The stable module category inside the homotopy category, perfect exact sequences and equivalences

Von der Fakultät Mathematik und Physik der Universität Stuttgart
zur Erlangung der Würde eines Doktors der Naturwissenschaften
(Dr. rer. nat.) genehmigte Abhandlung

Vorgelegt von
Sebastian Nitsche
aus Bietigheim-Bissingen

Hauptberichter: Prof. Dr. Steffen König
Mitberichterin: Prof. Dr. Kiriko Kato
2. Mitberichter: Prof. Dr. Yuming Liu

Tag der mündlichen Prüfung: 20. Juli 2021

Institut für Algebra und Zahlentheorie der Universität Stuttgart

2021

Acknowledgements

First of all, I would like to thank my supervisor Steffen König for his guidance and support in the past four years. I am especially grateful for the helpful advice and the friendly work environment he provided. Moreover, I would like to thank Kiriko Kato and Yuming Liu for valuable comments and suggestions.

I also wish to thank all current and former members of the representation theory group and the LExMath in Stuttgart for the personal and scientific exchange and the welcoming atmosphere. Finally, I would like to thank my parents for their support and encouragement.

Contents

Abstract

We consider the functor from the stable module category to the homotopy category constructed by Kato in [18]. This functor gives an equivalence between the stable module category and a full subcategory $\mathcal{L}$ of the unbounded homotopy category of projective modules. Moreover, the functor induces a correspondence between distinguished triangles in the homotopy category and perfect exact sequences in the module category.

In general, the stable module category and the category $\mathcal{L}$ are not triangulated. We provide a description of a triangulated hull of $\mathcal{L}$ inside the homotopy category and discuss its Grothendieck group. We also construct a larger subcategory which is shown to be characteristic inside the homotopy category under suitable assumptions. Both subcategories coincide with $\mathcal{L}$ if and only if the algebra is self-injective. Furthermore, stable equivalence of Morita type are shown to preserve both subcategories.

Another focus is put on the relationship between stable equivalences and perfect exact sequences. On the one hand, we give sufficient conditions for a stable equivalence to preserve perfect exact sequences up to projective direct summands. A stable equivalence which preserves perfect exact sequences in this way is shown to induce a triangulated equivalence between the categories of stable Gorenstein-projective modules. On the other hand, given a stable equivalence that is induced by an exact functor, we provide various sufficient conditions under which the equivalence is a stable equivalence of Morita type. In particular, stable equivalences of Morita type arise from equivalences that are given by tensoring with an arbitrary bimodule on the level of the category $\mathcal{L}$.

Finally, we give a description of all algebras that can be obtained by deleting or inserting nodes via stable equivalences constructed by Koenig and Liu in [22].

Zusammenfassung

Wir betrachten den Funktor von der stabilen Modulkategorie in die Homotopiekategorie, der in [18] von Kato konstruiert wurde. Hierdurch ergibt sich eine Äquivalenz zwischen der stabilen Modulkategorie und einer vollen Teilkategorie $\mathcal{L}$ der unbeschränkten Homotopiekategorie von projektiven Moduln. Der Funktor induziert außerdem eine Korrespondenz zwischen den ausgezeichneten Dreiecken in der Homotopiekategorie und den perfekt exakten Sequenzen in der Modulkategorie.

Die stabile Modulkategorie und die Kategorie $\mathcal{L}$ sind im Allgemeinen nicht trianguliert. Wir geben eine Beschreibung einer triangulierten Hülle von $\mathcal{L}$ innerhalb der Homotopiekategorie an und untersuchen ihre Grothendieck Gruppe. Weiterhin konstruieren wir eine größere Teilkategorie, die unter geeigneten Annahmen eine charakteristische Teilkategorie der Homotopiekategorie ist. Beide Teilkategorien stimmen genau dann mit der Kategorie $\mathcal{L}$ überein, wenn die Algebra selbstinjektiv ist. Zusätzlich zeigen wir, dass stabile Äquivalenzen vom Morita Typ beide Teilkategorien erhalten.

Ein weiterer Schwerpunkt wird auf den Zusammenhang zwischen stabilen Äquivalenzen und perfekt exakten Sequenzen gelegt. Zum Einen werden hinreichende Bedingungen angegeben, unter welchen eine stabile Äquivalenz perfekt exakte Sequenzen, bis auf projektive direkte Summanden, erhält. Es wird gezeigt, dass eine stabile Äquivalenz, die perfekt exakte Sequenzen in diesem Sinne erhält, eine triangulierte Äquivalenz zwischen den Kategorien der stabilen Gorenstein-projektiven Moduln induziert. Zum Anderen betrachten wir stabile Äquivalenzen, die von einem exakten Funktor induziert werden. Wir geben verschiedene hinreichende Bedingungen an, unter welchen solche Äquivalenzen eine stabile Äquivalenz vom Morita Typ sind. Insbesondere führen Äquivalenzen auf der Ebene der Kategorie $\mathcal{L}$, welche durch ein Tensorprodukt mit einem beliebigen Bimodul gegeben sind, zu stabilen Äquivalenzen vom Morita Typ.

Schließlich konstruieren wir alle Algebren, die durch das Streichen oder Einsetzen von Knoten entstehen können. Dies benutzt die Beschreibung solcher stabiler Äquivalenzen von Koenig und Liu in [22].

Introduction

To study the representation theory of finite dimensional algebras A and B, one often considers equivalences between three associated categories. The strongest of these is given by Morita equivalences between the categories of finitely generated modules $\mathrm{mod}\, A$ and $\mathrm{mod}\, B$. A weaker link is provided by equivalences between the derived categories $\mathcal{D}(\mathrm{mod}\, A)$ and $\mathcal{D}(\mathrm{mod}\, B)$. Finally, we have stable equivalences between the stable module categories $\underline{\mathrm{mod}}\, A$ and $\underline{\mathrm{mod}}\, B$. Thereby, the stable module category is defined by quotienting out morphisms factoring through projective modules. While there is a theory for derived equivalences developed by Keller([20]) and Rickard([37]) generalizing the Morita theory for module categories, so far no such theory is known for stable equivalences. Unlike the others, the stable module category in general is neither abelian nor triangulated. In contrast, Morita equivalences preserve the abelian structure of the module category and derived equivalences preserve the triangulated structure of $\mathcal{D}(\mathrm{mod}\, A)$.

As such, a stable equivalence often preserves relatively few properties. In particular, a stable equivalence does not need to be induced by a functor on the level of the module category. Therefore, several smaller, more specific classes of stable equivalences have been studied. Motivated by results for self-injective algebras and group algebras, Broué introduced the class of stable equivalences of Morita type; cf. [7]. These are induced by exact functors between the module categories which, under mild assumptions, form an adjoint pair. Stable equivalences of Morita type have been shown to preserve many properties of the algebra; see for example [24,26,35,42].

If A is a self-injective algebra, the stable module category has a triangulated structure with triangles induced by short exact sequences. Rickard ([38]) and Keller-Vossieck ([21]) have shown independently that derived equivalent self-injective algebras are stably equivalent of Morita type. In this way, stable equivalences arise naturally for self-injective algebras. Conversely, results by Asashiba in [2] and Dugas in [12] show that for self-injective algebras of finite representation type, every stable equivalence induces a derived equivalence and thus a stable equivalence of Morita type. Furthermore, a different result by Rickard in [39] states that for self-injective algebras every stable equivalence that is induced by an exact functor between the module categories is isomorphic to a stable equivalence of Morita type.

For arbitrary finite dimensional algebras it remains an open problem whether stable equivalences induced by exact functors are of Morita type. Progress has been made by Dugas and Martinez-Villa in [13], who have shown that a stable equivalence induced by tensoring with a bimodule ${}_AM_B$ which is projective on both sides is of Morita type if $\mathrm{Hom}_A(M, A)$ is projective over B. In a different direction, Liu and Xi provide several methods to construct stable equivalences of Morita type from given ones; cf. [28–30]. In practice, it remains difficult to determine whether two algebras are stably equivalent of Morita type.

For general stable equivalences the obstruction to nice properties is often given by the existence of a node. Nodes are non-projective, non-injective simple modules S where the middle term of the almost split sequence starting in S is projective. This has been studied by Auslander and Reiten in [4] and later Martínez-Villa in [32, 33]. By excluding algebras with nodes, stable equivalences preserve most short exact sequences and almost split sequences up to projective direct summands; cf. [4] and [33]. Furthermore, Martínez-Villa has shown in [33] that stable equivalences in this setting preserve the global and dominant dimension of algebras as well as the stable Grothendieck group. Note that stable equivalences which are induced by two exact functors preserve nodes; cf. [27].

Finally, a complementary class of stable equivalences is given by deleting nodes from the algebra. In [31], Martínez-Villa has shown that every algebra is stably equivalent to an algebra without nodes in this way. He and Montaño-Bermúdez study and extend stable equivalences which are induced by node deletion or node insertion in [34]. Using a different approach, Koenig and Liu show in [22] that such stable equivalences can be described by bimodules which are projective on one side.

In this thesis, we consider other concepts and categories associated to the stable module category and examine when they are preserved by a stable equivalence. This is based on the following work by Kiriko Kato who gives a description of the stable module category inside the homotopy category.

As seen above, there seems to be a close connection between the stable module category and the derived category in case that A is self-injective. A similar approach for rings which are not necessarily self-injective has been to study the relationship of the stable module category with the homotopy category of projective modules $\mathcal{K}(\mathrm{proj}\, A)$. In the context of commutative rings, Yoshino introduces in [43] an equivalence between the stable category of modules with finite projective dimension and a full subcategory of the homotopy category. A similar technique was used by Amasaki in [1]. In [18], Kato extends this equivalence to a functor from the stable module category to the unbounded homotopy category of projective modules. This results in a full subcategory $\mathcal{L}_A$ of the homotopy category which is equivalent to the stable module category. Furthermore this equivalence $\underline{\mathrm{mod}}\, A \xrightarrow{\sim} \mathcal{L}_A$ provides a correspondence between the distinguished triangles in $\mathcal{K}(\mathrm{proj}\, A)$ and so called perfect exact sequences in $\mathrm{mod}\, A$.

Here, a short exact sequence is called perfect exact if the induced sequence under the functor $\mathrm{Hom}_A(-, A)$ is exact as well. Using these concepts, Kato constructs a weak kernel and a weak cokernel for the stable module category and contrasts this with the abelian structure of $\operatorname{mod} A$. Later, in [19], the same methods are utilized by her to characterize morphisms which are stably equivalent to a monomorphism.

Although these results where introduced for modules over commutative rings, the main techniques still work for modules over general finite dimensional algebras. This provides the basis for most of the results in this thesis, in particular we will make use of the equivalence $\underline{\operatorname{mod}}\, A \to \mathcal{L}_A$. However, we focus more on stable equivalences, perfect exact sequences and the triangulated structure of the homotopy category.

For the latter, we discuss the category $\mathcal{L}_A \subset \mathcal{K}(\operatorname{proj} A)$ in situations where the stable module category is not triangulated. In this setting, $\mathcal{L}_A$ still contains or is contained in triangulated subcategories of $\mathcal{K}(\operatorname{proj} A)$. An example of this is the homotopy category of totally acyclic complexes $\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$. The corresponding objects in $\underline{\operatorname{mod}}\, A$ are the Gorenstein-projective modules. The framework of perfect exact sequences provides an intrinsic description of totally acyclic complexes inside $\mathcal{L}_A$; cf. Lemma 4.38. Furthermore, the equivalence $\underline{\operatorname{mod}}\, A \to \mathcal{L}_A$ by Kato restricts to the known triangulated equivalence between the category $\underline{\operatorname{Gproj}}\, A$ of stable Gorenstein-projective modules and $\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$; cf. Lemma 4.40. On the other hand, $\mathcal{L}_A$ can be enlarged to a triangulated category inside $\mathcal{K}(\operatorname{proj} A)$. We define two triangulated subcategories of $\mathcal{K}(\operatorname{proj} A)$ as perpendicular categories such that they have $\mathcal{L}_A$ as a subcategory; cf. Definition 4.1. The category $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ has objects which do not have non-zero morphisms to bounded complexes of projective-injective modules. The category $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ has objects which do not have non-zero morphisms to bounded complexes of strongly projective-injective modules. In summary, we will obtain the following chain of subcategories. With the exception of $\underline{\operatorname{mod}}\, A \simeq \mathcal{L}_A$, all of these are triangulated categories.

$$\begin{tikzcd}
\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A) \arrow[r, hook] & \mathcal{L}_A \arrow[r, hook] & \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) \arrow[r, hook] & \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A) \arrow[r, hook] & \mathcal{K}(\operatorname{proj} A) \\
\underline{\operatorname{Gproj}}\, A \arrow[r, hook] \arrow[u, "\wr"] & \underline{\operatorname{mod}}\, A \arrow[u, "\wr"]
\end{tikzcd}$$

While $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ is a larger category than $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$, it is closed under a functor $\nu_{\mathcal{K}}$ which is induced by the Nakayama functor ν_A; cf. Definition 4.22. If A has finite global dimension $\nu_{\mathcal{K}}$ is equivalent to the derived Nakayama functor between the bounded homotopy categories $\mathcal{K}^b(\operatorname{proj} A) \to \mathcal{K}^b(\operatorname{proj} A)$. Our result is as follows.

Theorem A (Theorem 4.11, Theorem 4.26)**.** *Let A be a finite dimensional k-algebra.*

(1) The category $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ is the smallest triangulated subcategory of $\mathcal{K}(\operatorname{proj} A)$ that contains $\mathcal{L}_A$ and is closed under isomorphisms.

(2) The category $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ is the smallest triangulated subcategory of $\mathcal{K}(\operatorname{proj} A)$ that contains $\mathcal{L}_A$ and is closed under $\nu_{\mathcal{K}}$ and under isomorphisms.

As an application of the first result, we discuss the Grothendieck group of the triangulated category $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$. Via the equivalence $\underline{\operatorname{mod}}\, A \to \mathcal{L}_A$, we obtain an alternative Grothendieck group $\mathrm{G}_0^{\mathcal{P}}(A)$ of $\underline{\operatorname{mod}}\, A$; cf. Definition 4.15. In contrast to the known stable Grothendieck group, $\mathrm{G}_0^{\mathcal{P}}(A)$ is defined via perfect exact sequences instead of short exact sequences. As such, it can be non-zero even for algebras of finite global dimension. Regarding the second result, a theorem by Fang, Hu and Koenig ([14, Theorem 4.3]) implies that $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ is a characteristic subcategory of $\mathcal{K}^b(\operatorname{proj} A)$ if A has finite global dimension and ν-dominant dimension at least 1; cf. Corollary 4.32. We also provide an extension of this consequence for algebras of arbitrary global dimension; cf. Theorem 4.35.

In case that A is a self-injective algebra, $\underline{\operatorname{mod}}\, A$ is already a triangulated category. Therefore, $\mathcal{L}_A$ is a triangulated subcategory of $\mathcal{K}(\operatorname{proj} A)$ and all previously mentioned subcategories coincide. In this sense, the above constructions are compatible with the existing structure of $\underline{\operatorname{mod}}\, A$ and $\mathcal{K}(\operatorname{proj} A)$. We show, that this characterizes the property of A to be self-injective.

Theorem B (Theorem 4.45)**.** *The following are equivalent for a finite dimensional algebra A.*

(1) A is self-injective.

(2) $\mathcal{L}_A$ is a triangulated subcategory of $\mathcal{K}(\operatorname{proj} A)$.

(3) $\mathcal{L}_A = \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$.

(4) $\mathcal{L}_A$ is closed under taking shifts in $\mathcal{K}(\operatorname{proj} A)$.

(5) $\mathcal{L}_A = \mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$.

If one of the above conditions holds, the functor $\underline{\operatorname{mod}}\, A \to \mathcal{L}_A$ is an equivalence of triangulated categories. Furthermore, we have $\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A) = \mathcal{L}_A = \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) = \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$.

The remaining part of this work studies different classes of stable equivalences. Our main focus is on stable equivalences that preserve the property of short exact sequences to be perfect exact. Necessarily, such a stable equivalence cannot be induced by deleting or inserting a node. The following result provides sufficient conditions in which this property is preserved. We say that a morphism $f : X \to Y$ in $\operatorname{mod} A$ has finite depth, if $f \notin \operatorname{rad}^n(X,Y)$ for some $n \in \mathbb{Z}_{\geqslant 1}$. See Definition 3.13 for more details.

Theorem C (Theorem 3.19)**.** *Let $\alpha : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$ be a stable equivalence.*

Suppose given a perfect exact sequence $0 \to X \xrightarrow{f} Y \oplus P \xrightarrow{g} Z \to 0$ without split summands where X has no node as a direct summand, $P \in \mathrm{proj}\, A$ and Y has no projective direct summand.

Suppose that $f\,p$ and $g\,\pi$ have finite depth for every projection p onto an indecomposable direct summand of Y and every projection π onto an indecomposable direct summand of Z. Then there exists a perfect exact sequence

$$0 \to \alpha(X) \xrightarrow{\tilde{f}} \alpha(Y) \oplus \tilde{P} \xrightarrow{\tilde{g}} \alpha(Z) \to 0$$

in $\mathrm{mod}\, B$ with $\tilde{P} \in \mathrm{proj}\, B$ such that $\underline{\tilde{f}} \simeq \alpha(\underline{f})$ and $\underline{\tilde{g}} \simeq \alpha(\underline{g})$.

In particular, this provides conditions on the algebras A and B under which every stable equivalence preserves perfect exact sequences in this way. While the assumption on the depth of f and g is necessary for our proof, it seems unclear whether a similar result holds in a more general setting. However, if a stable equivalence preserves perfect exact sequences it also preserves the stable category of Gorenstein-projective modules and the Grothendieck group $\mathrm{G}_0^{\mathcal{P}}(A)$ introduced above.

Theorem D (Corollary 3.20, Theorem 4.42, Theorem 4.17)**.** *Let $\alpha : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$ be a stable equivalence between finite dimensional algebras without nodes. Consider the following conditions.*

(1) Let $0 \to X \xrightarrow{f} Y \oplus P \xrightarrow{g} Z \to 0$ be a perfect exact sequence in $\mathrm{mod}\, A$ without split summands where $P \in \mathrm{proj}\, A$ and Y has no projective direct summand. Then there exists a perfect exact sequence

$$0 \to \alpha(X) \xrightarrow{\tilde{f}} \alpha(Y) \oplus \tilde{P} \xrightarrow{\tilde{g}} \alpha(Z) \to 0$$

in $\mathrm{mod}\, B$ with $\tilde{P} \in \mathrm{proj}\, B$ such that $\underline{\tilde{f}} \simeq \alpha(\underline{f})$ and $\underline{\tilde{g}} \simeq \alpha(\underline{g})$.

(2) The equivalence α induces a triangulated equivalence $\mathcal{K}_{\mathrm{tac}}(\mathrm{proj}\, A) \to \mathcal{K}_{\mathrm{tac}}(\mathrm{proj}\, B)$. This induces a triangulated equivalence $\underline{\mathrm{Gproj}}\, A \to \underline{\mathrm{Gproj}}\, B$.

(3) The equivalence α induces an isomorphism $\mathrm{G}_0^{\mathcal{P}}(A) \to \mathrm{G}_0^{\mathcal{P}}(B)$.

If condition (1) holds for α and its quasi-inverse, conditions (2) and (3) hold. If A and B have finite representation type, all three conditions hold.

For the stronger class of stable equivalences of Morita type the above results hold without any assumptions on the finite dimensional algebras A and B. In particular, such stable equivalences map perfect exact sequences to perfect exact sequences. We show the following further results for the subcategories of $\mathcal{K}(\mathrm{proj}\, A)$ introduced above.

Theorem E (Theorem 5.8). *Suppose ${}_AM_B$ and ${}_BN_A$ are bimodules that induce a stable equivalence of Morita type such that M and N do not have any non-zero projective bimodule as direct summand.*

(1) Applying $-\otimes_A M$ componentwise induces an equivalence of categories $\mathcal{L}_A \to \mathcal{L}_B$.
If A and B are self-injective, this is an equivalence of triangulated categories.

(2) Applying $-\otimes_A M$ componentwise induces an equivalence of triangulated categories

$$\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) \to \mathcal{H}_{\mathcal{P}}(\operatorname{proj} B).$$

(3) Applying $-\otimes_A M$ componentwise induces an equivalence of triangulated categories

$$\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A) \to \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} B).$$

Note that any stable equivalence induces an equivalence between $\mathcal{L}_A$ and $\mathcal{L}_B$. Yet, in general, there is no explicit description of this induced functor inside $\mathcal{K}(\operatorname{proj} A)$. On the other hand, if we know that a bimodule M induces an equivalence $-\otimes_A M : \mathcal{L}_A \to \mathcal{L}_B$, we can show that M and $\operatorname{Hom}_B(M, B)$ induce a stable equivalence of Morita type; cf. Theorem 5.13. This builds upon the result of Dugas and Martínez-Villa in [13] mentioned above. As an application, we find new conditions under which a stable equivalence that is induced by an exact functor is of Morita type. The last three conditions will be shown using perfect exact sequences.

Theorem F (Theorem 5.19). *Let A and B be finite dimensional algebras whose semisimple quotients are separable. Suppose given a bimodule M which is projective as left A- and as right B-module such that $-\otimes_A M$ induces a stable equivalence $\underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$. If one of the following conditions holds, M and $\operatorname{Hom}_B(M, B)$ induce a stable equivalence of Morita type between A and B.*

(i) The functor $-\otimes_A M$ induces an equivalence $\mathcal{L}_A \to \mathcal{L}_B$.

(ii) The homology $\mathrm{H}_k((F^\bullet \otimes_A M)^*)$ vanishes for $F^\bullet \in \mathcal{L}_A$ and $k \geqslant 0$.

(iii) There exist natural isomorphisms $\nu_B(P \otimes_A M) \simeq \nu_A(P) \otimes_A M$ for all $P \in \operatorname{proj} A$.

(iv) There exists a natural isomorphism $M \otimes_B \mathrm{D}B \simeq \mathrm{D}A \otimes_A M$ of right B-modules.

(v) The algebras A and B have no nodes. At least one of A or B has dominant dimension at least 1 and finite representation type. Moreover, for all simple A-modules S whose injective hull is not projective, the image $S \otimes_A M$ is an indecomposable B-module.

(*vi*) *The algebras A and B have no nodes. At least one of A or B is a Nakayama algebra. Moreover, for all simple A-modules S whose injective hull is not projective, the image $S \otimes_A M$ is an indecomposable B-module.*

(*vii*) *The algebras A and B have dominant dimension at least 1. There is a bimodule ${}_BL_A$ which is projective as left B- and right A-module and which induces the inverse stable equivalence. Moreover, for all simple A-modules S whose injective hull is not projective, the image $S \otimes_A M$ is an indecomposable B-module.*

Finally, we consider stable equivalences that are induced by either deleting or inserting a node. For all stable equivalences discussed so far, nodes were either excluded by assumption or preserved by the equivalence. Koenig and Liu provide an explicit description of stable equivalences that are induced by gluing idempotents corresponding to a simple projective and a simple injective module; cf. [22]. We provide a construction that describes all algebras that can be obtained from a given algebra in this way; cf. Theorem 6.11.

This thesis is structured as follows. The first chapter, Chapter 1, provides a summary of definitions and results important for the later chapters. At the beginning, a list of often used notations and conventions is included. Afterwards, we focus on the stable module category and on the homotopy category of projective modules. The chapter concludes with a short section about projective-injective modules. Note that a list of symbols with short explanations is attached at the end of this thesis.

Chapter 2 focuses on the equivalence $\underline{\mathrm{mod}}\, A \to \mathcal{L}_A$ and on perfect exact sequences. This collects and adapts Kato's results in [18] and [19]. We include modified versions of the proofs given by Kato and often fill in several details. Moreover, we provide further technical properties of perfect exact sequences. The next chapter, Chapter 3, is dedicated to show that certain perfect exact sequences are preserved by stable equivalences as stated in Theorem C. The following Chapter 4 covers the triangulated categories $\mathcal{K}_{\mathrm{tac}}(\mathrm{proj}\, A)$, $\mathcal{H}_{\mathcal{P}}(\mathrm{proj}\, A)$ and $\mathcal{H}_{\mathrm{stp}}(\mathrm{proj}\, A)$. In particular, Theorem A and Theorem B are proven. Furthermore, the result of the previous chapter is used to verify parts (2) and (3) of Theorem D. Chapter 5 discusses stable equivalences of Morita type in more detail. In the first section, we apply this to the categories of the previous chapter and give the proof of Theorem E. The rest of the chapter provides conditions under which a stable equivalence is of Morita type; cf. Theorem F. Finally, the case of stable equivalences induced by gluing a simple injective and a simple projective vertex of a quiver algebra is treated in Chapter 6. This part is mostly independent of the previous chapters.

The last chapter, Chapter 7, is dedicated to some extended examples. Each section of this chapter focuses on one or two algebras in greater detail. At the beginning of every section, several facts about the algebras are collected. Furthermore, examples during a section may reference results stated in earlier parts of the same section. However, every section of Chapter 7 can be read independently. Throughout this thesis, we often point to specific parts of this chapter intended to be read as an example for the current topic. On the other hand, this chapter can also be read as a self-contained part at the end of the thesis.

Chapter 1

Preliminaries

In this chapter, we introduce the main definitions and notations used throughout this thesis. We focus on the stable module category (Section 1.2) and on the homotopy category (Section 1.3). At the end of this chapter is a short section about projective-injective modules. We begin with a list of general notations and conventions which are used later without further comment.

The general setup for every chapter is the following. Let k be a field. Let A and B be finite dimensional k-algebras. We assume that A and B have no semisimple summands. For some chapters, we impose additional assumptions.

1.1 Notation

We use the following notation and conventions.

- By an A-module we understand a right A-module, if not specified otherwise. We denote the category of right A-modules by $\operatorname{Mod} A$. The full subcategory of projective modules is denoted by $\operatorname{Proj} A$. The full subcategory of injective modules is denoted by $\operatorname{Inj} A$.

- We denote the category of finitely generated right A-modules by $\operatorname{mod} A$. Similarly for $\operatorname{proj} A$ and $\operatorname{inj} A$. The corresponding categories of finitely generated left A-modules are denoted by A-mod, A-proj and A-inj respectively. If not specified otherwise, all modules are assumed to be finitely generated.

- Let X, Y and Z be sets. We write morphisms on the right. That is, given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $x \in X$ we denote the image of x under f by xf and the composite of f and g by $X \xrightarrow{fg} Z$.

- Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories. We write functors on the left. That is, given functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{E}$ we denote the image of an object C under $\mathcal{F}$ by $\mathcal{F}(C)$ and the composite of $\mathcal{F}$ and $\mathcal{G}$ by $\mathcal{C} \xrightarrow{\mathcal{G}\circ\mathcal{F}} \mathcal{E}$.

- For an A-module X and $n \in \mathbb{Z}_{\geqslant 0}$, we write $X^{\oplus n}$ for the direct sum of n copies of X.
- We often write morphisms between direct sums of modules as matrices. That is, given morphisms $f_{i,j} : X_i \to Y_j$ between A-modules X_1, X_2, Y_1 and Y_2, we write
$$f := \begin{pmatrix} f_{1,1} & f_{1,2} \\ f_{2,1} & f_{2,2} \end{pmatrix} : X_1 \oplus X_2 \to Y_1 \oplus Y_2$$
for the morphism f with $(x_1, x_2)f = (x_1 f_{1,1} + x_2 f_{2,1}, x_1 f_{1,2} + x_2 f_{2,2})$ for $x_1 \in X_1$ and $x_2 \in X_2$.
- Let u and v be two idempotent elements of A. We identify along
$$\begin{aligned} \mathrm{Hom}_A(uA, vA) &\xrightarrow{\sim} vAu \\ f &\mapsto uf \\ (uy \mapsto vxuy) &\leftarrow\!\shortmid vxu \end{aligned}$$
- For morphisms $X \xrightarrow{f} Y$ and $X' \xrightarrow{f'} Y'$ we write $f \simeq f'$ if there exist isomorphisms α, β such that the following diagram commutes.
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$
- Let X be an A-module. We often write $1 = 1_X$ for the identity map id_X on X.
- We denote the composition length of an A-module X by $l(X)$.
- We say that a short exact sequence $\eta : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ has a split summand if $\eta \neq 0$ and there exists a decomposition $\eta \simeq \eta_1 \oplus \eta_2$ into the direct sum of two short exact sequences $\eta_i : 0 \to X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i \to 0$ for $i = 1, 2$ such that η_1 is a split exact sequence. That is, if there exist isomorphisms φ_1, φ_2 and φ_3 such that the following diagram commutes with η_1 a split exact sequence.
$$\begin{tikzcd} \eta : & 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "\wr", "\varphi_1"'] & Y \arrow[r, "g"] \arrow[d, "\wr", "\varphi_2"'] & Z \arrow[r] \arrow[d, "\wr", "\varphi_3"'] & 0 \\ \eta_1 \oplus \eta_2 : & 0 \arrow[r] & X_1 \oplus X_2 \arrow[r, "\begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}"] & Y_1 \oplus Y_2 \arrow[r, "\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}"] & Z_1 \oplus Z_2 \arrow[r] & 0 \end{tikzcd}$$
- Let $\mathcal{A}$ be an additive category and $\mathcal{S}$ be a full subcategory of $\mathcal{A}$. We write ${}^{\perp}\mathcal{S}$ for the full subcategory of $\mathcal{A}$ consisting of all objects X in $\mathcal{A}$ such that $\mathrm{Hom}_{\mathcal{A}}(X, Z) = 0$ for all $Z \in \mathcal{S}$. Analogously, we define $\mathcal{S}^{\perp}$.

We recall the definition of several important functors and introduce relevant notation.

(1) The k-duality $\operatorname{Hom}_k(-,k) : \operatorname{mod} A \to A\text{-mod}$ will be denoted by $\mathrm{D}(-) := \operatorname{Hom}_k(-,k)$. Note that $\mathrm{D}(-)$ is exact and takes projective modules to injective modules and vice versa.

(2) The functor $\operatorname{Hom}_A(-,A) : \operatorname{mod} A \to A\text{-mod}$ will be denoted by $(-)^* := \operatorname{Hom}_A(-,A)$ Note that $(-)^*$ restricts to an equivalence $(-)^* : \operatorname{proj} A \to A\text{-proj}$.

(3) We define the *Nakayama functor* $\nu_A : \operatorname{mod} A \to \operatorname{mod} A$ as the composite $\nu(-) := \mathrm{D}((-)^*)$. We sometimes write ν instead of ν_A if there is no ambiguity. Note that ν restricts to an equivalence $\operatorname{proj} A \to \operatorname{inj} A$. The quasi-inverse is given by $\nu_A^{-1} = (\mathrm{D}(-))^*$.

For $P \in \operatorname{proj} A$ and $X \in \operatorname{mod} A$, we have a natural isomorphism

$$\operatorname{Hom}_A(X, \nu P) \simeq \mathrm{D}\operatorname{Hom}_A(P, X).$$

(4) Let $X \in \operatorname{mod} A$ with projective presentation $P^{-1} \to P^0 \to X$. Then the syzygy $\Omega(X) = \operatorname{Ker}(P^0 \to X)$ defines a functor $\Omega : \underline{\operatorname{mod}} A \to \underline{\operatorname{mod}} A$. The transpose of X given by $\operatorname{Tr}(X) := \operatorname{Cok}((P^0)^* \to (P^{-1})^*)$ defines a duality $\underline{\operatorname{mod}} A \to A\text{-}\underline{\operatorname{mod}}$.

1.2 Stable module category

We recall the definition of the stable module category and collect some basic properties. Additionally, some elementary proofs are included.

Definition 1.1. The *stable module category* $\underline{\operatorname{mod}} A$ is the category with the same objects as $\operatorname{mod} A$ and with morphisms $\underline{\operatorname{Hom}}_A(X,Y) := \operatorname{Hom}_A(X,Y)/\operatorname{PHom}_A(X,Y)$ for $X, Y \in \operatorname{mod} A$. A morphism $f : X \to Y$ is an element of $\operatorname{PHom}_A(X,Y)$ if there exists a $P \in \operatorname{proj} A$ such that f factors through P.

Remark 1.2. Let $X, Y \in \operatorname{mod} A$ and $f, g \in \operatorname{Hom}_A(X,Y)$.

(1) We sometimes write $\underline{f}$ for the image of f in $\underline{\operatorname{Hom}}_A(X,Y)$. However, we often denote morphisms in $\underline{\operatorname{mod}} A$ by f as well, if there is no ambiguity.

We sometimes write $X \overset{\mathrm{st}}{\simeq} Y$ or $f \overset{\mathrm{st}}{\simeq} g$ if $X \simeq Y$ or $f \simeq g$ in $\underline{\operatorname{mod}} A$ respectively.

(2) Let $P \in \operatorname{proj} A$. We have $P \simeq 0$ in $\underline{\operatorname{mod}} A$ via stable and mutually inverse isomorphisms $P \to 0$ and $0 \to P$.

(3) Note that $f = 0$ in $\underline{\mathrm{mod}}\, A$ if and only if f factors through the projective cover $p : P \to Y$ of Y.

In fact, if f factors through a projective module Q via $\alpha : X \to Q$ and $\beta : Q \to Y$, then we obtain a morphism $\beta' : P \to Q$ with $f = \alpha\beta = \alpha\beta' p$.

$$\begin{tikzcd} & Q \arrow[dl, dashed, "\beta'"'] \arrow[d, "\beta"] \\ P \arrow[r, "p"] & Y \end{tikzcd}$$

We have the following basic characterization of isomorphic modules and morphisms in the stable module category. We follow the proof given in [19, Lemma 2.3 and 2.6] and fill in some details.

Lemma 1.3. *Suppose given $X, Y, X', Y' \in \mathrm{mod}\, A$ and morphisms $f : X \to X'$ and $g : Y \to Y'$.*

(1) We have $X \overset{\mathrm{st}}{\simeq} Y$ if and only if there exist $P, Q \in \mathrm{proj}\, A$ such that $X \oplus Q \simeq Y \oplus P$.

(2) We have $f \overset{\mathrm{st}}{\simeq} g$ if and only if there exist $P, P', Q, Q' \in \mathrm{proj}\, A$, morphisms $\tilde{f}$ and $\tilde{g}$ which restrict to f and g respectively and isomorphisms φ, φ' such that the following diagram commutes.

$$\begin{tikzcd} X \oplus Q \arrow[r, "\tilde{f}"] \arrow[d, "\varphi", "\wr"'] & X' \oplus Q' \arrow[d, "\varphi'", "\wr"'] \\ Y \oplus P \arrow[r, "\tilde{g}"] & Y' \oplus P' \end{tikzcd}$$

Proof. Ad (1). Suppose that $\alpha : X \to Y$ is a stable isomorphism with inverse β in $\underline{\mathrm{mod}}\, A$. Then $\alpha\,\beta - \mathrm{id}_X$ factors through the projective cover P of X and $\beta\,\alpha - \mathrm{id}_Y$ factors through the projective cover Q of Y. Hence there exist morphisms

$$\begin{array}{ll} s_X : X \to P & t_X : P \twoheadrightarrow X \\ s_Y : Y \to Q & t_Y : Q \twoheadrightarrow Y \end{array}$$

such that $\alpha\,\beta + s_X\,t_X = \mathrm{id}_X$ and $\beta\,\alpha + s_Y\,t_Y = \mathrm{id}_Y$. Moreover, there exist morphisms $a : P \to Q$ and $b : Q \to P$ such that the following diagrams commute.

$$\begin{tikzcd} & P \arrow[dl, dashed, "a"'] \arrow[d, "t_X\,\alpha"] \\ Q \arrow[r, two heads, "t_Y"] & Y \end{tikzcd} \qquad \begin{tikzcd} & Q \arrow[dl, dashed, "b"'] \arrow[d, "t_Y\,\beta"] \\ P \arrow[r, two heads, "t_X"] & X \end{tikzcd}$$

We show that $\begin{pmatrix} \alpha & s_X \\ t_Y & -b \end{pmatrix} : X \oplus Q \to Y \oplus P$ and $\begin{pmatrix} \beta & s_Y \\ t_X & -a \end{pmatrix} : Y \oplus P \to X \oplus Q$ are mutually inverse isomorophisms. We have the following.

$$\begin{pmatrix} \alpha & s_X \\ t_Y & -b \end{pmatrix}\begin{pmatrix} \beta & s_Y \\ t_X & -a \end{pmatrix} = \begin{pmatrix} \alpha\beta + s_X t_X & \alpha s_Y - s_X a \\ t_Y\beta - b\,t_X & t_Y s_Y + b\,a \end{pmatrix} = \begin{pmatrix} \mathrm{id}_X & \alpha s_Y - s_X a \\ 0 & t_Y s_Y + b\,a \end{pmatrix}$$
$$\begin{pmatrix} \beta & s_Y \\ t_X & -a \end{pmatrix}\begin{pmatrix} \alpha & s_X \\ t_Y & -b \end{pmatrix} = \begin{pmatrix} \beta\alpha + s_Y t_Y & \beta s_X - s_Y b \\ t_X\alpha - a\,t_Y & t_X s_X + a\,b \end{pmatrix} = \begin{pmatrix} \mathrm{id}_Y & \beta s_X - s_Y b \\ 0 & t_X s_X + a\,b \end{pmatrix}$$

Using that P and Q are the projective covers of X and Y respectively, the equations

$$(t_Y s_Y + b\,a)t_Y = t_Y(\mathrm{id}_Y - \beta\alpha) + t_Y\beta\alpha = t_Y$$
$$(t_X s_X + a\,b)t_X = t_X(\mathrm{id}_X - \alpha\beta) + t_X\alpha\beta = t_X$$

imply that $t_Y s_Y + b\,a = \mathrm{id}_Q$ and $t_X s_X + a\,b = \mathrm{id}_P$. With this, we obtain

$$t_X(\alpha s_Y - s_X a) = a\,t_Y s_Y - t_X s_X a = a - a\,b\,a - a + a\,b\,a = 0$$
$$t_Y(\beta s_X - s_Y b) = b\,t_X s_X - t_Y s_Y b = b - b\,a\,b - b + b\,a\,b = 0$$

so that $\alpha s_Y - s_X a = 0$ and $\beta s_X - s_Y b = 0$ since t_X and t_Y are surjective.

Ad (2). We only have to show that $f \overset{\mathrm{st}}{\simeq} g$ implies the existence of the diagram above.

By assumption, we have stable isomorphisms α and α' such that the following diagram commutes in $\underline{\mathrm{mod}}\, A$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha", "\wr"'] & X' \arrow[d, "\alpha'", "\wr"'] \\ Y \arrow[r, "g"] & Y' \end{tikzcd}$$

Part (1) now provides isomorphisms $\varphi : X \oplus Q \to Y \oplus P$ and $\varphi' : X' \oplus Q' \to Y' \oplus P'$ with $P, P', Q, Q' \in \operatorname{proj} A$. We use the notation of part (1). Since $\alpha\, g = f\, \alpha'$ in $\underline{\mathrm{mod}}\, A$, we also have a morphism $u : X \to Q'$ such that $\alpha\, g - f\, \alpha' = u\, t_{Y'}$. Finally, using that Q is projective, we obtain a morphism $v : Q \to Q'$ such that the following diagram commutes.

$$\begin{tikzcd} & Q \arrow[d, "t_Y\, g"] \arrow[dl, dashed, "v"'] \\ Q' \arrow[r, "t_{Y'}"'] & Y' \end{tikzcd}$$

Together, we can define a morphism $\left(\begin{smallmatrix} f & u \\ 0 & v \end{smallmatrix}\right) : X \oplus Q \to X' \oplus Q'$ which restricts to f. Now,

$$\tilde{g} := \varphi^{-1}\begin{pmatrix} f & u \\ 0 & v \end{pmatrix}\varphi' = \begin{pmatrix} \beta & s_Y \\ t_X & -a \end{pmatrix}\begin{pmatrix} f & u \\ 0 & v \end{pmatrix}\begin{pmatrix} \alpha' & s_{X'} \\ t_{Y'} & -b' \end{pmatrix} : Y \oplus P \to Y' \oplus P'$$

restricts to the following morphism $Y \to Y'$.

$$\beta\, f\, \alpha' + \beta\, u\, t_{Y'} + s_Y\, v\, t_{Y'} = \beta\, f\, \alpha' + \beta(\alpha\, g - f\, \alpha') + s_Y\, t_Y\, g = \beta\alpha\, g + (\mathrm{id}_Y - \beta\,\alpha)g = g$$

This gives the claimed commutative diagram.

$$\begin{tikzcd}
X \oplus Q \arrow[r, "\begin{pmatrix} f & u \\ 0 & v \end{pmatrix}"] \arrow[d, "\varphi", "\wr"'] & X' \oplus Q' \arrow[d, "\varphi'", "\wr"'] \\
Y \oplus P \arrow[r, "\varphi^{-1}\begin{pmatrix} f & u \\ 0 & v \end{pmatrix}\varphi'"] & Y' \oplus P'
\end{tikzcd}$$

□

We also note the following characterization of a stable isomorphism.

Lemma 1.4. *Suppose given a surjective morphism $Y \xrightarrow{f} Z$ in $\operatorname{mod} A$.*

Then f is a stable isomorphism if and only if f is a split epimorphism with projective kernel.

Proof. Suppose that f is a stable isomorphism. By Lemma 1.3.(1), there exist projective modules P and Q such that $Y \oplus Q \simeq Z \oplus P$. Let $K := \mathrm{Ker}(f)$. We obtain the following morphism of short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d, dashed] & Y \oplus Q \arrow[r] \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"'] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r] & Y \arrow[r, "f"] & Z \arrow[r] & 0
\end{tikzcd}$$

Since $Y \oplus Q \simeq Z \oplus P$, the upper sequence splits and we obtain that the lower sequence is also split. Furthermore, $K \oplus Z \simeq Y$ is a direct summand of $Y \oplus Q \simeq Z \oplus P$. Hence, K is a direct summand of P and therefore projective.

On the other hand, suppose that f is a split epimorphism with projective kernel P. The split exact sequence

$$0 \to P \to Y \xrightarrow{f} Z \to 0$$

yields $Y \simeq Z \oplus P$. By Lemma 1.3.(1), we obtain that f is a stable isomorphism. □

In specific circumstances we can use short exact sequences to induce stable isomorphisms.

Lemma 1.5. *([19, Lemma 2.14]) Suppose given a morphism between two short exact sequence in* $\operatorname{mod} A$.

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r, "f_1"] \arrow[d, "\alpha"] & Y_1 \arrow[r, "g_1"] \arrow[d, "\beta"] & Z_1 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & X_2 \arrow[r, "f_2"] & Y_2 \arrow[r, "g_2"] & Z_2 \arrow[r] & 0
\end{tikzcd}$$

Assume that γ is an isomorphism and that β is a stable isomorphism. Then α is a stable isomorphism.

If additionally X_1 and X_2 have no projective direct summand, then α and β are isomorphisms.

Proof. We follow the proof in [19, Lemma 2.14, part (1)].

Let $\rho : P \to X_2$ be the projective cover of X_2 and $Q = \mathrm{Ker}(X_1 \oplus P \to X_2)$. Consider the following commutative diagram with exact rows and columns. Note that the exactness of the middle column follows from the assumption that γ is an isomorphism.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Q \arrow[r, equal] \arrow[d, "u"] & Q \arrow[r] \arrow[d, "v"] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_1 \oplus P \arrow[r, "\begin{pmatrix} f_1 & 0 \\ 0 & 1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \alpha \\ \rho \end{pmatrix}"] & Y_1 \oplus P \arrow[r, "\begin{pmatrix} g_1 \\ 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \beta \\ \rho f_2 \end{pmatrix}"] & Z_1 \arrow[r] \arrow[d, "\gamma", "\wr"'] & 0 \\
0 \arrow[r] & X_2 \arrow[r, "f_2"] \arrow[d] & Y_2 \arrow[r, "g_2"] \arrow[d] & Z_2 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By Lemma 1.4, we have that $\begin{pmatrix} \beta \\ \rho f_2 \end{pmatrix}$ is a split epimorphism with Q projective, since β is a stable isomorphism. Hence v is a split monomorphism which implies that u is a split monomorphism as well. By Lemma 1.4, we obtain that α is a stable isomorphism.

If X_1 and X_2 have no projective direct summand, then α is an isomorphism in $\mathrm{mod}\, A$. Using that β is part of a morphism of short exact sequences, this implies that β is an isomorphism as well. □

The following results can be found in [45, Propositions 5.1.8 and 5.1.10].

Proposition 1.6. *We have the following for a self-injective algebra A.*

(1) The syzygy functor $\Omega : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, A$ is a self-equivalence of categories.

(2) The category $\underline{\mathrm{mod}}\, A$ is triangulated with suspension functor Ω^{-1} and distinguished triangles isomorphic to those induced by short exact sequences.

1.3 Homotopy category of complexes

We start with some notation and conventions for the category of complexes $\mathcal{C}(\operatorname{mod} A)$. An element $F^\bullet = (F^k)_{k\in\mathbb{Z}} \in \mathcal{C}(\operatorname{mod} A)$ will be written as a cochain complex with differential $(d^k)_{k\in\mathbb{Z}} := (d_F^k)_{k\in\mathbb{Z}}$ as follows.

$$\cdots \to F^{-2} \xrightarrow{d^{-2}} F^{-1} \xrightarrow{d^{-1}} F^0 \xrightarrow{d^0} F^1 \xrightarrow{d^1} F^2 \to \cdots$$

We denote the cohomology of $F^\bullet$ in degree $k \in \mathbb{Z}$ by $\mathrm{H}^k(F^\bullet) := \operatorname{Ker} d^k / \operatorname{Im} d^{k-1}$. We say that $F^\bullet$ is exact in degree $k \in \mathbb{Z}$, if $\mathrm{H}^k(F^\bullet) = 0$.

The shift $[1]$ of a complex $F^\bullet \in \mathcal{C}(\operatorname{mod} A)$ is given as $F^\bullet[1] := (F^{k+1})_{k\in\mathbb{Z}}$ with differentials given by $(-d_F^k)_{k\in\mathbb{Z}}$. This yields an autoequivalence $[n] : \mathcal{C}(\operatorname{mod} A) \to \mathcal{C}(\operatorname{mod} A)$ for all $n \in \mathbb{Z}$.

For $n \in \mathbb{Z}$, truncation $\tau_{\leqslant n} F^\bullet$ of $F^\bullet$ is defined as follows.

$$\cdots \to F^{n-2} \xrightarrow{d^{n-2}} F^{n-1} \xrightarrow{d^{n-1}} F^n \to 0 \to 0 \to \cdots$$

We often abbreviate $F^{\leqslant n} := \tau_{\leqslant n} F^\bullet$ and similarly for $F^{\geqslant n} := \tau_{\geqslant n} F^\bullet$. We also use the notation $F^{\leqslant n}$ to indicate that $F^k = 0$ for $k > n$. An A-module X will be identified with the complex $X \in \mathcal{C}(\operatorname{mod} A)$ consisting of X concentrated in degree zero.

By componentwise application, the equivalence $(-)^* = \operatorname{Hom}_A(-, A) : \operatorname{proj} A \to A\text{-proj}$ can be extended to the following equivalence.

$$(-)^* : \mathcal{C}(\operatorname{proj} A) \to \mathcal{C}(A\text{-proj}) : F^\bullet \to F_\bullet^* = F^{\bullet,*}$$

Here, we write $(F_k^*)_{k\in\mathbb{Z}} := (F^{k,*})_{k\in\mathbb{Z}} := \big((F^k)^*\big)_{k\in\mathbb{Z}}$ as the chain complex with differentials $d_k^{F^*} := d_F^{k,*} := \big(d_F^k\big)^*$ for $k \in \mathbb{Z}$.

$$\cdots \to F_2^* \xrightarrow{d_1^*} F_1^* \xrightarrow{d_0^*} F_0^* \xrightarrow{d_{-1}^*} F_{-1}^* \xrightarrow{d_{-2}^*} F_{-2}^* \to \cdots$$

We denote the homology of $F_\bullet^*$ in degree $k \in \mathbb{Z}$ by $\mathrm{H}_k(F_\bullet^*) = \operatorname{Ker}(d_{k-1}^*) / \operatorname{Im}(d_k^*)$. In this sense, we use both chain complexes and cochain complexes in our notation. However, we reserve the notation of chain complexes for dualized cochain complexes.

Similarly to $(-)^*$, the functors D and ν also induce equivalences $\mathrm{D} : \mathcal{C}(\operatorname{mod} A) \to \mathcal{C}(A\text{-mod})$ and $\nu : \mathcal{C}(\operatorname{proj} A) \to \mathcal{C}(\operatorname{inj} A)$ respectively.

Now, we introduce notation for the homotopy category and the derived category of complexes. We are mainly interested in the homotopy category of unbounded complexes of projective modules $\mathcal{K}(\operatorname{proj} A)$. Let $\mathcal{A}$ be an additive subcategory of $\operatorname{mod} A$.

The *homotopy category* $\mathcal{K}(\mathcal{A})$ is the category with complexes in $\mathcal{C}(\mathcal{A})$ as objects and homotopy equivalence classes of morphisms of complexes as morphisms. Recall that a morphism $f^\bullet : F^\bullet \to G^\bullet$ in $\mathcal{K}(\mathcal{A})$ is said to be homotopic to zero, if there exist morphisms $h^k : F^k \to G^{k-1}$ for $k \in \mathbb{Z}$ such that $h^k d_G^{k-1} + d_F^k h^{k+1} = f^k$. The morphism $h^\bullet = (h^k)_{k\in\mathbb{Z}}$ will be called a *homotopy* with homotopy maps h^k. Two morphisms $f^\bullet$ and $g^\bullet$ in $\mathrm{Hom}_{\mathcal{A}}(F^\bullet, G^\bullet)$ are called homotopy equivalent if $f^\bullet - g^\bullet$ is homotopic to zero.

If $\mathcal{A}$ is an abelian category, the *derived category* $\mathcal{D}(\mathcal{A})$ is the localization of the homotopy category at the class of quasi-isomorphisms. Recall that a morphism of complexes $f^\bullet : F^\bullet \to G^\bullet$ is called a *quasi-isomorphism* if the induced morphisms $\mathrm{H}^k(f^\bullet) : \mathrm{H}^k(F^\bullet) \to \mathrm{H}^k(G^\bullet)$ is an isomorphism for all $k \in \mathbb{Z}$.

We write $\mathcal{C}^+(\mathcal{A})$, $\mathcal{K}^+(\mathcal{A})$ and $\mathcal{D}^+(\mathcal{A})$ for the subcategory consisting of left bounded complexes in $\mathcal{C}(\mathcal{A})$, $\mathcal{K}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ respectively. Similarly, we write $\mathcal{C}^-(\mathcal{A})$, $\mathcal{K}^-(\mathcal{A})$ and $\mathcal{D}^-(\mathcal{A})$ for right bounded complexes. The subcategory of left and right bounded complexes is denoted by $\mathcal{C}^b(\mathcal{A})$, $\mathcal{K}^b(\mathcal{A})$ or $\mathcal{D}^b(\mathcal{A})$. By $\mathcal{C}^{+,b}(\mathcal{A})$, $\mathcal{K}^{+,b}(\mathcal{A})$ and $\mathcal{D}^{+,b}(\mathcal{A})$ we denote the subcategory of left bounded complexes that are bounded in cohomology. Finally, the subcategories $\mathcal{C}^{+,b^*}(\mathrm{proj}\, A)$, $\mathcal{K}^{+,b^*}(\mathrm{proj}\, A)$ and $\mathcal{D}^{+,b^*}(\mathrm{proj}\, A)$ consist of the left bounded complexes $F^\bullet$ with bounded homology $\mathrm{H}_\bullet(F_\bullet^*)$. The analogue categories for right bounded complexes are defined similarly.

Definition 1.7. A complex $F^\bullet \in \mathcal{K}(\mathrm{proj}\, A)$ is said to be *totally acyclic,* if $\mathrm{H}^k(F^\bullet) = 0$ and $\mathrm{H}_k(F_\bullet^*) = 0$ for all $k \in \mathbb{Z}$. The full subcategory of totally acyclic complexes in $\mathcal{K}(\mathrm{proj}\, A)$ is denoted by $\mathcal{K}_{\mathrm{tac}}(\mathrm{proj}\, A)$.

Definition 1.8. Let $f^\bullet : F^\bullet \to G^\bullet$ be a morphism in $\mathcal{C}(\mathrm{mod}\, A)$. The *mapping cone* $C(f)^\bullet$ *of* $f^\bullet$ is given by the complex $(F^{k+1} \oplus G^k)_{k\in\mathbb{Z}}$ with differential $d_C^\bullet$ defined as follows for $k \in \mathbb{Z}$.

$$d_C^k := \begin{pmatrix} -d_F^{k+1} & f^{k+1} \\ 0 & d_G^k \end{pmatrix}$$

Let $f^\bullet : F^\bullet \to G^\bullet$ in $\mathcal{C}(\mathrm{mod}\, A)$ with mapping cone $C(f)^\bullet \in \mathcal{C}(\mathrm{mod}\, A)$. We have the following short exact sequence in $\mathcal{C}(\mathrm{mod}\, A)$.

$$0 \to G^\bullet \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} C(f)^\bullet \xrightarrow{\begin{pmatrix} -1 \\ 0 \end{pmatrix}} F^\bullet[1] \to 0$$

We will often need information about the vanishing of cohomology or homology of $C(f)^\bullet$.

Lemma 1.9. *Suppose given* $f^\bullet : F_1^\bullet \to F_2^\bullet$ *in* $\mathcal{K}(\mathrm{proj}\, A)$.

(1) *Assume that there exist* $l_1, l_2 \in \mathbb{Z}$ *with* $\mathrm{H}^{\leqslant l_i}(F_i^\bullet) = 0$ *for* $i = 1, 2$. *Let* $l := \min(l_1 - 1, l_2)$. *We have* $\mathrm{H}^{\leqslant l}(C(f)^\bullet) = 0$.

(2) Assume that there exist $r_1, r_2 \in \mathbb{Z}$ *with* $\mathrm{H}_{\geqslant r_i}((F_i)^*_\bullet) = 0$ *for* $i = 1, 2$.
Let $r := \max(r_1 - 1, r_2)$. *We have* $\mathrm{H}_{\geqslant r}(C(f)^*_\bullet) = 0$.

Proof. We have the following componentwise split exact sequence in $\mathcal{C}(\operatorname{proj} A)$.

$$0 \to F_2^\bullet \to C(f)^\bullet \to F_1^\bullet[1] \to 0$$

This induces a long exact sequence of cohomology.

$$\cdots \to \mathrm{H}^k(F_1^\bullet) \to \mathrm{H}^k(F_2^\bullet) \to \mathrm{H}^k(C(f)^\bullet) \to \mathrm{H}^{k+1}(F_1^\bullet) \to \mathrm{H}^{k+1}(F_2^\bullet) \to \cdots$$

Since the short exact sequence is componentwise split exact, we also obtain a long exact sequence of homology.

$$\cdots \to \mathrm{H}_{k+1}((F_2)^*_\bullet) \to \mathrm{H}_{k+1}((F_1)^*_\bullet) \to \mathrm{H}_k(C(f)^*_\bullet) \to \mathrm{H}_k((F_2)^*_\bullet) \to \mathrm{H}_k((F_1)^*_\bullet) \to \cdots$$

We have that $\mathrm{H}^{k+1}(F_1^\bullet) = 0$ for $k \leqslant l_1 - 1$ and $\mathrm{H}^k(F_2^\bullet) = 0$ for $k \leqslant l_2$. Hence, the first long exact sequence implies that $\mathrm{H}^k(C(f)^\bullet) = 0$ for $k \leqslant \min(l_1 - 1, l_2)$.

We have that $\mathrm{H}_{k+1}((F_1)^*_\bullet) = 0$ for $k \geqslant r_1 - 1$ and $\mathrm{H}_k((F_2)^*_\bullet) = 0$ for $k \geqslant r_2$. Hence, the second long exact sequence implies that $\mathrm{H}_k(C(f)^*_\bullet) = 0$ for $k \geqslant \max(r_1 - 1, r_2)$. □

The following results can be found in [45, Proposition 3.5.25 and 3.5.40].

Proposition 1.10. *Let* $\mathcal{A}$ *be an additive category. Let* $\mathcal{A}'$ *be an abelian category.*

The homotopy category $\mathcal{K}(\mathcal{A})$ *and the derived category* $\mathcal{D}(\mathcal{A}')$ *are triangulated categories with suspension functor* [1] *and distinguished triangles isomorphic to triangles of the following form.*

$$F^\bullet \xrightarrow{f^\bullet} G^\bullet \to C(f)^\bullet \to$$

Note that all subcategories of $\mathcal{K}(\mathcal{A})$ discussed above are triangulated subcategories. In particular, $\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$ is a triangulated category.

We recall the following equivalences of triangulated categories; cf. [45, Proposition 3.5.43].

Theorem 1.11. *There exist triangulated equivalences between the following categories.*

(1) $\mathcal{D}^-(\operatorname{mod} A) \simeq \mathcal{K}^-(\operatorname{proj} A)$ *and* $\mathcal{D}^+(\operatorname{mod} A) \simeq \mathcal{K}^+(\operatorname{inj} A)$.

(2) $\mathcal{D}^b(\operatorname{mod} A) \simeq \mathcal{K}^{-,b}(\operatorname{proj} A)$ *and* $\mathcal{D}^b(\operatorname{mod} A) \simeq \mathcal{K}^{+,b}(\operatorname{inj} A)$.

(3) $\mathcal{D}^b(\operatorname{mod} A) \simeq \mathcal{K}^b(\operatorname{proj} A)$ *and* $\mathcal{D}^b(\operatorname{mod} A) \simeq \mathcal{K}^b(\operatorname{inj} A)$ *if* $\operatorname{gldim} A < \infty$.

We will need in the future that the perpendicular category of a triangulated category is triangulated.

Lemma 1.12. *Suppose given a triangulated category $\mathcal{T}$ and a full triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$. Then ${}^{\perp}\mathcal{S}$ is a triangulated subcategory of $\mathcal{T}$.*

Proof. Suppose given $X \in {}^{\perp}\mathcal{S}$. For $k \in \mathbb{Z}$ and $S \in \mathcal{S}$ we have

$$\mathrm{Hom}_{\mathcal{T}}(X[k], S) \simeq \mathrm{Hom}_{\mathcal{T}}(X, S[-k]) = 0$$

since $S[-k] \in {}^{\perp}\mathcal{S}$. Thus, $X[k] \in {}^{\perp}\mathcal{S}$.

Suppose given a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h}$ in $\mathcal{T}$ with $X, Y \in {}^{\perp}\mathcal{S}$. We show that $Z \in {}^{\perp}\mathcal{S}$. Let $v : Z \to S$ be a morphism in $\mathcal{T}$ with $S \in \mathcal{S}$. This induces a morphism of triangles via a morphism $u : Y \to S$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "0"] & Y \arrow[r, "g"] \arrow[d, dashed, "u"] & Z \arrow[r, "h"] \arrow[d, "v"] & X[1] \arrow[d, "0"] \\ 0 \arrow[r] & S \arrow[r, "\mathrm{id}_S"] & S \arrow[r] & 0 \end{tikzcd}$$

Since $Y \in {}^{\perp}\mathcal{S}$, we obtain that $g\,v = u = 0$. This induces another morphism of triangles via a morphism $w : X[1] \to S$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed] & Y \arrow[r, "g"] \arrow[d, "0"] & Z \arrow[r, "h"] \arrow[d, "v"] & X[1] \arrow[d, dashed, "w"] \\ S[-1] \arrow[r] & 0 \arrow[r] & S \arrow[r, "\mathrm{id}_S"] & S \end{tikzcd}$$

Since $X[1] \in {}^{\perp}\mathcal{S}$, we obtain that $v = h\,w = 0$. This shows that $Z \in {}^{\perp}\mathcal{S}$. □

Finally, we recall the definition of the Grothendieck group of a triangulated category.

Definition 1.13. Let $\mathcal{T}$ be a triangulated category.

Let L be the free abelian group generated by the isomorphism classes of objects of $\mathcal{T}$. Let R be the subgroup of L generated by the classes

$$[X] - [Y] + [Z]$$

where $X \to Y \to Z \to$ is a distinguished triangle.

The *Grothendieck group $G_0(\mathcal{T})$ of $\mathcal{T}$* is defined as the quotient L/R.

Remark 1.14. Let $X \in \mathcal{K}(\mathrm{mod}\,A)$. Note that $[X] = -[X[1]]$ in $G_0(\mathcal{K}(\mathrm{mod}\,A))$. In fact, we have a distinguished triangle $X \to 0 \to X[1] \to$ so that $[X] + [X[1]] = 0$.

1.4 Projective-injective modules

In this section we discuss concepts related to A-modules which are both projective and injective. The full subcategory of projective-injective A-modules will be denoted by $\mathcal{P}_A$. Furthermore, we will need the following special class of projective-injective modules.

Definition 1.15. A module $Z \in \operatorname{mod} A$ is called *strongly projective-injective* if $\nu^k Z$ is projective for all $k \in \mathbb{Z}_{\geqslant 0}$. The full subcategory of strongly projective-injective A-modules will be denoted by $\operatorname{stp} A$.

Note that strongly projective-injective modules are projective-injective. In fact, for an indecomposable module $P \in \operatorname{proj} A$, the module $\nu(P)$ is indecomposable injective. Thus, $\nu^k(P)$ is projective-injective for $k \geqslant 1$ if $P \in \operatorname{stp} A$. Since A is finite dimensional, there are only finitely many indecomposable projective modules. Thus, there exist $m, n \in \mathbb{Z}_{\geqslant 0}$ with $m < n$ such that $\nu^m(P) \simeq \nu^n(P)$. Applying ν^{-m}, we obtain $P \simeq \nu^{n-m}(P) \in \mathcal{P}_A$. If $P \in \operatorname{stp} A$ is not indecomposable, we have seen that all indecomposable direct summands of P are strongly projective-injective. Hence, P is strongly projective-injective as well.

We state the definitions of two homological dimensions. The first, dominant dimension, was introduced by Nakayama. The latter has been introduced in [14] by Fang, Hu and Koenig.

Definition 1.16. Let $0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$ be a minimal injective resolution of A.

(1) The *dominant dimension* of A is defined as the largest $d \in \mathbb{Z}_{\geqslant 0}$ such that I^k is projective-injective for all $k < d$. We set $d = \infty$ if I^k is projective-injective for all $k \geqslant 0$. We denote the dominant dimension of A by $\operatorname{domdim} A$.

(2) The *ν-dominant dimension* of A is defined as the largest $d \in \mathbb{Z}_{\geqslant 0}$ such that I^k is strongly projective-injective for all $k < d$. We set $d = \infty$ if I^k is strongly projective-injective for all $k \geqslant 0$. We denote the ν-dominant dimension of A by $\nu\text{-}\operatorname{domdim} A$.

We will use these two dimensions mainly for the following properties.

Remark 1.17.

(1) Let $\operatorname{domdim} A \geqslant 1$. In this case, every projective module can be embedded into a projective-injective module. Thus, we have ${}^{\perp}(\operatorname{proj} A) = {}^{\perp}\mathcal{P}_A$.

(2) Let $\nu\text{-}\operatorname{domdim} A \geqslant 1$. In this case, every projective-injective module is strongly projective-injective. In fact, consider the embedding $I \hookrightarrow Z$ for a projective-injective module I and $Z \in \operatorname{stp} A$. Since I is injective, this morphisms splits and I is strongly projective-injective as a direct summand of Z.

As a direct consequence, we have $\operatorname{domdim} A = \nu\text{-}\operatorname{domdim} A$ and ${}^{\perp}(\operatorname{proj} A) = {}^{\perp}(\operatorname{stp} A)$.

Chapter 2

Stable module category and homotopy category

Let k be a field. Let A be a finite dimensional k-algebra without semisimple summands.

In this chapter, we discuss some of the concepts which where introduced by Kiriko Kato in [18] and [19] in the context of commutative rings. However, many of her results still hold for non-commutative finite dimensional k-algebras.

In the first section, we look at the construction of a functor from the stable module category to the homotopy category. This functor will restrict to an equivalence $\mathcal{F} : \underline{\mathrm{mod}}\, A \to \mathcal{L}_A$ with some full subcategory $\mathcal{L}_A$ of $\mathcal{K}(\mathrm{proj}\, A)$. In [18], Kato uses this equivalence to define a weak kernel and weak cokernel in the stable module category.

In the second section, we consider a special class of short exact sequences, called perfect exact sequences. These are short exact sequences that remain exact under the functor $\mathrm{Hom}_A(-, A)$. Importantly, a perfect exact sequence corresponds to a distinguished triangle in $\mathcal{K}(\mathrm{proj}\, A)$ via the equivalence $\mathcal{F}$. In [19], Kato characterizes morphisms which are stably equivalent to a monomorphism via the cohomology of the mapping cone in the image of $\mathcal{F}$. Such a morphism is always part of a unique perfect exact sequence.

Finally, the last section is dedicated to perfect exact sequences with projective middle term and their connection to the shift in $\mathcal{K}(\mathrm{proj}\, A)$.

2.1 A functor to the homotopy category

This section is dedicated to the construction of a functor $\mathcal{F} : \underline{\mathrm{mod}}\, A \to \mathcal{K}(\mathrm{proj}\, A)$. This functor restricts to an equivalence $\mathcal{F} : \underline{\mathrm{mod}}\, A \to \mathcal{L}_A$ with quasi-inverse $\mathrm{H}^0(\tau_{\leqslant 0}(-))$, where $\mathcal{L}_A$ is a subcategory of $\mathcal{K}(\mathrm{proj}\, A)$ which we introduce below. Throughout this section, we follow [19].

Definition 2.1 (Kato)**.** Let $\mathcal{L}_A$ be the full subcategory of $\mathcal{K}(\operatorname{proj} A)$ defined as follows.

$$\mathcal{L}_A = \{F^\bullet \in \mathcal{K}(\operatorname{proj} A) \mid \mathrm{H}^{<0}(F^\bullet) = 0,\, \mathrm{H}_{\geqslant 0}(F^*_\bullet) = 0\}$$

We often write $\mathcal{L} := \mathcal{L}_A$.

Note that a complex $F^\bullet$ in $\mathcal{L}_A$ can be truncated via $\tau_{\leqslant 0}(-)$ to a projective resolution of some A-module. Similarly, the truncation $\tau_{\geqslant -1}(F^*_\bullet)$ is a projective resolution of some left A-module. The latter will be used to verify that $\mathrm{H}^0(\tau_{\leqslant 0}(-))$ is an equivalence as is illustrated in the next example.

Example 2.2. Let A be the quiver algebra over k given by

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$$

with relation $\alpha\,\beta\,\gamma = 0$. The algebra has the following indecomposable projective modules. We also note their images under $(-)^*$.

$$P_1 := \begin{smallmatrix}1\\2\\3\end{smallmatrix}, \quad P_2 := \begin{smallmatrix}2\\3\\4\end{smallmatrix}, \quad P_3 := \begin{smallmatrix}3\\4\end{smallmatrix}, \quad P_4 := 4, \qquad P_1^* = 1, \quad P_2^* = \begin{smallmatrix}2\\1\end{smallmatrix}, \quad P_3^* = \begin{smallmatrix}3\\2\\1\end{smallmatrix}, \quad P_4^* = \begin{smallmatrix}4\\3\\2\end{smallmatrix}$$

The following two complexes are an element of $\mathcal{L}_A$. We note the degree above the complexes.

$$\begin{array}{ccccccc}
 & -3 & -2 & -1 & 0 & 1 & 2 \\
F^\bullet: & & 0 \longrightarrow & P_4 \xrightarrow{\beta\gamma(-)} & P_2 \xrightarrow{\alpha(-)} & P_1 \longrightarrow & 0 \\
G^\bullet: & 0 \longrightarrow & P_4 \xrightarrow{\gamma(-)} & P_3 \xrightarrow{\alpha\beta(-)} & P_1 \longrightarrow & 0 &
\end{array}$$

On the other hand, the minimal projective resolution $0 \to P_4 \to P_2 \to 0$ of $\mathrm{H}^0(\tau_{\leqslant 0} F^\bullet) = \begin{smallmatrix}2\\3\end{smallmatrix}$ is not an element of $\mathcal{L}_A$. Consider the lift of the morphism $\begin{smallmatrix}2\\3\end{smallmatrix} \to \begin{smallmatrix}1\\2\end{smallmatrix}$ to a morphism between the minimal projective resolutions of $\mathrm{H}^0(\tau_{\leqslant 0} F^\bullet) = \begin{smallmatrix}2\\3\end{smallmatrix}$ and $\mathrm{H}^0(\tau_{\leqslant 0} G^\bullet) = \begin{smallmatrix}1\\2\end{smallmatrix}$.

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & P_4 \arrow[r, "\beta\gamma(-)"] \arrow[d, "0"] & P_2 \arrow[r] \arrow[d, "\alpha(-)"] & 0 \arrow[d] \\
0 \arrow[r] & P_4 \arrow[r, "\gamma(-)"] & P_3 \arrow[r, "\alpha\beta(-)"] & P_1 \arrow[r] & 0
\end{tikzcd}$$

This morphism is non-zero in $\mathcal{K}(\operatorname{proj} A)$. In contrast, there is no non-zero morphism between $F^\bullet$ and $G^\bullet$ in $\mathcal{K}(\operatorname{proj} A)$. This corresponds to the fact that the morphism $\begin{smallmatrix}2\\3\end{smallmatrix} \to \begin{smallmatrix}1\\2\end{smallmatrix}$ factors through the projective module P_1.

The following is the key result needed for the main theorem of this section. We give an expanded version of the proof found in [19, Lemma 2.8], filling in several details.

Lemma 2.3. ([18, Lemma 2.1])

Let $f^\bullet$ *be a morphism in* $\mathcal{K}(\operatorname{proj} A)$*. Then* $\underline{\mathrm{H}^0(\tau_{\leqslant 0}\, f^\bullet)} = 0$ *in* $\underline{\operatorname{mod}}\, A$ *if* $f^\bullet = 0$ *in* $\mathcal{K}(\operatorname{proj} A)$*.*

Let $f^\bullet$ *be a morphism in* $\mathcal{L}_A$*. Then* $\underline{\mathrm{H}^0(\tau_{\leqslant 0}\, f^\bullet)} = 0$ *in* $\underline{\operatorname{mod}}\, A$ *if and only if* $f^\bullet = 0$ *in* $\mathcal{K}(\operatorname{proj} A)$*.*

Proof. Suppose given $f^\bullet : F^\bullet \to G^\bullet$ in $\mathcal{K}(\operatorname{proj} A)$ such that $f^\bullet = 0$ in $\mathcal{K}(\operatorname{proj} A)$. We show that $\mathrm{H}^0(\tau_{\leqslant 0}\, f^\bullet)$ factors through the projective module F^1.

By assumption, there exists a homotopy $h^\bullet : F^\bullet \to G^\bullet[-1]$ with $f^k = d_F^k\, h^{k+1} + h^k\, d_G^{k-1}$ for $k \in \mathbb{Z}$. We define the following two morphisms in $\mathcal{C}(\operatorname{mod} A)$.

$$\begin{tikzcd}
\tau_{\leqslant 0}\, F^\bullet \arrow[d, "\eta^\bullet"'] & \cdots \arrow[r] & F^{-2} \arrow[r, "d_F^{-2}"] \arrow[d] & F^{-1} \arrow[r, "d_F^{-1}"] \arrow[d] & F^0 \arrow[r] \arrow[d, "d_F^0"] & 0 \arrow[r] \arrow[d] & \cdots \\
F^1 \arrow[d, "\varphi^\bullet"'] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & F^1 \arrow[r] \arrow[d, "h^1"] & 0 \arrow[r] \arrow[d] & \cdots \\
\tau_{\leqslant 0}\, G^\bullet & \cdots \arrow[r] & G^{-2} \arrow[r, "d_G^{-2}"] & G^{-1} \arrow[r, "d_G^{-1}"] & G^0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

We verify that $\tau_{\leqslant 0}\, f^\bullet = \eta^\bullet \varphi^\bullet$ in $\mathcal{K}(\operatorname{proj} A)$ via the homotopy $\tau_{\leqslant 0} h^\bullet$.

$$\begin{tikzcd}
\tau_{\leqslant 0}\, F^\bullet \arrow[d, "(\tau_{\leqslant 0}\, f^\bullet) - \eta^\bullet\, \varphi^\bullet"'] & \cdots \arrow[r] & F^{-2} \arrow[r, "d_F^{-2}"] \arrow[d] & F^{-1} \arrow[r, "d_F^{-1}"] \arrow[d] \arrow[dl, "h^{-1}"'] & F^0 \arrow[r] \arrow[d] \arrow[dl, "h^0"'] & 0 \arrow[r] \arrow[d] \arrow[dl, "0"'] & \cdots \\
\tau_{\leqslant 0}\, G^\bullet & \cdots \arrow[r] & G^{-2} \arrow[r, "d_G^{-2}"] & G^{-1} \arrow[r, "d_G^{-1}"] & G^0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

For $k \leqslant -1$ we have by definition of $h^\bullet$ that

$$f^k - \eta^k \varphi^k = f^k = d_F^k\, h^{k+1} + h^k\, d_G^{k-1}$$

For $k = 0$ we have by definition of $\eta^\bullet$ and $\varphi^\bullet$ that

$$f^0 - \eta^0 \varphi^0 = d_F^0\, h^1 + h^0\, d_G^{-1} - d_F^0\, h^1 = h^0\, d_G^{-1}.$$

Therefore, we have

$$\mathrm{H}^0\big(\tau_{\leqslant 0}\, f^\bullet\big) = \mathrm{H}^0\big(\eta^\bullet \varphi^\bullet)\big) = \mathrm{H}^0\big(\eta^\bullet\big)\, \mathrm{H}^0\big(\varphi^\bullet\big)$$

so that $\mathrm{H}^0(\tau_{\leqslant 0}\, f^\bullet)$ factors through the projective module F^1.

Conversely, suppose given $f^\bullet : F^\bullet \to G^\bullet$ in $\mathcal{L}$ such that $\underline{\mathrm{H}^0(\tau_{\leqslant 0}\, f^\bullet)} = 0$. First, we construct homotopy maps $s^{k+1} : F^{k+1} \to G^k$ for $k \leqslant -1$ and $t^{k+1} : F^{k+1} \to G^k$ for $k \geqslant 1$. In a second step, we will use these maps to show that $f^\bullet = 0$.

By assumption, $\mathrm{H}^0(\tau_{\leqslant 0}\, f^\bullet)$ factors through the projective cover $p : P \to \mathrm{H}^0(\tau_{\leqslant 0}\, G^\bullet)$. Suppose given a morphism $\alpha : \mathrm{H}^0(\tau_{\leqslant 0}\, F^\bullet) \to P$ in $\operatorname{mod} A$ with $\alpha\, p = \mathrm{H}^0(\tau_{\leqslant 0}\, f^\bullet)$. Write $G^0 \xrightarrow{\rho} \mathrm{H}^0(\tau_{\leqslant 0}\, G^\bullet)$ for the canonical surjection. Since P is projective, p factors through G^0.

$$
\begin{tikzcd}
 & \mathrm{H}^0(\tau_{\leqslant 0}\, F^\bullet) \arrow[d, "\alpha"] \\
 & P \arrow[d, "p"] \arrow[dl, dashed, "\exists\beta"'] \\
G^0 \arrow[r, two heads, "\rho"] & \mathrm{H}^0(\tau_{\leqslant 0}\, G^\bullet)
\end{tikzcd}
$$

Write $g := \alpha\, \beta : \mathrm{H}^0(\tau_{\leqslant 0}\, F^\bullet) \to G^0$, so that $\mathrm{H}^0(\tau_{\leqslant 0}\, f^\bullet) = g\, \rho$ factors through G^0. Note that $\tau_{\leqslant 0}\, F^\bullet$ and $\tau_{\leqslant 0}\, G^\bullet$ are projective resolutions of $\mathrm{H}^0(\tau_{\leqslant 0}\, F^\bullet)$ and $\mathrm{H}^0(\tau_{\leqslant 0}\, G^\bullet)$ respectively. Therefore, the morphism g and ρ lift to morphisms of complexes $g^\bullet$ and $\rho^\bullet$ such that we have $\mathrm{H}^0(\tau_{\leqslant 0}\, g^\bullet) = g$ and $\mathrm{H}^0(\tau_{\leqslant 0}\, \rho^\bullet) = \rho$ respectively.

$$
\begin{tikzcd}
\tau_{\leqslant 0}\, F^\bullet \arrow[d, "g^\bullet"'] & \cdots \arrow[r] & F^{-2} \arrow[r, "d_F^{-2}"] \arrow[d] & F^{-1} \arrow[r, "d_F^{-1}"] \arrow[d] \arrow[ddl, dashed, "s^{-1}"'] & F^0 \arrow[r] \arrow[d, "g^0"] \arrow[ddl, dashed, "s^0"'] & 0 \arrow[r] \arrow[d] & \cdots \\
G^0 \arrow[d, "\rho^\bullet"'] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & G^0 \arrow[r] \arrow[d, "\rho^0"] & 0 \arrow[r] \arrow[d] & \cdots \\
\tau_{\leqslant 0}\, G^\bullet & \cdots \arrow[r] & G^{-2} \arrow[r, "d_G^{-2}"] & G^{-1} \arrow[r, "d_G^{-1}"] & G^0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

By construction, we have $\tau_{\leqslant 0}\, f^\bullet = g^\bullet\, \rho^\bullet$ in $\mathcal{K}(\operatorname{proj} A)$. Hence, there exist homotopy maps $s^{k+1} : F^{k+1} \to G^k$ for $k \leqslant -1$ such that

$$
f^k = f^k - g^k\, \rho^k = s^k\, d_G^{k-1} + d_F^k\, s^{k+1}, \qquad \text{for } k \leqslant -1.
$$

Now we construct homotopy maps $t^{k+1} : F^{k+1} \to G^k$ for $k \geqslant 1$. Note that $\underline{\mathrm{H}^0(\tau_{\leqslant 0}\, f^\bullet)^*} = 0$. Since $\mathrm{H}_{\geqslant 0}(F_\bullet^*) = 0$, we have natural isomorphisms

$$
\mathrm{H}^0(\tau_{\leqslant 0}\, F^\bullet)^* \simeq \operatorname{Ker}(F_0^* \to F_{-1}^*) \simeq \operatorname{Cok}(F_2^* \to F_1^*) = \mathrm{H}_1(\tau_{\geqslant 1} F_\bullet^*)
$$

so that $\mathrm{H}^0(\tau_{\leqslant 0}\, f^\bullet)^* \simeq \mathrm{H}_1(\tau_{\geqslant 1} f_\bullet^*)$. As above, we obtain that $\mathrm{H}_1(\tau_{\geqslant 1} f_\bullet^*)$ factors through F_1^* via morphisms $\tilde{g} : \mathrm{H}_1(\tau_{\geqslant 1} G_\bullet^*) \to F_1^*$ and $\tilde{\rho} : F_1^* \to \mathrm{H}_1(\tau_{\geqslant 1} F_\bullet^*)$. These maps lift to morphisms of complexes $\tilde{g}_\bullet$ and $\tilde{\rho}_\bullet$ such that $\mathrm{H}_1(\tau_{\geqslant 1} \tilde{g}_\bullet) = \tilde{g}$ and $\mathrm{H}_1(\tau_{\geqslant 1} \tilde{\rho}_\bullet) = \tilde{\rho}$ respectively.

Consider the following commutative diagram.

$$\begin{tikzcd}
\tau_{\geqslant 1} G^*_\bullet \arrow[d, "\tilde{g}_\bullet"'] & \cdots \arrow[r] & G^*_3 \arrow[r, "d_G^{2,*}"] \arrow[d] & G^*_2 \arrow[r, "d_G^{1,*}"] \arrow[d] \arrow[ddl, dashed, "t_3^*"'] & G^*_1 \arrow[r] \arrow[d, "\tilde{g}_1"] \arrow[ddl, dashed, "t_2^*"'] & 0 \arrow[r] \arrow[d] & \cdots \\
F^*_1 \arrow[d, "\tilde{\rho}_\bullet"'] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & F^*_1 \arrow[r] \arrow[d, "\tilde{\rho}_1"] & 0 \arrow[r] \arrow[d] & \cdots \\
\tau_{\geqslant 1} F^*_\bullet & \cdots \arrow[r] & F^*_3 \arrow[r, "d_F^{2,*}"] & F^*_2 \arrow[r, "d_F^{1,*}"] & F^*_1 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

By construction, we have $\tau_{\geqslant 1} f^*_\bullet = \tilde{g}_\bullet\, \tilde{\rho}_\bullet$ in $\mathcal{K}(\operatorname{proj} A)$. Hence, there exist homotopy maps $t^*_{k+1} : G^*_k \to F^*_{k+1}$ for $k \geqslant 1$ such that

$$f^*_k = f^*_k - \tilde{g}_k\, \tilde{\rho}_k = d_G^{k-1,*}\, t^*_k + t^*_{k+1}\, d_F^{k,*}, \qquad \text{for } k \geqslant 2.$$

Applying $(-)^*$, we obtain homotopy maps $t^{k+1} : F^{k+1} \to G^k$ for $k \geqslant 1$ such that

$$f^k = t^k\, d_G^{k-1} + d_F^k\, t^{k+1}, \qquad \text{for } k \geqslant 2.$$

For the final step, let $h^k := s^k$ for $k \leqslant 0$ and $h^k := t^k$ for $k \geqslant 2$. We already have that $f^k = h^k\, d_G^{k-1} + d_F^k\, h^{k+1}$ for $k \in \mathbb{Z} \setminus \{0, 1\}$. The situation can be visualized as follows.

$$\begin{tikzcd}
F^\bullet \arrow[d, "f^\bullet"] & \cdots \arrow[r] & F^{-2} \arrow[r, "d_F^{-2}"] \arrow[d, "f^{-2}"] & F^{-1} \arrow[r, "d_F^{-1}"] \arrow[d, "f^{-1}"] \arrow[dl, "h^{-1}"'] & F^0 \arrow[r, "d_F^0"] \arrow[d, "f^0"] \arrow[dl, "h^0"'] & F^1 \arrow[r, "d_F^1"] \arrow[d, "f^1"] & F^2 \arrow[r, "d_F^2"] \arrow[d, "f^2"] \arrow[dl, "h^2"'] & F^3 \arrow[r] \arrow[d, "f^3"] \arrow[dl, "h^3"'] & \cdots \\
G^\bullet & \cdots \arrow[r] & G^{-2} \arrow[r, "d_G^{-2}"'] & G^{-1} \arrow[r, "d_G^{-1}"'] & G^0 \arrow[r, "d_G^0"'] & G^1 \arrow[r, "d_G^1"'] & G^2 \arrow[r, "d_G^2"'] & G^3 \arrow[r] & \cdots
\end{tikzcd}$$

We define a morphism $\tilde{f}^\bullet : F^\bullet \to G^\bullet$ as follows.

$$\begin{aligned}
\tilde{f}^k &:= 0, && \text{for } k \leqslant -1 \\
\tilde{f}^0 &:= f^0 - h^0\, d_G^{-1} && \\
\tilde{f}^1 &:= f^1 - d_F^1\, h^2 && \\
\tilde{f}^k &:= 0, && \text{for } k \geqslant 2
\end{aligned}$$

We verify that $\tilde{f}^\bullet$ is a morphism of complexes.

$$\begin{aligned}
d_F^{-1}\, \tilde{f}^0 &= d_F^{-1}\, f^0 - d_F^{-1}\, h^0\, d_G^{-1} = f^{-1} d_G^{-1} - (f^{-1} - h^{-1}\, d_G^{-2}) d_G^{-1} = 0 \\
d_F^0\, \tilde{f}^1 &= d_F^0 (f^1 - d_F^1\, h^2) = d_F^0\, f^1 = f^0\, d_G^0 = (f^0 - h^0\, d_G^{-1}) d_G^0 = \tilde{f}^0\, d_G^0 \\
\tilde{f}^1\, d_G^1 &= f^1\, d_G^1 - d_F^1\, h^2\, d_G^1 = d_F^1\, f^2 - d_F^1 (f^2 - d_F^2\, h^3) = 0
\end{aligned}$$

By construction, the morphisms $\tilde{f}^\bullet$ and $f^\bullet$ are equal in $\mathcal{K}(\mathrm{proj}\, A)$.

$$\begin{tikzcd}
F^\bullet \arrow[d, "\tilde{f}^\bullet"] & \cdots \arrow[r] & F^{-1} \arrow[r, "d_F^{-1}"] \arrow[d, "0"] & F^0 \arrow[r, "d_F^0"] \arrow[d, "\tilde{f}^0"] & F^1 \arrow[r, "d_F^1"] \arrow[d, "\tilde{f}^1"] \arrow[dl, dashed, "u^1"'] & F^2 \arrow[r] \arrow[d, "0"] & \cdots \\
G^\bullet & \cdots \arrow[r] & G^{-1} \arrow[r, "d_G^{-1}"] & G^0 \arrow[r, "d_G^0"] & G^1 \arrow[r, "d_G^1"] & G^2 \arrow[r] & \cdots
\end{tikzcd}$$

We have $\tilde{f}^{0,*}\, d_F^{-1,*} = d_G^{-1,*}\, \tilde{f}^{-1,*} = 0$, so that $\tilde{f}^{0,*}$ factors through $\mathrm{Ker}(d_F^{-1,*}) = \mathrm{Im}(d_F^{0,*})$. Using that $G^{0,*}$ is projective, we obtain a morphism $u^{1,*} : G^{0,*} \to F^{1,*}$ such that $\tilde{f}^{0,*} = u^{1,*}\, d_F^{0,*}$. Applying $(-)^*$, we get that $\tilde{f}^0 = d_F^0\, u^1$.

We define a morphism $\tilde{\tilde{f}}^\bullet : F^\bullet \to G^\bullet$ as follows.

$$\begin{aligned}
\tilde{\tilde{f}}^k &:= 0, && \text{for } k \leqslant 0 \\
\tilde{\tilde{f}}^1 &:= \tilde{f}^1 - u^1\, d_G^0 && \\
\tilde{\tilde{f}}^k &:= 0, && \text{for } k \geqslant 2
\end{aligned}$$

We verify that $\tilde{\tilde{f}}^\bullet$ is a morphism of complexes.

$$\begin{aligned}
d_F^0\, \tilde{\tilde{f}}^1 &= d_F^0\, \tilde{f}^1 - d_F^0\, u^1\, d_G^0 = \tilde{f}^0\, d_G^0 - \tilde{f}^0\, d_G^0 = 0 \\
\tilde{\tilde{f}}^1\, d_G^1 &= \tilde{f}^1\, d_G^1 - u^1\, d_G^0\, d_G^1 = 0
\end{aligned}$$

By construction, the morphisms $\tilde{\tilde{f}}^\bullet$ and $\tilde{f}^\bullet$ are equal in $\mathcal{K}(\mathrm{proj}\, A)$.

$$\begin{tikzcd}
F^\bullet \arrow[d, "\tilde{\tilde{f}}^\bullet"] & \cdots \arrow[r] & F^{-1} \arrow[r, "d_F^{-1}"] \arrow[d, "0"] & F^0 \arrow[r, "d_F^0"] \arrow[d, "0"] & F^1 \arrow[r, "d_F^1"] \arrow[d, "\tilde{\tilde{f}}^1"] & F^2 \arrow[r, "d_F^2"] \arrow[d, "0"] \arrow[dl, dashed, "v^2"'] & F^3 \arrow[r] \arrow[d, "0"] \arrow[dl, dashed, "v^3"'] & \cdots \\
G^\bullet & \cdots \arrow[r] & G^{-1} \arrow[r, "d_G^{-1}"] & G^0 \arrow[r, "d_G^0"] & G^1 \arrow[r, "d_G^1"] & G^2 \arrow[r, "d_G^2"] & G^3 \arrow[r] & \cdots
\end{tikzcd}$$

We have that $\tilde{\tilde{f}}^{1,*}\, d_F^{0,*} = d_G^{0,*}\, \tilde{\tilde{f}}^{0,*} = 0$. As above, we obtain a morphism $v^{2,*} : G^{1,*} \to F^{2,*}$ such that $\tilde{\tilde{f}}^{1,*} = v^{2,*}\, d_F^{1,*}$. Applying $(-)^*$, this results in $\tilde{\tilde{f}}^1 = d_F^1\, v^2$. Moreover, we have that $d_G^{1,*}\, v^{2,*}\, d_F^{1,*} = d_G^{1,*}\, \tilde{\tilde{f}}^{1,*} = 0$. Again, we obtain a morphism $v^{3,*} : G^{2,*} \to F^{3,*}$ such that $d_G^{1,*}\, v^{2,*} = v^{3,*}\, d_F^{2,*}$. Applying $(-)^*$, this results in $v^2\, d_G^1 = d_F^2\, v^3$.

Letting $v^k := 0$ for $k \neq 2, 3$, we constructed a homotopy $v^\bullet : F^\bullet \to G^\bullet[-1]$ such that the following holds for all $k \in \mathbb{Z}$.

$$\tilde{\tilde{f}}^k = v^k\, d_G^{k-1} + d_F^k\, v^{k+1}$$

In conclusion, $f^\bullet = \tilde{f}^\bullet = \tilde{\tilde{f}}^\bullet = 0$ in $\mathcal{K}(\mathrm{proj}\, A)$. □

The next lemma provides the construction of the functor $\mathcal{F} : \underline{\mathrm{mod}}\, A \to \mathcal{K}(\mathrm{proj}\, A)$. We follow the proof given in [19, Lemma 2.9] with some added details.

Lemma 2.4. ([18, Proposition 2.3 and 2.4])

Suppose given $X, Y \in \underline{\mathrm{mod}}\, A$ and a morphism $f \in \underline{\mathrm{Hom}}_A(X, Y)$.

(1) There exists a complex $F_X^\bullet \in \mathcal{L}_A$ such that

$$\mathrm{H}^0(\tau_{\leqslant 0}\, F_X^\bullet) \overset{\mathrm{st}}{\simeq} X.$$

Furthermore, $F_X^\bullet$ is uniquely determined in $\mathcal{K}(\mathrm{proj}\, A)$ by $X \in \underline{\mathrm{mod}}\, A$ up to isomorphism.

(2) There exists a morphism $f^\bullet \in \mathrm{Hom}_{\mathcal{K}(\mathrm{proj}\, A)}(F_X^\bullet,\, F_Y^\bullet)$ such that

$$\mathrm{H}^0(\tau_{\leqslant 0}\, f^\bullet) \overset{\mathrm{st}}{\simeq} f.$$

Furthermore, $f^\bullet$ is uniquely determined in $\mathcal{K}(\mathrm{proj}\, A)$ by $\underline{f}$ up to isomorphism.

Proof. Ad (1). Let $P^\bullet \in \mathcal{K}(\mathrm{proj}\, A)$ be a projective resolution of X.

$$P^\bullet : \qquad \cdots \to P^{-2} \xrightarrow{d_P^{-2}} P^{-1} \xrightarrow{d_P^{-1}} P^0$$

Let $Q^\bullet \in \mathcal{K}(A\text{-proj})$ be a projective resolution of $\mathrm{Tr}(X) = \mathrm{Cok}(d_{-1}^{P^*})$ such that we have the following exact sequence.

$$\cdots \to Q^{-2} \xrightarrow{d_Q^{-2}} Q^{-1} = P_0^* \xrightarrow{d_Q^{-1} = d_{-1}^{P^*}} Q^0 = P_{-1}^* \to \mathrm{Tr}(X) \to 0$$

Applying $(-)^*$, we obtain the following complex in $\mathcal{K}(\mathrm{proj}\, A)$.

$$Q_0^* \xrightarrow{d_{-1}^{Q^*}} Q_{-1}^* \xrightarrow{d_{-2}^{Q^*}} Q_{-2}^* \xrightarrow{d_{-3}^{Q^*}} Q_{-3}^* \to \cdots$$

We define a complex $F_X^\bullet \in \mathcal{K}(\mathrm{proj}\, A)$ via

$$F_X^k := \begin{cases} P^k\,, & k \leqslant -1 \\ Q_{-1-k}^*\,, & k \geqslant 0 \end{cases} \qquad\qquad d_F^k := \begin{cases} d_P^k\,, & k \leqslant -1 \\ d_{-2-k}^{Q^*}\,, & k \geqslant 0\,. \end{cases}$$

Since we have $d_P^{-1} = d_{-1}^{Q^*}$, this is in fact a complex. Note that $F_X^\bullet$ can be visualized as follows.

$$\begin{tikzcd} \cdots \arrow[r] & F_X^{-2} \arrow[r, "d_F^{-2}"] \arrow[d, equal] & F_X^{-1} \arrow[r, "d_F^{-1}"] \arrow[d, equal] & F_X^0 \arrow[r, "d_F^0"] \arrow[d, equal] & F_X^1 \arrow[r, "d_F^1"] \arrow[d, equal] & F_X^2 \arrow[r] \arrow[d, equal] & \cdots \\ \cdots \arrow[r] & P^{-2} \arrow[r, "d_P^{-2}"] & P^{-1} \arrow[r, "d_P^{-1}"] & P^0 = Q_{-1}^* \arrow[r, "d_{-2}^{Q^*}"] & Q_{-2}^* \arrow[r, "d_{-3}^{Q^*}"] & Q_{-3}^* \arrow[r] & \cdots \end{tikzcd}$$

By construction, we have $\tau_{\leqslant 0} F_X^\bullet = \tau_{\leqslant 0} P^\bullet$ so that $\mathrm{H}^0(\tau_{\leqslant 0} F_X^\bullet) \simeq X$ and $\mathrm{H}^k(F_X^\bullet) = 0$ for $k < 0$. Moreover, we have $\tau_{\geqslant -1} F_X^{\bullet,*} = \tau_{\leqslant 0} Q^\bullet$ so that $\mathrm{H}_k(F_X^{\bullet,*}) = \mathrm{H}^{-1-k}(Q^\bullet) = 0$ for $k \geqslant 0$.

It remains to show that $F_X^\bullet$ is uniquely determined in $\mathcal{K}(\mathrm{proj}\, A)$ by X up to isomorphism. However, we first show the existence of $f^\bullet$ in part (2).

Ad (2). Let f in $\mathrm{mod}\, A$ be a lift of $\underline{f}$. Let $F^\bullet := F_X^\bullet$ and $G^\bullet := F_Y^\bullet$ be elements of $\mathcal{L}$ as constructed above.

Since $\tau_{\leqslant 0} F^\bullet$ and $\tau_{\leqslant 0} G^\bullet$ are projective resolutions, we can lift f to a morphism of complexes $\varphi^\bullet$.

$$\begin{tikzcd}
\tau_{\leqslant 0} F^\bullet \arrow[d, "\varphi^\bullet"] & \cdots \arrow[r] & F^{-2} \arrow[r] \arrow[d, "\varphi^{-2}"] & F^{-1} \arrow[r] \arrow[d, "\varphi^{-1}"] & F^0 \arrow[r] \arrow[d, "\varphi^0"] & 0 \arrow[r] \arrow[d] & \cdots \\
\tau_{\leqslant 0} G^\bullet & \cdots \arrow[r] & G^{-2} \arrow[r] & G^{-1} \arrow[r] & G^0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Similarly, we can lift $\mathrm{Tr}(f) : \mathrm{Tr}(Y) \to \mathrm{Tr}(X)$ to a morphism of complexes $\tau_{\geqslant -1} G_\bullet^* \xrightarrow{\rho_\bullet^*} \tau_{\geqslant -1} F_\bullet^*$. Applying $(-)^*$ yields a morphism of complexes $\rho^\bullet : \tau_{\geqslant -1} F^\bullet \to \tau_{\geqslant -1} G^\bullet$.

$$\begin{tikzcd}
\tau_{\geqslant -1} G_\bullet^* \arrow[d, "\rho_\bullet^*"] & \cdots \arrow[r] & G_1^* \arrow[r] \arrow[d, "\rho_1^*"] & G_0^* \arrow[r] \arrow[d, "\rho_0^*"] & G_{-1}^* \arrow[r] \arrow[d, "\rho_{-1}^*"] & 0 \arrow[r] \arrow[d] & \cdots \\
\tau_{\geqslant -1} F_\bullet^* & \cdots \arrow[r] & F_1^* \arrow[r] & F_0^* \arrow[r] & F_{-1}^* \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

By construction of $\mathrm{Tr}(f)$, we may choose $\rho^{-1} = \varphi^{-1}$ and $\rho^0 = \varphi^0$. Combining both, we obtain a morphism of cochain complexes $f^\bullet : F^\bullet \to G^\bullet$ via $\tau_{\leqslant 0} f^\bullet := \tau_{\leqslant 0}\varphi^\bullet$ and $\tau_{\geqslant 1} f^\bullet := \tau_{\geqslant 1}\rho^\bullet$. We have $\mathrm{H}^0(\tau_{\leqslant 0} f^\bullet) = \mathrm{H}^0(\tau_{\leqslant 0}\varphi^\bullet) \simeq f$ by construction.

We conclude the proof by showing that $f^\bullet$ and $F_X^\bullet$ are unique up to isomorphism in $\mathcal{K}(\mathrm{proj}\, A)$.

Suppose given $F^\bullet$ and $G^\bullet$ in $\mathcal{L}_A$ with $\mathrm{H}^0(\tau_{\leqslant 0} F^\bullet) \overset{\mathrm{st}}{\simeq} X$ and $\mathrm{H}^0(\tau_{\leqslant 0} G^\bullet) \overset{\mathrm{st}}{\simeq} X$ respectively. By adding a trivial complex $\cdots \to 0 \to P^0 \xrightarrow{1} P^1 \to 0 \to \cdots$ as a direct summand in $\mathcal{K}(\mathrm{proj}\, A)$ if necessary, we may assume that $\mathrm{H}^0(\tau_{\leqslant 0} F^\bullet) \simeq \mathrm{H}^0(\tau_{\leqslant 0} G^\bullet)$ in $\mathrm{mod}\, A$.

Let $\varphi : \mathrm{H}^0(\tau_{\leqslant 0} F^\bullet) \to \mathrm{H}^0(\tau_{\leqslant 0} G^\bullet)$ be an isomorphism with inverse ρ. Since projective resolutions are unique up to isomorphism in $\mathcal{K}(\mathrm{proj}\, A)$, the construction of part (2) yields morphisms $\varphi^\bullet$ and $\rho^\bullet$ in $\mathcal{K}(\mathrm{proj}\, A)$ such that $\underline{\mathrm{H}^0(\tau_{\leqslant 0}(\varphi^\bullet\, \rho^\bullet))} = \underline{\mathrm{id}_X}$ and $\underline{\mathrm{H}^0(\tau_{\leqslant 0}(\rho^\bullet\, \varphi^\bullet))} = \underline{\mathrm{id}_X}$. By Lemma 2.3, we can conclude that $\varphi^\bullet\, \rho^\bullet = \mathrm{id}_{F^\bullet}$ and $\rho^\bullet\, \varphi^\bullet = \mathrm{id}_{G^\bullet}$ in $\mathcal{K}(\mathrm{proj}\, A)$. Hence, $F^\bullet \simeq G^\bullet$ in $\mathcal{K}(\mathrm{proj}\, A)$.

Now, suppose given $f_1^\bullet$ and $f_2^\bullet$ in $\mathcal{L}_A$ with $\underline{\mathrm{H}^0(\tau_{\leqslant 0}\, f_1^\bullet)} \simeq \underline{f}$ and $\underline{\mathrm{H}^0(\tau_{\leqslant 0}\, f_2^\bullet)} \simeq \underline{f}$ in $\underline{\mathrm{mod}}\, A$ respectively. We have isomorphisms $\underline{\varphi}$ and $\underline{\psi}$ in $\underline{\mathrm{mod}}\, A$ such that $\underline{\varphi \mathrm{H}^0(\tau_{\leqslant 0}\, f_1^\bullet)\psi} = \underline{\mathrm{H}^0(\tau_{\leqslant 0}\, f_2^\bullet)}$. As above, φ and ψ lift to isomorphisms $\varphi^\bullet$ and $\psi^\bullet$ in $\mathcal{K}(\mathrm{proj}\, A)$. We obtain

$$\mathrm{H}^0(\tau_{\leqslant 0}\, f_2^\bullet) = \mathrm{H}^0(\tau_{\leqslant 0}\, \varphi^\bullet)\mathrm{H}^0(\tau_{\leqslant 0}\, f_1^\bullet)\mathrm{H}^0(\tau_{\leqslant 0}\, \psi^\bullet) = \mathrm{H}^0(\tau_{\leqslant 0}\, (\varphi^\bullet f_1^\bullet \psi^\bullet))$$

in $\underline{\mathrm{mod}}\, A$ so that $\varphi^\bullet f_1^\bullet \psi^\bullet = f_2^\bullet$ by Lemma 2.3. Thus, $f_1^\bullet \simeq f_2^\bullet$ in $\mathcal{K}(\mathrm{proj}\, A)$. □

Remark 2.5. For $X \in \operatorname{mod} A$ we fix $F_X^\bullet \in \mathcal{L}_A$ such that $\tau_{\leqslant 0} F_X^\bullet$ is the minimal projective resolution of X and $\tau_{\geqslant -1} F_X^{\bullet,*}$ is the minimal projective resolution of $\operatorname{Tr}(X)$. In this case, $\tau_{\geqslant 1} F_X^{\bullet,*}$ is the minimal projective resolution of X^*. Note that the minimal projective resolution of $P \in \operatorname{proj} A$ is the complex with P concentrated in degree 0. Thus, $F_P^k = 0$ for $k \neq 0, 1$ and $F_P^\bullet = (\cdots \to 0 \to P \to P \to 0 \to \cdots)$.

For $X \xrightarrow{f} Y$ in $\operatorname{mod} A$ we fix $F_X^\bullet \xrightarrow{f^\bullet} F_Y^\bullet$ as a lift of f in $\mathcal{K}(\operatorname{proj} A)$ with $F_X^\bullet$ and $F_Y^\bullet$ as above. In particular,

$$\begin{aligned} \mathrm{H}^0(\tau_{\leqslant 0} F_X^\bullet) &\simeq X \\ \mathrm{H}^0(\tau_{\leqslant 0} F_Y^\bullet) &\simeq Y \\ \mathrm{H}^0(\tau_{\leqslant 0} f^\bullet) &\simeq f \end{aligned}$$

even if X or Y have projective direct summands. Moreover, F_X^0 is the projective cover of X and $F_X^{1,*}$ the projective cover of X^*. If X is simple, $\nu(F_X^0)$ is the injective hull of X.

The results so far are summarized in the following theorem given in [18, Theorem 2.6]. In the future, we will often use this equivalence without further comment.

Theorem 2.6 (Kato)**.** *The mapping $X \mapsto F_X^\bullet$ defines a functor $\mathcal{F} : \underline{\operatorname{mod}}\, A \to \mathcal{K}(\operatorname{proj} A)$. The functor $\mathcal{F}$ restricts to an equivalence*

$$\mathcal{F} : \underline{\operatorname{mod}}\, A \xrightarrow{\sim} \mathcal{L}_A$$

with quasi-inverse $\mathrm{H}^0(\tau_{\leqslant 0}(-)) : \mathcal{L}_A \to \underline{\operatorname{mod}}\, A$*.*

Proof. Note that $\mathcal{F}$ and $\mathrm{H}^0(\tau_{\leqslant 0}(-))$ are well-defined by Lemma 2.3. Let $\underline{f}$ and $\underline{g}$ in $\underline{\operatorname{mod}}\, A$. We verify that $\mathcal{F}(\underline{f})\mathcal{F}(\underline{g}) = \mathcal{F}(\underline{fg})$. We have

$$\mathrm{H}^0\big(\tau_{\leqslant 0}\, \mathcal{F}(\underline{fg})\big) = \underline{fg} = \underline{f}\,\underline{g} = \mathrm{H}^0\big(\tau_{\leqslant 0}\, \mathcal{F}(\underline{f})\big)\, \mathrm{H}^0\big(\tau_{\leqslant 0}\, \mathcal{F}(\underline{g})\big) = \mathrm{H}^0\big(\tau_{\leqslant 0}\, (\mathcal{F}(\underline{f})\mathcal{F}(\underline{g}))\big).$$

By Lemma 2.3, we obtain $\mathcal{F}(\underline{f})\mathcal{F}(\underline{g}) = \mathcal{F}(\underline{fg})$. In conclusion, this shows that $\mathcal{F}$ defines a functor $\mathcal{F} : \underline{\operatorname{mod}}\, A \to \mathcal{K}(\operatorname{proj} A)$ via the construction of Lemma 2.4.

As chosen in Remark 2.5, we have a natural transformation $\mathrm{H}^0(\tau_{\leqslant 0}(-)) \circ \mathcal{F} \simeq \mathrm{id}_{\underline{\operatorname{mod}}\, A}$. On the other hand, we also have a transformation $\eta : \mathcal{F} \circ \mathrm{H}^0(\tau_{\leqslant 0}(-)) \simeq \mathrm{id}_{\mathcal{L}_A}$ by Lemma 2.4.(1). It remains to show that η is natural. Let $f^\bullet : F^\bullet \to G^\bullet$ in $\mathcal{L}_A$.

$$\begin{array}{ccc} \mathcal{F}(\mathrm{H}^0(\tau_{\leqslant 0} F^\bullet)) & \xrightarrow{\eta_{F^\bullet}} & F^\bullet \\ {\scriptstyle \mathcal{F}(\mathrm{H}^0(\tau_{\leqslant 0} f^\bullet))}\big\downarrow & & \big\downarrow{\scriptstyle f^\bullet} \\ \mathcal{F}(\mathrm{H}^0(\tau_{\leqslant 0} G^\bullet)) & \xrightarrow{\eta_{G^\bullet}} & G^\bullet \end{array}$$

By Remark 2.5, $\mathcal{F}(\mathrm{H}^0(\tau_{\leqslant 0} F^\bullet))$ is a complex which is taken to $\mathrm{H}^0(\tau_{\leqslant 0} F^\bullet)$ under the functor $\mathrm{H}^0(\tau_{\leqslant 0}(-))$. Similarly for $G^\bullet$. Thus, applying $\mathrm{H}^0(\tau_{\leqslant 0}(-))$ to the diagram yields

$$\begin{aligned}
&\mathrm{H}^0\big(\tau_{\leqslant 0}(\eta_{F^\bullet} f^\bullet)\big) = \mathrm{H}^0(\tau_{\leqslant 0}\,\eta_{F^\bullet})\,\mathrm{H}^0(\tau_{\leqslant 0} f^\bullet) = \mathrm{H}^0(\tau_{\leqslant 0} f^\bullet)\\
&\mathrm{H}^0\big(\tau_{\leqslant 0}(\mathcal{F}(\mathrm{H}^0(\tau_{\leqslant 0} f^\bullet))\eta_{G^\bullet})\big) = \mathrm{H}^0\big(\tau_{\leqslant 0}(\mathcal{F}(\mathrm{H}^0(\tau_{\leqslant 0} f^\bullet)))\big)\,\mathrm{H}^0(\tau_{\leqslant 0}\,\eta_{G^\bullet}) = \mathrm{H}^0(\tau_{\leqslant 0} f^\bullet).
\end{aligned}$$

Now, we obtain $\eta_{F^\bullet} f^\bullet = \mathcal{F}(\mathrm{H}^0(\tau_{\leqslant 0} f^\bullet))\eta_{G^\bullet}$ by Lemma 2.4.(2). Note that the identity on a module X is lifted to the identity on $F_X^\bullet$. □

Example in Chapter 7. A calculation of the functor $\mathcal{F}$ and the category $\mathcal{L}_B$ can be found in Example 7.1 for the algebra B of Section 7.1.

By Lemma 1.9 we have the following properties of $C(f)^\bullet$ for a morphism $f^\bullet$ in $\mathcal{L}$.

Remark 2.7. Suppose given $F^\bullet, G^\bullet \in \mathcal{L}_A$.

(1) Let $f^\bullet : F^\bullet \to G^\bullet$ in $\mathcal{L} \subseteq \mathcal{K}(\mathrm{proj}\,A)$. Then $\mathrm{H}^k(C(f)^\bullet) = 0$ for $k < -1$ and $\mathrm{H}_k(C(f)^*_\bullet) = 0$ for $k \geqslant 0$. In particular, $C(f)^\bullet \in \mathcal{L}$ if and only if $\mathrm{H}^{-1}(C(f)^\bullet) = 0$.

(2) Let $f^\bullet : G^\bullet \to F^\bullet[1]$ in $\mathcal{K}(\mathrm{proj}\,A)$. Then $C(f)^\bullet[-1] \in \mathcal{L}$.

The mapping cone $C(f)^\bullet$ can be used to characterize properties of a morphism f in $\underline{\mathrm{mod}}\,A$. In [19, Theorem 3.9], Kato shows that a morphism is stably equivalent to a monomorphism if and only if $\mathrm{H}^{-1}(C(f)^\bullet) = 0$. In [18, Definition and Lemma 3.1], the mapping cone is used to define a weak kernel and a weak cokernel in $\underline{\mathrm{mod}}\,A$.

Definition 2.8 (Kato)**.** Suppose given a morphisms $f \in \mathrm{mod}\,A$.

We define the *pseudo-kernel* $\underline{\mathrm{Ker}}(f)$ and the *pseudo-cokernel* $\underline{\mathrm{Cok}}(f)$ of f as

$$\begin{aligned}
\underline{\mathrm{Ker}}(f) &:= \mathrm{H}^0\big(\tau_{\leqslant 0}(C(f)^\bullet[-1])\big)\\
\underline{\mathrm{Cok}}(f) &:= \mathrm{H}^0\big(\tau_{\leqslant 0} C(f)^\bullet\big)
\end{aligned}$$

respectively. Both are uniquely determined in $\underline{\mathrm{mod}}\,A$ by $\underline{f}$ up to isomorphism.

For a morphism f in $\underline{\mathrm{mod}}\,A$, we have the following distinguished triangle in $\mathcal{K}(\mathrm{proj}\,A)$.

$$C(f)^\bullet[-1] \xrightarrow{u^\bullet} F_X^\bullet \xrightarrow{f^\bullet} F_Y^\bullet \xrightarrow{v^\bullet} C(f)^\bullet$$

This induces morphisms $u := \mathrm{H}^0(\tau_{\leqslant 0} u^\bullet) : \underline{\mathrm{Ker}}(f) \to X$ and $v := \mathrm{H}^0(\tau_{\leqslant 0} v^\bullet) : Y \to \underline{\mathrm{Cok}}(f)$. The following lemma shows that $\underline{\mathrm{Ker}}(f)$ and $\underline{\mathrm{Cok}}(f)$ have the properties of a weak kernel and weak cokernel respectively.

Lemma 2.9. ([18, Lemma 3.3 and 3.5])

Let $f : X \to Y$ be a morphism in $\underline{\mathrm{mod}}\, A$.

(1) We have $u\, f = 0$ and $f\, v = 0$ in $\underline{\mathrm{mod}}\, A$.

(2) Suppose given $t : T \to X$ with $t\, f = 0$ in $\underline{\mathrm{mod}}\, A$. There exists a morphism $h \in \underline{\mathrm{Hom}}_A(T, \underline{\mathrm{Ker}}(f))$ such that $h\, u = t$ in $\underline{\mathrm{mod}}\, A$.

(3) Suppose given $t : Y \to T$ with $f\, t = 0$ in $\underline{\mathrm{mod}}\, A$. There exists a morphism $h \in \underline{\mathrm{Hom}}_A(\underline{\mathrm{Cok}}(f), T)$ such that $v\, h = t$ in $\underline{\mathrm{mod}}\, A$.

Proof. Ad (1). We have $u^\bullet f^\bullet = 0$ and $f^\bullet v^\bullet = 0$. By Lemma 2.3, we obtain that $u\, f = 0$ and $f\, v = 0$ in $\underline{\mathrm{mod}}\, A$.

Ad (2). We have a morphism $t^\bullet : F_T^\bullet \to F_X^\bullet$ with $t^\bullet f^\bullet = 0$ by Theorem 2.6. This induces a morphism of distinguished triangles.

$$\begin{tikzcd}
C(f)[-1] \arrow[r, "u^\bullet"] & F_X^\bullet \arrow[r, "f^\bullet"] & F_Y^\bullet \arrow[r] & {} \\
F_T^\bullet \arrow[u, dashed, "h^\bullet"] \arrow[r, "1"] & F_T^\bullet \arrow[u, "t^\bullet"] \arrow[r] & 0 \arrow[u] \arrow[r] & {}
\end{tikzcd}$$

Let $h := \mathrm{H}^0(\tau_{\leqslant 0}\, h^\bullet)$. By Lemma 2.3 we obtain $h\, u = t$ in $\underline{\mathrm{mod}}\, A$. Part (3) is shown similarly. □

We will return to the pseudo-kernel and pseudo-cokernel at the end of the next section.

2.2 Perfect exact sequences

In this section, we introduce a special class of short exact sequences. We will see that a perfect exact sequence corresponds to a distinguished triangle in $\mathcal{K}(\mathrm{proj}\, A)$ via the equivalence $\mathcal{F}$. In later chapters, we will use perfect exact sequences mainly in the context of stable equivalences.

Definition 2.10. A short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathrm{mod}\, A$ is called *perfect exact* if the induced sequence $0 \to X^* \to Y^* \to Z^* \to 0$ is exact in A-mod.

Example 2.11. (1) Let A be self-injective. Since $\mathrm{Hom}_A(-, A)$ is an exact functor, every short exact sequence is perfect exact.

(2) A short exact sequence $0 \to X \to Y \to Z \to 0$ is perfect exact if the induced morphism $\mathrm{Hom}_A(Y, A) \to \mathrm{Hom}(X, A)$ is surjective. In particular, this holds if $X^* = 0$.

(3) Let A be hereditary. Then $\mathrm{Hom}_A(X,A) = 0$ for all $X \in \mathrm{mod}\, A$ not projective. Thus, every short exact sequence starting in a non-projective module is perfect exact by (2).

(4) A short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is perfect exact, if $\underline{f} \neq 0$ and g is irreducible.

In fact, using that g is irreducible, a morphism $p : X \to A$ induces either a morphism $\alpha : A \to Y$ with $p\,\alpha = f$ or a morphism $\beta : Y \to A$ with $f\,\beta = p$. However, in the first case we obtain $\underline{f} = 0$.

(5) An almost split sequence $0 \to X \xrightarrow{f} E_X \xrightarrow{g} \tau^{-1}(X) \to 0$ is perfect exact if and only if X is not projective.

In fact, a morphism $p : X \to A$ induces a morphism $\beta : E_X \to A$ with $f\,\beta = p$ if and only if p is not split. However, this holds if and only if X is not projective, since the starting term of an almost split sequence is indecomposable.

Recall that $\mathcal{P}_A$ denotes the category of projective-injective A-modules. In case that the dominant dimension of A is at least 1, a module $X \in {}^{\perp}\mathcal{P}_A$ satisfies $X^* = 0$. In particular, short exact sequences with middle term in ${}^{\perp}\mathcal{P}_A$ are perfect exact.

Lemma 2.12. *Suppose that* $\mathrm{domdim}\, A \geqslant 1$. *Let* $Y \in {}^{\perp}\mathcal{P}_A$. *Let* Y' *be a submodule of* Y.

We have $(Y')^* = 0$ *and every short exact sequence* $0 \to X \to Y' \to Z \to 0$ *in* $\mathrm{mod}\, A$ *is a perfect exact sequence.*

Proof. Let Y' be a submodule of Y. Since the embedding $Y' \hookrightarrow Y$ is injective, the condition $Y \in {}^{\perp}\mathcal{P}$ implies that Y' is contained in ${}^{\perp}\mathcal{P}$ as well. Using that $\mathrm{domdim}\, A \geqslant 1$, we obtain that $\mathrm{Hom}(Y', A) = 0$.

Suppose given a short exact sequence $0 \to X \to Y' \to Z \to 0$ in $\mathrm{mod}\, A$. Since X is isomorphic to a submodule of Y, we have seen above that $X^* = 0$. Thus, the result follows from Example 2.11.(2). $\square$

The following lemma can be useful to check if a short exact sequence can be perfect exact. If the starting morphism of a perfect exact sequence factors through an indecomposable projective module P, then P must be a direct summand of the middle term.

Lemma 2.13. *Suppose given a perfect exact sequence* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$. *Let* $P \in \mathrm{proj}\, A$ *be indecomposable. Suppose that* $\iota : X' \hookrightarrow X$ *is the embedding of a direct summand of* X.

If $\iota\, f = u\, v$ *with* $u : X' \to P$ *and* $v : P \to Y$ *then* v *is a split monomorphism.*

Proof. Let $\pi : X \to X'$ with $\iota\pi = \mathrm{id}_X$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, shift right, "\pi"'] & Y \arrow[r, "g"] & Z \\
X' \arrow[u, shift right, "\iota"'] \arrow[d, "u"'] & & \\
P \arrow[uur, "v"] & &
\end{tikzcd}$$

Since $Y^* \to X^*$ is surjective, we obtain a morphism $\sigma : Y \to P$ with $f\,\sigma = \pi\, u$. Thus, we have $u = \iota\,\pi\, u = \iota\, f\,\sigma = u\, v\,\sigma$. Inductively, we obtain $u = u(v\,\sigma)^n$ for all $n \in \mathbb{Z}_{\geqslant 0}$. Since P is indecomposable, $v\,\sigma$ is either an automorphism or nilpotent. However, if $u = u(v\,\sigma)^n = 0$ for some n, we obtain $f = \pi\, u\, v = 0$. A contradiction. Thus, $v\,\sigma$ is an automorphism and v is a split monomorphism. □

Example in Chapter 7. For the algebra A in Section 7.4 we discuss two short exact sequences that are not perfect exact in Example 7.13. The first does not satisfy the condition of the previous lemma. In the second perfect exact sequence, the starting morphism does not factor through a projective module. In particular, we see that not every short exact sequence satisfying the conditions of the previous lemma is perfect exact.

The situation is better for Nakayama algebras.

Lemma 2.14. *Let A be a Nakayama algebra. Suppose given a short exact sequence $\eta : 0 \to X \overset{f}{\to} Y \overset{g}{\to} Z \to 0$ without split summands.*

The sequence η is perfect exact if the following holds for all embeddings $\iota : X' \to X$ of indecomposable direct summands of X and for all projections $\pi : Y \to Y'$ onto indecomposable direct summands of Y. If $\iota\, f\,\pi$ factors through an indecomposable projective A-module P via $i : P \to Y'$, then i is a split monomorphism.

Proof. Suppose given a non-zero morphism $u : X \to P$ with $P \in \operatorname{proj} A$. Since A is a Nakayama algebra, there exists a projective-injective module Q with an embedding $i : P \hookrightarrow Q$.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[ddl, dashed, "v"] & Z \arrow[r] & 0 \\
& P \arrow[d, "i"'] & & & \\
& Q & & &
\end{tikzcd}$$

Since Q is injective, we obtain a morphism $v : Y \to Q$ such that $f\,v = u\,i$. It remains to show that there exists a morphisms $w : Y \to P$ with $w\,i = v$. If i is a split monomorphism, P is injective and we are done. It suffices to consider indecomposable projective modules P and Q with a non-split embedding $i : P \hookrightarrow Q$.

Claim. Suppose P and Q are indecomposable projective modules with $P = \operatorname{rad}(Q)$. Then there exists a morphism $w : Y \to P$ with $w\,i = v$.

$$\begin{tikzcd}
 & & Y \arrow[dl, dashed, "w"'] \arrow[d, "v"] & & \\
0 \arrow[r] & \operatorname{rad}(Q) \arrow[r, hook, "i"] & Q \arrow[r, "p"] & Q/\operatorname{rad}(Q) \arrow[r] & 0
\end{tikzcd}$$

Note that $Q/\operatorname{rad}(Q)$ is simple. If v is not surjective, $v\,p$ must be zero. Hence, there exists a $w : Y \to P$ as claimed. Suppose that v is surjective and thus Q a direct summand of Y. Then there exists an embedding $\iota : X' \to X$ of an indecomposable direct summand X' of X such that $\iota\, f\, v$ factors through $\operatorname{rad}(Q)$. Since $\operatorname{rad}(Q)$ is indecomposable projective, the assumption implies that $\operatorname{rad}(Q)$ is a direct summand of Y. A contradiction. Thus, v can not be surjective and the claim holds.

Now, assume that $P = \operatorname{rad}^n(Q) \in \operatorname{proj} A$ with Q indecomposable projective. We finish the proof by induction on n. The case $n = 1$ holds by the claim above. Since A is a Nakayama algebra, $\operatorname{rad}^k(Q)$ must be projective for all $1 \leqslant k \leqslant n$. By induction hypothesis, $v : Y \to Q$ factors through $\operatorname{rad}^{n-1}(Q)$. Using the claim for $\operatorname{rad}^{n-1}(Q)$ instead of Q, we obtain that this morphism factors through $\operatorname{rad}(\operatorname{rad}^{n-1}(Q)) = \operatorname{rad}^n(Q) = P$ and we are done. □

We note that the projective summand of the middle term in a perfect exact sequence is uniquely determined by the induced sequence in $\underline{\operatorname{mod}}\, A$.

Lemma 2.15. *Suppose given two perfect exact sequences* $0 \to X \xrightarrow{f_1} Y_1 \xrightarrow{g_1} Z \to 0$ *and* $0 \to X \xrightarrow{f_2} Y_2 \xrightarrow{g_2} Z \to 0$ *in* $\operatorname{mod} A$.

If there exists a stable isomorphism $\beta : Y_1 \to Y_2$ *such that* $\underline{\beta}\, \underline{g_2} = \underline{g_1}$ *in* $\underline{\operatorname{mod}}\, A$*, then the two sequences are isomorphic.*

Proof. Using that both sequences are perfect exact, we may assume that X and Z have no projective direct summand. Otherwise, both sequences have an isomorphic split sequence as a direct summand.

Let $Y_i = M_i \oplus P_i$ with $P_i \in \operatorname{proj} A$ and M_i without projective direct summands for $i = 1, 2$. The stable isomorphism β induces an isomorphism $\beta' : M_1 \xrightarrow{\sim} M_2$ such that $\beta'\, g_2$ is equal to the restriction of g_1 to M_1. Since P_1 is projective, there exists a morphism $\left(\begin{smallmatrix} a & b \end{smallmatrix}\right) : P_1 \to M_2 \oplus P_2$ such that the following diagram commutes with α induced by the universal property of the kernel.

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f_1"] \arrow[d, "\alpha"] & M_1 \oplus P_1 \arrow[r, "g_1"] \arrow[d, "\begin{pmatrix} \beta' & 0 \\ a & b \end{pmatrix}"] & Z \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X \arrow[r, "f_2"] & M_2 \oplus P_2 \arrow[r, "g_2"] & Z \arrow[r] & 0 \end{tikzcd}$$

By Lemma 1.5 we obtain that these two sequences are isomorphic. □

Remark 2.16. Let $0 \to X \xrightarrow{(f\ u)} Y \oplus P \xrightarrow{\begin{pmatrix} g \\ v \end{pmatrix}} Z \to 0$ be a perfect exact sequence with $P \in \operatorname{proj} A$ and Y without projective direct summand. Then the previous lemma shows that P is uniquely determined up to isomorphism by the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$.

We also need the following observation in the future. Let $0 \to X \xrightarrow{(f\ u)} Y \oplus M \xrightarrow{\begin{pmatrix} v \\ w \end{pmatrix}} Z \to 0$ be a short exact sequence in $\operatorname{mod} A$. If $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is another short exact sequence in $\operatorname{mod} A$, the two sequences are isomorphic since A is finite dimensional.

For our purposes, the next two results provide the key property of a perfect exact sequence. Under the equivalence $\mathcal{F} : \underline{\operatorname{mod}} A \to \mathcal{L}_A$, perfect exact sequences in $\operatorname{mod} A$ correspond to distinguished triangles in $\mathcal{K}(\operatorname{proj} A)$.

Lemma 2.17. *The following are equivalent for a sequence of complexes $P^\bullet \xrightarrow{f^\bullet} Q^\bullet \xrightarrow{g^\bullet} R^\bullet$ in $\mathcal{K}(\operatorname{proj} A)$.*

(1) There exists a short exact sequence $0 \to P^\bullet \xrightarrow{s^\bullet} Q_1^\bullet \xrightarrow{t^\bullet} R^\bullet \to 0$ in $\mathcal{C}(\operatorname{proj} A)$ and an isomorphism $\varphi^\bullet : Q_1^\bullet \to Q^\bullet$ in $\mathcal{K}(\operatorname{proj} A)$ such that $s^\bullet \varphi^\bullet = f^\bullet$ and $\varphi^\bullet g^\bullet = t^\bullet$ in $\mathcal{K}(\operatorname{proj} A)$.

(2) The sequence $P^\bullet \xrightarrow{f^\bullet} Q^\bullet \xrightarrow{g^\bullet} R^\bullet \to$ is a distinguished triangle in $\mathcal{K}(\operatorname{proj} A)$.

Proof. Suppose that $P^\bullet \xrightarrow{f^\bullet} Q^\bullet \xrightarrow{g^\bullet} R^\bullet \xrightarrow{h^\bullet} P^{\bullet+1}$ is a distinguished triangle in $\mathcal{K}(\operatorname{proj} A)$.

Write $C^\bullet := C(h^\bullet)[-1] \in \mathcal{C}(\operatorname{proj} A)$. By assumption, we have the following exact sequence.

$$0 \to P^\bullet \xrightarrow{(0\ 1)^\bullet} C^\bullet \xrightarrow{\begin{pmatrix} -1 \\ 0 \end{pmatrix}^\bullet} R^\bullet \to 0$$

By construction of $C^\bullet$ as the shifted mapping cone of $h^\bullet$, there is an isomorphism $C^\bullet \xrightarrow{\varphi^\bullet} Q^\bullet$ such that $s^\bullet \varphi^\bullet = f^\bullet$ and $\varphi^\bullet g^\bullet = t^\bullet$ in $\mathcal{K}(\operatorname{proj} A)$.

Conversely, suppose that there exists such an exact sequence $0 \to P^\bullet \xrightarrow{f^\bullet} Q_1^\bullet \xrightarrow{g^\bullet} R^\bullet \to 0$ in $\mathcal{C}(\operatorname{proj} A)$. Since R^k is projective for all $k \in \mathbb{Z}$ the sequence splits in every degree via a morphism $\sigma^k : R^k \to Q_1^k$, that is $\sigma^k g^k = \mathrm{id}_R^k$. Furthermore, there exists a morphism $\chi^k : R^k \to P^{k+1}$ such that the following diagram commutes.

$$\begin{tikzcd} & R^k \arrow[dl, "\chi^k"'] \arrow[d, "d_R^k\,\sigma^{k+1} - \sigma^k\, d_{Q_1}^k"] \\ P^{k+1} \arrow[r, "f^{k+1}"'] & \mathrm{Im}(f^{k+1}) \end{tikzcd}$$

Note that $\mathrm{Im}(d_R^k\,\sigma^{k+1} - \sigma^k\, d_{Q_1}^k) \subseteq \mathrm{Im}(f^{k+1})$ since $(d_R^k\,\sigma^{k+1} - \sigma^k\, d_{Q_1}^k)g^{k+1} = d_R^k - \sigma^k\, g^k\, d_R^k = 0$. For $k \in \mathbb{Z}$ we have

$$\begin{aligned} \left(\chi^k\, d_P^{k+1} + d_R^k\,\chi^{k+1}\right) f^{k+2} &= \chi^k\, f^{k+1}\, d_{Q_1}^{k+1} + d_R^k\left(d_R^{k+1}\,\sigma^{k+2} - \sigma^{k+1}\, d_{Q_1}^{k+1}\right) \\ &= d_R^k\,\sigma^{k+1}\, d_{Q_1}^{k+1} - \sigma^k\, d_{Q_1}^k\, d_{Q_1}^{k+1} - d_R^k\,\sigma^{k+1}\, d_{Q_1}^{k+1} = 0 \end{aligned}$$

so that $\chi^\bullet\, d_{P[1]}^\bullet = -\chi^\bullet\, d_P^\bullet[1] = d_R^\bullet\,\chi^\bullet[1]$ since $f^\bullet$ is injective. Thus, $\chi^\bullet : R^\bullet \to P^\bullet[1]$ is a morphism of complexes.

The differential of $C^\bullet := C(\chi^\bullet)[-1] = R^\bullet \oplus P^\bullet$ is then given by

$$d_C^\bullet := \begin{pmatrix} d_R^\bullet & -\chi^\bullet \\ 0 & d_P^\bullet \end{pmatrix}.$$

We show that the following is an isomorphism of short exact sequences with $\sigma^\bullet = (\sigma^k)_{k\in\mathbb{Z}}$.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & P^\bullet \arrow[r, "\begin{pmatrix} 0 & 1 \end{pmatrix}^\bullet"] \arrow[d, equal] & C^\bullet \arrow[r, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}^\bullet"] \arrow[d, "\begin{pmatrix} \sigma^\bullet \\ f^\bullet \end{pmatrix}"] & R^\bullet \arrow[r] \arrow[d, equal] & 0 \arrow[d] \\ 0 \arrow[r] & P^\bullet \arrow[r, "f^\bullet"] & Q_1^\bullet \arrow[r, "g^\bullet"] & R^\bullet \arrow[r] & 0 \end{tikzcd}$$

By construction of $\sigma^\bullet$, this is a commutative diagram. It remains to show that $\begin{pmatrix} \sigma^\bullet \\ f^\bullet \end{pmatrix}$ is a morphism of complexes. We calculate as follows for $k \in \mathbb{Z}$.

$$\begin{pmatrix} d_R^k & -\chi^k \\ 0 & d_P^k \end{pmatrix}\begin{pmatrix} \sigma^{k+1} \\ f^{k+1} \end{pmatrix} = \begin{pmatrix} d_R^k\,\sigma^{k+1} - \chi^k\, f^{k+1} \\ d_P^k\, f^{k+1} \end{pmatrix} = \begin{pmatrix} \sigma^k \\ f^k \end{pmatrix} d_{Q_1}^k$$

In conclusion, $R^\bullet \xrightarrow{\chi^\bullet} P^\bullet[1] \to C^\bullet[1] \to R^\bullet[1]$ is a distinguished triangle which induces the following distinguished triangle.

$$P\bullet \to Q_1^\bullet \to R^\bullet \xrightarrow{\chi^\bullet} P^\bullet[1]$$

By assumption, this triangle is isomorphic to the sequence $P^\bullet \xrightarrow{f^\bullet} Q^\bullet \xrightarrow{g^\bullet} R^\bullet \xrightarrow{\chi^\bullet} P^\bullet[1]$ in $\mathcal{K}(\mathrm{proj}\, A)$. □

The following proposition is based on [19, Proposition 3.6]. We will give a modified and expanded version of the proof found in [18, Lemma 2.7].

Proposition 2.18 (Kato)**.** *Suppose given a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\operatorname{mod} A$ such that Y has no projective direct summand. The following are equivalent.*

(1) There exists a projective module P and morphisms p and q such that

$$0 \to X \xrightarrow{\left(\begin{smallmatrix} f & p \end{smallmatrix}\right)} Y \oplus P \xrightarrow{\left(\begin{smallmatrix} g \\ q \end{smallmatrix}\right)} Z \to 0$$

is a perfect exact sequence in $\operatorname{mod} A$.

(2) There exists a short exact sequence $0 \to F_X^\bullet \xrightarrow{s^\bullet} G^\bullet \xrightarrow{t^\bullet} F_Z^\bullet \to 0$ in $\mathcal{C}(\operatorname{proj} A)$ and an isomorphism $\varphi^\bullet : G^\bullet \to F_Y^\bullet$ in $\mathcal{K}(\operatorname{proj} A)$ such that $s^\bullet \varphi^\bullet = f^\bullet$ and $\varphi^\bullet g^\bullet = t^\bullet$ in $\mathcal{K}(\operatorname{proj} A)$.

(3) The sequence $F_X^\bullet \xrightarrow{f^\bullet} F_Y^\bullet \xrightarrow{g^\bullet} F_Z^\bullet \to$ is a distinguished triangle in $\mathcal{K}(\operatorname{proj} A)$.

Remark 2.19. Suppose that $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is a perfect exact sequence in $\operatorname{mod} A$. Let $F_X^\bullet \xrightarrow{f^\bullet} F_Y^\bullet \xrightarrow{g^\bullet} F_Z^\bullet \to$ be the induced sequence in $\mathcal{L}_A$ obtained by applying the functor $\mathcal{F}$ of Theorem 2.6. The proposition above now states that this is a distinguished triangle in $\mathcal{K}(\operatorname{proj} A)$.

On the other hand, suppose that $X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \xrightarrow{h^\bullet} X^\bullet[1]$ is a distinguished triangle in $\mathcal{K}(\operatorname{proj} A)$ such that $X^\bullet$, $Y^\bullet$ and $Z^\bullet$ are elements of $\mathcal{L}_A$. We have seen that this induces a short exact sequence $0 \to X^\bullet \xrightarrow{s^\bullet} C(h)^\bullet[-1] \xrightarrow{t^\bullet} Z^\bullet \to 0$ in $\mathcal{C}(\operatorname{proj} A)$. During the proof of the proposition we will show the following. Applying $\mathrm{H}^0(\tau_{\leqslant 0}(-))$, the quasi-inverse of $\mathcal{F}$, induces a perfect exact sequence $0 \to \mathrm{H}^0(\tau_{\leqslant 0} X^\bullet) \to \mathrm{H}^0\big(\tau_{\leqslant 0}(C(h)^\bullet)\big) \to \mathrm{H}^0(\tau_{\leqslant 0} Z^\bullet) \to 0$ in $\operatorname{mod} A$. Furthermore, we have $\mathrm{H}^0\big(\tau_{\leqslant 0}(C(h)^\bullet[-1])\big) \simeq \mathrm{H}^0(\tau_{\leqslant 0} Y^\bullet)$ in $\underline{\operatorname{mod}} A$.

By Remark 2.5 and Remark 2.16, these constructions are mutually inverse up to isomorphism in $\operatorname{mod} A$ and $\mathcal{K}(\operatorname{proj} A)$ respectively. If the conditions in the proposition hold, the projective module P in (1) is uniquely determined up to isomorphism. Furthermore, we have that $\mathrm{H}^0(\tau_{\leqslant 0} G^\bullet) \simeq Y \oplus P$ and $G^\bullet = C(h)^\bullet[-1]$ where $h^\bullet : F_Z^\bullet \to F_X^\bullet[1]$.

Proof of Proposition 2.18. The equivalence of (2) and (3) is shown in Lemma 2.17.

Ad (1) ⇒ (2). We repeat the construction of Lemma 2.4 with a specific projective resolution of $Y' := Y \oplus P$. Suppose given minimal projective resolutions $P^{\leqslant 0}$ of X and $Q^{\leqslant 0}$ of Z.

$$\cdots \to P^{-2} \xrightarrow{d_X^{-2}} P^{-1} \xrightarrow{d_X^{-1}} P^0 \to 0$$

$$\cdots \to Q^{-2} \xrightarrow{d_Z^{-2}} Q^{-1} \xrightarrow{d_Z^{-1}} Q^0 \to 0$$

The horseshoe lemma gives us the following projective resolution of $Y' = Y \oplus P$.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P^{-1} \arrow[r, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] \arrow[d, "d_X^{-1}"] & P^{-1}\oplus Q^{-1} \arrow[r, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] \arrow[d, "d_Y^{-1}"] & Q^{-1} \arrow[r] \arrow[d, "d_Z^{-1}"] & 0 \\
0 \arrow[r] & P^{0} \arrow[r, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] \arrow[d, "\pi_X"] & P^{0}\oplus Q^{0} \arrow[r, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \alpha \\ \beta \end{pmatrix}"] & Q^{0} \arrow[r] \arrow[d, "\pi_Z"] & 0 \\
0 \arrow[r] & X \arrow[r, "f"] & Y' \arrow[r, "g"] & Z \arrow[r] & 0
\end{tikzcd}$$

with $d_Y^k = \begin{pmatrix} d_X^k & 0 \\ \sigma^k & d_Z^k \end{pmatrix}$ and a morphism $\sigma^k : Q^k \to P^{k+1}$ for $k \leqslant -1$ such that $d_Y^{k-1} d_Y^k = 0$.

Similarly, we construct projective resolutions of X^*, $(Y')^*$ and Z^* in A-mod

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Q^{2} \arrow[r, "\begin{pmatrix} 0 & 1 \end{pmatrix}"] \arrow[d, "\tilde{d}_Z^{1}"] & P^{2}\oplus Q^{2} \arrow[r, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] \arrow[d, "\tilde{d}_Y^{1}"] & P^{2} \arrow[r] \arrow[d, "\tilde{d}_X^{1}"] & 0 \\
0 \arrow[r] & Q^{1} \arrow[r, "\begin{pmatrix} 0 & 1 \end{pmatrix}"] \arrow[d, "p_Z"] & P^{1}\oplus Q^{1} \arrow[r, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} a \\ b \end{pmatrix}"] & P^{1} \arrow[r] \arrow[d, "p_X"] & 0 \\
0 \arrow[r] & Z^* \arrow[r, "g^*"] & (Y')^* \arrow[r, "f^*"] & X^* \arrow[r] & 0
\end{tikzcd}$$

with $\tilde{d}_Y^k = \begin{pmatrix} \tilde{d}_X^k & \tilde{\sigma}^k \\ 0 & \tilde{d}_Z^k \end{pmatrix}$ and a morphism $\tilde{\sigma}^k : P^{k+1} \to Q^k$ for $k \geqslant 1$ such that $\tilde{d}_Y^{k+1} \tilde{d}_Y^k = 0$.

Applying $\mathrm{Hom}_A(-, A)$ to the second diagram and combining it with the first yields

$$\begin{tikzcd}
\cdots \arrow[r] & P^{-1} \arrow[r, "d_X^{-1}"] \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & P^{0} \arrow[r, "d_X^{0}"] \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & P^{1,*} \arrow[r, "\tilde{d}_X^{1,*}"] \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & P^{2,*} \arrow[r] \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & \cdots & F_X^\bullet \arrow[d, "s^\bullet"] \\
\cdots \arrow[r] & P^{-1}\oplus Q^{-1} \arrow[r, "d_Y^{-1}"] \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & P^{0}\oplus Q^{0} \arrow[r, "d_Y^{0}"] \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & P^{1,*}\oplus Q^{1,*} \arrow[r, "\tilde{d}_Y^{1,*}"] \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & P^{2,*}\oplus Q^{2,*} \arrow[r] \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & \cdots & G^\bullet \arrow[d, "t^\bullet"] \\
\cdots \arrow[r] & Q^{-1} \arrow[r, "d_Z^{-1}"] & Q^{0} \arrow[r, "d_Z^{0}"] & Q^{1,*} \arrow[r, "\tilde{d}_Z^{1,*}"] & Q^{2,*} \arrow[r] & \cdots & F_Z^\bullet
\end{tikzcd}$$

with $d_X^0 := \pi_X\, p_X^*,\ d_Z^0 := \pi_Z\, p_Z^*$ and

$$d_Y^0 := \begin{pmatrix} \alpha a^* & \alpha b^* \\ \beta a^* & \beta b^* \end{pmatrix} = \begin{pmatrix} \pi_X\, f a^* & \pi_X\, f\, g\, p_Z^* \\ \beta\, a^* & \beta\, g\, p_Z^* \end{pmatrix} = \begin{pmatrix} d_X^0 & 0 \\ \sigma^0 & d_Z^0 \end{pmatrix}$$

where $\sigma^0 := \beta a^*$. Write $d_X^k := \tilde{d}_X^{k,*},\ d_Y^k := \tilde{d}_Y^{k,*}, d_Z^k := \tilde{d}_Z^{k,*}$ and $\sigma^k := \tilde{\sigma}^{k,*}$ for $k \geqslant 1$.

We obtain the complexes

$$F_X^k = \begin{cases} P^k, & k \leqslant 0 \\ P^{k,*}, & k > 0 \end{cases} \qquad G^k := \begin{cases} P^k \oplus Q^k, & k \leqslant 0 \\ P^{k,*} \oplus Q^{k,*}, & k > 0 \end{cases} \qquad F_Z^k = \begin{cases} Q^k, & k \leqslant 0 \\ Q^{k,*}, & k > 0 \end{cases}$$

with the differentials defined above. By construction, all three complexes are elements of $\mathcal{L}$. Additionally, we obtain two morphisms of complexes

$$s^\bullet := \begin{pmatrix} 1 & 0 \end{pmatrix}^\bullet : F_X^\bullet \to G^\bullet \quad \text{and} \quad t^\bullet := \begin{pmatrix} 0 \\ 1 \end{pmatrix}^\bullet : G^\bullet \to F_Z^\bullet.$$

We have the following.

$$\begin{aligned} &\mathrm{H}^0\left(\tau_{\leqslant 0} F_X^\bullet\right) \simeq X && \mathrm{H}^0\left(\tau_{\leqslant 0} s^\bullet\right) \simeq f \\ &\mathrm{H}^0\left(\tau_{\leqslant 0} G^\bullet\right) \simeq Y \oplus P \overset{\mathrm{st}}{\simeq} Y && \mathrm{H}^0\left(\tau_{\leqslant 0} t^\bullet\right) \simeq g \\ &\mathrm{H}^0\left(\tau_{\leqslant 0} F_Z^\bullet\right) \simeq Z && \end{aligned}$$

Moreover, $0 \to F_X^\bullet \xrightarrow{s^\bullet} G^\bullet \xrightarrow{t^\bullet} F_Z^\bullet \to 0$ is an exact sequence of complexes. The stable isomorphism $Y \oplus P \to Y$ lifts to an isomorphism $\varphi^\bullet : G^\bullet \to F_Y^\bullet$ in $\mathcal{K}(\operatorname{proj} A)$ with $\underline{\mathrm{H}^0(\tau_{\leqslant 0}\varphi^\bullet)} \simeq \underline{\mathrm{id}_Y}$. Lemma 2.3 now shows that $s^\bullet\, \varphi^\bullet = f^\bullet$ and $\varphi^\bullet\, g^\bullet = t^\bullet$ in $\mathcal{K}(\operatorname{proj} A)$.

Ad (2) ⇒ (1). The short exact sequence $0 \to F_X^\bullet \xrightarrow{s^\bullet} G^\bullet \xrightarrow{t^\bullet} F_Z^\bullet \to 0$ induces the following short exact sequence by applying $\tau_{\leqslant 0}(-)$.

$$0 \to \tau_{\leqslant 0} F_X^\bullet \xrightarrow{\tau_{\leqslant 0} s^\bullet} \tau_{\leqslant 0} G^\bullet \xrightarrow{\tau_{\leqslant 0} t^\bullet} \tau_{\leqslant 0} F_Z^\bullet \to 0$$

This yields a short exact sequence of cohomology. Note that $\mathrm{H}^{-1}(\tau_{\leqslant 0}\, F_Z^\bullet) = 0$ since $F_Z^\bullet \in \mathcal{L}$.

$$0 \to X \xrightarrow{s} H^0(\tau_{\leqslant 0} G^\bullet) \xrightarrow{t} Z \to 0$$

Since F_Z^k is projective for all $k \in \mathbb{Z}$, the sequence $0 \to F_X^\bullet \xrightarrow{s^\bullet} G^\bullet \xrightarrow{t^\bullet} F_Z^\bullet \to 0$ splits in every degree. Therefore, $0 \to F_Z^{\bullet,*} \xrightarrow{t^{\bullet,*}} G^{\bullet,*} \xrightarrow{s^{\bullet,*}} F_X^{\bullet,*} \to 0$ is also exact. As above, the sequence

$$0 \to \tau_{\geqslant 1} F_Z^{\bullet,*} \xrightarrow{\tau_{\geqslant 1} t^{\bullet,*}} \tau_{\geqslant 1} G^{\bullet,*} \xrightarrow{\tau_{\geqslant 1} s^{\bullet,*}} \tau_{\geqslant 1} F_X^{\bullet,*} \to 0$$

induces a short exact sequence of homology using that $\mathrm{H}_2(F_X^{\bullet,*}) = 0$ since $F_X^\bullet \in \mathcal{L}$.

$$0 \to Z^* \to \mathrm{H}_1(\tau_{\geqslant 1} G^{\bullet,*}) \to X^* \to 0$$

Since $G^\bullet \in \mathcal{L}$, we have $\mathrm{H}_1(\tau_{\geqslant 1} G^{\bullet,*}) = \mathrm{Cok}(d_G^{1,*}) = \mathrm{Ker}(d_G^{-1,*}) \simeq \mathrm{Cok}(d_G^{-1})^* = \mathrm{H}^0(\tau_{\leqslant 0}\, G^\bullet)^*$. Therefore,

$$0 \to X \xrightarrow{s} H^0(\tau_{\leqslant 0} G^\bullet) \xrightarrow{t} Z \to 0$$

is a perfect exact sequence.

We know that $H^0(\tau_{\leqslant 0} G^\bullet) \overset{\mathrm{st}}{\simeq} Y$. Furthermore, via this ismorphism we have $\underline{s} \simeq \underline{f}$ and $\underline{t} \simeq \underline{g}$. Using that Y has no projective direct summand, Lemma 2.15 shows that the above sequence is isomorphic to

$$0 \to X \xrightarrow{\begin{pmatrix} f & p \end{pmatrix}} Y \oplus P \xrightarrow{\begin{pmatrix} g \\ q \end{pmatrix}} Z \to 0$$

for a $P \in \mathrm{proj}\, A$ unique up to isomorphism. In particular, we have $H^0(\tau_{\leqslant 0} G^\bullet) \simeq Y \oplus P$. □

Example in Chapter 7. For the algebra B in Section 7.1 we discuss a perfect exact sequences and its corresponding distinguished triangle in Example 7.2.

At the end of this section, we return to the pseudo-kernel and pseudo-cokernel of Definition 2.8 and discuss their relationship with perfect exact sequences. We start with two short exact sequences containing the pseudo-kernel given in [18]. For the proof of part (2), we follow [18, Lemma 3.6.(1)].

Lemma 2.20. *Suppose given a morphism $g : Y \to Z$ in $\mathrm{mod}\, A$. We have the following short exact sequences.*

(1) $0 \to \underline{\mathrm{Ker}}(g) \to Y \oplus F_Z^0 \xrightarrow{\begin{pmatrix} g \\ \pi \end{pmatrix}} Z \to 0$ *with $\pi : F_Z^0 \to Z$ the natural projection.*

(2) $0 \to \mathrm{Ker}(g) \to \underline{\mathrm{Ker}}(g) \to L \to 0$ *with* $L = \mathrm{Ker}(F_Z^0 \to \mathrm{Cok}(g)) \overset{\mathrm{st}}{\simeq} \Omega(\mathrm{Cok}(g))$.

In particular, $\underline{\mathrm{Ker}}(g) \simeq \mathrm{Ker}(g) \oplus F_Z^0$ *if g is surjective and* $\underline{\mathrm{Ker}}(g) \simeq \mathrm{Ker}(F_Z^0 \to \mathrm{Cok}(g))$ *if g is injective.*

Proof. The distinguished triangle $F_Y^\bullet \xrightarrow{g^\bullet} F_Z^\bullet \xrightarrow{h^\bullet} C(g)^\bullet \to$ induces a short exact sequence of complexes $0 \to C(g)^\bullet[-1] \to C(h)^\bullet[-1] \to F_Z^\bullet \to 0$. Applying $\mathrm{H}^0(\tau_{\leqslant 0}\,(-))$ to this sequence, we obtain a short exact sequence $0 \to \underline{\mathrm{Ker}}(g) \to \mathrm{H}^0(\tau_{\leqslant 0}\,(C(h)^\bullet[-1])) \to Z \to 0$ via the long exact cohomology sequence. We show that $\mathrm{H}^0(\tau_{\leqslant 0}\,(C(h)^\bullet[-1])) \simeq Y \oplus F_Z^0$.

We have $\mathrm{H}^0(\tau_{\leqslant 0}\,(C(h)^\bullet[-1])) = (F_Z^0 \oplus F_Y^0 \oplus F_Z^{-1})/\operatorname{Im}(d_{C(h)}^{-2})$ with

$$d_{C(h)}^{-2} = \begin{pmatrix} -d_Z^{-1} & 0 & 1 \\ 0 & -d_Y^{-1} & g^{-1} \\ 0 & 0 & d_Z^{-2} \end{pmatrix}.$$

We have mutually inverse isomorphism $\varphi : \mathrm{H}^0(\tau_{\leqslant 0}\,(C(h)^\bullet[-1])) \to F_Z^0 \oplus F_Y^0/\operatorname{Im}(d_Y^{-1})$ and $\psi : F_Z^0 \oplus F_Y^0/\operatorname{Im}(d_Y^{-1}) \to \mathrm{H}^0(\tau_{\leqslant 0}\,(C(h)^\bullet[-1]))$ in $\operatorname{mod} A$ defined as follows.

$$\varphi := \begin{pmatrix} 1 & 0 \\ g^0 & 1 \\ d_Z^{-1} & 0 \end{pmatrix}, \qquad \psi := \begin{pmatrix} 1 & 0 & 0 \\ -g^0 & 1 & 0 \end{pmatrix}$$

Using that $F_Y^0/\operatorname{Im}(d_Y^{-1}) \simeq Y$, part (1) follows.

We obtain the following commutative diagram with exact rows and columns. Let L be the kernel of the morphism $F_Z^0 \to \operatorname{Cok}(g)$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Ker}(g) \arrow[r] \arrow[d, dashed] & Y \arrow[r] \arrow[d] & \operatorname{Im}(g) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{\operatorname{Ker}}(g) \arrow[r] \arrow[d, dashed] & Y \oplus F_Z^0 \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & F_Z^0 \arrow[r] \arrow[d] & \operatorname{Cok}(g) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We obtain a short exact sequence $0 \to \operatorname{Ker}(g) \to \underline{\operatorname{Ker}}(g) \to L \to 0$. Recall that F_Z^0 is the projective cover of Z. If g is surjective, we have $L \simeq F_Z^0$ and the sequence splits. $\square$

In general, the short exact sequence in (1) of Lemma 2.20 is not perfect exact. The situation is different for the pseudo-cokernel. The following is a special case of [19, Theorem 3.9] restricted to injective morphisms. We give a modified proof adapted to this situation.

Proposition 2.21 (Kato)**.** *Suppose given an injective morphism $f : X \to Y$ in $\operatorname{mod} A$.*

We have $\mathrm{H}^{-1}(C(f)^\bullet) = 0$. Furthermore, there exists a perfect exact sequence

$$0 \to X \xrightarrow{(f\ d)} Y \oplus F_X^1 \to \underline{\operatorname{Cok}}(f) \to 0.$$

Proof. Let $Z := \mathrm{Cok}(f)$ and denote the induced short exact sequence as follows.

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0.$$

The morphisms f and g induce two distinguished triangles in $\mathcal{K}(\operatorname{proj} A)$ and a morphism of complexes $\psi^\bullet : C(f)^\bullet \to F_Z^\bullet$ making the following diagram commutative.

$$\begin{tikzcd}
F_X^\bullet \arrow[r, "f^\bullet"] \arrow[d, dashed, "\varphi^\bullet"] & F_Y^\bullet \arrow[r, "(0\ 1)"] \arrow[d, equal] & C(f)^\bullet \arrow[r] \arrow[d, "\psi^\bullet = \binom{0}{g^\bullet}"] & {} \\
C(g)^\bullet[-1] \arrow[r] & F_Y^\bullet \arrow[r, "g^\bullet"] & F_Z^\bullet \arrow[r] & {}
\end{tikzcd}$$

We obtain a morphism $\varphi^\bullet : F_X^\bullet \to C(g)^\bullet[-1]$ with $\mathrm{H}^0(\tau_{\leqslant 0}\, \varphi^\bullet) : X \to \underline{\mathrm{Ker}}(g)$. Since g is surjective, $\mathrm{H}^0(\tau_{\leqslant 0}\, \varphi^\bullet)$ is a stable isomorphism by Lemma 2.20. In particular, $F_X^\bullet \simeq F^\bullet_{\underline{\mathrm{Ker}}(g)}$ in $\mathcal{K}(\operatorname{proj} A)$. Note that $\tau_{\leqslant 0}\, F^\bullet_{\underline{\mathrm{Ker}}(g)}$ and $\tau_{\leqslant 0}\, (C(g)^\bullet[-1])$ are projective resolutions of isomorphic modules and thus isomorphic themselves. As a result, we can assume that $\varphi^k = \mathrm{id}$ for $k \leqslant 0$ up to isomorphism in $\mathcal{K}(\operatorname{proj} A)$. In particular, we have $\mathrm{H}^k(C(\varphi)^\bullet) = 0$ for $k \leqslant 0$.

Consider the following commutative diagram of distinguished triangles.

$$\begin{tikzcd}
F_X^\bullet \arrow[r, "f^\bullet"] \arrow[d, "\varphi^\bullet"] & F_Y^\bullet \arrow[r, "(0\ 1)"] \arrow[d, equal] & C(f)^\bullet \arrow[r] \arrow[d, "\psi^\bullet"] & {} \\
C(g)^\bullet[-1] \arrow[r] \arrow[d] & F_Y^\bullet \arrow[r, "g^\bullet"] \arrow[d] & F_Z^\bullet \arrow[r] \arrow[d] & {} \\
C(\varphi)^\bullet \arrow[r] & 0 \arrow[r] & C(\psi)^\bullet \arrow[r] & {}
\end{tikzcd}$$

We obtain $C(\varphi)^\bullet[1] \simeq C(\psi)^\bullet$ so that $\mathrm{H}^k(C(\psi)^\bullet) = 0$ for $k < 0$. Now, the distinguished triangle

$$F_Z^\bullet \to C(\psi)^\bullet \to C(f)^\bullet[1] \to$$

provides $\mathrm{H}^{-1}(C(f)^\bullet) = 0$ by Lemma 1.9. This implies $\mathrm{H}^{-1}(C(f)^\bullet) \in \mathcal{L}_A$; cf. Remark 2.7. In conclusion, we have a distinguished triangle

$$F_X^\bullet \xrightarrow{f^\bullet} F_Y^\bullet \to C(f)^\bullet \xrightarrow{h^\bullet} F_X^\bullet[1]$$

in $\mathcal{L}_A$ which induces a perfect exact sequence

$$0 \to X \to \mathrm{H}^0(\tau_{\leqslant 0}\, (C(h)^\bullet[-1])) \to \underline{\mathrm{Cok}}(f) \to 0$$

in $\operatorname{mod} A$ by Proposition 2.18. It remains to show that $\mathrm{H}^0(\tau_{\leqslant 0}\, (C(h)^\bullet[-1])) \simeq Y \oplus F_X^1$.

We have $\mathrm{H}^0(\tau_{\leqslant 0}\,(C(h)^\bullet[-1])) = (F_X^1 \oplus F_Y^0 \oplus F_X^0)/\operatorname{Im}(d_{C(h)}^{-1})$ with

$$d_{C(h)}^{-1} = \begin{pmatrix} d_X^0 & -f^0 & -1 \\ 0 & -d_Y^{-1} & 0 \\ 0 & 0 & d_X^{-1} \end{pmatrix}.$$

We have mutually inverse isomorphism $\varphi : \mathrm{H}^0(\tau_{\leqslant 0}\,(C(h)^\bullet[-1])) \to F_X^1 \oplus F_Y^0/\operatorname{Im}(d_Y^{-1})$ and $\psi : F_X^1 \oplus F_Y^0/\operatorname{Im}(d_Y^{-1}) \to \mathrm{H}^0(\tau_{\leqslant 0}\,(C(h)^\bullet[-1]))$ in $\operatorname{mod} A$ defined as follows.

$$\varphi := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ d_X^0 & -f^0 \end{pmatrix}, \qquad \psi := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Using that $F_Y^0/\operatorname{Im}(d_Y^{-1}) \simeq Y$, the result follows. □

Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence in $\operatorname{mod} A$. We have seen that every monomorphism can be completed to a perfect exact sequence.

$$0 \to X \xrightarrow{\left(\begin{smallmatrix} f & d \end{smallmatrix}\right)} Y \oplus F_X^1 \to \underline{\operatorname{Cok}}(f) \to 0.$$

This induces a morphism $c : \underline{\operatorname{Cok}}(f) \to Z$ such that the following diagram commutes.

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\left(\begin{smallmatrix} f & d \end{smallmatrix}\right)"] \arrow[d, equal] & Y \oplus F_X^1 \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] & \underline{\operatorname{Cok}}(f) \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0 \end{tikzcd}$$

The morphism c can be used to characterize perfect exact sequences. The short exact sequence in the following lemma and its proof can be found in [18, Lemma 3.6.(2)].

Lemma 2.22. *Suppose given a short exact sequence* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *in* $\operatorname{mod} A$.

There exists a short exact sequence of the form

$$0 \to F_X^1 \to \underline{\operatorname{Cok}}(f) \xrightarrow{c} Z \to 0.$$

Moreover, $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *is a perfect exact sequence if and only if* $\underline{\operatorname{Cok}}(f)$ *and* $\operatorname{Cok}(f)$ *are stably isomorphic in* $\underline{\operatorname{mod}}\, A$.

Proof. Let $L := \operatorname{Ker}(c)$. The following commutative diagram with exact rows and columns shows that $L \simeq F_X^1 \in \operatorname{proj} A$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & F_X^1 \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right)"] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r, "\left(\begin{smallmatrix} f & d \end{smallmatrix}\right)"] \arrow[d, equal] & Y \oplus F_X^1 \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] & \underline{\operatorname{Cok}}(f) \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Let $\varphi : \underline{\operatorname{Cok}}(f) \to Z$ be a stable isomorphism. Recall that $\underline{\operatorname{Cok}}(f) = \mathrm{H}^0(\tau_{\leqslant 0} C(f)^\bullet)$. In the distinguished triangle $F_X^\bullet \to F_Y^\bullet \to C(f)^\bullet \to$, we have $C(f)^\bullet \in \mathcal{L}_A$ by Proposition 2.21. Now, φ induces an isomorphism $\varphi^\bullet : C(f)^\bullet \to F_Z^\bullet$ in $\mathcal{L}_A$ so that $F_X^\bullet \to F_Y^\bullet \to F_Z^\bullet \to$ is also a distinguished triangle via a morphism $\tilde{g}^\bullet : F_Y^\bullet \to F_Z^\bullet$.

$$\begin{tikzcd}
F_X^\bullet \arrow[r, "f^\bullet"] \arrow[d, equal] & F_Y^\bullet \arrow[r] \arrow[d, equal] & C(f)^\bullet \arrow[r] \arrow[d, "\varphi^\bullet", "\wr"'] & {} \\
F_X^\bullet \arrow[r, "f^\bullet"] & F_Y^\bullet \arrow[r, "\tilde{g}^\bullet"] & F_Z^\bullet \arrow[r] & {}
\end{tikzcd}$$

By Proposition 2.18, we obtain a perfect exact sequence $0 \to X \xrightarrow{\left(\begin{smallmatrix} f & p \end{smallmatrix}\right)} Y \oplus P \to Z \to 0$ in $\operatorname{mod} A$. Using Remark 2.16 we see that this perfect exact sequence is isomorphic to the short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$.

On the other hand, if $0 \to X \to Y \to Z \to 0$ is perfect exact, then

$$F_X^\bullet \xrightarrow{f^\bullet} F_Y^\bullet \to F_Z^\bullet \to$$

is a distinguished triangle with $F_Z^\bullet \simeq C(f)^\bullet$ in $\mathcal{L}_A$. Hence, $\underline{\operatorname{Cok}}(f) = \mathrm{H}^0(\tau_{\leqslant 0} C(f)^\bullet) \overset{\mathrm{st}}{\simeq} Z$ via c by Lemma 2.3. □

Example in Chapter 7. For the algebra B in Section 7.1 we calculate the pseudo-cokernel of a morphism and the corresponding short exact sequences in Example 7.3. Furthermore, we illustrate that the perfect exact sequence in Proposition 2.21 can be quite different from the original short exact sequence in case that $\underline{\operatorname{Cok}}(f)$ and $\operatorname{Cok}(f)$ are not stably isomorphic.

For more details on the pseudo-kernel and pseudo-cokernel see [18] and [19, Section 2]. Moreover, both are used in [19, Section 4] to further characterize morphisms which are stably equivalent to a monomorphism. In the context of commutative Gorenstein rings, Kato shows that a morphism is stably equivalent to a monomorphism if and only if its kernel is a submodule of a free module. We will go in a different direction and focus on perfect exact sequences.

2.3 Perfect exact sequences with projective middle term

Perfect exact sequences with projective middle term are of special importance for us. The corresponding distinguished triangle is induced by the shift in $\mathcal{K}(\operatorname{proj} A)$. The next lemma focuses on the ending term of such a perfect exact sequence.

Lemma 2.23. *The following are equivalent for $Z \in \operatorname{mod} A$.*

(1) $\operatorname{Ext}^1_A(Z, A) = 0$

(2) There exists a perfect exact sequence $0 \to X \to P \to Z \to 0$ with $P \in \operatorname{proj} A$.

(3) Every short exact sequence ending in Z is perfect exact.

(4) $F^\bullet_Z[-1] \in \mathcal{L}_A$

Proof. Suppose given a short exact sequences $0 \to X \to Y \to Z \to 0$ in $\operatorname{mod} A$. Applying $\operatorname{Hom}_A(-, A)$, we obtain the following exact sequence.

$$0 \to \operatorname{Hom}_A(Z, A) \to \operatorname{Hom}_A(Y, A) \to \operatorname{Hom}_A(X, A) \to \operatorname{Ext}^1_A(Z, A) \to \operatorname{Ext}^1_A(Y, A)$$

Thus, the short exact sequence is perfect exact if $\operatorname{Ext}^1_A(Z, A) = 0$. If in addition Y is projective, then the converse holds as well. Note that there always is a short exact sequence of the form $0 \to X \to P \to Z \to 0$ with P the projective cover of Z.

We verify the equivalence of (2) and (4). Consider the following distinguished triangle in $\mathcal{K}(\operatorname{proj} A)$.

$$F^\bullet_Z[-1] \to 0 \to F^\bullet_Z \to$$

By Proposition 2.18, $F^\bullet_Z[-1] \in \mathcal{L}_A$ if and only if we have a perfect exact sequence

$$0 \to X \to P \to Z \to 0$$

in $\operatorname{mod} A$ with $P \in \operatorname{proj} A$ and $X := \operatorname{H}^0(\tau_{\leqslant 0} F^\bullet_Z[-1])$. In this case, $F^\bullet_Z[-1] \simeq F^\bullet_X \in \mathcal{L}_A$. □

Now, we consider the starting term of a perfect exact sequence with projective middle term. We are mainly interested in the case of simple modules.

Recall that F_X^0 is the projective cover of $X \in \operatorname{mod} A$. If $F_X^1 = 0$, the complex $F_X^\bullet$ is the minimal projective resolution of X.

Lemma 2.24. *The following are equivalent for $X \in \operatorname{mod} A$.*

(1) $F_X^\bullet[1] \in \mathcal{L}_A$.

(2) $\mathrm{H}^0(F^\bullet) = 0$.

(3) There exists an embedding $X \hookrightarrow A$.

(4) There exists a perfect exact sequence $0 \to X \to P \to Z \to 0$ in $\operatorname{mod} A$ with $P \in \operatorname{proj} A$.

Furthermore, (1), (2), (3) and (4) imply the following equivalent conditions.

(5) $X^* \neq 0$.

(6) $\nu(X) \neq 0$.

(7) $F_X^1 \neq 0$.

If X is simple, all seven conditions are equivalent.

Proof. If $F_X^\bullet[1] \in \mathcal{L}_A$, we have $\mathrm{H}^0(F_X^\bullet) = 0$. In this case, $X = \operatorname{Cok}(d_X^{-1}) \simeq \operatorname{Im}(d_X^0)$ which embeds into $F_X^1 \in \operatorname{proj} A$.

Suppose that $f : X \hookrightarrow A$ is injective. By Proposition 2.21, we obtain a perfect exact sequence of the following form.

$$0 \to X \to A \oplus F_X^1 \to \underline{\operatorname{Cok}}(f) \to 0$$

Since $A \oplus F_X^1 \in \operatorname{proj} A$, part (4) follows.

By Proposition 2.18, a perfect exact sequence

$$0 \to X \to P \to Z \to 0$$

induces the following distinguished triangle in $\mathcal{L}_A$.

$$F_X^\bullet \to 0 \to F_Z^\bullet \to$$

In this case, $F_X^\bullet[1] \simeq F_Z^\bullet \in \mathcal{L}_A$. This shows the equivalence of (1), (2), (3) and (4).

There is nothing to show for the implication (3) $\Rightarrow$ (5). Since $\nu(X) = \mathrm{D}\operatorname{Hom}_A(X, A) = 0$ if and only if $\operatorname{Hom}_A(X, A) = 0$, conditions (5) and (6) are equivalent. Using that $\tau_{\geqslant 1} F_X^{\bullet,*}$ is the minimal projective resolution of X^*, we obtain the equivalence of (5) and (7).

If X is simple, condition (5) implies condition (3). $\square$

If it exists, there is only one perfect exact sequence without split summands starting in a module X which has projective middle term. A more general result for morphisms stably equivalent to a monomorphism has been shown as part of [19, Theorem 3.9].

Lemma 2.25. *Suppose given $X \in \operatorname{mod} A$ such that there exists a perfect exact sequence starting in X with projective middle term.*

The short exact sequence $0 \to X \xrightarrow{d} F^1_X \to \operatorname{Cok}(d) \to 0$ is perfect exact in $\operatorname{mod} A$. Every perfect exact sequence starting in X with projective middle term is isomorphic to a direct sum of this sequence and a split exact sequence of projective modules.

Proof. By assumption, Lemma 2.24 shows that $F^\bullet_X[1] \in \mathcal{L}_A$. As a consequence, we have that $X = \operatorname{Cok}(d^{-1}_X) \simeq \operatorname{Im}(d^0_X)$ is a submodule of F^1_X. In particular, $0 \to X \xrightarrow{d} F^1_X \to \operatorname{Cok}(d) \to 0$ is a perfect exact sequence since $d^* : F^{1,*}_X \to X^*$ is surjective.

Suppose given a perfect exact sequence $0 \to X \xrightarrow{f} P \xrightarrow{g} Z \to 0$ in $\operatorname{mod} A$ with $P \in \operatorname{proj} A$. Recall that $d^* : F^{1,*}_X \to X^*$ is the projective cover of X^*. We obtain a morphism $s : F^1_X \to P$ with $d\,s = f$. We also have that $f^* : P^* \to X^*$ is surjective. Therefore, $s^* : P^* \to F^{1,*}_X$ is a split epimorphism which implies that $s : F^1_X \to P$ is a split monomorphism. This induces a morphism of short exact sequences and the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "d"] \arrow[d, equal] & F^1_X \arrow[r] \arrow[d, "s"] & \operatorname{Cok}(d) \arrow[r] \arrow[d, dashed, "t"] & 0 \\
0 \arrow[r] & X \arrow[r, "f"] \arrow[d] & P \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \operatorname{Cok}(s) \arrow[r] & \operatorname{Cok}(t) \arrow[r] & 0
\end{tikzcd}$$

We obtain an isomorphism $\operatorname{Cok}(s) \simeq \operatorname{Cok}(t)$. Note that $\operatorname{Cok}(s)$ is projective as a direct summand of P. Moreover, since s is a split monomorphism, so is t. □

Finally, we note the following characterization of an algebra with positive dominant dimension. Recall that $\nu(F^0_S)$ is the injective hull of S if $S \in \operatorname{mod} A$ is simple.

Lemma 2.26. *Let $S \neq 0$ be a simple A-module. Then $\nu(F^0_S) \notin \mathcal{P}_A$ if $\nu(S) = 0$. Moreover,* $\operatorname{domdim} A \geqslant 1$ *if and only if $\nu(S) = 0$ for all simple A-modules $S \neq 0$ with $\nu(F^0_S) \notin \mathcal{P}_A$.*

Proof. Suppose $\nu(F^0_S) \in \mathcal{P}_A$. Then $\operatorname{Hom}_A(S, \nu(F^0_S)) \neq 0$ with $\nu(F^0_S) \in \operatorname{proj} A$. By Lemma 2.24 we obtain that $\nu(S) \neq 0$.

Now, suppose that $\operatorname{domdim} A \geqslant 1$ and let S be a simple A-module with $\nu(S) \neq 0$. Then there exists a $P \in \operatorname{proj} A$ with $\operatorname{Hom}_A(S, P) \neq 0$ by Lemma 2.24. By assumption, P embeds into a projective-injective module Z. However, this means that the injective hull $\nu(F_S^0)$ of S is a direct summand of $Z \in \mathcal{P}_A$.

Conversely, suppose that $\nu(S) = 0$ for all simple A-modules $S \neq 0$ with $\nu(F_S^0) \notin \mathcal{P}_A$. Suppose given $Q \in \operatorname{proj} A$ not injective with injective hull νP for some $P \in \operatorname{proj} A$. For every S in $\operatorname{soc}(Q) = \operatorname{soc}(\nu P)$ we have that $\nu(S) \neq 0$ by Lemma 2.24 and therefore $\nu(F_S^0) \in \mathcal{P}_A$ by assumption. Hence, $\nu(P) = \bigoplus_{S \mid \operatorname{soc}(\nu(P))} \nu(F_S^0) \in \mathcal{P}_A$ which was the injective hull of Q. We obtain $\operatorname{domdim} A \geqslant 1$. $\square$

Example in Chapter 7. We illustrate some of the previous results in Example 7.4 for the algebra B of Section 7.1.

Chapter 3

Perfect exact sequences and stable equivalences

Let k be a field. Let A and B be finite dimensional k-algebras without semisimple summands. Throughout this chapter, we will denote the almost split sequence starting in an indecomposable non-injective A-module X as follows.

$$0 \to X \to E_X \to \tau^{-1}(X) \to 0$$

This chapter is dedicated to examine what happens to perfect exact sequences under stable equivalences $\underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$. Being perfect exact can be seen as a property of a given sequence in $\mathrm{mod}\, A$. For later use, we introduce the following shortened notion for stable equivalences that preserve this property.

Definition 3.1. Let $\eta : 0 \to X \xrightarrow{f} Y \oplus P \xrightarrow{g} Z \to 0$ be a perfect exact sequence in $\mathrm{mod}\, A$ without split summands such that P is projective and Y has no projective direct summand. We say that a functor $\alpha : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$ *preserves the perfect exact sequence* η if there exists a perfect exact sequence

$$0 \to \alpha(X) \xrightarrow{\tilde{f}} \alpha(Y) \oplus \tilde{P} \xrightarrow{\tilde{g}} \alpha(Z) \to 0$$

in $\mathrm{mod}\, B$ with $\tilde{P} \in \mathrm{proj}\, B$ such that $\underline{\tilde{f}} \simeq \alpha(\underline{f})$ and $\underline{\tilde{g}} \simeq \alpha(\underline{g})$ in $\underline{\mathrm{mod}}\, B$.

We will see later that stable equivalences of Morita type preserve perfect exact sequences; cf. Lemma 5.5. For now, we can show that a stable equivalence preserves perfect exact sequences with projective middle term if the stable equivalence and its quasi-inverse are induced by an exact functor. Furthermore, an exact functor preserves arbitrary perfect exact sequences if and only if it preserves the pseudo-cokernel discussed in the previous chapter; cf. Definition 2.8. In this case, the exact functor maps perfect exact sequences to perfect exact sequences; cf. Remark 2.16.

Proposition 3.2. *Let ${}_AM_B$ be a bimodule which is projective as left A- and right B-module such that $- \otimes_A M : \operatorname{mod} A \to \operatorname{mod} B$ is an exact functor which induces a stable equivalence $\underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$.*

(1) The following are equivalent.

(i) The functor $- \otimes_A M$ preserves perfect exact sequences with projective middle term.

(ii) For all $Z \in \operatorname{mod} A$ we have $\operatorname{Ext}^1_B(Z \otimes_A M, B) = 0$ if $\operatorname{Ext}^1_A(Z, A) = 0$.

(2) If there is a bimodule ${}_BL_A$ which is projective as left B- and right A-module and which induces the inverse stable equivalence, then the equivalent conditions of part (1) hold.

(3) The functor $- \otimes_A M$ preserves a perfect exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ if and only if there exists a stable isomorphism $\underline{\operatorname{Cok}}(f \otimes M) \overset{\text{st}}{\simeq} \underline{\operatorname{Cok}}(f) \otimes_A M$ in $\underline{\operatorname{mod}}\, B$.

Proof. Ad (1). Suppose given $Z \in \operatorname{mod} A$ with $\operatorname{Ext}^1(Z, A) = 0$. We have seen in Lemma 2.23 that in this case there exists a perfect exact sequence ending in Z with projective middle term. Thus, (i) implies that there also is a perfect exact sequence in $\operatorname{mod} B$ with ending term $Z \otimes_A M$ and projective middle term. Using Lemma 2.23 again, we obtain $\operatorname{Ext}^1_B(Z \otimes_A M, B) = 0$. This shows the implication (i) $\Rightarrow$ (ii).

On the other hand, suppose that $0 \to X \to P \to Z \to 0$ is a perfect exact sequence in $\operatorname{mod} A$ with $P \in \operatorname{proj} A$. Then $\operatorname{Ext}^1(Z, A) = 0$ so that (ii) implies $\operatorname{Ext}^1_B(Z \otimes_A M, B) = 0$. Now, by Lemma 2.23, every short exact sequence ending in $Z \otimes_A M$ is perfect exact. In particular, this holds for the induced short exact sequence $0 \to X \otimes_A M \to P \otimes_A M \to Z \otimes_A M \to 0$. Therefore, the implication from (ii) to (i) holds as well.

Ad (2). We show condition (ii) of part (1). Suppose given $Z \in \operatorname{mod} A$ with $\operatorname{Ext}^1(Z, A) = 0$. We write $Z' := Z \otimes_A M \in \operatorname{mod} B$. Let $0 \to B \to Y' \xrightarrow{g'} Z' \to 0$ be a short exact sequence in $\operatorname{Ext}^1_B(Z \otimes_A M, B)$. Since $- \otimes_B L$ is exact, we obtain the following short exact sequence.

$$0 \to B \otimes_B L \to Y' \otimes_B L \xrightarrow{g' \otimes L} Z' \otimes_B L \to 0$$

Note that $Z' \otimes_B L = Z \otimes_A M \otimes_B L \simeq Z$ in $\underline{\operatorname{mod}}\, A$ and $B \otimes_B L_A \simeq L_A \in \operatorname{proj} A$. Using that $\operatorname{Ext}^1_A(Z, A) = 0$, this implies that $g' \otimes_B L$ is a split epimorphism with projective kernel. By Lemma 1.4, we obtain that $g' \otimes_B L$ is a stable isomorphism. As a consequence, $g' \otimes_B L \otimes_A M$ is a stable isomorphism as well. Using that $g' \otimes_B L \otimes_A M \simeq g'$ in $\underline{\operatorname{mod}}\, B$ and that g' is surjective, we obtain that g' is a split epimorphism by Lemma 1.4. In conclusion, $\operatorname{Ext}^1_B(Z \otimes_A M, B) = 0$.

Ad (3). Suppose that $\eta : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is a perfect exact sequence. Using Lemma 2.22, $- \otimes_A M$ preserves the perfect exact sequence if and only if there is a stable isomorphism $\underline{\operatorname{Cok}}(f \otimes M) \overset{\text{st}}{\simeq} \operatorname{Cok}(f \otimes M)$. The same lemma provides a stable isomorphism $\varphi : \underline{\operatorname{Cok}}(f) \to Z$

in $\underline{\operatorname{mod}} A$ since η is perfect exact. This induces a stable isomorphism $\underline{\operatorname{Cok}}(f) \otimes_A M \overset{\text{st}}{\simeq} Z \otimes_A M$. Using that $- \otimes_A M$ is right exact, we have the following.

$$\underline{\operatorname{Cok}}(f) \otimes_A M \overset{\text{st}}{\simeq} Z \otimes_A M \simeq \operatorname{Cok}(f) \otimes_A M \simeq \operatorname{Cok}(f \otimes M)$$

In conclusion, we have that $\underline{\operatorname{Cok}}(f \otimes M) \overset{\text{st}}{\simeq} \operatorname{Cok}(f \otimes M)$ if and only if $\underline{\operatorname{Cok}}(f \otimes M) \overset{\text{st}}{\simeq} \underline{\operatorname{Cok}}(f) \otimes_A M$ in $\underline{\operatorname{mod}} B$ □

For an arbitrary stable equivalence to preserve perfect exact sequences, we have to at least exclude short exact sequences that start with a node; see also Example 7.15. We recall the definition of a node. See [31] for more details.

Definition 3.3. A simple A-module S is called a *node* if it is neither projective nor injective and the middle term E_S of the almost split sequence starting in S is projective.

We will use the following immediate characterization of a node; see also [4, Proposition 2.5].

Lemma 3.4. *Suppose given an almost split sequence with X not injective.*

$$0 \to X \xrightarrow{f} E_X \xrightarrow{g} Z \to 0$$

(1) We have $\underline{f} = 0$ in $\underline{\operatorname{mod}} A$ if and only if X or E_X is projective.

We have $\underline{g} = 0$ in $\underline{\operatorname{mod}} A$ if and only if Z or E_X is projective.

(2) Suppose that X is simple and not projective. Then X is a node if and only if $\underline{f} = 0$.

Proof. Ad (1). It immediately follows that $\underline{f} = 0$ if X or E_X is projective. Suppose that $\underline{f} = 0$. Then there exists a projective module $P \in \operatorname{proj} A$ and morphisms $u : X \to P$ and $v : P \to E_X$ such that $f = u\,v$. Since f is irreducible, either u is a split monomorphism or v a split epimorphism. Thus, either X or E_X is projective.

Similarly we obtain that $\underline{g} = 0$ if and only if Z or E_X is projective since g is irreducible as well.

Ad (2). Since X is neither projective nor injective, X is a node if and only if E_X is projective. The result now follows from part (1). □

We also use that a node cannot be a direct summand of the middle term in an almost split sequence.

Lemma 3.5. *Suppose given an almost split sequence $0 \to X \xrightarrow{f} E_X \xrightarrow{g} Z \to 0$ in $\operatorname{mod} A$.*

The middle term E_X does not have a node as a direct summand.

Proof. Assume that $\iota : N \hookrightarrow E_X$ is is the embedding of a direct summand such that N is a node. We have an almost split sequence

$$0 \to N \xrightarrow{s} E_N \xrightarrow{t} \tau^{-1}(N) \to 0$$

starting in N with E_N projective. The morphism $\iota\, g : N \to Z$ factors through s via a morphism $u : E_N \to Z$, that is $\iota\, g = s\, u$. Since $\iota\, g$ and s are irreducible, we obtain that u is a split monomorphism. Thus, E_N is a projective direct summand of Z. A contradiction. □

In [4, Proposition 3.5], Auslander and Reiten provide the following result for the case of short exact sequences using functor categories. This was later generalized to a larger class of short exact sequences in [33, Theorem 1.7] by Martínez-Villa.

Theorem 3.6 (Auslander, Reiten)**.** *Let $\alpha : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$ be a stable equivalence.*

Let $0 \to X \xrightarrow{f} Y \oplus P \xrightarrow{g} Z \to 0$ be a short exact sequence in $\mathrm{mod}\, A$ without split summands such that X is indecomposable, $P \in \mathrm{proj}\, A$ and Y has no projective direct summand.

If X is not a node and not projective, there exists a short exact sequence

$$0 \to \alpha(X) \xrightarrow{\tilde{f}} \alpha(Y) \oplus \tilde{P} \xrightarrow{\tilde{g}} \alpha(Z) \to 0$$

in $\mathrm{mod}\, B$ with $\tilde{P} \in \mathrm{proj}\, B$ such that $\alpha(f) \simeq \tilde{f}$ and $\alpha(g) \simeq \tilde{g}$ in $\underline{\mathrm{mod}}\, B$.

We aim to prove a similar result for perfect exact sequences. However, our method follows an algorithmic approach. A perfect exact sequence will be linked to an associated almost split sequence by a series of intermediate perfect exact sequences. For this series to end, we additionally have to assume a finiteness condition on the morphisms in the perfect exact sequence.

3.1 Construction of perfect exact sequences

In this section, we give two methods to construct perfect exact sequences from existing ones. We start with a construction via pushout and pullback which will be used to merge a perfect exact sequence with an almost split sequence.

Lemma 3.7. *Suppose given a short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathrm{mod}\, A$.*

(1) Let $0 \to X \xrightarrow{u} U \xrightarrow{v} V \to 0$ be a short exact sequence in $\mathrm{mod}\, A$ such that $f = u\, \alpha$ via a morphism $\alpha : U \to Y$. Then there exists a short exact sequence such that the following

diagram commutes.

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & Z \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & U \arrow[r, "\begin{pmatrix} \alpha & v \end{pmatrix}"] & Y \oplus V \arrow[r, "\begin{pmatrix} g \\ \beta \end{pmatrix}"] & Z \arrow[r] & 0 \end{tikzcd}$$

If $0 \to X \xrightarrow{u} U \xrightarrow{v} V \to 0$ *and the upper row of this diagram are perfect exact, then so is the lower row.*

(2) Let $0 \to U \xrightarrow{u} V \xrightarrow{v} Z \to 0$ *be a short exact sequence in* $\operatorname{mod} A$ *such that* $g = \alpha\, v$ *via a morphism* $\alpha : Y \to V$*. Then there exists a short exact sequence such that the following diagram commutes.*

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\begin{pmatrix} \beta & f \end{pmatrix}"] \arrow[d, equal] & U \oplus Y \arrow[r, "\begin{pmatrix} u \\ \alpha \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & V \arrow[r] \arrow[d, "v"] & 0 \\ 0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0 \end{tikzcd}$$

If $0 \to U \xrightarrow{u} V \xrightarrow{v} Z \to 0$ *and the lower row of this diagram are perfect exact, then so is the upper row.*

Proof. Ad (1). For now, we only show the existence of the short exact sequence. We verify that the following is a pushout-square.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] \\ U \arrow[r, "\begin{pmatrix} \alpha & v \end{pmatrix}"] & Y \oplus V \end{tikzcd}$$

By assumption, this diagram commutes. Suppose given $T \in \operatorname{mod} A$ together with morphism $t_1 : Y \to T$ and $t_2 : U \to T$ such that $f\, t_1 = u\, t_2$. We construct $\varphi : Y \oplus V \to T$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] \arrow[ddr, bend left, "t_1"] & \\ U \arrow[r, "\begin{pmatrix} \alpha & v \end{pmatrix}"] \arrow[drr, bend right, "t_2"'] & Y \oplus V \arrow[dr, dashed, "\varphi"] & \\ & & T \end{tikzcd}$$

We have $u(t_2 - \alpha\, t_1) = u\, t_2 - f\, t_1 = 0$. Hence, there exists a unique $\beta : V \to T$ such that $v\, \beta = t_2 - \alpha\, t_1$.

$$\begin{tikzcd}
X \arrow[r, "u"] & U \arrow[r, "v"] \arrow[d, "t_2 - \alpha\, t_1"] & V \arrow[dl, dashed, bend left, "\beta"] \\
 & T &
\end{tikzcd}$$

This yields a unique $\varphi = \begin{pmatrix} t_1 \\ \beta \end{pmatrix}$ such that the diagram above commutes. In conclusion, letting $T := Z$, the pushout-square induces the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U \arrow[r, "\begin{pmatrix} \alpha & v \end{pmatrix}"] & Y \oplus V \arrow[r, "\begin{pmatrix} g \\ \beta \end{pmatrix}"] & Z \arrow[r] & 0
\end{tikzcd}$$

Ad (2). For now, we only show the existence of the short exact sequence. We verify that the following is a pullback-square.

$$\begin{tikzcd}
U \oplus Y \arrow[r, "\begin{pmatrix} u \\ \alpha \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & V \arrow[d, "v"] \\
Y \arrow[r, "g"] & Z
\end{tikzcd}$$

By assumption, this diagram commutes. Suppose given $T \in \operatorname{mod} A$ together with morphism $t_1 : T \to V$ and $t_2 : T \to Y$ such that $t_1\, v = t_2\, g$. We construct $\varphi : T \to U \oplus Y$ such that the following diagram commutes.

$$\begin{tikzcd}
T \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "t_1"] \arrow[ddr, bend right, "t_2"'] & & \\
 & U \oplus Y \arrow[r, "\begin{pmatrix} u \\ \alpha \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & V \arrow[d, "v"] \\
 & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

We have $(t_1 - t_2\alpha)v = t_1 v - t_2 g = 0$. Hence, there exists a unique $\beta : T \to U$ such that $\beta\, u = t_1 - t_2\alpha$.

$$\begin{tikzcd}
U \arrow[r, "u"] & V \arrow[r, "v"] & Z \\
 & T \arrow[u, "t_1 - t_2\alpha"'] \arrow[ul, dashed, bend left, "\beta"] &
\end{tikzcd}$$

This yields a unique $\varphi = \begin{pmatrix} \beta & t_2 \end{pmatrix}$ such that the diagram above commutes. In conclusion, letting

$T := X$, the pullback-square induces the following commutative diagram with exact rows.

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\begin{pmatrix}\beta & f\end{pmatrix}"] \arrow[d, equal] & U \oplus Y \arrow[r, "\begin{pmatrix}u \\ \alpha\end{pmatrix}"] \arrow[d, "\begin{pmatrix}0 \\ 1\end{pmatrix}"] & V \arrow[r] \arrow[d, "v"] & 0 \\ 0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0 \end{tikzcd}$$

Perfect exact sequences. Suppose that $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is a perfect exact sequence. That is, $0 \to Z^* \xrightarrow{g^*} Y \xrightarrow{f^*} X^* \to 0$ is exact in A-mod. If additionally, $0 \to V^* \xrightarrow{v^*} U^* \xrightarrow{u^*} X^* \to 0$ is exact, we can apply (2) for left A-modules to obtain the following exact sequence.

$$0 \to Z^* \xrightarrow{\begin{pmatrix}\beta^* & g^*\end{pmatrix}} V^* \oplus Y^* \xrightarrow{\begin{pmatrix}v^* \\ \alpha^*\end{pmatrix}} U^* \to 0$$

Hence, $0 \to U \xrightarrow{\begin{pmatrix}\alpha & v\end{pmatrix}} Y \oplus V \xrightarrow{\begin{pmatrix}g \\ \beta\end{pmatrix}} Z \to 0$ is perfect exact. Similarly, if $0 \to U \xrightarrow{u} V \xrightarrow{v} Z \to 0$ is perfect exact, applying part (1) for left A-modules yields the following perfect exact sequence.

$$0 \to X \xrightarrow{\begin{pmatrix}\beta & f\end{pmatrix}} U \oplus Y \xrightarrow{\begin{pmatrix}u \\ \alpha\end{pmatrix}} V \to 0$$

This concludes the proof. □

The next construction via the snake lemma will be used to reverse the process of the previous lemma.

Lemma 3.8. *The following holds.*

(1) Suppose given two short exact sequences in $\operatorname{mod} A$ *of the following form.*

$$0 \to X \xrightarrow{\begin{pmatrix}s & \iota\end{pmatrix}} U \oplus P \xrightarrow{\begin{pmatrix}t \\ \pi\end{pmatrix}} V \to 0$$
$$0 \to U \xrightarrow{\begin{pmatrix}v & t\end{pmatrix}} Y \oplus V \xrightarrow{\begin{pmatrix}g \\ w\end{pmatrix}} Z \to 0$$

Then $0 \to X \xrightarrow{\begin{pmatrix}sv & \iota\end{pmatrix}} Y \oplus P \xrightarrow{\begin{pmatrix}g \\ -\pi w\end{pmatrix}} Z \to 0$ *is a short exact sequence. If the given two sequences are perfect exact, then so is this sequence.*

(2) Suppose given two short exact sequences in mod A *of the following form.*

$$0 \to U \xrightarrow{\left(\begin{smallmatrix} s & \iota \end{smallmatrix}\right)} V \oplus P \xrightarrow{\left(\begin{smallmatrix} t \\ \pi \end{smallmatrix}\right)} Z \to 0$$

$$0 \to X \xrightarrow{\left(\begin{smallmatrix} f & v \end{smallmatrix}\right)} Y \oplus U \xrightarrow{\left(\begin{smallmatrix} w \\ s \end{smallmatrix}\right)} V \to 0$$

Then $0 \to X \xrightarrow{\left(\begin{smallmatrix} f & -v\iota \end{smallmatrix}\right)} Y \oplus P \xrightarrow{\left(\begin{smallmatrix} wt \\ \pi \end{smallmatrix}\right)} Z \to 0$ *is a short exact sequence. If the given two sequences are perfect exact, then so is this sequence.*

Proof. Ad (1). For now, we only show the existence of the short exact sequence. Note that there is an isomorphism of sequences

$$\begin{tikzcd}
0 \arrow[r] & U \oplus P \arrow[r, "\left(\begin{smallmatrix} v & t & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)"] \arrow[d, equal] & Y \oplus V \oplus P \arrow[r, "\left(\begin{smallmatrix} g \\ w \\ 0 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \pi & 1 \end{smallmatrix}\right)", "\wr"'] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U \oplus P \arrow[r, "\left(\begin{smallmatrix} v & t & 0 \\ 0 & \pi & 1 \end{smallmatrix}\right)"] & Y \oplus V \oplus P \arrow[r, "\left(\begin{smallmatrix} g \\ w \\ -\pi w \end{smallmatrix}\right)"] & Z \arrow[r] & 0
\end{tikzcd}$$

so that the lower sequence is exact as well. Consider the following commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X \arrow[d, "\left(\begin{smallmatrix} s & \iota \end{smallmatrix}\right)"] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & U \oplus P \arrow[r, equal] \arrow[d, "\left(\begin{smallmatrix} v & t & 0 \\ 0 & \pi & 1 \end{smallmatrix}\right)"] & U \oplus P \arrow[r] \arrow[d, "\left(\begin{smallmatrix} t \\ \pi \end{smallmatrix}\right)"] & 0 \\
0 \arrow[r] & Y \oplus P \arrow[r, "\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)"] \arrow[d, equal] & Y \oplus V \oplus P \arrow[r, "\left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} g \\ w \\ -\pi w \end{smallmatrix}\right)"] & V \arrow[r] \arrow[d] & 0 \\
 & Y \oplus P \arrow[r, "\left(\begin{smallmatrix} g \\ -\pi w \end{smallmatrix}\right)"] & Z \arrow[r] & 0 &
\end{tikzcd}$$

The snake lemma yields a short exact sequence

$$0 \to X \xrightarrow{\left(\begin{smallmatrix} sv & \iota \end{smallmatrix}\right)} Y \oplus P \xrightarrow{\left(\begin{smallmatrix} g \\ -\pi w \end{smallmatrix}\right)} Z \to 0.$$

Ad (2). For now, we only show the existence of the short exact sequence. Note that there is an isomorphism of sequences

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\begin{pmatrix} f & v & 0 \end{pmatrix}"] \arrow[d, equal] & Y \oplus U \oplus P \arrow[r, "\begin{pmatrix} w & 0 \\ s & 0 \\ 0 & 1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\iota \\ 0 & 0 & 1 \end{pmatrix}", "\wr"'] & V \oplus P \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r, "\begin{pmatrix} f & v & -v\iota \end{pmatrix}"] & Y \oplus U \oplus P \arrow[r, "\begin{pmatrix} w & 0 \\ s & \iota \\ 0 & 1 \end{pmatrix}"] & V \oplus P \arrow[r] & 0
\end{tikzcd}$$

so that the lower sequence is exact as well. Consider the following commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & X \arrow[r, "\begin{pmatrix} f & -v\iota \end{pmatrix}"] \arrow[d, "\begin{pmatrix} f & v & -v\iota \end{pmatrix}"] & Y \oplus P \arrow[d, equal] & \\
0 \arrow[r] & U \arrow[r, "\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} s & \iota \end{pmatrix}"] & Y \oplus U \oplus P \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} w & 0 \\ s & \iota \\ 0 & 1 \end{pmatrix}"] & Y \oplus P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V \oplus P \arrow[r, equal] \arrow[d, "\begin{pmatrix} t \\ \pi \end{pmatrix}"] & V \oplus P \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & Z \arrow[r] & 0 \arrow[r] & 0 &
\end{tikzcd}$$

The snake lemma yields a short exact sequence

$$0 \to X \xrightarrow{\begin{pmatrix} f & -v\iota \end{pmatrix}} Y \oplus P \xrightarrow{\begin{pmatrix} wt \\ \pi \end{pmatrix}} Z \to 0.$$

Perfect exact sequences. Suppose that in part (1) the sequences

$$0 \to V^* \xrightarrow{\begin{pmatrix} t^* & \pi^* \end{pmatrix}} U^* \oplus P^* \xrightarrow{\begin{pmatrix} s^* \\ \iota^* \end{pmatrix}} X^* \to 0$$

$$0 \to Z^* \xrightarrow{\begin{pmatrix} g^* & w^* \end{pmatrix}} Y^* \oplus V^* \xrightarrow{\begin{pmatrix} v^* \\ t^* \end{pmatrix}} U^* \to 0$$

are exact. We can apply part (2) for left A-modules to obtain the short exact sequence

$$0 \to Z^* \xrightarrow{\begin{pmatrix} g^* & -w^*\pi^* \end{pmatrix}} Y^* \oplus P^* \xrightarrow{\begin{pmatrix} v^*s^* \\ \iota^* \end{pmatrix}} X^* \to 0.$$

Hence, the sequence $0 \to X \to Y \oplus P \to Z \to 0$ of part (1) is perfect exact.

Now, suppose that in part (2) the sequences

$$0 \to Z^* \xrightarrow{\begin{pmatrix} t^* & \pi^* \end{pmatrix}} V^* \oplus P^* \xrightarrow{\begin{pmatrix} s^* \\ \iota^* \end{pmatrix}} U^* \to 0$$

$$0 \to V^* \xrightarrow{\begin{pmatrix} w^* & s^* \end{pmatrix}} Y^* \oplus U^* \xrightarrow{\begin{pmatrix} f^* \\ v^* \end{pmatrix}} X^* \to 0$$

are exact. We can apply part (1) for left A-modules to obtain the short exact sequence

$$0 \to Z^* \xrightarrow{\begin{pmatrix} t^*w^* & \pi^* \end{pmatrix}} Y^* \oplus P^* \xrightarrow{\begin{pmatrix} f^* \\ -\iota^*v^* \end{pmatrix}} X^* \to 0.$$

Hence, the sequence $0 \to X \to Y \oplus P \to Z \to 0$ of part (2) is perfect exact. □

3.2 Perfect exact sequences and almost split sequences

We aim to show that certain perfect exact sequences are preserved by stable equivalences $\underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$. We proceed as follows.

First, we construct a chain of perfect exact sequences $\eta_0 \to \eta_1 \to \cdots \to \eta_l$ in $\mathrm{mod}\, A$ such that η_l is a direct sum of almost split sequences. By remembering the steps taken during this construction, we can reconstruct a perfect exact sequence in $\mathrm{mod}\, B$ corresponding to the original perfect exact sequence η_0. This is done by using almost split sequences during each step of the construction, which are preserved by a stable equivalence between algebras without nodes. An example of the construction done in this section is given in Example 7.5.

In order for this chain to be finite, we need to assume some condition on the morphisms in the perfect exact sequence. This condition is satisfied for all morphisms if A is of finite representation type.

In case that the starting term of the perfect exact sequence is not indecomposable, we need the following remark.

Remark 3.9. Suppose given $X \in \mathrm{mod}\, A$ without injective direct summands. Let $X = \bigoplus_i X_i$ be the decomposition of X into indecomposable direct summands.

We denote the direct sum of all almost split sequences starting in the X_i as follows.

$$0 \to X \xrightarrow{s} E_X \xrightarrow{t} T(X) \to 0$$

In particular, $E_X = \bigoplus_i E_{X_i}$ and $T(X) = \bigoplus_i \tau^{-1}(X_i)$.

Recall that almost split sequences with X not projective are perfect exact; cf. Example 2.11.(5). We use Lemma 3.7 to combine a perfect exact sequence with such almost split sequences.

Lemma 3.10. *Suppose given a perfect exact sequence* $\eta : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *in* $\operatorname{mod} A$ *without split summands.*

Recall from Remark 3.9 the sequence $0 \to X \xrightarrow{s} E_X \xrightarrow{t} T(X) \to 0$. *Then there exists a perfect exact sequence*

$$\tilde{\eta} : 0 \to E_X \xrightarrow{\begin{pmatrix} v & t \end{pmatrix}} Y \oplus T(X) \xrightarrow{\begin{pmatrix} g \\ w \end{pmatrix}} Z \to 0$$

in $\operatorname{mod} A$ *such that* $f = s\,v$. *We often denote this sequence by* $\tilde{\eta}$.

Proof. By assumption, X has neither injective nor projective direct summands, otherwise the given exact sequence would have a split direct summand. Hence, there exists a perfect exact sequence that is the direct sum of almost split sequences starting in direct summands of X; cf. Remark 3.9.

$$0 \to X \xrightarrow{s} E_X \xrightarrow{t} T(X) \to 0$$

Moreover, f is not a split morphism. Therefore, f factors through s via a morphism $E_X \xrightarrow{v} Y$, that is $f = s\,v$.

By Lemma 3.7.(1), we obtain that $0 \to E_X \xrightarrow{\begin{pmatrix} v & t \end{pmatrix}} Y \oplus T(X) \xrightarrow{\begin{pmatrix} g \\ w \end{pmatrix}} Z \to 0$ is a perfect exact sequence with some morphism $w : T(X) \to Z$. □

Construction 3.11. Suppose given a perfect exact sequence $\eta_0 : 0 \to X_0 \xrightarrow{f_0} Y_0 \xrightarrow{g_0} Z_0 \to 0$ in $\operatorname{mod} A$ without split summands.

We construct perfect exact sequences η_k recursively. Let $k \geqslant 0$ such that η_k has no split summands. Recall the perfect exact sequence $\tilde{\eta}_k$ from Lemma 3.10 and the morphism of short exact sequences from Lemma 3.7. We define η_{k+1} to be the sequence obtained from $\tilde{\eta}_k$ by removing all split summands. Then η_{k+1} is a perfect exact sequence without split summands and a direct summand of $\tilde{\eta}_k$. In particular, we have the projection of the short exact sequence onto its direct summand. In general, the middle morphism is not the natural projection.

$$\begin{tikzcd}
\eta_k \arrow[d] & 0 \arrow[r] & X_k \arrow[r, "f_k"] \arrow[d, "s_k"] & Y_k \arrow[r, "g_k"] \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & Z_k \arrow[r] \arrow[d, equal] & 0 \\
\tilde{\eta}_k \arrow[d] & 0 \arrow[r] & E_{X_k} \arrow[r, "\begin{pmatrix} v_k & t_k \end{pmatrix}"] \arrow[d, two heads] & Y_k \oplus T(X_k) \arrow[r, "\begin{pmatrix} g_k \\ w_k \end{pmatrix}"] \arrow[d, two heads] & Z_k \arrow[r] \arrow[d, two heads, "\pi_k"] & 0 \\
\eta_{k+1} & 0 \arrow[r] & X_{k+1} \arrow[r, "f_{k+1}"] & Y_{k+1} \arrow[r, "g_{k+1}"] & Z_{k+1} \arrow[r] & 0
\end{tikzcd}$$

Note that for all k, the module Z_{k+1} is a direct summand of Z_k and consequently also of Z_0.

Furthermore, if η_k is an almost split sequence, $\tilde{\eta}_k$ will be a split sequence and therefore $\eta_{k+1} = 0$. In fact, in this case v_k is an isomorphism.

In general, this construction will not terminate, as the following example shows.

Example 3.12. Let k be an algebraically closed field. We consider the Kronecker algebra A given by the following quiver.

$$1 \rightrightarrows 2$$

Since A is hereditary, every short exact sequence starting in a non-projective module is perfect exact; cf. Example 2.11. We denote the indecomposable A-modules by their dimension vector. Let $n \in \mathbb{Z}_{\geqslant 0}$. The preprojective component of the Auslander-Reiten quiver of A consists of the modules with dimension vector $\binom{n}{n+1}$. The preinjective component consists of the modules with dimension vector $\binom{n+1}{n}$. Finally, a module with dimension vector $\binom{n}{n}$ is in the regular component. In particular, the indecomposable projective modules are given by $P_1 = \binom{1}{2}$ and $P_2 = \binom{0}{1}$. Recall that the almost split sequence starting in a module in the preprojective component is given by the following.

$$0 \to \binom{n}{n+1} \xrightarrow{s} \binom{n+1}{n+2} \oplus \binom{n+1}{n+2} \xrightarrow{t} \binom{n+2}{n+3} \to 0$$

We consider the following perfect exact sequence in $\operatorname{mod} A$. Note that the starting and middle term of this sequence are in different components of the Auslander-Reiten sequence.

$$\eta_0 : \quad 0 \to \binom{2}{3} \xrightarrow{f_0} \binom{3}{3} \xrightarrow{g_0} \binom{1}{0} \to 0$$

We show by induction, that the perfect exact sequence η_n of Construction 3.11 is given as follows.

$$\eta_n : \quad 0 \to \binom{n+2}{n+3}^{\oplus n+1} \to \binom{3}{3} \oplus \binom{n+3}{n+4}^{\oplus n} \to \binom{1}{0} \to 0$$

In fact, we have the following construction step from n to $n+1$ with notation as in Construction 3.11.

$$\begin{tikzcd}
\eta_n : & 0 \arrow[r] & \binom{n+2}{n+3}^{\oplus n+1} \arrow[r, "f_n"] \arrow[d, "s_n"] & \binom{3}{3} \oplus \binom{n+3}{n+4}^{\oplus n} \arrow[r] \arrow[d, hook, "(1\ 0)"] & \binom{1}{0} \arrow[r] \arrow[d, equal] & 0 \\
\tilde{\eta}_n : & 0 \arrow[r] & \binom{n+3}{n+4}^{\oplus 2n+2} \arrow[r, "(v_n\ t_n)"] \arrow[d, two heads] & \left(\binom{3}{3} \oplus \binom{n+3}{n+4}^{\oplus n}\right) \oplus \binom{n+4}{n+5}^{\oplus n+1} \arrow[r] \arrow[d, two heads] & \binom{1}{0} \arrow[r] \arrow[d, equal] & 0 \\
\eta_{n+1} : & 0 \arrow[r] & \binom{n+3}{n+4}^{\oplus n+2} \arrow[r, "f_{n+1}"] & \binom{3}{3} \oplus \binom{n+4}{n+5}^{\oplus n+1} \arrow[r] & \binom{1}{0} \arrow[r] & 0
\end{tikzcd}$$

For all $n \geqslant 0$, we have that f_{n+1} is not irreducible. Thus, the perfect exact sequence η_n cannot be almost split for any $n \geqslant 0$.

Now, consider the following perfect exact sequence.

$$\eta_0: \quad 0 \to \binom{2}{3} \xrightarrow{f_0} \binom{3}{4} \xrightarrow{g_0} \binom{1}{1} \to 0$$

This time, f_0 is an irreducible morphism in the preprojective component. However, the ending term is still in a different component of the Auslander-Reiten quiver. Similarly to the induction above, we can show that η_n of Construction 3.11 is given as follows.

$$\eta_n: \quad 0 \to \binom{n+2}{n+3} \to \binom{n+3}{n+4} \to \binom{1}{1} \to 0$$

Again, η_n is not an almost split sequence for any $n \geqslant 0$.

As seen above, we need a condition on both f_0 and g_0 for Construction 3.11 to terminate with an almost split sequence. This condition will be given via the radical of $\operatorname{mod} A$.

Definition 3.13. The *radical* $\operatorname{rad}(\operatorname{mod} A)$ of $\operatorname{mod} A$ has the same objects as $\operatorname{mod} A$ with morphisms $f \in \operatorname{rad}_A(X, Y)$ if gfh is not an isomorphism for all $g \in \operatorname{Hom}_A(Z, X)$, $h \in \operatorname{Hom}_A(Y, Z)$ and $Z \in \operatorname{mod} A$ indecomposable. Recursively, we can define $\operatorname{rad}^0(X, Y) := \operatorname{Hom}_A(X, Y)$ and $\operatorname{rad}_A^n(X, Y) := \{fg : f \in \operatorname{rad}_A(X, Z) \text{ and } g \in \operatorname{rad}_A^{n-1}(Z, Y) \text{ for a } Z \in \operatorname{mod} A\}$ for $n \in \mathbb{Z}_{\geqslant 1}$.

Let $f : X \to Y$ be a morphism in $\operatorname{mod} A$. Following [9], we say that f has *depth* $n \geqslant 0$, if $f \in \operatorname{rad}_A^n(X, Y)$, but $f \notin \operatorname{rad}^{n+1}(X, Y)$. In case that $f \in \operatorname{rad}_A^n(X, Y)$ for all $n \geqslant 0$, we set $\operatorname{depth} f = \infty$.

For more details on the radical see [6, Section V.7]. We list some properties that are important for our purposes.

Remark 3.14. Let $X, Y \in \operatorname{mod} A$.

(1) The n-th radical $\operatorname{rad}_A^n(X, Y)$ is a two-sided ideal for $n \in \mathbb{Z}_{\geqslant 0}$. Furthermore, we have

$$\operatorname{rad}_A^n(X, Y) \subseteq \operatorname{rad}_A^{n-1}(X, Y) \subseteq \cdots \subseteq \operatorname{rad}_A^2(X, Y) \subseteq \operatorname{rad}_A(X, Y).$$

(2) Suppose that $f : X \to Y$ has depth zero. Then there exists a $Z \in \operatorname{mod} A$ indecomposable and morphisms $g : Z \to X$, $h : Y \to Z$ such that gfh is an isomorphism. In particular, g and h split so that Z is a common direct summand of X and Y.

(3) Let $f : X \to Y$ be a morphism with $\operatorname{depth} f = n$. For all morphisms $g : X' \to X$ and $h : Y \to Y'$ in $\operatorname{mod} A$ we have $\operatorname{depth}(gfh) \geqslant n$ since the n-th radical is an ideal.

(4) Suppose that X or Y is indecomposable. An irreducible morphism $f : X \to Y$ has depth 1. If both X and Y are indecomposable, $f : X \to Y$ is irreducible if and only if $\operatorname{depth} f = 1$.

Furthermore, a direct sum of irreducible morphisms still has depth 1. In fact, every non-zero restriction to an indecomposable direct summand is irreducible and thus in the radical.

(5) If A is of finite representation type, every morphism in $\operatorname{mod} A$ has finite depth; cf. [6, Theorem 7.7].

We will assume that $f_0\, p$ has finite depth for every projection p onto an indecomposable direct summand. The next result shows that this property gets passed on to all f_k for $k \geqslant 0$.

Lemma 3.15. *Suppose given a perfect exact sequence* $\eta_0 : 0 \to X_0 \xrightarrow{f_0} Y_0 \xrightarrow{g_0} Z_0 \to 0$ *in* $\operatorname{mod} A$ *without split summands. We use the notation of Construction 3.11.*

Suppose that $f_0\, p_0$ *has finite depth for every projection* $p_0 : Y_0 \to Y_0'$ *onto an indecomposable direct summand* Y_0' *of* Y_0.

Then $f_k\, p_k$ *has finite depth for all* $k \geqslant 0$ *and every projection* $p_k : Y_k \to Y_k'$ *onto an indecomposable direct summand* Y_k' *of* Y_k.

Proof. We proceed by induction on k and show that $\operatorname{depth}(f_{k+1}\, p_{k+1})$ is finite for $k \geqslant 0$. We use the following expanded notation from Construction 3.11.

$$\begin{tikzcd}
X_k \arrow[r, "f_k"] \arrow[d, "s_k"] & Y_k \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] \\
E_{X_k} \arrow[r, "\begin{pmatrix} v_k & t_k \end{pmatrix}"] \arrow[d, two heads, "\rho_k"] & Y_k \oplus T(X_k) \arrow[d, two heads, "\begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix}"] \\
X_{k+1} \arrow[r, "f_{k+1}"] & Y_{k+1}
\end{tikzcd}$$

Recall that $0 \to X_k \xrightarrow{s_k} E_{X_k} \xrightarrow{t_k} T(X_k) \to 0$ is a direct sum of almost split sequences for all $k \geqslant 0$. Since $\begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix}$ is split, Y_{k+1}' is either an indecomposable direct summand of Y_k or an indecomposable direct summand of $T(X_k)$.

Suppose that Y_{k+1}' is a direct summand of $T(X_k)$. In this case, we have $\rho_k\, f_{k+1}\, p_{k+1} = t_k \psi_k p_{k+1}$. Using that $\psi_k p_{k+1}$ is a split epimorphism, $t_k \psi_k p_{k+1}$ is irreducible so that $\operatorname{depth}(t_k \psi_k p_{k+1}) = 1$. This implies $\operatorname{depth}(f_{k+1}\, p_{k+1}) \leqslant \operatorname{depth}(\rho_k\, f_{k+1}\, p_{k+1}) = \operatorname{depth}(t_k \psi_k p_{k+1}) = 1$.

Suppose that Y_{k+1}' is a direct summand of Y_k. In this case, we have $s_k\, \rho_k\, f_{k+1}\, p_{k+1} = f_k\, \varphi_k\, p_{k+1}$. Using that $\varphi_k p_{k+1}$ is a split epimorphism, we know that $\operatorname{depth}(f_k\, \varphi_k\, p_{k+1}) < \infty$ by induction hypothesis. This implies $\operatorname{depth}(f_{k+1}\, p_{k+1}) \leqslant \operatorname{depth}(s_k \rho_k\, f_{k+1}\, p_{k+1}) = \operatorname{depth}(f_k\, \varphi_k\, p_{k+1}) < \infty$. □

We aim to show that Construction 3.11 ends with an almost split sequence under the assumption that $f_0\,p$ and $g_0\,\pi$ have finite depth for every projection p and π onto an indecomposable direct summand. We use the assumption on f_0 to show that the middle morphism in the construction will eventually be an element of the radical. Together with the assumption on g_0 this guarantees that we arrive at a split sequence. Finally, we will use that η_l is an almost split sequence if and only if $\tilde{\eta}_l$ is a split sequence.

Lemma 3.16. *Suppose given a perfect exact sequence* $\eta_0 : 0 \to X_0 \xrightarrow{f_0} Y_0 \xrightarrow{g_0} Z_0 \to 0$ *in* $\operatorname{mod} A$ *without split summands. Suppose that* $f_0\,p$ *and* $g_0\,\pi$ *have finite depth for every projection* p *onto an indecomposable direct summand of* Y_0 *and every projection* π *onto an indecomposable direct summand of* Z_0.

Then there exists an $l \in \mathbb{Z}_{\geqslant 0}$ *such that* η_l *in Construction 3.11 is a direct sum of almost split sequences.*

Proof. We first prove that there exists an $l \in \mathbb{Z}_{\geqslant 0}$ such that $\tilde{\eta}_l$ is a split sequence. Construction 3.11 yields the following sequence of morphisms of short exact sequences. We aim to show that there is an $l \geqslant 0$ such that $g_0\,\pi_0 \cdots \pi_l = 0$.

$$
\begin{tikzcd}
\eta_0 \arrow[d] & 0 \arrow[r] & X_0 \arrow[r, "f_0"] \arrow[d, "s_0"] & Y_0 \arrow[r, "g_0"] \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & Z_0 \arrow[r] \arrow[d, equal] & 0 \\
\tilde{\eta}_0 \arrow[d] & 0 \arrow[r] & E_{X_0} \arrow[r, "\begin{pmatrix} v_0 & t_0 \end{pmatrix}"] \arrow[d, two heads] & Y_0 \oplus T(X_0) \arrow[r, "\begin{pmatrix} g_0 \\ w_0 \end{pmatrix}"] \arrow[d, two heads] & Z_0 \arrow[r] \arrow[d, two heads, "\pi_0"] & 0 \\
\eta_1 \arrow[d] & 0 \arrow[r] & X_1 \arrow[r, "f_1"] \arrow[d, "s_1"] & Y_1 \arrow[r, "g_1"] \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & Z_1 \arrow[r] \arrow[d, equal] & 0 \\
\tilde{\eta}_1 \arrow[d] & 0 \arrow[r] & E_{X_1} \arrow[r, "\begin{pmatrix} v_1 & t_1 \end{pmatrix}"] \arrow[d, two heads] & Y_1 \oplus T(X_1) \arrow[r, "\begin{pmatrix} g_1 \\ w_1 \end{pmatrix}"] \arrow[d, two heads] & Z_1 \arrow[r] \arrow[d, two heads, "\pi_1"] & 0 \\
\eta_2 \arrow[d] & 0 \arrow[r] & X_2 \arrow[r, "f_2"] \arrow[d] & Y_2 \arrow[r, "g_2"] \arrow[d] & Z_2 \arrow[r] \arrow[d, equal] & 0 \\
\vdots & & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

For $k \geqslant 0$, let $\varphi_k : Y_k \to Y_{k+1}$ be the morphism given by the sequence above. Assume that $g_0\,\pi_0 \cdots \pi_k = \varphi_0 \cdots \varphi_k\, g_{k+1} \neq 0$ for $k \geqslant 0$. In particular, φ_k is non-zero for all $k \geqslant 0$.

Assume that for all $N \geqslant 0$ there exists a $k \geqslant N$ such that $\varphi_N \cdots \varphi_k \in \operatorname{rad}(Y_N, Y_{k+1})$. Thus, for all $n \geqslant 0$ there exists a $k \geqslant 0$ such that $g_0\pi_0 \cdots \pi_k = \varphi_0 \cdots \varphi_k g_{k+1} \in \operatorname{rad}^n(Y_0, Z_{k+1})$. On the other hand, $g_0\pi_0 \cdots \pi_k$ is non-zero for all k and Z_0 has only finitely many indecomposable direct summands. Using that π_k is a split epimorphism, this implies that there must exist a $k' \geqslant 0$ such that $g_0\pi_0 \cdots \pi_k \simeq g_0\pi_0 \cdots \pi_{k'}$ for all $k \geqslant k'$. Thus, we have a projection π onto an indecomposable direct summand Z' of Z_0 such that $g_0\,\pi \in \mathrm{rad}^n(Y_0, Z')$ for all n. However, $g_0\,\pi$ has finite depth by assumption. *A contradiction.*

Therefore, there exists an $N \geqslant 0$ such that for all $k \geqslant N$ we have $\varphi_N \cdots \varphi_k \notin \mathrm{rad}(Y_N, Y_{k+1})$. By Lemma 3.15, there exists an $1 \leqslant m < \infty$ such that $\mathrm{depth}(f_N\,p) < m$ for all projections p of Y_N onto an indecomposable direct summand.

We know that the composite $\varphi := \varphi_N\varphi_{N+1}\cdots\varphi_{N+m-1}$ is neither zero, nor in the radical $\mathrm{rad}_A(Y_N, Y_{N+m})$. Thus, there exists an indecomposable non-zero module $M \in \mathrm{mod}\,A$ and morphisms $i : M \to Y_N$ and $p : Y_{N+m} \to M$ such that the composite

$$M \xrightarrow{i} Y_N \xrightarrow{\varphi} Y_{N+m} \xrightarrow{p} M$$

is an isomorphism. In particular, $\varphi\,p$ is split so that $\mathrm{depth}(f_N\,\varphi\,p) < m$. By commutativity of the diagram, we have that $f_N\,\varphi\,p$ factors through s_k for $N \leqslant k \leqslant (N+m-1)$. However, as a direct sum of irreducible morphism, s_k has depth 1 for $k \geqslant 0$. We obtain $\mathrm{depth}(f_N\,\varphi\,p) \geqslant m$, a contradiction.

In conclusion, there exists a minimal $l \geqslant 0$ such that $\varphi_0 \cdots \varphi_l\, g_{l+1} = g_0\,\pi_0 \cdots \pi_l$ is zero. Since η_0 is not split, the epimorphism g_0 is non-zero and we obtain $\pi_0 \cdots \pi_l = 0$. However, this is a surjection of Z onto its direct summand Z_{l+1}. This implies $Z_{l+1} = 0$ and thus $\eta_{l+1} = 0$ since η_{l+1} has no split summands. By construction, this means that $\tilde{\eta}_l$ is a split sequence. It remains to show, that η_l is an almost split sequence. Suppose that $\tilde{\eta}_l$ is split.

$$\tilde{\eta}_l : 0 \to E_{X_l} \xrightarrow{\begin{pmatrix} v_l & t_l \end{pmatrix}} Y_l \oplus T(X_l) \xrightarrow{\begin{pmatrix} g_l \\ w_l \end{pmatrix}} Z_l \to 0$$

We obtain that $E_{X_l} \oplus Z_l \simeq Y_l \oplus T(X_l)$. However, Z_l is not a direct summand of Y_l since η_l is not a split sequence by construction. Hence, Z_l is a direct summand of $T(X_l)$. Furthermore, $0 \to X_l \to E_{X_l} \to T(X_l) \to 0$ has no split direct summands as a direct sum of almost split sequences. Hence, E_{X_l} and $T(X_l)$ have no common direct summand.

In conclusion, this results in $Z_l \simeq T(X_l)$ and $Y_l \simeq E_{X_l}$ via some isomorphism $\psi : E_{X_l} \to Y_l$ with $\begin{pmatrix} \psi & 0 \end{pmatrix} \simeq \begin{pmatrix} v_l & t_l \end{pmatrix}$. Since $s_l \begin{pmatrix} v_l & t_l \end{pmatrix} = \begin{pmatrix} f_l & 0 \end{pmatrix}$, we obtain an isomorphism of short exact sequences.

$$\begin{tikzcd}
 & 0 \arrow[r] & X_l \arrow[r, "s_l"] \arrow[d, equal] & E_{X_l} \arrow[r, "t_l"] \arrow[d, "\psi", "\wr"'] & T(X) \arrow[r] \arrow[d, dashed, "\wr"'] & 0 \\
\eta_l : & 0 \arrow[r] & X_l \arrow[r, "f_l"] & Y_l \arrow[r, "g_l"] & Z_l \arrow[r] & 0
\end{tikzcd}$$

Hence, η_l is the direct sum of all almost split sequences starting in direct summands of X_l. $\square$

The following result is a reformulation of [4, Proposition 2.4] using Lemma 3.4.(2).

Proposition 3.17 (Auslander-Reiten). *Let* $\underline{\mathrm{mod}}\, A \xrightarrow{\alpha} \underline{\mathrm{mod}}\, B$ *be a stable equivalence.*

Let $0 \to X \xrightarrow{f} Y \oplus P \xrightarrow{g} Z \to 0$ *be an almost split sequence in* $\mathrm{mod}\, A$*, where* X *is not a node and not projective,* $P \in \mathrm{proj}\, A$ *and* Y *does not have projective direct summands.*

Then there exists an almost split sequence in $\mathrm{mod}\, B$

$$0 \to \alpha(X) \xrightarrow{\tilde{f}} \alpha(Y) \oplus \tilde{P} \xrightarrow{\tilde{g}} \alpha(Z) \to 0$$

where $\tilde{P}$ *is projective such that* $\underline{\tilde{f}} \simeq \alpha(\underline{f})$ *and* $\underline{\tilde{g}} \simeq \alpha(\underline{g})$ *in* $\underline{\mathrm{mod}}\, B$*.*

Inductively, we aim to construct perfect exact sequences in $\mathrm{mod}\, B$ corresponding to η_k for $0 \leqslant k \leqslant l$. The next lemma will be used as the induction step.

Lemma 3.18. *Let* $\underline{\mathrm{mod}}\, A \xrightarrow{\alpha} \underline{\mathrm{mod}}\, B$ *be a stable equivalence.*

Suppose given a perfect exact sequence $\eta : 0 \to X \xrightarrow{f} Y \oplus P \xrightarrow{g} Z \to 0$ *in* $\mathrm{mod}\, A$ *without split summands, where* $P \in \mathrm{proj}\, A$ *and* Y *has no projective summand. Suppose that* X *has no node as a direct summand.*

Assume furthermore, that there exists a $\tilde{Q} \in \mathrm{proj}\, B$ *such that*

$$0 \to \alpha(E_X) \xrightarrow{(\tilde{v}\ \tilde{t})} (\alpha(Y) \oplus \tilde{Q}) \oplus \alpha(T(X)) \xrightarrow{\begin{pmatrix} \tilde{g} \\ \tilde{w} \end{pmatrix}} \alpha(Z) \to 0$$

is a perfect exact sequence in $\mathrm{mod}\, B$ *where* $\underline{\tilde{v}} \simeq \alpha(\underline{v})$ *and* $\underline{\tilde{g}} \simeq \alpha(\underline{g})$*; cf. Lemma 3.10.*

Then there exist $\tilde{P} \in \mathrm{proj}\, B$ *and a perfect exact sequence in* $\mathrm{mod}\, B$

$$0 \to \alpha(X) \xrightarrow{\tilde{f}} \alpha(Y) \oplus \tilde{P} \xrightarrow{\tilde{g}} \alpha(Z) \to 0$$

with $\underline{\tilde{f}} \simeq \alpha(\underline{f})$ *and* $\underline{\tilde{g}} \simeq \alpha(\underline{g})$ *in* $\underline{\mathrm{mod}}\, B$*.*

Proof. Note that X has no projective direct summand, since the given perfect exact sequence has no split direct summands. Recall that in this case, we have the perfect exact sequence $0 \to X \xrightarrow{s} E_X \xrightarrow{t} T(X) \to 0$; cf. Remark 3.9. By Proposition 3.17, there exists an $\tilde{R} \in \mathrm{proj}\, B$ such that

$$0 \to \alpha(X) \xrightarrow{(\tilde{s}\ \tilde{\iota})} \alpha(E_X) \oplus \tilde{R} \xrightarrow{\begin{pmatrix} \tilde{t} \\ \tilde{\pi} \end{pmatrix}} \alpha(T(X)) \to 0$$

is a perfect exact sequence in $\mathrm{mod}\, B$ with $\underline{\tilde{s}} = \alpha(\underline{s})$. By assumption, we have the following perfect exact sequence in $\mathrm{mod}\, B$ with $\underline{\tilde{v}} = \alpha(\underline{v})$ and $\underline{\tilde{g}} = \alpha(\underline{g})$.

$$0 \to \alpha(E_X) \xrightarrow{(\tilde{v}\ \tilde{t})} (\alpha(Y) \oplus \tilde{Q}) \oplus \alpha(T(X)) \xrightarrow{\begin{pmatrix} \tilde{g} \\ \tilde{w} \end{pmatrix}} \alpha(Z) \to 0$$

Lemma 3.8.(1) now provides the following perfect exact sequence.

$$0 \to \alpha(X) \xrightarrow{(\tilde{s}\tilde{v}\ \tilde{\iota})} (\alpha(Y) \oplus \tilde{Q}) \oplus \tilde{R} \xrightarrow{(\tilde{g}\ -\tilde{\pi}\tilde{w})} \alpha(Z) \to 0$$

We have $\tilde{s}\,\tilde{v} \simeq \alpha(\underline{s})\alpha(\underline{v}) = \alpha(\underline{s\,v}) = \alpha(\underline{f})$ and $\tilde{g} \simeq \alpha(\underline{g})$ in $\underline{\mathrm{mod}}\,B$; cf. Lemma 3.10. □

We are now ready to prove the main result of this chapter.

Theorem 3.19. *Let $\alpha : \underline{\mathrm{mod}}\,A \to \underline{\mathrm{mod}}\,B$ be a stable equivalence.*

Suppose given a perfect exact sequence $0 \to X \xrightarrow{f} Y \oplus P \xrightarrow{g} Z \to 0$ without split summands where X has no node as a direct summand, $P \in \mathrm{proj}\,A$ and Y has no projective direct summand.

Suppose that $f\,p$ and $g\,\pi$ have finite depth for every projection p onto an indecomposable direct summand of Y and every projection π onto an indecomposable direct summand of Z. Then there exists a perfect exact sequence

$$0 \to \alpha(X) \xrightarrow{\tilde{f}} \alpha(Y) \oplus \tilde{P} \xrightarrow{\tilde{g}} \alpha(Z) \to 0$$

in $\mathrm{mod}\,B$ with $\tilde{P} \in \mathrm{proj}\,B$ such that $\underline{\tilde{f}} \simeq \alpha(\underline{f})$ and $\underline{\tilde{g}} \simeq \alpha(\underline{g})$ in $\underline{\mathrm{mod}}\,B$.

Proof. We denote the given perfect exact sequence by η_0 and use the notation of Construction 3.11. By Lemma 3.16 there exists an $l \in \mathbb{Z}_{\geqslant 0}$ and perfect exact sequences η_k for $1 \leqslant k \leqslant l$ such that η_l is a direct sum of almost split sequences. Furthermore, η_{k+1} is a direct summand of the sequence $\tilde{\eta}_k$.

$$\tilde{\eta}_k : 0 \to E_{X_k} \to Y_k \oplus T(X_k) \to Z \to 0$$

By assumption, X_0 has no node as a direct summand. Let $k \geqslant 1$ and assume that X_k has a node as a direct summand. Since X_k is a direct summand of $E_{X_{k-1}}$, the node is also a direct summand of $E_{X_{k-1}}$. However, by Lemma 3.5 the middle term of an almost split sequence has no nodes as direct summand. A contradiction. Thus X_k has no node as a direct summand for all $0 \leqslant k \leqslant l$.

We verify by induction on $0 \leqslant k \leqslant l$ that the assertion holds for η_k.

Let $k = l$. Then the given perfect exact sequence is a direct sum of almost split sequences and the claim holds by Proposition 3.17.

Let $0 \leqslant k < l$. Suppose that the assertion holds for η_{k+1}. We know that $\tilde{\eta}_k$ is the direct sum of η_{k+1} and a split exact sequence. Therefore, α preserves the perfect exact sequence $\tilde{\eta}_k$ as well. As a consequence, we can apply Lemma 3.18 to obtain that α preserves the perfect exact sequence η_k and its morphisms.

In conclusion, the assertion holds for all perfect exact sequences η_k with $0 \leqslant k \leqslant l$. In particular, it holds for η_0. □

Example in Chapter 7. In Example 7.5 we give an explicit example for the construction used in the proof of the theorem.

As seen in Example 3.12, the assumption on the depth of f and g is needed for our proof of this theorem. However, it seems unclear whether this assumption is really necessary for the result to hold.

With regard to Definition 3.1, we have the following corollary using Remark 3.14.

Corollary 3.20. *Let A and B be finite dimensional algebras without nodes. Suppose that A and B have finite representation type.*

Then every stable equivalence $\alpha : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$ and its quasi-inverse preserve perfect exact sequences.

The constructions of this chapter can also be used to characterize perfect exact sequences.

Remark 3.21. Suppose that A and B have finite representation type. We use the notation of Construction 3.11 and Lemma 3.16. Let η_0 be a short exact sequence in $\mathrm{mod}\, A$ without split summands.

$$\eta_0 : \quad 0 \to X_0 \xrightarrow{f_0} Y_0 \xrightarrow{g_0} Z_0 \to 0$$

The short exact sequence η_0 is perfect exact if and only if X_k has no projective direct summand for all $0 \leqslant k \leqslant l$.

In fact, we have seen that η_k is a perfect exact sequence without split summands if η_0 is perfect exact. Thus, X_k cannot have a projective direct summand. Conversely, suppose that X_k has no projective direct summand for $0 \leqslant k \leqslant l$. Then η_l is a perfect exact sequence; cf. Example 2.11. Let $0 \leqslant k < l$. Inductively, we may assume that η_{k+1} is a perfect exact sequence. Now, applying Lemma 3.8.(1) to the following two perfect exact sequences shows that η_0 is perfect exact.

$$\begin{array}{llllllllll} & 0 \longrightarrow X_k \xrightarrow{\;s_k\;} E_{X_k} \xrightarrow{\;t_k\;} T(X_k) \longrightarrow 0 \\ \tilde{\eta}_k : & 0 \longrightarrow E_{X_k} \xrightarrow{\begin{pmatrix} v_k & t_k \end{pmatrix}} Y_k \oplus T(X_k) \xrightarrow{\begin{pmatrix} g_k \\ w_k \end{pmatrix}} Z_k \longrightarrow 0 \end{array}$$

Here we used that η_{k+1} is a direct sum of $\tilde{\eta}_k$ and a split exact sequence.

Chapter 4

Triangulated subcategories inside the homotopy category

Let k be a field. Let A and B be finite dimensional k-algebras without semisimple direct summands. In general, the stable module category $\underline{\mathrm{mod}}\, A$ is not triangulated. However, we have seen that $\underline{\mathrm{mod}}\, A$ is equivalent to the category $\mathcal{L}_A$ which is a full subcategory of the triangulated category $\mathcal{K}(\mathrm{proj}\, A)$; cf. Theorem 2.6. Note that for arbitrary algebras $\mathcal{L}_A$ is not even closed under taking shifts. This can be seen, for instance, in the setting of Example 2.2.

In this chapter, we discuss several triangulated categories that are related to $\mathcal{L}_A$. In the first two sections, we characterize the smallest triangulated subcategory of $\mathcal{K}(\mathrm{proj}\, A)$ that contains $\mathcal{L}_A$ and is closed under isomorphisms. Moreover, we discuss its Grothendieck group. In Section 4.3, we extend the category of Section 4.1, to a triangulated category closed under an equivalence induced by the Nakayama functor. Afterwards, we consider the triangulated category of stable Gorenstein-projective modules. This category is equivalent to the category of totally acyclic modules $\mathcal{K}_{\mathrm{tac}}(\mathrm{proj}\, A)$, which is the largest subcategory of $\mathcal{L}_A$ that is triangulated. In a final section, we specialize to self-injective algebras. In this case, $\underline{\mathrm{mod}}\, A$ is already triangulated and all these categories coincide. We will see that $\mathcal{L}_A$ is closed under taking shifts inside $\mathcal{K}(\mathrm{proj}\, A)$ if and only if A is self-injective.

Two extended examples for the categories discussed in this chapter are given in Section 7.2 and Section 7.3. Occasionally, we refer to specific parts of these examples.

Recall that $\mathcal{P}_A$ denotes the category of projective-injective A-modules. The subcategory of strongly projective-injective A-modules is denoted by $\mathrm{stp}\, A$. We begin by defining the following subcategories with a left bound on cohomology and a right bound on homology.

Definition 4.1. We denote by $\mathcal{H}(\mathrm{proj}\, A)$ the full subcategory of $\mathcal{K}(\mathrm{proj}\, A)$ consisting of all complexes $F^\bullet \in \mathcal{K}(\mathrm{proj}\, A)$ such that there exist $l, r \in \mathbb{Z}$ with $\mathrm{H}^{<l}(F^\bullet) = 0$ and $\mathrm{H}_{\geqslant r}(F^*_\bullet) = 0$.

We denote by $\mathcal{H}_{\mathcal{P}}(\mathrm{proj}\, A)$ the full subcategory of $\mathcal{H}(\mathrm{proj}\, A)$ consisting of all complexes in ${}^\perp\mathcal{K}^b(\mathcal{P}_A) = \{F^\bullet \in \mathcal{K}(\mathrm{proj}\, A) : \mathrm{Hom}_{\mathcal{K}(\mathrm{proj}\, A)}(F^\bullet, Z^\bullet) = 0 \text{ for all } Z^\bullet \in \mathcal{K}^b(\mathcal{P}_A)\}$.

We denote by $\mathcal{H}_{\mathrm{stp}}(\mathrm{proj}\, A)$ the full subcategory of $\mathcal{H}(\mathrm{proj}\, A)$ consisting of all complexes in ${}^\perp\mathcal{K}^b(\mathrm{stp}\, A) = \{F^\bullet \in \mathcal{K}(\mathrm{proj}\, A) : \mathrm{Hom}_{\mathcal{K}(\mathrm{proj}\, A)}(F^\bullet, Z^\bullet) = 0 \text{ for all } Z^\bullet \in \mathcal{K}^b(\mathrm{stp}\, A)\}$.

Remark 4.2. (1) Note that $\mathcal{H}(\operatorname{proj} A)$ is a triangulated subcategory of $\mathcal{K}(\operatorname{proj} A)$. By construction, $\mathcal{H}(\operatorname{proj} A)$ is closed under taking shifts. By Lemma 1.9, we see that for a morphism $f : F^\bullet \to G^\bullet$ in $\mathcal{H}(\operatorname{proj} A)$, the distinguished triangle $F^\bullet \to G^\bullet \to C(f)^\bullet \to$ lies in $\mathcal{H}(\operatorname{proj} A)$.

By Lemma 1.12, we obtain that $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ and $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ are triangulated subcategories of $\mathcal{H}(\operatorname{proj} A)$. Furthermore, $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ and $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ are closed under isomorphisms in $\mathcal{K}(\operatorname{proj} A)$.

(2) We have a chain of subcategories $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) \subseteq \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A) \subseteq \mathcal{H}(\operatorname{proj} A) \subseteq \mathcal{K}(\operatorname{proj} A)$. Furthermore, $\mathcal{L}_A \subseteq \mathcal{H}(\operatorname{proj} A)$ letting $l = 0$ and $r = 0$ in the definition of $\mathcal{H}(\operatorname{proj} A)$. In this sense, the boundary conditions of $\mathcal{H}(\operatorname{proj} A)$ can be seen as a weaker version of those in $\mathcal{L}_A$. They will be used in Lemma 4.7. In particular, the smallest triangulated subcategory of $\mathcal{K}(\operatorname{proj} A)$ that contains $\mathcal{L}_A$ must be contained in $\mathcal{H}(\operatorname{proj} A)$.

(3) In general, complexes in $\mathcal{H}(\operatorname{proj} A)$ are neither left bounded nor right bounded. However, we have $\mathcal{H}(\operatorname{proj} A) \simeq \mathcal{K}^b(\operatorname{proj} A)$ if and only if $\operatorname{gldim} A < \infty$.

In fact, every complex in $\mathcal{H}(\operatorname{proj} A)$ can be truncated on the right to obtain a projective resolution in $\operatorname{mod} A$. Thus, the complex must split eventually, if it is unbounded on the left and $\operatorname{gldim} A < \infty$. Similarly, every complex in $\mathcal{H}(\operatorname{proj} A)$ can be truncated on the left to obtain a projective resolution in A-mod after applying $(-)^*$. Moreover, every projective resolution of a left or right A-module occurs in this way.

(4) Note that $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) = \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A) = \mathcal{H}(\operatorname{proj} A)$ if A has no projective-injective modules. In particular, we have that $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) = \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A) \simeq \mathcal{K}^b(\operatorname{proj} A)$ is the bounded derived category of A if additionally $\operatorname{gldim} A < \infty$. The same holds for $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ and $\mathcal{H}(\operatorname{proj} A)$ if A has no strongly projective-injective modules.

The categories discussed in this chapter can be visualized as follows. Note that the inclusion $\mathcal{L}_A \hookrightarrow \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ will be verified in Lemma 4.4. In general, this chain of subcategories has a proper inclusion at every position; see also Example 7.9. However, we will show later in Theorem 4.45 that $\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A) = \mathcal{L}_A = \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) = \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ if and only if A is self-injective.

$$\begin{tikzcd}
\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A) \arrow[r, hook] & \mathcal{L}_A \arrow[r, hook] & \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) \arrow[r, hook] & \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A) \arrow[r, hook] & \mathcal{H}(\operatorname{proj} A) \arrow[r, hook] & \mathcal{K}(\operatorname{proj} A) \\
\underline{\operatorname{Gproj}} A \arrow[u, "\wr"] \arrow[r, hook] & \underline{\operatorname{mod}} A \arrow[u, "\wr"', "\mathcal{F}"]
\end{tikzcd}$$

In general, with the exception of $\mathcal{L}_A$, none of these categories are preserved by a stable equivalence; cf. Example 7.15. We will see in Chapter 5 that the situation is different for stable equivalences of Morita type.

For now, we discuss $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ in more detail in the first section of this chapter. Later, in Section 4.3, we return to the category $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$.

4.1 A triangulated hull in $\mathcal{K}(\operatorname{proj} A)$

The aim of this section is to show that $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ is the smallest triangulated subcategory of $\mathcal{K}(\operatorname{proj} A)$ that contains $\mathcal{L}_A$ and is closed under isomorphisms. In order to prove this, we have to verify that $\mathcal{L}_A$ is contained in $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ and that a complex $F^\bullet \in \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ is an element of any triangulated subcategory of $\mathcal{K}(\operatorname{proj} A)$ that contains $\mathcal{L}_A$ and is closed under isomorphisms. The first assertion follows from the next two results. For the second assertion, we then proceed as follows.

Initially, we observe that a complex is in ${}^\perp\mathcal{K}^b(\mathcal{P}_A)$ if and only if its cohomology is in ${}^\perp\mathcal{P}_A$; cf. Lemma 4.5. Next, we reduce the problem in Lemma 4.7 to projective resolutions of modules in ${}^\perp\mathcal{P}_A$. As a further reduction step, we see in Lemma 4.9 that it is enough to consider simple modules in ${}^\perp\mathcal{P}_A$. Finally, we show in Lemma 4.10 that the assertion holds for projective resolutions of simple modules in ${}^\perp\mathcal{P}_A$.

We start with the following lemma. In an exact degree, a complex in $\mathcal{K}(\operatorname{mod} A)$ has no non-zero morphism to a projective module or from an injective module. The same holds for the dual complex in $\mathcal{K}(A\text{-proj})$.

Lemma 4.3. *Let $F^\bullet \in \mathcal{K}(\operatorname{mod} A)$ and $k \in \mathbb{Z}$.*

(1) If $\mathrm{H}^k(F^\bullet) = 0$ then $\operatorname{Hom}_{\mathcal{K}(\operatorname{mod} A)}(F^\bullet, Z[-k]) = 0$ for $Z \in \operatorname{inj} A$.

(2) If $\mathrm{H}^k(F^\bullet) = 0$ then $\operatorname{Hom}_{\mathcal{K}(\operatorname{mod} A)}(Z[-k], F^\bullet) = 0$ for $Z \in \operatorname{proj} A$.

Now, assume that $F^\bullet \in \mathcal{K}(\operatorname{proj} A)$.

(1') If $\mathrm{H}_k(F^*_\bullet) = 0$ then $\operatorname{Hom}_{\mathcal{K}(\operatorname{proj} A)}(Z[-k], F^\bullet) = 0$ for $Z \in \operatorname{proj} A$ with $Z^* \in A\text{-inj}$.

(2') If $\mathrm{H}_k(F^*_\bullet) = 0$ then $\operatorname{Hom}_{\mathcal{K}(\operatorname{proj} A)}(F^\bullet, Z[-k]) = 0$ for $Z \in \operatorname{proj} A$.

Proof. Ad (1) and (2). Suppose given a morphism of complexes $f^\bullet : F^\bullet \to Z[-k]$. In particular, we have $d^{k-1} f^k = 0$.

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & F^{k-1} & \xrightarrow{d^{k-1}} & F^k & \xrightarrow{d^k} & F^{k+1} & \longrightarrow & \cdots & \qquad & F^\bullet \\
 & & \big\downarrow{\scriptstyle f^{k-1}} & & \big\downarrow{\scriptstyle f^k} & & \big\downarrow{\scriptstyle f^{k+1}} & & & & \big\downarrow{\scriptstyle f^\bullet} \\
\cdots & \longrightarrow & 0 & \longrightarrow & Z & \longrightarrow & 0 & \longrightarrow & \cdots & & Z[-k]
\end{array}$$

By assumption, we have that $\operatorname{Ker} d^k = \operatorname{Im} d^{k-1} \subseteq \operatorname{Ker} f^k$. Thus, there exists a morphism $g : F^k/\operatorname{Ker} d^k \to Z$ such that the following diagram commutes.

$$\begin{tikzcd}
F^k \arrow[dr, "\pi"] \arrow[drr, bend left, "d^k"] \arrow[ddr, bend right, "f^k"'] & & \\
 & F^k/\operatorname{Ker} d^k \arrow[r, hook, "\widetilde{d^k}"] \arrow[d, "g"] & F^{k+1} \\
 & Z &
\end{tikzcd}$$

Since Z is injective, there exists a morphism $h : F^{k+1} \to Z$ with $\widetilde{d^k}\, h = g$. We obtain

$$d^k\, h = \pi\, \widetilde{d^k}\, h = \pi\, g = f^k$$

so that $f^\bullet = 0$. This shows part (1). Part (2) follows dually.

Ad (1') and (2'). Since $\operatorname{Hom}_{\mathcal{K}(\operatorname{proj} A)}(Z[-k], F^\bullet) \simeq \operatorname{Hom}_{\mathcal{K}(A\text{-proj})}(F^*_\bullet, Z^*[-k])$, part $(1')$ follows from part (1) applied to left A-modules.

Similarly, since $\operatorname{Hom}_{\mathcal{K}(\operatorname{proj} A)}(F^\bullet, Z[-k]) \simeq \operatorname{Hom}_{\mathcal{K}(A\text{-proj})}(Z^*[-k], F^*_\bullet)$, part $(2')$ follows from part (2) applied to left A-modules. □

We extend the previous result to morphisms between complexes. In particular, this lemma shows that $\mathcal{L}_A$ is contained in $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$.

Lemma 4.4. *Let $F^\bullet \in \mathcal{K}(\operatorname{proj} A)$.*

(1) If $\mathrm{H}^\bullet(F^\bullet) = 0$, then $\operatorname{Hom}_{\mathcal{K}(\operatorname{mod} A)}(F^\bullet, Z^\bullet) = 0$ for all $Z^\bullet \in \mathcal{K}^b(\operatorname{inj} A)$.

(2) If $\mathrm{H}_\bullet(F^*_\bullet) = 0$, then $\operatorname{Hom}_{\mathcal{K}(\operatorname{proj} A)}(F^\bullet, Z^\bullet) = 0$ for all $Z^\bullet \in \mathcal{K}^b(\operatorname{proj} A)$.

(1') If $\mathrm{H}_\bullet(F^*_\bullet) = 0$, then $\operatorname{Hom}_{\mathcal{K}(\operatorname{proj} A)}(Z^\bullet, F^\bullet) = 0$ for all $Z^\bullet \in \mathcal{K}^b(\operatorname{proj} A)$ with $Z^*_\bullet \in \mathrm{K}^b(A\text{-}\operatorname{inj})$.

(2') If $\mathrm{H}^\bullet(F^\bullet) = 0$, then $\operatorname{Hom}_{\mathcal{K}(\operatorname{proj} A)}(Z^\bullet, F^\bullet) = 0$ for all $Z^\bullet \in \mathcal{K}^b(\operatorname{proj} A)$.

(3) If $F^\bullet \in \mathcal{L}_A$, then $F^\bullet \in \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$.

Proof. At first we show the following claim.

Claim. Let $Z^\bullet \in \mathcal{K}^b(\operatorname{mod} A)$. We have $\operatorname{Hom}_{\mathcal{K}(\operatorname{mod} A)}(F^\bullet, Z^\bullet) = 0$ if $\operatorname{Hom}_{\mathcal{K}(\operatorname{mod} A)}(F^\bullet, Z^k) = 0$ for all $k \in \mathbb{Z}$.

Suppose that $Z^\bullet$ is non-zero. Since $Z^\bullet$ is bounded, there exists an $l \in \mathbb{Z}$ such that $Z^l \neq 0$ and $Z^k = 0$ for $k < l$. Moreover, there exists an $r \in \mathbb{Z}$ such that $Z^r \neq 0$ and $Z^k = 0$ for $k > r$.

We proceed by induction on the number of non-zero terms of $Z^\bullet$.

Suppose that $l = r$. Then $Z^\bullet = Z^l$ and $\operatorname{Hom}_{\mathcal{K}(\operatorname{mod} A)}(F^\bullet, Z^l) = 0$ by assumption.

Suppose that $l < r$. By induction, we can assume that $\operatorname{Hom}_{\mathcal{K}(\operatorname{mod} A)}(F^\bullet, Z^{\leqslant r-1}) = 0$. Thus, there exist homotopy maps $h^k : F^k \to Z^{k-1}$ for $k \leqslant r$ such that

$$h^k\, d_Z^{k-1} + d_F^k\, h^{k+1} = f^k, \quad \text{for } k < r.$$

Consider the following diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & F^{r-2} \arrow[r, "d_F^{r-2}"] \arrow[d, "f^{r-2}"] & F^{r-1} \arrow[r, "d_F^{r-1}"] \arrow[d, "f^{r-1}"] \arrow[dl, "h^{r-1}"'] & F^r \arrow[r, "d_F^r"] \arrow[d, "f^r"] \arrow[dl, "h^r"'] & F^{r+1} \arrow[r] \arrow[d] \arrow[dl, dashed, "h^{r+1}"'] & \cdots & F^\bullet \arrow[d, "f^\bullet"] \\
\cdots \arrow[r] & Z^{r-2} \arrow[r, "d_Z^{r-2}"] & Z^{r-1} \arrow[r, "d_Z^{r-1}"] & Z^r \arrow[r, "d_Z^r"] & 0 \arrow[r] & \cdots & Z^\bullet
\end{tikzcd}$$

Note that

$$d_F^{r-1}\left(f^r - h^r\, d_Z^{r-1}\right) = f^{r-1}\, d_Z^{r-1} - f^{r-1}\, d_Z^{r-1} + h^{r-1}\, d_Z^{r-2}\, d_Z^{r-1} = 0$$

so that $f^r - h^r\, d_Z^{r-1}$ induces a morphism of complexes $F^\bullet \to Z^r$. However, we have that $\operatorname{Hom}_{\mathcal{K}(\operatorname{mod} A)}(F^\bullet, Z^r) = 0$ by assumption. This yields a homotopy map $h^{r+1} : F^{r+1} \to Z^r$ with

$$d_F^r\, h^{r+1} = f^r - h^r\, d_Z^{r-1} \;\Leftrightarrow\; d_F^r\, h^{r+1} + h^r\, d_Z^{r-1} = f^r.$$

In conclusion, we obtain $\operatorname{Hom}_{\mathcal{K}(\operatorname{mod} A)}(F^\bullet, Z^\bullet) = 0$.

By Lemma 4.3.$(1, 2')$, we have $\operatorname{Hom}_{\mathcal{K}(\operatorname{mod} A)}(F^\bullet, Z^k) = 0$ for $k \in \mathbb{Z}$ in the situation of part (1) and (2) respectively. Hence, part (1) and (2) follow from the claim above. Note that for $Z^\bullet \in \mathcal{K}(\operatorname{proj} A)$, we have $\operatorname{Hom}_{\mathcal{K}(\operatorname{proj} A)}(Z^\bullet, F^\bullet) = 0$ if and only if $\operatorname{Hom}_{\mathcal{K}(A\text{-proj})}(F^*_\bullet, Z^*_\bullet) = 0$. Thus, $(1')$ and $(2')$ follow from the versions of (1) and (2) for left A-modules respectively.

Finally, let $F^\bullet \in \mathcal{L}_A$ and $Z^\bullet \in \mathcal{K}^b(\mathcal{P}_A)$. By definition, we have $\mathrm{H}^k(F^\bullet) = 0$ for $k < 0$ and $\mathrm{H}_k(F^*_\bullet) = 0$ for $k \geqslant 0$. In particular, $F^\bullet \in \mathcal{H}(\operatorname{proj} A)$. Furthermore, Lemma 4.3.(1) shows that $\operatorname{Hom}_{\mathcal{K}(\operatorname{proj} A)}(F^\bullet, Z^k) = 0$ for $k < 0$ and Lemma 4.3.$(2')$ shows that $\operatorname{Hom}_{\mathcal{K}(\operatorname{proj} A)}(F^\bullet, Z^k) = 0$ for $k \geqslant 0$. Now, the claim above gives $\operatorname{Hom}_{\mathcal{K}(\operatorname{proj} A)}(F^\bullet, Z^\bullet) = 0$ so that $F^\bullet \in {}^\perp\mathcal{K}^b(\mathcal{P}_A)$. Together, we obtain $F^\bullet \in \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ which shows part (3). $\square$

The next several results aim to show that a complex $F^\bullet \in \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ is contained in any triangulated subcategory of $\mathcal{K}(\operatorname{proj} A)$ that contains $\mathcal{L}_A$ and is closed under isomorphisms. We start with the following important observation about complexes in $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ and $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$.

Lemma 4.5. *Let $F^\bullet \in \mathcal{K}(\operatorname{mod} A)$.*

(1) $F^\bullet \in {}^\perp\mathcal{K}^b(\mathcal{P}_A)$ if and only if $\mathrm{H}^k(F^\bullet) \in {}^\perp\mathcal{P}_A$ for all $k \in \mathbb{Z}$.

(2) $F^\bullet \in {}^\perp\mathcal{K}^b(\operatorname{stp} A)$ if and only if $\mathrm{H}^k(F^\bullet) \in {}^\perp(\operatorname{stp} A)$ for all $k \in \mathbb{Z}$.

(1') $F^\bullet \in \mathcal{K}^b(\nu^{-1}\mathcal{P}_A)^\perp$ if and only if $\mathrm{H}^k(F^\bullet) \in (\nu^{-1}\mathcal{P}_A)^\perp$ for all $k \in \mathbb{Z}$.

(2') $F^\bullet \in \mathcal{K}^b(\operatorname{stp} A)^\perp$ if and only if $\mathrm{H}^k(F^\bullet) \in (\operatorname{stp} A)^\perp$ for all $k \in \mathbb{Z}$.

Proof. Let $\mathcal{I}$ be a full subcategory of $\operatorname{inj} A$. We show that we have $F^\bullet \in {}^\perp\mathcal{K}^b(\mathcal{I})$ if and only if $\mathrm{H}^k(F^\bullet) \in {}^\perp\mathcal{I}$ for all $k \in \mathbb{Z}$. Letting $\mathcal{I} = \mathcal{P}_A$ we obtain part (1) and letting $\mathcal{I} = \operatorname{stp} A$ we obtain part (2).

Suppose that $F^\bullet \in {}^\perp\mathcal{K}^b(\mathcal{I})$. We fix a $k \in \mathbb{Z}$ with $\mathrm{H}^k(F^\bullet) \neq 0$. Let $Z \in \mathcal{I}$ and suppose given a morphism $f : \mathrm{H}^k(F^\bullet) \to Z$.

Consider the following commutative diagram. The morphism α exists since Z is injective.

$$\begin{tikzcd}
\cdots \arrow[r] & F^{k-1} \arrow[r, "d^{k-1}"] & F^k \arrow[rr, "d^k"] \arrow[dr, two heads, "\pi"] & & F^{k+1} \arrow[r] & \cdots \\
& \operatorname{Ker} d^k \arrow[ur, hook, "\iota"] \arrow[dr, two heads, "p"] & & F^k/\operatorname{Im} d^{k-1} \arrow[dl, dashed, bend left, "\alpha"] & & \\
& & \mathrm{H}^k(F^\bullet) \arrow[ur, hook] \arrow[d, "f"] & & & \\
& & Z & & &
\end{tikzcd}$$

Since $d^{k-1}\,\pi\,\alpha = 0$, this yields a morphism of complexes $\pi\,\alpha : F^\bullet \to Z[-k]$. By assumption, there exists a homotopy map $h : F^{k+1} \to Z$ with $d^k\,h = \pi\,\alpha$. We have

$$0 = \iota\, d^k\, h = \iota\,\pi\,\alpha = p\,f$$

so that $f = 0$.

Conversely, let $\mathrm{H}^k(F^\bullet) \in {}^\perp\mathcal{I}$ for all $k \in \mathbb{Z}$. Suppose given a morphism of complexes $F^\bullet \xrightarrow{f^\bullet} Z^\bullet$ with $Z^\bullet$ in $\mathcal{K}^b(\mathcal{I})$. Let $r \in \mathbb{Z}$ be maximal such that $Z^r \neq 0$. By applying a shift $[-r]$ we may assume that $r = 0$. We show that $f^\bullet = 0$ by induction on the number of non-zero terms of $Z^\bullet$.

Suppose that $Z^k = 0$ for $k \neq 0$. We have $d_F^{-1} f^0 = f^{-1}\, d_Z^{-1} = 0$ so that there exists a morphism $\alpha : F^0/\operatorname{Im} d_F^{k-1} \to H^0(F^\bullet)$ with $f^0 = \pi\,\alpha$. This results in a morphism $g = i\,\alpha : \mathrm{H}^0(F^\bullet) \to Z^0$

such that the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & F^{-1} \arrow[rr, "d_F^{-1}"] & & F^0 \arrow[rr, "d_F^0"] \arrow[dr, two heads, "\pi"] \arrow[ddd, bend right, "f^0"] & & F^1 \arrow[r] & \cdots \\
& & \operatorname{Ker} d^0 \arrow[ur, hook, "\iota"] \arrow[dr, two heads, "p"] & & F^0/\operatorname{Im} d_F^{-1} \arrow[ddl, dashed, "\alpha"] & & \\
& & & \mathrm{H}^0(F^\bullet) \arrow[ur, hook, "i"] \arrow[d, dashed, "g"] & & & \\
\cdots \arrow[r] & 0 \arrow[rr, "d_Z^{-1}"] & & Z^0 \arrow[rr, "d_Z^0"] & & 0 \arrow[r] & \cdots
\end{tikzcd}$$

By assumption, $g = 0$. Hence $\iota\, f^0 = \iota\,\pi\,\alpha = p\,i\,\alpha = p\,g = 0$. This yields a morphism of complexes as follows.

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \operatorname{Ker} d^0 \arrow[r, "\iota"] \arrow[d] & F^0 \arrow[r, "d^0"] \arrow[d, "f^0"] & F^1 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & Z^0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Since Z^0 is injective, this morphism must be zero in $\mathcal{K}(\operatorname{mod} A)$ by Lemma 4.3.(1) so that there exists a morphism $h : F^1 \to Z^0$ with $d^0 h = f^0$. This implies that $f^\bullet : F^\bullet \to Z^\bullet$ is zero as well.

For the induction step, we consider the complex $\tau_{<0} Z^\bullet = Z^{<0}$. By induction hypothesis, we may assume that $\operatorname{Hom}_{\mathcal{K}(\operatorname{mod} A)}(F^\bullet, Z^{<0}) = 0$. Hence there exist homotopy maps $h^k : F^k \to Z^{k-1}$ for $k \leqslant 0$ such that $h^{k-1}\, d_Z^{k-2} + d_F^{k-1} h^k = f^{k-1}$.

$$\begin{tikzcd}
\cdots \arrow[r] & F^{-2} \arrow[r, "d_F^{-2}"] \arrow[d, "f^{-2}"] & F^{-1} \arrow[r, "d_F^{-1}"] \arrow[d, "f^{-1}"] \arrow[dl, "h^{-1}"'] & F^0 \arrow[r, "d_F^0"] \arrow[d, "f^0"] \arrow[dl, "h^0"'] & F^1 \arrow[r] \arrow[d, "f^1"] & \cdots \\
\cdots \arrow[r] & Z^{-2} \arrow[r, "d_Z^{-2}"] & Z^{-1} \arrow[r, "d_Z^{-1}"] & Z^0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Note that

$$d_F^{-1}\left(f^0 - h^0\, d_Z^{-1}\right) = f^{-1}\, d_Z^{-1} - f^{-1}\, d_Z^{-1} + h^{-1}\, d_Z^{-2}\, d_Z^{-1} = 0$$

so that $f^0 - h^0\, d_Z^{-1}$ induces a morphism of complexes $F^\bullet \to Z^0$.

Now we are in the same situation as above and we can conclude that there exists a morphism $F^1 \xrightarrow{h^1} Z^0$ such that $d_F^0 h^1 = f^0 - h^0\, d_Z^{-1}$. However, this yields $h^0\, d_Z^{-1} + d_F^0 h^1 = f^0$. Thus $f^\bullet = 0$.

It remains to show part (1') and part (2'). Let $\mathcal{C}$ be either $\nu^{-1}\mathcal{P}_A$ or $\operatorname{stp} A$. In both cases, $\mathcal{I} := \mathrm{D}\,\mathcal{C}$ is a full subcategory of A-inj. In particular, we can apply the arguments above for $\mathcal{I}$.

Suppose given $Z^\bullet \in \mathcal{K}^b(\mathcal{C})$. Note that $\mathrm{D}\, Z^\bullet \in \mathcal{K}^b(\mathcal{I})$ and

$$\mathrm{Hom}_{\mathcal{K}(\mathrm{mod}\, A)}(Z^\bullet, F^\bullet) \simeq \mathrm{Hom}_{\mathcal{K}(A\text{-mod})}(\mathrm{D}\, F^\bullet, \mathrm{D}\, Z^\bullet).$$

Thus, the arguments above for left A-modules show that there exists a $Z^\bullet \in \mathcal{K}^b(\mathcal{C})$ with $\mathrm{Hom}_{\mathcal{K}(\mathrm{mod}\, A)}(Z^\bullet, F^\bullet) \neq 0$ if and only if there is a ${}_A Z \in \mathcal{I}$ with $\mathrm{Hom}_A({}_A\mathrm{H}^k(\mathrm{D}\, F^\bullet), {}_A Z) \neq 0$. Since $\mathrm{D}(-)$ is exact, we have

$$\mathrm{Hom}_A({}_A\mathrm{H}^k(\mathrm{D}\, F^\bullet), {}_A Z) \simeq \mathrm{Hom}_A({}_A(\mathrm{D}\, \mathrm{H}^k(F^\bullet)), {}_A Z) \simeq \mathrm{Hom}_A((\mathrm{D}\, Z)_A, \mathrm{H}^k(F^\bullet)_A).$$

Using that $\mathrm{D}\, Z \in \mathrm{D}\, \mathcal{I} \simeq \mathcal{C}$, we are done. □

For a given complex $F^\bullet \in \mathcal{H}(\mathrm{proj}\, A)$, we want to construct a complex in $\mathcal{L}$ which is related to $F^\bullet$ via distinguished triangles; cf. Lemma 4.7. This is done by removing non-zero cohomology of $F^\bullet$ with projective resolutions. Using the boundary conditions in the definition of $\mathcal{H}(\mathrm{proj}\, A)$, there are only finitely many positions we have to consider until we arrive at a complex in $\mathcal{L}$. The distinguished triangles that arise during the proof also give a way to calculate the class of $F^\bullet$ in the Grothendieck group of $\mathcal{H}_{\mathcal{P}}(\mathrm{proj}\, A)$.

Because it will be needed later, we first state the induction step in a more general lemma.

Lemma 4.6. *Suppose given $F^\bullet \in \mathcal{H}(\mathrm{proj}\, A)$. Let $k \in \mathbb{Z}$ be minimal with $\mathrm{H}^k(F^\bullet) \neq 0$. Let H be a submodule of $\mathrm{H}^k(F^\bullet)$ with $P^\bullet$ a projective resolution of H. Then there exists a distinguished triangle $P^\bullet[-k] \to F^\bullet \to C^\bullet \to$ such that the following holds.*

(1) We have $\mathrm{H}^j(C^\bullet) = 0$ for $j < k$ and $\tau_{\geqslant k} C^\bullet = \tau_{\geqslant k} F^\bullet$.

(2) There is a short exact sequence $0 \to H \to \mathrm{H}^k(F^\bullet) \to \mathrm{H}^k(C^\bullet) \to 0$. In particular, we have $\mathrm{H}^k(C^\bullet) = 0$ if $H = \mathrm{H}^k(F^\bullet)$.

(3) Suppose that $F^\bullet \in \mathcal{L}_A$ with $k = 0$. Then we also have $C^\bullet \in \mathcal{L}_A$. In this setting, we have a short exact sequence $0 \to H \to \mathrm{H}^0(\tau_{\leqslant 0}\, F^\bullet) \to \mathrm{H}^0(\tau_{\leqslant 0}\, C^\bullet) \to 0$.

Proof. We have an injective morphism $f : H \to \mathrm{Cok}\, d_F^{k-1}$ since H is a submodule of $\mathrm{H}^k(F^\bullet)$ which embedds into $\mathrm{Cok}\, d_F^{k-1}$. Since k is minimal such that $\mathrm{H}^k(F^\bullet) \neq 0$, we know that $\mathrm{H}^j(F^\bullet) = 0$ for $j < k$. Hence, we can lift f to a morphism of complexes $f^\bullet : P^{\leqslant 0}[-k] \to F^\bullet$. In particular, we have $f^0\, d_F^k = 0$ since f factors through $\mathrm{H}^k(F^\bullet)$.

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & P^{-2} & \xrightarrow{d_P^{-2}} & P^{-1} & \xrightarrow{d_P^{-1}} & P^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \qquad P^{\leqslant 0}[-k] \\
 & & \downarrow{\scriptstyle f^{-2}} & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^{0}} & & \downarrow & & \downarrow & & \qquad \downarrow{\scriptstyle f^\bullet} \\
\cdots & \longrightarrow & F^{k-2} & \xrightarrow{d_F^{k-2}} & F^{k-1} & \xrightarrow{d_F^{k-1}} & F^k & \xrightarrow{d_F^{k}} & F^{k+1} & \xrightarrow{d_F^{k+1}} & F^{k+2} & \longrightarrow & \cdots \qquad F^\bullet
\end{array}$$

We write $C^\bullet := C(f)^\bullet$ for the mapping cone. We denote the natural projections by $\pi_P : P^0 \to H$ and by $\pi_F : F^k \to \operatorname{Cok} d_F^{k-1}$. Note that $\tau_{\geqslant k} C^\bullet = \tau_{\geqslant k} F^\bullet$. Moreover $\mathrm{H}^j(C^\bullet) = 0$ for $j \leqslant k-2$ by Lemma 1.9. For part (1), it remains to show that $\mathrm{H}^j(C^\bullet) = 0$ for $j = k-1$.

Let $(x,y) \in \operatorname{Ker} d_C^{k-1} \subseteq P^0 \oplus F^{k-1}$. Then $(x,y)\begin{pmatrix} f^0 \\ d_F^{k-1} \end{pmatrix} = 0$, i.e. $x f^0 = -y\, d_F^{k-1}$. We have

$$0 = y\, d_F^{k-1} \pi_F = -x f^0 \pi_F = -x\, \pi_P\, f$$

which implies $x\, \pi_P = 0$ since f is injective. Thus, there exists an element $q \in P^{-1}$ with $q\, d_P^{-1} = x$. Moreover,

$$\big(q f^{-1} + y\big)\; d_F^{k-1} = q\, d_P^{-1} f^0 + y\, d_F^{k-1} = x f^0 + y\, d_F^{k-1} = -y\, d_F^{k-1} + y\, d_F^{k-1} = 0$$

so that we obtain $p \in F^{k-2}$ with $p\, d_F^{k-2} = q f^{-1} + y$. We calculate

$$(-q, p)\begin{pmatrix} -d_P^{-1} & f^{-1} \\ 0 & d_F^{k-2} \end{pmatrix} = \big(x, -q f^{-1} + p\, d_F^{k-2}\big) = \big(x, -q f^{-1} + q f^{-1} + y\big) = (x, y)\,.$$

The distinguished triangle $P^\bullet[-k] \to F^\bullet \to C^\bullet \to$ induces a long exact sequence of cohomology.

$$\mathrm{H}^{k-1}(C^\bullet) \to \mathrm{H}^k(P^\bullet[-k]) \to \mathrm{H}^k(F^\bullet) \to \mathrm{H}^k(C^\bullet) \to 0$$

We have seen above, that $\mathrm{H}^{k-1}(C^\bullet) = 0$. Using that $\mathrm{H}^k(P^\bullet[-k]) = \mathrm{H}^0(P^\bullet) \simeq H$, we obtain a short exact sequence $0 \to H \to \mathrm{H}^k(F^\bullet) \to \mathrm{H}^k(C^\bullet) \to 0$. This shows part (2).

For part (3), suppose that $F^\bullet \in \mathcal{L}_A$ and $k = 0$. We have $\mathrm{H}^j(C^\bullet) = 0$ for $j < 0$ by part (1). Using Lemma 1.9.(2), we obtain that $C^\bullet = C(f)^\bullet \in \mathcal{L}_A$. Since $\tau_{\geqslant 0} C^\bullet = \tau_{\geqslant 0} F^\bullet$, we have

$$N := \operatorname{Cok}\big(\mathrm{H}^0(F^\bullet) \hookrightarrow \mathrm{H}^0(\tau_{\leqslant 0}\, F^\bullet)\big) \simeq \operatorname{Im}(d_F^0) \simeq \operatorname{Im}(d_C^0) \simeq \operatorname{Cok}\big(\mathrm{H}^0(C^\bullet) \hookrightarrow \mathrm{H}^0(\tau_{\leqslant 0}\, C^\bullet)\big).$$

Let $\tilde{H} \simeq \operatorname{Ker}\big(\mathrm{H}^0(\tau_{\leqslant 0}\, F^\bullet) \to \mathrm{H}^0(\tau_{\leqslant 0}\, C^\bullet)\big)$. Consider the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H & \longrightarrow & \mathrm{H}^0(F^\bullet) & \longrightarrow & \mathrm{H}^0(C^\bullet) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{H} & \longrightarrow & \mathrm{H}^0(\tau_{\leqslant 0}\, F^\bullet) & \longrightarrow & \mathrm{H}^0(\tau_{\leqslant 0}\, C^\bullet) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & N & = & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We obtain $H \simeq \tilde{H}$ and the short exact sequence $0 \to H \to \mathrm{H}^0(\tau_{\leqslant 0}\, F^\bullet) \to \mathrm{H}^0(\tau_{\leqslant 0}\, C^\bullet) \to 0$. $\square$

Lemma 4.7. *Let $\mathcal{T}$ be a triangulated subcategory of $\mathcal{K}(\operatorname{mod} A)$ that contains $\mathcal{L}$ and is closed under isomorphisms. The following holds for a complex $F^\bullet \in \mathcal{H}(\operatorname{proj} A)$ with integers $l \leqslant r$ such that $\mathrm{H}^{<l}(F^\bullet) = 0$ and $\mathrm{H}_{\geqslant r}(F^*_\bullet) = 0$.*

(1) If $\mathcal{T}$ contains the minimal projective resolution of $\mathrm{H}^k(F^\bullet)$ for all $k < r$, then $F^\bullet \in \mathcal{T}$.

(2) We abbreviate $G^\bullet := \mathcal{F}(\mathrm{H}^r(\tau_{\leqslant r} F^\bullet)) \in \mathcal{L}_A$. For $l \leqslant k < r$ let $P^\bullet_k$ be the minimal projective resolution of $\mathrm{H}^k(F^\bullet)$. We have $[F^\bullet] = \sum_{k=l}^{r-1} (-1)^k [P^\bullet_k] + (-1)^r [G^\bullet]$ in $G_0(\mathcal{H}(\operatorname{proj} \mathcal{A}))$.

Proof. Suppose that $F^\bullet \neq 0$. By definition of $\mathcal{H}(\operatorname{proj} A)$, there always exist $l, r \in \mathbb{Z}$ with $\mathrm{H}^{<l}(F^\bullet) = 0$ and $\mathrm{H}_{\geqslant r}(F^*_\bullet) = 0$. We can choose $r \in \mathbb{Z}$ such that $l \leqslant r$. We proceed by induction on $N := r - l$.

If $N = 0$, that is $l = r$, then $\mathrm{H}^{<r}(F^\bullet) = 0$ and $\mathrm{H}_{\geqslant r}(F^*_\bullet) = 0$. Thus, $F^\bullet[r] \in \mathcal{L}$ and we have $F^\bullet \in \mathcal{T}$. Furthermore, $[F^\bullet] = (-1)^r [F^\bullet[r]] = (-1)^r [G^\bullet]$ since $\mathrm{H}^r(\tau_{\leqslant r} F^\bullet) = \mathrm{H}^0\big(\tau_{\leqslant 0}(F^\bullet[r])\big)$.

We consider the case $N > 0$, that is $l < r$. By Lemma 4.6, we have a distinguished triangle

$$P^\bullet[-l] \to F^\bullet \to C^\bullet \to$$

with $P^\bullet$ the minimal projective resolution of $\mathrm{H}^l(F^\bullet)$. Moreover, $\mathrm{H}^j(C^\bullet) = 0$ for $j \leqslant l$ and $\tau_{\geqslant l} C^\bullet = \tau_{\geqslant l} F^\bullet$. In particular, this means $\mathrm{H}^{<l+1}(C^\bullet) = 0$ and $\mathrm{H}_{\geqslant r}(C^*_\bullet) = 0$. Hence, by induction, we have $C^\bullet \in \mathcal{T}$ and $[C^\bullet] = \sum_{k=l+1}^{r-1} (-1)^k [P^\bullet_k] + (-1)^r [G^\bullet]$ since $\mathrm{H}^k(C^\bullet) \simeq \mathrm{H}^k(F^\bullet)$ for $k \geqslant l+1$. Using that $P^\bullet[-l] \to F^\bullet \to C^\bullet \to$ is a distinguished triangle with $P^\bullet \in \mathcal{T}$, we conclude that $F^\bullet \in \mathcal{T}$ since $\mathcal{T}$ is closed under isomorphisms. Moreover, we have that

$$[F^\bullet] = (-1)^l [P^\bullet] + [C^\bullet] = \sum_{k=l}^{r-1} (-1)^k [P^\bullet_k] + (-1)^r [G^\bullet]. \qquad \square$$

Recall that we aim to show that a complex $F^\bullet \in \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ is contained in any triangulated subcategory that contains $\mathcal{L}$ and is closed under isomorphisms. The previous results state that it is enough to consider projective resolutions of modules in ${}^\perp\mathcal{P}$. In a next step, we further reduce this to composition factors of such modules. Again, we additionally obtain a formula for the class in the Grothendieck group. Note that no further steps are necessary in case that $\operatorname{domdim} A \geqslant 1$. In this context, a module $X \in {}^\perp\mathcal{P}$ already satisfies $X^* = 0$. By Lemma 2.24.(5,7), we obtain that $F^\bullet_X \in \mathcal{L}$ is the minimal projective resolution of X.

The next lemma will be used with $\mathcal{I} = \mathcal{P}_A$ in this section, as well as with $\mathcal{I} = \operatorname{stp} A$ in Section 4.3.

Lemma 4.8. *Let $\mathcal{I}$ be a full subcategory of* $\operatorname{inj} A$. *The following are equivalent for $X \in \operatorname{mod} A$.*

(i) $X \in {}^\perp\mathcal{I}$.

(ii) $S \in {}^\perp\mathcal{I}$ for every composition factor S of X.

Proof. Ad (i) $\Rightarrow$ (ii). Let S be a composition factor of X. Suppose given an injective module I together with a morphism $S \xrightarrow{f} I$.

Since S is a composition factor of X, there exists a submodule M of X such that $S \hookrightarrow X/M$. Using that I is injective, we obtain a morphism $X/M \xrightarrow{g} I$ such that the following diagram commutes.

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d, "f"'] & X/M \arrow[dl, dashed, "g"] \\ I & \end{tikzcd}$$

If f is non-zero then the composite map $X \twoheadrightarrow X/M \xrightarrow{g} I$ is non-zero as well.

Ad (ii) $\Rightarrow$ (i). Suppose given an A-module I together with a non-zero morphism $f : X \to I$.

Let S be in the socle of $\operatorname{Im} f \simeq X/\operatorname{Ker} f$. Then S is a composition factor of X and the composite $S \hookrightarrow \operatorname{Im} f \hookrightarrow I$ is non-zero. $\square$

Lemma 4.9. *Let $\mathcal{T}$ be a triangulated subcategory of $\mathcal{K}(\operatorname{mod} A)$ that is closed under isomorphisms.*

Suppose X is an A-module with minimal projective resolution $P^{\leqslant 0}$. The following holds.

(1) If $\mathcal{T}$ contains the minimal projective resolution of every composition factor S of X, then we have $P^{\leqslant 0} \in \mathcal{T}$.

(2) Let $n := l(X)$ and suppose $Q_i^{\leqslant 0}$ are the minimal projective resolutions of the composition factors of X. Then $[P^{\leqslant 0}] = \sum_{i=1}^{n} [Q_i^{\leqslant 0}]$ in $G_0(\mathcal{H}(\operatorname{proj} \mathcal{A}))$.

Proof. We show the assertions by induction on the length of X. Let $l(X) = 1$. Then X is simple and $P^{\leqslant 0} \in \mathcal{T}$ by assumption.

Let $l(X) > 1$. Then there exist A-modules S and Y with $l(S) = 1$ and $l(Y) < l(X)$ such that there is a short exact sequence

$$0 \to S \to X \to Y \to 0.$$

By assumption and induction respectively, the minimal projective resolutions $Q^{\leqslant 0}$ of S and $R^{\leqslant 0}$ of Y are contained in $\mathcal{T}$. Furthermore, assertion (2) holds for $R^{\leqslant 0}$.

Using the horseshoe lemma, the short exact sequence of modules induces a short exact sequence of complexes with $\tilde{P}^{\leqslant 0} \simeq P^{\leqslant 0}$ in $\mathcal{K}(\operatorname{proj} A)$.

$$0 \to Q^{\leqslant 0} \to \tilde{P}^{\leqslant 0} \to R^{\leqslant 0} \to 0$$

By Lemma 2.17, we have a distinguished triangle

$$Q^{\leqslant 0} \to P^{\leqslant 0} \to R^{\leqslant 0} \to$$

so that $P^{\leqslant 0} \in \mathcal{T}$ and $[P^{\leqslant 0}] = [Q^{\leqslant 0}] + [R^{\leqslant 0}]$. $\square$

The following lemma is the last we need to prove the main theorem of this section. Note that the short exact sequence starting in the simple module is not perfect exact in general; see also Example 7.11.

Lemma 4.10. *Suppose S is a simple A-module with minimal projective resolution $P^{\leqslant 0}$. Let I be the injective hull of S together with a short exact sequence $0 \to S \to I \to C \to 0$ in $\operatorname{mod} A$. If $S \in {}^{\perp}\mathcal{P}_A$, then there exists a distinguished triangle $P^{\leqslant 0} \to F_I^\bullet \to F_C^\bullet \to$ in $\mathcal{K}(\operatorname{proj} A)$. In particular, $P^{\leqslant 0} \in \mathcal{T}$ for any triangulated subcategory $\mathcal{T}$ of $\mathcal{K}(\operatorname{proj} A)$ that contains $\mathcal{L}$ and is closed under isomorphisms.*

Proof. We extend $P^{\leqslant 0}$ to an element $P^\bullet \in \mathcal{L}$ such that $P^\bullet \simeq F_S^\bullet$.

In case $P^1 = 0$, we obtain $P^{\leqslant 0} = P^\bullet \in \mathcal{L} \subseteq \mathcal{T}$. In this case, $S^* = 0$ and $0 \to S \to I \to C \to 0$ is a perfect exact sequence. The result now follows from Proposition 2.18. Hence, suppose that $P^1 \neq 0$ for the remainder of the proof.

Let I be the injective hull of S with embedding $f : S \hookrightarrow I$. Consider $F_I^\bullet \in \mathcal{L}$. By assumption, I is not projective so that $F_I^\bullet$ is non-zero. Our aim is to construct a morphism of complexes $f^\bullet : P^{\leqslant 0} \to F_I^\bullet$ with $C(f)^\bullet \in \mathcal{L}$.

$$\begin{tikzcd}
\cdots \arrow[r] & P^{-2} \arrow[r, "d_P^{-2}"] \arrow[d, "f^{-2}"] & P^{-1} \arrow[r, "d_P^{-1}"] \arrow[d, "f^{-1}"] & P^0 \arrow[r] \arrow[d, "f^0"] & 0 \arrow[r] \arrow[d] & \cdots & P^{\leqslant 0} \arrow[d, "f^\bullet"] \\
\cdots \arrow[r] & F_I^{-2} \arrow[r, "d_F^{-2}"] & F_I^{-1} \arrow[r, "d_F^{-1}"] & F_I^0 \arrow[r, "d_F^0"] & F_I^1 \arrow[r] & \cdots & F_I^\bullet
\end{tikzcd}$$

The morphism d_F^0 factors through I via a morphism $i : I \to F_I^1$. Since S is simple, the composite map $S \xrightarrow{f} I \xrightarrow{i} F_I^1$ is either injective or zero. If fi is injective, there exists a morphism $p : F_I^1 \to I$ with $f = (fi)p$ since I is an injective module. As a consequence, we have $f = f(i\,p) = f(i\,p)^n$ for all $n > 0$. Since S is simple, I is indecomposable and thus the composite $i\,p$ is either an automorphism or $(i\,p)^n$ is zero for some $n > 0$. If $i\,p$ is an automorphism, i is split so that I is projective-injective as a direct summand of F_I^1. However, I is not projective-injective by assumption. If $(i\,p)^n = 0$, we also have $f = f(i\,p)^n = 0$. A contradiction in both cases. Thus, the composite fi cannot be injective and must be zero.

This yields $S \hookrightarrow \operatorname{Ker}(i) \simeq \operatorname{Ker}\big(I \to \operatorname{Im} d_F^0\big) \simeq \operatorname{Ker}\big(\mathrm{H}^0(\tau_{\leqslant 0} F_I^\bullet) \to F_I^0/\operatorname{Ker}(d_F^0)\big) \simeq \mathrm{H}^0(F_I^\bullet)$. Since f is non-zero, S is isomorphic to a submodule of $\mathrm{H}^0(F_I^\bullet)$. Using Lemma 4.6, we obtain a distinguished triangle $P^{\leqslant 0} \xrightarrow{f^\bullet} F_I^\bullet \to C(f)^\bullet \to$ with $C(f)^\bullet \simeq F_C^\bullet \in \mathcal{L}_A$. $\square$

We are now ready to show the main result of this section.

Theorem 4.11. *Let A be a finite dimensional k-algebra.*

The category $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ is the smallest triangulated subcategory of $\mathcal{K}(\operatorname{proj} A)$ that contains the category $\mathcal{L}_A$ and is closed under isomorphisms.

Proof. By Lemma 4.4.(3), we have that $\mathcal{L} \subseteq \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$. By Remark 4.2.(1), $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ is a triangulated subcategory of $\mathcal{K}(\operatorname{proj} A)$ that is closed under isomorphisms. Together, we obtain that $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ is a triangulated category containing $\mathcal{L}$.

Suppose that $\mathcal{T}$ is another triangulated subcategory of $\mathcal{K}(\operatorname{mod} A)$ that contains $\mathcal{L}$ and is closed under isomorphisms. We show that $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) \subseteq \mathcal{T}$.

Recall that $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) \subseteq {}^{\perp}\mathcal{K}^b(\mathcal{P}_A)$. By Lemma 4.5 and Lemma 4.7 it suffices to show that the minimal projective resolution of every A-module $X \in {}^{\perp}\mathcal{P}_A$ is an element of $\mathcal{T}$. Moreover, by Lemma 4.9 it suffices to show that the minimal projective resolution of every composition factor of X is an element of $\mathcal{T}$. Let S be such a composition factor. Then S is an element of ${}^{\perp}\mathcal{P}_A$ by Lemma 4.8. Using Lemma 4.10, we now obtain that the minimal projective resolution of S is an element of $\mathcal{T}$. $\square$

Instead of defining $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ as a subcategory of ${}^{\perp}\mathcal{K}^b(\mathcal{P}_A)$, we also can consider right perpendicular categories.

Remark 4.12. The following are equivalent for a complex $F^\bullet \in \mathcal{K}(\operatorname{proj} A)$.

(1) $F^\bullet \in {}^{\perp}\mathcal{K}^b(\mathcal{P}_A)$.

(2) $H^k(F^\bullet) \in {}^{\perp}\mathcal{P}_A$ for all $k \in \mathbb{Z}$.

(3) $H^k(F^\bullet) \in (\nu^{-1}\mathcal{P}_A)^{\perp}$ for all $k \in \mathbb{Z}$.

(4) $F^\bullet \in \mathcal{K}^b(\nu^{-1}\mathcal{P}_A)^{\perp}$.

In particular, we have $F^\bullet \in \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ if and only if $F^\bullet \in \mathcal{K}^b(\nu^{-1}\mathcal{P}_A)^{\perp}$ and $F^\bullet \in \mathcal{H}(\operatorname{proj} A)$.

In fact, the equivalence of (1) and (2), as well as the equivalence of (3) and (4) were shown in Lemma 4.5. The equivalence of (2) and (3) follows from the natural isomorphism

$$\operatorname{Hom}_A(X, \nu P) \simeq \operatorname{D}\operatorname{Hom}_A(P, X)$$

for all $P \in \operatorname{proj} A$ and $X \in \operatorname{mod} A$.

Examples in Chapter 7. We visualize the categories $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ and $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ in Example 7.10 of Section 7.3. Note that we will discuss $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ in more detail later in Section 4.3.

In Example 7.11, we explicitly follow the steps in the proof of Theorem 4.11 and illustrate the constructions done in this section.

We close with a characterization of $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ inside $\mathcal{H}(\operatorname{proj} A)$ in case that A has dominant dimension at least one.

Remark 4.13. Let $\operatorname{domdim} A \geqslant 1$ and $F^\bullet \in \mathcal{H}(\operatorname{proj} A)$. Then $F^\bullet \in \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ if and only if $\nu(\mathrm{H}^k(F^\bullet)) = 0$ for all $k \in \mathbb{Z}$.

In fact, we have $F^\bullet \in \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ if and only if $\mathrm{H}^k(F^\bullet) \in {}^\perp\mathcal{P}_A$ for all $k \in \mathbb{Z}$ by Lemma 4.5.(1). However, under the assumption $\operatorname{domdim} A \geqslant 1$, we have ${}^\perp\mathcal{P}_A = {}^\perp \operatorname{proj} A$. Thus, $\mathrm{H}^k(F^\bullet) \in {}^\perp\mathcal{P}_A$ if and only if $(\mathrm{H}^k(F^\bullet))^* = 0$.

4.2 Grothendieck group

We recall the definition of the stable Grothendieck group as stated in [33].

Definition 4.14. Let L be the free abelian group generated by the isomorphism classes of objects in $\operatorname{mod} A$ without projective direct summands. Let R be the subgroup of L generated by the classes

$$[X] - [Y] + [Z]$$

where $0 \to X \oplus P \to Y \oplus Q \to Z \to 0$ is a short exact sequence with $P, Q \in \operatorname{proj} A$ and where X and Y may be zero.

The *stable Grothendieck group* $G_0^{\mathrm{st}}(A)$ *of* A is defined as the quotient L/R.

Martínez-Villa has shown in [33, Theorem 2.1] that stably equivalent algebras without nodes and without semisimple summands have isomorphic stable Grothendieck groups.

We consider the Grothendieck group $G_0(\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A))$ of the triangulated category $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$. Using the equivalence $\mathcal{F} : \underline{\operatorname{mod}}\, A \to \mathcal{L}_A$, we obtain a Grothendieck group $G_0^{\mathcal{P}}(A)$ for the stable module category which is defined via perfect exact sequences. We show that $G_0^{\mathcal{P}}(A)$ is invariant under stable equivalences which preserve perfect exact sequences.

Definition 4.15. Let L be the free abelian group generated by the isomorphism classes of objects in $\operatorname{mod} A$ without projective direct summands. Let R' be the subgroup of L generated by the classes

$$[X] - [Y] + [Z]$$

where $0 \to X \to Y \oplus P \to Z \to 0$ is a perfect exact sequence with $P \in \operatorname{proj} A$.

The group $G_0^{\mathcal{P}}(A)$ is defined as the quotient L/R'.

The next theorem follows from the results provided in Section 4.1.

Theorem 4.16. *The equivalence* $\mathcal{F} : \underline{\mathrm{mod}}\, A \to \mathcal{L}_A$ *induces an isomorphism*

$$G_0^{\mathcal{P}}(A) \simeq G_0(\mathcal{H}_{\mathcal{P}}(\mathrm{proj}\, A)).$$

Proof. By Proposition 2.18, every perfect exact sequence $0 \to X \to Y \oplus P \to Z \to 0$ in $\mathrm{mod}\, A$ with P projective induces a distinguished triangle $F_X^\bullet \to F_Y^\bullet \to F_Z^\bullet \to$ in $\mathcal{L} \subseteq \mathcal{H}_{\mathcal{P}}(\mathrm{proj}\, A)$ and vice versa. Hence, the natural map $\sigma : G_0^{\mathcal{P}}(A) \to G_0(\mathcal{H}_{\mathcal{P}}(\mathrm{proj}\, A))$ given by $[X] \mapsto [F_X^\bullet]$ is well-defined. Since split exact sequences are perfect exact, σ is a group homomorphism.

Suppose given $X \in \mathrm{mod}\, A$. Let $S_1, \ldots, S_n$ be the composition factors of X. For $1 \leqslant k \leqslant n$, let $S_k \hookrightarrow I_k$ be the injective hull of S_k together with a short exact sequence $0 \to S_k \to I_k \to C_k \to 0$ in $\mathrm{mod}\, A$. Note that the modules I_k and C_k are uniquely determined by X up to isomorphism. Throughout the proof, we write $I_X := \bigoplus_{k=1}^n I_k$ and $C_X := \bigoplus_{k=1}^n C_k$ for a given A-module X. If X is the zero module, we set $I_X = 0$ and $C_X = 0$.

Claim 1. Let $X \in {}^{\perp}\mathcal{P}_A$ with minimal projective resolution $P^\bullet$. We have $([I_X] - [C_X])\sigma = [P^\bullet]$.

Proof of claim 1. Let $Q_k^\bullet$ be the minimal projective resolution of S_k for $1 \leqslant k \leqslant n$. By Lemma 4.9.(2), we have $[P^\bullet] = \sum_{k=1}^n [Q_k]$. For all $1 \leqslant k \leqslant n$ we additionally have that $[Q_k^\bullet] = [F_{I_k}^\bullet] - [F_{C_k}^\bullet]$ by Lemma 4.10. Together, we obtain

$$([I_X] - [C_X])\sigma = \sum_{k=1}^n [I_k]\sigma - [C_k]\sigma = \sum_{k=1}^n [F_{I_k}^\bullet] - [F_{C_k}^\bullet] = \sum_{k=1}^n [Q_k] = [P^\bullet].$$

This proves the claim.

Suppose given a complex $G^\bullet \in \mathcal{H}_{\mathcal{P}}(\mathrm{proj}\, A)$. Let $l \in \mathbb{Z}$ such that $\mathrm{H}^{<l}(G^\bullet) = 0$. Let $r \in \mathbb{Z}_{\geqslant l}$ such that $\mathrm{H}_{\geqslant r}(G_\bullet^*) = 0$. We aim to define a map

$$\tilde{\sigma}' : \mathcal{H}_{\mathcal{P}}(\mathrm{proj}\, A) \to G_0^{\mathcal{P}}(A) : G^\bullet \mapsto \sum_{k=l}^{r-1} (-1)^k ([I_{\mathrm{H}^k(G^\bullet)}] - [C_{\mathrm{H}^k(G^\bullet)}]) + (-1)^r [\mathrm{H}^r(\tau_{\leqslant r} G^\bullet)].$$

Claim 2. Suppose given $F_X^\bullet \in \mathcal{L}_A$ for $X \in \mathrm{mod}\, A$ with $\mathrm{H}^0(F_X^\bullet)$ non-zero. Suppose given a submodule H of $\mathrm{H}^0(F_X^\bullet)$ together with a short exact sequence $0 \to H \to X \to N \to 0$. We have $[X] = [I_H] - [C_H] + [N]$ in $G_0^{\mathcal{P}}(A)$.

If $H = \mathrm{H}^0(F_X^\bullet)$, then we have $[X] = [\mathrm{H}^0(\tau_{\leqslant 0}\, F_X^\bullet)] = [I_H] - [C_H] - [\mathrm{H}^1(\tau_{\leqslant 1} F_X^\bullet)]$ in $G_0^{\mathcal{P}}(A)$.

Proof of claim 2. We proceed by induction on the number of composition factors of H.

Suppose that $S := H$ is simple. Let $I \in \mathrm{mod}\, A$ be the injective hull of S. Since S is a submodule of X, there exists an injective module $J \in \mathrm{mod}\, A$ such that $I \oplus J$ is the injective hull of X. Let $P^\bullet$ be the minimal projective resolution of S. By Lemma 4.6.(2,3), we have the following distinguished triangle

$$P^\bullet \xrightarrow{u^\bullet} F_X^\bullet \to F_N^\bullet \to$$

and a short exact sequence $0 \to S \to \mathrm{H}^0(F_X^\bullet) \to \mathrm{H}^0(F_N^\bullet) \to 0$. Additionally, we have the following distinguished triangle by Lemma 4.10 with $w^\bullet : P^\bullet \to F_I^\bullet$ induced by the embedding $S \hookrightarrow I$. Note that $S \in {}^\perp\mathcal{P}_A$ since $\mathrm{H}^0(F_X^\bullet) \in {}^\perp\mathcal{P}_A$.

$$P^\bullet \xrightarrow{\begin{pmatrix} w^\bullet & 0 \end{pmatrix}} F_I^\bullet \oplus F_J^\bullet \to F_C^\bullet \oplus F_J^\bullet \to$$

The embedding $S \hookrightarrow I \oplus J$ factors through the embedding $S \hookrightarrow X$ via an injective morphism $X \to I \oplus J$. This induces a morphism of complexes $(v_1^\bullet\ \ v_2^\bullet) : F_X^\bullet \to F_I^\bullet \oplus F_J^\bullet$ such that $u^\bullet(v_1^\bullet\ \ v_2^\bullet) = (w^\bullet\ \ 0)$ in $\mathcal{K}(\operatorname{proj} A)$. Let $K^\bullet := C\big((v_1\ \ v_2)\big)^\bullet$ be its mapping cone. Note that $K^\bullet \in \mathcal{L}_A$ by Proposition 2.21. We have the following distinguished triangle.

$$F_X^\bullet \xrightarrow{\begin{pmatrix} v_1^\bullet & v_2^\bullet \end{pmatrix}} F_I^\bullet \oplus F_J^\bullet \to K^\bullet \to$$

Now, the octahedral axiom gives another distinguished triangle.

$$F_N^\bullet \to F_C^\bullet \oplus F_J^\bullet \to K^\bullet \to$$

By Proposition 2.18 the two triangles above induce perfect exact sequences such that

$$[X] - [N] = [I] + [J] - [\mathrm{H}^0(\tau_{\leqslant 0} K^\bullet)] - ([C] + [J] - [\mathrm{H}^0(\tau_{\leqslant 0} K^\bullet)]) = [I] - [C].$$

This verifies the claim in the case that $S = H$ is simple. Now, suppose that H has $n > 1$ many composition factors. Let $0 \to U \to H \to T \to 0$ be a short exact sequence in $\operatorname{mod} A$ with T a simple module. Let $\tilde{X} := \operatorname{Cok}(U \to X)$. By induction, we may assume that we have $[X] = [I_U] - [C_U] + [\tilde{X}]$ in $G_0^{\mathcal{P}}(A)$. Consider the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r] \arrow[d, equal] & H \arrow[r] \arrow[d] & T \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & U \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \tilde{X} \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & 0 \arrow[r] & N \arrow[r, equal] & N \arrow[r] & 0
\end{tikzcd}$$

Since $0 \to H \to X \to N \to 0$ is a short exact sequence, we obtain another short exact sequence $0 \to T \to \tilde{X} \to N \to 0$. By Lemma 4.6.(2,3), we have a short exact sequence $0 \to U \to \mathrm{H}^0(F_X^\bullet) \to \mathrm{H}^0(F_{\tilde{X}}^\bullet) \to 0$. Thus, T is a submodule of $\mathrm{H}^0(F_{\tilde{X}}^\bullet)$. Using that T is simple, we can show as above that $[\tilde{X}] = [I_T] - [C_T] + [N]$. Since $[I_H] = [I_U] + [I_T]$ and $[C_H] = [C_U] + [C_T]$, combining all equations yields $[X] = [I_H] - [C_H] + [N]$ in $G_0^{\mathcal{P}}(A)$.

If $H = \mathrm{H}^0(F_X^\bullet)$, we have that $\mathrm{H}^0(F_N^\bullet) = 0$ and $\tau_{\geqslant 0} F_X^\bullet = \tau_{\geqslant 0} F_N^\bullet$ by Lemma 4.6.(1,2). Using Lemma 2.24.(2,4), we obtain $[N] = -[\mathrm{H}^1(\tau_{\leqslant 1} F_N^\bullet)] = -[\mathrm{H}^1(\tau_{\leqslant 1} F_X^\bullet)]$ in $G_0^{\mathcal{P}}(A)$. In conclusion, $[X] = [I_H] - [C_H] + [N] = [I_H] - [C_H] - [\mathrm{H}^1(\tau_{\leqslant 1} F_X^\bullet)]$. This proves the claim.

Claim 3. Suppose given a complex $G^\bullet \in \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$. Let $l \in \mathbb{Z}$ such that $\mathrm{H}^{<l}(G^\bullet) = 0$. Let $r_1 \in \mathbb{Z}_{\geqslant l}$ such that $\mathrm{H}_{\geqslant r_1}(G_\bullet^*) = 0$. We write $X_k := \mathrm{H}^k(\tau_{\leqslant k} G^\bullet)$. Then the element

$$\sum_{k=l}^{r_1-1} (-1)^k([I_{\mathrm{H}^k(G^\bullet)}] - [C_{\mathrm{H}^k(G^\bullet)}]) + (-1)^{r_1}[X_{r_1}]$$

in $G_0^{\mathcal{P}}(A)$ is independent of the choice of l and r_1 provided $\mathrm{H}^{<l}(G^\bullet) = 0$ and $\mathrm{H}_{\geqslant r_1}(G_\bullet^*) = 0$. That is, for every $r_2 \geqslant r_1$, we have

$$(-1)^{r_1}[X_{r_1}] = \sum_{k=r_1}^{r_2-1} (-1)^k([I_{\mathrm{H}^k(G^\bullet)}] - [C_{\mathrm{H}^k(G^\bullet)}]) + (-1)^{r_2}[X_{r_2}].$$

Proof of claim 3. The independence of l of the sum above follows from $I_{\mathrm{H}^k(G^\bullet)} = 0$ and $C_{\mathrm{H}^k(G^\bullet)} = 0$ for $k < l$. We show that

$$0 = \sum_{k=r_1}^{r_2-1} (-1)^k([I_{\mathrm{H}^k(G^\bullet)}] - [C_{\mathrm{H}^k(G^\bullet)}]) + (-1)^{r_2}[X_{r_2}] - (-1)^{r_1}[X_{r_1}].$$

by induction on $r_2 \in \mathbb{Z}_{\geqslant r_1}$. For $r_2 = r_1$ there is nothing to show. For the induction step, we may assume that the equation holds for some $r_2 \geqslant r_1$. We obtain

$$\begin{aligned} &\sum_{k=r_1}^{r_2} (-1)^k([I_{\mathrm{H}^k(G^\bullet)}] - [C_{\mathrm{H}^k(G^\bullet)}]) + (-1)^{r_2+1}[X_{r_2+1}] - (-1)^{r_1}[X_{r_1}] \\ &= (-1)^{r_2}([I_{\mathrm{H}^{r_2}(G^\bullet)}] - [C_{\mathrm{H}^{r_2}(G^\bullet)}]) + (-1)^{r_2+1}[X_{r_2+1}] - (-1)^{r_2}[X_{r_2}]. \end{aligned}$$

Note that $\mathrm{H}_{\geqslant r_2}(G_\bullet^*) = 0$ and $\mathrm{H}_{\geqslant r_2}((F_{X_{r_2}}^\bullet[-r_2])^*) = \mathrm{H}_{\geqslant 0}((F_{X_{r_2}}^\bullet)^*) = 0$. Furthermore, we have $\mathrm{H}^{r_2}(\tau_{\leqslant r_2}(F_{X_{r_2}}^\bullet[-r_2])) = \mathrm{H}^0(\tau_{\leqslant 0} F_{X_{r_2}}^\bullet) \simeq X_{r_2} = \mathrm{H}^{r_2}(\tau_{\leqslant r_2} G^\bullet)$. Since projective resolutions are unique up to isomorphism in $\mathcal{K}(\operatorname{proj} A)$, this implies that $\tau_{\geqslant r_2} G^\bullet \simeq \tau_{\geqslant r_2}(F_{X_{r_2}}^\bullet[-r_2])$ in $\mathcal{K}(\operatorname{proj} A)$. In particular, we have $\mathrm{H}^{r_2}(G^\bullet) \simeq \mathrm{H}^{r_2}(F_{X_{r_2}}^\bullet[-r_2]) = \mathrm{H}^0(F_{X_{r_2}}^\bullet)$. Using claim 2, we obtain

$$\begin{aligned} [I_{\mathrm{H}^{r_2}(G^\bullet)}] - [C_{\mathrm{H}^{r_2}(G^\bullet)}] &= [\mathrm{H}^0(\tau_{\leqslant 0} F_{X_{r_2}}^\bullet)] + [\mathrm{H}^1(\tau_{\leqslant 1} F_{X_{r_2}}^\bullet)] \\ &= [\mathrm{H}^{r_2}(\tau_{\leqslant r_2} G^\bullet)] + [\mathrm{H}^{r_2+1}(\tau_{\leqslant r_2+1} G^\bullet)] \\ &= [X_{r_2}] + [X_{r_2+1}]. \end{aligned}$$

Thus, we have

$$\begin{aligned}(-1)^{r_2}([I_{\mathrm{H}^{r_2}(G^\bullet)}] - [C_{\mathrm{H}^{r_2}(G^\bullet)}]) + (-1)^{r_2+1}[X_{r_2+1}] - (-1)^{r_2}[X_{r_2}] = \\ (-1)^{r_2}([X_{r_2}] + [X_{r_2+1}]) + (-1)^{r_2+1}[X_{r_2+1}] - (-1)^{r_2}[X_{r_2}] = \ 0.\end{aligned}$$

This proves the claim.

Suppose given a complex $G^\bullet \in \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$. Let $l \in \mathbb{Z}$ such that $\mathrm{H}^{<l}(G^\bullet) = 0$. Let $r \in \mathbb{Z}_{\geqslant l}$ such that $\mathrm{H}_{\geqslant r}(G^*_\bullet) = 0$. We define a map $\tilde{\sigma}'$ as follows.

$$\tilde{\sigma}' : \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) \to G_0^{\mathcal{P}}(A) : G^\bullet \mapsto \sum_{k=l}^{r-1} (-1)^k([I_{\mathrm{H}^k(G^\bullet)}] - [C_{\mathrm{H}^k(G^\bullet)}]) + (-1)^r[\mathrm{H}^r(\tau_{\leqslant r} G^\bullet)].$$

By claim 3 this definition is independent of the choice of l and r.

Claim 4. Let $G^\bullet \to K^\bullet \to L^\bullet \to$ be a distinguished triangle in $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$. We have that $([G^\bullet] - [K^\bullet] + [L^\bullet])\tilde{\sigma}' = 0$ in $G_0^{\mathcal{P}}(A)$.

Proof of claim 4. Let $l \in \mathbb{Z}$ such that $\mathrm{H}^{<l}(G^\bullet) = 0$, $\mathrm{H}^{<l}(K^\bullet) = 0$ and $\mathrm{H}^{<l}(L^\bullet) = 0$. Let $r \in \mathbb{Z}_{\geqslant l}$ such that $\mathrm{H}_{\geqslant r}(G^*_\bullet) = 0$, $\mathrm{H}_{\geqslant r}(K^*_\bullet) = 0$ and $\mathrm{H}_{\geqslant r}(L^*_\bullet) = 0$. By Lemma 2.17, we have a split short exact sequence of complexes $0 \to \tau_{\leqslant r} G^\bullet \to \tau_{\leqslant r} \tilde{K}^\bullet \to \tau_{\leqslant r} L^\bullet \to 0$ with $\tilde{K}^\bullet \simeq K^\bullet$ in $\mathcal{K}(\operatorname{proj} A)$. Note that $[\mathrm{H}^r(\tau_{\leqslant r}\tilde{K}^\bullet)] = [\mathrm{H}^r(\tau_{\leqslant r} K^\bullet)]$. This results in the following long exact sequence of cohomology.

$$0 \to \mathrm{H}^l(G^\bullet) \to \mathrm{H}^l(K^\bullet) \to \cdots \to \mathrm{H}^{r-1}(L^\bullet) \xrightarrow{\delta} \mathrm{H}^r(\tau_{\leqslant r} G^\bullet) \to \mathrm{H}^r(\tau_{\leqslant r}\tilde{K}^\bullet) \to \mathrm{H}^r(\tau_{\leqslant r} L^\bullet) \to 0$$

Recall that for $X \in \operatorname{mod} A$ the modules I_X and C_X are uniquely determined up to isomorphism by the composition factors of X. The long exact sequence of cohomology now implies that

$$\sum_{k=l}^{r-1}(-1)^k([I_{\mathrm{H}^k(G^\bullet)}] - [I_{\mathrm{H}^k(K^\bullet)}] + [I_{\mathrm{H}^k(L^\bullet)}] - [C_{\mathrm{H}^k(G^\bullet)}] + [C_{\mathrm{H}^k(K^\bullet)}] - [C_{\mathrm{H}^k(L^\bullet)}]) + (-1)^r([I_{\mathrm{Im}(\delta)}] - [C_{\mathrm{Im}(\delta)}]) = 0$$

We write $X := \mathrm{H}^r(\tau_{\leqslant r} G^\bullet)$ and $\tilde{X} := \mathrm{H}^r(\tau_{\leqslant r} G^\bullet)/\operatorname{Im}(\delta) \simeq \operatorname{Ker}\big(\mathrm{H}^r(\tau_{\leqslant r}\tilde{K}^\bullet) \to \mathrm{H}^r(\tau_{\leqslant r} L^\bullet)\big)$. Note that $\operatorname{Im}(\delta) \simeq \operatorname{Cok}\big(\mathrm{H}^{r-1}(K^\bullet) \to \mathrm{H}^{r-1}(L^\bullet)\big) \hookrightarrow \mathrm{H}^r(G^\bullet) \simeq \mathrm{H}^0(F^\bullet_X)$. Thus, Lemma 4.6.(1,3) implies $\tau_{\geqslant 0} F^\bullet_X = \tau_{\geqslant 0} F^\bullet_{\tilde{X}}$ so that $X^* \simeq \tilde{X}^*$. Moreover, $[\mathrm{H}^r(\tau_{\leqslant r} G^\bullet)] = [X] = [I_{\mathrm{Im}(\delta)}] - [C_{\mathrm{Im}(\delta)}] + [\tilde{X}]$ by claim 2. Together with the above, we obtain

$$\begin{aligned}
&([G^\bullet] - [K^\bullet] + [L^\bullet])\tilde{\sigma}' \\
= &\sum_{k=l}^{r-1}(-1)^k([I_{\mathrm{H}^k(G^\bullet)}] - [I_{\mathrm{H}^k(K^\bullet)}] + [I_{\mathrm{H}^k(L^\bullet)}] - [C_{\mathrm{H}^k(G^\bullet)}] + [C_{\mathrm{H}^k(K^\bullet)}] - [C_{\mathrm{H}^k(L^\bullet)}]) \\
&+ (-1)^r([\mathrm{H}^r(\tau_{\leqslant r} G^\bullet)] - [\mathrm{H}^r(\tau_{\leqslant r} K^\bullet)] + [\mathrm{H}^r(\tau_{\leqslant r} L^\bullet)]) \\
= &- (-1)^r([I_{\mathrm{Im}(\delta)}] - [C_{\mathrm{Im}(\delta)}]) + (-1)^r([\mathrm{H}^r(\tau_{\leqslant r} G^\bullet)] - [\mathrm{H}^r(\tau_{\leqslant r} K^\bullet)] + [\mathrm{H}^r(\tau_{\leqslant r} L^\bullet)])
\end{aligned}$$

$$= (-1)^r([\tilde{X}] - [X] + [\mathrm{H}^r(\tau_{\leqslant r} G^\bullet)] - [\mathrm{H}^r(\tau_{\leqslant r} K^\bullet)] + [\mathrm{H}^r(\tau_{\leqslant r} L^\bullet)])$$
$$= (-1)^r([\tilde{X}] - [\mathrm{H}^r(\tau_{\leqslant r} \tilde{K}^\bullet)] + [\mathrm{H}^r(\tau_{\leqslant r} L^\bullet)])$$

It remains to show that the short exact sequence $0 \to \tilde{X} \to \mathrm{H}^r(\tau_{\leqslant r}\tilde{K}^\bullet) \to \mathrm{H}^r(\tau_{\leqslant r} L^\bullet) \to 0$ is perfect exact. The componentwise split sequence $0 \to \tau_{\geqslant r+1} L^*_\bullet \to \tau_{\geqslant r+1}\tilde{K}^*_\bullet \to \tau_{\geqslant r+1} G^*_\bullet \to 0$ induces the following short exact sequence.

$$0 \to \mathrm{H}_{r+1}(\tau_{\geqslant r+1} L^*_\bullet) \to \mathrm{H}_{r+1}(\tau_{\geqslant r+1}\tilde{K}^*_\bullet) \to \mathrm{H}_{r+1}(\tau_{\geqslant r+1} G^*_\bullet) \to 0$$

Recall that $\mathrm{H}_{\geqslant r}(L^*_\bullet) = 0$. Thus, we have

$$\mathrm{H}_{r+1}(\tau_{\geqslant r+1} L^*_\bullet) \simeq \mathrm{Cok}(L^*_{r+2} \to L^*_{r+1}) \simeq \mathrm{Ker}(L^*_r \to L^*_{r-1}) \simeq \mathrm{H}^r(\tau_{\leqslant r} L^\bullet)^*.$$

Similarly for the other two terms. This results in the following short exact sequence.

$$0 \to \mathrm{H}^r(\tau_{\leqslant r} L^\bullet)^* \to \mathrm{H}^r(\tau_{\leqslant r}\tilde{K}^\bullet)^* \to X^* \to 0$$

Since $X^* \simeq \tilde{X}^*$, we obtain that $0 \to \tilde{X} \to \mathrm{H}^r(\tau_{\leqslant r}\tilde{K}^\bullet) \to \mathrm{H}^r(\tau_{\leqslant r} L^\bullet) \to 0$ is a perfect exact sequence. This proves the claim.

Using claim 4, the map $\tilde{\sigma}'$ induces a map

$$\tilde{\sigma} : G_0(\mathcal{H}_{\mathcal{P}}(\mathrm{proj}\, A)) \to G_0^{\mathcal{P}}(A) : [G^\bullet] \mapsto \sum_{k=l}^{r-1} (-1)^k([I_{\mathrm{H}^k(G^\bullet)}] - [C_{\mathrm{H}^k(G^\bullet)}]) + (-1)^r[\mathrm{H}^r(\tau_{\leqslant r} G^\bullet)].$$

We show that $\sigma\,\tilde{\sigma} = \mathrm{id}_{G_0^{\mathcal{P}}(A)}$ and $\tilde{\sigma}\,\sigma = \mathrm{id}_{G_0(\mathcal{H}_{\mathcal{P}}(\mathrm{proj}\, A))}$. Then σ and $\tilde{\sigma}$ are mutually inverse isomorphisms.

For $X \in \mathrm{mod}\, A$, we have $[X]\sigma\,\tilde{\sigma} = [F^\bullet_X]\tilde{\sigma} = [\mathrm{H}^0(\tau_{\leqslant 0}\, F^\bullet_X)] = [X]$ since we can choose $l = r = 0$ in the definition of $\tilde{\sigma}$. On the other hand, suppose given $G^\bullet \in \mathcal{H}_{\mathcal{P}}(\mathrm{proj}\, A)$. Let $P^\bullet_k$ be the minimal projective resolution of $\mathrm{H}^k(G^\bullet)$ for $l \leqslant k \leqslant r-1$ where $l, r \in \mathbb{Z}$ with $\mathrm{H}^{<l}(G^\bullet) = 0$ and $\mathrm{H}_{\geqslant r}(G^*_\bullet) = 0$. Let $X_r := \mathrm{H}^r(\tau_{\leqslant r} G^\bullet)$. Using claim 1, we have $([I_{\mathrm{H}^k(G^\bullet)}] - [C_{\mathrm{H}^k(G^\bullet)}])\sigma = [P^\bullet_k]$. By Lemma 4.7.(2), we have $[G^\bullet] = \sum_{k=l}^{r-1}(-1)^k[P^\bullet_k] + (-1)^r[F^\bullet_{X_r}]$. Together, we obtain

$$\begin{aligned}[G^\bullet]\tilde{\sigma}\,\sigma &= \sum_{k=l}^{r-1} (-1)^k([I_{\mathrm{H}^k(G^\bullet)}] - [C_{\mathrm{H}^k(G^\bullet)}])\sigma + (-1)^r[X_r]\sigma \\ &= \sum_{k=l}^{r-1} (-1)^k[P^\bullet_k] + (-1)^r[F^\bullet_{X_r}] = [G^\bullet].\end{aligned}$$ □

Recall that a stable equivalence $\underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$ preserves perfect exact sequences if A and B are of finite representation type and have no nodes; cf. Definition 3.1 and Corollary 3.20.

Theorem 4.17. *Let $\alpha : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$ be a stable equivalence such that α and its quasi-inverse preserve perfect exact sequences. Then α induces an isomorphism $G_0^{\mathcal{P}}(A) \to G_0^{\mathcal{P}}(B)$.*

Proof. By assumption, every perfect exact sequence $0 \to X \to Y \oplus P \to Z \to 0$ in $\mathrm{mod}\, A$ with P projective induces a perfect exact sequence

$$0 \to \alpha(X) \to \alpha(Y) \oplus P' \to \alpha(Z) \to 0$$

with $P' \in \mathrm{proj}\, B$.

Hence, the natural map $G_0^{\mathcal{P}}(A) \to G_0^{\mathcal{P}}(B)$ given by $[X] \mapsto [\alpha(X)]$ is well-defined. Since α is an equivalence, this map is an isomorphism. □

Note that $G_0^{\mathrm{st}}(A) = 0$ if $\mathrm{gldim}\, A < \infty$. In fact, we have $[\Omega(X)] = -[X]$ by setting $Y = 0$ in the definition of $G_0^{\mathrm{st}}(A) = 0$. If $\Omega^n(X)$ is projective for some $n \geqslant 1$, we obtain $[X] = 0$.

In general, $G_0^{\mathcal{P}}(A)$ can be non-zero even in case of finite global dimension; cf. Example 7.14. Moreover, $G_0^{\mathrm{st}}(A)$ and $G_0^{\mathcal{P}}(A)$ are not isomorphic, even for algebras of infinite global dimension. See Example 7.7 for more details. However, the following holds.

Remark 4.18. We have a surjective group homomorphism

$$G_0^{\mathcal{P}}(A) \to G_0^{\mathrm{st}}(A) : [X] \mapsto [X]_{\mathrm{st}}\,.$$

If A is self-injective, this is an isomorphism.

In fact, every perfect exact sequence in the definition of $G_0^{\mathcal{P}}(A)$ is also a short exact sequence of the form stated in the definition of $G_0^{\mathrm{st}}(A)$. If A is self-injective, every short exact sequence is perfect exact. After potentially removing a split exact sequence starting in a projective module, every short exact sequence is of the form as stated in Definition 4.15.

We close this section with a remark on generating systems of $G_0^{\mathcal{P}}(A)$.

Remark 4.19. By construction, $G_0^{\mathcal{P}}(A)$ is generated by the indecomposable modules in $\underline{\mathrm{mod}}\, A$. Thus, $G_0(\mathcal{H}_{\mathcal{P}}(\mathrm{proj}\, A))$ is generated by $[F^\bullet]$ for $F^\bullet \in \mathcal{L}$ indecomposable by Theorem 4.16.

However, in general $G_0^{\mathcal{P}}(A)$ is not generated by the non-projective simple modules in $\underline{\mathrm{mod}}\, A$; cf. Example 7.14. In comparison, every simple minded system over A is a generating system of $G_0^{\mathrm{st}}(A)$ as was shown in [23, Lemma 2.3]. In particular, this holds for the simple A-modules.

4.3 Nakayama closure

In general, the triangulated category $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ discussed in Section 4.1 is neither characteristic in $\mathcal{K}(\operatorname{proj} A)$, nor in $\mathcal{H}(\operatorname{proj} A)$. In particular, for algebras of finite global dimension, the category is not closed under the derived Nakayama functor. However, the category $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ can be enlarged to the category $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ by replacing the projective-injective modules with strongly projective-injective modules.

In this section, we consider an equivalence $\nu_{\mathcal{K}} : \mathcal{K}^{+,b^*}(\operatorname{proj} A) \to \mathcal{K}^{-,b}(\operatorname{proj} A)$ induced by the Nakayama functor $\nu_A : \operatorname{proj} A \to \operatorname{inj} A$ where $\nu_A(P) = \mathrm{D}(P^*) = \mathrm{D}\operatorname{Hom}_A(P, A)$. In case that $\operatorname{gldim} A < \infty$, we retrieve the derived Nakayama functor $\mathcal{K}^b(\operatorname{proj} A) \to \mathcal{K}^b(\operatorname{proj} A)$. Our aim is to show that $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ is the smallest triangulated subcategory of $\mathcal{K}(\operatorname{proj} A)$ that contains $\mathcal{L}_A$ and is closed under $\nu_{\mathcal{K}}$ and under isomorphisms. Assuming that A can be embedded into a strongly projective-injective module, we give conditions under which $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ is characteristic in $\mathcal{H}(\operatorname{proj} A)$.

For the main proof, we will be able to reuse most of the results of Section 4.1. The main new technical result in Lemma 4.24 shows that a projective resolution of a simple module in ${}^{\perp}(\operatorname{stp} A)$ is contained in any triangulated subcategory that contains $\mathcal{L}$ and is closed under $\nu_{\mathcal{K}}$ and under isomorphisms.

For now, we start with a lemma on the Nakayama functor, which will be needed later. Note that in general the Nakayama functor on $\operatorname{mod} A$ is not fully faithful.

Lemma 4.20. *Let X be an A-module and $Z \in \mathcal{P}_A$. Then* $\operatorname{Hom}_A(\nu^{-1}X, Z) \simeq \operatorname{Hom}_A(X, \nu Z)$ *as k-vector spaces.*

Proof. Let $I^\bullet \in \mathcal{K}^+(\operatorname{inj} A)$ be an injective presentation of X.

$$X \hookrightarrow I^0 \xrightarrow{d_I} I^1$$

Applying ν^{-1} componentwise, we obtain a sequence $Q^\bullet \in \mathcal{K}^+(\operatorname{proj} A)$.

$$\nu^{-1}X \overset{\iota}{\hookrightarrow} Q^0 \xrightarrow{d_Q} Q^1$$

Note that we have $X = \operatorname{Ker}(d_I)$ and $\nu^{-1}X = \operatorname{Ker}(d_Q)$ since ν^{-1} is left exact.

Claim. We have $\operatorname{Hom}_A(X, Z) \simeq \operatorname{Hom}_{\mathcal{K}(\operatorname{inj} A)}(I^\bullet, Z)$ and $\operatorname{Hom}_A(\nu^{-1}X, Z) \simeq \operatorname{Hom}_{\mathcal{K}(\operatorname{proj} A)}(Q^\bullet, Z)$ for $Z \in \operatorname{inj} A$.

Suppose given $Y \in \operatorname{mod} A$ and a sequence $0 \to Y \to C^0 \to C^1$ in $\operatorname{mod} A$ with $\operatorname{Ker}(C^0 \xrightarrow{d} C^1)$. This gives a complex $C^\bullet \in \mathcal{K}(\operatorname{mod} A)$ with $C^k = 0$ for $k \notin \{0, 1\}$. We show that this implies $\operatorname{Hom}_A(Y, Z) \simeq \operatorname{Hom}_{\mathcal{K}(\operatorname{mod} A)}(C^\bullet, Z)$.

Let f be a non-zero morphism in $\mathrm{Hom}_A(Y,Z)$. Since Z is injective, there exists a morphism $\varphi^0 : C^0 \to Z$ such that the following diagram commutes.

$$\begin{tikzcd} Y \arrow[r, hook, "\iota"] \arrow[d, "f"] & C^0 \arrow[r, "d"] \arrow[dl, "\varphi^0"] & C^1 \\ Z & & \end{tikzcd}$$

Set $\varphi^k = 0$ for all $k \neq 0$. We obtain $\varphi^\bullet \in \mathrm{Hom}_{\mathcal{K}(\mathrm{mod}\,A)}(C^\bullet, Z)$.

Let $C^0 \xrightarrow{\tilde{\varphi}^0} Z$ be another morphism such that $f = \iota\,\tilde{\varphi}^0$. Then $\iota(\varphi^0 - \tilde{\varphi}^0) = 0$ and we obtain the following morphism of complexes.

$$\begin{tikzcd} Y \arrow[r, "\iota"] \arrow[d] & C^0 \arrow[r, "d"] \arrow[d, "\varphi^0 - \tilde{\varphi}^0"] & C^1 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] & 0 \end{tikzcd}$$

By Lemma 4.3.(1) this yields $\varphi^0 - \tilde{\varphi}^0 = 0$ so that $f\varphi^0 = \tilde{\varphi}^0$. Thus, $f \mapsto f\psi := \varphi^\bullet$ defines a k-linear map

$$\mathrm{Hom}_A(Y,Z) \xrightarrow{\psi} \mathrm{Hom}_{\mathcal{K}(\mathrm{mod}\,A)}(C^\bullet, Z).$$

It remains to show that ψ is an isomorphism. Suppose that $\varphi^\bullet = f\psi = 0$ in $\mathcal{K}(\mathrm{mod}\,A)$. In this case, there exists a morphism $h : C^1 \to Z$ such that $d\,h = \varphi^0$. However, this implies

$$0 = \iota\,d\,h = \iota\,\varphi^0 = f$$

so that ψ is injective. Now, suppose given a morphism $\varphi^\bullet \in \mathrm{Hom}_{\mathcal{K}(\mathrm{mod}\,A)}(C^\bullet, Z)$. Setting $f := \iota\,\varphi^0 \in \mathrm{Hom}_A(Y,Z)$, we obtain $f\psi = \varphi^\bullet$ so that ψ is surjective. This concludes the proof of the claim.

We obtain the following sequence of isomorphisms using that $\nu : \mathcal{K}(\mathrm{proj}\,A) \to \mathcal{K}(\mathrm{inj}\,A)$ is an equivalence.

$$\begin{aligned} \mathrm{Hom}_A(\nu^{-1}X, Z) &\simeq \mathrm{Hom}_{\mathcal{K}(\mathrm{proj}\,A)}(Q^\bullet, Z) \\ &\simeq \mathrm{Hom}_{\mathcal{K}(\mathrm{inj}\,A)}(I^\bullet, \nu Z) \\ &\simeq \mathrm{Hom}_A(X, \nu Z) \end{aligned}$$

□

The next lemma provides one part of the functor $\nu_{\mathcal{K}} : \mathcal{K}^{+,b^*}(\mathrm{proj}\,A) \to \mathcal{K}^{-,b}(\mathrm{proj}\,A)$. Recall that the equivalence $\nu : \mathrm{proj}\,A \to \mathrm{inj}\,A$ given by the Nakayama functor $\nu(-) = D(-)^*$ induces an equivalence $\mathcal{K}(\mathrm{proj}\,A) \to \mathcal{K}(\mathrm{inj}\,A)$ via $F^\bullet \mapsto (\nu F^k)_{k\in\mathbb{Z}}$.

Lemma 4.21. *The equivalence of triangulated categories*

$$\nu : \mathcal{K}(\operatorname{proj} A) \xrightarrow{\sim} \mathcal{K}(\operatorname{inj} A)$$

induced by the Nakayama functor restricts to an equivalence of triangulated categories

$$\mathcal{K}^{+,b^*}(\operatorname{proj} A) \xrightarrow{\sim} \mathcal{K}^{+,b}(\operatorname{inj} A).$$

Proof. Recall that $\nu(X) = \mathrm{D}(X^*)$ for an A-module X. Suppose given $P^\bullet \in \mathcal{K}^{+,b^*}(\operatorname{proj} A)$. Thus, we have $P^*_\bullet \in \mathrm{K}^{-,b}(A\text{-proj})$ and we obtain $\mathrm{D}(P^*_\bullet) = \nu P^\bullet \in \mathcal{K}^{+,b}(\operatorname{inj} A)$ since D is exact.

Recall that $\nu^{-1}(X) = \big(\mathrm{D}(X)\big)^*$ for an A-module X. Suppose given $I^\bullet \in \mathcal{K}^{+,b}(\operatorname{inj} A)$. Applying $\mathrm{D}(-)$ to $I^\bullet$ componentwise, we obtain a complex $\mathrm{D}\, I^\bullet \in \mathrm{K}^{-,b}(A\text{-proj})$ since D is exact. In particular $\mathrm{D}\, I^\bullet$ is bounded in cohomology so that $\big(\mathrm{D}\, I^\bullet\big)^* = \nu^{-1} I^\bullet \in \mathcal{K}^{+,b^*}(\operatorname{proj} A)$.

In conclusion, ν restricts to an equivalence $\mathcal{K}^{+,b^*}(\operatorname{proj} A) \xrightarrow{\sim} \mathcal{K}^{+,b}(\operatorname{inj} A)$. □

Recall that the canonical functors $\mathcal{K}^+(\operatorname{mod} A) \to \mathcal{D}^+(\operatorname{mod} A)$ and $\mathcal{K}^-(\operatorname{mod} A) \to \mathcal{D}^-(\operatorname{mod} A)$ induce equivalences of triangulated categories.

$$\mathcal{K}^{+,b}(\operatorname{inj} A) \xrightarrow{\sim} \mathrm{D}^b(\operatorname{mod} A) \xrightarrow{\sim} \mathcal{K}^{-,b}(\operatorname{proj} A)$$

Composing these equivalences with the one from Lemma 4.21 yields an equivalence of triangulated categories $\mathcal{K}^{+,b^*}(\operatorname{proj} A) \to \mathcal{K}^{+,b}(\operatorname{inj} A) \to \mathcal{K}^{-,b}(\operatorname{proj} A)$.

Definition 4.22. We denote the above composite of equivalences by

$$\nu_{\mathcal{K}} :\ \mathcal{K}^{+,b^*}(\operatorname{proj} A) \xrightarrow{\sim} \mathcal{K}^{-,b}(\operatorname{proj} A).$$

We say that a triangulated subcategory $\mathcal{T}$ of $\mathcal{K}(\operatorname{mod} A)$ is *closed under* $\nu_{\mathcal{K}}$ if the restriction of $\nu_{\mathcal{K}}$ to the full subcategory with objects in $\mathcal{T} \cap \mathcal{K}^{+,b^*}(\operatorname{proj} A)$ has an essential image in $\mathcal{T}$.

Remark 4.23. Recall that $\mathcal{H}(\operatorname{proj} A)$ is the full subcategory of $\mathcal{K}(\operatorname{proj} A)$ consisting of all complexes $F^\bullet \in \mathcal{K}(\operatorname{proj} A)$ such that there exist $l, r \in \mathbb{Z}$ with $\mathrm{H}^{<l}(F^\bullet) = 0$ and $\mathrm{H}_{\geqslant r}(F^*_\bullet) = 0$. In particular, $\mathcal{K}^{+,b^*}(\operatorname{proj} A)$ and $\mathcal{K}^{-,b}(\operatorname{proj} A)$ are subcategories of $\mathcal{H}(\operatorname{proj} A)$.

(1) Let $\operatorname{gldim} A < \infty$. Then $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A) = {}^\perp\mathcal{K}^b(\operatorname{stp} A) \cap \mathcal{K}^b(\operatorname{proj} A)$ and $\nu_{\mathcal{K}}$ is equivalent to the derived Nakayama functor $\mathcal{K}^b(\operatorname{proj} A) \xrightarrow{\sim} \mathcal{K}^b(\operatorname{proj} A)$.

(2) If A is self-injective, we will see that $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A) = \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) = \mathcal{L}_A$; cf. Theorem 4.45. Furthermore, the restriction of $\nu_{\mathcal{K}}$ to $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A) \cap \mathcal{K}^{+,b^*}(\operatorname{proj} A)$ is zero, since all non-zero complexes in $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ are unbounded. Thus, $\mathcal{L}_A$ is trivially closed under $\nu_{\mathcal{K}}$ in this case.

We aim to show that $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ is the smallest triangulated subcategory of $\mathcal{K}(\operatorname{proj} A)$ that contains $\mathcal{L}_A$ and is closed under $\nu_{\mathcal{K}}$ and under isomorphisms. It remains to verify that a projective resolution $P^{\leqslant 0}$ of a simple module $S \in {}^{\perp}(\operatorname{stp} A)$ is an element of every triangulated category that contains $\mathcal{L}$ and is closed under $\nu_{\mathcal{K}}$ and under isomorphisms. Note that no further steps are necessary, if ν-domdim $A \geqslant 1$. In this case a module $X \in {}^{\perp}(\operatorname{stp} A)$ satisfies $X^* = 0$. By Lemma 2.24.(5,7), we obtain that $F_X^\bullet \in \mathcal{L}_A$. Our strategy for the general proof is as follows.

We consider a complex $Q^\bullet \in \mathcal{K}(\operatorname{proj} A)$ such that $\nu_{\mathcal{K}}(Q^{>0}) \simeq P^{\leqslant 0}$ and $Q^{\leqslant 0}$ is a projective resolution of $\nu^{-1}S$. Furthermore, $\nu^{-1}S$ embeds into Q^1. If Q^1 is not injective, the result follows from Lemma 4.10. Otherwise, we inductively construct new simple modules S_k which embed into $\nu^{-k}(Q^1)$. By assumption, $\nu(Q^1)$ cannot be strongly projective-injective, so that this procedure terminates with a $\nu^{-k}(Q^1)$ which is not injective.

Lemma 4.24. *Let $\mathcal{T}$ be a triangulated subcategory of $\mathcal{K}(\operatorname{proj} A)$ that contains $\mathcal{L}$ and is closed under $\nu_{\mathcal{K}}$ and under isomorphisms.*

Suppose S is a simple A-module with minimal projective resolution $P^{\leqslant 0}$. If $S \in {}^{\perp}(\operatorname{stp} A)$, then $P^{\leqslant 0} \in \mathcal{T}$.

Proof. Suppose that S is injective. Then we have $S^* = 0$, otherwise S is projective and thus strongly projective-injective, a contradiction. Hence, $F_S^\bullet$ is the minimal projective resolution of S which is contained in $\mathcal{L}_A$.

For the remainder of the proof we assume that S is not injective. Let $S \hookrightarrow \nu Q^1 \to \nu Q^2$ be the minimal injective presentation of S with $Q^1, Q^2 \in \operatorname{proj} A$.

Extend $Q^1 \to Q^2$ to an element $Q^\bullet[2] \in \mathcal{L}$. Then the truncation $Q^{\leqslant 0}$ is the minimal projective resolution of $\nu^{-1}S = \operatorname{Ker}(Q^1 \to Q^2)$.

$$\begin{tikzcd} \cdots \arrow[r] & Q^{-1} \arrow[r] & Q^0 \arrow[rr] \arrow[dr, two heads] & & Q^1 \arrow[r] & Q^2 \arrow[r] & \cdots \\ & & & \nu^{-1}S \arrow[ur, hook] & & & \end{tikzcd}$$

Note that $Q^{>0} \in \mathcal{K}^{+,b^*}(\operatorname{proj} A)$. Applying $\nu_{\mathcal{K}}$ yields $\nu_{\mathcal{K}}(Q^{>0}) \simeq P^{\leqslant 0}$ in $\mathcal{K}^{-,b}(\operatorname{proj} A)$ since $\nu Q^{>0}$ is an injective resolution of S. Hence, $Q^{>0} \in \mathcal{T}$ implies $P^{\leqslant 0} \in \mathcal{T}$ since $\mathcal{T}$ is closed under $\nu_{\mathcal{K}}$. Moreover, we have the distinguished triangle

$$Q^{>0} \to Q^\bullet \to Q^{\leqslant 0} \to Q^{>0}[1]$$

so that $Q^{>0} \in \mathcal{T}$ if and only if $Q^{\leqslant 0} \in \mathcal{T}$ since $\mathcal{T}$ is closed under isomorphisms. It remains to show that $Q^{\leqslant 0} \in \mathcal{T}$. If $\nu^{-1}S = 0$, we have $Q^{\leqslant 0} = 0$ and we are done at this point.

Note that $\dim_k \operatorname{Hom}_A(S, \nu Q^1) = 1$ and $\dim_k \operatorname{Hom}_A(S, I) = 0$ for any injective module $I \not\simeq \nu Q^1$ since S is simple. Suppose that νQ^1 is projective-injective. Otherwise $S \in {}^{\perp}\mathcal{P}$ and thus $P^{\leqslant 0} \in \mathcal{T}$ by Lemma 4.10. By assumption, νQ^1 is not strongly projective-injective.

Claim. It suffices to consider the case that $Q^1 \in \operatorname{proj} A$ is not injective.

Assume that $Q^1 \in \mathcal{P}_A$. For every $Z \in \mathcal{P}_A$ with $Z \not\simeq Q^1$, that is $\nu Z \not\simeq \nu Q^1$, we have

$$\dim_k \operatorname{Hom}_A(\nu^{-1}S, Q^1) = \dim_k \operatorname{Hom}_A(S, \nu Q^1) = 1$$
$$\dim_k \operatorname{Hom}_A(\nu^{-1}S, Z) = \dim_k \operatorname{Hom}_A(S, \nu Z) = 0$$

by Lemma 4.20. Therefore, there exists a unique composition factor S_0 of $\nu^{-1}S$ with an embedding into Q^1. Furthermore, every other composition factor S' of $\nu^{-1}S$ lies in ${}^{\perp}\mathcal{P}_A$.

By Lemma 4.10 this means that the minimal projective resolution of every composition factor which is not isomorphic to S_0 is an element of $\mathcal{T}$. Therefore, by Lemma 4.9, the minimal projective resolution $Q^{\leqslant 0}$ of $\nu^{-1}S$ is an element of $\mathcal{T}$ if the minimal projective resolution of S_0 is an element of $\mathcal{T}$. Note that $S_0 \in {}^{\perp}\operatorname{stp} A$. If not, then Q^1 must be strongly projective-injective and therefore νQ^1 as well so that $S \notin {}^{\perp}(\operatorname{stp} A)$.

Now we can repeat the process described above for S_0 instead of S and Q^1 instead of νQ^1. Inductively, for $k \geqslant 0$, this results in a simple module $S_k \in {}^{\perp}(\operatorname{stp} A)$ which is a composition factor of $\nu^{-1}(S_{k-1})$. Furthermore, S_k embeds into $\nu^{-k}(Q^1)$. Since A is finite dimensional and Q^1 is not strongly projective-injective, there exists a $k \in \mathbb{Z}$ such that $\nu^{-k}(Q^1)$ is projective but not injective. Thus, it suffices to show that the minimal projective resolution of S_k is in $\mathcal{T}$. This concludes the proof of the claim.

As a result, we can assume that Q^1 is not injective. We show that $\nu^{-1}S \in {}^{\perp}\mathcal{P}_A$. If not, there is a $Z \in \mathcal{P}_A$ such that $\dim_k \operatorname{Hom}(\nu^{-1}S, Z) \neq 0$. However, using that νQ_1 is the injective hull of S, we have

$$\dim_k \operatorname{Hom}(\nu^{-1}S, Z) \simeq \dim_k \operatorname{Hom}(S, \nu Z) = 0$$

since $Z \not\simeq Q^1 \notin \mathcal{P}_A$, that is $\nu Z \not\simeq \nu Q^1$. This yields $\nu^{-1}S \in {}^{\perp}\mathcal{P}_A$. Hence, the minimal projective resolution $Q^{\leqslant 0}$ of $\nu^{-1}S$ is an element of $\mathcal{T}$ by Lemma 4.10. □

We give an example of the procedure used in the proof of Lemma 4.24.

Example 4.25. Let A be the quiver algebra over k given by

$$\begin{tikzcd}
1 \arrow[r, "\alpha"] & 2 \arrow[r, "\varepsilon"] \arrow[d, "\beta"] & 5 \\
4 \arrow[u, "\delta"] & 3 \arrow[l, "\gamma"] &
\end{tikzcd}$$

with relations $\beta\gamma\delta = \gamma\delta\alpha = \delta\alpha\beta = \alpha\varepsilon = 0$. The algebra has the following indecomposable projective modules. We also note their images under the functor $(-)^*$.

$$P_1 := \begin{smallmatrix} 1 \\ 2 \\ 3 \\ 4 \end{smallmatrix}, \qquad P_2 := \begin{smallmatrix} & 2 & \\ 3 & & 5 \\ & 4 & \end{smallmatrix}, \qquad P_3 := \begin{smallmatrix} 3 \\ 4 \\ 1 \end{smallmatrix}, \qquad P_4 := \begin{smallmatrix} 4 \\ 1 \\ 2 \end{smallmatrix}, \qquad P_5 := 5$$

$$P_1^* = \begin{smallmatrix} 1 \\ 4 \\ 3 \end{smallmatrix}, \qquad P_2^* = \begin{smallmatrix} 2 \\ 1 \\ 4 \end{smallmatrix}, \qquad P_3^* = \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}, \qquad P_4^* = \begin{smallmatrix} 4 \\ 3 \\ 2 \\ 1 \end{smallmatrix}, \qquad P_5^* = \begin{smallmatrix} 5 \\ 2 \end{smallmatrix}$$

We have the orbit $P_3 = \nu(P_1) = \nu^2(P_4) = \nu^3(P_2)$ under the Nakayama functor with $\nu(P_3)$ not projective. Therefore, P_1, P_3 and P_4 are projective-injective but not strongly projective-injective. Let $\mathcal{T}$ be a triangulated subcategory of $\mathcal{K}(\operatorname{proj} A)$ that contains $\mathcal{L}$ and is closed under $\nu_{\mathcal{K}}$ and under isomorphisms. We have $\nu_{\mathcal{K}} : \mathcal{K}^b(\operatorname{proj} A) \to \mathcal{K}^b(\operatorname{proj} A)$, since A has finite global dimension. We aim to show that the minimal projective resolution of the simple module $S := 1 = \operatorname{soc}(P_3)$ is in $\mathcal{T}$.

The minimal injective presentation of S is given by $(P_3 \to P_1) = (\nu(P_1) \to \nu(P_4))$. Thus, we set $Q^1 := P_1$ and extend $P_1 \xrightarrow{d} P_4$ to the following element in $\mathcal{L}_A$ denoted by $Q^\bullet[2]$ in the proof above.

$$0 \to P_5 \to P_2 \to P_1 \to P_3 \to P_1 \xrightarrow{d} P_4 \to P_2 \to P_4 \to P_3 \to 0$$

In particular, we have $\nu^{-1}S = \operatorname{Ker}(d) = \begin{smallmatrix} 3 \\ 4 \end{smallmatrix}$. It now suffices to show that the minimal projective resolution of $\nu^{-1}S$ is an element of $\mathcal{T}$.

Since $Q^1 = P_1$ is injective, we inductively construct new simple modules S_k which embed into $\nu^{-k}(Q^1)$. This terminates with $\nu^{-2}(Q^1) = P_2$ which is not injective.

We set $S_0 := 4$ as the unique composition factor of $\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}$ that embeds into $P_1 = \nu^0(Q^1) \in \mathcal{P}_A$. Note that the other composition factor 3 is an element of ${}^\perp\mathcal{P}_A$. Thus, it suffices to show that the minimal projective resolution of S_0 is an element of $\mathcal{T}$; cf. Lemma 4.10.

We repeat the steps above for S_0. The minimal injective presentation $\nu(P_4) \to \nu(P_3)$ results in the following complex in $\mathcal{L}_A$.

$$0 \to P_5 \to P_2 \to P_1 \to P_3 \oplus P_5 \to P_2 \to P_4 \xrightarrow{d_0} P_3 \to 0$$

We have $\nu^{-1}(S_0) = \operatorname{Ker}(d_0) = 2 = \operatorname{soc}(P_4)$.

We set $S_1 := 2$, which embeds into $P_4 = \nu^{-1}(Q^1) \in \mathcal{P}_A$. The minimal injective presentation $\nu(P_2) \to \nu(P_1)$ results in the following complex in $\mathcal{L}_A$.

$$0 \to P_5 \to P_2 \xrightarrow{d_1} P_1 \to P_3 \to P_2 \to P_4 \to P_3 \to 0$$

We have $\nu^{-1}(S_1) = \operatorname{Ker}(d_1) = 5 = P_5$ in the socle of P_2.

We set $S_2 := P_5$, which embeds into $P_2 = \nu^{-2}(Q^1) \notin \mathcal{P}_A$. Thus, the minimal projective resolution of S_2 is an element of $\mathcal{T}$ by Lemma 4.10.

We are now ready to prove the main result of this section.

Theorem 4.26. *Let A be a finite dimensional k-algebra.*

The category $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ is the smallest triangulated subcategory of $\mathcal{K}(\operatorname{proj} A)$ that contains $\mathcal{L}_A$ and is closed under $\nu_{\mathcal{K}}$ and under isomorphisms.

Proof. Note that $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) \subseteq \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A) \subseteq \mathcal{H}(\operatorname{proj} A)$ and $\operatorname{stp} A \subseteq \mathcal{P}_A$. By Lemma 4.4.(3), we have that $\mathcal{L} \subseteq \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) \subseteq \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$. Furthermore, $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ is a triangulated subcategory of $\mathcal{K}(\operatorname{proj} A)$ that is closed under isomorphisms by Remark 4.2.

We show that $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ is closed under $\nu_{\mathcal{K}}$. Let $F^\bullet \in \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A) \cap \mathcal{K}^{+,b^*}(\operatorname{proj} A)$. Since $\nu_{\mathcal{K}}\left(\mathcal{K}^b(\operatorname{stp} A)\right) \simeq \mathcal{K}^b(\operatorname{stp} A)$, we have $\nu_{\mathcal{K}} F^\bullet \in {}^\perp\mathcal{K}^b(\operatorname{stp} A)$. Since $\mathcal{K}^{-,b}(\operatorname{proj} A) \cap {}^\perp\mathcal{K}^b(\operatorname{stp} A)$ is contained in $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$, we obtain $\nu_{\mathcal{K}}(F^\bullet) \in \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$. In conclusion, $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ is a triangulated category that contains $\mathcal{L}$ and is closed under $\nu_{\mathcal{K}}$ and under isomorphisms.

Suppose that $\mathcal{T}$ is another triangulated subcategory of $\mathcal{K}(\operatorname{proj} A)$ that contains $\mathcal{L}$ and is closed under $\nu_{\mathcal{K}}$ and under isomorphisms. We show that $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A) \subseteq \mathcal{T}$.

By Lemma 4.5 and Lemma 4.7 it suffices to show that the minimal projective resolution of every A-module $X \in {}^\perp(\operatorname{stp} A)$ is an element of $\mathcal{T}$. Moreover, by Lemma 4.9 it suffices to show that the minimal projective resolution of every composition factor of X is an element of $\mathcal{T}$. Let S be such a composition factor. Then S is an element of ${}^\perp(\operatorname{stp} A)$ by Lemma 4.8. Using Lemma 4.24, we now obtain that the minimal projective resolution of S is an element of $\mathcal{T}$. □

Similarly as for $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$, we can characterize $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ using right perpendicular categories; cf. Remark 4.12.

Remark 4.27. The following are equivalent for a complex $F^\bullet \in \mathcal{K}(\operatorname{proj} A)$.

(1) $F^\bullet \in {}^\perp\mathcal{K}^b(\operatorname{stp} A)$.

(2) $H^k(F^\bullet) \in {}^\perp(\operatorname{stp} A)$ for all $k \in \mathbb{Z}$.

(3) $H^k(F^\bullet) \in (\operatorname{stp} A)^\perp$ for all $k \in \mathbb{Z}$.

(4) $F^\bullet \in \mathcal{K}^b(\operatorname{stp} A)^\perp$.

In particular, we have $F^\bullet \in \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ if and only if $F^\bullet \in \mathcal{K}^b(\operatorname{stp} A)^\perp$ and $F^\bullet \in \mathcal{H}(\operatorname{proj} A)$.

We also note the following for the Auslander-Reiten quiver of $\mathcal{D}^b(\operatorname{mod} A)$.

Remark 4.28. Suppose that A has finite global dimension. An Auslander-Reiten triangle in $\mathcal{H}(\operatorname{proj} A) \simeq \mathcal{K}^b(\operatorname{proj} A)$ is of the following form; cf. [15, Theorem 1.4].

$$\nu(F^\bullet)[-1] \to G^\bullet \to F^\bullet \to$$

Thus, $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ is the union of some Auslander-Reiten components in $\mathcal{K}^b(\operatorname{proj} A)$.

Example in Chapter 7. We explicitly describe all indecomposable complexes of $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ and $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ in Example 7.10 for the algebra A of Section 7.3. We also visualize both categories together with their subcategory $\mathcal{L}_A$.

We observe the following for the case of $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) = \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$. The second part is a direct consequence of Lemma 2.26.

Lemma 4.29. *The following holds.*

(1) $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) = \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ *if and only if* $\mathcal{P}_A = \operatorname{stp} A$.

(2) ν-domdim $A \geqslant 1$ *if and only if* $\mathcal{P}_A = \operatorname{stp} A$ *and* $\nu(S) = 0$ *for all simple* A*-modules* S *with* $\nu(F^0_S) \notin \mathcal{P}_A$.

Proof. Ad (1). Suppose that $\mathcal{P}_A = \operatorname{stp} A$. Then we have ${}^\perp\mathcal{K}^b(\mathcal{P}_A) = {}^\perp\mathcal{K}^b(\operatorname{stp} A)$ so that $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) = \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$.

On the other hand, suppose that $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) = \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$. In particular, this means that ${}^\perp\mathcal{K}^b(\mathcal{P}_A) = {}^\perp\mathcal{K}^b(\operatorname{stp} A)$. Assume that $Z \in \mathcal{P}_A$ is indecomposable and not an element of $\operatorname{stp} A$. Then we have $\operatorname{soc}(Z) \not\simeq \operatorname{soc}(Z')$ for all $Z' \in \operatorname{stp} A$ indecomposable. We obtain $\operatorname{Hom}_A(\operatorname{soc}(Z), Z') = 0$ for all $Z' \in \operatorname{stp} A$ indecomposable. Hence, $\operatorname{soc}(Z) \in {}^\perp(\operatorname{stp} A)$. Using Lemma 4.5, the assumption ${}^\perp\mathcal{K}^b(\mathcal{P}_A) = {}^\perp\mathcal{K}^b(\operatorname{stp} A)$ implies that $\operatorname{soc}(Z) \in {}^\perp\mathcal{P}_A$. However, we have $\operatorname{Hom}_A(\operatorname{soc}(Z), Z) \neq 0$, a contradiction.

Ad (2). Suppose that ν-domdim $A \geqslant 1$. Then $\mathcal{P}_A = \operatorname{stp} A$ and domdim $A \geqslant 1$ by Remark 1.17.(2). Now, Lemma 2.26 implies that $\nu(S) = 0$ for all simple A-modules S with $\nu(F^0_S) \notin \mathcal{P}_A$.

Conversely, suppose that $\mathcal{P}_A = \operatorname{stp} A$. Then ν-domdim A = domdim A. With Lemma 2.26 we have domdim $A \geqslant 1$. □

Let e be a basic idempotent element in A such that $\operatorname{add}(eA) = \operatorname{stp} A$. The algebra eAe is called an *associated self-injective algebra*; cf. [13, Section 4]. We give a characterization of $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ inside $\mathcal{H}(\operatorname{proj} A)$ using the algebra eAe.

Lemma 4.30. *Let e be a basic idempotent element in A such that* $\operatorname{add}(eA) = \operatorname{stp} A$ *and let* $F^\bullet \in \mathcal{K}(\operatorname{mod} A)$. *Then* $F^\bullet \in {}^\perp\mathcal{K}^b(\operatorname{stp} A)$ *if and only if* $(Fe)^\bullet \in \mathcal{K}(\operatorname{mod} eAe)$ *is acyclic.*

Suppose that $F^\bullet \in \mathcal{H}(\operatorname{proj} A)$. *Then* $F^\bullet \in \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ *if and only if* $(Fe)^\bullet \in \mathcal{K}(\operatorname{mod} eAe)$ *is acyclic.*

Proof. We use that $(Fe)^\bullet \in \mathcal{K}(\operatorname{mod} eAe)$ is acyclic if and only if $\mathrm{H}^k\big((Fe)^\bullet\big) = 0$ for all $k \in \mathbb{Z}$.

Since Ae is a projective left A-module, the functor $- \otimes_A Ae$ is exact. Thus, we have that $\mathrm{H}^k\big((Fe)^\bullet\big) \simeq \mathrm{H}^k(F^\bullet)e$. By assumption, $\nu(eA) \simeq eA$ so that

$$\mathrm{H}^k(F^\bullet)e \simeq \operatorname{Hom}_A\big(eA,\, \mathrm{H}^k(F^\bullet)\big) \simeq \mathrm{D}\operatorname{Hom}_A\big(\mathrm{H}^k(F^\bullet), \nu(eA)\big) \simeq \mathrm{D}\operatorname{Hom}_A\big(\mathrm{H}^k(F^\bullet),\, eA\big).$$

Note that $\operatorname{Hom}_A\big(\mathrm{H}^k(F^\bullet),\, eA\big) = 0$ if and only if $\mathrm{H}^k(F^\bullet) \in {}^\perp(\operatorname{stp} A) = {}^\perp\big(\operatorname{add}(eA)\big)$. Furthermore, we have $\mathrm{H}^k(F^\bullet) \in {}^\perp(\operatorname{stp} A)$ for all $k \in \mathbb{Z}$ if and only if we have $F^\bullet \in {}^\perp\mathcal{K}^b(\operatorname{stp} A)$ by Lemma 4.5. □

Example in Chapter 7. We illustrate the associated self-injective algebra and its connection to the category $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ in Example 7.12 of Section 7.3.

In [14], Fang, Hu and Koenig show that derived equivalences between two algebras restrict to derived equivalences between their associated self-injective subalgebras, provided the two given algebras have ν-dominant dimension at least one. Their result is based on the following theorem.

Theorem 4.31. ([14, Theorem 4.3]) *Let A and B be derived equivalent k-algebras, both of ν-dominant dimension at least* 1. *Then any derived equivalence* $\mathcal{D}^b(\operatorname{mod} A) \xrightarrow{\sim} \mathcal{D}^b(\operatorname{mod} B)$ *restricts to an equivalence of triangulated subcategories* $\mathcal{K}^b(\operatorname{stp} A) \xrightarrow{\sim} \mathcal{K}^b(\operatorname{stp} B)$.

Recall that $\mathcal{H}(\operatorname{proj} A) \simeq \mathcal{K}^b(\operatorname{proj} A) \simeq \mathcal{D}^b(\operatorname{mod} A)$, if $\operatorname{gldim} A < \infty$. We have the following corollary for our situation.

Corollary 4.32. *Let A and B be derived equivalent k-algebras, both of ν-dominant dimension at least* 1. *Assume that A and B have finite global dimension.*

Then any derived equivalence $\mathcal{K}^b(\operatorname{proj} A) \to \mathcal{K}^b(\operatorname{proj} B)$ *restricts to an equivalence of triangulated subcategories* $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A) \to \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} B)$.

In particular, $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A) = \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ *is a characteristic subcategory of* $\mathcal{K}^b(\operatorname{proj} A)$.

We aim to state a similar result for equivalences $\mathcal{H}(\operatorname{proj} A) \to \mathcal{H}(\operatorname{proj} B)$ without a restriction on the global dimension of A. We follow the same strategy used in [14]. As the main tool for proving the above theorem, they provide the following proposition.

Proposition 4.33. ([14, Proposition 4.2])

Let $\mathscr{X}_A := \{P^\bullet \in \mathcal{K}^b(\operatorname{stp} A) | P^\bullet \simeq \nu_A(P^\bullet)$ *in* $\mathcal{D}^b(\operatorname{mod} A)\}$. *Suppose that* A *has* ν*-dominant dimension at least* 1.

Then $\mathcal{K}^b(\operatorname{stp} A)$ *is the smallest triangulated full subcategory of* $\mathcal{K}^b(\operatorname{proj} A)$ *that contains* $\mathscr{X}_A$ *and is closed under taking direct summands.*

Furthermore, we need a way to restrict an equivalence $\mathcal{H}(\operatorname{proj} A) \to \mathcal{H}(\operatorname{proj} B)$ to bounded complexes. For this, we adapt the characterization of $\mathcal{K}^b(\operatorname{proj} A)$ inside $\mathcal{K}^{-,b}(\operatorname{proj} A)$. A complex $X^\bullet \in \mathcal{K}^{-,b}(\operatorname{proj} A)$ is an element of $\mathcal{K}^b(\operatorname{proj} A)$ if and only if for all $Y^\bullet \in \mathcal{K}^{-,b}(\operatorname{proj} A)$ there exists an $N \in \mathbb{Z}$ such that $\operatorname{Hom}_{\mathcal{K}(\operatorname{proj} A)}(X^\bullet, Y^\bullet[-n]) = 0$ if $n < N$.

Lemma 4.34. *Let* $X^\bullet \in \mathcal{H}(\operatorname{proj} A)$.

(1) We have $X^\bullet \in \mathcal{K}^{+,b^*}(\operatorname{proj} A)$ *if for all* $Y^\bullet \in \mathcal{H}(\operatorname{proj} A)$ *there exists an* $N \in \mathbb{Z}$ *such that* $\operatorname{Hom}_{\mathcal{H}(\operatorname{proj} A)}(X^\bullet, Y^\bullet[-n]) = 0$ *if* $n < N$.

(2) We have $X^\bullet \in \mathcal{K}^{-,b}(\operatorname{proj} A)$ *if for all* $Y^\bullet \in \mathcal{H}(\operatorname{proj} A)$ *there exists an* $N \in \mathbb{Z}$ *such that* $\operatorname{Hom}_{\mathcal{H}(\operatorname{proj} A)}(Y^\bullet[n], X^\bullet) = 0$ *if* $n < N$.

(3) Suppose that $X \in \mathcal{K}^b(\operatorname{stp} A)$. *Let* $Y^\bullet \in \mathcal{H}(\operatorname{proj} A)$. *Then there exists an* $N \in \mathbb{Z}$ *such that* $\operatorname{Hom}_{\mathcal{H}(\operatorname{proj} A)}(Y^\bullet[n], X^\bullet) = 0$ *and* $\operatorname{Hom}_{\mathcal{H}(\operatorname{proj} A)}(X^\bullet, Y^\bullet[-n]) = 0$ *if* $n < N$.

Proof. Ad (1). It suffices to show, that $X^\bullet$ is bounded on the left.

We assume that $X^\bullet$ is unbounded on the left. Since $X^\bullet \in \mathcal{H}(\operatorname{proj} A)$, there exists an $N_0 \in \mathbb{Z}$ such that $\operatorname{H}^n(X^\bullet) = 0$ for $n < N_0$. Furthermore, there exists an $N < N_0$ such that $\operatorname{Ker}(d_X^n)$ is not projective for $n < N$. Otherwise the complex $X^\bullet$ would be isomorphic to a complex which is left bounded by removing split direct summands.

Let $\{S_1, \ldots, S_l\}$ be a complete set of pairwise non-isomorphic simple A-modules. We write $S := \bigoplus_{i=1}^l S_i$. Let $Y^\bullet$ be the minimal projective resolution of S. Note that $Y^\bullet$ is an element of $\mathcal{H}(\operatorname{proj} A)$. We show that $\operatorname{Hom}_{\mathcal{H}(\operatorname{proj} A)}(X^\bullet, Y^\bullet[-n]) \neq 0$ for $n < N$.

Let $n < N$. We write $X := \operatorname{H}^n(\tau_{\leqslant n} X^\bullet) = \operatorname{Ker}(d_X^{n+1}) \in \operatorname{mod} A$ using that $n + 1 \leqslant N < N_0$. Suppose that f is the composite of the natural projection $p : X \to X/\operatorname{rad}(X)$ and the natural embedding $X/\operatorname{rad}(X) \to S$. Since $\operatorname{H}^k(X^\bullet) = 0$ and $\operatorname{H}^k(Y^\bullet[-n]) = 0$ for $k < n$, the morphism

f lifts to a morphism $f^\bullet$ of complexes.

$$\begin{tikzcd}
X^\bullet \arrow[dddd, "f^\bullet"] & \cdots \arrow[r] & X^{n-1} \arrow[r] \arrow[ddd, "f^{-1}"] & X^n \arrow[rr] \arrow[ddd, "f^0"] \arrow[dr, two heads] & & X^{n+1} \arrow[r] \arrow[ddd] & \cdots \\
& & & & X \arrow[ur, hook] \arrow[d, "f"] & & \\
& & & & S & & \\
Y^\bullet[-n] & \cdots \arrow[r] & Y^{-1} \arrow[r] & Y^0 \arrow[rr] \arrow[ur, two heads] & & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Assume that $f^\bullet = 0$. By Lemma 2.3.(1), this implies that f factors through the projective cover P of $X/\operatorname{rad}(X) = \operatorname{Im}(f)$. Since p is surjective, we obtain a morphism $g : P \to X$.

$$\begin{tikzcd}
& P \arrow[d] \arrow[dl, dashed, "g"'] \\
X \arrow[r, "p"] & X/\operatorname{rad}(X)
\end{tikzcd}$$

Using that $X/\operatorname{rad}(X) \simeq P/\operatorname{rad}(P)$, we obtain that g is surjective so that X is a direct summand of P. This is a contradiction to the choice of $\operatorname{Ker}(d_X^{n+1}) = X$ as non-projective. Therefore, the morphism $f^\bullet : X^\bullet \to Y^\bullet[-n]$ is non-zero.

Ad (2). We have $X^\bullet \in \mathcal{K}^{-,b}(\operatorname{proj} A)$ if and only if $X^*_\bullet \in \mathcal{K}^{+,b^*}(A\text{-proj})$.

We rename $X^*_\bullet$ as $U^\bullet$ via $U^k := X^*_{-k} \in \mathcal{K}^{+,b^*}(A\text{-proj})$ so that $U^\bullet[1]$ shifts the complex to the left. In contrast, $X^*_\bullet[1]$ is a shift to the right.

$$\begin{aligned}
U^\bullet = X^*_\bullet : &\qquad \cdots \longrightarrow U^{-1} = X^*_1 \longrightarrow U^0 = X^*_0 \longrightarrow U^1 = X^*_{-1} \longrightarrow \cdots \\
U^\bullet[1] = X^*_\bullet[-1] : &\qquad \cdots \longrightarrow U^{-1} = X^*_1 \longrightarrow U^0 = X^*_0 \longrightarrow U^1 = X^*_{-1} \longrightarrow \cdots
\end{aligned}$$

Since part (1) also holds for left A-modules, it suffices to show that for all $V^\bullet \in \mathcal{H}(A\text{-proj})$ there exists an $N \in \mathbb{Z}$ such that we have $\operatorname{Hom}_{\mathcal{H}(A\text{-proj})}(U^\bullet, V^\bullet[-n]) = 0$ if $n < N$.

Let $V^\bullet \in \mathcal{H}(A\text{-proj})$ and write $V^k = Y^*_{-k} \in \mathcal{H}(A\text{-proj})$ for a complex $Y^\bullet \in \mathcal{H}(\operatorname{proj} A)$. By assumption, we have an $N \in \mathbb{Z}$ such that the following holds for $n < N$.

$$0 = \operatorname{Hom}_{\mathcal{H}(A\text{-proj})}(Y^\bullet[n], X^\bullet) \simeq \operatorname{Hom}_{\mathcal{H}(A\text{-proj})}(X^*_\bullet, Y^*_\bullet[n]) = \operatorname{Hom}_{\mathcal{H}(A\text{-proj})}(U^\bullet, V^\bullet[-n])$$

Ad (3). Suppose given $X^\bullet \in \mathcal{K}^b(\operatorname{stp} A)$ and $Y^\bullet \in \mathcal{H}(\operatorname{proj} A)$. Without loss of generality we may assume that $X^k = 0$ for $k > 0$. Let $l \in \mathbb{Z}_{\geqslant 0}$ such that $X^k = 0$ for $k < -l$.

Let $N_1 \in \mathbb{Z}$ such that $\mathrm{H}^n(Y^\bullet) = 0$ for all $n < N_1$. Using that $X^{>0} = 0$, Lemma 4.4.(1) implies that $\mathrm{Hom}_{\mathcal{H}(\mathrm{proj}\,A)}(Y^\bullet[n], X^\bullet) = 0$ if $n < N_1$.

$$\begin{tikzcd}
Y^\bullet[n] \arrow[d] & \cdots \arrow[r] & Y^{n-l-1} \arrow[r] \arrow[d] & Y^{n-l} \arrow[r] \arrow[d] & Y^{n-l+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & Y^{n-1} \arrow[r] \arrow[d] & Y^{n} \arrow[r] \arrow[d] & Y^{n+1} \arrow[r] \arrow[d] & \cdots \\
X^\bullet & & 0 \arrow[r] & X^{-l} \arrow[r] & X^{l-1} \arrow[r] & \cdots \arrow[r] & X^{-1} \arrow[r] & X^0 \arrow[r] & 0 &
\end{tikzcd}$$

Let $N_2 \in \mathbb{Z}$ such that $\mathrm{H}_{-n}(Y_\bullet^*) = 0$ for $n < N_2$. Using that $X^k = 0$ for $k < -l$, Lemma 4.4.(1') implies that $\mathrm{Hom}_{\mathcal{H}(\mathrm{proj}\,A)}(X^\bullet, Y^\bullet[-n+l]) = 0$ if $n < N_2$. Note that $X_\bullet^* \in \mathcal{K}^b(A\text{-inj})$ since $X^\bullet \in \mathcal{K}^b(\mathrm{stp}\,A)$.

$$\begin{tikzcd}
X^\bullet \arrow[d] & & 0 \arrow[r] \arrow[d] & X^{-l} \arrow[r] \arrow[d] & X^{-l+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & X^{-1} \arrow[r] \arrow[d] & X^0 \arrow[r] \arrow[d] & 0 \arrow[d] & \\
Y^\bullet[-n+l] & \cdots \arrow[r] & Y^{-n-1} \arrow[r] & Y^{-n} \arrow[r] & Y^{-n+1} \arrow[r] & \cdots \arrow[r] & Y^{-n+l-1} \arrow[r] & Y^{-n+l} \arrow[r] & Y^{-n+l+1} \arrow[r] & \cdots
\end{tikzcd}$$

Let $N := \min\{N_1, N_2 - l\}$. Then $\mathrm{Hom}_{\mathcal{H}(\mathrm{proj}\,A)}(Y^\bullet[n], X^\bullet) = 0$ and $\mathrm{Hom}_{\mathcal{H}(\mathrm{proj}\,A)}(X^\bullet, Y^\bullet[-n]) = 0$ if $n < N$. □

In [38, Proposition 5.2] Rickard has shown that any standard derived equivalence commutes with the Nakayama functor. It seems unclear whether the same holds true for an equivalence between $\mathcal{H}(\mathrm{proj}\,A)$ and $\mathcal{H}(\mathrm{proj}\,B)$. Therefore, we add a further assumption in contrast to Theorem 4.31. In case that $\mathrm{gldim}\,A < \infty$, we are in the situation of Theorem 4.31 where the additional steps of the following theorem are not needed.

Theorem 4.35. *Suppose given two finite dimensional k-algebras A and B both with ν-dominant dimension at least 1. Let $\alpha : \mathcal{H}(\mathrm{proj}\,A) \to \mathcal{H}(\mathrm{proj}\,B)$ be a triangulated equivalence such that there is a natural isomorphism $\nu_B(\alpha(X^\bullet)) \simeq \alpha(\nu_A X^\bullet)$ for all $X \in \mathcal{K}^b(\mathrm{stp}\,A)$.*

Then α restricts to an equivalence $\mathcal{K}^b(\mathrm{stp}\,A) \simeq \mathcal{K}^b(\mathrm{stp}\,B)$. Moreover, α restricts to an equivalence between $\mathcal{H}_{\mathcal{P}}(\mathrm{proj}\,A) = \mathcal{H}_{\mathrm{stp}}(\mathrm{proj}\,A)$ and $\mathcal{H}_{\mathcal{P}}(\mathrm{proj}\,B) = \mathcal{H}_{\mathrm{stp}}(\mathrm{proj}\,B)$.

Proof. Suppose given $X^\bullet \in \mathcal{K}^b(\mathrm{stp}\,A)$. By Lemma 4.34.(3), there is an $N \in \mathbb{Z}$ such that we have $\mathrm{Hom}_{\mathcal{H}(\mathrm{proj}\,A)}(Y^\bullet[n], X^\bullet) = 0$ and $\mathrm{Hom}_{\mathcal{H}(\mathrm{proj}\,A)}(X^\bullet, Y^\bullet[-n]) = 0$ for all $n < N$ and $Y^\bullet \in \mathcal{H}(\mathrm{proj}\,A)$.

Let $n < N$ and $Z^\bullet \in \mathcal{H}(\mathrm{proj}\,B)$. Since α is an equivalence, there exists a $Y^\bullet \in \mathcal{H}(\mathrm{proj}\,A)$ with $\alpha(Y^\bullet) = Z^\bullet$. We have the following.

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{H}(\mathrm{proj}\,B)}(Z^\bullet[n], \alpha(X^\bullet)) &\simeq \mathrm{Hom}_{\mathcal{H}(\mathrm{proj}\,A)}(Y^\bullet[n], X^\bullet) = 0 \\
\mathrm{Hom}_{\mathcal{H}(\mathrm{proj}\,B)}(\alpha(X^\bullet), Z^\bullet[-n]) &\simeq \mathrm{Hom}_{\mathcal{H}(\mathrm{proj}\,A)}(X^\bullet, Y^\bullet[-n]) = 0.
\end{aligned}$$

Hence, by Lemma 4.34.(1, 2), we obtain $\alpha(X^\bullet) \in \mathcal{K}^b(\mathrm{proj}\,B)$.

Now, suppose that $X^\bullet \in \mathscr{X}_A$; cf. Proposition 4.33. By assumption, we have $\alpha(X^\bullet) \simeq \alpha(\nu_A(X^\bullet)) \simeq \nu_B(\alpha(X^\bullet))$, which implies that $\alpha(X^\bullet) \in \mathscr{X}_B$. In conclusion, we have shown, that $\alpha(\mathscr{X}_A) \subseteq \mathscr{X}_B$. By Proposition 4.33, we therefore obtain $\alpha(\mathcal{K}^b(\operatorname{stp} A)) \subseteq \mathcal{K}^b(\operatorname{stp} B)$.

Let β be a quasi-inverse of α. Repeating the arguments from above, we similarly obtain that $\beta(\mathcal{K}^b(\operatorname{stp} B)) \subseteq \mathcal{K}^b(\operatorname{stp} A)$. Together, we can conclude that α induces an equivalence $\mathcal{K}^b(\operatorname{stp} A) \simeq \mathcal{K}^b(\operatorname{stp} B)$.

Recall that $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ is the full subcategory of $\mathcal{H}(\operatorname{proj} A)$ with objects in ${}^\perp\mathcal{K}^b(\operatorname{stp} A)$. Hence, α also induces an equivalence $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A) \simeq \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} B)$. Since ν-domdim $A \geqslant 1$ and ν-domdim $B \geqslant 1$, we have $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A) = \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ and $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} B) = \mathcal{H}_{\mathcal{P}}(\operatorname{proj} B)$; cf. Lemma 4.29. □

4.4 Stable Gorenstein-projective modules

So far, we have discussed triangulated categories that contain $\mathcal{L}_A$. On the other hand, $\underline{\operatorname{mod}}\, A$ always contains the triangulated category of stable Gorenstein-projective modules. This category has a close connection with the homotopy category of totally acylic complexes inside $\mathcal{K}(\operatorname{proj} A)$; cf. Definition 1.7. We begin with the definition of Gorenstein-projective modules.

Definition 4.36. An A-module X is said to be *Gorenstein-projective* if there exists a totally acyclic complex $F^\bullet \in \mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$ such that $\mathrm{H}^0(\tau_{\leqslant 0} F^\bullet) \simeq X$.

Let $\operatorname{Gproj} A$ be the full subcategory of $\operatorname{mod} A$ consisting of Gorenstein-projective modules. Let $\underline{\operatorname{Gproj}}\, A$ be the full subcategory of $\underline{\operatorname{mod}}\, A$ consisting of Gorenstein-projective modules. Note that a projective A-module P is Gorenstein-projective via the complex $0 \to P \to P \to 0$.

The following lemma collects some facts about the category of Gorenstein-projective modules which can be found in [11, Section 2.1]. The first property implies that every short exact sequence in $\operatorname{Gproj} A$ is perfect exact; cf. Lemma 2.23.

Lemma 4.37.

(1) $\operatorname{Ext}^1_A(X, A) = 0$ *for* $X \in \operatorname{Gproj} A$.

(2) The syzygy functor $\Omega : \underline{\operatorname{Gproj}}\, A \to \underline{\operatorname{Gproj}}\, A$ *is a self-equivalence of categories.*

(3) The category $\underline{\operatorname{Gproj}}\, A$ *is triangulated with suspension* Ω^{-1} *and distinguished triangles isomorphic to those induced by short exact sequences.*

Note that $\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$ is contained in $\mathcal{L}_A$. Using this, a Gorenstein-projective module X can be characterized via its image $F^\bullet_X$ in $\mathcal{L}_A$.

Lemma 4.38. *The following are equivalent for $X \in \operatorname{mod} A$.*

(1) X is Gorenstein-projective.

(2) $F^\bullet_X \in \mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$.

(3) $F^\bullet_X[k] \in \mathcal{L}_A$ for all $k \in \mathbb{Z}$.

Proof. Suppose that X is Gorenstein-projective. Assume that X is not projective, otherwise $F^\bullet_X \simeq 0$ in $\mathcal{L}$.

By definition, there exists a totally acyclic complex $P^\bullet \in \mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$ such that $\mathrm{H}^0(\tau_{\leqslant 0} P^\bullet) \simeq X$. However, such a complex $P^\bullet$ is an element of $\mathcal{L}$. Therefore, we have that $F^\bullet_X \simeq P^\bullet$ by Theorem 2.6, since $\mathrm{H}^0(\tau_{\leqslant 0} P^\bullet) \simeq X = \mathrm{H}^0(\tau_{\leqslant 0} F^\bullet_X)$. We obtain that F_X is totally acyclic.

Recall that $\mathrm{H}^0(\tau_{\leqslant 0}\, F^\bullet_X) = X$. Hence, (2) implies (1). Furthermore, a totally acyclic complex $F^\bullet$ satisfies $\mathrm{H}^k(F^\bullet) = 0$ and $\mathrm{H}_k(F^*_\bullet) = 0$ for all $k \in \mathbb{Z}$. Thus, (2) also implies (3).

Now, suppose that $F^\bullet_X[k] \in \mathcal{L}$ for all $k \in \mathbb{Z}$. We show that $F^\bullet_X$ is a totally acyclic complex. Using that $\mathcal{L}_A := \{F^\bullet \in \mathcal{K}(\operatorname{proj} A) \mid \mathrm{H}^{<0}(F^\bullet) = 0,\ \mathrm{H}_{\geqslant 0}(F^*_\bullet) = 0\}$, we obtain the following for all $k \in \mathbb{Z}$.

$$\begin{aligned} \mathrm{H}^k(F^\bullet_X) &= \mathrm{H}^{-1}(F^\bullet_X[k+1]) = 0 \\ \mathrm{H}_k((F^*_X)_\bullet) &= \mathrm{H}_0((F^*_X)_\bullet[k]) = \mathrm{H}_0\left((F^\bullet_X[k])^*\right) = 0 \end{aligned}$$

In conclusion, $F^\bullet_X \in \mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$. □

Example in Chapter 7. In Example 7.8 the totally acyclic complexes in $\mathcal{K}(\operatorname{proj} A)$ for the algebra A of Section 7.2 are calculated using the previous lemma.

Remark 4.39. The category $\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$ is the largest subcategory of $\mathcal{L}_A$ that is triangulated as a subcategory of $\mathcal{K}(\operatorname{proj} A)$.

In fact, a triangulated category is closed under shifts. However, a complex $F^\bullet$ in $\mathcal{L}_A$ with $F^\bullet_X[k] \in \mathcal{L}_A$ for all $k \in \mathbb{Z}$ is totally acyclic by Lemma 4.38.

We recover that the category of stable Gorenstein-projective modules is equivalent to $\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$. See [8, Theorem 4.4.1] or [25, Proposition 7.2] for different approaches.

Lemma 4.40. *The equivalence $\mathcal{F} : \underline{\operatorname{mod}}\, A \to \mathcal{L}_A$ restricts to an equivalence of triangulated categories*

$$\underline{\operatorname{Gproj}}\, A \xrightarrow{\sim} \mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A).$$

Proof. Lemma 4.38 shows that $\mathcal{F}$ restricts to an equivalence $\underline{\mathrm{Gproj}}\, A \to \mathcal{K}_{\mathrm{tac}}(\mathrm{proj}\, A)$. It remains to show that this is a triangulated functor.

Using Lemma 4.37.(1), we know that every short exact sequence in $\mathrm{Gproj}\, A$ is a perfect exact sequence by Lemma 2.23. By Proposition 2.18, the functor $\mathcal{F}$ maps perfect exact sequences to distinguished triangles in $\mathcal{K}(\mathrm{proj}\, A)$ and therefore preserves triangles. In particular, a perfect exact sequence

$$0 \to \Omega(X) \to P \to X \to 0$$

with $X \in \mathrm{Gproj}\, A$ and $P \in \mathrm{proj}\, A$ corresponds to the following triangle.

$$F^\bullet_{\Omega(X)} \to 0 \to F^\bullet_X \to$$

Thus, we have a natural isomorphism $F^\bullet_X[-1] \simeq F^\bullet_{\Omega(X)}$ so that $\mathcal{F}$ commutes with the shift. □

We note that condition (3) in Lemma 4.38 can be expressed via the existence of perfect exact sequences with projective middle term. Recall that a stable equivalence $\underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$ preserves perfect exact sequences if A and B are of finite representation type and have no nodes; cf. Definition 3.1 and Corollary 3.20. Furthermore, every stable equivalence induced by an exact functor preserves perfect exact sequences with projective middle term if the inverse equivalence is also induced by an exact functor; cf. Proposition 3.2.

Lemma 4.41. *Let $\alpha : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$ be a stable equivalence such that α and its quasi-inverse preserve perfect exact sequences with projective middle term. Suppose given $X \in \mathrm{mod}\, A$ and $k \in \mathbb{Z}$.*

We have $F^\bullet_X[k] \in \mathcal{L}_A$ if and only if $F^\bullet_{\alpha(X)}[k] \in \mathcal{L}_B$.

Proof. If $F^\bullet_X[1] \in \mathcal{L}_A$, there exists a $Y \in \mathrm{mod}\, A$ such that $F^\bullet_Y \simeq F^\bullet_X[1]$ in $\mathcal{K}(\mathrm{proj}\, A)$. We have a distinguished triangle $F^\bullet_X \to 0 \to F^\bullet_Y \to$ in $\mathcal{K}(\mathrm{proj}\, A)$. By Proposition 2.18, the triangle induces a perfect exact sequence $0 \to X \to P \to Y \to 0$ with some $P \in \mathrm{proj}\, A$. By assumption, we obtain a perfect exact sequence $0 \to \alpha(X) \to Q \to \alpha(Y) \to 0$ with some $Q \in \mathrm{proj}\, B$. By Proposition 2.18 the sequence induces a distinguished triangle in $\mathcal{L}_B$.

$$F_{\alpha(X)} \to 0 \to F_{\alpha(Y)} \to$$

We obtain that $F_{\alpha(X)}[1] \simeq F_{\alpha(Y)} \in \mathcal{L}_B$. Swapping the roles of X and Y shows that $F^\bullet_X[-1] \in \mathcal{L}_A$ implies $F_{\alpha(X)}[-1] \in \mathcal{L}_B$ as well. Inductively, we have that $F^\bullet_X[k] \in \mathcal{L}_A$ implies $F^\bullet_{\alpha(X)}[k] \in \mathcal{L}_B$ for all $k \in \mathbb{Z}$.

Let $\beta : \underline{\mathrm{mod}}\, B \to \underline{\mathrm{mod}}\, A$ be the quasi-inverse of α. The same argument as above yields $F^\bullet_{\beta(\alpha(X))}[k] \in \mathcal{L}_A$ if $F_{\alpha(X)}[k] \in \mathcal{L}_B$. Since $\beta(\alpha(X)) \simeq X$ in $\underline{\mathrm{mod}}\, A$, Lemma 2.4.(1) shows that $F^\bullet_X[k] \simeq F^\bullet_{\beta(\alpha(X))}[k] \in \mathcal{L}_A$ for $k \in \mathbb{Z}$. □

In the setting of the previous lemma, a stable equivalence restricts to an equivalence between the stable categories of Gorenstein-projective modules. If α additionally preserves arbitrary perfect exact sequences, this restriction is a triangulated equivalence.

Theorem 4.42. *Let $\alpha : \underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$ be a stable equivalence such that α and its quasi-inverse preserve perfect exact sequences. Then α restricts to a triangulated equivalence*

$$\underline{\operatorname{Gproj}}\, A \to \underline{\operatorname{Gproj}}\, B\,.$$

Proof. Let $X \in \underline{\operatorname{Gproj}}\, A$. By Lemma 4.38, we have that $F^\bullet_X[k] \in \mathcal{L}_A$ for all $k \in \mathbb{Z}$. Using Lemma 4.41 and Lemma 4.38 again, we obtain that $\alpha(X) \in \underline{\operatorname{Gproj}}\, B$. In conclusion, α restricts to an equivalence

$$\alpha : \underline{\operatorname{Gproj}}\, A \to \underline{\operatorname{Gproj}}\, B\,.$$

It remains to show that this is a triangulated functor.

By Lemma 4.37.(1) all short exact sequences of Gorenstein-projective modules are perfect exact, so that α preserves short exact sequences and thus distinguished triangles. We show that $\alpha(\Omega(X)) \simeq \Omega(\alpha(X))$ in $\underline{\operatorname{Gproj}}\, B$ for all $X \in \underline{\operatorname{Gproj}}\, A$.

Let $0 \to \Omega(X) \to P \to X \to 0$ be a short exact sequence without split summands where P is the projective cover of X. We know that this sequence lies in $\operatorname{Gproj} A$ and therefore must be a perfect exact sequence. By assumption, we obtain a perfect exact sequence

$$0 \to \alpha(\Omega(X)) \to \tilde{P} \to \alpha(X) \to 0$$

in $\operatorname{Gproj} B$ with $\tilde{P} \in \operatorname{proj} B$. By Proposition 2.18, this induces the following distinguished triangle.

$$F^\bullet_{\alpha(\Omega(X))} \to 0 \to F^\bullet_{\alpha(X)} \to$$

Thus, we have a natural isomorphism $F^\bullet_{\alpha(\Omega(X))} \simeq F^\bullet_{\alpha(X)}[-1]$. On the other hand, consider the short exact sequence $0 \to \Omega(\alpha(X)) \to Q \to \alpha(X) \to 0$ with Q the projective cover of $\alpha(X)$ in $\operatorname{mod} B$. As above, this is a perfect exact sequence and therefore induces the following distinguished triangle.

$$F^\bullet_{\Omega(\alpha(X))} \to 0 \to F^\bullet_{\alpha(X)} \to$$

Thus, we also have a natural isomorphism $F^\bullet_{\alpha(X)}[-1] \simeq F^\bullet_{\Omega(\alpha(X))}$. Together, we obtain a natural isomorphism $F^\bullet_{\alpha(\Omega(X))} \simeq F^\bullet_{\Omega(\alpha(X))}$ in $\mathcal{L}_A$. Finally, this induces the claimed natural isomorphism $\alpha(\Omega(X)) \simeq \Omega(\alpha(X))$ in $\underline{\operatorname{Gproj}}\, B$. □

We have seen that a stable equivalence which preserves perfect exact sequences induces an equivalence on the level of $\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$. It seems unclear, whether such an equivalence induces an equivalence on the level of $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ or $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$. However, we will see in Theorem 5.8 that this holds for stable equivalences of Morita type. We close this section with a short comparison of some triangulated categories connected to $\underline{\operatorname{mod}}\, A$.

Remark 4.43. We have a chain of subcategories

$$\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A) \subseteq \mathcal{L}_A \subseteq \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) \subseteq \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A).$$

The categories $\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$, $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ and $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ are triangulated for all finite dimensional algebras. In particular, $\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$ is a triangulated subcategory of $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$. In contrast to $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$, we have that $\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$ is zero if $\operatorname{gldim} A < \infty$.

By a theorem of Beligiannis, the category $\mathcal{K}_{\mathrm{tac}}(\operatorname{Proj} A)$ is compactly generated. In fact, let $I^\bullet$ be an injective resolution of A, then

$$\mathcal{K}_{\mathrm{tac}}(\operatorname{Proj} A) \simeq \left\{ F^\bullet \in \mathcal{K}(\operatorname{Proj} A) \;\middle|\; \begin{matrix} \operatorname{Hom}_{\mathcal{K}(\operatorname{Proj} A)}(A[n], F^\bullet) = 0 \\ \operatorname{Hom}_{\mathcal{K}(\operatorname{Proj} A)}(\nu^{-1} I^\bullet[n], F^\bullet) = 0 \end{matrix} \quad \text{for } n \in \mathbb{Z} \right\}$$

where A and $\nu^{-1}I^\bullet$ are compactly generated in $\mathcal{K}(\operatorname{Proj} A)$. See [11, Appendix B] for more details. As an analogue, we have the following isomorphism using Remark 4.12

$$\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) \simeq \left\{ F^\bullet \in \mathcal{H}(\operatorname{proj} A) \middle| \operatorname{Hom}_{\mathcal{H}(\operatorname{proj} A)}(Z[n], F^\bullet) \text{ for } n \in \mathbb{Z} \right\}$$

with Z the direct sum of all indecomposable projective modules in $\nu^{-1}\mathcal{P}_A \subseteq \operatorname{proj} A$. In particular, Z is a direct summand of A. Furthermore, Z is compactly generated in $\mathcal{K}(\operatorname{Proj} A)$ since $Z^* \in \mathcal{K}^{-,b}(A\text{-proj})$; cf. [11, Lemma B.0.3] which uses [17, Theorem 2.4]. However, $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ is not closed under taking direct sums and thus cannot be compactly generated. The same holds for both $\mathcal{H}(\operatorname{proj} A)$ and $\mathcal{H}(\operatorname{Proj} A)$.

Finally, we shortly mention the singularity category $\mathcal{D}_{\mathrm{sg}}(A) \simeq \mathcal{K}^{-,b}(\operatorname{proj} A)/\mathcal{K}^b(\operatorname{proj} A)$. Recall that $\mathcal{D}_{\mathrm{sg}}(A)$ is a triangulated category as well. If A is self-injective, there exist isomorphisms $\mathcal{D}_{\mathrm{sg}}(A) \simeq \underline{\operatorname{mod}}\, A \simeq \mathcal{L}_A$; cf. [36, Theorem 2.1]. Similarly as for $\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$, we have that $\mathcal{D}_{\mathrm{sg}}(A)$ is zero if $\operatorname{gldim} A < \infty$. Moreover, if A is a Gorenstein algebra, we have $\mathcal{D}_{\mathrm{sg}}(A) \simeq \underline{\operatorname{Gproj}}\, A$; cf. [8, Theorem 4.4.1].

4.5 Self-injective algebras

In this short section, we discuss the case of self-injective algebras. Recall that the category $\underline{\operatorname{mod}}\, A$ is triangulated, if A is self-injective.

Lemma 4.44. *Let A be self-injective.*

Then $\mathcal{L}_A$ is a triangulated subcategory of $\mathcal{K}(\mathrm{proj}\, A)$ and $\mathcal{F} : \underline{\mathrm{mod}}\, A \to \mathcal{L}_A$ is an equivalence of triangulated categories.

Proof. If A is self-injective, $\mathrm{Hom}_A(-, A)$ is exact so that we have

$$\begin{aligned} \mathrm{H}^{<0}(F^\bullet) = 0 &\Leftrightarrow \mathrm{H}^{<0}(F^*_\bullet) = 0 \\ \mathrm{H}^{\geqslant 0}(F^*_\bullet) = 0 &\Leftrightarrow \mathrm{H}^{\geqslant 0}(F^\bullet) = 0. \end{aligned}$$

Together, we obtain that $\mathcal{L}_A = \{F^\bullet \in \mathrm{K}(\mathrm{proj}\, A) \mid \mathrm{H}^k(F^\bullet) = 0,\ k \in \mathbb{Z}\}$. Hence, $\mathcal{L}_A$ is closed under shifts. As another consequence, a morphism $f^\bullet : F^\bullet \to G^\bullet$ in $\mathcal{L}_A$ is a quasi-isomorphism and therefore $C(f)^\bullet \in \mathcal{L}_A$. In conclusion, $\mathcal{L}_A$ is a triangulated subcategory of $\mathcal{K}(\mathrm{proj}\, A)$. It remains to show that $\mathcal{F}$ is triangulated.

Suppose given $X \in \underline{\mathrm{mod}}\, A$ with $F^\bullet_{\Omega^{-1}(X)} \in \mathcal{L}_A$. Then

$$\cdots \to F^{-2}_{\Omega^{-1}(X)} \to F^{-1}_{\Omega^{-1}(X)} \to F^{0}_{\Omega^{-1}(X)} \to 0$$

is a projective resolution of $\Omega^{-1}(X)$. Therefore, we have that $\mathrm{H}^{-1}\left(\tau_{\leqslant -1} F^\bullet_{\Omega^{-1}(X)}\right) \overset{\mathrm{st}}{\simeq} X$ so that $F^\bullet_X \cong F^\bullet_{\Omega^{-1}(X)}[-1]$ or equivalently $F^\bullet_X[1] \cong F^\bullet_{\Omega^{-1}(X)}$. Hence, $\mathcal{F}$ commutes with the shift.

If A is self-injective, every short exact sequence is perfect exact. Moreover, every distinguished triangle in $\underline{\mathrm{mod}}\, A$ is induced by a short exact sequence. Therefore, Proposition 2.18 shows that $\mathcal{F}$ maps distinguished triangles in $\underline{\mathrm{mod}}\, A$ to distinguished triangles in $\mathcal{L}_A$. $\square$

We recall the objects of the following full subcategories of $\mathcal{K}(\mathrm{proj}\, A)$.

$$\begin{aligned}
\mathcal{H}(\mathrm{proj}\, A) &= \{F^\bullet \in \mathcal{K}(\mathrm{proj}\, A) \mid \exists\, l, r \in \mathbb{Z} \text{ with } \mathrm{H}^{<l}(F^\bullet) = 0,\ \mathrm{H}_{\geqslant r}(F^*_\bullet) = 0\} \\
\mathcal{H}_{\mathrm{stp}}(\mathrm{proj}\, A) &= \mathcal{H}(\mathrm{proj}\, A) \cap {}^\perp\mathcal{K}^b(\mathrm{stp}\, A) \\
\mathcal{H}_{\mathcal{P}}(\mathrm{proj}\, A) &= \mathcal{H}(\mathrm{proj}\, A) \cap {}^\perp\mathcal{K}^b(\mathcal{P}_A) \\
\mathcal{L}_A &= \{F^\bullet \in \mathcal{K}(\mathrm{proj}\, A) \mid \mathrm{H}^{<0}(F^\bullet) = 0,\ \mathrm{H}_{\geqslant 0}(F^*_\bullet) = 0\} \\
\mathcal{K}_{\mathrm{tac}}(\mathrm{proj}\, A) &= \{F^\bullet \in \mathcal{K}(\mathrm{proj}\, A) \mid \mathrm{H}^k(F^\bullet) = 0,\ \mathrm{H}_k(F^*_\bullet) = 0 \text{ for } k \in \mathbb{Z}\}
\end{aligned}$$

The connection between these categories can be visualized as follows. By Lemma 4.40, the diagram is commutative.

$$\begin{tikzcd}
\mathcal{K}_{\mathrm{tac}}(\mathrm{proj}\, A) \arrow[r, hook] & \mathcal{L}_A \arrow[r, hook] & \mathcal{H}_{\mathcal{P}}(\mathrm{proj}\, A) \arrow[r, hook] & \mathcal{H}_{\mathrm{stp}}(\mathrm{proj}\, A) \arrow[r, hook] & \mathcal{H}(\mathrm{proj}\, A) \arrow[r, hook] & \mathcal{K}(\mathrm{proj}\, A) \\
\underline{\mathrm{Gproj}}\, A \arrow[r, hook] \arrow[u, "\wr"] & \underline{\mathrm{mod}}\, A \arrow[u, "\wr", "\mathcal{F}"']
\end{tikzcd}$$

Note that $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ and $\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$ are triangulated categories for all finite dimensional algebras. In general, $\mathcal{L}_A$ is not a triangulated category and all inclusions are proper; cf. Example 7.9. However, all these categories coincide if and only if A is self-injective.

Theorem 4.45. *The following are equivalent for a finite dimensional algebra A.*

(1) A is self-injective.

(2) $\mathcal{L}_A$ is a triangulated subcategory of $\mathcal{K}(\operatorname{proj} A)$.

(3) $\mathcal{L}_A = \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$.

(4) $\mathcal{L}_A$ is closed under taking shifts in $\mathcal{K}(\operatorname{proj} A)$.

(5) $\mathcal{L}_A = \mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$.

If one of the above conditions holds, $\mathcal{F} : \underline{\operatorname{mod}}\, A \to \mathcal{L}_A$ is an equivalence of triangulated categories. Furthermore, we have $\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A) = \mathcal{L}_A = \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) = \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$.

Proof. It was shown in Lemma 4.44 that condition (2) holds if A is self-injective. The implication (2) $\Rightarrow$ (3) holds by Theorem 4.11. Since $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ is a triangulated subcategory of $\mathcal{K}(\operatorname{proj} A)$, condition (3) implies condition (4). Furthermore, we have seen the equivalence of conditions (4) and (5) in Lemma 4.38.(2,3).

We verify the implication (5) $\Rightarrow$ (1). An algebra A is self-injective if and only if every finitely generated module is reflexive; cf. [6, IV. Proposition 3.4]. Let $X \in \operatorname{mod} A$. We show that X is reflexive, that is $(X^*)^* \simeq X$. We have

$$X^* = \big(\mathrm{H}^0(\tau_{\leqslant 0} F_X^{\bullet})\big)^* \simeq \mathrm{H}_0(\tau_{\leqslant 0}\, F_X^{\bullet,*}) \simeq \mathrm{H}_1(\tau_{\geqslant 1} F_X^{\bullet,*})$$

since $(-)^* = \operatorname{Hom}_A(-, A)$ is left exact and $F_X^{\bullet} \in \mathcal{L}_A$.

$$\begin{tikzcd}[column sep=small] \cdots \arrow[r] & F_2^* \arrow[r] & F_1^* \arrow[rr] \arrow[dr, two heads] & & F_0^* \arrow[r] & F_{-1}^* \arrow[r] & \cdots \\ & & & X^* \arrow[ur, hook] & & & \end{tikzcd}$$

Using that $\mathrm{H}^0(F_X^{\bullet}) = 0$ since $F_X^{\bullet} \in \mathcal{L}_A = \mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$, we similarly obtain

$$(X^*)^* = (\mathrm{H}_1(\tau_{\geqslant 1} F_X^{\bullet,*}))^* \simeq \mathrm{H}^1(\tau_{\geqslant 1} F_X^{\bullet}) \simeq \mathrm{H}^0(\tau_{\leqslant 0} F_X^{\bullet}) = X.$$

In conclusion, all conditions are equivalent. It was shown in Lemma 4.44 that $\mathcal{F} : \underline{\operatorname{mod}}\, A \to \mathcal{L}_A$ is an equivalence of triangulated categories if A is self-injective. Furthermore, in this case we have $\operatorname{stp} A = \mathcal{P}_A = \operatorname{proj} A$ so that $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A) = \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$. $\square$

Finally, we have the following consequence for the stable Grothendieck group.

Remark 4.46. If A is self-injective, we have the following sequence of isomorphisms.

$$G_0^{\mathrm{st}}(A) \simeq G_0(\underline{\mathrm{mod}}\,A) \simeq G_0(\mathcal{L}_A) = G_0(\mathcal{H}_{\mathcal{P}}(\mathrm{proj}\,A)) \simeq G_0^{\mathcal{P}}(A)$$

The first isomorphism is shown in [40, Proposition 1.1]. The second is induced by the equivalence $\mathcal{F} : \underline{\mathrm{mod}}\,A \to \mathcal{L}_A$, while the last isomorphism is shown in Theorem 4.16. We have already seen in Remark 4.18 that $G_0^{\mathrm{st}}(A)$ and $G_0^{\mathcal{P}}(A)$ are isomorphic if A is self-injective.

Chapter 5

Stable equivalences of Morita type

Let k be a field. Let A and B be finite dimensional k-algebras without semisimple summands.

In general, stable equivalences fail to preserve many homological properties of finite dimensional algebras. The situation is better for stable equivalences induced by exact functors between $\operatorname{mod} A$ and $\operatorname{mod} B$. That is, if the equivalence is given by $- \otimes_A M$ with an A-B-bimodule which is projective as left A- and as right B-module. An important class of such equivalences are stable equivalences of Morita type.

At the beginning of this chapter, we discuss stable equivalences of Morita type in more detail. In particular, we will see that such equivalences preserve perfect exact sequences. As the main result of the first section, we show that stable equivalences of Morita type induce equivalences on the level of $\mathcal{L}_A$, $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ and $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$. These equivalences are given by componentwise application of $- \otimes_A M$.

In the second section, we start with an equivalence $\mathcal{L}_A \to \mathcal{L}_B$ given by $- \otimes_A M$ for an arbitrary bimodule M. As we will see, this is enough to induce a stable equivalence of Morita type. This provides a way to determine if a stable equivalence which is induced by an exact functor is of Morita type. In the final section, we use this result to give conditions under which an exact functor that induces an equivalence $\underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$ is already a stable equivalence of Morita type. This is done using results of previous chapters about perfect exact sequences.

We recall the definition and collect some properties of stable equivalences of Morita type.

Definition 5.1 (Broué)**.** Let ${}_AM_B$ and ${}_BN_A$ be bimodules such that ${}_AM$, M_B, ${}_BN$ and N_A are projective. We say that M and N induce a *stable equivalence of Morita type* if

$${}_AM \otimes_B N_A \simeq A \oplus P \text{ and } {}_BN \otimes_A M_B \simeq A \oplus Q$$

as bimodules such that ${}_AP_A$ and ${}_BQ_B$ are projective bimodules.

We note two properties of stable equivalences of Morita type with regards to projective modules.

Remark 5.2. (1) Suppose given an A-B-bimodule M such that M_B is a projective B-module. Then $P \otimes_A M_B$ is a projective B-module for all $P \in \operatorname{proj} A$. In fact, P is a direct summand of $A^{\oplus n}$ for some $n \in \mathbb{Z}_{\geqslant 1}$. Moreover, we have the following isomorphism of right B-modules.

$$A^{\oplus n} \otimes_A M \simeq M^{\oplus n}$$

Together, we obtain that $P \otimes_A M$ is a direct summand of $M^{\oplus n}$ which is projective as a right B-module. In conclusion, $P \otimes_A M \in \operatorname{proj} B$.

(2) Suppose given a projective bimodule ${}_A P_A$. Then $X \otimes_A P_A$ is a projective A-module for all $X \in \operatorname{mod} A$. In fact, we have

$$X \otimes_A ({}_A A \otimes_k A_A) \simeq X \otimes_k A$$

for all $X \in \operatorname{mod} A$. Note that ${}_A P_A$ is a direct summand of $({}_A A \otimes_k A_A)^{\oplus n}$ for some $n \in \mathbb{Z}_{\geqslant 1}$. Since $X \otimes_k A$ is projective in $\operatorname{mod} A$, so is $X \otimes_A P$.

The functors given by a stable equivalence of Morita type form an adjoint pair. This result was first shown in [13, Corollary 3.1] for algebras whose semisimple quotients are separable.

Lemma 5.3. *([10, Lemma 4.1]) Suppose ${}_A M_B$ and ${}_B N_A$ are bimodules that induce a stable equivalence of Morita type such that M and N do not have any non-zero projective bimodule as direct summand.*

The functor $- \otimes_A M$ is left and right adjoint to $- \otimes_B N$. Furthermore, $\operatorname{Hom}_B(M, B) \simeq N$ *as B-A-bimodules and* $\operatorname{Hom}_A(N, A) \simeq M$ *as A-B-bimodules.*

Following [13], we state several consequences of this lemma. We also include the respective proofs.

Lemma 5.4. *Suppose ${}_A M_B$ and ${}_B N_A$ are bimodules that induce a stable equivalence of Morita type such that M and N do not have any non-zero projective bimodule as direct summand.*

The following holds for $X \in \operatorname{mod} A$.

(1) $X \otimes_A M$ is injective as a B-module if $X \in \operatorname{inj} A$.

(2) $X \otimes_A M$ is projective-injective as a B-module if $X \in \mathcal{P}_A$.

(3) There exists a natural isomorphism of left B-modules.

$$(X \otimes_A M_B)^* \simeq {}_B N \otimes_A X^*$$

(4) There exists a natural isomorphism of B-modules.

$$\nu_B(X \otimes_A M_B) \simeq \nu_A(X) \otimes_A M_B$$

(5) $X \otimes_A M$ is strongly projective-injective as a B-module if $X \in \operatorname{stp} A$.

Proof. Ad (1) and (2). We use that $- \otimes_A M$ is right adjoint to $- \otimes_B N$; cf. Lemma 5.3.

If $X \in \operatorname{mod} A$ is injective, $\operatorname{Hom}_B(-, X \otimes_A M) \simeq \operatorname{Hom}_A(- \otimes_B N, X)$ is an exact functor since $- \otimes_B N$ is exact. Thus, $X \otimes_A M \in \operatorname{inj} B$.

If $X \in \operatorname{proj} A$, then $X \otimes_A M$ is a projective B-module since $M_B \in \operatorname{proj} B$. Therefore, we have $X \otimes_A M \in \mathcal{P}_B$ if $X \in \mathcal{P}_A$.

Ad (3). Recall that $\operatorname{Hom}_B(M, B) \simeq N$ by Lemma 5.3. We have the following sequence of natural isomorphisms. For the last isomorphism, we use that N_A is projective and X_A finitely generated.

$$\begin{aligned}
(X \otimes_A M_B)^* &= \operatorname{Hom}_B(X \otimes_A M_B, {}_BB_B) \\
&\simeq \operatorname{Hom}_A(X_A, \operatorname{Hom}_B({}_AM_B, {}_BB_B)) \\
&\simeq \operatorname{Hom}_A(X_A, {}_BN_A) \\
&\simeq {}_BN \otimes_A X^*
\end{aligned}$$

Ad (4). Recall that $\operatorname{Hom}_A(N, A) \simeq M$ by Lemma 5.3. Applying $\mathrm{D}(-)$ to part (3) yields the following sequence of natural isomorphisms.

$$\begin{aligned}
\nu_B(X \otimes_A M_B) &= \operatorname{Hom}_k((X \otimes_A M_B)^*, k) \\
&\simeq \operatorname{Hom}_k({}_BN \otimes_A X^*, k) \\
&\simeq \operatorname{Hom}_A({}_BN_A, \operatorname{Hom}_k(X^*, k)) \\
&\simeq \operatorname{Hom}_A({}_BN_A, \nu_A(X)) \\
&\simeq \nu_A(X) \otimes_A \operatorname{Hom}_A({}_BN_A, {}_AA_A) \\
&\simeq \nu_A(X) \otimes_A M_B
\end{aligned}$$

Ad (5). Let $X \in \operatorname{stp} A$. Using part (4), we have that $\nu_B^k(X \otimes_A M_B) \simeq \nu_A^k(X) \otimes_A M_B \in \operatorname{proj} B$ for $k \in \mathbb{Z}$. Thus, $X \otimes_A M_B \in \operatorname{stp} B$. □

We are now able to show that a stable equivalence of Morita type preserves perfect exact sequences.

Lemma 5.5. *Suppose ${}_AM_B$ and ${}_BN_A$ are bimodules that induce a stable equivalence of Morita type such that M and N do not have any non-zero projective bimodule as direct summand.*

If $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *is a perfect exact sequence in* $\operatorname{mod} A$*, then*

$$0 \to X \otimes_A M \xrightarrow{f \otimes M} Y \otimes_A M \xrightarrow{g \otimes M} Z \otimes_A M \to 0$$

is a perfect exact sequence in $\operatorname{mod} B$*. Similarly, the functor* $- \otimes_B N$ *maps perfect exact sequences in* $\operatorname{mod} B$ *to perfect exact sequences in* $\operatorname{mod} A$*.*

Proof. Let $\eta : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a perfect exact sequence in $\operatorname{mod} A$. Since $- \otimes_A M$ is an exact functor, $0 \to X \otimes_A M \xrightarrow{f \otimes M} Y \otimes_A M \xrightarrow{g \otimes M} Z \otimes_A M \to 0$ is a short exact sequence in $\operatorname{mod} B$. By Lemma 5.4.(3) there exist isomorphisms such that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & (Z \otimes_A M)^* \arrow[r, "(g \otimes M)^*"] \arrow[d, "\wr"] & (Y \otimes_A M)^* \arrow[r, "(f \otimes M)^*"] \arrow[d, "\wr"] & (X \otimes_A M)^* \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & N \otimes_A Z^* \arrow[r, "N \otimes g^*"] & N \otimes_A Y^* \arrow[r, "N \otimes f^*"] & N \otimes_A X^* \arrow[r] & 0
\end{tikzcd}$$

Since $N \otimes_A -$ is an exact functor and η perfect exact, the lower sequence is exact. This implies that the upper sequence is exact as well. Consequently,

$$0 \to X \otimes_A M \xrightarrow{f \otimes M} Y \otimes_A M \xrightarrow{g \otimes M} Z \otimes_A M \to 0$$

is a perfect exact sequence. □

5.1 Induced equivalences in $\mathcal{K}(\operatorname{proj} A)$

In this section we aim to show that a stable equivalence of Morita type $\underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$ induces equivalences between $\mathcal{L}_A \to \mathcal{L}_B$, $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) \to \mathcal{H}_{\mathcal{P}}(\operatorname{proj} B)$ and $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A) \to \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} B)$. The next lemma collects some preliminary results in this direction. For parts (3) and (4), recall that $\operatorname{stp} A \subseteq \mathcal{P}_A$.

Lemma 5.6. *Suppose* ${}_AM_B$ *and* ${}_BN_A$ *are bimodules that induce a stable equivalence of Morita type such that* M *and* N *do not have any non-zero projective bimodule as direct summand. Write* ${}_AM \otimes_B N_A \simeq A \oplus P$ *as bimodules with* ${}_AP_A$ *projective. The following holds.*

(1) $X \otimes_A M \in {}^{\perp}\mathcal{P}_B$ *if* $X \in {}^{\perp}\mathcal{P}_A$*.*

(2) $X \otimes_A M \in {}^{\perp}(\operatorname{stp} A)$ *if* $X \in {}^{\perp}(\operatorname{stp} A)$*.*

(3) $X \otimes_A P_A \in \operatorname{stp} A$ *for all* $X \in \operatorname{mod} A$*.*

(4) $X \otimes_A M \otimes_B N \simeq X$ *if* $X \in {}^{\perp}(\operatorname{stp} A)$*.*

Proof. Ad (1) and (2). For $Z \in \mathcal{P}_B$ or $Z \in \operatorname{stp} B$ we have with Lemma 5.3 that

$$\operatorname{Hom}_B(X \otimes_A M, Z) \simeq \operatorname{Hom}_A(X, Z \otimes_B N) = 0$$

since $Z \otimes_B N \in \mathcal{P}_A$ or $Z \otimes_B N \in \operatorname{stp} A$ respectively by Lemma 5.4.(2,5).

Ad (3). We follow [16, Lemma 3.1]. With Lemma 5.4.(4), we have

$$\begin{aligned}\nu(A) \oplus \nu(P) &\simeq \nu(A \oplus P)\\ &\simeq \nu(A \otimes_A M \otimes_B N)\\ &\simeq \nu(A) \otimes_A M \otimes_B N\\ &\simeq \nu(A) \otimes_A (A \oplus P)\\ &\simeq \nu(A) \oplus \nu(A) \otimes_A P\end{aligned}$$

so that $\nu(P) \simeq \nu(A) \otimes_A P$. Since $\nu(A)$ is finitely generated, there exists an $n \geqslant 0$ and a surjection $A^{\oplus n} \twoheadrightarrow \nu(A)$. This induces a surjection $A^{\oplus n} \otimes_A P \twoheadrightarrow \nu(A) \otimes_A P$ with $A^{\oplus n} \otimes_A P \simeq P^{\oplus n}$. By Remark 5.2.(2), we have that $\nu(A) \otimes_A P$ is projective. Hence, $\nu(P)$ is a direct summand of $P^{\oplus n}$ and we obtain that $P \in \operatorname{stp} A$.

Let $X \in \operatorname{mod} A$. It remains to show that $X \otimes_A P_A \in \operatorname{stp} A$. As above, there exists an n and a surjection $A^{\oplus n} \twoheadrightarrow X$. By applying $- \otimes_A P$, we obtain $P^{\oplus n} \twoheadrightarrow X \otimes_A P_A$. Thus, $X \otimes_A P_A$ is a direct summand of $P^{\oplus n} \in \operatorname{stp} A$; cf. Remark 5.2.(2).

Ad (4). We have

$$X \otimes_A M \otimes_B N \simeq X \otimes_A (A_A \oplus P_A) \simeq X \oplus (X \otimes_A P_A).$$

By part (3) we know that $X \otimes_A P_A \in \operatorname{stp} A$. On the other hand, we have $X \otimes_A M \in {}^{\perp}(\operatorname{stp} B)$ by part (2). Similarly, we also have that $Y \otimes_B N \in {}^{\perp}(\operatorname{stp} A)$ for all $Y \in {}^{\perp}(\operatorname{stp} B)$. Together we obtain $X \otimes_A M \otimes_B N \in {}^{\perp}(\operatorname{stp} A)$. Thus, $X \otimes_A P_A$ must be zero and $X \otimes_A M \otimes_B N \simeq X$. □

The next lemma will be used to show that a stable equivalence of Morita type induces functors $\mathcal{L}_A \to \mathcal{L}_B$ and $\mathcal{H}(\operatorname{proj} A) \to \mathcal{H}(\operatorname{proj} B)$.

Lemma 5.7. *Suppose ${}_AM_B$ and ${}_BN_A$ are bimodules that induce a stable equivalence of Morita type such that M and N do not have any non-zero projective bimodule as direct summand.*

The following holds for a complex $F^\bullet \in \mathcal{K}(\operatorname{proj} A)$ and all $k \in \mathbb{Z}$.

(1) $\mathrm{H}^k(F^\bullet \otimes_A M) = 0$ *if* $\mathrm{H}^k(F^\bullet) = 0$.

(2) $\mathrm{H}_k\big((F^\bullet \otimes_A M)^*\big) = 0$ *if* $\mathrm{H}_k(F^*_\bullet) = 0$.

Proof. Using that $- \otimes_A M$ is an exact functor, we have $\mathrm{H}^k(F^\bullet \otimes_A M) \simeq \mathrm{H}^k(F^\bullet) \otimes_A M$. Using that $N \otimes_A -$ is an exact functor, we have with Lemma 5.4.(3) that

$$\mathrm{H}_k\big((F^\bullet \otimes_A M)^*\big) \simeq \mathrm{H}_k(N \otimes_A F_\bullet^*) \simeq N \otimes_A \mathrm{H}_k(F_\bullet^*).$$

Now, the result follows since $- \otimes_A M$ and $N \otimes_B -$ are additive functors. □

We have already seen in Theorem 4.42 that every stable equivalence that preserves perfect exact sequences induces an equivalence on the level of $\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$. In particular, this holds for stable equivalences of Morita type. Now, we can show that a stable equivalence of Morita type induces equivalences on the level of some of the other categories discussed in Chapter 4.

Theorem 5.8. *Suppose ${}_AM_B$ and ${}_BN_A$ are bimodules that induce a stable equivalence of Morita type such that M and N do not have any non-zero projective bimodule as direct summand.*

(1) Applying $- \otimes_A M$ componentwise induces an equivalence of categories $\mathcal{L}_A \to \mathcal{L}_B$.

If A and B are self-injective, this is an equivalence of triangulated categories.

(2) Applying $- \otimes_A M$ componentwise induces an equivalence of triangulated categories

$$\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) \to \mathcal{H}_{\mathcal{P}}(\operatorname{proj} B).$$

(3) Applying $- \otimes_A M$ componentwise induces an equivalence of triangulated categories

$$\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A) \to \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} B).$$

Proof. Ad (1). By Remark 5.2.(1), $- \otimes_A M$ induces a functor $\mathcal{K}(\operatorname{proj} A) \to \mathcal{K}(\operatorname{proj} B)$ by componentwise application. Now, Lemma 5.7 shows that $- \otimes_A M$ induces a well-defined functor $\mathcal{L}_A \to \mathcal{L}_B$. Consider the following diagram.

$$\begin{tikzcd}
\underline{\operatorname{mod}}\, A \arrow[r, "-\otimes_A M", "\sim"'] \arrow[d, "\mathcal{F}"', "\wr"] & \underline{\operatorname{mod}}\, B \arrow[d, "\wr"', "\mathcal{F}"] \\
\mathcal{L}_A \arrow[r, "-\otimes_A M"] & \mathcal{L}_B
\end{tikzcd}$$

Recall that the quasi-inverse of $\mathcal{F}$ is given by $\mathrm{H}^0(\tau_{\leqslant 0}(-))$; cf. Theorem 2.6. Let $F^\bullet \in \mathcal{L}_A$. Since $- \otimes_A M$ is exact, we have $\mathrm{H}^0\left(\tau_{\leqslant 0}\left(F^\bullet \otimes_A M\right)\right) \cong \mathrm{H}^0\left(\tau_{\leqslant 0} F^\bullet\right) \otimes_A M$ so that the diagram commutes. This shows that $- \otimes_A M$ induces an equivalence of categories $\mathcal{L}_A \to \mathcal{L}_B$.

Furthermore, if A and B are self-injective, the equivalence $\mathcal{F}$ is triangulated by Lemma 4.44. Thus, this diagram shows that the functor $- \otimes_A M$ induces an equivalence of triangulated categories $\mathcal{L}_A \to \mathcal{L}_B$.

Ad (2) and (3). By Lemma 5.7, the functor $- \otimes_A M$ induces a functor $\mathcal{H}(\operatorname{proj} A) \to \mathcal{H}(\operatorname{proj} B)$. Since $- \otimes_A M$ is applied componentwise, this is a triangulated functor.

Suppose given $F^\bullet \in \mathcal{H}_\mathcal{P}(\operatorname{proj} A)$. We verify that $F^\bullet \otimes_A M \in {}^\perp\mathcal{K}^b(\mathcal{P}_B)$. By Lemma 4.5 it suffices to show that $\mathrm{H}^k(F^\bullet \otimes_A M) \simeq \mathrm{H}^k(F^\bullet) \otimes_A M$ is an element of ${}^\perp\mathcal{P}_B$ for $k \in \mathbb{Z}$. However, this holds by Lemma 5.6.(1) since $\mathrm{H}^k(F^\bullet) \in {}^\perp\mathcal{P}_A$ by Lemma 4.5. In conclusion, $- \otimes_A M$ induces a functor $\mathcal{H}_\mathcal{P}(\operatorname{proj} A) \to \mathcal{H}_\mathcal{P}(\operatorname{proj} B)$.

Similarly, $- \otimes_A M$ induces a functor $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A) \to \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} B)$. It remains to show that these are equivalences of triangulated categories. Recall that $\mathcal{H}_\mathcal{P}(\operatorname{proj} A)$ is contained in $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$.

Let $F^\bullet \in \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ with $r \in \mathbb{Z}$ such that $\mathrm{H}_{\geqslant r}(F^*_\bullet) = 0$. We verify by induction on $N := |\{j \in \mathbb{Z}_{<r} \,|\, \mathrm{H}^j(F^\bullet) \neq 0\}|$ that we have a natural isomorphism $F^\bullet \otimes_A M \otimes_B N \simeq F^\bullet$. Since $F^\bullet \in \mathcal{H}(\operatorname{proj} A)$, we know that $\mathrm{H}^\bullet(F^\bullet)$ is left bounded and therefore $N < \infty$. We write $G^\bullet := F^\bullet \otimes_A M \otimes_B N$.

Let $N = 0$. Then $F^\bullet[r] \in \mathcal{L}_A$ and the assertion holds by part (1) since $- \otimes_A M$ commutes with the shift.

Let $N > 0$ and $k \in \mathbb{Z}_{<r}$ minimal such that $\mathrm{H}^k(F^\bullet) \neq 0$. By Lemma 4.6, we have a distinguished triangle

$$P^\bullet[-k] \to F^\bullet \to C^\bullet \to$$

with $P^\bullet$ a projective resolution of $\mathrm{H}^k(F^\bullet)$. Moreover, $\mathrm{H}^j(C^\bullet) = 0$ for $j \leqslant k$ and $\tau_{\geqslant k} C^\bullet = \tau_{\geqslant k} F^\bullet$.

Applying $- \otimes_A M \otimes_B N$ to this triangle, we obtain a new distinguished triangle.

$$P^\bullet[-k] \otimes_A M \otimes_B N \to G^\bullet \to C^\bullet \otimes_A M \otimes_B N \to$$

By Lemma 5.6.(4), we have $\mathrm{H}^k(F^\bullet) \otimes_A M \otimes_B N \simeq \mathrm{H}^k(F^\bullet)$ since $\mathrm{H}^k(F^\bullet) \in {}^\perp(\operatorname{stp} A)$ by Lemma 4.5. Hence, we have a natural isomorphism $P^\bullet \otimes_A M \otimes_B N \simeq P^\bullet$ in $\mathcal{K}(\operatorname{proj} A)$ and obtain the following distinguished triangle.

$$P^\bullet[-k] \to G^\bullet \to C^\bullet \otimes_A M \otimes_B N \to$$

By induction, we can assume that there is a natural isomorphism $C^\bullet \otimes_A M \otimes_B N \simeq C^\bullet$. This induces another distinguished triangle.

$$\begin{tikzcd}
P^\bullet[-k] \arrow[r] \arrow[d, equal] & G^\bullet \arrow[r] \arrow[d, dashed] & C^\bullet \arrow[r] \arrow[d, equal] & {} \\
P^\bullet[-k] \arrow[r] & F^\bullet \arrow[r] & C^\bullet \arrow[r] & {}
\end{tikzcd}$$

The induced morphism now yields a natural isomorphism $G^\bullet = F^\bullet \otimes_A M \otimes_B N \simeq F^\bullet$. □

Note that in general, a stable equivalence of Morita type does not induce an equivalence between $\mathcal{H}(\operatorname{proj} A)$ and $\mathcal{H}(\operatorname{proj} B)$. Similarly, it does not induce an equivalence between $\mathcal{K}(\operatorname{proj} A)$ and $\mathcal{K}(\operatorname{proj} B)$. This is discussed at the end of Example 7.6.

5.2 Functors in $\mathcal{K}(\operatorname{proj} A)$ inducing stable equivalences

In Theorem 5.8.(1) we have seen that a stable equivalence of Morita type induces an equivalence on the level of $\mathcal{L}$ given by tensoring with a bimodule M. We aim to show that any equivalence $-\otimes_A M : \mathcal{L}_A \to \mathcal{L}_B$ with an arbitrary bimodule ${}_B M_A$ induces a stable equivalence of Morita type $\underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$. The proof is based on the following theorem by Dugas and Martínez-Villa.

Theorem 5.9. ([13, Theorem 2.9]) *Let A and B be finite dimensional k-algebras whose semisimple quotients are separable. Suppose that ${}_A M_B$ is projective as left A- and as right B-module such that $- \otimes_A M$ induces a stable equivalence $\underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$.*

If ${}_B N_A := \operatorname{Hom}_A(M, A)$ is projective over B, then M and N induce a stable equivalence of Morita type between A and B.

Remark 5.10. Suppose given a projective bimodule ${}_A P_A$. Then $X \otimes_A P_A$ is a projective A-module for all $X \in \operatorname{mod} A$; cf. Remark 5.2.(2). The converse does not hold in general.

However, it does hold, if we assume that the semisimple quotients of A and B are separable; cf. [5, Corollary 3.1] and also [13, Theorem 2.8]. This separability assumption is satisfied in the following cases among others.

- k is a perfect field.
- A and B are given by quivers with relations.

We need a slightly different version of the above theorem, where ${}_B N_A = \operatorname{Hom}_B(M, B)$ instead of ${}_B N_A = \operatorname{Hom}_A(M, A)$.

Corollary 5.11. *Let A and B be finite dimensional k-algebras whose semisimple quotients are separable. Suppose that ${}_A M_B$ is projective as left A- and as right B-module such that $- \otimes_A M$ induces a stable equivalence $\underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$.*

If ${}_B N_A := \operatorname{Hom}_B(M, B)$ is projective over A, then M and N induce a stable equivalence of Morita type between A and B.

Proof. Using that M_B is projective, we have the following sequence of natural isomorphisms for all $Y \in \operatorname{mod} B$.

$$\begin{aligned}\operatorname{Hom}_B(X \otimes_A M, Y_B) &\simeq \operatorname{Hom}_A(X_A, \operatorname{Hom}_B(M_B, Y_B)_A) \simeq \operatorname{Hom}_A(X_A, Y \otimes_B \operatorname{Hom}_B(M, B)_A) \\ &\simeq \operatorname{Hom}_A(X_A, Y \otimes_B N_A)\end{aligned}$$

Thus, $-\otimes_A M$ is left adjoint to $-\otimes_B N$. By a result of Auslander and Kleiner in [3, Proposition 1.1], we obtain that $-\otimes_A M$ is left adjoint to $-\otimes_B N$ in $\underline{\operatorname{mod}}\, A$ since $-\otimes_A M$ and $-\otimes_B N$ take projective modules to projective modules. Hence, $-\otimes_B N : \underline{\operatorname{mod}}\, B \to \underline{\operatorname{mod}}\, A$ is the quasi-inverse of $-\otimes_A M : \underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$. In particular, $-\otimes_A N$ induces a stable equivalence.

We set ${}_B\tilde{M}_A := {}_BN_A$ and ${}_A\tilde{N}_B := \operatorname{Hom}_B(N, B)$. Then ${}_A\tilde{N}_B \simeq {}_AM_B$ as bimodules and $\tilde{M}$ and $\tilde{N}$ are are projective on both sides. The result follows by applying Theorem 5.9 to $\tilde{M}$ and $\tilde{N}$ while switching the role of A and B. $\square$

Note that ${}_BN_A$ is projective over A if and only if ${}_BN \otimes_A -$ is an exact functor. We aim to use that a complex $F^\bullet \in \mathcal{L}_A$ can be thought of as a projective resolution $F^{\leqslant 0}$ in $\operatorname{mod} A$ and a projective resolution $F^*_{\geqslant -1}$ in A-mod. Additionally, we need an analogue of Lemma 5.4.(3) under slightly different assumptions.

Lemma 5.12. *Suppose that M is an A-B-bimodule. Let ${}_BN_A := \operatorname{Hom}_B({}_AM_B, B)$.*

For every $P \in \operatorname{proj} A$ there exists a natural isomorphism of left B-modules

$$(P \otimes_A M_B)^* \simeq {}_BN \otimes_A P^*.$$

Proof. We have the following natural isomorphism of left B-modules.

$$\begin{aligned}(P \otimes_A M_B)^* &= \operatorname{Hom}_B(P \otimes_A M_B, {}_BB_B)\\ &\simeq \operatorname{Hom}_A(P_A, \operatorname{Hom}_B(M_B, {}_BB_B))\\ &= \operatorname{Hom}_A(P_A, {}_BN_A)\end{aligned}$$

We show hat $\operatorname{Hom}_A(P_A, {}_BN_A) \simeq {}_BN \otimes_A P^*$ using that P_A is projective and that N_A is finitely generated.

Since ${}_BB \otimes_k A_A$ is projective as a right A-module and since P_A is finitely generated, we have the following natural isomorphism of left B-modules.

$$\operatorname{Hom}_A(P_A, {}_BB \otimes_k A_A) \simeq {}_BB \otimes_k A_A \otimes_A P^*$$

Moreover, $\operatorname{Hom}_A(P_A, {}_BB \otimes_k A_A^{\oplus n}) \simeq {}_BB \otimes_k A^{\oplus n} \otimes_A P^*$ for all $n \in \mathbb{Z}_{\geqslant 1}$.

Let $N_A = \langle g_1, \dots, g_n \rangle$ be a minimal generating system of N as a right A-module with $n \in \mathbb{Z}_{\geqslant 1}$. Consider the following surjective map.

$$\begin{aligned} B \otimes_k A^{\oplus n} &\xrightarrow{\varphi} {}_BN_A\\ b \otimes (a_1, \dots, a_n) &\mapsto b \sum_{k=1}^{n} g_k a_k \end{aligned}$$

Note that φ is a morphism of B-A-bimodules, so that $\mathrm{Ker}(\varphi)$ is also a B-A-bimodule. In particular, $\mathrm{Ker}(\varphi)$ is an A-submodule of $B \otimes_k A^{\oplus n}$. Since A is finite dimensional, $\mathrm{Ker}(\varphi)$ is finitely generated as a right A-module. Therefore, there exists an $m \in \mathbb{Z}_{\geqslant 1}$ and a surjective morphism $B \otimes_k A^{\oplus m} \to \mathrm{Ker}(\varphi)$ as above.

We obtain a presentation $B \otimes_k A^{\oplus m} \to B \otimes_k A^{\oplus n} \to {}_B N_A \to 0$ of N via bimodules. Consider the following commutative diagram with exact rows. For the upper row we have used that P_A is projective.

$$\begin{tikzcd}
\mathrm{Hom}_A(P, B \otimes_k A^{\oplus m}) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_A(P, B \otimes_k A^{\oplus n}) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_A(P, N) \arrow[r] \arrow[d, dashed, "\wr"] & 0 \\
B \otimes_k A^{\oplus m} \otimes_A P^* \arrow[r] & B \otimes_k A^{\oplus n} \otimes_A P^* \arrow[r] & N \otimes_A P^* \arrow[r] & 0
\end{tikzcd}$$

By the above, the two morphisms on the left are isomorphisms. Therefore, we obtain a natural isomorphism of left B-modules $\mathrm{Hom}_A(P_A, {}_B N_A) \simeq {}_B N \otimes_A P^*$. □

We are now ready to state the main result of this section.

Theorem 5.13. *Let A and B be finite dimensional k-algebras whose semisimple quotients are separable.*

Suppose given a bimodule ${}_A M_B$ such that applying $- \otimes_A M$ componentwise induces an equivalence $\mathcal{L}_A \xrightarrow{\sim} \mathcal{L}_B$. Let ${}_B N_A := \mathrm{Hom}_B(M, B)$. Then M and N induce a stable equivalence of Morita type between A and B.

Proof. We show that M is projective as left A- and as right B-module and we show that N is projective as left B- and right A-module. Since $- \otimes_A M$ maps projective A-modules to projective B-modules, we have that $M \in \mathrm{proj}\, B$. Moreover, this means that $N \in B$-proj.

Let $X \in \mathrm{mod}\, A$. Suppose given a projective resolution $P^\bullet \in \mathcal{K}(\mathrm{proj}\, A)$ of X. Then $\tau_{\leqslant 0}\, F_X^\bullet \simeq P^\bullet$ in $\mathcal{K}(\mathrm{proj}\, A)$. Using that $- \otimes_A M$ is a right exact functor with image in $\mathcal{L}_B$, we obtain that $F_X^\bullet \otimes_A M \simeq F_{X \otimes M}^\bullet$. Hence, $P^\bullet \otimes_A M \simeq \tau_{\leqslant 0}\, (F_X^\bullet \otimes_A M) \simeq \tau_{\leqslant 0}\, F_{X\otimes M}^\bullet$ is a projective resolution of $X \otimes_A M$. Thus, we have $\mathrm{Tor}_i^A(X, M) \simeq \mathrm{H}^{-i}(P^\bullet \otimes_A M) = 0$ for all $i \geqslant 1$. This implies that M is projective as a left A-module.

Let Y be a left A-module. Suppose given a projective resolution $Q^\bullet$ of Y in $\mathrm{K}(A\text{-proj})$. There exists an $X \in \mathrm{mod}\, A$ such that $\mathrm{Tr}\, X = Y$. Then $\tau_{\geqslant -1} F_X^{\bullet;*} \simeq Q^\bullet$ in $\mathrm{K}(A\text{-proj})$. By Lemma 5.12 we have that $\tau_{\geqslant -1}(N \otimes_A F_X^{\bullet;*}) \simeq \tau_{\geqslant -1}(F_X^\bullet \otimes_A M)^*$ as complexes.

$$\begin{tikzcd}
\cdots \arrow[r] & N \otimes_A F_1^* \arrow[r] \arrow[d, "\wr"] & N \otimes_A F_0^* \arrow[r] \arrow[d, "\wr"] & N \otimes_A F_{-1}^* \arrow[r] \arrow[d, "\wr"] & N \otimes_A \mathrm{Tr}(X) \arrow[r] \arrow[d, dashed, "\wr"] & 0 \\
\cdots \arrow[r] & (F^1 \otimes_A M)^* \arrow[r] & (F^0 \otimes_A M)^* \arrow[r] & (F^{-1} \otimes_A M)^* \arrow[r] & \mathrm{Tr}(X \otimes_A M) \arrow[r] & 0
\end{tikzcd}$$

Since $F_X^\bullet \otimes_A M \in \mathcal{L}_B$, we have that $\tau_{\geqslant -1}(F_X^\bullet \otimes_A M)^*$ is a projective resolution of $\operatorname{Tr}(X \otimes M)$. Hence

$$N \otimes_A Q^\bullet \simeq \tau_{\geqslant -1}(N \otimes_A F_X^{\bullet,*}) \simeq \tau_{\geqslant -1}(F_X^\bullet \otimes_A M)^*$$

is a projective resolution of $N \otimes_A \operatorname{Tr}(X)$. Thus, we have $\operatorname{Tor}_i^A(N, \operatorname{Tr}(X)) \simeq \mathrm{H}^{-i}(N \otimes_A Q^\bullet) = 0$ for $i \geqslant 0$. This implies that N is projective as a right A-module.

Since the functor $- \otimes_A M$ induces an equivalence $\mathcal{L}_A \xrightarrow{\sim} \mathcal{L}_B$ and is exact, this also induces an equivalence $\underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$. Now, Corollary 5.11 shows that M and N induce a stable equivalence of Morita type between A and B. □

The above result can be useful to check if a stable equivalence induced by an exact functor $- \otimes_A M$ is a stable equivalence of Morita type. More precisely, one needs to check if $\mathrm{H}_k\big((F^\bullet \otimes_A M)^*\big) = 0$ for $F^\bullet \in \mathcal{L}_A$ and $k \geqslant 0$. We also state the following consequence which can be used in a similar way. Recall that if M and $\operatorname{Hom}_B(M, B)$ do not have any non-zero projective bimodule as direct summand and if they induce a stable equivalence of Morita type, then $\nu_B(X \otimes_A M) \simeq \nu_A(X) \otimes_A M$ for every $X \in \operatorname{mod} A$; cf. Lemma 5.4.(4).

Corollary 5.14. *Let A and B be finite dimensional k-algebras whose semisimple quotients are separable. Let ${}_AM_B$ be a bimodule which is projective as left A- and as right B-module such that $- \otimes_A M$ induces a stable equivalence $\underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$.*

Then M and $\operatorname{Hom}_B(M, B)$ induce a stable equivalence of Morita type between A and B if one of the following equivalent conditions holds.

(1) There exist natural isomorphisms $\nu_B(P \otimes_A M) \simeq \nu_A(P) \otimes_A M$ of right B-modules for every $P \in \operatorname{proj} A$.

(2) There exists a natural isomorphism $M \otimes_B \mathrm{D}B \simeq \mathrm{D}A \otimes_A M$ of right B-modules.

Proof. Suppose that condition (1) holds. Let $F^\bullet \in \mathcal{L}_A$. Note that $- \otimes_A M$ is exact since ${}_AM$ is projective. By assumption, we have the following for $k \geqslant 0$.

$$\begin{aligned}
& \mathrm{H}_k((F^\bullet \otimes_A M)^*) = 0 \\
\Leftrightarrow\ & \mathrm{H}^k(\nu_B(F^\bullet \otimes_A M)) = 0 \\
\Leftrightarrow\ & \mathrm{H}^k(\nu_A(F^\bullet) \otimes_A M) = 0 \\
\Leftrightarrow\ & \mathrm{H}^k(\nu_A(F^\bullet)) \otimes_A M = 0 \\
\Leftrightarrow\ & \mathrm{H}_k(F_\bullet^*) \otimes_A M = 0
\end{aligned}$$

The last equation holds, since $F^\bullet \in \mathcal{L}_A$. As a result, we have $F^\bullet \otimes_A M \in \mathcal{L}_B$ and $- \otimes_A M$ induces an equivalence $\mathcal{L}_A \to \mathcal{L}_B$. By Theorem 5.13, we obtain that M and $\operatorname{Hom}_B(M, B)$ induce a stable equivalence of Morita type between the algebras A and B.

It remains to show the equivalence of (1) and (2). We have the following natural isomorphisms of right B-modules.

$$\begin{aligned}
&\nu_A(A) \otimes_A M \simeq \mathrm{D}\,\mathrm{Hom}_A(A, A) \simeq \mathrm{D}A \otimes_A M \\
&\mathrm{D}(M \otimes_B \mathrm{D}B) = \mathrm{Hom}_k(M \otimes_B \mathrm{D}B, k) \simeq \mathrm{Hom}_B(M, \mathrm{Hom}_k(\mathrm{D}B, k)) \simeq \mathrm{Hom}_B(M, B) = M_B^*
\end{aligned}$$

Using the above, we see that condition (1) implies condition (2) by letting $P = A$.

$$\mathrm{D}A \otimes_A M \simeq \nu_A(A) \otimes_A M \simeq \nu_B(A \otimes_A M) \simeq \nu_B(M) \simeq M \otimes_B \mathrm{D}B.$$

Since every projective A-module is a direct summand of $A^{\oplus n}$ for some $n \in \mathbb{Z}$, this also shows that (2) implies (1). $\square$

5.3 Stable equivalences induced by exact functors

Suppose that ${}_B M_A$ is a bimodule which is projective as left A- and as right B-module such that $- \otimes_A M$ induces a stable equivalence $\underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$.

For self-injective algebras, Rickard has shown in [39, Theorem 3.2] that such a functor is isomorphic to a stable equivalence of Morita type. Dugas and Martínez-Villa provide the following generalization for arbitrary algebras which satisfy the separability condition. A stable equivalence that is induced by an exact functor $- \otimes_A M$ is of Morita type if and only if $\mathrm{Hom}_A(M, A)$ is projective on both sides. We have already made use of this result in the previous section; cf. Theorem 5.9.

We aim to give other sufficient conditions for $- \otimes_A M$ to be a stable equivalence of Morita type. In order to use our previous results, we will need to assume that $- \otimes_A M$ preserves perfect exact sequences with projective middle term. Furthermore, we will assume that A has positive dominant dimension in order to ensure that the cohomology of a complex in $\mathcal{L}$ vanishes under the functor $(-)^*$.

In order to use Theorem 5.13, we show that $- \otimes_A M$ induces an equivalence on $\mathcal{L}$. Since $- \otimes_A M$ is exact, it remains to check that $\mathrm{H}_k((F^\bullet \otimes_A M)^*) = 0$ vanishes for $F^\bullet \in \mathcal{L}$ in non-negative degrees. The following theorem by Yoshino provides a way to relate $\mathrm{H}_k(F_\bullet^*)$ with $(\mathrm{H}^k(F^\bullet))^*$ and $\mathrm{Ext}_A^1\big(\mathrm{Cok}(d_F^k), A\big)$. We give a modified version of the proof adapted to our notation.

Theorem 5.15. ([44, Theorem 2.3]) *Suppose given $F^\bullet \in \mathcal{K}(\mathrm{proj}\, A)$ and $M \in \mathrm{mod}\, A$.*

For all $k \in \mathbb{Z}$ there exists an exact sequence

$$0 \to \mathrm{Ext}_A^1\big(\mathrm{Cok}(d_F^k), M\big) \to \mathrm{H}^k\big(\mathrm{Hom}_A(F^\bullet, M)\big) \to \mathrm{Hom}_A\big(\mathrm{H}^k(F^\bullet), M\big) \to \mathrm{Ext}_A^2\big(\mathrm{Cok}(d_F^k), M\big).$$

Proof. Let $k \in \mathbb{Z}$. Note that $\mathrm{H}^k(F^\bullet) = \operatorname{Ker} d^k / \operatorname{Im} d^{k-1}$ and $\operatorname{Cok} d^{k-1} = F^k / \operatorname{Im} d^{k-1}$. Applying $\operatorname{Hom}_A(-, M)$ to the short exact sequence

$$0 \to \mathrm{H}^k(F^\bullet) \to \operatorname{Cok} d^{k-1} \xrightarrow{p} \operatorname{Im} d^k \to 0,$$

we obtain the following exact sequence.

$$0 \to \operatorname{Hom}(\operatorname{Im} d^k, M) \xrightarrow{(p,M)} \operatorname{Hom}_A(\operatorname{Cok} d^{k-1}, M) \to \operatorname{Hom}_A(\mathrm{H}^k(F^\bullet), M) \to \operatorname{Ext}^1(\operatorname{Im} d^k, M)$$

Since F^{k+1} is projective, the short exact sequence $0 \to \operatorname{Im} d^k \to F^{k+1} \to \operatorname{Cok} d^k \to 0$ yields that $\operatorname{Ext}^1(\operatorname{Im} d^k, M) \simeq \operatorname{Ext}^2(\operatorname{Cok} d^k, M)$. Thus, we have an exact sequence

$$0 \to \operatorname{Cok}\big((p, M)\big) \to \operatorname{Hom}_A(\mathrm{H}^k(F^\bullet), M) \to \operatorname{Ext}^2(\operatorname{Cok} d^k, M).$$

It remains to show the existence of a short exact sequence

$$0 \to \operatorname{Ext}^1_A(\operatorname{Cok} d^k, M) \to \mathrm{H}^k(\operatorname{Hom}_A(F^\bullet, M)) \to \operatorname{Cok}\big((p, M)\big) \to 0.$$

Applying $\operatorname{Hom}_A(-, M)$ to the exact sequence $F^{k-1} \xrightarrow{d^{k-1}} F^k \to \operatorname{Cok} d^{k-1} \to 0$, we obtain the following exact sequence.

$$0 \to \operatorname{Hom}_A(\operatorname{Cok} d^{k-1}, M) \to \operatorname{Hom}_A(F^k, M) \xrightarrow{(d^{k-1},M)} \operatorname{Hom}_A(F^{k-1}, M)$$

Hence, we have an isomorphism $\lambda : \operatorname{Ker}(d^{k-1}, M) \xrightarrow{\sim} \operatorname{Hom}_A(\operatorname{Cok} d^{k-1}, M)$.
Furthermore, the short exact sequence $0 \to \operatorname{Im} d^k \xrightarrow{i} F^{k+1} \to \operatorname{Cok} d^k \to 0$ gives rise to the exact sequence

$$\operatorname{Hom}_A(F^{k+1}, M) \xrightarrow{(i,M)} \operatorname{Hom}_A(\operatorname{Im} d^k, M) \to \operatorname{Ext}^1(\operatorname{Cok} d^k, M) \to 0,$$

using that F^{k+1} is projective. Moreover, since d^k factors through $\operatorname{Im} d^k$ via i, we obtain that (d^k, M) factors through (i, M).

$$\begin{tikzcd}
\operatorname{Hom}_A(F^{k+1}, M) \arrow[r, "(d^k,M)"] \arrow[d, "(i,M)"] & \operatorname{Hom}_A(F^k, M) \\
\operatorname{Hom}_A(\operatorname{Im} d^k, M) \arrow[ur, hook]
\end{tikzcd}$$

Therefore, we have $\operatorname{Im}\big((i, M)\big) = \operatorname{Im}\big((d^k, M)\big)$ which yields the following short exact sequence, induced from the sequence above.

$$0 \to \operatorname{Im}\big((d^k, M)\big) \to \operatorname{Hom}_A(\operatorname{Im} d^k, M) \to \operatorname{Ext}^1(\operatorname{Cok} d^k, M) \to 0$$

In conclusion, we constructed the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathrm{Ext}^1(\mathrm{Cok}\, d^k, M) \arrow[d, equal] & \\
0 \arrow[r] & \mathrm{Im}\big((d^k, M)\big) \arrow[r] \arrow[d] & \mathrm{Hom}_A(\mathrm{Im}\, d^k, M) \arrow[r] \arrow[d, "(p,M)"] & \mathrm{Ext}^1(\mathrm{Cok}\, d^k, M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}\big((d^{k-1}, M)\big) \arrow[r, "\lambda", "\sim"'] \arrow[d] & \mathrm{Hom}_A(\mathrm{Cok}\, d^{k-1}, M) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & \mathrm{H}^k\big(\mathrm{Hom}_A(F^\bullet, M)\big) \arrow[r] & \mathrm{Cok}\big((p, M)\big) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

The snake lemma now provides the desired short exact sequence.

$$0 \to \mathrm{Ext}^1_A(\mathrm{Cok}\, d^k, M) \to \mathrm{H}^k(\mathrm{Hom}_A(F^\bullet, M)) \to \mathrm{Cok}\big((p, M)\big) \to 0 \qquad \square$$

Recall that $(\mathrm{H}^k(F^\bullet))^* = 0$ for $F^\bullet \in \mathcal{L}_A$ and $k \in \mathbb{Z}$ if $\mathrm{domdim}\, A \geqslant 1$, as we have seen in Remark 4.13. By Lemma 2.24, the vanishing of S^* for a simple module S is invariant under stable equivalences that preserve perfect exact sequences with projective middle term.

Lemma 5.16. *Let $Y \in \mathrm{mod}\, A$ such that every short exact sequence $0 \to X \to Y' \to S \to 0$ with Y' a submodule of Y and S a simple A-module is a perfect exact sequence.*

Then $Y^* = 0$ if and only if $S^* = 0$ for all composition factors S of Y.

Proof. We proceed by induction on the length of Y. There is nothing to show for $l(Y) = 1$ so we assume $l(Y) > 1$. There exists a short exact sequence

$$0 \to X \to Y \to S \to 0$$

with S a simple A-module and $l(X) < l(Y)$. By assumption, this sequence is perfect exact. This implies that

$$0 \to S^* \to Y^* \to X^* \to 0$$

is a short exact sequence. Thus $Y^* = 0$ if and only if $X^* = 0$ and $S^* = 0$. Since X is a submodule of Y, we are done by induction. $\square$

Recall that a complex $F^\bullet$ in $\mathcal{L}_A$ satisfies $\mathrm{H}^k(F^\bullet) \in {}^\perp\mathcal{P}_A$ for all $k \in \mathbb{Z}$. If $\mathrm{domdim}\, A \geqslant 1$, the assumptions of the lemma above hold for the cohomology of $F^\bullet$ by Lemma 2.12. We also have seen that an exact functor $- \otimes_A M$ preserves perfect exact sequences with projective middle term if and only if $\mathrm{Ext}^1(Z, A) = 0$ implies $\mathrm{Ext}^1_B(Z \otimes_A M, B) = 0$ for all $Z \in \mathrm{mod}\, A$; cf. Proposition 3.2.(1). We are now ready to prove the main result of this section.

Proposition 5.17. *Let A and B be finite dimensional k-algebras whose semisimple quotients are separable. Assume that A and B have dominant dimension at least* 1*.*

Suppose given a bimodule ${}_AM_B$ which is projective as left A- and as right B-module such that $-\otimes_A M$ induces a stable equivalence $\underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$. Assume furthermore that the following conditions hold.

(1) The stable equivalence $-\otimes_A M$ and its quasi-inverse preserve perfect exact sequences with projective middle term.

(2) For all simple A-modules S whose injective hull is not projective, the image $S\otimes_A M$ is a simple B-module.

Then M and $\mathrm{Hom}_B(M,B)$ induce a stable equivalence of Morita type between A and B.

Proof. Suppose given $F^\bullet \in \mathcal{L}_A$. Using that $-\otimes_A M$ is an exact functor, it remains to show that $\mathrm{H}_k((F^\bullet\otimes_A M)^*) = 0$ for $k \geqslant 0$. In this case, the assertion follows from Theorem 5.13. By Theorem 5.15, the vanishing of $\mathrm{H}_k((F^\bullet\otimes_A M)^*)$ is implied by $\mathrm{Ext}^1_B(\mathrm{Cok}(d^k_{F\otimes M}), B) = 0$ and $\mathrm{H}^k(F^\bullet\otimes_A M)^* = 0$. We fix a $k \geqslant 0$.

We show that $\mathrm{Ext}^1_B(\mathrm{Cok}(d^k_{F\otimes M}), B) = 0$. Since $F^\bullet \in \mathcal{L}_A$, we have $\mathrm{H}_k(F^*_\bullet) = 0$. The exact sequence in Theorem 5.15 now implies that $\mathrm{Ext}^1_A(\mathrm{Cok}(d^k_F), A) = 0$. By Proposition 3.2.(1) and assumption (1), we obtain that $\mathrm{Ext}^1_B(\mathrm{Cok}(d^k_F)\otimes_A M, B) = 0$. Using that $-\otimes_A M$ is exact, we additionally have that

$$\mathrm{Cok}(d^k_F)\otimes_A M \simeq \mathrm{Cok}(d^k_F\otimes_A M) = \mathrm{Cok}(d^k_{F\otimes M}).$$

This results in $\mathrm{Ext}^1_B(\mathrm{Cok}(d^k_{F\otimes M}), B) = 0$.

We show that $\mathrm{H}^k(F^\bullet\otimes_A M)^* = 0$. By Lemma 4.5, we have $\mathrm{H}^k(F^\bullet) \in {}^\perp\mathcal{P}_A$. In particular, we have $\mathrm{H}^k(F^\bullet)^* = 0$ since $\mathrm{domdim}\, A \geqslant 1$. It suffices to show the following claim.

Claim. Let $X \in \mathrm{mod}\, A$ with $X \in {}^\perp\mathcal{P}_A$. Then $(X\otimes_A M)^* = 0$.

We prove the claim by induction on the length $l := l(X)$ of X. Since $-\otimes_A M$ is exact, we have $l = l(X) = l(X\otimes_A M)$. Note that we have $X^* = 0$ since $\mathrm{domdim}\, A \geqslant 1$ by assumption. Moreover, the assumptions of Lemma 5.16 are satisfied by Lemma 2.12. In particular, we have $S^* = 0$ for all composition factors S of X. Furthermore, $S\otimes_A M$ is a simple B-module by assumption (2) since $\nu_A(S) = 0$ implies $\nu_A(F^0_S) \notin \mathcal{P}_A$; cf. Lemma 2.26.

Let $l = 1$ so that X and $X\otimes_A M$ are simple modules. Thus, $(X\otimes_A M)^* = 0$ if and only if $X^* = 0$ by Lemma 2.24.(4,5) since $-\otimes_A M$ preserves perfect exact sequences with projective middle term by assumption (1). We have seen above, that $S^* = 0$ for all composition factors S of X.

Let $l > 1$. Suppose given a composition factor S of X together with a short exact sequence

$$0 \to U \to X \to S \to 0\,.$$

By Lemma 2.12, we have $U^* = 0$ and this is a perfect exact sequence. Using that $l(U) < l$, we can assume that $(U \otimes_A M)^* = 0$ by induction. Thus, the induced short exact sequence

$$0 \to U \otimes_A M \to X \otimes_A M \to S \otimes_A M \to 0$$

is perfect exact in $\operatorname{mod} B$. In particular, applying $(-)^*$, we obtain a short exact sequence in B-mod with $(U \otimes_A M)^* = 0$.

$$0 \to (U \otimes_A M)^* \to (X \otimes_A M)^* \to (S \otimes_A M)^* \to 0$$

As in the case $l = 1$, we also have $(S \otimes_A M)^* = 0$. This shows that $(X \otimes_A M)^* = 0$. □

Let S be a simple module whose injective hull is not projective. For algebras without nodes, a stable equivalence maps S up to projective direct summands to a simple module. This follows from a result by Martínez-Villa in [33, Proposition 2.4]. We slightly adapt his proof to show the following analogue for stable equivalences that are induced by an exact functor.

Lemma 5.18. *Let ${}_AM_B$ be a bimodule that is projective as left A- and as right B-module such that $- \otimes_A M$ induces a stable equivalence $\underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$. Suppose that the inverse stable equivalence is also induced by an exact functor.*

Let S be a non-projective simple A-module with injective hull I such that I is not projective. We have $S \otimes_A M \simeq S' \oplus P$ such that S' is a simple B-module and $P \in \operatorname{proj} B$.

Proof. The stable equivalence $-\otimes_A M$ induces a one-to-one correspondence between the isomorphism classes of indecomposable non-projective modules in $\operatorname{mod} A$ and in $\operatorname{mod} B$. We denote this correspondence by α'. Let $\pi : I \twoheadrightarrow I/S$ be the natural projection, which is an irreducible morphism. Since I is not projective, we know that $\underline{\pi} \neq 0$ in $\underline{\operatorname{mod}}\, A$. By [6, Lemma X.1.2], we obtain that the morphism $\alpha'(\pi) : \alpha'(I) \to \alpha'(I/S)$ which is induced by $\pi \otimes M$ is irreducible.

Using that the stable equivalence and its quasi-inverse are induced by an exact functor, $\alpha'(I)$ is an indecomposable injective and non-projective B-module; cf. [27, Lemma 3.5]. Thus, $S' := \operatorname{soc}(\alpha'(I))$ is a simple B-module. We have $S' \subseteq \operatorname{Ker}(\alpha'(\pi))$ since $\pi \otimes M$ is not a stable isomorphism. This induces a morphism

$$f : \alpha'(I)/S' \to \alpha'(I/S)$$

such that $\pi' f = \alpha'(\pi)$ with π' the natural projection $\pi' : \alpha'(I) \twoheadrightarrow \alpha'(I)/S'$. However, $\alpha'(\pi)$ is irreducible and thus f must be a split epimorphism. Now, consider the natural projection $\pi' : \alpha'(I) \twoheadrightarrow \alpha'(I)/S'$. Let β' be the inverse of the correspondence α'. As above, we obtain that

$$f' : I/S \to \beta'(\alpha'(I)/S')$$

is a split epimorphism. As a consequence, $\alpha'(I/S)$ is a direct summand of $\alpha'(I)/S'$. Together with the split epimorphism f, this results in $\alpha'(I)/S' \simeq \alpha'(I/S)$.

Write $I \otimes_A M \simeq \alpha'(I) \oplus P$ and $(I/S) \otimes_A M \simeq \alpha'(I/S) \oplus Q$ with $P, Q \in \operatorname{proj} B$. Consider the following commutative diagram with C the cokernel of the induced morphism $S' \to S \otimes_A M$.

$$\begin{tikzcd}
0 \arrow[r] & S' \arrow[r] \arrow[d] & \alpha'(I) \arrow[r] \arrow[d] & \alpha'(I/S) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S \otimes_A M \arrow[r] \arrow[d] & I \otimes_A M \arrow[r] \arrow[d] & (I/S) \otimes_A M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C \arrow[r] & P \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

Since $- \otimes_A M$ is exact and $\alpha'(I/S) \simeq \alpha'(I)/S'$, all rows are short exact sequences. In particular, the bottom row splits since Q is projective. Thus, C is projective as well and we obtain $S \otimes_A M \simeq S' \oplus C$. □

We summarize the results of the last two sections and include situations in which the assumptions are satisfied.

Theorem 5.19. *Let A and B be finite dimensional k-algebras whose semisimple quotients are separable. Suppose given a bimodule ${}_AM_B$ which is projective as left A- and as right B-module such that $- \otimes_A M$ induces a stable equivalence $\underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$. If one of the following conditions holds, M and $\operatorname{Hom}_B(M, B)$ induce a stable equivalence of Morita type between A and B.*

(i) The functor $- \otimes_A M$ induces an equivalence $\mathcal{L}_A \to \mathcal{L}_B$.

(ii) The homology $\mathrm{H}_k((F^\bullet \otimes_A M)^*)$ vanishes for $F^\bullet \in \mathcal{L}_A$ and $k \geqslant 0$.

(iii) There exist natural isomorphisms $\nu_B(P \otimes_A M) \simeq \nu_A(P) \otimes_A M$ for all $P \in \operatorname{proj} A$.

(iv) There exists a natural isomorphism $M \otimes_B \mathrm{D}B \simeq \mathrm{D}A \otimes_A M$ of right B-modules.

(v) The algebras A and B have no nodes. At least one of A or B has dominant dimension at least 1 and finite representation type. Moreover, for all simple A-modules S whose injective hull is not projective, the image $S \otimes_A M$ is an indecomposable B-module.

(*vi*) *The algebras A and B have no nodes. At least one of A or B is a Nakayama algebra. Moreover, for all simple A-modules S whose injective hull is not projective, the image $S \otimes_A M$ is an indecomposable B-module.*

(*vii*) *The algebras A and B have dominant dimension at least 1. There is a bimodule ${}_BL_A$ which is projective as left B- and right A-module and which induces the inverse stable equivalence. Moreover, for all simple A-modules S whose injective hull is not projective, the image $S \otimes_A M$ is an indecomposable B-module.*

Proof. If condition (i) holds, we have seen in Theorem 5.13 that M and $\operatorname{Hom}_B(M, B)$ induce a stable equivalence of Morita type between A and B. Let $F^\bullet \in \mathcal{L}_A$. Since $- \otimes_A M$ is an exact functor, we know that $\mathrm{H}^k(F^\bullet \otimes_A M) = 0$ for $k \leqslant -1$. Thus, condition (ii) implies condition (i). By Corollary 5.14, condition (iii) and (iv) also imply condition (i). The last three conditions (v), (vi) and (vii) are a consequence of Proposition 5.17 using the following additional results. Since $- \otimes_A M : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$ is a stable equivalence, A is of finite representation type if and only if B is of finite representation type. Moreover, by [33, Theorem 2.3], α preserves the dominant dimension if A and B have no nodes. Note that a Nakayama algebra is of finite representation type and has dominant dimension at least 1. In (v) and (vi) we now use that a stable equivalence between algebras without nodes and of finite representation type preserves perfect exact sequences by Corollary 3.20. In the setting of part (vii), perfect exact sequences with projective middle term are preserved by Proposition 3.2. Finally, for a simple A-module S, we have that $S \otimes_A M$ is isomorphic to a direct sum of a simple module and a projective module by [33, Proposition 2.4] in the setting of part (v) and (vi) and by Lemma 5.18 in the setting of part (vii). If $S \otimes_A M$ is indecomposable, $S \otimes_A M$ must be isomorphic to a simple B-module. Therefore, both assumptions of Proposition 5.17 are satisfied if condition (v), (vi) or (vii) holds. □

Remark 5.20. Suppose that ${}_AM_B$ is a bimodule such that $- \otimes_A M$ induces an equivalence $\mathcal{L}_A \to \mathcal{L}_B$ as in part (i) of the previous theorem. Let S be a simple A-module with $S^* = 0$. If $\operatorname{domdim} A \geqslant 1$, this holds for simple A-modules whose injective hull is not projective. Then $F_S^{\leqslant 0} = F_S^\bullet \in \mathcal{L}_A$ is a projective resolution of $S = \mathrm{H}^0(\tau_{\leqslant 0} F_S^\bullet)$; cf. Lemma 2.24. Thus, $F_S^\bullet \otimes_A M \in \mathcal{L}_B$ is a projective resolution of $S \otimes_A M$. In particular, $(S \otimes_A M)^* = 0$ by Lemma 2.24 and we obtain that $S \otimes M$ has no projective direct summand. Thus, $S \otimes_A M$ is indecomposable.

Suppose ${}_AM_B$ and ${}_BN_A$ are bimodules that induce a stable equivalence of Morita type. If ${}_AM_B$ and ${}_BN_A$ are indecomposable as bimodules, we even have that $S \otimes M$ is indecomposable for all simple A-modules S; cf. [23, Lemma 4.4].

It seems unclear whether the assumption in the previous theorem on the image $S \otimes_A M$ of a simple A-module can be dropped if we assume that ${}_AM_B$ is an indecomposable bimodule.

Example in Chapter 7. The algebras A and B in Section 7.1 are stably equivalent of Morita type. In Example 7.6, we give a bimodule that induces a stable equivalence. Using the results of this chapter, we verify that this is a stable equivalence of Morita type.

Chapter 6

Stable equivalences with nodes

Let k be a field. Let $A = kQ/I$ and $B = k\tilde{Q}/\tilde{I}$ be finite dimensional quiver algebras given by quivers Q and $\tilde{Q}$ and by admissible ideals I and $\tilde{I}$ respectively. We assume that A and B have no semisimple summands.

Starting with a finite dimensional algebra B with nodes, Martínez-Villa constructed an algebra A without nodes so that A is stably equivalent to B; cf. [31, Theorem 2.10]. More generally, he considered algebras which are stably equivalent and which can be obtained from each other by either deleting or inserting a node. See also [34].

Inserting a node can be described with the following process. Let $e_1, \ldots, e_n, u, v$ be a complete set of primitive idempotents in A. We say that B is obtained from A by gluing the primitive idempotents u and v if B is generated by $e_1, \ldots, e_n, u + v$ and all arrows in A. This induces a *radical embedding* $f : B \hookrightarrow A$, that is, an injective algebra monomorphism $f : B \hookrightarrow A$ with $\mathrm{rad}(Bf) = \mathrm{rad}(A)$. Now, the simple B-module corresponding to $u + v$ is a node. Here, we use the following characterization of a node.

Lemma 6.1. ([31, Lemma 1])

Let S be a simple A-module with projective cover P. The following are equivalent.

(1) S is either injective or a node.

(2) For all non-isomorphisms $f : P_1 \to P$ and $g : P \to P_2$ with P_1 and P_2 indecomposable projective A-modules, we have $f\,g = 0$.

(3) S is not a composition factor of $\mathrm{rad}(P_0)/\operatorname{soc}(P_0)$ for any indecomposable projective A-module P_0.

Let B be an algebra that is obtained from A by a finite number of steps of gluing a simple projective vertex and a simple injective vertex. In [22], Koenig and Liu used a different approach than Martínez-Villa to construct bimodules that induce a stable equivalences between A and B in this setting. We aim to give an explicit description of all algebras that can be obtained in this way. The following is an excerpt of [22, Theorem 4.12].

Theorem 6.2 (Koenig, Liu). *Let $A = kQ/I$ and $B = k\tilde{Q}/\tilde{I}$ be two finite dimensional algebras such that there is a radical embedding $f : B \hookrightarrow A$. Consider the following conditions.*

(1) A and B are stably equivalent.

(2) B is obtained from A by a finite number of steps of gluing a simple projective vertex and a simple injective vertex.

(3) There exists a pair of bimodules which induce inverse stable equivalences between $\underline{\mathrm{mod}}\, A$ and $\underline{\mathrm{mod}}\, B$.

Then (2) implies (3) and thus also implies (1). Under the assumption of the Auslander-Reiten conjecture, all three conditions are equivalent. In particular, if A or B has finite representation type, then all three conditions are equivalent.

Recall that the Auslander-Reiten conjecture states that two stably equivalent finite dimensional algebras have the same number of non-isomorphic non-projective simple modules; cf. [6, Conjecture 5, page 409]. The Auslander-Reiten conjecture was proven for algebras of finite representation type by Martínez-Villa in [32].

An extended example of the constructions in the next two sections can be found in Example 7.16. Throughout this chapter, we use the following notation.

We denote the number of isomorphism classes of simple A modules by s and the number of isomorphism classes of simple B modules by t. Let $\{P_1, \ldots P_s\}$ and $\{Q_1, \ldots Q_t\}$ be a complete set of non-isomorphic indecomposable projective A-modules and B-modules respectively. For $i \in [1, s]$, we denote the simple top of P_i by S_i. For $i \in [1, t]$, we denote the simple top of Q_i by T_i. Thus, $\{S_1, \ldots S_s\}$ and $\{T_1, \ldots T_t\}$ are a complete set of non-isomorphic simple A-modules and B-modules respectively.

We write $[n, m] = \{z \in \mathbb{Z} | n \leqslant z \leqslant m\}$ for n, $m \in \mathbb{Z}$. Given $n, m \in \mathbb{Z}$, we write $\delta_{m,n} = 1 \in k$ if $m = n$ and $\delta_{m,n} = 0 \in k$ if $m \neq n$. For $n \in \mathbb{Z}_{\geqslant 1}$ and $i \in [1, n]$ we denote by e_i the $n \times n$ matrices having entry 1 at position (i, i) and entry 0 elsewhere.

Recall that we write morphisms between direct sums of modules as matrices. We extend the usual notation for matrix algebras as follows. In particular, we will allow multiplication of morphisms that are not composable and thus we sometimes have to add a direct summand isomorphic to k on the diagonal. Let $n \in \mathbb{Z}_{\geqslant 1}$ and suppose given A-modules X_i and Y_i for $i \in [1, n]$. For $i, j \in [1, n]$ let $V_{i,j}$ be the k-vector space defined as follows.

$$V_{i,j} := \begin{cases} \mathrm{Hom}_A(X_i, Y_j) & \text{if } i \neq j \\ \mathrm{Hom}_A(X_i, X_i) & \text{if } i = j \text{ and } X_i = Y_i \neq 0 \\ k \oplus \mathrm{Hom}_A(X_i, Y_i) & \text{if } i = j \text{ and either } X_i \neq Y_i \text{ or } X_i = Y_i = 0 \end{cases}$$

We often abbreviate $k \oplus \mathrm{Hom}_A(X_i, Y_i) \simeq k$ by k if $X_i = Y_i = 0$. Now, we have the following multiplication of elements in $(V_{i,j})_{i,j\in[1,n]}$ induced by multiplication of matrices.

$$(v_{i,j})_{i,j\in[1,n]} \cdot (w_{i,j})_{i,j\in[1,n]} := \left(\sum_{l=1}^{n} v_{i,l} \cdot w_{l,j}\right)_{i,j\in[1,n]} \quad \text{for } (v_{i,j})_{i,j\in[1,n]}, (w_{i,j})_{i,j\in[1,n]} \in (V_{i,j})_{i,j\in[1,n]}$$

It remains to define $v_{i,l} \cdot w_{l,j}$ for $i, j, l \in [1, n]$. The idea is to use composition of morphisms while setting the composite of non-composable morphisms to zero. Let $v \in V_{i,l}$ and $w \in V_{l,j}$.

In case that $i \neq j$, we define $v \cdot w \in V_{i,j} = \mathrm{Hom}_A(X_i, Y_j)$ as follows.

$$v \cdot w := \begin{cases} 0 & \text{if} \quad v = f \in \mathrm{Hom}_A(X_i, Y_l) & \text{and} \quad w = g \in \mathrm{Hom}_A(X_l, Y_j) \text{ with } Y_l \neq X_l \\ f\, g & \text{if} \quad v = f \in \mathrm{Hom}_A(X_i, Y_l) & \text{and} \quad w = g \in \mathrm{Hom}_A(X_l, Y_j) \text{ with } Y_l = X_l \\ x\, g & \text{if} \quad v = x + f \in k \oplus \mathrm{Hom}_A(X_i, Y_l) & \text{and} \quad w = g \in \mathrm{Hom}_A(X_l, Y_j) \\ f\, y & \text{if} \quad v = f \in \mathrm{Hom}_A(X_i, Y_l) & \text{and} \quad w = y + g \in k \oplus \mathrm{Hom}_A(X_l, Y_j) \end{cases}$$

In case that $i = j$ with $X_i = Y_i \neq 0$, we define $v \cdot w \in V_{i,j} = \mathrm{Hom}_A(X_i, X_i)$ as follows.

$$v \cdot w := \begin{cases} 0 & \text{if} \quad v = f \in \mathrm{Hom}_A(X_i, Y_l) \quad \text{and} \quad w = g \in \mathrm{Hom}_A(X_l, Y_i) \text{ with } Y_l \neq X_l \\ f\, g & \text{if} \quad v = f \in \mathrm{Hom}_A(X_i, Y_l) \quad \text{and} \quad w = g \in \mathrm{Hom}_A(X_l, Y_i) \text{ with } Y_l = X_l \end{cases}$$

In case that $i = j$ and either $X_i \neq Y_i$ or $X_i = Y_i = 0$, we define $v \cdot w \in V_{i,j} = k \oplus \mathrm{Hom}_A(X_i, Y_i)$ as follows.

$$v \cdot w := \begin{cases} 0 + 0 & \text{if } v = f \in \mathrm{Hom}_A(X_i, Y_l) & \text{and} \quad w = g \in \mathrm{Hom}_A(X_l, Y_j) \text{ with } Y_l \neq X_l \\ 0 + f\, g & \text{if } v = f \in \mathrm{Hom}_A(X_i, Y_l) & \text{and} \quad w = g \in \mathrm{Hom}_A(X_l, Y_j) \text{ with } Y_l = X_l \\ x\, y + (f\, y + x\, g) & \text{if } v = x + f \in k \oplus \mathrm{Hom}_A(X_i, Y_l) & \text{and} \quad w = y + g \in k \oplus \mathrm{Hom}_A(X_l, Y_j) \end{cases}$$

In this way, we obtain a matrix algebra $(V_{i,j})_{i,j\in[1,n]}$. The identity of $(V_{i,j})_{i,j\in[1,n]}$ is given by the diagonal matrix $(v_{i,j})_{i,j}$ with $v_{i,j} = 0$ for $i \neq j$ and $v_{i,i} = \mathrm{id}_{X_i}$ if $X_i = Y_i \neq 0$ and $v_{i,i} = 1 + 0 \in k \oplus \mathrm{Hom}_A(X_i, Y_i)$ otherwise.

We illustrate this with an example. Let X, Y and Z be non-zero A-modules which are pairwise non-isomorphic. For the elements

$$\begin{pmatrix} x + a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x' + a' & b' \\ c' & d' \end{pmatrix} \in \begin{pmatrix} k \oplus \mathrm{Hom}_A(X, Y) & \mathrm{Hom}_A(X, Z) \\ \mathrm{Hom}_A(Z, Y) & \mathrm{Hom}_A(Z, Z) \end{pmatrix}$$

we have

$$\begin{pmatrix} x + a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x' + a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} x\, x' + x\, a' + a\, x' + b\, c' & x\, b' + b\, d' \\ c\, x' + d\, c' & d\, d' \end{pmatrix}.$$

6.1 Algebras obtained by gluing idempotents

Consider an algebra A with simple projective and simple injective modules. We construct a new algebra $E_\sigma(A)$ which is obtained by gluing pairs of simple projective vertices and simple injective vertices. These pairs will be described by an injective map σ.

Definition 6.3. Suppose given $\mathcal{J} \subseteq \{j \in [1,s] \mid S_j \in \operatorname{proj} A\}$ together with an injective map

$$\sigma : \mathcal{J} \hookrightarrow \{i \in [1,s] \mid S_i \in \operatorname{inj} A\}$$

so that $P_j = S_j$ is a simple projective module and $S_{j\sigma}$ is a simple injective module for $j \in \mathcal{J}$. Let $P := \bigoplus_{r \in [1,s] \setminus (J \cup J\sigma)} P_r$ so that $A_A = \bigoplus_{j \in \mathcal{J}} (P_j \oplus P_{j\sigma}) \oplus P$.

We define the following matrix algebra corresponding to σ.

$$E_\sigma(A) := \begin{pmatrix} \big(\delta_{i,j}\, k \oplus \operatorname{Hom}_A(P_i, P_{j\sigma})\big)_{i,j \in \mathcal{J}} & \big(\operatorname{Hom}_A(P_i, P)\big)_{i \in \mathcal{J}} \\ \big(\operatorname{Hom}_A(P, P_{j\sigma})\big)_{j \in \mathcal{J}} & \operatorname{End}_A(P) \end{pmatrix}$$

Since A has no semisimple summand, we have that $j\sigma \notin \mathcal{J}$ for $j \in \mathcal{J}$, that is $\mathcal{J} \cap \mathcal{J}\sigma = \emptyset$. In particular, $P_{j\sigma} \not\simeq P_i$ for $i,j \in \mathcal{J}$. Thus, the first $|\mathcal{J}|$ columns of this matrix correspond to indecomposable projective $E_\sigma(A)^{\mathrm{op}}$-modules whose simple top is the node obtained by gluing a simple projective and a simple injective vertex; cf. Lemma 6.1.(2). In total, the number of non-isomorphic indecomposable projective $E_\sigma(A)^{\mathrm{op}}$-modules is

$$|\mathcal{J}| + |\,[1,s] \setminus (J \cup J\sigma)| = s - |\mathcal{J}|.$$

Remark 6.4. Let $i,j \in \mathcal{J}$. Note that $P_{i\sigma}^*$ is a simple left A-module since $S_i \in \operatorname{inj} A$ and thus $\nu P_{i\sigma} \simeq S_{i\sigma}$. Using that P_j and $P_{i\sigma}^*$ are simple and that $i\sigma \neq j$, we have the following.

$$\begin{aligned} \operatorname{Hom}_A(P_{i\sigma}, P_j) &= 0 \\ \operatorname{Hom}_A(P_{i\sigma}, P) &\simeq \operatorname{Hom}_A(P^*, P_{i\sigma}^*) = 0 \\ \operatorname{Hom}_A(P, P_j) &= 0 \\ \operatorname{Hom}_A(P_i, P_j) &\simeq \delta_{i,j}\, k \\ \operatorname{Hom}_A(P_{i\sigma}, P_{j\sigma}) &\simeq \operatorname{Hom}_A(P_{j\sigma}^*, P_{i\sigma}^*) \simeq \delta_{i,j}\, k \end{aligned}$$

This can be used to rewrite the endomorphism algebra of A.

$$\operatorname{End}_A(A) \simeq \begin{pmatrix} \big(\operatorname{Hom}_A(P_i \oplus P_{i\sigma}, P_j \oplus P_{j\sigma})\big)_{i,j \in \mathcal{J}} & \big(\operatorname{Hom}_A(P_i \oplus P_{i\sigma}, P)\big)_{i \in \mathcal{J}} \\ \big(\operatorname{Hom}_A(P, P_j \oplus P_{j\sigma})\big)_{j \in \mathcal{J}} & \operatorname{End}_A(P) \end{pmatrix}$$

$$\simeq \begin{pmatrix} \begin{pmatrix} \mathrm{Hom}_A(P_i, P_j) & \mathrm{Hom}_A(P_i, P_{j\sigma}) \\ \mathrm{Hom}_A(P_{i\sigma}, P_j) & \mathrm{Hom}_A(P_{i\sigma}, P_{j\sigma}) \end{pmatrix}_{i,j\in\mathcal{J}} & \begin{pmatrix} \mathrm{Hom}_A(P_i, P) \\ \mathrm{Hom}_A(P_{i\sigma}, P) \end{pmatrix}_{i\in\mathcal{J}} \\ \begin{pmatrix} \mathrm{Hom}_A(P, P_j) & \mathrm{Hom}_A(P, P_{j\sigma}) \end{pmatrix}_{j\in\mathcal{J}} & \mathrm{End}_A(P) \end{pmatrix}$$

$$\simeq \begin{pmatrix} \begin{pmatrix} \delta_{i,j}\, k & \mathrm{Hom}_A(P_i, P_{j\sigma}) \\ 0 & \delta_{i,j}\, k \end{pmatrix}_{i,j\in\mathcal{J}} & \begin{pmatrix} \mathrm{Hom}_A(P_i, P) \\ 0 \end{pmatrix}_{i\in\mathcal{J}} \\ \begin{pmatrix} 0 & \mathrm{Hom}_A(P, P_{j\sigma}) \end{pmatrix}_{j\in\mathcal{J}} & \mathrm{End}_A(P) \end{pmatrix}$$

On the other hand, we have the following by rewriting $\big(\delta_{i,j}\, k \oplus \mathrm{Hom}_A(P_i, P_{j\sigma})\big)_{i,j\in\mathcal{J}}$ as an upper triangular matrix.

$$E_\sigma(A) \simeq \left\{ \begin{pmatrix} \begin{pmatrix} \delta_{i,j}\, x_j & a_{i,j} \\ 0 & \delta_{i,j}\, x_j \end{pmatrix}_{i,j\in\mathcal{J}} & \begin{pmatrix} b_i \\ 0 \end{pmatrix}_{i\in\mathcal{J}} \\ \begin{pmatrix} 0 & c_j \end{pmatrix}_{j\in\mathcal{J}_k} & d \end{pmatrix} \;\middle|\; \begin{array}{lr} x_j \in k & \text{for } j \in \mathcal{J} \\ a_{i,j} \in \mathrm{Hom}_A(P_i, P_{j\sigma}),\ b_i \in \mathrm{Hom}_A(P_i, P) & \text{for } i, j \in \mathcal{J} \\ c_j \in \mathrm{Hom}_A(P, P_{j\sigma}) & \text{for } j \in \mathcal{J} \\ d \in \mathrm{End}_A\,(P) & \end{array} \right\}$$

Together, we obtain a radical embedding $E_\sigma(A)^{\mathrm{op}} \hookrightarrow \mathrm{End}_A(A)^{\mathrm{op}} \simeq A$.

We aim to show that every algebra which is obtained from A by a finite number of steps of gluing a simple projective vertex and a simple injective vertex is of the form $E_\sigma(A)^{\mathrm{op}}$ for some σ as in Definition 6.3. As a first result, we show that $E_\sigma(A)^{\mathrm{op}}$ can be obtained in this way.

Lemma 6.5. *Suppose given $\mathcal{J}$ and σ as in Definition 6.3.*

There exists a radical embedding $E_\sigma(A)^{\mathrm{op}} \hookrightarrow A$ such that $E_\sigma(A)^{\mathrm{op}}$ is obtained from A by a finite number of steps of gluing a simple projective vertex and a simple injective vertex.

In particular, $E_\sigma(A)^{\mathrm{op}}$ and A are stably equivalent.

Proof. By Remark 6.4 there exists a radical embedding $E_\sigma(A)^{\mathrm{op}} \hookrightarrow A$.

Let $\mathcal{J} = \{j_1, \ldots, j_l\}$ for some $1 \leqslant l \leqslant s$. For $0 \leqslant r \leqslant l$ we write $\mathcal{J}_r := \{j_1, \ldots, j_r\} \subseteq \mathcal{J}$ and

$$\tilde{P}_r := \bigoplus_{p\in[r+1,l]} (P_p \oplus P_{p\sigma}) \oplus P.$$

Note that $\mathcal{J}_0 = \emptyset$ and $\tilde{P}_0 = A_A$ on the one hand and $\mathcal{J}_l = \mathcal{J}$ and $\tilde{P}_l = P$ on the other hand. Consider the following algebra for $r \in [0, l]$ with pairwise identical diagonal entries in the first $2r$ columns.

$$A_r := \left\{ \begin{pmatrix} \begin{pmatrix} \delta_{i,j}\, x_j & a_{i,j} \\ 0 & \delta_{i,j}\, x_j \end{pmatrix}_{i,j\in\mathcal{J}_r} & \begin{pmatrix} b_i \\ 0 \end{pmatrix}_{i\in\mathcal{J}_r} \\ \begin{pmatrix} 0 & c_j \end{pmatrix}_{j\in\mathcal{J}_r} & d \end{pmatrix} \;\middle|\; \begin{array}{lr} x_j \in k & \text{for } j \in \mathcal{J}_r \\ a_{i,j} \in \mathrm{Hom}_A(P_i, P_{j\sigma}),\ b_i \in \mathrm{Hom}_A(P_i, \tilde{P}_r) & \text{for } i, j \in \mathcal{J}_r \\ c_j \in \mathrm{Hom}_A(\tilde{P}_r, P_{j\sigma}) & \text{for } j \in \mathcal{J}_r \\ d \in \mathrm{End}_A(\tilde{P}_r) & \end{array} \right\}$$

We obtain the following chain of subalgebras.

$$E_\sigma(A)^{\mathrm{op}} = A_l^{\mathrm{op}} \subseteq A_{l-1}^{\mathrm{op}} \subseteq \cdots \subseteq A_1^{\mathrm{op}} \subseteq A_0^{\mathrm{op}} = \mathrm{End}_A(A)^{\mathrm{op}} \simeq A$$

Fix $r \in [0, l-1]$ and consider the inclusion $A_{r+1}^{\mathrm{op}} \subseteq A_r^{\mathrm{op}}$. Notice that A_{r+1}^{op} is obtained from A_r^{op} by gluing the simple projective vertex corresponding to column $2r+1$ and the simple injective vertex corresponding to row $2r+2$.

$$A_r = \left\{ \left(\begin{array}{ccc} \begin{pmatrix} \delta_{i,j}\, x_j & a_{i,j} \\ 0 & \delta_{i,j}\, x_j \end{pmatrix}_{i,j\in\mathcal{J}_r} & \begin{pmatrix} 0 & a_{i,r+1} \\ 0 & 0 \end{pmatrix}_{i\in\mathcal{J}_r} & \begin{pmatrix} b_i \\ 0 \end{pmatrix}_{i\in\mathcal{J}_r} \\ \begin{pmatrix} 0 & a_{r+1,j} \\ 0 & 0 \end{pmatrix}_{j\in\mathcal{J}_r} & \begin{pmatrix} x & a_{r+1,r+1} \\ 0 & y \end{pmatrix} & \begin{pmatrix} b_{r+1} \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & c_j \end{pmatrix}_{j\in\mathcal{J}_r} & \begin{pmatrix} 0 & c_{r+1} \end{pmatrix} & d \end{array} \right) \middle| \begin{array}{lr} x, y, x_j \in k & \text{for } j \in \mathcal{J}_r \\ a_{i,j} \in \mathrm{Hom}_A(P_i, P_{j\sigma}) & \text{for } i,j \in \mathcal{J}_{r+1} \\ b_i \in \mathrm{Hom}_A(P_i, \tilde{P}_{r+1}) & \text{for } i \in \mathcal{J}_{r+1} \\ c_j \in \mathrm{Hom}_A(\tilde{P}_{r+1}, P_{j\sigma}) & \text{for } j \in \mathcal{J}_{r+1} \\ d \in \mathrm{End}_A(\tilde{P}_{r+1}) & \end{array} \right\}$$

$$A_{r+1} = \left\{ \left(\begin{array}{ccc} \begin{pmatrix} \delta_{i,j}\, x_j & a_{i,j} \\ 0 & \delta_{i,j}\, x_j \end{pmatrix}_{i,j\in\mathcal{J}_r} & \begin{pmatrix} 0 & a_{i,r+1} \\ 0 & 0 \end{pmatrix}_{i\in\mathcal{J}_r} & \begin{pmatrix} b_i \\ 0 \end{pmatrix}_{i\in\mathcal{J}_r} \\ \begin{pmatrix} 0 & a_{r+1,j} \\ 0 & 0 \end{pmatrix}_{j\in\mathcal{J}_r} & \begin{pmatrix} x_{r+1} & a_{r+1,r+1} \\ 0 & x_{r+1} \end{pmatrix} & \begin{pmatrix} b_{r+1} \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & c_j \end{pmatrix}_{j\in\mathcal{J}_r} & \begin{pmatrix} 0 & c_{r+1} \end{pmatrix} & d \end{array} \right) \middle| \begin{array}{lr} x_j \in k & \text{for } j \in \mathcal{J}_{r+1} \\ a_{i,j} \in \mathrm{Hom}_A(P_i, P_{j\sigma}) & \text{for } i,j \in \mathcal{J}_{r+1} \\ b_i \in \mathrm{Hom}_A(P_i, \tilde{P}_{r+1}) & \text{for } i \in \mathcal{J}_{r+1} \\ c_j \in \mathrm{Hom}_A(\tilde{P}_{r+1}, P_{j\sigma}) & \text{for } j \in \mathcal{J}_{r+1} \\ d \in \mathrm{End}_A(\tilde{P}_{r+1}) & \end{array} \right\}$$

This shows that $E_\sigma(A)^{\mathrm{op}}$ is obtained from A by a finite number of steps of gluing a simple projective vertex and a simple injective vertex. With Theorem 6.2 we obtain that $E_\sigma(A)^{\mathrm{op}}$ and A are stably equivalent. □

Lemma 6.6. *Let $A = kQ/I$ and $B = k\tilde{Q}/\tilde{I}$ be two finite dimensional algebras such that there is a radical embedding $B \hookrightarrow A$. Suppose that B is obtained from A by a finite number of steps of gluing a simple projective vertex and a simple injective vertex.*

Then there exist $\mathcal{J}$ and σ as in Definition 6.3 such that $E_\sigma(A)^{\mathrm{op}} \simeq B$.

Proof. By assumption, we have a finite sequence of subalgebras

$$B \simeq A_l \subseteq A_{l-1} \subseteq \cdots \subseteq A_1 \subseteq A_0 = A$$

where each A_{r+1} is obtained from A_r by gluing a sink v_{r+1} and a source w_{r+1} for $0 \leqslant r \leqslant l-1$. Let $u_1, \ldots, u_p, v_{r+1}, w_{r+1}$ be a complete set of primitive orthogonal idempotents of A_r with some $p \in \mathbb{N}$. Then A_{r+1} is the subalgebra of A_r generated by $u_1, \ldots, u_p, v_{r+1} + w_{r+1}$ and $\mathrm{rad}(A_r)$.

The simple A_{r+1}-module corresponding to the vertex $v_{r+1} + w_{r+1}$ is a node. In particular, $v_{r+1} + w_{r+1}$ is neither a sink nor a source in A_{r+1}. Thus, a complete set of primitive orthogonal idempotents of A is of the form $u_1, \ldots u_q, v_1, w_1, \ldots, v_l, w_l$ for some $q \in \mathbb{N}$. Moreover, a complete set of primitive orthogonal idempotents of A_l is of the form $u_1, \ldots u_q, v_1 + w_1, \ldots, v_l + w_l$ where v_r is a sink and w_r is a source in $A_0 = A$ for $1 \leqslant r \leqslant l$.

Note that A has $s = q + 2l$ many non-isomorphic indecomposable projective modules. By reordering, we can choose $\mathcal{J} := [1, l] \subseteq [1, s]$ such that $P_j = v_j A$ for $j \in \mathcal{J}$. Let $\sigma : \mathcal{J} \hookrightarrow [1, s]$ such that $P_{j\sigma} = w_j A$ for $j \in \mathcal{J}$. We write $P_A := (u_1 + \cdots + u_q)A_A$ so that

$$A_A = \bigoplus_{j\in\mathcal{J}} (P_j \oplus P_{j\sigma}) \oplus P$$

as in Definition 6.3. By Remark 6.4, we have that

$$\mathrm{End}_A(A_0) = \mathrm{End}_A(A) \simeq \begin{pmatrix} \begin{pmatrix} \delta_{i,j}\, k & \mathrm{Hom}_A(P_i, P_{j\sigma}) \\ 0 & \delta_{i,j}\, k \end{pmatrix}_{i,j\in\mathcal{J}} & \begin{pmatrix} \mathrm{Hom}_A(P_i, P) \\ 0 \end{pmatrix}_{i\in\mathcal{J}} \\ \begin{pmatrix} 0 & \mathrm{Hom}_A(P, P_{j\sigma}) \end{pmatrix}_{j\in\mathcal{J}} & \mathrm{End}_A(P) \end{pmatrix}.$$

From this, we obtain A_l by gluing the sink v_r and the source w_r for each $1 \leqslant r \leqslant l$ which correspond to the columns $2r-1$, $2r$ of the matrix algebra.

$$\mathrm{End}_{A_l}(A_l) \simeq \left\{ \begin{pmatrix} \begin{pmatrix} \delta_{i,j}\, x_j & a_{i,j} \\ 0 & \delta_{i,j}\, x_j \end{pmatrix}_{i,j\in\mathcal{J}} & \begin{pmatrix} b_i \\ 0 \end{pmatrix}_{i\in\mathcal{J}} \\ \begin{pmatrix} 0 & c_j \end{pmatrix}_{j\in\mathcal{J}_k} & d \end{pmatrix} \;\middle|\; \begin{array}{ll} x_j \in k & \text{for } j \in \mathcal{J} \\ a_{i,j} \in \mathrm{Hom}_A(P_i, P_{j\sigma}) & \text{for } i, j \in \mathcal{J} \\ b_i \in \mathrm{Hom}_A(P_i, P) & \text{for } i \in \mathcal{J} \\ c_j \in \mathrm{Hom}_A(P, P_{j\sigma}) & \text{for } j \in \mathcal{J} \\ d \in \mathrm{End}_A(P) & \end{array} \right\}$$

By Remark 6.4, this is precisely $E_\sigma(A)$. Hence, $B \simeq A_l \simeq \mathrm{End}_{A_l}(A_l)^{\mathrm{op}} \simeq E_\sigma(A)^{\mathrm{op}}$. □

6.2 Algebras obtained by deleting nodes

Consider an algebra B with nodes. We construct a new algebra $E_\mathcal{N}(B)$ such that B is obtained from $E_\mathcal{N}(B)$ by gluing pairs of simple projective and simple injective vertices. This process can be seen as deleting nodes from the algebra B; cf. [34].

Definition 6.7. Suppose given a subset $\mathcal{N} \subseteq [1, t]$ such that $T_n \in \operatorname{mod} B$ is a node for $n \in \mathcal{N}$. Let $Q := \bigoplus_{r\in[1,t]\setminus\mathcal{N}} Q_r$.

We define the following matrix algebra corresponding to $\mathcal{N}$.

$$E_\mathcal{N}(B) := \begin{pmatrix} \begin{pmatrix} \delta_{m,n}\, k & \mathrm{Hom}_B(T_m, Q_n) \\ 0 & \delta_{m,n}\, k \end{pmatrix}_{m,n\in\mathcal{N}} & \begin{pmatrix} \mathrm{Hom}_B(T_m, Q) \\ 0 \end{pmatrix}_{m\in\mathcal{N}} \\ \begin{pmatrix} 0 & \mathrm{Hom}_B(Q, Q_n) \end{pmatrix}_{n\in\mathcal{N}} & \mathrm{End}_B(Q) \end{pmatrix}$$

Note that for $1 \leqslant n \leqslant |\mathcal{N}|$ the simple $E_{\mathcal{N}}(B)^{\mathrm{op}}$-module corresponding to the column $2n-1$ is projective and the simple $E_{\mathcal{N}}(B)^{\mathrm{op}}$-module corresponding to the column $2n$ is injective. In total, the number of non-isomorphic indecomposable projective $E_{\mathcal{N}}(B)^{\mathrm{op}}$-modules is

$$2|\mathcal{N}| + |[1,t] \setminus \mathcal{N}| = t + |\mathcal{N}|.$$

Remark 6.8. Using that T_m is a node for $m \in \mathcal{N}$ we have the following for all $m, n \in \mathcal{N}$.

$$\begin{aligned} \mathrm{Hom}_B(Q_m, Q) &\simeq \mathrm{Hom}_B(T_m, Q) \\ \mathrm{rad}\big(\mathrm{Hom}_B(Q_m, Q_n)\big) &\simeq \mathrm{Hom}_B(T_m, Q_n) \\ \mathrm{Hom}_B(Q_m, Q_n) &\simeq \left\{ \begin{pmatrix} x & a \\ 0 & x \end{pmatrix} \middle|\, x \in k,\ a \in \mathrm{Hom}_B(T_m, Q_n) \right\} \end{aligned}$$

This can be used to rewrite the endomorphism algebra of B.

$$\begin{aligned} \mathrm{End}_B(B) &\simeq \begin{pmatrix} \big(\mathrm{Hom}_B(Q_m, Q_n)\big)_{m,n\in\mathcal{N}} & \big(\mathrm{Hom}_B(Q_m, Q)\big)_{m\in\mathcal{N}} \\ \big(\mathrm{Hom}_B(Q, Q_n)\big)_{n\in\mathcal{N}} & \mathrm{End}_B(Q) \end{pmatrix} \\ &\simeq \left\{ \begin{pmatrix} \begin{pmatrix} \delta_{m,n}\, x_n & a_{m,n} \\ 0 & \delta_{m,n}\, x_n \end{pmatrix}_{m,n\in\mathcal{N}} & \begin{pmatrix} b_m \\ 0 \end{pmatrix}_{m\in\mathcal{N}} \\ \begin{pmatrix} 0 & c_n \end{pmatrix}_{n\in\mathcal{N}} & d \end{pmatrix} \middle|\, \begin{array}{ll} x_n \in k & \text{for } n \in \mathcal{N} \\ a_{m,n} \in \mathrm{Hom}_B(T_m, Q_n) & \text{for } m, n \in \mathcal{N} \\ b_m \in \mathrm{Hom}_B(T_m, Q) & \text{for } m \in \mathcal{N} \\ c_n \in \mathrm{Hom}_B(Q, Q_n) & \text{for } n \in \mathcal{N} \\ d \in \mathrm{End}_B(Q) & \end{array} \right\}. \end{aligned}$$

We obtain a radical embedding $B \simeq \mathrm{End}_B(B)^{\mathrm{op}} \hookrightarrow E_{\mathcal{N}}(B)^{\mathrm{op}}$.

Now, consider both algebras A and B together. We aim to show that there exists a σ as in Definition 6.3 such that $E_\sigma(A)^{\mathrm{op}} \simeq B$ if and only if there exists an $\mathcal{N}$ as in Definition 6.7 such that $E_{\mathcal{N}}(B)^{\mathrm{op}} \simeq A$.

Lemma 6.9. *Let $A = kQ/I$ be a finite dimensional algebra. Let $\mathcal{J} \subseteq [1, s]$ and suppose given an injective map $\sigma : \mathcal{J} \hookrightarrow [1, s]$ such that $S_j \in \mathrm{proj}\, A$ and $S_{j\sigma} \in \mathrm{inj}\, A$ for $j \in \mathcal{J}$.*

Let $B := E_\sigma(A)^{\mathrm{op}}$ and $\mathcal{N} = [\,1, |\mathcal{J}|\,]$. Then $E_{\mathcal{N}}(B)^{\mathrm{op}}$ is isomorphic to A as an algebra.

Proof. We abbreviate $l := |\mathcal{J}|$ and $E := E_\sigma(A)$. In particular, $B = E^{\mathrm{op}}$. Note that the matrix description of E in Definition 6.3 has $l+1$ columns. For $1 \leqslant j \leqslant l$ we have the indecomposable projective E^{op}-modules Q_j corresponding to the first l columns of this matrix. Furthermore, we have the projective E^{op}-module Q corresponding to the column $l+1$ of this matrix.

Recall that the simple B-module T_j with projective cover Q_j is a node for $1 \leqslant j \leqslant l$. Thus, $E_{\mathcal{N}}(B)^{\mathrm{op}}$ is well-defined. We show that $E_{\mathcal{N}}(B) \simeq \mathrm{End}_A(A)$.

Let $\mathcal{J} = \{j_1, \ldots, j_l\}$ using the ordering of $\mathcal{J}$. The choice $\mathcal{N} = [1, |\mathcal{J}|] = [1, l]$ induces a bijection $\mathcal{N} \to \mathcal{J}$ via $n \mapsto j_n$. Now, the equivalence of right B-modules and left E-modules induces the following isomorphisms for $m, n \in \mathcal{N} = [1, l]$. We also use that T_m is a node; cf. Remark 6.8.

$$\begin{aligned}
\mathrm{Hom}_B(T_m, Q_n) &\simeq \mathrm{rad}\big(\mathrm{Hom}_B(Q_m, Q_n)\big) \simeq \mathrm{rad}\big(\mathrm{Hom}_E(Ee_m, Ee_n)\big) \simeq e_m\,\mathrm{rad}(E)e_n \\
&\simeq \mathrm{Hom}_A(P_{j_m}, P_{j_n\sigma}) \\
\mathrm{Hom}_B(T_m, Q) &\simeq \mathrm{Hom}_B(Q_m, Q) \simeq \mathrm{Hom}_E(Ee_m, Ee_{l+1}) \simeq e_m E e_{l+1} \simeq \mathrm{Hom}_A(P_{j_m}, P) \\
\mathrm{Hom}_B(Q, Q_n) &\simeq \mathrm{Hom}_E(Ee_{l+1}, Ee_n) \simeq e_{l+1} E e_n \simeq \mathrm{Hom}_A(P, P_{j_n\sigma}) \\
\mathrm{End}_B(Q) &\simeq \mathrm{End}_E(Ee_{l+1}) \simeq e_{l+1} E e_{l+1} \simeq \mathrm{End}_A(P)
\end{aligned}$$

We obtain the following sequence of isomorphisms.

$$\begin{aligned}
E_{\mathcal{N}}(B) &= \begin{pmatrix} \begin{pmatrix} \delta_{m,n}\,k & \mathrm{Hom}_B(T_m, Q_n) \\ 0 & \delta_{m,n}\,k \end{pmatrix}_{m,n\in\mathcal{N}} & \begin{pmatrix} \mathrm{Hom}_B(T_m, Q) \\ 0 \end{pmatrix}_{m\in\mathcal{N}} \\ \begin{pmatrix} 0 & \mathrm{Hom}_B(Q, Q_n) \end{pmatrix}_{n\in\mathcal{N}} & \mathrm{End}_B(Q) \end{pmatrix} \\
&\simeq \begin{pmatrix} \begin{pmatrix} \delta_{j_m,j_n}\,k & \mathrm{Hom}_A(P_{j_m}, P_{j_n\sigma}) \\ 0 & \delta_{j_m,j_n}\,k \end{pmatrix}_{m,n\in\mathcal{N}} & \begin{pmatrix} \mathrm{Hom}_A(P_{j_m}, P) \\ 0 \end{pmatrix}_{m\in\mathcal{N}} \\ \begin{pmatrix} 0 & \mathrm{Hom}_A(P, P_{j_n\sigma}) \end{pmatrix}_{n\in\mathcal{N}} & \mathrm{End}_A(P) \end{pmatrix} \\
&\simeq \begin{pmatrix} \begin{pmatrix} \delta_{i,j}\,k & \mathrm{Hom}_A(P_i, P_{j\sigma}) \\ 0 & \delta_{i,j}\,k \end{pmatrix}_{i,j\in\mathcal{J}} & \begin{pmatrix} \mathrm{Hom}_A(P_i, P) \\ 0 \end{pmatrix}_{i\in\mathcal{J}} \\ \begin{pmatrix} 0 & \mathrm{Hom}_A(P, P_{j\sigma}) \end{pmatrix}_{j\in\mathcal{J}} & \mathrm{End}_A(P) \end{pmatrix}
\end{aligned}$$

By Remark 6.4, we have

$$\mathrm{End}_A(A) \simeq \begin{pmatrix} \begin{pmatrix} \delta_{i,j}\,k & \mathrm{Hom}_A(P_i, P_{j\sigma}) \\ 0 & \delta_{i,j}\,k \end{pmatrix}_{i,j\in\mathcal{J}} & \begin{pmatrix} \mathrm{Hom}_A(P_i, P) \\ 0 \end{pmatrix}_{i\in\mathcal{J}} \\ \begin{pmatrix} 0 & \mathrm{Hom}_A(P, P_{j\sigma}) \end{pmatrix}_{j\in\mathcal{J}} & \mathrm{End}_A(P) \end{pmatrix}$$

which is isomorphic to $E_{\mathcal{N}}(B)$ by the above. □

Lemma 6.10. *Suppose given a subset $\mathcal{N} \subseteq [1, t]$ such that $T_n \in \mathrm{mod}\,B$ is a node for $n \in \mathcal{N}$. Let $A := E_{\mathcal{N}}(B)^{\mathrm{op}}$ as well as $\mathcal{J} = \{2n - 1 \in \mathbb{N} \mid n \in [1, |\mathcal{N}|]\} = \{1, 3, 5, \ldots, 2|\mathcal{N}| - 1\}$ and*

$$\sigma : \mathcal{J} \to [2, 2|\mathcal{N}|] : j \mapsto j + 1.$$

Then $E_\sigma(A)^{\mathrm{op}}$ is isomorphic to B as an algebra.

Proof. We abbreviate $l := |\mathcal{N}|$ and $E := E_{\mathcal{N}}(B)$. In particular, $A = E^{\mathrm{op}}$. Note that the matrix description of E in Definition 6.7 has $2l+1$ columns. For $1 \leqslant r \leqslant 2l$ we have the indecomposable projective E^{op}-modules P_r corresponding to the first $2l$ columns of this matrix. Furthermore, we have the projective E^{op}-module P corresponding to the column $2l+1$ of this matrix.

Recall that the simple A-module S_{2r-1} with projective cover P_{2r-1} is projective for $1 \leqslant r \leqslant l$. Moreover, the simple A-module S_{2r} with projective cover P_{2r} is injective for $1 \leqslant r \leqslant l$. Thus, $E_\sigma(A)^{\mathrm{op}}$ is well-defined. We show that $E_\sigma(A) \simeq \mathrm{End}_B(B)$.

Let $\mathcal{N} = \{n_1, \dots, n_l\}$ using the ordering of $\mathcal{N}$. The choice of $\mathcal{J}$ and σ gives two bijections $\mathcal{J} \to \mathcal{N}$ via $j \mapsto n_{\frac{j+1}{2}}$ and $\mathcal{J}\sigma \to \mathcal{N}$ via $j\sigma = j+1 \mapsto n_{\frac{j+1}{2}}$.

Let $i, j \in \mathcal{J}$ with $m := n_{\frac{i+1}{2}} \in \mathcal{N}$ and $n := n_{\frac{j+1}{2}} \in \mathcal{N}$. The equivalence between right A-modules and left E-modules induces the following isomorphisms.

$$\begin{aligned}
&\mathrm{Hom}_A(P_i, P_{j\sigma}) = \mathrm{Hom}_A(P_i, P_{j+1}) \simeq \mathrm{Hom}_E(Ee_i, Ee_{j+1}) \simeq e_i E e_{j+1} \simeq \mathrm{Hom}_B(T_m, Q_n)\\
&\mathrm{Hom}_A(P_i, P) \simeq \mathrm{Hom}_E(Ee_i, Ee_{2l+1}) \simeq e_i E e_{2l+1} \simeq \mathrm{Hom}_B(T_m, Q)\\
&\mathrm{Hom}_A(P, P_{j\sigma}) = \mathrm{Hom}_A(P, P_{j+1}) \simeq \mathrm{Hom}_E(Ee_{2l+1}, Ee_{j+1}) \simeq e_{2l+1} E e_{j+1} \simeq \mathrm{Hom}_B(Q, Q_m)\\
&\mathrm{End}_A(P) \simeq \mathrm{End}_E(Ee_{2l+1}) \simeq e_{2l+1} E e_{2l+1} = \mathrm{End}_B(Q)
\end{aligned}$$

We obtain the following sequence of isomorphisms.

$$\begin{aligned}
E_\sigma(A) &= \begin{pmatrix} \big(\delta_{i,j}\, k \oplus \mathrm{Hom}_A(P_i, P_{j\sigma})\big)_{i,j\in\mathcal{J}} & \big(\mathrm{Hom}_A(P_i, P)\big)_{i\in\mathcal{J}} \\ \big(\mathrm{Hom}_A(P, P_{j\sigma})\big)_{j\in\mathcal{J}} & \mathrm{End}_A(P) \end{pmatrix}\\
&\simeq \begin{pmatrix} \big(\delta_{m,n}\, k \oplus \mathrm{Hom}_B(T_m, Q_n)\big)_{m,n\in\mathcal{N}} & \big(\mathrm{Hom}_B(T_m, Q)\big)_{m\in\mathcal{N}} \\ \big(\mathrm{Hom}_B(Q, Q_n)\big)_{n\in\mathcal{N}} & \mathrm{End}_B(Q) \end{pmatrix}\\
&\simeq \begin{pmatrix} \big(\mathrm{Hom}_B(Q_m, Q_n)\big)_{m,n\in\mathcal{N}} & \big(\mathrm{Hom}_B(Q_m, Q)\big)_{m\in\mathcal{N}} \\ \big(\mathrm{Hom}_B(Q, Q_n)\big)_{n\in\mathcal{N}} & \mathrm{End}_B(Q) \end{pmatrix} && \text{(Remark 6.8)}\\
&\simeq \mathrm{End}_B(B)
\end{aligned}$$

□

We can now state the main result of this chapter.

Theorem 6.11. *Let $A = kQ/I$ and $B = k\tilde{Q}/\tilde{I}$ be two finite dimensional algebras. The following are equivalent.*

(1) There exists a radical embedding $B \hookrightarrow A$ such that B is obtained from A by a finite number of steps of gluing a simple projective vertex and a simple injective vertex.

(2) There exists σ as in Definition 6.3 such that $E_\sigma(A)^{\mathrm{op}} \simeq B$.

(3) There exists $\mathcal{N}$ as in Definition 6.7 such that $E_{\mathcal{N}}(B)^{\mathrm{op}} \simeq A$.

If one of the conditions holds, A and B are stably equivalent.

Proof. The implication (1) $\Rightarrow$ (2) is shown in Lemma 6.6. The converse (2) $\Rightarrow$ (1) is shown in Lemma 6.5. The implication (2) $\Rightarrow$ (3) is shown in Lemma 6.9. The converse (3) $\Rightarrow$ (2) is shown in Lemma 6.10. Finally, Theorem 6.2 shows that A and B are stably equivalent if condition (1) holds. $\square$

The next corollary is a consequence of Theorem 6.2. A radical embedding $B \hookrightarrow A$ implies that B is obtained from A by a finite number of steps of gluing two primitive idempotents; cf. [41, Example 3]. If A and B are stably equivalent and the Auslander-Reiten conjecture holds, these primitive idempotents must correspond to a simple projective vertex and a simple injective vertex; cf. [22, Proposition 4.11].

Corollary 6.12. *Let $A = kQ/I$ and $B = k\tilde{Q}/\tilde{I}$ be two finite dimensional algebras. Under the assumption of the Auslander-Reiten conjecture, the following are equivalent.*

(1) There is a radical embedding $B \hookrightarrow A$ such that A and B are stably equivalent.

(2) There exists σ as in Definition 6.3 such that $E_{\sigma}(A)^{\mathrm{op}} \simeq B$.

(3) There exists $\mathcal{N}$ as in Definition 6.7 such that $E_{\mathcal{N}}(B)^{\mathrm{op}} \simeq A$.

Let $B = k\tilde{Q}/\tilde{I}$ be a finite dimensional algebra and $\mathcal{N}$ as in Definition 6.7. We have seen in Theorem 6.11 that $E_{\mathcal{N}}(B)^{\mathrm{op}}$ and B are stably equivalent. If we choose $\mathcal{N}$ such that $n \in \mathcal{N}$ for every node $T_n \in \operatorname{mod} B$, then $E_{\mathcal{N}}(B)^{\mathrm{op}}$ is an algebra without nodes stably equivalent to B. Thus, we recover for our setting that every algebra is stably equivalent to an algebra without nodes. This has been shown by Martínez-Villa in [31, Theorem 2.10.(a,c)] using a different method.

Suppose that Γ is the triangular matrix algebra without nodes given in [31, Theorem 2.10.(a)] which is stably equivalent to B. Then there is a radical embedding $B \hookrightarrow \Gamma$; see [22, Remark after Theorem 2.10] for more details. Under the assumption of the Auslander-Reiten conjecture, Γ is isomorphic to $E_{\mathcal{N}}(B)^{\mathrm{op}}$. In fact, by Corollary 6.12 there exists $\mathcal{N}'$ as in Definition 6.7 such that $E'_{\mathcal{N}}(B)^{\mathrm{op}} \simeq \Gamma$. The next lemma and Theorem 6.11 show that $\Gamma \simeq E'_{\mathcal{N}}(B)^{\mathrm{op}} \simeq E_{\mathcal{N}}(B)^{\mathrm{op}}$.

Lemma 6.13. *Let $A = kQ/I$ and $B = k\tilde{Q}/\tilde{I}$ be two finite dimensional algebras. Suppose that B is obtained from A by a finite number of steps of gluing a simple projective vertex and a simple injective vertex. Let $\mathcal{N} \subset [1, t]$ such that there exists an $n \in \mathcal{N}$ with $T \simeq T_n$ for every node T of B.*

If A has no nodes, then $E_{\mathcal{N}}(B)^{\mathrm{op}} \simeq A$.

Proof. By Theorem 6.11, there exists a set $\mathcal{N}'$ as in Definition 6.7 such that $E_{\mathcal{N}'}(B)^{\mathrm{op}} \simeq A$. We show that $\mathcal{N} = \mathcal{N}'$. Then $A \simeq E_{\mathcal{N}'}(B)^{\mathrm{op}} \simeq E_{\mathcal{N}}(B)^{\mathrm{op}}$.

We have $\mathcal{N}' \subseteq \mathcal{N}$ by definition of $\mathcal{N}$. Assume given an $r \in \mathcal{N}$ with $r \notin \mathcal{N}'$.

Let $Q_r \in \operatorname{proj} B$ be the projective module which has the node T_r as simple top. By Lemma 6.1, we have $f\,g = 0$ for all non-isomorphisms $f : Q_i \to Q_r$ and $g : Q_r \to Q_j$ with $i, j \in [1, t]$. Since $r \notin \mathcal{N}'$, there is a column $l > |\mathcal{N}'|$ in the following matrix description of $E_{\mathcal{N}'}(B)^{\mathrm{op}}$ corresponding to the projective module Q_r.

$$E_{\mathcal{N}'}(B) := \begin{pmatrix} \begin{pmatrix} \delta_{m,n}\, k & \mathrm{Hom}_B(T_m, Q_n) \\ 0 & \delta_{m,n}\, k \end{pmatrix}_{m,n\in\mathcal{N}'} & \begin{pmatrix} \mathrm{Hom}_B(T_m, Q_j) \\ 0 \end{pmatrix}_{m\in\mathcal{N}',\, j\in[1,t]\setminus\mathcal{N}'} \\ \begin{pmatrix} 0 & \mathrm{Hom}_B(Q_i, Q_n) \end{pmatrix}_{i\in[1,t]\setminus\mathcal{N}',\, n\in\mathcal{N}'} & \begin{pmatrix} \mathrm{Hom}_B(Q_i, Q_j) \end{pmatrix}_{i,j\in[1,t]\setminus\mathcal{N}'} \end{pmatrix}$$

Let P_l be the indecomposable projective $E_{\mathcal{N}'}(B)^{\mathrm{op}}$-module corresponding to this column. Recall that $\mathrm{Hom}_B(T_m, Q_r) \simeq \mathrm{Hom}_B(Q_m, Q_r)$ for $m \in \mathcal{N}'$; cf. Remark 6.8. By the above and Lemma 6.1.(2), the simple top of P_l is a node in $E_{\mathcal{N}'}(B)^{\mathrm{op}}$. A contradiction, since we assumed that $A \simeq E_{\mathcal{N}'}(B)^{\mathrm{op}}$ has no nodes. □

Finally, we give a remark about iterating the constructions of this chapter in an arbitrary order.

Remark 6.14. Let $A = kQ/I$ and $B = k\tilde{Q}/\tilde{I}$ be two finite dimensional algebras. We say that A is obtained from B by deleting nodes if B is obtained from A by gluing a simple projective vertex and a simple injective vertex.

Under the assumption of the Auslander-Reiten conjecture the following are equivalent.

(1) There exists an algebra $C = kQ_C/I_C$ together with radical embeddings $A \hookrightarrow C$ and $B \hookrightarrow C$ such that A, B and C are pairwise stably equivalent.

(2) The algebra B is obtained from A by a finite number of steps of either deleting a node or gluing a simple projective vertex and a simple injective vertex in any order.

(3) There exist $\mathcal{N} \subseteq [1, s]$ and $\mathcal{N}' \subseteq [1, t]$ as in Definition 6.7 such that $E_{\mathcal{N}}(A)^{\mathrm{op}} \simeq E_{\mathcal{N}'}(B)^{\mathrm{op}}$ as algebras.

It follows from Theorem 6.2 that (1) implies (2). The equivalence of (1) and (3) is a consequence of Corollary 6.12. Suppose that B is obtained from A by either deleting a node or gluing a simple projective vertex and a simple injective vertex. By Lemma 6.13, A and B can be embedded into a unique algebra without nodes which is obtained from A and B via sets $\mathcal{N}$ and $\mathcal{N}'$ as in Definition 6.7. This holds for every step in the situation of (2). Thus, (2) implies (3).

Example in Chapter 7. In Example 7.16 we construct all algebras B that are stably equivalent to the algebra A in Section 7.4 such that there is a radical embedding $B \hookrightarrow A$. In particular, we calculate the matrix algebras $E_\sigma(A)$ and $E_{\mathcal{N}}(B)$ defined in this chapter.

Chapter 7

Examples

Let A and B be two finite dimensional algebras given by quivers with admissible relations. As such, the semisimple quotients of A and B are separable.

Every section in this chapter is a self-contained example intended to illustrate different results of the previous chapters. However, we often reference calculations done previously during the same section. Throughout this chapter we use the following notation.

For a vertex i of the quiver, we denote the indecomposable projective A-module corresponding to i by P_i and the indecomposable projective B-module corresponding to i by Q_i. By abuse of notation, the corresponding simple module is sometimes denoted by S_i in both $\operatorname{mod} A$ and $\operatorname{mod} B$.

Let α be an arrow from vertex i to j in the quiver of A. Right multiplication by α gives a morphism between the indecomposable projective left-modules $P_i^* \to P_j^*$. By abuse of notation, we denote this morphism by α as well. On the other hand, left multiplication by α gives a morphism between the indecomposable projective modules $P_j \to P_i$. We denote the morphism given by left multiplication with the arrow α, β, γ, δ or ε by a, b, c, d or e respectively. The same notation is used for morphisms in B.

7.1 Algebras stably equivalent of Morita type

In this section we take a closer look at two algebras A and B found in [29, Example 1] which are stably equivalent of Morita type. Another focus will be on the equivalence $\mathcal{F}$ of Chapter 2 and perfect exact sequences.

In Example 7.1 we construct the image of a morphism under the functor $\mathcal{F} : \underline{\operatorname{mod}}\, B \to \mathcal{L}_B$ as done in the proof of Lemma 2.4. The next four examples discuss perfect exact sequences. First, we take a look at some perfect exact sequences and their corresponding distinguished triangles in Example 7.2; cf. Proposition 2.18. As stated in Proposition 2.21, the pseudo-cokernel induces a perfect exact sequence as well. This is discussed in Example 7.3. In Example 7.4, we specifically consider perfect exact sequences with projective middle term. Finally, in Example 7.5, we explicitly follow the construction in Chapter 3 to show that a perfect exact sequence in $\operatorname{mod} A$ is preserved under a stable equivalence $\underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$. In the last example of this section, Example 7.6, we verify that A and B are stably equivalent of Morita type using the results in Chapter 5 and discuss some properties of this equivalence.

We consider two algebras A and B given by the following quivers and relations.

Quiver of A

$$\begin{tikzcd} 1 \arrow[rr, "\alpha"] & & 2 \arrow[dl, "\beta"] \\ & 3 \arrow[ul, "\gamma"] & \end{tikzcd}$$

Quiver of B

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\delta"] \arrow[r, shift left, "\beta"] & 3 \arrow[l, shift left, "\gamma"] \end{tikzcd}$$

Relations of A

$$\alpha\,\beta\,\gamma\,\alpha = \gamma\,\alpha\,\beta = 0$$

Relations of B

$$\alpha\,\beta = \alpha\,\delta = \gamma\,\delta = 0 \text{ and } \delta\,\alpha = \beta\,\gamma$$

The algebra A has the following indecomposable projective modules. We also note their images under the functor $(-)^*$.

$$P_1 := \begin{smallmatrix}1\\2\\3\\1\end{smallmatrix}, \quad P_2 := \begin{smallmatrix}2\\3\\1\\2\end{smallmatrix}, \quad P_3 := \begin{smallmatrix}3\\1\\2\end{smallmatrix}, \qquad P_1^* := \begin{smallmatrix}1\\3\\2\\1\end{smallmatrix}, \quad P_2^* := \begin{smallmatrix}2\\1\\3\\2\end{smallmatrix}, \quad P_3^* := \begin{smallmatrix}3\\2\\1\end{smallmatrix}$$

The algebra B has the following indecomposable projective modules. We also note their images under the functor $(-)^*$.

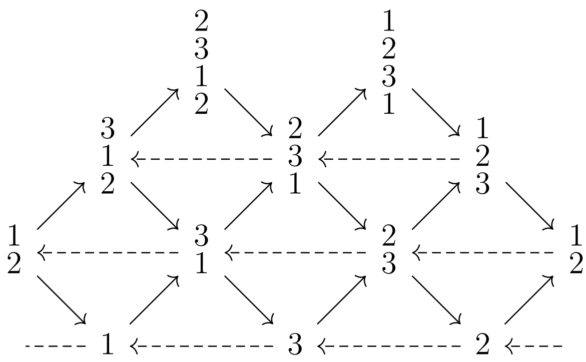
$$Q_1 := \begin{smallmatrix}1\\2\end{smallmatrix}, \quad Q_2 := \begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix}, \quad Q_3 := \begin{smallmatrix}3\\2\\3\end{smallmatrix}, \qquad Q_1^* := \begin{smallmatrix}1\\2\end{smallmatrix}, \quad Q_2^* := \begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix}, \quad Q_3^* := \begin{smallmatrix}3\\2\\3\end{smallmatrix}$$

The following table collects some properties of A and B.

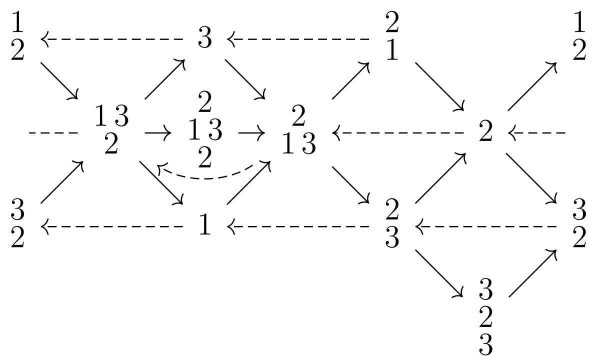
Property	A	B
Nakayama algebra	yes	no
$\operatorname{gldim} A$	4	4
Indecomposable projective-injective modules	P_1, P_2	Q_2, Q_3
Indecomposable strongly projective-injective modules	P_1, P_2	Q_2, Q_3
$\operatorname{domdim} A$	3	3
ν-$\operatorname{domdim} A$	3	3
Nodes	none	none

The Auslander-Reiten quivers of A and B can be written as follows.

Auslander-Reiten quiver of A



Auslander-Reiten quiver of B



For now, we concentrate on the algebra B. We will return to the algebra A later, when discussing properties of stable equivalences between A and B.

Example 7.1. Let $X := \begin{smallmatrix}2\\1\,3\end{smallmatrix}$ and $Y := \begin{smallmatrix}3\\2\end{smallmatrix}$ in $\operatorname{mod} B$. We aim to construct the image of the non-zero morphism $f : X \to Y$ under the equivalence $\mathcal{F} : \underline{\operatorname{mod}}\, B \to \mathcal{L}_B$ as in Lemma 2.4. Since f factors through the projective module Q_3, we expect $f^\bullet = \mathcal{F}(f)$ to be homotopic to zero in $\mathcal{L}_B \subseteq \mathcal{K}^b(\operatorname{proj} B)$. The minimal projective resolution of X is given by the following complex.

$$0 \to Q_1 \xrightarrow{d} Q_2 \xrightarrow{\begin{pmatrix} a & c \end{pmatrix}} Q_1 \oplus Q_3 \xrightarrow{\begin{pmatrix} d \\ -b \end{pmatrix}} Q_2 \xrightarrow{ad} Q_2 \to 0$$

We have the exact sequence $0 \to X^* \to Q_2^* \to Q_2^* \to \operatorname{Tr}(X) \to 0$ with $X^* = \begin{smallmatrix}1\,3\\2\end{smallmatrix}$ and $\operatorname{Tr}(X) = \begin{smallmatrix}2\\1\,3\end{smallmatrix}$ in B-mod. We extend $\delta\alpha : Q_2^* \to Q_2^*$ to a minimal projective resolution of $\operatorname{Tr}(X)$ in B-mod.

$$0 \to Q_1^* \xrightarrow{\alpha} Q_2^* \xrightarrow{\begin{pmatrix} \delta & -\beta \end{pmatrix}} Q_1^* \oplus Q_3^* \xrightarrow{\begin{pmatrix} \alpha \\ \gamma \end{pmatrix}} Q_2^* \xrightarrow{\delta\alpha} Q_2^* \to 0$$

Applying $(-)^*$ and combining the resulting complex with the projective resolution from above, we obtain the complex $F^\bullet_{\begin{smallmatrix}2\\1\,3\end{smallmatrix}} = F^\bullet_X = \mathcal{F}(X) \in \mathcal{L}_B$.

$$\begin{array}{ccccccccccccccccccccc}
F_X^{-5} & \xrightarrow{d_X^{-5}} & F_X^{-4} & \xrightarrow{d_X^{-4}} & F_X^{-3} & \xrightarrow{d_X^{-3}} & F_X^{-2} & \xrightarrow{d_X^{-2}} & F_X^{-1} & \xrightarrow{d_X^{-1}} & F_X^{0} & \xrightarrow{d_X^{0}} & F_X^{1} & \xrightarrow{d_X^{1}} & F_X^{2} & \xrightarrow{d_X^{2}} & F_X^{3} & \xrightarrow{d_X^{3}} & F_X^{4} \\
\| & & \| & & \| & & \| & & \| & & \| & & \| & & \| & & \| & & \| \\
0 & \longrightarrow & Q_1 & \xrightarrow{d} & Q_2 & \xrightarrow{\begin{pmatrix} a & c \end{pmatrix}} & Q_1 \oplus Q_3 & \xrightarrow{\begin{pmatrix} d \\ -b \end{pmatrix}} & Q_2 & \xrightarrow{ad} & Q_2 & \xrightarrow{\begin{pmatrix} a & c \end{pmatrix}} & Q_1 \oplus Q_3 & \xrightarrow{\begin{pmatrix} d \\ -b \end{pmatrix}} & Q_2 & \xrightarrow{a} & Q_1 & \longrightarrow & 0
\end{array}$$

Similarly, we obtain the complex $F^\bullet_{\begin{smallmatrix}3\\2\end{smallmatrix}} = F^\bullet_Y = \mathcal{F}(Y)$.

$$\begin{array}{ccccccccccccccc}
F_Y^{-4} & \xrightarrow{d_Y^{-4}} & F_Y^{-3} & \xrightarrow{d_Y^{-3}} & F_Y^{-2} & \xrightarrow{d_Y^{-2}} & F_Y^{-1} & \xrightarrow{d_Y^{-1}} & F_Y^{0} & \xrightarrow{d_Y^{0}} & F_Y^{1} & \xrightarrow{d_Y^{1}} & F_Y^{2} & \xrightarrow{d_Y^{2}} & F_Y^{3} \\
\| & & \| & & \| & & \| & & \| & & \| & & \| & & \| \\
0 & \longrightarrow & Q_1 & \xrightarrow{d} & Q_2 & \xrightarrow{c} & Q_3 & \xrightarrow{bc} & Q_3 & \xrightarrow{b} & Q_2 & \xrightarrow{a} & Q_1 & \longrightarrow & 0
\end{array}$$

Now, we can lift the morphism $f : X \to Y$ to a morphism between the projective resolutions $\tau_{\leqslant 0} F^\bullet_X$ of X and $\tau_{\leqslant 0} F^\bullet_Y$ of Y.

$$\begin{tikzcd}
\tau_{\leqslant 0} F^\bullet_X \arrow[d] & 0 \arrow[r] & Q_1 \arrow[r, "d"] \arrow[d, "0"] & Q_2 \arrow[r, "\begin{pmatrix} a & c \end{pmatrix}"] \arrow[d, "0"] & Q_1 \oplus Q_3 \arrow[r, "\begin{pmatrix} d \\ -b \end{pmatrix}"] \arrow[d, "0"] & Q_2 \arrow[r, "ad"] \arrow[d, "0"] & Q_2 \arrow[d, "c"] \\
\tau_{\leqslant 0} F^\bullet_Y & 0 \arrow[r] & 0 \arrow[r] & Q_1 \arrow[r, "d"] & Q_2 \arrow[r, "c"] & Q_3 \arrow[r, "bc"] & Q_3
\end{tikzcd}$$

This induces a morphism $\mathrm{Tr}(f) : \mathrm{Tr}(Y) \to \mathrm{Tr}(X)$ in B-mod. As before, we can lift this map to a morphism between the projective resolutions $\tau_{\geqslant 1} F^{\bullet,*}_Y$ of Y^* and $\tau_{\geqslant 1} F^{\bullet,*}_X$ of X^*. We choose the first two morphisms as in the lift $\tau_{\leqslant 0} F^\bullet_X \to \tau_{\leqslant 0} F^\bullet_Y$ of f.

$$\begin{tikzcd}
\tau_{\geqslant 1} F^{\bullet,*}_Y \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & Q_1^* \arrow[r, "\alpha"] \arrow[d, "0"] & Q_2^* \arrow[r, "\beta"] \arrow[d, "\begin{pmatrix} 0 & \beta \end{pmatrix}"] & Q_3^* \arrow[r, "\gamma\beta"] \arrow[d, "\gamma"] & Q_3^* \arrow[d, "0"] \\
\tau_{\geqslant 1} F^{\bullet,*}_X & 0 \arrow[r] & Q_1^* \arrow[r, "\alpha"] & Q_2^* \arrow[r, "\begin{pmatrix} \delta & -\beta \end{pmatrix}"] & Q_1^* \oplus Q_3^* \arrow[r, "\begin{pmatrix} \alpha \\ \gamma \end{pmatrix}"] & Q_2^* \arrow[r, "\delta\alpha"] & Q_2^*
\end{tikzcd}$$

In conclusion, we obtain the morphism $f^\bullet = \mathcal{F}(f) : F^\bullet_X \to F^\bullet_Y$.

$$\begin{tikzcd}
F^\bullet_X \arrow[d, "f^\bullet"] & 0 \arrow[r] & Q_1 \arrow[r, "d"] \arrow[d, "0"] & Q_2 \arrow[r, "\begin{pmatrix} a & c \end{pmatrix}"] \arrow[d, "0"] & Q_1 \oplus Q_3 \arrow[r, "\begin{pmatrix} d \\ -b \end{pmatrix}"] \arrow[d, "0"] & Q_2 \arrow[r, "ad"] \arrow[d, "0"] & Q_2 \arrow[r, "\begin{pmatrix} a & c \end{pmatrix}"] \arrow[d, "c"] & Q_1 \oplus Q_3 \arrow[r, "\begin{pmatrix} d \\ -b \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 \\ b \end{pmatrix}"] & Q_2 \arrow[r, "a"] \arrow[d, "0"] & Q_1 \arrow[r] \arrow[d, "0"] & 0 \\
F^\bullet_Y & 0 \arrow[r] & 0 \arrow[r] & Q_1 \arrow[r, "d"] & Q_2 \arrow[r, "c"] & Q_3 \arrow[r, "bc"] & Q_3 \arrow[r, "b"] & Q_2 \arrow[r, "a"] & Q_1 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

We have a homotopy $h^\bullet : F^\bullet_X \to F^\bullet_Y[-1]$ with $h^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : Q_1 \oplus Q_3 \to Q_3$ and $h^k = 0$ for $k \neq 1$. Thus, $f^\bullet = 0$ in $\mathcal{K}(\operatorname{proj} B)$ as expected since $\underline{f} = 0$ in $\underline{\operatorname{mod}}\, B$.

Similarly, we can construct all images under $\mathcal{F}$ for indecomposable non-projective B-modules. This results in the following list of all indecomposable complexes in the category $\mathcal{L}_B$. Note that we group complexes which are connected by applying a shift in $\mathcal{L}_B$.

degree : $-5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$

$$F^\bullet_{\substack{2\\3}}[3] = F^\bullet_{\substack{13\\2}}[2] = F^\bullet_2[1] = F^\bullet_1 : \quad 0 \longrightarrow Q_1 \xrightarrow{d} Q_2 \xrightarrow{\begin{pmatrix} a & c \end{pmatrix}} Q_1 \oplus Q_3 \xrightarrow{\begin{pmatrix} d \\ -b \end{pmatrix}} Q_2 \xrightarrow{a} Q_1 \longrightarrow 0$$

$$F^\bullet_3[2] = F^\bullet_{\substack{3\\2}}[1] = F^\bullet_{\substack{2\\1}} : \quad 0 \longrightarrow Q_1 \xrightarrow{d} Q_2 \xrightarrow{c} Q_3 \xrightarrow{bc} Q_3 \xrightarrow{b} Q_2 \xrightarrow{a} Q_1 \longrightarrow 0$$

$$F^\bullet_{\substack{2\\13}} : \quad 0 \longrightarrow Q_1 \xrightarrow{d} Q_2 \xrightarrow{\begin{pmatrix} a & c \end{pmatrix}} Q_1 \oplus Q_3 \xrightarrow{\begin{pmatrix} d \\ -b \end{pmatrix}} Q_2 \xrightarrow{ad} Q_2 \xrightarrow{\begin{pmatrix} a & c \end{pmatrix}} Q_1 \oplus Q_3 \xrightarrow{\begin{pmatrix} d \\ -b \end{pmatrix}} Q_2 \xrightarrow{a} Q_1 \longrightarrow 0$$

The categories $\underline{\operatorname{mod}}\, B$ and $\mathcal{L}_B$ can be visualized as follows. In particular, the right hand side contains information about possible shifts in $\mathcal{L}_B$. The dashed lines indicate zero relations.

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & 3 \arrow[rr, dash, dashed] \arrow[dr] & & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[dr] & & \\
{} \arrow[r, dash, dashed] & \begin{smallmatrix}1\,3\\2\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & \begin{smallmatrix}2\\1\,3\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & 2 \arrow[dr] \arrow[r, dash, dashed] & {} \\
\begin{smallmatrix}3\\2\end{smallmatrix} \arrow[ur] \arrow[rr, dash, dashed] & & 1 \arrow[ur] \arrow[rr, dash, dashed] & & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] \arrow[rr, dash, dashed] & & \begin{smallmatrix}3\\2\end{smallmatrix}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny, row sep=small]
 & & F^\bullet_3 \arrow[rr, dash, dashed] \arrow[dr] & & F^\bullet_3[2] \arrow[dr] & & \\
{} \arrow[r, dash, dashed] & F^\bullet_{\begin{smallmatrix}2\\3\end{smallmatrix}}[1] \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & F^\bullet_{\begin{smallmatrix}2\\1\,3\end{smallmatrix}} \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & F^\bullet_{\begin{smallmatrix}2\\3\end{smallmatrix}}[2] \arrow[dr] \arrow[r, dash, dashed] & {} \\
F^\bullet_3[1] \arrow[ur] \arrow[rr, dash, dashed] & & F^\bullet_{\begin{smallmatrix}2\\3\end{smallmatrix}}[3] \arrow[ur] \arrow[rr, dash, dashed] & & F^\bullet_{\begin{smallmatrix}2\\3\end{smallmatrix}} \arrow[ur] \arrow[rr, dash, dashed] & & F^\bullet_3[1]
\end{tikzcd}
$$

We continue with an example about perfect exact sequences in $\operatorname{mod} B$.

Example 7.2. Consider the following short exact sequence in $\operatorname{mod} B$.

$$\eta_1 : 0 \to 3 \to \begin{smallmatrix}2\\3\end{smallmatrix} \to 2 \to 0$$

A direct calculation shows that η_1 is a perfect exact sequence. In fact, we obtain the following exact sequence in B-mod after applying $(-)^*$ to η_1.

$$(\eta_1)^* : 0 \to \begin{smallmatrix}1\\2\end{smallmatrix} \to \begin{smallmatrix}1\,3\\2\end{smallmatrix} \to 3 \to 0$$

For instance, we have $\left(\begin{smallmatrix}2\\3\end{smallmatrix}\right)^* \simeq \operatorname{Ker}(Q_2^* \to Q_1^*) \simeq \begin{smallmatrix}1\,3\\2\end{smallmatrix}$. By Proposition 2.18 we therefore obtain an induced distinguished triangle in $\mathcal{K}(\operatorname{proj} B)$.

$$F^\bullet_3 \to F^\bullet_{\begin{smallmatrix}2\\3\end{smallmatrix}} \to F^\bullet_2 \to$$

We can use the information about the Auslander-Reiten quiver of B and the category $\mathcal{L}_B$ to verify this result in a different way.

The sequence $0 \to \begin{smallmatrix}2\\3\end{smallmatrix} \to 2 \oplus Q_3 \to \begin{smallmatrix}3\\2\end{smallmatrix} \to 0$ is an almost split sequence with non-projective starting term. Thus, it is a perfect exact sequence; cf. Example 2.11. We obtain the following distinguished triangle in $\mathcal{L}_B$.

$$F^\bullet_{\begin{smallmatrix}2\\3\end{smallmatrix}} \to F^\bullet_2 \to F^\bullet_{\begin{smallmatrix}3\\2\end{smallmatrix}} \to$$

This induces the distinguished triangle

$$F^\bullet_3 \to F^\bullet_{\begin{smallmatrix}2\\3\end{smallmatrix}} \to F^\bullet_2 \to$$

since $F^\bullet_{\begin{smallmatrix}3\\2\end{smallmatrix}}[-1] = F^\bullet_3 \in \mathcal{L}_B$. We obtain a perfect exact sequence

$$0 \to 3 \to \begin{smallmatrix}2\\3\end{smallmatrix} \oplus Q \to 2 \to 0$$

with $Q \in \operatorname{proj} B$. Since the simple module 3 is not projective, this sequence is isomorphic to the short exact sequence

$$\eta_1 : 0 \to 3 \to \begin{smallmatrix}2\\3\end{smallmatrix} \to 2 \to 0$$

by Lemma 1.5. In particular, $Q \simeq 0$ and η_1 is perfect exact.

Now, consider the following short exact sequence in $\operatorname{mod} B$.

$$\eta_2 : 0 \to 2 \to \begin{smallmatrix}1&&3\\&2&\end{smallmatrix} \to 1 \oplus 3 \to 0$$

A direct calculation shows that η_2 is not a perfect exact sequence. In fact,

$$(\eta_2)^* : 0 \to 0 \oplus 3 \to \begin{smallmatrix}2\\3\end{smallmatrix} \to \begin{smallmatrix}1\\2\end{smallmatrix}$$

is not a short exact sequence in B-mod. Note that $2 \to \begin{smallmatrix}1&&3\\&2&\end{smallmatrix}$ factors through the projective module Q_1. Thus, Lemma 2.13 verifies as well that η_2 cannot be a perfect exact sequence.

Example 7.3. We consider the pseudo-cokernel of the injective morphism $f : 2 \to \begin{smallmatrix}1&&3\\&2&\end{smallmatrix}$ in $\operatorname{mod} B$. Let $X := 2$ and $Y := \begin{smallmatrix}1&&3\\&2&\end{smallmatrix}$. We calculate as follows using that $f^\bullet \simeq 0$ in $\mathcal{K}^b(\operatorname{proj} A)$ since f factors through Q_1.

$$\underline{\mathrm{Cok}}(f) = \mathrm{H}^0(\tau_{\leqslant 0}\, C(f)^\bullet) \simeq \mathrm{Cok}\left(Q_2 \oplus Q_2 \xrightarrow{\left(\begin{smallmatrix}-a&0&0\\0&a&c\end{smallmatrix}\right)} Q_1 \oplus Q_1 \oplus Q_3 \right) \simeq 1 \oplus \begin{smallmatrix}1&&3\\&2&\end{smallmatrix}$$

We have $F_X^1 = Q_1 = \begin{smallmatrix}1\\2\end{smallmatrix}$ and $\mathrm{Cok}(f) = 1 \oplus 3$. We write $\pi : \begin{smallmatrix}1\\2\end{smallmatrix} \to 1$ and $f' : \begin{smallmatrix}1\\2\end{smallmatrix} \to \begin{smallmatrix}1&&3\\&2&\end{smallmatrix}$ for the respective non-zero morphism. This results in the following short exact sequence of the form $0 \to F_X^1 \to \underline{\mathrm{Cok}}(f) \to \mathrm{Cok}(f) \to 0$ given in Lemma 2.22.

$$0 \to \begin{smallmatrix}1\\2\end{smallmatrix} \xrightarrow{\left(\begin{smallmatrix}\pi & f'\end{smallmatrix}\right)} 1 \oplus \begin{smallmatrix}1&&3\\&2&\end{smallmatrix} \to 1 \oplus 3 \to 0$$

In particular, we have that $\underline{\mathrm{Cok}}(f)$ and $\mathrm{Cok}(f)$ are not stably isomorphic. Thus, Lemma 2.22 shows that

$$\eta_2 : 0 \to 2 \xrightarrow{f} \begin{smallmatrix}1&&3\\&2&\end{smallmatrix} \to 1 \oplus 3 \to 0$$

is not a perfect exact sequence as we have already seen in the previous example. However, we can extend f to a perfect exact sequence starting in X by Proposition 2.21. This perfect exact sequence of the form $0 \to X \to F_X^1 \oplus Y \to \underline{\mathrm{Cok}}(f) \to 0$ is given by

$$0 \to 2 \xrightarrow{\left(\begin{smallmatrix}d & f\end{smallmatrix}\right)} \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1&&3\\&2&\end{smallmatrix} \xrightarrow{\left(\begin{smallmatrix}\pi & f'\\0 & -1\end{smallmatrix}\right)} 1 \oplus \begin{smallmatrix}1&&3\\&2&\end{smallmatrix} \to 0$$

where $f = d\,f'$ with d induced by $d_X^0 : F_X^0 \to F_X^1$. This perfect exact sequence is isomorphic to the direct sum of a perfect exact sequence and a split exact sequence.

$$0 \to 2 \xrightarrow{\left(\begin{smallmatrix} d & 0 \end{smallmatrix}\right)} \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\,3\\2\end{smallmatrix} \xrightarrow{\left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right)} 1 \oplus \begin{smallmatrix}1\,3\\2\end{smallmatrix} \to 0$$

Note that f no longer explicitly occurs in this short exact sequence.

Example 7.4. We illustrate some of the results for perfect exact sequences with projective middle term in Section 2.3. Let S_i be the simple module with projective cover Q_i for $1 \leqslant i \leqslant 3$. Note that the dominant dimension of B is positive.

We have $S_1^* = 0$, $S_2^* = \begin{smallmatrix}1\\2\end{smallmatrix}$ and $S_3^* = 3$ in B-mod. As stated in Lemma 2.26, the projective cover Q_1 of S_1 is not injective since $S_1^* = 0$. On the other hand, we have Q_2, $Q_3 \in \mathcal{P}_B$. Moreover, $S_1^* = 0$ implies $F_{S_1}^1 = 0$ so that $F_{S_1}^\bullet \in \mathcal{L}_B$ is the minimal projective resolution of S_1. In particular, $F_{S_1}^\bullet[1] \notin \mathcal{L}_B$ as seen above.

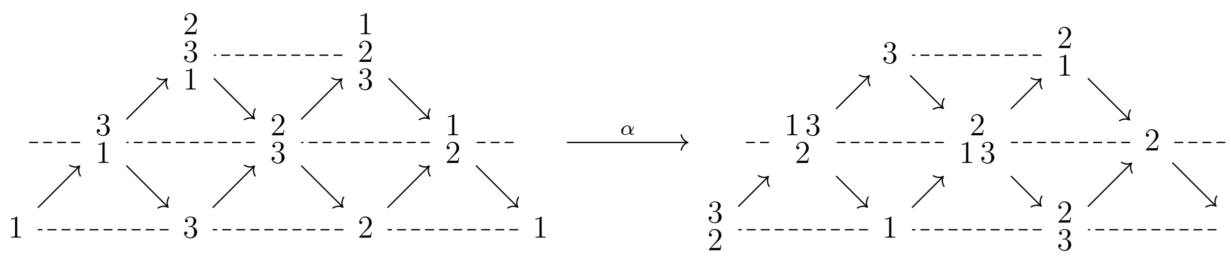
In contrast, $S_3^* \neq 0$ so that $F_{S_3}^\bullet[1] \in \mathcal{L}_B$ by Lemma 2.24. In fact, $F_{S_3}^\bullet[1] = F^\bullet_{\begin{smallmatrix}3\\2\end{smallmatrix}}$ and the distinguished triangle $F_{S_3}^\bullet \to 0 \to F^\bullet_{\begin{smallmatrix}3\\2\end{smallmatrix}} \to$ in $\mathcal{K}(\operatorname{proj} B)$ induces a perfect exact sequence with projective middle term $Q_3 = F_{S_3}^1$; cf. Lemma 2.25.

$$0 \to 3 \to Q_3 \to \begin{smallmatrix}3\\2\end{smallmatrix} \to 0$$

On the other hand, for the non-simple module $X := \begin{smallmatrix}2\\1\,3\end{smallmatrix}$ we also have $X^* = \begin{smallmatrix}1\,3\\2\end{smallmatrix} \neq 0$. Equivalently, this means $F_X^1 = Q_1 \oplus Q_3 \neq 0$. However, $F_X^\bullet[1] \notin \mathcal{L}_B$ since $\mathrm{H}^0(F_X^\bullet) \neq 0$. While we have a non-zero morphism $X \to Q_1 \oplus Q_3$, it is not injective. Thus, the implication (5) $\Rightarrow$ (4) in Lemma 2.24 does not hold in general.

Example 7.5. We give an example for the construction done in Section 3.2 about perfect exact sequences and stable equivalences. Since A is a Nakayama algebra, we often use the following notation for morphisms between indecomposable modules in $\operatorname{mod} A$ during this example. By abuse of notation, the embedding of a submodule is denoted by f and the projection onto a quotient module by g. We mainly want to distinguish which morphisms are zero or the identity.

We have a stable equivalence $\alpha : \underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$ induced by the Auslander-Reiten quivers.



Note that both A and B have no nodes and are of finite representation type. In particular, the assumptions of Corollary 3.20 hold. We aim to show that the perfect exact

$$\eta_0 : 0 \to \begin{smallmatrix}2\\3\\1\end{smallmatrix} \xrightarrow{f} \begin{smallmatrix}1\\2\\3\\1\end{smallmatrix} \xrightarrow{g} 1 \to 0$$

in $\operatorname{mod} A$ is preserved by the stable equivalence α using the constructions of Chapter 3. The short exact sequence η_0 is in fact perfect exact by Lemma 2.14.

Following the proof of Theorem 3.19, we first construct a series of perfect exact sequences. We use the notation of Construction 3.11. For the construction step $\eta_0 \rightsquigarrow \tilde{\eta}_0$ we need the following almost split sequence.

$$\chi_0 : \qquad 0 \to \begin{smallmatrix}2\\3\\1\end{smallmatrix} \xrightarrow{\begin{pmatrix} f & g\end{pmatrix}} \begin{smallmatrix}1\\2\\3\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \xrightarrow{\begin{pmatrix} -g \\ f\end{pmatrix}} \begin{smallmatrix}1\\2\\3\end{smallmatrix} \to 0$$

For the construction step $\eta_1 \rightsquigarrow \tilde{\eta}_1$ we need the following almost split sequence.

$$\chi_1 : \qquad 0 \to \begin{smallmatrix}2\\3\end{smallmatrix} \xrightarrow{\begin{pmatrix} f & g\end{pmatrix}} \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus 2 \xrightarrow{\begin{pmatrix} -g \\ f\end{pmatrix}} \begin{smallmatrix}1\\2\end{smallmatrix} \to 0$$

This results in the following chain of morphisms ending in the almost split sequence η_2. Note that the composite $\begin{smallmatrix}1\\2\\3\\1\end{smallmatrix} \to \begin{smallmatrix}1\\2\end{smallmatrix}$ of the middle morphisms in the chain is non-zero.

$$\begin{tikzcd}
\eta_0 : & 0 \arrow[r] & \begin{smallmatrix}2\\3\\1\end{smallmatrix} \arrow[r, "f"] \arrow[d, "\begin{pmatrix} f & g\end{pmatrix}"] & \begin{smallmatrix}1\\2\\3\\1\end{smallmatrix} \arrow[r, "g"] \arrow[d, "\begin{pmatrix} 1 & 0\end{pmatrix}"] & 1 \arrow[r] \arrow[d, equal] & 0 \\
\tilde{\eta}_0 : & 0 \arrow[r] & \begin{smallmatrix}1\\2\\3\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[r, "\begin{pmatrix} 1 & -g \\ 0 & f\end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 \\ 1\end{pmatrix}"] & \begin{smallmatrix}1\\2\\3\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r, "\begin{pmatrix} g & g\end{pmatrix}"] \arrow[d, "\begin{pmatrix} g \\ 1\end{pmatrix}"] & 1 \arrow[r] \arrow[d, equal] & 0 \\
\eta_1 : & 0 \arrow[r] & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[r, "f"] \arrow[d, "\begin{pmatrix} f & g\end{pmatrix}"] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r, "g"] \arrow[d, "\begin{pmatrix} 1 & 0\end{pmatrix}"] & 1 \arrow[r] \arrow[d, equal] & 0 \\
\tilde{\eta}_1 : & 0 \arrow[r] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus 2 \arrow[r, "\begin{pmatrix} 1 & -g \\ 0 & f\end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 \\ 1\end{pmatrix}"] & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[r, "\begin{pmatrix} g & g\end{pmatrix}"] \arrow[d, "\begin{pmatrix} g \\ 1\end{pmatrix}"] & 1 \arrow[r] \arrow[d, equal] & 0 \\
\eta_2 : & 0 \arrow[r] & 2 \arrow[r, "f"] & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[r, "g"] & 1 \arrow[r] & 0
\end{tikzcd}$$

All occurring almost split sequences have non-projective starting terms. Thus, they are preserved by the stable equivalence α; cf. Proposition 3.17. By abuse of notation, we denote the perfect exact sequence in $\operatorname{mod} B$ corresponding to a perfect exact sequence η in $\operatorname{mod} A$ by $\alpha(\eta)$.

Inductively, we now construct perfect exact sequences in $\operatorname{mod} B$ starting with η_2. The sequence $\alpha(\eta_2)$ is the almost split sequence in $\operatorname{mod} B$ starting in $\alpha(2) = \begin{smallmatrix}2\\3\end{smallmatrix}$.

$$\alpha(\eta_2): \qquad 0 \to \begin{smallmatrix}2\\3\end{smallmatrix} \to 2 \oplus \begin{smallmatrix}3\\2\\3\end{smallmatrix} \to \begin{smallmatrix}3\\2\end{smallmatrix} \to 0$$

Moreover, we have the almost split sequence $\alpha(\chi_1)$ in $\operatorname{mod} B$ corresponding to the almost split sequence χ_1.

$$\alpha(\chi_1): \qquad 0 \to \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \to \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \to 2 \to 0$$

We want to use Lemma 3.18 applied to η_1 and $\alpha(\tilde{\eta}_1)$. In order to obtain a perfect exact sequence corresponding to $\tilde{\eta}_1$, we have to form the direct sum of the split sequence consisting of the module $\begin{smallmatrix}2\\1\end{smallmatrix} = \alpha\begin{pmatrix}1\\2\\3\end{pmatrix}$ and the sequence $\alpha(\eta_2)$. We also rearrange the modules to fit the ordering in Lemma 3.18.

$$\alpha(\tilde{\eta}_1): \qquad 0 \to \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \to \left(\begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\3\end{smallmatrix}\right) \oplus 2 \to \begin{smallmatrix}3\\2\end{smallmatrix} \to 0$$

Now, we can use Lemma 3.8 as in the proof of Lemma 3.18 to obtain a perfect exact sequence in $\operatorname{mod} B$ corresponding to η_1.

$$\alpha(\eta_1): \qquad 0 \to \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \to \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\3\end{smallmatrix} \to \begin{smallmatrix}3\\2\end{smallmatrix} \to 0$$

In the notation of Lemma 3.8.(1) this equates to the following assignment.

$$\begin{array}{llll} X := \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} & U := \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} & P := 0 & V := 2 \\ U = \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} & Y := \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\3\end{smallmatrix} & V = 2 & Z := \begin{smallmatrix}2\\3\end{smallmatrix} \end{array}$$

For the next step, we have the almost split sequence $\alpha(\chi_0)$ in $\operatorname{mod} B$ corresponding to the almost split sequence χ_0.

$$\alpha(\chi_0): \qquad 0 \to 3 \to \begin{smallmatrix}&2&\\1&&3\end{smallmatrix} \to \begin{smallmatrix}2\\1\end{smallmatrix} \to 0$$

Again, we want to use Lemma 3.18. This time, $\tilde{\eta}_0$ is the direct sum of η_1 and a split sequence consisting of the projective module P_1. Thus, we have $\alpha(\tilde{\eta}_0) \simeq \alpha(\eta_1)$. Applying Lemma 3.8, we obtain a perfect exact sequence in $\operatorname{mod} B$ corresponding to η_0.

$$\alpha(\eta_0): \qquad 0 \to 3 \to \begin{smallmatrix}3\\2\\3\end{smallmatrix} \to \begin{smallmatrix}3\\2\end{smallmatrix} \to 0$$

A direct calculation verifies that $\alpha(\eta_0)$ is in fact a perfect exact sequence. Note that $\alpha(\eta_0)$ has a projective middle term, as was the case for η_0.

Before continuing with the next example, we collect all indecomposable complexes in the categories $\mathcal{L}_A$ and $\mathcal{L}_B$. Note that the complexes in $\mathcal{L}_B$ have already been listed in Example 7.1.

The category $\mathcal{L}_A$ consists of the following indecomposable complexes.

$$
\begin{array}{rl}
\text{degree}: & -5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \\
\hline
F^\bullet_{2}[3] = F^\bullet_{\substack{3\\1}}[2] = F^\bullet_{\substack{1\\2}}[1] = F^\bullet_{3}: & 0 \longrightarrow P_3 \xrightarrow{b} P_2 \xrightarrow{ac} P_3 \xrightarrow{ba} P_1 \xrightarrow{c} P_3 \longrightarrow 0 \\
F^\bullet_{\substack{2\\3\\1}}[2] = F^\bullet_{1}[1] = F^\bullet_{\substack{1\\2\\3}}: & 0 \longrightarrow P_3 \xrightarrow{b} P_2 \xrightarrow{acb} P_2 \xrightarrow{a} P_1 \xrightarrow{cba} P_1 \xrightarrow{c} P_3 \longrightarrow 0 \\
F^\bullet_{\substack{2\\3}}: & 0 \longrightarrow P_3 \xrightarrow{b} P_2 \xrightarrow{ac} P_3 \xrightarrow{ba} P_1 \xrightarrow{cb} P_2 \xrightarrow{ac} P_3 \xrightarrow{ba} P_1 \xrightarrow{c} P_3 \to 0
\end{array}
$$

The category $\mathcal{L}_B$ consists of the following indecomposable complexes.

$$
\begin{array}{rl}
\text{degree}: & -5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \\
\hline
F^\bullet_{\substack{2\\3}}[3] = F^\bullet_{\substack{13\\2}}[2] = F^\bullet_{2}[1] = F^\bullet_{1}: & 0 \longrightarrow Q_1 \xrightarrow{d} Q_2 \xrightarrow{\begin{pmatrix} a & c \end{pmatrix}} Q_1 \oplus Q_3 \xrightarrow{\begin{pmatrix} d \\ -b \end{pmatrix}} Q_2 \xrightarrow{a} Q_1 \longrightarrow 0 \\
F^\bullet_{3}[2] = F^\bullet_{\substack{3\\2}}[1] = F^\bullet_{\substack{2\\1}}: & 0 \longrightarrow Q_1 \xrightarrow{d} Q_2 \xrightarrow{c} Q_3 \xrightarrow{bc} Q_3 \xrightarrow{b} Q_2 \xrightarrow{a} Q_1 \longrightarrow 0 \\
F^\bullet_{\substack{2\\13}}: & 0 \longrightarrow Q_1 \xrightarrow{d} Q_2 \xrightarrow{\begin{pmatrix} a & c \end{pmatrix}} Q_1 \oplus Q_3 \xrightarrow{\begin{pmatrix} d \\ -b \end{pmatrix}} Q_2 \xrightarrow{ad} Q_2 \xrightarrow{\begin{pmatrix} a & c \end{pmatrix}} Q_1 \oplus Q_3 \xrightarrow{\begin{pmatrix} d \\ -b \end{pmatrix}} Q_2 \xrightarrow{a} Q_1 \to 0
\end{array}
$$

Example 7.6. It was shown in [29, Example 1] that A and B are stably equivalent of Morita type. We aim to verify this with our techniques. We have a functor $\mathcal{G} : \operatorname{proj} A \to \operatorname{proj} B$ given by the following.

$$
\begin{array}{llll}
P_2 \xrightarrow{a} P_1 & \mapsto & Q_2 \oplus Q_3 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & bc \end{pmatrix}} Q_2 \oplus Q_3 \qquad & P_2 \xrightarrow{ac} P_3 \mapsto Q_2 \oplus Q_3 \xrightarrow{\begin{pmatrix} a & c \\ 0 & bc \end{pmatrix}} Q_1 \oplus Q_3 \\
P_3 \xrightarrow{b} P_2 & \mapsto & Q_1 \oplus Q_3 \xrightarrow{\begin{pmatrix} d & 0 \\ -b & 1 \end{pmatrix}} Q_2 \oplus Q_3 \qquad & P_3 \xrightarrow{ba} P_1 \mapsto Q_1 \oplus Q_3 \xrightarrow{\begin{pmatrix} d & 0 \\ -b & bc \end{pmatrix}} Q_2 \oplus Q_3 \\
P_1 \xrightarrow{c} P_3 & \mapsto & Q_2 \oplus Q_3 \xrightarrow{\begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix}} Q_1 \oplus Q_3 \qquad & P_1 \xrightarrow{cb} P_2 \mapsto Q_2 \oplus Q_3 \xrightarrow{\begin{pmatrix} 0 & c \\ -b & 1 \end{pmatrix}} Q_2 \oplus Q_3 \\
P_2 \xrightarrow{acb} P_2 & \mapsto & Q_2 \oplus Q_3 \xrightarrow{\begin{pmatrix} 0 & c \\ 0 & bc \end{pmatrix}} Q_2 \oplus Q_3 \qquad & P_1 \xrightarrow{cba} P_1 \mapsto Q_2 \oplus Q_3 \xrightarrow{\begin{pmatrix} 0 & 0 \\ -b & bc \end{pmatrix}} Q_2 \oplus Q_3
\end{array}
$$

Let $M_B := \mathcal{G}(A) = (Q_2 \oplus Q_3) \oplus (Q_2 \oplus Q_3) \oplus (Q_1 \oplus Q_3)$. We write $\lambda_x : A \to A$ for the left multiplication with $x \in A$. Then M has a left A-module structure given by $x \cdot m := m\mathcal{G}(\lambda_x)$ for $x \in A$ and $m \in M$. Together, we obtain a bimodule ${}_AM_B$. We show that the functor $- \otimes_A M : \operatorname{mod} A \to \operatorname{mod} B$ induces a functor $\mathcal{L}_A \to \mathcal{L}_B$ by componentwise application.

Consider the complex $F^\bullet_3 \in \mathcal{L}_A$.

$$F^\bullet_3 \; : \; 0 \to P_3 \xrightarrow{b} P_2 \xrightarrow{ac} P_3 \xrightarrow{ba} P_1 \xrightarrow{c} P_3 \to 0$$

We have the following morphism $\varphi : F_3^\bullet \otimes_A M \to F_1^\bullet$ in $\mathcal{K}(\operatorname{proj} B)$.

$$\begin{tikzcd}
0 \arrow[r] & Q_1 \oplus Q_3 \arrow[r, "\begin{pmatrix} d & 0 \\ -b & 1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & Q_2 \oplus Q_3 \arrow[r, "\begin{pmatrix} a & c \\ 0 & bc \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 \\ b \end{pmatrix}"] & Q_1 \oplus Q_3 \arrow[r, "\begin{pmatrix} d & 0 \\ -b & bc \end{pmatrix}"] \arrow[d, equal] & Q_2 \oplus Q_3 \arrow[r, "\begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & Q_1 \oplus Q_3 \arrow[r] \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & 0 \\
0 \arrow[r] & Q_1 \arrow[r, "d"] & Q_2 \arrow[r, "\begin{pmatrix} a & c \end{pmatrix}"] & Q_1 \oplus Q_3 \arrow[r, "\begin{pmatrix} d \\ -b \end{pmatrix}"] & Q_2 \arrow[r, "a"] & Q_1 \arrow[r] & 0
\end{tikzcd}$$

Conversely, we do have the following morphism $\psi : F_1^\bullet \to F_3^\bullet \otimes_A M$ in $\mathcal{K}(\operatorname{proj} B)$.

$$\begin{tikzcd}
0 \arrow[r] & Q_1 \arrow[r, "d"] \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & Q_2 \arrow[r, "\begin{pmatrix} a & c \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & Q_1 \oplus Q_3 \arrow[r, "\begin{pmatrix} d \\ -b \end{pmatrix}"] \arrow[d, equal] & Q_2 \arrow[r, "a"] \arrow[d, "\begin{pmatrix} 1 & -c \end{pmatrix}"] & Q_1 \arrow[r] \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & 0 \\
0 \arrow[r] & Q_1 \oplus Q_3 \arrow[r, "\begin{pmatrix} d & 0 \\ -b & 1 \end{pmatrix}"] & Q_2 \oplus Q_3 \arrow[r, "\begin{pmatrix} a & c \\ 0 & bc \end{pmatrix}"] & Q_1 \oplus Q_3 \arrow[r, "\begin{pmatrix} d & 0 \\ -b & bc \end{pmatrix}"] & Q_2 \oplus Q_3 \arrow[r, "\begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix}"] & Q_1 \oplus Q_3 \arrow[r] & 0
\end{tikzcd}$$

We have $\psi\varphi = \mathrm{id}_{F_1^\bullet}$ and a direct calculation shows that $\varphi\psi = \mathrm{id}_{F_3^\bullet \otimes_A M}$ in $\mathcal{K}(\operatorname{proj} B)$. As a result, $F_3^\bullet \otimes_A M \simeq F_1^\bullet \in \mathcal{L}_B$. Similarly, we have isomorphisms $F^\bullet_{\substack{1\\2\\3}} \otimes_A M \simeq F^\bullet_{\substack{2\\1}}$ and $F^\bullet_{\substack{2\\3}} \otimes_A M \simeq F^\bullet_{\substack{2\\1\,3}}$ in $\mathcal{L}_B$.

Since $- \otimes_A M$ commutes with the shift, this results in a functor $- \otimes_A M : \mathcal{L}_A \to \mathcal{L}_B$. In particular, $- \otimes_A M : \operatorname{mod} A \to \operatorname{mod} B$ preserves projective resolutions and therefore is an exact functor; see also the proof of Theorem 5.13. It remains to show that $- \otimes_A M$ induces a stable equivalence. Then, M and $\operatorname{Hom}_B(M, B)$ induce a stable equivalence of Morita type by Theorem 5.13.

The irreducible morphism $f : 2 \to \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$ in $\operatorname{mod} A$ can be lifted to a morphism between the projective presentations.

$$\begin{tikzcd}
P_3 \arrow[r, "b"] \arrow[d, equal] & P_2 \arrow[r, two heads] \arrow[d, "a"] & 2 \arrow[r] \arrow[d, "f"] & 0 \\
P_3 \arrow[r, "ba"] & P_1 \arrow[r, two heads] & \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \arrow[r] & 0
\end{tikzcd}$$

Applying $\mathcal{G}$ to this projective presentation and taking the cokernel gives the morphism $f \otimes M$ since $- \otimes_A M$ is exact.

$$\begin{tikzcd}
Q_1 \oplus Q_3 \arrow[r, "\begin{pmatrix} d & 0 \\ -b & 1 \end{pmatrix}"] \arrow[d, equal] & Q_2 \oplus Q_3 \arrow[r, two heads] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & bc \end{pmatrix}"] & \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \arrow[r] \arrow[d, "f \otimes M"] & 0 \\
Q_1 \oplus Q_3 \arrow[r, "\begin{pmatrix} d & 0 \\ -b & bc \end{pmatrix}"] & Q_2 \oplus Q_3 \arrow[r, two heads] & 2 \oplus Q_3 \arrow[r] & 0
\end{tikzcd}$$

We see that $f \otimes M$ is isomorphic in $\underline{\mathrm{mod}}\, B$ to the unique non-zero morphism $\begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \to 2$. Similarly, we can check this for all other irreducible morphisms. Together, we obtain that $- \otimes_A M \simeq \alpha$, where α is the stable equivalence induced by the Auslander-Reiten quiver we discussed in Example 7.5. This shows that $- \otimes_A M$ is a stable equivalence.

If we already know that $- \otimes_A M$ is a stable equivalence, then M and $\mathrm{Hom}_B(M, B)$ induce a stable equivalence of Morita type if one of the conditions of Theorem 5.19.(2) is satisfied. We have seen above, that $- \otimes_A M$ induces an equivalence $\mathcal{L}_A \to \mathcal{L}_B$. Moreover, A is a Nakayama algebra without nodes and $S_3 \otimes_A M$ is simple in $\mathrm{mod}\, B$. Finally, we also have natural isomorphisms $\nu_B(P \otimes_A M) \simeq \nu_A(P) \otimes_A M$. In fact, we have the following.

$$\nu_B(P_i \otimes_A M) \simeq \nu_B(Q_2 \oplus Q_3) \simeq Q_2 \oplus Q_3 \simeq P_i \otimes_A M \simeq \nu_A(P_i) \otimes_A M \qquad \text{for } i = 1, 2$$

$$\nu_B(P_3 \otimes_A M) \simeq \nu_B(Q_1 \oplus Q_3) \simeq \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus Q_3 \simeq \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \otimes_A M \simeq \nu_A(P_3) \otimes_A M$$

By Theorem 5.8, a stable equivalence of Morita type induces triangulated equivalences $\mathcal{H}_{\mathcal{P}}(\mathrm{proj}\, A) \simeq \mathcal{H}_{\mathcal{P}}(\mathrm{proj}\, B)$ and $\mathcal{H}_{\mathrm{stp}}(\mathrm{proj}\, A) \simeq \mathcal{H}_{\mathrm{stp}}(\mathrm{proj}\, B)$ via componentwise application of $- \otimes_A M$. However, $- \otimes_A M$ does not induce an equivalence between $\mathcal{H}(\mathrm{proj}\, A)$ and $\mathcal{H}(\mathrm{proj}\, B)$. In fact, we have $P_3 \in \mathcal{H}(\mathrm{proj}\, A)$ concentrated in degree zero with $\mathrm{End}_{\mathcal{H}(\mathrm{proj}\, A)}(P_3) \simeq k$. On the other hand, we have $\mathrm{End}_{\mathcal{H}(\mathrm{proj}\, B)}(P_3 \otimes M) \simeq \mathrm{End}_{\mathcal{H}(\mathrm{proj}\, B)}(Q_1 \oplus Q_3) \not\simeq k$. The same argument shows that $- \otimes_A M$ does not induce an equivalence between $\mathcal{K}(\mathrm{proj}\, A)$ and $\mathcal{K}(\mathrm{proj}\, B)$.

At the end of this example, we return to the perfect exact sequence η_0 of Example 7.5.

$$\eta_0 : \qquad 0 \to \begin{smallmatrix} 2 \\ 3 \\ 1 \end{smallmatrix} \to \begin{smallmatrix} 1 \\ 2 \\ 3 \\ 1 \end{smallmatrix} \to 1 \to 0$$

Applying $- \otimes_A M$, we obtain the following perfect exact sequence.

$$\eta_0 \otimes_A M : \qquad 0 \to 3 \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix} \to \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \\ 3 \end{smallmatrix} \to \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \to 0$$

This sequence is the direct sum of the split exact sequence $0 \to Q_2 \xrightarrow{1} Q_2 \to 0$ and the following perfect exact sequence we constructed in Example 7.5.

$$0 \to 3 \to \begin{smallmatrix} 3 \\ 2 \\ 3 \end{smallmatrix} \to \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \to 0$$

7.2 Algebra with infinite global dimension

In this section we discuss some of the results which are specific to an algebra with infinite global dimension. We start in Example 7.7 by comparing the stable Grothendieck group $G_0^{\mathrm{st}}(A)$ with the group $G_0^{\mathcal{P}}(A)$ introduced in Definition 4.15. The next Example 7.8 is dedicated to calculate the stable category of Gorenstein-projective modules of A. The last example of this section, Example 7.9, compares the different triangulated subcategories of $\mathcal{K}(\operatorname{proj} A)$ discussed in Chapter 4. In particular, we list complexes which show that all of these categories can be different.

Let A be the quiver algebra over k given by the quiver

$$\begin{tikzcd}
1 \arrow[rr, "\alpha"] & & 2 \arrow[dl, "\beta"] \\
& 3 \arrow[ul, "\gamma"] &
\end{tikzcd}$$

with relations $\beta\,\gamma\,\alpha = \gamma\,\alpha\,\beta = 0$. The algebra has the following indecomposable projective modules. We also note their images under $(-)^*$.

$$P_1 := \begin{smallmatrix}1\\2\\3\\1\end{smallmatrix}, \qquad P_2 := \begin{smallmatrix}2\\3\\1\end{smallmatrix}, \qquad P_3 := \begin{smallmatrix}3\\1\\2\end{smallmatrix}, \qquad\qquad P_1^* := \begin{smallmatrix}1\\3\\2\\1\end{smallmatrix}, \qquad P_2^* := \begin{smallmatrix}2\\1\\3\end{smallmatrix}, \qquad P_3^* := \begin{smallmatrix}3\\2\\1\end{smallmatrix}$$

We collect some properties of A.

Property	A
Nakayama algebra	yes
gldim A	∞
Indecomposable projective-injective modules	P_1, P_3
Indecomposable strongly projective-injective modules	P_1
domdim A	2
ν-domdim A	0
Nodes	none

The Auslander-Reiten quiver of A can be written as follows.

$$\begin{tikzcd}
& & & & \begin{smallmatrix}1\\2\\3\\1\end{smallmatrix} \arrow[dr] & & \\
& \begin{smallmatrix}3\\1\\2\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}2\\3\\1\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[ll, dashed] \arrow[dr] & \\
\begin{smallmatrix}1\\2\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}3\\1\end{smallmatrix} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[ll, dashed] \\
{} & 1 \arrow[l, dash, dashed] \arrow[ur] & & 3 \arrow[ll, dashed] \arrow[ur] & & 2 \arrow[ll, dashed] \arrow[ur] & {} \arrow[l, dashed]
\end{tikzcd}$$

The category $\mathcal{L}_A$ contains the following indecomposable complexes. Note that we group complexes which are connected by applying a shift in $\mathcal{L}_A$.

degree :	-5	-4	-3	-2	-1	0	1	2	3
$F^\bullet_{\substack{1\\2\\3}}$:			$0 \longrightarrow$	$P_2 \xrightarrow{a}$	$P_1 \xrightarrow{cba}$	$P_1 \xrightarrow{c}$	$P_3 \longrightarrow$	0	
$F^\bullet_{\substack{1\\2}}[1] = F^\bullet_3$:	$\cdots \longrightarrow P_2 \xrightarrow{ac}$	$P_3 \xrightarrow{b}$	$P_2 \xrightarrow{ac}$	$P_3 \xrightarrow{ba}$	$P_1 \xrightarrow{c}$	$P_3 \longrightarrow$	0		
$F^\bullet_1[1] = F^\bullet_{\substack{2\\3}}$:			$0 \longrightarrow$	$P_2 \xrightarrow{a}$	$P_1 \xrightarrow{cb}$	$P_2 \xrightarrow{ac}$	$P_3 \xrightarrow{b}$	$P_2 \xrightarrow{ac}$	$P_3 \rightarrow \cdots$
$F^\bullet_{\substack{3\\1}}[1] = F^\bullet_2$:	$\cdots \longrightarrow P_3 \xrightarrow{b}$	$P_2 \xrightarrow{ac}$	$P_3 \xrightarrow{b}$	$P_2 \xrightarrow{ac}$	$P_3 \xrightarrow{b}$	$P_2 \xrightarrow{ac}$	$P_3 \xrightarrow{b}$	$P_2 \xrightarrow{ac}$	$P_3 \rightarrow \cdots$

The categories $\underline{\mathrm{mod}}\, A$ and $\mathcal{L}_A$ can be visualized as follows. In particular, the right hand side contains information about possible shifts in $\mathcal{L}_A$. The dashed lines indicate zero relations.

$$\begin{tikzcd}[column sep=small]
 & & & \substack{1\\2\\3} \arrow[dr] & & & \\
 & \substack{3\\1} \arrow[dr] \arrow[rr, dash, dashed] & & \substack{2\\3} \arrow[u] \arrow[dr] \arrow[rr, dash, dashed] & & \substack{1\\2} \arrow[dr] & \\
1 \arrow[ur] \arrow[rr, dash, dashed] & & 3 \arrow[ur] \arrow[rr, dash, dashed] & & 2 \arrow[ur] \arrow[rr, dash, dashed] & & 1
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & & & F^\bullet_{\substack{1\\2\\3}} \arrow[dr] & & & \\
 & F^\bullet_{\substack{3\\1}} \arrow[dr] \arrow[rr, dash, dashed] & & F^\bullet_1[1] \arrow[u] \arrow[dr] \arrow[rr, dash, dashed] & & F^\bullet_{\substack{1\\2}} \arrow[dr] & \\
F^\bullet_1 \arrow[ur] \arrow[rr, dash, dashed] & & F^\bullet_{\substack{1\\2}}[1] \arrow[ur] \arrow[rr, dash, dashed] & & F^\bullet_{\substack{3\\1}}[1] \arrow[ur] \arrow[rr, dash, dashed] & & F^\bullet_1
\end{tikzcd}$$

Example 7.7. We aim to calculate the group $G_0^{\mathcal{P}}(A)$ of Definition 4.15. Recall that we have $G_0^{\mathcal{P}}(A) \simeq G_0(\mathcal{H}_{\mathcal{P}}(\mathrm{proj}\, A))$ by Theorem 4.16. In case that $F^\bullet[1] \in \mathcal{L}_A$ for $F^\bullet \in \mathcal{L}_A$, we have the following distinguished triangle in $\mathcal{L}_A$.

$$F^\bullet \to 0 \to F^\bullet[1] \to$$

By Proposition 2.18, we obtain a corresponding perfect exact sequence with projective middle term. This implies $[X] = -[Y]$ in $G_0^{\mathcal{P}}(A)$ if $F^\bullet_X[1] \simeq F^\bullet_Y \in \mathcal{L}_A$. Using this, the perfect exact sequence

$$0 \to \substack{2\\3} \to \substack{1\\2\\3} \to 1 \to 0$$

with $F^\bullet_1[1] = F^\bullet_{\substack{2\\3}}$ yields $\begin{bmatrix}2\\3\end{bmatrix} = -[S_1]$ and $\begin{bmatrix}1\\2\\3\end{bmatrix} = 0$. Furthermore, we have $\begin{bmatrix}1\\2\end{bmatrix} = -[S_3]$ and $\begin{bmatrix}3\\1\end{bmatrix} = -[S_2]$ since $F^\bullet_{\substack{1\\2}}[1] = F^\bullet_3$ and $F^\bullet_{\substack{3\\1}}[1] = F^\bullet_2$. The perfect exact sequence

$$0 \to S_1 \to \substack{3\\1} \to S_3 \to 0$$

implies $\begin{bmatrix}3\\1\end{bmatrix} = [S_1] + [S_3]$. In conclusion, we obtain the following.

$$\begin{aligned}\begin{bmatrix}1\\2\end{bmatrix} &= -[S_3]\\ \begin{bmatrix}2\\3\end{bmatrix} &= -[S_1]\\ \begin{bmatrix}3\\1\end{bmatrix} &= -[S_2] = [S_1] + [S_3]\\ \begin{bmatrix}1\\2\\3\end{bmatrix} &= 0.\end{aligned}$$

Thus, $G_0^{\mathcal{P}}(A)$ is generated by the classes of the simple modules S_1 and S_3.

We want to compare this with the stable Grothendieck group $G_0^{\mathrm{st}}(A)$. By Remark 4.18, we have a surjective group homomorphism $G_0^{\mathcal{P}}(A) \to G_0^{\mathrm{st}}(A)$. In particular, the above equations still hold in $G_0^{\mathrm{st}}(A)$. Additionally, we know that the class in $G_0^{\mathrm{st}}(A)$ of every module with finite projective dimension is zero. Hence, we obtain $\begin{bmatrix}2\\3\end{bmatrix}_{\mathrm{st}} = [S_1]_{\mathrm{st}} = 0$ in $G_0^{\mathrm{st}}(A)$. This results in the following.

$$\begin{aligned}\begin{bmatrix}3\\1\end{bmatrix}_{\mathrm{st}} &= -\begin{bmatrix}1\\2\end{bmatrix}_{\mathrm{st}} = -[S_2]_{\mathrm{st}} = [S_3]_{\mathrm{st}}\\ \begin{bmatrix}1\\2\\3\end{bmatrix}_{\mathrm{st}} &= \begin{bmatrix}2\\3\end{bmatrix}_{\mathrm{st}} = [S_1]_{\mathrm{st}} = 0\end{aligned}$$

Thus, $G_0^{\mathrm{st}}(A)$ is already generated by the class of the simple module S_3.

Example 7.8. We calculate the category of stable Gorenstein-projective modules.

We have $F_2^\bullet[2k] \simeq F_2^\bullet \in \mathcal{L}_A$ for all $k \in \mathbb{Z}$. By Lemma 4.38, we obtain that $F_2^\bullet$ and $F_{\substack{3\\1}}^\bullet$ are indecomposable complexes in $\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$. None of the other complexes are periodic, so that these are the only ones. In particular, we have that the indecomposable modules of $\underline{\operatorname{Gproj}} A$ are given by 2 and $\substack{3\\1}$ with no non-zero morphism between them.

Note that A is a Gorenstein algebra. Thus, $\mathcal{D}_{\mathrm{sg}}(A)$ is equivalent to $\underline{\operatorname{Gproj}} A$.

Example 7.9. Consider the following inclusions of categories.

$$\begin{tikzcd}
\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A) \arrow[r, hook] & \mathcal{L}_A \arrow[r, hook] & \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) \arrow[r, hook] & \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A) \arrow[r, hook] & \mathcal{H}(\operatorname{proj} A) \arrow[r, hook] & \mathcal{K}(\operatorname{proj} A) \\
\underline{\operatorname{Gproj}} A \arrow[r, hook] \arrow[u, "\wr"] & \underline{\operatorname{mod}} A \arrow[u, "\wr", "\mathcal{F}"']
\end{tikzcd}$$

Since $(0) \subset \operatorname{stp} A \subset \mathcal{P}_A \subset \operatorname{proj} A$ are proper subsets, the above inclusions must be proper as well; see also Theorem 4.45 and Lemma 4.29. We verify this by constructing a complex for each category.

Recall the indecomposable elements of $\mathcal{L}_A$ listed above. In Example 7.8 we have already seen that $\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$ is a proper subset of $\mathcal{L}_A$. On the other hand, every shift of $F^\bullet_{\substack{1\\2\\3}}$ is still an element of $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ but no longer an element of $\mathcal{L}_A$. However, not every complex in $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ is obtained by shifting a complex in $\mathcal{L}_A$.

Consider the following complexes in $\mathcal{K}(\operatorname{proj} A)$.

$$\begin{array}{ll}
G_1^\bullet : & 0 \longrightarrow P_2 \xrightarrow{a} P_1 \xrightarrow{cb} P_2 \xrightarrow{ac} P_3 \xrightarrow{ba} P_1 \xrightarrow{c} P_3 \longrightarrow 0 \\
G_2^\bullet : & \cdots \longrightarrow P_3 \xrightarrow{b} P_2 \xrightarrow{ac} P_3 \xrightarrow{b} P_2 \xrightarrow{ac} P_3 \xrightarrow{b} P_2 \longrightarrow 0 \\
G_3^\bullet : & \cdots \longrightarrow P_3 \xrightarrow{b} P_2 \xrightarrow{ac} P_3 \xrightarrow{b} P_2 \xrightarrow{ac} P_3 \xrightarrow{ba} P_1 \longrightarrow 0 \\
G_4^\bullet : & \cdots \longrightarrow P_1 \xrightarrow{cba} P_1 \xrightarrow{cba} P_1 \xrightarrow{cba} P_1 \xrightarrow{cba} P_1 \xrightarrow{cba} P_1 \longrightarrow \cdots
\end{array}$$

A direct calculation yields that $\mathrm{H}^k(G_1^\bullet) \in \{0, S_3\}$ for all $k \in \mathbb{Z}$. Since $S_3 \in {}^\perp\mathcal{P}_A$, we have $G_1^\bullet \in \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$; cf. Lemma 4.5. However, neither $G_1^\bullet$ nor a shift thereof is an element of $\mathcal{L}_A$.

For the next complex, we have $\mathrm{H}^k(G_2^\bullet) \in \{0, S_2\}$ for all $k \in \mathbb{Z}$ where S_2 occurs only as cohomology in the degree of the last non-negative term. Since $S_2 \in {}^\perp(\operatorname{stp} A)$ but $S_2 \notin {}^\perp\mathcal{P}_A$, we have $G_2^\bullet \in \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ and $G_2^\bullet \notin \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$; cf. Lemma 4.5.

The complex $G_3^\bullet$ has $\begin{smallmatrix}1\\2\end{smallmatrix}$ as non-zero cohomology in a single degree. Since $\begin{smallmatrix}1\\2\end{smallmatrix} \notin {}^\perp(\operatorname{stp} A)$, we obtain $G_3^\bullet \notin \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ as before. However, we have $\mathrm{H}^k(G_3^\bullet) = 0$ in all other degrees and $G_3^\bullet$ is right bounded, so that $G_3^\bullet$ is an element of $\mathcal{H}(\operatorname{proj} A)$.

Finally, we have $\mathrm{H}^k(G_4^\bullet) = \begin{smallmatrix}2\\3\end{smallmatrix}$ for all k. Thus, $G_4^\bullet \in \mathcal{K}(\operatorname{proj} A)$ is not an element of $\mathcal{H}(\operatorname{proj} A)$. Note that the truncation $\tau_{\leqslant 0}\, G_4^\bullet$ and the truncation $\tau_{\geqslant 0} G_4^\bullet$ are not an element of $\mathcal{H}(\operatorname{proj} A)$ as well. However, we have $G_4^\bullet \in {}^\perp\mathcal{K}^b(\operatorname{stp} A)$ which affirms that the restriction to the category $\mathcal{H}(\operatorname{proj} A)$ is necessary in the definition of the smallest triangulated subcategory that contains $\mathcal{L}_A$.

7.3 Triangulated categories inside $\mathcal{K}(\operatorname{proj} A)$

The main purpose of this section is to calculate the categories $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ and $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ of Chapter 4 explicitly. This is done in Example 7.10. In the next Example 7.11, we follow the constructions in Section 4.1 which where used to show the minimality of $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$. Finally, we illustrate the associated self-injective algebra of A and its connection to the category $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ in Example 7.12.

Let A be the quiver algebra over k given by the quiver

$$\begin{tikzcd}
 & 1 \arrow[dl, "\alpha"'] & & & \\
2 \arrow[rr, "\beta"] & & 3 \arrow[ul, "\gamma"'] \arrow[r, "\delta"] & 4 \arrow[r, "\varepsilon"] & 5
\end{tikzcd}$$

with relations $\alpha\,\beta = \gamma\,\alpha = \beta\,\delta = \delta\,\varepsilon = 0$. The algebra has the following indecomposable projective modules. We also note their images under $(-)^*$.

$$P_1 := \begin{smallmatrix}1\\2\end{smallmatrix}, \qquad P_2 := \begin{smallmatrix}2\\3\\1\end{smallmatrix}, \qquad P_3 := \begin{smallmatrix}&3&\\1&&4\end{smallmatrix}, \qquad P_4 := \begin{smallmatrix}4\\5\end{smallmatrix}, \qquad P_5 := 5$$

$$P_1^* := \begin{smallmatrix}1\\3\\2\end{smallmatrix}, \qquad P_2^* := \begin{smallmatrix}2\\1\end{smallmatrix}, \qquad P_3^* := \begin{smallmatrix}3\\2\end{smallmatrix}, \qquad P_4^* := \begin{smallmatrix}4\\3\end{smallmatrix}, \qquad P_5^* := \begin{smallmatrix}5\\4\end{smallmatrix}$$

We collect some properties of A.

Property	A
Nakayama algebra	no
gldim A	5
Indecomposable projective-injective modules	P_1, P_2, P_4
Indecomposable strongly projective-injective modules	P_1, P_2
domdim A	0
ν-domdim A	0
Nodes	S_1, S_2, S_4

The Auslander-Reiten quiver of A can be written as follows.

$$\begin{tikzcd}[column sep=small]
 & \begin{smallmatrix}4\\5\end{smallmatrix} \arrow[dr] & & & & \begin{smallmatrix}2\\3\\1\end{smallmatrix} \arrow[dr] & & & & \\
5 \arrow[ur] & & 4 \arrow[ll, dashed] \arrow[dr] & & \begin{smallmatrix}3\\1\end{smallmatrix} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ll, dashed] \arrow[dr] & & & \\
 & & & \begin{smallmatrix}&3&\\1&&4\end{smallmatrix} \arrow[ur] \arrow[dr] & & 3 \arrow[ll, dashed] \arrow[ur] & & 2 \arrow[ll, dashed] \arrow[dr] & & 1 \arrow[ll, dashed] \\
 & & 1 \arrow[ur] & & \begin{smallmatrix}3\\4\end{smallmatrix} \arrow[ll, dashed] \arrow[ur] & & & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[ur] &
\end{tikzcd}$$

The category $\mathcal{L}_A$ contains the following indecomposable complexes.

$$
\begin{array}{llllllllllll}
\text{degree}: & -6 & & -5 & & -4 & & -3 & & -2 & & -1 \quad 0 \quad 1 \quad 2 \quad 3 \\
\hline
\end{array}
$$

$$F^\bullet_{\substack{3\\4}}: \quad 0 \longrightarrow P_5 \xrightarrow{e} P_4 \xrightarrow{d} P_3 \xrightarrow{b} P_2 \xrightarrow{a} P_1 \xrightarrow{c} P_3 \longrightarrow 0$$

$$F^\bullet_{\substack{2\\3}}: \quad 0 \longrightarrow P_5 \xrightarrow{e} P_4 \xrightarrow{d} P_3 \xrightarrow{b} P_2 \xrightarrow{a} P_1 \xrightarrow{cb} P_2 \xrightarrow{a} P_1 \xrightarrow{c} P_3 \longrightarrow 0$$

$$F^\bullet_3: \quad 0 \longrightarrow P_5 \xrightarrow{e} P_4 \xrightarrow{d} P_3 \xrightarrow{\begin{pmatrix} b & 0 \end{pmatrix}} P_2 \oplus P_5 \xrightarrow{\begin{pmatrix} a & 0 \\ 0 & e \end{pmatrix}} P_1 \oplus P_4 \xrightarrow{\begin{pmatrix} c \\ d \end{pmatrix}} P_3 \longrightarrow 0$$

Furthermore, we have $F^\bullet_4[4] = F^\bullet_{\substack{3\\1}}[3] = F^\bullet_2[2] = F^\bullet_1[1] = F^\bullet_{\substack{3\\4}}$.

The categories $\underline{\operatorname{mod}} A$ and $\mathcal{L}_A$ can be visualized as follows. In particular, the right hand side contains information about possible shifts in $\mathcal{L}_A$. The dashed lines indicate zero relations.

$$
\begin{tikzcd}
4 & \substack{3\\1} \arrow[dr] \arrow[rr, dash, dashed] & & \substack{2\\3} \arrow[dr] & \\
& & 3 \arrow[ur] \arrow[rr, dash, dashed] & & 2 \\
1 & \substack{3\\4} \arrow[ur] & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
F^\bullet_4 & F^\bullet_4[1] \arrow[dr] \arrow[rr, dash, dashed] & & F^\bullet_{\substack{2\\3}} \arrow[dr] & \\
& & F^\bullet_3 \arrow[ur] \arrow[rr, dash, dashed] & & F^\bullet_4[2] \\
F^\bullet_4[3] & F^\bullet_4[4] \arrow[ur] & & &
\end{tikzcd}
$$

Example 7.10. We aim to calculate the categories $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ and $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$. Since A has finite global dimension, both categories are contained in $\mathcal{K}^b(\operatorname{proj} A) \simeq \mathcal{H}(\operatorname{proj} A)$. First, we construct all complexes in $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$. Afterwards we can identify those that are an element of $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$.

A complex $F^\bullet \in \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ satisfies $\mathrm{H}^k(F^\bullet) \in {}^\perp(\operatorname{stp} A)$. The following indecomposable modules are an element of ${}^\perp(\operatorname{stp} A)$. These are precisely those that have no composition factor isomorphic to S_1 or S_2.

$$3, \quad \substack{3\\4}, \quad 4, \quad \substack{4\\5}, \quad 5$$

Similarly, we obtain the indecomposable modules in ${}^\perp\mathcal{P}_A$ which we will need later.

$$3, \quad \substack{3\\4}, \quad 4$$

We will use an integer $n \geqslant 1$ to index some of our complexes. We write $(P_2 \xrightarrow{a} P_1)^{\to n}$ for the periodic complex

$$P_2 \xrightarrow{a} P_1 \xrightarrow{cb} P_2 \xrightarrow{a} P_1 \xrightarrow{cb} \cdots \xrightarrow{cb} P_2 \xrightarrow{a} P_1$$

with P_2 appearing n-times. Furthermore, we write $P \xrightarrow{f} (P_2 \xrightarrow{a} P_1)^{\to 0} \xrightarrow{g} P_2 \oplus Q$ for the complex $P \xrightarrow{\begin{pmatrix} f & 0 \end{pmatrix}} P_2 \oplus Q$ where $f : P \to P_2$ and $g : P_1 \to P_2 \oplus Q$ are morphisms in $\operatorname{proj} A$.

We have the following indecomposable complexes in $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ with $n \geqslant 1$.

degree :	-8	-7	-6	-5	-4	-3	-2	-1	0	1
$P_5^\bullet$:								0	P_5	0
$P_4^\bullet$:								0	P_4	0
$S^\bullet$:							0	P_5	P_4	0

$$P_5^\bullet : \quad 0 \longrightarrow P_5 \longrightarrow 0$$

$$P_4^\bullet : \quad 0 \longrightarrow P_4 \longrightarrow 0$$

$$S^\bullet : \quad 0 \longrightarrow P_5 \xrightarrow{e} P_4 \longrightarrow 0$$

$$X_1(n)^\bullet : \quad 0 \longrightarrow P_3 \xrightarrow{b} (P_2 \xrightarrow{a} P_1)^{\to n} \xrightarrow{c} P_3 \longrightarrow 0$$

$$X_2(n)^\bullet : \quad 0 \longrightarrow P_4 \xrightarrow{d} P_3 \xrightarrow{b} (P_2 \xrightarrow{a} P_1)^{\to n} \xrightarrow{c} P_3 \longrightarrow 0$$

$$X_3(n)^\bullet : \quad 0 \longrightarrow P_5 \xrightarrow{e} P_4 \xrightarrow{d} P_3 \xrightarrow{b} (P_2 \xrightarrow{a} P_1)^{\to n} \xrightarrow{c} P_3 \longrightarrow 0$$

$$Y_1(n)^\bullet : \quad 0 \longrightarrow P_3 \xrightarrow{b} (P_2 \xrightarrow{a} P_1)^{\to(n-1)} \xrightarrow{cb} P_2 \xrightarrow{(a\ 0)} P_1 \oplus P_4 \xrightarrow{\binom{c}{d}} P_3 \longrightarrow 0$$

$$Y_2(n)^\bullet : \quad 0 \longrightarrow P_4 \xrightarrow{d} P_3 \xrightarrow{b} (P_2 \xrightarrow{a} P_1)^{\to(n-1)} \xrightarrow{cb} P_2 \xrightarrow{(a\ 0)} P_1 \oplus P_4 \xrightarrow{\binom{c}{d}} P_3 \longrightarrow 0$$

$$Y_3(n)^\bullet : \quad 0 \longrightarrow P_5 \xrightarrow{e} P_4 \xrightarrow{d} P_3 \xrightarrow{b} (P_2 \xrightarrow{a} P_1)^{\to(n-1)} \xrightarrow{cb} P_2 \xrightarrow{(a\ 0)} P_1 \oplus P_4 \xrightarrow{\binom{c}{d}} P_3 \longrightarrow 0$$

$$Z_1(n)^\bullet : \quad 0 \longrightarrow P_3 \xrightarrow{b} (P_2 \xrightarrow{a} P_1)^{\to(n-1)} \xrightarrow{(cb\ 0)} P_2 \oplus P_5 \xrightarrow{\begin{pmatrix} a & 0 \\ 0 & e \end{pmatrix}} P_1 \oplus P_4 \xrightarrow{\binom{c}{d}} P_3 \longrightarrow 0$$

$$Z_2(n)^\bullet : \quad 0 \longrightarrow P_4 \xrightarrow{d} P_3 \xrightarrow{b} (P_2 \xrightarrow{a} P_1)^{\to(n-1)} \xrightarrow{(cb\ 0)} P_2 \oplus P_5 \xrightarrow{\begin{pmatrix} a & 0 \\ 0 & e \end{pmatrix}} P_1 \oplus P_4 \xrightarrow{\binom{c}{d}} P_3 \longrightarrow 0$$

$$Z_3(n)^\bullet : \quad 0 \longrightarrow P_5 \xrightarrow{e} P_4 \xrightarrow{d} P_3 \xrightarrow{b} (P_2 \xrightarrow{a} P_1)^{\to(n-1)} \xrightarrow{(cb\ 0)} P_2 \oplus P_5 \xrightarrow{\begin{pmatrix} a & 0 \\ 0 & e \end{pmatrix}} P_1 \oplus P_4 \xrightarrow{\binom{c}{d}} P_3 \longrightarrow 0$$

It remains to show that every indecomposable complex in $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ is isomorphic to a shift of an entry of this list. First, we note that this holds for the elements of $\mathcal{L}_A$. More precisely, the indecomposable complexes in $\mathcal{L}_A$ are isomorphic to one of the following complexes.

$$X_3(1)^\bullet,\ X_3(1)^\bullet[-1],\ X_3(1)^\bullet[-2],\ X_3(1)^\bullet[-3],\ X_3(1)^\bullet[-4],\ X_3(2)^\bullet[-2],\ Z_3(1)^\bullet$$

Let $F^\bullet \in \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$. By Lemma 4.6, there exists a projective resolution $P^\bullet$ of a cohomology in ${}^\perp(\operatorname{stp} A)$ and a complex $C^\bullet \in \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ such that there is a distinguished triangle of the following form.

$$P^\bullet[-k] \to F^\bullet \to C^\bullet \to$$

In this triangle, $k \in \mathbb{Z}$ is chosen such that $\mathrm{H}^i(F^\bullet) = 0$ for $i < k$. Then we have $\mathrm{H}^i(C^\bullet) = 0$ for $i \leqslant k$. As in the proof of Lemma 4.7, we inductively arrive at a complex $C^\bullet$ with $C^\bullet[r] \in \mathcal{L}_A$ for some $r \in \mathbb{Z}$. For the purposes of this example, we will use the shifted triangle $C^\bullet[-1] \to P^\bullet[-k] \to F^\bullet \to$.

Let $P^\bullet$ be a projective resolution of a module in ${}^\perp(\operatorname{stp} A)$. Let $C^\bullet$ be a complex appearing in the list above. In particular, this includes all indecomposable complexes in $\mathcal{L}_A$. Let $k \in \mathbb{Z}$ such that $\mathrm{H}^i(C^\bullet) = 0$ for $i \leqslant k+1$. That is, $C^\bullet$ is exact in every degree smaller or equal to k and also exact in degree $k+1$. Consider the following distinguished triangle.

$$C^\bullet \xrightarrow{f^\bullet} P^\bullet[-k] \to C(f)^\bullet \to$$

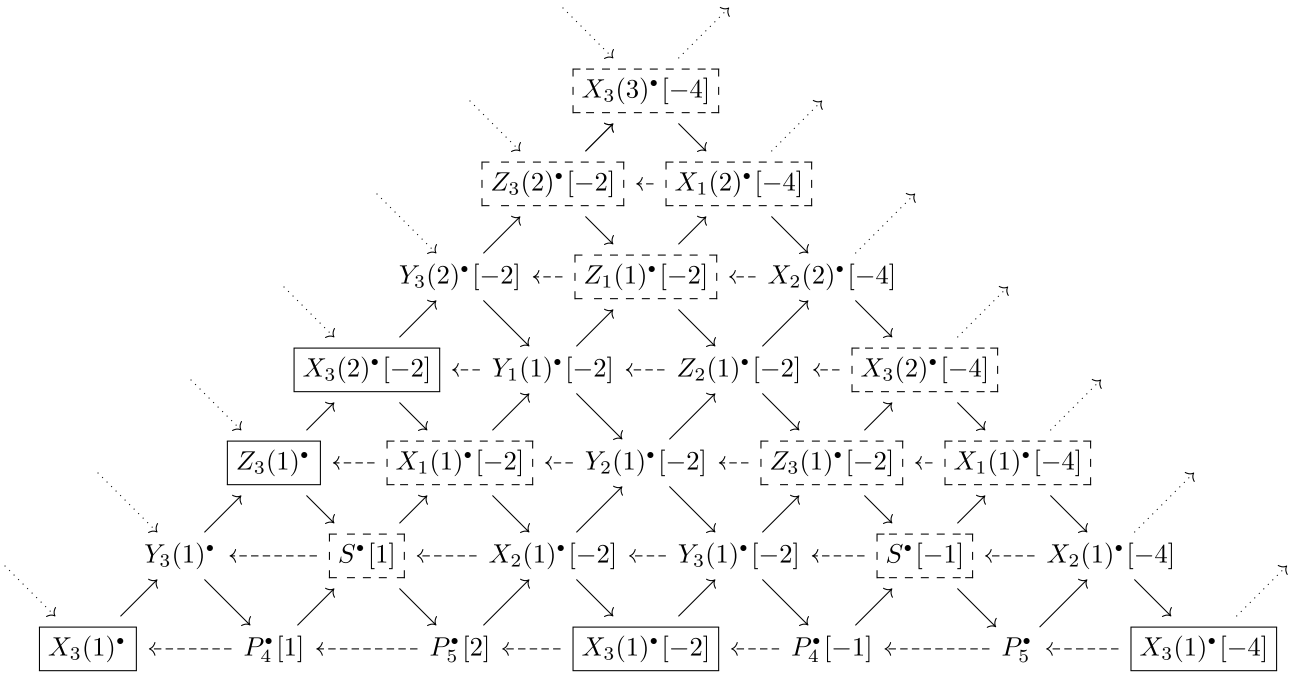
For every possible choice of $P^\bullet$, $C^\bullet$, k and $f^\bullet$, a direct calculation shows that $C(f)^\bullet$ is isomorphic to a direct sum of complexes appearing in the list above. Thus, every indecomposable complex $F^\bullet \in \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ is isomorphic to a shift of a complex in this list.

The following complexes are the indecomposable elements of $\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ up to shifts. These are precisely the complexes appearing in the list above with cohomology in ${}^\perp\mathcal{P}_A$.

$$S^\bullet,\ X_1(n)^\bullet,\ X_3(n)^\bullet,\ Z_1(n)^\bullet,\ Z_3(n)^\bullet \qquad \text{for } n \geqslant 1$$

One of the Auslander-Reiten components in $\mathcal{K}^b(\operatorname{proj} A)$ containing $\mathcal{L}_A$ can be written as follows. The other component is given by a shift of [1]. Both components together form precisely the category $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$; cf. also Remark 4.28.

Complexes $F^\bullet \in \mathcal{L}_A$ are marked as $\boxed{F^\bullet}$. Complexes $F^\bullet \in \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$ are marked as $\boxed{F^\bullet}$ (dashed).



Example 7.11. We use the methods provided in Section 4.1 to verify that the complex

$$F^\bullet := X_1(1)^\bullet : 0 \to P_3 \xrightarrow{b} P_2 \xrightarrow{a} P_1 \xrightarrow{c} P_3 \to 0$$

is an element of every triangulated subcategory $\mathcal{T}$ of $\mathcal{K}^b(\operatorname{proj} A)$ that contains $\mathcal{L}_A$. By going through the following steps in reverse order, this also provides a way to construct $F^\bullet$ via complexes in $\mathcal{L}_A$. Note that in this example, several steps that were necessary for the general case are either simplified or not necessary at all.

Step 1. It suffices to consider projective resolutions of modules in ${}^\perp\mathcal{P}_A$; cf. Lemma 4.7.

We have $\mathrm{H}^0(F^\bullet) = \begin{smallmatrix}3\\4\end{smallmatrix}$, $\mathrm{H}^{-3}(F^\bullet) = S_4$ and $\mathrm{H}^k(F^\bullet) = 0$ for $k \notin \{0, -3\}$. Furthermore, we have $\mathrm{H}_0(F^*_\bullet) = 0$. In particular, note that $\mathrm{H}^k(F^\bullet) \in {}^\perp\mathcal{P}_A$ as stated in Lemma 4.5.

In the notation of Lemma 4.7, we may choose $r = 0$. Then $k = -3$ is the only degree in $\mathbb{Z}_{<r}$ with non-zero cohomology. The minimal projective resolution of S_4 is given by

$$S^\bullet : 0 \to P_5 \xrightarrow{e} P_4 \to 0\,.$$

We have the following distinguished triangle in $\mathcal{K}^b(\operatorname{proj} A)$; cf. Lemma 4.6.

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & P_5 \arrow[r, "e"] \arrow[d] & P_4 \arrow[r] \arrow[d, "d"] & 0 \arrow[d] & & & & S^\bullet[3] \arrow[d, "f^\bullet"] \\
 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & P_3 \arrow[r, "b"] \arrow[d] & P_2 \arrow[r, "a"] \arrow[d, equal] & P_1 \arrow[r, "c"] \arrow[d, equal] & P_3 \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] & F^\bullet \arrow[d] \\
0 \arrow[r] & P_5 \arrow[r, "e"] & P_4 \arrow[r, "d"] & P_3 \arrow[r, "b"] & P_2 \arrow[r, "a"] & P_1 \arrow[r, "c"] & P_3 \arrow[r] & 0 & C(f^\bullet)
\end{tikzcd}$$

In this case, we already have $C(f^\bullet) = F^\bullet_{\begin{smallmatrix}3\\4\end{smallmatrix}} \in \mathcal{L}$. No further construction is necessary during this step. It remains to consider the projective resolution $S^\bullet$ of S_4.

Step 2. It suffices to consider projective resolutions of simple modules in ${}^\perp\mathcal{P}A$; cf. Lemma 4.9.

Only one projective resolution was used in step 1. In this case, nothing needs to be done during this step, since $S^\bullet$ is already a projective resolution of a simple module.

Step 3. All projective resolutions of simple modules $S \in {}^\perp\mathcal{P}_A$ are in $\mathcal{T}$; cf. Lemma 4.10.

Consider the projective resolution $S^\bullet$ of S_4. The injective hull of S_4 is given by $I := \begin{smallmatrix}3\\4\end{smallmatrix}$. Note that I is not projective since S_4 is an element of ${}^\perp\mathcal{P}_A$. We obtain the following distinguished triangle.

$$\begin{tikzcd}
 & & & & 0 \arrow[r] \arrow[d] & P_5 \arrow[r, "e"] \arrow[d, "0"] & P_4 \arrow[r] \arrow[d, "d"] & 0 \arrow[d] & S^\bullet \arrow[d, "g^\bullet"] \\
0 \arrow[r] \arrow[d] & P_5 \arrow[r, "e"] \arrow[d, equal] & P_4 \arrow[r, "d"] \arrow[d, equal] & P_3 \arrow[r, "b"] \arrow[d, equal] & P_2 \arrow[r, "a"] \arrow[d, "\begin{pmatrix}1 & 0\end{pmatrix}"] & P_1 \arrow[r, "c"] \arrow[d, "\begin{pmatrix}1 & 0\end{pmatrix}"] & P_3 \arrow[r] \arrow[d, equal] & 0 \arrow[d] & F^\bullet_I \arrow[d] \\
0 \arrow[r] & P_5 \arrow[r, "e"] & P_4 \arrow[r, "d"] & P_3 \arrow[r, "\begin{pmatrix}b & 0\end{pmatrix}"] & P_2 \oplus P_5 \arrow[r, "\begin{pmatrix}a & 0\\ 0 & e\end{pmatrix}"] & P_1 \oplus P_4 \arrow[r, "\begin{pmatrix}c\\ d\end{pmatrix}"] & P_3 \arrow[r] & 0 & C(g^\bullet)
\end{tikzcd}$$

We have $C(g^\bullet) \simeq F^\bullet_3 \in \mathcal{L}_A$. Note that $0 \to 4 \to \begin{smallmatrix}3\\4\end{smallmatrix} \to 3 \to 0$ is not a perfect exact sequence, since the first morphism factors through the projective module $P_3 = \begin{smallmatrix}&3&\\1&&4\end{smallmatrix}$.

In conclusion, we have the following two distinguished triangles with $F^\bullet_{\begin{smallmatrix}3\\4\end{smallmatrix}}, F^\bullet_3 \in \mathcal{L}_A$.

$$S^\bullet \xrightarrow{g^\bullet} F^\bullet_{\begin{smallmatrix}3\\4\end{smallmatrix}} \to F^\bullet_3 \to \qquad \text{This triangle implies } S^\bullet \in \mathcal{T}.$$

$$S^\bullet[3] \xrightarrow{f^\bullet} F^\bullet \to F^\bullet_{\begin{smallmatrix}3\\4\end{smallmatrix}} \to \qquad \text{This triangle implies } F^\bullet \in \mathcal{T}.$$

Example 7.12. We discuss the associated self-injective algebra of A. Consider the following complex $F^\bullet := X_2(1)^\bullet \in \mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$.

$$0 \to P_4 \xrightarrow{d} P_3 \xrightarrow{b} P_2 \xrightarrow{a} P_1 \xrightarrow{c} P_3 \to 0$$

Let $e = e_1 + e_2$ be the sum of the primitive idempotents corresponding to the strongly projective-injective modules P_1 and P_2. The algebra eAe is isomorphic to the quiver algebra A' given by the quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha'"] & 2 \arrow[l, shift left, "\beta'"] \end{tikzcd}$$

and relations $\alpha'\,\beta' = \beta'\,\alpha' = 0$. We denote the indecomposable projective modules by $P'_1 := \begin{smallmatrix}1\\2\end{smallmatrix}$ and $P'_2 := \begin{smallmatrix}2\\1\end{smallmatrix}$. Let S'_1 be the simple module in $\operatorname{mod} A'$ corresponding to P'_1. We obtain the following complex $(Fe)^\bullet \in \mathcal{K}^b(\operatorname{mod} eAe)$.

$$\begin{tikzcd} (Fe)^{-4} \arrow[r] \arrow[d, "\wr"] & (Fe)^{-3} \arrow[r] \arrow[d, "\wr"] & (Fe)^{-2} \arrow[r] \arrow[d, "\wr"] & (Fe)^{-1} \arrow[r] \arrow[d, "\wr"] & (Fe)^{0} \arrow[r] \arrow[d, "\wr"] & (Fe)^{1} \arrow[d, "\wr"] \\ 0 \arrow[r] & S'_1 \arrow[r] & P'_2 \arrow[r] & P'_1 \arrow[r] & S'_1 \arrow[r] & 0 \end{tikzcd}$$

As stated by Lemma 4.30, we see that $(Fe)^\bullet$ is exact in every degree. On the other hand, it seems difficult to construct $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$ by lifting exact complexes in $\mathcal{K}(\operatorname{mod} eAe)$ to projective complexes in $\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$.

7.4 Algebras stably equivalent by deleting nodes

In this section, we discuss two stably equivalent algebras A and B. The algebra A has no nodes, whereas the algebra B has two nodes. We mainly aim to illustrate the constructions of Chapter 6 in Example 7.16. In Example 7.15 we additionally take a look at some properties discussed in the previous chapters that are not preserved by stable equivalences induced by deleting nodes.

Independently of this, Example 7.13 provides two short exact sequences that are not perfect exact. This is done in context of Lemma 2.13. Moreover, we show in Example 7.14 that the non-projective simple modules of A do not generate the group $G_0^{\mathcal{P}}(A)$.

We consider two algebras A and B given by the following quivers and relations.

Quiver of A

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] \arrow[d, "\varepsilon"] & 3 \\ 4 \arrow[r, "\gamma"] & 5 \arrow[r, "\delta"] & 6 \end{tikzcd}$$

Quiver of B

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"] \arrow[r, "\varepsilon"] & 5 \arrow[r, shift left, "\gamma"] & 6 \arrow[l, shift left, "\delta"] \end{tikzcd}$$

Relations of A

$$\alpha\,\varepsilon = \gamma\,\delta = 0$$

Relations of B

$$\alpha\,\varepsilon = \gamma\,\delta = \beta\,\alpha = \delta\,\gamma = 0$$

The algebra A has the following indecomposable projective modules. We also note their images under the functor $(-)^*$.

$$P_1 := \begin{smallmatrix}1\\2\\3\end{smallmatrix}, \quad P_2 := \begin{smallmatrix} & 2 & \\ 3 & & 5 \\ & & 6\end{smallmatrix}, \quad P_3 := 3, \quad P_4 := \begin{smallmatrix}4\\5\end{smallmatrix}, \quad P_5 := \begin{smallmatrix}5\\6\end{smallmatrix}, \quad P_6 := 6$$

$$P_1^* := 1, \quad P_2^* := \begin{smallmatrix}2\\1\end{smallmatrix}, \quad P_3^* := \begin{smallmatrix}3\\2\\1\end{smallmatrix}, \quad P_4^* := 4, \quad P_5^* := \begin{smallmatrix} & 5 & \\ 2 & & 4\end{smallmatrix}, \quad P_6^* := \begin{smallmatrix}6\\5\\2\end{smallmatrix}$$

The algebra B has the following indecomposable projective modules. We also note their images under the functor $(-)^*$.

$$Q_1 := \begin{smallmatrix}1\\2\\1\end{smallmatrix}, \quad Q_2 := \begin{smallmatrix} & 2 & \\ 1 & & 5 \\ & & 6\end{smallmatrix}, \quad Q_5 := \begin{smallmatrix}5\\6\end{smallmatrix}, \quad Q_6 := \begin{smallmatrix}6\\5\end{smallmatrix}$$

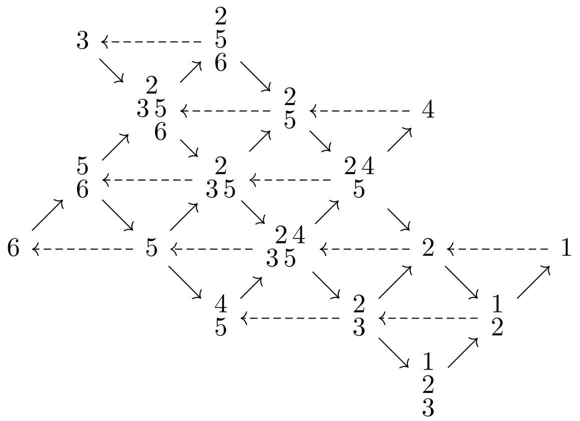
$$Q_1^* := \begin{smallmatrix}1\\2\\1\end{smallmatrix}, \quad Q_2^* := \begin{smallmatrix}2\\1\end{smallmatrix}, \quad Q_5^* := \begin{smallmatrix} & 5 & \\ 2 & & 6\end{smallmatrix}, \quad Q_6^* := \begin{smallmatrix}6\\5\\2\end{smallmatrix}$$

The following table collects some properties of A and B.

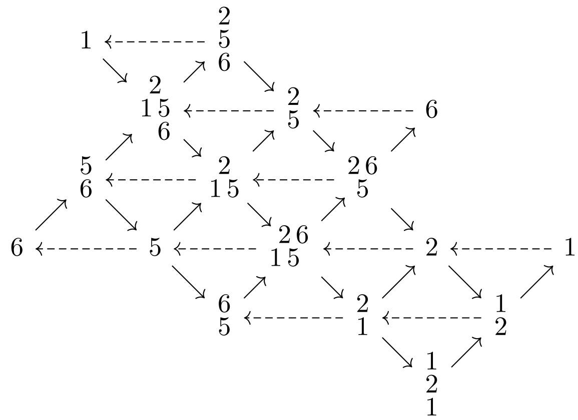
Property	A	B
Nakayama algebra	no	no
gldim A	2	∞
Indecomposable projective-injective modules	P_1	Q_1
Indecomposable strongly projective-injective modules	none	Q_1
domdim A	0	0
Nodes	none	S_1, S_6

The Auslander-Reiten quivers of A and B can be written as follows.

Auslander-Reiten quiver of A



Auslander-Reiten quiver of B



Example 7.13. We discuss two short exact sequences which are not perfect exact.

Consider the following short exact sequence in $\operatorname{mod} A$.

$$\eta_1: \qquad 0 \to 5 \to \begin{smallmatrix}2\,4\\5\end{smallmatrix} \to \begin{smallmatrix}2\\4\end{smallmatrix} \to 0$$

Using Lemma 2.13, we can show that η_1 is not a perfect exact sequence. The morphism $5 \to \begin{smallmatrix}2\,4\\5\end{smallmatrix}$ factors through the projective module P_4. However, P_4 is not a direct summand of the middle term of η_1.

A characterization using the left morphism of the short exact sequence does exist for Nakayama algebras; cf. Lemma 2.14. In general, there are short exact sequences whose left morphism is non-zero in $\underline{\operatorname{mod}}\, A$ but which are not perfect exact. The following short exact sequence is an example of this.

$$\eta_2: \qquad 0 \to 5 \to \begin{smallmatrix}2\\3\,5\end{smallmatrix} \to \begin{smallmatrix}2\\3\end{smallmatrix} \to 0$$

We see that η_2 is not perfect exact since $(\begin{smallmatrix}2\\3\,5\end{smallmatrix}^* \to 5^*) \simeq (1 \to 4)$ is not surjective. In contrast to η_1, the morphism $5 \to \begin{smallmatrix}2\\3\,5\end{smallmatrix}$ does not factor through a projective module.

Example 7.14. The non-projective simple modules of A are not a generating system of $G_0^{\mathcal{P}}(A)$. In fact, consider the module $X := \begin{smallmatrix}2\\5\\6\end{smallmatrix}$. Since X is injective, every short exact sequence starting in X is split. Furthermore, every morphism ending in X factors through the projective module P_2. Thus, the only non-split short exact sequence with X as middle or ending term is

$$0 \to S_3 \to P_2 \to X \to 0.$$

However, this is not a perfect exact sequence since $P_3 = S_3$ is projective.

In conclusion, the class of X must be an element of every generating system of $G_0^{\mathcal{P}}(A)$; see also Remark 4.19. In particular, $G_0^{\mathcal{P}}(A)$ is non-zero.

We compare this with the group $G_0^{\mathcal{P}}(B)$. In $\operatorname{mod} B$, we have the following almost split sequence ending in $X' = \begin{smallmatrix}2\\5\\6\end{smallmatrix}$.

$$0 \to S_1 \to Q_2 \to X' \to 0$$

This time, the starting term S_1 is not projective in $\operatorname{mod} B$ so that this is a perfect exact sequence. Therefore, we have $[X'] = -[S_1]$ in $G_0^{\mathcal{P}}(B)$.

Example 7.15. We detail some of the differences between $\operatorname{mod} A$ and $\operatorname{mod} B$. Let α be the stable equivalence $\underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$ induced by the Auslander-Reiten quivers above. As a consequence of the previous example, α cannot induce an isomorphism between $G_0^{\mathcal{P}}(A)$ and $G_0^{\mathcal{P}}(B)$. This is because S_1 is a node in $\operatorname{mod} B$.

Consider the following perfect exact sequence in $\operatorname{mod} B$. In fact, this is an almost split sequence with non-projective starting term; cf. Example 2.11.

$$0 \to 6 \to \begin{smallmatrix}5\\6\end{smallmatrix} \to 5 \to 0$$

However, the simple module S_6 in $\operatorname{mod} B$ is a node. Thus, the stable equivalence α does not preserve this perfect exact sequence. In fact, there is no short exact sequence starting in $S_4 \in \operatorname{mod} A$, where $\alpha(S_4) \overset{\text{st}}{\simeq} S_6 \in \underline{\operatorname{mod}}\, B$.

Since $\operatorname{gldim} A < \infty$, the category $\mathcal{L}_A$ consists of bounded complexes. On the other hand, we have $\operatorname{gldim} B = \infty$ so that $\mathcal{L}_B$ contains some unbounded complexes. In particular, there cannot exist an equivalence $\mathcal{L}_A \to \mathcal{L}_B$ which is induced by an exact functor $\operatorname{mod} A \to \operatorname{mod} B$. Moreover, A has no strongly projective-injective modules, whereas $\mathcal{P}_B = \operatorname{stp} B$ is non-zero. Finally, there is a totally acyclic complex $\cdots \to Q_5 \to Q_6 \to Q_5 \to Q_6 \to \cdots$ in $\operatorname{mod} B$ while there cannot exist a totally acyclic complex in $\operatorname{mod} A$. In summary, we have the following two chains of subcategories, where all inclusions are proper.

$$\begin{aligned}
&0 = \mathcal{K}_{\text{tac}}(\operatorname{proj} A) \subset \mathcal{L}_A \subset \mathcal{H}_{\mathcal{P}}(\operatorname{proj} A) \subset \mathcal{H}_{\text{stp}}(\operatorname{proj} A) = \mathcal{H}(\operatorname{proj} A) \simeq \mathcal{K}^b(\operatorname{proj} A)\\
&0 \subset \mathcal{K}_{\text{tac}}(\operatorname{proj} B) \subset \mathcal{L}_B \subset \mathcal{H}_{\mathcal{P}}(\operatorname{proj} B) = \mathcal{H}_{\text{stp}}(\operatorname{proj} B) \subset \mathcal{H}(\operatorname{proj} B)
\end{aligned}$$

In particular, α does not induce an equivalence between $\mathcal{K}_{\text{tac}}(\operatorname{proj} A)$ and $\mathcal{K}_{\text{tac}}(\operatorname{proj} B)$.

Example 7.16. In Chapter 6, we have seen how to construct stably equivalent algebras by deleting nodes or by gluing a simple projective vertex and a simple injective vertex. We aim to list all algebras which can be obtained from A or B by repeating these steps for a finite number of times in any order. Note that over an algebraically closed field, the Auslander-Reiten conjecture holds for A and B since both are of finite representation type.

We start by constructing the algebra A from B as described in Definition 6.7. Let $\mathcal{N} = \{1, 6\}$

corresponding to the nodes S_1 and S_6 in $\operatorname{mod} B$. We obtain the following matrix algebra.

$$E_{\mathcal{N}}(B) = \begin{pmatrix} k & \operatorname{Hom}_B(S_1,Q_1) & 0 & \operatorname{Hom}_B(S_1,Q_6) & \operatorname{Hom}_B(S_1,Q_2) & \operatorname{Hom}_B(S_1,Q_5) \\ 0 & k & 0 & 0 & 0 & 0 \\ 0 & \operatorname{Hom}_B(S_6,Q_1) & k & \operatorname{Hom}_B(S_6,Q_6) & \operatorname{Hom}_B(S_6,Q_2) & \operatorname{Hom}_B(S_6,Q_5) \\ 0 & 0 & 0 & k & 0 & 0 \\ 0 & \operatorname{Hom}_B(Q_2,Q_1) & 0 & \operatorname{Hom}_B(Q_2,Q_6) & \operatorname{Hom}_B(Q_2,Q_2) & \operatorname{Hom}_B(Q_2,Q_5) \\ 0 & \operatorname{Hom}_B(Q_5,Q_1) & 0 & \operatorname{Hom}_B(Q_5,Q_6) & \operatorname{Hom}_B(Q_5,Q_2) & \operatorname{Hom}_B(Q_5,Q_5) \end{pmatrix}$$

As a k-vector space this is isomorphic to the following matrix.

$$E_{\mathcal{N}}(B) \simeq \begin{pmatrix} k & k & 0 & 0 & k & 0 \\ 0 & k & 0 & 0 & 0 & 0 \\ 0 & 0 & k & 0 & k & k \\ 0 & 0 & 0 & k & 0 & 0 \\ 0 & k & 0 & 0 & k & 0 \\ 0 & 0 & 0 & k & k & k \end{pmatrix}$$

We label the columns and rows of this matrix by 1 to 6. Then $E_{\mathcal{N}}(B)^{\mathrm{op}}$ is isomorphic to the quiver algebra given by the quiver

$$\begin{tikzcd} 2 \arrow[r, "\alpha"] & 5 \arrow[r, "\beta"] \arrow[d, "\varepsilon"] & 1 \\ 4 \arrow[r, "\gamma"] & 6 \arrow[r, "\delta"] & 3 \end{tikzcd}$$

with relations $\alpha\,\varepsilon = \gamma\,\delta = 0$. Thus, we recover $E_{\mathcal{N}}(B)^{\mathrm{op}} \simeq A$ as algebras. Up to equivalence, this is the unique algebra without nodes stably equivalent to B such that there is a radical embedding $B \hookrightarrow A$; cf. Lemma 6.13.

Now, we construct the algebra B from A as described in Definition 6.3. Let $\mathcal{J} := \{3,\, 6\}$ and $\sigma : \mathcal{J} \to \{1, 4\}$ with $3\sigma = 1$ and $6\sigma = 4$. This corresponds to gluing the simple projective vertex 3 with the simple injective vertex 1 and the simple projective vertex 6 with the simple injective vertex 4. We obtain the following matrix algebra.

$$E_\sigma(A) = \begin{pmatrix} k \oplus \operatorname{Hom}_A(P_3,P_1) & \operatorname{Hom}_A(P_3,P_4) & \operatorname{Hom}_A(P_3,P_2) & \operatorname{Hom}_A(P_3,P_5) \\ \operatorname{Hom}_A(P_6,P_1) & k \oplus \operatorname{Hom}_A(P_6,P_4) & \operatorname{Hom}_A(P_6,P_2) & \operatorname{Hom}_A(P_6,P_5) \\ \operatorname{Hom}_A(P_2,P_1) & \operatorname{Hom}_A(P_2,P_4) & \operatorname{Hom}_A(P_2,P_2) & \operatorname{Hom}_A(P_2,P_5) \\ \operatorname{Hom}_A(P_5,P_1) & \operatorname{Hom}_A(P_5,P_4) & \operatorname{Hom}_A(P_5,P_2) & \operatorname{Hom}_A(P_5,P_5) \end{pmatrix}$$

As a k-vector space this is isomorphic to the following matrix.

$$E_\sigma(A) \simeq \begin{pmatrix} k^2 & 0 & k & 0 \\ 0 & k & k & k \\ k & 0 & k & 0 \\ 0 & k & k & k \end{pmatrix}$$

We label the columns and rows of this matrix by 1 to 4. Then $E_\sigma(A)^{\mathrm{op}}$ is isomorphic to the quiver algebra given by the quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 3 \arrow[l, shift left, "\beta"] \arrow[r, "\varepsilon"] & 2 \arrow[r, shift left, "\gamma"] & 4 \arrow[l, shift left, "\delta"] \end{tikzcd}$$

with relations $\alpha\,\varepsilon = \gamma\,\delta = \beta\,\alpha = \delta\,\gamma = 0$. Thus, we recover $E_\sigma(A)^{\mathrm{op}} \simeq B$ as algebras.

The following are all other possible choices for $\mathcal{J}$ and σ together with the resulting algebra $E_\sigma(A)^{\mathrm{op}}$. In each case, we only give the corresponding quiver and relations.

$\mathcal{J} := \{3,\, 6\}$ with $3\sigma = 4$ and $6\sigma = 1$:

$$\begin{tikzcd} 2 \arrow[r, "\alpha"] & 3 \arrow[r, "\beta"] \arrow[d, "\varepsilon"] & 1 \arrow[dl, "\gamma"] \\ & 4 \arrow[ul, "\delta"] & \end{tikzcd}$$

with relations $\alpha\,\varepsilon = \gamma\,\delta = \beta\,\gamma = \delta\,\alpha = 0$

$\mathcal{J} := \{3\}$ with $3\sigma = 1$:

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"] \arrow[d, "\varepsilon"] & \\ 3 \arrow[r, "\gamma"] & 4 \arrow[r, "\delta"] & 5 \end{tikzcd}$$

with relations $\alpha\,\varepsilon = \gamma\,\delta = \beta\,\alpha = 0$

$\mathcal{J} := \{3\}$ with $3\sigma = 4$:

$$\begin{tikzcd} 2 \arrow[r, "\alpha"] & 3 \arrow[d, "\varepsilon"] \arrow[dl, "\beta"'] & \\ 1 \arrow[r, "\gamma"] & 4 \arrow[r, "\delta"] & 5 \end{tikzcd}$$

with relations $\alpha\,\varepsilon = \gamma\,\delta = \beta\,\gamma = 0$

$\mathcal{J} := \{6\}$ with $6\sigma = 1$:

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] \arrow[d, "\varepsilon"] & 3 \\ 4 \arrow[r, "\gamma"] & 5 \arrow[ul, "\delta"] & \end{tikzcd}$$

with relations $\alpha\,\varepsilon = \gamma\,\delta = \delta\,\alpha = 0$

$\mathcal{J} := \{6\}$ with $6\sigma = 4$:

$$\begin{tikzcd} 2 \arrow[r, "\alpha"] & 3 \arrow[r, "\beta"] \arrow[d, "\varepsilon"] & 4 \\ & 5 \arrow[r, shift left, "\delta"] & 1 \arrow[l, shift left, "\gamma"] \end{tikzcd}$$

with relations $\alpha\,\varepsilon = \gamma\,\delta = \delta\,\gamma = 0$

By Remark 6.14, this is a complete list of all algebras C that are stably equivalent to A and B such that C is obtained from A or B by a finite number of steps of either deleting a node or gluing a simple projective vertex and a simple injective vertex in any order.

Bibliography

[1] M. Amasaki. Homogeneous prime ideals and graded modules fitting into long Bourbaki sequences. *J. Pure Appl. Algebra*, 162(1):1–21, 2001.

[2] H. Asashiba. On a lift of an individual stable equivalence to a standard derived equivalence for representation-finite self-injective algebras. *Algebr. Represent. Theory*, 6(4):427–447, 2003.

[3] M. Auslander and M. Kleiner. Adjoint functors and an extension of the Green correspondence for group representations. *Adv. Math.*, 104(2):297–314, 1994.

[4] M. Auslander and I. Reiten. Representation theory of artin algebras. VI. A functorial approach to almost split sequences. *Comm. Algebra*, 6(3):257–300, 1978.

[5] M. Auslander and I. Reiten. On a theorem of E. Green on the dual of the transpose. In *Representations of finite-dimensional algebras (Tsukuba, 1990)*, volume 11 of *CMS Conf. Proc.*, pages 53–65. Amer. Math. Soc., Providence, RI, 1991.

[6] M. Auslander, I. Reiten, and S. Smalø. *Representation theory of Artin algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. Corrected reprint of the 1995 original.

[7] M. Broué. Equivalences of blocks of group algebras. In *Finite-dimensional algebras and related topics (Ottawa, ON, 1992)*, volume 424 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 1–26. Kluwer Acad. Publ., Dordrecht, 1994.

[8] R.O. Buchweitz. Maximal Cohen-Macaulay modules and Tate Cohomology over Gorenstein rings. 1987.

[9] C. Chaio and S. Liu. A note on the radical of a module category. *Comm. Algebra*, 41(12):4419–4424, 2013.

[10] H. Chen, S. Pan, and C. Xi. Inductions and restrictions for stable equivalences of Morita type. *J. Pure Appl. Algebra*, 216(3):643–661, 2012.

[11] X. Chen. Gorenstein homological algebra of artin algebras. *arXiv:1712.04587*, 2017.

[12] A. Dugas. Tilting mutation of weakly symmetric algebras and stable equivalence. *Algebr. Represent. Theory*, 17(3):863–884, 2014.

[13] A. S. Dugas and R. Martínez-Villa. A note on stable equivalences of Morita type. *J. Pure Appl. Algebra*, 208(2):421–433, 2007.

[14] M. Fang, W. Hu, and S. Koenig. On derived equivalences and homological dimensions. *Journal fuer die reine und angewandte Mathematik*, 2021(770):59 – 85, 01 Jan. 2021.

[15] D. Happel. Auslander-Reiten triangles in derived categories of finite-dimensional algebras. *Proc. Amer. Math. Soc.*, 112(3):641–648, 1991.

[16] W. Hu and C. Xi. Derived equivalences and stable equivalences of Morita type, II. *Rev. Mat. Iberoam.*, 34(1):59–110, 2018.

[17] P. Jørgensen. The homotopy category of complexes of projective modules. *Adv. Math.*, 193(1):223–232, 2005.

[18] K. Kato. Stable module theory with kernels. *Math. J. Okayama Univ.*, 43:31–41, 2001.

[19] K. Kato. Morphisms represented by monomorphisms. *J. Pure Appl. Algebra*, 208(1):261–283, 2007.

[20] B. Keller. Deriving DG categories. *Ann. Sci. École Norm. Sup. (4)*, 27(1):63–102, 1994.

[21] B. Keller and D. Vossieck. Sous les catégories dérivées. *C. R. Acad. Sci. Paris Sér. I Math.*, 305(6):225–228, 1987.

[22] S. Koenig and Y. Liu. Gluing of idempotents, radical embeddings and two classes of stable equivalences. *J. Algebra*, 319(12):5144–5164, 2008.

[23] S. Koenig and Y. Liu. Simple-minded systems in stable module categories. *Q. J. Math.*, 63(3):653–674, 2012.

[24] H. Krause. Representation type and stable equivalence of Morita type for finite-dimensional algebras. *Math. Z.*, 229(4):601–606, 1998.

[25] H. Krause. The stable derived category of a Noetherian scheme. *Compos. Math.*, 141(5):1128–1162, 2005.

[26] Y. Liu. On stable equivalences of Morita type for finite dimensional algebras. *Proc. Amer. Math. Soc.*, 131(9):2657–2662, 2003.

[27] Y. Liu. On stable equivalences induced by exact functors. *Proc. Amer. Math. Soc.*, 134(6):1605–1613, 2006.

[28] Y. Liu and C. Xi. Constructions of stable equivalences of Morita type for finite dimensional algebras. II. *Math. Z.*, 251(1):21–39, 2005.

[29] Y. Liu and C. Xi. Construction of stable equivalences of Morita type for finite-dimensional algebras. I. *Trans. Amer. Math. Soc.*, 358(6):2537–2560, 2006.

[30] Y. Liu and C. Xi. Constructions of stable equivalences of Morita type for finite-dimensional algebras. III. *J. Lond. Math. Soc. (2)*, 76(3):567–585, 2007.

[31] R. Martínez-Villa. Algebras stably equivalent to l-hereditary. In *Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979)*, volume 832 of *Lecture Notes in Math.*, pages 396–431. Springer, Berlin, 1980.

[32] R. Martínez-Villa. The stable equivalence for algebras of finite representation type. *Comm. Algebra*, 13(5):991–1018, 1985.

[33] R. Martínez-Villa. Properties that are left invariant under stable equivalence. *Comm. Algebra*, 18(12):4141–4169, 1990.

[34] R. Martínez-Villa and G. Montaño Bermúdez. Node deletion and stably equivalent Koszul algebras. *Comm. Algebra*, 33(8):2569–2585, 2005.

[35] Z. Pogorzaly. A new invariant of stable equivalences of Morita type. *Proc. Amer. Math. Soc.*, 131(2):343–349, 2003.

[36] J. Rickard. Derived categories and stable equivalence. *J. Pure Appl. Algebra*, 61(3):303–317, 1989.

[37] J. Rickard. Morita theory for derived categories. *J. London Math. Soc. (2)*, 39(3):436–456, 1989.

[38] J. Rickard. Derived equivalences as derived functors. *J. London Math. Soc. (2)*, 43(1):37–48, 1991.

[39] J. Rickard. Some recent advances in modular representation theory. In *Algebras and modules, I (Trondheim, 1996)*, volume 23 of *CMS Conf. Proc.*, pages 157–178. Amer. Math. Soc., Providence, RI, 1998.

[40] H. Tachikawa and T. Wakamatsu. Cartan matrices and Grothendieck groups of stable categories. *J. Algebra*, 144(2):390–398, 1991.

[41] C. Xi. On the finitistic dimension conjecture. I. Related to representation-finite algebras. *J. Pure Appl. Algebra*, 193(1-3):287–305, 2004.

[42] C. Xi. Stable equivalences of adjoint type. *Forum Math.*, 20(1):81–97, 2008.

[43] Y. Yoshino. Maximal Buchsbaum modules of finite projective dimension. *J. Algebra*, 159(1):240–264, 1993.

[44] Y. Yoshino. Homotopy categories of unbounded complexes of projective modules. *arXiv:1805.05705v4*, 2019.

[45] A. Zimmermann. *Representation theory. A homological algebra point of view*, volume 19 of *Algebra and Applications*. Springer, Cham, 2014.

List of Symbols

Symbol	Description	Location
$(-)^*$	Functor $\mathrm{Hom}_A(-, A) : \mathrm{mod}\, A \to A\text{-mod}$	Section 1.1
$\underline{\mathrm{Cok}}(f)$	Pseudo-cokernel of a morphism f in $\mathrm{mod}\, A$ defined as $\underline{\mathrm{Cok}}(f) = \mathrm{H}^0(\tau_{\leqslant 0}\, C(f)^\bullet)$ with $f^\bullet = \mathcal{F}(f)$	Definition 2.8
$\mathcal{C}(\mathrm{mod}\, A)$	Category of complexes	Section 1.3
$C(f)^\bullet$	Mapping cone of a morphism $f^\bullet$ in $\mathcal{C}(\mathrm{mod}\, A)$	Definition 1.8
$\mathrm{depth}\, f$	Depth of a morphism f in $\mathrm{mod}\, A$	Definition 3.13
$\mathcal{D}(\mathrm{mod}\, A)$	Derived category	Section 1.3
$\mathrm{domdim}\, A$	Dominant dimension of A	Definition 1.16
$\mathcal{D}_{\mathrm{sg}}(A)$	Singularity category	Remark 4.43
D	Duality $\mathrm{Hom}_k(-, k) : \mathrm{mod}\, A \to A\text{-mod}$	Section 1.1
E_X	Middle term of the almost split sequence starting in an A-module X	Chapter 3
$E_{\mathcal{N}}(A)$	Matrix algebra obtained from A by deleting nodes	Definition 6.7
$E_\sigma(A)$	Matrix algebra obtained from A by gluing idempotents	Definition 6.3
$\mathcal{F}$	Equivalence $\mathcal{F} : \underline{\mathrm{mod}}\, A \to \mathcal{L}_A$	Theorem 2.6
$f^\bullet : F_X^\bullet \to F_Y^\bullet$	Image in $\mathcal{K}(\mathrm{proj}\, A)$ under $\mathcal{F}$ of $f : X \to Y$ in $\mathrm{mod}\, A$	Remark 2.5
$\mathrm{Gproj}\, A$	Category of Gorenstein-projective modules	Definition 4.36
$\underline{\mathrm{Gproj}}\, A$	Stable category of Gorenstein-projective modules	Definition 4.36
$G_0(\mathcal{T})$	Grothendieck group of a triangulated category $\mathcal{T}$	Definition 4.15
$G_0^{\mathcal{P}}(A)$	Alternative Grothendieck group of A defined via perfect exact sequences	Definition 4.14
$G_0^{\mathrm{st}}(A)$	Stable Grothendieck group of A	Definition 4.14

$\mathrm{H}^k(F^\bullet)$	Cohomology of a complex $F^\bullet$ in degree k	Section 1.3
$\mathcal{H}(\operatorname{proj} A)$	Full subcategory of $\mathcal{K}(\operatorname{proj} A)$ with objects $F^\bullet$ such that there exist $l, r \in \mathbb{Z}$ with $\mathrm{H}^{<l}(F^\bullet) = 0$ and $\mathrm{H}_{\geqslant r}(F^*_\bullet) = 0$	Definition 4.1
$\mathcal{H}_{\mathcal{P}}(\operatorname{proj} A)$	Full subcategory of $\mathcal{H}(\operatorname{proj} A)$ with objects in ${}^\perp\mathcal{K}^b(\mathcal{P}_A)$	Definition 4.1
$\mathcal{H}_{\mathrm{stp}}(\operatorname{proj} A)$	Full subcategory of $\mathcal{H}(\operatorname{proj} A)$ with objects in ${}^\perp\mathcal{K}^b(\operatorname{stp} A)$	Definition 4.1
$\mathcal{K}(\operatorname{mod} A)$	Homotopy category	Section 1.3
$\mathcal{K}_{\mathrm{tac}}(\operatorname{proj} A)$	Homotopy category of totally acyclic complexes	Definition 1.7
$\underline{\mathrm{Ker}}(f)$	Pseudo-kernel of a morphism f in $\operatorname{mod} A$ defined as $\underline{\mathrm{Ker}}(f) = \mathrm{H}^0\big(\tau_{\leqslant 0}\,(C(f)^\bullet[-1])\big)$	Definition 2.8
$\mathcal{L}_A$	Essential image of the functor $\mathcal{F} : \underline{\mathrm{mod}} \to \mathcal{K}(\operatorname{proj} A)$	Definition 2.1
$l(X)$	Composition length of an A-module X	Section 1.1
$\underline{\mathrm{mod}}\, A$	Stable module category	Definition 1.1
ν_A	Nakayama functor $\big(\mathrm{D}(-)\big)^* : \operatorname{mod} A \to \operatorname{mod} A$	Section 1.1
$\nu_{\mathcal{K}}$	Functor $\nu_{\mathcal{K}} : \mathcal{K}^{+,b^*}(\operatorname{proj} A) \to \mathcal{K}^{-,b}(\operatorname{proj} A)$ which is induced by the Nakayama functor	Definition 4.22
ν-domdim A	ν-dominant dimension of A	Definition 1.16
$\mathcal{P}_A$	Subcategory of projective-injective A-modules	Section 1.4
${}^\perp\mathcal{S}$	Full subcategory of $\mathcal{A}$ with objects X such that $\mathrm{Hom}_{\mathcal{A}}(X^\bullet, Y^\bullet) = 0$ for $Y^\bullet \in \mathcal{S}$	Section 1.1
$\mathrm{rad}_A(X, Y)$	Morphisms in the radical of $\operatorname{mod} A$	Definition 3.13
$\operatorname{stp} A$	Subcategory of strongly projective-injective A-modules	Definition 1.15
$T(X)$	Ending term in the direct sum of all almost split sequences starting in an A-module X	Remark 3.9
$\mathrm{Tr}(X)$	Transpose of an A-module X	Section 1.1
$\tau_{\leqslant k}$	Brutal truncation of a complex	Section 1.3
$\mathscr{X}_A$	Full subcategory of $\mathcal{K}^b(\operatorname{stp} A)$ with objects $P^\bullet$ such that $P^\bullet \simeq \nu_A(P^\bullet)$	Proposition 4.33