Desarguesian and geometric right ℓ-groups

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Zusammenfassung

Eine rechte ℓ-Gruppe ist eine Gruppe mit einer Verbandsstruktur, welche invariant unter Rechtsmultiplikation ist. Rechte ℓ-Gruppen sind eine starke Verallgemeinerung verschiedener Klassen geordneter Gruppen, zum Beispiel der Garside-Gruppen.

Rechte ℓ-Gruppen mit einer modularen noetherschen Verbandsstruktur wurden von Rump in der Arbeit [Rum15] als Verallgemeinerung der Strukturgruppen bestimmter Lösungen der mengentheoretischen Yang-Baxter-Gleichung untersucht. Ein zentrales Resultat in dieser Arbeit ist die Feststellung, dass der Radikalfilter im negativen Kegel einer jeden modularen noetherschen rechten ℓ-Gruppe ein dualer geometrischer Verband ist, in welchem die Relationen der Gruppe in Form einer Blockbeschriftung codiert sind. Dieses Resultat zeigt den *lokalen* geometrischen Charakter dieser Klasse rechter ℓ-Gruppen auf.

Wir wollen diese Untersuchungen hier fortführen. Ziel dieser Arbeit sind zwei Beiträge zur globalen Strukturtheorie modularer noetherscher rechter ℓ-Gruppen:

Im ersten Teil dieser Arbeit beschränken wir uns auf desarguesche rechte ℓ-Gruppen. Hierbei nennen wir eine rechte ℓ-Gruppe G *desarguesch*, sofern sie modular und noethersch ist und zusätzlich eine starke Ordnungseinheit $s \in G$ existiert, sodass das starke Ordnungsintervall $[s^{-1}, e] \subseteq G$ ein desarguescher Verband ist. Für den Fall, dass dessen Länge $\delta := l([s^{-1}, e])$ mindestens 4 ist, beweisen wir, dass die Verbandsstruktur von G eine Koordinatisierung durch den Verband aller R-Gitter in Q^δ zulässt, wobei Q ein geeigneter vollständiger diskreter Bewertungskörper mit Bewertungsring R ist. Hieraus leiten wir eine projektive Darstellung von G als Gruppe semilinearer Automorphismen des Moduls ${}_RQ^\delta$ ab.

Im zweiten Teil weiten wir unsere Untersuchungen auf allgemeine modulare noethersche rechte ℓ-Gruppen mit starker Ordnungseinheit aus. Wir beweisen, dass mit jeder solche Gruppe ein kanonischer geometrischer Verband assoziiert werden kann, auf dem die Gruppe nichttrivial vermöge Verbandsautomorphismen operiert, selbst wenn die Gruppe keine Koordinatisierung innerhalb eines Untermodulverbands zulässt. Damit wird eine *globale* Aussage über den geometrischen Charakter modularer noetherscher rechter ℓ-Gruppen getroffen, welcher im Lokalen bereits durch das oben genannte Resultat von Rump benannt wird. Zuletzt zeigen wir, dass diese kanonische Operation eine Verallgemeinerung der Permutationsoperation der Strukturgruppe einer Lösung der mengentheoretischen Yang-Baxter-Gleichung ist.

Abstract

A right ℓ-group is a group with a lattice structure that is invariant under right-multiplication. Right ℓ-groups are a vast generalization of several classes of ordered groups, for example of Garside groups.

Right ℓ-groups with a modular noetherian lattice structure have been investigated by Rump in the paper [Rum15] as a generalization of the structure groups of certain solutions of the set-theoretic Yang-Baxter equation. A central result in that paper is the observation that the radical filter in each modular noetherian right ℓ-group is a dual geometric lattice in which the relations of the group are encoded in form of a block labelling. This result demonstrates the *local* geometric character of this class of right ℓ-groups.

We want to continue these investigations. In this work, we intend to make two contributions to the global structure theory of modular noetherian right ℓ-groups:

In the first part of this work, we restrict ourselves to desarguesian right ℓ-groups. Here, we call a right ℓ-group G *desarguesian*, if it is modular and noetherian and is there is additionally a strong order unit $s \in G$ such that the strong order interval $[s^{-1}, e] \subseteq G$ is a desarguesian lattice. In case that those length $\delta := l([s^{-1}, e])$ is at least 4, we prove that the lattice structure of G admits a coordinatization by the lattice of all R-lattices in Q^δ where Q is a complete discrete valuation field with valuation ring R. From this, we derive a projective representation of G as a group of semilinear automorphisms of the module ${}_RQ^\delta$.

In the second part we extend our investigations to general modular noetherian right ℓ-groups with strong order unit. We prove that with each such group a canonical geometric lattice can be associated, on which the group acts nontrivally via lattice automorphisms, even when the group does not admit a coordinatization within a submodule lattice. This makes a *global* statement on the geometric character of modular, noetherian right ℓ-groups which locally has already been specified by the above result of Rump.

At last, we show that this canonical action is a generalization of the permutation action by the structure group of a solution of the set-theoretic Yang-Baxter equation.

Contents

Introduction

Garside groups

This work deals with a connection between ordered groups and geometry. To be more precise, it is about a connection between Garside theory and projective geometry.

To begin with, we explain what *Garside theory*, in the classical sense, is about.

For an integer $n \geq 1$, the *braid group on n strands*, denoted by B_n, is defined as the group generated by the elements $\sigma_1, \ldots, \sigma_{n-1}$ under the relations

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i \quad (|i-j| \geq 2) \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \quad (1 \leq i \leq n-1). \end{aligned}$$

The generators $\sigma_1, \ldots, \sigma_{n-1}$ generate a submonoid B_n^+, whose elements are called the *positive braids*. The pair (B_n, B_n^+) has particularly nice properties, which we will explain now.

Garside discovered that B_n is what is now called a *Garside group* with the *Garside monoid* B_n^+ ([Gar69]). We give the necessary definitions, going back to Dehornoy and Paris ([DP99])

Definition. A *Garside monoid* is a monoid M with a distinguished element $\Delta \in M$ - the *Garside element* - such that the following axioms hold:

i) M is left- and right-cancellative,

ii) M is a lattice under left-divisibility; M is also a lattice under right-divisibility,

iii) the set of right divisors of Δ coincides with the set of left divisors of Δ. This set is finite and generates M.

iv) M is *atomic* in the following sense: for each $m \in M$ there is an integer n such that for each factorization $m = x_1 x_2 \ldots x_k$ with all $x_i \neq e$, we have $k \leq n$.

A group G is called a *Garside group* if it is a left group of fractions for some Garside monoid M, meaning that M is a submonoid of G and each element $g \in G$ can be written as $g = m_1^{-1} m_2$ for some $m_1, m_2 \in M$.

If G is a Garside group which is a left group of fractions for the Garside monoid M, then $g \leq h :\Leftrightarrow gh^{-1} \in M$ defines a partial order relation on G, under which $M = \{g \in G : g \leq e\}$. This relation is invariant under right-multiplication in the sense that for all $g, g', h \in G$, we have the implication $g \leq g' \Rightarrow gh \leq g'h$.

However, this systematic definition of a Garside groups already stands in the middle of a long road that started with Garside's discovery. His approach then was restricted to the braid groups; however, he saw that similar structures can be found in several other groups. After Garside's fundamental discovery, Brieskorn and Saito found out that all spherical Artin groups carry a Garside structure ([BS72]).

Until now, we have just explained what a Garside group *is*, and that some interesting groups are Garside. Is the Garside concept of any use aside from being interesting?

It turns out that Garside groups are very well-behaved in manifold ways. Let us name some major examples:

1) The word problem is solvable for Garside groups ([Deh02]). In particular, it is solvable for braid groups and, more generally, for spherical Artin groups.

2) The conjugacy problem is solvable for Garside groups (see [Deh02]). This was first made possible for braid groups by Garside's approach and also generalized to Garside groups in the mentioned work of Brieskorn and Saito.

3) Garside groups have finite homomological dimension ([DL03]). They even have a finite-dimensional $K(\pi, 1)$ [CMW04].

Several ideas in „classical“ Garside theory have been generalized to other settings. Many different examples of how Garsidean structures arise in mathematics can be found in Part II of the beautiful book of Dehornoy et al. ([Deh15]).

Chapter XIII of the book is titled „Set-theoretic solutions of the Yang-Baxter equation“. This sounds interesting, so let's have a look at this chapter!

A mysterious equation appears on the second page of the chapter. For a finite dimensional vector space ${}_K V$ over some (commutative) field K, we call a K-linear automorphism

$$R : V \otimes V \to V \otimes V$$

a *solution* of the *Yang-Baxter equation* if the following operator equation is fulfilled in $\mathrm{End}(V \otimes V \otimes V)$:

$$(R \otimes \mathrm{id})(\mathrm{id} \otimes R)(R \otimes \mathrm{id}) = (\mathrm{id} \otimes R)(R \otimes \mathrm{id})(\mathrm{id} \otimes R) \qquad \text{(YBE)}$$

A solution R of (YBE) is called *involutive* if $R^2 = \mathrm{id}$.

This equation - for which there also is a parameter-dependent form - arose in the work of the physicists Yang and Baxter and is one of the main reasons for the discovery of *quantum groups* ([Dri88]).

Given a solution $R : V \otimes V \to V \otimes V$ of the Yang-Baxter equation, one can define, for any positiver integer n, on the iterated tensor product $V^{\otimes n}$ the operators $R^i : V^{\otimes n} \to V^{\otimes n}$ $(1 \leq i \leq n-1)$ by

$$R^i = \mathrm{id}^{\otimes(i-1)} \otimes R \otimes \mathrm{id}^{\otimes(n-i-1)}.$$

These operators fulfill the relations

$$\begin{aligned} R^i R^j &= R^j R^i \quad (|i-j| \geq 2) \\ R^i R^{i+1} R^i &= R^{i+1} R^i R^{i+1} \quad (1 \leq i \leq n-1). \end{aligned}$$

Hence, they define distinguished representations of the braid groups B_n, therefore providing important knot invariants, such as the Jones polynomial ([Tur88]).

For this and other (more detailed) information on the Yang-Baxter equation, we refer the reader to the book of Kassel ([Kas95]) and the survey of Jimbo ([Jim89]).

Drinfeld suggested several problems concerning quantum groups ([Dri92]), one of which is the study of *set-theoretic* solutions of the Yang-Baxter equation. Set-theoretic solutions are given by a set X - usually finite -, together with a bijection $r : X \times X \to X$ such that the identity

$$(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r) \tag{SYBE}$$

holds for the respective maps on $X \times X \times X$. Similar to the linear theory, such a solution is called *involutive* if $r^2 = \mathrm{id}$. Define two binary operations $X \times X \to X$ via

$$r(x, y) =: ({}^x y, x^y).$$

We call r *non-degenerate* if for each $x \in X$, the assignment $y \mapsto {}^x y$ is a self-bijection of X, and if also for each $y \in X$, the assigment $x \mapsto x^y$ is a self-bijection of X.

Each solution of (SYBE) can be linearized in order to produce a solution of (YBE). Unlike the first quantum groups, it produces solutions which are not deformations of the trivial solution given by $R(x \otimes y) = y \otimes x$ ([Dri92]).

An important step towards a systematic study of (SYBE) was made by Etingof, Schedler and Soloviev who constructed for each non-degenerate set-theoretic solution $r : X \to X$ a *structure group* $G(X, r)$ ([ESS99]). This group is defined by generators and relations as follows:

$$G(X, r) := \langle X | xy = {}^x y x^y \rangle_{\mathrm{gr}} .$$

Similarly, a *structure monoid* for r is defined by:

$$S(X,r) := \langle X | xy = {}^{x}y x^{y} \rangle_{\mathrm{mon}} .$$

Both $G(X,r)$ and $S(X,r)$ have a right action on X (for which there is also a left-handed counterpart) which is defined on the generators by

$$\begin{aligned} X \times G(X,r) &\to X \\ (x,y) &\mapsto x^{y} \quad (y \in X). \end{aligned}$$

We would like to mention that $S(X,r)$ and its action on X are the set-theoretic counterpart of the *FRT-construction* which associates with each (linear) solution $R : V \otimes V \to V \otimes V$ of (YBE) a cobraided bialgebra $A(R)$ and an $A(R)$-comodule structure on V such that $R : V \otimes V \to V \otimes V$ becomes a comodule map (see [FRT88] or [Kas95, VIII.6]).

The work of Etingof et al. started off an avalanche of results concerning the relationship between algebraic structures and the set-theoretic Yang-Baxter equation. One achievement was Rump's idea to introduce *braces* to study non-degenerate involutive set-theoretic solutions ([Rum07]), which was followed by the work of Bachiller et al. who developed methods to construct *all* non-degenerate and involutive set-theoretic solutions from braces ([BCJ16]).

To go back to our original topic - what does this have to do with Garside theory? This question is answered by the following theorem of Chouraqui (see [Cho10]):

Theorem. *Let $r : X \to X$ be a non-degenerate, involutive solution of the Yang-Baxter where X is a finite set. Then $S(X,r)$ is a Garside monoid. The canonical monoid homomorphism $S(X,r) \hookrightarrow G(X,r)$ makes $G(X,r)$ a Garside group with the Garside monoid $S(X,r)$.*

Furthermore, Chouraqui proved that there is a Garside element Δ in $S(X,r)$ which can be expressed as a product of n atoms (which are exactly the generators from X) and provided criteria for deciding which Garside groups are structure groups. Chouraqui's results were generalized by Rump, whose results we will now explain in the next subsection.

From now on, we will mainly focus on Rump's results on the SYBE. But we want to note that recently, Chouraqui made a very remarkable discovery regarding the order-theoretic nature of the SYBE - in [Cho16] she showed that the topological space of left-orderings of $G(X,r)$ is a Cantor set when (X,r) is a non-degenerate involutive solution of the SYBE which is additionally *retractable*.

Geometric right ℓ-groups

This subsection summarizes some results from the article [Rum15] in a nutshell.

Rump used the framework of *right ℓ-groups* to formulate and prove his Garside-theoretical results. We first explain what these objects are:

Definition. 1) A right ℓ-group is a group G with two binary operations $\wedge, \vee$ such that G is a lattice under $\wedge, \vee$ and such that for all $x, y, z \in G$ we have

$$(x \vee y)z = xz \vee yz$$
$$(x \wedge y)z = xz \wedge yz.$$

2) Let G be a right ℓ-group. An element $s \in G$, $s > e$ is called *normal* if for all $x, y \in G$, we have the equivalence

$$x \leq y \Leftrightarrow sx \leq sy.$$

3) Let G be a right ℓ-group. An element $s \in G$ is called a *strong order unit* if for each $g \in G$ there is an integer k such that $s^k \geq g$.

4) A right ℓ-group G is called *noetherian* if each subset $M \subseteq G$ which is bounded from below contains a minimal element and each subset $N \subseteq G$ which is bounded from above contains a maximal element.

This framework is already sufficient for doing *some* Garside theory in G (for example, constructing left- and right-normal factorizations).

Rump proved (see [Rum15]) that each structure group of an involutive, non-degenerate (and, possibly, infinite) solution of the set-theoretic Yang-Baxter equation is a noetherian right ℓ-group which, as a lattice, is distributive. He also showed a remarkable converse to this result.

Theorem. *[Rum15, Theorem 2] The structure groups of involutive non-degenerate - not necessarily finite - solutions of the set-theoretic Yang-Baxter equation are exactly the modular noetherian right ℓ-groups with a duality.*

We will not explain the notion of *duality* here, which belongs to Rump's theory of *L-Algebras* (see also [Rum20] for an account of the theory).

A corollary of this theorem is that the distributive noetherian right ℓ-groups with a strong order unit are exactly the structure groups of finite involutive non-degenerate solutions of the set-theoretic Yang-Baxter equation. We give a brief sketch of how this correspondence is set up.

If G is a right ℓ-group, we can define on $G^- := e^{\downarrow} := \{g \in G : g \leq e\}$ the $\rightarrow$-operation which is defined by

$$g \rightarrow h := e \wedge hg^{-1} = (g \wedge h)g^{-1}.$$

The pair $(G^-, \rightarrow)$ can be shown to be a *self-similar* L-Algebra (see [Rum08a]).

We now introduce some notation. We set

$$X(G^-) := \{x \in G^- : x \prec e\},$$

the set of *dual atoms* in G^-. Here, „$x \prec e$“ means that no further elements are between x and e). We furthermore define $\tilde{X}(G^-) := X(G^-) \cup \{e\}$.

When G is a modular noetherian right ℓ-group, the $\rightarrow$-operation behaves in a special way: in general, $X(G^-)$ is not closed under the $\rightarrow$-operation; neither is $\tilde{X}(G^-)$. However, when G is a modular (or just lower semimodular) lattice, the following happens: Let $x, y \in X(G^-)$. In case that $x = y$, we have $x \rightarrow y = e$. In the other case that $x \neq y$, we have $x \vee y = e$, and therefore, $y \prec x \vee y$. Modularity now implies that $x \wedge y \prec x$ which is equivalent to $x \rightarrow y = (x \wedge y)x^{-1} \prec e$. It follows that $x \rightarrow y \in X(G^-)$, in this case.

Therefore, in case that G is modular, the subset $\tilde{X}(G^-) \subseteq X(G^-)$ is closed under $\rightarrow$, and it can be proved that G^- and G can be reconstructed from the $\rightarrow$-operation on $\tilde{X}(G^-)$ as its *self-similar closure* ([Rum08a, Section 3]).

If G is noetherian, distributive and has a strong order unit, it can be shown that for all $x \in X(G^-)$, there is a unique element $D(x) \in X(G^-)$ which is „missed“ by the mapping $X(G^-) \rightarrow X(G^-)$; $y \mapsto (x \rightarrow y)$, that is

$$\{x \rightarrow y : y \in X(G^-)\} = X(G^-) \setminus \{D(x)\}.$$

This is the duality in case of a noetherian distributive right ℓ-group with strong order unit. With the map $D : X(G^-) \rightarrow X(G^-)$; $x \mapsto D(x)$, one can define a binary operation $\circ$ on $X(G^-)$ by

$$x \circ y = \begin{cases} D(x) & x = y \\ x \rightarrow y & x \neq y \end{cases}$$

For all $x \in X$, the map $\sigma_x : X \rightarrow X$; $y \mapsto x \circ y$ then is a bijection, and one can show that the map

$$\begin{aligned} r : X \times X &\rightarrow X \times X \\ (x, y) &\mapsto ((\sigma_x^{-1}(y)) \circ y, \sigma_x^{-1}(y)) \end{aligned}$$

is a finite involutive nondegenerate solution of equation (SYBE) with $G(X, r) \cong G$. On the other hand, such a solution r can always be reconstructed from $G(X, r)$ in the way described above.

In order to generalize the correspondence between finite involutive non-degenerate solutions of the SYBE and distributive noetherian right ℓ-groups with strong order unit, it turns out that one should not look at equation (SYBE), but at the algebras $(\tilde{X}(G^-), \rightarrow)$ which can still be defined when G is modular and noetherian.

In [Rum15], it is also proved that the same construction provides a one-to-one correspondence between modular noetherian right ℓ-groups and *non-degenerate* L-Algebras. This shows that modular noetherian right ℓ-groups are derived from a Yang-Baxter-like structure. Thus, it is reasonable to view them as a generalization of the structure groups of involutive non-degenerate solutions to (SYBE).

Now, we want to focus on an important observation of Rump considering the local structure of modular noetherian right ℓ-groups.

Theorem. *Let G be a modular noetherian right ℓ-group such that $\bigwedge X(G^-)$ exists. Then $s = \big(\bigwedge X(G^-)\big)^{-1}$ is a strong order unit of G and $[s^{-1}, e]$ is a dual modular geometric lattice.*

Here, we mean by a *geometric* lattice a bounded below, upper semimodular, atomistic lattice - and the adjective *dual* means that it is isomorphic to the dual of such a lattice. The theorem motivates the following definition:

Definition. Let G be noetherian right ℓ-group G for which $\bigwedge X(G^-) =: s^{-1}$ exists. Then s is called *geometric* when G is lower semimodular and $[s^{-1}, e]$ is dually atomistic.

Rump's theorem tells us that each modular noetherian right ℓ-group G is geometric whenever the meet $\bigwedge X(G^-)$ exists. In fact, each modular noetherian right ℓ-group is geometric in a more general sense when taking into account the cases where $\bigwedge X(G^-)$ does not exist.

In order to really understand the implications of Rump's theorem, one needs to know the structure theory of modular geometric lattices (see [Grä11, V.3.; V.5.], for example). Desarguesian lattices make up a big class of modular geometric lattices. Here, a lattice is called *desarguesian* whenever it is isomorphic to some lattice of the form $L({}_DV)$, the lattice of D-linear subspaces of a left D-vector space ${}_DV$, where D is some (possibly skew) field. Desarguesian lattices are irreducible, which means that they cannot be represented as a direct product of lattices in a nontrivial way.

Every modular geometric lattice of finite length is isomorphic to a direct product of finitely many irreducible ones. These can be described as follows:

1) The irreducible modular geometric lattices of length at least 4 are desarguesian, by a lattice-theoretic variant of the Veblen-Young theorem.

2) The irreducible modular geometric lattices of length 3 are either desarguesian or isomorphic to the lattice of linear subspaces in an exceptional projective plane.

3) The irreducible modular geometric lattices of length 2 are *degenerate*; these consist of a top element 1, a bottom element 0, and only atoms in between.

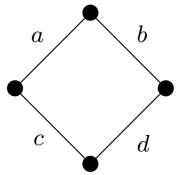
Rump's theorem states that if G is a modular noetherian right ℓ-group where $s := (\bigwedge X(G^-))^{-1}$ exists, the interval $[s^{-1}, e]$ is a dual modular geometric lattice of finite length. In this case, the strong order interval $[s^{-1}, e]$ is isomorphic to a direct product of dual irreducible modular geometric lattices of finite length; these can be described by the characterization above. Hence, we can say quite a lot about the lattice structure of $[s^{-1}, e]$.

What about the structure of G? We have already mentioned that the $\rightarrow$-operation on $\tilde{X}(G^-)$ determines G^- and G. Hence, it should be possible to derive G^- from suitable data in the dual modular geometric lattice $[s^{-1}, e]$ which properly contains $\tilde{X}(G^-)$. This is indeed true!

Rump proved that this can be achieved by considering (non-degenerate) *block labellings*. To construct such a labelling, take the Hasse diagram of the lattice $[s^{-1}, e]$. For each covering $x \prec y$ there is an edge between x and y - label this edge with the label $y \to x := xy^{-1} \in X(G^-)$. It can be proved that G^- and G can be reconstructed from these data.

Vice versa, Rump provided a characterization of block labellings of which is essentially combinatorial. We explain his construction in the dual modular geometric case:

Consider the following rhombic configurations in the Hasse diagram of a dual modular geometric lattice L.



Here, the black dots stand for elements of L. Two elements are connected by an edge if one covers the other. The Hasse diagram is drawn so that the larger element in a covering relation is always positioned above the smaller one.)

Take a set A and label each vertex of the Hasse diagram of L with an element of A while respecting the following rules:

1) Each $a \in A$ occurs exactly once as the label of an edge $x - e$ where $x \in X(G^-)$.

2) For each pair $a, b \in A$ with $a \neq b$ there is a configuration as above, and c, d are uniquely determined by a and b. There is no such configuration with $a = b$.

3) For each pair $c, d \in A$ with $c \neq d$ there is a configuration as above, and a, b are uniquely determined by c and d. There is no such configuration with $c = d$.

The group G (resp. the monoid G^-) can then be described as the group (resp. monoid) generated by A under all relations $ca = db$ where a, b, c, d are the labels of some rhombic configuration in our labelling.

A Garside group can be regarded as a noetherian right ℓ-group G where $s = (\bigwedge X(G^-))^{-1}$ is definable and the interval $[s^{-1}, e]$ is a finite set. Using Rump's results, we can use the following procedure to construct any modular Garside group:

I) Take a finite direct product of irreducible modular geometric lattices.

II) Add labels to the resulting lattice in a way that is consistent with the rules described above.

III) Read off the relations of G from this labelling.

Rump uses this procedure to construct a block labelling for the Hasse diagram of $L({}_{\mathbb{F}_2}\mathbb{F}_2^3)$. The group resulting from this can be regarded as the second[1] official example of a non-distributive modular Garside group. He then poses some problems on block labellings, part of which are the following:

1) Which dual modular geometric lattices admit a non-degenerate block labelling?

2) Does every finite desarguesian lattice admit a block labelling?

3) Is there a geometric interpretation of block labellings of desarguesian lattices?

We will only give an answer to the second question (in the Appendix). But these questions lead to the study of what we would like to call *desarguesian* right ℓ-groups.

Desarguesian right ℓ-groups

Since desarguesian lattices are well-understood, it makes sense to consider non-degenerate block labellings of desarguesian lattices. We call the respective right ℓ-groups *desarguesian*, which we precisely define as follows:

Definition. A modular noetherian right ℓ-group G is called *desarguesian* when $\bigwedge X(G^-) =: s^{-1}$ exists and the strong order interval $[s^{-1}, e]$ is a desarguesian lattice

Our first systematic examples of block labellings are the following.

Consider the desarguesian geometry $L({}_{\mathbb{R}}\mathbb{R}^n)$ which is equipped with the unary operation $U \mapsto U^\top$ of taking orthogonal complements with respect to the standard inner product. As our set of labels we take

$$A := \mathbb{P}(\mathbb{R}^n) := \{U \in L({}_{\mathbb{R}}\mathbb{R}^n) : \dim_{\mathbb{R}} U = 1\}$$

and label the vertex $U - V$ (where $U \prec V$) with the element $U^\top \cap V \in A$. A similar idea works for every vector space with a hermitean, anisotropic bilinear form.

This idea was immediately generalized by Rump to general orthomodular lattices (see [Rum17b]), leading to the constructing of a canonical structure group

[1] Alexander Thumm called my attention to the fact that the braid group B_3 is a modular lattice under the Garside structure whose Garside monoid is generated by the BKL-generators ([BKL98]).

for every orthomodular lattice which can be shown to be a right ℓ-group with strong order unit (which is not necessarily modular).

Rump's work enabled us to prove that the described block labellings of $L({}_{\mathbb{R}}\mathbb{R}^n)$ produce the class of *pure paraunitary groups* ([Die19]). These are defined as

$$\mathrm{PPU}(b) := \{A(t) \in \mathbb{R}\left[t, t^{-1}\right]^{n\times n} : A(t^{-1})^\top A(t) = 1; A(1) = 1\}$$

where b is the standard inner product on $n \times n$. The negative cone of the respective lattice order on $\mathrm{PPU}(b)$ is then given by $\mathrm{PPU}(b)^- = \mathrm{PPU}(b) \cap \mathbb{R}\left[t^{-1}\right]^{n\times n}$ and a strong order unit is given by $t \cdot I_n$. This construction can be generalized to arbitrary von Neumann algebras in a meaningful way (see [Die20]).

We describe our second family of non-degenerate block-labellings (which is explained in detail in Appendix A):

Let $\mathbb{F}_q$ be the field with $q = p^k$ elements where p is some prime number. We realize $\mathbb{F}_q^n$ as the $\mathbb{F}_q$-vector space ${}_{\mathbb{F}_q}\mathbb{F}_{q^n}$ and define for any $U \in L({}_{\mathbb{F}_q}\mathbb{F}_{q^n})$ an operator

$$\begin{aligned} \rho_U : \mathbb{F}_{q^n} &\to \mathbb{F}_{q^n} \\ x &\mapsto \prod_{y\in U}(x-y) \end{aligned}$$

It is not hard to show that the operator ρ_U is an $\mathbb{F}_q$-linear mapping. Using these operators we construct on $L({}_{\mathbb{F}_q}\mathbb{F}_{q^n})$ a block labelling with the label set

$$A := \mathbb{P}({}_{\mathbb{F}_q}\mathbb{F}_{q^n}) = \{U \in L({}_{\mathbb{F}_q}\mathbb{F}_{q^n}) : \dim_{\mathbb{F}_q} U = 1\}.$$

In the Hasse diagram of $L({}_{\mathbb{F}_q}\mathbb{F}_{q^n})$, we label the vertex $U - V$ (where $U \prec V$) with the element $\rho_U(V) \in A$. This also results in a non-degenerate block labelling.

The resulting desarguesian right ℓ-group can be realized as a multiplicative subgroup in a skew field. This skew field is the quotient field of the *twisted polynomial ring* $\mathbb{F}_{q^n}[x, \mathfrak{q}]$ which is the ring generated by the elements of $\mathbb{F}_{q^n}$ together with a variable x, such that the relations $xa = \mathfrak{q}(a)x$ are fulfilled for all $a \in \mathbb{F}_{q^n}$. Here, $\mathfrak{q}$ is defined as $\mathfrak{q}(a) = a^q$. (more detailed explanations can be found in the Appendix).

Surprisingly, both constructions of desarguesian right ℓ-groups followed a similar pattern, in that they were built from some kind of polynomial ring which was embedded in a somewhat bigger ring (a skew field resp. a matrix ring over a ring of Laurent polynomials). So we wondered if this phenomenon was due to our ignorance of other possible - non-ring-theoretic - constructions or if these two examples already reflected the universal truth regarding the nature of desarguesian right ℓ-groups.

We will see that these two examples reflect quite a lot of the true nature of general desarguesian right ℓ-groups. One objective of this work is to describe to what extent desarguesian right ℓ-groups are constructable by using ring-theory.

What is this work about?

We have two major goals which we will describe here:

Characterization of desarguesian right ℓ-groups of dimension $\delta \geq 4$

Our first result will concern the characterization of desarguesian right ℓ-groups of dimension $\delta \geq 4$. We will define a *desarguesian* right ℓ-group as a modular, noetherian right ℓ-group G with a strong order unit s such that the strong order interval $[s^{-1}, e]$ is isomorphic to some projective geometry $L({}_K K^\delta)$. Here, $L({}_K K^\delta)$ is defined as the lattice of K-linear subspaces of the vector space ${}_K K^\delta$, where K is some (skew) field and $\delta \geq 1$ is an integer.

We will focus on the cases where G is a desarguesian right ℓ-group of dimension $\delta \geq 4$. We prove that in these cases, one can find a (possibly skew) complete discrete valuation field Q with valuation ring R such that there is a lattice isomorphism $G \cong \mathrm{Lat}({}_R Q^\delta)$ - the latter being the lattice of R-lattices in Q^δ. By an *R-lattice* in an R-module ${}_R M$, we mean a finitely generated, essential R-submodule of ${}_R M$. This coordinatization shows that each desarguesian right ℓ-group of dimension $\delta \geq 4$ is a regular automorphism group on some lattice of the form $\mathrm{Lat}({}_R Q^\delta)$, where R, Q are as above.

From this result, we will also derive some kind of projective representation of G: we show that G can be realized as a complement to the subgroup $\mathrm{P\Gamma L}({}_R R^\delta) \leq \mathrm{P\Gamma L}({}_R Q^\delta)$. Here, for an R-module ${}_R M$ we denote by $\mathrm{P\Gamma L}({}_R M)$ the permutation group of $L({}_R M)$ induced by the semilinear automorphisms of ${}_R M$.

We state our first major result.

Theorem (Theorem 2.5.17)**.** *Each desarguesian right ℓ-group G of dimension $\delta \geq 4$ is a regular automorphism group of $\mathrm{Lat}({}_R Q^\delta)$ for some complete discrete valuation field Q with valuation ring R.*

Vice versa, each regular automorphism group G of some lattice $\mathrm{Lat}({}_R Q^\delta)$ ($\delta \geq 1$, R, Q as above) is a desarguesian right ℓ-group of dimension δ.

In terms of the groups $\mathrm{P\Gamma L}$, this result can also be stated as follows:

Theorem (Theorem 2.6.6)**.** *Each desarguesian right ℓ-group G of dimension $\delta \geq 4$ is a complement of the subgroup $\mathrm{P\Gamma L}({}_R R^\delta) \leq \mathrm{P\Gamma L}({}_R Q^\delta)$ for some complete discrete valuation field Q with valuation ring R.*

Vice versa, each complement G of some subgroup $\mathrm{P\Gamma L}({}_R R^\delta) \leq \mathrm{P\Gamma L}({}_R Q^\delta)$ ($\delta \geq 1$, R, Q as above) is a desarguesian right ℓ-group of dimension δ.

This result says that „most“ desarguesian right ℓ-groups can - at least in theory - be constructed ring-theoretically. This shows that the constructions for desarguesian right ℓ-groups we have mentioned above, are instances of an (almost) general principle.

We also have a ring-theoretic description of a significant part of the most primitive modular geometric right ℓ-groups (*and* modular Garside groups), which are those whose strong order interval is indecomposable. It is very probable that our approach can *not* be extended to all of these groups (see below). However, combined with Rump's decomposition theorem for noetherian right ℓ-groups ([Rum17a, Theorem 2]), one has a description of *many* modular geometric right ℓ-groups as iterated crossed products of the primitive groups.

Construction of a geometric action for modular geometric right ℓ-groups

Our second result concerns the „geometricity“ of all modular geometric right ℓ-groups with a strong order unit. If G is a desarguesian right ℓ-group of dimension $\delta \geq 4$, then, by the first result, G can be realized as a complement of the subgroup $\mathrm{P\Gamma L}({}_R R^\delta) \leq \mathrm{P\Gamma L}({}_R Q^\delta)$. There is a canonical homomorphism $\mathrm{P\Gamma L}({}_R Q^\delta) \to \mathrm{P\Gamma L}({}_Q Q^\delta)$ by which each such realization can be turned into an action of G on the modular geometric lattice $L({}_Q Q^\delta)$.

We show that the lattice $L({}_Q Q^\delta)$ can directly be constructed from G as the dual of the lattice $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ of *saturated* filters in G^-. This is a subposet (but *not* a sublattice) of the lattice $\mathrm{Fil}(G^-)$ of filters in G^-.

We call a filter $F \subseteq G$ *saturated* when the intervals $[G_{F,i+1}, G_{F,i}]$ have the same length for all $i \geq 0$. Here, the elements $G_{F,i}$ are defined as

$$G_{F,i} := \min(F \cap (s^{-i})^{\uparrow}) \quad (i \geq 0).$$

We will prove that $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is a modular geometric lattice when G is a modular geometric right ℓ-group.

We will describe the G-action on $\mathrm{Lat}({}_Q Q^\delta)$ within $\mathrm{Fil}(G^-)_{\mathrm{sat}}$, using the L-Algebra operation $\to$ in G^-. Moreover, we will show that for $F \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$ and $g \in G^+$ (!), the set $F^g := g^{-1} \to F \subseteq G^-$ is again a saturated filter, and that this defines a right G-action by lattice automorphisms of $\mathrm{Fil}(G^-)_{\mathrm{sat}}$.

This is our second major result:

Theorem (Theorem 3.3.4, Theorem 3.4.6)**.** *For each modular geometric right ℓ-group G, the subset* $\mathrm{Fil}(G^-)_{\mathrm{sat}} \subseteq \mathrm{Fil}(G^-)$ *is a modular geometric lattice under the induced order. There is a unique right G-action by lattice automorphisms of* $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ *under which we have $F^g = g^{-1} \to F$ for all $g \in G^+$.*

This shows that *all* modular geometric right ℓ-groups are „geometric“ in a *global* sense; namely, they act non-trivially on a modular geometric lattice by a canonical action. In particular, this is a statement about all modular Garside groups!

We will also see that, using this canonical geometric action, the permutation action of a structure group for a non-degenerate involutive solution of (SYBE) can be generalized to modular geometric right ℓ-groups.

The result also shows why there *might* be no easy characterization of all geometric right ℓ-group, at least not in ring-theoretic terms. A mere ring-theoretic description of modular geometric right ℓ-groups will lead to the lattice $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ being a direct product of desarguesian geometric lattices. However, there is no evidence for or against the appearance of non-desarguesian geometries in the theory.

The structure of this work

In Chapter 1, we introduce the lattice-theoretic machinery needed later in our analysis of desarguesian right ℓ-groups.

In Section 1.1, we will cite some general results about modular lattices. Regarding general modular lattices, we will mainly focus on independence and dual independence. Also, for a left module ${}_RM$ over some (unital) ring R, we discuss the lattice $L({}_RM)$ of R-submodules of ${}_RM$.

In Section 1.2, we present two coordinatization theorems, which are

1) The theorem of Veblen and Young (Theorem 1.2.4), which gives a lattice-theoretic characterization of the *desarguesian* lattices $L({}_KK^\delta)$ where K is a field and $\delta \geq 4$.
2) The theorem of Inaba (Theorem 1.2.8), which identifies certain *primary* lattices as being isomorphic to lattices of the form $L({}_RM)$ where R is a *completely uniserial primary* ring and ${}_RM$ a finitely generated left R-module.

In Section 1.3, we discuss semilinear maps between modules and look at the permutation groups $\mathrm{P\Gamma L}({}_RM)$ induced by the action of semilinear isomorphisms on the lattice $L({}_RM)$. Also, we introduce Camillo's generalization of the fundamental theorem of projective geometry (Theorem 1.3.6) which states that all automorphisms of the lattices ${}_RM$ are induced by semilinear isomorphisms, provided some mild extra conditions on ${}_RM$ are fulfilled.

The main goal of Chapter 2 will be the structure theory of desarguesian right ℓ-groups of dimension $\delta \geq 4$, as explained in the preceding section.

In Section 2.1, we begin by discussing Rump's notion of *right ℓ-group* in as groups G with a lattice-structure that is right-invariant, meaning that for all $x, y, z \in G$, we have

$$(x \vee y)z = xz \vee yz$$
$$(x \wedge y)z = xz \wedge yz.$$

Equivalently, a right ℓ-group can be seen as a lattice on which some group of lattice automorphisms acts regularly from the right.

We will then talk about *strong order units* which are elements $s > e$ such that *left*-multiplication by s is isotone and each element $g \in G$ is dominated some power s^k. Strong order units provide a framework in which right- and left-normal factorizations can be defined without any chain or finiteness conditions. For reasons that will become clear later, we will look at factorizations in the negative cone $G^- = \{g \in G : g \leq e\}$ with all factors in $[s^{-1}, e]$. We define a *right-normal* factorization (with respect to s) of some element $g \in G^-$ as a factorization

$$g = g_k g_{k-1} \dots g_1$$

such that all $g_i \in [s^{-1}, e]$, $g_i \neq e$ and it is furthermore right-maximal in that for all $1 \leq i \leq k-1$ there is *no* nontrivial factorization $g_{i+1} = g'g''$ such that $g''g_i \in [s^{-1}, e]$. We prove that such factorizations - and their respective left-handed counterparts - always exist and are unique (Proposition 2.1.17, Proposition 2.1.29). Part of this has already been done by Rump in the framework of L-Algebras ([Rum15]). However, our approach leads to explicit formulae for the right (and left)-normal factors which we will need later.

In Section 2.2, we proceed by focussing on right ℓ-groups G which are modular lattices and are *noetherian*, meaning that each bounded below (resp. above) subset of G contains a minimal (resp. maximal) element - we formulate this in terms of chain conditions. For these groups, we present *Rump's theorem* (Theorem 2.2.3) which says that for all modular noetherian right ℓ-groups there is a strong-order unit s with $s^{-1} = \bigwedge X(G^-)$ (if the latter exists) and that the strong order interval $[s^{-1}, e]$ is a modular geometric lattice. Consequently, we define a *desarguesian* right ℓ-group of dimension δ as a modular noetherian right ℓ-group where this strong order interval is isomorphic to a desarguesian lattice, that is, the lattice $L({}_K K^\delta)$ for some finite-dimensional vector space K^δ over a (skew) field K.

In Section 2.3, we analyze factorizations in G^- when G is a modular noetherian right ℓ-group. It turns out that under these conditions, each $g \in G^-$ has factorizations $g = x_n x_{n-1} \dots x_1$ with all $x_i \prec e$ (Proposition 2.3.1), where the integer n is only dependent on g. We will see that $d(g) := n$ provides a (well-defined) degree function on G^- which can be extended to a degree homomorphism $d : G \to \mathbb{Z}$ with several nice properties (Proposition 2.3.2).

Continuing, we deal with the combinatorics of right- and left-normal factorizations in G^- when G is a modular geometric right ℓ-group where the strong order unit $s^{-1} := \bigwedge X(G^-)$ exists. It turns out that the degree function is *extremely* well-behaved under this condition, which will be of central importance in this work. To name a few examples:

i) (Proposition 2.3.9) For $g \in G^-$ with the right-normal factorization $g = g_k g_{k-1} \dots g_1$ we have $d(g_k) \leq d(g_{k-1}) \leq \dots \leq d(g_1)$. This will be stated in terms of the *index function* $\iota_i^s(g) := d(g_i)$.

ii) (Proposition 2.3.11) If $g = g_k g_{k-1} \dots g_1$ is a right-normal factorization with $d(g_1) = d(g_k)$, this factorization is left-normal as well. Under similar conditions, left-normality implies right-normality.

iii) (Proposition 2.3.12) Given a right-normal factorization $g = g_k g_{k-1} \dots g_1$ and a left-normal factorization $g = h_1 \dots h_{l-1} h_l$ we have $k = l$ and $d(g_i) = d(h_i)$ for $1 \leq i \leq k$.

iv) (Proposition 2.3.14) If $g = g_k g_{k-1} \dots g_1$ is a right-normal factorization, then g is meet-irreducible in G^- if and only if $d(g_k) = 1$.

The full power of these properties will be seen in Chapter 3. In this chapter, we will only need the last property (which builds on some of the foregoing statements).

In the following section, Section 2.4, we prove that each interval in a desarguesian right ℓ-group is a primary lattice (Proposition 2.4.1). Furthermore, we prove that each interval $[s^{-k}, e]$ ($k \geq 1$) has a dual basis consisting of δ dually independent cochains in G^- (Proposition 2.4.7). These results are still valid when one only assumes that $[s^{-1}, e]$ is an indecomposable modular geometric lattice. However, when $\delta \geq 4$, Inaba's coordinatization theorem can be applied in order to show that for each k, there is some lattice isomomorphism $[s^{-k}, e] \cong L({}_{R_k} R_k^\delta)$ where R_k is a completely primary uniserial ring of length k (Proposition 2.4.8). So G^- can be coordinatized „piecewise“.

In the following section, Section 2.5, we assume that G is a desarguesian right ℓ-group of dimension $\delta \geq 4$. We show that the piecewise coordinatizations of $[s^{-k}, e]$ can be compiled to provide a coordinatization of G^- by the lattice $\mathrm{Lat}({}_R R^\delta)$, where R is some complete discrete valuation ring (possibly noncommutative) and $\mathrm{Lat}({}_R R^\delta)$ is the lattice of R-lattices in R^δ - i.e. the finitely generated, essential R-submodules of ${}_R R^\delta$ (Theorem 2.5.15). To be more precise, we show that the inverse limit $R := \varprojlim R_k$ is such a ring. We then show that this leads to a coordinatization of G by the lattice $\mathrm{Lat}({}_R Q^\delta)$ where Q is the valuation field for the valuation ring R (Theorem 2.5.16). This proves that each desarguesian right ℓ-group of dimension $\delta \geq 4$ is a regular group of lattice automorphisms of $\mathrm{Lat}({}_R Q^\delta)$ where Q is a complete discrete valuation ring with valuation ring R. We will also show that, vice versa, each regular automorphism group of such a lattice is a desarguesian right ℓ-group (Theorem 2.5.17).

In the last section of this chapter, Section 2.6, we will provide a projective representation for desarguesian right ℓ-groups of dimension $\delta \geq 4$. We will first show that each lattice automorphism of $G^- \cong \mathrm{Lat}({}_R R^\delta)$ is induced by a semilinear automorphism of ${}_R R^\delta$ (Proposition 2.6.1). We then prove that, similarly, each lattice automorphism of $G \cong \mathrm{Lat}({}_R Q^\delta)$ is induced by a semilinear automorphism of ${}_R Q^\delta$ (Proposition 2.6.5). We show that we can consider $\mathrm{Aut}(\mathrm{Lat}({}_R R^\delta)$ as a subgroup of $\mathrm{Aut}(\mathrm{Lat}({}_R Q^\delta)$ and that this embedding is reflected in the embedding $\mathrm{P\Gamma L}({}_R R^\delta) \leq \mathrm{P\Gamma L}({}_R Q^\delta)$ (Proposition 2.6.4). Since the former group is a point stabilizer under the action of $\mathrm{P\Gamma L}({}_R Q^\delta)$ on $\mathrm{Lat}({}_R Q^\delta)$, we deduce that each desarguesian right ℓ-group of dimension $\delta \geq 4$ is isomorphic to a complement to the subgroup $\mathrm{P\Gamma L}({}_RR^\delta) \leq \mathrm{P\Gamma L}({}_RQ^\delta)$ - and that we can similarly consider such complements as desarguesian right ℓ-groups of dimension δ (Theorem 2.6.6).

In the preceding chapter, we have shown that there exists a representation of G as a subgroup of $\mathrm{P\Gamma L}({}_RQ^\delta)$. Under the canonical homomorphism $\mathrm{P\Gamma L}({}_RQ^\delta) \to \mathrm{P\Gamma L}({}_QQ^\delta)$, this leads to a right action of G on the modular geometric lattice $L({}_QQ^\delta)$.

In Chapter 3, we build up a framework under which this representation can be derived from a canonical *geometric action* which exists for *all* modular geometric right ℓ-groups. This shows that modular geometric right ℓ-groups G show their „geometric character“ not only in the geometricity of the „small“ strong order interval $[s^{-1}, e]$, but also in the geometricity of a „bigger“ geometric lattice associated with G.

We proceed as follows:

First, we show in Section 3.1 how to construct $L({}_QQ^\delta)$ from $\mathrm{Lat}({}_RR^\delta)$ when Q is a complete discrete valuation field with valuation ring R:

We will first identify each $U \in L({}_QQ^\delta)$ with the R-submodule $U \cap R^\delta \subseteq R^\delta$. It can be shown that this provides an order-preserving bijection between $L({}_QQ^\delta)$ and $L({}_RR^\delta)_{\mathrm{sat}}$, the subposet of saturated R-submodules $M \subseteq R^\delta$ where *saturated* means that R^δ/M is a torsion-free R-module (Proposition 3.1.1).

We then set up an antiisomorphism between $L({}_RR^\delta)$ and $\mathrm{Fil}(\mathrm{Lat}({}_RR^\delta))$, the lattice of filters in $\mathrm{Lat}({}_RR^\delta)$ (Proposition 3.1.4). This antiisomorphism is provided by the following assignments which are inverse to each other:

$$\begin{aligned} L({}_RR^\delta) &\longleftrightarrow \mathrm{Fil}(\mathrm{Lat}({}_RR^\delta)) \\ M &\longmapsto \{A \in \mathrm{Lat}({}_RR^\delta) : M \subseteq A\} \\ \bigwedge F &\longleftarrow\!\shortmid F. \end{aligned}$$

Given an isomorphism $\varphi : G^- \xrightarrow{\sim} \mathrm{Lat}({}_RQ^\delta)$, we then determinine the filters in $\mathrm{Fil}(G^-)$ whose respective filters in $\mathrm{Fil}(\mathrm{Lat}({}_RR^\delta))$ belong to a saturated submodule of ${}_RR^\delta$. It will turn out that the correct definition of a *saturated* filter in $\mathrm{Fil}(G^-)$ is as follows:

Definition. $F \in \mathrm{Fil}(G^-)$ is *saturated* if $d(g_{F,i})$ takes the same value for all $i \geq 1$ where $g_i := G_{F,i}G_{F,i-1}^{-1}$ with $G_{F,i} := \min(F \cap (s^{-i})^{\uparrow})$.

We denote the subposet of saturated filters in $\mathrm{Fil}(G^-)$ as $\mathrm{Fil}(G^-)_{\mathrm{sat}}$. In this case (!), it is clear that $\mathrm{Fil}(G^-)_{\mathrm{sat}}$, being antiisomorphic to $L({}_QQ^\delta)$, is a modular geometric lattice.

We will then show that the right-action of G on $L({}_QQ^\delta)$ translates to $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ as the unique G-action under which $F^g = g^{-1} \to F$ for all $g \in G^+$. Here, for single elements $x, y \in G^-$, the arrow-operation is defined as $x \to y := (x \wedge y)x^{-1} = e \wedge yx^{-1}$, and $x \to F$ denotes the image of F under the map $y \mapsto (x \to y)$ (Theorem 3.1.6).

Therefore, we have provided a description of the G-action on $L({}_Q Q^\delta)$ in mere terms of right ℓ-groups. The subsequent sections are dedicated to the proof that this construction still makes sense when G is an arbitrary modular geometric right ℓ-group with a strong order unit.

In Section 3.2, we continue the study of combinatorics for left- and right-normal factorizations in G^-. We first show that for each filter F in G^-, the elements $g_{F,i}$ as defined above resemble an „infinite“ right-normal factorization and that right-normal expressions of infinite length point to filters in G^- (Proposition 3.2.5).

We then define for a filter F the index sequence $\iota_i^f(F) := d(g_{F,i})$ $(i \geq 1)$. Here, the upperscript-f stands for $f$ilter. On principal filters, this index sequence coincides with the index sequence defined in the chapter before (Proposition 3.2.2). The index sequence has, for example, the following nice properties:

1) (Proposition 3.2.6) For each $F \in \mathrm{Fil}(G^-)$, the sequence $\iota_i^f(F)$ is nonincreasing.

2) (Proposition 3.2.11) A filter $F \in \mathrm{Fil}(G^-)$ is an infinite chain if and only if $\iota_i^f(F) = 1$ holds for all i.

We call a filter $F \in \mathrm{Fil}(G^-)$ *saturated of degree* d if $\iota_i^f(F)$ is constant with $\iota_i^f(F) = d$ for all $i \geq 1$.

Moreover, we will show that the property of being saturated is stable under joins of filters (Theorem 3.2.16) (but *not* under taking meets) and that each saturated filter is a join of filters that are saturated of degree 1 (Proposition 3.2.15). Using this, we will prove that $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is a lattice under the partial order induced from $\mathrm{Fil}(G^-)$ and give explicit descriptions of the respective lattice operations (Theorem 3.2.17).

We will show that each $F \in \mathrm{Fil}(G^-)$ can be *saturated* in the following sense: if we set $d := \lim\limits_{i\to\infty} \iota_i^f(F)$, then there is a unique filter $F_{\mathrm{sat}} \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$ of degree d such that $F_{\mathrm{sat}} \subseteq F$. We will see that F_{sat} is the unique maximal saturated filter that is contained in F (Proposition 3.2.18).

In Section 3.3, we show that $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is a modular geometric lattice. By the results of the preceding section, we already know that this lattice is atomistic. We show that the assignment $\mathrm{Fil}(G^-) \to \mathrm{Fil}(G^-)_{\mathrm{sat}};\ F \mapsto F_{\mathrm{sat}}$ is a lattice homomorphism, under which $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is an epimorphic image of $\mathrm{Fil}(G^-)$ (Proposition 3.3.2). Since the latter is a modular lattice, this proves that $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is a modular atomistic lattice, that is, a modular geometric lattice (Theorem 3.3.4).

In Section 3.4 we finally prove that for all $g \in G^-$, $F \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$, the filter $g \to F$ is again saturated (Proposition 3.4.2) and the map $F \mapsto (g \to F)$ defines a lattice automorphism of $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ (Proposition 3.4.5). We then show that there is a unique right G-action on $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ under which $F^g = g^{-1} \to F$ holds for all $F \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$ and $g \in G^+$ (Theorem 3.4.6).

In the last section Section 3.5, we show that if $G := G(X, r)$ is a structure group for an involutive non-degenerate solution $r : X \times X \to X \times X$ of the set-theoretic Yang-Baxter equation, then $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is a finite Boolean lattice with $\delta := |X|$ atoms (Proposition 3.5.12) and that the geometric action constructed of G on $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is essentially the permutation action of G on X (Theorem 3.5.14).

In Appendix A, we construct for each finite cyclic field extension L/K a desarguesian right ℓ-group G with strong order unit s such that there is an isomorphism of lattices $[s^{-1}, e] \cong L({}_K L)$ (Theorem A.0.1). In particular, we will demonstrate that every finite desarguesian geometry is a Garside interval in some modular Garside group (Corollary A.0.2, Corollary A.0.3).

We then put this construction in the context of the results of Chapter 2 by providing a matrix representation that realizes G as a complement of $\mathrm{P\Gamma L}({}_R R^\delta) \leq \mathrm{P\Gamma L}({}_R Q^\delta)$ for a complete discrete valuation field Q with valuation ring R.

In the other Appendix we collect some problems on modular geometric right ℓ-groups which we find interesting and which we would like to solve or see solved.

Chapter 1

A lattice-theoretic toolbox

As the title indicates, the aim of this chapter is to provide certain results on lattices. These are partially ordered sets in which each pair of elements has a least upper bound and a greatest lower bound.

An important type of lattice is $L({}_RM)$, which is defined as the set of submodules of a left module ${}_RM$, ordered by inclusion. Given two submodules $A, B \in L({}_RM)$, the least upper bound is their sum $A + B$ whereas the greatest lower bound is their intersection $A \cap B$. This type of lattice has the additional property of being *modular*. Modularity is defined by an equation that reflects the second isomorphism theorem for modules: $(A + B)/C \cong A/(B \cap C)$ whenever $A, B, C \in L({}_RM)$ and $C \subseteq A$.

General results on lattices and, in particular, modular lattices, will be given in Section 1.1.

However, not every modular lattice is a submodule lattice. For example, a modular lattice without a least and a greatest element can not be isomorphic to any lattice $L({}_RM)$. However, certain weak extra requirements force a modular lattice to be isomorphic to some lattice $L({}_RM)$. It turns out that in many cases, these lattice-theoretic conditions reflect algebraic properties of the ring R resp. the module ${}_RM$. The study of these connections is the subject of *coordinatization theory*.

In Section 1.2, we will present two coordinatization theorems: the first theorem of Veblen and Young, that characterizes the lattices $L({}_RM)$ where R is a skew field and ${}_RM$ is an R-vector spaces of dimension ≥ 4. The other coordinatization theorem is Inaba's theorem that gives conditions for a lattice to be isomorphic to $L({}_RM)$ where M is a finitely generated module over some completely primary uniserial ring R.

A related question is the study of lattice-isomorphisms between two submodule lattices $L({}_RM)$, $L({}_SN)$. One possibility of constructing a lattice isomorphism

$L({}_RM) \xrightarrow{\sim} L({}_SN)$ is to take a Morita equivalence ${}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ that maps M to N. As an equivalence preserves subobjects, this induces the desired lattice isomorphism.

Another way of getting lattice isomorphisms $L({}_RM) \xrightarrow{\sim} L({}_SM)$ is by considering *semilinear isomorphisms* between ${}_RM$ and ${}_SN$. Here, a semilinear isomorphism is a bijection $f : M \to N$ that is a homomorphism of abelian groups that preserves the scalar multiplication up to a ring isomorphism $\alpha : R \to S$, meaning that $f(rm) = \alpha(r)f(m)$ holds for all $r \in R$, $m \in M$. Such a map necessarily induces a lattice isomorphism $L({}_RM) \xrightarrow{\sim} L({}_SN)$.

In Section 1.3, we will focus on semilinear maps between modules and the lattice maps induced by them. In particular, we will state Camillo's theorem that gives conditions under which an isomorphism between submodule lattice is induced by a semilinear isomorphism.

The aim of this chapter is not to generate any new knowledge but to provide a lattice-theoretic background for the following chapters where we will make heavy use of these results on modular lattices.

1.1 Modular lattices

We assume that the reader is familiar with the most basic order theoretic notions. For a comprehensive account of the elements of lattice theory, see, for example, [Grä11].

We start with a convention: if L is a set with a partial order relation $\leq$, then for each pair $x, y \in L$ with $x \leq y$, we define the (closed) interval between x and y as

$$[x, y] := \{a \in L : x \leq a \leq y\}.$$

Now, let L be a set with a partial order relation $\leq$. Given a subset $X \subseteq L$, we call an element $z \in L$ a *meet* of X if, for all $a \in L$, we have the equivalence

$$(\forall x \in X : a \leq x) \Leftrightarrow (a \leq z).$$

If such an element z exists, it is unique and in this case we define $\bigwedge X := z$. For elements $x, y \in L$ - not necessarily distinct - we also write $x \wedge y := \bigwedge\{x, y\}$, if the meet exists.

Similarly, we call an element $z \in L$ a *join* of X if, for all $a \in L$, we have the equivalence

$$(\forall x \in X : a \geq x) \Leftrightarrow (a \geq z).$$

If such an element z exists, it must be unique as well, in which case we define $\bigvee X := z$. For elements $x, y \in L$ we also write $x \vee y := \bigvee\{x, y\}$ if the join exists.

Definition 1.1.1. A *lattice* is a pair $(L, \leq)$ where L is a set and $\leq$ is a partial order on L such that for all $x, y \in L$, both the meet $x \wedge y$ and the join $x \vee y$ exist.

The condition of $(L, \leq)$ being a lattice means that $\wedge$ and $\vee$ define binary operations on L (instead of being defined for *some* pairs only).

In a lattice L, the binary operations $\wedge, \vee$ obey the following rules which hold for any choice of $x, y, z \in L$:

$$x \wedge x = x \tag{Id1}$$
$$x \vee x = x \tag{Id2}$$
$$(x \wedge y) \wedge z = x \wedge (y \wedge z) \tag{As1}$$
$$(x \vee y) \vee z = x \vee (y \vee z) \tag{As2}$$
$$x \wedge y = y \wedge x \tag{Co1}$$
$$x \vee y = y \vee x \tag{Co2}$$
$$x \wedge (x \vee y) = x \tag{Ab1}$$
$$x \vee (x \wedge y) = x. \tag{Ab2}$$

The axioms (Id1), (Id2) say that $\wedge$ and $\vee$ are idempotent, (As1), (As2) express their associativity and (Co1), (Co2) tell us that they are commutative. (Ab1), (Ab2) are the so-called *absorption identities.*

Due to their omnipresence, we will not cite the usage of the lattice identities in what follows. Neither will we prove them here, although the proofs are very easy (or can be looked up in [Grä11, I,1.8]).

Remark 1.1.2. 1) In a lattice L, the binary operations $\wedge, \vee$ determine the partial order $\leq$ in the sense that we have the equivalences

$$x \leq y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y.$$

2) Given a non-ordered set L with binary operations $\wedge, \vee$ which fulfill the eight axioms listed above, then L can be given a canonical partial order $\leq$ defined by

$$x \leq y :\Leftrightarrow x \wedge y = x$$
(or, equivalently, by $x \leq y :\Leftrightarrow x \wedge y = y$).

Under this partial order, L becomes a lattice whose binary meet and join operations are exactly $\wedge$ and $\vee$. This is the only partial order on L with this property ([Grä11, I,1.10]). This implies that the set of axioms above gives a characterization of lattices by equations only.

To avoid repetitions, we will always assume that L denotes a lattice with the partial order $\leq$ and the lattice operations $\wedge$ and $\vee$.

If L, M are lattices, we call a map $f : L \to M$ a *homomorphism* of lattices, if the equalities

$$f(x \wedge_L y) = f(x) \wedge_M f(y)$$
$$f(x \vee_L y) = f(x) \vee_M f(y)$$

hold for all $x, y \in L$. If f is also bijective, then we call f an *isomorphism* of lattices. Note that in this case, the inverse $f^{-1} : M \to L$ is also a lattice isomorphism.

We call L *bounded from above* if there is an element $1 \in L$ such that for all $x \in L$ we have $x \leq 1$. This can be more fancily expressed by saying that $1 = \bigwedge \emptyset$ exists. Dually, we say that L is *bounded from below* if there is an element $0 \in L$ such that for all $x \in L$, we have $x \geq 0$ which is the same as saying that $0 = \bigvee \emptyset$ exists.

These conditions clearly are equivalent to 1 and 0 (if they exist) being neutral with respect to $\wedge$ resp. $\vee$, meaning that they can be expressed algebraically (without regard to $\leq$) by

$$1 \wedge x = x \tag{Bo1}$$

$$0 \vee x = x. \tag{Bo2}$$

We call L *bounded* if it is bounded from both above and below.

For elements $x, y \in L$, we say that y *covers* x if $x < y$ and there is no $z \in L$ such that $x < z < y$. In this case, we write $x \prec y$. If L is bounded from below (above), we call the $x \in L$ with $x \succ 0$ (resp. $x \prec 1$) *atoms* (resp. *dual atoms*). In the case of L being bounded from above, we denote the set of dual atoms by $X(L)$[1].

If L is bounded from below and every element of L is the finite join of some atoms, we call L *atomistic*. Dually, if L is bounded from above and every element is the finite meet of some dual atoms, we call L *dually atomistic*.

Let $x \in L$. We call x *meet-irreducible* if $y \wedge z$ $(y, z \in L)$ implies $x = y$ or $x = z$. Dually, we call x *join-irreducible* if $x = y \vee z$ implies $x = y$ or $x = z$. Note that meet-irreducibility of an element $x \in L$ implies that there is at most one $y \in L$ with $y \prec x$. Similarly, a join-irreducible element covers at most one element.

We call a subset $X \subseteq L$ a *chain* if any two elements of X are comparable, meaning that for all $x, y \in X$ either $x \leq y$ or $y \leq x$. For a lattice L, we define its *length* $l(L)$ as

$$l(L) = \sup\{|X| - 1 \,:\, X \text{ is a chain in } L\}$$

where we use the convention $\mathfrak{c} - 1 := \mathfrak{c}$ if $\mathfrak{c}$ is an infinite cardinal.

If L is bounded from below (above), we call an element $x \in L$ a *chain* (resp. *cochain*) if the subset $[0, x]$ (resp. $[x, 1]$) is a chain.

We call a sequence $(x_i)_{i \geq 0}$ with all $x_i \in L$ *ascending* (resp. *descending*) if $i \leq j$ implies $x_i \leq x_j$ $(x_i \geq x_j)$. We say L fulfills the *ascending* (resp. *descending*) *chain condition* if every ascending (resp. descending) sequence $(x_i)_{i \geq 0}$ in L becomes

[1] We will not need the dual notion here.

stationary at some point, meaning that there is an integer i such that $x_j = x_i$ for all $j \geq i$.

It is easy to see that if L is bounded from below and fulfills the descending chain condition, then for each $x \in L$ with $x \neq 0$ there is some atom under x. A similar statement is valid with dual atoms instead of atoms if L is assumed to be bounded from above and fulfilling the ascending chain condition.

Let $x \in L$. If the sublattice $x^{\uparrow} := \{y \in L : y \geq x\}$ satisfies the descending chain condition, we define the *socle* of x as

$$\mathrm{Soc}(x) := \bigvee\{y \in x : y \succ x\}$$

if the join exists[2]. Dually, if the sublattice $x^{\downarrow} := \{y \in L : y \geq x\}$ satisfies the descending chain condition, we define the *radical* of x as

$$\mathrm{Rad}(x) := \bigwedge\{y \in x : y \prec x\},$$

if existent.

We call L *modular* if for all $x, y, z \in L$ we have the implication

$$(x \leq z) \Rightarrow ((y \wedge z) = (x \vee y) \wedge z)\,.$$

The following lemma is a useful characterization of modularity. We will use it freely in this work, meaning that we will not refer to it every time we are using it.

Lemma 1.1.3 (Diamond Lemma). *For a lattice L, the following conditions are equivalent:*

i) L is modular.

ii) For all $a, b \in L$ the map

$$\begin{aligned} \varphi_b : [a, a \vee b] &\to [a \wedge b, b] \\ x &\mapsto x \wedge b \end{aligned}$$

is an isomorphism of lattices.

iii) For all $a, b \in L$ the map

$$\begin{aligned} \psi_a : [a \wedge b, b] &\to [a, a \vee b] \\ y &\mapsto y \vee a. \end{aligned}$$

is an isomorphism of lattices.

[2] This may look weird to a reader familiar with module theory. To provide some intuition: in submodule lattices $L({}_RM)$, - which we will discuss later - the „classical“ socle, defined as the sum of all simple submodules, is $\mathrm{Soc}(0)$ in our notation as long as ${}_RM$ is artinian.

In particular, for any $a, b \in L$, L being modular, $[a, a \vee b] \cong [a \wedge b, b]$[3]

Proof. See [Grä11, Theorem 348]. □

An important class of modular lattices is the class of *submodule lattices*

Let R be a ring with unity and ${}_RM$ a left R-module. Then $L({}_RM)$ is defined as the set of R-submodules of M. The set $L({}_RM)$ is canonically ordered by the inclusion relation $\subseteq$. Given arbitrary submodules $A, B \in L({}_RM)$, there is a greatest R-submodule contained in A and B which is their intersection $A \cap B$. On the other hand, their sum $A + B$ is the smallest R-submodule containing A and B. This shows that $L({}_RM)$ is a lattice with binary meet and join operations $\cap$ and $+$.

One can easily see that the lattice $L({}_RM)$ is bounded with top element $1_{L({}_RM)} = M$ and bottom element $0_{L({}_RM)} = 0$, the zero submodule.

When speaking of $L({}_RM)$, we will always assume that it comes with the lattice structure described above. When doing calculations in $L({}_RM)$, we will frequently use module-theoretic notation $(\subseteq, \cap, +, 0, M)$ instead of lattice-theoretic notation $(\leq, \wedge, \vee, 0, 1)$ in order to increase readability.

It is easily seen that submodule lattices $L({}_RM)$ are always modular[4]: To see that, let $A, B, C \in L({}_RM)$ be given, where $A \subseteq C$.

Now $A+(B\cap C) \subseteq (A+B)\cap C$ holds since $A \subseteq A+B, C$ and $B\cap C \subseteq A+B, C$ do [5]. On the other hand, given $z \in (A + B) \cap C$, we can write $z = a + b$ with $a \in A, b \in B$. Since $a \in A \subseteq C$ and $a + b = z \in C$, it follows that $b = (a+b)-a \in C$. Therefore, $b \in B\cap C$, which proves that $z = a+b \in A+(B\cap C)$. We infer that $(A + B) \cap C \subseteq A + (B \cap C)$. Both inclusions together imply that $(A + B) \cap C = A + (B \cap C)$, the modular identity, holds.

The following class of submodule lattices - the class of *desarguesian* lattices - will be of central interest in this work.

Definition 1.1.4. A lattice L is called *desarguesian* if there is a skew field D and a left vector space ${}_DV$ such that $L \cong L({}_DV)$ as lattices.

We call an element $x \in L$ *distributive*, if the identity

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

holds for all $y, z \in L$.

Using the notion of distributivity, we can define the class of *primary* lattices which will play an important role in this work.

[3] In this work, we will primarily work with this property of modular lattices.

[4] We note here that the concept of modularity - and its name - is derived from the behaviour of submodule lattices.

[5] In fact, the inequality $x \vee (y \wedge z) \leq (x \vee y) \wedge z$ is valid in each lattice!

Definition 1.1.5. Let L be a modular lattice. We call L *primary* if any interval $[x, y]$ is either a chain or contains no distributive[6] elements aside from x, y when considered as a lattice on its own.

Using the notion of *perspectivity*, one can formulate a simple criterion for primarity: if L is bounded from below, we call two elements $x, y \in L$ *perspective* if there is a $z \in L$ with $x \wedge z = y \wedge z = 0$ and $x \vee z = y \vee z$.

An important example is the following

Lemma 1.1.6. *In a desarguesian lattice L, any two atoms are perspective.*

Proof. Let $L = L({}_DV)$. It is not hard to see that any atom in L is of the form Dv with $0 \neq v \in V$ and vice versa. So, let Dv, Dw be atoms in L where $v, w \neq 0$.

If $Dv = Dw$, we have $Dv \cap 0 = Dw \cap 0 = 0$ and $Dv + 0 = Dv = Dw = Dw + 0$.

If $Dv \neq Dw$, then Dv, Dw are perspective via $D(v + w)$ which follows from

$$\begin{aligned} Dv \cap D(v+w) &= Dw \cap D(v+w) = 0 \\ Dv + D(v+w) &= Dv + Dw = Dw + D(v+w). \end{aligned}$$

In both cases we see that Dv, Dw are perspective. □

Proposition 1.1.7. *If L is modular and of finite length[7] then L is primary if and only if in each interval $[a, b]$ ($a, b \in L$), any two atoms are perspective (with respect to $[a, b]$).*

Proof. See [Ina48, Theorem 45]. □

The notion of basis resp. dual basis plays an important role in coordinatization:

Definition 1.1.8. Let L be a bounded modular lattice.

A subset $X \subseteq L$ is called *independent* if for any two subsets $A, B \subseteq X$ with $A \cap B = \emptyset$ we have $(\bigvee A) \wedge (\bigvee B) = 0$. We call X *spanning* if $\bigvee X = 1$. We call an independent, spanning subset $X \subseteq L$ a *basis* if each element of X is join-irreducible.

Dually, we call a subset $X \subseteq L$ *dually independent* if for any two subsets $A, B \subseteq X$ with $A \cap B = \emptyset$ we have $(\bigwedge A) \vee (\bigwedge B) = 1$. We call X *dually spanning* if $\bigwedge X = 0$. If each element in a in dually independent, dually spanning subset $X \subseteq L$ is meet-irreducible, we call X a *dual basis*.

[6] We note here that in Inaba's original article such elements are called *neutral*. Today, the notion of neutrality is given by another identity which turns out to be equivalent to distributivity whenever L is modular [Grä11, Chapter III, Section 2.].

[7] In the cited article, Inaba assumes all lattices to be of finite length, see [Ina48, page 48].

Example 1.1.9. If R is an arbitrary ring and $\delta \geq 0$ an integer, then $L({}_R R^\delta)$ has the spanning independent set $X = \{Re_1, \dots Re_\delta\}$ where e_i denotes the i'th basis vector of R^δ.

X needs not be a basis of $L({}_R R^\delta)$ since Re_i need not always be join-irreducible. However, X *is* a basis when R is a local ring: we clearly have an isomorphism ${}_R Re_i \cong {}_R R$ and each proper R-submodule of ${}_R R$ is contained in $\mathfrak{m}$, the unique maximal submodule of ${}_R R$. Therefore, each R-submodule of R^δ which is properly contained in Re_i is also contained in $\mathfrak{m}e_i$. Thus, Re_i can never be the sum of R-submodules $A, B \subsetneq Re_i$. Therefore, Re_i a join-irreducible element of $L({}_R R^\delta)$.

In case that L is modular, there is the following characterization of (dual) independence:

Proposition 1.1.10. *If L is a modular lattice which is bounded from below (resp. above), then a finite set $\{x_1, \dots, x_\delta\}$ is (dually) independent if and only if for all $1 \leq i \leq \delta - 1$:*

$$(x_1 \vee \ldots \vee x_i) \wedge x_{i+1} = 0$$
$$(\text{resp. } (x_1 \wedge \ldots \wedge x_i) \vee x_{i+1} = 1).$$

Proof. See [Grä11, Theorem 360]. □

Proposition 1.1.11. *Let L be a bounded modular lattice, then if a finite subset $X = \{x_1, \dots, x_\delta\} \subseteq L$ is spanning and independent (resp. dually spanning and dually independent) then the set $X' = \{x'_1, \dots, x'_\delta\}$, where*

$$x'_i = \bigvee (X \setminus \{x_i\})$$
$$(\text{resp. } x'_i = \bigwedge (X \setminus \{x_i\}) \quad)$$

is dually spanning and dually independent (resp. spanning and independent) in X.

A basis (resp. dual basis) X consists only of chains (resp. cochains) if and only if X' consists only of cochains (resp. chains).

Furthermore, $l([0, x_i]) = l([x'_i, 1])$ (resp. $l([x_i, 1]) = l([0, x'_i])$) for $1 \leq i \leq \delta$.

Proof. We only prove that half of the proposition where X is spanning and independent, since the other half is proved analogously.

Let $X \subseteq L$ be independent and spanning. First of all, if $A, B \subseteq X$ are arbitrary (i.e. not necessarily disjoint), we clearly have $\bigvee(A \cap B) \subseteq \bigvee A$, therefore the modular equation implies

$$\begin{aligned}\left(\bigvee A\right) \wedge \left(\bigvee B\right) &= \left(\bigvee A\right) \wedge \left(\left(\bigvee (B \setminus A)\right) \vee \left(\bigvee (A \cap B)\right)\right) \\ &= \left(\left(\bigvee A\right) \wedge \left(\bigvee (B \setminus A)\right)\right) \vee \left(\bigvee (A \cap B)\right) \\ &= 0 \vee \left(\bigvee (A \cap B)\right) = \bigvee (A \cap B).\end{aligned}$$

From these considerations it follows that the map $\mathcal{P}(X) \to L$ given by $A \mapsto \bigvee A$ is an embedding of the lattice $\mathcal{P}(X)$ (ordered by inclusion, with intersection and union as meet and join operations) as a sublattice of L where $f(\emptyset) = 0_L$ and $f(X) = 1_L$.

Given $A \subseteq X$, we write $A' := \{x_i' : x_i \in A\}$ where $x_1', x_2', \ldots, x_\delta'$ are defined as in the statement of the proposition. Then

$$\bigwedge A' = \bigwedge_{x_i \in A} \left(\bigvee (X \setminus \{x_i\})\right) = \bigvee \left(\bigcap_{x_i \in A} (X \setminus \{x_i\})\right) = \bigvee (X \setminus A) \,.$$

This implies that if $A \cap B = \emptyset$, then

$$\begin{aligned}\left(\bigwedge A'\right) \vee \left(\bigwedge B'\right) &= \left(\bigvee (X \setminus A)\right) \vee \left(\bigvee (X \setminus B)\right) \\ &= \bigvee ((X \setminus A) \cup (X \setminus B)) \\ &= \bigvee (X \setminus (A \cap B)) \\ &= \bigvee X = 1_L.\end{aligned}$$

So X' is indeed dually independent. Finally, X' is also dually spanning since

$$\bigwedge X' = \bigvee (X \setminus X) = \bigvee \emptyset = 0_L.$$

Given a basis $X = \{x_1, \ldots, x_\delta\} \subseteq L$, we have for any $1 \leq i \leq \delta$:

$$\left[x_i', 1_L\right] = \left[x_i', x_i' \vee x_i\right] \cong \left[x_i' \wedge x_i, x_i\right] = [0_L, x_i] \,,$$

so if X consists of chains only, then X' consists of cochains only. Furthermore, from these isomorphisms it becomes clear that the considered intervals have the same length. □

1.2 Coordinatization theorems

In this section, we will be concerned about *coordinatization* of modular lattices. First of all, we explain what coordinatization is about.

We have defined lattices as ordered sets with certain additional properties and demonstrated that we could also have defined them as algebraic structures fulfilling a finite set of identities. This has the consequence that lattices, in general, are quite abstract objects and therefore somewhat hard to get a grip on.

However, there are certain classes of lattices which stem from more familiar structures, for example the lattices of open sets in a topological space or desarguesian lattices. The disadvantage of these classes of lattices is that they are derived from structures which are not of a lattice-theoretical nature a priori.

Representation theorems show that some abstract classes of lattices are equivalent to some classes of lattices which are derived from structures outside of lattice theory. For example, the *Stone representation theorem* tells us that each Boolean lattice is isomorphic to the lattice of clopen sets in some totally disconnected compact Hausdorff space. This is one example of a *topological representation theorem*, another one being the *Priestley representation theorem* which realizes distributive lattices inside certain ordered topological spaces. In [Grä11, II,5.], several topological representation theorems are discussed (together with many references to related literature). Since the lattice of open sets in a topological space is necessarily distributive, topological representation theorems are restricted to classes of distributive lattices.

Good representation theorems for modular lattices are generally only achievable after restricting to subclasses of the class of modular lattices. These theorems typically connect classes of modular lattices with certain classes of submodule lattices or sublattices thereof. A representation theorem might, for example, tell us which modular lattices are desarguesian (which would be particularly nice since we know linear algebra very well).

Such a representation theorem is called a *coordinatization theorem.* More precisely, a coordinatization theorem is of the following form:

Theorem (Form of coordinatization theorem). *Given a lattice L with certain properties, there is a ring R and a left R-module ${}_RM$ with certain properties such that L can be embedded as a certain sublattice of $L({}_RM)$.*

We state an example of a very prominent coordinatization theorem.

Theorem (von Neumann). *([vNH98, Theorem 14.1]) If L is modular, complemented and of order $n \geq 4$ then there is an isomorphism of lattices $L \cong \overline{L}({}_RR) \subseteq L({}_RR)$ where R is a von Neumann regular ring and $\overline{L}({}_RR)$ is the lattice of its principal left[8] ideals.*

The reader will find the necessary terminology in the cited literature.

However, in this work, we will use coordinatization theory only as a black box. The reader who is interested in learning about coordinatization is referred to the survey [Day83] which also contains a vast literature list.

[8] Since we do not work with right modules here we decided to use the „opposite“ version of von Neumann's theorem.

1.2.1 Veblen-Young coordinatization

We will later need the coordinatization theorem of Veblen and Young, which can be seen as a special case of the von Neumann coordinatization theorem cited above. The theorem tells us in a precise way that a projective geometry of dimension at least 3 admits the introduction of coordinates from a skew field.

We first need to introduce a somewhat weaker version of modularity, namely, *semimodularity.* Semimodularity comes in two flavours:

Definition 1.2.1. A lattice L is called *upper* (resp. *lower*) *semimodular* if for all $a, b, c \in L$ we have the implication

$$a \prec b \Rightarrow (a \vee c \prec b \vee c \text{ or } a \vee c = b \vee c)$$
$$(\text{resp. } a \prec b \Rightarrow (a \wedge c \prec b \wedge c \text{ or } a \wedge c = b \wedge c)).$$

Note that modularity implies both upper and lower semimodularity: the covering relation $a \prec b$ is equivalent to $[a, b]$ having only two elements. So if L is modular and $a, b, c \in L$ are such that $a \prec b$, then by the diamond lemma,

$$[a \vee c, b \vee c] = [a \vee c, a \vee b \vee c] \cong [\underbrace{(a \vee c) \wedge b}_{\geq a}, b] \subseteq [a, b].$$

So $[a \vee c, b \vee c]$ has either one or two elements, therefore $a \vee c = b \vee c$ or $a \vee c \prec b \vee c$. This proves upper semimodularity. Lower semimodularity is proved in a dual way.

We will now define the class of bounded *geometric* lattices ([Grä11, Chapter V, Section 3]). Generally, geometric lattices are of huge importance in - not exclusively - the theory of combinatorial geometries.

Definition 1.2.2. A bounded[9] lattice L is called (*dually*) *geometric* if it is upper (lower) semimodular and (dually) atomistic.

Caution! All geometric lattices considered in this work are assumed to be bounded!

Note that a modular geometric lattice L is the same as an atomistic modular lattice. In this work, we will only be interested in the modular case, for which there exists quite a nice structure theory. We cite the necessary results without proof.

We call a lattice *directly indecomposable* if it is not isomorphic to a non-trivial direct product of lattices.

Theorem 1.2.3. *Each geometric lattice L is isomorphic to a direct product of finitely many directly indecomposable geometric lattices.*

[9]There is also a somewhat more technical notion of (dual) geometricity for lattices which are not necessarily bounded from above (resp. below), see also the cited literature.

Proof. [Grä11, Theorem 393] tells us that L decomposes as a direct product of directly indecomposable geometric lattices. Since we are demanding our geometric lattices to be bounded, they are of finite length (since 1_L is a join of atoms and all intervals in an upper semimodular lattice are of the same length [Grä11, Theorem 374]). So each direct decomposition has finitely many factors. □

We cite, also without proof, a coordinatization theorem which can essentially be traced back to Veblen and Young's characterization of projective geometries of dimension at least 3.

Theorem 1.2.4. *If L is a directly indecomposable, modular geometric lattice with $l(L) \geq 4$, then L is desarguesian.*

Proof. [Grä11, Corollary 435]. □

We want to remark here that von Neumann's coordinatization theorem is a really vast generalization of Theorem 1.2.4. However, the latter coordinatization theorem completely suffices for our purposes and can be used more flexibly.

1.2.2 Inaba coordinatization

Before we can introduce the Inaba coordinatization theorem, which will be a central tool here, we need some ring-theoretic definitions. The rings considered in the coordinatization theory of primary lattices are the ones which are *completely primary uniserial* [JM69, Definition 6.6][10]

Definition 1.2.5. A ring R (with unity) is called *completely primary uniserial* (*cpu*, for short), if it is local with maximal ideal $\mathfrak{m}$ and every left- or right ideal $I \subseteq R$ is of the form $I = \mathfrak{m}^i$ for some $i \geq 0$.

The *length* of a cpu ring R is the smallest integer k with $\mathfrak{m}^k = (0)$. Note that such an integer must exist by definition since (0) is an ideal in R.

From the definition, we can easily deduce:

Corollary 1.2.6. *Each left or right ideal in a cpu ring R is both-sided. Furthermore, there is a $\pi \in R$ such that $\mathfrak{m} = \pi R = R\pi$. Given such an element $\pi \in R$, the ideals of R are given by $\pi^i R = R\pi^i$ $(1 \leq i \leq k)$ where k is the length of R.*

We will call an element $\pi \in R$ with the stated properties a *uniformizer* of R.

[10] We cite the article of Monk and Jonsson for the definition since Inaba does not give a definition.

Proof. The ideal $\mathfrak{m}$ is the Jacobson radical of R and therefore both-sided. It follows that each power $\mathfrak{m}^i$ is both-sided as well. Since every left or right ideal has this form, the first statement follows easily.

If R is of length 1 then $\mathfrak{m} = (0)$, thus $\pi = 0$ is a uniformizer, in this case. If R is of length > 1 then $\mathfrak{m}^2 \subsetneq \mathfrak{m}$. In this case, any $\pi \in \mathfrak{m} \setminus \mathfrak{m}^2$ is a uniformizer of R: given such a π, if we assumed that $\pi R \subsetneq \mathfrak{m}$ this would imply $\pi R \subseteq \mathfrak{m}^2$, contradicting the choice of π. Therefore, $\pi R = \mathfrak{m}$ and, symmetrically, $R\pi = \mathfrak{m}$.

Let k be the length of R. By definition, each ideal of R is of the form $\mathfrak{m}^i$ $(1 \leq i \leq k)$. Let $\pi \in R$ be a uniformizer, then, since $\pi R = R\pi$, these can also be written as $\mathfrak{m}^i = (\pi R)^i = \pi^i R = R\pi^i$. □

Example 1.2.7. We list a few examples of cpu rings:

1) For a prime p and a positive integer k, the ring $\mathbb{Z}/(p^k)$ has the maximal ideal $\mathfrak{m} = (p)$ and is cpu of length k.

2) If K is a skew field and k is a positive integer, the ring $K[x]/(x^k)$ has the maximal ideal (x) and is cpu of length k. In particular, each field is a cpu ring of length 1. Vice versa, each cpu ring of length 1 is a local ring with $\mathfrak{m} = (0)$ and therefore a field.

3) Here is a more exotic example: taking a skew field K and a field automorphism $\sigma : K \to K$, we can form the *twisted polynomial ring*

$$K[x, \sigma] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;:\; a_i \in K, a_i \neq 0 \text{ for finitely many } i \right\}$$

with the „obvious“ addition

$$\left(\sum_{i=0}^{\infty} a_i x^i \right) + \left(\sum_{i=0}^{\infty} b_i x^i \right) = \sum_{i=0}^{\infty} (a_i + b_i) x^i$$

and the „twisted“ multiplication

$$\left(\sum_{i=0}^{\infty} a_i x^i \right) \cdot \left(\sum_{j=0}^{\infty} b_j x^j \right) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} a_i \cdot \sigma^i(b_{n-i}) \right) x^n.$$

which amounts to saying that x commutes with the elements of K up to a twist by σ, i.e. $xk = \sigma(k)x$ holds for all $k \in K$.

Given a positive integer k, the element x^k is normal in $R = K[x, \sigma]$, meaning that $Rx^k = x^k R$. This shows that $(x^k) = Rx^k$ is a both-sided ideal in R. Furthermore, it can be shown that the only left- or right ideals above (x^k) are

$$(x^k) \subset (x^{k-1}) \subset \ldots \subset (x^1) \subset (x^0) = R.$$

Therefore, $R/(x^k)$ has the maximal ideal $\mathfrak{m} = (x)$ and is cpu of length k.

We can now state Inaba's theorem which will be of central importance in this work.

Theorem 1.2.8. *Let L be a primary lattice with a basis $y_1, \ldots, y_\delta$ ($\delta \geq 4$) where additionally $l([0, y_1]) \geq l([0, y_2]) \geq \ldots l([0, y_\delta])$.*

If $l([0, y_1]) = \ldots = l([0, y_4]) = k$ there is a cpu ring R of length k and a submodule ${}_RM \subseteq {}_RR^\delta$ such that there is a lattice isomorphism $L \cong L({}_RM)$.

If π is an uniformizer of R and $e_1, \ldots, e_\delta$ denote the canonical basis elements of R^δ, then M can be chosen as the submodule $M = \oplus_{i=1}^{\delta} \pi^{k-l([0,y_i])} e_i \subseteq R^\delta$.

Proof. See [Ina48, Theorem 86]. □

We will actually need the following direct corollary of Inaba's coordinatization theorem:

Corollary 1.2.9. *Notation and conditions as in Theorem 1.2.8. In case that $l([0, y_i]) = k$ for* all *i, we have an isomorphism $L \cong L({}_RR^\delta)$.*

1.3 Semilinear maps

Before starting, we make clear that all rings considered in this work are unital.

We now discuss a certain class of module maps which can be seen as module homomorphisms with a „twist“:

Definition 1.3.1. Let ${}_RM$, ${}_SN$ be left modules over the rings R, S. We call a map $f : M \to N$ *semilinear* if f is a homomorphism of abelian groups and there is a ring homomorphism $\alpha : R \to S$ such that $f(rm) = \alpha(r)f(m)$ for all $r \in R$, $m \in M$.

If the homomorphism α is specified, we call f an *α-semilinear* map. If both $f : M \to N$ and $\alpha : R \to S$ are bijective, we call f a *semilinear isomorphism*.

It is easy to see that if some map $f : {}_RM \to {}_SN$ is an α-semilinear isomorphism, then f^{-1} is an α^{-1}-semilinear isomorphism.

Each semilinear map $f : {}_RM \to {}_SN$ induces a map

$$\begin{aligned} f_* : L({}_RM) &\to L({}_SN) \\ A &\mapsto S \cdot f(A). \end{aligned}$$

This map, however, is not a homomorphism of lattices, in general (it *is*, however, a homomorphism of $\vee$-semilattices!).

There is also an induced map into the other direction

$$\begin{aligned} f^* : L({}_SN) &\to L({}_RM) \\ B &\mapsto f^{-1}(B) \end{aligned}$$

which *is* a lattice homomorphism.

Let R be a ring and ${}_RM$ a left R-module. We then define $\Gamma\mathrm{L}({}_RM)$ as the group of all semilinear isomorphisms $f : {}_RM \to {}_RM$ under composition. Also, we denote by $\mathrm{GL}({}_RM)$ the group of all module isomorphisms $f : {}_RM \xrightarrow{\sim} {}_RM$ under composition. We furthermore set $\mathrm{P}\Gamma\mathrm{L}({}_RM) = \Gamma\mathrm{L}({}_RM)/N$ where

$$N = \{f \in \Gamma\mathrm{L}({}_RM) \mid \forall A \in L({}_RM) : f(A) = A\}\,.$$

So, $\mathrm{P}\Gamma\mathrm{L}({}_RM)$ is the group of permutations of $L({}_RM)$ consisting of all lattice automorphisms induced by semilinear isomorphisms.

Lemma 1.3.2. *Let R be a ring and ${}_RM$ be a* faithful *left R-module. Then*

i) if $\varphi : {}_RM \to {}_RM$ is an α-semilinear isomorphism, then the automorphism α is uniquely determined by φ.

ii) if $\varphi, \psi : {}_RM \to {}_RM$ are α- resp. β-semilinear isomorphisms, then $\varphi\psi$ is an $\alpha\beta$-semilinear isomorphism.

Proof. i) If φ is both α and β-semilinear, then for all $r \in R, m \in M$,

$$0 = \varphi(rm) - \varphi(rm) = \alpha(r)\varphi(m) - \beta(r)\varphi(m) = (\alpha(r) - \beta(r))\varphi(m).$$

Since $\varphi(m)$ can be any element of M and ${}_RM$ was assumed to be faithful, this implies that $\alpha(r) - \beta(r) = 0$ for all r, proving $\alpha = \beta$.

ii) For all $r \in R, m \in M$, we have

$$(\varphi\psi)(rm) = \varphi(\beta(r)\psi(m)) = (\alpha\beta)(r) \cdot (\varphi\psi)(m).$$

□

Proposition 1.3.3. *Let δ be a nonnegative integer and R be an arbitrary ring. Then there is a semidirect decomposition $\Gamma\mathrm{L}({}_RR^\delta) = \mathrm{GL}({}_RR^\delta) \rtimes \mathrm{Aut}(R)$. Here the latter factor is the automorphism group of R as a ring.*

Proof. We define a map $p : \Gamma\mathrm{L}({}_RR^\delta) \to \mathrm{Aut}(R)$ by letting $p(\varphi) := \alpha$ if φ is α-linear. By Lemma 1.3.2, p is well-defined (part i)) and a group homomorphism (part ii)).

Furthermore, we define

$$\iota : \mathrm{Aut}(R) \to \Gamma\mathrm{L}({}_R R^\delta)$$
$$\alpha \mapsto \iota_\alpha$$
$$\text{where } \iota_\alpha(r_1, \dots, r_\delta) = (\alpha(r_1), \dots, \alpha(r_\delta)).$$

For $\alpha, \beta \in \mathrm{Aut}(R)$, it is easily seen that $\iota_\alpha \iota_\beta = \iota_{\alpha\beta}$, therefore ι is a group homomorphism. Since for any $\alpha \in \mathrm{Aut}(R)$ the map ι_α is clearly α-semilinear, we see that $p\iota = \mathrm{id}_{\mathrm{Aut}(R)}$. We conclude that ι is a section of p which embeds $\mathrm{Aut}(R)$ as a subgroup of $\Gamma\mathrm{L}({}_R R^\delta)$.

Furthermore, the id_R-semilinear elements of $\Gamma\mathrm{L}({}_R R^\delta)$ are exactly the automorphisms of ${}_R R^\delta$, i.e. $\ker(p) = \mathrm{GL}({}_R R^\delta)$. A standard group-theoretic argument now shows that $\Gamma\mathrm{L}({}_R R^\delta) = \mathrm{GL}({}_R R^\delta) \rtimes \mathrm{Aut}(R)$. □

Proposition 1.3.4. *For any ring R and any integer $\delta > 1$, we have $\mathrm{P}\Gamma\mathrm{L}({}_R R^\delta) = \Gamma\mathrm{L}({}_R R^\delta)/R^\times$ where we embed the unit group $R^\times$ as the subgroup of $\Gamma\mathrm{L}({}_R R^\delta)$ consisting of the maps μ_r (where $r \in R^\times$) which are given by*

$$\mu_r : R^\delta \to R^\delta$$
$$m \mapsto rm.$$

Remark 1.3.5. Note that for fixed $r \in R^\times$, the map $\mu_r : R^\delta \to R^\delta$ is in general not linear (this only holds when r is central in R) but semilinear: with the ring automorphism $\gamma_r(s) := rsr^{-1}$ of R, we calculate for all $s \in R$, $m \in R^\delta$:

$$\mu_r(sm) = rsm = rsr^{-1}rm = \gamma_r(s)\mu_r(m).$$

Proof. As in the definition of $\mathrm{P}\Gamma\mathrm{L}({}_R M)$, let N be the group of all semilinear automorphisms of R^δ fixing all elements of $L({}_R R^\delta)$.

Clearly, for every μ_r ($r \in R^\times$) and $A \in L({}_R R^\delta)$ we have $\mu_r(A) = rA = A$, therefore $R^\times \subseteq N$.

Let $f \in N$, i.e., we have $f(A) = A$ for all $A \in L({}_R R^\delta)$. Denoting by $e_1, \dots, e_\delta$ the elements of the standard basis of R^δ, f restricts to semilinear automorphisms of the submodules Re_i ($1 \leq i \leq \delta$), that is, $f(re_i) = g_i(r)e_i$ with $g_i \in \Gamma\mathrm{L}({}_R R)$.

Let $1 < i \leq \delta$. Since f maps the submodule $R(e_1 + e_i)$ to itself,

$$g(r(e_1 + e_i)) = g_1(r)e_1 + g_i(r)e_i \in R(e_1 + e_i)$$

which implies $g_1(r) = g_i(r)$. Therefore $g := g_1 = g_2 = \ldots = g_\delta$.

If $g(1) = r$, then for all $a \in R$ we have

$$g(e_1 + ae_2) = re_1 + g(a)e_2 \in R(e_1 + ae_2)$$

which shows that $g(a) = ra$. Therefore, $f(m) = rm$ for all $m \in M$. Since f is a bijection, it is necessary that $r \in R^\times$. We conclude that $f = \mu_r$ with $r \in R^\times$. This proves that also $N \subseteq R^\times$. Therefore, $N = R^\times$. □

The traditional version of the fundamental theorem of projective geometry tells us that if ${}_KV, {}_LW$ are left vector spaces of dimension 3 or greater over skew fields K, L, then any lattice isomorphism $\varphi : L({}_KV) \xrightarrow{\sim} L({}_LW)$ is of the form $\varphi = f_*$ for some semilinear isomorphism $f : {}_KV \to {}_LW$. In particular, $K \cong L$ and $\dim_K V = \dim_L W$.

In the following, we need a slight generalization of the finite-dimensional version of the fundamental theorem which applies to cpu rings. We cite here - without proof - Camillo's generalization of the fundamental theorem:

Theorem 1.3.6. *Let R be a uniserial ring and let $\varphi : L({}_RR^\delta) \to L({}_SS^\delta)$ be a lattice isomorphism such that for some index i $(1 \leq i \leq \delta)$, the submodule $\varphi(Re_i) \subseteq S^\delta$ is isomorphic to ${}_SS$, then $\varphi = f_*$ for some semilinear isomorphism $f : {}_RR^\delta \to {}_SS^\delta$.*

Proof. This is part of [Cam84, Corollary 6.1]. □

To be capable to apply Theorem 1.3.6 in the way we want to, we first need a few lemmata. Recall that a module ${}_RM$ is called *cyclic* if there is an element $x \in M$ such that $Rx = M$ which is the same as saying that ${}_RM$ is isomorphic to a factor of ${}_RR$.

Lemma 1.3.7. *Let R be a noetherian local ring and ${}_RM$ be a finitely generated module. Then an element $A \in L({}_RM)$ is join-irreducible if and only if ${}_RA$ is a cyclic module.*

Proof. Since R is a noetherian ring and ${}_RM$ is finitely generated, the module ${}_RM$ is noetherian, which is the same as saying that $L({}_RM)$ fulfills the ascending chain condition.

In the special case that $A = 0$, the stated equivalence is easily checked. Following that, we can therefore assume that $A \neq 0$.

Let $0 \neq A \in L({}_RM)$ be join-irreducible. Since $L({}_RM)$ fulfills the ascending chain condition (and since there are submodules under A), there is a $B \in L({}_RM)$ such that $B \prec A$. If there was another $B' \in L({}_RM)$ with $B \prec A$, then $B + B' = A$, contradicting the join-irreducibility of A. Thus, every $C \in L({}_RM)$ with $C \subsetneq A$ also fulfills $C \subseteq B$. Take an arbitrary $x \in A \setminus B$, then $Rx \subseteq A$ but Rx can not be contained in B. It follows that $Rx = A$, i.e. A is cyclic.

Let now $0 \neq A \in L({}_RM)$ be cyclic, then ${}_RA$ is isomorphic to a factor module of ${}_RR$, and so, it has the unique maximal proper submodule $\mathfrak{m}A$ (which also exists due to our assumption that $A \neq 0$). Each $B \in L({}_RM)$ with $B \subsetneq A$ also lies in $\mathfrak{m}A$. It follows that ${}_RA$ can not be a join of two of its proper R-submodules. So, A is join-irreducible in $L({}_RM)$. □

Proposition 1.3.8. *Let R, S be cpu rings and $m, n \geq 3$ integers. Then for any lattice isomorphism $\varphi : L({}_RR^m) \xrightarrow{\sim} L({}_SS^n)$ there is a semilinear isomorphism $f : {}_RR^m \to {}_SS^n$ such that $\varphi = f_*$. In particular, $R \cong S$ and $m = n$.*

Proof. Let $e_1, \ldots, e_m$ be the canonical basis of ${}_RR^m$, then each submodule Re_i ($1 \leq i \leq m$) is join-irreducible in $L({}_RR^m)$ by Lemma 1.3.7. Therefore, the elements $Re_1, \ldots, Re_m$ form a basis of $L({}_RR^m)$

As φ is a lattice isomorphism, the elements $\varphi(Re_1), \ldots, \varphi(Re_m) \in L({}_SS^n)$ form a basis of $L({}_SS^n)$ as well. By Lemma 1.3.7, there are $e_i' \in S^n$ ($1 \leq i \leq m$) such that $\varphi(Re_i) = Se_i'$ for all i. That the Se_i' form a lattice-theoretic basis means that $S^n = \bigoplus_{i=1}^m Se_i'$ and each summand is an indecomposable S-module. The module ${}_SS^n$ clearly has finite length, so the Krull-Schmidt theorem [Jac89, p.115] implies that $m = n$ and ${}_SSe_i' \cong {}_SS$ for all $1 \leq i \leq m$.

From Theorem 1.3.6 now follows that $\varphi = f_*$ for some semilinear isomorphism $f : {}_RR^m \to {}_SS^n$. □

From Proposition 1.3.4 we can now deduce:

Corollary 1.3.9. *If R is a cpu ring and $\delta \geq 3$, then* $\mathrm{Aut}(L({}_RR^\delta)) \cong \mathrm{P\Gamma L}({}_RR^\delta)$.

Chapter 2

Desarguesian right ℓ-groups

The aim of this chapter is the characterization of desarguesian right ℓ-groups of dimension $\delta \geq 4$.

A right ℓ-group is defined as a group G, together with a partial order that makes G a lattice and that is invariant under right-multiplication. Equivalently, a right ℓ-group can be seen as a pointed lattice, together with a regular group of lattice automorphisms. Several classes of ordered groups are right ℓ-groups, for example, Garside groups [Deh02] and „classical" (both-sided) ℓ-groups (see, for example, [Dar94]).

In this chapter, we are mainly interested in right ℓ-groups G with a strong order unit s. A strong order unit is an element $s > e$ such that

i) left-multiplication by s is a lattice automorphism,

ii) s is *archimedean* in the sense that each element of G is dominated by some power of G.

Right ℓ-groups with strong order unit can be seen as a generalization of Garside groups (where the Garside element takes the place of the strong order unit) - as we will see, nothing more than a strong order unit is needed in order to define and construct right- and left-normal forms for right ℓ-groups.

In Section 2.1, we will prove that in a right ℓ-group with strong order unit, each element $g \leq e$ has unique right-normal and left-normal factorizations while providing useful formulae for these factorizations (Proposition 2.1.17, Proposition 2.1.29).

In Section 2.2, we will define noetherian right ℓ-groups as those which fulfill certain chain conditions. We will then present Rump's theorem that each modular noetherian right ℓ-group with strong order unit has a distinguished strong order interval $[s^{-1}, e]$ that is a modular dual geometric lattice. This motivates the notion of *modular geometric* right ℓ-group. We then define the class of *desarguesian* right ℓ-groups as those modular geometric right ℓ-groups with a desarguesian strong order interval.

In Section 2.3, we will investigate the combinatorics of right- and left-normal factorizations in modular geometric right ℓ-groups. We will prove that each modular geometric right ℓ-group G has a unique degree homomorphism $d : G \to \mathbb{Z}$ such that $d(g) = l([g, e])$ whenever $g \leq e$ (Proposition 2.3.2).

The degree homomorphism is well-behaved on right-normal factorizations: if $g \leq e$ has the right-normal factorization $g = g_k g_{k-1} \ldots g_1$, we show that always $d(g_1) \geq d(g_2) \geq \ldots d(g_k)$ (Proposition 2.3.9).

Besides several other results on the behaviour of the degree in right- and left-normal factorizations, we will furthermore prove that an element $g \leq e$ is meet-irreducible in G^- if and only if the right-normal factorization of g only contains factors of degree 1 (Proposition 2.3.14). This result will enable us later to construct dual bases for intervals in G^-.

In Section 2.4, we will apply these results in order to prove that if G is a desarguesian right ℓ-group of dimension δ, then for each integer k, the interval $[s^{-k}, e]$ is a primary lattice with a basis consisting of δ elements of length k (Proposition 2.4.7). If $\delta \geq 4$, this implies, by Inaba's coordinatization theorem, that the intervals $[s^{-k}, e]$ are isomorphic to $L({}_{R_k}R_k^\delta)$ where R_k is a cpu ring of length k (Proposition 2.4.8). Thus, we have achieved a local coordinatization of G.

In Section 2.5,, we deduce a global coordinatization theorem for desarguesian right ℓ-groups of dimension $\delta \geq 4$. As G is neither bounded from below nor from above, G cannot be realized as a lattice $L({}_RM)$. However, it is possible to realize G within $L({}_RQ^\delta)$ where Q is some complete discrete valuation field with valuation ring R. To be more precise, we will prove that G, as a lattice, is isomorphic to the sublattice $\mathrm{Lat}({}_RQ^\delta) \subseteq L({}_RQ^\delta)$ consisting of R-lattices in ${}_RQ^\delta$ (Theorem 2.5.16).

In Section 2.6, we will show that each lattice automorphism of $\mathrm{Lat}({}_RQ^\delta)$ is induced by a semilinear automorphism of ${}_RQ^\delta$ that comes from a unique equivalence class in $\mathrm{P\Gamma L}({}_RQ^\delta)$ (Proposition 2.6.5). Furthermore, we will prove that the group $\mathrm{P\Gamma L}({}_RR^\delta)$ is a point stabilizer under the action of $\mathrm{P\Gamma L}({}_RQ^\delta)$ on $\mathrm{Lat}({}_RQ^\delta)$ (Proposition 2.6.4).

As the group G acts regularly on the lattice structure of G, this leads to a realization as a complement of the subgroup $\mathrm{P\Gamma L}({}_RR^\delta) \leq \mathrm{P\Gamma L}({}_RQ^\delta)$ whenever G is desarguesian of dimension $\delta \geq 4$. We will also show that the complements of $\mathrm{P\Gamma L}({}_RR^\delta) \leq \mathrm{P\Gamma L}({}_RQ^\delta)$ are desarguesian right ℓ-groups of dimension δ whenever Q is a discrete valuation field with valuation ring R (Theorem 2.6.6). We have therefore obtained a complete group-theoretic characterization of desarguesian right ℓ-groups of dimension $\delta \geq 4$.

2.1 Right ℓ-groups

In this section, we discuss right ℓ-groups, that is, groups with a lattice structure that is invariant under right-multiplication. We put an emphasis on right ℓ-groups with strong order unit which are a generalization of Garside groups. It turns out that a remarkable feature of Garside groups - namely the existence of right- and left-normal factorizations - is also present in these groups.

Definition 2.1.1. A *right-ordered group* is a pair $(G, \leq)$ where G is a group and $\leq$ is a partial order on G that is *right-invariant*, meaning that for all $x, y, z \in G$ we have the implication

$$x \leq y \Rightarrow xz \leq yz.$$

A *right lattice-ordered group* (*right ℓ-group*, for short) is a right ordered group $(G, \leq)$ such that G is a lattice under $\leq$.

When speaking of a group G as a right-ordered resp. right lattice-ordered group, we will always implicitly mean a pair $(G, \leq)$ confirming to the above definition.

Example 2.1.2. 1) The easiest example of a right ℓ-group is $(\mathbb{Z}, \leq)$, the additive group of integers with $\leq$ being the standard order of integers. The respective lattice operations are given by

$$a \wedge b = \min\{a, b\}, \quad a \vee b = \max\{a, b\}.$$

More generally, let X be a set and $\mathbb{Z}^{(X)}$ the free abelian group over X. We denote the elements of $\mathbb{Z}^{(X)}$ as $a =: \sum_{x \in X} a_x[x]$ where $a_x \neq 0$ for only finitely many $x \in X$. The coordinatewise order on $\mathbb{Z}^{(X)}$ given by

$$a \leq b :\Leftrightarrow a_x \leq b_x \ (x \in X)$$

is a right-invariant order such that $(\mathbb{Z}^{(X)}, \leq)$ is a right ℓ-group. The respective lattice operations are

$$a \wedge b = \sum_{x \in X} \min\{a_x, b_x\}[x],$$
$$a \vee b = \sum_{x \in X} \max\{a_x, b_x\}[x].$$

2) Let R be a Dedekind domain and K its field of fractions, then the set of fractional ideals,

$$J_K = \{I \in L(_R K) \ : \ I \text{ finitely generated}, \ I \neq 0\}.$$

is a group under the multiplication of fractional ideals ([Neu92, Chap. I, Prop. 3.8]) which is defined as

$$I_1 \cdot I_2 = \left\{ \sum_i a_i b_i : a_i \in I_1, b_i \in I_2 \right\}$$

The group J_K is ordered by inclusion. Furthermore, inclusion of fractional ideals is preserved under multiplication. The pair $(J_K, \subseteq)$ is then a right ℓ-group whose lattice operations are inherited by $L({}_RK)$, i.e.

$$I_1 \wedge I_2 = I_1 \cap I_2 \,, \quad I_1 \vee I_2 = I_1 + I_2 .$$

3) Let X be a topological space. Then $C(X, \mathbb{R})$, the additive group of real-valued continuous functions, is a right ℓ-group under the partial order given by

$$f \leq g :\Leftrightarrow f(x) \leq g(x) \; (x \in X).$$

Meet and join of two elements $f, g \in C(X, \mathbb{R})$ are then given by

$$(f \wedge g)(x) = \min\{f(x), g(x)\},$$
$$(f \vee g)(x) = \max\{f(x), g(x)\}.$$

4) Let G be an arbitrary group. With the *trivial order* $\leq$, defined by $x \leq y :\Leftrightarrow x = y$, the pair $(G, \leq)$ is a right-ordered group. However, if $|G| > 1$, this is not a right ℓ-group since $x \wedge y$ and $x \vee y$ only exist when $x = y$.

We remark here that all right ℓ-groups in this example are commutative which implies that the partial order is also invariant under left-multiplication. It is rather tricky to find examples of right ℓ-groups that are not also left ℓ-groups. At the end of this section, we will present such a family of right ℓ-groups, the *pure paraunitary groups.*

It turns out that the right-invariance of the partial order in a right ℓ-groups implies that the respective lattice operations are right-invariant, as well.

Proposition 2.1.3. *If G is a right ℓ-group, then all $x, y, z \in G$ fulfill the equations*

$$(x \wedge y)z = xz \wedge yz$$
$$(x \vee y)z = xz \vee yz$$

where $\wedge, \vee$ are the respective meet and join operations of G as a lattice.

Proof. Let G be a right ℓ-group and $z \in G$. Then the map $\rho_z : G \to G \; ; \; g \mapsto gz$ is an isomorphism of ordered sets with inverse $\rho_z^{-1} : G \to G \; ; \; g \mapsto gz^{-1}$. Therefore, ρ_z is also an isomorphism of lattices, implying the above equations. □

Remark 2.1.4. One could also define a right ℓ-group as a group G with additional binary operations $\wedge, \vee$ such that G is a lattice under these operations and the equations from Proposition 2.1.3 are fulfilled. Using the lattice operations, the partial order on G can then be expressed by

$$x \leq y \Leftrightarrow x \vee y = y.$$

Therefore, $x \leq y$ implies that $xz \vee yz = (x \vee y)z = yz$, that is, $xz \leq yz$. This in turn means that right-invariance can also be expressed by the equations of Proposition 2.1.3 in the case that G is a lattice.

This shows that one can define right ℓ-groups by equations involving group and lattice operations only. This makes the class of right ℓ-groups what is called a *variety* - or *equational class* - in universal algebra. See [Grä08] for details.

Remark 2.1.5. There is another way to look at right ℓ-groups: each group G acts regularly on itself from the right via $G \times G \to G$; $(g, h) \mapsto gh$. If G is a right ℓ-group, we can consider the left factor as a lattice only - then the left factor G becomes a lattice on which G acts regularly from the right via lattice automorphisms. On the other hand, given a regular automorphism group G of some lattice L, the lattice structure of L can be pulled back to G in a way that makes G a right ℓ-group. We will make use of this philosophy later (Theorem 2.5.17).

Each right-ordered group contains two distinguished subsets, namely its *positive cone*

$$G^+ = \{g \in G : g \geq e\}$$

and its *negative cone*

$$G^- = \{g \in G : g \leq e\}.$$

Either of these two subsets determines the other, since, by right-invariance, we have the equivalence $g \geq e \Leftrightarrow e \geq g^{-1}$. Therefore, $(G^+)^{-1} = G^-$ and $(G^-)^{-1} = G^+$. Furthermore, G^- is a submonoid of G: if $g, h \in G^-$, then $gh \leq eh = h \leq e$, so $gh \in G^-$ as well. Similarly, one sees that G^+ is a submonoid of G.

If G is a (not necessarily ordered) group, a submonoid $P \subseteq G$ is called *pure* if $P \cap P^{-1} = \{e\}$. If G is a right-ordered group, then its positive cone G^+ is in fact a pure submonoid of G since

$$G^+ \cap (G^+)^{-1} = G^+ \cap G^- = \{e\}.$$

Similar reasoning shows that G^- is a pure submonoid of G as well.

This proves one direction of the following fundamental result on the characterization of right-invariant orders on a group:

Proposition 2.1.6. *If G is a right-ordered group, then G^+ and G^- are pure submonoids of G. Vice versa, given a group G, each pure submonoid P is the positive (resp. negative) cone of a unique right-invariant order on G which is given by*

$$g \leq h :\Leftrightarrow hg^{-1} \in P$$

$$(\text{resp. } g \leq h :\Leftrightarrow gh^{-1} \in P).$$

Proof. See [KM96, Theorem 1.5.1.], for the (easy) proof. □

Let M be a monoid. We call M a *left Ore* monoid [Deh15, Definition 3.10][1], if both of the following conditions - the so-called *left Ore conditions* - are fulfilled:

[1]In the monograph [Deh15], most general definitions and results are stated in a categorical framework. However, in this work we only work with the special case of monoids and groups - which can be seen as categories resp. groupoids with one object. Therefore, every „categorical" statement cited from this monograph will be restricted to monoids and groups here.

i) The monoid M is *left-cancellative*, that is, for all $a, b, c \in M$ we have the implication $ab = ac \Rightarrow b = c$.

ii) There are common left-multiples in M, that is, for all $a, b \in M$ there are $c, d \in M$ such that $ca = db$.

If M is a monoid, we call a group G a *group of left* (resp. *right*) *fractions* if each $g \in G$ is expressible as $g = a^{-1}b$ (resp. $g = ab^{-1}$) for some elements $a, b \in M$.

The following theorem shows the interplay between left groups of fractions and the left Ore conditions:

Theorem 2.1.7. *Let M be a monoid. Then M has a group of left fractions if and only if M satisfies the left Ore conditions.*

Proof. See [Deh15, Proposition 3.11]. □

We will only use this important result in Appendix A. However, it would have felt unnatural not to mention it after introducing groups of left fractions and the left Ore conditions.

We can show that a right ℓ-group G is a group of left- and right fractions for G^-:

Proposition 2.1.8. *A right ℓ-group G is a group of left fractions for G^-. Also, G is a group of right fractions for G^-.*

Proof. For the first statement, let $g \in G$. We set $g_2 := e \wedge g \leq e$. From $g_2 \leq g$ and right-invariance, we deduce $e \leq gg_2^{-1}$ resp. $g_1 := g_2g^{-1} = (gg_2^{-1})^{-1} \leq e$. Therefore, $g_1, g_2 \in G^-$ and $g_1^{-1}g_2 = gg_2^{-1}g_2 = g$. Thus, any element of G is expressible as a left fraction of elements in G^-.

For the second statement, take again some $g \in G$. We now set $g_2' = (e \vee g)^{-1} \leq e$. From $g \leq e \vee g$ and right-invariance, it follows that $g_1' := g(e \vee g)^{-1} \leq e$. We then have $g_1', g_2' \leq e$ and $g_1' g_2'^{-1} = g(e \vee g)^{-1}(e \vee g) = g$. This shows that G is a group of right fractions of elements in G^-. □

If G is a left group of fractions for a monoid M, then the relations in G are essentially determined by the relations in M. This vague statement is specified by the following lemma:

Lemma 2.1.9. *Let G and H be groups which are groups of left fractions for the left Ore monoids $M \subseteq G$ and $N \subseteq H$. If $f' : M \to N$ is a homomorphism of monoids, then f' can be extended uniquely to a homomorphism of groups $f : G \to H$ such that $f|_M = f'$. This homomorphism is given by $f(g_2^{-1}g_1) = f'(g_2)^{-1}f'(g_1)$ for all $g_1, g_2 \in M$.*

Proof. See [Deh15, Chapter II, Lemma 3.13]. □

The fact that G is a group of fractions for G^-, now implies the following proposition which we will refer to several times in this work:

Proposition 2.1.10. *If G is a right ℓ-group and H is an arbitrary group, then any monoid homomorphism $f' : G^- \to H$ can uniquely be extended to a group homomorphism $f : G \to H$ with $f|_{G^-} = f'$ by setting $f(g_2^{-1}g_1) := f'(g_2)^{-1}f'(g_1)$ for all $g_1, g_2 \in G^-$.*

Proof. First of all, the monoid G^- is left-cancellative since it is a submonoid of a group. Furthermore, for any $g_1, g_2 \in G^-$ one can find $h_1, h_2 \in G^-$ such that $h_1g_1 = h_2g_2$ - choose $h_1 = (g_1 \wedge g_2)g_1^{-1}$ and $h_2 = (g_1 \wedge g_2)g_2^{-1}$. Therefore, the monoid G^- fulfills the left Ore conditions. Furthermore G is its group of left fractions by Proposition 2.1.8.

Each group H is clearly a left Ore monoid and a group of left fractions of itself. Due to Lemma 2.1.9, the monoid homomorphism f' can be extended to a group homomorphism f in the stated way, so the proposition follows. □

If G is a right ℓ-group, we can define the following residuation operation on G^-:

$$\begin{aligned} \to: G^- \times G^- &\to G^- \\ (g, h) &\mapsto g \to h := e \wedge hg^{-1} \end{aligned}$$

In the following proposition, we collect some important properties of the $\to$-operation. We would also like to remark that we will not use all of them - however, some of them are of such a great importance that they deserve to be mentioned here.

Proposition 2.1.11. *For all $x, y, z \in G^-$ we have:*

$$x \to x = x \to e = e \tag{S1a}$$
$$e \to x = x \tag{S1b}$$
$$(x \to y)x = x \wedge y \tag{S2}$$
$$(x \wedge y) \to z = (x \to y) \to (x \to z) \tag{S3}$$
$$x \to (y \wedge z) = (x \to y) \wedge (x \to z) \tag{S4}$$
$$xy \to z = x \to (y \to z) \tag{S5}$$
$$x \to yz = ((z \to x) \to y)(x \to z) \tag{S6}$$
$$x \to y = e \Leftrightarrow x \leq y. \tag{S7}$$

Remark 2.1.12. We note here that the equations (S1a), (S1b), (S3) and (S4) together tell us that G^- is a *semibrace* ([Rum08b, Definition 3]) under $\wedge$ and $\to$.

Furthermore, (S3) - together with the fact that $x \wedge y = y \wedge x$ - implies the *cycloid equation* (see [Rum08a, Section 1]):

$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z).$$

Proof of Proposition 2.1.11. The equations (S1a), (S1b), (S2) and (S7) are trivial or obvious. We will use them in the proofs of the remaining equations.

In order to prove (S3), we calculate

$$\begin{aligned} ((x \wedge y) \to (x \wedge z))(x \wedge y) &\overset{(S2)}{=} (x \wedge y) \wedge (x \wedge z) \\ &= x \wedge y \wedge z \\ &= (x \wedge y) \wedge z \\ &\overset{(S2)}{=} ((x \wedge y) \to z)(x \wedge y) \end{aligned}$$

and cancel the factor $x \wedge y$ from the right.

Equation (S4) is proven by the calculation:

$$\begin{aligned} x \to (y \wedge z) &= e \wedge (y \wedge z)x^{-1} \\ &= e \wedge yx^{-1} \wedge zx^{-1} \\ &= (e \wedge yx^{-1}) \wedge (e \wedge zx^{-1}) \\ &= (x \to y) \wedge (x \to z). \end{aligned}$$

(S5): we calculate (from now on without mentioning every application of (S2))

$$\begin{aligned} (x \to (y \to z))xy &= (x \wedge (y \to z))y \\ &= xy \wedge (y \to z)y \\ &= xy \wedge y \wedge z \\ &= (x \wedge e)y \wedge z \\ &= xy \wedge z \\ &= (xy \to z)xy \end{aligned}$$

and cancel xy from the right.

(S6): We calculate

$$\begin{aligned} ((z \to x) \to y)(x \to z)x &= ((z \to x) \to y)(x \wedge z) \\ &= ((z \to x) \to y)(z \to x)z \\ &= ((z \to x) \wedge y)z \\ &= (z \to x)z \wedge yz \\ &= z \wedge x \wedge yz \\ &= x \wedge (e \wedge y)z \\ &= x \wedge yz \\ &= (x \to yz)x \end{aligned}$$

and cancel x from the right. □

Strong order units in right ℓ-groups generalize the same-named entities (cf. [Dar94, Definition 7.4.]) in classical ℓ-groups which are defined as groups with a lattice order that is both right- and left-invariant. If G is a (classical) ℓ-group, then an element $s \in G^+$ is called a strong order unit if for each $g \in G^+$ there is an integer k such that $g \leq s^k$. Adding a normality condition, strong order units can be generalized in such a way to right ℓ-groups that they capture some of the properties of Garside elements in Garside groups. We will see later that these properties suffice for defining right- and left-normal factorizations in right ℓ-groups with respect to a strong order unit.

Definition 2.1.13. ([Rum15, Section 6.]) Let G be a right ℓ-group. We call an element $s \in G^+$ *normal* if for all $x, y \in G$, $s(x \vee y) = sx \vee sy$ and $s(x \wedge y) = sx \wedge sy$ (i.e. the lattice structure of G is also invariant under *left*-multiplication by s).

We call an element s of a right ℓ-group G a *strong order unit* if s is normal and for every $g \in G$ there is an integer k such that $g \leq s^k$.

If s is a normal element of a right ℓ-group G, then the map $G \to G$; $g \mapsto sg$ is a bijective lattice homomorphism, i.e. an isomorphism of lattices. This immediately implies:

Corollary 2.1.14. *Let G be a right ℓ-group and $s \in G$ a normal element of G. Then for any integer k and arbitrary $x, y \in G$ we have the equalities*

$$s^k(x \vee y) = s^k x \vee s^k y,$$
$$s^k(x \wedge y) = s^k x \wedge s^k y.$$

Due to its simplicity and since we will use this property very often in calculations we will mostly not refer to the corollary while using it.

We list some important properties of strong order units:

Proposition 2.1.15. *Let G be a right ℓ-group and $s \in G$ a strong order unit. Then the following holds:*

i) For all $g \in G$, we have $g \leq s^k$ if and only if $s^{-k} \leq g^{-1}$.

ii) For any $g \in G$ there is an integer k such that $s^{-k} \leq g$.

iii) Let k be a nonnegative integer. Let elements $g_i \in G$ $(1 \leq i \leq k)$ be given such that $g_i \leq s$ $(1 \leq i \leq k)$. Then $g_k g_{k-1} \dots g_1 \leq s^k$.

iv) Let k be a nonnegative integer. Let elements $g_i \in G$ $(1 \leq i \leq k)$ be given such that $g_i \geq s^{-1}$ $(1 \leq i \leq k)$. Then $g_k g_{k-1} \dots g_1 \geq s^{-k}$.

Convention: We decided to write products in G in the form $g_k g_{k-1} \dots g_1$. This may be misleading when $k < 3$ (probably even when $k = 3$). To emphasize what is meant in these cases:

$$\begin{aligned} k = 0: &\quad g_k g_{k-1} \dots g_1 := e, \\ k = 1: &\quad g_k g_{k-1} \dots g_1 := g_1, \\ k = 2: &\quad g_k g_{k-1} \dots g_1 := g_2 g_1, \\ k = 3: &\quad g_k g_{k-1} \dots g_1 := g_3 g_2 g_1. \end{aligned}$$

Proof of Proposition 2.1.15. i) $g \leq s^k \Leftrightarrow e \leq s^k g^{-1} \Leftrightarrow s^{-k} \leq g^{-k}$.

ii) For any $g \in G$, there is an integer k such that $g^{-1} \leq s^k$. Using i) gives $s^{-k} \leq g$.

iii) By induction on k: For $k = 0$, the product is empty and the statement - which then reads $e \leq e$ - is trivial in this case. Now assume that we have $g_1, g_2, \dots, g_k, g_{k+1} \leq s$ and $g_k g_{k-1} \dots g_1 \leq s^k$ already holds. Then, $s g_k g_{k-1} \dots g_1 \leq s^{k+1}$. From $g_{k+1} \leq s$ we deduce that $g_{k+1} g_k g_{k-1} \dots g_1 \leq s g_k g_{k-1} \dots g_1$. Together, this implies $g_{k+1} g_k \dots g_1 \leq s^{k+1}$.

iv) This is proven in the same manner as the statement above.

□

We have already encountered groups of fractions which play an important role in Garside theory. The presence of a strong order unit now enables us to use another important Garside theoretic concept, namely right- and left-normal factorizations (see [Deh15, Chapter I, Proposition 2.4] for the prototype). In Garside theory, these provide computable normal forms whose good combinatorial behaviour (see [Deh15, Chapter III]) is one of the main reasons that the computational aspects of Garside groups are quite well-understood.

Definition 2.1.16. Let G be a right ℓ-group with a strong order unit s. For an element $g \in G^-$, a *right-normal factorization* (with respect to s) of g consists of elements $g_1, g_2, \dots, g_k \in \left[s^{-1}, e\right]$ $(k \geq 0)$ with $g_i \neq e$ $(1 \leq i \leq k)$ such that

$$g = g_k g_{k-1} \cdots g_1$$

and this factorization is *right-maximal* in the following sense:

For $1 \leq i \leq k-1$, there is no $h < e$ such that $g_{i+1} \leq h$ and $h g_i \geq s^{-1}$.

The integer k is called the *length* of the right-normal factorization.

Let G be a right ℓ-group with a strong order unit s and let $g \in G^-$. Suppose that we are given a factorization $g = g_k g_{k-1} \dots g_1$ with all $g_i \in [s^{-1}, e]$, $g_i \neq e$ and let $1 \leq i \leq k-1$. Note that $h g_i \geq s^{-1}$ if and only if $h \geq s^{-1} g_i^{-1}$. From this, we can see that the inequalities $hg_i \geq s^{-1}$ and $h \geq g_{i+1}$ are fulfilled exactly when $h \geq s^{-1}g_i^{-1} \vee g_{i+1}$. So h is forced to be trivial if and only if the right hand side of the latter inequality is equal to e.

It follows that the condition that $g = g_k g_{k-1} \dots g_1$ is a right-maximal factorization can also be expressed equationally as

$$s^{-1}g_i^{-1} \vee g_{i+1} = e \quad (1 \leq i \leq k-1). \tag{2.1}$$

It can be shown:

Proposition 2.1.17. *Let G be a right ℓ-group with strong order unit s. Then each $g \in G^-$ has a unique right-normal factorization $g = g_k g_{k-1} \dots g_1$ with respect to s. Its length k is the minimal integer with $s^{-k} \leq g$ and the factors are given by*

$$g_i = (g \vee s^{-i})(g \vee s^{-i+1})^{-1} \tag{2.2}$$

for $1 \leq i \leq k$.

Proof. With the elements g_i given as in the statement of the proposition, we first show that $g_i \in [s^{-1}, e]$ for all $i \geq 1$, meaning that

$$s^{-1} \leq (g \vee s^{-i})(g \vee s^{-i+1})^{-1} \leq e$$

which is equivalent to

$$s^{-1}(g \vee s^{-i+1}) \leq g \vee s^{-i} \leq g \vee s^{-i+1}.$$

The right inequality follows from $s^{-i} \leq s^{-i+1}$. Using the normality of s we can „expand" the left term to $s^{-1}g \vee s^{-i}$. Then, the left inequality follows from $s^{-1}g \leq g$ which is equivalent to $s^{-1} \leq e$.

When the g_i are defined by (2.2), we have $g \vee s^{-i} = g_i g_{i-1} \dots g_1$. Furthermore, there is an integer k with the properties from the proposition: taking k as the least integer with $g^{-1} \leq s^k$, we also have $s^{-k} \leq g$, due to normality. Fixing this k, we get $g = g_k g_{k-1} \dots g_1$.

This factorization is right-maximal: assume that there was an index $1 \leq j < k$, such that there exists an element $h \in [s^{-1}, e]$ with $g_{j+1} \leq h < e$ and $s^{-1} \leq hg_j \leq e$, then by setting:

$$g_i' = \begin{cases} hg_j & i = j \\ g_{j+1}h^{-1} & i = j+1 \\ g_j & i \notin \{j, j+1\} \end{cases}$$

we have $s^{-1} \leq g_i' \leq e$ $(1 \leq i \leq k)$, $g_j' < g_j$ and $g = g_k' g_{k-1}' \dots g_1'$, therefore

$$g \leq g_j' g_{j-1}' \dots g_1' = g_j' g_{j-1} \dots g_1 < g_j g_{j-1} \dots g_1.$$

By part iv) of Proposition 2.1.15,

$$s^{-j} \leq g_j' g_{j-1}' \dots g_1' < g_j g_{j-1} \dots g_1.$$

Therefore, $g'_j g'_{j-1} \dots g'_1 \geq g \vee s^{-j}$, which contradicts $g_j g_{j-1} \dots g_1 = g \vee s^{-j}$. We have thus shown that the constructed factorization is right-maximal. Right-maximality also implies that $g_i \neq e$ $(1 \leq i \leq k)$, so we have indeed constructed a right-normal factorization of g.

Finally, we show that this is is the only right-maximal factorization of g. For this, we first show that for any right-maximal factorization $g = h_l h_{l-1} \dots h_1$ we have $h_1 = g_1 = g \vee s^{-1}$.

From $h_1 \geq s^{-1}$ and $h_1 \geq g$ we see that $h_1 \geq g \vee s^{-1} = g_1$. If we assume that $h_1 > g_1$, then $h_1 \to g_1 < e$, by (S7). By (S4), $h_1 \to g_1 \geq h_1 \to s^{-1} = s^{-1}h_1^{-1}$. Right-maximality tells us that $s^{-1}h_1^{-1} \vee h_2 = e$, consequently $(h_1 \to g_1) \vee h_2 = e$. Together with $h_1 \to g_1 < e$ this implies, by (S7), that $h_2 \to (h_1 \to g_1) < e$. An iteration of the argument, together with (S5), shows that:

$$g \to g_1 = (h_l h_{l-1} \dots h_1) \to g_1 = h_l \to (h_{l-1} \to (\dots \to (h_1 \to g_1)\dots)) < e$$

which contradicts $g \leq g_1$, due to (S7). Therefore, $h_1 = g_1$ which implies that two right-normal factorizations of the same element in G^- have coinciding last terms (if there are any).

Therefore $gg_1^{-1} = g_k g_{k-1} \dots g_2 = h_l h_{l-1} \dots h_2$. These are two right-normal factorizations for gg_1^{-1} which shows $g_2 = h_2$. An iteration of the argument finally proves that $k = l$ and $g_i = h_i$ $(1 \leq i \leq k)$. □

This proof tells us something about the length k of a right-normal factorization:

Corollary 2.1.18. *Let $g \in G^-$ and $g = g_k g_{k-1} \dots g_1$ its right-normal factorization with respect to s. Then k is the smallest integer with the property that $s^{-k} \leq g$.*

We define the *length* of an element $g \in G^-$ as $\lambda(g) = k$, where k is the smallest integer with $s^{-k} \leq g$. The preceding discussion makes clear that $\lambda(g)$ is also the length of the unique right-maximal factorization of g with respect to s.

We will later need that λ is *subadditive* in the following sense:

Proposition 2.1.19. *For any elements $g, h \in G^-$, $\lambda(gh) \leq \lambda(g) + \lambda(h)$.*

Proof. Let $k = \lambda(g)$, $l = \lambda(h)$. Then $s^{-(k+l)} \leq s^{-k}h \leq gh$, which shows that $\lambda(gh) \leq k + l = \lambda(g) + \lambda(h)$. □

The following corollary will come in handy later:

Corollary 2.1.20. *Let $g \in G^-$. If there is* any *factorization*

$$g = h_{k'} h_{k'-1} \dots h_1$$

with all $h_j \in \left[s^{-1}, e\right]$, then the right-normal factorization of g has the length $k \leq k'$.

Proof. Let g be as in the statement. Then, by part iv) of Proposition 2.1.15, $s^{-k'} \leq g$. Corollary 2.1.18 implies that the right-normal factorization must then be of length $\leq k'$. $\square$

Another useful corollary is:

Corollary 2.1.21. *If s is a strong order unit of the right ℓ-group G and $x \prec e$, then $s^{-1} \leq x$.*

Proof. Let $x = g_k g_{k-1} \dots g_1$ be the right-normal factorization. $x \prec e$, however, implies that only one factor is not equal to e. Therefore, $k = 1$ and $g_1 = x$, implying $s^{-1} \leq x$. $\square$

This immediately leads to the following observation:

Corollary 2.1.22. *Let s be a strong order unit of the right ℓ-group G. Then $s^{-1} \leq \bigwedge X(G^-) = \mathrm{Rad}(e)$, if the latter exists.*

In the next chapter we will see that, conversely, under the assumption of certain chain conditions in G, $s := (\mathrm{Rad}(e))^{-1}$ is a strong order unit (if $\mathrm{Rad}(e)$ exists).

We will now give some examples of right-normal factorizations:

Example 2.1.23. 1) Let X be a finite set. In the right ℓ-group $(\mathbb{Z}^{(X)}, \geq)$ (Example 2.1.2,1)), the element $s = \sum_{x \in X} [x]$ is a strong order unit.

The respective strong order interval is then given by

$$[-s, 0] = \left\{ \sum_{x \in X} \varepsilon_x [x] : \varepsilon_x \in \{-1, 0\} \right\}$$

It can be shown that a factorization

$$a = a_k + a_{k+1} + \ldots + a_1$$

with all $a_i \in [-s, 0] \setminus \{0\}$ is right-normal if and only if $a_k \geq a_{k-1} \geq \ldots \geq a_1$. As an easy illustrative example, take $X = \{1, 2, 3\}$ and identify $\mathbb{Z}^3 = \mathbb{Z}^{(X)}$. Then $s = (1, 1, 1)$ is a possible strong order unit of $\mathbb{Z}^3$ with respective strong order interval

$$\begin{aligned}[-s, 0] = \{&(0,0,0), (-1,0,0), (0,-1,0), (-1,-1,0), \\ &(0,0,-1), (-1,0,-1), (0,-1,-1), (-1,-1,-1)\}.\end{aligned}$$

The right-normal factorization of the element $(-4, -1, -2) \in (\mathbb{Z}^3)^-$ is then

$$(-4, -1, -2) = (-1, 0, 0) + (-1, 0, 0) + (-1, 0, -1) + (-1, -1, -1).$$

2) Let X be a compact topological space. Then $C(X, \mathbb{R})$ (Example 2.1.2,3)) is a right ℓ-group with a strong order unit s given by $s(x) = 1$: the normality of s is immediate from the commutativity of $C(X, \mathbb{R})$. On the other hand, each function $g \in C(X, \mathbb{R})$ is bounded by the compactness of X, so there is always an integer k such that $g \leq k \cdot s$. The respective strong order interval is

$$[-s, 0] = \{g \in C(X, \mathbb{R}) : -1 \leq g(x) \leq 0 \ (x \in X)\}$$

One can show that a factorization

$$g = g_k + g_{k+1} + \ldots + g_1$$

with all $g_i \in [-s, 0] \setminus \{0\}$ is right-normal if and only if for each $x \in X$, the values $g_i(x)$ $(1 \leq i \leq k)$ are increasing and $-1 < g_j(x) < 0$ holds for at most one index $1 \leq j \leq k$.

We close this chapter with a discussion about the dual order on a right ℓ-group G. If G is a right ℓ-group (or, more general, a right-ordered group), the *dual order*, defined by

$$x \lesssim y :\Leftrightarrow y^{-1} \leq x^{-1}$$

is the unique *left*-invariant lattice order (resp. left-invariant order) on G with G^- as its negative elements. The corresponding lattice operations are easily seen to be given by

$$x \curlyvee y = (x^{-1} \wedge y^{-1})^{-1}$$
$$x \curlywedge y = (x^{-1} \vee y^{-1})^{-1}.$$

Remark 2.1.24. We give some intuition for these definitions. We have already seen that for $x, y \in G$, the relation $y \leq x$ is equivalent to $yx^{-1} \in G^-$, meaning that x *right-divides* y. Similarly, for $x, y \in G$, the relation $y \lesssim x$ is equivalent to y being *left-divisible* by x in the sense that $x^{-1}y \in G^-$.

Note also that $g \leq e \Leftrightarrow e \leq g^{-1} \Leftrightarrow g \lesssim e$, therefore we have

$$G^- = \{g \in G : g \leq e\} = \{g \in G : g \lesssim e\},$$

i.e. the elements negative with respect to $\leq$ are the same as the elements negative with respect to $\lesssim$.

It is important to note that since $(G, \lesssim)$ is a left-(lattice-)ordered group, the pair $(G^{op}, \lesssim)$ (G^{op} being the group G with reversed multiplication $g \cdot_{op} h := h \cdot g$) is a right-(lattice)-ordered group.

It turns out that the property of being a strong order unit is preserved when going over to the dual order:

Proposition 2.1.25. *Let $(G, \leq)$ be a right ℓ-group with a strong order unit s. Then s is also a strong order unit of the right ℓ-group $(G^{op}, \lesssim)$.*

Proof. We first show that s is normal in $(G^{op}, \lesssim)$. For arbitrary $g, h \in G$, we have the implications

$$g \lesssim h \Rightarrow g^{-1} \geq h^{-1} \Rightarrow s^{-1}g^{-1} \geq s^{-1}h^{-1} \Rightarrow gs \lesssim hs \Rightarrow s \cdot_{op} g \lesssim s \cdot_{op} h.$$

Also, for any $g \in G$, there is an integer k such that $g \leq s^k$. Then

$$g \leq s^k \Leftrightarrow s^{-k}g \leq e \Leftrightarrow s^{-k} \leq g^{-1} \Leftrightarrow g \lesssim s^k,$$

which proves that s is a strong order unit in $(G^{op}, \lesssim)$. □

Proposition 2.1.26. *Let G be a right ℓ-group with strong order unit s. Then for all positive integers k, $\left[e, s^k\right]_{\leq} = \left[e, s^k\right]_{\lesssim}$ and $\left[s^{-k}, e\right]_{\leq} = \left[s^{-k}, e\right]_{\lesssim}$.*

Proof. In the proof of Proposition 2.1.25 we have already shown that $g \leq s^k \Leftrightarrow g \lesssim s^k$. In the same manner, one proves that $s^{-k} \leq g \Leftrightarrow s^{-k} \lesssim g$.

Furthermore, we have $e \leq g \Leftrightarrow g^{-1} \leq e \Leftrightarrow e \lesssim g$ and $g \leq e \Leftrightarrow e \leq g^{-1} \Leftrightarrow g \lesssim e$. □

We can now define left-normal factorizations:

Definition 2.1.27. Let G be a right ℓ-group with a strong order unit s. Let $g \in G^-$. We call a factorization $g = h_1 \ldots h_{k-1}h_k$ with $h_i \in \left[s^{-1}, e\right]$ and $h_i \neq e$ $(1 \leq i \leq k)$ *left-normal* with respect to s if the respective factorization $g = h_k \cdot_{op} h_{k-1} \cdot_{op} \ldots \cdot_{op} h_1$ is right-normal with respect to s in $(G^{op}, \lesssim)$.

We can directly deduce the following properties of left-normal factorizations:

Proposition 2.1.28. *Let G be a right ℓ group with strong order unit s. Let $g \in G^-$.*

i) A factorization $g = h_1 \ldots h_{k-1}h_k$ with all $h_i \in \left[s^{-1}, e\right]$ and $h_i \neq e$ is left-normal if and only if it is left-maximal *in the following sense:*

For $1 \leq i \leq k-1$ there is no $f \in G^-$ with $f \neq e$ such that $h_{i+1} \lesssim f$ and $h_i f \gtrsim s^{-1}$.

ii) A factorization $g = h_1 \ldots h_{k-1}h_k$ with all $h_i \in \left[s^{-1}, e\right]$ and $h_i \neq e$ is left-normal if and only if

$$h_{i+1} \curlyvee h_i^{-1}s^{-1} = e. \tag{2.3}$$

for all $1 \leq i \leq k-1$ or, equivalently,

$$s^{-1}h_{i+1}^{-1} \wedge h_i = s^{-1}. \tag{2.4}$$

Proof. Part i) follows from applying Definition 2.1.16 to the group $(G^{op}, \lesssim)$. In the same way, one deduces ii) from (2.1). The only nontrivial part is the equivalence between (2.4) and (2.3) which is shown as follows:

$$h_{i+1} \curlyvee h_i^{-1} s^{-1} = e \Leftrightarrow h_{i+1}^{-1} \wedge sh_i = e \Leftrightarrow s^{-1} h_{i+1}^{-1} \wedge h_i = s^{-1}$$

where in the last equivalence we used that s is normal. □

Proposition 2.1.29. *Let G be a right ℓ-group with strong order unit s. Then each $g \in G^-$ has a unique left-normal factorization $g = h_1 \dots h_{k-1} h_k$ with all $h_i \in [s^{-1}, e]$ $(1 \le i \le k)$, where k is the smallest integer with the property that $s^{-k} \le g$ and where the factors are given by*

$$h_i = (s^{-i+1} \curlyvee h)^{-1} (s^{-i} \curlyvee h). \tag{2.5}$$

Proof. As $(G^{op}, \lesssim)$ is a right ℓ-group with strong order unit s (Proposition 2.1.25), we have for any $g \in G^-$ the unique right-normal factorization $h = h_k \cdot_{op} h_{k-1} \cdot_{op} \dots \cdot_{op} h_1$ whose factors are given by $h_i = (s^{-i} \curlyvee h) \cdot_{op} (s^{-i+1} \curlyvee h)^{-1}$ $(1 \le i \le k)$, with k being the smallest integer with $s^{-k} \lesssim g$. This, however, is the same as saying that $h = h_1 \dots h_{k-1} h_k$ is the unique left-normal factorization whose factors are given by $h_i = (s^{-i+1} \curlyvee h)^{-1} (s^{-i} \curlyvee h)$ and k is minimal with the property that $s^{-k} \le g$ (Proposition 2.1.26).

Furthermore, Proposition 2.1.26 tells us all $h_i \in [s^{-1}, e]$. □

By Proposition 2.1.17, the length k of the right-normal factorization of some $g \in G^-$ is also the minimal integer with $s^{-k} \le g$. As a corollary we get:

Corollary 2.1.30. *Let G be a right ℓ-group with strong order unit s. For $g \in G^-$, the length $\lambda(g)$ is the common length of the right-normal and the left-normal factorization of g with respect to s.*

This corollary is a very primitive symmetry property in the sense that it tells us that if G is a right ℓ-group with strong order unit s, then for each element $g \in G^-$, the right-normal factorization of g has the same length as the left-normal factorization of g. We will see in Section 2.3 that if G is modular and noetherian, then very strong symmetries occur between right- and left-normal factorizations.

Before going to the next chapter, we illustrate these concepts in the special case of *pure paraunitary groups* which have been investigated by the author in [Die19].

Example 2.1.31. Let K be a commutative field and $K[t, t^{-1}]$ the respective ring of (finite) Laurent series. For a positive integer n, we look at the K-vector space $V := K^n$ which we assume to be equipped with a symmetric K-bilinear form[2] $b : V \times V \to V$ which is *anisotropic* in the sense that for all $v \in V$ one has the implication $b(v, v) = 0 \Rightarrow v = 0$).

[2] Our construction also works in the more general case when b is hermitean with respect to an involutive automorphism of K. However, we decided to look at only the symmetric version to keep things easier.

Using all this data, a right ℓ-group can be constructed as follows:

Since b is bilinear, we can write $b(v, w) = v^\top Aw$ for some symmetric matrix $A \in K^{n \times n}$. Here, the upperscript-$\top$ denotes the transpose of a matrix. We furthermore think of the vectors $v, w \in K^n$ as column vectors. Since b, as an anisotropic bilinear form, is necessarily non-degenerate, the matrix A is invertible.

Recall that one can always associate a *unitary group* with b, which is defined as

$$\mathrm{U}(b) := \{M \in K^{n \times n} : M^\top AM = A\}.$$

Checking the group axioms is straightforward. The unitary group $\mathrm{U}(b)$ can now be „extended“ in the following way: writing the elements of the matrix ring $K\left[t, t^{-1}\right]^{n \times n}$ suggestively as matrix valued functions $M(t)$, we can associate with b a *paraunitary group* which is defined as

$$\mathrm{PU}(b) := \left\{M(t) \in K\left[t, t^{-1}\right]^{n \times n} : M(t^{-1})^\top AM(t) = A\right\}.$$

Under matrix multiplication, $\mathrm{PU}(b)$ is also a group which can be checked as easily as in the case of the unitary group $\mathrm{U}(b)$.

There is the (not completely canonical[3]) homomorphism

$$\begin{aligned} \varepsilon_1 : \mathrm{PU}(b) &\to \mathrm{U}(b) \\ M(t) &\mapsto M(1) \end{aligned}$$

whose kernel is the *pure paraunitary group* $\mathrm{PPU}(b) := \ker(\varepsilon_1)$. The group $\mathrm{PPU}(b)$ contains the multiplicative submonoids

$$\begin{aligned} &\mathrm{PPU}(b)^- := \mathrm{PPU}(b) \cap K\left[t^{-1}\right]^{n \times n} \\ \text{and} \quad &\mathrm{PPU}(b)^+ := \mathrm{PPU}(b) \cap K\left[t\right]^{n \times n}. \end{aligned}$$

By definition, these consist of the elements in $\mathrm{PPU}(b)$ whose matrix entries only have nonpositive resp. nonnegative exponents (seen as Laurent polynomials).

One can show that $\mathrm{PPU}(b)^-$ is the negative cone of a right-invariant order on $\mathrm{PPU}(b)$: for any $M(t) \in \mathrm{PPU}(b)^-$ we have

$$M(t)^{-1} = A^{-1} M(t^{-1})^\top A \in \mathrm{PPU}(b)^+.$$

It follows that if both $M(t)$ and $M(t)^{-1}$ lie in $\mathrm{PPU}(b)^-$, then $M(t)$ is constant, i.e. $M(t) \in K^{n \times n} \subseteq K\left[t, t^{-1}\right]^{n \times n}$. Since $M(1) = 1$, this is only possible when $M = 1$. Therefore $\mathrm{PPU}(b)^-$ is a pure submonoid of $\mathrm{PPU}(b)$ and, by Proposition 2.1.6, the negative cone of a unique right-invariant order.

It turns out that under this order, $\mathrm{PPU}(b)$ is a right ℓ-group ([Die19, Theorem 3]). We do not give details of the (somewhat lengthy) proof here and refer the reader to the cited article.

[3] The other one specializes $t = -1$.

Under this lattice order, the element $t := t \cdot I_n$ - where $I_n \in K\left[t, t^{-1}\right]^{n\times n}$ is the identity matrix - is a strong order unit of $\mathrm{PPU}(b)$. The proof of this fact is pretty straightforward: since t commutes with each element of $K\left[t, t^{-1}\right]^{n\times n}$, left multiplication by t is the same as right multiplication by t, so t is a normal element in $\mathrm{PPU}(b)$, for sure. It is also a strong order unit: given $M(t) \in \mathrm{PPU}(b)$, take k large enough that $t^k M(t)^{-1} \in \mathrm{PPU}(b)^+$. Then

$$M(t) \leq t^k M(t)^{-1} M(t) = t^k.$$

For a K-subspace $U \subseteq V$, we define

$$U^* := \{w \in V : \forall v \in V : b(v, w) = 0\} \in L({}_K V).$$

For each such U, we have the direct decomposition $V = U \oplus U^*$, due to b being anisotropic[4]. Therefore, for each $U \in L({}_K V)$ there is a projection operator π_U with regard to this decomposition which is given by

$$\begin{aligned} \pi_U : V &\to V \\ u_1 + u_2 &\mapsto u_1 \quad (u_1 \in U, u_2 \in U^*). \end{aligned}$$

Let $P_U \in K^{n\times n}$ be the representing matrix of π_U. Then for each $U \in L({}_K V)$, one can define an element

$$p_U := P_U + (1 - P_U)t^{-1} = P_U + P_{U^*} t^{-1} \in \mathrm{PPU}(b).$$

It turns out that under the strong order unit t, the strong order interval in $\mathrm{PPU}(b)$ has the form

$$[t^{-1}, 1] = \{p_U : U \in L({}_K V)\}$$

([Die19, Lemma 12]). In fact, we have the equivalence $p_U \leq p_W \Leftrightarrow U \leq W$, which implies that the assignment $U \mapsto p_U$ defines an isomorphism of lattices $L({}_K V) \cong [t^{-1}, 1]$ ([Die19, Lemma 14]). The property of having a desarguesian strong order interval is part of the definition of *desarguesian* right ℓ-groups. We will give the definition in the next section and will show that the groups $\mathrm{PPU}(b)$ constitute a huge class of desarguesian right ℓ-groups.

We close this section with some words on factorizations in $\mathrm{PPU}(b)^-$: a calculation shows that $t^{-1} p_U^{-1} = p_{U^*}$ for all $U \in L({}_K V)$, so a factorization

$$M = p_{U_k} p_{U_{k-1}} \dots p_{U_1}$$

with $U_1, \dots, U_k \in L({}_K V)$ is right-normal if and only if $U_i^* + U_{i+1} = V$ holds for $1 \leq i \leq k - 1$.

One can show that on $[t^{-1}, 1]$, the opposite order $\lesssim$ coincides with $\leq$. It therefore follows that a factorization

$$M = p_{W_1} \dots p_{W_{k-1}} p_{W_k}$$

[4] One can show that the complementation $U \mapsto U^*$ makes $L({}_K V)$ an orthomodular lattice; this has already been observed by Birkhoff and von Neumann ([BvN36, Section 14.]).

with $W_1, \dots, W_k \in L({}_KV)$ is *left*-normal if and only if $W_i^* + W_{i+1} = V$ holds for $1 \le i \le k-1$.

In signal processing, paraunitary matrices and their factorization theory are well-known over the ground field $K = \mathbb{R}$. Here, typically, the vector space $V = \mathbb{R}^n$ is considered with the standard inner product given by $b(v,w) = v^\top w$ and the elements of $\mathrm{PU}(b)^+ := \mathrm{PU}(b) \cap k\,[t]^{n\times n}$ are regarded as $n \times n$-*FIR-lossless matrices.* Some elementary results concerning existence and uniqueness of right-normal factorizations can be found under [Vai92, Sections 14.4.2., 14.4.4]). What seems to be new is the fact that the factorization theory of paraunitary groups comes from a Garsidean structure.

2.2 Modular right ℓ-groups

In this section we discuss more special classes of right ℓ-groups.

Definition 2.2.1. A right ℓ-group G^- is *noetherian* if G^- fulfills the ascending chain condition and G^+ fulfills the descending chain condition.

Since $\le$ is right-invariant, this is the same as saying that every ascending chain $g_1 \le g_2 \le \dots$ (all $g_i \in G$) with $\{g_i : i \ge 1\}$ bounded from above becomes stationary at some point *and* every descending chain $g_1 \ge g_2 \ge \dots$ (all $g_i \in G$) with $\{g_i : i \ge 1\}$ bounded from below becomes stationary at some point, as well.

Example 2.2.2. 1) For an arbitrary set X, the right ℓ-group $\mathbb{Z}^{(X)}$ is noetherian (Example 2.1.2, 1)). This can be seen as follows: with the degree function

$$\begin{aligned} d : \mathbb{Z}^{(X)} &\to \mathbb{Z} \\ a &\mapsto -\sum_{x\in X} a_x, \end{aligned}$$

we have the implication $a < b \Rightarrow d(a) > d(b)$. In order to prove the ascending chain condition, let $a_1 \le a_2 \le \dots$ be an ascending sequence in $\mathbb{Z}^{(X)}$ that is bounded from above by b. Then, the sequence of degrees $d(a_1) \ge d(a_2) \ge \dots$ is bounded from below by $d(b)$. It follows that $d(a_i) > d(a_{i+1})$ can hold for only finitely many $i \ge 1$, therefore $a_i < a_{i+1}$ holds for only finitely many $i \ge 1$, which proves the ascending chain condition. The descending chain condition is proved similarly.

2) The right ℓ-group $(J_K, \subseteq)$ (Example 2.1.2, 2)) is noetherian. We prove the ascending chain condition: let $I_1 \subseteq I_2 \subseteq \dots$ be an ascending sequence in J_K that is bounded from above by an element $I \in J_K$. By right-invariance, we can assume that $I = R$. In this case, $I_1 \subseteq I_2 \subseteq \dots$ is an ascending sequence of ideals in R that must eventually become stationary by the mere definition of a Dedekind domain.

The group J_K is commutative, so we have the equivalence $I \subseteq I' \Leftrightarrow I^{-1} \supseteq I'^{-1}$ for all $I, I' \in J_K$. By this equivalence, the ascending chain condition implies the descending chain condition as well.

3) For a non-empty topological space X, the right ℓ-group $C(X, \mathbb{R})$ (see Example 2.1.2, 3)) is not noetherian. This can be seen by considering the descending sequence $f_n(x) = \frac{1}{n}$ which is bounded from below by 0 but does not become stationary.

Recall that for a lattice L, we have made the following definitions: if for an element $x \in L$, the sublattice $x^{\downarrow}$ fulfills the ascending chain condition then we have defined $\mathrm{Rad}(x) = \bigwedge\{y \in L : y \prec x\}$ whenever the meet exists. Dually, if the sublattice $x^{\uparrow}$ fulfills the descending chain condition, we defined its socle as $\mathrm{Soc}(x) = \bigvee\{y \in L : y \succ x\}$ whenever the join exists. Also recall that we have defined the length $l(L)$ as the supremum of the cardinals $|X| - 1$ where X ranges over all chains $X \subseteq L$.

The following observation of Rump is the starting point for this piece of research:

Theorem 2.2.3. *Let G be a modular noetherian right ℓ-group. If* $\mathrm{Rad}(e)$ *exists, then $s := (\mathrm{Rad}(e))^{-1}$ is a strong order unit of G. Furthermore, the interval $[s^{-1}, e]$ is a modular dual geometric lattice under the order induced from G.*

Proof. See [Rum15, Proposition 15] for a proof that s is indeed a strong order unit and [Rum15, Proposition 14] for the second part. □

Remark 2.2.4. We note here that Rump has actually proved slightly more: he showed that if G is a modular noetherian right ℓ-group, then the sublattice

$$\{a \in G^- \ : \ \exists x_1, \ldots, x_n \in X(G^-) \ : a \geq x_1 \wedge \ldots \wedge x_n\}$$

is dually geometric in a more general sense ([Grä11, Definition 389]). In this work, however, we will only deal with bounded geometric lattices. In order to avoid lengthy formulations, we therefore decided to call everything *geometric* here what should, in a more general setting, probably be called *bounded geometric*.

This motivates the following definition ([Rum15, Section 7])

Definition 2.2.5. A *geometric* right ℓ-group is a lower semimodular, right ℓ-group G with a strong order unit s such that $[s^{-1}, e]$ is a dual geometric lattice. Its length $\delta := l([s^{-1}, e])$ is called the *dimension* of G.

By Theorem 2.2.3, a modular, noetherian right ℓ-group is geometric if $\mathrm{Rad}(e)$ exists. In this case, $s^{-1} = \mathrm{Rad}(e)$. If s' is any strong order unit of G^-, Corollary 2.1.22 implies that $s'^{-1} \leq s^{-1}$. If we demand $[s'^{-1}, e]$ to be dually geometric, s'^{-1} must be a meet of dual atoms. For s^{-1} is the meet of *all* dual atoms, we have $s'^{-1} \geq s^{-1}$ from which we finally deduce that $s'^{-1} = s^{-1}$. This shows that the strong order unit s in Definition 2.2.5 is unique, and so is the dimension δ of G.

Lemma 2.2.6. *Let G be a modular geometric right ℓ-group where* $\mathrm{Rad}(e)$ *exists. Then* $\mathrm{Soc}(e)$ *also exists in G and* $\mathrm{Soc}(e) = s$ *where* $s := (\mathrm{Rad}(e))^{-1}$.

Proof. We first show that the interval $\left[s^{-1}, e\right]$ has a dually spanning, dually independent set $Y \subseteq X(G^-)$ of finite size. We claim that, in fact, one can take this set Y as a maximal dually independent set of dual atoms: since G is noetherian, and since an arbitrary intersection of elements of G is bounded from below by s^{-1}, Y is finite and $\bigwedge Y \geq s^{-1}$. If $\bigwedge Y > s^{-1}$, then there must be a $z \in X(G^-)$ with $z \not\geq \bigwedge Y$. So, by Proposition 1.1.10, $Y \cup \{z\}$ is a dually independent set containing Y, contradicting the maximality of Y. It follows that $\bigwedge Y = s^{-1}$.

As $Y =: \{y_1, \dots, y_\delta\}$ is dually spanning, dually independent with $l([y_i, e]) = 1$ $(1 \leq i \leq \delta)$, there is also a spanning, independent subset $Y' = \{y_1', \dots, y_\delta'\} \subseteq \left[s^{-1}, e\right]$ with $l(\left[s^{-1}, y_i'\right]) = 1$ $(1 \leq i \leq \delta)$, by Proposition 1.1.11.

This means that all $y_i' \succ s^{-1}$ and $\bigvee Y' = e$. By right-invariance, $y_i's \succ e$ $(1 \leq i \leq \delta)$ and $\bigvee(Y's) = s$.

Now assume that there was some $z' \succ e$ with $z' \not\leq s$, then $Y's \cup \{z'\}$ would be a spanning, independent subset of $[e, t]$ where $t := \bigvee(Y's \cup \{z'\}) > s$. Dually to the argument above, we could then find a subset $W \subseteq X([e, t])$ with $\bigwedge W = e$, which could be right-translated to the subset $Wt^{-1} \subseteq X(\left[t^{-1}, e\right]) \subseteq X(G^-)$. This set would have the property that $s^{-1} \leq \bigwedge(Wt^{-1}) = t^{-1}$. By Theorem 2.2.3, s is normal, which implies

$$s^{-1} \leq t^{-1} \Leftrightarrow ss^{-1}t \leq st^{-1}t \Leftrightarrow t \leq s,$$

which would contradict $t > s$.

Therefore, each $z' \succ e$ lies under s, so noetherianity of G implies that $\mathrm{Soc}(e)$ exists and $\mathrm{Soc}(e) \leq s$. We have already seen that there are elements $y_1', \dots, y_\delta' \succ e$ such that $y_1' \vee \ldots \vee y_\delta' = s$, which implies that we must have $\mathrm{Soc}(e) = s$. □

Since the covering relation is invariant under right-multiplication by elements of G, we immediately conclude:

Proposition 2.2.7. *If G is a modular geometric right ℓ-group where* $\mathrm{Rad}(e)$ *exists, then for arbitrary $g \in G$, we have* $\mathrm{Rad}(g) = s^{-1}g$ *and* $\mathrm{Soc}(g) = sg$ *where* $s := (\mathrm{Rad}(e))^{-1}$.

We recall that every modular geometric lattice is a direct product of irreducible modular geometric lattices (Theorem 1.2.3). In view of this result, we can say that in order to understand the structure of modular geometric right ℓ-groups, it makes sense to first look at the most „primitive“ ones: these are the right ℓ-groups G with a strong order unit s such that $\left[s^{-1}, e\right]$ is an irreducible modular geometric lattice. Since there exist exceptional projective planes (which are complicated), we should even restrict ourselves a little further, namely by just looking at the examples where $\left[s^{-1}, e\right]$ is a modular geometric lattice of length at least 4. In this case, the Veblen-Young theorem (Theorem 1.2.4) tells us that $\left[s^{-1}, e\right]$ is a desarguesian lattice. We see that if $l([s^{-1}, e])$ is of dimension at least 4, the condition of being dually geometric and modular forces $[s^{-1}, e]$ to be isomorphic to an element of a really well-understood class of lattices. So we might expect to be able to say a lot about G in this case, too.

This motivates the central definition of this work:

Definition 2.2.8. A *desarguesian* right ℓ-group is a modular, noetherian right ℓ-group with a strong order unit s such that $\left[s^{-1}, e\right]$ is a desarguesian lattice.

Remark 2.2.9. Recall that every desarguesian lattice is geometric, so by the discussion following Definition 2.2.5, a desarguesian right ℓ-group can have only one strong order unit s with $\left[s^{-1}, e\right]$ desarguesian - which is $s = (\mathrm{Rad}(e))^{-1}$. When talking about desarguesian right ℓ-groups, we assume our strong order unit s to be this special element.

In the following sections of this chapter, we will see that our intuition has not failed us - quite a lot can be said when G is desarguesian with a strong order interval of length at least 4! However, before dedicating ourselves to this task, we close this section by demonstrating that pure paraunitary groups are desarguesian. This shows that there are plenty of desarguesian right ℓ-groups.

Example 2.2.10. We have seen that the groups $\mathrm{PPU}(b)$ introduced at the end of the last chapter carry a right-invariant lattice-order induced by the negative cone $\mathrm{PPU}(b)^{-}$. It can be shown that this order is in fact modular [Die19, Theorem 4].

We show that $\mathrm{PPU}(b)$ is also noetherian: the interval $[t^{-1}, 1] \cong L({}_K V)$ has length n, i.e. in $[t^{-1}, 1]$ there is a maximal chain of length n with top element 1 and bottom element t^{-1}. Since for each $k \in \mathbb{Z}$, the interval $[t^{k-1}, t^k]$ is an isomorphic copy of $[t^{-1}, 1]$, it contains a maximal chain, also of length n, with top element t^k and bottom element t^{k-1}.

In order to get a maximal chain in the interval $[t^k, t^l]$ $(k \leq l)$, one just takes one maximal chain per interval $[t^i, t^{i+1}]$ $(k \leq i \leq l-1)$ and takes the union over all these chains in order to get a maximal chain of length $(l-k) \cdot n$. If there is a finite-length maximal chain in a modular lattice, then all maximal chains have the same length which follows from the proof of [Grä11, Theorem 374]. In particular, $[t^k, t^l]$ has finite length.

Since every interval in $\mathrm{PPU}(b)$ lies in some interval $[t^k, t^l]$ $(k \leq l)$, it follows that $\mathrm{PPU}(b)$ is noetherian. Therefore, $\mathrm{PPU}(b)$ is a modular, noetherian right ℓ-group.

Together with the isomorphism $[t^{-1}, 1] \cong L({}_K V)$, we see that $\mathrm{PPU}(b)$ is a desarguesian right ℓ-group. Its dimension is $\delta = l(L({}_K V)) = n$.

2.3 Factorizations in modular noetherian right ℓ-groups

In this section we discuss the influence of noetherianity and modularity on the factorization theory in right ℓ-groups. Note that, in particular, the results of this section apply to all modular Garside groups.

We begin with a result which involves only noetherianity and is a variation on a classical theme from divisibility theory. Recall that a right ℓ-group G is noetherian when every bounded descending or ascending sequence in G becomes stationary.

Proposition 2.3.1. *([Rum15, Proposition 1.]) Let G be a noetherian right ℓ-group. Then every element $g \in G^-$ admits a factorization $g = x_k x_{k-1} \dots x_1$ where $x_i \prec e$ $(1 \le i \le k)$.*

Proof. If $g = e$, the statement is clear. We may therefore assume that $g < e$.

Set $g_1 := g$. There is always an $x_1 \prec e$ such that $g_1 \le x_1$, else we could construct an infinite chain $g_1 = h_1 < h_2 < h_3 < \dots$ with all $h_i < e$ $(i \ge 1)$ - this would contradict the noetherian condition.

Fix such an x_1. We set $g_2 := g_1 x_1^{-1} \in G^-$ and proceed inductively as follows: when g_i has been constructed, we stop if $g_i = e$. Else we have $g_i < e$. In this case, we choose an x_i with $g \le x_i \prec e$. We then set $g_{i+1} = g_i x_i^{-1}$.

This process must stop at some index. If not, one sees that for each index i we would have the inequalities $e \le g_{i+1}^{-1} = x_i g_i^{-1} < g_i^{-1}$, thus, this process would result in a descending chain $g_1^{-1} > g_2^{-1} > \dots$, which would be bounded from below by e; this would also contradict noetherianity.

If we take k as the greatest integer with $g_k < e$, then our construction implies that $g x_1^{-1} x_2^{-1} \dots x_k^{-1} = g_{k+1} = e$ and $x_i \in X(G^-)$ $(1 \le i \le k)$. Therefore, $g = x_k x_{k-1} \dots x_1$. □

We have just seen that if G is a noetherian right ℓ-group, every element $g \in G^-$ can be written as a product of dual atoms in G^-. If G is also modular, then these factorizations are particularly well-behaved. This has the effect that a canonical degree function can be defined on G.

Proposition 2.3.2. *Let G be a modular noetherian right ℓ-group. Then there is a unique group homomorphism $d : G \to \mathbb{Z}$ - the* degree *homomorphism - such that $d(g) = l([g, e])$ for all $g \in G^-$. For $g \in G^-$, the degree $d(g)$ is the (unique) integer k such that there is a factorization $g = x_k x_{k-1} \dots x_1$ with $x_i \prec e$ $(1 \le i \le k)$.*

Remark 2.3.3. The reader may be confused by our decision of choosing d such that the elements of G^- are mapped to $\mathbb{Z}_0^+$. However, in our analysis of G we will mainly be focused on the structure of G^-, in particular on the factorization theory in G^- which looks way better with d being non-negative on G^-.

Proof. We first define a map $\tilde{d} : G^- \to \mathbb{Z}$ by $\tilde{d}(g) := l([g, e])$.

Let $g \in G^-$. Any maximal chain $e = y_0 \succ y_1 \succ \dots \succ y_k = g$ $(k = l([g, e]))$ produces a factorization $g = x_k x_{k-1} \dots x_1$ with $x_i \prec e$ via $x_i = y_i y_{i-1}^{-1}$ $(1 \le i \le k)$. Similarly, every factorization $g = x_k x_{k-1} \dots x_1$ with $x_i \prec e$ $(1 \le i \le k)$ produces a maximal chain $e = y_0 \succ y_1 \succ \ldots \succ y_k = g$ in $[g, e]$ via $y_i = x_i x_{i-1} \ldots x_1$ $(0 \leq i \leq k)$.

By modularity, all chains in $[g, e]$ have the same length ([Grä11, Theorem 374]); so, all factorizations of g into dual atoms must have the same common length, which proves the second part of the proposition.

Let $g, h \in G^-$ and set $k = \tilde{d}(g)$, $l = \tilde{d}(h)$. Then there are $x_1, \ldots, x_k \prec e$ and $y_1, \ldots y_l \prec e$ such that $g = x_k x_{k-1} \ldots x_1$ and $h = y_l y_{l-1} \ldots y_1$. Therefore, $gh = x_k x_{k-1} \ldots x_1 y_l y_{l-1} \ldots y_1$ is a factorization into $k + l$ dual atoms in G^-, implying that $\tilde{d}(gh) = k + l = \tilde{d}(g) + \tilde{d}(h)$ for all $g, h \in G^-$.

Therefore, $\tilde{d} : G^- \to \mathbb{Z}$ is a homomorphism of monoids. By Proposition 2.1.10, there is a unique group homomorphism $d : G \to \mathbb{Z}$ such that $d|_{G^-} = \tilde{d}$. □

Remark 2.3.4. If G is a geometric right ℓ-group with strong order unit $s = (\mathrm{Rad}(e))^{-1}$, we can express the dimension δ of G by means of d as

$$\delta = l([s^{-1}, e]) = d(s^{-1}).$$

There is also a useful *parallelogram identity* for d:

Proposition 2.3.5. *Let G be a modular, noetherian right ℓ-group. For all $g, h \in G$, we have*

$$d(g \wedge h) + d(g \vee h) = d(g) + d(h).$$

Proof. We have the following chain of lattice isomorphisms:

$$\left[(g \wedge h)g^{-1}, e\right] \cong [g \wedge h, g] \cong [h, g \vee h] \cong \left[h(g \vee h)^{-1}, e\right],$$

the middle one being a diamond isomorphism, the other ones coming from right-invariance. This shows that $d((g \wedge h)g^{-1}) = d(h(g \vee h)^{-1})$. Due to Proposition 2.3.2, this implies $d(g \wedge h) - d(g) = d(h) - d(g \vee h)$, from which the statement follows. □

Another useful inequality is the following:

Proposition 2.3.6. *Let G be a modular, noetherian right ℓ-group. For all $g \in G^-$ we have*

$$g \geq s^{-d(g)}. \tag{2.6}$$

Proof. Let $g = x_{d(g)} x_{d(g)-1} \ldots x_1$ be a factorization into elements $x_i \prec e$ $(1 \leq i \leq d(g))$. By Corollary 2.1.21, $x_i \geq s^{-1}$ $(1 \leq i \leq d(g))$. Now part iv) of Proposition 2.1.15 implies $g \geq s^{-d(g)}$. □

If our right ℓ-groups additionally carry a strong order unit, we can introduce the following very useful tool:

Definition 2.3.7. Let G be a modular, noetherian right ℓ-group with a strong order unit s. We define the *index sequence* (with respect to s) of an element $g \in G^-$ as

$$\iota^s : G^- \times \mathbb{Z}^+ \to \mathbb{Z}_0^+$$
$$(g, i) \mapsto \iota_i^s(g) := d(g \vee s^{-i}) - d(g \vee s^{-i+1}).$$

We will suppress the upperscript-s whenever there is no ambiguity concerning the strong order unit s.

By Proposition 2.3.2,

$$d(g \vee s^{-i}) - d(g \vee s^{-i+1}) = d((g \vee s^{-i})(g \vee s^{-i+1})^{-1}).$$

By the proof of Proposition 2.1.17,

$$g_i = (g \vee s^{-i})(g \vee s^{-i+1})^{-1}$$

where $g = g_k g_{k-1} \dots g_1$ is the right-normal factorization of g. We conclude:

Proposition 2.3.8. *Let G be a modular, noetherian right ℓ-group with strong order unit s. If $g \in G^-$ has the right-normal factorization $g = g_k g_{k-1} \dots g_1$,*

$$\iota_i(g) = \begin{cases} d(g_i) & 1 \le i \le k \\ 0 & i > k \end{cases}$$

The following nice property of the index sequence will be indispensable in our analysis of factorizations in modular noetherian right ℓ-groups.

Proposition 2.3.9. *Let G be a modular, noetherian right ℓ-group with a strong order unit s. For any $g \in G^-$, the sequence $\iota_i(g)$ is non-increasing in i.*

Proof. Let $g \in G^-$, and let $g = g_k g_{k-1} \dots g_1$ be the right-normal factorization. When in the range $1 \le i \le k$, the values $\iota_i(g) = d(g_i)$ (see Proposition 2.3.8) are *not* non-increasing, there is at least one index i such that $d(g_i) < d(g_{i+1})$. Recall equation (2.1), which tells us that right-maximality means that $s^{-1} g_i^{-1} \vee g_{i+1} = e$ for all $1 \le i \le k-1$. We show that this equation cannot hold when $d(g_i) < d(g_{i+1})$:

Recall that $\delta = d(s^{-1})$ and note that $s^{-1} g_i^{-1} \wedge g_{i+1} \ge s^{-1}$. Using this, together with Proposition 2.3.5, we can calculate

$$d(s^{-1} g_i^{-1} \vee g_{i+1}) = d(s^{-1} g_i^{-1}) + \underbrace{d(g_{i+1})}_{> d(g_i)} - \underbrace{d(s^{-1} g_i^{-1} \wedge g_{i+1})}_{\le \delta}$$
$$> (\delta - d(g_i)) + d(g_i) - \delta = 0.$$

This implies $s^{-1} g_i^{-1} \vee g_{i+1} < e$, so the factorization would not be right-maximal.

It follows that necessarily $d(g_i) \ge d(g_{i+1})$ resp. $\iota_i(g) \ge \iota_{i+1}(g)$ holds for all $1 \le i \le k-1$. For all remaining indices i - that is, $i > k-1$ - the latter inequality is trivial since $\iota_{i+1}(g) = 0$ in this case (Proposition 2.3.8). $\square$

It turns out that the index sequence is able to measure certain lattice-theoretic properties in G^-. These are captured by certain homogeneity properties in right-normal factorizations which we will now define.

Definition 2.3.10. Let G be a modular, noetherian right ℓ-group with a strong order unit s. For a non-negative integer d we call an element $g \in G^-$ *homogeneous of degree* d when in the right-normal factorization $g = g_k \dots g_1$ we have $d(g_i) = d$ for all $1 \leq i \leq k$.

The following interesting property of homogeneous elements will show its full importance in our analysis of saturated filters in Chapter 3. In this chapter, only the special case $d = 1$ will be relevant.

Proposition 2.3.11. *Let G be a modular, noetherian right ℓ-group with a strong order unit s. Let $g \in G^-$ be homogeneous of degree d with right-normal factorization $g = g_k \dots g_1$. Then this factorization is left-normal as well.*

More generally, let $g = g_k g_{k-1} \dots g_1$ be any factorization where for all $1 \leq i \leq k$, we have $e \neq g_i \in G^-$ and $d(g_i) = d$ with some fixed integer d. Then this factorization is left-normal, if and only if it is right-normal.

Proof. We first assume that $g = g_k g_{k-1} \dots g_1$ is right-normal. By equation (2.1), this is the same as saying that $s^{-1} g_i^{-1} \vee g_{i+1} = e$ for $1 \leq i \leq k-1$.

Using Proposition 2.3.5 and Proposition 2.3.2, we calculate

$$\begin{aligned} d(s^{-1} g_i^{-1} \wedge g_{i+1}) &= d(s^{-1} g_i^{-1}) + d(g_{i+1}) - d(\underbrace{s^{-1} g_i^{-1} \vee g_{i+1}}_{=e}) \\ &= d(s^{-1}) - d(g_i) + d(g_{i+1}) - 0 \\ &= d(s^{-1}) - d + d = d(s^{-1}). \end{aligned}$$

Since $s^{-1} g_i^{-1} \wedge g_{i+1} \geq s^{-1}$, this implies $s^{-1} g_i^{-1} \wedge g_{i+1} = s^{-1}$ which is equivalent to left-maximality (see equation (2.4) - note that the factorization in the referred corollary is read from the left to the right, so the interchanged indices here are *not* an error!).

Similarly, $s^{-1} g_i^{-1} \wedge g_{i+1} = s^{-1}$ implies that $d(s^{-1} g_i^{-1} \vee g_{i+1}) = 0$ and thus, $s^{-1} g_i^{-1} \vee g_{i+1} = e$. This means that a left-maximal factorization with factors of constant degree is also right-maximal. $\square$

There is another left-right-symmetry between left-normal and right-normal factorizations which we will need later:

Proposition 2.3.12. *Let G be a modular noetherian right ℓ-group with strong order unit s. For $g \in G^-$, let the right-normal factorization be*

$$g = g_k g_{k-1} \cdots g_1$$

and the left-normal factorization be

$$g = h_1 \dots h_{l-1} h_l.$$

Then $k = l$ and $d(g_i) = d(h_i)$ for all $1 \leq i \leq k$.

Proof. By Proposition 2.1.17, k is the smallest integer with $s^{-k} \leq g$ and by Proposition 2.1.29, l is the smallest integer with $s^{-l} \leq g$. This shows that $k = l$.

Since d is a homomorphism (Proposition 2.3.2), it suffices to show that

$$d(g_j \dots g_1) = d(h_1 \dots h_j)$$

for $1 \leq j \leq k$.

The expression for the right-normal factors g_i from Proposition 2.1.17 implies $g_j \dots g_1 = g \vee s^{-j}$. Using Proposition 2.1.29 in the same way on the left-normal factors, we get:

$$\begin{aligned}
d(h_1 \dots h_j) &= d(g \curlyvee s^{-j}) \\
&= d((g^{-1} \wedge s^j)^{-1}) \\
&= -d(g^{-1} \wedge s^j) \\
&= d(g^{-1} \vee s^j) - d(g^{-1}) - d(s^j) && \text{(Proposition 2.3.5)} \\
&= d(s^j (s^{-j} \vee g) g^{-1}) - d(g^{-1}) - d(s^j) && \text{(by normality of } s\text{)} \\
&= d(s^j) + d(s^{-j} \vee g) + d(g^{-1}) - d(g^{-1}) \\
&\quad - d(s^j) \\
&= d(s^{-j} \vee g) \\
&= d(g_j \dots g_1).
\end{aligned}$$

□

Corollary 2.3.13. *Let G be a modular noetherian right ℓ-group with strong order unit s. Let $g \in G^-$ and let $g = h_l h_{l-1} \dots h_1$ be some - not necessarily normal - factorization with $h_1, \dots, h_l \in [s^{-1}, e]$. Then $\iota_1(g) \geq \max_{1 \leq i \leq l} d(h_i)$.*

Proof. Choose an index j such that $d(h_j) = \max_{1 \leq i \leq l} d(h_i)$. Set $h := h_j h_{j-1} \dots h_1$ and let $h = f_1 \dots f_k$ be the left-normal factorization of h. Clearly $f_1 \lesssim h_j$, therefore $d(h_j) \leq d(f_1)$. By Proposition 2.3.12, $d(f_1) = d(h'_1)$ where h'_1 is the right-most term in the right-normal factorization of h. Then

$$h'_1 \geq h_j h_{j-1} \dots h_1 \geq h_l h_{l-1} \dots h_1 = g,$$

If $g = g_k g_{k-1} \dots g_1$ is the right-normal factorization of g then $h'_1 \geq g_1$. We conclude that

$$\iota_1(g) = d(g_1) \geq d(h'_1) = d(f_1) \geq d(h_j) = \max_{1 \leq i \leq l} d(h_i).$$

□

The following proposition tells us that in a modular geometric right ℓ-group, the meet-irreducible elements in G^- are exactly the elements which are homogeneous of degree 1.

Proposition 2.3.14. *Let G be a modular geometric right ℓ-group with strong order unit $s = (\mathrm{Rad}(e))^{-1}$. Then an element $g \in G^-$ is meet-irreducible in G^- if and only if $\iota_i(g)$ takes only the values 0 and 1.*

Proof. Let $g < e$ be meet-reducible. Then there are $g', g'' \succ g$ with $g = g' \wedge g''$. We show that the element $\gamma := g(g' \vee g'')^{-1}$ fulfills $\gamma \geq s^{-1}$: by modularity, $g' \vee g'' \succ g', g''$. Therefore, $g'(g' \vee g'')^{-1}, g''(g' \vee g'')^{-1} \prec e$ and

$$\gamma = (g' \wedge g'')(g' \vee g'')^{-1} = g'(g' \vee g'')^{-1} \wedge g''(g' \vee g'')^{-1}.$$

which is bounded from below by s^{-1}, since γ is an intersection of dual atoms and, by Corollary 2.1.21, all dual atoms are above s^{-1}.

By modularity, we have $[g', g' \vee g''] \cong [\underbrace{g' \wedge g''}_{=g}, g'']$. Since $[g, g'']$ has only two elements, due to $g \prec g''$, the same is true for the first interval. This shows that $g' \prec g' \vee g''$. From the resulting covering relations $g \prec g' \prec g' \vee g''$ we deduce that $d(g' \vee g'') = d(g) - 2$.

This implies $g' \vee g'' \geq s^{-d(g)+2}$ (Proposition 2.3.6), therefore

$$g = \gamma(g' \vee g'') \geq s^{-1}(g' \vee g'') \geq s^{-1}s^{-d(g)+2} = s^{-d(g)+1}.$$

But then, in the right-normal factorization $g = g_k g_{k-1} \ldots g_1$ we have $k \leq d(g) - 1$, which implies $d(g_i) > 1$ for at least one $1 \leq i \leq k$.

On the other hand, let $g \in G^-$ be meet-irreducible with right-normal factorization $g = g_k g_{k-1} \ldots g_1$. We want to show that $d(g_i) = 1$ $(1 \leq i \leq k)$. By Proposition 2.3.9, it suffices to show that $d(g_1) = 1$. Let $g = h_1 \ldots h_k$ be the left-normal factorization. h_1 must be meet-irreducible. If not, there would exist $h, h' \in G^-$ with $h, h' > h_1$ and $h_1 = h \wedge h'$ which would imply

$$h_1 h_2 \ldots h_k = (h \wedge h')h_2 \ldots h_k = hh_2 \ldots h_k \wedge h'h_2 \ldots h_k$$

despite $h_1 h_2 \ldots h_k \neq hh_2 \ldots h_k\ ,\ h'h_2 \ldots h_k$. Since $\left[s^{-1}, e\right]$ is dually geometric, $h_1 \in X(G^-)$. Therefore, $1 = d(h_1) = d(g_1)$, where we used Proposition 2.3.12. $\square$

Corollary 2.3.15. *Let G and s be as in Proposition 2.3.14. Then every meet-irreducible $g \in G^-$ is a cochain and vice versa. In particular, g is a cochain if and only if $\iota_i(g)$ only takes the values 0 and 1.*

Proof. Let $g \in G^-$ be meet-irreducible and $g = g_k g_{k-1} \ldots g_1$ be the right-normal factorization. Due to noetherianity, there is a unique $g' \in G^-$ with $g \prec g'$. The meet-irreducibility of g implies that any $h \in G^-$ with $g < h$ must fulfill $g' \leq h$. Since $d(g_k) = 1$ (Proposition 2.3.14), we must have $g' = g_{k-1} \ldots g_1 \succ$

$g_k g_{k-1} \dots g_1$. As g' has only normal factors of degree 1, the element g' is meet-irreducible and we can repeat the argument (except when $g' \neq e$) until we get the unique chain $g \prec g' \prec g'' \prec \dots \prec e$. Therefore, g is a cochain.

On the other hand, each cochain in a lattice is clearly meet-irreducible, so the other direction is trivial. □

We close this section with our *drosophila*, the group $\mathrm{PPU}(b)$. How does our factorization theory look in this case?

Example 2.3.16. Let us take a look at the modular, noetherian right ℓ-group $G = \mathrm{PPU}(b)$ with the strong order unit t. By the isomorphism $[t^{-1}, 1] \cong L({}_K V)$ we can calculate for any $U \in L({}_K V)$ that

$$d(p_U) = l([p_U, \underbrace{p_V}_{=1}]) = l([U, V]) = n - \dim_K U.$$

Let now $M \in \mathrm{PPU}(b)$ have the right-normal factorization

$$M = p_{U_k} p_{U_{k-1}} \cdots p_{U_1}$$

with $U_1, \dots, U_k \in L({}_K V)$. We have seen in the last chapter that right-normality is equivalent to $V = U_i^* + U_{i+1}$ for $1 \leq i \leq k-1$. Using the fact that $\dim_K U_i^* = n - \dim_K U_i$, this implies for $1 \leq i \leq k-1$ that

$$\begin{aligned} n \leq \dim_K U_i^* + \dim_K U_{i+1} &\Rightarrow n - \dim_K U_{i+1} \leq n - \dim_K U_i \\ &\Rightarrow d(p_{U_{i+1}}) \leq d(p_{U_i}) \\ &\Rightarrow \iota_{i+1}(M) \leq \iota_i(M). \end{aligned}$$

which is exactly Proposition 2.3.9 in the case of the group $\mathrm{PPU}(b)$.

If this factorization is additionally homogeneous, that is, $\dim_K U_1 = \dots = \dim_K U_k$, the condition $U_i^* + U_{i+1} = V$ can only be fulfilled when $U_i^* \cap U_{i+1} = 0$, since in case of homogeneity, we have for all $1 \leq i \leq k-1$

$$\dim_K U_i^* + \dim_K U_{i+1} = n - \dim_K U_i + \dim_K U_{i+1} = n.$$

Taking orthogonal complements in the equality $U_i^* \cap U_{i+1} = 0$, we get $U_i + U_{i+1}^* = V$ which is exactly the condition for the factorization $p_{U_k} p_{U_{k-1}} \cdots p_{U_1}$ to be left-normal. In a similar way, left-normality implies right-normality.

Since we believe there is no easy way to convert a right-normal factorization into a left-normal factorization in the case of $\mathrm{PPU}(b)$, we do not demonstrate the symmetry of the lengths in the left- and the right-normal factorization of an element which is predicted by Proposition 2.3.12. It is not known to the author if the researchers working with paraunitary matrices are aware of such a result or if it is of any use to them.

2.4 The lattice structure of desarguesian right ℓ-groups

In this section, we look at the aspects of the lattice structure of G^- - where G is a desarguesian right ℓ-group - which will be relevant for applying Inaba's coordinatization theorem (Theorem 1.2.8).

It is relatively easy to show:

Proposition 2.4.1. *Let G be a desarguesian right ℓ-group. Then every interval in G^- is a primary lattice. In particular, every interval $\left[s^{-k}, e\right] \subseteq G^-$ $(k \geq 1)$ is primary.*

Proof. By Proposition 1.1.7, we need to show that for arbitrary elements $a, b \in G^-$ with $a \leq b$, any two $h, h' \in [a, b]$ with $h, h' \succ a$ are perspective in the sublattice $[a, b]$.

Using Proposition 2.2.7, we can see that h, h' are clearly contained in the interval $[a, \operatorname{Soc}(a) \wedge b] = [a, sa \wedge b]$. However, $[a, sa \wedge b]$ is an interval in $[a, sa] \cong [s^{-1}, e]$, which is desarguesian. Since each interval in a desarguesian lattice is desarguesian itself, we deduce that $[a, \operatorname{Soc}(a) \wedge b]$ is desarguesian. By Lemma 1.1.6, the elements h, h' are perspective in $[a, \operatorname{Soc}(a) \wedge b]$. Therefore, they are perspective in $[a, b]$ as well. $\square$

We now investigate the existence of dual bases. We begin with the following lemma:

Lemma 2.4.2. *Let L be bounded from above and modular and let the subset $\{a, y_1, \ldots, y_n\} \subseteq L$ be dually independent. If $b \in L$ fulfills $a \vee b = 1$, then the set $\{(a \wedge y_i) \vee b : 1 \leq i \leq n\}$ is dually independent as well.*

Proof. Set $Y := \bigwedge_{i=1}^{n} y_i$. The elements $y_1, \ldots, y_n$ are dually independent in $[Y, 1]$. We also have the diamond isomorphism $[Y, 1] = [Y, Y \vee a] \xrightarrow{\sim} [Y \wedge a, a]$ which is given by the map $y \mapsto y \wedge a$. It maps the dually independent elements y_i $(1 \leq i \leq n)$ from $[Y, 1]$ to the elements $a \wedge y_i$ $(1 \leq i \leq n)$ in $[Y \wedge a, a]$ which are therefore dually independent in $[Y \wedge a, a]$.

A dually independent set keeps being dually independent after replacing some elements by bigger ones, so the elements $(y_i \wedge a) \vee (a \wedge b)$ $(1 \leq i \leq n)$ are still dually independent in $[a \wedge Y, a]$. In particular, they are dually independent in $[a \wedge b, a]$.

We have the diamond isomorphism $[a \wedge b, a] \xrightarrow{\sim} [b, a \vee b] = [b, 1]$ given by $x \mapsto x \vee b$. The images of the dually indepedent elements $(y_i \wedge a) \vee (a \wedge b)$ $(1 \leq i \leq n)$ can be determined as follows:

$$(y_i \wedge a) \vee (a \wedge b) \mapsto (y_i \wedge a) \vee (a \wedge b) \vee b = (y_i \wedge a) \vee b.$$

Therefore, the elements $(y_i \wedge a) \vee b$ are dually independent in $[b, 1]$. In particular, they are dually independent in L. $\square$

We continue with a useful lemma concerning cochains in lattices:

Lemma 2.4.3. *Let L be a bounded from above, modular lattice and $x_1, \ldots, x_n \in L$ cochains. Setting $X = x_1 \wedge \ldots \wedge x_n$, there are at most n dually independent dual atoms above X.*

Proof. We use induction on n: for $n = 1$, this is trivial. Now let $n > 1$ and assume that the statement has already been proven for all $k < n$.

Assume to the contrary that there were $n + 1$ dually independent dual atoms above X, say $y_1, \ldots, y_{n+1}$. Set $X' = x_1 \wedge \ldots \wedge x_{n-1}$.

We first show that we can then find i, j with $1 \leq i < j \leq n + 1$ such that $(y_i \wedge y_j) \vee X' = 1$: clearly, not all y_i can be above X' since this would contradict the induction hypothesis. Therefore, there is an index l with $1 \leq l \leq n + 1$ such that $y_l \ngeq X'$. There is another index m with $1 \leq m \leq n+1$ such that $(y_l \wedge y_m) \vee X' = 1$. Otherwise, for all $m \neq l$, we would have $y'_m := (y_l \wedge y_m) \vee X' < 1$. Note also that $y_l \ngeq X'$ implies $y'_m > y_l \wedge y_m$. Since modularity implies $l([y_l \wedge y_m], 1) = 2$, we get for all $m \neq l$ the covering relation $y'_m \prec 1$.

But then the elements y'_m $(m \neq l)$ would form a dually independent set of n dual atoms in $[X', 1]$, by Lemma 2.4.2. This would be a contradiction to the induction hypothesis. So there is an $m \neq l$ with $(y_l \wedge y_m) \vee X' = 1$. By indexing properly, we can take the desired $i < j$ such that $\{i, j\} = \{k, l\}$, and the claim is proven.

Set $Y := y_i \wedge y_j$, then

$$[Y, 1] = \left[Y, Y \vee X'\right] \cong \left[Y \wedge X', X'\right] \subseteq \left[X, X'\right],$$

where the inclusion follows from $Y \geq X$. But $[X, X']$ is a chain, which we can see by the consideration that

$$\left[X, X'\right] = \left[X' \wedge x_n, X'\right] \cong \left[x_n, X' \vee x_n\right] \subseteq [x_n, 1] .$$

Therefore, $[X, X']$ can have at most one dual atom - this, however, would contradict the fact that $[X, X']$ contains a sublattice isomorphic to $[Y, 1]$ which has at least two different dual atoms. $\square$

Recall that if a lattice L is bounded from above and fulfills the ascending chain condition, then for each cochain $y \in L$ $(y \neq 1)$ there is exactly one $x \in X(L)$ with $x \geq y$. This justifies the notion of a *unique dual atom lying over a cochain* if the ambient lattice fulfills the ascending chain condition.

Lemma 2.4.4. *Let L be a bounded from above, modular, lattice which fulfills the ascending chain condition. Let $y_1, \ldots y_n \in L$ be cochains. If $e \succ x_i \geq y_i$ $(1 \leq i \leq n)$ are the unique dual atoms lying over the y_i then $y_1, \ldots, y_n$ are dually independent if and only if $x_1, \ldots, x_n$ are.*

Proof. If, under the stated conditions, the elements $y_1, \ldots, y_n$ are dually independent, then the elements $x_1, \ldots, x_n$ are dually independent as well, since dual independence is preserved under replacing elements by possibly larger ones.

Now assume that $x_1, \ldots, x_n$ are dually independent. We prove that for any $I \subseteq \{1, \ldots, n\}$ and $1 \leq j \leq n$ such that $j \notin I$ we have $Y := \left(\bigwedge_{i \in I} y_i\right) \vee y_j = 1$.

If we had $Y < 1$, then, in particular $y_j \leq Y < 1$, which would imply $Y \leq x_j$. With $X := \bigwedge_{i \in I} y_i$, this would imply that there are at least $|I| + 1$ dually independent dual atoms above X, namely the elements x_i $(i \in I)$ and x_j, thus contradicting Lemma 2.4.3. □

Our next task in this section will be the construction of cochains in desarguesian right ℓ-groups. The preceding work will then enable us to show that suitable sets of these cochains are dual bases of intervals in desarguesian right ℓ-groups.

Lemma 2.4.5. *Let G be a right ℓ-group with strong order unit s. Then for all $g \in G$, we have the equivalence $g \in [s^{-1}, e] \Leftrightarrow s^{-1}g^{-1} \in [s^{-1}, e]$.*

Proof. Due to right-invariance and normality, we have the equivalences

$$g \leq e \Leftrightarrow e \leq g^{-1} \Leftrightarrow s^{-1} \leq s^{-1}g^{-1},$$

$$s^{-1} \leq g \Leftrightarrow e \leq sg \Leftrightarrow g^{-1}s^{-1} \leq e.$$

□

Lemma 2.4.6. *Let G be a desarguesian right ℓ-group of dimension δ. Then for every cochain $y \in G^-$, there is a cochain $y' \in G^-$ such that $y' \prec y$.*

Proof. Let $y = g_k g_{k-1} \ldots g_1$ be the right-normal factorization. By Proposition 2.3.14, $d(g_i) = 1$ $(1 \leq i \leq k)$. Recall equation (2.1) which expresses the right-maximality of the factorization as

$$g_{i+1} \vee s^{-1}g_i^{-1} = e \,(1 \leq i \leq k-1).$$

For $1 \leq i \leq k-1$, we have $s^{-1} \leq g_i < e$, so the previous lemma implies $s^{-1} < s^{-1}g_i^{-1} \leq e$. In particular, $s^{-1}g_i^{-1} \in [s^{-1}, e]$.

The interval $\left[s^{-1}, e\right]$, being desarguesian, is dually atomistic. As $s^{-1}g_k^{-1} > s^{-1}$, there must be an element $g_{k+1} \prec e$ with $g_{k+1} \ngeq s^{-1}g_k^{-1}$. For such an element g_{k+1}, we necessarily have $g_{k+1} \vee s^{-1}g_k^{-1} = e$. It follows that the expression $g_{k+1}g_k \ldots g_1$ is right-maximal.

Since $g_i \prec e$ $(1 \leq i \leq k+1)$, all factors in this expression have degree 1, so Corollary 2.3.15 tells us that $y' := g_{k+1}g_k \ldots g_1$ is a cochain. Furthermore, $y' \prec y$. □

Proposition 2.4.7. *If G is a desarguesian right ℓ-group of dimension δ, then for each $k > 0$ the interval $\left[s^{-k}, e\right]$ has a basis $y'_1, \ldots, y'_\delta$ with $l([s^{-k}, y'_i]) = k$ $(1 \le i \le \delta)$.*

Proof. $\left[s^{-1}, e\right]$ is a desarguesian lattice of dimension δ. Therefore, there are elements $x_i \prec e$ $(1 \le i \le \delta)$ which form a dually spanning, dually independent set in $\left[s^{-1}, e\right]$.

Let $k > 0$. By an iteration of Lemma 2.4.6, we can find cochains $y_1, \ldots, y_\delta$ with $y_i \le x_i$ and $d(y_i) = k$ for each $1 \le i \le \delta$.

For each $1 \le i \le \delta$, the element x_i is the unique dual atom above y_i. Since the elements x_i $(1 \le i \le \delta)$ are dually independent, Lemma 2.4.4, implies that the elements y_i $(1 \le i \le \delta)$ are dually independent as well.

Since $y_1, \ldots, y_\delta$ are dually independent, $d(y_1 \wedge \ldots \wedge y_\delta) = k\delta$ which can be shown by an iteration of the parallelogram identity (Proposition 2.3.5). By the estimate given by Proposition 2.3.6, $y_i \ge s^{-d(y_i)} = s^{-k}$ $(1 \le i \le \delta)$. Since $d(s^{-k}) = k\delta$, this shows that $y_1 \wedge \ldots \wedge y_\delta = s^{-k}$. Therefore, $y_1, \ldots, y_\delta$ form a dually spanning, dually independent subset of $\left[s^{-k}, e\right]$. Since the y_i $(1 \le i \le \delta)$ are also cochains, they form a dual basis of $\left[s^{-k}, e\right]$. Now Proposition 1.1.11 implies that $[s^{-k}, e]$ also has a basis $y'_1, \ldots, y'_\delta$ such that for $1 \le i \le \delta$, we have $l([s^{-k}, y'_i]) = l([y_i, e]) = k$. □

What we have shown in this section now enables us to prove:

Proposition 2.4.8. *If G is a desarguesian right ℓ-group of dimension $\delta \ge 4$, then for each positive integer k, there is a cpu ring R_k of length k such that there is an isomorphism of lattices $\left[s^{-k}, e\right] \cong L({}_{R_k}R_k^\delta)$.*

Proof. By Proposition 2.4.1, $\left[s^{-k}, e\right]$ is primary, and by Proposition 2.4.7, has a basis of size δ with all basis elements of length k. Since $\delta \ge 4$, the proposition follows from Inaba's theorem (Theorem 1.2.8). □

2.5 Coordinatization of desarguesian right ℓ-groups

Now that we know that each desarguesian right ℓ-group of dimension at least 4 is *locally* isomorphic to a submodule lattice (Proposition 2.4.7), we are left with putting this result into a *global* context. This will be the aim of the current section.

Before proving one our main theorems, we need to provide a few ring-theoretic preliminaries, part of which can be found in Gubareni's article [Gub11]. Of importance for us are the definition of a noncommutative discrete valuation field resp. -ring and the ideal structure of discrete valuation rings. Furthermore, we will use Lemma 2.5.6 from this article. The article does not cover completeness -

however, our discussion of completeness is a straightforward generalization of the commutative theory which, for example, is concerned with power series rings and p-adic numbers.

Definition 2.5.1. Let Q be a - possibly noncommutative - field. A *discrete valuation* on Q is a surjective function $v : Q \to \mathbb{Z} \cup \{+\infty\}$ such that the following properties hold for any $x, y \in Q$.

$$v(x) = +\infty \Leftrightarrow x = 0 \tag{2.7}$$
$$v(xy) = v(x) + v(y) \tag{2.8}$$
$$v(x+y) \geq \min\{x, y\} \tag{2.9}$$

(where we assume that $+\infty$ behaves in the obvious way when taking sums or minima). If Q is a field and v is a valuation on Q we call the pair (Q, v) a *discrete valuation field* (*dvf*, for short). If no ambiguity can arise from looking at different valuations (which we will not do), we will simply say that Q is a discrete valuation field while actually meaning the pair (Q, v). The valuation of Q will always be denoted by v (except when we say otherwise).

Each dvf Q carries a metric given by[5]

$$\mathrm{dist} : Q \times Q \to \mathbb{R}$$
$$\mathrm{dist}(x, y) = \begin{cases} 0 & x = y \\ 2^{-v(x-y)} & x \neq y \end{cases}.$$

We call a dvf Q *complete* if Q is complete under this metric, i.e. if every Cauchy sequence (with respect to dist) in Q converges.

Example 2.5.2. 1) Let $p \in \mathbb{Z}$ be a prime number. Then each integer $m \in \mathbb{Z}$ can be uniquely decomposed as $m = p^k \cdot a$ with $p \nmid a$ in which case one defines $v_p(m) := k$. By setting $v_p(0) := +\infty$ and $v_p\left(\frac{m}{n}\right) := v_p(m) - v_p(n)$, the field $\mathbb{Q}$ becomes a dvf with valuation v_p. Its respective dvr is the subring

$$R = \left\{\frac{m}{n} \in \mathbb{Q} : m, n \in \mathbb{Z}, p \nmid n\right\}.$$

2) Let K be a (possibly skew) field and let $\sigma : K \to K$. Then the field of *twisted Laurent series* over K is defined as

$$K((x, \sigma)) := \left\{ \sum_{i=-\infty}^{\infty} a_i x^i \ : \ a_i \in K \ ; \ \exists i_0 \in \mathbb{Z}\ \forall i \leq i_0 : a_i = 0 \right\}.$$

where the addition is obvious and the multiplication is defined as

$$\left(\sum_{i=-\infty}^{\infty} a_i x^i\right)\left(\sum_{j=-\infty}^{\infty} b_j x^j\right) = \sum_{n=-\infty}^{\infty}\left(\sum_{i=-\infty}^{\infty} a_i \sigma^i(b_{n-i})\right) x^n.$$

[5] Note that the number 2 in the definition of dist is random in a sense - any real number greater than 1 will work here.

Then $K((x,\sigma))$ is a complete dvf under the valuation

$$v\left(\sum_{i=-\infty}^{\infty} a_i x^i\right) = \min\{i \in \mathbb{Z} : a_i \neq 0\}.$$

The respective dvr is the ring of *twisted power series* over K,

$$K[[x,\sigma]] := \left\{\sum_{i=-\infty}^{\infty} a_i x^i \in K((x)) : a_i = 0 \text{ for } i < 0\right\}.$$

Note that for $\sigma = \mathrm{id}_K$, we have $K((x,\sigma)) = K((x))$, the (untwisted) field of Laurent series over K, and $K[[x,\sigma]] = K[[x]]$, the (untwisted) ring of power series over K.

Remark 2.5.3. For the reader's convenience, we express convergence and the Cauchy property in terms of the valuation v: if $(x_i)_{i\geq 1}$ is a sequence in Q, then x_i converges to an element $x \in Q$ if

$$\lim_{i\to\infty} v(x - x_i) = +\infty.$$

The sequence x_i is a Cauchy sequence if

$$\forall k \in \mathbb{Z}\ \exists n_k \in \mathbb{Z}\ \forall i,j \in \mathbb{Z}\ :\ (i,j \geq n_k) \Rightarrow (v(x_i - x_j) \geq k)\,.$$

Note that if Q is a dvf, the subset

$$R := v^{-1}(\mathbb{Z}_0^+ \cup \{+\infty\}) \subseteq Q$$

is easily seen to be a subring of Q. Furthermore, the subset

$$\mathfrak{m} := v^{-1}(\mathbb{Z}^+ \cup \{+\infty\}) \subseteq R$$

is an ideal in R.

An element $x \in R$ is included in $R \setminus \mathfrak{m} = v^{-1}(0)$ if and only if $v(x) = 0$. For such an element x we have $v(x^{-1}) = -v(x) = 0$, which implies that $x^{-1} \in R$. Since this shows that each element in $R \setminus \mathfrak{m}$ is invertible, the ideal $\mathfrak{m}$ must be the unique maximal ideal of R. This argument shows that R is a local ring.

These arguments motivate the following definitions:

Definition 2.5.4. A *discrete valuation ring* (*dvr*, for short) is a ring of the form

$$R = v^{-1}(\mathbb{Z}_0^+ \cup \{+\infty\})$$

where v is the valuation of some dvf Q. In this case, the *valuation ideal* of R is the ideal

$$\mathfrak{m} = v^{-1}(\mathbb{Z}^+ \cup \{+\infty\}) \subseteq R.$$

If Q is a complete dvf, we call R a *complete* dvr.

If R is a dvr, we say that an element $\pi \in R$ is a *uniformizer* of R if $R\pi = \mathfrak{m}$ (equivalently, $\pi R = \mathfrak{m}$ resp. $\pi \in \mathfrak{m} \setminus \mathfrak{m}^2$).

The ideal structure of dvrs is particularly easy to describe:

Proposition 2.5.5. *Let R be a dvr, then:*

i) Each left or right ideal $I \subseteq R$ is of the form $I = I_k := \{x \in R : v(x) \geq k\}$ for some $k \in \mathbb{Z}_0^+ \cup \{+\infty\}$.

ii) Each left ideal in R is also a right ideal and vice versa.

iii) Each right (resp. left) ideal is a principal right (resp. left) ideal.

iv) For any fixed choice of a uniformizer $\pi \in R$, the left (resp. right) ideals of R are (0) and $R\pi^k$ (resp. $\pi^k R$) where k is a nonnegative integer.

Proof. i) Let $I \subseteq R$ be a left ideal. Let $x \in I$ be such that $k := v(x)$ is as small as possible. From the definition of I_k, it is immediate that then $I \supseteq I_k$

If $k = +\infty$, we have $I = (0) = I_\infty$.

Now assume that $k \in \mathbb{Z}_0^+$. If $y \in R$ is such that $v(y) \geq k$, then $v(yx^{-1}) = v(y) - v(x) = v(y) - k \geq 0$. Therefore, $yx^{-1} \in R$. Since I is a left ideal, it must contain $(yx^{-1})x = y$. Therefore, $I = I_k$.

The version concerning right ideals is proven symmetrically.

ii) follows directly from part i).

iii) the proof of part i) shows that if I is a right (left) ideal, each $x \in I$ such that $v(x)$ is minimal fulfills $Rx = xR = I$.

iv) Note that an element $\pi \in R$ is a uniformizer if and only if $v(\pi) = 1$. Let π now be any uniformizer. If $I \neq (0)$, then $I = I_k$ for some integer k. Since $v(\pi^k) = k \cdot v(\pi) = k$, we have $\pi^k \in I_k$ and π^k has minimal valuation amongst all elements in I_k. So the element x in the proof of part i) can be chosen to be $x = \pi^k$.

□

Dvrs can be characterized without referring to an ambient dvf:

Lemma 2.5.6. *Let R be a noncommutative local ring where $\mathfrak{m} = aR = Ra$ for some non-nilpotent $a \in R$ and $\bigcap_{k=1}^\infty \mathfrak{m}^k = (0)$ then R is a dvr.*

Proof. See [Gub11, Proposition 4]. □

Suppose that we are given a ring R_k for each positive integer k. Furthermore, for each pair of integers $k \geq l$, let $\varphi_{k,l} : R_k \to R_l$ be a ring homomorphism such that all these homomorphisms are compatible in the sense that for every triple $k \geq l \geq m$, we have $\varphi_{l,m}\varphi_{k,l} = \varphi_{k,m}$. Note that such an inverse system is already completely determined by the homomorphisms $\varphi_i := \varphi_{i+1,i}$. This can be seen by expressing $\varphi_{k,l}$ as

$$\varphi_{k,l} = \varphi_{l+1,l}\varphi_{l+2,l+1} \cdots \varphi_{k,k-1} = \varphi_l \varphi_{l+1} \cdots \varphi_{k-1}.$$

In the same manner, each collection of surjective homomorphisms $\varphi_i : R_{i+1} \to R_i$ $(i \geq 1)$ uniquely defines an inverse system of rings with $\varphi_{i+1,i} = \varphi_i$ (take the above formula as the definition of $\varphi_{k,l}$!).

Recall that for an inverse system of rings, the *inverse limit* is defined as

$$\begin{aligned}\varprojlim R_k &:= \left\{ (x_k)_{k\geq 1} \in \prod_{k=1}^{\infty} R_k \ : \ \forall k \geq l : \varphi_{k,l}(x_k) = x_l \right\} \\ &= \left\{ (x_k)_{k\geq 1} \in \prod_{k=1}^{\infty} R_k \ : \ \forall i \geq 1 : \varphi_i(x_{i+1}) = x_i \right\},\end{aligned}$$

which again is a ring.

We can now state the following criteria for completeness:

Proposition 2.5.7. *Let Q be a dvf and R its respective dvr. Then the following statements are equivalent:*

i) Q is complete.

ii) R is complete under the induced metric.

iii) Given the inverse system with $R_k = R/\pi^k R$ $(k \geq 1)$ and, for $k \geq l$, the maps

$$\begin{aligned}\varphi_{k,l} &: R_k \to R_l \\ x + \pi^k R &\mapsto x + \pi^l R,\end{aligned}$$

then the canonical ring morphism

$$\begin{aligned}f : R &\to \varprojlim R_k \\ x &\mapsto (x + \pi^k R)_{k\geq 1}\end{aligned}$$

is an isomorphism.

Proof. *i)* $\Rightarrow$ *ii)*: We have $R = \{x \in Q : \mathrm{dist}(x, 0) \leq 1\}$. Therefore, R can be characterized as the closed 1-ball around 0 in Q. By a standard topological argument, R is closed in Q. This shows that if Q is complete (with respect to dist), then R is also complete under the induced metric.

ii) $\Leftarrow$ *i)*: Assume that R is complete and let $(x_i)_{i\geq 1}$ be a Cauchy sequence in Q.

We claim that the sequence of integers $v(x_i)$ is bounded from below. If we assume that this was not the case, then there is a subsequence $(x_{i_j})_{j\geq 1}$ such that

$\lim_{j\to\infty} v(x_{i_j}) = -\infty$ and $v(x_{i_j})$ is strictly decreasing. However, for any pair of integers $j' > j \geq 1$, we then have

$$v(x_{i_{j'}}) = v(x_{i_j} + (x_{i_{j'}} - x_{i_j})) \geq \min\{v(x_{i_j}), v(x_{i_{j'}} - x_{i_j})\},$$

which implies that $v(x_{i_{j'}} - x_{i_j}) \leq v(x_{i_{j'}})$ since $v(x_{i_{j'}}) < v(x_{i_j})$. However, the sequence $v(x_{i_j})$ is strictly decreasing, so x_{i_j} can not be a Cauchy sequence - which shows that the sequence x_i also cannot be one.

We can therefore say that there is an integer c such that $v(x_i) \geq c$ for all i. Let π be a uniformizer, then $v(\pi^{-c}x_i) = -c + v(x_i) \geq 0$ resp. $\pi^{-c}x_i \in R$ for all i. Furthermore, the sequence $(\pi^{-c}x_i)_{i\geq 1}$ is a Cauchy sequence. Since R is complete, this sequence converges, say, to x. The sequence x_i then converges to $\pi^c x$.

Thus, we have proven that Q is complete.

ii) $\Rightarrow$ *iii)*: Assume that R is complete and let $(x_k + \pi^k R)_{k\geq 1} \in \varprojlim R/\pi^k R$. Since $x_l + \pi^l R = x_k + \pi^l R$ holds for all pairs of integers $k \geq l \geq 1$, we get that $x_k - x_l \in \pi^l R$ in this case. This is the same as saying that $v(x_k - x_l) \geq l$ for all $k \geq l$. It follows that x_k is a Cauchy sequence in R and therefore has a limit, say, $x \in R$.

We now show that, in fact, $f(x) = (x_k + \pi^k R)_{k\geq 1}$: let $k \geq 1$. Then, there is an integer n_k such that for all $i \geq n_k$ we have $v(x - x_i) \geq k$ resp. $x + \pi^k R = x_i + \pi^k R$. Choose $i \geq \max\{k, n_k\}$, then

$$x + \pi^k R = x_i + \pi^k R = x_k + \pi^k R.$$

It follows that $f(x) = (x + \pi^k R)_{k\geq 1} = (x_k + \pi^k R)_{k\geq 1}$. Therefore, we have shown that f is surjective.

If $x \in \ker(f)$, then for all $k \geq 1$ we have $x \in \pi^k R$ resp. $v(x) \geq k$. This is only possible when $v(x) = \infty$ resp. $x = 0$. So, f is also injective.

iii) $\Rightarrow$ *ii)*: Assume that the map $f : R \to \varprojlim R/\pi^k R$ from above is an isomorphism and let $(x_i)_{i\geq 1}$ be an arbitrary Cauchy sequence in R. We have to show that x_i converges in R. To say that the sequence $(x_i)_{i\geq 1}$ is Cauchy, is the same as saying that for all $k \geq 1$ there is an integer n_k such that for all $i, j \geq n_k$ we have $x_i + \pi^k R = x_j + \pi^k R$.

For $k \geq 1$, set $y_k := x_{n_k}$. If $k \geq l$ then clearly $n_k \geq n_l$, which shows that $y_l + \pi^l R = y_k + \pi^l R$ in this case. We infer that $(y_k + \pi^k R)_{k\geq 1} \in \varprojlim R/\pi^k R$.

By assumption, there is an element $y \in R$ such that $f(y) = (y_k + \pi^k R)_{k\geq 1}$, i.e. $y + \pi^k R = y_k + \pi^k R$ holds for all $k \geq 1$. For $k \geq 1$, let n_k be as above, then for all $i \geq n_k$ we have

$$x_i + \pi^k R = y_k + \pi^k R = y + \pi^k R$$

resp. $v(x_i - y) \geq k$. This shows that $\lim_{i\to\infty} x_i = y$. In particular, $(x_i)_{i\geq 1}$ converges in R. $\square$

Assume that R is a dvr and $\pi \in R$ a uniformizer. It is not hard to see from Proposition 2.5.5 that for each $k \geq 1$, the ring $R/\pi^k R$ is a cpu ring of length k. We will now see that certain families of cpu rings are indeed derived from dvrs.

Suppose now that we are given, for each positive integer k, a cpu ring R_k of length k. Furthermore, for each pair of integers $k \geq l$, let a *surjective* homomorphism of rings $\varphi_{k,l} : R_k \to R_l$ be given such that the rings R_k, together with the homomorphisms $\varphi_{k,l}$ define an inverse system of rings. In this case, the inverse limit $R := \varprojlim R_k$ can be defined. The resulting ring can be shown to possess very nice properties:

Proposition 2.5.8. *If the rings R_k and the $\varphi_{k,l}$ are as above, then R is a complete dvr.*

Proof. Note that each R_k is a local ring. Let $\mathfrak{m}_k \subseteq R_k$ be its respective maximal ideal.

We note here that for all pairs $k \geq l$, we have $\varphi_{k,l}^{-1}(\mathfrak{m}_l) = \mathfrak{m}_k$, since $\varphi_{k,l}^{-1}(\mathfrak{m}_l)$ must be a maximal ideal in R_k and there is only one. On the other hand, we also have $\varphi_{k,l}(\mathfrak{m}_k) = \varphi_{k,l}(\varphi_{k,l}^{-1}(\mathfrak{m}_l)) = \mathfrak{m}_l$ since $\varphi_{k,l}$ is surjective. This shows that for each $(x_k)_{k\geq 1} \in R$, we have the equivalence

$$(\forall k \geq 1 : x_k \in \mathfrak{m}_k) \Leftrightarrow (\exists k \geq 1 : x_k \in \mathfrak{m}_k) \,. \tag{2.10}$$

Define the ideal $\mathfrak{m} \subseteq R$ by

$$\mathfrak{m} := \{(x_k)_{k\geq 1} \in R \,:\, \forall k \geq 1 : x_k \in \mathfrak{m}_k\} \,.$$

We show that $\mathfrak{m}$ is the unique maximal ideal of R: if $x = (x_k)_{k\geq 1} \in R \setminus \mathfrak{m}$, then $x_k \in R_k \setminus \mathfrak{m}_k$ for all $k \geq 1$, by (2.10). Therefore, x_k is invertible in R_k for each k, and so x is invertible in R. This proves that R is local with maximal ideal $\mathfrak{m}$.

Now let $\pi = (\pi_k)_{k\geq 1}$ be such that $\pi_2 \in \mathfrak{m}_2 \setminus (0)$. Since R_2 is cpu of length 2, we have $\mathfrak{m}_2^2 = (0)$, which implies that $\pi_k \in \mathfrak{m}_k \setminus \mathfrak{m}_k^2$ for all $k \geq 2$. This shows that π_k is a uniformizer in R_k for $k \geq 2$. Note the special case $k = 1$: under our choice of π, we have $\pi_1 = 0$ which is also a uniformizer of R_1 which is cpu of length 1.

We now prove that π generates the ideal $\mathfrak{m}$ both as a left and as a right ideal: let $x = (x_k)_{k\geq 1} \in \mathfrak{m}$. For each $k \geq 1$, we have $x_k \in \mathfrak{m}_k$, by definition, so there is some $r_k \in R_k$ with $r_k \pi_k = x_k$ (note that $(r_k)_{k\geq 1}$ does not necessarily lie in R!). For $k \geq 1$, apply the homomorphism $\varphi_k = \varphi_{k+1,k}$ to the equality $r_{k+1}\pi_{k+1} = x_{k+1}$:

$$\begin{aligned} \varphi_k(r_{k+1})\varphi_k(\pi_{k+1}) &= \varphi_k(x_{k+1}) \\ \Leftrightarrow \varphi_k(r_{k+1})\pi_k &= x_k . \end{aligned}$$

For $k \geq 1$ we set $s_k := \varphi_k(r_k)$. We now have $s_k \pi_k = x_k$.

We show that $(s_k)_{k\geq 1} \in R$. Subtracting the equation $r_k \pi_k = x_k$ gives us

$$(s_k - r_k)\pi_k = 0,$$

showing that $(s_k - r_k) \in \pi_k^{k-1} R_k$, the smallest non-zero ideal in R_k. For $k > 1$, the homomorphism φ_{k-1} annihilates this ideal, which implies $\varphi_{k-1}(s_k - r_k) = 0$, therefore $\varphi_{k-1}(s_k) = \varphi_{k-1}(r_k) = s_{k-1}$. We have shown that $s = (s_k)_{k \geq 1} \in R$ and $s\pi = x$.

Therefore, $R\pi = \mathfrak{m}$. By a symmetric argument, one also proves $\pi R = \mathfrak{m}$.

Since π_k is a uniformizer in R_k for each $k \geq 1$, we have $\pi_k^{k-1} \neq 0$. In particular $\pi^k \neq 0$ for all k, so π is non-nilpotent. However, for all $k \geq 1$, we also have $\mathfrak{m}_k^k = (0)$ in R_k which shows that $\bigcap_{k=1}^{\infty} \mathfrak{m}^k = (0)$.

To summarize what we have shown: the ring R is local and there is a non-nilpotent element $\pi \in R$ such that $\mathfrak{m} = \pi R = R\pi$. Furthermore, π itself is non-nilpotent. From Gubareni's theorem (Lemma 2.5.6), it follows that R is a discrete valuation domain.

Completeness of R finally follows from our construction, due to part iii) of Proposition 2.5.7. $\square$

Let R be a left noetherian ring and $_RM$ a left R-module. We call a submodule $A \subseteq M$ an *R-lattice* in M if A is finitely generated and *essential* in M, meaning that for every submodule $0 \neq B \subseteq M$, one also has $A \cap B \neq 0$ ([MR01, 2.2.1]). Since R is noetherian, intersections and sums of finitely generated submodules are also finitely generated. Also, the property of being essential is preserved under taking sums and intersections. Therefore, under the conditions imposed on R and $_RM$, the R-lattices in M form a sublattice of $L(_RM)$ which we denote by $\mathrm{Lat}(_RM)$.

Note that by Proposition 2.5.5, each dvr R is a principal left ideal domain. In what follows, we will freely make use of the well-known fact that a torsion-free, finitely generated left module over such a ring is free.

Proposition 2.5.9. *Let R be a complete dvr for the dvf Q and let $\pi \in R$ be a uniformizer. Fix some nonnegative integer δ.*

i) An R-submodule $A \subseteq R^\delta$ is an R-lattice if and only if there is an integer n such that $\pi^n M \subseteq R^\delta$.

ii) Let δ be a nonnegative integer. Then an R-submodule $A \subseteq Q^\delta$ is an R-lattice if and only if there are integers $m \leq n$ such that $\pi^n R^\delta \subseteq A \subseteq \pi^m R^\delta$ where π is a uniformizer of R.

Proof. i) Let $e_1, \dots e_\delta$ be the canonical basis elements of R^δ. If $A \subseteq R^\delta$ is an R-lattice, then there are integers n_i $(1 \leq i \leq \delta)$ such that $\pi^{n_i} e_i \in A$ since $Re_i \cap A \neq (0)$. Taking $n = \max_{1 \leq i \leq \delta} n_i$ we get $\pi^n e_i \in A$ for all $1 \leq i \leq \delta$, therefore

$$\pi^n R^\delta = \sum_{i=1}^{\delta} R\pi^n e_i \subseteq \sum_{i=1}^{\delta} R\pi^{n_i} e_i \subseteq A.$$

On the other hand, the submodule $\pi^n R^\delta \subseteq R^\delta$ is essential for any n, due to ${}_R R^\delta$ being torsion-free. Therefore, every submodule $A \subseteq R^\delta$ which contains $\pi^n R^\delta$ for some n is also essential. As a submodule of a finitely generated module over a noetherian ring, such an A is finitely generated as well.

ii) Let A be an R-lattice in Q^δ. If $e_1, \ldots, e_\delta$ denote the canonical basis elements of R^δ, there are integers n_i $(1 \leq i \leq \delta)$ such that $\pi^{n_i} e_i \in A$, since $Re_i \cap A \neq (0)$. As in part i) we can then find an n such that $\pi^n R^\delta \subseteq A$.

Since A is finitely generated, $A = Ra_1 + \ldots + Ra_l$ for some $a_1, \ldots, a_l \in Q^\delta$. For $1 \leq j \leq l$, write $a_j = \sum_{i=1}^{\delta} r_{ij} e_i$. Let m be the minimum of all $v(r_{ij})$ where $1 \leq i \leq \delta$ and $1 \leq j \leq l$. Then all r_{ij} are contained in $\pi^m R$, therefore $a_j \in \pi^m R^\delta$ $(1 \leq j \leq l)$, from which we infer that $A \subseteq \pi^m R^\delta$.

On the other hand, since for each integer m, $\pi^m R^\delta$ is isomorphic to R^δ as an R-module, $\pi^m R^\delta$ is finitely generated. Therefore, any R-submodule of Q^δ which is contained in some submodule $\pi^m R^\delta$ is also finitely generated, due to R being noetherian. Furthermore, $\pi^n R^\delta$ is essential in Q^δ for each n, since Q^δ is torsion-free. So, every submodule of Q^δ containing some $\pi^n R^\delta$ is also essential in Q^δ. Together, these two observations prove the other direction. $\square$

We have already talked about inverse systems of rings and their inverse limits. We will also need to define direct systems of lattices and their direct limits.

Suppose that we are given, for each $k \geq 1$, a lattice L_k. Furthermore, for $k \leq l$, let a lattice homomorphism $\iota_{k,l} : L_k \to L_l$ be given such that all these homomorphisms are compatible in the sense that for all triples $k \leq l \leq m$, we have $\iota_{l,m}\iota_{k,l} = \iota_{k,l}$. We will call such a system a *direct system* of lattices. Note that, again, the maps $\iota_k := \iota_{k,k+1}$ suffice for defining such a system.

For such a system, we can define the *direct limit* as

$$\lim_{\to} L_k := \left(\coprod_{k=1}^{\infty} L_k \right) / \sim$$

Here $\coprod_{i=1}^{\infty} L_k := \bigcup_{i=1}^{\infty} L_k \times \{k\}$, on which the following equivalence relation is defined:

$$(x, i) \sim (y, j) \Leftrightarrow \exists k \geq i, j : \iota_{i,k}(x) = \iota_{j,k}(y).$$

Let $L := \lim\limits_{\to} L_k$. By construction, for each $k \geq 1$, there is the canonical map

$$\begin{aligned} \iota_k : L_k &\to L \\ x &\mapsto [(x, k)] \end{aligned}$$

so that we have $\iota_j = \iota_k \iota_{j,k}$ for all $k \geq j \geq 1$. We note that if all $\iota_{j,k} : L_j \to L_k$ $(j \leq k)$ are embeddings, then all $\iota_k : L_k \to L$ are also embeddings.

One can uniquely define lattice operations $\wedge, \vee$ on L such that all $\iota_k : L_k \to L$ ($k \geq 1$) become lattice homomorphisms, namely:

$$[(x,j)] \wedge [(y,k)] = \begin{cases} [(\iota_{j,k}(x) \wedge y, k)] & j \leq k \\ [(x \wedge \iota_{k,j}(y), k)] & j > k, \end{cases}$$

$$[(x,j)] \vee [(y,k)] = \begin{cases} [(\iota_{j,k}(x) \vee y, k)] & j \leq k \\ [(x \vee \iota_{k,j}(y), k)] & j > k. \end{cases}$$

The lattice L, together with the homomorphisms $\iota_k : L_k \to L$, fulfill the following universal property:

Given a lattice M and a family of lattice homomorphisms $\alpha_k : L_k \to M$ ($k \geq 1$) such that $\alpha_k \iota_{j,k} = \alpha_j$ for all $j \leq k$, there is a unique lattice homomorphism $\alpha : L \to M$ such that $\alpha \iota_k = \alpha_k$ for all $k \geq 1$. This homomorphism is given by $\alpha([(x,k)]) = \alpha_k(x)$ ($k \geq 1, x \in L_k$).

We have not given any proofs in our discussion of direct limits. All constructions and statements are valid for more general algebraic structures and can be found - with proofs - in any good book about universal algebra (see, for example [Grä08, §21.]).

Example 2.5.10. Let G be a a right ℓ-group with strong order unit s. Then we have a lattice isomorphism

$$G^- \cong \lim_{\to} \, [s^{-k}, e]$$

where the maps $\iota_{k,l} : [s^{-k}, e] \to [s^{-l}, e]$ are given by inclusion. This is due to the fact that

$$G^- = \bigcup_{i=1}^{\infty} [s^{-k}, e],$$

which holds since s is a strong order unit.

More generally, let L be a lattice together with sublattices $L_k \subseteq L$ ($k \geq 1$) such that $L_j \subseteq L_k$ for all $k \geq j$ and $\bigcup_{k=1}^{\infty} L_k = L$. Then the lattices L_k, together with the natural inclusions $\iota_{j,k} : L_j \to L_k$, form a direct system of lattices with $\lim_{\to} L_k \cong L$. A canonical isomorphism $\lim_{\to} L_k \xrightarrow{\sim} L$ is given by $[(x,k)] \mapsto x$.

We explain the importance of this seemingly innocent example: if G is a desarguesian right ℓ-group of dimension $\delta \geq 4$, then, by Proposition 2.4.8, each interval $[s^{-k}, e] \subseteq G^-$ ($k \geq 1$) can be coordinatized by a lattice $L({}_{R_k}R_k^\delta)$ where R_k is a cpu-ring of length k. We therefore get a lattice isomorphism $G^- \cong \lim_{\to} L({}_{R_k}R_k^\delta)$ when the lattice homomorphisms $L({}_{R_k}R_k^\delta) \to L({}_{R_l}R_l^\delta)$ ($k \leq l$) are chosen appropriately. We will see that all these inclusions can be described by certain semilinear maps - this will enable us to piece the different „local“ coordinatizations of the intervals $[s^{-k}, e]$ together to one unified „global“ coordinatization of G^-.

The next lemma is in a sense *folklore*. Recall that for a semilinear map $f : {}_RM \to {}_SN$ between left modules ${}_RM$, ${}_SN$, the map $f^* : L({}_SN) \to L({}_RM)$ was defined by $f^*(B) = f^{-1}(B)$.

Lemma 2.5.11. *Let ${}_RM$ be an arbitrary module and let $A \in L({}_RM)$. If*

$$\begin{aligned} f : M &\to M/A \\ x &\mapsto x + A \end{aligned}$$

is the canonical quotient map, then $f^* : L({}_R(M/A)) \to L({}_RM)$ is an embedding of $L({}_R(M/A))$ as the sublattice $[A, M] \subseteq L({}_RM)$.

Proof. It is well-known that the map $p : L({}_RM) \to L({}_R(M/A))$ given by $B \mapsto f(B)$ restricts to an order-isomorphism between the lattice of submodules of ${}_RM$ which contain A - which is the interval $[A, M]$ - and the lattice of submodules of ${}_R(M/A)$. The map f^* is its inverse. $\square$

Lemma 2.5.12. *Let $f : {}_RM \to {}_SN$ be a α-semilinear surjection where $\alpha : R \to S$ is also surjective. Then $f^* : L({}_SN) \to L({}_RM)$ identifies $L({}_SN)$ with the interval $[\ker(f), M] \subseteq L({}_RM)$.*

Proof. By restriction of scalars via α, we can regard N as R-module. With this module structure, the map $\iota_N := \mathrm{id}_N : {}_RN \to {}_SN$ becomes an α-semilinear bijection (but not a semilinear isomorphism, in general). Therefore, there is a unique epimorphism of R-modules $g : {}_RM \twoheadrightarrow {}_RN$ such that $f = \iota_N g$. Let now $p : {}_RM \twoheadrightarrow {}_R(M/\ker(f))$ be the canonical projection. Then there is a unique isomorphism of R-modules $\tilde{g} : {}_R(M/\ker(f)) \xrightarrow{\sim} {}_RN$ such that $g = \tilde{g}p$. This situation is illustrated by the left diagram below.

$$\begin{tikzcd}
{}_RM \arrow[rr, "f"] \arrow[d, two heads, "p"'] \arrow[drr, two heads, "g"] & & {}_SN \\
{}_R(M/\ker(f)) \arrow[rr, "\sim", "\tilde{g}"'] & & {}_RN \arrow[u, "\iota_N"']
\end{tikzcd}
\qquad
\begin{tikzcd}
L({}_RM) & & L({}_SN) \arrow[ll, hook', "f^*"', "\sim"] \arrow[d, "\wr", "\iota_N^*"'] \\
L({}_R(M/\ker(f))) \arrow[u, hook, "p^*"] & & L({}_RN) \arrow[ll, "\sim"', "\tilde{g}^*"] \arrow[ull, hook', "g^*"']
\end{tikzcd}$$

The induced lattice maps then go the other way round, as can be seen in the right diagram. Since $\alpha : R \to S$ is surjective, the map $\iota_N^* : L({}_SN) \to L({}_RN)$ is an isomorphism, and $\tilde{g}^*$ is trivially an isomorphism. By Lemma 2.5.11, the map p^* identifies its domain with the interval $[\ker(f), M] \subseteq L({}_RM)$, and so does f^*. $\square$

Let R be a complete dvr and let $\pi \in R$ be a uniformizer. We set $R_k := R/\pi^k R$. In what follows, we will identify R_k^δ and $R^\delta/\pi^k R^\delta$, since under restriction of scalars we clearly have the isomorphism ${}_R R_k^\delta \cong {}_R(R^\delta/\pi^k R^\delta)$. This justifies the identification $L({}_{R_k} R_k^\delta) = L({}_R R_k^\delta)$. For integers $i \leq j$, define $f_{j,i}$ by

$$f_{j,i} : \begin{array}{ccc} R_j^\delta & \to & R_i^\delta \\ x + \pi^j R^\delta & \mapsto & x + \pi^i R^\delta. \end{array} \tag{2.11}$$

Note that by Lemma 2.5.11, the maps $f_{j,i}^* : L({}_R R_i^\delta) \to L({}_R R_j^\delta)$ $(i \leq j)$ embed $L({}_R R_i^\delta)$ as an interval into $L({}_R R_j^\delta)$ such that

$$f_{j,i}(1_{L({}_R R_i^\delta)}) = 1_{L({}_R R_j^\delta)}$$

(where we write 1_L for the top element of a lattice L that it bounded from above). Therefore, the lattices $L({}_R R_k^\delta)$ can be considered as intervals of the direct limit $L := \varinjlim L({}_R R_k^\delta)$. Under these embeddings the top elements are identified in the sense that $1_{L({}_R R_k^\delta)} = 1_L$ for all k.

We can now show:

Proposition 2.5.13. *Let R be a complete dvr. For any nonnegative integer δ we have an isomorphism of lattices:*

$$\mathrm{Lat}({}_R R^\delta) \cong \varinjlim L({}_R R_k^\delta)$$

where the embeddings $g_{i,j} : L({}_R R_i^\delta) \to L({}_R R_j^\delta)$ $(i \leq j)$ are given by $g_{i,j} = f_{j,i}^*$ where the $f_{j,i}$ are defined by (2.11).

Proof. We define $\alpha : \varinjlim L({}_R R_k^\delta) \to \mathrm{Lat}({}_R R^\delta)$ by the maps[6] $\alpha_i := f_i^* : L({}_R R_i^\delta) \to \mathrm{Lat}({}_R R^\delta)$ where we take f_i as the canonical factor map

$$f_i : R^\delta \to R_i^\delta$$
$$a \mapsto a + \pi^i R^\delta.$$

where π is a uniformizer of R.

This map is well-defined: if M is an R-submodule of R_k^δ, then $\pi^k R^\delta \subseteq f_k^*(M)$, since $\pi^k R^\delta = 0 + \pi^k R^\delta$ is contained in every submodule of R_k^δ. By part i) of Proposition 2.5.9, $f_k^*(M) \in \mathrm{Lat}({}_R R^\delta)$. It is clear that $f_{j,i} f_j = f_i$, so we also have $\alpha_j g_{i,j} = \alpha_i$. Therefore, we can really define a lattice homomorphism α in the stated way.

The map α is easily seen to be order-preserving: given $A, B \in L({}_R R_k^\delta)$ with $A \subseteq B$, we also have $\alpha_k(A) = f_k^{-1}(A) \subseteq f_k^{-1}(B) = \alpha_k(B)$.

We show that α is surjective: given $A \in \mathrm{Lat}({}_R R^\delta)$, take a positive integer k with $\pi^k R^\delta \subseteq A$ - this integer k is guaranteed to exist by part i) of Proposition 2.5.9. Then

$$f_k^{-1}(f_k(A)) = A + \ker(f_k) = A + \pi^k R^\delta = A,$$

[6] Of course, here, the range is restricted to $\mathrm{Lat}({}_R R^\delta) \subseteq L({}_R R^\delta)$.

so A lies in the image of $\alpha_k = f_k^*$, and thus, in the image of α.

Our map α is also injective: since all maps $g_{i,j} : L({}_R R_i^\delta) \to L({}_R R_j^\delta)$ are embeddings, the lattices $L({}_R R_i^\delta)$ can be regarded as sublattices of $\varinjlim L({}_R R_i^\delta)$ such that

$$\bigcup_{i=1}^{\infty} L({}_R R_i^\delta) = \varinjlim L({}_R R_k^\delta).$$

In order to prove that α is injective, it therefore suffices that each $\alpha_i = f_i^*$ is injective. This, however, follows directly from Lemma 2.5.11.

Let $A, A' \in \varinjlim L({}_R R_k^\delta)$ be represented by elements $B \in L({}_R R_k^\delta)$, $B' \in L({}_R R_l^\delta)$ for certain integers $k, l > 0$. We may assume that $l \geq k$; in this case, A is represented by $C = g_{i,j}(B) \in L({}_{R_l} R_l^\delta)$ as well. If $\alpha(A) = \alpha(A')$, then $f_l^{-1}(C) = f_l^{-1}(B')$. Since f_l is surjective, $C = B'$, therefore $A = A'$.

We have shown that α is a bijective lattice homomorphism, that is, an isomorphism of lattices. $\square$

We also need to understand the radical and socle operations in $\mathrm{Lat}({}_R R^\delta)$ where R is a dvr:

Proposition 2.5.14. *i) Let R be a local ring with maximal ideal $\mathfrak{m}$ and let ${}_R M$ be an R-module.*

Let $A \in L({}_R M)$ be such that the sublattice $A^\uparrow$ fulfills the descending chain condition and $A^\downarrow$ fulfills the ascending chain condition [7]. We then have the equalities:

$$\mathrm{Rad}(A) = \mathfrak{m}A$$
$$\mathrm{Soc}(A) = \mathfrak{m}^{-1}A$$

where $\mathfrak{m}^{-1}A = \{x \in M : \mathfrak{m}x \subseteq A\}$.

ii) Let δ be a positive integer, R a complete dvr with respective dvf Q and $\pi \in R$ a uniformizer. In $\mathrm{Lat}({}_R Q^\delta)$ we have for any $A \in \mathrm{Lat}({}_R Q^\delta)$ that $\mathrm{Rad}(A) = \pi A$ and $\mathrm{Soc}(A) = \pi^{-1}A$.

iii) Let δ be a positive integer, R a cpu ring and $\pi \in R$ a uniformizer. In $L({}_R R^\delta)$ we have for any $A \in \mathrm{Lat}({}_R Q^\delta)$ that $\mathrm{Rad}(A) = \pi A$ and $\mathrm{Soc}(A) = \pi^{-1}A$ where $\pi^{-1}A := \{x \in R^\delta : \pi x \in A\}$.

Proof. (i) It is well-known that, under the stated conditions, $\mathrm{Rad}(A) = \mathfrak{m}A$. Nevertheless, we give a proof: if $B \prec A$ holds in $L({}_R M)$, then A/B is simple and therefore is annihilated by $\mathfrak{m}$. It follows that $\mathfrak{m}A \subseteq B$. Therefore, $\mathfrak{m}A \subseteq \mathrm{Rad}(A)$. On the other hand, the R-module structure of $A/\mathfrak{m}A$ descends to an $R/\mathfrak{m}$-module structure. Since $R/\mathfrak{m}$ is a field, $A/\mathfrak{m}A$ is a semisimple $R/\mathfrak{m}$-module, and thus, a

[7] Else, the lattice-theoretic radicals and socles would be nonsensical.

$$\begin{tikzcd}
L_k \arrow[r, dashed, "g_k^*", "\sim"'] \arrow[d, "\psi_k"', "\wr"] \arrow[rr, dashed, bend left, "f_k^*"] & L({}_{S_k}S_k^\delta) \arrow[r, hook, "P_k^*"] \arrow[dl, "\sim", "\alpha_k"'] & L_{k+1} \arrow[d, "\psi_{k+1}", "\wr"'] \\
\left[s^{-k}, e\right] \arrow[rr, hook, "\iota_k"'] & & \left[s^{-(k+1)}, e\right]
\end{tikzcd}
\qquad
\begin{tikzcd}
R_k & S_k \arrow[l, "\gamma_k"'] & R_{k+1} \arrow[l, "\Pi_k"'] \arrow[ll, bend right, "\varphi_k"'] \\
R_k^\delta & S_k^\delta \arrow[l, "g_k"] & R_{k+1}^\delta \arrow[l, "P_k"] \arrow[ll, bend left, "f_k"]
\end{tikzcd}$$

Figure 2.1: The objects and maps constructed in the proof of Theorem 2.5.15.

semisimple R-module. Therefore, the intersection of all maximal submodules of $A/\mathfrak{m}A$ is $(0) + \mathfrak{m}A$, implying the inclusion $\mathrm{Rad}(A) \subseteq \mathfrak{m}A$.

$\mathrm{Soc}(A) = \mathfrak{m}^{-1}A$ is proven in almost the same way: if $B \succ A$ in $L({}_RM)$, then B/A is simple and therefore is annihilated by $\mathfrak{m}$. It follows that $\mathfrak{m}B \subseteq A$, resp. $B \subseteq \mathfrak{m}^{-1}A$. Therefore, $\mathrm{Soc}(A) \subseteq \mathfrak{m}^{-1}A$. As above, the R-module structure of $\mathfrak{m}^{-1}A/A$ descends to a $R/\mathfrak{m}$-module structure showing that $\mathfrak{m}^{-1}A/A$ is a semisimple R-module and therefore the sum of its simple submodules, implying the inclusion $\mathfrak{m}^{-1}A \subseteq \mathrm{Soc}(A)$.

(ii) For $A \in \mathrm{Lat}({}_RQ^\delta)$, the sublattices $A^\downarrow$, $A^\uparrow$ fulfill the ascending resp. descending chain conditions since every interval in $\mathrm{Lat}({}_RQ^\delta)$ is of finite length; this can be derived from Proposition 2.5.9 and the fact that for $m \le n$, the R-module $(\pi^n R^\delta)/(\pi^m R^\delta)$ has finite composition length, being a finite sum of copies of the finite-length R-module $(\pi^n R)/(\pi^m R)$.

Analogously to part i), one can now prove that $\mathrm{Rad}(A) = \mathfrak{m}A$ and $\mathrm{Soc}(A) = \mathfrak{m}^{-1}A$. Since $\mathfrak{m} = \pi R$, we have

$$\mathrm{Rad}(A) = \pi R A = \pi A$$

and

$$\mathrm{Soc}(A) = \{x \in Q^\delta : \pi R x \subseteq A\} = \{x \in Q^\delta : Rx \subseteq \pi^{-1}A\} = \pi^{-1}A.$$

(iii) If the ring R is cpu, then ${}_RR^\delta$ is of finite length, so the conditions of part i) are fulfilled. Rewriting the formulae for $\mathrm{Soc}(A)$, $\mathrm{Rad}(A)$ from part i) in terms of π can then be done in the same way as in part ii). $\square$

We can finally prove:

Theorem 2.5.15. *If G is a desarguesian right ℓ-group of dimension $\delta \geq 4$, then there is a complete dvr R such that there is an isomorphism of lattices $G^- \cong \operatorname{Lat}({}_R R^\delta)$.*

Proof. For each $k > 0$, we fix a cpu ring R_k of length k and some lattice isomorphism $\psi_k : L({}_{R_k} R_k^\delta) \xrightarrow{\sim} [s^{-k}, e]$, which exists by Proposition 2.4.8. In the following, we abbreviate $L_k := L({}_{R_k} R_k^\delta)$. Also, for each $k > 0$, fix some uniformizer $\pi_k \in R_k$.

Our first step is to show that there is a system of surjective ring homomorphisms $\varphi_k : R_{k+1} \to R_k$ $(k > 0)$ together with φ_k-semilinear maps $f_k : R_{k+1}^\delta \to R_k^\delta$ such that the induced maps $f_k^* : L_k \to L_{k+1}$ result in a direct system of lattices, whose limit $\varinjlim L_k$ is isomorphic to G^-, as a lattice. The commutative diagrams in Figure 2.1 illustrate this construction and can be used to keep track of all relevant maps.

In G, we have $\operatorname{Soc}(s^{-(k+1)}) = s^{-k}$ for all $k \geq 0$ (Proposition 2.2.7). Therefore, each element $x \succ s^{-(k+1)}$ also fulfills $x \leq s^{-k}$. In particular, all these $x \in G^-$, so we also have the equality $\operatorname{Soc}(s^{-(k+1)}) = s^{-k}$ in G^-. On the other hand, using Proposition 2.5.14, we see that in L_{k+1}, we have

$$\operatorname{Soc}(0_{L_{k+1}}) = \pi_{k+1}^{-1}(0) = \pi_{k+1}^k R_{k+1}^\delta.$$

We set $S_k := R_{k+1}/\pi_{k+1}^k R_{k+1}$ and let $\Pi_k : R_{k+1} \to S_k$ be the canonical projection map. Furthermore, we let $P_k : R_{k+1}^\delta \to S_k^\delta$ be the Π_k-semilinear map which acts as Π_k on the coordinates of R_{k+1}^δ. The map P_k is surjective and its kernel is given by

$$\ker P_k = \pi_{k+1}^k R_{k+1} = \operatorname{Soc}(0_{L_{k+1}}).$$

Therefore, $P_k^* : L({}_{S_k} S_k^\delta) \hookrightarrow L_{k+1}$ embeds $L({}_{S_k} S_k^\delta)$ as the interval

$$\left[\operatorname{Soc}(0_{L_{k+1}}), R_{k+1}^\delta\right] \subseteq L_{k+1}$$

(Lemma 2.5.12). Since ψ_{k+1} is an isomorphism,

$$\psi_{k+1}(\operatorname{Soc}(0_{L_{k+1}})) = \operatorname{Soc}(\psi_{k+1}(0_{L_{k+1}})) = \operatorname{Soc}(s^{-(k+1)}) = s^{-k}.$$

In $\left[s^{-(k+1)}, e\right]$ we have $\operatorname{Soc}(s^{-(k+1)}) = s^{-k}$. There is an isomorphism $\alpha_k : [s^{-k}, e] \to L({}_{S_k} S_k^\delta)$ such that $\psi_{k+1} P_k^* = \iota_k \alpha_k$ since the lattice homomorphisms ι_k and $\psi_{k+1} P_k^*$ both embed their respective domains as the sublattice $[s^{-k}, e] \subseteq \left[s^{-(k+1)}, e\right]$

Since $\alpha_k^{-1}\psi_k$ is an isomorphism between L_k and $L({}_{S_k} S_k^\delta)$ there is, by the fundamental theorem of projective geometry for cpu-rings (Proposition 1.3.8), an isomorphism $\gamma_k : S_k \to R_k$ and a β_k-semilinear isomorphism $g_k : S_k^\delta \to R_k^\delta$ such that $g_k^* = \alpha_k^{-1}\psi_k$, i.e. g_k^* makes the left diagram in Figure 2.1 commute.

$$\begin{tikzcd}
L_1 \arrow[r, "f_1^*"] \arrow[d, "\psi_1"', "\wr"] & L_2 \arrow[r] \arrow[d, "\psi_2"', "\wr"] & \cdots \arrow[r] & L_k \arrow[r, "f_k^*"] \arrow[d, "\psi_k"', "\wr"] & L_{k+1} \arrow[r] \arrow[d, "\psi_{k+1}"', "\wr"] & \cdots & \varinjlim L_k \arrow[d, "\wr"', "\psi"] \\
\left[s^{-1}, e\right] \arrow[r, "\iota_1"] & \left[s^{-2}, e\right] \arrow[r] & \cdots \arrow[r] & \left[s^{-k}, e\right] \arrow[r, "\iota_k"] & \left[s^{-(k+1)}, e\right] \arrow[r] & \cdots & G^-
\end{tikzcd}$$

Figure 2.2: Construction of the isomorphism $\psi : \varinjlim L_k \to G^-$.

If we define $\varphi_k := \gamma_k \Pi_k$ and $f_k := g_k P_k$, then f_k is a φ_k-semilinear surjection with $\iota_k \psi_k = \psi_{k+1} f_k^*$. The embeddings $f_k : L_k \to L_{k+1}$ thus produce a direct system of lattices with

$$\varinjlim L_k \cong \varinjlim \left[s^{-k}, e\right] \cong G^-.$$

Here, the isomorphism $\psi : \varinjlim L_k \to G^-$ is provided by the maps ψ_k as is illustrated in Figure 2.2.

Since γ_k is an isomorphism and Π_k a projection, $\varphi_k : R_{k+1} \to R_k$ is surjective. It follows from Proposition 2.5.8, that $R := \varprojlim R_k$ is a complete dvr. Furthermore,

$$M := \varprojlim R_k^\delta = \left\{ (m_k)_{k\geq 1} \in \prod_{k=1}^{\infty} R_k^\delta \ : \ \forall k \geq 1 : f_k(m_{k+1}) = m_k \right\}$$

is naturally a left R-module with scalar multiplication given by $(r_k)_{k\geq 1} \cdot (m_k)_{k\geq 1} = (r_k m_k)_{k\geq 1}$.

We now show that ${}_R M \cong {}_R R^\delta$ (this is not completely trivial, since the connecting maps between the different R_k^δ are not given a priori by reduction of coefficients). There are clearly elements $x^1, \dots, x^\delta \in R_1^\delta$ which form a basis of R_1^δ (recall that a cpu ring of length 1 is a field!). We now choose $m^1, \dots, m^\delta \in M$ with $m_1^i = x^i$ $(1 \leq i \leq \delta)$. Our aim is now to show that $m^1, \dots, m^\delta$ form a basis of ${}_R M$.

Fix some integer $k \geq 1$. We clearly have $\ker(f_{k,1}) = \pi_k R_k^\delta = \mathrm{Rad}(R_k^\delta)$, so $f_{k,1} : R_k^\delta \to R_1^\delta$ provides an isomorphism of R_k-modules $\overline{f}_{k,1} : R_k^\delta / \mathrm{Rad}(R_k^\delta) \xrightarrow{\sim} R_1^\delta$. Here, we see R_1^δ as an R_k-module via restriction of scalars by $\varphi_{k,1} : R_k \to R_1$. Since the elements $f_{k,1}(m_k^i) = m_1^i$ $(1 \leq i \leq \delta)$ generate R_1^δ, the elements m_k^i $(1 \leq i \leq \delta)$ generate R_k^δ, by Nakayama's lemma. The module R_k^δ has composition length $\delta \cdot k$, so the m_k^i $(1 \leq i \leq \delta)$ must form a basis of R_k^δ for any fixed k.

Given any element $m = (m_k)_{k\geq 1} \in M$, we can therefore find for each $k \geq 1$ unique elements $r_k^i \in R_k$ $(1 \leq i \leq \delta)$ such that $m_k = \sum_{i=1}^{\delta} r_k^i m_k^i$. For each pair of integers $k \geq l \geq 1$, we then have

$$\begin{aligned} m_l &= f_{k,l} m_k \\ \Rightarrow \sum_{i=1}^{\delta} r_l^i m_l^i &= f_{k,l}\left(\sum_{i=1}^{\delta} r_k^i m_k^i\right) \\ &= \sum_{i=1}^{\delta} \varphi_{k,l}(r_k^i) f_{k,l}(m_k^i) \\ &= \sum_{i=1}^{\delta} \varphi_{k,l}(r_k^i) m_k^l. \end{aligned}$$

This proves that $\varphi_{k,l}(r_k^i) = r_l^i$. Therefore, for each $1 \leq i \leq \delta$, setting $r^i := (r_k^i)_{k \geq 1}$ defines an element of R. These $r^i \in R$ are the unique elements in R with the property that $\sum_{i=1}^{\delta} r^i m^i = m$. Since this shows that every element in M is uniquely expressible as an R-linear combination of the elements $m_1, \ldots, m_\delta$, we have ${}_RM \cong {}_RR^\delta$.

Let $\pi \in R$ be a uniformizer. By restriction of scalars, we can make for $k \geq 1$ the identification $L_k = L({}_RM_k)$ where $M_k := M/\pi^k M$. We have seen that ${}_RM \cong {}_RR^\delta$. From Proposition 2.5.13, we can therefore conclude that $G^- \cong \varinjlim L_k \cong \mathrm{Lat}({}_RR^\delta)$. $\square$

This result also allows us to describe the structure of G as a lattice:

Theorem 2.5.16. *Let Q be a complete dvf with dvr R and let G be a desarguesian right ℓ-group of dimension $\delta \geq 1$. Any lattice isomorphism $\varphi^- : G \xrightarrow{\sim} \mathrm{Lat}({}_RR^\delta)$ from Theorem 2.5.15 can be uniquely extended to an isomorphism of lattices $\varphi : G \xrightarrow{\sim} \mathrm{Lat}({}_RQ^\delta)$.*

In particular, if G is a desarguesian right ℓ-group of dimension $\delta \geq 4$, then there is a complete dvf Q with dvr R such that $G \cong \mathrm{Lat}({}_RQ^\delta)$.

Proof. Let an isomorphism $\varphi^- : G^- \to \mathrm{Lat}({}_RR^\delta)$ as in the first statement be given.

In G, we have, by Proposition 2.2.7, $\mathrm{Soc}(g) = sg$ and $\mathrm{Rad}(g) = s^{-1}g$ for every $g \in G$. Let $\pi \in R$ be a uniformizer, then in $\mathrm{Lat}({}_RQ^\delta)$ we similarly have, by Proposition 2.5.14, that $\mathrm{Rad}(A) = \pi A$ and $\mathrm{Soc}(A) = \pi^{-1}A$ for each $A \in \mathrm{Lat}({}_RQ^\delta)$.

For any $g \in G^-$ and any integer $k \geq 0$, we want to define $\varphi(s^k g) := \pi^{-k}\varphi^-(g)$. This is the only possibility to extend φ^-, since for any $g \in G$ we must have $\varphi(sg) = \varphi(\mathrm{Soc}(g)) = \mathrm{Soc}(\varphi(g)) = \pi^{-1}\varphi(g)$ which implies $\varphi(s^k g) = \pi^{-k}\varphi(g)$ for every $k \geq 0$. Note that this also shows $\varphi^-(s^{-k}g) = \pi^k\varphi^-(g)$ for $g \in G^-$, $k \geq 0$.

We show that this map is indeed well-defined: Suppose that we have $s^k g = s^l g'$ where $g, g' \in G^-$, $k, l \geq 0$. We may assume that $l \geq k$. Then $g' = s^{-(l-k)}g$, therefore

$$\pi^{-l}\varphi^{-}(g') = \pi^{-l}\varphi^{-}(s^{-(l-k)}g) = \pi^{-l}\pi^{l-k}\varphi^{-}(g) = \pi^{-k}\varphi^{-}(g),$$

so our definition is indeed independent of the specific choice of k and g.

We show that φ is surjective: let $A \in \mathrm{Lat}({}_RQ^\delta)$. Then $A = \pi^{-k}B$ for some $B \in \mathrm{Lat}({}_RR^\delta)$ by Proposition 2.5.9. Take $g \in G^-$ with $\varphi^-(g) = B$, then $\varphi(s^k g) = \pi^{-k}\varphi^-(g) = \pi^{-k}B = A$.

If $l \geq k \geq 0$ and $g, g' \in G^-$, we have the equivalences:

$$\begin{aligned}\varphi(s^k g) \leq \varphi(s^l g') &\Leftrightarrow \pi^{-k}\varphi^-(g) \leq \pi^{-l}\varphi^-(g') \\ &\Leftrightarrow \varphi^-(g) \leq \pi^{-(l-k)}\varphi^-(g') = \varphi^-(s^{l-k}g') \\ &\Leftrightarrow g \leq s^{l-k}g' \Leftrightarrow s^k g \leq s^l g'.\end{aligned}$$

This shows that $\varphi : G \to \mathrm{Lat}({}_RQ^\delta)$ is an embedding. Taking surjectivity into account as well, we see that φ is an equivalence of ordered sets. It follows that φ is an isomorphism of lattices.

The second part is now an easy corollary from the first part and Theorem 2.5.15 which states that if G is desarguesian of dimension $\delta \geq 4$, then a suitable isomorphism $\varphi^- : G \to \mathrm{Lat}({}_RR^\delta)$ exists. □

We conclude:

Theorem 2.5.17. *Each desarguesian right ℓ-group of dimension $\delta \geq 4$ can be regarded as a regular group of lattice-automorphisms of* $\mathrm{Lat}({}_RQ^\delta)$*, where Q is some complete dvf and R is the corresponding dvr.*

On the other hand, any regular group G of automorphisms on $\mathrm{Lat}({}_RQ^\delta)$ *- where R and Q are as above, and $\delta \geq 1$ - is a desarguesian right ℓ-group of dimension δ under the right-invariant order with the negative cone*

$$G^- = \{g \in G : (R^\delta)^g \subseteq R^\delta\}.$$

Proof. Theorem 2.5.16 tells us that if G is a desarguesian right ℓ-group of dimension $\delta \geq 4$, then there is a complete dvr R with respective dvf Q such that there is an isomorphism of lattices $\varphi : G \xrightarrow{\sim} \mathrm{Lat}({}_RQ^\delta)$. Since G acts regularly on itself by lattice automorphisms via right-multiplication, G can be regarded as a regular group of automorphisms of $\mathrm{Lat}({}_RQ^\delta)$, with the right action given by

$$\begin{aligned}\mathrm{Lat}({}_RQ^\delta) \times G &\to \mathrm{Lat}({}_RQ^\delta) \\ (A, g) &\mapsto A^g = \varphi(\varphi^{-1}(A) \cdot g) \\ &\quad \left(\Leftrightarrow \varphi^{-1}(A^g) = \varphi^{-1}(A) \cdot g\right).\end{aligned}$$

From the last formulation it becomes clear that $\mathrm{Lat}({}_RQ^\delta)$ and G are isomorphic G-sets via the isomorphism φ. Since each group acts regularly on itself by right-multiplication, the action on $\mathrm{Lat}({}_RQ^\delta)$ is regular as well.

On the other hand, if G is a regular group of automorphisms of some $\mathrm{Lat}({}_RQ^\delta)$, then the map φ defined as

$$\begin{aligned}\varphi : G &\to \mathrm{Lat}({}_RQ^\delta)\\ g &\mapsto (R^\delta)^g\end{aligned}$$

is a bijection (note that we regard R^δ as an element of $\mathrm{Lat}({}_RQ^\delta)$ in this definition). Defining a lattice structure on G via

$$\begin{aligned}x \vee y &:= \varphi^{-1}(\varphi(x) + \varphi(y)) \Leftrightarrow (R^\delta)^{x\vee y} = (R^\delta)^x + (R^\delta)^y\\ x \wedge y &:= \varphi^{-1}(\varphi(x) \cap \varphi(y)) \Leftrightarrow (R^\delta)^{x\wedge y} = (R^\delta)^x \cap (R^\delta)^y\end{aligned}$$

It is clear that G and $\mathrm{Lat}({}_RQ^\delta)$ are isomorphic lattices via the isomorphism φ. Also, G becomes a right ℓ-group under these lattice operations: for all $x, y, z \in G$,

$$\begin{aligned}(R^\delta)^{(x\vee y)z} &= ((R^\delta)^{x\vee y})^z\\ &= ((R^\delta)^x + (R^\delta)^y)^z\\ &= (R^\delta)^{xz} + (R^\delta)^{yz}\\ &= (R^\delta)^{xz\vee yz},\end{aligned}$$

so $(x \vee y)z = xz \vee yz$ follows. $(x \wedge y)z = xz \wedge yz$ is proven similarly.

We also have the equivalences $g \in G^- \Leftrightarrow g \vee e = e \Leftrightarrow (R^\delta)^g + (R^\delta) = R^\delta \Leftrightarrow (R^\delta)^g \subseteq R^\delta$.

Furthermore, G fulfills the ascending and descending chain conditions since every interval $[A, B] \subseteq \mathrm{Lat}({}_RQ^\delta)$ has finite length. This follows from Proposition 2.5.9 and the fact that for $n \geq m$, the R-module $(\pi^m R^\delta)/(\pi^n R^\delta)$ has finite composition length.

It remains to show that $[\mathrm{Rad}(e), e]$ is desarguesian. By Proposition 2.5.14, we get the following isomorphisms of lattices:

$$[\mathrm{Rad}(e), e] \cong [\mathrm{Rad}(R^\delta), R^\delta] \cong L\left({}_R(R^\delta/(\pi R^\delta))\right) \cong L\left({}_{R_1}R_1^\delta\right).$$

Since $R_1 = R/\mathfrak{m}$ is a field, the statement follows. □

Example 2.5.18. One can prove that $\mathrm{PPU}(b)$ is isomorphic, as a lattice, to the lattice

$$\begin{aligned}\mathcal{L} := &\left\{B \in L({}_{K[t^{-1}]}K\left[t, t^{-1}\right]^n) : \exists k, l \in \mathbb{Z} : t^k \cdot K\left[t^{-1}\right]^n \subseteq B \subseteq t^l \cdot K\left[t^{-1}\right]^n\right\}\\ &\subseteq L({}_{K[t^{-1}]}K\left[t, t^{-1}\right]^n)\end{aligned}$$

([Die19, Lemma 11]). On this lattice, $\mathrm{PPU}(b)$ acts regularly from the right via $B^M = M(t^{-1})^{-1}B$ $(B \in \mathcal{L}, M \in \mathrm{PPU}(b))$.

Since $K\left[t^{-1}\right]$ is not a complete dvr yet, we complete it to $R := K[[t^{-1}]]$, which is one. It can be shown that, by tensoring with R, one gets a lattice isomorphism

$$\begin{aligned} \mathcal{L} &\to \mathrm{Lat}({}_RQ^n) \\ B &\mapsto R \otimes_{K\left[t^{-1}\right]} B \subseteq Q^n, \end{aligned}$$

where Q is the quotient field of R. Note here that $Q = R \otimes_{K\left[t^{-1}\right]} K\left[t, t^{-1}\right]$ follows from the fact that $K\left[t, t^{-1}\right]$ and Q are localizations of $K\left[t^{-1}\right]$ resp. R at the same multiplicative subset, which is the one generated by t^{-1}. To get an idea how the lattice isomorphism is proved, take a look at Appendix A.

Actually, the isomorphism of $\mathrm{PPU}(b)$ and $\mathcal{L}$ is the central idea of the proof that $\mathrm{PPU}(b)$ is a modular lattice or a lattice at all. So one could say that, until now, we told the story about $\mathrm{PPU}(b)$ backwards. But since this is a legitimate way to tell a story ([Lee00]), we chose to do so.

2.6 A projective representation

Theorem 2.5.17 tells us that each desarguesian right ℓ-group of dimension $\delta \geq 4$ has a projective representation in a lattice-theoretic sense. The aim of this section is to describe this in module-theoretic terms.

We first determine the automorphism group of the lattice $\mathrm{Lat}({}_RR^\delta)$ in case that R is a complete dvr and $\delta \geq 3$.

Proposition 2.6.1. *Let R be a complete dvr, $\pi \in R$ a uniformizer thereof and $\delta \geq 3$. Then for every lattice automorphism $\varphi \in \mathrm{Aut}(\mathrm{Lat}({}_RR^\delta))$ there is an $f \in \Gamma\mathrm{L}({}_RR^\delta)$ such that $\varphi = f_*$. This f is unique up to multiplication by an element of $R^\times$. In particular, $\mathrm{Aut}(\mathrm{Lat}({}_RR^\delta)) \cong \mathrm{P}\Gamma\mathrm{L}({}_RR^\delta)$.*

Proof. φ must pointwisely fix the terms of the radical series $\mathrm{Rad}^k({}_RR^\delta) = \pi^k R^\delta$ (Proposition 2.5.14), so for each $k \geq 1$, φ restricts to an automorphism φ_k of $\left[\pi^k R^\delta, R^\delta\right] \cong L({}_R(R^\delta/\pi^k R^\delta)) \cong L({}_{R_k}R_k^\delta)$. Since furthermore $\mathrm{Lat}({}_RR^\delta) \cong \varinjlim L({}_{R_k}R_k^\delta)$ (Proposition 2.5.13), we have an isomorphism

$$\mathrm{Aut}(\mathrm{Lat}({}_RR^\delta)) \cong \varprojlim \mathrm{Aut}(L({}_{R_k}R_k^\delta)).$$

By Proposition 1.3.8, and Proposition 1.3.4, we have for each $k \geq 1$ an exact sequence

$$1 \to R_k^\times \xrightarrow{\iota_k} \Gamma\mathrm{L}({}_{R_k}R_k^\delta) \xrightarrow{\gamma_k} \mathrm{Aut}(L({}_{R_k}R_k^\delta)) \to 1. \tag{2.12}$$

Using Proposition 2.5.14, one sees that for all $f \in \Gamma\mathrm{L}({}_RR^\delta)$ and all $k \geq 1$,

$$f(\pi^k R^\delta) = f_*(\mathrm{Soc}^k(R^\delta)) = \mathrm{Soc}^k(f_*(R^\delta)) = \pi^k R^\delta.$$

Therefore $f : R^\delta \to R^\delta$ descends to a semilinear isomorphism

$$\begin{aligned} f_k : R_k^\delta &\to R_k^\delta \\ x + \pi^k R^\delta &\mapsto f(x) + \pi^k R^\delta. \end{aligned}$$

It is clear from the construction that $(f_k)_{k\geq 1} \in \varprojlim \Gamma\mathrm{L}({}_{R_k}R_k^\delta)$. Vice versa, an easy calculation[8] shows that each element of $\varprojlim \Gamma\mathrm{L}({}_{R_k}R_k^\delta)$ defines an element of $\Gamma\mathrm{L}({}_R R^\delta)$; together, both arguments prove that we have an isomorphism

$$\begin{aligned} g : \Gamma\mathrm{L}({}_R R^\delta) &\to \varprojlim \Gamma\mathrm{L}({}_{R_k}R_k^\delta) \\ f &\mapsto (g_k(f))_{k\geq 1} \end{aligned}$$

where $g_k(f) := f_k$, as defined above.

Let integers $l \geq k \geq 1$ be given. With the maps

$$\begin{aligned} g_{l,k} : \Gamma\mathrm{L}({}_{R_l}R_l^\delta) &\to \Gamma\mathrm{L}({}_{R_k}R_k^\delta) \\ (g_{l,k}(f))\,(x + \pi^l R^\delta) &= f(x) + \pi^k R^\delta \end{aligned}$$

we have

$$f_*|_{L({}_{R_k}R_k^\delta)} = (g_{l,k}(f))_*$$

for all $f \in \Gamma\mathrm{L}({}_{R_l}R_l^\delta)$. We are therefore allowed to take the limit over the homomorphisms γ_k and get a homomorphism $\gamma : \Gamma\mathrm{L}({}_R R^\delta) \to \mathrm{Aut}(\mathrm{Lat}({}_R R^\delta))$.

The homomorphisms $g_{l,k} : \Gamma\mathrm{L}({}_{R_l}R_l^\delta) \to \Gamma\mathrm{L}({}_{R_k}R_k^\delta)$ restrict to homomorphisms $R_l^\times \to R_k^\times$ which are reduction modulo π^k. Under these maps, we have $\varprojlim R_k^\times \cong R^\times$.

In order to show that each element of $\mathrm{Aut}(\mathrm{Lat}({}_R R^\delta))$ is given by a semilinear isomorphism, we must only show that the limit over the exact sequences (2.12) again is an exact sequence, which by our preceding arguments is

$$1 \to R^\times \xrightarrow{\iota} \Gamma\mathrm{L}({}_R R^\delta) \xrightarrow{\gamma} \mathrm{Aut}(\mathrm{Lat}({}_R R^\delta)) \to 1.$$

This follows from the fact that the maps $R_l^\times \to R_k^\times$ are all surjective and therefore form a *Mittag-Leffler system* [MS82, Chapter II,6.2]. The same exact sequence proves that two elements of $\Gamma\mathrm{L}({}_R R^\delta)$ define the same automorphism of $\mathrm{Lat}({}_R R^\delta)$ if and only if they are equal up to multiplication by an element of $R^\times$. $\square$

Before we can determine $\mathrm{Aut}(\mathrm{Lat}({}_R Q^\delta))$, we need to locate $\mathrm{Aut}(\mathrm{Lat}({}_R R^\delta))$ in it. We start by first locating $\Gamma\mathrm{L}({}_R R^\delta)$ in $\Gamma\mathrm{L}({}_R Q^\delta)$:

[8] or taking an inverse limit over suitable isomorphisms $\Gamma\mathrm{L}({}_{R_k}R_k^\delta) \cong \mathrm{GL}({}_{R_k}R_k^\delta) \rtimes \mathrm{Aut}(R_k)$, as provided by Proposition 1.3.3

Proposition 2.6.2. *Let Q be a dvf with dvr R and $\delta \geq 1$. Let $f : {}_R R^\delta \to {}_R R^\delta$ be a semilinear isomorphism. Then there exists a unique semilinear isomorphism $\overline{f} : {}_R Q^\delta \to {}_R Q^\delta$ such that $\overline{f}|_{R^\delta} = f$.*

On the other hand, each semilinear isomorphism $f : {}_R Q^\delta \to {}_R Q^\delta$ with $f(R^\delta) = R^\delta$ restricts to a semilinear automorphism of ${}_R R^\delta$.

In particular, we can regard $\Gamma\mathrm{L}({}_R R^\delta)$ as the subgroup of all $f \in \Gamma\mathrm{L}({}_R Q^\delta)$ with $f(R^\delta) = R^\delta$.

For the proof we need an easy lemma:

Lemma 2.6.3. *Let Q be a dvf with dvr R and let $\alpha : R \to R$ be a ring automorphism. Then there exists a unique field automorphism $\overline{\alpha} : Q \to Q$ with $\overline{\alpha}|_R = \alpha$.*

Proof. The multiplicative monoid $R \setminus \{0\}$, being a submonoid of the group $Q^\times$, is clearly left-cancellative. Also, for each $r \in R \setminus \{0\}$, there are $s \in R \setminus \{0\}$ and $k \geq 0$ such that $sr = \pi^k$ where π is some uniformizer. From this one easily deduces that for any $r_1, r_2 \in R \setminus \{0\}$, there are $s_1, s_2 \in R \setminus \{0\}$ such that $s_1 r_1 = s_2 r_2$, i.e. $R \setminus \{0\}$ is a left Ore monoid.

Also, $Q^\times$ is a group of left fractions for $R \setminus \{0\}$: if $q \in Q^\times$ fulfills $v(q) := -k$ with $k > 0$, then we can write $q = (\pi^k)^{-1}(\pi^k q)$. Here, we have $\pi^k q \in R \setminus \{0\}$, since $v(\pi^k q) = k - k = 0$. If $v(q) \geq 0$ we have $q \in R \setminus \{0\}$ so we can simply write $q = 1^{-1} q$.

By Lemma 2.1.9, the automorphism $\alpha : R \setminus \{0\} \to R \setminus \{0\}$ extends uniquely to a multiplicative isomorphism $\overline{\alpha} : Q^\times \to Q^\times$ which is given by $\overline{\alpha}(r^{-1}s) := \alpha(r)^{-1}\alpha(s)$ for $r, s \in R \setminus \{0\}$. Note that this expression is also valid for $s = 0$, so we define

$$\begin{aligned} &\overline{\alpha} : Q \to Q \\ &r^{-1}s \mapsto \alpha(r)^{-1}\alpha(s) \quad (\text{for } r \in R \setminus \{0\}, s \in R). \end{aligned}$$

This still defines a multiplicative isomorphism, which can be seen by taking into account the cases where $s = 0$.

The map $\overline{\alpha}$ is also an additive isomorphism: let $q_1, q_2 \in Q$, then there is an integer $k \geq 0$ and $r_1, r_2 \in R$ such that $q_1 = \pi^{-k} r_1$, $q_2 = \pi^{-k} r_2$ where π is an uniformizer. It follows that

$$\begin{aligned} \overline{\alpha}(q_1 + q_2) &= \overline{\alpha}(\pi^{-k}(r_1 + r_2)) \\ &= \alpha(\pi^k)^{-1}\alpha(r_1 + r_2) \\ &= \alpha(\pi^k)^{-1}\alpha(r_1) + \alpha(\pi^k)^{-1}\alpha(r_2) \\ &= \overline{\alpha}(\pi^{-k} r_1) + \overline{\alpha}(\pi^{-k} r_2) \\ &= \overline{\alpha}(q_1) + \overline{\alpha}(q_2). \end{aligned}$$

$\square$

Proof of Proposition 2.6.2. The second part of the proposition is clear; it is also clear how the first and the second together imply the third, so we only prove the first.

Assume that $f : {}_RR^\delta \to {}_RR^\delta$ is α-semilinear. Let $\overline{\alpha} : Q \to Q$ be the extension of α which is guaranteed to exist by Lemma 2.6.3. Let $e_1, \ldots, e_\delta$ be the canonical basis of ${}_RR^\delta$, then f can be written as

$$f\left(\sum_{i=1}^{\delta} r_i e_i\right) = \sum_{i=1}^{\delta} \alpha(r_i) f(e_i).$$

with all $r_i \in R$.

Under inclusion, $e_1, \ldots, e_\delta$ also form a Q-basis for ${}_QQ^\delta$. We can therefore define

$$\overline{f} : Q^\delta \to Q^\delta$$
$$\sum_{i=1}^{\delta} k_i e_i \mapsto \sum_{i=1}^{\delta} \overline{\alpha}(k_i) f(e_i)$$

where all $k_i \in Q$. This mapping is an α-semilinear self-map of ${}_RQ^\delta$ with $\overline{f}|_{R^\delta} = f$. There is no other possibility to define such a map: given $k = r^{-1}s \in Q^\times$, where $r, s \in R \setminus \{0\}$, we must have for each $1 \le i \le \delta$, that

$$\alpha(s) f(e_i) = f(se_i) = \overline{f}(rke_i) = \alpha(r)\overline{f}(ke_i)$$
$$\Leftrightarrow \overline{f}(ke_i) = \alpha(r)^{-1}\alpha(s) f(e_i) = \overline{\alpha}(k) f(e_i).$$

By additivity, one can then easily see that $\overline{f}$ is also unique on sums $\sum_{i=1}^{\delta} k_i e_i$.

Furthermore, $\overline{f}$ is an α-semilinear automorphism of ${}_RQ^\delta$, since $\overline{f}$ can be regarded as an $\overline{\alpha}$-semilinear automorphism of ${}_QQ^\delta$. The latter can be seen as follows: the elements $f(e_i)$ $(1 \le i \le \delta)$ form a basis of ${}_RR^\delta$, so they also form a Q-basis of ${}_QQ^\delta$ by inclusion. Therefore, there is a unique inverse f^{-1} which is the unique $\overline{\alpha}^{-1}$-semilinear map which maps $f(e_i) \mapsto e_i$ $(1 \le i \le \delta)$. Clearly, this inverse is also an α-semilinear isomorphism of ${}_RQ^\delta$. □

In a very similar way, $\mathrm{GL}({}_RR^\delta)$ lies in $\mathrm{GL}({}_RQ^\delta)$:

Proposition 2.6.4. *Let Q be a dvf with dvr R and $\delta \ge 1$. For each $\varphi \in \mathrm{Aut}(\mathrm{Lat}({}_RR^\delta))$, there is a unique $\overline{\varphi} \in \mathrm{Aut}(\mathrm{Lat}({}_RQ^\delta))$ such that $\overline{\varphi}|_{\mathrm{Lat}({}_RR^\delta)} = \varphi$.*

On the other hand, for each $\psi \in \mathrm{Aut}(\mathrm{Lat}({}_RQ^\delta))$ with $\psi(R^\delta) = R^\delta$, we have $\psi|_{\mathrm{Lat}({}_RR^\delta)} \in \mathrm{Aut}(\mathrm{Lat}({}_RR^\delta))$.

In particular, we can identify $\mathrm{GL}({}_RR^\delta) = \mathrm{Stab}\ (R^\delta) \subseteq \mathrm{Aut}(\mathrm{Lat}({}_RQ^\delta))$.

Proof. Existence: Let $\varphi \in \mathrm{Aut}(\mathrm{Lat}({}_RR^\delta))$. By Proposition 2.6.1, there is a semilinear isomorphism $f \in \mathrm{P\Gamma L}({}_RR^\delta)$ such that $f_* = \varphi$. Let $\overline{f} : {}_RQ^\delta \to {}_RQ^\delta$

be its unique extension which exists by Proposition 2.6.2. Then $\overline{\varphi} := \overline{f}_* \in \mathrm{Aut}(\mathrm{Lat}({}_RQ^\delta))$ and also $\overline{\varphi}|_{\mathrm{Lat}({}_RR^\delta)} = f_* = \varphi$.

Uniqueness: Let $\overline{\varphi}$ be an extension of φ to $\mathrm{Lat}({}_RQ^\delta)$. Using Proposition 2.5.14, we see that for any $B \in \mathrm{Lat}({}_RQ^\delta)$ and $k \geq 0$, we have $\mathrm{Soc}^k(B) = \pi^{-k}B$, where π is some uniformizer of R. Since every $A \in \mathrm{Lat}({}_RQ^\delta)$ is contained in $\pi^{-k}R^\delta$ for some $k \geq 0$ (Proposition 2.5.9), we can write $A = \pi^{-k}B$ with $B \in \mathrm{Lat}({}_RR^\delta)$. We then have

$$\overline{\varphi}(A) = \overline{\varphi}(\mathrm{Soc}^k(B)) = \mathrm{Soc}^k(\overline{\varphi}(B)) = \mathrm{Soc}^k(\varphi(B)),$$

so there can be at most one extension $\overline{\varphi}$.

On the other hand it is clear that each automorphism of $\mathrm{Lat}({}_RQ^\delta)$, which fixes R^δ, restricts to an automorphism of the sublattice $(R^\delta)^\downarrow = \mathrm{Lat}({}_RR^\delta)$. ☐

The following proposition is now an easy consequence:

Proposition 2.6.5. *Let R be a complete dvr for the dvf Q and $\delta \geq 3$. Then for every lattice automorphism $\varphi \in \mathrm{Aut}(\mathrm{Lat}({}_RQ^\delta))$, there is an $h \in \Gamma\mathrm{L}({}_RQ^\delta)$ such that $\varphi = h_*$. This h is unique up to multiplication by an element of $R^\times$. In particular,* $\mathrm{Aut}(\mathrm{Lat}({}_RQ^\delta)) \cong \mathrm{P}\Gamma\mathrm{L}({}_RQ^\delta)$.

Proof. Let $\varphi \in \mathrm{Aut}(\mathrm{Lat}({}_RQ^\delta))$. Set $A := \varphi(R^\delta)$. Since A is an R-lattice in Q^δ and R is a left principal ideal domain, there is some R-basis $a_1, \ldots, a_\delta$ of A. The unique Q-linear map $g : Q^\delta \to Q^\delta$ defined by $g(e_i) = a_i$ $(1 \leq i \leq \delta)$ is R-linear, invertible and fulfills $g(R^\delta) = A$.

Thus, $(g_*^{-1}\varphi)(R^\delta) = R^\delta$ and $g_*^{-1}\varphi$ restricts to an automorphism of $\mathrm{Lat}({}_RR^\delta)$. By Proposition 2.6.1 there is a semilinear isomorphism $f : {}_RR^\delta \to {}_RR^\delta$ - unique up to multiplication by an element of $R^\times$ - such that

$$f_* = (g_*^{-1}\varphi)|_{\mathrm{Lat}({}_RR^\delta)}.$$

If the semilinear isomorphism $\overline{f} : {}_RQ^\delta \to {}_RQ^\delta$ is its extension (Proposition 2.6.2), then $\overline{f}_*$ also extends f_*. By the uniqueness part of Proposition 2.6.4, we have $\overline{f}_* = g_*^{-1}\varphi$. This shows that $\varphi = g_*\overline{f}_* = (g\overline{f})_*$. So we can take $h = g\overline{f}$, which is the unique choice, up to multiplication by a unit. ☐

We can now prove:

Theorem 2.6.6. *For each desarguesian right ℓ-group G of dimension $\delta \geq 4$ there is a complete dvf Q with dvr R such that there is a subgroup $H \leq \mathrm{P}\Gamma\mathrm{L}({}_RQ^\delta)$ with $H \cong G$ that is a* complement *of the subgroup $\mathrm{P}\Gamma\mathrm{L}({}_RR^\delta) \leq \mathrm{P}\Gamma\mathrm{L}({}_RQ^\delta)$, meaning that $H \cap \mathrm{P}\Gamma\mathrm{L}({}_RR^\delta) = 1$ and $H \cdot \mathrm{P}\Gamma\mathrm{L}({}_RR^\delta) = \mathrm{P}\Gamma\mathrm{L}({}_RQ^\delta)$.*

Vice versa, each complement H of the subgroup $\mathrm{P}\Gamma\mathrm{L}({}_RR^\delta) \leq \mathrm{P}\Gamma\mathrm{L}({}_RQ^\delta)$ is isomorphic to the underlying group of a desarguesian right ℓ-group.

Proof. We have already shown (Theorem 2.5.17) that for $\delta \geq 4$, the desarguesian right ℓ-groups of dimension δ are exactly the groups G which act regularly by lattice automorphisms of $\mathrm{Lat}({}_RQ^\delta)$. Such a right action is equivalent to a homomorphism[9] $\rho : G^{\mathrm{op}} \to \mathrm{Aut}(\mathrm{Lat}({}_RQ^\delta))$ that embeds G^{op} as a complement $H := \rho(G^{\mathrm{op}})$ of the subgroup $\mathrm{Stab}(R^\delta) \leq \mathrm{Aut}(\mathrm{Lat}({}_RQ^\delta))$.

However, we have $\mathrm{Aut}(\mathrm{Lat}({}_RQ^\delta)) \cong \mathrm{P\Gamma L}({}_RQ^\delta)$ (Proposition 2.6.5) and the lattice $R^\delta \in \mathrm{Lat}({}_RQ^\delta)$ is fixed by the subgroup $\mathrm{Stab}(R^\delta) = \mathrm{Aut}(\mathrm{Lat}({}_RR^\delta)) = \mathrm{P\Gamma L}({}_RR^\delta)$ (Proposition 2.6.1 and Proposition 2.6.4).

Since $G \cong G^{\mathrm{op}}$ for every group, the result follows. □

Let Q be a complete dvf with respective dvr R. Furthermore, let $\delta \geq 1$.

Given a complement H of the subgroup $\mathrm{P\Gamma L}({}_RR^\delta) \leq \mathrm{P\Gamma L}({}_RQ^\delta)$, we construct a regular right action on $\mathrm{Lat}({}_RQ^\delta)$ via

$$\begin{aligned}\mathrm{Lat}({}_RQ^\delta) \times H &\to \mathrm{Lat}({}_RQ^\delta)\\ (A, f) &\mapsto A^f := f^{-1}(A).\end{aligned}$$

By Theorem 2.5.17 , this regular action defines on H the structure of a desarguesian right ℓ-group of dimension δ with the negative cone

$$H^- := \{f \in H : (R^\delta)^f \subseteq R^\delta\} = \{f \in H : R^\delta \subseteq f(R^\delta)\}.$$

Theorem 2.6.6 tells us that each desarguesian right ℓ-group of dimension $\delta \geq 4$ can be constructed in this way.

However, for the construction of a desarguesian right ℓ-group, it often suffices to start with a slightly smaller - but still transitive! - group of lattice automorphisms of $\mathrm{Lat}({}_RQ^\delta)$, e.g. $\mathrm{GL}({}_QQ^\delta)/R^\times$. Each regular subgroup of such a group is then a desarguesian right ℓ-group of dimension δ.

The desarguesian right ℓ-groups $\mathrm{PPU}(b)$ can already be constructed within the groups $\mathrm{GL}({}_{K[t,t^{-1}]}K\left[t,t^{-1}\right]^n)$. Also, in the Appendix, we will construct a further series of desarguesian right ℓ-groups within the left quotient fields of skew polynomial rings.

It is not a coincidence that we could construct two large families of desarguesian right ℓ-groups by ring-theoretic methods. We have proven that, in fact, these constructions point to a general phenomenon which can be summarized by the following statement:

Most desarguesian right ℓ-groups are of a ring-theoretic nature!

[9] Note that our automorphism groups act from the left, therefore a right action of G translates into a homomorphism with domain G^{op}.

Example 2.6.7. We close with a few words on the example PPU(b). Recall that we have defined a lattice $\mathcal{L}$ as

$$\begin{aligned}\mathcal{L} = &\left\{B \in L({}_{K\left[t^{-1}\right]}K\left[t,t^{-1}\right]^n) : \exists k,l \in \mathbb{Z} : t^k \cdot K\left[t^{-1}\right]^n \subseteq B \subseteq t^l \cdot K\left[t^{-1}\right]^n\right\}\\ &\subseteq L({}_{K\left[t^{-1}\right]}K\left[t,t^{-1}\right]^n).\end{aligned}$$

Under the transitive right action

$$\begin{aligned}\mathcal{L} \times \mathrm{PPU}(b) &\to \mathcal{L}\\ (B,M) &\mapsto B^M := M(t^{-1})^{-1}B,\end{aligned}$$

the group PPU(b) acts regularly on $\mathcal{L}$.

Let $R = K[[t^{-1}]]$ be the negative Laurent ring and Q its quotient field. Since we have canonical inclusions $K\left[t^{-1}\right] \hookrightarrow R$ and $K\left[t,t^{-1}\right] \hookrightarrow Q$, we can regard PPU$(b)$ as a subgroup of $\mathrm{GL}({}_QQ^n)$.

Recall from the last section that $\mathcal{L} \cong \mathrm{Lat}({}_RQ^n)$ via $B \mapsto R \otimes_{k\left[t^{-1}\right]} B$. The group $\mathrm{GL}({}_QQ^n)$ acts from the right on $\mathrm{Lat}({}_RQ^n)$ via

$$\begin{aligned}\mathrm{Lat}({}_RQ^n) \times \mathrm{GL}({}_QQ^n) &\to \mathrm{Lat}({}_RQ^n)\\ (B,M) &\mapsto B^M = M^{-1}B.\end{aligned}$$

Under this action we have the stabilizer

$$\mathrm{Stab}(R^n) = \mathrm{GL}({}_RR^n).$$

By the inclusion $\mathrm{GL}({}_{K\left[t,t^{-1}\right]}K\left[t,t^{-1}\right]^n) \subseteq \mathrm{GL}({}_QQ^n)$, the group PPU$(b)$ can also be regarded as a subgroup of $\mathrm{GL}({}_QQ^n)$. Since the isomorphism $\mathcal{L} \cong \mathrm{Lat}({}_RQ^n)$ comes from the same scalar extension $K\left[t,t^{-1}\right] \subseteq R$, the group PPU$(b)$ also acts regularly on $\mathrm{Lat}({}_RQ^n)$, too. We therefore get the decomposition

$$\mathrm{GL}({}_QQ^n) = \mathrm{PPU}(b) \cdot \mathrm{GL}({}_RR^n)\ ;\ \mathrm{PPU}(b) \cap \mathrm{GL}({}_RR^n) = 1.$$

If we map PPU(b) one step further into $\mathrm{P\Gamma L}({}_RQ^n)$, which acts on $\mathrm{Lat}({}_RQ^n)$ with stabilizer

$$\mathrm{Stab}(R^n) = \mathrm{P\Gamma L}({}_RR^n),$$

we finally get the decomposition predicted by Theorem 2.6.6, namely

$$\mathrm{P\Gamma L}({}_RQ^n) = \mathrm{PPU}(b) \cdot \mathrm{P\Gamma L}({}_RR^n)\ ;\ \mathrm{PPU}(b) \cap \mathrm{P\Gamma L}({}_RR^n) = 1.$$

Chapter 3

The geometric action

Rump's theorem (Theorem 2.2.3) tells us that if G is a modular noetherian right ℓ-group where $s^{-1} = \mathrm{Rad}(e)$ exists, then the strong order interval $[s^{-1}, e]$ is a dually geometric modular lattice. By this theorem, we know part of the local lattice structure in these right ℓ-groups. We may ask if there is a *global* counterpart to Rump's theorem in the sense that a (dually) geometric lattice can be associated with G that captures part of the global structure in G.

By Theorem 2.5.16 we know that if G is desarguesian right ℓ-group of dimension $\delta \geq 4$, then there is a lattice isomorphism $G \cong \mathrm{Lat}({}_RQ^\delta)$ where Q is some complete dvf and R is the respective dvr. In this case, one might associate with G the desarguesian lattice $L({}_QQ^\delta)$ - however, this choice appears artificial and there does not seem to be an immediate connection to the lattice structure in G. One further drawback is that this choice is only possible when G admits a coordinatization in terms of some lattice $\mathrm{Lat}({}_RQ^\delta)$ - which is not guaranteed when G is a general modular geometric right ℓ-group with strong order unit.

In Section 3.1, we show: *If G admits a such a coordinatization, then it is possible to construct $L({}_QQ^\delta)$ without regard to the coordinatization.* This is done as follows:

The central idea is that for a complete dvr R, the lattice $L({}_RR^\delta)$ can be reconstructed from the sublattice $\mathrm{Lat}({}_RR^\delta)$: there is an order-reversing one-to-one correspondence between filters in $\mathrm{Lat}({}_RR^\delta)$ and submodules of ${}_RR^\delta$ which is established by mapping a filter to the intersection of its elements (Proposition 3.1.4). Given a coordinatization $G^- \cong \mathrm{Lat}({}_RR^\delta)$, the lattice of filters $\mathrm{Fil}(G^-)$ can therefore be identified with the lattice $L({}_RR^\delta)$.

The map $L({}_QQ^\delta) \to L({}_RR^\delta)$; $U \mapsto U \cap R^\delta$ is an order-embedding whose image is the subposet of *saturated* submodules of ${}_RR^\delta$ (Proposition 3.1.1). Here, an R-submodule $A \subseteq {}_RR^\delta$ is called saturated when R^δ/A is torsionfree.

Under the lattice isomorphism $L({}_RR^\delta) \cong \mathrm{Fil}(G^-)$, there is therefore an order-reversing embedding $L({}_QQ^\delta) \to \mathrm{Fil}(G^-)$. Calling the images of this embedding the *saturated filters*, we give combinatorial conditions for a filter in G^- to be saturated. In the following section, these conditions will be generalized in order to give a definition of saturated filters in an arbitrary modular geometric right ℓ-group.

The group $\mathrm{P\Gamma L}({}_RQ^\delta)$ can be shown to act by automorphisms of $L({}_QQ^\delta)$. By means of the representation $\rho : G \to \mathrm{P\Gamma L}({}_RQ^\delta)$ obtained in Theorem 2.6.6, this leads to a group action of G by lattice automorphisms of $L({}_QQ^\delta)$. Calling this action the *geometric action* of G, we realize the geometric action - also without regard to the coordinatization - as an action on $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ (Theorem 3.1.6).

In the following three sections, we will show that for *each* modular geometric right ℓ-group G, the poset of saturated filters in G^-, ordered by inclusion, is a modular geometric lattice. The strategy is as follows:

In Section 3.2, we look at filters in G^- where G is a modular geometric right ℓ-group. We define *infinite* right-normal factorizations for filters in G^- that specialize to the right-normal factorizations from Section 2.1 when applied to principal filters. In this way, several results from Section 2.3 carry over to filters. We will then focus on *saturated* filters, that is, filters whose infinite right-normal factors are of a constant degree.

We will prove that each saturated filter is a join of saturated filters of degree 1 (Proposition 3.2.15) which are the minimal saturated filters of nonzero degree. Furthermore, we will prove that the join (but not the meet) of saturated filters is again saturated (Theorem 3.2.16) and that each filter contains a unique maximal saturated filter (Proposition 3.2.18). Thus, $\mathrm{Fil}(G^-)_{\mathrm{sat}}$, the poset of saturated filters in G^-, is an atomistic lattice (Theorem 3.2.17).

In Section 3.3, we study the saturation map $\mathrm{sat} : \mathrm{Fil}(G^-) \to \mathrm{Fil}(G^-)_{\mathrm{sat}}$ and show that it is a lattice homomorphism (Proposition 3.3.2). As an epimorphic image of a modular lattice, $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is itself a modular lattice. Together with the results from Section 3.2, this proves that $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is indeed a modular geometric lattice (Theorem 3.3.4).

In Section 3.4, we finally construct the geometric action of G by lattice automorphisms of $\mathrm{Fil}(G^-)_{\mathrm{sat}}$.

In these three sections, we will make heavy use of the results on the mechanics of right- and left-normal factorizations obtained in Section 2.1 and Section 2.3.

We then turn to the case of distributive geometric right ℓ-groups - by theorems of Chouraqui [Cho10] and Rump [Rum15], these are exactly the structure groups of certain solutions to the set-theoretic Yang-Baxter equation. The lattice structure of these right ℓ-groups is rather uniform: if G is a distributive geometric right ℓ-group of dimension δ, then there is a lattice isomorphism $G \cong \mathbb{Z}^\delta$. Furthermore, with $s^{-1} := \mathrm{Rad}(e)$, the strong order interval $[s^{-1}, e]$ is a Boolean lattice with δ atoms.

In Section 3.5, we prove that in the distributive case, the lattice $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is also a Boolean lattice with δ atoms (Proposition 3.5.12) and that G acts on these atoms via the permutation action associated with a set-theoretic solution.

3.1 Motivation

One crucial result of Chapter 2 was Theorem 2.5.17 which told us that for each desarguesian right ℓ-group G of dimension $\delta \geq 4$ there is a regular right action

$$\begin{aligned} \mathrm{Lat}({}_RQ^\delta) \times G &\to \mathrm{Lat}({}_RQ^\delta) \\ (A, g) &\mapsto A^g, \end{aligned}$$

Q being a suitable dvf with respective dvr R, such that for each $g \in G$, the map $A \mapsto A^g$ is a lattice automorphism.

Fix an element $g \in G$. By Proposition 2.6.5, we can find an $f_g \in \Gamma\mathrm{L}({}_RQ^\delta)$, unique up to multiplication by an element of $R^\times$, such that $A^g = f_g(A)$ for all $A \in \mathrm{Lat}({}_RQ^\delta)$.

The map $f_g \in \Gamma\mathrm{L}({}_RQ^\delta)$ is α-semilinear for a ring automorphism $\alpha \in \mathrm{Aut}(R)$. Recall that $\alpha : R \to R$ can uniquely be extended to an automorphism $\overline{\alpha} : Q \to Q$ by setting $\overline{\alpha}(r^{-1}s) = \alpha(r)^{-1}\alpha(s)$ $(r, s \in R, r \neq 0)$ (Lemma 2.6.3).

Let us look at the map $f_g : Q^\delta \to Q^\delta$. Let $r, s \in R$ and $r \neq 0$, then for all $m \in R^\delta$:

$$\begin{aligned} \alpha(s) f_g(m) = f_g(s \cdot m) &= \alpha(r) \cdot f_g(r^{-1}s \cdot m) \\ \Rightarrow f_g(r^{-1}s \cdot m) &= \alpha(r)^{-1}\alpha(s) f_g(m) = \overline{\alpha} f_g(m). \end{aligned}$$

We see that $f_g : Q^\delta \to Q^\delta$ is automatically $\overline{\alpha}$-semilinear with respect to the Q-vector space structure of Q^δ. Thus, f_g defines a unique element $\overline{f}_g \in \Gamma\mathrm{L}({}_QQ^\delta)$. Remember that $\overline{f}_g$ is unique up to multiplication by an element of $R^\times$, and therefore defines a unique element of $\mathrm{P}\Gamma\mathrm{L}({}_QQ^\delta)$.

For each $g \in G$, we fix such an element $\overline{f}_g \in \Gamma\mathrm{L}({}_QQ^\delta)$. With these, we get a well-defined right action

$$\begin{aligned} L({}_QQ^\delta) \times G &\to L({}_QQ^\delta) \\ (U, g) &\mapsto U^g := \overline{f}_g(U). \end{aligned}$$

We show that this really is a right-action: It is clear that $U^e = U$ for all $U \in L({}_QQ^\delta)$. Also, for all $g, h \in G$, the elements $f_h f_g$, $f_{gh} \in \Gamma\mathrm{L}({}_RQ^\delta)$ are equivalent up to multiplication by an element $r \in R^\times$, and so are $\overline{f}_h \overline{f}_g$ and $\overline{f}_{gh}$. This shows that

$$U^{gh} = \overline{f}_{gh}(U) = r \cdot (\overline{f}_h \overline{f}_g)(U) = (U^g)^h.$$

It is clear that the single maps $U \mapsto U^g$ $(g \in G)$ are lattice automorphisms of $L({}_QQ^\delta)$.

We have just defined a right-action of G on the desarguesian projective geometry $L({}_QQ^\delta)$. We would like to call this action the *geometric action* of G.

The aim of this section is to give a construction of this action in purely lattice-theoretic terms. This will allow us to generalize it to a broader class of modular noetherian right ℓ-groups (i.e. modular geometric right ℓ-groups) which are not necessarily desarguesian - or which are desarguesian of dimension $\delta < 4$. It will turn out that *each* modular geometric right ℓ-group G has a canonical geometric action that generalizes the geometric action we have just constructed for the special case of a desarguesian right ℓ-group of dimension $\delta \geq 4$. Specifically, we want to find a solution to the following problem:

The problem.

Let G be a desarguesian right ℓ-group of dimension $\delta \geq 4$ and let $\varphi : G \overset{\sim}{\to} \mathrm{Lat}({}_RQ^\delta)$ a lattice isomorphism where Q is a complete dvf with respective dvr R.

Reconstruct the geometric action $L({}_QQ^\delta) \times G \to L({}_QQ^\delta)$ *without* regard to the coordinatization by R and Q. More precisely: using only the right ℓ-group structure of G, construct a geometric lattice L_G together with a right action $L_G \times G \to L_G$ such that there is an isomorphism $L_G \overset{\sim}{\to} L({}_QQ^\delta)$ of G-sets that is also an isomorphism of lattices.

The solution.

Step I. We focus on the following problem first: given the lattice $\mathrm{Lat}({}_RR^\delta)$, we want to construct the lattice $L({}_QQ^\delta)$ in purely lattice-theoretic terms (i.e. without referring to R or ${}_RR^\delta$).

First of all, we need to find suitable submodules of ${}_RR^\delta$ that represent the subspaces of ${}_QQ^\delta$: for this sake, we first think of R^δ as an R-submodule of Q^δ. If $U \subseteq Q^\delta$ is a Q-subspace, then one verifies quickly that $A := U \cap R^\delta$ is what is called a *saturated* submodule of R^δ. This means that for all $r \in R, m \in R^\delta$, we have the equivalence $rm \in A \Leftrightarrow (r = 0 \text{ or } m \in A)$. On the other hand, if A is a saturated submodule of ${}_RR^\delta$, then $Q \cdot A \subseteq Q^\delta$ is a subspace of ${}_QQ^\delta$ where we define

$$Q \cdot A := \{q_1x_1 + \ldots + q_nx_n \,:\, q_1, \ldots, q_n \in Q;\; x_1, \ldots, x_n \in A\}.$$

Denote by $L({}_RR^\delta)_{\mathrm{sat}} \subseteq L({}_RR^\delta)$ the subposet of all saturated submodules in ${}_RR^\delta$. We then have the following correspondence:

Proposition 3.1.1. *The map $L({}_QQ^\delta) \to L({}_RR^\delta)_{\mathrm{sat}};\ U \mapsto U \cap R^\delta$ is an order-equivalence with inverse $A \mapsto Q \cdot A$.*

Proof. It is clear that both maps are order-preserving. It remains to show that they are inverse to each other.

We first show $Q \cdot (U \cap R^\delta) = U$ for all $U \in L({}_Q Q^\delta)$. Clearly, $Q \cdot (U \cap R^\delta) \subseteq U$. Let $x \in U$; then $\pi^k x \in U \cap R^\delta$ for some integer k. It follows that $x = \pi^{-k}(\pi^k x) \in Q \cdot U \cap R^\delta$. This proves the other inclusion.

We now show that $(Q \cdot A) \cap R^\delta = A$ for all $A \in L({}_R R^\delta)_{\mathrm{sat}}$. It is clear that $(Q \cdot A) \cap R^\delta \supseteq A$. On the other hand, let $x \in (Q \cdot A) \cap R^\delta$. Then there are $q_1, \ldots, q_n \in Q$, $x_1, \ldots, x_n \in A$ such that $x = q_1 x_1 + \ldots + q_n x_n$. There is an integer k such that $\pi^k q_i \in R$ for all $1 \leq i \leq n$. Therefore, $\pi^k x = (\pi^k q_1) x_1 + \ldots + (\pi^k q_n) x_n \in A$. Since A is saturated, we also have $x \in A$, thus proving the other inclusion. □

Note that the condition that $A \subseteq R^\delta$ is a saturated R-submodule is equivalent to ${}_R(R^\delta / A)$ being torsion-free.

If $A, B \subseteq R^\delta$ are saturated, then clearly $A \cap B$ is as well. However, the sum of saturated submodules need not be saturated, as the following example shows:

Example 3.1.2. Let K be an arbitrary field and take $R = K[[t]]$. In the R-module ${}_R R^2$, the submodules $A = Re_1$ and $B = R(e_1 + te_2)$ are saturated. However, $A + B$ is not saturated, since $te_2 = (e_1 + te_2) - e_1 \in A + B$, but $e_2 \notin A + B$.

Yet, we can define the *saturation* of any $A \in L({}_R R^\delta)$ as

$$A_{\mathrm{sat}} = \{x \in R^\delta \ : \ \exists r \in R \setminus \{0\} : rx \in A\}$$

The submodule A_{sat} clearly is the smallest saturated submodule of ${}_R R^\delta$ which contains A. This implies that even when the sum $A+B$ of saturated R-submodules $A, B \subseteq R^\delta$ is not saturated, there is a smallest saturated submodule containing $A + B$, namely $(A + B)_{\mathrm{sat}}$.

This shows that $L({}_R R^\delta)_{\mathrm{sat}}$ becomes a lattice under the inherited partial order, which is the order given by inclusion of submodules (however, $L({}_R R^\delta)_{\mathrm{sat}}$ is *not* a sublattice of $L({}_R R^\delta)$, in general). The respective lattice operations in $L({}_R R^\delta)_{\mathrm{sat}}$ are given for $A, B \in L({}_R R^\delta)_{\mathrm{sat}}$ by

$$\begin{aligned} A \vee B &= (A + B)_{\mathrm{sat}}, \\ A \wedge B &= A \cap B. \end{aligned}$$

Furthermore, saturating submodules is well-behaved in the following sense:

Proposition 3.1.3. *The saturation map*

$$\begin{aligned} \mathrm{sat} : L({}_R R^\delta) &\to L({}_R R^\delta)_{\mathrm{sat}} \\ A &\mapsto A_{\mathrm{sat}} \end{aligned}$$

is a lattice homomorphism.

Proof. Let $A, B \in L({}_RR^\delta)$. Since $A_{\mathrm{sat}} \cap B_{\mathrm{sat}}$ is saturated and contains $A \cap B$, we clearly have $A \cap B \subseteq A_{\mathrm{sat}} \cap B_{\mathrm{sat}}$. On the other hand, given any $x \in A_{\mathrm{sat}} \cap B_{\mathrm{sat}}$, there are nonnegative integers k, l such that $\pi^k x \in A$ and $\pi^l x \in B$, where $\pi \in R$ is a uniformizer. Without loss of generality, we may assume $k \leq l$, then $\pi^l x = \pi^{l-k} \cdot (\pi^k x) \in A$, implying that $\pi^l x \in A \cap B$. Therefore, $x \in (A \cap B)_{\mathrm{sat}}$. We conclude that $A_{\mathrm{sat}} \cap B_{\mathrm{sat}} \subseteq (A \cap B)_{\mathrm{sat}}$.

This shows that $(A \cap B)_{\mathrm{sat}} = A_{\mathrm{sat}} \cap B_{\mathrm{sat}} = A_{\mathrm{sat}} \wedge B_{\mathrm{sat}}$.

On the other hand, $(A_{\mathrm{sat}} + B_{\mathrm{sat}})_{\mathrm{sat}}$ is saturated and clearly contains $A + B$; therefore, $(A + B)_{\mathrm{sat}} \subseteq (A_{\mathrm{sat}} + B_{\mathrm{sat}})_{\mathrm{sat}}$. Now let $x \in (A_{\mathrm{sat}} + B_{\mathrm{sat}})_{\mathrm{sat}}$. Then there is an integer $k \geq 0$ such that $\pi^k x \in A_{\mathrm{sat}} + B_{\mathrm{sat}}$. Take $a \in A_{\mathrm{sat}}$, $b \in B_{\mathrm{sat}}$ such that $\pi^k x = a + b$. Let $l \geq 0$ be such that $\pi^l a \in A$ and $\pi^l b \in B$. Then

$$\pi^{l+k} x = \pi^l (\pi^k x) = \pi^l (a + b) \in A + B,$$

which shows that $x \in (A + B)_{\mathrm{sat}}$. Therefore, $(A_{\mathrm{sat}} + B_{\mathrm{sat}})_{\mathrm{sat}} \subseteq (A + B)_{\mathrm{sat}}$.

We conclude that $(A + B)_{\mathrm{sat}} = (A_{\mathrm{sat}} + B_{\mathrm{sat}})_{\mathrm{sat}} = A_{\mathrm{sat}} \vee B_{\mathrm{sat}}$. □

We now know that $L({}_QQ^\delta)$ can be regarded as a certain subposet of $L({}_RR^\delta)$. However, this answer is not satisfactory yet:

First of all, we still do not really know how to determine if an element $A \in L({}_RR^\delta)$ is saturated or not, at least not without taking into account the module structure of ${}_RR^\delta$. However, we want to describe the subposet $L({}_RR^\delta)_{\mathrm{sat}} \subseteq L({}_RR^\delta)$ in lattice-theoretic terms only (remember that our aim is to provide a geometric action for modular noetherian right ℓ-groups that are not necessarily coordinatizable over a dvr!). The second problem is that we do not really „have“ $L({}_RR^\delta)$ but only the sublattice $\mathrm{Lat}({}_RR^\delta)$.

In the next step, we address the latter problem.

Step II. It turns out we can describe the lattice $L({}_RR^\delta)$ in terms of *filters* in $\mathrm{Lat}({}_RR^\delta)$:

Let L be a lattice. Recall that for an element $x \in L$, we have set $x^\uparrow := \{y \in L : y \geq x\}$. For $X \subseteq L$, we define $X^\uparrow = \bigcup_{x \in X} x^\uparrow$. A nonempty subset $X \subseteq L$ is called a *filter* if $X^\uparrow = X$ and X is closed under taking meets.

We denote the set of all filters in L by $\mathrm{Fil}(L)$. The set $\mathrm{Fil}(L)$ becomes a lattice under set-theoretic inclusion. The respective lattice operations are given for $F_1, F_2 \in \mathrm{Fil}(L)$ by (see [Grä11, I,3.4])

$$F_1 \wedge F_2 = F_1 \cap F_2 \tag{3.1}$$

$$F_1 \vee F_2 = \{x_1 \wedge x_2 : x_1 \in F_1, x_2 \in F_2\}^\uparrow . \tag{3.2}$$

Each R-submodule $A \subseteq R^\delta$ defines a filter $\mathcal{F}(A) \subseteq \mathrm{Lat}({}_RR^\delta)$ by:

$$\mathcal{F}(A) = \left\{ B \in \mathrm{Lat}({}_RR^\delta) : A \subseteq B \right\}.$$

On the other hand, each filter $F \subseteq \mathrm{Lat}({}_RR^\delta)$ also gives us a R-submodule $\mathcal{S}(F) \subseteq R^\delta$ which is given by

$$\mathcal{S}(F) := \bigcap_{A \in F} A$$

where the intersection is taken in R^δ.

It turns out that $\mathcal{F}$ and $\mathcal{S}$ are inverses to each other:

Proposition 3.1.4. *If R is a complete dvr and $\delta \geq 1$, then*

i) for all $A \in L({}_RR^\delta)$, we have $\mathcal{S}(\mathcal{F}(A)) = A$.

ii) for all $F \in \mathrm{Fil}(\mathrm{Lat}({}_RR^\delta))$, we have $F = \mathcal{F}(\mathcal{S}(F))$.

Proof. i) It is clear that $A \subseteq \mathcal{S}(\mathcal{F}(A))$. Let $\pi \in R$ be a uniformizer. For each $k \geq 0$, we set $A_k := A + \pi^k R^\delta$ which is an element of $\mathrm{Lat}({}_RR^\delta)$ by Proposition 2.5.9.

We are done when we prove that $A = \bigcap_{k=0}^\infty A_k$. Let $p : R^\delta \to R^\delta/A$; $x \mapsto x + A$ be the canonical projection. In $M := R^\delta/A$ we have $\bigcap_{k=0}^\infty \pi^k M = (0)$, since M is finitely generated over the noetherian ring R, and $N := \bigcap_{k=0}^\infty \pi^k M$, as a submodule thereof, is finitely generated as well. Since $\mathfrak{m}N = \pi N = N$, we have $N = (0)$ by Nakayama's lemma[1]. As $p^{-1}(\pi^k M) = A_k$, the statement follows.

ii) Let $F \in \mathrm{Fil}(\mathrm{Lat}({}_RR^\delta))$. For each $k \geq 0$, we define

$$F_k := \min\Big((\pi^k R^\delta)^\uparrow \cap F\Big) \in \mathrm{Lat}({}_RR^\delta).$$

We note that F is uniquely determined by the elements F_k, since each element of F necessarily lies above $\pi^k R^\delta$ for some integer k (Proposition 2.5.9), such that

$$F = \{F_k : k \geq 0\}^\uparrow.$$

We would like to show that there is an $A \in L({}_RR^\delta)$ such that $F_k = A + \pi^k R^\delta$ for all k.

For an integer $k \geq 1$, let $R_k := R/\pi^k R$. By completeness, we have a natural isomorphism: ${}_RR^\delta \cong \varprojlim {}_RR_k^\delta$. We define, for $k \geq 1$, the R-submodule $U_k := F_k + \pi^k R^\delta \subseteq R_k^\delta$. Note that for all pairs $l \geq k$, we have $F_k = F_l + \pi^k R^\delta$; therefore, the reduction maps ${}_RR_l^\delta \to {}_RR_k^\delta$ also induce surjective module maps ${}_RU_l \twoheadrightarrow {}_RU_k$. Defining $U := \varprojlim {}_RU_k$, the inclusions ${}_RU_k \hookrightarrow {}_RR_k^\delta$ merge together to an inclusion ${}_RU \hookrightarrow {}_RR^\delta$. Let A be the image of this inclusion. By construction, we have $A_k := A + \pi^k R^\delta = F_k$.

We want to show that $\mathcal{F}(A) = F$. Note that for all $k \geq 1$, the lattice A_k is the least element in $\mathcal{F}(A)$ which contains $\pi^k R^\delta$. Since every element in $\mathrm{Lat}({}_RR^\delta)$

[1] Alternatively, one could prove this statement directly by writing M as a sum of cyclic R-modules.

lies above an element of the form $\pi^k R^\delta \in \mathrm{Lat}({}_R R^\delta)$ (Proposition 2.5.9), it follows that

$$\mathcal{F}(A) = \{A_k : k \geq 1\}^\uparrow = \{F_k : k \geq 1\}^\uparrow = F.$$

This shows that $\mathcal{F} : L({}_R R^\delta) \to \mathrm{Fil}(\mathrm{Lat}({}_R R^\delta))$ is surjective. From part i) follows the injectivity of $\mathcal{F}$. So, $\mathcal{F}$ is bijective and $\mathcal{S}$ is its inverse. □

Since both $\mathcal{F}$ and $\mathcal{S}$ are order-reversing bijection we have the following corollary:

Corollary 3.1.5. $\mathcal{F} : L({}_R R^\delta) \to \mathrm{Fil}(\mathrm{Lat}({}_R R^\delta))$ *is an antiisomorphism of lattices.*

Step III. We call a filter $F \in \mathrm{Fil}(\mathrm{Lat}({}_R R^\delta))$ *saturated* if it is of the form $F = \mathcal{F}(A)$ with $A \in L({}_R R^\delta)_{\mathrm{sat}}$. How can we decide if A is saturated by just looking at $\mathcal{F}(A)$?

We have already encountered the sequence of R-lattices $A_k = A + \pi^k R^\delta$ $(k \geq 0)$. Each A_k is the least element in $\mathcal{F}(A)$ containing $\pi^k R^\delta$. We have already seen (Proposition 2.5.14) that $\pi^k R^\delta = \mathrm{Rad}^k(R^\delta)$, so the series A_k can be derived from $\mathcal{F}(A)$ in lattice-theoretic terms.

Let $A \subseteq R^\delta$ be an R-submodule. We now look at $M := R^\delta / A$, which is a finitely generated module over the principal left ideal domain R. Therefore, ${}_R M$ is isomorphic to a finite sum of cyclic left R-modules:

$${}_R M \cong {}_R M' := {}_R R^N \oplus \bigoplus_{i=1}^{\infty} {}_R R_i^{n_i},$$

where $R_i := R/\pi^i R$. We have for all $k \geq 0$ that

$$\begin{aligned} \pi^k M' &= \pi^k R^N \oplus \bigoplus_{i=1}^{\infty} \pi^k R_i^{n_i} \\ &= \pi^k R^N \oplus \bigoplus_{i=k+1}^{\infty} (\pi^k R/\pi^i R)^{n_i}. \end{aligned}$$

From this and the isomorphism of R-modules ${}_R(\pi^k R/\pi^{k+1} R) \cong {}_R R_1$, we derive for all $k \geq 0$, that

$${}_R\left((\pi^k M')/(\pi^{k+1} M')\right) \cong {}_R R_1^{\tilde{N}_k}$$

where $\tilde{N}_k = N + \sum_{i=k+1}^{\infty} n_i$, which is also the length of this factor module. We see that $\tilde{N}_k$ is constant in k if and only if $n_i = 0$ for all $i \geq 1$, which is equivalent to M' and $M = R^\delta / A$ being torsion-free resp. to A being a saturated submodule of R^δ.

The submodules $A_k \subseteq R^\delta$ all contain A which is the kernel of the canonical projection $p : R^\delta \to M$, so p produces isomorphisms

$${}_R(A_k/A_{k+1}) \cong {}_R\left(p(A_k)/p(A_{k+1})\right) = {}_R\left(\pi^k M/\pi^{k+1} M\right)$$

for all $k \geq 0$. This argument shows that A is saturated if and only if all modules A_k/A_{k+1} $(k \geq 0)$ have the same length.

Thus, we have shown that a filter $F \in \mathrm{Fil}(\mathrm{Lat}({}_R R^\delta))$ is saturated if and only if the sequence of lengths $l([F_{k+1}, F_k])$ is a constant for $k \geq 0$, where we set

$$F_k = \min(F \cap (\pi^k R^\delta)^\uparrow) = A_k^\uparrow.$$

Given a lattice isomorphism $\varphi : G^- \xrightarrow{\sim} \mathrm{Lat}({}_R R^\delta)$, we have, by Proposition 2.5.14 and Proposition 2.2.7, the equalities

$$\varphi^{-1}(\pi^k R^\delta) = \varphi^{-1}(\mathrm{Rad}^k(R^\delta)) = \mathrm{Rad}^k(\varphi^{-1}(R^\delta)) = \mathrm{Rad}^k(e) = s^{-k}$$

If we are furthermore given a filter $F' \in \mathrm{Fil}(G^-)$ with $\varphi(F') = F$, then

$$\begin{aligned} \varphi^{-1}(F_k) &= \min\left(\varphi^{-1}(F) \cap \varphi^{-1}(\pi^k R^\delta)\right) \\ &= \min\left(F' \cap (s^{-k})^\uparrow\right) =: G_{F',k}. \end{aligned}$$

And $l([F_{k+1}, F_k]) = l([G_{F',k+1}, G_{F',k}]) = d(G_{F',k+1} G_{F',k}^{-1})$.

If we call a filter $F \in \mathrm{Fil}(G^-)$ *saturated* if $\varphi(F) \in \mathrm{Fil}(\mathrm{Lat}({}_R R^\delta))$ is saturated, then saturatedness of F is equivalent to $d(g_{F,k})$ being constant where we define $g_{F,k} := G_{F,k} G_{F,k-1}^{-1}$ for $k \geq 1$.

We now know that an antiisomorphism of ordered sets between $L({}_Q Q^\delta)$ and $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ - the poset of saturated filters in G^- - is induced by the composition of the maps

$$L({}_Q Q^\delta) \xrightarrow{U \mapsto U \cap R^\delta} L({}_R R^\delta) \xrightarrow{\mathcal{F}} \mathrm{Fil}(\mathrm{Lat}({}_R R^\delta)) \xrightarrow{\varphi^{-1}} \mathrm{Fil}(G^-).$$

This implies that $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is a lattice under inclusion of filters in the case considered here (for a general modular geometric right ℓ-group this will be considerably harder to show). Note that $L({}_R R^\delta)_{\mathrm{sat}}$ is not a sublattice of $L({}_R R^\delta)$, so neither is the subset $\mathrm{Fil}(G^-)_{\mathrm{sat}}$. However, since each R-submodule of R^δ is contained in a minimal saturated one, we can at least tell that each filter in G^- contains a maximal saturated one.

Now there is a final question we have to answer in this special case: We have seen that there is a right action $\mathrm{P\Gamma L}({}_Q Q^\delta) \times G \to \mathrm{P\Gamma L}({}_Q Q^\delta)$ such that G acts by lattice automorphisms. Therefore, there also is a right G-action by automorphisms of $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ under which the constructed lattice antiisomorphism between $L({}_Q Q^\delta)$ and $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ becomes an isomorphism of G-sets. How is this action constructed without taking advantage of the coordinatization?

Step IV. We first construct a suitable right action of the submonoid G^- by automorphisms of $\mathrm{Fil}(G^-)_{\mathrm{sat}}$. This action then uniquely determines an action of G on $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ by Proposition 2.1.10.

First, let $f \in \Gamma\mathrm{L}({}_RQ^\delta)$, be a semilinear isomorphism with $f(R^\delta) \subseteq R^\delta$. Note that each element of $\Gamma\mathrm{L}({}_RQ^\delta)$ is automatically a semilinear automorphism of ${}_QQ^\delta$, and so is f.

We claim that for all $U \in L({}_QQ^\delta)$, we have the following equality in $\mathrm{Lat}({}_RR^\delta)$:

$$\mathcal{F}(U \cap R^\delta) = \{f^{-1}(A) \cap R^\delta : A \in \mathcal{F}(f(U) \cap R^\delta)\}. \tag{3.3}$$

First of all, we show that the set on the right is a filter. Let $B = f^{-1}(A) \cap R^\delta$ with $A \in \mathcal{F}(f(U) \cap R^\delta)$. Then we have the following isomorphisms of intervals in $\mathrm{Lat}({}_RQ^\delta)$:

$$\begin{aligned} [A, f(R^\delta) + A] &\overset{M \mapsto M \cap f(R^\delta)}{\longrightarrow} [f(R^\delta) \cap A, f(R^\delta)] \\ &\overset{f^{-1}(\ldots)}{\longrightarrow} \quad [f^{-1}(A) \cap R^\delta, R^\delta] = [B, R^\delta]. \end{aligned}$$

This shows that every $B' \in \mathrm{Lat}({}_RR^\delta)$ is of the form $B' = f^{-1}(A') \cap R^\delta$ with $B' \in \mathcal{F}(f(U) \cap R^\delta)$. Therefore, the set on the right is indeed a filter in $\mathrm{Lat}({}_RR^\delta)$.

We then calculate the intersection of all elements in the filter which is

$$\begin{aligned} \bigcap_{A \in \mathcal{F}(f(U) \cap R^\delta)} (f^{-1}(A) \cap R^\delta) &= f^{-1}\left(\bigcap_{A \in \mathcal{F}(f(U) \cap R^\delta)} A\right) \cap R^\delta \\ &= f^{-1}(f(U) \cap R^\delta) \cap R^\delta \qquad \text{(Proposition 3.1.4)} \\ &= U \cap \underbrace{f^{-1}(R^\delta) \cap R^\delta}_{=R^\delta} \\ &= U \cap R^\delta. \end{aligned}$$

By Proposition 3.1.4, the considered filters are equal, which proves the equality.

Assume now that we are given a lattice isomorphism $\varphi : G \to \mathrm{Lat}({}_RQ^\delta)$ where $\varphi(g'g) = \varphi(g')^g$ for all $g, g' \in G$. Fix some $g \in G^-$ and chose an $f_g \in \Gamma\mathrm{L}({}_RQ^\delta)$ such that $A^g = f_g(A)$ for all $A \in \mathrm{Lat}({}_RQ^\delta)$ (which exists by Proposition 2.6.5). Furthermore, take $F \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$. What is an appropriate definition for a filter F^g that is saturated?

The first idea would be to define F^g as the biggest saturated filter contained in the filter $(F \cdot g)^\uparrow$. Such a filter should exist, since every submodule of ${}_RR^\delta$ is contained in a biggest saturated one and we have the order-antiisomorphism $\mathcal{F} : L({}_RR^\delta) \to \mathrm{Fil}(\mathrm{Lat}({}_RR^\delta))$, which - after composing with φ^{-1} - translates saturated submodules of ${}_RR^\delta$ to saturated filters in $\mathrm{Fil}(G^-)$.

However, the ideas sketched here lead to an easier-to-work-with group operation on $\mathrm{Fil}(G^-)$: Let $g \in G^-$ and $U \in L({}_QQ^\delta)$. In (3.3), we put $f = f_g$ and assume that $\mathcal{F}(U \cap R^\delta) = \varphi(F)$ for a suitable $F \in \mathrm{Fil}(G^-)$. Then $\varphi(F^g)$ must defined in a way such that it becomes equal to $\mathcal{F}(U^g \cap R^\delta)$ where we set $U^g := f_g(U)$. Writing the elements of $\mathrm{Lat}({}_R R^\delta)$ as $\varphi(g')$ with $g' \in G^-$, and noting that $\varphi(e) = R^\delta$, this results in the following equation:

$$\varphi(F) = \{f_g^{-1}\left(\varphi(g')\right) \cap \varphi(e) : \varphi(g') \in \varphi(F^g)\}.$$

We can now write $f_g^{-1}(\varphi(g')) = \varphi(g'g^{-1})$ and take φ^{-1} on both sides (not forgetting that φ is a lattice isomorphism):

$$\begin{aligned} F &= \{g'g^{-1} \wedge e : g' \in F^g\} \\ &= \{g \to g' : g' \in F^g\} \\ &=: g \to F^g. \end{aligned}$$

From the fact that G acts via lattice automorphisms on $\mathrm{Lat}({}_Q Q^\delta)$, it follows that the maps $F \mapsto (g \to F)$ on $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ are lattice automorphisms for all $g \in G^-$.

There is a unique group action $\mathrm{Fil}(G^-)_{\mathrm{sat}} \times G \to \mathrm{Fil}(G^-)_{\mathrm{sat}}; (F, g) \mapsto F^g$ such that $F^{g^{-1}} = g \to F$ for all $g \in G^-$.

We explain the argument: denote by γ the lattice antiautomorphism $L({}_Q Q^\delta) \to \mathrm{Fil}(G^-)_{\mathrm{sat}}$. We have shown that $(g^{-1}) \to \gamma(U) = \gamma(U^g)$ for all $g \in G^+$ - in fact, we have *constructed* the action in a way such that γ becomes an isomorphism of G-sets.

Therefore, defining $F^g = (g^{-1}) \to F$ for $g \in G^+$ already defines a right action of the monoid G^+ by automorphisms of $\mathrm{Fil}(G^-)_{\mathrm{sat}}$. The right action induces a monoid homomorphism $G^+ \to \mathrm{Aut}(\mathrm{Fil}(G^-)_{\mathrm{sat}})^{\mathrm{op}}$ which has, by Proposition 2.1.10, a unique extension to the whole of G - namely the one induced by the right action given by $\gamma(U^g) = \gamma(U)^g$ for all $U \in L({}_Q Q^\delta)$, $g \in G$.

We have shown:

Theorem 3.1.6. *Let G be a desarguesian right ℓ-group of dimension $\delta \geq 4$ with the lattice isomorphism $\varphi : G \to \mathrm{Lat}({}_R Q^\delta)$, where R is a complete dvr with dvf Q.*

Equip $L({}_Q Q^\delta)$ with the right G-action

$$\begin{aligned} L({}_Q Q^\delta) \times G &\to L({}_Q Q^\delta) \\ (A, g) &\mapsto \overline{f}_g(A) \end{aligned}$$

where for each $g \in G$, the map $\overline{f}_g \in \Gamma\mathrm{L}({}_Q Q^\delta)$ is defined by the property that $\overline{f}_g(\varphi(h)) = \varphi(hg)$ for all $h \in G$.

Equip $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ *with the unique right G-action $(F, g) \mapsto F^g$ such that we have $g \to F^g = F$ for all $g \in G^-$, $F \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$.*

Then the map

$$\begin{aligned} \gamma : L({}_Q Q^\delta) &\to \mathrm{Fil}(G^-)_{\mathrm{sat}} \\ U &\mapsto \varphi^{-1}\left(\mathcal{F}(A \cap R^\delta)\right) \end{aligned}$$

is an antiisomorphism of lattices which also is an isomorphism of G-sets.

Since the right action of G on $L({}_QQ^\delta)$ - which is a modular geometric lattice - is by lattice automorphisms, this property directly can be translated to $\mathrm{Fil}(G^-)_{\mathrm{sat}}$, which, being antiisomorphic to $L({}_QQ^\delta)$, is modular geometric itself.

Corollary 3.1.7. *Let G be a desarguesian right ℓ-group of dimension $\delta \geq 4$. Then $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is a modular geometric lattice on which G acts by lattice automorphisms via $(F, g) \mapsto F^g$.*

It turns out that this statement is actually true for *each* modular geometric right ℓ-group - there is always a canonical action of G on the lattice of saturated filters in its negative cone, and this lattice is a modular geometric lattice.

We will later see that this *geometric action* in case of structure groups of the involutive set-theoretic Yang-Baxter equation (which make up another big class of modular geometric right ℓ-groups) is a well-known concept.

3.2 Saturated filters in G^-

Recall that if L is a lattice, then for each $x \in L$, $x^\uparrow$ is a filter which we call the *principal* filter over x. Furthermore, note that for a sequence $x_1 \geq x_2 \geq \ldots$ with all $x_i \in L$, the set $\{x_i : i \geq 1\}^\uparrow$ is always a filter in L.

From now on, G will be a modular geometric right ℓ-group where the special strong order unit s given by $s^{-1} = \mathrm{Rad}(e)$ resp. $s = \mathrm{Soc}(e)$ exists. We assume that G and s are fixed throughout until the end of Section 3.4 (except when we say otherwise).

We begin by generalizing the index sequence of an element $g \in G^-$ (Definition 2.3.7) to filters in G^-:

Definition 3.2.1. Let $F \in \mathrm{Fil}(G^-)$. We define the *principal sequence* as

$$G_{F,i} = \min(F \wedge (s^{-i})^\uparrow) \quad (i \geq 0),$$

the *right-normal factors* as

$$g_{F,i} = G_{F,i} G_{F,i-1}^{-1} \quad (i \geq 1)$$

and the *index sequence* as[2]

$$\iota_i^f(F) = d(g_{F,i}) \quad (i \geq 1).$$

The following proposition ensures that our definitions of right-normal factors and an index function are indeed a generalization from elements to filters:

[2]The upperscript-f indicates that this is the index sequence for $f$ilters.

Proposition 3.2.2. *If the element $g \in G^-$ has the right-normal decomposition $g = g_k g_{k-1} \dots g_1$, then, with $F := g^\uparrow$, we have*

$$g_{F,i} = \begin{cases} g_i & i \leq k \\ e & i > k \end{cases}.$$

In particular, $\iota_i^f(g^\uparrow) = \iota_i(g)$ for all i.

Proof. First of all,

$$G_{F,i} = \min((s^{-i})^\uparrow \wedge g^\uparrow) = \min\left((s^{-i} \vee g)^\uparrow\right) = s^{-i} \vee g.$$

Using formula (2.2) for the right-normal factors of g, we get for $1 \leq i \leq k$ that

$$g_{F,i} = G_{F,i} G_{F,i-1}^{-1} = (s^{-i} \vee g)(s^{-i+1} \vee g)^{-1} = g_i.$$

For all $i > k$, we have $s^{-i} \vee g = g$ (Proposition 2.1.17), so the other case is settled as well.

Finally, for all $i \geq 1$, we have

$$\iota_i^f(g^\uparrow) = d(G_{F,i} G_{F,i-1}^{-1}) = d(s^{-i} \vee g) - d(s^{-i+1} \vee g) = \iota_i(g).$$

□

Proposition 3.2.3. *For all $F \in \mathrm{Fil}(G^-)$, we have $F = \{G_{F,i} : i \geq 0\}^\uparrow$.*

Proof. Set $\mathcal{G} = \{G_{F,i} : i \geq 0\}^\uparrow$. Clearly, $\mathcal{G} \subseteq F$ since all $G_{F,i}$ lie in F. Let now $g \in F$, and let i be some integer such that $g \geq s^{-i}$, then $g \geq G_{F,i}$, so $g \in \mathcal{G}$. It follows that also, $F \subseteq \mathcal{G}$. □

Proposition 3.2.4. *Let $k \geq 1$ be an integer. Then $\iota_i(G_{F,k}) = \iota_i^f(F)$ for $1 \leq i \leq k$.*

Proof. For $1 \leq i \leq k$ we have

$$\begin{aligned} G_{F,i}^\uparrow &= F \wedge (s^{-i})^\uparrow \\ &= F \wedge (s^{-k})^\uparrow \wedge (s^{-i})^\uparrow \\ &= G_{F,k}^\uparrow \wedge (s^{-i})^\uparrow \\ &= (G_{F,k} \vee s^{-i})^\uparrow, \end{aligned}$$

implying $G_{F,i} = G_{F,k} \vee s^{-i}$. Using this, we calculate

$$\begin{aligned} \iota_i^f(F) &= d(G_{F,i}) - d(G_{F,i-1}) \\ &= d(G_{F,k} \vee s^{-i}) - d(G_{F,k} \vee s^{-i+1}) \\ &= \iota_i(G_{F,k}). \end{aligned}$$

□

Proposition 3.2.5. *Let $F \in \mathrm{Fil}(G^-)$. Then each term $g_{F,k}g_{F,k-1}\dots g_{F,1}$ $(k \geq 0)$ is a right-normal factorization (with probably some leading terms being equal to e). On the other hand, for each sequence of elements $g_i \in [s^{-1}, e]$ $(i \geq 1)$ such that for each k, the expression $g_k g_{k-1}\dots g_1$ is a right-normal factorization in G^- (where we again allow leading e's), there is a unique $F \in \mathrm{Fil}(G^-)$ such that $g_{F,i} = g_i$ for all i.*

Proof. Let $F \in \mathrm{Fil}(G^-)$. Then for each $k \geq 0$, $G_{F,k} = g_{F,k}g_{F,k-1}\dots g_{F,1}$. For any $1 \leq i \leq k$, we have

$$G^{\uparrow}_{F,i} = F \wedge (s^{-i})^{\uparrow} = (F \wedge (s^{-k})^{\uparrow}) \wedge (s^{-i})^{\uparrow} = G^{\uparrow}_{F,k} \wedge (s^{-i})^{\uparrow} = (G_{F,k} \vee s^{-i})^{\uparrow}$$

from which it becomes clear that $G_{F,k} \vee s^{-i} = G_{F,i} = g_i g_{i-1}\dots g_1$ for all $1 \leq i \leq k$. This is, however, the characterizing property of the right-normal factors of $G_{F,k}$, as you can see in equation (2.2).

On the other hand, let a sequence of $g_i \in [s^{-1}, e]$ be given such that every expression $g_k g_{k-1}\dots g_1$ is right-normal. Defining $G_k := g_k g_{k-1}\dots g_1$ for $k \geq 0$, it must apply for all pairs $0 \leq i \leq k$ that $G_k \vee s^{-i} = G_i$ - also by equation (2.2).

Defining a filter by $F := \{G_k : k \geq 0\}^{\uparrow}$, it follows that $G_{F,k} = F \wedge (s^{-k})^{\uparrow} = G_k$ for all k. Therefore, $g_{F,i} = G_i G_{i-1}^{-1} = g_i$ for all $i \geq 1$. ☐

We have established a one-to-one correspondence between filters in G^- and what we would like to call *infinite right-normal expressions*, that is, sequences of elements $g_i \in [s^{-1}, e]$ $(i \geq 1)$ so that each tail $g_k g_{k-1}\dots g_1$ is a right-normal factorization. This generalizes the fact that there is a one-to-one correspondence between the elements of G^- (resp. their principal filters) and finite right-normal factorizations.

We can also directly generalize Proposition 2.3.9 to

Proposition 3.2.6. *For any $F \in \mathrm{Fil}(G^-)$, $\iota^f_i(F)$ is non-decreasing in i.*

Proof. This follows directly from Proposition 3.2.4 and Proposition 2.3.9. ☐

Another important role will be played by a filter-theoretic generalization of the arrow-operation $\to$ as defined in Section 2.1 (see 43). Recall that this operation was originally by $g \to h = e \wedge hg^{-1}$ where $g, h \in G^-$.

For $F \in \mathrm{Fil}(G^-)$ and $g \in G^-$, we define the arrow-operation by

$$g \to F = \{g \to f : f \in F\}. \tag{3.4}$$

This operation indeed is a generalization of the original $\to$-operation:

Proposition 3.2.7. *For all $g, h \in G^-$, we have $g \to (h^{\uparrow}) = (g \to h)^{\uparrow}$.*

Proof. Fix some elements $g, h \in G^-$. Recall equation (S4) from Proposition 2.1.11 (see page 43) which tells us that $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$ holds for all $x, y, z \in G^-$. It follows that the mapping $G^- \to G^-; x \mapsto (g \to x)$ is monotone, and therefore, $g \to h^\uparrow \subseteq (g \to h)^\uparrow$. Using the diamond lemma and right-invariance, we see that the composition

$$[h, g \vee h] \overset{x \mapsto g \wedge x}{\longrightarrow} [g \wedge h, g] \overset{x \mapsto x g^{-1}}{\longrightarrow} [(g \vee h) g^{-1}, e] = [g \to h, e]$$

which is $x \mapsto (g \to x)$, is an isomorphism of lattices. Since $[h, g \vee h] \subseteq h^\uparrow$, we get $g \to h^\uparrow = (g \to h)^\uparrow$. □

It is also important that $g \to F$ is not just some „random“ subset of G^-:

Proposition 3.2.8. *For $F \in \mathrm{Fil}(G^-)$ and $g \in G^-$, we have $g \to F \in \mathrm{Fil}(G^-)$.*

Proof. F is nonempty, and so is $g \to F$. We have $F^\uparrow = F$. By the above proposition, $(g \to F)^\uparrow = g \to F^\uparrow = g \to F$. This proves that $g \to F$ is an upset.

Now take $x_1, x_2 \in (g \to F)$. Let $h_1, h_2 \in F$ be such that $x_1 = g \to h_1$, $x_2 = g \to h_2$. Since F is a filter, $h_1 \wedge h_2 \in F$. By Proposition 2.1.11, (S4), we have

$$x_1 \wedge x_2 = (g \to h_1) \wedge (g \to h_2) = g \to (h_1 \wedge h_2) \in (g \to F).$$

We conclude that $g \to F$ is a filter. □

We now come to the central definition of this chapter:

Definition 3.2.9. We call a filter $F \in \mathrm{Fil}(G^-)$ *saturated* if $\iota_i^f(F)$ is constant in i. We call this constant the *degree* of the saturated filter. We denote by $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ the set of saturated filters in G^-.

Example 3.2.10. Let $n \geq 1$ and consider the right ℓ-group $(\mathbb{Z}^n, \leq)$ with the coordinatewise order. In Example 2.1.23,1) we have seen that $s = (1, \ldots, 1)$ is a strong order unit with strong order interval $[-s, 0] = \{-1, 0\}^n$. Furthermore, the coordinatewise lattice order on $\mathbb{Z}^n$ is distributive. Therefore, $\mathbb{Z}^n$ is a modular geometric right ℓ-group.

Writing $a = (a^{(1)}, \ldots, a^{(n)})$, the degree function on $\mathbb{Z}^n$ is

$$d(a) = -\sum_{i=1}^{n} a^{(i)}.$$

An expression $a_k + a_{k-1} + \ldots + a_1$ with all $a_i \in [-s, 0] \setminus \{0\}$ $(1 \leq i \leq k)$ is right-normal if and only if $a_k \geq a_{k-1} \geq \ldots \geq a_1$. Then $d(a_1) = \ldots = d(a_k)$ can only hold if $a_1 = \ldots = a_k$. This consideration implies that all saturated filters in $\mathbb{Z}^n$ are of the form $\{i \cdot a : i \geq 0\}^\uparrow$ for some element $a \in [-s, 0]$. If we denote by $J \subseteq \{1, 2, \ldots, n\}$ the set of all indices k such that $a^{(k)} = 0$, then

$$\{i \cdot a : i \geq 0\}^\uparrow = \{a \in \mathbb{Z}^n \ : \ \forall k \in J : a^{(k)} = 0\}.$$

Therefore, the saturated filters in $(\mathbb{Z}^n, \leq)$ can be identified with the subsets of $\{1, \ldots, n\}$. We will see later that in each distributive geometric right ℓ-group, the saturated filters form a Boolean lattice (Proposition 3.5.12).

We will show later that $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is closed under taking joins of filters (Theorem 3.2.16). But some preliminary work has to be done until we arrive at this result.

The most basic nontrivial saturated filters are clearly the ones which are of degree 1. It turns out that they have a really simple lattice-theoretic structure:

Proposition 3.2.11. *A filter $F \in \mathrm{Fil}(G^-)$ is saturated of degree 1 if and only if F is an infinite chain.*

Proof. By Proposition 3.2.3, we have $F = \bigcup_{k \geq 0} G_{F,k}^{\uparrow}$. Also, $G_{F,k}^{\uparrow} \subseteq G_{F,k+1}^{\uparrow}$ holds for all $k \geq 0$. So F is a chain if and only if each $G_{F,k}$ is a cochain. By Corollary 2.3.15, this is equivalent to $\iota_i(G_{F,k}) \in \{0, 1\}$ for all $1 \leq i \leq k$ resp. $\iota_i^f(F) \in \{0, 1\}$ by Proposition 3.2.4.

We clearly have $g_{F,i} = e$ for some index i if and only if F is a principal filter. The property of *not* being principal is clearly equivalent to $\iota_i^f(F) > 0$ for all $i \geq 1$. So a filter F is an infinite chain if and only if $\iota_i^f(F) = 1$ for all $i \geq 1$, meaning that F is saturated of degree 1. $\square$

Lemma 3.2.12. *If $F_1, \ldots, F_k \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$ are of degree 1, then $F_1 \vee \ldots \vee F_k \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$.*

Proof. The lemma will follow from the following statement:

If $F_1, \ldots, F_k \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$ are of degree 1 and furthermore $F_{j+1} \not\subseteq F_1 \vee \ldots \vee F_j$ for $1 \leq j \leq k-1$, then $F_1 \vee \ldots \vee F_k$ is saturated of degree k.

For $k = 1$, this is trivial. Now assume that we have proved it for $k = k_0$ and let $F_1, \ldots, F_{k_0}, F_{k_0+1}$ fulfill the conditions of the stronger statement.

First of all, for all $1 \leq j \leq k_0 + 1$, the elements of the principal sequences, $G_{F_j,i}$, are cochains in G^- for all $i \geq 1$. We can write

$$F := F_1 \vee \ldots \vee F_{k_0} \vee F_{k_0+1} = \{G_{F_1,i} \wedge \ldots \wedge G_{F_{k_0},i} \wedge G_{F_{k_0+1},i} : i \geq 1\}^{\uparrow}.$$

By Lemma 2.4.3, there cannot be more than k_0+1 dually independent dual atoms in G^- above an element of the form $G_{F_1,i} \wedge \ldots \wedge G_{F_{k_0},i} \wedge G_{F_{k_0+1},i}$. Since $\left[s^{-1}, e\right]$ is dually geometric, we infer that

$$d\left((G_{F_1,i} \wedge \ldots \wedge G_{F_{k_0},i} \wedge G_{F_{k_0+1},i}) \vee s^{-1}\right) \leq k_0 + 1$$

for all $i \geq 1$. This follows from the fact that in a dually geometric lattice L, one can find for an $x \in L$ a set of $l([x, 1])$ dually independent dual atoms above x; and

$\left[s^{-1}, e\right]$ *is*, in fact, dually geometric by Rump's theorem (Theorem 2.2.3)! This argument proves $\iota_1^f(F) \leq k_0 + 1$.

We set $F' := F_1 \vee \ldots \vee F_{k_0}$. By our induction hypothesis, F' is saturated of degree k_0. This implies that $d(G_{F',i}) = k_0 \cdot i$ for all $i \geq 0$. Similarly $d(G_{F_{k_0+1},i}) = i$ for all $i \geq 0$.

Since F_{k_0+1} is an infinite chain (Proposition 3.2.11,), we see that at a certain height, say i_0, the elements $G_{F_{k_0+1},i}$ will „leave" F', meaning that for all $i \geq i_0$ we have

$$G_{F_{k_0+1},i} \vee G_{F',i} = G_{F_{k_0+1},i_0} \vee G_{F',i_0} =: g.$$

Using the parallelogram identity (Proposition 2.3.5), we deduce that for $i \geq i_0$,:

$$\begin{aligned} d(G_{F_{k_0+1},i} \wedge G_{F',i}) &= d(G_{F_{k_0+1},i}) + d(G_{F',i}) - d(g) \\ &= i + k_0 \cdot i - d(g) \\ &= (k_0 + 1) \cdot i - d(g). \end{aligned}$$

Also, $G_{F_{k_0+1},i} \wedge G_{F',i} \geq s^{-i}$, which implies for $i \geq i_0$ that

$$\begin{aligned} \sum_{m=1}^{i} \iota_m^f(F) &= d(G_{F,i}) \\ &\geq d(G_{F_{k_0+1},i} \wedge G_{F',i}) \\ &= (k_0 + 1) \cdot i - d(g). \end{aligned}$$

Since $\iota_1^f(F) \leq k_0 + 1$ and $\iota_m(F)$ is a nonincreasing sequence in m by Proposition 3.2.6, such a behaviour is only possible when $\iota_m(F) = k_0 + 1$ for all $m \geq 1$, meaning that F is saturated of degree $k_0 + 1$. This completes the induction step. $\square$

Lemma 3.2.13. *Let $F \in \mathrm{Fil}(G^-)$ be saturated of degree d, then for all $g \in F$, the filter $g \to F$ is saturated of degree d.*

We remind the reader that for an element $g \in G^-$ we have defined $\lambda(g)$ as the length k of the right-normal factorization $g = g_k g_{k-1} \ldots g_1$ (here, of course, we demand $g_k \neq e$). Recall (Corollary 2.1.18) that $\lambda(g)$ can be characterized as the smallest integer k such that $g \geq s^{-k}$.

Proof. For $d = 0$, this is trivial, so we may assume that $d > 0$.

Under the given assumptions, $g \to F = \{h \in G^- : hg \in F\}$. Take an integer k large enough that $G_{F,k} \leq g$, then

$$g \to F = \bigcup_{l \geq k} (G_{F,l} \cdot g^{-1})^{\uparrow}.$$

Denote the sequence of right-normal factors of $g \to F$ by h_i $(i \geq 1)$. Since $\iota_i^f(g \to F) = d(h_i)$, we want to show that $d(h_i) = d$ for all i.

First of all, we show that $d(h_1) \le d$: Assume that $d(h_1) > d$. Let g_1' be the rightmost factor in the right-normal factorization of $h_1 g$. Then Corollary 2.3.13 tells us that $d(g_1') \ge d(h_1) > d$. Since $g_1' \ge h_1 g \in F$, we have $g_{F,1} \le g_1'$, so

$$\iota_1^f(F) = d(g_{F,1}) \ge d(g_1') > d$$

which is a contradiction. This proves $d(h_1) \le d$.

Let the sequence of h_i be as before. Let k be any positive integer. We now look at $h_k h_{k-1} \dots h_1$. Take l large enough that $G_{F,l} g^{-1} \le h_k h_{k-1} \dots h_1$ (this is possible since $g \to F$ is the union of the upsets of the $G_{F,i} g^{-1}$). Then the right-normal factorization of $G_{F,l} g^{-1}$ ends in $h_k h_{k-1} \dots h_1$ (in $g \to F$ there is no smaller element which is $\ge s^{-k}$). This property will still hold if we enlarge l.

We clearly have $\lambda(G_{F,l}) = l$, which implies that $G_{F,l} \ge s^{-l}$. Using right-invariance and the normality of s, we get $G_{F,l} g^{-1} \ge s^{-l} g^{-1} \ge s^{-l}$. Therefore, $\lambda(G_{F,l} g^{-1}) \ge l$.

It is clear that

$$d(G_{F,l} g^{-1}) = \left(\sum_{i=1}^{l} d(g_{F,i}) \right) - d(g) = l \cdot d - d(g).$$

On the other hand,

$$d(G_{F,l} g^{-1}) = \sum_{i=1}^{\lambda(G_{F,l} g^{-1})} \iota_i(G_{F,l} g^{-1}).$$

From $\iota_1(G_{F,l} g^{-1}) = d(h_1) \le d$ and the monotonicity of the ι-sequence (Proposition 2.3.9), we know that $\iota_i(G_{F,l} g^{-1}) \le d$ for all i.

Assume that $d(h_k) < d$ for some integer k, then $\iota_i(G_{F,l} g^{-1}) \le d - 1$ holds by monotonicity for all $i > k$; but then:

$$\begin{aligned}
l \cdot d - d(g) &= d(G_{F,l} g^{-1}) \\
&= \sum_{i=1}^{\lambda(G_{F,l} g^{-1})} \iota_i(G_{F,l} g^{-1}) \\
&= \sum_{i=1}^{k} d(h_i) + \sum_{i=k+1}^{\lambda(G_{F,l} g^{-1})} \iota_i(G_{F,l} g^{-1}) \\
&< k \cdot d + (\lambda(G_{F,l} g^{-1}) - k)(d-1) \\
&\le k \cdot d + l \cdot (d-1).
\end{aligned}$$

which leads to a contradiction if we choose l large enough. Therefore, $d(h_k) = d$. Since $d(h_1) \le d$ also, we must have $d(h_1) = \dots d(h_k) = d$. □

Lemma 3.2.14. *Let $F \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$ and $x \in X(G^-) \cap F$. Then there is a saturated filter $F' \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$ of degree 1 such that $F' \subseteq F$ and $x \in F'$.*

Proof. By Proposition 3.2.5, we must construct a sequence of elements $g_i \prec e$ ($i \geq 1$) with $g_1 = x$ such that for each $k \geq 1$, the expression $g_k g_{k-1} \dots g_1$ is right-normal and $g_k g_{k-1} \dots g_1 \in F$.

Assume that for some integer $k \geq 1$, we have already found elements $g_1, \dots, g_k \prec e$ with such that $g_k g_{k-1} \dots g_1$ is a right-normal expression and $g_k g_{k-1} \dots g_1 \in F$. We now want to find a further element $g_{k+1} \in X(G^-)$ such that $g_{k+1} g_k \dots g_1$ is right-normal and $g_{k+1} g_k \dots g_1 \in F$.

Since g_{k+1} must be a dual atom, the only non-trivial right-divisor of g_{k+1} in such an expression can be g_{k+1} itself. Therefore, g_{k+1} can *not* be attached to $g_k \dots g_1$ if and only if $g_{k+1} g_k \in \left[s^{-1}, e\right] \Leftrightarrow g_{k+1} \in [s^{-1} g_k^{-1}, e]$.

We define $g := g_k \dots g_1$. By Lemma 3.2.13, the filter

$$\tilde{F} := g \to F = \{h \in G^- : hg \in F\}$$

(note that $g \in F$!) is saturated of degree d, i.e. $d(G_{\tilde{F},1}) = d$. We claim that there is an $h \in \tilde{F} \cap X(G^-)$ such that $h \notin [s^{-1} g_k^{-1}, e]$.

Now assume that $h \in [s^{-1} g_k^{-1}, e]$ for all $h \in \tilde{F} \cap X(G^-)$. Since $[s^{-1}, e]$ is dually geometric,

$$G_{F',1} = \bigwedge \left(\tilde{F} \cap X(G^-)\right) \geq s^{-1} g_k^{-1}.$$

It follows that $G_{\tilde{F},1} g_k \geq s^{-1}$.

Since $G_{\tilde{F},1} g_k g_{k-1} \dots g_1 \in F$, we have

$$G_{\tilde{F},1} g_k \in (g_{k-1} \dots g_1) \to F,$$

but $d(G_{\tilde{F},1} g_k) = d+1$. The latter filter is saturated of degree d by Lemma 3.2.13, so all elements lying in $[s^{-1}, e] \cap F$ have degree at most d. Contradiction!

So there must exist an $h \prec e$ such that $h g_k \notin [s^{-1}, g_k]$ and $h g_k g_{k-1} \dots g_1 \in F$. Setting $g_{k+1} := h$ we have $g_{k+1} g_k \dots g_1 \in F$ and $g_{k+1} g_k \dots g_1$ still is a right-normal expression.

By induction, we end up with a sequence $(g_i)_{i \geq 1}$ such that for all $k \geq 0$, $g_k g_{k-1} \dots g_1$ is a right-normal factorization of some element of F. By Proposition 3.2.5, and its proof, the filter $F' := \{g_k g_{k-1} \dots g_1 : k \geq 0\}^{\uparrow}$ has $g_{F',i} = g_i$ for all $i \geq 1$. Therefore, F' is saturated of degree 1.

Since $g_k g_{k-1} \dots g_1 \in F$ for all k, by construction, we also have $F' \subseteq F$. We also remember that $x = g_1 \in F'$, thus finishing the proof. □

Proposition 3.2.15. *Each saturated filter $F \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$ of degree k is the join of some $F_1, \dots, F_k \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$ of degree* 1.

Proof. We have $G_{F,1} \in [s^{-1}, e]$, the latter being a dual geometric lattice. Thus, we can find dually independent $x_1, \dots, x_k \prec e$ such that $G_{F,1} = x_1 \wedge \dots \wedge x_k$. By

Lemma 3.2.14, there exist saturated filters $F_1, \ldots, F_k \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$ of degree 1, such that for $1 \leq i \leq k$, we have $G_{F_i,1} = x_i$ and $F_i \subseteq F$.

Let $j \geq 1$. Then all $G_{F_i,j}$ $(1 \leq i \leq k)$ are cochains in G^- and x_i is the unique dual atom of G^- lying above $G_{F_i,j}$. By Lemma 2.4.4, the elements $G_{F_i,j}$, where i ranges from 1 to k, are dually independent. We conclude that

$$d(G_{F_1,j} \wedge \ldots \wedge G_{F_k,j}) = k \cdot j = d(G_{F,j}).$$

Since we have $G_{F_i,j} \geq s^{-j}$ for all i, and these elements all lie in F, we have

$$G_{F_1,j} \wedge \ldots \wedge G_{F_k,j} \geq G_{F,j}.$$

We have seen that both sides have the same degree, so we have in fact equality here.

We conclude that

$$F_1 \vee \ldots \vee F_k = \{G_{F_1,j} \wedge \ldots \wedge G_{F_k,j} : j \geq 1\}^{\uparrow} = \{G_{F,j} : j \geq 1\}^{\uparrow} = F.$$

□

We can finally prove:

Theorem 3.2.16. $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ *is closed under taking joins of filters.*

Proof. Let $F, F' \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$. By Proposition 3.2.15, there are saturated filters $F_1, \ldots, F_k, F'_1, \ldots, F'_l \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$ of degree 1 such that $F = F_1 \vee \ldots \vee F_k$ and $F' = F'_1 \vee \ldots \vee F'_l$.

Then $F \vee F' = F_1 \vee \ldots \vee F_k \vee F'_1 \vee \ldots \vee F'_l$ is a join of saturated filters of degree 1, so by Lemma 3.2.12, $F \vee F'$ itself is saturated. □

By Theorem 3.2.16, each $F \in \mathrm{Fil}(G^-)$ contains a unique maximal saturated filter

$$F_{\mathrm{sat}} = \bigvee \{F' \in \mathrm{Fil}(G^-)_{\mathrm{sat}} : F' \subseteq F\}. \tag{3.5}$$

This join is actually infinite, so it is probably not immediately clear why we are allowed to apply the theorem. But note that under proper inclusion of saturated filters, the degree strictly increases, so there cannot be infinite chains of saturated filters. So after joining a suitable finite set of saturated filters contained in F, nothing new will happen.

We conclude:

Theorem 3.2.17. $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ *is a lattice (but not necessarily a sublattice!) under the partial order inherited from* $\mathrm{Fil}(G^-)$. *The respective lattice operations are given for* $F_1, F_2 \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$ *by*

$$F_1 \sqcup F_2 := F_1 \vee F_2 \tag{3.6}$$

$$F_1 \sqcap F_2 := (F_1 \wedge F_2)_{\mathrm{sat}}. \tag{3.7}$$

Proof. Let $F_1, F_2 \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$. Since $F_1 \vee F_2$ is saturated by Theorem 3.2.16, this is clearly the unique minimal saturated filter containing F_1 and F_2. On the other hand, $(F_1 \wedge F_2)_{\mathrm{sat}}$ is clearly the unique maximal saturated filter contained in $F_1 \wedge F_2$. Therefore, it is also the unique maximal saturated filter lying in F_1 and F_2. $\square$

From now on, we keep the symbols $\sqcup, \sqcap$ for the join and meet operations in $\mathrm{Fil}(G^-)_{\mathrm{sat}}$.

One can tell more about the degree of the saturation:

Proposition 3.2.18. *Let $F \in \mathrm{Fil}(G^-)$ and set $d := \lim_{i\to\infty} \iota_i^f(F)$. Then F_{sat} is saturated of degree d.*

To give the proof, we need the following technical lemma:

Lemma 3.2.19. *Let $g \in G^-$ and let $j \geq k \geq 0$ be integers. Furthermore, let $l \leq k$. Then in the left-normal factorizations*

$$\begin{aligned} g \vee s^{-k} &= h_1 \dots h_{k-1} h_k \\ g \vee s^{-j} &= h'_1 \dots h'_{j-1} h'_j \end{aligned}$$

(where we take the missing right terms to be e if the factorizations are shorter than k resp. j) we have

$$h_{k-l+1} h_{k-l+2} \dots h_k \geq h'_{j-l+1} h'_{j-l+2} \dots h'_j.$$

Proof. Using formula (2.5) for the left-normal factors, we calculate

$$\begin{aligned} h'_{j-l+1} h'_{j-l+2} \dots h'_j &= (h'_1 \dots h'_{j-l})^{-1} (g \vee s^{-j}) \\ &= ((g \vee s^{-j}) \curlyvee s^{-(j-l)})^{-1} (g \vee s^{-j}) \\ &= ((g \vee s^{-j})^{-1} \wedge s^{j-l})(g \vee s^{-j}) \\ &= e \wedge s^{j-l}(g \vee s^{-j}) \\ &= e \wedge (s^{j-l} g \vee s^{-l}). \end{aligned}$$

Similarly, $h_{k-l+1} h_{k-l+2} \dots h_k = e \wedge (s^{k-l} g \vee s^{-l})$. Since $s^k \geq s^j$, we also have $s^{k-l} g \geq s^{j-l} g$; therefore, $e \wedge (s^{k-l} g \vee s^{-l}) \geq e \wedge (s^{j-l} g \vee s^{-l})$ which proves the lemma. $\square$

Proof of Proposition 3.2.18. For each $k \geq 0$, we take the *left*-normal factorization

$$G_{F,k} = h_1^{(k)} \dots h_{k-1}^{(k)} h_k^{(k)}.$$

The right-normal factorization of this element is given by

$$G_{F,k} = g_{F,k} g_{F,k-1} \dots g_{F,1}.$$

By Proposition 2.3.12, $d(h_i^{(k)}) = d(g_{F,i}) = \iota_i^f(F)$ for $1 \leq i \leq k$.

Let k_0 be the minimal integer with $\iota_{k_0}^f(F) = d$, then $\iota_i^f(F) = d$ for all $i \geq k_0$.

Let $l \geq 1$ be an arbitrary integer. We want to see how the „tail"

$$H_{k,l} := h_{k-l+1}^{(k)} \dots h_k^{(k)}$$

behaves. If $k \geq k_0 + l - 1$ then $k - l + 1 \geq k_0$, implying that $d(h_i^{(k)}) = d$ for all $i \geq k - l + 1$. Therefore, $d(H_{k,l}) = l \cdot d$ for all $k \geq k_0 + l - 1$. By Lemma 3.2.19, $H_{j,l} \leq H_{k,l}$ holds for $j \geq k$; this shows that $H_{k,l} = H_{k_0+l-1,l}$ for all $k \geq k_0 + l - 1$.

For all $i \geq 1$, the i-th term from the right in the left-normal factorizations of $G_{F,k}$ stabilizes in the following way: for $k \geq k_0 + i - 1$, we have $h_{k-i+1}^{(k)} = h_{k_0}^{(k_0+i-1)}$. Also

$$d(h_{k_0}^{(k_0+i-1)}) = d(H_{k_0+i-1,i} H_{k_0+i-1,i-1}^{-1}) = d \cdot i - d \cdot (i-1) = d.$$

For $i \geq 1$, we set $g_i' := h_{k_0}^{(k_0+i-1)}$. With this definition, we have $d(g_i') = d$ for all i.

For every $l \geq 0$, the product $g_l' g_{l-1}' \dots g_1'$ is then the right-most part of a left-normal factorization of some $G_{F,k}$ and therefore, a left-normal expression. Since all g_i' have degree d, this is factorization is homogeneous and, by Proposition 2.3.11, is also a right-normal expression.

By Proposition 3.2.5, the filter

$$F' := \{g_l' g_{l-1}' \dots g_1' : l \geq 0\}^{\uparrow}$$

has $g_{F',i} = g_i'$ for all i. The filter F' is clearly saturated of degree d. Since every product $g_l' g_{l-1}' \dots g_1'$ is equal to $H_{k,l} \geq G_{F,k}$, for big enough k, we also have $g_l' g_{l-1}' \dots g_1' \in F$. This shows that $F' \subseteq F$. Since F' is saturated, we clearly have $F' \subseteq F_{\text{sat}}$.

Assume that $F' \subsetneq F_{\text{sat}}$. Then F_{sat} would be saturated of a degree $d' > d$. For all $k \geq 0$ we have $G_{F,k} \leq G_{F_{\text{sat}},k}$, which would imply $d(G_{F,k}) \geq d(G_{F_{\text{sat}},k})$. However, we have

$$d(G_{F,k}) = \sum_{i=1}^{k} \iota_i^f(F) = k \cdot d + O(1)$$

and, similarly, $d(G_{F_{\text{sat}},k}) = kd'$, which would result in a contradiction for large enough k.

We conclude that $F' = F_{\text{sat}}$. $\square$

3.3 The lattice structure of $\mathrm{Fil}(G^-)_{\mathrm{sat}}$

In this section we will show that when G is a modular geometric right ℓ-group, then $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is a modular geometric lattice.

We will prove the result by showing that the saturation map

$$\begin{aligned} \mathrm{sat} : \mathrm{Fil}(G^-) &\to \mathrm{Fil}(G^-)_{\mathrm{sat}} \\ F &\mapsto F_{\mathrm{sat}} \end{aligned}$$

is a lattice homomorphism. It will be easy to show that $\mathrm{Fil}(G^-)$ is a modular lattice; Thus, $\mathrm{Fil}(G^-)_{\mathrm{sat}}$, as the image of a modular lattice under a homomorphism, is also modular.

We begin with an easy result:

Lemma 3.3.1. *If G is a modular geometric right ℓ-group then* $\mathrm{Fil}(G^-)$ *is modular.*

Before we give a proof sketch, we give a brief explanation about terminology and notation:

If L is a lattice, then the opposite lattice $\tilde{L}$ is the set L with the opposite order defined by $x \tilde{\leq} y :\Leftrightarrow y \leq x$ which is a lattice under the operations

$$\begin{aligned} x \tilde{\vee} y &= x \wedge y \\ x \tilde{\wedge} y &= x \vee y. \end{aligned}$$

We call a subset $I \subseteq L$ an *ideal* if I is nonempty, closed under taking meets and we have $I^{\downarrow} = I$. This is the same as saying that I is a filter in $\tilde{L}$. Similarly, a filter in L is an ideal in $\tilde{L}$.

It can be shown that $\mathrm{Id}(L)$, the set of ideals in L, is a lattice under the partial order given by inclusion of ideals, and the respective lattice operations are given for $I_1, I_2 \in \mathrm{Id}(L)$ by

$$\begin{aligned} I_1 \wedge I_2 &= I_1 \cap I_2 \\ I_1 \vee I_2 &= \{x_1 \vee x_2 : x_1 \in I_1, x_2 \in I_2\}^{\downarrow}. \end{aligned}$$

Proof of Lemma 3.3.1. The lattice G^- is modular. Therefore, in $\tilde{G^-}$, the dualized modular condition holds, i.e. $x \tilde{\geq} z \Rightarrow (x \tilde{\wedge} (y \tilde{\vee} z)) = (x \tilde{\wedge} y) \tilde{\vee} z$ for all $x, y, z \in \tilde{G^-}$. However, this is the condition for $\tilde{G^-}$ to be modular.

$\mathrm{Id}(\tilde{G^-})$ is modular, since $\tilde{G^-}$ is, by [Grä11, Lemma 59]. Trivially, $\mathrm{Fil}(G^-) \cong \mathrm{Id}(\tilde{G^-})$, so the lattice $\mathrm{Fil}(G^-)$ is modular as well. □

It turns out that saturation process is well-behaved in the following way:

Proposition 3.3.2. *The saturation map*

$$\begin{aligned} \mathrm{sat} : \mathrm{Fil}(G^-) &\to \mathrm{Fil}(G^-)_{\mathrm{sat}} \\ F &\mapsto F_{\mathrm{sat}} \end{aligned}$$

is a lattice homomorphism

For the proof, we need the following lemma:

Lemma 3.3.3. *i) For each $F \in \mathrm{Fil}(G^-)$, the sublattice $[F_{\mathrm{sat}}, F] \subseteq \mathrm{Fil}(G^-)$ has finite length.*

ii) Let $F, F' \in \mathrm{Fil}(G^-)$ and $F' \subseteq F$. If $F' \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$ and the lattice $[F', F]$ is of finite length, then $F_{\mathrm{sat}} = F'$.

Proof. (i) Let $d := \lim_{i\to\infty} \iota_i^f(F)$. Then by Proposition 3.2.18, F_{sat} is saturated of degree d.

Let j_0 be the greatest integer with $\iota_{j_0}^f(F) > d$. For all $i \geq j_0$, we can say that

$$d(G_{F,i}) = \sum_{j=1}^{i} \iota_j^f(F) = (i - j_0) \cdot d + \sum_{j=1}^{j_0} \iota_j^f(F) = (i - j_0) \cdot d + d(G_{F,i_0}). \qquad (*)$$

Since F_{sat} is contained in F, we have for all $i \geq j_0$ that $G_{F_{\mathrm{sat}},i} \vee G_{F,j_0} = G_{F_{\mathrm{sat}},j_0}$. Furthermore, for all $i \geq j_0$ the inequality $G_{F_{\mathrm{sat}},i} \wedge G_{F,j_0} \geq G_{F,i}$ holds for the same reason. But this is actually an equality since

$$\begin{aligned} d(G_{F_{\mathrm{sat}},i} \wedge G_{F,j_0}) &= d(G_{F_{\mathrm{sat}},i}) + d(G_{F,j_0}) - d(\underbrace{G_{F_{\mathrm{sat}},i} \vee G_{F,j_0}}_{=G_{F_{\mathrm{sat}},j_0}}) \\ &= d \cdot i + d(G_{F,i_0}) - d \cdot j_0 \\ &= d \cdot (i - j_0) + d(G_{F,i_0}) \overset{(*)}{=} d(G_{F,i}). \end{aligned}$$

Therefore,

$$\begin{aligned} G_{F,j_0}^{\uparrow} \vee F_{\mathrm{sat}} &= \{G_{F,j_0} \wedge G_{F_{\mathrm{sat}},i} : i \geq 0\}^{\uparrow} \\ &= \{G_{F,j_0} \wedge G_{F_{\mathrm{sat}},i} : i \geq j_0\}^{\uparrow} \\ &= \{G_{F,i} : i \geq j_0\}^{\uparrow} = F. \end{aligned}$$

By the modularity of $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ (Lemma 3.3.1), we have diamond isomorphisms in the lattice $\mathrm{Fil}(G^-)$. Therefore,

$$[F_{\mathrm{sat}}, F] = [F_{\mathrm{sat}}, G_{F,j_0}^{\uparrow} \vee F_{\mathrm{sat}}] \cong [G_{F,j_0}^{\uparrow} \wedge F_{\mathrm{sat}}, G_{F,j_0}^{\uparrow}] \subseteq [e^{\uparrow}, G_{F,j_0}^{\uparrow}].$$

The last interval is of finite length since it is antiisomorphic to $[G_{F,j_0}, e] \subseteq G^-$.

(ii) Assume that $[F', F]$ is of finite length with $F' \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$, but $F' \subsetneq F_{\mathrm{sat}}$. Then $[F', F_{\mathrm{sat}}]$ is also of finite length. The filters F' and F_{sat} must be saturated of different degree, so $G_{F',1} > G_{F_{\mathrm{sat}},1}$. Therefore, there is an $x \prec e$ with $x \in F_{\mathrm{sat}} \setminus F'$. By Lemma 3.2.14, there exists a saturated filter $E \subsetneq F_{\mathrm{sat}}$ of degree 1 such that $G_{E,1} = x$.

The interval $[\{e\}, E] \subseteq \mathrm{Fil}(G^-)$ is an infinite chain by Proposition 3.2.11 and we have $F' \wedge E = \{e\}$. Using diamond isomorphisms, we see that in $\mathrm{Fil}(G^-)$, we have the isomorphisms

$$[\{e\}, E] = [E \wedge F', E] \cong [F', F' \vee E] \subseteq [F', F_{\mathrm{sat}}].$$

This, however, shows the existence of an infinite chain in $[F', F_{\mathrm{sat}}]$ which is a contradiction to this interval being of finite length. It follows that $F' = F_{\mathrm{sat}}$. □

Proof of Proposition 3.3.2. Let $F, F' \in \mathrm{Fil}(G^-)$. Then the intervals $[F_{\mathrm{sat}}, F]$, $[F'_{\mathrm{sat}}, F'] \subseteq \mathrm{Fil}(G^-)$ are of finite length, due to part i) of Lemma 3.3.3. We show that $[F_{\mathrm{sat}} \vee F'_{\mathrm{sat}}, F \vee F']$ has finite length: in the modular lattice $\mathrm{Fil}(G^-)$, we have

$$[F \vee F'_{\mathrm{sat}}, F \vee F'] = [F \vee F'_{\mathrm{sat}}, (F'_{\mathrm{sat}} \vee F) \vee F'] \cong [(F'_{\mathrm{sat}} \vee F) \wedge F', F'] \subseteq [F'_{\mathrm{sat}}, F'],$$

which shows that $[F \vee F'_{\mathrm{sat}}, F \vee F']$ is of finite length. Similarly, the consideration

$$\begin{aligned}[F'_{\mathrm{sat}} \vee F_{\mathrm{sat}}, F'_{\mathrm{sat}} \vee F] &= [F'_{\mathrm{sat}} \vee F_{\mathrm{sat}}, (F_{\mathrm{sat}} \vee F'_{\mathrm{sat}}) \vee F] \\ &\cong [(F_{\mathrm{sat}} \vee F'_{\mathrm{sat}}) \wedge F, F] \\ &\subseteq [F_{\mathrm{sat}}, F]\end{aligned}$$

shows that $[F'_{\mathrm{sat}} \vee F_{\mathrm{sat}}, F'_{\mathrm{sat}} \vee F]$ is of finite length, as well. Therefore, the interval $[F_{\mathrm{sat}} \vee F'_{\mathrm{sat}}, F \vee F']$ is of finite length, due to modularity.

Also, $F_{\mathrm{sat}} \sqcup F'_{\mathrm{sat}} = F_{\mathrm{sat}} \vee F'_{\mathrm{sat}}$ is saturated by Theorem 3.2.16, so part ii) of Lemma 3.3.3 implies that $F_{\mathrm{sat}} \sqcup F'_{\mathrm{sat}} = (F \vee F')_{\mathrm{sat}}$.

By an argument as above, the interval $[F_{\mathrm{sat}} \wedge F'_{\mathrm{sat}}, F \wedge F'] \subseteq \mathrm{Fil}(G^-)$ has finite length. By part i) of Lemma 3.3.3, the interval $[(F_{\mathrm{sat}} \wedge F'_{\mathrm{sat}})_{\mathrm{sat}}, F_{\mathrm{sat}} \wedge F'_{\mathrm{sat}}] \subseteq \mathrm{Fil}(G^-)$ is also of finite length. By modularity, we conclude that the interval

$$[F_{\mathrm{sat}} \sqcap F'_{\mathrm{sat}}, F \wedge F'] = [(F_{\mathrm{sat}} \wedge F'_{\mathrm{sat}})_{\mathrm{sat}}, F \wedge F']$$

has finite length as well, so we must have $F_{\mathrm{sat}} \sqcap F'_{\mathrm{sat}} = (F \wedge F')_{\mathrm{sat}}$ by part ii) of Lemma 3.3.3. □

Let L be a modular lattice. In the modular identity, $x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z$ $(x, y, z \in L)$, we can eliminate the implication arrow as follows: The condition $x \leq z$ is equivalent to $z = x \vee a$ for some $a \in L$, so we can restate the modularity of L in the form $x \vee (y \wedge (x \vee a)) = (x \vee y) \wedge (x \vee a)$ (for $x, y, a \in L$). Since modularity of a lattice (and of course, the condition of being a lattice itself) can be expressed by equations in $\wedge, \vee$ only, modular lattices form what is called a *variety* of lattices [Grä11, Chapter I, Section 5.1]. In particular, this means that modularity is preserved in epimorphic images of modular lattices.

Theorem 3.3.4. $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ *is a modular geometric lattice.*

Proof. $\mathrm{Fil}(G^-)$ is a modular lattice (Lemma 3.3.1) and $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is an epimorphic image thereof (Proposition 3.3.2). This shows that $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is modular. By Proposition 3.2.15, each saturated filter is a join of saturated filters of degree 1. These are the atoms of $\mathrm{Fil}(G^-)_{\mathrm{sat}}$, so $\mathrm{Fil}(G^-)$ is also atomistic. □

3.4 The geometric action

We have seen in the last section that $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is a modular geometric lattice. In this section, we will construct a canonical action of G on $\mathrm{Fil}(G^-)_{\mathrm{sat}}$, thus completing our analysis of saturated filters in the general case.

First, we show that for every $g \in G^-$, the map μ_g, defined as

$$\begin{aligned} \mu_g : \mathrm{Fil}(G^-) &\to \mathrm{Fil}(G^-) \\ F &\mapsto (g \to F), \end{aligned}$$

takes saturated filters to saturated filters. In order to show this, we first prove that each μ_g is a homomorphism of $\vee$-semilattices:

Lemma 3.4.1. *Let $g \in G^-$ and $F_1, F_2 \in \mathrm{Fil}(G^-)$. Then*

$$g \to (F_1 \vee F_2) = (g \to F_1) \vee (g \to F_2).$$

Proof. First of all, $F_1 \vee F_2 = \bigcup_{x_1 \in F_1; x_2 \in F_2} (x_1 \wedge x_2)^{\uparrow}$.

Since μ_g respects meets (equation (S4)) and maps $x^{\uparrow}$ onto $(\mu_g(x))^{\uparrow} = (g \to x)^{\uparrow}$ for all $x \in G^-$ (Proposition 3.2.7), we get that

$$\begin{aligned} g \to (F_1 \vee F_2) &= g \to \left(\bigcup_{x_1 \in F_1; x_2 \in F_2} (x_1 \wedge x_2)^{\uparrow} \right) \\ &= \bigcup_{x_1 \in F_1; x_2 \in F_2} \left(g \to (x_1 \wedge x_2)^{\uparrow} \right) \\ &= \bigcup_{x_1 \in F_1; x_2 \in F_2} \left((g \to (x_1 \wedge x_2))^{\uparrow} \right) \\ &= \bigcup_{x_1 \in F_1; x_2 \in F_2} \left(((g \to x_1) \wedge (g \to x_2))^{\uparrow} \right) \\ &= (g \to F_1) \vee (g \to F_2). \end{aligned}$$

□

We now prove that μ_g restricts to a self-map of $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ for all $g \in G^-$:

Proposition 3.4.2. *For all $g \in G^-$ and $F \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$, we have $g \to F \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$.*

Proof. If F is saturated of degree 0, we have $F = \{e\}$. For all $g \in G^-$, $g \to \{e\} = \{e\}$, so this case is clear.

If F is saturated of degree 1, then F is an infinite chain (Proposition 3.2.11). Write $F = \{G_{F,k} : k \geq 1\}^\uparrow$. Then $g \to F = \{g \to G_{F,k} : k \geq 1\}^\uparrow$ since μ_g is monotone.

For each $k \geq 0$, $G_{F,k}^\uparrow$ is a chain. Therefore, $[G_{F,k}, g \vee G_{F,k}] \subset G_{F,k}^\uparrow$ is also a chain. Using the diamond lemma and right-invariance, we have the isomorphisms

$$[G_{F,k}, g \vee G_{F,k}] \cong [G_{F,k} \wedge g, g] \cong [(G_{F,k} \wedge g)g^{-1}, e] = (g \to G_{F,k})^\uparrow, \qquad (*)$$

which proves that all $(g \to G_{F,k})^\uparrow$ $(k \geq 0)$ are chains. Since $G_{F,k}^\uparrow \subseteq G_{F,k+1}^\uparrow$ holds for all k, and $g \to F$ is the union of all sets $G_{F,k}^\uparrow$, the filter $g \to F$ is a chain.

$g \to F$ is in fact an infinite chain: take k_0 such that $s^{-k_0} \leq g$. Then for all $k \geq k_0$, we have $g \vee G_{F,k} = g \vee s^{-k_0} \vee G_{F,k} = g \vee G_{F,k_0}$. Using the isomorphism $(*)$, we can show that for these k,

$$\begin{aligned} l((g \to G_{F,k})^\uparrow) &= l([G_{F,k}, g \vee G_{F,k}]) \\ &= l([G_{F,k}, G_{F,k_0}]) \\ &= d(G_{F,k}) - d(G_{F,k_0}) \\ &= k - k_0, \end{aligned}$$

so $g \to F$ contains arbitrary long intervals.

We now come to the general case: let $F \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$. By Proposition 3.2.15, we can write $F = F_1 \vee \ldots \vee F_k$ with saturated filters $F_1, \ldots, F_k \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$ of degree 1. By Lemma 3.4.1,

$$g \to F = g \to (F_1 \vee \ldots \vee F_k) = (g \to F_1) \vee \ldots \vee (g \to F_k),$$

and since all operands $g \to F_i$ $(1 \leq i \leq k)$ are saturated of degree 1 (by the degree-1-case above), their join $g \to F$ is also saturated by Theorem 3.2.16. $\square$

We want to show that the operation $(g, F) \mapsto (g \to F)$ defines a left monoid action of G^- on $\mathrm{Fil}(G^-)_{\mathrm{sat}}$:

First of all, we show that $(g, F) \mapsto (g \to F)$ defines a left action of G^- on $\mathrm{Fil}(G^-)$:

Recall that for all $x \in G^-$, we have $e \to x = x$ ((S1a)). This immediately implies that $e \to F = F$ for all $F \in \mathrm{Fil}(G^-)$.

Also recall equation (S5), which tells us that $xy \to z = x \to (y \to z)$ for all $x, y, z \in G^-$. Therefore, for all $g, h \in G^-$, $F \in \mathrm{Fil}(G^-)$, we calculate

$$\begin{aligned}(gh) \to F &= \{(gh) \to x : x \in F\} \\ &= \{g \to (h \to x) : x \in F\} \\ &= g \to \{h \to x : x \in F\} \\ &= g \to (h \to F).\end{aligned}$$

So $\mathrm{Fil}(G^-)$ indeed becomes a left G-set under this operation. By Proposition 3.4.2, each μ_g $(g \in G^-)$ maps $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ to itself, so the restriction to $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is indeed the stated left action of G^- on $\mathrm{Fil}(G^-)_{\mathrm{sat}}$.

This is, however, more than just some monoid action on some set, as we will now see:

Recall that we have called an element $g \in G^-$ *homogeneous* when in the right-normal factorization $g = g_k g_{k-1} \dots g_1$, we have $d(g_1) = \dots = d(g_{k-1}) = d(g_k)$.

Furthermore, in the following, for any pair $x, y \in G$, we write $x^y := y^{-1}xy$.

Lemma 3.4.3. *Let $g \in G^-$ be homogeneous with the right-normal factorization $g = g_k g_{k-1} \dots g_1$. Then $s^{-1} \to g = g_{k-1}^s \dots g_1^s$.*

Proof. First of all, $g_1 = g \vee s^{-1}$ (see equation (2.2)).

We show that $s^{-1} \wedge g = s^{-1} g_{k-1} \dots g_1$: since $s^{-1} \leq g_k$, we already know that

$$s^{-1} g_{k-1} \dots g_1 \leq g_k g_{k-1} \dots g_1.$$

Therefore, $s^{-1} \wedge g \geq s^{-1} g_{k-1} \dots g_1$.

Using the parallelogram identity (Proposition 2.3.5), we calculate

$$\begin{aligned}d(g \wedge s^{-1}) &= d(g) + d(s^{-1}) - d(g \vee s^{-1}) \\ &= d(g) + d(s^{-1}) - d(g_1) \\ &= d(g) + d(s^{-1}) - d(g_k) \\ &= d(s^{-1} g_k^{-1} g) \\ &= d(s^{-1} g_{k-1} \dots g_1),\end{aligned}$$

which implies that, in fact, $s^{-1} \wedge g = s^{-1} g_{k-1} \dots g_1$. We conclude that

$$\begin{aligned}s^{-1} \to g &= (s^{-1} \wedge g)s \\ &= s^{-1} g_{k-1} \dots g_1 s \\ &= g_{k-1}^s \dots g_1^s.\end{aligned}$$

□

Lemma 3.4.4. *The map $\mu_{s^{-1}} : \mathrm{Fil}(G^-)_{\mathrm{sat}} \to \mathrm{Fil}(G^-)_{\mathrm{sat}}$ is a lattice automorphism.*

Proof. Recall that $F = \{G_{F,k} : k \geq 1\}^{\uparrow}$ and note that the right-normal factorizations $G_{F,k} = g_{F,k} g_{F,k-1} \ldots g_{F,1}$ are homogeneous since F is saturated.

By the above lemma, for all $k \geq 1$, we have $s^{-1} \to G_{F,k} = g^s_{F,k-1} \cdots g^s_{F,1} = G^s_{F,k-1}$. Therefore,

$$\begin{aligned} s^{-1} \to F &= \{s^{-1} \to G_{F,k} : k \geq 1\}^{\uparrow} \\ &= \{G^s_{F,k-1} : k \geq 1\}^{\uparrow} \\ &= \left(\{G_{F,k} : k \geq 0\}^{\uparrow}\right)^s = F^s. \end{aligned}$$

So $\mu_{s^{-1}}(F) = F^s$ for all $F \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$.

Conjugation by s is a lattice automorphism of G, so the map $F \mapsto F^s$ is also a lattice automorphism of $\mathrm{Fil}(G^-)$. It remains to show that $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is invariant under conjugation by s: in order to prove this, let $F \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$. Then for all $k \geq 0$,

$$\begin{aligned} G_{F^s,k} &= \min(F^s \wedge (s^{-k})^{\uparrow}) \\ &= \min(F^s \vee ((s^{-k})^{\uparrow})^s) \\ &= \left(\min(F \vee (s^{-k})^{\uparrow})\right)^s \\ &= G^s_{F,k}. \end{aligned}$$

Since $d(G^s_{F,k}) = d(G_{F,k})$ $(k \geq 0)$, we have for all $i \geq 1$ that

$$\iota^f_i(F^s) = d(G_{F^s,i}) - d(G_{F^s,i-1}) = d(G_{F,i}) - d(G_{F,i-1}) = \iota^f_i(F),$$

so F^s is also saturated. By a similar argument, one proves that $\mu^{-1}_{s^{-1}}(F) = F^{s^{-1}}$ is also saturated.

Therefore, $F \mapsto F^s$ restricts to a lattice automorphism of $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ which is exactly $\mu_{s^{-1}}$. ☐

This simple lemma already suffices to prove:

Proposition 3.4.5. *For each $g \in G^-$, the map $\mu_g : \mathrm{Fil}(G^-)_{\mathrm{sat}} \to \mathrm{Fil}(G^-)_{\mathrm{sat}}$ is a lattice automorphism.*

Proof. Let $g \in G^-$. Since each μ_g clearly respects inclusion of filters, it is at least order-preserving. Now take an integer k with $s^{-k} \leq g$. Then $s^{-k}g^{-1} \leq e$ and $g^{-1}s^{-k} \leq e$.

Then $\mu_{s^{-k}g^{-1}} \mu_g = \mu_g \mu_{g^{-1}s^{-k}} = \mu_{s^{-k}}$. Since $\mu_{s^{-k}}$ is a lattice automorphism (Lemma 3.4.4) and $\mu_{s^{-k}g^{-1}}$ and $\mu_{g^{-1}s^{-k}}$ are order-preserving, it follows that μ_g is an order-equivalence and, thus, a lattice automorphism. ☐

We can now prove the desired result which generalizes Corollary 3.1.7 to all modular geometric right ℓ-groups.

Theorem 3.4.6. *There is a unique right action*

$$\begin{aligned} \mathrm{Fil}(G^-)_{\mathrm{sat}} \times G &\to \mathrm{Fil}(G^-)_{\mathrm{sat}} \\ (F, g) &\mapsto F^g \end{aligned}$$

such that $F \mapsto F^g$ is a lattice automorphism for all $g \in G$ and such that for all $g \in G^-$ we have $F^{g^{-1}} = \mu_g(F)$.

Proof. By Proposition 3.4.5, there is a monoid homomorphism given by

$$\begin{aligned} \mu : G^- &\to \mathrm{Aut}(\mathrm{Fil}(G^-)_{\mathrm{sat}}) \\ g &\mapsto \mu_g \end{aligned}$$

which has a unique extension to a group homomorphism

$$\overline{\mu} : G \to \mathrm{Aut}(\mathrm{Fil}(G^-)_{\mathrm{sat}})$$

(Proposition 2.1.10). Defining $F^g := \overline{\mu}_g^{-1}(F)$ for $g \in G$ now gives the desired right action. ☐

3.5 Case study: Structure groups of the set-theoretic Yang-Baxter equation

Geometric right ℓ-groups have been studied by Rump ([Rum15]) as a generalization of certain structure groups of the set-theoretic Yang-Baxter equation. This was motivated by a result of Chouraqui ([Cho10]) showing that the structure groups of involutive non-degenerate solution of the Yang-Baxter equation are in fact Garside groups with a distributive lattice structure. Vice versa, each such Garside group is a structure group for such a solution.

Since the introduction, we have not talked about the Yang-Baxter equation anymore. However, we have proved that *all* modular geometric right ℓ-groups act on a modular geometric lattice, not just the desarguesian ones. So it is only natural to go „back to the roots“ to see what Theorem 3.3.4 and Theorem 3.4.6 mean for the structure groups of solutions of the Yang-Baxter equation. It will turn out that the action on the saturated filters is a well-known invariant of the solution.

Let X be a finite set and $R : X \times X \to X \times X$ a map. From such a map R we can always derive two binary operations on X, written as $(x, y) \mapsto {}^y x$ and $(x, y) \mapsto y^x$, which are given by $R(x, y) = ({}^y x, y^x)$. For all $y \in X$, this defines a map

$$\begin{aligned} \rho_y : X &\to X \\ x &\mapsto x^y. \end{aligned}$$

Similarly, for all $x \in X$, we have a map

$$\begin{aligned} \lambda_x : X &\to X \\ y &\mapsto {}^x y. \end{aligned}$$

Definition 3.5.1. Let X be a finite set. A map $R : X \times X \to X \times X$ is called

i) *non-degenerate*, if the maps ρ_y, λ_x are bijective for any $x, y \in X$,

ii) *involutive*, if $R^2 = \mathrm{id}_{X \times X}$,

iii) a *solution* of the *set-theoretic Yang-Baxter equation* (*SYBE*, for short), if it fulfills the equation

$$R^{12} R^{23} R^{12} = R^{23} R^{12} R^{23}, \tag{YB}$$

where R^{ij} is the self-map of $X \times X \times X$ which acts as R on the i-th and j-th coordinate and keeps the remaining coordinate untouched.

Among the best-understood solutions of the SYBE are those which are involutive and non-degenerate, and no other solutions will be regarded here. For this reason, when speaking of a *solution of the SYBE*, we will always mean a non-degenerate, involutive solution.

We express (YB) in terms of the operations ${}^y z$, y^z. For all $x, y, z \in X$ we calculate

$$\begin{aligned} (R^{12} R^{23} R^{12})(x, y, z) &= (R^{12} R^{23}) \left({}^x y, x^y, z\right) \\ &= R^{12} \left({}^x y, {}^{(x^y)} z, (x^y)^z\right) \\ &= \left({}^{({}^x y)} \left({}^{(x^y)} z\right), ({}^x y)^{\left({}^{(x^y)} z\right)}, (x^y)^z\right). \end{aligned}$$

Similarly, we calculate

$$(R^{23} R^{12} R^{23})(x, y, z) = \left({}^x ({}^y z), \left({}^{x^{({}^y z)}}\right) (y^z), \left(x^{({}^y z)}\right)^{(y^z)}\right).$$

(YB) states that these terms must be the same, which results in the following three equations:

$$\begin{aligned} {}^{({}^x y)} \left({}^{(x^y)} z\right) &= {}^x ({}^y z) &\text{(YB1)} \\ ({}^x y)^{\left({}^{(x^y)} z\right)} &= \left({}^{x^{({}^y z)}}\right) (y^z) &\text{(YB2)} \\ (x^y)^z &= \left(x^{({}^y z)}\right)^{(y^z)} &\text{(YB3)} \end{aligned}$$

This already gives an impression of the complexity of the equation (YB).

We can also make the involutive condition $R^2 = \mathrm{id}_{X \times X}$ explicit. For $x, y \in X$, we have

$$R^2(x, y) = R\left({}^x y, x^y\right) = \left({}^{({}^x y)} (x^y), ({}^x y)^{(x^y)}\right),$$

so the involutiveness reads

$$^{(^x y)}(x^y) = x \tag{I1}$$
$$(^x y)^{(x^y)} = y. \tag{I2}$$

Example 3.5.2. For every finite set X, the map given by $R(x, y) = (y, x)$ $(x, y \in X)$ is a solution of the SYBE, which is called the *flip* (or, the *trivial* solution) for obvious reasons. Other examples of solutions can be constructed by using *cycle sets* (see below).

Algebraic invariants - associative or nonassociative - are among the most important tools for studying the SYBE. In the following, we will talk about some very basic associative invariants.

With each solution of the SYBE, one can associate a *structure monoid*, a *structure group* and a *permutation group*. These algebraic structures already appeared in the landmark paper of Etingof, Schedler and Soloviev [ESS99] and are defined as follows:

Definition 3.5.3. Let (X, R) be a solution of the SYBE.

i) The *structure monoid* $S(X, R)$ is the monoid defined by generators and relations as
$$S(X, R) := \langle X \mid xy = {}^x y x^y \quad (x, y \in X) \rangle_{\text{mon}} .$$

ii) The *structure group* $G(X, R)$ is the group defined by generators and relations as
$$G(X, R) := \langle X \mid xy = {}^x y x^y \quad (x, y \in X) \rangle_{\text{gr}} .$$

iii) The *permutation group* $\mathcal{G}(X, R)$ is the subgroup of $\mathrm{S}(X)$ generated by all ρ_y $(y \in X)$, i.e.
$$\mathcal{G}(X, R) := \langle \rho_y : y \in X \rangle \subseteq \mathrm{S}_X,$$
where S_X denotes the symmetric group on the set X.

The structure group $G(X, R)$ is connected with the permutation group $\mathcal{G}(X, R)$ by the right action given by

$$\begin{aligned} X \times G(X, R) &\to X \\ (x, y) &\mapsto x^y \quad (\text{for } x, y \in X). \end{aligned}$$

Since this action is only defined on the generators $y \in X \subseteq G(X, R)$, it should be checked if this is consistent with the relations holding in $G(X, R)$. These are $yz = {}^y z y^z$ for $y, z \in Z$. We should therefore have for all $x, y, z \in Z$ the equality

$$(x^y)^z = (x^{(^y z)})^{(y^z)},$$

which is exactly (YB3).

Therefore, we have indeed defined a right-action of $G(X,R)$ on X. Since $x^y = \rho_y(x)$ and $G(X,R)$ is generated by X, the permutation group $\mathcal{G}(X,R)$ is the image of the homomorphism $G(X,R)^{\mathrm{op}} \to \mathrm{S}_X$ resulting from the group action.

The following results of Chouraqui and Rump connect solutions of the SYBE with geometric right ℓ-groups (resp. Garside groups).

Theorem 3.5.4. *Let (X,R) be a solution of the SYBE. Then the canonical monoid homomorphism $S(X,R) \hookrightarrow G(X,R)$ is an embedding which identifies $S(X,R)$ with the negative cone $G(X,R)^-$ of a right-invariant lattice-order on $G(X,R)$. This lattice-order makes $G(X,R)$ a distributive geometric right ℓ-group with $X(G(X,R)^-) = X$.*

Vice versa, each distributive geometric right ℓ-group G is isomorphic to some group $G(X,R)$ where (X,R) is a solution of the SYBE.

Proof. For the fact that $G(X,R)$ is a right ℓ-group such with $X(G(X,R)^-) = X$, see [Cho10, Theorem 3.3]. The proof that $G(X,R)$ is a distributive geometric right ℓ-group can be found in [Rum15, Theorem 2, Proposition 7]. For the result that each distributive geometric right ℓ-group is isomorphic to some structure group, see also [Rum15, Theorem 2] (which treats a more general case). □

Remark 3.5.5. We remind the reader that we have restricted our notion of geometricity to right ℓ-groups with a strong order unit. However, Rump's notion of geometricity embraces cases where there is no strong order unit. One central result in his work is the characterization of the right ℓ-groups coming from solutions of the SYBE where the underlying set X does not need to be finite - these are the modular, noetherian right ℓ-groups with a *duality*. The duality assumption implies that such a right ℓ-group is distributive. These results can be found in the article [Rum15].

Remark 3.5.6. In fact, if $|X| =: \delta$, then the right ℓ-group $G(X,R)$ „looks like“ $\mathbb{Z}^\delta$. We explain what we mean by this and sketch a proof.

We have already seen that $G(X,R)$ acts from the right on X. Denote this action by x^y. We now extend X to $\mathbb{Z}^X$, the abelian group of $\mathbb{Z}$-linear combinations of formal elements of X. Writing the elements of $\mathbb{Z}^X$ as $\sum_{x\in X} n_x[x]$ (all $n_x \in \mathbb{Z}$), we can define a right-action of $G(X,R)$ by

$$\begin{aligned} \cdot : \mathbb{Z}^X \times X &\to \mathbb{Z}^X \\ (\sum_{x\in X} n_x[x], y) &\mapsto [y] + \sum_{x\in X} n_x\,[x^y]\,. \end{aligned}$$

This definition is indeed compatible with the group relations $yz = {}^y z y^z$, as can be seen from the calculations

$$\left(\left(\sum_{x\in X} n_x[x]\right)\cdot y\right)\cdot z = \left([y] + \sum_{x\in X} n_x\,[x^y]\right)\cdot z$$

$$= [z] + [y^z] + \sum_{x\in X} n_x \left[(x^y)^z\right].$$

$$\left(\left(\sum_{x\in X} n_x[x]\right)\cdot {}^y z\right)\cdot y^z = \left([{}^y z] + \sum_{x\in X} n_x \left[x^{({}^y z)}\right]\right)\cdot y^z$$

$$= [y^z] + \left[({}^y z)^{(y^z)}\right] + \sum_{x\in X} n_x \left[\left(x^{({}^y z)}\right)^{(y^z)}\right]$$

$$= [y^z] + [z] + \sum_{x\in X} n_x \left[(x^y)^z\right] \quad \text{by (YB3), (I2).}$$

We see that this definition results in a well-defined group action. It can be proven that this action is regular on $\mathbb{Z}^\delta$. This is already stated in [ESS99, Proposition 2.5.] in terms of 1-cocycles.

We now equip $\mathbb{Z}^X$ with the lattice order given by

$$\sum_{x\in X} n_x[x] \leq \sum_{x\in X} n'_x[x] \quad :\Leftrightarrow \quad \forall x\in X : n'_x \leq n_x,$$

which is the opposite coordinatewise order and easily seen to be distributive. The action of $G(X,R)$ is by lattice-automorphisms of $\mathbb{Z}^X$, since every element of $G(X,R)$ acts as a permutation of the coordinates followed by a translation. Also, one can show that $0\cdot g \leq 0 \Leftrightarrow g \in S(X,R)$ (see the proof of [Rum15, Theorem 2]). Therefore, the opposite coordinatewise order of $\mathbb{Z}^X$ is isomorphic to the right-invariant lattice order of $G(X,R)$ defined by the negative cone $S(X,R)$.

We now give a definition of a *cycle set* - an algebraic structure which has been introduced by Rump ([Rum05]) in order to parametrize solutions of the SYBE:

Definition 3.5.7. A *cycle set* is a finite set X with a binary operation $\circ : X\times X \to X$ such that

i) for all $x,y,z\in X$, we have

$$(x\circ y)\circ(x\circ z) = (y\circ x)\circ(y\circ z), \tag{C}$$

ii) for all $x\in X$, the map $\sigma_x : X\to X$; $y\mapsto x\circ y$ is a bijection.

We call a cycle-set X *non-degenerate* if the square map $X\to X$; $x\mapsto x\circ x$ is bijective.

Example 3.5.8. 1) On each set X, the operation $x\circ y = y$ $(x,y\in X)$ defines a non-degenerate cycle set structure.

2) On the set $X=\{1,2,3,4\}$, there is a nontrivial cycle set structure on X, which is defined by the multiplication table

$\circ$	1	2	3	4
1	4	2	3	1
2	3	1	4	2
3	1	3	2	4
4	2	4	1	3

This example is borrowed from [Ven16].

3) A systematic way of constructing many nontrivial cycle sets - the so-called *linear cycle sets* - is by the use of *braces* ([Rum07]).

Rump proved that cycle sets parametrize solutions of the SYBE in the following sense:

Theorem 3.5.9. *When $R : X \times X \to X \times X$; $(x, y) \mapsto ({}^x y, x^y)$ is a solution of the SYBE, then X becomes a nondegenerate cycle set under the binary operation*

$$x \circ y = \rho_x^{-1}(y).$$

Vice versa, any nondegenerate cycle set X defines a solution $R : X \times X \to X \times X$ of the SYBE via

$$R(x, y) = ({}^x y, x^y) =: ((\sigma_x^{-1}(y)) \circ y, \sigma_x^{-1}(y))$$

Proof. See [Rum05, Proposition 1]. □

Using the correspondence given by Theorem 3.5.9, we can rewrite the relations $xy = {}^x y x^y$ of the structure monoid $S(X)$ resp. the structure group $G(X)$ as follows:

We set $y = a$. Let $b \in X$. Since $\rho_y : X \to X$ is bijective, there is a unique $x \in X$ such that $x^y = b$, namely $x = \rho_y^{-1}(b) = a \circ b$. Furthermore, ${}^x y = x^y \circ y = b \circ a$. With these substitutions, the relations of $S(X, R)$ resp. $G(X, R)$ read as $(a \circ b)a = (b \circ a)b$ $(a, b \in X)$.

For the rest of this section, we assume that $G = G(X, R)$ is the right ℓ-group with the negative cone $G^- = S(X, R)$ where $R : X \times X \to X \times X$ is a solution of the SYBE. Also we let $\mathcal{G} = \mathcal{G}(X, R)$ be the permutation group of the solution. We assume that $\circ$ is the cycle set operation of X which is associated with R. As before, $s \in G$ will be the strong order unit defined by $s^{-1} = \mathrm{Rad}(e)$.

Note that the strong order interval $[s^{-1}, e] \subseteq G$ is distributive (Theorem 3.5.4) and dually atomistic. Therefore, $[s^{-1}, e]$ is a finite Boolean algebra whose dual atoms are the elements of X, s^{-1} is the bottom element and e is the top element. From the Boolean property, we also infer that the dimension of G, as a geometric right ℓ-group, is $l([s^{-1}, e]) = |X| = \delta$ (which justifies our choice to denote the quantity $|X|$ by δ).

We want to determine the saturated filters in G^-. To achieve this goal, we need to understand what the right-normal factorizations of constant degree look like. We start with some lemmata.

As X is embedded as the set of dual atoms in G^-, we sometimes apply cycle-set operations to these elements. Note that we can not (and will not) do that with other elements of G.

Proposition 3.5.10. *For all $x, y \in X$ we have*

$$x \to y = \begin{cases} x \circ y & x \neq y \\ e & x = y. \end{cases}$$

Proof. The case when $x = y$ is clear. So we assume that $x \neq y$. Since $x \neq y$, we have $d(x \wedge y) > d(x) = 1$. On the other hand, the relation $(x \circ y)x = (y \circ x)y$ shows that $((x \circ y)x) \leq x \wedge y$. But $d((x \circ y)x) = 2$; therefore, $(x \circ y)x = x \wedge y$. We conclude that $x \to y = (x \wedge y)x^{-1} = x \circ y$. □

Lemma 3.5.11. *For each $g \in [s^{-1}, e]$, there is a unique element $D(g) \in [s^{-1}, e]$ such that $d(D(g)) = d(g)$ and $D(g)g$ is a right-normal expression (here we allow trivial factors in case that $g = e$). If $x \in X$, then $D(x) = x \circ x$.*

We need this property of G only as a lemma here. However, in the bigger framework of modular geometric right ℓ-groups, the existence of such a duality actually *characterizes* the structure groups of the SYBE, as has been proven by Rump ([Rum15, Theorem 2.]).

Proof. For $g = e$, we simply take $D(e) = e$. Now let $g < e$. An factorization $g'g$ with $g, g' \in [s^{-1}, e]$ is right-normal if and only if $g' \vee (s^{-1}g^{-1}) = e$, by (2.1). Assume that $g' \in [s^{-1}, e]$ is such that $g'g$ is a right-normal factorization.

We have $d(s^{-1}g^{-1}) = \delta - d(g)$. If additionally $d(g') = d(g)$, then we calculate, using the parallelogram identity (Proposition 2.3.5):

$$d(s^{-1}g^{-1} \wedge g') = d(s^{-1}g^{-1}) + d(g') - d((s^{-1}g^{-1}) \vee g') = d(g) + \delta - d(g) - 0 = \delta.$$

Since $s^{-1}g^{-1} \wedge g' \geq s^{-1}$, this calculation implies that $s^{-1}g^{-1} \wedge g' = s^{-1}$, so g' must be a complement to $s^{-1}g^{-1}$ in $[s^{-1}, e]$. Since this interval, as a lattice, is Boolean, there is exactly one such $g' \in [s^{-1}, e]$. With this g', we have $D(g) = g'$.

Let now $x \in X$. We claim that $(x \circ x)x$ is a cochain in G^-. Assume to the contrary that other dual atoms lie above $(x \circ x)x$, then there is another $y \in X$

with $y \neq x$ such that $D(x)x < y$. This implies that

$$\begin{aligned}
e &= ((x \circ x)x) \to y && \text{Proposition 2.1.11, (S7)}\\
&= (x \circ x) \to (x \to y) && \text{Proposition 2.1.11, (S5)}\\
&= (x \circ x) \to \underbrace{(x \circ y)}_{\neq (x \circ x)} && \text{(Proposition 3.5.10)}\\
&= (x \circ x) \circ (x \circ y) && \text{(Proposition 3.5.10 again)}\\
&\in X.
\end{aligned}$$

which is a contradiction. We have shown that $(x \circ x)x$ is a cochain. By Corollary 2.3.15 , $(x \circ x)x$ is a right-normal expression, from which we deduce that $D(x) = x \circ x$. □

From the lemma, we can deduce:

Proposition 3.5.12. *For each $g \in [s^{-1}, e]$ there is a unique saturated filter $F =: F(g) \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$ such that $g_{F,1} = g$. This filter is given by*

$$F(g) = \{D^{k-1}(g) \dots D(g)g : k \geq 0\}^{\uparrow}.$$

Furthermore, the assignment $[s^{-1}, e] \to \mathrm{Fil}(G^-)_{\mathrm{sat}}$; $g \mapsto F(g)$ is an antiisomorphism of lattices. In particular, $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is a Boolean lattice with the atoms $F(x)$ $(x \in X)$.

Proof. It is easy that the only filter $F \in \mathrm{Fil}(G^-)_{\mathrm{sat}}$ with $\iota_1^f(F) = e$ is $F = \{e\}$. Now assume that $g < e$. By Proposition 3.2.5, a filter F in G^- is given by a sequence $(g_i)_{i \geq 1}$, where all $g_i \in [s^{-1}, e]$, such that for each k, the expression $g_k g_{k-1} \dots g_1$ is right-normal. In particular, the expression $g_{i+1}g_i$ is right-normal for every $i \geq 1$.

Since this filter is saturated if and only if $d(g_i)$ is a constant, the lemma above implies that $g_{i+1} = D(g_i)$. By induction, this means $g_i = D^{i-1}(g_1)$. Given $g_1 = g$, this is only possible when $g_i = D^{i-1}(g)$.

The lemma also tells that for the sequence $(g_i)_{i \geq 1}$ defined by $g_i = D^{i-1}(g)$, the expressions $g_k g_{k-1} \dots g_1$ are indeed right-normal for all $k \geq 0$. Thus, this sequence defines a saturated filter $F := F(g)$ with $g_{F,1} = g$, and there is no other such filter.

We conclude that the assignment $[s^{-1}, e] \to \mathrm{Fil}(G^-)_{\mathrm{sat}}$; $g \mapsto F(g)$ is a bijection with inverse $F \mapsto \iota_1^f(F)$.

The map $F \mapsto \iota_1^f(F) = \min(F \wedge s^{\uparrow})$ is clearly order-reversing. We prove that $g \mapsto F(g)$ is order-reversing as well: given $g \in [s^{-1}, e]$, we write $g = x_1 \wedge \dots \wedge x_k$ where $\{x_1, \dots, x_k\} = X \cap g^{\uparrow}$.

The filter $F(g)$ is saturated and contains the elements x_i, so we can find, by Lemma 3.2.14, saturated filters $F_i \subseteq F(g)$ $(1 \leq i \leq k)$ of degree 1 such that $\iota_1^f(F_i) = x_i$. By uniqueness, $F_i = F(x_i)$.

The filter $F(x_1) \vee \ldots \vee F(x_k)$ is saturated, clearly contains $g = x_1 \wedge \ldots \wedge x_k$, and is contained in $F(g)$ itself. It follows that $F(g) = F(x_1) \vee \ldots \vee F(x_k)$. We have therefore shown that for all $g \in [s^{-1}, e]$,

$$F(g) = \bigvee_{x \in X \cap g^{\uparrow}} F(x).$$

If $g, g' \in [s^{-1}, e]$ fulfill $g \leq g'$, then $X \cap g^{\uparrow} \supseteq X \cap g'^{\uparrow}$, which implies that $F(g) \supseteq F(g')$ in this case. So, $g \mapsto F(g)$ is also order-reversing and thus, an antiisomorphism of lattices.

Since $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is antiisomorphic to the Boolean lattice $[s^{-1}, e]$, it is a Boolean lattice itself. $\square$

From now on, for $g \in [s^{-1}, e]$, we denote by $F(g)$ the unique saturated filter F „ending in g“, i.e. the one with $g_{F,1} = g$.

We can now calculate the arrow operation of G^- on $\mathrm{Fil}(G^-)_{\mathrm{sat}}$:

Lemma 3.5.13. *For all $x, y \in X$, we have $x \to F(y) = F(x \circ y)$.*

Proof. By Proposition 3.5.12, we have $F(y) = \{y,\ D(y)y,\ D^2(y)D(y)y,\ \ldots\}^{\uparrow}$.

If $x \neq y$, then

$$F' := x \to F(y) = \{x \to y,\ x \to D(y)y,\ \ldots\}^{\uparrow}.$$

By the proof of Proposition 3.4.2, $x \to F(y)$ is saturated of degree 1. Since $x \to F$ contains $x \to y = x \circ y \neq e$ (here we are using Proposition 3.5.10), we have $g_{F',1} = x \circ y$. By Proposition 3.5.12, this implies $F' = F(x \circ y)$, in this case.

If $x = y$, we have

$$\begin{aligned} x \to F(y) &= \{x \to x,\ x \to D(x)x,\ x \to D^2(x)D(x)x,\ \ldots\}^{\uparrow} \\ &= \{e,\ D(x),\ D^2(x)D(x),\ \ldots\}^{\uparrow} \\ &= F(D(x)) = F(x \circ x), \end{aligned}$$

where $D(x) = x \circ x$ comes from Lemma 3.5.11. $\square$

It turns out that the geometric action of G on the lattice $\mathrm{Fil}(G)_{\mathrm{sat}}$ of saturated filters is equivalent to the permutation action given on page 126.

Theorem 3.5.14. *Let G act on X by the permutation action (page 126) and on $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ by the geometric action. Then the map $X \to \mathrm{Fil}(G^-)_{\mathrm{sat}};\ y \mapsto F(y)$ is a morphism of G-sets that identifies X with the set of atoms in $\mathrm{Fil}(G^-)_{\mathrm{sat}}$.*

Proof. $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is a finite Boolean algebra with the atoms $F(y)$ $(y \in X)$ (Proposition 3.5.12). This shows that X is mapped bijectively on the set of atoms in $\mathrm{Fil}(G^-)_{\mathrm{sat}}$.

Using Lemma 3.5.13 and the cycle set correspondence Theorem 3.5.9, we get for any $x, y \in X$ that

$$F(y)^{x^{-1}} = x \to F(y) = F(x \circ y) = F(\rho_x^{-1}(y))$$

which implies that for all $x, y \in X$, $F(y)^x = F(\rho_x(y))$. Since X generates G, the theorem follows. □

Since automorphism of a finite Boolean algebra is already determined by a permutation of its atoms, we can infer from this theorem:

Corollary 3.5.15. *The permutation group $\mathcal{G}$ is isomorphic to the image of the group homomorphism $G^{\mathrm{op}} \to \mathrm{Aut}(\mathrm{Fil}(G^-)_{\mathrm{sat}})$ induced by the geometric action.*

We have therefore found a meaningful generalization of the permutation group associated with a (non-degenerate and involutive) solution of the SYBE:

Definition. Let G be a modular geometric right ℓ-group. The *permutation group* $\Pi(G)$ is the image of the group homomorphism $G^{\mathrm{op}} \to \mathrm{Aut}(\mathrm{Fil}(G^-)_{\mathrm{sat}})$ induced by the geometric action.

It may be unusual to close this work with a definition. However, we can not say much interesting about these groups yet.

Suppose that G is realizable as a complement H of $\mathrm{P\Gamma L}({}_R R^\delta) \leq \mathrm{P\Gamma L}({}_R Q^\delta)$, where Q is a complete dvf with dvr R. Then $\Pi(G)$ is isomorphic to the image of H under the canonical homomorphism $\mathrm{P\Gamma L}({}_R Q^\delta) \to \mathrm{P\Gamma L}({}_Q Q^\delta)$. Identifying $\mathrm{P\Gamma L}({}_R Q^\delta) = \mathrm{\Gamma L}({}_R Q^\delta)/R^\times$, the kernel of this homomorphism is the subgroup $Q^\times R^\times \leq \mathrm{\Gamma L}({}_R Q^\delta)/R^\times$. In this case, it follows that we have an isomorphism

$$\Pi(G) \cong H/(H \cap Q^\times R^\times).$$

Note that the image of $Q^\times$ under the map $\mathrm{\Gamma L}({}_R Q^\times) \twoheadrightarrow \mathrm{P\Gamma L}({}_R Q^\times)$ is isomorphic to $Q^\times/R^\times$ which is an infinite cyclic subgroup. This implies that either $\Pi(G) \cong H$ or $\Pi(G)$ is isomorphic to a quotient of H by an infinite cyclic subgroup.

However, more general statements about the groups $\Pi(G)$ would be desirable.

So, this is the definition. *What are the theorems?*

Appendix A

Finite desarguesian geometries as strong order intervals

We want to prove that each finite desarguesian geometry is a strong order interval in a suitable right ℓ-group, thus answering a question of Rump [Rum15, p.507].

In what follows, L is a commutative field and $\sigma : L \to L$ is a field automorphism. Furthermore, we set $K := \{x \in L : \sigma(x) = x\}$. We assume that $\delta := \dim_K(L) < \infty$, so that L/K is a cyclic field extension of finite degree.

We repeat the construction of the ring $L[x, \sigma]$ from Section 1.2: the ring $L[x, \sigma]$ is defined as the set of all expressions $\sum_{i=0}^{\infty} a_i x^i$ with $a_i \in L$ for all $i \geq 0$ and at most finitely many $a_i \neq 0$.

For $p, q \in L[x, \sigma]$ with $p = \sum_{i=0}^{\infty} a_i x^i$, $q = \sum_{i=0}^{\infty} b_i x^i$, sum and product are defined by

$$p + q = \sum_{i=0}^{\infty} (a_i + b_i) x^i \quad ,$$
$$p \cdot q = \sum_{n=0}^{\infty} \left(a_i \cdot \sigma^i(b_{n-i}) \right) x^n.$$

It can be shown that these operations turn $L[x, \sigma]$ into a ring. These so-called *twisted polynomial rings* belong to the slightly broader class of *skew polynomial rings* ([MR01, Chapter 1,§2]).

We define a *degree function* deg on $L[x, \sigma]$ by

$$\begin{aligned} \deg : L[x, \sigma] &\to \{-\infty\} \cup \mathbb{Z}_0^+ \\ p = \sum_{i=0}^{\infty} a_i x^i &\mapsto \begin{cases} -\infty & p = 0 \\ \max\{i \in \mathbb{Z}_0^+ : a_i \neq 0\} & p \neq 0. \end{cases} \end{aligned}$$

Note that for all $p, q \in L[x, \sigma]$, we have the identity $\deg(pq) = \deg(p) + \deg(q)$ which can be proven in the same way as in the „untwisted“ case of polynomial rings over a commutative field. Similar to the classical case, this identity implies that $L[x, \sigma]$ is (noncommutative) domain.

Let Ω be a well-ordered set and R a noncommutative ring. Furthermore, let $\deg : R \to \Omega$ be a function. We call R *left-euclidean* ([Bru73]) with respect to the *degree function* deg, if for any $s, q \in R$ with $q \neq 0$, there are $p, r \in R$ such that $s = p \cdot q + r$ and $\deg(r) < \deg(q)$.

We call a noncommutative unital ring R a *principal left ideal* ring (*pli*-ring, for short) if every left ideal $I \subseteq R$ is of the form $I = Rx$ with $x \in R$.

If R is left-euclidean then R is also pli. This is proven as in the commutative case: given a nonzero left ideal $I \subseteq R$, take any $q \in I$ with $\deg(q)$ as small as possible. If $R \setminus Rq$ is nonempty, pick some element $s \in R \setminus Rq$. Then there are $p, r \in R$, $\deg(r) < \deg(q)$ such that $s = p \cdot q + r$. However, then $r = s - p \cdot q \in I$, but $\deg(r) < \deg(q)$, contradicting the choice of q. Therefore, $I = Rq$.

Besides the degree function, we will make use of the valuation function ν on $L[x, \sigma]$, which is defined as

$$\begin{aligned} \nu : L[x,\sigma] &\to \mathbb{Z}_0^+ \cup \{+\infty\} \\ p = \sum_{i=0}^{\infty} a_i x^i &\mapsto \begin{cases} +\infty & p = 0 \\ \min\{i \in \mathbb{Z}_0^+ : a_i \neq 0\} & p \neq 0 \end{cases} \end{aligned}$$

For any $p, q \in L[x, \sigma]$, we can easily see that $\nu(pq) = \nu(p) + \nu(q)$ and $\nu(p + q) \geq \min\{v(p), v(q)\}$, so ν is a valuation on $L[x, \sigma]$.

We set $L[x, \sigma]_0 := L[x, \sigma] \setminus (x) = \nu^{-1}(0)$ and define on $L[x, \sigma]_0$ the function

$$\begin{aligned} c : L[x,\sigma]_0 &\to L^\times \\ \sum_{i=0}^{\infty} a_i x^i &\mapsto a_0. \end{aligned}$$

It is easy to see that for all $p, q \in L[x, \sigma]_0$, we have $c(pq) = c(p)c(q)$.

Let R be a noncommutative domain. We call an overring $Q \supseteq R$ a *ring of left quotients* if every element $r \in R \setminus \{0\}$ has an inverse $r^{-1} \in Q$ and every element $q \in Q$ can be expressed as $q = r^{-1}s$ with $r, s \in R$. It can be shown that if R is a left noetherian domain, then R always has a ring of left quotients Q which is a field (see [MR01, Corollary 2.1.14 and Theorem 2.1.15]). In particular, this applies to left pli-domains.

We can now state and (re-)prove some facts on $L[x, \sigma]$, part of which are well-known (see, for example, [MR01, 1.§2])

Lemma. *Let $S = L[x, \sigma]$ where L is a commutative field and $\sigma : L \to L$ is a field automorphism such that $\delta := \dim_K(L)$ is finite.*

i) The ring S is left euclidean with respect to the degree function deg. *In particular, S is a pli domain.*

ii) The ring S has a field of left quotients Q. Furthermore, there exist unique homomorphisms

$$\overline{\deg} : Q^\times \to \mathbb{Z}$$
$$\overline{\nu} : Q^\times \to \mathbb{Z}$$

such that for all $p \in S \setminus \{0\}$, we have $\overline{\deg}(p) = \deg(p)$ and $\overline{\nu}(p) = \nu(p)$.

iii) With $Q_0 := \overline{\nu}^{-1}(0)$, there exists a unique homomorphism $\overline{c} : Q_0 \to L^\times$ such that $\overline{c}(p) = c(p)$ for all $p \in S_0$.

iv) The unit group of S is $S^\times = \{p \in S : \deg(p) = 0\} = L^\times$ where we regard L as a subset of S.

v) The centre of S is

$$Z(S) = K[x^\delta] := \{\sum_{i=0}^{\infty} k_i x^{i \cdot \delta} \in S : k_i \in K\}.$$

Furthermore, S is a free left (and right) $Z(S)$-module of dimension δ^2.

vi) Let $I \subseteq {}_SQ$ be a finitely generated S-submodule with $\min\{\overline{\nu}(I) : p \in Q\} = 0$. *Then there is a unique $p \in Q_0$ such that $I = Sp$ and $\overline{c}(p) = 1$.*

On the other hand, for each $p \in Q_0$, the submodule $Sp =: I \subseteq {}_SQ$ always fulfills $\min\{\overline{\nu}(I) : p \in Q\} = 0$.

vii) There is an isomorphism of rings $S/S(1 - x^\delta) \cong \mathrm{End}_K(L)$ where we regard L as a K-vector space.

Proof. i) Let $q \in S$ be fixed where $q \neq 0$. We set $m = \deg(q)$. We want to show that for each $s \in S$ there are $p, r \in S$ such that $s = pq + r$ and $\deg(r) < m$.

When $\deg(s) =: n \leq m - 1$, we can trivially take $p = 0$, $r = s$.

Now assume that $\deg(s) = n \geq m$ and that for each $s' \in S$ with $\deg(s') < n$, there are $p', r' \in S$ such that $s' = p'q + r'$ and $\deg(r') < m$.

We write

$$q = \sum_{i=0}^{m} a_i x^i$$
$$s = \sum_{i=0}^{n} b_i x^i.$$

We have $a_m \neq 0$, so the element $b_n(\sigma^{n-m}(a_m))^{-1}x^{m-n} \in S$ is defined. Since we have

$$b_n(\sigma^{n-m}(a_m))^{-1}x^{m-n} \cdot a_m x^m = b_n \sigma^{n-m}(a_m)^{-1}\sigma^{m-n}(a_m)x^{n-m}x^m = b_n x^n,$$

the n-th coefficient of $b_n(\sigma^{n-m}(a_m))^{-1}x^{m-n} \cdot q$ is equal to b_n, so the element

$$s' := s - \left(b_n(\sigma^{n-m}(a_m))^{-1}x^{m-n}\right) q$$

fulfills $\deg(s') < n$. By our assumption, there are $p', r' \in S$ such that $s' = p'q + r'$ and $\deg(r') < m$. Therefore,

$$\begin{aligned} s &= \left(b_n(\sigma^{n-m}(a_m))^{-1}x^{m-n}\right) q + s' \\ &= \left(b_n(\sigma^{n-m}(a_m))^{-1}x^{m-n}\right) q + p'q + r' \\ &= \left(b_n(\sigma^{n-m}(a_m))^{-1}x^{m-n} + p'\right) q + r'. \end{aligned}$$

Since $\deg(r') < m$, the induction step is complete.

ii) The fact that S has a field of left quotients follows from the discussion preceding the lemma together with part i), since a left-euclidean ring is necessarily a pli domain.

Since $Q^\times$ is a group of left fractions for $S \setminus \{0\}$, the latter is a left Ore monoid, by Ore's theorem (Theorem 2.1.7). Therefore, the extensions $\overline{\deg}, \overline{\nu}$ exist due to Proposition 2.1.10.

iii) We show that Q_0 is a group of left fractions for S_0. The existence of $\overline{c}$ then follows from the same argument as in ii).

Q_0 clearly is a group. Let $q \in Q_0$; then there are $p_1, p_2 \in S$ such that $q = p_1^{-1}p_2$. Since $\overline{\nu}(q) = 0$, these must fulfill $\nu(p_1) = \nu(p_2) =: k$. With this k, the elements p_1, p_2 can be written as $p_1 = x^k p_1'$, $p_2 = x^k p_2'$ where $p_1', p_2' \in S_0$. Then

$$q = p_1^{-1}p_2 = (x^k p_1')^{-1}(x^k p_2') = p_1'^{-1}p_2',$$

which proves that q is a left quotient of elements from S_0.

iv) Given $p, q \in S$ with $pq = 1$, we have $\deg(p) + \deg(q) = \deg(1) = 0$. This is only possible when $\deg(p) = \deg(q) = 0$, which is the case if and only if $p, q \in \{ax^0 : a \in L^\times\}$. On the other hand, given $p = ax^0 \in S$ with $0 \neq a \in L$, the element $a^{-1}x^0$ is clearly an inverse to p.

v) The ring S is generated by x together with the elements of L, so it suffices to determine which elements of S commute with x and the elements of L. Let $p = \sum_{i=0}^{\infty} a_i x^i \in Z(S)$. Now p commutes with x if and only if

$$xp = px \Leftrightarrow \sum_{i=0}^{\infty} \sigma(a_i)x^{i+1} = \sum_{i=0}^{\infty} a_i x^{i+1}$$

which is the case if and only if $a_i \in K$ for all $i \geq 0$. So we know that all coefficients of p are in K. For all $b \in L$, we have

$$bp = pb \Leftrightarrow \sum_{i=0}^{\infty} ba_i x^i = \sum_{i=0}^{\infty} a_i \sigma^i(b) x^i,$$

which shows that for all $i \geq 0$, we either have that $\sigma^i(b) = b$ for all $b \in L$ or that $a_i = 0$. Since σ is of order δ, this implies that $a_i \neq 0$ can only hold if $\delta | i$. This proves that $p \in Z(S)$ only if $p \in K[x^\delta]$. That $p \in K[x^\delta]$ is sufficient for p to be in $Z(S)$ follows from the same considerations.

Now let $b_1, b_2, \ldots, b_\delta$ be a basis of the vector space ${}_K L$. We then have the direct sum decompositions

$$S = \bigoplus_{i=0}^{\delta-1} L[x^\delta] x^i = \bigoplus_{i=0}^{\delta-1} \left(\bigoplus_{j=1}^{\delta} K[x^\delta] b_j x^i \right),$$

which shows that the δ^2 elements $b_j x^i$ $(0 \leq i \leq \delta - 1,\ 1 \leq j \leq \delta)$ form a basis for the (left) $K[x^\delta]$-module S. Since $K[x^\delta] = Z(S)$, the same elements also work as a basis for S as a right $K[x^\delta]$-module.

vi) Let the left ideal $I \subseteq {}_S Q$ be finitely generated and nonzero. Since ${}_S Q$ is torsion-free and S is pli, we have an isomorphism of S-modules ${}_S I \cong {}_S S^k$ for some integer k. By [MR01, 2.2.11], k must be equal to 1.

It follows that $I = Sp'$ for some p'. Now assume that $\min\{\overline{\nu}(p) : p \in I\} = 0$, too. For all $s \in S$, we have $\overline{\nu}(sp') = \nu(s) + \overline{\nu}(p') \geq \overline{\nu}(p')$, hence $\overline{\nu}(p') = 0$.

For $p \in Q$, we have $Sp = I$ if and only if $p = a \cdot p'$ with $a \in S^\times = L^\times$ (part iv)). For such an a, we calculate $\overline{c}(ap) = c(a)\overline{c}(p) = a \cdot \overline{c}(p)$. So, if we set $a := (\overline{c}(p))^{-1}$, then $p := ap'$ can be the only element in Q such that $\overline{c}(p) = 1$ and $I = Sp$.

The other statement is trivial.

vii) We make L a left S-module by defining a scalar multiplication $\cdot : S \times L \to L$ by

$$\left(\sum_{i=0}^{\infty} a_i x^i \right) \cdot c = \sum_{i=0}^{\infty} a_i \cdot \sigma^i(c).$$

It is clear that $1 \cdot c = c$ holds for all $c \in L$. The distributivity laws are also immediate. To see that $p \cdot (q \cdot c) = (pq) \cdot c$ for all $p, q \in S$, $c \in L$, we assume that $p = ax^i$ and $q = bx^j$ for some $a, b \in L$ and integers $i, j \geq 0$. In this case, we have

$$p \cdot (q \cdot c) = ax^i \cdot (bx^j \cdot c) = a \cdot \sigma^i(b \cdot \sigma^j(c))$$

$$= a \cdot \sigma^i(b) \cdot \sigma^{i+j}(c) = (a \cdot \sigma^i(b) x^{i+j}) \cdot c = (pq) \cdot c$$

and this extends to all $p, q \in S$ by bilinearity. So, L is indeed an S-module under this scalar multiplication.

For $p \in S$, we define $m_p : L \to L; x \mapsto p \cdot x$. Since K lies in $Z(S)$ (part v)), each m_p is a K-linear map. Therefore, these maps define a canonical ring homomorphism $m : S \to \mathrm{End}_K(L); p \mapsto m_p$.

We claim that $\ker(m) = S(1 - x^\delta)$ and m is surjective.

Since K is the fixed field of L under σ and $\dim_K(L) = \delta$, the automorphism σ must be of order δ. Therefore, we have for all $c \in L$ that $(1-x^\delta)c = 1 \cdot c - \sigma^\delta(c) = c - c = 0$. This implies $1 - x^\delta \in \ker(m)$ and thus, $S(1 - x^\delta) \subseteq \ker(m)$.

Now let $s \in \ker(m)$. Then by left euclidicity, we can find $p, r \in S$ such that $s = p \cdot (1 - x^\delta) + r$ and $\deg(r) < \deg(1 - x^\delta) = \delta$. Since $p \cdot (1 - x^\delta) \in S$, we also have $r \in S$.

Now write $r = \sum_{i=0}^{\delta-1} a_i x^i$. Then $\mu_r = \sum_{i=0}^{\delta-1} a_i \sigma^i$. However, $\sigma^0, \sigma^1, \ldots, \sigma^{\delta-1}$ are distinct automorphisms of L, and a classical theorem of Dedekind ([Jac85, p.291]) tells us that they are linearly independent over L (seen as a subring of $\mathrm{End}_K(L)$), hence all $a_i = 0$ $(0 \leq i \leq \delta - 1)$. This shows that $r \in S(1 - x^\delta)$. We have proven that $\ker(m) = S(1 - x^\delta)$.

Since $\dim(\mathrm{End}_K(L)) = \delta^2$ and $\dim_K(L) = \delta$, we have $\dim_L(\mathrm{End}_K(L)) = \delta$. The elements $\sigma^0, \sigma^1, \ldots, \sigma^{\delta-1}$ are linearly independent over L, and must therefore span, with respect to L, a δ-dimensional L-subspace of $\mathrm{End}_K(L)$ which must be equal to $\mathrm{End}_K(L)$ itself, i.e. $\oplus_{i=0}^{\delta-1} L\sigma^i = \mathrm{End}_K(L)$. The left-hand-side of this equality is clearly in the image of m, from which we deduce that m is surjective.

It follows that $\mathrm{End}_K(L) \cong S/S(1 - x^\delta)$. □

We now fix some notations from the lemma and the discussions before:

L - a commutative field

σ - an automorphism $\sigma : L \to L$ of finite order

K - the fixed field of σ, $K = \{a \in L : \sigma(a) = a\}$

δ $= \dim_K L$ (also, the order of σ).

S $= L[x, \sigma]$

Q - a field of left quotients for S,

ν, deg, c - we will abuse notation by denoting the extensions $\overline{\nu}, \overline{\deg}, \overline{c}$ simply by $\nu, \deg, c$.

Q_0 $= \nu^{-1}(0)$.

Keeping our notations regarding K, L and δ, we can now prove:

Theorem A.0.1. *There is a desarguesian right ℓ-group with strong order unit s such that there is a lattice isomorphism $[s^{-1}, e] \cong L({}_K K^\delta)$.*

Proof. We divide the proof into several steps.

Step I: Constructing the lattice

We look at Q as a left module over the subring S and define the lattice

$$\mathcal{L} := \left\{ M \in L({}_S Q) : \exists k, l \in \mathbb{Z} : S(1-x^\delta)^k \subseteq M \subseteq S(1-x^\delta)^l \right\} \subseteq L({}_S Q).$$

$\mathcal{L}$, as a sublattice of $L({}_S Q)$, is clearly modular. Since for all integers $k \geq l$ the S-module $S(1-x^\delta)^l / S(1-x^\delta)^k \cong S/S(1-x^\delta)^{k-l}$ has finite length $\delta \cdot (l-k)$, each bounded below (resp. above) descending (resp. ascending) chain must become stationary.

Step II: Definition of G

Note that each $M \in \mathcal{L}$ is contained in some S-submodule $S(1-x^\delta)^k$, which is isomorphic to ${}_S S$. Since S is pli, this implies that M is cyclic. For all $k \in \mathbb{Z}$, we have

$$\min\{\nu(p) : p \in S(1-x^\delta)^k \setminus \{0\}\} = 0,$$

so by part vi) of the lemma, for each $M \in \mathcal{L}$, we can find a unique $p \in Q$ with $c(p) = 1$ and $M = Sp$.

We define

$$G = \{p \in Q_0 : c(p) = 1, Sp \in \mathcal{L}\}.$$

and further define a partial order on G as follows: for $p, q \in G$ we let $p \leq q :\Leftrightarrow Sp \subseteq Sq$. This clearly makes G a modular lattice since the assignment $G \to \mathcal{L}; p \mapsto Sp$ is bijective by what we have already discussed, and $\mathcal{L}$, as a sublattice of the modular lattice $L({}_S Q)$, is modular as well.

Step III: G is a group

We claim that G is a subgroup of $Q^\times$:

We have $S = S \cdot 1$, so $S(1-x^\delta)^0 \leq S \leq S(1-x^\delta)^0$. Also, $c(1) = 1$, from which $1 \in G$ follows.

Let $p_1, p_2 \in G$ and let integers k_1, k_2, l_1, l_2 be given such that $S(1-x^\delta)^{k_1} \subseteq Sp_1 \subseteq S(1-x^\delta)^{l_1}$ and $S(1-x^\delta)^{k_2} \subseteq Sp_2 \subseteq S(1-x^\delta)^{l_2}$. Then

$$\begin{aligned} Sp_1p_2 &\subseteq Sp_1Sp_2 \\ &\subseteq S(1-x^\delta)^{l_1} S(1-x^\delta)^{l_2} \\ &= S(1-x^\delta)^{l_1}(1-x^\delta)^{l_2} = S(1-x^\delta)^{l_1+l_2}, \end{aligned}$$

where we have used that $(1-x^\delta)^l$ is central in Q for every $l \in \mathbb{Z}$ - this follows from part v) of the lemma. On the other hand, we also have

$$\begin{aligned} Sp_1p_2 &\supseteq S(1-x^\delta)^{k_1}p_2 \\ &= Sp_2(1-x^\delta)^{k_1} \\ &\supseteq S(1-x^\delta)^{k_2}(1-x^\delta)^{k_1} \\ &= S(1-x^\delta)^{k_1k_2}. \end{aligned}$$

Furthermore, $c(p_1p_2) = c(p_1)c(p_2) = 1$. We conclude that $p_1p_2 \in G$.

Finally, for $p \in G$, we find $k, l \in \mathbb{Z}$ such that

$$\begin{aligned} S(1-x^\delta)^k \subseteq Sp \subseteq S(1-x^\delta)^l &\Rightarrow S(1-x^\delta)^k p^{-1} \subseteq S \subseteq S(1-x^\delta)^l p^{-1} \\ &\Rightarrow (Sp^{-1}(1-x^\delta)^k \subseteq S) \wedge (S \subseteq Sp^{-1}(1-x^\delta)^l) \\ &\Rightarrow (Sp^{-1} \subseteq S(1-x^\delta)^{-k}) \wedge (S(1-x^\delta)^{-l} \subseteq Sp^{-1}). \end{aligned}$$

Also, we have $c(p^{-1}) = c(p)^{-1} = 1$; therefore, $p^{-1} \in G$ as well.

Step IV: Right-invariance

The order relation $\leq$ on G is also right-invariant: given $p_1, p_2, q \in G$, we have the following chain of implications:

$$p_1 \leq p_2 \Rightarrow Sp_1 \subseteq Sp_2 \Rightarrow Sp_1q \subseteq Sp_2q \Rightarrow p_1q \leq p_2q.$$

Step V: G has a strong order unit

Next, we show that $s := (1-x^\delta)^{-1}$ is a strong order unit.

Clearly, $1 - x^\delta \in G$; therefore, $s \in G$ as well. Also, $s = (1-x^\delta)^{-1}$ is central in Q; thus, we have the equivalences $p \leq q \Leftrightarrow ps \leq qs \Leftrightarrow sp \leq sq$, proving that s is normal. Furthermore, for every $p \in G$, we find an integer l with $Sp \leq S(1-x^\delta)^l \Leftrightarrow p \leq s^{-l}$, finally proving that s is a strong order unit.

Step VI: G is desarguesian

At last, we determine the strong order interval. Using part vii) of the lemma, we have

$$\begin{aligned} [s^{-1}, 1] &\cong [Ss^{-1}, S] \\ &= [S(1-x^\delta), S] \\ &\cong L({}_S(S/S(1-x^\delta))) \\ &\cong L\left({}_{\mathrm{End}_K(L)}\mathrm{End}_K(L)\right) \\ &\cong L({}_K L) \cong L({}_K K^\delta). \end{aligned}$$

where the isomorphism before the last one follows from the Morita equivalence between K and $\mathrm{End}_K(L)$. We have also seen in Step I that G is modular and noetherian. So we conclude that G is a desarguesian right ℓ-group. $\square$

For a prime power $q = p^k$, we denote by $\mathbb{F}_q$ the field with q elements.

Corollary A.0.2. *Let $q = p^k$ be some prime power. For any integer $\delta \geq 1$, there exists a modular geometric right ℓ-group G with strong order unit s such that we have an isomorphism of lattices $[s^{-1}, e] \cong L({}_{\mathbb{F}_q}\mathbb{F}_q^\delta)$.*

Proof. Set $L = \mathbb{F}_{q^\delta}$ and take the automorphism $\sigma : L \to L;\ a \mapsto a^q$. Then

$$K = \{a \in L : a^q = a\} = \mathbb{F}_q.$$

By Theorem A.0.1 there exists a modular geometric right ℓ-group with a strong order unit s such that $[s^{-1}, e] \cong L({}_{\mathbb{F}_q}\mathbb{F}_q^\delta)$. □

If a right ℓ-group G has a strong order unit with finite strong order interval, then G is even a Garside group. Since $L({}_{\mathbb{F}_q}\mathbb{F}_q^\delta)$ is finite, we can put our corollary into another nice form:

Corollary A.0.3. *Each finite desarguesian geometry is a strong order interval of some Garside group.*

The question arises how Theorem A.0.1 fits into the projective representation theory in Chapter 2.

We sketch how to get a matrix representation for the right ℓ-groups constructed in the proof of Theorem A.0.1:

Note that we have constructed G as the subgroup of all elements $p \in Q_0$ such that $c(p) = 1$ and Sp is nested between some S-submodules of Q which are of the form $S(1 - x^\delta)^k$.

By part v) of the lemma, the center of S is $Z(S) = K[x^\delta]$ and S is a free $Z(S)$-module of rank δ^2. From $Z(S)$, we construct the commutative ring

$$\hat{Z}(S) = K[[1 - x^\delta]] := \lim_{\leftarrow} Z(S)/(Z(S)(1 - x^\delta)^k).$$

This is a complete dvr with maximal ideal $\mathfrak{m} = \tilde{Z}(S)(1 - x^\delta)$. The ring $\hat{S} := \hat{Z}(S) \otimes_{Z(S)} S$ is then a δ^2-dimensional algebra over $\hat{Z}(S)$ and we have the homomorphism given by $t_S : S \to \hat{S};\ p \mapsto 1 \otimes p$.

Using part vii) of the lemma, we get the following isomorphisms of rings:

$$\hat{S}/\mathfrak{m}\hat{S} \cong S/S(1 - x^\delta) \cong \mathrm{End}_K(K^\delta).$$

Before continuing we insert the following useful lemma:

Lemma. *Let R be a commutative complete dvr with maximal ideal $\mathfrak{m}$ and residue field $k := R/\mathfrak{m}$. Furthermore, let A be an R-algebra which is finitely generated and free as an R-module. If we have an isomorphism of k-algebras*

$$A/\mathfrak{m}A \cong \mathrm{End}_k(k^\delta)$$

where $\delta \geq 0$ is some integer, then we also have an isomorphism of R-algebras

$$A \cong \mathrm{End}_R(R^\delta).$$

Proof. Let $V = k^\delta$. If $\tilde{A} := A/\mathfrak{m}A \cong \mathrm{End}_k(k^\delta)$, then the k-vector space V can be made a left $\tilde{A}$-module on which $\tilde{A}$ acts as a full matrix ring. Let ${}_{\tilde{A}}V$ be such a module.

There is an idempotent $\tilde{e} \in \tilde{A}$ such that $\tilde{A}/\tilde{A}\tilde{e} \cong {}_{\tilde{A}}V$. Since R is a complete dvr, this idempotent can be lifted modulo $\mathfrak{m}$, meaning that one can find an idempotent $e \in A$ such that $\tilde{e} = e + \mathfrak{m}A$ (see [Eis95, Corollary 7.5]). Let $M := A/Ae$; then ${}_AM$ is a projective A-module with ${}_{\tilde{A}}(M/\mathfrak{m}M) \cong {}_{\tilde{A}}V$. Since ${}_RA$ is free, this implies that ${}_RM \cong {}_RR^\delta$.

The scalar multiplications induce a ring homomorphism $A \to \mathrm{End}_R(M)$ which gives, after composing with the reduction map $\mathrm{End}_R(M) \to \mathrm{End}_k(V)$, an epimorphism $A \twoheadrightarrow \mathrm{End}_k(V)$. Since A is finitely generated, Nakayama's lemma tells us that $A \to \mathrm{End}_R(M)$ is already an epimorphism.

From $A/\mathfrak{m}A \cong \mathrm{End}_k(k^\delta)$, it follows that A is free over R of rank δ^2, which is also the free rank of $\mathrm{End}_R(M)$ over R. We infer that the map $A \to \mathrm{End}_R(M)$ is an isomorphism; therefore, A is isomorphic to $\mathrm{End}_R(R^\delta)$. $\square$

We now know that $\hat{S} \cong \mathrm{End}_{\hat{Z}(S)}(\hat{Z}(S)^\delta)$.

We continue by defining the lattices

$$\begin{aligned} \mathcal{L}^- &= \{M \in L({}_SS) : \exists k \in \mathbb{Z} : S(1-x^\delta)^k \subseteq M\} \\ \hat{\mathcal{L}}^- &= \{M \in L({}_{\hat{S}}\hat{S}) : \exists k \in \mathbb{Z} : \hat{S}(1-x^\delta)^k \subseteq M\}. \end{aligned}$$

Since we have an isomorphism of rings $S/S(1-x^\delta)^k \cong \hat{S}/\hat{S}(1-x^\delta)^k$ for all integers k, we have an isomorphism of lattices

$$\begin{aligned} \mathcal{L}^- &\xrightarrow{\sim} \hat{\mathcal{L}}^- \\ I &\mapsto \hat{I} = \hat{Z}(S) \otimes_{Z(S)} I \end{aligned}$$

Furthermore, $t_S : S \to \hat{S}$ restricts to an embedding $\varepsilon := t_S|_{G^-}$. For all $p \in G^-$, we then have

$$\hat{S} \cdot \varepsilon(p) = (\hat{Z}(S) \otimes S) \cdot (1 \otimes p) = \hat{Z}(S) \otimes (S \cdot p) = \widehat{S \cdot \varepsilon(p)}.$$

By construction, for each $I \in \mathcal{L}^-$, there is a $p \in G^-$ such that $I = Sp$. It follows that for all $J \in \hat{\mathcal{L}}^-$, there is a unique $p \in G^-$ such that $J = \hat{S}\varepsilon(p)$.

Let now $\mathrm{Reg}(\hat{S})$ denote the monoid of all regular elements of $\hat{S}$ under multiplication. Let $x \in \mathrm{Reg}(\hat{S})$. There is an integer k such that $\hat{S}(1-x^\delta)^k \subseteq \hat{S}x$ - this follows from the fact that $\hat{S}$ is a full matrix ring over the dvr $\hat{Z}(S)$ where $1-x^\delta$ is a uniformizer. Therefore, we can find a unique $p \in G^-$ such that $\hat{S}x = \hat{S}\varepsilon(p)$. This proves that

$$\mathrm{Reg}(\hat{S}) = \hat{S}^\times \cdot \varepsilon(G^-) \quad ; \quad S^\times \cap \mathrm{Reg}(\varepsilon(G^-)) = 1.$$

Letting $\hat{Q}$ be the quotient field of $\hat{Z}(S)$, we construct the ring

$$\widehat{QS} := \hat{Q} \otimes_{\hat{Z}(S)} \hat{S} \cong \mathrm{End}_{\hat{Q}}(\hat{Q}^{\delta}).$$

The unit group $\widehat{QS}^{\times}$ is a group of left fractions for the monoid $\mathrm{Reg}(\hat{S})$ since an invertible matrix over $\hat{Q}$ can always be written as a fraction of a regular matrix over $\hat{Z}(S)$ and an element of $\hat{Z}(S)$. With Proposition 2.1.10, we infer that the embedding $\varepsilon : G^{-} \hookrightarrow \hat{S}^{\times}$ can be extended to an embedding $\tilde{\varepsilon} : G \hookrightarrow \widehat{QS}^{\times}$. For this embedding we have

$$\widehat{QS}^{\times} = \hat{S}^{\times} \cdot \tilde{\varepsilon}(G) \quad ; \quad \hat{S}^{\times} \cap \tilde{\varepsilon}(G) = 1$$

Recalling the isomorphisms $\hat{S} \cong \mathrm{End}_{\hat{Z}(S)}(\hat{Z}(S)^{\delta})$; $\widehat{QS} \cong \mathrm{End}_{\hat{Q}}(\hat{Q}^{\delta})$, we get that G is isomorphic to a complement of the subgroup $\mathrm{GL}({}_{\hat{Z}(S)}\hat{Z}(S)^{\delta}) \leq \mathrm{GL}({}_{\hat{Q}}\hat{Q}^{\delta})$.

Applying the natural homomorphism $\mathrm{GL}({}_{\hat{Q}}\hat{Q}^{\delta}) \to \mathrm{P\Gamma L}({}_{\hat{Z}(S)}\hat{Q}^{\delta})$, we finally get the decomposition predicted by Theorem 2.6.6.

Appendix B

Problem session

In this appendix, we collect some - in our opinion - interesting problems arising from this work.

Problem 1. *Which desarguesian geometries are strong order intervals?*

Given a field K and an integer $\delta \geq 1$, is there a desarguesian right ℓ-group G with strong order unit s, such that we have an isomorphism of lattices $[s^{-1}, e] \cong L({}_K K^\delta)$?

We could answer this question in two cases:

1) The results in Appendix 1 tell us that such a group exists when there is a cyclic Galois extension L/K of degree δ.

2) The groups $\mathrm{PPU}(b)$ discussed in Chapter 2 show that such a group exists, for K commutative, when there is a symmetric, anisotropic K-bilinear form $b : K^\delta \times K^\delta \to K$. As is indicated, it suffices to assume that b is hermitean and anisotropic with respect to some involution of K [Die19, Remark 5].

   It can even be proven [DRZ19, Section 6] that it suffices that K is some (possible non-commutative) field and $b : K^\delta \times K^\delta \to K$ is an anisotropic sesquilinear form with respect to some involutive antiautomorphism K. It is not necessary to assume that b is symmetric (or that the antiautomorphism of K is involutive) to show that even in this case, an appropriate desarguesian right ℓ-group exists with strong order interval $[s^{-1}, e] \cong L({}_K K^\delta)$ exists.

   However, in this case no matrix representation is known, as is the case for $\mathrm{PPU}(b)$. It is probable that these groups can only be obtained in a „twisted" form, i.e. lying in some group of the form $\mathrm{P\Gamma L}({}_R Q^\delta)$, as predicted by Theorem 2.6.6.

No examples have been found where such a group does *not* exist. Are there any finite-dimensional desarguesian geometries which can not arise as a strong order interval?

Are there any such groups with $K = \overline{\mathbb{F}}_p$, the algebraic closure of the finite field with p elements?

Problem 2. *Are there „exotic“ modular geometric right ℓ-groups?* Theorem 3.3.4 tells us that $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is a modular geometric lattice if G is a modular geometric right ℓ-group.

Until now, we essentially know two classes of modular geometric right ℓ-groups, namely:

1) Structure groups for the set-theoretic Yang-Baxter equation. In this case, $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is a Boolean lattice and therefore, decomposes as a direct product of copies of the two-element Boolean algebra $\{0, 1\}$.

2) Complements of the subgroup $\mathrm{P\Gamma L}({}_R R^\delta) \le \mathrm{P\Gamma L}({}_R Q^\delta)$ where R is a complete dvr with dvf Q. In this case, $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is the desarguesian lattice $L({}_Q Q^\delta)$, by Theorem 3.1.6.

In all of these cases, $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ decomposes as a direct product of desarguesian lattices or two-element Boolean algebras.

However, are there modular geometric right ℓ-groups G where one indecomposable direct factor of $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is a non-desarguesian modular lattice, that is, the lattice associated with a non-desarguesian plane?

Could it even be possible for a modular geometric right ℓ-group G that $[s^{-1}, e]$ is desarguesian but $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ is not? (note that Theorem 2.5.16 does not make a statement about desarguesian right ℓ-groups of dimension $\delta = 3$).

Problem 3. *Is there a general coordinatization theorem for modular geometric right ℓ-groups?*

In a sense, this problem is related to the problem before. Theorem 2.5.16 tells us that the lattice of G, G being desarguesian of dimension $\delta \geq 4$, can be coordinatized by a complete dvr R. In this case, the „trinity“ ,

$$[s^{-1}, e] - G^- - \mathrm{Fil}(G^-)_{\mathrm{sat}}$$

is a lattice-theoretical reflection of the ring-theoretic trinity

$$R/\mathfrak{m} - R - Q$$

where R is a complete dvr with maximal ideal $\mathfrak{m}$ and quotient field Q.

If G is an „exotic“ modular geometric right ℓ-group, then $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ can not be coordinatized over a field, but over a so-called *ternary field*. Ternary fields are nonassociative generalizations of fields which are used to coordinatizate non-desarguesian planes.

In this case, $\mathrm{Fil}(G^-)_{\mathrm{sat}}$ and $[s^{-1}, e]$ can be coordinatized over a ternary field. However, what is the middle part of the trinity for a coordinatization theorem which includes the exotic modular geometric right ℓ-groups?

Problem 4. *Is there an easy way to recognize the coordinatization?*

A modular geometric right ℓ-group G is generated by the dual atoms in G^-, that is $X(G^-)$. Rump proved in [Rum15] that a modular geometric right ℓ-group G and its negative cone G^- are completely determined by the quadratic relations

$$(x \to y)x = (y \to x)y \quad \big(x, y \in X(G^-)\big)\,.$$

The operation $\to$ on $X(G^-) \cup \{e\}$ is algebraized by what is called an *L-Algebra* (see [Rum08a]).

For a Garside group G, the set $X(G^-)$ is finite by definition and there are only finitely many relations. When G is desarguesian of dimension $\delta \geq 4$, Theorem 2.5.16 tells us that the lattice of G can be coordinatized over a complete dvr R which is in this sense determined by the quadratic relations in $X(G^-)$ resp. the L-algebra $X(G^-) \cup \{e\}$.

How can R be derived from these relations?

Problem 5. *Are there desarguesian right ℓ-groups of p*-adic type*?*

Until now, we have only seen desarguesian right ℓ-groups where the lattice structure of G is coordinatized over some power series ring: in the case of $\mathrm{PPU}(\mathfrak{b})$ (which has been discussed in Chapter 2), this is the power series ring $R = K[[t^{-1}]]$. The desarguesian right ℓ-groups constructed in the proof of Theorem A.0.1 are coordinatized by the power series ring $R = K[[1 - x^\delta]]$.

Are there examples where R is a dvr which does not contain its residue field? For example, does a desarguesian right ℓ-group G exist which is coordinatized by the dvr $R = \mathbb{Z}_p$, the p-adic integers? The residue field $R/\mathfrak{m} \cong \mathbb{F}_p$ is finite, so these groups would provide further examples of desarguesian Garside groups, if existent!

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Ich erkläre hiermit, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen benutzt habe. Alle Stellen dieser Arbeit, die dem Wortlaut, dem Sinn oder der Argumentation nach anderen Werken entnommen sind (einschließlich des World Wide Web und anderer elektronischer Text- und Datensammlungen), habe ich unter Angabe der Quellen vollständig kenntlich gemacht.