) = \begin{pmatrix} r_A r_B - 3r_A^2 A_{\text{front}} - r_A^3 - (A_{\text{front}} + r_A) r_\psi^2 \\ \partial_x (r_A r_B - 3r_A^2 A_{\text{front}} - r_A^3 - (A_{\text{front}} + r_A) r_\psi^2) \\ 2\gamma \partial_x (r_A r_C) \\ 2\partial_x \left(\dfrac{\partial_\xi A_{\text{front}}}{A_{\text{front}}} \dfrac{r_A}{A_{\text{front}} + r_A} r_\psi + \dfrac{r_C r_\psi}{A_{\text{front}} + r_A} \right) \end{pmatrix}.$$

Similar to [ES02], it turns out that $\mathcal{N}_1$, which is linear in r, can be treated as an additional nonlinearity in appropriately weighted spaces.

4.3.1. Weighted operator and linear estimates

As discussed in the introduction, we introduce a weighted variable $w_j(\xi,t) = e^{\beta\xi} r_j(\xi + ct, t)$ for $j \in \{A, C, B\}$ and $w_\psi(\xi,t) = e^{\beta_\psi \xi} r_\psi(\xi + ct, t)$ with $\xi = x - ct$ and weights $\beta, \beta_\psi > 0$ to be determined later. In the co-moving frame, these variables satisfy

$$\begin{aligned}
\partial_t w_A &= \partial_\xi^2 w_A + (c - 2\beta)\partial_\xi w_A + (\beta^2 - c\beta + 1) w_A + A_{\text{front}} w_B + B_{\text{front}} w_A \\
&\quad + w_B r_A - 3A_{\text{front}}^2 w_A - 3A_{\text{front}} w_A r_A - r_A^2 w_A - e^{(\beta-\beta_\psi)\xi} A_{\text{front}} w_\psi r_\psi - w_A r_\psi^2,
\end{aligned} \tag{4.18a}$$

$$\begin{aligned}
\partial_t w_C &= \partial_\xi^2 w_C + (c - 2\beta)\partial_\xi w_C + (\beta^2 - c\beta + 1) w_C + (\partial_\xi - \beta)(A_{\text{front}} w_B + B_{\text{front}} w_A) \\
&\quad + (\partial_\xi - \beta)(w_B r_A - 3A_{\text{front}}^2 w_A - 3A_{\text{front}} w_A r_A - r_A^2 w_A - w_A r_\psi^2) \\
&\quad - e^{(\beta-\beta_\psi)\xi} (\partial_\xi - \beta_\psi)(A_{\text{front}} w_\psi r_\psi),
\end{aligned} \tag{4.18b}$$

$$\begin{aligned}
\partial_t w_B &= \mu \partial_\xi^2 w_B + (c - \mu 2\beta)\partial_\xi w_B + (\mu\beta^2 - c\beta) w_B \\
&\quad + 2\gamma(\partial_\xi - \beta)(A_{\text{front}} w_C + C_{\text{front}} w_A + w_C r_A),
\end{aligned} \tag{4.18c}$$

$$\partial_t w_\psi = \partial_\xi^2 w_\psi + (c - 2\beta_\psi)\partial_\xi w_\psi + (\beta_\psi^2 - c\beta_\psi) w_\psi + 2(\partial_\xi - \beta_\psi)\left(\frac{\partial_\xi A_{\text{front}} + r_C}{A_{\text{front}} + r_A} w_\psi\right). \tag{4.18d}$$

We abbreviate the above system (4.18) by

$$\partial_t w = L_\beta w + \mathcal{N}_\beta(w, r), \tag{4.19}$$

where L_β contains the linear and $\mathcal{N}_\beta$ the nonlinear terms. Note that, since $\partial_x r_A = r_C$ we obtain $(\partial_\xi - \beta) w_A = w_C$ and thus, we restrict L_β to the set $\{w = (w_A, w_C, w_B, w_\psi)^T \mid (\partial_\xi - \beta) w_A = w_C\}$ in what follows. To prove the stability result, we use the following property on the linear operator L_β, which we prove in Section 4.4.

Lemma 4.3.1. *Let $\beta > 0$, $\mu > 0$ and $c > c_{min}(\mu)$, see* (4.4)*, such that*

$$\max(\beta^2 - c\beta + 1, \mu\beta^2 - c\beta) < 0. \tag{4.20}$$

Then, there exists a $\beta_\psi^0 > 0$, depending on c, such that for every $\beta_\psi \in (0, \beta_\psi^0)$ the operator $L_\beta : \mathcal{D}(L_\beta) \to L^2$ is a densely defined, closed, sectorial operator with domain $\mathcal{D}(L_\beta) \subset L^2$. Furthermore, there exists a $\gamma_0 > 0$ and a $\kappa > 0$ such that for all $\gamma \in (-\gamma_0, \gamma_0)$ holds

1. $\mathrm{Re}(\sigma(L_\beta)) \leq -\kappa$ *and*

2. *there exists a* $\kappa \geq \tilde{\kappa} > 0$ *such that the estimates*

$$\left\| e^{tL_\beta} w \right\|_{L^2} \lesssim e^{-\tilde{\kappa} t} \left\| w \right\|_{L^2}, \tag{4.21}$$

$$\left\| e^{tL_\beta} \partial_\xi w \right\|_{L^2} \lesssim e^{-\tilde{\kappa} t} (1 + t^{-1/2}) \left\| w \right\|_{L^2}, \tag{4.22}$$

hold for any $t > 0$ *and* $w \in L^2$.

Remark 4.3.2. In particular, Lemma 4.3.1 implies that there is no eigenvalue at $\lambda = 0$, which would be expected since the original system (4.17) is translational invariant. Therefore, we formally have that

$$L_\beta \begin{pmatrix} \partial_\xi A_{\mathrm{front}} e^{\beta\xi} \\ \partial_\xi C_{\mathrm{front}} e^{\beta\xi} \\ \partial_\xi B_{\mathrm{front}} e^{\beta\xi} \\ 0 \end{pmatrix} = 0.$$

However, we cannot expect the derivative of $(A_{\mathrm{front}}, C_{\mathrm{front}}, B_{\mathrm{front}})$ to decay fast enough such that the potential eigenfunction is bounded. This follows by insertion of $(A, B)(t, x) = (\tilde{A}, \tilde{B})(x - ct)$ into (4.1) and considering the eigenvalues of the linearisation about the origin in the resulting first-order ODE-system, see (4.37). In this system, the origin is a stable, hyperbolic fixed-point and the largest eigenvalue is given by

$$\lambda_{\max} := \max\left(-\frac{c}{\mu}, \frac{1}{2}\left(-c + \sqrt{c^2 - 4} \right) \right).$$

In particular, we have $|\lambda_{\max}| < \beta$, if β satisfies (4.20). Hence, we expect that generically $(A_{\mathrm{front}}, C_{\mathrm{front}}, B_{\mathrm{front}})$ converges to the origin slower than $e^{-\beta\xi}$ for $\xi \to \infty$. Thus, the formal eigenfunction is unbounded and in particular not in L^2. □

Remark 4.3.3. We also highlight that we can choose β and β_ψ in Lemma 4.4.4 such that $e^{(\beta-\beta_\psi)\xi} A_{\mathrm{front}}$ and $e^{(\beta-\beta_\psi)\xi}(\partial_\xi - \beta_\psi) A_{\mathrm{front}}$ in (4.18) are bounded. To show this we note that it is sufficient to prove boundedness for $\xi \to \infty$. Using Proposition 4.A.1 we therefore need to choose the weights β and β_ψ such that

$$\beta - \beta_\psi - \frac{1}{2}(c - \sqrt{c^2 - 4}) \leq 0. \tag{4.23}$$

Utilizing that (4.20) is equivalent to

$$\frac{1}{2}(c - \sqrt{c^2 - 4}) < \beta < \min\left(\frac{c}{\mu}, \frac{1}{2}(c + \sqrt{c^2 - 4})\right),$$

condition (4.23) can be satisfied by choosing β close to $(c - \sqrt{c^2 - 4})/2$. □

4.3.2. Nonlinear stability

We stress that Lemma 4.3.1 only makes statements on estimates in L^2 instead of H^m. However, this is enough to obtain the necessary results in H^m. For that we prove a nonlinear damping estimate similar to [JZ11a, JZ11b], which controls the H^m-norm of any solution w of (4.19) by its L^2-norm and the H^m-norm of the initial data $w|_{t=0}$ as long as all variables stay sufficiently small in H^m.

Lemma 4.3.4. *Let $m \in \mathbb{N}$ and assume that $w|_{t=0} \in H^m$ and that there exists some $T > 0$ such that the H^m-norm of r and w remains sufficiently bounded for all $0 \le t \le T$. In particular, assume that there exists a constant $K_{r,T}$ such that $\sup_{0\le t\le T} \|r(t)\|_{H^m} \le K_{r,T}$. Then there exist constants $\theta_m > 0$ and $K < \infty$ depending on $K_{r,T}$ such that*

$$\|w(t)\|_{H^m}^2 \le e^{-\theta_m t} \|w|_{t=0}\|_{H^m}^2 + K \int_0^t e^{-\theta_m(t-s)} \|w(s)\|_{L^2}^2 \, ds \tag{4.24}$$

for all $0 \le t \le T$.

Proof. We take the L^2 scalar product of (4.19) with the test function $\vartheta = \sum_{j=0}^m (-1)^j \partial_\xi^{2j} w$, i.e.

$$\langle \vartheta, \partial_t w\rangle_{L^2} = \langle \vartheta, L_\beta w + \mathcal{N}_\beta(w, r)\rangle_{L^2}.$$

By choice of the test function, it holds that $2\langle \vartheta, \partial_t w\rangle_{L^2} = \partial_t \|w\|_{H^m}^2$ and $\left\langle \vartheta, \partial_\xi^2 w\right\rangle_{L^2} = -(\|w\|_{H^m}^2 + \left\|\partial_\xi^{m+1} w\right\|_{L^2}^2)$. Furthermore, for $f \in C_b^{m+1}$, $n \in \{0,1,2\}$, $0 \le j \le m$ and $i_1, i_2, i_3 \in \{A, C, B, \psi\}$ we estimate with Young's inequality

$$\begin{aligned}\left\langle (-1)^j \partial_\xi^{2j} w_{i_1}, f w_{i_2} r_{i_3}^n\right\rangle_{L^2} = \left\langle \partial_\xi^j w_{i_1}, \partial_\xi^j (f w_{i_2} r_{i_3}^n)\right\rangle_{L^2} &\lesssim \left\|\partial_\xi^j w_{i_1}\right\|_{L^2}^2 + \|f\|_{C_b^m}^2 \|w_{i_2}\|_{H^m}^2 \|r_{i_3}\|_{H^m}^{2n} \\ &\lesssim \left\|\partial_\xi^j w_{i_1}\right\|_{L^2}^2 + \|f\|_{C_b^m}^2 K_{r,T}^{2n} \|w_{i_2}\|_{H^m}^2,\end{aligned}$$

where we used the fact that H^m is a Banach algebra for $m > 1/2$ and that w and r are assumed to be small in H^m. Similarly, for any $\varepsilon > 0$ holds that

$$\begin{aligned}\left\langle (-1)^j \partial_\xi^{2j} w_{i_1}, \partial_\xi (f w_{i_2} r_{i_3}^n) \right\rangle_{L^2} &= - \left\langle \partial_\xi^{j+1} w_{i_1}, \partial_\xi^j (f w_{i_2} r_{i_3}^n) \right\rangle_{L^2} \\ &\lesssim \varepsilon \left\| \partial_\xi^{j+1} w_{i_1} \right\|_{L^2}^2 + \frac{1}{\varepsilon} \left\| f \right\|_{C_b^m}^2 K_{r,T}^{2n} \left\| w_{i_2} \right\|_{H^m}^2 ,\end{aligned}$$

where we used Young's inequality. Using Remark 4.3.3 and the fact that r_A decays faster than A_{front} for $\xi \to \infty$ since w_A is small in H^m and thus in L^∞ we can treat all but one terms in (4.18) using the above estimate. To treat the remaining term, let $\chi_0 \in [0,1]$ be a smooth cut-off function with $\chi_0(\xi) = 0$ for $\xi \leq -1$ and $\chi_0(\xi) = 1$ for $\xi \geq 1$. Then, we write

$$\begin{aligned}&\left\langle (-1)^j \partial_\xi^{2j} w_\psi, \partial_\xi \left(\frac{r_C}{A_{\text{front}} + r_A} w_\psi \right) \right\rangle_{L^2} \\ &\quad = - \left\langle \partial_\xi^{j+1} w_\psi, \partial_\xi^j \left(\chi_0 \frac{w_C w_\psi}{A_{\text{front}} e^{\beta\xi} + w_A} + (1-\chi_0) \frac{r_C w_\psi}{A_{\text{front}} + r_A} \right) \right\rangle_{L^2} .\end{aligned}$$

Since r_C is small in H^m and thus small in L^∞ for all $0 \leq t \leq T$, we obtain that $(A_{\text{front}} + r_A)|_{\xi \in (-\infty,1]}$ is bounded away from zero. Furthermore, using Proposition 4.A.1 and the assumption that w_A is small in H^m we find that $(A_{\text{front}} e^{\beta\xi} + w_A)|_{\xi \in [-1,\infty)}$ is bounded away from zero. Thus, we estimate

$$\left\langle (-1)^j \partial_\xi^{2j} w_\psi, \partial_\xi \left(\frac{r_C}{A_{\text{front}} + r_A} w_\psi \right) \right\rangle_{L^2} \lesssim \frac{1}{\varepsilon} (\|w_C\|_{H^m} + \|r_c\|_{H^m})^2 \|w_\psi\|_{H^m}^2 + \varepsilon \left\| \partial_\xi^{j+1} w_\psi \right\|_{L^2}^2$$

for some small $\varepsilon > 0$, using Young's inequality.

Hence, combining the above estimates and choosing $\varepsilon > 0$ sufficiently small, there exists a $\tilde{\theta}_m > 0$ and a constant $\tilde{K} > 0$, both depending on $K_{r,T}$, such that

$$\partial_t \|w(t)\|_{H^m}^2 \leq -\tilde{\theta}_m \left\| \partial_\xi^{m+1} w(t) \right\|_{L^2}^2 + \tilde{K} \|w(t)\|_{H^m}^2$$

for all $0 \leq t \leq T$. Then, using Sobolev interpolation for every $\tilde{\varepsilon} > 0$ exists a $K_{\tilde{\varepsilon}} > 0$ such that

$$\|g\|_{H^m}^2 \leq \tilde{\varepsilon} \left\| \partial_\xi^{m+1} g \right\|_{L^2}^2 + K_{\tilde{\varepsilon}} \|g\|_{L^2}^2 . \tag{4.25}$$

Inserting this into the above equation and choosing $\tilde{\varepsilon}$ small enough we find that there exists a $\theta_m > 0$ and a constant $K > 0$ such that

$$\partial_t \|w(t)\|_{H^m}^2 \leq -\theta_m \|w(t)\|_{H^m}^2 + K \|w(t)\|_{L^2}^2$$

for all $0 \leq t \leq T$. Again, both constants depend on $K_{r,T}$. Then, the statement follows by applying Gronwall's inequality. □

Remark 4.3.5. Using the nonlinear damping estimate Lemma 4.3.4 and the semigroup estimates Corollary 4.2.4 and Lemma 4.3.1, we obtain local existence and uniqueness of the coupled (r, w)-system (4.17), (4.19) in H^m using standard fixed point arguments. □

Using Lemma 4.3.1 we can now prove the nonlinear stability. To this end, we recall that the coupled (r, w)-system reads as

$$\begin{aligned} \partial_t r &= Lr + \mathcal{N}_1(r) + \mathcal{N}_2(r), \\ \partial_t w &= L_\beta w + \mathcal{N}_\beta(w, r), \end{aligned}$$

which are given in (4.17) and (4.19), respectively. Recalling the nonlinear stability of the invading state discussed in Section 4.2, we rewrite this system by introducing the transformed variable $p := S^{-1}r$ with S given in (4.11). Thus, we consider the transformed (p, w)-system which is given by

$$\partial_t p = \Lambda p + S^{-1}\mathcal{N}_1(Sp) + S^{-1}\mathcal{N}_2(Sp), \tag{4.26a}$$

$$\partial_t w = L_\beta w + \mathcal{N}_\beta(w, Sp), \tag{4.26b}$$

where Λ is given in Lemma 4.2.1. As in (4.14) we introduce

$$\begin{aligned} \eta_A(t) &:= \sup_{0\leq s\leq t} (1+s)^{1/2} \|p_A(s)\|_{H^m}, \\ \eta_C(t) &:= \sup_{0\leq s\leq t} (1+s)^{1/2} \|p_C(s)\|_{H^m}, \\ \eta_B(t) &:= \sup_{0\leq s\leq t} (1+s)^{1/4} \|p_B(s)\|_{H^m}, \\ \widetilde{\eta_B}(t) &:= \sup_{0\leq s\leq t} (1+s)^{1/2} \|S_{CB}p_B(s)\|_{H^m}, \\ \eta_\psi(t) &:= \sup_{0\leq s\leq t} (1+s)^{1/4} \|p_\psi(s)\|_{H^m}. \end{aligned}$$

Additionally, for the weighted variable we define

$$\mu_j(t) := \sup_{0\le s\le t} e^{\bar{\kappa}s} \|w_j(s)\|_{H^m}$$

for $j \in \{A, C, B, \psi\}$ and $2\bar{\kappa} \in (0, \min(\theta_m, 2\tilde{\kappa}))$ with $\tilde{\kappa} > 0$ from Lemma 4.3.1 and θ_m from Lemma 4.3.4. Similar to Section 4.2 the goal of this section is to show that $E(t)$ and $E_w(t)$ given by

$$E(t) := \eta_A(t) + \eta_C(t) + \eta_B(t) + \widetilde{\eta_B}(t) + \eta_\psi(t), \tag{4.27}$$

$$E_w(t) := \mu_A(t) + \mu_C(t) + \mu_B(t) + \mu_\psi(t) \tag{4.28}$$

are uniformly bounded for all time.

We first obtain an estimate for $E_w(t)$, which can be done similarly to Lemma 4.2.9. Therefore, we rewrite (4.26b) using Duhamel's formula as

$$w(t) = e^{tL_\beta} w|_{t=0} + \int_0^t e^{(t-s)L_\beta} \mathcal{N}_\beta(w(s), Sp(s))\, ds$$

for all $t \ge 0$. Using (4.21), (4.22), Lemma 4.3.4, the estimate

$$e^{\bar{\kappa}t} \int_0^t e^{-\tilde{\kappa}(t-s)} \max(1, (t-s)^{-1/2}) e^{-\bar{\kappa}s}\, ds < \infty$$

and the fact that $\mathcal{N}_\beta(w, Sp)$ is linear with respect to w we obtain the following result.

Proposition 4.3.6. *Fix $m \in \mathbb{N}$. Then, let $\beta > 0$, $\beta_\psi > 0$, $\mu > 0$, $\gamma > -\mu$ and $c > c_{min}(\mu)$ such that Lemma 4.3.1 applies and* (4.23) *holds. Furthermore, assume that there exists a $T > 0$ such that $w(t)$ and $r(t) = Sp(t)$ are small in H^m for all $0 \le t \le T$. Then, $E_w(t)$ satisfies the estimate*

$$E_w(t) \le K_w(\|w|_{t=0}\|_{H^m} + E_w(t)E(t) + E_w(t)^2), \tag{4.29}$$

with $K_w < \infty$.

Proof. Similar to Lemma 4.2.9, using (4.21), (4.22) and $\bar{\kappa} < \tilde{\kappa}$ we obtain the estimate

$$\|w(t)\|_{L^2} \lesssim e^{-\bar{\kappa}t} \left(\|w|_{t=0}\|_{L^2} + E_w(t)E(t) + E_w(t)^2\right),$$

since r is small in H^m. The only additional challenge originates from the $w_\psi(\partial_\xi A_{\text{front}} +$

$r_C)/(A_{\text{front}} + r_A)$-term in the w_ψ-equation. However, this can be handled using a similar splitting argument as in Lemma 4.3.4, which leads to the quadratic contribution $E_w(t)^2$. Note that the main challenge of Lemma 4.2.9 – the polynomially decaying variable – does not appear here since we have exponential decay of the semigroup generated by L_β, see Lemma 4.3.1. Then, using the damping estimate Lemma 4.3.4 yields the statement, since we have that

$$\int_0^t e^{-\theta_m(t-s)} e^{-2\bar{\kappa}s}\, ds = \mathcal{O}(e^{-2\bar{\kappa}t}),$$

since $2\bar{\kappa} < \theta_m$ by assumption. □

The previous Proposition 4.3.6 provides the necessary estimate to deal with the weighted equation in the nonlinear iteration. Thus, it remains to obtain an estimate for $E(t)$. Here, the most critical part is $\mathcal{N}_1$, which is dealt with using an interplay of the non-weighted and the weighted variable, exploiting the expected exponential temporal decay of the weighted variable. To make this precise we analyse $\mathcal{N}_1(r)$ in more detail, recalling that $r = Sp$. A closer inspection reveals that all terms in $\mathcal{N}_1$ are of the form $f(\xi)r_j(x,t)$, $j \in \{A, C, B, \psi\}$, where f decays exponentially fast for $\xi \to -\infty$. Ideally, if $f(\xi)e^{-\beta_j\xi} = \mathcal{O}(1)$ for $\xi \to -\infty$ we would write $f(\xi)r_j(x,t) = f(\xi)w_j(\xi,t)e^{-\beta_j\xi}$ and use that w decays exponentially, at least on the linear level. However, we can only verify this for the ψ-part of $\mathcal{N}_1$, for which f is given by

$$f_\psi = \frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}}.$$

Since we can choose β_ψ close to zero, we obtain that $f_\psi(\xi)e^{-\beta_\psi\xi}$ is bounded for $\xi \to -\infty$. For the remaining part, we cannot expect that a similar statement is true since its in general false for the KPP-equation, which corresponds to the case $\gamma = 0$, $r_B = r_\psi = 0$ in our setting. In the case of the KPP-equation the asymptotic behavior of the fronts for $\xi \to -\infty$ can be explicitly calculated, see [Sat76] and we obtain that $(1 - A_{\text{front}})e^{-\beta\xi}$ is unbounded for any appropriate choice of β. However, it turns out that we do not need that $f(\xi)e^{-\beta\xi}$ is bounded. Instead we use the following result.

Lemma 4.3.7. *Let $m \in \mathbb{N}_0$, $\tilde{c} > 0$ be small and $f \in C_b^m$ with $f(\xi) = \mathcal{O}(e^{\tilde{\theta}\xi})$ for $\xi \to -\infty$ for some $\tilde{\theta} > 0$. Then, for all $\delta > 0$ exists a constant $K_{\tilde{c},\delta} < \infty$ such that*

$$\|f(x-ct)r_j(x,t)\|_{H^m(x)} \leq \delta e^{-\tilde{\theta}\tilde{c}t} \|r_j(x,t)\|_{H^m(x)} + K_{\tilde{c},\delta} e^{\beta_j \tilde{c}t} \|w_j(\xi,t)\|_{H^m(\xi)}$$

for any $j \in \{A, C, B, \psi\}$.

Proof. First note that since $\partial_\xi = \partial_x$ we have $\|g(x-ct)\|_{C_b^m(x)} = \|g(\xi)\|_{C_b^m(\xi)}$ if $g \in C_b^m$. Next, let χ_{ξ_0} be a smooth cut-off function with $\chi_{\xi_0} \in [0,1]$, $\chi_{\xi_0}(\xi) = 0$ for $\xi > \xi_0 + 1$ and $\chi_{\xi_0}(\xi) = 1$ for $\xi < \xi_0 - 1$ for a fixed $\xi_0 \in \mathbb{R}$. Then, we write for $j \in \{A, C, B, \psi\}$

$$\begin{aligned} f(\xi) r_j(x,t) &= \chi_{\xi_0 - \tilde{c}t}(\xi) f(\xi) r_j(x,t) + (1 - \chi_{\xi_0 - \tilde{c}t}(\xi)) f(\xi) e^{-\beta_j \xi} w_j(\xi, t) \\ &= \chi_{\xi_0 - \tilde{c}t}(\xi) f(\xi) r_j(x,t) + (1 - \chi_{\xi_0 - \tilde{c}t}(\xi)) f(\xi) e^{-\beta_j(\xi + \tilde{c}t)} e^{\beta_j \tilde{c}t} w_j(\xi, t). \end{aligned}$$

The lemma is proven if the estimates

$$\begin{aligned} \|\chi_{\xi_0 - \tilde{c}t}(\xi) f(\xi)\|_{C_b^m(\xi)} &\leq \delta e^{-\tilde{\theta}\tilde{c}t}, \\ \sup_{t \geq 0} \left\| (1 - \chi_{\xi_0 - \tilde{c}t}(\xi)) f(\xi) e^{-\beta_j(\xi + \tilde{c}t)} \right\|_{C_b^m(\xi)} &\leq K_{\tilde{c},\delta} \end{aligned}$$

hold. To prove these inequalities we introduce

$$\begin{aligned} \mathcal{U}_{\xi_0}(t) &= \operatorname{supp}(\chi_{\xi_0 - \tilde{c}t}) = \{\xi \in \mathbb{R} : \xi \leq \xi_0 - \tilde{c}t + 1\}, \\ \tilde{\mathcal{U}}_{\xi_0}(t) &= \operatorname{supp}(1 - \chi_{\xi_0 - \tilde{c}t}) = \{\xi \in \mathbb{R} : \xi \geq \xi_0 - \tilde{c}t - 1\}. \end{aligned}$$

Then, the first estimate follows from

$$\|\chi_{\xi_0 - \tilde{c}t}(\xi) f(\xi)\|_{C_b^m} \leq \|\chi_{\xi_0 - \tilde{c}t}\|_{C_b^m} \|f(\xi)\|_{C_b^m(\mathcal{U}_{\xi_0 - \tilde{c}t}(t))} \leq C_f e^{\tilde{\theta}(\xi_0 + 1)} e^{-\tilde{\theta}\tilde{c}t} = \delta e^{-\tilde{\theta}\tilde{c}t},$$

which holds for any small $\delta > 0$ by choosing $\xi_0 = \xi_0(\delta) < 0$ small enough. Furthermore, we obtain the second estimate by

$$\begin{aligned} \sup_{t \geq 0} \left\| (1 - \chi_{\xi_0 - \tilde{c}t}(\xi)) f(\xi) e^{-\beta_j(\xi + \tilde{c}t)} \right\|_{C_b^m} &\leq \|f\|_{C_b^m} \sup_{t \geq 0} \left\| e^{-\beta_j(\xi + \tilde{c}t)} \right\|_{C_b^m(\tilde{\mathcal{U}}(\chi_{\xi_0 - \tilde{c}t}(t)))} \\ &\leq K(1 + \beta_j^m) \sup_{t \geq 0} e^{-\beta_j(\xi_0 - \tilde{c}t - 1 + \tilde{c}t)} \\ &= K(1 + \beta_j^m) e^{-\beta_j(\xi_0 - 1)} < \infty. \end{aligned}$$

Note that for $\xi_0 \to -\infty$, this bound tends to $+\infty$. □

We now comment on why this is enough to treat $\mathcal{N}_1$ as a nonlinear term in the stability analysis. The above Lemma 4.3.7 states that we can estimate every term in $\mathcal{N}_1$ from above by the H^m-norm of r with an arbitrarily small, exponentially decaying prefactor and the

H^m-norm of w with a prefactor which grows exponentially fast. However, using the linear stability estimates in Lemma 4.3.1 and the nonlinear damping estimate Lemma 4.3.4 we can expect that the H^m-norm of w shows exponential decay. Thus, by choosing $\tilde{c} > 0$ small enough it is, at least heuristically, possible to obtain an exponentially decaying estimate, which allows to treat $\mathcal{N}_1$ as a nonlinearity, see the proof of the subsequent Lemma 4.3.8 for details.

We now have all necessary results to prove the boundedness of $E(t)$ and $E_w(t)$ given in (4.27) and (4.28), respectively. In turn this implies the decay of solutions of the (p, w)-system (4.26). Thus, the main result reads as follows.

Lemma 4.3.8. *Fix $m \in \mathbb{N}$ and let $\mu > 0$, $\gamma > -\mu$ and $c > c_{min}(\mu)$, see* (4.4). *Then, choose $\beta > 0$ and $\beta_\psi > 0$ such that*

$$\begin{aligned}
&\max(\beta^2 - c\beta + 1, \mu\beta - c) < 0,\\
&\beta_\psi \in (0, \beta_\psi^0),\\
&\beta \leq \frac{1}{2}(c - \sqrt{c^2 - 4}) + \beta_\psi
\end{aligned}$$

with β_ψ^0 from Lemma 4.3.1. Then, there exists a $\varepsilon > 0$ and a $\bar{\kappa} > 0$ such that if

$$\begin{aligned}
&\|(p_A, p_C, p_B, p_\psi)|_{t=0}\|_{H^m \times H^m \times (L^1 \cap H^m) \times (L^1 \cap H^m)} < \varepsilon,\\
&\|(w_A, w_C, w_B, w_\psi)|_{t=0}\|_{H^m \times H^m \times H^m \times H^m} < \varepsilon,
\end{aligned}$$

and $\gamma \in (-\gamma_0, \gamma_0)$ with γ_0 from Lemma 4.3.1, there exists a constant $K < \infty$ such that

$$E(t) + E_w(t) \leq K$$

for all $t \geq 0$. In particular, this yields

$$\begin{aligned}
&\|p_A(t)\|_{H^m}, \|p_C(t)\|_{H^m} \lesssim (1+t)^{-1/2},\\
&\|p_B(t)\|_{H^m}, \|p_\psi(t)\|_{H^m} \lesssim (1+t)^{-1/4},\\
&\|w(t)\|_{H^m} \lesssim e^{-\bar{\kappa} t}
\end{aligned}$$

for all $t \geq 0$.

Proof. Since the system (4.26) is semilinear and using the semigroup estimates (4.21), (4.22) for L_β provided in Lemma 4.3.1, the semigroup estimates for Λ from Corollary

4.2.4 and the nonlinear damping estimate Lemma 4.3.4, we obtain local existence and uniqueness in H^m by standard arguments. In particular, E and E_w are continuous with respect to time as long as they remain small. Hence, for sufficiently small initial data, there exists a time $T > 0$ such that

$$\sup_{0\leq t\leq T} E(t) \leq \frac{1}{2K_w}. \tag{4.30}$$

In particular, since $\hat{S}$ is bounded in Fourier space, this yields that $Sp(t)$ is uniformly bounded in H^m for all $0 \leq t \leq T$. Therefore, we can apply Proposition 4.3.6 and obtain

$$E_w(t) \leq 2K_w(\|w|_{t=0}\|_{H^m} + E_w(t)^2)$$

for all $0 \leq t \leq T$, which yields that

$$E_w(t) \leq \tilde{K}_w\varepsilon \tag{4.31}$$

for all $0 \leq t \leq T$. Then, we consider the Duhamel formula for (4.26a)

$$p(t) = e^{t\Lambda}p|_{t=0} + \int_0^t e^{(t-s)\Lambda}S^{-1}\mathcal{N}_1(Sp(s))\,ds + \int_0^t e^{(t-s)\Lambda}S^{-1}\mathcal{N}_2(Sp(s))\,ds.$$

We estimate the two integral terms separately. For this, we note that $(A_{\text{front}} - 1)$, C_{front} and B_{front} satisfy the assumptions of Lemma 4.3.7 for some $\theta > 0$. Hence, for $\tilde{c} > 0$ small we estimate the integral term for $\mathcal{N}_1$ using Lemmas 4.2.1 and 4.3.7. Similar to Lemma 4.2.9 we focus on the polynomially decaying part of the estimates for p_B, $S_{CB}p_B$ and p_ψ since all other components have exponential decay on the linear level, which simplifies the estimates. Therefore, let P_j, $j = A, C, B, \psi$ be the projection onto the j-th component as in Corollary 4.2.4.

We start by estimating p_B. Using Lemma 4.2.1, Corollary 4.2.4, Lemma 4.3.7 and $r = Sp$ we find

$$\begin{aligned}
&\int_0^t \left\|P_B e^{(t-s)\Lambda}S^{-1}\mathcal{N}_1(Sp(s))\right\|_{H^m} ds \\
&\quad\lesssim \int_0^t e^{-\theta(t-s)}\max(1,(t-s)^{-1/2})\left(\delta e^{-\tilde{\theta}\tilde{c}s}\|p(s)\|_{H^m} + K_{\tilde{c},\delta}e^{\beta\tilde{c}s}\|w(s)\|_{H^m}\right) ds \\
&\qquad + \max(1,2|\gamma|)\int_0^t (t-s)^{-1/2}\left(\delta e^{-\tilde{\theta}\tilde{c}s}\|p(s)\|_{H^m} + K_{\tilde{c},\delta}e^{\beta\tilde{c}s}\|w(s)\|_{H^m}\right) ds
\end{aligned}$$

$$\lesssim \max(1, 2|\gamma|)(1+t)^{-1/2}(\delta E(t) + K_{\tilde{c},\delta} E_w(t))$$

Following the proof of Lemma 4.2.5 and using that $(A_{\text{front}}, C_{\text{front}}, B_{\text{front}})$ are uniformly bounded in C_b^m, we obtain for $\mathcal{N}_2$ that

$$\int_0^t \left\| P_B e^{(t-s)\Lambda} S^{-1} \mathcal{N}_2(Sp(s)) \right\|_{H^m} ds \lesssim (1+t)^{-1/4} E(t)^2.$$

Similarly, we estimate $S_{CB} p(t)$ in H^m. For the $\mathcal{N}_1$ estimate we use that S_{CB} is a bounded map from H^m to H^m and thus, we obtain the same estimate as above for $p_B(t)$. For $\mathcal{N}_2$ we use as in Lemma 4.2.5 that $|\hat{S}_{CB}(k)| = \mathcal{O}(|k|)$ for $|k| \to 0$, which gives an additional $(1+t-s)^{-1/2}$ in the semigroup estimates.

Finally, we estimate $p_\psi(t)$ in H^m. For the $\mathcal{N}_1$-term we use that $\partial_\xi A_{\text{front}}/A_{\text{front}}$ satisfies the assumptions of Lemma 4.3.7 and thus proceeding as above we find

$$\int_0^t \left\| P_\psi e^{(t-s)\Lambda} S^{-1} \mathcal{N}_1(Sp(s)) \right\|_{H^m} ds \lesssim (1+t)^{-1/2}(\delta E(t) + K_{\tilde{c},\delta} E_w(t)).$$

To estimate the $\mathcal{N}_2$-contribution, let χ_0 be a cut-off function with $\chi_0 \in [0,1]$, $\chi_0(\xi) = 0$ for $\xi \leq -1$ and $\chi_0(\xi) = 1$ for $\xi \geq 1$. Then, proceeding with a similar splitting as in the proof of Lemma 4.3.4 we find

$$\begin{aligned}
&\int_0^t \left\| P_\psi e^{(t-s)\Lambda} S^{-1} \mathcal{N}_2(Sp(s)) \right\|_{H^m} ds \\
&\quad = \int_0^t \left\| e^{(t-s)\partial_x^2} 2\partial_x \left(\frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}} \frac{r_\psi(s) r_A(s)}{A_{\text{front}} + r_A(s)} + \frac{r_C(s) r_\psi(s)}{A_{\text{front}} + r_A(s)} \right) \right\|_{H^m} ds \\
&\quad \lesssim \int_0^t (t-s)^{-1/2} \left\| \chi_0 \left(\frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}} \frac{w_A(s) r_\psi(s)}{A_{\text{front}} e^{\beta\xi} + w_A(s)} + \frac{w_C(s) r_\psi(s)}{A_{\text{front}} e^{\beta\xi} + w_A(s)} \right) \right\|_{H^m} ds \\
&\qquad + \int_0^t \left\| e^{(t-s)\partial_x^2} \partial_x \left((1-\chi_0) \frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}} \frac{r_A r_\psi}{A_{\text{front}} + r_A} \right) \right\|_{H^m} ds \\
&\qquad + \int_0^t \left\| e^{(t-s)\partial_x^2} \partial_x \left((1-\chi_0) \frac{r_C r_\psi}{A_{\text{front}} + r_A} \right) \right\|_{H^m} ds \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

To estimate I_1, we use that since $w_A(s)$ is small in H^m for $0 \leq s \leq T$, $A_{\text{front}} e^{\beta\xi} + w_A(s)$ is bounded away from zero on $[-1, \infty)$ for all $0 \leq s \leq T$. This yields

$$I_1 \lesssim \int_0^t (t-s)^{-1/2} (\|w_A(s)\|_{H^m} + \|w_C(s)\|_{H^m}) \|r_\psi(s)\|_{H^m} ds \lesssim (1+t)^{-1/2} E_w(t) E(t).$$

For I_2 we use that $\partial_\xi A_{\text{front}}/A_{\text{front}}$ satisfies the assumption of Lemma 4.3.7 and that $A_{\text{front}} + r_A$ is bounded away from zero on $(-\infty, 1]$, which leads to

$$\begin{aligned} I_2 &\lesssim \int_0^t (t-s)^{-1/2} \|r_A(s)\|_{H^m} \left\| \frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}} r_\psi(s) \right\|_{H^m} ds \\ &\lesssim (1+t)^{-1/2}(\delta E(t)^2 + K_{\tilde{c},\delta} E_w(t) E(t)). \end{aligned}$$

Finally, we obtain for I_3 that

$$I_3 \lesssim (1+t)^{-1/4} E(t)^2$$

similar to the proof of Lemma 4.2.5 by using the improved decay rate for $S_{CB} p_B$. Combining the estimates for I_1, I_2 and I_3 then yields

$$\int_0^t \left\| P_\psi e^{(t-s)\Lambda} S^{-1} \mathcal{N}_2(Sp(s)) \right\|_{H^m} ds \lesssim (1+t)^{-1/4} (E(t) E_w(t) + E(t)^2).$$

Putting the above estimates together and using the linear stability estimates in Corollary 4.2.4 we obtain

$$\begin{aligned} E(t) &\le K(\varepsilon + \delta E(t) + K_{\tilde{c},\delta} E_w(t) + E(t) E_w(t) + E(t)^2) \\ &\le K(\varepsilon + (\delta + \tilde{K}_w \varepsilon) E(t) + K_{\tilde{c},\delta} \tilde{K}_w \varepsilon + E(t)^2) \end{aligned}$$

for all $0 \le t \le T$, where we used (4.31). Since $\delta > 0$ and $\varepsilon > 0$ are arbitrary, this yields

$$\begin{aligned} E(t) &\le 2K(\varepsilon + K_{\tilde{c},\delta} \tilde{K}_w \varepsilon + E(t)^2) \\ &\le \tilde{K}(\varepsilon + E(t)^2) \end{aligned}$$

for all $0 \le t \le T$. In particular, by choosing $\varepsilon > 0$ small enough and using continuous induction, we can extend the argument to $T = \infty$, which proves the statement. □

Using that S, S^{-1} are bounded transformations in H^m, see Lemma 4.2.1, we obtain the stability of fronts in the system (4.1) with respect to real perturbations similarly to the proof of Theorem 4.2.6.

Theorem 4.3.9. *Fix $m \in \mathbb{N}$ and let $\mu > 0$, $\gamma > -\mu$ and $c > c_{min}(\mu)$. Then, choose*

$\beta > 0$ and $\beta_\psi > 0$ such that

$$\begin{aligned}
&\max(\beta^2 - c\beta + 1, \mu\beta - c) < 0,\\
&\beta_\psi \in (0, \beta_\psi^0),\\
&\beta \leq \frac{1}{2}(c - \sqrt{c^2 - 4}) + \beta_\psi,
\end{aligned}$$

with β_ψ^0 from Lemma 4.3.1. Then, there exists a $\varepsilon > 0$ and a $\bar{\kappa} > 0$ such that if

$$\begin{aligned}
&\|(r_A, r_B, r_\psi)|_{t=0}\|_{(W^{1,1}\cap H^{m+1})\times(L^1\cap H^m)\times(L^1\cap H^m)} < \varepsilon,\\
&\left\|(r_A, r_B)|_{t=0} e^{\beta x}\right\|_{H^{m+1}\times H^m} + \left\|r_\psi|_{t=0} e^{\beta_\psi x}\right\|_{H^m} < \varepsilon,
\end{aligned}$$

and $\gamma \in (-\gamma_0, \gamma_0)$ with γ_0 from Lemma 4.3.1 it holds that

$$\begin{aligned}
&\|(r_A, r_B, r_\psi)(t)\|_{H^{m+1}\times H^m\times H^m} \lesssim (1+t)^{-1/4},\\
&\left\|(r_A, r_B)(t) e^{\beta(x-ct)}\right\|_{H^{m+1}\times H^m} + \left\|r_\psi(t) e^{\beta_\psi(x-ct)}\right\|_{H^m} \lesssim e^{-\bar{\kappa}t}
\end{aligned}$$

for all $t \geq 0$.

4.4. Spectral and linear stability of the weighted operator

We now provide a proof of Lemma 4.3.1. The approach is based on perturbation arguments and estimates on the spectrum in the case $\gamma = 0$.

4.4.1. Spectral estimates for L_β for γ close to zero

Recall that the weighted operator L_β is given by

$$L_\beta w = \begin{pmatrix} \partial_\xi^2 w_A + (c - 2\beta)\partial_\xi w_A + (\beta^2 - c\beta + 1)w_A + A_{\text{front}} w_B - 3A_{\text{front}}^2 w_A \\ \partial_\xi^2 w_C + (c - 2\beta)\partial_\xi w_C + (\beta^2 - c\beta + 1)w_C + (\partial_\xi - \beta)(A_{\text{front}} w_B - 3A_{\text{front}}^2 w_A) \\ \mu\partial_\xi^2 w_B + (c - 2\mu\beta)\partial_\xi w_B + (\mu\beta^2 - c\beta)w_B \\ \partial_\xi^2 w_\psi + (c - 2\beta)\partial_\xi w_\psi + (\beta_\psi^2 - c\beta_\psi)w_\psi + 2(\partial_\xi - \beta_\psi)(\partial_\xi A_{\text{front}}/A_{\text{front}})w_\psi \end{pmatrix}$$

$$
\begin{aligned}
&+ \begin{pmatrix} B_{\text{front}} w_A \\ (\partial_\xi - \beta)(B_{\text{front}} w_A) \\ 2\gamma(\partial_\xi - \beta)(A_{\text{front}} w_C + C_{\text{front}} w_A) \\ 0 \end{pmatrix} \\
=: &\begin{pmatrix} \widetilde{L_\beta} & 0 \\ 0 & L_\psi \end{pmatrix} w + \begin{pmatrix} P(\gamma) & 0 \\ 0 & 0 \end{pmatrix} w
\end{aligned}
$$

for $w = (w_A, w_C, w_B, w_\psi) \in \mathcal{D}(L_\beta)$ and $\widetilde{L_\beta}, P(\gamma)$ operate on (w_A, w_C, w_B), while L_ψ operates on w_ψ. Using the diagonal structure of L_β, we can analyze the spectra of $\widetilde{L_\beta} + P(\gamma)$ and L_ψ separately.

For $\widetilde{L_\beta} + P(\gamma)$ we use perturbation arguments. Since the $(A_{\text{front}}, C_{\text{front}}, B_{\text{front}})$ are uniformly bounded and B_{front} vanishes uniformly in C^1 for $\gamma \to 0$, there exists constants $a(\gamma), b(\gamma)$, which vanish for $\gamma \to 0$ and satisfy

$$
\|P(\gamma)\tilde{w}\|_{L^2} \leq a(\gamma) \|\tilde{w}\|_{L^2} + b(\gamma) \left\|\widetilde{L_\beta}\tilde{w}\right\|_{L^2} \tag{4.32}
$$

for all $\tilde{w} \in \mathcal{D}(\widetilde{L_\beta})$. Thus, by [Kat66, Theorem IV.3.17] the spectrum of $\widetilde{L_\beta} + P(\gamma)$ is close to the spectrum of $\widetilde{L_\beta}$ if γ is close to zero. Additionally, we have the following result on the spectrum of $\widetilde{L_\beta}$.

Lemma 4.4.1. *Let $\beta > 0$, $\mu > 0$ and $c > c_{min}(\mu)$, see* (4.4)*, and suppose that there exists a $\kappa > 0$ such that*

$$
\max(\beta^2 - c\beta + 1, \mu\beta^2 - c\beta) \leq -\kappa < 0.
$$

Then, $\mathrm{Re}(\sigma(\widetilde{L_\beta})) \leq -\kappa$.

Proof. The proof is done in two steps: first, we analyze the essential spectrum and afterwards the point spectrum (we refer to [KP13] for a definition).

To control the essential spectrum, we note that it is sufficient to consider the asymptotic operators

$$
\widetilde{L_{\beta,\pm}}(\partial_\xi) := \lim_{\xi \to \pm\infty} \widetilde{L_\beta}(\xi, \partial_\xi).
$$

Since $(A_{\text{front}}, B_{\text{front}}, \psi_{\text{front}})$ convergences exponentially fast to its rest states we can bound the essential spectrum of $\widetilde{L_\beta}$ by the essential spectrum of the asymptotic operators, see

[San02]. We first consider $\widetilde{L_{\beta,+}}$, which is given by

$$\widetilde{L_{\beta,+}} = \begin{pmatrix} \partial_\xi^2+(c-2\beta)\partial_\xi+\beta^2-c\beta+1 & 0 & 0 \\ 0 & \partial_\xi^2+(c-2\beta)\partial_\xi+\beta^2-c\beta+1 & 0 \\ 0 & 0 & \mu\partial_\xi^2+(c-2\mu\beta)\partial_\xi+\mu\beta^2-c\beta \end{pmatrix},$$

since $(A_{\text{front}}, B_{\text{front}}, \psi_{\text{front}})(\xi) \to (0,0,0)$ for $\xi \to +\infty$. Applying Fourier transform then yields

$$\sigma(\widetilde{L_{\beta,+}}) = \bigcup_{k\in\mathbb{R}} \{-k^2 + (c-2\beta)ik + \beta^2 - c\beta + 1, -\mu k^2 + (c-2\mu\beta)ik + \mu\beta^2 - c\beta\}.$$

Similarly, using $(A_{\text{front}}, B_{\text{front}}, \psi_{\text{front}})(\xi) \to (1,0,0)$ for $\xi \to -\infty$ we obtain

$$\widetilde{L_{\beta,-}} = \begin{pmatrix} \partial_\xi^2+(c-2\beta)\partial_\xi+\beta^2-c\beta-2 & 0 & 1 \\ 0 & \partial_\xi^2+(c-2\beta)\partial_\xi+\beta^2-c\beta-2 & \partial_\xi-\beta \\ 0 & 0 & \mu\partial_\xi^2+(c-2\mu\beta)\partial_\xi+\mu\beta^2-c\beta \end{pmatrix},$$

where we used that all $w \in \mathcal{D}(L_\beta)$ satisfy $w_C = (\partial_\xi - \beta)w_A$. Noting that $\widetilde{L_{\beta,-}}$ is an upper triangular matrix, we obtain using Fourier transform that

$$\sigma(\widetilde{L_{\beta,-}}) = \bigcup_{k\in\mathbb{R}} \{-k^2 + (c-2\beta)ik + \beta^2 - c\beta - 2, -\mu k^2 + (c-2\mu\beta)ik + \mu\beta^2 - c\beta\}.$$

In particular, this yields that $\mathrm{Re}(\sigma_{\text{ess}}(\widetilde{L_\beta})) \leq -\kappa$.

Secondly, we control the point spectrum. Therefore, consider the eigenvalue problem

$$\lambda w = \widetilde{L_\beta} w \tag{4.33}$$

for $w \in \mathcal{D}(\widetilde{L_\beta})$. We show that this eigenvalue problem (4.33) has no nontrivial solution if $\mathrm{Re}(\lambda) > -\kappa$. First note that the w_B-equation reads as

$$\lambda w_B = \mu\partial_\xi^2 w_B + (c-2\beta)\partial_\xi w_B + (\mu\beta^2 - c\beta)w_B =: \mathcal{L}_B w_B$$

and is decoupled from the remaining system. In particular, any $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -\kappa$ is in the resolvent set of $\mathcal{L}_B$ and hence any solution of (4.33) satisfies $w_B = 0$. Thus, the w_A-equation reads as

$$\lambda w_A = \partial_\xi^2 w_A + (c-2\beta)\partial_\xi w_A + (\beta^2 - c\beta + 1)w_A - 3A_{\text{front}}^2 w_A =: \mathcal{L}_A w_A.$$

Using $-3A_{\text{front}}^2 < 0$ and comparison principle, see e.g. [ES00], we find that $\mathrm{Re}(\sigma(\mathcal{L}_A)) \leq -\kappa$ and thus, $w_A = 0$. Since all $w \in \mathcal{D}(\widetilde{L_\beta})$ satisfy $w_C = (\partial_\xi - \beta)w_A$, this also implies $w_C = 0$. Hence, (4.33) has only the trivial solution in $\mathcal{D}(\widetilde{L_\beta})$ if $\mathrm{Re}(\lambda) > -\kappa$. This concludes the proof. □

Remark 4.4.2. It is worth pointing out that the condition on the choice of weights is determined by the (weighted) linearization $\widetilde{L_{\beta,+}}$ about the unstable origin. Therefore, the same weights, which are used to stabilize $\widetilde{L_\beta}$ also can be used to stabilize the origin. □

It remains to analyze the spectrum of L_ψ. Therefore, we prove the following result.

Lemma 4.4.3. *Let $c > 2$ then there exists a $\beta_\psi^0 > 0$, depending on c, such that for all $\beta_\psi \in (0, \beta_\psi^0)$ there exists a $\gamma_0 > 0$ such that for all $\gamma \in (-\gamma_0, \gamma_0)$ holds that*

$$\mathrm{Re}(\sigma(L_\psi)) < -\kappa_\psi < 0$$

for some $\kappa_\psi = \kappa_\psi(c, \beta_\psi, \gamma_0) > 0$.

Proof. Similar to the analysis of $\widetilde{L_\beta} + P(\gamma)$ we first set $\gamma = 0$. Let $w_\psi \in \mathcal{D}(L_\psi)$ with $\|w_\psi\|_{L^2} = 1$. Then, we estimate

$$\begin{aligned}
\mathrm{Re}\,\langle L_\psi w_\psi, w_\psi\rangle_{L^2} &= -\|\partial_\xi w_\psi\|_{L^2}^2 + (\beta_\psi^2 - c\beta_\psi) + 2\,\mathrm{Re}\left\langle (\partial_\xi - \beta_\psi)\left(\frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}} w_\psi\right), w_\psi \right\rangle_{L^2} \\
&= -\|\partial_\xi w_\psi\|_{L^2}^2 + (\beta_\psi^2 - c\beta_\psi) \\
&\quad + 2\left\langle \left(-\beta_\psi \frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}} + \frac{1}{2}\partial_\xi \frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}}\right) w_\psi, w_\psi \right\rangle_{L^2} \\
&\leq -\|\partial_\xi w_\psi\|_{L^2}^2 + \beta_\psi^2 - c\beta_\psi + \sup\left(-2\beta_\psi \frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}} + \partial_\xi \frac{\partial_\xi A_{\text{front}}}{A_{\text{front}}}\right) \\
&=: -\|\partial_\xi w_\psi\|_{L^2}^2 + \beta_\psi^2 - c\beta_\psi + \sup(f)
\end{aligned}$$

It remains to calculate the supremum of f. Therefore, we note that for $\gamma = 0$, $\partial_\xi A_{\text{front}}/A_{\text{front}}$ is negative and monotonically decreasing, see Proposition 4.A.3. This yields with Proposition 4.A.1 that

$$\sup(f) \leq \beta_\psi(c - \sqrt{c^2 - 4}).$$

Therefore, we obtain that

$$\mathrm{Re}(\sigma(L_\psi)) \leq \beta_\psi^2 - \beta_\psi\sqrt{c^2-4}$$

and thus, for $c > 2$, there exists a β_ψ^0 such that $\mathrm{Re}(\sigma(L_\psi)) < -\kappa_\psi$ for all $\beta_\psi \in (0, \beta_\psi^0)$ with $\kappa_\psi = \kappa_\psi(\beta_\psi) > 0$. Then, using perturbation arguments, we obtain the result also for γ close to zero. □

Now, using Lemmas 4.4.1 and 4.4.3, (4.32) and applying [Kat66, Theorem IV.3.17] we obtain the spectral stability of the full weighted operator.

Lemma 4.4.4. *Let $\mu > 0$ and $c > c_{min}(\mu)$. There exists a $\beta_\psi^0 > 0$ such that for all $\beta > 0$, $\beta_\psi > 0$ satisfying*

$$\begin{aligned}
&\max(\beta^2 - c\beta + 1, \mu\beta^2 - c\beta) < 0\\
&\beta_\psi \in (0, \beta_\psi^0)
\end{aligned}$$

exists a $\tilde{\kappa} > 0$ such that there exists a $\gamma_0 > 0$ such that for all $\gamma \in (-\gamma_0, \gamma_0)$ holds

$$\mathrm{Re}(\sigma(L_\beta)) \leq -\frac{\tilde{\kappa}}{2}.$$

Remark 4.4.5. Using similar arguments, we can also deduce that L_β is a closed operator and in particular sectorial in L^2. □

4.4.2. Linear stability estimates

After proving the spectral stability of L_β in the previous section, we now focus on the linear stability. In particular, we aim to prove the linear stability estimates (4.21), (4.22), which we restate below. Therefore, we fix parameters $\beta > 0$, $\beta_\psi > 0$, $\mu > 0$ and $c > c_{\min}(\mu)$ such that Lemma 4.4.4 applies. In this case, L_β is a closed, sectorial operator in L^2 and hence, standard semigroup theory implies that L_β generates an analytic semigroup in L^2, see e.g. [EN00], which we denote by e^{tL_β} for all $t \geq 0$. We now prove that the estimates

$$\left\| e^{tL_\beta} w \right\|_{L^2} \lesssim e^{-\kappa t} \left\| w \right\|_{L^2}, \tag{4.34}$$

$$\left\| e^{tL_\beta} \partial_\xi w \right\|_{L^2} \lesssim (1 + t^{-1/2}) e^{-\kappa t} \left\| w \right\|_{L^2}, \tag{4.35}$$

hold for some $\kappa > 0$ and all $w \in L^2$, $t > 0$.

For the first estimate, we recall two results from semigroup theory, see e.g. [EN00]. First, we define the growth bound of a strongly continuous semigroup with generator L by

$$\eta_0(L) := \inf\{\eta \in \mathbb{R} : \exists M < \infty \text{ such that } \left\| e^{tL} \right\| \leq M e^{\eta t}\}.$$

Note that this is well-defined for all strongly continuous semigroups and that the growth bound is connected to the spectral radius of e^{tL} by

$$\eta_0(L) = \frac{1}{t} \log(r(e^{tL})),$$

for some fixed $t > 0$, where $r(e^{tL}) := \sup\{|\lambda| : \lambda \in \sigma(e^{tL})\}$, see [EN00, Proposition IV.2.2]. Furthermore, if L is a sectorial operator, then the Spectral Mapping Theorem holds, i.e.

$$\sigma(e^{tL}) \setminus \{0\} = \{e^{\lambda t} : \lambda \in \sigma(L)\},$$

see [EN00, Corollary IV.3.12]. Using these results, we obtain the first linear stability estimate (4.34) by using that L_β is a sectorial operator in L^2 and that $\mathrm{Re}(\sigma(L_\beta))$ is strictly negative.

We now consider the second linear stability estimate (4.35). To prove this, we introduce operators T, containing the terms with constant coefficients and the remainder S, such that $L_\beta = T + S$. Next, we introduce the operators S_1, S_2 such that

$$S\partial_\xi w = (\partial_\xi S_1)w + S_2 w$$

for $w \in H^1$. We note that S, S_1 and S_2 are T-bounded – in fact, S_2 is a bounded operator in L^2 – with constants $\tilde{a}, \tilde{b}$, see [Kat66, Chapter IV.1.1] for a definition. The main result leading to (4.35) is the following resolvent estimate.

Lemma 4.4.6. *There exists a sector* $\mathcal{S}(w_0, \varphi) = \{\lambda \in \mathbb{C} : \arg(\lambda - w_0) \in (\pi - \varphi, \pi + \varphi)\} \subset \mathbb{C}$, $w_0 \in \mathbb{R}$, $\varphi \in (0, \pi)$ *such that*

$$\left\| R(\lambda, L_\beta)\partial_\xi w \right\|_{L^2} \leq \delta^{-1} \left\| R(\lambda, T)\partial_\xi \right\|_{L^2 \to L^2} \left\| w \right\|_{L^2} + \frac{1-\delta}{\delta^2} \left\| R(\lambda, T) \right\|_{L^2 \to L^2} \left\| w \right\|_{L^2},$$

for all $\lambda \in \mathbb{C} \setminus \mathcal{S}(w_0, \varphi)$ and all $w \in H^1$. Here, $R(\lambda, \mathcal{L}) := (\lambda - \mathcal{L})^{-1}$ is the resolvent map, which is defined on the resolvent set of $\mathcal{L}$.

Proof. For any chosen $\delta \in (0,1)$ we choose a sector $\mathcal{S}(w_0, \varphi)$ such that

$$\tilde{a} \left\| R(\lambda, T) \right\|_{L^2 \to L^2} + \tilde{b} \left\| T R(\lambda, T) \right\|_{L^2 \to L^2} < 1 - \delta,$$

which in particular yields

$$\left\| J R(\lambda, T) \right\|_{L^2 \to L^2} \leq (1 - \delta)$$

for $J \in \{S, S_1, S_2\}$, since J is T-bounded. This is possible using that T is a constant coefficient, second-order operator and thus $R(\lambda, T)$ and $TR(\lambda, T)$ can be estimates using Fourier transform. Hence, following the proof of [Kat66, Theorem IV.3.17], we have that for all $\lambda \in \mathbb{C} \setminus \mathcal{S}(w_0, \varphi)$ holds that $\left\| SR(\lambda, T) \right\|_{L^2 \to L^2} < 1 - \delta$ and that

$$R(\lambda, L_\beta) = R(\lambda, T)(1 - SR(\lambda, T))^{-1} = R(\lambda, T) \sum_{n=0}^{\infty} (SR(\lambda, T))^n.$$

Furthermore, we have that $\partial_\xi S_1$ is T-bounded, since it is a second order differential operator with smooth coefficients. This yields that $\left\| \partial_\xi S_1 R(\lambda, T) \right\|_{L^2 \to L^2} < K < \infty$ and especially,

$$\left\| \partial_\xi (S_1 R(\lambda, T))^n \right\|_{L^2 \to L^2} \leq K \left\| S_1 R(\lambda, T) \right\|_{L^2 \to L^2}^{n-1} < K(1 - \delta)^{n-1}$$

for all $n \geq 1$. Hence, the series $\sum_{n=1}^{\infty} \partial_\xi (S_1 R(\lambda, T))^n$ is well-defined and a bounded operator mapping $L^2 \to L^2$.

Now let $w \in H^1$ be arbitrary. Then, in particular $\sum_{n=1}^{N} (S_1 R(\lambda, T))^n w$ converges in H^1 for $N \to \infty$ and thus, since $\partial_\xi : H^1 \to L^2$ is closed this yields

$$\partial_\xi \sum_{n=0}^{\infty} (S_1 R(\lambda, T))^n w = \sum_{n=0}^{\infty} \partial_\xi (S_1 R(\lambda, T))^n w.$$

Hence, it holds that

$$R(\lambda, L_\beta) \partial_\xi w = R(\lambda, T) \sum_{n=0}^{\infty} (SR(\lambda, T))^n \partial_\xi w$$

$$
\begin{aligned}
&= R(\lambda, T) \sum_{n=0}^{\infty} \Bigg[\partial_\xi (S_1 R(\lambda, T))^n w \\
&\qquad\qquad\qquad + \sum_{j=0}^{n-1} (SR(\lambda, T))^{n-1-j} S_2 R(\lambda, T) (S_1 R(\lambda, T))^j w \Bigg] \\
&= R(\lambda, T) \partial_\xi \sum_{n=0}^{\infty} (S_1 R(\lambda, T))^n w \\
&\quad + R(\lambda, T) \sum_{n=0}^{\infty} \sum_{j=0}^{n-1} (SR(\lambda, T))^{n-1-j} S_2 R(\lambda, T) (S_1 R(\lambda, T))^j w,
\end{aligned}
$$

where we used the definition of S_1, S_2. Using the choice of $\mathcal{S}(w_0, \varphi)$ and the fact that $JR(\lambda, T)$, $J \in \{S, S_1, S_2\}$ are bounded operators on $L^2 \to L^2$, we can estimate that

$$
\begin{aligned}
&\sum_{j=0}^{n-1} \left\| (SR(\lambda, T))^{n-1-j} S_2 R(\lambda, T) (S_1 R(\lambda, T))^j \right\|_{L^2 \to L^2} \\
&\qquad \leq n(\tilde{a} \left\| R(\lambda, T) \right\|_{L^2 \to L^2} + \tilde{b} \left\| TR(\lambda, T) \right\|_{L^2 \to L^2})^n \\
&\qquad < n(1 - \delta)^n,
\end{aligned}
$$

which finally yields that

$$
\left\| R(\lambda, L_\beta) \partial_\xi w \right\|_{L^2} \leq \delta^{-1} \left\| R(\lambda, T) \partial_\xi \right\|_{L^2 \to L^2} \left\| w \right\|_{L^2} + \frac{1 - \delta}{\delta^2} \left\| R(\lambda, T) \right\|_{L^2 \to L^2} \left\| w \right\|_{L^2}
$$

for all $w \in H^1$. This concludes the proof. $\square$

Note that the above result is not enough to proceed yet since $\mathcal{S}(w_0, \varphi)$, in general, contains a subset of $\mathbb{C}^+$. Therefore, we fix $\gamma \in (-\gamma_0, \gamma_0)$ with γ_0 from Lemma 4.4.4 with corresponding $\tilde{\kappa} > 0$. Then, we show that for all $\lambda \in \Omega := \mathcal{S}(w_0, \varphi) \cap \{\mathrm{Re}(\lambda) > -\tilde{\kappa}/4\}$ there exist a constant $K < \infty$ such that

$$
\left\| R(\lambda, L_\beta) \partial_\xi w \right\|_{L^2} \leq K \left\| w \right\|_{L^2}. \tag{4.36}
$$

Therefore, we use that for $\lambda_1, \lambda_2 \in \rho(L_\beta)$ with $|\lambda_1 - \lambda_2|$ small enough, the resolent $R(\lambda_2, L_\beta)$ can be expanded into a Neumann series such that

$$
R(\lambda_2, L\beta) = \sum_{n=0}^{\infty} (\lambda_2 - \lambda_1)^n R(\lambda_1, L_\beta)^{n+1}
$$

holds, where the sum converges absolutely. Recalling that $\mathrm{Re}(\sigma(L_\beta)) < -\tilde{\kappa}/2$ using Lemma 4.4.4, we can bound $R(\lambda, L_\beta)$ uniformly on Ω since it is bounded away from the spectrum of L_β. Furthermore, using the fact that Ω is compact, we can find a finite set of balls $\mathcal{B}_l$, which covers Ω. Then, by choosing the $\mathcal{B}_l$ small enough, we can estimate

$$\|R(\lambda_2, L_\beta)\partial_\xi w\|_{L^2} \le \|R(\lambda_1, L_\beta)\partial_\xi w\|_{L^2} \sum_{n=0}^{\infty} |\lambda_2 - \lambda_1|^n \|R(\lambda_1, L_\beta)\|^n_{L^2\to L^2}$$

for all $\lambda_1, \lambda_2 \in \mathcal{B}_l$ for some l. Then using Lemma 4.4.6, we obtain (4.36) for all $\lambda \in \Omega$.

Now, using the above Lemma 4.4.6 and (4.36), we can derive (4.35) by noting that L_β is a sectorial operator and thus, the semigroup is defined by

$$e^{tL_\beta} w = \frac{1}{2\pi i} \int_\Gamma e^{t\lambda} R(\lambda, L_\beta) w \, d\lambda,$$

for a curve Γ in the resolvent set of L_β going from $\infty e^{i(\pi/2+\tilde{\delta})}$ to $\infty e^{-i(\pi/2+\tilde{\delta})}$ for some $\tilde{\delta} \in (0, \vartheta)$, with ϑ from Lemma 4.4.6. Furthermore, we choose Γ such that $\mathrm{Re}(\Gamma) \le -\tilde{\kappa}/4$. In particular, for all $\lambda \in \Gamma + \tilde{\kappa}/4$ we have the estimate

$$\|R(\lambda, T)\partial_\xi\|_{L^2\to L^2} \lesssim |\lambda|^{-1/2} \text{ and } \|R(\lambda, T)\|_{L^2\to L^2} \lesssim |\lambda|^{-1},$$

which can be checked using Fourier analysis, since T is a constant coefficient operator and explicitly given. Using the resolvent estimate provided in Lemma 4.4.6 and the fact that T generates a strongly continuous semigroup yields

$$\left\| e^{tL_\beta} \partial_\xi w \right\|_{L^2} \lesssim e^{-(\tilde{\kappa}/4)t} \int_{\Gamma+\tilde{\kappa}/4} |e^{\lambda t}| (|\lambda|^{-1/2} + |\lambda|^{-1}) \, d\lambda \, \|w\|_{L^2} \lesssim (1 + t^{-1/2}) e^{-\tilde{\kappa}t/4} \|w\|_{L^2},$$

which proves the linear stability estimate (4.35) for all $w \in H^1$. Furthermore, using that H^1 is dense in L^2 the estimate also holds for $w \in L^2$. This concludes the proof of Lemma 4.3.1.

Remark 4.4.7. We hightlight that it is sufficient to show (4.36) on Ω since the crucial part of the semigroup estimate comes from the decay of the resolvent for $|\mathrm{Re}(\lambda)| \to \infty$. □

4.5. Spectral stability for $|\gamma|$ large

As pointed out in the motivation of Theorem 4.1.9, we recall that the nonlinear analysis in Section 4.3.2 does not require that γ is close to zero. Therefore, we now study the spectral properties of the weighted operator L_β in the case that γ is not close to zero with the goal of verifying Assumption (A2). While the presence of unstable essential spectrum can be excluded similarly as in the previous section, the presence of unstable point spectrum is still possible. To this end, we employ numerical Evans function computation, see e.g. [Bar09, Bar14, BHLL18] to show that there is no unstable point spectrum and indeed also a spectral gap between the imaginary axis and the spectrum. Here, we use the Matlab implementation in STABLAB [BHLZ15]. This leads to similar spectral results as in Section 4.4.

Since we do not have an analytical existence result for traveling front solutions of (4.1) except for γ close to zero, we additionally assume their existence, see Assumption (A1). However, the fronts can be found numerically as in Section 2.4.1, see Figure 2.4. Furthermore, we make the following assumption, which can also be verified numerically.

(A3) The quotient $f_\psi = \partial_\xi A_{\text{front}}/A_{\text{front}}$ is negative and monotonically decreasing.

Now, we recall that the weighted operator L_β has a diagonal structure and is given by

$$L_\beta = \begin{pmatrix} \widetilde{L_\beta} + P(\gamma) & 0 \\ 0 & L_\psi \end{pmatrix},$$

see Section 4.4. Using Assumption (A3) we find that Lemma 4.4.3 still holds true for γ not close to zero. Hence, we can restrict to studying the spectral properties of $\mathcal{L}_\beta := \widetilde{L_\beta} + P(\gamma)$.

4.5.1. Spectral estimates

In order to deploy numerical methods, we have to exclude the presence of unstable eigenvalues on the complex plane except for a compact subset. Then, we can use numerical methods to study the existence of eigenvalues on this compact domain. As a first step, we now derive bounds on the spectrum of $\mathcal{L}_\beta$. These are not sophisticated enough to restrict the spectrum to the negative complex half-plane, however, they are sufficient for our purposes.

We decompose $\mathcal{L}_\beta = T + S$, where T is a diagonal, constant-coefficient operator and S is a T-bounded operator, that is, the domain of T is a subset of the domain of S (i.e. $\mathcal{D}(T) \subset \mathcal{D}(S)$) and there exist constants $a, b > 0$ such that

$$\|Sw\|_{L^2} \leq a\,\|w\|_{L^2} + b\,\|Tw\|_{L^2}$$

for all w in the domain of T, see [Kat66]. Here, T and S are given by

$$Tw = \begin{pmatrix} \partial_\xi^2 w_A + (c-2\beta)\partial_\xi w_A + (\beta^2 - c\beta + 1)w_A \\ \partial_\xi^2 w_C + (c-2\beta)\partial_\xi w_C + (\beta^2 - c\beta + 1)w_C \\ \mu\partial_\xi^2 w_B + (c-2\mu\beta)\partial_\xi w_B + (\mu\beta^2 - c\beta)w_B \end{pmatrix},$$
$$Sw = \begin{pmatrix} A_{\text{front}} w_B + (B_{\text{front}} - 3A_{\text{front}}^2)w_A \\ (\partial_\xi - \beta)(A_{\text{front}} w_B + (B_{\text{front}} - 3A_{\text{front}}^2)w_A) \\ 2\gamma(\partial_\xi - \beta)(A_{\text{front}} w_C + C_{\text{front}} w_A) \end{pmatrix}.$$

To estimate the spectrum of $\mathcal{L}_\beta$, we use that the spectrum of the constant coefficient operator T can be calculated explicitly via Fourier analysis. Then, we apply [Kat66, Theorem IV.3.17], which implies that all $\lambda \in \rho(T)$ – the resolvent set of T – satisfying

$$a\,\|R(\lambda, T)\|_{L^2\to L^2} + b\,\|TR(\lambda, T)\|_{L^2\to L^2} < 1$$

are in the resolvent set of $\mathcal{L}_\beta = T + S$. Here, $R(\lambda, T) = (\lambda - T)^{-1}$ is the resolvent mapping. Additionally, if there is at least one such λ, then $\mathcal{L}_\beta$ is closed. By explicit calculation of $R(\lambda, T)$ and $TR(\lambda, T)$ we then obtain the following result.

Lemma 4.5.1. *For all $c > 2$, $\mu > 0$, $\gamma > -\mu$ and $\beta > 0$ exists a sector $\mathcal{S}(\tilde{\theta}, \tilde{\omega}_0) := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < \tilde{\omega}_0$ and $\tilde{\theta} + \pi/2 < |\arg(\lambda)|\}$ for some $\tilde{\omega}_0 \in \mathbb{R}$ and $\tilde{\theta} \in (0, \pi/2)$ such that $\mathcal{L}_\beta$ is a closed operator in L^2 and $\sigma(\mathcal{L}_\beta) \subset \mathcal{S}(\tilde{\omega}_0, \tilde{\theta})$.*

Note that in general $\tilde{\omega}_0$ will be large and positive and thus, the usage of the sector $\mathcal{S}$ is not particularly useful in our setting. However, we can improve the bound on the spectrum by estimating the numerical range $\mathcal{R}_{L^2}(\mathcal{L}_\beta)$, given by

$$\mathcal{R}_{L^2}(\mathcal{L}_\beta) := \{\langle \mathcal{L}_\beta w, w\rangle_{L^2} : w \in \mathcal{D}(\mathcal{L}_\beta), \|w\|_{L^2} = 1\},$$

see [KP13, Equation (4.1.11)]. Also recall that $w \in \mathcal{D}(\mathcal{L}_\beta)$ implies that $(\partial_\xi - \beta)w_A = w_C$. Then, we find the following estimate.

Lemma 4.5.2. *Let $\mu > 0$, $c > 0$, $\gamma > -\mu$ and $\beta > 0$. For all $w \in \mathcal{D}(\mathcal{L}_\beta)$ holds*

$$\begin{aligned}
\mathrm{Re}(\langle \mathcal{L}_\beta w, w \rangle_{L^2}) &\leq -\min(1,\mu) \left\| \partial_\xi w \right\|_{L^2}^2 + a_1 \left\| \partial_\xi w \right\|_{L^2} + a_0, \\
|\,\mathrm{Im}(\langle \mathcal{L}_\beta w, w \rangle_{L^2})| &\leq \tilde{a}_1 \left\| \partial_\xi w \right\|_{L^2} + \tilde{a}_0,
\end{aligned}$$

where the constants are given by

$$\begin{aligned}
a_0 &= \max(\beta^2 - c\beta + 1, \mu\beta^2 - c\beta) + \max(0, \sup_{\xi \in \mathbb{R}} (B_{\mathit{front}}(\xi) - 3A_{\mathit{front}}^2(\xi))), \\
&\quad + \frac{1}{2} \left(\left\| A_{\mathit{front}} + 2\gamma \partial_\xi C_{\mathit{front}} \right\|_\infty + \left\| \partial_\xi (B_{\mathit{front}} - 3A_{\mathit{front}}^2 \right\|_\infty \right) \\
&\quad + \frac{1}{2} \left\| (\partial_\xi - \beta) A_{\mathit{front}} + 2\gamma (C_{\mathit{front}} - \beta A_{\mathit{front}}) \right\|_\infty \\
a_1 &= \min \left[(\sqrt{3}|c - 2\beta| + |c - 2\mu\beta|, \sqrt{3} \max(|c - 2\beta|, |c - 2\mu\beta|) \right] \\
&\quad + \sqrt{3} |1 - 2\gamma| \left\| A_{\mathit{front}} \right\|_\infty, \\
\tilde{a}_0 &= \frac{1}{2} \left(\left\| A_{\mathit{front}} - 2\gamma \partial_\xi C_{\mathit{front}} \right\|_\infty + \left\| \partial_\xi (B_{\mathit{front}} - 3A_{\mathit{front}}^2 \right\|_\infty \right) \\
&\quad + \frac{1}{2} \left\| (\partial_\xi - \beta) A_{\mathit{front}} - 2\gamma (C_{\mathit{front}} - \beta A_{\mathit{front}}) \right\|_\infty, \\
\tilde{a}_1 &= \min \left[(\sqrt{3}|c - 2\beta| + |c - 2\mu\beta|, \sqrt{3} \max(|c - 2\beta|, |c - 2\mu\beta|) \right] \\
&\quad + \sqrt{3} |1 + 2\gamma| \left\| A_{\mathit{front}} \right\|_\infty.
\end{aligned}$$

In particular, the complement of the numerical range $\mathcal{R}_{L^2}(\mathcal{L}_\beta)$ in $\mathbb{C}$ is a connected set.

Proof. We recall that $\left\| (w_A, w_C, w_B) \right\|_{L^2}^2 = \left\| w_A \right\|_{L^2}^2 + \left\| w_C \right\|_{L^2}^2 + \left\| w_B \right\|_{L^2}^2$. Furthermore, using $w_C = (\partial_\xi - \beta) w_A$ we can rewrite

$$\begin{aligned}
(\partial_\xi - \beta)((B_{\text{front}} - 3A_{\text{front}}^2) w_A) &= w_A \partial_\xi (B_{\text{front}} - 3A_{\text{front}}^2) + (B_{\text{front}} - 3A_{\text{front}}^2) w_C, \\
(\partial_\xi - \beta)(A_{\text{front}} w_C + C_{\text{front}} w_A) &= ((\partial_\xi - \beta) A_{\text{front}} + C_{\text{front}}) w_C + A_{\text{front}} \partial_\xi w_C + \partial_\xi C_{\text{front}} w_A
\end{aligned}$$

in $\mathcal{L}_\beta$, which turns out to improve the constants in the estimates. Then, using integration by parts and the fact that the front solutions are real-valued, we obtain for $w = (w_A, w_C, w_B)^T$, with $\left\| w \right\|_{L^2} = 1$ that

$$\begin{aligned}
\langle \mathcal{L}_\beta w, w \rangle_{L^2} &= \sum_{j = \{A, C, B\}} \langle (L_\beta w)_j, w_j \rangle_{L^2} \\
&= -(\left\| \partial_\xi w_A \right\|_{L^2}^2 + \left\| \partial_\xi w_C \right\|_{L^2}^2 + \mu \left\| \partial_\xi w_B \right\|_{L^2}^2) + (\beta^2 - c\beta + 1)(\left\| w_A \right\|_{L^2}^2 + \left\| w_C \right\|_{L^2}^2)
\end{aligned}$$

$$
\begin{aligned}
&+ (\mu\beta^2 - c\beta) \left\|w_B\right\|_{L^2}^2 + \sum_{j\in\{A,C\}} \left\langle (B_{\text{front}} - 3A_{\text{front}}^2) w_j, w_j \right\rangle + \langle A_{\text{front}} w_B, w_A \rangle \\
&+ \overline{\langle 2\gamma \partial_\xi C_{\text{front}} w_B, w_A \rangle} + (c - 2\beta)(\langle \partial_\xi w_A, w_A \rangle + \langle \partial_\xi w_C, w_C \rangle) \\
&+ (c - 2\mu\beta) \langle \partial_\xi w_B, w_B \rangle + \left\langle \partial_\xi (B_{\text{front}} - 3A_{\text{front}}^2) w_A, w_C \right\rangle + \langle (\partial_\xi - \beta) A_{\text{front}} w_B, w_C \rangle \\
&+ \overline{\langle 2\gamma (C_{\text{front}} - \beta A_{\text{front}}) w_B, w_C \rangle} + \langle A_{\text{front}} \partial_\xi w_B, w_C \rangle - 2\gamma \overline{\langle A_{\text{front}} \partial_\xi w_B, w_C \rangle}
\end{aligned}
$$

We additionally use that $\|w_i\|_{L^2} \|w_j\|_{L^2} \leq 1/2$ and $\|w_A\|_{L^2} + \|w_C\|_{L^2} + \|w_B\|_{L^2} \leq \sqrt{3}$, since $\|w\|_{L^2} = 1$. Then, estimating the real part and the absolute value of the imaginary part of $\langle \mathcal{L}_\beta w, w\rangle$ separately yields the conjectured estimates. □

Since $\mathcal{L}_\beta$ is a closed operator in L^2 and $(\mathbb{C} \setminus \mathcal{R}_{L^2}(\mathcal{L}_\beta)) \cap \rho(\mathcal{L}_\beta)$ is non-empty due to Lemma 4.5.1, we can apply [Kat66, Theorem V.3.2], which yields the following statement.

Lemma 4.5.3. *For all $c > 2$, $\mu > 0$, $\gamma > -\mu$ and $\beta > 0$, the L^2-spectrum of $\mathcal{L}_\beta$ is contained in a parabolic region of the complex plane. In particular, $\sigma(\mathcal{L}_\beta) \subset \mathcal{R}_{L^2}(\mathcal{L}_\beta)$.*

4.5.2. Estimates of the essential spectrum of $\mathcal{L}_\beta$

As in the case γ close to zero (Section 4.4) we use that the coefficients of $\mathcal{L}_\beta$ converge exponentially fast to their limit states for $\xi \to \pm\infty$. Hence, we introduce

$$\mathcal{L}_\beta^{\pm} := \lim_{\xi\to\pm\infty} \mathcal{L}_\beta.$$

Then, following [San02] the essential spectrum of $\mathcal{L}_\beta$ can be bounded be the the spectrum of the respective asymptotic operators. Since the front $(A_{\text{front}}, C_{\text{front}}, B_{\text{front}})$ vanishes for $\xi \to +\infty$ the operator $\mathcal{L}_\beta^+$ is diagonal and thus, utilizing Fourier transform we find

$$\sigma(\mathcal{L}_\beta^+) = \bigcup_{k\in\mathbb{R}} \{-k^2 + (c-2\beta)ik + \beta^2 - c\beta + 1, -\mu k^2 + (c - 2\mu\beta)ik + \mu\beta^2 - c\beta\}.$$

For $\xi \to -\infty$ we recall that $(A_{\text{front}}, C_{\text{front}}, B_{\text{front}})$ tends to $(1,0,0)$. Thus, we find

$$\mathcal{L}_\beta^- = \begin{pmatrix} \partial_\xi^2 + (c-2\beta)\partial_\xi + \beta^2 - c\beta - 2 & 0 & 1 \\ 0 & \partial_\xi^2 + (c-2\beta)\partial_\xi + \beta^2 - c\beta - 2 & \partial_\xi - \beta \\ 0 & 2\gamma(\partial_\xi - \beta) & \mu\partial_\xi^2 (c-2\mu\beta)\partial_\xi + \mu\beta^2 - c\beta \end{pmatrix}.$$

Note that since $\gamma \neq 0$, this is not an upper triangular operator. However, we can still calculate the eigenvalues of this operator in Fourier space, which are given by

$$\begin{aligned}\lambda_1 &= \beta^2 - c\beta - 2,\\ \lambda_2^{\pm} &= \frac{1}{2}\left(-2 + 2ic(k+i\beta) - (k+i\beta)^2(1+\mu)\right)\\ &\quad \pm \frac{1}{2}\sqrt{4 + (k+i\beta)^4(-1+\mu)^2 - 4(k+i\beta)^2(-1+2\gamma+\mu)},\end{aligned}$$

for $k \in \mathbb{R}$. Although the eigenvalues can be explicitly calculated, for practical purposes we estimate the eigenvalues using Gerschgorin circle theorem and find

$$\mathrm{Re}(\mathcal{L}_\beta^-) \leq \max\left(\beta^2 - c\beta - 1, \beta^2 + (1-c)\beta - \frac{7}{4}, \mu\beta^2 - c\beta + 2\beta|\gamma| + \frac{|\gamma|^2}{\mu}\right).$$

Combining this with the spectrum of $\mathcal{L}_\beta^+$ we arrive at the following result.

Lemma 4.5.4. *Fix $\mu > 0$, $\gamma \in \mathbb{R}$, $\beta > 0$ and $c > c_{min}^{\mu,\gamma}(\beta)$ with*

$$c_{min}^{\mu,\gamma}(\beta) := \max\left(\beta + \frac{1}{\beta}, \beta + 1 - \frac{7}{4\beta}, \mu\beta + 2|\gamma| + \frac{|\gamma|^2}{\mu\beta}\right).$$

Then, there exists a $\delta = \delta(\mu, \gamma, c, \beta) > 0$ such that $\mathrm{Re}(\sigma_{ess}(\mathcal{L}_\beta)) \leq -\delta$.

Remark 4.5.5. Note that there are parameter values (γ, μ) such that the essential spectrum of $\mathcal{L}_\beta^-$, i.e. the weighted linearization about the invading state, is unstable for all weights $\beta > 0$ such that $\mathrm{Re}(\sigma(\mathcal{L}_\beta^+)) < 0$. This can be seen by evaluating the eigenvalues λ_2^+ at Fourier number $k = 0$, which yields

$$\lambda_2^+ = \frac{1}{2}\left(-2 - 2c\beta + \beta^2(1+\mu) + \sqrt{4 + \beta^4(\mu-1)^2 + 4\beta^2(-1+2\gamma+\mu)}\right).$$

Thus, for any $\mu > 1$ and $\gamma > 0$ we find

$$\mathrm{Re}(\lambda_2^+) \geq \frac{1}{2}\beta^2(1+\mu) - c\beta + \sqrt{1 + 2\gamma\beta^2},$$

which is positive for any fixed $\beta > 0$ if γ is sufficiently large. In particular, since $\mathcal{L}_\beta^+$ is independent of γ, the condition $\mathrm{Re}(\sigma(\mathcal{L}_\beta^+)) < 0$ yields that $\beta > \beta_c > 0$ for some β_c independent of γ. Thus, we find a $\gamma_0 > 0$ such that $\mathrm{Re}(\lambda_2^+) > 0$ for all admissible weights

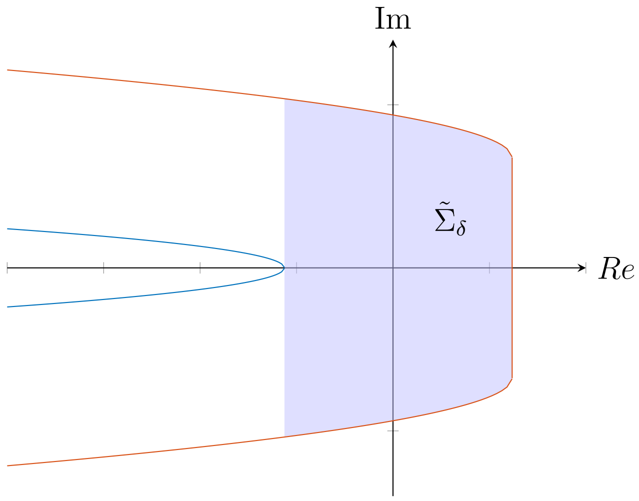

Figure 4.6.: Spectral boundary for $\mathcal{L}_\beta$. The essential spectrum lies to the left of the blue curve, while the orange curve marks the bound on the numerical range. Therefore, the set $\tilde{\Sigma}_\delta$ contains only isolated eigenvalues.

β. However, we highlight that we can circumvent this issue by using the refined weights

$$\omega(\xi) = \begin{cases} e^{\beta^+\xi}, & \xi \geq 1 \\ e^{\beta^-\xi}, & \xi \leq -1 \end{cases},$$

for $\beta^+, \beta^- > 0$, instead of $\omega(\xi) = e^{\beta\xi}$. □

4.5.3. Numerical analysis of the point spectrum

Fix now μ, c, γ and β as in Lemma 4.5.4. Then, using Lemmas 4.5.2, 4.5.3 and 4.5.4 we obtain the following setting. First, there exists a δ such that $\mathrm{Re}(\sigma_{\mathrm{ess}}(\mathcal{L}_\beta)) \leq -\delta$. Second, the compact set $\tilde{\Sigma}_\delta := \mathcal{R}_{L^2}(\mathcal{L}_\beta) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\delta\}$, see Figure 4.6, contains at most isolated point spectrum. And third, there is no spectrum with real part larger than $-\delta$ in the complement of $\tilde{\Sigma}_\delta$. Summarizing, to obtain spectral stability, it suffices to study the presence of point spectrum in $\tilde{\Sigma}_\delta$. Since $\tilde{\Sigma}_\delta$ is a compact set, this problem is accessible to numerical treatment and as discussed above, we use numerical Evans function computation. For an introduction to the method and to the Evans function itself we refer to [Bar09, BHLL18] and [San02, KP13], respectively. The main idea of the method is that there exists an analytic function $\mathcal{E} : \mathbb{C}_{\mathrm{Re}(\lambda)>-\delta} \to \mathbb{C}$, called Evans function, such that

$$\mathcal{E}(\lambda) = 0 \iff \lambda \in \sigma_{\mathrm{pt}}(\mathcal{L}_\beta),$$

which can be constructed using spatial dynamics. That is, the point spectrum can be characterized by the roots of an analytic function. Thus, the number of eigenvalues in $\tilde{\Sigma}_\delta$ equals the winding number

$$\omega_\Gamma(\mathcal{E}) = \frac{1}{2\pi i}\int_\Gamma \frac{\mathcal{E}'(\lambda)}{\mathcal{E}(\lambda)}\, d\lambda,$$

where Γ is a simple, closed and positively oriented curve enclosing the domain $\tilde{\Sigma}_\delta$. Here, we use the implementation available in STABLAB [BHLZ15] for our numerical validation.

To apply this method to our setting, we numerically calculate the traveling front solutions $(A_{\mathrm{front}}, B_{\mathrm{front}})$ of (4.1) for given parameters μ, γ and c. Inserting the traveling front ansatz $(A, B)(x, t) = (A_{\mathrm{front}}, B_{\mathrm{front}})(\xi)$ with $\xi = x - ct$ into (4.1), integrating the conservation law once and writing the resulting system as a first-order ODE in ξ, we find

$$\begin{aligned}
\partial_\xi A_{\mathrm{front}} &= C_{\mathrm{front}},\\
\partial_\xi C_{\mathrm{front}} &= -cC_{\mathrm{front}} - A_{\mathrm{front}} - A_{\mathrm{front}}B_{\mathrm{front}} + A_{\mathrm{front}}^3,\\
\partial_\xi B_{\mathrm{front}} &= \frac{1}{\mu}(-cB_{\mathrm{front}} - 2\gamma A_{\mathrm{front}}C_{\mathrm{front}}).
\end{aligned}$$

We then construct a heteroclinic orbit connecting $(1, 0, 0)$ to $(0, 0, 0)$ for this system by approximating an initial point on the unstable manifold of $(1, 0, 0)$ using the unstable eigenspace of the linearisation about $(1, 0, 0)$ and solving the ODE system forward in ξ. With this solution, we can also set the numerical infinity in the algorithm, i.e. a point where the front is numerically close enough (in our test, we require that the distance is smaller than 10^{-5}) to the asymptotic states. Since in the most general setting, the problem is dependent on four parameters (μ, γ, c, β), we restrict the numerical computations to two particularly interesting scenarios.

Scenario 1 – fixed $\mu = 1$ and varying (γ, c). In the first case, we fix $\mu = 1$. This allows for the fixed choice of $\beta = c/2$. We then fix

$$\gamma \in \mathcal{I}_\gamma^1 := \{-0.1 + 0.01k,\ k = 0, \ldots, 20\}.$$

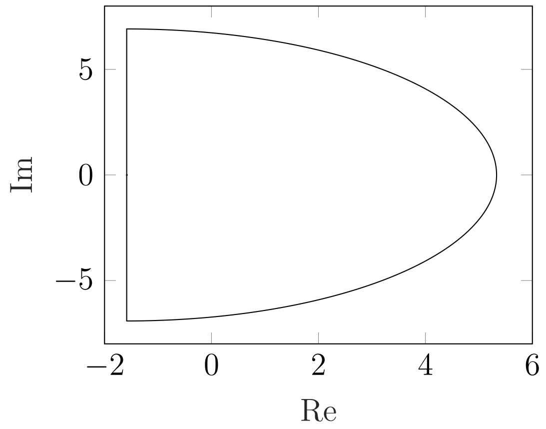

Figure 4.7.: Curve enclosing $\tilde{\Sigma}_\delta$ for $\mu = 1$, $\gamma = -0.01$, $c = 3.2159$ and $\beta = c/2$.

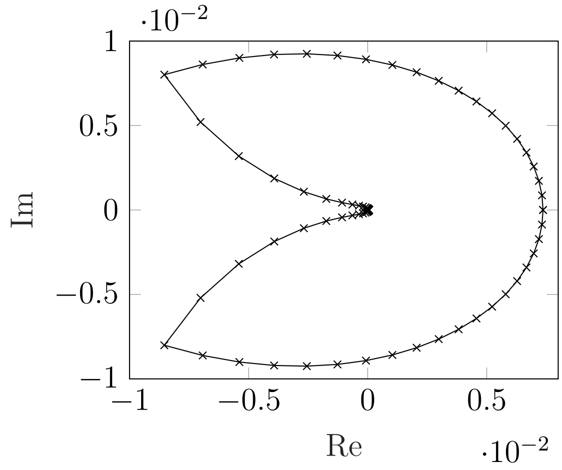

Figure 4.8.: Evans function for $\mu = 1$, $\gamma = -0.01$, $c = 3.2159$ and $\beta = c/2$.

Since $|\gamma| \leq 0.1$ we find that $c_{\min}^{1,\gamma}(1) = 2$. Next, for fixed μ, γ, β we numerically estimate $c_{\max}(\gamma)$, that is, the velocity such that $\tilde{\Sigma}_\delta$ strictly lies in the left complex half-plane. This is done using the explicit estimates provided in Lemma 4.5.3 on the numerical range. Then, we define

$$\mathcal{I}_c^1(\gamma) := \left\{ 2.01 + \frac{c_{\max}(\gamma) - 2.01}{59},\ k = 0, \ldots, 59 \right\}.$$

For every $c \in \mathcal{I}_c^1(\gamma)$ we use a semicircular shaped curve to enclose $\tilde{\Sigma}_\delta$, see Figure 4.7, and calculate the Evans function on that curve using STABLAB, see 4.8. Finally, we calculate the winding number, which vanishes for all tested parameters.

Scenario 2 – fixed (c, β) **and varying** (μ, γ)**.** In the second scenario, we check the presence of eigenvalues in $\tilde{\Sigma}_\delta$ in the case $\mu \neq 1$, but for fixed velocity c close to the "minimal" velocity $c_{\min}^{\mu,\gamma}$ from Lemma 4.5.4. Therefore, we define

$$\mathcal{I}_\mu^2 := \{0.3, 0.44, 0.58, 0.72, 0.86, 1, 1.556, 2.3717, 3.5146, 5\},$$
$$\mathcal{I}_\gamma^2 := \{-0.3 + 0.012k,\ k = 0, \ldots, 50\}.$$

For all parameters $(\mu, \gamma) \in \mathcal{I}_\mu^2 \times \mathcal{I}_\gamma^2$ we calculate a $\beta > 0$ such that $c_{\min}^{\mu;\gamma}(\beta)$ is minimal and then perform the numerical analysis using $c = c_{\min}^{\mu;\gamma}(\beta) + 0.1$. As in the first scenario, for all tested parameters the winding number vanishes.

We summarize the numerical experiments in the following result.

Lemma 4.5.6. *Fix μ, γ, c and β such that either*

$$\mu = 1,\ \gamma \in \mathcal{I}_\gamma^1,\ c \in \mathcal{I}_c^1(\gamma),\ \beta = c/2, \tag{S1}$$

$$\mu \in \mathcal{I}_\mu^2,\ \gamma \in \mathcal{I}_\gamma^2,\ c = \inf_{\beta>0} c_{min}^{\mu;\gamma}(\beta) + 0.1,\ \beta = \operatorname*{argmin}_{\beta>0} c_{min}^{\mu;\gamma}(\beta), \tag{S2}$$

with $c_{min}^{\mu;\gamma}(\beta)$ from Lemma 4.5.4. Then, there exists a $\delta > 0$ such that $\tilde{\Sigma}_\delta \cap \sigma(\mathcal{L}_\beta) = \emptyset$. In particular, this yields $\mathrm{Re}(\sigma(\mathcal{L}_\beta)) \leq -\delta$.

Remark 4.5.7 (Computational environment and runtime)**.** We comment briefly on the setup used for the numerical calculations. All computations were performed in STABLAB [BHLZ15] using Matlab version R2017a [MAT17]. The computations ran on a Debian desktop with 4 cores. The computations for each set of parameters took on average approximately 41 seconds (min. 17 seconds, max. 227 seconds) and the test ran with 1770 sets of parameters. □

Remark 4.5.8. Note that the above result can be made rigorous using validated numerics, that is numerics using interval arithmetic instead of floating-point arithmetic, see e.g. [Bri00, Bar14]. The idea of interval arithmetic is to work with intervals instead of standard floating-point numbers. The intervals and the corresponding arithmetic operators are then implemented in such a way that it can be guaranteed that the analytical result of any operation is contained in the resulting interval. Since the winding number takes only values in the integers, a discrete set, it is then enough to show that the winding number lies in an interval, which only contains one integer to obtain an analytic result. □

Remark 4.5.9. An alternative method to numerically locate the point spectrum has been suggested recently in [BLRM14]. This method is based on Keldysh' theorem, see e.g. [Bey12], and works by calculating contour integrals of solutions to resolvent equations. One particular advantage of this method compared to the Evans function is that it can also be applied analyze the stability in multi-dimensional settings. Except for special cases, the Evans function cannot be used for such systems, since it is tied to the concept of spatial dynamics. We want to highlight that recently, a generalization of the concept of spatial dynamics to multi-dimensional settings has been developed, see [BCJ+19a, BCJ+19b, Bec20], although it is an open question if this can be used to generalize the Evans function. □

The result Lemma 4.5.6 gives the spectral stability of $\mathcal{L}_\beta$ in the case of γ outside a neighborhood of zero under suitable assumptions on the parameters. Additionally, recalling the Assumptions (A1) and (A3), we obtain the spectral stability of the full weighted operator L_β, that is we verified the spectral assumption (A2). Thus, Theorem 4.1.9 can be applied to obtain the nonlinear stability of invading fronts for large $|\gamma|$.

Appendix

4.A. Properties of front solutions

The aim of this section is to provide some important analytical properties of the front solutions of (4.1).

4.A.1. Decay properties of front solutions

We study the asymptotic spatial behavior of the front, i.e. how fast the front approaches its asymptotic rest state for $\xi \to +\infty$. Therefore, recall that the front solutions $A_{\text{front}}, B_{\text{front}}$ are real stationary solutions of the spatial dynamics formulation of (4.1) in a co-moving frame, i.e. for $c > 2$ we have

$$\begin{aligned}
0 &= \partial_\xi^2 A_{\text{front}} + c\partial_\xi A_{\text{front}} + A_{\text{front}} + A_{\text{front}} B_{\text{front}} - A_{\text{front}}^3, \\
0 &= \mu \partial_\xi^2 B_{\text{front}} + c\partial_\xi B_{\text{front}} + \gamma \partial_\xi^2 (A_{\text{front}}^2).
\end{aligned}$$

After integrating the B_{front} equation once, we write this system as a first order system in ξ we obtain

$$\partial_\xi A_{\text{front}} = \widetilde{A_{\text{front}}}, \tag{4.37a}$$

$$\partial_\xi \widetilde{A_{\text{front}}} = -c\widetilde{A_{\text{front}}} - A_{\text{front}} - A_{\text{front}} B_{\text{front}} + A_{\text{front}}^3, \tag{4.37b}$$

$$\partial_\xi B_{\text{front}} = -\frac{1}{\mu}(cB_{\text{front}} + 2\gamma \widetilde{A_{\text{front}}} A_{\text{front}}). \tag{4.37c}$$

To calculate the decay rate to zero for $\xi \to +\infty$, we linearize about $(0,0,0)$ and find the linear operator

$$\mathcal{L}_0 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -c & 0 \\ 0 & 0 & -\frac{c}{\mu} \end{pmatrix}.$$

The eigenvalues of $\mathcal{L}_0$ are given by

$$\lambda_\pm = -\frac{c}{2} \pm \frac{1}{2}\sqrt{c^2-4} \text{ and } \lambda_3 = -\frac{c}{\mu}.$$

Note in particular, that for $c > 2$ all eigenvalues are simple and due to the $\widetilde{A_{\text{front}}} A_{\text{front}}$ nonlinearity in (4.37c) we have the following result.

Proposition 4.A.1. *Let $c > 2$, $\mu > 0$. Then it holds that*

$$|A_{front}(\xi)| \lesssim e^{\lambda_+ \xi},\ |\partial_\xi A_{front}(\xi)| \lesssim e^{\lambda_+ \xi},\ \textit{and}\ |B_{front}(\xi)| \lesssim e^{\max(\lambda_3, 2\lambda_+)\xi}$$

for $\xi \to +\infty$.

Remark 4.A.2. Note that the condition on the weight β in Lemma 4.4.4 is equivalent to $\beta \in (-\lambda_+, -\lambda_-)$ and $\beta \in (0, -\lambda_3)$. Hence we require in particular that the perturbation decays faster than the front for $\xi \to \infty$. □

4.A.2. Properties of $\partial_\xi A_{\text{front}}/A_{\text{front}}$

We now prove the results for the logarithmic derivative of A_{front}, which are needed in the spectral analysis of L_ψ in Lemma 4.4.3.

Proposition 4.A.3. *Let $c > 0$ and $\gamma = 0$. Then, $f_\psi := \partial_\xi A_{front}/A_{front}$ is negative, monotonically decreasing and satisfies*

$$\lim_{\xi\to\infty} f(\xi) = \lambda_+ = -\frac{c}{2} + \frac{1}{2}\sqrt{c^2-4}.$$

Proof. We first note that the asymptotic limiting property is a well-known result for fronts of the Ginzburg-Landau equation, see e.g. [Sat76]. Furthermore, since A_{front} is

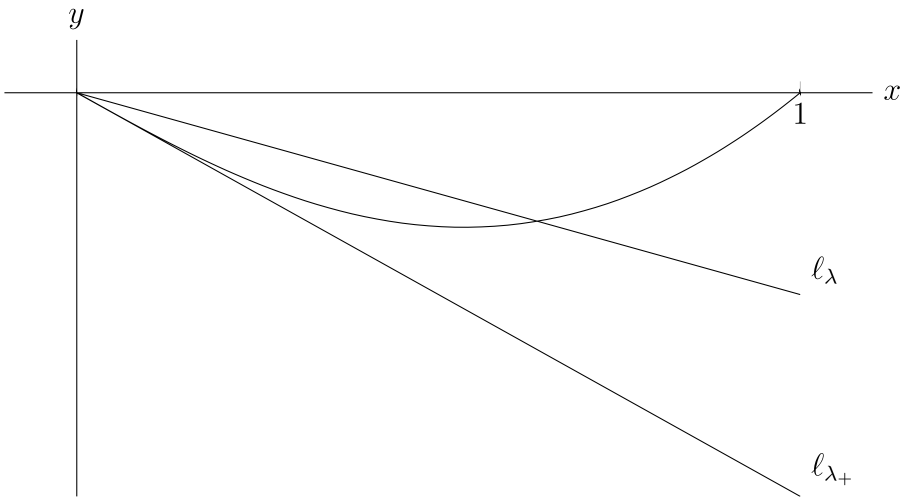

Figure 4.A.1.: Phase plane depicting the heteroclinic orbit as well as ℓ_{λ_+} and ℓ_λ for $\lambda \in (\lambda_+, 0)$.

monotonically decreasing, f_ψ is negative. Thus, it remains to prove that f_ψ is monotonically decreasing. For that we use phase plane analysis. Therefore, recall that for $\gamma = 0$, A_{front} satisfies $\partial_\xi^2 A_{\text{front}} = -c\partial_\xi A_{\text{front}} - A_{\text{front}} + A_{\text{front}}^3$. Writing this as a first order system we obtain

$$\partial_\xi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -cy - x + x^3 \end{pmatrix},$$

which exhibits a heterclinic orbit (x, y) connecting $(1, 0)$ to $(0, 0)$. Next, for $\lambda \in [\lambda_+, 0)$ we define $\ell_\lambda := \lambda x$. We now show that $f_\psi = y/x$ intersects each ℓ_λ exactly once for $\lambda \in (\lambda_+, 0)$, see Figure 4.A.1.

The idea of the proof is to show that for all $\lambda \in (\lambda_+, 0)$ exists exactly one $x_\lambda \in (0, 1)$ such that

$$\frac{dy}{dx}|_{y=\ell_\lambda} - \lambda = 0, \tag{4.38}$$

i.e. the slope of the vector field coincides with the slope of ℓ_λ. Since $f_\psi \to \lambda_+$ for $\xi \to \infty$ and $f_\psi \to 0$ for $\xi \to -\infty$, f_ψ has to intersect each ℓ_λ at least once. Now suppose that f_ψ is non-monotone. Then, there exists a $\lambda \in (\lambda_+, 0)$ such that f_ψ intersects with ℓ_λ at least three times. However, this contradicts the fact that (4.38) is only satisfies for exactly one

$x_\lambda \in (0,1)$. Hence, f_ψ is monotonically decreasing.

It remains to show that (4.38) is satisfied for exactly one $x_\lambda \in (0,1)$. This is a direct calculation.

$$\frac{dy}{dx}\Big|_{y=\ell_\lambda} - \lambda = -c - \frac{1-x^2}{\lambda} - \lambda = 0 \quad \iff \quad x_\lambda = \sqrt{\lambda^2 + c\lambda + 1},$$

which is well-defined since $0 < \lambda^2 + c\lambda + 1 \leq 1$ for all $\lambda \in (\lambda_+, 0)$. $\square$

5. Discussion and outlook

We now discuss some open questions and give an outlook on possible future research topics.

Modulating traveling fronts for convection problems with a free surface

Recalling the existence of modulating traveling fronts for the Swift-Hohenberg equation with an additional conservation law in Chapter 2, a natural follow-up question is if a similar existence result can be obtained for the original physical problems, which motivated the model problem. In particular, this includes convection problems with a free surface such as the Bénard-Maragnoni problem, see Figure 1.3. In [Zim14], the modified Ginzburg-Landau system (1.3) has been rigorously justified as an amplitude equation for the Bénard-Marangoni problem, which indicates that the dynamic close to the onset of instability is similar to that of the Swift-Hohenberg equation coupled to a conservation law. Notably, this includes the modulating traveling front solutions. Although arguing along these lines gives a good intuition as to why modulating traveling front solutions should be present, a rigorous proof as done in Chapter 2 is still open and likely presents a significant challenge, in particular due to the presence of the free surface.

Nonlinear stability of modulating traveling fronts

In Chapter 4, the stability analysis of invading fronts in the Ginzburg-Landau equation with an additional conservation law was motivated by the fact that it serves as an amplitude equation for the Swift-Hohenberg equation (and more generally patter-forming systems admitting a conservation law structure). However, although this gives an indication for the stability of modulating traveling fronts constructed in Chapter 2 a proof of this is still open. Studying the stability of the modulating traveling fronts directly instead of relying on the amplitude equation is of particular importance since there can be effects which are not captured by the amplitude equation but originate from (small) higher order terms which are neglected in the derivation. A precedent for such a situation is the phase locking in a cubic-quintic Swift-Hohenberg equation with slightly subcritical nonlinearity, which has already been predicted in [Pom86] and requires beyond all order asymptotics to be captured, see [SU17, Remark 10.9.7.] and the references therein. In the case of invading fronts, one possible effect caused by small higher order terms is the presence of an anomalous spreading speed as studied for example in [Hol14, Hol16].

We now discuss possible challenges which have to be addressed in a stability analysis of modulating traveling fronts constructed in Chapter 2. Following the ideas in Chapter 4, we start by discussing stability of the periodic solutions established in Lemma 2.2.1. In recent years there were many results regarding the diffusive stability of periodic solutions in systems of conservation laws and we refer to [JZ10, JZ11a, BJN$^+$13] for details. However, there are still open questions regarding the stability of Turing patterns in parabolic systems of conservation laws, see [BJZ18]. Nevertheless, for some settings similar to ours spectral stability has already been discussed in [MC00, Suk16] using Floquet-Bloch theory and we expect that similar calculations apply in our case making the formal calculations in Section 2.B rigorous. Therefore, we conjecture that stability of the periodic solutions from Lemma 2.2.1 can be established in the parameter regime obtained in Section 2.B.

Now we turn to the modulating traveling fronts. The main problem for establishing stability of these solutions is the fact that they connect an unstable state, the origin, to a (possibly) stable state, the periodic solution. As in Chapter 4 this requires the use of exponentially weighted spaces to stabilize the origin. With these ideas, in the case of the Swift-Hohenberg equation [ES02] and the Taylor-Couette problem [ES00] the nonlinear stability of sufficiently fast (i.e. supercritical) modulating traveling fronts has been established using Floquet-Bloch theory to analyze the spectral stability and renormalization group theory to close the nonlinear argument. Whether these ideas can also be used to establish stability in the case of an additional conservation law is open, though. Especially, following the discussion in [BJN$^+$13], it is unclear if renormalization group theory can still be used to close the argument or if other tools such as direct estimates used in Chapter 4 are necessary.

Critical spreading speed for invading fronts in a modified Ginzburg-Landau system

In Chapter 4 we established the nonlinear stability of sufficiently fast invading fronts in a Ginzburg-Landau equation with an additional conservation law. Here, a front is deemed sufficiently fast if its spreading speed c satisfies $c > c_{\min}(\mu)$, see (4.4). A natural follow-up question is if stability can also be established for slower fronts. For this discussion, we consider the cases $\mu \in (0, 2)$ and $\mu > 2$ separately.

First, we consider the case $\mu \in (0, 2)$, in which $c_{\min}(\mu) = 2$. For $c < c_{\min}(\mu)$, there are no exponential weights which stabilize the origin and hence, we expect the invading fronts to be unstable. However, for $c = c_{\min}(\mu)$ we can chose exponential weights such that the origin is spectrally stable in exponentially weighted spaces with spectrum touching the imaginary axis. In the scalar case such as a real Ginzburg-Landau equation and the Fisher-KPP equation, the stability of these so-called critical fronts is well understood, see e.g. [Kir92, Gal94, FH19a]. The proof relies on obtaining sufficiently good polynomial decay estimates for the semigroup in appropriately chosen weighted spaces. There are also recent advances to prove the stability of critical fronts in the system case using pointwise estimates, see [FH19b]. However, these techniques require some structure of the equations which allows to exclude the presence of unstable point spectrum and in particular the presence of an embedded eigenvalue at $\lambda = 0$ in an appropriately weighted space. Therefore, to apply these methods to the invading fronts in the Ginzburg-Landau equation with an additional conservation law, a more precise analysis of the decay rates of the invading fronts and of the weighted operator is necessary.

Second, we consider the case $\mu > 2$, in which $c_{\min}(\mu) > 2$. Similar to the first case, we also find that for $c = c_{\min}(\mu)$, the origin is spectrally stable for optimally chosen exponential weights used in Chapter 4. Surprisingly, it is likely possible to obtain stability of the origin in exponentially weighted spaces even if $c \in (2, c_{\min}(\mu))$ using more refined exponential weights. In essence this is due to the absence of a linear coupling in the modified Ginzburg-Landau system (4.1). It will be a part of future research to turn these ideas into a rigorous proof.

The above discussion of stability of the critical front and its spreading speed is also connected to the question what the naturally selected spreading speed is. That is, does sufficiently steep initial data evolve into a traveling front and if it does, which speed is selected for the front. Similar questions have been answered for the Fisher-KPP equation see e.g. [AW78, EvS00]. However, the proofs usually rely on the use of a comparison principle, see also [Hol16]. Therefore, existing techniques seem insufficient to obtain a similar result for the case of a Ginzburg-Landau equation with an additional conservation law and such an extension is still an open question. Finally, we point out that the same question regarding the naturally selected spreading speed is also open for modulating traveling fronts, even in the case of non-conserved pattern-forming systems.

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